

THE STABILITY OF ELASTIC CYLINDRICAL PIPES CONTAINING
INVISCID FLUID IN UNIFORM MOTION

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by

Leonard Kanangwe Shayo

Mathematics Department
Imperial College of Science and Technology
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ABSTRACT

The work in this thesis is concerned with the stability of elastic cylindrical pipes conveying inviscid fluid in uniform motion.

Infinitely long and finite length pipes are studied. For the finite length pipes the cases of simply supported, clamped-clamped and clamped-free ends are considered. For the simply supported case analytical expressions are obtained for the critical stability conditions in the limits of small and large radius to length ratio, while for the clamped-clamped pipes only the limit of small radius to length ratio is considered. For the clamped-free case the importance of the flow conditions beyond the free end of the pipe is examined by considering several flow models. The limiting case of long cantilevers is also considered and it is shown that the point relationship between pressure and pipe-wall deformation, which is well known for the beam mode, also applies for the shell modes.

In both cases comparison is made with previous numerical results. The connection between the finite and the infinite pipe analyses is demonstrated by comparing the finite pipe results with results obtained from a travelling and a standing wave analysis on infinitely long pipes.

The effects (on stability) of the boundary layer adjacent to the pipe wall are studied briefly by considering a simple shear flow in an infinitely long pipe.

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LIST OF SYMBOLS

A_{nk}, B_{nk}, C_{nk}	Amplitudes for the axial, circumferential and radial displacements respectively
b	$(1 - M^2)^{1/2} a$
c	Speed of sound in fluid (also used as wavespeed in Chapter Eight)
D	Flexural stiffness $(Eh^3 / \{12(1 - \nu^2)\})$
E	Young's Modulus
$F(\lambda, k)$	Defined by (3.9)
$F_n(x), G_n(x)$	Coordinate functions - see (4.7)
$g_k(\lambda)$	$I_k(\lambda) / \{\lambda I_k'(\lambda)\}$
h	Thickness of pipe wall
$I_k(\lambda), K_k(\lambda)$	Modified Bessel functions of the first and second kind respectively, of order k and argument λ
$J_k(\lambda)$	Bessel function of the first kind, of order k and argument λ
k	Number of circumferential waves (integer)
L	Length of finite pipe or distance between periodic supports on infinitely long elastic pipe
M	Mach number, U/c
N	Number of terms in the truncated modal solution for the pipe wall deformation
m, n	Number of longitudinal half waves on a pipe of length L (integer)

$P_{mk}(x)$	Perturbation pressure associated with the (m,k) mode - defined by (2.11)
$P_{mn,kl}$	Generalized pressures - defined by (2.14)
P_{mnj}	See (4.24)
p	Perturbation pressure
q	$\rho_m R^2 (1 - \nu^2) / E$
R	Radius of undeformed pipe
(r, θ, x)	Cylindrical Polar Coordinates
s	Foundation radial stiffness
t	Time coordinate
U, U_∞	Undisturbed flow speed in the axial direction
u, v, w	Axial, circumferential and radial displacements of an element of the pipe wall respectively (also used as perturbation velocity components in Chapter Eight)
$W_n(x)$	Coordinate functions for the radial displacement - see (2.10)
ϕ	Perturbation velocity potential
α	Radius ratio, R/L
β	$h^2 / \{12R^2\}$
ρ	Fluid density
μ	Mass ratio, $\rho R / \{\rho_m h\}$
ν	Poisson's ratio
ω	Circular frequency
ω^*	Dimensionless frequency, $q^{1/2} \omega$
ω_0, ω_{0n}	Free wave frequency, defined by (3.8) and (3.24) respectively
κ	Structural damping coefficient

λ	Longitudinal mode number (also used as variable in Fourier transforms)
Λ	Defined after (3.2)
ρ_m	Density of pipe material
ϵ_k	Defined after (3.22)
δ_{mn}	Kronecker delta (= 1 for $m = n$, 0, otherwise)
σ_n	Longitudinal wave number for n^{th} mode, $n\pi\alpha$.
$\frac{\rho U^2 R^3}{D}$	Stability parameter (dynamic pressure). R is replaced by L in the case of large radius pipes
$\frac{D}{Dt}$	Defined after (2.7)
∇_1^4	Defined after (2.2)

CHAPTER 1

GENERAL INTRODUCTION

§1.1 Introduction

Studies on elastic pipes containing fluid may be traced back to the eighteenth century in connection with physiological, engineering and acoustic applications. Boulanger [1] has given a review of some of the earlier contributions in studies relating to the propagation of shocks in fluid conveying pipes with a view to determining the thickness ratio required to withstand the shock, pulsatile flow of blood in arteries, and the propagation of sound in fluids contained in elastic pipes. The physiological aspect of flows in elastic pipes has been studied extensively by McDonald [2] who also gives a comprehensive review of earlier work.

The motivation for studies leading to the present work concerning the stability of elastic pipes conveying fluid originated from fluid induced vibrations which were observed in the Trans-Arabian oil pipe line in the 1950's. The pipe was periodically supported above ground and alternate spans vibrated out of phase. The interesting aspect, which led Ashley and Haviland [3] to begin the first investigation, was that the vibrations were of small amplitude and of short duration. This was contrary to the commonly observed phenomenon of a wind inducing violent vibrations in other cylindrical beams. Since then, a number of publications, on the problem of elastic pipes conveying fluid and other related problems, have appeared and the technological significance for such studies has increased considerably.

The problem of stability of elastic pipes conveying fluid, like all problems of the stability of panels in contact with a flowing fluid, is governed by perturbation equations stating the balance between inertial, structural, fluid, and external forces. In general these equations are highly complicated and various simplifications are often necessary. In most of the available publications the undisturbed state of the pipe-fluid system is taken to be steady (although a number of recent papers have considered the effects of a superimposed oscillatory component), and the fluid incompressible and inviscid. Moreover one of the first three of the following combination of structural and fluid theories is employed:

- (i) Linear beam theory with slug flow
- (ii) Linear shell theory with linearized potential flow theory
- (iii) Non-linear beam theory with slug flow
- (iv) Non-linear shell theory with linearized potential flow theory.

The last case is the most difficult and complete but it has not been considered for pipes with an internal flow. A brief review of some of the previous work is now presented.

§1.2 Analysis using linear beam theory

In this category are included all the analyses which are based on an equation of motion, for the deformed pipe, which is derived in the manner of the classical linearized beam equation.

The first work is that of Ashley and Haviland [3] who considered the damping effect of a flowing fluid on the vibrations of simply supported pipes. The governing equation (this refers to the equation of motion excluding the boundary conditions) employed did not adequately allow for the effects of the flowing fluid on the pipe motion. In

particular the effects of the fluid inertia due to the curvature of the pipe, when in motion, was not represented as was demonstrated by later analyses (see, for example, [4]). The main conclusion of [3] is that the flowing fluid has a damping effect on the vibrations of the pipe. This conclusion is not correct and is partly due to the aforementioned inadequacy of the governing equation.

Housner [4] re-examined the problem considered in [3] by deriving a more correct equation of motion using Hamilton's principle and employed a Fourier series solution. The analysis showed that a simply supported undamped pipe develops resonance when the frequency of the disturbed flow coincides with a natural frequency of the pipe and, consequently, amplitudes of unlimited magnitude will occur. Specifically the pipe would buckle, like a beam carrying a lateral load, when the flow speed attains a certain critical value.

Long [5] considered the transverse vibration, for the fundamental mode, of pipes with simply supported, fixed, fixed-simple and fixed-free ends, using the same equation as Housner and a power series solution. He considered values of the flow velocity well below the critical buckling value obtained by Housner and found that in all cases an increase in the flow velocity resulted in a decrease in the frequency. Moreover, in the case of the fixed-free pipe, a decaying vibration resulted.

Benjamin [6] considered a chain of articulated rigid pipes conveying fluid. He observed that the motion of the pipes is independent of fluid friction, and presented an argument to the effect that this characteristic applies for continuously flexible pipes. By considering the stability of infinitesimal lateral motions he found that when both ends of the chain are fixed, the system loses stability by buckling but when the downstream end is free while the upstream end is fixed, the system loses stability

by flutter (oscillatory instability), as the flow velocity increases beyond a certain critical value. He derived an equation of motion similar to that of Housner by using Hamilton's principle and letting the number of pipes in the chain tend to infinity.

Stein and Tobriner [7] considered the stability of infinitely long pipes by employing Fourier-Laplace transforms. They found that, in the absence of external forces and damping, an infinitely long pipe is unstable for all non-zero flow speeds.

The stability of cantilevered pipes has been studied in several publications by Paidoussis and others [8-10]. Gregory and Paidoussis [8] re-established the earlier results of Benjamin [6] that the vibrations of the pipe are independent of fluid friction and that the free cantilever loses stability by flutter. Moreover they found that the second axial mode is the most unstable for all the cases they considered and that the effect of the flow, for velocities below the critical flutter value, is to damp the vibrations (see also [5]). The cantilever pipe with flow has also been studied as a model for other problems (see, for example, [11]), most of these being of the follower-force type.

The publications mentioned so far are concerned with steady flows. The stability of pipes containing pulsatile flow has been studied by Chen [12], Ginsberg [13] and by Paidoussis and Issid [14]. In [12] and [13] a simply supported pipe is studied and it is shown that pulsating flow can cause parametric resonance resembling that of a column subjected to periodic axial loads. The regions of instability are found to increase with the relative magnitude of the oscillatory component of the mean flow as compared with the uniform component: if the amplitude of the oscillatory component is small compared with that of the uniform component then the corrections to the stability conditions

are small and the use of a steady flow analysis is justified in many cases.

In [14] clamped-clamped and cantilever pipes are studied and results similar to those of [12] and [13] are obtained.

The analyses of [6,8,9,10] are accompanied by experimental work [15,16,9,10]. Moreover, some of the above publications (see, for example [10]) include the effects of gravity.

§1.3 Analysis using linear shell theory

For pipes which are not too long and whose walls are sufficiently thin, an analysis employing shell theory is necessary because instabilities associated with circumferential mode numbers, other than the first, may be the most important in determining stability.

Among the earlier workers, Niordson [17] is the first to use a formulation in terms of shell equations. However he considered only the case of one circumferential wave (the beam mode) and employed the long wavelength limit to the shell equation (the beam equation), together with potential flow theory for the fluid pressure. He showed that the governing equation (for the beam mode) reduces to that derived earlier by Housner when only the leading term is retained in the long pipe expansion for the fluid pressure. He studied the case of simply supported ends and found that the natural frequencies decrease with increasing flow velocity and that eventually the pipe loses stability by buckling, in agreement with the earlier predictions.

Paidoussis and Denise [18] studied the stability of clamped-clamped and cantilever pipes using the Flügge shell equations and a travelling wave analysis. They found that clamped-clamped pipes lose stability by buckling while cantilever pipes lose stability by flutter. For the clamped-clamped pipes they also found that flutter follows static

divergence for only a small increment in the flow velocity. The earlier conclusions regarding the effect of flow on the natural frequencies for lateral vibrations are found to extend to vibrations with more circumferential waves (shell modes).

The stability of simply supported and clamped-clamped pipes has been studied by Weaver and Unny [19] and by Weaver and Myklatun [20] respectively, using the Flügge-Kempner shell equation [21] and Galerkin's method. The results of [20] are similar to those of [18] but agreement is only qualitative possibly because of a numerical error in the analysis of [20] and because the travelling wave analysis employed in [18] is not strictly applicable to pipes of short length: these comments will be elaborated on later in this thesis.

Kornecki [22] claims to have proved that cylindrical shells (and two dimensional flat panels) constrained to zero displacement at their ends, and exposed to subsonic flow, lose stability by divergence. This result agrees with all predictions available.

§1.4 Analysis using nonlinear beam theory

Little attention has been paid to this case. Thurman and Mote [23] considered the weakly nonlinear fundamental and second periods of transverse vibration of a simply supported pipe under the assumptions that the pipe remains elastic and that the vibrations are finite but small enough for a perturbation analysis to be valid. They found that the flow reduces all the natural periods of oscillation and increases the relative importance of nonlinear terms in the equation of motion.

As yet no nonlinear stability analysis has been published but, for infinitesimal disturbances, nonlinear effects may be expected to become important only for flow velocities above the critical predicted by the linear theory which, therefore, is sufficient for predicting the stability

boundary. However, since there is no evidence that the linear stability boundary is a lower bound for all possible disturbances, a nonlinear analysis is still desirable especially in cases where there is significant disagreement between experimental and linear theoretical results. In this thesis only infinitesimal disturbances will be considered.

§1.5 On the present work

In this thesis the stability of thin-walled pipes conveying inviscid fluid is considered by using linear shell theory together with linearized potential flow theory. An attempt is made to relate a number of unanswered questions such as the connection between travelling or standing wave results for infinitely long pipes and results for a finite pipe (the interest being in the relevance of the infinite pipe results), and the significance of the environment, or conditions beyond the ends of a finite pipe, on the stability conditions. Particular attention is given to the behaviour of the fluid discharging from the end of a cantilever. The conditions of static divergence and oscillatory instability are examined and the question of whether inplane inertia effects might be important, for a cantilever, is considered. Finally an attempt is made to apply analytical methods which enable results to be presented in clear and compact form where the effects of changing various parameters can be readily estimated.

The layout of the work now follows.

In Chapter Two the governing equations and boundary conditions to be used later are presented.

Chapter Three is concerned with the infinitely long pipes. Expressions for the stability conditions are obtained but detailed discussion is deferred to a later chapter where similar expressions are obtained.

With the exception of the travelling wave analysis on infinitely long pipes considered in Chapter Three, the Galerkin method of solution is used. In Chapter Four expressions for the weighted integral of the fluid pressures (the generalized pressures), which arise in the Galerkin procedure, are derived, in the cases of the finite pipes for later reference.

The stability of simply supported pipes conveying compressible fluid in uniform subsonic motion is considered in Chapter Five, by finding asymptotic forms for the generalized pressures determined in Chapter Four. The analysis is based on the Flügge-Kempner shell equation.

In Chapter Six the analysis of Chapter Five is extended to the case of clamped-clamped pipes of small radius to length ratio. Comparison is made with the earlier results of Weaver and Myklatun [20], and an error in that work is found to be significant.

Chapter Seven deals with the stability of cantilevered pipes conveying incompressible fluid. An attempt to employ the Flügge-Kempner shell equation is abandoned after it is found that the procedure cannot give sufficiently accurate predictions for the natural frequencies of an empty cantilever. This is replaced by a generalization of the Galerkin method which employs Flügge's shell equations. Various approaches (in connection with the flow beyond the free end) to the problem are considered and the results are compared.

The final chapter, eight, gives a brief consideration of boundary layer effects: it is found that a stabilizing effect is a possibility. Some concluding remarks, concerning the main results obtained in the previous chapters, are also presented and topics for further study are suggested.

CHAPTER 2

GENERAL FORMULATION

§2.1 Introduction

In this chapter the basic equations and boundary conditions, describing the stability of thin-walled elastic cylindrical pipes containing inviscid fluid in uniform motion, are given for reference in the later parts of this thesis. Attention is restricted to linear theories for the general case of a system which, in the undisturbed state, consists of a uniform circular cylindrical pipe of radius R , thickness h (which is small compared with R) and length L (which may be infinite), conveying a fluid of density ρ , and being made up of isotropic material of density ρ_m , Young's modulus E and Poisson's ratio ν . The effects of gravity are not included and the fluid is taken to completely fill the pipe. The equations describing the behaviour of such a system, with regard to infinitesimal disturbances, are quite familiar and their derivation is not included in this work.

§2.2 Shell Equations

The disturbed state of the pipe may be defined by a deformation vector (w, v, u) in cylindrical polar coordinates (r, θ, x) , where r and x are the non-dimensional (with respect to L , except for travelling waves on infinitely long pipes where the non-dimensionalization is with respect to R) radial and axial coordinates while θ is the circumferential coordinate. This represents the displacement of points on the middle surface of the pipe wall from its mean position. In the case of a thin-walled pipe the radial displacement w (positive outwards) satisfies a linear equation of the form

$$\mathcal{L}\{u, v, w\} + \rho_m h \frac{\partial^2 w}{\partial t^2} = Q_w, \quad (2.1)$$

where $\mathcal{L}$ is a spatial differential operator and Q_w is the loading in the direction of w . The displacements u and v satisfy equations similar to (2.1).

A number of linear shell theories exist (see, for example, [24]), the difference between them being the form of the operator $\mathcal{L}$ in (2.1) and the related equations. In this work Flügge's version of the theory, which is widely used elsewhere, is employed. However, the use of any other theory would be equivalent to replacing a single factor in the results for the stability condition: this may be expected to introduce only a slight modification.

Flügge's coupled equations, without imposed midplane stresses, have the form

$$\begin{aligned} & \alpha^2 \frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1+\nu}{2} \alpha \frac{\partial^2 v}{\partial x \partial \theta} \\ & + \nu \alpha \frac{\partial w}{\partial x} + \beta \left[\frac{1-\nu}{2} \frac{\partial^2 u}{\partial \theta^2} - \alpha^3 \frac{\partial^3 w}{\partial x^3} \right. \\ & \left. + \frac{1-\nu}{2} \alpha \frac{\partial^3 w}{\partial x \partial \theta^2} \right] - q \frac{\partial^2 u}{\partial t^2} = 0, \\ & \frac{1+\nu}{2} \alpha \frac{\partial^2 u}{\partial x \partial \theta} + \frac{\partial^2 v}{\partial \theta^2} + \frac{1-\nu}{2} \alpha^2 \frac{\partial^2 v}{\partial x^2} \\ & + \frac{\partial w}{\partial \theta} + \beta \left[\frac{3}{2} (1-\nu) \alpha^2 \frac{\partial^2 v}{\partial x^2} \right. \\ & \left. - \frac{3-\nu}{2} \alpha^2 \frac{\partial^3 w}{\partial x^2 \partial \theta} \right] - q \frac{\partial^2 v}{\partial t^2} = 0, \end{aligned}$$

$$\begin{aligned}
& v\alpha \frac{\partial u}{\partial x} + \frac{\partial v}{\partial \theta} + w + \beta \left[\frac{1-v}{2} \alpha \frac{\partial^3 u}{\partial x \partial \theta^2} \right. \\
& - \alpha^3 \frac{\partial^3 u}{\partial x^3} - \frac{3-v}{2} \alpha^2 \frac{\partial^3 v}{\partial x^2 \partial \theta} + \nabla_1^4 w \\
& \left. + 2 \frac{\partial^2 w}{\partial \theta^2} + w \right] + q \frac{\partial^2 w}{\partial t^2} = \frac{R^2 (1-v^2)}{Eh} p
\end{aligned} \tag{2.2}$$

where

$$\nabla_1^4 = \alpha^4 \frac{\partial^4}{\partial x^4} + 2\alpha^2 \frac{\partial^4}{\partial x^2 \partial \theta^2} + \frac{\partial^4}{\partial \theta^4},$$

$$q = \rho_m R^2 (1-v^2)/E,$$

$$\beta = \frac{h^2}{12R^2},$$

$$\alpha = R/L \quad (\text{or } 1 \text{ for travelling waves since the}$$

non-dimensionalization is then with respect to R),

and p is the radial surface loading per unit area of the middle surface (same as the perturbation pressure). When the inplane inertia terms are neglected, equations (2.2) can be decoupled to give a single equation for w (the Flügge-Kempner equation) together with expressions giving u and v in terms of w [21]. In the case $\beta \ll 1$ the Flügge-Kempner equation takes the form

$$\frac{D}{R} \mathcal{L}\{w\} + \rho_m h \frac{\partial^2 w}{\partial t^2} = p, \tag{2.3}$$

where

$$\begin{aligned}
 \mathcal{L}\{w\} = & \nabla_1^4 w + \nabla_1^{-4} [12(1-\nu^2) \left(\frac{R}{h}\right)^2 \alpha^4 \frac{\partial^4 w}{\partial x^4} \\
 & + 2(2-\nu) \alpha^2 \frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{\partial^4 w}{\partial \theta^4} \\
 & + 2\nu \alpha^6 \frac{\partial^6 w}{\partial x^6} + 6\alpha^4 \frac{\partial^6 w}{\partial x^4 \partial \theta^2} \\
 & + 2(4-\nu) \alpha^2 \frac{\partial^6 w}{\partial x^2 \partial \theta^4} + 2 \frac{\partial^6 w}{\partial \theta^6}] .
 \end{aligned}$$

A term $\kappa \frac{\partial w}{\partial t}$ may be added on the left hand side of (2.3) to give an estimate of structural damping effects.

Equations (2.2) or (2.3) can be solved once the boundary conditions on the displacements and the radial loading, p , are known. To use (2.3) the neglect of inplane inertia terms must be justified.

§2.3 Displacement boundary conditions

In this work, pipes with simply supported, clamped-clamped, and clamped-free, ends are considered. Thus the boundary conditions of interest are:

at a simply supported end,

$$v = w = \frac{\partial u}{\partial x} = \frac{\partial^2 w}{\partial x^2} = 0; \quad (2.4)$$

at a clamped end,

$$u = v = w = \frac{\partial w}{\partial x} = 0; \quad (2.5)$$

at a free end,

$$\alpha \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial \theta} + vw - \beta \alpha^2 \frac{\partial^2 w}{\partial x^2} = 0,$$

$$\frac{\partial u}{\partial \theta} + (1 + 3\beta) \alpha \frac{\partial v}{\partial x} - 3\beta \alpha \frac{\partial^2 w}{\partial x \partial \theta} = 0,$$

$$\alpha^3 \frac{\partial^3 w}{\partial x^3} + (2-v) \alpha \frac{\partial^3 w}{\partial x \partial \theta^2} - \frac{3-v}{2} \alpha \frac{\partial^2 v}{\partial x \partial \theta}$$

$$+ \frac{1-v}{2} \frac{\partial^2 u}{\partial \theta^2} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

$$\alpha \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial \theta} - v \frac{\partial^2 w}{\partial \theta^2} - \alpha^2 \frac{\partial^2 w}{\partial x^2} = 0. \quad (2.6)$$

Reference [25] may be consulted for the derivation and physical significance of these conditions.

§2.4 The fluid pressure

The undisturbed flow is taken to be homentropic, with a constant velocity (0,0,U). Thus the disturbed flow is irrotational and may be characterized by a perturbation velocity potential ϕ which satisfies the linearized convected wave equation (see, for example, [26])

$$\nabla^2 \phi = \frac{1}{c^2} \frac{D^2 \phi}{Dt^2}, \quad (2.7)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{U}{L} \frac{\partial}{\partial x}$$

denotes convective differentiation, c is the speed of sound in the fluid, and ∇^2 is the Laplacian operator (L must be replaced by R when considering travelling waves on infinitely long pipes).

The perturbation pressure is related to the velocity potential through the linearized unsteady Bernoulli equation [26]:

$$p = -\rho \frac{D\phi}{Dt}. \quad (2.8)$$

In general, a pipe conveying fluid is surrounded by a stationary fluid and the loading on the pipe wall is the difference between the internal and external pressures as given by (2.8), with $U = 0$ for the external fluid. Thus the external fluid reduces the inertia effect of the internal fluid as well as allowing a certain amount of energy to be radiated away from the pipe when the fluid is compressible. However, if the external fluid density is small compared with the internal fluid density, or if the time derivative in (2.8) is zero, the behaviour of the external fluid may be neglected. This is taken to be the case in this thesis.

Equations (2.7) and (2.8) together with appropriate boundary conditions are sufficient for the determination of p .

§2.5 Boundary conditions

The pipe wall is taken to be impermeable to the fluid. Thus the boundary conditions for the flow include the condition that ϕ is bounded together with the kinematic condition that the fluid must remain in contact with the pipe wall at all times. If the equation of the pipe wall, S , is given by $f(r, \theta, x, t) = 0$, the latter condition is equivalent to

$$L \frac{\partial f}{\partial t} + \left(\frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, LU + \frac{\partial \phi}{\partial x} \right) \cdot \nabla f = 0 \text{ on } S.$$

In the present case, $f = \alpha - r + w/L$ so that, on neglecting non-linear terms in the perturbation variables,

$$\frac{\partial w}{\partial t} - \frac{1}{L} \left(\frac{\partial \phi}{\partial r} - U \frac{\partial w}{\partial x} \right) = 0 \quad \text{on } r = \alpha,$$

and the required boundary condition follows:

$$\frac{1}{L} \frac{\partial \phi}{\partial r} = \frac{Dw}{Dt}. \quad (2.9)$$

§2.6 Outline of the solution procedure

A harmonic time dependence, in the form $\exp(-i\omega t)$, is introduced for the perturbation variables and a characteristic equation for the frequency, ω , which is in general complex ($\omega = \omega_r + i\omega_i$), is determined. In the linear theory unsteady oscillations will grow or decay with time depending on whether ω_i is positive or negative, the case $\omega_i = 0$ corresponding to sustained oscillations of constant-amplitude. Thus $\omega_i > 0$ corresponds to instability, this being of oscillatory (flutter) type if in addition $\omega_r \neq 0$; otherwise it is of divergence type.

The Galerkin method (or its generalization) is employed throughout except for the analysis of travelling waves on infinitely long pipes presented in §3.2. In the application of the Galerkin method a modal solution for the radial deformation w is introduced in the form

$$w = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} C_{nk} e^{-i\omega t} W_n(x) \cos k\theta, \quad (2.10)$$

where the C_{nk} 's are unknown generalized coordinates and the coordinate functions $W_n(x) \cos k\theta$ satisfy all the displacement boundary conditions for w . The perturbation pressure, p , is then determined from (2.7) to (2.9):

$$p = \frac{1}{L} \sum_{n,k} C_{nk} e^{-i\omega t} P_{nk}(x, \theta). \quad (2.11)$$

For general descriptive purposes, equation (2.3) may be used with (2.10) and (2.11) to give

$$\begin{aligned} \sum_{n,k} C_{nk} e^{-i\omega t} \left[\frac{D}{3R} \mathcal{L}\{W_n(x) \cos k\theta\} \right. \\ \left. - R \rho_m h \omega^2 W_n(x) \cos k\theta - \alpha P_{nk}(x, \theta) \right] = 0 \end{aligned} \quad (2.12)$$

Equation (2.12) is then multiplied by $F_m(x) \cos \ell\theta$ and integrated over the surface, S , of the flexible pipe. The weighting functions $F_m(x)$ are to be specified: they correspond to the part of $W_m(x)$ associated with the flexible pipe. This gives

$$\begin{aligned} \sum_{n,k} C_{nk} e^{-i\omega t} \left[\int_S \left(\frac{D}{3R} \mathcal{L}\{W_n(x) \cos k\theta\} \right. \right. \\ \left. \left. - R \rho_m h \omega^2 W_n(x) \cos k\theta \right) F_m(x) \cos \ell\theta \, dS \right. \\ \left. - \alpha P_{mn,kl} \right] = 0, \end{aligned} \quad (2.13)$$

$$\ell = 0, 1, 2, \dots,$$

$$m = 1, 2, 3, \dots,$$

where the $P_{mn,kl}$ are the generalized pressures defined by

$$P_{mn,kl} = \iint_S P_{nk}(x, \theta) F_m(x) \cos l\theta \, dS. \quad (2.14)$$

The required equation for ω follows from the requirement of a non-trivial solution for the C_{nk} 's.

The generalized Galerkin method involves some modification of this procedure but its discussion is postponed until §7.3.

CHAPTER 3

INFINITELY LONG PIPES

§3.1 Introduction

In this chapter, travelling waves, and standing waves on pipes with periodic supports, are considered. The analysis, considering both beam and shell modes, does not appear to have been published before although several authors (see, for example, [27,28]) have included an internal fluid (whose velocity is later set to zero) in the mathematical formulation of the external flow problem. In connection with the pipe without supports Stein and Tobriner [7] have employed beam theory together with Fourier-Laplace transforms: instability is predicted for all flow velocities but this conclusion has practical limitations as will be indicated in this chapter.

The analysis of infinitely long pipes is of interest in this work partly as an introduction to the study of finite pipes (Chapters 4 to 6) and partly because a direct connection can be made between the results here and the results for long but finite pipes. The travelling wave analysis is presented in §3.2 while the standing wave analysis is presented in §3.3.

The Flügge-Kempner equation (2.3) is employed in this and the next three chapters. However, in the case of travelling waves, the coupled Flügge's equations (2.2) can easily be employed also. The frequency equations obtained from (2.3) and (2.2) can be shown to be identical when $O(\beta)$ terms are neglected.

§3.2 Travelling waves

The travelling wave solution of equation (2.7) is taken in the form

$$\phi = \phi(r) e^{i(k\theta + \lambda x - \omega t)}, \quad (3.1)$$

where λ is the axial wavenumber. Upon substitution in (2.7) the function ϕ is found to satisfy the modified Bessel equation:

$$\frac{d^2\phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} - \left(\frac{k^2}{r^2} + \Lambda^2\right)\phi = 0, \quad (3.2)$$

where

$$\Lambda^2 = \lambda^2 - \left(\frac{\omega R}{c} - M\lambda\right)^2,$$

M being the flow Mach number. The general solution of (3.2) is

$$\phi(r) = A(\lambda) I_k(r\Lambda) + B(\lambda) K_k(r\Lambda), \quad (3.3)$$

where A and B are arbitrary while $I_k(r\Lambda)$ and $K_k(r\Lambda)$ are the modified Bessel functions of the first and second kind respectively. Since $K_k(r\Lambda)$ is unbounded at $r = 0$, B must be zero.

In view of boundary condition (2.9) and the travelling wave form (3.1), the pipe radial deflection, w , is taken in the form

$$w = C_{\lambda k} e^{i(k\theta + \lambda x - \omega t)}, \quad (3.4)$$

where $C_{\lambda k}$ is an unknown amplitude. Equations (2.9), (3.1), (3.3) and (3.4) give

$$\phi(r) = iC_{\lambda k} (\omega R - U\lambda) \frac{I_k(r\lambda)}{\Lambda I_k'(\Lambda)} . \quad (3.5)$$

The perturbation pressure on the pipe wall may now be obtained from (2.8), (3.1), and (3.5):

$$p = -\frac{\rho}{R} (\omega R - U\lambda)^2 \omega g_k(\Lambda) , \quad (3.6)$$

where

$$g_k(\Lambda) = \frac{I_k(\Lambda)}{\Lambda I_k'(\Lambda)} .$$

When (3.6) is substituted in (2.3) with damping and (3.4) is employed, a characteristic equation for the frequency ω is obtained:

$$R^2 (\omega^2 - \omega_0^2) + \frac{iR^2 \kappa}{\rho_m h} \omega + \mu (\omega R - U\lambda)^2 g_k(\Lambda) = 0 , \quad (3.7)$$

where $\mu = \frac{R\rho}{h\rho_m}$ is the mass ratio, κ is the structural damping coefficient and ω_0 is the free wave frequency given by

$$\omega_0^2 = \frac{D}{\rho_m h R^4} F(\lambda, k) , \quad (3.8)$$

with

$$\begin{aligned} F(\lambda, k) = & (\lambda^2 + k^2)^2 + \frac{1}{(\lambda^2 + k^2)^2} [12(1-\nu^2) \frac{R^2}{h^2} \lambda^4 \\ & + 2(2-\nu)\lambda^2 k^2 + k^4 - 2\nu\lambda^6 - 6\lambda^4 k^2 \\ & - 2(4-\nu)\lambda^2 k^4 - 2k^6] . \end{aligned} \quad (3.9)$$

Since equation (3.7) is transcendental in ω , an explicit solution is not possible in general. An expression suitable for an iterative procedure may be obtained by regarding g_k as known and solving the resulting quadratic:

$$\omega = \left[\mu U \lambda g_k - \frac{i R \kappa}{2 \rho_m h} \pm \sqrt{\omega_0^2 R^2 (1 + \mu g_k) - \mu U^2 \lambda^2 g_k - \frac{R \kappa}{\rho_m h} \left(-\frac{R \kappa}{4 \rho_m h} + i \mu U \lambda g_k \right)} \right] / [(1 + \mu g_k) R]. \quad (3.10)$$

In the case of incompressible flow, equation (3.10) gives ω explicitly and the stability condition can be determined by considering the expression under the radical. Thus when damping is present ($\kappa > 0$), instability corresponds to the case when the absolute value of the imaginary part of the square root is greater than $R\kappa/(2\rho_m h)$. This is the case when $\mu U^2 \lambda^2 g_k > \omega_0^2 R^2$ so that the stability boundary is defined by $\mu U^2 \lambda^2 g_k = \omega_0^2 R^2$. That value of ω which corresponds to instability is then seen to be zero at the point of instability so that the stability condition is a divergence condition. But for a divergence condition, the critical condition can be obtained, without restricting consideration to incompressible flows, by setting $\omega = 0$ in (3.7). Thus the stability condition for compressible flows, when damping is present, is $\mu U^2 \lambda^2 g_k = \omega_0^2 R^2$, $\omega = 0$, or, on using (3.8), and g_k as defined after (3.6) with $\omega = 0$,

$$\frac{\rho U_{cd}^2 R^3}{D} = \frac{(1-M^2)^{1/2} F(\lambda, k) I_k'(\{1-M^2\}^{1/2} \lambda)}{\lambda I_k(\{1-M^2\}^{1/2} \lambda)}, \quad (3.11)$$

where the subscripts cd denote critical divergence condition and M is the flow Mach number.

In the case of incompressible flow with no damping ($\kappa = 0$) the stability condition corresponds to the vanishing of the quantity under the radical in (3.10) and is given by

$$\frac{\rho U_{cf}^2 R^3}{D} = \{1 + \mu g_k(\lambda)\} F(\lambda, k) / \{\lambda^2 g_k(\lambda)\},$$

$$\omega_{cf} = \frac{\mu U_{cf} \lambda g_k(\lambda)}{\{1 + \mu g_k(\lambda)\} R}, \quad (3.12)$$

where the subscripts cf denote critical flutter condition. The expression for ω_{cf} shows that the critical wavespeed of the disturbance, $\omega_{cf} R/\lambda$, is less than the critical flow speed and approaches it asymptotically as $\mu \rightarrow \infty$. A similar result has been obtained previously for the external flow and the infinite flat plate problems [29]. The expression for $\rho U_{cf}^2 R^3/D$ must be minimized numerically with respect to both λ and k in order to obtain the most critical mode.

It can be seen from (3.12) that in the limit $\mu \rightarrow 0$, $\omega_{cf} \rightarrow 0$ and the stability condition becomes the divergence condition of (3.11). Moreover, since g_k is positive for incompressible fluids, the divergence condition is a lower bound to the stability condition (3.12) for any μ and, in that sense, the divergence condition may be regarded as supplying a conservative estimate of the stability boundary for non-zero μ .

Now

$$F(\lambda) \sim \left\{ \begin{array}{ll} \lambda^4, & \lambda \gg 1, \\ (1 - v^2)/\beta, & k = 0 \\ (1 - v^2)\lambda^4/\beta, & k = 1 \\ (k^2 - 1)^2, & k > 1 \end{array} \right\}, \quad \lambda \ll 1,$$

$$g_k(\lambda) \sim \left\{ \begin{array}{ll} 1/\lambda, & \lambda \gg 1 \\ 2/\lambda^2, & k = 0 \\ 1/k, & k \geq 1 \end{array} \right\}, \quad \lambda \ll 1,$$

and, on using (3.11),

$$\frac{\rho U_{cd}^2 R^3}{D} \sim \left\{ \begin{array}{ll} (1 - v^2)/2\beta & k = 0 \\ (1 - v^2)\lambda^2/\beta, & k = 1 \\ k(k^2 - 1)^2/\lambda^2, & k > 1 \\ \lambda^3, & \lambda \gg 1. \end{array} \right\}, \quad \lambda \ll 1, \quad (3.13)$$

Expressions (3.13) show that the beam mode ($k = 1$) is unstable to long wavelengths ($\lambda \ll 1$) as soon as there is any flow. This result has been obtained before using beam theory [7] but its significance is limited by the fact that, in practice, the wavelength is bounded by the finiteness of the system. When $k \neq 1$, the expressions also show that a minimum for $\rho U_{cd}^2 R^3/D$ occurs for a finite value of λ . The determination of the most critical mode for $k \neq 1$, by finding the minimum value of $\rho U_{cd}^2 R^3/D$ with respect to both k and λ , is considered in §5.4 where a result similar to (3.11) is obtained for finite but long pipes.

The analysis of this section predicts that all shell mode instabilities are preceded by the beam mode instability which occurs for infinite wavelengths as soon as there is any fluid motion. Such a prediction is likely to be irrelevant for finite pipes and indicates the necessity of introducing boundaries or supports to obtain results of practical significance.

§3.3 Standing waves on pipes with periodic supports

A straightforward model, with considerable practical significance, restricting long wavelength behaviour of the pipe is supplied by introducing periodic supports. With L denoting the distance between consecutive supports, the whole pipe may be regarded as an infinite array of identical pipes connected together at evenly spaced support positions. In this section only simply supported connections are considered.

As indicated earlier, the solution procedure employs the classical Galerkin method. By taking the origin at a support position and using sine functions for the radial deflection, w , the simple support boundary conditions (2.4) are automatically satisfied at each support. Thus the modal solution takes the form

$$w = \sum_{n,k} C_{nk} e^{-i\omega t} \sin n\pi x \cos k\theta, \quad (3.14)$$

where the C_{nk} 's are unknown generalized coordinates. The perturbation velocity potential consistent with (2.9) and (3.14) may be expressed as a Fourier series in x :

$$\phi = \sum_{n,k} e^{-i\omega t} \{X_n(r) \sin n\pi x + Y_n(r) \cos n\pi x\} \cos k\theta. \quad (3.15)$$

The result of substituting (3.15) in the convected wave equation (2.7) is a system of differential equations of the form

$$\frac{d^2 \psi_{n\pm}}{dr^2} + \frac{1}{r} \frac{d\psi_{n\pm}}{dr} - \left(\frac{k^2}{r^2} + \Lambda_{n\pm}^2 \right) \psi_{n\pm} = 0,$$

$$n = 1, 2, 3, \dots, \quad (3.16)$$

where

$$\psi_{n\pm} = Y_n \pm iX_n$$

$$\Lambda_{n\pm}^2 = n^2 \pi^2 - (Mn\pi \pm \omega L/c)^2 \quad (3.17)$$

Since (3.16) and (3.2) are identical, the solution for $\psi_{n\pm}$ is

$$\psi_{n\pm} = A_{\pm} I_k(\Lambda_{n\pm} r), \quad (3.18)$$

where the constants $A_{\pm}$ are to be determined. Expressions (3.17) and (3.18) obtain

$$Y_n(r) = \frac{1}{2} \{A_+ I_k(\Lambda_{n+} r) + A_- I_k(\Lambda_{n-} r)\}$$

$$X_n(r) = \frac{1}{2i} \{A_+ I_k(\Lambda_{n+} r) - A_- I_k(\Lambda_{n-} r)\}, \quad (3.19)$$

and on using (3.14), (3.15) together with boundary condition (2.9),

$$A_{\pm} = \frac{n\pi U \pm \omega L}{\Lambda_{n\pm} I_k'(\Lambda_{n\pm} \alpha)} C_{nk} \quad (3.20)$$

The perturbation velocity potential is now completely specified by (3.15), (3.19) and (3.20). The perturbation pressure is obtained from (2.8) and (3.15): it can be expressed in the form (2.11) with

$$\begin{aligned}
 P_{nk} = \frac{\rho}{L} [& \{ i\omega L X_n(\alpha) + Un\pi Y_n(\alpha) \} \sin n\pi x \\
 & + \{ i\omega L Y_n(\alpha) - Un\pi X_n(\alpha) \} \cos n\pi x] \cos k\theta.
 \end{aligned} \tag{3.21}$$

Expression (3.21) may be substituted in (2.14) and the generalized pressures determined using $F_m(x) = \sin m\pi x$ and integrating over one bay of the pipe. Thus

$$\begin{aligned}
 P_{mn,kl} &= 0, \quad k \neq l \\
 P_{mn,kk} &= m\pi i \rho \epsilon_k \{ \omega L Y_n(\alpha) \\
 &\quad + iUn\pi X_n(\alpha) \} B_{mn}, \\
 P_{nn,kk} &= \frac{\pi}{2} \rho \epsilon_k \{ i\omega L X_n(\alpha) + Un\pi Y_n(\alpha) \},
 \end{aligned} \tag{3.22}$$

where

$$\begin{aligned}
 B_{mn} &= \frac{1 - (-)^{m+n}}{(m^2 - n^2)\pi}, \\
 \epsilon_k &= \begin{cases} 2, & k = 0, \\ 1, & k > 0. \end{cases}
 \end{aligned}$$

The characteristic equation for ω is obtained by evaluating (2.13), using (3.22) and $W_n(x) = \sin n\pi x$, and requiring a non-trivial solution for the C_{nk} 's. Thus, from (2.13),

$$\begin{aligned}
\sum_n \left[\left\{ \frac{LR}{\mu} (\omega^2 - \omega_{Om}^2) C_{mk} + i\omega L X_m(\alpha) \right. \right. \\
\left. \left. + U m \pi Y_m(\alpha) \right\} \delta_{mn} + 2i m \{ \omega L Y_n(\alpha) \right. \\
\left. \left. + i U n \pi X_n(\alpha) \right\} B_{mn} \right] = 0, \\
m = 1, 2, 3, \dots,
\end{aligned} \tag{3.23}$$

where ω_{Om} represents the free wave frequency for the m^{th} mode and is given by

$$\omega_{Om}^2 = \frac{D}{\rho_m h R^4} F(m\pi\alpha, k), \tag{3.24}$$

with F defined by (3.9). Equations (3.23) constitute an infinite system of homogeneous linear equations for the C_{nk} 's. For a non-trivial solution the determinant of the coefficients must be zero. Thus

$$\det(a_{mn}) = 0, \tag{3.25}$$

where

$$\begin{aligned}
a_{mm} = & \omega^2 L^2 \left[\frac{1}{\mu} + \frac{1}{2} \{ g_k(\Lambda_{n+}\alpha) + g_k(\Lambda_{n-}\alpha) \} \right] \\
& + \omega L U n \pi \{ g_k(\Lambda_{n+}\alpha) - g_k(\Lambda_{n-}\alpha) \} \\
& + \frac{1}{2} U^2 n^2 \pi^2 \{ g_k(\Lambda_{n+}\alpha) + g_k(\Lambda_{n-}\alpha) \} \\
& - \frac{1}{\mu} \omega_{On}^2 L^2,
\end{aligned}$$

$$\begin{aligned}
a_{mn} = & \text{im}[(\omega_{Ln}^2 + U_n^2 \pi^2) \{g_k(\Lambda_{n+} \alpha) - g_k(\Lambda_{n-} \alpha)\} \\
& + 2Un\pi \omega L \{g_k(\Lambda_{n+} \alpha) + g_k(\Lambda_{n-} \alpha)\}] B_{mn},
\end{aligned}$$

with ω_{On}^2 given by (3.24) and g_k as defined after (3.6).

The general procedure consists of solving (3.25) truncated to, say, order N , which corresponds to terminating the modal series (3.14) at N terms. Convergence is checked by increasing N and observing the change in the smaller eigenvalues ω . However in the limit $\mu \rightarrow 0$, $\omega = 0$ corresponds to a divergence stability boundary: the off-diagonal terms in the characteristic determinant (3.25) vanish and the solution, without truncation, is

$$\frac{\rho U_{cd}^2 R^3}{D} = \frac{(1 - M^2)^{1/2} F(n\pi\alpha, k) I_k'(bn\pi)}{n\pi\alpha I_k(bn\pi)}, \quad (3.26)$$

where $b = (1 - M^2)^{1/2} \alpha$. The result is the same as (3.11) with λ replaced by an effective wavenumber $n\pi\alpha$. For supports which are a large distance apart, α is small but non-zero and, as n takes integer values, $n\pi\alpha$ is seen to vary almost continuously. Thus, apart from the upper limit of wavelength now imposed, the discussion of the preceding section is relevant here and further remarks are again set aside until §5.4 where a similar result is obtained for the finite-length simply supported pipe.

CHAPTER 4

FINITE LENGTH PIPES: GENERALIZED PRESSURES

§4.1 Introduction

In this chapter expressions for the generalized pressures are obtained for reference later in the analysis of simply supported, clamped-clamped and clamped-free (cantilever) pipes. The mathematical formulations employed to obtain these expressions are suggested by the fact that if the fluid problem is considered in general with the outer boundary conditions (for example the tube wall displacement) specified everywhere, then the solution (satisfying the finiteness condition on the axis) for the perturbation pressure may be determined from the governing equations. Typically, in the analysis of flexible boundary-fluid interaction problems, the displacement of the boundary is specified apart from arbitrary constants which can be determined when the boundary-fluid interaction is specified (for example by the dynamic equation for the boundary). For a finite pipe, the wall displacements beyond the ends are needed. In the former chapter (§3.3) these are specified through periodicity conditions. The formulation used here considers each end of the simply supported and the clamped-clamped pipes to be connected to a semi-infinite rigid pipe of the same radius so that the whole pipe becomes infinitely long.

In the case of the cantilever, the flow beyond the free end, when the pipe is surrounded by fluid (as is usually the case), is yet another interaction problem. The fluid-fluid interaction problem is well known: the boundary between the flowing and the stationary fluid is a vortex sheet which is unstable [30-32]. The experimentally observed instability is a spatial growth downstream [31]. In considering the cantilever-fluid

interaction problem it is mathematically convenient to avoid this downstream instability: if the external fluid is comparatively of negligible density the two interactions may be considered well decoupled. The effect on the stability of the cantilever may be expected to be small because physically the flow conditions well beyond the free end cannot have a strong influence on the dynamics of the cantilever.

Paidoussis and Denise [18] avoided consideration of the vortex sheet instability by employing a travelling wave solution, thus implicitly considering the cantilever as part of an infinitely long flexible pipe. Their solution is more appropriate for long cantilevers since the effect of the details of the flow beyond the ends may then be expected to be small. In the present method of solution any instability beyond the end of the pipe is suppressed by introducing appropriate stable models of the downstream flow, while the upstream flow is modelled by considering the clamped end to be connected to a semi-infinite rigid pipe. It may be expected that these models will generate the same stability conditions for long pipes but that differences may emerge as the pipe becomes shorter. There appears to be no way of distinguishing the relevance of these models other than by reference to a particular experiment or the exact solution of the problem (which does not exist).

In addition to the models mentioned in the previous paragraph, two other approximations are considered. The first is an asymptotic approximation for long pipes while the second (cantilever continuation) is a variation of the travelling wave analysis of [18]. It is expected that a comparison of the stability results corresponding to the various approximations for the generalized pressures will indicate the significance of the details of the flow beyond the ends of the cantilever.

The layout of the rest of this chapter is in five sections. In §4.2 general expressions for the generalized pressures are obtained. The forms taken by these expressions for the support conditions of interest in this work are presented in §4.3 to §4.5. In §4.6 the expressions corresponding to the cantilever continuation model are obtained.

§4.2 General expression for the generalized pressures

In this section a solution for the radial deflection, w , in $-\infty < x < \infty$ is used together with Fourier transform theory to determine the generalized pressures $P_{mn,k\ell}$. The transform pair is defined by

$$\begin{aligned} f^*(\lambda) &= \int_{-\infty}^{\infty} e^{i\lambda x} f(x) dx, \\ f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} f^*(\lambda) d\lambda, \end{aligned} \quad (4.1)$$

where $f^*(\lambda)$ is the Fourier transform of $f(x)$. Thus the transform of the convected wave equation (2.7) is

$$\frac{d^2 \phi^*}{dr^2} + \frac{1}{r} \frac{d\phi^*}{dr} - \left(\frac{k^2}{r^2} + \Lambda^2 \right) \phi^* = 0, \quad (4.2)$$

where

$$\Lambda^2 = \lambda^2 - \left(\frac{\omega L}{c} + M\lambda \right)^2.$$

Since (4.2) and (3.2) are identical, the solution is obtained from (3.3) with $B = 0$:

$$\phi^* = A(\lambda) I_k(r\Lambda) \cos k\theta. \quad (4.3)$$

To determine $A(\lambda)$ the transform of boundary condition (2.9) is employed:

$$\left(\frac{\partial \phi^*}{\partial r}\right)_{r=\alpha} = -i(\omega L + \lambda U)w^*(\lambda) . \quad (4.4)$$

Thus $A(\lambda)$ may be eliminated from (4.3), using (4.4), to give

$$\phi^* = -i(\omega L + \lambda U)w^*(\lambda) \frac{I_k(r\lambda)}{\lambda I_k'(\alpha\lambda)} ,$$

and, upon inversion,

$$\phi = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-i\lambda x} (\omega L + \lambda U)w^*(\lambda) \frac{I_k(r\lambda)}{\lambda I_k'(\alpha\lambda)} d\lambda . \quad (4.5)$$

The perturbation pressure is obtained from the Bernoulli equation, (2.8), which with (4.5), gives

$$p = \frac{\rho}{2\pi L} \int_{-\infty}^{\infty} e^{-i\lambda x} (\omega L + \lambda U)^2 w^*(\lambda) \frac{I_k(r\lambda)}{\lambda I_k'(\alpha\lambda)} d\lambda . \quad (4.6)$$

It is assumed that the properties of $w^*(\lambda)$ are such that the differentiation under the integral, performed to obtain (4.6), is valid.

In applying the Galerkin method the radial deflection w is taken in the form (2.10) with

$$W_n(x) = \begin{cases} 0, & x < 0, \\ F_n(x), & 0 \leq x \leq 1, \\ G_n(x), & x > 1. \end{cases} \quad (4.7)$$

The functions F_n and G_n will be specified later. Thus the transform of w is

$$w^*(\lambda) = \sum_{n,k} C_{nk} [F_n^*(\lambda) + G_n^*(\lambda)] \cos k\theta, \quad (4.8)$$

where

$$\begin{aligned} F_n^*(\lambda) &= \int_0^1 e^{i\lambda x} F_n(x) dx, \\ G_n^*(\lambda) &= \int_1^\infty e^{i\lambda x} G_n(x) dx. \end{aligned} \quad (4.9)$$

When (4.8) is inserted in (4.6) the pressure on the pipe wall may be expressed in the form (2.11) with

$$\begin{aligned} P_{nk}(x, \theta) &= \frac{\alpha \rho}{2\pi} \cos k\theta \int_{-\infty}^{\infty} e^{-i\lambda x} (\omega L + \lambda U)^2 [F_n^*(\lambda) \\ &\quad + G_n^*(\lambda)] g_k(\alpha \lambda) d\lambda, \end{aligned} \quad (4.10)$$

where g_k is defined after (3.6). The required expression for the generalized pressures is obtained by performing the integration specified in (2.14) after substituting for P_{nk} from (4.10) and interchanging the order of integration with respect to x and λ . This gives

$$\begin{aligned} P_{mn, kk} &= \frac{\alpha}{2} \rho \epsilon_k \int_{-\infty}^{\infty} (\omega L + \lambda U)^2 F_m^*(-\lambda) [F_n^*(\lambda) \\ &\quad + G_n^*(\lambda)] g_k(\alpha \lambda) d\lambda, \\ P_{mn, k\ell} &= 0, \quad k \neq \ell. \end{aligned} \quad (4.11)$$

The forms taken by (4.11) will now be considered in §4.3 to §4.5 for the support conditions of interest in this work.

§4.3 Pipes with simply supported ends

Following the discussion in §4.1 and in order to satisfy the simply supported boundary conditions (2.4) with a rigid pipe downstream, the following forms are taken:

$$F_n(x) = \sin n\pi x, \quad (4.12)$$

$$G_n(x) = 0.$$

From (4.9) and (4.12),

$$F_n^*(\lambda) = \frac{n\pi\{1 - (-)^n e^{i\lambda}\}}{n^2\pi^2 - \lambda^2},$$

$$G_n^*(\lambda) = 0. \quad (4.13)$$

Expression (4.11), after substituting for F_n^* and G_n^* from (4.13), becomes

$$P_{mn,kk} = \frac{\alpha}{2} mn\pi^2 \rho \epsilon_k \int_{-\infty}^{\infty} \frac{(\omega L + \lambda U)^2}{(m^2\pi^2 - \lambda^2)(n^2\pi^2 - \lambda^2)} \{ [1 + (-)^{m+n}\{1 - (-)^m \cos \lambda\} + \{(-)^m - (-)^n\} i \sin \lambda] g_k(\alpha\lambda) d\lambda. \quad (4.14)$$

When either $\omega = 0$ or the fluid is incompressible, (4.14) becomes a semi-infinite integral.

§4.4 Pipes with clamped ends

The form for w satisfying the clamped boundary conditions (2.5) at both ends of the pipe, with a rigid pipe downstream, is taken as (4.7) with

$$\begin{aligned} F_n(x) &= 1 - \cos 2n\pi x, \\ G_n(x) &= 0. \end{aligned} \tag{4.15}$$

The functions $F_n(x)$ in (4.15) are not strictly appropriate since they can only approximate the symmetric modes (about $x = 1/2$). However, antisymmetric modes in fact play restricted roles in determining the stability condition (as will be demonstrated later). A form for $F_n(x)$ satisfying the clamped boundary conditions, and representing both symmetric and antisymmetric modes, is

$$F_n(x) = \cos (n+1)\pi x - \cos (n-1)\pi x. \tag{4.16}$$

With (4.15), the transforms (4.9) give

$$F_n^*(\lambda) = \frac{4n^2 \pi^2 i (1 - e^{i\lambda})}{\lambda (4n^2 \pi^2 - \lambda^2)}, \tag{4.17}$$

$$G_n^*(\lambda) = 0,$$

while, with (4.16),

$$F_n^*(\lambda) = \frac{4n\pi^2 i \lambda \{1 + (-)^n e^{i\lambda}\}}{\{(n+1)^2 \pi^2 - \lambda^2\} \{(n-1)^2 \pi^2 - \lambda^2\}}. \tag{4.18}$$

Expressions (4.11) and (4.17) give

$$P_{mn,kk} = 16m^2 n^2 \pi^4 \alpha \rho \epsilon_k \int_{-\infty}^{\infty} g_k(\alpha \Lambda) (\omega L + \lambda U)^2 \frac{1 - \cos \lambda}{\lambda^2 (4m^2 \pi^2 - \lambda^2) (4n^2 \pi^2 - \lambda^2)} d\lambda, \quad (4.19)$$

which becomes a semi-infinite integral when either $\omega = 0$ or the fluid is incompressible. A result similar to (4.19) may be obtained using (4.18) instead of (4.17).

§4.5 Pipes with clamped-free ends: flow models downstream

4.5.1 Introduction

In this section the forms taken by (4.11), when the flow is incompressible ($M = 0$), are presented for two models of the flow downstream. An asymptotic approximation for long pipes is also considered.

The function $F_n(x)$ satisfying the clamped end boundary condition (2.5), but not necessarily satisfying the free end boundary conditions (2.6), is taken in the form

$$F_n(x) = \cos \lambda_n x - \cosh \lambda_n x + B_n (\sin \lambda_n x - \sinh \lambda_n x). \quad (4.20)$$

The λ_n 's and the coefficients B_n are specified in chapter seven where the reason for taking the form (4.20) will become clear. In the following sections attention will be directed towards the specification of G_n .

4.5.2 Linear form for $G_n(x)$

The downstream flow is modelled by envisaging a servomechanism which controls a semi-infinite pipe, stretching to downstream infinity, in such a way that its end is constantly adjusted to coincide with the free end of the cantilever. Thus all the fluid ejected from the end of the cantilever passes into the downstream pipe, whose wall displacement decreases linearly in the downstream direction, and a downstream instability is circumvented.

The mathematical representation of this model is the function $G_n(x)$ defined by

$$G_n(x) = \begin{cases} \frac{L_0 - x}{L_0 - 1} F_n(1), & 1 < x \leq L_0, \\ 0, & x > L_0, \end{cases} \quad (4.21)$$

where L_0 is arbitrarily large.

With (4.20) and (4.21), the transforms (4.9) give

$$\begin{aligned} F_n^*(\lambda) &= F_{nc}(\lambda) + iF_{ns}(\lambda), \\ G_n^*(\lambda) &= F_n(1) \left\{ \frac{e^{i\lambda} - e^{i\lambda L_0}}{\lambda^2 (L_0 - 1)} - \frac{e^{i\lambda}}{i\lambda} \right\}, \end{aligned} \quad (4.22)$$

where

$$F_{nc}(\lambda) = \int_0^1 F_n(x) \cos \lambda x \, dx,$$

and on integrating by parts using (4.20),

$$\begin{aligned}
 &= \frac{\lambda^3}{\lambda^4 - \lambda_n^4} [F_n(1) \sin \lambda + \frac{1}{\lambda} F_n'(1) \cos \lambda \\
 &\quad - \frac{1}{\lambda^2} F_n''(1) \sin \lambda - \frac{1}{\lambda^3} F_n'''(1) \cos \lambda \\
 &\quad + \frac{1}{\lambda^3} F_n'''(0)] , \\
 F_{ns}(\lambda) &= \int_0^1 F_n(x) \sin \lambda x \, dx \\
 &= \frac{\lambda^3}{\lambda^4 - \lambda_n^4} [-F_n(1) \cos \lambda + \frac{1}{\lambda} F_n'(1) \sin \lambda \\
 &\quad + \frac{1}{\lambda^2} F_n''(1) \cos \lambda - \frac{1}{\lambda^3} F_n'''(1) \sin \lambda \\
 &\quad - \frac{1}{\lambda^2} F_n''(0)] . \tag{4.23}
 \end{aligned}$$

Here a dash denotes derivative with respect to x . Expressions (4.22)

may be substituted in (4.11) to give

$$P_{mn,kk} = \alpha \rho \epsilon_k (P_{mn1} \omega^2 L^2 + 2i\omega L U P_{mn2} + U^2 P_{mn3}) , \tag{4.24}$$

where

$$\begin{aligned}
 P_{mn1} &= \int_0^\infty [F_{mc}(\lambda) Q_n(\lambda) + F_{ms}(\lambda) R_n(\lambda)] g_k(\alpha \lambda) d\lambda, \\
 P_{mn2} &= - \int_0^\infty [F_{ms}(\lambda) Q_n(\lambda) - F_{mc}(\lambda) R_n(\lambda)] \lambda g_k(\alpha \lambda) d\lambda, \\
 P_{mn3} &= \int_0^\infty [F_{mc}(\lambda) Q_n(\lambda) + F_{ms}(\lambda) R_n(\lambda)] \lambda^2 g_k(\alpha \lambda) d\lambda, \tag{4.25}
 \end{aligned}$$

with

$$Q_n(\lambda) = F_{nc}(\lambda) + F_n(1) \left\{ \frac{\cos \lambda - \cos \lambda L_0}{\lambda^2 (L_0 - 1)} - \frac{\sin \lambda}{\lambda} \right\}$$

$$R_n(\lambda) = F_{ns}(\lambda) + F_n(1) \left\{ \frac{\sin \lambda - \sin \lambda L_0}{\lambda^2 (L_0 - 1)} + \frac{\cos \lambda}{\lambda} \right\}.$$

The limit $L_0 \rightarrow \infty$ may be taken after evaluating the integrals. However, for the purpose of numerical integration, L_0 is assigned an arbitrarily large value such that further increase does not alter the numerical values of the integrals (to the accuracy required).

4.5.3 Oscillatory form for $G_n(x)$

In appendix A the solution of Laplace's equation (obtained from (2.7) when $c = \infty$) for ϕ in $x > 1$, satisfying a zero pressure condition on $r = \alpha$, is shown to be a sum of oscillatory (with respect to x) modes together with rapidly decaying exponential modes whose effect is localized near the free end of the pipe. As a result, an argument is given there to the effect that an oscillatory form for $G_n(x)$ may be employed to model the flow beyond the free end. The form taken is

$$G_n(x) = F_n(1) e^{i \frac{\omega L}{U}(x-1)}. \quad (4.26)$$

This represents a straightforward kinematic effect of fluid issuing from the free end of the cantilever with no downstream constraint.

In using Fourier transforms with (4.26), $\text{Im } \omega$ is considered to be positive (see appendix A for further discussion). Thus, on using (4.20) and (4.26) the transforms (4.9) give $F_n^*(\lambda)$ as defined in (4.22) together with

$$G_n^*(\lambda) = \frac{iUF_n(1)e^{i\lambda}}{\lambda U + \omega L} \quad (4.27)$$

When $F_n^*(\lambda)$ from (4.22) and $G_n^*(\lambda)$ from (4.27) are substituted in (4.11) the generalized pressures may be expressed in the form (4.24) with

$$\begin{aligned} P_{mn1} &= \int_0^\infty [F_{mc}(\lambda)F_{nc}(\lambda) + F_{ms}(\lambda)F_{ns}(\lambda)]g_k(\alpha\lambda)d\lambda, \\ P_{mn2} &= \int_0^\infty [F_{mc}(\lambda)\{F_{ns}(\lambda) + \frac{1}{2\lambda}F_n(1)\cos\lambda\} \\ &\quad - F_{ms}(\lambda)\{F_{nc}(\lambda) - \frac{1}{2\lambda}F_n(1)\sin\lambda\}]\lambda g_k(\alpha\lambda)d\lambda, \\ P_{mn3} &= \int_0^\infty [F_{mc}(\lambda)\{\lambda F_{nc}(\lambda) - F_n(1)\sin\lambda\} \\ &\quad + F_{ms}(\lambda)\{\lambda F_{ns}(\lambda) + F_n(1)\cos\lambda\}]\lambda g_k(\alpha\lambda)d\lambda, \end{aligned} \quad (4.28)$$

where F_{nc} and F_{ns} are given by (4.23).

4.5.4 Long pipe approximation

In this section it is assumed that $G_n(x)$ is such that $F_n^*(\lambda) + G_n^*(\lambda) \sim \lambda^{-m}$, $m \geq 2$, when $\lambda \gg 1$ (the cases considered in sections 4.5.2 and 4.5.3 satisfy this condition). For any $G_n(x)$ satisfying the above condition, and for a sufficiently long pipe (α small), the generalized pressures defined by (4.11) may be expanded in ascending powers of α . The details of how to obtain the expansion are given in appendix B where the case of simply supported pipes is considered. In the present case only the leading term (the long pipe approximation) is required and it follows that $G_n(x)$ does not have to be specified explicitly. Moreover, it is more informative to obtain the approximation by first employing the analysis of Appendix B on (4.5)

rather than on (4.11). In the case of (4.5), the leading term in the expansion for $\phi(\alpha, \theta, x)$ may be obtained by replacing $g_k(\alpha\lambda)$ by its small argument form α/k , whence

$$\begin{aligned}\phi(\alpha, \theta, x) &= \frac{\alpha}{2\pi i k} \int_{-\infty}^{\infty} e^{-i\lambda x} (\omega L + \lambda U) w^*(\lambda) d\lambda \\ &= \frac{\alpha}{k} (L \frac{\partial}{\partial t} + U \frac{\partial}{\partial x}) w(x),\end{aligned}$$

and on using (2.8),

$$p(\alpha, \theta, x) = -\frac{\alpha \rho}{kL} (L \frac{\partial}{\partial t} + U \frac{\partial}{\partial x})^2 w(x). \quad (4.29)$$

The result (4.29) has been employed by several authors to study the beam mode ($k = 1$) instability. The present analysis demonstrates that the beam expression for the pressure on the pipe wall, modified by a factor k^{-1} , may also be employed to study the shell mode instability. Thus the result extends the point relationship between pressure and displacement, which is often assumed in beam mode analyses, to the shell modes.

When $w(x)$ in (4.29) is replaced by its modal expansion (2.10), using $W_n(x) = F_n(x)$, p may be expressed in the form (2.11) with

$$P_{nk}(x, \theta) = \frac{-\alpha \rho}{k} (-i\omega L + U \frac{d}{dx})^2 F_n(x) \cos k\theta. \quad (4.30)$$

The generalized pressures obtained from (2.14) and (4.30) may be expressed in the form (4.24) using

$$\begin{aligned}
 P_{mn1} &= \frac{\pi}{k} \int_0^1 F_m(x) F_n(x) dx, \\
 P_{mn2} &= \frac{\pi}{k} \int_0^1 F_m(x) F_n'(x) dx, \\
 P_{mn3} &= - \frac{\pi}{k} \int_0^1 F_n''(x) F_m(x) dx.
 \end{aligned} \tag{4.31}$$

§4.6 Pipes with Clamped-Free ends: Cantilever continuation

Paidoussis and Denise [18] employed a sum of travelling wave solutions and, in satisfying the boundary conditions, obtained a determinantal equation for the frequencies. Thus the pipe is defined (implicitly) to deflect beyond the ends of the section under consideration. Following this idea, the definition of the cantilever functions, satisfying the boundary conditions, may be continued beyond the ends of the flexible pipe, thus specifying the deflection everywhere (there are problems with regard to finiteness of the functions at large distances but these can be handled quite easily as far as the pressure in the pipe in question is concerned). This is equivalent to

$$G_n(x) = F_n(x),$$

with a similar specification upstream of the clamped end. In order to avoid existence problems associated with the Fourier transform approach of §4.2, a formal separation of variables procedure is applied to (2.7), leading to a solution of the form

$$\phi = \sum \phi_{nk}(r, x) \cos k\theta, \quad (4.32)$$

where

$$\begin{aligned} \phi_{nk}(r, x) = & I_k(r\lambda_n) [A_1 \sin \lambda_n x + A_2 \cos \lambda_n x] \\ & + J_k(r\lambda_n) [A_3 \cosh \lambda_n x + A_4 \sinh \lambda_n x]. \end{aligned}$$

Here $J_k(r\lambda_n)$ is the Bessel function of the first kind. The fact that ϕ is bounded at $r = 0$ has been used to eliminate terms involving the Bessel functions of the second kind. The constants A_j are determined by using boundary conditions (2.9) with ϕ replaced by ϕ_{nk} and w by $F_n(x)$.

Thus

$$\begin{aligned} A_1 &= -C_{nk} (U\lambda_n + i\omega L B_n) / \{\lambda_n I_k'(\alpha\lambda_n)\} \\ A_2 &= C_{nk} (B_n U\lambda_n - i\omega L) / \{\lambda_n I_k'(\alpha\lambda_n)\} \\ A_3 &= C_{nk} (i\omega L - B_n U\lambda_n) / \{\lambda_n J_k'(\alpha\lambda_n)\} \\ A_4 &= C_{nk} (i\omega L B_n - U\lambda_n) / \{\lambda_n J_k'(\alpha\lambda_n)\}. \end{aligned} \quad (4.33)$$

Expressions (4.32) and (4.33) together with the Bernoulli equation (2.8) obtain p in the form (2.11) with

$$\begin{aligned} P_{nk}(x, \theta) = & \rho \cos k\theta [\omega^2 L^2 f_{1nk}(x) + 2i\omega L U f_{2nk}(x) \\ & + U^2 f_{3nk}(x)], \end{aligned} \quad (4.34)$$

where

$$f_{1nk} = - (B_n \sinh \lambda_n x + \cosh \lambda_n x) J_{nk}$$

$$+ (\cos \lambda_n x + B_n \sin \lambda_n x) I_{nk}$$

$$f_{2nk} = -\lambda_n \{ (B_n \cosh \lambda_n x + \sinh \lambda_n x) J_{nk}$$

$$+ (\sin \lambda_n x - B_n \cos \lambda_n x) I_{nk} \}$$

$$f_{3nk} = \lambda_n^2 \{ (\cosh \lambda_n x + B_n \sinh \lambda_n x) J_{nk}$$

$$+ (\cos \lambda_n x + B_n \sin \lambda_n x) I_{nk} \}.$$

Here

$$I_{nk} = \frac{I_k(\alpha \lambda_n)}{\lambda_n I_k'(\alpha \lambda_n)}, \quad J_{nk} = \frac{J_k(\alpha \lambda_n)}{\lambda_n J_k'(\alpha \lambda_n)}.$$

The generalized pressures follow from (2.14), and (4.34). They take the form (4.24) with

$$P_{mnj} = \frac{\pi}{\alpha} \int_0^1 F_m(x) f_{jnk}(x) dx, \quad (4.35)$$

where $F_m(x)$ is defined by (4.20) and the f_{jnk} are defined after (4.34).

CHAPTER 5

FINITE LENGTH PIPES WITH SIMPLY SUPPORTED ENDS

§5.1 Introduction

In this chapter the title problem is considered for subsonic compressible flow. The stability of simply supported pipes was first studied by Housner [4] using beam theory and slug flow. Niordson [17] is the first to consider compressible flow but, although he formulated the problem in terms of shell equations, he obtained the critical divergence condition only in the limiting case of an infinitely long pipe (for which compressibility of the fluid has no effect) vibrating in the beam mode. His conclusion regarding stability is therefore identical to that of [4].

Weaver and Unny [19] considered both beam and shell modes for the case of incompressible flow using the Flügge-Kempner equation, potential flow theory and Galerkin's method. However, considerable numerical computation is required to generate the stability boundaries, and a simple study of the parametric dependence of the critical condition is not possible. In the present work analytical results for the stability conditions are obtained. The analysis employs asymptotic forms for the generalized pressures, for long and short pipes, together with the Flügge-Kempner equation to determine the critical condition for divergence instability. The connection between the stability conditions for infinite and finite length pipes is considered by comparison with the travelling and standing wave analysis results of §3.2 and §3.3 respectively. In the earlier case of standing waves on an infinite array of finite pipes (§3.3), the conditions of simple supports for each pipe mean that the pipes are isolated structurally

and so any difference between the results of §3.3 and those obtained later in this chapter is due purely to altered fluid effects. Thus the comparison will indicate the significance of conditions beyond the ends of the pipe with regard to stability.

The layout of the chapter is in four sections. In §5.2 the characteristic equation is obtained. This is then employed in §5.3 to §5.5 to determine the stability conditions for long and short pipes. In §5.3 an analysis is presented for pipes of small radius to length ratio on the assumption that the number of axial half waves is small. It is found that in order to determine the most critical condition for all circumferential mode numbers an analysis valid for a large number of axial half waves is necessary: this is presented in §5.4. In §5.5 pipes of large radius to length ratio are considered. The limiting case of infinite radius and finite length corresponds to the large aspect ratio flat plate problem considered by Ellen [33].

§5.2 The characteristic equation

When the substitution $W_m(x) = F_m(x) = \sin m\pi x$ (§4.3) is made in (2.13), and (4.11) and (4.14) employed for the generalized pressures $P_{mn,k\ell}$, the integrals may be evaluated to give a system of homogeneous linear equations for the C_{nk} 's:

$$\sum_{n,k} C_{nk} a_{mn} = 0, \quad (5.1)$$

$$m = 1, 2, 3, \dots,$$

where

$$a_{mn} = \left[\frac{D}{3} F(n\pi\alpha, k) - \rho_m h R \omega^2 \right] \frac{\pi}{2} \epsilon_k \delta_{mn} - \alpha P_{mn, kk}, \quad (5.2)$$

with F defined by (3.9). The requirement of a non-trivial solution for the C_{nk} 's gives the characteristic equation

$$\det (a_{mn}) = 0 . \quad (5.3)$$

The solutions of this characteristic determinant, (5.3), for incompressible flow ($M = 0$) have been found numerically by truncation and using the lowest two longitudinal modes in (2.10) [19]. In this chapter analytical results for the divergence stability conditions are obtained: it seems to be generally true that, for surfaces with fixed boundaries, the subsonic flow stability conditions are associated with divergence criteria: $\omega = 0$ at the point of instability. Expression (4.14) then shows that $P_{mn,kk} = 0$ when $m + n$ is odd. Moreover, for the cases to be considered in this chapter and for $m + n$ even, $P_{mn,kk}$ is comparatively small when $m \neq n$, as shown in appendices B and D. Thus the effect of the off-diagonal terms on the value of the determinant (5.3) is small and will be neglected: the stability analysis of individual modes may be made independently of the rest. Thus for the (n,k) mode the stability condition may be obtained by equating a_{nn} in (5.2) to zero. This gives

$$\frac{\rho U_R^3}{D} = \frac{(1 - M^2)^{1/2} F(n\pi\alpha, k)}{4n\pi\alpha Q_{nk}} , \quad (5.4)$$

where

$$Q_{nk} = b n \int_0^\infty \frac{\lambda^2 \{1 - (-1)^n \cos \lambda\}}{(n^2 \pi^2 - \lambda^2)^2} g_k(b\lambda) d\lambda ,$$

$$b = (1 - M^2)^{1/2} \alpha . \quad (5.5)$$

The critical divergence boundary is defined by the minimum of the right hand side of (5.4) with respect to both n and k . This is the subject of sections 5.3 to 5.5. Structural damping is not included since it has no effect on a divergence boundary.

§5.3 Long Pipes ($b \ll 1$)

5.3.1 Asymptotic expansion for the stability parameter $(\rho U^2 R^3 / D)$

In appendix B it is shown that for disturbances of sufficiently long wavelengths (n small) several terms in the asymptotic expansion of Q_{nk} , equation (5.5), can be determined. Quite briefly the procedure involves making a small argument approximation for $g_k(b\lambda)$ over the lower part of the integration range. Over the remainder the term $(n^2 \pi^2 - \lambda^2)^{-2}$ is expanded in ascending powers of $n\pi/\lambda$, this being valid when n is sufficiently small. The results are:

$$Q_{nk} = \left\{ \begin{array}{l} (2bn\pi)^{-1} + bn\pi/16 + O(b^2), \quad k = 0, \\ \\ \frac{bn\pi}{4k} \left[1 + \frac{4}{\pi} k F_1 b - \frac{n^2 \pi^2}{2k(k+1)} b^2 \right. \\ \\ \quad \left. + 8n^2 \pi k F_2 b^3 + \frac{(4+3k)n^4 \pi^4}{8k^2(k+1)^2(k+2)} b^4 \right. \\ \\ \quad \left. + O(b^5) \right], \quad k \geq 1, \end{array} \right. \quad (5.6)$$

where

$$\begin{aligned}
 F_1(k) &= \int_0^{\infty} \left[\frac{I_k(x)}{x I_{k-1}(x)} - \frac{1}{k} \right] \frac{dx}{x^2} \\
 F_2(k) &= \int_0^{\infty} \left[\frac{I_k(x)}{x I_{k-1}(x)} - \frac{1}{k} + \frac{x^2}{2k^2(k+1)} \right] \frac{dx}{x^4} .
 \end{aligned} \tag{5.7}$$

Some numerical values for F_1 and F_2 are given in table 5.1.

In previous beam mode ($k = 1$) stability analyses the fluid loading at divergence is generally contained in a term of the form $\rho U^2 w_{xx}$ which represents the inertia forces due to the curvature of the deformed pipe. This term generates only the leading term in (5.6), suggesting that beam theory may be expected to give good predictions of the stability conditions when $bn\pi$ is sufficiently small.

The expansion for $\rho U^2 R^3/D$ is obtained from (5.4) after substituting for Q_{nk} from (5.6) and using the expansion for F in ascending powers of $n\pi\alpha$:

$$\begin{aligned}
 F(n\pi\alpha, k) &= (k^2 - 1)^2 + 2n^2\pi^2\alpha^2[k^2 + \nu - 2 \\
 &\quad + (1 - \nu)/k^2] + n^4\pi^4\alpha^4[1 \\
 &\quad + \{12(1 - \nu^2)R^2/h^2 + 4k^2(1 - \nu) \\
 &\quad + 4\nu - 5\}/k^4] + O(n^6\pi^6\alpha^6) .
 \end{aligned} \tag{5.8}$$

This gives

$$\frac{\rho U^2 R^3}{D} = 12(1 - M^2)(1 - v^2)(R/h)^2 + O(n^2 \pi^2 \alpha^2), \quad k = 0, \quad (5.9)$$

$$= 12(1 - v^2)(R/h)^2 n^2 \pi^2 \alpha^2 + O(n^3 \pi^3 \alpha^3), \quad k = 1, \quad (5.10)$$

$$= k(k^2 - 1)^2 / (n^2 \pi^2 \alpha^2) + O\left(\frac{1}{n\pi\alpha}\right), \quad k > 1, \quad (5.11)$$

where only the leading terms in the limit $b \rightarrow 0$ have been evaluated to determine the most critical circumferential mode. When (5.9) and (5.10) are compared in the limit $b \rightarrow 0$, the beam mode ($k = 1$) is seen to be more unstable. Thus for long pipes it is sufficient to consider $k \geq 1$ only. The beam and shell mode stability conditions will now be examined separately in more detail.

5.3.2 Beam mode stability condition

Equation (5.10) shows that, with $b \rightarrow 0$, the most critical condition when $k = 1$ is associated with $n = 1$. In order to obtain a better approximation for $\rho U^2 R^3 / D$, a few higher order terms must be included in the expansion. Now

$$\begin{aligned} F(n\pi\alpha, 1) &= \frac{n^4 \pi^4 \alpha^4}{(n^2 \pi^2 \alpha^2 + 1)^2} [12(1 - v^2)R^2/h^2 + 1 \\ &\quad + 2n^2 \pi^2 \alpha^2 (2 - v) + n^4 \pi^4 \alpha^4]. \end{aligned} \quad (5.12)$$

For $n\pi\alpha$ small, the last two terms in (5.12) may be neglected. The resulting expression for $\rho U^2 R^3 / D$ at critical when Q_{11} is substituted in (5.4), using (5.6) and (5.12), is

$$\begin{aligned}
\frac{\rho U_{cd}^2 R^3}{D} = & \frac{12R^2}{h^2} (1 - v^2) \pi^2 \alpha^2 \left[1 + \frac{4}{\pi} F_1 (1 - M^2)^{1/2} \alpha \right. \\
& + (7 + M^2) \frac{\pi^2}{4} \alpha^2 + 8\pi (1 - M^2)^{1/2} \{F_1 \\
& + (1 - M^2) F_2\} \alpha^3 + \frac{\pi^4}{96} \{55 + 34M^2 \\
& \left. + 7M^4\} \alpha^4 + O(\alpha^5) \right]^{-1}.
\end{aligned} \tag{5.13}$$

In obtaining (5.9), (5.10) and (5.11) it was necessary to expand $F(n\pi\alpha, k)$. However, by retaining the inverse in (5.13), this expansion is not made and the result appears to have a wider range of applicability.

The leading term in (5.13) gives a result identical to that of beam theory [4]. This is as expected, in view of the discussion after (5.7) and the fact that beam theory and shell theory are equivalent when only the leading term is taken in the long axial wavelength limit approximation to the shell theory [34].

Since both the fluid and shell equations generate terms of higher order in the approximation, the factor in square brackets in (5.13) contains corrections from both equations for finite length pipes. On denoting this factor by $f_1(\alpha)$, the coefficients may be evaluated, using F_1 and F_2 from table 5.1, to obtain

$$\begin{aligned}
f_1(\alpha) = & [1 - 0.945(1 - M^2)^{1/2} \alpha \\
& + 2.467(7 + M^2)\alpha^2 - (1 - M^2)^{1/2}(13.348 \\
& + 5.301 M^2)\alpha^3 + O(\alpha^4)]^{-1}.
\end{aligned} \tag{5.14}$$

This result shows that compressibility ($M > 0$) has a destabilizing effect.

In reference [19] a result similar to (5.13) is obtained but without the shell equation corrections: this corresponds to replacing the factor $(n^2 \pi^2 \alpha^2 + 1)^{-2}$ in (5.12) by unity and evaluating the integral Q_{nk} in (5.5) numerically. The effect of neglecting the shell equation corrections here is to replace the function f_1 in (5.14) by f_2 where

$$f_2(\alpha) = [1 - 0.945(1 - M^2)^{1/2} \alpha - 2.467(1 - M^2) \alpha^2 + 5.301(1 - M^2)^{3/2} \alpha^3 + O(\alpha^4)]^{-1} . \quad (5.15)$$

The variation of f_1 and f_2 with L/R , when $M = 0$, is shown in figure 5.1. For f_1 , expression (5.12) is employed without any approximation in order to obtain a correct comparison. In both cases the four term approximation to the fluid pressure is used. It can be seen that f_1 approaches unity from below whereas f_2 approaches unity from above. A comparison of the f_2 curve with results from figure 6 of [19] (shown by the crosses in figure 5.1), gives good agreement for α as large as 0.5. Thus the correction factor of [19] is misleading and it would appear more appropriate to include shell equation corrections to obtain an accurate appreciation of the variation of the stability condition with length ratio.

More discussion of (5.13) is given after considering the shell mode stability conditions.

5.3.3 Effect of added stiffness to Beam condition

The loss of stiffness when $k = 1$ is an important characteristic of the beam mode instability, as can be seen by comparing (5.8), (5.10) and (5.11). In view of this rather special character, the question

arises as to how the stability condition would be modified if this loss of stiffness did not occur. Consider, for example, the case when the pipe is supported on the outside by continuous elastic springs of radial stiffness, s , giving added stiffness to the pipe wall. The effect may be examined by adding a term sw/R^3 on the left hand side of (2.3). The result is that the function F in (5.8) is replaced by $F + sR/D$ and (5.10) becomes

$$\frac{\rho U^2 R^3}{D} = \frac{sR}{n^2 \pi^2 \alpha^2 D} + O\left(\frac{1}{n\pi\alpha}\right). \quad (5.16)$$

Since (5.16) and (5.11) are similar, it is no longer necessary to consider the beam mode, when $s > 0$, and the shell mode separately. Thus the discussion in the next section concerning the shell mode applies for the beam mode also when $s > 0$.

5.3.4 Shell mode stability condition

When only the leading term (in the limit $\alpha \rightarrow 0$) is retained in (5.11), the most critical condition for $k \geq 2$ is seen to be associated with $n \rightarrow \infty$. Since the expressions (5.6) are valid only for small values of n , the result suggests a re-examination of the problem for $b \rightarrow 0$, $n \rightarrow \infty$, with $bn = O(1)$. The case corresponds to short waves on long pipes and is considered next. A similar situation is encountered in the analysis of flat plates of small aspect ratio [33].

§5.4 Short waves on Long Pipes ($b \ll 1$, $bn \sim 1$)

5.4.1 Asymptotic expansion for the stability parameter ($\rho U^2 R^3/D$)

In appendix C a technique similar to the Laplace method in the theory of asymptotic expansions [35] is employed to derive the leading two terms in the asymptotic expansion for Q_{nk} . The result is

$$Q_{nk} \sim \frac{1}{4} \frac{I_k(bn\pi)}{I_k'(bn\pi)} [1 + bf_k(bn\pi)] \quad (5.17)$$

where

$$f_k(\gamma_n) = \frac{2}{\pi G_k(\gamma_n)} [(-1)^n \{1 - (1 + k^2/\gamma_n^2) G_k^2(\gamma_n)\} + \int_0^\infty \frac{G_k'(\gamma_n x) - G_k'(\gamma_n)}{x^2 - 1} dx]$$

with $G_k(x) = I_k(x)/I_k'(x)$.

When (5.17) is substituted in (5.4) the required expansion for $\rho U^2 R^3/D$ is obtained. Thus, to leading order,

$$\frac{\rho U^2 R^3}{D} = \frac{F(n\pi\alpha, k) I_k'(bn\pi) (1 - M^2)^{1/2}}{n\pi\alpha I_k(bn\pi)} [1 + bf_k(bn\pi)]^{-1}. \quad (5.18)$$

This result is now used to determine the stability condition for $k > 1$, $s = 0$ and for $k \geq 1$, $s > 0$.

5.4.2 Shell mode stability condition

This is given by the minimum of the right hand side of (5.18) with respect to n and k . Since $f_k(bn\pi)$ involves an integral to be evaluated numerically for each value of b , n and k , and since $\rho U^2 R^3/D$ must be minimized numerically, the amount of numerical computations is large when

the first two terms in (5.18) are considered simultaneously, and the simplicity of the asymptotic approach is lost. However an approximation to the minimum $\rho U_{cd}^2 R^3 / D$ may be obtained from the leading term alone and, if necessary, the correction due to the second term evaluated later by considering a small region around the minimum given by the leading term. To check whether the correction should be considered, $bf_k(bn\pi)$ may be evaluated for the critical values of n and k as determined from the leading term analysis to see if it is significant. Thus, from (5.18), the first approximation is

$$\frac{\rho U_{cd}^2 R^3}{D} = (1 - M^2)^{1/2} \min_{(n,k)} \left\{ \frac{F(n\pi\alpha, k) I_k'(bn\pi)}{n\pi\alpha I_k(bn\pi)} \right\}. \quad (5.19)$$

As before, this result shows that compressibility has a destabilizing effect. Moreover the stability condition given by (5.19) is identical to that of infinitely long pipes obtained in chapter three, equations (3.11) and (3.26), when either viscous damping is present or when the mass ratio, μ , tends to zero. Thus, when either of these conditions prevail, results obtained from a standing or a travelling wave analysis on finite pipes (see, for example, [18]) may be expected to be quite good when α is sufficiently small and provision is made for the limitation imposed on the axial wavelength by the finiteness of the pipe. The discussion which follows now is relevant to the long pipes considered in chapter three as well as the configuration under investigation in this chapter.

In order to minimize the amount of numerical computations in this work, equation (5.19) is considered for incompressible flow, $M = 0$ (other values of M equally well may be considered but the choice $M = 0$ allows comparison with earlier work). Two procedures for evaluating $\rho U_{cd}^2 R^3 / D$ are employed. The first treats $\sigma_n = \alpha n\pi$ as a continuous

variable in analogy with the travelling wave analysis of §3.2 while the second treats σ_n as a discrete variable, corresponding to the integer values of n . For a given α , the correction to (5.19) due to the second term in (5.18) is evaluated only in the case σ_n treated as discrete variable.

Case (i): σ_n continuous

When α is small, σ_n may be considered a continuous variable and $\rho U_{cd}^2 R^3/D$ in (5.19) minimized with respect to σ_n and k . The procedure consists of choosing numerical values for k , h/R , and v , followed by numerically evaluating $\rho U_{cd}^2 R^3/D$ for various values of σ_n , starting from a small value which is then increased in small steps. From the results, the interval enclosing the minimum of $\rho U_{cd}^2 R^3/D$ in the σ_n axis is located and the procedure is repeated for this interval using smaller steps in the σ_n increments. The fact that a minimum occurs at a finite σ_n has already been established in §3.2 by considering the limits $\sigma_n \rightarrow \infty$ and $\sigma_n \rightarrow 0$.

The above procedure is equivalent to plotting curves similar to those of figure 5.2 and reading $\rho U_{cd}^2 R^3/D$ as the smallest minimum, in this case corresponding to $k = 2$. The variation of $\rho U_{cd}^2 R^3/D$ with σ_n is shown in figure 5.3 for $k = 2$, and $h/R = 0.04, 0.06$, and 0.1 : as h/R increases, $\rho U_{cd}^2 R^3/D$ decreases while the corresponding σ_n increases. The variation of $\rho U_{cd}^2 R^3/D$ with h/R is shown in figure 5.4 for $k = 2, 3$ and 4 . Similar results show that the smallest minimum, and hence the most critical condition, always corresponds to $k = 2$. Numerical values for $\rho U_{cd}^2 R^3/D$ when $k = 2$ and h/R between 0.001 and 0.5 are given in table 5.2 under the heading "Travelling waves".

Case ii: σ_n discrete and specified

In this case $\rho U^2 R^3/D$ is evaluated for a given radius ratio, α , and each σ_n ($n = 1, 2, \dots$) until the minimum value, $\rho U_{cd}^2 R^3/D$, is obtained. The calculation is repeated for each k to determine the smallest minimum.

Some results obtained in this way are given in table 5.2 for $\alpha = 0.05, 0.1, 0.2$ and 0.5 . It can be seen that the critical condition is determined by progressively larger values of k as h/R decreases, $\rho U_{cd}^2 R^3/D$ increasing as before. For the values of α investigated the critical condition always corresponds to $n = 1$ or 2 .

5.4.3 The most critical mode

The actual stability condition, when considering all possible modes, must be obtained by comparing the beam mode ($s = 0$: $k = 1$, $n = 1$) results of 5.3.2 with the shell mode results of 5.4.2.

A comparison of (5.13) and (5.19), in the limit $\alpha \rightarrow 0$, shows that for a given h/R and for sufficiently small α , the beam mode is the most critical. The results given in table 5.2 indicate that as α decreases the range of h/R for which the beam mode is most critical increases. Thus, for a given h/R , there is a minimum value, α_t , of α above which shell mode instability is most critical and below which beam mode instability is most critical. This transition value is given approximately by equating the leading term in (5.13) with $\rho U_{cd}^2 R^3/D$ obtained from (5.19):

$$\alpha_t \approx \frac{h}{2\pi R} \left\{ \frac{\rho U_{cd}^2 R^3}{3(1 - \nu^2)D} \right\}^{1/2}. \quad (5.20)$$

The result (5.20) also implies that for a given α , there is a transition value of h/R above which beam mode instability is most critical and below which shell mode instability is most critical. Figure 5.5 shows the variation of α_t with h/R : it can be seen that $\alpha_t R/h \approx 1$ over a greater part of the curve. A similar result is discussed in [36].

5.4.4 Accuracy of the results and comparison with previous work

The accuracy of the asymptotic analysis is considered by comparison with the numerical results of Weaver and Unny [19] and also by comparing the numerical and asymptotic evaluations of Q_{nk} as defined by (5.5). The stability parameter, V , defined by equation (11) of [19] is related to the stability parameter employed in this work by

$$V^2 = \frac{7.85}{\pi^4 \alpha^2} \frac{h}{R} \left\{ \frac{\rho U^2 R^3}{D} \right\}. \quad (5.21)$$

Numerical values for V obtained from (5.21), (5.13) and (5.19) for various values of h/R (within the range considered in [19]), together with results from figure 4 of [19], are plotted in figure 5.6. In obtaining the results, the right hand side of (5.19) is minimized with respect to the discrete values of σ_n .

In figure 5.6 the results for $\alpha = 0.05$ coincide with the corresponding results from figure 4 of [19]. This is as expected since for this value of α the beam mode is the most critical, with $n = 1$, and the asymptotic expression (5.13) gives a very good approximation for $\rho U_{cd}^2 R^3 / D$. For the same reason the results for $\alpha = 0.1$ and $h/R \geq 0.1$ are also coincident, while the results for $\alpha = 0.2$ and $h/R \geq 0.3$ compare very well. The comparison for $\alpha = 0.5$ and $h/R > 0.1$ show significant differences because for $\alpha = 0.5$ the asymptotic approximation

(5.13) is not accurate enough. For the rest of the curves the comparison is good except in those regions which are associated with the transition values of h/R at which the most critical circumferential mode number increases (by unity) as h/R decreases. It is interesting that the results obtained from using only the leading term in (5.18) compare so well with the numerical results despite the small values of the longitudinal mode number, n , associated with instability (table 5.2). However, if the beam mode instability is excluded, for example by tethering, the most critical value of n is large (table 5.3) and an analysis using the first few terms in the modal solution, as in [19], will not be accurate.

The asymptotic and numerical evaluations of the integral Q_{nk} (equation (5.5)) are compared in tables 5.4 to 5.6 for $k = 1$ and 2. The comparison for higher values of k is similar to that for $k = 2$. The comparison for Q_{11} (table 5.4) is quite good even for α as large as 0.25. The results for Q_{n2} (tables 5.5 and 5.6) suggest that when σ_n is small, expression (5.6) gives slightly better approximations when compared with (5.17). For large values of σ_n (table 5.6) expression (5.17) gives better approximations while the accuracy of (5.6) deteriorates with increasing σ_n . Nevertheless an analysis using (5.6) gives values for $\rho U_{cd}^2 R^3/D$ which differ only by less than 5% from the results in table 5.2 (when $\alpha \leq 0.1$) and predicts the same values for the critical n and k .

The correction to (5.19) due to the second term in (5.17) is small when α is small (tables 5.5 and 5.6). This is as expected since the correction term represents effects due to the flow conditions beyond the ends of the pipe (see discussion in 5.4.2).

From the preceding considerations it may be concluded that the stability condition for simply supported, thin-walled, pipes is

essentially a single mode phenomenon when α is small (< 0.2) and may be obtained with sufficient accuracy (which increases with decreasing α) using (5.13) and (5.19). The numerical results suggest that expression (5.6) may be employed with (5.4), instead of (5.19), when $k > 1$. This is by far the simpler procedure since it avoids the need to evaluate the Bessel functions in (5.19).

5.4.5 The case of added stiffness

When $s > 0$ the critical condition for all values of k is given by (5.19) with F replaced by $F + SR/D$. This represents an increase in the stiffness of the pipe and hence a stabilizing effect which increases with s . Since $\sigma_n g_k(\sigma_n)$ decreases with σ_n for small σ_n , the minimum value of $\rho U^2 R^3/D$ ($= \{F + SR/D\}/\{\sigma_n^2 g_k(\sigma_n)\}$) when $s > 0$ is associated with a value of σ_n which is greater than, or equal to, the corresponding value when $s = 0$. Thus the value of n at instability may be raised as s increases from zero.

When $k > 1$, the variation of $\rho U^2 R^3/D$ with σ_n is similar to that when $s = 0$. However, when $k = 1$, $\rho U^2 R^3/D \rightarrow \infty$ (instead of zero when $s = 0$) as $\sigma_n \rightarrow 0$, and, therefore, it has a minimum for σ_n small when s is small. This critical value of σ_n , $\approx \{s/(hRE)\}^{1/4}$, may be obtained with sufficient accuracy by considering the small σ_n approximation to (5.19)

For a given value of s and h/R , and for sufficiently small α , the beam mode is still the most critical. However, if s is considered to be increasing gradually, the critical value of $\rho U^2 R^3/D$ for $k = 1$ is seen to increase until it becomes larger than the critical value corresponding to the most critical shell mode. The point at which equality occurs defines the transition value for sR/D above which beam mode instability gives way to shell mode instability. Table 5.7

gives some values for sR/D at transition for various values of h/R .

The preceding discussion shows that whereas setting s to zero gives a beam mode instability with $n = 1$ when α is small, the value of n at instability, for s greater than zero, is large but decreases as α increases for h/R fixed. Moreover, whether the beam mode or the shell mode is more critical depends on α when $s = 0$ and only on s when α is small: large values of α or s lead to shell mode instability.

§5.5 Large radius pipes ($b \gg 1$)

5.5.1 Asymptotic expansion for the stability parameter ($\rho U^2 L^3/D$)

When $b \gg 1$, the pipe is short and, for k small, elements of the pipe wall may be expected to respond in a way similar to that of a flat plate. Thus the relevant stability parameter here is $\rho U^2 L^3/D$.

In appendix D the asymptotic expansion of Q_{nk} (equation (5.5)) for large b is determined by employing the large-argument expansion of the Bessel function over part of the integration range while the factor $(n^2 \pi^2 - k^2)^{-2}$ is expanded in powers of $k/n\pi$ over the remaining part. The result is

$$\begin{aligned}
 Q_{nk} \quad b \gg 1 &= \frac{1}{2n\pi^2} [(-)^n - 1 + n\pi \operatorname{Si}(n\pi)] + \frac{1}{8n\pi b} \\
 &+ \frac{1}{n^3 \pi^4 b^2} [\{1 - (-)^n\} F_3(k) \\
 &+ \frac{3 - 4k^2}{16} \{n\pi \operatorname{Si}(n\pi) + 2 \operatorname{Cin}(n\pi) \\
 &+ [1 - (-)^n] (3 \log 2 - 1 - 2\gamma)\}] + O(b^{-3}) , \quad (5.22)
 \end{aligned}$$

where γ is Euler's constant and the functions $F_3(k)$, $\operatorname{Si}(n\pi)$ and $\operatorname{Cin}(n\pi)$ are defined in appendix D.

The expansion for $\rho U^2 L^3 / D$ is obtained by using (5.4), (5.22) together with the large α expansion for $F(n\pi\alpha, k)$: on using (3.9),

$$F(n\pi\alpha, k) = n^4 \pi^4 \alpha^4 + \beta_1 n^2 \pi^2 \alpha^2 + \beta_2 + O(n\pi\alpha)^{-2}, \quad (5.23)$$

where

$$\begin{aligned} \beta_1 &= 2(k^2 - v) + \frac{12L^2(1 - v^2)}{n^2 \pi^2 h^2}, \\ \beta_2 &= k^2(k^2 + 4v - 6) - \frac{24L^2(1 - v^2)}{n^2 \pi^2 h^2}. \end{aligned}$$

Expressions (5.4), (5.22) and (5.23) lead to

$$\frac{\rho U^2 L^3}{D} = 2(1 - M^2)^{1/2} n^4 \pi^4 Y_1 \left[1 - \frac{Y_1}{b} + \frac{Y_2}{b^2} + O(b^{-3}) \right], \quad (5.24)$$

where

$$\begin{aligned} Y_1 &= \frac{\pi}{4} \{ (-)^n - 1 + n\pi \operatorname{Si}(n\pi) \}^{-1} \\ Y_2 &= Y_1^2 + \frac{1-M^2}{n^2 \pi^2} \beta_1 - \frac{Y_1}{2n^2 \pi} [\{1 - (-)^n\} F_3(k) \\ &\quad + \frac{3-4k^2}{16} \{n\pi \operatorname{Si}(n\pi) + 2 \operatorname{Cin}(n\pi) \\ &\quad + [1 - (-)^n] (3 \log 2 - 1 - 2\gamma)\}]. \end{aligned} \quad (5.25)$$

5.5.2 The most critical condition

This is obtained by minimizing the right hand side of (5.24) with respect to n and k . With respect to n it may be noted that the leading term in (5.24) is minimum when $n = 1$ and, since $b \gg 1$, this determines the critical condition. The circumferential mode number affects the divergence condition first through terms of $O(b^{-2})$. Thus the most critical value of k can be determined by considering the coefficient of b^{-2} when $n = 1$. When this coefficient is evaluated numerically, its minimum value, and hence the most critical condition, is found to correspond to $k = 1$.

With $n = k = 1$, (5.24) reduces to

$$\begin{aligned} \frac{\rho U_{cd}^2 L^3}{D} = & 40.08(1 - M^2)^{1/2} [1 - 0.206b^{-1} \\ & + (1 - M^2)\{0.206(1 - \nu) \\ & + 0.123(1 - \nu^2)L^2/h^2 \\ & - 3.373/(1 - M^2)\}b^{-2} + O(b^{-3})] . \end{aligned} \quad (5.26)$$

As might be expected the limit $b \rightarrow \infty$ gives the same result as that of simply-supported flat plates of large aspect ratio [33]. There are no published results available for comparison with (5.26) but it may be noted that the negative sign of the second term suggests a destabilizing effect of increasing curvature. Moreover, the fact that circumferential modal effects appear only through fluid dynamic terms of second order in b^{-1} suggests a weak dependence of the instability on this particular modal characteristic.

The result corresponding to (5.26) for an infinite array of simply supported pipes (equation (3.26)) is, to leading order,

$$\begin{aligned} \frac{\rho U_{cd}^2 L^3}{D} &= (1 - M^2)^{1/2} \pi^3 \\ &= 31.01 (1 - M^2)^{1/2} . \end{aligned} \tag{5.27}$$

This is the result for an infinite array of two dimensional panels [29].

CHAPTER 6

PIPES WITH CLAMPED ENDS

§6.1 Introduction

The stability analysis of clamped-clamped pipes conveying inviscid, incompressible fluid, using shell theory, has been considered by Paidoussis and Denise [18] and by Weaver and Myklatun [20]. In [18] a combination of travelling wave solutions for the pipe wall deformation, satisfying the clamped boundary conditions, and for the perturbation pressure, is employed with Flügge's shell equations and potential flow theory to obtain a pair of coupled determinantal equations for the dimensionless frequency and the axial wavenumber as unknowns. These equations are solved numerically to determine the stability conditions. In [20] the Flügge-Kempner shell equation, potential flow theory, and Galerkin's method are employed and divergence stability conditions are obtained numerically using a truncated solution for the pipe wall deformation. Thus, as for the simply supported pipe problem considered in [19], considerable numerical work is required to obtain the stability conditions.

In this chapter the asymptotic analysis of chapter 5 is extended to include the present case of pipes with clamped ends. The method of solution is similar: starting from (2.13) the characteristic equation for the frequency, ω , is obtained using $W_n(x) = F_n(x) = 1 - \cos 2n\pi x$ together with the expressions for the generalized pressures $P_{mn,kl}$ given by (4.19). Analytical expressions for the divergence stability conditions are obtained by finding asymptotic forms for $P_{mn,kl}$. The use of the modal forms, $W_n(x)$, is examined and the importance of the off-diagonal structural and pressure terms is investigated.

§6.2 The characteristic equation

When the substitution $W_n(x) = F_n(x) = 1 - \cos 2n\pi x$ (§4.4) is made in (2.13), the integrals may be evaluated to give

$$\sum_{n,k} C_{nk} a_{mn,k} = 0, \quad (6.1)$$

$$m = 1, 2, 3, \dots; \quad k = 0, 1, 2, \dots;$$

where

$$a_{mn,k} = \left[\frac{D}{R^3} \{ (k^2 - 1)^2 + \frac{1}{2} F(2n\pi\alpha, k) \delta_{mn} \} - \rho_m h R \omega^2 (1 + \frac{1}{2} \delta_{mn}) \right] \pi \epsilon_k - \alpha P_{mn, kk}, \quad (6.2)$$

with $P_{mn, kk}$ defined by (4.19) and F by (3.9). The modal functions $F_n(x)$ are identical to those employed in [20] but equation (16) of [20], which corresponds to (6.2), is apparently incorrect.

The requirement of a non-trivial solution for the C_{nk} 's in (6.1) gives the characteristic equation

$$\det(a_{mn,k}) = 0. \quad (6.3)$$

The solution of (6.3) for the stability conditions will now be reduced to the determination of the minimum eigenvalue of a simple matrix.

§6.3 An equivalent eigenvalue problem

The off-diagonal generalized pressure terms in (6.3) are small when $\omega = 0$ (corresponding to the divergence condition) and b is small (see appendix E). These terms may therefore be neglected, whence (6.3) becomes

$$\det \left[\mathbf{B} - \frac{\rho U^2 R^3}{D} \mathbf{I} \right] = 0, \quad (6.4)$$

where $\mathbf{I}$ is the identity matrix and the elements b_{mn} of the matrix $\mathbf{B}$ are given by

$$b_{mn} = \begin{cases} 2(k^2 - 1)^2, & m \neq n \\ \frac{F(2n\pi\alpha, k) + 2(k^2 - 1)^2}{16n\pi\alpha Q_{nk}}, & m = n, \end{cases} \quad (6.5)$$

with

$$Q_{nk} = 4\alpha n^3 \pi^2 \int_0^\infty \frac{1 - \cos \lambda}{(4n^2 \pi^2 - \lambda^2)^2} g_k(b\lambda) d\lambda. \quad (6.6)$$

The asymptotic expansions for Q_{nk} , corresponding to (5.6) and (5.17), are derived in Appendix E by using procedures similar to those of Appendices B and C. The results are:

$$Q_{nk} = \frac{1}{4(1-M^2)^{1/2}} \begin{cases} \frac{3}{2bn\pi} + \frac{bn\pi}{4} + O(bn\pi)^2 & k = 0, \\ \frac{bn\pi}{k} \left[1 - \frac{2b^2 n^2 \pi^2}{k(k+1)} + 16kb^3 n^2 \pi F_2(k) \right. \\ \quad \left. + \frac{2(4+3k)b^4 n^4 \pi^4}{k^2(k+1)^2(k+2)} + O(bn\pi)^5 \right] & k > 0, \end{cases} \quad (6.7)$$

$$Q_{nk} \sim \frac{I_k(2bn\pi)}{8(1-M^2)^{1/2} I_k'(2bn\pi)} [1 + bf_k(2bn\pi)] , \quad (6.8)$$

$b \ll 1$
 $bn \gg 1$

where F_2 is defined by (5.7) and tabulated in table 5.1, while f_k is given by

$$f_k(\gamma_n) = \frac{2\gamma_n^2}{\pi} \frac{I_k'(\gamma_n)}{I_k(\gamma_n)} \left\{ G_k'(\gamma_n) - \frac{1}{\gamma_n} G_k(\gamma_n) \right. \\ \left. + \int_0^\infty \frac{G_k'(\gamma_n x) - G_k'(\gamma_n)}{1 - x^2} dx \right\} ,$$

with $G_k(x) = \{g_k(x) - 1/k\}/x$.

The determination of the critical condition is thus reduced to finding the minimum eigenvalue $\rho U^2 R^3/D$ of the matrix B whose elements b_{mn} are given by (6.5) and (6.7) or (6.8): the conditions regarding the use of these expressions are similar to those discussed in 5.4.4 and will be considered later.

§6.4 The stability conditions and discussion of the results

6.4.1 Beam mode stability condition

In the case of the beam mode ($k = 1$) equation (6.4) can be solved exactly, since only diagonal terms remain, to give

$$\frac{\rho U^2 R^3}{D} = \frac{F(2n\pi\alpha, 1)}{16n\pi\alpha Q_{n1}} , \quad (6.9)$$

for some n . The most critical condition is determined by minimizing $\rho U^2 R^3/D$ with respect to n . To do this an equation similar to (5.10) may be derived from (6.7) and (6.9), and the most critical n determined in the limit $b \rightarrow 0$, whence $n = 1$ determines the most critical condition, with

$$\frac{\rho U_{cd}^2 R^3}{D} = \frac{F(2\alpha\pi, 1)}{4\alpha^2 \pi^2} [1 - \pi^2 b^2 + 16\pi F_2(1)b^3 + \frac{7\pi^4}{6} b^4 + O(b^5)]^{-1}, \quad (6.10)$$

and on retaining only the leading term in the expression (5.12) for F (the term containing R^2/h^2),

$$\begin{aligned} &= \frac{48R^2}{h^2} (1 - \nu^2) \pi^2 \alpha^2 [1 + (7 + M^2) \pi^2 \alpha^2 \\ &\quad + 16(1 - M^2)^{3/2} F_2(1) \pi \alpha^3 \\ &\quad + \{55 + (7M^2 - 8)M^2\} \frac{\pi}{6} \alpha^4 + O(\alpha^5)]^{-1}. \end{aligned} \quad (6.11)$$

The expressions in square brackets in (6.10) and (6.11) correspond to the "end flow factor" of [20] (without and with the shell equation correction respectively). A consideration similar to that for the simply supported pipes (discussion following (5.14) and (5.15)) leads to the same conclusion: the neglect of the shell equation correction, as in equation (19) of [20], gives erroneous deductions regarding stability. More discussion of (6.11) is given later after considering the shell mode stability condition.

6.4.2 Shell mode stability condition

When $k > 1$, the critical condition must be determined numerically using (6.4) truncated to, say, order N . Expression (6.8) may be used for all n although, strictly speaking, (6.7) is more appropriate when n is small (see also discussion in 5.4.4): a comparison of the numerical and asymptotic evaluations of Q_{nk} (see for example, tables 6.1 and 6.2) shows that expression (6.8) gives good approximations, even for $n = 1$, when α is moderately small.

To determine the most critical condition the leading term in (6.8) is used with (6.4) truncated to order N where N is such that, for fixed k , the minimum eigenvalue $\rho U^2 R^3 / D$, does not change significantly with increasing N . By varying k , the results may be compared and the smallest minimum eigenvalue, $\rho U_{cd}^2 R^3 / D$, obtained.

In this chapter numerical results are presented in terms of the stability parameter V which is defined by

$$V^2 = \frac{h^3}{\pi^2 R^3} \left\{ \frac{\rho U^2 R^3}{D} \right\} \quad (6.12)$$

and corresponds to the parameter employed in [20]. The variation of the minimum value of V with N is shown in figure 6.1 for $k = 2$ and 3 and for $N = 1, 2$ and 3. The curve for $k = 1$, obtained from (6.10) is also given. It can be seen that there is a wide range of L/R where N needs to be larger than 1 for accurate results and that the critical k increases with decreasing L/R . The determination of the most critical condition, considering both beam and shell modes, involves finding the lower envelope of these curves.

6.4.3 The most critical mode

Figure 6.2 gives a sample of the stability boundaries for $L/R = 5, 10, 20$ and 45 as h/R varies from 0.01 to 0.2. These values are chosen in order to facilitate comparison with the results of Weaver and Myklatun [20], which are also given in the figure and will be discussed later. The critical values of k are given and N is taken to give the accuracy needed for the figure. It can be seen that the critical k increases with decreasing L/R and/or h/R : for sufficiently large L/R or h/R the beam mode is the most critical. Thus the results are qualitatively similar to those of the simply supported pipes (see 5.4.3).

6.4.4 Accuracy of the results and comparison with previous work

The asymptotic results obtained using (6.4), (6.7), (6.8) and (6.12) are compared with the numerical results of Weaver and Myklatun [20] in figure 6.2 for $L/R = 5$ and 10. It can be seen that there is some disagreement for larger values of h/R . A comparison of figure 4 of [20] and figure 6.1 also shows significant disagreement whenever $N > 1$.

The off-diagonal generalized pressure terms, which are neglected in the present analysis but not in [20], have been shown to be small (appendix E) and, therefore, are not expected to affect the stability results significantly. This is confirmed by an analysis similar to that of [20] in which the generalized pressures are evaluated numerically: the results with and without the off diagonal pressure terms agree to three significant figures and do not differ from the results obtained using (6.4), (6.8) and (6.10) when $L/R > 5$.

In an attempt to resolve this confusing disagreement the results from figure 4 of [20] are compared with results obtained using (6.4) without the off-diagonal stiffness terms and shown in figure 6.3. The generalized pressures are evaluated numerically in order to follow as closely as possible the calculations of [20]. It can be seen from this figure that the agreement is good except around the points where curves corresponding to different values of N (for fixed k) intersect (if the off-diagonal stiffness terms are neglected and if the off-diagonal pressure terms are insignificant, a single mode analysis is implied and the sharp intersections between the curves for different values of N is expected). This suggests that the off-diagonal stiffness terms are not included in the analysis of [20] (as implied there by equation (16)) and so the numerical results presented there have small but significant errors. However, a full explanation of the differences

in the numerical results cannot be found.

In figure 6.4 results from figure 7 of [18] and figure 8 of [20] are compared with results obtained using (6.4), (6.8) and (6.10). It can be seen that the present results agree well with those of [18] especially for large L/R . This is as expected since the travelling wave results of [18] may be considered to be sufficiently accurate for long pipes. The comparison indicates the connection between the infinite and the finite pipe analyses as for the simply supported pipe considered in the previous chapter.

The results of [20] differ significantly from the other two (figure 6.4) for large L/R before ^{the} beam mode dominates. Moreover, the difference increases with L/R : this is rather peculiar but is just what is expected if the off-diagonal stiffness terms are neglected because the significance of these terms in determining the stability condition increases with L/R .

In figure 6.2 the curves corresponding to $L/R = 5$ and $L/R = 10$ from [20] may be compared with the curves of the present analysis. The former curves cross near $h/R = 0.09$ and $h/R = 0.115$ while the latter do not cross. This characteristic is unexpected and is apparently due to the neglect of the off-diagonal stiffness terms in the earlier work.

6.4.5 Analysis using symmetric and antisymmetric modes

Figure 6.4 shows slight differences between the present results and those of [18] only in certain ranges of L/R . Since the present results are only based on symmetric modes (about $x = 1/2$), while the results of [18], with the travelling wave analysis, have no such bias, the results based on a general modal form, (4.16), are presented in figure 6.5 and compared with results from figure 7 of [18]. It can be seen that the agreement is quite good except for small L/R when the travelling wave

analysis may be expected to be strictly inapplicable.

The results obtained using (4.15) and (4.16) are also compared in figure 6.5 (the number of terms needed, in the truncated solution, to obtain the most critical condition is almost doubled when (4.16) is employed instead of (4.15)). It may be concluded, from this comparison, that the deflection at instability is predominantly symmetric with respect to $x = 1/2$ but that antisymmetric effects may become important for certain ranges of L/R .

6.4.6 Conclusions

In this chapter the stability of clamped-clamped pipes has been considered and results for the divergence stability conditions obtained by employing asymptotic forms for the generalized pressures. In the case $k = 1$ (beam mode) an analytical expression for the critical condition is obtained and shown to be quite accurate by comparison with previous numerical results. In the case $k > 1$ (shell mode) the off-diagonal generalized pressures are shown to be unimportant for long pipes and the problem is reduced to the determination of the smallest eigenvalue for a matrix whose elements are simple analytical expressions.

An error in the earlier analysis of Weaver and Myklatun [20] is corrected, and the results are found to agree with those of Paidoussis and Denise [18] except for small L/R when their results are not expected to be accurate enough (see chapter seven for more discussion of this point). The conclusions of [20] are, nevertheless, qualitatively correct with the most critical circumferential mode number decreasing as h/R and/or L/R increases. Moreover, the agreement with the travelling wave results of [18] indicates the relationship between the Flügge-Kempner equation and Flügge's equations (when the inertia terms are neglected in

both equations) as well as the connection between the infinite pipe and the finite pipe analysis results similar to that observed in the previous chapter.

Finally the most critical mode is shown to be essentially symmetric (as, indeed, is the case for simply supported pipes). The advantage of using modal forms which do not give off-diagonal structural terms, as with the simply supported pipe of the previous chapter, is obvious but it seems quite unlikely that this fact can be exploited in the present case.

CHAPTER 7

CLAMPED-FREE PIPES

§7.1 Introduction

In this chapter the stability of cantilevered pipes is considered by employing a generalization of the Galerkin method. This approach is one of several alternatives to the classical Ritz and Galerkin procedures developed in recent years specifically in connection with certain problems in the study of the stability of elastic systems subjected to non-conservative forces, for which the classical methods are not easily applicable. A comprehensive review of the development and the theoretical background of some of these alternatives can be found in several publications by Anderson and others (see [38] and references therein).

Benjamin [6], and Paidoussis and others [8-10], have shown that horizontal cantilevers, when vibrating in the beam mode, lose stability by flutter while, with gravity present, vertical cantilevers lose stability by divergence, flutter or both. Paidoussis and Denise have given experimental evidence of the significance of shell mode instabilities [37] and extended the beam mode results by employing Flügge's shell theory together with a combination of travelling wave solutions satisfying the clamped-free boundary conditions [18]. Their results demonstrate coupled mode oscillatory instability for flow velocities above a certain critical value and a damping effect for velocities below the critical.

The need to consider the cantilever problem using a Galerkin-type procedure is stimulated partly by the results of [18] and the results of the previous chapters on pipes with supported ends. There it is shown

that the stability condition for finite but long pipes is similar to the condition for infinitely long pipes obtained by a travelling wave or a standing wave analysis. Thus, extrapolating from the previous conclusions, the travelling wave results of [18] may be expected to be satisfactory for long pipes but to have increasing error as the pipe becomes shorter.

The main disadvantage of the Galerkin method, which makes it difficult to apply in the present problem, is that it requires the approximating functions to satisfy all the boundary conditions. Flügge's equations (2.2) and boundary conditions (2.6) are strongly coupled and, consequently, suitable approximating modes are necessarily complicated even for the Flügge-Kempner equation (2.3). With the possibility of coupled mode oscillatory instability, the inadequacy of the Flügge-Kempner equation becomes a serious problem: the neglect of the inplane inertia terms may be no longer easily justified. In this work the limitations of the Flügge-Kempner equation and the Galerkin method will be considered first in §7.2 by evaluating the natural frequencies in the absence of any fluid.

In §7.3 the characteristic equation for ω is obtained using the generalized Galerkin method: from Flügge's equations (2.2) an equivalent variational problem, incorporating the boundary conditions (2.6), is derived and the beam functions are employed to approximate the displacements u , v and w . In this formulation the effect of the flowing fluid also appears as the generalized pressures considered in Chapter 4.

Section 7.4 is concerned with the numerical solution of the characteristic equation and the discussion of the results. The expressions for the generalized pressures, corresponding to the different models for the flow beyond the ends of the cantilever (see §4.5 and §4.6), are used and the stability results are compared. More results are obtained using the linear profile model of 4.5.2.

§7.2 The limitations of the Flügge-Kempner equation

7.2.1 The approximating series

When the inplane inertia terms are neglected in the Flügge equations (2.2), u and v can be shown to be given in terms of w [21] by the relations

$$\begin{aligned} u &= \beta \left(\alpha^5 \frac{\partial^5 \Psi}{\partial x^5} - \alpha \frac{\partial^5 \Psi}{\partial x \partial \theta^4} \right) + \alpha \frac{\partial^3 \Psi}{\partial x \partial \theta^2} - \nu \alpha^3 \frac{\partial^3 \Psi}{\partial x^3} \\ v &= 2\beta \left(\alpha^4 \frac{\partial^5 \Psi}{\partial x^4 \partial \theta} + \alpha^2 \frac{\partial^5 \Psi}{\partial x^2 \partial \theta^3} \right) - (2 + \nu) \alpha^2 \frac{\partial^3 \Psi}{\partial x^2 \partial \theta} \\ &\quad - \frac{\partial^3 \Psi}{\partial \theta^3}, \end{aligned} \quad (7.1)$$

where $\Psi = \nabla_1^{-4} w$, with ∇_1 defined after (2.2), and terms of $O(\beta^2)$ have been neglected. When only the leading terms are retained in (7.1) in the limits $\alpha \rightarrow 0$ and $\beta \rightarrow 0$, Ψ may be replaced by $k^{-4} w$ so that, to leading order,

$$\begin{aligned} u &= -\frac{\alpha}{k^2} \frac{\partial w}{\partial x}, \\ w &= -\frac{\partial v}{\partial \theta}, \end{aligned} \quad (7.2)$$

where $\cos k\theta$ -dependence for w is assumed. Thus once w is specified, u and v are determined.

The relations (7.2), apart from arbitrary constants, have been employed in several studies involving vibration of cylindrical pipes (see, for example, [39]) and will be used in this chapter. Generally the solution for the deformation of the pipe is taken in the form

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \sum_{n,k} \begin{pmatrix} A_{nk} \alpha F_n'(x) \cos k\theta \\ B_{nk} F_n(x) \sin k\theta \\ C_{nk} F_n(x) \cos k\theta \end{pmatrix}, \quad (7.3)$$

where the A_{nk} 's, B_{nk} 's and C_{nk} 's are generalized coordinates, and a dash denotes differentiation with respect to x . In this section the A_{nk} 's and B_{nk} 's together with the $F_n(x)$ are determined so that each term (in curly brackets) on the right-hand side of (7.3) satisfies the cantilever boundary conditions (2.5) and (2.6).

One condition to be satisfied by $F_n(x)$ is obtained by subtracting the first and fourth equations in (2.6) and setting $w = F_n(x) \cos k\theta$.

This gives

$$\alpha^2 F_n''(1) - b_1 F_n(1) = 0, \quad (7.4)$$

where

$$b_1 = v(k^2 - 1),$$

and β has been neglected in comparison with unity. Similarly the rest of (2.6), after using (7.3) and (7.4), gives

$$A_{nk} = -\frac{1}{2k^2 - 1} C_{nk},$$

$$B_{nk} = -\frac{k}{2k^2 - 1} C_{nk},$$

$$\alpha^2 F_n'''(1) - b_2 F_n'(1) = 0, \quad (7.5)$$

where

$$b_2 = (2 - \nu) (k^2 - 1)$$

It remains to choose $F_n(x)$ to satisfy conditions (2.5), (7.4) and (7.5). These conditions are similar to the Donnell conditions on w (equivalent to $b_1 = \nu k^2$, $b_2 = (2 - \nu)k^2$) but have the property that they reduce to the beam conditions when $k = 1$.

By analogy with the cantilever beam problem, the following form is taken:

$$F_n(x) = \cos \lambda_n x - \cosh \lambda_n x - B_n (\sin \lambda_n x - \sinh \lambda_n x), \quad (7.6)$$

where λ_n and B_n are constants to be determined. In order to satisfy (7.4),

$$B_n = \frac{b_1 (\cos \lambda_n - \cosh \lambda_n) + \gamma_n^2 (\cos \lambda_n + \cosh \lambda_n)}{b_1 (\sin \lambda_n - \sinh \lambda_n) + \gamma_n^2 (\sin \lambda_n + \sinh \lambda_n)}, \quad (7.7)$$

with

$$\gamma_n = \alpha \lambda_n.$$

When (7.6) is substituted in (7.5) and (7.7) is used, the resulting equation may be simplified to give

$$\begin{aligned} \gamma_n^4 (1 + \cos \lambda_n \cosh \lambda_n) + \gamma_n^2 (b_2 - b_1) \sin \lambda_n \sinh \lambda_n \\ + b_1 b_2 (1 - \cos \lambda_n \cosh \lambda_n) = 0. \end{aligned} \quad (7.8)$$

Equation (7.8) is transcendental in λ_n and may be solved for the roots

$$\lambda_n, \quad n = 1, 2, 3, \dots \quad (7.9)$$

This completes the specification of the coordinate functions to be used with the Flügge-Kempner equation. When $k = 1$, b_1 and b_2 are zero and (7.8) becomes the characteristic equation for a clamped-free beam for which the λ_n 's are well-known, but $k = 2$ and 3 are of particular interest in the shell stability problem.

7.2.2 The frequency equation

The form for w given by (7.3), (7.6), (7.7) and (7.8) may be substituted in the Flügge-Kempner equation (2.3) to give

$$\sum_{n,k} C_{nk} [S_{nk}(x) - \frac{12R^2}{h^2} \omega^{*2} F_n(x)] \cos k\theta = 0, \quad (7.10)$$

where

$$S_{nk}(x) = F(\gamma_n, k) \{ \cos \lambda_n x - B_n \sin \lambda_n x \} \\ - F(i\gamma_n, k) \{ \cosh \lambda_n x - B_n \sinh \lambda_n x \},$$

$$\omega^* = \left\{ \frac{\rho_m R^2 (1 - \nu^2)}{E} \right\}^{1/2} \omega$$

and F is defined by (3.9). Applying the Galerkin method, (7.10) is multiplied by $F_m(x) \cos l\theta$ and the result integrated over the tube surface to give

$$\sum_{n,k} C_{nk} \epsilon_k \delta_{kl} [S_{mnk} - \omega^{*2} F_{mn}] = 0, \quad (7.11)$$

where

$$S_{mnk} = \int_0^1 S_{nk}(x) F_m(x) dx,$$

$$F_{mn} = \frac{12R^2}{h^2} \int_0^1 F_n(x) F_m(x) dx.$$

The requirement of a non-trivial solution for the C_{nk} 's obtains the frequency equation in the form

$$\det(\underline{F}^{-1} \underline{S} - \omega^2 \underline{I}) = 0 \quad (7.12)$$

where

$$\underline{F} = (F_{mn}), \quad \underline{S} = (S_{mn})$$

and $\underline{I}$ is the identity matrix.

7.2.3 The approximate natural frequencies

Equation (7.12) may be solved by standard procedures, using a truncated solution, to obtain approximations to the natural frequencies of the lower order modes. Table 7.1 gives a sample of the results when the first five terms are retained in (7.3). For each L/R the first row gives the results from (7.12) while the third row gives the results from an analysis similar to that for the cubic case of Sharma [39] (and are identical to those given in table 1 therein); the origin of the second row will be discussed later.

When the shell mode results are compared it is seen that for small values of k (<5 in the table) the n^{th} axial mode results from the first row seem to approximate the $(n+1)^{\text{th}}$ mode results from the third row,

while for large values of k (≥ 5 in the table) the first two axial mode results compare reasonably well. In general, however, the results are far from accurate and it may be concluded that either the inplane inertia terms, neglected here but included in [39], are important or the modal functions are so far from the actual modes that a large number of them would be necessary for adequate accuracy. Moreover, for $k > 1$, the results for the lower circumferential modes seem to indicate that the modal functions cannot approximate the first axial mode. The significance of the inplane inertias, as well as the relevance of the modal functions, will be examined later. At this stage the approach employing the Flügge-Kempner equation must be considered unsatisfactory. In the next section the problem is re-examined using the generalized Galerkin method.

§7.3 The generalized Galerkin method

7.3.1 The governing equation

In this approach the natural boundary conditions (2.6) are incorporated in the governing equation so that the functions used to approximate the deformation of the pipe need not satisfy these conditions (the shell equations and boundary conditions are satisfied in the mean). The governing equation is obtained by equating to zero the result of adding the sum of the weighted integrals of Flügge's equations (2.2), over the surface S of the cantilever, to particular multiples of boundary conditions (2.6). The weighting functions are the variations δu , δv and $-\delta w$ for the first, the second and the third of (2.2) respectively, while the boundary condition multipliers are chosen in such a way that a certain variational equation is satisfied when the fluid loading is zero. Thus, denoting the multipliers by B_j , $j = 1, 2, 3, 4$, the above procedure gives

$$\begin{aligned}
& \int_0^{2\pi} d\theta \int_0^1 \left[\left\{ \alpha^2 \frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2} (1+\beta) \frac{\partial^2 u}{\partial \theta^2} + \omega^*{}^2 u \right\} \delta u \right. \\
& + \frac{1+\nu}{2} \alpha \left\{ \frac{\partial^2 u}{\partial x \partial \theta} \delta v + \frac{\partial^2 v}{\partial x \partial \theta} \delta u \right\} + \nu \alpha \left\{ \frac{\partial w}{\partial x} \delta u \right. \\
& - \frac{\partial u}{\partial x} \delta w \left. \right\} - \beta \alpha^3 \left\{ \frac{\partial^3 w}{\partial x^3} \delta u - \frac{\partial^3 u}{\partial x^3} \delta w \right\} \\
& + \frac{1-\nu}{2} \beta \alpha \left\{ \frac{\partial^3 w}{\partial x \partial \theta^2} \delta u - \frac{\partial^3 u}{\partial x \partial \theta^2} \delta w \right\} \\
& + \left\{ \frac{\partial^2 v}{\partial \theta^2} + \frac{1-\nu}{2} \alpha^2 (1+3\beta) \frac{\partial^2 v}{\partial x^2} + \omega^*{}^2 v \right\} \delta v \\
& + \frac{\partial w}{\partial \theta} \delta v - \frac{\partial v}{\partial \theta} \delta w - \frac{3-\nu}{2} \beta \alpha^2 \left\{ \frac{\partial^3 w}{\partial x^2 \partial \theta} \delta v \right. \\
& - \frac{\partial^3 v}{\partial x^2 \partial \theta} \delta w \left. \right\} - \left\{ (1+\beta)w + \beta (\nabla_1^4 w + 2 \frac{\partial^2 w}{\partial \theta^2}) \right. \\
& \left. - \omega^*{}^2 w + \frac{q}{\rho_m h} p(w) \right\} \delta w \Big] dx + \sum_{j=1}^4 B_{1j}^* = 0, \tag{7-13}
\end{aligned}$$

where

$$\begin{aligned}
B_{11}^* &= B_1 \left\{ \alpha \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial \theta} + \nu w - \beta \alpha^2 \frac{\partial^2 w}{\partial x^2} \right\}_{x=1} \\
B_{12}^* &= B_2 \left\{ \frac{\partial u}{\partial \theta} + (1+3\beta) \alpha \frac{\partial v}{\partial x} - 3\beta \alpha \frac{\partial^2 w}{\partial x \partial \theta} \right\}_{x=1} \\
B_{13}^* &= B_3 \left\{ \alpha^3 \frac{\partial^3 w}{\partial x^3} + (2-\nu) \alpha \frac{\partial^3 w}{\partial x \partial \theta^2} - \frac{3-\nu}{2} \alpha \frac{\partial^2 v}{\partial x \partial \theta} \right. \\
& \left. + \frac{1-\nu}{2} \frac{\partial^2 u}{\partial \theta^2} - \alpha^2 \frac{\partial^2 u}{\partial x^2} \right\}_{x=1}
\end{aligned}$$

$$B_{14}^* = B_4 \left\{ \alpha \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial \theta} - v \frac{\partial^2 w}{\partial \theta^2} - \alpha^2 \frac{\partial^2 w}{\partial x^2} \right\}_{x=1}.$$

Integration by parts, assuming (2.5) to be satisfied at the clamped end $x = 0$, gives

$$\begin{aligned} \delta E^* - \frac{q}{\rho_m h} \iint p(w) \delta w \, dx \, d\theta \\ + \sum_{j=1}^4 (B_{1j}^* + B_{2j}^*) = 0, \end{aligned} \quad (7.14)$$

where

$$E^* = T - V,$$

$$T = \frac{1}{2} \omega^2 \int_0^{2\pi} d\theta \int_0^1 (u^2 + v^2 + w^2) dx$$

$$V = \frac{1}{2} \int_0^{2\pi} d\theta \int_0^1 \left[\left(\alpha \frac{\partial u}{\partial x} \right)^2 + \left(v + \frac{\partial v}{\partial \theta} \right)^2 \right.$$

$$\left. + \frac{1-v}{2} \left(\alpha \frac{\partial v}{\partial x} + \frac{\partial u}{\partial \theta} \right)^2 + 2v\alpha \frac{\partial u}{\partial x} \left(\frac{\partial v}{\partial \theta} + w \right) \right.$$

$$\left. + \beta \left[\left(\alpha^2 \frac{\partial^2 w}{\partial x^2} \right)^2 + \left(w + \frac{\partial^2 w}{\partial \theta^2} \right)^2 - 2\alpha^3 \frac{\partial u}{\partial x} \frac{\partial^2 w}{\partial x^2} \right] \right.$$

$$\left. - 2v\alpha^2 \frac{\partial^2 w}{\partial x^2} \left(\frac{\partial v}{\partial \theta} - \frac{\partial^2 w}{\partial \theta^2} \right) \right.$$

$$\left. - \frac{1-v}{2} \left\{ 6\alpha^2 \frac{\partial v}{\partial x} \frac{\partial^2 w}{\partial x \partial \theta} - 4 \left(\alpha \frac{\partial^2 w}{\partial x \partial \theta} \right)^2 \right\} \right]$$

$$- 3 \left(\alpha \frac{\partial v}{\partial x} \right)^2 - \left(\frac{\partial u}{\partial \theta} \right)^2 - 2 \alpha \frac{\partial^2 w}{\partial x \partial \theta} \frac{\partial u}{\partial \theta} \}] dx$$

$$B_{21}^* = \alpha \int_0^{2\pi} \left[\left\{ \alpha \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial \theta} + v w - \beta \alpha^2 \frac{\partial^2 w}{\partial x^2} \right\} \delta u \right]_{x=1} d\theta$$

$$B_{22}^* = \frac{1-v}{2} \alpha \int_0^{2\pi} \left[\left\{ \frac{\partial u}{\partial \theta} + (1+3\beta) \alpha \frac{\partial v}{\partial x} - 3\beta \alpha \frac{\partial^2 w}{\partial x \partial \theta} \right\} \delta v \right]_{x=1} d\theta$$

$$B_{23}^* = \beta \alpha \int_0^{2\pi} \left[\left\{ \alpha^2 \frac{\partial^2 u}{\partial x^2} - \frac{1-v}{2} \frac{\partial^2 u}{\partial \theta^2} - \alpha^3 \frac{\partial^3 w}{\partial x^3} \right. \right. \\ \left. \left. - (2-v) \alpha \frac{\partial^3 w}{\partial x \partial \theta^2} + \frac{3-v}{2} \alpha \frac{\partial^2 v}{\partial x \partial \theta} \right\} \delta w \right]_{x=1} d\theta$$

$$B_{24}^* = \beta \alpha \int_0^{2\pi} \left[\left\{ \alpha^2 \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial \theta^2} - v \frac{\partial v}{\partial \theta} \right. \right. \\ \left. \left. - \alpha \frac{\partial u}{\partial x} \right\} \delta \left(\alpha \frac{\partial w}{\partial x} \right) \right]_{x=1} d\theta$$

(7.15)

The choice $B_{1j}^* = -B_{2j}^*$ is now made so that (7.14) becomes

$$\delta E^* - \frac{\alpha}{\rho_m h} \int_0^{2\pi} d\theta \int_0^1 p(w) \delta w dx = 0. \quad (7.16)$$

Equation (7.16) is the governing equation for the generalized Galerkin procedure.

Equations (7.13) to (7.16) imply that when $p = 0$, the variation in E^* gives Flügge's equations (2.2) together with boundary conditions (2.5) and (2.6). Thus, if suitable functions are chosen for the variation in E^* , the boundary conditions are automatically included. With $p \neq 0$, equations (7.16) shows how to add in the effect of fluid pressure. In

this chapter the beam functions, corresponding to $k = 1$ in (7.6) to (7.8), will be employed, together with (7.3) and (7.16), to determine the stability conditions. However, in order to check the analysis, the natural frequencies will be determined first using (7.16) with $p = 0$: this form of the equation has been employed in previous frequency analyses (see, for example, [39]).

7.3.2 The characteristic equation

When (7.3) is substituted in (7.16) using (7.15) for E^* and (2.11) for $p(w)$, the integrals may be evaluated and the variations taken to give

$$\frac{\partial E^*}{\partial A_{mk}} = \frac{\partial E^*}{\partial B_{mk}} = \frac{\partial E^*}{\partial C_{mk}} + \frac{q}{R\rho} \frac{h}{\underline{m}} \frac{\alpha}{2\pi} \sum_n C_{nk} P_{mn,kk} = 0,$$

$$m = 1, 2, 3, \dots, \quad (7.17)$$

where $P_{mn,kk}$ are the generalized pressures defined by (2.14) and considered in chapter 4 for different forms of the flow beyond the ends of the cantilever. The expression for δE^* , derived in Appendix F may now be substituted in (7.17) and (4.24) employed for $P_{mn,kk}$ to give an infinite system of homogeneous linear equations for the A_{nk} 's, B_{nk} 's and C_{nk} 's. The characteristic equation for ω is then obtained by requiring a non-trivial solution. When the modal representation (7.3) is truncated at N terms, the corresponding system of equations may be expressed in matrix form

$$\underline{A}\underline{x} = 0, \quad (7.18)$$

where A is a matrix of order $3N \times 3N$ and x is a column vector, of dimension $3N$, defined by

$$\underline{A} = (a_{mn}),$$

$$\underline{x}^T = (x_1, x_2, \dots, x_{3N}),$$

with T denoting the transpose, and

$$a_{mn} = \left\{ \begin{array}{l} \alpha^4 \lambda_n^4 \delta_{mn} + \left\{ \frac{1-\nu}{2} k^2 (1+\beta) - \omega^{*2} \right\} J_{mn}, \quad 1 \leq n \leq N, \\ \\ \nu k I_{nm} - \frac{1-\nu}{2} k J_{mn}, \quad N < n \leq 2N, \\ \\ \nu I_{nm} + \beta \left(\frac{1-\nu}{2} k^2 J_{mn} - \alpha^4 \lambda_n^4 \delta_{mn} \right), \quad 2N < n \leq 3N \end{array} \right\} \quad 1 \leq m \leq N,$$

$$\left\{ \begin{array}{l} k(\nu I_{mn} - \frac{1-\nu}{2} J_{mn}), \quad 1 \leq n \leq N, \\ \\ (k^2 - \omega^{*2}) \delta_{mn} + \frac{1-\nu}{2} (1+3\beta) J_{mn}, \quad N < n \leq 2N, \\ \\ k[\delta_{mn} + \beta \{ \frac{3}{2} (1-\nu) J_{mn} - \nu I_{mn} \}], \quad 2N < n \leq 3N \end{array} \right\} \quad N < m \leq 2N,$$

$$\left\{ \begin{array}{l} \nu I_{mn} + \beta \left(\frac{1-\nu}{2} k^2 J_{mn} - \alpha^4 \lambda_n^4 \delta_{mn} \right), \quad 1 \leq n \leq N, \\ \\ k[\delta_{mn} + \beta \{ \frac{3}{2} (1-\nu) J_{mn} - \nu I_{nm} \}], \quad N < n \leq 2N, \\ \\ \{ 1 + \beta (k^2 - 1)^2 \} \delta_{mn} + \beta [\alpha^4 \lambda_n^4 \delta_{mn} \\ + k^2 \{ 2(1-\nu) J_{mn} - \nu (I_{mn} + I_{nm}) \}] \end{array} \right\} \quad 2N < m \leq 3N.$$

$$\left[\begin{array}{l} -\omega^{*2} \left(\delta_{mn} + \frac{\mu}{\alpha\pi} P_{mn1} \right) \\ -\frac{2i\omega^*}{\pi} P_{mn2} \left(\mu\beta \frac{\rho U^2 R^3}{D} \right)^{1/2} \\ -\frac{\alpha\beta}{\pi} \frac{\rho U^2 R^3}{D} P_{mn3}, \quad 2N < n \leq 3N \end{array} \right]$$

$$x_n = \left\{ \begin{array}{ll} A_{nk}, & 1 \leq n \leq N, \\ B_{nk}, & N < n \leq 2N, \\ C_{nk}, & 2N < n \leq 3N. \end{array} \right. \quad (7.19)$$

Here

$$I_{mn} = \alpha^2 \int_0^1 F_m(x) F_n''(x) dx,$$

$$J_{mn} = \alpha^2 F_m(1) F_n'(1) - I_{mn},$$

where a dash denotes differentiation with respect to x .

The requirement of a non-trivial solution gives the characteristic equation

$$\det \underline{A} = 0. \quad (7.20)$$

The determination of the critical stability condition involves solving (7.20) for the frequencies, ω , as the stability parameter $\rho U^2 R^3/D$ is varied. An obvious difficulty is the size of the determinant. The computing time can be greatly reduced if an approximation to the critical $\rho U^2 R^3/D$ is obtained first by neglecting the inplane inertias.

When these inertias are neglected the size of the characteristic determinant can be reduced to $N \times N$.

To obtain the reduced determinant, equation (7.18) is rewritten in the form

$$\begin{bmatrix} \underline{A}_1 & \underline{B}_1 & \underline{C}_1 \\ \underline{A}_2 & \underline{B}_2 & \underline{C}_2 \\ \underline{A}_3 & \underline{B}_3 & \underline{C}_3 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \end{bmatrix} = \underline{0}, \quad (7.21)$$

where the matrices $\underline{A}_j$, $\underline{B}_j$ and $\underline{C}_j$ are of order $N \times N$ and $\underline{x}_j$ are column vectors of order N , and are given by

$$\begin{aligned} \underline{A}_1 &= (a_{mn})_{\omega=0} & \underline{A}_2 &= (a_{m+N \ n}) \\ \underline{A}_3 &= (a_{m+2N \ n}) & \underline{B}_1 &= (a_{m \ n+N}) \\ \underline{B}_2 &= (a_{m+N \ n+N})_{\omega=0} & \underline{B}_3 &= (a_{m+2N \ n+N}) \\ \underline{C}_1 &= (a_{m \ n+2N}) & \underline{C}_2 &= (a_{m+N \ n+2N}) \\ \underline{C}_3 &= (a_{m+2N \ n+2N}) & \underline{x}_1^T &= (x_1 \ x_2 \ \dots \ x_N) \\ \underline{x}_2^T &= (x_{N+1} \ x_{N+2} \ \dots \ x_{2N}) \\ \underline{x}_3^T &= (x_{2N+1} \ x_{2N+2} \ \dots \ x_{3N}) , \end{aligned} \quad (7.22)$$

with a_{mn} and x_n being defined by (7.19), and T denoting transpose.

Now x_1 and x_2 may be eliminated from (7.21) to give

$$\underline{B} \underline{x}_3 = 0, \quad (7.23)$$

where

$$\begin{aligned} \underline{B} = & \underline{A}_3 (\underline{B}_1^{-1} \underline{A}_1 - \underline{B}_2^{-1} \underline{A}_2)^{-1} (\underline{B}_2^{-1} \underline{C}_2 - \underline{B}_1^{-1} \underline{C}_1) \\ & + \underline{B}_3 (\underline{A}_1^{-1} \underline{B}_1 - \underline{A}_2^{-1} \underline{B}_2)^{-1} (\underline{A}_2^{-1} \underline{C}_2 - \underline{A}_1^{-1} \underline{C}_1) + \underline{C}_3. \end{aligned}$$

The characteristic equation neglecting inplane inertias follows from

(7.23) by requiring a non-trivial solution for $\underline{x}_3$:

$$\det \underline{B} = 0 \quad (7.24)$$

7.3.3 An equivalent eigenvalue problem

Although the characteristic equation (7.20) or (7.24) may be solved for the critical $\rho U^2 R^3/D$ using Routh's criteria, the procedure becomes very cumbersome when N is large. However, since the matrices involved are quadratic in ω^* , an equivalent eigenvalue problem may be obtained. Consider, for example, (7.24). The matrix B can be rewritten

$$\underline{B} = \underline{C} + \omega^* \underline{D} + \omega^{*2} \underline{E} \quad (7.25)$$

where

$$\underline{C} = \underline{B} (\omega = 0)$$

and the $N \times N$ matrices $\underline{D}$ and $\underline{E}$ do not involve ω and are given by

$$\underline{D} = (d_{mn}), \quad \underline{E} = (e_{mn}),$$

with

$$\begin{aligned} d_{mn} &= -\frac{2i}{\pi} P_{m+2N} P_{n+2N} \left(\mu \beta \frac{\rho U^2 R^3}{D} \right)^{1/2}, \\ e_{mn} &= -\left(\delta_{mn} + \frac{\mu}{\alpha \pi} P_{m+2N} P_{n+2N} \right). \end{aligned} \quad (7.26)$$

Equation (7.23), with $\underline{B}$ in the form (7.25), may be multiplied on the left by $\underline{E}^{-1}$ to give

$$\underline{E}^{-1} \underline{D} \underline{y} + \underline{E}^{-1} \underline{C} \underline{x}_3 = -\omega^* \underline{y} \quad (7.27)$$

where

$$\underline{y} = \omega^* \underline{x}_3. \quad (7.28)$$

Equations (7.27) and (7.28) are equivalent to

$$[\underline{M} - \omega^* \underline{I}] \begin{bmatrix} \underline{y} \\ \underline{x}_3 \end{bmatrix} = \underline{0} \quad (7.29)$$

where

$$\underline{M} = \begin{bmatrix} -\underline{E}^{-1} \underline{D} & -\underline{E}^{-1} \underline{C} \\ \underline{I} & \underline{0} \end{bmatrix} \quad (7.30)$$

and I is the identity matrix. Equation (7.29) is the required eigenvalue problem corresponding to (7.24). A similar equation corresponding to (7.20) may be obtained: the matrix $\underline{M}$ is of order $6N \times 6N$ and is defined by (7.30) with

$$\underline{C} = \underline{A} (\omega = 0), \quad (7.31)$$

$\underline{A}$ being defined after (7.18), and instead of (7.26),

$$\begin{aligned} d_{mn} &= \begin{cases} 0, & m, n < 2N+1 \\ -\frac{2i}{\pi} (\mu\beta \frac{\rho U_R^3}{D})^{1/2} P_{mn2}, & 2N+1 \leq m, n \leq 3N \end{cases} \\ e_{mn} &= \begin{cases} -J_{mn}, & 1 \leq m, n \leq N, \\ -\delta_{mn}, & N < n \leq 2N, \\ -(\delta_{mn} + \frac{\mu}{\alpha\pi} P_{mni}), & 2N < m, n \leq 3N, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (7.32)$$

§7.4 Numerical solution of the eigenvalue problem and discussion of the results

7.4.1 Introduction

The rest of this chapter is mainly concerned with the determination of the smallest value of $\rho U^2 R^3/D$ above which at least one of the roots for ω has a positive imaginary part. This critical value will be obtained by numerical means (for details see appendix G). The natural frequencies will be obtained first in order to determine the significance of the inplane inertia terms neglected in the Flügge-Kempner equation and the relevance of the modal functions employed in §7.2.

7.4.2 The approximate natural frequencies

While it was instructive in §7.2 to check the accuracy of the method employing the Flügge-Kempner equation by evaluating the approximate natural frequencies, the accuracy of the present method, at least in predicting these frequencies, is known [39, 40].

The results are compared in tables 7.1 and 7.2. In table 7.1 the third row, for each k , is obtained from a single mode analysis using (7.29) to (7.32) as in [39]. However a coupled mode analysis gives slightly better results as can be seen from table 7.2.

From table 7.1, a comparison of the second and third row results shows that the inplane inertias have a large effect in some cases (see, for example, $k = n = 1$ where the change is about 35%) and therefore cannot be neglected. It may be noted that if the relations (7.2) to (7.5) are approximately satisfied, then the effects of neglecting these inertias, which corresponds to neglecting u and v in the kinetic energy term, T , defined after (7.14), may be expected to be large, especially when $k = 1$, and to decrease with increasing k and/or L/R .

A comparison of the first and second row results (for $k > 1$), both of which correspond to a coupled mode analysis with $N = 5$ and the inplane inertias neglected, shows that the large differences between the results obtained from (7.12) and the results of [39] (third row) is mainly due to the choice of the modal functions.

For non-zero values of $\rho U^2 R^3/D$ the frequency parameter ω^* is generally complex due to fluid damping. As $\rho U^2 R^3/D$ is increased from zero the frequency whose imaginary part first becomes positive varies in the manner typical of that shown in figure 7.1 (the results correspond to the linear profile model for the flow beyond the free end of the cantilever - see 4.5.2). Since the real part of ω^* in this case is non-zero at the point of instability (when the imaginary part changes sign) the instability is of the flutter type. The results obtained later show that provided N is not too small the stability boundary is always of the flutter type, in agreement with previous results.

7.4.3 Convergence of the solution

The critical value of $\rho U^2 R^3/D$ depends on the number N of terms retained in the truncated modal solution (7.3). In the conventional Galerkin method the calculated results converge to the correct results as N increases, provided that suitable modal functions are employed, and it may be expected that this is the case with the present method. However, numerical difficulties may arise for N large because of ill-conditioning and excessive time being needed to retain precision with larger matrices. A certain amount of judgement is needed to exclude spurious instabilities arising because of numerical errors with a large number of terms.

Table 7.3 gives the critical $\rho U^2 R^3/D$ as N varies from 2 to 6. The beam functions employed with (7.3) preclude the consideration of $N = 1$.

In fact the first modal form cannot become unstable because the fluid leaves the free end of the cantilever in such a way as to always oppose the deflection of the pipe. Benjamin [6] has shown that the slope of the deformed shape of the cantilever at the free end, when vibrating in the beam mode at the point of instability, has the opposite sign of the deflection. This is physically consistent with instability since the fluid exerts a force at the free end tending to increase the deflection.

The results for $k = 1$ show that the difference between consecutive values decrease at first as N increases but increase later as N becomes larger (see, for example, $L/R = 5, 7.5$). This characteristic is also present in the shell mode results ($k = 2, L/R = 5$) and appears to be associated with the problems mentioned earlier in connection with large values of N . Thus no objective convergence can be said to exist in general but if N is limited to values below the one above which numerical errors cause fluctuations in the critical $\rho U^2 R^3/D$, the results in table 7.3 may be taken as an approximation to the correct values (the results in table 7.3 do not include the effect of the inplane inertias; this is considered later).

For large L/R the results also indicate that N must be large before good accuracy is obtained. This is particularly the case for the shell modes (see, for example, $k = 3, L/R > 15$). A similar characteristic was encountered earlier for the simply supported and the clamped-clamped pipes. Other results for fixed L/R and various values of h/R and μ indicate that the variation of $\rho U^2 R^3/D$ with N is similar to that in table 7.3, suggesting that accuracy is more or less independent of h/R and μ . Thus the value of N giving the approximate critical $\rho U^2 R^3/D$ for a given L/R may be determined from a table similar to 7.3. The results in the rest of this work are obtained for a set value of N .

7.4.4 Effects of flow behaviour beyond the ends of the pipe

Table 7.4 gives values for the critical $\rho U^2 R^3/D$ corresponding to the different solutions for the flow beyond the ends of the cantilever (see §4.5 and §4.6). The inplane inertia effects are neglected but they are not expected to be important in this comparison.

It can be seen that the difference between the results is small when α is small: the flow details beyond the ends can have only a small effect on the pressure fluctuations inside a long pipe. For $\alpha < 0.1$ when $k = 1$, or $\alpha < 0.2$ when $k > 1$, the long pipe approximation (4.31) to the generalized pressures may be employed. This result is significant because it shows that the simple point relationship, between fluid pressure and pipe wall deformation, which has often been employed for the beam mode may be used also for the shell modes.

It can also be seen that, of the four solutions, only the linear profile and the oscillatory profile give similar results for all values of α considered. Since these solutions are quite different it may be inferred that they represent better models for the flow beyond the free end for all values of α . In the rest of this chapter results will be obtained using the linear profile model with a rigid pipe upstream of the clamped end.

7.4.5 Effect of inplane inertias

The need to take N large when α is small is very inconvenient because, if the inplane inertia terms are not neglected, the order of the matrix $\tilde{M}$ in the eigenvalue problem defined by (7.29) to (7.32) will be large. The inplane inertia terms have often been assumed to have a negligible effect on the stability condition for other configurations but the results of 7.4.2 seem to suggest that these inertias may be important.

Figures 7.2 to 7.4 give a comparison of the results with and without inplane inertia effects. In obtaining the results including the inplane inertias it is found that, in certain cases (when α is large and k is small) the analysis indicates instability as soon as there is any flow. However, an examination of the frequency spectrum shows that this characteristic is associated with some frequencies (among the first $4N$ with the largest real parts and therefore corresponding to the inplane deformations) whose imaginary part is always positive and almost independent of variations in $\rho U^2 R^3/D$. The instabilities indicated by these frequencies are spurious and may be suppressed by the inclusion of a small amount of damping.

Figures 7.2 and 7.3 show the variation of the critical $\rho U^2 R^3/D$ with L/R , when $N = 4$, for $k = 1$ and $k = 2$ respectively. It can be seen that the results without inplane inertia effects are larger and the difference varies from 15% to 10% when $k = 1$, and from 3% to 20% when $k = 2$, as L/R varies from 7.5 to 30. Figure 7.4 shows the effects of neglecting the inplane inertias as $\mu/(\mu+2)$ varies from C to 0.8 (this mass ratio parameter correspond to the parameter β used in [8]). The results for $k = 1$ show that the neglect of the inplane inertia terms may give differences of up to 120% (see for example $\mu/(\mu+2) = 0.2$). For small or large values of μ , the differences are small. Results corresponding to other values of L/R are similar to those of figure 7.4.

From the above discussion it may be concluded that the inplane inertias are important for moderate values of μ , but may be neglected when μ is sufficiently small or large.

7.4.6 Effect of thickness ratio

Table 7.5 gives the critical $\rho U^2 R^3/D$ for different values of h/R and $L/R = 10, 15$ and 20 . The results for $L/R = 20$ are plotted in figure 7.5.

For the three circumferential modes considered and for small h/R , the $k = 3$ mode is the most critical. However, as h/R increases the instability passes down to $k = 2$ and then to $k = 1$. This means that there are transition values, for h/R when L/R is fixed and for L/R when h/R is fixed, at which the circumferential mode number associated with instability changes from one value to another. The transition takes place at a smaller h/R when L/R is large.

7.4.7 Effect of length ratio and comparison with previous work

In figure 7.6 results obtained from the present analysis, including the inplane inertias, are compared with results from figure 8 of [18]. The stability parameter, $\bar{U}$, used in [18] is given by

$$\bar{U}^2 = \frac{h^2}{12R^2\mu} \left\{ \frac{\rho U^2 R^3}{D} \right\}. \quad (7.33)$$

The comparison for $k = 1$ shows that the present analysis gives slightly higher values for the critical $\bar{U}$. However the agreement is quite good, especially for large L/R as expected, and the small differences may be attributed to the effect of the external fluid on the results of [18] as well as the different flow conditions beyond the ends of the pipe.

The results for $k = 2$ also compare quite well (considering the effects mentioned above) except for $13 < L/R < 22$ where there are significant differences, with the present results being smaller. The comparison for $k = 3$ is similar: there is good agreement for $L/R > 18$ but significant differences for smaller L/R . Thus, for the shell modes and for certain

values of L/R , the differences are larger than may be expected from the effects mentioned in the previous paragraph, and a full explanation requires an examination of the modal functions employed in [18] together with a check on the convergence of the iterative procedure. The results are, nevertheless, qualitatively similar: the most critical circumferential mode number decreases with increasing L/R .

CHAPTER 8

DISCUSSION AND CONCLUSIONS

§8.1 Introduction

In this chapter some general conclusions from the previous chapters, together with limits and possible extensions of the analysis, are discussed.

Among the factors which have been excluded from the foregoing and which might be influential in modifying the stability results are departures from a uniform axial flow in the form of a small swirl component, unsteadiness in the mean flow and/or effects due to the boundary layer near the tube wall. The latter would entail a full-scale study in its own right but to make a qualitative assessment of the boundary layer, a simplified model in the form of an inviscid shear flow profile in close proximity to the wall offers a relatively straightforward approach but one with obvious limitations since it cannot examine the Tollmien-Schlichting wave instabilities. In the following section a brief study is made of the significance of the stability parameter corrections caused by the simple model of the fluid boundary layer: the stability of infinitely long elastic pipes is considered for the case when the mean flow velocity $(0, 0, U)$ is a constant ($U = U_\infty$) except over a thin layer of thickness δ (non-dimensionalized with respect to R), adjacent to the pipe wall, where it varies with r , from zero on the wall to U_∞ on the edge of the layer. In the present analysis, as in previous plate analyses (see, for example [41]), the effects of viscosity appear only in determining the mean flow profile but, for simplicity, the perturbations are considered inviscid (i.e. the inviscid shear flow model). Moreover, it will also be assumed that the amplitude of the pipe wall

deflection is small compared with the shear layer thickness, thus allowing perturbation quantities to be evaluated on the mean, rather than the actual, position of the pipe wall.

§8.2 Shear layer effects

8.2.1 The governing equation

The system considered here is that of §3.2 except that the fluid is incompressible and the mean flow speed, U , is given by

$$U = \begin{cases} U_{\infty}, & 0 \leq r \leq 1-\delta, \\ U(r), & 1-\delta \leq r \leq 1, \end{cases} \quad (8.1)$$

where δ is the non-dimensional (with respect to R) thickness of the shear layer adjacent to the pipe wall and U_{∞} is a constant. The perturbed flow velocity $\underline{u}$ and pressure P may be expressed as a sum of the mean flow values and a perturbation velocity (w, v, u), in cylindrical polar coordinates (r, θ, x) , and pressure p :

$$\begin{aligned} \underline{u} &= (w, v, u+U), \\ P &= p_0 + p, \end{aligned} \quad (8.2)$$

where p_0 is the mean flow pressure. In this subsection an equation for the determination of the perturbation pressure, p , is derived. To do this, the continuity and inviscid momentum equations are employed. Thus, with (8.2), the continuity equation gives

$$\frac{\partial u}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (rw) + \frac{1}{r} \frac{\partial v}{\partial \theta} = 0, \quad (8.3)$$

and the momentum equations are

$$R \frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + w \frac{\partial U}{\partial r} = - \frac{1}{\rho} \frac{\partial p}{\partial x}$$

$$R \frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial r}$$

$$R \frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} = - \frac{1}{r\rho} \frac{\partial p}{\partial \theta} \quad (8.4)$$

A travelling wave solution is taken in the form

$$\{u, v, w, p\} = e^{i(\lambda x - \omega t)} \{\hat{u} \cos k\theta, \hat{v} \sin k\theta, \hat{w} \cos k\theta, \hat{p} \cos k\theta\}, \quad (8.5)$$

which, on substituting in (8.3) and (8.4), gives

$$i\lambda \hat{u} + \frac{1}{r} \frac{d}{dr} (r\hat{w}) + \frac{k}{r} \hat{v} = 0 \quad (8.6)$$

$$i\hat{u}(\lambda U - R\omega) + \hat{w}U' = - \frac{1}{\rho} i\lambda \hat{p}, \quad (8.7)$$

$$i\hat{w}(\lambda U - R\omega) = - \frac{1}{\rho} \hat{p}', \quad (8.8)$$

$$i\hat{v}(\lambda U - R\omega) = - \frac{k}{r\rho} \hat{p}, \quad (8.9)$$

where a dash denotes differentiation with respect to r .

After eliminating $\hat{u}$, $\hat{v}$ and $\hat{w}$ from (8.6) to (8.9), and simplifying, the following equation is obtained:

$$\frac{d^2 p}{dr^2} + \left(\frac{1}{r} - \frac{2\lambda U'}{\lambda U - R\omega} \right) \frac{dp}{dr} - \left(\frac{k^2}{r^2} + \lambda^2 \right) p = 0. \quad (8.10)$$

Equation (8.10) is essentially Rayleigh's equation for the perturbation pressure. In [30] the corresponding equation for $\hat{w}$ (equation (2.16) therein), to which (8.10) reduces when (8.8) is employed, is derived but the form (8.10) is simpler to use in this context. The rest of this section is concerned with the solution of (8.10) and the related stability condition.

8.2.2 The perturbation pressure

In the uniform stream $U = U_\infty$ (see (8.1)), $U' = 0$ and (8.10) reduces to the modified Bessel equation with solution (bounded at $r = 0$)

$$p = A I_k(r\lambda), \quad (8.11)$$

where A is an arbitrary constant to be determined.

For the shear layer $1 - \delta \leq r \leq 1$, introduce the variable

$$\xi = \frac{1 - r}{\delta}$$

so that (8.10) becomes

$$\frac{d^2 p}{d\xi^2} - \left\{ \frac{\delta}{1 - \xi\delta} + \frac{2}{U - c} \frac{dU}{d\xi} \right\} \frac{dp}{d\xi} - \left\{ \frac{k^2 \delta^2}{(1 - \xi\delta)^2} + \lambda^2 \delta^2 \right\} p = 0, \quad (8.12)$$

where $c = \omega R/\lambda$.

To determine the pressure on the pipe wall, a wall deformation is taken in the form

$$y(x, \theta, t) = B e^{i(\lambda x - \omega t)} \cos k\theta \quad (8.13)$$

where B is non-dimensional. In appendix H the solution of (8.12) is obtained for $\delta \ll 1$ under the assumptions $\lambda\delta \ll 1$, $B \ll \delta$ and $U \neq c$ anywhere in the shear layer (i.e. there is no critical layer). The result is, to $O(\delta)$,

$$\begin{aligned} p(0) = & B\rho\{U_\infty - c\}^2 \frac{\lambda I_k(\lambda)}{I_k'(\lambda)} \left[1 - \lambda\delta \frac{I_k'(\lambda)}{I_k(\lambda)} \right. \\ & - \int_0^1 \left(\frac{U-c}{U_\infty-c}\right)^2 d\xi - (k^2 + \lambda^2) \left\{ \frac{I_k(\lambda)}{\lambda I_k'(\lambda)} \right\}^2 \left\{1 \right. \\ & \left. \left. - \int_0^1 \left(\frac{U_\infty-c}{U-c}\right)^2 d\xi \right\} \right] . \end{aligned} \quad (8.14)$$

8.2.3 The stability condition

In this section it is assumed that the boundary layer only modifies the numerical value of the critical dynamic pressure and does not alter the type of instability (divergence or oscillatory) predicted by the uniform flow analysis. It is also assumed that either the mass ratio is small, or damping is present, so that the stability boundary, in the present case, is a divergence boundary (see §3.2). Since $c = 0$ at instability, the integral in the curly brackets in (8.14) exists only if $U \sim \xi^\gamma$, for $\xi \ll 1$, where $\gamma < 0.5$. This peculiarity is due to the neglect of the viscous terms in (8.4): these terms are important in the critical layer which, in the case $c = U(0) = 0$, is adjacent to the pipe wall. In

this section it is assumed that $\gamma < 0.5$. The stability condition is then obtained by substituting for p from (8.14) in (2.3) with $c = 0$ and using (8.13) for the radial deflection w . This gives, to $O(\delta)$,

$$\begin{aligned} \frac{\rho U_{cd}^2 R^3}{D} = & \frac{F(\lambda, k) I_k'(\lambda)}{\lambda I_k(\lambda)} \left[1 + \lambda \delta \frac{I_k'(\lambda)}{I_k(\lambda)} \left(1 - \int_0^1 \left(\frac{U}{U_\infty} \right)^2 d\xi \right) \right. \\ & \left. + (k^2 + \lambda^2) \left\{ \frac{I_k(\lambda)}{\lambda I_k'(\lambda)} \right\}^2 \left\{ \int_0^1 \left(\frac{U_\infty}{U} \right)^2 d\xi - 1 \right\} \right], \end{aligned} \quad (8.15)$$

where F is defined by (3.9).

As expected the leading term in (8.15) is identical to (3.11) when $M = 0$. Since the integral in the curly brackets is greater than unity while the other integral is less than unity, the second term in (8.15) shows that, whatever its flow profile, the shear layer has a stabilizing effect. This result is interesting because it implies that the inviscid analysis gives conservative stability criteria. In this respect the result is similar to that of flat plates (in supersonic flows - see, for example, [41]).

For the purpose of estimating the order of the correction term in (8.15), the "1/7 power law" velocity profile may be employed. Thus, on setting $U/U_\infty = \xi^{1/7}$ and assuming that the values of λ at instability are small for the first few values of k (see, for example, 5.4.2 where the critical values of λ correspond to the σ_n giving the minima in figure 5.2), expression (8.15) gives

$$\frac{\rho U_{cd}^2 R^3}{D} \sim \frac{F(\lambda, k)}{\lambda^2 k^{-1}} \left[1 + \frac{28}{45} k \delta \right]. \quad (8.16)$$

§8.3 General conclusions

In this thesis the stability of pipes conveying fluid is considered for various support boundary conditions. Although the main results are discussed in the various chapters, some appear to be quite general, in the sense that they apply for all boundary conditions considered, and are summarized below.

First, the point relationship between pressure and displacement, commonly assumed for the beam mode ($k = 1$), is extended to the shell mode ($k > 1$):

$$p = - \frac{R\rho}{k} \left(\frac{\partial}{\partial t} + \frac{U}{L} \frac{\partial}{\partial x} \right)^2 w .$$

The accuracy of this result, which corresponds to using only the leading term in the asymptotic expansion for the generalized pressures when L/R is large, depends on the support boundary conditions and increases with L/R . Moreover, all the cases investigated suggest that the accuracy also increases with k , showing that the result is even more appropriate for the shell modes.

The second result is that, for long pipes, sufficiently accurate stability results may be obtained by employing asymptotic approximations for the generalized pressures, thereby greatly reducing the amount of numerical computations. In the case of clamped-free pipes only the leading term in the approximation (corresponding to the point relationship discussed above) is determined: the stability results for $L/R > 10$ ($k = 1$) and for $L/R > 5$ ($k > 1$) differ by less than 10% from results obtained by other means. For simply supported and clamped-clamped pipes, four terms in the approximation are obtained and shown to give good agreement with previous numerical results.

The third result is that the stability condition obtained from an infinite length analysis (for example a travelling wave analysis) on finite pipes tends to that obtained from a finite length analysis as L/R increases. In this respect the behaviour of pipes containing flowing fluid is similar to that of flat plates of finite width, with one side exposed to a flowing fluid [42].

Finally the most critical circumferential mode number increases with decreasing h/R and/or L/R , the beam mode being the most critical for sufficiently large h/R and/or L/R . The results presented for a cantilever are not comprehensive, being based, for checking purposes, on pipe parameters selected by previous investigators, but the procedures discussed will be relevant to other situations for which the calculations could be readily carried out.

§8.4 Some topics for further study

It is clear from the discussion in chapter 1 and the results of the other chapters that new aspects of pipe flow phenomena are continually arising and stimulating studies in this field. A variety of new problems may be formulated by considering different support boundary conditions, structural properties of the pipe material, geometries of the undisturbed pipe, properties of the flowing fluid and conditions for the medium surrounding the pipe. Moreover, in each case, a linear or a nonlinear (structural) theory may be employed. Below are a few problems which do not appear to have been studied.

In connection with pipe geometry, previous analyses have considered only straight pipes. Since other pipe geometries are also encountered in practice, and since the straight configuration is not necessarily the most stable, it will be interesting to study the stability of pipes with

initial deformations. Such studies are also suggested by the results of a recent publication by Bishop and Fawzy [43] which studied, theoretically and experimentally, the novel effects of a nozzle on the end (previously considered in, for example, [9]) and the harmonic response of hanging cantilever pipes. It is found (experimentally) that the system passes through a deflected (static) equilibrium state for a large range of flow velocities below the critical, thus suggesting the need for a theoretical investigation of the stability of other pipe geometries.

As indicated in chapter 1, the case of unsteady mean flow has only been considered using linear beam theory (the existence of parametric and combination resonances has been demonstrated experimentally by Paidoussis and Issid [44]). Thus an obvious extension is to employ shell theory. Moreover the study of a mean flow velocity along the length of the pipe, for both the steady and the unsteady problems, may be extended by considering a mean flow velocity with both an azimuthal and an axial component.

The situation regarding nonlinear (structural) analysis is also discussed in chapter 1: there is no nonlinear stability analysis. A nonlinear analysis is necessary in order to study the limit cycle behaviour of the pipe for flow velocities above the critical as determined from the linear theory.

APPENDIX A

The fluid pressure beyond the free end of an idealized model for the cantilever ($x > 1$, $M = 0$).

A separable solution of Laplace's equation

$$\nabla^2 \phi = 0, \quad (\text{A1})$$

which is the incompressible form of (2.7), is taken in the form

$$\phi = e^{-i\omega t} \phi(r) X(x) \cos k\theta. \quad (\text{A2})$$

Substitution in (A1) gives

$$\phi''(x) + \frac{1}{x} \phi'(x) - \left(\frac{k^2}{2} + \sigma^2\right) \phi(x) = 0,$$

$$X''(x) + \sigma^2 X(x) = 0, \quad (\text{A3})$$

where σ is the separation constant. The solution to (A3) which is bounded at $r = 0$ and at $x = \infty$ and which is outgoing at $x = \infty$ (causality) is

$$\begin{aligned} \phi &= A I_k(\sigma r), \\ X &= e^{i\sigma_1 x}, \end{aligned} \quad (\text{A4})$$

where A is an arbitrary constant to be determined and

$$\sigma_1 = \pm \sigma \left\{ \begin{array}{ll} \sigma \gtrless 0 & (\text{causality}) \\ \text{or} & \\ \text{Im } \sigma \gtrless 0, & (\text{boundedness}), \end{array} \right. \quad (\text{A5})$$

provided that when σ is complex, $\text{Re } \sigma_1 > 0$ (causality).

The radial deflection, w , is taken as

$$w = C_k e^{i(\sigma_1 x - \omega t)} \cos k\theta. \quad (\text{A6})$$

Expressions (2.8), (2.9), (A2), (A4) and (A6) give

$$p = \frac{\rho}{L} C_k e^{i(\sigma_1 x - \omega t)} (\sigma_1 U - \omega L)^2 \frac{I_k(\sigma r)}{\sigma I_k'(\sigma \alpha)} \cos k\theta. \quad (\text{A7})$$

Assuming that there is no fluid in $r > \alpha$, the continuity of the pressure p requires the right hand side of (A7) to vanish when $r = \alpha$. Since, for a non-trivial solution, $C_k \neq 0$, this gives

$$\sigma_1 U - \omega L = 0, \quad (\text{A8})$$

or

$$I_k(\sigma \alpha) = 0. \quad (\text{A9})$$

The solution given by (A7) and (A8) corresponds to the case when p is zero for $x > 1$, $0 \leq r \leq \alpha$. Since the velocity potential is also zero, the restrictions (A5), which are equivalent to requiring $\text{Im } \omega$ to be positive in view of (A8), are no longer necessary. The form (A6) then represents a kinematic effect, of the motion of the cantilever, on the fluid issuing from the free end: unstable oscillations of the cantilever correspond to oscillations of the fluid which decay with downstream distance (which represents an earlier time), and vice versa. This is precisely what is expected from physical considerations.

The roots of (A9) are purely imaginary and are given by

$$\sigma = \pm i\sigma_{kj}, \quad j = 1, 2, 3, \dots, \quad (\text{A10})$$

where the σ_{kj} 's are the positive numbers satisfying

$$J_k(\sigma_{kj}) = 0.$$

Expressions (A5) and (A10) give

$$\sigma_1 = i\sigma_{kj}, \quad j = 1, 2, 3, \dots,$$

and the general solution corresponding to (A6) is

$$w = \sum_{k,j} C_{kj} e^{-\sigma_{kj} x - i\omega t} \cos k\theta, \quad (A11)$$

where the C_{kj} 's are arbitrary constants.

The solution for w in $x > 1$ is then given by adding (A6), with σ_1 defined by (A8), and (A11). However, since the σ_{kj} 's are large (> 3.8) and increase rapidly with j , the latter represents effects which appear to be localized near the free end of the pipe. Such effects are not expected to be important in determining the pipe stability and may therefore be ignored. Thus the form (A6) may be used as a model for the flow downstream of the free end.

In applying Fourier transforms with (A6), $\text{Im } \omega$ may be taken to be positive but the resulting expressions for the generalized pressures interpreted as applicable for all values of ω . This procedure is quite consistent since the solution for p when $\text{Im } \omega > 0$ may be used to determine the stability boundary. Moreover, this approach is convenient because the generalized pressures are evaluated numerically and the formal method employing Fourier transforms in the complex plane (requiring a modification of (4.11), with $\text{Im } \lambda > -\text{Im } \frac{\omega L}{U}$) is not applicable.

APPENDIX B

I. Asymptotic expansion of Q_{nk} (equation (5.5)): $b \ll 1$

Let $T_0 \gg 1$ be such that $bT_0 \ll 1$ so that $I_k(bT_0)$ may be expanded in ascending powers of bT_0 and $(n^2\pi^2 - \lambda^2)^{-2}$ may be expanded in powers of λ^{-2} for $\lambda \geq T_0$.

Consider the integral Q_{nk} and write

$$Q_{nk} = n(Q_{nk1} + Q_{nk2}) \quad (B1)$$

where

$$Q_{nk1} = \int_0^{T_0} g_{nk}(\lambda) d\lambda, \quad Q_{nk2} = \int_{T_0}^{\infty} g_{nk}(\lambda) d\lambda, \quad (B2)$$

$$g_{nk}(\lambda) = \frac{\lambda \{1 - (-)^n \cos \lambda\}}{(n^2\pi^2 - \lambda^2)^2} \frac{I_k(b\lambda)}{I_k'(b\lambda)}.$$

To evaluate Q_{nk1} replace the Bessel quotient by its small argument expansion:

$$\begin{aligned} \frac{I_k(b\lambda)}{I_k'(b\lambda)} &= b\lambda \left[\frac{1}{k} - \frac{b^2\lambda^2}{2k^2(k+1)} + \frac{(4+3k)b^4\lambda^4}{8k^3(k+1)^2(k+2)} \right. \\ &\quad \left. + O(b^6\lambda^6) \right], \quad k > 0, \end{aligned}$$

$$\frac{I_0(b\lambda)}{I_0'(b\lambda)} = \frac{2}{b\lambda} \left(1 + \frac{b^2\lambda^2}{8} + \dots \right). \quad (B3)$$

Case (i): $k > 0$

Expressions (B2) and (B3) give

$$Q_{nkl} = \frac{b}{k} \left[I_1 - \frac{b^2}{2k(k+1)} I_2 + \frac{(4+3k)b^4}{8k^2(k+1)^2(k+2)} I_3 + O(b^6) \right], \quad (B4)$$

where

$$\begin{aligned} I_1 &= \int_0^{T_0} \frac{\lambda^2 \{1 - (-)^n \cos \lambda\}}{(n^2 \pi^2 - \lambda^2)^2} d\lambda \\ &= \int_0^{\infty} \frac{\lambda^2 \{1 - (-)^n \cos \lambda\}}{(n^2 \pi^2 - \lambda^2)^2} d\lambda \\ &\quad - \int_{T_0}^{\infty} \{1 - (-)^n \cos \lambda\} \left\{ \frac{1}{\lambda^2} + \frac{2n^2 \pi^2}{\lambda^4} \right. \\ &\quad \left. + \frac{3n^4 \pi^4}{\lambda^6} + O(\lambda^{-8}) \right\} d\lambda \\ &= \frac{\pi}{4} - \frac{1}{T_0} - \frac{(-)^n \sin T_0}{T_0^2} - \frac{2}{T_0^3} \left\{ \frac{n^2 \pi^2}{3} - (-)^n \cos T_0 \right\} \\ &\quad + O(T_0^{-4}). \end{aligned} \quad (B5)$$

Some of the integrals are tabulated in reference [45]. The expansion in powers of $n\pi/\lambda$ is valid only if $T_0 > n\pi$ so that, in view of the assumption $bT_0 \ll 1$, the number n must satisfy $bn\pi \ll 1$. This is equivalent to restricting consideration to sufficiently long axial wavelengths

$$\begin{aligned}
I_2 &= \int_0^{T_0} \frac{\lambda^4 \{1 - (-)^n \cos \lambda\}}{(n^2 \pi^2 - \lambda^2)^2} d\lambda \\
&= \int_0^{T_0} \{1 - (-)^n \cos \lambda\} \left\{1 - \frac{n^2 \pi^2}{n^2 \pi^2 - \lambda^2}\right\}^2 d\lambda \\
&= T_0 - (-)^n \sin T_0 + \int_0^\infty \{1 - (-)^n \cos \lambda\} \left\{ \frac{n^4 \pi^4}{(n^2 \pi^2 - \lambda^2)^2} \right. \\
&\quad \left. - \frac{2n^2 \pi^2}{n^2 \pi^2 - \lambda^2} \right\} d\lambda \\
&\quad - \int_{T_0}^\infty \{1 - (-)^n \cos \lambda\} \left\{ \frac{n^4 \pi^4}{\lambda^4} \left(1 + \frac{2n^2 \pi^2}{\lambda^2} + \dots\right) \right. \\
&\quad \left. + \frac{2n^2 \pi^2}{\lambda^2} \left(1 + \frac{n^2 \pi^2}{\lambda^2} + \dots\right) \right\} d\lambda \\
&= \frac{n^2 \pi^3}{4} + T_0 - (-)^n \sin T_0 - \frac{2n^2 \pi^2}{T_0} + O(T_0^{-2}). \tag{B6}
\end{aligned}$$

$$\begin{aligned}
I_3 &= \int_0^{T_0} \frac{\lambda^6 \{1 - (-)^n \cos \lambda\}}{(n^2 \pi^2 - \lambda^2)^2} d\lambda \\
&= \int_0^{T_0} \{1 - (-)^n \cos \lambda\} \left\{ \lambda^2 + \frac{n^4 \pi^4 \lambda^2}{(n^2 \pi^2 - \lambda^2)^2} \right. \\
&\quad \left. - \frac{2n^4 \pi^4}{n^2 \pi^2 - \lambda^2} + 2n^2 \pi^2 \right\} d\lambda \\
&= \frac{T_0^3}{3} - (-)^n T_0^2 \sin T_0 + 2T_0 \{n^2 \pi^2 - (-)^n \cos T_0\} \\
&\quad + \frac{n^4 \pi^5}{4} + 2(-)^n \{1 - n^2 \pi^2\} \sin T_0 + O(T_0^{-1}). \tag{B7}
\end{aligned}$$

For Q_{nk2} (eqn B2), the factor $(n^2\pi^2 - \lambda^2)^{-2}$ may be expanded in ascending powers of λ^{-1} and (B3) employed:

$$\begin{aligned}
 Q_{nk2} &= \int_{T_0}^{\infty} \{1 - (-)^n \cos \lambda\} \left\{ \frac{1}{\lambda^3} + \frac{2n^2\pi^2}{\lambda^5} \right. \\
 &\quad \left. + \frac{3n^4\pi^4}{\lambda^7} + \dots \right\} \frac{I_k(b\lambda)}{I_k'(b\lambda)} d\lambda \\
 &= J_3 + 2n^2\pi^2 J_5 + \dots \\
 &\quad - (-)^n [H_3 + 2n^2\pi^2 H_5 + \dots],
 \end{aligned} \tag{B8}$$

where

$$\begin{aligned}
 J_3 &= \int_{T_0}^{\infty} \frac{I_k(b\lambda)}{I_k'(b\lambda)} \frac{d\lambda}{\lambda^3} = b^2 \int_{bT_0}^{\infty} \frac{I_k(x)}{I_k'(x)} \frac{dx}{x^3} \\
 &= b^2 [F_1(k) + \frac{1}{k} \int_{bT_0}^{\infty} \frac{1}{x^2} dx \\
 &\quad - \int_0^{bT_0} \left\{ \frac{I_k(x)}{x I_k'(x)} - \frac{1}{k} \right\} \frac{dx}{x^2}] \\
 &= b^2 [F_1(k) + \frac{1}{kbT_0} + \frac{bT_0}{2k^2(k+1)} \\
 &\quad - \frac{(4 + 3k)b^3 T_0^3}{24k^3(k+1)^2(k+2)} + O(bT_0)^5],
 \end{aligned} \tag{B9}$$

$$F_1(k) = \int_0^{\infty} \left\{ \frac{I_k(x)}{x I_k'(x)} - \frac{1}{k} \right\} \frac{dx}{x^2}. \tag{B10}$$

$$\begin{aligned}
J_5 &= \int_{T_0}^{\infty} \frac{I_k(b\lambda)}{I_k'(b\lambda)} \frac{d\lambda}{\lambda^5} = b^4 \int_{bT_0}^{\infty} \frac{I_k(x)}{I_k'(x)} \frac{dx}{x^5} \\
&= b^4 \left[\int_{bT_0}^{\infty} \left\{ \frac{I_k(x)}{x I_k'(x)} - \frac{1}{k} + \frac{x^2}{2k^2(k+1)} \right\} \frac{dx}{x^4} \right. \\
&\quad \left. + \int_{bT_0}^{\infty} \left\{ \frac{1}{k} - \frac{x^2}{2k^2(k+1)} \right\} \frac{dx}{x^4} \right] \\
&= b^4 \left[F_2(k) - \frac{(4+3k)bT_0}{8k^3(k+1)^2(k+2)} + o(bT_0)^3 \right. \\
&\quad \left. + \frac{1}{3kb^3T_0^3} - \frac{1}{2k^2(k+1)} \frac{1}{bT_0} \right], \tag{B11}
\end{aligned}$$

$$F_2(k) = \int_0^{\infty} \left\{ \frac{I_k(x)}{x I_k'(x)} - \frac{1}{k} + \frac{x^2}{2k^2(k+1)} \right\} \frac{dx}{x^4}. \tag{B12}$$

$$J_{\underline{m}} = \int_{T_0}^{\infty} \frac{I_k(b\lambda)}{I_k'(b\lambda)} \frac{d\lambda}{\lambda^m} = o(T_0^{-5}). \tag{B13}$$

$$H_m = \int_{T_0}^{\infty} f_m(\lambda) d\lambda, \quad f_m(\lambda) = \frac{\cos \lambda}{\lambda^m} \frac{I_k(b\lambda)}{I_k'(b\lambda)}.$$

Let a be such that $baT_0 \gg 1$ and write

$$H_m = H_{m1} + H_{m2} \tag{B14}$$

where

$$H_{m1} = \int_{T_0}^{aT_0} f_m(\lambda) d\lambda, \quad H_{m2} = \int_{aT_0}^{\infty} f_m(\lambda) d\lambda. \quad (B15)$$

To evaluate H_{m1} , differentiate with respect to T_0 and employ the large argument expansion for the Bessel quotient together with (B3):

$$\frac{I_k(z)}{I_k'(z)} \sim \frac{e^z}{\sqrt{2\pi z}} \left\{ 1 - \frac{4k^2-1}{8z} + \frac{(4k^2-1)(4k^2-9)}{2(8z)^2} + \dots \right\}, \quad z \gg 1, \quad (B16)$$

so that

$$\frac{I_k(b\lambda)}{I_k'(b\lambda)} \sim 1 + \frac{1}{2b\lambda} + \frac{3-4k^2}{8b^2\lambda^2} + \dots, \quad b\lambda \gg 1. \quad (B17)$$

$$\begin{aligned} \frac{\partial H_{31}}{\partial T_0} &= af_3(aT_0) - f_3(T_0) \\ &= \frac{a \cos(aT_0)}{(aT_0)^3} \left[1 + \frac{1}{baT_0} + \frac{3-4k^2}{8b^2a^2T_0^2} + \dots \right] \\ &\quad - \frac{b \cos T_0}{T_0^2} \left[\frac{1}{k} - \frac{b^2T_0^2}{2k^2(k+1)} \right. \\ &\quad \left. + \frac{(4+3k)b^4T_0^4}{8k^3(k+1)^2(k+2)} + \dots \right]. \end{aligned} \quad (B18)$$

Expression (B18) may be integrated to give

$$\begin{aligned} H_{31} &= \frac{\sin(aT_0)}{a^3T_0^3} + O(aT_0)^{-4} \\ &\quad - \frac{b}{k} \left\{ \frac{\sin T_0}{T_0^2} - \frac{2 \cos T_0}{T_0^3} + O(T_0^{-4}) \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{b^3 \sin T_0}{2k^2(k+1)} - \frac{(4+3k)b^5}{8k^3(k+1)^2(k+2)} \{T_0^2 \sin T_0 \\
& + 2T_0 \cos T_0 - 2 \sin T_0\}. \tag{B19}
\end{aligned}$$

A similar procedure gives

$$H_{51} = \frac{-b^3}{2k^2(k+1)} \left\{ \frac{(4+3k)b^2 \sin T_0}{4k(k+1)(k+2)} + O(T_0^{-2}) \right\} \tag{B20}$$

$$H_{\substack{m1 \\ m \geq 7}} = O(T_0^{-6}). \tag{B21}$$

To evaluate H_{m2} (equation (B15)) employ (B17)

$$H_{m2} = \int_{aT_0}^{\infty} \frac{\cos \lambda}{\lambda^m} \left\{ 1 + \frac{1}{b\lambda} + \frac{3-4k^2}{8b^2\lambda^2} + \dots \right\} d\lambda.$$

Thus, on integrating by parts,

$$H_{32} = -\frac{\sin(aT_0)}{a^3 T_0^3} + O(aT_0)^{-4},$$

$$H_{\substack{m2 \\ m \geq 5}} = O(aT_0)^{-5}. \tag{B22}$$

Case (ii): $k = 0$

From (B2),

$$Q_{n01} = \int_0^{T_0} \frac{\lambda \{1 - (-)^n \cos \lambda\}}{(n^2 \pi^2 - \lambda^2)^2} \frac{I_0(b\lambda)}{I_0'(b\lambda)} d\lambda$$

and on using (B3),

$$= \frac{2}{b} [I_1 + \frac{b^2}{8} I_2 + O(b^3)] \quad (B23)$$

where

$$\begin{aligned} I_1 &= \int_0^{T_0} \frac{1 - (-)^n \cos \lambda}{(n^2 \pi^2 - \lambda^2)^2} d\lambda \\ &= \frac{1}{n^2 \pi^2} \int_0^{T_0} \{1 - (-)^n \cos \lambda\} \left\{ \frac{d}{d\lambda} \left(\frac{\lambda}{n^2 \pi^2 - \lambda^2} \right) - \frac{\lambda^2}{(n^2 \pi^2 - \lambda^2)^2} \right\} d\lambda \\ &= \frac{1}{2n^2 \pi^2} \left[\frac{T_0 \{1 - (-)^n \cos T_0\}}{n^2 \pi^2 - T_0^2} - (-)^n \int_0^{T_0} \frac{\lambda \sin \lambda}{n^2 \pi^2 - \lambda^2} d\lambda \right. \\ &\quad \left. + \int_0^{T_0} \frac{1 - (-)^n \cos \lambda}{n^2 \pi^2 - \lambda^2} d\lambda \right] \\ &= \frac{1}{2n^2 \pi^2} \left[\frac{\pi}{2} + O(T_0^{-1}) \right] . \end{aligned}$$

$$\begin{aligned}
I_2 &= \int_0^{T_0} \frac{\lambda^2 \{1 - (-)^n \cos \lambda\}}{(n^2 \pi^2 - \lambda^2)^2} d\lambda \\
&= \int_0^{T_0} \{1 - (-)^n \cos \lambda\} \left\{ \frac{n^2 \pi^2}{(n^2 \pi^2 - \lambda^2)^2} - \frac{1}{n^2 \pi^2 - \lambda^2} \right\} d\lambda \\
&= n^2 \pi^2 I_1 + O(T_0^{-1}).
\end{aligned}$$

Also, from (B2),

$$Q_{n02} = \int_{T_0}^{\infty} \frac{\lambda \{1 - (-)^n \cos \lambda\}}{(n^2 \pi^2 - \lambda^2)^2} \frac{I_0(b\lambda)}{I_0'(b\lambda)} d\lambda$$

and on expanding $(n^2 \pi^2 - \lambda^2)^{-2}$,

$$\begin{aligned}
&= \int_{T_0}^{\infty} \frac{1 - (-)^n \cos \lambda}{\lambda^3} \left\{ 1 + \frac{2n^2 \pi^2}{\lambda^2} + \dots \right\} \frac{I_0(b\lambda)}{I_0'(b\lambda)} d\lambda \\
&= \frac{2}{3bT_0^3} + O(T_0^{-1}).
\end{aligned} \tag{B24}$$

Equations (B1), (B4), (B8), (B23) and (B24) give Q_{nk} in the form (5.6).

II. Expansion for $P_{mn,kk}$, $m \neq n$

From (4.14) with $\omega = 0$,

$$P_{mn,kk} = \begin{cases} \frac{2m\pi^2}{(1-M^2)^{1/2}} U^2 \rho \varepsilon_k Q_{mnk}, & m+n \text{ even,} \\ 0, & m+n \text{ odd,} \end{cases} \tag{B25}$$

where

$$Q_{mnk} = n \int_0^{\infty} \frac{\{1 - (-)^m \cos \lambda\} \lambda}{O(m^2 \pi^2 - \lambda^2)(n^2 \pi^2 - \lambda^2)} \frac{I_k(b\lambda)}{I_k'(b\lambda)} d\lambda.$$

The previous analysis may be employed to give, for $m \neq n$ and for $k > 0$,

$$Q_{mnk} = n(Q_{mnk1} + Q_{mnk2}), \quad (B26)$$

where

$$\begin{aligned} Q_{mnk1} &= \frac{b}{k} \int_0^{T_0} \frac{\{1 - (-)^m \cos \lambda\} \lambda^2}{(m^2 \pi^2 - \lambda^2)(n^2 \pi^2 - \lambda^2)} \left\{1 - \frac{b^2 \lambda^2}{2k(k+1)} + \dots\right\} d\lambda \\ &= \frac{b}{k} [Q_1 - \frac{b^2}{2k(k+1)} Q_2 + \dots], \end{aligned} \quad (B27)$$

$$\begin{aligned} Q_{mnk2} &= \int_{T_0}^{\infty} \frac{\{1 - (-)^m \cos \lambda\} \lambda}{(m^2 \pi^2 - \lambda^2)(n^2 \pi^2 - \lambda^2)} \frac{I_k(b\lambda)}{I_k'(b\lambda)} d\lambda \\ &= \int_{T_0}^{\infty} \frac{1 - (-)^m \cos \lambda}{\lambda^3} \left\{1 + \frac{(m^2 + n^2) \pi^2}{\lambda^2} + \dots\right\} \frac{I_k(b\lambda)}{I_k'(b\lambda)} d\lambda \\ &= b^2 [F_1(k) + \frac{b T_0}{2k^2(k+1)} + O(b T_0)^2] + \frac{b}{k T_0} \\ &\quad + O(T_0^{-2}) - (-)^m \frac{b^3 \sin T_0}{2k^2(k+1)}, \end{aligned} \quad (B28)$$

with

$$\begin{aligned}
 Q_1 &= \int_0^{T_0} \frac{\lambda^2 \{1 - (-)^m \cos \lambda\}}{(m^2 \pi^2 - \lambda^2)(n^2 \pi^2 - \lambda^2)} d\lambda \\
 &= - \int_{T_0}^{\infty} \frac{1 - (-)^m \cos \lambda}{\lambda^2} \left\{1 + \frac{(m^2 + n^2) \pi^2}{\lambda^2} + \dots\right\} d\lambda \\
 &= - \frac{1}{T_0} + O(T_0^{-2}),
 \end{aligned}$$

$$\begin{aligned}
 Q_2 &= \int_0^{T_0} \frac{1 - (-)^m \cos \lambda}{(m^2 \pi^2 - \lambda^2)(n^2 \pi^2 - \lambda^2)} \lambda^4 d\lambda \\
 &= \int_0^{T_0} \{1 - (-)^m \cos \lambda\} \left\{ \frac{m^2 n^2 \pi^4}{(m^2 \pi^2 - \lambda^2)(n^2 \pi^2 - \lambda^2)} + 1 \right. \\
 &\quad \left. - \frac{m^2 \pi^2}{m^2 \pi^2 - \lambda^2} - \frac{n^2 \pi^2}{n^2 \pi^2 - \lambda^2} \right\} d\lambda \\
 &= T_0 - (-)^m \sin T_0 + O(T_0^{-1}),
 \end{aligned}$$

and $F_1(k)$ defined by (B10).

Expressions (B26), (B27), and (B28) give, to leading order,

$$Q_{mnk} = nb^2 F_1(k) \quad (B29)$$

Now $Q_{nnk} = Q_{nk}$ as given by (5.6). Thus, on using (B25) and (B29) for $m \neq n$,

$$\frac{P_{mn,kk}}{P_{nn,kk}} = \begin{cases} O(b), & m + n \text{ even,} \\ 0, & m + n \text{ odd.} \end{cases} \quad (B30)$$

The result (B30) shows that the off-diagonal generalized pressures are small compared to the diagonal pressures when b is small.

APPENDIX C

Asymptotic expansion for Q_{nk} when $bn \sim 1$, $b \ll 1$.

Consider the integral Q_{nk} from (5.5) and introduce the variable $x = \lambda/n\pi$ to get

$$Q_{nk} = \frac{1}{n\pi^2} \int_0^\infty \frac{x\{1 - (-)^n \cos n\pi x\}}{(1 - x^2)^2} H_k(bn\pi x) dx, \quad (C1)$$

where

$$H_k(x) = I_k(x)/I_k'(x). \quad (C2)$$

Integration by parts once gives

$$\begin{aligned} Q_{nk} &= \frac{(-)^n}{2\pi} \int_0^\infty \frac{\sin n\pi x}{x^2 - 1} H_k(bn\pi x) dx \\ &+ \frac{b}{2\pi} \int_0^\infty \frac{1 - (-)^n \cos n\pi x}{x^2 - 1} H_k'(bn\pi x) dx. \end{aligned} \quad (C3)$$

The second integral in (C3) can be rewritten as

$$\begin{aligned} \int_0^\infty \frac{1 - (-)^n \cos n\pi x}{x^2 - 1} H_k'(bn\pi x) dx &= \int_0^\infty \frac{H_k'(bn\pi x) - H_k'(bn\pi)}{x^2 - 1} \\ &- (-)^n \int_0^\infty \frac{\cos n\pi x}{x^2 - 1} \{H_k'(bn\pi x) - H_k'(bn\pi)\} dx, \end{aligned}$$

and on integrating the second term by parts once,

$$\begin{aligned}
 &= \int_0^{\infty} \frac{H_k'(bn\pi x) - H_k'(bn\pi)}{x^2 - 1} dx \\
 &\quad - \frac{(-)^n}{n\pi} \left[2 \int_0^{\infty} \frac{x \sin n\pi x}{(x^2 - 1)^2} \{H_k'(bn\pi x) - H_k'(bn\pi)\} dx \right. \\
 &\quad \left. - bn\pi \int_0^{\infty} \frac{\sin n\pi x}{x^2 - 1} H_k''(bn\pi x) dx \right]. \tag{C4}
 \end{aligned}$$

Now $\sin n\pi x$ oscillates rapidly in the range $0 \leq x < \infty$ since, by assumption, $n \sim b^{-1} \gg 1$. Consequently, for the first integral in (C3) and the last two integrals in (C4) cancellations may be expected over most of the range except in a small region around $x = 1$ in which the integrands remain stationary. Consider, for example, the integrand for the first integral in (C3).

$$\frac{\sin n\pi x}{x^2 - 1} \sim \frac{n\pi(-)^n}{2} \quad \text{when } |x - 1| \ll 1.$$

The factor $H_k(bn\pi x)$ is a slowly varying positive function and so the integrand does not oscillate in a small region around $x = 1$. An examination of the rest of the integration range shows that the point $x = 1$ is the only one with this characteristic: the factor multiplying $\sin n\pi x$ is one signed on either side of the point $x = 1$ and the oscillations of $\sin n\pi x$ persist. Thus a good approximation may be obtained by expanding $H_k(bn\pi x)$ in a Taylor series around $x = 1$. Hence for the first integral in (C3),

$$\begin{aligned}
& \int_0^{\infty} \frac{\sin n\pi x}{x^2 - 1} H_k(bn\pi x) \sim H_k(bn\pi) \int_0^{\infty} \frac{\sin n\pi x}{x^2 - 1} dx \\
& + bn\pi H_k'(bn\pi) \int_0^{\infty} \frac{\sin n\pi x}{x + 1} dx \\
& = \frac{(-)^n \pi}{2} H_k(bn\pi) + b[1 - (1 + \frac{k^2}{b^2 n^2 \pi^2}) H_k^2(bn\pi)] \\
& + o(b^2) .
\end{aligned} \tag{C5}$$

From (C4),

$$\begin{aligned}
& \int_0^{\infty} \frac{x \sin n\pi x}{(x^2 - 1)^2} \{H_k'(bn\pi x) - H_k'(bn\pi)\} dx \\
& \sim \frac{bn\pi}{2} H_k''(bn\pi) \int_0^{\infty} \frac{x \sin n\pi x}{x^2 - 1} dx \\
& = \frac{(-)^n}{4} bn\pi^2 H_k''(bn\pi) .
\end{aligned}$$

$$\begin{aligned}
& \int_0^{\infty} \frac{\sin n\pi x}{x^2 - 1} H_k''(bn\pi x) dx \sim H_k''(bn\pi) \int_0^{\infty} \frac{\sin n\pi x}{x^2 - 1} dx \\
& = \frac{(-)^n}{2} \pi H_k''(bn\pi) + o(b) .
\end{aligned} \tag{C6}$$

Expressions (C3), (C4), (C5) and (C6) obtain Q_{nk} in the form (5.17).

APPENDIX D

I. Asymptotic expansion for Q_{nk} when $b \gg 1$

A procedure similar to that of appendix B is employed using the large argument asymptotic expansion for the Bessel quotient given by (B17).

Let $\epsilon \ll 1$ be such that $b\epsilon \gg 1$ and, on using (5.5), write

$$Q_{nk} = Q_{nk1} + Q_{nk2}, \quad (D1)$$

where

$$\begin{aligned} Q_{nk1} &= \int_0^\epsilon g_{nk}(\lambda) d\lambda, \\ Q_{nk2} &= \int_\epsilon^\infty g_{nk}(\lambda) d\lambda, \end{aligned} \quad (D2)$$

with

$$g_{nk}(\lambda) = \frac{n\lambda\{1 - (-)^n \cos \lambda\}}{(n^2\pi^2 - \lambda^2)^2} \frac{I_k(b\lambda)}{I_k'(b\lambda)}.$$

For Q_{nk1} the factor $(n^2\pi^2 - \lambda^2)^{-2}$ may be expanded in ascending powers of $\lambda/n\pi$. This gives

$$\begin{aligned} Q_{nk1} &= \frac{1}{n^3\pi^4} \int_0^\epsilon \lambda\{1 - (-)^n \cos \lambda\} \left\{1 + \frac{2\lambda^2}{n^2\pi^2} + \frac{3\lambda^4}{n^4\pi^4} \right. \\ &\quad \left. + \frac{4\lambda^6}{n^6\pi^6} + \dots \right\} \frac{I_k(b\lambda)}{I_k'(b\lambda)} d\lambda. \end{aligned} \quad (D3)$$

When $\cos \lambda$ is replaced by its small λ expansion, the terms in (D3) involve integrals of the form

$$J_m = \int_0^\epsilon \lambda^m \frac{I_k(b\lambda)}{I_k'(b\lambda)} d\lambda. \quad (D4)$$

To evaluate these integrals, introduce (B17): a procedure similar to that employed in appendix B for the case $b \ll 1$ may then be employed.

Thus, for example,

$$\begin{aligned} J_1 &= \frac{1}{b^2} \int_0^{b\epsilon} \frac{x I_k(x)}{I_k'(x)} dx \\ &= \frac{1}{b^2} [F_3(k) - \int_{b\epsilon}^\infty x \left\{ \frac{I_k(x)}{I_k'(x)} - 1 - \frac{1}{2x} - \frac{3 - 4k^2}{8x^2 + 1} \right\} dx \\ &\quad + \int_0^{b\epsilon} \left\{ x + \frac{1}{2} + \frac{(3 - 4k^2)x}{8x^2 + 1} \right\} dx], \end{aligned} \quad (D5)$$

where

$$F_3(k) = \int_0^\infty \left\{ \frac{I_k(x)}{I_k'(x)} - 1 - \frac{1}{2x} - \frac{3 - 4k^2}{8x^2 + 1} \right\} x dx. \quad (D6)$$

When (B17) is employed for the integral between $b\epsilon$ and ∞ , (D3) may be evaluated to give, finally,

$$\begin{aligned} Q_{nk1} &= \frac{\{1 - (-)^n\}}{b^2 n^3 \pi^4} [F_3(k) + \frac{b^2 \epsilon^2}{2} + \frac{3 - 4k^2}{16} \log(8b^2 \epsilon^2 + 1) \\ &\quad + \frac{b\epsilon}{2} + o(\frac{1}{b\epsilon})]. \end{aligned} \quad (D7)$$

For Q_{nk2} (defined by (D2)), (B17) may be used to give

$$Q_{nk2} = n \int_{\epsilon}^{\infty} \frac{\lambda \{1 - (-)^n \cos \lambda\}}{(n^2 \pi^2 - \lambda^2)^2} \left[1 + \frac{1}{2b\lambda} + \frac{3 - 4k^2}{8b^2 \lambda^2} + \dots \right] d\lambda,$$

and on evaluating the integrals,

$$= n \left[I_1 + \frac{1}{2b} I_2 + \frac{3 - 4k^2}{8b^2} I_3 + O(b^{-3}) \right], \quad (D8)$$

where

$$I_1 = \int_{\epsilon}^{\infty} \frac{\lambda \{1 - (-)^n \cos \lambda\}}{(n^2 \pi^2 - \lambda^2)^2} d\lambda = \frac{1}{2} \left[\frac{(-)^n - 1}{n^2 \pi^2} + \frac{1}{n\pi} \text{Si}(n\pi) \right] - \frac{1 - (-)^n}{2n^4 \pi^4} \epsilon^2 + O(\epsilon^3)$$

$$I_2 = \int_{\epsilon}^{\infty} \frac{1 - (-)^n \cos \lambda}{(n^2 \pi^2 - \lambda^2)^2} d\lambda = \frac{1}{4n^2 \pi} - \frac{1 - (-)^n}{n^4 \pi^4} \epsilon + O(\epsilon^2)$$

$$I_3 = \int_{\epsilon}^{\infty} \frac{1 - (-)^n \cos \lambda}{\lambda (n^2 \pi^2 - \lambda^2)^2} d\lambda = \frac{1}{2n^3 \pi^3} \left\{ \frac{(-)^n - 1}{n\pi} + \text{Si}(n\pi) \right\}$$

$$+ \frac{1}{n^4 \pi^4} \{ \log(n\pi) - \text{Ci}(n\pi) - \log \epsilon$$

$$+ (-)^n \text{Ci}(\epsilon) \} + O(\epsilon)$$

Here

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt,$$

$$\text{Ci}(x) = \gamma + \log x - \text{Cin } x$$

where

$$\text{Cin}(x) = \int_0^x \frac{1 - \cos t}{t} dt,$$

and Euler's constant, $\gamma \approx 0.5772$.

Expressions (D1), (D7) and (D8) give Q_{nk} in the form (5.22).

II. Approximation for $P_{mn,kk}$ when $m \neq n$

A procedure similar to that employed above may be used to show that

$$\frac{P_{mn,kk}}{P_{nn,kk}} = \begin{cases} O(b^{-1}), & m+n \text{ even,} \\ 0, & m+n \text{ odd,} \end{cases} \quad (\text{D9})$$

However, it is sufficient to note that the leading term in the expansion for $P_{nn,kk}$, given by setting $Q_{nnk} = Q_{nk}$ in (B25) and using 5.22), comes from the integral I_1 in (D8), and that when $m \neq n$, this integral is replaced by (see (B25))

$$\int_{\epsilon}^{\infty} \frac{\lambda \{1 - (-)^n \cos \lambda\}}{(m^2 \pi^2 - \lambda^2)(n^2 \pi^2 - \lambda^2)} d\lambda = O(\epsilon). \quad (\text{D10})$$

This term will cancel with a term from the lower integration range and the leading term in the expansion for the generalized pressures, when $m \neq n$ and $m + n$ even, is thus given by a term corresponding to that containing I_2 in (D8) and the result (D9) follows.

APPENDIX E

I. Asymptotic expansion for Q_{nk} when $b \ll 1$ (eqn (6.6))

Using the notation of appendix B,

$$Q_{nk} = (Q_{nk1} + Q_{nk2}) 4n^3 \pi^2 \alpha \quad (E1)$$

where

$$Q_{nk1} = \int_0^{T_0} g_{nk}(\lambda) d\lambda, \quad Q_{nk2} = \int_{T_0}^{\infty} g_{nk}(\lambda) d\lambda, \quad (E2)$$

with

$$g_{nk} = \frac{1 - \cos \lambda}{(4n^2 \pi^2 - \lambda^2)^2} \frac{I_k(b\lambda)}{b\lambda I_k'(b\lambda)}, \quad T_0 \gg 2n\pi.$$

(i) $k > 1$

On using (B3),

$$Q_{nk1} = \frac{1}{k} \left[I_1 - \frac{b^2}{2k(k+1)} I_2 + \frac{(4+3k)b^4}{8k^2(k+1)^2(k+2)} I_3 + o(b^6) \right] \quad (E3)$$

where

$$\begin{aligned} I_1 &= \int_0^{T_0} \frac{1 - \cos \lambda}{(4n^2 \pi^2 - \lambda^2)^2} d\lambda = \int_0^{\infty} \frac{1 - \cos \lambda}{(4n^2 \pi^2 - \lambda^2)} d\lambda - \int_{T_0}^{\infty} \frac{1 - \cos \lambda}{(4n^2 \pi^2 - \lambda^2)^2} d\lambda \\ &= \frac{1}{16n^2 \pi} - \int_{T_0}^{\infty} \frac{1 - \cos \lambda}{\lambda^4} \left[1 + \frac{8n^2 \pi^2}{\lambda^2} + \frac{48n^4 \pi^4}{\lambda^4} + \dots \right] d\lambda \end{aligned}$$

$$= \frac{1}{16n^2\pi} - \frac{1}{3T_0^3} + o(T_0^{-4}) . \quad (E4)$$

$$\begin{aligned} I_2 &= \int_0^{T_0} \frac{\lambda^2(1 - \cos \lambda)}{(4n^2\pi^2 - \lambda^2)^2} d\lambda = \frac{\pi}{4} - \int_{T_0}^{\infty} \frac{\lambda^2(1 - \cos \lambda)}{(4n^2\pi^2 - \lambda^2)^2} d\lambda \\ &= \frac{\pi}{4} - \frac{1}{T_0} + o(T_0^{-2}) \end{aligned} \quad (E5)$$

$$\begin{aligned} I_3 &= \int_0^{T_0} \frac{\lambda^4(1 - \cos \lambda)}{(4n^2\pi^2 - \lambda^2)^2} d\lambda \\ &= \int_0^{T_0} (1 - \cos \lambda) \left\{ 1 + \frac{16n^4\pi^4}{(4n^2\pi^2 - \lambda^2)^2} - \frac{8n^2\pi^2}{4n^2\pi^2 - \lambda^2} \right\} d\lambda \\ &= n^2\pi^3 + T_0 - \sin T_0 + o(T_0^{-1}) \end{aligned} \quad (E6)$$

Also, from (E2),

$$\begin{aligned} Q_{nk2} &= \int_{T_0}^{\infty} \frac{1 - \cos \lambda}{\lambda^4} \left\{ 1 + \frac{8n^2\pi^2}{\lambda^2} + \frac{48n^4\pi^4}{\lambda^4} + \dots \right\} \frac{I_k(b\lambda)}{b\lambda I_k'(b\lambda)} d\lambda \\ &= J_4 - H_4 + 8n^2\pi^2(J_6 - H_6) + \dots, \end{aligned} \quad (E7)$$

where

$$\begin{aligned} J_m &= \int_{T_0}^{\infty} \frac{I_k(b\lambda)}{b\lambda I_k'(b\lambda)} \frac{d\lambda}{\lambda^m} \\ H_m &= \int_{T_0}^{\infty} \frac{\cos \lambda}{\lambda^m} \frac{I_k(b\lambda)}{b\lambda I_k'(b\lambda)} d\lambda. \end{aligned}$$

As with (B8),

$$\begin{aligned}
 J_4 &= \int_{T_0}^{\infty} \left[\frac{I_k(b\lambda)}{b\lambda I'_k(b\lambda)} - \frac{1}{k} + \frac{b^2 \lambda^2}{2k^2(k+1)} \right] \frac{d\lambda}{\lambda^4} \\
 &\quad + \int_{T_0}^{\infty} \left[\frac{1}{k} - \frac{b^2 \lambda^2}{2k^2(k+1)} \right] \frac{d\lambda}{\lambda^4} \\
 &= b^3 F_2(k) + \frac{1}{k} \left[\frac{1}{3T_0^3} - \frac{b^2}{2k(k+1)} \frac{1}{T_0} \right. \\
 &\quad \left. - \frac{(4+3k)b^4 T_0}{8k^2(k+1)^2(k+2)} + O(b^6 T_0^3) \right], \tag{E8}
 \end{aligned}$$

$F_2(k)$ being defined by (B12).

$$\begin{aligned}
 J_{\underline{m}} &= \int_{T_0}^{\infty} \frac{I_k(b\lambda)}{b\lambda I'_k(b\lambda)} \frac{d\lambda}{\lambda^m} \\
 &= O(b^5). \\
 H_4 &= - \frac{(4+3k)b^4 \sin T_0}{8k^3(k+1)^2(k+2)} + O(T_0^{-4}) \\
 H_{\underline{m}} &= O(T_0^{-6}) \tag{E9}
 \end{aligned}$$

(ii) $k = 0$

$$\begin{aligned}
 Q_{n01} &= \int_0^{T_0} \frac{1 - \cos \lambda}{(4n^2 \pi^2 - \lambda^2)^2} \frac{I_0(b\lambda)}{b\lambda I_0'(b\lambda)} d\lambda \\
 &= \frac{2}{b^2} [I_1 + \frac{b^2}{8} I_2 + O(b^3)] \quad (E10)
 \end{aligned}$$

where

$$\begin{aligned}
 I_1 &= \int_0^{T_0} \frac{1 - \cos \lambda}{(4n^2 \pi^2 - \lambda^2)^2} \frac{d\lambda}{\lambda^2} \\
 &= \frac{1}{4n^2 \pi^2} \int_0^{T_0} (1 - \cos \lambda) \left[\frac{1}{4n^2 \pi^2} \left(\frac{1}{\lambda^2} + \frac{1}{4n^2 \pi^2 - \lambda^2} \right) \right. \\
 &\quad \left. + \frac{1}{(4n^2 \pi^2 - \lambda^2)^2} \right] d\lambda \\
 &= \frac{3}{64n^4 \pi^3} + O(T_0^{-1}) \\
 I_2 &= \int_0^{T_0} \frac{1 - \cos \lambda}{(4n^2 \pi^2 - \lambda^2)^2} d\lambda \\
 &= \frac{1}{16n^2 \pi} + O(T_0^{-1})
 \end{aligned}$$

$$Q_{n02} = \int_{T_0}^{\infty} \frac{1 - \cos \lambda}{(4n^2 \pi^2 - \lambda^2)^2} \frac{I_0(b\lambda)}{b\lambda I_0'(b\lambda)} d\lambda$$

$$\begin{aligned}
&= \int_{T_0}^{\infty} \frac{1 - \cos \lambda}{\lambda^4} \left[1 + \frac{8n^2 \pi^2}{\lambda^2} + \frac{48n^4 \pi^4}{\lambda^4} + \dots \right] \frac{I_0(b\lambda)}{b\lambda I_0'(b\lambda)} d\lambda \\
&= O(b^{-2} T_0^{-5}).
\end{aligned} \tag{E11}$$

Expressions (E1), (E3), (E7), (E10) and (E11) obtain Q_{nk} in the form (6.7).

II. Approximation for $P_{mn,kk}$, $m \neq n$

From (4.19) with $\omega = 0$,

$$P_{mn,kk} = 8m\pi^2 \rho U^2 \epsilon_k Q_{mnk}, \tag{E12}$$

where

$$Q_{mnk} = 4mn^2 \pi^2 \alpha \int_0^{\infty} \frac{1 - \cos \lambda}{(4m^2 \pi^2 - \lambda^2)(4n^2 \pi^2 - \lambda^2)} \frac{I_k(b\lambda)}{b\lambda I_k'(b\lambda)} d\lambda. \tag{E13}$$

The expansion for Q_{mnk} may be obtained by expressing (E13) in the form (E1) and proceeding as before: the integrals I_1 and I_2 in (E3) are replaced by

$$\int_0^{T_0} \frac{1 - \cos \lambda}{(4m^2 \pi^2 - \lambda^2)(4n^2 \pi^2 - \lambda^2)} d\lambda = O(T_0^{-3})$$

and

$$\int_0^{T_0} \frac{\lambda^2 (1 - \cos \lambda)}{(4m^2 \pi^2 - \lambda^2)(4n^2 \pi^2 - \lambda^2)} d\lambda = O(T_0^{-1})$$

respectively. Thus the leading term in the expansion for Q_{mnk} comes from a term like $J_4 (= O(b^3))$ in (E7). Since $P_{nn,kk}$ is given by (E12) with Q_{mnk} replaced by Q_{nk} from (6.7), the result is that

$$\frac{P_{mn,kk}}{P_{nn,kk}} = O(b^3) \quad (E14)$$

III. Asymptotic expansion for Q_{nk} when $bn \sim 1$, $b \ll 1$.

Introduce the variable $x = \lambda/2n\pi$ in (6.6) to get

$$Q_{nk} = \frac{\alpha}{2\pi} \int_0^\infty \frac{1 - \cos 2n\pi x}{(1 - x^2)^2} g_k(\gamma_n x) dx,$$

where

$$\gamma_n = 2bn\pi.$$

Let

$$G_k(x) = \frac{g_k(x) - 1/k}{x}. \quad (E15)$$

Then

$$Q_{nk} = \alpha(Q_{nk1} + Q_{nk2}) \quad (E16)$$

where

$$\begin{aligned}
 Q_{nk1} &= \frac{1}{2\pi k} \int_0^{\infty} \frac{1 - \cos 2n\pi x}{(1 - x^2)^2} dx \\
 &= \frac{\gamma_n}{8bk} \\
 Q_{nk2} &= \frac{\gamma_n}{2\pi} \int_0^{\infty} \frac{x(1 - \cos 2n\pi x)}{(1 - x^2)^2} G_k(\gamma_n x) dx. \quad (E17)
 \end{aligned}$$

To evaluate Q_{nk2} the procedure of appendix C may be employed. Thus

$$\begin{aligned}
 Q_{nk2} &= \frac{\gamma_n^2}{4\pi b} \left[- \int_0^{\infty} \frac{\sin 2n\pi x}{1 - x^2} G_k(\gamma_n x) dx \right. \\
 &\quad \left. - b \int_0^{\infty} \frac{1 - \cos 2n\pi x}{1 - x^2} G_k'(\gamma_n x) dx \right] \\
 &\sim \frac{-\gamma_n^2}{4\pi b} [G_k(\gamma_n) \int_0^{\infty} \frac{\sin 2n\pi x}{1 - x^2} dx \\
 &\quad - \gamma_n G_k'(\gamma_n) \int_0^{\infty} \frac{\sin 2n\pi x}{1 + x} dx \\
 &\quad + b \int_0^{\infty} \frac{G_k'(\gamma_n x) - G_k'(\gamma_n)}{1 - x^2} dx \\
 &\quad + b \gamma_n G_k''(\gamma_n) \int_0^{\infty} \frac{\cos 2n\pi x}{1 + x} dx]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{\gamma_n^2}{4\pi b} [G_k(\gamma_n) \text{Si}(2n\pi) \\
&\quad + \gamma_n G_k'(\gamma_n) \left\{ \frac{\pi}{2} - \text{Si}(2n\pi) \right\} \\
&\quad + b \gamma_n G_k''(\gamma_n) \text{Ci}(2n\pi) \\
&\quad - b \int_0^\infty \frac{G_k'(\gamma_n x) - G_k'(\gamma_n)}{1 - x^2} dx],
\end{aligned}$$

and on using the asymptotic forms for the Si and Ci functions,

$$\begin{aligned}
&= \frac{\gamma_n^2}{4\pi b} \left[\frac{\pi}{2} G_k(\gamma_n) + b \left\{ G_k'(\gamma_n) - \frac{1}{\gamma_n} G_k(\gamma_n) \right. \right. \\
&\quad \left. \left. + b \int_0^\infty \frac{G_k'(\gamma_n x) - \bar{G}_k'(\gamma_n)}{1 - x^2} dx \right\} + O(b^2) \right]. \tag{E18}
\end{aligned}$$

Expressions (E15) to (E18) give Q_{nk} in the form (6.8).

APPENDIX F

Evaluation of δE^* (equation (7.14))

The modal solution (7.3) for the pipe deformation is substituted in (7.14) and the variations taken to obtain, for example,

$$\begin{aligned} & \delta \int_0^{2\pi} d\theta \int_0^1 u^2 dx \\ &= \delta \sum_{n,k,m,\ell} A_{m\ell} A_{nk} \alpha^2 \iint F_m'(x) F_n'(x) \cos k\theta \cos \ell\theta \, dx d\theta \\ &= 2\pi \alpha^2 \sum_{m,n,k} \epsilon_k A_{nk} \delta A_{mk} \int_0^1 F_m'(x) F_n'(x) dx, \end{aligned}$$

where a dash denotes differentiation with respect to x . The result after evaluating the variations for all the terms in the expressions for δT and δV is

$$\begin{aligned} \delta T &= \pi \omega^2 \sum_{m,n,k} \epsilon_k [A_{nk} \delta A_{mk} \alpha^2 \int_0^1 F_m'(x) F_n'(x) dx \\ &\quad + (B_{nk} \delta B_{mk} + C_{nk} \delta C_{mk}) \int_0^1 F_m(x) F_n(x) dx] \end{aligned} \quad (F1)$$

$$\begin{aligned} \delta V &= \pi \sum_{m,n,k} \epsilon_k [V_{1mnk} \delta A_{mk} + V_{2mnk} \delta B_{mk} \\ &\quad + V_{3mnk} \delta C_{mk}], \end{aligned} \quad (F2)$$

where

$$\begin{aligned}
 V_{1mnk} = & \alpha^4 (A_{nk} - C_{nk} \beta) \int_0^1 F_n''(x) F_m''(x) dx \\
 & + \frac{1-\nu}{2} k \alpha^2 \{ k(1+\beta) A_{nk} - B_{nk} \\
 & + k\beta C_{nk} \} \int_0^1 F_n'(x) F_m'(x) dx \\
 & + \nu \alpha^2 (kB_{nk} + C_{nk}) \int_0^1 F_n(x) F_m''(x) dx.
 \end{aligned}$$

$$\begin{aligned}
 V_{2mnk} = & k(kB_{nk} + C_{nk}) \int_0^1 F_n(x) F_m(x) dx \\
 & + \frac{1-\nu}{2} \alpha^2 \{ (1+3\beta) B_{nk} - kA_{nk} \\
 & + 3k\beta C_{nk} \} \int_0^1 F_n'(x) F_m'(x) dx \\
 & + \nu \alpha^2 k(A_{nk} - \beta C_{nk}) \int_0^1 F_m(x) F_n''(x) dx.
 \end{aligned}$$

$$\begin{aligned}
 V_{3mnk} = & \{ C_{nk} + kB_{nk} + \beta(k^2 - 1)^2 C_{nk} \} \int_0^1 F_n(x) F_m(x) dx \\
 & + \beta \alpha^4 (C_{nk} - A_{nk}) \int_0^1 F_m''(x) F_n''(x) dx \\
 & - \nu k \alpha^2 (B_{nk} + kC_{nk}) \int_0^1 F_n(x) F_m''(x) dx \\
 & + \frac{1-\nu}{2} k \alpha^2 \{ 3B_{nk} + 4kC_{nk} + kA_{nk} \} \int_0^1 F_n'(x) F_m'(x) dx \\
 & + \nu \alpha^2 (A_{nk} - \beta k^2 C_{nk}) \int_0^1 F_n''(x) F_m(x) dx.
 \end{aligned} \tag{F3}$$

The cantilevered beam boundary conditions, $F_n(0) = F_n'(0) = F_n''(1) = F_n'''(1) = 0$, together with the orthogonality property of the beam functions may be used to give

$$\begin{aligned} \alpha^2 \int_0^1 F_m'(x) F_n'(x) dx &= \alpha^2 F_m'(1) F_n'(1) - I_{mn}, \\ \int_0^1 F_m(x) F_n(x) dx &= \delta_{mn}, \\ \int_0^1 F_n''(x) F_m''(x) dx &= \lambda_n^4 \delta_{mn}, \end{aligned} \quad (F4)$$

where

$$I_{mn} = \alpha^2 \int_0^1 F_m(x) F_n''(x) dx.$$

Expressions (F3) and (F4) obtain

$$V_{jn+k} = X_{mnj}^* A_{nk} + Y_{mnj}^* B_{nk} + Z_{mnj}^* C_{nk}, \quad (F5)$$

where

$$X_{mn1}^* = \alpha^4 \lambda_n^4 \delta_{mn} + \frac{1-\nu}{2} k^2 (1+\beta) J_{mn}$$

$$Y_{mn1}^* = \nu k I_{nm} - \frac{1-\nu}{2} k J_{mn}$$

$$Z_{mn1}^* = \nu I_{nm} + \beta \left\{ \frac{1-\nu}{2} k^2 J_{mn} - \alpha^4 \lambda_n^4 \delta_{mn} \right\}$$

$$X_{mn2}^* = k \left\{ \nu I_{nm} - \frac{1-\nu}{2} J_{mn} \right\}$$

$$y_{mn2}^* = k^2 \delta_{mn} + \frac{1-v}{2} (1+3\beta) J_{mn}$$

$$z_{mn2}^* = k[\delta_{mn} + \beta\{\frac{3}{2}(1-v)J_{mn} - vI_{mn}\}]$$

$$x_{mn3}^* = vI_{mn} + \beta\{\frac{1-v}{2} k^2 J_{mn} - \alpha^4 \lambda_n^4 \delta_{mn}\}$$

$$y_{mn3}^* = k[\delta_{mn} + \beta\{\frac{3}{2}(1-v)J_{mn} - vI_{nm}\}]$$

$$z_{mn3}^* = \{1 + \beta(k^2 - 1)^2\} \delta_{mn} + \beta[\alpha^4 \lambda_n^4 \delta_{mn} + k^2\{2(1-v)J_{mn} - v(I_{mn} + I_{nm})\}].$$

(F6)

Here

$$J_{mn} = \alpha^2 F_m(1) F_n'(1) - I_{mn}.$$

Also, from (F1),

$$\begin{aligned} \delta T = \pi \omega^*^2 \sum_{m,n,k} \epsilon_k [A_{nk} J_{mn} \delta A_{mk} \\ + (B_{nk} \delta B_{mk} + C_{nk} \delta C_{mk}) \delta_{mn}] \end{aligned} \quad (F7)$$

Expressions (F2), (F5), (F6) and (F7) give

$$\begin{aligned}\delta E^* = & - \pi \sum_{m,n,k} \epsilon_k [(X_{mn1}^A + Y_{mn1}^B + Z_{mn1}^C) \delta A_{mk} \\ & + (X_{mn2}^A + Y_{mn2}^B + Z_{mn2}^C) \delta B_{mk} \\ & + (X_{mn3}^A + Y_{mn3}^B + Z_{mn3}^C) \delta C_{mk}],\end{aligned}$$

(F8)

where

$$X_{mnj} = X_{mnj}^* - \omega^{*2} \delta_{mn} \delta_{j1}$$

$$Y_{mnj} = Y_{mnj}^* - \omega^{*2} \delta_{mn} \delta_{j2}$$

$$Z_{mnj} = Z_{mnj}^* - \omega^{*2} \delta_{mn} \delta_{j3}.$$

APPENDIX G

Numerical computations

The numerical results in this thesis have been obtained by using the Imperial College CDC 6400 computer. The programs for the computations were written in FORTRAN. A simple flow chart for the problem considered in chapter 7 is presented at the end of this section: the flow charts for the problems considered in the other chapters are much simpler and are not discussed.

The generalized pressure integrals, defined by (4.25) or (4.28), were evaluated numerically using Chebyshev-Gauss Quadrature formula:

$$\int_{-1}^1 \frac{f(z)}{\sqrt{1-z^2}} dz = \frac{\pi}{m} \sum_{j=1}^m f\left(\cos \frac{2j-1}{2m} \pi\right), \quad (G1)$$

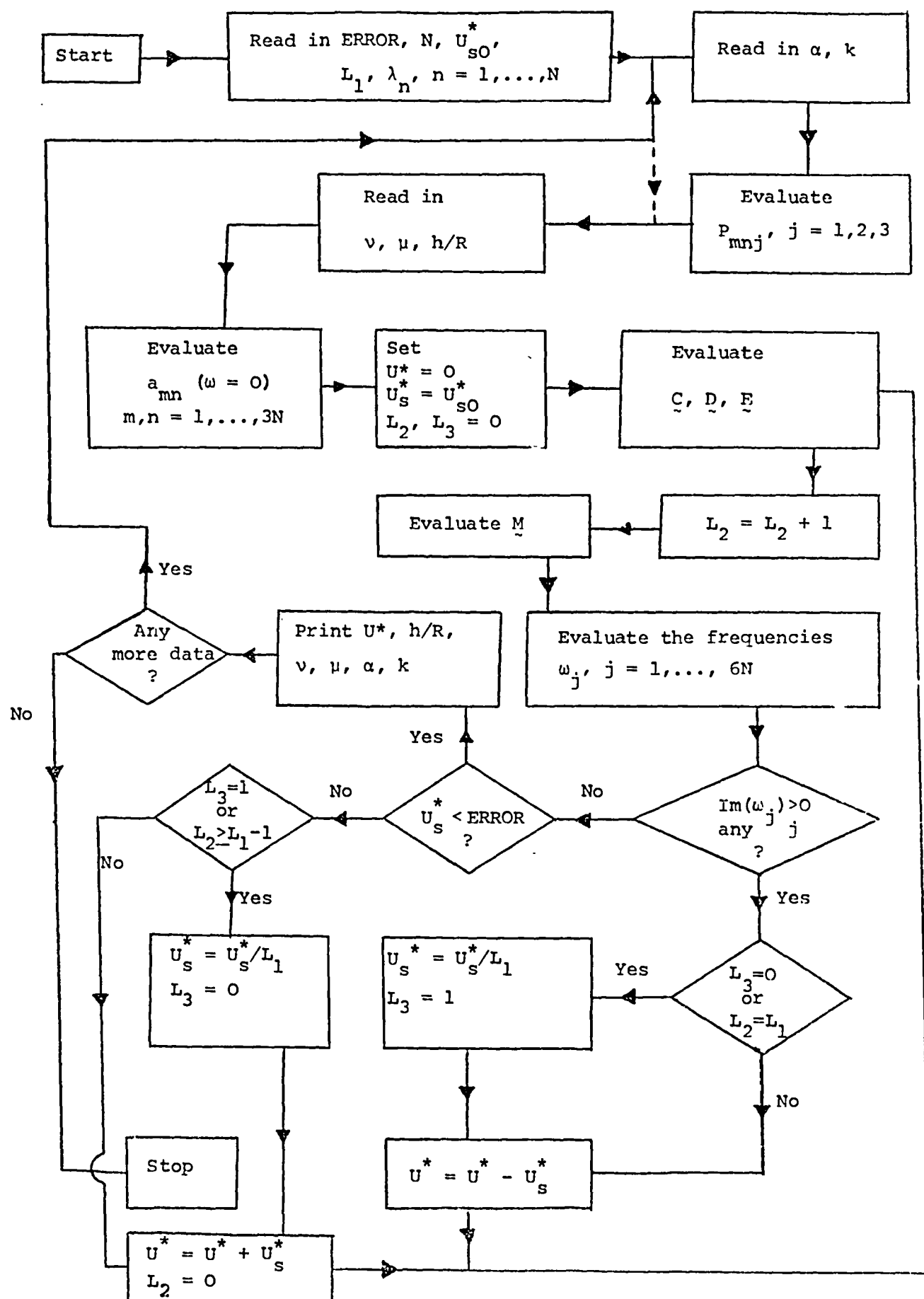
where an error term has been neglected and where the semi-infinite range for the P_{mnj} integrals is transformed to $(-1,1)$ by using the variable $z = (\lambda+1)/(\lambda-1)$. To check (G1) for convergence the value of m may be doubled and the integrals re-evaluated until the difference between two consecutive evaluations becomes less than the accuracy required. Convergence is achieved faster if the factor $g_k(\alpha\lambda)$ is replaced by $g_k(\alpha\lambda) - g_k(\alpha\lambda_n)$ and the rest of the integral evaluated analytically. However this procedure takes a lot of computer time but for the range of α considered in this work, sufficiently accurate results are obtained much more readily when a fixed value of m ($= 250$) is taken.

After evaluating the P_{mnj} 's and setting $\rho U^2 R^3/D$ to a given value, the entries for the determinant (7.20) or (7.24) are determined using (7.19). From either (7.20) or (7.24) the eigenvalue problem in the form (7.29) is obtained and the eigenvalues ω^* determined as functions of

$\rho U_R^2/D$ using standard eigenvalue routines.

In the flow chart below, $U^* = \rho U_R^2/D$ while U_s^* denotes the increments in U^* and U_{s0}^* denotes the initial value of U_s^* . L_1 is an integer, greater than 1, which controls the number of points, on the U^* -axis, examined for instability. The other variables are defined in the text (see list of symbols). When $N = 4$ it takes approximately 30 seconds to evaluate the generalized pressure integrals and 4.5 seconds to generate the critical dynamic pressure when the inplane inertias are neglected, but 23 seconds when the inplane inertias are not neglected.

Flow Chart for the Computer Program



APPENDIX H

The perturbation pressure in the shear layer

The solution of (8.12) is obtained by a perturbation analysis for a thin shear layer. For sufficiently small δ , $p(\xi)$ may be expanded in ascending powers of δ :

$$p = p_0 + \delta p_1 + \delta^2 p_2 + \dots \quad (H1)$$

When (H1) is substituted in (8.12) (assuming that $U \neq c$ anywhere in the shear layer), the coefficients of δ^j may be equated to zero to give, to $O(\delta^2)$,

$O(\delta^0)$:

$$\frac{d^2 p_0}{d\xi^2} - \frac{2}{U - c} \frac{dU}{d\xi} \frac{dp_0}{d\xi} = 0 \quad (H2)$$

$O(\delta)$:

$$\frac{d^2 p_1}{d\xi^2} - \frac{2}{U - c} \frac{dU}{d\xi} \frac{dp_1}{d\xi} = \frac{dp_0}{d\xi} \quad (H3)$$

$O(\delta^2)$:

$$\begin{aligned} \frac{d^2 p_2}{d\xi^2} - \frac{2}{U - c} \frac{dU}{d\xi} \frac{dp_2}{d\xi} &= \frac{dp_1}{d\xi} \\ &+ \xi \frac{dp_0}{d\xi} + (k^2 + \lambda^2) p_0 \end{aligned} \quad (H4)$$

In obtaining (H2), (H3) and (H4) it is assumed that λ and k are not too large compared with unity. In fact it will be necessary to impose the condition $\lambda\delta \ll 1$: in view of the results in chapters 3 and 5 it may be expected that the value of λ at instability is small enough for this condition to be applicable.

Determination of p_0

Equation (H2) is equivalent to

$$\frac{d}{d\xi} \left\{ \frac{1}{(U - c)^2} \frac{dp_0}{d\xi} \right\} = 0,$$

so that

$$p_0(\xi) = B_{00} \int_{\xi}^{\xi} (U - c)^2 d\xi + B_{01}, \quad (H5)$$

where B_{00} and B_{01} are arbitrary constants to be determined from the boundary conditions of continuity of pressure and radial component of velocity at $\xi = 1$.

For the continuity of pressure, equations (8.11) and (H1) may be employed; in using (8.11), A must be regarded as a function of, and therefore expanded in powers of, δ :

$$A = A_0 + \delta A_1 + \delta^2 A_2 + \dots, \quad (H6)$$

thus accounting for the modification of the mean flow perturbation by the shear layer.

This gives

$$\begin{aligned}
 & (A_0 + \delta A_1 + \delta^2 A_2 + \dots) \{ I_k(\lambda) - \lambda \delta I_k'(\lambda) \\
 & + \frac{1}{2} \lambda^2 \delta^2 I_k''(\lambda) + \dots \} \\
 & = p_0(1) + \delta p_1(1) + \delta^2 p_2(1) + \dots,
 \end{aligned}$$

and on equating coefficients of δ^j to zero,

$$p_0(1) = A_0 I_k(\lambda) \quad (H7)$$

$$p_1(1) = A_1 I_k(\lambda) - A_0 \lambda I_k'(\lambda) \quad (H8)$$

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Similarly, for the continuity of the radial component of velocity, equations (8.8), (8.11), (H1) and (H6) may be employed to give

$$p_0'(1) = 0 \quad (H9)$$

$$p_1'(1) = -A_0 \lambda I_k'(\lambda) \quad (H10)$$

$$p_2'(1) = A_0 \lambda^2 I_k''(\lambda) - A_1 \lambda I_k'(\lambda) \quad (H11)$$

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Equations (H5), (H7) and (H9) give

$$p_0(\xi) = A_0 I_k(\lambda). \quad (H12)$$

Determination of p_1

In view of (H12), (H3) is identical to (H2) and so $p_1(\xi)$ may be expressed in the form (H5) and conditions (H8) and (H10) employed to give

$$p_1(\xi) = A_1 I_k(\lambda) - A_0 \lambda I_k'(\lambda) \left\{ 1 - \int_{\xi}^1 \left(\frac{U - c}{U_{\infty} - c} \right)^2 d\xi \right\}. \quad (H13)$$

Determination of p_2

On using (H12) and (H13), (H4) becomes

$$\begin{aligned} \frac{d^2 p_2}{d\xi^2} - \frac{2}{U - c} \frac{dU}{d\xi} \frac{dp_2}{d\xi} = & - A_0 \lambda I_k'(\lambda) \left\{ \frac{U - c}{U_{\infty} - c} \right\}^2 \\ & + (k^2 + \lambda^2) A_0 I_k(\lambda). \end{aligned} \quad (H14)$$

Only $p_2'(\xi)$ is required in determining the perturbation pressure to $O(\delta)$: equation (H14) may be integrated once and (H11) employed to give

$$\begin{aligned} \frac{dp_2}{d\xi} = & \left(\frac{U - c}{U_{\infty} - c} \right)^2 [A_0 (k^2 + \lambda^2) I_k(\lambda) \left\{ 1 - \int_{\xi}^1 \left(\frac{U_{\infty} - c}{U - c} \right)^2 d\xi \right\} \\ & - A_0 \lambda I_k'(\lambda) \xi - A_1 \lambda I_k'(\lambda)]. \end{aligned} \quad (H15)$$

Determination of A to $O(\delta)$

The constants A_j in (H6) are related to the pipe wall deformation through the kinematic boundary condition equivalent to (2.9). Thus, denoting the radial wall deformation by $y(x)$ and taking the non-dimensional form

$$y(x) = B e^{i(\lambda x - \omega t)}, \quad (H16)$$

the kinematic boundary condition is

$$R \frac{\partial y}{\partial t} + U \frac{\partial y}{\partial x} = w, \quad \xi = 0, \quad (\text{H17})$$

where w is given by (8.5). In using (H17) it is assumed that $B \ll \delta$ but is otherwise arbitrary. Equations (8.8), (H1), (H12), (H16) and (H17) then give

$$-\lambda^2 \rho c^2 B = p_1'(0) + \delta p_2'(0) + \dots,$$

and on equating coefficients of δ^j to zero,

$$p_1'(0) = -\lambda^2 \rho c^2 B,$$

$$p_j'(0) = 0, \quad j > 1. \quad (\text{H18})$$

Equations (H13), (H15) and (H18) give

$$A_0 = \frac{\lambda \rho B}{I_k'(\lambda)} (U_\infty - c)^2,$$

$$A_1 = A_0 (k^2 + \lambda^2) \frac{I_k(\lambda)}{\lambda I_k'(\lambda)} \left\{ 1 - \int_0^1 \left(\frac{U_\infty - c}{U - c} \right)^2 d\xi \right\}. \quad (\text{H19})$$

Equations (H1), (H12), (H13) and (H19) give $p(0)$, to $O(\delta)$, in the form (8.14).

Table 5.1

Numerical values for the integrals F_1 and F_2 defined by equations (5.7).

k	$F_1(k)$	$F_2(k)$
1	-0.7423	0.2110
2	-0.2105	0.0210
3	-0.0972	0.0044
4	-0.0551	0.0018
5	-0.0362	0.0008
6	-0.0261	0.0004
7	-0.0193	0.0002
8	-0.0149	0.0001
9	-0.0118	0.0001
10	-0.0096	0.0001

Table 5.2

Values of $\rho U^2 R^3/D$ at divergence for simply supported pipes of various radius ratios, α , and thickness ratios, h/R ($\nu = 0.3$, $M = 0$). The integers in brackets are the corresponding longitudinal, n , and circumferential, k , mode numbers at instability. The dashed border separates beam from shell mode instabilities. V is the stability parameter of [19].

$\rho U^2 R^3/D$		$\rho U^2 R^3/D$ $V (n,k)$			
h/R	Travelling waves	$\alpha = 0.05$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.5$
0.001	1.01×10^4	1.78×10^4 23.9 (1,3)	2.59×10^4 14.5(1,4)	3.87×10^4 8.8(1,6)	6.04×10^4 4.4(1,9)
0.003	3.85×10^3	4.45×10^3 20.7(1,2)	6.36×10^3 12.4(1,3)	9.62×10^3 7.6(1,4)	1.55×10^4 3.9(1,7)
0.005	1.99×10^3	2.08×10^3 18.3(1,2)	3.57×10^3 12.0(1,3)	5.01×10^3 7.1(1,4)	8.00×10^3 3.6(1,6)
0.007	1.42×10^3	1.42×10^3 17.9(1,2)	2.80×10^3 12.6(1,3)	3.58×10^3 7.1(1,3)	5.26×10^3 3.4(1,5)
0.009	1.11×10^3	1.15×10^3 18.3(1,2)	1.79×10^3 11.4(1,2)	2.38×10^3 6.6(1,3)	3.77×10^3 3.3(1,5)
0.01	995	1.07×10^3 18.6(1,2)	1.49×10^3 10.9(1,2)	2.03×10^3 6.4(1,3)	3.33×10^3 3.3(1,5)
0.03	337	301 17.0(1,1)	338 9.0(1,2)	571 5.9(1,2)	893 2.9(1,4)
0.05	203	108 13.2(1,1)	243 9.9(2,2)	243 4.9(1,2)	415 2.6(1,3)
0.07	147	55.2 11.2(1,1)	153 9.3(2,2)	153 4.6(1,2)	282 2.5(1,3)
0.09	115	33.4 9.8(1,1)	116 9.2(2,2)	116 4.6(1,2)	219 2.5(1,2)
0.1	105	27.2 9.3(1,1)	101 9.0(1,1)	105 4.6(1,2)	183 2.4(1,2)
0.3	43.0	3.08 5.5(1,1)	11.5 5.3(1,1)	33.3 4.5(1,1)	27.8 1.6(1,1)
0.5	34.0	1.16 4.3(1,1)	4.32 4.2(1,1)	12.7 3.6(1,1)	12.1 1.4(1,1)

Table 5.3

Values of $\rho U^2 R^3/D$ (simply supported pipes) at divergence for various circumferential mode numbers, k , and thickness ratios, h/R . The critical longitudinal mode numbers, n , are given in brackets. ($\alpha = 0.05$, $\nu = 0.3$, $M = 0$)

h/R	$\rho U^2 R^3/D$ (n)				
	k	2	3	4	5
0.001		3.41×10^4 (1)	1.78×10^4 (1)	2.59×10^4 (2)	3.24×10^4 (3)
0.005		2.08×10^3 (1)	3.57×10^3 (2)	5.01×10^3 (4)	6.47×10^3 (6)
0.010		1.07×10^3 (1)	1.78×10^3 (3)	2.55×10^3 (6)	3.29×10^3 (9)
0.030		338 (2)	616 (5)	892 (11)	1.22×10^3 (17)
0.050		204 (3)	378 (7)	578 (14)	866 (22)
0.070		150 (3)	279 (9)	458 (17)	750 (24)
0.100		105 (4)	210 (11)	384 (19)	690 (27)

Table 5.4

Comparison of numerical and asymptotic evaluations of the integral Q_{nl} using equations (5.5) and (5.6). ($M = 0$)

n	α	Q_{nl}	
		Numerical Integration equation (5.5)	Asymptotic Approximation equation (5.6)
1	0.0400	0.0301	0.0301
2		0.0596	0.0596
3		0.0877	0.0877
1	0.0667	0.0486	0.0486
2		0.0943	0.0944
3		0.1351	0.1357
1	0.1000	0.0696	0.0696
2		0.1311	0.1318
3		0.1797	0.1858
1	0.1333	0.0883	0.0884
2		0.1604	0.1643
3		0.2099	0.2431
1	0.2000	0.1190	0.1202
2		0.2002	0.2411
3		0.2411	0.5773
1	0.2500	0.1368	0.1412
2		0.2181	0.3621
3		0.2505	1.3958

Table 5.5

Comparison of numerical and asymptotic evaluation of the integral

Q_{n2} (M = 0)

n	α	Num. integr. Eqn (5.5)	Eqn (5.6)	Asymptotic Approximation Equation (5.17)	
				1 term	2 terms
1	0.0400	0.0154	0.0154	0.0157	0.0157
2		0.0306	0.0306	0.0313	0.0312
3		0.0456	0.0456	0.0466	0.0463
1	0.0667	0.0252	0.0252	0.0261	0.0259
2		0.0498	0.0498	0.0516	0.0510
3		0.0734	0.0735	0.0701	0.0690
1	0.1333	0.0480	0.0480	0.0516	0.0504
2		0.0924	0.0926	0.0990	0.0949
3		0.1309	0.1321	0.1393	0.1319
1	0.2000	0.0682	0.0683	0.0761	0.0724
2		0.1266	0.1283	0.1393	0.1292
3		0.1705	0.1842	0.1846	0.1682
1	0.2500	0.0817	0.0819	0.0934	0.0865
2		0.1470	0.1529	0.1642	0.1464
3		0.1907	0.2411	0.2078	0.1854

Table 5.6

Comparison of numerical and asymptotic evaluations of the integral Q_{n2} .

($\alpha = 0.1, M = 0$)

n	$n\pi\alpha$	Num. integr. Eqn (5.5)	Eqn (5.6)	Asymptotic Approximation Equation (5.17)	
				1 term	2 terms
1	0.3142	0.0369	0.0369	0.0390	0.0384
2	0.6283	0.0721	0.0722	0.0761	0.0742
3	0.9425	0.1045	0.1047	0.1098	0.1061
4	1.2566	0.1330	0.1340	0.1393	0.1338
5	1.5708	0.1573	0.1610	0.1642	0.1571
6	1.8850	0.1775	0.1879	0.1846	0.1764
7	2.1991	0.1940	0.2193	0.2010	0.1921
8	2.5133	0.2071	0.2619	0.2139	0.2049
9	2.8274	0.2176	0.3252	0.2240	0.2150
10	3.1416	0.2258	0.4221	0.2318	0.2231

Table 5.7

Values of $\rho U^2 R^3/D$ (simply supported pipes) at divergence for sR/D
(see 5.4.5) above and below the transition value. ($\nu = 0.3, M = 0$)

h/R	sR/D	k = 1		k = 2	
		$n\pi\alpha$	$\rho U^2 R^3/D$	$n\pi\alpha$	$\rho U^2 R^3/D$
0.001	2.53	0.02	11248	0.07	11241
	2.51		11154		11227
0.010	3.06	0.07	1152.1	0.21	1150
	3.04		1146.3		1148.2
0.030	3.16	0.13	387.0	0.37	386.4
	3.14		385.1		385.9
0.050	3.29	0.17	235	0.49	234.3
	3.26		234		234.0
0.100	3.23	0.32	126	0.71	118.9
	3.20		113.2		118.7

Table 6.1

Comparison of numerical and asymptotic evaluations of the integral Q_{n1} using equations (6.6) and (6.7). ($M = 0$)

n	α	Numerical Integration equation (6.6)	Asymptotic Approximation equation (6.7)
1	0.0400	0.1238	0.1238
2		0.2371	0.2373
3		0.3327	0.3346
1	0.0667	0.2013	0.2014
2		0.3615	0.3657
3		0.4666	0.5123
1	0.1000	0.2890	0.2901
2		0.4707	0.5212
3		0.5511	1.0630
1	0.1333	0.3651	0.3710
2		0.5388	0.8157
3		0.5868	3.2131

Table 6.2

Comparison of numerical and asymptotic evaluation of Q_{n2} using equations (6.6), (6.7) and the leading term in (6.8). ($M = 0$)

n	α	Numerical Int. equation (6.6)	Asymptotic Approximation	
			equation (6.7)	equation (6.8)
1	0.0400	0.0625	0.0625	0.0625
2		0.1232	0.1232	0.1231
3		0.1803	0.1803	0.1801
1	0.0667	0.1033	0.1033	0.1032
2		0.1985	0.1986	0.1980
3		0.2799	0.2814	0.2786
1	0.1000	0.1524	0.1525	0.1521
2		0.2806	0.2823	0.2786
3		0.3739	0.3923	0.3692
1	0.1333	0.1989	0.1991	0.1980
2		0.3479	0.3579	0.3429
3		0.4380	0.5435	0.4277

Table 7.1

Approximate natural frequencies for $h/R = 0.026$ ($= 1.63/63.5$), $\nu = 0.28$, $L/R = 7.91$ ($= 502/63.5$), $E = 2.1 \times 10^{11}$ N/m², $\rho_m = 7.8 \times 10^3$ kg/m³: these constants are taken from [39]. The first row, for each k , gives results from §7.2, while the second and third row give results from §7.3 without and with the inplane inertia effects respectively; $N = 5$ except for the third row which corresponds to a single mode analysis.

k	Natural frequencies, $\omega/2\pi$ (Hz)				
	n	1	2	3	4
1		737.2	4290.5	10542	11394
		676.3	3111.0	6332.7	8571.0
		500.2	2496.1	5076.5	7347.4
2		1039.7	2363.2	4009.8	8453.1
		354.3	1072.6	2545.2	4204.5
		319.5	1019.7	2398.9	3963.2
3		951.9	1504.6	2394.5	3677.1
		811.4	970.5	1557.7	2482.0
		769.9	930.4	1515.4	2428.3
4		1549.3	1735.3	2138.9	2771.7
		1510.5	1569.6	1775.6	2200.5
		1465.8	1525.0	1730.3	2158.0
5		2433.4	2521.6	2708.8	3018.7
		2413.1	2453.6	2561.1	2771.0
		2367.1	2409.2	2513.4	2722.6
6		3532.4	3594.5	3711.6	3900.5
		3517.0	3552.3	3633.4	3773.6
		3470.3	3508.8	3586.6	3724.3
7		4835.6	4890.0	4984.3	5131.7
		4821.8	4855.4	4927.8	5043.8
		4774.6	4812.4	4882.2	4996.0

Table 7.2

Approximate values for the frequency parameter $\omega^*{}^2$ obtained from the generalized Galerkin method. The first and second row, for each L/R, gives the results obtained from a single mode and a coupled mode analysis (N = 4) respectively, with the inplane inertias included. The third row gives the results of an exact analysis [40]. ($\nu = 0.3$, $h/R = 0.01$)

L/R	k	n	Natural frequency parameter, $\omega^*{}^2$		
			1	2	3
5.30	2		7.85×10^{-4}	2.17×10^{-2}	0.102
			7.22×10^{-4}	1.71×10^{-2}	8.59×10^{-2}
			$7. \times 10^{-4}$	-	-
10.90			1.03×10^{-4}	1.60×10^{-3}	1.02×10^{-2}
			1.01×10^{-4}	1.45×10^{-3}	9.22×10^{-3}
			1×10^{-4}	-	-
27.30			6.12×10^{-5}	1.03×10^{-4}	3.75×10^{-4}
			6.11×10^{-5}	1.01×10^{-4}	3.63×10^{-4}
			-	1×10^{-4}	-
45.60			6.02×10^{-5}	6.58×10^{-5}	1.02×10^{-4}
			6.02×10^{-5}	6.55×10^{-5}	1.01×10^{-4}
			-	-	$1. \times 10^{-4}$
1.39	4		1.16×10^{-2}	0.186	0.459
			1.01×10^{-2}	0.133	0.408
			0.01	-	-
4.34			1.92×10^{-3}	6.42×10^{-3}	2.93×10^{-2}
			1.91×10^{-3}	5.86×10^{-3}	2.62×10^{-2}
			1.9×10^{-3}	-	-
18.90			1.77×10^{-3}	1.79×10^{-3}	1.91×10^{-3}
			1.77×10^{-3}	1.79×10^{-3}	1.91×10^{-3}
			-	-	1.9×10^{-3}

Table 7.3

Variation of $\rho U^2 R^3/D$ at instability as a function of N , the number of terms in the truncated solution (equation (7.3)). Inplane inertia effects are not included. ($\nu = 0.5$, $M = 0$, $\mu = 0.06$, $h/R = 0.0227$; spurious divergence instabilities are denoted by an asterisk).

L/R	N	$\rho U^2 R^3/D$		
		k	1	2 3
5.0	2		9100	1320 497
	3		4500	1060 458
	4		4020	1030 451
	5		3770	1010 449
	6		3100*	988 445
7.5	2		4380	673 298
	3		3080	614 300
	4		2850	603 304
	5		2840	595 306
	6		2690	588 307
10.0	2		2750	413 294
	3		2210	396 338
	4		2120	390 411
	5		2100	386 414
	6		2030	384 414
15.0	2		1390	217 4210*
	3		1270	219 295
	4		1240	216 287
	5		1230	216 299
	6		1210	216 305

Table 7.3 (continued)

L/R	N	k	$\rho U^2 R^3 / D$		
			1	2	3
20.0	2		835	172	7110*
	3		801	185	1930*
	4		786	208	319
	5		776	206	328
	6		770	214	305
30.0	2		390	1620*	15500*
	3		389	201	3970*
	4		381	197	1920*
	5		378	192	292
	6		375	190	301

Table 7.4

Values of $\rho U^2 R^3 / D$ at instability obtained from the different formulations for the flow beyond the ends of the cantilever. Inplane inertia effects are not included. ($\nu = 0.5$, $M = 0$, $h/R = 0.0227$, $\mu = 0.06$, $N = 4$)

k	L/R	Cantilever Continuation - §4.6	Long pipe Approximation - 4.5.4	Linear form for $G_n(x)$ - 4.5.2	Oscillatory form for $G_n(x)$ - 4.5.3
1	5	1840	2730	4020	3860
	10	2110	2020	2120	2180
	15	1220	1200	1240	1280
	20	772.0	765.0	786.0	804.0
2	5	1010	964.0	1030	1070
	10	383.0	379.0	390.0	401.0
	15	215.0	214.0	218.0	222.0
	20	211.0	212.0	208.0	210.0
3	5	444.0	434.0	451.0	467.0
	10	404.0	400.0	411.0	394.0
	15	275.0	273.0	287.0	287.0
	20	319.0	318.0	319.0	315.0

Table 7.5

Values of $\rho U^2 R^3 / D$ at instability for various values of h/R and L/R . Inplane inertia effects are not included. ($\nu = 0.5$, $M = 0$, $\mu = 0.06$; the integers in brackets indicate the value of N)

h/R	k	$\rho U^2 R^3 / D$		
		L/R (N)	10 (4)	15 (4) 20 (5)
0.001	1		1.09×10^6	6.40×10^5 4.00×10^5
	2		1.90×10^5	9.21×10^4 5.30×10^4
	3		6.02×10^4	2.79×10^4 1.60×10^4
0.005	1		4.37×10^4	2.56×10^4 1.60×10^4
	2		7630	3720 2170
	3		2520	1350 1360
0.010	1		1.09×10^4	6400 4000
	2		1920	954 586
	3		732	812 662
0.030	1		1210	712 444
	2		233	147 196
	3		300	235 256 (6)
0.050	1		437	256 160
	2		99.8	124 92.6
	3		162	183 (5) 169 (6)
0.080	1		171	100 62.5
	2		69.7	66.0 71.7
	3		129	129 (5) 140 (6)
0.100	1		110	64.2 40.1
	2		70.6	49.4 57.6
	3		107	108 (5) 140 (6)
0.300	1		12.6	7.30 4.6
	2		28.1	22.6 22.3
	3		75.5	80.1 (5) 321 (6)

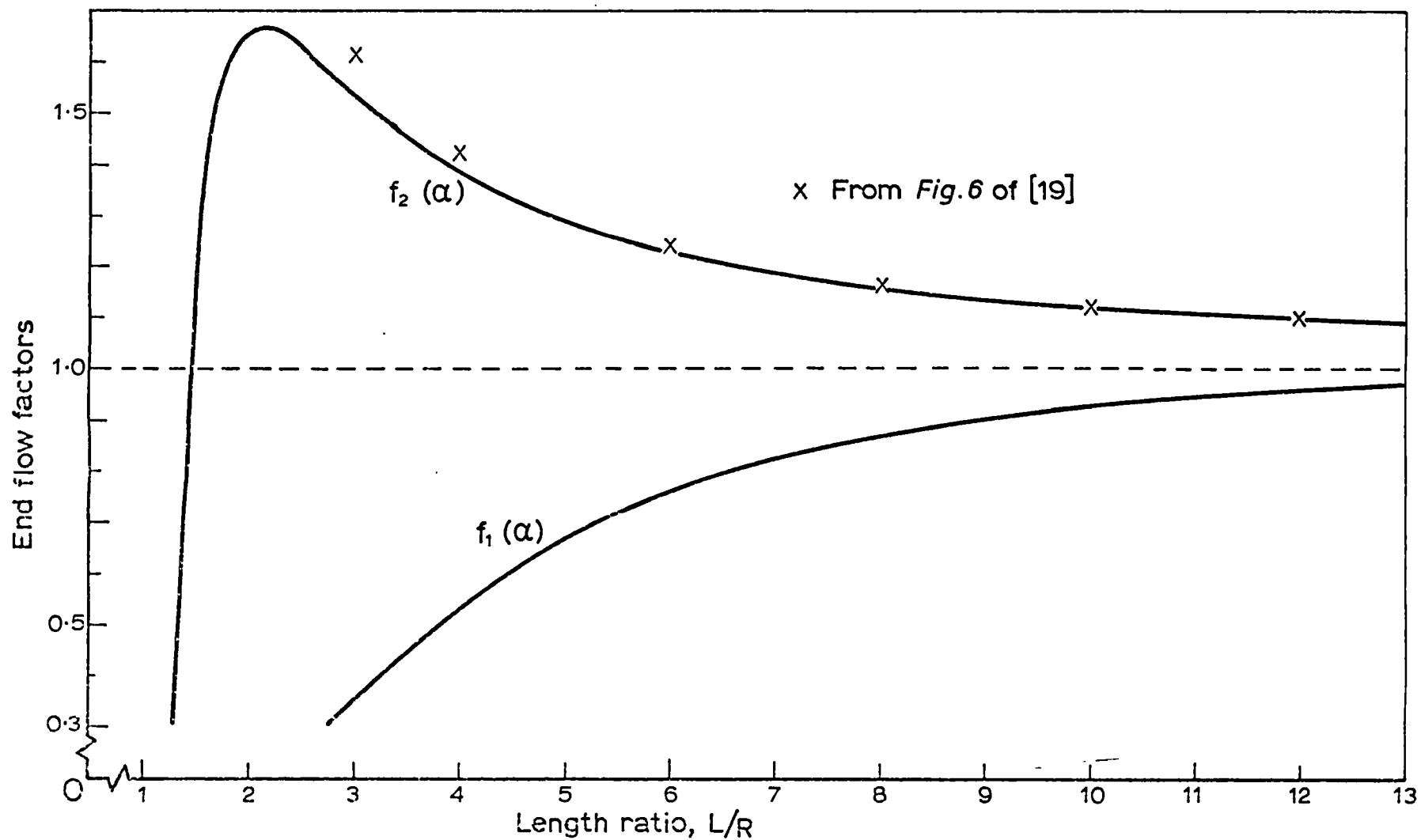


Fig. 5.1 Variation of the end flow factor $f_1(\alpha)$ and $f_2(\alpha)$ (defined by (5.14) and (5.15) respectively) with length ratio. ($\nu=0.3$, $M=0$)

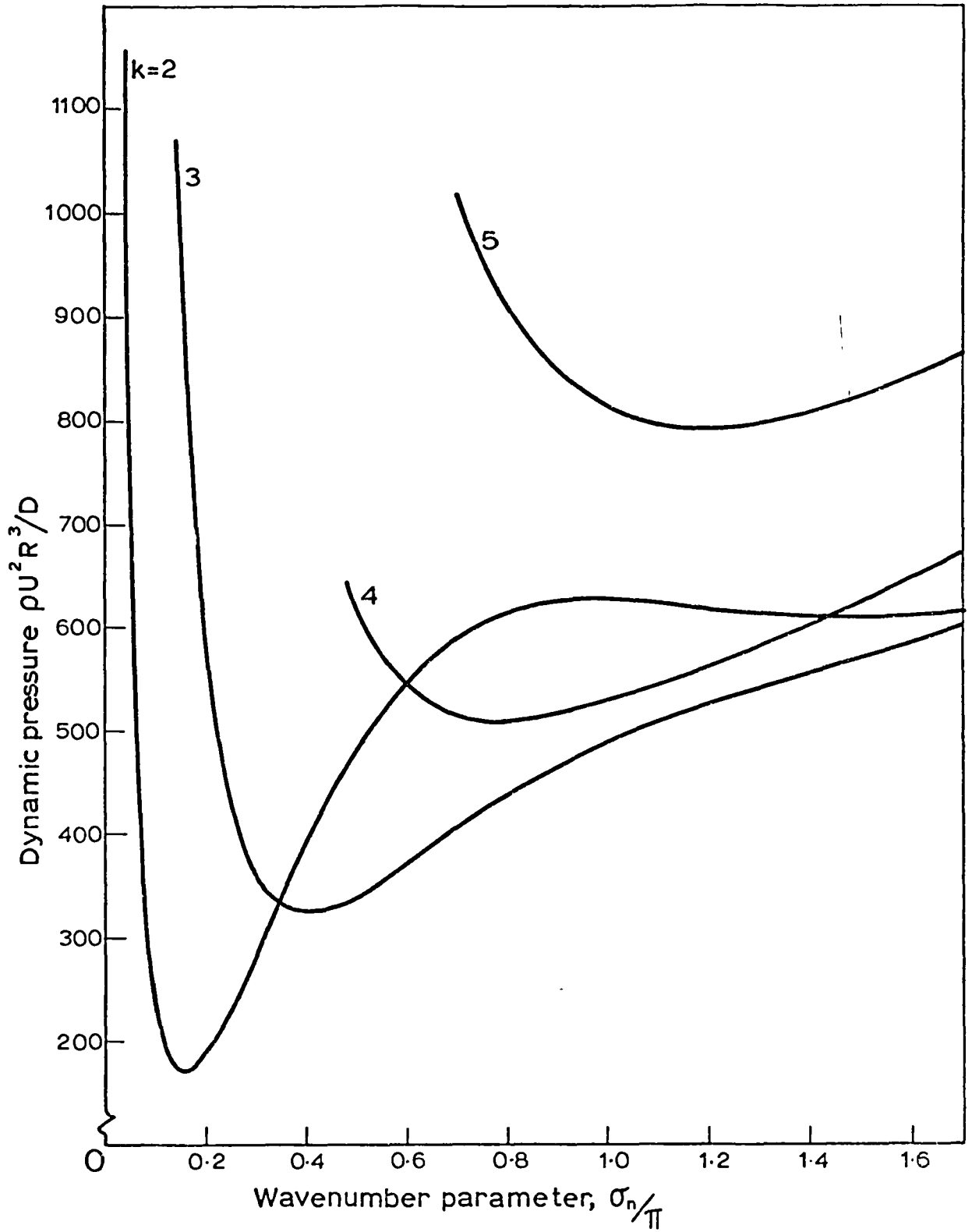


Fig. 5.2 Variation of $\rho U^2 R^3/D$ with σ_n for simply supported pipes.
($\nu=0.3$, $M=0$, $h/R=0.06$)

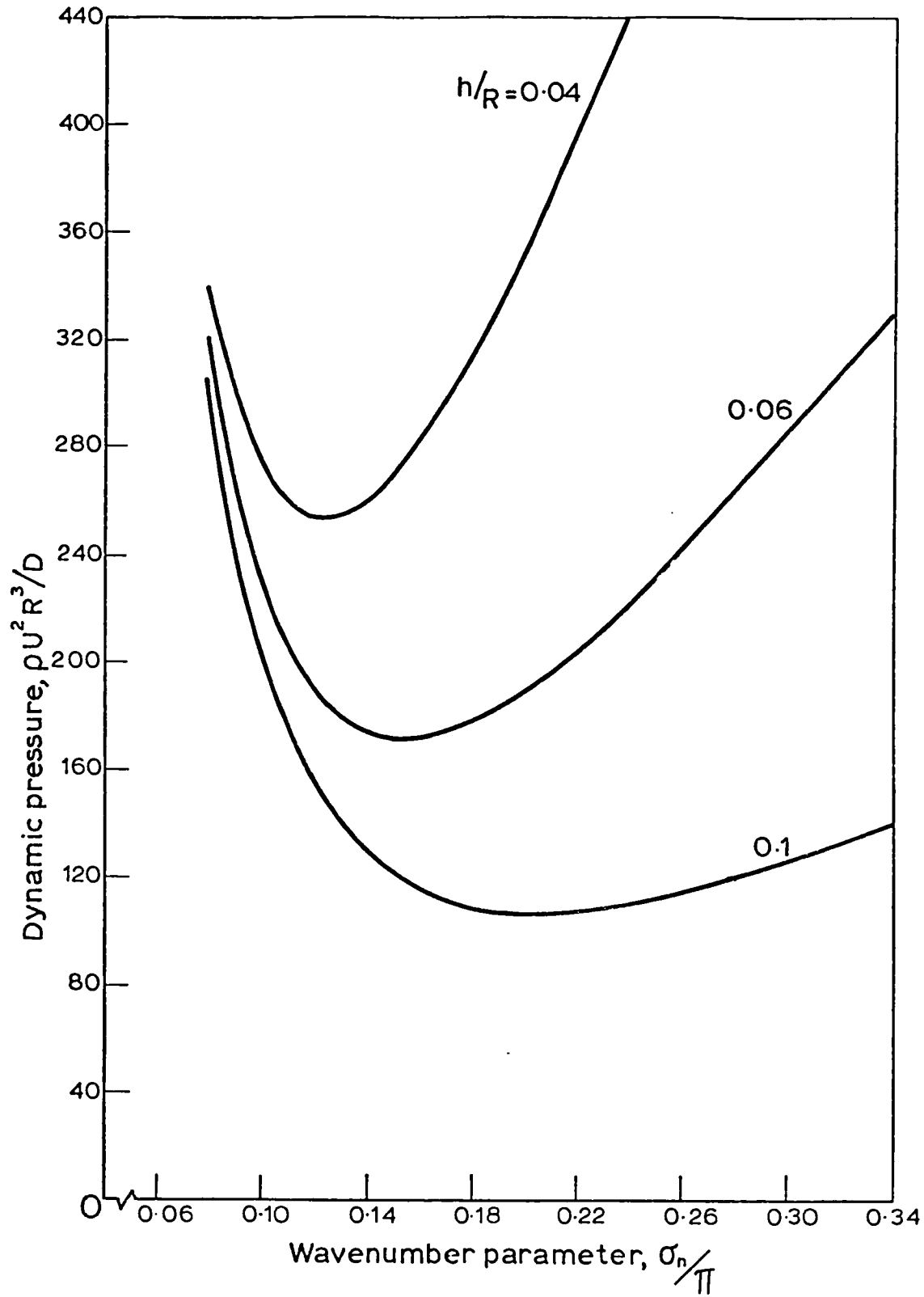


Fig. 5.3 Variation of $\rho U^2 R^3/D$ with σ_n for simply supported pipes.
($k=2, \nu=0.3, M=0$)

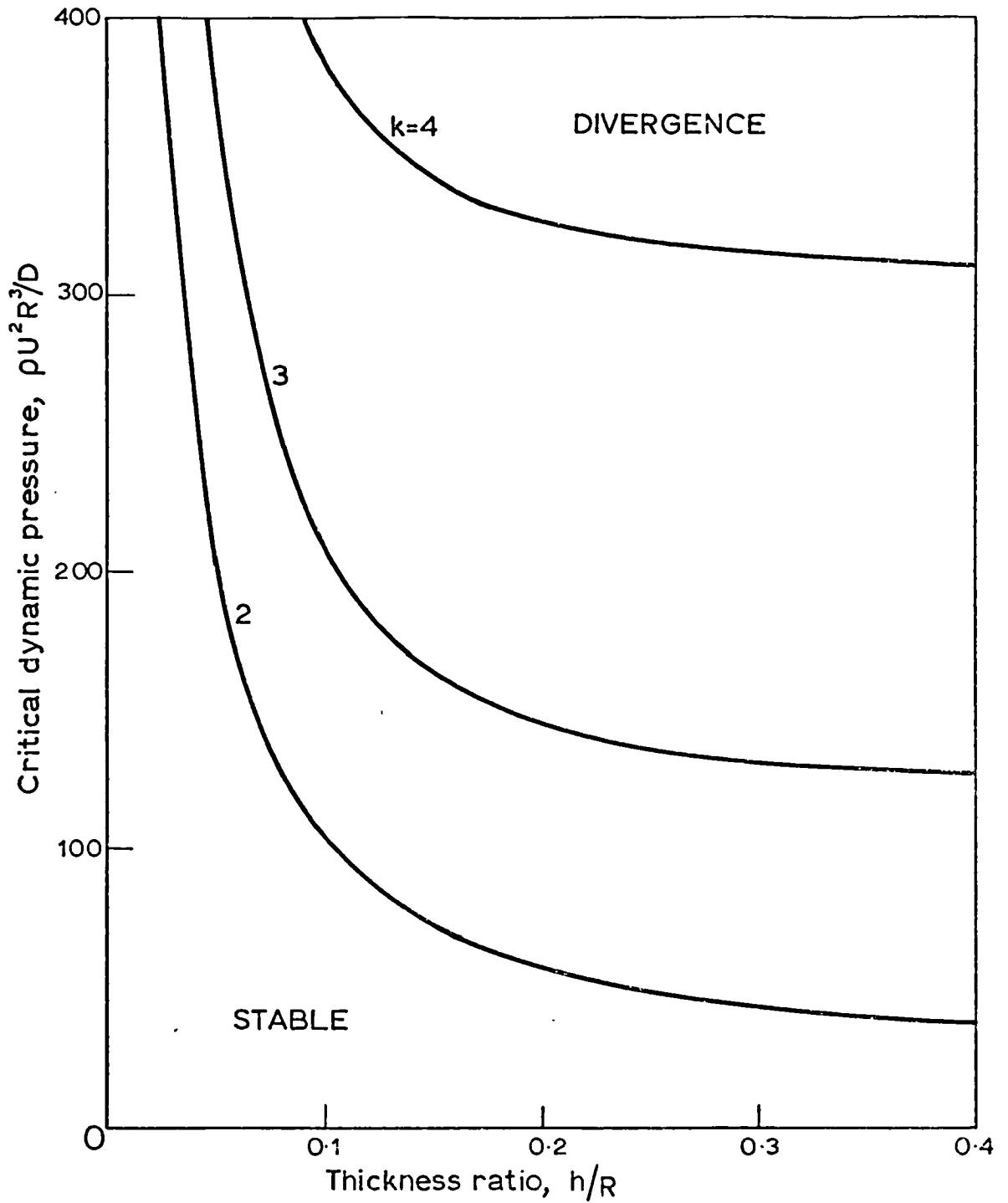


Fig. 5.4 Typical variation of pU^2R^3/D , at instability, for simply supported pipes when σ_n is treated as a continuous variable and the minima for each value of the thickness ratio, h/R , taken from curves similar to those of Fig. 5.3 ($\nu=0.3, M=0$)

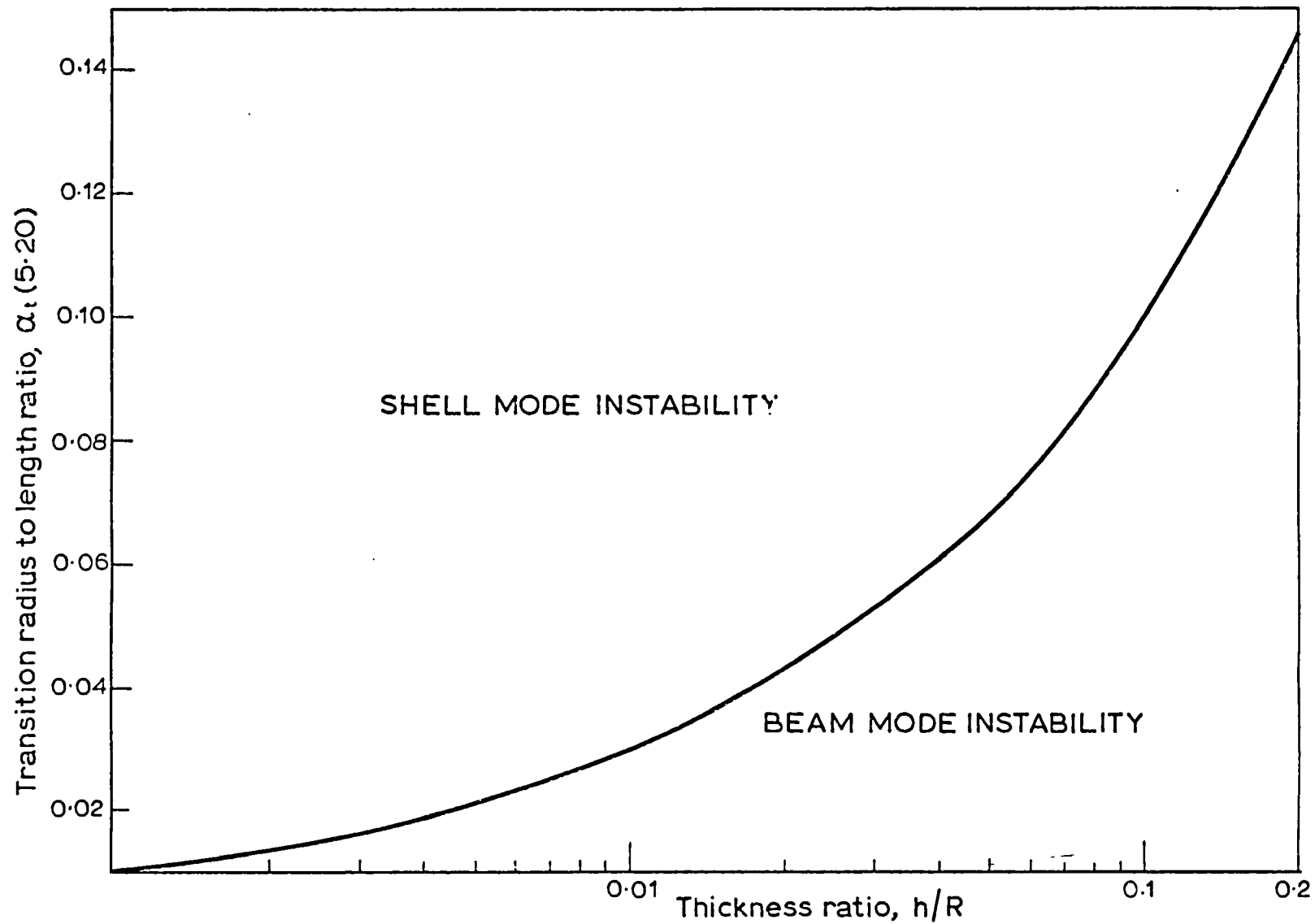


Fig. 5-5 The variation of the transition value, α_t , of the radius ratio, α , with thickness ratio, h/R , separating beam mode from shell mode instabilities for simply supported pipes.
($\nu = 0.3$, $M = 0$)

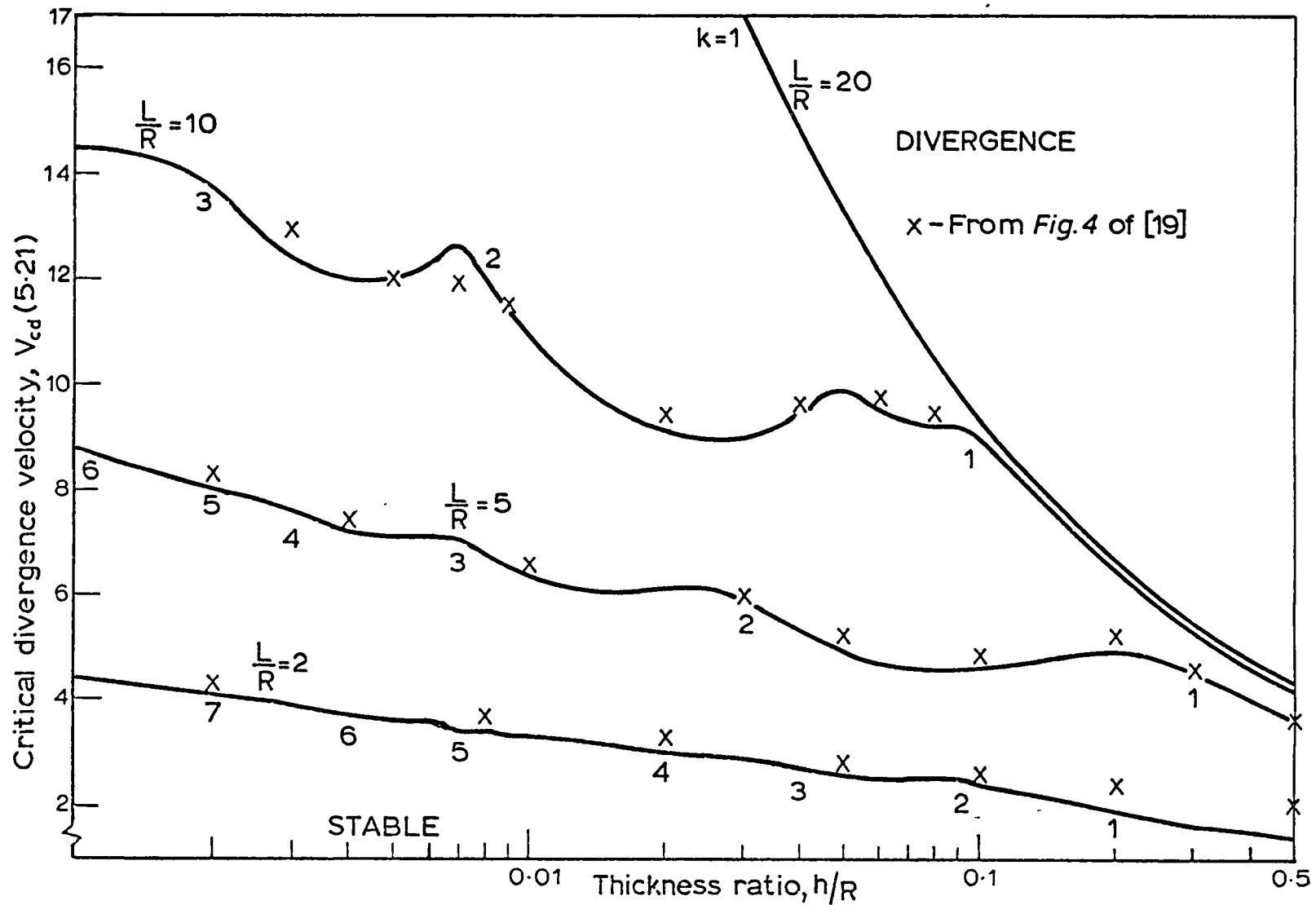


Fig. 5.6 Stability boundaries of simply supported pipes. The numbers on the curves indicate the critical circumferential mode. ($\nu=0.3$, $M=0$)

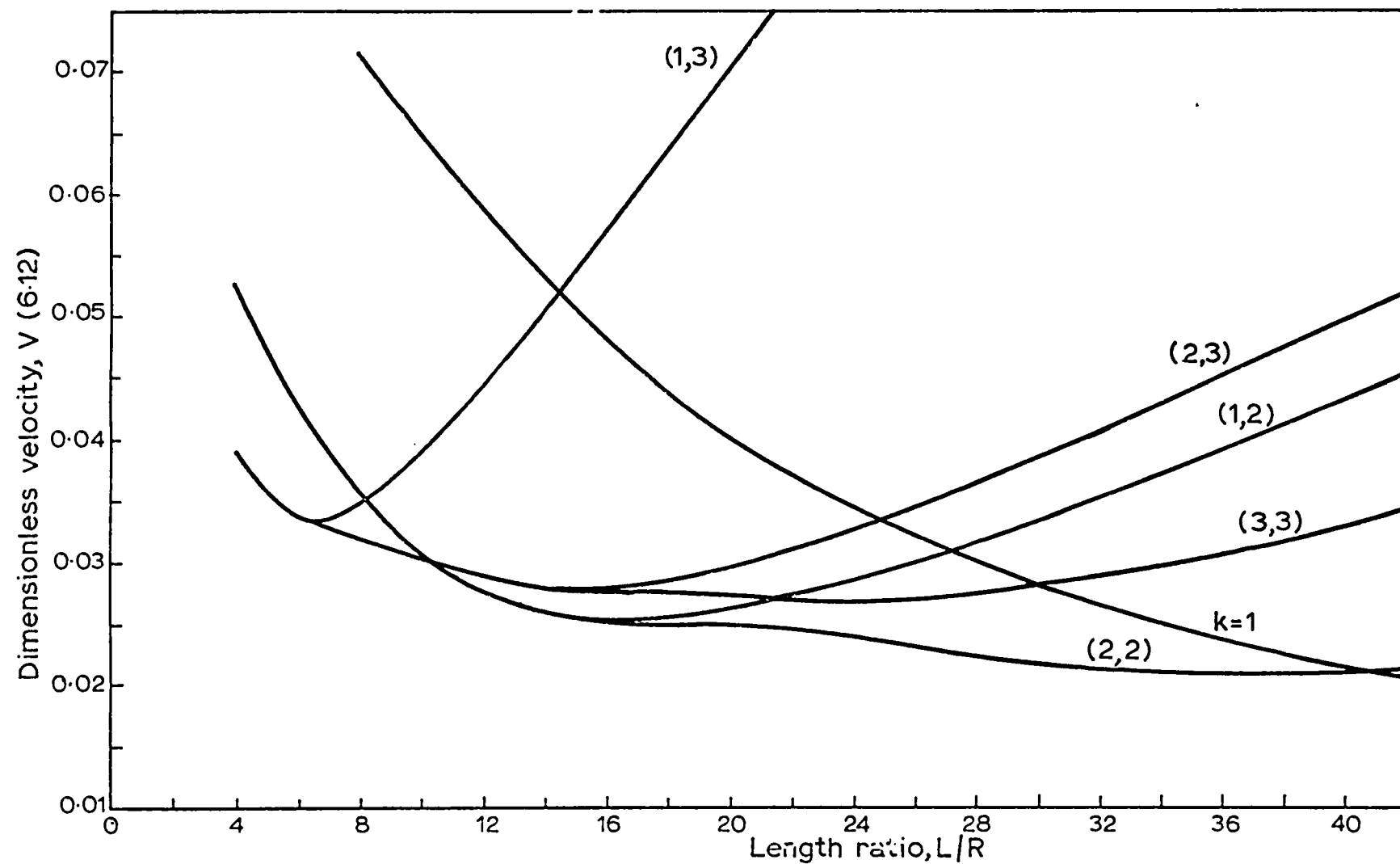


Fig.6.1 Typical variation of the minimum value of the dimensionless velocity, V , with length ratio, L/R , for clamped-clamped pipes and for different values of (N, k) . ($V=0.45, M=0, h/R=0.02$)

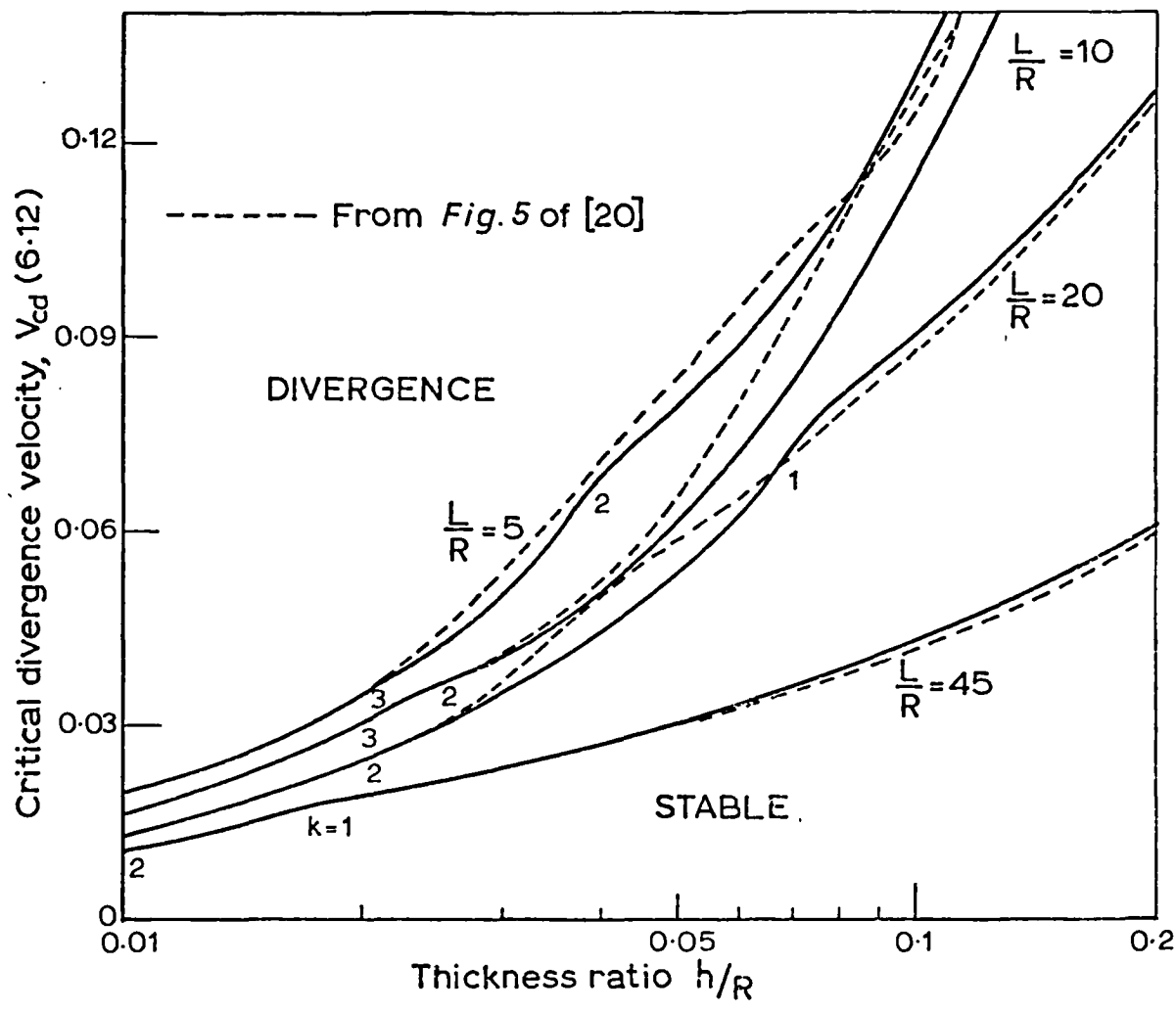


Fig. 6.2 Stability boundaries for clamped-clamped pipes.
($\nu=0.45, M=0$)

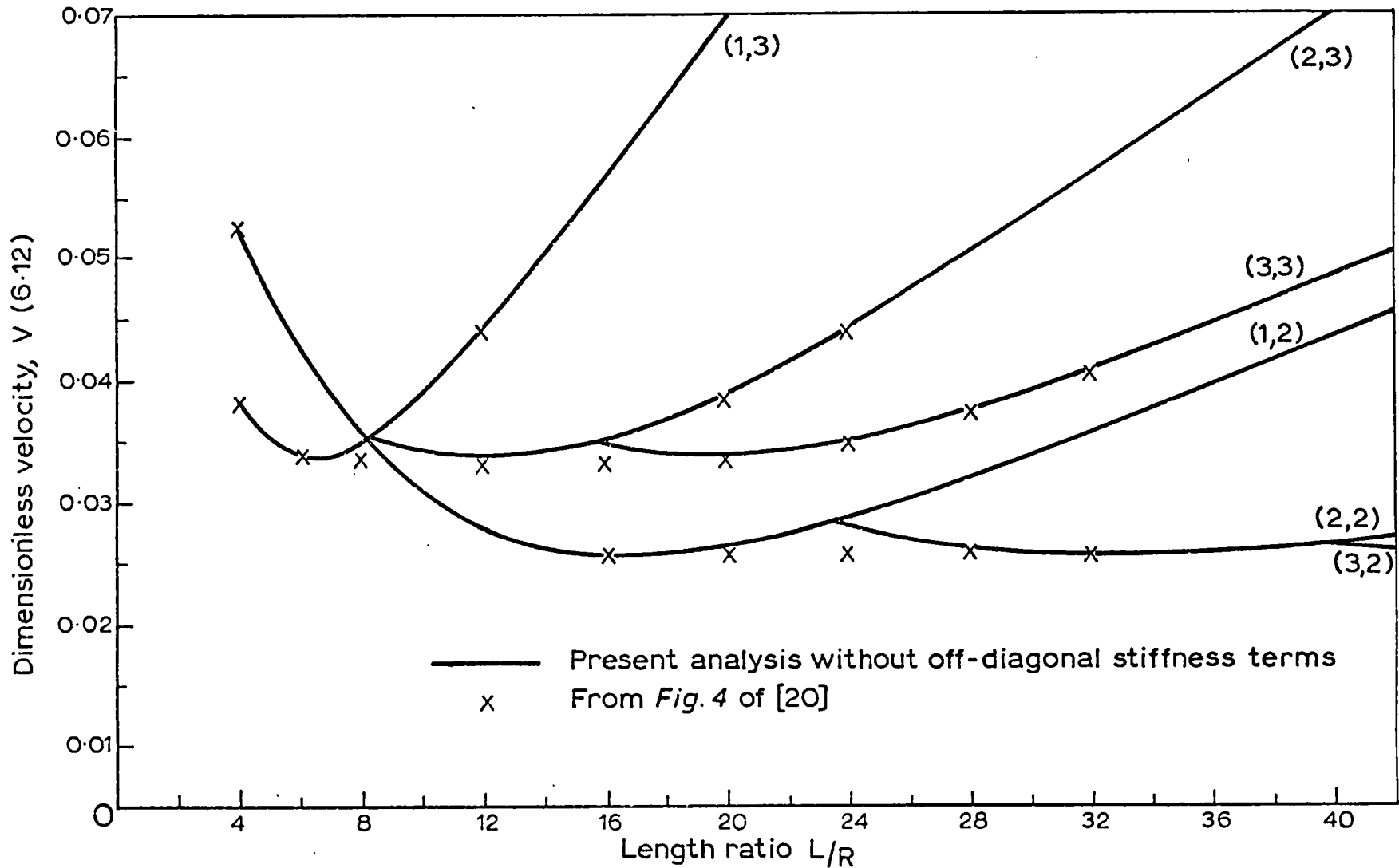


Fig. 6.3 Convergence of the solution for clamped-clamped pipes when the off-diagonal stiffness terms are neglected in the characteristic determinant. The numbers in brackets represent (n,k) . ($V=0.45$, $M=0$, $h/R=0.02$)

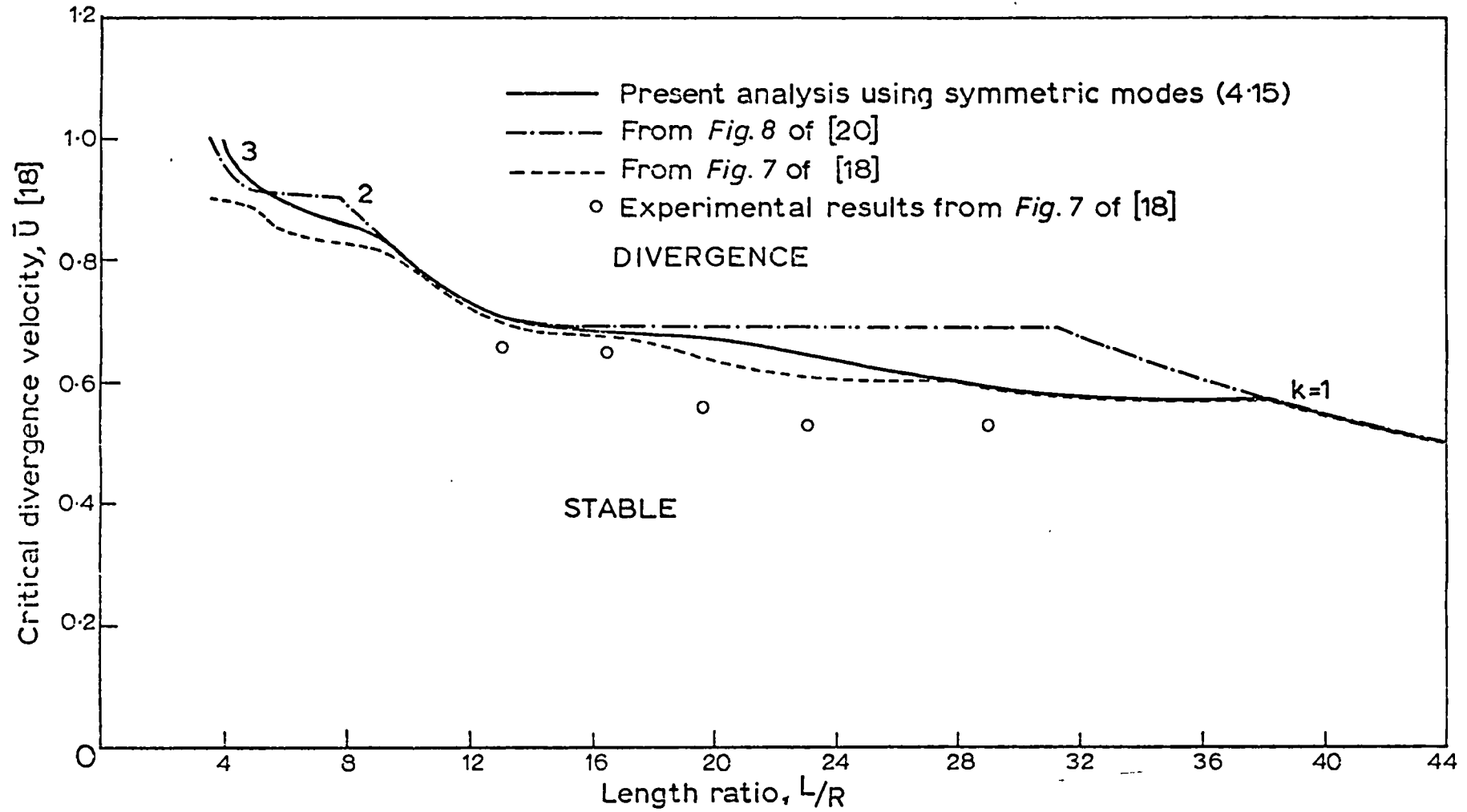


Fig. 6.4 Stability boundaries of clamped-clamped pipes.

$$(v=0.5, M=0, h/R=0.0227, \bar{U}^2 = \frac{h^2}{12R^2\mu} \left\{ \frac{\rho U^2 R^3}{D} \right\})$$

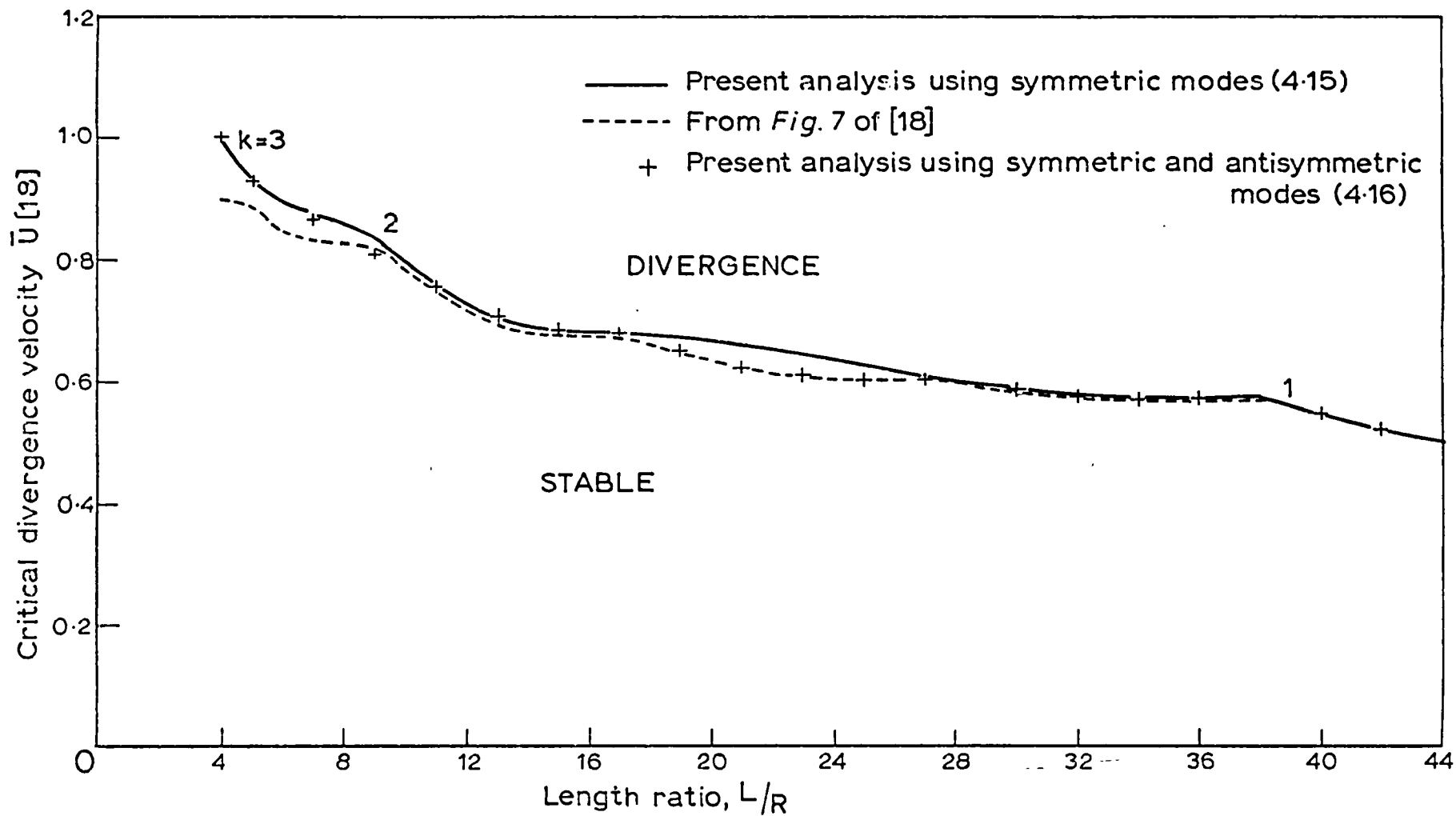


Fig. 6.5 Stability boundaries of clamped-clamped pipes obtained by employing symmetric and antisymmetric modes.

$$(\nu=0.5, M=0, h/R=0.0227, \mu=0.06, \bar{U}^2 = \frac{h^2}{12R^2\mu} \left\{ \frac{\rho U^2 R^3}{D} \right\})$$

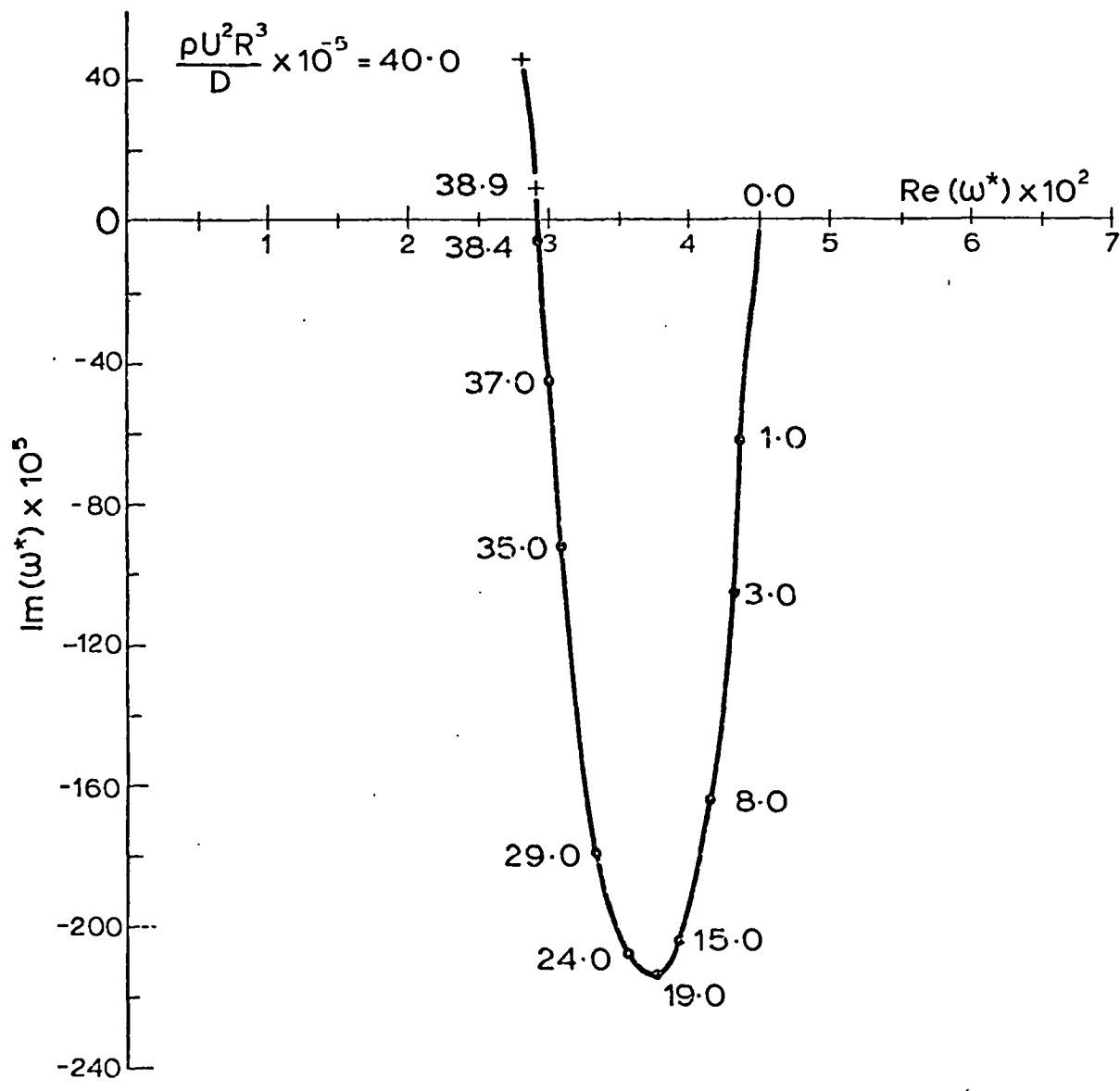


Fig. 7.1 Typical variation of the most unstable frequency, ω^* , for cantilever pipes.
 ($\nu=0.5$, $M=0$, $h/R=0.0227$, $\mu=0.06$, $L/R=20$, $k=1$, $N=4$; no inplane inertias)

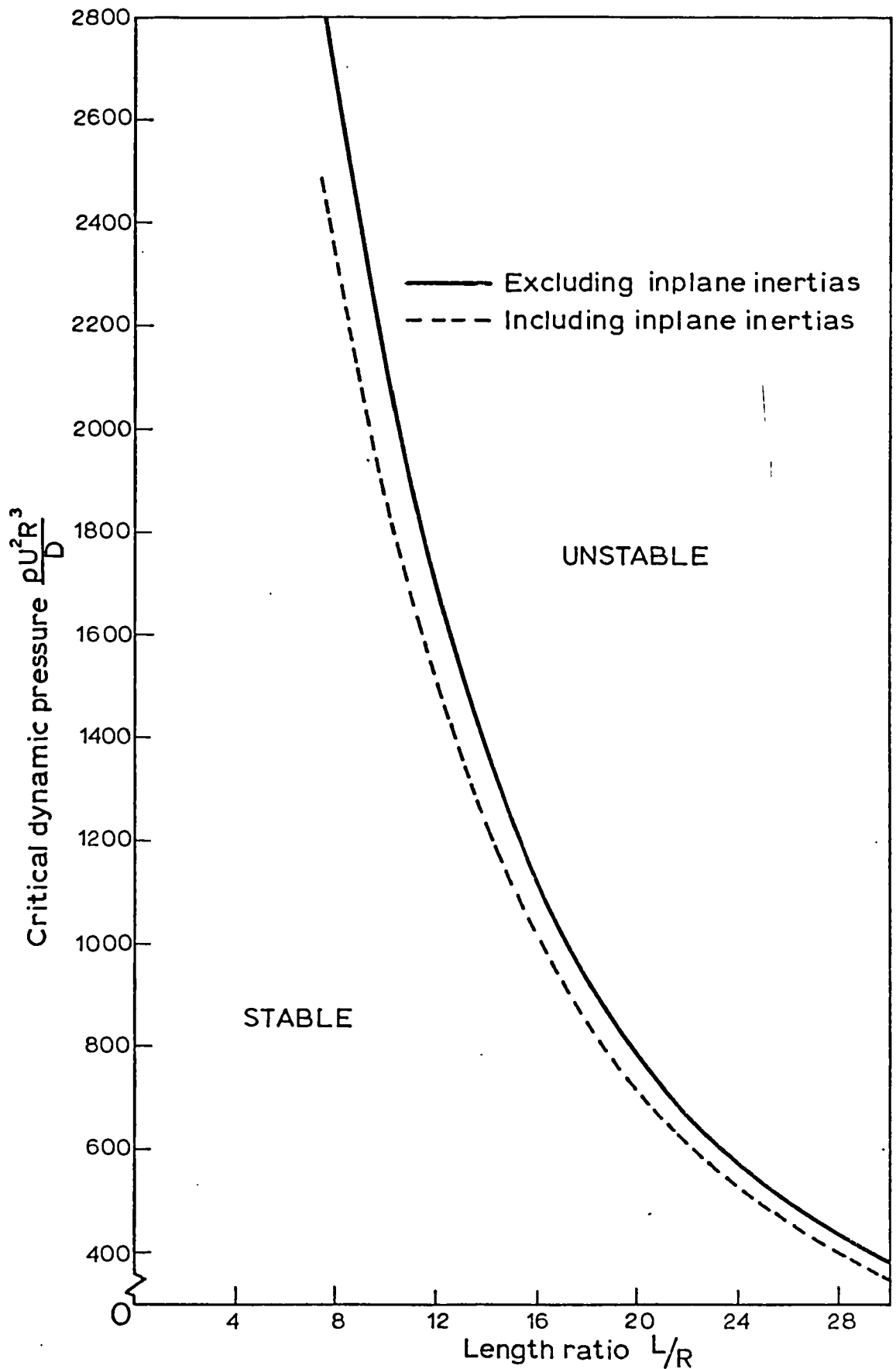


Fig. 7.2 The critical dynamic pressure as a function of length ratio, showing the effects of inplane inertias for cantilever pipes. ($\nu=0.5$, $M=0$, $h/R=0.0227$, $\mu=0.06$, $k=1$, $N=4$).

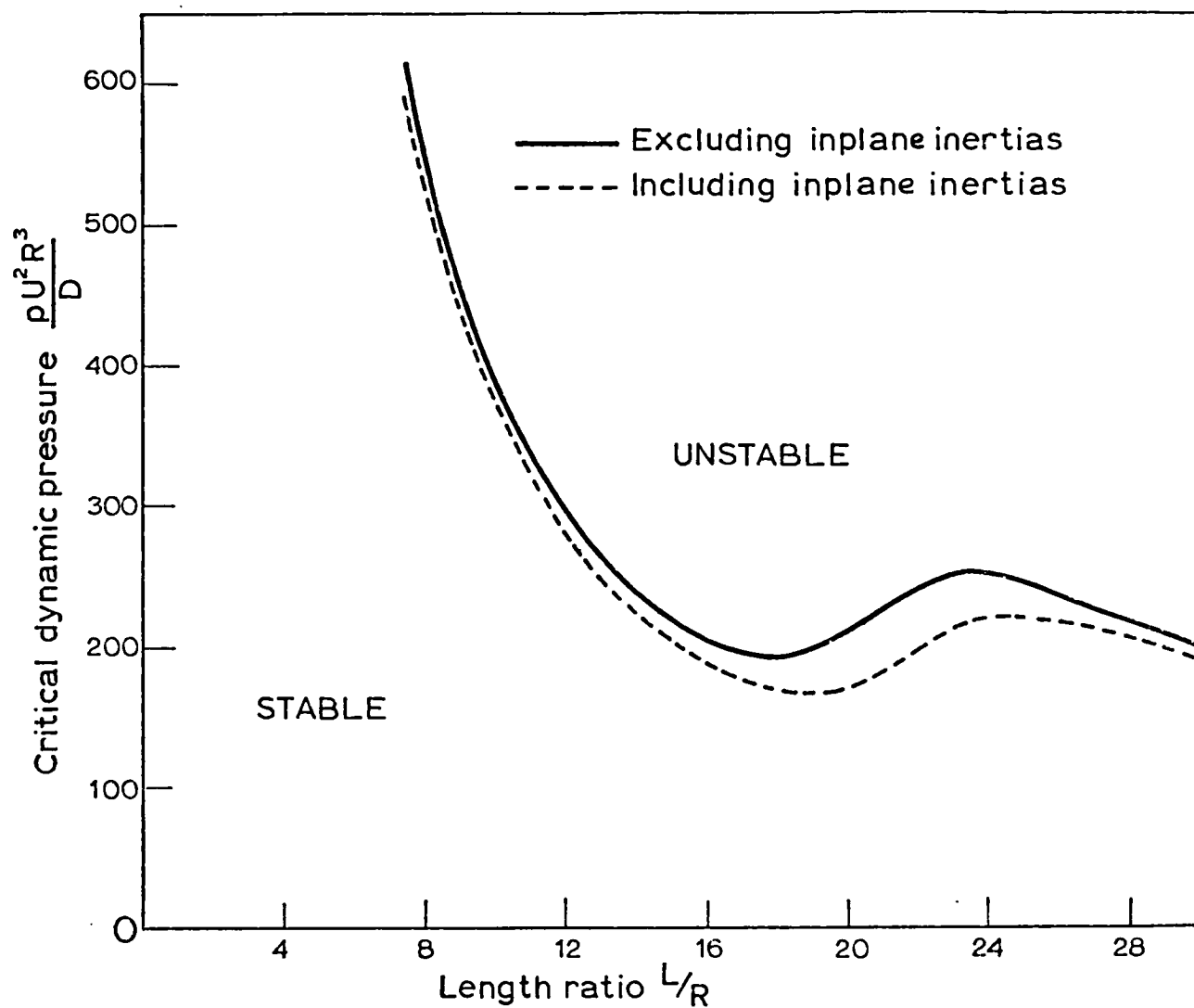


Fig. 7.3 The critical dynamic pressure as a function of length ratio, showing the effects of inplane inertias for cantilever pipes.

($V=0.5$, $M=0$, $h/R=0.0227$, $\mu=0.06$, $k=2$,
 $N=4$)

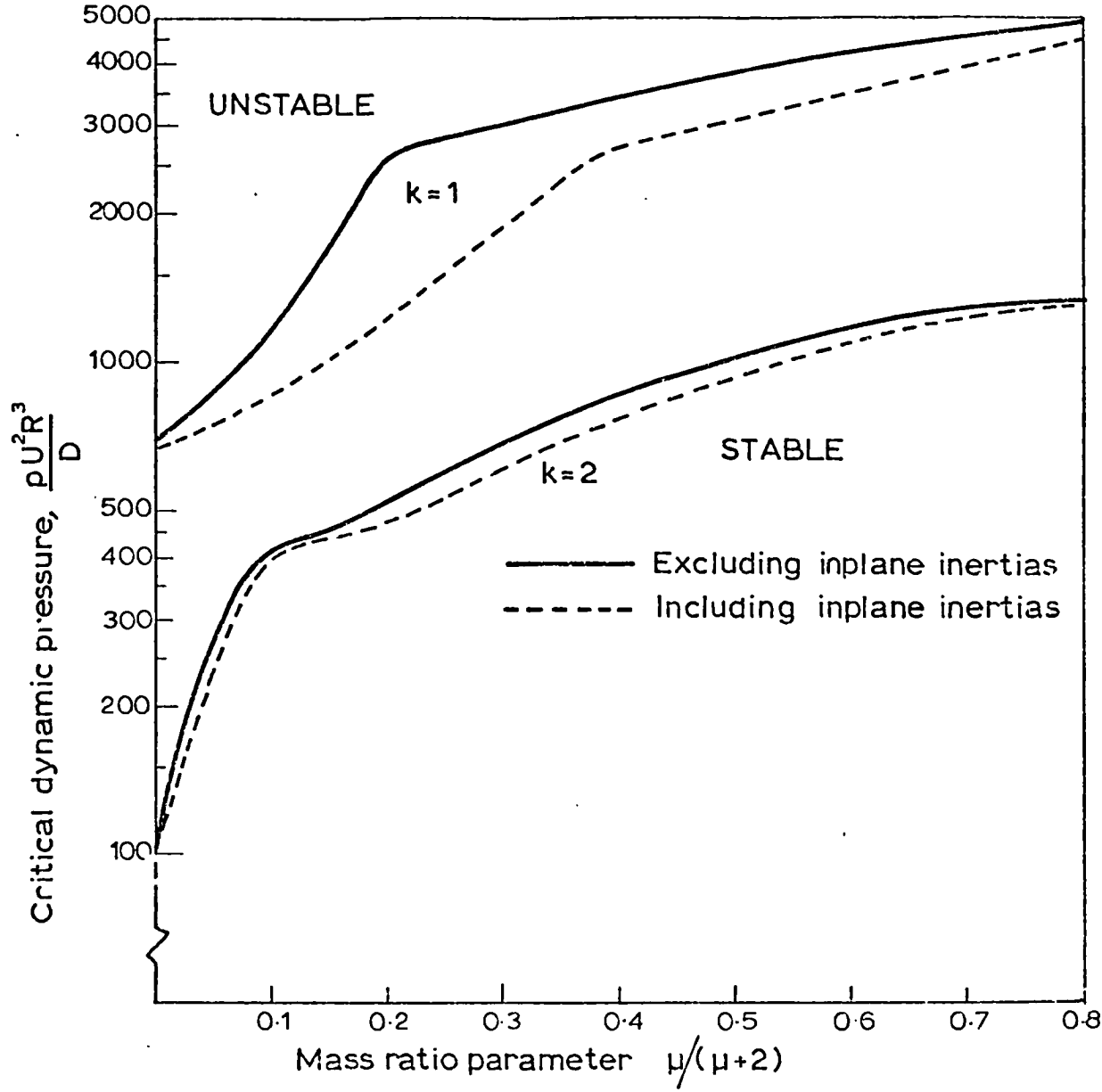


Fig. 7.4 The critical dynamic pressure as a function of the mass ratio parameter, showing the effect of inplane inertias for cantilever pipes. ($\nu=0.5$, $M=0$, $h/R=0.0227$, $L/R=20$, $N=5$)

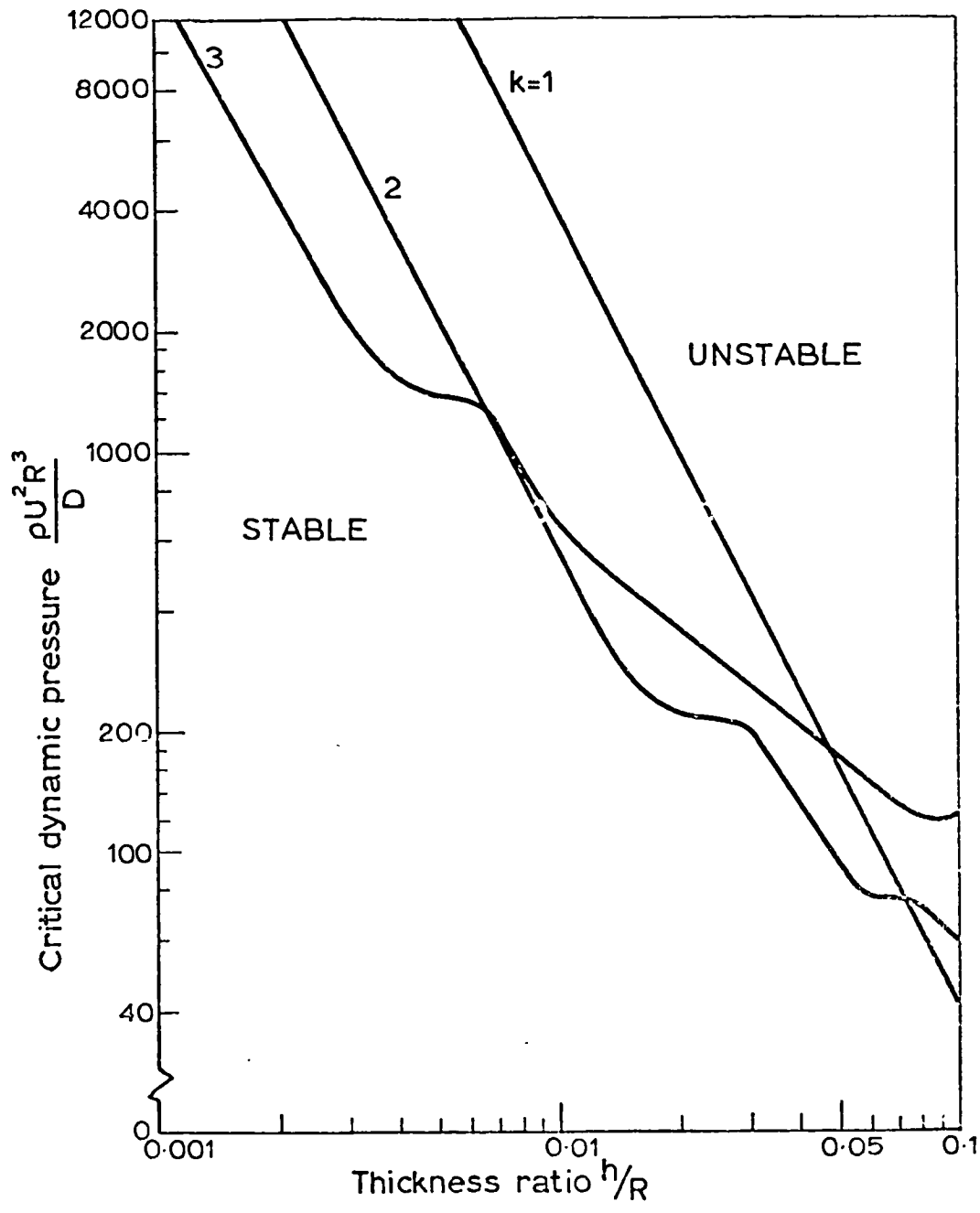


Fig. 7.5 Variation of the critical dynamic pressure with thickness ratio for cantilever pipes. The results are taken from Table 7.5.
($\nu=0.5$, $M=0$, $\mu=0.06$, $L/R=20$; no inplane inertias)

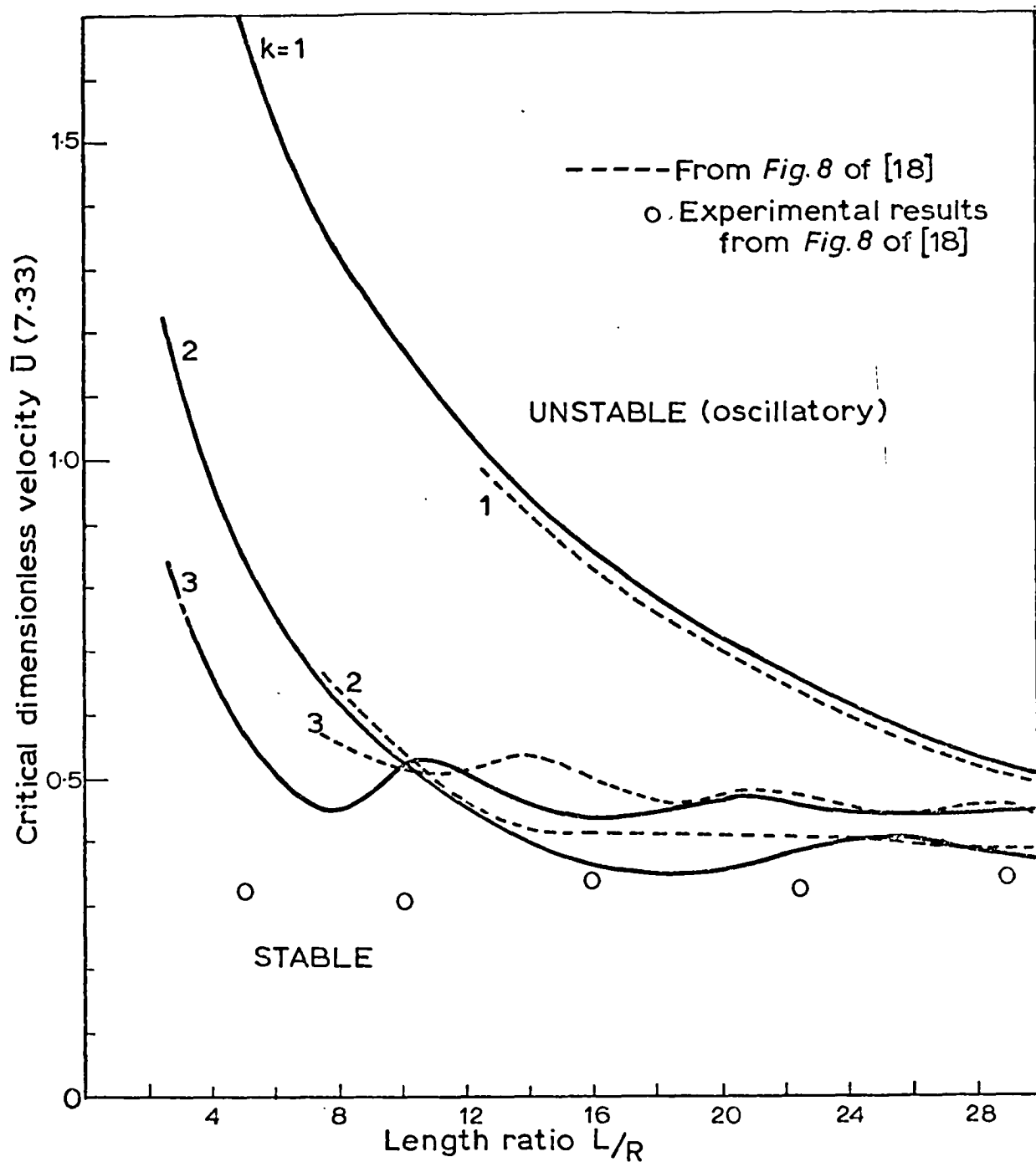


Fig. 7-6 Stability boundaries for cantilever pipes.
 ($\nu=0.5$, $M=0$, $h/R=0.0227$, $\mu=0.06$)

REFERENCES

1. A. BOULANGER, 1913 Université de Lille, Travaux et Mémoires N.S. II Vols. 1-3, 1903-25. Étude sur la Propagation des Ondes Liquides dans les Tuyaux Élastiques.
2. D.A. McDONALD, 1960 Blood Flow in Arteries. London. Edward Arnold (Publishers)
3. H. ASHLEY and G. HAVILAND, 1950 Journal of Applied Mechanics 17, 229-232. Bending vibrations of a pipeline containing flowing fluid.
4. G.W. HOUSNER, 1952 Journal of Applied Mechanics 19, 205-208. Bending vibrations of a pipeline containing flowing fluid.
5. R.H. LONG, Jr., 1955 Journal of Applied Mechanics 22, 65-68. Experimental and theoretical study of transverse vibration of a tube containing flowing fluid.
6. T. BROOKE BENJAMIN, 1961 Proceedings of the Royal Society (London) A, 261, 457-486. Dynamics of a system of articulated pipes conveying fluid. I. Theory.
7. R.A. STEIN and M.W. TOBRINER, 1970 Journal of Applied Mechanics 37, 906-916. Vibration of pipes containing flowing fluids.
8. R.W. GREGORY and M.P. PAIDOUSSIS, 1966 Proceedings of the Royal Society (London) A, 293, 512-527. Unstable oscillations of tubular cantilevers conveying fluid. I. Theory.
9. M.P. PAIDOUSSIS, 1970 The Journal of Mechanical Engineering Science 12, 85-103. Dynamics of tubular cantilevers conveying fluid.
10. M.P. PAIDOUSSIS and E.B. DEKSNIS, 1970 The Journal of Mechanical Engineering Science 12, 288-300. Articulated models of cantilevers conveying fluid: the study of a paradox.

11. S. NEMAT-NASSER, S.N. PRASAD and G. HERRMANN, 1966 American Institute of Aeronautics and Astronautics Journal 4, 1276-1280. Destabilizing effect of velocity-dependent forces in non-conservative continuous systems.
12. S.-S. CHEN, 1971 Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, 97, 1469-1485. Dynamic stability of a tube conveying fluid.
13. J.H. GINSBERG, 1973 International Journal of Engineering Science 11, 1013-1024. The Dynamic stability of a pipe conveying a pulsatile flow.
14. M.P. PAIDOUSSIS and N.T. ISSID, 1974 Journal of Sound and Vibration 33, 267-294. Dynamic stability of pipes conveying fluid.
15. T. BROOKE BENJAMIN, 1961 Proceedings of the Royal Society (London) A, 261, 487-499. Dynamics of a system of articulated pipes conveying fluid. II. Experiments.
16. R.W. GREGORY and M.P. PAIDOUSSIS, 1966 Proceedings of the Royal Society (London) A, 293, 528-542. Unstable oscillation of tubular cantilevers conveying fluid. II. Experiments.
17. F.I. NIORDSON, 1953 Transactions of the Royal Institute of Technology Stockholm, Sweden, No.73. Vibrations of a cylindrical tube containing flowing fluid.
18. M.P. PAIDOUSSIS and J.-P. DENISE, 1972 Journal of Sound and Vibration 20, 9-26. Flutter of thin cylindrical shells conveying fluid.
19. D.S. WEAVER and T.E. UNNY 1973 Journal of Applied Mechanics 40, 48-52. Dynamic stability of fluid-conveying pipes.
20. D.S. WEAVER and B. MYKLATUN, 1973 Journal of Sound and Vibration 31, 399-410. On the stability of thin pipes with an internal flow.

21. J. KEMPNER, 1955 Journal of Applied Mechanics 22,
Transactions of the ASME 77, 117-118. Remarks on Donnell's
Equations.
22. A. KORNECKI, 1974 Journal of Sound and Vibration 32, 251-263.
Static and dynamic instability of panels and cylindrical shells
in subsonic potential flow.
23. A.L. THURMAN and C.D. MOTE, Jr. 1969 Journal of Engineering
for Industry 91, 1147-1155. Nonlinear oscillation of a cylinder
containing a flowing fluid.
24. R.L. BISPLINGHOFF and H. ASHLEY, 1962 Principles of Aero-
elasticity. John Wiley and Sons, Inc. New York, London.
25. W. FLÜGGE, 1962 Stresses in Shells, Berlin: Springer-Verlag.
26. R.L. BISPLINGHOFF, H. ASHLEY and R.L. HALFMAN, 1955
Aeroelasticity, Addison-Wesley Publishing Co., Inc., Cambridge,
Mass.
27. R.W. LEONARD and J.M. HEDGEPEETH, 1957 NACA TR 1302. On panel
flutter and divergence of infinitely long unstiffened and
ring-stiffened thin-walled circular cylinders.
28. V.V. BOLOTIN, 1963 Nonconservative Problems of the Theory of
Elastic Stability. A Pergamon Press Book, The Macmillan Co.,
New York.
29. J.W. MILES, 1956 Journal of The Aeronautical Sciences 23,
771-780. On the Aerodynamic Instability of Thin Panels.
30. G.K. BATCHELOR and A.E. GILL, 1962 Journal of Fluid Mechanics
14, 529-551. Analysis of the stability of axisymmetric jets.
31. A. MICHALKE, 1965 Journal of Fluid Mechanics 23, 521-544.
On spatially growing disturbances in an inviscid shear layer.
32. S.A. ORSZAG and S.C. CROW, 1970 Studies in Applied Mathematics
49, 167-181. Instability of a Vortex sheet Leaving a Semi-
Infinite Plate.

33. C.H. ELLEN, 1973 Journal of Applied Mechanics 40, 68-72.
The Stability of Simply Supported Rectangular Surfaces in Uniform Subsonic Flow.
34. A KORNECKI, 1970 Journal of Sound and Vibration 14, 1-6.
A note on beam-type vibrations of circular cylindrical shells.
35. A. ERDELYI, 1956 Asymptotic Expansion, Dover Publications, Inc.
36. D.S. WEAVER, 1972 Journal of Sound and Vibration 22, 247-248.
On the flutter of thin cylindrical shells conveying fluid.
37. M.P. PAIDOUSSIS and J.-P. DENISE, 1971 Journal of Sound and Vibration 16, 459-463. Flutter of cylindrical shells conveying fluid.
38. G.L. ANDERSON, 1973 Journal of Sound and Vibration 27, 279-296.
Application of a Variational Method to Dissipative, Non-conservative problems of Elastic Stability.
39. C.B. SHARMA, 1974 Journal of Sound and Vibration 35, 55-76.
Calculation of natural frequencies of fixed-free circular cylindrical shells.
40. G.B. WARBURTON, 1970 Journal of Sound and Vibration 11, 335-338.
Natural frequencies of thin cantilever cylindrical shells.
41. E.H. DOWELL, 1971 Institute of Aeronautics and Astronautics Journal 9, 834-841. Generalized Aerodynamic Forces on a Flexible Plate Undergoing Transient Motion in a Shear Flow with an Application to Panel Flutter.
42. C.H. ELLEN, 1972 Institute of Aeronautics and Astronautics Journal 10, 1716-1717. Stability of Clamped Rectangular Plates in Uniform Subsonic Flow.

43. R.E.D. BISHOP and I. FAWZY, 1976 Philosophical Transactions of The Royal Society of London, 284, 1-47. Free and forced oscillation of a vertical tube containing a flowing fluid.
44. M.P. PAIDOUSSIS and N.T. ISSID, 1976 Journal of Applied Mechanics 43, 198-202. Experiments on Parametric Resonances of Pipes containing Pulsatile Flow.
45. I.S. GRADSHTEYN and I.M. RYZHIK, 1965 Tables of Integrals, Series and Products, Academic Press, New York and London.