

VISCOUS/INVISCID MATCHING USING
IMBEDDED NAVIER-STOKES EQUATIONS

by

Hong Sun Ryou

Department of Aeronautics
Imperial College
University of London

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To my wife Gina

Thanks to God, I am grateful you are with me.

ABSTRACT

The conventional method of calculating flows past aerofoils involves the solution of an elliptic inviscid equation for the outer flow, to provide the boundary condition for a solution of parabolic thin shear layer equations in the viscous/turbulent part of the flow. The viscous part in turn affects the inner boundary condition for the inviscid equation and the two sets of boundary conditions must be matched iteratively.

It is well known that the thin shear equations become inaccurate once the normal pressure gradient across the shear layer becomes significant as in the trailing edge region. If the normal pressure gradient is significant, the upstream influence within the shear layer is also likely to be significant and the shear layer equations must be regarded as elliptic. The iterative solution of these elliptic equations can be imbedded in the viscous/inviscid matching loop.

The present work is to develop a calculation procedure based on the Mahgoub and Bradshaw method, to improve accuracy, and to compute the complete flow over aerofoils. In order to deal with the geometric singularity at a sharp trailing edge, and to improve accuracy near the trailing edge, a transformation to "inviscid wedge flow" coordinates is used. The Bradshaw-Ferriss-Atwell turbulence model is used for the present boundary layer calculation. For the near wake region, which has an important effect on the pressure distribution along the aerofoil, Bradshaw(1972)'s superposition hypothesis is used

by changing the appropriate coefficient value.

Complete aerofoil calculations, including laminar boundary layer calculation (assuming negligible $\partial p/\partial y$) and full treatment of the wake, have been done for incompressible flow. The Hess and Smith surface source distribution method was used for the inviscid flow calculation and Thwaites' integral method for the laminar boundary layer calculation. Finally several cases have been tested including Nakayama's asymmetric aerofoil Model A and good agreement with measurements was obtained. A compressible version of the shear layer calculation is fully coded but the accompanying inviscid program is not yet satisfactory.

Note: The Ph.D thesis by Mahgoub (1978) presents the strategy of the method at length : the reader requiring detailed information should consult Mahgoub's thesis (available on microfiche from Imperial college) as well as the present one.

CONTENTS

	<u>Page</u>
<u>Abstract</u>	i
<u>Acknowledgements</u>	iii
<u>Contents</u>	iv
Nomenclature:	
(i) List of Abbrevtion	viii
(ii) List of Definition	ix
(iii) List of Symbols	x
<u>Chapter 1. Introduction</u>	
1.1 Purpose of the present work	1
1.2 Review of previous work	4
1.2.1 Introduction	4
1.2.2 Methods using the boundary layer equation	6
1.2.3 Use of Normal Momentum Equation	8
1.2.4 Turbulence Models	12
<u>Chapter 2. Shear layer calculation</u>	
2.1 The Shear Layer along the Aerofoil	14
2.1.1 Physical Description and Governing Equations	14
2.1.2 The Solution procedure	17
2.2 The Wake Region	23
2.2.1 Physical Description and Governing Equations	23
2.2.2 The Solution procedure for the Wake	27
2.3 The numerical scheme for One Streamwise Sweep of the Calculation	29
2.4 The Transformation Region	36
2.5 Description of the Program	37
2.5.1 The Boundary Layer	38

2.5.2 The Wake	48
<u>Chapter 3 The Potential-Flow Calculation</u>	
3.1 Description of the Potential Flow Calculation	71
3.1.1 The Normal velocity Boundary condition	71
3.1.2 Calculation of the Pressure	
Coefficient for a Nonlifting Aerofoil	74
3.1.3 The Kutta Condition for Lifting Aerofoil	75
3.1.4 The Calculation of the Stagnation Point	80
3.2 Description of Program	81
<u>Chapter 4 Viscous / Inviscid Matching Procedure</u>	
4.1 Description and Numerical Details of	
the Matching Procedure	87
4.2 Description of Program for Matching Procedure	97
4.3 Description of Operation/User's Guide for	
Viscous/Inviscid Iteration Cycle	101
<u>Chapter 5 Test Cases</u>	112
5.1 Flat plate.	112
5.2 Symmetrical aerofoil	113
5.3 Asymmetrical aerofoil.	116
<u>Chapter 6 Conclusion</u>	162
<u>Reference</u>	165
<u>Appendix 1 Derivation of the Navier-Stokes Equations</u>	
<u>in Orthogonal Coordinates Using General</u>	
<u>Tensor Analysis</u>	178
A1.1 General Definitions and Derivations	178
A1.2 Derivation of the Navier-Stokes Equations	182
<u>Appendix 2 Derivation of Reynolds Stress</u>	
<u>Transport Equations and the Turbulent</u>	
<u>Kinetic Energy Equations Using</u>	
<u>General Tensor Analysis</u>	191

A2.1 Derivation of the Reynolds Stress Transport Equations	192
A2.2 Derivation of the Turbulent Kinetic Energy Equation	200
<u>Appendix 3 Derivation of Equations, and Definitions for the Shear Layer Calculation</u>	
A3.1 Derivation of Characteristic Equations	203
A3.2 Derivation of Equations along the Characteristics	206
A3.3 Finite Difference Forms for the V and p Calculations	207
A3.4 Method of Solution for the Wall Boundary Condition in Boundary Layer	213
A3.5 Spalding-Chi Skin Friction Formula	220
A3.6 Transfer U, V and τ Value Between (s,n) and (φ, ψ) Coordinate System	2
<u>Appendix 4 Data for Viscous/Inviscid Interaction Cycle and Miscellaneous Derivation</u>	
A4.1 Input Data	222
A4.2 Derivation of the Expression for the Velocity Induced by a Straight Line Source Element	228
<u>Appendix 5 Namelist for Variables used in the Computer Program</u>	
A5.1 Namelist for Variables in the Boundary Layer Calculation Program TURB	231
A5.2 Namelist for Variables in the Shear Layer of the Wake Calculation Program WAKE	240
A5.3 Namelist for Variables in the potential Flow Calculation Program AMOS	245
A5.4 Namelist for Variables in the Matching Program	

MATCH	249
<u>Appendix 6</u>	
A6.1 Procedure File for Running Comuter Program	254
A6.2 Example of Input and Output Data for Asymmetric Aerofoil	256

NOMENCLATURE(i) List of Abbreviation

AMOS	A.M.O.Smith (Smith and Hess).
BFA	Bradshaw-Ferriss-Atwell.
BL	Boundary Layer.
CFL	Courant-Friedrichs-Lewy.
LE	Leading Edge.
MB	Mahgoub-Bradshaw.
MS	Matching Surface.
PNS	Parabolised Navier-Stokes Equation.
PPNS	Partially Parabolised Navier-Stokes Equation.
RS	Real Surface.
SL	Shear Layer.
TE	Trailing Edge.
TSL	Thin Shear Layer.

(ii) List of Definitions

Nodes	These refer to either real surface or matching surface nodes.
Initial	Initial conditions refer to conditions just before the first viscous/inviscid iteration is performed.
Iteration	This refers to one complete viscous/inviscid interaction cycle, including potential flow, shear layer including wake and matching calculation.
Run	This indicates one potential flow calculation (program AMOS), one application of the matching procedure (program MATCH).
Shear Layer calculation station	These indicates calculation station for the shear layer.
Starting	Starting conditions at the first station ($s=s_0$) of the turbulent shear layer calculation. In wake it indicate the first station of wake just after trailing edge.
Sweep	This refers to <u>one streamwise calculation</u> of the shear layer.

(iii) List of Symbols

A', F', G', K', H'	Functions of the coefficients of partial differential equations.
$A_1, B_1, C_1, D_1, E_1,$ A_2, B_2, C_2, D_2, E_2	Coefficient of partial differential equations.
A_{pn}	A matrix of coefficients of the source element; this is known as the matrix of influence coefficients. Its elements represent the normal velocity induced by a unit source density at the n th element, on the p th element and at the mid-point of the p th element.
a_l	This is equal to $-\frac{\overline{uv}}{q^2}$ for the SL (see equation 2.5).
a^i	A general contravariant tensor of order one.
a_i	A general covariant tensor of order one.
$a(i)$	The physical component of a first order tensor.
a^{ij}	A general contravariant tensor of order two.
a_{ij}	A general covariant tensor of order two.
B_{pn}	A matrix of coefficients of the source elements; its elements represent the tangential velocity induced by a unit source density at the n th. element, on the p th. element and at the mid-point of the p th. element.
b_p	The normal velocity induced by the

	undisturbed stream velocity U_∞ at the mid-point of the pth. element.
b_p^*	Value of b_p for $U_\infty=1$.
C	The additive constant in the logarithmic law for the inner layer of the turbulent SL.
C_1	A constant used in representing $\overline{\rho u^2}$: $C_1 = \frac{\overline{\rho u^2}}{-\overline{\rho uv}}$
C_2	A constant used in representing $\overline{\rho v^2}$: $C_2 = \frac{\overline{\rho v^2}}{-\overline{\rho uv}}$
C_f	The skin friction coefficient, defined as $C_f = \frac{\tau_w}{\frac{1}{2} \rho U_e^2}$
C_p	The pressure coefficient, $(p - p_\infty) / \frac{1}{2} \rho U_\infty^2$
D	A function of R_θ used in the Spalding-Chi skin friction formula.
ds	Element of arc length in (s,n) coordinate system.
$d\phi$	Element of arc length in (ϕ, ψ) coordinate system.
$\hat{e}$	Extra strain rate correction in turbulent kinetic energy equation.
F_1, F_2, F_3	Functions used in applying the wall boundary condition for the shear layer calculation.
f_p	The tangential velocity induced by the

	undisturbed stream velocity U_∞ at the mid point of the pth. element.
f_p^*	Value of f_p for $U_\infty=1$.
G	An empirical function for the shear layer calculation: defined as
	$\frac{(\overline{p^2 v} + \frac{1}{2} \overline{q^2 v})}{(-uv)(-uv)_{\max}^{1/2}}$
$\overline{G}$	$G(\tau_m/\rho)^{1/2}$
G^{ij}	The Generation of Reynolds stress.
g_{rs}	The slope of the MS at any point (also at a node).
g_{rsJ}	Approximated slope of the RS at the Jth. node.
g^{ij}	The covariant metric tensor.
g_{ij}	The contravariant component of the metric tensor.
H	The shape parameter δ^*/θ for the laminar BL and also for the turbulent SL.
H_{ms}	Height of the MS from the RS in the normal direction.
h_1, h_2	The metric values.
h_i	A factor related to the physical component of a tensor of first order defined as $(g_{ii})^{1/2}$.
K	Von Karman's constant.
k	General node index.
k_m	Index of node on MS where the speed (total velocity magnitude) is a minimum.
L	Empirical length scale defined as $(-\overline{u v})^{3/2}/\epsilon$ used to model ϵ in the

	turbulent kinetic energy equation.
l_{ref}	A reference length used in scaling all lengths.
M	Source strength of point source.
M^{ij}	The mean transport of Reynolds stress.
MT^{ij}	Turbulent mass flux times mean acceleration.
m_n	Source density of the nth. element for the case of unit stream velocity only.
m_n'	Source density of nth. element for the case of vorticity distribution only.
m_p	Source density of the pth. element for the case of unit stream velocity only.
N	The total number of source elements.
n	The coordinate direction normal to the real surface, in the s-n coordinate system (equal to y in plane flow).
P	Total pressure.
P_e	Total pressure outside SL (on the MS).
p^{ij}	The turbulent transport of Reynolds stress by pressure fluctuations.
p	The two dimensional mean pressure field.
p'	The fluctuating pressure.
$P'(s)$	The shifted pressure distribution in the SL, at a constant normal distance from the RS; it is given by; $p(x,y) _{y=const} + P_{e,new} - P_{e,old} \quad .$
p_1	The value of p at the first mesh point in the SL calculation.

p_e	External pressure on the matching surface at SL calculation station or at MS nodes. In wake region the external pressure on upper part of MS.
p_{em}	External pressure on bottom part of MS at the element mid points of wake.
$p_{e,new}$	The current sweep value of p_e at the SL calculation stations.
$p_{e,old}$	The previous sweep value of p_e at the SL calculation stations.
p_{ref}	A reference pressure.
p_w	The wall pressure.
q_p	Tangential velocity on MS.
$q_p^{(1)}$	Tangential velocity distribution for flow case (1) of unit stream velocity.
$q_p^{(2)}$	Tangential velocity distribution for flow case (2) of unit vorticity distribution.
q_p	The nett velocity tangential to the pth element at it's mid point.
$q_p^{(i)}$	The value of q_p for flow case (i), $U_\infty = 1$.
$q_p^{(ii)}$	The value of q_p for flow case (ii) of unit vorticity distribution.
q_{np}	The tangential velocity induced by the nth. element at the mid point of the pth element.
R	The radius of curvature of the body surface (the real surface) and also used as constant in eddy viscosity term in the

	inner wake.
R_e	Reynolds number.
$R_{e\theta}$	The Reynolds number based on momentum thickness, defined as $R_{e\theta} = \frac{U_e \theta}{\nu}$.
$R_{e\theta, tr}$	The value of $R_{e\theta}$ at transition.
R^{ij}	Re-distribution of Reynolds stress by pressure fluctuations.
S	The total arc length around the MS.
s	The streamwise coordinate direction (arc length on RS) in the s-n coordinate system (equal to x in plane flow).
s'	Arc length measured along the MS, starting from the LE.
s_0	The value of s at the start of the turbulent SL calculation (i.e. at the first station).
s_1, s_2	Values of s' at which tangents drawn to the tangential velocity curve with s' , intersect the s' -axis.
s_l	The value of s at the end of turbulent SL calculation.
T^{ij}	The total turbulent transport of Reynolds stress.
t	Time variable.
U	The streamwise component of the mean velocity.
U_∞	The undisturbed stream velocity (external flow only).
U_1	The value of U at the first mesh point from

U_e	The value of the U-velocity (parallel to the $\underline{RS}$) on the matching surface, from the turbulent SL calculation. It is also used to describe the "free-stream" velocity for the laminar BL calculation.
U_{fut}	The value of U at the future station $x+\Delta x$ in the U, τ calculation.
U_{int}	The value of U interpolated at the position where the characteristic intersects a turbulent SL calculation station.
U_{JCL}	The streamwise velocity at the centre point in the wake.
U_{mid}	The value of U at the mid point of a characteristic segment.
U_{ref}	The reference velocity used in scaling all velocities.
U^i	The mean velocity component in the i direction.
u	The streamwise component of the turbulent fluctuating velocity.
$\overline{-\rho u^2}$	Reynolds normal stress in the streamwise direction.
$\overline{-\rho uv}$	Reynolds shear stress.
u_τ	The "friction velocity", defined as $(\tau_w/\rho)^{1/2}$.
u^i	The turbulent fluctuating velocity tensor in the i-direction.
V	The normal component of the mean velocity.
V_1	The value of V at the first mesh point from the surface in the SL calculation.

V_{1n}	Velocity induced by nth. source element, in the direction parallel to the nth. element.
V_{2n}	Velocity induced by the nth source element, in the direction normal to the nth-element.
V_e	The value of the V-velocity (normal to the RS) on the matching surface, from the turbulent SL calculation. It is also used to describe the "external" V velocity for the laminar BL calculation.
V_{fut}	The value of V at the future station $x+\Delta x$ in the U, τ calculation.
V_{int}	The value of V interpolated at the position where the characteristic intersects a turbulent SL calculation station.
V_{JCL}	The streamwise velocity at the centre point in the wake.
V_{mid}	The value of V at the mid point of a characteristic segment.
V_{ms}	Velocity normal to the MS
V_p	Nett velocity normal to the pth. element at its mid point.
$V_p^{(i)}$	The value of V_p for flow case (i), $U_\infty = 1$.
$V_p^{(ii)}$	The value of V_p for flow case (ii) of unit vorticity distribution.
v_{ij}	The viscous destruction of Reynolds stress.
V_{np}	The velocity induced by the nth. element at the mid point of the pth. element and in the direction normal to the pth. element.
V_p	The velocity induced by all the elements at the mid point of the pth. element and

	normal to it.
V_r	Radial velocity induced by a point source.
v	The normal component of the turbulent fluctuating velocity.
$-\overline{\rho v^2}$	Reynolds normal stress in the normal direction.
X, Y	X, Y fixed with their origin at a specified point on the RS. The X -axis is made to pass through the leading and trailing edges of the aerofoil.
X_J	Coordinates of the J th. node on the RS, relative to the (X, Y) axes.
X_{ms}, Y_{ms}	Coordinates of a point (and also a node) on the MS, relative to the (X, Y) axes.
X_{rs}, Y_{rs}	Coordinates of a point (and also a node) on the RS, relative to X, Y axes.
x	Streamwise coordinate direction for plane flow.
x, y	Axes parallel and perpendicular to the n th. element.
x^i	Coordinate distance tensor.
y	Normal coordinate direction for plane flow.
α'	This is equal to Angle of incidence; this is the angle made by the undisturbed stream velocity to the X -axis.
α_0	This is equal to $\frac{\tau - \tau_w}{\Delta y}$, Which is a finite difference form of $\partial \tau / \partial y$ at the wall.
α_{ms}	Angle made at a point (and also a node) on the MS, to the X -axis.
α_{rs}	Angle made at a point (and also a node) on

the RS, to the X-axis.

β

This is equal to

$$\sqrt{[\alpha^2 + 2a_1\{\tau + (\bar{u}^2 - \bar{v}^2)\frac{v}{u}\}]}$$

Γ

The total circulation around the MS.

γ

Vorticity distribution on MS. Also vorticity of a line vortex.

γ_n

Vorticity of the nth element.

Δn

The spacing between grid points (mesh points) in the normal-direction for the shear layer calculation in (s,n) coordinate system.

Δq

Nett tangential velocity difference at the TE.

$\Delta q^{(1)}$

The tangential velocity difference at the TE for flow case (1).

$\Delta q^{(2)}$

The tangential velocity difference at the TE for flow case(2).

Δs

The calculating step between stations in the streamwise-direction, for the shear layer calculation in (s,n) coordinate system. Also referred to as the s-step.

$\Delta s_{\max}$

The maximum permissible value of s, given by the CFL criterion.

Δs_n

Length of the nth. MS element.

δ

The normal distance of the matching surface from the real (body) surface (in the normal-direction).

δ^*

The displacement thickness of the laminar BL and an integral for the turbulent shear layer.

δ_{05}

The value of y (the height from the

	surface) at which τ falls to .05 of its maximum value in the SL; roughly equal to δ_{995} .
δ_{coles}	The boundary layer thickness obtained from the Coles velocity profile family for the start of the turbulent SL calculation.
δ_{pn}	The Kroenecker delta in cartesian tensor notation.
δ_i^j	The Kroenecker delta in generalised tensor notation.
ϵ	The viscous dissipation of turbulent kinetic energy.
λ_+	The tangent of the "outgoing" characteristic angle.
λ_-	The tangent of the "ingoing" characteristic angle.
η	This is $\left \frac{g_{rs}}{\sqrt{1 + g_{rs}^2}} \right $
θ	The momentum thickness of the laminar BL and an integral for the turbulent SL.
θ_0	The value of θ at the start of the turbulent SL calculation (at $s = s_0$).
θ_n	The angle made by the nth element to the X-axis.
θ_p	Angle made by the pth element to the X-axis.
κ	The curvature of the body surface.
ν	The kinematic viscosity.
ν_t	Eddy viscosity in the inner wake.
ξ	This is $\left \frac{g_{rs} H_{ms}}{\sqrt{1 + g_{rs}^2}} \right $
ρ	The mean density (assumed constant for incompressible flow).

σ_{np}	Acute angle between the nth and pth elements.
τ	Reynolds shear stress = $-\overline{uv}$.
τ_1	The value of τ at the first mesh point from the surface for the SL calculation.
τ_{fut}	The value of τ at the future station, in the U, τ calculation.
τ_{int}	The value of τ interpolated at the position where the characteristic intersects a station.
τ_m, τ_{max}	Maximum Reynolds shear stress at given station.
τ_{mid}	The value of τ at the mid point of the characteristic segment.
τ_w	The wall shear stress.
τ^+	The Reynolds shear stress in the upper part of wake.
τ^-	The Reynolds shear stress in the lower part of wake.
τ^{ij}	The Reynolds stress tensor (second order).
v_p	Specified velocity normal to the pth element at its mid point.
v_p^*	The value of v_p normalised with the free stream velocity; i.e. $v_p^* = v_p / U_\infty$.
ϕ'	A function used in the characteristic equation (see equation A3.17)
ϕ	A general characteristic angle denoting one of the inclined characteristics.
ϕ_{in}	The angle made by the "ingoing" characteristic, with the streamwise

- direction.
- Φ_{out} The angle made by the "outgoing" characteristic, with the streamwise direction.
- Ψ A complicated function which is the coefficient of dx in the equations along the inclined characteristics, for the shear layer calculation (see equation (A3.17)).
- Ω An under-relaxation factor.

Chapter 1. Introduction

1.1 Purpose of the Present Work

The classical method of viscous-inviscid matching is to assume that the viscous effects are confined to the boundary layers on a solid body and the wake behind it so that the outer flow-field can be assumed to be inviscid. For brevity we refer to the boundary layer and wake together as the "shear layer". The solution of an "inviscid equation" for the outer flow provides the boundary conditions of the parabolic "thin shear layer equations" (or boundary layer equations) in the viscous or turbulent part of the flow; the presence of the retarded viscous region, in turn, affects the inner boundary conditions for the inviscid equation. The calculation is then iterated between inviscid flow solutions and thin shear layer solutions until convergence is reached.

It is well known that the thin shear layer equations become inaccurate once the normal pressure gradient across the shear layer becomes significant, as it does near the trailing edge of an aerofoil. Sometimes the normal pressure gradient can be related to the curvature of a solid surface or a known streamline (in which case the equations remain parabolic) but if the curvature is changing rapidly as near the TE or on highly curved surfaces like bends in ducts or highly cambered aerofoils, this procedure is not reliable. If the normal pressure gradient is significant, the upstream influence within the shear layer is also likely to be

significant and elliptic equations must be used to describe it. In brief, some form of the normal-component momentum equation must be included.

The obvious choice for the matching position between viscous and inviscid flow is the shear layer edge, which is the true boundary between "viscous" and "inviscid" regions. According to Prandtl's boundary layer theory, the viscous effect on the inviscid flow can be taken into account more simply by using the displacement thickness δ^* . Here the matching can be performed at two different positions. One is at the "displacement surface", formed by the displacement thickness added to the real solid body. The other is at the solid body (aerofoil) where the displacement effect of boundary layer and wake is represented, in the inviscid calculation, by a transpiration velocity. But if the normal pressure gradient is significant, U_e varies with y . Hence the displacement of the streamlines varies with y so that there is no unique definition of δ^* . Several higher-order approximations have been suggested, but the only exact option is to match at (or slightly above) the shear layer edge, using the true velocity normal to the matching surface as the inner boundary condition on the inviscid-flow calculation.

Mahgoub and Bradshaw (1979) and Chen and Bradshaw (1982) described a method of combining the iterative scheme required to match shear layer calculations to inviscid flow calculations with the iterative scheme required to solve elliptic equations for the shear layer (obtained by including the normal momentum equation). In the shear layer calculation upstream influence is allowed to propagate inside the shear layers (at one x step per iteration)

through suitable differencing of the $\partial p / \partial x$ term, and the variation of pressure across the shear layer is calculated through normal momentum equation. Global iteration is used to propagate upstream influence through the external (elliptic) inviscid calculation too. Thus the full time-averaged Navier-Stokes equations (or Reynolds equations), or any desired simplification of them which includes upstream influence, could be solved for the shear layer only, avoiding the large computing times and costs required to solve full time-averaged Navier-Stokes equations for the whole flow field. Their work is restricted to shear layers without separation, because the given pressure distribution from the inviscid flow calculation on the matching surface for shear layer calculation can not propagate strong upstream influence.

The closest methods to ours is Rubin's (1982) - developed for laminar flows, but only extendible to turbulent flows. Rubin and Lin (1980) and Rubin (1982, 1984) solved Parabolised Navier-Stokes equations, where the longitudinal diffusion terms are omitted from the Navier-Stokes equations including normal momentum equation (in our calculations longitudinal diffusion is included, but only approximately. Upstream influence is propagated through the $\partial p / \partial x$ term, as in Mahgoub and Bradshaw method. Because upstream influence is propagated only via pressure velocity interaction, not through the external inviscid calculation, global pressure relaxation by multiple marching sweeps of the shear layer calculation is used to propagate upstream influence. Their work is restricted to attached flows or small laminar separation regions and does not include wakes or complete aerofoil calculations.

The present work is the continuation of that of

Mahgoub and Bradshaw (1979) and Chen and Bradshaw (1982). The purpose is to develop a calculation procedure based on Mahgoub and Bradshaw's method to improve accuracy, and to compute the complete flow over the aerofoil. In order to deal with the geometric singularity at the sharp trailing edge as well as improve accuracy near it, a transformation to "inviscid wedge flow" coordinates is used there. The turbulence model of Bradshaw, Ferriss and Atwell (1967) is used again. For the near wake region, which has an important effect on the pressure distribution along the aerofoil, Bradshaw(1972)'s superposition hypothesis is used.

Complete aerofoil calculations, including laminar boundary layer calculations (assuming negligible $\partial p / \partial y$) and full treatment of the wake, have been done for incompressible flows. The Hess and Smith surface source distribution method was used for the inviscid flow calculation and Thwaites' integral method for laminar boundary layer calculation. Finally several cases have been tested including Nakayama's aerofoil A (1984,1985) and good agreement with measurement was obtained. A compressible version of the shear layer calculation is fully coded but the accompanying inviscid program is not yet satisfactory.

1.2 Review of previous work

1.2.1 Introduction

It is generally accepted that the Navier-Stokes equations adequately represent fluid flow phenomena. To solve the full Navier-Stokes equations for turbulence, even the large capacity and high speed of present computers are adequate only for very simple low-Reynolds number flows. In

the majority of problems in the aero industry, the Reynolds number is very large. Hence the flow field can be divided into a viscous region ("viscous" by convention include "turbulence") and an inviscid region. Since Prandtl's first order boundary layer theory, the classical procedure for calculating the flow around an aerofoil has been that the calculation of the inviscid flow gives the pressure distribution for the boundary layer, and the boundary layer calculation gives the inner boundary condition for the inviscid flow calculation (Cebeci and Bradshaw, 1977). Hence the viscous and inviscid flow calculations are performed separately and iterated until convergence is reached. This is called a "weak interaction" (Lock and Firmin, 1982). This method for calculating the interaction between the viscous and inviscid parts of the flow is based on a direct hierarchy between the viscous and inviscid regions which applies as long as the disturbance to the inviscid flow due to the viscous displacement effects are small. The formal justification for this method is obtained by the triple deck theory which is obtained from asymptotic expansion of the Navier-Stokes equations (Stewartson, 1974). However, the flows over an aerofoil involve both a weak overall interaction arising from displacement effects and wake curvature effects, as well as local strong interaction arising from boundary separation, shock boundary interaction, or rapid flow acceleration immediately aft of the trailing edge. In such situations, viscous displacements cause substantial changes in local inviscid pressure fields and the normal pressure gradient term in the shear layer equations, which is assumed zero in classical method can not be ignored. This is called "strong

interaction" and the classical hierarchy in the structure of the flow breaks down, leading to inaccuracy, instability or even singularities in the classical method. The concept of an inner viscous region and an outer inviscid region still applies (Schmatz and Hirschel, 1986), but the coupling mechanism which connects the viscous and inviscid flow equations is not directly coupled. Recently, Lock and Williams (1987) reviewed general viscous-inviscid interaction techniques.

During the past ten years, substantial research has been directed in different ways to treat strong pressure-displacement interaction in integral method or differential method. Despite the advantage of integral method which use "equivalent inviscid flow" method (Lock and Firmin, 1982 and Lock and Williams, 1987), the differential method have been developed fast because accuracy is better and speed and capacity of computer have been increased.

1.2.2 Methods using the boundary layer equations

The streamwise pressure gradient term is modified as pressure-displacement interaction boundary condition formulation elliptically and interactively using Cauchy integral or transpiration velocity or Hilbert integral. Hence the inviscid flow or the surface pressure is allowed to adjust simultaneously with the boundary layer growth. Catherall and Mangler(1966) are the pioneers who demonstrated that the Goldstein singularity at separation can be avoided by prescribing the displacement thickness and not the pressure for the solution of the boundary layer equations. The inviscid equations are solved explicitly with this prescribed surface pressure to obtain a displacement thickness distribution through Cauchy's

integral equation from thin airfoil theory (Carter and Wornom 1975). This process is continued until convergence is reached. Boundary layer equations could be solved for the prescribed skin friction (Klineberg and Steger 1974 and Carter 1974) or prescribed transverse component of velocity at the boundary (Klineberg and Steger 1973, and Tai 1974) instead of displacement thickness. This coupling mechanism is called inverse method (Carter and Wornom, 1975; Arieli and Murphy, 1979). Le Balleur (1978,1980,1984), Carter (1978,1981), Kwon and Pletcher (1979,1984) and Davis and Carter (1984) developed the semi-inverse procedure. The viscous and inviscid flow equations are solved separately for a prescribed displacement thickness and a new displacement thickness is then determined by a relaxation formula composed of the viscous and inviscid predictions for the pressure along the displacement body. Vasta and Verdon (1984) calculated near trailing edge region using semi-inverse method including wake curvature effect. But Melnik (1983) pointed that this semi-inverse method needed a significant underrelaxation factor. The semi-inverse method solved the viscous and inviscid flow equations explicitly at each global iteration. For strongly interactive flows, the viscous and inviscid regions are locally coupled. Hence Veldman (1979) developed the quasi-simultaneous method which accounts for this strong coupling through localized implicit treatment between viscous and inviscid flows. In the quasi-simultaneous procedure, a local representation of inviscid flow is solved simultaneously with a streamwise sweep of the viscous equations to yield, at a given level of global iteration, a streamwise distribution of the surface pressure and displacement thickness which mutually satisfies

the viscous flow equations and the local inviscid relation. Hence the velocity field is written as the sum of the inviscid velocity and the perturbation velocity due to displacement thickness from interacting equation through Hilbert's integral equation between pressure and displacement. Global iteration is performed on the inviscid flow over the displacement body to account for the elliptic influence of the entire displacement body (Wai and Yoshihara, 1980; Veldman, 1983; Le Balleur, 1984; Davis and Werle, 1981; Davis, 1984; Edwards and Carter, 1986; Cebeci and Clark, 1984; Cebeci, Chen, Chang and Peavey, 1986). For the internal flow Strawn, Ferziger and Kline (1984) used mass conservation law as interacting formula between pressure and displacement. These methods avoid breakdown or singularity in strong interaction but still use the classical BL approximation, e.g. $\partial p / \partial y = 0$. In addition to the numerical aspects, modeling of the physics of flow was considered by the triple-deck theory (Stewartson, 1969; Messiter, 1970; Melnik and Chow, 1975). This triple-deck equation provides an interactive coupling between boundary layer and inviscid flow regions (Stewartson, 1974; Melnik, 1980). Triple deck allows $\partial p / \partial y$ in the "inviscid" rotational outer flow (middle deck) but rigorous only for lower flow at very high R_e . In turbulent flow the qualitative application is to neglect shear stress in the outer layer in the short triple-deck region - not really much help especially if you need the shear stress again further downstream.

1.2.3 Use of normal momentum equation

The second modern approach to strong interaction

acknowledges that the matching between viscous and inviscid flow is ambiguous in the region where the normal pressure gradient is significant. The definition of displacement thickness is not unique because the displacement thickness varies with y if U_e varies with y . Hence several researchers have developed approximate equations for solving viscous flow problems which are nearly boundary-layer type flows, but include some additional terms in the equations themselves or (in the boundary conditions) according to the physical behavior of the flow problems concerned. These approximate equations are called "Parabolised Navier-Stokes Equations (PNS)" or "Partially-Parabolised Navier-Stokes Equations (PPNS)". Here the pressure-displacement interaction of the streamwise pressure gradient term is transmitted through the normal momentum equation (which allows p , and therefore $\partial p / \partial x$, to vary with y). The advantage of these approximate equations compared to the full Navier-Stokes equations is that they can be solved very efficiently with boundary layer type marching techniques which require much less computer storage than Navier-Stokes solvers—because if the flow is unseparated, you need only store p , not U and V . In the early 1970's, Rubin and Lin (1971) and Lin and Rubin (1973) applied this scheme to the hypersonic viscous flow calculation and Werle and Vasta (1974) did the same for supersonic flow. The treatment of the pressure as dependent variable in the streamwise momentum equation introduces an "elliptic-like" effect that is reflected in the appearance of the departure solutions associated with forward marching procedure of the purely parabolic behavior of the boundary layer equations. This pressure interaction introduces a finite upstream influence zone and Lubard and

Helliwell (1974), Lin and Rubin (1973), and Davis and Rubin (1980) showed that there is a restriction on the marching step size for stability, which is generally larger than the shear layer thickness, if backward differencing for the streamwise pressure gradient term is used. Rubin (1980), Rubin and Lin (1980,1982), Israeli and Lin (1982), Rubin and Reddy(1983), Khosla and Rubin (1979,1982) and Khosla and Lai (1983) applied forward differencing for the streamwise pressure gradient term to remove the restriction of streamwise step size: this forward difference formulation introduces upstream influence by using the downstream and local pressure values. The local pressure value allows a coupling between pressure and velocity to provide a free pressure interaction and this forward differencing leads to a global computational procedure which includes not only the parabolic marching step but also the elliptic relaxation step for the pressure. That is, the calculation involves multiple "sweeps" of a marching/relaxation procedure. Rubin and Reddy (1983), Israeli and Lin (1982) applied multi-grid schemes and introduced a source term in the streamwise momentum equation to improve accuracy and convergence. But they had to adjust the outer boundary conditions for the multiple global sweep, so they applied the boundary conditions far from the body and fixed the pressure value. Brown (1984) provided upstream influence by calculating the pressure at a point one step upstream from the main calculation point and performing a global iteration for updating boundary conditions from the inviscid flow calculation. He also used second order backward differencing for the streamwise gradient term, including pressure, because forward differencing for the streamwise

pressure gradient term is not consistent with backward differencing for the other streamwise gradient term. Mahgoub and Bradshaw (1979) applied central differencing for the streamwise pressure gradient term by using previous sweep values, and calculate pressure one step upstream from the main calculation point to propagate upstream influence. The upstream influence is propagated two steps inside the boundary layer thickness through $\partial p/\partial x$ and $\partial v/\partial x$ terms for one sweep and the multiple viscous/inviscid matching sweep provide upstream influence through the elliptic inviscid flow equations too. But allowing upstream influence propagation over two steps inside boundary layer induces wiggles. Chen and Bradshaw (1982) applied this method to compressible flows. They used forward differencing for the streamwise pressure gradient term and calculated pressure at the same point as the other variables. They also used an iteration at the same point by using newly-calculated pressure for forward differencing to couple pressure and velocity locally. But calculation of velocity and pressure at same point suppresses the upstream influence inside the boundary layer. Recently, Raven and Hoekstra (1985) applied the updating boundary condition from inviscid flow calculation as viscous/inviscid interaction effect and Halim and Hafez (1986) patched the stream function and vorticity function at the patching line between viscous and inviscid flow for the global sweep. Contrary to the PNS method where the pressure is calculated from the normal momentum equation, Patankar and Spalding (1972), Pratap and Spalding (1975,1976) introduce a bulk streamwise pressure gradient correction in order to ensure conservation of global mass flow for internal flow (PPNS). Thus they solved the Poisson

equation derived from continuity equation for pressure decoupled from the velocity calculation. Pressure calculation is lagged to velocity calculation. A global sweep is performed to correct pressure distribution and reduce continuity error. Upstream influence inside boundary layer is propagated via forward differencing of $\partial p / \partial x$ term as in the PNS method. Chen and Patel (1984) apply PPNS to thick boundary layers. But as in the PNS method, they faced boundary condition problems during the global sweep. Cousteix, Saint-Victor, and Houdeville (1985) and Pouagare and Lakshminarayana (1985) applied the simultaneous implicit coupling between pressure and velocity by using PPNS equation. Recently, McNally and Sockol (1985) and Mehta, Chang and Cebeci (1985) reviewed the general computational methods for viscous-inviscid interaction.

1.2.4 Turbulence models

Except for interactions with shock waves and other special cases such as strong curvature effects, turbulent modeling for boundary layers is in a satisfactory state, but **in the wake** region a complex flow develops which is very important in aerodynamic problems. The turbulence modeling for this region has become the subject of investigation during last several years. The near wake is strongly dependent on the initial boundary layers at trailing edge (El-Assar and Page, 1969; Chevray and Kovasznay, 1969; Agrawal, Pande and Prakash, 1977; Alber, 1980). Recently, Nakayama (1985) showed for highly asymmetric case that there is strong normal pressure gradient near the wake region and the mixing of the two merging boundary layers controls the turbulence mechanism. Morel and Torda (1974) and Huffman

and Ng (1977) developed numerical scheme based on the "interaction hypothesis" (Bradshaw, Dean and McEligot (1973) for the near wake region. Andreopoulos and Bradshaw (1980) pointed that interaction between the two merging boundary layers occurred mainly by the small scale fine-grained mixing in the inner wake region, and the eddy viscosity assumption for the mixed fluid in the inner wake region could be used. Bromilow (1978) and Vicatos (1979) applied the Bradshaw-Ferriss-Atwell (1967) model including the eddy viscosity assumption for the inner wake to the near wake calculation. Waskiewicz, Shang and Hankey (1980) used the relaxed eddy viscosity model for the wake region. Bonnet, Delville and Garem (1986) measured for the longitudinal velocity correlations and pointed that there are coherent structures. More advanced models for shear layers have been developed, e.g. Hanjalic and Launder (1972) used transport equations for all the Reynolds stresses and the turbulent energy dissipation rate, and Launder, Reece and Rodi (1975) developed a better approximation of the pressure-strain rate. However application of these advanced turbulence models to the wake region is still not complete and is still a subject for discussion.

Chapter 2. Shear Layer Calculation

In this chapter, the equations used in the shear layer calculation of the present method including the wake region will be presented, their derivation being left to appendices. For simplicity we discuss incompressible flow first. The SL calculation is divided into three regions in the present numerical method. One is from the LE to a point of just upstream of the TE region. The second is the transformation region near the TE. The third is the wake downstream of the transformation region. In the transformation region the orthogonal curvilinear (ϕ, ψ) coordinate system is used, while elsewhere an orthogonal semi-curvilinear (s, n) coordinate system is used. For turbulence modeling purposes we shall divide the turbulent shear layer calculation into two parts: one is along the aerofoil (boundary layer region downstream of transition) and the other is the wake region. Thwaites' integral method is used for the laminar boundary layer: at present, transition is specified at $Re_\theta = 600$ or at an input value of s/c , but any desired transition criterion could be inserted.

For the compressible flow the governing equations are derived in appendices: the calculation of temperature is performed in one extra subroutine TEMPRO. The detailed numerical procedure will be described in the following section. Finally, a description of the Fortran program will be presented.

2.1 The Shear Layer along the Aerofoil

2.1.1. Physical Description and Governing Equations

The equations of flow for the two-dimensional shear layers are written in a body-fitted orthogonal curvilinear (s,n) coordinate system as used by Mahgoub and Bradshaw (1979) outside of transformation region, and the orthogonal curvilinear (φ,ψ) coordinate system obtained by using "wedge flow transformation" is used near TE region (see Fig.1). Their derivation using general tensor analysis is given in appendices. The governing equations are as follows :

Continuity equation :

$$h_2 \frac{\partial u}{\partial x} + h_1 \frac{\partial v}{\partial y} + u \frac{\partial h_2}{\partial x} + v \frac{\partial h_1}{\partial y} = 0 \quad (2.1)$$

The streamwise momentum equation :

$$\begin{aligned} h_2 u \frac{\partial u}{\partial x} + h_1 v \frac{\partial u}{\partial y} - h_1 \frac{\partial \tau}{\partial y} - \left\{ v^2 - (\bar{u}^2 - \bar{v}^2) \right\} \frac{\partial h_2}{\partial x} \\ + (uv - 2\tau) \frac{\partial h_1}{\partial y} - h_2 \left(\frac{\partial p}{\partial x} + \frac{\partial u^2}{\partial x} \right) = 0 \end{aligned} \quad (2.2)$$

Since the normal pressure gradient is significant near TE, we solve the normal momentum equation:

$$\begin{aligned} u h_2 \frac{\partial v}{\partial x} + h_1 v \frac{\partial v}{\partial y} - h_2 \frac{\partial \tau}{\partial x} + u \left(v \frac{\partial h_2}{\partial x} - u \frac{\partial h_1}{\partial y} \right) \\ + h_1 \frac{\partial p}{\partial y} + h_1 \frac{\partial \bar{v}^2}{\partial y} - 2\tau \frac{\partial h_2}{\partial x} - (\bar{u}^2 - \bar{v}^2) \frac{\partial h_1}{\partial y} = 0 \end{aligned} \quad (2.3)$$

The turbulent kinetic energy equation for $1/2 \overline{q^2}$:

$$\begin{aligned} h_2 u \frac{\partial}{\partial x} \left(\frac{1}{2} \overline{q^2} \right) + v h_1 \frac{\partial}{\partial y} \left(\frac{1}{2} \overline{q^2} \right) \\ = \tau \left(h_1 \frac{\partial u}{\partial y} + h_2 \frac{\partial v}{\partial x} - v \frac{\partial h_2}{\partial x} - u \frac{\partial h_1}{\partial y} \right) \\ + \bar{u}^2 \left(h_2 \frac{\partial u}{\partial x} + v \frac{\partial h_1}{\partial y} \right) + \bar{v}^2 \left(h_1 \frac{\partial v}{\partial y} + u \frac{\partial h_2}{\partial x} \right) \\ - \frac{\partial}{\partial x} \left(h_2 \overline{u p} + \frac{1}{2} \overline{q^2 u} \right) - \frac{\partial}{\partial y} h_1 \left(\overline{v p} + \frac{1}{2} \overline{q^2 v} \right) - \epsilon \end{aligned} \quad (2.4)$$

where all variables are nondimensionalised by U_∞ and c , $\tau = -\overline{u v}$ and ϵ is the dissipation of turbulent kinetic energy

and h_1, h_2 are metric values.

For the turbulence modeling of equation (2.4), using the BFA one-equation turbulence model (Bradshaw, Ferriss, and Atwell 1967) we define an empirical function

$$a_1 = \frac{\tau}{q^2} \quad \text{for the turbulent kinetic energy}$$

and

$$G = \frac{\overline{p'v} + \frac{1}{2} \overline{q^2 v}}{\tau^{1/2}_{\max} \tau} \quad \text{for the turbulent diffusion term}$$

($\overline{p'v} + \frac{1}{2} \overline{q^2 v}$). The turbulent dissipation term, which has dimensions (velocity)³/length, is modelled as $\epsilon = \tau^{3/2}/L$ where L is the length scale of the large or energy containing eddies of turbulence. We use the algebraic fit to experimental data given by Bradshaw (1969) for the length scale L/δ .

In the present program we put $a_1 = 0.15$ throughout whole flow field. This is adequate for the near wake but data show an increase in the far wake.

Since we shall solve these equations only outside the viscous sublayer, the viscous diffusion terms in the turbulent kinetic energy equation and in the two momentum equations have been neglected. We include normal turbulent stress terms because they have significant effects for flows with large difference between the streamline direction and the direction of axes, such as occur near the TE. We assume that $\overline{u^2} = C_1 \overline{u v}$ and $\overline{v^2} = C_2 \overline{u v}$, where C_1 and C_2 are experimental constants. The turbulent kinetic energy equation is reduced to a modelled turbulent shear stress equation by the above assumptions and empirical substitutions. Hence the turbulent shear stress equation can be written as follows :

$$\begin{aligned}
u h_2 \frac{\partial}{\partial x} \left(\frac{\tau}{2a_1} \right) + v h_1 \frac{\partial}{\partial y} \left(\frac{\tau}{2a_1} \right) &= \tau \left(h_1 \frac{\partial u}{\partial y} + h_2 \frac{\partial v}{\partial x} - u \frac{\partial h_1}{\partial y} - v \frac{\partial h_2}{\partial x} \right) \\
+ \tau e - c_1 \tau \left(h_2 \frac{\partial u}{\partial x} + v \frac{\partial h_1}{\partial y} \right) - c_2 \tau \left(h_1 \frac{\partial v}{\partial y} + u \frac{\partial h_2}{\partial x} \right) \\
- \frac{\partial}{\partial y} (h_1 \hat{G} \tau) - \tau^{3/2} / L
\end{aligned} \tag{2.5}$$

where $\tau = -(u \ v)$ and $\hat{G} = G \tau_{\max}$.

2.1.2 The Solution Procedure

The complete set of equations (2.1), (2.2), (2.3) and (2.5) is elliptic if we solve these equations for U , V , τ and p simultaneously. But if we assume $\partial p / \partial s$ terms are known, the continuity, streamwise momentum, turbulent shear stress equations are hyperbolic (three first-order PDEs) in U , V and τ . Thus we can solve for U , V and τ using the same method of characteristics as in the original BFA method. We find (Appendix 3) that there are three characteristic directions: one is normal to the surface,

$$\frac{dy}{dx} = 0 \tag{2.6}$$

and the other two are as follows:

$$\frac{h_2 dy}{h_1 dx} = \frac{v + \alpha \pm \beta}{u} \tag{2.7}$$

where $\alpha = a_1 \left\{ \hat{G} + \frac{(c_1 - c_2)}{u} \tau \right\}$

$$\beta = \sqrt{\alpha^2 + 2 a_1 \tau \left\{ 1 + \frac{v}{u} (c_1 - c_2) \right\}}$$

The equations (2.1), (2.2) and (2.5) can be expressed along the three characteristic directions (see Appendix 3).

$$\begin{aligned}
h_1 \left(\frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y} \right) + h_2 \frac{\partial p}{\partial x} - (u^2 + v^2) \frac{\partial h_2}{\partial x} \\
- h_1 \frac{\partial \tau}{\partial y} + h_2 c_1 \frac{\partial \tau}{\partial x} - 2 \tau \frac{\partial h_1}{\partial y} + (c_1 - c_2) \tau \frac{\partial h_2}{\partial x} \\
= 0
\end{aligned} \tag{2.8}$$

along the normal characteristic direction, and

$$F' dU + [A'\lambda - G'] d\tau = -H' + \lambda K' \quad (2.9)(2.10)$$

where $A' = \frac{h_2^2 U^2}{2 a_1}$

$$F' = -h_1 h_2 [U\tau - v(c_1 - c_2)\tau]$$

$$G' = \frac{h_1 h_2 UV}{2 a_1}$$

$$\begin{aligned} H' = h_1 v [& \tau \{ -u \frac{\partial h_1}{\partial y} - v \frac{\partial h_2}{\partial x} + \hat{e} + h_2 \frac{\partial v}{\partial x} - \tau^{1/2}/L - \frac{\partial}{\partial y} (h_1 \hat{G}) \} \\ & - (c_1 - c_2) \tau v \frac{\partial h_1}{\partial y}] + h_1 \tau [v - (c_1 - c_2) \tau] \frac{\partial h_2}{\partial x} \\ & - (uv - 2\tau) \frac{\partial h_1}{\partial y} + h_2 (\frac{\partial p}{\partial x} + c_1 \frac{\partial \tau}{\partial x}) \\ K' = h_2 u [& \tau \{ -u \frac{\partial h_1}{\partial y} - v \frac{\partial h_2}{\partial x} + \hat{e} + h_2 \frac{\partial v}{\partial x} - \tau^{1/2}/L \\ & - \frac{\partial}{\partial y} (h_1 \hat{G}) \} - (c_1 - c_2) \tau v \frac{\partial h_1}{\partial y}] - h_2 (c_1 - c_2) \tau [\\ & v^2 - (c_1 - c_2) \tau] \frac{\partial h_2}{\partial x} - (uv - 2\tau) \frac{\partial h_1}{\partial y} + h_2 (\frac{\partial p}{\partial x} \\ & - c_1 \frac{\partial \tau}{\partial x}) \end{aligned}$$

$$\lambda = \frac{h_1}{h_2} [\frac{v + \alpha \pm \beta}{U}]$$

along the two inclined characteristic directions,

where $\hat{e}$ is an extra strain rate correction to the turbulence model equation (Bradshaw, 1973 and 1976). The advantage of the method of characteristics is that it is explicit and simpler than the implicit method commonly used for parabolic boundary layer equations. Hence no iteration is required and a simple step size criterion that ensures stability is known. The complicated wall boundary conditions can be satisfied independently of the main calculation. The normal momentum equation is solved separately for p as a function of y . This calculated pressure distribution is used in the next global sweep to give the streamwise pressure gradient which allows pressure-velocity interaction along the normal direction. The upstream influence is propagated through pressure inside the shear layer where the normal pressure

gradient is significant. Hence the calculation procedure can be set up in two parts.

- (i) The U , τ and V calculation for given $p(x,y)$.
- (ii) The pressure calculation from the normal momentum equation, using newly calculated U, V and τ values.

The V calculation is decoupled from U and τ calculation as in the BFA method. After calculation of U and τ from two inclined characteristic equations, we can calculate V values from the normal characteristic equation using the same pressure gradient term. These newly calculated U, V and τ values are used for the normal momentum equation (2.3) to calculate p . Hence we can set up the procedure as:

- (i) The U and τ calculation for a given V and p .
- (ii) The V and p calculation for newly calculated U and τ .

For a few points near the wall, where the variables change most rapidly along the streamwise direction, the U and τ calculation is improved by repeating each step, evaluating coefficients at the mid-points of the characteristic segments by interpolation of the first guess solution (predictor-corrector).

This two part calculation procedure is mathematically equivalent to one step of a line-relaxation procedure for solving the elliptic Navier Stokes' equation but only the pressure need be stored for the next sweep, and it will be seen below that even the pressure storage can be greatly simplified. However the main economy in the whole method is the imbedding of one sweep of the multiple-sweep elliptic shear layer iteration in each pass through the viscous-inviscid matching loop.

Now we consider the boundary and initial conditions necessary for the shear layer calculation.

(1) Boundary conditions

(i) The external boundary condition :

As mentioned before, at the matching surface the pressure distribution is needed. The detailed pressure calculation is described in §3.1. The inviscid calculation actually gives U_{ms} , the outer boundary condition on U . It is related to p_{ms} by Bernoulli's equation in shear layer axes,

$$U_{ms}^2 = 2(p - p_{ms}) / \rho - V_{ms}^2$$

where V_{ms} is known from the previous sweep or the initial guess and is usually small compared with U_{ms} .

(ii) The wall boundary condition

The general boundary condition at wall is $U=0$ and $V=V_w$ (for transpiration through the wall) or $V=0$ for the usual solid walls. But the equations above are not solved within viscous sublayer, the boundary condition at a wall being satisfied by matching the solution to the logarithmic "law of the wall" at the first mesh point above the wall. The detailed wall boundary calculation will be shown in §2.3.

(iii) The downstream boundary condition

Because the equations which we solve are elliptic we need downstream boundary conditions, which must as always be non-critical. The calculation includes the wake region, so we applied the downstream boundary condition at about $0.5c$ downstream of the TE, hereafter referred as "the end of the

wake". The boundary condition is $p(s_\ell, n) = p(s_{(\ell-1)}, n)$ or $\partial p / \partial x(s_\ell, n) = 0$. In the present program we applied $\partial p / \partial s|_{s=s_\ell} = 0$ as the downstream boundary condition, where s_ℓ is the end of the wake.

(2) The initial condition

- i) the pressure at the shear layer edge at the initial MS is given by the inviscid flow calculation
- ii) The initial condition for U , τ and V

The flow is laminar from the stagnation point (LE) to the transition point, so we apply Thwaites' integral equation from the LE to the transition point where the criterion ($Re_\theta = 600$) is met. The U and τ profiles are evaluated as the zero-pressure gradient profile for the known θ and Re_θ at the transition point: the V profile is calculated from the normal characteristic equation (2.8), and we assume that $\partial p / \partial y$ is neglected so that $p(s_0, n) = p_e(s_0, n)$.

(3) Calculation procedure for one streamwise step

Now we shall consider the calculation procedure for one streamwise step of the shear layer calculation with the above mentioned boundary and initial conditions.

- (i) Specify the pressure $p(s, n$ or $\varphi, \psi)$ everywhere, as the values calculated on the previous sweep (or take $p = p_e$, or some better guess, on the first sweep) and specify U and τ at $s = s_0$.
- (ii) Calculate the U and τ at the "future" station $s + \Delta s$ from equations (2.9) and (2.10) -(initially using "upstream" coefficient and then updating).

(iii) Calculate V at the future station from equation (2.8) by using newly calculated U and τ .

(iv) Calculate the p at the half-step $s+1/2\Delta s$ point from equation (2.3) by using newly calculated U , V and τ (This allows central differences in s to be used in the $\partial p/\partial y$ equation).

The equations which we solve are elliptic, so we need elliptic-type boundary condition for the set as a whole, although we split the equations into "elliptic" and "hyperbolic" parts, i.e. a part which allows upstream influence (the V equation) and a part that does not. This is why the downstream boundary conditions are supplied for the SL calculation. The upstream influence is propagated by three different terms in the U equation: longitudinal stress gradient or longitudinal convection or pressure. In fairly thin, unseparated shear layers (see Bradshaw 1973), the main factor which propagates the upstream influence is pressure. Upstream influence can be propagated through the external flow by the inviscid calculation, because the equation which we solve is elliptic (Laplace's equation, represented by the panel method). But for strong interaction regions, e.g. near the TE, upstream influence must also be allowed to propagate inside the viscous flow. In order to represent this effect in the finite difference scheme, we use forward differencing for the pressure gradient term, using values from the previous sweep for downstream points. The direct effect of upstream influence within the shear layer will only extend a few boundary layer thicknesses upstream: therefore although it only propagates upstream at one step per sweep, convergence for a given external pressure distribution would in principle be achieved in only a few

sweeps (actually this problem is unstable). Longer-range upstream influence propagates via the inviscid calculation and is "instantaneous", so the whole calculation converges in only $O(10)$ sweeps.

2.2 The Wake Region

2.2.1 Physical Description and Governing Equations

The near wake region plays a very important role in the calculation of aerodynamic parameters along the aerofoil, because the displacement surface has less incidence and less camber than the aerofoil, and the shape of the displacement surface near the TE is responsible for the reduction of overall circulation around the aerofoil.

As mentioned in the Introduction, the wake could be modeled as two interacting turbulent shear layers using Bradshaw's superposition hypothesis (1973) which assumes that the interaction is represented by a time-sharing process, in which shear stresses of opposite sign can occur at the same place (but not at the same time) without noticeable change in their turbulent structure. The original assumption was the transfer of fluid from one set of large eddies to the other by "fine-grained" mixing has negligible effect. But according to Andreopoulos's experimental work (1980), the shear-layer interaction in the wake initially takes place between the two inner layers whose eddy size was previously confined to be proportional to the distance from the wall in the low speed sides of the boundary layers. This is different from duct or jet flows in which the merging occurs by meeting of the high speed sides of the shear layers. The interaction of the wake involves significant "fine-grained" mixing but structure

changes are confined to the inner wake region. The additional "fine-grained" shear stress in the inner wake region can be legitimately represented by an eddy viscosity, because the length scale of mixing is much smaller compared to the width of the inner wake.

The basis of the present method is that profiles for the positive and negative shear stresses are calculated independently, using the turbulent shear stress transport equation of the BFA method (equation 2.5), assuming that the mean velocities are common and that small scale fine-grained mixing is included by using an eddy viscosity model in the inner wake. Hence the total shear stress can be given by (see Fig.2):

$$\tau = \tau^+ - \tau^- + \nu_t \frac{\partial u}{\partial y} \quad (2.11)$$

For convenience we define τ^+ to be a positive quantity, so the shear stress in the lower part of the wake is (τ^-). We ignore the $\partial V / \partial x$ term in the rate of strain, as curvature effects on ν_t (also ignored provisionally) will be more important.

The parameters G and L in equation (2.5), which are defined above, are evaluated in terms of τ^+ and τ^- separately (the value of the parameter a_1 is taken as 0.15 as in the boundary layer). The value of G is calculated taking into account the peak value of the shear stresses τ^+ and τ^- independently in the subroutine TPARA and TPARAM.

The equations for the U, V and p calculations are the same as in the boundary layer.

(1) Length scale equation

In the turbulent energy or shear-stress transport

equation, a length scale is needed for the turbulence dissipation term. In the present wake program, the length scale equation used is based on Bradshaw and Unsworth (1974 and 1976). An empirical transport equation for a representative length scale L in the two-dimensional thin shear layer can be written in dimensional corrected form by assuming the u_τ is a representative velocity scale for the turbulent motion, as follows:

$$h_2 u \frac{\partial L}{\partial x} + h_1 v \frac{\partial L}{\partial y} = C_1 h_1 L \frac{\partial u}{\partial y} - h_1 \frac{\partial}{\partial y} (L v_1) + C_{s1} (h_1 h_2)^{1/2} \tau^{1/2} \quad (2.12)$$

where we use C_1 for $\overline{u^2}/\tau$ and C_{s1} are dimensionless, but not necessarily constant and v_1 is the turbulent transport velocity. The turbulent velocity v_1 may be specified by a gradient-diffusion assumption or by an algebraic equation analogous to that used for the turbulent transport velocity in the shear layer equation, $2a_1 G \tau_{\max}$. Other workers' length-scale (dissipation) equations imply that the value of C_1 was approximately zero, thus we put zero for C_1 . If v_1 is known, the equations (2.12) is hyperbolic, with a single characteristic angle and we obtain characteristic equation along this characteristic line.

$$\frac{dL}{dx} = \frac{(h_1 h_2)^{1/2} \tau^{1/2} C_{s1}}{\quad} \quad (2.12')$$

$$\xi = \tan^{-1} \left(\frac{h_1 (v + v_1)}{h_2 u} \right) \quad (2.13)$$

The value of C_{s1} is varied in the outer region as a function of $\tau_{\max}/U_e^2$ and the rate of divergence of the streamlines in the (x, y) plane;

$$C_{s1} = C_{s1s} \left(1 + 4 \frac{\partial y}{\partial x} \right) \quad (2.14)$$

where C_{s1s} is constant and input as 0.15 (see Bradshaw and Unsworth 1974).

This length scale equation (2.12') is not coupled with equations (2.1), (2.2), (2.3) and (2.5), because the gradient of length scale does not appear in the later equations. Thus the length scale equation can be solved independently. An analogous equation will be incorporated in the boundary-layer calculation later to deal with highly-curved flows.

(2) Eddy Viscosity Calculation

The eddy viscosity for the "fine-grained" shear stress is defined from experimental work (Andreopoulos and Bradshaw 1980). From that work, the viscosity is represented as parabolic fit in the inner wake region. Thus the empirical form used in the calculation is

$$\nu_t = Ru_T DI \left\{ 1 - \left(\frac{y}{DI} \right)^2 \right\} \quad (2.15)$$

where u_T is the velocity scale, R is a dimensionless constant and DI is the half-width of the inner wake. R is obtained from the experimental results and taken to be 0.15 in the present calculation. The gradient of shear stress can be represented as follows:

$$\frac{\partial \nu_t}{\partial y} \frac{\partial}{\partial y} \left(\frac{u}{h_1} \right) + \frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} \left(\frac{u}{h_1} \right) \right) \nu_t \quad (2.16)$$

(3) The Width Calculation

In the eddy viscosity formula, the inner wake half-width DI is needed. This is calculated by using the "outgoing" characteristics from the TE. The equation is:

$$\frac{\partial DI}{\partial x} = \frac{h_1 v - (2a_1 u_T)^{1/2}}{h_2 u} \quad (2.17)$$

We neglect v and use $\tau_{\max}$ to give the friction velocity in

this equation: we obtain

$$\frac{\partial D_1}{\partial x} = \frac{\tau_{\max}^{1/2}}{h_2 U} \quad (2.18)$$

2.2.2 The Solution Procedure for the Wake

The solution procedure is similar to the boundary layer region, but there are detailed differences.

(1) Length scale calculation

As mentioned above, the length scale equation (2.12) is decoupled. The characteristic equation (2.12') is solved by approximating mid-point values at the current station.

(2) U and τ calculation

The U and τ calculations are performed by using "ingoing" and "outgoing" characteristic equations (2.9) and (2.10). The U and τ calculation procedure is performed separately for the upper and lower parts of the wake. A detailed description is presented in §2.3.

(3) V and p calculation

In the wake, there are no boundary conditions on a solid surface, so the boundary condition for V is not known.

In order to calculate V , given the latest estimate of the pressure difference across the wake from the inviscid calculation, we rearrange the normal characteristic equation (2.8) and integrate. We can obtain the integrated equation (see §2.3).

$$\left(\frac{V}{U}\right)_y - \left(\frac{V}{U}\right)_0 = - \int_0^y \frac{F(U, V, p)}{h_1 U^2} dy \quad (2.19)$$

Because the V calculation is decoupled with the U and τ calculation, we write this equation as

$$v = \frac{V_{cl}}{U_{cl}} F_1(y) + F_2(y) \quad (2.20)$$

where V_{cl} and U_{cl} are values at the centre line of the MS for the wake. Firstly we assume $V_{cl}=0$, and evaluate $F_2(y)$ for the upper and lower parts of the wake. After evaluating $F_2(y)$ we insert equation (2.20) into the normal momentum equation (2.3) and integrate to obtain a quadratic equation for V_{cl} . This V_{cl} is inserted into equation (2.20), so we can obtain the value of V at any position (see Appendix 3). The solution procedure is as follows.

- (i) The initial pressure distribution for all edges of the wake providing U_e as external boundary condition on U , is given from the inviscid flow calculation, and the initial values of U , V and τ are given from the boundary layer calculation at the TE.
- (ii) Calculate the inner width from equation (2.18).
- (iii) Calculate the eddy viscosity and the gradient of shear stress from equations (2.15) and (2.16).
- (iv) Calculate length scale from equation (2.12').
- (v) Calculate the U and τ values at the future station by using two characteristic equations for the upper and lower part of the wake separately.
- (vi) Calculate the V values by using the newly calculated U and τ at the future station (see Appendix 3).
- (vii) Calculate the pressure p half a step upstream of the U calculation point, by using the newly calculated U , V and τ . The step (ii)-(vii) are repeated until the end of the wake, $s=s_l$, where the downstream boundary condition is applied as $\partial p / \partial x(s_l, n) = 0$.

2.3 Numerical Scheme for One Streamwise Sweep of the Calculation

In this section we shall describe the numerical scheme used in the main part of the shear layer calculation including the wake. The governing equations for boundary layer region and wake are the same, except for some additional terms, and the length-scale equation, in the wake. Hence the numerical procedures for the boundary layer region and the wake region will be described separately only where necessary.

Since the shear layer calculation is based on the Mahgoub and Bradshaw method, we will mainly point out the changes made in the present method from the Mahgoub and Bradshaw method, but a brief explanation of the Mahgoub and Bradshaw method will be given as needed. For full details of the numerical method the reader is referred to Mahgoub and Bradshaw (1979) or Bradshaw and Unsworth (1974 and 1976).

(1) The calculation grid system

The domain of the shear layer calculation including the wake and the calculation grid system is shown in Fig.1. The flow field is divided by a series of straight lines normal to the body surface along the aerofoil and normal to the centre line of the MS in the wake, except in the transformation region: these are the s-stations. A series of curved constant-n lines is then drawn, parallel to the body surface or to the centre line of the MS in the wake. In the "transformation region" near the TE a fully-curvilinear (ϕ, ψ) coordinate system is used (see

§2.4).

The index i is used for the grid points along the s -direction from the transition point to the starting point of the transformation, and from the last point of the transformation in the wake to the downstream face of the MS, $s=s_\ell$. The s -spacing is determined as the calculation proceeds, by using the CFL stability condition, and increases as the shear layer thickness increases. The s -coordinate of any grid point is denoted by s_i and the streamwise spacing is $\Delta s_i = s_i - s_{i-1}$, referred to as the s -step. In the transformation region, the coordinate of any point is denoted as ϕ , and $\Delta\phi = \phi_i - \phi_{i-1}$ is the streamwise spacing.

For any given s -station the grid points are equally spaced in the n direction (ψ in the transformation region) and are termed "mesh points". The index j is used for normal direction and Δn is the constant n -step. These mesh lines run from the wall to the MS on the aerofoil, and for the wake from the lower to the upper MS where external boundary conditions are supplied. Hence $j=0$ represents the position at the wall in the aerofoil and $j=j_{cl}$ represents the centre line of the MS in the wake. The number of points between the wall and the MS is an integer independent of s , but the value of the normal coordinate n at the MS depends on s . In the transformation region, the s -step is reduced: a smooth reduction of s -spacing is needed as we approach the transformation region, and in this program the streamwise spacing is reduced parabolically between $s=0.9c$ and the start of the transformation region ($s/c > 1$ at TE, but $x/c=1$).

(2) The U and τ calculation

The equations solved to obtain U and τ at any station with given V and $p(s, n$ or $\phi, \psi)$ are (2.9) and (2.10) along the two inclined characteristic directions (see Fig. 3). For a full description of the method see Bradshaw and Unsworth (1974 and 1976) or Mahgoub (1978). AM and BM represents the segments of "ingoing" and "outgoing" characteristics at any point M where the U and τ values are calculated. The inclination of characteristics is

$$\frac{h_1 dy}{h_2 dx} = \frac{V + \alpha \pm \beta}{U}$$

at point M in Fig. 3. C is the mid-point of the "ingoing" characteristic segment AM and C' is the mid-point of the "outgoing" characteristic segment BM. Equations (2.9) and (2.10) are solved along AM and BM and can be written as :

$$\tau [1 - \frac{V}{U}(C_1 - C_2)] dU - \frac{1}{2a_1} (\alpha - \beta) d\tau - \frac{\phi}{h_1 h_2 U} dx = 0 \quad (2.22)$$

$$\tau [1 - \frac{V}{U}(C_1 - C_2)] dU - \frac{1}{2a_1} (\alpha + \beta) d\tau - \frac{\phi}{h_1 h_2 U} dx = 0 \quad (2.23)$$

Equation (2.22) is solved along AM while equation (2.23) is solved along BM. We can write (2.22) and (2.23) in finite difference form (see Fig. 3) as

$$\tau_{mid} [1 - \frac{V_{mid}}{U_{mid}} (C_1 - C_2)] (U_{fut} - U_{int}) - \frac{1}{2a_1} (\alpha \pm \beta) (\tau_{fut} - \tau_{int}) - \frac{\phi \Delta x}{h_1 h_2 U_{mid}} = 0 \quad (2.24), (2.25)$$

where U_{fut} and τ_{fut} are the values of U and τ at the "future" station and U_{int} and τ_{int} are values of U and τ at A and B, obtained by parabolic interpolation at the "current" station; since the values of U and τ are known at the current station at the mesh point, the values of U and τ at A and B can be obtained. For the terms which are not derivatives, we centre at the mid-points C and C' in Fig. 3.

For the "predictor" step, U_{mid} , V_{mid} and τ_{mid} are approximated by values at the current station. We solve the equations (2.24) and (2.25) along AM and BM to obtain U_{fut} and τ_{fut} values at the "future" station. This procedure continues from the first mesh point above the wall to the MS. To improve the accuracy, we use a "corrector" step at the same station. After obtaining U_{fut} , τ_{fut} and V_{fut} values for U_{mid} , τ_{mid} and V_{mid} equal to the averages of current values and newly-predicted values are used. The solution procedure along AM and BM is repeated to obtain "corrected" U_{fut} and τ_{fut} at the future station. In the original BFA method the update was done only in the inner layer, but here we continue from the first point above the wall to the MS to improve interaction between pressure and velocity. For the values of U , τ and V at the first point, see the wall boundary calculation later in this section.

The main differences between the MB method and the present method are as follows:

Firstly, the staggered grid is used for improving numerical accuracy and stability. Secondly the (ϕ, ψ) coordinate system is used to improve numerical accuracy by making a smooth change from the (s, n) coordinates of the boundary layer region to the (s, n) coordinates of the wake, for an aerofoil with finite TE angle.

In the MB method central differencing is used for the $\partial p / \partial x$ term to provide upstream influence in the U and V calculation. The pressure p is calculated at one step behind (upstream of) the velocity-calculation point so as to use the same central differencing for $\partial v / \partial x$, the dominant term in the normal momentum equation. But Mahgoub (1978) obtained diverged values after several sweeps, because in

the viscous/inviscid matching procedure central differencing for $\partial p/\partial x$ and $\partial v/\partial x$ cannot dissipate small wiggles (Issa 1983) and propagates upstream influence too rapidly. In Chen and Bradshaw (1982) U , V , τ and p are calculated at the same station in the strong interaction region. For the $\partial p/\partial x$ term, forward differencing scheme is used to take account of upstream influence, and they insist on iteration at the same station for the pressure-velocity coupling, in which the newly-calculated pressure distribution is used for the U, V and τ calculation to get remove inconsistency between pressure and velocity. But if the newly calculated p value at current station and previous sweep value at the future station are used for the $\partial p/\partial x$ term in the momentum equation, the inconsistency between the U, V, τ and pressure value exists in the present sweep and the upstream influence is lagged by one sweep inside the boundary layer. In the present program we iterate viscous/inviscid matching procedure by using global pressure relaxation, and allow upstream influence inside the shear layer to remove this inconsistency by using staggered grid points for pressure and forward differencing for the $\partial p/\partial x$ term. The centering point is shown in Fig. 6: the pressure grid points are half a step upstream of the associated velocity and shear stress grid points. Upstream influence is propagated also through $\partial v/\partial x$ term in the pressure calculation. We use different centering points for the streamwise momentum equation and the continuity equation, for unconditional stability (V.Dedoussis, private communication and PhD thesis 1987). For the finite differencing of $\partial p/\partial x$ we use a forward differencing scheme. Hence the pressure gradient is:

$$\left(\frac{\partial p}{\partial x} \right)_{i,j} = \left(\frac{p_{i+1} - p_i}{\Delta x} \right)_{,j} \quad (2.26)$$

The upstream influence is propagated half an s-step per sweep via the $\partial p / \partial x$.

(3) V and p Calculation

We solve the vertical characteristic equation (2.7) to obtain V , and solve the normal momentum equation (2.3) to obtain p , at any station (see Fig. 5), having the newly calculated U and τ at that station. As shown in Fig. 3, given the newly calculated U and τ , we calculated V by solving along the vertical characteristic. After obtaining the U , τ and V values at the future station, we solve the normal momentum equation (2.3) for $p - p_e$.

Now we describe the finite difference scheme applied to solve the equations (2.3) and (2.8).

(i) For the normal derivative of any quantity appearing in the equations, we use central differencing centred at point A (see Fig. 5). For instance $\partial p / \partial y$ is approximated by

$$\frac{\partial p}{\partial y} = \frac{p_j - p_{j-1}}{\Delta y}$$

(ii) For the streamwise derivative of any quantity we use backward differencing (i.e. differencing centred at the point A). For instance, $\frac{\partial \tau}{\partial x}$ is approximated by

$$\frac{\partial \tau}{\partial x} = \frac{\tau_i - \tau_{i-1}}{\Delta x}$$

But for the streamwise pressure gradient term, we use forward differencing to propagate the upstream influence. Also in the normal momentum equation for p , $\partial V / \partial x$ is the dominant term, representing the streamline curvature, so in equation (2.3) we use forward differencing centred at Y (see Fig. 6). Because the newly calculated V at x_i is used for the streamwise normal velocity gradient term in calculating p , the upstream influence is propagated half a step through

$\partial V/\partial x$ term inside shear layer. Hence as mentioned in previous subsection the upstream influence propagated one step inside shear layer for one sweep.

The finite difference equations for V is as follows:

$$h_{2i,j-\frac{1}{2}} \frac{1}{2} (v_{ij} + v_{i,j-1}) \left(-\frac{u_{ij} - u_{i,j-1}}{\Delta y} \right) - h_{1i,j-\frac{1}{2}} \frac{1}{2} (u_{ij} + u_{i,j-1}) \times \left(\frac{v_{ij} - v_{i,j-1}}{\Delta y} \right) = (E2 - UE1)_{i,j} \quad (2.27)$$

Thus we can obtain the V values at the future station. The solution procedure continues from the first mesh point above the wall to the MS.

After calculating V we calculate p at the half-step downstream point with the boundary condition at the MS given by the inviscid calculation (see Chapter 3): this calculation proceed from the MS to the wall. The newly calculated p is underrelaxed by using underrelaxation factor .09.

In the wake region, a boundary condition for V is not given, as mentioned in §2.2. We integrate the normal momentum equation (2.3) by assuming the velocity distribution as shown in equation (2.20)-see §2.2.

(4) The wall boundary calculation

As seen in the previous sections, the U, V and τ calculations start from the first mesh point above the wall, assuming the values U_1 , V_1 and τ_1 (where the suffix "1" denotes the first mesh point and "w" for the wall). Hence the purpose of the wall boundary calculation is to obtain the values of U_1 , V_1 , τ_1 and τ_w given $U=0$ and $V=0$ at wall.

This is performed by using three equations:

- (i) The logarithmic law of the wall for the inner layer of the turbulent shear layer, applied at the first mesh point (with an extra term (Townsend 1961) in cases where $\partial \tau / \partial y$ is

significant).

(ii) The streamwise momentum equation integrated in the normal direction (y) from the wall to the first mesh point by using a one-fifth power law assumption for the U velocity profile .

(iii) The equation along the segment of the "ingoing" characteristic which ends at the first mesh point (see Fig. 4).

These three equations for U_1, τ_1 and V_1 are nonlinear and are solved iteratively using Newton's method. Since V_1 and the angle of the ingoing characteristic are needed in this calculation, values at previous station are used as a first guess, as in the U and τ calculation.

After obtaining converged solutions for U_1, τ_1 and τ_w , we evaluate V_1 by integrating the continuity equation (2.11) in the y -direction from the wall to the first mesh point, again taking a one-fifth power law for U profile. The newly calculated V values are used in another pass through Newton's method to obtain new U_1, τ_1 and τ_w values. The new values of U_1, τ_1 and τ_w are used to obtain a V , and so on.

The detailed calculation procedure is described in Appendix 3.

2.4 Transformation region

In the transformation region, the curvilinear orthogonal coordinate system (φ, ψ) is used as mentioned in the introduction. The equations are the same as in the (s, n) coordinate system except for different metric values (and an extra metric value because the constant- φ lines are curved while the constant- s lines are straight). The

solution procedure is also the same as in (s,n) coordinate system.

In order to start the calculation a transfer of values of variables from (s,n) coordinates to the (φ,ψ) coordinate system is needed. A patching scheme between (s,n) and (φ,ψ) coordinate system by using quartic polynomial fits over one streamwise step is used to smooth out the transition (see Chapter 4). However the curved φ -line at the start of the (φ,ψ) region does not match the straight s -line at the patching point exactly. To remove this difficulty we used interpolation at the patching point in the (φ,ψ) coordinate system (see Fig. 6). Calculations are performed at two s -coordinates around the patching point and all the values are transferred into (φ,ψ) coordinate system by the tensor relations, and then interpolated at the patching point. At the downstream end of the transformation region, the reverse procedure is performed from (φ,ψ) coordinate to (s,n) coordinates, i.e calculations at two values of φ .

2.5 Description of the program

In this section, a detailed description of the program for the boundary layer and wake calculation is presented. The relevant program listing, namelist and flow charts will be given in Appendix 5. Some completely new subroutines like, TRANSF for the transformation, have been added to the program of Mahgoub and Bradshaw (1979) and the others have been extensively rewritten for ease of understanding, so that nearly all the code has been written by the present author. For a full description of the original BFA program the reader is referred to Bradshaw and

Unsworth (1974,1976) and Mahgoub and Bradshaw(1979).

We will describe the boundary layer program in section 2.5.1 and the wake program in section 2.5.2. It is convenient to describe the main routine first and then to describe the subroutines.

2.5.1 The Boundary Layer

(1) The main routine of TURB

The main flow chart of the calculation procedure is shown in Fig. 9. The main U, τ and V calculation procedure is executed twice in each sweep, once for each aerofoil surface. After reading some input variables and initialising the calculation, the subroutine INITIAL, which includes subroutine INPUTV and subroutine START, is called. Subroutine START includes subroutine SYNTH, in which the laminar boundary layer calculation is performed by Thwaites' integral method and the initial "synthetic" profiles of U, τ and θ etc. at the transition point are set up, using Coles' profile family. Given the synthetic U and τ profiles, the V profiles are determined by using the normal characteristic equation, and except on the first sweep the stored pressure field in the SL is shifted to make it consistent, at the matching surface, with the pressure distribution from the newest inviscid calculation (the old and new values of p_{ms} being different until the calculation converges). We write

$$p(s,n)=p_{ms}(s)+[p(s,n)-p_{ms}(s)]_{\text{stored}} \quad (2.28)$$

The calculation loop now begins and continues until the TE is reached. The boundary thickness and the τ_{max} value for scaling L,G in the turbulence model are calculated from subroutine TPARA. The two characteristic angles to be

used for the future station are calculated from subroutine TANCAL. The subroutine PRINT prints out all the required variables. We then calculate the integral parameters like θ in subroutine INTPAR, and calculate the new s-step (XSTEP) and s (X) for the "future" station in subroutine CFL. In the subroutine MATSURF the matching surface height is adjusted and we calculate the tangential and normal velocity (UEXT,VEXT) on the new MS. Then boundary values for the future station are provided by the subroutine BVALUES, and metric values and $\partial p / \partial x$ values are set up in the subroutine TIMSET. The U, V and τ values at the first mesh point for the future station are calculated by applying the wall boundary condition (extended logarithmic law) in the subroutine WALLBC which also checks separation. If separation occurs, the program stops: in principle, the inviscid pressure distribution could be arbitrarily made less adverse and the calculation re-started.

The U and τ profiles at the future station are calculated in the subroutine NEWPRO, and the V profile in subroutine VPRESS. We next re-calculate characteristic angles by using newly calculated U, V and τ and re-calculate U, V and τ , using those new characteristic angles, in another pass through NEWPRO and VPRESS. The pressure is calculated half a step upstream of the "future" station, using newly calculated U, V and τ , and the coefficients of a 5th-order polynomial fit to the pressure profile $p(y)$ are evaluated in the subroutine VPRESS. The underrelaxed pressure ($\Omega=.09$) is stored for the next sweep. When the transformation region is approached, interpolation of the variables is done in the subroutine TRANSF. In the transformation region the (φ, ψ) coordinate system is applied

by using the "inviscid wedge flow" transformation. The calculation procedure continues until the TE. When the TE is reached, the program stores the data for the wake calculation, including parameters for transformation and the inviscid/viscous interaction, in the subroutine DUMTAP. This completes the calculation for the upper side of aerofoil, and the procedure is repeated on the lower side.

(2) The subroutine BVALUES

This subroutine involves the pressure distribution at the MS for the shear layer calculation. The SL calculation nodes are not same as inviscid calculation nodes, so we interpolate the pressure for SL calculation (see §4.1). If $I(8) \neq 0$, then it calls the subroutine ROUGH for the roughness effect on the logarithmic velocity profile law.

(3) The subroutine CFL

This subroutine evaluates the streamwise step (XSTEP) from the Courant-Friedrichs-Lewy stability criterion for hyperbolic partial differential equations. Near the TE region the (φ, ψ) coordinate system is used and the streamwise step is reduced to set up more grid points there, for accuracy rather than for stability. Hence in order to match with s-step from the CFL condition to the smaller value in the transformation region, we reduce the s-step parabolically beyond the .9c point.

(4) The subroutine CONSTS

This subroutine evaluates the coefficient values of the 5th order polynomial fit for pressure distribution along

the normal direction ,given the boundary value p_e and the $\partial p/\partial y$ values on the profile.

(5) The subroutine DPDXA

This subroutine gives the streamwise pressure gradient. The ETA parameter controls the differencing scheme. If $ETA=1.$, the differencing is forward with respect to the s-step being calculated; if $ETA=.5$, the differencing is quarter point central (half of the central differencing and forward differencing); if $ETA=0.$, the differencing is central; if $ETA=-1.$, the differencing is backward.

(6) The subroutine INPUTV

This subroutine is the initialising subroutine, as the name implies, and is called in the subroutine INITIAL. The subroutine reads in variables and parameters for the shear layer calculation and transformation, from the data files BLDATT and BLDATT1 for the upper surface and BLDATB and BLDATB1 for the lower surface. These disc files have been written by other routines in the viscous/inviscid calculation procedure. One of the inputs is the pressure distribution of p_e on the MS, which is given at the mid points between the "nodes" of the inviscid calculation and stored in the array PEXTA. To evaluate $\partial p/\partial x$ at the TE, we include p_e at the first point in the wake, from the data file COEPPY for the upper side and COEPPYB for the lower side.

(7) The subroutine INTPAR

This subroutine evaluates the displacement thickness

and momentum thickness, using the correction of Coles (Kline et al 1968) for the viscous sublayer.

(8) The subroutine MATSURF

This subroutine calculates the new MS height for the next sweep based on the boundary layer thickness. In the (φ, ψ) coordinate system we at present fix the height at about 1.4 times the boundary layer thickness at the end of the (s, n) on the first sweep. This subroutine also gives the tangential and normal velocity values at the new MS.

(9) The subroutine METRICS

The subroutine evaluates the metric values for the (s, n) coordinate system and the (φ, ψ) coordinate system.

(10) The subroutine NEWPRO

This subroutine calculates the U and τ profiles at the future station (XXSTEP). Firstly, the calculation evaluates the U_{int} , τ_{int} and V_{int} values at the current station (X) at the point where the "ingoing" and "outgoing" characteristics through the KM th point at the future station intersects the normal axis (s or φ axis) at the current station. The U , τ and V values at the mid-points of the characteristics are approximated by interpolated values at the same n (or ψ) at the current station. Then by using the inclined characteristic equations the U and τ values at the KM th point at the future station are obtained (the calculation is just the solution of two simultaneous linear algebraic equations). After calculating the new V velocity this procedure is repeated by using the newly calculated U , τ and V values. The calculation proceeds from the second

mesh point above the wall to the edge of the shear layer, which is defined in the calculation as the mesh point where $U > .995U_e$ and $\tau < .0001U_e^2$ are both satisfied. Above this thickness we apply the "edge clip" condition on U and τ to prevent the velocity defect and shear stress from becoming negative due to (small) finite difference errors (see Mahgoub and Bradshaw 1979).

(11) The subroutine PHIXO

This subroutine evaluates the value of φ and $\partial\varphi/\partial x$ consistent to the X_0 value in the (φ, ψ) coordinate system (see Fig.38).

(12) The subroutine PRINT

This subroutine prints out profiles of all the variables, and lists of parameters, for instance C_f , δ^* and θ .

(13) The subroutine ROUGHS

This subroutine evaluates the additive constant in the logarithmic velocity profile law for a rough surface.

(14) The subroutine SPLINE

This subroutine is used for cubic spline interpolation of given function values. The function values are stored in the array XX and YY . The spline calls the NAG library routines E02BAF and E02BCF which fit a cubic spline through the given function values and calculate the interpolated values and first, and second derivative values.

Other computer centres are likely to have access to the NAG library or an equivalent mathematical library.

(15) The subroutine START

This subroutine calls the subroutine SYNTH for initial value of variables at the transition point. Except on the first sweep, the pressure profiles are shifted as mentioned at the start of this section.

The next section initialises the values of U , τ , C_1 and C_2 etc. and calculates the V profile at the starting station of turbulent shear layer by using U and τ profiles from subroutine SYNTH.

(16) The subroutine SYNTH

This is the subroutine which calculates the laminar boundary layer calculation by using Thwaites' integral method from the LE to a station, $s=s_0$, say, where a transition criterion is satisfied (in the present program we use $R_{e\theta, tr}=600$ but a criterion based on pressure gradient, free-stream turbulence etc could be inserted at the user's choice). The initial U and τ profiles for the turbulent shear layer calculation are evaluated by using Coles' velocity profile family. The Thwaites calculation loop is performed twice as explained below.

The section starts to perform the integration loop for the laminar calculation in the streamwise direction with XF (the future station) and X (the current station). The value of θ and δ^* are calculated and the shape factor (H_1) is evaluated from an empirical formula. This then allows us to compute the V_e distribution from equation (2.1). In the first pass the MS is not known, so the V_e calculation is delayed until the second pass. The new MS based on the momentum thickness is calculated. The laminar calculation stops at the station where the transition criterion is matched.

The skin friction coefficient C_f is either input or calculated from the Spalding-Chi empirical formula for zero pressure gradient for calculating $C_f/2$ at $s=s_0$. If input $A(5)=0$, then this formula is used. If $A(5) \neq 0$, then we use $A(5)$ value as $C_f/2$.

The next section calculates the initial profiles of U and τ for the turbulent boundary layer calculation given from Thwaites' method (strictly only δ_{coles} is needed, to choose a MS for the interval $0 < s < s_0$) and the second path starts. The V_e value at the MS is calculated in this second path and the U and τ profiles at $s=s_0$ are set up again.

(17) The subroutine TANCAL

This subroutine evaluates the "ingoing" and "outgoing" characteristic angles from equation (2.8) or (2.9) at any station and stores them in the arrays TANAFU and TANBFU.

(18) The subroutine TEMPRO This subroutine calculates the temperature for compressible flow over an adiabatic wall. It is used in compressible flow only.

(20) Ths subroutine TIMSET

This subroutine sets up the streamwise pressure gradient values at each y from subroutine DPDXA, and metric values from subroutine METRICS, at any station and stores them in the arrays. These arrays help to save computer time because the stored values are used throughout one sweep without recalculating.

(21) The subroutine TPARA

This subroutine calculate δ_{05} for the scaling of L and G in the turbulence model and includes the low Reynolds number effect on length scale as in the BFA program.

(22) The subroutine TRANSF

This subroutine calculates the initial value of U , τ and V for the (φ, ψ) coordinate system by using interpolation and transformation rules as mentioned in section §2.4.

(23) The subroutine VPRESS

This subroutine contains the V and p calculation inside the matching surface. The first section sets up the variables needed for solving the normal characteristic equation (2.7), to save calculation time. The next section calculates the V profile from equation (2.7) along the vertical characteristic. This calculation proceeds from the second mesh point along the normal direction to the edge of the MS. The V value at the first point is given from the subroutine WALLBC, called before subroutine VPRESS. For the pressure gradient term $\partial p / \partial x$ the forward difference is used.

The $\partial p / \partial y = 0$ assumption is used in the first sweep (the "centrifugal" approximation $\partial p / \partial y = U^2 / R$ could easily be used), but in general stored pressure values from the previous sweep are used.

The next section starts by setting up the metric values and calculating the pressure profile, given the boundary condition p_e at the MS. The calculation proceeds from the edge of the MS to the first mesh point above the wall. The pressure at the wall is calculated in subroutine WALLAC. After getting the pressure distribution we

calculate the coefficient values $\partial p / \partial y$ at the five points where the interpolation fit is applied, by using spline interpolation, and store in the array ADPDY. This is more accurate than fitting p and involves less work. Then we call the subroutine CONSTS to get coefficients for the 5th-order polynomial fit and underrelax the pressure distribution consistent with the velocity underrelaxation for the viscous/inviscid interaction.

(24) The subroutine WALLBC

This subroutine calculates U_1 , V_1 , τ_1 and α at the first mesh point for the future station. Here $\alpha = \Delta \tau / y = (\tau - \tau_w) / y$. The first main section of this subroutine makes a first guess of U_1 , τ_w and α at the first mesh point for the future station by using U , τ_w and α values at the current station, and sets up the metric values. The next section calculates F_1 , F_2 , F_3 and their derivative values, needed in the Newton's iteration method for the three variables U_1 , τ_w and α . Thereafter, the iteration is repeated until convergence is reached (See Appendix 3).

The last section is used to evaluate better values of V_1 and the "ingoing" characteristic angle.

(25) The subroutine XSTEPS

This subroutine calculates the streamwise step value (XSTEP) by using the CFL stability condition from the subroutine CFL, except in the TE region. In the TE region the (ϕ, ψ) coordinate system is used, and the streamwise step is fixed by using ϕ value from subroutine PHIXO to consider more grids, because the streamwise gradients of variables are

severe in this region.

(26) The function PHI

This function calculates the deviation from the logarithmic velocity profile law at the first mesh point, due to finite $\partial \tau / \partial y$.

(27) The function PVALU

This function calculates $p(y)$ at arbitrary y by using the 5th -order polynomial assumption.

(28) The function SIMPSON

This function is used to integrate a given function FR specified at equal intervals, using Simpson's rule.

(29) The function VALUE

This is used in the interpolation of a function given at unequal intervals. If IOPT=1, a linear interpolation is performed, while if IOPT=2, a quadratic interpolation is performed by using Lagrange's formula (Hamming (1962) and Hayes(1970)) and if IOPT=3, a logarithmic interpolation is performed.

(30) The Block Data subprogram

This contains tables of values of the empirical function G, L and normal derivative of G . This also contains the value of parameters which are used in the calculation of coefficients of 5th polynomial distribution of pressure.

2.5.2 The wake

In this section, the wake program is described in detail. The calculation procedure is almost the same as in

the boundary layer: hence the subroutines which are the same as in the boundary layer program are not described.

The wake program for the CDC machine is divided into two "overlays" to save core space. In the overlay (overl,1,0), the initial data for the main wake calculation are set up. The main routine including subroutines, is given in the overlay (overl,2,0) after initialising. A flow diagram and data flow are shown in Fig.10 and Fig.11.

(1) The subroutine START (see Fig.12)

This subroutine initialises the values of variables and parameters obtained from the turbulent boundary layer calculation at the TE, and from the viscous/inviscid interaction procedure. There are 5 data files in this subroutine. Data flow is shown in Fig. 11.

The START subroutine begins by reading data for the transformation from (s,n) coordinates to (φ, ψ) coordinates which is written in TURB, and values of constants for the eddy-viscosity (in the inner wake region) and the length scale equation, all from the data file "INWAK". The subroutine then reads in the I and A arrays of constants to control the SL calculation, exactly as in the BL calculation.

The U,V and τ profiles, the n-step, and the number of nodes and coefficients for the pressure profile $p(n)$ or $p(\psi)$ in the normal direction at the TE, are derived from results for the upper- and lower-surface turbulent boundary layers, in data files "TETOP" and "TEBOT". Then the pressure distribution at the MS and the geometric coordinates of the MS in the wake region are read from data file "WAKDAT". The pressure distribution obtained in the

previous sweep (or the initial guess in the case of first sweep) is read in data file "WAKDAT1". After reading in the pressure distribution, it is shifted, as in boundary layer, to match the new p_{ms} from the inviscid calculation: recall that the $\partial p / \partial n$ (or $\partial p / \partial \psi$) is not altered. The contents of these data files are shown in Appendix 5.

(2) The main routine for wake calculation

The first section sets up the initial profiles of U, V and τ , which are read in subroutine START, for the future station. The values of $I(13)$ is read: if $I(13)=0$, then the values of variables and parameters at the "current" station are stored in the subroutine DUMP for the continuing calculation procedure, while if $I(13)=1$, the program reads the values of variables and parameters which were stored at given last calculation point in the subroutine DUMP.

The program then sets up the τ_{max} and δ_{05} for scaling of L, G and defines the new upper and lower MS heights, relative to the old lower surface. If the MS has not changed but the shear layer has grown beyond the MS, we then correct the V velocity component on the MS using the displacement surface argument which calculates the mass flux and extrapolates the velocity.

The next section sets up the turbulent transport velocity for the length scale, and the length scale L itself, for the upper and lower parts of the wake at the TE.

All the program variables for the lower part of the wake have subscript "M" (for "minus"), so for example the transport velocities for length scale are V_1 and V_{1m} . **The**

two inclined characteristic angles are calculated and the shear stress profile "peakiness parameter" and other constant

values needed in Bradshaw and Unsworth's (1974) length scale equation are obtained from the subroutines TPARA and TPARAM for the upper and lower parts respectively.

The next section calculates the Δs step value in the (φ, ψ) coordinate system and Δs -step for (s, n) coordinate system, from the subroutine CFL, to satisfy CFL stability condition. We set up the last calculation station of the program, which is $.5c$ from the TE in our program. The eddy viscosity and the derivative of shear stress in the inner wake region are calculated for the "future" station in the subroutine EDDY. The widths (DI and DIM) of the upper and lower parts of inner wake are calculated for the "future" station in the subroutine INWIDTH, and we print out the values of variables and parameters in the subroutine PRINT.

The next section sets up the pressure values at the matching surface (except on the first sweep, where we assume that $\partial p / \partial y = (p_e - p_{em}) / (y_{mst} - y_{msl})$ for the first sweep).

The next section calculates the U and τ profiles for the "future" station for the upper and lower parts of the wake, in the subroutines NEWPRO and NEWPRM respectively. The length scale equation is solved for the future station in the subroutines RLPRO and RLPRM, again for the upper and lower parts of wake respectively. The eddy viscosity (ν_t) is calculated for the future station by using the newly calculated values (DI and DIM) in the subroutine EDDY, which then evaluates $(\nu_t \frac{\partial U}{\partial y})$, the corresponding term in the U -component momentum equation: this term is regarded as a source term rather than a genuine derivative, but is small enough that the calculation remains stable. After calculating the U and τ profiles, V is calculated in the subroutine VPRESS. As stated in section 2.2.2 the actual

boundary condition on v is different from and more complicated than in the boundary layer region, but the code is straightforward. We use the same prediction and correction scheme as thin boundary region calculation. Hence the calculation procedure for U , τ and V is performed as a boundary layer calculation. Then VPRESS calculates P by solving normal momentum equation, evaluates the coefficient values for the 5th order polynomial fit to the pressure, and stores the underrelaxed pressure, as in the boundary layer calculation.

The transfer of variables from (φ, ψ) to (s, n) coordinates to patch the calculation procedure at the end of the transformation region is done in the subroutine TRANSF. At the last station (the downstream end of the MS), the program stops the wake calculation and stores the data for the next sweep, and for the viscous/inviscid interaction at the MS, in the data files MATDATW, WAKDAT1, COEPPY and COEPPYB.

(3) The subroutine CFL

As in the boundary layer program: the largest characteristic angle anywhere in the wake (upper or lower part) is used to set the s -step.

(4) The subroutine CONSTS

As in boundary layer program: evaluates the coefficient for the 5th order polynomial fit of pressure based on lower MS.

(5) The subroutine EDDY

This subroutine calculates eddy viscosity

$(\nu_t = Ru_t(DI+DIM))$ inside the inner wake region. First, the proportional constant value based on the inner width is evaluated by calling INWIDTH. Then the metric values are evaluated from the subroutine METRICS and we calculate the turbulent shear stress and derivative values inside the inner wake region by using the experimental formula(2.15).

(6) The subroutine INWIDTH

This subroutine calculates the "inner wake" width by using equation (2.17): it is used for calculating eddy viscosity in subroutine EDDY.

(7) The subroutine METRICS

As in the boundary layer program: calculate the metric values.

(8) The subroutine NEWPRO

This subroutine calculates U and τ values at the future station for the upper part of wake. The procedure is the same as in the boundary layer, except for the addition, as a source term, of the "eddy viscosity" contribution to shear stress by fine grained mixing in the inner wake region (see Fig.13).

(9) The subroutine NEWPRM

This subroutine calculates U and τ values at the future station for the lower part of wake: recall that the shear stress profiles τ^+ and τ^- actively overlap. The procedure is same as the subroutine NEWPRO. The difference is that in the last section of the subroutine the centre-line position is shifted to allow wake curvature.

(10) The subroutine PHIXO

As in the boundary layer program: evaluates the metric value of φ coordinates and is called in subroutine METRICS.

(11) The subroutine PRINT

As in boundary layer program: print out the variables as required.

(12) The subroutine RLPRO

This subroutine calculates the length scale profile at the future station for the upper part of the wake (positive shear stress) from equation (2.13)-(see Fig.14). First, the turbulent transport velocity (V_L) is calculated at each mesh point across the wake. Then the calculation evaluates U_{int} , V_{int} , τ_{int} and L_{int} at the current station at the point the characteristic of the length equation (2.12') through the k th point at the future station intersects the normal axis (s or φ axis) at the current station. The U , τ and L values at the mid-point of the characteristic (2.12') are approximated by mean values (see Fig.3).

The next section corrects the values of CSL based on the effect of streamline divergence $\partial V / \partial y$. Then the new length scale (L) values are calculated at each point along the normal direction at the future station.

(13) The subroutine RLPRM

This subroutine calculates the length scale in the lower part of the wake (negative shear stress) using the

same procedure as RLPRO. The displacement thickness, momentum thickness and shape factor are also evaluated in this subroutine.

(14) The subroutine SPLINE

As same as boundary layer program.

(15) The subroutine STORE

This subroutine stores values of all the variables at the current station on disc.

(16) The subroutine TANCAL

As in the boundary layer program, this subroutine calculates the "ingoing" and "outgoing" characteristic angles. The characteristic angles in the upper and lower part of the wake are defined as in Fig.7.

(17) The subroutine TEMPRO

This subroutine calculates the temperature in the adiabatic compressible flow by using Crocco's formula (see Cebeci and Bradshaw 1984).

(18) The subroutine TPARA

This subroutine calculates the peakiness parameter needed in the length scale calculation for the upper part of the wake (positive shear stress).

(19) The subroutine TPARAM

This subroutine calculates the peakiness parameter needed in the length scale calculation for the lower part of

the wake (negative shear stress).

(20) The subroutine TRANSF

As in the boundary layer this subroutine transfers the variables U , τ and V from (φ, ψ) coordinate to (s, n) coordinate at the patching point.

(21) The subroutine VPRESS (see Fig.15)

This subroutine calculates the V and p profiles at the future station. It differs from the corresponding subroutine in the boundary layer region because the "boundary condition" on V is now that the pressure difference across the wake, calculated within the shear layer program, shall match the pressure difference calculated by the inviscid program (this is roughly a condition on $\partial V / \partial x$). First we integrate the vertical characteristic equation (2.8) to get the V velocity profile, $V(J)$, assuming that $V(JCLF)=0$ (where $JCLF$ is the point of centre of the wake) and store it in the arrays F_1 and F_2 which are such that $V(J)=F_1(J)V(JCLF)+F_2(J)$. This calculation is performed for the upper and lower part of the wake from the centre point separately.

The next section approximates the U , τ and V values at the current station where the y value is same as the future station, for the upper and lower parts of the wake and inserts these values into the integral equation (2.19). Then we obtain equation for $V(JCLF)$ and we insert this value in the formula $V(J)=F_1(J)V(JCLF)+F_2(J)$ to give the V profile across the wake at the future station.

The last section of VPRESS for calculating pressure profiles is controlled by the value of IN . If $IN=1$, the p

calculation is by-passed and p is assumed equal to p_e , while if $IN=2$, then p is found by solving the normal momentum equation (2.3) with p_{em} as the top and bottom edge pressures. The V profile has already been adjusted so that integrating $\partial p/\partial y$ from the bottom p_{em} ends up at the top p_e approximately - actually we ignore any remaining difference between $p_{e,upper}$ and $p_{e,lower}$. After calculating the pressure profile (half-step upstream of U and τ - staggered grid) the coefficients of the 5th-order polynomial fit are found from subroutine CONSTS. Then underrelaxation of the pressure profile is performed in order to accelerate convergence in the inviscid/viscous interaction.

(22) the functions

(i) DGORD

This function evaluates $\partial G/\partial y$ which is used in the shear stress equations.

(ii) DPDXA

This function gives the streamwise pressure gradient value ($\partial p/\partial x$) as in the boundary layer program.

(iii) GRAD

This function calculates the gradient of any variable FR at equal intervals over a specified range.

(iv) GORD

This function gives the empirical function G , also for the shear stress equation.

(v) ORDIN

This function is an interpolation function for any variable FA at a given range. The interpolation is controlled by the sign of the interval DX. If $DX > 0$, then interpolation is linear, while if $DX < 0$ it is quadratic.

(vi) PVALU

As in the boundary layer program this function evaluates the pressure profile by using a 5th order polynomial.

(vii) SIMPSN

As in the boundary layer program this is used to integrate a given function FR with equal intervals, using Simpson's rule.

(viii) VALUE and VALUM

As in the boundary layer program this is used in the interpolation of a function given at unequal intervals.

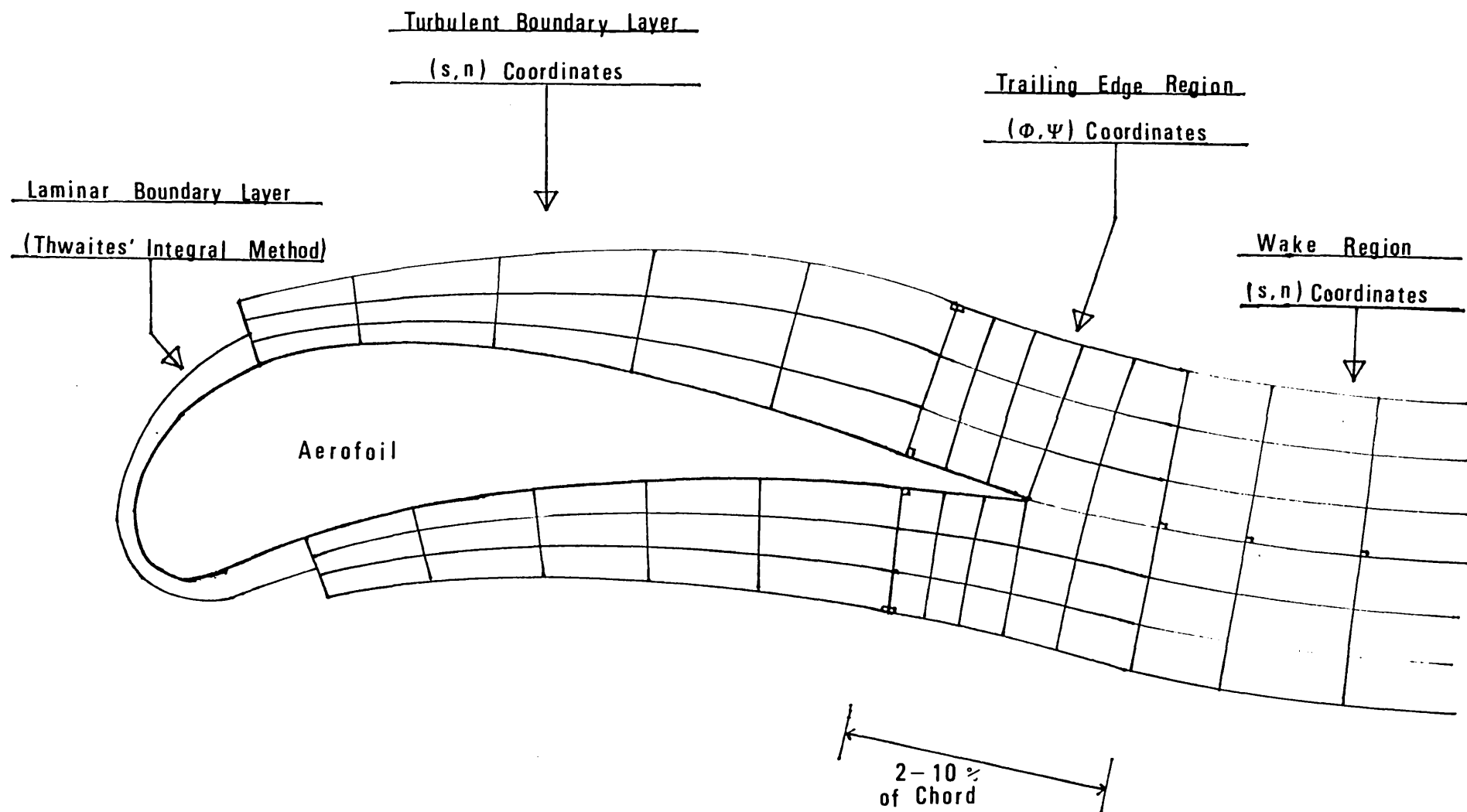


Fig 1 The Shear Layer Coordinates

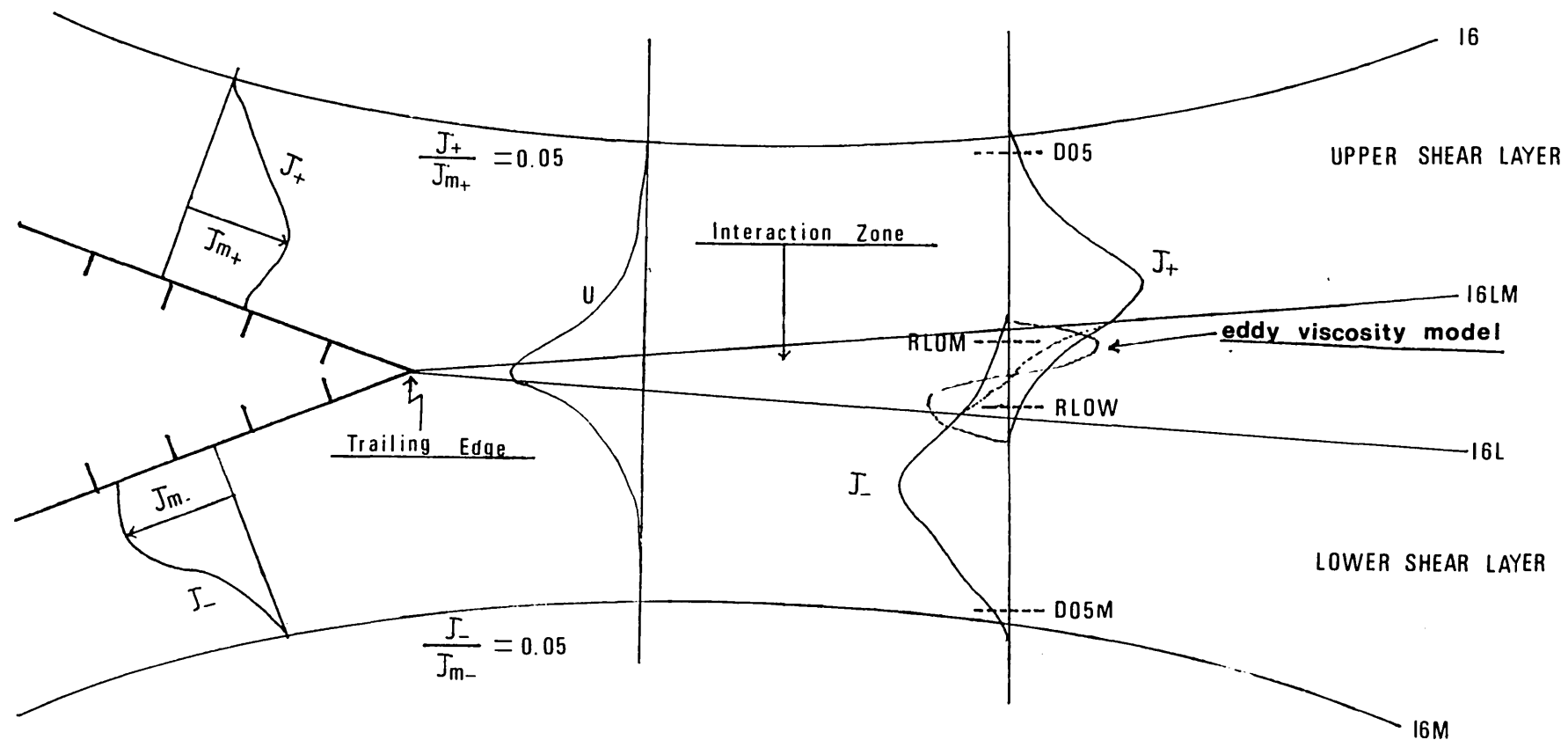


Fig 2 Shear Layer Superposition in the Wake

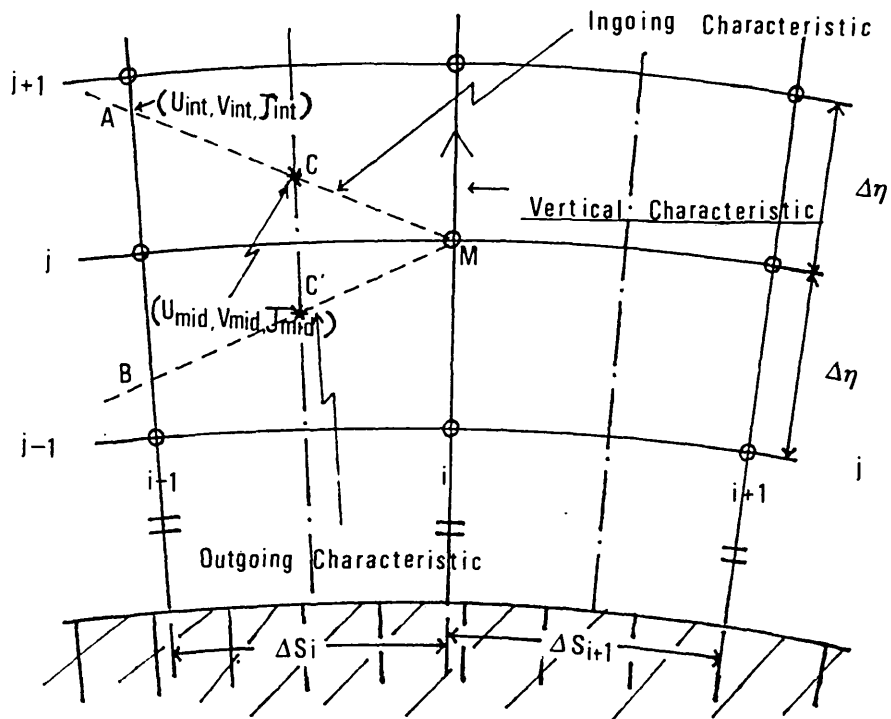


Fig 3 U, T Characteristic Calculation

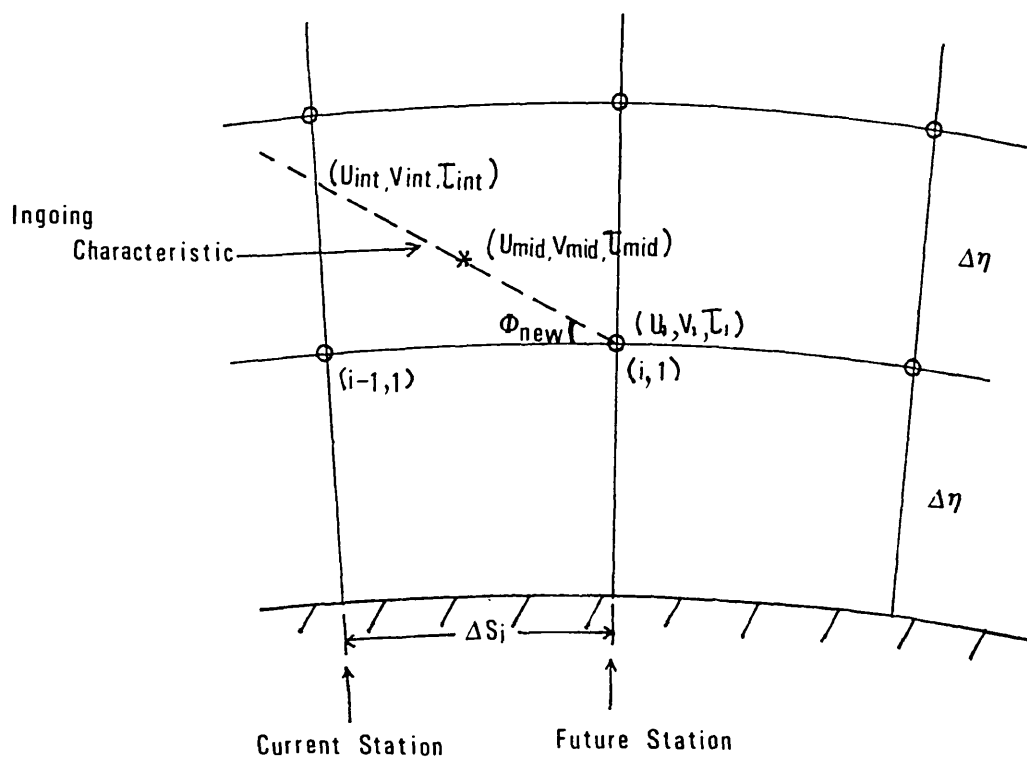


Fig 4 Scheme for Wall Boundary Calculation

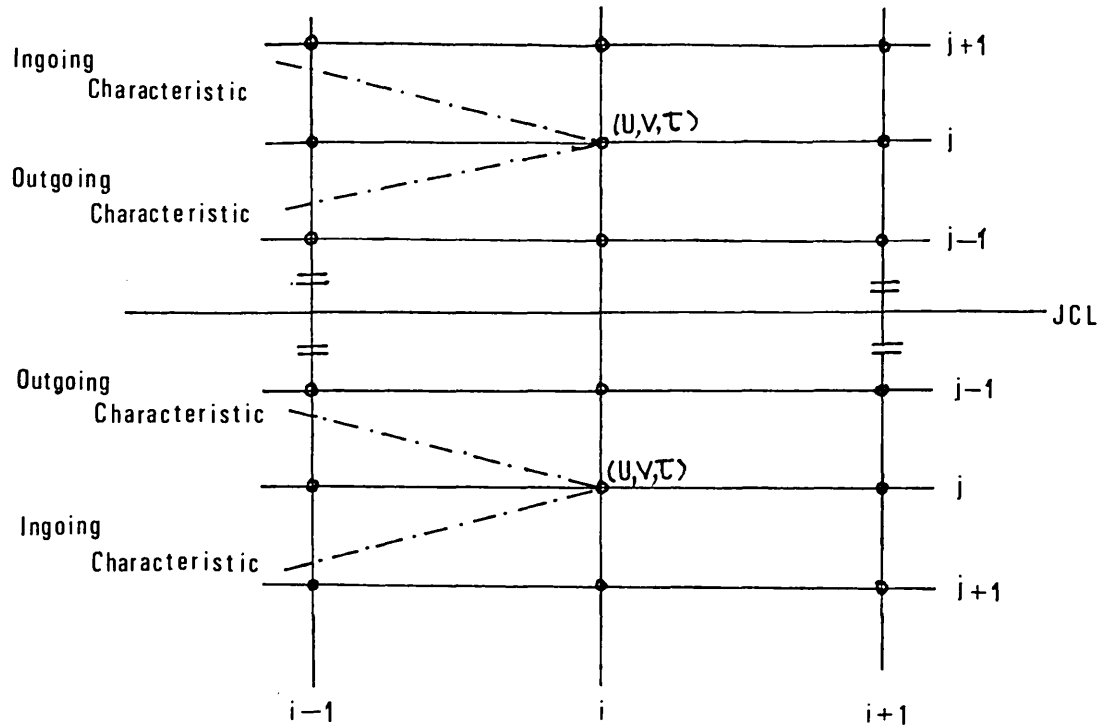


Fig 7 Ingoing and Outgoing Characteristics in Wake

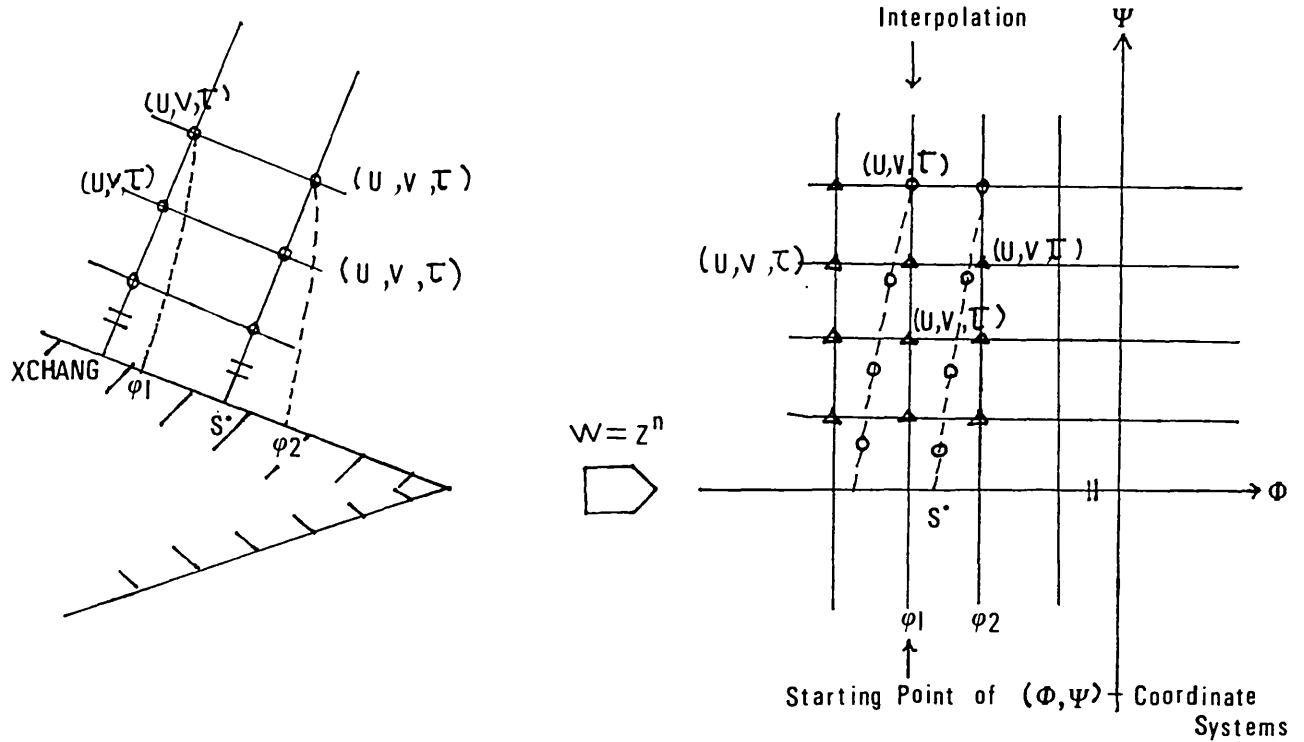


Fig 8 The Transformation of U, V, τ into (ϕ, ψ) System

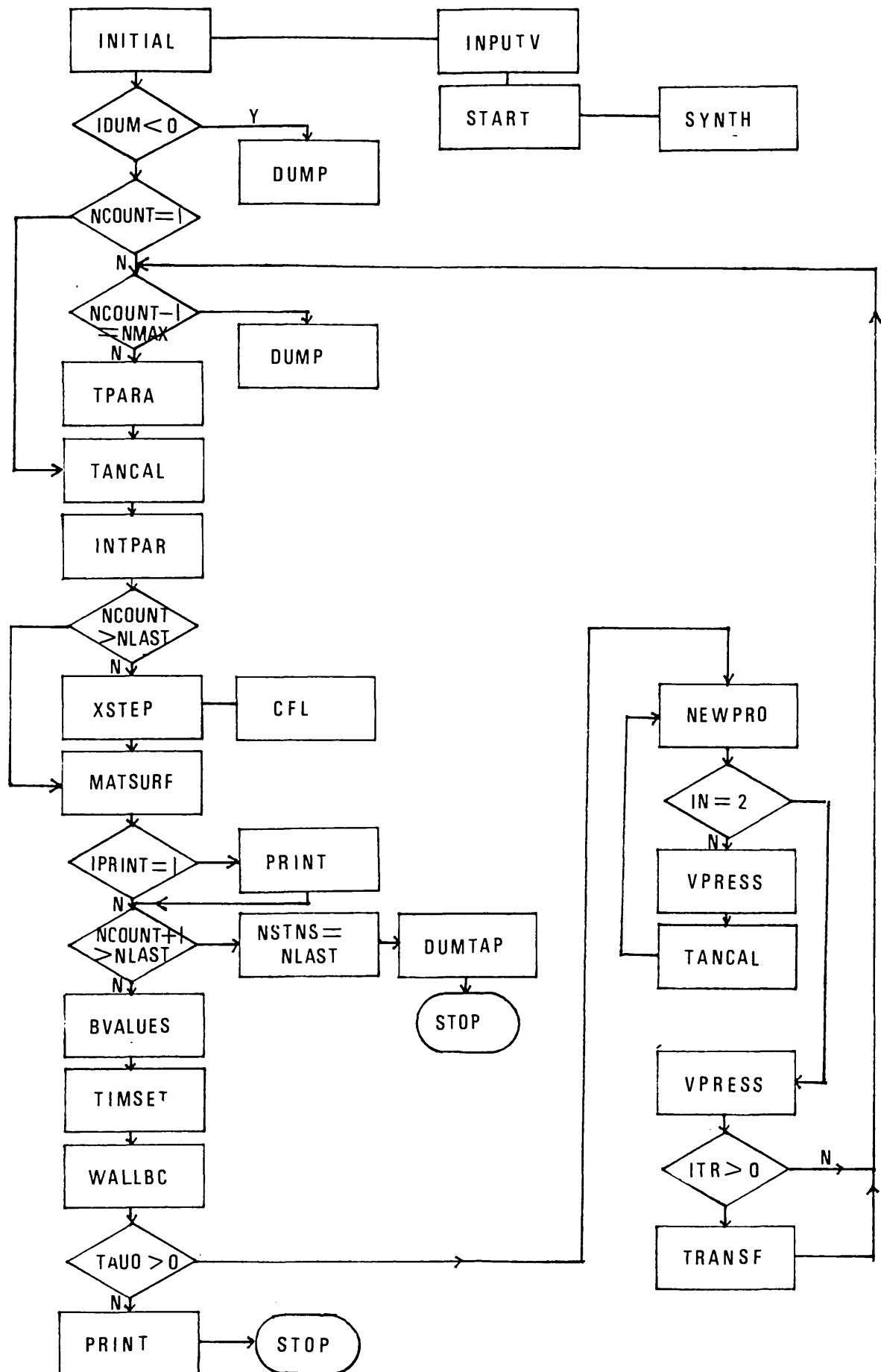


Fig 9 The Flow Chart of Turb

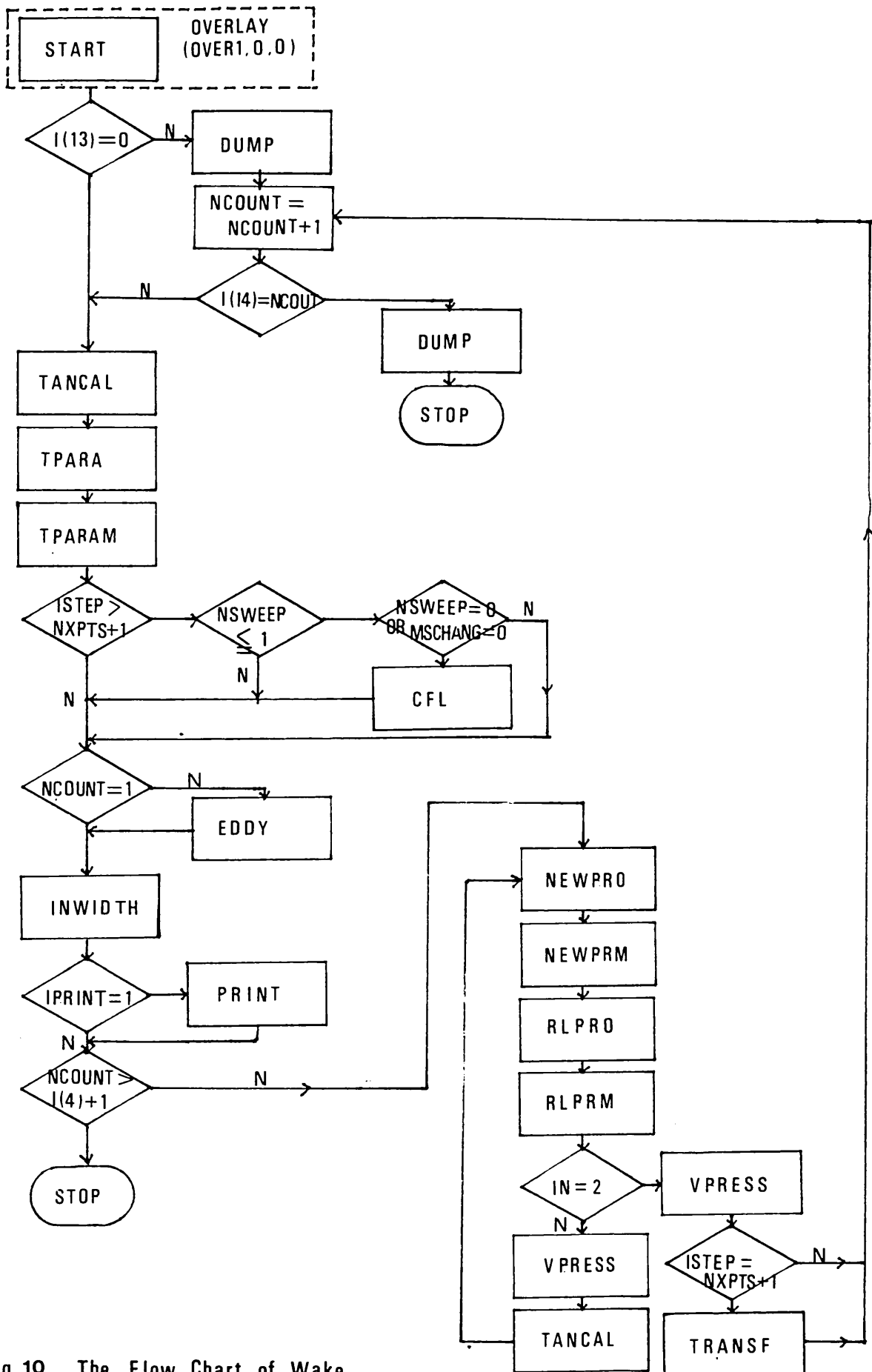


Fig 10 The Flow Chart of Wake

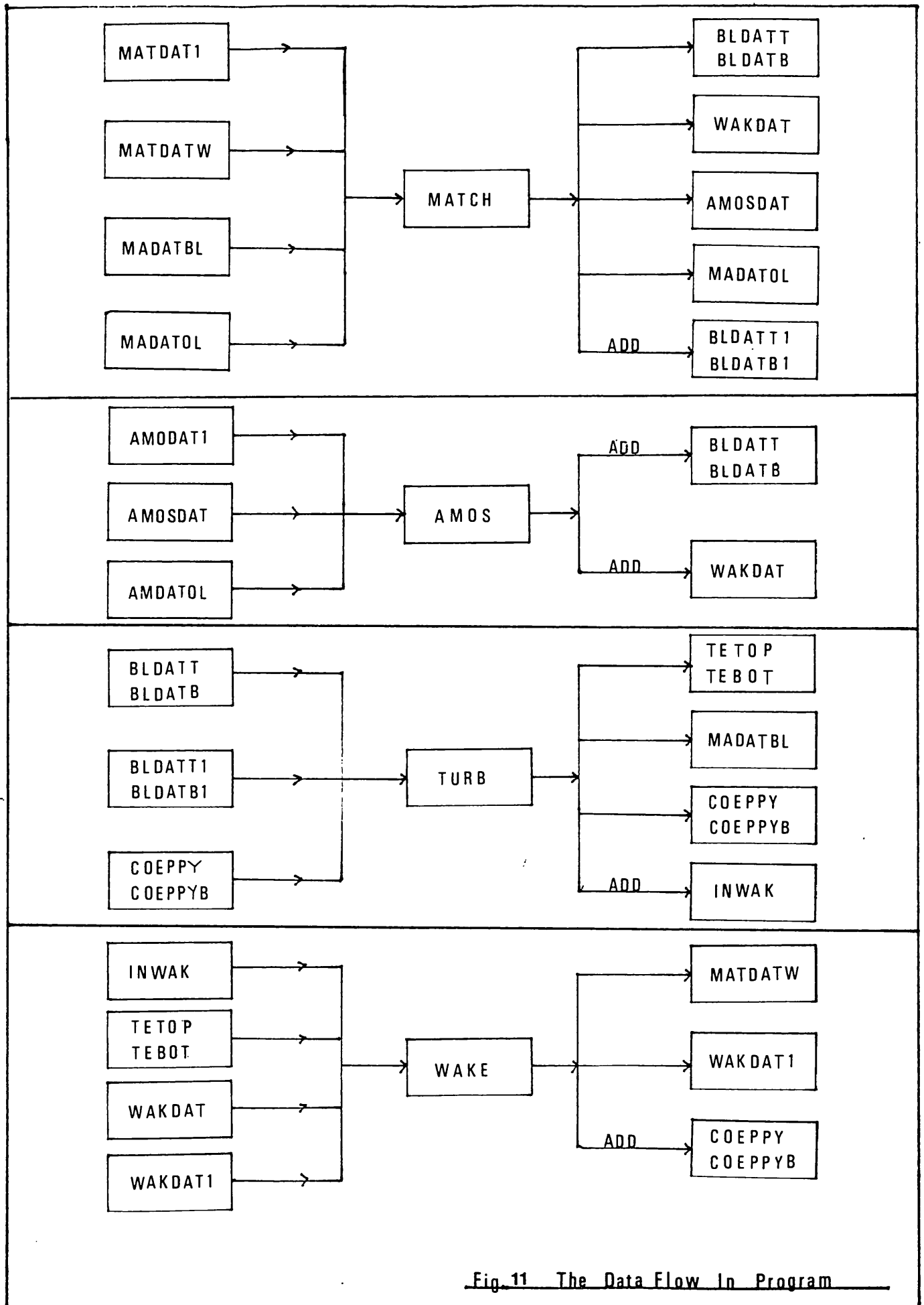
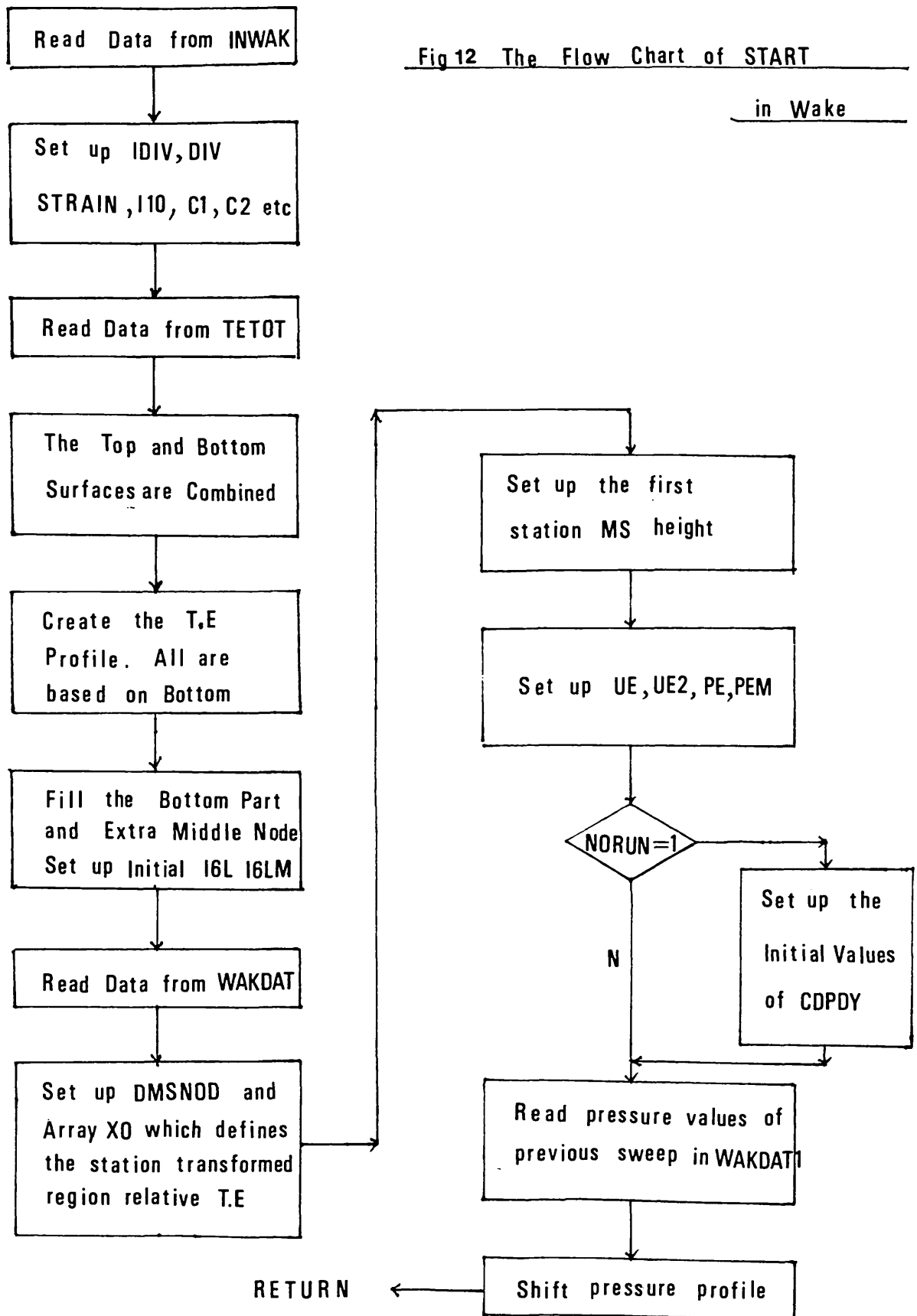


Fig. 11 The Data Flow In Program



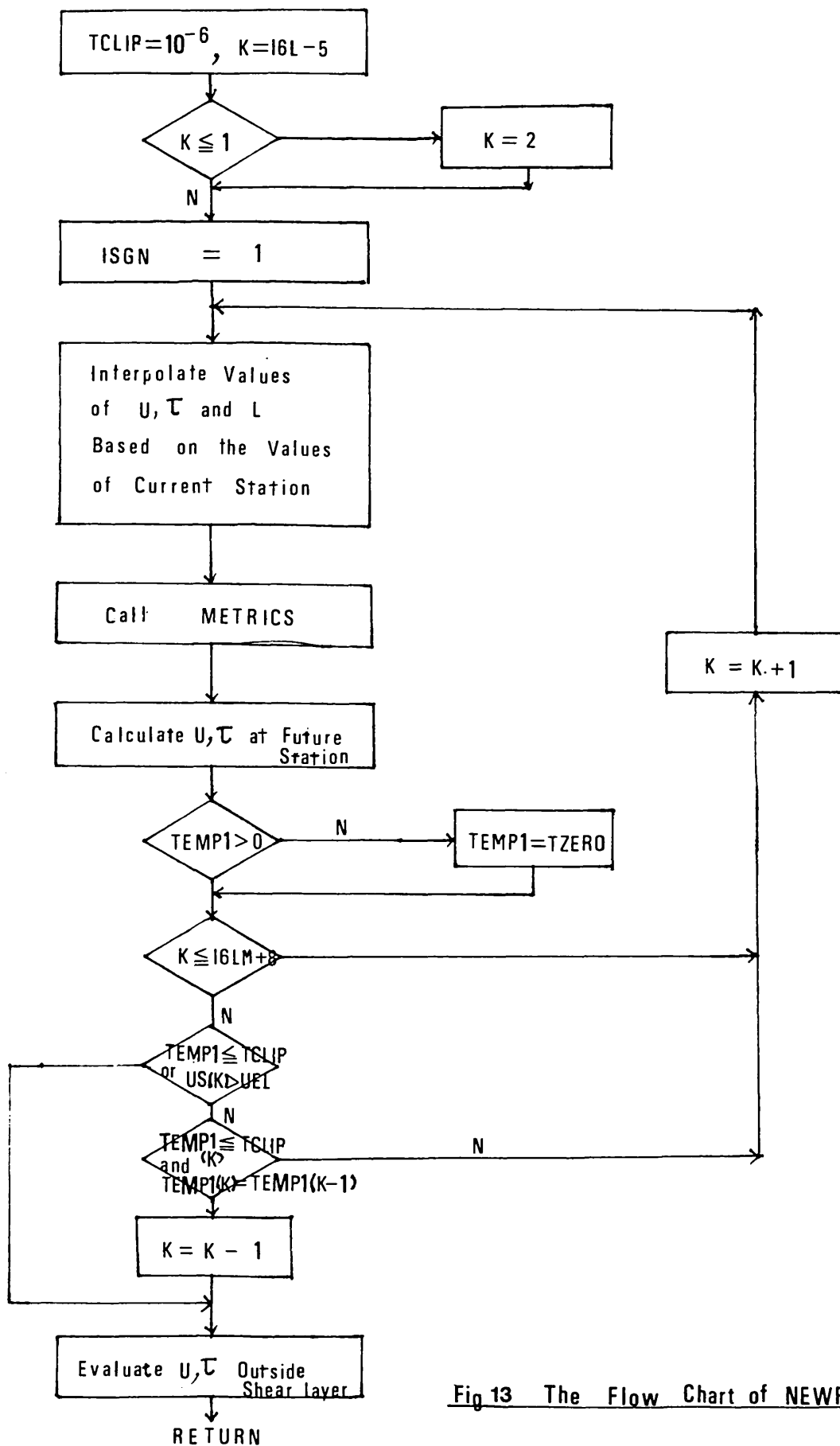


Fig 13 The Flow Chart of NEWPRO in Wake

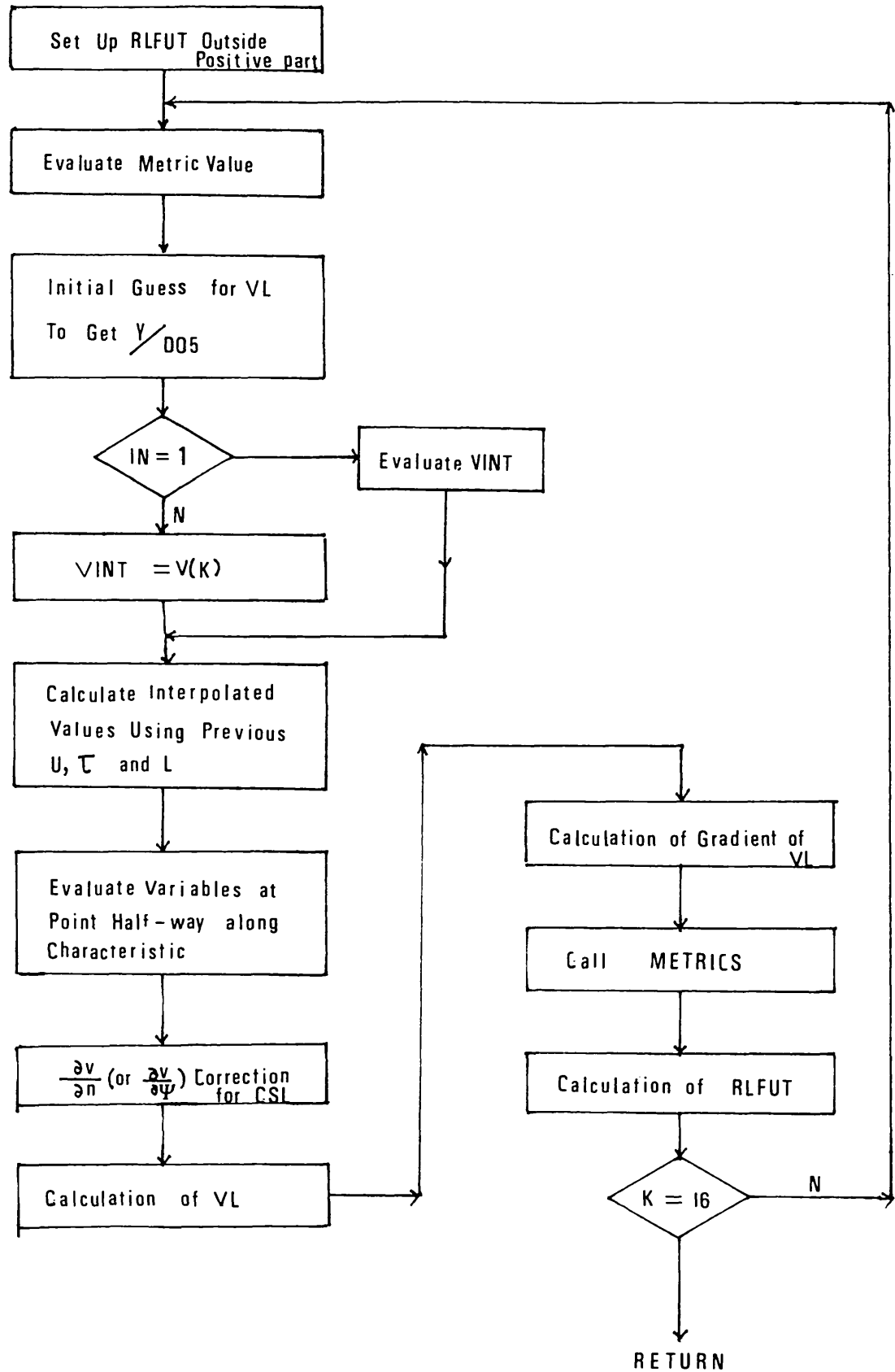
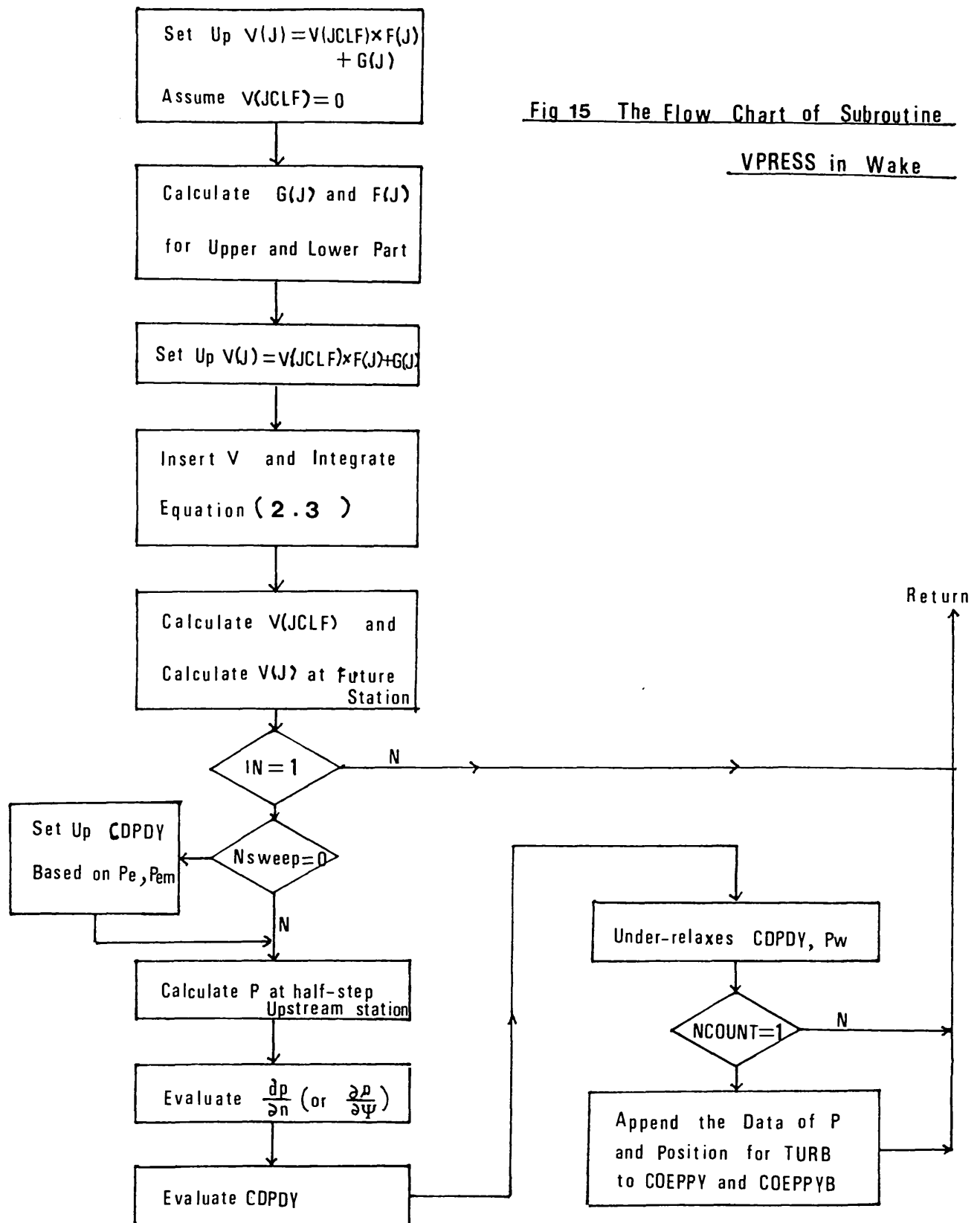


Fig 14 The Flow Chart of RLPRO



CHAPTER 3. THE POTENTIAL FLOW CALCULATION

3.1 Description of the Potential Flow Calculation

The inviscid calculation for incompressible flow uses the surface source singularity method of Hess and Smith (1967). The flow around a body is approximated by an unknown source distribution (or vortices-Wilkinson (1968)) on the MS with uniform oncoming flow. In order to calculate the unknown sources (or vorticity), the boundary condition on the component of velocity normal to the MS, and the Kutta-Zhukovskii condition at the downstream end of the MS in the wake are applied ; $U_{upper} = U_{lower}$ far downstream in the wake - It is in fact not a critical assumption (see §3.1.3). Then the integral equation for the unknown source distribution is solved in finite difference form. Then we can obtain the tangential velocity distribution on the MS, and the pressure distribution $p(s')$ from Bernoulli's equation, by using the calculated source distribution.

The surface source singularity distribution on the MS is approximated by segments on which the source density is constant. Hess (1975) showed that the first-order method, with source density constant, is not accurate in concave corners, but is quite accurate on mainly convex bodies, except for thin cusped aerofoils. The use of a first-order method does not incur serious errors for typical shapes of the MS, but a higher-order method could easily be substituted.

3.1.1 The normal velocity boundary condition

We divide the MS into a series of straight line segments (source "elements"). Therefore the MS is approximated by a large polygon (see Fig.16). The source density m_n is assumed constant along each segment, changing discontinuously between segments. Now we examine the velocity normal to the pth element induced by the nth-element. The velocity at a point P induced by a point source of strength M is given (see Fig.17) as follows

$$V_r = M/2\pi r \quad \text{in the radial direction.}$$

where r is the distance from the point source to the point P. The velocity induced by line source segment AB is given (see Fig.18) by equation (3.1) and (3.2): the detailed derivation is given in Appendix 4.

$$V_{1n} = \frac{m_n}{4\pi} \log \left\{ \frac{(\bar{x} + 1/2 \Delta s_n)^2 + \bar{y}^2}{(\bar{x} - 1/2 \Delta s_n)^2 + \bar{y}^2} \right\} \quad (3.1)$$

$$V_{2n} = \frac{m_n}{2\pi} \tan^{-1} \left\{ \frac{\bar{y} \Delta s_n}{\bar{x}^2 + \bar{y}^2 - 1/4 (\Delta s_n)^2} \right\} \quad (3.2)$$

where V_{1n} is in the direction AB and V_{2n} is normal to AB. Next we consider the more general problem of the MS around the aerofoil divided into such elements. The velocity on the pth element induced by the nth element is given by equations (3.1) and (3.2) by the same analysis as above (see Fig.20). We set up global axes with the X-axis is defined as the line passing through the LE and TE. The arc length around the MS is defined from the LE and is taken positive in the clockwise direction. The undisturbed stream velocity is defined as U_∞ at an angle α' to the X-axis (see Fig.19).

In order to impose the boundary condition $V_p = v_p$ at the mid-point of the pth element, we examine all the contributions to the velocity normal to that element (see

Fig.20): for the present we neglect the circulation.

(i) The normal velocity component induced by all the other elements on the pth element. The normal velocity induced by the nth element is given by:

$$V_{np} = V_{2n} \cos \sigma_{np} - V_{1n} \sin \sigma_{np} \quad (3.3)$$

where σ_{np} is the angle between the pth and nth elements. Thus the total normal velocity component induced by all other elements is given by:

$$V_p^{(i)} = \sum_{n=1}^N (V_{2n} \cos \sigma_{np} - V_{1n} \sin \sigma_{np}) \quad (3.4)$$

(ii) The normal velocity component induced by the pth element on itself. The velocity is given by $m_p/2$ in the outward direction (see Appendix 4).

(iii) The component of the undisturbed velocity in the direction normal to the pth element is given by:

$$V_p^{(ii)} = -U_{\infty} \sin(\theta_p - \alpha') \quad (3.5)$$

where θ_p is the inclination angle respect to the X-axis.

The net normal component velocity becomes:

$$\begin{aligned} V_p &= V_p^{(i)} + V_p^{(ii)} \\ &= \frac{1}{2}m_p + \sum_{n=1}^N (V_{2n} \cos \sigma_{np} - V_{1n} \sin \sigma_{np}) \\ &\quad - U_{\infty} \sin(\theta_p - \alpha') \end{aligned} \quad (3.6)$$

By substitution of the expression of V_{1n} and V_{2n} from equations (3.1) and (3.2), we can obtain the normal component velocity:

$$V_p = \sum_{n=1}^N \left(\frac{1}{2} \delta_{pn} + A_{pn} \right) m_n - b_p \quad (3.7)$$

where δ_{np} is the Kronecker delta and $A_{pp}=0$ for all p. The matrix form of this equation can be written as:

$$\tilde{v}_p = \tilde{A}_{pn} \tilde{m}_n - \tilde{b}_p \quad (3.8)$$

with summation over the dummy index n , and $\tilde{A}_{pn}$ is a two dimensional array which depends only on geometry and not on α' and v_p .

If we apply the boundary condition $v_p = v_p$, we obtain

$$\tilde{A}_{pn} \tilde{m}_n - \tilde{b}_p = \tilde{v}_p$$

or

$$\tilde{A}_{pn} = \tilde{b}_p + \tilde{v}_p \quad (3.9)$$

The source distribution $\tilde{m}_n$ can be obtained by inversion of the matrix of coefficients $\tilde{A}_{pn}$ (called the "influence coefficient matrix")

$$\tilde{m}_n = \tilde{A}_{pn}^{-1} (\tilde{b}_p + \tilde{v}_p) \quad (3.10)$$

where $\tilde{A}_{pn}^{-1}$ is the inverse matrix:

3.1.2 Calculation of the pressure coefficient for a nonlifting aerofoil

The pressure coefficient at the p th element is defined by

$$C_p = \frac{p - p_\infty}{\frac{1}{2} \rho U_\infty^2}$$

using Bernoulli's equation.

$$C_p = 1 - \left(\frac{q_p^2 + v_p^2}{U_\infty^2} \right) \quad (3.11)$$

where v_p is the specified normal component velocity and q_p is the tangential velocity at the mid-point of the p th element. There are three contributions to the tangential component velocity at the p th element, as shown in Fig.20.

(i) The tangential velocity induced by all other elements on the pth element: the tangential velocity induced by the nth element on the pth element is given by

$$q_{np} = v_{1n} \cos \sigma_{np} + v_{2n} \sin \sigma_{np} \quad (3.12)$$

The total tangential velocity induced by all other elements becomes:

$$q_p^{(i)} = \sum_{n=1}^N (v_{1n} \cos \sigma_{np} + v_{2n} \sin \sigma_{np}) \quad (3.13)$$

(ii) The tangential component of the undisturbed stream velocity at the pth element is given by

$$q_p^{(ii)} = u_{\infty} \cos(\theta_p - \alpha) \quad (3.14)$$

Thus the net tangential component velocity at the pth element can be obtained as follows:

$$\begin{aligned} q_p &= q_p^{(i)} + q_p^{(ii)} \\ &= \sum_{n=1}^N (v_{1n} \cos \sigma_{np} + v_{2n} \sin \sigma_{np}) + u_{\infty} \cos(\theta_p - \alpha) \end{aligned} \quad (3.15)$$

or by the matrix form

$$\tilde{q}_p = \tilde{B}_{pn} \tilde{m}_n + \tilde{f}_p \quad (3.15')$$

where $\tilde{B}_{pn}$ is a two dimensional array which only depends on the geometry. By inserting the equation (3.10) for $\tilde{m}_n$, we can obtain

$$\tilde{q}_p = \tilde{B}_{pn} \tilde{A}_{pn} \tilde{b}_p + \tilde{f}_p \quad (3.16)$$

Thus we calculate the pressure coefficient on the MS by inserting equation (3.16) into (3.11).

3.1.3 The Kutta condition for lifting aerofoil

In the previous sections we described the application of the normal velocity boundary condition and

the calculation of the pressure distribution around the MS for a non-lifting body, ignoring circulation. However, the general aerofoil problem is a lifting case, and therefore we consider how to solve this problem by using the singularity method.

For the lifting aerofoil, circulation exists around the MS, so we must add a vorticity distribution around the MS. This vorticity distribution has the appropriate strength to satisfy the Kutta condition at the end of MS in the wake (see Fig.21). Hence the solution procedure is divided into two steps:

(i) Firstly, we calculate the tangential surface velocity distribution $q^{(i)}(s')$ on the MS for stream velocity $U_\infty = 1$ at an angle of incidence. This induces a finite difference in tangential velocity at the end of wake between upper and lower part of the MS.

(ii) We evaluate the tangential surface velocity distribution $q^{(ii)}(s')$ on the MS for a constant unit vorticity distribution ($\gamma_n = 1$ for the n th element) on each element around the MS. This induces a finite difference $q^{(ii)}$ in tangential velocity at the end of wake (see Fig.25).

Since the calculation of the tangential surface velocity is performed by linear equations we can write, for the total tangential surface velocity distribution $q(s)$ of undisturbed uniform velocity U_∞ and circulation Γ ,

$$q(s') = U_\infty q^{(i)}(s') + \frac{\Gamma}{S} q^{(ii)}(s') \quad (3.17)$$

where S is total arc length around MS.

The total tangential velocity difference which is given from (i) and (ii) is zero at the end of the wake. So we have

$$\Delta q = u_{\infty} \Delta q^{(i)} + \frac{\Gamma}{S} \Delta q^{(ii)} = 0 \quad (3.18)$$

and this equation gives the amount of circulation.

Now we consider how to calculate the tangential velocity distribution $q^{(ii)}(s')$ for circulation. From elementary fluid mechanics it is known that the velocity induced by a point vortex of strength γ_n located at A, is the same as that induced by a point source of strength $m_n = \gamma_n$ located at same point A but with its direction rotated by 90° in the clockwise sense relative to that of the source (see Fig. 22). Thus, the induced velocity values V_{1n} and V_{2n} at the mid-point of the pth element, due to a vortex line of density strength γ_n at the nth element (see Fig. 23) are the same as equations (3.1) and (3.2) except that $m_n = \gamma_n$ and that the direction of the velocities are rotated clockwise by 90° .

The vorticity distribution induces a normal component velocity distribution at each element as mentioned above. Since the normal velocity boundary condition is satisfied in the first step of the calculation (see §3.1.1), the normal velocity induced by vorticity should be zero. In order to satisfy this condition, we add a new source distribution m_n' to each element of the MS. Therefore, we have:

i) Normal velocity induced at Pth element by the vorticity distribution γ_n (see Fig. 23)

$$\begin{aligned} v_p^{(i)} &= \sum_{n=1}^N (-v_{1n} \cos \sigma_{np} - v_{2n} \sin \sigma_{np}) \frac{4\pi}{\gamma_n} \\ &= -\tilde{B}_{pn} \frac{\tilde{\gamma}_n}{4\pi} \end{aligned} \quad (3.19)$$

ii) Normal velocity induced at the pth element by the source distribution m_n'

$$v_p^{(ii)} = \tilde{A}_{pn} \frac{\tilde{m}_n'}{4\pi} \quad (3.20)$$

Since the net normal velocity induced at the mid-point of the Pth element must be zero, we get:

$$v_p^{(i)} + v_p^{(ii)} = \tilde{A}_{pn} \frac{\tilde{m}_n'}{4\pi} - \tilde{B}_{pn} \frac{\tilde{\gamma}_n}{4\pi} = 0 \quad (3.21)$$

thus

$$\tilde{m}_n' = \tilde{A}_{pn}^{-1} \tilde{B}_{pn} \tilde{\gamma}_n \quad (3.22)$$

Since $\gamma_n = 1$

$$\tilde{m}_n' = \tilde{A}_{pn}^{-1} \sum_{n=1}^N \tilde{B}_{pn} \quad (3.23)$$

If we consider the total tangential velocity due to the vorticity γ_n and the added source distribution m_n' , we obtain:

$$\begin{aligned} q^{(ii)} &= \sum_{n=1}^N \left(v_{2n} \cos \sigma_{np} - v_{1n} \sin \sigma_{np} \right) \frac{\gamma_n}{4\pi} \\ &= \tilde{A}_{pn} \frac{\tilde{\gamma}_n}{4\pi} \end{aligned} \quad (3.24)$$

Substituting the m_n' from equation (3.22) yields

$$q^{(ii)} = \tilde{B}_{pn} \frac{\tilde{m}_n'}{4\pi} + \tilde{A}_{pn} \frac{\tilde{\gamma}_n}{4\pi} \quad (3.25)$$

where, since all the γ_n 's are equal to unity,

$$\tilde{A}_{pn} \frac{\tilde{\gamma}_n}{4\pi} = \sum_{n=1}^N \frac{\tilde{A}_{pn}}{4\pi} \quad (3.26)$$

and

$$q^{(ii)} = \tilde{B}_{pn} \tilde{A}_{pn}^{-1} \sum_{n=1}^N \frac{\tilde{B}_{pn}}{4\pi} + \sum_{n=1}^N \frac{\tilde{A}_{pn}}{4\pi} \quad (3.27)$$

Now the tangential component velocity due to the source distribution with $U_\infty=1$ is given by :

$$q_p^{(i)} = \tilde{B}_{pn} \tilde{A}_{pn}^{-1} \tilde{b}_p^* + \tilde{f}_p^* \quad (3.28)$$

where $\tilde{b}_p^*$ and $\tilde{f}_p^*$ are $\tilde{b}_p$ and $\tilde{f}_p$ as defined by equations (3.8) and (3.9) respectively, but $U_\infty=1$. Hence the total tangential velocity at the mid-point of the p th element is given by

$$\begin{aligned} \tilde{q}_p = U_\infty & \left(\tilde{B}_{pn} \tilde{A}_{pn}^{-1} \tilde{b}_p + \tilde{f}_p \right) \\ & + \frac{\Gamma}{S} \left(\tilde{B}_{pn} \tilde{A}_{pn}^{-1} \sum_{n=1}^N \frac{B_{pn}}{4\pi} + \sum_{n=1}^N \frac{A_{pn}}{4\pi} \right) \end{aligned} \quad (3.29)$$

As a check on the calculation, we can compute the total normal velocity component at the mid-point of the p th element:

From equation (3.17), we have

$$\hat{v}_p = U_\infty \hat{v}_p^{(i)} + \frac{\Gamma}{S} \hat{v}_p^{(ii)} \quad (3.30)$$

From equations (3.8),

$$\hat{v}_p^{(i)} = \tilde{A}_{pn} \frac{\tilde{m}_n}{4\pi} - \tilde{b}_p^* \quad (3.31)$$

and from equations (3.19) and (3.20) we obtain

$$\hat{v}_p^{(ii)} = \tilde{A}_{pn} \frac{\tilde{m}_n}{4\pi} - \tilde{B}_{pn} \frac{\tilde{v}_n}{4\pi} \quad (3.32)$$

Substituting equations (3.31) and (3.32) into (3.30) we obtain

$$\begin{aligned} \hat{v}_p = U_\infty & \left(\tilde{A}_{pn} \frac{\tilde{m}_n}{4\pi} - \tilde{b}_p^* \right) \\ & + \frac{\Gamma}{S} \left(\tilde{A}_{pn} \frac{\tilde{m}_n}{4\pi} - \tilde{B}_{pn} \frac{\tilde{v}_n}{4\pi} \right) \end{aligned} \quad (3.33)$$

Then using equations (3.23) and (3.10) for $\tilde{m}_n'$ and $\tilde{m}_n$ we obtain

$$\begin{aligned} v_p = U_\infty & \left(\tilde{A}_{pn} \tilde{A}_{pn}^{-1} \tilde{b}_p^* - \tilde{b}_p^* \right) \\ & + \frac{\Gamma}{S} \left(\tilde{A}_{pn} \tilde{A}_{pn}^{-1} \tilde{B}_{pn} \frac{\tilde{v}_n}{4\pi} - \tilde{B}_{pn} \frac{\tilde{v}_n}{4\pi} \right) = 0 \end{aligned} \quad (3.34)$$

which shows that the calculated normal velocity component at the mid-point of the p th element is that which was specified

as boundary condition.

The pressure coefficient C_p is, as before

$$C_p = 1 - \left(\frac{q_p^2 + v_p^2}{U_\infty^2} \right) \quad (3.35)$$

3.1.4 The calculation of the stagnation point

The position of the stagnation point on the MS is found by first searching for the node (K_m say) near the front of the MS at which the velocity is a minimum (see Fig.24). Since on the top surface q is in the positive s' -direction and on the bottom surface it is in the negative s' -direction, the tangential velocity q changes sign from the top to the bottom surface of the MS. From Fig.24 which shows the typical plot of velocity q for the arc length s' in the region of minimum surface velocity, we find the intersection ($s'=s_1'$) of the straight line drawn through nodes K_m and K_{m+1} and also the intersection ($s'=s_2'$) line drawn through nodes K_{m-1} and K_m . The position of the stagnation point is then given by taking the average of two intersection positions, i.e.

$$(s_1' + s_2')/2 \quad (3.36)$$

This point is then used to start the laminar boundary layer calculation.

3.2 Description of Program

The main changes to the Department-standard AMOS program have been in the input and output arrangements. Also, we use overlay to save core memory. The reader is referred to Mahgoub (1978) for a detailed description.

(1) The main overlay (OVAMOS,0,0)

This contains the main control statements for the whole calculation. The chord length, and the angle of attack, in degrees, are read from data file AMODAT1. Then the subroutine COORD is called to calculate the arc length along the MS and the inclination angles of the potential flow calculation elements.

(2) The primary overlay (OVAMOS,1,0)

This is responsible for setting up the influence coefficient matrices A and B.

(3) The primary overlay (OVAMOS,2,0)

This is the overlay containing the inversion of the influence coefficient matrix A. The matrix inversion subroutine MINV is included.

(4) The primary overlay (OVAMOS,3,0)

This overlay contains the calculation of the velocities, tangential and normal to the MS at the element mid-points. It also calculates the circulation around the MS in the lifting case. The pressure coefficient is also calculated at the element mid-points. $C_p = 1$ at the stagnation point.

(5) The subroutine COORD

This subroutine starts by reading some parameters, for instance, the number of MS elements. The (X,Y) coordinates and normal velocity at the MS mid-points are read next, and the arc lengths of mid-points and the slope of elements on the MS are evaluated.

(6) The subroutine MINV

This subroutine computes the inverse of a matrix stored in the array A. The inversion uses the Gaussian elimination method. On return from the subroutine, the inverse matrix is stored in A by overwriting it.

(7) The function VALUE

This function is identical with that in the boundary layer program (see §2.5).

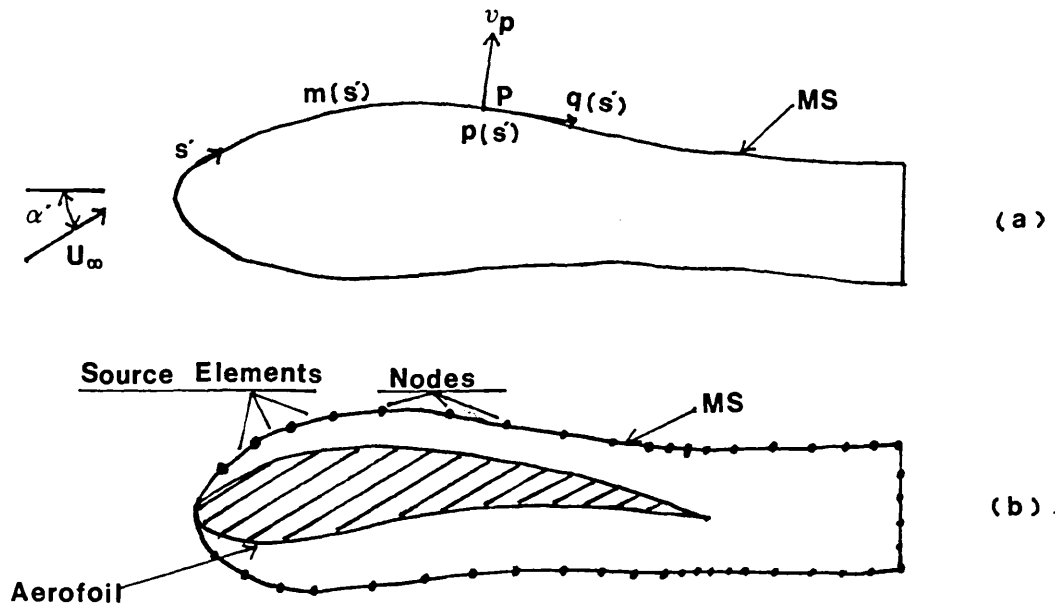


Fig 16 Hess and Smith Surface Source Distribution Method

(a) Continuous Source Distribution

(b) Discrete Source Distribution

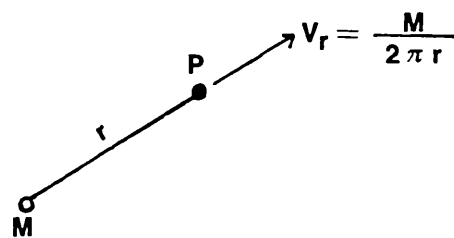


**Fig 17 Velocity Induced
by a Point Source**

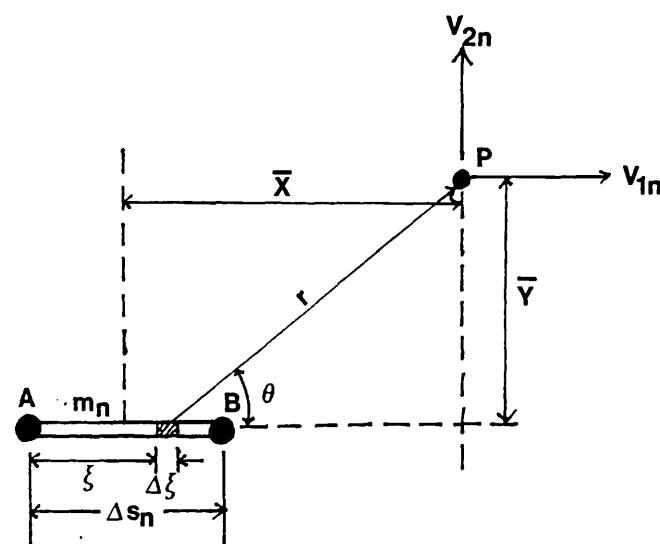


Fig 18 Velocity Field Induced by a Straight Line Source Element

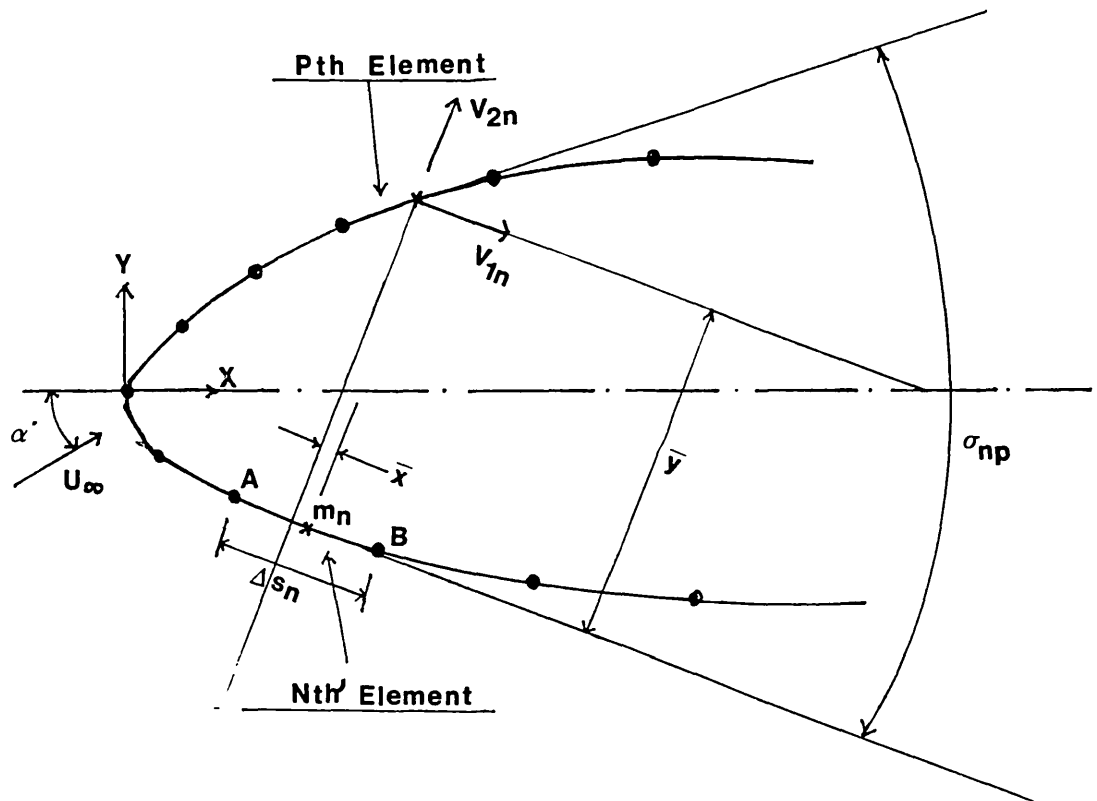


Fig 19 Velocity Field Induced by Nth Element on Pth Element

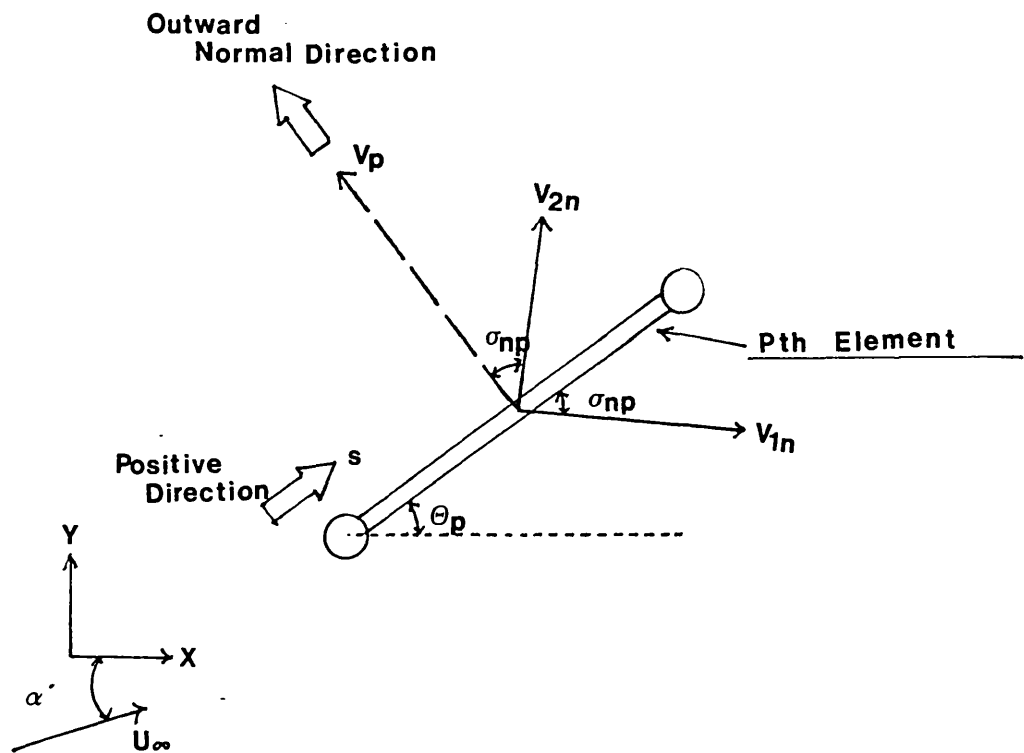


Fig 20 Velocity Induced on Pth Element due to Nth Element

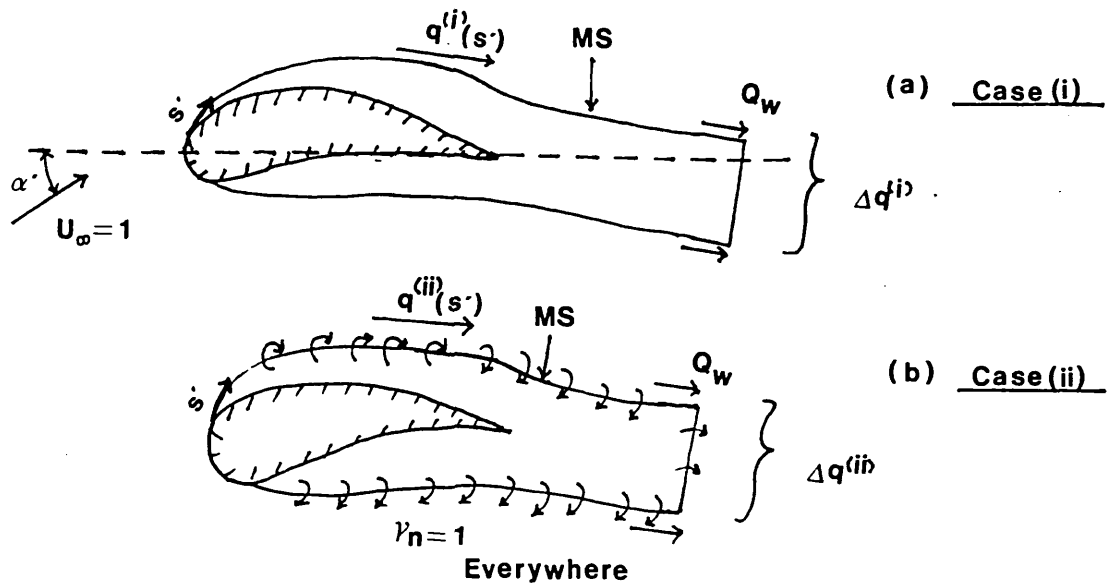


Fig 21 Solution Procedure for Lifting Aerofoil

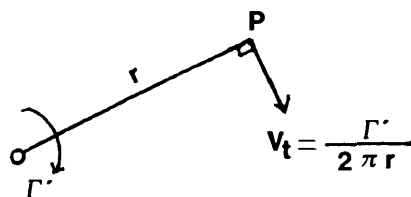


Fig 22 Velocity Induced
by a Line Vortex

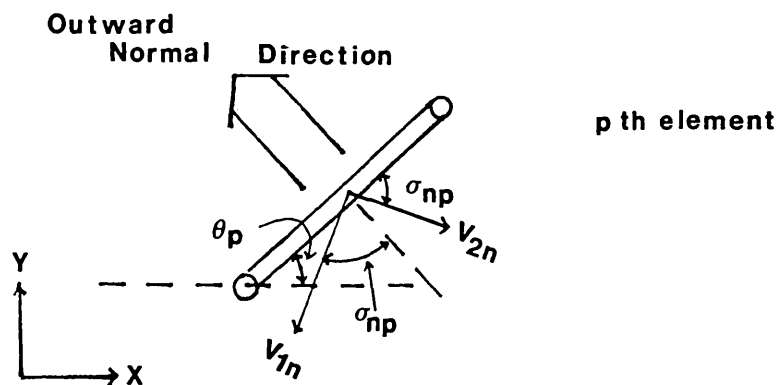


Fig 23 Velocity Induced on pth element due to Vortex at
nth element

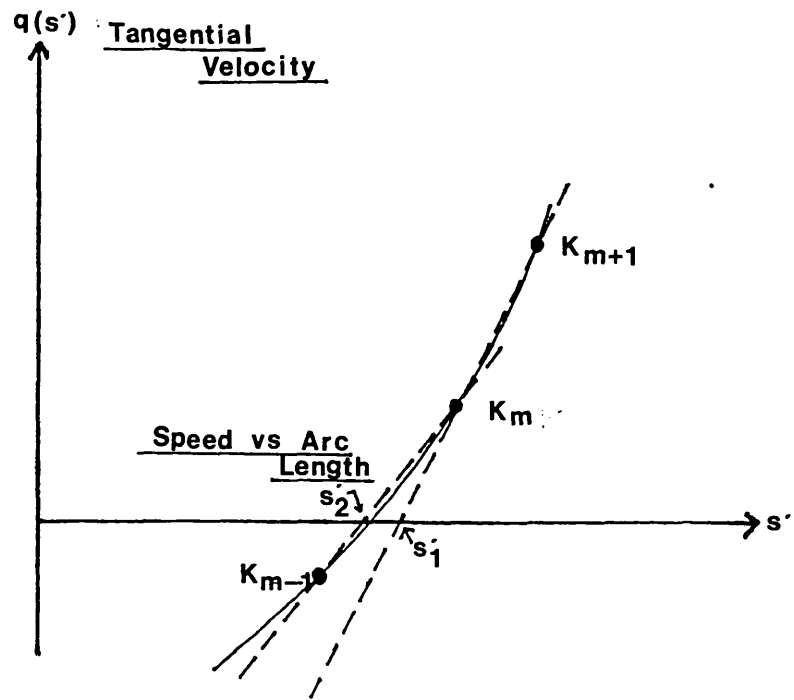


Fig 24 Calculation of Stagnation Point Position

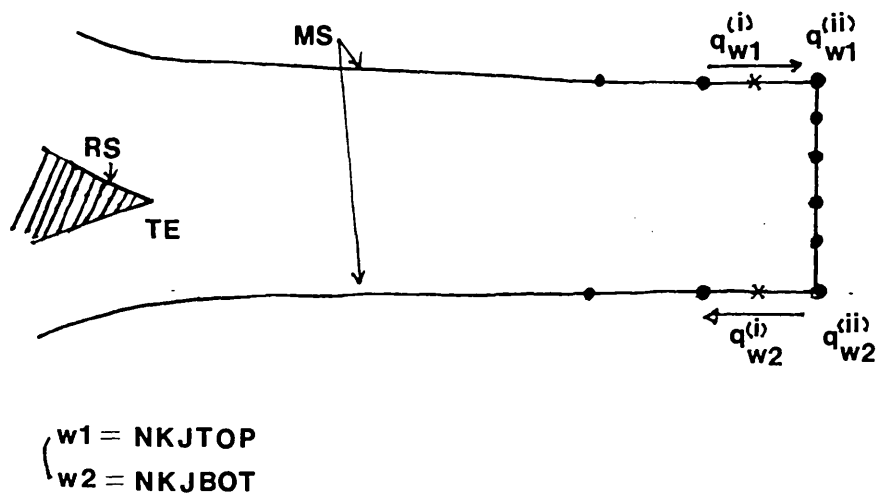


Fig 25 Application of Kutta-Zhukovskii Condition at the
End of Wake

Chapter 4. The Viscous/inviscid Matching Procedure

4.1 Description and Numerical Details of The Matching Procedure

As mentioned in the Introduction, the flow can be divided into two different domains : (1) A shear layer (including the wake), where we solve shear layer equations according to the method of chapter 2, and (2) A potential flow, where the flow is solved by using- for instance - the Hess and Smith surface source distribution method. These two solutions must be properly matched in order to obtain a meaningful solution. The matching surface for the interacting viscous/inviscid method using the thin shear layer equations can be chosen as the solid boundary, if displacement effects are represented by a transpiration velocity or as the "displacement surface" (body thickness plus displacement thickness: (Lock 1980, Lock and Firmin 1980). In the present method we place the matching surface at or above the boundary layer edge because the equations we actually solve include the normal momentum equation and, as mentioned in the introduction, the standard definition of δ^* is not valid. The quantities which are matched at the MS are the velocity normal to the MS (V_{ms}) and the pressure (p_e); The normal velocity (V_{ms}) is obtained from the shear layer calculation (including the wake) which is performed in region I (see Fig.26) from the real surface RS up to the MS.

V_{ms} is a function of s' , the arc length on the MS. The pressure distribution on the MS ($p_e(s')$) is obtained from the potential flow calculation which is performed in region II (see Fig.26); strictly from U_{ms} , hence $p_{ms} = P - 1/2(U_{ms}^2 +$

V_{ms}^2) where V_{ms} is small compared to U_{ms} . The matching procedure is the same in principle as Mahgoub (1978) but now includes a wake calculation and the "wedge flow transformation" in the TE region.

(1) Definition of the matching surface, excluding TE region

Because the s-n coordinate system is used in the shear layer calculation, except near the TE, it is convenient to define the MS by specifying the height (value of n) $n=H_{ms}(s)$ of the MS for any point on the RS (in the near wake region we use the wedge transformation rule for the (ϕ, ψ) coordinate system). The trailing edge bisector line is taken as the actual RS line: thus the MS is defined as a normal distance from the RS which is, in general, a function of s (the arc length along the RS)-(see Fig. 27). Therefore if we draw the normal lines through the point C_{ms} on the RS (with coordinate X_{rs}, Y_{rs}), we can define the point Q_{ms} on MS at the height H_{ms} (with coordinate X_{ms}, Y_{ms}). The α_{rs} and α_{ms} are the inclinations of the RS and MS, respectively, to the X-axis.

As shown in Fig.26 we need the equation of the normal through the point C_{ms} on the RS to obtain X_{ms} and Y_{ms} . The slope of the normal to the RS is $-1/G_{rs}$, where G_{rs} is the slope of the RS which is equal to $\tan \alpha_{rs}$. Therefore the equation of normal through C_{ms} is given

$$Y = -1/G_{rs} X + c \quad (4.1)$$

where c is a constant to be determined.

We insert the point coordinates into this equation

$$Y_{ms} = -1/G_{rs} X_{ms} + c \quad (4.2)$$

$$Y_{rs} = -1/G_{rs} X_{rs} + c \quad (4.3)$$

and

$$H_{ms}^2 = (x_{ms} - x_{rs})^2 + (y_{ms} - y_{rs})^2 \quad (4.4)$$

From equation (4.2) and (4.3) we can obtain

$$y_{ms} - y_{rs} = -\frac{1}{G_{rs}} (x_{ms} - x_{rs}) \quad (4.5)$$

We insert equation(4.5) into (4.4), and obtain

$$x_{ms} = x_{rs} \pm \frac{G_{rs} H_{ms}}{\sqrt{1 + G_{rs}^2}} \quad (4.6)$$

and

$$y_{ms} = y_{rs} \mp \frac{H_{ms}}{\sqrt{1 + G_{rs}^2}} \quad (4.7)$$

or alternatively

$$x_{ms} = x_{rs} \pm \xi \quad (4.8)$$

$$y_{ms} = y_{rs} \mp \eta \quad (4.9)$$

For the upper surface we take

$$x_{ms} = x_{rs} - \xi \quad y_{ms} = y_{rs} + \eta$$

and for the lower surface

$$x_{ms} = x_{rs} + \xi$$

$$y_{ms} = y_{rs} - \eta$$

Thus, given the slope of the RS and the required height H_{ms} from the RS (in the n-direction) we can find the coordinates of the MS (see Fig.26) from the nodes of the RS.

Since the potential flow calculation divides the surface on which the pressure is calculated (MS) into discrete straight line elements (see chapter 3) between defined "nodes", it is convenient to define the RS by a series of nodes and then define the nodes for the MS by means of the normals to RS at the RS nodes (see Fig.27). The RS slope at the Jth node is

obtained by taking the average of the slopes of the two segments on the RS which meet at the Jth node :

$$G_{rsJ} = \frac{1}{2} \left\{ \frac{y_{J+1} - y_J}{x_{J+1} - x_J} + \frac{y_J - y_{J-1}}{x_J - x_{J-1}} \right\} \quad (4.10)$$

where the X and Y coordinates refer to the nodes on the RS. Hence we define the MS nodes by specifying $H_{ms}(s)$ for all RS nodes (as shown in Fig.27).

(2) The matching surface in the near-TE region

The trailing edge region, which is defined roughly as the region containing about 5 percent of the airfoil chord and about the same distance in the wake from TE, needs special treatment. If curvilinear orthogonal (s,n) coordinates are used in the whole flow field, a singularity in the coordinate system (multi-valued s) occurs at the TE because of the discontinuity of the RS (s -axis) slope. It is necessary to provide a smooth transition from the (s,n) coordinate system of the boundary layer region to the (s,n) coordinate system of the wake. It would be inefficient to introduce a nonorthogonal coordinate system for this small region alone, simply to accommodate the singularity in geometry. Instead, we use the "wedge flow" transformation of potential theory to provide an orthogonal (but singular) grid and to improve numerical accuracy. The transformation $W = Z^n$ maps the whole of the W -plane (the "computational plane") to a region $-n/\pi < \theta < n/\pi$ in the Z -plane (the "physical plane"), and rectangular Cartesian coordinates (ϕ, ψ) , where $W = \phi + i\psi$, map to orthogonal curvilinear coordinates as shown in Fig.28.

As shown in Fig.28, we set up the XO, YO coordinate system and we define $Z = XO + iYO (=r(\cos \theta + i \sin \theta))$ for any

point in this system. Then we can obtain

$$\begin{aligned}
 W &= z^n \\
 &= (r(\cos \theta + i \sin \theta))^n \\
 &= r^n (\cos n\theta + i \sin n\theta)
 \end{aligned} \tag{4.11}$$

Therefore,

$$\begin{aligned}
 \varphi &= r^n \cos n\theta \\
 \psi &= r^n \sin n\theta
 \end{aligned} \tag{4.12}$$

The constant ψ -lines are streamlines of inviscid flow past the TE.

The transformed "streamlines" all asymptote to the wedge surface at an infinite distance upstream of the TE. However it is necessary for the streamwise coordinate lines to become straight and parallel - or at least to match the constant- n lines in a finite distance from the TE. This is arranged by patching the (s,n) system of the turbulent boundary layer and orthogonal curvilinear (φ,ψ) system of the trailing edge region together by using polynomial streamwise curves (see Fig.29). For conformity with the constant- n lines of the turbulent boundary layer, these curves have to match slope and curvature at the upstream end. Similarly downstream they must have the same slope and curvature as the wake coordinate lines (see Fig.29). In order to satisfy these minimum conditions quartic curves are used. If the slope of the $\psi=\text{const}$ curve at B is y' and the second derivative there is y'' , then the equation of the quartic is

$$y = y_1 + ax^4 + bx^3$$

and the boundary conditions are

$$\begin{aligned}
 \text{at } x_0=0 \quad y_0 &= y_1 \\
 \partial y_0 / \partial x_0 &= 0 \\
 \partial^2 y_0 / \partial x_0^2 &= 0
 \end{aligned}$$

$$\text{at } XO=L \quad YO=y_2$$

We find that, inserting the conditions at B ($XO=L$) :

$$\begin{aligned} YO'' &= 12aL^2 + 6bL \\ YO' &= 4aL^3 + 3bL^2 \end{aligned} \quad (4.13)$$

Hence we can obtain

$$\begin{aligned} a &= (LYO'' - 2YO')/4L^3 \\ b &= (3YO' - LYO'')/3L^2 \end{aligned} \quad (4.14)$$

$$y_2 = y_1 + aL^4 + bL^3 \quad (4.15)$$

We substitute equations (4.13) and (4.14) into (4.15)

$$y_1 = y_2 - \frac{1}{4}(LYO'' - 2YO') - \frac{1}{3}(3YO' - LYO'') \quad (4.16)$$

We need the value of YO'' and YO' to obtain the value of y_1 . As shown in Fig.31, $\psi=0$ lines are radially directed from the TE at angle π/n . Comparing (XO, YO) and (r, α) as the coordinate transformation, the inviscid flow field will now take a rotated form as follows :

$$W = \varphi + i\psi = r^n [\cos n(\alpha - \beta) + i \sin n(\alpha - \beta)]$$

so everything is shifted around the axis by β .

Expanding the cosine and sine terms ;

$$\begin{aligned} W &= r^n [(\cos n\alpha \cos n\beta + \sin n\alpha \sin n\beta) \\ &\quad + i(\sin n\alpha \cos n\beta - \cos n\alpha \sin n\beta)] \\ &= r^n [(\cos n\alpha + i \sin n\alpha) \\ &\quad (\cos n\beta - \sin n\beta)] \end{aligned}$$

$$\begin{aligned} \partial W / \partial Z &= n r^{n-1} [\{\cos (n-1)\alpha + i \sin (n-1)\alpha\} \{\cos n\beta - i \sin n\beta\}] \\ &= n r^{n-1} [\cos \{(n-1)\alpha - n\beta\} + i \sin \{(n-1)\alpha - n\beta\}] \\ &= -U + iV \end{aligned}$$

So we can obtain,

$$V/U = -\tan((n-1)\alpha - n\beta)$$

Considering the angle in terms of θ ,

$$\alpha = \pi - \theta, \quad \beta = \pi - \frac{\pi}{n}$$

$$V/U = \tan[(n-1)\theta]$$

For the second derivative (see Fig.30),

$$\begin{aligned} \left. \frac{\partial^2 y}{\partial x^2} \right|_{\psi} &= \frac{\partial \{\tan[(n-1)\theta]\}}{\partial \theta} \frac{\partial \theta}{\partial x} \\ \frac{\partial \{\tan[(n-1)\theta]\}}{\partial \theta} &= (n-1) [\sec^2(n-1)\theta] \\ \text{and} \quad \Delta C &= r \Delta \theta = H \sin(\theta + \alpha) \\ H &= \frac{\Delta x}{\cos \alpha} \\ \frac{\Delta \theta}{\Delta x} &= \frac{1}{r} \frac{\sin(\theta + \alpha)}{\cos \alpha} \\ \left. \frac{\partial^2 y}{\partial x^2} \right|_{\psi} &= \frac{1}{r} \frac{\sin(\alpha + \theta)}{\cos \alpha} (n-1) \sec^2(n-1)\theta \end{aligned}$$

We would prefer the patching curves to have no turning point in the region $0 < x_0 < L$. The turning point is found to be at

$$\frac{x_{0t}}{L} = \frac{LY_0'' - 3Y_0'}{LY_0'' - 2Y_0'}$$

If we plot x_{0t}/L vs the parameter LY_0''/Y_0' (see Fig.32), we find that the necessary and sufficient condition for no turning points in the region $0 < x_0 < L$ is $LY_0''/Y_0' < 3$. This condition forms the basis for choosing the initial φ -curves ($\varphi = \varphi_0$). This required quartic curve turns out to be highly degenerate and the whole coordinate transformation is simple to program. At present the coordinate system is symmetrical about the TE bisector (see Fig.33), which is slightly inefficient since the lower boundary layer is usually thinner.

(3) The initial guess for the matching surface

As mentioned before, we have to guess the initial MS for the first iteration of the interaction calculation; then, during the first sweep of the shear layer calculation

a new MS based on the δ^* or I6 value will be computed and this new MS value will be used for the next sweep (see Fig.33). Once MS changes become small the MS is fixed for subsequent sweeps. The initial MS is determined by using a parabolic fit from LE to the patching point of trailing edge region in the present program.

(4) The viscous/inviscid interaction procedure

In this section we describe how the matching procedure is applied numerically. The detailed flow chart and data flow are shown in Fig.35.

As mentioned in the introduction to this chapter, the external pressure distribution p_e is obtained from the potential flow calculation and used for shear layer calculation, while the shear calculation in turn produces the normal velocity distribution V_e (normal to the RS: in the transformation region the calculated U_e and V_e values are already parallel and normal to the MS). But the normal velocity V_{ms} values (normal to the MS) are needed in the potential calculation. Generally the MS and RS are not parallel and in that case (see Fig.34) V_{ms} is obtained from U_e and V_e through the expression :

$$V_{ms} = V_e \cos(\alpha_{ms} - \alpha_{rs}) - U_e \sin(\alpha_{ms} - \alpha_{rs}) \quad (4.17)$$

Now, the potential flow calculation calculates p_e (strictly, U_e) at the middle point of the MS elements, while V_{ms} for the potential flow calculation is obtained at the MS nodes, using equation (4.17) from the values of U_e and V_e at those nodes. In general there will be more shear layer calculation nodes than potential calculation nodes (see Fig.34), so that in order to obtain the U_e and V_e at the potential calculation nodes we need to interpolate from U_e

and V_e as given at shear layer calculation nodes. Moreover since the V_{ms} values at the mid-points of the elements of the MS are required for potential flow calculation, we need to interpolate from the V_{ms} calculated at the nodes.

(5) The general viscous/inviscid matching procedure

The general viscous/inviscid matching procedure is as follows:

- (i) Make an initial guess of the MS and of the velocity V_{ms} normal to the MS at the mid-points of the elements of the MS.
- (ii) Perform the potential flow calculation in region II (see Fig.31) to obtain the tangential velocity U_e and hence the external pressure distribution p_e on the MS at the mid-point of the elements, with $p_e = p - \frac{1}{2} (U_{ms}^2 + V_{ms}^2)$.
- (iii) Using the external pressure distribution p_e , in the SL calculation (including the wake) find V_e at the shear layer nodes, and calculate the pressure distribution across the viscous region by solving the normal momentum equation using the newly calculated U, V and τ values described in §2.3. Then change the MS height to be just outside the viscous layer if needed.
- (iv) Calculate the new V_{ms} on the MS at the mid-point from U_e and V_e values given at SL calculation stations by using interpolation. With these new V_{ms} values go to (ii) to recalculate potential flow and repeat steps (ii), (iii) and (iv) until convergence is achieved (see Fig.36).

(6) The underrelaxation and shifting of pressure

If we use the value of V_{ms} and p_e at the MS as matching quantities without underrelaxation, we obtain

oscillating values of V_{ms} near TE. In order to obtain convergent values, the underrelaxation for P_e and V_e will be required. If $V_{ms}^{(q-1)}$ is the value obtained from the $(q-1)$ th sweep and $V_{ms}^{(q)}$ is the value calculated from the q th sweep, then the actual value used as output from the q th sweep is

$$V_{ms} = \Omega V_{ms}^{(q)} + (1-\Omega) V_{ms}^{(q-1)} \quad (4.18)$$

where Ω is the "underrelaxation factor".

At present we use 0.3 as the underrelaxation factor for V_{ms} . The V_{ms} values from the SL calculation are used to determine the source distribution and this source distribution is in turn used for the pressure distribution on the MS. In order to accelerate convergence, the pressure values are underrelaxed. At present the underrelaxation factor of .3 is used. The new p_e value is

$$p_e = \Omega p_e^n + (1-\Omega) p_e^o \quad (4.19)$$

The pressure distribution within the viscous flow is underrelaxed as described in §2.3.

In the viscous/inviscid matching procedure, the iteration cycle gives the new pressure distribution p_e at MS for SL calculation. Since p_e changes from one sweep to the next, it will not match the pressure profile calculated within the shear layer from the previous sweep. We simply regard the SL calculation as providing $\partial p/\partial y$ and p_e as providing the constant of integration in deriving p from $\partial p/\partial y$. Hence we shift the pressure distribution while keeping the same $\partial p/\partial y$.

$$p = p + p_{e \text{ new}} - p_{e \text{ old}} \quad (4.20)$$

,so

$$p = p_e - \int \partial p / \partial y dy. \quad (4.21)$$

4.2 Description of Program for Matching Procedure.

We now describe the program used in the viscous/inviscid matching procedure. The matching program has been entirely rewritten in order to accomodate the new TE scheme. The detailed matching procedure, described in detail in §4.1, is similar to the original method (Mahgoub 1978). The matching is performed separately for each surface as described in §4.1. Below, we shall provide a brief description of each routine. The reader is referred to the relevant program listing (Appendix 6) and namelist (Appendix 5), to Fig.37 which shows the numerical implementation of the matching procedure of viscous/inviscid matching procedure.

(1) The Main Routine of MATCH

At the beginning of the run, as shown in Fig.35, some fixed data are read in together with the RS coordinates at the nodes. After the first sweep, the variables (mainly H_{ms} and U_e and V_e) obtained from the SL calculation (including the wake) are read in from data files MADATBL and MATDATW. These data are given at the SL calculation stations, and interpolated to obtain values at the inviscid calculation nodes.

In the next section of the routine, we calculate the coordinates of the RS and MS in the TE region in MESHTe. The matching surface for the boundary layer region and wake region are estimated, for the first sweep by using a

parabolic fit for the boundary layer region and lines parallel to the TE bisector line for the wake (see Fig. 39).

After setting up the matching surface the subroutine VMATCH is called to evaluate the velocity normal to the MS at the inviscid element mid-points. Generally the inviscid node points are not coincident with the shear layer node points and also the MS and the RS are not parallel. Hence by interpolating and resolving V_e and U_e (which are in s-n coordinates related to the RS) the normal velocity V_{ms} (normal to the MS) is evaluated at the mid-points of elements and then underrelaxed ready for the next inviscid flow calculation in VMATCH. The data which are needed for other programs are stored in disc data files (see Fig.11).

(2) The subroutine ARCLLEN

This subroutine calculates the arc length along the RS or MS surface, given the global coordinates of the surface. It assumes the surface is divided into piecewise linear segments, which is consistent with the potential flow calculation. This approximation could be improved if a higher order potential flow routine were used.

(3) The subroutine CURVAT

This subroutine evaluates the curvature of any surface, using a three-point parabolic fit: it accepts unequal intervals in the global coordinates in X.

(4) The subroutine GRADNT

This subroutine calculates the gradient of any surface, again using a three-point parabolic approximation.

(5) The subroutine MATCHSD

This subroutine matches the second derivative of the streamwise coordinate curves in the TE region. It ensures that the quartic curves generated in the patching region (see Fig.38) have the same second derivative as the curves generated by the transformation at the junction where they meet (which is the curve $\varphi = \varphi_0$).

(6) The subroutine MESHTE

This subroutine generates the coordinates of the MS in the TE region, given the coordinates of the RS there (see Fig.39). The coordinates of the MS are calculated relative to local coordinates, XLOC, YLOC for a typical "quartic", as shown in Fig.38, and then they are transformed to refer to global coordinates. Note that the calculation stations for both the boundary layer region and wake are made to coincide with the inviscid nodes in the TE region.

The subroutine starts by evaluating a minimum value of φ (the streamwise orthogonal curvilinear coordinate), and, by introducing a safety factor, the initial value of φ , is arrived at (see Fig.38). The value of XLOC on the RS, corresponding to φ_0 , is set to XLO. At this position we locate the first node inside the TE region: the choice of position for the rest of the nodes (a total of NXPTS elements in each "quartic") is entirely optional.

(7) The subroutine NORMAL

This subroutine calculates the coordinates (XSS, YSS) of a point which is a normal distance RNORM away from the surface whose global coordinates are designated by (XS, YS)-(see Fig.27).

(8) The subroutine PARAB

This subroutine fits a parabola through two points (whose coordinates are given), with zero slope specified at one of these points. The two points through which the parabola is passed have coordinates $(X(1), Y(1))$ and $(X(NP), YNP)$, where NP is the total number of points on the parabola. The X is a specified array and Y will contain corresponding values on the parabola. If IOPT=1, the parabola has zero slope at $(X(1), Y(1))$, while IOPT=2, then it has zero slope at $(X(NP), YNP)$.

(9) The subroutine VMATCH

This subroutine calculates the velocity normal to the MS at the MS element mid-points, given the velocities U_e (parallel) and V_e (normal) to the RS at the SL calculation node stations. In the first section we assume normal velocity as zero value for first sweep.

In the next section we calculate V_{ms} by using U_e and V_e which is written in the data files MADATBL and MATDATW in the SL calculation. For the laminar SL calculation region, the calculated values are given at MS node stations. The U_e and V_e from turbulent SL calculations are interpolated to get U_e and V_e at MS node stations. After obtaining U_e and V_e at MS node stations, V_{ms} are calculated by using equation (4.17). For the near TE region, the U_e and V_e values which are obtained from SL calculation are already normal to MS (see section §4.1).

(10) The subroutine XYMS

This subroutine contains the main code for

calculating the coordinates of the MS, given the RS coordinates. The coordinates of the MS are referred to global axes (X,Y). The generation of (X,Y) for the MS is done for the boundary layer region, TE region and wake region separately.

(11) The function

(i) SPLINE

This function is the same as in §2.5.

(ii) VALUE

This function is the same as in §2.5.

(iii) Y1PSI

This function gives the values of y_1 (see Fig.29), given the values of ψ , the normal ordinate.

4.3 Description of Operation/User's Guide for Viscous/Inviscid Iteration Cycle

In this section we shall describe the practical computer implementation of the viscous/inviscid iteration cycle, described in §4.1, on the ICCC Cyber system running under NOS 2. The flow diagram of iteration cycle is shown in Fig.37.

The running procedure of the viscous/inviscid iteration cycle has been arranged to facilitate calculation.

Each of the separation calculation program like potential program (AOMS), the shear layer program (TURB and WAKE), and matching program (MATCH) can be executed one after the other since all the "getting" and "replacing" of the variable data files are performed inside the program automatically, so

that no additional control statements are required. For a library function like SPLINE which is a NAG library routine, the command to execute is as follows :

```
Libfile,NAG11F/D.
```

```
Libfile,NAG11F
```

```
Library,NAG11F/A.
```

For the operation of the iteration cycle, the user merely needs to create fixed data files MATDAT1, AMODAT1, and MATDATW, BLDATT1/BLDATB1 (for the upper and lower boundary region), and INWAK as "local" disc files. The relevant namelist is given in Appendix 5 and we shall show how to run the program on the CDC computer at Imperial College in Appendix 6.

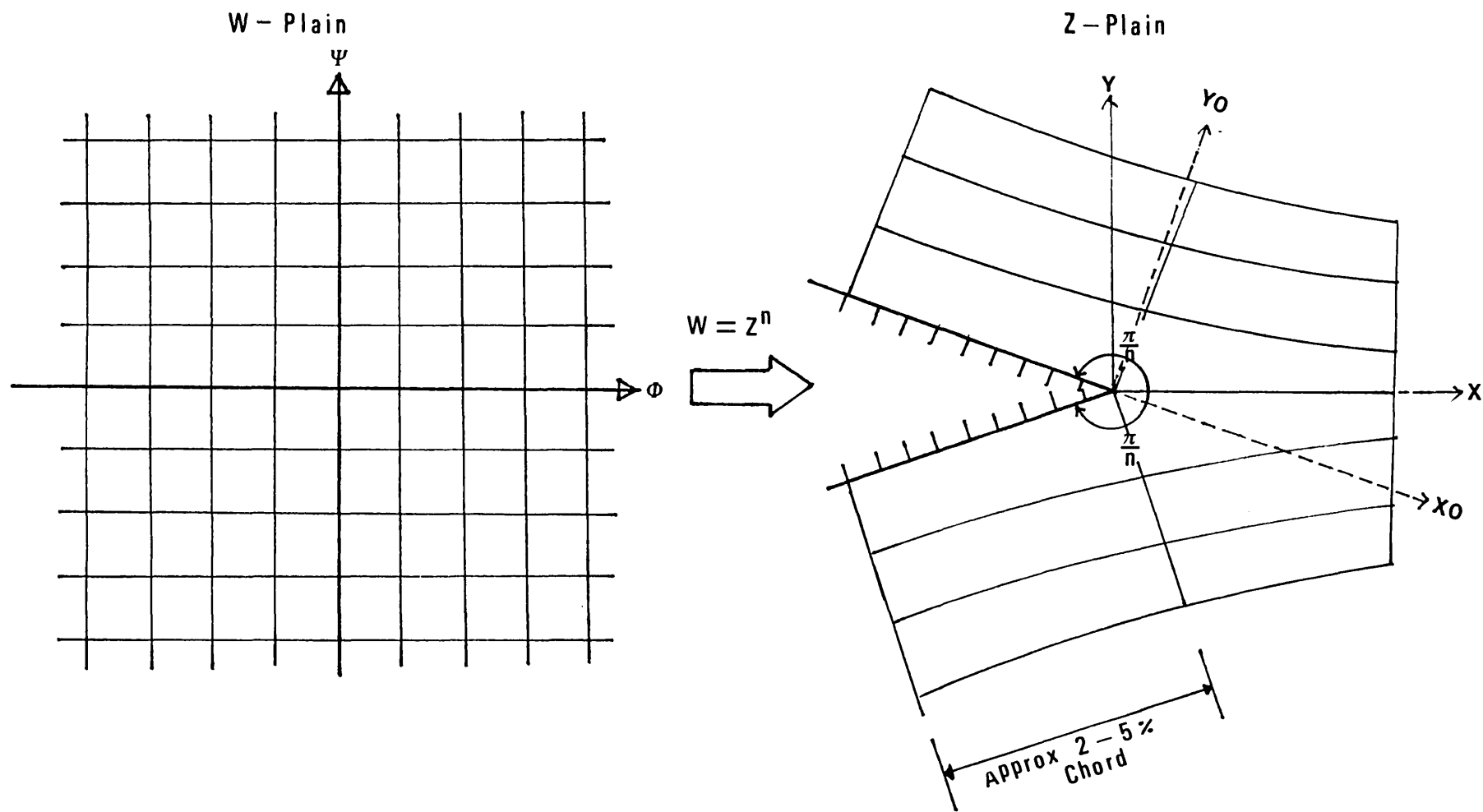


Fig 28 Coordinate Transformation

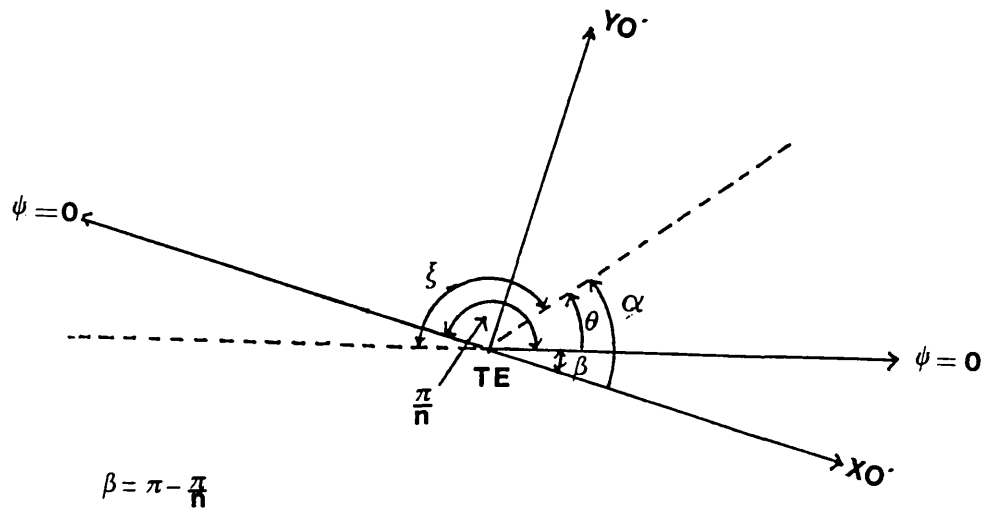


Fig 31 The Coordinate Systems

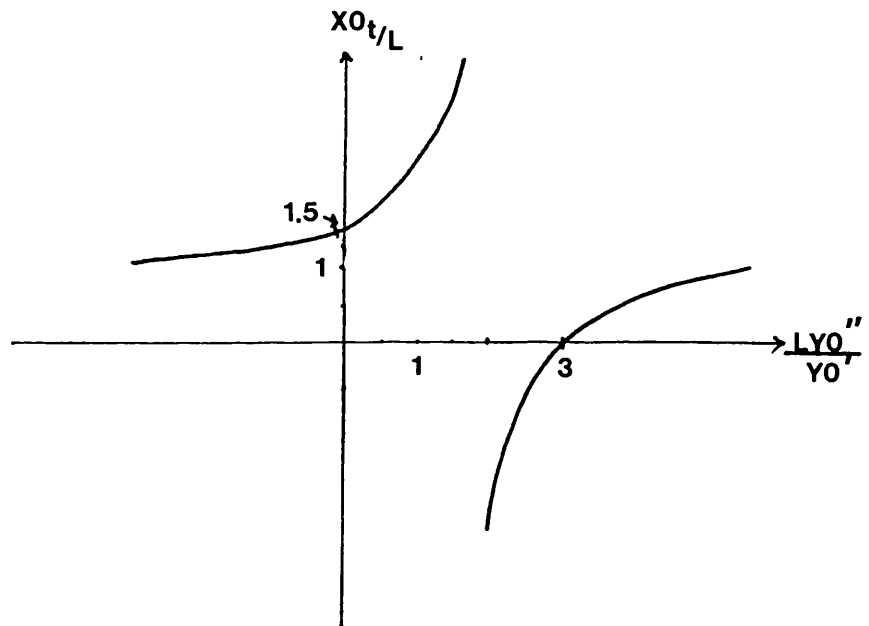


Fig 32 $\frac{X0t}{L}$ vs $\frac{LY0''}{Y0'}$

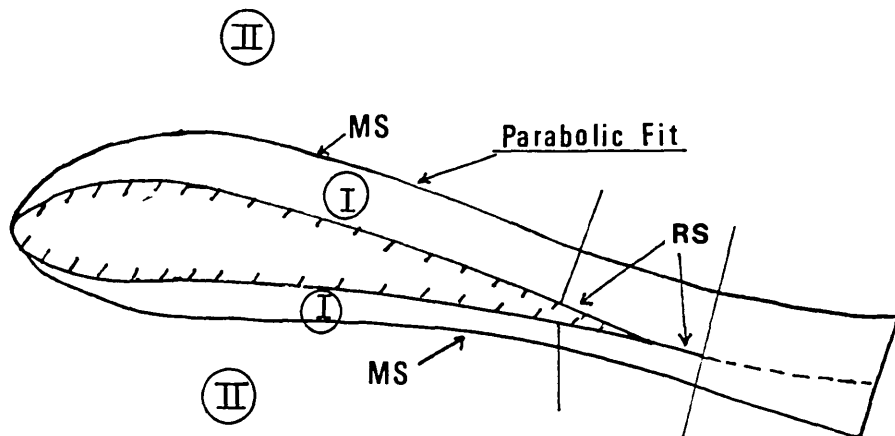


Fig 33 Initial Guess for Matching Surface

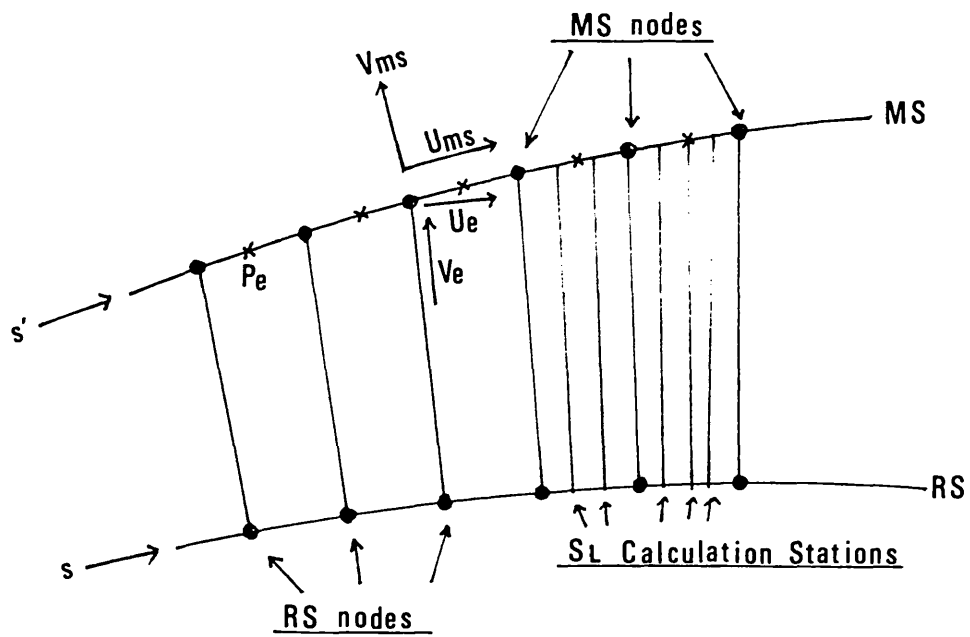


Fig 34 Numerical Scheme for Viscous Inviscid Matching

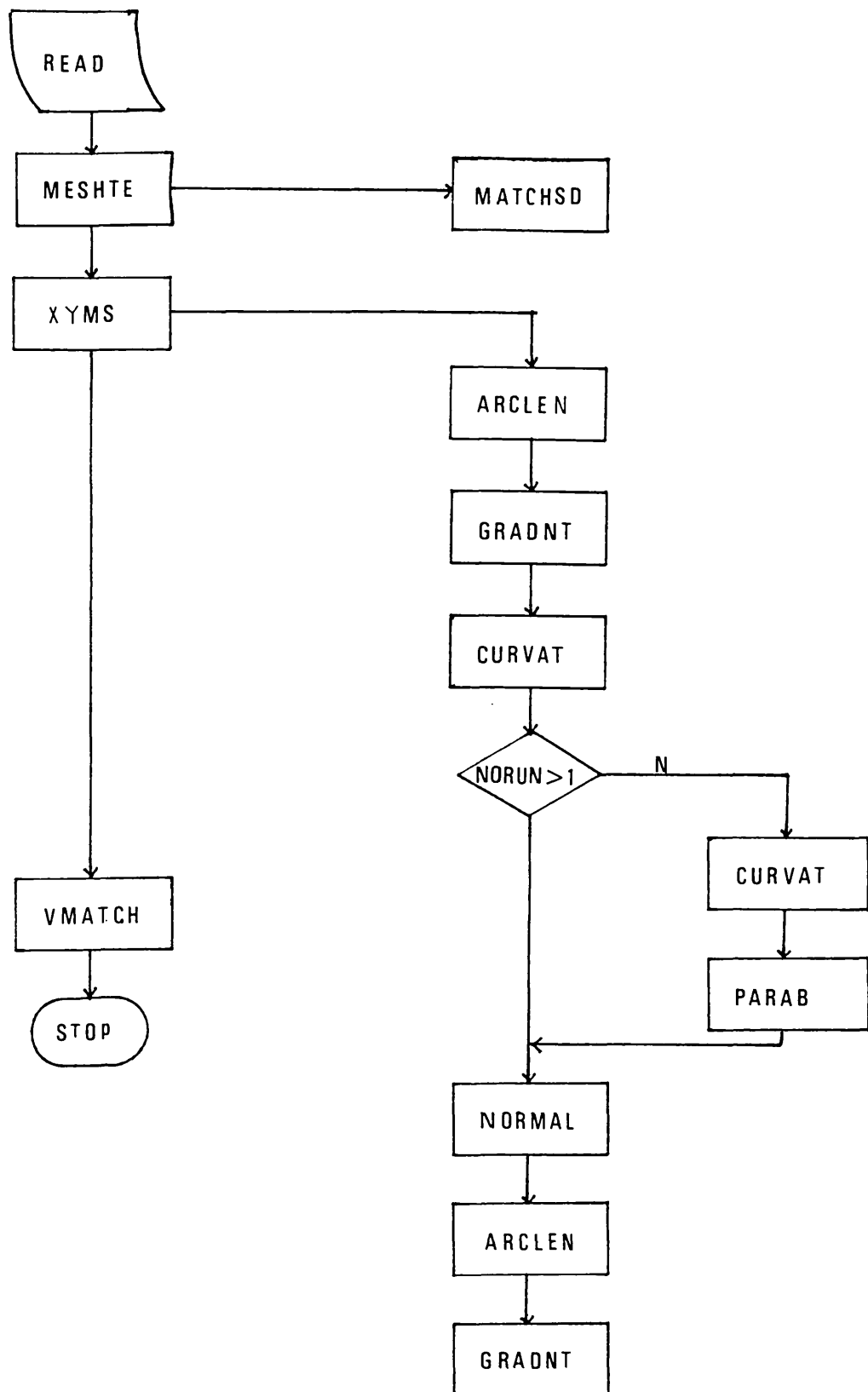


Fig 35 The Flow Chart of Match

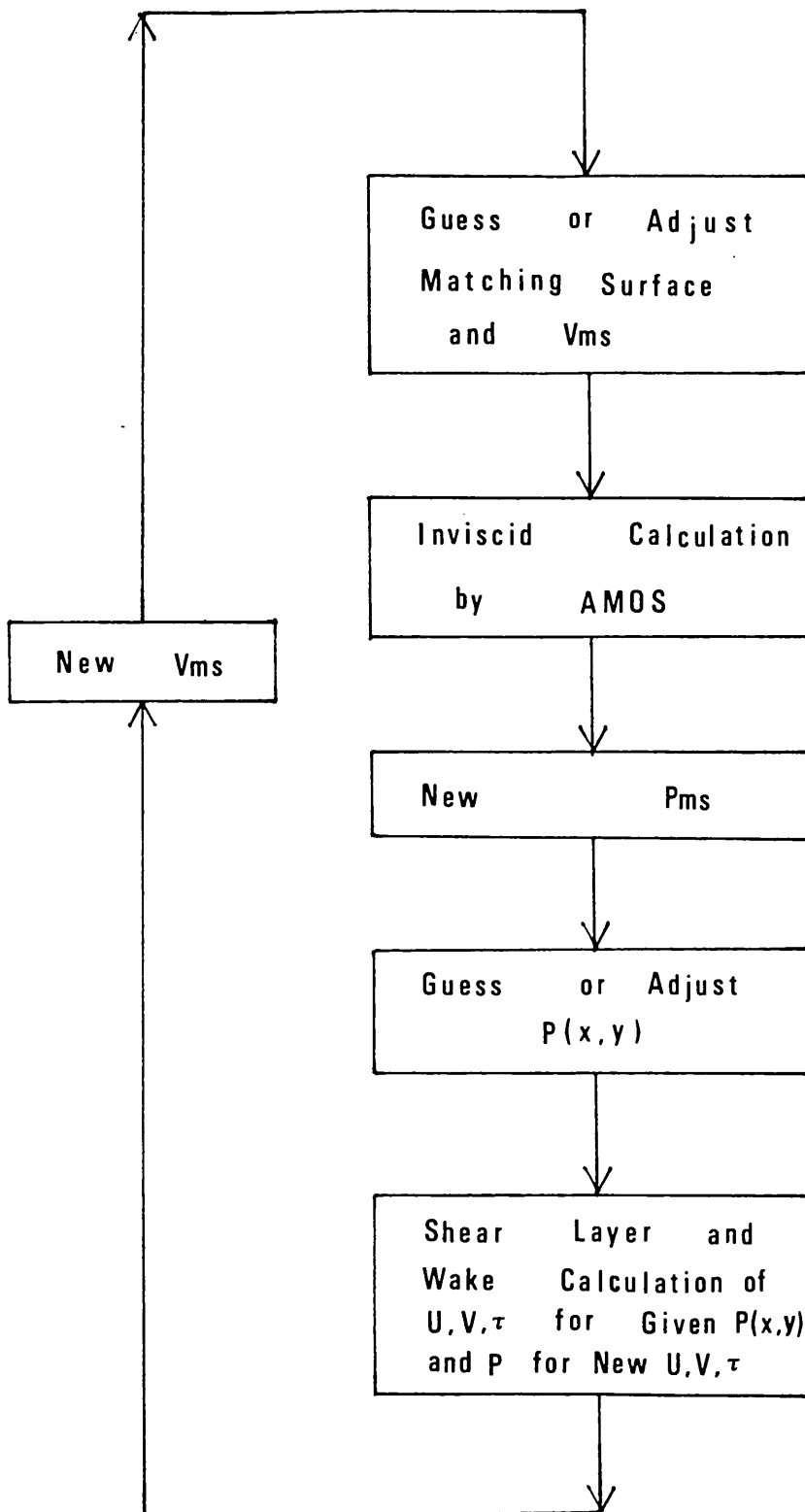


Fig 36 The Flow Chart of Matching Process

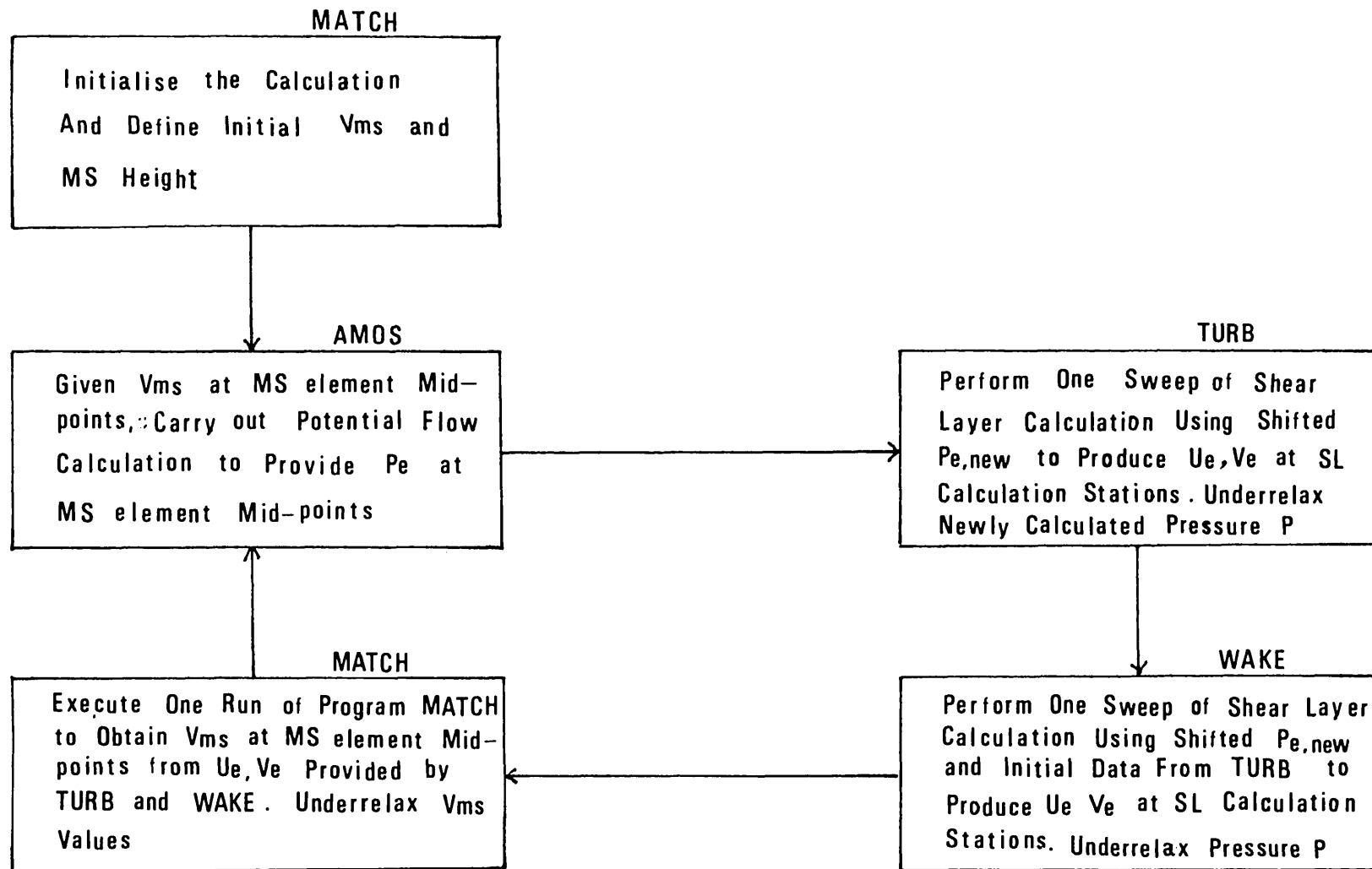
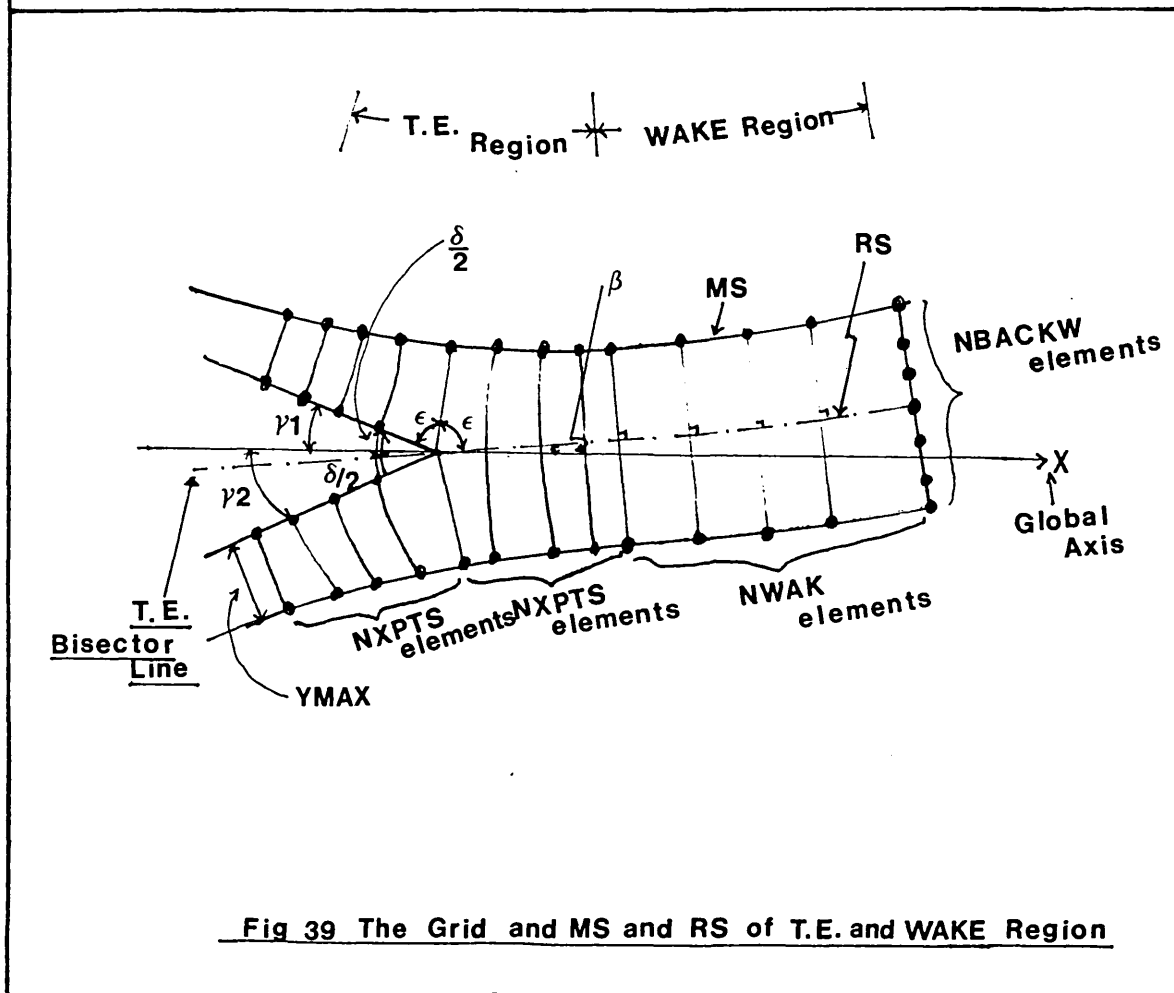


Fig 37 Flow Diagram of Viscous Inviscid Interaction Cycle



Chapter 5. Test Cases

In this section we shall consider several test cases to which the present method has been applied. The first case is the flow along the flat plate with different Reynolds numbers. The second is the symmetrical flow along an aerofoil: one is Hebbar's aerofoil (1986) and the other is the RAE101 aerofoil. The third is asymmetrical flow along the aerofoil. In the present program we choose Nakayama's model A (1984, 1985) as a asymmetrical flow test case.

The purpose of these test cases is to show the improved shear calculation method and the adequacy of the viscous/inviscid interaction calculation technique for a strong interaction region. Besides, experimental data is available for the above cases for verification of the eddy viscosity turbulence model in the inner wake region. In all cases the undisturbed velocity and aerofoil chord are used as reference velocity and length respectively. Note that typically only 78 nodes were used in the inviscid flow calculation, and that since the matching surface was held at the first guess (the wake MS being simply parallel-sided) the velocity normal to it varies in an erratic way even if nearby streamlines are quite smooth. Also $p_{ms} - p_{rs}$ depends on MS height.

5.1 Flat Plate

As the first test case we choose the flat plate. The heights of the matching surface and the real surface are shown in Fig.40. The Reynolds numbers of the tests are

6.8×10^6 and 5×10^6 . The transition point is forced by Reynolds number based on momentum thickness (the present program uses $Re_\theta = 600$), but any alternative transition criterion could be inserted. The transition point of flat plate is .15c. Figures 41 to 46 show the convergence of the present method, i.e., the converged wall pressure, the converged pressure and normal velocity at the MS. Generally we can obtain converged results in 12-15 sweeps within a tolerance of 10^{-2} in normal velocity at MS. The computing time for one iteration of viscous/inviscid interaction is approximately 50 sec. on the CDC 855 at Imperial College. The pressure values around .3c in Fig.43 does not seem to be changed between sweeps, but Fig. 43-A shows the variation of pressure value between sweeps. One obvious reason is that the pressure around that point is very close to the converged one. Another reason, as explained in chapter 4.1.5 and chapter 4.1.6, is that the newly calculated inviscid pressure distribution from new V_{ms} at the MS, which will be used as boundary conditions for next sweep shear layer calculation, are not matched with pressure distribution at MS used in previous viscous/inviscid interaction. Therefore we have to shift pressure distribution by following formula:

$$p(y) = p(y) + p_e^n - p_e^o$$

Fig.47 shows the pressure difference between MS and wall near the trailing edge. The pressure variation along the normal direction is quite big so that the conventional shear layer calculation method in which normal pressure gradient is assumed to be zero, is not adequate in this region.

The predicted shape factors and Reynolds shear stress in the wake (Fig.48 and Fig.49) are compared with experimental results (Andreopoulos 1980). The agreement in the near wake is quite satisfactory, which clearly shows that the eddy viscosity model (partly based on Andreopoulos' data) is indeed acceptable in near wake region. The predicted Reynolds shear stress increases in the far wake, because the eddy viscosity model in the inner wake region is not appropriate model for the far wake region. A more physically meaningful turbulence model is needed for the far wake region. Since the effect of the wake on the pressure distribution on an aerofoil is confined in the near wake region, the results are acceptable.

Fig 50 and 51 show the predicted skin friction along the flat plate, and Reynolds shear stress across the wake versus distance from TE. The C_f value near the TE slightly increases due to the favorable pressure gradient (Fig.47).

5.2 Symmetric aerofoil

As the second test case, we choose aerofoil studied by Hebbar to compare calculated results with that available from experiment in near wake region (1986) and the RAE 101 aerofoil (Brebner and Bagley, 1956). The coordinates of symmetric aerofoil (Hebbar 1986) are given in table 1. Note that results are plotted against s/c , not x/c , and s/c is slightly greater than unity at the TE.

(1) Hebbar's aerofoil

The convergence of the pressure at the wall and at the MS, and of the normal velocity at the MS, is shown in Fig. 52 to 54. The convergence rate is very fast and

monotonic. The computing time is just 40 sec. for one iteration of viscous/inviscid interaction. The Reynolds number of the test is 1×10^6 and the calculated pressure distributions at the wall and at the MS are shown in Fig. 55. From the LE to the transition point ($=.6615c$) we assumed the normal pressure gradient is zero, i.e. $\partial p / \partial y = 0$, so there is no difference between the pressure at the edge of the boundary layer and that at the wall. The Fig. 55A shows the pressure distribution across the wake. The predicted pressure distribution, especially in the region $.1c - .3c$ and $.7c - .9c$, does not agree with experimental work well (Hebbbar 1986). This may be caused by the obviously very rough coordinate values given by Hebbbar (1986), as shown in Table 1. From $.15c$ (90 mm) to $.17c$ (105 mm) and from $.2c$ (120 mm) to $.325c$ (195 mm) the Y coordinates are same. It gives an odd aerofoil shape. The results show a noticeable normal pressure gradient inside the shear layer near the TE (around $.03$). It also shows that the normal pressure gradient decreases along the wake region. Note that the s value of the RS and MS are same, because we use s-n coordinate based on RS except near TE. Fig. 55 shows that at transition, $s = .96c$, there is a rapid increase (that is, a jump) in the calculated MS pressure coefficient which is odd. This jump can be attributed to the patching method (see Chapter 2.4) employed at transition to transform (s,n) coordinates system to (ϕ, ψ) coordinates system at a very small downstream position. It also applies to other example on symmetrical aerofoil. The experimental results are given only up to $x = .95c$ so that we could not compare calculated results with experimental ones.

The predicted shape factor and velocity on the centre line of the MS in the wake are shown in Fig.56 and Fig.57. The agreement is very good except very near the TE.

This is because we use - provisionally - a rather crude estimate of the streamwise velocity and shear stress on the wake centre line at the first x-step after TE. in the current calculation these values are assumed to be .8 times the average U and τ values of upper and lower shear calculation at the first grid point above wall at TE, that

$$U = .8(U_u(1) + U_l(1))/2$$

$$\tau = .8(\tau_u(1) + \tau_l(1))/2$$

where U_u and τ_u are U and τ values at first grid point above the wall in the upper shear layer at TE and U_l and τ_l are the values in the lower shear layer.

The maximum Reynolds shear stress along the streamwise direction is shown in Fig.58, and the calculated Reynolds shear stress and streamwise velocity at centre of the MS are shown in Fig. 59 and Fig.60. The predicted Reynolds shear stress in near wake increases with x , but decreases in Hebbar's experimental result. In the far wake region, the actual mean strain is decreasing because of releasing wall boundary from TE: hence the production term in the turbulent kinetic energy equation is decreasing, and thus Reynolds shear stress should be decreased. There is some doubts about experimental results. Also it shows that the eddy viscosity model in the far wake region is not adequate.

(2) RAE 101 aerofoil

Fig.61 shows the coordinates of the MS and RS. The Reynolds number of test is 3×10^6 and the transition point is

around .36c. The convergence of the calculation is shown in Figs.62 to 64: convergence of pressure at the wall, and convergence of normal velocity and pressure at the MS. As mentioned in part (1) of this section, the computing time is about 40 sec. per viscous/inviscid interaction iteration on the CDC 855.

The predicted pressure distributions at the wall and at the MS are compared to experimental results in Fig. 65. Fig 65A shows the pressure distribution across the wake. This shows that the normal pressure gradient near TE is quite large (about .025) and also shows that the normal pressure gradient decreases along the wake region. The normal pressure gradients along the normal direction can be neglected except near the TE, where the streamlines are significantly curved and the viscous ("displacement") effect on inviscid flow is strong. Compared to experimental results, the present method underestimates the pressure distribution. The main reason maybe partly stem from the roughly given coordinates and partly restriction on the number of panels which could be used in inviscid calculation (due to the computing storage limitation). The predicted skin friction along the aerofoil, and the shape factor and the streamwise velocity on the centre line of the wake are shown in Figs.66 to 68.

5.3 Asymmetric aerofoil

We shall apply our present method to an asymmetric aerofoil for which Nakayama (1985) did experiments and calculations at McDonnell Douglas Company. Because the results, and even the aerofoil shape, are "company confidential", we could not obtain exact coordinates of the

aerofoil. Therefore the coordinates used in this calculation are obtained from the enlarged drawing of the figure of aerofoil given in the published paper, but the coordinates of the aerofoil deduced from the drawing are not very accurate.

The Reynolds number of the test is 1.2×10^6 and the transition point of the upper part of the shear layer is .695c and that of the lower part is .653c. The shape of the MS and RS are shown in Fig.69, but several kinks near LE and TE are found, and it is very difficult to smooth out coordinate values derived from the small scale published sketch. Convergence of the calculation is shown in Figs.70 to 72: in which are plotted the convergence of the pressure at the wall and the pressure and normal velocity on the MS. It takes 12 sweeps and the computing time is around 600 sec. on the CDC 855 at Imperial College to obtain converged value within tolerance 10^{-2} in normal velocity at MS.

The predicted pressure distribution of present method and experimental results (Nakayama 1985,1984) for upper and lower part of aerofoil are shown in Fig.73-A and Fig.73-B. Except near LE, the predicted results are in quite good agreement with experimental results. In the LE region, the inviscid calculation result shows better quantitative agreement with the experimental result as compared to the predicted one. Note that the inviscid pressure result is obtained by using RS coordinates, while the present C_p values are obtained by using MS coordinates. The discrepancies in the predicted result are contributed partly by the eccentricity of the stagnation point (see chapter 3.1.4) and partly by the neglect of curvature effect which is the same as neglecting the normal pressure

gradient. The improper choice of the MS near LE is also responsible for the discrepancies. The MS is set up by parabolic fit from LE to the patching point of turbulent BL in the present program. Since the patching point is quite far from LE, the MS obtained from the parabolic curve fit may be improper near LE because the real BL thickness has infinite slope at the stagnation point whereas the parabolic fit gives a finite gradient. Hence, there is a possibility that the error between MS and boundary layer thickness may induce error in the C_p calculations. To get proper MS which fits laminar stagnation point BL at $s=0$ and the turbulent BL at $s=s_{int}$ and varies proportional to $s^{1/2}$, we calculate momentum thickness by using Thwaites formula and get δ_{turb} and then get MS from following formula:

$$y_{ms} = \theta_1(s) \frac{\delta_{turb}(s = s_{trans})}{\theta_1(s = s_{trans})}$$

Another area which may contribute towards discrepancies is the inaccuracy in obtaining the aerofoil coordinates (the data used were extracted directly from published graphs because the original data are classified as company confidential).

The stagnation point is .0027c along the upper surface in the aerofoil at present calculation. Since the edge pressure in LE region is obtained at the MS, the neglect of normal pressure gradient means that the calculated wall pressure in the present method is greater than the actual value. This can be easily shown by using centrifugal approximation for normal pressure gradient in the normal momentum equation:

$$\partial p / \partial y = U^2 / R$$

where R is the radius of curvature of a streamline and U is the streamwise velocity. Since R in the LE is positive, $\partial p / \partial y$ is positive as well. This means that the pressure at MS is greater than the pressure at the wall. This suggests that a better approximation for the normal pressure gradient is needed in the LE region to get better result. For example, the difference of pressure coefficient between edge and wall in the LE region, which is about .05 on the lower surface and .03 on the upper surface of the aerofoil should be used instead of $\partial p / \partial y = 0$ (see Fig.73A and 73B). Note that although the present program shows some discrepancies near LE (low Reynolds numbers or laminar flow region), its overall effect on the aerofoil performance is small because the discrepancies are confined only to a small region of the whole computational domain. Also the effect would be negligible at flight R_e where transition occurs near the LE.

Generally, a viscous flow has smaller lift than predicted by inviscid flow calculations. Although our pressure coefficient results agree fairly well with experiment, the viscous flow calculation gives more lift than the inviscid flow calculation, as shown in Fig.73A and 73B. We use the wedge flow transformation to generate the mesh near TE and we applied the Kutta condition at $1.5c$ from LE. The upper and lower matching surfaces are parallel to the TE bisector. The calculated velocity on the MS in the upper surface of the wake is larger than in the lower surface, which means larger circulation than in inviscid flow where the velocities on the upper and lower surface are assumed same and this in turn means larger lift coefficient.

For a thin layer of curved inviscid flow we have : $\Delta p / \rho U^2 = \kappa(\Delta z)$, where Δz is wake thickness and κ is curvature of streamline. Near $x=1.1c$ the $\Delta p / \rho U^2 = .023$ suggesting a flow curvature of more than .38. This is much more than measured ($=.075$). Hence calculated ΔU ($=.023$) is larger than experimental result (Fig.74) which the ΔU is almost zero. It appears that this overestimation of curvature near TE may be caused by a bug or coding error in the wake program : further work is planned to check this.

Note that the pressure difference between inviscid flow and the present calculation is quite big near the TE. The output of inviscid flow results stops at the upstream of the TE because the tangential velocity calculation is performed at the mid-point of the element (see ch 3). At the TE ($s=1.023c$ in Fig.73A and $s=1.003c$ in Fig. 73B) the C_p value of inviscid calculation is equal to unity. The streamwise velocity at MS and skin friction are shown in Figs.74 and 75

with experimental results. The present method shows overestimation of the streamwise velocity on both surface near the TE - for instance, the difference of normalised streamwise velocity around $x=.7c$ is .015. The measured boundary layer thickness on the upper surface is .015c and our matching surface height is .03c near $x=.7c$. Hence the edge velocity as defined in the experimental work is at $y=.015c$ and our streamwise velocity is at $y=.03c$. The normal pressure gradient of our calculation is $\partial p / \partial (y/h) = .01$ where h is MS height and p is nondimensionalised by ρU^2 , so $p = .01(y/h) + p_w$. Hence $C_{p,015c} = 2(.01/2) + C_{pw}$

$$C_{p,03c} = 2(.01) + C_{pw}$$

From Bernoulli's equation, $\Delta U = \sqrt{1 - C_{p,05c}} - \sqrt{1 - C_{p,03c}} = .005$.

The difference of streamwise velocity between calculation

and experiment is still .01 which is 1 percent of error. In the experiment there may be measurement error.

It also shows the predicted streamwise velocity on the lower surface is larger than that on the upper surface near TE, but the trends are correct: probably the difference stems from the kink in the lower-surface coordinates near TE as shown in Fig.69: again, we regret that accurate coordinates for this vital test case are not available.

The calculated skin friction, the shape factor, and the Reynolds shear stress and streamwise velocity at the centre line of the MS in the wake are shown in Figs.76 to 79. Fig.80 shows the pressure distribution at $X=1.05c$. There is difference between calculated (by Nakayama, 1984 and present method) and experimental results. Probably, it is caused partly from short wake calculation region : we stop the wake calculation up to .5c from TE. Fig. 79 shows variation of minimum streamwise velocity. Fig.81 shows the pressure across the wake at different streamwise distance from TE. It shows consistence with streamwise velocity difference across the wake (see Fig.79 and Fig.74). The normal pressure gradient is increased in the near wake then gradually disappears further downstream - maybe this increase of normal pressure gradient near wake is caused by panel size in the inviscid flow calculation. The panel size is dense near TE, but after .1c from TE in the wake the size is almost doubled due to restrictions of computer memory and this may have caused the wiggle in the pressure calculation near wake as shown in Fig.81 (private communication from Dr.J.M.R. Graham). It shows that the normal pressure gradient ($\partial p / \partial y$) is large in the inner wake. Hence our results strongly suggest that the normal pressure gradient should be accounted for near the TE, in any calculation.

Table 5.1 Coordinates of Symmetric Aerofoil (Ref 70)

X (mm)	Y (mm)	X (mm)	Y (mm)
0.	0.	315	31.0
15	14.5	330	30.0
30	23.5	345	28.5
45	28.5	360	27.5
60	32	375	26.5
75	34.5	390	25.0
90	36.5	405	23.5
105	36.5	420	21.5
120	38.0	435	20.5
135	38.0	450	19.0
150	38.0	465	17.0
165	38.0	480	15.5
180	38.0	495	14.0
195	38.0	510	12.0
210	37.5	525	10.5
225	37.0	540	9.0
240	36.0	555	7.0
255	35.0	570	4.5
270	34.0	585	2.5
285	33.5	600	.75
300	32.5	leading edge radius=5.5mm	

Table 5.2 Coordinates of Asymmetric Aerofoil (Ref 70)

	Upper	Lower
X	Y	Y
0.	0.	.0
.005	.0127	-.0072
.013	.0186	-.0127
.025	.0259	-.0182
.055	.037	-.0233
.096	.0470	-.029
.147	.0538	-.0335
.206	.0591	-.0366
.273	.065	-.0388
.346	.0678	-.0383
.422	.0673	-.0341
.5	.0638	-.0295
.578	.0583	-.0251
.645	.0526	-.0214
.715	.0441	-.0175
.78	.0347	-.0139
.839	.0252	-.0108
.875	.0193	-.0088
.906	.0142	-.0071
.93	.0101	-.0058
.95	.0069	-.0048
.965	.0048	-.0033
.982	.0025	-.0017
1.0	0.	

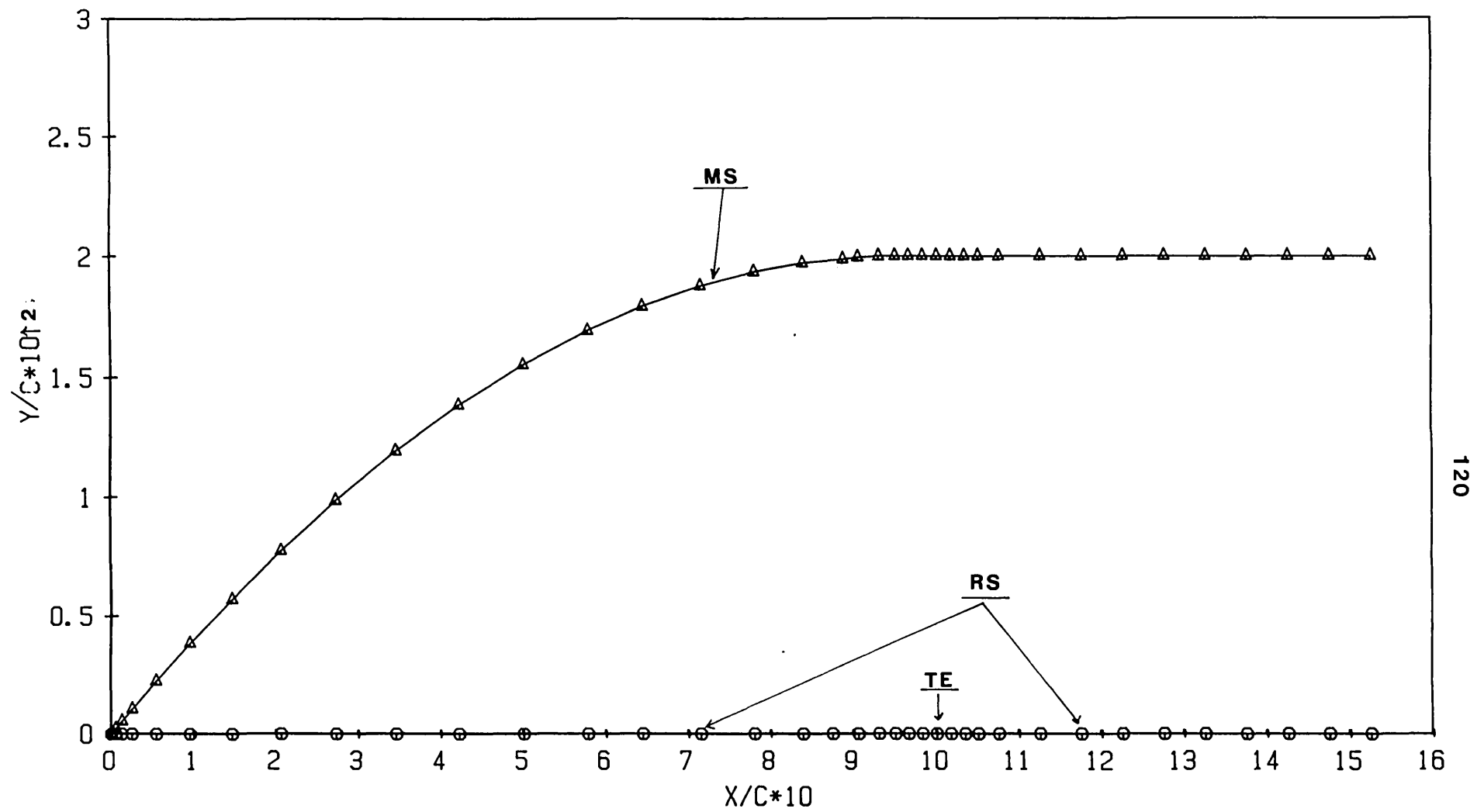


Fig. 40 COORDINATES OF RS AND MS OF FLAT PLATE

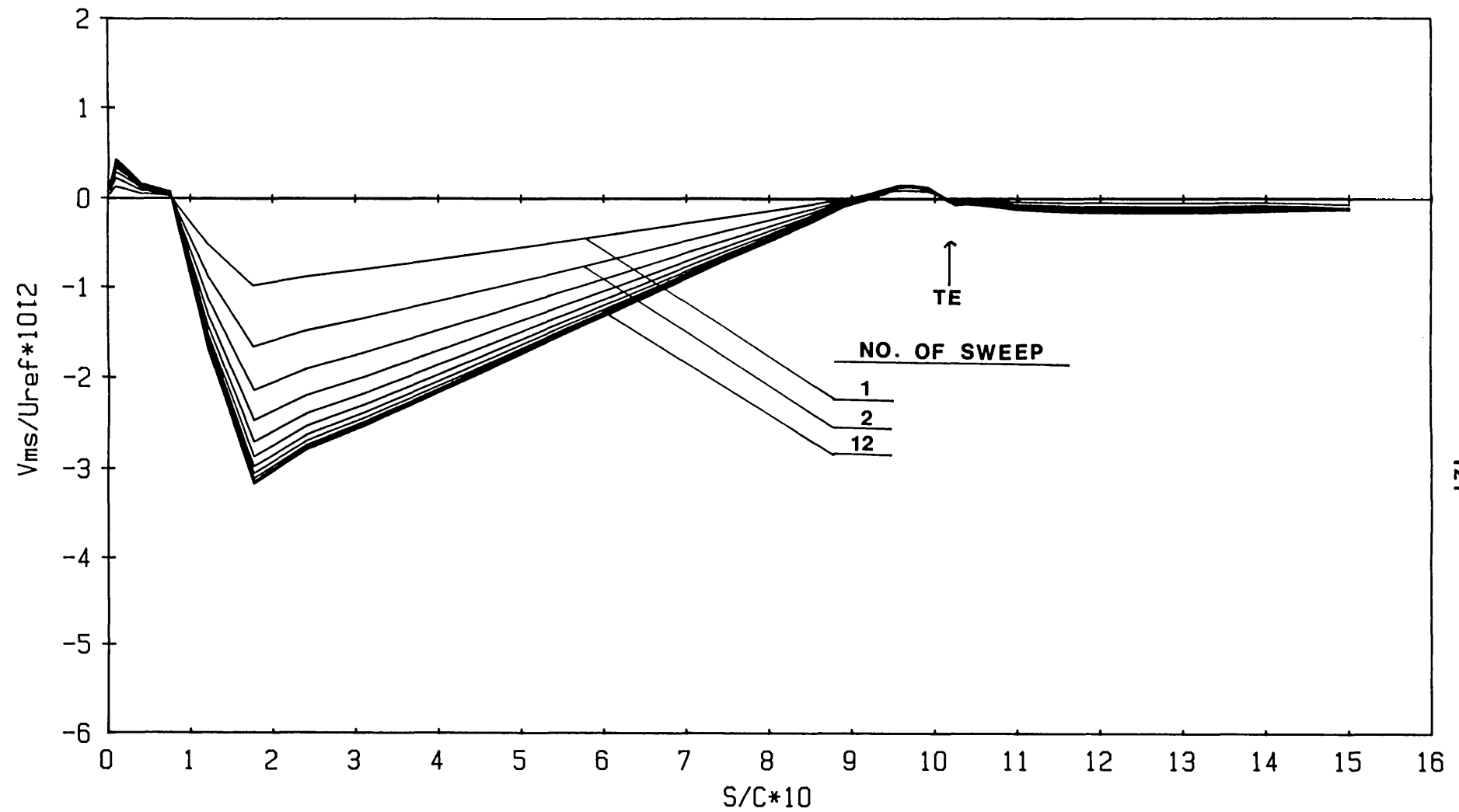


Fig. 41 CONVERGENCE OF NORMAL VELOCITY AT MS OF FLAT PLATE ($Re=6.8 \cdot 10^6$)

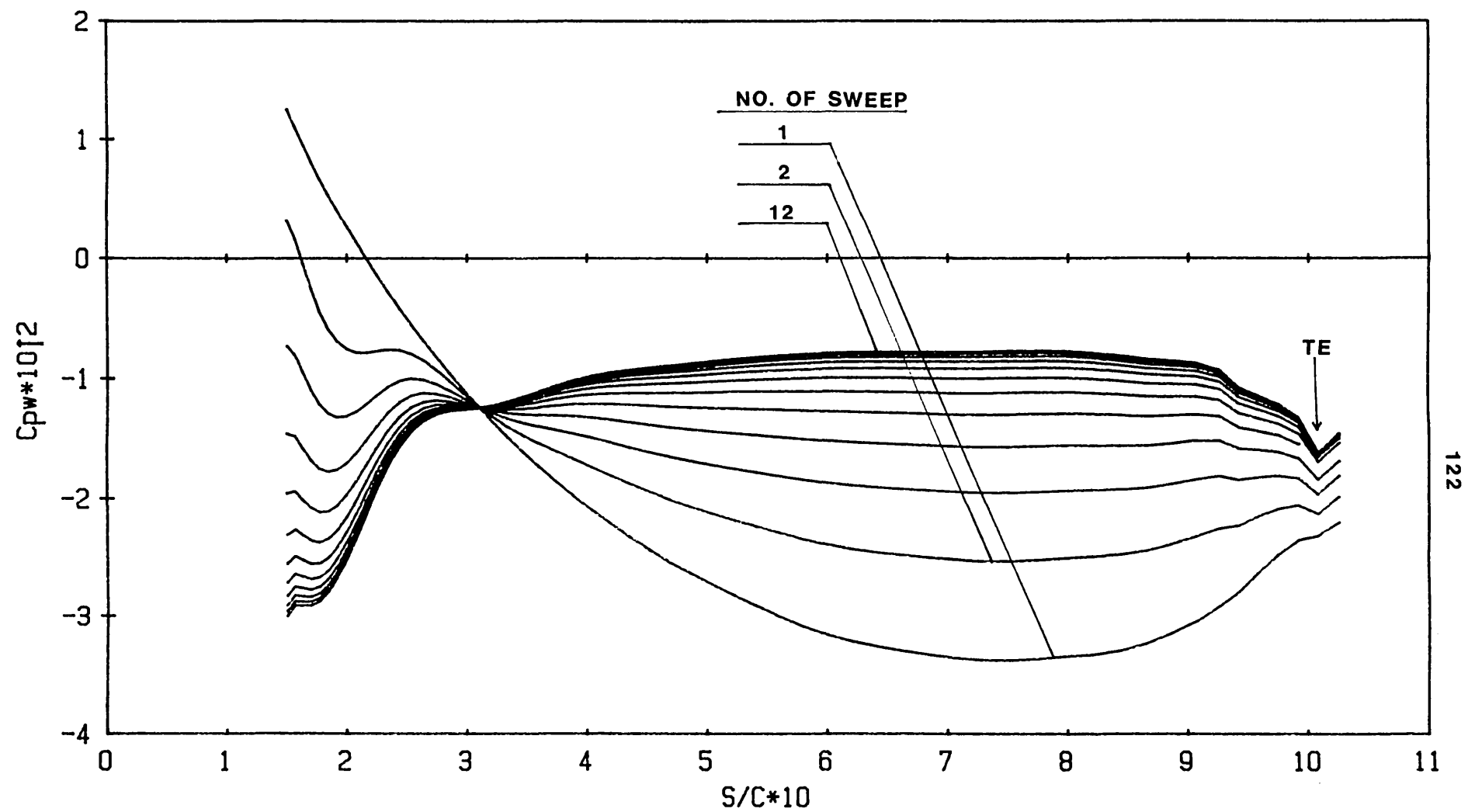


Fig.42 CONVERGENCE OF C_p AT WALL OF FLAT PLATE ($Re=6.8 \times 10^6$)

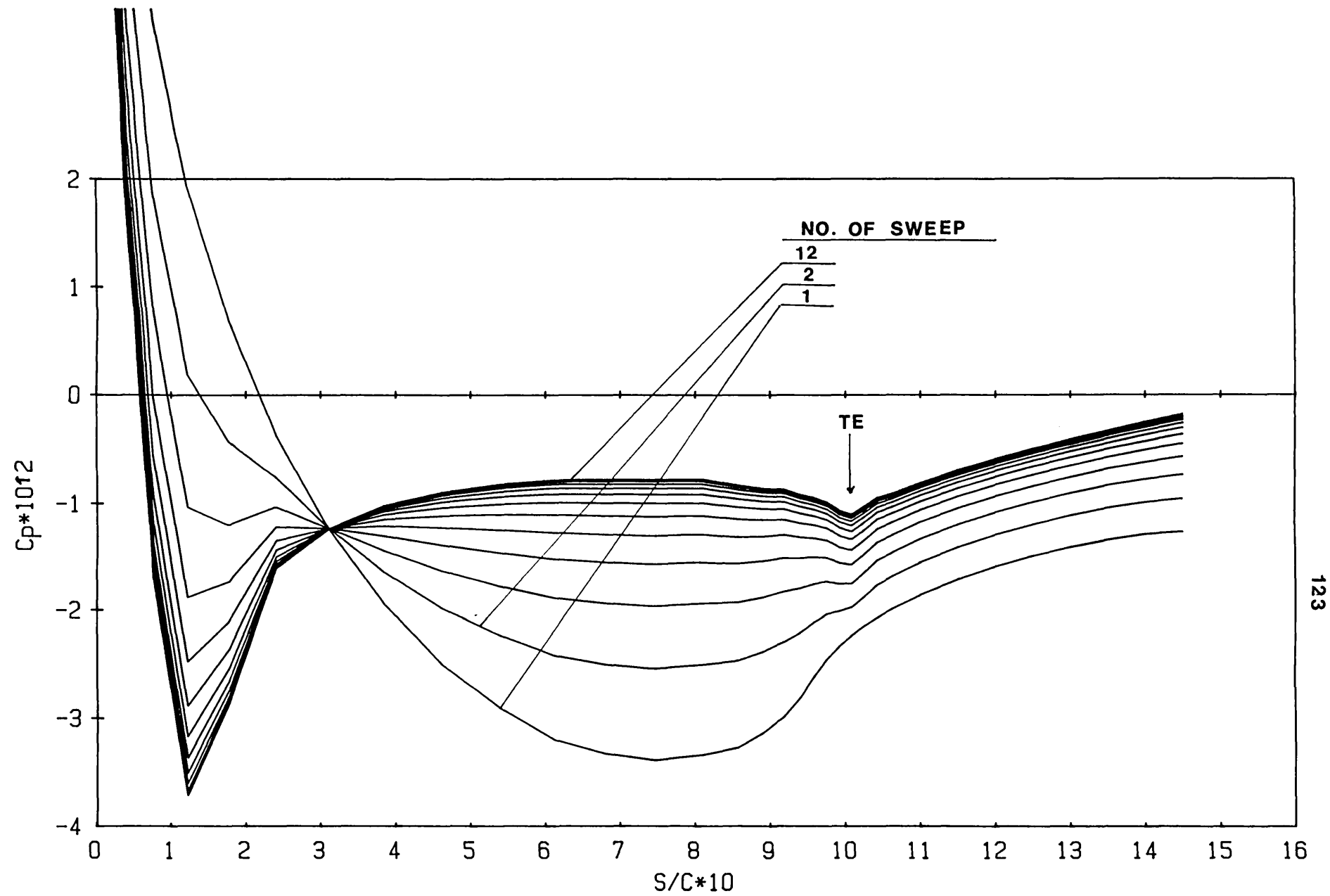


Fig. 43 CONVERGENCE OF C_p AT MS OF FLAT PLATE ($Re = 6.8 \cdot 10^6$)

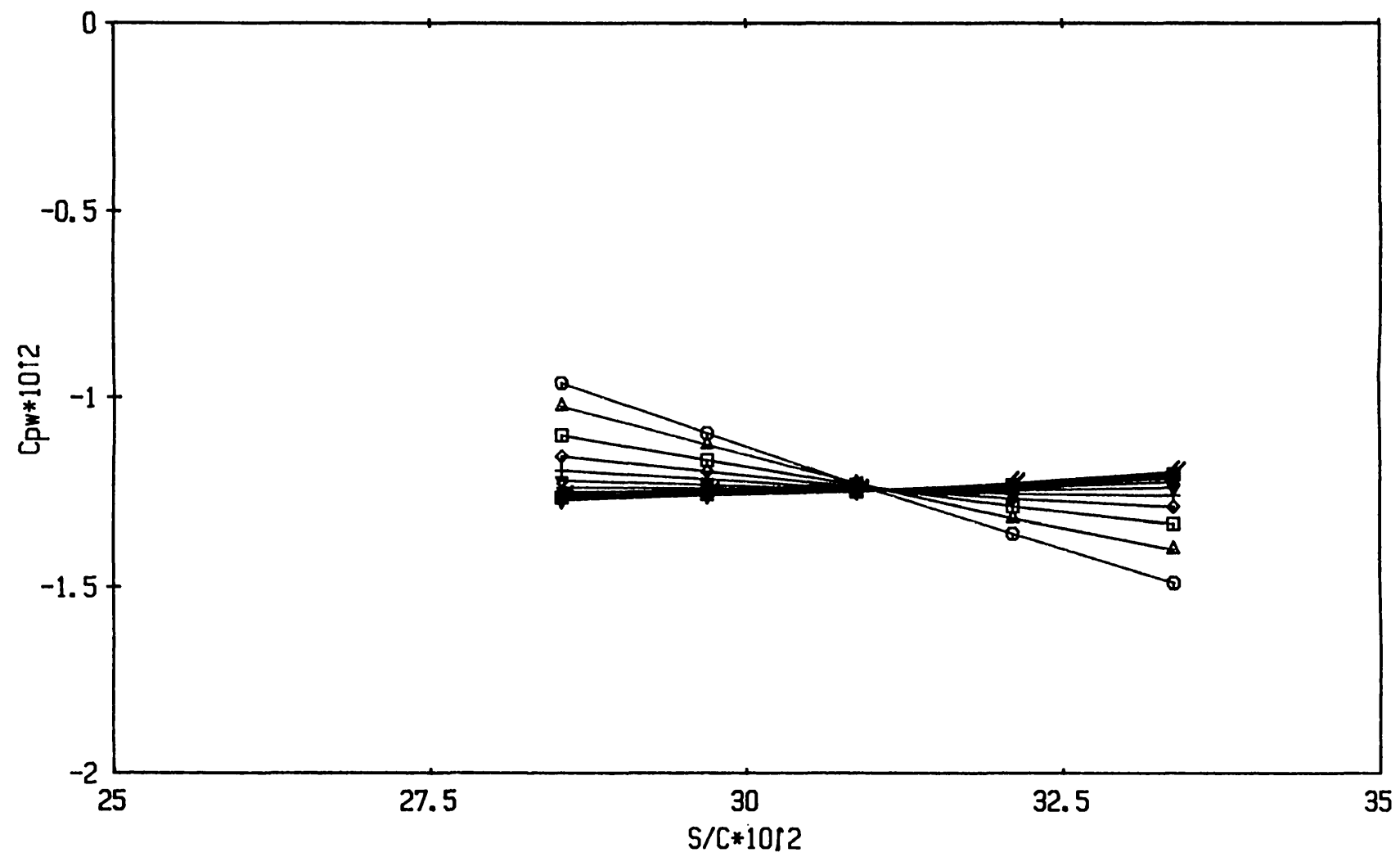


Fig.43A PRESSURE DISTRIBUTION AT WALL NEAR .3C

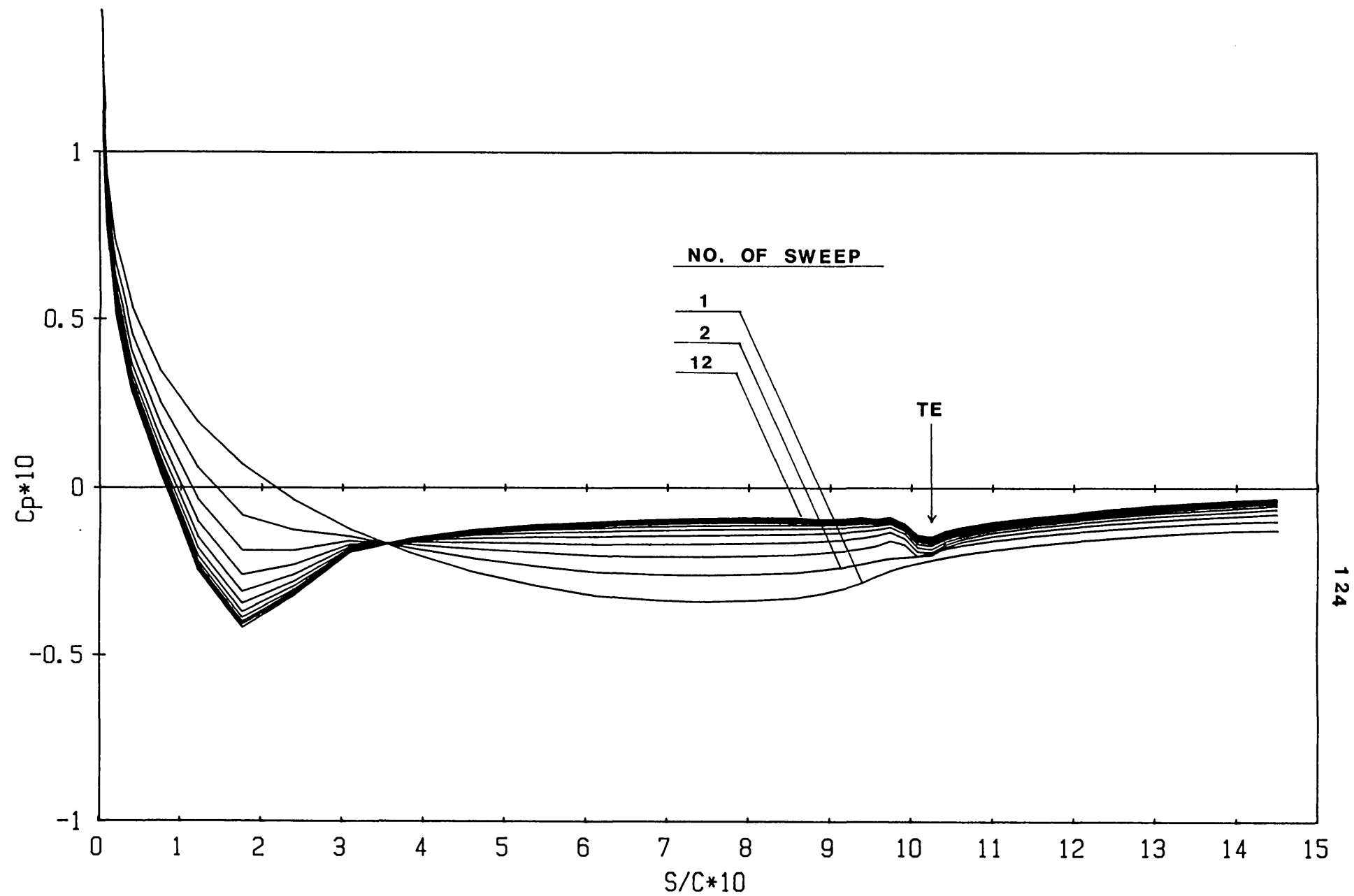


Fig. 44 CONVERGENCE OF C_p AT MS OF FLAT PLATE ($Re=5 \cdot 10^6$)

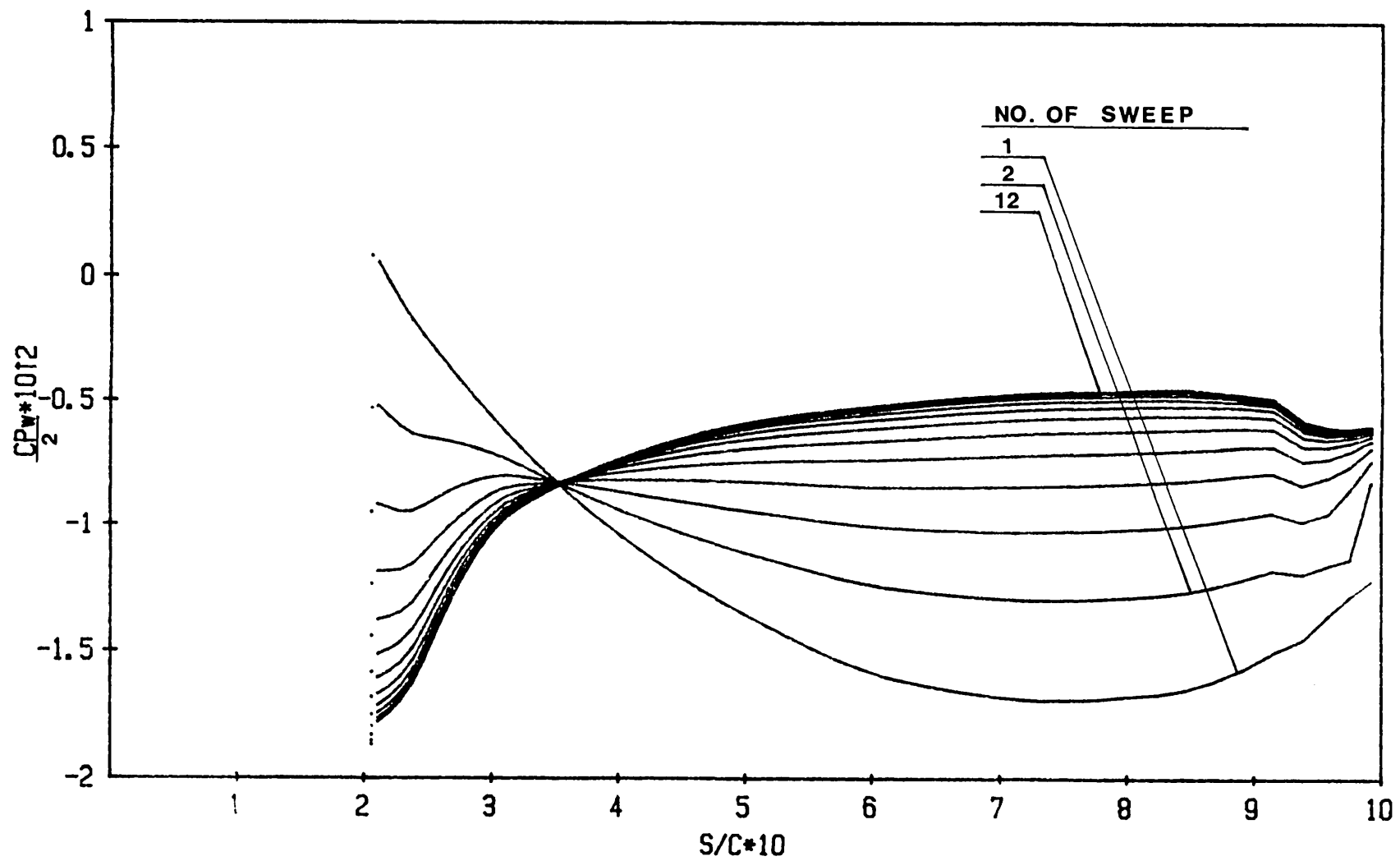


Fig.45 PRESSURE DISTRIBUTION AT WALL FOR FLAT PLATE(1-12 SWEEP) $Re = 5.0 \times 10^6$

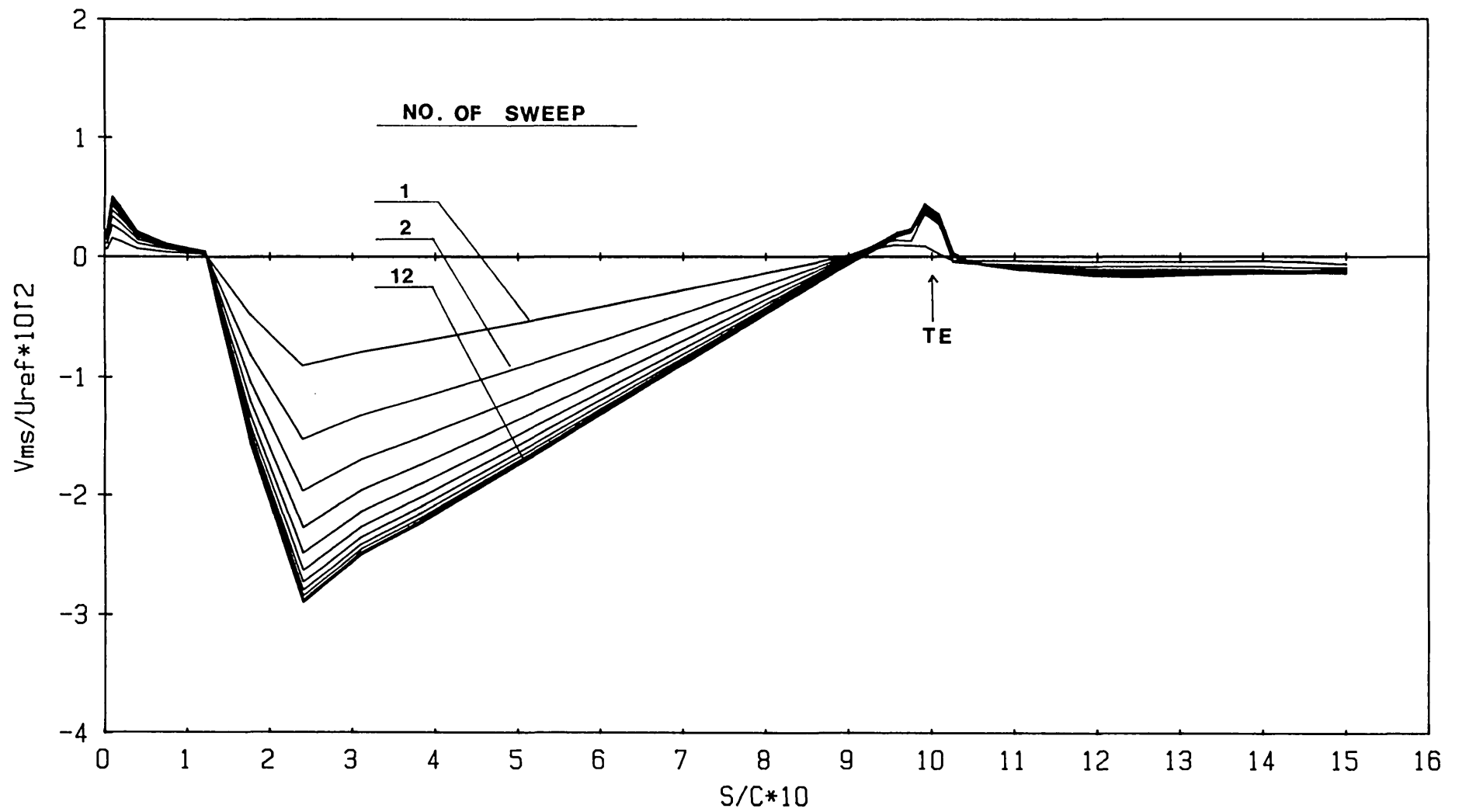


Fig.46 CONVERGENCE OF NORMAL VELOCITY AT MS OF FLAT PLATE ($Re=5.0 \cdot 10^6$)

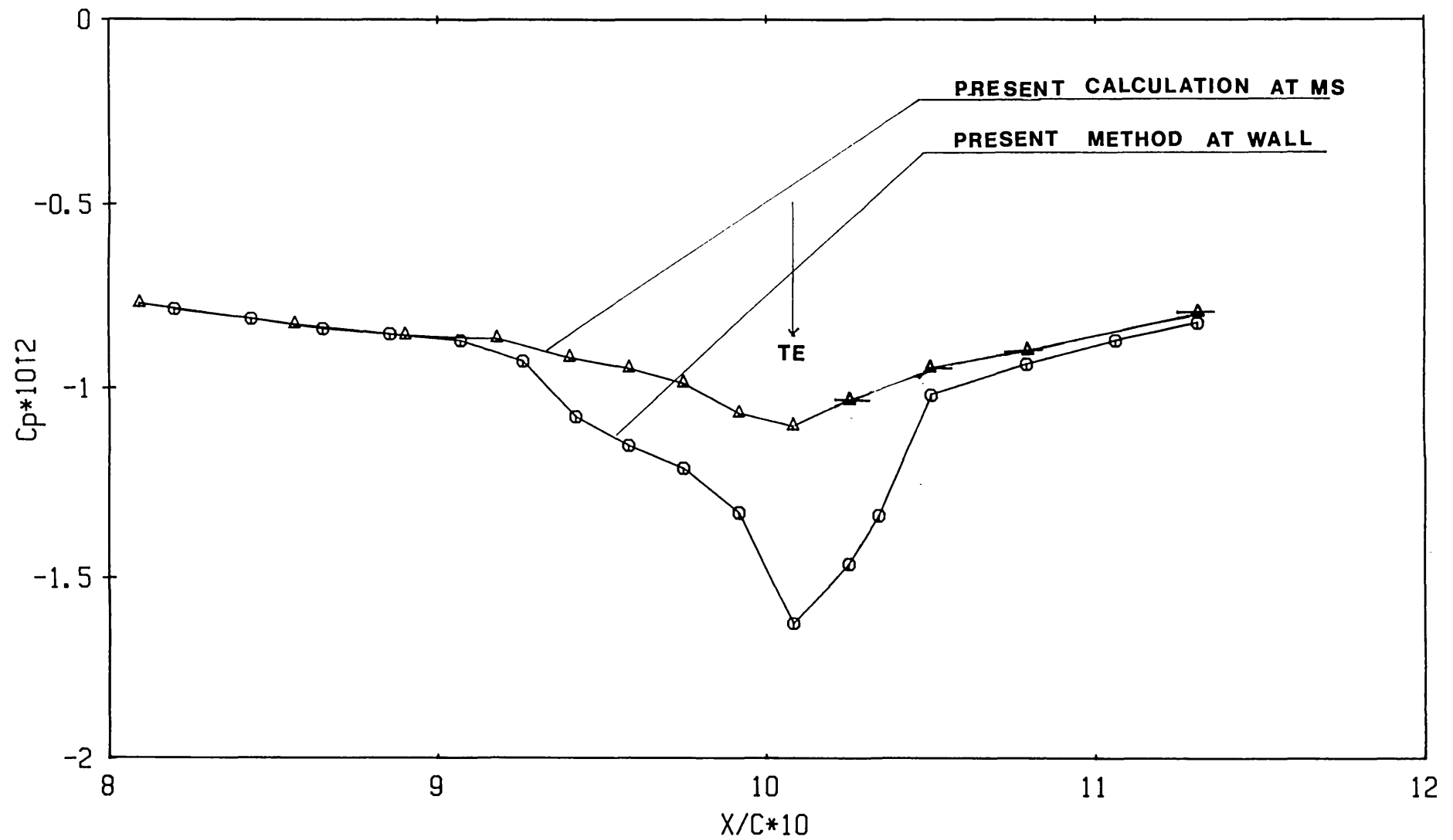


Fig.47 PRESSURE DISTRIBUTION NEAR TE OF FLAT PLATE ($Re=6.8 \cdot 10^6$)

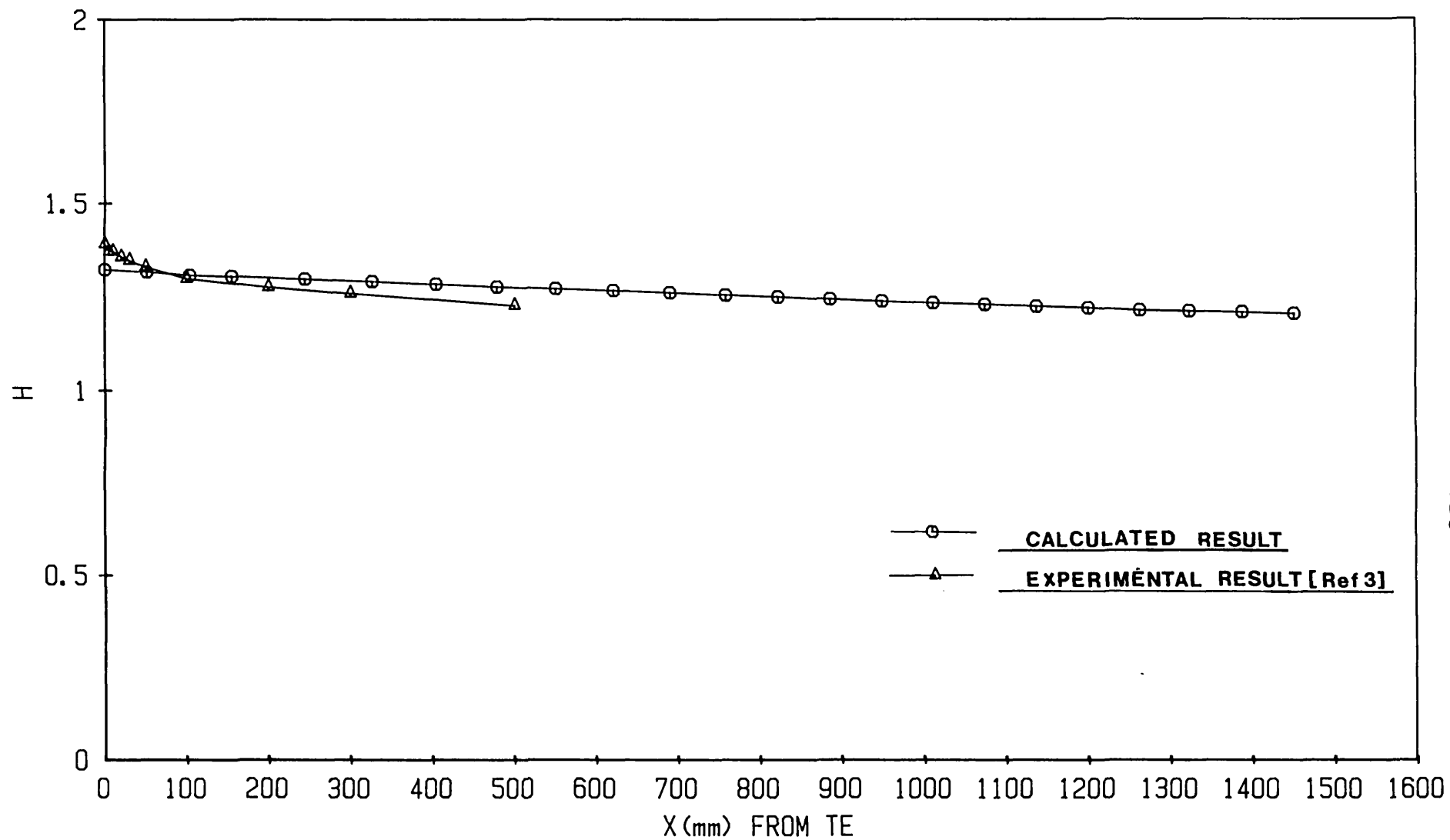


Fig. 48 SHAPE FACTOR OF FLAT PLATE ($Re = 6.8 \times 10^6$)

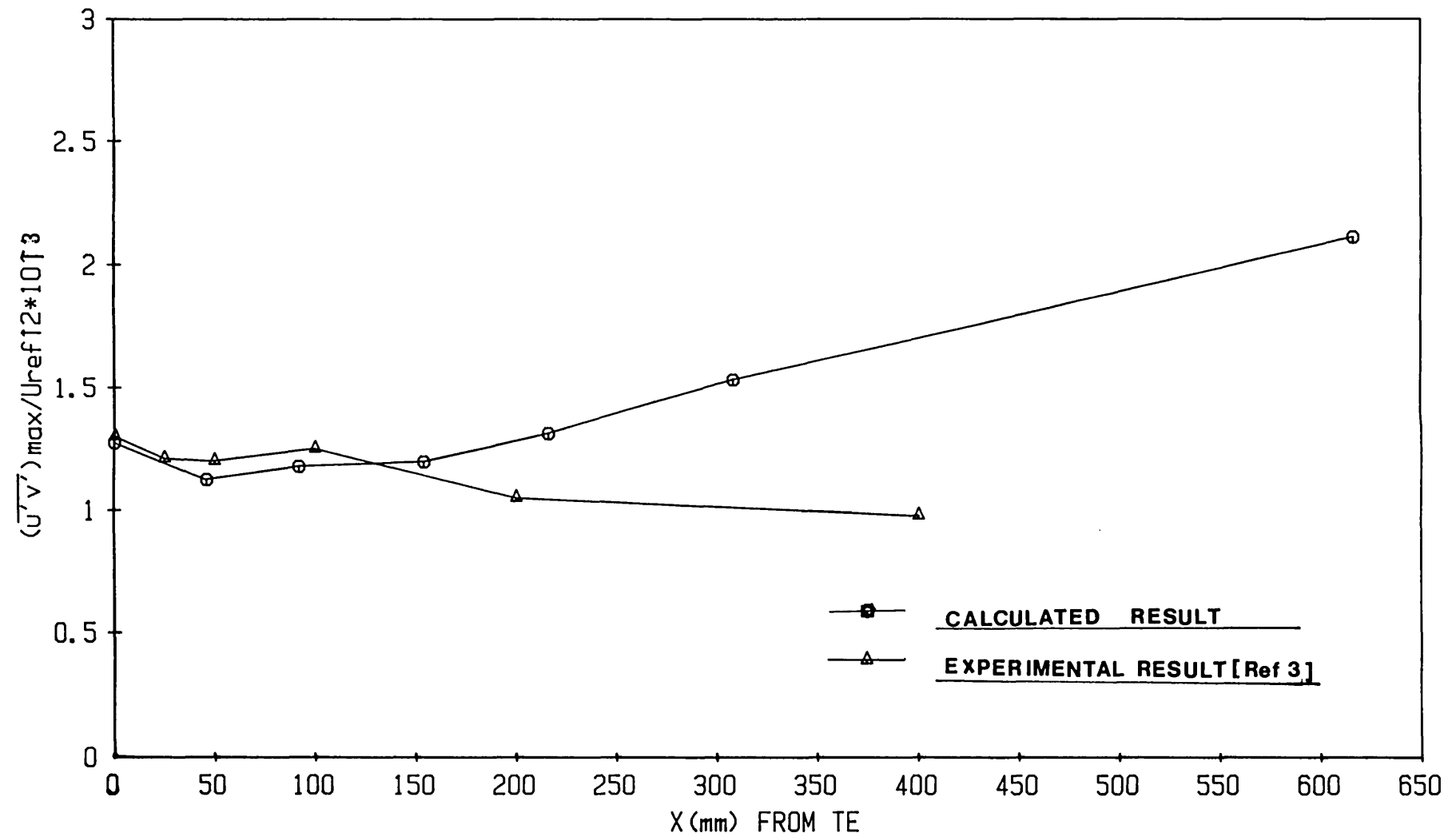


Fig.49 MAXIMUM SHEAR STRESS IN WAKE OF FLAT PLATE ($Re=6.8 \cdot 10^6$)

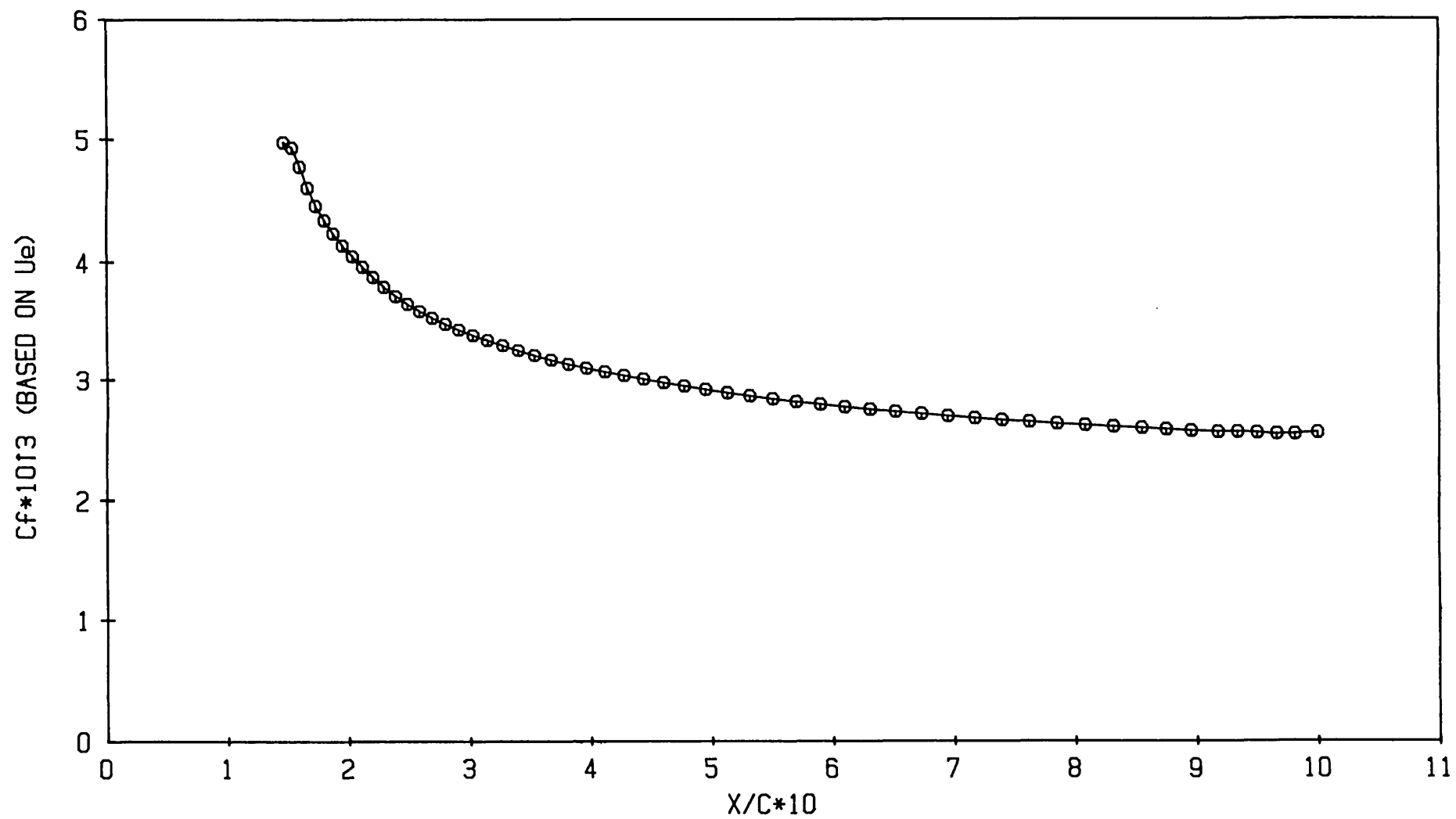


Fig. 50 SKIN FRICTION OF FLAT PLATE ($Re=6.8 \cdot 10^6$)

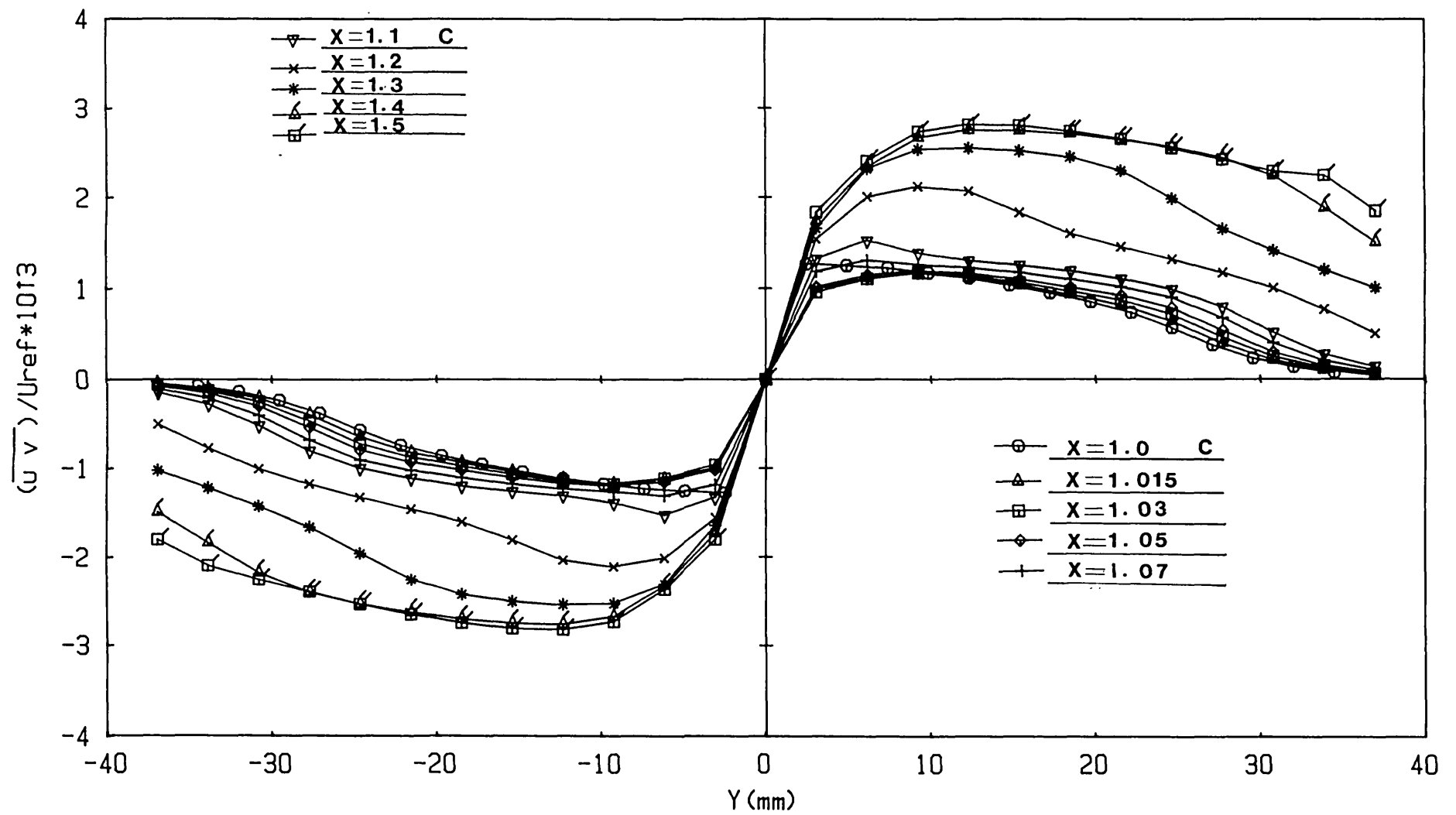


Fig. 51 REYNOLDS SHEAR STRESS IN WAKE OF FLAT PLATE ($Re=6.8 \cdot 10^6$)

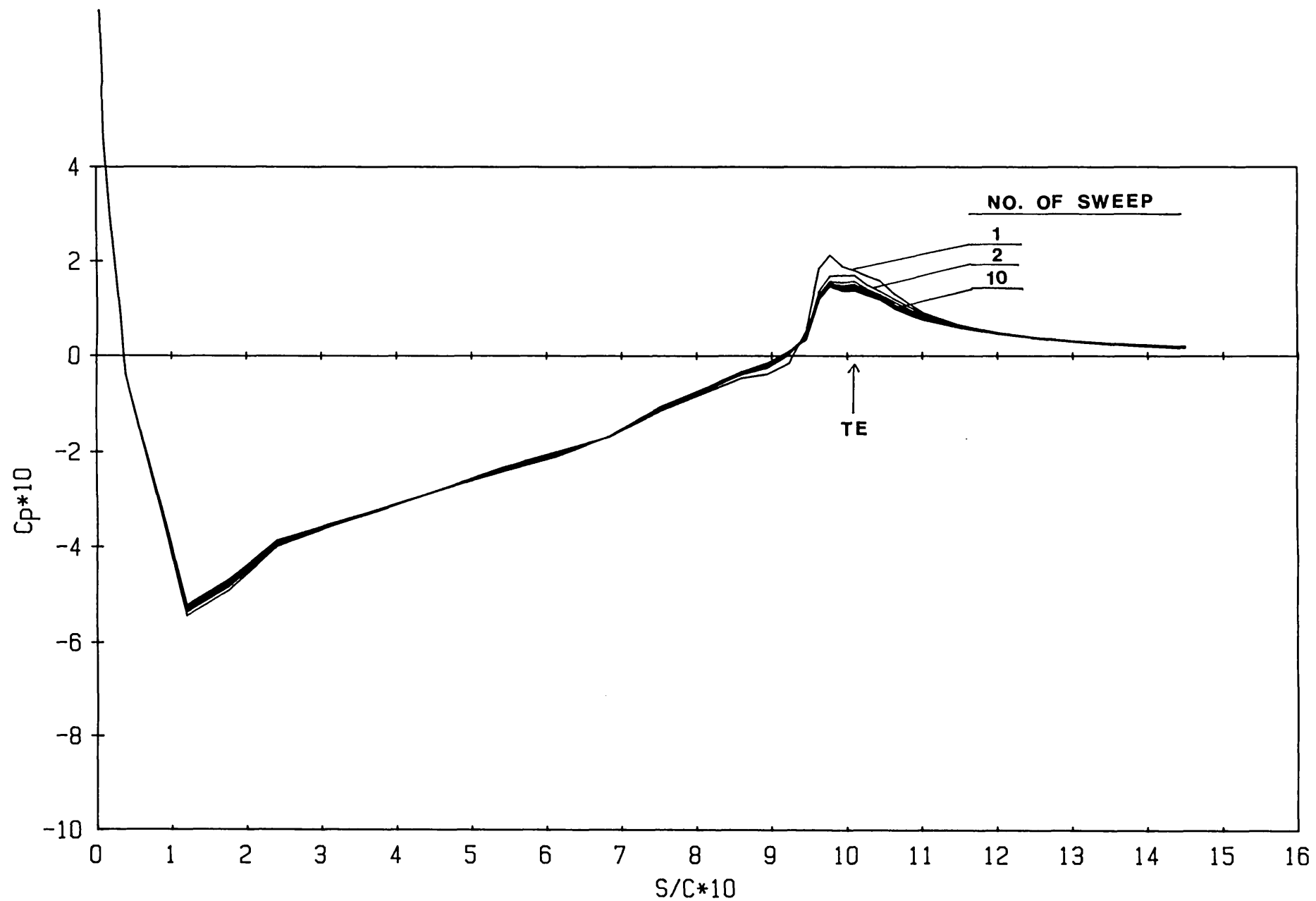


Fig.52 CONVERGENCE OF C_p AT MS OF HEBBAR'S SYMMETRICAL AEROFOIL

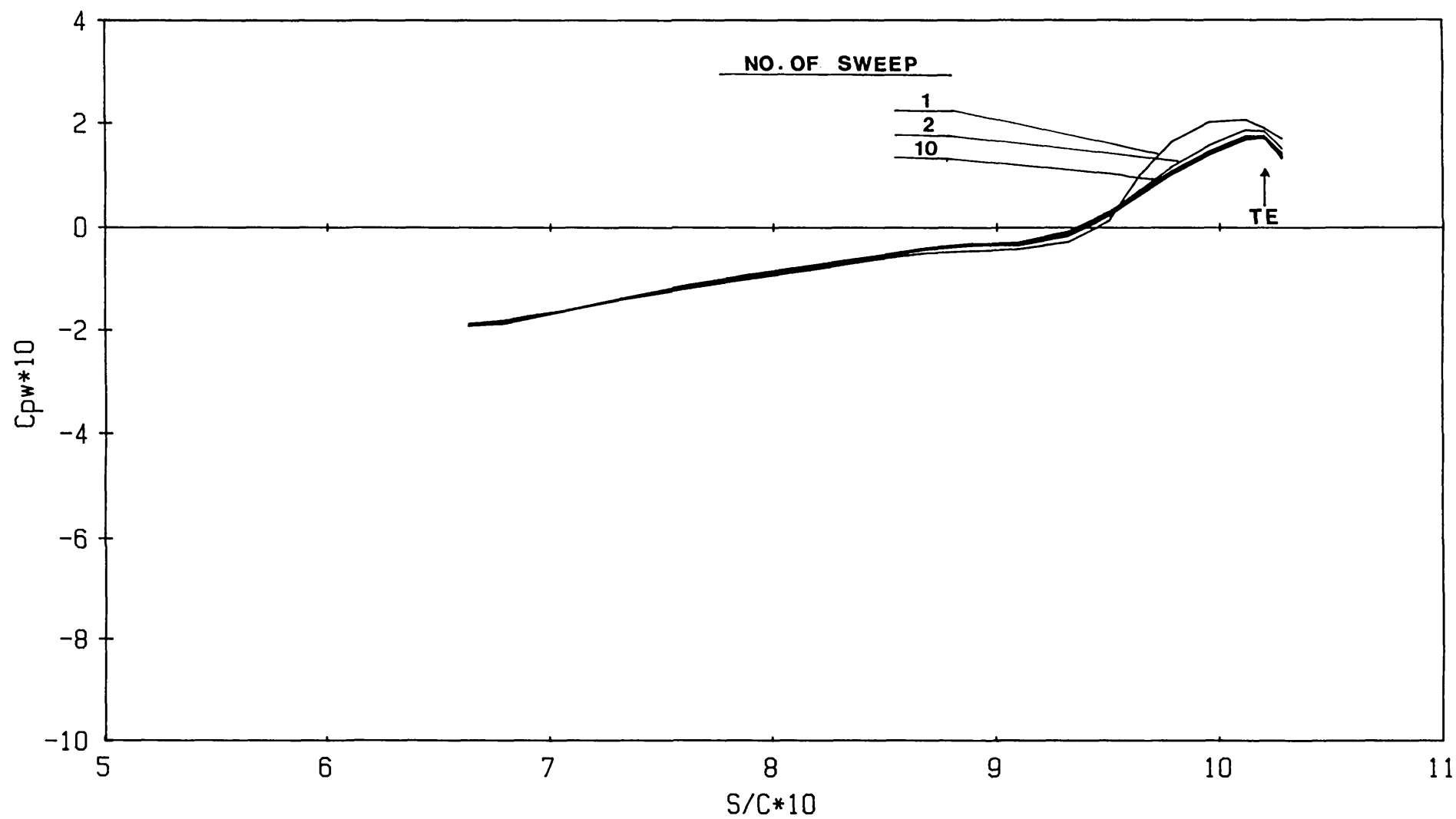


Fig. 53 CONVERGENCE OF C_p AT WALL OF HEBBAR'S AEROFOIL

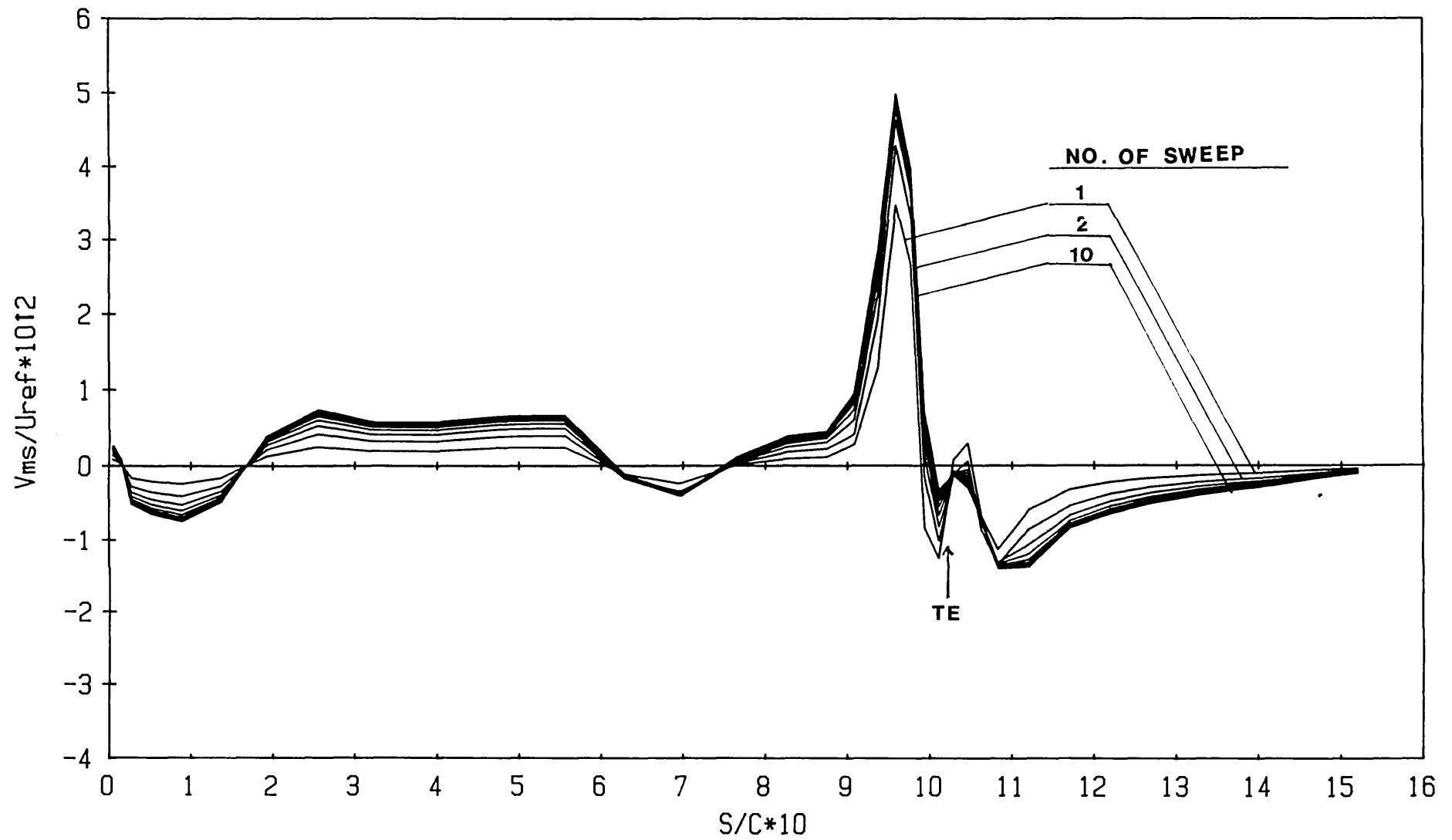


Fig.54 CONVERGENCE OF NORMAL VELOCITY AT MS OF HEBBAR'S AEROFOIL.

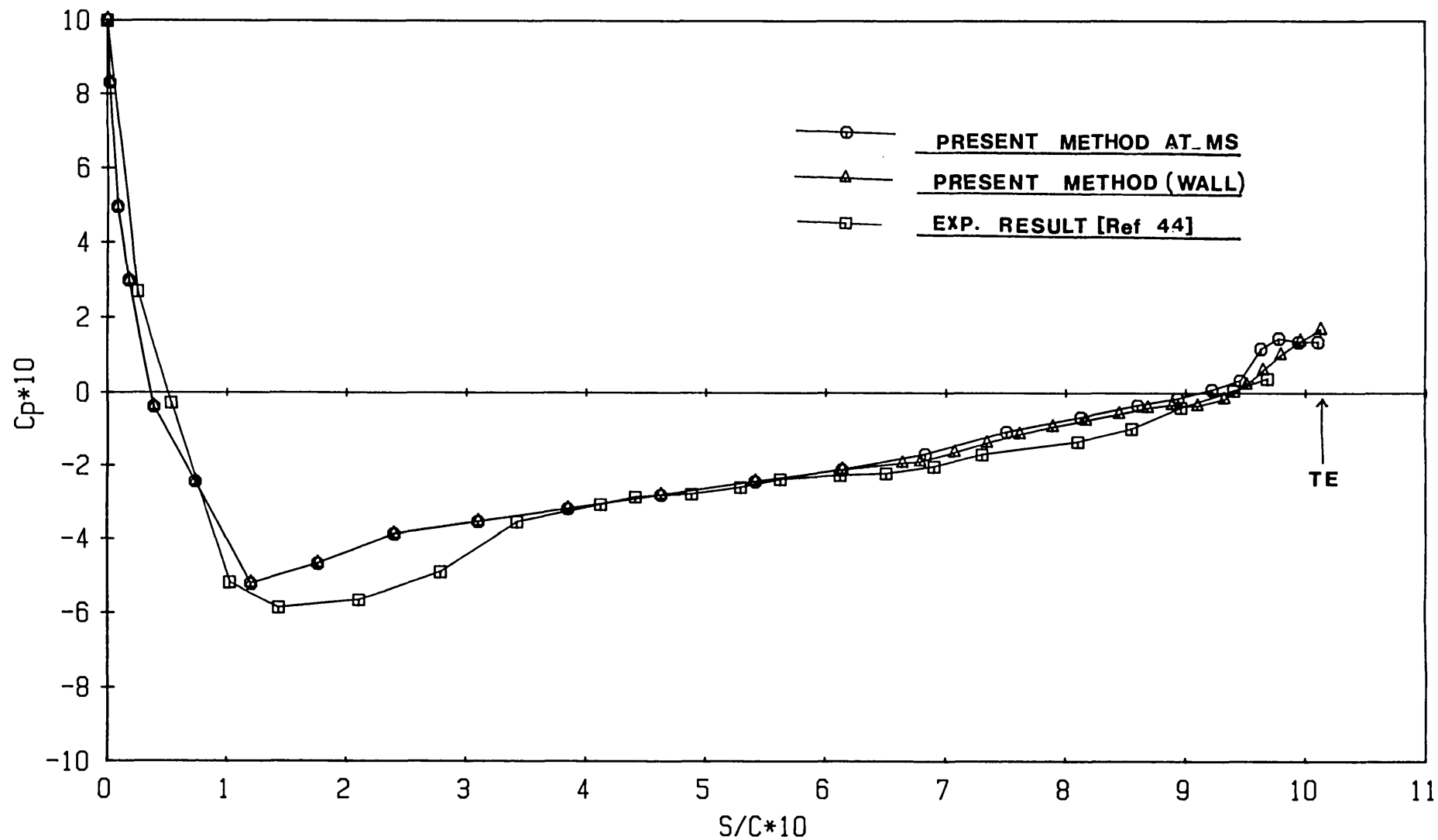


Fig.55 CHORDWISE STATIC PRESSURE DISTRIBUTION ON ... HEBBAR' AEROFOIL

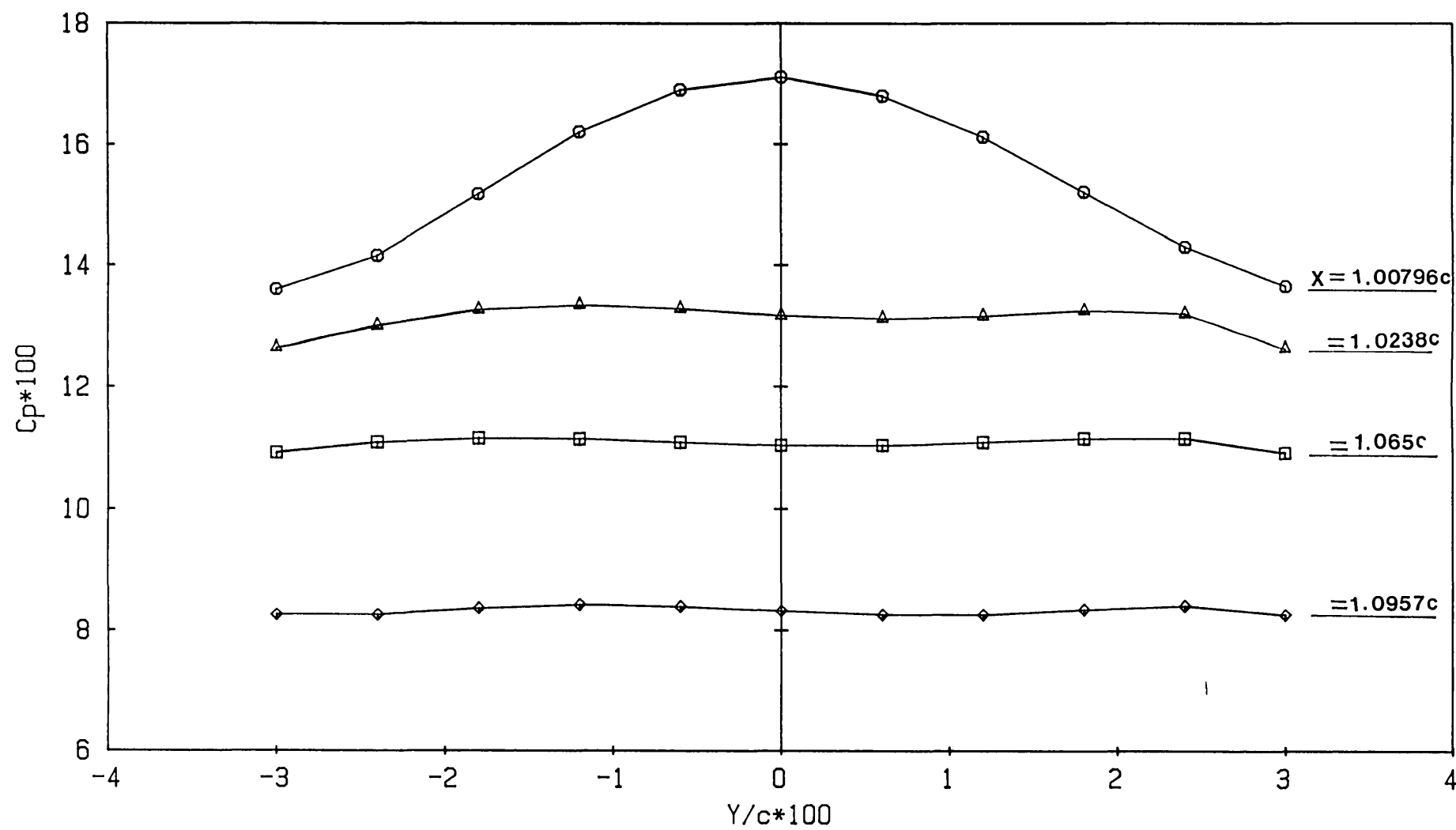


FIG. 55A C_p DISTRIBUTION NEAR WAKE

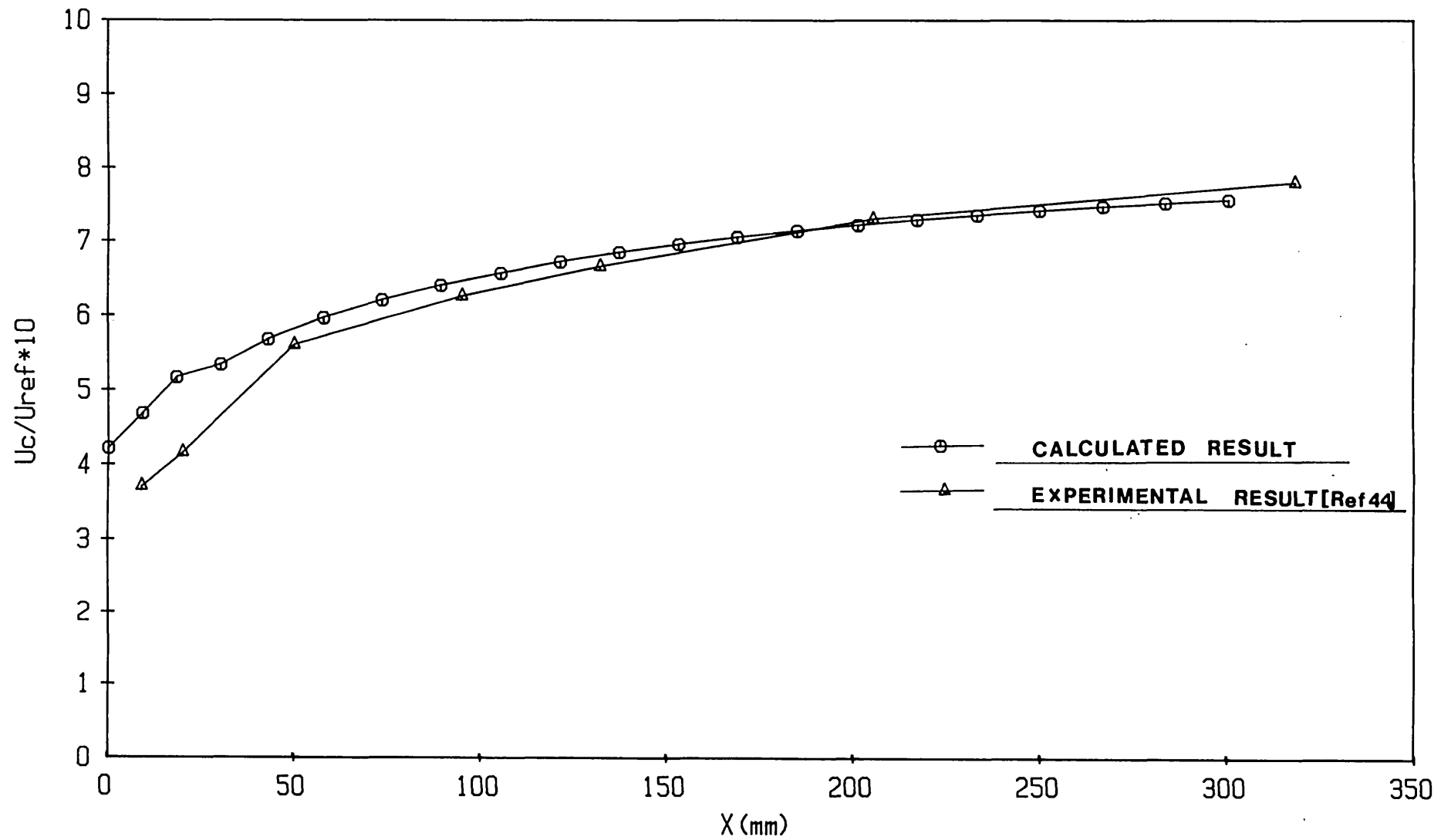


Fig. 57 U_c VELOCITY DISTRIBUTION IN WAKE (HEBBAR'S AEROFOIL)

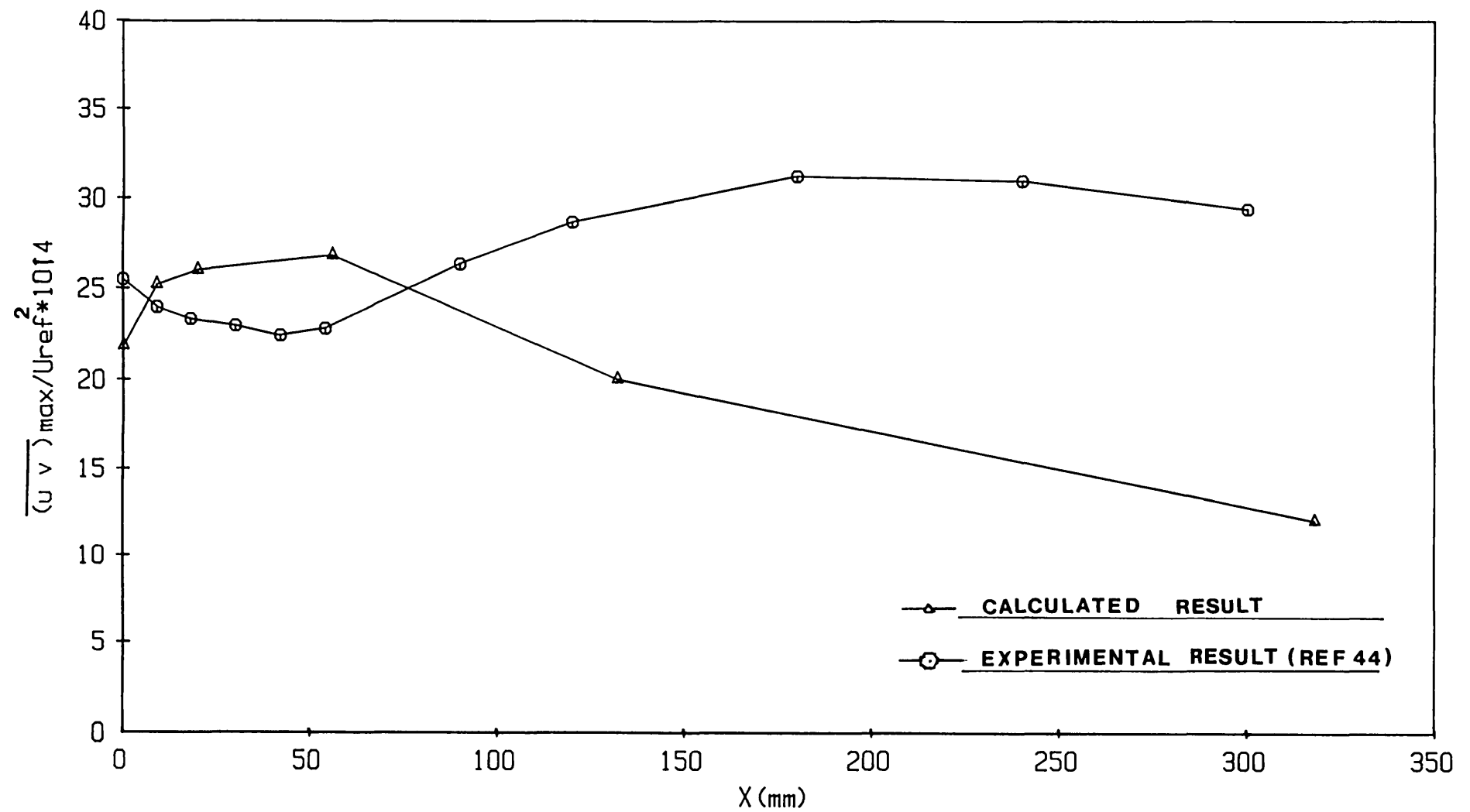


Fig. 58 MAXIMUM SHEAR STRESS DISTRIBUTION IN WAKE (HEBBAR'S AEROFOIL)

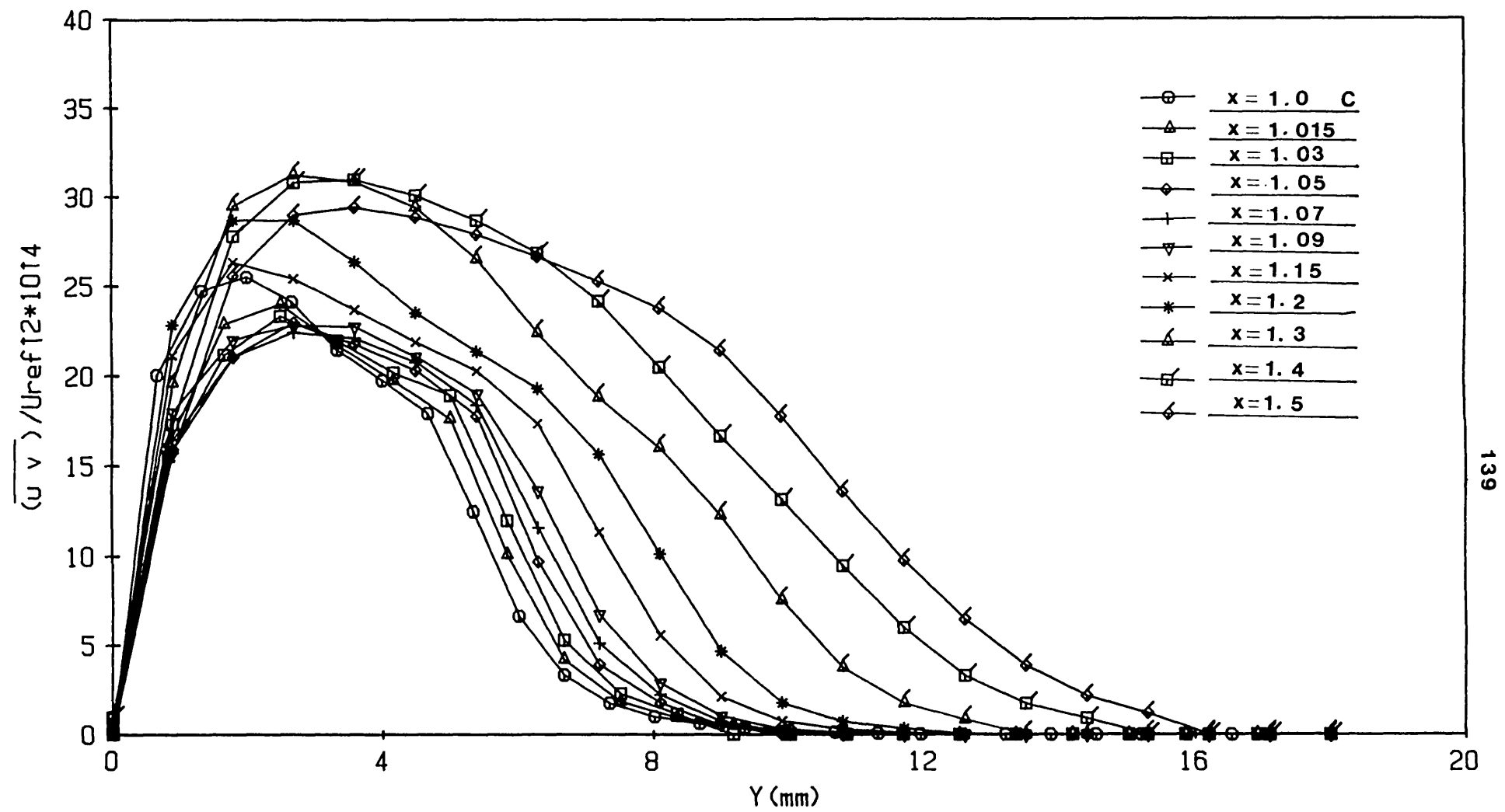


Fig. 59 REYNOLDS SHEAR STRESS IN WAKE (Ref. 44)

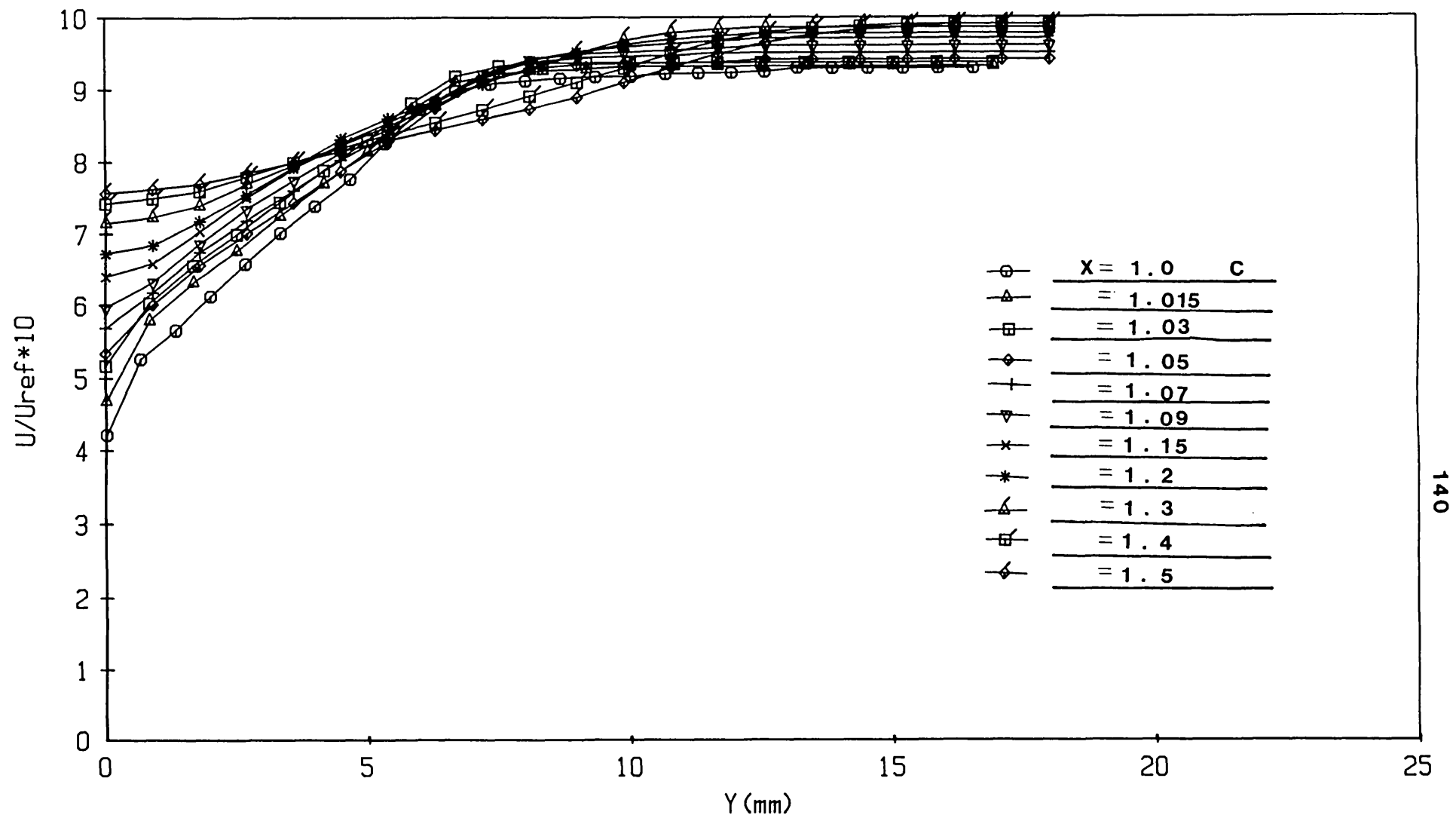


Fig.60 U VELOCITY DISTRIBUTION IN WAKE(HEBBAR'S AEROFOIL)

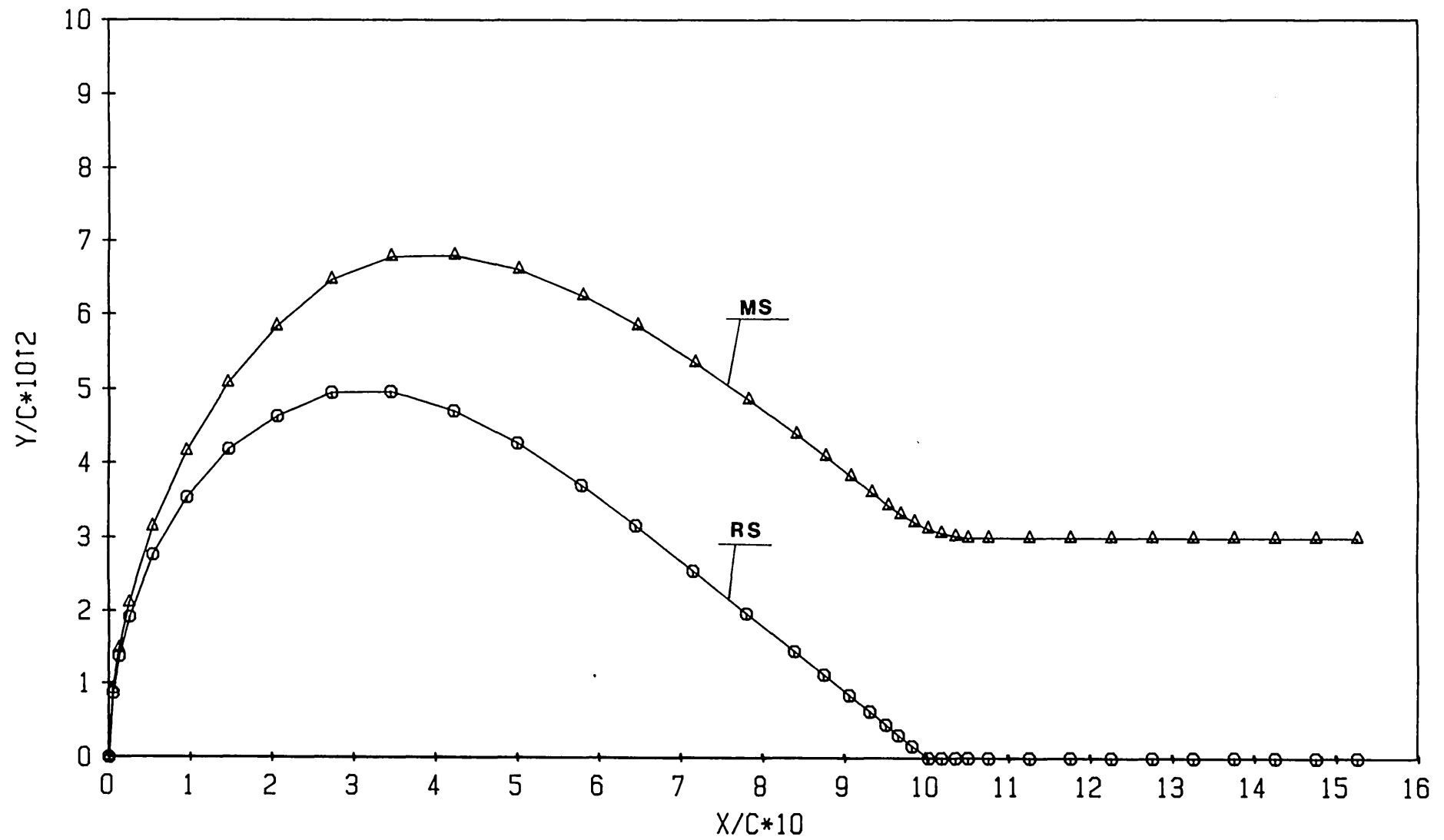


Fig.61 COORDINATES OF RS AND MS OF RAE101

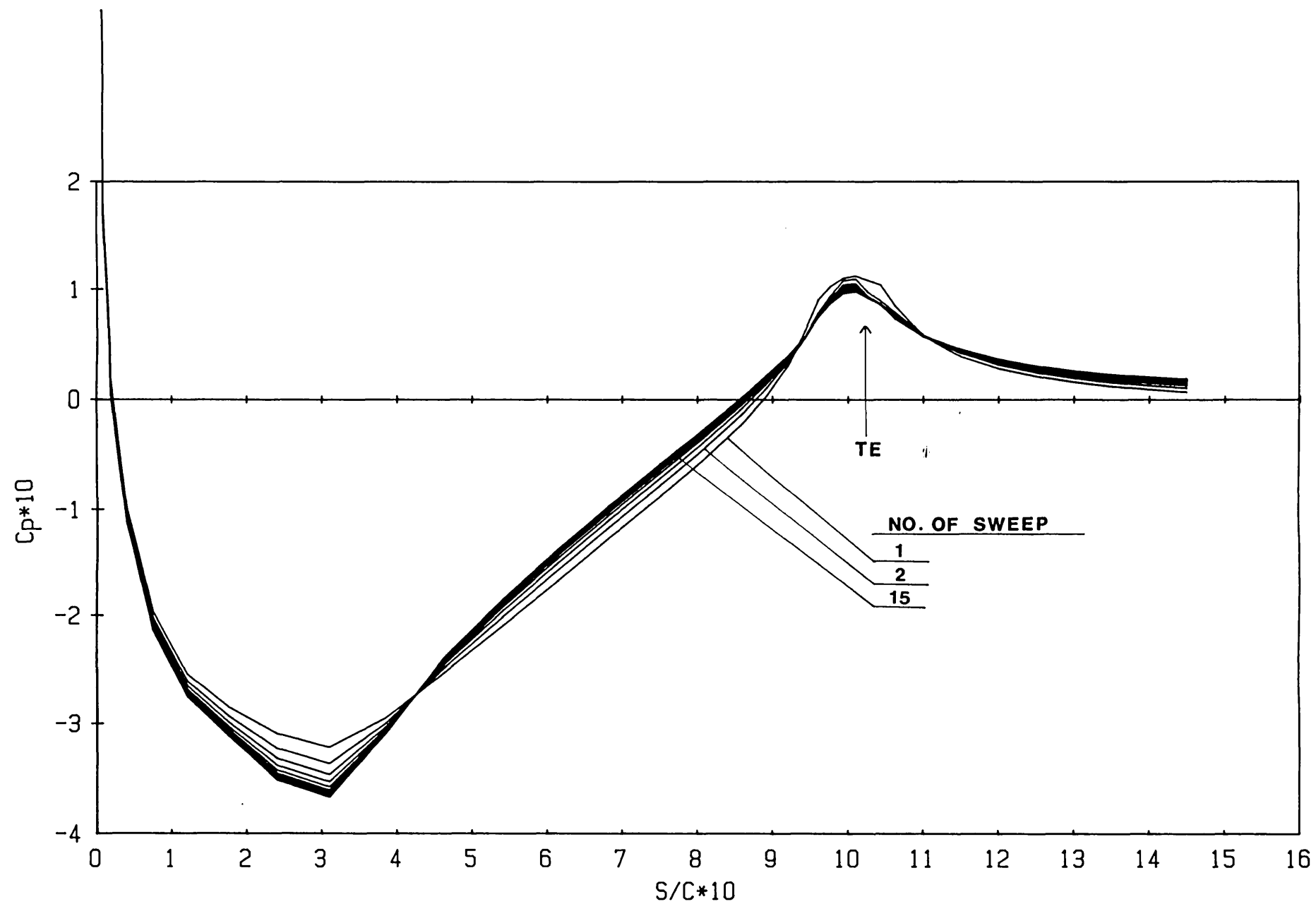


Fig.62 CONVERGENCE OF C_p AT MS OF RAE101 ($\alpha=0^\circ$)

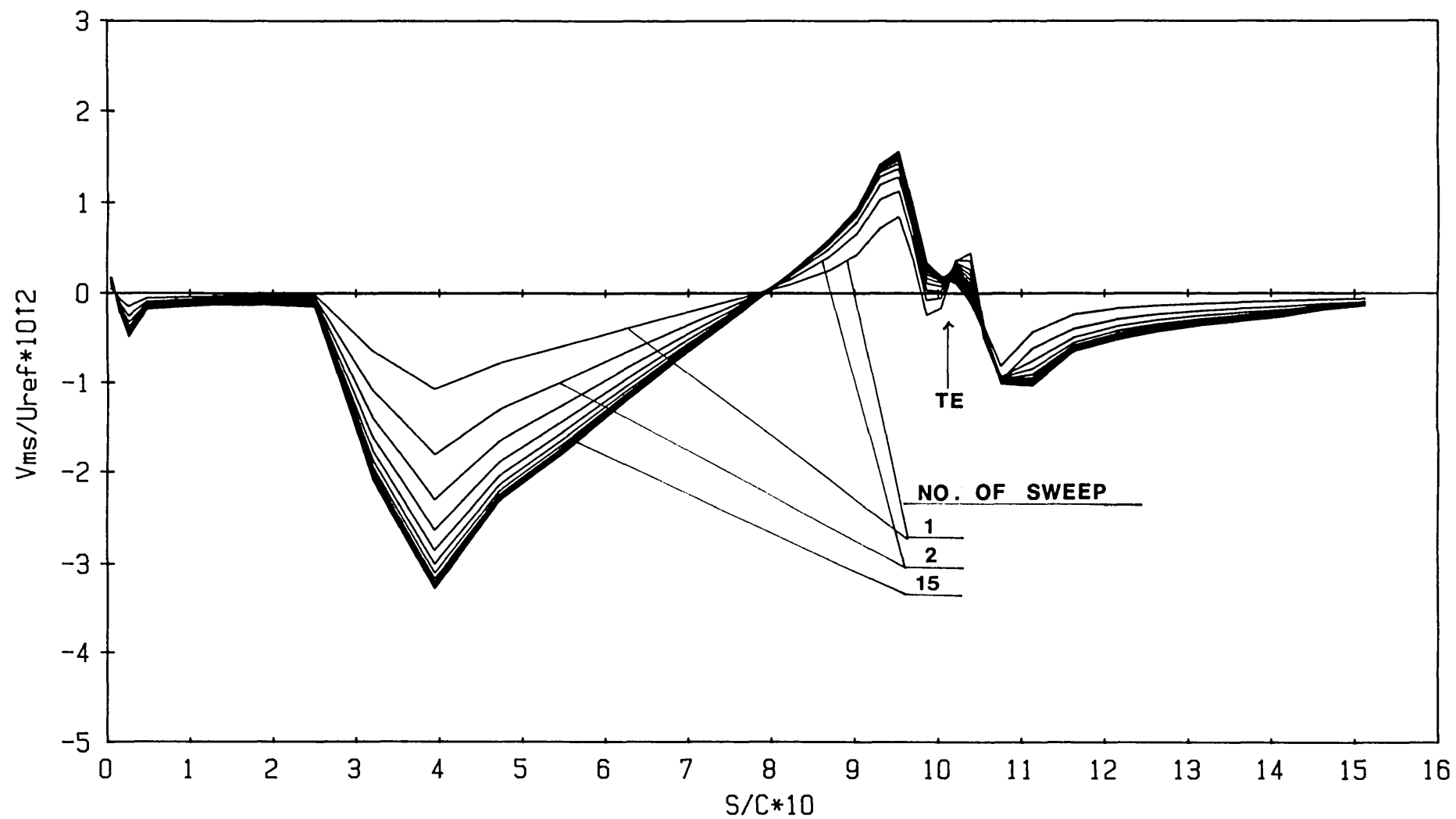


Fig. 63 CONVERGENCE OF NORMAL VELOCITY AT MS OF RAE101 ($\alpha=0^\circ$)

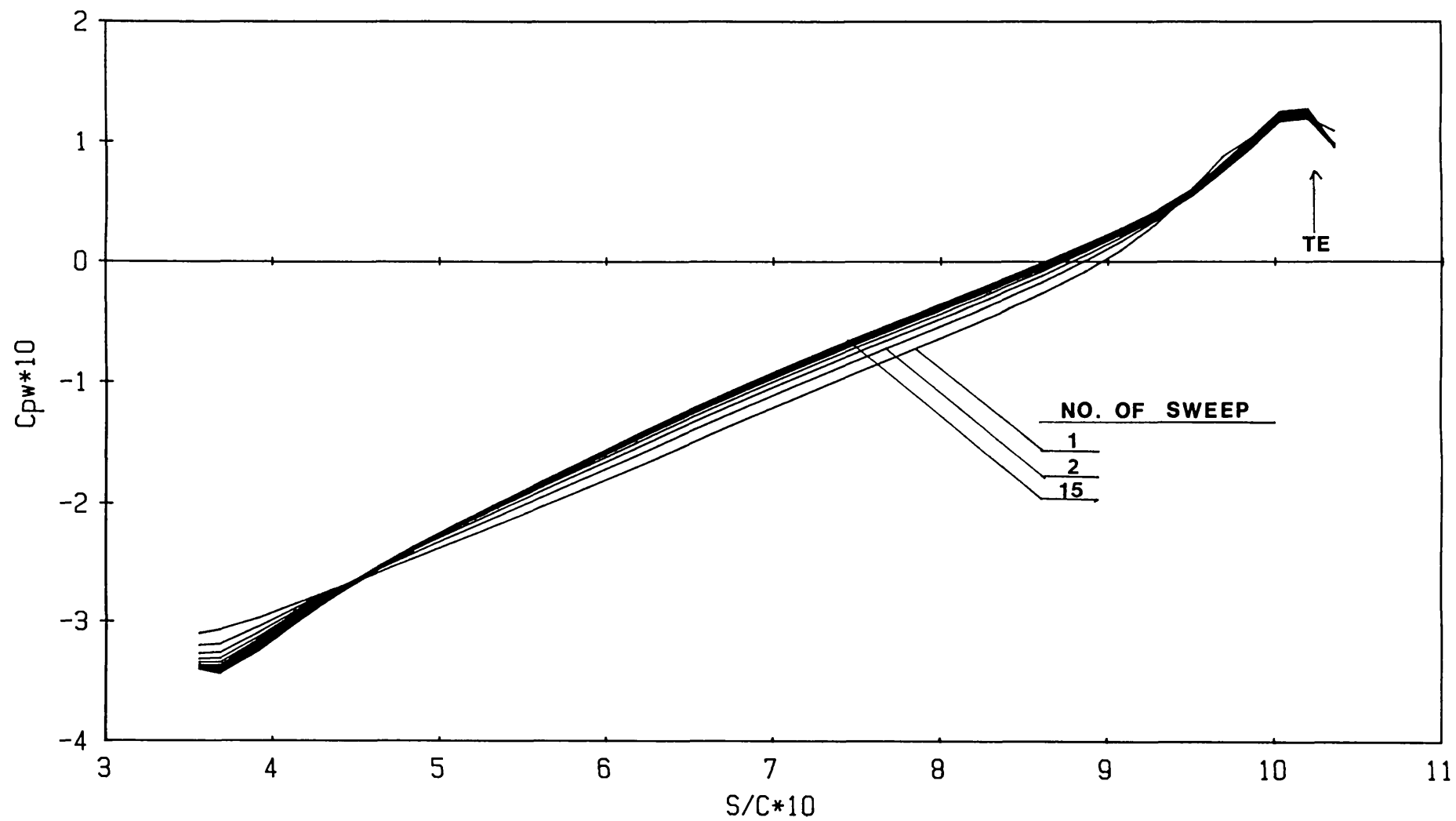


Fig. 64 CONVERGENCE OF C_p AT WALL OF RAE101 ($\alpha = 0^\circ$)

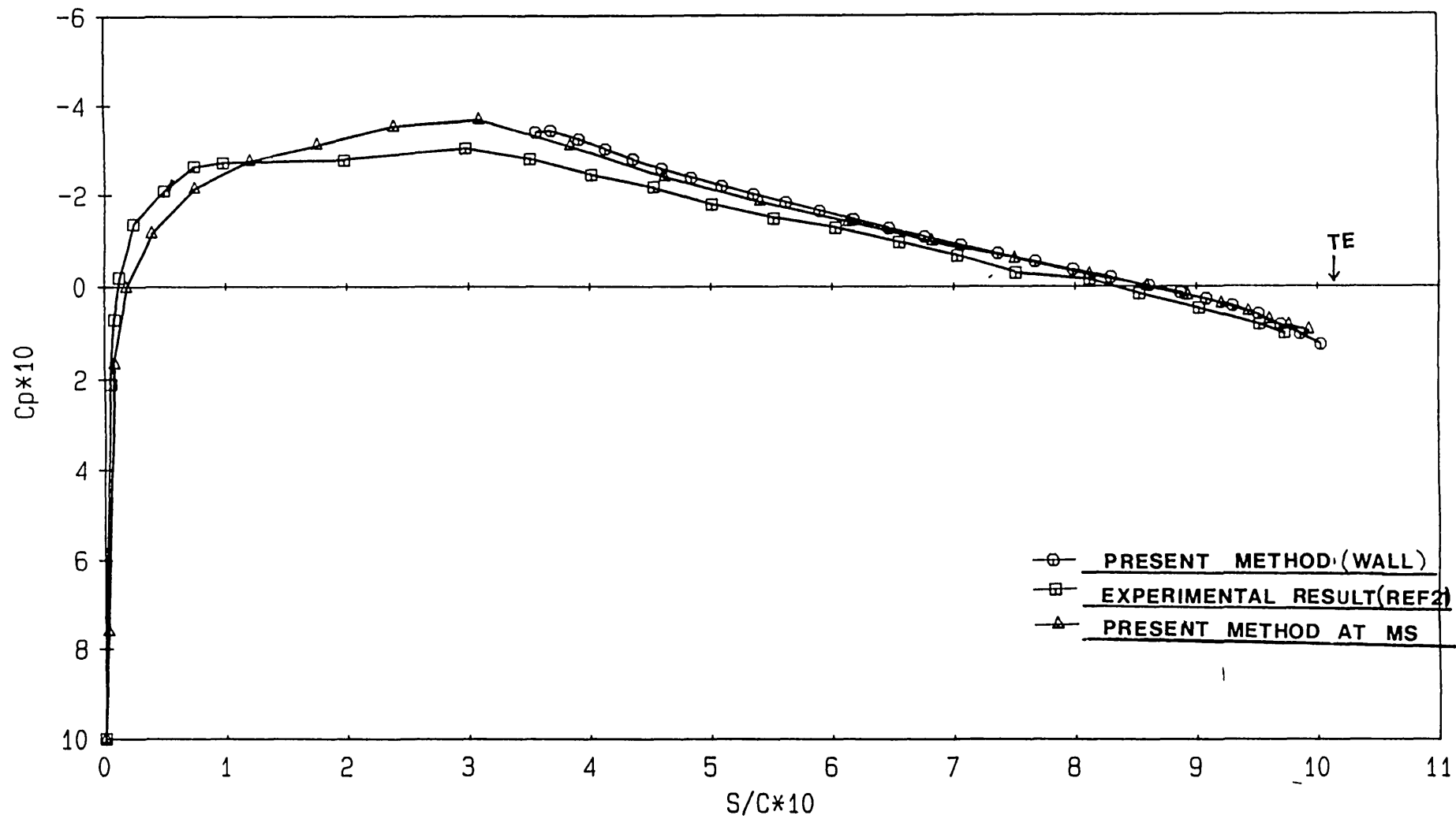


Fig.65 PRESSURE DISTRIBUTION OF RAE101 (REF 21)

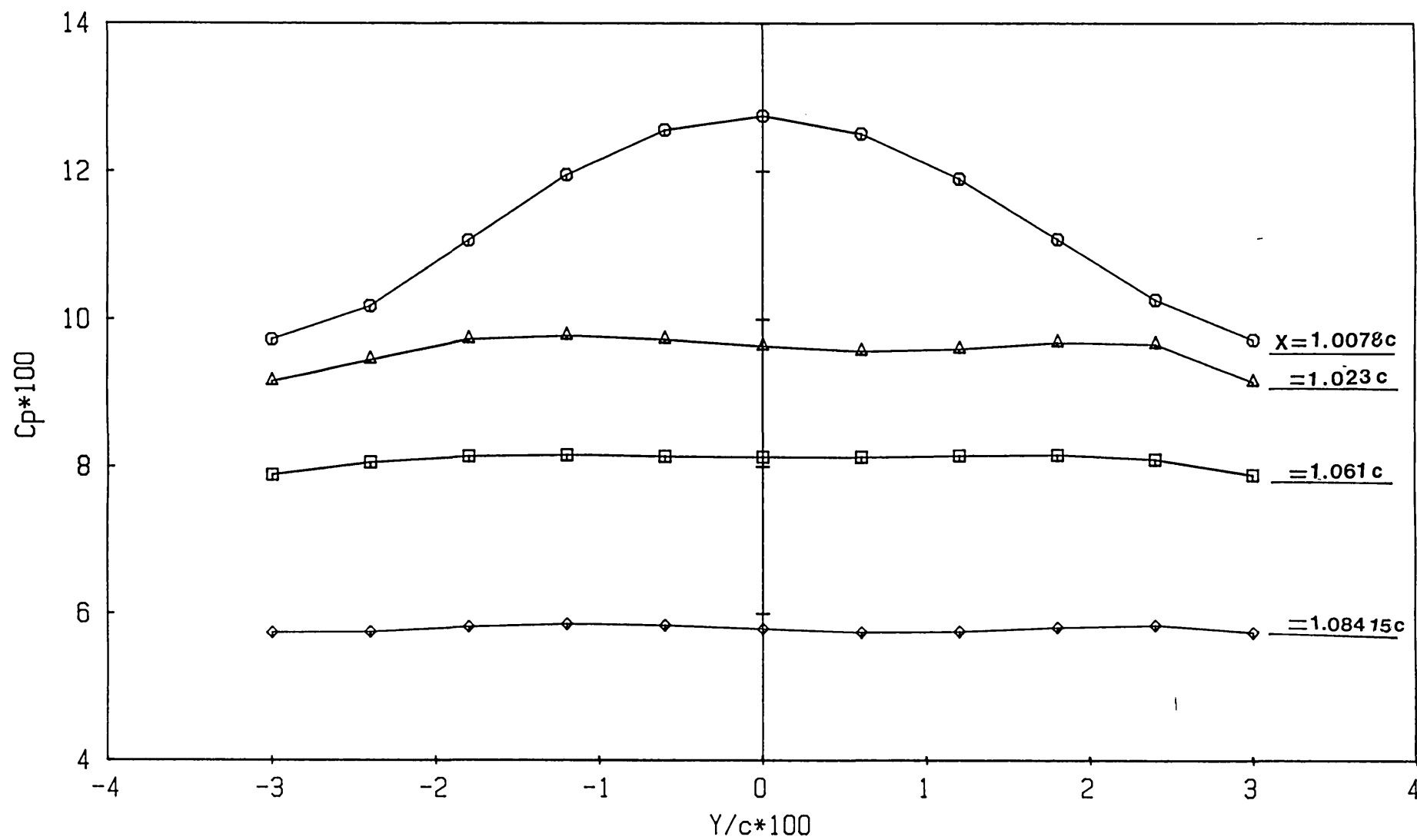


FIG. 65A C_p DISTRIBUTION NEAR WAKE

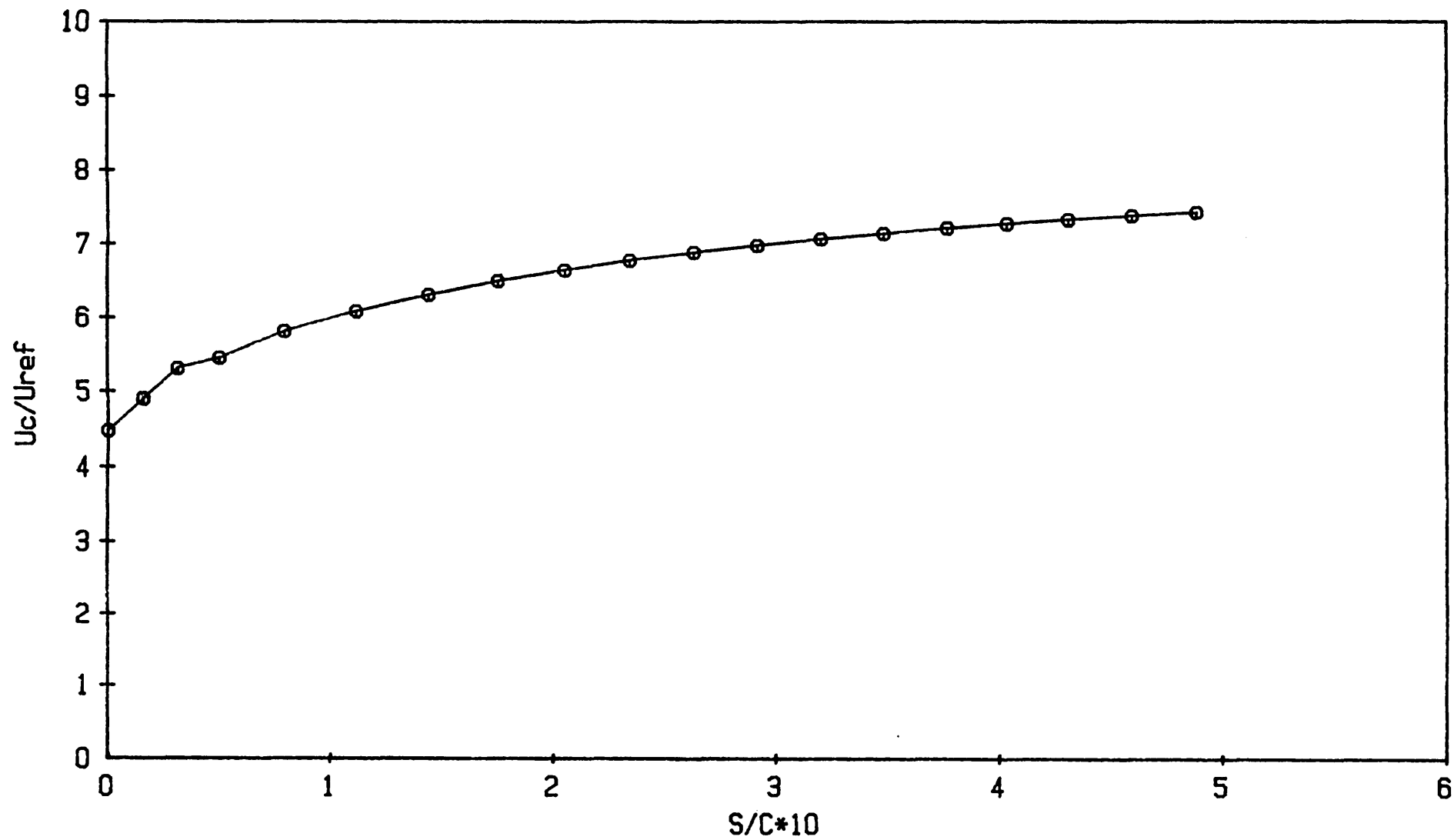


Fig. 66 U_c VELOCITY DISTRIBUTION IN THE WAKE REGION OF RAE101

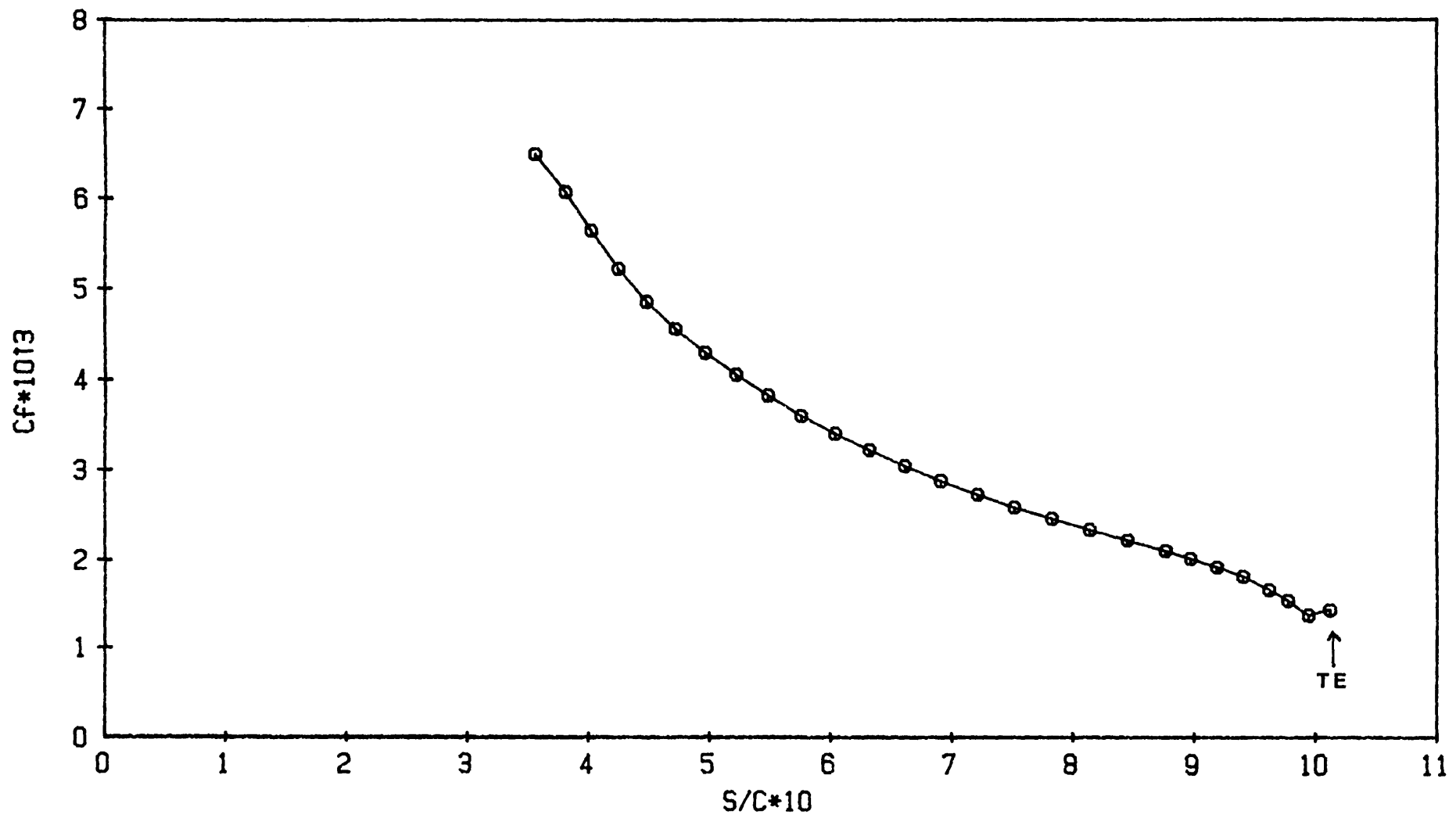


Fig. 67 SKIN FRICTION ALONG THE RAE101 AIRFOIL

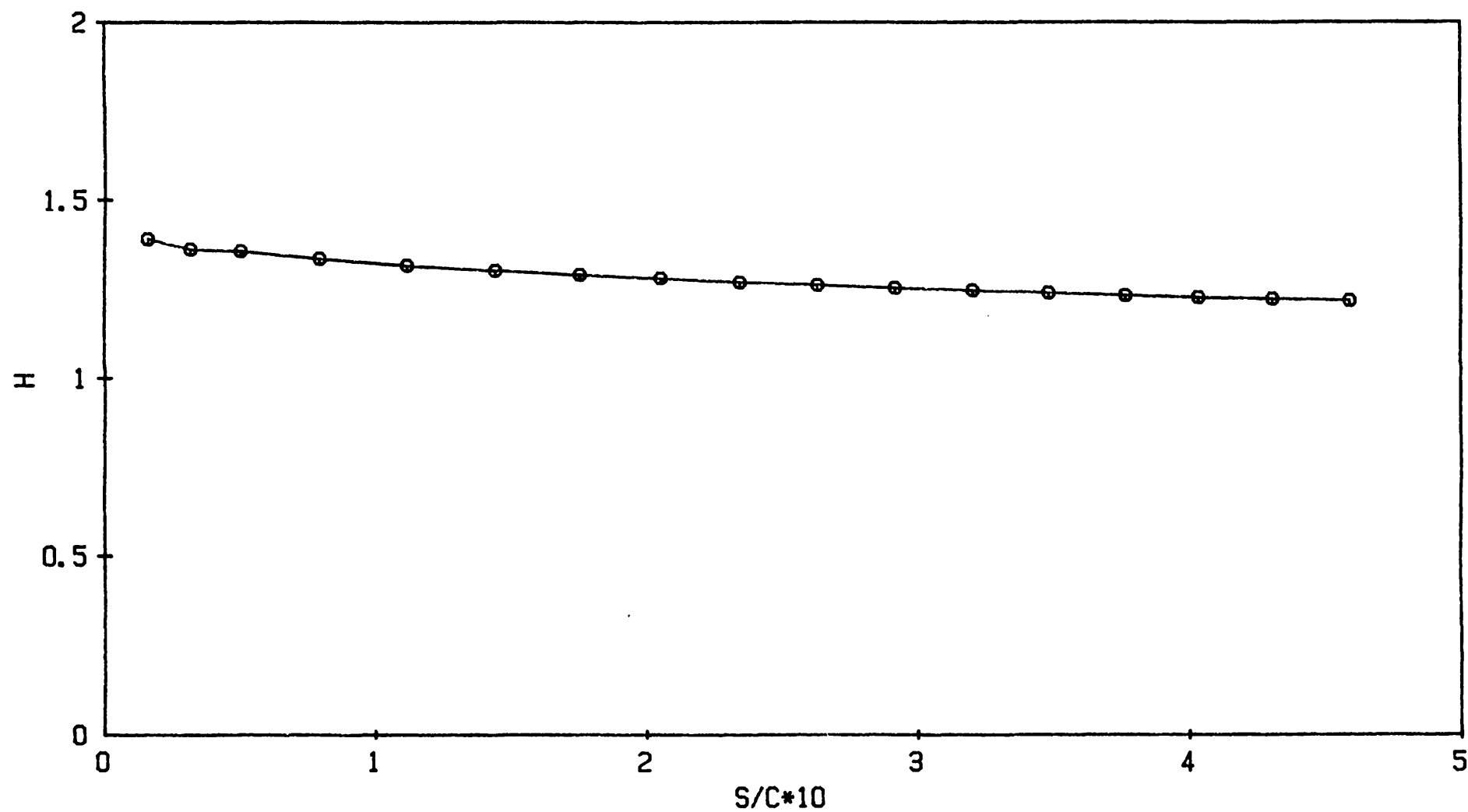


Fig. 68 SHAPE FACTOR VALUE IN THE WAKE REGION

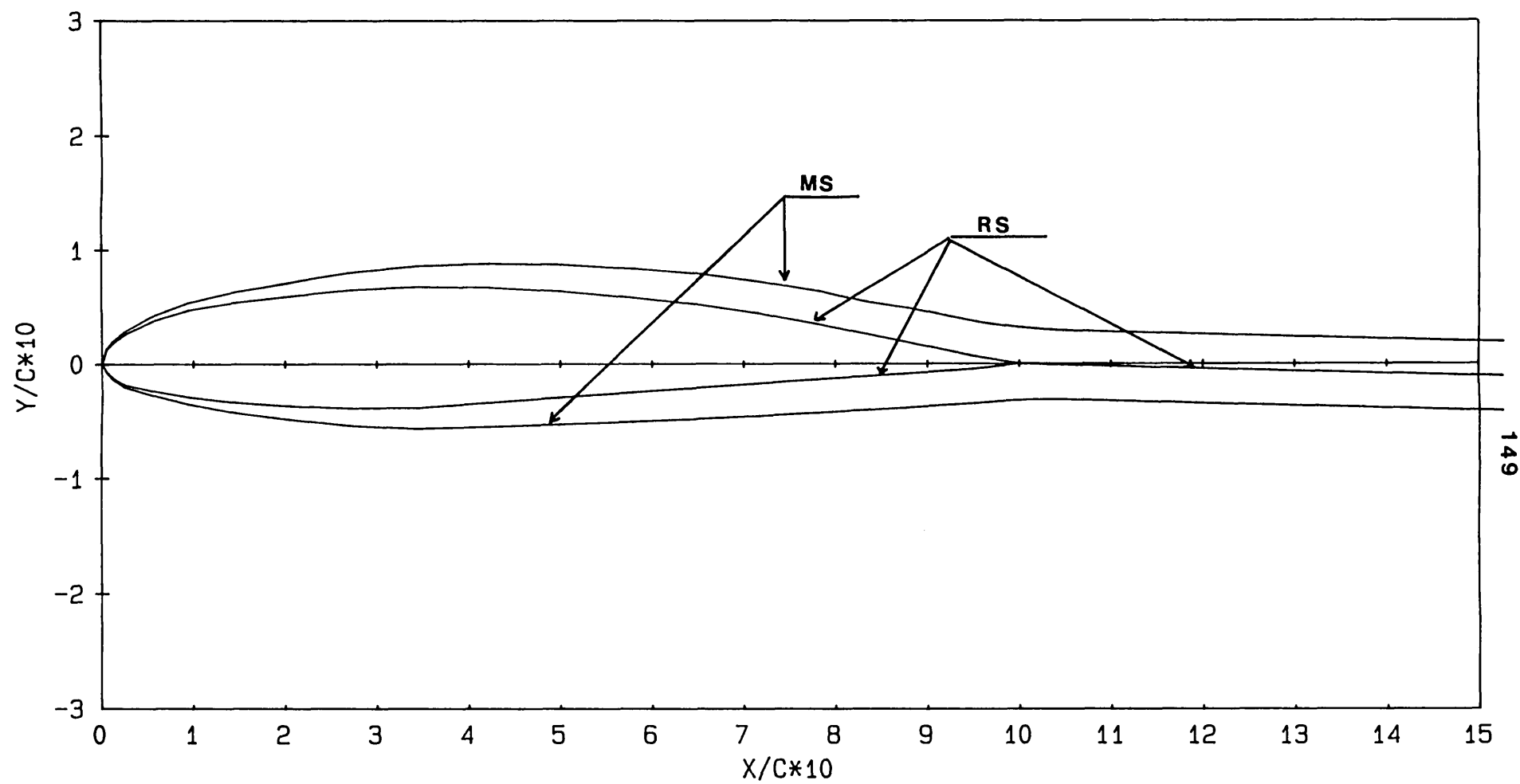


Fig.69 (X-Y) COORDINATE OF MS AND RS (ASYMMETRIC AEROFOIL-NAKAYAMA'S MODEL A)

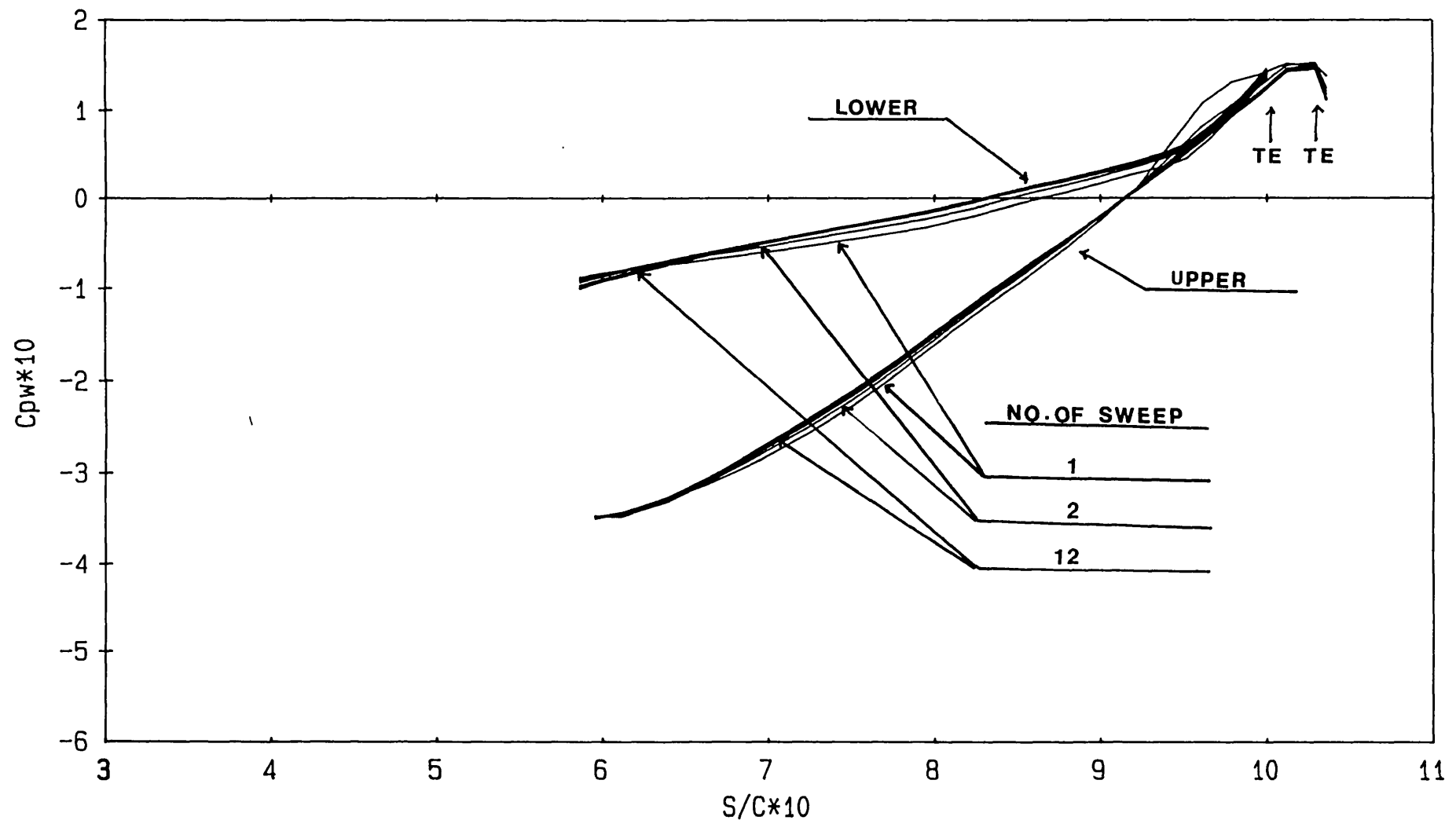


Fig.70 CONVERGENCE OF C_{pw} AT WALL (ASYMMETRIC AEROFOIL-NAKAYAMA'S MODEL A)

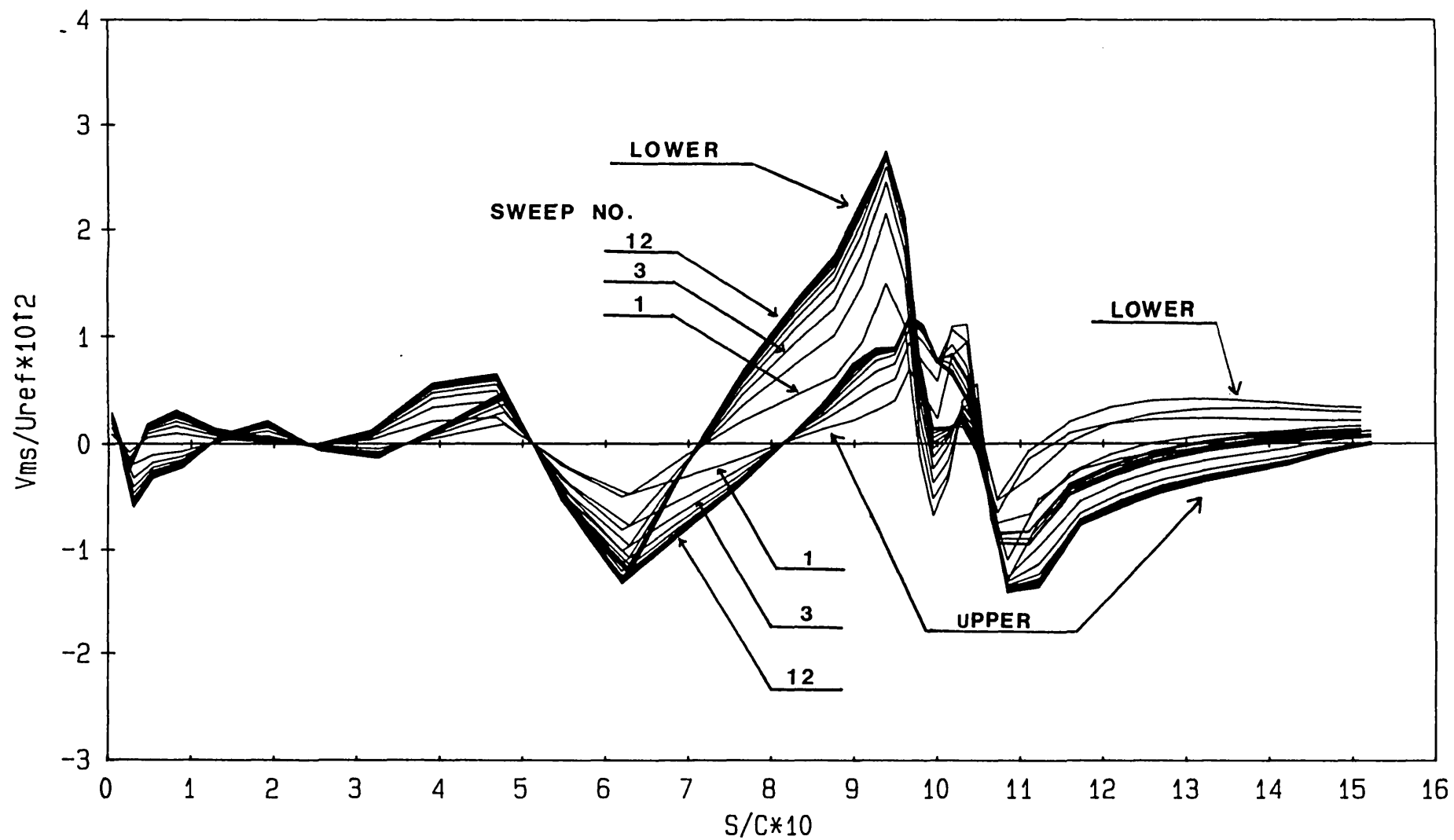


Fig. 71 CONVERGENCE OF V_{ms} (ASYMMETRIC AEROFOIL-NAKAYAMA'S MODEL A)

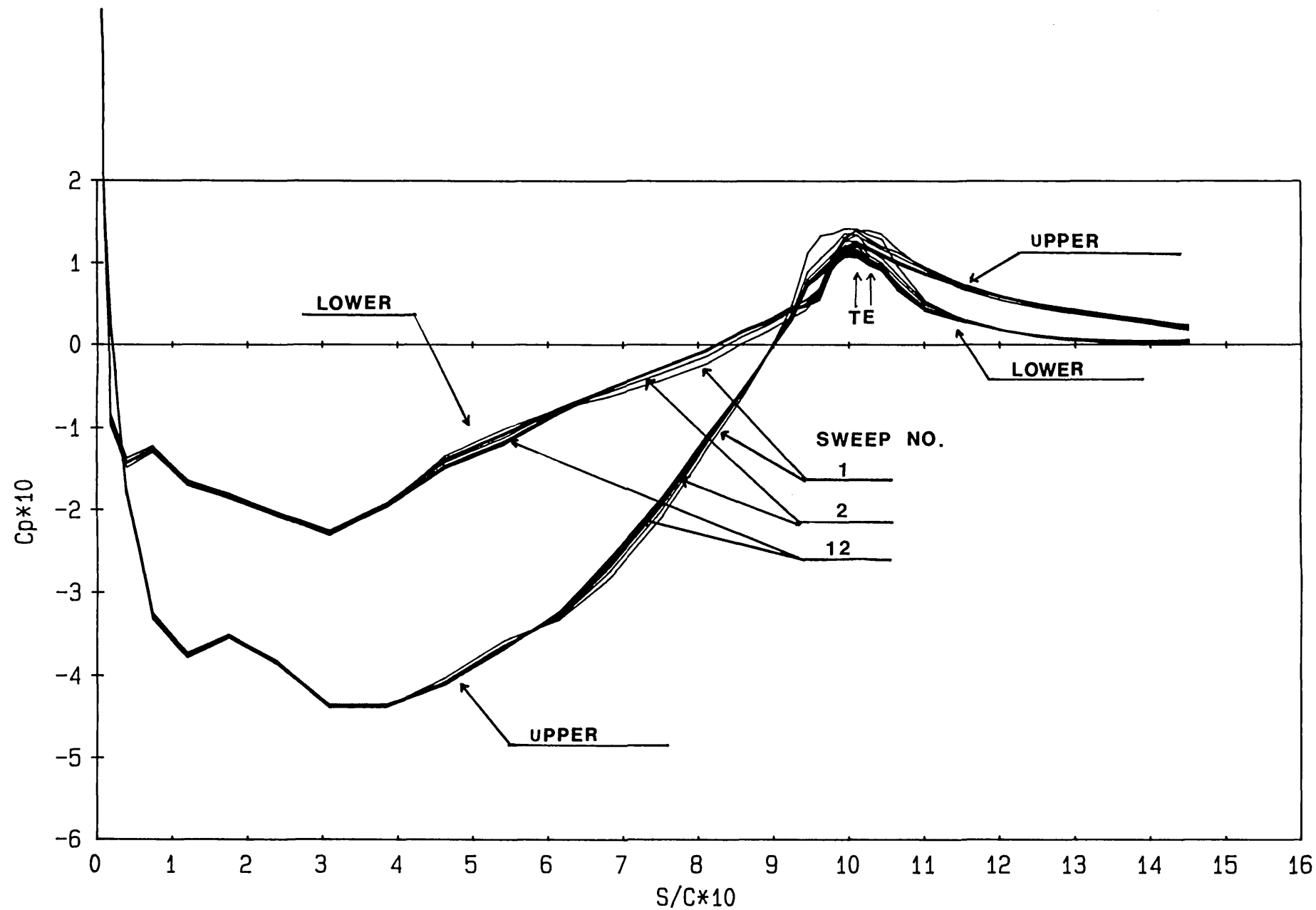


Fig. 72 CONVERGENCE OF C_p AT MS (ASYMMETRIC AEROFOIL-NAKAYAMA MODEL A)

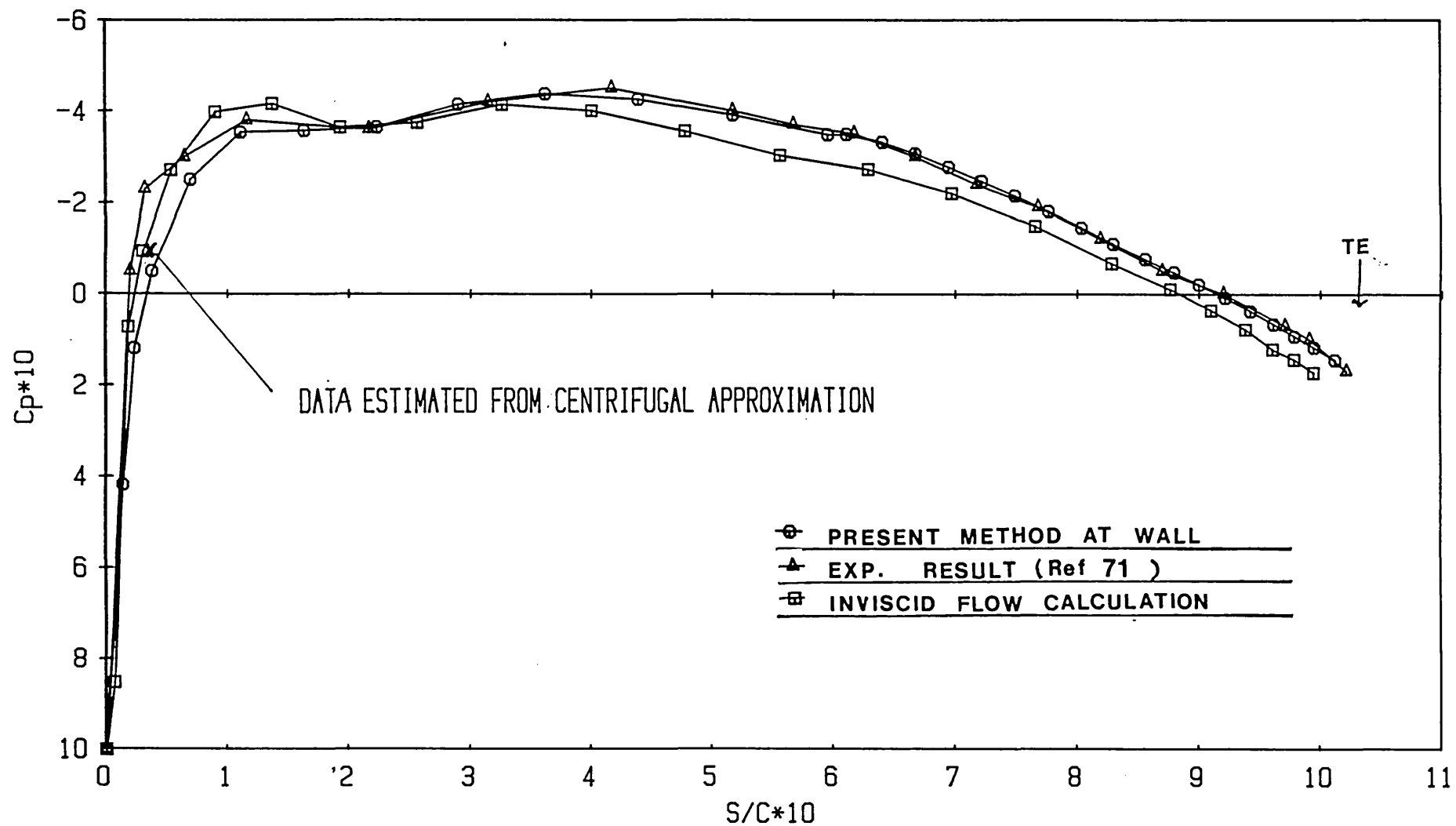
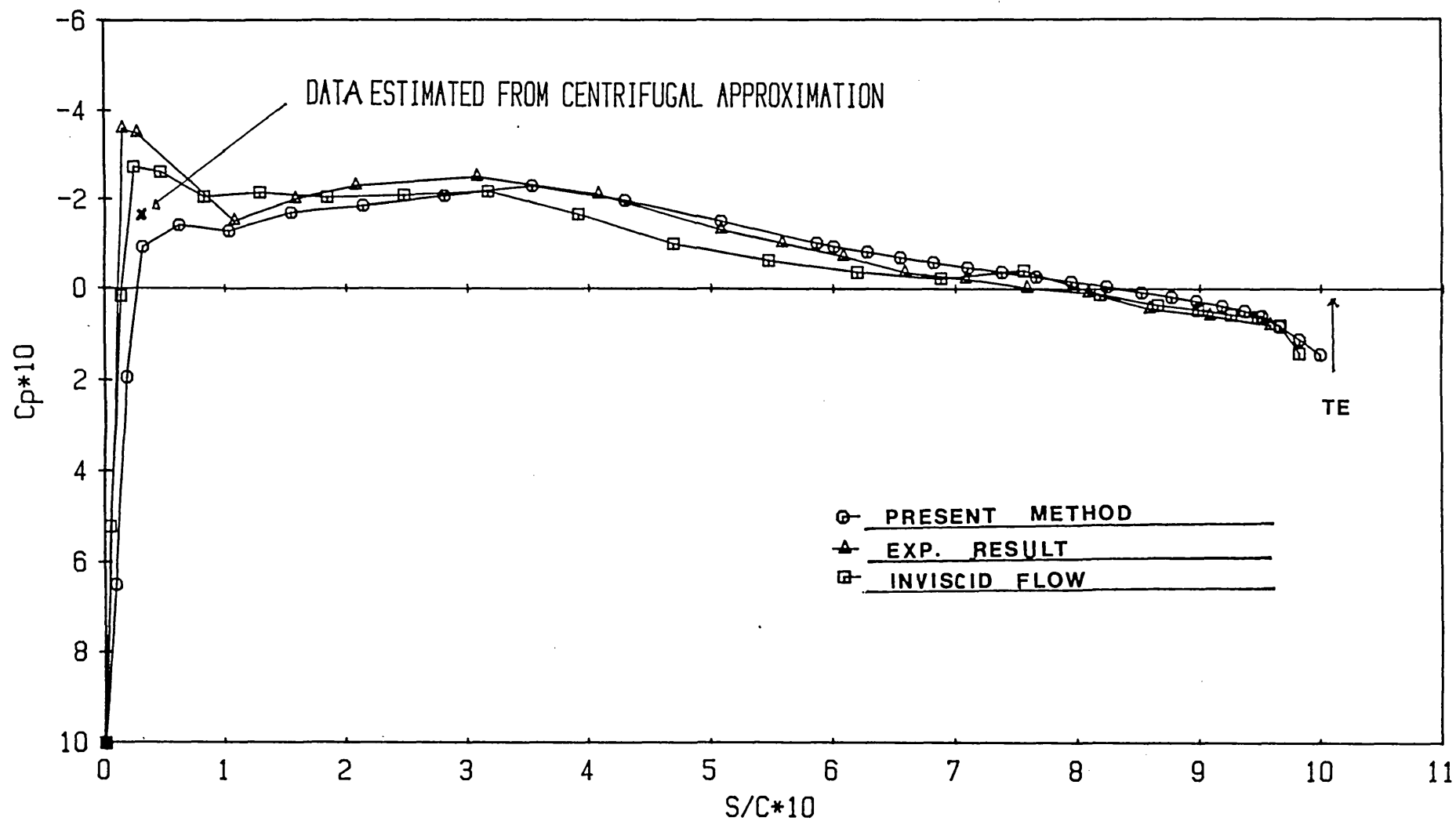


Fig. 73A C_p DISTRIBUTION FOR UPPER PART OF ASYMMETRIC AEROFOIL (Ref. 71)



153-B

Fig.73B C_p DISTRIBUTION FOR LOWER PART OF ASYMMETRIC AEROFOIL (Ref. 71)

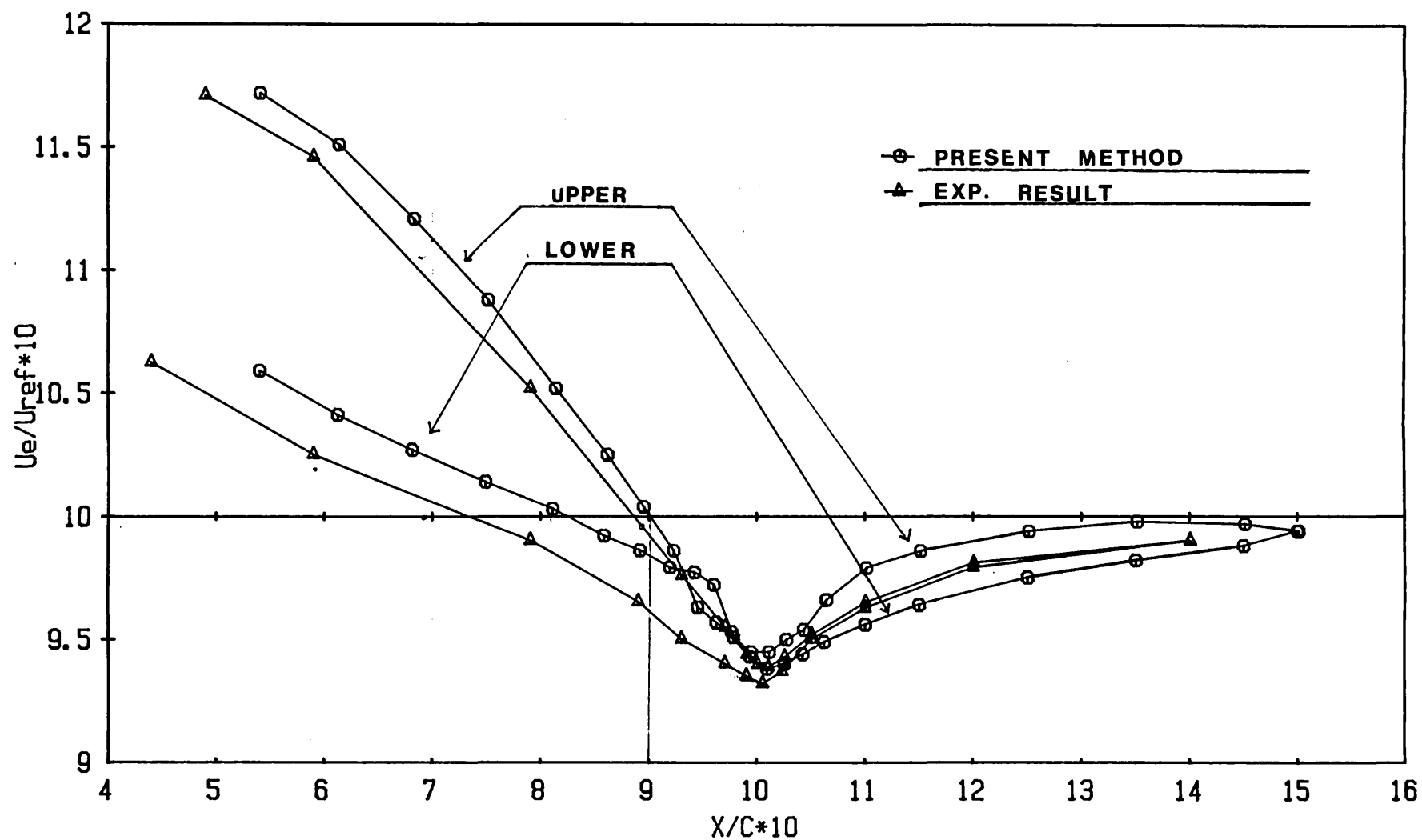


Fig. 74 U_e DISTRIBUTION OF ASYMMETRIC AEROFOIL (Ref. 71)

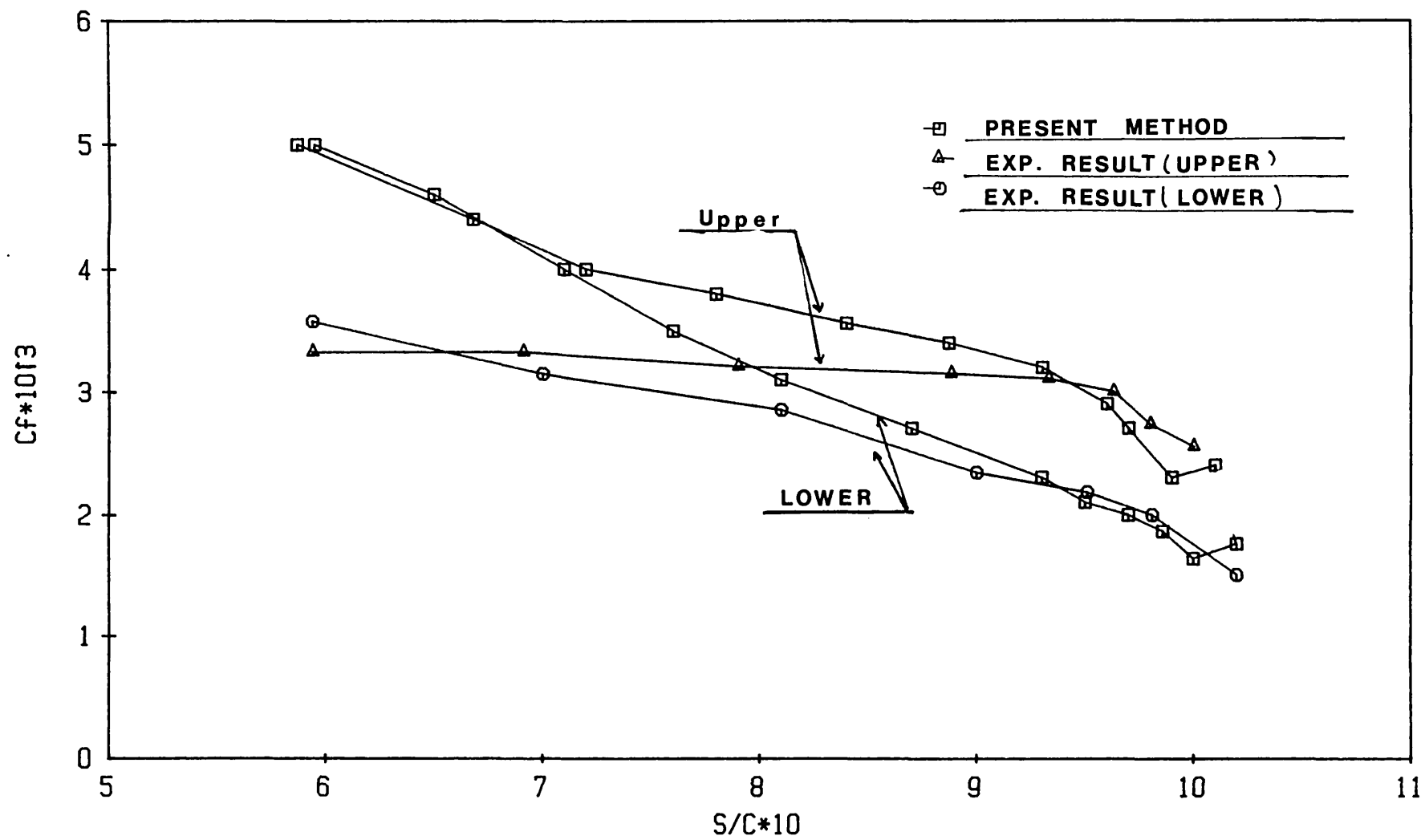


Fig. 75 SKIN FRICTION OF ASYMMETRIC AEROFOIL (Ref 71)

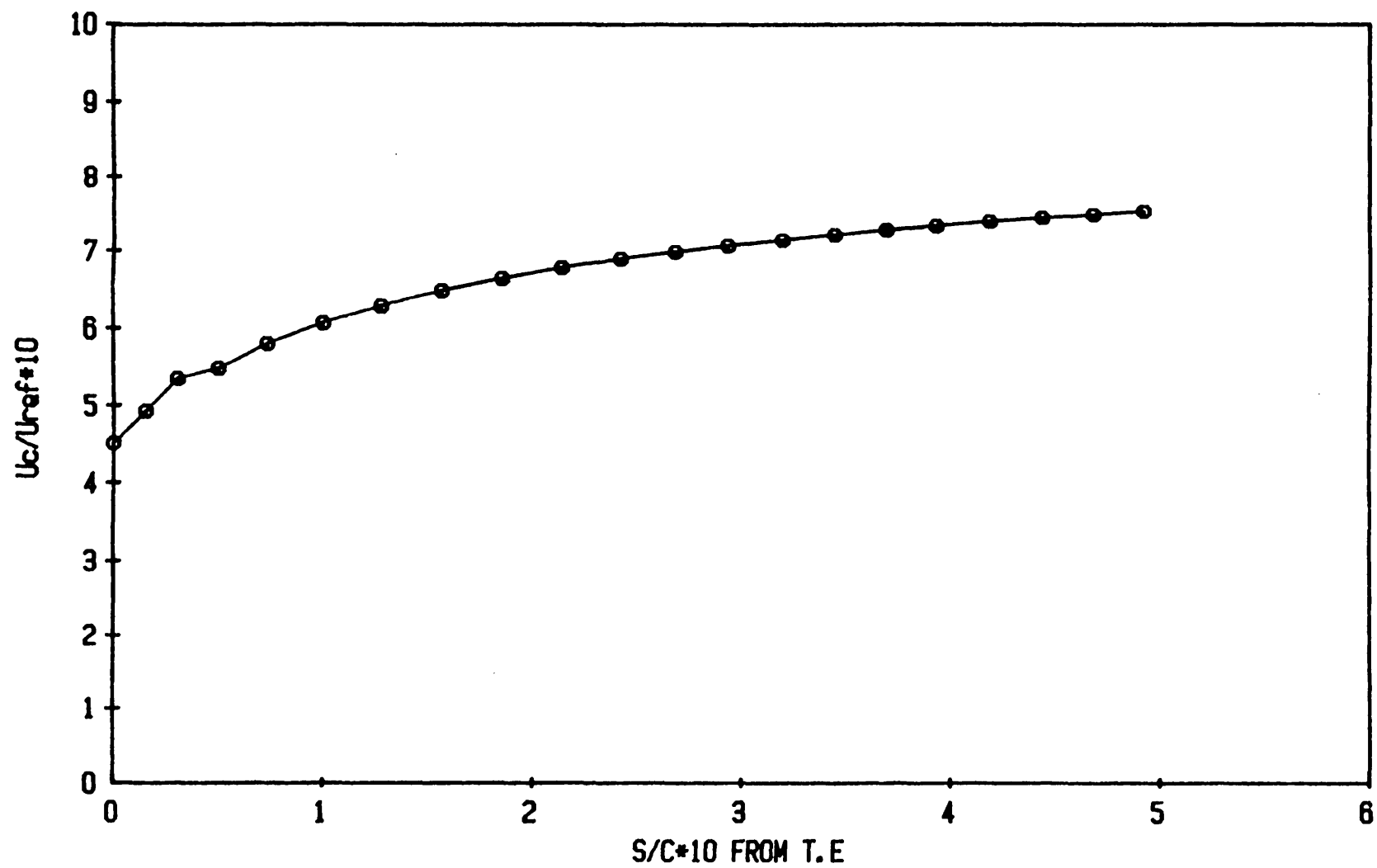


Fig.76 U_c DISTRIBUTION ALONG THE WAKE (ASYMM. AIRFOIL-MODEL A)

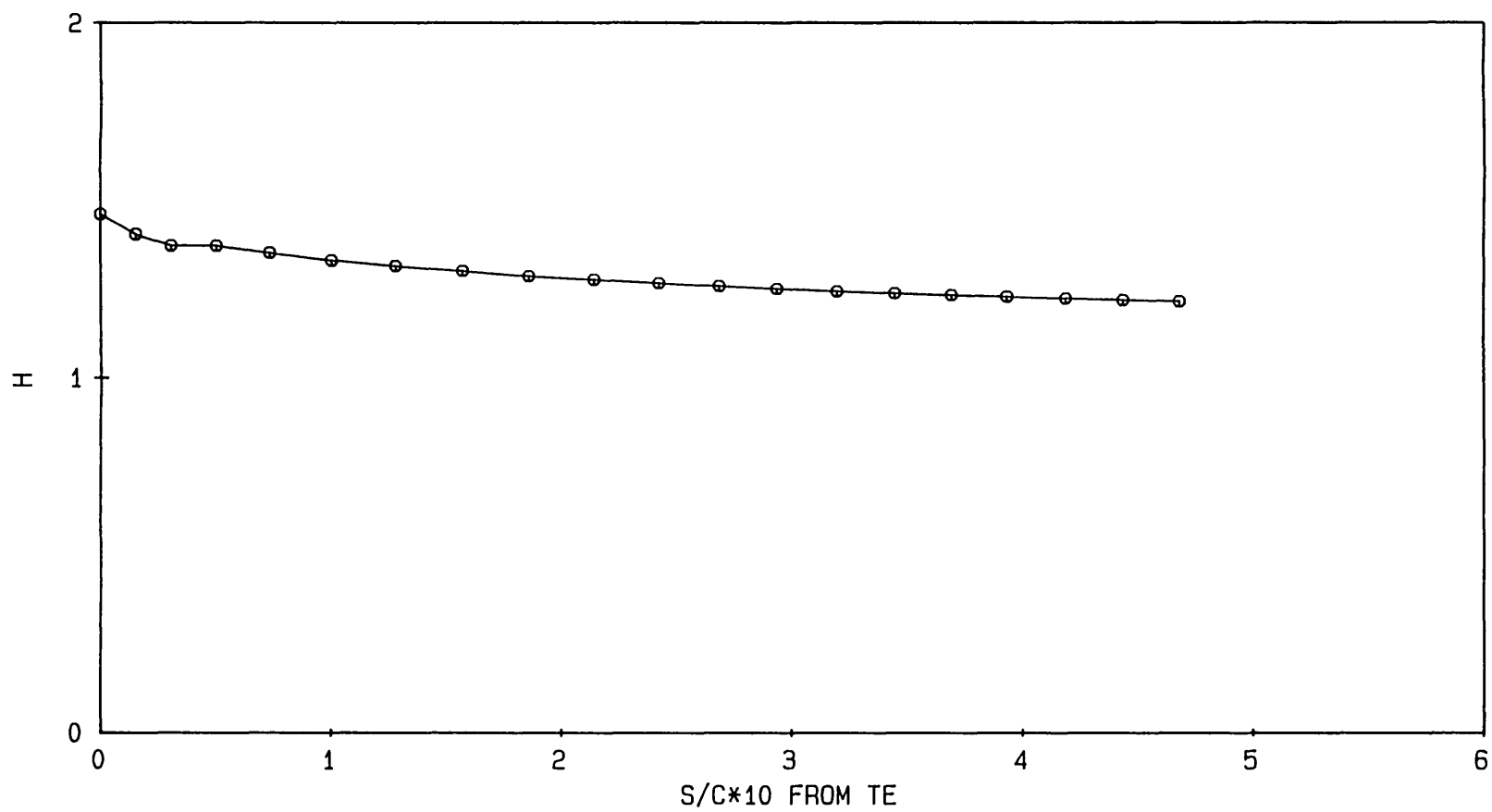


Fig.77 SHAPE FACTOR ALONG THE WAKE (ASYMMETRIC AEROFOIL-REF 71)

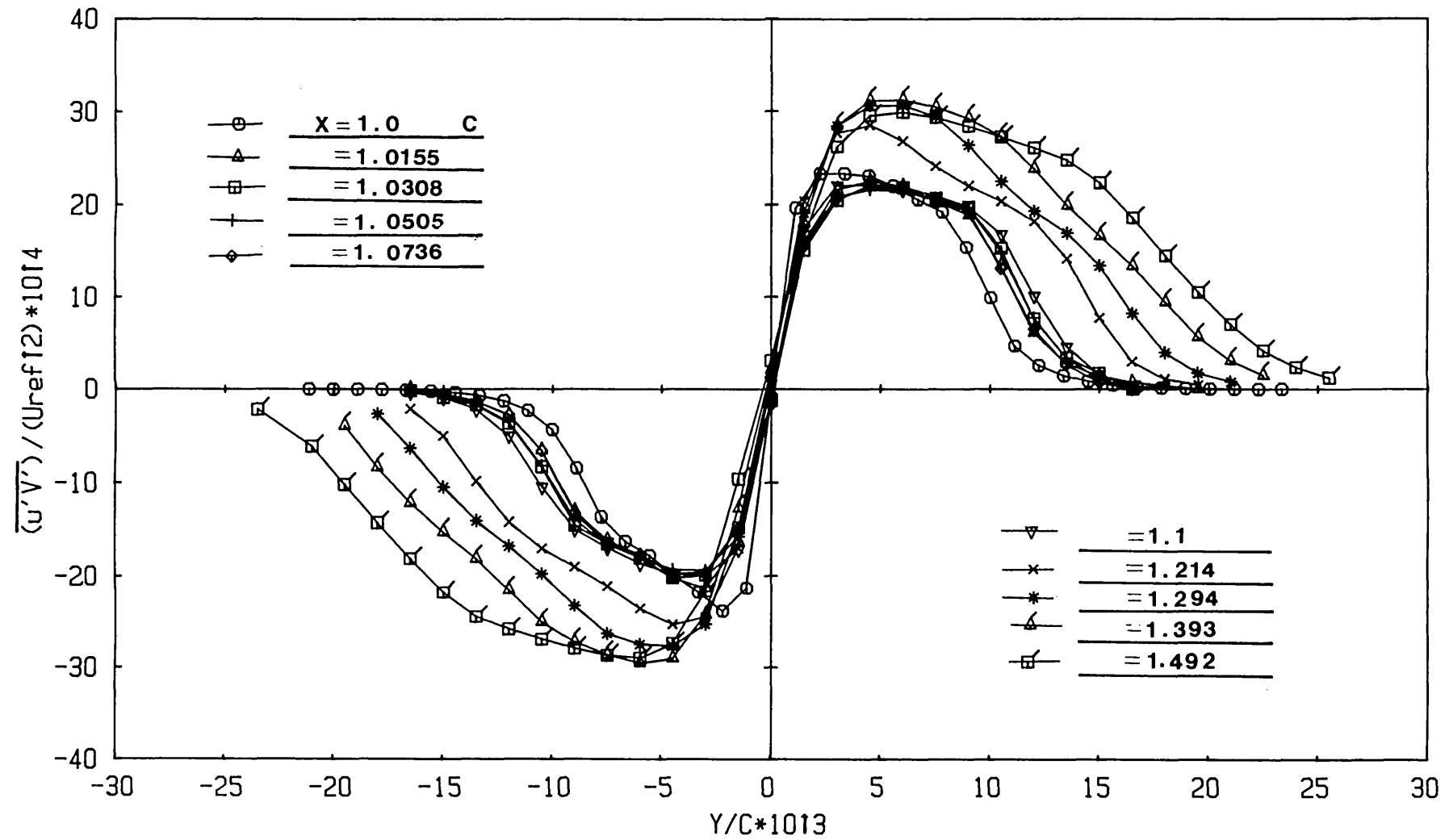


Fig.78 SHEAR STRESS DISTRIBUTION IN WAKE (NAKAYAMA' MODEL A)

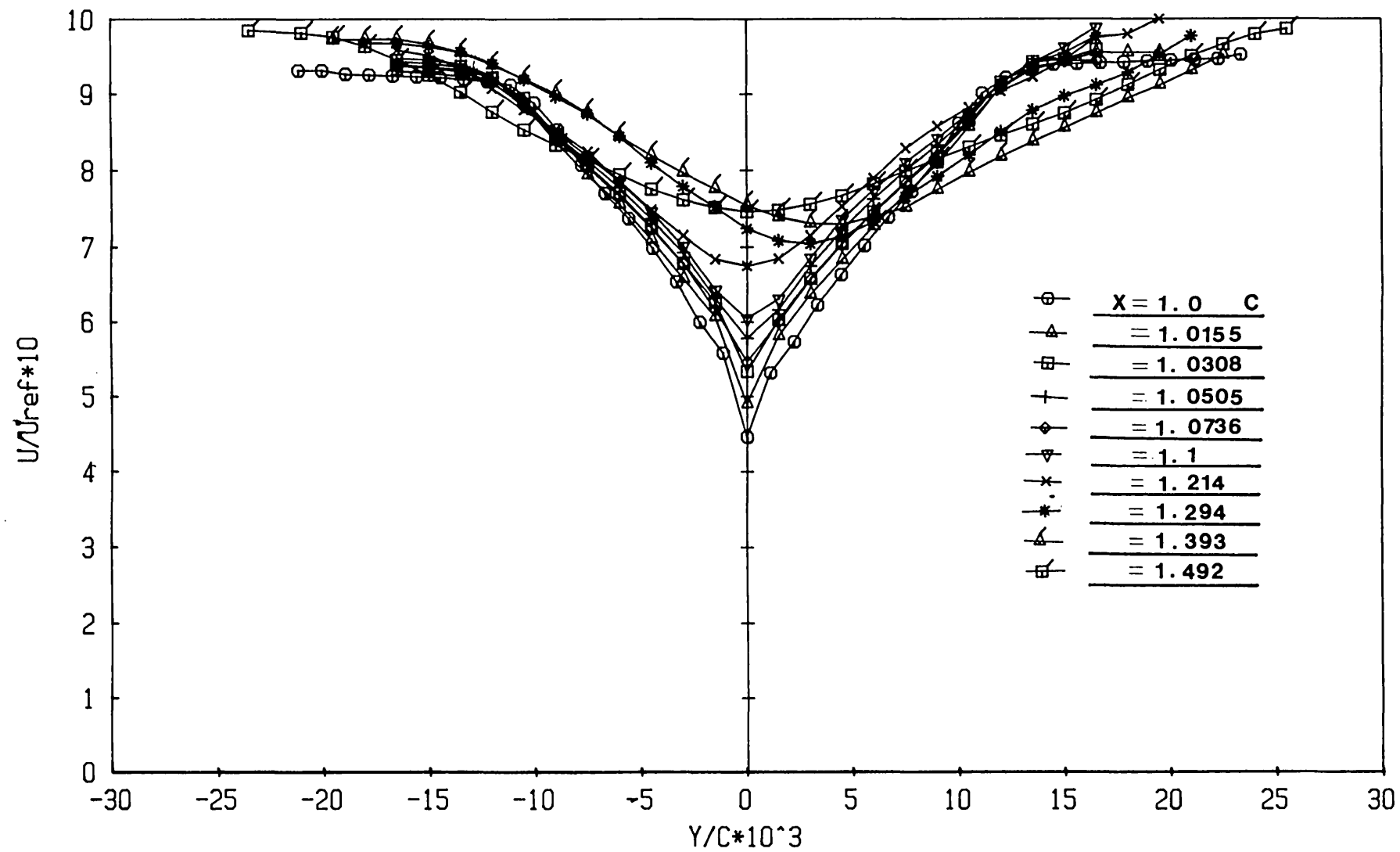


Fig.79 U VELOCITY DISTRIBUTION IN WAKE (Ref.71)

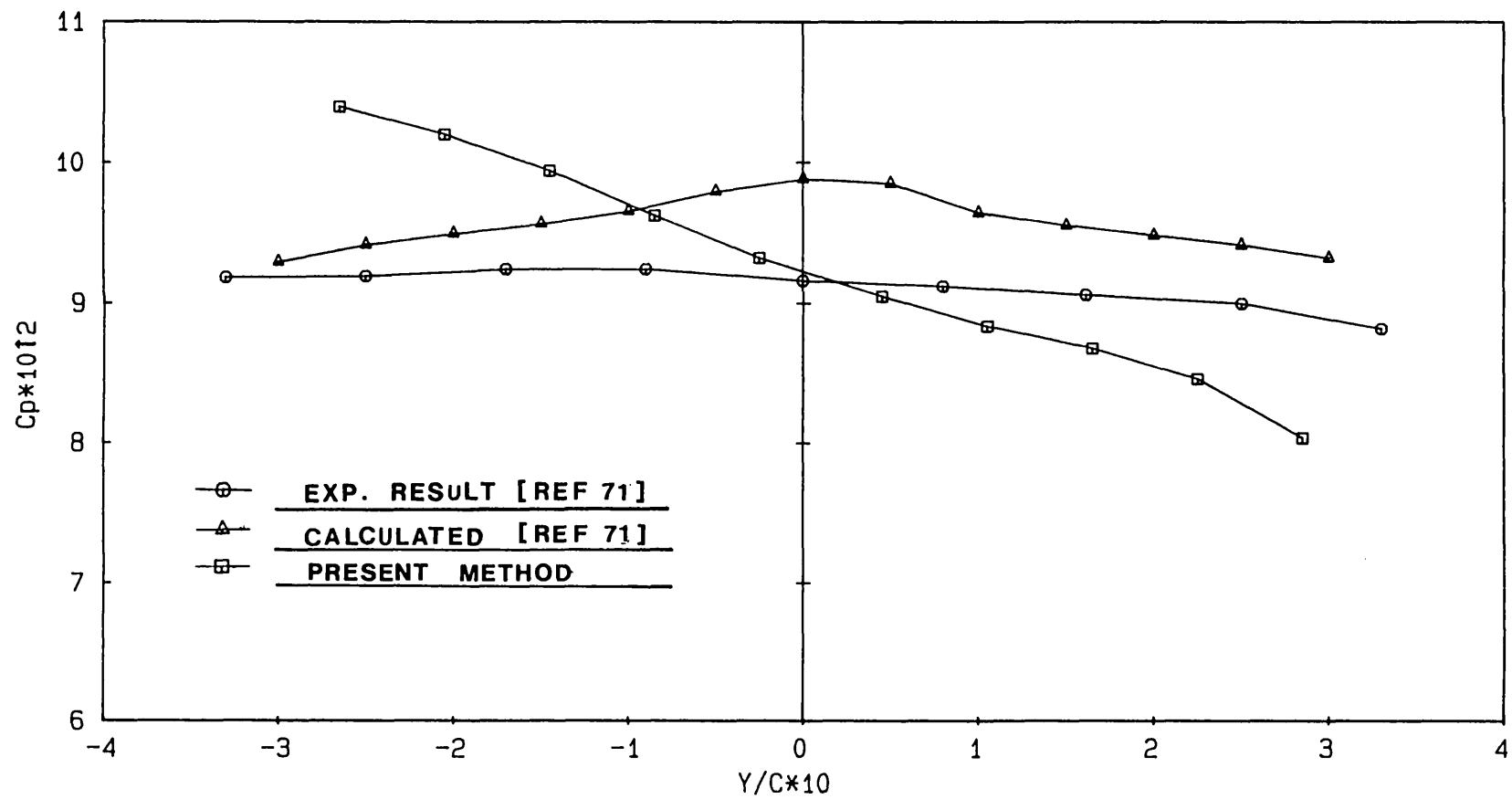


Fig. 80 C_p OF ASYMMETRIC AEROFOIL AT $X/C=1.05$ (REF 71)

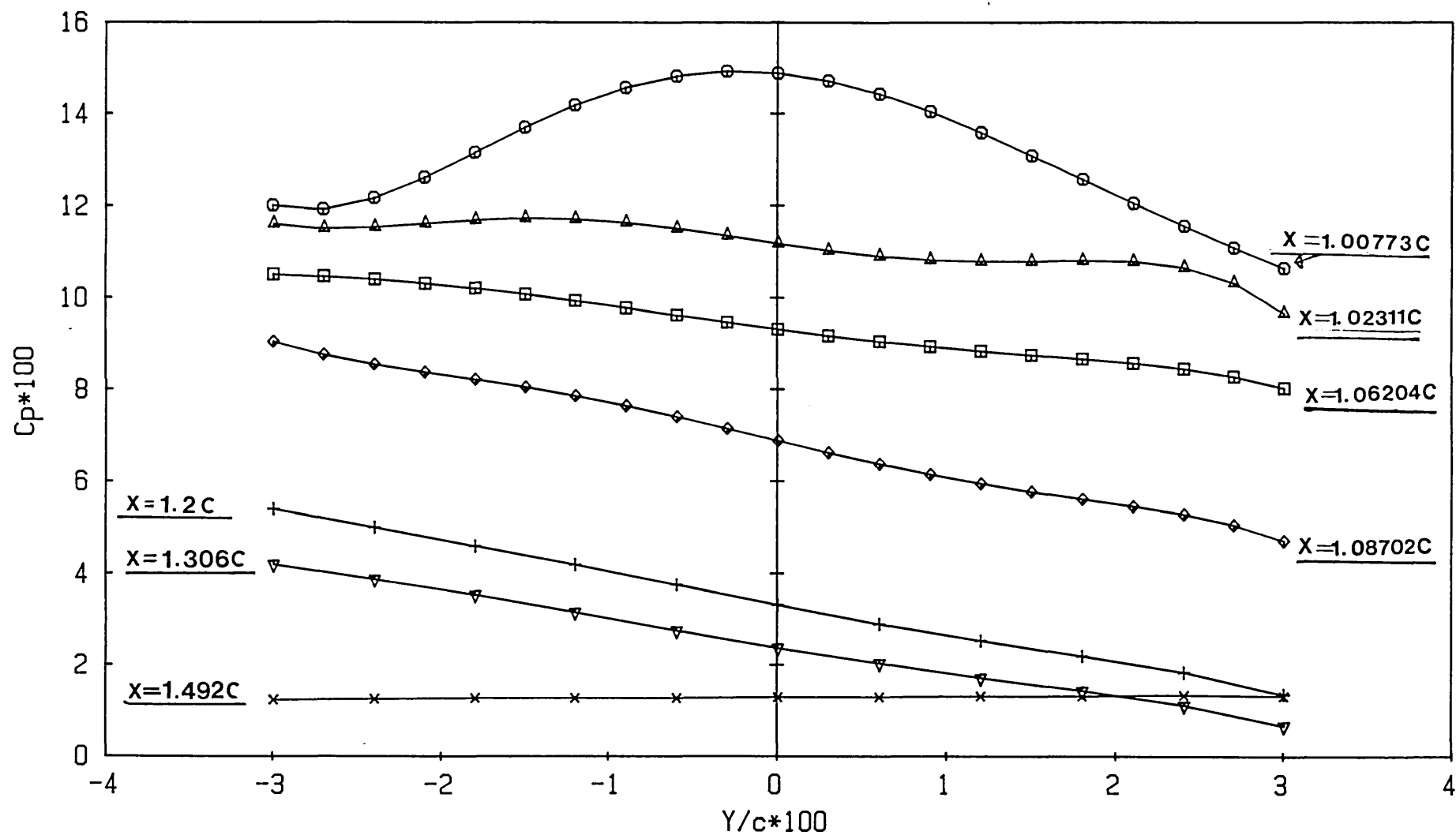


Fig.81 C_p DISTRIBUTION ACROSS THE WAKE (NAKAYAMA MODEL A)

Chapter 6. Conclusions

(1) The conventional viscous/inviscid interaction method (assuming $\partial p/\partial y=0$) is not adequate, and breaks down for flows which have large normal pressure gradients, especially near the trailing edge of lifting aerofoil. If normal pressure gradient is significant, the upstream influence within the shear layer is also likely to be significant and elliptic equations must be used to describe it. Hence some form of the normal-component momentum equation must be included. If we solve the full Navier-Stokes equations in this region, it costs much computing time. This leads us to search for new techniques to represent elliptic effects within the shear layer without solving full Navier-Stokes equations.

(2) The present method overcomes the problem as mentioned above by

- (i) matching viscous and inviscid flow just outside the shear layer edge
- (ii) combining the iterative scheme, which is required in the global matching of shear layer calculation and inviscid flow calculation, with the iterative scheme to solve elliptic equations for the shear layer. This is done by including the normal momentum equation, rather than matching on a displacement surface (or equivalent), and simply ignoring $\partial p/\partial y$. Hence upstream influence is propagated within the shear layer through the $\partial p/\partial x$

term in the streamwise momentum equation and $\partial V/\partial x$ term in the normal momentum equation.

If we use central differencing for $\partial p/\partial x$ and $\partial V/\partial x$ terms and calculate pressure at one step upstream station (Mahgoub 1978), pressure and velocity interaction cannot be occurred (Issa, 1984) and calculation diverges due to too much propagation of upstream influence inside shear layer: the upstream influence is propagated two steps in one sweep of the shear layer calculation instead of one. If one calculates U , V and p at the same point and then uses forward differencing for $\partial p/\partial x$ term and backward differencing for $\partial V/\partial x$ term (Chen 1982), the subsequent calculation of pressure and velocity is not consistent because $\partial V/\partial x$ is also a dominant term in the normal momentum equation. In order to remove this inconsistency Chen had to iterate pressure and velocity calculation at the same station within one sweep shear layer calculation until convergence was reached. This internal iteration costs a lot of computing time. In order to overcome these problems, we used staggered grid and forward differencing for $\partial p/\partial x$ and $\partial V/\partial x$ terms. Hence upstream influence is propagated one step inside shear layer. Because we use the same forward differencing scheme for $\partial p/\partial x$ and $\partial V/\partial x$ terms, the calculation of pressure and velocity is therefore consistent. It also costs much less computing time.

(3) The present method is economical and cost-effective as well as time-saving compared to full Navier-Stokes equation solvers. The computer storage is greatly reduced since only pressure is stored and further reduction is

brought about by a simple curve fit of the pressure. However its application is restricted to non-reversed flow.

(4) The convergence of the whole viscous/inviscid interaction cycle is very fast and usually monotonic. Generally we can obtain convergence in about 12 iteration cycles. The calculation results show large normal pressure gradient near the trailing edge region. Relaxation factor of .09 is used for the pressure within the shear layer calculation. On the other hand, relaxation factors of .3 in V_{ms} , the velocity component normal to the matching surface, and .3 for edge pressure are being used to accelerate the convergence.

(5) The usage of a body-fitted co-ordinate system resulted in either a nonorthogonal grid or a singularity problem occuring at the trailing edge. The wedge transformation employed near the trailing edge gives a singular but orthogonal grid near the trailing edge.

(6) The near wake region is very important for distribution of pressure along aerofoil. According to Andreopoulos (1980), the interaction between the two merging boundary layers is dominated mainly by the small scale fine-grained mixing in the inner wake region. As such the eddy viscosity model in the inner wake, based on Bradshaw (1973)'s superposition hypothesis, is used and has proved to be very successful since it is in good quantitative agreement with experimental data (Andreopoulos, 1980, Hebbar, 1986). It therefore demonstrates that the eddy viscosity is

an acceptable turbulence model for the inner wake in most engineering fields. But it is not accurate for the far wake region. Generally a more sophisticated and physical turbulence models for the far wake region would be needed if accurate wake results were required for, say, turbomachinery calculations.

(7) Since the normal pressure gradient is not small near LE, some approximation for the normal pressure gradient, e.g. the centrifugal approximation, would be favoured.

(8) This viscous/inviscid interaction method can be extended to compressible flow very easily by adding a temperature calculation. In the three dimensional flow case, the boundary layer program for infinite swept wing exists in Bradshaw's group. Hence it is helpful to have V_{ms} directly for viscous and inviscid matching.

(9) The velocity variation across the wake are shown to be inconsistent for asymmetric aerofoil, so there is a error in wake calculation : further check of the wake calculation is needed.

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Appendix 1.Derivation of the Navier-Stokes Equations in
Orthogonal Coordinates Using General Tensor Analysis

In this appendix we shall derive the Navier-Stokes equations in orthogonal coordinates using generalised tensor form. The reader is referred to Sokolonikoff (1966) for a general tensor analysis, while for a more advanced and applied to fluid mechanics see Aris (1967) and for a detailed derivation see Mahgoub (1978).

A1.1 General definitions and derivations

In general tensor transformation rule, there are two different basic types of tensors. These are the covariant tensor written with suffix a_i and the contravariant tensor written with superfix a^i . The difference between these two types of tensors is the transformation rule between coordinates. If we convert from coordinate system x^i to the other system $\bar{x}^i$, the transformation rule for covariant tensor is

$$\bar{a}_i = \frac{\partial x^i}{\partial \bar{x}^i} a_i \quad (\text{A1.1})$$

while for contravariant tensors it is

$$\bar{a}^i = \frac{\partial \bar{x}^i}{\partial x^j} a^j \quad (\text{A1.2})$$

The coordinate lengths are contravariant tensor because they transform according to the contravariant rule i.e. by the chain rule for partial differentiation.

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j} dx^j \quad (\text{A1.3})$$

Similarly, the time derivative of length (=velocity) is a contravariant tensor.

The arc length ds^2 is defined using covariant metric

tensor g_{ij} by

$$ds^2 = g_{ij} dx^i dx^j \quad (A1.4)$$

The contravariant component of the metric tensor can be defined by

$$g_{ij} g^{jk} = \delta_i^k$$

where $\delta_i^k = 1$ if $i=k$ and $\delta_i^k = 0$ if $i \neq k$.

By using metric value we can obtain contravariant component of a covariant tensor a_i as

$$a^j = g^{ij} a_i \quad (A1.5)$$

Conversely, the covariant component of a contravariant tensor a^i can be obtained.

$$a_j = g_{jk} a^k \quad (A1.6)$$

Now, we consider the partial derivative value $\partial a^i / \partial x^j$ of any contravariant tensor quantity a^i , it can be shown that this partial derivative does not transform under the tensor rule (A1.1) and (A1.2). We therefore have to define the "covariant derivative" of a^i as

$$a^i{}_{;j} = \frac{\partial a^i}{\partial x^j} + \left\{ \begin{matrix} i \\ k j \end{matrix} \right\} a^k \quad (A1.7)$$

and similarly for a covariant tensor

$$a_{i,j} = \frac{\partial a_i}{\partial x^j} - \left\{ \begin{matrix} k \\ j i \end{matrix} \right\} a_k \quad (A1.8)$$

where the symbol $\left\{ \begin{matrix} l \\ m n \end{matrix} \right\}$ is the Christoffel symbol of the second kind and is defined by

$$\left\{ \begin{matrix} l \\ m n \end{matrix} \right\} = \frac{1}{2} g^{lr} \left(\frac{\partial g_{mr}}{\partial x^n} + \frac{\partial g_{nr}}{\partial x^m} - \frac{\partial g_{mn}}{\partial x^r} \right) \quad (A1.9)$$

The derivatives defined by equations (A1.7) and (A1.8) are tensor quantities and transform under the tensor rules.

Similarly, derivatives of higher order tensors can be defined

$$a^{ij}_{,k} = \frac{\partial a^{ij}}{\partial x^k} + \{^i_{lk}\} a^{lj} + \{^j_{lk}\} a^{il} \quad (A1.10)$$

$$a_{ij,k} = \frac{\partial a_{ij}}{\partial x^k} - \{^l_{ki}\} a_{lj} - \{^l_{kj}\} a_{il} \quad (A1.11)$$

Since tensor represents the physical quantities, we have to distinguish between a tensor and its physical components. The physical j -component of a contravariant tensor a^j is defined as

$$a(j) = (g_{jj})^{1/2} a^j \quad (A1.12)$$

where summation over the index j is not performed. Similarly, the physical j -component of a covariant tensor a_j is defined as

$$a(j) = (g_{jj})^{1/2} g^{ij} a_i \quad (A1.13)$$

The tensor analysis outlined above apply for transformations in general coordinate systems. However, for orthogonal coordinate systems (i.e. curvilinear orthogonal (s,n) coordinate system or orthogonal (φ, ψ) coordinate system - near trailing edge region in this program), following simplification can be applied:

(i) The metric tensor $g_{ij}=0$ for $i \neq j$, so that only components $g_{ii}=h_i^2$, say, are non-zero. Similarly, $g^{ij}=0$ for $i \neq j$ and $g^{ii}=1/h_i^2$.

(ii) This reduces Christoffel symbol of the second kind as follows:

$$\text{If } \left\{^p_{qr}\right\} = \frac{1}{2} g^{pk} \left(\frac{\partial g_{qk}}{\partial x^r} + \frac{\partial g_{rk}}{\partial x^q} - \frac{\partial g_{qr}}{\partial x^k} \right)$$

then,

$$\text{if } p=q=r, \quad \left\{^q_{qq}\right\} = \frac{1}{2g_{qq}} \frac{\partial g_{qq}}{\partial x^q} = \frac{1}{h_q} \frac{\partial h_q}{\partial x^q}$$

(A1.14)

If $p=q \neq r$,

$$\left\{ \begin{matrix} q \\ q \ r \end{matrix} \right\} = \frac{1}{2g_{qq}} \frac{\partial g_{qq}}{\partial x^r} = \frac{1}{h_q} \frac{\partial h_q}{\partial x^r} \quad (\text{A1.15})$$

If $p \neq q=r$,

$$\left\{ \begin{matrix} p \\ r \ r \end{matrix} \right\} = -\frac{1}{2g_{pp}} \frac{\partial g_{rr}}{\partial x^p} = -\frac{h_r}{h_p^2} \frac{\partial h_r}{\partial x^p} \quad (\text{A1.16})$$

If $p \neq q \neq r$, then $\left\{ \begin{matrix} p \\ q \ r \end{matrix} \right\} = 0$.

(iii) The physical j -component of a contravariant tensor a^i is as before $a(j)=h_j a^j$. Similarly, for a covariant tensor $a(j)=a_j/h_j$.

Now we apply the above rules to our (s,n) coordinate and (φ, ψ) coordinate system (in two dimensions). The element of arc length in orthogonal coordinate system can be written

$$dS^2 = g_{11} dx_1^2 + g_{22} dx_2^2 \quad (\text{A1.17})$$

Using simplification for orthogonal coordinate system we obtain

$$dS^2 = h_1^2 ds^2 + h_2^2 dn^2$$

For (s,n) coordinate system,

$$h_1 = (1+n/R)$$

and

$$h_2 = 1 \quad (\text{A1.18})$$

Similarly, for (φ, ψ) coordinate system

$$dS^2 = h_1^2 d\varphi^2 + h_2^2 d\psi^2$$

From wedge transformation rule we obtain (see Fig.28)

$$\varphi = r^n \cos n\theta$$

$$\psi = r^n \sin n\theta$$

$$\begin{aligned} h_1 &= (\partial x / \partial \varphi)^2 + (\partial y / \partial \varphi)^2 \\ &= 1/n (\varphi^2 + \psi^2)^{1/2} (1/n - 1) \end{aligned}$$

$$\begin{aligned} h_2 &= (\partial x / \partial \psi)^2 + (\partial y / \partial \psi)^2 \\ &= 1/n (\varphi^2 + \psi^2)^{1/2} (1/n - 1) \end{aligned} \quad (\text{A1.19})$$

The only difference between (s,n) coordinate system and

(φ, ψ) coordinate system is h_1 and h_2 metric values. Hence we use h_1 and h_2 metric values without inserting particular values when deriving every equation. The Christoffel symbols of second kind can be written as

$$\begin{aligned}
 \left\{ \begin{matrix} 1 \\ 1 \ 1 \end{matrix} \right\} &= \frac{1}{h_1} \frac{\partial h_1}{\partial x^1} \\
 \left\{ \begin{matrix} 2 \\ 2 \ 2 \end{matrix} \right\} &= \frac{1}{h_2} \frac{\partial h_2}{\partial x^2} \\
 \left\{ \begin{matrix} 1 \\ 1 \ 2 \end{matrix} \right\} &= \left\{ \begin{matrix} 1 \\ 2 \ 1 \end{matrix} \right\} = \frac{1}{h_1} \frac{\partial h_1}{\partial x^2} \\
 \left\{ \begin{matrix} 2 \\ 2 \ 1 \end{matrix} \right\} &= \left\{ \begin{matrix} 2 \\ 1 \ 2 \end{matrix} \right\} = \frac{1}{h_2} \frac{\partial h_2}{\partial x^1} \\
 \left\{ \begin{matrix} 1 \\ 2 \ 2 \end{matrix} \right\} &= -\frac{h_2}{h_1^2} \frac{\partial h_2}{\partial x^1} \\
 \left\{ \begin{matrix} 2 \\ 1 \ 1 \end{matrix} \right\} &= -\frac{h_1}{h_2^2} \frac{\partial h_1}{\partial x^2}
 \end{aligned}
 \tag{A1.20}$$

A1.2 Derivation of the Navier-Stokes equations

As mentioned previous section, the velocity is defined as contravariant tensor and stress is also contravariant tensor. If we define the turbulent Reynolds stress tensor as

$$\tau^{ij} = \overline{\tau^{ij}} = -\rho u^i u^j$$

(where $\tau^{ij} = \tau^{ji}$) then the i -component momentum equation for compressible steady flow can be written

$$(\rho u^j) u^i_{,j} = -g^{ij} p_{,j} + \tau^{ij}_{,j} + \mu g^{jk} u^i_{,jk} \tag{A1.21}$$

where p is the mean pressure and μ is the viscosity.

The mean continuity equation can be written

$$(\rho u^i)_{,i} = 0 \tag{A1.22}$$

and the energy equation is

$$(\rho h u^i)_{,i} = u^i p_{,i} + \overline{u^i p_{,i}} + \tau^{ij} u^i_{,j} + \mu g^{ik} u^i_{,k} - q_{,i} \quad (A1.23)$$

In the following sections we shall derive the above three equations term by term in (s, n) and (ϕ, ψ) coordinate system from their generalised tensor form given above. The equation will be derived in terms of physical components of the quantities, which are defined as follows:

physical component	tensor quantity
U, u	$U^1 = \frac{U}{h_1}, \quad u^1 = \frac{u}{h_1}$
V, v	$U^2 = \frac{V}{h_2}, \quad u^2 = \frac{v}{h_2}$
p	p
$-\overline{\rho u^2}$	$\tau^{11} = -\frac{\overline{\rho u^2}}{h_1^2}$
$-\overline{\rho v^2}$	$\tau^{22} = -\frac{\overline{\rho v^2}}{h_2^2}$
$-\overline{\rho uv}$	$\tau^{21} = \tau^{12} = -\frac{\overline{\rho uv}}{h_1 h_2}$
T	T
H	H
h	n
ρ	ρ
dx, dy	dx^1, dx^2

(I) The continuity equation

From equation (A1.22) the continuity equation is

$(\rho U^i)_{,i} = 0$ and using the definition (A1.7) of the covariant derivative of a contravariant tensor, this can be written as:

$$(\rho U^i)_{,i} = \frac{\partial \rho U^i}{\partial x^i} + \left\{ \begin{matrix} i \\ k \ i \end{matrix} \right\} (\rho U^k) \quad (\text{A1.24})$$

Inserting the physical components directly, we can obtain:

$$\begin{aligned} & \frac{\partial}{\partial x} \left(\frac{\rho U}{h_1} \right) + \frac{\partial}{\partial y} \left(\frac{\rho V}{h_2} \right) + \left\{ \begin{matrix} 1 \\ 1 \ 1 \end{matrix} \right\} \left(\frac{\rho U}{h_1} \right) + \left\{ \begin{matrix} 1 \\ 2 \ 1 \end{matrix} \right\} \left(\frac{\rho V}{h_2} \right) \\ & + \left\{ \begin{matrix} 2 \\ 1 \ 2 \end{matrix} \right\} \left(\frac{\rho U}{h_1} \right) + \left\{ \begin{matrix} 2 \\ 2 \ 2 \end{matrix} \right\} \left(\frac{\rho V}{h_2} \right) = 0 \end{aligned} \quad (\text{A1.25})$$

Using the Christoffel symbols given by equation (A1.20) this reduces to

$$\begin{aligned} & \frac{\partial}{\partial x} \left(\frac{\rho U}{h_1} \right) + \frac{\partial}{\partial y} \left(\frac{\rho V}{h_2} \right) + \frac{1}{h_1} \frac{\partial h_1}{\partial x} \frac{\rho U}{h_1} + \\ \text{or} \quad & \frac{1}{h_1} \frac{\partial h_1}{\partial y} \frac{\rho V}{h_2} + \frac{1}{h_2} \frac{\partial h_2}{\partial x} \frac{\rho U}{h_1} + \frac{1}{h_2} \frac{\partial h_2}{\partial y} \frac{\rho V}{h_2} = 0 \\ & \frac{\partial}{\partial x} (h_2 U) + \frac{\partial}{\partial y} (h_1 V) = -h_2 U \frac{1}{\rho} \frac{\partial \rho}{\partial x} - h_1 V \frac{1}{\rho} \frac{\partial \rho}{\partial y} \end{aligned} \quad (\text{A1.26})$$

where $h_1 = 1 + \frac{n}{R}$, $h_2 = 1$ for (s,n) coordinate system

$$\begin{aligned} h_1 &= \frac{1}{n} (\varphi^2 + \psi^2)^{1/2} (1/n - 1) \\ h_2 &= \frac{1}{n} (\varphi^2 + \psi^2)^{1/2} (1/n - 1) \end{aligned} \quad \text{for } (\varphi, \psi) \text{ coordinate system.}$$

(II) The momentum equation

We shall derive each of terms of the momentum equation (A1.21) separately, for clarity.

(a) The acceleration term

This is the term

$$(\rho U^j)_{U,j} = \rho U^j \frac{\partial U^j}{\partial x^j} + \left\{ \begin{matrix} j \\ i \ j \end{matrix} \right\} \rho U^j U^i \quad (\text{A1.27})$$

(i) For i=1 (the streamwise direction)

$$\begin{aligned} & \frac{\rho U}{h_1} \left[\frac{\partial}{\partial x} \left(\frac{U}{h_1} \right) + \frac{1}{h_1} \frac{\partial h_1}{\partial x} \frac{U}{h_1} + \frac{1}{h_1} \frac{\partial h_1}{\partial y} \frac{V}{h_2} \right] \\ & + \frac{\rho V}{h_2} \left[\frac{\partial}{\partial y} \left(\frac{U}{h_1} \right) + \frac{1}{h_1} \frac{\partial h_1}{\partial y} \frac{U}{h_1} - \frac{1}{h_1} \frac{\partial h_2}{\partial x} \frac{V}{h_1} \right] \end{aligned} \quad (\text{A1.28})$$

(ii) For i=2 (the normal direction)

$$\begin{aligned} & \frac{\rho U}{h_1} \left[\frac{\partial}{\partial x} \left(\frac{v}{h_2} \right) - \frac{U}{h_2^2} \frac{\partial h_1}{\partial y} + \frac{1}{h_2} \frac{\partial h_2}{\partial x} \frac{v}{h_2} \right] \\ & - \frac{\rho V}{h_2} \left[\frac{\partial}{\partial y} \left(\frac{v}{h_2} \right) + \frac{1}{h_2} \frac{\partial h_2}{\partial x} \frac{U}{h_1} + \frac{1}{h_2} \frac{\partial h_2}{\partial y} \frac{v}{h_2} \right] \end{aligned} \quad (A1.29)$$

(b) The pressure gradient term

This is the term

$$-g^{ij} p_{,j} = -g^{ii} \frac{\partial p}{\partial x^i} = -\frac{1}{h_i^2} \frac{\partial p}{\partial x^i} \quad (A1.30)$$

Since the covariant derivative of scalar is just its partial derivative and since $g^{ij}=0$ for $i \neq j$,

(i) For i=1

$$-\frac{1}{h_1^2} \frac{\partial p}{\partial x} \quad (A1.31)$$

(ii) For i=2

$$-\frac{1}{h_2^2} \frac{\partial p}{\partial y} \quad (A1.32)$$

(c) The Reynolds stress Gradient term

This is the term

$$\tau_{,j}^{ij} = \frac{\partial \tau^{ij}}{\partial x^j} - \left\{ \begin{matrix} i \\ 1 \ j \end{matrix} \right\} \tau^{1j} + \left\{ \begin{matrix} j \\ 1 \ i \end{matrix} \right\} \tau^{i1} \quad (A1.33)$$

from the covariant derivative of a second order tensor, equation (A1.10).

(i) For i=1

$$\frac{\partial \tau^{1j}}{\partial x^j} + \left\{ \begin{matrix} 1 \\ 1 \ j \end{matrix} \right\} \tau^{1j} + \left\{ \begin{matrix} j \\ 1 \ 1 \end{matrix} \right\} \tau^{11}$$

Since

$$\begin{aligned} \left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\} \tau^{11} &= \left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\} \tau^{11} + 2 \left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\} \tau^{12} + \left\{ \begin{matrix} 1 \\ 2 \end{matrix} \right\} \tau^{22} \\ &= \frac{1}{h_1} \frac{\partial h_1}{\partial x} \tau^{11} + \frac{2}{h_1} \frac{\partial h_1}{\partial y} \tau^{12} - \frac{h_2}{h_1^2} \frac{\partial h_2}{\partial x} \tau^{22} \end{aligned}$$

and

$$\begin{aligned} \left\{ \begin{matrix} j \\ 1 \end{matrix} \right\} \tau^{11} &= \left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\} \tau^{11} + \left\{ \begin{matrix} 1 \\ 2 \end{matrix} \right\} \tau^{12} + \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} \tau^{11} + \left\{ \begin{matrix} 2 \\ 2 \end{matrix} \right\} \tau^{12} \\ &= \frac{1}{h_1} \frac{\partial h_1}{\partial x} \tau^{11} + \frac{1}{h_1} \frac{\partial h_1}{\partial y} \tau^{12} + \frac{1}{h_2} \frac{\partial h_2}{\partial x} \tau^{11} \\ &\quad + \frac{1}{h_2} \frac{\partial h_2}{\partial y} \tau^{12} \end{aligned}$$

$$\begin{aligned} \therefore \frac{\partial}{\partial x} \left(\frac{-\rho u^2}{h_1^2} \right) + \frac{\partial}{\partial y} \left(\frac{-\rho v u}{h_1 h_2} \right) + \frac{2}{h_1} \frac{\partial h_1}{\partial x} \frac{-\rho u^2}{h_1^2} \\ + \frac{3}{h_1} \frac{\partial h_1}{\partial y} \frac{-\rho u v}{h_1 h_2} + \frac{1}{h_2} \frac{\partial h_2}{\partial y} \frac{-\rho u v}{h_1 h_2} + \left(\frac{\rho u^2}{h_1^2} \frac{1}{h_2} - \frac{\rho v^2 h_2}{h_2^2 h_1^2} \right) \frac{\partial h_2}{\partial x} \end{aligned} \quad (A1.34)$$

(ii) For i=2

$$\frac{\partial \tau^{2j}}{\partial x^j} + \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} \tau^{1j} + \left\{ \begin{matrix} j \\ 1 \end{matrix} \right\} \tau^{21}$$

Since

$$\begin{aligned} \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} \tau^{1j} \\ = -\frac{h_1}{h_2^2} \frac{\partial h_1}{\partial y} \tau^{11} + 2 \frac{1}{h_2} \frac{\partial h_2}{\partial x} \tau^{12} + \frac{1}{h_2} \frac{\partial h_2}{\partial y} \tau^{22} \end{aligned}$$

and

$$\begin{aligned} \left\{ \begin{matrix} j \\ 1 \end{matrix} \right\} \tau^{21} &= \frac{1}{h_1} \frac{\partial h_1}{\partial x} \tau^{21} + \frac{1}{h_2} \frac{\partial h_2}{\partial x} \tau^{21} + \frac{1}{h_1} \frac{\partial h_1}{\partial y} \tau^{22} \\ &\quad + \frac{1}{h_2} \frac{\partial h_2}{\partial y} \tau^{22} \end{aligned}$$

$$\begin{aligned} \therefore \frac{\partial}{\partial x} \left(\frac{-\rho u v}{h_1 h_2} \right) + \frac{\partial}{\partial y} \left(\frac{-\rho v^2}{h_2^2} \right) - \frac{h_1}{h_2^2} \frac{\partial h_1}{\partial y} \frac{-\rho u^2}{h_1^2} + \frac{-\rho u v}{h_1 h_2} \left(\frac{3}{h_2} \frac{\partial h_2}{\partial x} + \frac{1}{h_1} \frac{\partial h_1}{\partial x} \right) \\ + \frac{-\rho v^2}{h_2^2} \left(\frac{2}{h_2} \frac{\partial h_2}{\partial y} - \frac{1}{h_1} \frac{\partial h_1}{\partial y} \right) \end{aligned} \quad (A1.35)$$

(d) The viscous stress gradient term

The i-component of this term is given by

$$\mu g^{jk} u_{,jk}^i$$

and since $g^{jk}=0$ for $j \neq k$, then

$$\mu g^{kk} u_{,kk}^i = \mu \left[\frac{u_{,11}^i}{h_1^2} + \frac{u_{,22}^i}{h_2^2} \right] \quad (\text{A1.36})$$

Firstly, we shall deduce a general expression for the quantity $u_{,jk}^i$.

By the definition of covariant differentiation

$$\begin{aligned} u_{,jk}^i &= (u_{,j}^i)_{,k} = \frac{\partial u_{,j}^i}{\partial x^k} - \left\{ \begin{matrix} i \\ k \ j \end{matrix} \right\} u_{,j}^i + \left\{ \begin{matrix} i \\ j \ k \end{matrix} \right\} u_{,j}^i \\ &= \frac{\partial}{\partial x^k} \left[\frac{\partial u^i}{\partial x^j} + \left\{ \begin{matrix} i \\ j \ l \end{matrix} \right\} u^l \right] - \left\{ \begin{matrix} i \\ k \ j \end{matrix} \right\} \left[\frac{\partial u^i}{\partial x^j} + \left\{ \begin{matrix} i \\ j \ m \end{matrix} \right\} u^m \right] \\ &\quad + \left\{ \begin{matrix} i \\ j \ k \end{matrix} \right\} \left[\frac{\partial u^i}{\partial x^j} + \left\{ \begin{matrix} i \\ j \ m \end{matrix} \right\} u^m \right] \end{aligned}$$

which reduces to

$$\begin{aligned} u_{,jk}^i &= \frac{\partial^2 u^i}{\partial x^j \partial x^k} + \left\{ \begin{matrix} i \\ j \ l \end{matrix} \right\} \frac{\partial u^l}{\partial x^k} - \left\{ \begin{matrix} i \\ k \ j \end{matrix} \right\} \frac{\partial u^i}{\partial x^j} + \left\{ \begin{matrix} i \\ j \ k \end{matrix} \right\} \frac{\partial u^i}{\partial x^j} \\ &\quad + u^m \frac{\partial}{\partial x^k} \left\{ \begin{matrix} i \\ j \ m \end{matrix} \right\} - u^m \left\{ \begin{matrix} i \\ k \ j \end{matrix} \right\} \left\{ \begin{matrix} i \\ j \ m \end{matrix} \right\} + u^m \left\{ \begin{matrix} i \\ j \ k \end{matrix} \right\} \left\{ \begin{matrix} i \\ j \ m \end{matrix} \right\} \end{aligned}$$

Now for $j=k$ this becomes

$$\begin{aligned} u_{,kk}^i &= \frac{\partial^2 u^i}{\partial x^k \partial x^k} + 2 \left\{ \begin{matrix} i \\ j \ k \end{matrix} \right\} \frac{\partial u^j}{\partial x^k} - \left\{ \begin{matrix} i \\ k \ k \end{matrix} \right\} \frac{\partial u^i}{\partial x^j} \\ &\quad + u^m \frac{\partial}{\partial x^k} \left\{ \begin{matrix} i \\ k \ m \end{matrix} \right\} - \left\{ \begin{matrix} i \\ k \ k \end{matrix} \right\} \left\{ \begin{matrix} i \\ j \ m \end{matrix} \right\} u^m + \left\{ \begin{matrix} i \\ j \ k \end{matrix} \right\} \left\{ \begin{matrix} i \\ k \ m \end{matrix} \right\} u^m \quad (\text{A1.37}) \end{aligned}$$

(i) For $i=1$

$$\mu \left[\frac{u_{,11}^1}{h_1^2} + \frac{u_{,22}^1}{h_2^2} \right]$$

Inserting $i=1$ in (A1.37) and using the definitions of the Christoffel symbols given by equations (A1.20) we obtain after some algebra

$$\begin{aligned} u_{,11}^1 &= \frac{\partial^2 u^1}{\partial x^2} + \frac{1}{h_1} \frac{\partial h_1}{\partial x} \frac{\partial u^1}{\partial x} + 2 \frac{1}{h_1} \frac{\partial h_1}{\partial y} \frac{\partial u^2}{\partial x} \\ &\quad + \frac{h_1}{h_2^2} \frac{\partial h_1}{\partial y} \frac{\partial u^2}{\partial y} \end{aligned}$$

$$\begin{aligned}
& + u^1 \frac{\partial}{\partial x} \left(\frac{1}{h_1} \frac{\partial h_1}{\partial x} \right) + u^2 \left[\frac{\partial}{\partial x} \left(\frac{1}{h_1} \frac{\partial h_1}{\partial y} \right) - \frac{h_2}{h_1} \frac{\partial h_2}{\partial x} \frac{\partial h_1}{\partial y} \right. \\
& \left. + \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial x} \frac{\partial h_1}{\partial y} \right] \\
\text{and} \quad u_{,22}^1 &= \frac{\partial^2 u^1}{\partial y^2} + \frac{\partial u^1}{\partial y} \left[\frac{2}{h_1} \frac{\partial h_1}{\partial y} - \frac{1}{h_2} \frac{\partial h_2}{\partial y} \right] + \frac{\partial u^2}{\partial y} \frac{2h_2}{h_1^2} \frac{\partial h_2}{\partial x} \\
& + \frac{\partial u^1}{\partial x} \frac{h_2}{h_1^2} \frac{\partial h_2}{\partial x} + u^1 \frac{\partial}{\partial y} \left(\frac{1}{h_1} \frac{\partial h_1}{\partial y} \right) + u^2 \frac{\partial}{\partial y} \left(\frac{h_2}{h_1^2} \frac{\partial h_2}{\partial x} \right) \\
& + u^1 \left[\frac{-1}{h_1^2} \frac{\partial h_2}{\partial x} \frac{\partial h_2}{\partial x} + \frac{1}{h_1^2} \frac{\partial h_1}{\partial y} \frac{\partial h_1}{\partial y} - \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial y} \frac{\partial h_1}{\partial y} + \frac{h_2}{h_1^3} \frac{\partial h_1}{\partial x} \frac{\partial h_2}{\partial x} \right]
\end{aligned}$$

Therefore

$$\begin{aligned}
\therefore \frac{1}{h_1^2} & \left[\frac{\partial^2}{\partial x^2} \left(\frac{u}{h_1} \right) + \frac{\partial}{\partial x} \left(\frac{u}{h_1^2} \frac{\partial h_1}{\partial x} + \frac{v}{h_1 h_2} \frac{\partial h_1}{\partial y} \right) \right. \\
& \left. + \frac{\partial h_1}{\partial y} \left(\frac{h_1}{h_2^2} \frac{\partial}{\partial y} \left(\frac{u}{h_1} \right) + \frac{1}{h_1} \frac{\partial}{\partial x} \left(\frac{v}{h_2} \right) \right) \right] \\
& + \frac{1}{h_2^2} \left[\frac{\partial^2}{\partial y^2} \left(\frac{u}{h_1} \right) + \frac{\partial}{\partial y} \left(\frac{u}{h_1^2} \frac{\partial h_1}{\partial y} - \frac{v}{h_1^2} \frac{\partial h_2}{\partial x} \right) \right. \\
& + \frac{h_2}{h_1^3} \frac{\partial h_2}{\partial x} \frac{\partial u}{\partial x} - \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial y} \frac{\partial u}{\partial y} + \frac{1}{h_1^2} \frac{\partial h_1}{\partial y} \frac{\partial u}{\partial y} \\
& \left. + \frac{h_2}{h_1 h_2} \frac{\partial h_2}{\partial x} \frac{\partial h_2}{\partial y} - \frac{1}{h_1^2} \frac{\partial h_2}{\partial x} \frac{\partial v}{\partial y} - \frac{u}{h_1^3} \left(\frac{\partial h_2}{\partial x} \right)^2 \right]
\end{aligned} \quad (A1.38)$$

(ii) For $i=2$

Similarly

$$\begin{aligned}
& \frac{1}{h_1^2} \left[\frac{\partial^2}{\partial x^2} \left(\frac{v}{h_2} \right) - 2 \frac{h_1}{h_2^2} \frac{\partial h_1}{\partial y} \frac{\partial}{\partial x} \left(\frac{u}{h_1} \right) + 2 \frac{1}{h_2} \frac{\partial h_2}{\partial x} \frac{\partial}{\partial x} \left(\frac{v}{h_2} \right) \right. \\
& - \frac{1}{h_1} \frac{\partial h_1}{\partial x} \frac{\partial}{\partial x} \left(\frac{v}{h_2} \right) + \frac{1}{h_2^2} \frac{\partial h_1}{\partial y} \frac{\partial}{\partial y} \left(\frac{v}{h_2} \right) - \frac{u}{h_1} \frac{1}{x} \left(\frac{1}{h_2^2} \frac{h_1}{y} \right) \\
& - \frac{v}{h_2} \left\{ \frac{\partial}{\partial x} \left(\frac{1}{h_2} \frac{\partial h_2}{\partial x} \right) - \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial x} \frac{\partial h_2}{\partial x} + \frac{h_1}{h_2^3} \frac{\partial h_1}{\partial y} \frac{\partial h_2}{\partial y} \right. \\
& \left. \left. - \left(\frac{1}{h_2} \frac{\partial h_1}{\partial y} \right)^2 + \left(\frac{1}{h_2} \frac{\partial h_2}{\partial x} \right)^2 \right\} \right] + \frac{1}{h_2^2} \left[\frac{\partial^2}{\partial y^2} \left(\frac{v}{h_2} \right) + \frac{2}{h_2} \frac{\partial h_2}{\partial x} \right. \\
& \frac{\partial}{\partial y} \left(\frac{u}{h_1} \right) - \frac{1}{h_2} \frac{\partial h_2}{\partial y} \frac{\partial}{\partial y} \left(\frac{v}{h_2} \right) + \frac{h_2}{h_1^2} \frac{\partial h_2}{\partial x} \frac{\partial}{\partial x} \left(\frac{v}{h_2} \right) - \frac{u}{h_1} (A1.39) \\
& \left. \left\{ \frac{\partial}{\partial y} \left(\frac{1}{h_2} \frac{\partial h_2}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{h_2} \frac{\partial h_2}{\partial y} \right) - \frac{h_1}{h_2^3} \frac{\partial h_2}{\partial x} \frac{\partial h_1}{\partial y} - \left(\frac{1}{h_2} \frac{\partial h_2}{\partial x} \right)^2 \right\} \right]
\end{aligned}$$

(III) The energy equation

This is

$$\begin{aligned}
& (\rho C_p T u^i)_{,i} \\
& = u^i g^{ij} p_{,j} - \tau^{ij} u_{,j}^i + k (g^{ij} T_{,j})_{,i} + \overline{u^i p_{,i}} - \overline{(\rho u^i h')_{,i}} \quad (A1.40)
\end{aligned}$$

$$\begin{aligned}
(\rho C_p T u^i)_{,i} &= \frac{\partial}{\partial x} (\rho C_p T \frac{u}{h_1}) + \frac{\partial}{\partial y} (\rho C_p T \frac{v}{h_2}) \\
& + \frac{u}{h_1^2} \frac{\partial h_1}{\partial x} \rho C_p T + \rho C_p T \frac{v}{h_1 h_2} \frac{\partial h_1}{\partial y} \\
& + \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial x} \rho C_p T + \rho C_p T \frac{1}{h_2^2} \frac{\partial h_2}{\partial y}
\end{aligned}$$

$$u^i p_{,i} = \frac{u}{h_1} \frac{\partial p}{\partial x} + \frac{v}{h_2} \frac{\partial p}{\partial y}$$

$$\tau^{ij} u_{,i} = \mu g^{jk} u_{,k} u_{,j} = \phi \quad (A1.41)$$

$$\begin{aligned} \therefore \frac{\partial}{\partial x} \left(\rho C_p T \frac{u}{h_1} \right) + \frac{\partial}{\partial y} \left(\rho C_p T \frac{v}{h_2} \right) + \rho C_p T \frac{1}{h_2^2} \frac{\partial h_2}{\partial y} \\ + \rho C_p T \frac{v}{h_1 h_2} \frac{\partial h_1}{\partial y} + \rho C_p T \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial x} + \rho C_p T \frac{u}{h_1^2} \frac{\partial h_1}{\partial x} \\ = \frac{u}{h_1} \frac{\partial p}{\partial x} + \frac{v}{h_2} \frac{\partial p}{\partial y} + k (g^{ij} T_{,j})_{,i} + \phi \end{aligned}$$

(A1.42)

Thus the complete Navier-Stokes equations for compressible steady flow are obtained:

The continuity equation:

$$\rho \left[\frac{\partial}{\partial x} (h_2 u) + \frac{\partial}{\partial y} (h_1 v) \right] = -h_2 u \frac{\partial \rho}{\partial x} - h_1 v \frac{\partial \rho}{\partial y} \quad (A1.43)$$

The streamwise momentum equation:

$$\begin{aligned} \rho u \left[h_2 \frac{\partial u}{\partial x} + \frac{\partial h_1}{\partial y} v \right] + \bar{\rho} v \left[h_1 \frac{\partial u}{\partial y} - \frac{\partial h_2}{\partial x} v \right] \\ = -h_2 \frac{\partial p}{\partial x} - 2 \frac{\partial h_1}{\partial y} \bar{\rho} u v - h_1 \frac{\partial}{\partial y} (\bar{\rho} u^2) \\ - \frac{\partial h_2}{\partial x} (\bar{\rho} u^2 - \bar{\rho} v^2) - h_2 \frac{\partial}{\partial x} (\bar{\rho} u^2) \end{aligned} \quad (A1.44)$$

The normal momentum equation:

$$\begin{aligned} \rho u \left[h_2 \frac{\partial v}{\partial x} - \frac{\partial h_1}{\partial y} u \right] + \bar{\rho} v \left[h_1 \frac{\partial v}{\partial y} + \frac{\partial h_2}{\partial x} u \right] \\ = -h_1 \frac{\partial p}{\partial y} - \left[h_2 \frac{\partial}{\partial x} (\bar{\rho} u v) + h_1 \frac{\partial}{\partial y} (\bar{\rho} v^2) \right] \\ + 2 \frac{\partial h_2}{\partial x} \bar{\rho} u v + \frac{\partial h_1}{\partial y} (\bar{\rho} v^2 - \bar{\rho} u^2) \end{aligned} \quad (A1.45)$$

The energy equation:

$$\begin{aligned} u \left[h_2 \frac{\partial v}{\partial x} - u \frac{\partial h_1}{\partial y} \right] + \frac{\rho v}{\rho} \left[h_1 \frac{\partial v}{\partial y} + \frac{\partial h_2}{\partial x} u \right] \\ = -\frac{\tau}{\gamma p} \frac{\partial p}{\partial y} h_1 + \frac{\partial \tau}{\partial x} h_2 + \left[-\frac{\partial p}{\partial x} \frac{1}{p} - \frac{\partial \tau}{\partial x} \frac{1}{\tau} \right] h_2 \tau \end{aligned}$$

$$+ h_1 \frac{\partial \overline{\rho u^2}}{\partial y} - \overline{\rho h_1 v^2} \left[\frac{1}{\rho} \frac{\partial \rho}{\partial y} - \frac{1}{\tau} \frac{\partial \tau}{\partial y} \right]$$

(A1.46)

The Navier-Stokes equation for incompressible flow can be obtained by putting $d\rho=0$. This reduces to the same as Mahgoub (1978).

Appendix 2.Derivation of the Reynolds Stress Transport Equations and the Turbulent Kinetic Energy Equations Using General Tensor Analysis

The instantaneous i-component momentum equation can be written in generalised tensor form as:

$$\rho \frac{\partial u^i}{\partial t} + (\rho u^k)_{,k} = -g^{ik} p_{,k} + \tau^{ik}_{,k} + \mu g^{kl} u^i_{,kl} \quad (A2.1)$$

and for j-component momentum equation is as follows:

$$\rho \frac{\partial u^j}{\partial t} - (\rho u^k)_{,k} = -g^{jk} p_{,k} + \tau^{jk}_{,k} + \mu g^{kl} u^j_{,kl} \quad (A2.2)$$

where $\tau = -\overline{\rho uv}$.

If we multiply (A2.1) by the fluctuating velocity u^j and multiply the equation (A2.2) by u^i , then add and take time averaging, we obtain

$$\begin{aligned} & \overline{\rho \left[u^j \frac{\partial u^i}{\partial t} - u^i \frac{\partial u^j}{\partial t} \right] + \rho u^k (u^j u^i_{,k} + u^i u^j_{,k})} \\ & = \overline{-g^{ik} u^j p_{,k} + g^{jk} u^i p_{,k}} \\ & \quad + \overline{u^j \tau^{ik}_{,k} + u^i \tau^{jk}_{,k}} + \overline{\mu g^{kl} (u^j u^i_{,kl} - u^i u^j_{,kl})} \end{aligned} \quad (A2.3)$$

If we assume stationary motion

$$\frac{\partial}{\partial t} () = 0$$

and we insert $U = \bar{U} + u$ then we can obtain

$$\begin{aligned} & \bar{u}^j_{,k} \left[\bar{\rho} \bar{u}^i \bar{u}^k + \bar{u}^k \bar{\rho}' u^i + \bar{\rho}' u^k u^i \right] + \bar{u}^i_{,k} \left[\bar{\rho} u^k u^j + \bar{u}^k \bar{\rho}' u^j + \bar{\rho}' u^k u^j \right] \\ & + (\bar{u}^i u^j)_{,k} \left[\bar{u}^k \bar{\rho} + \bar{u}^k \bar{\rho}' \right] + \bar{\rho} u^k (u^i u^j)_{,k} + \bar{u}^k \bar{\rho}' (u^i u^j)_{,k} \\ & = \left[g^{ik} (\bar{\rho}' u^j)_{,k} + g^{jk} (\bar{\rho}' u^i)_{,k} \right] + \overline{g^{ik} \bar{\rho}' u^j_{,k} + g^{jk} \bar{\rho}' u^i_{,k}} \\ & \quad + \overline{u^j \tau^{ik}_{,k} + u^i \tau^{jk}_{,k}} + \overline{\mu g^{kl} (u^j u^i_{,kl} + u^i u^j_{,kl})} \\ & \therefore - \underbrace{\left[\bar{\rho} \bar{u}^k + \bar{\rho}' u^k \right] (u^i u^j)_{,k}}_I = \underbrace{\left[\bar{\rho} \bar{u}^i u^k + \bar{\rho}' u^i u^k \right] \bar{u}^j_{,k} + \left[\bar{\rho} u^j u^k + \bar{\rho}' u^j u^k \right] \bar{u}^i_{,k}}_{II} \\ & \quad + \underbrace{- \bar{\rho}' [(g^{ik} u^j_{,k}) + g^{jk} u^i_{,k}]}_{III} \end{aligned}$$

$$\begin{aligned}
& + \overline{\rho u^k u^i u^j}_{,k} + \overline{\rho' u^k u^i u^j}_{,k} + \overline{g^i u^j \rho'}_{,k} + \overline{g^i (u^j \rho')}_{,k} \quad V \\
& + u^k \left[\overline{\rho u^i u^j}_{,k} + \overline{\rho u^j u^i}_{,k} \right] + \mu \overline{g^{kl} (u^j u^i_{,kl} + u^i u^j_{,kl})} \quad VI
\end{aligned}
\tag{A2.4}$$

The terms denoted by I, II, III, IV, V and VI are the mean transport, generation, redistribution by pressure fluctuations, turbulent mass flux times mean acceleration term, turbulent transport and viscous transport/destruction terms of the Reynolds stress $(-\overline{\rho u^i u^j})$, respectively. Thus we can write (A2.3) as follows:

$$M^{ij} = G^{ij} + R^{ij} + T^{ij} + MT^{ij} + V^{ij}$$

where M^{ij} , G^{ij} , R^{ij} , MT^{ij} , T^{ij} and V^{ij} denotes terms I, II, III, IV, V and VI respectively. We shall now derive the Reynolds stress transport equation and turbulent kinetic energy equation term by term.

A2.1 Derivation of the Reynolds stress transport equations

(1) The mean transport term ($=M^{ij}$)

This term represents the transport of Reynolds stress by the mean velocity field. We can write it as

$$-M^{ij} = \left[\overline{\rho u^k} + \overline{\rho' u^k} \right] \overline{(u^i u^j)_{,k}} \tag{A2.5}$$

using equation (A1.10) for the covariant derivative of a second order contravariant tensor formula. For simplicity, let $\tau^{ij} = (-\overline{\rho u^i u^j})$ (and this is symmetric tensor for i and j).

We can then write (A2.5) as:

$$\begin{aligned}
-M^{ij} = & \left[\overline{\rho u^1} + \overline{\rho' u^1} \right] \left[\frac{\partial (u^i u^j)}{\partial x} + \left\{ \begin{matrix} i \\ 1, 1 \end{matrix} \right\} \overline{(u^1 u^j)} + \left\{ \begin{matrix} j \\ 1, 1 \end{matrix} \right\} \overline{(u^i u^1)} \right] \\
& + \left[\overline{\rho u^2} + \overline{\rho' u^2} \right] \left[\frac{\partial (u^i u^j)}{\partial y} + \left\{ \begin{matrix} i \\ 1, 2 \end{matrix} \right\} \overline{(u^1 u^j)} + \left\{ \begin{matrix} j \\ 1, 2 \end{matrix} \right\} \overline{(u^i u^1)} \right]
\end{aligned}
\tag{A2.6}$$

Note the metric values h_1 and h_2 are as follows:

$$h_1 = 1 + \frac{n}{R}, \quad h_2 = 1 \quad \text{for } (s, n) \text{ coordinate system}$$

$$h_1 = h_2 = \frac{1}{n} (\phi^2 + \psi^2)^{\frac{1}{2}(n-1)} \text{ for } (\phi, \psi) \text{ coordinate system.}$$

(i) For i=1, j=1

From equation (A2.6)

$$\begin{aligned} -M^{11} = & \left[\bar{\rho} \frac{\bar{u}}{h_1} + \bar{\rho}' \frac{\bar{u}}{h_1} \right] \left[\frac{\partial}{\partial x} \left(\frac{\bar{u}^2}{h_1^2} \right) + \frac{1}{h_1} \frac{\partial h_1}{\partial x} \frac{\bar{u}^2}{h_1^2} + \frac{1}{h_1} \frac{\partial h_1}{\partial y} \frac{(\bar{u}v)}{h_1 h_2} \right. \\ & + \left. \frac{1}{h_1} \frac{\partial h_1}{\partial x} \frac{\bar{u}^2}{h_1^2} + \frac{1}{h_1} \frac{\partial h_1}{\partial y} \frac{(\bar{u}v)}{h_1 h_2} \right] \\ & + \left[\bar{\rho} \frac{\bar{v}}{h_2} + \bar{\rho}' \frac{\bar{v}}{h_2} \right] \left[\frac{\partial}{\partial y} \left(\frac{\bar{u}^2}{h_1^2} \right) + \frac{1}{h_1} \frac{\partial h_1}{\partial y} \frac{\bar{u}^2}{h_1^2} \right. \\ & - \left. \frac{(\bar{u}v)}{h_1^3} \frac{\partial h_2}{\partial x} + \frac{1}{h_1} \frac{\partial h_1}{\partial y} \frac{\bar{u}^2}{h_1^2} - \frac{(\bar{u}v)}{h_1^3} \frac{\partial h_2}{\partial x} \right] \end{aligned} \quad (A2.7)$$

(ii) For i=1, j=2

From equation (A2.6)

$$\begin{aligned} -M^{12} = & \left[\bar{\rho} \frac{\bar{u}}{h_1} + \bar{\rho}' \frac{\bar{u}}{h_1} \right] \left[\frac{\partial}{\partial x} \left(\frac{(\bar{u}v)}{h_1 h_2} \right) + \frac{1}{h_1} \frac{\partial h_1}{\partial x} \frac{(\bar{u}v)}{h_1 h_2} \right. \\ & + \left. \frac{1}{h_1} \frac{\partial h_1}{\partial y} \frac{\bar{v}^2}{h_2^2} - \frac{h_1}{h_2^2} \frac{\partial h_1}{\partial y} \frac{\bar{u}^2}{h_1^2} + \frac{1}{h_2} \frac{\partial h_2}{\partial x} \frac{(\bar{u}v)}{h_1 h_2} \right] \\ & + \left[\bar{\rho} \frac{\bar{v}}{h_2} + \bar{\rho}' \frac{\bar{v}}{h_2} \right] \left[\frac{\partial}{\partial y} \left(\frac{(\bar{u}v)}{h_1 h_2} \right) + \frac{1}{h_1} \frac{\partial h_1}{\partial y} \frac{(\bar{u}v)}{h_1 h_2} \right. \\ & - \left. \frac{h_2}{h_1^2} \frac{\partial h_2}{\partial x} \frac{\bar{v}^2}{h_2^2} + \frac{1}{h_2} \frac{\partial h_2}{\partial x} \frac{\bar{u}^2}{h_1^2} + \frac{1}{h_2} \frac{\partial h_2}{\partial y} \frac{(\bar{u}v)}{h_1 h_2} \right] \end{aligned} \quad (A2.8)$$

(iii) For i=2, j=2

From equation (A2.6)

$$\begin{aligned} -M^{22} = & \left[\bar{\rho} \frac{\bar{u}}{h_1} + \bar{\rho}' \frac{\bar{u}}{h_1} \right] \left[\frac{2\bar{u}v}{h_2^3} \frac{\partial h_1}{\partial y} + \frac{\partial}{\partial x} \left(\frac{\bar{v}^2}{h_2^2} \right) \right. \\ & + \left. \frac{2\partial h_2}{\partial x} \frac{\bar{v}^2}{h_2^3} \right] \\ & + \left[\bar{\rho} \frac{\bar{v}}{h_2} + \bar{\rho}' \frac{\bar{v}}{h_2} \right] \left[\frac{\partial}{\partial y} \left(\frac{\bar{v}^2}{h_2^2} \right) + \frac{2(\bar{u}v)}{h_1 h_2^2} \frac{\partial h_2}{\partial x} \right. \\ & + \left. \frac{2\bar{v}^2}{h_2^3} \frac{\partial h_2}{\partial y} \right] \end{aligned} \quad (A2.9)$$

(2) The generation term (=G^{ij})

This term represents the generation of Reynolds

stress by the mean flow and is written

$$G^{ij} = \left[\bar{\rho} \overline{u^i u^k} + \overline{\rho' u^i u^k} \right] U_{,k}^j + \left[\bar{\rho} \overline{u^j u^k} + \overline{\rho' u^j u^k} \right] U_{,k}^i \quad (A2.10)$$

(i) For i=1, j=1

Using equation (A2.10)

$$\begin{aligned} \frac{G^{11}}{2} &= \left[\bar{\rho} \overline{u^1 u^1} + \overline{\rho' u^1 u^1} \right] U_{,1}^1 \\ &\quad + \left[\bar{\rho} \overline{u^1 u^2} + \overline{\rho' u^1 u^2} \right] U_{,2}^1 \end{aligned} \quad (A2.11)$$

Using the Christoffel symbols given by equation (A1.20) this reduces to

$$\begin{aligned} &= \left[\bar{\rho} \frac{\overline{u^2}}{h_1^2} + \overline{\rho' \frac{u^2}{h_1^2}} \right] \left[\frac{\partial}{\partial x} \left(\frac{\bar{U}}{h_1} \right) + \frac{1}{h_1} \frac{\partial h_1}{\partial x} \frac{\bar{U}}{h_1} + \frac{1}{h_1} \frac{\partial h_1}{\partial y} \frac{\bar{V}}{h_2} \right] \\ &\quad + \left[\bar{\rho} \frac{\overline{uv}}{h_1 h_2} + \overline{\rho' \frac{uv}{h_1 h_2}} \right] \left[\frac{\partial}{\partial y} \left(\frac{\bar{U}}{h_1} \right) + \frac{1}{h_1} \frac{\partial h_1}{\partial y} \frac{\bar{U}}{h_1} - \frac{\bar{V}}{h_1^2} \frac{\partial h_2}{\partial x} \right] \end{aligned} \quad (A2.11')$$

(ii) For i=1, j=2

Using equation (A2.10)

$$\begin{aligned} G^{12} &= \left[\bar{\rho} \frac{\overline{u^2}}{h_1^2} + \overline{\rho' \frac{u^2}{h_1^2}} \right] \left[\frac{\partial}{\partial x} \left(\frac{\bar{V}}{h_2} \right) - \frac{1}{h_2^2} \frac{\partial h_1}{\partial y} \frac{\bar{U}}{h_1} + \frac{1}{h_2} \frac{\partial h_2}{\partial x} \frac{\bar{V}}{h_2} \right] \\ &\quad + \left[\bar{\rho} \frac{\overline{uv}}{h_1 h_2} + \overline{\rho' \frac{uv}{h_1 h_2}} \right] \left[\frac{\partial}{\partial y} \left(\frac{\bar{V}}{h_2} \right) + \frac{\bar{U}}{h_1 h_2} \frac{\partial h_2}{\partial x} + \frac{\bar{V}}{h_2^2} \frac{\partial h_2}{\partial y} \right] \\ &\quad + \left[\bar{\rho} \frac{\overline{uv}}{h_1 h_2} + \overline{\rho' \frac{uv}{h_1 h_2}} \right] \left[\frac{\partial}{\partial x} \left(\frac{\bar{U}}{h_1} \right) + \frac{\bar{U}}{h_1^2} \frac{\partial h_1}{\partial x} + \frac{\bar{V}}{h_1 h_2} \frac{\partial h_1}{\partial y} \right] \\ &\quad + \left[\bar{\rho} \frac{\overline{v^2}}{h_2^2} + \overline{\rho' \frac{v^2}{h_2^2}} \right] \left[\frac{\partial}{\partial y} \left(\frac{\bar{U}}{h_1} \right) + \frac{\bar{U}}{h_1^2} \frac{\partial h_1}{\partial y} - \frac{\bar{V}}{h_1^2} \frac{\partial h_2}{\partial x} \right] \end{aligned} \quad (A2.12)$$

(iii) For i=2, j=2

From equation (A2.10)

$$\begin{aligned} \frac{G^{22}}{2} &= \left[\bar{\rho} \frac{\overline{uv}}{h_1 h_2} + \overline{\rho' \frac{uv}{h_1 h_2}} \right] \left[\frac{\partial}{\partial x} \left(\frac{\bar{V}}{h_2} \right) - \frac{\bar{U}}{h_2^2} \frac{\partial h_1}{\partial y} \right. \\ &\quad \left. + \frac{\bar{V}}{h_2^2} \frac{\partial h_2}{\partial x} \right] + \left[\bar{\rho} \frac{\overline{v^2}}{h_2^2} + \overline{\rho' \frac{v^2}{h_2^2}} \right] \left[\frac{\partial}{\partial y} \left(\frac{\bar{V}}{h_2} \right) \right. \end{aligned}$$

$$+ \frac{\bar{U}}{h_1 h_2} \frac{\partial h_2}{\partial x} + \frac{\bar{V}}{h_2^2} \frac{\partial h_2}{\partial y} \Big]$$

(A2.13)

(3) The pressure-redistribution term ($=R^{ij}$)

This term represents redistribution of the Reynolds stress over the three coordinate direction, tending to make the turbulence isotropic. We can write it as:

$$R^{ij} = -\overline{\rho'(g^{ik} u_{,k}^j + g^{jk} u_{,k}^i)}$$

(A2.14)

For the orthogonal coordinate system $g^{jk}=0$ for $j \neq k$.

$$R^{ii} = -\overline{\rho'(g^{ii} u_{,i}^i + g^{jj} u_{,j}^i)}$$

(A2.15)

(i) For $i=1, j=1$

Using equation (A2.15)

$$\begin{aligned} \frac{R^{11}}{2} &= - \overline{\rho'(g^{11} u_{,1}^1 + g^{11} u_{,1}^1)} \\ &= - \overline{\rho' \frac{1}{h_1^3} \left(\frac{\partial u}{\partial x} + \frac{v}{h_2} \frac{\partial h_1}{\partial y} \right)} \end{aligned}$$

(A2.16)

(ii) For $i=1, j=2$

Using equation (A2.15)

$$\begin{aligned} R^{12} &= - \overline{\rho'(g^{11} u_{,1}^2 + g^{22} u_{,2}^1)} \\ &= - \overline{\rho' \frac{1}{h_1^2 h_2^2} \left[h_2 \frac{\partial v}{\partial x} - \frac{\partial h_1}{\partial y} u + h_1 \frac{\partial u}{\partial y} - \frac{\partial h_2}{\partial x} v \right]} \end{aligned}$$

(A2.17)

(iii) For $i=2, j=2$

Using equation (A2.15)

$$\frac{R^{22}}{2} = - \overline{\rho'(g^{22} u_{,2}^2)}$$

$$= - \rho' \frac{1}{h_2^3} \left(\frac{\partial v}{\partial y} + \frac{\partial h_2}{\partial x} \frac{u}{h_1} \right) \quad (\text{A2.18})$$

(4) The turbulent transport term

This term represents the transport of Reynolds stress by the turbulence via the pressure and velocity fluctuation. We shall represent the transport by the pressure fluctuations by TP^{ij} and that by velocity fluctuations by TV^{ij} , so that

$$T^{ij} = TP^{ij} + TV^{ij} \quad (\text{A2.19})$$

(a) For TP^{ij}

(i) For $i=1, j=1$

From equation (A2.4)

$$\begin{aligned} \frac{TP^{11}}{2} &= g^{11}(\overline{u^1 p'})_{,1} \\ &= \frac{1}{h_1^3} \left[\frac{\partial}{\partial x} (\overline{u p'}) + \frac{1}{h_2} \frac{\partial h_1}{\partial y} \overline{v p'} \right] \end{aligned}$$

(A2.20)

(ii) For $i=1, j=2$

From equation (A2.4)

$$\begin{aligned} TP^{12} &= g^{11}(\overline{u^2 p'})_{,1} + g^{22}(\overline{u^1 p'})_{,2} \\ &= \frac{1}{h_1^2 h_2^2} \left[h_2 \frac{\partial}{\partial x} (\overline{v p'}) - \frac{\partial h_1}{\partial y} \overline{u p'} \right. \\ &\quad \left. + h_1 \frac{\partial}{\partial y} (\overline{u p'}) - \frac{\partial h_2}{\partial x} \overline{v p'} \right] \end{aligned}$$

(A2.21)

(iii) For $i=2, j=2$

$$\begin{aligned}
\frac{TP}{2}^{22} &= g^{22}(\overline{u^2 p'})_{,2} \\
&= \frac{1}{h_2^3} \left[\frac{\partial}{\partial y} (\overline{v p'}) + \frac{1}{h_1} \frac{\partial h_2}{\partial x} (\overline{u p'}) \right]
\end{aligned}
\tag{A2.22}$$

(b) For TV^{ij}

(i) For i=1, j=1

From equation (A2.4)

$$\begin{aligned}
TV^{11} &= \bar{\rho} \frac{u}{h_1^3 h_2} \left[h_2 \frac{\partial u^2}{\partial x} + 2 \frac{\partial h_1}{\partial y} uv \right] \\
&\quad + \frac{\bar{U}}{h_1^3 h_2} \rho' \left[h_2 \frac{\partial u^2}{\partial x} + 2 \frac{\partial h_1}{\partial y} uv \right] \\
&\quad + \bar{\rho} \frac{v}{h_1^3 h_2} \left[h_1 \frac{\partial u^2}{\partial y} - 2 \frac{\partial h_2}{\partial x} uv \right] \\
&\quad + \frac{\bar{V}}{h_1^3 h_2} \left[\rho' \left[h_1 \frac{\partial u^2}{\partial y} - 2 \frac{\partial h_2}{\partial x} uv \right] \right]
\end{aligned}
\tag{A2.23}$$

(ii) For i=1, j=2

From equation (A2.4)

$$\begin{aligned}
TV^{12} &= \bar{\rho} \frac{u}{h_1^2 h_2^2} \left[h_2 \frac{\partial}{\partial x} (uv) + v^2 \frac{\partial h_1}{\partial y} - u^2 \frac{\partial h_1}{\partial y} \right] \\
&\quad + \frac{\bar{U}}{h_1^2 h_2^2} \rho' \left[h_2 \frac{\partial}{\partial x} (uv) + v^2 \frac{\partial h_1}{\partial y} - u^2 \frac{\partial h_1}{\partial y} \right] \\
&\quad - \bar{\rho} \frac{v}{h_1 h_2^2} \left[h_1 \frac{\partial}{\partial y} (uv) - v^2 \frac{\partial h_2}{\partial x} + u^2 \frac{\partial h_2}{\partial x} \right] \\
&\quad - \frac{\bar{V}}{h_1^2 h_2^2} \rho' \left[h_1 \frac{\partial}{\partial y} (uv) - \frac{\partial h_2}{\partial x} v^2 + u^2 \frac{\partial h_2}{\partial x} \right]
\end{aligned}
\tag{A2.24}$$

(iii) For i=2, j=2

From equation (A2.4)

$$\begin{aligned}
TV^{22} &= \bar{\rho} \frac{u}{h_1 h_2^3} \left[h_2 \frac{\partial v}{\partial x} - 2 \frac{\partial h_1}{\partial y} uv \right] \\
&\quad + \frac{\bar{U}}{h_1 h_2^3} \rho' \left[h_2 \frac{\partial v}{\partial x} - 2 \frac{\partial h_1}{\partial y} uv \right] \\
&\quad + \bar{\rho} \frac{v}{h_1 h_2^3} \left[h_1 \frac{\partial v}{\partial y} + 2 \frac{\partial h_2}{\partial x} uv \right]
\end{aligned}$$

$$+ \frac{\bar{v}}{h_1 h_2^3} \overline{\rho' \left[h_1 \frac{\partial v^2}{\partial y} + 2 \frac{\partial h_2}{\partial x} uv \right]} \quad (A2.25)$$

(5) Turbulent mass flux times mean acceleration

This term represents turbulent mass flux by mean flow acceleration.

$$MT^{ij} = \left[\overline{\rho' u^i} \bar{u}_{,k}^j + \overline{\rho' u^j} \bar{u}_{,k}^i \right] \bar{u}^k \quad (A2.26)$$

(i) For i=1, j=1

Using equation (A2.26)

$$\begin{aligned} \frac{MT^{11}}{2} &= \frac{\overline{\rho' u}}{h_1^3 h_2} \left[h_2 \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial h_1}{\partial y} \right] \bar{u} \\ &\quad - \frac{\overline{\rho' u}}{h_1^3 h_2} \left[h_1 \frac{\partial \bar{u}}{\partial y} - \bar{v} \frac{\partial h_2}{\partial x} \right] \bar{v} \end{aligned}$$

(A2.27)

(ii) For i=1, j=2

Using equation (A2.26)

$$\begin{aligned} MT^{12} &= \frac{\overline{\rho' u}}{h_1^2 h_2^2} \left[\bar{u} \left(h_2 \frac{\partial \bar{v}}{\partial x} - \bar{u} \frac{\partial h_1}{\partial y} \right) \right. \\ &\quad \left. + \bar{v} \left(h_1 \frac{\partial \bar{v}}{\partial y} + \bar{u} \frac{\partial h_2}{\partial x} \right) \right] \\ &\quad + \frac{\overline{\rho' v}}{h_1^2 h_2^2} \left[\bar{u} \left(h_2 \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial h_1}{\partial y} \right) \right. \\ &\quad \left. + \bar{v} \left(h_1 \frac{\partial \bar{u}}{\partial y} - \bar{v} \frac{\partial h_2}{\partial x} \right) \right] \end{aligned}$$

(A2.28)

(iii) For i=2, j=2

Using equation (A2.26)

$$\begin{aligned} \frac{MT^{22}}{2} &= \frac{\overline{\rho'v}}{h_1 h_2^3} \left[\bar{U} \left(h_1 \frac{\partial \bar{v}}{\partial x} - \bar{U} \frac{\partial h_1}{\partial y} \right) \right. \\ &\quad \left. + \bar{v} \left(h_1 \frac{\partial \bar{v}}{\partial y} + \bar{U} \frac{\partial h_2}{\partial x} \right) \right] \end{aligned}$$

(A2.29)

(6) The Viscous dissipation term

The viscous dissipation terms can be written as $-\epsilon^{11}$, $-\epsilon^{12}$ and $-\epsilon^{22}$. Since the viscous dissipation term will be approximated physically, we shall not derive the detailed expression (see Mahgoub (1978) and Chen(1982)).

Now we can write the transport equations for the Reynolds stresses $-\overline{\rho u^2}$, $-\overline{\rho v^2}$, $-\overline{\rho uv}$ in (s, n) or (φ, ψ) coordinate system for two dimensional flow, as follows:

(1) $-\overline{\rho u^2}$ equation

$$\begin{aligned} & - \left[(\bar{\rho} \bar{U} + \bar{\rho}' u) \left\{ h_2 \frac{\partial}{\partial x} \left(\frac{1}{2} u^2 \right) + \frac{\partial h_1}{\partial y} \overline{uv} \right\} - [\bar{\rho} \bar{v} + \bar{\rho}' v] \left[h_1 \frac{\partial}{\partial y} \left(\frac{1}{2} v^2 \right) \right. \right. \\ & \quad \left. \left. - \frac{\partial h_2}{\partial x} \overline{uv} \right] \right] = \overline{[\bar{\rho} u^2 + \bar{\rho}' u^2]} \left[h_2 \frac{\partial \bar{U}}{\partial x} + \frac{\partial h_1}{\partial y} \bar{v} \right] - (\bar{\rho} \overline{uv} + \bar{\rho}' \overline{uv}) \left[h_1 \frac{\partial \bar{U}}{\partial y} - \bar{v} \frac{\partial h_2}{\partial x} \right] \\ & \quad - \left(h \frac{\partial u}{\partial x} + \frac{\partial h}{\partial y} v \right) \rho' + h_2 \frac{\partial}{\partial x} (\overline{u \rho'}) + \frac{\partial h_1}{\partial y} \overline{v \rho'} \\ & \quad + \overline{[\bar{\rho} u \{ h_2 \frac{\partial}{\partial x} (\frac{1}{2} u^2) + \frac{\partial h_1}{\partial y} uv \}]} + \bar{U} \overline{[\rho' \{ h_2 \frac{\partial}{\partial x} (\frac{1}{2} u^2) + \frac{\partial h_1}{\partial y} uv \}]} \\ & \quad + \overline{[\bar{\rho} v \{ h_1 \frac{\partial}{\partial y} (\frac{1}{2} v^2) + \frac{\partial h_2}{\partial x} uv \}]} + \bar{v} \overline{[\rho' \{ h_1 \frac{\partial}{\partial y} (\frac{1}{2} v^2) + \frac{\partial h_2}{\partial x} uv \}]} \\ & \quad + \overline{\rho' u} \left[h_2 \frac{\partial \bar{U}}{\partial x} + \frac{\partial h_1}{\partial y} \bar{v} \right] \bar{U} + \overline{\rho' u} \left[h_1 \frac{\partial \bar{U}}{\partial y} + \frac{\partial h_2}{\partial x} \bar{v} \right] \bar{v} + \epsilon_{11} \end{aligned}$$

(A2.30)

(2) $-\overline{\rho v^2}$ equation

$$\begin{aligned} & - \left[\bar{\rho} \bar{U} + \bar{\rho}' u \right] \left[h_2 \frac{\partial}{\partial x} \left(\frac{1}{2} v^2 \right) - \frac{\partial h_1}{\partial y} \overline{uv} \right] - (\bar{\rho} \bar{v} + \bar{\rho}' v) \left[h_1 \frac{\partial}{\partial y} \left(\frac{1}{2} v^2 \right) + \right. \\ & \quad \left. + \frac{\partial h_2}{\partial x} \overline{uv} \right] = (\bar{\rho} \overline{uv} + \bar{\rho}' \overline{uv}) \left[h_2 \frac{\partial \bar{v}}{\partial x} - \frac{\partial h_1}{\partial y} \bar{U} \right] + (\bar{\rho} \overline{v^2} + \bar{\rho}' \overline{v^2}) \left[\right. \\ & \quad \left. h_1 \frac{\partial \bar{v}}{\partial y} + \frac{\partial h_2}{\partial x} \bar{U} \right] - \overline{[\rho' (h_1 \frac{\partial v}{\partial y} + \frac{\partial h_2}{\partial x} u)]} + h_1 \frac{\partial}{\partial y} (\overline{v \rho'}) + \frac{\partial h_2}{\partial x} (\overline{u \rho'}) \\ & \quad - \bar{\rho}' u \left[h_2 \frac{\partial}{\partial x} \left(\frac{1}{2} v^2 \right) + \frac{\partial h_1}{\partial y} uv \right] - \bar{U} \rho' \left[h_2 \frac{\partial}{\partial x} \left(\frac{1}{2} v^2 \right) + \frac{\partial h_1}{\partial y} uv \right] \end{aligned}$$

$$\begin{aligned}
& + \bar{\rho} \overline{v \left(h_1 \frac{\partial}{\partial y} \left(\frac{1}{2} v^2 \right) + \frac{\partial h_2}{\partial x} uv \right)} + v \overline{\rho' \left\{ \frac{\partial h_2}{\partial x} uv + h_1 \frac{\partial}{\partial y} \left(\frac{1}{2} v^2 \right) \right\}} \\
& + \overline{\rho' v} \left[\left(h_1 \frac{\partial v}{\partial x} + \frac{\partial h_1}{\partial y} u \right) u + \left(h_1 \frac{\partial v}{\partial y} + \frac{\partial h_2}{\partial x} u \right) v \right] + \epsilon_{22}
\end{aligned} \tag{A2.31}$$

(3) $-\rho uv$ equation

$$\begin{aligned}
& - (\bar{\rho} \bar{u} + \bar{\rho}' \bar{u}) \left[h_2 \frac{\partial}{\partial x} (\bar{uv}) - \frac{\partial h_1}{\partial y} (u^2 - v^2) \right] - (\bar{\rho} \bar{v} + \bar{\rho}' \bar{v}) \left[h_2 \frac{\partial}{\partial y} (\bar{uv}) + \frac{\partial h_2}{\partial x} (u^2 - v^2) \right] \\
& = (\bar{\rho} \bar{u}^2 + \bar{\rho}' \bar{u}^2) \left(h_2 \frac{\partial \bar{v}}{\partial x} - \frac{\partial h_1}{\partial y} \bar{u} \right) + (\bar{\rho} \bar{u} \bar{v} + \bar{\rho}' \bar{u} \bar{v}) \left(h_1 \frac{\partial \bar{v}}{\partial y} + \frac{\partial h_2}{\partial x} \bar{u} + \frac{\partial \bar{u}}{\partial x} h_2 \right. \\
& \quad \left. + \frac{\partial h_1}{\partial y} \bar{v} \right) + (\bar{\rho} \bar{v}^2 + \bar{\rho}' \bar{v}^2) \left(h_1 \frac{\partial \bar{u}}{\partial y} - \frac{\partial h_2}{\partial x} \bar{v} \right) - \overline{\rho' \left(h_2 \frac{\partial v}{\partial x} - \frac{\partial h_1}{\partial y} u + h_1 \frac{\partial u}{\partial y} - \frac{\partial h_2}{\partial x} v \right)} \\
& \quad + \left[h_2 \frac{\partial}{\partial x} (\bar{v} \rho') - \frac{\partial h_1}{\partial y} (\bar{u} \rho') + h_1 \frac{\partial}{\partial y} (\bar{u} \rho') - \frac{\partial h_2}{\partial x} (\bar{v} \rho') \right] - \bar{\rho} \bar{u} \left[h_2 \frac{\partial}{\partial x} (\bar{uv}) + v^2 \frac{\partial h_1}{\partial y} \right. \\
& \quad \left. - u^2 \frac{\partial h_1}{\partial y} \right] - \bar{u} \bar{\rho}' \left[h_2 \frac{\partial}{\partial x} (\bar{uv}) + v^2 \frac{\partial h_1}{\partial y} - u^2 \frac{\partial h_1}{\partial y} \right] - \bar{\rho} \bar{v} \left[h_1 \frac{\partial}{\partial y} (\bar{uv}) + \frac{\partial h_2}{\partial x} (u^2 - v^2) \right] \\
& \quad - \bar{v} \bar{\rho}' \left[h_1 \frac{\partial}{\partial y} (\bar{uv}) + \frac{\partial h_2}{\partial x} (u^2 - v^2) \right] + \bar{\rho}' \bar{u} \left[\bar{u} \left\{ h_2 \frac{\partial \bar{v}}{\partial x} - \bar{u} \frac{\partial h_1}{\partial y} \right\} + \right. \\
& \quad \left. \bar{v} \left\{ h_1 \frac{\partial \bar{v}}{\partial y} + \bar{u} \frac{\partial h_2}{\partial x} \right\} \right] + \bar{\rho}' \bar{v} \left[\bar{u} \left\{ h_2 \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial h_1}{\partial y} \right\} + \bar{v} \left\{ h_1 \frac{\partial \bar{u}}{\partial y} - \bar{v} \frac{\partial h_2}{\partial x} \right\} \right] \tag{A2.32}
\end{aligned}$$

A2.2 Derivation of turbulent kinetic energy equation

We could derive the turbulent kinetic energy equation from its generalised tensor form given by equation (A2.4) in a similar way to the derivation of the Reynolds stress equations. However, as we have already derived the normal Reynolds stress $(-\rho u^2, -\rho v^2)$ equations we could deduce the equation for turbulent kinetic energy $1/2 \overline{q^2} = 1/2 (\overline{u^2} + \overline{v^2} + \overline{w^2})$ by adding the $(-\rho u^2, -\rho v^2)$ equations, (A2.30) and (A2.31), and the $-\rho w^2$ equation.

The $-\rho w^2$ equation in simple 2-dimensional flow is just

$$\frac{D(\overline{\rho w^2})}{Dt} = \overline{\rho' \frac{\partial w}{\partial z}} - \frac{\partial}{\partial y} \left(\frac{1}{2} \overline{w^2 v} \right) - \epsilon_{33}$$

where ϵ_{33} is the only extra bits.

$$\begin{aligned}
& - (\bar{\rho} \bar{u} + \bar{\rho}' \bar{u}) \left\{ h_2 \frac{\partial}{\partial x} \left(\frac{1}{2} \overline{q^2} \right) \right\} + (\bar{\rho} \bar{v} + \bar{\rho}' \bar{v}) \left\{ h_1 \frac{\partial}{\partial y} \left(\frac{1}{2} \overline{q^2} \right) \right\} \\
& = (\bar{\rho} \bar{uv} + \bar{\rho}' \bar{uv}) \left\{ h_2 \frac{\partial \bar{v}}{\partial x} + \frac{\partial h_1}{\partial y} \bar{u} + h_1 \frac{\partial \bar{u}}{\partial y} + \frac{\partial h_2}{\partial x} \bar{v} \right\} + (\bar{\rho} \bar{u}^2 + \bar{\rho}' \bar{u}^2) \\
& \quad \left\{ h_2 \frac{\partial \bar{u}}{\partial x} + \frac{\partial h_1}{\partial y} \bar{v} \right\} + (\bar{\rho} \bar{v}^2 + \bar{\rho}' \bar{v}^2) \left\{ h_1 \frac{\partial \bar{v}}{\partial y} + \frac{\partial h_2}{\partial x} \bar{u} \right\}
\end{aligned}$$

$$\begin{aligned}
& - \overline{\rho' \left\{ h_1 \frac{\partial v}{\partial y} + h_2 \frac{\partial u}{\partial x} + \frac{\partial h_1}{\partial y} v + \frac{\partial h_2}{\partial x} u + h_3 \frac{\partial w}{\partial z} \right\}} \\
& + h_1 \frac{\partial}{\partial y} (\overline{v \rho'}) + h_2 \frac{\partial}{\partial x} (\overline{u \rho'}) + \frac{\partial h_1}{\partial y} (\overline{v \rho'}) + \frac{\partial h_2}{\partial x} (\overline{u \rho'}) \\
& + \overline{\bar{\rho} u \left[h_2 \frac{\partial}{\partial x} \left(\frac{1}{2} q^2 \right) \right]} + \overline{\bar{u} \rho' \left(h_2 \frac{\partial}{\partial x} \left(\frac{1}{2} q^2 \right) \right)} \\
& + \overline{\bar{\rho} v \left[h_1 \frac{\partial}{\partial y} \left(\frac{1}{2} q^2 \right) \right]} + \overline{\bar{v} \rho' \left[h_1 \frac{\partial}{\partial y} \left(\frac{1}{2} q^2 \right) \right]} \\
& + \overline{\rho' u \left[\bar{u} \left(h_2 \frac{\partial \bar{u}}{\partial x} + \frac{\partial h_1}{\partial y} \bar{v} \right) + \left(h_1 \frac{\partial \bar{u}}{\partial y} + \frac{\partial h_2}{\partial x} \bar{v} \right) \bar{v} \right]} \\
& + \overline{\rho' v \left[\bar{u} \left(h_1 \frac{\partial \bar{v}}{\partial x} + \frac{\partial h_1}{\partial y} \bar{u} \right) + \left(h_1 \frac{\partial \bar{v}}{\partial y} + \frac{\partial h_2}{\partial x} \bar{u} \right) \bar{v} \right]} \\
& + \epsilon_{11} + \epsilon_{22} + \epsilon_{33}
\end{aligned}$$

Hence

$$\begin{aligned}
& (\bar{\rho} \bar{u} + \overline{\rho' u}) h_2 \frac{\partial}{\partial x} \left(\frac{1}{2} q^2 \right) + (\bar{\rho} \bar{v} + \overline{\rho' v}) h_1 \frac{\partial}{\partial y} \left(\frac{1}{2} q^2 \right) \\
& = - \left[\bar{\rho} \bar{u}^2 + \overline{\rho' u^2} + \overline{\rho' u \bar{u}} \right] \left(h_2 \frac{\partial \bar{u}}{\partial x} + \frac{\partial h_1}{\partial y} \bar{v} \right) \\
& \quad - \left[\bar{\rho} \bar{v}^2 + \overline{\rho' v^2} + \overline{\rho' v \bar{v}} \right] \left(h_1 \frac{\partial \bar{v}}{\partial y} + \frac{\partial h_2}{\partial x} \bar{u} \right) \\
& \quad - \left[\bar{\rho} \bar{u} \bar{v} + \overline{\rho' u v} + \overline{\rho' u \bar{v}} \right] \left(h_1 \frac{\partial \bar{u}}{\partial y} + \frac{\partial h_2}{\partial x} \bar{v} \right) \\
& \quad - \left[\bar{\rho} \bar{u} \bar{v} + \overline{\rho' u v} + \overline{\rho' u \bar{u}} \right] \left(h_2 \frac{\partial \bar{v}}{\partial x} + \frac{\partial h_1}{\partial y} \bar{u} \right) \\
& + \overline{\rho' \left(\frac{\partial}{\partial y} (h_1 v) + \frac{\partial}{\partial x} (h_2 u) + \frac{\partial}{\partial z} (h_3 w) \right)} - \frac{\partial}{\partial y} (\overline{h_1 v \rho'}) - \frac{\partial}{\partial x} (\overline{h_2 u \rho'}) - \frac{\partial}{\partial z} (\overline{h_3 w \rho'}) \\
& + \bar{\rho} \left[h_2 \bar{u} \frac{\partial}{\partial x} \left(\frac{1}{2} q^2 \right) + h_1 \bar{v} \frac{\partial}{\partial x} \left(\frac{1}{2} q^2 \right) \right] \\
& - \bar{u} h_2 \overline{\rho' \frac{\partial}{\partial y} \left(\frac{1}{2} q^2 \right)} + h_1 \bar{v} \overline{\rho' \frac{\partial}{\partial y} \left(\frac{1}{2} q^2 \right)} - \epsilon
\end{aligned}$$

(A2.33)

Appendix 3.Derivation of Equations, and Definitions for the Shear Layer Calculation

In this appendix we shall derive equations which is used in incompressible shear layer calculations. Firstly, we write down the set of partial differential equations solved: these are the continuity equation (A1.43), the streamwise momentum equation (A1.44), the normal momentum equation (A1.45) and turbulent kinetic energy equation (A1.46) for incompressible flow. We have also inserted the BFA turbulence model assumptions into the turbulent kinetic energy equation as mentioned in §1 of the thesis.

Hence we have the following set of equations:

$$h_2 \frac{\partial U}{\partial x} + h_1 \frac{\partial V}{\partial y} = -U \frac{\partial h_2}{\partial x} - V \frac{\partial h_1}{\partial y} \quad (A3.1)$$

$$\begin{aligned} & h_2 \frac{\partial U^2}{\partial x} + h_1 \frac{\partial UV}{\partial y} + 2UV \frac{\partial h_1}{\partial x} + U^2 \frac{\partial h_2}{\partial x} \\ & - h_1 \frac{\partial \tau}{\partial y} - 2\tau \frac{\partial h_1}{\partial y} - V^2 \frac{\partial h_2}{\partial x} - (\overline{u^2} - \overline{v^2}) \frac{\partial h_2}{\partial x} \\ & + h_2 \left(\frac{\partial p}{\partial x} + \frac{\partial \overline{u^2}}{\partial x} \right) = 0 \end{aligned} \quad (A3.2)$$

$$\begin{aligned} & U h_2 \frac{\partial V}{\partial x} + h_1 V \frac{\partial V}{\partial y} - h_2 \frac{\partial \tau}{\partial x} - U \left(V \frac{\partial h_2}{\partial x} - U \frac{\partial h_1}{\partial y} \right) \\ & + h_1 \frac{\partial p}{\partial y} + h_1 \frac{\partial \overline{v^2}}{\partial y} - 2\tau \frac{\partial h_2}{\partial x} - (\overline{u^2} - \overline{v^2}) \frac{\partial h_1}{\partial y} \\ & = 0 \end{aligned} \quad (A3.3)$$

$$\begin{aligned} & h_2 U \frac{\partial}{\partial x} \left(\frac{\tau}{2a_1} \right) + h_1 V \frac{\partial}{\partial y} \left(\frac{\tau}{2a_1} \right) \\ & = \tau \left(h_1 \frac{\partial U}{\partial y} + h_2 \frac{\partial V}{\partial x} - U \frac{\partial h_1}{\partial y} - V \frac{\partial h_2}{\partial x} \right) + \tau \hat{e} \\ & - c_1 \tau \left(h_2 \frac{\partial U}{\partial x} + V \frac{\partial h_1}{\partial y} \right) - c_2 \tau \left(h_1 \frac{\partial V}{\partial y} + U \frac{\partial h_2}{\partial x} \right) \end{aligned}$$

$$-\frac{\partial}{\partial y} (h_1 \hat{G} \tau) - (\tau)^{3/2}/L$$

(A3.4)

$$\begin{aligned} \text{where } a_1 &= \frac{\tau}{q^2}, \quad \tau = \frac{\tau}{\rho} = \overline{uv} \\ \overline{u^2} &= c_1 \tau \\ \overline{v^2} &= c_2 \tau \\ \hat{G} &= G(\tau_m)^{1/2} \end{aligned}$$

Also, the streamwise turbulent transport term $-\partial/\partial x (\overline{p'u} + \frac{1}{2} \overline{q^2 u})$ has been neglected in comparison with normal-component turbulent transport term $-\partial/\partial y (\overline{p'v} + \frac{1}{2} \overline{q^2 v})$. The latter has been modelled by assuming that $(\overline{p'v} + \frac{1}{2} \overline{q^2 v}) = G(\frac{y}{\delta})(\tau_m)^{1/2} \tau$, where G is empirical function of y/δ containing a further empirical factor $(\tau_m/\rho U_e^2)^{0.5}$. The viscous dissipation term has been assumed equal to $(\tau)^{3/2}/L(\frac{y}{\delta})$, where L is an length scale (empirical or calculated). we have also included an extra strain rate correction $\hat{e}$ to the turbulence model same as BFA method.

A3.1 Derivation of Characteristic Equation

If we set $\xi = c$ is a characteristic curve, and we insert this derivative formula into equations (A3.1), (A3.2) and (A3,4), we can obtain equations as follows ;

$$h_2 \frac{\partial \xi}{\partial x} \frac{\partial u}{\partial \xi} + \frac{\partial \xi}{\partial y} \frac{\partial v}{\partial \xi} h_1 = -u \frac{\partial h_2}{\partial x} - v \frac{\partial h_1}{\partial y} = E1 \quad (A3.5)$$

$$\begin{aligned} & u h_2 \frac{\partial \xi}{\partial x} \frac{\partial u}{\partial \xi} + h_1 v \frac{\partial \xi}{\partial y} \frac{\partial u}{\partial \xi} - h_1 \frac{\partial \xi}{\partial y} \frac{\partial \tau}{\partial \xi} \\ &= -2uv \frac{\partial h_1}{\partial x} + u^2 \frac{\partial h_2}{\partial x} + 2\tau \frac{\partial h_1}{\partial y} + v^2 \frac{\partial h_2}{\partial x} + (c_1 - c_2) \tau \frac{\partial h_2}{\partial x} \\ & - h_2 \left(\frac{\partial p}{\partial x} + c_1 \frac{\partial \tau}{\partial x} \right) = E2 \end{aligned} \quad (A3.6)$$

$$\begin{aligned}
& \frac{h_2 U}{2a_1} \frac{\partial \xi}{\partial x} \frac{\partial \tau}{\partial \xi} + \left(\frac{h_1 v}{2a_1} + h_1 \hat{G} \right) \frac{\partial \xi}{\partial y} \frac{\partial \tau}{\partial \xi} + h_2 c_1 \tau \frac{\partial \xi}{\partial x} \frac{\partial U}{\partial \xi} \\
& + h_1 c_2 \tau \frac{\partial \xi}{\partial y} \frac{\partial v}{\partial \xi} - \tau h_1 \frac{\partial \xi}{\partial y} \frac{\partial U}{\partial \xi} = \tau h_2 \frac{\partial v}{\partial x} - \tau \left(U \frac{\partial h_1}{\partial y} + v \frac{\partial h_2}{\partial x} \right) \\
& + \tau \hat{e} - \tau \left(c_1 v \frac{\partial h_1}{\partial y} + c_2 U \frac{\partial h_2}{\partial x} \right) - (\tau)^{3/2} / L = E3
\end{aligned}
\tag{A3.7}$$

The terms in right hand side are treated as "known": thus we are trying to find the directions of the characteristics in solving for the three variables U , v and τ assuming the terms in the right hand are given. In particular, we assume that the pressure gradient is known (from the previous guess for the pressure array) and the normal Reynolds stresses $\overline{u^2}$, $\overline{v^2}$ are given. We also assume the total extra strain rate is known.

In order to find the characteristic directions we have to solve a determinant for $\frac{\partial U}{\partial \xi}$, $\frac{\partial v}{\partial \xi}$, $\frac{\partial \tau}{\partial \xi}$. We write down the equations (A3.5) to (A3.7) in turn as follows:

$$\begin{vmatrix}
\frac{\partial U}{\partial \xi} & \frac{\partial v}{\partial \xi} & \frac{\partial \tau}{\partial \xi} \\
h_2 \frac{\partial \xi}{\partial x} & h_1 \frac{\partial \xi}{\partial y} & 0 \\
U h_2 \frac{\partial \xi}{\partial x} + h_1 v \frac{\partial \xi}{\partial y} & 0 & -h_1 \frac{\partial \xi}{\partial y} \\
c_1 \tau h_2 \frac{\partial \xi}{\partial x} - \tau h_1 \frac{\partial \xi}{\partial y} & \tau c_2 h_1 \frac{\partial \xi}{\partial y} & \frac{h_2 U}{2a_1} \frac{\partial \xi}{\partial x} + \left(\frac{h_1 v}{2a_1} + h_1 \hat{G} \right) \frac{\partial \xi}{\partial y}
\end{vmatrix}
= E1 \tag{A3.5}$$

$$\begin{vmatrix}
U h_2 \frac{\partial \xi}{\partial x} + h_1 v \frac{\partial \xi}{\partial y} & 0 & -h_1 \frac{\partial \xi}{\partial y} \\
c_1 \tau h_2 \frac{\partial \xi}{\partial x} - \tau h_1 \frac{\partial \xi}{\partial y} & \tau c_2 h_1 \frac{\partial \xi}{\partial y} & \frac{h_2 U}{2a_1} \frac{\partial \xi}{\partial x} + \left(\frac{h_1 v}{2a_1} + h_1 \hat{G} \right) \frac{\partial \xi}{\partial y}
\end{vmatrix}
= E2 \tag{A3.6}$$

$$\begin{vmatrix}
c_1 \tau h_2 \frac{\partial \xi}{\partial x} - \tau h_1 \frac{\partial \xi}{\partial y} & \tau c_2 h_1 \frac{\partial \xi}{\partial y} & \frac{h_2 U}{2a_1} \frac{\partial \xi}{\partial x} + \left(\frac{h_1 v}{2a_1} + h_1 \hat{G} \right) \frac{\partial \xi}{\partial y}
\end{vmatrix}
= E3 \tag{A3.7}$$

Expanding determinant gives

$$\begin{aligned}
& h_1 \frac{\partial \xi}{\partial y} \left(h_1 \frac{\partial \xi}{\partial y} \right) \left[-c_1 \tau h_2 \frac{\partial \xi}{\partial x} + \tau h_1 \frac{\partial \xi}{\partial y} \right] \\
& + h_1 \frac{\partial \xi}{\partial y} h_2 \tau c_2 h_1 \frac{\partial \xi}{\partial y} \frac{\partial \xi}{\partial x} - h_1 \frac{\partial \xi}{\partial y} \left(U h_2 \frac{\partial \xi}{\partial x} + h_1 v \frac{\partial \xi}{\partial y} \right) \times
\end{aligned}$$

$$\left[\left(\frac{h_2 U}{2a_1} \frac{\partial \xi}{\partial x} \right) + \left(\frac{h_1 v}{2a_1} + h_1 \hat{G} \right) \frac{\partial \xi}{\partial y} \right] = 0 \quad (\text{A3.8})$$

so that we obtain three characteristic directions as follows:

$$h_1 \frac{\partial \xi}{\partial y} = 0$$

and (A3.9)

$$\begin{aligned} & h_1 \frac{\partial \xi}{\partial y} \left(-c_1 \tau h_2 \frac{\partial \xi}{\partial x} + \tau h_1 \frac{\partial \xi}{\partial y} \right) + c_2 \tau h_1 h_2 \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial y} \\ & - \left(U h_2 \frac{\partial \xi}{\partial x} + h_1 v \frac{\partial \xi}{\partial y} \right) \left[\frac{U h_2}{2a_1} \frac{\partial \xi}{\partial x} + \left(\frac{h_1 v}{2a_1} + h_1 \hat{G} \right) \frac{\partial \xi}{\partial y} \right] = 0 \end{aligned} \quad (\text{A3.10})$$

where

$$d\xi^2 = h_1^2 dx^2 + h_2^2 dy^2$$

We rearrange equation (A3.10) we obtain

$$\begin{aligned} & \left(\frac{h_2 dy}{h_1 dx} \right)^2 \frac{U^2}{2a_1} + \frac{h_2 d}{h_1 d} \left[(c_1 - c_2) \tau + \frac{Uv}{a_1} \right. \\ & \left. + U \hat{G} \right] - \tau + v \left(\frac{v}{2a_1} + \hat{G} \right) = 0 \end{aligned} \quad (\text{A3.11})$$

The solution of equation (A3.11) is

$$\frac{h_2 dy}{h_1 dx} = \frac{v + \alpha \pm \beta}{U} \quad (\text{A3.12})$$

where

$$\begin{aligned} \alpha &= a_1 \left[\hat{G} + \frac{c_1 - c_2}{U} \tau \right] \\ \beta &= \sqrt{\alpha^2 + 2a_1 \tau \left(1 + \frac{(c_1 - c_2)v}{U} \right)} \end{aligned}$$

Thus equation (A3.9) gives the vertical characteristic and equation (A3.12) gives the inclination of two inclined characteristics (the "ingoing" and "outgoing" characteristics).

A3.2 Derivation of Equation along the Characteristics

Firstly, we shall consider along the vertical characteristic; since the characteristic direction is normal, we can obtain this equation by substituting the continuity equation (A3.5) into the streamwise momentum equation (A3.6) to obtain

$$U E1 - U h_1 \frac{\partial v}{\partial y} + h_1 v \frac{\partial U}{\partial y} + h_1 \frac{\partial \tau}{\partial y} = E2$$

or

$$h_1 v \frac{\partial U}{\partial y} - U h_2 \frac{\partial v}{\partial y} + h_1 \frac{\partial \tau}{\partial y} = (E2 - U E1) \quad (A3.13)$$

where the derivative in the normal direction are now total derivatives and is used to solve for v .

We shall now derive the equations along the two inclined characteristics to solve for U and τ . We assume that two partial differential equations (A3.6) and (A3.7) are written in the form:

$$A_1 \frac{\partial U}{\partial x} + B_1 \frac{\partial U}{\partial y} + C_1 \frac{\partial \tau}{\partial x} + D_1 \frac{\partial \tau}{\partial y} = E2 \quad (A3.14)$$

$$A_2 \frac{\partial U}{\partial x} + B_2 \frac{\partial U}{\partial y} + C_2 \frac{\partial \tau}{\partial x} + D_2 \frac{\partial \tau}{\partial y} = E3 \quad (A3.15)$$

where

$$A_1 = U h_2, \quad A_2 = (C_1 - C_2) \tau h_2$$

$$B_1 = h_1 v, \quad B_2 = -\tau h_1$$

$$C_1 = 0, \quad C_2 = \frac{U h_2}{2a_1}$$

$$D_1 = -h_1$$

$$D_2 = -h_1 \left(\frac{v}{2a_1} + \hat{G} \right)$$

If we insert the characteristic angle into equations (A3.14) and (A3.15) we obtain

$$F' dU + [A' \lambda + G'] d\tau = (K' \lambda + H') dx \quad (A3.16)$$

where

$$F' = \tau h_1 h_2 (U + (C_1 - C_2) V)$$

$$A' = - \frac{U^2 h_2^2}{2 a_1}$$

$$G' = \frac{h_1 h_2 U V}{2 a_1}$$

$$H' = h_1 (V E_2 + \tau E_1)$$

$$K' = (C_1 - C_2) \tau h_2 E_1 - U h_2 E_2$$

Hence

$$\begin{aligned} & \tau \left[1 + (C_1 - C_2) \frac{V}{U} \right] dU - \frac{\alpha \pm \beta}{2 a_1} d\tau \\ &= \frac{\psi}{U} dx \end{aligned} \quad (A3.17)$$

where

$$\psi = \frac{\lambda K' + H'}{h_1 h_2}$$

A3.3 Finite Difference Forms for the V and p Calculation

We shall derive here the finite difference form of the equations used for the V and p calculation, using finite difference scheme described in §2.3. The equations solved are vertical characteristic equation (A3.13) and the normal momentum equation (A3.3).

(a) Solution for V

(i) boundary layer

From the equation (A3.13), we centred all y-derivatives at the point B, so that

$$\frac{\partial U}{\partial y} = \frac{U_{i,j} - U_{i,j-1}}{\Delta y} ; \quad \frac{\partial v}{\partial y} = \frac{v_{i,j} - v_{i,j-1}}{\Delta y}$$

$$\frac{\partial \tau}{\partial y} = \frac{\tau_{i,j} - \tau_{i,j-1}}{\Delta y}$$

The variables which do not have derivatives will be centred at the point. For example τ and $C_1 \frac{\partial \tau}{\partial x}$ are known in U and τ calculation, so that we use simple backward difference for it. The backward differencing is used for $\partial p / \partial x$ term by using previous sweep value.

With these assumption, the finite difference form of the equation along the vertical characteristic, which is used to obtain V at any station i, becomes:

$$h_{i,j-\frac{1}{2}} \frac{1}{2} (v_{i,j} + v_{i,j-1}) \left(\frac{U_{i,j} - U_{i,j-1}}{\Delta y} \right) - h_{i,j-\frac{1}{2}} \frac{1}{2} (U_{i,j} + U_{i,j-1}) \times$$

$$\left(\frac{v_{i,j} - v_{i,j-1}}{\Delta y} \right) = (E_2 - U E_1)_{i,j}$$

(A3.18)

Hence $V_{i,j}$ can be found at the jth mesh point, given $V_{i,j-1}$ mesh point, both at the ith station point.

(ii) wake region

In the wake region the boundary condition for V value is replaced by a pressure value. In order to calculate V in wake region we rearrange the vertical characteristic equation (A3.13) as follows:

$$h_1 \left[\bar{v} \frac{\partial \bar{U}}{\partial y} - \bar{U} \frac{\partial \bar{v}}{\partial y} \right] = -h_2 \frac{\partial p}{\partial x} + (\bar{U}^2 + \bar{v}^2) \frac{\partial h_2}{\partial x} + h_1 \frac{\partial \tau}{\partial y}$$

$$- h_2 \frac{\partial \bar{U}^2}{\partial x} + \tau \frac{\partial h_1}{\partial y} - (\bar{U}^2 - \bar{v}^2) \frac{\partial h_2}{\partial x} \quad (A3.19)$$

After integrating equation (A3.13) along the normal direction, we obtain

$$\left(\frac{\bar{v}(y)}{\bar{u}(y)} \right) - \left(\frac{\bar{v}(JCL)}{\bar{u}(JCL)} \right) = - \int_0^y \frac{f(u, \tau, p)}{h_1 \bar{u}^2} dy \quad (A3.20)$$

where $\bar{v}(JCL)/\bar{u}(JCL)$ is constant value and JCL is centre point of wake. If we assume $\bar{v}(JCL)=0$ initially, we obtain $\bar{v}$ as follows:

$$\bar{v}_i(y) = - \bar{u}(y) \int_0^y \frac{f(u, \tau, p)}{h_1 \bar{u}^2} dy = F_2(y)$$

Once $\bar{v}(JCL)$ is defined, we set $\bar{v}$ value as follows:

$$\begin{aligned} \bar{v}(y) &= \bar{v}(JCL) \frac{\bar{u}(y)}{\bar{u}(JCL)} + \bar{v}_i(y) \\ &= \bar{v}(JCL) \alpha + \bar{v}_i(y) \\ &= \frac{\bar{v}(JCL)}{\bar{u}(JCL)} F_1(y) + F_2(y) = v_1 F_1(y) + F_2(y) \end{aligned} \quad (A3.21)$$

We substitute this $\bar{v}$ value into normal momentum equation and integrate as follows:

$$\begin{aligned} & \int_0^\Delta (h_2 u \frac{\partial \bar{v}}{\partial x}) dy + \int_0^\Delta (h_1 v \frac{\partial \bar{v}}{\partial y}) dy - \int_0^\Delta (h_2 \frac{\partial \tau}{\partial x}) dy \\ & + \int_0^\Delta [u (v \frac{\partial h_2}{\partial x} - u \frac{\partial h_1}{\partial y})] dy + \int_0^\Delta (h_1 \frac{\partial p}{\partial y}) dy + \int_0^\Delta (h_1 \frac{\partial \bar{v}^2}{\partial y}) dy \\ & - \int_0^\Delta (2\tau \frac{\partial h_2}{\partial x}) dy - \int_0^\Delta (\bar{u}^2 - \bar{v}^2) \frac{\partial h_1}{\partial y} dy = 0 \end{aligned} \quad (A3.22)$$

Now, we integrate equation (A3.22) term by term as follows:

$$\begin{aligned} T1 &= \int_0^\Delta (h_2 u \frac{\partial \bar{v}}{\partial x}) dy = \frac{1}{\Delta x} \int_0^\Delta h_2 u (v - v_{OLD}) dy \\ &= \frac{1}{\Delta x} \left[v_1 \int_0^\Delta h_2 u F_1 dy + \int_0^\Delta h_2 u (F_2 - v_{OLD}) dy \right] \end{aligned} \quad (A3.23)$$

$$\begin{aligned} T2 &= \int_0^\Delta h_1 v \frac{\partial \bar{v}}{\partial y} dy \\ &= \frac{1}{2} \left[(h_1 v^2) \Big|_0^\Delta - \int_0^\Delta \frac{\partial h_1}{\partial y} v^2 dy \right] \\ &= \frac{1}{2} \left\{ v_1^2 [(h_1 F_1^2)_\Delta - (h_1 F_1^2)_0] + v_1 [(2 F_1 F_2 h_1)_\Delta - (2 F_1 F_2 h_1)_0] + (h_1 F_2^2)_\Delta \right\} \end{aligned}$$

$$\begin{aligned}
& - (h_1 F_2^2)_0 - v_1^2 \int_0^\Delta \frac{\partial h_1}{\partial y} F_1^2 dy - 2v_1 \int_0^\Delta \frac{\partial h_1}{\partial y} F_1 F_2 dy \\
& - \int_0^\Delta \frac{\partial h_1}{\partial y} F_2^2 dy \} \quad (A3.24)
\end{aligned}$$

$$T3 = - \int_0^\Delta (h_2 \frac{\partial \tau}{\partial x}) dy = - \frac{1}{\Delta x} \int_0^\Delta h_2 (\tau_f - \tau_c) dy \quad (A3.25)$$

$$\begin{aligned}
T4 &= \int_0^\Delta uv \frac{\partial h_2}{\partial x} dy - \int_0^\Delta (u^2 \frac{\partial h_1}{\partial y}) dy \\
&= v_1 \int_0^\Delta \frac{\partial h_2}{\partial x} u F_1 dy + \int_0^\Delta \frac{\partial h_2}{\partial x} u F_2 dy \\
&\quad - \int_0^\Delta (u^2 \frac{\partial h_1}{\partial y}) dy \quad (A3.26)
\end{aligned}$$

$$T5 = \int_0^\Delta (h_1 \frac{\partial p}{\partial y}) dy = (h_1 p)_\Delta - (h_1 p)_0 - \int_0^\Delta (\frac{\partial h_1}{\partial y} p) dy \quad (A3.27)$$

$$T6 = \int_0^\Delta (h_1 \frac{\partial \bar{v}^2}{\partial y}) dy = (h_1 \bar{v}^2)_\Delta - (h_1 \bar{v}^2)_0 - \int_0^\Delta (\frac{\partial h_1}{\partial y} \bar{v}^2) dy \quad (A3.28)$$

$$T7 = - \int_0^\Delta (2\tau \frac{\partial h_2}{\partial x}) dy \quad (A3.29)$$

$$T8 = - \int_0^\Delta (\bar{u}^2 - \bar{v}^2) \frac{\partial h_1}{\partial y} dy \quad (A3.30)$$

Hence we can obtain equation for v_1 as follows:

$$AAv_1^2 + BBv_1 + CC = 0 \quad (A3.22)$$

where

$$\begin{aligned}
AA &= \left[\frac{h_1 F_1^2}{2} \right]_\Delta - \left[\frac{h_1 F_1^2}{2} \right]_0 \\
&\quad - \frac{1}{2} \int_0^\Delta \frac{\partial h_1}{\partial y} F_1^2 dy \\
BB &= [F_1 F_2 h_1]_\Delta - [F_1 F_2 h_1]_0 - \int_0^\Delta \frac{\partial h_1}{\partial y} F_1 F_2 dy \\
&\quad + \frac{1}{\Delta x} \int_0^\Delta h_2 u F_1 dy + \int_0^\Delta \frac{\partial h_2}{\partial x} u F_1 dy \\
CC &= \frac{1}{\Delta x} \int_0^\Delta h_2 u (F_2 - v_0) dy + \left[\frac{h_1 F_2^2}{2} \right]_\Delta - \left[\frac{h_1 F_2^2}{2} \right]_0 \\
&\quad - \int_0^\Delta \frac{\partial h_1}{\partial y} \frac{F_2^2}{2} dy - \int_0^\Delta \frac{\partial h_2}{\partial x} u F_2 dy - \int_0^\Delta u^2 \frac{\partial h_1}{\partial y} dy
\end{aligned}$$

$$+ T5 + [h_1 \bar{v}^2]_{\Delta} - [h_1 \bar{v}^2]_0 + T7 + T8 + T9$$

$$\begin{aligned} \text{where } T9 &= - \int_0^{\Delta} \frac{\partial h_1}{\partial y} \bar{v}^2 dy - \int_0^{\Delta} (\bar{u}^2 - \bar{v}^2) \frac{\partial h_1}{\partial y} dy \\ &= - \int_0^{\Delta} \frac{\partial h_1}{\partial y} dy \end{aligned}$$

After obtaining v_1 values we can obtain V values by substituting v_1 into equation (A3.21).

(b) Solution for p

We calculate p from normal momentum equation (A3.3).

The differencing scheme is similar to Mahgoub (1978). We use the backward differencing for streamwise derivative terms, for example, $\frac{\partial \tau}{\partial x} = \frac{\tau_{ij} - \tau_{i-1,j}}{\Delta x}$. The term $(c_1 - c_2)\tau$ is assumed as known value. Hence the normal momentum equation becomes:

$$\begin{aligned} & h_{2,i+\frac{1}{2},j} \frac{1}{2} (u_{ij} + u_{i,j-1}) \left(\frac{\partial v}{\partial x} \right)_{i,j-\frac{1}{2}} + h_{1,i,j-\frac{1}{2}} \frac{1}{2} (v_{i,j} + v_{i-1,j}) \times \\ & \left(\frac{v_{ij} - v_{i,j-1}}{\Delta y} \right) - h_{2,i,j-\frac{1}{2}} \frac{\tau_{ij} - \tau_{i-1,j}}{\Delta x} - \frac{1}{4} (u_{ii} + u_{i,i-1}) (v_{ij} + v_{i,i-1}) \times \\ & \left(\frac{\partial h}{\partial x} \right)_{i,j-\frac{1}{2}} + \frac{1}{4} (u_{ij} + u_{i,j-1})^2 \left(\frac{\partial h_1}{\partial y} \right)_{i,j-\frac{1}{2}} + h_{1,i,j-\frac{1}{2}} \frac{p_{ij} - p_{i,j-1}}{\Delta y} \\ & + h_{1,i,j-\frac{1}{2}} c_2 \frac{\tau_{ij} - \tau_{i,j-1}}{\Delta y} - (\tau_{ij} + \tau_{i,j-1}) \left(\frac{\partial h_2}{\partial x} \right)_{i,j-\frac{1}{2}} - (c_1 - c_2) \frac{1}{2} \\ & (\tau_{ij} + \tau_{i,j-1}) \left(\frac{\partial h_1}{\partial y} \right)_{i,j-\frac{1}{2}} = 0 \end{aligned}$$

(A3.23)

(c) Solution for p_w from p_1 in boundary layer

From equation (A3.23) we can solve for p at the $(J-1)$ th. mesh point from the value of p at the J th mesh point. Because the only boundary condition for p is external pressure p_e on MS (say, when $J=N, p = p_{eN}$), we can

solve for p_{ij} across the shear layer down to the first mesh point where p_{ij} is p_1 , say. We shall now consider obtaining p_w from p_1 at any given station: we shall assume that U is given by a fifth power law profile between the wall and the first mesh point (see Cebeci and Bradshaw 1977), so that $U/U_1 = (y/\Delta y)^{1/5}$, where U_1 is streamwise velocity at the first mesh point. If we insert this into the continuity equation

$$(A3.1) \quad \frac{\partial}{\partial x} \left(h_2 U_1 \left(\frac{y}{\Delta y} \right)^{1/5} \right) - \frac{\partial}{\partial y} (h_1 v) = 0$$

and integrate with respect to y from the wall to the point y , we obtain

$$h_1 v(y) = -\frac{5}{6} h_2 \frac{\partial U_1}{\partial x} \frac{y^{6/5}}{(\Delta y)^{1/5}} \quad (A3.24)$$

for $0 < y < \Delta y$.

We then insert $U/U_1 = (y/\Delta y)^{1/5}$ and Equation (A3.24) into normal momentum equation (A3.3) and integrate with respect to y from the wall to the first mesh point. We shall perform this term by term.

$$\begin{aligned} (i) \quad h_2 \int_0^{\Delta y} \frac{\partial (uv)}{\partial x} dy &= h_2 \frac{\partial}{\partial x} \int_0^{\Delta y} (uv) dy \\ &= -\frac{25}{144} \frac{h_2}{h_1} \frac{\partial^2 U_1^2}{\partial x^2} \\ &= \text{Neglect} \end{aligned} \quad (A3.25)$$

We neglect second order term, because it is proportional to y^2 .

$$(ii) \quad \int_0^{\Delta y} h_2 \frac{\partial \tau}{\partial x} dy = \frac{1}{2} h_{2m} (\tau_1 + \tau_w) \frac{\Delta y}{\Delta x} \quad (A3.26)$$

$$(iii) \quad \int_0^{\Delta y} 2uv \frac{\partial h_2}{\partial x} dy$$

$$\begin{aligned}
&= 2 \frac{\partial h_2}{\partial x} \int_0^{\Delta y} u_1 \frac{y^{1/5}}{\Delta y^{1/5}} \left(-\frac{5}{6} \frac{h_2}{h_1} \frac{\partial u_1}{\partial x} - \frac{(y)^{6/5}}{(\Delta y)^{1/5}} \right) dy \\
&= -\frac{25}{72} \frac{h_2}{h_1} \frac{\partial h_2}{\partial x} \frac{\partial u_1^2}{\partial x} \Delta y
\end{aligned} \tag{A3.27}$$

$$(iv) \quad \int_0^{\Delta y} h_1 \frac{\partial C_2 \tau}{\partial y} dy = C_2 h_{1m} (\tau_1 - \tau_0) \tag{A3.28}$$

$$(v) \quad \int_0^{\Delta y} (C_1 - C_2) \tau \frac{\partial h_1}{\partial y} dy = (C_1 - C_2) \tau_{av} (h_{11} - h_{10}) \tag{A3.29}$$

$$(vi) \quad \int_0^{\Delta y} 2 \tau \frac{\partial h_1}{\partial x} dy = 2 \tau_{av} \left(\frac{\partial h_1}{\partial x} \right)_m \Delta y \tag{A3.30}$$

$$(vii) \quad \int_0^{\Delta y} h_1 \frac{\partial v^2}{\partial y} dy = h_1 v(1)^2 \tag{A3.31}$$

$$(viii) \quad \int_0^{\Delta y} h_1 \frac{\partial p}{\partial y} dy = h_{1m} (p_1 - p_w) \tag{A3.32}$$

$$(ix) \quad \int_0^{\Delta y} (v^2 - u^2) \frac{\partial h_1}{\partial y} dy = \Delta y \left(v_{av}^2 \left(\frac{\partial h_1}{\partial y} \right)_m - \frac{5}{7} u_1^2 \left(\frac{\partial h_1}{\partial y} \right)_m \right) \tag{A3.33}$$

Therefore if we insert these terms (A3.25) to (A3.33) into equation (A3.3) we can obtain a direct equation for $(p_w - p_1)$

Hence we can calculate complete pressure profile in boundary layer.

A3.4 Method of solution for the wall boundary condition in boundary layer

In this section we describe the method of

application of the wall boundary conditions for the boundary layer. As stated in §2.3 . we seek to solve U_1 , τ_1 , τ_w and V_1 for the first mesh point, given the wall boundary conditions $U=0$ and $V=0$. We can obtain the values of U_1 , τ_1 and τ_w by solving the following three non-linear equations iteratively by Newton's method:

(i) The logarithmic law for the inner layer, written as an equation $F1=0$.

(ii) The equation along the ingoing characteristic which passes through the first mesh point (see Fig. 4), written as an equation $F2=0$.

(iii) The streamwise momentum equation, with a fifth power law U-velocity profile between the wall and the first mesh point, integrated from the wall to the first mesh point: this is written as $F3=0$.

We shall derive the functions $F1$, $F2$ and $F3$ separately and we will show the method of solution of the equations for $F1$, $F2$ and $F3$ by Newton's method. For τ_1 we use $\alpha_0 = \frac{\tau_1 - \tau_w}{\Delta y}$. Therefore, we solve for U_1 , α_0 and τ_w and we write $F1$, $F2$ and $F3$ in terms of these variables accordingly.

(i) The function $F1$

This is the logarithmic law for the inner layer written at the first mesh point.

$$F1(U_1, \alpha_0, \tau_w) = U_1 - u_\tau \left[\frac{1}{K} \left\{ \ln \frac{u_\tau \Delta y}{\nu} + \psi' \left(\frac{y}{\delta} \right) \right\} + c \right] = 0 \quad (A3.34)$$

where ψ' is a function of y/δ only, c is the additive constant for the inner layer logarithmic law and $u_\tau^2 = \frac{\tau_w}{\rho}$.

(ii) The function $F2$

This is the equation along the "ingoing" characteristic which passes through the first mesh point. This is equation (A3.17) with α, β coefficient for $d\tau$: thus corresponding to the characteristic angle ϕ_{in} where $\tan\phi_{in} = (\frac{h_1}{h_2} \frac{V+\alpha-\beta}{U})$. Thus

$$F'dU + [A'\lambda - G']d\tau = (-H' + \lambda K')dx$$

Also G and $\partial G/\partial y$ are equated to zero near the wall, so that

$$\alpha = a_1 (C_1 - C_2) \frac{\tau}{U}$$

and

$$\beta = \sqrt{\alpha^2 + 2a_1\tau \left\{ 1 + (C_1 - C_2) \frac{V}{U} \right\}}$$

Hence the $F2$ can be written as follows:

$$F2 = F'(U_{fut}(1) - U_{int}) + (A'\lambda - G')(\tau_{fut}(1) - \tau_{int})$$

$$-(-H' + K'\lambda) = 0 \quad (A3.35)$$

where U_{int}, τ_{int} are the values of U and τ interpolated at the point where the ingoing characteristic meets the previous station. The values of the various variables are written in the same way as for U and τ calculation (see §2.3).

(iii) The function F3

This is defined by inserting one-fifth power law for U profile into the streamwise momentum equation and then integrating it from the wall to the first mesh point. Inserting $U/U_1 = (y/\Delta y)^{1/5}$ into continuity equation and integrating from the wall to the first mesh point we can obtain

$$\partial/\partial x \left(\int_0^y h_2 U dy \right) = - \int_0^y h_2 U_1 \left(\frac{y}{y_1} \right)^{1/5} dy$$

$$= h_2 \frac{\partial u_1}{\partial x} \frac{y^{6/5}}{(y_1)^{1/5}} \quad (A3.36)$$

$$h_1 v = - \frac{5}{6} h_2 \frac{\partial u_1}{\partial x} \frac{y^{6/5}}{(y_1)^{1/5}} \quad (A3.37)$$

We can write the streamwise momentum equation in the form

$$\begin{aligned} h_2 \frac{\partial u^2}{\partial x} + h_1 \frac{\partial uv}{\partial y} + 2uv \frac{\partial h_1}{\partial y} + u^2 \frac{\partial h_2}{\partial x} - h_1 \frac{\partial \tau}{\partial y} - 2\tau \frac{\partial h_1}{\partial y} \\ - v^2 \frac{\partial h_2}{\partial x} + \tau(C_1 - C_2) \frac{\partial h_2}{\partial x} + h_2 \left(\frac{\partial p}{\partial x} + \frac{\partial C_1 \tau}{\partial x} \right) \end{aligned} \quad (A3.38)$$

We integrate streamwise momentum equation from the wall to the first mesh point term by term as follows:

$$\begin{aligned} \int_0^{y_1} h_2 \frac{\partial u^2}{\partial x} dy &= \frac{5}{7} h_{2m} \frac{\partial u_1^2}{\partial x} \Delta y \\ \int_0^{y_1} h_1 \frac{\partial uv}{\partial y} dy &= h_1 u_1 v_1 \\ \int_0^{y_1} 2uv \frac{\partial h_1}{\partial y} dy &= - \frac{5}{144} \frac{\partial u_1^2}{\partial x} \frac{h_2}{h_1} (\Delta y)^2 \\ \int_0^{y_1} h_1 \frac{\partial \tau}{\partial y} dy &= h_{1m} (\tau_1 - \tau_w) \\ \int_0^{y_1} -2\tau \frac{\partial h_1}{\partial y} dy &= -2 \left(\frac{\partial h_1}{\partial y} \right)_m \tau_{av} (\Delta y)^2 \\ \int_0^{y_1} (C_1 - C_2) \tau \frac{\partial h}{\partial x} dy &= (C_1 - C_2) \tau_{av} \left(\frac{\partial h}{\partial x} \right)_m \Delta y \\ \int_0^{y_1} \left(u^2 \frac{\partial h_2}{\partial x} - v^2 \frac{\partial h_2}{\partial x} \right) dy &= \left(\frac{\partial h_2}{\partial x} \right)_m \left(\frac{5}{7} u_1^2 \Delta y \right) - \left(\frac{\partial h_2}{\partial x} \right)_m v_{av}^2 \Delta y \\ \int_0^{y_1} h_2 \left(\frac{\partial p}{\partial x} + \frac{\partial C_1 \tau}{\partial x} \right) dy &= h_{2m} \left(\frac{\Delta p}{\Delta x} + \frac{C_1 \Delta \tau}{\Delta x} \right) \Delta y \end{aligned}$$

After substituting these values into equation (A3.38), we can obtain

$$\begin{aligned}
F3 = & -h_{1m} \alpha_0 + 5h_{2m} \frac{\partial U_1^2}{\partial x} \left(\frac{1}{7} - \frac{1}{12} h_{1m} \right) + \frac{5}{7} \frac{\partial h_{2m}}{\partial x} U_1^2 \\
& + h_{2m} \left(\frac{\Delta p}{\Delta x} - \frac{C_1 \Delta \tau}{\Delta x} \right) + \tau_0 \left(-2 \frac{\partial h_{1m}}{\partial y} + \right. \\
& \left. (C_1 - C_2) \frac{\partial h_{2m}}{\partial x} \right)
\end{aligned}
\tag{A3.39}$$

(vi) Newton's method for U_1, α_0, τ_w

We have three simultaneous non-linear equations of the form

$$\begin{aligned}
F1(U_1, \alpha_0, \tau_w) &= 0 \\
F2(U_1, \alpha_0, \tau_w) &= 0 \\
F3(U_1, \alpha_0, \tau_w) &= 0
\end{aligned}
\tag{A3.40}$$

We guess the values U_1 , α_0 and τ_w (we can use the values at the previous station). We then increase dU_1 , $d\alpha_0$ and $d\tau_w$ for each U_1 , α_0 and τ_w . These produce corresponding increments $dF1$, $dF2$ and $dF3$ in $F1$, $F2$ and $F3$: these increments are given as follows:

$$\begin{aligned}
dF1 &= \partial F1 / \partial U_1 dU_1 + \partial F1 / \partial \alpha_0 d\alpha_0 + \partial F1 / \partial \tau_w d\tau_w \\
dF2 &= \partial F2 / \partial U_1 dU_1 + \partial F2 / \partial \alpha_0 d\alpha_0 + \partial F2 / \partial \tau_w d\tau_w \\
dF3 &= \partial F3 / \partial U_1 dU_1 + \partial F3 / \partial \alpha_0 d\alpha_0 + \partial F3 / \partial \tau_w d\tau_w
\end{aligned}
\tag{A3.41}$$

We shall now derive derivative values term by term.

From the $F1$ function (equation (A3.34)) we can write

$$\begin{aligned}
F1 = & U_{fut}(1) - \frac{u_\tau}{k} \left[\left\{ \ln \frac{y u_\tau}{\nu} + kC \right\} + 2 \ln \left\{ \frac{2}{(1+z)^{1/2} + 1} \right\} \right. \\
& \left. - 2 \left\{ (1+z)^{1/2} - 1 \right\} \right]
\end{aligned}$$

$$\partial F1 / \partial U_1 = 1$$

$$\begin{aligned}
\partial F1 / \partial \tau_w = & -1/2 k u_\tau \left\{ \ln \{ y \sqrt{\tau_w} / \nu \} + C \right\} - \partial / \partial \tau_w \left[2 \left\{ (1+z)^{1/2} - 1 \right\} + 2 \ln \left\{ 2 / ((1+z)^{1/2} + 1) \right\} \right]
\end{aligned}$$

$$\begin{aligned}
&= - \frac{1}{2ku_\tau} \frac{UFUT(1)}{u_\tau} - \frac{1}{2ku_\tau} \\
&\quad - \frac{\alpha_0 y}{ku_\tau} \frac{1}{\tau_w} \frac{1}{\{(1+z)^{1/2}+1\}} \\
&= - \frac{1}{2k} \frac{UFUT(1)}{\tau_w} - \frac{1}{2ku_\tau} - \frac{y}{ku_\tau} \frac{1}{\tau_w} \\
&\quad \frac{1}{\{(1+z)^{1/2}+1\}} = E \\
\partial F1/\partial \alpha_0 &= \partial F1/\partial z \partial z/\partial \alpha_0 \\
&= \frac{\sqrt{\tau_w}}{k} \frac{y}{\tau_w} \frac{1}{[(1+z)^{1/2}+1]} \\
&= \frac{y}{k\sqrt{\tau_w}} \frac{1}{[(1+z)^{1/2}+1]} = D
\end{aligned}$$

From the equation (A3.35) we can write

$$F2 = \tau_{fut(1)} - \tau_{int} + \left(- \frac{2a_1\tau}{(\alpha-\beta)} - \frac{2a_1\tau}{(\alpha-\beta)} \frac{V}{U} (C_1 - C_2) \right)$$

$$(U_{fut(1)} - U_{int}) - \frac{2a_1}{h_1 h_2 U (\alpha-\beta)} (\lambda K' - H')$$

$$\begin{aligned}
\partial F2/\partial U_1 &= - \frac{2a_1\tau}{\alpha-\beta} \left(1 + (C_1 - C_2) \frac{V}{U} \right) \\
&= B
\end{aligned}$$

$$\partial F2/\partial \alpha_0 = 1$$

$$\partial F2/\partial \tau_w = y$$

From the equation (A3.39) we can write

$$\begin{aligned}
F3 &= -h_{1m}\alpha_0 - \frac{5}{12}h_{2m}\frac{\partial U_1^2}{\partial x} + \frac{\partial h_{1m}}{\partial x}U_1^2\frac{5}{7} - \frac{\partial h_{2m}}{\partial x}v_{av}^2 \\
&\quad + h_{2m}\left(\frac{\Delta p}{\Delta x} - \frac{C_1\Delta\tau}{\Delta x}\right) + \tau_{av}\left\{\frac{\partial h_{1m}}{\partial y}2 + (C_1 - C_2)\frac{\partial h_{2m}}{\partial x}\right\}
\end{aligned}$$

$$\begin{aligned}
\partial F3/\partial U_1 &= 5 \left(\frac{1}{7} - \frac{1}{12}h_{1m} \right) h_{2m} \frac{2UFUT(1)}{\Delta x} \\
&\quad + \frac{5}{7} \frac{\partial h_{2m}}{\partial x} 2UFUT(1) = A
\end{aligned}$$

$$\partial F3/\partial \tau_w = 0$$

$$\partial F3/\partial \alpha_0 = -h_{1m}$$

Thus we insert these values into equation (A3.41) and we solve these equations for dU_1 , $d\tau_w$ and $d\alpha_0$.

$$dU = \frac{\begin{vmatrix} F1 & \partial F1/\partial \alpha_0 & \partial F1/\partial \tau_w \\ F2 & \partial F2/\partial \alpha_0 & \partial F2/\partial \tau_w \\ F3 & \partial F3/\partial \alpha_0 & \partial F3/\partial \tau_w \end{vmatrix}}{\begin{vmatrix} \partial F1/\partial U_1 & \partial F1/\partial \alpha_0 & \partial F1/\partial \tau_w \\ \partial F2/\partial U_1 & \partial F2/\partial \alpha_0 & \partial F2/\partial \tau_w \\ \partial F3/\partial U_1 & \partial F3/\partial \alpha_0 & \partial F3/\partial \tau_w \end{vmatrix}}$$

The Determinant value of denominator is as follows:

$$\begin{aligned} \text{Det} &= AD - h_{1m} BE - AE\Delta y + h_{1m} \\ &= h_{1m}(1 - BE) + A(D - E\Delta y) \end{aligned}$$

Hence we can obtain dU_1 value

$$dU_1 = (-F3D + h_{1m}F2E + F3E\Delta y) / \text{Det}$$

In a similar manner

$$d\alpha_0 = (-AF1 - F3BE + AF + F3) / \text{Det}$$

$$d\tau_w = (-F3\Delta y - F2AD + F1h_{1m}B + AF1\Delta y - F2h_{1m} + F3BD) / \text{Det}$$

Now, for a solution $F1+dF1=0$, $F2+dF2=0$ and $F3+dF3=0$ so that these three equations form a set of three linear algebraic equations for the increment dU_1 , $d\alpha_0$ and $d\tau_w$, which we solve. This improves our original guess for U_1 , α_0 and τ_w and we insert these improved values into the three equations and we obtain better estimates for dU_1 , $d\alpha_0$ and $d\tau_w$. This procedure is iterated until the values of U_1 , α_0 and τ_w converge to within prescribed tolerance.

Re-calculation of V and ingoing characteristic angle

We recalculate the value of V_1 and also the angle of the ingoing characteristic, to improve on the values of U_1 , τ_1 and τ_w calculated by Newton's method. V_1 is obtained by integrating the continuity equation from the wall to the first mesh point assuming $U/U_\tau = f(u_\tau y/\nu)$. This gives:

$$h_1 V = -u_1 \frac{\Delta y}{u_\tau} - \frac{\partial u_\tau}{\partial x}$$

Now, the ingoing characteristic angle at the first mesh point is given by equation (A3.12), so that

$$\frac{h_2 dy}{h_1 dx} = \frac{V_1 + \alpha_1 - \beta_1}{U_1} \quad (\text{A3.43})$$

where $\alpha_1 = a_1 (C_1 - C_2) \frac{\tau_1}{U_1}$

$$\beta_1 = \sqrt{\alpha_1^2 + 2a_1\tau_1 \left\{ 1 + \frac{V_1}{U_1} (C_1 - C_2) \right\}}$$

A3.5 Spalding-Chi skin friction formula

This empirical formula may be used to give the starting value of the skin friction coefficient C_f at the start of the turbulent shear layer calculation. It used the "overlap law" for skin friction in the form

$$\sqrt{\frac{2}{C_f}} = \frac{U_e}{u_\tau} = \frac{1}{k} \log_e \frac{U_e \delta^*}{\nu} + D(R_{e\theta}) \quad (\text{A3.44})$$

where $D(R_{e\theta})$ is an empirical function of $R_{e\theta}$ as follows:

$$D(R_{e\theta}) = 4.84 - 4.88 / (1 + 7.14 \times 10^{-4} R_{e\theta})^3 \quad (\text{A3.45})$$

The second term allows empirically for low $R_{e\theta}$ effects.

$$\delta^* / \theta = 1 / (1 - 6.8 u_\tau / U_e) \quad (\text{A3.46})$$

is an empirical formula used to obtain the shape factor δ^* / θ for a turbulent boundary layer: it is equivalent to the statement that Clauser's parameter G is 6.8.

The procedure for finding C_f is: given θ and $R_{e\theta}$, firstly to guess a value (for example the value given by the simple formula $u_\tau / U_e = C_f / 2 = .113 R_{e\theta}^{-1/8}$). We then insert this into (A3.46) to obtain δ^* and insert this in turn into (A3.44) to give better value of U_e / u_τ (and hence C_f). This value of u_τ / U_e is then used in (A3.46) and so on: in principle the procedure is repeated until C_f has converged. However, in present program we have found it sufficient to obtain just one improvement on the initial guess for C_f .

A3.6 Transfer of U, V and τ at the patching point

The transformation from (s,n) coordinate to (φ, ψ) coordinate system at the patching point in the boundary layer and from (φ, ψ) coordinate to (s,n) coordinate in the wake is formulated.

Firstly, we consider Reynolds shear stress term. From tensor transfer formula, we obtain shear stress values.

$$\tau^{ij} = l_{ip} l_{jq} \tau^{pq} \quad (\text{A3.47})$$

where

$$l_{ip} = \frac{\partial x^i}{\partial x^p} \quad , \quad l_{jq} = \frac{\partial x^j}{\partial x^q}$$

After inserting physical component for tensor quantities, we can obtain shear stress in different coordinate system as follows;

$$\tau_{\varphi\psi} = \left\{ -\sin[2(n-1)\alpha] \frac{C_1 - C_2}{2} + \cos[2(n-1)\alpha] \right\} \tau \quad (\text{A3.48})$$

Similarly, we can obtain U and V values at different coordinate system as follows;

$$U_{\varphi} = U \cos[(n-1)\alpha] + V \sin[(n-1)\alpha] \quad (\text{A3.49})$$

$$V_{\psi} = -U \sin[(n-1)\alpha] + V \cos[(n-1)\alpha] \quad (\text{A3.50})$$

Appendix 4. Data for Viscous/inviscid Interaction
Cycle and Miscellaneous derivation

A4.1 Input Data

1. Data file AMDATOL (format 8E15.6)

TY(J): The array containing the influence coefficients.
 It is calculated at first sweep and fixed.

2. Data file AMODAT1 (format F7.3)

ALP : The angle of attack (degree).
 CHD : The chord length (=1)

3. Data file AMOSDAT

(1) NORUN,NCH,NXPTS,NTE,NCHW,NENDW,NTBW,N,MSCHNO,MSCHNG
 (format 16I5)

(2) X(J),Y(J) (format 2E15.6)

The array containing the coordinates of
 nodes relative to global axis.

(3) VMS(J) (format 8E15.6)

The array containing normal velocity normal to MS
 at element mid-point).

4. Data file BLDATT/BLDATB

(1) NSTART (format 16I5)

The total number of nodes from LE to transition
 point to turbulent flow.

(2) NORUN,NCH,NXPTS,NTE,NCHW,NENDW,NBTW,N,MSCHNO,MSCHNG
 (format 16I5)

(3) DXTE,YMAX,RIN (format 8E15.6)

(4) YMSNOD(J) (format 8E15.6)

(5) XO(J),YLOC(J) format (2E15.6)

(4) PEXTA(J) (format 8E15.6)

An array containing pressure distribution on MS.

5. Data file BLDAT1/BLDATB1

(1) I(J) (format 16I5)

An input array containing control parameters ,
controlling the options in the program.

(1) spare;first digit is carriage control
character. If I(1) is negative, profile
printing is suppressed.

(2) =1, profile input.

=0, synthetic starting profiles.

(3) number of x steps between each call PRINT;if
negative, (-I(3)) value of x are read into
array XP and PRINT is called at these values.

(4) total number of steps to be executed.

(5) total number of boundary values (U_e etc.)

(6) number of points on input profiles (U, T).

(7) =1, separate dU_e/dx input.

=0, dU_e/dx from three point parabolic fit.

(8) =0, there is no Roughness.

=1, considering the effect of Roughness.

(9) =1, longitudinal curvature correction.

=0, else.

(10) =0, no lateral divergence.

=1, x_0 input.

=2, body of revolution radius input.

=3, $1/(x-x_0)$.

(11) maximum number of points on each profile; if
set to 0, program sets $I(11)=1.5I(6)$.

(12) =1, distributed suction/injection.

=0, else.

(13) <0, all values are stored in subroutine DUMP.

(14) maximum number of executed stations.

(2) A(J) (format 7F10.2)

An array specifying scales and starting conditions.

(1) y_{step}/l_{ref} ; if synthetic starting profiles are used, A(1) is initial θ/l_{ref} .

(2) (streamwise spacing of shear layer)/ l_{ref} .

(3) Initial x/l_{ref} .

(4) $U_{ref} l_{ref} / \nu$; if synthetic starting profiles are used, A(4) is initial $U_{e\theta} / \nu$.

(5) initial $C_f/2$.

(6) The coefficient used to evaluate XSTEP from CFL.

(7) The under-relaxation factor for pressure.

(8)=ETA , control value for differencing scheme.

(3) ROUGH(J), if I(8)=0 (format 7F10.2)

(4) XP(J), if I(3)<0 (format 16I5)

The station subroutine PRINT is called.

(5) SRST(J), CURV(J) (format 2E15.6)

6. Data file CPDAT (format 2E10.4)

S(J) An array containing potential calculation points.

Y(J) An array containing pressure distribution at potential flow calculation points.

7. Data file COEPPY/COEPPYB

(1) NSTNS (format 16I5)

(2) CDPDY(J) (format 6E15.6)

(3) PRESSW(J) (format 6E15.6)

An array containing pressure at wall in boundary layer.

(4) XCAL(J) (format 8E15.6)

(5) XPPCAL(J) (format 8E15.6)

The shear layer calculation station at which pressure calculated.

(6) YMSOLD(J) (format 8E15.6)

(7) ADPDY(J) (format 8E15.6)

An array containing gradient of pressure profile at TE.

(8) PE (format 8E15.5)

External pressure on upper MS at first shear layer calculation station of the wake.

(9) ADPDY(J) (format 8E15.5)

An array containing gradient of pressure profile at first shear layer calculation station of the wake.

(10) XCALW (format 8E15.5)

The first shear layer calculation station of the wake.

(11) PRESSW (format 8E15.5)

pressure at centre point of the wake at first shear layer calculation station.

8. Data file INWAK

(1) CEDDY (format F8.6)

(2) US(J) (format 8A4)

An array containing characters for a title.

(1) I(J) (format 16I5)

An input array containing control parameters , controlling the options in the program.

(1) spare; first digit is carriage control character. If I(1) is negative, profile printing is suppressed.

(2) =1, profile input.

(4) $U_{ref}^{l_{ref}}/\nu$; if synthetic starting profiles are used, A(4) is initial $U_{e\theta}/\nu$.

(5) initial $C_f/2$.

(6) The coefficient used to evaluate XSTEP from CFL.

(7) The under-relaxation factor for pressure.

(5) CSLS,ETA (format 7F10.2)

(6) DXTE,YMAX,RIN,PSMX,YMIN (format 6E15.6)

9. Data file MADATBL (for boundary layer)

(1) NSTART (format 16I5)

(2) NSTNS (FORMAT 16I5)

(3) SRSCAL(J) (FORMAT 8E15.6)

(4) UE(J),VE(J) (format 2E15.6)

(5) RNMS(J) (format 8E15.6)

the MS height of boundary layer.

10. Data file MADATOL

(1) XRS(J),YRS(J) (format 2E15.6)

(2) VMSOLD(J) (format 8E15.6)

11. Data file MATDAT1 (free format)

(1) NCH,NXPTS,NWAK,NBACKW

(2) DELTA,BETA,XC,DXTE

(3) XRS(J),YRS(J) (J=1,NCH)

(4) YMAX

(5) XRS(J),YRS(J) (J=NCHW,NENDW)

12. Data file MATDATW

(1) NORUN (format 16I5)

(2) NAP (format 16I5)

Control constant

(3) YMAX (format 8E15.6)

(4) NSTNSW (format 16I5)

(5) XCALW(J) (format 8E15.6)

(6) UE(J),VE(J) (format 2E15.6)

An array containing the streamwise and normal component velocity at shear layer calculation point of the wake.

(7) RNMS(J) (format 8E15.6)

An array containing the new MS height of the wake.

13. Data file MBLDAT

XPCAL(J),PRESSW(J) (format 2E10.4)

An array containing the pressure calculation station and wall pressure.

14. Data file TETOP/TEBOT

(1) JIT/B,NCT/B,YSTPT/B (format 3E15.6)

JIT/B number of points along the normal direction at TE.

NCT/B number of total boundary layer calculation point.

YSTPT/B normal spacing at TE.

(2) TANA/M,TANAFU/M,TANB/M (format 6E15.6)

U, τ and V profiles at TE.

(3) CDPDYT/B (format 3E15.6)

the coefficient of 5th polynomial fit of pressure profile at TE.

15. Data file VMSDAT (format 2E15.6)

SMSMID(J),VMST(J) or SMSNOD(J),VMS(J)

an array containing mid-point of potential flow

calculation element and normal component velocity
on MS.

16. Data file WAKDAT

(1) NORUN,NCH,NXPTS,NTE,NCHW,NENDW,NBTW,N,MSCHNO,MSCHNG
(format 16I5)

(2) SS1(J) (format 8E15.6)

An array containing distance from TE to shear
layer calculation station.

(3) CURVA(J) (format 8E15.6)

(4) TANB(J) (format 8E15.6)

An array containing y values of upper MS of the
wake.

(5) TANAFU(J) (format 8E15.6)

An array containing y values of lower MS of the
wake.

(4) PEXTA(J),PEXTAM(J) (format 8E15.6)

An array containing pressure distribution of upper
and lower MS.

17. Data file WAKDAT1

(1) NSTW (format 16I5)

total number of shear layer calculation station of
the wake.

(2) CDPDY(J) (format 6E15.6)

(3) XCALW(J) (format 8E15.6)

(4) XPPCALW(J) (format 8E15.6)

(5) YMST(J),YMSL(J) (format 8E15.6)

A4.2 Derivation of the expresion for the velocity induced by a straight line source element

We shall derive velocities induced by a single

straight line source element of source density m_n at a point P (see Fig.15). $\bar{x}$ is the distance between P and the mid-point of the element, in the direction parallel to the element AB, while $\bar{y}$ is the distance of P from the mid-point of element, in the direction normal to AB. If we consider a small segment d on AB, whose radial distance from P is r , and if we take this segment to be small enough so that it approximates a point source, then the radial velocity induced by the segment at P is $m_n d\xi / 2\pi r$, where the equivalent source strength of the point is $m_n d\xi$. Therefore the components of velocity parallel and normal to AB are $m_n d\xi / 2\pi r \cos\theta$ and $m_n d\xi / 2\pi r \sin\theta$, respectively. Hence the total velocities, parallel and normal to AB, induced by the whole source element AB are:

$$v_{1n} = \int_{-\frac{\Delta s_n}{2}}^{\frac{\Delta s_n}{2}} m_n d\xi / 2\pi r \cos\theta \quad (A4.1)$$

and

$$v_{2n} = \int_{-\frac{\Delta s_n}{2}}^{\frac{\Delta s_n}{2}} m_n d\xi / 2\pi r \sin\theta \quad (A4.2)$$

respectively.

$$\text{Now, } r^2 = (\bar{x} - \xi)^2 + \bar{y}^2$$

$$\cos\theta = (\bar{x} - \xi)/r, \quad \sin\theta = \bar{y}/r$$

$v_{1n} = m_n / 2\pi \int_{-\frac{\Delta s_n}{2}}^{\frac{\Delta s_n}{2}} (\bar{x} - \xi) d\xi / ((\bar{x} - \xi)^2 + \bar{y}^2)$, where m_n has been taken outside the integral, since it is assumed constant.

$$v_{1n} = m_n / 4\pi \log_e [\{ (\bar{x} + \Delta s_n / 2)^2 + \bar{y}^2 \} / \{ (\bar{x} - \Delta s_n / 2)^2 + \bar{y}^2 \}] \quad (A4.3)$$

and

$$v_{2n} = m_n / 4\pi \tan^{-1} [\bar{y} \Delta s_n / \{ \bar{x}^2 + \bar{y}^2 - 1/4 (\Delta s_n)^2 \}] \quad (A4.4)$$

Appendix 5.Namelist for Variables used in the Computer Program

- A5.1 Namelist for Variables in the Boundary Layer
 Calculation Program TURB
- A5.2 Namelist for Variables in the Shear Layer of
 the Wake calculation Program WAKE
- A5.3 Namelist for Variables in the Potential Flow
 Calculation Program AMOS
- A5.4 Namelist for Variables in the Matching Program
 MATCH

A5.1

A	An input array containing scales and starting conditions for the SL calculation (see Appendix 4).
A2	steamwise-spacing of boundary values (except for pressure and MS height boundary values); this is equal to A(2) (see Appendix 4).
A7	The under relaxation factor.
Al0	a_1 usually 0.15.
ADPDX	The coefficient of normal gradient of 5th polynomial fit of pressure distribution.
ALPHA	α .
ALPHA0	$= \frac{\tau_1 - \tau_w}{\Delta y}$, the shear stress gradient at the wall (strictly, the average between $y = 0$ and $y = \Delta y$).
BDASH	$= \alpha - \beta$.
BETA	β .
C1,C2	The constants C_1, C_2 relating the normal Reynolds stress to the Reynolds shear stress.
C1MC2	$= C_1 - C_2$.
CADD	The additive constant C in the inner layer logarithmic law for the U velocity profile.
CDPDY,CDPDYT	The coefficient of 5th polynomial fit of pressure distribution.
CURV	The curvature at the current station in (s,n) coordinate system.
CURVA	Array containing curvature boundary values.
D05	δ_{05} .

DA	$d\alpha_0$ in the application of the wall boundary condition.
DEL	The surface on which Ve is calculated in the laminar BL calculation, at a given station (or node).
DELO	The value of DEL at the previous station (or node) in the laminar BL calculation.
DELSTA	The displacement thickness .
DELTA	The MS height above the RS.
DGA	An array containing a table of value of $\delta_{05} \frac{\partial}{\partial y} \left[\frac{G}{(\tau_m / U_e^2)^{1/2}} \right]$
DPDX	The streamwise pressure gradient $\partial p / \partial x$.
DPINT	The integral of $\partial p / \partial x$ from the wall to the first mesh point.
DT	$d\tau_1$ in the application of the wall boundary condition.
DTW	$d\tau_w$ in the application of the wall boundary condition.
DU	du_1 in the application of the wall boundary condition.
DU2DUS	The value of $\overline{du^2} / \partial x$.
DU2INT	The integral of $\overline{du^2} / \partial x$ from the wall to the first mesh point.
DVDX	The streamwise gradient of V ($=\partial V / \partial x$).
ETA	The control constant of differencing.
F1,F2,F3	The functions $F1, F2, F3$ which are used in wall boundary condition.
FACMS	A factor multiplying (x) to determine the surface on which Ve is to be evaluated, for

the first sweep only of the laminar BL calculation. This factor is equal to $(2\Delta y + \delta_{\text{coles}} / \theta)$.

GA An array containing a table of values of

$$\frac{G}{(\tau_m / U_e^2)^{1/2}}$$

HE1 The shape parameter H

I An input array containing control parameters, controlling options in the program (see Appendix 4).

I5 The number of boundary values (except for pressure and MS height boundary values).

I6 Equal to I(6), the number of mesh points for which the U, τ calculation is performed.

I61 Equal to I6+1.

I62 Equal to I6+2; this is the number of the mesh point that lies on the MS.

I6COLE The value of I6 for the first turbulent SL calculation station, at which $y = \delta_{\text{Coles}}$.

I10 Equal to the input parameter I(10) (see Appendix 4).

IDIV Set to zero to suppress allowance for the extra strain rate effect of lateral divergence.

ISTEP The control constant: if $ISTEP > 0$, calculations are performed in the (ϕ, ψ) coordinate system.

ISURF The control constant: $ISURF=1$ for upper surface, $ISURF=2$ for lower surface.

ITR The control constant: if $ITR > 0$, then transfer the U, V and τ values from (s, n)

	coordinate system to (φ, ψ) coordinate system.
JI62	The total number of mesh in the normal direction at the future station.
JI620	The total number of mesh in the normal direction at the current station.
NCOUNT	A count of the number of steps executed.
NSTART	The RS node number for the start of the turbulent shear layer calculation.
NSTNS	The number of calculation stations.
NSWEEP	A count of the number of sweeps of the SL calculation.
NXP	The number of stations in array XP.
OLTAUO	The value of τ_w at the previous station.
OMEGA	The under relaxation factor for the pressure.
PE	The external pressure on the MS at the turbulent SL calculation stations, and at the nodes for the laminar BL calculation.
PEFUT	The external pressure at the station where $s=s_0+\Delta s$, for the first sweep only.
PEM	The external pressure profile at U, τ calculation station.
PEOLD	The external pressure on the MS at the previous station.
PEXTA	An array of pressure boundary value.
PHIT	An array containing φ value.
PRESSW	An array containing the wall pressure distribution.
PSIT	An array containing the ψ value.
PW	The wall pressure at the current station.
PWOLD	The wall pressure at the previous station.

RETRAN	The value of Re_θ at which the transition criterion is applied.
RI6	A real variable equal to I6.
RLAMB	The parameter $\frac{\theta^2}{\nu} \partial U_e / \partial x$ for the laminar BL calculation.
ROUGH	Dimensionless surface roughness.
ROUGHA	Array of roughness boundary value.
RTAU	The Reynolds number based on u_τ .
RTHETA	The Reynolds number based on θ .
SMID	An array containing p calculation station value.
STRAIN	The extra rate correction $\hat{\epsilon}$ in the turbulence model; it is not used, but merely set to zero in the program.
T	The value of θ/δ_{coles} from the synthetic U velocity profile for the start of the turbulent SL calculation.
TANA	An array containing the tangents of the "outgoing" characteristic angles at station.
TANAFU	An array containing the tangents of the "outgoing" characteristic angles at the future station.
TANB	An array containing the tangents of the "ingoing" characteristic angles at station.
TANBFU	An array containing the tangents of the "ingoing" characteristic angles at the future station.
TAU	An array containing the profile at the current station.
TAUO	The wall shear stress

TFUT	An array containing the τ profile at the future station.
TFUTT	An array containing τ profile at the patching station.
THETA	The value of momentum thickness obtained by integrating the U velocity profile.
TINT	Equal to τ_{int} which is the value of interpolated at the beginning of a characteristic segment.
TOR	The value of τ at the mid point of a characteristic segment.
TRAN	This is equal to $R_{e\theta}/R_{e\theta,tran}$.
TRANO	The value of TRAN at the previous station (or node) in the laminar BL calculation.
U	An array containing the U profile at the current station.
UDUDX	The value of $U_e \partial U_e / \partial x$ at the current station.
UDUDXO	The value of $U_e \partial U_e / \partial x$ at the previous station.
UE	U_e at the current station.
UE2	U_e at the current station.
UEF	The value of U_e at the next (or future) station (or node) in the laminar BL calculation.
UEFUT	The value of U_e at the station where $s = s_0 + \Delta y$, for the first sweep only.
UEOLD	U_e at the previous station.
UEXT	Array containing values of U ; at nodes if in laminar BL region and at SL calculation stations if in turbulent SL region.

UFUT	An array containing the U profile at the future station $x+\Delta x$.
UFUTT	An array containing U profile at the patching point (see §2.4)
UINT	The value of U, which is the interpolated value of U at the beginning of a characteristic segment.
UL	The value of U at the mid point of a characteristic segment.
US1,US2	A scratch array used in intermediate operations.
UTAU	u_{τ} .
V	An array containing the V profile at the current station.
V1	The value of V at the first mesh point.
VEXT	Array containing values of V ; at nodes if in laminar BL region and at SL calculation stations if in turbulent SL region.
VINT	The value V, which is the value of V interpolated at the beginning of a characteristic segment.
VL	The value of V at the mid point of a characteristic segment.
VOLD	An array containing the V profile at the previous station.
VT	An array containing V at the patching station.
X	The value of s (or ϕ) for the current calculation station.
XCAL	Array containing streamwise-positions of SL calculation stations.

XCHANG	The station where (φ, ψ) coordinate system starts.
XCURV	Array containing s-positions for the curvature boundary value.
XF	The value of s at the next (or future) station (or node) for the laminar BL calculation.
XMAX	The maximum value of x for all boundary values except the pressure and MS height.
XNOD	The potential flow calculation nodes which is given from AMOS program.
XP	An input integer array containing numbers of stations at which information is printed.
XPCAL	The value of x for pressure calculation.
XSTEP	The streamwise-step.
XXSTEP	The value of x for the future station.
YMSNEW	An array containing new MS height (in the normal direction): at nodes if in laminar BL region and at SL calculation station if in turbulent SL region.
YMSNOD	An array containing new MS height (in the normal direction) at nodes.
YMSOLD	An array containing MS height of previous sweep (in the normal direction): at nodes if in laminar BL region and at SL calculation station if in turbulent SL region.
YSD	$\Delta y / \delta_{05}$.
YSTP	The vertical spacing y of the grid (or mesh) points at the current station.
YSTPFU	The vertical spacing y of the grid (or

mesh) points at the future station.

201

Shift of L (length scale) profile in
empirical allowance for low Reynolds number
effect.

A5.2

A	An array containing scales and starting conditions for the SL calculation.
A10	$=a_1$, usually equals .15.
CDI	$=\sqrt{2a_1}$.
CDPDY,CDPDYO	An array containing the Coeffs. for the Polynomial 5th fit in pressure along the normal direction.
CEDDY	Constant used in eddy viscosity formula.
CSLP,CSLPM	Corrected coefficient in length scale equation for positive and negative SL of wake.
CSLS,CSLSM	Coefficient used in length scale equation for positive and negative SL of wake.
CURVA	An array containing curvature value at RS nodes.
DIV	The lateral divergence term.
D05,D05M	The positons at which the Reynolds shear stress equals the δ_{05} of maxmium value of positive and negative SL of wake.
DI,DIM	The width of inner layer of positive and negative shear layers.
DMSNOD	An array containing the height of the wake at inviscid node points.
DMSOLD	An array containing the height of wake at SL calculation station in previous sweep.
DMSNEW	An array containing the height of wake at SL calculation station in current sweep.
DPDX	The streamwise pressure gradient.
DPWDX	The streamwise wall pressure gradient.
DTAUDY	The normal Reynolds Stress gradient.
DUDY	The normal gradient of streamwise velocity.

F2P,F2PM	Correction factor of coefficient in length scale equation for positive and negative SL of wake.
HE1	Shape factor, $=\delta^*/\theta$.
I6	The position at outer edge of the positive shear layer.
I61	=I6+1.
I62	=I6+2.
I6L	The position at inner edge of the positive shear layer.
I6L1	=I6L-1.
I6L2	=I6L-2.
I6LM	The position at the inner edge of negative shear layer.
I6LM1	=I6LM+1.
I6LM2	=I6LM+2.
I6M	The position at the outer edge of negative shear layer.
I6M1	=I6M-1.
I6M2	=I6M-2.
IDUM	=I(13)
ISTEP	Constant: if ISTEP > 4, (s,n) coordinate system is used.
JCL	The position of centre point of the wake at current position.
JCLF	The position of centre point of the wake at future position.
JCLMK	The position of centre point of the wake at current position.
JI62	The number of point along the normal

	direction at current station.
NSTW	The number of SL calculation station.
PE2	p_e^2 .
PEA	The peakness parameter in the expression of VL.
PEXTA, PEXTAM	An array containing the pressure distribution at the node point on the MS for positive and negative SL of wake.
RK, RKM	Indices where $\tau = \tau_{\max}$ for positive and negative SL of wake.
RL, RLM	The length scale of positive and negative shear layers at current station.
RLA	An array containing a table of values of $L/\delta_{0.5}$.
RLFUT, RLFUM	The value of length scale of positive and negative shear layers at future station.
RLOW, RLOM	Indices where $\tau = .05\tau_{\max}$ which are inner edge of inner wake for positive and negative SL of wake.
RMF, RMMF	The maximum shear stress for positive and negative SL of wake at future station.
RM, RMM	The maximum shear stress for positive and negative SL of wake at current station.
RM05, RMM05	$= 0.05\tau_{\max}$.
TANAM	An array containing the tangents of the "Out-going" Characteristic angles at current station for negative SL of wake.
TANAFUM	An array containing the tangents of the "Out-going" Characteristic angles at further station for negative SL of wake.
TANBM	An array containing the tangents of the "In-going" Characteristic angles at current

	station for negative SL of wake.
TANBFUM	An array containing the tangents of the "In-going" Characteristic angles at further station for negative SL of wake.
TAM	An array containing the Reynolds shear stress at current station for negative SL of wake.
TEMP1	An array containing the previous sweep values of CDPDY at previous station.
TFUTM	An array containing the Reynolds shear stress at further station for negative SL of wake.
UEL	Streamwise velocity at SL calculation station on MS edge for lower part of wake.
UEXTT,UEXTL	An array containing values of U_e of at the SL calculation stations.
UFUM	An array containing the streamwise velocity profile at the future station for negative SL of wake.
VIS	The value of gradient of shear stress in the inner wake.
VL,VLM	The turbulent transport velocity for positive and negative SL of wake.
XCALW	An array containing the stations of wake calculation stations.
XPCALW,XPPCALW	An array containing the stations where pressure calculation are done.
XPE	An array containing the nodes where the pressure distribution from inviscid calculation are given.
YASTP	The vertical spacing of the grid point where p calculation is performed.

YCL	The y value of centre of wake.
YMST,YMSL	The y value of new MS height of positive and negative wake.
YOMSA	The y value where 5th polynomial fit of pressure distribution are performed.
YSDM	y/δ_{05} for negative SL of wake.

A5.3

A	The two dimensional array (or matrix) of influence coefficient A_{pn} .
ALP	The angle of incidence (in degree) of the aerofoil.
AP	The X-coordinates difference between mid-points of the MS elements.
AQ	The Y-coordinates difference between mid-points of the MS elements.
AX	The parallel distance between two mid-points of the MS elements, $=X$.
AY	The normal distance between two mid-points of the MS elements, $=Y$.
B	The two dimensional array (or matrix) of coefficients of B_{pn} .
CHD	The arefoil chord length.
CL	The lift coefficient of aerofoil.
CD	The drag coefficient of aerofoil.
CP	An array containing values of pressure coefficient $C_p = (p - p_{ref}) / (1/2 * U_{ref}^2)$ at the mid-point of MS elements.
DG	An array containing the element source density mn' .
DS	An array containing the length of the MS elements.
GAP	The value of circulation Γ/S required to impose the Kutta-Zhukovskii condition for a lifting aerofoil.
N2	$=N^2$.
NKJBOT	The lower MS element which the

	Kutta-Zhukovskii condition is applied (see Fig.25).
NKJTOP	The upper MS element which the Kutta-Zhukovskii condition is applied (see Fig.25).
POS	An array containing the cosine value of TH.
POG	An array containing the sine value of TH.
PA	The velocity parallel to the source element induced by a source (or vorticity) element.
PB	The velocity normal to the source element induced by a source (or vorticity) element.
R	An array containing the X-coordinates of the mid-points of the MS elements. Later in this program, it is also used as an array containing the pressure value of previous sweep.
S	An array containing the Y-coordinates of the mid-points of the MS elements. Later in this program, it is also used as an array containing the X-coordinates of the mid-points of the MS elements.
SIG	The angle between source elements.
SP	The total arc length around the MS.
SS	The array containing the arc length at the mid-points of the MS elements.
STAG	The arc length at the stagnation point on the MS.
SUM	The total value of C_p normal to the X-axis (finally).
SV	An array containing the total value (finally calculated) of tangential velocity at the

	mid-points of the MS elements.
SVM	The total value of C_p parallel to the X-axis (finally).
TH	An array containing the angles which the MS elements make with X-axis.
TSA	The total tangential velocity at the mid-point of the upper MS element nearest to the end of wake, for flow case (ii) of unit vorticity distribution.
TSB	The total tangential velocity at the mid-point of the lower MS element nearest to the end of wake, for flow case (ii) of unit vorticity distribution.
TT	The angle of incidence (in radians) of the aerofoil.
TVBN	The total tangential velocity at the mid-point of the lower MS element near the end of wake, for flow case (i) of unit stream velocity ($U_\infty = 1$).
TVTN	The total tangential velocity at the mid-point of the upper MS element near the end of wake, for flow case (i) of unit stream velocity ($U_\infty = 1$).
TY	An array containing the inverse of the influence coefficient matrix A_{pn} .
VC	An array containing the total induced velocities normal to the MS elements at their mid-points. Later in this program it is also used as an array containing the calculated element source density m_n .
X	An array containing the X-coordinates of the

MS nodes.

Y AN array containing the Y-coordinates of the MS nodes. Later in this program, it is also used as an array containing the velocities normal to MS at the mid-points of MS elements and used as an array containing the C_p values at the mid-point of MS nodes for shear layer calculation.

A5.4

ALPLIM	$=\tan^{-1}(YMAX/DXTE)$.
BETA	The angle between bisectors of TE angle and X axis.
CURVA,CURVT	An array containing the curvature of RS at the nodes for bottom and top sides.
CURVW	An array containing the curvature of RS of wake at nodes.
DELTA	The TE angle.
DS	An array containing the length of the MS elements (as in AMOS).
DX	ΔX : Δs in (s,n) coordinate and $\Delta\varphi$ in (φ,ψ) coordinate system.
DXTE	The distance from TE which the "wedge flow" transformation is applied.
DY	ΔY : Δn in (s,n) coordinate and $\Delta\psi$ in (φ,ψ) coordinate sysem.
EPS	$=(\pi-\delta/2)/2$.
GRAD	An array containing the gradient of the RS at the nodes (relative to X,Y axis).
GRADM	An array containing the gradient of the MS at the nodes (relative to X,Y axis).
ISURF	The value of the control parameter; If =1, the top side. =2, the bottom side.
MSCHNG	The control parameter; if =1, the matching surface is changed. =2, the matching surface is not changed.
MSCHNO	The fixed value, =1.

N	The total number of nodes around the MS.
NBACKW	The number of nodes across the wake at the end of wake.
NCH	The number of nodes from LE to the starting point of transformation.
NCHW	The number of nodes from LE to the last point of transformation in the wake.
NENDW	The number of nodes from LE to the end of wake.
NORUN	The number of runs of the matching program.
NSTART,	The node number indicating the start of turbulent SL calculation region for top and bottom sides.
NSTART1	
NSTNS	The number of nodes of shear layer calculation of boyndary layer region.
NSTNSW	The number of nodes of shear layer calculation of wake region.
NTE	The number of elements for the RS from LE to TE.
NSTOT	The total number of nodes of shear layer calculation.
NXPTS	The half number of elements inside the transformation region.
NWAK	The number of nodes of wake except the transformation region.
PH	φ value
PHIO	The value which patching between (s,n) and (φ, ψ) coordinate is applied.
PSI	AN array containing the ψ values.
OMEGA	The underrelaxation factor.
RLIM	$= \sqrt{(YMAX^2 + DXTE^2)}$.

RIN	$=1/n$
RN	$= n.$
RNMS,RNMST	An array containing the new MS height values at the turbulent SL calculation stations for top and bottom sides: for the laminar BL region they are assumed to be at the nodes.
SMSCAL	An array containing the arc length of the MS at the SL calculation stations.
SMSMID	An array containing the arc length of the mid-point of MS nodes.
SMSMOD	An array containing the arc length of the MS at the nodes.
SRS,SRST	An array containing the arc length of the RS at the nodes for top and bottom sides.
SRSCAL	An array containing the arc length of the RS at the SL calculation stations.
UE	An array containing the velocities parallel to the RS at the turbulent SL calculation stations.
URS	The velocities parallel to the RS at the nodes.
VE	An array containing the velocities normal to the RS at the turbulent SL calculation stations.
VMS,VMST	An array containing the velocities normal to the MS at the mid-point of nodes.
VMSNOD	An array containing the velocities normal to the MS at the nodes.
VMSOLD	An array containing the velocities of the previous sweep normal to the MS at the nodes.

VRS	The velocities normal to the RS at the nodes.
XC	The chord length.
XCALW	An array containing the arc length of the RS from TE in the wake.
XLOC,XLOCT	An array containing the arc length of MS nodes in (s,n) coordinate system from TE for the top and bottom sides.
XMS,XMST	An array containing the X-coordinates value of MS at nodes for the top and bottom sides.
XO	An array containing the value of arc length from TE consistent to the value of the (ϕ, ψ) coordinate system.
XRS,XRST	An array containing the X-coordinate values of RS at nodes for top and bottom sides.
YLOC,YLOCT	An array containing the height in (s,n) coordinate system of MS nodes for top and bottom sides.
YMAX	The value of height of MS at the starting point of transformation.
YMS,YMST	An array containing the Y-coordinate values of MS at nodes for top and bottom sides.
YRS,YRST	An array containing the coordinate values of RS related at nodes for top and bottom sides.

Appendix 6.A6.1 Procedure file for running Computer ProgramA6.2 Example of Input and Output for Asymmetric aerofoil

Appendix A6.1 Procedure file for running computer program

In this section we introduce procedure file for running computer program for one iteration in CDC 850 at Imperial College. It could be different for different computer.

```

GET,MATCH
GET,AMOS
GET,TURB
FETCH(WAKE)
FTN5(I=MATCH,L=A,B=BINM)
FTN5(I=AMOS,L=A,B=BINA)
FTN5(I=TURB,L=A,B=BTURB)
FTN5(I=WAKE,L=A,B=BWAKE)
SAVE,BINM
SAVE,BINA
SAVE,BTURB
SAVE,BWAKE
LIBRARY,NAG11F/D.
LIBFILE(NAG11F)
LIBRARY,NAG11F/A.
GET,BINM
GET,BINA
GET,BTURB
GET,BWAKE
SETTL,100000
BINM(,,,,,TAPE6)
BINA(,,,TAPE6)
GET,BLDATT
GET,BLDATB
GET,BLDATT1

```

GET,BLDATB1
GET,WAKDAT
PACK,BLDATT
PACK,BLDATB
PACK,BLDATT1
PACK,BLDATB1
PACK,WAKDAT
REPLACE,BLDATT
REPLACE,BLDATB
REPLACE,BLDATT1
REPLACE,BLDATB1
REPLACE,WAKDAT
GET,BTURB
BTURB(,,,,,TAPE6)
GET,MADATBL
PACK,MADATBL
REPLACE,MADATBL
GET,BWAKE
BWAKE(,,,,,TAPE6)
GET,MATDATW
GET,COEPPY
GET,COEPPYB
PACK,COEPPY
PACK,COEPPYB
PACK,MATDATW
REPLACE,COEPPY
REPLACE,COEPPYB
REPLACE,MATDATW

A6.2 Examples of Input and Output Data for Asymmetric Aerofoil

A6.2.1 Input Data

(1) MATDAT1

21,3,10,5

0.231693,-.021071,1.,0.050

	0	0
	.005000	.012700
	.012500	.018640
	.025000	.025850
	.054500	.036960
	.095500	.047020
	.146500	.053780
	.206100	.059140
	.273000	.064950
	.345500	.067800
	.421800	.067250
	.500000	.063810
	.578200	.05828
	.644500	.052550
	.715000	.04407
	.780000	.034670
	.838600	.025170
	.874500	.0193
	.905500	.014210
	.930200	.010147
	.950000	.006889
	.03	
1.05044	-.001063	
1.07544	-.00159	
1.12544	-.002644	
1.17544	-.003697	
1.22544	-.004751	
1.27544	-.005805	
1.32544	-.006858	
1.37544	-.007912	
1.42544	-.008966	
1.47544	-.01002	
1.52544	-.01107	
	0	0
	.005000	.0072
	.012500	.0127
	.025000	.01822
	.054500	.02330
	.095500	.0290
	.146500	.0335
	.206100	.036550
	.273000	.03876
	.345500	.03834
	.421800	.03405
	.500000	.029450
	.578200	.02505
	.644500	.02135

.715000	.01749
.780000	.013933
.838600	.0107530
.874500	.008811
.905500	.00714
.930200	.00581
.950000	.004753

(2) MATDATW

1
1
.300000E-01

(3) AMODAT1

1.,1.

(4) BLDATB1

1986	0	1	0	0	10	0	0	0	0	0	0	0	300
0.		0.		0.		1.2E6		0.		1.		.3	
1.													

(5) BLDATT1

1986	0	1	0	0	10	0	0	0	0	0	0	0	300
0.			0.		0.		1.2E6		0.		1.		.3
1.													

(6) INWAK

0.070000

FLAT PLATE RUN

1984	0	1	118	0	10	0	0	0	0	0	0	0	40	0
0.			0.		0.		1.2E6		0.		1.		.3	
0.15			1.											
.500000E-01			.300000E-01			.963125E+00		.278291E-01		.278291E-01		.278291E-01		
.500000E-01			.300000E-01			.963125E+00		.278291E-01		.278291E-01		.278291E-01		
.500000E-01			.300000E-01			.963125E+00		.278291E-01		.278291E-01		.278291E-01		
.500000E-01			.300000E-01			.963125E+00		.278291E-01		.278291E-01		.278291E-01		

A6.2.2 Output Data

(1) BLDATB

14														
6	21	3	24	27	37	42	78	1	0					
.500000E-01			.300000E-01			.963125E+00								
.000000E+00			.546040E-03			.111990E-02		.195280E-02		.373478E-02				
.610263E-02		.887658E-02		.118941E-01										
.150020E-01		.180383E-01		.208675E-01		.233719E-01		.254755E-01						

.269453E-01	.281632E-01	.290246E-01			
.295687E-01	.297969E-01	.299263E-01	.299842E-01	.300000E-01	
.154780E-01	.303190E-01				
.327390E-01	.308796E-01				
.500000E-01	.316665E-01				
.100000E+01	.876536E+00	.489468E+00	.189645E+00	.489511E-01	
.208031E-01	-.495545E-01	-.868816E-01			
-.121088E+00	-.155686E+00	-.138236E+00	-.930521E-01	-.749774E-01	
-.578130E-01	-.357781E-01	-.110654E-01			
.650129E-02	.287833E-01	.390325E-01	.486626E-01	.600086E-01	
.678507E-01	.911434E-01	.118391E+00			
.136816E+00					

(2) BLDATT

13									
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.000000E+00		.837597E-03		.141765E-02		.228152E-02		.412252E-02	
.648896E-02	.921942E-02	.121834E-01							
.152382E-01	.182161E-01	.209862E-01		.234429E-01		.255124E-01			
.269422E-01	.281878E-01	.290320E-01							
.295716E-01	.297969E-01	.299252E-01		.299861E-01		.300000E-01			
.154780E-01	.303190E-01								
.327390E-01	.308796E-01								
.500000E-01	.316665E-01								
.100000E+01	.665886E+00	-.143525E+00		-.270715E+00		-.402303E+00			
-.507008E+00	-.509912E+00	-.460051E+00							
-.481355E+00	-.517509E+00	-.502748E+00		-.466851E+00		-.414589E+00			
-.366854E+00	-.291754E+00	-.211374E+00							
-.125235E+00	-.700048E-01	-.234913E-01		.121780E-01		.671903E-01			
.740638E-01	.823102E-01	.936622E-01							
.942184E-01									

(3) COEPPYB

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-.237031E-01					
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-.195247E-01					
.933075E-03	-.303371E-02	.388437E-02	-.233257E-02	.738907E-03	
-.145682E-01					
.249636E-03	-.100839E-02	.167067E-02	-.111137E-02	.461345E-03	
-.987343E-02					
.198305E-03	-.878759E-03	.147002E-02	-.146340E-02	.306506E-03	
-.567660E-02					
.265557E-03	-.114247E-02	.168820E-02	-.196239E-02	.245486E-03	
-.100955E-02					
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.483124E-02					
-.105390E-02	.304885E-02	-.289977E-02	.158389E-02	-.980028E-04	
.976368E-02					
-.645551E-03	.158023E-02	-.111177E-02	-.338347E-03	-.373073E-04	
.140174E-01					

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.185817E-02	-.511139E-02	.833566E-02	-.102704E-01	.581178E-03
.258723E-01				
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.148915E-01				
-.724552E-01	.157234E+00	-.892778E-01	-.554860E-01	-.630211E-02
.904212E-01				
-.148678E+00	.433300E+00	-.471567E+00	.261982E+00	-.308688E-01
.131056E-02				
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.976637E-01				
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.874984E+00	.895511E+00	.917261E+00	.938033E+00	.958806E+00
.974053E+00	.990999E+00	.100881E+01		
.652856E+00	.667764E+00	.696558E+00	.724738E+00	.753537E+00
.782573E+00	.811620E+00	.840550E+00		
.864973E+00	.885247E+00	.906386E+00	.927647E+00	.948420E+00
.966430E+00	.982526E+00	.999903E+00		
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(4) COEPPY

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-.141849E+00				
-.149284E-01	.447688E-01	-.512965E-01	.267426E-01	.239220E-02
-.124200E+00				
-.140859E-01	.423095E-01	-.485687E-01	.253769E-01	.137302E-02
-.105223E+00				

-.112056E-01	.338208E-01	-.388248E-01	.210545E-01	.453204E-03
-.853323E-01				
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-.440738E-02	.125206E-01	-.132496E-01	.567891E-02	-.452104E-03
-.486447E-01				
-.689862E-02	.182579E-01	-.173607E-01	.505160E-02	-.643448E-03
-.329985E-01				
-.100151E-01	.281082E-01	-.277497E-01	.114578E-01	-.142545E-02
-.200511E-01				
-.121118E-01	.320407E-01	-.294703E-01	.641684E-02	-.122189E-02
-.320418E-02				
-.167933E-01	.466205E-01	-.521322E-01	.266165E-01	-.323385E-02
.736462E-02				
-.398942E+00	.110302E+01	-.108929E+01	.440869E+00	-.664318E-01
.442532E-01				
-.209102E+00	.594237E+00	-.637757E+00	.334460E+00	-.511132E-01
.630804E-02				
-.337821E+00	.911608E+00	-.863874E+00	.297348E+00	-.570095E-01
.909030E-01				
-.167072E+00	.469700E+00	-.479102E+00	.207971E+00	-.330454E-01
.483414E-01				
-.196385E+00	-.197876E+00	-.189103E+00	-.175642E+00	-.159090E+00
-.141849E+00				
-.124200E+00	-.105223E+00	-.853323E-01	-.662091E-01	-.486447E-01
-.329985E-01				
-.200511E-01	-.320418E-02	.736462E-02	.442532E-01	.630804E-02
.909030E-01				
.483414E-01				
.595353E+00	.626281E+00	.653443E+00	.680631E+00	.707877E+00
.735109E+00	.762272E+00	.789253E+00		
.814827E+00	.840556E+00	.866640E+00	.886829E+00	.907758E+00
.930043E+00	.950441E+00	.970839E+00		
.986333E+00	.100328E+01	.102084E+01		
.595353E+00	.610817E+00	.639862E+00	.667037E+00	.694254E+00
.721493E+00	.748691E+00	.775763E+00		
.802040E+00	.827691E+00	.853598E+00	.876735E+00	.897294E+00
.918901E+00	.940242E+00	.960640E+00		
.978586E+00	.994807E+00	.101206E+01		
.000000E+00	.837597E-03	.141765E-02	.228152E-02	.412252E-02
.648896E-02	.921942E-02	.121834E-01		
.152382E-01	.182161E-01	.209862E-01	.234429E-01	.255124E-01
.262085E-01	.267746E-01	.273044E-01		
.277882E-01	.282203E-01	.285973E-01	.289279E-01	.292008E-01
.294356E-01	.296329E-01	.297572E-01		
.298588E-01	.299391E-01	.299856E-01	.300000E-01	.278291E-01
.278291E-01	.278291E-01			
.103878E-01	.536918E-01	.737970E-01	.847640E-01	.935061E-01

(5) MADATOL

.105044E+01	.106300E-02
.107544E+01	.159000E-02
.112544E+01	.264400E-02
.117544E+01	.369700E-02
.122544E+01	.475100E-02
.127544E+01	.580500E-02
.132544E+01	.685800E-02
.137544E+01	.791200E-02

.142544E+01	.896600E-02			
.147544E+01	.100200E-01			
.152544E+01	.110700E-01			
.211510E-02	-.941917E-04	-.194799E-02	.575313E-04	.306178E-03
.277222E-02	.321625E-02	.315534E-03		
-.520004E-04	.269816E-02	.576807E-02	-.504161E-02	-.110040E-01
-.918438E-03	.706342E-02	.126510E-01		
.161743E-01	.206570E-01	.280268E-01	.197345E-01	.987042E-02
.472647E-02	-.734974E-04	.956686E-03		
-.351732E-03	-.659715E-02	-.161654E-01	-.132725E-01	-.775199E-02
-.597504E-02	-.435186E-02	-.309891E-02		
-.202509E-02	-.847286E-03	.118296E-02	.249082E-02	
.205407E-02	-.142078E-02	-.553975E-02	-.158814E-02	.565619E-03
-.230338E-03	-.624913E-03	-.119211E-02		
-.255232E-03	.385620E-02	.514316E-02	.369686E-02	-.458375E-02
-.932835E-02	-.440840E-02	-.148193E-03		
.307494E-02	.467567E-02	.805419E-02	.443586E-02	.193634E-01
.129411E-01	-.361185E-02	.467945E-02		
.350890E-02	-.225447E-02	-.112680E-01	-.103517E-01	-.663312E-02
-.529032E-02	-.442572E-02	-.409975E-02		
-.419496E-02	-.442550E-02	-.498319E-02	-.568293E-02	

(6) MADATBL

13				
19				
.000000E+00	.136488E-01	.232161E-01	.376464E-01	.691692E-01
.111385E+00	.162831E+00	.222672E+00		
.289824E+00	.362380E+00	.438682E+00	.516957E+00	
.595353E+00	.626281E+00	.653443E+00	.680631E+00	.707877E+00
.735109E+00	.762272E+00	.789253E+00		
.814827E+00	.840556E+00	.866640E+00	.886829E+00	.907758E+00
.930043E+00	.950441E+00	.970839E+00		
.986333E+00	.100328E+01	.102084E+01		
.000000E+00	.000000E+00			
.944318E+00	.508565E-02			
.109501E+01	-.527959E-02			
.114783E+01	.675456E-03			
.120752E+01	-.467417E-03			
.123020E+01	.122601E-02			
.121610E+01	.535891E-02			
.121191E+01	.224430E-02			
.122669E+01	-.146626E-02			
.122994E+01	.148413E-02			
.121942E+01	.476644E-02			
.120027E+01	.837388E-02			
.118016E+01	.823243E-02			
.116989E+01	.118162E-01			
.115833E+01	.165819E-01			
.114509E+01	.190698E-01			
.113136E+01	.197024E-01			
.111750E+01	.205819E-01			
.110242E+01	.227888E-01			
.108575E+01	.250938E-01			
.106911E+01	.250030E-01			
.105429E+01	.233632E-01			
.104074E+01	.237695E-01			
.102678E+01	.279636E-01			

.101352E+01	.234910E-01			
.999554E+00	.360443E-01			
.978816E+00	.220867E-01			
.964101E+00	.835951E-01			
.960111E+00	-.786257E-01			
.954994E+00	.546263E-01			
.951872E+00	-.178382E-01			
.000000E+00	.837597E-03	.141765E-02	.228152E-02	.412252E-02
.648896E-02	.921942E-02	.121834E-01		
.152382E-01	.182161E-01	.209862E-01	.234429E-01	.255124E-01
.269394E-01	.281854E-01	.290275E-01		
.295672E-01	.297936E-01	.299236E-01	.299858E-01	.300000E-01

(7) TETOP

25	19	.111316E-02		
.527431E+00	-.387648E-02	.204743E-02	.556696E+00	-.560629E-02
.251857E-02				
.610738E+00	-.663415E-02	.243774E-02	.652000E+00	-.770039E-02
.241146E-02				
.690010E+00	-.869398E-02	.232496E-02	.726932E+00	-.962161E-02
.216410E-02				
.761316E+00	-.105435E-01	.198686E-02	.795975E+00	-.114045E-01
.176028E-02				
.842117E+00	-.120204E-01	.122430E-02	.887443E+00	-.125655E-01
.614665E-03				
.913133E+00	-.132147E-01	.309320E-03	.923978E+00	-.138404E-01
.177009E-03				
.928146E+00	-.144057E-01	.103650E-03	.930378E+00	-.149353E-01
.595616E-04				
.932368E+00	-.154348E-01	.335044E-04	.934213E+00	-.158933E-01
.194167E-04				
.935945E+00	-.163020E-01	.112806E-04	.937456E+00	-.166552E-01
.585086E-05				
.940899E+00	-.169917E-01	.236740E-05	.943860E+00	-.172710E-01
.469851E-06				
.951872E+00	-.175968E-01	.139277E-07	.951872E+00	-.177297E-01
.139277E-07				
.951872E+00	-.178174E-01	.139277E-07	.951872E+00	-.178564E-01
.139277E-07				
.951872E+00	-.178382E-01	.139277E-07		
-.167072E+00	.469700E+00	-.479102E+00		
.207971E+00	-.330454E-01	.483414E-01		

(8) TEBOT

25	16	.111316E-02		
.595622E+00	.106047E-02	.190298E-02	.624192E+00	.147755E-02
.276758E-02				
.693490E+00	.441567E-02	.215362E-02	.739619E+00	.662522E-02
.189672E-02				
.778054E+00	.866866E-02	.165179E-02	.813169E+00	.105782E-01
.141835E-02				
.862931E+00	.128951E-01	.925827E-03	.902885E+00	.149797E-01
.433204E-03				
.915009E+00	.162647E-01	.201305E-03	.918360E+00	.171941E-01
.112776E-03				

.920717E+00	.180022E-01	.669740E-04	.923070E+00	.187298E-01
.380989E-04				
.925392E+00	.193803E-01	.213970E-04	.927322E+00	.199508E-01
.113500E-04				
.928877E+00	.204426E-01	.511843E-05	.930518E+00	.208649E-01
.171203E-05				
.932771E+00	.212289E-01	.388103E-06	.938694E+00	.216055E-01
.587054E-07				
.943700E+00	.218935E-01	.126397E-07	.942648E+00	.219798E-01
.105356E-07				
.942648E+00	.220407E-01	.105356E-07	.942648E+00	.220697E-01
.105356E-07				
.942648E+00	.220940E-01	.105356E-07	.942648E+00	.221522E-01
.105356E-07				
.942648E+00	.222968E-01	.105356E-07		
-.231542E+00	.596343E+00	-.505242E+00		
.108496E+00	-.116308E-01	.976637E-01		
(9) WAKDAT1				
21				
-.538858E+00	.160302E+01	-.170578E+01	.698327E+00	-.658936E-01
.573195E-01				
-.889466E+00	.214478E+01	-.180198E+01	.585832E+00	-.537018E-01
.576100E-01				
-.556340E+00	.128861E+01	-.100287E+01	.341945E+00	-.854148E-01
.555360E-01				
-.246158E+00	.555948E+00	-.403828E+00	.995798E-01	-.235783E-01
.540180E-01				
-.252071E+00	.614906E+00	-.514270E+00	.172449E+00	-.537008E-01
.485423E-01				
-.154409E+00	.360327E+00	-.284731E+00	.906490E-01	-.458482E-01
.441860E-01				
-.109491E+00	.245603E+00	-.182329E+00	.533507E-01	-.393548E-01
.395413E-01				
-.906918E-01	.196340E+00	-.136820E+00	.363502E-01	-.359209E-01
.356300E-01				
-.797560E-01	.167039E+00	-.109674E+00	.262258E-01	-.338537E-01
.326563E-01				
-.535809E-01	.100902E+00	-.518261E-01	.589897E-02	-.302777E-01
.301448E-01				
-.375277E-01	.584186E-01	-.123986E-01	-.922654E-02	-.267009E-01
.279748E-01				
-.426681E-02	-.240971E-01	.583672E-01	-.333089E-01	-.226433E-01
.260334E-01				
.128431E-01	-.695716E-01	.100754E+00	-.495424E-01	-.188947E-01
.242649E-01				
.329344E-01	-.123404E+00	.151296E+00	-.687402E-01	-.148798E-01
.227030E-01				
.585012E-01	-.193696E+00	.219957E+00	-.965007E-01	-.918058E-02
.212897E-01				
.737835E-01	-.231823E+00	.252760E+00	-.107197E+00	-.630976E-02
.199521E-01				
.915525E-01	-.279445E+00	.297463E+00	-.124512E+00	-.196190E-02
.186815E-01				
.103742E+00	-.311184E+00	.324543E+00	-.134594E+00	.244329E-02
.175075E-01				

.111721E+00	-.332640E+00	.342792E+00	-.140513E+00	.751127E-02
.164061E-01				
.129218E+00	-.378717E+00	.385368E+00	-.152171E+00	.116007E-01
.154148E-01				
.000000E+00	.000000E+00	.000000E+00	.000000E+00	.000000E+00
.107135E-01				
.000000E+00	.180326E-01	.349785E-01	.504512E-01	.734807E-01
.100139E+00	.127642E+00	.156502E+00		
.184965E+00	.212468E+00	.240132E+00	.265983E+00	.293175E+00
.319080E+00	.344089E+00	.368193E+00		
.393192E+00	.418145E+00	.442852E+00	.466788E+00	.491065E+00
.516537E+00				
.901630E-02	.265056E-01	.427149E-01	.521172E-01	.868098E-01
.113891E+00	.142072E+00	.170734E+00		
.198716E+00	.226300E+00	.253057E+00	.279579E+00	.306128E+00
.331585E+00	.356141E+00	.380692E+00		
.405668E+00	.430498E+00	.454820E+00	.478926E+00	.503801E+00
.278291E-01	.278291E-01	.278291E-01	.300000E-01	.300000E-01
.300000E-01	.300000E-01	.300000E-01		
.300000E-01	.300000E-01	.300000E-01	.300000E-01	.300000E-01
.300000E-01	.300000E-01	.300000E-01		
.300000E-01	.300000E-01	.300000E-01	.300000E-01	.300000E-01
-.278291E-01	-.278291E-01	-.278291E-01	-.300000E-01	-.300000E-01
-.300000E-01	-.300000E-01	-.300000E-01		
-.300000E-01	-.300000E-01	-.300000E-01	-.300000E-01	-.300000E-01
-.300000E-01	-.300000E-01	-.300000E-01		
-.300000E-01	-.300000E-01	-.300000E-01	-.300000E-01	-.300000E-01

(10) WAKDAT

[illegible]