

ON THE EFFECT OF STOCHASTIC LIE TRANSPORT NOISE ON FLUID DYNAMIC EQUATIONS.

A THESIS PRESENTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY OF IMPERIAL COLLEGE LONDON

AND THE
DIPLOMA OF IMPERIAL COLLEGE

BY
AYTHAMI BETHENCOURT DE LEÓN

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE
180 QUEEN'S GATE, LONDON SW7 2BZ

JANUARY 2021

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

COPYRIGHT

The copyright of this thesis rests with the author and is made available under a Creative Commons Attribution Non-Commercial No Derivatives licence. Researchers are free to copy, distribute or transmit the thesis on the condition that they attribute it, that they do not use it for commercial purposes and that they do not alter, transform or build upon it. For any reuse or redistribution, researchers must make clear to others the licence terms of this work.

On the effect of stochastic Lie transport noise on fluid dynamic equations.

ABSTRACT

Motivated by a recent geometric approach for adding stochastic noise of “Lie transport” type to PDEs developed by Darryl Holm, we investigate the effect of stochastic “Lie transport” noise in deterministic fluid equations. More concretely, first we develop a tool which facilitates the rigorous treatment of these equations and we will need in our research. This tool comprises an extension of the Itô-Wentzell formula to allow for advection of k -forms as well as tensors. Afterwards, we proceed to ask whether addition of this type of noise can improve the solution properties of some well-known deterministic fluid equations of interest. In particular, (A) we study the solution properties of the inviscid Burgers’ equation with transport noise, establishing the local well-posedness of strong solutions, and the global well-posedness in the viscous case. We also present some shock formation results. Inspired by recent advances in the problem of well-posedness of PDEs by addition of stochastic noise, (B) we demonstrate a general well-posedness by noise result for spatially weak, strong in the probabilistic sense solutions to the equation of linear advection of k -forms with Lie transport noise, which is ill-posed in the deterministic case. Finally, (C) we consider a novel Lagrangian-averaged approach for including stochastic noise in fluid equations and prove that it permits establishing the global well-posedness of the Boussinesq equations, where global well-posedness in the deterministic case is a fundamental open problem in mathematics.

The following publications have arisen during the course of my PhD work:

Diego Alonso-Orán and Aythami Bethencourt de León, Stability, well-posedness and blow-up criterion for the Incompressible Slice Model, *Physica D: Nonlinear Phenomena* **(392)** (2019).⁹

Diego Alonso-Orán, Aythami Bethencourt de León and So Takao, The Burgers' equation with stochastic transport: shock formation, local and global existence of smooth solutions, *Nonlinear Differential Equations and Applications* **(26)** (2019).⁵⁵

Diego Alonso-Orán and Aythami Bethencourt de León, On the Well-posedness of stochastic Boussinesq equations with transport noise, *Journal of Nonlinear Science* **(30)** (2019).⁶

Aythami Bethencourt de León, Darryl D. Holm, Erwin Luesink and So Takao, Implications of Kunita-Itô-Wentzell formula for k -forms in stochastic fluid dynamics, *Journal of Nonlinear Science* **(30)** (2019).⁵⁸

Aythami Bethencourt de León and So Takao, (Revisited version of) Well-posedness by noise for linear advection of k -forms, *Submitted to Annals of Applied Probability* (2020).⁵⁹

Diego Alonso-Orán, Aythami Bethencourt de León, Darryl Holm and So Takao, Modelling the Climate and Weather of a 2D Lagrangian-Averaged Euler–Boussinesq Equation with Transport Noise, *Journal of Statistical Physics* **(179)** (2020).⁷

I wish to express my deepest gratitude towards several people who in one way or another have made this mathematical adventure more enjoyable and interesting, and significantly contributed to improving the present dissertation. First and foremost, I wish to sincerely thank my supervisor Darryl Holm for his constant support in difficult situations and all the invaluable lessons I have learned from him. Darryl belongs to those rare people who are really passionate about their job and that confers a special meaning to everything he does and is inspiring. I hope this will not be the end of our walks around Round Pond. I am incredibly obliged to my research collaborators. In particular, I thank So Takao for countless discussions and support. I also thank him for his amazing ideas and intuition, since our projects would not have been possible without them. Le doy unas gracias muy especiales a Diego Alonso porque además de colaborar conmigo y enseñarme muchísimo en un periodo en el que estaba un poco perdido, siempre me ha brindado su apoyo incondicional en momentos complicados. Gracias por ser tan buen amigo. Le doy las gracias a Marta de León por siempre ayudarme con dudas y correcciones, incluso cuando estaba de vacaciones o tenía muchas cosas que hacer. Le estoy muy agradecido a Víctor Jiménez por ofrecerse a ayudarme con las correcciones y mirar cada línea con cautela. I thank Oana Lang for selflessly having accepted to help me with corrections and for instructive mathematical discussions. I am very grateful to James Michael-Leahy for sharp observations, mathematical lessons, and lots of help. One does get the opportunity of meeting such bright mathematicians every day. I am very grateful to Wei Pan for all the mathematical advice he has gifted to me. In particular, I deeply thank him because he helped me complete a successful project during my summer at the Met Office. This brings me directly to Rachel Killick, my adored supervisor at the Met Office, whose presence that summer meant and changed everything. Rachel makes me have faith in humanity. Sería imposible expresar la gratitud que siento hacia mi familia y todo lo que les debo y jamás podré devolverles. Sin mi familia, el haber estudiado en Imperial College no sería ni siquiera la sombra de un sueño. Quiero también aprovechar para darle las gracias a Dani Sarmiento y Alejandro López por su ayuda con las correcciones. Mi periodo en Berlín sería inimaginable sin ellos y estará

eternamente vinculado a ellos. Je tiens à remercier Camille Letourneur car sa présence dans ma vie a été fondamentale pour moi. Je ne suis pas sûr que j'aurais rassemblé l'énergie pour commencer à écrire des articles sans elle. I thank the MPE CDT and its direction for making this possible, for this amazing PhD program and all the personalised attention. I thank the students of Cohort Alpha and Bravo for warm memories together. I also thank the secretary at Imperial College London, which is incredibly effective and has freed me from some important burdens. I thank Riccardo Passeggeri and Giorgios Chalivopoulos for all those evenings together that now have a place in my heart. Quiero darle las gracias a José Sabina por ayudarme con las correcciones y "perder" su tiempo resolviéndome dudas sobre problemas parabólicos de buena gana. Les estoy muy agradecido a Juan Carlos y a Edith Padrón porque todos estos años de doctorado me han apoyado. Le doy las gracias a los organizadores del PEJIM en Navidades, que es una idea genial para ponerse al día los unos con los otros. Ich danke Rubi Afzal, denn obwohl wir uns häufig streiten, hat sie mir sehr geholfen und mich inspiriert. Sie hat mir auch vieles beigebracht. Ya blagodaryu Nikolaya Krylova za to, chto on otvetil na moi pis'ma, nesmotrya na to, chto on menya ne znal, i dal mne tsennyy matematicheskiye sovery. Eto khorosho otzyvayetsya o nem, i ne vse matematiki yego urovnya sdelali by eto. Aunque no quería incluir a nadie con el que no hubiese tenido contacto directo durante mi doctorado en esta dedicatoria, tengo que hacer una excepción: gracias Pepe Méndez por ser tan buen profesor y persona. Nakonets, ya blagodaryu moyu samuyu obozhayemuyu Mariitu, chey svet osveshchayet t'mu v mire. I also greatly thank my examiners for taking the time to read my thesis (I would include names but I do not know who they are at the moment).

I acknowledge financial support from the EPSRC Centre for Doctoral Training, "Mathematics of Planet Earth". This has allowed me to fulfill my goal. I sincerely thank Ana Bela Cruzeiro and Xue-Mei Li for agreeing to be my examiners and taking the time to read my thesis.

CONTENTS

0	INTRODUCTION	1
0.1	Introduction and motivations	1
0.2	A review of the main results and comparison with available literature	6
1	PRELIMINARIES	28
1.1	Some basic Sobolev theory and Hölder spaces	29
1.2	Some Banach spaces of tensor fields	36
1.3	Some background in analysis of PDEs	42
1.4	Some background in stochastic analysis	45
2	THE KUNITA-ITÔ-WENTZELL (KIW) FORMULA FOR TENSORS	65
2.1	The KIW formula in the low regularity case	70
2.2	Applications of KIW in stochastic fluid dynamics	78
3	THE BURGERS' EQUATION WITH TRANSPORT NOISE	88
3.1	Local well-posedness	90
3.2	Blow-up criterion	108
3.3	Global well-posedness in the viscous case	112
3.4	Summary of results on shock formation	118
4	LINEAR ADVECTION OF k -FORMS WITH LIE TRANSPORT NOISE	121
4.1	Main result and brief discussion	124
4.2	Regularity of the flow	128
4.3	Existence of weak solutions	144

4.4	Uniqueness of weak solutions	159
4.5	The space-time diffusion case	170
5	LAGRANGIAN-AVERAGED BOUSSINESQ EQUATIONS WITH TRANSPORT NOISE	173
5.1	The stochastic Lagrangian-averaged Boussinesq equations . . .	173
5.2	Global well-posedness of the expectation system	179
5.3	Global well-posedness of the linear system	195
5.4	A closed equation for the statistical moments	198
6	SUMMARY AND OUTLOOK	205
	REFERENCES	224
	APPENDIX A SOME EXTRA BACKGROUND IN TENSORS	225
A.1	Background in tensors	225
A.2	Vector fields and differential forms	228
	APPENDIX B PROOF OF GENERALISED ESTIMATES FOR LINEAR OPERATORS	236

LIST OF FIGURES

1	Map of all the Southern Ocean Lagrangian surface drifters in the NOAA GDP Data Set (Lumpkin and Pazos, 2007). Each drifter corresponds to a color ¹⁴⁷ . The distribution of the drifters makes think that there is a stochastic component to their motion.	5
2.1	Fluid flow at time t_0 on a domain D . $u(t, x)$ denotes the velocity of the flow at time t and position x	79
2.2	The circulation is preserved when $F = 0$	84
4.1	Improved regularity of the flow $\phi_{s,t}$	136
A.1	Example of a diffeomorphism on the square.	232
A.2	The pull-back of a volume form.	233



INTRODUCTION

0.1 INTRODUCTION AND MOTIVATIONS

Parameterisations and stochastic parameterisations. Understanding climate change relies on having accurate atmospheric models that can be run on computers in a desirable time. In the models employed for this task, not all relevant physical processes can be included due to limitations in model complexity and computational time. Parameterisations comprise employing simplified functions in order to approximately capture the behavior of important physical processes that are not in the main equation/s. Parameterisations aim to be computationally cheap while representing the missing processes in a sufficiently accurate manner. Examples where they are often used are cloud convection and microphysics, understanding the descent rate of raindrops or certain thermodynamic processes. The idea of considering parameterisations to be stochastic is growing in popularity and seems to be very effective, since the theory of probability provides the ideal framework to model uncertainty. The importance of stochastic parameterisations in modern meteorology has been discussed in¹⁹, and several interesting applications can be found in the literature. For instance, see⁶³ for an application to cloud

dynamics,¹³⁹ for a stochastic parameterisation of mesoscale eddy dynamics, and¹³⁸ for the use of parameterisations in deep convection and its role in climate.

In this dissertation, we investigate the issue of how the solution properties of different partial differential equations (PDEs) may change after a stochastic noise has been added. One of our motivations is that noise could be interpreted as a parameterisation of certain physical processes and help capture uncertainty. To assist in this assessment, we develop some tools for the rigorous treatment of stochastic PDEs (SPDEs). The starting equations we consider have a geometric origin, and the noise of interest is of Lie transport type. This will be described in detail later.

Stochastic geometric mechanics. In particular, we are inspired by the methodology for adding stochastic noise to geometric equations developed by Darryl Holm in¹⁰², where a variational approach for introducing noise in equations in a fashion that respects the geometry of the system is developed. From a different perspective,¹⁰² is one of many papers that propose techniques to include stochasticity in the fundamental theory of geometric mechanics, since the classical theory does not account for the effects of noise although noise is ubiquitous in our macroscopic world. For a beautiful review of classical geometric mechanics, we point the reader to^{130,2}. A more applied exhaustive review that we recommend consists in the works^{100,101,104}. The dearth of publications about uncertainty and dissipative effects in the theory of geometric mechanics has encouraged several authors to develop ideas which permit extending the wisdom in the theory to the stochastic formalism. A fundamental classical work in this direction is²³, which provides a geometric formulation of Itô calculus, extends the concept of stochastic flow to manifolds, discusses stochastic variational calculus...A more modern interesting review is¹²¹, and⁵ provides a compilation of recent research articles on different related topics in stochastic geometric mechanics. We note that although the approach for introducing noise in¹⁰² is variational, mathematical formalisation is needed and we are rather interested in exploiting the final form of the equations that this formalism casts. Steps towards this for-

malisation in the framework of the theory of rough paths have been recently proposed in⁵¹. Variational principles are fundamental to the theory of geometric mechanics and the stochastic formalisation of variational principles in the finite-dimensional case has been pursued for instance in^{54,121}. Stochastic Euler-Poincaré variational principles from which one can derive dissipative equations such as Navier-Stokes are developed in¹³.

Stochastic fluid dynamics. Fluid dynamics is fundamental in our world due its endless applications in diverse areas such as weather forecasting, climate science, ocean dynamics, ship and aeronautical engineering, medicine, mitigation of natural disasters, modelling of extreme weather events, transport...Whereas it is mostly deterministic fluid PDEs that have been typically employed for this purpose, the idea of considering SPDEs to model the behaviour of a fluid has been lately popularised and put into practice. In particular, in weather forecasting, inclusion of stochastic noise can be a way of representing model uncertainty and turbulence. As an example, renowned meteorology institutions such as ECMWF or the Met Office have implemented probabilistic methods, obtaining encouraging results (see¹⁴⁴ for a stochastic kinetic energy backscatter scheme employed in the Met Office, and^{148,28} for applications of stochastic schemes to ensemble weather forecasting employed in the ECMWF). Moreover, the phenomenon of cloud formation is to this day poorly understood and parameterisations have become a tool for gaining better understanding about it (see for instance⁴¹ for a stochastic parameterisation of deep cumulus convection and⁴² for practical testing of the methodology developed in the previous work).

Our approach. These ideas motivate us to consider different relevant deterministic fluid PDEs and study how the behaviour of their solutions changes once noise has been added to them, from a theoretical viewpoint (that is, we are interested, for instance, in the existence and uniqueness of solutions to these equations, or in whether noise can somehow regularise their solutions). The range of equations that in principle could be considered is of course very broad and pinpointing the behaviour difference between a PDE and its

“stochastic counterpart” represents a very complex problem. Hence, we will merely consider different fluid PDE models which are physically relevant and investigate how noise affects them. As we mentioned, we are not interested in adding “any noise” and are motivated by the ideas in¹⁰². There, advective Lie transport noise dependent on the gradient of the solution is added in Stratonovich form, in a way that preserves the core geometry of the equation (in lay terms, noise is added at the physical level). The formalism presented in¹⁰² has encouraged multiple theoretical works. For instance, the pioneering work⁴⁹ investigates the local well-posedness of the 3D Euler equation with Lie derivative-type noise, whereas the global well-posedness of the 2D version is established in⁵², the Boussinesq equations are treated in⁶, and the Great Lake equations in⁵³. Shock formation in the Camassa-Holm equation with Lie transport noise is investigated in⁵⁰, and the stochastic Lagrangian dynamics proposed in¹⁰² are shown to arise naturally by taking the diffusive limit of a slow/fast decomposition of the deterministic Lagrangian flow in⁴⁷. Several practical applications of this formalism have also been shown in recent years. For instance, numerical testing and methodology for determining the diffusion coefficients are available in^{45,48} and a particle filter for the damped 2D Euler equation with forcing is developed in⁴⁶ as part of the same project.

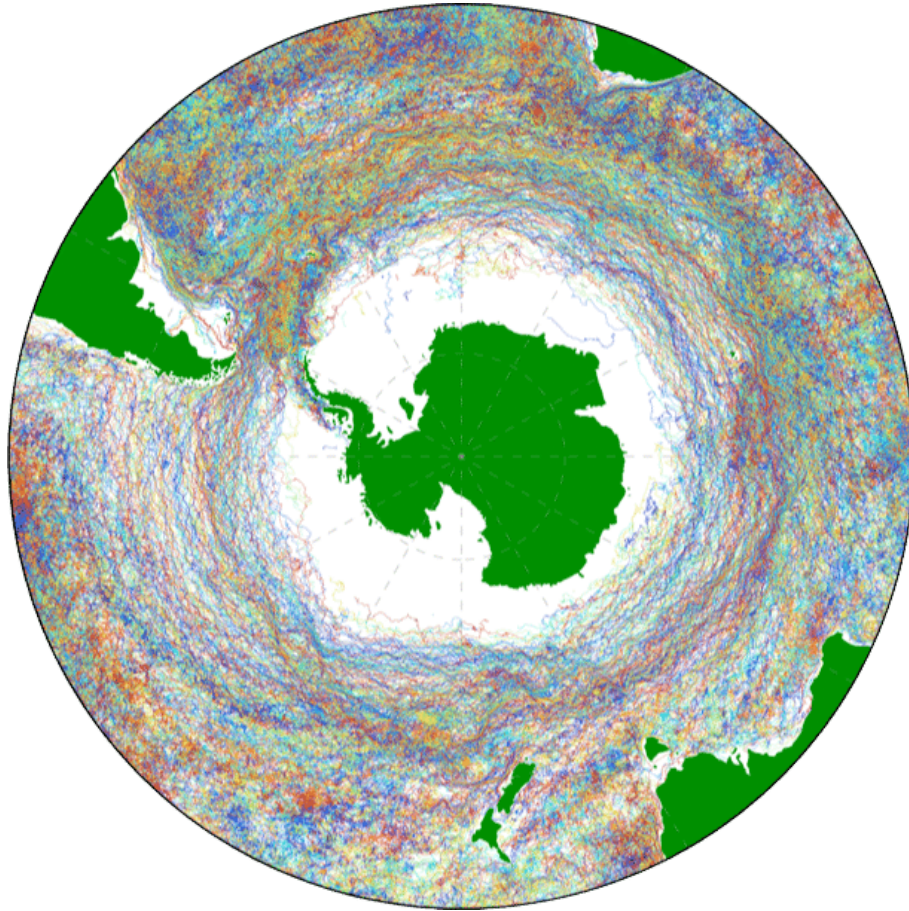


Figure 1: Map of all the Southern Ocean Lagrangian surface drifters in the NOAA GDP Data Set (Lumpkin and Pazos, 2007). Each drifter corresponds to a color¹⁴⁷. The distribution of the drifters makes think that there is a stochastic component to their motion.

Final note. The mathematical problems discussed in this dissertation form an important part of the research I have carried out together with my collaborators during my PhD period at Imperial College London.

0.2 A REVIEW OF THE MAIN RESULTS AND COMPARISON WITH AVAILABLE LITERATURE

We are interested in investigating the solution properties of relevant deterministic fluid PDEs upon including noise in the fluid transport operator.

Our general setting. Let us consider an SPDE of the form ^{*}

$$\begin{aligned} dU(t, x) + \mathcal{L}_{u(t, x)}U(t, x)dt + \sum_{i=1}^N \mathcal{L}_{\xi_i(t, x)}U(t, x) \circ dW_t^i = F(t, x)dt, \\ t > 0, \quad x \in \mathbb{R}^n, \end{aligned} \quad (0.2.1)$$

where

- $U(t, x)$ is a stochastic process that takes values in the space of differential k -forms on $\mathbb{R}^n$ (i.e., the natural objects that one can integrate over k -dimensional submanifolds in $\mathbb{R}^n$). However, we will often consider $U(t, x)$ to be a scalar-valued function (i.e., $k = 0$).
- $u(t, x)$ is a stochastic vector field and $\{\xi_i(t, x)\}_{i=1, \dots, N}$ is a family of deterministic vector fields. $F(t, x)$ is a forcing term, which we will normally consider to be zero and sometimes could be stochastic (i.e. $F(\omega, t, x)$).
- $\mathcal{L}_u U$ denotes the Lie derivative of a k -form U with respect to a vector field u , thereby generalising the notion of directional derivative to tensor fields (see Appendix A for supplementary notes on tensor calculus).
- $\{W_t^i\}_{i=1, \dots, N}$ is a family of independent Brownian motions and $\circ$ indicates that the stochastic integral is performed in Stratonovich form [†].

^{*}Whereas we write most SPDEs in this introduction in differential notation, it should be noted that the equations only make sense in integral form and this is purely notation.

[†]Most of the SPDEs in this introduction are written in Stratonovich form for simplicity purposes. This is rigorous under certain regularity assumptions. However, when establishing mathematical results we will sometimes be forced to employ the Itô formulation as it can be seen in the following chapters.

In some cases, we will alternatively consider equation (0.2.1) on the n -dimensional torus $\mathbb{T}^n$ instead of on $\mathbb{R}^n$, since it can be more convenient to perform Sobolev estimates. We note that equation (0.2.1) is a natural stochastic counterpart of a deterministic equation of the form

$$\partial_t U(t, x) + \mathcal{L}_{u(t, x)} U(t, x) = F(t, x), \quad (0.2.2)$$

since we have introduced the noise in Stratonovich form. The multiplicative noise in (0.2.1) makes the advection velocity stochastic, and allows (0.2.1) to retain the original form of advection equation of (0.2.2). This is, (0.2.1) can be expressed as $dU + \mathcal{L}_{d\tilde{u}} U = Fdt$, where now $d\tilde{u}(t, x) = u(t, x)dt + \sum_{i=1}^N \xi_i(t, x) \circ dW_t^i$ is a stochastic vector field with smooth in space and rough in time noise. The multiplicative noise term considered in (0.2.1) can be referred to as “transport noise”, since the noise appears in the transport velocity of the Lie derivative. In the general Lie derivative case, we will often refer to it as “Lie transport noise”.

As we have already noted, the multiplicative type of noise we consider in (0.2.1) was initially motivated by the stochastic dynamics proposed in¹⁰². An important feature of our equations and the equations in¹⁰² is that the noise included depends on the gradient of the solution and the solution itself. Before the publication of the latter paper, there were some works available in the literature where noise dependent on the gradient of the solution was added to fluid or geometric equations in similar manners (see^{39,132}). In particular, the geometric formalism proposed in³⁹ gives rise to fluid equations which contain dissipation, where the noise depends on the gradient of the solution. In some cases, there even seems to be intersection with the models proposed in¹⁰², baptised under the name “SALT”. In several recent works, the type of noise we propose has been used to stochastically parametrise unresolved scales in fluid models while retaining the essential physics of the system^{45,48,46,49,50,88,12,6,52,53}. When $U(t, x)$ is chosen to be a scalar-valued function, the noise added in (0.2.1) takes the pure transport form $\sum_{i=1}^N \xi_i(t, x) \cdot DU(t, x) \circ dW_t^i$ (known as linear transport noise) and it has also been shown to have a regularising effect on the solutions of certain PDEs. This latter phenomenon is known as well-posedness by noise and has

been studied recently in the literature (see for instance^{76,77,72,73,90,81,79,75}). We will comment on some of these last works in more detail in the paragraph titled **The equation of linear advection of k -forms with Lie transport noise**, which is related to our research in Chapter 4.

Remark 1. *As an example, we note that in the scalar case $k = 0$, the Lie derivative takes the form $\mathcal{L}_b f = b \cdot Df$, where $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $b : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and if $k = n$, $\mathcal{L}_b(\rho d^n x) = \text{div}(b\rho) d^n x$ is the Lie derivative of a volume-form with global coordinates $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ (see Appendix A for a short review of the Lie derivative $\mathcal{L}$). If in (0.2.1), we consider $k = n$, $u(t, x) = b(t, x)$ (for notation purposes), $U(t, x) = D(t, x)$ a density (modelled as a volume form $D(t, x) = \rho(t, x) d^n x$), and $F = 0$, we obtain the following equation in global coordinates*

$$d\rho(t, x) + \text{div}(b(t, x)\rho(t, x))dt + \sum_{i=1}^N \text{div}(\xi_i(t, x)\rho(t, x)) \circ dW_t^i = 0, \quad (0.2.3)$$

which is a natural stochastic equivalent of the deterministic continuity equation $\partial_t \rho + \text{div}(b\rho) = 0$. Moreover, a natural stochastic counterpart of the transport equation $\partial_t f + b \cdot Df = 0$ is given by

$$df(t, x) + b(t, x) \cdot Df(t, x)dt + \sum_{i=1}^N \xi_i(t, x) \cdot Df(t, x) \circ dW_t^i = 0, \quad (0.2.4)$$

which is recovered by plugging in (0.2.1) $k = 0$, $u(t, x) = b(t, x)$, $U(t, x) = f(t, x)$ (i.e., U is a tracer quantity transported by the fluid, modelled as a scalar function f , which is a zero-form). Equation (0.2.4) is equivalent to (0.2.3) when b and $\xi_i, i = 1, \dots, N$ are divergence-free. If $k = 0$, $n = 1$, $U(t, x) = u(t, x)$, we obtain a stochastic version of the Burgers' equation given by

$$du(t, x) + u(t, x)u_x(t, x)dt + \sum_{i=1}^N \xi_i(t, x)u_x(t, x) \circ dW_t^i = 0.$$

Moreover, by choosing $k = 1$, $U(t, x) = u(t, x) \cdot dx$ (a one-form with coordinates $u(t, x)$), $F = \nabla(p(t, x) - |u(t, x)|^2/2) \cdot dx$ (a stochastic forcing depen-

dent on the solution), we get a stochastic counterpart of the Euler equation in global coordinates

$$du(t, x) + u(t, x) \cdot \nabla u(t, x) dt + \sum_{i=1}^N \mathcal{L}_{\xi_i(t, x)} u(t, x) \circ dW_t^i = -\nabla p(t, x).$$

Another well-known example in physics is the advection of a magnetic field, commonly interpreted as a two-form $\widehat{B} = B \cdot dS$ in 3D ideal magnetohydrodynamics, where we have $\mathcal{L}_v \widehat{B} = \text{curl}(v \times B) \cdot dS$ (see^{103,80}. The first reference also provides many other examples of advected quantities in physics).

We note that $u = u(t, x)$ often represents the velocity of a fluid (for instance in the transport, continuity, Burgers', and Euler equations just mentioned). Equation (0.2.1) is related to an integral conservation law. More concretely, in Chapter 3 we will see that equation (0.2.1) with $F = 0$ arises naturally by assuming that the k -form U is (integrally) conserved along the solution of the SDE

$$d\phi_t(x) = u(t, \phi_t(x))dt + \sum_{i=1}^N \xi_i(t, \phi_t(x)) \circ dB_t^i, \quad \phi_0(x) = x. \quad (0.2.5)$$

Solutions to general SDEs of the type (0.2.5) have flow properties under certain regularity assumptions on the drift and diffusion vector fields (see Section 1.4 in the preliminaries for the definition of stochastic flow) and that is why we will often refer to equation (0.2.5) as the flow equation. We note that the stochastic aspect of the map ϕ_t represents uncertainty in the Lagrange-to-Euler map of the fluid¹⁰².

In order to treat some of the equations we are interested in, we need to develop a couple of new preparatory tools that we could not find in the literature. In Chapter 2, we introduce these tools and show some of their immediate applications. We discuss this point in the next paragraph.

The Kunita-Itô-Wentzell (KIW) formula for advection of k -forms.

Oftentimes, we are interested in understanding how a physical quantity evolves as it is advected or convected by a fluid. This is very useful if we

wish to derive momentum or continuity equations, circulation theorems, the Reynolds transport theorem, etc. It also finds applications, for instance, when establishing the existence and uniqueness of solutions to some equations. For a fixed time t , the Lagrange-to-Euler map φ_t sends a fluid particle at position X at time zero (initial configuration) to its position $\varphi_t(X)$ at time t . We refer to ⁵⁸ for explanations on the Lagrange-to-Euler map φ_t or to ^{130,2} for a more thorough insight. We also explain its meaning in more detail in Chapter 2. Consider a time-dependent stochastic tracer quantity K_t which we assume takes values on the k -forms (note that in the previous point, we showed some examples where physical quantities are interpreted as k -forms and ¹⁰³ provides more of them). It is important to understand how K_t changes along the fluid, which is mathematically expressed as $\varphi_t^* K_t$, where φ_t^* is the pull-back of φ_t . We note that in the scalar case, $\varphi_t^* f_t = f_t \circ \varphi_t$ is simple evaluation on the flow. We seek a chain rule for the quantity $\varphi_t^* K_t$, different versions of which we develop in Chapter 2. In particular, we will often adopt the notation $\varphi = \phi$, since we will assume it satisfies the flow equation

$$\phi_t(x) = x + \int_0^t b(s, \phi_s(x)) ds + \sum_{j=1}^N \int_0^t \xi_j(s, \phi_s(x)) \circ dB_s^j, \quad (0.2.6)$$

where B_t^j , $j = 1, \dots, N$ are independent Brownian motions. We are interested in the case where ϕ_t is not of very high regularity for reasons that will become clear later. We will refer to this chain rule as the Kunita-Itô-Wentzell (KIW) formula for k -forms. We can extend the KIW formula to tensor fields if the flow has enough regularity and probably also in the weak regularity case with some more work. We note that the difficulty in deriving our KIW formula resides in the facts that (1) both K_t and ϕ_t are stochastic and depend on time, and especially (2) we wish to consider flows ϕ_t of low regularity. In ^{118,120}, Kunita derived an extension of Itô's formula, showing that if T is a time-independent sufficiently regular (r, s) -tensor and ϕ_t is the flow of the SDE (0.2.6) with sufficiently regular coefficients, then an analogue

of Itô's formula holds for tensor fields, namely

$$\phi_t^* T(x) - T(x) = \int_0^t \phi_s^* (\mathcal{L}_b T)(x) ds + \sum_{j=1}^N \int_0^t \phi_s^* (\mathcal{L}_{\xi_j} T)(x) \circ dB_s^j,$$

In ¹¹⁹, a version of the classical Itô-Wentzell formula is proven, which we will later state in the preliminaries. More concretely, Kunita computes the equation for the evolution of a scalar function-valued stochastic process $f(t, x)$ along a semimartingale. In particular, when the semimartingale is taken to be a stochastic flow ϕ_t , the result states that for a sufficiently smooth scalar function-valued stochastic process $f(t, x)$, represented as [‡]

$$f(t, x) = f(0, x) + \int_0^t g(s, x) ds + \int_0^t h(s, x) \circ dW_s,$$

where W_t is Brownian motion, the equation describing the evolution of the process $f(t, x)$ via pull-back by ϕ_t reads

$$\begin{aligned} \phi_t^* f(t, x) &= f(0, x) + \int_0^t \phi_s^* g(s, x) ds + \int_0^t \phi_s^* h(s, x) \circ dW_s \\ &\quad + \int_0^t \phi_s^* (b \cdot Df)(s, x) ds + \sum_{j=1}^N \int_0^t \phi_s^* (\xi_j \cdot Df)(s, x) \circ dB_s^j, \end{aligned}$$

where ϕ_t is the flow of the SDE (0.2.6).

Remark 2. *Note that in the scalar case above, we have $\phi_t^* f(t, x) = f(t, \phi_t(x))$, so no differential geometric objects are needed in order to make sense of the Itô-Wentzell formula and we just stated it in terms of the pull-back for convenience.*

In Chapter 2, we prove the following extension of the two Itô-type theorems just mentioned.

Theorem 3 (Kunita-Itô-Wentzell (KIW) formula for tensor-valued processes in Stratonovich form. Smooth flow case). *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_t)$ be a*

[‡]Actually, the result in ¹¹⁹ permits considering much more general scalar function-valued semimartingales than the ones we consider, but we are interested in the present case.

filtered probability space satisfying the usual conditions. Let $T : \Omega \times [0, T] \rightarrow \Gamma(T^{(r,s)}(\mathbb{R}^n))$ be a tensor-valued semimartingale such that

- (a) $\mathbb{P}$ -a.s. T is continuous in (t, x) ,
- (b) $T(t, \cdot)$ is of class C^3 a.s. for every $t \in [0, T]$.

Moreover, let T have the following explicit expression

$$T(t, x) = T(0, x) + \int_0^t G(s, x) ds + \sum_{i=1}^M \int_0^t H_i(s, x) \circ dW_s^i, \quad t \in [0, T], \quad x \in \mathbb{R}^n,$$

where G, H_i are tensor-valued adapted processes satisfying

- (i) $\mathbb{P}$ -a.s. G, H_i are continuous in (t, x) ,
- (ii) $G(t, \cdot), H_i(t, \cdot)$ are of class C^2 a.s. for every $t \in [0, T]$,

and W_t^i are independent Brownian processes, $i = 1, \dots, M$. Finally, assume the representations

$$G(t, x) = G(0, x) + \int_0^t g(s, x) ds + \sum_{j=1}^L \int_0^t g_j(s, x) dM_s^j,$$

$$H_i(t, x) = H_i(0, x) + \int_0^t h_i(s, x) ds + \sum_{j=1}^L \int_0^t h_{ij}(s, x) dN_s^{ij}, \quad t \in [0, T], \quad x \in \mathbb{R}^n,$$

where g, g_j, h_i, h_{ij} are tensor-valued adapted processes such that

- (1) $\mathbb{P}$ -a.s. g, g_j, h_i, h_{ij} are continuous in (t, x) ,
- (2) $g(t, \cdot), g_j(t, \cdot), h_i(t, \cdot), h_{ij}(t, \cdot)$ are of class C^1 a.s. for every $t \in [0, T]$,

and M_t^j, N_t^{ij} are independent Brownian processes adapted to $(\mathcal{F}_t)_t$, $i = 1, \dots, M$, $j = 1, \dots, L$. Let ϕ_t be the continuous (semimartingale) $C^{4+\alpha'}$ -solution to the flow equation (0.2.6), with (deterministic) $b \in L^\infty([0, T]; C_b^{4+\alpha}(\mathbb{R}^n, \mathbb{R}^n))$, $\xi_j \in L^\infty([0, T]; C_b^{5+\alpha}(\mathbb{R}^n, \mathbb{R}^n)) \cap C^1([0, T]; C^2(\mathbb{R}^n, \mathbb{R}^n))$, $0 < \alpha' < \alpha$, and B_t^j be

independent Brownian motions adapted to $(\mathcal{F}_t)_t$, $j = 1, \dots, N$. Then $\mathbb{P}$ -a.s. the evolution of $T(t, \cdot)$ by the pull-back of the flow satisfies the equation

$$\begin{aligned} \phi_t^* T(t, x) &= T(0, x) + \int_0^t \phi_s^* G(s, x) ds + \sum_{i=1}^M \int_0^t \phi_s^* H_i(s, x) \circ dW_s^i \\ &+ \int_0^t \phi_s^* \mathcal{L}_b T(s, x) ds + \sum_{j=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_j} T(s, x) \circ dB_s^j, \quad t \in [0, T], \quad x \in \mathbb{R}^n. \end{aligned} \tag{0.2.7}$$

Remark 4. The notation $\phi_t^* T(t, x)$ can be confusing and should be interpreted as $(\phi_t^* T(t, \cdot))(x)$. This is, for a fixed time t , the pull-back at time t $\phi_t^* T(t, \cdot)$ of the tensor $T(t, \cdot) \in \Gamma(T^{(r,s)}(\mathbb{R}^n))$ is the new tensor $\phi_t^* T(t, \cdot) \in \Gamma(T^{(r,s)}(\mathbb{R}^n))$. The same applies to the other tensor-valued processes affected by the pull-back in (0.2.7).

Remark 5. In applications, one can express (0.2.7) by employing the (formal) differential notation

$$d(\phi_t^* T)(t, x) = \phi_t^* (dT + \mathcal{L}_{dX_t} T)(t, x),$$

where dX_t is the stochastic vector field

$$dX_t(x) = b(t, x)dt + \sum_{j=1}^N \xi_j(t, x) \circ dB_t^j.$$

In Chapter 2, we will also prove a more interesting version of Theorem 3 in the k -form case, where the coefficients of the flow ϕ_t are of low regularity, which we will need in Chapter 4 to show the global well-posedness of solutions to the equation of linear advection of k -forms with Lie transport noise. In Chapter 2 we also discuss some implications of Theorem 3 in fluid dynamics. The applications we present consist in (1) deriving advection equations by starting with integral conserved quantities, (2) showing how to establish stochastic circulation theorems when adding Lie transport noise to deterministic fluid equations, (3) demonstrating a stochastic version of Reynolds transport theorem. For the particular case in which K is a deterministic

k -form-valued process, some consequences in fluid dynamics have also been discussed previously in ^{34,142}.

Once these tools have been developed and some applications have been shown, in Chapter 4 we study a simple nonlinear fluid model with transport noise, namely the Burgers' equation, which we discuss in the next paragraph.

The Burgers' equation with transport noise. The Burgers' equation is a very simple nonlinear PDE, and it is well-known as a prototypical model of conservation equation whose solutions can develop shocks in the inviscid case. In particular, shocks are formed by the solution when the initial velocity profile has negative slope. The Burgers' equation is often employed as a toy model in fluid dynamics due to its simplicity and transport form. Moreover, it can be very helpful for understanding nonlinear and diffusive effects⁷¹. By substituting $k = 0$, $n = 1$, $U = u$ in equation (0.2.1) and assuming the vector fields ξ_i , $i = 1, \dots, N$ to be time-independent, we obtain a stochastic version of inviscid Burgers'

$$du(t, x) + u(t, x)\partial_x u(t, x)dt + \sum_{i=1}^N \xi_i(x)\partial_x u(t, x) \circ dW_t^i = 0, \quad (0.2.8)$$

where $t > 0$ and we assume $x \in \mathbb{T}$. As already noted in the general case, equation (0.2.8) conserves the transport form of the deterministic Burgers' equation. Taking into account that the solutions of deterministic (inviscid) Burgers' develop singularities in finite time, it is of interest to investigate the solutions behaviour in the stochastic case.

We note that the Burgers' equation is not a good model for turbulence because solutions are not chaotic⁵⁵. However, some authors have proposed stochastic perturbations of the Burgers' equation as a simple model for turbulence^{38,106}, which provides another motivation for studying this type of models. In Chapter 3, we establish the local well-posedness of strong solutions of equation (0.2.8) in Sobolev spaces and show a blow-up criterion that mimics the available one in the deterministic case. In fact, we show

our local well-posedness results and blow-up criterion for a noise of the form $Qu \circ dW_t$, where $Qu = a(x)\partial_x u + b(x)u$, which is more general than the (pure transport) noise in (0.2.8). We also consider the viscous Burgers' equation with stochastic transport, given by

$$du(t, x) + u(t, x)\partial_x u(t, x)dt + \sum_{i=1}^N \xi_i(x)\partial_x u(t, x) \circ dW_t^i = \nu u_{xx}(t, x)dt, \quad (0.2.9)$$

which fits the general formula (0.2.1) if we understand $\nu u_{xx}(t, x)$ as a stochastic forcing F . We prove global well-posedness of strong solutions of equation (0.2.9) in Sobolev spaces. Finally, we comment on some new shock formation results. In particular, we argue that when $N = 1$ and the diffusion vector field is chosen to be linear, shock formation occurs almost surely, as it does in the deterministic case. Moreover, for a general choice of the diffusion vector fields, formation of shocks cannot be avoided in expectation.

The question of whether noise can regularise PDEs has been treated in the literature, especially recently. First we note that in the ODE case, it is known that additive noise can serve to restore the well-posedness of an ill-behaved equation (see¹⁵⁴ for a general result). In the PDE case, there were no results available until recently. However, after significant attempts by the community,⁷⁶ provides the first example of well-posedness by addition of noise in the PDE case. More concretely, the authors consider the linear transport equation $\partial_t f(t, x) + b(t, x) \cdot Df(t, x) = 0$, which is ill-posed if the drift b is sufficiently irregular (of spatial α -Hölder regularity, $\alpha \in (0, 1)$), and demonstrate that under addition of the same noise we employ in (0.2.4) with $N = n$, $\xi_i = e_i$, $i = 1, \dots, N$, well-posedness of L^∞ -solutions, strong in the probability sense, is recovered. The work⁷⁶ was of so much inspiration to us that it motivated the research we will discuss in Chapter 4, in which we provide an extension of the result in⁷⁶ for a more general geometric version of the linear transport equation (0.2.4).

We note that for the Burgers' equation with additive space-time noise, there have been several works showing well-posedness^{20,56,33}. The techniques em-

ployed in these works are mainly based either on reformulating the equation by a change of variables or on studying its linear part. There have been some works considering multiplicative noise, but the main difference in our equation is that the multiplicative noise we consider in (0.2.8) depends on the gradient of the solution. Hence, the effect of the noise on the equation is completely different from either additive noise or multiplicative noise on only the solution itself. We will indeed observe that due to the gradient form of the noise, certain high order terms needing treatment appear in the a priori estimates. Recently, the same type of advective multiplicative noise has been introduced in the Euler equation^{49,52} and the Boussinesq system⁶, whose techniques we follow closely in our proof. The well-posedness analysis of a more general version of (0.2.8) has been considered in^{85,92,87}. However, these works deal with the well-posedness analysis of kinetic and entropy solutions, in contrast to the strong solutions in the classical sense which we consider.

After investigating the Burgers' equation, we proceed to study the equation of linear advection of k -forms, whose behaviour is extremely beautiful.

The equation of linear advection of k -forms with Lie transport noise. We consider an SPDE of the form

$$dK(t, x) + \mathcal{L}_b K(t, x)dt + \sum_{i=1}^N \mathcal{L}_{\xi_i} K(t, x) \circ dW_t^i = 0, \quad t > 0, \quad x \in \mathbb{R}^n, \quad (0.2.10)$$

which is (0.2.1) with $U(t, x) = K(t, x)$ a k -form-valued stochastic process, and $u(t, x) = b(t, x)$, $\xi_i(t, x)$, $i = 1, \dots, N$ time-dependent deterministic vector fields. From a physical viewpoint, our motivation for considering equation (0.2.10) resides in its importance in fluid modelling, where it represents the advection of a locally conserved quantity K along stochastic paths traversed by fluid particles. This is, (0.2.10) is equivalent to the stochastic local con-

servation law

$$\int_{\mathcal{D}} K(0, x) = \int_{\mathcal{D}(t)} K(t, x), \quad (0.2.11)$$

where $\mathcal{D} \subset \mathbb{R}^n$ is any smooth orientable k -dimensional manifold, and $\mathcal{D}(t) = \phi_t(\mathcal{D})$ is the image of $\mathcal{D}$ under a sufficiently smooth stochastic flow map satisfying (0.2.6) (more concretely, smooth enough so that the weak version of Theorem 3 we present in Chapter 2 can be applied), representing the trajectory of a stochastic fluid particle starting at initial position x . We show this result in Chapter 2. For example, if $K(t, x) = \rho(t, x)dx$ is the mass density, conservation of mass (0.2.11) implies that the continuity equation (0.2.3) is satisfied. If $K(t, x) = f(t, x)$ is a scalar field, (0.2.10) becomes (0.2.4).

As we already hinted in the previous point, our mathematical motivation for studying equation (0.2.10) are the groundbreaking results in⁷⁶. There, it is shown that the stochastic linear transport equation (more concretely, (0.2.4) with $N = n$, $\xi_i = e_i$, $i = 1, \dots, N$ the Euclidean basis vectors) admits a unique L^∞ -weak solution, strong in the probability sense, under the assumptions that the drift vector field $b(t, x)$ is bounded, globally α -Hölder continuous uniformly in time for some $\alpha \in (1/2, 1)$, with $\operatorname{div}(b) \in L^1_{loc}([0, T] \times \mathbb{R}^n)$. The result also holds for $\alpha \in (0, 1/2]$ in dimension one and in any dimension if the extra condition $\operatorname{div}(b) \in L^p([0, T] \times \mathbb{R}^n)$, $p > 2$, is enforced (note that uniqueness is the main difficulty here). In the deterministic case (i.e., when the diffusion vector fields are zero), one can construct examples where b satisfies the conditions specified above, but the linear transport equation admits infinitely many solutions. Hence, the stochastic linear transport equation is well-posed, whereas its deterministic counterpart is not. We stress the fact that the work⁷⁶ is very innovative because, as the authors themselves write in the abstract, it provides the first example of an ill-posed PDE that becomes well-posed under addition of stochastic noise.

Remark 6 (Motivation behind regularising effect). *It is very instructive to illustrate the main intuition behind addition of noise in (0.2.4) as a regularisation mechanism. As argued in⁷⁶, we note that in Itô form, this equation*

reads

$$df + \mathcal{L}_b f dt + \sum_{i=1}^N \mathcal{L}_{\xi_i} f dW_t^i = \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 f dt,$$

where $\mathcal{L}_b f = b \cdot Df$ is the Lie derivative of a scalar function (we have expressed it in geometric terms for convenience). We formally take expectations in the above equation, obtaining

$$\partial_t \mathbb{E}[f] + \mathcal{L}_b \mathbb{E}[f] = \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[f], \quad (0.2.12)$$

by taking into account that $b, \xi_i, i = 1, \dots, N$ are deterministic and the expectation of the martingale term vanishes. Now just observe that in equation (0.2.12), the term $\sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[f]$ acts as a generalised Laplacian if some ellipticity conditions are imposed on the diffusion coefficients, and can therefore induce a regularising effect. We note that this same argument cannot be applied to stochastic Burgers' (0.2.8), since in general, it is not true that $\mathbb{E}[\mathcal{L}_u u] = \mathcal{L}_{\mathbb{E}[u]} \mathbb{E}[u]$. This remark also motivated the research in our last chapter, in which we propose a strategy so that this last step can be carried out in the Boussinesq equations.

In Chapter 4, we extend the results in⁷⁶ by first observing that the natural generalisation of the linear transport equation considered in⁷⁶ is (0.2.10). This equation has a geometric structure that we exploit in order to achieve the desired extension. Indeed, we establish the well-posedness of (0.2.10), and therefore of all of the equations that belong to this class, for spatially $L^2 \cap L_{loc}^\infty$ weak solutions, strong in the probability sense, with drift vector field satisfying $b \in L^\infty([0, T]; C_b^\alpha(\mathbb{R}^n, \mathbb{R}^n))$, $0 < \alpha < 1$, of spatial class $W_{loc}^{1,2}$, and diffusions $\xi_i \in L^\infty([0, T])$ [§] (only dependent on time), $i = 1, \dots, N$ satisfying a uniform ellipticity condition. Without stochastic noise, it is known that one can construct a drift vector field b in the class above such that (0.2.10) admits infinitely many solutions. Thus, the (Lie) transport noise term in

[§]We actually consider diffusions which are semimartingales because we wish to work with the Stratonovich formulation sometimes, but if everything was done in Itô form, this would not be needed.

(0.2.10) has the effect of restoring the uniqueness of solutions under some general conditions. We also provide important insight in the case when the drift vector fields ξ_i are chosen to depend both on time and space, which will probably be the subject of a publication soon.

After⁷⁶ was published, there has been a growing interest in the phenomenon of well-posedness by noise of linear PDEs and there exist several related works in the literature (see for instance^{14,72,32,133,134,136,17,91,78}). We comment on some of the ones we consider most relevant. Concerning the stochastic linear transport equation,³² shows well-posedness for initial data $f_0 \in L^p(\mathbb{R}^n)$ in the space $L^\infty([0, T]; L^p(\mathbb{R}^n))$, $p \geq 1$ by making appropriate Sobolev assumptions on b and its divergence, and in⁷² it is shown that for initial condition $f_0 \in \cap_{r=1}^\infty W_{loc}^{1,r}(\mathbb{R}^n)$, a unique solution f_t can be constructed, where $\mathbb{P}(f_t \in \cap_{r=1}^\infty W_{loc}^{1,r}(\mathbb{R}^n)) = 1$, $t \in [0, T]$. This latter phenomenon is known as regularisation by addition of stochastic noise. ¶

In¹³⁴, it is demonstrated that well-posedness by transport noise is also applicable to the stochastic continuity equation (0.2.3), establishing the existence and uniqueness of L^∞ -solutions for vector fields $b \in L^q([0, T]; L^p(\mathbb{R}^n, \mathbb{R}^n))$, with $p, q \geq 2$ satisfying the so-called Ladyzhenskaya-Prodi-Serrin condition in the fluid dynamics' literature. In¹³⁶, it is shown that the same equation with drift vector field $b \in L^\infty([0, T]; (L^\infty \cap L^2)(\mathbb{R}))$ becomes well-posed in the space $L^\infty([0, T]; (L^1 \cap L^2)(\mathbb{R}))$.

In general, there have been many works studying the phenomenon of well-posedness and regularisation by noise in the case of the linear transport and continuity equations, and for a more insightful review of the available literature and many other references, we point the reader to⁸⁹.

Finally, we note that a key classical well-posedness result for the deterministic continuity equation with a non-smooth velocity field b was established in⁶¹, where existence and uniqueness of L^∞ -solutions is proven when $b \in L^1([0, T]; W_{loc}^{1,1}(\mathbb{R}^n, \mathbb{R}^n))$, $\operatorname{div}(b) \in L^1([0, T]; L^\infty(\mathbb{R}^n))$. In⁶¹, it is also shown that by adding a linear growth condition on b to the prior assump-

¶Although we could have used the notation K_t for the solution as in (0.2.10), we have employed f_t to be consistent with the notation in (0.2.4).

tions, existence and uniqueness holds for the deterministic counterpart of the linear transport equation. The literature following⁶¹ is wide and a remarkable extension to vector fields b of locally bounded variation was later achieved in¹¹. Another interesting extension is proposed in⁴⁰, where existence and uniqueness of the flow of the transport equation is proven for vector fields $b \in L^1([0, T]; W_{loc}^{1,1}(\sigma))$, where σ is a Gaussian measure, together with a global integrability condition on the exponential of b and its divergence. In our well-posedness by addition of noise results in Chapter 4, the existence of multiple solutions to (0.2.10) with the same initial value in the deterministic case is motivated by the fact that we do not assume such controls on the divergence of the drift b .

After having investigated the solution properties of the Burgers' equation and the linear advection of k -forms, we study a special averaged Boussinesq model.

Lagrangian-averaged Boussinesq equations with Lie transport noise.

Since Joseph Boussinesq derived the Boussinesq equations in 1872 as a response to a comment on wave translation made by John Scott Russell, the Boussinesq system has become a fundamental model in fluid dynamics. Apart from finding numerous applications in wave and soliton modelling, it has also been employed as a fundamental model in such diverse areas as soil morphology, dispersion of gases, dynamics of liquids, built-environment, the study of atmospheric fronts, etc (see for instance^{155,95,18}). The deterministic two-dimensional incompressible Boussinesq equations may be expressed as

$$\begin{aligned}\partial_t u + (u \cdot \nabla) u &= -\nabla p + g\theta\hat{y}, \\ \partial_t \theta + u \cdot \nabla \theta &= 0, \\ \nabla \cdot u &= 0, \quad t > 0, \quad (x, y) \in \mathbb{T}^2,\end{aligned}\tag{0.2.13}$$

although other versions of these equations are also commonly considered. We note that in equation (0.2.13), $u = (u_1, u_2)$ is the incompressible velocity field, p is the pressure, g is the (constant) acceleration of gravity, θ is the potential temperature, and $\nabla = (\partial_x, \partial_y)$. From a mathematical viewpoint, the

Boussinesq equations are extremely important because it is unknown whether they are globally well-posed in regular Sobolev spaces. In other words, it is not clear whether smooth initial conditions can develop singularities in finite time. Moreover, this question is directly related to the Navier-Stokes million dollar problem, since both equations share some key features. We consider a Lagrangian-averaged with Lie transport noise modification of the deterministic Boussinesq system as a suitable model for predicting climate dynamics⁷. The equations we investigate read

$$\begin{aligned}
du + \mathcal{L}_{\mathbb{E}[u]}u dt + \sum_{i=1}^N \mathcal{L}_{\xi_i}u \circ dW_t^i \\
= -d\mathbb{E} \left[p - |u|^2/2 \right] dt + g\mathbb{E}[\theta] dy dt - gy d(\theta - \mathbb{E}[\theta])dt, \\
d\theta + \mathcal{L}_{\mathbb{E}[u]}\theta dt + \sum_{i=1}^N \mathcal{L}_{\xi_i}\theta \circ dW_t^i = 0, \\
\operatorname{div}(\mathbb{E}[u]) = 0, \quad t > 0, \quad (x, y) \in \mathbb{T}^2, \quad (0.2.14)
\end{aligned}$$

where the velocity u is modelled as a one-form-valued stochastic process and the potential temperature θ is scalar-valued. Moreover, $\mathbb{E}[\cdot]$ represents the expected value of a physical variable at a particular time and space point and d is the exterior differential (introduced in Appendix A), which is simply the gradient operator ∇ when expressed in global coordinates, but we have chosen to write in geometric form for convenience. We observe that system (0.2.14) is non-local in probability space, since the expected velocity $\mathbb{E}[u]$ is assumed to replace the drift velocity u in the transport operator. This stochastic fluid model was derived by exploiting a novel idea introduced in⁶⁴, which consists in applying Lagrangian-averaging (LA) in probability space to the fluid equations governed by stochastic advection by Lie transport introduced in¹⁰². Model (0.2.14) has been proposed and studied in⁷, where its variational and Hamiltonian formulations are presented and the circulation theorem for its solutions is compared with the respective one in the Boussinesq system with Lie transport noise. We note that the Boussinesq equations with Lie transport noise have been studied in⁶. The general framework of LA equations with Lie advection noise has been developed in⁶⁵, where solutions

to a general class of equations containing the LA Euler equation with Lie transport noise are constructed. As explained in⁶⁵, the stochastic LA formalism we consider is closely connected with the framework in⁴³ for the 3D incompressible Navier-Stokes equations if we choose the double Lie derivative in (0.2.14) to be the Laplacian (i.e., $N = 2$, $\xi_1 = e_1$, $\xi_2 = e_2$). Besides its geometric richness, a motivation for considering this type of system consists in the same regularisation principle discussed in Remark 6 for the stochastic linear transport equation. In other words, observe that by rewriting equations (0.2.14) in Itô form and formally taking expectations, we obtain the system

$$\begin{aligned}\partial_t \mathbb{E}[u] + \mathcal{L}_{\mathbb{E}[u]} \mathbb{E}[u] &= -\mathbf{d} \left(\mathbb{E}[p] - \mathbb{E}[|u|^2/2] \right) + g \mathbb{E}[\theta] dy + \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[u], \\ \partial_t \mathbb{E}[\theta] + \mathcal{L}_{\mathbb{E}[u]} \mathbb{E}[\theta] &= \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[\theta].\end{aligned}$$

In the same fashion as we argued in Remark 6, we notice that the terms $\frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[u]$ and $\frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[\theta]$ can provide a regularising effect if ellipticity conditions are assumed. In particular, we prove that (5.1.2) admits a global strong solution in regular Sobolev spaces, thus providing a regularisation of its deterministic counterpart, whose behaviour is much more mysterious and involved. Moreover, we show that the statistical momenta $\mathbb{E}[(u - \mathbb{E}[u])^2]$ and $\mathbb{E}[(\theta - \mathbb{E}[\theta])^2]$ satisfy closed equations.

We stress that the main novel feature of equation (0.2.14) is that the full stochastic velocity u is assumed to be transported by its expected field $\mathbb{E}[u]$. This effects fundamental changes in the fluid Lagrange-to-Euler map. We also note that the Boussinesq equations with stochastic Lie transport (i.e., not Lagrangian-averaged, with velocity transported by the full velocity itself) inherit many features from the deterministic model and their well-posedness and a blow-up criterion are established in⁶. The convergence of the same stochastic equations to a deterministic viscous system is demonstrated in¹²⁵.

WHAT TO FIND IN EACH CHAPTER

Each chapter uses or builds on the contents of one of the journal articles I have co-authored during my PhD student term. However, there are many aspects from these articles that are not included here and vice versa, and this thesis contains significant improvements and revisions.

- Chapter 2 is based on⁵⁸ *Implications of Kunita-Itô-Wentzell formula for k -forms in stochastic fluid dynamics (with So Takao, Darryl Holm, and Erwin Luesink)*. We prove a geometric generalisation of the Itô-Wentzell formula and Itô formulas for tensors of Kunita, which we call the Kunita-Itô-Wentzell (KIW formula), under two different regularity considerations. We investigate some implications of the KIW formula in fluid dynamics. In particular, we apply KIW to derive the stochastic continuity equation starting from conservation of mass, and we show stochastic counterparts of the Kelvin's circulation and Reynolds transport theorems.
- Chapter 3 is based on⁸ *The Burgers' equation with stochastic transport: shock formation, local and global existence of smooth solutions (with So Takao and Diego Alonso-Orán)*. We add stochastic transport noise to the Burgers' equation in Stratonovich form. We establish the local existence and uniqueness of smooth solutions in the inviscid case and construct a blow-up criterion. In the viscous case, we prove global existence and uniqueness of smooth solutions. Finally, we briefly comment on new shock formation results, both almost surely and in expectation.
- Chapter 4 is based on⁵⁹ *Well-posedness by noise for linear advection of k -forms (with So Takao)*. We consider the linear advection of k -forms equation with Stratonovich Lie transport noise. We extend the results in⁷⁶ by showing that the stochastic linear advection of differential k -forms, which includes the linear transport and continuity equations as special cases, becomes well-posed by addition of noise. We also show that the noise considered in⁷⁶ can be chosen to be more general.

- Chapter 5 is based on⁷ *Modelling the Climate and Weather of a 2D Lagrangian-Averaged Euler-Boussinesq Equation with Transport Noise (with Diego Alonso-Orán, Darryl Holm, and So Takao)*. We apply a new averaging technique developed in⁶⁵ to provide a regularised stochastic version of the 2D incompressible Boussinesq equations. To illustrate the power of this new framework, we establish the global well-posedness of the new stochastic Boussinesq model. In the deterministic case, the issue of whether (smooth) solutions of the Boussinesq equations could develop singularities in finite time is still not well-understood. Moreover, we derive closed dynamical equations for the moments of the fluctuations of the velocity and the temperature in the proposed model.

LIST OF SYMBOLS

M	Configuration manifold (normally $\mathbb{R}^n$ or the torus)
TM	Tangent bundle
T^*M	Cotangent bundle
$\bigwedge^k(TM)$	k -th alternating tangent bundle
$\bigwedge^k(T^*M)$	k -th alternating cotangent bundle
$T^{(r,s)}M$	(r, s) -tensor bundle
Γ	Section of tensor bundle
$C^\infty(M)$	Smooth functions from M to $\mathbb{R}$
$\mathfrak{X}(M)$	Smooth vector fields on M
$\mathfrak{X}^k(M)$	Smooth k -vector fields on M
$\mathbf{u} = u_1 \wedge \dots \wedge u_k$	Smooth k -vector field (element of $\mathfrak{X}^k(M)$)
$\Omega^k(M)$	Smooth differential k -forms on M
$C^\infty(T^{(r,s)}(M))$	Smooth (r, s) -tensor fields on M
$\wedge$	Exterior product or minimum function
$\otimes$	Tensor product
$\sharp$	Sharp operator
$\flat$	Flat operator
τ_x^y	Parallel transport between the points x and y
φ^*	Pull-back by the diffeomorphism φ
φ_*	Push-forward by the diffeomorphism φ
$\mathcal{L}$	Lie derivative
$\mathcal{L}^w$	Weak Lie derivative
$\mathcal{L}^{dist}$	Distributional Lie derivative
d	Exterior derivative
d	Time differential or differential notation for stochastic integrals
dV	Volume element
$d^n x$	Volume element in an n -dimensional domain
$d\mathbf{S}$	Surface element $d\mathbf{S} = (dx \wedge dy, dy \wedge dz, dz \wedge dx)$
Ω	Sample space
ω	Random element of a sample space

$\mathbb{P}$	Probability measure
$\mathbb{E}$	Expectation
$(\mathcal{F}_t)_t$	Filtration
W_t	Brownian process (also denoted B_t)
dW_t	Itô integration with respect to Brownian motion
$\circ dW_t$	Stratonovich integration with respect to Brownian motion
$\widehat{dW_t}$	Backward Itô integration with respect to Brownian motion
$\widehat{\circ dW_t}$	Backward Stratonovich integration with respect to Brownian motion
K_t	k -form-valued stochastic process
M_t	Martingale
τ	Stopping time
$[\cdot, \cdot]$	Co-variation between two stochastic processes or commutator
$[\cdot]$	Quadratic variation of a stochastic process or seminorm
$\mathbb{T}^n$	n -dimensional torus
u	Velocity field or vector field
ξ	Diffusion vector field
curl	Curl operator (sometimes denoted by $\nabla \times (\cdot)$)
p	Pressure
ρ	Density
ρ^ϵ	Mollifier
$\widehat{f}$	Fourier transform of the function f
Λ^s	Fractional Laplacian $(-\Delta)^{s/2}$ of order s
$W^{k,p}$	Sobolev space
$W_{loc}^{k,p}$	Local Sobolev space
$\mathcal{X}$	Indicator function
$\lesssim$	Less or equal up to constants
$\langle \cdot, \cdot \rangle$	Inner product or dual pairing
$\langle \cdot, \cdot \rangle_{L^2}$	L^2 -pairing
$\langle \langle \cdot, \cdot \rangle \rangle_{L^2}$	L^2 -pairing between tensors
A^T	L^2 -adjoint of the operator A
X^*	Dual space of X
$\ \cdot\ _{W^{k,p}}$	Sobolev norm

$\ \cdot\ _{W_R^{k,p}}$	Sobolev norm on $B(0, R)$ (open ball)
H^k	Sobolev space $W^{k,2}$
$\dot{H}^k$	Homogeneous Sobolev space H^k
C^α	α -Hölder functions
C_b^α	Bounded α -Hölder functions
$C^{k,\alpha}$	C^k -functions with α -Hölder k -th derivative
$C_b^{k,\alpha}$	Bounded $C^{k,\alpha}$ -functions with bounded derivatives
$\mathcal{B}$	Borel sigma-algebra
$\{e_i\}_{i=1}^n$	Canonical basis or canonical basis of vector fields on $\mathbb{R}^n$
$\{\hat{x}, \hat{y}, \hat{z}\}$	Canonical basis of $\mathbb{R}^3$ (physical notation)
$\oint$	Line integral along a closed curve
J	Jacobian matrix
id	The identity matrix or operator (also I)
u_t	Solution u at time t ($(\cdot)_t$ never indicates a time derivative)
δ_{ij}	Kronecker delta (sometimes δ_i^j or δ^{ij})
$\mathbb{N}$	Natural numbers excluding zero
d	A distance
∞	$+\infty$

1

Preliminaries

In this chapter, we present some preliminary material that will be necessary for understanding our exposition and proving our main theorems. In Section 1.1, we introduce some basic Sobolev spaces, embeddings, inequalities, and the Biot-Savart law. We also introduce classes of bounded α -Hölder continuous functions that we will need. In Section 1.2, we define standard Banach spaces of tensor fields on the Euclidean space, which constitute the natural spaces where some of our equations are defined, and introduce the notions of mollification of tensors fields and weak Lie derivatives. In Section 1.3, we discuss some background in deterministic PDEs, such as parabolic estimates that will be employed during our proofs. In Section 1.4, we start by providing a short review of stochastic processes, and then present some stochastic results we will need such as a version of Fubini, various Itô theorems for the evolution of a quantity along a stochastic flow, compacity results, Burkholder-Davis-Gundy inequality...

1.1 SOME BASIC SOBOLEV THEORY AND HÖLDER SPACES

Sobolev spaces, embeddings, and inequalities. Let $n, m \in \mathbb{N}$ and $\mathbb{T}^n$ represent the n -dimensional torus (or periodic domain). Sobolev spaces on the periodic domain are defined as

$$W^{s,p}(\mathbb{T}^n, \mathbb{R}^m) = \{f \in L^p(\mathbb{T}^n, \mathbb{R}^m) : (I - \Delta)^{s/2} f \in L^p(\mathbb{T}^n, \mathbb{R}^m)\},$$

for any $s \in \mathbb{R}$ and $p \in [1, \infty]$, equipped with the norm $\|f\|_{W^{s,p}} = \|(I - \Delta)^{s/2} f\|_{L^p}$ (assume the convention $(I - \Delta)^0 = I$). Here, we denote by $(I - \Delta)^{s/2} f$ the function having Fourier transform $(1 + |\xi|^2)^{s/2} \widehat{f}(\xi)$, $\xi \in \mathbb{T}^n$, where $\widehat{f}$ represents the Fourier transform of f . Sometimes we will employ the notation $\Lambda^s = (-\Delta)^{s/2}$. The homogeneous Sobolev norm is defined by $\|f\|_{\dot{W}^{s,p}} = \|(-\Delta)^{s/2} f\|_{L^p}$. We recall that Sobolev spaces are Banach. Moreover, L^2 -based Sobolev spaces are Hilbert and may alternatively be denoted by $H^s = W^{s,2}$. We denote $W^{s,p}(\mathbb{T}^n) = W^{s,p}(\mathbb{T}^n, \mathbb{R})$.

We note that although we have defined our Sobolev spaces on the periodic domain $\mathbb{T}^n$, all the above definitions work equally on the Euclidean space $\mathbb{R}^n$ and can be naturally extended to an (even non-smooth!) open set $\Omega \subset \mathbb{R}^n$, but we do not need such generality for our purposes. In the particular case of $\mathbb{R}^n$, we say that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is of class $W_{loc}^{s,p}(\mathbb{R}^n, \mathbb{R}^m)$ if for every $U \in \mathcal{B}(\mathbb{R}^n)$ with compact closure, $\mathcal{X}_U f \in W^{s,p}(\mathbb{R}^n, \mathbb{R}^m)$, where $\mathcal{X}_U$ represents the characteristic function on U . In this case, we denote $\|f\|_{W^{s,p}(U)} = \|\mathcal{X}_U f\|_{W^{s,p}(\mathbb{R}^n)}$. In particular, if $U = B(0, R)$ is the open ball with centre zero and radius R , we employ the notation $\|f\|_{W_R^{s,p}}$.

Remark 7 (Alternative definition of Sobolev spaces⁶⁰). *The Sobolev space $W^{s,p}(\mathbb{T}^n, \mathbb{R}^m)$, $0 < s < 1$, can be alternatively defined as*

$$W^{s,p}(\mathbb{T}^n, \mathbb{R}^m) = \left\{ f \in L^p(\mathbb{T}^n, \mathbb{R}^m) : \frac{|f(x) - f(y)|}{|x - y|^{\frac{n}{p} + s}} \in L^p(\mathbb{T}^n \times \mathbb{T}^n, \mathbb{R}) \right\},$$

with norm

$$\|f\|_{W^{s,p}} = \left(\int_{\mathbb{T}^n} |f(x)|^p d^n x + \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} d^n x d^n y \right)^{\frac{1}{p}}.$$

Remark 8. Let $p \in (1, \infty)$. We note that since the gradient operator can be expressed as $\nabla = \Lambda R$, where R is the Riesz transform, and taking into account the fact that R is a bounded operator in L^p , we have

$$\|\nabla f\|_{L^p} = \|\Lambda R f\|_{L^p} = \|R \Lambda f\|_{L^p} \lesssim \|\Lambda f\|_{L^p}, \quad f \in W^{1,p}(\mathbb{T}^n, \mathbb{R}^m),$$

since R and Λ commute due to the multiplier property. By iteration of the argument, this implies that any derivative of order $k \in \mathbb{N}$ can be bounded by Λ^k in L^p . This motivates our definition of $\|\cdot\|_{W^{s,p}}$.

In our analysis, we will often work in Sobolev spaces and employ different forms of Sobolev embeddings. Hence, we list some key embeddings in one and two dimensions that we will make use of.

Lemma 9 (Sobolev embeddings in one dimension). *For $f : \mathbb{T} \rightarrow \mathbb{R}$ with zero mean, it holds*

$$\|f\|_{L^4} \lesssim \|f\|_{L^2}^{1/2} \|\partial_x f\|_{L^2}^{1/2}, \quad (1.1.1)$$

$$\|f\|_{L^\infty} \lesssim \|f\|_{H^{1/2+\epsilon}}, \quad \epsilon > 0, \quad (1.1.2)$$

$$\|\partial_x f\|_{L^4} \lesssim \|f\|_{L^\infty}^{1/2} \|\partial_{xx} f\|_{L^2}^{1/2}. \quad (1.1.3)$$

Lemma 10 (Sobolev embeddings in two dimensions). *Let $f : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ with zero mean. Then*

$$\|f\|_{L^3} \lesssim \|f\|_{L^2}^{2/3} \|\nabla f\|_{L^2}^{1/3}, \quad (1.1.4)$$

$$\|f\|_{L^4} \lesssim \|\nabla f\|_{L^2}. \quad (1.1.5)$$

We will also use the following embedding into the continuous functions.

Lemma 11 (Embedding into the Hölder continuous functions). *Let $n, m, j \in$*

$\mathbb{N} \cup \{0\}$ and $0 < \alpha < 1$ such that $0 < \alpha \leq m - n/2$. Then we have

$$H^{m+j}(\mathbb{T}^n) \subset C^{j+\alpha}(\mathbb{T}^n). \quad (1.1.6)$$

Remark 12. In (1.1.6), $C^{j+\alpha}$ is the usual notation for C^j functions with α -Hölder continuous k -th derivative (see Remark 20).

We also present the Kato-Ponce commutator inequality (see¹¹⁰).

Lemma 13 (¹¹⁰). Let $s > 0$, $f, g \in (H^s \cap W^{1,\infty})(\mathbb{T}^n)$, and $1 < p, p_2, p_3 < \infty$ with

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p}.$$

Then, the following two inequalities hold

$$\|[\Lambda^s, f]g\|_{L^p} \lesssim \|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}, \quad (1.1.7)$$

$$\|\Lambda^s(fg)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|\Lambda^s g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}, \quad (1.1.8)$$

where in (1.1.7), $[\Lambda^s, f]$ is the commutator $[\Lambda^s, f] = \Lambda^s f - f \Lambda^s$.

Remark 14. The above two lemmas are immediately true also for functions with multiple components.

We note that all these Sobolev formulas on the periodic domain can be extended to $\mathbb{R}^n$. For a more exhaustive review of theory of Sobolev spaces, we refer the reader to^{3,26}. We will also need a particular case of Young's inequality.

Lemma 15 (Peter-Paul's inequality). We remind that Young's inequality states that given $a, b \geq 0$ and $1 \leq p, q \leq \infty$ such that $1/p + 1/q = 1$, it holds

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

A particular case of this (known as the Peter-Paul's inequality) reads

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}, \quad (1.1.9)$$

for $\epsilon > 0$.

We present Young's convolution inequality.

Lemma 16 (Young's convolution inequality). *Let $p, q, r \geq 1$ satisfying the condition $1/p + 1/q + 1/r = 2$ and functions $f \in L^p(\mathbb{T}^n)$, $g \in L^q(\mathbb{T}^n)$, $h \in L^r(\mathbb{T}^n)$. Then we have*

$$\begin{aligned} & \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} f(x)g(x-y)h(y)d^nx d^ny \\ & \leq \left(\int_{\mathbb{T}^n} |f(x)|^p d^nx \right)^{\frac{1}{p}} \left(\int_{\mathbb{T}^n} |g(x)|^q d^nx \right)^{\frac{1}{q}} \left(\int_{\mathbb{T}^n} |h(x)|^r d^nx \right)^{\frac{1}{r}}. \end{aligned}$$

The last lemma also holds on $\mathbb{R}^n$ and for vector-valued functions. When deriving energy estimates, we will often need Grönwall's inequality. We present its differential and integral versions. Normally, when we refer to Grönwall's inequality in our future proofs, we refer to the differential version, which is the following lemma.

Lemma 17 (Grönwall's inequality in differential form). *Let $\beta : [0, T] \rightarrow \mathbb{R}$ and $u : [0, T] \rightarrow \mathbb{R}$ be continuous functions. Assume that u is differentiable in $(0, T)$ and*

$$u'(t) \leq \beta(t)u(t), \quad t \in (0, T).$$

Then we have the following bound for u

$$u(t) \leq u(0) \exp\left(\int_0^t \beta(s)ds\right), \quad t \in [0, T].$$

Lemma 18 (Grönwall's inequality in integral form). *Let $\alpha : [0, T] \rightarrow \mathbb{R}$, $\beta : [0, T] \rightarrow \mathbb{R}$, and $u : [0, T] \rightarrow \mathbb{R}$ be functions such that β and u are*

continuous with $\beta \geq 0$ and α is integrable. Moreover, assume that

$$u(t) \leq \alpha(t) + \int_0^t \beta(s)u(s)ds, \quad t \in [0, T].$$

Then

$$u(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(r)dr\right)ds, \quad t \in [0, T].$$

Moreover, if $\alpha \geq 0$,

$$u(t) \leq \alpha(t) \exp\left(\int_0^t \beta(s)ds\right), \quad t \in [0, T].$$

The Biot-Savart law. The Eulerian velocity u of a fluid is a fundamental object in fluid dynamics. Typically, a velocity is represented by a three-dimensional vector field $u : \mathbb{T}^3 \rightarrow \mathbb{R}^3$, $u = (u^1, u^2, u^3)$. If $u \in H^1(\mathbb{T}^3, \mathbb{R}^3)$, we define its curl (also denoted by $\nabla \times u$ or ω) as

$$\omega = \text{curl}(u) = (\partial_y u^3 - \partial_z u^2) e_1 + (\partial_z u^1 - \partial_x u^3) e_2 + (\partial_x u^2 - \partial_y u^1) e_3, \quad (1.1.10)$$

where e_1, e_2, e_3 are the Euclidean basis vectors. We note that when working with a two-dimensional velocity $v = (v^1, v^2)$, it can be naturally embedded in three dimensions as $u = (v^1, v^2, 0)$, and its curl (1.1.10) can be calculated as

$$\omega = \text{curl}(u) = 0e_1 + 0e_2 + (\partial_x v^2 - \partial_y v^1) e_3,$$

and hence identified with the scalar function $\omega = \partial_x v^2 - \partial_y v^1$. This will be implicitly assumed whenever we are working with a two-dimensional velocity, as it is often done in the fluids literature. Sometimes, the velocity u can be inverted in terms of the vorticity ω , and this is explained in the following general theorem.

Lemma 19 (Biot-Savart operator). *For any incompressible, mean-free $u \in$*

$H^2(\mathbb{T}^3, \mathbb{R}^3)$, the inverse of the curl operator (1.1.10) exists and is given by

$$u = \mathcal{K}\omega = \nabla \times (-\Delta)^{-1}\omega.$$

$\mathcal{K}$ is called the Biot-Savart operator. Moreover, if $s \in [0, \infty)$ and $u \in H^{s+1}(\mathbb{T}^3, \mathbb{R}^3)$, we have the inequality

$$\|u\|_{H^{s+1}} \lesssim \|\omega\|_{H^s}, \quad (1.1.11)$$

where we stress that in the last inequality, the constant can depend on s (for completion, one can check¹⁴⁵).

Note that we have chosen the periodic domain to introduce the Biot-Savart law, but similar theory is also available on the Euclidean space $\mathbb{R}^n$.

Bounded Hölder spaces in $\mathbb{R}^n$.⁷⁶ Let $\alpha \in (0, 1)$. We say that a bounded Borel-measurable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is (globally) α -Hölder continuous if

$$[f]_\alpha = \sup_{x \neq y \in \mathbb{R}^n} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

We denote the space of bounded α -Hölder continuous functions by $C_b^\alpha(\mathbb{R}^n)$, which is a Banach space endowed with the norm

$$\|f\|_{C_b^\alpha} = \|f\|_{L^\infty} + [f]_\alpha.$$

This definition can be straightforwardly extended to functions of multiple components, in which case we employ the notation $C_b^\alpha(\mathbb{R}^n, \mathbb{R}^m)$. For $k \in \mathbb{N}$, we define $C_b^{k+\alpha}(\mathbb{R}^n)$ to be the space of functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ having bounded continuous derivatives of order $i = 0, \dots, k$, and k -th derivatives of class $C_b^\alpha(\mathbb{R}^n)$. The corresponding norm is defined as

$$\|f\|_{C_b^{k+\alpha}} = \sum_{i=0}^k \|D^i f\|_{L^\infty} + [D^k f]_\alpha.$$

This definition can easily be extended to vector-valued functions. Let $T > 0$ and $\alpha \in (0, 1)$. We define the space of bounded α -Hölder continuous functions uniformly in time $L^\infty([0, T]; C_b^\alpha(\mathbb{R}^n))$ to be the set of bounded Borel-measurable functions $f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ with

$$[f]_{\alpha, T} = \sup_{t \in [0, T]} \sup_{x \neq y \in \mathbb{R}^n} \frac{|f(t, x) - f(t, y)|}{|x - y|^\alpha} < \infty.$$

This is a Banach space with respect to the norm $\|f\|_{C_b^\alpha, T} = \|f\|_{L_{t,x}^\infty} + [f]_{\alpha, T}$, where of course

$$\|f\|_{L_{t,x}^\infty} = \sup_{(t,x) \in [0, T] \times \mathbb{R}^n} |f(t, x)|.$$

Naturally, we denote $L^\infty([0, T]; C_b^\alpha(\mathbb{R}^n, \mathbb{R}^m))$ the space of vector fields $f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ having components in $L^\infty([0, T]; C_b^\alpha(\mathbb{R}^n))$.

Moreover, for $k \in \mathbb{N}$, we say that $f \in L^\infty([0, T]; C_b^{k+\alpha}(\mathbb{R}^n))$ if all spatial derivatives of order i are in $L^\infty([0, T]; C_b^\alpha(\mathbb{R}^n))$, $i = 0, 1, \dots, k$. The corresponding norm is defined as

$$\|f\|_{C_b^{k+\alpha}, T} = \sum_{i=0}^k \|D^i f\|_{L_{t,x}^\infty} + [D^k f]_{\alpha, T}.$$

This definition can be straightforwardly extended to functions of multiple components. All the previous spaces of functions can be defined similarly when $T = \infty$.

Remark 20 (α -Hölder spaces). *For $\alpha \in (0, 1)$, the common notation $C^\alpha(\mathbb{R}^n, \mathbb{R}^m)$ designs the space of vector-valued (globally) α -Hölder continuous functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$. For $k \in \mathbb{N}$, $C^{k+\alpha}(\mathbb{R}^n, \mathbb{R}^m)$ is the space of C^k -functions with α -Hölder continuous k -th derivative. These spaces shall not be confused with the spaces of bounded globally α -Hölder continuous functions $C_b^{k+\alpha}(\mathbb{R}^n, \mathbb{R}^m)$ defined above, which are Banach spaces.*

1.2 SOME BANACH SPACES OF TENSOR FIELDS

We work on the Euclidean space $\mathbb{R}^n$. We introduce some Banach spaces of (r, s) -tensor fields (for this, denote the space of (r, s) -tensor fields on $\mathbb{R}^n$ by $\Gamma(T^{(r,s)}(\mathbb{R}^n))$) that will be employed throughout this dissertation. For general background on tensors, see Appendix A. We only consider tensor fields on the Euclidean space $\mathbb{R}^n$ equipped with the standard Euclidean metric. However, these notions can be extended to general Riemannian manifolds (see^{146,15,16,62}).

L^p classes of tensor fields. For every $x \in \mathbb{R}^n$, one can induce a natural metric on $\Gamma(T^{(r,s)}(\mathbb{R}^n))$ by making use of the standard inner product on $\mathbb{R}^n$. We denote this metric by angle brackets $\langle \cdot, \cdot \rangle_x : T_x^{(r,s)}(\mathbb{R}^n) \times T_x^{(r,s)}(\mathbb{R}^n) \rightarrow \mathbb{R}$, and the corresponding norm is denoted by $|\cdot|_x : T_x^{(r,s)}(\mathbb{R}^n) \rightarrow [0, \infty)$, for $x \in \mathbb{R}^n$ (see Appendix A for more details about the construction of a bundle metric). Let $K \in \Gamma(T^{(r,s)}(\mathbb{R}^n))$ such that $\langle K, S \rangle$ is measurable, for every $S \in C^\infty(T^{(s,r)}(\mathbb{R}^n))$ ^{*}. For $p \in [1, \infty)$, we define the L^p -norm of K by

$$\|K\|_{L^p} = \left(\int_{\mathbb{R}^n} |K(x)|_x^p dx \right)^{\frac{1}{p}},$$

and for $p = \infty$, we define the supremum norm as

$$\|K\|_{L^\infty} = \sup_{x \in \mathbb{R}^n} |K(x)|_x.$$

For $p \in [1, \infty]$, the L^p -section of $T^{(r,s)}(\mathbb{R}^n)$, denoted by $L^p(T^{(r,s)}(\mathbb{R}^n))$, is the equivalence class of tensor fields K with $\|K\|_{L^p} < \infty$, where two tensor fields K_1, K_2 are identified if $\|K_1 - K_2\|_{L^p} = 0$ as it is done in the classic L^p spaces.

Given a Borel subset $U \subset \mathbb{R}^n$ with compact closure, we define the (local)

^{*}This is the same as saying that the global coordinates of K with respect to the canonical basis of vector fields are measurable.

L^p -norm of K on U by

$$\|K\|_{L^p(U)} = \left(\int_U |K(x)|_x^p d^n x \right)^{\frac{1}{p}},$$

for $p \in [1, \infty)$, and

$$\|K\|_{L^\infty(U)} = \sup_{x \in U} |K(x)|_x,$$

when $p = \infty$. If $\|K\|_{L^p(U)} < \infty$ for any subset $U \in \mathcal{B}(\mathbb{R}^n)$ with compact closure, then we say that K is of class $L_{loc}^p(T^{(r,s)}(\mathbb{R}^n))$. We will denote by $\|\cdot\|_{L_R^p}$ the local L^p -norm on the open ball $B(0, R)$.

In the special case $p = 2$, the space $L^2(T^{(r,s)}(\mathbb{R}^n))$ is furthermore equipped with an inner product $\langle\langle \cdot, \cdot \rangle\rangle$, defined by

$$\langle\langle F, G \rangle\rangle_{L^2} = \int_{\mathbb{R}^n} \langle F(x), G(x) \rangle_x d^n x,$$

for any $F, G \in L^2(T^{(r,s)}(\mathbb{R}^n))$, making it a Hilbert space. Moreover, for $F \in L^p(T^{(r,s)}(\mathbb{R}^n))$ and $G \in L^q(T^{(r,s)}(\mathbb{R}^n))$, where $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$, we have

$$\begin{aligned} \langle\langle F, G \rangle\rangle_{L^2} &= \int_{\mathbb{R}^n} \langle F(x), G(x) \rangle_x d^n x \\ &\leq \int_{\mathbb{R}^n} |F(x)|_x |G(x)|_x d^n x \leq \|F\|_{L^p} \|G\|_{L^q} < \infty, \end{aligned} \quad (1.2.1)$$

where we have made use of the usual Cauchy-Schwartz and Hölder's inequalities. We call (1.2.1) the Hölder's inequality for tensor fields.

C^k classes of tensor fields. We can also define the differentiability class C^k of tensor fields. If every component $K_{i_{r+1}, \dots, i_{r+s}}^{i_1, \dots, i_r}$ of an (r, s) -tensor field K (see Appendix A) is k times differentiable, where $k \in \mathbb{N}$, then we say that K is a C^k -section of the tensor bundle, denoted by $C^k(T^{(r,s)}(\mathbb{R}^n))$ (see²). Given an (r, s) -tensor field K of class C^1 , we define its derivative to be the

$(r, s + 1)$ -tensor field

$$\nabla K = \frac{\partial K_{i_{r+1}, \dots, i_{r+s}}^{i_1, \dots, i_r}}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{i_1} \otimes dx_{i_{r+1}} \otimes \dots \otimes dx_{i_{r+s}},$$

where Einstein's convention of summation over repeated indices is assumed. Analogously, we can define the k -th derivative $\nabla^k K$ for C^k -tensor fields K , which is an $(r, s + k)$ -tensor field. This notion is extensible to a general Riemannian manifold, where ∇ becomes the covariant derivative with respect to the Levi-Civita connection, but we will not work in this generality.

Sobolev classes of tensor fields. One can also define Sobolev spaces of tensor fields in a similar way one defines Sobolev spaces of functions. First we provide the notion of weak derivative of tensor fields.

Definition 21. Given $K \in L_{loc}^1(T^{(r,s)}(\mathbb{R}^n))$, we say $S \in L_{loc}^1(T^{(r,s+1)}(\mathbb{R}^n))$ is a weak derivative of K if it satisfies $\langle \langle S_i, \theta \rangle \rangle_{L^2} = - \langle \langle K, \nabla_i \theta \rangle \rangle_{L^2}$, for all $i = 1, \dots, n$, where

$$\begin{aligned} S_i(x) &= S_{i, i_{r+1}, \dots, i_{r+s}}^{i_1, \dots, i_r}(x) \frac{\partial}{\partial x_{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{i_{r+1}} \otimes \dots \otimes dx_{i_{r+s}} \\ &\in L_{loc}^1(T^{(r,s)}(\mathbb{R}^n)), \\ \nabla_i \theta(x) &= \frac{\partial \theta_{i_{r+1}, \dots, i_{r+s}}^{i_1, \dots, i_r}}{\partial x_i}(x) \frac{\partial}{\partial x_{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{i_{r+1}} \otimes \dots \otimes dx_{i_{r+s}} \\ &\in L_{loc}^1(T^{(r,s)}(\mathbb{R}^n)), \end{aligned}$$

for any test tensor field $^\dagger \theta \in C_0^\infty(T^{(r,s)}(\mathbb{R}^n))$, where as usual, sum over repeated indices is implied. We denote the weak derivative of a tensor field by $S = \nabla_w K$, where “ w ” is usually omitted if it is understood from the context.

In coordinates, this is the standard weak derivative component-wise. We can also generalise this notion to Riemannian manifolds by taking $\nabla_i \theta$ to be the

[†]Test tensor fields are smooth tensor fields with compact support, extending the usual notion of test functions.

i -th component of the covariant derivative with respect to the Levi-Civita connection. One can also define k -th weak derivatives in a similar manner. Finally, for $k \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$, we define the Sobolev norm $\|\cdot\|_{W^{k,p}}$ of a tensor field by

$$\|K\|_{W^{k,p}} = \sum_{i=0}^k \|\nabla_w^i K\|_{L^p},$$

and we say that K belongs to the Sobolev space $W^{k,p}(T^{(r,s)}(\mathbb{R}^n))$ if $\nabla_w^i K$ exist, $i = 1, \dots, k$, and $\|K\|_{W^{k,p}} < \infty$. We identify K_1, K_2 in $W^{k,p}(T^{(r,s)}(\mathbb{R}^n))$ if $\|K_1 - K_2\|_{W^{k,p}} = 0$ as usual. If the tensor field $\mathcal{X}_U K$ is of class $W^{k,p}$ for every $U \in \mathcal{B}(\mathbb{R}^n)$ with compact closure, then we say that K is of class $W_{loc}^{k,p}(T^{(r,s)}(\mathbb{R}^n))$, and we denote $\|K\|_{W^{k,p}(U)} = \|\mathcal{X}_U K\|_{W^{k,p}(\mathbb{R}^n)}$.

Mollification of tensor fields. First of all, we introduce distribution-valued tensor fields. Given the space $\mathcal{D}(T^{(r,s)}(\mathbb{R}^n)) = C_0^\infty(T^{(r,s)}(\mathbb{R}^n))$ of smooth (r, s) -tensor fields with compact support, a distribution-valued tensor field K is a continuous linear functional $K : \mathcal{D}(T^{(r,s)}(\mathbb{R}^n)) \rightarrow \mathbb{R}$. We denote the space of all distribution-valued tensor fields by $\mathcal{D}'(T^{(r,s)}(\mathbb{R}^n))$, which is equipped with the weak-* topology.

We wish to introduce a definition that explains how to mollify tensors fields, but before this, we provide the standard definition of mollifiers on the Euclidean space we will employ. Let $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^∞ -function such that $0 \leq \rho(x) \leq 1$, $\rho(x) = \rho(-x)$, $x \in \mathbb{R}^n$, $\int_{\mathbb{R}^n} \rho(x) dx = 1$, such that it is supported on $B(0, 1)$ and $\rho(x) = 1$, $x \in B(0, 1/2)$. For any $\epsilon > 0$, we define ρ^ϵ as $\rho^\epsilon(x) = \epsilon^{-n} \rho(x/\epsilon)$. Any function $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ can be approximated by its mollification. We denote $f^\epsilon = \rho^\epsilon * f$.

Lemma 22. *If $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then*

- $f^\epsilon \in C^\infty(\mathbb{R}^n)$, $\epsilon > 0$,
- $\|f^\epsilon\|_{L^p} \leq \|f\|_{L^p}$, $\epsilon > 0$,
- $f^\epsilon \rightarrow f$ in L^p , $\epsilon \rightarrow 0$,

- $f^\epsilon \rightarrow f$ a.e., $\epsilon \rightarrow 0$.

Remark 23. *The previous lemma can be automatically extended to approximate vector-valued functions $f \in L^p(\mathbb{R}^n, \mathbb{R}^m)$, $1 \leq p < \infty$. Moreover, by definition, functions of class L^p_{loc} can be approximated by mollifiers on Borel subsets U with compact closure.*

Definition 24 (Mollification of tensor fields). *Let $1 \leq p < \infty$ and consider a tensor field $K \in L^p(T^{(r,s)}(\mathbb{R}^n))$ and a mollifier (in the above sense) ρ^ϵ . We define the mollified tensor field K^ϵ as the (smooth) (r,s) -tensor field satisfying*

$$\begin{aligned} K^\epsilon(x)(\alpha^1(x), \dots, \alpha^r(x), v_1(x), \dots, v_s(x)) \\ = \int_{\mathbb{R}^n} \rho^\epsilon(x-y) K(y)(\tau_x^y \alpha^1(x), \dots, \tau_x^y \alpha^r(x), \tau_x^y v_1(x), \dots, \tau_x^y v_s(x)) dy, \end{aligned}$$

for any $\alpha^i \in \Omega^1(\mathbb{R}^n)$, $i = 1, \dots, r$, $v_j \in \mathfrak{X}(\mathbb{R}^n)$, $j = 1, \dots, s$, where τ_x^y denotes parallel transport from x to y along the geodesic with respect to the Euclidean metric. We remind that in the Euclidean case $\tau_x^y v(x) = v(x)$, $\tau_x^y \alpha(x) = \alpha(x)$ in global coordinates.

Lemma 25. *If $K \in L^p(T^{(r,s)}(\mathbb{R}^n))$, $1 \leq p < \infty$, then*

- $K^\epsilon \in C^\infty(T^{(r,s)}(\mathbb{R}^n))$, $\epsilon > 0$,
- $\|K^\epsilon\|_{L^p} \leq \|K\|_{L^p}$, $\epsilon > 0$,
- $K^\epsilon \rightarrow K$ in L^p , $\epsilon \rightarrow 0$,
- $K^\epsilon \rightarrow K$ a.e., $\epsilon \rightarrow 0$.

The previous lemma can of course be extended to approximate tensors of class L^p_{loc} on Borel subsets with compact closure. We employ the standard convolution notation $K^\epsilon = \rho^\epsilon * K$ to denote the mollification of a tensor field, since in the Euclidean case it corresponds to the usual notion of mollification in (global) coordinates. For a more general notion of mollifiers on manifolds, see⁸⁶.

Definition 26 (Mollification of distribution-valued tensor fields). *Given a distribution-valued tensor field $K \in \mathcal{D}'(T^{(r,s)}(\mathbb{R}^n))$ and a mollifier ρ^ϵ , we define the mollified tensor field K^ϵ as the (smooth) (r, s) -tensor field satisfying*

$$\langle\langle K^\epsilon, \theta \rangle\rangle_{L^2} = K(\rho^\epsilon * \theta),$$

for any test tensor field $\theta \in \mathcal{D}(T^{(r,s)}(\mathbb{R}^n))$.

Weak notions of Lie derivatives. We introduce weak notions of Lie derivatives in order to be able to define rigorously some of the operators appearing in this thesis. For the classical notion of Lie derivative, we point the reader to Appendix A.

Definition 27 (Weak Lie derivative). *For $K \in C^1(T^{(r,s)}(\mathbb{R}^n))$ and $b \in W_{loc}^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$, we say that $\mathcal{L}_b^w K$ is the Lie derivative of K with respect to b in the weak sense if the following limit exists[‡]:*

$$\mathcal{L}_b^w K = \lim_{\epsilon \rightarrow 0} \mathcal{L}_{b^\epsilon} K,$$

in L_{loc}^1 , where $\mathcal{L}_{b^\epsilon}$ is the Lie derivative with respect to the smooth vector field $b^\epsilon = \rho^\epsilon * b$. We will omit the superscript “w” if it is clear from the context.

The explicit formula for the Lie derivative of a C^1 -regular k -form K with respect to a smooth vector field $b \in \mathfrak{X}(\mathbb{R}^n)$ can be computed as (see¹³⁰ or Appendix A)

$$\begin{aligned} \mathcal{L}_b K(x)(v_1(x) \wedge \dots \wedge v_k(x)) &= b^l(x) \frac{\partial K_{i_1, \dots, i_k}}{\partial x_l}(x) v_1^{i_1}(x) \dots v_k^{i_k}(x) \\ &+ \sum_{j=1}^k K_{i_1, \dots, i_k}(x) \frac{\partial b^{i_j}}{\partial x_l}(x) v_1^{i_1}(x) \dots v_j^l \dots v_k^{i_k}(x), \quad v_1, \dots, v_k \in \mathfrak{X}(\mathbb{R}^n), \end{aligned} \tag{1.2.2}$$

where sum over repeated indices is implied and we have employed the upper indices v_i^j to refer to the j -th coordinate component of v_i . Its corresponding

[‡]The weak Lie derivative is well-defined up to sets of Lebesgue measure zero.

L^2 -adjoint operator applied to a smooth k -form θ can be computed to be

$$\begin{aligned} \mathcal{L}_b^T \theta(x)(v_1(x) \wedge \dots \wedge v_k(x)) &= -\frac{\partial}{\partial x_l} (b^l(x) \theta_{i_1, \dots, i_k}(x)) v_1^{i_1}(x) \dots v_k^{i_k}(x) \\ &\quad + \sum_{j=1}^k \delta^{i_j} \delta_{mp} \theta_{i_1, \dots, i_j, \dots, i_k}(x) \frac{\partial b^m}{\partial x_l}(x) v_1^{i_1}(x) \dots v_j^p \dots v_k^{i_k}(x), \end{aligned} \quad (1.2.3)$$

for all $v_1, \dots, v_k \in \mathfrak{X}(\mathbb{R}^n)$, where the low indices $\theta_{i_1, \dots, i_k}$ design the coordinates of the k -form θ . Taking into account Definition 27, we see that these formulas also hold for $b \in W_{loc}^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$ if the weak Lie derivative exists.

Definition 28 (Distributional Lie derivative). *For $K \in L_{loc}^\infty(T^{(r,s)}(\mathbb{R}^n))$ and $b \in W_{loc}^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$, we say that $\mathcal{L}_b^{dist} K \in \mathcal{D}'(T^{(r,s)}(\mathbb{R}^n))$ is a Lie derivative with respect to b in the sense of distributions if*

$$\mathcal{L}_b^{dist} K(\theta) = \langle \langle K, (\mathcal{L}_b^w)^T \theta \rangle \rangle_{L^2},$$

for any $\theta \in \mathcal{D}(T^{(r,s)}(\mathbb{R}^n))$, where $(\mathcal{L}_b^w)^T$ denotes the L^2 -adjoint of the weak Lie derivative $\mathcal{L}_b^w$. We will usually omit the superscript “dist” if it is understood from the context.

1.3 SOME BACKGROUND IN ANALYSIS OF PDES

Parabolic estimates. We state some results on the solvability of parabolic equations that we will need. The theorems we present can be found in more generality (for high-order parabolic systems with L^p in time, exclusively time-dependent coefficients) in²⁴, but we have adapted them to our purposes. The reference¹¹¹ can also be checked for completion. For general parabolic systems with measurable in time coefficients depending also on space, there are sharp results available in¹¹⁶.

Theorem 29. *Let $\alpha \in (0, 1)$, $\lambda > 0$, $b, f \in L^\infty([0, \infty); C_b^\alpha(\mathbb{R}^n, \mathbb{R}^n))$, and $\xi_i \in L^\infty([0, \infty))$, $i = 1, \dots, N$. Moreover, assume that there exist $0 < c < C < \infty$*

such that the uniform ellipticity condition

$$c|v|^2 \leq \sum_{i,j=1}^n \sum_{k=1}^N \xi_k^i(t) \xi_k^j(t) v_i v_j \leq C|v|^2, \quad t \geq 0, \quad v \in \mathbb{R}^n, \quad (1.3.1)$$

is satisfied. Then there exists a unique solution $u_\lambda \in L^\infty([0, \infty); C_b^{2+\alpha}(\mathbb{R}^n, \mathbb{R}^n))$ [§] of the parabolic system ¶

$$\partial_t u_\lambda(t, x) + Lu_\lambda(t, x) - \lambda u_\lambda(t, x) = f(t, x), \quad t > 0, \quad x \in \mathbb{R}^n, \quad (1.3.2)$$

where the operator L is defined by

$$Lu_\lambda = \sum_{i=1}^N \left(\frac{1}{2} \xi_i^j(t) \xi_i^k(t) \partial_j \partial_k u_\lambda + b^k(t, x) \partial_k u_\lambda \right),$$

and Einstein's convention of summing over repeated indices is assumed. We note that the notion of solution of equation (1.3.2) is not standard since the functions b, f are only measurable in time instead of continuous. Therefore, following¹¹⁶, we prescribe that $u_\lambda : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a solution to (1.3.2) if

$$u_\lambda(t, x) - u_\lambda(s, x) = \int_s^t [-Lu_\lambda(r, x) + \lambda u_\lambda(r, x) + f(r, x)] dr, \quad (1.3.3)$$

for all $0 \leq s < t$, $x \in \mathbb{R}^n$. Moreover, the following estimate holds:

$$\sup_{t \geq 0} \|u_\lambda(t, \cdot)\|_{C_b^{2+\alpha}} \lesssim \sup_{t \geq 0} \|f(t, \cdot)\|_{C_b^\alpha}.$$

Remark 30. The notation for the C_b^α -type spaces was introduced in Section 1.1.

Theorem 31. Let $\alpha \in (0, 1)$, $\lambda > 0$, $\tilde{f} \in L^\infty([0, \infty); C_b^\alpha(\mathbb{R}^n, \mathbb{R}^n))$, and $\xi_i \in L^\infty([0, \infty))$, $i = 1, \dots, N$. Moreover, assume the uniform ellipticity

[§]The time interval where the solution is defined includes zero. This is a subtlety which is not a mistake.

¶The observant reader might have noticed that the sign of λ here is the opposite from the one in²⁴. A change of variables $t \rightarrow -t$ solves the issue, since the results in²⁴ are proven for $t \in \mathbb{R}$.

condition (1.3.1) on the diffusion vector fields. Then the parabolic system

$$\partial_t u_\lambda(t, x) + \sum_{i=1}^N \xi_i^j(t) \xi_i^k(t) \partial_j \partial_k u_\lambda(t, x) - \lambda u_\lambda(t, x) = \tilde{f}(t, x), \quad t > 0, x \in \mathbb{R}^n, \quad (1.3.4)$$

has a unique solution $u_\lambda \in L^\infty([0, \infty); C_b^{2+\alpha}(\mathbb{R}^n, \mathbb{R}^n))$, where (1.3.4) is understood in the sense prescribed in (1.3.3). Moreover, there exists a well-defined fundamental solution $p(t, s, x)$, $0 < t < s$, $x \in \mathbb{R}^n$ such that the unique solution of system (1.3.4) can be explicitly constructed via application of the formula

$$u_\lambda(t, x) = G\tilde{f}(t, x) = \int_t^\infty \int_{\mathbb{R}^n} e^{-\lambda(s-t)} p(t, s, x - y) \tilde{f}(s, y) dy ds, \quad t > 0, x \in \mathbb{R}^n, \quad (1.3.5)$$

and the values of $u_\lambda(0, \cdot)$ are obtained by continuity. Furthermore p is infinitely differentiable in space and there exists $\epsilon > 0$ such that

$$|Dp(t, s, x)| \lesssim (s - t)^{-(n+1)/2} \exp\left(\frac{-\epsilon|x|^2}{s - t}\right), \quad (1.3.6)$$

for all $0 < s < t$, $x \in \mathbb{R}^n$.

Finally, we will need the following theorem, whose proof can be found in⁶. Similar results for a very general operator $\mathcal{P}$ were proven in^{96,97,99}, but our proof is independent.

Theorem 32. *Let $\mathcal{Q}$ be a linear differential operator of first order of the form*

$$\mathcal{Q} = \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i} + b(x),$$

acting on functions $f \in H^1(\mathbb{T}^n, \mathbb{R}^n)$. Then if $a_i, b \in W^{1,\infty}(\mathbb{T}^n)$, $i = 1, \dots, n$, and $f \in H^2(\mathbb{T}^n, \mathbb{R}^n)$, we have

$$\langle \mathcal{Q}^2 f, f \rangle_{L^2} + \langle \mathcal{Q} f, \mathcal{Q} f \rangle_{L^2} \lesssim \|f\|_{L^2}^2. \quad (1.3.7)$$

Moreover, if $a_i, b \in W^{s+1,\infty}(\mathbb{T}^n)$, $i = 1, \dots, n$, $f \in H^{s+2}(\mathbb{T}^n, \mathbb{R}^n)$, and $\mathcal{P}$ is a pseudodifferential operator of order $s \in [1, \infty)$, then

$$\langle \mathcal{P}Q^2f, \mathcal{P}f \rangle_{L^2} + \langle \mathcal{P}Qf, \mathcal{P}Qf \rangle_{L^2} \lesssim \|f\|_{H^s}^2. \quad (1.3.8)$$

1.4 SOME BACKGROUND IN STOCHASTIC ANALYSIS

Filtered probability spaces and stochastic processes: Itô processes and semimartingales. We start by reviewing certain basic stochastic concepts that we will use in this dissertation. Unfortunately, we cannot provide

a complete review of all the theory we employ due to space and time limitations, so the present paragraph does not aim to be comprehensive but rather to guide the reader who might not be familiar with stochastic calculus. Along the preliminaries and in Appendix A.1, we provide numerous references that exhaustively cover all the theory we employ and much more. In particular, for a general review of stochastic integration, we point the reader to ^{135,140}.

Let I be a totally ordered set. A filtration of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is an indexed family $(\mathcal{F}_t)_{t \in I}$ of sigma-algebras such that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$, $s \leq t$, $s, t \in I$. We call $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in I})$ a filtered probability space.

Let $(\mathcal{F}_t^+)_{t \in I}$ be the filtration defined as

$$\mathcal{F}_t^+ = \bigcap_{t < s} \mathcal{F}_s.$$

We say that $(\mathcal{F}_t)_{t \in I}$ is right-continuous when it coincides with $(\mathcal{F}_t^+)_{t \in I}$. We define

$$\mathcal{N}_{\mathbb{P}} = \{A \subset \Omega : A \subset B, \text{ for some } B \subset \mathcal{F} \text{ with } \mathbb{P}(B) = 0\}$$

the collection of all sets in the sample space contained in a set of zero probability measure. We say that $(\mathcal{F}_t)_{t \in I}$ is complete if $\mathcal{N}_{\mathbb{P}} \subset \mathcal{F}_t$, $t \in I$. We note that a filtration $(\mathcal{F}_t)_{t \in I}$ is complete if and only if $(\Omega, \mathcal{F}_t, \mathbb{P})$ is a complete measure space, for all $t \in I$. A filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in I})$ is said to satisfy the usual conditions if the filtration is right-continuous and

complete.

Consider $I = [0, T]$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$ a filtered probability space. A continuous stochastic process is a collection of random variables $X : \Omega \times [0, T] \rightarrow \mathbb{R}^n$ (where $\mathbb{R}^n$ is endowed with the Borel sigma-algebra $\mathcal{B}(\mathbb{R}^n)$). We say that the stochastic process $(X_t)_{t \in [0, T]}$ is adapted to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ if the random variables $X_t : \Omega \rightarrow \mathbb{R}^n$ are $\mathcal{F}_t$ -measurable^{||}, $t \in [0, T]$. A stochastic process $(X_t)_{t \in [0, T]}$ is said to be jointly measurable if $X : \Omega \times [0, T] \rightarrow \mathbb{R}^n$ is $\mathcal{F} \otimes \mathcal{B}([0, T])$ -measurable. Moreover, a stochastic process is called progressively measurable if for every $t \in [0, T]$, the map $X(\cdot) : \Omega \times [0, t] \rightarrow \mathbb{R}^n$ defined by

$$(\omega, s) \rightarrow X_s(\omega), \quad s \in [0, t],$$

is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$ -measurable, where $\mathcal{B}([0, t])$ denotes the set of Borel-measurable functions on $[0, t]$. This implies that $(X_t)_{t \in [0, T]}$ is adapted to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$. Let $P(\Omega \times [0, T])$ denote the power set of $\Omega \times [0, T]$ and consider $\mathcal{P}^* \subset P(\Omega \times [0, T])$ defined by

$$\mathcal{P}^* = \{f^{-1}(A) : A \in \mathcal{B}(\mathbb{R}^n)\},$$

where f is any left-continuous^{**} adapted stochastic process $f : \Omega \times [0, T] \rightarrow \mathbb{R}^n$. The predictable sigma-algebra $\mathcal{P}$ is defined to be the smallest sigma-algebra generated by $\mathcal{P}^*$ ^{††}. We say that a stochastic process is predictable if it is measurable with respect to $\mathcal{P}$. All predictable processes are adapted. In fact, we have the following hierarchy for stochastic processes.

$$\text{predictable} \subset \text{progressively measurable} \subset \text{adapted}$$

Moreover, every adapted process with left or right-continuous paths is progressively measurable and every adapted process with left-continuous paths

^{||}In lay terms, $\mathcal{F}_t$ is observable at time t .

^{**}Actually, one could also write continuous here, but it is more convenient and illustrative to write left-continuous.

^{††}In the literature, the smallest sigma-algebra generated by $\mathcal{P}^*$ is often denoted by $\sigma(\mathcal{P}^*)$.

is predictable. Let $(X_t)_{t \in [0, T]}$ be a stochastic process. The natural filtration with respect to $(X_t)_{t \in [0, T]}$ is $(\mathcal{F}_t^X)_{t \in [0, T]}$ defined by

$$\mathcal{F}_t^X = \sigma \{X_s^{-1}(A) : 0 \leq s \leq t, A \in \mathcal{B}(\mathbb{R}^n)\}, \quad 0 \leq t \leq T,$$

where σ denotes the smallest sigma-algebra containing all sets above. $\mathcal{F}_t^X$ is the smallest sigma-algebra with respect to which the stochastic process $(X_t)_{t \in [0, T]}$ is adapted.

Given two stochastic processes $(X_t)_{t \in [0, T]}$ and $(Y_t)_{t \in [0, T]}$, we say that they are indistinguishable (or equivalently, one is a version of the other) if $\mathbb{P}$ -a.s., $X_t = Y_t$, $t \in [0, T]$. $(X_t)_{t \in [0, T]}$ is said to be a modification of $(Y_t)_{t \in [0, T]}$ if for every $t \in [0, T]$, $X_t = Y_t$, *a.s.* Every version is a modification, but not the other way around. In particular, if $(X_t)_{t \in [0, T]}$ and $(Y_t)_{t \in [0, T]}$ have left or right-continuous trajectories, every modification is a version.

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$ be a filtered probability space and $\{W_t^i\}_{i=1}^N$, $t \in [0, T]$ be a collection of independent Brownian motions (i.e., the N -dimensional standard Brownian motion) adapted to the filtration. An Itô process is a stochastic process of the form

$$X_t = X_0 + \int_0^t f_s ds + \sum_{i=1}^N \int_0^t g_s^i dW_s^i, \quad t \in [0, T],$$

where:

- The initial value X_0 is $\mathcal{F}_0$ -measurable,
- $f : \Omega \times [0, T] \rightarrow \mathbb{R}^n$ is a progressively measurable process satisfying

$$\int_0^T |f_s| ds < \infty, \quad a.s.,$$

- $g^i : \Omega \times [0, T] \rightarrow \mathbb{R}^n$ are progressively measurable processes satisfying

$$\int_0^T |g_s^i|^2 ds < \infty, \quad a.s.,$$

for $i = 1, \dots, N$.

All Itô processes are continuous and adapted to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$.

A stochastic process $(X_t)_{t \in [0, T]}$ adapted to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ is said to be a martingale if

- $\mathbb{E}[|X_t|] < \infty, t \in [0, T],$
- $\mathbb{E}[X_t | \mathcal{F}_s] = X_s, \quad a.s., \quad 0 \leq s \leq t \leq T.$

A stopping time is a random variable $\tau : \Omega \rightarrow [0, T]$ (also known as a random time) such that the following condition is met ^{‡‡}

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t, \quad t \in [0, T].$$

An adapted stochastic process $(X_t)_{t \in [0, T]}$ is said to be a local martingale if there exists a sequence of stopping times $\{\tau_k\}_{k=1}^\infty$ such that

- $\{\tau_k\}_{k=1}^\infty$ converges to T , i.e.

$$\mathbb{P}\left(\omega : \lim_{k \rightarrow \infty} \tau_k(\omega) = T\right) = 1.$$

- For every $k \in \mathbb{N}$, the stopped process $X^{\tau_k} : \Omega \times [0, T] \rightarrow \mathbb{R}^n$ ^{§§} defined by

$$X^{\tau_k}(\omega, t) = X(\omega, \tau_k(\omega) \wedge t)$$

is a martingale.

An adapted stochastic process $(X_t)_{t \in [0, T]}$ is called a semimartingale if it can be decomposed into a sum of the type

$$X_t = M_t + A_t, \quad t \in [0, T],$$

^{‡‡}In lay terms, the event that the stopping time has already happened is observable.

^{§§}For a stopping time τ , the stopped process X^τ can be naturally denoted by $(X_{\tau \wedge t})_{t \in [0, T]}$ and we will employ this notation in Chapter 3.

where $(M_t)_{t \in [0, T]}$ is a local martingale and $(A_t)_{t \in [0, T]}$ is a càdlàg (right continuous with well-defined left limits, and therefore admitting “jumps”) adapted process of locally bounded variation. Semimartingales are extremely important in probability theory because they form the largest class of stochastic processes with respect to which the Itô integral can be defined. In this thesis, we will mainly consider continuous semimartingales. All the above definitions can be easily extended to the case $I = [0, \infty)$, where the notations $(X_t)_{t \geq 0}$ and $(\mathcal{F}_t)_{t \geq 0}$ will be employed. Stopping times of the type $\tau : \Omega \rightarrow [0, \infty]$ can also be considered.

An Itô process of the type

$$X_t = X_0 + \int_0^t f_s ds + \sum_{i=1}^N \int_0^t g_s^i dW_s^i, \quad t \in [0, T],$$

is a semimartingale (also in the case $T = \infty$). We will often work with random fields of the form

$$X(t, x) = X(0, x) + \int_0^t f(s, x) ds + \sum_{i=1}^N \int_0^t g_i(s, x) dW_s^i, \quad (1.4.1)$$

where $f, g_i, i = 1, \dots, N$ are functions from $\Omega \times [0, T] \times \mathbb{R}^m$ to $\mathbb{R}^n$ such that for every $x \in \mathbb{R}^m$, (1.4.1) is an Itô process. We note that this condition is not enough so that $\mathbb{P}$ -a.s. (1.4.1) is a well-defined function of (t, x) nor does it guarantee any regularity on x , so we will impose different spatial regularity assumptions on the variable x depending on the context ^{¶¶}. Finally, we note that we will sometimes work with stochastic differential equations of the type

$$K(t, x) = K(0, x) + \int_0^t F(s, x) ds + \sum_{i=1}^N \int_0^t G_i(s, x) dW_s^i, \quad t \in [0, T], \quad (1.4.2)$$

where $K(0, \cdot)$ is a tensor field and $F(s, \cdot), G_i(s, \cdot), i = 1, \dots, N$ are tensor-valued stochastic processes. We will only consider tensors on the Euclidean space so the equation can be interpreted in global coordinates if desired.

^{¶¶}For instance, continuity is a natural condition to ask.

This type of processes $K(t, \cdot)$ should be understood as natural extensions of the Euclidean-valued random fields we introduced above. Indeed, for $r, s \in \mathbb{N} \cup \{0\}$ such that $K(0, \cdot) \in \Gamma(T^{(r,s)}(\mathbb{R}^n))$, $F \in L^1([0, T]; \Gamma(T^{(r,s)}(\mathbb{R}^n)))$, $G_i \in L^2([0, T]; \Gamma(T^{(r,s)}(\mathbb{R}^n)))$, $i = 1, \dots, N$ and such that for every $x \in \mathbb{R}^n$, $\langle K(0, \cdot), S(\cdot) \rangle(x)$ is $\mathcal{F}_0$ -measurable, and $\langle F(t, \cdot), S(\cdot) \rangle(x)$, $\langle G_i(t, \cdot), S(\cdot) \rangle(x)$, $i = 1, \dots, N$, $t \in [0, T]$, are progressively measurable for every tensor field $S \in C^\infty(T^{(s,r)}(\mathbb{R}^n))$, we say that (1.4.2) holds if for every $S \in C^\infty(T^{(s,r)}(\mathbb{R}^n))$

$$\begin{aligned} \langle K(t, \cdot), S(\cdot) \rangle(x) &= \langle K(0, \cdot), S(\cdot) \rangle(x) + \int_0^t \langle F(s, \cdot), S(\cdot) \rangle(x) ds \\ &\quad + \sum_{i=1}^N \int_0^t \langle G_i(s, \cdot), S(\cdot) \rangle(x) dW_s^i, \end{aligned}$$

holds in the usual sense for every $x \in \mathbb{R}^n$, $t \in [0, T]$.

Throughout the remaining of this section, we assume that we are working with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in I})$ satisfying the usual conditions, where I might be $[0, T]$ or $[0, \infty)$, together with a family $\{W_t^i\}_{i=1}^N$ of independent Brownian processes adapted to the filtration. Sometimes we might consider two families of independent Brownian motions $\{W_t^i\}_{i=1}^N$ and $\{B_t^j\}_{j=1}^M$ adapted to the same filtration, or even more than two. For simplicity, we will not indicate that they are adapted to the same filtration, but this shall be understood. The semimartingales we consider are also adapted to the same filtration. The notation $(\mathcal{F}_t)_{t \in I}$ is usually dropped and one writes $(\mathcal{F}_t)_t$ instead since the set of indices I is often clear from the context.

Stochastic Fubini Theorem. We present a version of Fubini Theorem in the stochastic case appearing in ¹¹⁵.

Theorem 33 (Stochastic Fubini). *Let $T > 0$, and $g_t(x)$, $h_t^i(x)$, $i = 1, \dots, N$ be real-valued stochastic processes defined on $\Omega \times [0, T] \times \mathbb{R}^n$ measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n)$ (remember $\mathcal{P}$ is the predictable sigma-algebra) satisfying the following assumptions:*

- *There exists a set A of zero probability such that for every $(x, \omega) \in$*

$\mathbb{R}^n \times \Omega \setminus A$, we have

$$\int_0^T \left(|g_t(x)| + \sum_{i=1}^N |h_t^i(x)|^2 \right) dt < \infty.$$

- g_t and h_t^i , $i = 1, \dots, N$ also satisfy

$$\int_0^T \int_{\mathbb{R}^n} |g_t(x)| d^n x dt + \sum_{i=1}^N \int_0^T \left(\int_{\mathbb{R}^n} |h_t^i(x)|^2 d^n x \right)^{1/2} dt < \infty, \quad a.s.$$

Then there exists a Borel-measurable adapted continuous in time stochastic process $f : \Omega \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$, such that for any $x \in \mathbb{R}^n$ with probability one

$$f_t(x) = \int_0^t g_s(x) ds + \sum_{i=1}^N \int_0^t h_s^i(x) dW_s^i, \quad t \in [0, T]. \quad (1.4.3)$$

Furthermore, the stochastic integrals

$$\int_0^t \int_{\mathbb{R}^n} h_s^i(x) d^n x dW_s^i, \quad i = 1, \dots, N,$$

are well-defined, $t \in [0, T]$, and $\mathbb{P}$ -a.s. the following equality holds:

$$\int_{\mathbb{R}^n} f_t(x) d^n x = \int_0^t \int_{\mathbb{R}^n} g_s(x) d^n x ds + \sum_{i=1}^N \int_0^t \int_{\mathbb{R}^n} h_s^i(x) d^n x dW_s^i.$$

Stochastic flow of diffeomorphisms and Itô-type formulas. We are interested in working with the solution $\varphi_{s,t}$ of the equation ***

$$\varphi_{s,t}(x) = x + \int_s^t b(r, \varphi_{s,r}(x)) dr + \sum_{i=1}^N \int_s^t \xi_i(r, \varphi_{s,r}(x)) \circ dW_r^i, \quad 0 \leq s \leq t \leq T, \quad (1.4.4)$$

***It might seem unnatural to start by stating the flow equation in Stratonovich form since more regularity conditions will be needed than in Itô form, but this is the natural setting for our equations.

$x \in \mathbb{R}^n$, which turns out to define a stochastic flow of diffeomorphisms under rather weak assumptions on the coefficients¹²⁰. We note that for fixed $s \geq 0$ and $x \in \mathbb{R}^n$, (1.4.4) gives an SDE, so it can be understood as a set of SDEs. Many times we consider $s = 0$ in equation (1.4.4), obtaining the (simpler) flow equation (0.2.6). To explain exactly what we mean by “stochastic flow of diffeomorphisms”, we introduce an important definition which can be found in⁷⁶ and is inspired by^{22,120}.

Definition 34 (Stochastic flow of C^k -diffeomorphisms). *Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_t)$ be a filtered probability space satisfying the usual conditions together with a family of independent Brownian motions $\{W_t^i\}_{i=1}^N$, $t \in [0, T]$ (which we can interpret as standard N -dimensional Brownian motion) adapted to $(\mathcal{F}_t)_t$, and denote by $(\mathcal{F}_{s,t})_{s,t}$ the completed sigma-algebra generated by $\{W_u^i - W_r^i\}_{i=1}^N$, $s \leq r \leq u \leq t$, for each $0 \leq s < t \leq T$.*

Let $k \in \mathbb{N}$. An $\mathcal{F}_{s,t}$ -measurable random field $\varphi_{s,t} : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $0 \leq s \leq t \leq T$ solving equation (1.4.4) is called a stochastic flow of C^k -diffeomorphisms if $\mathbb{P}$ -a.s. the following properties are satisfied

- $\varphi_{s,r} = \varphi_{t,r} \circ \varphi_{s,t}$, for all $0 \leq s \leq t \leq r \leq T$,
- $\varphi_{s,s}(\cdot) = id_{\mathbb{R}^n}$, for all $0 \leq s \leq T$,
- $\varphi_{s,t}(\cdot)$ is a C^k -diffeomorphism, $0 \leq s \leq t \leq T$. Moreover, the spatial derivatives $D^i \varphi_{s,t}(x)$, $D^i \varphi_{s,t}^{-1}(x)$ are continuous in (s, t, x) , $i = 0, \dots, k$.

For $\alpha \in (0, 1)$, we define a $C^{k+\alpha}$ -flow of diffeomorphisms to be a C^k -flow of diffeomorphisms such that $\mathbb{P}$ -a.s. its k -th spatial derivative $D^k \varphi_{s,t}(x)$ is α -Hölder continuous, uniformly in (s, t) .

We will sometimes employ the following assumption on the drift and diffusion vector fields.

Assumption A1 (Flow equation in Stratonovich form). *Let $\alpha > 0$ and $k \in \mathbb{N}$ (i.e., at least $k \geq 1$!). Let $b \in L^\infty([0, T]; C_b^{k+\alpha}(\mathbb{R}^n, \mathbb{R}^n))$, $\xi_i \in L^\infty([0, T]; C_b^{k+1+\alpha}(\mathbb{R}^n, \mathbb{R}^n)) \cap C^1([0, T]; C^2(\mathbb{R}^n, \mathbb{R}^n))$ ^{†††}, $i = 1, \dots, N$ deterministic vector fields. Then (as a particular case of Theorem 3.4.7 in¹²⁰ and*

^{†††}The C^1 in time assumption is needed to convert from Itô to Stratonovich.

the results in¹¹⁹) (1.4.4) admits a unique solution $\varphi_{t,s}$, which is an adapted C^k -semimartingale^{†††} and a $C^{k+\alpha'}$ -stochastic flow of diffeomorphisms for any $0 < \alpha' < \alpha$.

We will need an extension of Itô's formula to tensor fields, which can be found in¹²⁰.

Theorem 35. *Let $K \in C^3(T^{(r,s)}(\mathbb{R}^n))$ and $T > 0$. Let Assumption A1 hold with $k = 4$ for the drift and diffusion vector fields in (1.4.4). Then Itô's second formula (i.e., Itô's formula with respect to the final time variable) for tensor fields states that $\mathbb{P}$ -a.s. for $0 \leq s \leq t \leq T$,*

$$(\varphi_{s,t})^* K - K = \int_s^t (\varphi_{s,r})^* (\mathcal{L}_b K) dr + \sum_{i=1}^N \int_s^t (\varphi_{s,r})^* (\mathcal{L}_{\xi_i} K) \circ dW_r^i, \quad (1.4.5)$$

$$(\varphi_{t,s})^* K - K = - \int_s^t (\varphi_{t,r})^* (\mathcal{L}_b K) dr - \sum_{i=1}^N \int_s^t (\varphi_{t,r})^* (\mathcal{L}_{\xi_i} K) \circ d\widehat{W}_r^i. \quad (1.4.6)$$

Moreover, Itô's first formula (i.e., Itô's formula with respect to the initial time variable) reads

$$(\varphi_{s,t})^* K - K = \int_s^t \mathcal{L}_b((\varphi_{r,t})^* K) dr + \sum_{i=1}^N \int_s^t \mathcal{L}_{\xi_i}((\varphi_{r,t})^* K) \circ d\widehat{W}_r^i, \quad (1.4.7)$$

$$(\varphi_{t,s})^* K - K = - \int_s^t \mathcal{L}_b((\varphi_{r,s})^* K) dr - \sum_{i=1}^N \int_s^t \mathcal{L}_{\xi_i}((\varphi_{r,s})^* K) \circ dW_r^i, \quad (1.4.8)$$

where $\circ d\widehat{W}_r^i$ denotes Stratonovich integration with respect to the backward Brownian motion.

Remark 36. *Equivalent formulas for the push-forward of the flow $(\varphi_{s,t})_*$ and $(\varphi_{t,s})_*$ can also be straightforwardly obtained by noting that $(\varphi)_* = (\varphi^{-1})^*$ (see Appendix A for some insight into these geometric objects).*

Remark 37. *As the observant reader might have noticed, the spatial regularity assumptions on the drift and diffusion vector fields in Theorem 35 seem*

^{†††}Following Kunita, a C^k -semimartingale is a continuous semimartingale such that its first k (spatial) derivatives are continuous semimartingales.

excessive. In Kunita's book¹²⁰, the theorem is stated in much more generality and is hence not optimised for our particular case. However, Theorem 35 holds in weaker conditions and we will provide proof (see for instance Theorem 47, which establishes a more general result).

The celebrated Itô-Wentzell formula explains how a stochastic process changes when it is evaluated along a semimartingale. This result is of extreme importance in stochastic fluid dynamics and its existence greatly inspired our research in Chapters 2, 3, and 4. In particular, we provide some extensions of it in Chapter 2. Proofs of different versions can be found in^{22,119,115,118}. More concretely, the classical theorems we state for continuous semimartingales are demonstrated in¹¹⁸, a version which computes the evaluation of a stochastic flow along a continuous semimartingale (and not the other way around as in Theorems 41 and 42!) is shown in²², and¹¹⁵ provides a generalisation of the classical Itô-Wentzell formula to distribution-valued processes.

Theorem 38 (Itô-Wentzell formula in Itô form¹¹⁸). *Let $f : \Omega \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a semimartingale satisfying*

- (a) $\mathbb{P}$ -a.s. f is continuous in (t, x) ,
- (b) $f(t, \cdot)$ is of class C^2 a.s. for every $t \in [0, T]$.

Moreover, let f have the explicit form

$$f(t, x) = f(0, x) + \sum_{i=1}^M \int_0^t g_i(s, x) dN_s^i, \quad t \in [0, T], \quad x \in \mathbb{R}^n,$$

where $g_i : \Omega \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are adapted processes satisfying

- (i) $\mathbb{P}$ -a.s. g_i are continuous in (t, x) ,
- (ii) $g_i(t, \cdot)$ are of spatial class C^1 a.s. for every $t \in [0, T]$,

and $N^i : \Omega \times [0, T] \rightarrow \mathbb{R}$ are a.s. continuous semimartingales, $i = 1, \dots, M$. Let $M = (M^1, \dots, M^n) : \Omega \times [0, T] \rightarrow \mathbb{R}^n$ be an a.s. continuous semimartin-

gale. Then $\mathbb{P}$ -a.s.

$$\begin{aligned} f(t, M_t) &= f(0, M_0) + \sum_{i=1}^M \int_0^t g_i(s, M_s) dN_s^i + \sum_{j=1}^n \int_0^t \frac{\partial f}{\partial x_j}(s, M_s) dM_s^j \\ &+ \sum_{i=1}^M \sum_{j=1}^n \int_0^t \frac{\partial g_i}{\partial x_j}(s, M_s) d[N^i, M^j]_s + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, M_s) d[M^i, M^j]_s, \end{aligned}$$

for all $t \in [0, T]$.

Remark 39. We note that as we already specified at the beginning of this section, all the semimartingales in the theorem above are considered to be adapted to the same filtration $(\mathcal{F}_t)_t$.

Theorem 40 (Itô-Wentzell formula in Stratonovich form¹¹⁸). Let $f : \Omega \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a semimartingale satisfying

- (a) $\mathbb{P}$ -a.s. f is continuous in (t, x) ,
- (b) $f(t, \cdot)$ is of class C^3 a.s. for every $t \in [0, T]$.

Moreover, let f have the explicit form

$$f(t, x) = f(0, x) + \sum_{i=1}^M \int_0^t g_i(s, x) \circ dN_s^i, \quad t \in [0, T], \quad x \in \mathbb{R}^n,$$

where g_i are adapted processes satisfying

- (i) $\mathbb{P}$ -a.s. g_i are continuous in (t, x) ,
- (ii) $g_i(t, \cdot)$ are of class C^2 a.s. for every $t \in [0, T]$,

and N_t^i are a.s. continuous semimartingales, $i = 1, \dots, M$. Moreover, we assume the representations

$$g_i(t, x) = g_i(0, x) + \sum_{j=1}^L \int_0^t h_{ij}(s, x) \circ dA_s^{ij}, \quad t \in [0, T], \quad x \in \mathbb{R}^n,$$

where h_{ij} are adapted processes satisfying (i),(ii) in Theorem 38, and A_t^{ij} are a.s. continuous semimartingales, $i = 1, \dots, M$, $j = 1, \dots, L$. Let $M_t = (M_t^1, \dots, M_t^n)$ be an a.s. continuous semimartingale. Then $\mathbb{P}$ -a.s.

$$f(t, M_t) = f(0, M_0) + \sum_{i=1}^M \int_0^t g_i(s, M_s) \circ dN_s^i + \sum_{j=1}^n \int_0^t \frac{\partial f}{\partial x_j}(s, M_s) \circ dM_s^j, \quad (1.4.9)$$

for all $t \in [0, T]$.

We want to apply Theorems 38 & 40 to semimartingales $M_t = \phi_t$ that are a solution to the flow equation. We first note that the (general) flow equation (1.4.4) can be expressed in Itô form as

$$\begin{aligned} \varphi_{s,t}(x) = x + \int_s^t \left(b + \frac{1}{2} \sum_{j=1}^N \xi_j \cdot \nabla \xi_j \right) (r, \varphi_{s,r}(x)) dr \\ + \sum_{j=1}^N \int_s^t \xi_j(r, \varphi_{s,r}(x)) dB_r^j, \end{aligned} \quad (1.4.10)$$

$0 \leq s \leq t \leq T$, $x \in \mathbb{R}^n$. If $s = 0$, we obtain the usual flow equation (0.2.6) in Itô form

$$\phi_t(x) = x + \int_0^t \left(b + \frac{1}{2} \sum_{j=1}^N \xi_j \cdot \nabla \xi_j \right) (s, \phi_s(x)) ds + \sum_{j=1}^N \int_0^t \xi_j(s, \phi_s(x)) dB_s^j, \quad (1.4.11)$$

$0 \leq t \leq T$.

Assumption A2 (Flow equation in Itô form). *Let $\alpha > 0$, $k \in \mathbb{N}$, and consider (deterministic) drift and diffusions $b \in L^\infty([0, T]; C_b^{k+\alpha}(\mathbb{R}^n, \mathbb{R}^n))$, $\xi_j \in L^\infty([0, T]; C_b^{k+1+\alpha}(\mathbb{R}^n, \mathbb{R}^n))$, $j = 1, \dots, N$. In this case (from¹¹⁹) (1.4.10) admits a unique solution $\varphi_{t,s}$, which is an adapted C^k -semimartingale and a $C^{k+\alpha'}$ -stochastic flow of diffeomorphisms for any $0 < \alpha' < \alpha$.*

^{§§§}In Itô form, C^1 regularity in time is not needed unless we desire to claim that the Itô and Stratonovich formulations are equivalent.

The following two theorems can be derived as corollaries of Theorem 38 and Theorem 40, respectively, and are of great interest to us. We will provide their proof in Chapter 2.

Theorem 41 (Itô-Wentzell formula for flows in Itô form). *Let $f : \Omega \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a semimartingale satisfying*

- (a) $\mathbb{P}$ -a.s. f is continuous in (t, x) ,
- (b) $f(t, \cdot)$ is of class C^2 a.s. for every $t \in [0, T]$.

Moreover, let f have the explicit form

$$f(t, x) = f(0, x) + \int_0^t g(s, x) ds + \sum_{i=1}^M \int_0^t h_i(s, x) dW_s^i, \quad t \in [0, T], \quad x \in \mathbb{R}^n,$$

where g, h_i are adapted processes satisfying

- (i) $\mathbb{P}$ -a.s. g, h_i are continuous in (t, x) ,
- (ii) $g(t, \cdot), h_i(t, \cdot)$ are of class C^1 a.s. for every $t \in [0, T]$,

and W_t^i are independent Brownian processes, $i = 1, \dots, M$. Finally, let ϕ_t be the (semimartingale) solution to the flow equation (1.4.11) with drift and diffusion vector fields satisfying Assumption A2 with $k = 1$. Then $\mathbb{P}$ -a.s.

$$\begin{aligned} f(t, \phi_t(x)) &= f(0, x) + \int_0^t g(s, \phi_s(x)) ds + \sum_{i=1}^M \int_0^t h_i(s, \phi_s(x)) dW_s^i \\ &\quad + \int_0^t (b \cdot \nabla f)(s, \phi_s(x)) ds + \sum_{j=1}^N \int_0^t (\xi_j \cdot \nabla f)(s, \phi_s(x)) dB_s^j \\ &\quad + \sum_{i=1}^M \sum_{j=1}^N \int_0^t (\xi_j \cdot \nabla h_i)(s, \phi_s(x)) d[W^i, B^j]_s \\ &\quad + \frac{1}{2} \sum_{j=1}^N \int_0^t (\xi_j \cdot \nabla (\xi_j \cdot \nabla f))(s, \phi_s(x)) ds, \end{aligned}$$

for all $t \in [0, T]$ and $x \in \mathbb{R}^n$.

Theorem 42 (Itô-Wentzell formula for flows in Stratonovich form). *Let $f : \Omega \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a semimartingale satisfying*

- (a) $\mathbb{P}$ -a.s. f is continuous in (t, x) ,
- (b) $f(t, \cdot)$ is of class C^3 a.s. for every $t \in [0, T]$.

Moreover, let f have the explicit form

$$f(t, x) = f(0, x) + \int_0^t g(s, x) ds + \sum_{i=1}^M \int_0^t h_i(s, x) \circ dW_s^i, \quad t \in [0, T], \quad x \in \mathbb{R}^n,$$

where g, h_i are adapted processes satisfying

- (i) $\mathbb{P}$ -a.s. g, h_i are continuous in (t, x) ,
- (ii) $g(t, \cdot), h_i(t, \cdot)$ are of class C^2 a.s. for every $t \in [0, T]$,

and W_t^i are independent Brownian processes, $i = 1, \dots, M$. Finally, assume the representations

$$g(t, x) = g(0, x) + \int_0^t a(s, x) ds + \sum_{j=1}^L \int_0^t a_j(s, x) \circ dM_s^j,$$

$$h_i(t, x) = h_i(0, x) + \int_0^t b_i(s, x) ds + \sum_{j=1}^L \int_0^t b_{ij}(s, x) \circ dN_s^{ij}, \quad t \in [0, T], \quad x \in \mathbb{R}^n,$$

where a, a_j, b_i, b_{ij} are adapted processes satisfying

- (1) $\mathbb{P}$ -a.s. a, a_j, b_i, b_{ij} are continuous in (t, x) ,
- (2) $a(t, \cdot), a_j(t, \cdot), b_i(t, \cdot), b_{ij}(t, \cdot)$ are of class C^1 a.s. for every $t \in [0, T]$,

and M_t^j, N_t^{ij} are independent Brownian processes, $i = 1, \dots, M, j = 1, \dots, L$. Finally, let ϕ_t be the (semimartingale) solution to the flow equation (0.2.6)

with coefficients satisfying Assumption A1 with $k = 1$. Then $\mathbb{P}$ -a.s.

$$\begin{aligned} f(t, \phi_t(x)) &= f(0, x) + \int_0^t g(s, \phi_s(x)) ds + \sum_{i=1}^M \int_0^t h_i(s, \phi_s(x)) \circ dW_s^i \\ &+ \int_0^t (b \cdot \nabla f)(s, \phi_s(x)) ds + \sum_{j=1}^N \int_0^t (\xi_j \cdot \nabla f)(s, \phi_s(x)) \circ dB_s^j, \end{aligned}$$

for all $t \in [0, T]$ and $x \in \mathbb{R}^n$.

We present another result we will need, whose proof we provide in Chapter 2. It comprises a geometric generalisation of Theorems 41 & 42 which we call the Kunita-Itô-Wentzell formula. First we present its Itô version and then its Stratonovich counterpart.

Theorem 43 (Kunita-Itô-Wentzell (KIW) formula for k -form-valued processes: Itô version). *Let $K : \Omega \times [0, T] \rightarrow \bigwedge^k(T^*\mathbb{R}^n)$ be a k -form-valued semimartingale satisfying*

- (a) $\mathbb{P}$ -a.s. K is continuous in (t, x) ,
- (b) $K(t, \cdot)$ is of class C^2 a.s. for every $t \in [0, T]$.

Moreover, let K have the explicit form

$$K(t, x) = K(0, x) + \int_0^t G(s, x) ds + \sum_{i=1}^M \int_0^t H_i(s, x) dW_s^i, \quad t \in [0, T], \quad x \in \mathbb{R}^n, \quad (1.4.12)$$

where G, H_i are k -form-valued adapted processes satisfying

- (i) $\mathbb{P}$ -a.s. G, H_i are continuous in (t, x) ,
- (ii) $G(t, \cdot), H_i(t, \cdot)$ are of class C^1 a.s. for every $t \in [0, T]$,

and W_t^i are independent Brownian motions, $i = 1, \dots, M$. Let ϕ_t be an adapted solution of equation (1.4.11), assumed to be a C^1 -diffeomorphism,

with (deterministic) $b \in L^\infty([0, T]; W_{loc}^{1,1}(\mathbb{R}^n, \mathbb{R}^n))$, $\xi_j \in L^\infty([0, T]; C_b^2(\mathbb{R}^n, \mathbb{R}^n))$, $j = 1, \dots, N$. Then $\mathbb{P}$ -a.s. the k -form-valued process K pulled back by the flow ϕ_t satisfies the following relation

$$\begin{aligned} \phi_t^* K(t, x) = & K(0, x) + \int_0^t \phi_s^* G(s, x) ds + \sum_{i=1}^M \int_0^t \phi_s^* H_i(s, x) dW_s^i \\ & + \int_0^t \phi_s^* \mathcal{L}_b K(s, x) ds + \sum_{j=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_j} K(s, x) dB_s^j \\ & + \sum_{i=1}^M \sum_{j=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_j} H_i(s, x) d[W^i, B^j]_s + \frac{1}{2} \sum_{j=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_j} \mathcal{L}_{\xi_j} K(s, x) ds, \end{aligned} \quad (1.4.13)$$

for all $t \in [0, T]$ and $x \in \mathbb{R}^n$.

Shall the pull-back notation $\phi_t^* K(t, x)$ be confusing, we point the reader to Remark 4 (take $T = K$).

Remark 44. The simplest manner of writing down expression (1.4.13) is doing it in Stratonovich form, which will be clear from the next theorem, although the price to pay for achieving a neater final expression is that stronger hypotheses will be required. Moreover, in order to provide a rigorous proof of Theorem 47, it is important first to do it in Itô form and then provide its Stratonovich equivalent.

Remark 45. We also stress that in Theorem 43, $\{W_t^i\}_{i=1}^M$ and $\{B_t^i\}_{i=1}^N$ denote two families of independent Brownian motion adapted to the same filtration, but these two families are not assumed to be independent between each other.

Remark 46. In Chapter 4, we will show general conditions under which the map ϕ_t solving (1.4.11) is a C^1 -flow of diffeomorphisms when the drift vector field has low regularity in time and space, so our assumption on the flow in Theorem 43 makes sense.

We present the Stratonovich version of Theorem 43 as a corollary, which we will also derive in Chapter 2.

Theorem 47 (Kunita-Itô-Wentzell (KIW) formula for k -form-valued processes: Stratonovich version). *Let $K : \Omega \times [0, T] \rightarrow \bigwedge^k(T^*\mathbb{R}^n)$ be a k -form-valued semimartingale satisfying*

- (a) $\mathbb{P}$ -a.s. K is continuous in (t, x) ,
- (b) $K(t, \cdot)$ is of class C^3 a.s. for every $t \in [0, T]$.

Moreover, let K have the explicit form

$$K(t, x) = K(0, x) + \int_0^t G(s, x) ds + \sum_{i=1}^M \int_0^t H_i(s, x) \circ dW_s^i, \quad t \in [0, T], \quad x \in \mathbb{R}^n, \quad (1.4.14)$$

where G, H_i are k -form-valued adapted processes satisfying

- (i) $\mathbb{P}$ -a.s. G, H_i are continuous in (t, x) ,
- (ii) $G(t, \cdot), H_i(t, \cdot)$ are of class C^2 a.s. for every $t \in [0, T]$,

and W_t^i are independent Brownian motions, $i = 1, \dots, M$. Finally, assume the representation

$$H_i(t, x) = H_i(0, x) + \int_0^t g_i(s, x) ds + \sum_{k=1}^L \int_0^t h_{ik}(s, x) dN_s^{ik}, \quad t \in [0, T], \quad x \in \mathbb{R}^n, \quad (1.4.15)$$

where g_i, h_{ik} are k -form-valued adapted processes such that

- (1) expression (1.4.15) satisfies (a), (b), (i), (ii) in Theorem 43,
- (2) $h_{ik}(t, \cdot)$ are of class C^2 a.s. for all $t \in [0, T]$,

and N_t^{ik} are independent Brownian motions, $i = 1, \dots, M, k = 1, \dots, L$. Let ϕ_t be the solution of equation (0.2.6), which is assumed to be a semimartingale and a C^1 -diffeomorphism, with (deterministic) $b \in L^\infty([0, T]; W_{loc}^{1,1}(\mathbb{R}^n, \mathbb{R}^n))$,

$\xi_j \in L^\infty([0, T]; C_b^2(\mathbb{R}^n, \mathbb{R}^n)) \cap C^1([0, T]; C^2(\mathbb{R}^n, \mathbb{R}^n))$, $j = 1, \dots, N$. Then $\mathbb{P}$ -a.s. the following holds

$$\begin{aligned} \phi_t^* K(t, x) &= K(0, x) + \int_0^t \phi_s^* G(s, x) ds + \sum_{i=1}^M \int_0^t \phi_s^* H_i(s, x) \circ dW_s^i \\ &\quad + \int_0^t \phi_s^* \mathcal{L}_b K(s, x) ds + \sum_{j=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_j} K(s, x) \circ dB_s^j, \end{aligned} \quad (1.4.16)$$

for all $t \in [0, T]$ and $x \in \mathbb{R}^n$.

Remark 48. In Theorem 47, assuming that the solution to (0.2.6) is a semimartingale is fundamental because due to the low regularity of the coefficients, this is not guaranteed anymore by^{119,120}.

Some useful stochastic tools. We state a version of the Burkholder-Davis-Gundy inequality, which can be found in¹⁰⁸. The Burkholder-Davis-Gundy inequality was originally proven in^{29,30}.

Theorem 49 (Burkholder-Davis-Gundy). *Given a continuous local martingale $X : [0, \infty) \rightarrow \mathbb{R}^n$, we have*

$$c_p \mathbb{E} \left[|X|_T^{p/2} \right] \leq \mathbb{E} \left[\left(\sup_{s \in [0, T]} |X_s| \right)^p \right] \leq C_p \mathbb{E} \left[|X|_T^{p/2} \right], \quad (1.4.17)$$

for any $T, p > 0$, where c_p, C_p are absolute constants depending on p .

The extension of the classical dominated convergence theorem to predictable processes is the following.

Theorem 50 (Dominated convergence for stochastic integrals). *Let $\{X_t^n\}_{n=1}^\infty$ be a family of real-valued predictable processes, $t \in [0, T]$, such that $X_t^n \rightarrow X_t$ pointwise, as $n \rightarrow \infty$. Moreover, let $|X_t^n| \leq Y_t$, $t \in [0, T]$, for some Itô integrable process Y_t , $n \in \mathbb{N}$. Then*

$$\int_0^T X_s^n dW_s \rightarrow \int_0^T X_s dW_s, \quad n \rightarrow \infty,$$

in probability.

We will sometimes need Itô's product rule.

Lemma 51 (Itô's product rule). *Let $(X_t^i)_{t \in [0, T]}$, $i = 1, \dots, N$ be real-valued continuous semimartingales. Then, we have the following:*

$$d \prod_{i=1}^N X_t^i = \sum_{i=1}^N \left(\prod_{l \neq i}^N X_t^l \right) dX_t^i + \frac{1}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^N \left(\prod_{l \neq j,k}^N X_t^l \right) d[X^j, X^k]_t, \quad t \in [0, T]. \quad (1.4.18)$$

In Stratonovich form, this formula takes the usual shape

$$d \prod_{i=1}^N X_t^i = \sum_{i=1}^N \left(\prod_{l \neq i}^N X_t^l \right) \circ dX_t^i, \quad t \in [0, T]. \quad (1.4.19)$$

Compactness theorems. Finally, we review some classical stochastic convergence tools. Let (E, d) be a separable metric space and $\mathcal{B}(E)$ the Borel sigma-algebra on E . Let $\mathcal{P}(E)$ denote the collection of all the probability measures that can be defined on $(E, \mathcal{B}(E))$. A set $\Gamma \subset \mathcal{P}(E)$ is said to be tight if, for every $\epsilon > 0$, there exists a compact subset $K_\epsilon \subset E$ such that

$$\mu(K_\epsilon) \geq 1 - \epsilon, \quad \mu \in \Gamma.$$

We say that a sequence of probability measures $\{\mu_n\}_{n=1}^\infty \subset \mathcal{P}(E)$ converges weakly to a probability measure μ if

$$\lim_{n \rightarrow \infty} \int_E \varphi d\mu_n = \int_E \varphi d\mu,$$

for all bounded continuous functions $\varphi : E \rightarrow \mathbb{R}$. Moreover, we say that a set $\Gamma \subset \mathcal{P}(E)$ is weakly compact if every sequence $\{\mu_n\}_{n=1}^\infty \subset \Gamma$ has a weakly convergent subsequence. The proofs of the following two classical results can be found in⁵⁷ and also appear in⁴⁹.

Theorem 52 (Prokhorov). *The collection $\Gamma \subset \mathcal{P}(E)$ is weakly compact if and only if it is tight.*

Theorem 53 (Skorokhod representation). *Let (E, d) be a complete separable metric space and $\{\mu_n\}_{n=1}^\infty$ a sequence of probability measures on E converging weakly to some measure μ . Then there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a sequence of E -valued random variables $\{X_n\}_{n=1}^\infty$ converging almost surely to a random variable X , where the laws of X_n and X are μ_n and μ , respectively, $n \in \mathbb{N}$.*

We state another important auxiliary result.

Theorem 54 (Gyöngy-Krylov lemma⁹⁸). *Let $\{X_n\}_{n=1}^\infty$ be a sequence of random variables taking values in a Polish space (E, d) endowed with the Borel sigma-algebra. Then X_n converges in probability to an E -valued random process, $n \rightarrow \infty$, if and only if, for every subsequence $\{X_{n_j}, X_{m_j}\}_{j=1}^\infty$, there exists a further subsequence that converges weakly to a random variable supported on the diagonal $\{(x, y) \in E \times E : x = y\}$.*

2

The Kunita-Itô-Wentzell (KIW) formula for tensors

This chapter focuses on demonstrating Theorem 3 and especially Theorem 43, which provides a similar result in lower regularity conditions and whose proof is more involved. We also prove some auxiliary results and corollaries. Moreover, we discuss some of the applications of these results in geometric fluid dynamics. The results we prove also permit making some of the derivations originally carried out in [102](#) mathematically more rigorous. Furthermore, Theorem 3 will help us establish the well-posedness by noise result for the equation of linear advection of k -forms, which is the main result in Chapter 4. Throughout this chapter, we fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_t)$ satisfying the usual conditions, together with two families $\{W_t^i\}_{i=1}^M, \{B_t^j\}_{j=1}^N$ of independent Brownian processes adapted to $(\mathcal{F}_t)_t$, which are not assumed to be independent between each other. Sometimes we will consider some extra families of independent Brownian motions, which we also assume adapted to $(\mathcal{F}_t)_t$. We start by proving Theorem 41 as a corollary of Theorem 38, which we will employ as an auxiliary result.

Proof of Theorem 41. Denote

$$f_1(t, x) = f(0, x) + \int_0^t g(s, x) ds,$$

$$f_2(t, x) = \sum_{i=1}^M \int_0^t h_i(s, x) dW_s^i.$$

First apply Theorem 38 with $f = f_1$, $M = 1$, $g_1 = g$, $N_t^1 = t$, and $M_t = \phi_t(x)$ satisfying the flow equation (1.4.11). Hence we obtain

$$\begin{aligned} f_1(t, \phi_t(x)) &= f_1(0, x) + \int_0^t g(s, \phi_s(x)) ds + \sum_{j=1}^n \int_0^t \frac{\partial f_1}{\partial x_j}(s, \phi_s(x)) b^j(s, \phi_s(x)) ds \\ &\quad + \frac{1}{2} \sum_{k=1}^N \sum_{i,j=1}^n \int_0^t \frac{\partial f_1}{\partial x_j}(s, \phi_s(x)) \left(\xi_k^i \frac{\partial \xi_k^j}{\partial x_i} \right) (s, \phi_s(x)) ds \\ &\quad + \sum_{k=1}^N \sum_{j=1}^n \int_0^t \frac{\partial f_1}{\partial x_j}(s, \phi_s(x)) \xi_k^j(s, \phi_s(x)) dB_s^k \\ &\quad + \frac{1}{2} \sum_{k=1}^N \sum_{i,j=1}^n \int_0^t \frac{\partial^2 f_1}{\partial x_i \partial x_j}(s, \phi_s(x)) (\xi_k^i \xi_k^j)(s, \phi_s(x)) ds, \end{aligned}$$

where we have taken into account that $d[B^k, B^l]_t = \delta_{kl} dt$, $d[B^k, \cdot]_t = 0$, $k, l = 1, \dots, N$. By noting that

$$\xi_k \cdot \nabla (\xi_k \cdot \nabla f) = \sum_{i,j=1}^n \xi_k^i \frac{\partial}{\partial x_i} \left(\xi_k^j \frac{\partial f}{\partial x_j} \right) = \sum_{i,j=1}^n \left(\xi_k^i \xi_k^j \frac{\partial^2 f}{\partial x_i \partial x_j} + \xi_k^i \frac{\partial \xi_k^j}{\partial x_i} \frac{\partial f}{\partial x_j} \right),$$

$k = 1, \dots, N$, we get

$$\begin{aligned} f_1(t, \phi_t(x)) &= f(0, x) + \int_0^t g(s, \phi_s(x)) ds + \int_0^t (b \cdot \nabla f_1)(s, \phi_s(x)) ds \\ &\quad + \sum_{k=1}^N \int_0^t (\xi_k \cdot \nabla f_1)(s, \phi_s(x)) dB_s^k + \frac{1}{2} \sum_{k=1}^N \int_0^t (\xi_k \cdot \nabla (\xi_k \cdot \nabla f_1))(s, \phi_s(x)) ds. \end{aligned}$$

Now apply again Theorem 38 with $f = f_2$, $g_i = h_i$, $N_t^i = W_t^i$, $i = 1, \dots, M$,

$M_t = \phi_t(x)$ to obtain

$$\begin{aligned}
f_2(t, \phi_t(x)) &= \sum_{i=1}^M \int_0^t h_i(s, \phi_s(x)) dW_s^i \\
&+ \int_0^t (b \cdot \nabla f_2)(s, \phi_s(x)) ds + \sum_{k=1}^N \int_0^t (\xi_k \cdot \nabla f_2)(s, \phi_s(x)) dB_s^k \\
&+ \sum_{i=1}^M \sum_{k=1}^N \int_0^t (\xi_k \cdot \nabla h_i)(s, \phi_s(x)) d[W^i, B^k]_s \\
&+ \frac{1}{2} \sum_{k=1}^N \int_0^t (\xi_k \cdot \nabla (\xi_k \cdot \nabla f_2))(s, \phi_s(x)) ds.
\end{aligned}$$

Therefore, by summing up the expressions obtained for f_1 and f_2 and taking into account the linearity of the operators $b \cdot \nabla$ and $\xi \cdot \nabla(\xi \cdot \nabla)$, together with the fact that $f = f_1 + f_2$, we conclude our result. $\square$

Theorem 42 can be equally derived by using Theorem 40 in a similar way, but the proof is easier and we omit it to avoid repetition. We proceed to show Theorem 3.

Proof of Theorem 3. For simplicity and without loss of generality, we treat the case $M = N = 1$. We consider the scalar function-valued stochastic process

$$F_t(x) = \langle T(t, x), (\phi_t)_* S(x) \rangle, \quad (2.0.1)$$

where we remind that $T : \Omega \times [0, T] \rightarrow \Gamma(T^{(r,s)}(\mathbb{R}^n))$ is a $\mathbb{P}$ -a.s. continuous tensor-valued semimartingale as indicated in the hypotheses, and we have applied dual pairing $\langle \cdot, \cdot \rangle$ with an arbitrary smooth tensor $S \in C^\infty(T^{(s,r)}(\mathbb{R}^n))$ (and hence independent of time). By the Stratonovich product rule (see (1.4.19)) *

$$dF_t(x) = \langle \odot T(t, x), (\phi_t)_* S(x) \rangle + \langle T(t, x), \odot((\phi_t)_* S(x)) \rangle.$$

*Observe that $\phi_t^* F_t(x) = \langle \phi_t^* T(t, x), S(x) \rangle$ is a continuous semimartingale since $T(t, x)$ and ϕ_t are continuous C^1 -semimartingales, so F_t is also a continuous semimartingale.

We apply Kunita's push-forward formula to the tensor S (i.e., use (1.4.8) with $s = 0$ and observe that $\phi_{t,0}^* = (\phi_{0,t})_* = (\phi_t)_*$) obtaining an expression for the term $d((\phi_t)_*S)$

$$(\phi_t)_*S - S = - \int_0^t \mathcal{L}_b((\phi_s)_*S)ds - \int_0^t \mathcal{L}_\xi((\phi_s)_*S) \circ dB_s.$$

Hence

$$\begin{aligned} dF_t(x) &= \langle G(t, x), (\phi_t)_*S(x) \rangle dt + \langle H(t, x), (\phi_t)_*S(x) \rangle \circ dW_t \\ &\quad - \langle T(t, x), \mathcal{L}_b((\phi_t)_*S)(x) \rangle dt - \langle T(t, x), \mathcal{L}_\xi((\phi_t)_*S)(x) \rangle \circ dB_t. \end{aligned}$$

Apply the Itô-Wentzell formula for flows in Stratonovich form (Theorem 42) to the (real-valued) stochastic process F_t (note that the assumptions are met thanks to the hypotheses of Theorem 3), and noticing that the Lie derivative in the scalar case is $\mathcal{L}_b f = b \cdot Df$ (see Appendix A), we obtain

$$\begin{aligned} &\phi_t^* F_t(x) \\ &= F_0(x) + \int_0^t \phi_s^* \langle G(s, x), (\phi_s)_*S(x) \rangle ds + \int_0^t \phi_s^* \langle H(s, x), (\phi_s)_*S(x) \rangle \circ dW_s \\ &\quad - \int_0^t \phi_s^* \langle T(s, x), \mathcal{L}_b((\phi_s)_*S)(x) \rangle ds - \int_0^t \phi_s^* \langle T(s, x), \mathcal{L}_\xi((\phi_s)_*S)(x) \rangle \circ dB_s \\ &\quad + \int_0^t \phi_s^* \mathcal{L}_b \langle T(s, x), (\phi_s)_*S(x) \rangle ds + \int_0^t \phi_s^* \mathcal{L}_\xi \langle T(s, x), (\phi_s)_*S(x) \rangle \circ dB_s. \\ &= F_0(x) + \int_0^t \langle \phi_s^* G(s, x), S(x) \rangle ds + \int_0^t \langle \phi_s^* H(s, x), S(x) \rangle \circ dW_s \\ &\quad + \int_0^t \mathcal{L}_b \langle \phi_s^* T(s, x), S(x) \rangle ds - \int_0^t \langle \phi_s^* T(s, x), \mathcal{L}_b S(x) \rangle ds \\ &\quad + \int_0^t \mathcal{L}_\xi \langle \phi_s^* T(s, x), S(x) \rangle \circ dB_s - \int_0^t \langle \phi_s^* T(s, x), \mathcal{L}_\xi S(x) \rangle \circ dB_s, \end{aligned}$$

where we have also taken into account the fact that $\mathcal{L}$ and ϕ_t^* commute and the identity $\phi_t^* \circ (\phi_t)_* = id$ (see Appendix A). Also, we employ the property

$\mathcal{L}_b \langle T, S \rangle = \langle \mathcal{L}_b T, S \rangle + \langle T, \mathcal{L}_b S \rangle$ to conclude

$$\begin{aligned} \phi_t^* F_t(x) &= F_0(x) + \int_0^t \langle \phi_s^* G(s, x), S(x) \rangle ds + \int_0^t \langle \phi_s^* H(s, x), S(x) \rangle \circ dW_s \\ &\quad + \int_0^t \langle \phi_s^* \mathcal{L}_b T(s, x), S(x) \rangle ds + \int_0^t \langle \phi_s^* \mathcal{L}_\xi T(s, x), S(x) \rangle \circ dB_s. \end{aligned} \quad (2.0.2)$$

Finally, by noticing that

$$\phi_t^* F_t(x) = \langle \phi_t^* T(t, x), S(x) \rangle,$$

and observing that formula (2.0.2) is valid for any smooth tensor $S \in C^\infty(T^{(s,r)}(\mathbb{R}^n))$, we conclude

$$\begin{aligned} \phi_t^* T(t, x) &= T(0, x) + \int_0^t \phi_s^* G(s, x) ds + \int_0^t \phi_s^* H(s, x) \circ dW_s \\ &\quad + \int_0^t \phi_s^* \mathcal{L}_b T(s, x) ds + \int_0^t \phi_s^* \mathcal{L}_\xi T(s, x) \circ dB_s. \end{aligned}$$

□

Remark 55 (KIW formula in the deterministic case). *We note that in the case the tensor-valued function $T(t, x)$ and the flow ϕ_t are chosen to be deterministic in Theorem 3, formula (0.2.7) becomes*

$$\frac{d}{dt}(\phi_t^* T)(t, x) = \phi_t^* \left(\dot{T} + \mathcal{L}_{\dot{\phi}_t \circ \phi_t^{-1}} T \right) (t, x), \quad (2.0.3)$$

where “ $\cdot$ ” represents derivative with respect to t [†]. This result is essential because it permits swapping integration and derivation when integrating along a moving framework, since the pull-back naturally acts as a change of coordinates as we illustrate in Section 2.2. Formula (2.0.3) is for instance a fundamental tool for proving Reynolds transport and Kelvin’s circulation theorems in the deterministic case.

We observe that, formally, equation (0.2.7) can be expressed in a similar

[†]For this we would need to assume, of course, that T and ϕ_t are differentiable in time are their equations are ODEs in the classical sense. Otherwise, as in the remark below, the formula only makes sense in the integral sense.

way to (2.0.3) by employing the differential notation

$$d(\phi_t^* T)(t, x) = \phi_t^* \left(dT + \mathcal{L}_{d\phi_t \circ \phi_t^{-1}} T \right) (t, x). \quad (2.0.4)$$

This can be useful in applications. We stress again that (2.0.4) is purely formal since the notation $d\phi_t$ only makes sense when it is expressed in integral form.

2.1 THE KIW FORMULA IN THE LOW REGULARITY CASE

This section is devoted to proving Theorem 43 and its Stratonovich counterpart Theorem 47, which relax the regularity conditions on the flow equation required in Theorem 3.

Proof of Theorem 43. We divide this proof into several steps.

Step 1. Setup and preparation. For simplicity and without loss of generality, we choose $N = M = 1$ so that the only diffusion vector field in the flow equation (1.4.11) is $\xi = (\xi^1, \dots, \xi^n)$. Moreover, we denote

$$\widehat{b} = b + \frac{1}{2} \sum_{j=1}^n \xi^j D_j \xi, \quad (2.1.1)$$

the drift term of (1.4.11). Equation (2.1.1) is expressed component-wise as

$$\widehat{b}^i = b^i + \frac{1}{2} \sum_{j=1}^n \xi^j D_j \xi^i, \quad i = 1, \dots, n.$$

We consider the following real-valued continuous $\mathcal{F}_t$ -semimartingale [‡]

$$F_t(x) = \langle \phi_t^* K(t, x), \mathbf{u}(x) \rangle = K_{i_1, \dots, i_k}(t, \phi_t(x)) \prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(t, x) u_\alpha^{j_\alpha}(x), \quad (2.1.2)$$

[‡]This is true because the (global) coordinates of K are semimartingales and also the (spatial) derivatives of ϕ_t (see (2.1.3) below).

for $t \in [0, T]$ and $x \in \mathbb{R}^n$. In (2.1.2), we assumed the k -form-valued stochastic process K has (global) coordinate expression

$$K(t, x) = K_{i_1, \dots, i_k}(t, x) dx_{i_1}(x) \wedge \dots \wedge dx_{i_k}(x),$$

where summation over repeated indices $i_\alpha = 1, \dots, n$, $\alpha = 1, \dots, k$ is implied and we employ the bold notation $\mathbf{u} = u_1 \wedge \dots \wedge u_k \in \mathfrak{X}^k(\mathbb{R}^n)$ to stress the fact that $\mathbf{u}$ is a k -vector field with components $u_\alpha \in \mathfrak{X}(\mathbb{R}^n)$ with coordinate expressions

$$u_\alpha(x) = u_\alpha^i(x)(\partial/\partial x_i)(x), \quad \alpha = 1, \dots, k,$$

(summation over repeated indices implied). This is, u_α , $\alpha = 1, \dots, k$ are k arbitrary smooth vector fields that we feed to the time-dependent k -form K via contraction of tensors.

Step 2. Equation for F_t in simple form. By Itô's product rule (see (1.4.18)), F_t satisfies the following equation

$$\begin{aligned} dF_t(x) = & \left(\prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(t, x) u_\alpha^{j_\alpha}(x) \right) dK_{i_1, \dots, i_k}(t, \phi_t(x)) \\ & + K_{i_1, \dots, i_k}(t, \phi_t(x)) d \left(\prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(t, x) u_\alpha^{j_\alpha}(x) \right) \\ & + d \left[K_{i_1, \dots, i_k}(\cdot, \phi(\cdot)), \prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(\cdot, x) u_\alpha^{j_\alpha}(x) \right]_t. \end{aligned}$$

From the Itô-Wentzell formula for flows (see Theorem 41), we have

$$\begin{aligned} dK_{i_1, \dots, i_k}(t, \phi_t(x)) = & G_{i_1, \dots, i_k}(t, \phi_t(x)) dt + H_{i_1, \dots, i_k}(t, \phi_t(x)) dW_t \\ & + (b \cdot \nabla K_{i_1, \dots, i_k})(t, \phi_t(x)) dt + (\xi \cdot \nabla K_{i_1, \dots, i_k})(t, \phi_t(x)) dB_t \\ & + (\xi \cdot \nabla H_{i_1, \dots, i_k})(t, \phi_t(x)) d[W, B]_t + \frac{1}{2} (\xi \cdot \nabla (\xi \cdot \nabla K_{i_1, \dots, i_k}))(t, \phi_t(x)) dt. \end{aligned}$$

Remark 56. *Actually, the coefficients of the flow equation (1.4.11) do not satisfy the assumptions of Theorem 41, but it is straightforward to check that our proof also works in the current regularity conditions. We assumed more regularity in Theorem 41 only to ensure that the flow equation has indeed a solution. If this last fact is assumed, then the regularity conditions on the drift and diffusion vector fields can easily be reduced (just inspect the proof of Theorem 41).*

By Itô's product rule (1.4.18), we can also obtain

$$\begin{aligned} d \left(\prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(t, x) u_\alpha^{j_\alpha}(x) \right) &= \sum_{p=1}^k \left(\prod_{\alpha \neq p}^k J_{j_\alpha}^{i_\alpha}(t, x) u_\alpha^{j_\alpha}(x) \right) u_p^{j_p}(x) dJ_{j_p}^{i_p}(t, x) \\ &+ \frac{1}{2} \sum_{\substack{p, q=1 \\ p \neq q}}^k \left(\prod_{\alpha \neq p, q}^k J_{j_\alpha}^{i_\alpha}(t, x) u_\alpha^{j_\alpha}(x) \right) u_p^{j_p}(x) u_q^{j_q}(x) d \left[J_{j_p}^{i_p}(\cdot, x), J_{j_q}^{i_q}(\cdot, x) \right]_t. \end{aligned}$$

We differentiate (1.4.11) with respect to x to arrive at

$$dJ_j^i(t, x) = D_t \widehat{b}^i(t, \phi_t(x)) J_j^l(t, x) dt + D_t \xi^i(t, \phi_t(x)) J_j^l(t, x) dB_t, \quad i, j = 1, \dots, n. \quad (2.1.3)$$

Step 3. Full equation for F_t . By using all the formulas obtained in Step 2, one checks that $F_t(x)$ can be expressed as

$$\begin{aligned} F_t(x) - F_0(x) &= \int_0^t \widehat{G}_s^1(x) ds + \int_0^t \widehat{G}_s^2(x) d[W., B.]_s \\ &+ \int_0^t \widehat{H}_s^1(x) dW_s + \int_0^t \widehat{H}_s^2(x) dB_s, \end{aligned} \quad (2.1.4)$$

where

$$\begin{aligned}
\widehat{G}_s^1(x) &= G_{i_1, \dots, i_k}(s, \phi_s(x)) \prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(s, x) u_\alpha^{j_\alpha}(x) \\
&+ b^l(s, \phi_s(x)) D_l K_{i_1, \dots, i_k}(s, \phi_s(x)) \prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(s, x) u_\alpha^{j_\alpha}(x) \\
&+ \sum_{p=1}^k K_{i_1, \dots, i_k}(s, \phi_s(x)) D_l b^{i_p}(s, \phi_s(x)) \left(\prod_{\alpha \neq p}^k J_{j_\alpha}^{i_\alpha}(s, x) u_\alpha^{j_\alpha}(x) \right) J_{j_p}^l(s, x) u_p^{j_p}(x) \\
&+ \frac{1}{2} \xi^l(s, \phi_s(x)) \xi^m(s, \phi_s(x)) D_{lm}^2 K_{i_1, \dots, i_k}(s, \phi_s(x)) \prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(s, x) u_\alpha^{j_\alpha}(x) \\
&+ \frac{1}{2} \xi^m(s, \phi_s(x)) D_m \xi^l(s, \phi_s(x)) D_l K_{i_1, \dots, i_k}(s, \phi_s(x)) \prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(s, x) u_\alpha^{j_\alpha}(x) \\
&+ \sum_{p=1}^k \left[\xi^m(s, \phi_s(x)) D_l \xi^{i_p}(s, \phi_s(x)) D_m K_{i_1, \dots, i_k}(s, \phi_s(x)) \right. \\
&\left. + \frac{1}{2} D_l \left[\xi^m(s, \phi_s(x)) D_m \xi^{i_p}(s, \phi_s(x)) \right] K_{i_1, \dots, i_k}(s, \phi_s(x)) \right] \\
&\times \left(\prod_{\alpha \neq p}^k J_{j_\alpha}^{i_\alpha}(s, x) u_\alpha^{j_\alpha}(x) \right) J_{j_p}^l(s, x) u_p^{j_p}(x) \\
&+ \frac{1}{2} \sum_{\substack{p, q=1 \\ p \neq q}}^k K_{i_1, \dots, i_k}(s, \phi_s(x)) D_l \xi^{i_p}(s, \phi_s(x)) D_m \xi^{i_q}(s, \phi_s(x)) \\
&\times \left(\prod_{\alpha \neq p, q}^k J_{j_\alpha}^{i_\alpha}(s, x) u_\alpha^{j_\alpha}(x) \right) J_{j_p}^l(s, x) J_{j_q}^m(s, x) u_p^{j_p}(x) u_q^{j_q}(x),
\end{aligned}$$

$$\begin{aligned}
\widehat{G}_s^2(x) &= \xi^l(s, \phi_s(x)) D_l H_{i_1, \dots, i_k}(s, \phi_s(x)) \prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(s, x) u_\alpha^{j_\alpha}(x) \\
&+ \sum_{p=1}^k H_{i_1, \dots, i_k}(s, \phi_s(x)) D_l \xi^{i_p}(s, \phi_s(x)) \left(\prod_{\alpha \neq p}^k J_{j_\alpha}^{i_\alpha}(s, x) u_\alpha^{j_\alpha}(x) \right) J_{j_p}^l(s, x) u_p^{j_p}(x),
\end{aligned}$$

$$\widehat{H}_s^1(x) = H_{i_1, \dots, i_k}(s, \phi_s(x)) \prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(s, x) u_\alpha^{j_\alpha}(x),$$

$$\begin{aligned} \widehat{H}_s^2(x) &= \xi^l(s, \phi_s(x)) D_l K_{i_1, \dots, i_k}(s, \phi_s(x)) \prod_{\alpha=1}^k J_{j_\alpha}^{i_\alpha}(s, x) u_\alpha^{j_\alpha}(x) \\ &+ \sum_{p=1}^k K_{i_1, \dots, i_k}(s, \phi_s(x)) D_l \xi^{i_p}(s, \phi_s(x)) \left(\prod_{\alpha \neq p}^k J_{j_\alpha}^{i_\alpha}(s, x) u_\alpha^{j_\alpha}(x) \right) J_{j_p}^l(s, x) u_p^{j_p}(x). \end{aligned}$$

Step 4. Conclusion. Finally, the explicit formula for the Lie derivative of a k -form (A.2.2) in our context reads

$$\begin{aligned} \mathcal{L}_\xi K(x)(u_1(x) \wedge \dots \wedge u_k(x)) &= \xi^l(x) D_l K_{i_1, \dots, i_k}(x) u_1^{i_1}(x) \dots u_k^{i_k}(x) \\ &+ \sum_{p=1}^k K_{i_1, \dots, i_k}(x) D_l \xi^{i_p}(x) u_1^{i_1}(x) \dots u_p^l \dots u_k^{i_k}(x), \end{aligned}$$

and by double application we also obtain

$$\begin{aligned} \mathcal{L}_\xi \mathcal{L}_\xi K(x)(u_1(x) \wedge \dots \wedge u_k(x)) &= \xi^l(x) D_l (\xi^m(x) D_m K_{i_1, \dots, i_k}(x)) u_1^{i_1}(x) \dots u_k^{i_k}(x) \\ &+ \sum_{p=1}^k \xi^l(x) D_l (K_{i_1, \dots, i_k}(x) D_m \xi^{i_p}(x)) u_1^{i_1}(x) \dots u_p^m \dots u_k^{i_k}(x) \\ &+ \sum_{p=1}^k \xi^l(x) D_m \xi^{i_p}(x) D_l K_{i_1, \dots, i_k}(x) u_1^{i_1}(x) \dots u_p^m \dots u_k^{i_k}(x) \\ &+ \sum_{\substack{p, q=1 \\ p \neq q}}^k K_{i_1, \dots, i_k}(x) D_l \xi^{i_p}(x) D_m \xi^{i_q}(x) u_1^{i_1}(x) \dots u_p^l u_q^m \dots u_k^{i_k}(x), \end{aligned}$$

so we conclude that

$$\begin{aligned}
\int_0^t \widehat{G}_s^1(x) ds &= \int_0^t \langle \phi_s^* G(s, x), \mathbf{u}(x) \rangle ds \\
&\quad + \int_0^t \langle \phi_s^* \mathcal{L}_b K(s, x), \mathbf{u}(x) \rangle ds + \frac{1}{2} \int_0^t \langle \phi_s^* \mathcal{L}_\xi \mathcal{L}_\xi K(s, x), \mathbf{u}(x) \rangle ds, \\
\int_0^t \widehat{G}_s^2(x) d[W, B]_s &= \int_0^t \langle \phi_s^* \mathcal{L}_\xi H(s, x), \mathbf{u}(x) \rangle d[W, B]_s, \\
\int_0^t \widehat{H}_s^1(x) dW_s &= \int_0^t \langle \phi_s^* H(s, x), \mathbf{u}(x) \rangle dW_s, \\
\int_0^t \widehat{H}_s^2(x) dB_s &= \int_0^t \langle \phi_s^* \mathcal{L}_\xi K(s, x), \mathbf{u}(x) \rangle dB_s,
\end{aligned}$$

where we remind $\mathbf{u} = u_1 \wedge \dots \wedge u_k$. Since $\mathbf{u}$ was chosen arbitrarily, we finish our proof here. $\square$

Now, we show how to prove the Stratonovich version of this theorem by utilising the Itô version that we have just demonstrated.

Proof of Theorem 47. We can convert (0.2.6) into Itô form as

$$\begin{aligned}
\phi_t(x) &= x + \int_0^t b(s, \phi_s(x)) ds + \sum_{j=1}^N \int_0^t \xi_j(s, \phi_s(x)) dB_s^j \\
&\quad + \sum_{j=1}^N [\xi_j(\cdot, \phi(\cdot)), B^j]_t, \quad (2.1.5)
\end{aligned}$$

which by application of (classical) Itô's Lemma to the functions $\xi_j(t, \phi_t(x))$, $j = 1, \dots, N$ (take into account the diffusion vector fields have regularity $C^1([0, T]; C^2(\mathbb{R}^n, \mathbb{R}^n))$) becomes (1.4.11) [§]. We can therefore assume that

[§]Note that since we have assumed that ϕ_t is a semimartingale, by Itô's Lemma $\xi_j(t, \phi_t(x))$ is also a semimartingale. Therefore the co-variation above is well-defined.

(1.4.11) is satisfied. In Itô form, (1.4.14) is given by

$$\begin{aligned} K(t, x) = K(0, x) &+ \int_0^t G(s, x) ds + \sum_{i=1}^M \int_0^t H_i(s, x) dW_s^i \\ &+ \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^L \int_0^t h_{ij}(s, x) d[W^i, N^{ij}]_s. \end{aligned}$$

Now, applying (1.4.13) to K , we obtain

$$\begin{aligned} \phi_t^* K(t, x) = K(0, x) &+ \int_0^t \phi_s^* G(s, x) ds + \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^L \int_0^t \phi_s^* h_{ij}(s, x) d[W^i, N^{ij}]_s \\ &+ \sum_{i=1}^M \int_0^t \phi_s^* H_i(s, x) dW_s^i + \int_0^t \phi_s^* \mathcal{L}_b K(s, x) ds + \sum_{i=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_i} K(s, x) dB_s^i \\ &+ \sum_{i=1}^M \sum_{j=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_j} H_i(s, x) d[W^i, B^j]_s + \frac{1}{2} \sum_{i=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_i} \mathcal{L}_{\xi_i} K(s, x) ds. \end{aligned} \tag{2.1.6}$$

Since H_i and $\mathcal{L}_{\xi_j} K$ satisfy the assumptions of Theorem 43 for $i = 1, \dots, M$, $j = 1, \dots, N$, applying (1.4.13) to H_i and $\mathcal{L}_{\xi_i} K$ respectively yields

$$\begin{aligned} \phi_t^* H_i(t, x) = H_i(0, x) &+ \int_0^t \phi_s^* g(s, x) ds + \sum_{j=1}^L \int_0^t \phi_s^* h_{ij}(s, x) dN_s^{ij} \\ &+ \int_0^t \phi_s^* \mathcal{L}_b H_i(s, x) ds + \sum_{j=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_j} H_i(s, x) dB_s^j \\ &+ \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^L \int_0^t \phi_s^* \mathcal{L}_{\xi_j} h_{ik}(s, x) d[N^{ik}, B^j]_s + \frac{1}{2} \sum_{j=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_j} \mathcal{L}_{\xi_j} H_i(s, x) ds, \end{aligned}$$

for $i = 1, \dots, M$ and

$$\begin{aligned}
\phi_t^* \mathcal{L}_{\xi_i} K(t, x) &= \mathcal{L}_{\xi_i} K(0, x) + \int_0^t \phi_s^* \mathcal{L}_{\xi_i} G(s, x) ds \\
&+ \frac{1}{2} \sum_{j=1}^M \sum_{k=1}^L \int_0^t \mathcal{L}_{\xi_i} h_{jk}(s, x) d[W^j, N^{jk}]_s + \sum_{j=1}^M \int_0^t \phi_s^* \mathcal{L}_{\xi_i} H_j(s, x) dW_s^j \\
&+ \int_0^t \phi_s^* \mathcal{L}_b \mathcal{L}_{\xi_i} K(s, x) ds + \sum_{j=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_j} \mathcal{L}_{\xi_i} K(s, x) dB_s^j \\
&+ \sum_{j=1}^N \sum_{k=1}^M \int_0^t \phi_s^* \mathcal{L}_{\xi_j} \mathcal{L}_{\xi_i} H_k(s, x) d[W^k, B^j]_s + \frac{1}{2} \sum_{j=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_j} \mathcal{L}_{\xi_j} \mathcal{L}_{\xi_i} K(s, x) ds,
\end{aligned}$$

for $i = 1, \dots, N$. Therefore, we conclude

$$\begin{aligned}
[\phi^* H_i(\cdot, x), W^i]_t &= \sum_{j=1}^L \int_0^t \phi_s^* h_{ij}(s, x) d[W^j, N^{ij}]_s \\
&+ \sum_{j=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_j} H_i(s, x) d[W^i, B^j]_s, \quad i = 1, \dots, M,
\end{aligned}$$

$$\begin{aligned}
[\phi^* \mathcal{L}_{\xi_i} K(\cdot, x), B^i]_t &= \sum_{j=1}^M \int_0^t \phi_s^* \mathcal{L}_{\xi_i} H_j(s, x) d[W^j, B^i]_s \\
&+ \int_0^t \phi_s^* \mathcal{L}_{\xi_i} \mathcal{L}_{\xi_i} K(s, x) ds, \quad i = 1, \dots, N.
\end{aligned}$$

Hence, formula (2.1.6) becomes

$$\begin{aligned}
\phi_t^* K(t, x) &= K(0, x) + \int_0^t \phi_s^* G(s, x) ds + \sum_{i=1}^M \int_0^t \phi_s^* H_i(s, x) \circ dW_s^i \\
&+ \int_0^t \phi_s^* \mathcal{L}_b K(s, x) ds + \sum_{i=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_i} K(s, x) \circ dB_s^i,
\end{aligned}$$

which is identical to (1.4.16). $\square$

Remark 57. *Theorem 47 confirms a posteriori a well-known rule of thumb*

in stochastic differential geometry, called by Malliavin the transfer principle^{70,129}. The transfer principle allows one to replace classical calculus with Stratonovich stochastic calculus in certain circumstances. In finite dimensions, this is admitted via Stratonovich differentials obeying the ordinary chain and product rules, and also via certain approximation results, although various regularity conditions would need to be added for SPDEs and we are interested in working under relatively low regularity conditions. See⁷⁰ for an extensive discussion of the transfer principle in the finite-dimensional situation.

2.2 APPLICATIONS OF KIW IN STOCHASTIC FLUID DYNAMICS

The aim of this section is showing how to employ the demonstrated KIW theorem(s) for deriving useful formulas in stochastic fluid dynamics. We note that our motivation here is not focusing on sharp regularity results, but rather showcasing the applicability of Theorem 3 (or its lower regularity version Theorem 43) in the derivation of fluid dynamic equations. The applicability of KIW is of course not limited to this type of formulas and in Chapter 4 we will show that KIW also provides one of the key tools for closing uniqueness estimates in an interesting problem of regularisation by noise for the equation of linear advection of k -forms. In the remaining of this chapter, we mainly follow the ideas in⁵⁸.

Derivation of advection equations from conservation laws. As explained in the introduction, in stochastic fluid dynamics one encounters several types of advected k -forms (and even tensors!) such as the momentum one-form density, mass density, magnetic field... We consider a fluid moving in a domain $D \subset \mathbb{R}^n$ and an arbitrary control volume $\mathcal{D}_0 \subset D$ which we assume to be a smooth orientable k -dimensional manifold. We denote the spatial labels in this domain by $X \in \mathcal{D}_0$. Moreover, we assume that fluid particles initially at point X evolve under the flow ϕ_t , satisfying the stochastic differential equation (0.2.6) with one noise term (for simplicity), which

we write as

$$d\phi_t(X) = b(t, \phi_t(X))dt + \xi(t, \phi_t(X)) \circ dW_t, \quad (2.2.1)$$

where the coefficients are considered to be smooth. The control volume at time t is given by $\mathcal{D}_t = \phi_t(\mathcal{D}_0)$, and the position at time t of the fluid particle initially at label X is denoted by $x(t, X) = \phi_t(X)$. Assuming that a smooth

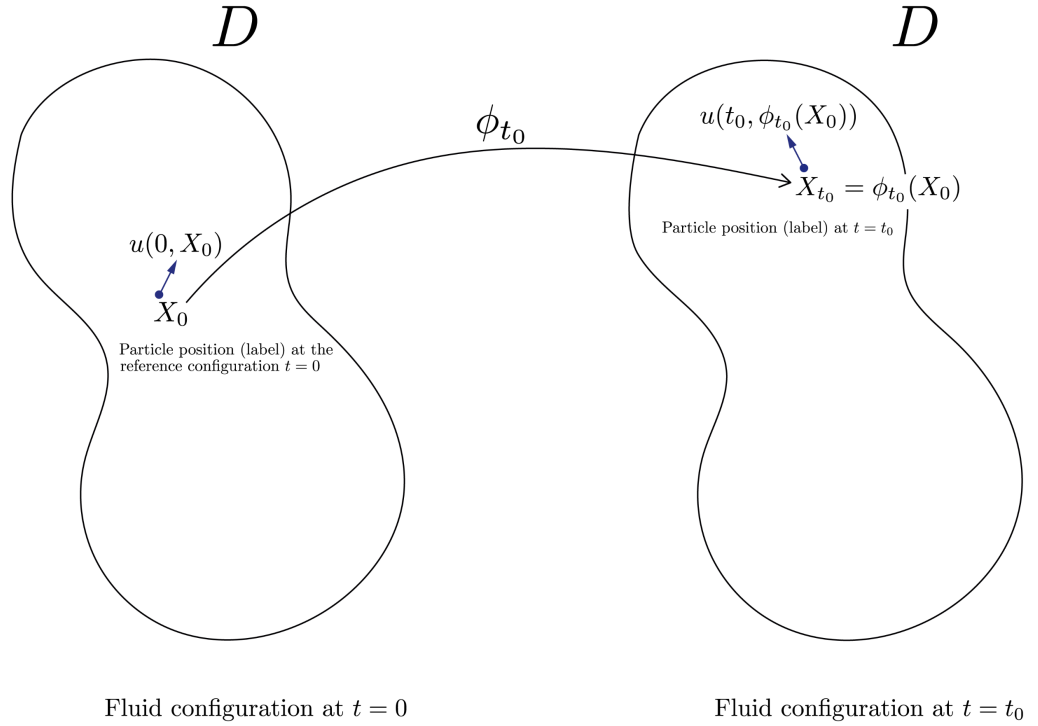


Figure 2.1: Fluid flow at time t_0 on a domain D . $u(t, x)$ denotes the velocity of the flow at time t and position x .

stochastic k -form α satisfies an advection equation of the form (1.4.14) and its integral over the control volume $\mathcal{D}_t$ is conserved with time, i.e.,

$$\int_{\mathcal{D}_t} \alpha(t, x) - \int_{\mathcal{D}_0} \alpha(0, X) = 0,$$

then the previous equation can be rewritten as

$$\int_{\mathcal{D}_0} ((\phi_t^* \alpha)(t, X) - \alpha(0, X)) = 0,$$

after changing variables. We can then apply the KIW formula for k -forms (see Theorem 3) to obtain

$$\begin{aligned} \int_{\mathcal{D}_0} ((\phi_t^* \alpha)(t, X) - \alpha(0, X)) \\ = \int_0^t \int_{\mathcal{D}_0} \phi_s^* (d\alpha + \mathcal{L}_b \alpha ds + \mathcal{L}_\xi \alpha \circ dW_s)(s, X) = 0, \end{aligned}$$

where we have also used Fubini theorem [¶]. Transforming back the coordinates yields

$$\int_0^t \int_{\mathcal{D}_s} (d\alpha(s, x) + \mathcal{L}_b \alpha(s, x) ds + \mathcal{L}_\xi \alpha(s, x) \circ dW_s) = 0,$$

which implies that α satisfies the SPDE

$$d\alpha(t, x) + \mathcal{L}_b \alpha(t, x) dt + \mathcal{L}_\xi \alpha(t, x) \circ dW_t = 0, \quad x \in \mathbb{R}^n, \quad (2.2.2)$$

since the control volume $\mathcal{D}_0$ was chosen arbitrarily. We apply this derivation to some examples.

Example 1: deriving the continuity equation (advection of density per unit mass). As an interesting application of the above, we use conservation of mass to derive a stochastic counterpart of the continuity equation on $\mathbb{R}^3$ (this can be straightforwardly extended to other dimensions but $\mathbb{R}^3$ is the most natural case from a physical point of view). Let $\rho(0, \mathbf{X})$ be the mass density ^{||} in the reference configuration and $\rho(t, \mathbf{x})$ the mass density at time t written in global coordinates, where we employ the bold font notation “ $\mathbf{x}$ ” and “ $\mathbf{X}$ ” to denote the coordinate expression of the points x and X

[¶]We stress again that technicalities are not the concern in this section.

^{||}We remind that the density is a real-valued function in coordinates, and a volume-form as an element.

respectively. Conservation of mass reads

$$\int_{\mathcal{D}_t} \rho(t, \mathbf{x}) d^3 \mathbf{x} - \int_{\mathcal{D}_0} \rho(0, \mathbf{X}) d^3 \mathbf{X} = 0.$$

By applying the argument above, we obtain (2.2.2) for the particular case of $\alpha(t, x) = \rho(t, \mathbf{x}) d^3 \mathbf{x}$, whose Lie derivative is expressed in coordinates as $\mathcal{L}_b \rho = \mathcal{L}_b(\rho(t, \mathbf{x}) d^3 \mathbf{x}) = \text{div}(b(t, \mathbf{x}) \rho(t, \mathbf{x})) d^3 \mathbf{x}$. Thus, we arrive at the stochastic continuity equation

$$d\rho(t, \mathbf{x}) + \text{div}(b(t, \mathbf{x}) \rho(t, \mathbf{x})) dt + \text{div}(\xi(t, \mathbf{x}) \rho(t, \mathbf{x})) \circ dW_t = 0 \quad (2.2.3)$$

in global coordinates. Similar equations can be derived for the advection of entropy per unit mass, which is a scalar function, and magnetic field, which is a two-form, as we did in ⁵⁸. We discuss how to calculate these in the following example.

Example 2: deriving advection of entropy per unit mass and advection of magnetic field. In magnetohydrodynamics (MHD), the real-valued function $S = S(\mathbf{x}, t)$ is typically employed to denote entropy per unit mass whereas $B = B(\mathbf{x}, t) \cdot d\mathbf{S}$ signifies magnetic field, with coordinates $B : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $\mathbf{x} \in \mathbb{R}^3$. We stress that the letter S is both being employed for entropy per unit mass and the surface element $d\mathbf{S}$ (the latter in bold font), which shall not be confused. The natural stochastic counterparts of the equations of advection of entropy per unit mass and magnetic field in MHD, respectively, read as follows

$$dS + \mathcal{L}_u S dt + \mathcal{L}_\xi S \circ dW_t = 0,$$

or expressed in coordinates

$$dS + u \cdot \nabla S dt + u \cdot \nabla \xi \circ dW_t = 0,$$

and

$$dB + \mathcal{L}_u B dt + \mathcal{L}_\xi B \circ dW_t = 0,$$

or expressed in coordinates

$$d(B \cdot d\mathbf{S}) = (\text{curl}(u \times B) \cdot d\mathbf{S})dt + (\text{curl}(\xi \times B) \cdot d\mathbf{S}) \circ dW_t = 0.$$

These equations can be derived through application of Theorem 3 by starting from the conservation laws

$$\int_{\mathcal{D}_t} B(t, \mathbf{x}) \cdot d\mathbf{S} - \int_{\mathcal{D}_0} B(0, \mathbf{X}) \cdot d\mathbf{S} = 0,$$

where $\mathcal{D}_0$ is a smooth orientable two-dimensional manifold, and

$$S(t, \mathbf{x}) - S(0, \mathbf{X}) = 0,$$

respectively. In the latter case, we have

$$S(t, \phi_t(\mathbf{X})) = S(0, \mathbf{X}),$$

which translates to conservation of entropy per unit mass along orbits. In this case, application of Theorem 3 becomes Itô-Wentzell for flows (Theorem 42).

Example 3: deriving Kelvin's circulation theorems. We are interested in showing that Theorem 3 can be applied for deriving a stochastic version of Kelvin's circulation theorems for geometric equations. Kelvin's circulation theorem is a statement of Newton's Force Law for the motion of distributions of mass on closed material loops. In the deterministic case, an illustrative example is the following. Consider the following forced Euler's equation written in geometric form

$$\partial_t(u \otimes \rho) = -\mathcal{L}_u(u \otimes \rho) + F(t, x),$$

where $u(t, x) = u(t, \mathbf{x}) \cdot d\mathbf{x}$ is an n -dimensional velocity, $F(t, x)$ is a one-form density, and the density $\rho(t, x) = \rho(t, \mathbf{x}) d^n \mathbf{x}$ satisfies the equation

$$\partial_t \rho + \mathcal{L}_u \rho = 0.$$

We note that $u \otimes \rho^{**}$ is $\rho(t, \mathbf{x}) u(t, \mathbf{x}) \cdot d\mathbf{x} \otimes d^n \mathbf{x}$ in global coordinates. Then if we define $v(t, x) = u(t, x)$, the following circulation law holds ^{††}

$$\begin{aligned} I_{det}(t) - I_{det}(0) &= \oint_{c_t} v(t, x) - \oint_{c_0} v(0, X) \\ &= \int_0^t \oint_{c_s} \rho^{-1}(s, x) F(s, x) ds. \end{aligned} \quad (2.2.4)$$

Here, c_0 represents a smooth fluid loop and $c_t = \phi_t(c_0)$ is the loop carried along by the deterministic flow at time t . The circulation result (2.2.4) explains that the total circulation is accounted by the integral of the forcing term times the inverse density.

^{**}In differential geometry, these objects are called one-form densities, and are the tensor product of a one form and a density.

^{††}We note that since ρ is a density and F is a one-form density, $\rho^{-1} F$ is a one-form.

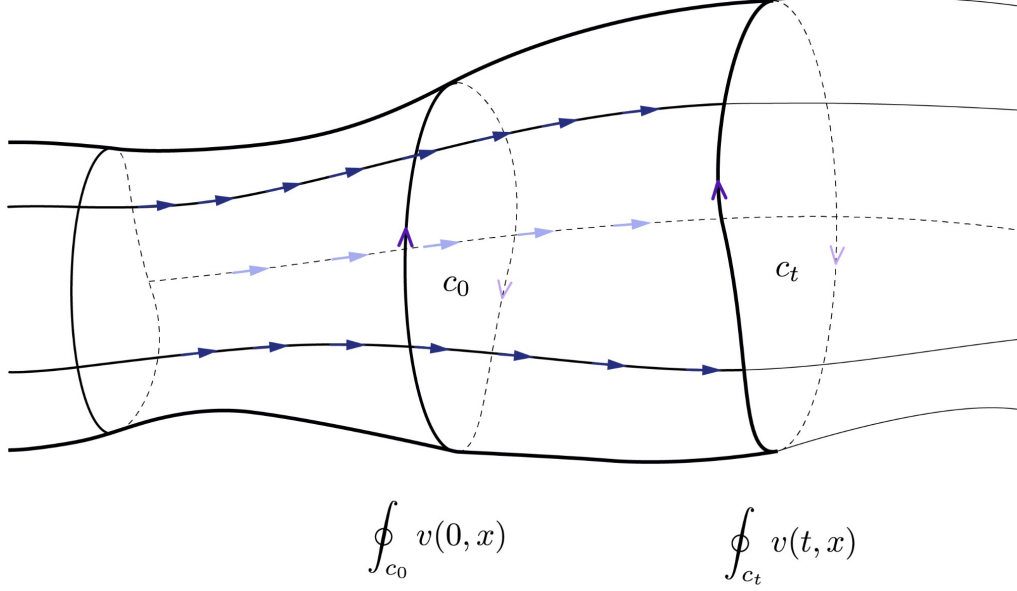


Figure 2.2: The circulation is preserved when $F = 0$.

In the remaining of this paragraph, we provide a proof of the natural stochastic counterpart of the above circulation theorem. Consider the stochastic forced Euler equation

$$d(u \otimes \rho) = -\mathcal{L}_u(u \otimes \rho)dt - \mathcal{L}_\xi(u \otimes \rho) \circ dB_t + F(t, x)dt, \quad (2.2.5)$$

where $u(t, x) = u(t, \mathbf{x}) \cdot d\mathbf{x}$ is an n -dimensional stochastic velocity, $\rho(t, x) = \rho(t, \mathbf{x})d^n\mathbf{x}$ is a stochastic mass density satisfying the equation

$$d\rho + \mathcal{L}_u\rho dt + \mathcal{L}_\xi\rho \circ dW_t = 0, \quad (2.2.6)$$

(which has the same form as the continuity equation we previously derived (2.2.3) written in geometric form), and $F(t, x)$ represents a space-time forcing which can be stochastic (following the philosophy of equation (0.2.1)). Then, upon defining the (stochastic) Kelvin-Noether quantity $v(t, x) = u(t, x)$ (see⁵⁸ for further insight), the following stochastic Kelvin's

circulation theorem holds

$$\begin{aligned} I_{sto}(t) - I_{sto}(0) &= \oint_{c_t} v(t, x) - \oint_{c_0} v(0, X) \\ &= \int_0^t \oint_{c_s} \rho^{-1}(s, x) F(s, x) ds. \end{aligned} \quad (2.2.7)$$

Remark 58. *Note that in the equality above, not only the velocity v is stochastic but also the fluid loop c_t becomes stochastic.*

We proceed to prove formula (2.2.7). From equation (2.2.5), we have

$$\begin{aligned} d(v \otimes \rho) &= dv \otimes \rho + v \otimes d\rho = -\mathcal{L}_u(v \otimes \rho)dt - \mathcal{L}_\xi(v \otimes \rho) \circ dW_t + F(t, x)dt \\ &= (-\mathcal{L}_u v dt - \mathcal{L}_\xi v \circ dW_t + \rho^{-1} F(t, x)dt) \otimes \rho - v \otimes (\mathcal{L}_u \rho dt + \mathcal{L}_\xi \rho \circ dW_t), \end{aligned}$$

where we have taken into account the stochastic product rule in Stratonovich form $d(v \otimes \rho) = dv \otimes \rho + v \otimes d\rho$ and Cartan's formula for Lie derivatives $\mathcal{L}(v \otimes \rho) = \mathcal{L}v \otimes \rho + v \otimes \mathcal{L}\rho$ (see Appendix A). Therefore, we deduce

$$dv(t, x) = -\mathcal{L}_u v(t, x)dt - \mathcal{L}_\xi v(t, x) \circ dW_t + \rho^{-1} F(t, x)dt, \quad (2.2.8)$$

since ρ satisfies the continuity equation (2.2.6). By changing variables, we can express the left-hand side of (2.2.7) as

$$\oint_{c_t} v(t, x) - \oint_{c_0} v(0, X) = \oint_{c_0} (\phi_t^* v(t, X) - v(0, X)).$$

By applying the KIW formula (Theorem 3) under the integral sign, we obtain

$$\begin{aligned} \oint_{c_0} (\phi_t^* v(t, X) - v(0, X)) &= \oint_{c_0} \int_0^t \phi_s^* (dv + \mathcal{L}_b v ds + \mathcal{L}_\xi v \circ dW_s) (s, X) \\ &= \int_0^t \oint_{c_s} (dv + \mathcal{L}_b v ds + \mathcal{L}_\xi v \circ dW_s) (s, x), \end{aligned}$$

where we have changed the coordinates back to the original domain and

applied Fubini. Finally, from the momentum equation (2.2.8), we conclude

$$\oint_{c_t} v(t, x) - \oint_{c_0} v(0, X) = \int_0^t \oint_{c_s} \rho^{-1} F(s, x) ds.$$

Remark 59. *From the stochastic Kelvin's circulation formula (2.2.7), we can deduce that if the fluid is not acted on by external forces, then the circulation $I_{sto}(t) = \oint_{c_t} v_t$ is preserved.*

For a more general circulation theorem in a geometric framework, we point to ⁵⁸.

Example 4: deriving Reynolds transport theorem. We can also derive a stochastic version of Reynolds transport theorem. We first remind the deterministic theorem, which in geometric form can be stated as follows.

Theorem 60 (Reynolds transport). *Let $\mathcal{D}_0 \subset \mathbb{R}^3$ be a smooth orientable manifold and $\phi_t : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ a smooth flow, so that the domain $\mathcal{D}_0$ at time t under the action of the flow becomes $\mathcal{D}_t = \phi_t(\mathcal{D}_0)$. Moreover, let $F(t, x) = F(t, \mathbf{x})d^3\mathbf{x}$ be a smooth volume form (in global coordinates). Then we have*

$$\frac{\partial}{\partial t} \left(\int_{\mathcal{D}_t} F(t, \mathbf{x}) d^3\mathbf{x} \right) = \int_{\mathcal{D}_t} \frac{\partial}{\partial t} F(t, \mathbf{x}) d^3\mathbf{x} + \int_{\partial\mathcal{D}_t} \langle u(t, \cdot), \vec{n} \rangle(\mathbf{x}) F(t, \mathbf{x}) d\mathbf{S},$$

where $u(t, x)$ represents the velocity of the surface and is understood as a one-form, $\vec{n}$ is the vector field normal to the surface $\partial\mathcal{D}_t$, and $\langle \cdot, \cdot \rangle$ represents dual pairing.

In the stochastic case, if we assume that our flow ϕ_t satisfies equation (2.2.1) and our control volume is of the form $\mathcal{D}_t = \phi_t(\mathcal{D}_0)$, we can carry out the

following derivation by application of KIW, Fubini, and divergence theorem

$$\begin{aligned}
\int_{\mathcal{D}_t} F(t, \mathbf{x}) d^3 \mathbf{x} &= \int_{\mathcal{D}_0} (\phi_t^* F)(t, \mathbf{X}) d^3 \mathbf{X} \\
&= \int_0^t \int_{\mathcal{D}_0} \phi_s^* (d(F d^3 \mathbf{X})) + \int_0^t \int_{\mathcal{D}_0} \phi_s^* (\mathcal{L}_b(F d^3 \mathbf{X}) ds + \mathcal{L}_\xi(F d^3 \mathbf{X}) \circ dW_s) \\
&= \int_0^t \int_{\mathcal{D}_t} d(F d^3 \mathbf{x}) + \int_0^t \int_{\mathcal{D}_t} (\mathcal{L}_b(F d^3 \mathbf{x}) ds + \mathcal{L}_\xi(F d^3 \mathbf{x}) \circ dW_s) \\
&= \int_0^t \int_{\mathcal{D}_t} d(F d^3 \mathbf{x}) + \int_0^t \int_{\mathcal{D}_t} (\operatorname{div}(bF) d^3 \mathbf{x} ds + \operatorname{div}(\xi F) d^3 \mathbf{x} \circ dW_s) \\
&= \int_0^t \int_{\mathcal{D}_t} d(F d^3 \mathbf{x}) + \int_0^t \int_{\partial \mathcal{D}_t} (\langle b, \vec{n} \rangle F d^3 \mathbf{x} ds + \langle \xi, \vec{n} \rangle F d^3 \mathbf{x} \circ dW_s),
\end{aligned}$$

where we have omitted the dependence of the functions on the variables $(t, \mathbf{x})$ (respectively $(t, \mathbf{X})$ when integrating along the fixed frame) in order to simplify notation. Hence

$$\begin{aligned}
&\int_{\mathcal{D}_t} F(t, \mathbf{x}) d^3 \mathbf{x} \\
&= \int_0^t \int_{\mathcal{D}_t} d(F d^3 \mathbf{x}) + \int_0^t \int_{\partial \mathcal{D}_t} (\langle b, \vec{n} \rangle F d^3 \mathbf{x} ds + \langle \xi, \vec{n} \rangle F d^3 \mathbf{x} \circ dW_s),
\end{aligned}$$

which provides a stochastic version of Theorem 60.

3

THE BURGERS' EQUATION WITH TRANSPORT NOISE

In this chapter, we consider the Burgers' equation with stochastic transport noise. We note that the results we establish are extensible to cylindrical noise (i.e. a countable number of diffusion terms instead of finitely many ones), but we will not do this here and instead point the reader to⁸. We fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_t)$ satisfying the usual conditions, together with a family $\{W_t^i\}_{i=1}^N$ of independent Brownian processes adapted to $(\mathcal{F}_t)_t$.

Different notions of solution. We start by introducing two different notions of solution of the 1D Burgers' equation with transport noise (0.2.8) that we will employ throughout this chapter. Sometimes we use the notation $u_t(x) = u(t, x)$ as it is common for stochastic processes. For the following definition to make sense, we assume $\xi \in W^{s+1, \infty}(\mathbb{T})$.

Definition 61 (Strong local solution). *A strong H^s -local solution (τ, u) of (0.2.8), $s \geq 2$, is a stochastic process $u : \Omega \times [0, \tau] \times \mathbb{T} \rightarrow \mathbb{R}$, where $\tau : \Omega \rightarrow [0, \infty)$ is a stopping time such that $u_{t \wedge \tau}$ is adapted to $(\mathcal{F}_t)_t$ and we have:*

- $\mathbb{P}$ -a.s., u has paths of class $C([0, \tau]; H^s(\mathbb{T}))$.
- It holds

$$u_{\tau'} - u_0 + \int_0^{\tau'} u_s \partial_x u_s ds + \sum_{i=1}^N \int_0^{\tau'} \xi_i \partial_x u_s dW_s^i = \frac{1}{2} \sum_{i=1}^N \int_0^{\tau'} (\xi_i \partial_x)^2 u_s ds,$$

* as an identity in $L^2(\mathbb{T})$, a.s., for bounded stopping times $\tau' \leq \tau$.

When the stopping time is clear from the context, we simply write that u is a solution.

Definition 62 (Maximal solution). A maximal solution $(\tau_{\max}, u)$ of (0.2.8) is a stochastic process $u : \Omega \times [0, \tau_{\max}) \times \mathbb{T} \rightarrow \mathbb{R}$, for a stopping time $\tau_{\max} : \Omega \rightarrow [0, \infty]$ [†], satisfying the following conditions:

- $\mathbb{P}(\tau_{\max} > 0) = 1$, where $\tau_{\max} = \lim_{n \rightarrow \infty} \tau_n$ for an increasing sequence of stopping times $\{\tau_n\}_{n=1}^\infty$.
- (τ_n, u) is a local solution in the sense of Definition 61, for all $n \in \mathbb{N}$.
- If (τ', u') is another pair satisfying the above two conditions and $u' = u$ on $[0, \tau' \wedge \tau_{\max})$, then $\tau' \leq \tau_{\max}$, a.s.

A maximal solution is said to be global if $\tau_{\max} = \infty$, a.s.

Definition 63 (Weak solution). A (spatially) weak solution (τ, u) of (0.2.8) is a stochastic process $u : \Omega \times [0, \tau] \times \mathbb{T} \rightarrow \mathbb{R}$, where $\tau : \Omega \rightarrow [0, \infty)$ is a stopping time, such that for any test function $\phi \in C^\infty(\mathbb{T})$, $\langle u_{t \wedge \tau}, \phi \rangle_{L^2}$ is adapted to $(\mathcal{F}_t)_t$ and it holds

- $\mathbb{P}$ -a.s. u has paths of class $C([0, \tau]; L^2(\mathbb{T}))$.

*Since s is common notation for the integration parameter for stochastic processes, we employ it as an integration parameter when we think it will not cause any confusion, although we have also used the notation s for H^s . When we suspect there might be confusion, we will employ a different notation.

[†]Unlike a stopping time for a local solution, a stopping time for a maximal solution is allowed to take the value infinity.

- It holds

$$\begin{aligned} \langle u_{\tau'}, \phi \rangle_{L^2} &= \langle u_0, \phi \rangle_{L^2} + \frac{1}{2} \int_0^{\tau'} \langle (u_s)^2, \partial_x \phi \rangle_{L^2} ds \\ &+ \sum_{i=1}^N \int_0^{\tau'} \langle u_s, \partial_x (\xi_i \phi) \rangle_{L^2} dW_s^i + \frac{1}{2} \sum_{i=1}^N \int_0^{\tau'} \langle u_s, \partial_x (\xi_i \partial_x (\xi_i \phi)) \rangle_{L^2} ds, \end{aligned} \quad (3.0.1)$$

a.s., for bounded stopping times $\tau' \leq \tau$.

Remark 64. *It can be shown that a local solution in the sense of Definition 61 is also a weak solution in the sense of Definition 63. Normally, when we talk about a solution to the stochastic Burgers' equation we mean it in the strong sense, otherwise we will use the words weak or mild (if it is a solution in the mild sense), or it will simply be clear from the context.*

3.1 LOCAL WELL-POSEDNESS

This section focuses on proving the local existence of strong solutions of the stochastic Burgers' equation (0.2.8). To be more specific, we establish the local well-posedness of a more general equation, which includes the latter as a special case, since the techniques employed in the proof we provide are essentially the same and we find it more convenient. We follow closely the ideas in⁴⁹, which comprise a natural stochastic extension of the classical deterministic methods appearing for instance in^{109,124}. The stochastic Burgers' equation we treat is the following one

$$du + u \partial_x u dt + \mathcal{Q} u dW_t = \frac{1}{2} \mathcal{Q}^2 u dt, \quad (3.1.1)$$

being $\mathcal{Q}$ a first order linear differential operator of the form

$$\mathcal{Q} = a(x) \partial_x + b(x),$$

where we assume that the coefficients $a, b \in W^{s+1,\infty}(\mathbb{T})$. Our proof is immediately extensible to the case

$$du + u\partial_x u dt + \sum_{i=1}^N \mathcal{Q}_i u dW_t^i = \frac{1}{2} \sum_{i=1}^N \mathcal{Q}_i^2 u dt,$$

with

$$\mathcal{Q}_i = a_i(x)\partial_x + b_i(x), \quad i = 1, \dots, N,$$

and even to the cylindrical noise case under the addition of a few assumptions (we comment on the cylindrical noise case at the end of the section). However, for simplicity, we focus on (3.1.1). The definitions of solution are just the obvious generalisation of the ones introduced above. The main theorem we will demonstrate is the following.

Theorem 65. *Let $u_0 \in H^s(\mathbb{T})$ for some $s > 2$ fixed, then there exists a (pathwise) unique H^s -maximal solution $(\tau_{\max}, u)$ of the extended stochastic Burgers' equation (3.1.1) in the sense of Definition 62 with initial data u_0 . Moreover, either $\tau_{\max} = \infty$ or $\limsup_{t \rightarrow \tau_{\max}} \|u(t)\|_{H^s} = \infty$, a.s.*

Remark 66 (Comparison with the deterministic case). *We note that in the deterministic case, it is sufficient to impose $s > 3/2$ in order to guarantee the local existence of strong solutions. However, in the stochastic case, equation (3.1.1) is only well-defined if $s \geq 2$ due to the presence of the second order Stratonovich-to-Itô correction term.*

Remark 67. *The case $\limsup_{t \rightarrow \tau_{\max}} \|u(t)\|_{H^s} = \infty$ indicates that a singularity forms at the random time $\tau_{\max}$. As mentioned in the introduction, this happens in the deterministic case when the slope of the initial profile is negative. In the last section of this chapter, we will comment on some shock formation results for the stochastic Burgers' equation without showing the proofs.*

We provide a complete outline of the proof of Theorem 65 but omit some details since our argument follows closely the approach developed in⁴⁹. However, the well-posedness proof presented here contains more details than the

sketch we carried out in⁸. We break up our argument into several steps for the sake of exposition clarity.

Pathwise uniqueness of local solutions. To show pathwise uniqueness of local solutions, one argues by contradiction as it is usually done in the PDE case. More concretely, one can prove that any two different local solutions to (3.1.1) defined up to a stopping time must coincide, as explained in the following proposition.

Proposition 68. *Let $s > 2$. Let $\tau : \Omega \rightarrow [0, \infty)$ be a stopping time, and $u_1, u_2 : \Omega \times [0, \tau] \times \mathbb{T} \rightarrow \mathbb{R}$ be two H^s -local solutions of (3.1.1) with the same initial data $u_0 \in H^s(\mathbb{T})$. Then $\mathbb{P}$ -a.s. $u_1 = u_2$ on $[0, \tau]$.*

Proof. We know that up to bounded stopping times $\tau' \leq \tau$, the following relation (which we write in differential notation for simplicity) is satisfied

$$du_j + u_j \partial_x u_j dt + Q u_j dW_t = \frac{1}{2} \mathcal{Q}^2 u_j dt, \quad j = 1, 2.$$

Defining the difference $\tilde{u} = u_1 - u_2$, we infer that

$$d\tilde{u} + u_1 \partial_x \tilde{u} dt + \tilde{u} \partial_x u_2 dt + Q \tilde{u} dW_t = \frac{1}{2} \mathcal{Q}^2 \tilde{u} dt.$$

Therefore, we can write (upon using Itô's lemma for $f(x) = x^2$ in the Hilbert space L^2 , which can be found in¹¹⁷) that

$$\begin{aligned} \frac{1}{2} d\langle \tilde{u}, \tilde{u} \rangle_{L^2} + \langle u_1 \partial_x \tilde{u}, \tilde{u} \rangle_{L^2} dt + \langle \tilde{u} \partial_x u_2, \tilde{u} \rangle_{L^2} dt + \langle Q \tilde{u}, \tilde{u} \rangle_{L^2} dW_t \\ = \frac{1}{2} \langle \mathcal{Q}^2 \tilde{u}, \tilde{u} \rangle_{L^2} dt + \frac{1}{2} \langle Q \tilde{u}, Q \tilde{u} \rangle_{L^2} dt. \end{aligned} \quad (3.1.2)$$

From inequality (1.3.7), we bound the last two terms as

$$\frac{1}{2} \langle \mathcal{Q}^2 \tilde{u}, \tilde{u} \rangle_{L^2} + \frac{1}{2} \langle Q \tilde{u}, Q \tilde{u} \rangle_{L^2} \lesssim \|\tilde{u}\|_{L^2}^2.$$

By using embedding (1.1.2), we can estimate the nonlinear terms in (3.1.2)

as follows

$$|\langle \tilde{u} \partial_x u_2, \tilde{u} \rangle_{L^2}| \lesssim \|u_2\|_{H^s} \|\tilde{u}\|_{L^2}^2,$$

and

$$|\langle u_1 \partial_x \tilde{u}, \tilde{u} \rangle_{L^2}| \lesssim \|u_1\|_{H^s} \|\tilde{u}\|_{L^2}^2,$$

where $s > 2$ and in the second inequality we have integrated by parts. Hence, we obtain

$$d\|\tilde{u}\|_{L^2}^2 + 2\langle Q\tilde{u}, \tilde{u} \rangle_{L^2} dW_t \lesssim (1 + \|u_1\|_{H^s} + \|u_2\|_{H^s}) \|\tilde{u}\|_{L^2}^2 dt, \quad (3.1.3)$$

up to bounded stopping times $\tau' \leq \tau$. For every $n \in \mathbb{N}$, define the stopping time [‡]

$$\tau_n = \inf \left\{ r \geq 0 : \left| \int_0^r \langle Q\tilde{u}, \tilde{u} \rangle_{L^2} dB_s \right| \geq n \right\}.$$

Then it is clear that $\tau_n \rightarrow \tau$ a.s. as $n \rightarrow \infty$ since $\int_0^{t \wedge \tau} \langle Q\tilde{u}, \tilde{u} \rangle_{L^2} dB_s$ is well-defined for any $t > 0$ due to the definition of solution and is hence finite almost surely, so $t \wedge \tau_n \rightarrow t \wedge \tau$, for every $t > 0$. Now, by choosing $t > 0$ and defining

$$Y_t = - \int_0^t (1 + \|u_1\|_{H^s} + \|u_2\|_{H^s}) ds,$$

for any $\omega \in \Omega$, we can rewrite equation (3.1.3) in Grönwall's form as [§]

$$\exp(Y_{t \wedge \tau_n}) \|\tilde{u}_{t \wedge \tau_n}\|_{L^2}^2 \lesssim - \int_0^{t \wedge \tau_n} \exp(Y_s) \langle Q\tilde{u}_s, \tilde{u}_s \rangle_{L^2} dW_s.$$

[‡]The quantity below is a stopping time by the Début theorem.

[§]We have stopped the process by τ_n so that we can apply expectation rigorously. Observe that $\left(\int_0^{t \wedge \tau_n} \langle Q\tilde{u}, \tilde{u} \rangle_{L^2} dB_s \right)_{t \geq 0}$ is a martingale for every $n \in \mathbb{N}$, but $\left(\int_0^{t \wedge \tau} \langle Q\tilde{u}, \tilde{u} \rangle_{L^2} dB_s \right)_{t \geq 0}$ is not necessarily a martingale.

By taking expectation in the above inequality, we obtain

$$\mathbb{E} [\exp(Y_{t \wedge \tau_n}) \|\tilde{u}_{t \wedge \tau_n}\|_{L^2}^2] \lesssim -\mathbb{E} \left[\int_0^{t \wedge \tau_n} \exp(Y_s) \langle Q\tilde{u}_s, \tilde{u}_s \rangle_{L^2} dW_s \right] = 0,$$

for every $n \in \mathbb{N}$ and $t > 0$. We conclude that

$$\mathbb{E} [\exp(Y_{t \wedge \tau_n}) \|\tilde{u}_{t \wedge \tau_n}\|_{L^2}^2] = 0.$$

This implies that for every $n \in \mathbb{N}$ and $t > 0$,

$$\exp(Y_{t \wedge \tau_n}) \|\tilde{u}_{t \wedge \tau_n}\|_{L^2}^2 = 0, \quad a.s.$$

Since $Y_{t \wedge \tau_n}$ is finite we obtain

$$\|\tilde{u}_{t \wedge \tau_n}\|_{L^2}^2 = 0, \quad a.s.,$$

and thus, $u_{1,t \wedge \tau_n} = u_{2,t \wedge \tau_n}$, a.s., $n \in \mathbb{N}$, $t > 0$. By taking $n \rightarrow \infty$ and using continuity of the trajectories, we obtain $u_{1,t \wedge \tau} = u_{2,t \wedge \tau}$, a.s. Finally, since t is arbitrary and solutions have continuous trajectories by definition, we conclude (the stronger property) that $\mathbb{P}$ -a.s. $u_1 = u_2$ on $[0, \tau]$. For this, it would have been enough to have left or right-continuous trajectories. $\square$

Existence and pathwise uniqueness of truncated maximal solutions.

Following the techniques in⁴⁹ for the Euler equation with Lie transport noise, for $r > 0$ we consider the truncated stochastic Burgers' equation

$$du_r + \theta_r(\|\partial_x u_r\|_{L^\infty}) u_r \partial_x u_r dt + Q u_r dW_t = \frac{1}{2} Q^2 u_r dt, \quad (3.1.4)$$

where $\theta_r : [0, \infty) \rightarrow [0, 1]$ is a smooth function such that [¶]

$$\theta_r(x) = \begin{cases} 1, & |x| \leq r, \\ 0, & |x| \geq 2r. \end{cases}$$

[¶]This type of functions are known as “cut-off” in the literature.

As we explain in the following lemma, local solutions of (3.1.1) can be constructed by restricting global solutions of (3.1.4) to a certain stopping time.

Lemma 69. *Fix $r > 0$ and $u_0 \in H^s(\mathbb{T})$, $s > 2$. Let $u_r : \Omega \times [0, \infty) \times \mathbb{T} \rightarrow \mathbb{R}$ be an H^s -global solution of (3.1.4) with initial data u_0 . Consider the stopping time*

$$\tau_r = \inf \left\{ t \geq 0 : \|u_r(t, \cdot)\|_{H^s} \geq \frac{r}{C} \right\},$$

where C is chosen in such a way that the following inequality holds:

$$\|\partial_x u_r\|_{L^\infty} \leq C \|u_r\|_{H^s},$$

which can be guaranteed thanks to the Sobolev embedding (1.1.2). Then (τ_r, u_r) is a local solution to the stochastic Burgers' equation (3.1.1).

Proof of Lemma 69. The proof is straightforward by construction. For any $t \in [0, \tau_r]$, we have that

$$\|\partial_x u_r\|_{L^\infty} \leq C \|u_r\|_{H^s} \leq r,$$

and therefore $\theta_r(\|\partial_x u_r\|_{L^\infty}) = 1$. □

Let us state the result which is the cornerstone for proving existence and uniqueness of maximal solutions of the stochastic Burgers' equation (3.1.1).

Proposition 70. *Given $r > 0$ and $u_0 \in H^s(\mathbb{T})$ for $s > 2$, there exists a path-wise unique H^s -global solution of the truncated stochastic Burgers' equation (3.1.4).*

We first check that once Proposition 70 is proven, Theorem 65 follows immediately by following the techniques in ⁴⁹. Indeed, by choosing $r = n \in \mathbb{N}$ in Lemma 69, we have that (τ_n, u_n) is a local solution of the stochastic Burgers' equation (3.1.1). Let us define $\tau_{max} = \lim_{n \rightarrow \infty} \tau_n$. This is well-defined because of the fact that for $n \geq m$, u_n coincides with u_m on $[0, \tau_m)$ due to uniqueness. We also define the function u on $[0, \tau_{max})$ by the relation $u = u_n$

on $[0, \tau_n)$, $n \in \mathbb{N}$.

We check that either $\tau_{max} = \infty$ or $\limsup_{s \rightarrow \tau_{max}} \|u(s)\|_{H^s} = \infty$, a.s. Indeed, assume that $\tau_{max} < \infty$. Then by continuity of u , for every $n \in \mathbb{N}$, there exists some stopping time $\tau'_n < \tau_n$ satisfying $|\tau'_n - \tau_n| \leq 1/n$ and

$$\|u(\tau'_n)\|_{H^s} \geq (n-1)/C.$$

Hence, we have that

$$\limsup_{s \rightarrow \tau_{max}} \|u(s)\|_{H^s} \geq \limsup_{n \rightarrow \infty} \|u(\tau'_n)\|_{H^s} = \infty.$$

Now we show that (τ_{max}, u) is a maximal solution. For this, assume by contradiction that there exists another solution (τ', u') such that $u' = u$ on $[0, \tau' \wedge \tau_{max})$, with $\tau' > \tau_{max}$ on a set with positive probability. This is only possible if $\tau_{max} < \infty$ and therefore, on the set $\{\tau' > \tau_{max}\}$ we have

$$\begin{aligned} \infty &= \limsup_{n \rightarrow \infty} \|u(\tau_n)\|_{H^s} = \limsup_{n \rightarrow \infty} \|u'(\tau_n)\|_{H^s} \\ &= \|u'(\tau_{max})\|_{H^s} < \infty, \end{aligned}$$

since u' is continuous on $[0, \tau')$, leading to a contradiction. We conclude that $\tau' \leq \tau_{max}$ and therefore (τ_{max}, u) is a maximal solution. Finally, suppose that there exists another maximal solution (τ, u^*) with the same initial condition u_0 . Due to the uniqueness result of Proposition 68, one deduces that $u = u^*$, on $[0, \tau \wedge \tau_{max})$. By a similar argument as the one presented above, the inequality $\tau_{max} < \tau$ cannot hold on any set with positive probability, so $\tau \leq \tau_{max}$.

From this point on, we focus our efforts on demonstrating Proposition 70. We note that in order to rigorously conclude the uniqueness result in Proposition 70, one would first need to prove uniqueness of solutions of the truncated equation (3.1.4). Since in this section of the dissertation we only wish to provide an overview of the techniques applied without going into excessive detail, and for simplicity reasons, we omit this proof and refer to the ideas in ^{49,6}. We remark that the proof is mainly based on H^s -estimates that follow a similar

strategy to the a priori ones we have already provided for uniqueness, but is more involved due to the presence of the cut-off term.

Global existence of solutions of the hyper-regularised equation. In order to show the global well-posedness of the truncated equation, we consider the following hyper-regularisation of (3.1.4)

$$du_r^\nu + \theta_r(\|\partial_x u_r^\nu\|_{L^\infty})u_r^\nu \partial_x u_r^\nu dt + \mathcal{Q}u_r^\nu dW_t = \nu \partial_{xx}^{s'} u_r^\nu dt + \frac{1}{2} \mathcal{Q}^2 u_r^\nu dt, \quad (3.1.5)$$

where $\nu > 0$ is a positive parameter and $s' = 2[s] + 1$. Notice that we have added dissipation in order to be able to perform our computations rigorously. Equation (3.1.5) is understood in the mild sense, i.e., as a solution to an integro-differential equation, which we discuss in the next step (see (3.1.6)).

Proposition 71. *For every $\nu, r > 0$ and initial data $u_0 \in H^s(\mathbb{T})$ for $s > 2$, there exists a pathwise unique global strong solution u_r^ν of equation (3.1.5) in the class $L^2(\Omega; C([0, T]; H^s(\mathbb{T})))$, for all $T > 0$. Moreover, its paths gain instant extra regularity $C([\delta, T]; H^{s+2}(\mathbb{T}))$, for every $T > \delta > 0$.*

Remark 72. *Instant regularisation mechanisms as in Proposition 71 are common in parabolic equations. We will provide some insight into this in Chapter 5.*

Proof of Proposition 71. The proof is based on a classical fixed point iteration argument which employs Duhamel's principle (see^{49,84}). We omit the subscripts ν and r throughout the proof for simplicity, but it should be kept in mind that our functions depend on those parameters. Given $u_0 \in H^s(\mathbb{T})$, consider the mild formulation of the hyper-regularised truncated equation (3.1.5)

$$u(t) = (\Upsilon u)(t), \quad (3.1.6)$$

where

$$\begin{aligned} (\Upsilon u)(t) &= e^{tA}u_0 - \int_0^t e^{(t-s)A}W_\theta u(s)ds \\ &\quad + \int_0^t e^{(t-s)A}Lu(s)ds - \int_0^t e^{(t-s)A}Ru(s)dW_s, \end{aligned}$$

$t > 0$ and we have employed the notation $A = \nu \partial_{xx}'$, $W_\theta u = \theta_r(\|\partial_x u\|_{L^\infty})u \partial_x u$, $Lu = \frac{1}{2}\mathcal{Q}^2 u$, and $Ru = \mathcal{Q}u$. Define the space $\mathcal{W}_T = L^2(\Omega; C([0, T]; H^s(\mathbb{T})))$. One can show that Υ is a contraction on $\mathcal{W}_T$ by following classical arguments as in⁴⁹. Therefore, by applying Picard's iteration, a local solution can be constructed. To extend it to a global one, it is sufficient to show that for any given $T > 0$ and initial data $u_0 \in H^s(\mathbb{T})$, $s > 2$, the following bound is available

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|u(t)\|_{H^s}^2 \right] \leq C(T), \quad (3.1.7)$$

for a finite constant $C(T) < \infty$, so that one can patch together each local solution to cover any finite time interval $[0, T]$. We will prove estimate (3.1.7) further below. Furthermore, by standard properties of the semigroup e^{tA} (cf.⁹⁴), one can prove that for positive times $T > \delta > 0$, each term in the mild equation (3.1.6) enjoys higher regularity, namely, $u \in L^2(\Omega; C([\delta, T]; H^{s+2}(\mathbb{T})))$. All the computations are omitted and can be carried out easily by mimicking the same ideas as in^{49,6}. $\square$

Limiting procedure and compactness argument. We show that the family of solutions $\{u_r^\nu\}_{\nu>0}$ to the hyper-regularised equation (3.1.5) is compact in some sense, which will enable us to extract a subsequence converging strongly to a solution of the truncated stochastic Burgers' equation (3.1.4) in a convenient space. The central idea for proving this is to show compactness of the probability laws of this family. Consequently, we demonstrate that these laws are tight in a suitable metric space. Let $T > 0$ and define the Polish space E by

$$E = C([0, T]; H^\beta(\mathbb{T})), \quad \beta \geq 2. \quad (3.1.8)$$

Assume that the laws of $\{u_r^\nu\}_{\nu>0}$ are tight in E . Then, Theorem 52 can be applied, allowing us to extract a weakly converging subsequence which we denote by $\{u_r^{1/n}\}_{n=1}^\infty$ without loss of generality. However, a stronger type of convergence is needed so that we can carry out our limiting argument successfully. To this purpose, we use the Gyöngy-Krylov lemma (Theorem 54), which guarantees that the sequence $\{u_r^{1/n}\}_{n=1}^\infty$ converges in probability if a diagonal assumption is met. This latter assumption can be checked by applying Theorem 53 to the sequence $\{u_r^{1/n}\}_{n=1}^\infty$ and finding a probability space and a family of probability measures $\{u_r^{1/n}\}_{n=1}^\infty$ with the same laws, where the convergence is now $\mathbb{P}$ -a.s.⁴⁹. Moreover, $\{u_r^{1/n}\}_{n=1}^\infty$ can be easily shown to satisfy equation (3.1.5) weakly. Although we have not gone into extreme detail at some of the points in this paragraph, this is based on standard and classical stochastic partial differential equations arguments. A more detailed explanation can be found in^{49, 93}.

We know that $\mathbb{P}$ -a.s. $\{u_r^{1/n}\}_{n=1}^\infty$ converges in E as $n \rightarrow \infty$. Let us denote its limit by u'_r . We claim that u'_r satisfies equation (3.1.4) in the weak sense explained in Definition 63. We show this now and then argue that since u'_r has paths of class $C([0, T]; H^\beta(\mathbb{T}))$, the equation is satisfied in the strong sense. Below, we drop the “prime” notation for simplicity purposes, but if we wanted to be completely rigorous we would need to write u' instead of u . We need to check weak convergence of $u_r^{1/n}$ in the equation

$$\begin{aligned} du_r^{1/n} + \theta_r(\|\partial_x u_r^{1/n}\|_{L^\infty}) u_r^{1/n} \partial_x u_r^{1/n} dt + Q u_r^{1/n} dW_t \\ = (1/n) \partial_{xx}^{s'} u_r^{1/n} dt + \frac{1}{2} Q^2 u_r^{1/n} dt. \end{aligned}$$

In other words, we want to establish that

$$\begin{aligned} \langle u_r^{1/n}, \phi \rangle_{L^2} &= \langle u_0, \phi \rangle_{L^2} + \frac{1}{2} \theta_r(\|\partial_x u_r^{1/n}\|_{L^\infty}) \int_0^t \langle (u_r^{1/n})^2, \partial_x \phi \rangle_{L^2} ds \\ &\quad - \int_0^t \langle u_r^{1/n}, Q^* \phi \rangle_{L^2} dW_s + \frac{1}{2} \int_0^t \langle u_r^{1/n}, (Q^2)^* \phi \rangle_{L^2} ds \\ &\quad + (1/n) \int_0^t \langle u_r^{1/n}, \partial_{xx}^{s'} \phi \rangle_{L^2} ds \end{aligned}$$

converges to

$$\begin{aligned}\langle u_r, \phi \rangle_{L^2} &= \langle u_0, \phi \rangle_{L^2} + \frac{1}{2} \theta_r (\|\partial_x u_r\|_{L^\infty}) \int_0^t \langle (u_r)^2, \partial_x \phi \rangle_{L^2} ds \\ &\quad - \int_0^t \langle u_r, Q^* \phi \rangle_{L^2} dW_s + \frac{1}{2} \int_0^t \langle u_r, (Q^2)^* \phi \rangle_{L^2} ds,\end{aligned}$$

for every $\phi \in C^\infty(\mathbb{T})$, where $t \in [0, T]$. We analyse the convergence of each term separately.

- The term on the left-hand side. We have $\mathbb{P}$ -a.s.

$$\langle u_r^{1/n}, \phi \rangle_{L^2} \rightarrow \langle u_r, \phi \rangle_{L^2},$$

for any $t \in [0, T]$, as $n \rightarrow \infty$, by observing that

$$|\langle u_r^{1/n} - u_r, \phi \rangle_{L^2}| \leq \|u_r^{1/n} - u_r\|_{L^2} \|\phi\|_{L^2} \rightarrow 0,$$

where we have taken into account pathwise convergence in $L^2(\mathbb{T})$.

- The viscosity term $(1/n) \partial_{xx}^{s'} u_r^{1/n}$. Use pathwise convergence in $L^1([0, T]; L^2(\mathbb{T})) \supset E$ to obtain that $\mathbb{P}$ -a.s. for any $t \in [0, T]$

$$\left| \frac{1}{n} \int_0^t \langle u_r^{1/n}, \partial_{xx}^{s'} \phi \rangle_{L^2} ds \right| \leq \frac{1}{n} \|\partial_{xx}^{s'} \phi\|_{L^2} \int_0^t \|u_r^{1/n}\|_{L^2} ds \rightarrow 0.$$

- The single derivative term $Q u_r^{1/n}$. We have that $\mathbb{P}$ -a.s.

$$\langle u_r^{1/n} - u_r, Q^* \phi \rangle_{L^2} \lesssim \langle u_r^{1/n} - u_r, \partial_x \phi \rangle_{L^2},$$

by taking into account that Q has bounded coefficients in $W^{1,\infty}(\mathbb{T})$. Hence by using pathwise convergence in $L^1([0, T]; L^2(\mathbb{T}))$, we get that $\mathbb{P}$ -a.s. for $t \in [0, T]$

$$\int_0^t \langle u_r^{1/n} - u_r, Q^* \phi \rangle_{L^2} ds \lesssim \|\phi\|_{H^1} \int_0^t \|u_r^{1/n} - u_r\|_{L^2} ds \rightarrow 0.$$

- The second order term $Q^2 u_r^{1/n}$. By similar arguments, we obtain that

$\mathbb{P}$ -a.s. for $t \in [0, T]$

$$\int_0^t \langle u_r^{1/n} - u_r, (Q^2)^* \phi \rangle_{L^2} ds \lesssim \|\phi\|_{H^2} \int_0^t \|u_r^{1/n} - u_r\|_{L^2} ds \rightarrow 0.$$

(use the boundedness of the coefficients of Q in $W^{2,\infty}(\mathbb{T})$).

- The nonlinear term $\theta_r(\|\partial_x u_r^{1/n}\|_{L^\infty}) u_r^{1/n} \partial_x u_r^{1/n}$. First, we observe that $\mathbb{P}$ -a.s.

$$\langle (u_r^{1/n})^2, \partial_x \phi \rangle_{L^2} \rightarrow \langle (u_r)^2, \partial_x \phi \rangle_{L^2},$$

in $L^\infty([0, T])$, by pathwise convergence in $C([0, T]; L^2(\mathbb{T})) \supset E$. We argue that

$$\theta_r(\|\partial_x u_r^{1/n}\|_{L^\infty}) \rightarrow \theta_r(\|\partial_x u_r\|_{L^\infty})$$

in $L^\infty([0, T])$. For this, take into account the fact that θ_r is bounded continuous and apply pathwise convergence in the space $C([0, T]; C^1(\mathbb{T}))$ (use embedding (1.1.6)), so $\mathbb{P}$ -a.s.

$$\theta_r(\|\partial_x u_r^{1/n}\|_{L^\infty}) \rightarrow \theta_r(\|\partial_x u_r\|_{L^\infty})$$

in $L^\infty([0, T])$. Hence $\mathbb{P}$ -a.s.

$$\begin{aligned} \int_0^t \theta_r(\|\partial_x u_r^{1/n}\|_{L^\infty}) \langle (u_r^{1/n})^2, \partial_x \phi \rangle_{L^2} ds \\ \rightarrow \int_0^t \theta_r(\|\partial_x u_r\|_{L^\infty}) \langle (u_r)^2, \partial_x \phi \rangle_{L^2} ds. \end{aligned}$$

- The local martingale term $Q u_r^{1/n} dW_t$. We have that

$$\int_0^t \langle u_r^{1/n} - u, Q^* \phi \rangle_{L^2} dW_s \rightarrow 0$$

in probability, by stochastic dominated convergence (see Theorem 50), where we have taken into account pathwise convergence in the space $L^2([0, T]; L^2(\mathbb{T}))$.

We have hence proven that solutions of equation (3.1.5) converge to solutions of (3.1.4) in the weak limit in the Polish space E (3.1.8). As already pointed, the first step to carry out these arguments relies on showing that the family of laws of $\{u_r^\nu\}_{\nu>0}$ is tight in E . This result is encapsulated in the following proposition.

Proposition 73. *Assume that for some $\alpha > 0$, $M \in \mathbb{N}$, there exist constants $C_1(T)$ and $C_2(T)$ such that*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|u_r^\nu(t)\|_{H^s}^4 \right] \leq C_1(T), \quad (3.1.9)$$

$$\mathbb{E} \left[\int_0^T \int_0^T \frac{\|u_r^\nu(t) - u_r^\nu(s)\|_{H^{-M}}^4}{|t - s|^{1+4\alpha}} dt ds \right] \leq C_2(T), \quad (3.1.10)$$

uniformly in ν . Then the sequence $\{u_r^\nu\}_{\nu>0}$ is tight in E .

Proof of Proposition 73. We employ the following lemma, which can be found in ⁴⁹, was originally proved in ¹⁴⁹, and constitutes a variation of the classical Aubin-Lions Lemma (see ¹²³).

Lemma 74. *Suppose that X, Y, Z are separable Hilbert spaces with continuous dense embedding $X \hookrightarrow Y \hookrightarrow Z$ such that there exists $\theta \in (0, 1)$ and $M > 0$ verifying*

$$\|v\|_Y \leq M \|v\|_X^{1-\theta} \|v\|_Z^\theta,$$

for all $v \in X$. Assume that $X \hookrightarrow Y$ is a compact embedding. Let $\alpha > 0$. Then

$$L^\infty([0, T]; X) \cap W^{\alpha, 4}([0, T]; Z) \hookrightarrow C([0, T]; Y)$$

is a compact embedding.

In Lemma 74 we select

$$X = H^s(\mathbb{T}), \quad Y = H^\beta(\mathbb{T}), \quad Z = H^{-M}(\mathbb{T}),$$

where $\beta \geq 2$ as specified before and we impose the extra condition $\beta < s$ so that the embedding of X into Y is compact. We also choose $\alpha \in (0, 1)$. Therefore we obtain that $L^\infty([0, T]; H^s(\mathbb{T})) \cap W^{\alpha, 4}([0, T]; H^{-M}(\mathbb{T}))$ is compactly embedded in E . By hypotheses (3.1.9)-(3.1.10)⁴⁹ (take into account Remark 7), the family of laws of $\{u_r^\nu\}_{\nu > 0}$ is $\mathbb{P}$ -a.s. supported on the space

$$E_0 = L^\infty([0, T]; H^s(\mathbb{T})) \cap W^{\alpha, 4}([0, T]; H^{-M}(\mathbb{T})),$$

and it suffices to prove that this family is tight in E_0 . For $A_1, A_2, A_3 > 0$, define the set

$$B = \left\{ f : [0, T] \times \mathbb{T} \rightarrow \mathbb{R} : \sup_{t \in [0, T]} \|f(t)\|_{H^s}^4 \leq A_1, \int_0^T \|f(t)\|_{H^{-M}}^4 dt \leq A_2, \right. \\ \left. \int_0^T \int_0^T \frac{\|f(t) - f(s)\|_{H^{-M}}^4}{|t - s|^{1+4\alpha}} dt ds \leq A_3 \right\},$$

which is relatively compact in E_0 . It is enough to show that for every ϵ , there exist $A_1, A_2, A_3 > 0$ such that $\mathbb{P}(u_r^\nu \in B^c) \leq \epsilon$. Fix $\epsilon > 0$. Invoking Markov's inequality and taking into account hypothesis (3.1.9), we have that

$$\mathbb{P} \left(\sup_{t \in [0, T]} \|u_r^\nu(t)\|_{H^s}^4 > A_1 \right) \leq \frac{\mathbb{E} [\sup_{t \in [0, T]} \|u_r^\nu(t)\|_{H^s}^4]}{A_1} \leq \frac{C_1(T)}{A_1},$$

and this is smaller than $\epsilon/3$ if we choose A_1 sufficiently large. Moreover, since $\|f\|_{H^{-M}} \lesssim \|f\|_{H^s}$,

$$\mathbb{P} \left(\int_0^T \|u_r^\nu(t)\|_{H^{-M}}^4 dt > A_2 \right) \leq \mathbb{P} \left(T \sup_{t \in [0, T]} \|u_r^\nu(t)\|_{H^{-M}}^4 > A_2 \right) \\ \lesssim \mathbb{P} \left(T \sup_{t \in [0, T]} \|u_r^\nu(t)\|_{H^s}^4 > A_2 \right) \leq \frac{C_1(T)T}{A_2},$$

which can also be made arbitrarily small. A similar argument applies to the set

$$\mathbb{P} \left(\int_0^T \int_0^T \frac{\|u_r^\nu(t) - u_r^\nu(s)\|_{H^{-M}}^4}{|t-s|^{1+4\alpha}} dt ds > A_3 \right),$$

by using hypothesis (3.1.10). Hence we conclude that $\mathbb{P}(u_r^\nu \in B^c) \leq \epsilon$ if we choose A_1, A_2, A_3 large enough, as desired. $\square$

Hypotheses estimates in Proposition 73. We need to show that hypotheses (3.1.9)-(3.1.10) hold. First, we prove that condition (3.1.9) implies condition (3.1.10). By following the techniques in⁴⁹ and using equation (3.1.5), one can obtain

$$\begin{aligned} \mathbb{E} [\|u_r^\nu(t) - u_r^\nu(s)\|_{H^{-M}}^4] &\lesssim (t-s)^3 \int_s^t \mathbb{E} [\theta_r(\|\partial_x u_r^\nu\|_{L^\infty})^4 \|u_r^\nu \partial_x u_r^\nu\|_{H^{-M}}^4] d\gamma \\ &+ (t-s)^3 \int_s^t \mathbb{E} [\nu \|\partial_{xx}^{s'} u_r^\nu\|_{H^{-M}}^4] d\gamma + (t-s)^3 \int_s^t \mathbb{E} [\|\mathcal{Q}^2 u_r^\nu\|_{H^{-1}}^4] d\gamma \\ &+ \mathbb{E} \left[\left\| \int_s^t \mathcal{Q} u_r^\nu dW_\gamma \right\|_{L^2}^4 \right], \end{aligned}$$

for $0 \leq s < t \leq T$. It is easy to infer that

$$\int_s^t \mathbb{E} [\theta_r(\|\partial_x u_r^\nu\|_{L^\infty})^4 \|u_r^\nu \partial_x u_r^\nu\|_{H^{-M}}^4] d\gamma \lesssim \int_s^t \mathbb{E} [\|u_r^\nu\|_{H^s}^4] d\gamma \leq A_1(t-s),$$

since

$$\|u_r^\nu \partial_x u_r^\nu\|_{H^{-M}} \lesssim \|\partial_x u_r^\nu\|_{L^\infty} \|u_r^\nu\|_{H^s},$$

where we have taken into account hypothesis (3.1.9). In a similar way, one can check that for $M = 3[s'] + 2$,

$$\int_s^t \mathbb{E} [\|\nu \partial_{xx}^{s'} u_r^\nu\|_{H^{-M}}^4] d\gamma \lesssim \int_s^t \mathbb{E} [\|u_r^\nu\|_{H^s}^4] d\gamma \leq A_2(t-s),$$

since

$$\|\partial_{xx}^{s'} u_r^\nu\|_{H^{-M}} \lesssim \|u_r^\nu\|_{H^s}.$$

By similar techniques, we have

$$\int_s^t \mathbb{E} [\|\mathcal{Q}^2 u_r^\nu\|_{H^{-1}}^4] d\gamma \leq A_3(t-s),$$

and (take into account Burkholder-Davis-Gundy as in ⁴⁹)

$$\mathbb{E} \left[\left\| \int_s^t \mathcal{Q} u_r^\nu dW_\gamma \right\|_{L^2}^4 \right] \leq A_4(t-s)^2.$$

Combining the above estimates, we deduce that

$$\mathbb{E} [\|u_r^\nu(t) - u_r^\nu(s)\|_{H^{-M}}^4] \leq A_5(T)(t-s)^2.$$

Hence for $0 < \alpha < 1/2$,

$$\begin{aligned} \mathbb{E} \left[\int_0^T \int_0^T \frac{\|u_r^\nu(t) - u_r^\nu(s)\|_{H^{-M}}^4}{|t-s|^{1+4\alpha}} dt ds \right] &\leq \mathbb{E} \left[\int_0^T \int_0^T \frac{A_5(T)}{|t-s|^{4\alpha-1}} dt ds \right] \\ &\leq C_1(T). \end{aligned}$$

We are left to prove that hypothesis (3.1.9) holds, i.e.

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|u_r^\nu(t)\|_{H^s}^4 \right] \leq C_2(T). \quad (3.1.11)$$

This is the most difficult part of the hypotheses estimates. We compute the evolution of the $\dot{H}^s$ -norm of u_r^ν . Indeed, the equation for the L^2 -norm of

$\Lambda^s u_r^\nu$ in (3.1.5) is

$$\begin{aligned}
\frac{1}{2}|\Lambda^s u_r^\nu(t)|_{L^2}^2 &= \frac{1}{2}|\Lambda^s u_r^\nu(0)|_{L^2}^2 \\
&\quad - \int_0^t \langle \theta_r(\|\partial_x u_r^\nu\|_{L^\infty}) \Lambda^s(u_r^\nu \partial_x u_r^\nu), \Lambda^s u_r^\nu \rangle_{L^2} d\gamma \\
&\quad - \int_0^t \langle \Lambda^s \mathcal{Q} u_r^\nu, \Lambda^s u_r^\nu \rangle_{L^2} dW_\gamma + \int_0^t \langle \nu \Lambda^s \partial_{xx}^{s'} u_r^\nu, \Lambda^s u_r^\nu \rangle_{L^2} d\gamma \\
&\quad + \frac{1}{2} \int_0^t \langle \Lambda^s \mathcal{Q}^2 u_r^\nu, \Lambda^s u_r^\nu \rangle_{L^2} d\gamma + \frac{1}{2} \int_0^t \langle \Lambda^s \mathcal{Q} u_r^\nu, \Lambda^s \mathcal{Q} u_r^\nu \rangle_{L^2} d\gamma,
\end{aligned} \tag{3.1.12}$$

for $t \in [0, T]$. The nonlinear term can be estimated as

$$\begin{aligned}
\int_{\mathbb{T}} \Lambda^s(u_r^\nu \partial_x u_r^\nu) \Lambda^s u_r^\nu dV &= \int_{\mathbb{T}} (u_r^\nu \Lambda^s \partial_x u_r^\nu) \Lambda^s u_r^\nu dV + \int_{\mathbb{T}} [\Lambda^s, u_r^\nu] \partial_x u_r^\nu \Lambda^s u_r^\nu dV \\
&= I_1 + I_2.
\end{aligned}$$

Integrating by parts, we note that I_1 can be rewritten and bounded as

$$|I_1| = \frac{1}{2} \left| \int_{\mathbb{T}} \partial_x u_r^\nu (\Lambda^s u_r^\nu)^2 dV \right| \leq \|\partial_x u_r^\nu\|_{L^\infty} \|\Lambda^s u_r^\nu\|_{L^2}^2.$$

I_2 can be estimated via the Kato-Ponce commutator estimate (1.1.7)

$$\begin{aligned}
|I_2| &\leq \|[\Lambda^s, u_r^\nu] \partial_x u_r^\nu\|_{L^2} \|\Lambda^s u_r^\nu\|_{L^2} \\
&\lesssim (\|\partial_x u_r^\nu\|_{L^\infty} \|\Lambda^{s-1} \partial_x u_r^\nu\|_{L^2} + \|\Lambda^s u_r^\nu\|_{L^2} \|\partial_x u_r^\nu\|_{L^\infty}) \|\Lambda^s u_r^\nu\|_{L^2} \\
&\lesssim \|\partial_x u_r^\nu\|_{L^\infty} \|\Lambda^s u_r^\nu\|_{L^2}^2.
\end{aligned}$$

Putting together the estimates for I_1 and I_2 , we get

$$\left| \int_{\mathbb{T}} \Lambda^s(u_r^\nu \partial_x u_r^\nu) \Lambda^s u_r^\nu dV \right| \lesssim \|\partial_x u_r^\nu\|_{L^\infty} \|\Lambda^s u_r^\nu\|_{L^2}^2.$$

We note that the dissipative term

$$\langle \nu \Lambda^s \partial_{xx}^{s'} u_r^\nu, \Lambda^s u_r^\nu \rangle_{L^2} = -\nu \|\Lambda^{s+2[s]+1} u_r^\nu\|_{L^2}^2 \leq 0$$

has the “good” sign so it can be dropped. The last two terms in (3.1.12) can be treated in a similar fashion by taking into account estimate (1.3.8). The previous techniques also serve to treat the evolution of $\|u_r^\nu\|_{L^2}$ (this is a much simpler case).

Denote

$$M_t = \int_0^t (\langle \mathcal{Q}u_r^\nu, u_r^\nu \rangle_{L^2} + \langle \Lambda^s \mathcal{Q}u_r^\nu, \Lambda^s u_r^\nu \rangle_{L^2}) dW_\gamma, \quad t \in [0, T].$$

We drop the parameter dependence on u_r^ν to make the notation simpler. By putting together the previous estimates, we have

$$\|u(t)\|_{H^s}^2 \lesssim \|u(0)\|_{H^s}^2 + |M_t| + \int_0^t \|u\|_{H^s}^2 d\gamma,$$

where the constant in the last inequality depends on r . Integrating the above expression against $\exp(t)$ and squaring as in⁴⁹, we obtain

$$\|u(t)\|_{H^s}^4 \lesssim \exp(t)(\|u(0)\|_{H^s}^4 + |M_t|^2).$$

By applying supremum and expectation on both sides of the equation above, we have

$$\mathbb{E} \left[\sup_{\gamma \in [0, t]} \|u(\gamma)\|_{H^s}^4 \right] \lesssim \exp(t) \left(\|u(0)\|_{H^s}^4 + \mathbb{E} \left[\sup_{\gamma \in [0, t]} |M_\gamma|^2 \right] \right), \quad (3.1.13)$$

where we remind that $t \in [0, T]$. We apply Burkholder-Davis-Gundy inequality (1.4.17) in order to obtain the bound

$$\mathbb{E} \left[\sup_{\gamma \in [0, t]} |M_\gamma|^2 \right] \lesssim \mathbb{E}[[M]_t], \quad (3.1.14)$$

where

$$[M]_t = \int_0^t (\langle \mathcal{Q}u, u \rangle_{L^2} + \langle \Lambda^s \mathcal{Q}u, \Lambda^s u \rangle_{L^2})^2 d\gamma.$$

Integrating by parts, we can derive

$$|\langle \Lambda^s \mathcal{Q}u, \Lambda^s u \rangle_{L^2}| \lesssim \|u\|_{H^s}^2, \quad (3.1.15)$$

as in the proof of (1.3.8) in Appendix B or in⁶. The constant in the last inequality depends on the $W^{s+1,\infty}$ -norm of the coefficients. Therefore,

$$\mathbb{E}[[M]_t] \lesssim \int_0^t \mathbb{E} \left[\sup_{r \in [0,t]} \|u(r)\|_{H^s}^4 \right] d\gamma. \quad (3.1.16)$$

Hence, combining (3.1.13)-(3.1.16), together with application of Grönwall's inequality yields

$$\mathbb{E} \left[\sup_{t \in [0,T]} \|u(t)\|_{H^s}^4 \right] \leq C_2(T).$$

3.2 BLOW-UP CRITERION

In this section, we derive a blow-up criterion for our Burgers' equation with generalised noise (3.1.1). First of all, we note that for the deterministic Burgers' equation

$$u_t + u\partial_x u = 0, \quad (3.2.1)$$

there is a well-known blow-up criterion available. In the deterministic case, the following theorem of local existence and uniqueness of strong solutions holds.

Theorem 75. *Let $u_0 \in H^s(\mathbb{T})$ with $s > 3/2$. Then there exists $T > 0$ and $u \in C([0, T]; H^s(\mathbb{T})) \cap C^1([0, T]; H^{s-1}(\mathbb{T}))$ solving equation (3.2.1). Moreover, if v is another function with the same properties, necessarily $u = v$.*

The above result is classic and numerous proofs are available in the literature (see¹¹³ for a related cutting edge result and the references therein). The blow-up criterion for equation (3.2.1) is the following.

Theorem 76 (Blow-up criterion for deterministic Burgers'). *Assume that $u_0 \in H^s(\mathbb{T})$, $s > 3/2$, $T^* > 0$, and $u : [0, T^*) \times \mathbb{T} \rightarrow \mathbb{R}$ is a local solution of (3.2.1). Then the following statements are equivalent*

- $\lim_{t \rightarrow T^*} \|u(t, \cdot)\|_{H^s} = \infty$.
- $\int_0^{T^*} \|\partial_x u(t, \cdot)\|_{L^\infty} dt = \infty$.

In the rest of this section, we focus on proving the following stochastic version of Theorem 76.

Theorem 77 (Blow-up criterion for stochastic Burgers'). *Assume that $u_0 \in H^s(\mathbb{T})$, $s > 2$, and $u : \Omega \times [0, \tau_{max}) \times \mathbb{T} \rightarrow \mathbb{R}$ is a maximal solution of (3.1.1). If $\tau_{max} < \infty$, then $\int_0^{\tau_{max}} \|\partial_x u(t, \cdot)\|_{L^\infty} dt = \infty$ a.s. Moreover, if $0 \leq \tau < \tau_{max}$ is a smaller stopping time, necessarily $\int_0^\tau \|\partial_x u(t, \cdot)\|_{L^\infty} dt < \infty$ a.s.*

Proof of Theorem 77. By following the argument in⁴⁹, we start by noting that it is clear that if $0 \leq \tau < \tau_{max}$ is a smaller stopping time, necessarily $\int_0^\tau \|\partial_x u(t, \cdot)\|_{L^\infty} dt < \infty$ a.s. This is guaranteed by the embedding

$$\|\partial_x u\|_{L^\infty} \lesssim \|u\|_{H^s}. \quad (3.2.2)$$

Proving $\int_0^{\tau_{max}} \|\partial_x u(t, \cdot)\|_{L^\infty} dt = \infty$ is more involved. For this, we consider the hyper-regularised truncated Burgers' equation (3.1.5) and define the following stopping times:

$$\tau^s = \lim_{n \rightarrow \infty} \tau_n^s, \quad \tau_n^s = \inf \{t \geq 0 : \|u_r^\nu(t, \cdot)\|_{H^s} \geq n\},$$

$$\tau^\infty = \lim_{n \rightarrow \infty} \tau_n^\infty, \quad \tau_n^\infty = \inf \left\{ t \geq 0 : \int_0^t \|\partial_x u_r^\nu(s, \cdot)\|_{L^\infty} ds \geq n \right\}.$$

We claim that $\tau^\infty = \tau^s$. By again making use of (3.2.2), it is easy to check $\tau^s \leq \tau^\infty$. For simplicity, we omit the subscripts ν and r throughout the proof.

Imitating the techniques from the previous section, we arrive at

$$\begin{aligned} & \frac{1}{2} d\|\Lambda^s u\|_{L^2}^2 + \theta(\|\partial_x u\|_{L^\infty}) \langle \Lambda^s(u \partial_x u), \Lambda^s u \rangle_{L^2} dt + \langle \Lambda^s \mathcal{Q}u, \Lambda^s u \rangle_{L^2} dW_t \\ &= \nu \langle \Lambda^{s+2s'} u, \Lambda^s u \rangle_{L^2} dt + \frac{1}{2} \langle \Lambda^s \mathcal{Q}^2 u, \Lambda^s u \rangle_{L^2} dt + \frac{1}{2} \langle \Lambda^s \mathcal{Q}u, \Lambda^s \mathcal{Q}u \rangle_{L^2} dt. \end{aligned}$$

By integration by parts and standard estimates (use (1.3.8) for estimating the last two terms), one gets

$$d\|\Lambda^s u\|_{L^2}^2 + 2 \langle \Lambda^s \mathcal{Q}u, \Lambda^s u \rangle_{L^2} dW_t \lesssim (1 + \|\partial_x u\|_{L^\infty}) \|\Lambda^s u\|_{L^2}^2 dt.$$

By also deriving a similar estimate for $s = 0$ (which follows in a simpler manner by application of the techniques in the previous section), we obtain

$$d\|u\|_{H^s}^2 + 2(\langle \mathcal{Q}u, u \rangle_{L^2} + \langle \Lambda^s \mathcal{Q}u, \Lambda^s u \rangle_{L^2}) dW_t \lesssim (1 + \|\partial_x u\|_{L^\infty}) \|u\|_{H^s}^2 dt. \quad (3.2.3)$$

Finally, we treat the stochastic term. Without loss of generality, we assume $\|u\|_{H^s} \geq \epsilon > 0$ (otherwise add a positive constant to this function) and by applying Itô's formula in L^2 ¹¹⁷ to the logarithm, we obtain

$$d \log(\|u\|_{H^s}^2) = \frac{d\|u\|_{H^s}^2}{\|u\|_{H^s}^2} - \frac{dN_t}{2(\|u\|_{H^s}^2)^2}, \quad (3.2.4)$$

where N_t is defined by

$$N_t = 4 \int_0^t (\langle \mathcal{Q}u, u \rangle_{L^2} + \langle \Lambda^s \mathcal{Q}u, \Lambda^s u \rangle_{L^2})^2 d\gamma.$$

Notice that we have the bound

$$|N_t| \lesssim \|u\|_{H^s}^2, \quad (3.2.5)$$

as we have already indicated before (see the proof of Proposition 32 in Appendix B or⁸), where the constant in the last inequality depends on the $W^{s+1,\infty}$ -norm of the coefficients of Q . Plug (3.2.3) and (3.2.5) into (3.2.4) to

derive the estimate

$$d\log(\|u\|_{H^s}^2) \lesssim \frac{(1 + \|\partial_x u\|_{L^\infty})\|u\|_{H^s}^2}{\|u\|_{H^s}^2} dt + dM_t, \quad (3.2.6)$$

where M_t is the local martingale

$$M_t = \int_0^t \frac{\langle \mathcal{Q}u, u \rangle_{L^2} + \langle \Lambda^s \mathcal{Q}u, \Lambda^s u \rangle_{L^2}}{\|u\|_{H^s}^2} dW_\gamma,$$

for any $t > 0$. Expressing (3.2.6) in integral form, we have

$$\log(\|u(t)\|_{H^s}^2) \lesssim \log(\|u(0)\|_{H^s}^2) + \int_0^t (1 + \|\partial_x u\|_{L^\infty}) d\gamma + \int_0^t dM_\gamma. \quad (3.2.7)$$

By applying Burkholder-Davis-Gundy inequality (see (1.4.17)), we can control the local martingale term by estimating

$$[M]_t = \int_0^t \frac{(\langle \mathcal{Q}u, u \rangle_{L^2} + \langle \Lambda^s \mathcal{Q}u, \Lambda^s u \rangle_{L^2})^2}{\|u\|_{H^s}^4} d\gamma \lesssim t.$$

Here, we have employed again the arguments in Appendix B to bound the numerator in the fraction above. Burkholder-Davis-Gundy then yields

$$\mathbb{E} \left[\sup_{s \in [0, t]} \left| \int_0^s dM_\gamma \right| \right] \lesssim \sqrt{t}. \quad (3.2.8)$$

Taking expectation in (3.2.7) and using estimate (3.2.8), we establish that for any $n, m \in \mathbb{N}$,

$$\mathbb{E} \left[\sup_{\gamma \in [0, \tau_n^\infty \wedge m]} \log(\|u(\gamma)\|_{H^s}^2) \right] \lesssim \log(\|u_0\|_{H^s}^2) + m(1 + n) + \sqrt{m} < \infty.$$

Therefore

$$\mathbb{E} \left[\log \left(\sup_{\gamma \in [0, \tau_n^\infty \wedge m]} \|u(\gamma)\|_{H^s}^2 \right) \right] < \infty,$$

which in particular means that for $n, m \in \mathbb{N}$, $\sup_{\gamma \in [0, \tau_n^\infty \wedge m]} \|u(\gamma)\|_{H^s}$ is finite

a.s. Hence

$$\mathbb{P} \left(\sup_{\gamma \in [0, \tau_n^\infty \wedge m]} \|u(\gamma)\|_{H^s}^2 < \infty \right) = 1, \quad (3.2.9)$$

for every $n, m \in \mathbb{N}$, which from⁴⁹ implies $\tau^\infty \leq \tau^s$. Therefore $\tau^\infty = \tau^s$. Recall that we have omitted subscripts throughout this proof but $u = u_r^\nu$. By application of Fatou's Lemma we also obtain (3.2.9) in the limit $\nu \rightarrow 0$, $r \rightarrow \infty$, and we can show that $\tau^\infty = \tau^s$ also holds in the limit, concluding our argument. The few steps missing can be checked in⁴⁹. $\square$

3.3 GLOBAL WELL-POSEDNESS IN THE VISCOUS CASE

We add viscosity $\nu > 0$ to the generalised stochastic Burgers' equation (3.1.1), obtaining the following viscous version of stochastic Burgers'

$$du + u\partial_x u dt + QudW_t = \frac{1}{2}Q^2u dt + \nu\partial_{xx}u dt. \quad (3.3.1)$$

We assume $Q = \xi(x)\partial_x$ with $\xi \in (W^{s+1,\infty} \cap H^4)(\mathbb{T})$, $s > 2$.

Remark 78. *Observe that from our assumption on Q , (3.3.1) is simply (0.2.8) with $N = 1$, but we wish to keep the Q -notation for convenience.*

In this section, we establish the global well-posedness of strong solutions of (3.3.1). More concretely, we prove the following theorem.

Theorem 79. *Let $u_0 \in H^s(\mathbb{T})$, $s > 2$. Then there exists a pathwise unique H^s -global solution of the viscous stochastic Burgers' equation (3.3.1).*

The proof follows the same strategy as the local existence proof for the Burgers' equation without viscosity (3.1.1) that we provided in Section 3.1. The most important part is proving the following a priori estimate. In this estimate, we assume that u is smooth enough, but the rigorous way to do this is to regularise the equation first as we did in Section 3.1.

Proposition 80. *Let $u_0 \in H^s(\mathbb{T})$, $T > 0$, and $u : \Omega \times [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ be a solution to equation (3.3.1) that we assume to be smooth enough. Then there exists a constant such that ^{||}*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|u(t)\|_{H^s}^2 \right] \leq C_*(T). \quad (3.3.2)$$

Once the a priori estimate (3.3.2) is established, one can repeat the arguments in Section 3.1 to obtain Theorem 79. However, since this is repetitive and tedious, we do not explicitly carry out these arguments here. From now on, we focus on proving Proposition 80.

Proof. We start by computing the evolution of the L^2 -norm of the solution u . First note that the viscosity term has the good sign since

$$\nu \langle \partial_{xx} u, u \rangle_{L^2} = -\nu \|\partial_x u\|_{L^2}^2.$$

By applying the same techniques as in Section 3.1 (take into account estimate (1.3.7)), we obtain

$$d\|u\|_{L^2}^2 + 2\langle \mathcal{Q}u, u \rangle_{L^2} dW_t + 2\nu \|\partial_x u\|_{L^2}^2 dt \lesssim \|u\|_{L^2}^2 dt,$$

and we have the bounds

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|u(t, \cdot)\|_{L^2}^2 \right] \leq C_1(T), \quad (3.3.3)$$

$$\mathbb{E} \left[\int_0^T \|\partial_x u(s, \cdot)\|_{L^2}^2 ds \right] \leq C_2(T). \quad (3.3.4)$$

Remark 81. *Bounds of the type (3.3.3) and (3.3.4) (together!) are typical in parabolic equations. They are in general available for derivatives of k -th order and often enable instant regularisation of solutions. See Chapter 5,*

^{||}The attentive reader might ask why the H^s -norm of the velocity in (3.3.2) is taken to the power of two instead of four as in (3.1.9), but the arguments we carry out together with the control we provide for the L^∞ -norm of u immediately yield the bound for the fourth power.

Section 5.2 for bounds of this type for a deterministic parabolic equation.

We compute the evolution of the $\dot{H}^s$ -norm of u

$$\begin{aligned} & \frac{1}{2} d\|\Lambda^s u\|_{L^2}^2 + \langle \Lambda^s(\mathcal{Q}u), \Lambda^s u \rangle_{L^2} dW_t \\ &= -\langle \Lambda^s(u\partial_x u), \Lambda^s u \rangle_{L^2} dt + \nu \langle \Lambda^s(\partial_{xx} u), \Lambda^s u \rangle_{L^2} dt \\ &+ \frac{1}{2} \langle \Lambda^s(\mathcal{Q}u), \Lambda^s(\mathcal{Q}u) \rangle_{L^2} dt + \frac{1}{2} \langle \Lambda^s(\mathcal{Q}^2 u), \Lambda^s u \rangle_{L^2} dt. \end{aligned} \quad (3.3.5)$$

Integrating by parts the first term of the right-hand side above and observing that $u\partial_x u = (1/2)\partial_x(u^2)$, we obtain

$$-\langle \Lambda^s(u\partial_x u), \Lambda^s u \rangle_{L^2} = \frac{1}{2} \langle \Lambda^{s-1}(\partial_x(u^2)), \Lambda^{s+1} u \rangle_{L^2}.$$

Hence

$$\begin{aligned} & |\langle \Lambda^s(u\partial_x u), \Lambda^s u \rangle_{L^2}| \leq \|\Lambda^{s-1}(\partial_x(u^2))\|_{L^2} \|\Lambda^{s+1} u\|_{L^2} \\ & \lesssim (\|u\|_{L^\infty} \|\Lambda^s u\|_{L^2}) \|\Lambda^{s+1} u\|_{L^2} \leq \frac{1}{2\nu} \|u\|_{L^\infty}^2 \|\Lambda^s u\|_{L^2}^2 + \frac{\nu}{2} \|\Lambda^{s+1} u\|_{L^2}^2, \end{aligned} \quad (3.3.6)$$

where we have employed Kato-Ponce (1.1.8) in the second inequality. The second term on the right-hand side after the equality of equation (3.3.5) can be integrated as

$$\nu \langle \Lambda^s(\partial_{xx} u), \Lambda^s u \rangle_{L^2} = -\nu \langle \Lambda^{s+1} u, \Lambda^{s+1} u \rangle_{L^2} = -\nu \|\Lambda^{s+1} u\|_{L^2}^2. \quad (3.3.7)$$

As usual, applying inequality (1.3.8) we also have the estimate

$$|\langle \Lambda^s(\mathcal{Q}u), \Lambda^s(\mathcal{Q}u) \rangle_{L^2} + \langle \Lambda^s(\mathcal{Q}^2 u), \Lambda^s u \rangle_{L^2}| \lesssim \|\Lambda^s u\|^2. \quad (3.3.8)$$

Putting together (3.3.6)-(3.3.8), one derives

$$\begin{aligned} & d\|\Lambda^s u\|_{L^2}^2 + 2\langle \Lambda^s(\mathcal{Q}u), \Lambda^s u \rangle_{L^2} dW_t + \nu \|\Lambda^{s+1} u\|_{L^2}^2 dt \\ & \lesssim \frac{1}{\nu} (1 + \|u\|_{L^\infty}^2) \|\Lambda^s u\|_{L^2}^2 dt. \end{aligned}$$

By applying expectation in the above equation and taking into account that

the expectation of the Itô integral vanishes due to the martingale property, one obtains

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} \|\Lambda^s u(t)\|_{L^2}^2 \right] &\lesssim \mathbb{E} [\|\Lambda^s u(0)\|_{L^2}^2] \\ &+ \frac{1}{\nu} \mathbb{E} \left[\int_0^T (1 + \|u(\gamma)\|_{L^\infty}^2) \|\Lambda^s u(\gamma)\|_{L^2}^2 d\gamma \right], \end{aligned} \quad (3.3.9)$$

and

$$\begin{aligned} \nu \mathbb{E} \left[\int_0^T \|\Lambda^{s+1} u(\gamma)\|_{L^2}^2 d\gamma \right] &\lesssim \mathbb{E} [\|\Lambda^s u(0)\|_{L^2}^2] \\ &+ \frac{1}{\nu} \mathbb{E} \left[\int_0^T (1 + \|u(\gamma)\|_{L^\infty}^2) \|\Lambda^s u(\gamma)\|_{L^2}^2 d\gamma \right]. \end{aligned} \quad (3.3.10)$$

Now we claim that the following maximum principle holds

$$\|u(t, \cdot)\|_{L^\infty} \leq \|u(0, \cdot)\|_{L^\infty}, \quad t \in [0, T], \quad (3.3.11)$$

which we show in Lemma 82. Once (3.3.11) is proven, we can infer from (3.3.9) and (3.3.10) together with Grönwall's inequality that there exist constants $A_1 = A_1(T)$, $A_2 = A_2(T)$, such that

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|\Lambda^s u(t, \cdot)\|_{L^2}^2 \right] \leq A_1, \quad \nu \mathbb{E} \left[\int_0^T \|\Lambda^{s+1} u(s, \cdot)\|_{L^2}^2 ds \right] \leq A_2.$$

This concludes the proof, so we are only left to show the maximum principle (3.3.11). $\square$

Lemma 82. *The maximum principle (3.3.11) is satisfied.*

Proof. Following the type of argument in ¹⁷, consider the SDE

$$dX_t = \xi(X_t) \circ dW_t, \quad (3.3.12)$$

and let $\psi_t(X_0)$ be the corresponding flow of (3.3.12) with initial condition $X_0 \in \mathbb{T}$, which is a semimartingale and a diffeomorphism of class C^2 pro-

vided $\xi \in C^{3+\alpha}(\mathbb{T})$, $\alpha > 0$ (see Assumption A1), which holds since from embedding (1.1.6) we have $H^4(\mathbb{T}) \subset C^{3+1/2}(\mathbb{T})$. By applying Itô-Wentzell in Stratonovich form (see Theorem 40), we evaluate u along $X_t = \psi_t(X_0)$, getting that

$$\begin{aligned} u(t, X_t) &= u(0, X_0) - \int_0^t u(s, X_s) \partial_x u(s, X_s) ds + \nu \int_0^t \partial_{xx} u(s, X_s) ds \\ &\quad - \int_0^t \xi(X_s) \partial_x u(s, X_s) \circ dW_s + \int_0^t \partial_x u(s, X_s) \circ dX_s, \end{aligned} \quad (3.3.13)$$

a.s., where the term in the last line

$$\int_0^t \xi(X_s) \partial_x u(s, X_s) \circ dW_s + \int_0^t \partial_x u(s, X_s) \circ dX_s = 0.$$

Consider the change of variables $w(t, X_0) = u(t, \psi_t(X_0))$ and by chain rule, obtain

$$\begin{aligned} \partial_x u(t, X_t) &= \frac{\partial \psi_t^{-1}}{\partial x}(\psi_t(X_0)) \frac{\partial w}{\partial X_0}(t, X_0), \\ \partial_{xx} u(t, X_t) &= \frac{\partial^2 \psi_t^{-1}}{\partial x^2}(\psi_t(X_0)) \frac{\partial w}{\partial X_0}(t, X_0) + \left(\frac{\partial \psi_t^{-1}}{\partial x}(\psi_t(X_0)) \right)^2 \frac{\partial^2 w}{\partial X_0^2}(t, X_0). \end{aligned}$$

Therefore, (3.3.13) is equivalent to the following PDE with random coefficients

$$\frac{\partial w}{\partial t} + \tilde{w} \frac{\partial w}{\partial X_0} = \tilde{\nu} \frac{\partial^2 w}{\partial X_0^2},$$

where

$$\begin{aligned} \tilde{w} &= \frac{\partial \psi_t^{-1}}{\partial x}(\psi_t(X_0)) u(t, \psi_t(X_0)) - \nu \frac{\partial^2 \psi_t^{-1}}{\partial x^2}(\psi_t(X_0)), \\ \tilde{\nu} &= \nu \left(\frac{\partial \psi_t^{-1}}{\partial x}(\psi_t(X_0)) \right)^2. \end{aligned}$$

We claim that w satisfies the maximum principle

$$\|w_t\|_{L^\infty} \leq \|w_0\|_{L^\infty}, \quad t > 0. \quad (3.3.14)$$

To prove (3.3.14), we follow the strategy in⁴⁴. If $w_0 = 0$, the claim is evident. Otherwise, for every $t \geq 0$, we define $X_t^0 \in \mathbb{T}$ to be such that $\|w_t\|_{L^\infty} = |w(t, X_t^0)|$ (i.e., the spatial point where $\|w_t\|_{L^\infty}$ is attained. Its existence is guaranteed because $\mathbb{T}$ is compact). We start by claiming that for all $t > 0$, $\|w_t\|_{L^\infty}$ is differentiable in time and

$$\frac{d}{dt}\|w_t\|_{L^\infty} = \frac{\partial w}{\partial t}(t, X_t^0). \quad (3.3.15)$$

First, we check that $\|w_t\|_{L^\infty}$ is differentiable in time. For this, note that for $h \geq 0$ from the mean value theorem we have

$$|\|w_{t+h}\|_{L^\infty} - \|w_t\|_{L^\infty}| \leq \|w_{t+h} - w_t\|_{L^\infty} \leq \max_{s \in [t, t+h]} |\partial_t w(s)| h,$$

and a similar argument can be applied if $h \leq 0$. Therefore $\|w_t\|_{L^\infty}$ is Lipschitz in time and from Rademacher's theorem a.e. differentiable.

We proceed to prove equality (3.3.15). Without loss of generality, we assume $w(t, X_t^0) \geq 0$, $t > 0$. We note that by taking small enough $h > 0$

$$\begin{aligned} \frac{d}{dt}\|w_t\|_{L^\infty} &= \lim_{h \rightarrow 0} \frac{\|w_{t+h}\|_{L^\infty} - \|w_t\|_{L^\infty}}{h} = \lim_{h \rightarrow 0} \frac{w(t+h, X_{t+h}^0) - w(t, X_t^0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{w(t+h, X_{t+h}^0) - w(t+h, X_t^0)}{h} + \lim_{h \rightarrow 0} \frac{w(t+h, X_t^0) - w(t, X_t^0)}{h} \\ &\geq \frac{\partial w}{\partial t}(t, X_t^0), \end{aligned}$$

for $t > 0$. Symmetrically, by choosing small enough $h > 0$ we also have

$$\begin{aligned} \frac{d}{dt}\|w_t\|_{L^\infty} &= \lim_{h \rightarrow 0} \frac{\|w_{t-h}\|_{L^\infty} - \|w_t\|_{L^\infty}}{-h} = \lim_{h \rightarrow 0} \frac{w(t-h, X_{t-h}^0) - w(t, X_t^0)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{w(t-h, X_{t-h}^0) - w(t-h, X_t^0)}{-h} + \lim_{h \rightarrow 0} \frac{w(t-h, X_t^0) - w(t, X_t^0)}{-h} \\ &\leq \frac{\partial w}{\partial t}(t, X_t^0), \end{aligned}$$

for $t > 0$. Therefore we conclude (3.3.15). To finish our argument, we observe

that

$$\frac{\partial w}{\partial t}(t, X_t^0) = -\tilde{w} \frac{\partial w}{\partial X_0}(t, X_t^0) + \tilde{v} \frac{\partial^2 w}{\partial X_0^2}(t, X_t^0) \leq 0.$$

Hence (3.3.14) is satisfied. Since ψ_t is a diffeomorphism, we have $\|u_t\|_{L^\infty} = \|w_t\|_{L^\infty}$ so the maximum principle also follows for u .

□

With this, we conclude the proof of Theorem 79.

Remark 83 (Stochastic Burgers' equation on the real line). *The main results we have shown in this chapter can easily be extended to the case when the Burgers' equations we have considered are defined on $\mathbb{R}$ instead of on the periodic domain $\mathbb{T}$, since the required techniques are almost identical.*

Remark 84 (The cylindrical noise case). *The arguments in this chapter could be extended to the cylindrical noise case*

$$du + u\partial_x u dt + \sum_{i=1}^{\infty} Q_i u \circ dW_t^i = 0,$$

where

$$Q_i = a_i(x)\partial_x + b_i(x), \quad i \in \mathbb{N},$$

with a little bit of extra work. Indeed, we note that estimates (1.3.7) and (1.3.8) are available in this case by enforcing a good control of the quantity

$$\sum_{i=1}^{\infty} (\|a_i\|_{H^{s+1}} + \|b_i\|_{H^{s+1}}).$$

3.4 SUMMARY OF RESULTS ON SHOCK FORMATION

In this section, we briefly comment on some shock formation results for the inviscid Burgers' equation for completion. We acknowledge that these results

belong more to So Takao than to us (see⁸ for the proofs and some more results). We work on the real line instead of on the torus because it is more illustrative. Consider equation (0.2.8) on the real line, i.e.

$$du(t, x) + \left(u(t, x)dt + \sum_{i=1}^N \xi_i(x) \circ dW_t^i \right) \partial_x u(t, x) = 0, \quad (3.4.1)$$

$t > 0$, $x \in \mathbb{R}$. Here we assume that ξ_i , $i = 1, \dots, N$ and u_0 are smooth for simplicity, but for sharper conditions we refer the reader to⁸. The following theorem discusses a case when almost sure formation of shocks occurs.

Theorem 85. *Let $\xi_1(x) = \alpha x$, $\xi_i(x) = 0$, $i = 2, \dots, N$, $\alpha \in \mathbb{R}$. If the initial profile u_0 has a negative slope, then $\tau_{max} < \infty$ almost surely. In other words, the unique maximal solution constructed in Section 3.1 forms a singularity in finite time.*

To conclude the results in this short section, we argue that for a broader class of noises, shock formation takes place in expectation provided the initial profile u_0 evinces a sufficiently negative slope. Moreover, no new shocks can develop from positive slopes. This type of “shock formation” is of course weaker than the almost surely one considered in the above theorem, but the present result allows for more general noise profiles.

Theorem 86. *Let X_t be a characteristic of inviscid Burgers’ such that $\partial_x u(0, X_0) = -\sigma < 0$ **. If*

$$\frac{1}{2} \sum_{i=1}^N ((\partial_x \xi_i(x))^2 - \xi_i(x) \partial_{xx} \xi_i(x)) > -\sigma, \quad x \in \mathbb{R},$$

then there exists $0 < t_* < \infty$ such that $\lim_{t \rightarrow t_*} \mathbb{E}[\partial_x u(t, X_t)] = -\infty$. Moreover, if $\partial_x u(0, X_0) \geq 0$ then $\partial_x u(t, X_t) < \infty$ a.s. for all $t > 0$.

Finally, there is an important note we wish to make. In⁷⁴, Chapter 5.1.4, it is argued that shock formation in Burgers’ cannot be prevented even by

**Note that X_0 is deterministic and also the initial data $u(0, x)$.

addition of a general cylindrical transport noise of the type

$$\sum_{i=1}^{\infty} \xi_i(x) \partial_x u(t, x) \circ dW_t^i.$$

However, there is a small mistake in the argument. The author writes the characteristic equation as an Itô SDE

$$X_t = X_0 + u(0, X_0)t + \sum_{i=1}^{\infty} \int_0^t \xi_i(X_s) dW_s^i, \quad (3.4.2)$$

which is a (local) martingale perturbation of crossing straight lines $X_0 + u(0, X_0)t$, since $\sum_{i=1}^{\infty} \int_0^t \xi_i(X_s) dW_s^i$ is a local martingale when X_t is regular enough and $\{\xi_i\}_{i=1}^{\infty}$ forms an orthonormal basis. However, the characteristic equation for the stochastic Burgers' equation (0.2.8) is not (3.4.2), and instead takes the Stratonovich form

$$X_t = X_0 + u(0, X_0)t + \sum_{i=1}^{\infty} \int_0^t \xi_i(X_s) \circ dW_s^i,$$

and therefore Flandoli's argument can be applied to the martingale term, but not to the additional drift term arising from the Stratonovich-to-Itô correction, which may affect shock formation.

4

Linear advection of k -forms with Lie transport noise

In this chapter, we consider the equation of Lie transport of k -forms with Lie transport noise defined in (0.2.10) and show existence and uniqueness of weak solutions, strong in the probability sense. This equation is ill-posed in the deterministic case, so our results provide a general geometric example of well-posedness of PDEs by addition of stochastic noise. Throughout this chapter, we fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_t)$ satisfying the usual conditions, together with a family $\{W_t^i\}_{i=1}^N$ of independent Brownian processes adapted to $(\mathcal{F}_t)_t$.

Different notions of solution. We state several notions of solutions of (0.2.10) along similar lines as we did for Burgers' in Chapter 3. We note that the solutions we consider are global. For questions regarding the notation we employ we point the reader to the preliminaries, and particularly to Section 1.2 for the geometric notation. Sometimes we will choose the notation $K_t(x) = K(t, x)$ for the solution of (0.2.10) or simply K at our convenience. For the following definitions to make sense, we enforce Assumption A3 (i)

(ii) on the drift and diffusion vector fields of (0.2.10).

Definition 87 (Classical solution). *A solution to equation (0.2.10) in the classical sense is a k -form-valued C^1 -semimartingale $^*K$ adapted to $(\mathcal{F}_t)_t$. Moreover, $\mathbb{P}$ -a.s. K has trajectories of class $C([0, T]; C^3(\bigwedge^k(T^*\mathbb{R}^n)))$ and for any k -vector fields $u = u_1 \wedge \dots \wedge u_k \in \mathfrak{X}^k(\mathbb{R}^n)$, $\mathbb{P}$ -a.s. it holds*

$$\begin{aligned} K_t(x)(u(x)) - K_0(x)(u(x)) + \int_0^t \mathcal{L}_b K_s(x)(u(x)) ds \\ + \sum_{i=1}^N \int_0^t \mathcal{L}_{\xi_i} K_s(x)(u(x)) \circ dW_s^i = 0, \quad x \in \mathbb{R}^n, \end{aligned} \quad (4.0.1)$$

$t \in [0, T]$. In Itô form, the semimartingale assumption can be dropped and (4.0.1) becomes

$$\begin{aligned} K_t(x)(u(x)) - K_0(x)(u(x)) + \int_0^t \mathcal{L}_b K_s(x)(u(x)) ds \\ + \sum_{i=1}^N \int_0^t \mathcal{L}_{\xi_i} K_s(x)(u(x)) dW_s^i - \frac{1}{2} \sum_{i=1}^N \int_0^t \mathcal{L}_{\xi_i}^2 K_s(x)(u(x)) ds = 0, \quad x \in \mathbb{R}^n, \end{aligned} \quad (4.0.2)$$

$t \in [0, T]$.

Remark 88. *In Definition 87, the semimartingale assumption in the Stratonovich formulation (4.0.1) is needed so that one can pass to the Itô formulation via the identity*

$$\begin{aligned} \sum_{i=1}^N \int_0^t \mathcal{L}_{\xi_i} K_s(x)(u(x)) \circ dW_s^i \\ = \sum_{i=1}^N \int_0^t \mathcal{L}_{\xi_i} K_s(x)(u(x)) dW_s^i + \sum_{i=1}^N [\mathcal{L}_{\xi_i} K(\cdot)(x)(u(x)), W^i]_t. \end{aligned}$$

This is possible because thanks to this assumption and the fact that ξ_i , $i = 1, \dots, N$ are semimartingales, the terms $\mathcal{L}_{\xi_i} K(\cdot)(x)(u(x))$, $i = 1, \dots, N$ ap-

*A C^k -semimartingale is a continuous semimartingale such that its first k (spatial) derivatives are also continuous semimartingales.

pearing in the covariation are semimartingales. The above assumption is not required in the Itô formulation.

Remark 89. In coordinates, choosing $u_j = \partial/\partial x_j$, $j = 1, \dots, k$, (4.0.1) is equivalent to saying that SPDE (0.2.10) is satisfied component-wise. The same of course applies to (4.0.2) by taking into account the Itô formulation of (0.2.10).

Definition 90 (Weak L^2 -solution). We say that a k -form-valued process $K \in \mathcal{P} \otimes \mathcal{B}(\bigwedge^k(T^*\mathbb{R}^n))$ satisfies equation (0.2.10) weakly in the L^2 -sense if

- $K \in L^2(\Omega \times [0, T] \times \bigwedge^k(T^*\mathbb{R}^n))$, which is defined to be the Hilbert space of k -form-valued stochastic processes with norm

$$\|K\|_{L^2(\Omega \times [0, T] \times \bigwedge^k(T^*\mathbb{R}^n))} = \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^n} K_s^2(x) d^n x ds \right]^{1/2} < \infty.$$

- For all $R > 0$, $\sup_{0 \leq t \leq T} \|K_t\|_{L_R^\infty} < \infty$, a.s.
- For any test k -form $\theta \in C_0^\infty(\bigwedge^k(T^*\mathbb{R}^n))$, the process $\langle\langle K_t, \theta \rangle\rangle_{L^2}$ has a continuous modification which is a semimartingale adapted to $(\mathcal{F}_t)_t$ and

$$\begin{aligned} \langle\langle K_t, \theta \rangle\rangle_{L^2} - \langle\langle K_0, \theta \rangle\rangle_{L^2} + \int_0^t \langle\langle K_s, \mathcal{L}_b^T \theta \rangle\rangle_{L^2} ds \\ + \sum_{i=1}^N \int_0^t \langle\langle K_s, \mathcal{L}_{\xi_i}^T \theta \rangle\rangle_{L^2} \circ dW_s^i = 0, \end{aligned}$$

for almost every $(\omega, t) \in \Omega \times [0, T]$. In Itô form, it is sufficient to impose that the process $\langle\langle K_t, \theta \rangle\rangle_{L^2}$ has a continuous $\mathcal{F}_t$ -adapted modification and

$$\begin{aligned} \langle\langle K_t, \theta \rangle\rangle_{L^2} - \langle\langle K_0, \theta \rangle\rangle_{L^2} + \int_0^t \langle\langle K_s, \mathcal{L}_b^T \theta \rangle\rangle_{L^2} ds \\ + \sum_{i=1}^N \int_0^t \langle\langle K_s, \mathcal{L}_{\xi_i}^T \theta \rangle\rangle_{L^2} dW_s^i - \frac{1}{2} \sum_{i=1}^N \int_0^t \langle\langle K_s, \mathcal{L}_{\xi_i}^T \mathcal{L}_{\xi_i}^T \theta \rangle\rangle_{L^2} ds = 0, \end{aligned} \tag{4.0.3}$$

for almost every $(\omega, t) \in \Omega \times [0, T]$.

We remind that $\langle\langle \cdot, \cdot \rangle\rangle_{L^2}$ stands for the L^2 -inner product of tensor fields defined in Section 1.2.

Remark 91. *In Definitions 87 and 90, we have introduced the concept of solution both in Stratonovich and Itô form because whereas the Stratonovich framework is the natural one for our equation, sometimes it is crucial to operate in Itô form.*

4.1 MAIN RESULT AND BRIEF DISCUSSION

In the preliminaries, Section 1.1, we defined the space of uniformly in time globally α -Hölder continuous functions $L^\infty([0, T]; C_b^\alpha(\mathbb{R}^n, \mathbb{R}^n))$, for $\alpha \in (0, 1)$. We say that a vector field b is in the class $L^\infty([0, T]; W_{loc}^{1,2}(\mathbb{R}^n, \mathbb{R}^n))$ if for every $R > 0$, $\mathcal{X}_{B(0,R)}b \in L^\infty([0, T]; W^{1,2}(\mathbb{R}^n, \mathbb{R}^n))$, i.e., for every $t \in [0, T]$, $\mathcal{X}_{B(0,R)}b(t, \cdot)$ is of class $W^{1,2}$ uniformly in time. Consider the (general) flow equation

$$\phi_{s,t}(x) = x + \int_s^t b(r, \phi_{s,r}(x))dr + \sum_{i=1}^N \int_s^t \xi_i(r) dW_r^i, \quad 0 \leq s \leq t \leq T. \quad (4.1.1)$$

As usually, the case $s = 0$ is of particular importance and carries the notation

$$\phi_t(x) = x + \int_0^t b(s, \phi_s(x))dr + \sum_{i=1}^N \int_0^t \xi_i(s) dW_s^i, \quad 0 \leq t \leq T. \quad (4.1.2)$$

We are mainly interested in the regularity properties of solutions to (4.1.2), but we will sometimes need the auxiliary more general flow equation (4.1.1) to prove the desired results. We will make the following assumptions on the drift and diffusion coefficients of (4.1.1).

Assumption A3.

(i) $b \in L^\infty([0, T]; C_b^\alpha(\mathbb{R}^n, \mathbb{R}^n))$ for some $\alpha \in (0, 1)$, $\xi_i \in L^\infty([0, T])$ (only

depending on time!), $i = 1, \dots, N$. We stress that these are deterministic.

(ii) $b \in L^\infty([0, T]; W_{loc}^{1,2}(\mathbb{R}^n, \mathbb{R}^n))$ and ξ_i , $i = 1, \dots, N$ are semimartingales.

(iii) There exist $0 < c < C < \infty$ such that the uniform ellipticity condition

$$c|v|^2 \leq \sum_{i,j=1}^n \sum_{k=1}^N \xi_k^i(t) \xi_k^j(t) v_i v_j \leq C|v|^2, \quad t \geq 0, \quad v \in \mathbb{R}^n, \quad (4.1.3)$$

is met.

The semimartingale requirement in Assumption A3 is not really needed, but we will employ it sometimes because in some proofs it is convenient to use the Stratonovich formulation of equation (0.2.10). If we always worked in the Itô side, this assumption would not be necessary; however, it is useful to simplify some computations while not losing rigor. We also note that we will strongly use the fact that $W_{loc}^{1,2}(\mathbb{R}^n, \mathbb{R}^n) \subset W_{loc}^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$. The main aim of this chapter is proving the following theorem.

Theorem 92. *Let $K_0 \in L^\infty(\bigwedge^k(T^*\mathbb{R}^n))$. Moreover, let Assumption A3 hold. Then equation (0.2.10) admits a unique (strong in the probability sense) weak solution K in the sense of Definition 90 with initial condition K_0 . Moreover, we can explicitly construct this solution via the formula $K_t(x) = (\phi_t)_* K_0(x)$, where we remind that $(\phi_t)_*$ represents the push-forward (in the sense of differential geometry) of the flow ϕ_t , assumed to be a solution of the flow equation (4.1.2).*

We stress that the class of equations (0.2.10) we consider in this chapter includes, for instance, the stochastic linear transport, continuity, and vector advection equations as particular cases. In other words, all of these equations have the same geometric interpretation as a differential k -form being advected by a stochastic vector field.

Main strategy of the proof. In order to prove our result, we will stick to the following lines. Given a flow ϕ_t satisfying (4.1.2) with smooth enough

drift and diffusion vector fields, the solution of (0.2.10) with smooth enough initial data K_0 is unique and can be expressed as the push-forward of the initial data, i.e.

$$K_t(x) = (\phi_t)_* K_0(x), \quad t \in [0, T], \quad x \in \mathbb{R}^n. \quad (4.1.4)$$

The main idea for showing the existence of a solution of the form (4.1.4) within our low regularity conditions is demonstrating that the classical smooth assumptions can gradually be reduced by employing mollifiers. For this, we generalise the characteristic techniques employed in⁷⁶. However, in order for our characteristic argument to work, we first need to prove that solutions of the flow equation (4.1.2) are regular enough under our assumptions. This is done in⁷⁶ for the particular case $\xi_i = e_i$, $i = 1, \dots, N$; however, we need to extend their arguments since we are considering a more general noise, which involves extra work and even the application of stronger techniques. Thus, generalising the flow-regularity results in⁷⁶ to allow for time-dependent diffusions, we prove the existence and uniqueness of a $C^{1+\alpha'}$ -flow of diffeomorphisms, $0 < \alpha' < \alpha$ (remember $\alpha \in (0, 1)$) that solves (4.1.1) under our regularity assumptions. One of the main points in the argument appearing in⁷⁶ is applying the Itô-Tanaka trick, which shows that a transformation of the flow $\phi_{s,t}$ satisfies an equation of the type (4.1.1) with more regular new coefficients $\tilde{b}, \tilde{\xi}_i$, yielding improved regularity of the flow. We also need sharp estimates on the solutions of parabolic equations for this task, since we are working under weak in time regularity conditions. Moreover, some extra regularity properties that do not appear in⁷⁶ need to be shown for the flow. The flow regularity arguments are carried out in Section 4.2. Once it is shown that a smooth enough flow can be constructed, we prove existence of solutions by extending the mollification arguments in⁷⁶ and adapting them to our geometric formalism. In this geometric case, the estimates become more involved and we need to take extra care because our solution is not necessarily of class L^∞ . In order to achieve our task, we constantly exploit the geometric structure of our equation. This is done in Section 4.3. Finally, we devote Section 4.4 to showing uniqueness of solutions. In⁷⁶, uniqueness is established via application of an extension of DiPerna-Lions commutator

argument (see⁶¹). Here, we encounter some extra difficulties because we are dealing with k -form-valued processes and the commutators appearing in our proofs are more general. The KIW formula (see Theorem 47) and generalisations of the DiPerna-Lions commutator estimates that we prove help tackle this issue.

Since the results we establish in this chapter are intended to be an extension of some of the results in⁷⁶, and especially targeting the readers who might be familiar with this last work, we point a few interesting aspects that we were forced to change due to the general geometric form of our equation and noise.

- (i) In order to treat the weak formulation of equation (0.2.10) in the sense of Definition 90, weak versions of Lie derivatives need to be developed, and objects such as mollifiers have to be extended to differential forms (or more generally, to tensor fields), fact which plays an important role in our analysis.
- (ii) When proving the existence of solutions, it is more convenient to work with L^2 -solutions rather than L^∞ -solutions as done in⁷⁶, since in general, the push-forward (4.1.4) contains derivatives of the backward flow ϕ_t^{-1} , which a priori is not clear how to bound in L^∞ , but can be bounded in L^2 .
- (iii) The additional assumption $b \in W_{loc}^{1,2}$ was made since the general form of $\mathcal{L}_b K$ requires a (weak) derivative on b , and it is also convenient when computing the commutator estimates in the uniqueness proof, since it avoids having to control the derivative of the Jacobian as done in⁷⁶. In other words, although we assume b admits a weak derivative in order to make sense of our equation, we do not need to control the Jacobian of the flow ϕ_t and can hence drop the assumptions on $\text{div}(b)$ required in⁷⁶.

4.2 REGULARITY OF THE FLOW

We state our main theorem establishing the regularity properties of the flow of equation (4.1.1). We note that the existence of strong unique path-wise solutions to a general class of SDEs containing (4.1.1) was established in¹⁵⁴, which also seems to provide the first example of well-posedness by addition of stochastic noise in the ODE case. Moreover, in the deterministic case (i.e., when the diffusion terms in (4.1.1) vanish), remarkable uniqueness and existence of L^∞ -solutions results under weak assumptions on the drift vector field have been shown in^{11,61}. A linear growth condition on the divergence of b is required in these works, which we do not assume here.

We extend the proof of the original flow regularity result in⁷⁶ to allow for the more general type of noise we consider in (0.2.10).

Theorem 93 (Flow regularity). *Consider equation (4.1.1) with drift vector field and diffusions satisfying (i) and (iii) in Assumption A3. Then we have the following:*

- (i) *For every $s \in [0, T]$, $x \in \mathbb{R}^n$, the SDE (4.1.1) has a unique continuous adapted solution $(\phi_{s,t}(x))_{t \in [s, T]}$. For simplicity, for fixed $s \geq 0$ and $x \in \mathbb{R}^n$, we might sometimes employ the notation $X_t = \phi_{s,t}(x)$, $t \in [s, T]$.*
- (ii) *$\phi_{s,t}$, $0 \leq s \leq t \leq T$, is a stochastic $C^{1+\alpha'}$ -flow of diffeomorphisms (see Definition 34), for any $0 < \alpha' < \alpha$. Furthermore, letting $r(x) = \sqrt{1 + |x|^2}$, $x \in \mathbb{R}^n$, for any $\alpha' \in (0, \alpha - n/p)$ with $\alpha p > n$ and $\epsilon > n/p$, we have the following estimate in the $r^{-\epsilon}$ -weighted Hölder norm of $D\phi_{s,t}$*

$$\sup_{0 \leq s \leq T} \mathbb{E} \left[\sup_{s \leq t \leq T} \left(\sup_{\substack{x, y \in \mathbb{R}^n \\ x \neq y}} \frac{|[r(x)]^{-\epsilon} D\phi_{s,t}(x) - [r(y)]^{-\epsilon} D\phi_{s,t}(y)|}{|x - y|^{\alpha'}} \right)^p \right] < \infty. \quad (4.2.1)$$

- (iii) *Let $\{b^n\}_{n=1}^\infty \subset L^\infty([0, T]; C_b^\alpha(\mathbb{R}^n, \mathbb{R}^n))$, $\{\xi_i^n\}_{n=1}^\infty \subset L^\infty([0, T])$, $i = 1, \dots, N$ be sequences of vector fields such that we have $b^n \rightarrow b$ in $L^\infty([0, T]; C_b^\alpha(\mathbb{R}^n, \mathbb{R}^n))$ and $\xi_i^n \rightarrow \xi_i$ in $L^\infty([0, T])$, $i = 1, \dots, N$, $n \rightarrow \infty$*

∞ , and $\{\phi_{s,t}^n\}_{n=1}^\infty$ be the corresponding sequence of associated stochastic flows. Then for any $p \geq 1$, we have

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^n} \sup_{0 \leq s \leq T} \mathbb{E} \left[\sup_{s \leq t \leq T} |\phi_{s,t}^n(x) - \phi_{s,t}(x)|^p \right] = 0, \quad (4.2.2)$$

$$\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}^n} \sup_{0 \leq s \leq T} \mathbb{E} \left[\sup_{s \leq t \leq T} |D\phi_{s,t}^n(x)|^p \right] < \infty, \quad (4.2.3)$$

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^n} \sup_{0 \leq s \leq T} \mathbb{E} \left[\sup_{s \leq t \leq T} |D\phi_{s,t}^n(x) - D\phi_{s,t}(x)|^p \right] = 0. \quad (4.2.4)$$

We proceed to establish the proof of Theorem 93, indicating which auxiliary tools are required to extend the results in ⁷⁶ to our more general case. Since our proof follows the techniques in ⁷⁶, the latter reference can sometimes be checked for completeness.

Proof of Theorem 93. Main idea. The essence of the argument is the following: we show that the solution of equation (4.1.1) is unique and can be expressed as a composition of functions which are regular enough (actually, they are remarkably regular taking into account our assumptions on the drift and diffusions!). More concretely, for fixed $s \leq t \leq T$, $x \in \mathbb{R}^n$, we prove that the process $Y_t = \Psi_\lambda(t, \phi_{s,t}(x)) = \phi_{s,t}(x) + \psi_\lambda(t, \phi_{s,t}(x))$, $t \geq s$, where ψ_λ is a solution of a parabolic equation on the extended domain $(0, \infty) \times \mathbb{R}^n$ depending on a parameter $\lambda \geq 0$, solves a flow equation of the type (4.1.1) with new coefficients $\tilde{b}, \tilde{\xi}_i$, $i = 1, \dots, N$ enjoying more regularity than the original ones. Moreover, this equation admits a smooth enough flow of diffeomorphisms $\varphi_{s,t}$ (this technique is called the Itô-Tanaka trick or Zvonkin transform. See ¹⁵⁷ for one of the original motivations). By applying sharp results treating the regularity of solutions of parabolic equations (some of the results we use have indeed been developed recently) and estimates for ψ_λ , which can be found in Section 1.3, we show that Ψ_λ enjoys the desired regularity and is invertible if λ is large enough. Finally, we prove that the regularity of Ψ_λ and $\varphi_{s,t}$ is inherited by the flow $\phi_{s,t}$, since $\phi_{s,t} = \Psi_\lambda^{-1} \circ \varphi_{s,t} \circ \Psi_\lambda$. Moreover, by employing the equation for Y_t and performing several estimates using Grönwall's and Burkholder-David-Gundy inequalities, we conclude (4.2.1). This yields points (i) and (ii). For point (iii), we take into account the previous steps

and closely follow the techniques in⁷⁶.

In order to facilitate the understanding of our argument, we break up this proof into five steps. In Step 1 and Step 2, we treat the regularity of the functions ψ_λ and Ψ_λ , whereas Step 3 is devoted to the Itô-Tanaka trick and the improved regularity of the flow. Step 4 is slightly technical and covers the derivation of estimate (4.2.1). This yields points (i) and (ii). Step 5 addresses the third and last point.

For simplicity and without loss of generality, we treat the case $N = 1$ (i.e., we consider one diffusion term $\xi = (\xi^1, \dots, \xi^n)$). Incorporating more noise terms is straightforward.

Step 1. First of all, we extend b and ξ to the whole real time line $[0, \infty)$ by simply defining

$$\begin{aligned} b(t, x) &= b(T, x), \quad t > T, \quad x \in \mathbb{R}^n, \\ \xi(t) &= \xi(T), \quad t > T. \end{aligned}$$

We note that trivially

$$b \in L^\infty([0, \infty); C_b^\alpha(\mathbb{R}^n, \mathbb{R}^n)), \quad \xi \in L^\infty([0, \infty)).$$

For $\lambda > 0$, consider the parabolic equation

$$\partial_t \psi_\lambda + L\psi_\lambda - \lambda \psi_\lambda = -b, \quad t > 0, \quad x \in \mathbb{R}^n, \quad (4.2.5)$$

with (assume Einstein's convention of summing over repeated indices, running from 1 to n)

$$L\psi_\lambda = \frac{1}{2} \xi^j(t) \xi^k(t) \partial_j \partial_k \psi_\lambda + b^k(t, x) \partial_k \psi_\lambda.$$

As in⁷⁶, we note that the notion of solution of (4.2.5) is not standard since b and ξ are only measurable in time instead of continuous and as in the

preliminaries, we prescribe that a function $\psi_\lambda : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ [†] in the space $L^\infty([0, \infty); C_b^{2+\alpha}(\mathbb{R}^n, \mathbb{R}^n))$ is a solution to (4.2.5) if

$$\psi_\lambda(t, x) - \psi_\lambda(s, x) = \int_s^t [-L\psi_\lambda(r, x) + \lambda\psi_\lambda(r, x) - b(r, x)] dr,$$

$0 \leq s < t, x \in \mathbb{R}^n$. From the above identity, it follows that ψ is Lipschitz continuous in time, although it also turns out to enjoy other regularity properties (see the results in¹¹⁶). Due to Theorem 29 (making the choice $f = -b$), there exists a unique solution ψ_λ of (4.2.5) in the space $L^\infty([0, \infty); C_b^{2+\alpha}(\mathbb{R}^n, \mathbb{R}^n))$. Moreover, Theorem 29 also yields the bound

$$\sup_{t \geq 0} \|\psi_\lambda(t, \cdot)\|_{C_b^{2+\alpha}} \lesssim \sup_{t \geq 0} \|b(t, \cdot)\|_{C_b^\alpha}. \quad (4.2.6)$$

Step 2. Construct the function $\Psi_\lambda : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ as

$$\Psi_\lambda(t, x) = x + \psi_\lambda(t, x), \quad t \geq 0, \quad x \in \mathbb{R}^n. \quad (4.2.7)$$

By means of the following lemma, originally appearing in⁷⁶, we can establish that the function Ψ_λ gains extra regularity with respect to ψ_λ .

Lemma 94. *For λ large enough such that we achieve the condition[‡]*

$$\|D\psi_\lambda\|_{L_{t,x}^\infty} < 1, \quad (4.2.8)$$

the following properties are satisfied:

- (i) *Uniformly in $t \in [0, \infty)$, Ψ_λ has bounded first and second spatial derivatives, and $D_x^2 \Psi_\lambda$ is globally α -Hölder continuous.*
- (ii) *For any $t \in [0, \infty)$, $\Psi_\lambda(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a non-singular diffeomorphism of class C^2 .*
- (iii) *Uniformly in $t \in [0, \infty)$, $\Psi_\lambda^{-1}(t, \cdot)$ has bounded first and second derivatives.*

[†]Zero is included here for convenience.

[‡]We employ the notation $L_{t,x}^\infty$ for the essential supremum in both t and x .

In order to be able to apply Lemma 94 to our function ψ_λ , we need to show that for large enough λ , ψ_λ satisfies assumption (4.2.8). In the original argument in ⁷⁶, the fact that $\|D\psi_\lambda\|_{L_{t,x}^\infty} < 1$ for large enough λ is established by means of an auxiliary lemma, which states that the solution ψ_λ of equation (4.2.5) satisfies $\|D\psi_\lambda\|_{L_{t,x}^\infty} \rightarrow 0$, as $\lambda \rightarrow \infty$. Hence we look for a solution of the parabolic equation (4.2.5). In the original argument ⁷⁶, this equation becomes (set $\xi_i = e_i$, $i = 1, \dots, N$)

$$\partial_t \psi_\lambda + \frac{1}{2} \Delta \psi_\lambda - \lambda \psi_\lambda = -b - b \cdot D\psi_\lambda = \tilde{f}, \quad t > 0, \quad x \in \mathbb{R}^n, \quad (4.2.9)$$

and it is argued that the general solution of (4.2.9) can be explicitly expressed as

$$\psi_\lambda(t, x) = \int_t^{+\infty} e^{-\lambda(r-t)} P_{r-t} \tilde{f}(r, \cdot)(x) dr = \int_0^{+\infty} e^{-\lambda s} P_s \tilde{f}(t+s, \cdot)(x) ds, \quad (4.2.10)$$

where P_t is the forward heat semigroup given by

$$P_t g(x) = \int_{\mathbb{R}^n} p(t, x-y) g(y) dy,$$

and the fundamental heat solution p is defined by the explicit formula

$$p(t, x) = (4\pi)^{-n/2} ((2t)^{-n/2}) \exp(-(t^{-1}|x|^2/2)),$$

where $t > 0$, $x \in \mathbb{R}^n$, and extended to $t = 0$ by continuity. We note that although there are various classical theorems that construct solutions of the type (4.2.10) to parabolic equations (see for instance ^{83,114,151}), the present representation (4.2.10) is however not classic due to the low regularity in time of the drift and diffusion and was established for the first time by Priola and Krylov in ¹¹⁶. The formula is established in Lemma 3.3 in ¹¹⁶ and is only proven in the Laplacian case. As the authors themselves write, they could not find a precise reference for it in the literature. After arguing that formula

(4.2.10) holds, in ⁷⁶ they use the fact that

$$\|DP_t g\|_{L_x^\infty} \leq Ct^{-1/2}\|g\|_{L_x^\infty}, \quad g \in C_b(\mathbb{R}^n, \mathbb{R}^n), \quad t > 0, \quad (4.2.11)$$

and differentiate in formula (4.2.10) to obtain

$$\|D\psi_\lambda\|_{L_{t,x}^\infty} \leq C\lambda^{-1/2}\|\tilde{f}\|_{L_{t,x}^\infty} \leq C\lambda^{-1/2}(\|b\|_{L_{t,x}^\infty} + \|b\|_{L_{t,x}^\infty}\|D\psi_\lambda\|_{L_{t,x}^\infty}),$$

and hence

$$(1 - C\lambda^{-1/2}(\|b\|_{L_{t,x}^\infty}))\|D\psi_\lambda\|_{L_{t,x}^\infty} \leq C\lambda^{-1/2}\|b\|_{L_{t,x}^\infty}.$$

From the last inequality, it follows that by choosing λ large enough, one can obtain the bound

$$\|D\psi_\lambda\|_{L_{t,x}^\infty} < 1,$$

and hence condition (4.2.8) is met. In the general case, we apply the representation formula (1.3.5) in Theorem 31. Hence

$$G\tilde{f}(t, x) = \int_t^\infty \int_{\mathbb{R}^n} e^{-\lambda(s-t)} p(t, s, x-y) \tilde{f}(s, y) dy dt, \quad t > 0, \quad x \in \mathbb{R}^n,$$

where $\tilde{f} = -b - b \cdot D\psi_\lambda$. From (1.3.6) (consider the change of variables $u = s - t$), it follows that

$$\begin{aligned} |DG\tilde{f}(t, x)| &\lesssim \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda u} u^{-(n+1)/2} \exp\left(\frac{-|x-y|^2}{u}\right) \tilde{f}(s, y) dy du \\ &\leq \|\tilde{f}\|_{L_{t,x}^\infty} \int_0^\infty \int_{\mathbb{R}^n} e^{-\lambda u} u^{-1/2} \exp[-|z|^2] dz du \\ &\lesssim \|\tilde{f}\|_{L_{t,x}^\infty} \int_0^\infty e^{-\lambda u} u^{-1/2} du = \|\tilde{f}\|_{L_{t,x}^\infty} \frac{1}{\lambda^{1/2}} \int_0^\infty e^{-r} r^{-1/2} dr \lesssim \frac{\|\tilde{f}\|_{L_{t,x}^\infty}}{\lambda^{1/2}}, \end{aligned}$$

where we have made the change of variables $y - x = z\sqrt{u}$ in the second

inequality and $r = \lambda u$ in the equality in the last line. Therefore

$$\|DG\tilde{f}\|_{L_{t,x}^\infty} \lesssim \frac{\|\tilde{f}\|_{L_{t,x}^\infty}}{\lambda^{1/2}}. \quad (4.2.12)$$

Now we can repeat the previous techniques to obtain (4.2.8). We note that generalisations of formula (4.2.11) are available even in the manifold case and for (stochastic!) heat-type equations if the drift and diffusion coefficients are chosen to be smooth enough, where now the heat semigroup becomes the sub-Markovian semigroup associated to a uniformly elliptic SDE. This is studied in⁶⁹, where formulas for the derivatives of the heat semigroup are provided, illustrating the powerful smoothing properties of the semigroup in a general context.[§]

Step 3. The crucial point for deriving Theorem 93 by making use of Step 1 and Step 2 consists in carrying out the so-called Itô-Tanaka trick. Define

$$\begin{aligned} \tilde{b}(t, y) &= \lambda \psi_\lambda(t, \Psi_\lambda^{-1}(t, y)), \\ \tilde{\xi}(t, y) &= \xi(t) D\Psi_\lambda(t, \Psi_\lambda^{-1}(t, y)), \end{aligned} \quad (4.2.13)$$

and consider, for $s \in [0, T]$, $y \in \mathbb{R}^n$, the following SDE in Itô form:

$$Y_t = y + \int_s^t \tilde{b}(u, Y_u) du + \int_s^t \tilde{\xi}(u, Y_u) dW_u, \quad t \in [s, T]. \quad (4.2.14)$$

We claim that equation (4.2.14) is equivalent to the (general) flow equation (4.1.1) in the following sense: if $\phi_{s,t}(x)$ solves (4.1.1), then $Y_t = \Psi_\lambda(t, \phi_{s,t}(x))$ solves (4.2.14) with $y = \Psi_\lambda(s, x)$. We show this by applying Itô's Lemma to the function Ψ_λ (we remind that this function was defined in (4.2.7)). However, one needs to be slightly careful because this function does not enjoy enough regularity in time to satisfy the assumptions of the classical Itô's Lemma. From⁷⁶ we have the following weak in time version of Itô's Lemma.

[§]However, in the general stochastic case the growth takes a more complicated form.

Lemma 95. Let $\alpha \in (0, 1)$ and $\Psi \in L^\infty([0, \infty); C_b^{2+\alpha}(\mathbb{R}^n))$, satisfying

$$\Psi(t, x) - \Psi(s, x) = \int_s^t v(r, x) dr, \quad 0 \leq s < t, \quad x \in \mathbb{R}^n,$$

with $v \in L^\infty([0, \infty); C_b^\alpha(\mathbb{R}^n))$. Let $(X_t)_{t \geq 0}$ be an Itô process of the form

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \xi(s, X_s) dW_s,$$

where $b, \xi : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Then $\mathbb{P}$ -a.s.

$$\begin{aligned} \Psi(t, X_t) &= \Psi(0, x) + \int_0^t (v + (D\Psi)^T b)(s, X_s) ds \\ &\quad + \frac{1}{2} \int_0^t \text{Tr} [\xi \xi^T (H_X \Psi)](s, X_s) ds + \int_0^t ((D\Psi)^T \xi)(s, X_s) dW_s, \end{aligned}$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$, where $H_X \Psi$ is the spatial Hessian matrix of Ψ .

Remark 96. Lemma 95 can be straightforwardly extended to allow for multiple diffusion terms and applied to functions $\Psi \in L^\infty([0, \infty); C_b^{2+\alpha}(\mathbb{R}^n, \mathbb{R}^n))$ component-wise.

Fix $s \geq 0$ and $x \in \mathbb{R}^n$. By application of Lemma 95 to $X_t = \phi_{s,t}(x)$, we observe

$$\begin{aligned} \Psi_\lambda(t, X_t) &= \Psi_\lambda(s, x) + \int_s^t v(r, X_r) dr + \int_s^t ((D\Psi_\lambda)^T b)(r, X_r) dr \\ &\quad + \frac{1}{2} \int_s^t \text{Tr} [\xi \xi^T (H_X \Psi_\lambda)](r, X_r) dr + \int_s^t ((D\Psi_\lambda)^T \xi)(r, X_r) dW_r, \end{aligned}$$

where $t \geq s$ and

- $v = \lambda \psi_\lambda - \frac{1}{2} \xi^i \xi^j \partial_i \partial_j \psi_\lambda - b^j \partial_j \psi_\lambda - b$,
- $(D\Psi_\lambda)^T b = b + b^j \partial_j \psi_\lambda$,
- $\frac{1}{2} \text{Tr} [\xi \xi^T (H_X \Psi_\lambda)] = \frac{1}{2} \xi^i \xi^j \partial_i \partial_j \psi_\lambda$,

which yields (4.2.14). We notice $\tilde{b}, \tilde{\xi} \in L^\infty([0, T]; C_b^{1+\alpha}(\mathbb{R}^n, \mathbb{R}^n))$ and by Assumption A2 with $k = 1$, equation (4.2.14) has a unique solution which

defines a $C^{1+\alpha'}$ -flow of diffeomorphisms $\varphi_{s,t}$, $0 \leq s \leq t \leq T$, for any $0 < \alpha' < \alpha$. As in ⁷⁶, owing to the regularity shown for ψ_λ in Step 1 and for Ψ_λ in Step 2, and using the fact that $\phi_{s,t} = \Psi_\lambda^{-1} \circ \varphi_{s,t} \circ \Psi_\lambda$, we conclude that $\phi_{s,t}$ enjoys the extra regularity specified in Theorem 93 and the solution to (4.1.1) is unique, which proves points (i) and (ii). The following diagram summarises the essence of the Itô-Tanaka trick.

$$\begin{array}{ccc}
 & \phi_{s,t} = \Psi_t^{-1} \circ \varphi_{s,t} \circ \Psi_s \in C^{1+\alpha'}, \quad \alpha' < \alpha & \\
 \mathbb{R}^n & \xrightarrow{\quad\quad\quad} & \mathbb{R}^n \\
 \downarrow \begin{array}{l} \text{Parabolic theory} \\ \Psi_s \in C^2 \\ \Psi = I + \psi_\lambda \end{array} & & \uparrow \Psi_s^{-1} \in C^2 \\
 \mathbb{R}^n & \xrightarrow{\quad\quad\quad} & \mathbb{R}^n \\
 \text{Itô-Tanaka} \quad \varphi_{s,t} \in C^{1+\alpha'}, \quad \alpha' < \alpha & & \\
 d\varphi_t(x) = \tilde{b}(t, \phi_t(x))dt + \sum \tilde{\xi}_i(t, \phi_t(x)) \circ dB_t^i & & \\
 \tilde{b} \in C_b^{1+\alpha}, \quad \tilde{\xi}_i \in C_b^{1+\alpha} & &
 \end{array}$$

The function $\phi_{s,t}$ enjoys improved regularity because it can be written as a composition of regular functions

Figure 4.1: Improved regularity of the flow $\phi_{s,t}$.

Step 4. We start by stating the following useful lemma, which can be found in¹²², where properties of flows of SDEs with stochastic coefficients are analysed.

Lemma 97. *Let $\alpha \in (0, 1]$ and $p \geq 1$ satisfying $\alpha p > n$. Moreover, assume there exists a positive constant C such that*

$$\mathbb{E} \left[\sup_{s \leq t \leq T} |D\phi_{s,t}(x)|^p \right] \leq C, \quad (4.2.15)$$

and

$$\mathbb{E} \left[\sup_{s \leq t \leq T} |D\phi_{s,t}(x) - D\phi_{s,t}(y)|^p \right] \leq C|x - y|^{\alpha p}, \quad (4.2.16)$$

(uniformly) for all $x, y \in \mathbb{R}^n$ and $0 \leq s \leq T$. Then for any $\alpha' \in (0, \alpha - n/p)$ and $\epsilon > n/p$, (4.2.1) holds.

Hence, in order to establish (4.2.1), it suffices to verify estimates (4.2.15) and (4.2.16).

Proof. For simplicity, we set $\xi = 1$, but our argument can be easily generalised to allow for L^∞ -diffusions. We divide the proof into three steps, since it involves a lengthy calculation.

Step 4.1. Since $Y_t(x) = \Psi_\lambda(t, \phi_{s,t}(x))$ satisfies the modified SDE (4.2.14), we have the following equation for its (spatial) derivative

$$\begin{aligned} DY_t(x) &= D\Psi_\lambda(s, x) - \lambda \int_s^t D\psi_\lambda(r, \phi_{s,r}(x)) D\Psi_\lambda^{-1}(r, Y_r(x)) DY_r(x) dr \\ &\quad + \int_s^t D^2\Psi_\lambda(r, \phi_{s,r}(x)) D\Psi_\lambda^{-1}(r, Y_r(x)) DY_r(x) dW_r, \end{aligned} \quad (4.2.17)$$

where $0 \leq s \leq t \leq T$, $x \in \mathbb{R}^n$. To prove the first bound (4.2.15), noting that $D\phi_{s,t}(x) = D\Psi_\lambda^{-1}(t, Y_t(x)) DY_t(x)$, we have

$$\mathbb{E} \left[\sup_{s \leq t \leq T} |D\phi_{s,t}(x)|^p \right] \leq \sup_{s \leq t \leq T} |D\Psi_\lambda^{-1}(t, Y_t(x))|^p \mathbb{E} \left[\sup_{s \leq t \leq T} |DY_t(x)|^p \right].$$

Since $D\Psi_\lambda^{-1}(t, \cdot)$ is bounded uniformly in time, we only need to show that $\mathbb{E} [\sup_{s \leq t \leq T} |DY_t(x)|^p] < \infty$, which can be proven directly from (4.2.17) by a standard argument using Grönwall's lemma and Burkholder-Davis-Gundy inequality (see (1.4.17)), together with the fact that $D\psi_\lambda, D\Psi_\lambda, D\Psi_\lambda^{-1}, D^2\Psi_\lambda$ are bounded uniformly in time.

Step 4.2. Next, we claim that

$$\mathbb{E} \left[\sup_{s \leq t \leq T} |\phi_{s,t}(x) - \phi_{s,t}(y)|^p \right] \lesssim |x - y|^{\beta p} \quad (4.2.18)$$

holds for any $\beta \in (0, 1]$. To verify this, we first note that

$$\begin{aligned} \mathbb{E} \left[\sup_{s \leq t \leq T} |\phi_{s,t}(x) - \phi_{s,t}(y)|^p \right] &= \mathbb{E} \left[\sup_{s \leq t \leq T} |\Psi_\lambda^{-1}(t, Y_t(x)) - \Psi_\lambda^{-1}(t, Y_t(y))|^p \right] \\ &\lesssim \mathbb{E} \left[\sup_{s \leq t \leq T} |Y_t(x) - Y_t(y)|^p \right], \end{aligned}$$

where we have taken into account that the Lipschitz seminorm $[\Psi_\lambda^{-1}(t, \cdot)]_{\text{Lip}}$ is bounded uniformly in time since $\Psi_\lambda^{-1} \in L^\infty([0, T]; C_b^2(\mathbb{R}^n, \mathbb{R}^n))$. By using the modified SDE (4.2.14), we also have

$$\begin{aligned} |Y_t(x) - Y_t(y)| &\leq |\Psi_\lambda(s, x) - \Psi_\lambda(s, y)| \\ &\quad + \lambda \int_s^t |\psi_\lambda(r, \Psi_\lambda^{-1}(r, Y_r(x))) - \psi_\lambda(r, \Psi_\lambda^{-1}(r, Y_r(y)))| dr \\ &\quad + \left| \int_s^t (D\Psi_\lambda(r, \Psi_\lambda^{-1}(r, Y_r(x))) - D\Psi_\lambda(r, \Psi_\lambda^{-1}(r, Y_r(y)))) dW_r \right|. \end{aligned}$$

Employing the fact that ψ_λ and Ψ_λ^{-1} are globally Lipschitz, we obtain

$$\begin{aligned} |Y_t(x) - Y_t(y)| &\leq |\Psi_\lambda(s, x) - \Psi_\lambda(s, y)| + C_1 \int_s^t |Y_r(x) - Y_r(y)| dr \\ &\quad + \sup_{s \leq u \leq t} \left| \int_s^u (D\Psi_\lambda(r, \Psi_\lambda^{-1}(r, Y_r(x))) - D\Psi_\lambda(r, \Psi_\lambda^{-1}(r, Y_r(y)))) dW_r \right|, \end{aligned}$$

where $C_1 = \lambda \sup_{s \leq t \leq T} [\psi_\lambda(t, \cdot)]_{\text{Lip}} \sup_{s \leq t \leq T} [\Psi_\lambda^{-1}(t, \cdot)]_{\text{Lip}}$ is a finite constant.

By applying Grönwall's inequality, we get

$$\begin{aligned}
|Y_t(x) - Y_t(y)| &\leq (|\Psi_\lambda(s, x) - \Psi_\lambda(s, y)| \\
&\quad + \sup_{s \leq u \leq t} \left| \int_s^u (D\Psi_\lambda(r, \Psi_\lambda^{-1}(r, Y_r(x))) - D\Psi_\lambda(r, \Psi_\lambda^{-1}(r, Y_r(y)))) \, dW_r \right| \\
&\quad \times \exp(C_1(T - s)).
\end{aligned}$$

Moreover, by using Burkholder-Davis-Gundy inequality (1.4.17), we have that for any $p \geq 1$, there exists a constant $C_p^* > 0$ such that

$$\begin{aligned}
&\mathbb{E} \left[\sup_{s \leq u \leq t} \left| \int_s^u (D\Psi_\lambda(r, \Psi_\lambda^{-1}(r, Y_r(x))) - D\Psi_\lambda(r, \Psi_\lambda^{-1}(r, Y_r(y)))) \, dW_r \right|^p \right] \\
&\leq C_p^* \mathbb{E} \left[\left(\int_s^t |D\Psi_\lambda(r, \Psi_\lambda^{-1}(r, Y_r(x))) - D\Psi_\lambda(r, \Psi_\lambda^{-1}(r, Y_r(y)))|^2 \, dr \right)^{p/2} \right] \\
&\leq C_2 \int_s^t \mathbb{E} \left[\sup_{s \leq u \leq r} |Y_u(x) - Y_u(y)|^p \right] \, dr.
\end{aligned}$$

¶ Hence, applying Grönwall's inequality to the function

$$f(t) = \mathbb{E} \left[\sup_{s \leq u \leq t} |Y_u(x) - Y_u(y)|^p \right], \quad t \in [0, T], \quad x, y \in \mathbb{R}^n,$$

we obtain the estimate

$$\mathbb{E} \left[\sup_{s \leq u \leq t} |Y_u(x) - Y_u(y)|^p \right] \leq C_3 |\Psi_\lambda(s, x) - \Psi_\lambda(s, y)|^p \lesssim |x - y|^{\beta p}, \quad (4.2.19)$$

for any $s \leq t \leq T$ and $\beta \in (0, 1]$ as expected, where C_3 can be explicitly computed to be $C_3 = \exp[C_2(T - s)\exp(pC_1(T - s))] < \infty$ and we have used the fact that the β -Hölder seminorm $[\Psi_\lambda(t, \cdot)]_\beta$ is uniformly bounded in time.

Step 4.3. Finally, we establish estimate (4.2.16). For this, employing a

¶ Explicitly, $C_2 = C_p^*(T - s)^{p-1} \sup_{s \leq t \leq T} [D\Psi_\lambda(t, \cdot)]_{\text{Lip}} \sup_{s \leq t \leq T} [\Psi_\lambda^{-1}(t, \cdot)]_{\text{Lip}}$.

similar argument as before, we start by noting that

$$\begin{aligned}
& \mathbb{E} \left[\sup_{s \leq t \leq T} |D\phi_{s,t}(x) - D\phi_{s,t}(y)|^p \right] \\
&= \mathbb{E} \left[\sup_{s \leq t \leq T} |D\Psi_\lambda^{-1}(t, Y_t(x))DY_t(x) - D\Psi_\lambda^{-1}(t, Y_t(y))DY_t(y)|^p \right] \\
&\lesssim \mathbb{E} \left[\sup_{s \leq t \leq T} |D\Psi_\lambda^{-1}(t, Y_t(x))|^p |DY_t(x) - DY_t(y)|^p \right] \\
&\quad + \mathbb{E} \left[\sup_{s \leq t \leq T} |DY_t(y)|^p |D\Psi_\lambda^{-1}(t, Y_t(x)) - D\Psi_\lambda^{-1}(t, Y_t(y))|^p \right] \\
&\lesssim \mathbb{E} \left[\sup_{s \leq t \leq T} |DY_t(x) - DY_t(y)|^p \right] \\
&\quad + \mathbb{E} \left[\sup_{s \leq t \leq T} |D\Psi_\lambda^{-1}(t, Y_t(x)) - D\Psi_\lambda^{-1}(t, Y_t(y))|^{2p} \right]^{1/2},
\end{aligned} \tag{4.2.20}$$

where we have applied Hölder's inequality in the last line and have taken into account the fact that $\mathbb{E} [\sup_{s \leq t \leq T} |DY_t(x)|^{2p}]$ is uniformly bounded as shown at the beginning of this proof. The second term in (4.2.20) can be treated easily as

$$\begin{aligned}
& \mathbb{E} \left[\sup_{s \leq t \leq T} |D\Psi_\lambda^{-1}(t, Y_t(x)) - D\Psi_\lambda^{-1}(t, Y_t(y))|^{2p} \right]^{1/2} \\
&\lesssim \mathbb{E} \left[\sup_{s \leq t \leq T} |Y_t(x) - Y_t(y)|^{2p} \right]^{1/2} \lesssim |x - y|^\alpha,
\end{aligned}$$

where we have used (4.2.19) with $\beta = \alpha$ and the fact that $D\Psi_\lambda^{-1}$ is globally Lipschitz uniformly in time. For the first term in (4.2.20), using (4.2.17) and defining

$$\begin{aligned}
P_t(x) &= D\psi_\lambda(t, \phi_{s,t}(x))D\Psi_\lambda^{-1}(t, Y_t(x)), \\
Q_t(x) &= D^2\Psi_\lambda(t, \phi_{s,t}(x))D\Psi_\lambda^{-1}(t, Y_t(x)), \\
M_t(x) &= \int_s^t Q_r(x)DY_r(x)dW_r,
\end{aligned}$$

we obtain the following inequality

$$\begin{aligned}
|DY_t(x) - DY_t(y)| &\leq |D\Psi_\lambda(s, x) - D\Psi_\lambda(s, y)| + |M_t(x) - M_t(y)| \\
&\quad + \lambda \int_s^t |P_r(x)DY_r(x) - P_r(y)DY_r(y)|dr \\
&\leq |D\Psi_\lambda(s, x) - D\Psi_\lambda(s, y)| + \sup_{s \leq r \leq t} |M_r(x) - M_r(y)| \\
&\quad + \lambda \left(\int_s^t |P_r(x)| |DY_r(x) - DY_r(y)|dr + \int_s^t |DY_r(y)| |P_r(x) - P_r(y)|dr \right).
\end{aligned}$$

Employing Burkholder-Davis-Gundy inequality (1.4.17), we also check that

$$\begin{aligned}
\mathbb{E} \left[\sup_{s \leq u \leq t} |M_u(x) - M_u(y)|^p \right] &\lesssim \int_s^t \mathbb{E} \left[\sup_{s \leq u \leq r} |DY_u(x) - DY_u(y)|^p \right] dr \\
&\quad + \mathbb{E} \left[\int_s^t |DY_r(y)|^p |Q_r(x) - Q_r(y)|^p dr \right].
\end{aligned}$$

By application of Grönwall's lemma as before, we arrive at

$$\begin{aligned}
\mathbb{E} \left[\sup_{s \leq u \leq t} |DY_u(x) - DY_u(y)|^p \right] &\lesssim |D\Psi_\lambda(s, x) - D\Psi_\lambda(s, y)|^p \\
&\quad + \mathbb{E} \left[\int_s^t |DY_r(y)|^p |P_r(x) - P_r(y)|^p dr \right] + \mathbb{E} \left[\int_s^t |DY_r(y)|^p |Q_r(x) - Q_r(y)|^p dr \right].
\end{aligned} \tag{4.2.21}$$

The first term in (4.2.21) can be controlled easily as

$$|D\Psi_\lambda(s, x) - D\Psi_\lambda(s, y)|^p \lesssim |x - y|^{\alpha p}.$$

For the second term in (4.2.21), applying Hölder's and Minkowski inequality

we obtain the control

$$\begin{aligned}
& \mathbb{E} \left[\int_s^t |DY_r(y)|^p |P_r(x) - P_r(y)|^p dr \right] \lesssim \mathbb{E} \left[\int_s^t |P_r(x) - P_r(y)|^{2p} dr \right]^{1/2} \\
& \lesssim \mathbb{E} \left[\sup_{s \leq r \leq t} |D\psi_\lambda(r, \phi_{s,r}(x))|^{2p} |D\Psi_\lambda^{-1}(r, Y_t(x)) - D\Psi_\lambda^{-1}(r, Y_t(y))|^{2p} \right]^{1/2} \\
& \quad + \mathbb{E} \left[\sup_{s \leq r \leq t} |D\Psi_\lambda^{-1}(r, Y_r(y))|^{2p} |D\psi_\lambda(r, \phi_{s,r}(x)) - D\psi_\lambda(r, \phi_{s,r}(y))|^{2p} \right]^{1/2} \\
& \lesssim \mathbb{E} \left[\sup_{s \leq r \leq t} |D\Psi_\lambda^{-1}(r, Y_r(x)) - D\Psi_\lambda^{-1}(r, Y_r(y))|^{2p} \right]^{1/2} \\
& \quad + \mathbb{E} \left[\sup_{s \leq r \leq t} |D\psi_\lambda(r, \phi_{s,r}(x)) - D\psi_\lambda(r, \phi_{s,r}(y))|^{2p} \right]^{1/2} \lesssim |x - y|^{\alpha p},
\end{aligned}$$

where in the last inequality, we have employed (4.2.18) and (4.2.19) with $\beta = \alpha$, together with the fact that $D\Psi_\lambda^{-1}, D\psi_\lambda$ are globally Lipschitz uniformly in time. Similarly, for the last term in (4.2.21), we get the bound

$$\begin{aligned}
& \mathbb{E} \left[\int_s^T |DY_r(y)|^p |Q_r(x) - Q_r(y)|^p dr \right] \lesssim \mathbb{E} \left[\int_s^t |Q_r(x) - Q_r(y)|^{2p} dr \right]^{1/2} \\
& \lesssim \mathbb{E} \left[\sup_{s \leq r \leq t} |D\Psi_\lambda^{-1}(r, Y_r(x)) - D\Psi_\lambda^{-1}(r, Y_r(y))|^{2p} \right]^{1/2} \\
& \quad + \mathbb{E} \left[\sup_{s \leq r \leq t} |D^2\Psi_\lambda(r, \phi_{s,r}(x)) - D^2\Psi_\lambda(r, \phi_{s,r}(y))|^{2p} \right]^{1/2} \\
& \lesssim \mathbb{E} \left[\sup_{s \leq r \leq t} |Y_r(x) - Y_r(y)|^{2\alpha p} \right]^{1/2} + \mathbb{E} \left[\sup_{s \leq r \leq t} |\phi_{s,r}(x) - \phi_{s,r}(y)|^{2\alpha p} \right]^{1/2} \\
& \lesssim |x - y|^{\alpha p},
\end{aligned}$$

where in the fourth line, we have employed the fact that the α -Hölder seminorms $[D\Psi_\lambda^{-1}(t, \cdot)]_\alpha, [D^2\Psi_\lambda(t, \cdot)]_\alpha$ are bounded uniformly in time and in the last line, we have applied (4.2.18) and (4.2.19) with $\beta = 1$. Hence, the estimate (4.2.16) holds and we can deduce the bound (4.2.1) by Lemma 97. $\square$

Step 5. To prove point (iii) in this lemma, for $n \in \mathbb{N}$, let ψ_λ^n be the solution of (4.2.5) with b replaced by b^n and ξ replaced by ξ^n , Ψ_λ^n the function (4.2.7) constructed from ψ_λ^n , and $\tilde{b}^n, \tilde{\xi}^n$ the vector fields (4.2.13) derived from ψ_λ^n and Ψ_λ^n .

By arguing as in⁷⁶, one can show that $\psi^n \rightarrow \psi$ in $L^\infty([0, T]; C_b^{2+\alpha}(\mathbb{R}^n, \mathbb{R}^n))$, and therefore $\tilde{b}^n \rightarrow \tilde{b}$, $\tilde{\xi}^n \rightarrow \tilde{\xi}$ in $L^\infty([0, T]; C_b^{1+\alpha}(\mathbb{R}^n, \mathbb{R}^n))$, $n \rightarrow \infty$, which allows us to deduce (4.2.2). Letting $\tilde{\varphi}_{s,t}^n$ be the flow of (4.2.14) with vector fields $\tilde{b}^n$ and $\tilde{\xi}^n$, we observe as in⁷⁶ that $D\tilde{\varphi}_{s,t}^n$ satisfies

$$D\tilde{\varphi}_{s,t}^n = I + \int_s^t D\tilde{b}^n(u, \tilde{\varphi}_{s,u}^n) D\tilde{\varphi}_{s,u}^n du + \int_s^t D\tilde{\xi}^n(u, \tilde{\varphi}_{s,u}^n) D\tilde{\varphi}_{s,u}^n dW_u.$$

By taking into account Grönwall's inequality, the uniform boundedness of $\{D\tilde{b}^n\}_{n=1}^\infty$, $\{D\tilde{\xi}^n\}_{n=1}^\infty$, and Burkholder-Davis-Gundy inequality (1.4.17), we obtain the estimate

$$\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}^n} \sup_{0 \leq s \leq T} \mathbb{E} \left[\sup_{s \leq t \leq T} |D\tilde{\varphi}_{s,t}^n(x)|^p \right] < \infty,$$

for any $p \geq 1$, and using the uniform boundedness of ψ_λ^n , Ψ_λ^n , and $(\Psi_\lambda^n)^{-1}$, we deduce (4.2.3). Finally, combining (4.2.3) and (4.2.2), we can obtain (4.2.4) by following the same procedure as in⁷⁶. $\square$

Remark 98. We note that Theorem 93 also holds for the inverse flow $\phi_{s,t}^{-1}$. Indeed, by using the flow properties we observe that

$$\phi_{s,t}(\phi_{s,t}^{-1}(y)) = \phi_{s,t}^{-1}(y) + \int_s^t b(r, \phi_{s,r}(\phi_{s,t}^{-1}(y))) dr + \sum_{j=1}^N \int_s^t \xi_j(r) dW_r^j,$$

which can be rewritten as

$$\phi_{s,t}^{-1}(y) = y - \int_s^t b(r, \phi_{r,t}^{-1}(y)) dr - \sum_{j=1}^N \int_s^t \xi_j(r) dW_r^j,$$

where $0 \leq s < t$, $y \in \mathbb{R}^n$, so the inverse flow satisfies a backward SDE with a sign change in the drift and diffusion coefficients.

As in⁷⁶, we present the following important consequences of Theorem 93 (iii).

Corollary 99. *Let $R > 0$ and $p \geq 1$. We have*

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{B(0,R)} \sup_{t \in [0,T]} \|\phi_t^n(x) - \phi_t(x)\|^p d^n x \right] &= 0, \\ \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{B(0,R)} \sup_{t \in [0,T]} \|D\phi_t^n(x) - D\phi_t(x)\|^p d^n x \right] &= 0. \end{aligned}$$

In particular

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^T \int_{B(0,R)} \|\phi_t^n(x) - \phi_t(x)\|^p d^n x dt \right] = 0, \quad (4.2.22)$$

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^T \int_{B(0,R)} \|D\phi_t^n(x) - D\phi_t(x)\|^p d^n x dt \right] = 0. \quad (4.2.23)$$

4.3 EXISTENCE OF WEAK SOLUTIONS

In this section, we prove the existence of solutions of the SPDE (0.2.10), and uniqueness will be demonstrated in the next section. First we establish the existence of weak solutions in the case when the drift vector field b is smooth in space. Then we extend these results to the case when b is only α -Hölder continuous and $W_{loc}^{1,2}$, uniformly in time, by means of a limiting argument.

Existence of solutions for smooth drift.

Proposition 100 (Existence of classical solutions for regular drift and initial condition). *Let $b \in L^\infty([0, T]; C_b^{4+\beta}(\mathbb{R}^n, \mathbb{R}^n))$, for some $\beta \in (0, 1)$, $\xi_i \in L^\infty([0, T])$, $i = 1, \dots, N$ be semimartingales^{||}, and $K_0 \in C^3(\bigwedge^k(T^*\mathbb{R}^n))$. Then there exists a classical solution to equation (0.2.10) of the form $K_t(x) = (\phi_t)_* K_0(x)$, where ϕ_t is the flow of (4.1.2).*

Proof. For the sake of simplicity, we assume without loss of generality that $N = 1$. Let $\phi_{s,t}$ be the solution to (4.1.1) from time s to time t , for $0 \leq s \leq$

^{||}This is purely a time-regularity condition as we discussed when introducing the notion of solutions at the beginning of the chapter. It should be kept in mind that ξ_i , $i = 1, \dots, N$ are deterministic.

$t \leq T$, which is a C^4 -stochastic flow of diffeomorphisms (this is a classical result, see Assumption A2 with $k = 3$). Then, the corresponding backward equation (see pages 113 and 117 in ¹²⁰) reads

$$\phi_{t,s}(x) = x - \int_s^t b(r, \phi_{t,r}(x)) dr - \int_s^t \xi(r) \circ \widehat{dW}_r,$$

where $\widehat{dW}_r$ denotes backward integration against W_r . By Itô's first formula on tensor fields (1.4.8), we obtain (choose $s = 0$) that $\mathbb{P}$ -a.s.

$$\phi_{t,0}^* K_0(x) = K_0(x) - \int_0^t \mathcal{L}_b(\phi_{r,0}^* K_0(x)) dr - \int_0^t \mathcal{L}_\xi(\phi_{r,0}^* K_0(x)) \circ dW_r, \quad (4.3.1)$$

$t \in [0, T]$. Setting $K_t(x) = \phi_{t,0}^* K_0(x) = (\phi_t)_* K_0(x)$, we see that (4.3.1) is identical to (0.2.10). We claim that K solves (0.2.10) in the sense of Definition 87. For this, we note that $\mathbb{P}$ -a.s. K has paths in the class $C([0, T]; C^3(\bigwedge^k(T^*\mathbb{R}^n)))$, since by definition

$$((\phi_t)_* K_0)_{j_1, \dots, j_k}(x) = (K_0)_{i_1, \dots, i_k}(\psi_t(x)) \frac{\partial \psi_t^{i_1}}{\partial x_{j_1}}(x) \cdots \frac{\partial \psi_t^{i_k}}{\partial x_{j_k}}(x), \quad (4.3.2)$$

in global coordinates, where $\psi_t = \phi_t^{-1}$ is the inverse flow and summation over repeated indices is assumed. Moreover, the semimartingale assumption in Definition 87 is met because ψ_t is a C^4 -semimartingale and hence K is a C^3 -semimartingale by the representation formula (4.3.2). Since we have shown that K is a strong solution, it is also a weak solution in the sense of Definition 90.

□

Next, we show the existence of weak solutions of the linear transport of k -forms (0.2.10) when the driving vector field b is smooth in space and the initial condition is of low regularity. First we state some useful lemmas.

Lemma 101. *Given a k -form $K \in L_{loc}^\infty(\bigwedge^k(T^*\mathbb{R}^n))$, a test k -form $\theta \in$*

$C_0^\infty(\bigwedge^k(T^*\mathbb{R}^n))$, and a C^1 -diffeomorphism ϕ , we have the identity

$$\int_{\mathbb{R}^n} \langle \phi_* K(x), \theta(x) \rangle d^n x = \int_{\mathbb{R}^n} \langle K(y), J\phi(y) (\phi^* \theta^\sharp(y))^\flat \rangle d^n y. \quad (4.3.3)$$

Here, for a k -form θ , the corresponding sharp k -vector field $\theta^\sharp$ is defined as

$$\theta^\sharp = \delta^{i_1 j_1} \dots \delta^{i_k j_k} \theta_{j_1, \dots, j_k}(x) \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_k}},$$

and similarly for a k -vector field v , the flat k -form $v^\flat$ is defined as

$$v^\flat = \delta_{i_1 j_1} \dots \delta_{i_k j_k} v^{j_1, \dots, j_k}(x) dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

Proof. By standard tensor identities, we have

$$\langle \phi_* K(x), \theta(x) \rangle = \phi_* K(x)(\theta^\sharp(x)) = K(\phi^{-1}(x))(\phi^* \theta^\sharp(\phi^{-1}(x))).$$

Hence, by the change of coordinates $y = \phi^{-1}(x)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \langle \phi_* K(x), \theta(x) \rangle d^n x &= \int_{\mathbb{R}^n} (K(\phi^{-1}(x))(\phi^* \theta^\sharp(\phi^{-1}(x)))) d^n x \\ &= \int_{\mathbb{R}^n} K(y)(\phi^* \theta^\sharp(y)) J\phi(y) d^n y \\ &= \int_{\mathbb{R}^n} \langle K(y), J\phi(y)(\phi^* \theta^\sharp(y))^\flat \rangle d^n y. \end{aligned}$$

For further comments on the geometric notation sharp/flat, we point the reader to Appendix A. $\square$

Remark 102. When changing coordinates in Lemma 101, we observe that the term $J\phi(y)(\phi^* \theta^\sharp(y))^\flat$ appears. For simplicity in our future calculations, we will sometimes denote it by $\tilde{\theta}$.

We introduce a lemma that we will employ several times.

Lemma 103. Let $\theta \in C_0^\infty(\bigwedge^k(T^*\mathbb{R}^n))$, ϕ_t be a C^1 -flow of diffeomorphism and $\tilde{\theta}_t = J\phi_t(y)(\phi_t^* \theta^\sharp(y))^\flat$. Moreover, assume that the drift vector field b satisfies Assumption A3 (ii). Then $\mathbb{P}$ -a.s. we have the following properties

(A) $\widetilde{\theta}_t$ has compact support, uniformly in $t \in [0, T]$.

(B) $\|\widetilde{\theta}\|_{L_{t,x}^\infty} < \infty$ ^{**}.

(C) $\|\widetilde{\mathcal{L}_b\theta}(t)\|_{L_x^2}$ is of class $L^\infty([0, T])$ (and hence $\|\widetilde{\mathcal{L}_b\theta}(t)\|_{L_x^1}$ is also of class $L^\infty([0, T])$).

Proof. By applying Lemma 101, we obtain the identity

$$\int_{\mathbb{R}^n} \langle (\phi_t)_* K(x), \theta(x) \rangle d^n x = \int_{\mathbb{R}^n} \langle K(y), \widetilde{\theta}_t(x) \rangle d^n y, \quad t \in [0, T].$$

We observe that by definition, we have the explicit formula

$$(\phi_t^* \theta^\sharp)_{j_1, \dots, j_k}(x) = \theta_{i_1, \dots, i_k}(\phi_t(x)) \frac{\partial \psi_t^{i_1}}{\partial x_{j_1}}(\phi_t(x)) \cdots \frac{\partial \psi_t^{i_k}}{\partial x_{j_k}}(\phi_t(x)) \quad (4.3.4)$$

for the pull-back of the k -vector field $\theta^\sharp$ in global coordinates. Since ϕ_t is a C^1 -flow of diffeomorphisms, both its (first order) derivatives and the derivatives of the inverse flow are $\mathbb{P}$ -a.s. continuous in (t, x) . Hence $\mathbb{P}$ -a.s., $\widetilde{\theta}_t$ is uniformly bounded on every ball $B(0, R')$, $R' > 0$, since every continuous function on a compact set reaches a maximum. ^{††} Also, since θ has support contained in some ball which we denote $B(0, R)$, we observe that the following set

$$\{\phi_t^{-1}(x) : x \in B(0, R), \quad t \in [0, T]\}$$

is bounded a.s. and we can hence choose $R^* : \Omega \rightarrow (0, \infty]$ ^{††} such that $R^* < \infty$ a.s. and

$$\{\phi_t^{-1}(x) : x \in B(0, R), \quad t \in [0, T]\} \subset B(0, R^*).$$

We can therefore conclude (A) and (B).

^{**} $L_{t,x}^\infty$ denotes the (essential) supremum in both t and x .

^{††} We note that in our case, since we consider the Euclidean metric, the k -vector field $\theta^\sharp$ has the same coordinate expression as θ , and the k -form $(\phi_t^* \theta^\sharp)^\flat$ has the same coordinate expression as $\phi_t^* \theta^\flat$.

^{††} More concretely, $R^* = \sup_{t \in [0, T], x \in B(0, R)} |\phi_t^{-1}(x)|$, which is measurable and hence a random variable.

Moreover, if b satisfies Assumption A3 (ii), property (A) is trivially available for $\widetilde{\mathcal{L}_b^T \theta}(t, x) = J\phi_t(x)(\phi_t^*(\mathcal{L}_b^T \theta)^\sharp)^\flat(x)$, $t \in [0, T]$, $x \in \mathbb{R}^n$, simply because the last expression is well-defined. Furthermore, by taking into account the explicit formula for the L^2 -adjoint of the Lie derivative (1.2.3), we conclude that the coordinate expression of $\mathcal{L}_b^T \theta$ only contains terms of the type: (1) derivatives of b times components of θ , (2) derivatives of θ times components of b . Since b is of class $L^\infty([0, T]; W^{1,2})$ and θ is of class $W^{1,\infty}$ (note that it is smooth with compact support), by application of Hölder's inequality for tensor fields we can obtain ^{§§}

$$\|\widetilde{\mathcal{L}_b \theta}(t)\|_{L_x^2} \lesssim \|J\phi_t\|_{L_{R^*}^\infty} \|D\psi_t\|_{L_{R^*}^\infty}^k \|\mathcal{L}_b \theta\|_{L_{R^*}^2}, \quad (4.3.5)$$

where we have employed the notation $\psi_t = \phi_t^{-1}$ and used the explicit formula for the pull-back of a k -vector field (4.3.4). The last term in (4.3.5) can be bounded as

$$\|\mathcal{L}_b \theta\|_{L_{R^*}^2} \lesssim \|Db\|_{L_{R^*}^2} \|\theta\|_{L_{R^*}^\infty} + \|b\|_{L_{R^*}^2} \|D\theta\|_{L_{R^*}^\infty} \leq \|b\|_{W_{R^*}^{1,2}} \|\theta\|_{W_{R^*}^{1,\infty}}. \quad (4.3.6)$$

Since $R^* < \infty$ a.s., all terms in (4.3.5) (take into account (4.3.6) for the last term) are a.s. in $L^\infty([0, T])$ so we can conclude (C). $\square$

Proposition 104 (Weak existence for weak initial condition and strong drift). *Let $b \in L^\infty([0, T]; C_b^{4+\beta}(\mathbb{R}^n, \mathbb{R}^n))$ for some $\beta \in (0, 1)$, $\xi_i \in L^\infty([0, T])$, $i = 1, \dots, N$ be semimartingales, and $K_0 \in L^\infty(\bigwedge^k(T^*\mathbb{R}^n))$. Then there exists a weak solution to equation (0.2.10) of the form $K_t(x) = (\phi_t)_* K_0(x)$.*

Remark 105. *We stress that b and ξ_i , $i = 1, \dots, N$ are deterministic.*

Proof. We assume without loss of generality that $N = 1$. Consider the mollified initial data $K_0^\epsilon = \rho^\epsilon * K_0$. By the mollifier properties $K_0^\epsilon \rightarrow K_0$, $\epsilon \rightarrow 0$, strongly in L_{loc}^p , for any $1 \leq p < \infty$, and therefore weakly. Also, $K_0^\epsilon \rightarrow K_0$ weak-* in L_{loc}^∞ . Moreover, by Proposition 100, $K_t^\epsilon(x) = (\phi_t)_* K_0^\epsilon(x)$ is the unique strong solution to (0.2.10) and therefore also a weak solution.

^{§§}We have used the notation L_x^2 to stress that the L^2 -norm should be understood in the spatial sense.

Hence, for any test k -form $\theta \in C_0^\infty(\bigwedge^k(T^*\mathbb{R}^n))$ (4.0.3) holds a.s., i.e.,

$$\begin{aligned} \langle \langle (\phi_t)_* K_0^\epsilon, \theta \rangle \rangle_{L^2} - \langle \langle K_0^\epsilon, \theta \rangle \rangle_{L^2} &= - \int_0^t \langle \langle (\phi_s)_* K_0^\epsilon, \mathcal{L}_b^T \theta \rangle \rangle_{L^2} ds \\ &\quad - \int_0^t \langle \langle (\phi_s)_* K_0^\epsilon, \mathcal{L}_\xi^T \theta \rangle \rangle_{L^2} dW_s + \frac{1}{2} \int_0^t \langle \langle (\phi_s)_* K_0^\epsilon, \mathcal{L}_\xi^T \mathcal{L}_\xi^T \theta \rangle \rangle_{L^2} ds, \end{aligned}$$

for all $t \in [0, T]$. We analyse term-by-term convergence in the last equality.

- By the weak convergence $K_0^\epsilon \rightarrow K_0$ in L_{loc}^2 , the second term on the left-hand side converges as

$$\langle \langle K_0^\epsilon, \theta \rangle \rangle_{L^2} \rightarrow \langle \langle K_0, \theta \rangle \rangle_{L^2}.$$

- We claim that the first term on the left-hand side converges a.s. as

$$\langle \langle (\phi_t)_* K_0^\epsilon, \theta \rangle \rangle_{L^2} = \langle \langle K_0^\epsilon, \tilde{\theta}_t \rangle \rangle_{L^2} \rightarrow \langle \langle K_0, \tilde{\theta}_t \rangle \rangle_{L^2} = \langle \langle (\phi_t)_* K_0, \theta \rangle \rangle_{L^2},$$

where we have employed the notation $\tilde{\theta}_t(x) = J\phi_t(x)(\phi_t^* \theta^\sharp)^\flat(x)$ as established in Lemma 101. For this, note that a.s.

$$|\langle \langle K_0^\epsilon - K_0, \tilde{\theta}_t \rangle \rangle_{L^2}| \leq \|K_0^\epsilon - K_0\|_{L_{R^*}^1} \|\tilde{\theta}_t\|_{L_{t,x}^\infty} \rightarrow 0,$$

where we have taken into account the following facts (i) $\tilde{\theta}_t$ has compact support contained in $B(0, R^*)$, uniformly in $t \in [0, T]$, where $R^* < \infty$ a.s., (ii) $\mathbb{P}$ -a.s. $\|\tilde{\theta}_t\|_{L_{t,x}^\infty} < \infty$ (use Lemma 103 (A) & (B) for (i) & (ii)), and (iii) $K_0^\epsilon \rightarrow K_0$ in $L^1(B(0, R^*))$, a.s. from the mollifier properties, since $R^* < \infty$ a.s.

- By the bounded convergence theorem, the first term on the right-hand side converges a.s. as

$$\begin{aligned} \int_0^t \langle \langle (\phi_s)_* K_0^\epsilon, \mathcal{L}_b^T \theta \rangle \rangle_{L^2} ds &= \int_0^t \langle \langle K_0^\epsilon, \widetilde{\mathcal{L}_b^T \theta}(s) \rangle \rangle_{L^2} ds \\ &\rightarrow \int_0^t \langle \langle K_0, \widetilde{\mathcal{L}_b^T \theta}(s) \rangle \rangle_{L^2} ds = \int_0^t \langle \langle (\phi_s)_* K_0, \mathcal{L}_b^T \theta \rangle \rangle_{L^2} ds. \end{aligned}$$

For this, note that a.s. we have

$$|\langle\langle K_0^\epsilon - K_0, \widetilde{\mathcal{L}_b^T \theta}(t) \rangle\rangle_{L^2}| \rightarrow 0,$$

pointwise in $t \in [0, T]$, since (i) from the mollifier properties $K_0^\epsilon \rightarrow K_0$ weak-* in L_{loc}^∞ , (ii) $\widetilde{\mathcal{L}_b^T \theta}(t)$ has compact support, uniformly in $t \in [0, T]$, (iii) $\widetilde{\mathcal{L}_b^T \theta}(t)$ is of spatial class L^1 (use Lemma 103 (C) for (ii) & (iii)). Moreover, we have the following bound in $L^\infty([0, T])$

$$|\langle\langle K_0^\epsilon, \widetilde{\mathcal{L}_b^T \theta}(t) \rangle\rangle_{L^2}| \leq \|K_0^\epsilon\|_{L^\infty} \|\widetilde{\mathcal{L}_b^T \theta}(t)\|_{L^1} \leq \|K_0\|_{L^\infty} \|\widetilde{\mathcal{L}_b^T \theta}(t)\|_{L^1},$$

since $K_0 \in L^\infty(\bigwedge^k(T^*\mathbb{R}^n))$ and $\|\widetilde{\mathcal{L}_b^T \theta}\|_{L^1} \in L^\infty([0, T])$ from Lemma 103 (C).

- Similarly, the Itô-correction term converges a.s. as

$$\frac{1}{2} \int_0^t \langle\langle (\phi_s)_* K_0^\epsilon, \mathcal{L}_\xi^T \mathcal{L}_\xi^T \theta \rangle\rangle_{L^2} ds \rightarrow \frac{1}{2} \int_0^t \langle\langle (\phi_s)_* K_0, \mathcal{L}_\xi^T \mathcal{L}_\xi^T \theta \rangle\rangle_{L^2} ds,$$

where we can use the strategies in the last step which simplify significantly since ξ only depends on t .

- Finally, the martingale term

$$\begin{aligned} \int_0^t \langle\langle (\phi_s)_* K_0^\epsilon, \mathcal{L}_\xi^T \theta \rangle\rangle_{L^2} dW_s &= \int_0^t \langle\langle K_0^\epsilon, \widetilde{\mathcal{L}_\xi^T \theta}(s) \rangle\rangle_{L^2} dW_s \\ &\rightarrow \int_0^t \langle\langle (\phi_s)_* K_0, \mathcal{L}_\xi^T \theta \rangle\rangle_{L^2} dW_s, \end{aligned}$$

in probability, by stochastic dominated convergence (Theorem 50), where we have applied the same strategy as in the third step, which again simplifies since ξ is only time-dependent. When applying bounded convergence we use $L^\infty([0, T])$ -integrability of $\langle\langle K_0^\epsilon, \widetilde{\mathcal{L}_\xi^T \theta}(t) \rangle\rangle_{L^2}$, which is more than enough.

To finish the proof, we observe that by definition

$$((\phi_t)_* K_0)_{j_1, \dots, j_k}(x) = (K_0)_{i_1, \dots, i_k}(\psi_t(x)) \frac{\partial \psi_t^{i_1}}{\partial x_{j_1}}(x) \cdots \frac{\partial \psi_t^{i_k}}{\partial x_{j_k}}(x), \quad (4.3.7)$$

in global coordinates, where we have employed the notation $\psi_t = \phi_t^{-1}$. From Theorem 93, $\mathbb{P}$ -a.s. the spatial derivatives of ψ_t are continuous in (t, x) . Hence, it holds $\sup_{0 \leq t \leq T} \|K_t\|_{L_R^\infty} < \infty$ for any $R > 0$. Furthermore, we have the bound

$$\begin{aligned} \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^n} |K_t(x)|^2 d^n x dt \right] &= \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^n} |(\phi_t)_* K_0(x)|^2 d^n x dt \right] \\ &\leq \|K_0\|_{L^\infty}^2 \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^n} |D\psi_t(x)|^{2k} d^n x dt \right] \\ &= \|K_0\|_{L^\infty}^2 \int_{\mathbb{R}^n} \mathbb{E} \left[\int_0^T |D\psi_t(x)|^{2k} dt \right] d^n x \\ &\leq \|K_0\|_{L^\infty}^2 \sup_{x \in \mathbb{R}^n} \mathbb{E} \left[\sup_{0 \leq t \leq T} |D\psi_t(x)|^{2k} \right] \leq C, \end{aligned} \quad (4.3.8)$$

where we have used Tonelli's theorem in the third line, applied property (4.2.3) (take into account Remark 98) in the last line, and also employed the fact that $K_0 \in L^\infty(\bigwedge^k(T^*\mathbb{R}^n))$. We have therefore shown that $K \in L^2(\Omega \times [0, T] \times \bigwedge^k(T^*\mathbb{R}^n))$. Hence, $K_t(x) = (\phi_t)_* K_0(x)$ is a weak solution to (0.2.10). $\square$

Remark 106. *The above proof would immediately work for the case $\xi = \xi(t, x)$ if we had a smooth enough solution to the flow equation (4.1.2) and assumed some spatial regularity on ξ .*

Existence of solutions for nonsmooth drift. We proceed to reduce the regularity on the drift.

Proposition 107 (Weak existence for weak initial condition and drift). *Let $b, \xi_i, i = 1, \dots, N$ satisfy Assumption A3 and $K_0 \in L^\infty(\bigwedge^k(T^*\mathbb{R}^n))$. Then there exists a weak solution to (0.2.10) of the form $K_t(x) = (\phi_t)_* K_0(x)$.*

Proof. For simplicity, we treat the case $N = 1$. First, we take the mollified

vector field $b^\epsilon = \rho^\epsilon * b$ and consider equation (0.2.10) with drift vector field b^ϵ instead of b . Let ϕ_t^ϵ be the smooth flow of (4.1.2) with drift vector field b^ϵ (see Assumption A2 with $k = 3$) and define $K_t^\epsilon(x) = (\phi_t^\epsilon)_* K_0(x)$, which is a weak solution of (0.2.10) by Proposition 104. By (4.2.3) applied to the inverse flow (take into account Remark 98), we have

$$\sup_{0 < \epsilon < 1} \sup_{x \in \mathbb{R}^n} \mathbb{E} \left[\sup_{s \leq t \leq T} |D\psi_t^\epsilon(x)|^p \right] < \infty,$$

for any $p \geq 1$. As at the end of the proof of the last proposition, by considering the change of coordinates $x = \phi_t^\epsilon(y)$ and taking into account property (4.2.3), we obtain

$$\mathbb{E} \left[\int_0^T \int_{\mathbb{R}^n} |K_t^\epsilon(x)|^2 dx dt \right] \leq C < \infty. \quad (4.3.9)$$

We have therefore shown that $K_t^\epsilon(x)$ is equibounded in the Hilbert space $L^2(\Omega \times [0, T] \times \bigwedge^k(T^*\mathbb{R}^n))$. Since every bounded sequence in a Hilbert space has a convergent subsequence, there exists a subsequence $\epsilon_n \rightarrow 0$ and some k -form-valued stochastic process $K \in L^2(\Omega \times [0, T] \times \bigwedge^k(T^*\mathbb{R}^n))$ such that K^{ϵ_n} converges weakly to K in $L^2(\Omega \times [0, T] \times \bigwedge^k(T^*\mathbb{R}^n))$. From now on, we shall write ϵ instead of ϵ_n for the sake of simplicity. We choose a test k -form $\theta \in C_0^\infty(\bigwedge^k(T^*\mathbb{R}^n))$. Note that the process $\langle\langle K^\epsilon, \theta \rangle\rangle_{L^2}$ is adapted by definition and converges weakly in $L^2(\Omega \times [0, T])$. Since $\langle\langle K^\epsilon, \theta \rangle\rangle_{L^2}$ is adapted and the set of adapted processes in $L^2(\Omega \times [0, T])$ is closed (and hence weakly closed), $\langle\langle K, \theta \rangle\rangle_{L^2}$ is adapted. Moreover, we claim that $\langle\langle K, \mathcal{L}_\xi^T \theta \rangle\rangle_{L^2} \in L^2(\Omega \times [0, T])$. Indeed, by applying Hölder's inequality for tensors and using bound (4.3.9), we have that

$$\begin{aligned} \mathbb{E} \left[\int_0^T \langle\langle K, \mathcal{L}_\xi^T \theta \rangle\rangle_{L^2}^2 dt \right] &\leq \mathbb{E} \left[\int_0^T \|K\|_{L_x^2}^2 \|\mathcal{L}_\xi^T \theta\|_{L_x^2}^2 dt \right] \\ &\leq \sup_{t \in [0, T]} \|\mathcal{L}_\xi^T \theta\|_{L_x^2}^2 \mathbb{E} \left[\int_0^T \|K\|_{L_x^2}^2 dt \right] \lesssim \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^n} |K|^2 dx dt \right] < \infty, \end{aligned} \quad (4.3.10)$$

where we have taken into account that $\xi \in L^\infty([0, T])$. Since $\langle\langle K, \mathcal{L}_\xi^T \theta \rangle\rangle_{L^2} \in L^2(\Omega \times [0, T])$, we must have that $\int_0^T \langle\langle K, \mathcal{L}_\xi^T \theta \rangle\rangle_{L^2}^2 ds < \infty$ a.s. so the Itô

integral is well-defined. We know that K^ϵ satisfies equation (4.0.3) and as in ⁷⁶, it can be checked that $\langle\langle K, \theta \rangle\rangle_{L^2}$ also satisfies (4.0.3), weakly in $L^2(\Omega \times [0, T])$, by using the weak convergence of K^ϵ in $L^2(\Omega \times [0, T] \times \bigwedge^k(T^*\mathbb{R}^n))$ and properties of the flow. Therefore we may pass to the weak $L^2(\Omega \times [0, T])$ -limit in the equation for K and (4.0.3) holds for almost every $(\omega, t) \in \Omega \times [0, T]$. $\square$

Until now, we have constructed a solution to our equation but we have not proven that there exists a solution with the particular form $K_t(x) = (\phi_t)_* K_0(x)$. To prove this, we require the following technical lemma.

Lemma 108. *Let b and $\xi_i, i = 1, \dots, N$ satisfy Assumption A3 and $\{b^m\}_{m=1}^\infty$ be a sequence of drift vector fields such that $b^m \rightarrow b$ in $L^\infty([0, T]; C_b^\alpha(\mathbb{R}^n, \mathbb{R}^n))$, $m \rightarrow \infty$. Moreover, let $\{\phi_t^m\}_{m=1}^\infty$ be the corresponding sequence of stochastic flows solving SDE (0.2.6) with drifts b^m and fixed diffusions $\xi_i, i = 1, \dots, N$. Let $p \geq 2$. Then for any multi-vector field $\vartheta \in C^\infty(\bigwedge^k(T\mathbb{R}^n))$ we have*

$$\mathbb{E} \left[\int_0^T \int_{B(0, R^*(m))} ((\phi_t^m)^* \vartheta(x) - (\phi_t)^* \vartheta(x))^p dx dt \right] \rightarrow 0, \quad m \rightarrow \infty, \quad (4.3.11)$$

where we have chosen $R^*(m) < \infty$ a.s. such that

$$\{(\phi_t^m)^{-1}(x), \phi_t^{-1}(x) : x \in B(0, R), t \in [0, T]\} \subset B(0, R^*(m)),$$

and we assume $B(0, R^*(m)) \subset B(0, R')$ with deterministic $R, R' < \infty$.

Proof. Let us adopt the notation $\|\cdot\|_{L_{\omega, t, R}^p}$ to denote the norm in the Banach space $L^p(\Omega \times [0, T] \times B(0, R))$. First consider the case $k = 1$ (i.e. ϑ is a vector field). Then, employing the notation $\psi_t = \phi_t^{-1}$ as usually, ϑ has the global coordinate expression

$$(\phi_t^* \vartheta)^i(x) = \vartheta^j(\phi_t(x)) \frac{\partial \psi_t^i}{\partial y_j}(\phi_t(x)),$$

and therefore by taking into account Tonelli's theorem we have

$$\begin{aligned}
& \|(\phi^m)^*\vartheta(\cdot) - (\phi)^*\vartheta(\cdot)\|_{L^p_{\omega,t,R^*(m)}}^p \\
& \leq \|\vartheta\|_{L^\infty_{R^*}}^p \mathbb{E} \left[\int_0^T \int_{B(0,R^*(m))} |D\psi_t^m(\phi_t^m(x)) - D\psi_t(\phi_t(x))|^p d^n x dt \right] \\
& \lesssim \mathbb{E} \left[\int_0^T \int_{B(0,R^*(m))} |D\psi_t^m(\phi_t^m(x)) - D\psi_t(\phi_t^m(x))|^p d^n x dt \right] \\
& + \mathbb{E} \left[\int_0^T \int_{B(0,R^*(m))} |D\psi_t(\phi_t^m(x)) - D\psi_t(\phi_t(x))|^p d^n x dt \right].
\end{aligned} \tag{4.3.12}$$

By considering the change of coordinates $y = \phi_t^m(x)$, we obtain convergence of the first integral in (4.3.12) by observing that

$$\begin{aligned}
& \|D\psi^m(\phi^m(\cdot)) - D\psi(\phi^m(\cdot))\|_{L^p_{\omega,t,R^*(m)}}^p \\
& \leq \mathbb{E} \left[\int_0^T \int_{B(0,R)} |D\psi_t^m(y) - D\psi_t(y)|^p J\psi_t^m(y) d^n y dt \right] \\
& \leq \|D\psi^m(\cdot) - D\psi(\cdot)\|_{L^{2p}_{\omega,t,R}}^p \|D\psi^m(\cdot)\|_{L^{2n}_{\omega,t,R}}^n \rightarrow 0,
\end{aligned}$$

where we have taken into account (4.2.23). To establish the convergence of the last term in (4.3.12), we set $r(x) = \sqrt{1 + |x|^2}$ and first note that

$$\begin{aligned}
& |D\psi_t(\phi_t^m(x)) - D\psi_t(\phi_t(x))| = |r(\phi_t(x))r^{-1}(\phi_t(x))(D\psi_t(\phi_t^m(x)) - D\psi_t(\phi_t(x)))| \\
& \leq |r(\phi_t(x))| (|r^{-1}(\phi_t^m(x))D\psi_t(\phi_t^m(x)) - r^{-1}(\phi_t(x))D\psi_t(\phi_t(x))| \\
& \quad + |D\psi_t(\phi_t^m(x))||r^{-1}(\phi_t^m(x)) - r^{-1}(\phi_t(x))|) \\
& \leq |r(\phi_t(x))| \left(C_1(\omega, t)|\phi_t^m(x) - \phi_t(x)|^{\alpha'} + C_2|D\psi_t(\phi_t^m(x))||\phi_t^m(x) - \phi_t(x)| \right),
\end{aligned}$$

where

$$C_1(\omega, t) = \sup_{\substack{x, y \in \mathbb{R}^n \\ x \neq y}} \frac{|r^{-1}(x)D\psi_{s,t}(x) - r^{-1}(y)D\psi_{s,t}(y)|}{|x - y|^{\alpha'}}$$

is finite $\mathbb{P}$ -a.s. for all $0 \leq t \leq T$ by property (4.2.1) of the flow and

$$C_2 = \sup_{\substack{x, y \in \mathbb{R}^n \\ x \neq y}} \frac{|r^{-1}(x) - r^{-1}(y)|}{|x - y|}$$

is finite since r^{-1} is globally Lipschitz. Hence, by applying Hölder's inequality in the whole space in the third line, we get

$$\begin{aligned} & \|D\psi(\phi^n(\cdot)) - D\psi(\phi(\cdot))\|_{L^p_{\omega, t, R^*(m)}}^p \\ & \leq \mathbb{E} \left[\int_0^T \int_{B(0, R')} |r(\phi_t(x))|^p \left(C_1(\cdot, t) |\phi_t^m(x) - \phi_t(x)|^{\alpha'} \right)^p d^n x dt \right] \\ & \quad + \mathbb{E} \left[\int_0^T \int_{B(0, R')} |r(\phi_t(x))|^p (C_2 |D\psi_t(\phi_t^m(x))| |\phi_t^m(x) - \phi_t(x)|)^p d^n x dt \right] \\ & \lesssim \|r(\phi(\cdot))\|_{L^{2p}_{\omega, t, R'}}^p \left(\mathbb{E} \left[\int_0^T \int_{B(0, R')} C_1^{2p}(\cdot, t) |\phi_t^m(x) - \phi_t(x)|^{2p\alpha'} d^n x dt \right]^{1/2} \right. \\ & \quad \left. + C_2^{2p} \mathbb{E} \left[\int_0^T \int_{B(0, R')} |D\psi_t(\phi_t^m(x))|^{2p} |\phi_t^m(x) - \phi_t(x)|^{2p} d^n x dt \right]^{1/2} \right) \\ & = \|r(\phi(\cdot))\|_{L^{2p}_{\omega, t, R'}}^p [(I) + (II)]. \end{aligned}$$

By our usual arguments, we have the control

$$\begin{aligned} \|r(\phi(\cdot))\|_{L^{2p}_{\omega, t, R'}}^p &= \mathbb{E} \left[\int_0^T \int_{B(0, R')} |r(y)|^{2p} J\phi_t(y) d^n y dt \right]^{1/2} \\ &\leq \|r\|_{L^\infty_{R'}}^{2p} \|D\phi(\cdot)\|_{L^n_{\omega, t, R'}}^{n/2} < \infty. \end{aligned}$$

Moreover, for term (I) we have the convergence

$$\begin{aligned} (I) &\lesssim \mathbb{E} \left[\sup_{0 \leq t \leq T} C_1^{2pq'}(\cdot, t) \right]^{1/2q'} \mathbb{E} \left[\int_0^T \int_{B(0, R')} |\phi_t^m(x) - \phi_t(x)|^{2\alpha'pp'} d^n x dt \right]^{1/2p'} \\ &\rightarrow 0, \end{aligned}$$

where we have used Hölder's inequality with p', q' such that $2\alpha'pp' \geq 1$ and $1/q' + 1/p' = 1$, taken into account (4.2.1) to control the weighted Hölder

norm $C_1(\omega, t)$, and employed (4.2.22). For the second term (II), we have

$$\begin{aligned} \text{(II)} &\lesssim \mathbb{E} \left[\int_0^T \int_{B(0, R')} |D\psi_t(y)|^{4p} J\phi_t^m(x) dy dt \right]^{1/4} \|\phi^m(\cdot) - \phi(\cdot)\|_{L^{4p}_{\omega, t, R'}}^p \\ &\leq \|D\psi(\cdot)\|_{L^{8p}_{\omega, t, R'}}^p \|D\phi^m(\cdot)\|_{L^{2n}_{\omega, t, R'}}^{n/4} \|\phi^m(\cdot) - \phi(\cdot)\|_{L^{4p}_{\omega, t, R'}}^p \rightarrow 0, \end{aligned}$$

where we have employed our usual strategies and the fact that $\|D\phi^m(\cdot)\|_{L^p_{\omega, t, R'}}$ is uniformly bounded for any $p \geq 1$ by property (4.2.3) of the flow. This concludes the convergence (4.3.11) when ϑ is a vector field. The general case can be treated by induction. □

The following lemma concludes this existence section.

Lemma 109. *There exists a weak solution to (0.2.10) of the form $K_t(x) = (\phi_t)_* K_0(x)$.*

Proof. Case 1. First we assume that K_0 is supported on $B(0, R)$. We will show that we can choose a sequence converging to a weak solution of (0.2.10), which we denote by K^ϵ for convenience, such that for every test k -form $\theta \in C_0^\infty(\bigwedge^k(T^*\mathbb{R}^n))$ we have $\langle\langle K^\epsilon - (\phi_t)_* K_0, \theta \rangle\rangle_{L^2} \rightarrow 0$, weakly in $L^2(\Omega \times [0, T])$. Indeed, we select a convergent (sub)sequence K^ϵ by applying the same argument we did in the previous proposition. We claim that K^ϵ converges to $(\phi_t)_* K_0$, weakly in $L^2(\Omega \times [0, T] \times \bigwedge^k(T^*\mathbb{R}^n))$. Fix a test k -form $\theta \in C_0^\infty(\bigwedge^k(T^*\mathbb{R}^n))$ which we assume without loss of generality to be supported on $B(0, R)$ and a test function $f \in L^2(\Omega \times [0, T])$. We denote $f(\omega, t) = f$ during the proof for simplicity of notation. Taking into account Lemma 101 in the second line and Hölder's inequality for tensor fields in the last step,

we have

$$\begin{aligned}
& \left| \int_{\Omega} \int_0^T \langle \langle K_t^\epsilon - (\phi_t)_* K_0, \theta \rangle \rangle_{L^2} f dt d\omega \right| \\
& \leq \int_{\Omega} \int_0^T \int_{B(0,R)} |(K_0 J \phi_t^\epsilon ((\phi_t^\epsilon)^* \theta^\sharp)^\flat - K_0 J \phi_t (\phi_t^* \theta^\sharp)^\flat) f| d^n y dt d\omega \\
& = \int_{\Omega} \int_0^T \int_{B(0,R)} |K_0 (J \phi_t^\epsilon ((\phi_t^\epsilon)^* \theta^\sharp)^\flat - J \phi_t^\epsilon (\phi_t^* \theta^\sharp)^\flat) f| d^n y dt d\omega \\
& + \int_{\Omega} \int_0^T \int_{B(0,R)} |K_0 (J \phi_t^\epsilon (\phi_t^* \theta^\sharp)^\flat - J \phi_t (\phi_t^* \theta^\sharp)^\flat) f| d^n y dt d\omega \\
& \leq \|K_0 f\|_{L_{\omega,t,R}^2} \|J \phi_t^\epsilon ((\phi_t^\epsilon)^* \theta^\sharp)^\flat - (\phi_t^* \theta^\sharp)^\flat\|_{L_{\omega,t,R}^2} \\
& \quad + \|K_0 f\|_{L_{\omega,t,R}^2} \|(J \phi_t^\epsilon - J \phi_t)(\phi_t^* \theta^\sharp)^\flat\|_{L_{\omega,t,R}^2}.
\end{aligned}$$

We claim that the last terms converge to zero. First we note

$$\|K_0 f\|_{L_{\omega,t,R}^2} \leq \|K_0\|_{L_R^\infty} \|f\|_{L_{\omega,t}^2} < \infty.$$

Moreover, we have

$$\|J \phi_t^\epsilon ((\phi_t^\epsilon)^* \theta^\sharp)^\flat - (\phi_t^* \theta^\sharp)^\flat\|_{L_{\omega,t,R}^2} \leq \|J \phi_t^\epsilon\|_{L_{\omega,t,R}^4} \|(\phi_t^\epsilon)^* \theta^\sharp - \phi_t^* \theta^\sharp\|_{L_{\omega,t,R}^4} \rightarrow 0,$$

where we have used (4.2.23) to control the Jacobian term together with Lemma 108 to obtain convergence of the second term. By applying similar techniques we can also obtain

$$\|(J \phi_t^\epsilon - J \phi_t)(\phi_t^* \theta^\sharp)^\flat\|_{L_{\omega,t,R}^2} \rightarrow 0.$$

Hence, our argument is finished in the case K_0 has compact support.

Case 2. Now we treat the general case $K_0 \in L^\infty(\bigwedge^k(T^*\mathbb{R}^n))$. We choose $\zeta \in C_0^\infty(\mathbb{R}^n)$ supported on $B(0, 1)$ such that $\zeta(x) = 1$, $x \in B(0, 1/2)$, and such that ζ takes values on $[0, 1]$. For every $n \in \mathbb{N}$, let $K_0^n(x) = K_0(x) \zeta(x/n)$, which is compactly supported. By taking into account the first part of the proof, we can construct a corresponding solution which we denote by K^n .

Therefore we have

$$\begin{aligned} \langle \langle K_t^n, \theta \rangle \rangle_{L^2} - \langle \langle K_0^n, \theta \rangle \rangle_{L^2} &= - \int_0^t \langle \langle K_s^n, \mathcal{L}_b^T \theta \rangle \rangle_{L^2} ds \\ &\quad - \int_0^t \langle \langle K_s^n, \mathcal{L}_\xi^T \theta \rangle \rangle_{L^2} dW_s + \frac{1}{2} \int_0^t \langle \langle K_s^n, \mathcal{L}_\xi^T \mathcal{L}_\xi^T \theta \rangle \rangle_{L^2} ds, \end{aligned} \quad (4.3.13)$$

$t \in [0, T]$, for any test k -form $\theta \in C_0^\infty(\bigwedge^k(T^*\mathbb{R}^n))$. We know that

$$K_t^n(x) = (\phi_t)_*(K_0(\cdot)\zeta(\cdot/n))(x) = (\phi_t)_*K_0(x)\zeta(\phi_t^{-1}(x)/n), \quad (4.3.14)$$

where $t \in [0, T]$ and $x \in \mathbb{R}^n$, by the product property of the push-forward

$$(\phi_t)_*(Kf)(x) = (\phi_t)_*K(x)(\phi_t)_*f(x),$$

and taking into account that $(\phi_t)_*(x) = f(\phi_t^{-1}(x))$, for scalar functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$. It is clear that the sequence K_t^n converges to $K_t = (\phi_t)_*K_0$ pointwise in (t, x) . We want to show that K_t also satisfies (4.3.13). Fix a test k -form $\theta \in C_0^\infty(\bigwedge^k(T^*\mathbb{R}^n))$ supported on $B(0, R)$. As in the proof Lemma 103, we choose $R^* < \infty$ a.s. such that

$$\{\phi_t^{-1}(x) : x \in B(0, R), t \in [0, T]\} \subset B(0, R^*).$$

Let ω belong to the set of probability one such that $R^* = R^*(\omega) < \infty$. Let $R' = \max(R, R^*)$. Then we have that for $n > 2R'$, $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{X}_{B(0, R)}K_t^n(x) &= \mathcal{X}_{B(0, R)}(\phi_t)_*K_0(x)\zeta(\phi_t^{-1}(x)/n) \\ &= \mathcal{X}_{B(0, R)}(\phi_t)_*K_0(x) = \mathcal{X}_{B(0, R)}K_t(x), \quad t \in [0, T], \quad x \in \mathbb{R}^n. \end{aligned}$$

Therefore $\mathcal{X}_{B(0, R)}K_t^n(x) = \mathcal{X}_{B(0, R)}K_t(x)$. We conclude that for $\mathbb{P}$ -a.s. ω , there exists $n = n(\omega)$ such that $K^n = K$ and hence the terms $\langle \langle K_t^n, \theta \rangle \rangle_{L^2}$, $\int_0^t \langle \langle K_s^n, \mathcal{L}_b^T \theta \rangle \rangle_{L^2} ds$, and $\frac{1}{2} \int_0^t \langle \langle K_s^n, \mathcal{L}_\xi^T \mathcal{L}_\xi^T \theta \rangle \rangle_{L^2} ds$ in (4.3.13) become $\langle \langle K_t, \theta \rangle \rangle_{L^2}$, $\int_0^t \langle \langle K_s, \mathcal{L}_b^T \theta \rangle \rangle_{L^2} ds$, $\frac{1}{2} \int_0^t \langle \langle K_s, \mathcal{L}_\xi^T \mathcal{L}_\xi^T \theta \rangle \rangle_{L^2} ds$. To finish the argument, we claim that

$$\int_0^t \langle \langle K_s^n, \mathcal{L}_\xi^T \theta \rangle \rangle_{L^2} dW_s \rightarrow \int_0^t \langle \langle K_s, \mathcal{L}_\xi^T \theta \rangle \rangle_{L^2} dW_s$$

in probability, $t \in [0, T]$. For this, it suffices to establish that

$$\mathbb{E} \left[\left(\int_0^t \langle \langle K_s^n - K_s, \mathcal{L}_\xi^T \theta \rangle \rangle_{L^2} dW_s \right)^2 \right] = \mathbb{E} \left[\int_0^t \langle \langle K_s^n - K_s, \mathcal{L}_\xi^T \theta \rangle \rangle_{L^2}^2 ds \right] \rightarrow 0, \quad (4.3.15)$$

where we have used Itô's isometry and employed the fact that L^2 -mean convergence implies convergence in probability. Claim (4.3.15) can be verified by applying dominated convergence in the space $\Omega \times [0, T]$ to the sequence of functions $\langle \langle K_s^n, \mathcal{L}_\xi^T \theta \rangle \rangle_{L^2}$, where the hypotheses for dominated convergence are satisfied since (i) clearly, a.s. $\langle \langle K_s^n - K_s, \mathcal{L}_\xi^T \theta \rangle \rangle_{L^2}^2 \rightarrow 0$, (ii) by using Hölder's inequality for tensors, we have the uniform bound

$$\langle \langle K_s^n, \mathcal{L}_\xi^T \theta \rangle \rangle_{L^2}^2 \leq \|K_s^n\|_{L_x^2}^2 \|\mathcal{L}_\xi^T \theta\|_{L_x^2}^2 \leq \|K_s\|_{L_x^2}^2 \|\mathcal{L}_\xi^T \theta\|_{L_x^2}^2 \lesssim \|K_s\|_{L_x^2}^2,$$

where we have taken into account that $|K_t^n(x)| \leq |K_t(x)|$ and have used that $\xi \in L^\infty([0, T])$. Moreover, (iii) $\|K_s\|_{L_x^2}^2 \in L^1(\Omega \times [0, T])$ by usual strategies as in the bound (4.3.8).

Finally, it can be easily checked that $\sup_{0 \leq t \leq T} \|K_t\|_{L_R^\infty} < \infty$ because $\mathbb{P}$ -a.s. ϕ_t^{-1} has continuous derivatives. $\square$

Remark 110. *The last proof is also immediately extensible to the case $\xi = \xi(t, x)$.*

4.4 UNIQUENESS OF WEAK SOLUTIONS

For convenience, we employ the notation $K(t, x) = K_t(x)$. In this section, we prove uniqueness of solutions to equation (0.2.10) under Assumption A3. More concretely, we establish the following theorem.

Theorem 111. *Under Assumption A3, there exists a unique weak solution of (0.2.10) for any initial condition $K_0 \in L^\infty(\bigwedge^k(T^*\mathbb{R}^n))$.*

For simplicity, we treat the case $N = 1$. As usually, from the linearity of the equation, it is sufficient to show that if $K(0, \cdot) = 0$, then $K(t, \cdot) = 0$

a.s., for every $t \in [0, T]$. We note that this property is slightly weaker than pathwise uniqueness because the set of measure zero in which $K(t, \cdot) = 0$ depends on t . Unfortunately, left or right-continuity is not available and we cannot prove a stronger version of this property (exactly as it occurs in⁷⁶). First, consider the mollified k -form $K^\epsilon(t, x) = (\rho^\epsilon * K(t, \cdot))(x)$ with initial condition $K(0, \cdot) = 0$, which by (0.2.10) satisfies

$$K^\epsilon(t, x) = - \int_0^t (\rho^\epsilon * \mathcal{L}_b K)(s, x) ds - \int_0^t (\rho^\epsilon * \mathcal{L}_\xi K)(s, x) \circ dW_s, \quad (4.4.1)$$

where $\mathcal{L}_b K$, $\mathcal{L}_\xi K$ are understood as Lie derivatives in the sense of distributions (see Definition 28) owing to the low spatial regularity of K and b . Since all the terms in (4.4.1) are now smooth in space, one can apply the KIW formula for k -forms in Stratonovich form (see Theorem 47) to obtain

$$\begin{aligned} \phi_t^* K^\epsilon(t, x) &= - \int_0^t \phi_s^* (\mathcal{L}_b K)^\epsilon(s, x) ds + \int_0^t \phi_s^* \mathcal{L}_b K^\epsilon(s, x) ds \\ &\quad - \int_0^t \phi_s^* (\mathcal{L}_\xi K)^\epsilon(s, x) \circ dW_s + \int_0^t \phi_s^* \mathcal{L}_\xi K^\epsilon(s, x) \circ dW_s \\ &= \int_0^t \phi_s^* [\mathcal{L}_b, \rho^\epsilon *] K(s, x) ds + \int_0^t \phi_s^* [\mathcal{L}_\xi, \rho^\epsilon *] K(s, x) \circ dW_s, \end{aligned} \quad (4.4.2)$$

where

$$[\mathcal{L}_b, \rho^\epsilon *] K = \mathcal{L}_b K^\epsilon - (\mathcal{L}_b K)^\epsilon \quad (4.4.3)$$

denotes the commutator between the linear operators $\mathcal{L}_b$ and $\rho^\epsilon * (\cdot)$, generalising the DiPerna-Lions commutator appearing in^{61,124}. We will show that the last line of (4.4.2) vanishes weakly as $\epsilon \rightarrow 0$, thus proving uniqueness of solutions.

Remark 112. *The observant reader might have noticed that an explanation is now in place. Indeed, the k -form-valued process K^ϵ in (4.4.1) enjoys regularity in space but not in time, and Theorem 47 was proven only for processes that are continuous in time. However, this is not a problem and there are several ways of solving this issue. First, as in⁷⁶, we remark that*

the Itô-Wentzell formula in Stratonovich form appearing in Kunita's book (Theorem 40) is not optimised with respect to the processes we are using as integrators (bounded variation processes and Brownian motion), but rather optimised with respect to integrating against continuous semimartingales in general. Since the integrands (i.e., the drift and diffusion) in (4.4.1) are uniformly bounded in time, adapted, and enjoy the required spatial regularity assumptions, in⁷⁶ it is argued that Itô-Wentzell in Stratonovich form (Theorem 40) can be applied. Now observe that the main tool required to establish Theorem 47 is Itô-Wentzell, or more concretely, Theorem 42, which consists in a particular case of Itô-Wentzell for flows. Hence we argue that we can apply Theorem 47 as long as we have (1) a uniform boundedness assumption in time, (2) adaptedness, and (3) the process enjoys (spatial) C^3 -regularity with regular enough in space coefficients. We also note that the Itô-Wentzell formula with low regularity in time Flandoli et al. refer to in⁷⁶ can be found for instance in¹¹⁵ for predictable processes.

We can also outline an alternative proof of KIW for predictable processes. Indeed, in⁵⁸ we show the KIW formula for continuous in time processes by employing a geometric generalisation of Krylov's argument in¹¹⁵. Our approach consists in mollifying the stochastic process K , and the key steps are applying Fubini and dominated convergence. Uniform in time assumptions for the derivatives and predicability (due to Theorem 33) are enough to do this and no continuity of K is required in the proof.

We prove the following lemma.

Lemma 113. *Given a stochastic k -form-valued process of the form*

$$\alpha(t, x) = \alpha(x) + \int_0^t G(s, x) ds + \sum_{i=1}^M \int_0^t H_i(s, x) dW_s^i,$$

satisfying the conditions in Theorem 43, and a stochastic flow

$$\phi_t(x) = x + \int_0^t \left(b + \frac{1}{2} \sum_{k=1}^N \xi_k \cdot \nabla \xi_k \right) (s, \phi_s(x)) ds + \sum_{k=1}^N \int_0^t \xi_k(s, \phi_s(x)) dB_s^k,$$

also satisfying the conditions in Theorem 43, we have

$$\begin{aligned} \int_0^t \phi_s^* \alpha(s, x) \circ dW_s^i &= \int_0^t \phi_s^* \alpha(s, x) dW_s^i \\ &+ \frac{1}{2} \int_0^t \phi_s^* H_i(s, x) ds + \frac{1}{2} \sum_{k=1}^N \int_0^t \phi_s^* \mathcal{L}_{\xi_k} \alpha(s, x) d[W^i, B^k]_s, \quad i = 1, \dots, M. \end{aligned}$$

Proof. For simplicity, we consider the case $M = N = 1$. Note that

$$\int_0^t \phi_s^* \alpha(s, x) \circ dW_s = \int_0^t \phi_s^* \alpha(s, x) dW_s + \frac{1}{2} [\phi^* \alpha(\cdot, x), W]_t. \quad (4.4.4)$$

Applying the KIW formula for k -forms (Theorem 43), we obtain

$$\begin{aligned} \phi_t^* \alpha(t, x) &= \alpha(0, x) + \int_0^t \phi_s^* G(s, x) ds + \int_0^t \phi_s^* H(s, x) dW_s \\ &+ \int_0^t \phi_s^* \mathcal{L}_b \alpha(s, x) ds + \int_0^t \phi_s^* \mathcal{L}_\xi \alpha(s, x) dB_s \\ &+ \int_0^t \phi_s^* \mathcal{L}_\xi H(s, x) d[W, B]_s + \frac{1}{2} \int_0^t \phi_s^* \mathcal{L}_\xi \mathcal{L}_\xi \alpha(s, x) ds, \quad (4.4.5) \end{aligned}$$

so we conclude

$$[\phi^* \alpha(\cdot, x), W]_t = \int_0^t \phi_s^* H(s, x) ds + \int_0^t \phi_s^* \mathcal{L}_\xi \alpha(s, x) d[W, B]_s,$$

which yields the result by taking into account (4.4.4). $\square$

The following is a straightforward corollary.

Corollary 114. *Let Assumption A3 be satisfied. For a k -form-valued process $K(t, \cdot)$ satisfying (0.2.10) and a stochastic flow ϕ_t solving (4.1.2) with $N = 1$, we have the following identity*

$$\begin{aligned} \int_0^t \phi_s^* [\mathcal{L}_\xi, \rho^\epsilon] K(s, x) \circ dW_s &= \int_0^t \phi_s^* [\mathcal{L}_\xi, \rho^\epsilon] K(s, x) dW_s \\ &+ \frac{1}{2} \int_0^t \phi_s^* [\mathcal{L}_\xi, [\mathcal{L}_\xi, \rho^\epsilon]] K(s, x) ds. \end{aligned}$$

Proof. Just note that by the previous lemma, the correction term is

$$\frac{1}{2} \int_0^t \phi_s^* H(s, x) ds + \frac{1}{2} \int_0^t \phi_s^* \mathcal{L}_\xi [\mathcal{L}_\xi, \rho^\epsilon] K(s, x) d[W., W.]_s,$$

where

$$H(s, x) = -[\mathcal{L}_\xi, \rho^\epsilon] \mathcal{L}_\xi K(s, x).$$

□

Remark 115. *This proof is also valid for the case when $\xi = \xi(t, x)$ under some regularity assumptions in space.*

By Corollary 114, we conclude that the commutator (4.4.2) can be expressed in Itô form as

$$\begin{aligned} \phi_t^* K^\epsilon(t, x) &= \int_0^t \phi_s^* [\mathcal{L}_b, \rho^\epsilon] K(s, x) ds + \int_0^t \phi_s^* [\mathcal{L}_\xi, \rho^\epsilon] K(s, x) dW_s \\ &\quad + \frac{1}{2} \int_0^t \phi_s^* [\mathcal{L}_\xi, [\mathcal{L}_\xi, \rho^\epsilon]] K(s, x) ds. \end{aligned} \tag{4.4.6}$$

We start by treating the drift part of the commutator evaluated at a test k -form θ

$$\int_{\mathbb{R}^n} \langle [\mathcal{L}_b, \rho^\epsilon] K(x), \theta(x) \rangle d^n x = \int_{\mathbb{R}^n} \langle \mathcal{L}_b K^\epsilon(x) - (\mathcal{L}_b K)^\epsilon(x), \theta(x) \rangle d^n x. \tag{4.4.7}$$

We prove a generalisation to k -forms of the classical DiPerna-Lions-type commutator estimates (see^{61,124,76,74}) for terms of the form (4.4.7).

Lemma 116. *Given a k -form $K \in L_{loc}^\infty(\wedge^k(T^*\mathbb{R}^n))$, a vector field $b \in W_{loc}^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$, and a test function $\theta \in C_0^\infty(\wedge^k(T^*\mathbb{R}^n))$ supported on $B(0, R)$ for some $R > 0$, then for $\epsilon < 1$, we have the following estimate on the commutator (4.4.7):*

$$\left| \int_{\mathbb{R}^n} \langle [\mathcal{L}_b, \rho^\epsilon] K(x), \theta(x) \rangle d^n x \right| \lesssim \|\theta\|_{L_R^\infty} \|K\|_{L_{R+1}^\infty} \|b\|_{W_{R+1}^{1,1}}.$$

Moreover, $\langle [\mathcal{L}_b, \rho^\epsilon *] K(x), \theta(x) \rangle$ converges to zero in L_{loc}^1 as $\epsilon \rightarrow 0$.

Proof. By the definition of the commutator (4.4.7), we have

$$\begin{aligned} \int_{\mathbb{R}^n} \langle [\mathcal{L}_b, \rho^\epsilon *] K(x), \theta(x) \rangle d^n x &= \int_{\mathbb{R}^n} \langle \mathcal{L}_b K^\epsilon(x), \theta(x) \rangle d^n x \\ &\quad - \int_{\mathbb{R}^n} \langle (\mathcal{L}_b K)^\epsilon(x), \theta(x) \rangle d^n x. \end{aligned} \quad (4.4.8)$$

The first integral becomes

$$\begin{aligned} \int_{\mathbb{R}^n} \langle \mathcal{L}_b K^\epsilon(x), \theta(x) \rangle d^n x &= \int_{\mathbb{R}^n} \langle K^\epsilon(x), \mathcal{L}_b^T \theta(x) \rangle d^n x \\ &= \int_{\mathbb{R}^n} \delta^{i_1 j_1} \dots \delta^{i_k j_k} K_{i_1, \dots, i_k}^\epsilon(x) (\mathcal{L}_b^T \theta)_{j_1, \dots, j_k}(x) d^n x \\ &= \int_{\mathbb{R}^n} K^\epsilon(x) (e_{i_1}, \dots, e_{i_k}) (\mathcal{L}_b^T \theta)^{i_1, \dots, i_k}(x) d^n x, \end{aligned}$$

where $(\mathcal{L}_b^T \theta)^{i_1, \dots, i_k} = \delta^{i_1 j_1} \dots \delta^{i_k j_k} (\mathcal{L}_b^T \theta)_{j_1, \dots, j_k}$, which by (1.2.3) can be expressed as

$$(\mathcal{L}_b^T \theta)^{i_1, \dots, i_k}(x) = -\frac{\partial}{\partial x_l} (b^l(x) \theta^{i_1, \dots, i_k}(x)) + \sum_{j=1}^k \theta^{i_1, \dots, l, \dots, i_k}(x) \frac{\partial b^{i_j}}{\partial x_l}(x). \quad (4.4.9)$$

Hence by Definition 24, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \langle \mathcal{L}_b K^\epsilon(x), \theta(x) \rangle d^n x &= - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho^\epsilon(x-y) K_{i_1, \dots, i_k}(y) \frac{\partial}{\partial x_l} (b^l(x) \theta^{i_1, \dots, i_k}(x)) d^n y d^n x \\ &\quad + \sum_{j=1}^k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho^\epsilon(x-y) K_{i_1, \dots, i_k}(y) \theta^{i_1, \dots, l, \dots, i_k}(x) \frac{\partial b^{i_j}}{\partial x_l}(x) d^n y d^n x \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{\partial \rho^\epsilon}{\partial x_l}(x-y) K_{i_1, \dots, i_k}(y) b^l(x) \theta^{i_1, \dots, i_k}(x) \right. \\ &\quad \left. + \sum_{j=1}^k \rho^\epsilon(x-y) K_{i_1, \dots, i_k}(y) \frac{\partial b^{i_j}}{\partial x_l}(x) \theta^{i_1, \dots, l, \dots, i_k}(x) \right) d^n y d^n x. \end{aligned}$$

For the last integral in (4.4.8), we take into account the property $D_y \rho^\epsilon(x -$

$y) = -D_x \rho^\epsilon(x - y)$ and apply the definition of mollifiers on distribution-valued k -forms (see Definition 26), obtaining

$$\begin{aligned} \int_{\mathbb{R}^n} \langle (\mathcal{L}_b K)^\epsilon(x), \theta(x) \rangle d^n x &= \int_{\mathbb{R}^n} \langle K(y), \mathcal{L}_b^T(\rho^\epsilon * \theta)(y) \rangle d^n y \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{\partial \rho^\epsilon}{\partial x_l}(x - y) K_{i_1, \dots, i_k}(y) b^l(y) \theta^{i_1, \dots, i_k}(x) \right. \\ &\quad \left. - \rho^\epsilon(x - y) K_{i_1, \dots, i_k}(y) \frac{\partial b^l}{\partial y_l}(y) \theta^{i_1, \dots, i_k}(x) \right) d^n y d^n x \\ &\quad + \sum_{j=1}^k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho^\epsilon(x - y) K_{i_1, \dots, i_k}(y) \frac{\partial b^{i_j}}{\partial y_l}(y) \theta^{i_1, \dots, l, \dots, i_k}(x) d^n y d^n x. \end{aligned}$$

Combining the above expressions, we derive

$$\begin{aligned} \int_{\mathbb{R}^n} \langle [\mathcal{L}_b, \rho^\epsilon *] K(x), \theta(x) \rangle d^n x &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \epsilon^{-n} \frac{\partial}{\partial y_l} \rho \left(\frac{y - x}{\epsilon} \right) K_{i_1, \dots, i_k}(y) \left(\frac{b^l(y) - b^l(x)}{\epsilon} \right) \theta^{i_1, \dots, i_k}(x) d^n y d^n x \\ &\quad + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho^\epsilon(x - y) \left(\frac{\partial b^l}{\partial y_l}(y) \theta^{i_1, \dots, i_k}(x) K_{i_1, \dots, i_k}(y) \right. \\ &\quad \left. - \sum_{j=1}^k \left(\frac{\partial b^{i_j}}{\partial y_l}(y) - \frac{\partial b^{i_j}}{\partial x_l}(x) \right) \theta^{i_1, \dots, l, \dots, i_k}(x) K_{i_1, \dots, i_k}(y) \right) d^n y d^n x \\ &= \int_{\mathbb{R}^n} I_1^\epsilon(x) d^n x + \int_{\mathbb{R}^n} I_2^\epsilon(x) d^n x. \end{aligned}$$

Applying Young's convolution inequality (see Lemma 16), we have

$$\left| \int_{\mathbb{R}^n} I_2^\epsilon(x) d^n x \right| \lesssim \|\theta\|_{L_R^\infty} \|K\|_{L_{R+1}^\infty} \|b\|_{W_{R+1}^{1,1}},$$

since $\epsilon < 1$, where we have also taken into account that $\|\rho^\epsilon\|_{L^1} = 1$, independently of ϵ , and that θ has support in $B(0, R)$. To obtain an analogous estimate for the first integral, we follow the proof of Lemma 2.3 in ¹²⁴. First, we note that by Taylor's theorem

$$b^l(y) - b^l(x) = \int_0^1 (y - x) \cdot D b^l(x + \lambda(y - x)) d\lambda,$$

hence,

$$\begin{aligned} I_1^\epsilon(x) &= \epsilon^{-n} \int_{B(x,\epsilon)} \frac{\partial}{\partial y_l} \rho \left(\frac{y-x}{\epsilon} \right) K_{i_1, \dots, i_k}(y) \left(\frac{b^l(y) - b^l(x)}{\epsilon} \right) \theta^{i_1, \dots, i_k}(x) d^n y \\ &= \int_{B(0,1)} \frac{\partial}{\partial w_l} \rho(w) K_{i_1, \dots, i_k}(x + \epsilon w) \left(\int_0^1 w \cdot Db^l(x + \lambda \epsilon w) d\lambda \right) \theta^{i_1, \dots, i_k}(x) d^n w, \end{aligned}$$

and therefore

$$|I_1^\epsilon(x)| \leq \|D\rho\|_{L_1^\infty} \|K\|_{L_{R+1}^\infty} \|\theta\|_{L_R^\infty} \int_{B(0,1)} \int_0^1 |Db(x + \lambda \epsilon w)| d\lambda d^n w.$$

Finally, by observing that

$$\int_{B(0,1)} \int_0^1 |Db(x + \lambda \epsilon w)| d\lambda d^n w = |Db| * \chi^\epsilon(x),$$

where

$$\chi^\epsilon(z) = \int_0^1 \frac{1}{(\epsilon\lambda)^n} \mathcal{X}_{B(0,\epsilon\lambda)}(z) d\lambda,$$

and that $\|\chi^\epsilon\|_{L_\epsilon^1} = \text{meas}(B(0,1))$, we obtain

$$\begin{aligned} \int_{B(0,R)} |I_1^\epsilon(x)| d^n x &\leq \|D\rho\|_{L_1^\infty} \|K\|_{L_{R+1}^\infty} \|\theta\|_{L_R^\infty} \|Db\|_{L_{R+1}^1} \|\chi^\epsilon\|_{L_\epsilon^1} \\ &\lesssim \|K\|_{L_{R+1}^\infty} \|\theta\|_{L_R^\infty} \|b\|_{W_{R+1}^{1,1}}, \quad (4.4.10) \end{aligned}$$

where we have made use of Young's convolution inequality (Lemma 16) and the constant in the last inequality is independent of K, b, θ and ϵ .

We now show the L_{loc}^1 convergence of $\langle [\mathcal{L}_b, \rho^\epsilon *] K(x), \theta(x) \rangle$. First, check that

$$I_2^\epsilon(x) \rightarrow \langle K(x), \theta(x) \rangle \text{div}(b(x))$$

in L_{loc}^1 as $\epsilon \rightarrow 0$. Next, taking into account the density of C_0^∞ functions in $W_{loc}^{1,1}$, we consider a sequence of C_0^∞ -vector fields $\{b_\delta\}_{\delta>0}$ such that $b_\delta \rightarrow b$ in

$W_{loc}^{1,1}$. Since $b_\delta \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, we have

$$\begin{aligned}
& I_1^\epsilon[b_\delta](x) \\
&= \epsilon^{-n} \int_{B(x, \epsilon)} \frac{\partial}{\partial y_l} \rho \left(\frac{y-x}{\epsilon} \right) K_{i_1, \dots, i_k}(y) \left(\frac{b_\delta^l(y) - b_\delta^l(x)}{\epsilon} \right) \theta^{i_1, \dots, i_k}(x) d^n y \\
&= \int_{B(0,1)} \frac{\partial \rho}{\partial w_l}(w) K_{i_1, \dots, i_k}(x + \epsilon w) \left(\frac{b_\delta^l(x + \epsilon w) - b_\delta^l(x)}{\epsilon} \right) \theta^{i_1, \dots, i_k}(x) d^n w \\
&\rightarrow \langle K(x), \theta(x) \rangle \int_{B(0,1)} w_i \frac{\partial b_\delta^j}{\partial x_i}(x) \frac{\partial \rho}{\partial w_j}(w) d^n w = -\langle K(x), \theta(x) \rangle \operatorname{div}(b_\delta(x)),
\end{aligned}$$

in L_{loc}^1 as $\epsilon \rightarrow 0$. Then, by the linearity of I_1^ϵ in the b -argument and using estimate (4.4.10), we have

$$\begin{aligned}
\|I_1^\epsilon[b] + \langle K, \theta \rangle \operatorname{div}(b)\|_{L_R^1} &\leq \|I_1^\epsilon[b] - I_1^\epsilon[b_\delta]\|_{L_R^1} \\
&\quad + \|I_1^\epsilon[b_\delta] + \langle K, \theta \rangle \operatorname{div}(b_\delta)\|_{L_R^1} + \|\langle K, \theta \rangle \operatorname{div}(b) - \langle K, \theta \rangle \operatorname{div}(b_\delta)\|_{L_R^1} \\
&\lesssim \|K\|_{L_{R+1}^\infty} \|\theta\|_{L_R^\infty} \left(\|b - b_\delta\|_{W_{R+1}^{1,1}} + \|\operatorname{div}(b) - \operatorname{div}(b_\delta)\|_{L_R^1} \right) \\
&\quad + \|I_1^\epsilon[b_\delta] + \langle K, \theta \rangle \operatorname{div}(b_\delta)\|_{L_R^1} \rightarrow 0,
\end{aligned}$$

as $\delta, \epsilon \rightarrow 0$. Hence $I_1^\epsilon + I_2^\epsilon \rightarrow -\langle K, \theta \rangle \operatorname{div}(b) + \langle K, \theta \rangle \operatorname{div}(b) = 0$ in L_{loc}^1 as $\epsilon \rightarrow 0$. $\square$

Corollary 117. *Let ϕ be a C^1 -diffeomorphism, $K \in L_{loc}^\infty(\wedge^k(T^*\mathbb{R}^n))$, $b \in W_{loc}^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$, and $\theta \in C_0^\infty(\wedge^k(T^*\mathbb{R}^n))$ a test k -form with support contained in $B(0, R)$. Then*

$$\left| \int_{\mathbb{R}^n} \langle \phi^*[\mathcal{L}_b, \rho^\epsilon *] K(x), \theta(x) \rangle d^n x \right| \lesssim \|\tilde{\theta}\|_{L_{R^*}^\infty} \|K\|_{L_{R^*+1}^\infty} \|b\|_{W_{R^*+1}^{1,1}}, \quad (4.4.11)$$

where $R^* < \infty$ a.s. is chosen such that

$$\{\phi^{-1}(x), x \in B(0, R)\} \subset B(0, R^*).$$

Moreover, it holds

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} \langle \phi^*[\mathcal{L}_b, \rho^\epsilon *] K(x), \theta(x) \rangle d^n x = 0.$$

Proof. Using Lemma 101, we get

$$\left| \int_{\mathbb{R}^n} \langle \phi^* [\mathcal{L}_b, \rho^\epsilon *] K(x), \theta(x) \rangle d^n x \right| = \left| \int_{\mathbb{R}^n} \langle [\mathcal{L}_b, \rho^\epsilon *] K(y), \tilde{\theta}(y) \rangle d^n y \right|,$$

where $\tilde{\theta}(y) = J\phi^{-1}(y)(\phi_*\theta^\sharp(y))^\flat$. Now, applying Lemma 116, we obtain (4.4.11). $\square$

Moreover, by integration by parts, one can also check that

$$\int_{\mathbb{R}^n} \phi^* \langle [\mathcal{L}_\xi, [\mathcal{L}_\xi, \rho^\epsilon *]] K(x), \theta(x) \rangle = 0, \quad (4.4.12)$$

for any C^1 -diffeomorphism ϕ , since ξ is only time-dependent.

We are now ready to prove our uniqueness result.

Proof of Theorem 111. We showed earlier by applying the KIW formula that the pull-back of the mollified k -form-valued process K^ϵ satisfies (4.4.6). We observe that in order to prove the uniqueness of solutions to equation (0.2.10), it suffices to show that $\phi_t^* K(t, \cdot)$ is zero in the weak sense for all $t \in [0, T]$, a.s. if $K(0, \cdot) = 0$, since ϕ_t is a C^1 -diffeomorphism for all $t \in [0, T]$.

Fixing a test function $\theta \in C_0^\infty(\bigwedge^k(T^*\mathbb{R}^n))$ with support in $B(0, R)$, taking into account (0.2.10), and using stochastic Fubini (Theorem 33), we have

$$\begin{aligned} \int_{\mathbb{R}^n} \langle \phi_t^* K^\epsilon(t, x), \theta(x) \rangle d^n x &= \int_0^t \int_{\mathbb{R}^n} \langle \phi_s^* [\mathcal{L}_b, \rho^\epsilon *] K(s, x), \theta(x) \rangle d^n x ds \\ &\quad + \int_0^t \int_{\mathbb{R}^n} \langle \phi_s^* [\mathcal{L}_\xi, \rho^\epsilon *] K(s, x), \theta(x) \rangle d^n x dW_s, \end{aligned} \quad (4.4.13)$$

where we have employed (4.4.12). We note that stochastic Fubini can be applied since by (4.4.11) and taking into account Lemma 103 (A) & (B), we have the bound

$$\left| \int_{\mathbb{R}^n} \langle \phi_s^* [\mathcal{L}_b, \rho^\epsilon *] K(s, x), \theta(x) \rangle d^n x \right| \lesssim \|\tilde{\theta}_s\|_{L_{R^*}^\infty} \|K_s\|_{L_{R^*+1}^\infty} \|b_s\|_{W_{R^*+1}^{1,1}}, \quad (4.4.14)$$

uniformly in ϵ , where $R^* < \infty$ a.s. is chosen such that

$$\{\phi_t^{-1}(x), x \in B(0, R), 0 \leq t \leq T\} \subset B(0, R^*),$$

where almost surely, we know that $\|\tilde{\theta}\|_{L_{R^*}^\infty} \in L^\infty([0, T])$. Moreover, by the definition of solution we also have $\|K\|_{L_{R^*+1}^\infty} \in L^\infty([0, T])$ a.s. and from assumption $\|b\|_{W_{R^*+1}^{1,1}} \in L^\infty([0, T])$ a.s., so the left-hand side of (4.4.14) is almost surely in $L^\infty([0, T])$. We check term-by-term convergence in (4.4.13).

- The left-hand side. We claim that

$$\begin{aligned} \int_{\mathbb{R}^n} \langle \phi_t^* K^\epsilon(t, x), \theta(x) \rangle d^n x &= \int_{\mathbb{R}^n} \langle K^\epsilon(t, y), \tilde{\theta}_t(y) \rangle d^n y \\ &\rightarrow \int_{\mathbb{R}^n} \langle K(t, y), \tilde{\theta}_t(y) \rangle d^n y = \int_{\mathbb{R}^n} \langle \phi_t^* K(t, x), \theta(x) \rangle d^n x. \end{aligned}$$

For this, we note that a.s.

$$\int_{\mathbb{R}^n} \langle K^\epsilon(t, y) - K(t, y), \tilde{\theta}_t(y) \rangle d^n y \leq \|K^\epsilon(t, \cdot) - K(t, \cdot)\|_{L_{R^*}^1} \|\tilde{\theta}_t\|_{L_{R^*}^\infty} \rightarrow 0,$$

pointwise in t , since K^ϵ converges to K in L_{loc}^1 .

- The first term on the right-hand side. By the arguments we employed above to justify Fubini, we have that

$$\left| \int_{\mathbb{R}^n} \langle \phi_s^* [\mathcal{L}_b, \rho^\epsilon *] K(s, x), \theta(x) \rangle d^n x \right| \lesssim \|\tilde{\theta}_s\|_{L_{R^*}^\infty} \|K_s\|_{L_{R^*+1}^\infty} \|b_s\|_{W_{R^*+1}^{1,1}},$$

where a.s. all of the individual terms on the right-hand side are of class L^∞ . Hence, by dominated convergence we obtain

$$\lim_{\epsilon \rightarrow 0} \int_0^t \int_{\mathbb{R}^n} \langle \phi_s^* [\mathcal{L}_b, \rho^\epsilon *] K(s, x), \theta(x) \rangle d^n x ds = 0.$$

- The martingale term on the right-hand side. Similarly, by the stochas-

tic dominated convergence theorem (Theorem 50), we have

$$\lim_{\epsilon \rightarrow 0} \left| \int_0^t \int_{\mathbb{R}^n} \langle \phi_s^* [\mathcal{L}_\xi, \rho^\epsilon *] K(s, x), \theta(x) \rangle d^n x dW_s \right| = 0,$$

in probability.

□

4.5 THE SPACE-TIME DIFFUSION CASE

We have proven our well-posedness result for equations of the type (0.2.10) (consider $N = 1$ for simplicity), with drift vector field in the class

$$b \in L^\infty([0, T]; C_b^\alpha(\mathbb{R}^n, \mathbb{R}^n)) \cap L^\infty([0, T]; W_{loc}^{1,2}(\mathbb{R}^n, \mathbb{R}^n)), \quad \text{for some } \alpha \in (0, 1),$$

and diffusion $\xi \in L^\infty([0, T])$ a semimartingale only dependent on time. First we note that the semimartingale assumption could have been dropped had we only worked in the Itô formulation. However, we wished to simplify some computations, since the Stratonovich side is the natural setting for our equation. A natural question to ask is whether our strategy would also be extensible to the case $\xi = \xi(t, x)$. For this, we recall that our argument has three important parts: (1) showing enough regularity of the solutions of the flow equation (4.1.1), (2) proving existence of weak solutions of (0.2.10), and (3) establishing the uniqueness of solutions. By inspecting our proofs of (2), it is easy to check that the existence results follow almost immediately if we incorporate a space-time-dependent diffusion vector field $\xi(t, x)$ given that (1) is provided and once some spatial regularity is assumed (actually, we wrote the proofs in a way that permits generalising the results very easily to the space-time case). Extending (3) is more complicated, since in the time-dependent case, the Stratonovich-to-Itô correction of the commutator evaluated in a test k -form θ via L^2 -pairing

$$\frac{1}{2} \int_{\mathbb{R}^n} \phi_t^* \langle [\mathcal{L}_\xi, [\mathcal{L}_\xi, \rho^\epsilon *]] K(x), \theta(x) \rangle \quad (4.5.1)$$

vanishes. In the general case, (4.5.1) becomes very convoluted and we need an estimate of the type of the one we provided in Corollary 117 for the correction in order to obtain uniqueness. After doing some computations, this seems rather counter-intuitive because some high order terms appear, which in principle seem too singular to treat. However, by a different extension of DiPerna-Lions argument, we are able to explore some cancellations and establish an estimate of the type

$$\left| \int_{\mathbb{R}^n} \langle \phi^*[\mathcal{L}_\xi, [\mathcal{L}_\xi, \rho^\epsilon *]] K(x), \theta(x) \rangle dx \right| \lesssim \|\tilde{\theta}\|_{L_R^\infty} \|K\|_{L_{R+1}^\infty} \left(\|\xi\|_{L_{R+1}^\infty} \|\nabla^2 \xi\|_{L_{R+1}^1} + \|\nabla \xi\|_{L_{R+1}^\infty} \|\nabla \xi\|_{L_{R+1}^1} \right),$$

which we can then successfully apply to obtain uniqueness. Point (1) however seems to be the most mathematically challenging one. To be more concrete, we are able to imitate almost our entire argument in the flow section in the general case except for the key fact that desired parabolic results (as in Theorem 31) are not available for equations with form

$$\partial_t u_\lambda(t, x) + \xi^j(t, x) \xi^k(t, x) \partial_j \partial_k u_\lambda(t, x) - \lambda u_\lambda(t, x) = \tilde{f}(t, x), \quad t > 0, x \in \mathbb{R}^n,$$

with coefficients $\tilde{f}(t, x), \xi(t, x)$ being merely L^∞ -regular in time. In particular, representation formulas of the form

$$Gf(t, x) = \int_t^\infty \int_{\mathbb{R}^n} e^{-\lambda(s-t)} p(t, s, x - y) \tilde{f}(s, y) dy ds$$

fail not available, and gradient estimates similar to

$$|Dp(t, s, x)| \lesssim (s - t)^{-(n+1)/2} \exp\left(\frac{-\epsilon|x|^2}{s - t}\right)$$

do not seem to be in the literature either, although a general existence and uniqueness theorem is provided in¹¹⁶ for only measurable in time coefficients. When ξ is chosen only to depend on space, related gradient estimates for unbounded ξ can be found in^{21,82} in the case $\lambda = 0, \tilde{f} = 0$.

Our conclusion is that new sharp parabolic estimates would need to be devel-

oped, and for instance, the time-dependent coefficients results in²⁴ that we strongly use here would need to be generalised. Finally, in the case when the drift and diffusion vector fields are taken to be continuous in time, we believe that the results in (1) can be proven by taking into account Chapter 9 in⁸³ since we are working with vector fields in the space $L^\infty([0, T]; C_b^\alpha(\mathbb{R}^n, \mathbb{R}^n))$, which enjoy appropriate uniform properties. This will likely be the subject of a future publication.

5

Lagrangian-averaged Boussinesq equations with transport noise

5.1 THE STOCHASTIC LAGRANGIAN-AVERAGED BOUSSINESQ EQUATIONS

The deterministic 2D incompressible Boussinesq equations are

$$\begin{aligned}\partial_t u + (u \cdot \nabla) u &= -\nabla p + g\theta \hat{y}, \\ \partial_t \theta + u \cdot \nabla \theta &= 0, \\ \nabla \cdot u &= 0, \quad t > 0, \quad \mathbf{x} = (x, y) \in \mathbb{T}^2.\end{aligned}\tag{5.1.1}$$

Here $u = (u_1, u_2)$ is the incompressible velocity field, p is the pressure, g is the acceleration due to gravity, θ corresponds to the potential temperature, or buoyancy, which is transported by the fluid, and $\hat{y}$ is the unitary basis vector in the y direction. The Boussinesq equations (5.1.1) are fundamental in meteorology. Among other aspects, these equations are used to model the process of front formation. They are also employed for the study of large scale atmospheric and oceanic flows, built environment, and dispersion of dense gases^{137,143}. From a mathematical viewpoint, the Boussinesq equations re-

tain some key features of the well-known Euler and Navier-Stokes equations, as for instance, the vortex stretching mechanism for $\nabla\theta \times \widehat{y}$, as explained in⁴ and first remarked in¹²⁷. See³⁶ for insight into the relation between Boussinesq and the axisymmetric 3D Euler equations. The Boussinesq equations have attracted considerable attention in the PDE community and local existence results (starting with the pioneering results in³¹), regularity criteria (see³⁷ for a classical local existence of smooth solutions result and blow-up criterion), as well as global existence results for its viscous counterpart^{35,107} are available. The fundamental issue of whether classical solutions of the 2D incompressible Boussinesq equations can develop finite-time singularities remains an outstanding open problem which seems to be out of reach¹⁵⁶. A partial answer to the question has been provided in⁶⁷, where for a particular choice of domain, it is proven that solutions become singular in finite time. Numerical studies have given reason to believe that singularities form indeed in finite time^{141,126}.

In this chapter, we consider a Lagrangian-averaged with transport noise modification of the deterministic Boussinesq system (5.1.1) as a suitable model for predicting “climate” dynamics. The model we consider was introduced and studied in⁷, arising from a rich geometric framework developed in⁶⁵. The equations of interest read

$$\begin{aligned} du + \mathcal{L}_{\mathbb{E}[u]}u dt + \sum_{i=1}^N \mathcal{L}_{\xi_i}u \circ dW_t^i \\ = -d\mathbb{E} \left[p - |u|^2/2 \right] dt + g\mathbb{E}[\theta] dy dt - gy d(\theta - \mathbb{E}[\theta])dt, \\ d\theta + \mathcal{L}_{\mathbb{E}[u]}\theta dt + \sum_{i=1}^N \mathcal{L}_{\xi_i}\theta \circ dW_t^i = 0, \\ \operatorname{div}(\mathbb{E}[u]) = 0, \quad t > 0, \quad \mathbf{x} \in \mathbb{T}^2, \end{aligned} \quad (5.1.2)$$

where we remind that u is modelled as a one-form-valued stochastic process and θ is a scalar-function-valued stochastic process. We stress that the symbols d and $\mathbf{d}$ are not to be confused, since the former is the “differential” notation for stochastic differential equations that we have been using throughout this dissertation, and the latter is the exterior derivative

$df = \nabla f \cdot d\mathbf{x}$ (i.e., the gradient operator in global coordinates written in geometric form, which is convenient to us at the moment). We will refer to system (5.1.2) as LA Boussinesq equations with transport noise or stochastic LA Boussinesq equations. We have introduced our system in Stratonovich form because it is more natural, but all the theory will be developed in the Itô side.

Remark 118. *Writing $df = \nabla f \cdot d\mathbf{x}$ is of course an abuse of notation since the first f is a one-form on $\mathbb{T}^2$ and the second one refers to its global coordinate expression. We hope no confusion is caused.*

We stress that in equations (5.1.2), we have employed the notation $\mathcal{L}_\xi$ to indicate Lie derivative of a one-form $u = u \cdot d\mathbf{x}$ (abuse of notation) with respect to the vector field ξ . As written in Appendix A, the Lie derivative of a one-form u with respect to a vector field ξ on $\mathbb{R}^n$ takes the shape

$$\mathcal{L}_\xi u = \sum_{j=1}^n (\xi^j \partial_j u + u_j \nabla \xi^j)$$

in global coordinates. We could have written equations (5.1.2) in explicit form by using the formula above, but we have chosen to do it geometrically for convenience.

Remark 119 (Divergence-free condition on the expectation of the velocity). *We note that although, at first sight, the more restrictive divergence-free condition $\operatorname{div}(u) = 0$ might seem more natural to consider than our current condition $\operatorname{div}(\mathbb{E}[u])$, it would make equations (5.1.2) ill-posed. This is due to the presence of the term $d\mathbb{E}[p - |u|^2/2]$, which imposes the pressure to be deterministic. Further insight into this will be provided once we present our approach for solving equations (5.1.2).*

Upon (formally) passing to the Itô formulation, the LA Boussinesq system

(5.1.2) transforms into

$$\begin{aligned}
& du + \mathcal{L}_{\mathbb{E}[u]}u dt + \sum_{i=1}^N \mathcal{L}_{\xi_i}u dW_t^i \\
&= -d\mathbb{E}[p - |u|^2/2] dt + g\mathbb{E}[\theta] dy dt - gy d(\theta - \mathbb{E}[\theta]) dt + \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 u dt, \\
& d\theta + \mathcal{L}_{\mathbb{E}[u]}\theta dt + \sum_{i=1}^N \mathcal{L}_{\xi_i}\theta dW_t^i = \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \theta dt, \\
& \operatorname{div}(\mathbb{E}[u]) = 0,
\end{aligned} \tag{5.1.3}$$

where we have employed the notation $\mathcal{L}^2$ for the composition of Lie derivatives $\mathcal{L}_\xi^2 f = \mathcal{L}_\xi(\mathcal{L}_\xi f)$. We formally take expectations at both sides of the equations above, obtaining the following deterministic equations for the evolution of the expectations

$$\begin{aligned}
& \partial_t \mathbb{E}[u] + \mathcal{L}_{\mathbb{E}[u]}\mathbb{E}[u] = -d(\mathbb{E}[p] - \mathbb{E}[|u|^2/2]) + g\mathbb{E}[\theta] dy + \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[u], \\
& \partial_t \mathbb{E}[\theta] + \mathcal{L}_{\mathbb{E}[u]}\mathbb{E}[\theta] = \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[\theta].
\end{aligned} \tag{5.1.4}$$

Remark 120. *An uncommon feature of equations (5.1.4) is that they do not preserve the mean on the torus. This will play a role in our analysis.*

We define the vorticity of the fluid by $\omega = (\nabla \times u) \cdot \widehat{z}$ as explained in the preliminaries (we note that u in the formula above should be understood in coordinates). It is straightforward to compute the LA Boussinesq equations with transport noise in curl form as

$$\begin{aligned}
& d\omega + \mathcal{L}_{\mathbb{E}[u]}\omega dt + \sum_{i=1}^N \mathcal{L}_{\xi_i}\omega \circ dW_t^i = g\partial_x \theta dt, \\
& d\theta + \mathcal{L}_{\mathbb{E}[u]}\theta dt + \sum_{i=1}^N \mathcal{L}_{\xi_i}\theta \circ dW_t^i = 0.
\end{aligned}$$

We stress that since ω is a scalar quantity for incompressible planar flow, its

Lie derivative is to be understood as the Lie derivative of a scalar function, i.e., $\mathcal{L}_\xi \omega = \xi \cdot \nabla \omega$ (see Appendix A), where we are again abusing notation. The corresponding equations for the expectation of ω are given by

$$\begin{aligned}\partial_t \mathbb{E}[\omega] + \mathcal{L}_{\mathbb{E}[u]} \mathbb{E}[\omega] &= g \partial_x \mathbb{E}[\theta] + \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[\omega], \\ \partial_t \mathbb{E}[\theta] + \mathcal{L}_{\mathbb{E}[u]} \mathbb{E}[\theta] &= \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[\theta].\end{aligned}\tag{5.1.5}$$

We will devote the next two sections to establishing the global existence and uniqueness of smooth solutions to equations (5.1.3). We start by providing some important definitions. As usually, we fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_t)$ satisfying the usual conditions, together with a family $\{W_t^i\}_{i=1}^N$ of independent Brownian processes adapted to $(\mathcal{F}_t)_t$.

Definition 121 (Global solution). *A stochastic process $(u, \theta) : \Omega \times [0, \infty) \times \mathbb{T}^2 \rightarrow \mathbb{R}^2 \times \mathbb{R}$ is called a strong global solution to the stochastic LA Boussinesq equations (5.1.3) if (u_t, θ_t) is adapted to $(\mathcal{F}_t)_t$ and the following is satisfied :*

- $\mathbb{P}$ -a.s. (u, θ) has paths of class $C([0, \infty); H^2(\mathbb{T}^2, \mathbb{R}^2) \times H^3(\mathbb{T}^2, \mathbb{R}))$.
- It holds

$$\begin{aligned}u_t - u_0 + \int_0^t \mathcal{L}_{\mathbb{E}[u]} u \, ds + \sum_{i=1}^N \int_0^t \mathcal{L}_{\xi_i} u \, dW_s^i + \int_0^t f \, ds \\ = \frac{1}{2} \sum_{i=1}^N \int_0^t \mathcal{L}_{\xi_i}^2 u \, ds - g y \int_0^t d\theta \, ds, \\ \theta_t - \theta_0 + \int_0^t \mathcal{L}_{\mathbb{E}[u]} \theta \, ds + \sum_{i=1}^N \int_0^t \mathcal{L}_{\xi_i} \theta \, dW_s^i = \frac{1}{2} \sum_{i=1}^N \int_0^t \mathcal{L}_{\xi_i}^2 \theta \, ds,\end{aligned}\tag{5.1.6}$$

as identities in $L^2(\mathbb{T}^2, \mathbb{R}^2)$ and $L^2(\mathbb{T}^2, \mathbb{R})$, respectively, for all $t > 0$. Above, we have employed the notation

$$f = d\mathbb{E}[p - |u|^2/2 - g\theta y] + g\mathbb{E}[\theta] \, dy.$$

We seek to establish the following theorem.

Theorem 122. *Let $(u_0, \theta_0) \in H^2(\mathbb{T}^2, \mathbb{R}^2) \times H^3(\mathbb{T}^2, \mathbb{R})$ with $(\mathbb{E}[u_0], \mathbb{E}[\theta_0]) \in H^5(\mathbb{T}^2, \mathbb{R}^2) \times H^4(\mathbb{T}^2, \mathbb{R})$. Assume that $\xi_i \in L^\infty([0, \infty); C^{4+\alpha}(\mathbb{T}^2, \mathbb{R}^2))$, $i = 1, \dots, N$, for some $0 < \alpha < 1$, and there is $\lambda > 0$ such that*

$$\sum_{i=1}^N \sum_{j,k=1}^2 \xi_i^j(t, \mathbf{x}) \xi_i^k(t, \mathbf{x}) v_j v_k \geq \lambda |v|^2, \quad v \in \mathbb{R}^2, \quad (5.1.7)$$

for all $t \geq 0$ and $\mathbf{x} \in \mathbb{T}^2$. Then there exists a pathwise unique global solution of the stochastic LA Boussinesq equations (5.1.3) in the sense of Definition 121.

Remark 123. Condition (5.1.7) generalises a fundamental property of the Laplace operator Δ , which can be recovered in the double Lie derivative terms of equations (5.1.5) by choosing $N = 2$, $\xi_i = e_i$, $i = 1, 2$.

The proof of Theorem 122 strongly depends on the following fact, which we state as a separate result and consists in finding a smooth enough solution to the (deterministic) expectation equations (5.1.4).

Theorem 124. *Let $(U_0, \Theta_0) \in H^{s+1}(\mathbb{T}^2, \mathbb{R}^2) \times H^s(\mathbb{T}^2, \mathbb{R})$, $s \geq 2$ and $\xi_i \in L^\infty([0, \infty); W^{s+1,\infty}(\mathbb{T}^2, \mathbb{R}^2))$, $i = 1, \dots, N$ satisfying (5.1.7). Then there exists a unique global solution $(U, \Theta) \in C([0, \infty); H^{s+1}(\mathbb{T}^2, \mathbb{R}^2) \times H^s(\mathbb{T}^2, \mathbb{R}))$ to equations*

$$\begin{aligned} \partial_t U + \mathcal{L}_U U &= -d\tilde{P} + g\Theta dy + \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 U, \\ \partial_t \Theta + \mathcal{L}_U \Theta &= \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \Theta. \end{aligned} \quad (5.1.8)$$

Remark 125. We note that the existence results provided in both Theorem 122 and Theorem 124 could be sharper. However, we are not interested in treating the sharpest case and rather wish to illustrate the fact that by applying the stochastic LA approach to the Boussinesq equations, global existence and uniqueness of smooth solutions in Sobolev spaces is guaranteed. Hence the stochastic LA approach provides a powerful regularisation mechanism.

We comment on the main idea of our global existence proof.

Main strategy of the proof. We start by solving for the variables $\mathbb{E}[u]$ and $\mathbb{E}[\theta]$ in the equations for the expectation (5.1.4). Since the expectation equations are not mean-preserving, we split our variables into their mean on the torus and their deviation from the mean. Taking into account that the mean-free part of $\mathbb{E}[u]$ is incompressible, we project the momentum equation onto its vorticity formulation by applying the curl operator, derive appropriate energy estimates for the new system, and then recover the mean-free part of $\mathbb{E}[u]$ by means of the Biot-Savart law (for a short explanation of the Biot-Savart law, see Section 1.1 in the preliminaries). Moreover, the deviation from the mean part can be estimated in terms of the mean and this allows us to close our energy estimates. The pressure term $\mathbb{E}[p - |u|^2/2]$ can be recovered by solving a Poisson problem. This argument is detailed in Section 5.2. After having shown that the equations for the mean have a unique solution, we plug the already solved deterministic variables $\mathbb{E}[u]$, $\mathbb{E}[\theta]$, and $\mathbb{E}[p - |u|^2/2]$ into the main equations (5.1.3). We note that these become stochastic linear equations with a forcing. We find an explicit solution for u and θ by applying a characteristics method. Once this is done, we can recover the expected pressure as $\mathbb{E}[p] = \mathbb{E}[p - |u|^2/2] + \mathbb{E}[|u|^2/2]$. We also establish uniqueness via common SPDE methods.

5.2 GLOBAL WELL-POSEDNESS OF THE EXPECTATION SYSTEM

In this section, we provide the proof of Theorem 124. We divide our argument into three steps: first, we employ energy methods to obtain a priori estimates of the Sobolev norms of (Ω, Θ) , where Ω is the vorticity of U . The mean-free part of the velocity can be recovered from Ω via the Biot-Savart law. Second, we prove a bound for the evolution of the average of U on the torus, since equations (5.1.8) are not mean-preserving. This permits establishing that solutions exist in the required spaces. To conclude, we argue that the solutions we constructed are also unique. We note that the global existence of solutions of the Boussinesq equations with dissipation has been

investigated in the literature. For instance, ³⁵ shows the global well-posedness of the Boussinesq system with either zero diffusivity or zero viscosity. More partial viscosity results were later provided for instance in ^{4,1}.

Step 1: A priori estimates. Since equations (5.1.8) do not preserve the mean due to the double Lie derivative term, we will take into account the following decomposition of our variables into their mean and deviation from the mean

$$\begin{aligned} U &= \bar{U} + V, \quad \Theta = \bar{\Theta} + W, \\ \bar{U} &= \int_{\mathbb{T}^2} U dV, \quad \bar{\Theta} = \int_{\mathbb{T}^2} \Theta dV, \end{aligned}$$

where V, W are mean-free. Equations (5.1.8) are

$$\begin{aligned} \partial_t U + \mathcal{L}_U U &= -d\tilde{P} + \Theta dy + \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 U, \\ \partial_t \Theta + \mathcal{L}_U \Theta &= \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \Theta, \end{aligned} \tag{5.2.1}$$

where we have set $g = 1$ without loss of generality. We apply the curl operator to the momentum equation (take into account that the curl commutes with the Lie derivative), obtaining the following coupled system for the (expected) vorticity $\Omega = (\nabla \times U) \cdot \hat{z}$ and the (expected) potential temperature, which are scalars *

$$\begin{aligned} \partial_t \Omega + \mathcal{L}_U \Omega &= \partial_x \Theta + \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \Omega, \\ \partial_t \Theta + \mathcal{L}_U \Theta &= \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \Theta. \end{aligned} \tag{5.2.2}$$

*Although throughout this section we will be working with variables that represent expectations of the original variables, we will refer to Ω and Θ as the vorticity and potential temperature for simplicity.

We can derive the following equation for the evolution of the mean of U by integrating in space the momentum equation in (5.2.1)

$$\partial_t \bar{U} = \int_{\mathbb{T}^2} \Theta \widehat{y} dV + \frac{1}{2} \sum_{i=1}^N \int_{\mathbb{T}^2} \mathcal{L}_{\xi_i}^2 V dV + \frac{1}{2} \sum_{i=1}^N \int_{\mathbb{T}^2} \mathcal{L}_{\xi_i}^2 \bar{U} dV, \quad (5.2.3)$$

where we have taken into account that

$$\int_{\mathbb{T}^2} \mathcal{L}_U U dV = 0$$

since U is divergence-free. Assuming that $(\Omega(t), \Theta(t))$ is smooth and satisfies (5.2.2), we can start to derive a priori estimates. In particular, for $s \geq 2$, we show the bound

$$\sup_{t \in [0, T]} (\|\Omega(t)\|_{H^s}^2 + \|\Theta(t)\|_{H^s}^2) \leq C(T) < \infty, \quad T > 0, \quad (5.2.4)$$

where $C(T) = C(\|\xi_i\|_{L^\infty([0, \infty); W^{s+1, \infty})}, \lambda, \|\Omega_0\|_{H^s}, \|\Theta_0\|_{H^s}, T)^\dagger$ and we remind that λ is the ellipticity constant specified in (5.1.7). Since we will be dragging along several constants, we assume for simplicity of the calculations that $\|\Lambda^s V\|_{L^2} \leq \|\Lambda^{s-1} \Omega\|_{L^2}$, although there should be a constant in the last inequality (see (1.1.11)).

We begin by providing the L^2 -estimate. We multiply the second equation in (5.2.2) with Θ and integrate over $\mathbb{T}^2$ to obtain

$$\frac{1}{2} \frac{d}{dt} \|\Theta\|_{L^2}^2 + \int_{\mathbb{T}^2} \mathcal{L}_U \Theta \Theta dV = \frac{1}{2} \sum_{i=1}^N \int_{\mathbb{T}^2} \mathcal{L}_{\xi_i}^2 \Theta \Theta dV.$$

Taking into account the incompressibility condition $\operatorname{div}(U) = 0$, the second term on the left-hand side above vanishes. Moreover, by expanding the

[†]Here, we have employed the notation $L^\infty([0, \infty); W^{s+1, \infty})$ for $L^\infty([0, \infty); W^{s+1, \infty}(\mathbb{T}^2, \mathbb{R}^2))$, which we will keep along this subsection for simplicity.

double Lie derivative operator, we get

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^N \int_{\mathbb{T}^2} \mathcal{L}_{\xi_i}^2 \Theta \Theta dV &= \frac{1}{2} \sum_{i=1}^N \sum_{j,k=1}^2 \int_{\mathbb{T}^2} \left(a_i^{jk} \partial_j \partial_k \Theta + b_i^j \partial_j \Theta \right) \Theta dV \\ &= \frac{1}{2} (I_1 + I_2), \end{aligned}$$

with coefficients $a_i^{jk} = \xi_i^j \xi_i^k$ and $b_i^j = (\xi_i \cdot \nabla) \xi_i^j$, $i = 1, \dots, N$, $j, k \in \{1, 2\}$, where

$$\begin{aligned} I_1 &= - \sum_{i=1}^N \sum_{j,k=1}^2 \int_{\mathbb{T}^2} a_i^{jk} (\partial_j \Theta) (\partial_k \Theta) dV - \sum_{i=1}^N \sum_{j,k=1}^2 \int_{\mathbb{T}^2} (\partial_j a_i^{jk}) (\partial_k \Theta) \Theta dV, \\ I_2 &= \sum_{i=1}^N \sum_{j,k=1}^2 \int_{\mathbb{T}^2} b_i^j (\partial_j \Theta) \Theta dV. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Theta\|_{L^2}^2 + \frac{1}{2} \sum_{i=1}^N \sum_{j,k=1}^2 \int_{\mathbb{T}^2} a_i^{jk} (\partial_j \Theta) (\partial_k \Theta) dV \\ = - \frac{1}{2} \sum_{i=1}^N \sum_{j,k=1}^2 \int_{\mathbb{T}^2} (\partial_j a_i^{jk}) (\partial_k \Theta) \Theta dV + I_2, \end{aligned}$$

and applying the uniform ellipticity condition (5.1.7) and Hölder's inequality, we arrive at

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\Theta\|_{L^2}^2 + \frac{\lambda}{2} \|\nabla \Theta\|_{L^2}^2 \\ &\leq \sum_{i=1}^N \sum_{j,k=1}^2 \|\nabla a_i^{jk}\|_{L_{t,x}^\infty} \|\nabla \Theta\|_{L^2} \|\Theta\|_{L^2} + \sum_{i=1}^N \sum_{j,k=1}^2 \|b_i^j\|_{L_{t,x}^\infty} \|\nabla \Theta\|_{L^2} \|\Theta\|_{L^2}. \end{aligned}$$

Applying Peter-Paul's inequality (see (1.1.9)), we find

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\Theta\|_{L^2}^2 + \frac{\lambda}{2} \|\nabla \Theta\|_{L^2}^2 \\ &\leq \sum_{i=1}^N \sum_{j,k=1}^2 \left(\|\nabla a_i^{jk}\|_{L_{t,x}^\infty} + \|b_i^j\|_{L_{t,x}^\infty} \right) \left(\frac{\|\Theta\|_{L^2}^2}{2\epsilon} + \frac{\epsilon}{2} \|\nabla \Theta\|_{L^2}^2 \right), \end{aligned}$$

and choosing

$$\epsilon = \frac{\lambda}{2} \left(\sum_{i=1}^N \sum_{j,k=1}^2 \left(\|\nabla a_i^{jk}\|_{L_{t,x}^\infty} + \|b_i^j\|_{L_{t,x}^\infty} \right) \right)^{-1},$$

we obtain

$$\frac{1}{2} \frac{d}{dt} \|\Theta\|_{L^2}^2 + \frac{\lambda}{4} \|\nabla \Theta\|_{L^2}^2 \leq C_0 \|\Theta\|_{L^2}^2,$$

where $C_0 = C_0(\|\xi_i\|_{L^\infty([0,\infty);W^{1,\infty})}, \lambda)$. By Grönwall's inequality, we conclude

$$\sup_{t \in [0,T]} \|\Theta\|_{L^2}^2 \leq 2C_0 \|\Theta_0\|_{L^2}^2 e^T, \quad \int_0^T \|\nabla \Theta\|_{L^2}^2 d\tau \leq \frac{8}{\lambda} C_0^2 \|\Theta_0\|_{L^2}^2 e^T.$$

In other words

$$\sup_{t \in [0,T]} \|\Theta\|_{L^2}^2 \leq C_1, \quad \int_0^T \|\nabla \Theta\|_{L^2}^2 d\tau \leq C_1, \quad (5.2.5)$$

where $C_1 = C_1(C_0, \lambda, \|\Theta_0\|_{L^2}, T)$. We note that since along this proof we are merely interested in deriving universal bounds and are not aiming to find optimal constants for our bounds, we will omit the explicit applications of Grönwall's Lemma from now on, simply indicating on which variables each constant depends. To compute the evolution of the L^2 -norm of Ω , we multiply the first equation in (5.2.2) by Ω and integrate, getting

$$\frac{1}{2} \frac{d}{dt} \|\Omega\|_{L^2}^2 + \int_{\mathbb{T}^2} (\mathcal{L}_U \Omega) \Omega dV = \int_{\mathbb{T}^2} (\partial_x \Theta) \Omega dV + \frac{1}{2} \sum_{i=1}^N \int_{\mathbb{T}^2} (\mathcal{L}_{\xi_i}^2 \Omega) \Omega dV.$$

By the incompressibility condition, the second term on the left-hand side vanishes, and taking into account the ellipticity condition (5.1.7) satisfied by the double Lie derivative operator, we integrate by parts and apply Hölder's

inequality as before, obtaining

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Omega\|_{L^2}^2 + \frac{\lambda}{2} \|\nabla \Omega\|_{L^2}^2 \\ & \leq \sum_{i=1}^N \sum_{j,k=1}^2 \left(\left(\|\nabla a_i^{j,k}\|_{L_{t,x}^\infty} + \|b_i^j\|_{L_{t,x}^\infty} \right) \|\Omega\|_{L^2} \|\nabla \Omega\|_{L^2} + \|\Theta\|_{L^2} \|\nabla \Omega\|_{L^2} \right). \end{aligned}$$

Once again using Peter-Paul's inequality (1.1.9) and the fact that $\|\nabla V\|_{L^2} \leq \|\Omega\|_{L^2}$ by the Biot-Savart inequality (1.1.11), we get

$$\frac{1}{2} \frac{d}{dt} \|\Omega\|_{L^2}^2 + \frac{\lambda}{4} \|\nabla \Omega\|_{L^2}^2 \leq C_2 (\|\Omega\|_{L^2}^2 + \|\Theta\|_{L^2}^2),$$

where $C_2 = C_2(\|\xi_i\|_{L^\infty([0,\infty);W^{1,\infty})}, \lambda)$. Finally, by Grönwall's inequality together with bound (5.2.5), we conclude

$$\sup_{t \in [0,T]} \|\Omega\|_{L^2}^2 \leq C_3, \quad \int_0^T \|\nabla \Omega\|_{L^2}^2 d\tau \leq C_3, \quad (5.2.6)$$

where $C_3 = C_3(C_1, C_2, \lambda, \|\Omega_0\|_{L^2}, T)$.

Next, we derive the a priori estimate for the $\dot{H}^s$ -norm of the variables Θ and Ω , $s \geq 2$. We prove this recursively and assume without loss of generality that

$$\sup_{t \in [0,T]} \|\Lambda^{s-1} \Theta\|_{L^2}^2 < \infty, \quad \int_0^T \|\Lambda^s \Theta\|_{L^2}^2 d\tau < \infty, \quad (5.2.7)$$

$$\sup_{t \in [0,T]} \|\Lambda^{s-1} \Omega\|_{L^2}^2 < \infty, \quad \int_0^T \|\Lambda^s \Omega\|_{L^2}^2 d\tau < \infty. \quad (5.2.8)$$

We justify assumptions (5.2.8) & (5.2.7) at the end of the proof. We start by computing the evolution of the $\dot{H}^s$ -norm of Θ . Integrating by parts in the

second equation in (5.2.2), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^s \Theta\|_{L^2}^2 &= - \int_{\mathbb{T}^2} \Lambda^s (\bar{U} \cdot \nabla \Theta) \Lambda^s \Theta dV \\ &\quad - \int_{\mathbb{T}^2} \Lambda^s (V \cdot \nabla \Theta) \Lambda^s \Theta dV + \frac{1}{2} \sum_{i=1}^N \int_{\mathbb{T}^2} \Lambda^s \mathcal{L}_{\xi_i}^2 \Theta \Lambda^s \Theta dV. \end{aligned} \quad (5.2.9)$$

By noting that $\bar{U}$ does not depend on the spatial variable, integrating by parts, and using the divergence-free condition, we get

$$\begin{aligned} - \int_{\mathbb{T}^2} \Lambda^s (\bar{U} \cdot \nabla \Theta) \Lambda^s \Theta dV &= - \frac{1}{2} \int_{\mathbb{T}^2} \bar{U} \cdot \nabla (|\Lambda^s \Theta|^2) dV \\ &= - \frac{1}{2} \int_{\mathbb{T}^2} \nabla \cdot (\bar{U} (|\Lambda^s \Theta|^2)) dV = 0. \end{aligned}$$

For the second term on the right-hand side of (5.2.9), we can do the following

$$\begin{aligned} - \int_{\mathbb{T}^2} \Lambda^s (V \cdot \nabla \Theta) \Lambda^s \Theta dV &= - \int_{\mathbb{T}^2} V \cdot \nabla \Lambda^s \Theta \Lambda^s \Theta dV \\ &\quad - \int_{\mathbb{T}^2} [\Lambda^s, V \cdot] \nabla \Theta \Lambda^s \Theta dV, \end{aligned}$$

where the first term on the right-hand side

$$\int_{\mathbb{T}^2} V \cdot \nabla \Lambda^s \Theta \Lambda^s \Theta dV = \frac{1}{2} \int_{\mathbb{T}^2} V \cdot \nabla |\Lambda^s \Theta|^2 dV$$

vanishes. We treat the second term by applying Hölder's inequality with $p = 3/2$, $q = 3$, and Kato-Ponce (1.1.7) with $p = 3/2$, $p_i = 3$, $i = 1, \dots, 4$, $f = V$, $g = \nabla \Theta$ in the last line, i.e.

$$\begin{aligned} \left| \int_{\mathbb{T}^2} [\Lambda^s, V \cdot] \nabla \Theta \Lambda^s \Theta dV \right| &\leq \|[\Lambda^s, V \cdot] \nabla \Theta\|_{L^{3/2}} \|\Lambda^s \Theta\|_{L^3} \\ &\lesssim (\|\nabla V\|_{L^3} \|\Lambda^{s-1} \nabla \Theta\|_{L^3} + \|\Lambda^s V\|_{L^3} \|\nabla \Theta\|_{L^3}) \|\Lambda^s \Theta\|_{L^3} \\ &\lesssim \|\Lambda^s V\|_{L^3} \|\Lambda^s \Theta\|_{L^3}^2. \end{aligned} \quad (5.2.10)$$

For the third term on the right-hand side of (5.2.9), we have

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^N \int_{\mathbb{T}^2} \Lambda^s \mathcal{L}_{\xi_i}^2 \Theta \Lambda^s \Theta dV &= \frac{1}{2} \sum_{i=1}^N \int_{\mathbb{T}^2} \mathcal{L}_{\xi_i}^2 \Lambda^s \Theta \Lambda^s \Theta dV \\ &\quad - \frac{1}{2} \sum_{i=1}^N \int_{\mathbb{T}^2} [\mathcal{L}_{\xi_i}^2, \Lambda^s] \Theta \Lambda^s \Theta dV. \end{aligned}$$

The commutator term can be bounded using Hölder's inequality

$$\begin{aligned} \left| \frac{1}{2} \sum_{i=1}^N \int_{\mathbb{T}^2} [\mathcal{L}_{\xi_i}^2, \Lambda^s] \Theta \Lambda^s \Theta dV \right| &\leq \sum_{i=1}^N \|[\mathcal{L}_{\xi_i}^2, \Lambda^s] \Theta\|_{L^2} \|\Lambda^s \Theta\|_{L^2} \\ &\leq C_4 \|\Lambda^{s+1} \Theta\|_{L^2} \|\Lambda^s \Theta\|_{L^2}, \end{aligned} \quad (5.2.11)$$

since by a general result from harmonic analysis, the commutator $[\mathcal{L}_{\xi_i}^2, \Lambda^s]$ is an operator of order $s+1$ (cf. [152](#)). We note that $C_4 = C_4(\|\xi_i\|_{L^\infty([0,\infty);W^{1,\infty})})$. Imitating the same arguments we carried out in the L^2 -estimates using the uniform ellipticity condition, and putting together the estimates (5.2.10) & (5.2.11), we arrive at

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^s \Theta\|_{L^2}^2 + \frac{\lambda}{2} \|\Lambda^{s+1} \Theta\|_{L^2}^2 \\ \leq C_5 \left(\|\Lambda^s V\|_{L^3} \|\Lambda^s \Theta\|_{L^3}^2 + \|\Lambda^s \Theta\|_{L^2} \|\Lambda^{s+1} \Theta\|_{L^2} \right), \end{aligned} \quad (5.2.12)$$

where $C_5 = C_5(C_4, \|\xi_i\|_{L^\infty([0,\infty);W^{s,\infty})}, \lambda)$. Moreover, by Peter-Paul's inequality we have

$$\|\Lambda^s \Theta\|_{L^2} \|\Lambda^{s+1} \Theta\|_{L^2} \leq \frac{1}{2\delta} \|\Lambda^s \Theta\|_{L^2}^2 + \frac{\delta}{2} \|\Lambda^{s+1} \Theta\|_{L^2}^2, \quad (5.2.13)$$

for any $\delta > 0$, and the bound

$$\begin{aligned} \|\Lambda^s V\|_{L^3} \|\Lambda^s \Theta\|_{L^3}^2 &\leq \|\Lambda^s V\|_{L^3}^3 + \|\Lambda^s \Theta\|_{L^3}^3 \\ &\lesssim \|\Lambda^s V\|_{L^2}^2 \|\Lambda^{s+1} V\|_{L^2} + \|\Lambda^s \Theta\|_{L^2}^2 \|\Lambda^{s+1} \Theta\|_{L^2} \\ &\leq \frac{1}{2\epsilon} \|\Lambda^s V\|_{L^2}^2 \|\Lambda^s V\|_{L^2}^2 + \frac{\epsilon}{2} \|\Lambda^{s+1} V\|_{L^2}^2 \\ &\quad + \frac{1}{2\nu} \|\Lambda^s \Theta\|_{L^2}^2 \|\Lambda^s \Theta\|_{L^2}^2 + \frac{\nu}{2} \|\Lambda^{s+1} \Theta\|_{L^2}^2, \end{aligned} \quad (5.2.14)$$

where $\delta, \epsilon, \nu > 0$ will be chosen later and we have invoked Gagliardo-Nirenberg inequality (1.1.4) in the second line and Peter-Paul's (1.1.9) in the third one. Inserting (5.2.13) and (5.2.14) into (5.2.12), we obtain that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^s \Theta\|_{L^2}^2 + \frac{\lambda}{2} \|\Lambda^{s+1} \Theta\|_{L^2}^2 &\leq C_6 \left(\frac{1}{2\epsilon} \|\Lambda^s V\|_{L^2}^2 \|\Lambda^s V\|_{L^2}^2 + \frac{\epsilon}{2} \|\Lambda^{s+1} V\|_{L^2}^2 \right. \\ &\quad \left. + \frac{1}{2\nu} \|\Lambda^s \Theta\|_{L^2}^2 \|\Lambda^s \Theta\|_{L^2}^2 + \frac{\nu}{2} \|\Lambda^{s+1} \Theta\|_{L^2}^2 + \frac{1}{2\delta} \|\Lambda^s \Theta\|_{L^2}^2 + \frac{\delta}{2} \|\Lambda^{s+1} \Theta\|_{L^2}^2 \right), \end{aligned}$$

where $C_6 = C_6(C_5)$. Choosing $\epsilon = \nu = \delta = (\lambda/2)C_6^{-1}$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^s \Theta\|_{L^2}^2 + \frac{\lambda}{4} \|\Lambda^{s+1} \Theta\|_{L^2}^2 \\ \leq C_7 \left(\|\Lambda^s V\|_{L^2}^2 \|\Lambda^s V\|_{L^2}^2 + \|\Lambda^{s+1} V\|_{L^2}^2 + \|\Lambda^s \Theta\|_{L^2}^2 \|\Lambda^s \Theta\|_{L^2}^2 + \|\Lambda^s \Theta\|_{L^2}^2 \right), \end{aligned}$$

with $C_7 = C_7(C_6, \lambda)$. Noticing that $\|\Lambda^s V\|_{L^2} \leq \|\Lambda^{s-1} \Omega\|_{L^2}$, $\|\Lambda^{s+1} V\|_{L^2} \leq \|\Lambda^s \Omega\|_{L^2}$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^s \Theta\|_{L^2}^2 + \frac{\lambda}{4} \|\Lambda^{s+1} \Theta\|_{L^2}^2 \\ \leq C_7 \left(\|\Lambda^s \Theta\|_{L^2}^2 + \|\Lambda^{s-1} \Omega\|_{L^2}^2 \|\Lambda^{s-1} \Omega\|_{L^2}^2 + \|\Lambda^s \Omega\|_{L^2}^2 \right) \left(\|\Lambda^s \Theta\|_{L^2}^2 + 1 \right). \end{aligned}$$

Using the global bounds (5.2.7) (5.2.8) (notice that we have assumed a bound for the integral of the H^s -norm of our variables) and applying Grönwall's inequality with $\beta(t) = \|\Lambda^s \Theta\|_{L^2}^2 + \|\Lambda^{s-1} \Omega\|_{L^2}^2 \|\Lambda^{s-1} \Omega\|_{L^2}^2 + \|\Lambda^s \Omega\|_{L^2}^2$, we obtain

$$\sup_{t \in [0, T]} \|\Lambda^s \Theta\|_{L^2}^2 \leq C_8, \quad \int_0^T \|\Lambda^{s+1} \Theta\|_{L^2}^2 d\tau \leq C_8, \quad (5.2.15)$$

where $C_8 = C_8(C_7, \lambda, \|\Lambda^s \Theta_0\|_{L^2}, T) < \infty$. The evolution of the $\dot{H}^s$ -norm of the vorticity Ω is given by

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^s \Omega\|_{L^2}^2 - \frac{1}{2} \sum_{i=1}^N \int_{\mathbb{T}^2} \mathcal{L}_{\xi_i}^2 \Lambda^s \Omega \Lambda^s \Omega dV &= - \int_{\mathbb{T}^2} \Lambda^s (\bar{U} \cdot \nabla \Omega) \Lambda^s \Omega dV \\ &- \int_{\mathbb{T}^2} \Lambda^s (V \cdot \nabla \Omega) \Lambda^s \Omega dV + \int_{\mathbb{T}^2} \Lambda^s \partial_x \Theta \Lambda^s \Omega dV - \frac{1}{2} \sum_{i=1}^N \int_{\mathbb{T}^2} [\mathcal{L}_{\xi_i}^2, \Lambda^s] \Omega \Lambda^s \Omega dV \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{5.2.16}$$

Note that $I_1 = 0$ and the bounds for I_2 and I_4 can be obtained by applying the same arguments we have previously employed. Hence

$$|I_2| \lesssim \|\Lambda^s V\|_{L^3} \|\Lambda^s \Omega\|_{L^3}^2, \quad |I_4| \leq C_9 \|\Lambda^{s+1} \Omega\|_{L^2} \|\Lambda^s \Omega\|_{L^2},$$

where $C_9 = C_9(\|\xi_i\|_{L^\infty([0,\infty);W^{s+1,\infty})}, \lambda)$. For I_3 , by integrating by parts and using Hölder's inequality, we get

$$|I_3| = \left| \int_{\mathbb{T}^2} \Lambda^s \Theta \Lambda^s \partial_x \Omega dV \right| \leq \|\Lambda^s \Theta\|_{L^2} \|\Lambda^{s+1} \Omega\|_{L^2}.$$

The double Lie derivative term on the left-hand side of (5.2.16) can be manipulated as usual by using our ellipticity conditions. Hence we conclude

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^s \Omega\|_{L^2}^2 + \frac{\lambda}{2} \|\Lambda^{s+1} \Omega\|_{L^2}^2 &\leq C_{10} (\|\Lambda^{s+1} \Omega\|_{L^2} \|\Lambda^s \Omega\|_{L^2} \\ &+ \|\Lambda^s V\|_{L^3} \|\Lambda^s \Omega\|_{L^3}^2 + \|\Lambda^s \Theta\|_{L^2} \|\Lambda^{s+1} \Omega\|_{L^2}), \end{aligned}$$

where $C_{10} = C_{10}(C_9, \|\xi_i\|_{L^\infty([0,\infty);W^{s,\infty})}, \lambda)$. Taking into account Young's inequality, Gagliardo-Nirenberg, Peter-Paul's inequality as before, and the fact

that $\|\Lambda^{s+1}V\|_{L^2} \leq \|\Lambda^s\Omega\|_{L^2}$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^s\Omega\|_{L^2}^2 + \frac{\lambda}{2} \|\Lambda^{s+1}\Omega\|_{L^2}^2 &\leq C_{11} \left(\frac{1}{2\epsilon} \|\Lambda^s V\|_{L^2}^2 \|\Lambda^s V\|_{L^2}^2 + \frac{\epsilon}{2} \|\Lambda^{s+1}V\|_{L^2}^2 \right. \\ &+ \frac{1}{2\nu} \|\Lambda^s\Omega\|_{L^2}^2 \|\Lambda^s\Omega\|_{L^2}^2 + \frac{\nu}{2} \|\Lambda^{s+1}\Omega\|_{L^2}^2 + \frac{1}{2\delta} \|\Lambda^s\Omega\|_{L^2}^2 + \frac{\delta}{2} \|\Lambda^{s+1}\Omega\|_{L^2}^2 \\ &\left. + \frac{\mu}{2} \|\Lambda^{s+1}\Omega\|_{L^2}^2 + \frac{1}{2\mu} \|\Lambda^s\Theta\|_{L^2}^2 \right), \end{aligned}$$

where $C_{11} = C_{11}(C_{10})$. Choosing $\epsilon = \nu = \delta = \mu = (\lambda/2)C_{11}^{-1}$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^s\Omega\|_{L^2}^2 + \frac{\lambda}{4} \|\Lambda^{s+1}\Omega\|_{L^2}^2 \\ \leq C_{12} \left(\|\Lambda^{s-1}\Omega\|_{L^2}^2 \|\Lambda^{s-1}\Omega\|_{L^2}^2 + \|\Lambda^s\Omega\|_{L^2}^2 (1 + \|\Lambda^s\Omega\|_{L^2}^2) + \|\Lambda^s\Theta\|_{L^2}^2 \right), \end{aligned}$$

where $C_{12} = C_{12}(C_{11}, \lambda)$. Applying Grönwall's inequality and noticing that we have the global bounds (5.2.7), (5.2.8), we find that

$$\sup_{t \in [0, T]} \|\Lambda^s\Omega\|_{L^2}^2 \leq C_{13}, \quad \int_0^T \|\Lambda^{s+1}\Omega\|_{L^2}^2 d\tau \leq C_{13}, \quad (5.2.17)$$

where $C_{13} = C_{13}(C_8, C_{12}, \lambda, \|\Lambda^s\Omega_0\|_{L^2}, T) < \infty$. Thus, we have shown the a priori estimate (5.2.4).

Remark 126 (Justification of assumptions (5.2.7) and (5.2.8)). *First we note that if $s = k \in \mathbb{N}$, $k \geq 2$, our assumptions can be justified because we have shown (5.2.15) and (5.2.17). Hence, by induction, it follows that our variables (U, Θ) enjoy regularity H^k , $k \in \mathbb{N}$, $k \geq 2$. This automatically justifies the assumption for any real number $s \geq 2$, since $H^k \subset H^s$, for $k = \lceil s \rceil$.*

Step 2: Mean growth control. We show how to control the growth of the mean part of U in terms of V, Θ and the initial mean value $\bar{U}_0$. This estimate is essential in order to establish the uniqueness of solutions, as we

will see later on. Integrating (5.2.3) in time, we get

$$\begin{aligned}\bar{U}(t) &= \bar{U}_0 + \int_0^t \int_{\mathbb{T}^2} \Theta \hat{y} dV d\tau \\ &\quad + \frac{1}{2} \sum_{i=1}^N \int_0^t \int_{\mathbb{T}^2} \mathcal{L}_{\xi_i}^2 V dV d\tau + \frac{1}{2} \sum_{i=1}^N \int_0^t \int_{\mathbb{T}^2} \mathcal{L}_{\xi_i}^2 \bar{U} dV d\tau.\end{aligned}\quad (5.2.18)$$

We treat the terms below in (5.2.18). Using the fact that $\bar{U}$ does not depend on the spatial variable, we have

$$\frac{1}{2} \sum_{i=1}^N \int_0^t \left| \int_{\mathbb{T}^2} \mathcal{L}_{\xi_i}^2 \bar{U} dV \right| d\tau \leq D_1 \int_0^t |\bar{U}| d\tau.$$

By integrating by parts twice and applying the divergence theorem, we also have

$$\frac{1}{2} \sum_{i=1}^N \int_0^t \left| \int_{\mathbb{T}^2} \mathcal{L}_{\xi_i}^2 V dV \right| d\tau \leq D_2 \int_0^t \|V\|_{L^2} d\tau.$$

We note that $D_1 = D_1(\|\xi_i\|_{W^{2,\infty}}) < \infty$ and $D_2 = D_2(\|\xi_i\|_{W^{2,\infty}}) < \infty$. Hence, we deduce

$$|\bar{U}(t)| \leq D_3 \left(|\bar{U}_0| + \int_0^t (\|V\|_{L^2} + \|\Theta\|_{L^2} + |\bar{U}|) d\tau \right),$$

with $D_3 = D_3(D_1, D_2)$. By invoking Grönwall's integral inequality (Lemma 18), we conclude that

$$\begin{aligned}|\bar{U}(t)| &\leq D_3 \left(|\bar{U}_0| + \int_0^t (\|V\|_{L^2} + \|\Theta\|_{L^2}) d\tau \right) \\ &\quad + D_3 \int_0^t \left(|\bar{U}_0| + \int_0^s (\|V\|_{L^2} + \|\Theta\|_{L^2}) d\tau \right) ds (e^t),\end{aligned}$$

and hence

$$\sup_{t \in [0, T]} |\bar{U}(t)| \leq D_4 \left(|\bar{U}_0| + \left(\sup_{t \in [0, T]} \|V\|_{L^2} + \sup_{t \in [0, T]} \|\Theta\|_{L^2} \right) \right), \quad (5.2.19)$$

where $D_4 = D_4(D_3, T)$.

Remark 127. Notice that since equation (5.2.18) is linear, one can easily imitate the previous argument to show that equation (5.2.19) also holds for the difference between two solutions $\bar{U}^1$ and $\bar{U}^2$, which we denote by $\tilde{\bar{U}} = \bar{U}^1 - \bar{U}^2$, this is

$$\sup_{t \in [0, T]} |\tilde{\bar{U}}(t)| \leq D_4 \left(|\tilde{\bar{U}}_0| + \left(\sup_{t \in [0, T]} \|\tilde{V}\|_{L^2} + \sup_{t \in [0, T]} \|\tilde{\Theta}\|_{L^2} \right) \right), \quad (5.2.20)$$

where $\tilde{V} = V^1 - V^2$ and $\tilde{\Theta} = \Theta^1 - \Theta^2$.

Step 3: Uniqueness of solutions. It suffices to prove that any two different solutions of (5.2.1) with the same initial data must be equal. Let (U^1, Θ^1) and (U^2, Θ^2) be two solutions to equations (5.2.1) with respective vorticities Ω^1 and Ω^2 and decompositions $U^1 = \bar{U}^1 + V^1$, $U^2 = \bar{U}^2 + V^2$ (see the beginning of this section for the decomposition of the velocity into its mean-free part and deviation from the mean), sharing the same initial data (U_0, Θ_0) , $\Omega_0 = (\nabla \times U_0) \cdot \hat{z}$. Defining the differences $\tilde{U} = U^1 - U^2$, $\tilde{\Theta} = \Theta^1 - \Theta^2$, $\tilde{\Omega} = \Omega^1 - \Omega^2$, $\tilde{\bar{U}} = \bar{U}^1 - \bar{U}^2$, $\tilde{V} = V^1 - V^2$, we infer that

$$\begin{aligned} \partial_t \tilde{\Omega} + \mathcal{L}_{U^1} \tilde{\Omega} + \mathcal{L}_{\tilde{V}} \Omega^2 &= \partial_x \tilde{\Theta} + \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \tilde{\Omega}, \\ \partial_t \tilde{\Theta} + \mathcal{L}_{U^1} \tilde{\Theta} + \mathcal{L}_{\tilde{V}} \Theta^2 &= \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \tilde{\Theta}. \end{aligned} \quad (5.2.21)$$

We will provide appropriate bounds for the L^2 -norm of the solutions of equations (5.2.21). In order to control the nonlinear terms $\mathcal{L}_{\tilde{V}} \Omega^2$ and $\mathcal{L}_{\tilde{V}} \Theta^2$, we will apply the identity

$$\int_{\mathbb{T}^2} \tilde{U} \cdot \nabla f^2 \tilde{f} dV = - \int_{\mathbb{T}^2} \tilde{U} \cdot \nabla \tilde{f} f^2 dV, \quad (5.2.22)$$

with $f = \Omega$ and $f = \Theta$ respectively. We first note that by using Hölder's, Peter-Paul's inequalities, and the Sobolev embedding (1.1.5), (5.2.22) can be

estimated as

$$\begin{aligned}
\left| \int_{\mathbb{T}^2} \tilde{U} \cdot \nabla \tilde{f} f^2 dV \right| &\leq \|\tilde{U}\|_{L^4} \|\nabla \tilde{f}\|_{L^2} \|f^2\|_{L^4} \\
&\leq \frac{1}{2\epsilon} \|\tilde{U}\|_{L^4}^2 \|f^2\|_{L^4}^2 + \frac{\epsilon}{2} \|\nabla \tilde{f}\|_{L^2}^2 \lesssim \frac{1}{2\epsilon} \|\tilde{U}\|_{H^1}^2 \|f^2\|_{H^1}^2 + \frac{\epsilon}{2} \|\nabla \tilde{f}\|_{L^2}^2,
\end{aligned} \tag{5.2.23}$$

where we have assumed that f enjoys enough regularity. By integrating (5.2.21) against $\tilde{\Omega}$ and $\tilde{\Theta}$, we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\tilde{\Omega}\|_{L^2}^2 + \int_{\mathbb{T}^2} \mathcal{L}_{\tilde{U}} \Omega^2 \tilde{\Omega} dV &= \int_{\mathbb{T}^2} \partial_x \tilde{\Theta} \tilde{\Omega} dV + \frac{1}{2} \sum_{i=1}^N \int_{\mathbb{T}^2} \mathcal{L}_{\xi_i}^2 \tilde{\Omega} \tilde{\Omega} dV, \\
\frac{1}{2} \frac{d}{dt} \|\tilde{\Theta}\|_{L^2}^2 + \int_{\mathbb{T}^2} \mathcal{L}_{\tilde{U}} \Theta^2 \tilde{\Theta} dV &= \frac{1}{2} \sum_{i=1}^N \int_{\mathbb{T}^2} \mathcal{L}_{\xi_i}^2 \tilde{\Theta} \tilde{\Theta} dV.
\end{aligned}$$

By using the ellipticity condition (5.1.7) as we have done several times in this section (see for instance the L^2 -estimate for θ), and applying the recently derived estimate (5.2.23), we obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left(\|\tilde{\Omega}\|_{L^2}^2 + \|\tilde{\Theta}\|_{L^2}^2 \right) + \frac{\lambda}{2} \left(\|\nabla \tilde{\Omega}\|_{L^2}^2 + \|\nabla \tilde{\Theta}\|_{L^2}^2 \right) &\lesssim \frac{1}{2\mu} \|\tilde{\Theta}\|_{L^2}^2 + \frac{\mu}{2} \|\nabla \tilde{\Omega}\|_{L^2}^2 \\
&+ \frac{1}{2\epsilon} \|\tilde{U}\|_{H^1}^2 \|\Omega^2\|_{H^1}^2 + \frac{\epsilon}{2} \|\nabla \tilde{\Omega}\|_{L^2}^2 + \frac{1}{2\delta} \|\tilde{U}\|_{H^1}^2 \|\Theta^2\|_{H^1}^2 + \frac{\delta}{2} \|\nabla \tilde{\Theta}\|_{L^2}^2.
\end{aligned}$$

By choosing $\mu = \delta = \epsilon = \lambda/4$, we get

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left(\|\tilde{\Omega}\|_{L^2}^2 + \|\tilde{\Theta}\|_{L^2}^2 \right) + \frac{\lambda}{4} \left(\|\nabla \tilde{\Omega}\|_{L^2}^2 + \|\nabla \tilde{\Theta}\|_{L^2}^2 \right) \\
\lesssim \|\tilde{\Theta}\|_{L^2}^2 + \|\tilde{U}\|_{H^1}^2 \left(\|\Omega^2\|_{H^1}^2 + \|\Theta^2\|_{H^1}^2 \right).
\end{aligned}$$

Integrating the last expression in time, we obtain

$$\begin{aligned}
\|\tilde{\Omega}\|_{L^2}^2 + \|\tilde{\Theta}\|_{L^2}^2 &\lesssim \left(\|\tilde{\Omega}_0\|_{L^2}^2 + \|\tilde{\Theta}_0\|_{L^2}^2 \right) \\
&+ \int_0^t \left(\|\tilde{U}\|_{H^1}^2 + \|\tilde{\Theta}\|_{L^2}^2 \right) \left(\|\Omega^2\|_{H^1}^2 + \|\Theta^2\|_{H^1}^2 + 1 \right) d\tau. \tag{5.2.24}
\end{aligned}$$

Moreover, since $\tilde{U} = \tilde{\tilde{U}} + \tilde{V}$, we have (take into account (5.2.20) and the

Biot-Savart inequality (1.1.11))

$$\begin{aligned}
\|\tilde{U}\|_{H^1} &= \|\tilde{\bar{U}} + \tilde{V}\|_{L^2} + \|\nabla \tilde{V}\|_{L^2} \\
&\lesssim \left| \tilde{\bar{U}}_0 \right| + \sup_{t \in [0, T]} \left(\|\tilde{V}\|_{L^2} + \|\tilde{\Theta}\|_{L^2} \right) + \|\nabla \tilde{V}\|_{L^2} \\
&\lesssim \left| \tilde{\bar{U}}_0 \right| + \sup_{t \in [0, T]} \left(\|\tilde{\Omega}\|_{L^2} + \|\tilde{\Theta}\|_{L^2} \right).
\end{aligned}$$

Therefore

$$\|\tilde{U}\|_{H^1}^2 \lesssim \left| \tilde{\bar{U}}_0 \right|^2 + \sup_{t \in [0, T]} \left(\|\tilde{\Omega}\|_{L^2}^2 + \|\tilde{\Theta}\|_{L^2}^2 \right).$$

Plugging this in (5.2.24) and denoting

$$X(t) = \sup_{s \in [0, t]} \left(\|\tilde{\Omega}\|_{L^2}^2 + \|\tilde{\Theta}\|_{L^2}^2 \right)$$

yields

$$X(t) \lesssim X(0) + \int_0^t (X(\tau) + 1) \left(\|\Omega^2\|_{H^1}^2 + \|\Theta^2\|_{H^1}^2 + 1 \right) d\tau.$$

Applying Grönwall's inequality we can obtain

$$X(t) \lesssim (X(0) + 1) \exp \left\{ \|\Omega_0^2\|_{H^1}^2 + \|\Theta_0^2\|_{H^1}^2 + 1 \right\},$$

where we have used the energy estimates derived in Step 1 for the H^k -norm of our variables, $k = 1$. Since we have assumed that $U_0^1 = U_0^2$ and $\Theta_0^1 = \Theta_0^2$, we have $X(0) = 0$, $\tilde{\bar{U}}_0 = 0$, and hence $\Omega^1(t) = \Omega^2(t)$ and $\Theta^1(t) = \Theta^2(t)$, $t > 0$. From Biot-Savart (19) we obtain $V^1 = V^2$, and from (5.2.20) we have $\bar{U}^1 = \bar{U}^2$. Finally, taking into account the decomposition $U^1 = \bar{U}^1 + V^1$, $U^2 = \bar{U}^2 + V^2$, we conclude $U^1(t) = U^2(t)$, $t > 0$.

After having derived the required a priori estimates, to establish the existence of strong solutions, it suffices to apply a standard Galerkin approximation. We do not provide the details here as this is considered standard in the PDE literature.

Remark 128 (Bootstrapping the regularity of the solutions to equations (5.2.2)). Due to the parabolic character of equations (5.2.2) (cf. ¹⁵³), we can bootstrap the regularity of the variables Ω, Θ and show that for any $t_0 > 0$, we have that $(\Omega(t), \Theta(t)) \in C^\infty([t_0, \infty) \times \mathbb{T}^2, \mathbb{R}^2 \times \mathbb{R})$. The strategy consists in noting that for initial data in $(\Omega_0^k, \Theta_0^k) \in H^k(\mathbb{T}^2, \mathbb{R}^2 \times \mathbb{R})$, where $k \in \mathbb{N}$, $k \geq 2$, there exists $C = C(\|\xi_i\|_{W^{k+1,\infty}}, \lambda, \|\Omega_0^k\|_{H^k}, \|\Theta_0^k\|_{H^k}, T)$ such that

$$\sup_{t \in [0, T]} \|\Omega\|_{H^k} \leq C, \quad \int_0^T \|\Omega\|_{H^{k+1}} d\tau \leq C,$$

$$\sup_{t \in [0, T]} \|\Theta\|_{H^k} \leq C, \quad \int_0^T \|\Theta\|_{H^{k+1}} d\tau \leq C,$$

as we have demonstrated in Step 1 (see bounds (5.2.15) and (5.2.17)). In other words, any solution to (5.2.2) with initial condition in $(\Omega_0^k, \Theta_0^k) \in H^k(\mathbb{T}^2, \mathbb{R}^2 \times \mathbb{R})$ instantly gains spatial regularity $H^{k+1}(\mathbb{T}^2, \mathbb{R}^2 \times \mathbb{R})$. Hence by starting with initial condition $(\Omega_0, \Theta_0) \in H^2(\mathbb{T}^2, \mathbb{R}^2 \times \mathbb{R})$ and choosing an increasing sequence of times $\{t_k\}_{k=2}^\infty$ with $t_k \rightarrow t_0$, $k \rightarrow \infty$, where $(U(t_k), \Theta(t_k)) \in H^k(\mathbb{T}^2, \mathbb{R}^2 \times \mathbb{R})$, we obtain that $(U(t_0), \Theta(t_0)) \in C^\infty(\mathbb{T}^2, \mathbb{R}^2 \times \mathbb{R})$. For the time regularity, note that for any $t_0 > 0$, we have shown that the right-hand side of equations (5.2.2) has regularity $C([t_0, \infty); C^\infty(\mathbb{T}^2, \mathbb{R}^2 \times \mathbb{R}))$, so $(\Omega(t), \Theta(t)) \in C^1([t_0, \infty); C^\infty(\mathbb{T}^2, \mathbb{R}^2 \times \mathbb{R}))$ and we can apply this argument iteratively.

Remark 129. To recover the modified pressure term $\tilde{P}$, we apply divergence to the momentum equation in (5.1.8), obtaining the following Poisson equation

$$-\Delta \tilde{P} = \nabla \cdot \mathcal{L}_U U - \partial_y \Theta - \frac{1}{2} \sum_{i=1}^N \nabla \cdot \mathcal{L}_{\xi_i}^2 U, \quad (5.2.25)$$

where we have taken into account the incompressibility condition $\nabla \cdot U = 0$. This Poisson equation differs from the usual, since the double Lie derivative term on the right-hand side does not vanish. Inverting the Laplacian and noticing that the right-hand side of equation (5.2.25) is of class $H^1(\mathbb{T}^2)$, we

can recover the modified pressure term and $\tilde{P} \in H^3(\mathbb{T}^2)$ (see²⁵). By bootstrapping arguments, we have

$$\tilde{P} \in C^\infty([t_0, \infty) \times \mathbb{T}^2),$$

for any $t_0 > 0$. We avoid writing the explicit form of the modified pressure term $\tilde{P}$, which is given by the convolution with the periodic Newtonian potential, since this goes beyond the purpose of the present discussion, cf.¹²⁸.

5.3 GLOBAL WELL-POSEDNESS OF THE LINEAR SYSTEM

We complete the proof of our main theorem (see Theorem 122). Set $g = 1$. Let $(u_0, \theta_0) \in H^2(\mathbb{T}^2, \mathbb{R}^2) \times H^3(\mathbb{T}^2, \mathbb{R})$ with $(\mathbb{E}[u_0], \mathbb{E}[\theta_0]) \in H^5(\mathbb{T}^2, \mathbb{R}^2) \times H^4(\mathbb{T}^2, \mathbb{R})$. First we observe that $(u, \theta) \in H^2(\mathbb{T}^2, \mathbb{R}^2) \times H^3(\mathbb{T}^2, \mathbb{R})$ is a solution to (5.1.3) if and only if it is a solution to the system

$$\begin{aligned} du + \mathcal{L}_U u dt + \sum_{i=1}^N \mathcal{L}_{\xi_i} u dW_t^i \\ = \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 u dt - d\tilde{P} dt + \Theta \widehat{y} dt - y d(\theta - \Theta) dt, \\ d\theta + \mathcal{L}_U \theta dt + \sum_{i=1}^N \mathcal{L}_{\xi_i} \theta dW_t^i = \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \theta dt, \end{aligned} \quad (5.3.1)$$

where (U, Θ) is the only solution of (5.2.1) with initial data $(U_0, \Theta_0) = (\mathbb{E}[u_0], \mathbb{E}[\theta_0])$. Indeed, it is straightforward to check that once (5.3.1) is solved, $U = \mathbb{E}[u]$, $\Theta = \mathbb{E}[\theta]$, $\tilde{P} = (\mathbb{E}[p] - \mathbb{E}[|u|^2/2])$, since the solution to (5.1.8) is unique. Hence, it is sufficient to find a solution of (5.3.1) in the required spaces. We divide the proof into two steps: first we show the existence of solutions by explicitly constructing them employing the characteristics of the system. Next, we show the uniqueness of solutions by performing a standard energy estimate.

Step 1: Existence of solutions via characteristics. Consider the char-

acteristics of system (5.3.1)

$$\begin{aligned} dX_t &= U(t, X_t)dt + \sum_{i=1}^N \xi_i(t, X_t) \circ dW_t^i \\ &= \left(U(t, X_t) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^2 \xi_i^j \partial_j \xi_i(t, X_t) \right) dt + \sum_{i=1}^N \xi_i(t, X_t) dW_t^i. \end{aligned} \quad (5.3.2)$$

[‡] Invoking Theorem 124, we have $U \in C([0, \infty); H^5(\mathbb{T}^2))$, and therefore $U \in C([0, \infty); C^{3, \alpha'}(\mathbb{T}^2))$, for $0 < \alpha' < 1$, due to the Sobolev embedding (1.1.6) (take $n = 2$, $m = 2$, $j = 3$). Therefore from Assumption A2, (5.3.2) admits a unique global solution and its flow $\phi_{s,t}$ is $C^{3, \alpha}$ regular for every $0 < \alpha < \alpha'$ (cf. ¹²⁰).

We claim that an explicit solution to the θ -equation in (5.3.1) is

$$\theta(t, \mathbf{x}) = \theta_0(\phi_t^{-1}(\mathbf{x})), \quad (5.3.3)$$

where we have employed the shorthand notation ϕ_t to represent $\phi_{0,t}$. Furthermore, one can verify that $\theta_t \in H^3(\mathbb{T}^2)$ owing to the $C^{3, \alpha}$ regularity of the flow $\phi_{s,t}$ and the fact that $\theta_0 \in H^3(\mathbb{T}^2)$. To show that the formula for θ in (5.3.3) solves the second equation in (5.3.1), we consider the mollified initial condition $\theta_0^\epsilon = \rho^\epsilon * \theta_0$. By Itô's second formula for the push-forward (1.4.8), we obtain

$$\begin{aligned} \theta_0^\epsilon(\phi_t^{-1}(\mathbf{x})) &= \theta_0^\epsilon(\mathbf{x}) - \int_0^t \mathcal{L}_U(\theta_0^\epsilon(\phi_s^{-1}(\mathbf{x}))) ds \\ &\quad - \frac{1}{2} \sum_{i=1}^N \int_0^t \mathcal{L}_{\xi_i}^2(\theta_0^\epsilon(\phi_s^{-1}(\mathbf{x}))) ds - \sum_{i=1}^N \int_0^t \mathcal{L}_{\xi_i}(\theta_0^\epsilon(\phi_s^{-1}(\mathbf{x}))) dW_s^i, \end{aligned}$$

and due to the mollifier properties, $\theta_0^\epsilon \rightarrow \theta_0$ as $\epsilon \rightarrow 0$ in H^3 , which implies that $\theta(t, \mathbf{x}) = \theta_0(\phi_t^{-1}(\mathbf{x}))$ solves the θ -equation in (5.3.1). Furthermore, since

[‡]The Stratonovich equation in the first line should only be understood formally. We are working in Itô, but the equation looks more natural in Stratonovich form.

the inverse map satisfies the equation

$$\begin{aligned} d\phi_t^{-1}(\mathbf{x}) = \int_0^t & \left(-U(s, \phi_s^{-1}(\mathbf{x})) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^2 (\xi_i^j \partial_j \xi_i)(s, \phi_s^{-1}(\mathbf{x})) \right) ds \\ & - \sum_{i=1}^N \int_0^t \xi_i(s, \phi_s^{-1}(\mathbf{x})) dW_s^i, \end{aligned}$$

it enjoys $C^{3,\alpha}$ regularity. By applying similar techniques, an explicit solution can also be constructed for the momentum equation in (5.3.1) (see⁷ for more details).

Step 2: Pathwise uniqueness. By the linearity of system (5.3.1), it suffices to verify that if $(u_0, \theta_0) = (0, 0)$, then $\mathbb{P}$ -a.s. $(u_t, \theta_t) = (0, 0)$, $t > 0$. By taking into account the techniques we used for deriving the estimate for $\|\theta_t\|_{L^2}^2$ in Step 1 of the previous section, we can obtain

$$d\|\theta_t\|_{L^2}^2 \leq C_0 \|\theta_t\|_{L^2}^2 - dM_t, \quad (5.3.4)$$

with

$$M_t = \sum_{i=1}^N \int_0^t \langle \mathcal{L}_{\xi_i} \theta, \theta \rangle_{L^2}(s) dW_s^i.$$

For $n \in \mathbb{N}$, consider the stopping time

$$\tau_n = \inf \left\{ t \geq 0 : \left| \sum_{i=1}^N \int_0^t \langle \mathcal{L}_{\xi_i} \theta, \theta \rangle_{L^2}(s) dW_s^i \right| \geq n \right\}.$$

Since the solution is global and M_t is well-defined for all $t > 0$, we must have $\tau_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$. Now, by defining

$$Y_t = -C_0 \int_0^t \|\theta_s\|_{L^2}^2 ds,$$

we can rewrite equation (5.3.4) in Grönwall's form as

$$d(\exp(Y_t) \|\theta_t\|_{L^2}^2) \leq -\exp(Y_t) dM_t,$$

and therefore by writing the stopped process in integral form we obtain

$$\exp(Y_{t \wedge \tau_n}) \|\theta_{t \wedge \tau_n}\|_{L^2}^2 \leq - \sum_{i=1}^N \int_0^{t \wedge \tau_n} \exp(Y_s) \langle \mathcal{L}_{\xi_i} \theta, \theta \rangle_{L^2}(s) dW_s^i, \quad n \in \mathbb{N}, \quad t > 0.$$

By taking expectation in the equation above, one obtains

$$\mathbb{E} [\exp(Y_{t \wedge \tau_n}) \|\theta_{t \wedge \tau_n}\|_{L^2}^2] \leq - \sum_{i=1}^N \mathbb{E} \left[\int_0^{t \wedge \tau_n} \exp(Y_s) \langle \mathcal{L}_{\xi_i} \theta, \theta \rangle_{L^2}(s) dW_s^i \right] = 0.$$

We conclude

$$\mathbb{E} [\exp(Y_{t \wedge \tau_n}) \|\theta_{t \wedge \tau_n}\|_{L^2}^2] = 0, \quad n \in \mathbb{N}, \quad t > 0.$$

This implies that for every $n \in \mathbb{N}$ and $t > 0$,

$$\exp(Y_{t \wedge \tau_n}) \|\theta_{t \wedge \tau_n}\|_{L^2}^2 = 0, \quad a.s.$$

Since $Y_{t \wedge \tau_n}$ is finite, we can then deduce that $\theta_{t \wedge \tau_n} = 0$, for every $n \in \mathbb{N}$ and $t > 0$. From $\tau_n \rightarrow \infty$ as $n \rightarrow \infty$ we obtain that for every $t > 0$, $\theta_t = 0$ a.s. Finally, since the trajectories are continuous, we obtain that $\mathbb{P}$ -a.s. $\theta = 0$. Moreover, by uniqueness of the expectation equations we have $f = d\mathbb{E}[p - |u|^2/2 - \theta y] + \mathbb{E}[\theta] \hat{y} = 0$, and hence $\mathbb{P}$ -a.s. $u = 0$.

5.4 A CLOSED EQUATION FOR THE STATISTICAL MOMENTS

In this section, we show that we can derive closed equations for the statistical moments of solutions to the stochastic LA Boussinesq equations (5.1.2). We assume that we are working with regular enough solutions. We note that we will employ both the Stratonovich and Itô formulations of our equations since it is convenient. For the two formulations to be equivalent, some regularity is needed [§], but here we are mainly interested in illustrating that closed

[§]To be more concrete, for a Stratonovich equation to be rewritten in Itô form, one can assume that both the integrand and integrator are semimartingales, as we did for instance in Chapter 4.

equations can be derived for the covariance fields and the exact required regularity is not our main concern. However, everything could be done in Itô form.

Letting $u' = u - \mathbb{E}[u]$ and $\theta' = \theta - \mathbb{E}[\theta]$ be the fluctuations of the variables from their expectations, we remind that the covariance fields are defined by $U^{(2)} = \mathbb{E}[(u')^2]$ and $\Theta^{(2)} = \mathbb{E}[(\theta')^2]$, where we have employed the notation $(u')^2 = u' \otimes u'$, $(\theta')^2 = \theta' \otimes \theta'$. First observe that the fluctuations satisfy the SPDEs (written in Stratonovich form)

$$\begin{aligned} du' + \mathcal{L}_{\mathbb{E}[u]}u' dt + \sum_{i=1}^N \mathcal{L}_{\xi_i} u' \circ dW_t^i &= -\frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[u] dt - gy d\theta' dt, \\ d\theta' + \mathcal{L}_{\mathbb{E}[u]}\theta' dt + \sum_{i=1}^N \mathcal{L}_{\xi_i} \theta' \circ dW_t^i &= -\frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[\theta] dt, \end{aligned} \quad (5.4.1)$$

where we have taken into account the linearity of the Lie derivative operator and used equations (5.1.2) and (5.1.4). Before continuing with the proof, we present a useful auxiliary lemma.

Lemma 130. *The Stratonovich integral*

$$\sum_{i=1}^N \int_0^t ((\mathcal{L}_{\xi_i} u) \otimes u' + u' \otimes (\mathcal{L}_{\xi_i} u)) \circ dW_s^i$$

can be rewritten in Itô form as

$$\begin{aligned} \sum_{i=1}^N \int_0^t ((\mathcal{L}_{\xi_i} u) \otimes u' + u' \otimes (\mathcal{L}_{\xi_i} u)) dW_s^i \\ - \frac{1}{2} \sum_{i=1}^N \int_0^t ((\mathcal{L}_{\xi_i}(\mathcal{L}_{\xi_i} u)) \otimes u' + 2(\mathcal{L}_{\xi_i} u) \otimes (\mathcal{L}_{\xi_i} u) + u' \otimes \mathcal{L}_{\xi_i}(\mathcal{L}_{\xi_i} u)) ds. \end{aligned}$$

Proof. Assume for simplicity that $N = 1$. We have that

$$\begin{aligned} \int_0^t ((\mathcal{L}_{\xi} u) \otimes u' + u' \otimes (\mathcal{L}_{\xi} u)) \circ dW_s &= \int_0^t ((\mathcal{L}_{\xi} u) \otimes u' + u' \otimes (\mathcal{L}_{\xi} u)) dW_s \\ &\quad + \frac{1}{2} ([(\mathcal{L}_{\xi} u) \otimes u', W]_t + [u' \otimes (\mathcal{L}_{\xi} u), W]_t). \end{aligned}$$

Now, taking the Lie derivative $\mathcal{L}_\xi$ on both sides of the momentum equation in (5.1.2), we get

$$\begin{aligned} d(\mathcal{L}_\xi u) + \mathcal{L}_\xi \mathcal{L}_{\mathbb{E}[u]} u dt + \mathcal{L}_\xi(\mathcal{L}_\xi u) \circ dW_t \\ = -d\mathcal{L}_\xi \mathbb{E} [p - |u|^2/2] dt + g\mathcal{L}_\xi \mathbb{E} [\theta] \widehat{y} dt - g\mathcal{L}_\xi (y d(\theta - \mathbb{E} [\theta])) dt. \end{aligned} \quad (5.4.2)$$

Using (5.4.2), (5.4.1), and the stochastic Leibniz rule $d(S \otimes T) = (\odot dS) \otimes T + S \otimes (\odot dT)$, we obtain that the diffusion term in the equation for $d((\mathcal{L}_\xi u) \otimes u')$ is

$$-((\mathcal{L}_\xi \mathcal{L}_\xi u) \otimes u' + (\mathcal{L}_\xi u) \otimes (\mathcal{L}_\xi u)) \circ dW_t,$$

and similarly, the diffusion term in the equation for $d(u' \otimes (\mathcal{L}_\xi u))$ is

$$-((\mathcal{L}_\xi u) \otimes (\mathcal{L}_\xi u) + u' \otimes (\mathcal{L}_\xi \mathcal{L}_\xi u)) \circ dW_t.$$

We can therefore conclude that

$$\begin{aligned} [(\mathcal{L}_\xi u) \otimes u', W.]_t &= - \int_0^t (\mathcal{L}_\xi(\mathcal{L}_\xi u) \otimes u' + (\mathcal{L}_\xi u) \otimes (\mathcal{L}_\xi u)) ds, \\ [u' \otimes (\mathcal{L}_\xi u), W.]_t &= - \int_0^t ((\mathcal{L}_\xi u) \otimes (\mathcal{L}_\xi u) + u' \otimes \mathcal{L}_\xi(\mathcal{L}_\xi u)) ds, \end{aligned}$$

from which our claim follows. $\square$

By applying the stochastic product rule in Stratonovich form (1.4.19), we observe that

$$\begin{aligned} d(u')^2 &= (\odot du') \otimes u' + u' \otimes (\odot du') \\ &= -\mathcal{L}_{\mathbb{E}[u]} u' \otimes u' dt - \sum_{i=1}^N \mathcal{L}_{\xi_i} u \otimes u' \circ dW_t^i - \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[u] \otimes u' dt - gy d\theta' \otimes u' dt \\ &\quad - u' \otimes \mathcal{L}_{\mathbb{E}[u]} u' dt - \sum_{i=1}^N u' \otimes \mathcal{L}_{\xi_i} u \circ dW_t^i - \frac{1}{2} \sum_{i=1}^N u' \otimes \mathcal{L}_{\xi_i}^2 \mathbb{E}[u] dt - gy u' \otimes d\theta' dt. \end{aligned}$$

By taking into account the Leibniz property of the Lie derivative, i.e., $\mathcal{L}_\xi(S \otimes T) = \mathcal{L}_\xi S \otimes T + S \otimes \mathcal{L}_\xi T$, for any smooth tensors S and T , we can rewrite

the last expression as

$$\begin{aligned}
& d(u')^2 + \mathcal{L}_{\mathbb{E}[u]}(u')^2 dt + \frac{1}{2} \sum_{i=1}^N (\mathcal{L}_{\xi_i}^2 \mathbb{E}[u] \otimes u' + u' \otimes \mathcal{L}_{\xi_i}^2 \mathbb{E}[u]) dt \\
& + gy(d\theta' \otimes u' + u' \otimes d\theta') = - \sum_{i=1}^N (\mathcal{L}_{\xi_i} u \otimes u' + u' \otimes \mathcal{L}_{\xi_i} u) \circ dW_t^i \\
& = \frac{1}{2} \sum_{i=1}^N \left(\mathcal{L}_{\xi_i}^2 u \otimes u' + 2(\mathcal{L}_{\xi_i} u)^2 + u' \otimes \mathcal{L}_{\xi_i}^2 u \right) dt - \sum_{i=1}^N (\mathcal{L}_{\xi_i} u \otimes u' + u' \otimes \mathcal{L}_{\xi_i} u) dW_t^i \\
& = \frac{1}{2} \sum_{i=1}^N \underbrace{\left(\mathcal{L}_{\xi_i}^2 u' \otimes u' + 2(\mathcal{L}_{\xi_i} u')^2 + u' \otimes \mathcal{L}_{\xi_i}^2 u' \right)}_{= \mathcal{L}_{\xi_i}(\mathcal{L}_{\xi_i}(u')^2)} dt + \sum_{i=1}^N (\mathcal{L}_{\xi_i} \mathbb{E}[u])^2 dt \\
& \quad - \sum_{i=1}^N (\mathcal{L}_{\xi_i} u \otimes u' + u' \otimes \mathcal{L}_{\xi_i} u) dW_t^i \\
& + \frac{1}{2} \sum_{i=1}^N (\mathcal{L}_{\xi_i}^2 \mathbb{E}[u] \otimes u' + 2(\mathcal{L}_{\xi_i} \mathbb{E}[u] \otimes \mathcal{L}_{\xi_i} u' + \mathcal{L}_{\xi_i} u' \otimes \mathcal{L}_{\xi_i} \mathbb{E}[u]) + u' \otimes \mathcal{L}_{\xi_i}^2 \mathbb{E}[u]) dt,
\end{aligned} \tag{5.4.3}$$

where in the second equality we have taken into account the Stratonovich-to-Itô correction calculated in Lemma 130, and in the last equality, we have expanded the Stratonovich-to-Itô correction term using $u = u' + \mathbb{E}[u]$ and the linearity of the Lie derivative. Taking expectations on both sides of (5.4.3) and noting that the expectation of the Itô integral vanishes by the martingale property and $\mathbb{E}[u'] = \mathbb{E}[\theta'] = 0$ by definition, we obtain

$$\begin{aligned}
\partial_t U^{(2)} + \mathcal{L}_{\mathbb{E}[u]} U^{(2)} &= \sum_{i=1}^N \left(\frac{1}{2} \mathcal{L}_{\xi_i}^2 U^{(2)} + (\mathcal{L}_{\xi_i} \mathbb{E}[u])^2 \right) \\
&\quad - gy \mathbb{E}[d\theta' \otimes u' + u' \otimes d\theta']. \tag{5.4.4}
\end{aligned}$$

Actually, equations of the type (5.4.4) hold in much more generality for stochastic LA equations (see⁷). Carrying out the same procedure for the variable θ , we conclude that the following system of equations holds for the

covariance fields

$$\begin{aligned}
\partial_t U^{(2)} + \mathcal{L}_{\mathbb{E}[u]} U^{(2)} &= \sum_{i=1}^N \left(\frac{1}{2} \mathcal{L}_{\xi_i}^2 U^{(2)} + (\mathcal{L}_{\xi_i} \mathbb{E}[u])^2 \right) \\
&\quad - gy \mathbb{E}[\mathbf{d}\theta' \otimes u' + u' \otimes \mathbf{d}\theta'], \\
\partial_t \Theta^{(2)} + \mathcal{L}_{\mathbb{E}[u]} \Theta^{(2)} &= \sum_{i=1}^N \left(\frac{1}{2} \mathcal{L}_{\xi_i}^2 \Theta^{(2)} + (\mathcal{L}_{\xi_i} \mathbb{E}[\theta])^2 \right). \tag{5.4.5}
\end{aligned}$$

Clearly, system (5.4.5) is in principle not closed due to the presence of the term $\mathbb{E}[\mathbf{d}\theta' \otimes u' + u' \otimes \mathbf{d}\theta']$ in the equation for $U^{(2)}$. However, applying the exterior derivative $\mathbf{d}$ on both sides of the equation for θ in (5.1.2), the equation for its expectation in (5.1.4), and its corresponding fluctuation in (5.4.1), and noting that the exterior derivative and the Lie derivative commute as a consequence of Cartan's formula (see Appendix A), we obtain the following system for $\mathbf{d}\theta$, its expectation, and $\mathbf{d}\theta'$

$$\begin{aligned}
\mathbf{d}(\mathbf{d}\theta) + \mathcal{L}_{\mathbb{E}[u]} \mathbf{d}\theta \, dt + \sum_{i=1}^N (\mathcal{L}_{\xi_i} \mathbf{d}\theta) \circ dW_t^i &= 0, \\
\partial_t \mathbb{E}[\mathbf{d}\theta] + \mathcal{L}_{\mathbb{E}[u]} \mathbf{d}\mathbb{E}[\theta] &= \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[\mathbf{d}\theta], \\
\mathbf{d}(\mathbf{d}\theta') + \mathcal{L}_{\mathbb{E}[u]} \mathbf{d}\theta' \, dt + \sum_{i=1}^N (\mathcal{L}_{\xi_i} \mathbf{d}\theta) \circ dW_t^i &= -\frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[\mathbf{d}\theta] \, dt.
\end{aligned}$$

We claim that obtaining an equation for $\mathbb{E}[\mathbf{d}\theta' \otimes u' + u' \otimes \mathbf{d}\theta']$ closes system (5.4.5). Indeed, by taking into account the stochastic product rule, using equations (5.4.1), and applying the Leibniz property of the Lie derivative

$\mathcal{L}_\xi(S \otimes T) = \mathcal{L}_\xi S \otimes T + S \otimes \mathcal{L}_\xi T$, we have

$$\begin{aligned}
d(u' \otimes d\theta') &= u' \otimes (\circ d(d\theta')) + (\circ du') \otimes d\theta' \\
&= -\mathcal{L}_{\mathbb{E}[u]}(u' \otimes d\theta')dt - \sum_{i=1}^N (u' \otimes \mathcal{L}_{\xi_i}(d\theta) + \mathcal{L}_{\xi_i}u \otimes d\theta') \circ dW_t^i \\
&\quad - \frac{1}{2} \sum_{i=1}^N (u' \otimes \mathcal{L}_{\xi_i}^2 \mathbb{E}[d\theta] + \mathcal{L}_{\xi_i}^2 \mathbb{E}[u] \otimes d\theta') dt - gy(d\theta')^2 dt \\
&= -\mathcal{L}_{\mathbb{E}[u]}(u' \otimes d\theta')dt - \sum_{i=1}^N (u' \otimes \mathcal{L}_{\xi_i}(d\theta) + \mathcal{L}_{\xi_i}u \otimes d\theta') dW_t^i \\
&\quad - \frac{1}{2} \sum_{i=1}^N (u' \otimes \mathcal{L}_{\xi_i}^2 \mathbb{E}[d\theta] + \mathcal{L}_{\xi_i}^2 \mathbb{E}[u] \otimes d\theta') dt - gy(d\theta')^2 dt \\
&\quad + \frac{1}{2} \sum_{i=1}^N (u' \otimes \mathcal{L}_{\xi_i}^2(d\theta) + 2(\mathcal{L}_{\xi_i}u) \otimes (\mathcal{L}_{\xi_i}(d\theta)) + \mathcal{L}_{\xi_i}^2 u \otimes d\theta') dt.
\end{aligned} \tag{5.4.6}$$

By the Leibniz property of Lie derivative, we obtain

$$\begin{aligned}
\mathcal{L}_\xi^2(u' \otimes d\theta') &= \mathcal{L}_\xi(u' \otimes \mathcal{L}_\xi(d\theta') + \mathcal{L}_\xi(u') \otimes d\theta') \\
&= u' \otimes \mathcal{L}_\xi^2(d\theta') + 2(\mathcal{L}_\xi u') \otimes (\mathcal{L}_\xi(d\theta')) + \mathcal{L}_\xi^2 u' \otimes d\theta',
\end{aligned}$$

so the last term in (5.4.6) can be rewritten as

$$\begin{aligned}
&\frac{1}{2} \sum_{i=1}^N (u' \otimes \mathcal{L}_{\xi_i}^2(d\theta) + 2(\mathcal{L}_{\xi_i}u) \otimes (\mathcal{L}_{\xi_i}(d\theta)) + \mathcal{L}_{\xi_i}^2 u \otimes d\theta') dt \\
&= \frac{1}{2} \sum_{i=1}^N (u' \otimes \mathcal{L}_{\xi_i}^2(d\theta') + \mathcal{L}_{\xi_i}^2 u' \otimes d\theta' + u' \otimes \mathcal{L}_{\xi_i}^2(\mathbb{E}[d\theta]) + \mathcal{L}_{\xi_i}^2 \mathbb{E}[u] \otimes d\theta') dt \\
&\quad + \sum_{i=1}^N (\mathcal{L}_{\xi_i}u) \otimes (\mathcal{L}_{\xi_i}(d\theta)) \\
&= \frac{1}{2} \sum_{i=1}^N (\mathcal{L}_{\xi_i}^2(u' \otimes d\theta') + u' \otimes \mathcal{L}_{\xi_i}^2(\mathbb{E}[d\theta]) + \mathcal{L}_{\xi_i}^2 \mathbb{E}[u] \otimes d\theta') dt \\
&\quad + \sum_{i=1}^N ((\mathcal{L}_{\xi_i} \mathbb{E}[u]) \otimes (\mathcal{L}_{\xi_i}(\mathbb{E}[d\theta]))) + (\mathcal{L}_{\xi_i} \mathbb{E}[u]) \otimes (\mathcal{L}_{\xi_i}(d\theta')) + (\mathcal{L}_{\xi_i} u') \otimes (\mathcal{L}_{\xi_i}(\mathbb{E}[d\theta])).
\end{aligned}$$

Using the above expression, taking expectations on both sides of (5.4.6), and noting again that the expectation of the (local) martingale terms vanishes and $\mathbb{E}[u'] = \mathbb{E}[\theta'] = 0$, we get

$$\begin{aligned} \partial_t \mathbb{E}[u' \otimes d\theta'] + \mathcal{L}_{\mathbb{E}[u]} \mathbb{E}[u' \otimes d\theta'] &= \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[u' \otimes d\theta'] \\ &\quad - gy \mathbb{E}[(d\theta')^2] + \sum_{i=1}^N (\mathcal{L}_{\xi_i} \mathbb{E}[u]) \otimes (\mathcal{L}_{\xi_i}(\mathbb{E}[d\theta])). \end{aligned}$$

Similarly, we can obtain a PDE for $\mathbb{E}[d\theta' \otimes u']$, and putting them together we derive the following equation for $\mathbb{E}[d\theta' \otimes u' + u' \otimes d\theta']$

$$\begin{aligned} &\partial_t \mathbb{E}[d\theta' \otimes u' + u' \otimes d\theta'] + \mathcal{L}_{\mathbb{E}[u]} \mathbb{E}[d\theta' \otimes u' + u' \otimes d\theta'] \\ &= \frac{1}{2} \sum_{i=1}^N \mathcal{L}_{\xi_i}^2 \mathbb{E}[d\theta' \otimes u' + u' \otimes d\theta'] + \sum_{i=1}^N (\mathcal{L}_{\xi_i} \mathbb{E}[u]) \otimes (\mathcal{L}_{\xi_i}(\mathbb{E}[d\theta])) \\ &\quad + \sum_{i=1}^N (\mathcal{L}_{\xi_i}(\mathbb{E}[d\theta])) \otimes (\mathcal{L}_{\xi_i} \mathbb{E}[u]) - 2gy \mathbb{E}[(d\theta')^2]. \end{aligned}$$

6

Summary and outlook

In this thesis, we have investigated how transport noise in the Stratonovich sense can affect the solution properties of various fluid equations of interest. In Chapter 3, we have established that the inviscid Burgers' equation with transport noise is locally well-posed in regular Sobolev spaces H^s , $s > 2$. We have developed a blow-up criterion that imitates the one available in the deterministic case, and have proven a global well-posedness result in H^s , $s > 2$ for the solutions of viscous Burgers' with transport noise. We have also presented shock formation results and shown that in the cases we consider, shock formation cannot be avoided by addition of transport noise. Hence in the case of both the inviscid and viscous Burgers' equations, it is natural to conclude that addition of stochastic noise seems to leave unchanged the fundamental strong solution properties of the deterministic equations. This is also what we concluded in ⁶, except for the fact that the blow-up criterion in ⁶ is not as sharp as the deterministic one. In Chapter 4, we have illustrated how addition of Lie transport noise in the equation of linear advection of k -forms can greatly affect (and improve) the (weak) solution properties of this equation. Indeed, we have demonstrated that whereas the deterministic equation is in general not weakly well-posed in L^2 , the stochastic equation

becomes well-posed in this space. Our results comprise a general geometric extension of the novel well-posedness by addition of noise results in⁷⁶ for the linear transport equation. Our conclusion is that the beautiful regularisation mechanism appearing in⁷⁶ is available in a more general geometric class of equations. Also, in contradiction to our initial intuition, extending the results in⁷⁶ to our more general setting requires developing new geometric tools and mathematical strategies. Finally, we have applied a novel stochastic Lagrangian-averaging (LA) technique to construct a stochastic Boussinesq model. We stress that this technique should not be confused with addition of transport noise, since the key part of the strategy consists in substituting the stochastic transport velocity by its expectation. In the LA formulation, we have demonstrated that the Boussinesq equations become globally well-posed, while in the deterministic case this remains an important open question in the mathematical community. Moreover, we have derived closed equations for the covariance fields of the variables. Hence we conclude that the Lagrangian-averaging technique we propose provides a strong regularisation mechanism. In order to establish some of our results, we developed an important mathematical tool, which we call the Kunita-Itô-Wentzell (KIW) formula. This tool was a key element when showing uniqueness of solutions of stochastic linear transport of k -forms. Moreover, the KIW formula formula has other interesting applications in fluid dynamics that we have also illustrated. The KIW formula is therefore an instrument of great utility in stochastic fluid dynamics.

We comment on some future lines of research which we consider interesting:

- **Validity of the Kunita-Itô-Wentzell formula in broader contexts.** One of the cornerstones of this dissertation is the Kunita-Itô-Wentzell (KIW) formula, which extends the Itô-Wentzell formula to stochastic time-dependent tensor fields (see Theorems 3 & 43 for the Itô versions of KIW). We have also established a weak version of KIW for k -forms on the Euclidean space $\mathbb{R}^n$ (Theorem 43). Moreover, although we have not mentioned it in this thesis, we know that the KIW formula is valid on manifolds under strong assumptions on the coefficients of the flow equation, since Itô-Wentzell is available in the manifold case.

An interesting task would be to extend our weak version to general manifolds and tensors. For this purpose, there are several technical hurdles that need to be surpassed. However, we have good reasons to conjecture that our weak version of the KIW formula indeed holds on manifolds, since we are aware that this is true under certain regularity conditions. This generalisation would also be a key element for extending our well-posedness by noise results in Chapter 4 to the advection equation of tensors on general manifolds, which would be very interesting, since it would provide the first example of well-posedness of PDEs by addition of noise on manifolds in a beautiful geometric setting.

- **Extending Burgers' well-posedness and shock formation results.** Regarding the question of shock formation in the inviscid Burgers' equation with transport noise, it is natural to ask whether our results can be extended to show that shock formation occurs almost surely for more general types of noise than the linear one we considered. As we mentioned in Chapter 3, we know that shock formation occurs at least in expectation for a general class of noises, which leads us to ask whether the same result holds almost surely. Another interesting question is whether our global well-posedness result for viscous stochastic Burgers' can be extended for the same equation upon replacing the full Laplacian with a fractional Laplacian $(-\Delta)^\alpha$, $\alpha \in (0, 1)$. The main difficulty here is that in the stochastic case, the proof of the maximum principle (Proposition 82) employs pointwise chain rule, which is not available for the fractional Laplacian, being the latter a non-local operator. In the deterministic case, this question has been settled and it is known that the solution exhibits a very different behaviour depending on the value of α : for $\alpha \in [1/2, 1]$, the solution is global in time, and for $\alpha \in (0, 1/2)$, the solution develops singularities in finite time^{113,112}.
- **Well-posedness of advection of k -forms with more generality.** It would be interesting to investigate whether our Lie transport noise can regularise the equation as observed for the linear transport equation in⁷². Another possible idea would be to study whether our well-

posedness results can also be proven when the underlying space is a smooth manifold instead of the Euclidean space $\mathbb{R}^n$, which is a natural setting for our equation due to its coordinate-free nature. This would involve extending several intermediate results such as the parabolic and commutator estimates to manifolds. There is some hope regarding the first point since^{69,68} show that the smoothing mechanism of heat-type semigroups is available on manifolds even for SDEs if the coefficients are smooth enough. The second point would require, for instance, generalisation of the weak version of the KIW formula to manifolds (see the first bullet). It would also be natural to ask whether we can replace k -forms by general tensors. Moreover, we would like to explore further the case when the diffusion vector fields depend on space instead of only on time, especially when the regularity of the drift is low. Finally, it would be extremely interesting to investigate whether the type of well-posedness by noise results that we provide could be shown for nonlinear PDEs. This is considered state of the art and important progress in this direction in the case of the Navier-Stokes equation has recently been made in^{79,75}.

- **Well-posedness of stochastic fluid equations in more general conditions.** When proving existence and uniqueness results for the solutions of stochastic fluid equations, we have chosen to work on the torus or the Euclidean space as most authors do (as we do for instance in^{8,6}). In the deterministic case, the well-posedness of the Euler equations on manifolds was first demonstrated by Ebin and Marsden in⁶⁶ by means of a groundbreaking geometric argument. The well-posedness of the Boussinesq equations in general domains is discussed, for instance, in¹⁵⁰. Recently, existence of solutions of the Euler equation with additive noise on manifolds has been established in¹³¹ by extending the original ideas of Ebin and Marsden. Moreover, instead of limiting ourselves to the study of one equation with noise, an attractive idea is considering a more abstract formalism yielding the well-posedness of a large class of equations in the style of^{27,10}.

REFERENCES

- [1] ABIDI, H., AND HMIDI, T. On the global well-posedness for Boussinesq system. *Journal of Differential Equations* 233, 1 (2007), 199–220.
- [2] ABRAHAM, R., AND MARSDEN, J. *Foundations of mechanics*. The Benjamin/Cummings Publishing Company Reading, Massachusetts, 1978.
- [3] ADAMS, R., AND FOURNIER, J. *Sobolev Spaces*, vol. 140. Elsevier, 2003.
- [4] ADHIKARI, D., ET AL. Global regularity results for the 2D Boussinesq equations with partial dissipation. *Journal of Differential Equations* 260, 2 (2016), 1893–1917.
- [5] ALBEVERIO, S., CRUZEIRO, A. B., AND HOLM, D. D. *Stochastic Geometric Mechanics*. CIB, Lausanne, 2015.
- [6] ALONSO-ORÁN, D., AND DE LEÓN, A. B. On the Well-posedness of stochastic Boussinesq equations with transport noise. *Journal of Nonlinear Science* 30, 1 (2020), 175–224.
- [7] ALONSO-ORÁN, D., DE LEÓN, A. B., HOLM, D., AND TAKAO, S. Modelling the Climate and Weather of a 2D Lagrangian-Averaged Euler–Boussinesq Equation with Transport Noise. *Journal of Statistical Physics* 179 (2020), 1267–1303.
- [8] ALONSO-ORÁN, D., DE LEÓN, A. B., AND TAKAO, S. The Burgers’ equation with stochastic transport: shock formation, local and global existence of smooth solutions. *Nonlinear Differential Equations and Applications* 26, 6 (2019).

- [9] ALONSO-ORÁN, D., AND DE LEÓN, A. B. Stability, well-posedness and blow-up criterion for the Incompressible Slice Model. *Physica D: Nonlinear Phenomena* 392 (2019), 99–118.
- [10] ALONSO-ORÁN, D., ROHDE, C., AND TANG, H. A local-in-time theory for singular SDEs with applications to fluid models with transport noise. *arXiv:2010.09972* (2020).
- [11] AMBROSIO, L. Transport equation and Cauchy problem for BV vector fields. *Inventiones mathematicae* 158 (2004), 227–260.
- [12] ARNAUDON, A., HOLM, D., D., AND SOMMER, S. A Geometric Framework for Stochastic Shape Analysis. *Foundations of Computational Mathematics* 19, 3 (2019), 653–701.
- [13] ARNAUDON, M., CHEN, X., AND CRUZEIRO, A. B. Stochastic Euler-Poincaré reduction. *Journal of Mathematical Physics* 55, 8 (2014).
- [14] ATTANASIO, S., AND FLANDOLI, F. Renormalized solutions for stochastic transport equations and the regularization by bilinear multiplicative noise. *Communications in Partial Differential Equations* 36, 8 (2011), 1455–1474.
- [15] BAUER, M., HARMS, P., AND MICHOR, P. W. Sobolev metrics on shape space of surfaces. *Journal of Geometric Mechanics* 3, 4 (2012), 389–438.
- [16] BAUER, M., HARMS, P., AND MICHOR, P. W. Sobolev metrics on the manifold of all Riemannian metrics. *Journal of Differential Geometry* 94, 2 (2013), 187–208.
- [17] BECK, L., FLANDOLI, F., MASSIMILIANO, G., AND MARIO, M. Stochastic ODEs and stochastic linear PDEs with critical drift: regularity, duality and uniqueness. *Electronic Journal of Probability* 24 (2019), 1–72.
- [18] BERESTYCKI, H., CONSTANTIN, P., AND RYZHIK, L. Non-planar fronts in Boussinesq reactive flows. *Annales de l’Institut Henri Poincaré (C) Non Linear Analysis* 23, 4 (2006), 407–437.

- [19] BERNER, J., ET AL. Stochastic Parameterisations: Toward a New View of Weather and Climate Models. *Bulletin of the American Meteorological Society* 98, 3 (2017), 565–588.
- [20] BERTINI, L., CANCRINI, N., AND JONA-LASINIO, G. The Stochastic Burgers’ Equation. *Communications in Mathematical Physics* 165 (1994), 211–232.
- [21] BERTOLDI, M., AND FORNARO, S. Gradient estimates in parabolic problems with unbounded coefficients. *Studia Mathematica* 165, 3 (2004), 221–254.
- [22] BISMUT, J.-M. A generalized formula of Itô and some other properties of stochastic flows. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 55 (1981), 331–350.
- [23] BISMUT, J.-M. Mécanique aléatoire. In *Lecture Notes in Mathematics*, vol. 866. Springer-Verlag, 1981.
- [24] BOCCIA, S., AND KRYLOV, N. V. On the fundamental matrix solution for higher-order parabolic systems. *Proceedings of the St. Petersburg Mathematical Society* 15 (2014), 11–34.
- [25] BOYER, F., AND FABRIE, P. *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models*. Springer New York, 2013.
- [26] BREZIS, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, New York, 2011.
- [27] BRZEŹNIAK, Z., AND SLAVÍK, J. Well-Posedness of the 3D Stochastic Primitive Equations with Transport Noise. *arXiv:2008.00274* (2020).
- [28] BUIZZA, R., MILLEER, M., AND PALMER, T. N. Stochastic representation of model uncertainties in the ECMWF ensemble prediction system. *Quarterly Journal of the Royal Meteorological Society* 125, 560 (1999), 2887–2908.

- [29] BURKHOLDER, D., DAVIS, B., AND GUNDY, R. Integral inequalities for convex functions of operators on martingales. *Procedures of the 6th Berkley Symposium of Mathematical Statistics and Probability 2* (1972), 223–240.
- [30] BURKHOLDER, D. L., AND GUNDY, R. F. Extrapolation and interpolation for convex functions of operators on martingales. *Acta Mathematica* 124 (1970), 249–304.
- [31] CANNON, J. R., AND DiBENEDETTO, E. The initial value problem for the Boussinesq equations with data in L^p . In *Approximation methods for Navier-Stokes problems. Lecture Notes in Mathematics*, vol. 771. Springer, 1980, pp. 129–144.
- [32] CATUOGNO, P., AND OLIVERA, C. L^p -solutions of the stochastic transport equation. *Random Operators and Stochastic Equations* 21, 2 (2013), 125–134.
- [33] CATUOGNO, P., AND OLIVERA, C. Strong solution of the stochastic Burgers’ equation. *Applicable Analysis: An International Journal* 93, 3 (2013), 646–652.
- [34] CATUOGNO, P., AND STELMASTCHUK, S. N. A stochastic transport theorem. *Communications on Stochastic Analysis* 10, 1 (2016), 31–38.
- [35] CHAE, D. Global regularity for the 2D Boussinesq equations with partial viscosity terms. *Advances in Mathematics* 203, 2 (2006), 497–513.
- [36] CHAE, D., AND IMANUVILOV, O. Y. Generic Solvability of the Axisymmetric 3-D Euler Equations and the 2-D Boussinesq Equations. *Journal of Differential Equations* 156, 1 (1999), 1–17.
- [37] CHAE, D., AND NAM, H.-S. Local existence and blow-up criterion for the Boussinesq equations. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* 127 (1997), 935–946.

- [38] CHAMBERS, D. H., ET AL. Karhunen Lóeve expansion of Burgers' model of turbulence. *The Physics of Fluids* 31, 9 (1988), 2573–2582.
- [39] CHEN, X., CRUZEIRO, A. B., AND RATIU, T. S. Stochastic variational principles for dissipative equations with advected quantities. *arXiv:1506.05024* (2015).
- [40] CIPRIANO, F., AND CRUZEIRO, A. B. Flows associated with irregular $\mathbb{R}^d$ –vector fields. *Journal of Differential Equations* 219, 1 (2015), 183–201.
- [41] COHEN, B., AND CRAIG, G. Fluctuations in an Equilibrium Convective Ensemble. Part I: Theoretical Formulation. *Journal of the Atmospheric Sciences* 63, 8 (2006), 1996–2004.
- [42] COHEN, B., AND CRAIG, G. Fluctuations in an Equilibrium Convective Ensemble. Part II: Numerical Experiments. *Journal of the Atmospheric Sciences* 63 (2006), 2005–2015.
- [43] CONSTANTIN, P., AND IYER, G. A stochastic-Lagrangian approach to the Navier–Stokes equations in domains with boundary. *Annals of Applied Probability* 21, 4 (2011), 1466–1492.
- [44] CÓRDOBA, A., AND CÓRDOBA, D. A Maximum Principle Applied to Quasi-Geostrophic Equations. *Communications in Mathematical Physics* 249, 249 (2004), 511–528.
- [45] COTTER, C., ET AL. Modelling uncertainty using circulation-preserving stochastic transport noise in a 2-layer quasi-geostrophic model. *arXiv preprint arXiv:1802.05711* (2018).
- [46] COTTER, C., ET AL. A Particle Filter for Stochastic Advection by Lie Transport (SALT): A case study for the damped and forced incompressible 2D Euler equation. *arXiv:1907.11884* (2019).
- [47] COTTER, C., GOTTWALD, G., AND HOLM, D. Stochastic partial differential fluid equations as a diffusive limit of deterministic Lagrangian multi-time dynamics. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science* 473 (2017).

- [48] COTTER, C. J., CRISAN, D., HOLM, D. D., PAN, W., AND SHEVCHENKO, I. Numerically modelling stochastic Lie transport in fluid dynamics. *Multiscale Modelling & Simulation* 17, 1 (2019), 192–232.
- [49] CRISAN, D., FLANDOLI, F., AND HOLM, D. D. Solution Properties of a 3D Stochastic Euler Fluid Equation. *Journal of Nonlinear Science* 29, 3 (2019), 813–870.
- [50] CRISAN, D., AND HOLM, D. D. Wave breaking for the Stochastic Camassa–Holm equation. *Physica D: Nonlinear Phenomena* 376–377 (2018), 138–143. Special Issue: Nonlinear Partial Differential Equations in Mathematical Fluid Dynamics.
- [51] CRISAN, D., HOLM, D. D., LEAHY, J.-M., AND NILSSEN, T. Variational principles for fluid dynamics on rough paths. *arXiv: 2004.07829* (2020).
- [52] CRISAN, D., AND LANG, O. Well-posedness for a stochastic 2D Euler equation with stochastic Lie transport noise. *arXiv:1907.00451* (2019).
- [53] CRISAN, D., AND LANG, O. Local well-posedness for the great lake equation with transport noise. *arXiv:2003.03357* (2020).
- [54] CRUZEIRO, A. B., HOLM, D. D., AND RATIU, T. S. Momentum Maps and Stochastic Clebsch Action Principles. *Communications in Mathematical Physics* 357 (2018), 873–912.
- [55] DA PRATO, G., DEBUSSCHE, A., AND TEMAM, R. Stochastic Burgers’ equation. *Nonlinear Differential Equations and Applications* 1 (1994), 389–402.
- [56] DA PRATO, G., AND GATAREK, D. Stochastic Burgers equation with correlated noise. *Stochastics and Stochastic Reports* 52, 1–2 (2007), 29–41.
- [57] DA PRATO, G., AND ZABCZYK, J. Stochastic Equations in Infinite Dimensions. In *Encyclopedia of Mathematics and Its Applications*, vol. 44. Cambridge University Press, 1992.

- [58] DE LEÓN, A. B., HOLM, D. D., LUESINK, E., AND TAKAO, S. Implications of Kunita-Itô-Wentzell formula for k -forms in stochastic fluid dynamics. *arXiv:1903.07201* (2019).
- [59] DE LEÓN, A. B., AND TAKAO, S. Well-posedness by noise for linear advection of k -forms. *arXiv:1904.13319* (2019).
- [60] DI NEZZA, E., PALATUCCI, G., AND VALDINOCI, E. Hitchhiker’s guide to the fractional Sobolev spaces. *Bulletin des Sciences Mathématiques* 136, 5 (2012), 521–573.
- [61] DIPERNA, R. J., AND LIONS, P.-L. Ordinary differential equations, transport theory and Sobolev spaces. *Inventiones mathematicae* 98, 3 (1989), 511–547.
- [62] DODZIUK, J. Sobolev spaces of differential forms and de Rham–Hodge isomorphism. *Journal of Differential Geometry* 16, 1 (1981), 63–73.
- [63] DORRESTIJN, J., CROMMELIN, D. T., SIEBESMA, A. P., AND JONKER, H. J. J. Stochastic parameterization of shallow cumulus convection estimated from high-resolution model data. *Theoretical and Computational Fluid Dynamics* 27, 1 (2013), 133–148.
- [64] DRIVAS, T. D., AND HOLM, D. D. Circulation and Energy Theorem Preserving Stochastic Fluids. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* (2019), 1–39.
- [65] DRIVAS, T. D., HOLM, D. D., AND LEAHY, J.-M. Lagrangian averaged stochastic advection by Lie transport for fluids. *Journal of Statistical Physics* 179 (2020), 1304–1342.
- [66] EBIN, D. G., AND MARSDEN, J. Groups of Diffeomorphisms and the Motion of an Incompressible Fluid. *Annals of Mathematics* 92, 1 (1970), 102–163.
- [67] ELGINDI, T., AND JEONG, I.-J. Finite-Time Singularity Formation for Strong Solutions to the Axi-symmetric 3D Euler Equations. *Annals of PDE* 5, 16 (2019).

- [68] ELWORTHY, K. D., AND LI, X.-M. Differentiation of heat semigroups and applications. In *Probability Theory and Mathematical Statistics* (1994), TEV, Vilnius, p. 239–251.
- [69] ELWORTHY, K. D., AND LI, X.-M. Formulae for the derivatives of heat semigroups. *Journal of Functional Analysis* 125, 1 (1994), 252–286.
- [70] ÉMERY, M. On two transfer principles in stochastic differential geometry. In *Séminaire de probabilités XXIV, Lecture Notes in Mathematics*, vol. 1426. Springer, 1990.
- [71] EVANS, L. C. *Partial Differential Equations*, vol. 19. American Mathematical Society, 2010.
- [72] FEDRIZZI, E., AND FLANDOLI, F. Noise prevents singularities in linear transport equations. *Journal of Functional Analysis* 264, 6 (2013), 1329–1354.
- [73] FEDRIZZI, E., FLANDOLI, F., PRIOLA, E., AND VOVELLE, J. Regularity of stochastic kinetic equations. *Electronic Journal of Probability* 22, 48 (2017), 1–42.
- [74] FLANDOLI, F. Random Perturbation of PDEs and Fluid Dynamic Models. In *Lecture Notes in Mathematics*, vol. 2015. Springer, Berlin, 2011.
- [75] FLANDOLI, F., GALEATI, L., AND LUO, D. Scaling limit of stochastic 2D Euler equations with transport noises to the deterministic Navier-Stokes equations. *Journal of Evolution Equations* (2020).
- [76] FLANDOLI, F., GUBINELLI, M., AND PRIOLA, E. Well-posedness of the transport equation by stochastic perturbation. *Inventiones mathematicae* 180 (2010), 1–53.
- [77] FLANDOLI, F., GUBINELLI, M., AND PRIOLA, E. Full well-posedness of point vortex dynamics corresponding to stochastic 2D Euler equations. *Stochastic Processes and their Applications* 121, 7 (2011), 1445–1463.

- [78] FLANDOLI, F., AND LUO, D. Euler-Lagrangian approach to 3D stochastic Euler equations. *arXiv:1803.05319* (2018).
- [79] FLANDOLI, F., AND LUO, D. High mode transport noise improves vorticity blow-up control in 3D Navier-Stokes equations. *arXiv:1910.05742* (2019).
- [80] FLANDOLI, F., MAURELLI, M., AND NEKLYUDOV, M. Noise prevents infinite stretching of the passive field in a stochastic vector advection equation. *Journal of Mathematical Fluid Mechanics* 16, 4 (2014), 805–822.
- [81] FLANDOLI, F., AND OLIVERA, C. Well-posedness of the vector advection equations by stochastic perturbation. *Journal of Evolution Equations* 18, 2 (2018), 277–301.
- [82] FORNARO, S., METAFUNE, G., AND PRIOLA, E. Gradient estimates for Dirichlet parabolic problems in unbounded domains. *Journal of Differential Equations* 205, 2 (2004), 329–353.
- [83] FRIEDMAN, A. *Partial Differential Equations of Parabolic Type*. R.E. Krieger Publishing Company, Florida, 1983.
- [84] FRITZ, J. *Partial Differential Equations*. Springer-Verlag, New York, 1982.
- [85] FRIZ, P. K., AND GESS, B. Stochastic scalar conservation laws driven by rough paths. *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis* 33, 4 (2016), 933–963.
- [86] FUKUOKA, R. Mollifier Smoothing of tensor fields on differentiable manifolds and applications to Riemannian Geometry. *arXiv:math/0608230* (2006).
- [87] FUNAKI, T., GAO, Y., AND HILHORST, D. Uniqueness of the entropy solution of a stochastic conservation law with a Q-Brownian motion. *HAL Id:hal-02159743* (2020).

- [88] GAY-BALMAZ, F., AND HOLM, D. D. Stochastic Geometric Models with Non-stationary Spatial Correlations in Lagrangian Fluid Flows. *Journal of Nonlinear Science* 28, 3 (2018), 873–904.
- [89] GESS, B. Regularization and Well-Posedness by Noise for Ordinary and Partial Differential Equations. *Stochastic Partial Differential Equations and Related Fields* (2018), 43–67.
- [90] GESS, B., AND MAURELLI, M. Well-posedness by noise for scalar conservation laws. *Communications in Partial Differential Equations* 43, 12 (2018), 1702–1736.
- [91] GESS, B., AND SMITH, S. Stochastic continuity equations with conservative noise. *Journal de Mathématiques Pures et Appliquées* 128 (2019), 225–263.
- [92] GESS, B., AND SOUGANIDIS, P. E. Stochastic non-isotropic degenerate parabolic–hyperbolic equations. *Stochastic Processes and their Applications* 127, 9 (2017), 2961–3004.
- [93] GLATT-HOLTZ, N., AND VICOL, V. Local and global existence of smooth solutions for the stochastic Euler equation with multiplicative noise. *The Annals of Probability* 42 (2014), 80–145.
- [94] GOLDSTEIN, J. A. *Semigroups of linear operators and applications*. Dover Publications, New York, 1985.
- [95] GRAY, D. D., AND GIORGINI, A. The validity of the Boussinesq approximation for liquids and gases. *International Journal of Heat and Mass Transfer* 19, 5 (1976), 545–551.
- [96] GYÖNGY, I. On the approximation of stochastic partial differential equations II. *Stochastics and Stochastics Reports* 26, 3 (1989), 129–164.
- [97] GYÖNGY, I., AND KRYLOV, N. V. Stochastic partial differential equations with unbounded coefficients and applications III. *Stochastics and Stochastics Reports* 40, 1-2 (1992), 77–115.

- [98] GYÖNGY, I., AND KRYLOV, N. V. Existence of strong solutions for Itô's stochastic equations via approximations. *Probability Theory and Related Fields* 105, 2 (1996), 143–158.
- [99] GYÖNGY, I., AND KRYLOV, N. V. On the splitting-up method and stochastic partial differential equations. *Annals of Probability* 31, 2 (2003), 564–591.
- [100] HOLM, D. D. *Geometric mechanics: Part I: Dynamics and Symmetry*. Imperial College Press, London, 2008.
- [101] HOLM, D. D. *Geometric mechanics: Part II: Rotating, Translating and Rolling*. World Scientific Publishing Company, London, 2008.
- [102] HOLM, D. D. Variational principles for stochastic fluid dynamics. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 471, 2176 (2015).
- [103] HOLM, D. D., MARSDEN, J. E., AND RATIU, T. S. The Euler–Poincaré equations and semidirect products with applications to continuum theories. *Advances in Mathematics* 137, 1 (1998), 1–81.
- [104] HOLM, D. D., SCHMAH, T., AND STOICA, C. *Geometric mechanics and symmetry: from finite to infinite dimensions*. Oxford University Press, 2009.
- [105] HÖRMANDER, L. *The Analysis of Linear Partial Differential Operators III*. Springer-Verlag Berlin Heidelberg, 1985.
- [106] HOSOKAWA, I., AND YAMAMOTO, K. Turbulence in the randomly forced, one-dimensional Burgers flow. *Journal of Statistical Physics* 13, 3 (1975), 245–272.
- [107] HOU, T. Y., AND LI, C. Global well-posedness of the viscous Boussinesq equations. *Discrete and Continuous Dynamical Systems* 12, 1 (2005), 1–12.

- [108] KARATZAS, I., AND SHREVE, S. E. *Brownian motion and stochastic calculus. Second edition. Graduate Texts in Mathematics.* Springer-Verlag, New York, 1991.
- [109] KATO, T., AND LAI, C. Y. Nonlinear evolution equations and the Euler flow. *Journal of Functional Analysis* 56, 1 (1984), 15–28.
- [110] KATO, T., AND PONCE, G. Commutator estimates and the Euler and Navier-Stokes equations. *Communications in Pure and Applied Mathematics* 41, 7 (1988), 891–907.
- [111] KIJUNG, L. On a deterministic linear partial differential system. *Journal of Mathematical Analysis and Applications* 353, 1 (2009), 24–42.
- [112] KISELEV, A. Regularity and Blow up for Active Scalars. *The Mathematical Modelling of Natural Phenomena* 5, 4 (2010), 225–255.
- [113] KISELEV, A., NAZAROV, F., AND SHTERENBERG, R. Blow Up and Regularity for Fractal Burgers’ Equation. *Dynamics of Partial Differential Equations* 5, 3 (2008), 211–240.
- [114] KRYLOV, N. V. *Lectures on Elliptic and Parabolic Equations in Hölder Spaces.* American Mathematical Society, Providence, 1996.
- [115] KRYLOV, N. V. On the Itô–Wentzell formula for distribution-valued processes and related topics. *Probability Theory and Related Fields* 150, 1-2 (2011), 295–319.
- [116] KRYLOV, N. V., AND PRIOLA, E. Elliptic and parabolic second-order PDEs with growing coefficients. *Communications in Partial Differential Equations* 35, 1 (2009), 1–22.
- [117] KRYLOV, N. V., AND ROZOVSKII, B. L. Stochastic evolution equations. *Journal of Mathematical Sciences* 16, 249 (1981), 1233–1277.
- [118] KUNITA, H. Some extensions of Itô’s formula. In *Séminaire de probabilités XV, Lecture Notes in Mathematics*, vol. 850. Springer, Berlin, 1981.

- [119] KUNITA, H. Stochastic differential equations and stochastic flows of diffeomorphisms. In *Ecole d'été de probabilités de Saint-Flour XII-1982*. Springer, 1984, pp. 143–303.
- [120] KUNITA, H. *Stochastic flows and stochastic differential equations*, vol. 24. Cambridge University Press, Cambridge, 1997.
- [121] LÁZARO-CAMÍ, J. A. *Stochastic geometric mechanics*. Doctoral Thesis, 2008.
- [122] LEAHY, J.-M., AND MIKULEVIČIUS, R. On some properties of space inverses of stochastic flows. *Stochastic Partial Differential Equations: Analysis and Computations* 3, 4 (2015), 445–478.
- [123] LIONS, J. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris, 1969.
- [124] LIONS, P.-L. *Mathematical Topics in Fluid Mechanics: Incompressible Models*, vol. 1. Oxford University Press, Oxford, 1996.
- [125] LUO, D. Convergence of stochastic 2D inviscid Boussinesq equations with transport noise to a deterministic viscous system. *arXiv:2008.01434* (2020).
- [126] LUO, G., AND HOU, T. Y. Toward the Finite-Time Blowup of the 3D Axisymmetric Euler Equations: A Numerical Investigation. *Multiscale Modeling & Simulation* 12, 4 (2014), 1722–1776.
- [127] MAJDA, A. Vorticity and the mathematical theory of incompressible fluid flow. *Communications on Pure and Applied Mathematics* 39, S1 (1986), S187–S220.
- [128] MAJDA, A. J., AND BERTOZZI, A. L. *Vorticity and incompressible flow*, vol. 27. Cambridge University Press, Cambridge, 2002.
- [129] MALLIAVIN, P. From Ordinary Differential Equations to Stochastic Flow: The Transfer Principle. In *Stochastic Analysis. Grundlehren der mathematischen Wissenschaften (A Series of Comprehensive Studies in*

- Mathematics*) (1997), vol. 313, Springer, Berlin, Heidelberg, pp. 203–235.
- [130] MARSDEN, J. E., AND RATIU, T. S. *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems*. Springer Publishing Company, New York, 1994.
 - [131] MAURELLI, M., MODIN, K., AND SCHMEDING, A. Incompressible Euler equations with stochastic forcing: a geometric approach. *arXiv:1909.09982* (2019).
 - [132] MIKULEVICIUS, R., AND ROZOVSKII, B. On Equations of Stochastic Fluid Mechanics. In *Stochastics in Finite and Infinite Dimensions. Trends in Mathematics* (2001), Birkhäuser, Boston, MA, pp. 13–29.
 - [133] MOHAMMED, S.-E. A., NILSSEN, T. K., AND PROSKE, F. N. Sobolev differentiable stochastic flows for SDEs with singular coefficients: Applications to the transport equation. *The Annals of Probability* 43, 3 (2015), 1535–1576.
 - [134] NEVES, W., AND OLIVERA, C. Well-posedness for stochastic continuity equations with Ladyzhenskaya–Prodi–Serrin condition. *Nonlinear Differential Equations and Applications* 22, 5 (2015), 1247–1258.
 - [135] OKSENDAL, B. *Stochastic Differential Equations (3rd Edition): An Introduction with Applications*. Springer-Verlag, Berlin, Heidelberg, 1992.
 - [136] OLIVERA, C. Regularization by noise in one-dimensional continuity equation. *Potential Analysis* 5, 1 (2019), 23–35.
 - [137] PEDLOSKY, J. *Geophysical Fluid Dynamics*. Springer, New York, 1987.
 - [138] PLANT, R. S., AND CRAIG, G. C. A Stochastic Parameterization for Deep Convection Based on Equilibrium Statistics. *Journal of the Atmospheric Sciences* 65, 1 (2008), 87–105.

- [139] PORTA MANA, P., AND ZANNA, L. Toward a stochastic parameterization of ocean mesoscale eddies. *Ocean Modelling* 79 (2014), 1–20.
- [140] PROTTER, P. *Stochastic Integration and Differential Equations*. Springer-Verlag, Berlin, Heidelberg, 1992.
- [141] PUMIR, A., AND SIGGIA, E. D. Finite-time singularities in the axisymmetric three-dimension Euler equations. *Physical Review Letters* 68, 10 (1992), 1511–1514.
- [142] REZAKHANLOU, F. Stochastically symplectic maps and their applications to the Navier–Stokes equation. *Annales de l’Institut Henri Poincaré (C) Non Linear Analysis* 33, 1 (2016), 1–22.
- [143] RICHARDSON, L. *Weather prediction by numerical process*. Cambridge University Press, Second Edition, 2007.
- [144] SANCHEZ, C., WILLIAMS, K. D., AND COLLINS, M. Improved stochastic physics schemes for global weather and climate models. *Quarterly Journal of the Royal Meteorological Society* 142, 694 (2016), 147–159.
- [145] SCHOCHET, S. The point-vortex method for periodic weak solutions of the 2-D Euler equations. *Communications on Pure and Applied Mathematics* 49, 9 (1996), 911–965.
- [146] SCOTT, C. L^p theory of differential forms on manifolds. *Transactions of the American Mathematical Society* 347, 6 (1995), 2075–2096.
- [147] SEBILLE, E., ET AL. Lagrangian ocean analysis: Fundamentals and practices. *Ocean Modelling* 121 (2018), 49–75.
- [148] SHUTTS, G. A stochastic convective backscatter scheme for use in ensemble prediction systems. *Quarterly Journal of the Royal Meteorological Society* 141, 692 (2015), 2602–2616.
- [149] SIMON, J. Compact sets in the space $L^p(0, T; B)$. *Annali di Matematica Pura ed Applicata* 146 (1986), 65–96.

- [150] SIRAN, L., JIAHONG, W., AND KUN, Z. On the degenerate Boussinesq equations on surfaces. *Journal of Geometric Mechanics* 12 (2020), 107–140.
- [151] STINGA, P. R., AND TORREA, J. L. Hölder, Sobolev, weak-type and BMO estimates in mixed-norm with weights for parabolic equations. *arXiv:1808.01311* (2018).
- [152] TAYLOR, M. Pseudo Differential Operators. In *Lecture Notes in Mathematics*, vol. 416. Springer, Berlin, 1974.
- [153] TEMAM, R. *Navier-Stokes Equations: Theory and Numerical Analysis*. AMS Chelsea Publishing, Rhode Island, 2001.
- [154] VERETENNIKOV, A. J. On strong solutions and explicit formulas for solutions of stochastic integral equations. *Math. USSR-Sb.* 39, 3 (1981), 387–403.
- [155] VRETTOS, C. Time-harmonic Boussinesq problem for a continuously non-homogeneous soil. *Earthquake Engineering & Structural Dynamics* 20, 10 (1991), 961–977.
- [156] YUDOVICH, V. I. Eleven great problems of mathematical hydrodynamics. *Moscow Mathematical Journal* 3, 2 (2003), 711–737.
- [157] ZVONKIN, A. A transformation of the phase space of a diffusion process that removes the drift. *Mathematics of the USSR-Sbornik* 22, 1 (1974), 129–149.



SOME EXTRA BACKGROUND IN TENSORS

This appendix describes some fundamental concepts of differential geometry that we employ in this dissertation and is targeted to the reader who may not be very experienced with the basic elements of differential geometry. As usual, the goal is not providing a comprehensive review, and this appendix should instead be interpreted as a guide to understand some of the computations appearing in this thesis. Since in this dissertation we work either on the Euclidean space or the torus, the coordinates are global so the geometric objects can be conveniently expressed in global coordinates.

A.1 BACKGROUND IN TENSORS

Let $n \in \mathbb{N}$, V be a real finite-dimensional vector space with basis $\{e_1, \dots, e_n\}$, and V^* be its corresponding dual space with basis $\{e^1, \dots, e^n\}$, defined via the relation $\langle e^j, e_i \rangle = \delta_i^j$, $i, j \in \{1, \dots, n\}$, where $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{R}$ is the natural dual pairing $\langle \alpha, v \rangle = \alpha(v)$, $\alpha \in V^*$, $v \in V$. Given a vector $v \in V$ (called contravariant), we can express it as the sum of its components

$v = v^i e_i$ with the upper indices (where we have assumed Einstein's convention of summation over repeated indices), and given a covector $\alpha \in V^*$ (called covariant), we can express it as $\alpha = \alpha_i e^i$ with the lower indices.

Definition 131. Given a vector space V , its dual V^* , and r, s nonnegative integers such that at least one of them is nonzero, we define an (r, s) -tensor T as a multilinear map of the form

$$T : \underbrace{V^* \times \cdots \times V^*}_{r \text{ times}} \times \underbrace{V \times \cdots \times V}_{s \text{ times}} \rightarrow \mathbb{R}.$$

Fixing a basis $\{e_1, \dots, e_n\}$ of V and $\{e^1, \dots, e^n\}$ of V^* , we may interpret tensors as vectors with basis $\{e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{i_{r+1}} \otimes \cdots \otimes e^{i_{r+s}}\}_{i_k \in \{1, \dots, n\}, k \in \{1, \dots, r+s\}}$, where we define

$$\begin{aligned} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{i_{r+1}} \otimes \cdots \otimes e^{i_{r+s}} (e^{j_1}, \dots, e^{j_r}, e_{j_{r+1}}, \dots, e_{j_{r+s}}) \\ = \delta_{i_1}^{j_1} \cdots \delta_{i_r}^{j_r} \delta_{j_{r+1}}^{i_{r+1}} \cdots \delta_{j_{r+s}}^{i_{r+s}}, \end{aligned} \quad (\text{A.1.1})$$

when it acts on elements of the basis. We can extend definition (A.1.1) to any element in

$$\underbrace{V^* \times \cdots \times V^*}_{r \text{ times}} \times \underbrace{V \times \cdots \times V}_{s \text{ times}}$$

by linearity. Therefore, we can express T as

$$T = T_{i_{r+1}, \dots, i_{r+s}}^{i_1, \dots, i_r} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{i_{r+1}} \otimes \cdots \otimes e^{i_{r+s}},$$

where $T_{i_{r+1}, \dots, i_{r+s}}^{i_1, \dots, i_r} = T(e^{i_1}, \dots, e^{i_r}, e_{i_{r+1}}, \dots, e_{i_{r+s}})$ are its components, and we sum over repeated indices. We denote the space of (r, s) -tensors over V by $T_V^{(r,s)}$.

Given two tensors $T \in T_V^{(r,s)}$, $S \in T_V^{(s,r)}$, there is a natural dual pairing given by

$$\langle T, S \rangle = T_{i_{r+1}, \dots, i_{r+s}}^{i_1, \dots, i_r} T_{i_{s+1}, \dots, i_{r+s}}^{i_1, \dots, i_s},$$

where we sum over repeated indices as usually. Consider a symmetric, non-degenerate, positive definite bilinear form $g : V \times V \rightarrow \mathbb{R}$, which is a $(0, 2)$ -tensor whose components are given by $g_{ij} = g(e_i, e_j)$. We call g a metric tensor or just a metric. Fixing a vector $v \in V$, one can construct an associated covector $v^\flat \in V^*$ through the relation $v^\flat(w) = g(v, w)$, for any $w \in V$, and we define $g^\flat : V \rightarrow V^*$ to be the map $g^\flat : v \rightarrow v^\flat$, for $v \in V$. This procedure is called lowering the index, since the components of $v^\flat$ now have lower indices $((v^\flat)_i = g_{ij}v^j, i = 1, \dots, n)$. We also introduce the inverse operation $g^\sharp = (g^\flat)^{-1} : V^* \rightarrow V$, which maps a covector $\alpha \in V^*$ to a vector $\alpha^\sharp \in V$ by the relation $\alpha^\sharp = g^\sharp(\alpha)$. This is called raising the index. One can then define a $(2, 0)$ -tensor g^* by $g^*(\alpha, \beta) = g(\alpha^\sharp, \beta^\sharp)$, $\alpha, \beta \in V^*$, whose components we simply denote by $g^{ij} = g^*(e^i, e^j)$. We call g^* the cometric tensor or just the cometric. Combining the metric and cometric tensors, we can construct a metric on tensors, defined as follows (see for instance¹⁵).

Definition 132. *Given a metric tensor g on V , a bundle metric $\bar{g}$ on $T_V^{(r,s)}$ is defined by*

$$\bar{g}(T, S) = g_{i_1 j_1} \cdots g_{i_r j_r} g^{i_{r+1} j_{r+1}} \cdots g^{i_{r+s} j_{r+s}} T_{i_{r+1}, \dots, i_{r+s}}^{i_1, \dots, i_r} S_{j_{r+1}, \dots, j_{r+s}}^{j_1, \dots, j_r},$$

for any $S, T \in T_V^{(r,s)}$, where we stress that we are summing over repeated indices. This also provides us with a norm $|\cdot|_g$ on $T_V^{(r,s)}$, defined by

$$|T|_g = \sqrt{\bar{g}(T, T)}.$$

We note that this generalises the Frobenius inner product for matrices to general tensors.

From now on, we assume $V = \mathbb{R}^n$, since every finite-dimensional vector space is isomorphic to $\mathbb{R}^n$ and this is actually the case we treat in this dissertation. An (r, s) -tensor bundle on $\mathbb{R}^n$ is a tuple $(T^{(r,s)}(\mathbb{R}^n), \pi)$, where $T^{(r,s)}(\mathbb{R}^n)$ can be identified with $\mathbb{R}^n \times T_{\mathbb{R}^n}^{(r,s)}$, and π is the natural projection map $\pi : \mathbb{R}^n \times T^{(r,s)}(\mathbb{R}^n) \rightarrow \mathbb{R}^n$. For every point $x \in \mathbb{R}^n$, we have $\pi^{-1}(x) \cong T_{\mathbb{R}^n}^{(r,s)}$, which we call the fibres of the bundle. We will often denote the tensor bundle by $T^{(r,s)}(\mathbb{R}^n)$ instead of $(T^{(r,s)}(\mathbb{R}^n), \pi)$, omitting the projec-

tion map for simplicity. For example, taking $(r, s) = (1, 0)$ we get the tangent bundle $T\mathbb{R}^n$, and considering $(r, s) = (0, 1)$ we obtain the cotangent bundle $T^*\mathbb{R}^n$. Although we will not explain it here, the notion of tensor bundles can be generalised to manifolds and manifolds are the natural setting for a tensor bundle to be defined.

A section s of the tensor bundle $(T^{(r,s)}(\mathbb{R}^n), \pi)$ is a map $s : \mathbb{R}^n \rightarrow T^{(r,s)}(\mathbb{R}^n)$ such that $\pi \circ s : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the identity map. We denote the space of all sections by $\Gamma(T^{(r,s)}(\mathbb{R}^n))$ and the set of all smooth sections by $C^\infty(T^{(r,s)}(\mathbb{R}^n))$. Geometrically, a section is a map that assigns a unique tensor to each point in $\mathbb{R}^n$, hence defining a tensor field. Given two tensor fields $T \in \Gamma(T^{(r,s)}(\mathbb{R}^n))$, $S \in \Gamma(T^{(s,r)}(\mathbb{R}^n))$, there is a natural pairing $\langle \cdot, \cdot \rangle : \mathbb{R}^n \rightarrow \mathbb{R}$ given by $\langle T, S \rangle(x) = \langle T(x), S(x) \rangle$. In the next section, we describe in more detail two common examples of tensor fields, namely, vector fields and differential forms.

A.2 VECTOR FIELDS AND DIFFERENTIAL FORMS

Two fundamental examples of tensor fields (sections of the tensor bundle) are vector fields and differential one-forms, which we will discuss in more detail here.

VECTOR FIELDS. A vector field X is a $(1, 0)$ -tensor field $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (i.e., a section of the tangent bundle). We denote by $\mathfrak{X}(\mathbb{R}^n) = C^\infty(T\mathbb{R}^n)$ the set of all smooth vector fields on $\mathbb{R}^n$, where $T\mathbb{R}^n = T^{(1,0)}(\mathbb{R}^n)$.

DIFFERENTIAL ONE-FORMS. A differential one-form α is a $(0, 1)$ -tensor field $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$, (i.e., a section of the cotangent bundle). We denote by $\Omega^1(\mathbb{R}^n) = C^\infty(T^*\mathbb{R}^n)$ the set of all smooth differential one-forms on $\mathbb{R}^n$, where $T^*\mathbb{R}^n = T^{(0,1)}(\mathbb{R}^n)$.

As a standard convention, the usual basis of vector fields on $\mathbb{R}^n$ evaluated at a point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is expressed as $\{(\partial/\partial x_1)(x), \dots, (\partial/\partial x_n)(x)\}$ and its corresponding dual basis is denoted by $\{dx_1(x), \dots, dx_n(x)\}$. Note

that these are related by dual pairing via the identity $\langle \partial/\partial x_i, dx_j \rangle(x) = \langle (\partial/\partial x_i)(x), dx_j(x) \rangle = \delta_i^j$, $x \in \mathbb{R}^n$. In coordinates, a vector field X and a one-form α can be expressed explicitly as

$$X(x) = X^i(x) \frac{\partial}{\partial x_i}(x), \quad \alpha(x) = \alpha_i(x) dx_i(x),$$

$x \in \mathbb{R}^n$, where we have assumed Einstein's convention of summation over repeated indices. More generally, we can express an (r, s) -tensor field T in global coordinates as

$$T(x) = T_{i_{r+1}, \dots, i_{r+s}}^{i_1, \dots, i_r}(x) \frac{\partial}{\partial x_{i_1}}(x) \otimes \dots \otimes \frac{\partial}{\partial x_{i_r}}(x) \otimes dx_{i_{r+1}}(x) \otimes \dots \otimes dx_{i_{r+s}}(x),$$

for $x \in \mathbb{R}^n$.

MULTI-VECTOR FIELDS. Multi-vector fields generalise the notion of a vector field. Let $k \in \{0, \dots, n\}$. A k -vector field X on $\mathbb{R}^n$ is a $(k, 0)$ -tensor field such that for any permutation $\sigma \in S_k$ (here, S_k represents the symmetric group of order k), we have

$$X(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(k)}) = \text{sgn}(\sigma) X(\alpha_1, \dots, \alpha_k),$$

for all $\alpha_1, \dots, \alpha_k \in \Omega^1(\mathbb{R}^n)$, or in other words

$$X(x)(\alpha_{\sigma(1)}(x), \dots, \alpha_{\sigma(k)}(x)) = \text{sgn}(\sigma) X(x)(\alpha_1(x), \dots, \alpha_k(x)),$$

for all $\alpha_1, \dots, \alpha_k \in \Omega^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, where $\text{sgn}(\sigma)$ is the signature of the permutation $\sigma \in S_k$. In particular, this implies that X is a smooth section of the alternating $(k, 0)$ -tensor bundle $\bigwedge^k(T\mathbb{R}^n)$. We denote the space of k -vector fields by $\mathfrak{X}^k(\mathbb{R}^n)$. In coordinates, one can express $X \in \mathfrak{X}^k(\mathbb{R}^n)$ as

$$\begin{aligned} X(x) &= \sum_{i_1 < \dots < i_k}^n \sum_{\sigma \in S_k} \text{sgn}(\sigma) X_{i_1, \dots, i_k}(x) \frac{\partial}{\partial x_{i_{\sigma(1)}}}(x) \otimes \dots \otimes \frac{\partial}{\partial x_{i_{\sigma(k)}}}(x) \\ &= \sum_{i_1 < \dots < i_k}^n X_{i_1, \dots, i_k}(x) \frac{\partial}{\partial x_{i_1}}(x) \wedge \dots \wedge \frac{\partial}{\partial x_{i_k}}(x), \quad x \in \mathbb{R}^n. \end{aligned}$$

DIFFERENTIAL k -FORMS. Other fundamental examples of tensor fields in differential geometry are differential k -forms, which generalise the notion of a differential one-form. Let $k \in \{0, \dots, n\}$. A differential k -form α on $\mathbb{R}^n$ is a $(0, k)$ -tensor field such that for any permutation $\sigma \in S_k$ we have

$$\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \text{sgn}(\sigma) \alpha(v_1, \dots, v_k),$$

for all $v_1, \dots, v_k \in \mathfrak{X}(\mathbb{R}^n)$. We denote the space of smooth k -forms by $\Omega^k(\mathbb{R}^n) = C^\infty\left(\bigwedge^k(T^*\mathbb{R}^n)\right)$, which is a smooth section of the alternating $(0, k)$ -tensor bundle $\bigwedge^k(T^*\mathbb{R}^n)$. In coordinates, a k -form can be expressed as

$$\alpha(x) = \sum_{i_1 < \dots < i_k}^n \alpha_{i_1, \dots, i_k}(x) dx_{i_1}(x) \wedge \dots \wedge dx_{i_k}(x), \quad x \in \mathbb{R}^n. \quad (\text{A.2.1})$$

EXTERIOR PRODUCT. Given a k -form α and an l -form β , one can construct a $(k + l)$ -form $\alpha \wedge \beta$, defined as

$$\begin{aligned} (\alpha \wedge \beta)(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) \\ = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}), \end{aligned}$$

where $n = k + l$ and the operation “ $\wedge$ ” is called the exterior product.

NOTION OF INTEGRATION. Differential k -forms are especially important in differential geometry since they allow to extend the notion of integration to general oriented manifolds. For a k -form $\alpha \in \Omega^k(\mathbb{R}^n)$, we define its integral over $\mathbb{R}^k$ by

$$\int_{\mathbb{R}^k} \alpha = \sum_{i_1 < \dots < i_k}^n \int_{\mathbb{R}^k} \alpha_{i_1, \dots, i_k}(x) dx_{i_1} \dots dx_{i_k},$$

where $dx_{i_1} \dots dx_{i_k}$ is the Lebesgue measure. We note that all of the above notions can be defined in a coordinate-free manner on arbitrary k -dimensional smooth oriented manifolds (see for instance¹³⁰). We also review some elementary operations defined on tensor fields, and in particular, on vector fields

and k -forms.

EXTERIOR DERIVATIVE. Given a k -form $\alpha \in \Omega^k(\mathbb{R}^n)$, which in coordinates is written as $\alpha(x) = \alpha_{i_1, \dots, i_k}(x) dx_{i_1}(x) \wedge \dots \wedge dx_{i_k}(x)$, its exterior derivative $d\alpha$ is the $(k+1)$ -form

$$d\alpha(x) = \frac{\partial \alpha_{i_1, \dots, i_k}}{\partial x_l}(x) dx_l(x) \wedge dx_{i_1}(x) \wedge \dots \wedge dx_{i_k}(x), \quad x \in \mathbb{R}^n,$$

where we are summing as in (A.2.1) and also in $l = 1, \dots, n$. Hence, the exterior derivative is a map $d : \Omega^k(\mathbb{R}^n) \rightarrow \Omega^{k+1}(\mathbb{R}^n)$.

CONTRACTION. Given a vector field X and a k -form α with $k \geq 1$, we define the contraction of α with X , which we denote by $\iota_X \alpha$, as the $(k-1)$ -form

$$\iota_X \alpha(x) = X^l(x) \alpha_{l, i_2, \dots, i_k}(x) dx_{i_2}(x) \wedge \dots \wedge dx_{i_k}(x), \quad x \in \mathbb{R}^n,$$

where again we are summing as in (A.2.1) and also in $l = 1, \dots, n$. Given a vector field X , the contraction is a map $\iota_X : \Omega^k(\mathbb{R}^n) \rightarrow \Omega^{k-1}(\mathbb{R}^n)$.

PULL-BACK AND PUSH-FORWARD. Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth diffeomorphism. The push-forward $\varphi_* X$ of a vector field $X \in \mathfrak{X}(\mathbb{R}^n)$ with respect to φ is the vector field (i.e., an element of $\mathfrak{X}(\mathbb{R}^n)$) defined as

$$(\varphi_* X)(\varphi(x)) = X^i(x) \frac{\partial \varphi_j}{\partial x_i}(x) \frac{\partial}{\partial x_j}(\varphi(x)), \quad x \in \mathbb{R}^n,$$

where we are summing over repeated indices as usually, and the pull-back $\varphi^* \alpha$ of a one-form $\alpha \in \Omega^1(\mathbb{R}^n)$ with respect to φ is the one-form (i.e., an element of $\Omega^1(\mathbb{R}^n)$) defined as

$$(\varphi^* \alpha)(x) = \alpha_i(\varphi(x)) \frac{\partial \varphi_i}{\partial x_j}(x) dx_j(x), \quad x \in \mathbb{R}^n.$$

One can also define the pull-back of a vector field as $\varphi^*X = (\varphi^{-1})_*X$, and the push-forward of a one-form as $\varphi_*\alpha = (\varphi^{-1})^*\alpha$.

For a general smooth (r, s) -tensor field T , we define its pull-back φ^*T with respect to φ by

$$\begin{aligned} & (\varphi^*T)(x) (\alpha_1(x), \dots, \alpha_r(x), v_1(x), \dots, v_s(x)) \\ &= T(\varphi(x)) ((\varphi_*\alpha_1)(\varphi(x)), \dots, (\varphi_*\alpha_r)(\varphi(x)), (\varphi_*v_1)(\varphi(x)), \dots, (\varphi_*v_s)(\varphi(x))), \end{aligned}$$

for any $\alpha_1, \dots, \alpha_r \in \Omega^1(\mathbb{R}^n)$, $v_1, \dots, v_s \in \mathfrak{X}(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$. Similarly, we define the push-forward of an (r, s) -tensor field as $\varphi_*T = (\varphi^{-1})^*T$.

It follows immediately that $\varphi^* \circ \varphi_* = id$. For simplicity, we will employ the notations $\varphi^*T(x)$, $\varphi_*T(x)$ instead of $(\varphi^*T)(x)$, $(\varphi_*T)(x)$, respectively.

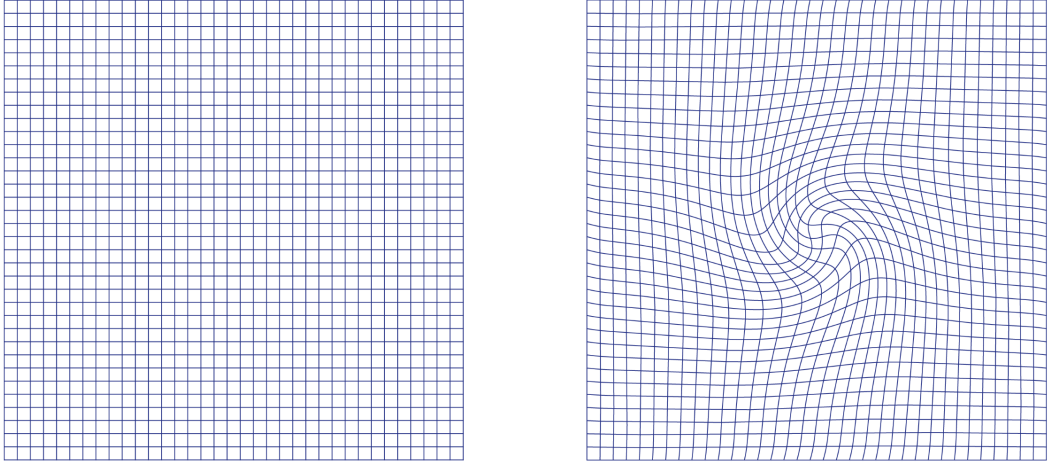


Figure A.1: Example of a diffeomorphism on the square.

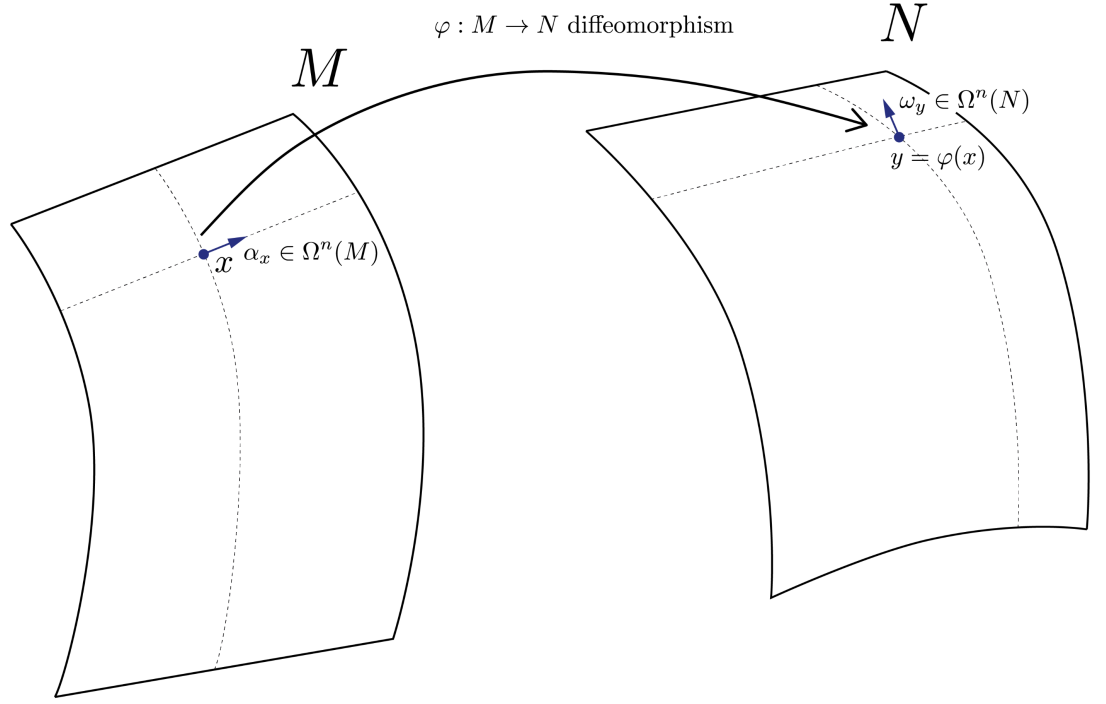


Figure A.2: The pull-back of a volume form.

LIE DERIVATIVE. Let $T \in C^\infty(T^{(r,s)}(\mathbb{R}^n))$, and φ_t be the (local) flow of the vector field $X \in \mathfrak{X}(\mathbb{R}^n)$, this is, the local solution to the ODE

$$\dot{\varphi}_t(x) = X(\varphi_t(x)), \quad \varphi_0(x) = x.$$

The Lie derivative of T with respect to X is the (r, s) -tensor field defined by the relation

$$(\mathcal{L}_X T)(x) = \lim_{t \rightarrow 0} \frac{1}{t} (\varphi_t^* T(x) - T(x)),$$

which is a map $\mathcal{L}_X : C^\infty(T^{(r,s)}(\mathbb{R}^n)) \rightarrow C^\infty(T^{(r,s)}(\mathbb{R}^n))$. We employ the notation $\mathcal{L}_X(x)$ instead of $(\mathcal{L}_X)(x)$ for simplicity. Below, we list the global coordinate expressions of some common examples of Lie derivatives on $\mathbb{R}^n$ that we will employ. In the following formulas, x should be understood as

any element in $\mathbb{R}^n$. We assume summation over repeated indices.

- For a function $f \in C^\infty(\mathbb{R}^n)$ and a vector field $u \in \mathfrak{X}(\mathbb{R}^n)$, we have

$$\mathcal{L}_u f(x) = u^i(x) \frac{\partial f}{\partial x_i}(x).$$

- Given two vector fields $u, v \in \mathfrak{X}(\mathbb{R}^n)$, we have

$$\mathcal{L}_u v(x) = [u, v](x) = \left(u^i(x) \frac{\partial v^j}{\partial x_i}(x) - v^i(x) \frac{\partial u^j}{\partial x_i}(x) \right) \frac{\partial}{\partial x_j}(x),$$

where $[\cdot, \cdot]$ is the standard commutator of vector fields (Lie bracket of vector fields).

- Given a one-form $\alpha \in \Omega^1(\mathbb{R}^n)$ and a vector field $u \in \mathfrak{X}(\mathbb{R}^n)$, we have

$$\mathcal{L}_u \alpha(x) = \left(u^j(x) \frac{\partial \alpha_i}{\partial x_j}(x) + \alpha_j(x) \frac{\partial u^j}{\partial x_i}(x) \right) dx_i(x).$$

- For a density $D \in \Omega^n(\mathbb{R}^n)$ with coordinate expression $\rho(x) dx^n$ and a vector field $u \in \mathfrak{X}(\mathbb{R}^n)$, we have

$$\mathcal{L}_u D(x) = \left(\frac{\partial \rho}{\partial x_i}(x) u^i(x) + \frac{\partial u^i}{\partial x_i}(x) \rho(x) \right) dx^n(x).$$

- In coordinates, the Lie derivative of a k -form $\alpha \in \Omega^k(\mathbb{R}^n)$ with respect to a vector field $X \in \mathfrak{X}(\mathbb{R}^n)$ is given by

$$\begin{aligned} \mathcal{L}_X \alpha(x)(v_1(x), \dots, v_k(x)) &= X^l(x) \frac{\partial \alpha_{i_1, \dots, i_k}}{\partial x_l}(x) v_1^{i_1}(x) \cdots v_k^{i_k}(x) \\ &+ \sum_{j=1}^k \alpha_{i_1, \dots, i_k}(x) \frac{\partial X^{i_j}}{\partial x_l}(x) v_1^{i_1}(x) \cdots v_j^l \cdots v_k^{i_k}(x), \quad v_1, \dots, v_k \in \mathfrak{X}(\mathbb{R}^n). \end{aligned} \tag{A.2.2}$$

For a k -form $\alpha \in \Omega^k(\mathbb{R}^n)$ and a vector field $X \in \mathfrak{X}(\mathbb{R}^n)$, we have the identity

$$\mathcal{L}_X \alpha = d(\iota_X \alpha) + \iota_X d\alpha.$$

This is called Cartan's magic formula.

We note that the Lie derivative commutes with the exterior derivative, with the pull-back, and with the push-forward. Moreover, for tensors $T, S \in C^\infty(T^{(r,s)}(\mathbb{R}^n))$, $X \in \mathfrak{X}(\mathbb{R}^n)$, we have the property

$$\mathcal{L}_X(T \otimes S) = \mathcal{L}_X T \otimes S + T \otimes \mathcal{L}_X S. \quad (\text{A.2.3})$$

Remark 133. *We have defined different geometric operations which require derivation such as the Lie derivative or the exterior derivative for smooth tensors, but this can be done of course under much weaker regularity assumptions. Indeed, differentiability is enough and even weak notions of some of these operations can be defined by merely imposing weak differentiability of the tensor components.*

B

Proof of generalised estimates for linear operators

In this appendix, we collect the proof of Proposition 32, which is very useful, for instance, when deriving Sobolev bounds for equations with transport noise depending on the gradient of the solution. This result was initially motivated by the H^2 -estimates for the three-dimensional Lie derivative appearing in⁴⁹ (see Lemmas 27 & 30). In our proof, we modify their argument to be more general. More precisely, we provide an extension of their result to higher or fractional order pseudodifferential operators $\mathcal{P}$ and linear differential operators Q of first order.

Proof. We start by establishing (1.3.7). Indeed, this follows after a straightforward computation, since

$$\langle \mathcal{Q}^2 f, f \rangle_{L^2} = \langle \mathcal{Q} f, \mathcal{Q}^* f \rangle_{L^2} = -\langle \mathcal{Q} f, \mathcal{Q} f \rangle_{L^2} + \langle \mathcal{Q} f, E f \rangle_{L^2}, \quad (\text{B.0.1})$$

where $\mathcal{Q}^*$ denotes the adjoint operator of $\mathcal{Q}$ under the L^2 -pairing and we

have employed the notation

$$\mathcal{Q}^* = -\mathcal{Q} + E \quad (\text{B.0.2})$$

where E is a zero order operator. (B.0.2) follows from the general theory of differential operators. Now just note that the last term on the right-hand side of (B.0.1) can be rewritten as

$$\begin{aligned} \langle \mathcal{Q}f, Ef \rangle_{L^2} &= -\langle f, \mathcal{Q}Ef \rangle_{L^2} + \langle f, E^2f \rangle_{L^2} \\ &= -\langle f, E\mathcal{Q}f \rangle_{L^2} - \langle f, T_0f \rangle_{L^2} + \langle f, E^2f \rangle_{L^2} \\ &= -\langle Ef, \mathcal{Q}f \rangle_{L^2} - \langle f, T_0f \rangle_{L^2} + \langle f, E^2f \rangle_{L^2}, \end{aligned}$$

since

$$\mathcal{Q}E - E\mathcal{Q} = [\mathcal{Q}, E] = T_0,$$

where T_0 is a zero order differential operator and we have employed the fact that $\langle Ef, g \rangle_{L^2} = \langle f, Eg \rangle_{L^2}$, for L^2 -integrable vector-valued functions f, g . Hence,

$$|\langle \mathcal{Q}^2f, f \rangle_{L^2} + \langle \mathcal{Q}f, \mathcal{Q}f \rangle_{L^2}| = (1/2)|\langle f, T_0f \rangle_{L^2} + \langle f, E^2f \rangle_{L^2}| \lesssim \|f\|_{L^2}^2.$$

Next we show inequality (1.3.8). We define

$$T_1 = \mathcal{P}\mathcal{Q} - \mathcal{Q}\mathcal{P} = [\mathcal{P}, \mathcal{Q}].$$

The classical theory of pseudodifferential operators states that the resulting commutator is of order s (c.f., ^{105,152}). Following the same idea, let us define

$$T_2 = T_1\mathcal{Q} - \mathcal{Q}T_1 = [T_1, \mathcal{Q}],$$

which is also an operator of order s for the same reason. Therefore,

$$\begin{aligned}\langle \mathcal{P}\mathcal{Q}^2f, \mathcal{P}f \rangle_{L^2} &= \langle (\mathcal{Q}\mathcal{P} + T_1)\mathcal{Q}f, \mathcal{P}f \rangle_{L^2} = \langle \mathcal{Q}\mathcal{P}\mathcal{Q}f, \mathcal{P}f \rangle_{L^2} + \langle T_1\mathcal{Q}f, \mathcal{P}f \rangle_{L^2} \\ &= -\langle \mathcal{P}\mathcal{Q}f, \mathcal{Q}\mathcal{P}f \rangle_{L^2} + \langle \mathcal{P}\mathcal{Q}f, E\mathcal{P}f \rangle_{L^2} + \langle T_1\mathcal{Q}f, \mathcal{P}f \rangle_{L^2} \\ &= -\langle \mathcal{P}\mathcal{Q}f, \mathcal{P}\mathcal{Q}f \rangle_{L^2} + \langle \mathcal{P}\mathcal{Q}f, T_1f \rangle_{L^2} + \langle T_1\mathcal{Q}f, \mathcal{P}f \rangle_{L^2} + \langle \mathcal{P}\mathcal{Q}f, E\mathcal{P}f \rangle_{L^2},\end{aligned}$$

where we have taken into account the definition of T_1 and (B.0.2). Therefore,

$$\begin{aligned}\langle \mathcal{P}\mathcal{Q}^2f, \mathcal{P}f \rangle_{L^2} + \langle \mathcal{P}\mathcal{Q}f, \mathcal{P}\mathcal{Q}f \rangle_{L^2} \\ = \langle \mathcal{P}\mathcal{Q}f, T_1f \rangle_{L^2} + \langle T_1\mathcal{Q}f, \mathcal{P}f \rangle_{L^2} + \langle \mathcal{P}\mathcal{Q}f, E\mathcal{P}f \rangle_{L^2}.\end{aligned}$$

Once again, manipulating the above equality, we obtain

$$\begin{aligned}\langle \mathcal{P}\mathcal{Q}^2f, \mathcal{P}f \rangle_{L^2} + \langle \mathcal{P}\mathcal{Q}f, \mathcal{P}\mathcal{Q}f \rangle_{L^2} \\ = \langle \mathcal{Q}\mathcal{P}f, T_1f \rangle_{L^2} + \langle T_1f, T_1f \rangle_{L^2} + \langle T_1\mathcal{Q}f, \mathcal{P}f \rangle_{L^2} + \langle \mathcal{P}\mathcal{Q}f, E\mathcal{P}f \rangle_{L^2} \\ = -\langle \mathcal{P}f, \mathcal{Q}T_1f \rangle_{L^2} + \langle T_1f, T_1f \rangle_{L^2} + \langle T_1\mathcal{Q}f, \mathcal{P}f \rangle_{L^2} \\ \quad + \langle \mathcal{P}\mathcal{Q}f, E\mathcal{P}f \rangle_{L^2} + \langle \mathcal{P}f, ET_1f \rangle_{L^2} \\ = \langle (T_1\mathcal{Q} - \mathcal{Q}T_1)f, \mathcal{P}f \rangle_{L^2} + \langle T_1f, T_1f \rangle_{L^2} \\ \quad + \langle \mathcal{P}\mathcal{Q}f, E\mathcal{P}f \rangle_{L^2} + \langle \mathcal{P}f, ET_1f \rangle_{L^2} \\ = \langle T_2f, \mathcal{P}f \rangle_{L^2} + \langle T_1f, T_1f \rangle_{L^2} + \langle \mathcal{P}f, ET_1f \rangle_{L^2} + \langle \mathcal{P}\mathcal{Q}f, E\mathcal{P}f \rangle_{L^2}.\end{aligned}$$

Notice that the last term on the right-hand side in the last equality seems to be singular as well. However, one can treat it as follows:

$$\begin{aligned}\langle \mathcal{P}\mathcal{Q}f, E\mathcal{P}f \rangle_{L^2} &= \langle (\mathcal{Q}\mathcal{P} + T_1)f, E\mathcal{P}f \rangle_{L^2} = \langle \mathcal{Q}\mathcal{P}f, E\mathcal{P}f \rangle_{L^2} + \langle T_1f, E\mathcal{P}f \rangle_{L^2} \\ &= -\langle \mathcal{P}f, \mathcal{Q}E\mathcal{P}f \rangle_{L^2} + \langle \mathcal{P}f, E^2\mathcal{P}f \rangle_{L^2} + \langle T_1f, E\mathcal{P}f \rangle_{L^2} \\ &= -\langle \mathcal{P}f, E\mathcal{Q}\mathcal{P}f \rangle_{L^2} - \langle \mathcal{P}f, T_0\mathcal{P}f \rangle_{L^2} + \langle \mathcal{P}f, E^2\mathcal{P}f \rangle_{L^2} + \langle T_1f, E\mathcal{P}f \rangle_{L^2} \\ &= -\langle E\mathcal{P}f, \mathcal{Q}\mathcal{P}f \rangle_{L^2} - \langle \mathcal{P}f, T_0\mathcal{P}f \rangle_{L^2} + \langle \mathcal{P}f, E^2\mathcal{P}f \rangle_{L^2} + \langle T_1f, E\mathcal{P}f \rangle_{L^2},\end{aligned}$$

where we have employed (B.0.2) and the commutators constructed above. Hence,

$$2\langle \mathcal{P}\mathcal{Q}f, E\mathcal{P}f \rangle_{L^2} = -\langle \mathcal{P}f, T_0\mathcal{P}f \rangle_{L^2} + \langle \mathcal{P}f, E^2\mathcal{P}f \rangle_{L^2} + 2\langle T_1f, E\mathcal{P}f \rangle_{L^2}.$$

Finally, by making use of Hölder's inequality, together with the fact that $T_1, T_2, \mathcal{P}$ are differential operators of order s , and E, T_0 are zero order operators, we conclude that

$$\begin{aligned} & \left| \langle \mathcal{P} \mathcal{Q}^2 f, \mathcal{P} f \rangle_{L^2} + \langle \mathcal{P} \mathcal{Q} f, \mathcal{P} \mathcal{Q} f \rangle_{L^2} \right| \\ &= \left| \langle T_2 f, \mathcal{P} f \rangle_{L^2} + \langle T_1 f, T_1 f \rangle_{L^2} + \langle \mathcal{P} f, E T_1 f \rangle_{L^2} \right. \\ & \quad \left. - \frac{1}{2} \langle \mathcal{P} f, T_0 \mathcal{P} f \rangle_{L^2} + \frac{1}{2} \langle \mathcal{P} f, E^2 \mathcal{P} f \rangle_{L^2} + \langle T_1 f, E \mathcal{P} f \rangle_{L^2} \right| \lesssim \|f\|_{H^s}^2. \end{aligned}$$

□

Remark 134. *It is easy to check that inequalities (2.9) and (2.10) in⁴⁹ represent a particular case of inequalities (1.3.7) and (1.3.8) in Proposition 32, respectively. Indeed, let $\mathcal{Q} = \mathcal{L}_{\xi_i}$ and $f : \mathbb{T}^3 \rightarrow \mathbb{R}^3$. Then we have $\mathcal{Q}^* = -\mathcal{Q}$, yielding (2.9). Inequality (2.10) follows by taking $\mathcal{Q} = \mathcal{L}_{\xi_i}$, $\mathcal{P} = \Lambda^s$. It is also worth noting that although we have proven our estimates for scalar-valued functions f defined on the periodic domain $\mathbb{T}^n$, they extend to the whole space $\mathbb{R}^n$ without the need of modifying the argument. Moreover, since all the commutator properties are also available on compact manifolds, we expect these estimates also to be available in that context.*