ON THE FORMULATION OF ANTI-LINEAR QUANTIZED FIELDS

by

PIYADASA WIMALASENA EPASINGHE

A thesis presented for the Degree of Doctor of Philosophy of The University of London and for the Diploma of Imperial College.

Department of Theoretical Physics,
Imperial College of Science and Technology,
London S.W. 7.

August 1965.

ABSTRACT

In Chapter 1, anti-linear operators in a complex Hilbert space are defined, and a theory of such operators is formulated on the same lines as that of linear operators. The theorems established for anti-linear operators are remarkably similar to the corresponding ones for linear operators. The most important result in this Chapter is the existence of an anti-automorphism, called "transposition" in the Algebra of Anti-Linear Operators, which is the counterpart of "adjoint" for Linear Operators. The entire procedure is co-ordinate independent.

Anti-linear Field Operators are introduced in Chapter 2, and axioms set up for a Theory of Anti-linear Fields. The axioms chosen are the same as those for Linear Fields, as given by Streater and Wightman.[14] The consequences due to the axioms are worked out using the Vacuum Expectation Functionals of the Theory. These come out as similar to those for Linear Fields in some cases, but in several cases, the differences are important and decisive, specially the spectral conditions.

In the first part of Chapter 3, a Reconstruction Theorem for a scalar symmetric field is proved whereby the Vacuum Expectation Functionals are shown to determine the field operator uniquely up to unitary equivalence. The method of establishing the theorem for a theory of several

multi-component fields is outlined in the second part of this Chapter.

The main result of this thesis is the last theorem of Chapter 4. A Theory of Free local Anti-linear Fields is defined, and the properties of the Vacuum Expectation Functionals which were derived in Chapter 2 are used together with the Reconstruction Theorem to prove that all the field components, in a Theory of Free local Fields must vanish irrespective of spin.

PREFACE

The research presented in this thesis was carried out in the Department of Theoretical Physics, Imperial College of Science and Technology, London, between the periods October 1961 to November 1962 and October 1963 to July 1965 under the supervision of Professor P.T. Matthews. The author wishes to express his gratitude to Professor P.T. Matthews, Professor A. Salam and Dr. R.F. Streater for their continued help, understanding, guidance and encouragement.

Except where stated in the text, the work described is original and has not been submitted for any other degree in this or any other University. The author wishes to thank all members of the Theoretical Physics Group who in some way contributed to the author's understanding of the subject. He also wishes to thank the University of Ceylon, Colombo, for granting him leave and acknowledges gratefully the allowance paid by them during his entire stay at Imperial College.

CONTENTS

	Page
ABSTRACT	1
PREFACE	3
CONTENTS	4
INTRODUCTION	5
CHAPTER 1.	13
Anti-linear Operators in a Hilbert Space	13
Transpose of an Anti-Linear Operator	30
Algebra of Anti-Linear Operators	33
Representation With Respect to a Basis in $\mathcal{H}$	37
Transformation under a Change of Basis	41
CHAPTER 2.	44
Anti-Linear Field Operators	44
Properties of the Vacuum Expectation Functionals	53
CHAPTER 3.	77
Reconstruction Theorem	77
Completion of the Space Z/Z_o	108
Reconstruction Theorem in a Theory with Several Fields each field having an Arbitrary number of components.	125
CHAPTER 4.	132
REFERENCES	162

INTRODUCTION

In any text book on Quantum Theory of Fields one would find that a basic assumption that is made is that the fields are linear operators in a complete, separable Hilbert Space $\mathcal{H}$. Although observables in a Field Theory are described by Hermitian, and therefore linear operators, one does not have to use only Hermitian Operators as field operators. This is evident from the fact that a spinless charged particle can be described by a complex (non-Hermitian) scalar field. The moment we try to have fields corresponding to half odd integer spin particles, it is found that it is necessary to have non-Hermitian, therefore non-observable fields.

Now, if we carry this argument one step further, we could say that a Quantum Theory of Fields could be set up using non-observable fields, provided that the algebra of the fields used contains linear Hermitian Operators. In this thesis, we are going to consider the possibility of setting up a Field Theory using Anti-linear Fields. For each $h \epsilon \mathcal{S}(R^4)$, $\varphi(h)$ is defined to be an Anti-linear Field, if $\varphi(h)\ (\alpha x + \beta y) = \bar{\alpha}\ \varphi(h)\ x + \bar{\beta}\ \varphi(h)\ y$, and $\varphi(\alpha h^{(1)} + \beta h^{(2)}) = \alpha\varphi(h^{(1)}) + \beta\varphi(h^{(2)})$ where α,β are complex numbers, x,y are elements of $\mathcal{H}$, and $h^{(1)}$, $h^{(2)}\ \epsilon\ \mathcal{S}(R^4)$, and $\bar{\alpha},\bar{\beta}$ are the complex conjugates of α,β respectively. As discussed in Chapter 1, the algebra of Anti-linear Operators contains a sub-algebra of Linear Operators which in turn contains Hermitian Operators.

Moreover, there is no 'a priori' reason as to why fields may not be anti-linear.

The distinct possibility that Anti-linear Fields may exist was first discussed by E.C.G. Stueckelberg et al.[38,39] He and his collaborators, in a series of papers,[40,41,42] discussed the problem of setting up Quantum Mechanics using a Hilbert Space over the field of Real numbers instead of the usual field of Complex Numbers. This, they argued ought to be possible, because, physics essentially deals with real numbers. They found that, if there is an operator J in the Real Hilbert Space (R.H.S.) such that J is anti-symmetric and satisfies the equation $J^2 = -1$, and if J commutes with every observable as well as with the orthogonal (corresponding to 'unitary' in a Complex Hilbert Space (C.H.S.)) representations of the ortho-chronous Lorentz Group, and if J anti-commutes with the representations of pseudo-chronous Lorentz transformations, then a Quantum Theory with an uncertainty principle could be set up. They also found that this Quantum Theory was isomorphic to the conventional Quantum Theory using a C.H.S. with the correspondence $J \leftrightarrow i$ ($i^2 = -1$ in C.H.S.).

When they went further and attempted[38] to have a Theory of Quantized Fields over a R.H.S., they found that fields fell into two classes, those which commute with J which they call fields of the 1st kind, and those which

anti-commute with J which are called fields of the 2nd kind. These fields of the 2nd kind naturally go over to anti-linear fields when one goes over to C.H.S.

In attempting to quantize fields of the 2nd kind by the usual procedure of taking commutation relations (C.R.) or anti-commutation relations (A.C.R.), they found that tensor fields (those fields which transform according to single valued representations of the Lorentz Group) gave rise to algebraic contradictions, whichever way they quantized. On the other hand, for spinor fields (those fields which transform according to the two-valued representations of the Lorentz Group), they found that (C.R)'s were again inconsistent, but that (A.C.R)'s were consistent to the extent that, they had solutions of the (A.C.R)'s.

Their procedure had three short-comings. Firstly, they had to use a Unitary Space, i.e. a finite dimensional linear vector space with a positive definite metric, and not a Hilbert Space, to make some of their manipulations mathematically meaningful. Secondly, the whole procedure is not mathematically rigorous. Thirdly, they did not see that the solutions of the A.C.R.'s violated the spectral conditions.

What we have attempted to do is to set up a mathematically rigorous theory of anti-linear fields in a

Complex Hilbert Space. For this purpose, we need the concept of "Anti-linear Operators" in a C.H.S. As this concept is new in mathematical literature, we had to study the properties of such quantities. We have done this as far as we need it in the rest of the thesis, in Chapter 1. It is a remarkable fact that almost every theorem we have established for Anti-linear Operators has a counterpart in the theory of linear operators. The most important result of this chapter is the Theorem 1.5 according to which, given any anti-linear operator A in $\mathcal{H}$ with domain D(A) dense in $\mathcal{H}$, $\exists$ a unique anti-linear operator A^t in $\mathcal{H}$ with domain $D(A^t)$, such that $(g,Af) = (f,A^tg)$, for all $f \in D(A)$ and $g \in D(A^t)$. This concept of going from A to A^t, which we call transposition corresponds to taking the adjoint of a linear operator. By Theorem 1.6, the operation of transposition is anti-automorphic. We also see that the Algebra generated by a set of Anti-linear Operators has a sub-algebra of Linear Operators containing Hermitian Operators.

In the 2nd chapter we define what we mean by an Anti-linear Field. To set up a physical theory using such fields we have to connect them with physics. This, we do, by postulating the axioms A0, A1, A2 and A3. These axioms are the same as those used by Streater and Wightman.[14] We argue that any physical theory of fields must satisfy these axioms. As it is our intention to study a Theory of any number

of multi-component fields, we do not specialize to the case of only one single-component field. Using Schwartz's Nuclear Theorem[19], we see that the Vacuum Expectation (V.E.) Functionals of the theory are tempered distributions. Then we study the consequences of the axioms on the V.E. Functionals, and find that the results are very similar to those of Linear Fields. But it must be emphasised that when they differ, the differences are important and decisive, specially the spectral condition.

As is usually done[29,30,31,32], we state and prove a Reconstruction Theorem for a Theory of a scalar symmetric field (a one-component field) in the first part of Chapter 3. In the first place, we do this, in spite of the fact that repeated attempts to construct tempered distributions satisfying those properties ended up in failure. In the second place, we give a complete proof, although the proof is on the same lines as that of Streater and Wightman[31], because of the essential differences between our theory and their linear theory, and for the sake of completeness. We also need the Reconstruction Theorem to prove that the vanishing of all the V.E. Functionals implies that the fields themselves must vanish. In the second part of this chapter, we indicate how exactly the theorem could be extended to a theory of several multi-component fields. The essential content of this chapter is that, given a set of tempered

distributions $V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots,x_\nu)$, $\nu=1,2,3,\ldots,$ $1 \leqslant \mu_1,\mu_2,\ldots,\mu_\nu \leqslant \Lambda(\Gamma)$, $\Lambda(\Gamma)$ being the total number of field components in the theory, satisfying the contents of Theorems 2.2, 2.3, 2.4, 2.5, 2.6 and 2.7, one could construct a set of fields whose V.E. Functionals are the above tempered distributions in such a way that this field theory satisfies all the axioms A0, A1, A2 and A3, and that any other theory having the same V.E. Functionals is unitary equivalent to the theory already constructed.

In the last Chapter, we restrict further, the possible fields by defining free local fields. As is customary we define free fields to be those satisfying the corresponding Klein Gordon equations. We make this concept rigorous by defining free fields as those such that the V.E. Functionals satisfy the corresponding Klein Gordon Equations. We also adopt the usual definition of locality where kinematically independent fields completely commute or completely anti-commute irrespective of whether the separation of their space-time points is space-like or not, and that kinematically dependent fields (if $\varphi(x)$ is symmetric then $\varphi(x)$, $\varphi(y)$ are kinematically dependent, whereas if $\varphi(x)$ is not symmetric, then $\varphi(x)$, $\varphi^t(y)$ are kinematically dependent) have commutators or anti-commutators which are c-number functions, vanishing as usual, when the separation of their space time points is spacelike. Since the discussion in this chapter

does not depend on how the fields transform under the Lorentz Group but only on how they transform under the translation subgroup of the Poincare Group, the concept of spin enters only through the (anti)-commutation relations. We see in Theorem 4.4, that C.R's ($\varepsilon = -1$) between symmetric field components give rise to algebraic contradictions unless their 2-point functions vanish, and that A.C.R's ($\varepsilon = +1$) are algebraically consistent but contradict the spectral conditions, again unless the 2-point functions vanish. Subsequently, we show that all field components in a Theory of Free Local Fields must vanish irrespective of whether C.R's or A.C.R's are used to define locality.

We would like to comment that, by defining Free Fields what we have essentially done is to have removed the origin from the support of $\tilde{W}^{(\nu)}_{\mu_1\mu_2\ldots\ldots\mu_\nu}(p_1,p_2,\ldots.p_{\nu-1})$. That is why we claim that a suitable definition of truncation would allow us to prove the non-existence of anti-linear fields without using the K.G. equation or locality, but by using only Microscopic Causality and the Jost Lehman Dyson[43] representation of a Causal Commutator. We also have made two assumptions which we were unable to prove. One is that there is only one state which is invariant under the translation subgroup of the Poincare Group.[33] This is highly plausiable from a physical point of view, because, a state of 4-momentum p transforms with a factor of the form $e^{ip.a}$ under

a translation a, and the state would be invariant if and only if p=0, and this is the unique vacuum state. The other assumption is what we have made in Theorem 4.4, that the 4-point function of a symmetric field component in a Theory of Free Local Fields can be expressed using its 2-point functions. This again seems to be plausible due to equations (4.8), (4.9) and (4.10). Both these results are true in a theory of Free Linear Fields.

CHAPTER 1

ANTI-LINEAR OPERATORS IN A HILBERT SPACE

It is convenient to start very generally with a Banach space instead of a Hilbert Space. We begin by defining a Banach Space.

Definition 1.1. A linear vector space X over the field of complex numbers is called a Banach Space,[44] if for each $x \in X$, $\exists$ a non-negative real number $||x||$ called the norm of x, such that

(i) $||x|| \geqslant 0$ for all $x \in X$, $||x|| = 0$ iff $x = 0$

(ii) $||x+y|| \leqslant ||x|| + ||y||$ for all $x, y \in X$.

(iii) $||\alpha x|| = |\alpha| . ||x||$, all $x \in X$, α any complex number.

(iv) If $\{x_\mu\}$ is a sequence of elements of X such that $||x_\mu - x_\nu|| \to 0$ as $\mu, \nu \to \infty$, then $\exists\, x \in X$ such that $||x_\mu - x|| \to 0$ as $\mu \to \infty$. (It follows that this x is unique)

In standard terminology, a Banach Space is a normed linear space over the complex field, which is also complete with respect to the metric $||x-y||$. In this chapter, simple English letters like .... $f, g, h, \ldots, x, y, z$, will always stand for elements of a Banach Space. The field of complex numbers will always be denoted by $\mathbb{C}$, and its elements by α, β, γ. $\lambda, \mu, \nu, \rho, \sigma$ will always stand for positive integers.

With this definition, a Complete Hilbert Space (defined in Ref. 2) as well as $\mathbb{C}$ are Banach Spaces, with the

following norms. If $x \in$ a Hilbert Space $\mathcal{H}$, $||x|| = (x,x)^{\frac{1}{2}}$. If $x \in \mathbb{C}$, $||x|| = |x|$.

Definition 1.2. A mapping (operator) A of a normed linear space X into a normed linear space Y is said to be anti-linear if $Ax \in Y$ for all $x \in X$, and $A(\alpha x_1 + \beta x_2) = \bar{\alpha}Ax_1 + \bar{\beta}Ax_2$ for all $x_1, x_2 \in X$, $\alpha, \beta \in \mathbb{C}$, where $\bar{\alpha}, \bar{\beta}$ are the complex conjugates of α, β respectively.

In this case, X is called the domain of A and denoted by D(A), and the set of all $Ax \in Y$ for all $x \in X$ is called the range of A and denoted R(A). Clearly R(A) is a linear space, a (not necessarily proper) subspace of Y. If R(A) = Y, then the mapping is said to be onto.

The basic difference between an anti-linear operator A and a linear operator $\tilde{L}$ (linear operators will always be denoted with the symbol ~ and 'operator' will always mean an anti-linear operator unless otherwise stated) is that if α is real and $i^2 = -1$, then $[A,\alpha]_- = [\tilde{L},\alpha]_- = [\tilde{L},i]_- = 0$, but $\{A,i\}_+ = 0$ in the usual notation.

Definition 1.3. The operator A is said to be continuous at $f \in D(A)$, if given $\varepsilon > 0$, $\exists\ \delta > 0$ such that for all $g \in D(A)$ satisfying $||f-g|| < \delta$, we have $||Af-Ag|| < \varepsilon$

Definition 1.4. The operator A is said to be bounded in D(A) if $\exists$ a positive constant C such that, for all $f \in D(A)$ satisfying $||f|| \neq 0$, $\dfrac{||Af||}{||f||} < C$.

If A is bounded, then $\frac{||Af||}{||f||}$ will clearly have a least upper bound[1] as f varies over D(A). This is denoted by $||A||$ and is called the norm of A. Without loss of generality, we can take $||A|| > 0$, because if $||A|| = 0$ ($||A||$ is necessarily non-negative), then $||Af|| = 0$ for all $f \in D(A)$, and hence $A = 0$, in D(A). Leaving out this trivial case, we have the result that

$$\frac{||Af||}{||f||} \leqslant ||A|| \quad \text{for all } f \in D(A),\ f \neq 0, \text{ and given } \varepsilon > 0,$$

$$\exists \text{ some } g \in D(A) \text{ with } g \neq 0 \text{ such that } \frac{||Ag||}{||g||} > ||A|| - \varepsilon$$

Now, we are in a position to prove the following theorems for an anti-linear operator A, which are also true for a linear operator.

Theorem 1.1 - A necessary and sufficient condition that A is continuous at every $f \in D(A)$ is that A is continuous at $0 \in D(A)$.

Proof: The necessity of the condition is trivially satisfied because $0 \in D(A)$.

To establish the sufficient condition, suppose A is continuous at $0 \in D(A)$.

Hence by definition 1.3, given any $\varepsilon > 0$, $\exists\ \delta > 0$ such that for all $h \in D(A)$ satisfying $||h|| < \delta$, we have

$||Ah|| < \varepsilon$.

Now let $f \in D(A)$. Then for all $g \in D(A)$ such that $||f-g|| < \delta$, we have $||Af-Ag|| = ||A(f-g)|| < \varepsilon$ since $h = f-g \in D(A)$. Hence, A is continuous at $f \in D(A)$, which concludes the proof of the whole theorem.

Theorem 1.2: A necessary and sufficient condition that A is continuous at every $f \in D(A)$ is that A is bounded in $D(A)$.

Proof: Suppose A is continuous at every $f \in D(A)$. Then obviously A is continuous at $0 \in D(A)$.

$\therefore$ Given $\varepsilon > 0$, $\exists\ \delta > 0$ such that for all $h \in D(A)$ satisfying $||h|| < \delta$, we have $||Ah|| < \varepsilon$.

Now take any arbitrary $f \in D(A)$ with $||f|| \neq 0$. Putting $h = \dfrac{\delta}{2||f||} f$, we see that $h \in D(A)$ and $||h|| = \dfrac{\delta}{2} < \delta$

$$\therefore ||Ah|| = ||A \cdot \frac{\delta}{2||f||} f|| = || \frac{\delta}{2||f||} Af||$$

$$= \frac{\delta}{2||f||} \cdot ||Af|| < \varepsilon$$

$$\therefore \frac{||Af||}{||f||} < \frac{2\varepsilon}{\delta}$$ which means that A is bounded in $D(A)$, since $\dfrac{2\varepsilon}{\delta}$ is independent of f.

To prove the other half of the theorem, suppose A is bounded in $D(A)$.

Then $\exists$ a constant $C > 0$ such that $\dfrac{||Ah||}{||h||} < C$ for

all $h \in D(A)$ satisfying $||h|| \neq 0$.

Let f be any arbitrary element of D(A). Given $\varepsilon > 0$, suppose $\exists$ $g \in D(A)$ such that $||f-g|| < \frac{\varepsilon}{C}$.

Then $||Af-Ag|| = ||A(f-g)|| < C||f-g|| < \varepsilon$.

i.e. given $\varepsilon > 0$, $\exists$ $\delta = \frac{\varepsilon}{C} > 0$ such that for all $g \in D(A)$ satisfying $||f-g|| < \delta$, we have $||Af-Ag|| < \varepsilon$

$\therefore$ A is continuous at f, which completes the proof.

Theorem 1.3: If A is bounded in D(A), and D(A) is not a Banach space but whose closure is $\overline{D(A)}$, then $\exists$ a unique anti-linear operator B such that (i) $D(B) = \overline{D(A)}$, (ii) B is bounded in $\overline{D(A)}$, (iii) $B = A$ in D(A), and (iv) $||B|| = ||A||$.

Proof: Let $f \in \overline{D(A)}$. Since D(A) is dense in $\overline{D(A)}$, $\exists$ a Cauchy sequence $\{f_\mu\}$ of elements of D(A) which converge to f.

i.e. given $\varepsilon > 0$, $\exists$ a positive integer μ_o such that $||f_{\mu_1} - f_{\mu_2}|| < \varepsilon$ for all $\mu_1, \mu_2 \geqslant \mu_o$, and $||f_{\mu_1} - f|| < \varepsilon$ for all $\mu_1 \geqslant \mu_o$.

Now consider the sequence $\{Af_\mu\}$. Then, $||Af_{\mu_1} - Af_{\mu_2}|| = ||A(f_{\mu_1} - f_{\mu_2})|| < C||f_{\mu_1} - f_{\mu_2}||$ since A is bounded in D(A), and $\exists$ a positive integer μ_o^1 such that $||f_{\mu_1} - f_{\mu_2}|| < \frac{\varepsilon}{C}$ for all $\mu_1, \mu_2 \geqslant \mu_o^1$

$$\therefore \quad ||\Lambda f_{\mu_1} - \Lambda f_{\mu_2}|| < \varepsilon \quad \text{for all } \mu_1, \mu_2 \geqslant \mu_o^1.$$

$\therefore \{\Lambda f_\mu\}$ is a Cauchy sequence in $R(\Lambda)$. If $\overline{R(\Lambda)}$ is the closure of $R(\Lambda)$, this sequence converges to an element $g \in \overline{R(\Lambda)}$. Now, we show that g is completely defined by Λ and f and is independent of the Cauchy sequence used to define f. For, suppose $\{f'_\mu\}$ is another Cauchy sequence of elements of $D(\Lambda)$ converging to f.

As in the previous case $\{\Lambda f'_\mu\}$ is also a Cauchy sequence of elements of $R(\Lambda)$, and hence converge to $g' \in \overline{R(\Lambda)}$.

$$\begin{aligned}
\text{Now, } ||g'-g|| &= ||g'-\Lambda f'_\mu + \Lambda f'_\mu - \Lambda f_\mu + \Lambda f_\mu - g|| \\
&\leqslant ||g'-\Lambda f'_\mu|| + ||\Lambda f'_\mu - \Lambda f_\mu|| + ||\Lambda f_\mu - g|| \\
&< ||g'-\Lambda f'_\mu|| + C||f'_\mu - f_\mu|| + ||\Lambda f_\mu - g|| \\
&= ||g'-\Lambda f'_\mu|| + C||f'_\mu - f + f - f_\mu|| + ||\Lambda f_\mu - g|| \\
&\leqslant ||g'-\Lambda f'_\mu|| + C||f'_\mu - f|| + C||f - f_\mu|| + ||\Lambda f_\mu - g|| \\
&< \frac{\varepsilon}{4} + C\,\frac{\varepsilon}{4C} + C.\frac{\varepsilon}{4C} + \frac{\varepsilon}{4} \quad \text{for all } \mu \geqslant \text{some } \mu_o'' \\
&= \varepsilon
\end{aligned}$$

$\therefore g = g'$, which proves that g is uniquely determined by f.

Let B be the mapping of f on g. i.e. $g = Bf$ Then clearly $D(B) = \overline{D(\Lambda)}$. Now, we prove that B is an anti-linear operator in $\overline{D(\Lambda)}$.

If h is also an element of $\overline{D(A)}$, $\exists$ a Cauchy sequence $\{h_\mu\}$ of element of D(A) which converges to h.

$\therefore$ $\{\alpha f_\mu + \beta h_\mu\}$ is a Cauchy sequence in D(A) which converges to $\alpha f + \beta h$ for all $\alpha, \beta \in \mathbb{C}$

$\therefore$ B($\alpha f + \beta h$) = Limit of the Cauchy Sequence $\{A(\alpha f_\mu + \beta h_\mu)\}$

$= \quad " \qquad " \qquad " \qquad \{\overline{\alpha}\, Af_\mu + \overline{\beta}\, Ah_\mu\}$

$= \overline{\alpha}Bf + \overline{\beta}Bh$

Hence B is an anti-linear operator of $\overline{D(A)}$ into $\overline{R(A)}$. Now, if $f \in D(A)$, then the sequence $\{f_\mu\}$ where $f_\mu = f$ is a Cauchy sequence in D(A) converging to f

$\therefore$ $\{Af_\mu\}$ with $f_\mu = f$ converges to Bf.

i.e. $Af = Bf$

$\therefore$ B = A in D(A)

The next thing to be proved is that B is bounded in D(B).

Let f be any element of D(B) such that $||f|| \neq 0$. Then $\exists$ a Cauchy sequence $\{f_\mu\}$ of elements of D(A) which converges to f.

Without loss of generality, we can assume that $||f_\mu|| \neq 0$ for all μ. (If $||f_\mu|| = 0$ for a finite number of

values of μ, we can drop those f_μ from the sequence, and the remaining sequence still converges to f. If $||f_\mu|| = 0$ for an infinite number of values of μ, then $||f|| = 0$ which is not the case).

Consider the sequence $\left\{ \frac{||Af_\mu||}{||f_\mu||} \right\}$.

$$\left| \frac{||Af_\mu||}{||f_\mu||} - \frac{||Bf||}{||f||} \right| = \left| \frac{||Af_\mu||}{||f_\mu||} - \frac{||Af_\mu||}{||f||} + \frac{||Af_\mu||}{||f||} - \frac{||Bf||}{||f||} \right|$$

$$\leqslant \left| \frac{||Af_\mu||}{||f_\mu||} - \frac{||Af_\mu||}{||f||} \right| + \left| \frac{||Af_\mu||}{||f||} - \frac{||Bf||}{||f||} \right|$$

$$= \frac{||Af_\mu||}{||f_\mu||} \cdot \frac{\left| ||f|| - ||f_\mu|| \right|}{||f||} + \frac{1}{||f||} \left| ||Af_\mu|| - ||Bf|| \right|$$

$$\leqslant \frac{||Af_\mu||}{||f_\mu||} \cdot \frac{||f - f_\mu||}{||f||} + \frac{1}{||f||} \left| \left| Af_\mu - Bf \right| \right|$$

$$\leqslant ||A|| \cdot \frac{1}{||f||} \cdot \frac{||f|| \cdot \varepsilon}{2||A||} + \frac{1}{2||f||} \cdot ||f|| \, \varepsilon \quad \text{for all } \mu \geqslant \text{some } \mu_o$$

because $\{f_\mu\}$ converges to f and $\{Af_\mu\}$ converges to Bf.

i.e. $\left| \frac{||Af_\mu||}{||f_\mu||} - \frac{||Bf||}{||f||} \right| < \varepsilon$ for all $\mu \geqslant \mu_o$

$\therefore \left\{ \frac{||Af_\mu||}{||f_\mu||} \right\}$ converges to $\frac{||Bf||}{||f||}$

But $\frac{||Af_\mu||}{||f_\mu||} \leqslant ||A||$ for all μ.

$\therefore \quad \frac{||Bf||}{||f||} \leqslant ||A||$

$\therefore$ B is bounded in D(B).

Further, we also will have $||B|| \leqslant ||A||$. If $||B|| \neq ||A||$, let $\varepsilon = \frac{||A||-||B||}{2} > 0$.

Then $\exists$ $f \in D(A)$ with $||f|| \neq 0$ such that $\frac{||Af||}{||f||} > ||A|| - \varepsilon$

$= \frac{||A||+||B||}{2} > ||B||$

But $Af = Bf$ since $f \in D(A)$

$\therefore \quad \frac{||Bf||}{||f||} > ||B||$ which is a contradiction

$\therefore \quad ||B|| = ||A||$.

The last thing to be proved is the uniqueness of B which goes as follows.

Suppose B_1 is another anti-linear operator with domain $\overline{D(A)}$, bounded in $\overline{D(A)}$ and equal to A in D(A). Then for each $f \in \overline{D(A)}$, $\exists$ a Cauchy sequence $\{f_\mu\}$ of

elements of $D(A)$ converging to f, and the sequence $\{B_1 f_\mu\}$ converges to $B_1 f$, since B_1 is bounded in $\overline{D(A)}$ and hence is continuous in $\overline{D(A)}$, by Theorem 1.2. But $B_1 f_\mu = A f_\mu$ since $f_\mu \in D(A)$, and $\{A f_\mu\}$ converges to Bf.

$\therefore$ $B = B_1$ in $\overline{D(A)}$. and hence B is unique. Thus, the theorem is completely established.

According to this theorem, if A is bounded in $D(A)$, and $D(A)$ is not complete, then the domain of A could be extended to cover the whole of $\overline{D(A)}$ in a unique way so as to preserve the boundedness, and hence the continuity and the norm of A.

In the particular case when A is a mapping from $\mathcal{H}$ into $\mathcal{H}$ where $\mathcal{H}$ is a Hilbert Space and $D(A)$ is not a dense subset of $\mathcal{H}$, this theorem gives the following result. To begin with, $\exists$ a unique[2] closed subspace S of $\mathcal{H}$ such that $\overline{D(A)} \perp S$ and $\mathcal{H} = \overline{D(A)} \oplus S$. Moreover, every element $f \in \mathcal{H}$ can be written in a unique fashion as $f = g + h$, where $g \in \overline{D(A)}$ $h \in S$. Hence, if B is any bounded anti-linear operator with $D(B) = S$, then we could extend the definition of A to the whole of $\mathcal{H}$ by defining $Af = Ag + Bh$ for arbitrary $f \in \mathcal{H}$, where $g \in \overline{D(A)}$ and $h \in S$ $(=D(B))$. This shows how arbitrarily the definition of A could be extended to the whole of $\mathcal{H}$, still maintaining continuity.

If $D(A)$ is not dense in $\mathcal{H}$ and A is unbounded, the definition of A could still be extended to a dense domain

in $\mathcal{H}$, as in the previous case, again in an infinite number of ways. Thus we see that we need consider only the following two possibilities.

1. A is bounded and $D(A) = \mathcal{H}$
2. A is unbounded and $D(A)$ is dense in $\mathcal{H}$.

For the further development of the theory of anti-linear operators from $\mathcal{H}$ into $\mathcal{H}$ we require the following theorem.

Theorem 1.4: Riesz's Theorem for an Anti-linear Functional

Suppose that for each $f \in \mathcal{H}$, $T(f) \in \mathbb{C}$ such that

(i) $T(\alpha g) = \bar{\alpha}\, T(g)$ for all $\alpha \in \mathbb{C}$ and all $g \in \mathcal{H}$

(ii) $T(g_1 + g_2) = T(g_1) + T(g_2)$ for all $g_1, g_2 \in \mathcal{H}$

(iii) For all $g \in \mathcal{H}$ with $||g|| \neq 0$, $\exists$ a positive constant C such that $\dfrac{|T(g)|}{||g||} < C$

Then $\exists$ a unique element $t \in \mathcal{H}$ such that $T(f) = (f,t)$ for all $f \in \mathcal{H}$.

Note: This theorem follows directly from that for Linear Functionals.[3] The proof is elementary and goes as follows.

Let $\bar{T}$ be the functional such that

$$\bar{T}(f) = \overline{T(f)} \text{ for all } f \in \mathcal{H}.$$

Then
$$\begin{aligned}
\bar{T}(\alpha f + \beta g) &= \overline{T(\alpha f + \beta g)} \\
&= \overline{T(\alpha f) + T(\beta g)} \\
&= \overline{\bar{\alpha}\, T(f) + \bar{\beta}\, T(g)} \\
&= \alpha\, \overline{T(f)} + \beta\, \overline{T(g)} \\
&= \alpha\, \bar{T}(f) + \beta\, \bar{T}(g)
\end{aligned}$$

$\therefore$ $\overline{T}$ is a linear functional on $\mathcal{H}$

Also, for all $g \in \mathcal{H}$ such that $||g|| \neq 0$,

$$\frac{|\overline{T}(g)|}{||g||} = \frac{|\overline{T(g)}|}{||g||} = \frac{|T(g)|}{||g||} < C$$

$\therefore$ $\overline{T}$ satisfies all the conditions sufficient for Riesz Theorem to hold.

$\therefore$ $\exists$ a unique element $t \in \mathcal{H}$ such that $\overline{T}(f) = (t,f)$ for all $f \in \mathcal{H}$

$\therefore$ $T(f) = (f,t)$, which proves Theorem 1,4. However, for completeness, we give a direct proof of the result following Riesz and Nagy.[3]

Proof: Since $\frac{|T(g)|}{||g||} < C$ for all $g \in \mathcal{H}$ with $||g|| \neq 0$,

$\frac{|T(g)|}{||g||}$ will have a least upper bound $||T||$[1] as g varies over $\mathcal{H}$.

But $\frac{|T(g)|}{||g||} = \left|T\left(\frac{1}{||g||}g\right)\right|$, and $\left|\left|\frac{1}{||g||}g\right|\right| = 1$

$\therefore$ $||T||$ is the least upper bound of $|T(f)|$ for all $f \in \mathcal{H}$ such that $||f|| = 1$. Without loss of generality we can take $||T|| > 0$.

(1.4.1) Now we show that there is an element $g^* \in \mathcal{H}$ such that $||g^*|| = 1$ and $T(g^*) = ||T||$.

Since $||T||$ is the least upper bound of $|T(f)|$ for all unit elements of $\mathcal{H}$, $\exists$ a unit $f_1 \in \mathcal{H}$ such that $|T(f_1)| > \frac{||T||}{2}$

If $|T(f_1)| \neq ||T||$, then $\varepsilon_2 = \frac{||T|| - |T(f_1)|}{2} > 0$

(1.4.2) Then $\exists$ a unit element $f_2 \in \mathcal{H}$ such that

$$|T(f_2)| > ||T|| - \varepsilon_2 = \frac{||T|| + |T(f_1)|}{2} > |T(f_1)|.$$

and $||T|| - |T(f_2)| < \varepsilon_2 < \frac{||T||}{2^2}$

If $|T(f_2)| \neq ||T||$, then $\varepsilon_3 = \frac{||T|| - |T(f_2)|}{2} > 0$

$\therefore$ $\exists$ a unit element $f_3 \in \mathcal{H}$ such that

(1.4.3) $$|T(f_3)| > ||T|| - \varepsilon_3 = \frac{||T|| + |T(f_2)|}{2} > |T(f_2)|$$

$\therefore$ and $||T|| - |T(f_3)| < \varepsilon_3 < \frac{||T||}{2^3}$

Proceeding in this manner, we will either have some $f_\nu \in \mathcal{H}$ of unit norm such that $|T(f_\nu)| = ||T||$, or else, we will have an infinite sequence $\{f_\mu\}$ of unit elements of $\mathcal{H}$ such that, $|T(f_{\mu+1})| > |T(f_\mu)|$,

and $0 < ||T|| - |T(f_\mu)| < \frac{||T||}{2^\mu}$, for all $\mu \geqslant 1$.

In the former case, let $T(f_\nu) = ||T||e^{i\theta}$ where θ is real. Then $g^* = e^{i\theta} f_\nu \in \mathcal{H}$, and $T(g^*) = T(e^{i\theta} f_\nu)$

$= e^{-i\theta} T(f_\nu) = ||T||$ by the property (i) of the functional T. Also $||g^*|| = ||f_\nu|| = 1$.

Hence, in this case $\exists$ a unit element $g^* \in \mathcal{H}$ such that $T(g^*) = ||T||$.

In the latter case, the sequence $\{|T(f_\mu)|\}$ is monotonic increasing and converges to $||T||$.

Since $|T(f_\mu)| > T(f_1) > \frac{||T||}{2} > 0$ for all $\mu \geqslant 2$, $T(f_\mu)$ can be uniquely expressed in the form $\rho_\mu e^{i\theta_\mu}$, where $\rho_\mu > 0$ and $-\pi < \theta_\mu \leqslant \pi$ for all $\mu \geqslant 1$.

Putting $g_\mu = e^{i\theta_\mu} f_\mu$, we have $T(g_\mu) = \rho_\mu = |T(f_\mu)|$.

$\therefore$ $\{T(g_\mu)\}$ is a real, monotonically increasing sequence converging to $||T||$.

Now, consider the sequence $\{g_\mu\}$ of elements of $\mathcal{H}$.

$$||g_{\mu_1} - g_{\mu_2}||^2 = 2||g_{\mu_1}||^2 + 2||g_{\mu_2}||^2 - ||g_{\mu_1} + g_{\mu_2}||^2$$

$$= 4 - ||g_{\mu_1} + g_{\mu_2}||^2, \text{ whereas}$$

$|T(g_{\mu_1}) + T(g_{\mu_2})| = |T(g_{\mu_1} + g_{\mu_2})| \leqslant ||T||\ ||g_{\mu_1} + g_{\mu_2}||$ showing that $||g_{\mu_1} + g_{\mu_2}||^2 \geqslant \dfrac{\left(T(g_{\mu_1}) + T(g_{\mu_2})\right)^2}{||T||^2}$

$$\therefore \quad ||g_{\mu_1} - g_{\mu_2}||^2 \leqslant 4 - \frac{\left(T(g_{\mu_1}) + T(g_{\mu_2})\right)^2}{||T||^2} \to 0 \text{ as } \mu_1, \mu_2 \to \infty$$

$\therefore$ $\{g_\mu\}$ is a Cauchy sequence in H. Since H is complete, there is $g^* \in H$ such that $||g^* - g_\mu|| \to 0$ as $\mu \to \infty$.

Now $||g^*|| = ||g^* - g_\mu + g_\mu|| \leqslant ||g^* - g_\mu|| + ||g_\mu||$

$= 1 + ||g^* - g_\mu|| \to 1$ as $\mu \to \infty$

$\therefore \quad ||g^*|| \leqslant 1$

On the other hand, $||g^*|| = ||g_\mu - g^* - g_\mu||$

$\geqslant | \, ||g_\mu|| - ||g_\mu - g^*|| \, | = |1 - ||g_\mu - g^*|| \, |$

$\to 1$ as $\mu \to \infty$ which says that $||g^*|| \geqslant 1$.

$\therefore \quad ||g^*|| = 1.$

Further $|T(g_\mu) - T(g^*)| = |T(g_\mu - g^*)| \leqslant ||T|| . ||g_\mu - g^*|| \to 0$ as $\mu \to \infty$.

$\therefore$ $\{T(g_\mu)\}$ converges to $T(g^*)$

$\therefore \quad T(g^*) = ||T||.$

(1.4.4) This establishes the result (1.4.1), that $\exists$ $g^* \in H$ where $||g^*|| = 1$ such that $T(g^*) = ||T||$.

Let $t = ||T||g^* \in H$. Then we prove that, for all $f \in H$, $T(f) = (f,t)$.

Case i : $f = g^*$

In this case, L.H.S. $= T(g^*) = ||T||$,

and the R.H.S. $= (g^*,t) = (g^*,||T||g^*) = ||T||.||g^*||^2 = ||T||$.

Hence $T(f) = (f,t)$.

Case ii : Suppose f satisfies $T(f) = 0$

If $||f|| = 0$, the result is obvious.

If $||f|| \neq 0$, $||T||^2 = (T(g^*) - \lambda T(f))^2$ for all λ

$$= (T(g^* - \bar{\lambda}f))^2$$

$$\leqslant ||T||^2 \, ||g^* - \bar{\lambda}f||^2$$

$$\therefore \quad ||g^* - \bar{\lambda}f||^2 \geqslant 1 \text{ for all } \lambda$$

i.e. $||g^*||^2 + \bar{\lambda}\lambda||f||^2 - \bar{\lambda}(g^*,f) - \lambda(f,g^*) \geqslant 1$

i.e. $\lambda\bar{\lambda}||f||^2 - \lambda(f,g^*) - \bar{\lambda}(g^*,f) \geqslant 0$ for all λ

Putting $\lambda = \frac{(g^*,f)}{||f||^2}$, we get

$$\frac{(g^*,f).(f,g^*)}{||f||^2}.\frac{}{||f||^2} ||f||^2 - \frac{(g^*,f)}{||f||^2}.(f,g^*) - \frac{(f,g^*)}{||f||^2}(g^*,f) \geqslant 0$$

i.e. $$-\frac{|(f,g^*)|^2}{||f||^2} \geqslant 0$$

$$\therefore \quad (f,g^*) = 0$$

$\therefore$ $(f,t) = (f,||T||g^*) = ||T||\ (f,g^*) = 0 = T(f)$

$\therefore$ $T(f) = (f,t)$ for all f satisfying $T(f) = 0$.

Case iii : Now, take any arbitrary $f \in H$

Then, $T\left(f - \frac{\overline{T(f)}}{||T||} g^*\right) = T(f) - T\left(\frac{\overline{T(f)}}{||T||} g^*\right)$ by property (ii) of T

$= T(f) - \frac{T(f)}{||T||}\ T(g^*)$ by property (i) of T

$= 0$

$\therefore$ By case (ii) $(f - \frac{\overline{T(f)}}{||T||} g^*, t) = 0$

i.e. $(f,t) - \frac{T(f)}{||T||}(g^*,t) = 0$

i.e. $(f,t) - \frac{T(f)}{||T||} \cdot (g^*,||T||g^*) = 0$

i.e. $T(f) = (f,t)$

Hence, for all $f \in H$, $T(f) = (f,t)$

To prove the uniqueness of t, suppose $T(f) = (f,t_1)$ for all $f \in H$ and some $t_1 \in H$.

$\therefore$ $(f,t-t_1) = 0$ for all $f \in H$

$\therefore$ $t = t_1$, which completes the proof of Riesz's Theorem.

TRANSPOSE OF AN ANTILINEAR OPERATOR.

If A is a given anti-linear operator from $\mathcal{H}$ into $\mathcal{H}$, by the discussion at the end of Theorem 1.3, its domain $D(A)$ can be assumed to be dense in $\mathcal{H}$. If A is bounded, then we can take $D(A) = \mathcal{H}$. We will not assume that A is bounded or unbounded in the following.

Let f be an arbitrary element of $D(A)$. Then (g, Af) is defined for all $g \in \mathcal{H}$. For convenience, we denote $(g, Af) = T_g(f) = S_f(g)$.

For fixed $g \in \mathcal{H}$, T_g can be regarded as an anti-linear operator which maps the normed linear space $D(A)$ into the Banach Space $\mathbb{C}$. On the other hand, for fixed $f \in D(A)$, S_f also defines an anti-linear mapping of $\mathcal{H}$ into $\mathbb{C}$.

If A is bounded in $D(A)$, and hence in $\mathcal{H}$, then using the relation $|(f,g)| \leqslant ||f||.||g||$ for all $f, g \in \mathcal{H}$, we see that T_g is continuous, at all f in $D(A)$ for fixed $g \in \mathcal{H}$, and S_f is continuous at all $g \in \mathcal{H}$ for fixed $f \in D(A)$. If A is unbounded, then T_g need not be continuous at all $f \in D(A)$ for fixed $g \in \mathcal{H}$, but S_f is necessarily continuous at all $g \in \mathcal{H}$ for fixed $f \in D(A)$.

Let D_f^t be the set of all $g \in \mathcal{H}$ such that T_g is continuous at $f \in D(A)$. Clearly $D_f^t = D_o^t$, and denoting each by D^t, we have the result that T_g is continuous at each $f \in D(A)$ for all $g \in D^t$. It is quite possible that D^t contains only the element 0. Leaving this aside as an

uninteresting case, we can easily see that D^t is a non-trivial linear subspace of $\mathcal{H}$.

So far T_g is meaningful only on $f \in D(A)$. Now, we show that this definition could be extended to include any f in the whole of $\mathcal{H}$ in a unique way, by using the continuity properties of $T_g(f)$.

For fixed $g \in D^t$, T_g is continuous at all $f \in D(A)$ and hence by Theorem 1.2, T_g is bounded in $D(A)$. Therefore, by Theorem 1.3, T_g could be uniquely defined on every element of $\mathcal{H}$, so as to be continuous at all $f \in \mathcal{H}$.

Hence, for each $g \in D^t$ and each $f \in \mathcal{H}$, $T_g(f)$ is a complex number, which obviously satisfies the following conditions:

(i) $T_g(\alpha f) = \bar{\alpha}\, T_g(f)$, for all $\alpha \in \mathbb{C}$, all $f \in \mathcal{H}$ and all $g \in D^t$

(ii) $T_g(f_1 + f_2) = T_g(f_1) + T_g(f_2)$ for all $f_1, f_2 \in \mathcal{H}$ and all $g \in D^t$

(iii) $T_{\alpha g}(f) = \bar{\alpha}\, T_g(f)$ for all $\alpha \in \mathbb{C}$, all $f \in \mathcal{H}$, and all $g \in D^t$

(iv) $T_{g_1+g_2}(f) = T_{g_1}(f) + T_{g_2}(f)$ for all $f \in \mathcal{H}$, and all $g_1, g_2 \in D^t$

Moreover, for given $g \in D^t$, $\exists\, C(g) > 0$ such that $\dfrac{|T_g(f)|}{||f||} < C(g)$ for all $f \in \mathcal{H}$ with $f \neq 0$.

For fixed $g \in D^t$, we now see that $T_g(f)$ satisfies all the conditions sufficient for Reisz's Theorem[3] to hold.

$\therefore$ $\exists$ a unique $g^t \in \mathcal{H}$ such that $T_g(f) = (f,g^t)$ for all $f \in \mathcal{H}$. In particular, given $g \in D^t$, $\exists$ a unique $g^t \in \mathcal{H}$ such that

$$(g,Af) = (f,g^t) \text{ for all } f \in D(A).$$

Let A^t denote the mapping of g onto g^t. i.e. $g^t = A^t g$

$\therefore$ $(g,Af) = (f,A^t g)$, for all $g \in D^t$ and all $f \in D(A)$.

Using the properties (iii) and (iv) of $T_g(f)$, it can easily be seen that A^t is an anti-linear operator with $D(A^t) = D^t$.

Definition 1.5: The anti-linear operator A^t defined above is called the transpose of A.

If $D(A) \subset D(A^t)$, and $A = A^t$ in $D(A)$, we say that A is symmetric, and we write $A \subset A^t$ or $A^t \supset A$.

If $D(A) = D(A^t)$, and $A = A^t$, then we say that A is self-transpositive. We condense the above results as follows.

Theorem 1.5: Suppose A is an anti-linear operator with $D(A)$ dense in $\mathcal{H}$. Then $\exists$ a linear space D^t such that it is the set of all g for which (g,Af) is continuous in f at each $f \in D(A)$, and $\exists$ an anti-linear operator A^t with $D(A^t)=D^t$, such that $(g,Af) = (f,A^t g)$ for all $f \in D(A)$ and all $g \in D^t$.

(i) If A is symmetric, then $D(A^t) \supset D(A)$, and $A^t = A$ in $D(A)$

(ii) If A is self-transpositive, then $D^t = D(A)$, and $(f,Ag) = (g,Af)$ for all $f,g \in D(A)$.

If A is bounded, then clearly $D^t = \mathcal{H}$, and we have $(f,Ag) = (g,A^t f)$ for all $f,g \in \mathcal{H}$.

Since A is bounded, $|(f,Ag)| \leqslant ||f||.||Ag||$

$\leqslant ||A||.||f||.||g||$

$\therefore |(g,A^t f)| \leqslant ||A||.||f||.||g||$ for all $f,g \in \mathcal{H}$.

Putting $g = A^t f$, we get $||A^t f||^2 \leqslant ||A||.||f||.||A^t f||$

Hence, if $||A^t f|| \neq 0$ and $||f|| \neq 0$, we get

$$\frac{||A^t f||}{||f||} \leqslant ||A||.$$

Since this is true even when $||A^t f|| = 0$, $||f|| \neq 0$, we see that A^t is bounded, and that $||A^t|| \leqslant ||A||$.

Now, by reversing the argument, $||A|| \leqslant ||A^t||$. Hence, if A is bounded, A^t is also bounded, and $||A|| = ||A^t||$. Further, $(A^t)^t = A$ in $\mathcal{H}$.

ALGEBRA OF ANTI-LINEAR OPERATORS.

Let $A_1, A_2, \ldots, A_\rho$ be a set of ρ anti-linear operators, with domains $D(A_1)$, $D(A_2)$, $\ldots\ldots$, $D(A_\rho)$ respectively. If $\alpha \in \mathbb{C}$, we define αA_μ by the relation $(\alpha A_\mu)f = \alpha(A_\mu f)$ for all $f \in D(A_\mu)$.

$\therefore$ $D(\alpha A_\mu) = D(A_\mu)$, and αA_μ is also an anti-linear operator. $A_\mu + A_\nu$ is defined as the operator such that $D(A_\mu + A_\nu) = D(A_\mu) \cap D(A_\nu)$ and for all $f \in D(A_\mu) \cap D(A_\nu)$,

$(\Lambda_\mu + \Lambda_\nu)f = \Lambda_\mu f + \Lambda_\nu f$. Clearly $\Lambda_\mu + \Lambda_\nu$ is also an anti-linear operator. When $\mu=\nu$, this definition is consistent with that of $2\Lambda_\mu$.

We define $\Lambda_\mu \Lambda_\nu$ by the relation $(\Lambda_\mu\Lambda_\nu)f = \Lambda_\mu(\Lambda_\nu f)$ for all $f \in D(\Lambda)$ such that $\Lambda_\nu f \in D(\Lambda_\mu)$. In this case, $\Lambda_\mu\Lambda_\nu$ is a linear operator.

It is also clear that $\Lambda_{\mu_1} \Lambda_{\mu_2} \Lambda_{\mu_3} \ldots \Lambda_{\mu_\nu}$, when defined, is a linear or an anti-linear operator according as ν is even or odd respectively.

Now, suppose $D(\Lambda) = D(\Lambda_1) \cap D(\Lambda_2) \cap D(\Lambda_3) \ldots \cap D(\Lambda_\rho)$ is a non-trivial subset of $\mathfrak{H}$, and $\Lambda_\nu \cdot D(\Lambda) \subseteq D(\Lambda)$ for all $\nu = 1,2,\ldots,\rho$. Then we can define the algebra generated by $\Lambda_1,\Lambda_2,\ldots,\Lambda_\rho$, as the set of all operators which are products of $\Lambda_1,\Lambda_2,\ldots,\Lambda_\rho$, and their finite linear combinations.

This algebra clearly contains a sub-algebra which consists only of linear operators. If $\Lambda_1^t, \Lambda_2^t,\ldots,\Lambda_\rho^t$, exist, then the set $\Lambda_1,\Lambda_2,\ldots,\Lambda_\rho,\Lambda_{\rho+1},\Lambda_{\rho+2},\ldots,\Lambda_{2\rho}$ of 2ρ operators, where $\Lambda_{\rho+\mu} = \Lambda_\mu^t$, $1 \leqslant \mu \leqslant \rho$ also would generate an algebra which we denote by $\mathcal{A}(\Lambda_\mu,\Lambda_\mu^t: \mu=1,\ldots\rho) \equiv \mathcal{A}$, provided of course, their domains satisfy a suitable condition.

In the following, we assume that each suffix μ,ν takes the values $1,2,\ldots,2\rho$.

If we define $\hat{\mu}$ by the relation $\hat{\mu}-1 = \mu+\rho-1 \pmod{2\rho}$, then $\hat{\mu}$ also takes the values $1,2,\ldots,2\rho$, and $\Lambda_\mu^t = \Lambda_{\hat{\mu}}$

Using this notation, we derive the following properties of the algebra $\mathcal{A}$, which we state as a theorem for future reference.

Theorem 1.6:

(i) If ν is odd, $\Lambda_{\mu_1} \Lambda_{\mu_2} \ldots \Lambda_{\mu_\nu}$ is an anti-linear operator and satisfies the relation $(\Lambda_{\mu_1} \Lambda_{\mu_2} \ldots \Lambda_{\mu_\nu})^t$

$= \Lambda_{\mu_\nu} \Lambda_{\mu_{\nu-1}} \ldots \Lambda_{\mu_1}$

(ii) If ν is even, $\Lambda_{\mu_1} \Lambda_{\mu_2} \ldots \Lambda_{\mu_\nu}$ is a linear operator, and satisfies the relation, $(\Lambda_{\mu_1} \Lambda_{\mu_2} \ldots \Lambda_{\mu_\nu})^*$

$= \Lambda_{\mu_\nu} \Lambda_{\mu_{\nu-1}} \ldots \Lambda_{\mu_1}$, where * denotes the adjoint.

(iii) If B_1, B_2 are anti-linear operators in $\mathcal{A}$, then

$$(\alpha B_1 + \beta B_2)^t = \alpha B_1^t + \beta B_2^t \text{ and belongs to } \mathcal{A}$$

Proof:

(i) Let f,g be two elements of the domain over which the operators of $\mathcal{A}$ are defined. If ν is odd, $\Lambda_{\mu_1} \Lambda_{\mu_2} \ldots \Lambda_{\mu_\nu}$ is clearly an anti-linear operator, and

$(f, \Lambda_{\mu_1} \Lambda_{\mu_2} \ldots \Lambda_{\mu_\nu} g) = (\Lambda_{\mu_2} \Lambda_{\mu_3} \ldots \Lambda_{\mu_\nu} g, \Lambda_{\mu_1}^t f)$ by Theorem 1.5

$= (\Lambda_{\mu_2}^t \Lambda_{\mu_1}^t f, \Lambda_{\mu_3} \Lambda_{\mu_4} \ldots \Lambda_{\mu_\nu} g)$ again using Theorem 1.5.

Proceeding similarly, we get, since ν is odd

$$(f, A_{\mu_1} A_{\mu_2} \ldots A_{\mu_\nu} g) = (g, A^t_{\mu_\nu} A^t_{\mu_{\nu-1}} \ldots A^t_{\mu_1} f)$$

But $(f, A_{\mu_1} A_{\mu_2} \ldots A_{\mu_\nu} g) = (g, (A_{\mu_1} A_{\mu_2} \ldots A_{\mu_\nu})^t f)$

$$\therefore (A_{\mu_1} A_{\mu_2} \ldots A_{\mu_\nu})^t = A^t_{\mu_\nu} A^t_{\mu_{\nu-1}} \ldots A^t_{\mu_1}$$

$$= A_{\mu_\nu} A_{\mu_{\nu-1}} \ldots A_{\mu_1}$$

(ii) If ν is even, $A_{\mu_1} A_{\mu_2} \ldots A_{\mu_\nu}$ is obviously linear, and $(f, A_{\mu_1} A_{\mu_2} \ldots A_{\mu_\nu} g) = (A^t_{\mu_\nu} A^t_{\mu_{\nu-1}} \ldots A^t_{\mu_1} f, g)$ by using Theorem 1.5 successively ν times.

But $(f, A_{\mu_1} A_{\mu_2} \ldots A_{\mu_\nu} g) = ((A_{\mu_1} A_{\mu_2} \ldots A_{\mu_\nu})^* f, g)$

where * denotes the adjoint.

$$\therefore (A_{\mu_1} A_{\mu_2} \ldots A_{\mu_\nu})^* = A^t_{\mu_\nu} A^t_{\mu_{\nu-1}} \ldots A^t_{\mu_1} = A_{\mu_\nu} A_{\mu_{\nu-1}} \ldots A_{\mu_1}.$$

(iii) $(f, (\alpha B_1 + \beta B_2) g) = (f, \alpha B_1 g) + (f, \beta B_2 g)$

$$= \alpha (f, B_1 g) + \beta (f, B_2 g)$$

$$= \alpha (g, B^t_1 f) + \beta (g, B^t_2 f), \text{ since}$$

B_1, B_2 are anti-linear and B^t_1 and B^t_2 exist. (The latter assertion follows from (i) and (ii)

$$= (g, \alpha B^t_1 f) + (g, \beta B^t_2 f)$$

$$= (g, (\alpha B^t_1 + \beta B^t_2) f)$$

$$\therefore (\alpha B_1 + \beta B_2)^t = \alpha B^t_1 + \beta B^t_2, \text{ and obviously belongs to } \mathcal{A}.$$

This completes the proof of the theorem.

REPRESENTATION WITH RESPECT TO A BASIS IN $\mathcal{H}$.

Let A be an anti-linear operator whose domain D(A) is dense in $\mathcal{H}$ which we also assume to be separable. Since D(A) is a subset of $\mathcal{H}$, D(A) itself is separable,[4] and therefore, there is a sequence $f_1, f_2, \ldots, f_\mu, \ldots$ of elements of D(A) which is dense in D(A).

Let f be an arbitrary element of $\mathcal{H}$. Since D(A) is dense in $\mathcal{H}$, $\exists$ g ϵ D(A) such that $||f-g|| < \frac{\varepsilon}{2}$, where $\varepsilon > 0$ is any prescribed quantity. Also, since $\{f_\mu\}$ is dense in D(A), $\exists$ some f_ν of this sequence such that $||g-f_\nu|| < \frac{\varepsilon}{2}$.

$$\therefore \quad ||f-f_\nu|| = ||f-g+g-f_\nu||$$

$$\leqslant ||f-g|| + ||g-f_\nu|| < \varepsilon.$$

$\therefore$ $\{f_\mu\}$ is dense in $\mathcal{H}$.

Let F be the linear manifold spanned by the elements $f_1, f_2, \ldots, f_\mu, \ldots$. i.e. the set of all elements which are finite linear combinations of the elements of the sequence $\{f_\mu\}$.[5] Then the closure of F is clearly $\mathcal{H}$, and there exists an orthonormal set $e_1, e_2, \ldots e_\mu, \ldots$ of elements of F which spans the same linear manifold as F.

Now, suppose h is any arbitrary element of $\mathcal{H}$. Then $\exists$ a Cauchy sequence $\{h_\mu\}$ of elements from the sequence

$\{f_\mu\}$ defined above and converging to h.

Since $h_\mu \in F$, we can write $h_\mu = \sum_{\nu=1}^{\infty} \alpha_{\mu\nu} \; e_\nu$

where, for each μ, $\alpha_{\mu\nu} = 0$ except for a finite number of values of ν, and $||h_\mu|| = \left[\sum_{\nu=1}^{\infty} |\alpha_{\mu\nu}|^2 \right]^{\frac{1}{2}}$.

Consider $||h_\mu - h_\nu|| = || \sum_{\rho=1}^{\infty} (\alpha_{\mu\rho} - \alpha_{\nu\rho}) \; e_\rho ||$

$$= \left[\sum_{\rho=1}^{\infty} |\alpha_{\mu\rho} - \alpha_{\nu\rho}|^2 \right]^{\frac{1}{2}} < \varepsilon \text{ for all } \mu,\nu \geqslant \Gamma_1(\varepsilon)$$

where $\varepsilon > 0$ is any prescribed quantity and $\Gamma_1(\varepsilon)$ is a suitable integer.

$\therefore$ $|\alpha_{\mu\rho} - \alpha_{\nu\rho}|^2 < \varepsilon$ for all $\mu,\nu \geqslant \Gamma_1(\varepsilon)$ whatever ρ is.

$\therefore$ $|\alpha_{\mu\rho} - \alpha_{\nu\rho}| < \varepsilon$ for all $\mu,\nu \geqslant \Gamma_1(\varepsilon^2)$

$\therefore$ $\alpha_{\mu\rho} \to \alpha_\rho$ as $\mu \to \infty$.

Further $(e_\mu, h_\nu) = \alpha_{\nu\mu}$ for all μ,ν

$\therefore$ $(e_\mu, h_\nu) \to \alpha_\mu$ as $\nu \to \infty$

But $|(e_\mu, h) - (e_\mu, h_\nu)| = |(e_\mu, h-h_\nu)| \leqslant ||e_\mu|| . ||h-h_\nu||$

$\to 0$ as $\nu \to \infty$

$\therefore$ $(e_\mu, h_\nu) \to (e_\mu, h)$ as $\nu \to \infty$

$\therefore$ $(e_\mu, h) = \alpha_\mu$

Hence $\sum_{\mu=1}^{\infty} |\alpha_\mu|^2$ is absolutely convergent, and

also $\sum_{\mu=1}^{\infty} \alpha_\mu e_\mu$ is convergent.[6]

Let $h_o = \sum_{\mu=1}^{\infty} \alpha_\mu e_\mu = \sum_{\mu=1}^{\infty} (e_\mu, h) e_\mu$

$\therefore$ $h-h_o$ is orthogonal to $e_1, e_2, \ldots, e_\mu, \ldots$ and hence to the whole of F. If k is any element of $\mathfrak{H}$, $\exists$ a Cauchy sequence $\{k_\mu\}$ of elements of F converging to k.

$\therefore$ $|(h-h_o, k) - (h-h_o, k_\mu)| = |(h-h_o, k-k_\mu)|$

$\leqslant ||h-h_o|| . ||k-k_\mu|| \to 0$ as $\mu \to \infty$

$\therefore$ $(h-h_o, k_\mu) \to (h-h_o, k)$ as $\mu \to \infty$

But $(h-h_o, k_\mu) = 0$ for all μ

$\therefore$ $(h-h_o, k) = 0$

Since k is arbitrary in $\mathcal{H}$, $h_o = h$.

$\therefore$ $h = \sum_{\mu=1}^{\infty} \alpha_\mu e_\mu$ which means that the orthonormal set $\{e_\mu\}$ which consists of elements of $D(A)$ is[45] a basis for the whole of $\mathcal{H}$.

In future, by the expression "an orthonormal basis", we always mean a basis of the above type.

Since $e_\mu \in D(A)$, Ae_μ is defined and belongs to $\mathcal{H}$ for all μ.

$\therefore$ $Ae_\mu = \sum_{\nu=1}^{\infty} \beta_{\mu\nu} e_\nu$, and hence $(e_\nu, Ae_\mu) = \beta_{\mu\nu}$

If $f \in D(A)$, we can write $f = \sum_{\mu=1}^{\infty} \pi_\mu e_\mu$, and

$$Af = \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \overline{\pi}_\mu \beta_{\mu\nu} e_\nu$$

If g is any other vector belonging to $\mathcal{H}$, we can write

$$g = \sum_{\rho=1}^{\infty} \gamma_\rho \, e_\rho$$

$$\therefore \quad (g, Af) = \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \overline{\pi}_\mu \, \beta_{\mu\nu} \, \overline{\gamma}_\nu$$

$$= \sum_{\nu=1}^{\infty} \sum_{\mu=1}^{\infty} \overline{\gamma}_\nu \, \beta^t_{\nu\mu} \, \overline{\pi}_\mu$$ assuming that the order of summation could be interchanged.

where $\beta^t_{\nu\mu} = \beta_{\mu\nu}$

$= (f, A^t g)$ where A^t is the anti-linear operator

defined by $A^t e_\mu = \sum_{\nu=1}^{\infty} \beta^t_{\mu\nu} \, e_\nu = \sum_{\nu=1}^{\infty} \beta_{\nu\mu} \, e_\nu$. The reason why we choose to call A^t the transpose of A is now evident.

TRANSFORMATION UNDER A CHANGE OF BASIS

With respect to the basis we had in the previous section, we have the results

$$g = \sum_{\mu=1}^{\infty} \gamma_\mu \, e_\mu, \quad f = \sum_{\mu=1}^{\infty} \pi_\mu \, e_\mu,$$

$$\text{and } (g, Af) = \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \overline{\pi}_\mu \, \beta_{\mu\nu} \, \overline{\gamma}_\nu$$

Let $e'_1, e'_2, \ldots, e'_\mu \ldots$ be another orthonormal basis for $\mathcal{H}$.

$$\therefore (e'_\mu, e'_\nu) = \delta_{\mu\nu}.$$

Also $$e'_\mu = \sum_{\nu=1}^{\infty} \sigma_{\mu\nu}\, e_\nu.$$

Using the previous equation, we get

$$\sum_{\rho=1}^{\infty} \bar{\sigma}_{\mu\rho}\, \sigma_{\nu\rho} = \delta_{\mu\nu}.$$

Moreover, $(e_\nu, e'_\mu) = \sigma_{\mu\nu}$.

$$\therefore e_\nu = \sum_{\mu=1}^{\infty} (e'_\mu, e_\nu)\, e'_\mu = \sum_{\mu=1}^{\infty} \bar{\sigma}_{\mu\nu}\, e'_\mu$$

Using the fact that $(e_\mu, e_\nu) = \delta_{\mu\nu}$, we also get

$$\sum_{\rho=1}^{\infty} \sigma_{\rho\mu}\, \bar{\sigma}_{\rho\nu} = \delta_{\mu\nu}$$

Hence if we define the operator L by $e'_\mu = Le_\mu$ for all μ, and by linearity everywhere else, we have $L = \tilde{L}$, and

$$\tilde{L}\tilde{L}^* = \tilde{L}^*\tilde{L} = 1.$$

Let $f = \sum_{\mu=1}^{\infty} \pi'_\mu\, e'_\mu$, and $g = \sum_{\mu=1}^{\infty} \gamma'_\mu\, e'_\mu$

$$\therefore \quad \pi_\nu = \sum_{\mu=1}^{\infty} \pi'_\mu \, \sigma_{\mu\nu}, \text{ and } \gamma_\nu = \sum_{\mu=1}^{\infty} \gamma'_\mu \, \sigma_{\mu\nu}$$

$$\text{If } Ae'_\mu = \sum_{\nu=1}^{\infty} \beta'_{\mu\nu} \, e'_\nu, \text{ then } (\xi, Af) = \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \bar{\pi}'_\mu \, \beta'_{\mu\nu} \, \bar{\gamma}'_\nu$$

$$\therefore \quad \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \bar{\pi}'_\mu \, \beta'_{\mu\nu} \, \bar{\gamma}'_\nu = \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \bar{\pi}_\mu \, \beta_{\mu\nu} \, \bar{\gamma}_\nu$$

$$\text{i.e.} \quad \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \bar{\pi}'_\mu \, \beta'_{\mu\nu} \, \bar{\gamma}'_\nu = \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \sum_{\lambda=1}^{\infty} \sum_{\rho=1}^{\infty} \bar{\pi}'_\lambda \, \bar{\sigma}_{\lambda\mu} \, \beta_{\mu\nu} \, \bar{\gamma}'_\rho \, \bar{\sigma}_{\rho\nu}$$

$$= \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \sum_{\lambda=1}^{\infty} \sum_{\rho=1}^{\infty} \bar{\pi}'_\mu \, \bar{\sigma}_{\mu\lambda} \, \beta_{\lambda\rho} \, \bar{\sigma}_{\nu\rho} \, \bar{\gamma}'_\nu$$

$$\therefore \quad \beta'_{\mu\nu} = \bar{\sigma}_{\mu\lambda} \, \beta_{\lambda\rho} \, \bar{\sigma}_{\nu\rho}$$

Using the matrix notation, and remembering that A is anti-linear, we can write the above relation as

$$\Lambda' = \bar{\tilde{L}} \, A \, \tilde{L}^t$$

Putting $\tilde{L}^t = \tilde{M}$, we get

$$\Lambda' = \tilde{M}^* A \, \tilde{M}, \text{ where}$$

$$\tilde{M} \, \tilde{M}^* = \tilde{M}^* \tilde{M} = 1.$$

This is the transformation law of an anti-linear operator under a change of basis.

CHAPTER 2

ANTI-LINEAR FIELD OPERATORS.

[8]As in the theory of linear quantised fields, we assume that the states of a physical system are described by unit rays in a Hilbert Space (which, by definition, is both complete and separable). We use the Heisenberg picture and hence, these states are independent of time. On the other hand, observables are defined by Hermitian operators which are necessarily linear and which bear the entire time dependence of the system. So the question arises as to whether a physical theory could be constructed using anti-linear operators. There is a lesson we have already learnt from linear fields that a field need not necessarily be an observable, e.g. a Dirac Field.[9] As long as we are able to construct Hermitian Operators by using these non-physical fields, a theory involving such non-physical fields becomes a possible candidate for a physical theory of fields. This is the exact situation we are facing in the case of anti-linear fields. By Theorem 1.6 we know that A^tA or AA^t is a linear Hermitian Operator, if A is an anti-linear operator in $\mathcal{H}$. In more general terms, the algebra generated by any set of anti-linear operators in $\mathcal{H}$ contains linear Hermitian Operators. Hence, if a mathematically consistent theory of fields could be set up using only anti-linear fields, it automatically becomes a physical theory, at least in principle.

Let $\varphi(x)$ be an anti-linear operator for each space time point $x \{= (x^{(0)}, \underset{\sim}{x}) = (x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)})\}$. As in the theory of linear fields, we make the assumption that only smeared fields give rise to well defined operators in $\mathcal{H}$.[10] i.e. we assume that $\varphi(f) = \int f(x)\varphi(x)dx$, $dx = dx^{(0)}d\underset{\sim}{x}$ $= dx^{(0)}dx^{(1)}dx^{(2)}dx^{(3)}$, is a well defined operator in $\mathcal{H}$ for each $f(x)$ suitably chosen. Note that, we make it a convention to write $f(x)$ before $\varphi(x)$ in the expression for $\varphi(f)$, so that $\varphi(f)$ is an anti-linear operator, which is linear in f. ie. $\varphi(\alpha f+\beta g) = \alpha\varphi(f)+\beta\varphi(g)$, $\alpha\ \beta \in \mathbb{C}$. The functions $f(x)$ for which $\varphi(f)$ is defined are called the test functions. Now, what are the test functions suitable for our purpose? Every experimental observation occupies a finite amount of time and a finite amount of space. i.e. a finite amount of space time. Hence, we would like the test functions to be reasonably smooth and of compact support.[11] For convenience, we take 'reasonable smoothness' to mean infinite differentiability. Hence, the space of test functions[12] ought to be $\mathcal{D}$. However, for mathematical convenience and for other physical reasons as mentioned by Wightman,[11] we assume that the space of test functions is $\mathcal{S}$ furnished with the usual topology. This is the set of all complex-valued infinitely differentiable functions which, together with their derivatives, approach zero at infinity faster than any power of the Euclidean distance.[13] Using

this definition of $\varphi(f)$, it turns out to be an unbounded operator in $\mathcal{H}$ and therefore, by the discussion after Theorem 1.3 the domain D_φ of $\varphi(f)$ can be taken to be dense in $\mathcal{H}$.

Now, let $\psi_1, \psi_2 \in D_\varphi$ and $f \in \mathcal{S}$. Then $(\psi_1, \varphi(f)\psi_2)$ is linear in f, and if we make the further assumption that this is also continuous in f, then $(\psi_1, \varphi(x)\psi_2$ becomes a tempered distribution.[13]

Following Streater and Wightman[14], we make the following assumptions.

A0 Assumptions of Relativistic Quantum Theory.

The states of the theory are described by unit rays in a separable, complete Hilbert space $\mathcal{H}$. The relativistic transformation law of the states is given by a continuous unitary representation $\tilde{U}(a,A)$ of the[15] inhomogeneous $SL(2,\mathbb{C})$.

Since $\tilde{U}(a,1)$ is unitary and continuous, it can be written as $\tilde{U}(a,1) = \exp(iP^\mu a_\mu)$ where P^μ is an unbounded self-adjoint operator, interpreted as the energy momentum operator of the theory. The operator $P^\mu P_\mu = m^2$ is interpreted as the square of the mass. The spectrum of P^μ lies in or on the plus cone (Spectral Condition).

There is a state Ψ_o, invariant under $\tilde{U}(a,A)$, i.e. $\tilde{U}(a,\Lambda)\Psi_o = \Psi_o$, and unique up to a constant phase factor (uniqueness of the vacuum).

A1 Assumptions about the Domain and Continuity of the Field.

For each test function $f \in \mathcal{S}$, defined on space time, there exists a set $\varphi_1^{(\gamma)}(f), \varphi_2^{(\gamma)}(f), \ldots, \varphi_{\nu(\gamma)}^{(\gamma)}(f)$; $\gamma = 1,2,\ldots,\Gamma$; of $\sum_{\gamma=1}^{\Gamma} \nu(\gamma)$ anti-linear operators. These operators, together with their transposes $\varphi_1^{(\gamma)}(f)^t$, $\varphi_2^{(\gamma)}(f)^t, \ldots, \varphi_{\nu(\gamma)}^{(\gamma)}(f)^t$; $\gamma = 1,2,\ldots,\Gamma$; are defined on a domain D of vectors dense in $\mathcal{H}$.[16] Furthermore, $\Psi_o \in D$, and $\tilde{U}(a,A)D \subset D$, $\varphi_\mu^{(\gamma)}(f)\, D \subset D$, $\varphi_\mu^{(\gamma)}(f)^t\, D \subset D$; $\mu = 1,2,\ldots,\nu(\gamma)$; $\gamma = 1,2,\ldots,\Gamma$.

If $\Psi_1, \Psi_2 \in D$, then $(\Psi_1, \varphi_\mu^{(\gamma)}(f)\Psi_2)$ is a continuous linear functional of f for all $\mu = 1,2,\ldots,\nu(\gamma)$; $\gamma = 1,2,\ldots,\Gamma$.

A2 Transformation law of the Field Components.

The equation

$$\tilde{U}(a,A)\varphi_\mu^{(\gamma)}(f)\tilde{U}(a,A)^{-1} = \sum_{\nu=1}^{\nu(\gamma)} S_{\mu\nu}^{(\gamma)}(A^{-1})\varphi_\nu^{(\gamma)}(\{a,A\}f),\ \gamma = 1,2,\ldots,\Gamma;$$

is valid when each side is applied to any vector in D. Here, $S^{(\gamma)}(A)$ is the corresponding $\nu(\gamma)$ dimensional matrix representation of SL(2,C), and $\{a,A\}f(x) = f(A^{-1}(x-a))$.

A3 Local Commutativity, sometimes called Microscopic Causality.

If the supports of f and g are such that $f(x)g(y) = 0$ whenever $(x-y)^2 \geqslant 0$, then

$$\varphi_\mu^{(\gamma)}(f)\varphi_\nu^{(\lambda)}(\bar{g})+\varepsilon(\gamma,\mu;\lambda,\nu)\varphi_\nu^{(\lambda)}(g)\varphi_\mu^{(\gamma)}(\bar{f}) = 0 \text{ in } D,$$

where $\varepsilon(\gamma,\mu;\lambda,\nu) = +1$ for all μ,ν or -1 for all μ,ν for each γ and λ. Here $\bar{f}$ and $\bar{g}$ stand for the complex conjugates of f and g respectively.

We also have

$$\varphi_\mu^{(\gamma)}(f)\varphi_\nu^{(\lambda)}(\bar{g})^t+\varepsilon(\gamma,\mu;\lambda,\nu)\varphi_\nu^{(\lambda)}(g)^t\varphi_\mu^{(\gamma)}(\bar{f}) = 0 \text{ in } D$$

under the same conditions, and two other similar relations obtained by transposition.

Note: It is important to notice the difference between this assumption and the corresponding one for linear fields as given by Streater and Wightman.[17] However, both are identical when written in terms of unsmeared fields.

So far nothing has been mentioned about the number of field operators we are going to have in our theory. For the theory to be a field theory as stated by Streater and Wightman[18], we must have sufficient fields in it so that its states can be uniquely characterised using the fields. With them, we too adopt the following definition.

Definition 2.1: A relativistic quantum theory satisfying axiom A0 with fields $\varphi_\mu^{(\gamma)}(f)$, $\mu = 1,2,\ldots,\nu(\gamma)$; $\gamma = 1,2,\ldots,\Gamma$

satisfying A1, A2, and A3 is a field theory if the vacuum state is cyclic for the smeared fields. That is, if polynomials in the smeared field components when applied to Ψ_o yield a set D_o of vectors dense in the Hilbert space of states.

Now, we are fully equipped to proceed with the discussion of the properties of a field theory. But, for convenience, we develop the following notation. This notation is almost indispensable when we try to establish the Reconstruction Theorem for a field theory involving a number of multi-component fields.

We first derive the transformation law for the transpose of $\varphi_\mu^{(\gamma)}(f)$, using A2. If $\Psi_1, \Psi_2 \in D$, then

$$(\Psi_1, \tilde{U}(a,A)\varphi_\mu^{(\gamma)}(f)\tilde{U}(a,A)^{-1}\Psi_2) = (\Psi_1, \sum_{\nu=1}^{\nu(\gamma)} S_{\mu\nu}^{(\gamma)}(A^{-1})\varphi_\nu^{(\gamma)}(\{a,A\}f)\Psi_2)$$

$$= \sum_{\nu=1}^{\nu(\gamma)} S_{\mu\nu}^{(\gamma)}(A^{-1})(\Psi_1, \varphi_\nu^{(\gamma)}(\{a,A\}f)\Psi_2)$$

Since $\tilde{U}(a,A)^{-1} = \tilde{U}(a,A)^*$, the L.H.S.

$$= (\tilde{U}(a,A)^*\Psi_1, \varphi_\mu^{(\gamma)}(f)\tilde{U}(a,A)^{-1}\Psi_2)$$

$$= (\tilde{U}(a,A)^{-1}\Psi_2, \varphi_\mu^{(\gamma)}(f)^t\tilde{U}(a,A)^*\Psi_1)$$

using the definition of the transpose of an operator

$$= (\Psi_2, \tilde{U}(a,A)\varphi_\mu^{(\gamma)}(f)^t\tilde{U}(a,A)^{-1}\Psi_1)$$

$$\therefore\ (\Psi_2, \tilde{U}(a,\Lambda)\varphi_\mu^{(\gamma)}(f)^t\tilde{U}(a,\Lambda)^{-1}\Psi_1)$$

$$= \sum_{\nu=1}^{\nu(\gamma)} S_{\mu\nu}^{(\gamma)}(\Lambda^{-1})(\Psi_1, \varphi_\nu^{(\gamma)}(\{a,\Lambda\}f)\Psi_2)$$

$$= \sum_{\nu=1}^{\nu(\gamma)} S_{\mu\nu}^{(\gamma)}(\Lambda^{-1})(\Psi_2, \varphi_\nu^{(\gamma)}(\{a,\Lambda\}f)^t\Psi_1)$$

$$= (\Psi_2, \sum_{\nu=1}^{\nu(\gamma)} S_{\mu\nu}^{(\gamma)}(\Lambda^{-1})\varphi_\nu^{(\gamma)}(\{a,\Lambda\}f)^t\Psi_1)$$

$$(2.1)\quad \therefore\ \tilde{U}(a,\Lambda)\varphi_\mu^{(\gamma)}(f)^t\tilde{U}(a,\Lambda)^{-1} = \sum_{\nu=1}^{\nu(\gamma)} S_{\mu\nu}^{(\gamma)}(\Lambda^{-1})\varphi_\nu^{(\gamma)}(\{a,\Lambda\}f)^t$$

This is the first result we wish to have. The next thing we do is to write the $2\sum_{\nu=1}^{\nu(\gamma)}\nu(\gamma)$ field components and their transposes as a single sequence in the following form using the definition $\left[\varphi_\mu^{(\gamma)}(f)\right]^t = \left[\varphi_\mu^{(\gamma)}\right]^t(f)$.

$$\varphi_1(f) = \varphi_1^{(1)}(f)$$

$$\varphi_2(f) = \varphi_2^{(1)}(f)$$

$$\cdots\cdots\cdots\cdots\quad \therefore\ \varphi_\rho(f) = \varphi_\rho^{(1)}(f),\ 1 \leqslant \rho \leqslant \nu(1)$$

$$\varphi_{\nu(1)}(f) = \varphi_{\nu(1)}^{(1)}(f)$$

$$\varphi_{\nu(1)+1}(f) = \varphi_1^{(1)}(f)^t$$

$$\varphi_{\nu(1)+2}(f) = \varphi_2^{(1)}(f)^t$$

$$\cdots\cdots\cdots\cdots\quad \therefore\ \varphi_\rho(f) = \varphi_{\rho-\nu(1)}^{(1)}(f)^t,$$

$$\varphi_{2\nu(1)}(f) = \varphi_{\nu(1)}^{(1)}(f)^t \qquad \nu(1)+1 \leqslant \rho \leqslant 2\nu(1)$$

$$\varphi_{2\nu(1)+1}(f) = \varphi_1^{(2)}(f)$$

$$\varphi_{2\nu(1)+2}(f) = \varphi_2^{(2)}(f)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \quad \therefore \quad \varphi_\rho(f) = \varphi_{\rho-2\nu(1)}^{(2)}(f),$$

$$2\nu(1)+1 \leqslant \rho \leqslant 2\nu(1)+\nu(2)$$

$$\varphi_{2\nu(1)+\nu(2)}(f) = \varphi_{\nu(2)}^{(2)}(f)$$

and so-on.

The last set we can write is as follows. Putting

$$\Lambda(\mu) = 2\sum_{\gamma=1}^{\mu} \nu(\gamma)$$

$$\varphi_{\Lambda(\Gamma-1)+1}(f) = \varphi_1^{(\Gamma)}(f)$$

$$\varphi_{\Lambda(\Gamma-1)+2}(f) = \varphi_2^{(\Gamma)}(f)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \quad \therefore \quad \varphi_\rho(f) = \varphi_{\rho-\Lambda(\Gamma-1)}^{(\Gamma)}(f),$$

$$\varphi_{\Lambda(\Gamma-1)+\nu(\Gamma)}(f) = \varphi_{\nu(\Gamma)}^{(\Gamma)}(f) \qquad \Lambda(\Gamma-1)+1 \leqslant \rho \leqslant \Lambda(\Gamma-1)+\nu(\Gamma)$$

$$\varphi_{\Lambda(\Gamma-1)+\nu(\Gamma)+1}(f) = \varphi_1^{(\Gamma)}(f)^t$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \quad \therefore \quad \varphi_\rho(f) = \varphi_{\rho-\Lambda(\Gamma-1)-\nu(\Gamma)}^{(\Gamma)}(f),$$

$$\varphi_{\Lambda(\Gamma)}(f) = \varphi_{\nu(\Gamma)}^{(\Gamma)}(f)^t \qquad \Lambda(\Gamma-1)+\nu(\Gamma)+1 \leqslant \rho \leqslant \Lambda(\Gamma).$$

(2.2) All this can be condensed in the following simple formulae.

$$\varphi_\nu^{(\gamma)}(f) = \varphi_{\nu+\Lambda(\gamma-1)}(f)$$

$$\nu = 1,2,\ldots,\nu(\gamma);\gamma=1,2,\ldots,\Gamma.$$

$$\varphi_\nu^{(\gamma)}(f)^t = \varphi_{\nu+\Lambda(\gamma-1)+\nu(\gamma)}(f)$$

where $\Lambda(\mu) = 2\sum_{\gamma=1}^{\mu} \nu(\gamma)$, $\mu \geqslant 1$; and $\Lambda(0) = 0$

Further, given $\varphi_\rho(f)$, $1 \leqslant \rho \leqslant \Lambda(\Gamma)$, $\exists$ a unique γ, $1 \leqslant \gamma \leqslant \Gamma$ such that $\Lambda(\gamma-1)+1 \leqslant \rho \leqslant \Lambda(\gamma)$. i.e. the suffix ρ lies in one and only one of the intervals $1 \leqslant \rho \leqslant 2\nu(1)$, $2\nu(1)+1 \leqslant \rho \leqslant 2\nu(1)+2\nu(2)$, $2\nu(1)+2\nu(2)+1 \leqslant \rho \leqslant 2\nu(1)+2\nu(2)+2\nu(3)$,.....,

$2\nu(1)+2\nu(2)+....+2\nu(\Gamma-1)+1 \leqslant \rho \leqslant 2\nu(1)+2\nu(2)+....+2\nu(\Gamma)$.

Then, if we define $\hat{\rho}$ by the relation,

(2.3) $\hat{\rho}-\Lambda(\gamma-1)-1 = \rho-\Lambda(\gamma-1)-1+\nu(\gamma) \pmod{2\nu(\gamma)}$,

it follows immediately that $1 \leqslant \hat{\rho} \leqslant \Lambda(\Gamma)$ and that

(2.4) $\varphi_\rho(f)^t = \varphi_{\hat{\rho}}(f)$,

and $\varphi_{\hat{\rho}}(f)^t = \varphi_\rho(f)$

Now, let $S(\Lambda) = S^{(1)}(\Lambda) \oplus S^{(1)}(\Lambda) \oplus S^{(2)}(\Lambda) \oplus S^{(2)}(\Lambda) \oplus \ldots \oplus S^{(\Gamma)}(\Lambda) \oplus S^{(\Gamma)}(\Lambda)$

i.e $S(\Lambda) = \overset{\Gamma}{\underset{\gamma=1}{\oplus}} \{S^{(\gamma)}(\Lambda) \oplus S^{(\gamma)}(\Lambda)\}$. Then $S(\Lambda)$ is a $\Lambda(\Gamma) \times \Lambda(\Gamma)$-matrix, and the transformation law $\Lambda 2$ of the fields takes the simple form

(2.5) $\tilde{U}(a,\Lambda)\varphi_\rho(f)\tilde{U}(a,\Lambda)^{-1} = \sum_{\sigma=1}^{\Lambda(\Gamma)} S_{\rho\sigma}(\Lambda^{-1})\varphi_\sigma(\{a,\Lambda\}f)$ for all ρ, $1 \leqslant \rho \leqslant \Lambda(\Gamma)$

At this stage we are in a position to proceed further with the theory.

Properties of the Vacuum Expectation Functionals.

For any $f \in \mathcal{S}(R^4)$, $\varphi_\rho(f) = \int f(x)\varphi_\rho(x)dx$ is a well defined unbounded anti-linear operator in $\mathcal{H}$ which is linear in f. By the assumption A1 concerning the domain D of the field operators, it follows immediately that the scalar product $(\Psi_o, \varphi_{\mu_1}(f_1)\varphi_{\mu_2}(f_2)....\varphi_{\mu_\nu}(f_\nu)\Psi_o)$ where $f_1, f_2, .., f_\nu \in \mathcal{S}(R^4)$ is perfectly well defined.

Remembering the last part of assumption A1 that $(\Psi_1, \varphi_\rho(f)\Psi_2)$ is a continuous linear functional in f for all $\Psi_1, \Psi_2 \in D$, and the contents of Theorem 1.6, we immediately see that $(\Psi_o, \varphi_{\mu_1}(f_1)\varphi_{\mu_2}(f_2)...\varphi_{\mu_\nu}(f_\nu)\Psi_o)$ is a functional in $f_1, f_2, ..., f_\nu$, separately continuous in each f_ρ when the others are fixed, and linear in f_ρ for odd values of ρ and anti-linear in f_ρ for even values of ρ. (Actually, as regards continuity, what we have is, that, it is continuous in each f_ρ for odd ρ and in $\bar{f}_\rho$ for even ρ. But continuity in f_ρ is identical with continuity in $\bar{f}_\rho$).

Therefore $\mathcal{S}^{(\nu)}_{\mu_1\mu_2...\mu_\nu}(f_1, f_2, f_3, f_4, ..., f_\nu)$

$$= (\Psi_o, \varphi_{\mu_1}(f_1)\varphi_{\mu_2}(\bar{f}_2)\varphi_{\mu_3}(f_3)\varphi_{\mu_4}(\bar{f}_4)...\varphi_{\mu_\nu}(f_\nu)\Psi_o), \ \nu \text{ odd}$$

$$= (\Psi_o, \varphi_{\mu_1}(f_1)\varphi_{\mu_2}(\bar{f}_2)\varphi_{\mu_3}(f_3)\varphi_{\mu_4}(\bar{f}_4)...\varphi_{\mu_\nu}(\bar{f}_\nu)\Psi_o), \ \nu \text{ even}$$

is a separately continuous multi-linear functional in $f_1, f_2, .., f_\nu \in \mathcal{S}(R^4)$.

Hence, using Schwartz's Nuclear Theorem[19], there exists a unique distribution G defined on test functions of the form $f_1 f_2 \ldots f_\nu$ where $f_1 f_2 \ldots f_\nu(x_1, x_2, \ldots x_\nu)$ $= f_1(x_1) f_2(x_2) \ldots f_\nu(x_\nu)$ such that $\mathcal{V}^{(\nu)}_{\mu_1 \mu_2 \ldots \mu_\nu}(f_1, f_2, \ldots, f_\nu)$ $= G_{\mu_1 \mu_2 \ldots \mu_\nu}(f_1 f_2 \ldots f_\nu)$

Since $G_{\mu_1 \mu_2 \ldots \mu_\nu}$ is a distribution in the above space of test functions, we can define

$$G_{\mu_1 \mu_2 \ldots \mu_\nu}(\alpha f_1 f_2 \ldots f_\nu + \beta g_1 g_2 \ldots g_\nu) = \alpha G_{\mu_1 \mu_2 \ldots \mu_\nu}(f_1 f_2 \ldots f_\nu) + \beta G_{\mu_1 \mu_2 \ldots \mu_\nu}(g_1 g_2 \ldots g_\nu)$$

where $\alpha, \beta \in \mathbb{C}$ and $f_\rho, g_\rho \in \mathcal{S}(R^4)$, $\rho = 1, 2, \ldots, \nu$.

Thus, if we denote the space of all functions of the form $f_1 f_2 \ldots f_\nu$ and their finite linear combinations by $\mathcal{S}_0(R^{4\nu})$, then we immediately see that $G_{\mu_1 \mu_2 \ldots \mu_\nu}$ is a distribution over the space $\mathcal{S}_0(R^{4\nu})$ of test functions.

But $\mathcal{S}_0(R^{4\nu})$ is dense in $\mathcal{S}(R^{4\nu})$. Therefore, by continuity, $G_{\mu_1 \mu_2 \ldots \mu_\nu}$ is a distributi on over the space $\mathcal{S}(R^{4\nu})$ of test functions, and hence is a tempered distribution. Denoting this by $V^{(\nu)}_{\mu_1 \mu_2 \ldots \mu_\nu}(x_1, x_2, \ldots x_\nu)$ or by $(\Psi_0, \varphi_{\mu_1}(x_1) \varphi_{\mu_2}(x_2) \ldots \varphi_{\mu_\nu}(x_\nu) \Psi_0)$ we write

$$\int f_1(x_1) f_2(x_2) \ldots f_\nu(x_\nu) . (\Psi_0, \varphi_{\mu_1}(x_1) \varphi_{\mu_2}(x_2) \ldots \varphi_{\mu_\nu}(x_\nu) \Psi_0) [dx]^\nu$$

$= (\Psi_o, \varphi_{\mu_1}(f_1)\varphi_{\mu_2}(\bar{f}_2)\varphi_{\mu_3}(f_3)\varphi_{\mu_4}(\bar{f}_4)\ldots\varphi_{\mu_\nu}(f_\nu)\Psi_o)$ if ν is odd

$= (\Psi_o, \varphi_{\mu_1}(f_1)\varphi_{\mu_2}(\bar{f}_2)\varphi_{\mu_3}(f_3)\varphi_{\mu_4}(\bar{f}_4)\ldots\varphi_{\mu_\nu}(\bar{f}_\nu)\Psi_o)$ if ν is even.

where $[dx]^\nu = dx_1 dx_2 \ldots dx_\nu$

We state all these results as a theorem.

Theorem 2.1: There exists a unique tempered distribution $V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots,x_\nu) = (\Psi_o, \varphi_{\mu_1}(x_1)\varphi_{\mu_2}(x_2)\ldots\varphi_{\mu_\nu}(x_\nu)\Psi_o)$ over test functions in $\mathcal{S}(R^{4\nu})$ such that

$$\int f_1(x_1) f_2(x_2)\ldots f_\nu(x_\nu) V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots,x_\nu)[dx]^\nu$$

$= (\Psi_o, \varphi_{\mu_1}(f_1)\varphi_{\mu_2}(\bar{f}_2)\varphi_{\mu_3}(f_3)\varphi_{\mu_4}(\bar{f}_4)\ldots\varphi_{\mu_\nu}(f_\nu)\Psi_o)$, if ν is odd

$= (\Psi_o, \varphi_{\mu_1}(f_1)\varphi_{\mu_2}(\bar{f}_2)\varphi_{\mu_3}(f_3)\varphi_{\mu_4}(\bar{f}_4)\ldots\varphi_{\mu_\nu}(\bar{f}_\nu)\Psi_o)$, if ν is even.

Now, following Streater and Wightman[20], we prove that expressions like

$$\int [dx]^\nu f(x_1,x_2,\ldots,x_\nu)\varphi_{\mu_1}(x_1)\varphi_{\mu_2}(x_2)\ldots\varphi_{\mu_\nu}(x_\nu)\Psi_o$$

can be interpreted as elements of $\mathcal{H}$ in a unique way.

In the first place there exist functions of the form

$$f^{(\Lambda)}(x_1,x_2,\ldots,x_\nu) = \sum_{\mu=1}^{\Lambda} f_1^{(\mu)}(x_1) f_2^{(\mu)}(x_2)\ldots f_\nu^{(\mu)}(x_\nu)$$ such

that $f^{(\Lambda)} \to f$ in $\mathcal{S}(R^{4\nu})$ as $\Lambda \to \infty$, where $f_\rho^{(\mu)} \in \mathcal{S}(R^4)$.

Now consider the state $\Psi^{(\Lambda)}_{\mu_1\mu_2\cdots\mu_\nu}=\left\{\sum_{\mu=1}^{\Lambda}\varphi_{\mu_1}(f_1^{(\mu)})\varphi_{\mu_2}(\bar{f}_2^{(\mu)})\cdots\right\}\Psi_o$

where φ_{μ_ρ} has, as argument, $f_\rho^{(\mu)}$ or $\bar{f}_\rho^{(\mu)}$ according as ρ is odd or even respectively. This state converges in norm, as is proved easily, as $\Lambda\to\infty$. This limit is defined to be

$$\int[dx]^\nu f(x_1,x_2,\ldots,x_\nu)\varphi_{\mu_1}(x_1)\varphi_{\mu_2}(x_2)\cdots\varphi_{\mu_\nu}(x_\nu)\Psi_o.$$

If the set of all such states is denoted by D_1, then the set D_o in the definition 2.1 is clearly a subset of D_1. This enables us to extend the domain D_o over which the field operators are defined to the larger domain D_1. Now, we proceed with the discussion of various properties of the vacuum expectation functionals $V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(x_1,x_2,\ldots x_\nu)$, and write them as theorems. Except for a different notation, the following theorems and their proofs are very similar to those given by Streater and Wightman.[21]

Theorem 2.2: The Relativistic Transformation Law

$$V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(\Lambda x_1+a,\Lambda x_2+a,\Lambda x_3+a,\ldots\ldots,\Lambda x_\nu+a)$$

$$=\sum_{\rho_1,\rho_2,\cdots\rho_\nu=1}^{\Lambda(\Gamma)}S_{\mu_1\rho_1}(\Lambda)\overline{S_{\mu_2\rho_2}(\Lambda)}S_{\mu_3\rho_3}(\Lambda)\overline{S_{\mu_4\rho_4}(\Lambda)}\cdots S_{\mu_\nu\rho_\nu}(\Lambda)V^{(\nu)}_{\rho_1\rho_2\cdots\rho_\nu}(x_1,x_2,\ldots,x_\nu)$$

if ν is odd

$$=\sum_{\rho_1,\rho_2,\cdots\rho_\nu=1}^{\Lambda(\Gamma)}S_{\mu_1\rho_1}(\Lambda)\overline{S_{\mu_2\rho_2}(\Lambda)}S_{\mu_3\rho_3}(\Lambda)\overline{S_{\mu_4\rho_4}(\Lambda)}\cdots\overline{S_{\mu_\nu\rho_\nu}(\Lambda)}V^{(\nu)}_{\rho_1\rho_2\cdots\rho_\nu}(x_1,x_2,\ldots,x_\nu)$$

if ν is even, where the left hand[22] side is that tempered distribution, which, on the function $f_1 f_2 f_3 \ldots f_\nu \in \mathcal{S}(R^{4\nu})$, gives $\mathcal{V}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(\{a,\Lambda\}f_1, \{a,\Lambda\}f_2, \ldots, \{a,\Lambda\}f_\nu)$. (see equation 2.6)

Proof: We will assume that ν is odd. The other case follows the same way.

$$\int f_1(x_1) f_2(x_2) \ldots f_\nu(x_\nu) V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(\Lambda x_1 + a, \Lambda x_2 + a, \ldots, \Lambda x_\nu + a)[dx]^\nu$$

$$= \mathcal{V}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(\{a,\Lambda\}f_1, \{a,\Lambda\}f_2, \ldots, \{a,\Lambda\}f_\nu) \text{ by definition}$$

$$= (\Psi_o, \varphi_{\mu_1}(\{a,\Lambda\}f_1)\varphi_{\mu_2}(\overline{\{a,\Lambda\}f_2})\varphi_{\mu_3}(\{a,\Lambda\}f_3)\varphi_{\mu_4}(\overline{\{a,\Lambda\}f_4}) \ldots \varphi_{\mu_\nu}(\{a,\Lambda\}f_\nu)\Psi_o)$$

by using equation 2.6

$$= (\Psi_o, \tilde{U}(b,B)^{-1}\tilde{U}(b,B)\varphi_{\mu_1}(\{a,\Lambda\}f_1)\tilde{U}(b,B)^{-1}\tilde{U}(b,B)\varphi_{\mu_2}(\overline{\{a,\Lambda\}f_2}) \ldots \tilde{U}(b,B)^{-1}\ldots\tilde{U}(b,B)\varphi_\nu(\{a,\Lambda\}f_\nu)\tilde{U}(b,B)^{-1}\tilde{U}(b,B)\Psi_o),$$ where

$\{b,B\}$ is any element from the inhomogeneous SL(2,C).

$$= \sum_{\rho_1,\rho_2,\ldots\rho_\nu=1}^{\Lambda(\Gamma)} S_{\mu_1\rho_1}(B^{-1})\overline{S_{\mu_2\rho_2}(B^{-1})}S_{\mu_3\rho_3}(B^{-1})\overline{S_{\mu_4\nu_4}(B^{-1})}\ldots S_{\mu_\nu\rho_\nu}(B^{-1})$$

$$\times (\Psi_o, \varphi_{\rho_1}(\{b,B\}\{a,\Lambda\}f_1)\varphi_{\rho_2}(\{b,B\}\overline{\{a,\Lambda\}f_2})\ldots\varphi_{\rho_\nu}(\{b,B\}\{a,\Lambda\}f_\nu)\Psi_o)$$

where we have used the fact that $\tilde{U}(b,B)$ is unitary and leaves Ψ_o invariant (assumption A0), and the equation 2.5 giving the

transformation law of the field components.

Here $\{b,B\}\{a,A\}f_\rho$ stands for $\{b,B\}g_\rho$ where $g_\rho = \{a,A\}f_\rho$.

$$\therefore\ g_\rho(x) = \{a,A\}f_\rho(x)$$
$$= f_\rho(A^{-1}(x-a))$$

$$\therefore\ \{b,B\}\{a,A\}f_\rho(x) = \{b,B\}g_\rho(x)$$
$$= g_\rho(B^{-1}(x-b))$$
$$= f_\rho(A^{-1}(B^{-1}(x-b)-a))$$
$$= f_\rho(A^{-1}B^{-1}x-A^{-1}B^{-1}b-A^{-1}a)$$

Hence, if we take $B^{-1} = A$, $b = -A^{-1}a$, and remember the fact that $\{a,A\}$ is a real Lorentz transformation, we have

$$\{b,B\}\{a,A\}f_\rho = f_\rho, \text{ and}$$
$$\{b,B\}\overline{\{a,A\}f_\rho} = \overline{f}_\rho.$$

$$\therefore \int f_1(x)f_2(x_2)\cdots f_\nu(x_\nu)V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(Ax_1+a,Ax_2+a,\ldots,Ax_\nu+a)[dx]^\nu$$

$$= \sum_{\rho_1,\rho_2,\cdots\rho_\nu}^{\Lambda(\Gamma)} S_{\mu_1\rho_1}(A)\overline{S_{\mu_2\rho_2}(A)}S_{\mu_3\rho_3}(A)\overline{S_{\mu_4\rho_4}(A)}\cdots S_{\mu_\nu\rho_\nu}(A)$$

$$\times (\Psi_o,\varphi_{\rho_1}(f_1)\varphi_{\rho_2}(\overline{f}_2)\varphi_{\rho_3}(f_3)\varphi_{\rho_4}(\overline{f}_4)\cdots\varphi_{\rho_\nu}(f_\nu)\Psi_o)$$

$$= \int f_1(x_1) f_2(x_2) \ldots f_\nu(x_\nu) \cdot \sum_{\rho_1,\rho_2,\ldots\rho_\nu=1}^{-\Lambda(\Gamma)} S_{\mu_1\rho_1}(\Lambda)\overline{S_{\mu_2\rho_2}(\Lambda)} S_{\mu_3\rho_3}(\Lambda)\overline{S_{\mu_4\rho_4}(\Lambda)} \ldots$$

$$\ldots S_{\mu_\nu\rho_\nu}(A) \cdot V^{(\nu)}_{\rho_1\rho_2\ldots\rho_\nu}(x_1,x_2,\ldots,x_\nu)[dx]^\nu .$$

Here, by continuity, we get the required relation as a relation between two temepered distributions over the space $\mathcal{S}(R^{4\nu})$ of test functions.

Theorem 2.3: Spectral Conditions.

There are tempered distributions $W^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(\xi_1,\xi_2,\ldots,\xi_{\nu-1})$ such that

$$W^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(\xi_1,\xi_2,\ldots,\xi_{\nu-1}) = V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots x_\nu) \text{ for } \nu \geqslant 2$$

where $\xi_\rho = x_\rho - x_{\rho+1}$, $1 \leqslant \rho \leqslant \nu-1$.

The Fourier transforms of W and V defined respectively by

$$\tilde{W}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(p_1,p_2,\ldots p_{\nu-1}) = \frac{1}{(2\pi)^{2\nu-2}} \int \exp\Big(i \sum_{\rho=1}^{\nu-1} p_\rho \cdot \xi_\rho\Big) W^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}$$

$$(\xi_1,\xi_2,\ldots,\xi_{\nu-1})[d\xi]^{\nu-1}$$

and

$$\tilde{V}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(p_1,p_2,\ldots,p_\nu) = \frac{1}{(2\pi)^{2\nu}} \int \exp\Big(i \sum_{\rho=1}^{\nu} p_\rho \cdot x_\rho\Big) V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}$$

$$(x_1,x_2,\ldots,x_\nu)[dx]^\nu$$

are tempered distributions, and are related by

$$\tilde{V}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(p_1,p_2,\ldots,p_\nu)=(2\pi)^2\delta\Big(\sum_{\rho=1}^{\nu}p_\rho\Big)\tilde{W}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(p_1,p_1+p_2,p_1+p_2+p_3,$$

$$\ldots,p_1+p_2+p_3+\ldots+p_{\nu-1}).$$

Further, we have the result that

$\tilde{W}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(-q_1,q_2,-q_3,q_4,\ldots,(-1)^{\nu-1}q_{\nu-1}) = 0$, unless each q_ρ, $1 \leqslant \rho \leqslant \nu-1$ lies in the energy momentum spectrum of the states

Proof: Using Theorem 2.2 and taking $A = 1$, we get

$$V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1+a,x_2+a,\ldots,x_\nu+a)=V^{(\nu)}_{\mu_1,\mu_2,\ldots,\mu_\nu}(x_1,x_2\ldots,x_\nu)$$

Putting $\xi_1 = x_1-x_2$, $\xi_2 = x_2-x_3,\ldots,$ $\xi_{\nu-1}= x_{\nu-1}-x_\nu$

$X = \frac{1}{\nu}(x_1+x_2+\ldots+x_\nu)$, we get $x_1,x_2,\ldots,x_\nu$ uniquely as linear functions of $\xi_1,\xi_2,\ldots,\xi_{\nu-1}$, X.

Therefore there is a distribution $T(X,\xi_1,\xi_2,\ldots,\xi_{\nu-1})$ such that

$$V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots,x_\nu) = T(X,\xi_1,\xi_2,\ldots,\xi_{\nu-1}).$$

Translation invariance of $V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots,x_\nu)$ implies that

$T(X, \xi_1,\xi_2,\ldots,\xi_{\nu-1}) = T(X + a, \xi_1,\xi_2,\ldots,\xi_{\nu-1})$ for all a.

Using the argument of Schwartz[23] it can easily be proved that there exists a tempered distribution

$$W^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(\xi_1,\xi_2,\ldots,\xi_{\nu-1}) \text{ such that}$$

$$T(X,\xi_1,\xi_2,\ldots,\xi_{\nu-1}) = W^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(\xi_1,\xi_2,\ldots,\xi_{\nu-1})$$

$$\therefore \quad V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(x_1,x_2,\ldots,x_\nu) = W^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(\xi_1,\xi_2,\ldots,\xi_{\nu-1})$$

which proves the first part of the theorem.

For a function $f \in \mathcal{S}(R^{4\nu})$ we define $\mathcal{F}f$ by the relation $(\mathcal{F}f)(p_1,p_2,\ldots,p_\nu) = \frac{1}{(2\pi)^{2\nu}} \int \exp\left(i\sum_{\mu=1}^{\nu} p_\mu \cdot x_\mu\right) f(x_1,\ldots,x_\nu) \left[dx\right]^{\nu}$.

Then by the Lemma on page 44 of PCT, $\mathcal{F}f \in \mathcal{S}(R^{4\nu})$.

For a tempered distribution T over test functions in $\mathcal{S}(R^{4\nu})$ we define $\mathcal{F}T$ by the equation

$$(\mathcal{F}T)(f) = T(\mathcal{F}f) \text{ for } f \in \mathcal{S}(R^{4\nu}).$$

Then, according to Theorem 2.2 of PCT, $\mathcal{F}T$ is itself a tempered distribution over $\mathcal{S}(R^{4\nu})$. The mapping denoted by $\mathcal{F}$ is called the Fourier Transform.

Using these results, it is obvious that the Fourier Transforms of $W^{(\nu)}_{\mu_1,\mu_2,\ldots,\mu_\nu}(\xi_1,\xi_2,\ldots,\xi_{\nu-1})$ and $V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(x_1,x_2,\ldots,x_\nu)$ respectively given by

$$\tilde{W}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(p_1,p_2,\ldots,p_{\nu-1}) = \frac{1}{(2\pi)^{2(\nu-1)}} \int \exp\left(i\sum_{\mu=1}^{\nu-1} p_\mu\cdot\xi_\mu\right)$$

$$W^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(\xi_1,\xi_2,\ldots,\xi_{\nu-1})[d\xi]^{\nu-1}$$

and

$$\tilde{V}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(p_1,p_2,\ldots,p_\nu) = \frac{1}{(2\pi)^{2\nu}} \int \exp\left(i\sum_{\mu=1}^{\nu} p_\mu\cdot x_\mu\right)$$

$$V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots,x_\nu)[dx]^\nu$$

are both tempered distributions.

If we change to new variables given by

$$\xi_1 = x_1 - x_2,$$

$$\xi_2 = x_2 - x_3,$$

$$\ldots\ldots\ldots\ldots,$$

$$\xi_{\nu-1} = x_{\nu-1} - x_\nu,$$

$$\xi_\nu = x_\nu, \text{ we get}$$

$$x_\mu = \sum_{\rho=\mu}^{\nu} \xi_\rho$$

$$\therefore \sum_{\mu=1}^{\nu} p_\mu\cdot x_\mu = \sum_{\mu=1}^{\nu} p_\mu\cdot\sum_{\rho=\mu}^{\nu}\xi_\rho = p_1\cdot\xi_1+(p_1+p_2)\cdot\xi_2+(p_1+p_2+p_3)\cdot\xi_3+\ldots$$

$$+ (p_1+p_2+\ldots+p_{\nu-1})\cdot\xi_{\nu-1}+\left(\sum_{\mu=1}^{\nu} p_\mu\right)\cdot\xi_\nu = \sum_{\mu=1}^{\nu}\xi_\mu\cdot\sum_{\rho=1}^{\mu} p_\rho$$

Using this result we get

$$\tilde{V}^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(p_1,p_2,\cdots,p_\nu) = \frac{1}{(2\pi)^{2\nu}} \int \exp\Big(i\sum_{\mu=1}^{\nu}\xi_\mu \cdot \sum_{\rho=1}^{\mu} p_\rho\Big)$$

$$V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(\xi_1,\xi_2,\cdots,\xi_{\nu-1})[d\xi]^\nu$$

$$= \frac{1}{(2\pi)^2}\cdot(2\pi)^4\delta\Big(\sum_{\mu=1}^{\nu} p_\mu\Big)\tilde{V}^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(p_1,p_1+p_2,\cdots,p_1+p_2+\cdots+p_{\nu-1})$$

i.e $\tilde{V}^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(p_1,p_2,\cdots,p_\nu)=(2\pi)^2\delta\Big(\sum_{\mu=1}^{\nu} p_\mu\Big)\tilde{V}^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(p_1,p_1+p_2,$

$$,\cdots,p_1+p_2+\cdots+p_{\nu-1}).$$

To prove the last part we use the result[24] that, if $\Phi,\Psi \in D_0$ (see definition 2.1) then, as a tempered distribution over $\mathcal{S}(R^4)$, $\int e^{-ip\cdot a}da(\Phi,\tilde{U}(a,1)\Psi) = 0$ unless p belongs to the (physical) spectrum of energy momentum of the states. A proof of this result when the physical spectrum is the forward cone $\bar{V}^+$ has been given by Uhlman.[25]

At this stage we adopt the following notation, for convenience.

If $f_\nu \in \mathcal{S}(R^4)$, then $f^o_\nu = f_\nu$ if ν is odd

$= \bar{f}_\nu$ if ν is even

The proof has to be broken down into two cases as follows.

Let $\Psi = \varphi_{\mu_{\rho+1}}(f^{\circ}_{\rho+1})\varphi_{\mu_{\rho+2}}(f^{\circ}_{\rho+2})\ldots\varphi_{\mu_{\nu}}(f^{\circ}_{\nu})\Psi_{\circ}$, and

$$\Phi = \varphi_{\hat{\mu}_{\rho}}(f^{\circ}_{\rho})\varphi_{\hat{\mu}_{\rho-1}}(f^{\circ}_{\rho-1})\ldots\varphi_{\hat{\mu}_{1}}(f^{\circ}_{1})\Psi_{\circ}$$

$$\therefore (\Phi,\tilde{U}(a,1)\Psi)=(\varphi_{\hat{\mu}_{\rho}}(f^{\circ}_{\rho})\varphi_{\hat{\mu}_{\rho-1}}(f^{\circ}_{\rho-1})\ldots\varphi_{\hat{\mu}_{1}}(f^{\circ}_{1})\Psi_{\circ},\tilde{U}(a,1)\Psi)$$

Case 1: Suppose ρ is even.

$$\therefore (\Phi,\tilde{U}(a,1)\Psi)=(\Psi_{\circ},\varphi_{\mu_1}(f^{\circ}_{1})\varphi_{\mu_2}(f^{\circ}_{2})\ldots\varphi_{\mu_{\rho}}(f^{\circ}_{\rho})\tilde{U}(a,1)\Psi),$$ by the definition of transposition and the index $\hat{\mu}$.

$$= \int g_1(x_1)g_2(x_2)\ldots g_{\nu}(x_{\nu})V^{(\nu)}_{\mu_1\mu_2\ldots\mu_{\nu}}(x_1,x_2,\ldots,x_{\nu})[dx]^{\nu},$$

where

$$g_{\mu} = f_{\mu},\ 1 \leqslant \mu \leqslant \rho$$

$$= \{a,1\}f_{\mu},\ \rho+1 \leqslant \mu \leqslant \nu$$

$$= \int f_1(x_1)f_2(x_2)\ldots f_{\nu}(x_{\nu})V^{(\nu)}_{\mu_1\mu_2\ldots\mu_{\nu}}(x_1,x_2,\ldots,x_{\rho},x_{\rho+1}+a,$$

$$\ldots,x_{\nu}+a)[dx]^{\nu}$$

But, if $\tilde{g} \in \mathcal{S}(R^4)$ is such that its support does not intersect the physical spectrum of energy momentum, then

$$\int \tilde{g}(p) e^{-ip.a} (\Phi, \tilde{U}(a,1)\Psi) da\, dp = 0$$

$$\text{i.e } 0 = \int \tilde{g}(p) e^{-ip.a} f_1(x_1) f_2(x_2) \ldots f_\nu(x_\nu) V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1, x_2, \ldots x_\rho,$$

$$x_{\rho+1}+a, \ldots, x_\nu+a)[dx]^\nu da.dp$$

$$\text{Hence by continuity } \int \tilde{g}(p) e^{-ip.a} V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1, x_2, \ldots, x_\rho,$$

$$x_{\rho+1}+a, \ldots, x_\nu+a) da.dp = 0,$$

as a distribution over $\mathcal{S}(R^{4\nu})$

$$\text{i.e. } \int \tilde{g}(p) e^{-ip.a} W^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(\xi_1, \xi_2, \ldots, \xi_{\rho-1}, \xi_\rho - a, \xi_{\rho+1}, \ldots, \xi_{\nu-1})$$

$$da.dp = 0$$

$$\text{i.e. } \int \tilde{g}(p) e^{-ip.(\xi_\rho+a)} W^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(\xi_1, \xi_2, \ldots, \xi_{\rho-1}, -a, \xi_{\rho+1}, \ldots \xi_{\nu-1})$$

$$da.dp = 0$$

$$\ldots \int \tilde{g}(p) e^{-ip.\xi_\rho - ip.a} \exp\Big(i \sum_{\sigma=1}^{\nu-1} q_\sigma.\xi_\sigma\Big) W^{(\nu)}_{\mu_1\mu_2\ldots\mu_{\nu-1}}(\xi_1, \xi_2, \ldots, \xi_{\rho-1},$$

$$-a, \xi_{\rho+1}, \ldots, \xi_{\nu-1}) da.dp[d\xi]^{\nu-1} = 0$$

i.e. $\int \tilde{g}(p) e^{-i(p-q_\rho)\cdot\xi_\rho} \tilde{\mathbb{V}}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(q_1,q_2,\ldots,q_{\rho-1},p,q_{\rho+1},$

$\ldots,q_{\nu-1})dp\,d\xi_\rho = 0$

i.e. $\tilde{g}(q_\rho)\tilde{\mathbb{V}}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(q_1,q_2,\ldots q_\rho,\ldots,q_{\nu-1})=0$

By the definition of $\tilde{g}$, this implies that

$\tilde{\mathbb{V}}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(q_1,q_2,\ldots,q_\rho,\ldots,q_{\nu-1}) = 0$ unless q_ρ

belongs to the physical spectrum, and this is true for all even ρ.

Case 2: Suppoe ρ is odd

Then $(\Phi,\tilde{U}(a,1)\Psi)=(\varphi_{\mu_1}(f_1^\circ)\varphi_{\mu_2}(f_2^\circ)\ldots\varphi_{\mu_\rho}(f_p^\circ)\tilde{U}(a,1)\varphi_{\mu_{\rho+1}}(f_{\rho+1}^\circ)$

$\ldots\varphi_{\mu_\nu}(f_\nu^\circ)\Psi_\circ,\Psi_\circ)$

$= \int g_1(x_1)g_2(x_2)\ldots g_\nu(x_\nu)V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,..,x_\nu)$

$[dx]^\nu$, where

$g_\mu=\overline{f}_\mu, \quad 1 \leqslant \mu \leqslant \rho$

$= \{a,1\}\overline{f}_\mu, \quad \rho+1 \leqslant \mu \leqslant \nu$

$= \int \overline{f_1(x_1)}\,\overline{f_2(x_2)}\ldots\overline{f_\nu(x_\nu)}\ \overline{V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots,x_\rho,x_{\rho+1}+a,}$

$\overline{\ldots,x_\nu+a)}[dx]^\nu$

$$\therefore \int e^{-ip\cdot a}(\Phi,\tilde{U}(a,1)\Psi)da$$

$$= \int e^{-ip\cdot a}\overline{f_1(x_1)}\,\overline{f_2(x_2)}\ldots\overline{f_\nu(x_\nu)}\,\overline{V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots,x_\rho,x_{\rho+1}+a,\ldots,x_\nu+a)}.da.[dx]^\nu$$

Hence $\int e^{ip\cdot a}V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots,x_\rho,x_{\rho+1}+a,\ldots,x_\nu+a)da = 0$

unless p belongs to the physical spectrum

By the same argument as in the previous case,

$$\tilde{g}(-q_\rho)\tilde{W}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(q_1,q_2,\ldots,q_\rho,\ldots,q_{\nu-1}) = 0$$

$$\therefore \tilde{W}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(q_1,q_2,\ldots,q_{\rho-1}\ -q_\rho,q_{\rho+1},\ldots,q_{\nu-1}) = 0 \text{ unless}$$

q_ρ lies in the physical spectrum, and this is true for all odd ρ.

Hence we have the final result that

$$\tilde{W}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(-p_1,p_2,-p_3,\ldots,(-1)^\rho p_\rho,\ldots,(-1)^{\nu-1}p_{\nu-1}) = 0$$

unless each p_ρ, $1 \leqslant \rho \leqslant \nu-1$, lies in the physical spectrum.

It is interesting to note that this is the first property of the V.E. Functionals of anti-linear fields which differs non-trivially from the corresponding one for linear fields.

Theorem 2.4:

$V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(x_1,x_2,\ldots,x_\nu)=V^{(\nu)}_{\hat{\mu}_\nu\hat{\mu}_{\nu-1}\cdots\hat{\mu}_1}(x_\nu,x_{\nu-1},\ldots,x_1)$ if ν is odd

$$= \overline{V^{(\nu)}_{\hat{\mu}_\nu\hat{\mu}_{\nu-1}\cdots\hat{\mu}_1}(x_\nu,x_{\nu-1},\ldots,x_1)} \text{ if } \nu \text{ is even}$$

Proof:

$$\int f_1(x_1)f_2(x_2)\cdots f_\nu(x_\nu)V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(x_1,x_2,\ldots,x_\nu)[dx]^\nu$$

$= (\Psi_\circ,\varphi_{\mu_1}(f_1^\circ)\varphi_{\mu_2}(f_2^\circ)\cdots\varphi_{\mu_\nu}(f_\nu^\circ)\Psi_\circ)$ using the notation of last theorem.

$= (\Psi_\circ,\varphi_{\hat{\mu}_\nu}(f_\nu^\circ)\varphi_{\hat{\mu}_{\nu-1}}(f_{\nu-1}^\circ)\cdots\varphi_{\hat{\mu}_1}(f_1^\circ)\Psi_\circ)$ if ν is odd

$= (\varphi_{\hat{\mu}_\nu}(f_\nu^\circ)\varphi_{\hat{\mu}_{\nu-1}}(f_{\nu-1}^\circ)\cdots\varphi_{\hat{\mu}_1}(f_1^\circ)\Psi_\circ,\Psi_\circ)$ if ν is even

Hence, $V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(x_1,x_2,\ldots,x_\nu)$

$= V^{(\nu)}_{\hat{\mu}_\nu\hat{\mu}_{\nu-1}\cdots\hat{\mu}_1}(x_\nu,x_{\nu-1},\ldots,x_1)$ if ν is odd

$= \overline{V^{(\nu)}_{\hat{\mu}_\nu\hat{\mu}_{\nu-1}\cdots\hat{\mu}_1}(x_\nu,x_{\nu-1},\ldots,x_1)}$ if ν is even

which is the result we had to establish.

Theorem 2.5: Local Commutativity

If Local Commutativity (Axiom A3) in terms of the notation in equation 2.2 takes the weaker form

$$\varphi_\mu(f)\varphi_\nu(\overline{g})+\varepsilon(\mu,\nu)\varphi_\nu(g)\varphi_\mu(\overline{f}) = 0,\ \varepsilon(\mu,\nu) = \varepsilon(\nu,\mu) = +1$$

or -1 for each pair (μ,ν), $\mu,\nu = 1,2,\ldots,\Lambda(\Gamma)$, where $f,g \in \mathcal{S}(R^4)$ are such that $f(x)g(y) = 0$ whenever $(x-y)^2 \geqslant 0$, then

$$V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\rho\,\mu_{\rho+1}\cdots\mu_\nu}(x_1,x_2,\ldots,x_\rho,x_{\rho+1},\ldots,x_\nu)$$

$$+\ \varepsilon(\mu_\rho,\mu_{\rho+1})V^{(\nu)}_{\mu_1\mu_2\cdots\mu_{\rho-1}\mu_{\rho+1}\mu_\rho\mu_{\rho+2}\cdots\mu_\nu}(x_1,x_2,\ldots,x_{\rho-1},$$

$$x_{\rho+1},x_\rho,x_{\rho+2},\ldots,x_\nu)$$

$$= 0 \text{ for all } (x_\rho - x_{\rho+1})^2 < 0,\quad 1 \leqslant \rho \leqslant \nu-1$$

Proof: Suppose $f_\rho(x_\rho)f_{\rho+1}(x_{\rho+1}) = 0$ whenever $(x_\rho - x_{\rho+1})^2 \geqslant 0$

$$\therefore\ \varphi_{\mu_\rho}(f^\circ_\rho)\varphi_{\mu_{\rho+1}}(f^\circ_{\rho+1})+\varepsilon(\mu_\rho,\mu_{\rho+1})\varphi_{\mu_{\rho+1}}(\overline{f}^\circ_{\rho+1})\varphi_{\mu_\rho}(\overline{f}^\circ_\rho) = 0.$$

(see notation just before the proof of the spectral condition)

Hence $(\Psi_\circ,\varphi_{\mu_1}(f^\circ_1)\varphi_{\mu_2}(f^\circ_2)\cdots\varphi_{\mu_{\rho-1}}(f^\circ_{\rho-1})\cdot[\varphi_{\mu_\rho}(f^\circ_\rho)\varphi_{\mu_{\rho+1}}(f^\circ_{\rho+1})$

$$+\ \varepsilon(\mu_\rho,\mu_{\rho+1})\varphi_{\mu_{\rho+1}}(\overline{f}^\circ_{\rho+1})\varphi_{\mu_\rho}(\overline{f}^\circ_\rho)]\varphi_{\mu_{\rho+2}}(f^\circ_{\rho+2})\cdots\varphi_{\mu_\nu}(f^\circ_\nu)\Psi_\circ)=0$$

i.e. $\int f_1(x_1)f_2(x_2)\ldots f_\rho(x_\rho)f_{\rho+1}(x_{\rho+1})\ldots f_\nu(x_\nu)[V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\rho\mu_{\rho+1}\ldots\mu_\nu}$

$(x_1,x_2,\ldots,x_\rho,x_{\rho+1},\ldots,x_\nu)+\varepsilon(\mu_\rho,\mu_{\rho+1})V^{(\nu)}_{\mu_1\mu_2\ldots\mu_{\rho-1}\mu_{\rho+1}\mu_\rho\mu_{\rho+2}\ldots\mu_\nu}$

$(x_1,x_2,\ldots,x_{\rho-1},x_{\rho+1},x_\rho,x_{\rho+2},\ldots x_\nu)]\,[dx]^\nu = 0$

Since functions of the form $f_1(x_1)f_2(x_2)\ldots f_\nu(x_\nu)$ and their finite linear combinations form a dense set in $\mathcal{S}(R^{4\nu})$, and any $f(x_1,x_2,\ldots,x_\rho,x_{\rho+1},\ldots,x_\nu) \in \mathcal{S}(R^{4\nu})$ vanishing for all $(x_\rho - x_{\rho+1})^2 \geqslant 0$ can be defined by a sequence of such finite linear combinations, we get the desired result.

The next result we require about the Vacuum Expectation Functionals is the restriction imposed on them by the positive definiteness of the scalar product in the underlying Hilbert Space $\mathcal{H}$.

Theorem 2.6: Positive Definiteness Condition

If $f^{(\rho)}_{\mu_1\mu_2\ldots\mu_\rho} \in \mathcal{S}(R^{4\rho})$ for each $\rho \geqslant 0$, $1 \leqslant \mu_1,\mu_2,\ldots,\mu_\rho \leqslant \Lambda(\Gamma)$ ($\mathcal{S}(R^0) = \mathbb{C}$) and $f^{(\rho)}_{\mu_1\mu_2\ldots\mu_\rho} \equiv 0$ except for a finite number of values of ρ, then

$$\sum_{\substack{\text{even}\\ \rho\geqslant 0}}^{\infty}\sum_{\substack{\text{all}\\ \sigma\geqslant 0}}^{\infty}\sum_{\mu,\nu}\int \overline{f^{(\rho)}_{\mu_1\mu_2\ldots\mu_\rho}(x_1x_2,\ldots,x_\rho)}\,W^{(\rho+\sigma)}_{\hat{\mu}_\rho\hat{\mu}_{\rho-1}\ldots\hat{\mu}_1\nu_1\nu_2\ldots\nu_\sigma}(x_\rho,$$

$$x_{\rho-1},\ldots,x_1,y_1,y_2,\ldots,y_\sigma)f^{(\sigma)}_{\nu_1\nu_2\ldots\nu_\sigma}(y_1,y_2,\ldots,y_\sigma)[dx]^\rho[dy]^\sigma$$

$$+ \sum_{\substack{\text{odd} \\ \rho \geqslant 1}}^{\infty} \sum_{\substack{\text{all} \\ \sigma \geqslant 0}}^{\infty} \sum_{\mu,\nu} \int \overline{f^{(\rho)}_{\mu_1\mu_2\cdots\mu_\rho}(x_1,x_2,\ldots,x_\rho)} \cdot$$

$$\times \overline{V^{(\rho+\sigma)}_{\mu_\rho\mu_{\rho-1}\cdots\mu_1\nu_1\nu_2\cdots\nu_\sigma}(x_\rho,x_{\rho-1},\ldots,x_1,y_1,y_2,\ldots,y_\sigma)}$$

$$f^{(\sigma)}_{\nu_1\nu_2\cdots\nu_\sigma}(y_1,y_2,\ldots,y_\sigma)[dx]^\rho[dy]^\sigma \geqslant 0$$

where $\sum_{\mu,\nu}$ stands for summation over $\mu_1,\mu_2,\ldots,\mu_\rho,\nu_1,\nu_2,\ldots,\nu_\sigma$ from 1 to $\Lambda(\Gamma)$.

Proof: The proof is straight forward and goes as follows.

Let $\Phi = f^{(o)}\Psi_o + \sum_{\rho=1}^{\infty} \sum_{\mu} \int f^{(\rho)}_{\mu_1\mu_2\cdots\mu_\rho}(x_1,x_2,\ldots,x_\rho)\varphi_{\mu_1}(x_1)\varphi_{\mu_2}(x_2)$

$$\ldots\varphi_{\mu_\rho}(x_\rho)[dx]^\rho \Psi_o$$

where $\sum_{\mu}$ stands for summation over $\mu_1,\mu_2,\ldots\mu_\rho$ from 1 to $\Lambda(\Gamma)$.

There are no difficulties of convergence, because only a finite number of terms in the sum $\sum_{\rho=0}^{\infty}$ are non zero. By the discussion preceding Theorem 2.2, Φ is a uniquely defined element of $\mathcal{H}$, and hence, $(\Phi,\Phi) \geqslant 0$.

If we write this out explicitly, and use the definition of the V.E. Functionals we get the above result.

It would have been very useful if we could define at this stage, a Truncated Vacuum Expectation Functional[27] by subtracting from the ordinary V.E. Functional the contribution coming from the origin in momentum space, in a suitable way. This would perhaps allow us to proceed as in Araki's paper[26] and then prove a theorem which is much stronger than what we would actually prove in Chapter 4. Unfortunately, we have failed to define "Truncation" in such a useful way. But, we have the following result which is sufficient to establish the uniqueness of the vacuum, in the next chapter.

Theorem 2.7: Cluster Decomposition Property

$$\lim_{\kappa \to \infty} V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\rho\,\mu_{\rho+1}\cdots\mu_\nu}(x_1,x_2,\ldots,x_\rho,x_{\rho+1}+\kappa a,\ldots,x_\nu+\kappa a)$$

$$= V^{(\rho)}_{\mu_1\mu_2\cdots\mu_\rho}(x_1,x_2,\ldots,x_\rho)\,V^{(\nu-\rho)}_{\mu_{\rho+1}\mu_{\rho+2}\cdots\mu_\nu}(x_{\rho+1},x_{\rho+2},\ldots,x_\nu)$$

$$\text{if } \rho \ (2 \leqslant \rho \leqslant \nu-1) \text{ is even}$$

$$= V^{(\rho)}_{\mu_1\mu_2\cdots\mu_\rho}(x_1,x_2,\ldots,x_\rho)\,\overline{V^{(\nu-\rho)}_{\mu_{\rho+1}\mu_{\rho+2}\cdots\mu_\nu}(x_{\rho+1},x_{\rho+2},\ldots,x_\nu)}$$

$$\text{if } \rho \ (1 \leqslant \rho \leqslant \nu-1) \text{ is odd,}$$

where a is a non-null 4-vector and the convergence is weak convergence[28] in $\mathcal{S}'$.

Proof: The theorem follows immediately from the following result of Borchers.[32]

If $\tilde{U}(a,\Lambda)$ is a continuous unitary representation of the Poincare Group, and Ψ is any state in $\mathcal{H}$, then $\tilde{U}(\kappa a,1)\Psi$ tends weakly to $\tilde{P}\Psi$ as $\kappa \to \infty$ if a is a non-null 4-vector, where $\tilde{P}$ is the projection operator on to the subspace of states which are invariant under translations.

This means that, if $\Phi \in \mathcal{H}$, then

$$\lim_{\kappa \to \infty} (\Phi,\tilde{U}(\kappa a,1)\Psi) = (\Phi,\tilde{P}\Psi).$$

But $\tilde{P}\Psi = (\Psi_o,\tilde{P}\Psi)\Psi_o$ if we assume that the only vector invariant under all translations is the vacuum.

$$= (\Psi_o,\Psi)\Psi_o$$

This result of Borchers is true whether the fields are linear or not, because, it does not use any particular realization of the Hilbert Space $\mathcal{H}$.

(2.7) $\therefore$ $\lim_{\kappa \to \infty}$ $(\Phi,\tilde{U}(\kappa a,1)\Psi) = (\Phi,\Psi_o)(\Psi_o,\Psi)$ if a is a non-null 4 vector

Now, let $\Psi = \varphi_{\mu_{\rho+1}}(f^o_{\rho+1})\varphi_{\mu_{\rho+2}}(f^o_{\rho+2})\ldots\varphi_{\mu_\nu}(f^o_\nu)\Psi_o$, and

$$\Phi = \varphi_{\lambda_\rho}(f^o_\rho)\varphi_{\lambda_{\rho-1}}(f^o_{\rho-1})\ldots\varphi_{\lambda_1}(f^o_1)\Psi_o$$

Case i: Suppose ρ is even

Then $(\Phi, \tilde{U}(\kappa a,1)\Psi)$

$= (\Psi_o, \varphi_{\mu_1}(g_1^o)\varphi_{\mu_2}(g_2^o)\ldots\varphi_{\mu_\nu}(g_\nu^o)\Psi_o)$, where

$$g_\sigma = f_\sigma, \quad 1 \leqslant \sigma \leqslant \rho$$

$$= \{\kappa a, 1\} f_\sigma, \quad \rho+1 \leqslant \sigma \leqslant \nu$$

$$= \int f_1(x_1) f_2(x_2)\ldots f_\nu(x_\nu) V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\rho\mu_{\rho+1}\ldots\mu_\nu}(x_1, x_2, \ldots, x_\rho,$$

$$x_{\rho+1}+\kappa a, x_{\rho+2}+\kappa a, \ldots, x_\nu+\kappa a)\ [dx]^\nu, \text{ and}$$

$$(\Phi, \Psi_o)(\Psi_o, \Psi) = (\overline{\Psi_o, \Phi})(\Psi_o, \Psi)$$

$$= \overline{\left[\int \bar{f}_\rho(x_\rho)\bar{f}_{\rho-1}(x_{\rho-1})\ldots\bar{f}_1(x_1) V^{(\rho)}_{\hat{\mu}_\rho\hat{\mu}_{\rho-1}\ldots\hat{\mu}_1}(x_\rho, x_{\rho-1}, \ldots x_1)[dx]^\rho\right]}$$

$$\cdot \left[\int f_{\rho+1}(x_{\rho+1}) f_{\rho+2}(x_{\rho+2})\ldots f_\nu(x_\nu) V^{(\nu-\rho)}_{\mu_{\rho+1}\mu_{\rho+2}\ldots\mu_\nu}(x_{\rho+1}, x_{\rho+2}, \ldots, x_\nu) \times \frac{[dx]^\nu}{[dx]^\rho}\right]$$

where $\dfrac{[dx]^\nu}{[dx]^\rho} = dx_{\rho+1}\cdot dx_{\rho+2}\ldots dx_\nu$.

$$= \int f_1(x_1) f_2(x_2)\ldots f_\nu(x_\nu) V^{(\rho)}_{\mu_1\mu_2\ldots\mu_\rho}(x_1, x_2, \ldots, x_\rho)$$

$$\cdot\ V^{(\nu-\rho)}_{\mu_{\rho+1}\mu_{\rho+2}\ldots\mu_\nu}(x_{\rho+1}, x_{\rho+2}, \ldots, x_\nu)[dx]^\nu \text{ by using}$$

theorem 2.4.

Hence, equation 2.7 gives

$$V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\rho\mu_{\rho+1}\cdots\mu_\nu}(x_1,x_2,\ldots,x_\rho,x_{\rho+1}+\kappa a,x_{\rho+2}+\kappa a,\ldots,x_\nu+\kappa a)$$

$$\to V^{(\rho)}_{\mu_1\mu_2\cdots\mu_\rho}(x_1,x_2,\ldots x_\rho)V^{(\nu-\rho)}_{\mu_{\rho+1}\mu_{\rho+2}\cdots\mu_\nu}(x_{\rho+1},x_{\rho+2},\ldots,x_\nu)$$

weakly as $\kappa \to \infty$.

Case ii. Suppose ρ is odd.

Then $(\Phi,\tilde{U}(\kappa a,1)\Psi)$

$= (\varphi_{\mu_1}(g_1^\circ)\varphi_{\mu_2}(g_2^\circ)\cdots\varphi_{\mu_\nu}(g_\nu^\circ)\Psi_\circ,\Psi_\circ)$ where g_σ is as before.

$$= \int f_1(x_1)f_2(x_2)\cdots f_\nu(x_\nu)\overline{V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\rho\mu_{\rho+1}\cdots\mu_\nu}(x_1,x_2,\ldots,x_\rho,x_{\rho+1}+\kappa a,x_{\rho+2}+\kappa a,\ldots,x_\nu+\kappa a)}[dx]^\nu, \text{ and}$$

$(\Phi,\Psi_\circ)(\Psi_\circ,\Psi)$

$= (\varphi_{\mu_1}(f_1^\circ)\varphi_{\mu_2}(f_2^\circ)\cdots\varphi_{\mu_\rho}(f_\rho^\circ)\Psi_\circ,\Psi)$

$(\Psi_\circ,\varphi_{\mu_{\rho+1}}(f_{\rho+1}^\circ)\varphi_{\mu_{\rho+2}}(f_{\rho+2}^\circ)\cdots\varphi_{\mu_\nu}(f_\nu^\circ)\Psi_\circ)$

$$= \int \bar{f}_1(x_1)\bar{f}_2(x_2)\cdots\bar{f}_\nu(x_\nu)\overline{V^{(\rho)}_{\mu_1\mu_2\cdots\mu_\rho}(x_1,x_2,\ldots,x_\rho)}$$

$$\cdot V^{(\nu-\rho)}_{\mu_{\rho+1}\mu_{\rho+2}\cdots\mu_\nu}(x_{\rho+1},x_{\rho+2},\ldots,x_\nu)[dx]^\nu$$

Hence, equation 2.7 gives

$$V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\rho\mu_{\rho+1}\cdots\mu_\nu}(x_1,x_2,\ldots,x_\rho,x_{\rho+1}+\kappa a,x_{\rho+2}+\kappa a,\ldots,x_\nu+\kappa a)$$

$$\to V^{(\rho)}_{\mu_1\mu_2\cdots\mu_\rho}(x_1,x_2,\ldots,x_\rho)V^{(\nu-\rho)}_{\mu_{\rho+1}\mu_{\rho+2}\cdots\mu_\nu}(x_{\rho+1},x_{\rho+2},\ldots,x_\nu)$$

weakly as $\kappa\to\infty$.

This completes the proof of the theorem, and also the results we need for proving the Theorems in the next 2 Chapters.

CHAPTER 3

RECONSTRUCTION THEOREM

In this chapter we are going to prove that a field theory can be completely described, up to a unitary transformation, by the Vacuum Expectation Functionals. That is, given a set of tempered distributions satisfying certain relevant conditions, a set of anti-linear fields, unique up to a unitary transformation, can be reconstructed such that the given tempered distributions are the V.E. Functionals of this set of fields. This enables us to replace the entire theory of fields which consists of unbounded operators in a Hilbert Space by the corresponding V.E. Functionals, which are much easier to handle.

Following Wightman[29] and others,[30,31,32] we establish the "Reconstruction Theroem" for a single component field, i.e. a symmetric scalar anti-linear field, in the first part of this chapter. The proof is on the same lines as given by Streater and Wightman,[31] but we give it completely in order to show the essential differences between our fields and their linear fields. In the second part of this chapter, we would show how the proof can be extended to the case of a theory with any number of fields, each field having any number of components. The relevant conditions as mentioned above, to be satisfied by the given tempered distributions are those given in Theorems 2.2, 2.3, 2.4, 2.5, 2.6 and 2.7, and to avoid

repetition, we would straight off specialize to the case $\Gamma = 1$, $\Lambda(\Gamma) = 1$. (See eq. 2.2).

Theorem 3.1: The Reconstruction Theorem.

Suppose $V^{(\nu)}(x_1,x_2,\ldots,x_\nu)$, $\nu = 1,2,3,\ldots$ are a given set of tempered distributions where $x_1,x_2,\ldots x_\nu$ are real 4-vectors, and satisfying the following conditions.

(a) Relativistic Transformation Law.

$$V^{(\nu)}(\Lambda x_1+a,\Lambda x_2+a,\ldots,\Lambda x_\nu+a)$$

$$= V^{(\nu)}(x_1,x_2,\ldots,x_\nu)$$ where a is any real 4-vector and $\Lambda \in L_+^{\uparrow}$

(b) Spectral Conditions.

$V^{(\nu)}(x_1,x_2,\ldots,x_\nu) = W^{(\nu}(\xi_1,\xi_2,\ldots,\xi_{\nu-1})$, $\xi_\rho = x_\rho - x_{\rho+1}$, $1 \leqslant \rho \leqslant \nu-1$, $\nu \geqslant 2$ whose existence follows from (a) satisfies the conditions,

$$\tilde{V}^{(\nu)}(p_1,p_2,\ldots,p_\nu) = (2\pi)^2\delta\left(\sum_{\rho=1}^{\nu} p_\rho\right)\tilde{W}^{(\nu)}(p_1,p_1+p_2,p_1+p_2+p_3$$

$$,\ldots,p_1+p_2+p_3+\ldots+p_{\nu-1}),$$ and

$\tilde{W}^{(\nu)}(-q_1,q_2,-q_3,\ldots,(-1)^\rho q_\rho,\ldots,(-1)^{\nu-1}q_{\nu-1}) = 0$ unless each $q_\rho \in \overline{V}_+$, $1 \leqslant \rho \leqslant \nu-1$, where

$$\tilde{V}^{(\nu)}(p_1,p_2,\ldots,p_\nu) = \frac{1}{(2\pi)^{2\nu}} \int \exp\left(i\sum_{\rho=1}^{\nu} p_\rho \cdot x_\rho\right) V^{(\nu)}(x_1,x_2,$$

$$\ldots x_\nu)\ [dx]^\nu,$$

$$\tilde{W}^{(\nu)}(p_1p_2,\ldots,p_{\nu-1}) = \frac{1}{(2\pi)^{2(\nu-1)}} \int \exp\left(i\sum_{\rho=1}^{\nu} p_\rho \cdot \xi_\rho\right) W^{(\nu)}(\xi_1,\xi_2,$$

$$\ldots \xi_{\nu-1})\ [d\xi]^{\nu-1}.$$

Note: We will prove in Chapter 4 by a very elementary method that $V_\mu^{(1)}(x_1) = C_\mu$, a constant for each μ, but this result is irrelevant for the following discussion.

(c) Symmetry Conditions

$$V^{(\nu)}(x_1,x_2,\ldots,x_\nu) = V^{(\nu)}(x_\nu,x_{\nu-1},\ldots,x_1) \text{ if } \nu \text{ is odd}$$

$$= \overline{V^{(\nu)}(x_\nu,x_{\nu-1},\ldots,x_1)} \text{ if } \nu \text{ is even}$$

(d) Local Commutativity

$$V^{(\nu)}(x_1,x_2,\ldots,x_\rho,x_{\rho+1},\ldots,x_\nu)+\varepsilon V^{(\nu)}(x_1,x_2,\ldots,x_{\rho-1},x_{\rho+1},$$

$$x_\rho,x_{\rho+2},x_{\rho+3},\ldots,x_\nu) = 0 \text{ for all } (x_\rho-x_{\rho+1})^2 < 0,$$

for each ρ, $1 \leqslant \rho \leqslant \nu-1$. $\nu \geqslant 2$, where $\varepsilon = +1$ or -1.

(e) Positive Definiteness Conditions

If $f^{(\nu)} \in \mathcal{S}(\mathbb{R}^{4\nu})$, $\nu = 0,1,2,\ldots$, is any sequence of functions such that $f^{(\nu)} \equiv 0$ except for a finite number of values of ν, then

$$\sum_{\substack{\text{even} \\ \rho \geqslant 0}}^{\infty} \sum_{\substack{\text{all} \\ \sigma \geqslant 0}}^{\infty} \int \overline{f^{(\rho)}(x_1,x_2,\ldots,x_\rho)} V^{(\rho+\sigma)}(x_\rho,x_{\rho-1},\ldots,x_1,y_1,y_2,\ldots y_\sigma)$$

$$\cdot f^{(\sigma)}(y_1,y_2,\ldots,y_\sigma)[dx]^\rho[dy]^\sigma$$

$$+ \sum_{\substack{\text{odd} \\ \rho \geqslant 1}}^{\infty} \sum_{\substack{\text{all} \\ \sigma \geqslant 0}}^{\infty} \int \overline{f^{(\rho)}(x_1,x_2,\ldots,x_\rho)}\ \overline{V^{(\rho+\sigma)}(x_\rho,x_{\rho-1}\ldots,x_1,y_1,y_2,\ldots y_\sigma)}$$

$$\cdot f^{(\sigma)}(y_1,y_2,\ldots,y_\sigma)[dx]^\rho[dy]^\sigma \geqslant 0, \text{ where}$$

$V^{(0)} = 1$, and the other cases when $\rho,\sigma = 0$ are given the obvious interpretation. (e.g. $f^{(0)}$ is a complex number)

(f) Cluster Decomposition Property

If a is a non-null 4-vector, then

$$V^{(\nu)}(x_1,x_2,\ldots,x_\rho,x_{\rho+1}+\kappa a,\ x_{\rho+2}+\kappa a,\ldots,x_\nu+\kappa a)$$

$$\to V^{(\rho)}(x_1,x_2,\ldots,x_\rho)V^{(\nu-\rho)}(x_{\rho+1},x_{\rho+2},\ldots,x_\nu) \text{ if } \rho \text{ is even}$$

$$\to V^{(\rho)}(x_1,x_2,x_3,\ldots,x_\rho)\overline{V^{(\nu-\rho)}(x_{\rho+1},x_{\rho+2},\ldots,x_\nu)} \text{ if } \rho \text{ is odd}$$

as $\kappa \to \infty$, where the convergence is weak.

[31]Then there exists a separable Hilbert Space $\mathcal{H}$, a continuous unitary representation $\tilde{U}(a,\Lambda)$ of the Poincare Group in $\mathcal{H}$, a unique state $\Psi_o \in \mathcal{H}$ invariant under $\tilde{U}(a,\Lambda)$, and for each $h \in \mathcal{S}(R^4)$ a symmetric scalar anti-linear field $\varphi(h)$ with domain D and representation of the Poincare Group, $\tilde{U}(a,\Lambda)$, such that

$$(\Psi_o, \varphi(h_1^o)\varphi(h_2^o)\cdots\varphi(h_\nu^o)\Psi_o) = \int h_1(x_1)h_2(x_2)\ldots h_\nu(x_\nu) \cdot V^{(\nu)}(x_1,x_2,\ldots x_\nu)[dx]^\nu .$$

Further, if $\mathcal{H}_1$ is a separable Hilbert Space, $\tilde{U}_1(a,\Lambda)$ is a continuous unitary representation of the Poincare Group in $\mathcal{H}_1$, Ψ_{01} is a unique element of $\mathcal{H}_1$ invariant under $\tilde{U}_1(a,\Lambda)$ and for each $h \in \mathcal{S}(R^4)$ $\varphi_1(h)$ is a symmetric scalar anti-linear field with domain D_1 and satisfying $(\Psi_{01}, \varphi_1(h_1^o)\varphi_1(h_2^o)..\varphi_1(h_\nu^o)\Psi_{01})$

$$= \int h_1(x_1)h_2(x_2)\ldots h_\nu(x_\nu)V^{(\nu)}(x_1,x_2,\ldots,x_\nu)[dx]^\nu$$

for each $\nu \geqslant 1$ and having Ψ_{01} as a cyclic vector, then there exists a unitary transformation $\tilde{T}$ of $\mathcal{H}$ on to $\mathcal{H}_1$ such that

$$\Psi_{01} = \tilde{T}\Psi_o, \quad \tilde{U}_1(a,\Lambda) = \tilde{T}U(a,\Lambda)\tilde{T}^{-1},$$

$$\varphi_1(h) = \tilde{T}\varphi(h)\tilde{T}^{-1}, \quad D_1 = \tilde{T}D .$$

Moreover, $\varphi(h)$ satisfies local commutativity, i.e. if $h(x)g(y)=0$ whenever $(x-y)^2 \geqslant 0$, then $\varphi(h)\varphi(\bar{g})+\varepsilon\varphi(g)\varphi(\bar{h})=0$,

and if $\tilde{U}(a,1) = \exp i(a_\mu P^\mu)$, then the eigen values of P^μ lie in the forward cone $\overline{V}^+$, i.e. the physical energy momentum spectrum is in the forward cone.

Proof: Let Z be the space of all infinite sequences of the form $F = (f_o, f_1, f_2, \ldots, f_\nu, \ldots)$ where $f_o \in \mathbb{C}$, and $f_\nu \in \mathcal{S}(R^{4\nu})$, and $f_\nu = 0$ except for a finite number of values of ν.

If $G = (g_o, g_1, g_2, \ldots, g_\nu, \ldots) \in Z$, we define

(i) $F = G$ if $f_\nu = g_\nu$ for all $\nu \geqslant 0$

(ii) $\alpha F = (\alpha f_o, \alpha f_1, \alpha f_2, \ldots, \alpha f_\nu, \ldots)$, $\alpha \in \mathbb{C}$ where

$$(\alpha f_\nu)(x_1, x_2, \ldots, x_\nu) = \alpha . f_\nu(x_1, x_2, \ldots, x_\nu), \quad \nu \geqslant 1$$

(iii) $F + G = (f_o+g_o, f_1+g_1, f_2+g_2, \ldots, f_\nu+g_\nu, \ldots)$

where $(f_\nu+g_\nu)(x_1, x_2, \ldots, x_\nu) = f_\nu(x_1, x_2, \ldots, x_\nu) + g_\nu(x_1, x_2, \ldots, x_\nu)$.

With these definitions, it is clear that Z is a linear space over the field of complex numbers, with $0 = (0,0,0,\ldots,0,$ as the zero element.

Z is clearly not finite dimensional, because, the sequence of elements of Z of the form

$$E_o = (1,0,0,0,\ldots,0,\ldots)$$

$$E_\nu = (\delta_{\nu o}, \delta_{\nu 1}, \delta_{\nu 2}, \ldots, \delta_{\nu\nu} f_\nu, 0, 0, \ldots), \quad \nu \geqslant 1$$

is linearly independent

If $F = (f_o, f_1, f_2, \ldots, f_\nu, \ldots)$,

$G = (g_o, g_1, g_2, \ldots, g_\nu, \ldots)$, belong to Z

we define an inner product (scalar product) in the space Z by

$$(3.1)\quad (F,G)=\sum_{\substack{\text{even}\\ \rho\geq 0}}^{\infty}\ \sum_{\substack{\text{all}\\ \sigma\geq 0}}^{\infty}\int \overline{f_\rho(x_1,x_2,\ldots,x_\rho)}\, V^{(\rho+\sigma)}(x_\rho,x_{\rho-1},$$

$$\ldots,x_1,y_1,y_2,\ldots,y_\sigma)\cdot g_\sigma(y_1,y_2,\ldots,y_\sigma)[dx]^\rho[dy]^\sigma$$

$$+\sum_{\substack{\text{odd}\\ \rho\geq 1}}^{\infty}\ \sum_{\substack{\text{all}\\ \sigma\geq 0}}^{\infty}\int \overline{f_\rho(x_1,x_2,\ldots,x_\rho)}\,\overline{V^{(\rho+\sigma)}(x_\rho,x_{\rho-1},\ldots,x_1,y_1,y_2,\ldots,y_\sigma)}$$

$$\cdot g_\sigma(y_1,y_2,\ldots,y_\sigma)[dx]^\rho[dy]^\sigma.$$

No problems of convergence arise, because, all except a finite number of terms in the above sums vanish due to the definition of Z. This inner product satisfies the following conditions.

(3.2) (i) $(\overline{F,G}) = (G,F)$

The first sum in the definition of (F,G) can be split into two sums as

$$\sum_{\substack{\text{even}\\ \rho\geq 0}}^{\infty}\ \sum_{\substack{\text{even}\\ \sigma\geq 0}}^{\infty} \quad + \quad \sum_{\substack{\text{even}\\ \rho\geq 0}}^{\infty}\ \sum_{\substack{\text{odd}\\ \sigma\geq 1}}^{\infty},$$ and the

second sum too can be broken up the same way.

Hence $(\overline{F,G})$

$$= \sum_{\substack{\text{even} \\ \rho \geq 0}}^{\infty} \sum_{\substack{\text{even} \\ \sigma \geq 0}}^{\infty} \int f_\rho(x_1, x_2, \ldots, x_\rho) \overline{V^{(\rho+\sigma)}(x_\rho, \ldots, x_1, y_1, \ldots, y_\sigma)} \, \overline{g_\sigma(y_1, \ldots, y_\sigma)} [dx]^\rho [dy]^\sigma$$

$$+ \sum_{\substack{\text{even} \\ \rho \geq 0}}^{\infty} \sum_{\substack{\text{odd} \\ \sigma \geq 1}}^{\infty} \int f_\rho(x_1, \ldots, x_\rho) \overline{V^{(\rho+\sigma)}(x_\rho, \ldots, x_1, y_1, \ldots, y_\sigma)} \, \overline{g_\sigma(y_1, \ldots, y_\sigma)} [dx]^\rho [dy]^\sigma$$

$$+ \sum_{\substack{\text{odd} \\ \rho \geq 1}}^{\infty} \sum_{\substack{\text{even} \\ \sigma \geq 0}}^{\infty} \int f_\rho(x_1, \ldots, x_\rho) V^{(\rho+\sigma)}(x_\rho, \ldots, x_1, y_1, \ldots, y_\sigma) \, \overline{g_\sigma(y_1, \ldots, y_\sigma)} [dx]^\rho (dy]^\sigma$$

$$+ \sum_{\substack{\text{odd} \\ \rho \geq 1}}^{\infty} \sum_{\substack{\text{odd} \\ \sigma \geq 1}}^{\infty} \int f_\rho(x_1, \ldots, x_\rho) V^{(\rho+\sigma)}(x_\rho, \ldots, x_1, y_1, \ldots, y_\sigma) \, \overline{g_\sigma(y_1, \ldots, y_\sigma)} [dx]^\rho [dy]^\sigma$$

In the first summation and the last summation, $\rho + \sigma$ is even, and in the other two summations, $\rho+\sigma$ is odd. Hence if we use (c) of Theorem 3.1, we can group together the first and the third summations, and also the 2nd and 4th summations. This gives, after interchanging the summations over ρ and σ,

$$(\overline{F,G}) = \sum_{\substack{\text{even}\\ \sigma\geqslant 0}}^{\infty} \sum_{\substack{\text{all}\\ \rho\geqslant 0}}^{\infty} \int \overline{g_\sigma(y_1,y_2,\ldots,y_\sigma)} V^{(\sigma+\rho)}(y_\sigma,\ldots,y_1,x_1,\ldots,x_\rho)$$

$$f_\rho(x_1,\ldots,x_\rho)[dy]^\sigma[dx]^\rho$$

$$+ \sum_{\substack{\text{odd}\\ \sigma\geqslant 1}}^{\infty} \sum_{\substack{\text{all}\\ \rho\geqslant 0}}^{\infty} \int \overline{g(y_1,y_2,\ldots,y_\sigma)} V^{(\sigma+\rho)}(y_\sigma,\ldots,y_1,x_1,\ldots,x_\rho)$$

$$f_\rho(x_1,\ldots,x_\rho)[dy]^\sigma[dx]^\rho$$

which is nothing but (G,F)

$$\therefore \quad (\overline{F,G}) = (G,F)$$

(3.2) (ii) $(F,F) \geqslant 0$.

This is obvious from (e) of Theorem 3.1.

(3.2) (iii) $(F,\alpha G) = \alpha(F,G)$, $\alpha \in \mathbb{C}$

Follows immediately from the defining equation (3.1).

(3.2) (iv) $(F,G^{(1)} + G^{(2)}) = (F,G^{(1)}) + (F,G^{(2)})$.

This also follows immediately from equation (3.1).

This inner product satisfies all the requirements of one in a linear vector space, except that it is not positive definite. i.e., although $(F,F) \geqslant 0$ for all $F \in Z$, it does not follow that $F = 0$ if $(F,F) = 0$. This matter will be put right shortly by a standard mathematical technique.

If $\{a,\Lambda\}$ is an element of the Poincare Group,

(3.3) let $\tilde{U}(a,\Lambda)$ be the mapping

$$\tilde{U}(a,\Lambda)F = \tilde{U}(a,\Lambda)(f_o,f_1,f_2,\ldots,f_\nu,\ldots)$$

$$= (f_o,\{a,\Lambda\}f_1,\{a,\Lambda\}f_2,\ldots,\{a,\Lambda\}f_\nu,\ldots)$$

where $(\{a,\Lambda\}f_\nu)(x_1,x_2,\ldots,x_\nu)=f_\nu(\Lambda^{-1}(x_1-a),\Lambda^{-1}(x_2-a),..,\Lambda^{-1}(x_\nu-a)$

Then $\tilde{U}(a,\Lambda)$ is clearly a mapping of Z into Z. It is also clear that this mapping is linear, and $(\tilde{U}(a,\Lambda)F,\ \tilde{U}(a,\Lambda)G)$

$$= \sum_{\substack{\text{even}\\ \rho\geq 0}}^{\infty}\sum_{\substack{\text{all}\\ \sigma\geq 0}}^{\infty}\int\overline{\{a,\Lambda\}f_\rho\ (x_1,x_2,..,x_\rho)}V^{(\rho+\sigma)}(x_\rho,..,x_1,y_1,..,y_\sigma)$$

$$\{a,\Lambda\}g_\sigma(y_1,..,y_\sigma)[dx]^\rho[dy]^\sigma$$

$$+ \sum_{\substack{\text{odd}\\ \rho\geq 1}}^{\infty}\sum_{\substack{\text{all}\\ \sigma\geq 0}}^{\infty}\int\overline{\{a,\Lambda\}f_\rho\ (x_1,x_2,..,x_\rho)}\,\overline{V^{(\rho+\sigma)}(x_\rho,..,x_1,y_1,..,y_\sigma)}$$

$$\{a,\Lambda\}g_\sigma(y_1,..,y_\sigma)[dx]^\rho[dy]^\sigma.$$

$$= \sum_{\substack{\text{even}\\ \rho\geq 0}}^{\infty}\sum_{\substack{\text{all}\\ \sigma\geq 0}}^{\infty}\int\overline{f_\rho(x_1,..,x_\rho)}V^{(\rho+\sigma)}(\Lambda x_\rho+a,\Lambda x_{\rho-1}+a,..,\Lambda x_1+a,\Lambda y_1+a,$$

$$\ldots,\Lambda y_\sigma+a).g_\sigma(y_1,y_2,..,y_\sigma)[dx]^\rho[dy]^\sigma$$

$$+ \sum_{\substack{\text{odd}\\ \rho\geq 1}}^{\infty}\sum_{\substack{\text{all}\\ \sigma\geq 0}}^{\infty}\int\overline{f_\rho(x_1,x_2,..,x_\rho)}\,\overline{V^{(\rho+\sigma)}(\Lambda x_\rho+a,\Lambda x_{\rho-1}+a,..,\Lambda x_1+a,}$$

$$\overline{\ldots,\Lambda y_\sigma+a)}.g_\sigma(y_1,y_2,..,y_\sigma[dx]^\rho[dy]^\sigma,$$

where we have used the definition as in the statement of Theorem 2.2.

Now, if we use (a) of Theorem 3.1, we immediately see that,

$$(\tilde{U}(a,\Lambda)F,\ \tilde{U}(a,\Lambda)G) = (F,G)$$

(3.4) Hence $\tilde{U}(a,\Lambda)$ is a unitary operator, and therefore, the mapping is really onto.

(3.5) If $\Psi_o = (1,0,0,..,0,..)$, then $\Psi_o \in Z$, and $\tilde{U}(a,\Lambda)\Psi_o=\Psi_o$. Now let $\tilde{U}(a_1,\Lambda_1)F = G$

$$\therefore\ g_\nu(x_1,x_2,..,x_\nu)=f_\nu(\Lambda_1^{-1}(x_1-a_1),\Lambda_1^{-1}(x_2-a_1),..,\Lambda_1^{-1}(x_\nu-a_1))$$

Hence, if $\tilde{U}(a_2,\Lambda_2)G = H$, then

$$h_\nu(x_1,x_2,..,x_\nu)=g_\nu(\Lambda_2^{-1}(x_1-a_2),\Lambda_2^{-1}(x_2-a_2)..,\Lambda_2^{-1}(x_\nu-a_2))$$

$$= f_\nu(y_1,y_2,..,y_\nu)$$

where

$$y_\rho=\Lambda_1^{-1}[\Lambda_2^{-1}(x_\rho-a_2)-a_1] = (\Lambda_2\Lambda_1)^{-1}[x_\rho-(a_2+\Lambda_2 a_1)]$$

$\therefore\ \tilde{U}(a_2,\Lambda_2)\tilde{U}(a_1,\Lambda_1)F = \tilde{U}(a_2+\Lambda_2 a_1,\Lambda_2\Lambda_1)F$. Since F is arbitrary, this gives

(3.6) $\tilde{U}(a_2,\Lambda_2)\tilde{U}(a_1,\Lambda_1)=\tilde{U}(a_2+\Lambda_2 a_1,\Lambda_2\Lambda_1)$ in Z which means that $\tilde{U}(a,\Lambda)$ is a unitary representation of the Poincare Group, in Z.

(3.7) Now, for each $h \in \mathcal{S}(\mathbb{R}^4)$, let $\varphi(h)$ be the mapping of Z into Z given by

$$\varphi(h)F = \varphi(h)(f_o, f_1, f_2, .., f_\nu, ..)$$

$$= (0, \overline{f}_o h,\ h \otimes \overline{f}_1,\ h \otimes \overline{f}_2, .., h \otimes \overline{f}_\nu, ..)$$

where $h \otimes \overline{f}_\nu(x_1, x_2, .., x_{\nu+1}) = h(x_1)\overline{f_\nu(x_2, x_3, .., x_{\nu+1})} \in \mathcal{S}(\mathbb{R}^{4\nu+4})$.

i.e. if $\varphi(h)F = G$

$$= (g_o, g_1, g_2, .., g_\nu, ..), \text{ then}$$

$g_\nu = h \otimes \overline{f}_{\nu-1}$ for all $\nu \geqslant 0$ if we define

$$h \otimes \overline{f}_{\nu-1} = \overline{f}_o h \quad \text{if } \nu = 1$$

$$= 0 \quad \text{if } \nu = 0$$

Since $g_o = 0$, it is clear that the mapping $\varphi(h)$ is not onto. This mapping obviously satisfies the following conditions in Z.

(3.8) (i) $\varphi(\alpha h+\beta g) = \alpha\varphi(h)+\beta\varphi(g)$, $\alpha, \beta \in \mathbb{C}$ and $h, g \in \mathcal{S}(\mathbb{R}^4)$

(3.8) (ii) $\varphi(h)(\alpha F+\beta G) = \overline{\alpha}\varphi(h)F+\overline{\beta}\varphi(h)G$, $\alpha, \beta \in \mathbb{C}$, $h \in \mathcal{S}(\mathbb{R}^4)$ and $F, G \in Z$

Hence, $\varphi(h)$ is an anti-linear operator of Z into Z which is linear in h.

(3.8) (iii) $\tilde{U}(a,\Lambda)\varphi(h)\tilde{U}(a,\Lambda)^{-1} = \varphi(\{a,\Lambda\}h)$ in Z, where

$$(\{a,\Lambda\}h)(x) = h(\Lambda^{-1}(x-a)) \in \mathcal{S}(\mathbb{R}^4).$$

This is easily proved as follows.

For arbitrary $F = (f_o, f_1, f_2, .., f_\nu, ...) \in Z$,

$\varphi(h)F = G$ as given above.

$\therefore \tilde{U}(a,\Lambda)\varphi(h)F = \tilde{U}(a,\Lambda)G$

$= (g_o, \{a,\Lambda\}g_1, \{a,\Lambda\}g_2, .., \{a,\Lambda\}g_\nu, ...)$

Now $\{a,\Lambda\}g_\nu(x_1 x_2, .., x_\nu) = g_\nu(\Lambda^{-1}(x_1-a), \Lambda^{-1}(x_2-a), .., \Lambda^{-1}(x_\nu-a))$

$= h(\Lambda^{-1}(x_1-a))\overline{f_{\nu-1}(\Lambda^{-1}(x_2-a), \Lambda^{-1}(x_3-a), .., \Lambda^{-1}(x_\nu-a))}, \nu \geqslant 2$

$= (\{a,\Lambda)h) \otimes (\{a,\Lambda\}\overline{f}_{\nu-1})(x_1, x_2, .., x_\nu)$

Also $g_o = 0$, and $\{a,\Lambda\}g_1(x) = \overline{f}_o(\{a,\Lambda\}h)(x)$

$\therefore \tilde{U}(a,\Lambda)\varphi(h)F = \varphi(\{a,\Lambda\}h).\tilde{U}(a,\Lambda)F.$

Note that the fact that a,Λ are real is very important for this argument.

$\therefore \tilde{U}(a,\Lambda)\varphi(h) = \varphi(\{a,\Lambda\}h)\tilde{U}(a,\Lambda)$

i.e. $\tilde{U}(a,\Lambda)\varphi(h)\tilde{U}(a,\Lambda)^{-1} = \varphi(\{a,\Lambda\}h)$

Hence the field $\varphi(h)$ transforms as a scalar under the Poincare Group.

(3.8) (iv) For all $F,G \in Z$, $(F,\varphi(h)G) = (G,\varphi(h)F)$.

For, $(G,\varphi(h)F)$

$$= \sum_{\substack{\text{even} \\ \rho \geqslant 0}}^{\infty} \sum_{\substack{\text{all} \\ \sigma \geqslant 0}}^{\infty} \int \overline{g_\rho(x_1, x_2, .., x_\rho)} V^{(\rho+\sigma)}(x_\rho, .., x_1, y_1, .., y_\sigma) \cdot h(y_1) \overline{f_{\sigma-1}(y_2, y_3, .., y_\sigma)} [dx]^\rho [dy]^\sigma$$

$$+ \sum_{\substack{\text{odd}\\ \rho\geqslant 1}}^{\infty} \sum_{\substack{\text{all}\\ \sigma\geqslant 0}}^{\infty} \int \overline{g_\rho(x_1,x_2\cdots,x_\rho)}\,\overline{v^{(\rho+\sigma)}(x_\rho,\cdots,x_1,y_1,\cdots,y_\sigma)}$$

$$\cdot h(y_1)\,\overline{f_{\sigma-1}(y_2,y_3,\cdots,y_\sigma)}\,[dx]^\rho[dy]^\sigma$$

In rewriting the above summations, we note that each integrand vanishes when $\sigma = 0$ and hence $\sum_{\substack{\text{all}\\ \sigma\geqslant 0}}^{\infty} \equiv \sum_{\substack{\text{all}\\ \sigma\geqslant 1}}^{\infty}$, and then break up each of the 2 sums into 2 parts as we did in the proof of the relation $(\overline{F,G}) = (G,F)$. We also replace ρ by $\rho-1$ in the integrands, and this alters the summations over ρ as follows

$$\sum_{\substack{\text{even}\\ \rho\geqslant 2}}^{\infty} \to \sum_{\substack{\text{odd}\\ \rho\geqslant 1}}^{\infty}, \qquad \sum_{\substack{\text{odd}\\ \rho\geqslant 1}}^{\infty} \to \sum_{\substack{\text{even}\\ \rho\geqslant 2}}^{\infty}$$

Moreover, we replace σ by $\sigma+1$. and this changes $\sum_{\substack{\text{all}\\ \sigma\geqslant 1}}^{\infty}$ to $\sum_{\substack{\text{all}\\ \sigma\geqslant 0}}^{\infty}$.

Finally, we change the variables of integration by

$$y_\mu = x_{\mu-1}, \quad 2 \leqslant \mu \leqslant \sigma+1$$

$$y_1 = w_1$$

$$x_\nu = w_{\nu+1} \quad 1 \leqslant \nu \leqslant \rho-1.$$

This gives us $(G,\varphi(h)F)$

$$= \sum_{\substack{\text{odd}\\ \rho\geq 1}}^{\infty} \sum_{\substack{\text{even}\\ \sigma\geq 0}}^{\infty} \int \overline{g_{\rho-1}(w_2,w_3,\ldots,w_\rho)}\, V^{(\rho+\sigma)}(w_\rho,w_{\rho-1},\ldots,w_2,w_1,$$

$$z_1,z_2,\ldots,z_\sigma).h(w_1)\overline{f_\sigma(z_1,z_2,\ldots,z_\sigma)}[dw]^\rho[dz]^\sigma$$

$$+ \sum_{\substack{\text{odd}\\ \rho\geq 1}}^{\infty} \sum_{\substack{\text{odd}\\ \sigma\geq 1}}^{\infty} \int \quad \text{(same integrand as above)}$$

$$+ \sum_{\substack{\text{even}\\ \rho\geq 2}}^{\infty} \sum_{\substack{\text{even}\\ \sigma\geq 0}}^{\infty} \int \overline{g_{\rho-1}(w_2,w_3,\ldots,w_\rho)}\, \overline{V^{(\rho+\sigma)}(w_\rho,w_{\rho-1},\ldots,w_1,z_1,z_2,\ldots,z_\sigma)}$$

$$.h(w_1)\overline{f_\sigma(z_1,z_2,\ldots,z_\sigma)}[dw]^\rho[dz]^\sigma$$

$$+ \sum_{\substack{\text{even}\\ \rho\geq 2}}^{\infty} \sum_{\substack{\text{odd}\\ \sigma\geq 1}}^{\infty} \int \quad \text{(same integrand as above).}$$

Since $h(w_1)\overline{g_{\rho-1}(w_2,w_3,\ldots,w_\rho)} = (h \otimes \overline{g}_{\rho-1})(w_1,w_2,\ldots,w_\rho)$

$= 0$ when $\rho = 0$, we can replace $\sum_{\substack{\text{even}\\ \rho\geq 2}}^{\infty}$ by $\sum_{\substack{\text{even}\\ \rho\geq 0}}^{\infty}$. Now, if we use (c) of Theorem 3.1, we can group together the 1st and the 3rd summations and also the

2nd and the 4th summations. Finally, by interchanging the order of summations, we get $(G,\varphi(h)F)$

$$= \sum_{\substack{\text{even} \\ \sigma \geqslant 0}}^{\infty} \sum_{\substack{\text{all} \\ \rho \geqslant 0}}^{\infty} \int \overline{f_\sigma(z_1,z_2,\ldots,z_\sigma)} V^{(\sigma+\rho)}(z_\sigma,z_{\sigma-1},\ldots,z_1,w_1,\ldots,w_\rho) \cdot (h \otimes \bar{g}_{\rho-1})(w_1,w_2,\ldots,w_\rho)[dz]^\sigma[dw]^\rho$$

$$+ \sum_{\substack{\text{odd} \\ \sigma \geqslant 1}}^{\infty} \sum_{\substack{\text{all} \\ \rho \geqslant 0}}^{\infty} \int \overline{f_\sigma(z_1,z_2,\ldots,z_\sigma)} \overline{V^{(\sigma+\rho)}(z_\sigma,z_{\sigma-1},\ldots,z_1,w_1,\ldots,w_\rho)} \cdot (h \otimes \bar{g}_{\rho-1})(w_1,w_2,\ldots,w_\rho)[dz]^\sigma[dw]^\rho.$$

$$= (F,\varphi(h)G$$

$$\text{i.e} \quad (F,\varphi(h)G) = (G,\varphi(h)F)$$

(3.8) (v) $\quad (\Psi_o,\varphi(h^{o(1)})\varphi(h^{o(2)}) \ldots \varphi(h^{o(\nu)})\Psi_o)$

$$= \int h^{(1)}(x_1)h^{(2)}(x_2)\ldots h^{(\nu)}(x_\nu)V^{(\nu)}(x_1,x_2,\ldots,x_\nu)[dx]^\nu$$

This follows immediately because

$$\varphi(h^{o(1)})\varphi(h^{o(2)})\ldots\varphi(h^{o(\nu)})\Psi_o = (g_o,g_1,g_2,\ldots,g_\rho,\ldots),$$

where $g_\rho = 0$ if $\rho \neq \nu$

and $g_\nu(x_1,x_2,\ldots,x_\nu)=h^{(1)}(x_1)h^{(2)}(x_2)\ldots h^{(\nu)}(x_\nu)$, because

$$\Psi_o = (1,0,0,\ldots,0,\ldots)$$

(3.8) (vi) If $h^{(1)}, h^{(2)} \in \mathcal{S}(\mathbb{R}^4)$ are such that $h^{(1)}(x)h^{(2)}(y)=0$ whenever $(x-y)^2 \geqslant 0$, then $(F,[\varphi(h^{(1)})\varphi(\bar{h}^{(2)})+\varepsilon\varphi(h^{(2)})\varphi(\bar{h}^{(1)})]G) = 0$

for all $F,G \in Z$.

$$(F,[\varphi(h^{(1)})\varphi(\bar{h}^{(2)})+\varepsilon\varphi(h^{(2)})\varphi(\bar{h}^{(1)})]G)$$

$$= \sum_{\substack{\text{even} \\ \rho \geqslant 0}}^{\infty} \sum_{\substack{\text{all} \\ \sigma \geqslant 0}}^{\infty} \int \overline{f_\rho(x_1,x_2,\ldots,x_\rho)} V^{(\rho+\sigma)}(x_\rho,x_{\rho-1},\ldots,x_1,y_1,\ldots,y_\sigma)$$

$$\cdot[h^{(1)}(y_1)h^{(2)}(y_2)+\varepsilon h^{(2)}(y_1)h^{(1)}(y_2)]$$

$$g_{\sigma-2}(y_3,y_4,\ldots,y_\sigma)[dx]^\rho[dy]^\sigma$$

$$+ \sum_{\substack{\text{odd} \\ \rho \geqslant 1}}^{\infty} \sum_{\substack{\text{all} \\ \sigma \geqslant 0}}^{\infty} \int \overline{f_\rho(x_1,x_2,\ldots,x_\rho)} \overline{V^{(\rho+\sigma)}(x_\rho,x_{\rho-1},\ldots,x_1,y_1,\ldots,y_\sigma)}$$

$$[h^{(1)}(y_1)h^{(2)}(y_2)+\varepsilon h^{(2)}(y_1)h^{(1)}(y_2)]$$

$$g_{\sigma-2}(y_3,y_4,\ldots,y_\sigma)[dx]^\rho[dy]^\sigma$$

(The integrands corresponding to $\sigma = 0,1$ are identically zero).

Now, if we interchange the variables y_1 and y_2 of integration in the terms containing the factor ε, and use the condition (d) of Theorem 3.1, we see that each double sum vanishes identically.

Hence the result is true.

We also have the following results for $\tilde{U}(a,\Lambda)$.

(3.9) (i) $$\text{Max}\ \frac{(\{\tilde{U}(a,\Lambda)-\tilde{U}(a_o,\Lambda_o)\}F,\{\tilde{U}(a,\Lambda)-\tilde{U}(a_o,\Lambda_o)\}F)^{\frac{1}{2}}}{(F,F)^{\frac{1}{2}}}$$

as F varies over Z, subject to $(F,F) \neq 0$ tends to zero as $a \to a_o$, $\Lambda \to \Lambda_o$.

This follows from the definition and from the properties of functions in $\mathcal{S}(R^{4\nu})$.

(3.9) (ii) For all $F,G \in Z$, $\frac{1}{(2\pi)^4}\int e^{-iq\cdot a}(F,\tilde{U}(a,1)G)da$ vanishes unless q lies in or on the forward cone $\overline{V}^+$.

As we will see very shortly, if $F, H \in Z$, then $|(F,H)| \leqslant (F,F)^{\frac{1}{2}}\ (H,H)^{\frac{1}{2}}$, although the scalar product in Z is only positive semi-definite.[35]

$\therefore\ |(F,\tilde{U}(a,1)G)| \leqslant ||F||.||G||$ because of equation (3.4), where we have written $||F|| = (F,F)^{\frac{1}{2}}$.

$\therefore\ (F,\tilde{U}(a,1)G)$ is bounded for all a, and by definition is also a continuous function of a.

Hence it is a tempered distribution over $\mathcal{S}(R^4)$.

$\therefore\ \frac{1}{(2\pi)^4}\int e^{-iq\cdot a}(F,\tilde{U}(a,1)G)da$ is also a tempered distribution and can be written as

$\frac{1}{(2\pi)^4}\ (\alpha+\overline{\beta})$, where

$$\alpha = \sum_{\substack{\text{even}\\ \rho\geqslant 0}}^{\infty} \sum_{\substack{\text{all}\\ \sigma\geqslant 0}}^{\infty} \int \overline{f_\rho(x_1,x_2,\ldots,x_\rho)} V^{(\rho+\sigma)}(x_\rho,x_{\rho-1},\ldots,x_1,y_1,\ldots,y_\sigma)$$

$$\cdot g_\sigma(y_1-a,y_2-a,\ldots,y_\sigma-a)e^{-iq\cdot a}\cdot da[dx]^\rho[dy]^\sigma$$

and

$$\beta = \sum_{\substack{\text{odd}\\ \rho\geqslant 1}}^{\infty} \sum_{\substack{\text{all}\\ \sigma\geqslant 0}}^{\infty} \int \overline{g_\sigma(y_1-a,y_2-a,\ldots,y_\sigma-a)} V^{(\rho+\sigma)}(x_\rho,x_{\rho-1},\ldots,x_1,y_1\ldots y_\sigma)$$

$$\cdot f_\rho(x_1,x_2,\ldots,x_\rho)e^{iq\cdot a}\, da[dx]^\rho[dy]^\sigma.$$

If we put $f_\rho(x_1,x_2,\ldots,x_\rho) = \frac{1}{(2\pi)^{2\rho}} \int \exp\left(-i\sum_{\mu=1}^{\rho} p_\mu\cdot x_\mu\right)$

$$\tilde{f}_\rho(p_1,p_2,\ldots,p_\rho)[dp]^\rho$$

and $g_\sigma(y_1,y_2,\ldots,y_\sigma) = \frac{1}{(2\pi)^{2\sigma}} \int \exp\left(-i\sum_{\mu=1}^{\sigma} q_\mu\cdot y_\mu\right)$

$$\tilde{g}_\sigma(q_1,q_2,\ldots,q_\sigma)[dq]^\sigma$$

we get

$$\alpha = \sum_{\substack{\text{even} \\ \rho \geqslant 0}}^{\infty} \sum_{\substack{\text{all} \\ \sigma \geqslant 0}}^{\infty} \frac{1}{(2\pi)^{2(\rho+\sigma)}} \int \overline{\tilde{f}_\rho(p_1,p_2,..,p_\rho)} . \tilde{g}_\sigma(q_1,q_2,..,q_\sigma) .$$

$$. \exp(+i \sum_{\mu=1}^{\rho} p_\mu . x_\mu - i \sum_{\mu=1}^{\sigma} q_\mu . y_\mu - iq.a) W^{(\rho+\sigma)}(x_\rho - x_{\rho-1}, x_{\rho-1} - x_{\rho-2},$$

$$\ldots, x_2 - x_1, x_1 - y_1 - a, y_1 - y_2, .., y_{\sigma-1} - y_\sigma)[dp]^\rho [dq]^\sigma [dx]^\rho [dy]^\sigma da$$

where we have changed the variables $y_\mu \to y_\mu + a$ for $1 \leqslant \mu \leqslant \sigma$, and used condition (b) of Theorem 3.1.

Now $$+ \sum_{\mu=1}^{\rho} p_\mu . x_\mu - \sum_{\mu=1}^{\sigma} q_\mu . y_\mu - q.a$$

$$= +p_\rho . (x_\rho - x_{\rho-1}) + (p_\rho + p_{\rho-1}) . (x_{\rho-1} - x_{\rho-2}) + (p_\rho + p_{\rho-1} + p_{\rho-2}) . (x_{\rho-2} - x_{\rho-3})$$

$$+ \ldots + (p_\rho + p_{\rho-1} + \ldots + p_2) . (x_2 - x_1) + P . (x_1 - y_1 - a) + (P-q) . a$$

$$+ (P - q_1) . (y_1 - y_2) + (P - q_1 - q_2) . (y_2 - y_3) + (P - q_1 - q_2 - q_3) . (y_3 - y_4)$$

$$+ \ldots + (P - q_1 - q_2 \ldots q_{\sigma-1}) . (y_{\sigma-1} - y_\sigma) + (P - Q) . y_\sigma ,$$

where $P = \sum_{\mu=1}^{\rho} p_\mu$, $Q = \sum_{\mu=1}^{\sigma} q_\mu$.

Now, if we change the variables $x_1, x_2, .., x_\rho$; $y_1, y_2, .., y_\sigma$; a, according to

$$\xi_\rho = x_\rho - x_{\rho-1}, \qquad \eta_1 = y_1 - y_2,$$

$$\xi_{\rho-1} = x_{\rho-1} - x_{\rho-2}, \qquad \eta_2 = y_2 - y_3,$$

$$\xi_{\rho-2} = x_{\rho-2} - x_{\rho-3}, \qquad \eta_3 = y_3 - y_4,$$

$$\cdots\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots,$$

$$\xi_2 = x_2 - x_1 \qquad \cdots\cdots\cdots\cdots, \ \cdots$$

$$\xi_1 = x_1 - y_1 - a \qquad \eta_{\sigma-1} = y_{\sigma-1} - y_\sigma,$$

$$\xi_o = a \qquad \eta_\sigma = y_\sigma,$$

we can do the integration over $x_1, x_2, .., x_\rho$; $y_1, y_2, .., y_\sigma$; a, and we get

$$(3.10) \quad \alpha = \sum_{\substack{\text{even} \\ \rho \geqslant 0}}^{\infty} \sum_{\substack{\text{all} \\ \sigma \geqslant 0}}^{\infty} \frac{1}{(2\pi)^2} \cdot \int (2\pi)^4 \delta\,(P-q) \cdot (2\pi)^4 \delta\,(P-Q) \cdot$$

$$\overline{\mathfrak{F}_\rho(p_1, p_2, .., p_\rho)}\, \tilde{g}_\sigma(q_1, q_2, .., q_\sigma)\, \mathfrak{W}^{(\rho+\sigma)}(p_\rho, p_\rho + p_{\rho-1},$$

$$p_\rho + p_{\rho-1} + p_{\rho-2}, .., P-p_1, P, P-q_1, P-q_1-q_2, .., P-Q+q_\sigma)\,[dp]^\rho [dq]^\sigma$$

where P occurs in the ρth place in the argument of $\mathfrak{W}^{(\rho+\sigma)}$, and the δ functions come from the integrations over ξ_o and η_σ.

Due to the presence of $\delta(P-q)$ in the integrand, the integration can be taken only over the hypersurface

$P-q = 0$, and if q is not in or on the forward cone, P cannot be in or on the forward cone. Since ρ is even, condition (b) of Theorem 3.1 implies that the tempered distribution $\tilde{W}^{(\rho+\sigma^-)}$ vanishes everywhere within the region of integration.

$$\therefore \quad \alpha = 0.$$

Now β is obtained from α by replacing $\bar{f}_\rho$ by f_ρ, g_σ by $\bar{g}_\sigma$, even $\rho \geqslant 0$ by odd $\rho \geqslant 1$ and q by $-q$. This simply has the effect of changing equation (3.10) by replacing even $\rho \geqslant 0$ by odd $\rho \geqslant 1$, and $p_1, p_2, \ldots, p_\rho, q_1, q_2, \ldots, q_\sigma, q$ by $-p_1, -p_2, \ldots, -p_\rho, -q_1, -q_2, \ldots, -q_\sigma, -q$ respectively.

By the same argument as in the previous case, we see that $\beta = 0$.

Hence if q does not lie in or on the forward cone

$\frac{1}{(2\pi)^4} \int e^{-iq\cdot a}(F, \tilde{U}(a,1)\Psi)da = 0$, which is the result we had to establish.

As mentioned earlier (preceding equation (3.3)) the scalar product in Z is only[35] positive semi-definite. This can be rectified as follows. Firstly we require the inequality $|(F,G)| \leqslant ||F||.||G||$ for all $F, G \in Z$.

(3.11) This is proved in reference 2, page 40, Theorem 1. It is important to note that the proof goes through even though the scalar product is only positive semi-definite.

Let Z_o be the set of all $F \in Z$ such that $||F|| = (F,F)^{\frac{1}{2}} = 0$. Then obviously $0 \in Z_o$.

If $F \in Z_o$, then $\alpha F \in Z_o$ for all $\alpha \in \mathbb{C}$, because $||\alpha F|| = (\alpha F, \alpha F)^{\frac{1}{2}} = |\alpha| . ||F|| = 0$. Also $(F,G) = 0$ for all $G \in Z$, since $|(F,G)| \leqslant ||F|| . ||G|| = 0$

Hence, if G also belongs to Z_o, then $(\alpha F+\beta G, \alpha F+\beta G) = |\alpha|^2 (F,F)+\bar{\alpha}\beta(F,G)+\alpha\bar{\beta}(G,F)+|\beta|^2 (G,G)$ by equations (3.2)(iii) and (3.2)(iv) $= 0$ implying that $\alpha F+\beta G \in Z_o$ for all $\alpha,\beta \in \mathbb{C}$.

(3.12) Hence Z_o, the set of all $F \in Z$ such that $(F,F) = 0$ is a subspace of Z and is such that if $F \in Z_o$, $G \in Z$, then $(F,G) = 0$

Now, given any $F \in Z$, let ΩF denote the set of all elements of Z such that if $G \in \Omega F$, then $G-F \in Z_o$. If we define $\Omega F = \Omega G$ iff ΩF and ΩG consist of the same elements of Z, then we can establish the following.

(3.13) (i) $\Omega 0 = Z_o$. This is obvious.

(3.13) (ii) $\Omega F = \Omega G$ iff $F-G \in Z_o$

Suppose $\Omega F = \Omega G$.

Since $F \in \Omega F$, we have $F \in \Omega G$

$\therefore F-G \in Z_o$.

Conversely, suppose $F-G \in Z_o$

If $H \in \Omega F$, then $H-F \in Z_o$

$\therefore H-G \in Z_o$ since Z_o is a sub-space

$\therefore H \in \Omega G$

Similarly, if $H \in \Omega G$, then $H \in \Omega F$

$\therefore \Omega F = \Omega G$

(3.13) (iii) ΩG and ΩF are either equal, i.e. $\Omega G = \Omega F$, or they are disjoint, i.e. they have no elements in common.

Suppose they are not disjoint. Then $\exists H \in Z$ such that $H \in \Omega F$, $H \in \Omega G$

$\therefore H-F \in Z_o$

$H-G \in Z_o$

$\therefore F-G \in Z_o$

Hence, by equation (3.13)(ii) $\Omega F = \Omega G$.

(3.13) (iv) If $G \in \Omega F$, then $\Omega G = \Omega F$.

This follows from equation (3.13)(ii)

ΩF is really the set of all elements of Z equivalent to F where equivalence is defined by $G \sim F$ is $G-F \in Z_o$

This is clearly an equivalence relation[36] because

(i) $F \sim F$

(ii) $F \sim G$ implies $G \sim F$

(iii) $F \sim G$, $G \sim H$ implies $F \sim H$.

$\therefore \Omega F$ is an equivalence class.

(3.14) Now, suppose we define

(i) $\alpha\Omega F = \Omega(\alpha F)$

(ii) $\Omega F+\Omega G = \Omega(F+G)$

Then we can see that $\Omega F+\Omega G$ is also the set of all elements of Z of the form F_1+G_1 where $F_1 \in \Omega F$, $G_1 \in \Omega G$.

For, if $F_1 \in \Omega F$, $G_1 \in \Omega G$, then

$$F_1-F \in Z_o$$

$$G_1-G \in Z_o$$

$$\therefore (F_1+G_1) - (F+G) \in Z_o$$

$$\therefore F_1+G_1 \in \Omega(F+G)$$

Further, if $H \in \Omega(F+G)$

$$H-(F+G) \in Z_o$$

$$\therefore H-G \in \Omega F$$

But $G \in \Omega G$

$$\therefore H = (H-G)+G \text{ where } H-G \in \Omega F,\ G \in \Omega G,$$

which establishes the result.

On the other hand, $\alpha\Omega F$ is the set of all elements of the form αF_1 where $F_1 \in \Omega F$ only if $\alpha \neq 0$, but if $\alpha = 0$, $\alpha\Omega F = Z_o$

For, suppose $\alpha \neq 0$.

If $G \in \alpha\Omega F = \Omega(\alpha F)$, then

$$G-\alpha F \in Z_o$$

$$\therefore \frac{1}{\alpha}G \in \Omega F$$

$$\therefore G = \alpha F_1 \text{ where } F_1 \in \Omega F.$$

If $G = \alpha F_1$ where $F_1 \in \Omega F$, then

$$F_1-F \in Z_o$$

$$\text{i.e. } \frac{1}{\alpha}G-F \in Z_o$$

$$\text{i.e. } G-\alpha F \in Z_o$$

$$\therefore G \in \Omega(\alpha F) = \alpha\Omega F.$$

If $\alpha = 0$, then obviously $\alpha\Omega F = Z_o$

Now, we can see that the set of all such equivalence classes forms a linear space over the field of complex numbers, with Z_o as the unique additive identity, because, in the first place

$$\begin{aligned} \Omega F+Z_o &= \Omega F+\Omega 0 && \text{(equation 3.13 (1))} \\ &= \Omega(F+0) && \text{(Defn. 3.14 (ii))} \\ &= \Omega F, \end{aligned}$$

and if $\Omega F+\Omega G = \Omega F$,

then $\Omega(F+G) = \Omega F$

Hence, using equation (3.13) (ii)

$$(F+G)-F \in Z_o$$

$$\therefore G \in Z_o$$

Then by equation (3.13) (iv), $\Omega G = Z_o$.

This linear space is denoted by Z/Z_o.

Suppose $F_1 \in \Omega F$, $G_1 \in \Omega G$

Then $(F_1,G_1) = (F+N_1, G+N_2)$ where

$$N_1 = F_1-F \in Z_o$$

and $$N_2 = G_1-G \in Z_o$$

$$= (F,G)$$ because of the properties of the scalar product in Z (equations 3.2) and equation (3.12).

Since this is independent of the respective elements chosen from ΩF and ΩG, it can be taken to be the definition of the scalar product $(\Omega F,\Omega G)$. This scalar product satisfies the following

(3.15) (i) $(\Omega F,\Omega F) = (F,F) \geqslant o$

(3.15) (ii) $(\Omega F,\Omega F) = 0$ implies $(F,F) = 0$

$\therefore F \in Z_o$

$\therefore \Omega F = Z_o$

(3.15) (iii) $(\overline{\Omega F,\Omega G}) = (\Omega G,\Omega F)$

(3.15) (iv) $(\Omega F,\alpha\Omega G) = (\Omega F,\Omega(\alpha G))$

$$= (F,\alpha G) = \alpha(F,G)$$

$$= \alpha(\Omega F,\Omega G), \quad \alpha \in \mathbb{C}$$

$$(3.15)(v)(\Omega F, \Omega G+\Omega H) = (\Omega F, \Omega(G+H))$$
$$= (F, G+H)$$
$$= (F,G)+(F,H)$$
$$= (\Omega F, \Omega G) + (\Omega F, \Omega F).$$

Hence Z/Z_o is a linear vector space with a positive definite metric, with Z_o as the unique null element.

It is obvious that we would not be able to prove that Z/Z_o is infinite dimensional, because, we have not attempted to discover the consequences for the V.E. Functionals due to the dimensionality of the underlying vector space being infinite.

The next thing to do is to extend the definitions of $\tilde{U}(a,\Lambda)$ and $\varphi(h)$ $(h \in \mathcal{S}(R^4)$ to the space Z/Z_o.

We define

$$(3.16)(i) \quad \tilde{U}(a,\Lambda)(\Omega F) = \Omega[\tilde{U}(a,\Lambda)F] \quad \text{and}$$

$$(3.16)(ii) \quad \varphi(h)(\Omega F) = \Omega[\varphi(h)F], \quad h \in \mathcal{S}(R^4).$$

For consistency we have to show that the above definitions are independent of the representative vector chosen from each equivalence class ΩF.

i.e., if $\Omega F = \Omega G$, or in other words, if $G \in \Omega F$, we must show that

$$\Omega[\tilde{U}(a,\Lambda)F] = \Omega[\tilde{U}(a,\Lambda)G], \text{ and}$$
$$\Omega[\varphi(h)F] = \Omega[\varphi(h)G].$$

The first relation is proved as follows.

$$||\tilde{U}(a,\Lambda)F - \tilde{U}(a,\Lambda)G|| = ||\tilde{U}(a,\Lambda)(F-G)||$$

$$= ||F-G|| = 0 \quad \text{since } G \in \Omega F$$

$$\therefore \tilde{U}(a,\Lambda)F - \tilde{U}(a,\Lambda)G \in Z_o$$

Hence, by equation (3.13) (ii),

$\Omega[\tilde{U}(a,\Lambda)F] = \Omega[\tilde{U}(a,\Lambda)G]$ which establishes the consistency of definition (3.16) (i). In this case we really have more. $\tilde{U}(a,\Lambda)(\Omega F)$ is the set of all $\tilde{U}(a,\Lambda)G$ when G varies over the whole of ΩF. This follows because if H is any vector belonging to the equivalence class $\Omega[\tilde{U}(a,\Lambda)F]$, then

$$||H-\tilde{U}(a,\Lambda)F|| = 0$$

$$\therefore ||\tilde{U}(a,\Lambda)^{-1}H-F|| = 0$$

i.e. $\tilde{U}(a,\Lambda)^{-1}H \in \Omega F$

i.e. given any H in $\tilde{U}(a,\Lambda)[\Omega F]$, $\exists$ $G = \tilde{U}(a,\Lambda)^{-1} H \in \Omega F$ such that $\tilde{U}(a,\Lambda)G = H$.

The second relation is proved as follows.

$$||\varphi(h)F-\varphi(h)G||^2 = (\varphi(h)(F-G), \varphi(h)(F-G))$$

$= (\varphi(h).\varphi(h)(F-G), (F-G)$ by the result in equation (3.8) (iv).

$\leqslant ||\varphi(h).\varphi(h)(F-G)||.||F-G||$ due to the inequality (3.11)

$= 0$ because $G \in \Omega F$.

$\therefore$ By equation (3.13) (ii), $\Omega[\varphi(h)F]=\Omega[\varphi(h)G]$ i.e. the definition (3.16) (ii) is independent of the representative vector chosen from ΩF. Unlike in the definition (3.16) (i), $\varphi(h)(\Omega F)$ is not the set of all $\varphi(h)G$ when G varies over ΩF, except when Z_o is one dimensional in which case the question does not arise.

For, suppose $\varphi(h)(\Omega F)$ is the set of all $\varphi(h)G$ when G varies over ΩF.

$\therefore$ given $H \in \varphi(h)(\Omega F)=\Omega[\varphi(h)F]$, $\exists\ G \in \Omega F$ such that $\varphi(h)G = H$

Let $K_o \in Z_o$.

$\therefore$ $H+K_o \in \varphi(h)(\Omega F) = \Omega[\varphi(h)F]$

Hence $\exists\ G^{(1)} \in \Omega F$ such that

$$\varphi(h)G^{(1)} = H+K_o$$

But $G^{(1)}, G \in \Omega F$

$\therefore$ $\exists\ L_o \in Z_o$ such that $G^{(1)} = G+L_o$

$\therefore$ $\varphi(h)(G+L_o) = H+K_o$

Hence, given $K_o \in Z_o$, $\exists\ L_o \in Z_o$ such that

$$\varphi(h)L_o = K_o.$$

i.e. $\varphi(h)$ maps Z_o onto Z_o.

Hence $\varphi(h_1).\varphi(h_2)\ldots\varphi(h_\nu)$ maps Z_o onto Z_o for all $\nu \geqslant 1$.

Now take arbitrary $K_o=(k_o,k_1,k_2,\ldots,k_\nu,\ldots) \in Z_o$. Let $\rho \geqslant 0$ be the smallest integer such that $k_\rho \neq 0$. Then $\exists\ L_o = (\ell_o,\ell_1,\ell_2,\ldots,\ell_\nu,\ldots) \in Z_o$ such that

$$\varphi(h_1)\varphi(h_2)\ldots\varphi(h_{\rho+1})L_o = K_o$$

But, by the definition of $\varphi(h)$, this implies that $k_\rho = 0$ which is a contradiction, unless the only vector belonging to Z_o is $0 = (0,0,0,\ldots,0,\ldots)$. Hence, unless Z_o is one dimensional, $\varphi(h)[\Omega\mathfrak{F}]$ cannot be the set of all $\varphi(h)G$ as G varies over $\Omega\mathfrak{F}$.

All the properties established for the operators $\tilde{U}(a,\Lambda)$ and $\varphi(h)$ in the space Z obviously hold in the space Z/Z_o too.

For example,

$\tilde{U}(a,\Lambda)\varphi(h)(\Omega F)=\tilde{U}(a,\Lambda)\Omega(\varphi(h)F)$ by definition (3.16) (ii)

$= \Omega(\tilde{U}(a,\Lambda)\varphi(h)F)$ " (3.16)(i)

$= \Omega(\varphi(\{a,\Lambda\}h)\tilde{U}(a,\Lambda)F)$ by equation (3.8) (iii)

$= \varphi(\{a,\Lambda\}h)\Omega(\tilde{U}(a,\Lambda)F)$ by definition (3.16) (ii)

$= \varphi(\{a,\Lambda\}h)\tilde{U}(a,\Lambda)(\Omega F)$ " (3.16) (i)

$\therefore\ \tilde{U}(a,\Lambda)\varphi(h)\tilde{U}(a,\Lambda)^{-1} = \varphi(\{a,\Lambda\}h)$ on Z/Z_o.

Now, we have a linear vector space Z/Z_o with a +ve definite metric and having $\Omega\Psi_o$ as a vacuum, and having certain operators $\tilde{U}(a,\Lambda)$ and $\varphi(h)$ defined on it. This space Z/Z_o can fall short of a Hilbert Space only by being not complete. In the following we will effect a completion

of the space Z/Z_o and see that the resulting space is also separable.

Completion of the Space Z/Z_o.

In this section we write F instead of ΩF whenever it would not lead to any confusion, i.e. capital letters at the beginning of the English alphabet would stand for elements of Z/Z_o.

Let Y be the set of all Cauchy sequences of the form $\Psi = (F_1, F_2, \ldots, F_\nu, \ldots)$. i.e. sequences of this form where $F_\rho \in Z/Z_o$, $\rho \geqslant 1$ such that $||F_\rho - F_\sigma|| \to 0$ as $\rho, \sigma \to \infty$.

If $\Phi = (G_1, G_2, \ldots, G_\nu, \ldots) \in Y$, then we define

$\alpha\Psi = (\alpha F_1, \alpha F_2, \ldots, \alpha F_\nu, \ldots)$, $\alpha \in \mathbb{C}$,

$\Psi + \Phi = (F_1 + G_1, F_2 + G_2, \ldots, F_\nu + G_\nu, \ldots)$, and

$\Phi = \Psi$ iff $G_\rho = F_\rho$, $\rho \geqslant 1$.

Now $||\alpha F_\rho - \alpha F_\sigma|| = |\alpha| \cdot ||F_\rho - F_\sigma|| \to 0$ as $\rho, \sigma \to \infty$, and

$$||F_\rho + G_\rho - F_\sigma - G_\sigma|| = ||(F_\rho - F_\sigma) + (G_\rho - G_\sigma)||$$

$$\leqslant ||F_\rho - F_\sigma|| + ||G_\rho - G_\sigma||$$

$$\to 0 \text{ as } \rho, \sigma \to \infty.$$

$\therefore$ $\alpha\Psi, \Psi + \Phi$ belong to Y.

The above inequality follows directly from (3.11)

Hence Y is a linear space over the complex field.

We define a scalar product over Y by

$$(\Psi,\Phi) = \lim_{\rho\to\infty} (F_\rho, G_\rho).$$

In the first place, for each ρ, (F_ρ, G_ρ) is a scalar, and

$$|(F_\rho,G_\rho)-(F_\sigma,G_\sigma)| = |(F_\rho,G_\rho)-(F_\rho,G_\sigma)+(F_\rho,G_\sigma)-(F_\sigma,G_\sigma)|$$

$$= |(F_\rho, G_\rho-G_\sigma)+(F_\rho-F_\sigma,G_\sigma)|$$

$$\leqslant |(F_\rho,G_\rho-G_\sigma)|+|(F_\rho-F_\sigma,G_\sigma)|$$

$$\leqslant ||F_\rho||.||G_\rho-G_\sigma||+||F_\rho-F_\sigma||.||G_\sigma||$$

But $||F_\rho-F_\sigma|| \geqslant \left|\,||F_\rho||-||F_\sigma||\,\right|$ for all ρ,σ.

$$\therefore \left|\,||F_\rho||-||F_\sigma||\,\right| \to 0 \text{ as } \rho,\sigma \to \infty$$

Hence, by the General Principle of convergence,

$||F_\rho||$ converges as $\rho \to \infty$

Similarly $||G_\sigma||$ converges as $\sigma \to \infty$.

$$\therefore \left|(F_\rho,G_\rho)-(F_\sigma,G_\sigma)\right| \to 0 \text{ as } \rho,\sigma \to \infty$$

$$\therefore \lim_{\rho\to\infty} (F_\rho,G_\rho) \text{ exists.}$$

In the second place, we have the following

(i) $(\Psi,\alpha\Phi) = \alpha(\Psi,\Phi)$

(ii) $(\overline{\Phi,\Psi}) = (\Psi,\Phi)$

(iii) $(\Phi,\Psi^{(1)}+\Psi^{(2)}) = (\Phi,\Psi^{(1)})+(\Phi,\Psi^{(2)})$.

(iv) $(\Phi,\Phi) \geqslant 0$

Hence, it is a suitable definition of scalar product. But, as in the case of Z, this scalar product may not be positive definite. This short coming can be rectified the same way by defining the linear space Y/Y_o, where Y_o is the subspace of all $\Phi \epsilon Y$ such that $(\Phi,\Phi) = 0$. i.e. Given $\Phi \epsilon Y$, we define $\Omega\Phi$ as the class of all elements of Y such that, if $\Psi \epsilon \Omega\Phi$, then $\Psi-\Phi \epsilon Y_o$. Denoting $Y/Y_o=\mathcal{H}$, $\mathcal{H}$ is a linear vector space over the field of complex numbers and having a positive definite scalar product. Now, we prove that this space $\mathcal{H}$ is complete, by the following indirect method.

We first of all define D to be the set of all elements of $\mathcal{H}$ of the form $\Omega\Phi_F$ where Φ_F is the sequence $(F,F,F,\ldots F,\ldots)$, $F \epsilon Z/Z_o$. This sequence is clearly a Cauchy sequence of elements of Z/Z_o and hence is an element of Y.

Now, for $\alpha \epsilon \mathbb{C}$, $\alpha\Omega\Phi_F = \Omega\alpha\Phi_F$ by definition (see equation (3.14) (i))

$$= \Omega\Phi_{\alpha F} \quad \text{since}$$

$$\alpha\Phi_F = (\alpha F,\alpha F,\ldots,\alpha F,\ldots) = \Phi_{\alpha F}$$

Also $\Omega\Phi_F+\Omega\Phi_G = \Omega(\Phi_F+\Phi_G)$

$= \Omega\Phi_{F+G}$

Hence D is a subspace of $\mathcal{H}$.

Now consider a Cauchy sequence of elements of D.

i.e. a sequence of the form $\Omega\Phi_F(1), \Omega\Phi_F(2), \ldots, \Omega_F(\rho), \ldots$

such that $||\Omega\Phi_F(\rho) - \Omega\Phi_F(\sigma)|| \to 0$ as $\rho,\sigma \to \infty$

i.e. $||\Omega\Phi_F(\rho)_{-F}(\sigma)|| \to 0$ as $\rho,\sigma \to \infty$

i.e. $||F^{(\rho)}_{-F}{}^{(\sigma)}|| \to 0$ as $\rho,\sigma \to \infty$

If Ψ is the sequence $(F^{(1)}, F^{(2)}, \ldots, F^{(\rho)}, \ldots)$ which is a Cauchy Sequence by the above condition, then Ψ is an element of Y.

$\therefore \Omega\Psi \in \mathcal{H}$.

Further $||\Omega\Phi_F(\rho) - \Omega\Psi|| = \underset{\sigma\to\infty}{\text{Lt}} ||F^{(\rho)}_{-F}{}^{(\sigma)}||$

which limit clearly exists, by the result established for the scalar product over Y.

We also know that the double limit $\underset{\rho,\sigma \to \infty}{\text{Lt}} ||F^{(\rho)}_{-F}{}^{(\sigma)}||$

exists and is equal to zero.

$\therefore \underset{\rho \to \infty}{\text{Lt}} ||\Omega\Phi_F(\rho) - \Omega\Psi|| = \underset{\rho \to \infty}{\text{Lt}} \underset{\sigma \to \infty}{\text{Lt}} ||F^{(\rho)}_{-F}{}^{(\sigma)}||$

exists and is zero.

Hence, the above Cauchy sequence $\Omega\Phi_F(1), \Omega\Phi_F(2), \ldots, \Omega\Phi_F(\rho), \ldots$ of elements of D converges to an element $\Omega\Psi$ of $\mathcal{H}$,

and this element is also unique as is easily proved.

By adjoining to D all such limit points of Cauchy sequences of elements of D, we get a set $\bar{D}$ satisfying the relation $D \subset \bar{D} \subseteq \mathcal{H}$. It is easy to show that $\bar{D} = \mathcal{H}$.

For, suppose $\Omega\Phi \in \mathcal{H}$ where $\Phi = (F_1, F_2, \ldots, F_\rho, \ldots)$ is a Cauchy sequence of elements of Z/Z_o.

i.e. $||F_\rho - F_\sigma|| \to 0$ as $\rho, \sigma \to \infty$

Consider the sequence $\Omega\Phi_F(1), \Omega\Phi_F(2), \ldots, \Omega\Phi_F(\rho), \ldots$

where $F^{(\rho)} = (F_\rho, F_\rho, F_\rho, \ldots, F_\rho, \ldots)$, $\rho \geqslant 1$.

All elements of this sequence belong to D, and because

$$||\Omega\Phi_F(\rho) - \Omega\Phi_F(\sigma)|| = ||F_\rho - F_\sigma|| \to 0 \text{ as } \rho, \sigma \to \infty,$$

it is a Cauchy sequence in D.

But $||\Omega\Phi_F(\rho) - \Omega\Phi|| \to 0$ as $\rho \to \infty$ as we provedearlier.

$\therefore \Omega\Phi \in \bar{D}$

Hence $\mathcal{H} \subseteq \bar{D}$

This shows that $\bar{D} = \mathcal{H}$.

Now, we prove that $\bar{D}$ is complete, thereby getting the result that $\mathcal{H}$ is complete.

Let $\Omega\Psi^{(1)}, \Omega\Psi^{(2)}, \ldots, \Omega\Psi^{(\rho)}, \ldots$ be a Cauchy sequence of elements of $\bar{D}$, and therefore, satisfying the condition

(3.17) $$||\Omega\Psi^{(\rho)} - \Omega\Psi^{(\sigma)}|| \to 0 \text{ as } \rho, \sigma \to \infty$$

Given any $\rho \geqslant 1$, we can find some $\rho' > \rho$ such that

(3.18) $$||\Omega\Psi^{(\rho)} - \Omega\Psi^{(\rho')}|| > 0$$

For, otherwise, $\exists$ some $\rho \geqslant 1$ such that for al $\rho' > \rho$, $||\Omega\Psi^{(\rho)} - \Omega\Psi^{(\rho')}|| = 0$, and since the scalar product is +ve definite in $\bar{D}$ $(=\mathcal{H})$, this means that $\Omega\Psi^{(\rho')} = \Omega\Psi^{(\rho)}$ for all $\rho' > \rho$, which in turn means that the above sequence of elements of $\bar{D}$ converges trivially to $\Omega\Psi^{(\rho)} \in \bar{D}$.

By the definition of $\bar{D}$, for each $\sigma \geqslant 1$, $\Omega\Psi^{(\sigma)}$ is either an element of D or is the limit point of a Cauchy sequence of elements of D. In either case, $\exists$ a Cauchy sequence $\Omega\Phi^{(\sigma)}_{G(1)}, \Omega\Phi^{(\sigma)}_{G(2)}, \ldots, \Omega\Phi^{(\sigma)}_{G(\rho)}, \ldots$ of elements of D converging to $\Omega\Psi^{(\sigma)}$.

$\therefore$ $||\Omega\Phi^{(\sigma)}_{G(\rho)} - \Omega\Psi^{(\sigma)}|| \to 0$ as $\rho \to \infty$, or by interchanging ρ and σ, we get

$$||\Omega\Phi^{(\rho)}_{G(\sigma)} - \Omega\Psi^{(\rho)}|| \to 0 \text{ as } \sigma \to \infty \text{ for each } \rho.$$

Combining this with (3.18) we see that, given ρ, $\exists$ ρ'' such that

$$||\Omega\Phi^{(\rho)}_{G(\sigma)} - \Omega\Psi^{(\rho)}|| < ||\Omega\Psi^{(\rho)} - \Omega\Psi^{(\rho')}|| \text{ for all } \sigma \geqslant \rho''.$$

(3.19) This relation being true for $\sigma=\rho''$ in particular, we have

$$||\Omega\Phi^{(\rho)}_{G(\rho'')} - \Omega\Psi^{(\rho)}|| < ||\Omega\Psi^{(\rho)} - \Omega\Psi^{(\rho')}|| \text{ for each } \rho \geqslant 1.$$

Now, we consider the sequence $\Omega\Phi^{(\rho)}_{G(\rho'')}$, $\rho = 1,2,3,\ldots,$ of elements of D.

Since $||\Omega\Phi^{(\rho)}_{G(\rho'')} - \Omega\Phi^{(\sigma)}_{G(\sigma'')}||$

$= ||\Omega\Phi^{(\rho)}_{G(\rho'')} - \Omega\Psi^{(\rho)} + \Omega\Psi^{(\rho)} - \Omega\Psi^{(\sigma)} + \Omega\Psi^{(\sigma)} - \Omega\Phi^{(\sigma)}_{G(\sigma'')}||$

$\leq ||\Omega\Phi^{(\rho)}_{G(\rho'')} - \Omega\Psi^{(\rho)}|| + ||\Omega\Psi^{(\rho)} - \Omega\Psi^{(\sigma)}|| + ||\Omega\Psi^{(\sigma)} - \Omega\Phi^{(\sigma)}_{G(\sigma'')}||$

$< ||\Omega\Psi^{(\rho)} - \Omega\Psi^{(\rho')}|| + ||\Omega\Psi^{(\rho)} - \Omega\Psi^{(\sigma)}|| + ||\Omega\Psi^{(\sigma)} - \Omega\Psi^{(\sigma')}||$

using inequality (3.19)

$\to 0$ as $\rho, \sigma \to \infty$ by equation (3.17), this sequence is a Cauchy sequence of elements of D. Hence, by the definition of $\bar{D}$, $\exists$ an element $\Omega\Psi \in \bar{D}$ which is the limit of this sequence.

i.e. $\exists\ \Omega\Psi \in \bar{D}$ such that

(3.20) $||\Omega\Psi - \Omega\Phi^{(\rho)}_{G(\rho'')}|| \to 0$ as $\rho \to \infty$.

Now, we see that the original Cauchy sequence $\Omega\Psi^{(1)}, \Omega\Psi^{(2)}, \ldots, \Omega\Psi^{(\rho)}, \ldots$ of elements of $\bar{D}$ also converges to $\Omega\Psi \in \bar{D}$, because,

$||\Omega\Psi^{(\rho)} - \Omega\Psi|| = ||\Omega\Psi^{(\rho)} - \Omega\Phi^{(\rho)}_{G(\rho'')} + \Omega\Phi^{(\rho)}_{G(\rho'')} - \Omega\Psi||$

$\leq ||\Omega\Psi^{(\rho)} - \Omega\Phi^{(\rho)}_{G(\rho'')}|| + ||\Omega\Phi^{(\rho)}_{G(\rho'')} - \Omega\Psi||$

$< ||\Omega\Psi^{(\rho)} - \Omega\Psi^{(\rho')}|| + ||\Omega\Phi^{(\rho)}_{G(\rho'')} - \Omega\Psi||$ by (3.19)

$\to 0$ as $\rho \to \infty$. by (3.17) and (3.20).

Since every Cauchy sequence of elements of $\bar{D}$ converges to an element of $\bar{D}$, $\bar{D}$ is complete.

Hence $\mathcal{H}$ is complete.

The field operator $\varphi(h)$ and the unitary operator $\tilde{U}(a,\Lambda)$ are defined, so far, only on Z/Z_o. But Z/Z_o and D are homomorphic, by the homomorphism $F \in Z/Z_o \rightarrow \Omega\Phi_F \in D$. (note the change in notation for elements of Z/Z_o). Hence, we can define $\varphi(h)$ and $\tilde{U}(a,\Lambda)$ on elements of D by

(3.21) (i) $\qquad \varphi(h)\Omega\Phi_F = \Omega\Phi_{\varphi(h)F}$

(3.21) (ii) $\qquad \tilde{U}(a,\Lambda)\Omega\Phi_F = \Omega\Phi_{\tilde{U}(a,\Lambda)F}$

It is quite a simple matter to prove, as we did once earlier, that these two definitions are independent of the representative sequence of the equivalence class $\Omega\Phi_F$. But D is a dense subspace of $\mathcal{H}$, because, given any $\Omega\Psi \in \mathcal{H}$, $\Omega\Psi$ is either an element of D or it belongs to $\bar{D}$. In either case, there is a Cauchy sequence of elements of D converging to $\Omega\Psi$, and hence D is dense in $\mathcal{H}$. Since $\tilde{U}(a,\Lambda)$ is unitary in D too, $||\tilde{U}(a,\Lambda)|| = 1$ and hence it is bounded. Hence by a theorem for bounded linear operators (see Theorem 1.3) $\tilde{U}(a,\Lambda)$ has a unique extension to $\mathcal{H}$. This is not necessary for $\varphi(h)$, because $\varphi(h)$ has been defined on a domain D which is dense in $\mathcal{H}$, and this domain D satisfies the conditions $\varphi(h)D \subset D$, $U(a,\Lambda)D \subset D$, $\Psi_o \in D$ (equation 3.22)

The properties of $\varphi(h)$ and $\tilde{U}(a,\Lambda)$ which we established on elements of Z, also hold on elements of their

domains, because, the process of going over to the equivalence classes is linear and isometric.

Explicitly, we have the following results

(3.6A) $\tilde{U}(a,\Lambda)$ is a unitary representation of the Poincare Group in $\mathcal{H}$.

(3.7A) For each $h \in \mathcal{S}(R^4)$, $\varphi(h)$ is an anti-linear operator of D into $\mathcal{H}$, and is linear in h.

(3.8A) (iii) $\tilde{U}(a,\Lambda)\varphi(h)\tilde{U}(a,\Lambda)^{-1} = \varphi(\{a,\Lambda\}h)$ when each side is applied on any element of D, where

$$\{a,\Lambda\}h(x) = h(\Lambda^{-1}(x-a)) \quad \in \mathcal{S}(R^4).$$

(3.8A)(iv) $\varphi(h)$ is symmetric.

Let $\Omega\Phi_F$, $\Omega\Phi_G \in D$.

$\therefore (\Omega\Phi_F, \varphi(h)\Omega\Phi_G) = (\Omega\Phi_F, \Omega\Phi_{\varphi(h)G})$ by equation (3.21) (i)

$= (\Phi_F, \Phi_{\varphi(h)G}) = (F, \varphi(h)G)$ by the definition of scalar product over Y.

$= (G, \varphi(h)F)$ by equation (3.8) (iv)

$= (\Omega\Phi_G, \varphi(h)\Omega\Phi_F)$

$= (\Omega\Phi_F, [\varphi(h)]^t\Omega\Phi_G)$ by the definition of transposition.

$\therefore$ Domain of $[\varphi(h)]^t \supset D$, and $\varphi(h) = [\varphi(h)]^t$ in D because $(\Omega\Phi_F, \varphi(h)\Omega\Phi_G)$ is continuous in $\Omega\Phi_F$.

$\therefore$ $\varphi(h)$ is symmetric.

(3.8A) (vi) If $h^{(1)},h^{(2)} \in \mathcal{S}(\mathbb{R}^4)$ are such that $h^{(1)}(x)h^{(2)}(y) = 0$ whenever $(x-y)^2 \geqslant 0$, then

$$\varphi(h^{(1)})\varphi(\bar{h}^{(2)})+\varepsilon\varphi(h^{(2)})\varphi(\bar{h}^{(1)}) = 0 \text{ in } \mathcal{H}.$$

This follows from equation (3.8) (vi) because D is dense in $\mathcal{H}$, and $\mathcal{H}$ is complete.

(3.9A) (a) $\tilde{U}(a,\Lambda)$ is strongly continuous in (a,Λ).

This follows from equation (3.9) (i)

(3.9A) (ii) The physical energy momentum spectrum lies in or on the Forward Cone $\bar{V}^+$.

For, by SNAG Theorem[34], $\tilde{U}(a,1) = \int e^{ip.a}\, d\tilde{E}(p)$

where $\tilde{E}(p)$ is a projection valued measure on momentum space. What we have to prove is that $\tilde{E}(S) = 0$ in $\mathcal{H}$ if the set S does not meet $\bar{V}^+$.

Let $\Omega\Phi_F,\Omega\Phi_G \in D$.

$\therefore$ by SNAG Theorem, $(\Omega\Phi_F,\tilde{U}(a,1)\Omega\Phi_G)=\int e^{ip.a}\, d(\Omega\Phi_F,\tilde{E}(p)\Omega\Phi_G)$

Using an argument which is similar to that used at the beginning of the proof of result (3.9) (ii)

$$(\Omega\Phi_F,\tilde{E}(p)\Omega\Phi_G)= \frac{1}{(2\pi)^4}\int e^{-ip.a}\,(F,\tilde{U}(a,1)G)da$$

$= 0$ unless $p \in \bar{V}^+$ by equation (3.9)(ii)

Since $\tilde{E}(p)$ is a projection, and D is dense in $\mathcal{H}$, this implies that $\tilde{E}(p) = 0$ in $\mathcal{H}$ if $p \notin \bar{V}^+$.

We found that $\Psi_o = (1,0,0,\ldots)$ is a vacuum in Z, and hence $\Omega\Psi_o$ is a vacuum in Z/Z_o. Hence the sequence $(\Omega\Psi_o,\Omega\Psi_o,\ldots,\Omega\Psi_o\ldots.)=\Phi_{\Omega\Psi_o}$ is a vacuum in Y, implying that

(3.22) $\Omega\Phi_{\Omega\Psi_o}$ is a vacuum in $\mathcal{H}$. For convenience, we denote this element of $\mathcal{H}$ by Ψ_o in future.

The next result we want to establish is the separability of $\mathcal{H}$. We already know that D is dense in $\mathcal{H}$. If D is separable, then it follows immediately that $\mathcal{H}$ is separable. D is separable because we can choose a sequence of elements of D as follows, such that this sequence is dense in D.

We beging with Z. Suppose we take the subset of elements of Z of the form $F = (f_o,f_1,f_2,\ldots,f_\nu,\ldots)$ where f_o is a rational number, and f_ν is of the form $P_\nu \cdot g$, where P_ν is a plynomial of arbitrary degree in $x_1,x_2,\ldots,x_\nu$ whose coefficients are complex numbers with their real and imaginary parts being rational numbers, and $g \in \mathcal{S}(R^{4\nu})$ is such that $g \equiv 1$ for all $x_1^2+x_2^2+\ldots+x_\nu^2 \leqslant \rho^2$, and $g = 0$ for all $x_1^2+x_2^2+\ldots+x_\nu^2 \geqslant (\rho+1)^2$ where ρ is an arbitrary positive integer, and $x_\mu^2 = (x_\mu^{(0)})^2+(x_\mu^{(1)})^2+(x_\mu^{(2)})^2+(x_\mu^{(3)})^2$, $1 \leqslant \mu \leqslant \nu$. The theorem that a countable set of countable sets is countable coupled with Weierstrass's Approximation Theorem will show us that Z is separable. Hence Z/Z_o is separable,

and so is D. Hence $\mathcal{H}$ is separable.

Now, we show that the vacuum is unique.

Suppose Ψ_o' is another vacuum in $\mathcal{H}$, linearly independent of Ψ_o. Since $\Psi_o' = (\Psi_o, \Psi_o')\Psi_o + \Psi_0^{\perp}$ where $(\Psi_o, \Psi_o^{\perp}) = 0$, $\Psi_o^{\perp}$ is also a vacuum. Hence, without loss of generality we can assume that $(\Psi_o, \Psi_o') = 0$, and also that $(\Psi_o', \Psi_o') = 1$, (Now we go back to the original notation for elements of Z)

Suppose $\Psi_o' \in D$. Then $\exists$ $F \in Z$ such that $F = (f_o, f_1, f_2, \ldots, f_\nu, \ldots)$ and

$$\Psi_o' = \Omega\Phi_{\Omega F}, \text{ where } \Phi_{\Omega F} = (\Omega F, \Omega F, \ldots, \Omega F, \ldots) \in Y.$$

$$\therefore \quad (\Psi_o', \tilde{U}(\kappa a, 1)\Psi_o') = (\Omega\Phi_{\Omega F}, \tilde{U}(\kappa a, 1)\Omega\Phi_{\Omega F})$$

$$= (\Omega\Phi_{\Omega F}, \Omega\Phi_{\tilde{U}(\kappa a, 1)\Omega F}) \text{ by definition (3.21) (ii)}$$

$$= (\Phi_{\Omega F}, \Phi_{\tilde{U}(\kappa a, 1)\Omega F}) \text{ by definition of scalar product in the equivalence classes.}$$

$$= (\Omega F, \tilde{U}(\kappa a, 1)\Omega F) \text{ by definition of scalar product in Y.}$$

$$= (\Omega F, \Omega\tilde{U}(\kappa a, 1)F) \text{ by } \quad " \quad \tilde{U}(\kappa a, 1) \text{ on } Z/Z_o$$

$= (F, \tilde{U}(\kappa a, 1)F)$, where a is any non-null real 4 vector and κ is a real constant

$$= \sum_{\substack{\text{even}\\ \mu\geq 0}}^{\infty} \sum_{\substack{\text{all}\\ \nu\geq 0}}^{\infty} \int \overline{f_\mu(x_1,x_2,\ldots,x_\mu)}\, V^{(\mu+\nu)}(x_\mu,x_{\mu-1},\ldots,x_1,y_1+\kappa a,$$

$$y_2+\kappa a,\ldots,y_\nu+\kappa a)\, f_\nu(y_1,y_2,\ldots,y_\nu)[dx]^\mu[dy]^\nu$$

$$+ \sum_{\substack{\text{odd}\\ \mu\geq 1}}^{\infty} \sum_{\substack{\text{all}\\ \nu\geq 0}}^{\infty} \int \overline{f_\mu(x_1,x_2,\ldots,x_\mu)}\, \overline{V^{(\mu+\nu)}(x_\mu,x_{\mu-1},\ldots,x_1,y_1+\kappa a}$$

$$\overline{y_2+\kappa a,\ldots,y_\nu+\kappa a)}\, f_\nu(y_1,y_2,\ldots,y_\nu)[dx]^\mu[dy]^\nu$$

$$\to \sum_{\substack{\text{even}\\ \mu\geq 0}}^{\infty} \sum_{\substack{\text{all}\\ \nu\geq 0}}^{\infty} \left\{ \int \overline{f_\mu(x_1,x_2,\ldots,x_\mu)}\, V^{(\mu)}(x_\mu,x_{\mu-1},\ldots,x_1)[dx]^\mu \right\}$$

$$\times \left\{ \int V^{(\nu)}(y_1,y_2,\ldots,y_\nu) f_\nu(y_1,y_2,\ldots,y_\nu)[dy]^\nu \right\}$$

$$+ \sum_{\substack{\text{odd}\\ \mu\geq 1}}^{\infty} \sum_{\substack{\text{all}\\ \nu\geq 0}}^{\infty} \left\{ \int \overline{f_\mu(x_1,x_2,\ldots,x_\mu)}\, \overline{V^{(\mu)}(x_\mu,x_{\mu-1},\ldots,x_1)}[dx]^\mu \right\}$$

$$\times \left\{ \int V^{(\nu)}(y_1,y_2,\ldots,y_\nu) f_\nu(y_1,y_2,\ldots,y_\nu)[dy]^\nu \right\}$$

as $\kappa \to \infty$ due to condition (f) of Theorem 3.1

$$= (\Psi_o', \Psi_o)(\Psi_o, \Psi_o') = 0$$

But the L.H.S. $= (\Psi_o', \Psi_o')$ since Ψ_o' is invariant under $\tilde{U}(a,\Lambda) = 1$.

This gives a contradiction. On the other hand, if Ψ_o' is not in D, $\exists\ \Omega\Phi_{\Omega F} \in D$ such that

$||\Psi_o' - \Omega\Phi_{\Omega F}|| < \varepsilon$ for any arbitrary $\varepsilon > 0$, because D is dense in $\mathcal{H}$.

$$\therefore \quad 1 = (\Psi_o', \Psi_o') = (\Psi_o', \tilde{U}(\kappa a, 1)\Psi_o')$$

$$= (\Psi_o' - \Omega\Phi_{\Omega F}, \tilde{U}(\kappa a, 1)\Psi_o') + (\Omega\Phi_{\Omega F}, \tilde{U}(\kappa a, 1)(\Psi_o' - \Omega\Phi_{\Omega F}))$$

$$+ (\Omega\Phi_{\Omega F}, \tilde{U}(\kappa a, 1)\Omega\Phi_{\Omega F})$$

$$\therefore \quad \left|1-(\Omega\Phi_{\Omega F}, \tilde{U}(\kappa a, 1)\Omega\Phi_{\Omega F})\right| \leqslant ||\Psi_o' - \Omega\Phi_{\Omega F}|| + ||\Omega\Phi_{\Omega F}||.||\Psi_o' - \Omega\Phi_{\Omega F}||$$

Taking the limit as $\kappa \to \infty$, we get

$\left|1-(F,\Psi_o)(\Psi_o,F)\right| \leqslant \varepsilon + ||\Omega\Phi_{\Omega F}||.\varepsilon$, where the Ψ_o here is the vector $(1,0,0,\ldots,0,\ldots) \in Z$.

But $(F,\Psi_o) = (\Omega\Phi_{\Omega F} - \Psi_o', \Psi_o)$, because $(\Psi_o', \Psi_o) = 0$

$$\therefore \quad |(F,\Psi_o)| < \varepsilon$$

$$\therefore \quad |1-(F,\Psi_o)(\Psi_o,F)| > 1 - \varepsilon^2$$

$$\therefore \quad 1 \leqslant \varepsilon^2 + \varepsilon + ||\Omega\Phi_{\Omega F}||.\varepsilon \text{ which is a contradiction}$$

for sufficiently small ε.

This shows that the vacuum is unique.

The only thing that we have to prove now is that any other theory with the same V.E. Functionals is unitary equivalent to the theory we constructed.

i.e. Suppose $\mathcal{H}_1, \tilde{U}_1(a,\Lambda), \varphi_1(h), \Psi_{01}$ define another field theory with the same V.E. Functionals.

Let $\tilde{T}$ be the mapping of D into $\mathcal{H}_1$ given by

$$\tilde{T}.\Omega\Phi_{\Omega F}=f_o\Psi_{01}+\left[\sum_{\mu=1}^{\infty}\int f_\mu(x_1,x_2,..,x_\mu)\varphi_1(x_1)\varphi_1(x_2)..\varphi_1(x_\mu)[dx]^\mu\right]\Psi_{01}$$

where $F = (f_o,f_1,f_2,..,f_\mu,...) \in Z$.

We denote the R.H.S. by $(\Omega\Phi_{\Omega F})^{(1)}$. This mapping $\tilde{T}$ satisfies the following.

(3.23) (i) $\tilde{T}$ is linear.

$$\text{For } \tilde{T}\{\alpha\Omega\Phi_{\Omega F}\} = \tilde{T}\{\Omega\Phi_{\alpha\Omega F}\} = \tilde{T}\{\Omega\Phi_{\Omega\alpha F}\} = \alpha.\tilde{T}\{\Omega\Phi_{\Omega F}\}$$

$$\text{and } \tilde{T}\{\Omega\Phi_{\Omega F}+\Omega\Phi_{\Omega G}\} = \tilde{T}\{\Omega\Phi_{\Omega(F+G)}\} = \tilde{T}\Omega\Phi_{\Omega F}+\tilde{T}\Omega\Phi_{\Omega G}$$

where $\alpha \in \mathbb{C}$ and $F,G \in Z$.

(3.23) (ii) $\tilde{T}$ is unitary.

$$\text{For } (\tilde{T}\Omega\Phi_{\Omega F},\tilde{T}\Omega\Phi_{\Omega G})$$

$$= \sum_{\mu,\nu=0}^{\infty} \left(\left\{\int f_\mu(x_1,x_2,..,x_\mu)\varphi_1(x_1)\varphi_1(x_2)...\varphi_1(x_\mu)[dx]^\mu\right\}\Psi_{01},\right.$$

$$\left.\left\{\int g_\nu(y_1,y_2,..,y_\nu)\varphi_1(y_1)\varphi_2(y_2)...\varphi_\nu(y_\nu)[dy]^\nu\right\}\Psi_{01}\right)$$

$$= \sum_{\substack{\text{even}\\ \mu\geqslant 0}}^{\infty} \sum_{\substack{\text{all}\\ \nu\geqslant 0}}^{\infty} \int \overline{f_\mu(x_1,x_2,\ldots,x_\mu)} V^{(\mu+\nu)}(x_\mu,x_{\mu-1},\ldots,x_1;y_1,y_2,\ldots y_\nu)$$

$$g_\nu(y_1,y_2,\ldots,y_\nu)[dx]^\mu[dy]^\nu$$

$$+ \sum_{\substack{\text{odd}\\ \mu\geqslant 1}}^{\infty} \sum_{\substack{\text{all}\\ \nu\geqslant 0}}^{\infty} \int \overline{f_\mu(x_1,x_2,\ldots,x_\mu)} V^{(\mu+\nu)}(x_\mu,x_{\mu-1},\ldots,x_1,y_1,y_2,\ldots,y_\nu)$$

$g_\nu(y_1,y_2,\ldots,y_\nu)[dx]^\mu[dy]^\nu$, because $\varphi_1(h)$ is symmetric, and the V.E. Functionals are the same in both theories.

$= (F,G) = (\Omega F, \Omega G) = (\Omega\Phi_{\Omega F}, \Omega\Phi_{\Omega G})$.

This also means that $\tilde{T}$ ($\sim$ because of linearity) has a unique extension to the whole of $\mathcal{H}$. Since $\mathcal{H}_1$ is also complete and separable, the mapping is really onto.

$\therefore$ $\tilde{T}\,\mathcal{H} = \mathcal{H}_1$. (This follows since Ψ_{01} is a cyclic vector in $\mathcal{H}_1$ with respect to $\varphi_1(h)$.

(3.23) (iii) $\tilde{T}\Psi_o = \Psi_{01}$. This follows immediately from the definition

(3.23) (iv) $\tilde{T}\,\varphi(h)\tilde{T}^{-1} = \varphi_1(h)$

$\tilde{T}\varphi(h).\Omega\Phi_{\Omega F} = \tilde{T}.\Omega\Phi_{\Omega\varphi(h)F}$, where

$$\varphi(h)F = (0, h\otimes\overline{f}_0, h\otimes\overline{f}_1, \ldots, h\otimes\overline{f}_\mu, \ldots)$$

$$= \varphi_1(h).\tilde{T}\Omega\Phi_{\Omega F}.$$

$$\therefore \quad \tilde{T}\varphi(h) = \varphi_1(h)\tilde{T} \text{ on } D.$$

(3.23) (v) $\quad \tilde{T}\tilde{U}(a,\Lambda)\tilde{T}^{-1} = \tilde{U}_1(a,\Lambda)$

$$\tilde{T}\tilde{U}(a,\Lambda).\Omega\Phi_{\Omega F} = \tilde{T}.\Omega\Phi_{\tilde{U}(a,\Lambda)\Omega F}$$

$$= \tilde{T}.\Omega\Phi_{\Omega}.\tilde{U}(a,\Lambda)F$$

$$= f_o\Psi_{01} + \sum_{\mu=1}^{\infty} \int \{a,\Lambda\} f_\mu(x_1,x_2,..,x_\mu)\varphi_1(x_1)\varphi_1(x_2)..\varphi_1(x_\mu)[dx]^\mu\Psi_{01}$$

$$= f_o\Psi_{01} + \sum_{\mu=1}^{\infty} \int f_\mu(x_1,..,x_\mu)\tilde{U}_1(a,\Lambda)\varphi_1(x_1)\tilde{U}_1(a,\Lambda)^{-1}\tilde{U}_1(a,\Lambda)\varphi_1(x_2)$$

$$\tilde{U}_1(a,\Lambda)^{-1}..\tilde{U}_1(a,\Lambda)\varphi_1(x_\mu)\tilde{U}_1(a,\Lambda)^{-1}.\Psi_{01}[dx]^\mu$$

$$= \tilde{U}_1(a,\Lambda).\tilde{T}\Omega\Phi_{\Omega F}$$

$$\therefore \tilde{T}\tilde{U}(a,\Lambda) = \tilde{U}_1(a,\Lambda)\tilde{T} \quad \text{on } D.$$

Since each is a unitary operator, this relation holds on the whole of $\mathcal{H}$.

This concludes the complete proof of the reconstruction theorem.

RECONSTRUCTION THEOREM IN A THEORY WITH SEVERAL FIELDS EACH FIELD HAVING AN ARBITRARY NUMBER OF COMPONENTS

In this section we are going to indicate as to how the proof given in the last section can be carried over to this case. The properties for the V.E. Functionals in this case are the contents of Theorems 2.2, 2.3, 2.4, 2.5, 2.6 and 2.7.

We are given a sequence of tempered distributions

$$V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots,x_\nu),\ \nu \geqslant 1,\ 1 \leqslant \mu_\rho \leqslant \Lambda(\Gamma),\ 1 \leqslant \rho \leqslant \nu,$$

having the above properties. Before we proceed any further, we have to introduce the following concept.

For each $\nu \geqslant 1$, consider the set of suffixes $(\mu_1,\mu_2,\ldots,\mu_\nu)$. Since each suffix takes any value between 1 and $\Lambda(\Gamma)$, there are $[\Lambda(\Gamma)]^\nu$ such sets of suffixes, for each ν. Now, suppose we write these $[\Lambda(\Gamma)]^\nu$ sets of suffixes in a sequence according to some rule. i.e. given $\nu \geqslant 1$, suppose we also have a fixed rule for each fixed ν according to which we arrange the $[\Lambda(\Gamma)]^\nu$ sets of suffixes in a row. We denote this rule according to which the sets of suffixes $(\mu_1,\mu_2,\ldots,\mu_\nu)$ are arranged by O_ν.

e.g. we may take O_ν to be the natural ordering defined as follows. In the natural ordering, $(\mu_1,\mu_2,\ldots,\mu_\nu)$ precedes $(\rho_1,\rho_2,\ldots,\rho_\nu)$ in each of the following cases.

(1) $\mu_1 < \rho_1$

(2) $\mu_1 = \rho_1, \mu_2 < \rho_2$

(3) $\mu_1 = \rho_1, \mu_2 = \rho_2, \mu_3 < \rho_3$

(4) .....

.....

(ν-1) $\mu_1 = \rho_1, \mu_2 = \rho_2, \ldots, \mu_{\nu-1} = \rho_{\nu-1}, \mu_\nu < \rho_\nu$.

If $f^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu} \in \mathcal{S}(R^{4\nu})$, then, we denote by $\{f\}_\nu$, the sequence of functions $f^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu} \in \mathcal{S}(R^{4\nu})$ obtained by arranging the $[\Lambda(\Gamma)]^\nu$ sets of suffixes $(\mu_1,\mu_2,\ldots,\mu_\nu)$ according to the rule O_ν.

e.g. if $\nu = 1$, and O_1 is the natural ordering, then $\{f\}_1 = \{f_1^{(1)}, f_2^{(1)}, f_3^{(1)}, \ldots, f^{(1)}_{\Lambda(\Gamma)}\}$, where $f_\mu^{(1)} \in \mathcal{S}(R^4)$ for each $1 \leqslant \mu \leqslant \Lambda(\Gamma)$.

Now, let Z be the set of all sequences of the form $F = (\{f\}_0, \{f\}_1, \{f\}_2, \ldots, \{f\}_\nu, \ldots)$, where $\{f\}_0$ is a sequence of only one element, which is a complex number, and $\{f\}_\nu \equiv (0,0,0,\ldots,0)$ except for a finite number of values of ν.

(3.24) We define (i) $\{f\}_\nu = \{g\}_\nu$ if $f^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu} = g^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}$ for all values of $(\mu_1,\mu_2,\ldots,\mu_\nu)$

(3.24) (ii) $\alpha\{f\}_\nu = \{h\}_\nu$ where

$$h^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu} = \alpha f^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}$$

(3.24) (iii) $\{f\}_\nu + \{g\}_\nu = \{h\}_\nu$, where

$$h^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu} = f^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu} + g^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}$$

(3.24) (iv) A scalar product $(\{f\}_\nu,\{g\}_\rho)$ by

$$(\{f\}_\nu,\{g\}_\rho) = \sum_{\mu\ \sigma}\int \overline{f^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(x_1,x_2,\ldots,x_\nu)V^{(\nu+\rho)}_{\mu_\nu\mu_{\nu-1}\cdots\mu_1\sigma_1\sigma_2\cdots\sigma_\rho}}$$

$$(x_\nu,x_{\nu-1},\ldots,x_1,y_1,y_2,\ldots,y_\rho)g^{(\rho)}_{\sigma_1\sigma_2\cdots\sigma_\rho}(y_1,y_2,\ldots y_\rho)[dx]^\nu[dy]^\rho$$

if ν is even,

$$= \sum_{\mu,\sigma}\int \overline{f^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(x_1,x_2,\ldots,x_\nu)} \cdot$$

$$\times \overline{V^{(\nu+\rho)}_{\mu_\nu\mu_{\nu-1}\cdots\mu_1\sigma_1\sigma_2\cdots\sigma_\rho}(x_\nu,x_{\nu-1},\ldots,x_1,y_1,y_2,\ldots,y_\rho)}$$

$$\times g^{(\rho)}_{\sigma_1\sigma_2\cdots\sigma_\rho}(y_1,y_2\ldots,y_\rho)[dx]^\nu[dy]^\rho \text{ if } \nu \text{ is odd,}$$

where the summation is over all $\mu_1,\mu_2,\ldots,\mu_\nu,\sigma_1,\sigma_2,\ldots,\sigma_\rho = 1,2,3,\ldots,\Lambda(\Gamma)$.

(3.24) (v) For each element $\{a,\Lambda\}$ of the Poincare Group, a linear operator $\tilde{U}(a,\Lambda)$ by the relation

$$\tilde{U}(a,\Lambda)\{f\}_\nu = \{g\}_\nu, \text{ where}$$

$$g^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(x_1,x_2,\ldots,x_\nu) = \sum_\rho S_{\rho_1\mu_1}(\Lambda^{-1})\overline{S_{\rho_2\mu_2}(\Lambda^{-1})}S_{\rho_3\mu_3}(\Lambda^{-1})$$

$$\ldots\overline{S_{\rho_4\mu_4}(\Lambda^{-1})}\ldots\overline{S_{\rho_\nu\mu_\nu}(\Lambda^{-1})}f^{(\nu)}_{\rho_1\rho_2\cdots\rho_\nu}((\Lambda^{-1}(x_1-a),\Lambda^{-1}(x_2-a),$$

$$,..,\Lambda^{-1}(x_\nu-a))$$

if ν is even

$$= \sum_\rho S_{\rho_1\mu_1}(\Lambda^{-1})\overline{S_{\rho_2\mu_2}(\Lambda^{-1})}S_{\rho_3\mu_3}(\Lambda^{-1})\overline{S_{\rho_4\mu_4}(\Lambda^{-1})}\ldots.$$

$$\ldots S_{\rho_\nu\mu_\nu}(\Lambda^{-1})f^{(\nu)}_{\rho_1\rho_2\cdots\rho_\nu}(\Lambda^{-1}(x_1-a),\Lambda^{-1}(x_2-a),\Lambda^{-1}(x_\nu-a)),$$

if ν is odd.

When $\nu = 0$, $\tilde{U}(a,\Lambda)\{f\}_o = \{f\}_o$

(3.24) (vi) For each $h \in \mathcal{S}(\mathbb{R}^4)$ and μ, $1 \leqslant \mu \leqslant \Lambda(\Gamma)$ an operator $\varphi_\mu(h)$ by

$$\varphi_\mu(h)\{f\}_\nu = \{g\}_{\nu+1} \qquad \text{where}$$

$$g_{\rho_1\rho_2\cdots\rho_{\nu+1}}(x_1,x_2,x_3,\ldots,x_{\nu+1}) = \delta_{\mu\rho_1}.h(x_1)\overline{f_{\rho_2\rho_3\cdots\rho_{\nu+1}}(x_2,x_3,}$$

$$\overline{\ldots,x_{\nu+1}}),$$

$\delta_{\mu\rho_1}$ being the Kronecker delta.

In terms of these definitions, we define the following operations in the space Z.

(3.25) (i) If $F = (\{f\}_0, \{f\}_1, \{f\}_2, \ldots, \{f\}_\nu, \ldots)$ and

$G = (\{g\}_0, \{g\}_1, \{g\}_2, \ldots, \{g\}_\nu, \ldots)$ belong to Z,

then $F = G$ iff $\{f\}_\nu = \{g\}_\nu$, $\nu \geqslant 0$

(3.25) (ii) If $\alpha \in \mathbb{C}$, then $\alpha F = G$ where

$$\{g\}_\nu = \alpha\{g\}_\nu, \quad \nu \geqslant 0$$

(3.25) (iii) $F + G = H$

$= (\{h\}_0, \{h\}_1, \{h\}_2, \ldots, \{h\}_\nu, \ldots)$ where

$$\{h\}_\nu = \{f\}_\nu + \{g\}_\nu, \quad \nu \geqslant 0$$

(3.25) (iv) A scalar product (F,G) by

$$(F,G) = \sum_{\mu,\nu=0}^{\infty} (\{f\}_\mu, \{g\}_\nu)$$

(3.25) (v) For each element $\{a,\Lambda\}$ of the Poincare Group,

$\tilde{U}(a,\Lambda)F = G$, where $\{g\}_\nu = \tilde{U}(a,\Lambda)\{f\}_\nu$, $\nu \geqslant 0$

(3.25) (vi) For each $h \in \mathcal{S}(\mathbb{R}^4)$ and μ, $1 \leqslant \mu \leqslant \Lambda(\Gamma)$,

$\varphi_\mu(h)F = G$, where

$$\{g\}_{\nu+1} = \varphi_\mu(h)\{f\}_\nu, \quad \nu \geqslant 0$$

$$\{g\}_0 = \{0\}_0$$

Defining $\Psi_o \in Z$ by

$\Psi_o = (\{1\}_o, \{0\}_1, \{0\}_2, \ldots, \{0\}_\nu, \ldots)$ it can easily be verified that Z is a linear vector space with a + ve semi-definite metric and having the following properties.

(3.26) (i) $\tilde{U}(a,\Lambda)$ is a continuous unitary representation of the Poincare Group in Z.

(3.26) (ii) For each $h \in \mathcal{S}(R^4)$ and μ, $1 \leqslant \mu \leqslant \Lambda(\Gamma)$, $\varphi_\mu(h)$ is an anti-linear operator in Z which is linear in h, and satisfies the following.

$$(F, \varphi_\mu(h)G) = (G, \varphi_\beta(h)F) \text{ for all } F,G \in Z.$$

$$(\Psi_o, \varphi_\mu(h_1^o)\varphi_{\mu_2}(h_2^o)\ldots\varphi_{\mu_\nu}(h_\nu^o)\Psi_o)$$

$$= \int h_1(x_1)h_2(x_2)\ldots h_\nu(x_\nu)V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots,x_\nu)[dx]^\nu$$

(3.26) (iii)
$$\tilde{U}(a,\Lambda)\varphi_\mu(h)\tilde{U}(a,\Lambda)^{-1} = \sum_{\nu=1}^{\Lambda(\Gamma)} S_{\mu\nu}(\Lambda^{-1})\varphi_\nu(\{a,\Lambda\}h), \text{ in Z.}$$

From this point onwards, the procedure that has to be adopted is the same as that for a single component (symmetric scalar) field. In proving the above results, we require the following results

(3.2) (i) $S_{\mu\nu}(\Lambda^{-1}) = S_{\beta\delta}(\Lambda^{-1})$

(3.2) (ii) $\sum_{\nu=1}^{\Lambda(\Gamma)} S_{\mu\nu}(A)S_{\nu\rho}(B) = S_{\mu\rho}(AB)$

The first result follows immediately from the definition of S (Just after equation 2.4) and the "capped" indices $\hat{\mu}$ (eqn.2.3) The second also follows immediately, because

$$S(A)S(B) = \left[\bigoplus_{\gamma=1}^{\Gamma}\left\{S^{(\gamma)}(A) \oplus S^{(\gamma)}(A)\right\}\right]\left[\bigoplus_{\gamma=1}^{\Gamma}\left\{S^{(\gamma)}(B) \oplus S^{(\gamma)}(B)\right\}\right]$$

$$= \bigoplus_{\gamma=1}^{\Gamma}\left\{S^{(\gamma)}(A)S^{(\gamma)}(B) \oplus S^{(\gamma)}(A)S^{(\gamma)}(B)\right\}$$

$$= \bigoplus_{\gamma=1}^{\Gamma}\left\{S^{(\gamma)}(AB) \oplus S^{(\gamma)}(AB)\right\}$$

$$= S(AB).$$

In particular, if $B = A^{-1}$, then we get

$$S(1) = 1.$$

This completes the Reconstruction Theorem for a field theory with any number of fields, each field having any number of components.

CHAPTER 4

The main result of this thesis is that free local anti-linear fields cannot exist. As we mentioned at the end of Chapter 2, we would have been able to prove that no anti-linear fields satisfying microscopic causality could exist, provided that we were able to have a suitable definition of "truncation". All attempts to define truncation in a way analogous to that of linear fields have ended up in failure. In this chapter, we are going to establish the main result referred to, but require the following preliminary results. We consider the most general possibility of a theory with any number of fields, each field having any number of components.

Theorem 4.1: If $\varphi_\mu(x)$ is any field component in the theory, then $V_\mu^{(1)}(x) = (\Psi_o, \varphi_\mu(x)\Psi_o) = C_\mu$, a constant.

Proof: Given $f \in \mathcal{S}(\mathbb{R}^4)$, consider the quantity

$$(\Psi_o, \varphi_\mu(f)\Psi_o) = \int f(x) V_\mu^{(1)}(x)\,dx$$

But $(\Psi_o, \varphi_\mu(f)\Psi_o) = (\tilde{U}(-a,1)\Psi_o, \varphi_\mu(f)\tilde{U}(-a,1)\Psi_o)$ for any 4-vector a

$$= (\Psi_o, \tilde{U}(a,1)\varphi_\mu(f)\tilde{U}(a,1)^{-1}\Psi_o) \text{ since } \tilde{U}(-a,1)$$

is unitary, and $\tilde{U}(a,1)^{-1} = \tilde{U}(-a,1)$

$$= (\Psi_o, \varphi_\mu(\{a,1\}f)\Psi_o)$$

$$= \int \{a,1\} f(x) V_\mu^{(1)}(x)dx$$

$$= \int f(x-a) V_\mu^{(1)}(x)dx$$

(4.1) $$\therefore \int \left[f(x)-f(x-a) \right] V_\mu^{(1)}(x)dx = 0$$

Let $f(x) = \frac{1}{(2\pi)^2} \int e^{-ip.x} \tilde{f}(p)\, dp.$

$$\therefore f(x-a) = \frac{1}{(2\pi)^2} \int e^{-ip.x}\, e^{ip.a}\, \tilde{f}(p)\, dp.$$

$$\therefore 0 = \frac{1}{(2\pi)^2} \int e^{-ip.x} [1-e^{+ip.a}] \tilde{f}(p) V_\mu^{(1)}(x)dx.dp$$

But $\frac{1}{(2\pi)^2} \int e^{-ip.x}\, V_\mu^{(1)}(x)dx = \tilde{V}_\mu^{(1)}(-p)$

$$\therefore 0 = \int [1-e^{ip.a}]\, \tilde{V}^{(1)}(-p).\tilde{f}(p)dp.$$

Since $\tilde{f} \in \mathcal{S}(R^4)$ is arbitrary, $[1-e^{ip.a}]\, \tilde{V}^{(1)}(-p) = 0$ as a tempered distribution over $\mathcal{S}(R^4)$.

Now, given a 4-vector $p \neq 0$, we can always find a 4-vector a such that $q.a \neq 0$ for all q in some neighbourhood of p.

$\therefore$ $\tilde{V}_\mu^{(1)}(-p)$ vanishes everywhere except in the neighbourhood of $p = 0$.

Since $\tilde{V}_\mu^{(1)}(-p)$ is a tempered distribution, this means that $\tilde{V}_\mu^{(1)}(p) = \sum_{|\alpha|\leqslant\rho} C_\mu^{(\alpha)} D^\alpha \delta(p)$, where ρ is some positive integer, $\alpha = (\alpha_o,\alpha_1,\alpha_2,\alpha_3)$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial p_o^{\alpha_o}\partial p_1^{\alpha_1}\partial p_2^{\alpha_2}\partial p_3^{\alpha_3}}$, $\alpha_o,\alpha_1,\alpha_2,\alpha_3 \geqslant 0$, $|\alpha| = \alpha_o+\alpha_1+\alpha_2+\alpha_3$, $p = (p_o,p_1,p_2,p_3)$ and $\delta(p)$ is the Dirac δ-function.[37]

$\therefore$ $V_\mu^{(1)}(x)$ is a polynomial in the components of x.

But, we can write equation (4.1) also as

$$\int f(x)[V_\mu^{(1)}(x)-V_\mu^{(1)}(x+a)]dx = 0$$

Since $f(x)$ is arbitrary, this means that $V_\mu^{(1)}(x) = V_\mu^{(1)}(x+a)$ for all 4-vectors a.

$\therefore$ $V_\mu^{(1)}(x) = C_\mu$, where C_μ is a constant.

It is interesting to note that for linear fields we can always take this constant to be zero by defining a new field $\hat{\varphi}_\mu(x) = \varphi_\mu(x) - C_\mu$. This will not work for anti-linear fields, because, if $C_\mu \neq 0$, then $\hat{\varphi}_\mu(x)$ is neither linear, nor anti-linear.

Theorem 4.2: If $\Delta(x,y) = \frac{1}{(2\pi)^2}\int\theta(-p^{(0)})\delta(p^2-m^2)e^{-ip.(x-y)}dp$ where m is a +ve constant, then the 12 tempered distributions

$\Delta(x_1,x_2)\Delta(x_3,x_4), \Delta(x_1,x_3)\Delta(x_2,x_4), \Delta(x_1,x_4)\Delta(x_2,x_3)$

$\Delta(x_2,x_1)\Delta(x_3,x_4), \Delta(x_3,x_1)\Delta(x_2,x_4), \Delta(x_4,x_1)\Delta(x_2,x_3),$

$\Delta(x_2,x_1)\Delta(x_4,x_3), \Delta(x_3,x_1)\Delta(x_4,x_2), \Delta(x_4,x_1)\Delta(x_3,x_2)$

$\Delta(x_1,x_2)\Delta(x_4,x_3), \Delta(x_1,x_3)\Delta(x_4,x_2), \Delta(x_1,x_4)\Delta(x_3,x_2),$

are linearly independent.

Proof: By the definition of $\Delta(x,y)$, we can define another tempered distribution $\Delta(\xi)$ by

$$\Delta(\xi) = \Delta(x,y), \text{ where } \xi = x-y.$$

(4.2) If we put $\tilde{\Delta}(p) = \theta(-p^{(0)})\delta(p^2-m^2)$, then we have

(4.3) $$\Delta(\xi) = \frac{1}{(2\pi)^2}\int \tilde{\Delta}(p)e^{-ip\cdot\xi}dp.$$

(4.4) $$\therefore \quad \tilde{\Delta}(p) = \frac{1}{(2\pi)^2}\int \Delta(\xi)e^{ip\cdot\xi}d\xi.$$

Let $\xi_1 = x_1-x_2$, $\xi_2 = x_2-x_3$, $\xi_3 = x_3-x_4$.

Then the 12 distributions in the Theorem can all be expressed using ξ_1,ξ_2,ξ_3 as arguments.

Hence, what we have to prove is that, if

(4.5) $\lambda_1\Delta(\xi_1)\Delta(\xi_3)+\mu_1\Delta(\xi_1+\xi_2)\Delta(\xi_2+\xi_3)+\nu_1\Delta(\xi_1+\xi_2+\xi_3)\Delta(\xi_2)$

$+ \lambda_2\Delta(-\xi_1)\Delta(\xi_3)+\mu_2\Delta(-\xi_1-\xi_2)\Delta(\xi_2+\xi_3)+\nu_2\Delta(-\xi_1-\xi_2-\xi_3)\Delta(\xi_2)$

$+ \lambda_3\Delta(-\xi_1)\Delta(-\xi_3)+\mu_3\Delta(-\xi_1-\xi_2)\Delta(-\xi_2-\xi_3)+\nu_3\Delta(-\xi_1-\xi_2-\xi_3)\Delta(-\xi_2)$

$+ \lambda_4\Delta(\xi_1)\Delta(-\xi_3)+\mu_4\Delta(\xi_1+\xi_2)\Delta(-\xi_2-\xi_3)+\nu_4\Delta(\xi_1+\xi_2+\xi_3)\Delta(-\xi_2) = 0$

for all ξ_1, ξ_2, ξ_3 where $\lambda_\rho, \mu_\rho, \nu_\rho$; $\rho = 1,2,3,4$ are constants, then $\lambda_\rho = \mu_\rho = \nu_\rho = 0$ for all $\rho = 1,2,3,4$.

Now $p_1 \cdot \xi_1 + p_2 \cdot \xi_2 + p_3 \cdot \xi_3$

$$= p_1 \cdot (\xi_1 + \xi_2) + (p_2 - p_1) \cdot (\xi_2 + \xi_3) + (p_3 + p_1 - p_2) \cdot \xi_3$$

$$= p_1 \cdot (\xi_1 + \xi_2 + \xi_3) + (p_2 - p_1) \cdot \xi_2 + (p_3 - p_1) \cdot \xi_3$$

Using these alternate forms for $p_1 \cdot \xi_1 + p_2 \cdot \xi_2 + p_3 \cdot \xi_3$, we can Fourier Transform equation (4.5). Together with equation (4.4), this gives us the following.

$$\lambda_1 \delta(p_2)\tilde{\Delta}(p_1)\tilde{\Delta}(p_3) + \mu_1 \delta(p_3 + p_1 - p_2)\tilde{\Delta}(p_1)\tilde{\Delta}(p_2 - p_1)$$

$$+ \nu_1 \delta(p_3 - p_1)\tilde{\Delta}(p_1)\tilde{\Delta}(p_2 - p_1)$$

$$+ \lambda_2 \cdot \delta(p_2)\tilde{\Delta}(-p_1)\tilde{\Delta}(p_3) + \mu_2 \delta(p_3 + p_1 - p_2)\tilde{\Delta}(-p_1)\tilde{\Delta}(p_2 - p_1)$$

$$+ \nu_2 \delta(p_3 - p_1)\tilde{\Delta}(-p_1)\tilde{\Delta}(p_2 - p_1)$$

$$+ \lambda_3 \delta(p_2)\tilde{\Delta}(-p_1)\tilde{\Delta}(-p_3) + \mu_3 \delta(p_3 + p_1 - p_2)\tilde{\Delta}(-p_1)\tilde{\Delta}(-p_2 + p_1)$$

$$+ \nu_3 \delta(p_3 - p_1)\tilde{\Delta}(-p_1)\tilde{\Delta}(-p_2 + p_1)$$

$$+ \lambda_4 \delta(p_2)\tilde{\Delta}(p_1)\tilde{\Delta}(-p_3) + \mu_4 \delta(p_3 + p_1 - p_2)\tilde{\Delta}(p_1)\tilde{\Delta}(-p_2 + p_1)$$

$$+ \nu_4 \delta(p_3 - p_1)\tilde{\Delta}(p_1)\tilde{\Delta}(-p_2 + p_1) = 0$$

This tempered distribution should give the value zero to every test function $f \in \mathcal{S}(\mathbb{R}^{12})$. Let the support of f

be in the neighbourhood of the point (q_1,q_2,q_3) chosen as follows

(i) $q_2 = 0$, $q_1^2 = q_3^2 = m^2$, $q_1^{(0)} < 0$, $q_3^{(0)} < 0$, $q_1 \neq q_3$.

($q_1^{(0)}$ is the time-component of the 4 vector q_1).

This gives us $\lambda_1 \int \delta(p_2)\tilde{\Delta}(p_1)\tilde{\Delta}(p_3) f(p_1,p_2,p_3).[dp]^3 = 0$

where we have used equation (4.2).

$$\therefore \quad \lambda_1 = 0$$

(ii) $q_2 = 0$, $q_1^2 = q_3^2 = m^2$, $q_1^{(0)} > 0$, $q_3^{(0)} < 0$, $q_1 + q_3 \neq 0$.

This gives $\lambda_2 = 0$

(iii) $q_2 = 0$, $q_1^2 = q_3^2 = m^2$, $q_1^{(0)} > 0$, $q_3^{(0)} > 0$, $q_1 \neq q_3$

Then $\lambda_3 = 0$

(iv) $q_2 = 0$, $q_1^2 = q_3^2 = m^2$, $q_1^{(0)} < 0$, $q_3^{(0)} > 0$, $q_1 + q_3 \neq 0$

Then $\lambda_4 = 0$

(v) $q_1^2 = m^2$, $q_1^{(0)} < 0$, $q_3 = q_1$, $(q_2-q_1)^2 = m^2$, $q_2^{(0)} - q_1^{(0)} < 0$, $q_2 \neq 2q_1$

Then $\nu_1 = 0$

(vi) $q_1^2 = m^2$, $q_1^{(0)} > 0$, $q_3 = q_1$, $(q_2-q_1)^2 = m^2$, $q_2^{(0)} - q_1^{(0)} < 0$

Then $\nu_2 = 0$

(vii) $q_1^2 = m^2$, $q_1^{(0)} > 0$, $q_3 = q_1$, $(q_2-q_1)^2 = m^2$, $q_2^{(0)} - q_1^{(0)} > 0$, $q_2 \neq 2q_1$

$$\therefore \quad \nu_3 = 0$$

(viii) $q_1^2 = m^2$, $q_1^{(0)}<0, q_3=q_1, (q_2-q_1)^2=m^2, q_2^{(0)}-q_1^{(0)}>0$

$\therefore \nu_4 = 0$

(ix) $q_1^2=m^2$, $q_1^{(0)}<0$, $q_2 = 2q_1$, $q_3 = q_1$

$\therefore \mu_1 = 0$

(x) $q_1^2 = m^2$, $q_1^{(0)}>0$, $q_2 = 0$, $q_3 = -q_1$

$\therefore \mu_2 = 0$

(xi) $q_1^2 = m^2$, $q_1^{(0)}>0$, $q_2 = 2q_1$, $q_3 = q_1$

$\therefore \mu_3 = 0$

Hence $\mu_4 = 0$.

This completes the proof that the 12 tempered distributions given in the statement of the Theorem are linearly independent.

Before we proceed further, we make the following definitions.

Definition 4.1: For each μ, $1 \leqslant \mu \leqslant \Lambda(\Gamma)$, if we have a positive constant $m_\mu = m_{\tilde{\mu}}$ such that

$\left[\Box_\rho+m_{\mu_\rho}^2\right] V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\rho\ldots\mu_\nu}(x_1,x_2,\ldots,x_\rho,\ldots,x_\nu) = 0$ for all ρ, $1 \leqslant \rho \leqslant \nu$, $\nu \geqslant 1$, where

$$\Box_\sigma = \frac{\partial^2}{\partial(x_\sigma^{(0)})^2} - \frac{\partial^2}{\partial(x_\sigma^{(1)})^2} - \frac{\partial^2}{\partial(x_\sigma^{(2)})^2} - \frac{\partial^2}{\partial(x_\sigma^{(3)})^2}$$, then

we say that the theory described by the above V.E. Functionals is a Theory of Free Fields.

Definition 4.2: If the fields $\varphi_\mu(h)$, $h \in \mathcal{S}(R^4)$, $1 \leqslant \mu \leqslant \Lambda(\Gamma)$ satisfy the following conditions, such a theory is called a Theory of Local Fields.

Let $\varepsilon(\mu,\nu) = \varepsilon(\nu,\mu) = \varepsilon(\hat{\mu},\hat{\nu}) = +1$ or -1 for each μ,ν, $1 \leqslant \mu, \nu \leqslant \Lambda(\Gamma)$.

(i) If $\varphi_\mu(h)$ is a symmetric field component, i.e. $[\varphi_\mu(h)]^t = \varphi_{\hat{\mu}}(h) \supset \varphi_\mu(h)$, then $\varphi_\mu(f)\varphi_\nu(\bar{g}) + \varepsilon(\mu,\nu)\varphi_\nu(g)\varphi_\mu(\bar{f}) = 0$, if $\nu \neq \mu,\hat{\mu}$ for all $f,g \in \mathcal{S}(R^4)$, and $\varphi_\mu(f)\varphi_\mu(\bar{g}) + \varepsilon(\mu,\mu)\varphi_\mu(g)\varphi_\mu(\bar{f})$ is a continuous linear functional on $f \otimes g \in \mathcal{S}(R^8)$.

(ii) If $\varphi_\mu(h)$ is not a symmetric field component, i.e $[\varphi_\mu(h)]^t \neq \varphi_\mu(h)$ but $[\varphi_\mu(h)]^t = \varphi_{\hat{\mu}}(h)$, then

$\varphi_\mu(f)\varphi_\nu(\bar{g}) + \varepsilon(\mu,\nu)\varphi_\nu(g)\varphi_\mu(\bar{f}) = 0$ if $\nu \neq \hat{\mu}$ for all $f,g \in \mathcal{S}(R^4)$, and $\varphi_\mu(f)\varphi_{\hat{\mu}}(\bar{g}) + \varepsilon(\mu,\hat{\mu})\varphi_{\hat{\mu}}(g)\varphi_\mu(\bar{f})$ is a continuous linear functional on $f \otimes g \in \mathcal{S}(R^8)$.

With these definitions, we have the following theorem.

Theorem 4.3: In a Theory of Free Local Fields, we have the following.

(i) $V_\mu^{(1)}(x) = C_\mu = 0$ for all μ, $1 \leqslant \mu \leqslant \Lambda(\Gamma)$.

(ii) If $\varphi_\mu(h)$ is a symmetric field component, then

$$V_{\mu\nu}^{(2)}(x_1,x_2) = 0 \text{ if } \nu \neq \mu,\hat{\mu}$$

$$= \lambda_{\mu\nu}\,\Delta(x_1,x_2) \text{ if } \nu = \mu,\hat{\mu}$$

(iii) If $\varphi_\mu(h)$ is not a symmetric field component, then

$$V_{\mu\nu}^{(2)}(x_1,x_2) = 0 \text{ if } \nu \neq \hat{\mu}$$

$$= \lambda_{\mu\hat{\mu}}\,\Delta(x_1,x_2) \text{ if } \nu = \hat{\mu}.$$

In this, $\lambda_{\mu\nu}$ are constants, and $\Delta(x_1,x_2)$ is as defined in Theorem 4.2 with the constant m as given in the definition of Free Fields. [We should really write $\Delta(x_1,x_2;m)$ instead of $\Delta(x_1,x_2)$, but this does not lead to any confusion].

Proof:

(i) Since the fields are free, $(\Box+m_\mu^2)V_\mu^{(1)}(x) = 0$

But from Theorem 4.1, $V_\mu^{(1)}(x) = C_\mu$, a constant. Hence, by taking a test function $f \in \mathcal{S}(\mathbb{R}^4)$ such that $\Box f(x) = 0$, and $\int f(x)dx \neq 0$, we get $C_\mu = 0$, because $m_\mu > 0$.

$$\therefore\ V_\mu^{(1)}(x) = 0 \text{ for all } \mu,\ 1 \leqslant \mu \leqslant \Lambda(\Gamma).$$

(ii) Suppose $[\varphi_\mu(h)]^t = \varphi_\mu(h) \supset \varphi_\mu(h)$, and $\nu \neq \mu, \hat{\mu}$.

Then by definition of a Theory of Local Fields,

$$\varphi_\mu(h)\varphi_\nu(\bar{g}) + \varepsilon(\mu,\nu)\varphi_\nu(g)\varphi_\mu(\bar{h}) = 0$$

Taking the V.E. Functionals of the above operator equation, we get $\int h(x)g(y)\left[V^{(2)}_{\mu\nu}(x,y) + \varepsilon(\mu,\nu)V^{(2)}_{\nu\mu}(y,x)\right]dx.dy = 0.$

$$\therefore V^{(2)}_{\mu\nu}(x,y) + \varepsilon(\mu,\nu)V^{(2)}_{\nu\mu}(y,x) = 0$$

$$\therefore W^{(2)}_{\mu\nu}(\xi) + \varepsilon(\mu,\nu)W^{(2)}_{\nu\mu}(-\xi) = 0 \text{ where } \xi = x-y.$$

Taking the Fourier Transform, we have

$$\hat{W}^{(2)}_{\mu\nu}(p) + \varepsilon(\mu,\nu)\hat{W}^{(2)}_{\nu\mu}(-p) = 0$$

From the spectral conditions in Theorem 2.3, we also get

$$\hat{W}^{(2)}_{\mu\nu}(-p) = 0 \text{ unless } p \in \overline{V^+};$$

$$\text{Hence } \hat{W}^{(2)}_{\mu\nu}(p) = 0 \text{ unless } p = 0$$

But, because the fields are free,

$$(\Box_1 + m_\mu^2)\, V^{(2)}_{\mu\nu}(x_1,x_2) = 0$$

By Fourier Transforming, this gives

$$(p^2 - m_\mu^2)\hat{W}^{(2)}_{\mu\nu}(p) = 0$$

$$\therefore \hat{W}^{(2)}_{\mu\nu}(p) = 0 \text{ even when } p = 0, \text{ since } m_\mu > 0.$$

$$\therefore V^{(2)}_{\mu\nu}(x_1,x_2) = 0.$$

Now, suppose $[\varphi_\mu(h)]^t = \varphi_{\hat{\mu}}(h) \supset \varphi_\mu(h)$ and $\nu = \mu$, or $\hat{\mu}$

Then by the definition 4.1,

$$(\Box_1 + m_\mu^2) V_{\mu\nu}^{(2)}(x_1, x_2) = 0$$

Putting $V_{\mu\nu}^{(2)}(x_1, x_2) = W_{\mu\nu}^{(2)}(x_1 - x_2) = \frac{1}{(2\pi)^2} \int \tilde{W}_{\mu\nu}(p) e^{-ip.(x_1 - x_2)} dp$,

we get $(p^2 - m_\mu^2) \tilde{W}_{\mu\nu}^{(2)}(p) = 0$

Together with the spectral conditions in Theorem (2.3) this gives $\tilde{W}_{\mu\nu}^{(2)}(p) = \lambda_{\mu\nu}\, \theta(-p^{(0)}) \delta(p^2 - m_\mu^2)$

$$= \lambda_{\mu\nu}\, \tilde{\Delta}(p)$$

$$\therefore \quad W_{\mu\nu}^{(2)}(\xi) = \lambda_{\mu\nu}\, \Delta(\xi)$$

$$\therefore \quad V_{\mu\nu}^{(2)}(x_1, x_2) = \lambda_{\mu\nu}\, \Delta(x_1, x_2).$$

(iii) If $[\varphi_\mu(h)]^t = \varphi_{\hat{\mu}}(h) \neq \varphi_\mu(h)$, let $\nu \neq \hat{\mu}$.

Then $\varphi_\mu(h)\varphi_\nu(\bar{g}) + \varepsilon(\mu,\nu)\varphi_\nu(g)\varphi_\mu(\bar{h}) = 0$.

Exactly as in the previous case, this gives us

$$V_{\mu\nu}^{(2)}(x_1, x_2) = 0.$$

On the other hand, if $\nu = \hat{\mu}$, then again we can proceed as in the previous case, and get

$$V_{\mu\nu}^{(2)}(x_1, x_2) = \lambda_{\mu\nu}\, \Delta(x_1, x_2).$$

This completes the proof of the Theorem.

Theorem 4.4: If $\varphi_\mu(h)$ is a symmetric field component in a Theory of Free Local Fields, and

$$V^{(4)}_{\mu\mu\mu\mu}(x_1,x_2,x_3,x_4) = \sum_\nu \lambda_{\nu_1\nu_2\nu_3\nu_4} V^{(2)}_{\mu\mu}(x_{\nu_1},x_{\nu_2}) V^{(2)}_{\mu\mu}(x_{\nu_3},x_{\nu_4})$$

where $\sum_\nu$ stands for the summation over ν_1,ν_2,ν_3,ν_4 from 1 to 4

and $\lambda_{\nu_1\nu_2\nu_3\nu_4} = \lambda_{\nu_3\nu_4\nu_1\nu_2} = 0$ if $(\nu_1\nu_2\nu_3\nu_4)$ is not a permutation of (1234), then $V^{(2)}_{\mu\mu}(x_1,x_2) = 0$.

Proof: We know from the definition of local fields that $\varphi_\mu(h)\varphi_\mu(\bar{g})+\varepsilon(\mu,\mu)\varphi_\mu(g)\varphi_\mu(\bar{h})$ is a continuous linear functional on $h \otimes g$.

Hence, by Schwartz's Nuclear Theorem,[19] there is a unique tempered distribution $G_\mu(x,y)$ such that

$$(4.6) \qquad \varphi_\mu(h)\varphi_\mu(\bar{g})+\varepsilon(\mu,\mu)\varphi_\mu(g)\varphi_\mu(\bar{h}) = \int h(x)g(y)G_\mu(x,y)dx.dy.$$

In the rest of the proof we can drop the suffix μ without any confusion.

Taking the V.E. Functional of equation (4.6), we get

$$\int h(x)g(y)[V^{(2)}(x,y)+\varepsilon V^{(2)}(y,x)]dx.dy = \int h(x)g(y)G(x,y)dx.dy$$

$$\therefore G(x,y) = V^{(2)}(x,y)+\varepsilon V^{(2)}(y,x)$$

$$(4.7) \qquad \therefore \varphi(h)\varphi(\bar{g})+\varepsilon\varphi(g)\varphi(\bar{h}) = \int h(x)g(y)[V^{(2)}(x,y) + \varepsilon V^{(2)}(y,x)]dx.dy.$$

Now, $\int h_1(x_1)h_2(x_2)h_3(x_3)h_4(x_4)[V^{(4)}(x_1,x_2,x_3,x_4)+\varepsilon V^{(4)}(x_2,x_1,x_3,x_4)][dx]^4$

$= (\Psi_o,\varphi(h_1)\varphi(\bar{h}_2)\varphi(h_3)\varphi(\bar{h}_4)\Psi_o)+\varepsilon(\Psi_o,\varphi(h_2)\varphi(\bar{h}_1)\varphi(h_3)\varphi(\bar{h}_4)\Psi_o)$

$= (\Psi_o,[\varphi(h_1)\varphi(\bar{h}_2)+\varepsilon\varphi(h_2)\varphi(\bar{h}_1)]\varphi(h_3)\varphi(\bar{h}_4)\Psi_o)$

$= \int h_1(x_1)h_2(x_2)[V^{(2)}(x_1,x_2)+\varepsilon V^{(2)}(x_2,x_1)]dx_1\cdot dx_2\cdot$

$\times \quad (\Psi_o,\varphi(h_3)\varphi(\bar{h}_4)\Psi_o)$

where we have used equation (4.7)

$= \int h_1(x_1)h_2(x_2)h_3(x_3)h_4(x_4)[V^{(2)}(x_1,x_2)+\varepsilon V^{(2)}(x_2,x_1)]$,

$\times \quad V^{(2)}(x_3,x_4)[dx]^4$

(4.8) $\therefore \quad V^{(4)}(x_1,x_2,x_3,x_4)-V^{(2)}(x_1,x_2)V^{(2)}(x_3,x_4)$

$= -\varepsilon[V^{(4)}(x_2,x_1,x_3,x_4)-V^{(2)}(x_2,x_1)V^{(2)}(x_3,x_4)]$

Again, by considering

$\int h_1(x_1)h_2(x_2)h_3(x_3)h_4(x_4)[V^{(4)}(x_1,x_2,x_3,x_4)+\varepsilon V^{(4)}(x_1,x_2,x_4,x_3)]$

$\times \quad [dx]^4$

we also get

(4.9) $V^{(4)}(x_1,x_2,x_3,x_4)-V^{(2)}(x_1,x_2)V^{(2)}(x_3,x_4)$

$= -\varepsilon[V^{(4)}(x_1,x_2,x_4,x_3)-V^{(4)}(x_1,x_2)V^{(4)}(x_4,x_3)]$

Moreover,

$$\int h_1(x_1)h_2(x_2)h_3(x_3)h_4(x_4)[V^{(4)}(x_1,x_2,x_3,x_4)+\varepsilon V^{(4)}(x_1,x_3,x_2,x_4)] \times [dx]^4$$

$$= (\Psi_o,\varphi(h_1)[\varphi(\bar{h}_2)\varphi(h_3)+\varepsilon\varphi(\bar{h}_3)\varphi(h_2)]\varphi(\bar{h}_4)\Psi_o)$$

$$= (\Psi_o,\varphi(h_1)\Big[\int \bar{h}_2(x_2)\bar{h}_3(x_3).\Big\{V^{(2)}(x_2,x_3)+\varepsilon V^{(2)}(x_3,x_2)\Big\} dx_2.dx_3\Big]\varphi(\bar{h}_4)\Psi_o)$$

$$= \int h_1(x_1)h_2(x_2)h_3(x_3)h_4(x_4).[V^{(2)}(x_2,x_3)+\varepsilon V^{(2)}(x_3,x_2)]V^{(2)}(x_1,x_4) \times [dx]^4$$

since $\varphi(h_1)$ is an anti-linear operator.

Hence, using Theorem (2.4) we get

$$(4.10) \qquad V^{(4)}(x_1,x_2,x_3,x_4)-\varepsilon V^{(2)}(x_2,x_3)V^{(2)}(x_1,x_4)$$

$$= -\varepsilon[V^{(4)}(x_1,x_3,x_2,x_4)-\varepsilon V^{(2)}(x_3,x_2)V^{(2)}(x_1,x_4)]$$

Equations (4.8), (4.9) and (4.10) indicate that there is some reason for the assumption made that the 4 point functional can be expressed in terms of the 2 point functionals. This, together with Theorem 4.3,(ii) will enable us to write

$$V^{(4)}(x_1,x_2,x_3,x_4)=\lambda^2[\alpha_1\Delta(x_1,x_2)\Delta(x_3,x_4)+\beta_1\Delta(x_1,x_3)\Delta(x_2,x_4)$$

$$+\gamma_1\Delta(x_1,x_4)\Delta(x_2,x_3)$$

$$+\alpha_2\Delta(x_2,x_1)\Delta(x_3,x_4)+\beta_2\Delta(x_3,x_1)\Delta(x_2,x_4)+\gamma_2\Delta(x_4,x_1)\Delta(x_2,x_3)$$

$$+\alpha_3\Delta(x_2,x_1)\Delta(x_4,x_3)+\beta_3\Delta(x_3,x_1)\Delta(x_4,x_2)+\gamma_3\Delta(x_4,x_1)\Delta(x_3,x_2)$$

$$+\alpha_4\Delta(x_1,x_2)\Delta(x_4,x_3)+\beta_4\Delta(x_1,x_3)\Delta(x_4,x_2)+\gamma_4\Delta(x_1,x_4)\Delta(x_3,x_2),$$

where λ is the factor coming from Theorem 4.3 (ii) and $\alpha_\mu,\beta_\mu,\gamma_\mu$, $\mu = 1,2,3,4$, are constants.

This expression and equations (4.8), (4.9) and (4.10) together with Theorem 4.2, give the following 3 sets of relations among the constants, λ, α_μ, β_μ, γ_μ, $\mu = 1,2,3,4$.

(4.11),(4.12)

(1)	$\lambda^2(\alpha_1-1)$	$= -\varepsilon\lambda^2\alpha_2$	$= -\varepsilon\lambda^2\alpha_4$	(2)
	$\lambda^2\beta_1$	$= -\varepsilon\lambda^2\gamma_1$	$= -\varepsilon\lambda^2\gamma_1$	
(11)	$\lambda^2\gamma_1$	$= -\varepsilon\lambda^2\beta_1$	$= -\varepsilon\lambda^2\beta_1$	
	$\lambda^2\alpha_2$	$= -\varepsilon\lambda^2(\alpha_1-1)$	$= -\varepsilon\lambda^2\alpha_3$	
(9)	$\lambda^2\beta_2$	$= -\varepsilon\lambda^2\gamma_4$	$= -\varepsilon\lambda^2\gamma_2$	
(7)	$\lambda^2\gamma_2$	$= -\varepsilon\lambda^2\beta_4$	$= -\varepsilon\lambda^2\beta_2$	(8)
	$\lambda^2\alpha_3$	$= -\varepsilon\lambda^2\alpha_4$	$= -\varepsilon\lambda^2\alpha_2$	
(5)	$\lambda^2\beta_3$	$= -\varepsilon\lambda^2\gamma_3$	$= -\varepsilon\lambda^2\gamma_3$	
	$\lambda^2\gamma_3$	$= -\varepsilon\lambda^2\beta_3$	$= -\varepsilon\lambda^2\beta_3$	
(3)	$\lambda^2\alpha_4$	$= -\varepsilon\lambda^2\alpha_3$	$= -\varepsilon\lambda^2(\alpha_1-1)$	
	$\lambda^2\beta_4$	$= -\varepsilon\lambda^2\gamma_2$	$= -\varepsilon\lambda^2\gamma_4$	
	$\lambda^2\gamma_4$	$= -\varepsilon\lambda^2\beta_2$	$= -\varepsilon\lambda^2\beta_4$	

(4.13)

$$
\begin{aligned}
(12)\quad \lambda^2\alpha_1 &= -\varepsilon\lambda^2\beta_1 \\
\lambda^2\beta_1 &= -\varepsilon\lambda^2\alpha_1 \\
\lambda^2(\gamma_1-\varepsilon) &= -\varepsilon\lambda^2\gamma_4 \\
\lambda^2\alpha_2 &= -\varepsilon\lambda^2\beta_2 \\
\lambda^2\beta_2 &= -\varepsilon\lambda^2\alpha_2 \\
\lambda^2\gamma_2 &= -\varepsilon\lambda^2\gamma_3 \\
\lambda^2\alpha_3 &= -\varepsilon\lambda^2\beta_3 \qquad (4) \\
\lambda^2\beta_3 &= -\varepsilon\lambda^2\alpha_3 \\
\lambda^2\gamma_3 &= -\varepsilon\lambda^2\gamma_2 \qquad (6) \\
\lambda^2\alpha_4 &= -\varepsilon\lambda^2\beta_4 \\
\lambda^2\beta_4 &= -\varepsilon\lambda^2\alpha_4 \\
(10)\quad \lambda^2\gamma_4 &= -\varepsilon\lambda^2(\gamma_1-\varepsilon)
\end{aligned}
$$

These 3 sets of relations give us the following chain of relations.

$$
\lambda^2(\alpha_1-1) = -\varepsilon\lambda^2\alpha_2 = -\varepsilon\lambda^2\alpha_4 = \lambda^2\alpha_3 = -\varepsilon\lambda^2\beta_3 = \lambda^2\gamma_3
$$
$$
(1) \qquad (2) \qquad (3) \qquad (4) \qquad (5)
$$

$$
= -\varepsilon\lambda^2\gamma_2 = \lambda^2\beta_4 = \lambda^2\beta_2 = -\varepsilon\lambda^2\gamma_4 = \lambda^2(\gamma_1-\varepsilon) = -\lambda^2\varepsilon(\beta_1+1)
$$
$$
(6) \qquad (7) \qquad (8) \qquad (9) \qquad (10) \qquad (11)
$$

$$
= \lambda^2(\alpha_1-\varepsilon)
$$
$$
(12)
$$

Hence, if $\varepsilon \neq 1$, i.e. if $\varepsilon = -1$, then $\lambda = 0$

Hence, if $\varepsilon = -1$, $V^{(2)}(x_1,x_2) = 0$.

Now, let us analyse the other case when $\varepsilon = 1$.

Then we have a solution with $\lambda \neq 0$, given by

$\alpha_1 - 1 = -\alpha_2 = -\alpha_4 = \alpha_3 = -\beta_3 = \gamma_3 = -\gamma_2 = \beta_4 = \beta_2 = -\gamma_4 = \gamma_1 - 1$

$= -\beta_1 - 1$.

Taking $-\gamma_4 = \alpha$, we get the following solution, as the only solution.

$$V^{(4)}(x_1,x_2,x_3,x_4) = \lambda^2[(\alpha+1)\Delta(x_1,x_2)\Delta(x_3,x_4) - (\alpha+1)\Delta(x_1,x_3)\Delta(x_2,x_4)$$

$$+(\alpha+1)\Delta(x_1,x_4)\Delta(x_2,x_3) - \alpha.\Delta(x_2,x_1)\Delta(x_3,x_4) + \alpha\Delta(x_3,x_1)\Delta(x_2,x_4)$$

$$-\alpha\Delta(x_4,x_1)\Delta(x_2,x_3) + \alpha\Delta(x_2,x_1)\Delta(x_4,x_3) - \alpha\Delta(x_3,x_1)\Delta(x_4,x_2)$$

$$+\alpha\Delta(x_4,x_1)\Delta(x_3,x_2) - \alpha\Delta(x_1,x_2)\Delta(x_4,x_3) + \alpha\Delta(x_1,x_3)\Delta(x_4,x_2)$$

$$-\alpha\Delta(x_1,x_4)\Delta(x_3,x_2)$$

$$= \lambda^2[\Delta(x_1,x_2)\Delta(x_3,x_4) - \Delta(x_1,x_3)\Delta(x_2,x_4) + \Delta(x_1,x_4)\Delta(x_2,x_3)]$$

$$+\lambda^2\alpha[\Delta(x_1,x_2) - \Delta(x_2,x_1)][\Delta(x_3,x_4) - \Delta(x_4,x_3)]$$

$$-\lambda^2\alpha[\Delta(x_1,x_3) - \Delta(x_3,x_1)][\Delta(x_2,x_4) - \Delta(x_4,x_2)]$$

$$+\lambda^2\alpha[\Delta(x_1,x_4) - \Delta(x_4,x_1)][\Delta(x_2,x_3) - \Delta(x_3,x_2)].$$

Putting $x_1 - x_2 = \xi_1$, $x_2 - x_3 = \xi_2$, $x_3 - x_4 = \xi_3$, we get

$$W^{(4)}(\xi_1,\xi_2,\xi_3)=\lambda^2[\Delta(\xi_1)\Delta(\xi_3)-\Delta(\xi_1+\xi_2)\Delta(\xi_2+\xi_3)+\Delta(\xi_1+\xi_2+\xi_3)\Delta(\xi_2)]$$

$$+\lambda^2\alpha[\Delta(\xi_1)-\Delta(-\xi_1)][\Delta(\xi_3)-\Delta(-\xi_3)]-\lambda^2\alpha[\Delta(\xi_1+\xi_2)-\Delta(-\xi_1-\xi_2)]$$

$$\times\quad[\Delta(\xi_2+\xi_3)-\Delta(-\xi_2-\xi_3)]$$

$$+\lambda^2\alpha[\Delta(\xi_1+\xi_2+\xi_3)-\Delta(-\xi_1-\xi_2-\xi_3)][\Delta(\xi_2)-\Delta(-\xi_2)].$$

We can Fourier Transform this expression using the fact that $p_1\cdot\xi_1+p_2\cdot\xi_2+p_3\cdot\xi_3$

$$=p_1\cdot(\xi_1+\xi_2)+(p_2-p_1)\cdot(\xi_2+\xi_3)+(p_3+p_1-p_2)\cdot\xi_3$$

$$=p_1\cdot(\xi_1+\xi_2+\xi_3)+(p_2-p_1)\cdot\xi_2+(p_3-p_1)\cdot\xi_3,\text{and getting}$$

(4.14)

$$\frac{1}{(2\pi)^2}\ \tilde{W}^{(4)}(p_1,p_2,p_3)=\lambda^2[\delta(p_2)\tilde{\Delta}(p_1)\tilde{\Delta}(p_3)-\delta(p_3+p_1-p_2)$$

$$\times\quad\tilde{\Delta}(p_1)\tilde{\Delta}(p_2-p_1)+\delta(p_3-p_1)\tilde{\Delta}(p_1)\tilde{\Delta}(p_2-p_1)]$$

$$+\lambda^2\alpha\ \delta(p_2)[\tilde{\Delta}(p_1)-\tilde{\Delta}(-p_1)][\tilde{\Delta}(p_3)-\tilde{\Delta}(-p_3)]-\lambda^2\alpha\ \delta(p_3+p_1-p_2)$$

$$\times\ [\tilde{\Delta}(p_1)-\tilde{\Delta}(-p_1)]$$

$$\times\ [\tilde{\Delta}(p_2-p_1)-\tilde{\Delta}(-p_2+p_1)]+\lambda^2\alpha\ \delta(p_3-p_1)[\tilde{\Delta}(p_1)-\tilde{\Delta}(-p_1)]$$

$$\times\ [\tilde{\Delta}(p_2-p_1)-\tilde{\Delta}(-p_2+p_1)].$$

(4.15) Since $(\Box_1+m^2)V^{(4)}(x_1,x_2,x_3,x_4) = 0$, we also see that $\tilde{W}^{(4)}(p_1,p_2,p_3)$ vanishes unless $p_1^2 = m^2$.

Equation (4.14) holds as a tempered distribution, and hence will be true for any test function in $\mathcal{S}(R^{12})$. Let $f \in \mathcal{S}(R^{12})$ have its support in the neighbourhood of the point (q_1,q_2,q_3) chosen as follows

(i) $q_1^2 = m^2, q_1^{(0)}>0;\quad q_2 = 0;\ q_3^2 = m^2,\ q_3^{(0)} >0;\ q_3-q_1 \neq 0$

By the spectral condition in Theorem (2.3), the L.H.S. of equation (4.14) vanishes, and the only term which survives on the R.H.S. is $+\lambda^2\alpha\,\delta(p_2)\tilde{\Delta}(-p_1)\tilde{\Delta}(-p_3)$

$$\therefore\ \lambda^2\alpha = 0$$

(ii) Now, let us take $q_1^2 = m^2,\ q_1^{(0)}< 0;\ q_3 = q_1$

$$(q_2-q_1)^2=m^2,\ q_2^{(0)}-q_1^{(0)} < 0,\ q_2 \neq 2q_1.$$

Such a choice is clearly possible and implies that q_2 cannot be in or on the forward cone.

Then, again by the spectral condition, the L.H.S. vanishes, and the only term which survives on the R.H.S. is $\lambda^2\delta(p_3-p_1)\tilde{\Delta}(p_1)\tilde{\Delta}(p_2-p_1)$.

$$\therefore\ \lambda^2 = 0$$

Hence, even in the case $\varepsilon = +1$, $V^{(2)}(x_1,x_2) = 0$ which completes the required proof.

(4.16) Before we proceed any further, we have to get the expression for

$$E = V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\rho\mu_{\rho+1}\cdots\mu_\nu}(x_1,x_2,\ldots,x_\nu)$$

$$+\varepsilon(\rho,\rho+1)V^{(\nu)}_{\mu_1\mu_2\cdots\mu_{\rho-1}\mu_{\rho+1}\mu_\rho\mu_{\rho+2}\cdots\mu_\nu}(x_1,x_2,\ldots,x_{\rho-1},x_{\rho+1},x_\rho,x_{\rho+2},\ldots,x_\nu)$$

in momentum space ir terms of $\tilde{W}^{(\nu)}_{\sigma_1\sigma_2\cdots\sigma_\nu}(p_1,p_2,\ldots,p_{\nu-1})$,

$$1 \leqslant \rho \leqslant \nu-1.$$

In terms of the variables $\xi_\sigma = x_\sigma - x_{\sigma+1}$, $1 \leqslant \sigma \leqslant \nu-1$,

$$E = W^{(\nu)}_{\mu_1\mu_2\cdots\mu_\rho\mu_{\rho+1}\cdots\mu_\nu}(\xi_1,\xi_2,\ldots,\xi_{\nu-1})$$

$$+\varepsilon(\rho,\rho+1)W^{(\nu)}_{\mu_1\mu_2\cdots\mu_{\rho-1}\mu_{\rho+1}\mu_\rho\mu_{\rho+2}\cdots\mu_\nu}(\xi_1,\xi_2,\ldots,\xi_{\rho-2},$$

$$,\xi_{\rho-1}+\xi_\rho,-\xi_\rho,\xi_\rho+\xi_{\rho+1},\xi_{\rho+2},\ldots,\xi_{\nu-1}),\ 2 \leqslant \rho\ \nu-2$$

$$= W^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(\xi_1,\ \xi_2,\ldots,\xi_{\nu-1})+\varepsilon(1,2)W^{(\nu)}_{\mu_2\mu_1\mu_3\mu_4\cdots\mu_\nu}(-\xi_1,$$

$$,\xi_1+\xi_2,\xi_3,\ldots,\xi_{\nu-1}) \text{ if } \rho = 1$$

$$= W^{(\nu)}_{\mu_1\mu_2\cdots\mu_{\nu-1}\mu_\nu}(\xi_1,\xi_2,\ldots,\xi_{\nu-1})$$

$$+\varepsilon(\nu-1,\nu)W^{(\nu)}_{\mu_1\mu_2\cdots\mu_{\nu-2}\mu_\nu\mu_{\nu-1}}(\xi_1,\xi_2,\ldots,\xi_{\nu-3},\xi_{\nu-2}+\xi_{\nu-1},-\xi_{\nu-1})$$

$$\text{if } \rho=\nu-1$$

Now $$\sum_{\sigma=1}^{\nu-1} p_\sigma \cdot \xi_\sigma = \sum_{\sigma=1}^{\rho-2} p_\sigma \cdot \xi_\sigma + \sum_{\sigma=\rho+2}^{\nu-1} p_\sigma \cdot \xi_\sigma + p_{\rho-1} \cdot \xi_{\rho-1} + p_\rho \cdot \xi_\rho + p_{\rho+1} \cdot \xi_{\rho+1}$$

if $2 \leqslant \rho \leqslant \nu-2$

$$= \sum_{\sigma=1}^{\rho-2} p_\sigma \cdot \xi_\sigma + \sum_{\sigma=\rho+2}^{\nu-1} p_\sigma \cdot \xi_\sigma + \ldots \ldots$$

$$p_{\rho-1} \cdot (\xi_{\rho-1} + \xi_\rho) + (p_\rho - p_{\rho-1} - p_{\rho+1}) \cdot \xi_\rho + p_{\rho+1} \cdot (\xi_\rho + \xi_{\rho+1})$$

if $2 \leqslant \rho \leqslant \nu-2$

If $\rho=1$, $$\sum_{\sigma=1}^{\nu-1} p_\sigma \cdot \xi_\sigma = p_1 \cdot \xi_1 + p_2 \cdot \xi_2 + \sum_{\sigma=3}^{\nu-1} p_\sigma \cdot \xi_\sigma$$

$$= (p_1 - p_2) \cdot \xi_1 + p_2 \cdot (\xi_1 + \xi_2) + \sum_{\sigma=3}^{\nu-1} p_\sigma \cdot \xi_\sigma \cdot$$

If $\rho=\nu-1$, $$\sum_{\sigma=1}^{\nu-1} p_\sigma \cdot \xi_\sigma = \sum_{\sigma=1}^{\nu-3} p_\sigma \cdot \xi_\sigma + p_{\nu-2} \cdot \xi_{\nu-2} + p_{\nu-1} \cdot \xi_{\nu-1}$$

$$= \sum_{\sigma=1}^{\nu-3} p_\sigma \cdot \xi_\sigma + p_{\nu-2} \cdot (\xi_{\nu-2} + \xi_{\nu-1}) + (p_{\nu-1} - p_{\nu-2}) \cdot \xi_{\nu-1}$$

Using these expressions, we can write the Fourier Transform $\tilde{E}$ of E as follows

$$(4.17)\ (i)\qquad \tilde{E} = \tilde{W}^{(\nu)}_{\mu_1\mu_2\cdots\mu_\rho\mu_{\rho+1}\cdots\mu_\nu}(p_1,p_2,\cdots,p_\rho,p_{\rho+1},\cdots,p_{\nu-1})$$

$$+\ \varepsilon(\rho,\rho+1)\tilde{W}^{(\nu)}_{\mu_1\mu_2\cdots\mu_{\rho-1}\mu_{\rho+1}\mu_\rho\mu_{\rho+2}\cdots\mu_\nu}(p_1,p_2,\cdots,p_{\rho-2},\ p_{\rho-1},$$

$$p_{\rho-1}-p_\rho+p_{\rho+1},p_{\rho+1},p_{\rho+2},\cdots,p_{\nu-1})\quad \text{if } 2\leqslant\rho\leqslant\nu-2$$

$$(4.17)\ (ii)\qquad \tilde{E} = \tilde{W}^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(p_1,p_2,\cdots,p_{\nu-1})$$

$$+\ \varepsilon(1,2)\tilde{W}^{(\nu)}_{\mu_2\mu_1\mu_3\mu_4\cdots\mu_\nu}(p_2-p_1,p_2,p_3,\cdots p_{\nu-1})$$

$$\text{if } \rho = 1$$

$$(4.17\ (iii)\qquad = \tilde{W}^{(\nu)}_{\mu_1\mu_2\cdots\mu_{\nu-1}\mu_\nu}(p_1,p_2,\cdots,p_{\nu-1})$$

$$+\ \varepsilon(\nu-1,\nu)\tilde{W}^{(\nu)}_{\mu_1\mu_2\cdots\mu_{\nu-2}\mu_\nu\mu_{\nu-1}}(p_1,p_2,\cdots,p_{\nu-3},p_{\nu-2},p_{\nu-2}-p_{\nu-1})$$

$$\text{if } \rho = \nu-1$$

With these expressions ready, we now state and prove the following theorem.

Theorem 4.5: If $\varphi_\mu(h)$ is not a symmetric field component in a Theory of free local fields, and two consecutive suffixes $\mu_\rho,\mu_{\rho+1}$ (or $\mu_{\rho-1},\mu_\rho$, $2 \leqslant \rho \leqslant \nu$) $1 \leqslant \rho \leqslant \nu-1$ in the V.E. Functional $V^{(\nu)}_{\mu_1\mu_2\ldots\mu_{\rho-1}\mu_\rho\mu_{\rho+1}\ldots\mu_\nu}(x_1,x_2,\ldots,x_\nu)$ such that $\mu_\rho = \mu$, and $\mu_{\rho+1} \neq \hat{\mu}$ ($\mu_{\rho-1} \neq \hat{\mu}$), then this V.E. Functional vanishes.

Furthermore, $V^{(2)}_{\mu\hat{\mu}}(x_1,x_2) = 0$.

Proof: Suppose $\mu_1 = \mu, \mu_2 \neq \hat{\mu}$ ($\mu_1 \neq \hat{\mu}, \mu_2 = \mu$)

Then $V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots,x_\nu)+\varepsilon(1,2)V^{(\nu)}_{\mu_2\mu_1\mu_3\ldots\mu_\nu}(x_2,x_1,x_3\ldots x_\nu)$

$$=(\Psi_o,[\varphi_{\mu_1}(x_1)\varphi_{\mu_2}(x_2)+\varepsilon(1,2)\varphi_{\mu_2}(x_2)\varphi_{\mu_1}(x_1)]\varphi_{\mu_3}(x_3)\ldots \varphi_{\mu_\nu}(x_\nu)\Psi_o)$$

where we have dropped the test functions for convenience. But, by the definition of a local Field Theory,

$$\varphi_{\mu_1}(x_1)\varphi_{\mu_2}(x_2)+\varepsilon(1,2)\varphi_{\mu_2}(x_2)\varphi_{\mu_1}(x_1) = 0$$

$$\therefore \quad V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots,x_\nu)+\varepsilon(1,2)V^{(\nu)}_{\mu_2\mu_1\mu_3\ldots\mu_\nu}(x_2,x_1,x_3,\ldots,x_\nu) = 0$$

Using equation (4.17) (ii), we get

$$\tilde{W}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(p_1,p_2,\ldots,p_{\nu-1})+\varepsilon(1,2)\tilde{W}^{(\nu)}_{\mu_2\mu_1\mu_3\ldots\mu_\nu}(p_2-p_1,p_2,p_3,\ldots p_{\nu-1}) = 0$$

By the spectral condition, the first term vanishes, and hence the second term also vanishes unless $p_1 \in \overline{V}^-, p_2 \in \overline{V}^+$.

$\therefore$ each one of them vanishes unless $p_1 = p_2 = 0$

But $(\Box_1+m_{\mu_1}^2)V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(x_1,x_2,\ldots,x_\nu) = 0$ which means that

$\tilde{W}^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(p_1,p_2,\ldots,p_{\nu-1}) = 0$ unless $p_1^2 = m_{\mu_1}^2 \quad (\neq 0)$

Hence $\tilde{W}^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(p_1,p_2,\ldots,p_{\nu-1}) = 0$

$\therefore$ $V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(x_1,x_2,\ldots,x_\nu) \equiv 0$

The case when $\mu_{\nu-1} = \mu$, $\mu_\nu \neq \Lambda$ $(\mu_\nu=\mu, \mu_{\nu-1}\neq\Lambda)$ is almost the same as above.

Now, suppose

$$\mu_\rho=\mu, \mu_{\rho+1} \neq \Lambda, \quad 2 \leqslant \rho \leqslant \nu-2 \quad (\mu_{\rho+1} = \mu, \mu_\rho \neq \Lambda)$$

Here again, we will see, as in the previous case, that

$$V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\rho\mu_{\rho+1}\cdots\mu_\nu}(x_1,x_2,\ldots,x_\nu)$$

$$+ \varepsilon(\rho,\rho+1)V^{(\nu)}_{\mu_1\mu_2\cdots\mu_{\rho-1}\mu_{\rho+1}\mu_\rho\mu_{\rho+2}\cdots\mu_\nu}(x_1,x_2,\ldots,x_{\rho-1},x_{\rho+1},$$

$$x_\rho,x_{\rho+2},\ldots,x_\nu) = 0$$

Using equation (4.17) (i) we get

$$\tilde{W}^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(p_1,p_2,\ldots,p_{\rho-1},p_\rho,p_{\rho+1},\ldots,p_{\nu-1})$$

$$+\varepsilon(\rho,\rho+1)\tilde{W}^{(\nu)}_{\mu_1\mu_2\cdots\mu_{\rho-1}\mu_{\rho+1}\mu_\rho\mu_{\rho+2}\cdots\mu_\nu}(p_1,p_2,\ldots,p_{\rho-1}$$

$$,p_{\rho-1}-p_\rho+p_{\rho+1},p_{\rho+1},\ldots,p_{\nu-1}) = 0$$

If ρ is even, the support of each one of the above two terms will be contained in the intersection of the regions

$$p_{\rho-1} \in \bar{V}^-, p_{\rho+1} \in \bar{V}^-, p_\rho \in \bar{V}^+, p_{\rho-1}-p_\rho+p_{\rho+1} \in \bar{V}^+,$$

and if ρ is odd, the region is

$p_{\rho-1} \in \bar{V}^+$, $p_{\rho+1} \in \bar{V}^+$, $p_\rho \in \bar{V}^-$, $p_{\rho-1}-p_\rho + p_{\rho+1} \in \bar{V}^-$ as follows from the spectral condition.

In either case, the region is $p_{\rho-1} = p_\rho = p_{\rho+1} = 0$

Hence $\tilde{W}^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(p_1,p_2,\ldots,p_{\rho-1},p_\rho,p_{\rho+1},\ldots,p_{\nu-1}) = 0$

unless $p_{\rho-1} = p_\rho = p_{\rho+1} = 0$

But $(\Box_\rho+m_{\mu_\rho}^2)V^{(\nu)}_{\mu_1\mu_2\cdots\mu_\rho\cdots\mu_\nu}(x_1,x_2,\ldots,x_\rho,x_{\rho+1},\ldots,x_\nu) = 0$

$\therefore \tilde{W}^{(\nu)}_{\mu_1\mu_2\cdots\mu_\nu}(p_1,p_2,\ldots,p_{\nu-1}) = 0$ unless $(p_\rho-p_{\rho-1})^2=m^2_{\mu_\rho}$ $(\neq 0)$

$\therefore$ $\hat{W}^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(p_1,p_2,\ldots,p_{\nu-1}) \equiv 0$, and hence

$$V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots,x_\nu) = 0.$$

To prove the 2nd part of the Theorem, we consider

$$V^{(4)}_{\mu\beta\mu\beta}(x_1,x_2,x_3,x_4).$$

$$V^{(4)}_{\mu\beta\mu\beta}(x_1,x_2,x_3,x_4)=V^{(4)}_{\mu\beta\mu\beta}(x_1,x_2,x_3,x_4)+\varepsilon(\mu,\beta)V^{(4)}_{\beta\mu\mu\beta}(x_2,x_1,x_3,x_4)$$

since the 2nd term is zero by the result already proved.

$$=(\Psi_o[\varphi_\mu(x_1)\varphi_\beta(x_2)+\varepsilon(\mu,\beta)\varphi_\beta(x_2)\varphi_\mu(x_1)]\varphi_\mu(x_3)\varphi_\beta(x_4)\Psi_o).$$

Here again, we have a result similar to equation (4.7) and together with the result (iii) of Theorem 4.3, we get,

$V^{(4)}_{\mu\beta\mu\beta}(x_1,x_2,x_3,x_4)$

(4.18)

$$= (\lambda_{\mu\beta})^2\Delta(x_1,x_2)\Delta(x_3,x_4)+\varepsilon(\mu,\beta)\lambda_{\mu\beta}\cdot\lambda_{\beta\mu}\cdot\Delta(x_2,x_1)\Delta(x_3,x_4)$$

We also can write $V^{(4)}_{\mu\beta\mu\beta}(x_1,x_2,x_3,x_4)$

$$= V^{(4)}_{\mu\beta\mu\beta}(x_1,x_2,x_3,x_4)+\varepsilon(\beta,\mu)V^{(4)}_{\mu\mu\beta\beta}(x_1,x_3,x_2,x_4)$$

$$= (\Psi_o,\varphi_\mu(x_1)[\varphi_\beta(x_2)\varphi_\mu(x_3)+\varepsilon(\beta,\mu)\varphi_\mu(x_3)\varphi_\beta(x_2)]\varphi_\beta(x_4)\Psi_o)$$

$$= [\overline{V^{(2)}_{\beta\mu}(x_2,x_3)}+\varepsilon(\beta,\mu)\overline{V^{(2)}_{\mu\beta}(x_3,x_2)}]V^{(2)}_{\mu\beta}(x_1,x_4) \text{ again}$$

because of an equation similar to (4.7)

$= [V^{(2)}_{\mu\hat{\mu}}(x_3,x_2)+\varepsilon(\hat{\mu},\mu)V^{(2)}_{\hat{\mu}\mu}(x_2,x_3)]V^{(2)}_{\mu\hat{\mu}}(x_1,x_4)$ due to

Theorem 2.4

$$(4.19) \quad = (\lambda_{\mu\hat{\mu}})^2\Delta(x_3,x_2)\Delta(x_1,x_4)+\varepsilon(\hat{\mu},\mu)\lambda_{\hat{\mu}\mu}\cdot\lambda_{\mu\hat{\mu}}\Delta(x_2,x_3)\Delta(x_1,x_4)$$

where we have used result (iii) of Theorem 4.3

Equations (4.18) and (4.19) together with Theorem 4.2 gives

$$\lambda_{\mu\hat{\mu}} = 0$$

$$\therefore \quad V^{(2)}_{\mu\hat{\mu}}(x_1,x_2) = 0$$

Theorem 4.6: If $\varphi_\mu(h)$ is a symmetric field component in a Theory of Free Local Fields, and $\exists$ two consecutive suffixes $\mu_\rho,\mu_{\rho+1}$, $1 \leqslant \rho \leqslant \nu-1$ $(\mu_\rho,\mu_{\rho-1}$, $2 \leqslant \rho \leqslant \nu)$ in the V.E. Functional

$$V^{(\nu)}_{\mu_1\mu_2\cdots\mu_{\rho-1}\mu_\rho\mu_{\rho+1}\cdots\mu_\nu}(x_1,x_2,\ldots,x_{\rho-1},x_\rho,x_{\rho+1},\ldots,x_\nu)$$

such that $\mu_\rho = \mu$, $\mu_{\rho+1} \neq \mu,\hat{\mu}$ $(\mu_{\rho-1} \neq \mu,\hat{\mu})$, then the V.E. Functional vanishes.

Proof: The proof is identical with that given for the first part of Theorem 4.5.

Theorem 4.7: If $\varphi_\mu(h)$ is a symmetric field component in a Theory of Free Local Fields, and satisfies the condition in Theorem 4.4, then $V^{(\nu)}_{\mu\mu\mu\ldots\mu}(x_1,x_2,x_3,\ldots,x_\nu) = 0$ for all $\nu \geqslant 1$.

Proof: $V^{(\nu)}_{\mu\mu\ldots\mu}(x_1,x_2,\ldots,x_\nu)+\varepsilon(\mu,\mu)V^{(\nu)}_{\mu\mu\ldots\mu}(x_2,x_1,x_3,\ldots,x_\nu)$

$= (\Psi_o,[\varphi_\mu(x_1)\varphi_\mu(x_2)+\varepsilon(\mu,\mu)\varphi_\mu(x_2)\varphi_\mu(x_1)]\varphi_\mu(x_3)\varphi_\mu(x_4)\ldots\varphi_\mu(x_\nu)\Psi_o)$

for all $\nu \geqslant 3$. Here again, we have dropped test functions for convenience.

$= [V^{(2)}_{\mu\mu}(x_1,x_2)+\varepsilon(\mu,\mu)V^{(2)}_{\mu\mu}(x_2,x_1)]V^{(\nu-2)}_{\mu\mu\ldots\mu}(x_3,x_4,\ldots,x_\nu)$

by using equation (4.7)

$= 0$ by Theorem 4.4.

Using equation (4.17) (ii), we get

$$\tilde{W}^{(\nu)}_{\mu\mu\ldots\mu}(p_1,p_2,\ldots,p_{\nu-1})+\varepsilon(\mu,\mu)\tilde{W}^{(\nu)}_{\mu\mu\ldots\mu}(p_2-p_1,p_2,p_3,\ldots,p_{\nu-1}) = 0$$

Together with the spectral condition, this means that

$\tilde{W}^{(\nu)}_{\mu\mu\ldots\mu}(p_1,p_2,\ldots,p_{\nu-1}) = 0$ unless $p_1 = p_2 = 0$, for all $\nu \geqslant 3$.

But $(\Box_1+m_\mu^2)V^{(\nu)}_{\mu\mu\ldots\mu}(x_1,x_2,\ldots,x_\nu) = 0$

$\therefore \tilde{W}^{(\nu)}_{\mu\mu\ldots\mu}(p_1,p_2,\ldots,p_{\nu-1}) = 0$ unless $p_1^2 = m_\mu^2$ $(\neq 0)$

$\therefore$ $V^{(\nu)}_{\mu\mu\ldots\mu}(x_1,x_2,\ldots,x_\nu) = 0$ for all $\nu \geqslant 1$, because the cases $\nu = 1,2$ are already covered by Theorems 4.3 and 4.4.

Theorem 4.8: If $\varphi_\mu(h)$ is not a symmetric field component in a Theory of Free Local Fields, then

$$0 = V^{(\nu)}_{\mu\hat{\mu}\mu\hat{\mu}\ldots\mu\hat{\mu}}(x_1,x_2,\ldots,x_\nu), \quad \nu \text{ even} \geqslant 2$$

$$= V^{(\nu)}_{\mu\hat{\mu}\mu\hat{\mu}\ldots\mu\hat{\mu}\mu}(x_1,x_2,\ldots,x_\nu), \quad \nu \text{ odd} \geqslant 1.$$

Proof: Let $\mu_\rho = \mu$ if ρ is odd

$= \hat{\mu}$ if ρ is even.

$$\therefore \quad V^{(\nu)}_{\mu_1\mu_2\mu_3\ldots\mu_\nu}(x_1,x_2,x_3,\ldots,x_\nu)+\varepsilon(\mu_1,\mu_2)V^{(\nu)}_{\mu_2\mu_1\mu_3\ldots\mu_\nu}(x_2,x_1,x_3,\ldots,x_\nu)$$

$$= (\Psi_o,[\varphi_{\mu_1}(x_1)\varphi_{\mu_2}(x_2)+(\mu_1,\mu_2)\varphi_{\mu_2}(x_2)\varphi_{\mu_1}(x_1)]\varphi_{\mu_3}(x_3)\varphi_{\mu_4}(x_4)\ldots\varphi_{\mu_\nu}(x_\nu)\Psi_o)$$

$$= [V^{(2)}_{\mu\hat{\mu}}(x_1,x_2)+\varepsilon(\mu,\hat{\mu})V^{(2)}_{\hat{\mu}\mu}(x_2,x_1)]V^{(\nu-2)}_{\mu_3\mu_4\ldots\mu_\nu}(x_3,x_4,\ldots,x_\nu)$$

because we have an equation similar to (4.7)

$= 0$ by Theorem 4.5.

Using equation (4.17) (ii), the spectral condition and the condition for the fields to be free, we get the desired result.

Theorem 4.9: In a Theory of Free Local Fields, all field components vanish, i.e. $\varphi_\mu(h) = 0$ for all μ, $1 \leqslant \mu \leqslant \Lambda(\Gamma)$, if all symmetric field components satisfy the condition for Theorem 4.4.

Proof: Let $V^{(\nu)}_{\mu_1\mu_2\ldots\mu_\nu}(x_1,x_2,\ldots,x_\nu)$, $\nu \geqslant 2$ be any V.E. Functional of the Theory. This can be one of two types.

(a) All the field components $\varphi_{\mu_1}(h_1),\varphi_{\mu_2}(h_2),..,\varphi_{\mu_\nu}(h_\nu)$ are symmetric

(b) $\exists$ at least one ρ, $1 \leqslant \rho \leqslant \nu$ such that $\varphi_{\mu_\rho}(h)$ is not symmetric.

The case (a) can be broken up into 2 parts as follows

(ai) For all ρ, $1 \leqslant \rho \leqslant \mu-1$, $\mu_\rho = \mu_{\rho+1}$ or $\hat{\mu}_{\rho+1}$

(aii) $\exists$ at least one ρ $1 \leqslant \rho \leqslant \nu-1$ such that $\mu_\rho \neq \mu_{\rho+1}, \hat{\mu}_{\rho+1}$

Similarly the case (b) can also be broken up into 2 parts as follows.

(bi) For every ρ given by (b), $\mu_\rho = \hat{\mu}_{\rho+1}$ $(1 \leqslant \rho \leqslant \nu-1)$

or $\mu_\rho = \hat{\mu}_{\rho-1}$ $(2 \leqslant \rho \leqslant \nu)$ or both

(bii) For some ρ given by (b), either $\mu_\rho \neq \hat{\mu}_{\rho+1}$ $(1 \leqslant \rho \leqslant \nu-1)$

or $\mu_\rho \neq \hat{\mu}_{\rho-1}$ $(2 \leqslant \rho \leqslant \nu)$.

All V.E. Functionals of type (ai) vanish by Theorem 4.7. All of type (aii) vanish due to Theorem 4.6. Those of types (bi) and (bii) vanish because of Theorems 4.5 and 4.8. Hence, all V.E. Functionals of a Theory of Free Local Fields vanish identically.

Hence, by the Reconstruction Theorem, all field components vanish identically.

REFERENCES

1. E.G. Phillips - A Course of Analysis pp 29-30.
2. John von Neumann - Mathematical Foundations of Quantum Mechanics - pp 13-14 and Theorem 10.
3. F. Riesz and B. Sz. Nagy - Functional Analysis pp 61.
4. Ref 2 - pp 57.
5. Ref 2 - pp 55 Theorem 8.
6. Ref 2 - pp 53, Theorem 6; pp 52, Theorem 5.
7. Ref 2 - pp 53, Corollary to Theorem 5.
8. R.F. Streater and A.S. Wightman - PCT, Spin and Statistics, and all that, pp 4 (Referred to as PCT in the text).
9. Ref 8 - pp 96.
10. Ref 8 - pp 97
11. A.S. Wightman - Lectures presented at a Seminar, Trieste, 16 July-25 August 1962 - pp 36, Section 2.2.
12. Ref 8 - pp 35
13. Ref 8 - pp 33
14. Ref 8 - Chapter 3, Section 3.1.
15. Ref 8 - Chapter 1, Section 1.3.
16. Ref 8 - pp 98, Discussion at the end of Assumption 1.
17. Ref 8 - pp 100.
18. Ref 8 - pp 100, para 4 onwards.
19. Ref 8 - pp 42 and Theorem 2.1.
20. Ref 8 - pp 106, 107.
21. Ref 8 - pp 106-113.

22. Ref 8 - pp 38.

23. Ref 8 - pp 39.

24. Ref 8 - pp 90.

25. A. Uhlman - Annals of Physics 13 (1961) pp 453-462.

26. A. Araki - Annals of Physics 11 (1960) pp 260-274.

27. D. Ruelle - Helv. Phys. Acta 35 (1962) pp 147-163.

28. Ref 8 - pp 111.

29. A.S. Wightman - Phys. Rev. 101 (1956) pp 860.

30. W. Schmidt and K. Baumann - Nuovo Cimento 4 (1956)pp 860.

31. Ref. 8 - pp 117.

32. H.J. Borchers - Nuovo Cimento 24 (1962) pp 215-236.

33. This is provable for linear fields as has been done in the preceding reference, page 222, Theorem 3. Since the proof of this uses the construction of $\mathcal{H}$ in terms of the fields, we cannot take over this proof for anti-linear fields. However, we assume the result on physical grounds.

34. Ref 8 - pp 92.

35. B.L. Van der Waerden - Modern Algebra, Vol II pp 127.

36. Ref 35 - Vol I, pp9.

37. I.M. Gelfand and G.E. Shilov - Generalized Functions Vol 2, page 119 Theorem 4 (French).

38. E.C.G. Stueckelberg, M. Guenin, C. Piron and H. Ruegg. Helv. Phys. Acta 34 (1961) pp 675-698.

39. E.C.G. Stueckelberg and M. Guenin
Helv. Phys. Acta 35 (1962) pp 673-

40. E.C.G. Stueckelberg
Helv. Phys. Acta 33 (1960) pp 727-752.

41. M. Guenin and E.C.G. Stueckelberg
Helv. Phys. Acta 34 (1961) pp 506-508.

42. E.C.G. Stueckelberg and M. Guenin
Helv. Phys. Acta 34 (1961) pp 621-628.

43. A.S. Wightman - Relations de dispersion et particules êlementaires - Edited by De Witt and R. Omnes.
Les Houches 1960, pp 291-

44. K. Hoffmann- Banach Spaces of Analytic Functions - Prentice Hall, Page 5.

45. N.I. Akhiezer and I.M. Glazman - Theory of Linear Operators in Hilbert Space - footnote on page 48.