Convergence in economic model predictive control with average constraints[☆]

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Abstract

In this paper, we thoroughly investigate various aspects of economic model predictive control with average constraints, i.e., constraints on average values of state and input variables. In particular, we first show that a certain time-varying output constraint has to be included into the MPC problem formulation in order to ensure fulfillment of these average constraints. Optimizing a general (possibly economic) performance criterion may result in a non-converging behavior of the corresponding closed-loop system. While such a behavior might be acceptable in some cases (such as the operation of chemical reactors), it may be undesirable for other types of applications. Hence as a second contribution, we provide a Lyapunov-like analysis to conclude that indeed asymptotic convergence to the optimal steady-state follows if the system satisfies a certain dissipativity condition. Finally, for the case that this dissipativity property is not satisfied but still a convergent behavior of the closed-loop is required, we examine two different methods how convergence can be enforced within an economic MPC setup by imposing additional average constraints on the system. In the first method, an additional average constraint is defined which results in the system being dissipative, while the second consists of imposing an additional even zero-moment average constraint. We illustrate our results with various examples.

Keywords: Economic model predictive control, Average constraints, Nonlinear systems

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1. Introduction

Economic model predctive control (MPC) is a recently introduced control strategy which, in contrast to standard tracking MPC, uses some general cost function in the design of the receding horizon controller which need not be positive definite with respect to any setpoint (or trajectory) to be tracked. The main motivation for studying this more general variant of MPC is that directly optimizing the true economics of a plant results in a cost function which is in general not related to any steady-state at all. Using the economic cost function directly in the design of the receding horizon controller can lead to a significant performance improvement compared to the standard state-of-the art approach, where the so-called Real Time Optimization layer (RTO for short) determines the economically optimal feasible steady-state, which is in turn used in a standard tracking MPC formulation (see, e.g., [3] for a further discussion on this issue). In [3, 4, 5, 6, 7, 8, 9], different economic MPC setups have been studied, using different assumptions and/or additional (terminal) constraints. Furthermore, some recent examples applying an economic MPC scheme can, e.g., be found in [10, 11, 12].

In this work, we consider economic MPC which besides pointwise in time constraints, as usual in MPC, also includes average constraints, i.e., constraints on average values of state and input variables. In standard tracking MPC, which guarantees convergence to an equilibrium, any asymptotic average of state and input variables is determined by the value at this equilibrium, so average constraints do not need further attention online (i.e., within the design of the receding horizon controller), but have to be taken into account offline as a static constraint in the RTO layer. On the other hand, in economic MPC, one of the key features is that the resulting closed-loop system may exhibit a non-converging behavior, but could, e.g., result in some periodic orbit or even a complex chaotic behavior. In such a case, constraints on average values of state and input variables become an interesting feature and have to be taken into account online, i.e., within the MPC algorithm.

In [3, Section V.B], a first economic MPC algorithm also ensuring satisfaction of asymptotic average constraints was presented, using a specific time-varying constraint in the optimization problem. A first contribution of this paper is to provide a more general verion of this time-varying constraint (see Sections 2.2 and 3) which still guarantees fulfillment of asymptotic average constraints. While in a setting with a terminal equality constraint (such as in [3, Section V.B]), this more general formulation can be used as a relaxation in order to improve performance, it turns out

that it is in fact necessary for ensuring recursive feasibility if a terminal cost and terminal region is used instead of a terminal equality constraint (see Section 3.2).

As mentioned above, an economic MPC algorithm might result in a non-converging behavior of the corresponding closed-loop system. While this can be acceptable in applications in which outputs are material/physical outflows with the possibility of being stored, in many other contexts convergence to an equilibrium is a requirement that cannot be sacrificed by trading it off with economics. Hence an important question is to examine under what conditions the closed-loop system converges to an equilibrium despite using a general (economic) cost function in the MPC algorithm. If no average constraints are present, it was shown for both settings with [3, 5] and without [7] additional terminal constraints that a specific dissipativity condition is sufficient for ensuring convergence to the optimal steady-state (or even asymptotic stability). For the case including average constaints, a first preliminary result was obtained in [3, Section VI] under a similar dissipativity condition; however, only convergence in a weak sense could be shown and no Lyapunov analysis was available. A second contribution of this paper (Section 4) is to use this dissipativity condition to prove asymptotic convergence of the closed-loop system using a Lyapunov-like analysis. Notably, the Lyapunov function we propose is different from the one used in case of no average constraints, but contains an additional term which directly corresponds to the present average constraints.

Finally, we consider the case where the above mentioned dissipativity condition is not satisfied, but still convergence of the closed-loop system is a required specification for the controller design. One method would be to modify the cost function and add a sufficiently large convex term such as to make the system dissipative, as proposed in [3]. However, the disadvantage of this method is that by doing this, the (transient) performance of the system might be significantly deteriorated. Hence as a third contribution of this paper (Section 5), we propose two methods how the requirement of closed-loop convergence can be achieved *without* having to sacrifice the goal of (economic) process optimization, i.e., without having to modify the economic performance criterion. Both methods consist of imposing an additional average constraint on the system, where the first (Section 5.1) is such that the additional average constraint results in the system being dissipative while the second (Section 5.2) enforces an even average moment of the closed loop system to be zero.

2. Preliminaries and problem setup

2.1. Notation

Let $\mathbb{I}_{\geq b}$ denote the set of integers greater or equal than b, and $\mathbb{R}$ the set of real numbers. The unit ball in $\mathbb{R}^n$ is denoted as B_1, i.e., $B_1 := \{x \in \mathbb{R}^n : |x| \leq 1\}$. The point to set distance from a point $x \in \mathbb{R}^n$ to a set $\mathbb{Y} \subseteq \mathbb{R}^n$ is defined as $|x|_{\mathbb{Y}} := \inf_{y \in \mathbb{Y}} |x - y|$. For a symmetric matrix $S \in \mathbb{R}^{n \times n}$, let $\lambda_{\min}(S)$ and $\lambda_{\max}(S)$ denote its mininum, respectively maximum, eigenvalue. As in [3], for any vector valued bounded signal $v : \mathbb{I}_{\geq 0} \to \mathbb{R}^{n_v}$, the set of asymptotic averages is defined as

$$Av[v] := \{\bar{v} \in \mathbb{R}^{n_v} : \ \exists t_n \to +\infty : \lim_{n \to \infty} \frac{\sum_{k=0}^{t_n} v(k)}{t_n + 1} = \bar{v}\}. \tag{1}$$

Note that $Av[v]$ is nonempty (as bounded sequences in $\mathbb{R}^{n_v}$ have limit points), but it need not be a singleton in general. Furthermore, in case that $Av[v]$ is a singleton, i.e., $Av[v] = \{\bar{v}\}$ for some $\bar{v} \in \mathbb{R}^{n_v}$, define for each $n \in \mathbb{I}_{\geq 2}$ the n-th average moment as $Av[(v - \bar{v})^n]$ where powers are intended componentwise.

2.2. Problem formulation

We consider discrete-time nonlinear systems of the form

$$x(t+1) = f(x(t), u(t)), \qquad x(0) = x_0, \tag{2}$$

with $t \in \mathbb{I}_{\geq 0}$, where $x \in \mathbb{X} \subseteq \mathbb{R}^n$ and $u \in \mathbb{U} \subseteq \mathbb{R}^m$. We assume that f is continuous in (x, u). The system is subject to (possibly coupled) state and input constraints

$$(x, u) \in \mathbb{Z}$$

for some compact set $\mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$. Furthermore, the system is subject to asymptotic average constraints, which are expressed in terms of an auxiliary output variable

$$y = h(x, u),$$

where $y \in \mathbb{R}^p$ and h is assumed to be continuous in (x, u). The average constraints are now given as

$$Av[y] \in \mathbb{Y}$$

for some closed convex set[2] $\mathbb{Y} \subseteq \mathbb{R}^p$.

In order to compute a control input to system (2) in a model predictive control framework, we equip the system (2) with a stage cost $\ell : \mathbb{Z} \to \mathbb{R}$ which is assumed to be continuous. The stage cost ℓ can be a general, possibly economic, cost function and need not satisfy any specific requirements such as convexity or positive definiteness with respect to any setpoint. In such a setting, the optimal control input might not result in a trajectory which converges to some steady-state, but may exhibit some more complex behavior such as a periodic orbit. Let (x_s, u_s) be the optimal steady-state defined as

$$\ell(x_s, u_s) = \min_{(x,u)\in\mathbb{Z}, h(x,u)\in\mathbb{Y}, x=f(x,u)} \ell(x, u). \tag{3}$$

Note that the minimum in (3) exists as ℓ and h are continuous, $\mathbb{Y}$ is closed and $\mathbb{Z}$ is compact. Furthermore, for simplicity we assume that (x_s, u_s) is unique; if this is not the case, in the following let (x_s, u_s) denote any of the steady-states satisfying (3).

In order to define the receding horizon control law, at each time t, the following optimization problem is solved:

$$\min_{\mathbf{u}(t)} \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)) + V_f(x(N|t)) \tag{4}$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad k \in \mathbb{I}_{0:N-1} \tag{5a}$$

$$(x(k|t), u(k|t)) \in \mathbb{Z}, \quad k \in \mathbb{I}_{0:N-1} \tag{5b}$$

$$x(N|t) \in \mathbb{X}_f(t), \quad x(0|t) = x(t) \tag{5c}$$

$$\sum_{k=0}^{N-1} h(x(k|t), u(k|t)) \in \mathbb{Y}_t \tag{5d}$$

where N is the prediction horizon, $V_f : \mathbb{X} \to \mathbb{R}$ is the terminal cost function which is assumed to be continuous, and for each $t \in \mathbb{I}_{\geq 0}$ the (possibly time-varying) terminal region $\mathbb{X}_f(t) \subseteq \mathbb{X}$ is some compact set containing x_s in its interior. We discuss later why we allow for a time-varying terminal region and how it can be defined appropriately. Constraint (5d), which will be used for ensuring

[2] If no average constraints are present, just take $h \equiv 0$ and $\mathbb{Y} = \{0\}$.

satisfaction of the average constraints, contains the time-varying output set $\mathbb{Y}_t$ which is recursively defined as

$$\mathbb{Y}_{t+1} := \mathbb{Y}_t \oplus \mathbb{Y} \oplus \overline{\mathbb{Y}}(t) \ominus h(x(t), u(t)), \quad \mathbb{Y}_0 = N\mathbb{Y} + \mathbb{Y}_{00}, \tag{6}$$

where $\mathbb{Y}_{00} \subseteq \mathbb{R}^p$ is an arbitrary compact set containing $h(x_s, u_s)$ and $\overline{\mathbb{Y}}(t)$ will be specified later. In (6) sums are in the sense of Pontryagin's set-sums and similarly subtractions. Notice the slight abuse of notation in the use of $\ominus$ followed by a vector when subtracting a singleton set. We remark that the recursion in (6) can be solved explicitly, yielding (thanks to convexity of $\mathbb{Y}$)

$$\mathbb{Y}_t = \mathbb{Y}_{00} \oplus (t+N)\mathbb{Y} \oplus \sum_{k=0}^{t-1} \overline{\mathbb{Y}}(k) \ominus \sum_{k=0}^{t-1} h(x(k), u(k)). \tag{7}$$

In (4)–(5), the sequences $\mathbf{u}(t) := \{u(0|t), \ldots, u(N-1|t)\}$ and $\mathbf{x}(t) := \{x(0|t), \ldots, x(N|t)\}$ are the predicted input and corresponding state sequences at time t. Denote the minimizer of problem (4)–(5) by[3] $\mathbf{u}^0(t)$ and the corresponding state sequence by $\mathbf{x}^0(t)$. Then, as usual in MPC, the first part of $\mathbf{u}^0(t)$ is applied to system (2) resulting in the closed-loop system

$$\begin{aligned} x(t+1) &= f(x(t), u^0(0|t)) \qquad x(0) = x_0, \\ y(t) &= h(x(t), u^0(0|t)) \end{aligned} \tag{8}$$

Furthermore, denote by $\mathcal{X}_N$ the set of states $x_0 \in \mathbb{X}$ such that problem (4)–(5) is feasible with $t = 0$ and $x(0) = x_0$.

Remark 1. *An economic MPC algorithm similar to* (4)–(5) *for a setting including average constraints was initially proposed in [3, Section V.B]. The main novel feature of our algorithm is that we use a more general definition of the time-varying output constraint set* $\mathbb{Y}_t$ *in* (6)*, including the set* $\overline{\mathbb{Y}}(t)$*. Furthermore, while a terminal equality constraint* $x(N|t) = x_s$ *was used in [3], here we adopt the more general terminal region and terminal penalty approach proposed in [5] (with a fixed terminal region) for an economic MPC setting without average constraints. It turns out that the more general definition of the set* $\mathbb{Y}_t$ *is not only a relaxation which can be used for (transient) performance improvement, but is in fact necessary when using a terminal region and terminal penalty approach. This will be discussed in more detail in Section 3.2.*

[3]For simplicity, we assume that $\mathbf{u}^0$ is unique. If this is not the case, just assign a unique constant selection map to select one of the multiple minima.

3. Properties of economic MPC with average constraints and terminal cost

In this section, we discuss various properties of the economic MPC algorithm introduced in the previous section. For the case of a terminal equality constraint, it was shown in [3] that the resulting closed-loop system (8) satisfies the pointwise in time constraints as well as the average constraints, and the average performance is at least as good as operation at the optimal steady-state (x_s, u_s). In Section 3.1, we show that this property can be maintained when using both a terminal region approach as well as the more general definition (6) of the output constraint set $\mathbb{Y}_t$ including the sets $\overline{\mathbb{Y}}(t)$. In Section 3.2, we then discuss how the sets $\mathbb{X}_f(t)$ and $\overline{\mathbb{Y}}(t)$ can be calculated.

3.1. Constraint satisfaction and average performance

In order to establish recursive feasibility and average performance bounds of the economic MPC algorith (4)–(5), we make the following assumption concerning the terminal cost V_f and terminal regions $\mathbb{X}_f(t)$.

Assumption 1. *There exists an auxiliary terminal control law $\kappa_f : \mathbb{X} \to \mathbb{U}$ and for each $t \in \mathbb{I}_{\geq 0}$, the terminal region $\mathbb{X}_f(t)$ is defined such that for all $x \in \mathbb{X}_f(t)$, the following is satisfied: (i) $(x, \kappa_f(x)) \in \mathbb{Z}$, (ii) $f(x, \kappa_f(x)) \in \mathbb{X}_f(t+1)$, and (iii)*

$$V_f(f(x, \kappa_f(x))) - V_f(x) \leq -\ell(x, \kappa_f(x)) + \ell(x_s, u_s). \tag{9}$$

If the terminal region is constant, i.e., $x \in \mathbb{X}_f(t) = \mathbb{X}_f$ for all $t \in \mathbb{I}_{\geq 0}$, then condition (ii) reduces to $\mathbb{X}_f$ being invariant under the control law $u = \kappa_f(x)$. Then, Assumption 1 is nothing but the standard assumption used in MPC within a terminal region / terminal cost framework, and many different methods have been proposed how V_f, κ_f and $\mathcal{X}_f$ can be computed such that Assumption 1 is satisfied in the context of tracking MPC (see, e.g., [13]). However, in economic MPC, these methods do not apply anymore as a general cost function ℓ is considered which is not necessarily positive definite with respect to x_s, and hence different techniques have to be used which possibly result in a terminal cost function V_f which is also not positive definite with respect to x_s. In [5], under the assumption that f and ℓ are twice continuously differentiable and the linearized system around (x_s, u_s) is stabilizable, it was shown how V_f, κ_f and $\mathcal{X}_f$ can be computed in an economic MPC setting such that Assumption 1 is satisfied with a constant terminal

region. In Section 3.2, we discuss how based on this construction, one can define appropriate time-varying terminal regions $\mathbb{X}_f(t)$ suited for our setup including average constraints, i.e., such that Assumption 1 is satisfied.

Besides Assumption 1, we furthermore need the following conditions concerning the sets $\overline{\mathbb{Y}}(t)$.

Assumption 2. *For each $t \in \mathbb{I}_{\geq 0}$, the set $\overline{\mathbb{Y}}(t)$ is such that $h(x, \kappa_f(x)) \in \mathbb{Y} \oplus \overline{\mathbb{Y}}(t)$ for all $x \in \mathbb{X}_f(t)$.*

Assumption 3. *The exist a constant $0 \leq \alpha < 1$ and a compact set $\overline{\mathbb{Y}}$ such that*

$$\sum_{k=0}^{t} \overline{\mathbb{Y}}(k) \subseteq t^{\alpha} \overline{\mathbb{Y}}. \tag{10}$$

Assumptions 2 and 3 will be crucial in establishing recursive feasibility of problem (4)–(5) and fulfilment of the average constraints. Furthermore, fulfilment of Assumption 3 with $\alpha = 0$ will be needed to establish convergence of the closed-loop system (8) to x_s later on (see Section 4). We will show in Section 3.2 how the sets $\overline{\mathbb{Y}}(t)$ can be calculated such that Assumptions 2 and 3 are satisfied.

Remark 2. *As discussed in Remark 1, in the setting of [3, Section V.B] with a terminal equality constraint, the time-varying output constraint set $\mathbb{Y}_t$ was defined as in (6) with $\overline{\mathbb{Y}}(t) = \{0\}$ for all $t \in \mathbb{I}_{\geq 0}$. Then, due to the terminal equality constraint $x(N|t) = x_s$ and the fact that $h(x_s, u_s) \in \mathbb{Y}$ according to (3), it follows that Assumptions 2 and 3 are immediately satisfied with $\alpha = 0$ (and $\overline{\mathbb{Y}} = \{0\}$).*

Theorem 1 (Constraint satisfaction and average performance). *Let $x_0 \in \mathcal{X}_N$, and suppose that Assumptions 1–3 are satisfied. Then the optimization problem (4)–(5) is feasible for all $t \in \mathbb{I}_{\geq 0}$ and the following is satisfied for the closed-loop system (8):*

$$(x(k), u(k)) \in \mathbb{Z} \quad \forall\ k \in \mathbb{I}_{\geq 0}, \tag{11}$$

$$Av[y] \subseteq \mathbb{Y}, \tag{12}$$

$$Av[\ell(x, u)] \subseteq (-\infty, \ell(x_s, u_s)]. \tag{13}$$

Proof: As usual in MPC, the proof is by induction. Suppose that the optimization problem (4)–(5) is feasible at time t. Then, at time $t+1$, consider the candidate input sequence $\tilde{\mathbf{u}}(t+1) := \{u^0(1|t), \ldots, u^0(N-1|t), \kappa_f(x^0(N|t))\}$ with corresponding candidate state sequence

$\tilde{\mathbf{x}}(t+1) := \{x^0(1|t), \ldots, x^0(N|t), f(x^0(N|t), \kappa_f(x^0(N|t)))\}$. These candidate sequences fulfill the constraints (5a)–(5c) due to Assumption 1 and the fact that the optimal solution at time t was feasible. Furthermore, due to Assumption 2 and the fact that the optimal solution at time t satisfies (5d), we obtain

$$\begin{aligned} & \sum_{k=0}^{N-1} h(\tilde{x}(k|t+1), \tilde{u}(k|t+1)) \\ = & \sum_{k=0}^{N-1} h(x^0(k|t), u^0(k|t)) + h(x^0(N|t), \kappa_f(x^0(N|t))) - h(x(t), u(t)) \\ \in & \mathbb{Y}_t \oplus \mathbb{Y} \oplus \overline{\mathbb{Y}}(t) \ominus \{h(x(t), u(t))\} = \mathbb{Y}_{t+1}. \end{aligned}$$

Hence the optimization problem (4)–(5) is feasible for all $t \in \mathbb{I}_{\geq 0}$, and from the definition of the receding horizon input to the system it follows that the closed-loop system (8) satisfies the state and input constraints (11).

Next, in order to show that the closed-loop system (8) satisfies the average constraints (12), consider the following. From (5d) and (7), it follows that at any time t

$$\sum_{k=0}^{t-1} h(x(k), u(k)) + \sum_{k=0}^{N-1} h(x(k|t), u(k|t)) \in \mathbb{Y}_{00} \oplus (t+N)\mathbb{Y} \oplus \sum_{k=0}^{t-1} \overline{\mathbb{Y}}(k), \tag{14}$$

for each predicted input and state sequences which are feasible at time t. Notice that, due to compactness of the set $\mathbb{Z}$, for any sequence of feasible predicted input and state sequences it holds:

$$\lim_{t \to +\infty} \frac{\sum_{k=0}^{N-1} h(x(k|t), u(k|t))}{t} = 0. \tag{15}$$

In particular then, taken any time sequence $t_n \to +\infty$ such that

$$\lim_{n \to +\infty} \frac{\sum_{k=0}^{t_n-1} h(x(k), u(k))}{t_n}$$

exists, it holds that

$$\begin{aligned} \lim_{n \to +\infty} \frac{\sum_{k=0}^{t_n-1} h(x(k), u(k))}{t_n} & \overset{(14),(15)}{\in} \lim_{n \to +\infty} \frac{\mathbb{Y}_{00} \oplus (t_n+N)\mathbb{Y} \oplus \sum_{k=0}^{t_n-1} \overline{\mathbb{Y}}(k)}{t_n} \\ & \overset{\text{Ass. 3}}{\subseteq} \lim_{n \to +\infty} \frac{\mathbb{Y}_{00} \oplus (t_n+N)\mathbb{Y} \oplus t_n^{\alpha}\overline{\mathbb{Y}}}{t_n} = \mathbb{Y}, \end{aligned} \tag{16}$$

which means that (12) is satisfied.

Finally, the average performance estimate (13) can be shown as in [5, Theorem 18] and [3, Theorem 5]. □

3.2. Calculating $\mathbb{X}_f(t)$ and $\overline{\mathbb{Y}}(t)$

We now discuss how the sets $\mathbb{X}_f(t)$ and $\overline{\mathbb{Y}}(t)$ can be calculated such that Assumptions 1–3 are satisfied. As noted in Remark 2, in the case of a terminal equality constraint $x(N|t) = x_s$ (where Assumption 1 is trivially satisfied with the constant terminal region $\mathbb{X}_f = \{x_s\}$ and $\kappa_f(x_s) = u_s$), one can choose $\overline{\mathbb{Y}}(t) = \{0\}$ for all $t \in \mathbb{I}_{\geq 0}$ and Assumptions 2 and 3 are immediately satisfied (with $\alpha = 0$). In this case, choosing $\overline{\mathbb{Y}}(t) \supsetneq \{0\}$ can be used to relax the output constraint (5d) in order to improve the transient performance of the system. Loosely speaking, this means that the system is allowed to spend some more time in a region where $h(x, u) \notin \mathbb{Y}$. For example, taking $\overline{\mathbb{Y}}(t) = (t^\alpha - (t-1)^\alpha)\overline{\mathbb{Y}}$ for some compact set $\overline{\mathbb{Y}}$ and some $0 \leq \alpha < 1$ results in

$$\mathbb{Y}_t = \mathbb{Y}_{00} \oplus (t+N)\mathbb{Y} \oplus t^\alpha \overline{\mathbb{Y}} \ominus \sum_{k=0}^{t-1} h(x(k), u(k))$$

according to (7), and both Assumptions 2 and 3 are satisfied.

We now turn our attention to the setting with a terminal region constraint as in (5c) instead of a terminal equality constraint. If a constant terminal region $\mathbb{X}_f$ can be found (e.g., with the methodology proposed in [5, Section 4]) such that $h(x, \kappa_f(x)) \in \mathbb{Y}$ for all $y \in \mathbb{X}_f$, then the same considerations as above apply, i.e., $\overline{\mathbb{Y}}(t) = \{0\}$ for all $t \in \mathbb{I}_{\geq 0}$ is a possible choice such that Assumptions 2 and 3 are satisfied with $\alpha = 0$; note that in such a case, there is also no need for a time-varying terminal region. On the other hand, if no constant termial region $\mathbb{X}_f$ can be found satisfying $\mathbb{X}_f \subseteq \mathbb{Y}$, things are inherently different as Assumption 2 is not satisfied anymore with $\overline{\mathbb{Y}}(t) = \{0\}$. Hence in this case, a nontrivial $\overline{\mathbb{Y}}(t)$ is not only beneficial in order to improve performance, but is first of all needed in order to establish recursive feasibility of the optimization problem (4)–(5) via Theorem 1. In the following, we will construct sets $\mathbb{X}_f(t)$ and $\overline{\mathbb{Y}}(t)$ satisfying Assumptions 1–3; this will be done on the basis of a constant terminal region $\mathbb{X}_f$ satisfying Assumption 1 (which, as mentioned above, can e.g. be calulated as proposed in [5]). Furthermore, for notational simplicity, for the remainder of this section we assume without loss of generality that $(x_s, u_s) = (0, 0)$. We make the following assumption on the constant terminal region $\mathbb{X}_f$ and the terminal auxiliary controller κ_f:

Assumption 4. *There exists a (constant) terminal region $\mathbb{X}_f$ satisfying Assumption 1, which is of the form $\mathbb{X}_f := \{x \in \mathbb{R}^n : V(x) \leq b_0\}$ for some $b_0 \geq 0$ and $V(x) := x^T P x$ with $P > 0$. Furthermore, the following is satisfied: $V(f(x, \kappa_f(x))) - V(x) \leq -x^T Q x$ for some $Q > 0$ and all $x \in \mathbb{X}_f$.*

Remark 3. *The approach proposed in [5] results in a (constant) terminal region $\mathbb{X}_f$ and a terminal auxiliary controller $\kappa_f(x)$ which is such that both Assumptions 1 and 4 are satisfied.* □

Now suppose that the terminal region $\mathbb{X}_f$ is given as specified in Assumption 4, and define $b_{\max} := \max_{\mathbb{X}_f \subseteq \mathbb{Y}} b_0$. DID NOT UNDERSTAND DEFINITION OF b_{max} Note that $b_{\max}$ is well defined and $b_{\max} \geq 0$ as the optimal steady-state satisfies $h(x_s, u_s) \in \mathbb{Y}$ by definition (see (3)). As noted above, for $b_0 \leq b_{\max}$, $\overline{\mathbb{Y}}(t) = \emptyset$ for all $t \in \mathbb{I}_{\geq 0}$ is a possible choice such that Assumptions 2 and 3 are satisfied (together with the constant terminal region $\mathbb{X}_f$). Hence in the following, we consider the case $b_{\max} < b_0$ and propose to gradually tighten the terminal region such that $\mathbb{X}_f(t) \subseteq \mathbb{Y}$ for $t \to \infty$. Namely, for each $t \in \mathbb{I}_{\geq 0}$, let

$$\mathbb{X}_f(t) := \{x \in \mathbb{R}^n : V(x) \leq b(t)\}, \tag{17}$$

where

$$b(t+1) = \max\left\{(1 - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)})b(t), b_{\max}\right\}, \qquad b(0) = b_0. \tag{18}$$

Proposition 1. *Suppose that a constant terminal region $\mathbb{X}_f$ is known satisfying both Assumptions 1 and 4. Then Assumption 1 is also satisfied for the time-varying terminal regions given by* (17)–(18)*.*

Proof: Fulfillment of items (i) and (iii) of Assumption 1 is inherited from the fact that these conditions were satisfied for the constant terminal region $\mathbb{X}_f$ as well. In order to verify item (ii), consider the following. As $\mathbb{X}_f(t) \subseteq \mathbb{X}_f(0) = \mathbb{X}_f$ for all $t \in \mathbb{I}_{\geq 0}$, by Assumption 4 we obtain that for all $x \in \mathbb{X}_f(t)$,

$$\begin{aligned} V(f(x, \kappa_f(x))) &\leq V(x) - x^T Q x \leq V(x) - \lambda_{\min}(Q)|x|^2 \\ &\leq V(x) - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} V(x) = (1 - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)})V(x) \leq (1 - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)})b(t). \end{aligned} \tag{19}$$

But this means according to (17)–(18) that $f(x, \kappa_f(x)) \in \mathbb{X}_f(t+1)$ for all $x \in \mathbb{X}_f(t)$, i.e., item (ii) of Assumption 1 is satisfied. □

Remark 4. *We note that using of time-varying terminal regions* (17) *leads only to very little additional computational effort compared to a fixed terminal region. Namely, the only additional computation which has to be performed at each time step is determining $b(t)$ via* (18)*, which is a simple algebraic operation.* □

Given the sets $\mathbb{X}_f(t)$, we can now define the sets $\overline{\mathbb{Y}}(t)$ such that Assumptions 2 and 3 are satisfied.

Proposition 2. *Suppose that for each $t \in \mathbb{I}_{\geq 0}$, the terminal region $\mathbb{X}_f(t)$ is given by (17)–(18). Furthermore, assume that where is point-set distance notation defined ? $|h(x, \kappa_f(x))|_{\mathbb{Y}} \leq \gamma(V(x))$ for all $x \in \mathbb{X}_f(0)$ with $\gamma(s) := as^r$ and some constants $a, r > 0$. Then the choice*

$$\overline{\mathbb{Y}}(t) = c\rho^t B_1 \tag{20}$$

with $c := ab_0^r$ and $\rho := (1 - \lambda_{\min}(Q)/\lambda_{\max}(P))^r$ satisfies Assumptions 2 and 3 with $\alpha = 0$ and $\overline{\mathbb{Y}} = c/(1-\rho)B_1$.

Proof: Let $t' := \inf_{\tau \in \mathbb{I}_{\geq 0}, b(\tau) = b_{\max}} \tau$. The definition of $\mathbb{X}_f(t)$ in (17)–(18) yields that for all[4] $t \in \mathbb{I}_{[0,t'-1]}$ and all $x \in \mathbb{X}_f(t)$,

$$V(x) \leq b(t) = b_0(1 - \lambda_{\min}(Q)/\lambda_{\max}(P))^t =: b_0\bar{\rho}^t.$$

Then, as $\mathbb{X}_f(t) \subseteq \mathbb{X}_f(0)$ for all $t \in \mathbb{I}_{[0,t'-1]}$, by assumption there exist constants $a, r > 0$ such that

$$|h(x, \kappa_f(x))|_{\mathbb{Y}} \leq \gamma(b_0\bar{\rho}^t) = ab_0^r(\bar{\rho}^r)^t =: c\rho^t$$

for all $x \in \mathbb{X}_f(t)$. Note that $\rho < 1$ and that this inequality is also valid for all $t \in \mathbb{I}_{\geq t'+1}$, as for all such t we have $|h(x, \kappa_f(x))|_{\mathbb{Y}} = 0$ for all $x \in \mathbb{X}_f(t)$ due to the definition of t'. But this implies that Assumption 2 is satisfied if the sets $\overline{\mathbb{Y}}(t)$ are defined as in (20) with $c := ab_0^r$ and $\rho := (1 - \lambda_{\min}(Q)/\lambda_{\max}(P))^r$. Furthermore, we obtain

$$\sum_{k=0}^{t} \overline{\mathbb{Y}}(k) = B_1 \sum_{k=0}^{t} c\rho^t \subseteq B_1 \sum_{k=0}^{\infty} c\rho^t = \frac{c}{1-\rho} B_1 =: \overline{\mathbb{Y}}. \tag{21}$$

But this imples that Assumption 3 is satisfied with $\alpha = 0$ and $\overline{\mathbb{Y}}$ from (21), which concludes the proof of the Proposition. □

Remark 5. *Assuming the existence of constants $a, r > 0$ such that $|h(x, \kappa_f(x)|_{\mathbb{Y}} \leq \gamma(V(x))$ for all $x \in \mathbb{X}_f(0)$ with $\gamma(s) := as^r$ is not a too big restriction and is satisfied for many output functions h.*

[4] If $t' = +\infty$, then the following holds for all $t \in \mathbb{I}_{\geq 0}$.

For example, this assumption is satisfied if $|h|_{\mathbb{Y}}$ is locally Lipschitz[5]*, due to compactness of $\mathbb{X}_f(0)$.* □

Remark 6. *In order to retain the benefits of a larger terminal region, the tightening of the terminal region via* (17)–(18) *can be relaxd as follows. Namely,* (18) *can be applied only after some finite time $\bar{t} > 0$, and the constant terminal region $\mathbb{X}_f(t) = \mathbb{X}_f(0)$ is used for $t \in \mathbb{I}_{[0,\bar{t}-1]}$. Then the above analysis holds true in a similar way, and again Assumptions 2 and 3 (with $\alpha = 0$) are satisfied.* □

4. Convergence of economic MPC with average constraints

While recursive feasibility, average performance and satisfaction of average constraints were shown in Theorem 1 for the economic MPC algorithm (4)–(5), the closed-loop system (8) does in general not converge to the optimal steady-state x_s. However, as discussed in the Introduction, for some applications, a convergent closed-loop behavior is indispensable. Hence in this section, we show that under certain conditions the economic MPC algorithm (4)–(5) indeed leads to a convergent closed-loop behavior. To this end, in the following we assume that the average constraint set $\mathbb{Y}$ is given by

$$\mathbb{Y} := \{y \in \mathbb{R}^p : y \leq 0\}. \tag{22}$$

Note that this is not a major restriction, as the output map h can be some general nonlinear function.[6]

For an economic MPC setting without average constraints, it was shown in [3, 5] that a certain dissipativity condition on the system (2) is sufficient for asymptotic stability of the closed-loop system (8). In the following, we show that for the setting including average constraints, a similar dissipativity condition is sufficient for asymptotic convergence of the closed-loop system (8) to x_s. The notion of dissipativity was introduced in [14] (for a discrete time version see [15]); we adapt it here to our setting including state and input constraints. To this end, for a set $\mathbb{W} \subseteq \mathbb{Z}$, denote by $\mathbb{W}_{\mathbb{X}}$ the projection of $\mathbb{W}$ on $\mathbb{X}$, i.e., $\mathbb{W}_{\mathbb{X}} := \{x \in \mathbb{X} : \exists u \in \mathbb{U} \ s.t. \ (x,u) \in \mathbb{W}\}$.

[5] If, for example, the set $\mathbb{Y}$ is given by $\mathbb{Y} := \{y \in \mathbb{R}^p : y \leq 0\}$ as in the following sections, this is satisfied if h is locally Lipschitz.

[6] In particular, this includes the setup in [3], where $\mathbb{Y}$ was supposed to be a polyhedron given by $\mathbb{Y} := \{y \in \mathbb{R}^p : Ay - b \leq 0\}$ for some $A \in \mathbb{R}^{n_c \times p}$ and $b \in \mathbb{R}^{n_c}$. Namely, if necessary, one can define $\tilde{y} = Ah(x,u) - b =: \tilde{h}(x,u)$ as a new auxiliary output variable and then use the set $\mathbb{Y}$ as in (22).

Definition 1. *The system* (2) *is dissipative on a set* $\mathbb{W} \subseteq \mathbb{Z}$ *with supply rate* $s : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ *if there exists a bounded storage function* $\lambda : \mathbb{W}_{\mathbb{X}} \to \mathbb{R}$ *such that the following inequality is satisfied for all* $(x, u) \in \mathbb{W}$*:*

$$\lambda(f(x,u)) - \lambda(x) \leq s(x,u). \tag{23}$$

If, in addition, for some positive definite[7] $\rho : \mathbb{W}_{\mathbb{X}} \to \mathbb{R}$ *it holds that for all* $(x, u) \in \mathbb{W}$

$$\lambda(f(x,u)) - \lambda(x) \leq -\rho(x) + s(x,u), \tag{24}$$

then system (2) *is strictly dissipative on* $\mathbb{W}$. □

Note that in the original definition [14, 15], the storage function λ is required to be nonnegative; we do not impose this assumption here but only require that it is bounded on the bounded set $\mathbb{W}$ in accordance with [3]. The dissipativity condition we impose is the following.

Assumption 5. *There exists a multiplier* $\bar{\lambda} \in [0, \infty)^{n_c}$ *such that system* (2) *is strictly dissipative on* $\mathbb{Z}$ *with supply rate* $s(x,u) := \ell(x,u) - \ell(x_s, u_s) + \bar{\lambda}^T h(x,u)$.

This dissipativity condition was introduced in the context of economic MPC in [3], and for a further discussion and analysis including robustness with respect to changes in the constraint set (and hence in (x_s, u_s)) as well as the role of the multiplier $\bar{\lambda}$, the interested reader is referred to [16, 17].

Concerning closed-loop convergence of economic MPC with average constraints, a first preliminary result was obtained in [3, Remark 6.5] (for the case of a terminal equality constraint), where it was noted that under Assumption 5, the solution of the closed-loop MPC system (8) satisfies

$$\liminf_{t \to \infty} |x(t) - x_s| = 0. \tag{25}$$

This result was proven by showing that Assumption 5 implies that operation of the system at steady-state is optimal, and combining this fact with the average performance result of Theorem 1, i.e., inequality (13). However, (25) only implies convergence of the closed-loop MPC solution in

[7] A function ρ is positive definite with respect to some point $\bar{x} \in \mathbb{X}$ if it is continuous, $\rho(\bar{x}) = 0$ and $\rho(x) > 0$ for all $x \in \mathbb{X}$ with $x \neq \bar{x}$. In the following, when speaking of strict dissipativity, we take $\bar{x} = x_s$, i.e., the function ρ is assumed to be positive definite with respect to the optimal steady-state x_s defined via (3).

a weak sense, and furthermore a Lyapunov-like stability analysis has not been available so far for economic MPC with average constraints. The main result of this section is to close this gap and prove asymptotic convergence of the closed-loop MPC solution to the optimal steady-state x_s by means of a Lyapunov-like analysis.

Theorem 2 (Convergence of economic MPC with average constraints). *Suppose that Assumptions 1 and 2, Assumption 3 with $\alpha = 0$, and Assumption 5 are satisfied. Then the solution of the MPC closed-loop system (8) asymptotically converges to x_s, i.e., $\lim_{t\to\infty} x(t) = x_s$.* □

Proof: Denote by $L(x,u) := \ell(x,u) - \ell(x_s,u_s) + \lambda(x) - \lambda(f(x,u))$ the rotated stage cost and by $\widetilde{V}_f(x) := V_f(x) - V_f(x_s) + \lambda(x) - \lambda(x_s)$ the rotated terminal cost, where λ is the storage function from (24). Let $\widetilde{V}_N^0(x(t))$ denote the optimal value function of problem (4)–(5) with ℓ and V_f in (4) replaced by L and $\widetilde{V}_f$, respectively. As was shown in [5, Lemma 14], the optimal solution to this modified optimization problem is identical to the optimal solution of the original problem (4)–(5), as the cost functions only differ by a constant term and the constraints are the same. In order to prove Theorem 2, we propose to use the following Lyapunov-like function, which we define along the solution of the closed-loop system (8):

$$V(t) = \widetilde{V}_N^0(x(t)) + w(t) \tag{26}$$

with

$$w(t) := \sup_{T\geq 0} \sum_{k=t}^{t+T} \bar{\lambda}^T y(k), \tag{27}$$

where the sequence $y(k)$ is the output along the solution of the closed-loop MPC system (8) from time t on.

The function w has the following properties. Firstly, we have that

$$w(t) \geq \bar{\lambda}^T y(t) = \bar{\lambda}^T h(x(t), u^0(0|t)) \geq \min_{(z,v)\in\mathbb{Z}} \bar{\lambda}^T h(z,v) =: \underline{w} > -\infty,$$

for all $t \in \mathbb{I}_{\geq 0}$, where the first inequality follows as $T = 0$ is allowed in the definition of w in (27), and the last inequality follows from compactness of $\mathbb{Z}$ and continuity of h. Secondly, from (5d), (7) and the fact that Assumption 3 is satisfied with $\alpha = 0$, we obtain that the closed-loop MPC system (8) satisfies

$$\sum_{k=0}^{N-1} h(x^0(k|t), u^0(k|t)) + \sum_{k=0}^{t-1} y(k) \in \mathbb{Y}_{00} \oplus \overline{\mathbb{Y}} \oplus (t+N)\mathbb{Y} \tag{28}$$

for all $t \in \mathbb{I}_{\geq 0}$. Hence, due to the definition of the set $\mathbb{Y}$ in (22), from (28) we obtain that for all $t \in \mathbb{I}_{\geq 0}$,

$$\sum_{k=0}^{t-1} y(k) \leq y_{00} + \bar{y} + y', \tag{29}$$

where the inequality holds component-wise and y_{00}, $\bar{y}$ and y' are defined component-wise as

$$[y_{00}]_i := \max_{y \in \mathbb{Y}_{00}} [y]_i, \quad [\bar{y}]_i := \max_{y \in \overline{\mathbb{Y}}} [y]_i, \quad [y']_i := N \max_{(x,u) \in \mathbb{Z}} -[h(x,u)]_i.$$

Note that y_{00}, $\bar{y}$ and y' are finite due to compactness of $\mathbb{Y}_{00}$, $\overline{\mathbb{Y}}$ and $\mathbb{Z}$ and continuity of h, respectively. Therefore, as $\bar{\lambda} \in [0, \infty)^{n_c}$, from (29) we obtain that the closed-loop MPC solution satisfies, for all $t \in \mathbb{I}_{\geq 0}$,

$$\sum_{k=0}^{t-1} \bar{\lambda}^T y(k) \leq \bar{\lambda}^T (y_{00} + \bar{y} + y').$$

Thus it follows that $w(0) \leq \bar{\lambda}^T (y_{00} + \bar{y} + y') < \infty$.

Next, consider the term $\widetilde{V}_N^0(x)$ in (26). Due to continuity of ℓ and V_f, boundedness of λ and compactness of $\mathbb{Z}$ and $\mathbb{X}_f$, there exist constants $-\infty < \underline{V}_f \leq \overline{V}_f < +\infty$ and $-\infty < \underline{L} \leq \overline{L} < +\infty$ such that $\underline{V}_f \leq \widetilde{V}_f(x) \leq \overline{V}_f$ for all $x \in \mathbb{X}_f$ and $\underline{L} \leq L(x,u) \leq \overline{L}$ for all $(x,u) \in \mathbb{Z}$. Hence $N\underline{L} + \underline{V}_f \leq \widetilde{V}_N^0(x) \leq N\overline{L} + \overline{V}_f$ for all $x \in \mathcal{X}_N$.

Combining the above, we obtain that for all $x_0 \in \mathcal{X}_N$, the function V satisfies $V(0) \leq \bar{\lambda}^T (y_{00} + \bar{y}) + N\overline{L} + \overline{V}_f < \infty$ and $V(t) \geq \underline{w} + N\underline{L} + \underline{V}_f > -\infty$ for all $t \in \mathbb{I}_{\geq 0}$. Now consider the evolution of V along the solution of the closed-loop system (8). First, by considering the candidate input and state sequences $\tilde{\mathbf{u}}(t+1)$ and $\tilde{\mathbf{x}}(t+1)$, respectively, from the proof of Theorem 1, we obtain that

$$\begin{aligned}
\widetilde{V}_N^0(x(t+1)) - \widetilde{V}_N^0(x(t)) &\leq L(x^0(N|t), \kappa_f(x^0(N|t))) - L(x(t), u^0(0|t)) \\
&\quad + \widetilde{V}_f(f(x^0(N|t), \kappa_f(x^0(N|t)))) - \widetilde{V}_f(x^0(N|t)) \\
&\leq -L(x(t), u^0(0|t)),
\end{aligned} \tag{30}$$

where the last inequality follows from Assumption 1 and the definitions of L and $\widetilde{V}_f$ (compare

also [5, Lemma 9]). With this, we obtain

$$\begin{aligned} V(t+1) - V(t) &= \widetilde{V}_N^0(x(t+1)) - \widetilde{V}_N^0(x(t)) + w(t+1) - w(t) \\ &\overset{(30)}{\leq} -L(x(t), u^0(0|t)) + w(t+1) - w(t) \\ &\overset{\text{Ass. } 5}{\leq} -\rho(x(t)) + \bar{\lambda}^T h(x(t), u^0(t)) + w(t+1) - w(t) \\ &= -\rho(x(t)) + \sup_{T \geq 1} \sum_{k=t}^{t+T} \bar{\lambda}^T h(x(k), u^0(k)) - \sup_{T \geq 0} \sum_{k=t}^{t+T} \bar{\lambda}^T h(x(k), u^0(k)) \\ &\leq -\rho(x(t)). \end{aligned} \tag{31}$$

where has the lambda term disappeared ? Summarizing the above, the sequence $V(t)$ is non-increasing in t, bounded from below and $V(0)$ is finite; hence it converges. But this implies according to (31) that $\sum_{t=0}^{\infty} \rho(x(t))$ converges, which by positive definiteness of ρ with respect to x_s and boundedness of $x(\cdot)$ in turn implies that $x(t)$ asymptotically converges to x_s, i.e., $\lim_{k\to\infty} x(k) = x_s$. □

Remark 7. *While in Theorem 2 we showed asymptotic convergence of the closed-loop solution to x_s, note that x_s is not necessarily asymptotically stable (i.e., lacks the Lyapunov stability property). This is in contrast to the setting without average constraints, where strict dissipativity leads to an asymptotically stable equilibrium x_s (see [3, Theorem 2]). Namely, the average constraints allow the system to initially "spend time" in a region of the state-space where it is not allowed on average. This means that even when starting at the optimal steady state x_s, the closed-loop MPC trajectory might not stay there (see Example 1). On the other hand, we note that if Assumption 5 is strengthened to hold with $\bar{\lambda} = 0$, then the same analysis as in [3, Section IV] for systems without average constraints can be used to conclude that x_s is asymptotically stable.* □

Remark 8. *We note that the function V (defined in (26)), which is used in the proof of Theorem 2, is different from the Lyapunov function used in economic MPC without average constraints [3, 5]. Namely, in these references, $\widetilde{V}_N^0(x)$ is used as a Lyapunov function, while here we need the additional term w defined by (27) in order to be able to conclude the decaying of V via (31).* □

Remark 9. *In Theorem 2, requiring Assumption 3 to hold with $\alpha = 0$ is crucial for establishing boundedness of $w(0)$ and hence $V(0)$. Namely, if Assumption 3 is only satisfied for some $\alpha > 0$,*

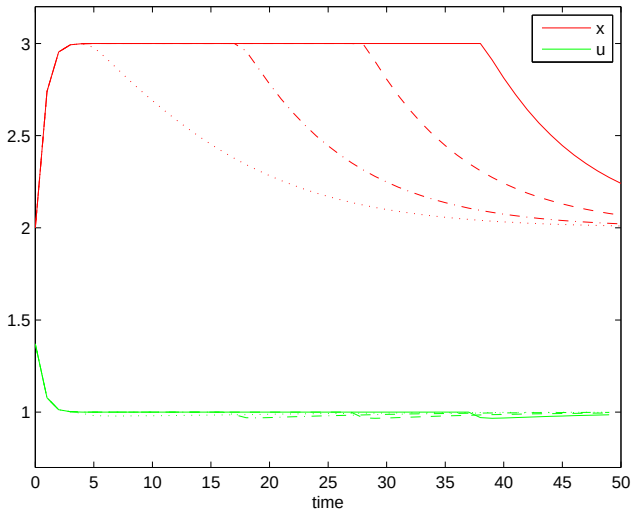


Figure 1: Closed-loop state and input trajectories of system (32) with $\bar{y} = 20$ (dotted lines), $\bar{y} = 40$ (dash-dotted lines), $\bar{y} = 60$ (dashed lines), and $\bar{y} = 80$ (solid lines).

then no finite upper bound for $w(0)$ can be found via (28)–(29). As discussed above, the sets $\overline{\mathbb{Y}}(t)$ can always be chosen such that Assumption 3 with $\alpha = 0$ is satisfied, if, for example, a terminal equality constraint is used, or $\mathbb{X}_f \subseteq \mathbb{Y}$, or if the tightening of the terminal region as proposed in Section 3.2 is applied. □

Example 1: Consider the system

$$x(k+1) = x(k)u(k) \tag{32}$$

with state and input constraint set $\mathbb{Z} = \mathbb{X} \times \mathbb{U} := [-10\ 10]^2$ and average constraint of the form (22) with $y = h(x,u) := 2x + u - 5$. The set of all feasible steady-state input pairs is given by $S := \{(x,u) : x = 0, u \in U\} \cup \{(x,u) : x \in X, u = 1\}$. The stage cost is chosen as $\ell(x,u) := (x-3)^2 + u^2$, and hence the optimal steady-state defined via (3) is given by $(x_s, u_s) = (2,1)$ with $\ell(2,1) = 2$. The terminal cost V_f and the sets $\mathbb{X}_f(t)$ and $\overline{\mathbb{Y}}(t)$ were calculated as described in Section 3.2 such that Assumptions 1–3 are satisfied. Furthermore, one can show that Assumption 5 is satisfied with $\bar{\lambda} = 1$ and $\lambda(x) = (3/2)x$, i.e., the system (32) is strictly dissipative with storage function $\lambda(x) = (3/2)x$ and supply rate $s(x,u) = \ell(x,u) - \ell(x_s,u_s) + \bar{\lambda}h(x,u)$. Figure 1 shows simulations of system (32) in closed-loop with the economic MPC algorithm (4)–(5), where $\mathbb{Y}_{00} = \{y \in \mathbb{R} : y \leq \bar{y}\}$ with different values of $\bar{y}$. One can see that the closed-loop system converges to x_s. However, x_s is not a Lyapunov

stable equilibrium point. Namely, although $x_0 = x_s = 2$, the closed-loop first moves away from x_s and approaches $x = 3$, before finally converging to $x_s = 2$. In fact, it turns out that without the average constraint, the optimal steady-state would be given by $(x, u) = (3, 1)$ with $\ell(3, 1) = 1 < \ell(2, 1)$, and the system would be strictly dissipative with supply rate $s(x, u) = \ell(x, u) - \ell(3, 1)$ and storage function $\lambda(x) = (2/3)x$. Hence the closed-loop system would asymptotically converge to $x = 3$ if no average constraint was present. Figure 1 can now be interpreted as follows. The average constraint initially allows the system to converge to $x = 3$, before forcing it to leave this steady-state again and to converge to $x_s = 2$, which is the best steady-state also fulfilling the average constraint. The amount of time how long the system is allowed to stay in a vicinity of $x = 3$ can be tuned by choosing the parameter $\bar{y}$ in $\mathbb{Y}_{00}$ accordingly (see Figure 1).

5. Using average constraints to enforce convergence of economic MPC

In the previous section, we have shown that under a dissipativity condition (Assumption 5), the closed-loop MPC system (8) converges to the optimal steady-state x_s. If this dissipativity assumption is not satisfied, then in general the closed-loop system might not have a convergent behavior, but can be some oscillating or even complex chaotic trajectory. While this can be acceptable in certain types of applications (for instance in the production of chemicals, or more generally for plants in which outputs are truly outflows which can easily be stored and dispatched at some later time) in other types of applications it is instead crucial to converge to a steady-state within a specified amount of time. In the following, we discuss different methods how one can enforce convergence of the closed-loop system (8) to the optimal steady-state x_s. In [3], one method was already proposed (in a setting without average constraints) how this goal can be achieved. Namely, the economic stage cost function ℓ can be modified by an additional term, which is positive definite with respect to (x_s, u_s), such that the system becomes dissipative, i.e., Assumption 5 is satisfied. In the following, we propose two methods to enforce convergence where the cost function need not be altered, but the original (economic) stage cost function ℓ is used in the MPC algorithm. Both methods are based on the idea to define an (additional) auxiliary output $y = h(x, u)$ and then impose an (additional) average constraint of the form (22) on the system. Note that the output map h must be such that the optimal steady-state (x_s, u_s) is not altered by the additional average constraint, and that the sets $\mathbb{X}_f(t)$ and $\overline{\mathbb{Y}}(t)$ might have to be adjusted according to Section 3.2.

5.1. Imposing average constraints to ensure dissipativity

In light of Section 4, the first approach consists of finding an output map h such that Assumption 5 is satisfied, i.e., such that the system becomes strictly dissipative. Then, by Theorem 2 we can conclude that the closed-loop system (8) converges to x_s as desired. We next show that such an output map h can always be found for the case of locally Lipschitz ℓ.

Proposition 3. *Assume that the stage cost function ℓ is locally Lipschitz. Then a possible choice for the auxiliary output map such that Assumption 5 is satisfied is given by $h(x,u) := |x - x_s| + |u - u_s|$.*

Proof: First, note that the given choice of h does not change the optimal steady-state (x_s, u_s), i.e., the optimization problem (3) has the same solution with the additional constraint $h(x,u) \leq 0$. As ℓ is locally Lipschitz and $\mathbb{Z}$ is compact, there exist constants $\ell_x, \ell_u \geq 0$ such that

$$|\ell(x',u') - \ell(x'',u'')| \leq \ell_x|x' - x''| + \ell_u|u' - u''|$$

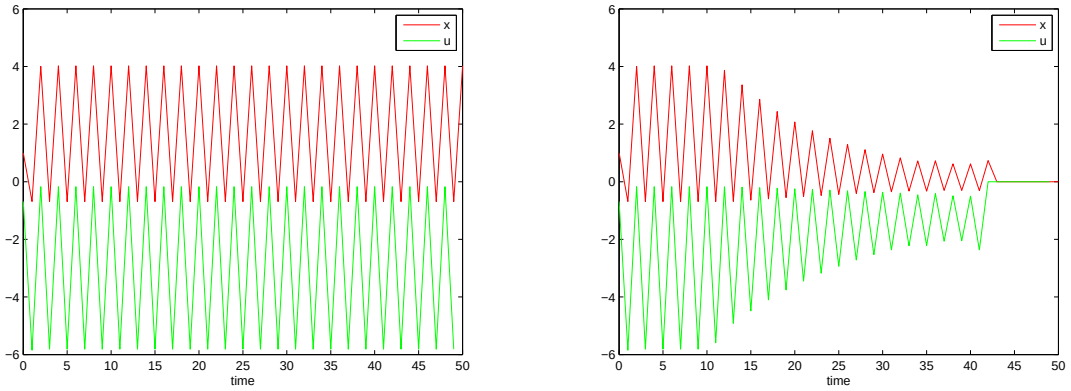
for all $(x',u') \in \mathbb{Z}$ and $(x'',u'') \in \mathbb{Z}$. Hence with $\bar{\lambda} := 1 + \max\{\ell_x, \ell_u\}$ we obtain

$$\begin{aligned} \ell(x,u) - \ell(x_s,u_s) + \bar{\lambda}h(x,u) &\geq -\ell_x|x - x_s| - \ell_u|u - u_s| + \bar{\lambda}(|x - x_s| + |u - u_s|) \\ &= |x - x_s| + |u - u_s| \geq |x - x_s| =: \rho(x) \end{aligned} \tag{33}$$

for all $(x,u) \in \mathbb{Z}$. But this implies that the system (2) is strictly dissipative on $\mathbb{Z}$ with supply rate $s(x,u) := \ell(x,u) - \ell(x_s,u_s) + \bar{\lambda}h(x,u)$ and storage function $\lambda(x) \equiv 0$, i.e., Assumption 5 is satisfied. □

Remark 10. *The specific auxiliary output map h proposed in Proposition 3 is only one possible choice ensuring strict dissipativity, and other auxiliary output maps h might exist such that Assumption 5 is satisfied (compare Example 2). Different choices of h will in general lead to a different behavior of the closed-loop system* (8)*. Hence the choice of h can be used together with the sets $\mathbb{Y}_{00}$ and $\overline{\mathbb{Y}}(t)$ in* (6) *to adjust the rate of convergence as well as the transient performance of the closed-loop system* (8)*.* □

Example 2: Consider again the system (32) of Example 1, but now with state and input constraint set $\mathbb{Z} = \mathbb{X} \times \mathbb{U} := [-10\ 10] \times [-10\ 0]$ and stage cost $\ell(x,u) = (x-5)^2 + 0.1(u-1)^2$.



(a) Without average constraint.

(b) With average constraint.

Figure 2: Closed-loop state and input trajectory of Example 2..

Again, the terminal cost V_f and the sets $\mathbb{X}_f(t)$ and $\overline{\mathbb{Y}}(t)$ were calculated as described in Section 3.2 such that Assumptions 1–3 are satisfied. The optimal steady-state defined via (3) is given by $(x_s, u_s) = (0, 0)$ with $\ell(0, 0) = 25.1$. One can show that the system is not optimally operated at steady-state; e.g., the two-periodic state and input sequences $\mathbf{x} = \{5, -0.6, 5, -0.6, \dots\}$ and $\mathbf{u} = \{-0.12, -8.33, -0.12, -8.33, \dots\}$, respectively, result in a lower average cost (≈ 18.4) than $\ell(0, 0) = 25.1$. Indeed, simulations show that the economic MPC algorithm (4)–(5) yields a two-periodic closed-loop solution (see Figure 2(a)).

In order to enforce convergence, we apply an additional average constraint of the form (22) with $y = h(x, u) := x + (1/50)u$. One can show that Assumption 5 is satisfied with $\bar{\lambda} = 10$ and $\lambda(x) \equiv 0$, and hence we can apply Theorem 2 to conclude that the closed-loop system converges to $x_s = 0$, which can also be seen in the simulation results (see Figure 2(b)).

5.2. Imposing zero-moment average constraints

The second method we propose is to impose an (additional) zero-moment average constraint. With such an additional average constraint, Assumption 5 is not necessarily satisfied; however, convergence to x_s can nevertheless be achieved. To this end, consider the following weaker notion of convergence.

Definition 2. *A sequence* $v(t)$, $t = \mathbb{I}_{\geq 0}$, *is* essentially converging *to* $\bar{v}$ *if the following is true:*

$$\forall\, \varepsilon > 0: \quad \limsup_{T \to +\infty} \frac{\mathit{card}(\{0 \leq t \leq T : |v(t) - \bar{v}| \geq \varepsilon\})}{T + 1} = 0.$$

To clarify this concept and illustrate a fundamental technical subtlety when trying to enforce convergence by means of zero-moment average constraints, consider the following discrete-time sequence:

$$v(t) = \begin{cases} 1 & \text{if } t = 2^n \text{ for } n \in \mathbb{I}_{\geq 0} \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

This sequence is not a convergent signal in the standard sense, i.e., is not asymptotically converging to its asymptotic average $\bar{v} = 0$; however, it is *essentially converging* to zero. Furthermore, it is easy to see that $v(t)$ has all average moments equal to zero, as

$$\lim_{T \to +\infty} \frac{\sum_0^T v(t)^n}{T+1} = 0$$

for all $n \in \mathbb{I}_{\geq 1}$. The following Lemma shows that zero *even* average moments are indeed enough to guarantee essential convergence of a sequence to its average value.

Lemma 1. *A sequence $v(t)$ with a zero even average moment, i.e., $Av[(v - \bar{v})^n] = \{0\}$ for some even $n \in \mathbb{I}_{\geq 2}$, is essentially converging to its average value $\bar{v}$.*

Proof: Let $\varepsilon > 0$ be arbitrary. The following equality follows from the definition of cardinality:

$$\text{card}(\{0 \leq t \leq T : |v(t) - \bar{v}| \geq \varepsilon\}) = \sum_{0 \leq t \leq T : |v(t) - \bar{v}| \geq \varepsilon} 1 \tag{35}$$

Consequently, for any even positive integer n:

$$\begin{aligned} \varepsilon^n \text{card}(\{0 \leq t \leq T : |v(t) - \bar{v}| \geq \varepsilon\}) &= \sum_{\{0 \leq t \leq T :\ |v(t) - \bar{v}| \geq \varepsilon\}} \varepsilon^n \\ \leq \sum_{\{0 \leq t \leq T :\ |v(t) - \bar{v}| \geq \varepsilon\}} |v(t) - \bar{v}|^n &\leq \sum_{t=0}^{T} |v(t) - \bar{v}|^n \end{aligned} \tag{36}$$

Thanks to (36) and the fact that the n-th average moment is zero for some even n by assumption, we may conclude

$$\limsup_{T \to +\infty} \frac{\text{card}(\{0 \leq t \leq T : |v(t) - \bar{v}| \geq \varepsilon\})}{T+1} \leq \limsup_{T \to +\infty} \frac{1}{\varepsilon^n} \frac{\Sigma_{t=0}^T |v(t) - \bar{v}|^n}{T+1} = 0, \tag{37}$$

i.e., the sequence $v(t)$ essentially converges to its average value $\bar{v}$. □

With the help of Lemma 1, we can now show that imposing an even zero-moment average constraint, i.e., using

$$y = h(x, u) = (x - x_s)^n \tag{38}$$

for some even $n \in \mathbb{I}_{\geq 2}$, results in a convergent closed-loop system.

Theorem 3. *Consider the economic MPC algorithm* (4)–(5) *with (additional) average constraint output* h *as in* (38) *and* $\mathbb{Y}$ *as in* (22)*. If Assumptions 1–3 are satisfied, then the closed-loop system* (8) *essentially converges to* x_s*. Furthermore, if Assumption 3 is satisfied with* $\alpha = 0$*, then the closed-loop system* (8) *asymptotically converges to* x_s*.*

Proof: As Assumptions 1–3 are satisfied, we can apply Theorem 1 to conclude that the average constraints $Av[y] \in \mathbb{Y}$ are satisfied for the closed-loop system (8), which translates into

$$\lim_{T\to\infty} \frac{\sum_{t=0}^{T} |x(t) - x_s|^n}{T+1} = 0,$$

i.e., the n-th average moment is zero. Hence we can apply Lemma 1 to conclude that the closed-loop system (8) essentially converges to x_s.

If Assumption 3 is satisfied with $\alpha = 0$, then one can show as in the proof of Theorem 2 that inequality (29) holds, which translates into

$$\lim_{T\to\infty} \sum_{t=0}^{T} |x(t) - x_s|^n < +\infty. \tag{39}$$

But this implies that $\lim_{t\to\infty} x(t) = x_s$, i.e., the closed-loop system (8) asymptotically converges to x_s. □

Remark 11. *Satisfaction of Assumption 3 with* $\alpha = 0$ *is crucial for establishing asymptotic convergence instead of only essential convergence. Namely, if Assumption 3 is only satisfied for some* $\alpha > 0$*, then one cannot necessarily ensure that* (39) *is satisfied. This is similar to the situation of Section 4, where asymptotic convergence of the closed-loop system* (8) *was established via dissipativity (compare Remark 9).* □

6. Application to a CSTR example

7. Conclusions

In this paper, we examined and discussed economic MPC including constraints on average values of variables. We provided an economic MPC algorithm within a terminal region / terminal cost setting which guarantees fulfillment of the average constraints, and we showed with a Lyapunov-like analysis that the resulting closed-loop system asymptotically converges to the optimal steady-state

if a certain dissipativity condition is satisfied. Furthermore, in case that convergence of the closed-loop system has to be enforced, we provided two methods how to achieve this without having to modify the (economic) cost function. Both methods are based on applying an additional average constraint, such that either the system is dissipative or an even averge moment is forced to be zero. In conclusion, we find that average constraints are a natural and useful tool in economic MPC, and we provided a thorough analysis of the various important theoretical aspects for their use.

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