

DEVELOPMENT AND APPLICATION OF
THE BOUNDARY INTEGRAL AND RIGID BLOCK METHODS
FOR GEOTECHNICS

by

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A thesis submitted to the University of London
(Imperial College of Science and Technology)
for the Degree of
Doctor of Philosophy in the Faculty of Engineering.

JUNE 1977

ABSTRACT

Two dimensional formulations of the boundary integral method for the solution of problems in elastostatics and groundwater seepage are presented. The formulations are written to handle both mixed and heterogeneous problems. The integral method for elastostatics is further extended to non-linear problems, such as slip and separation of joints and analysis of a no-tension material. The formulation for groundwater seepage also includes iterative techniques for locating the phreatic surface under both steady state and transient conditions. Application of the integral methods to two and three dimensional geotechnical problems in elastostatics and two dimensional problems in groundwater seepage are presented.

A quasi-rigid block method for analysing large movements in blocky systems is presented. The method assumes all general shaped blocks are rigid and that interactions between neighbouring blocks are communicated via block corners. The calculation of stresses within each block by coupling with the finite element method is also included. Application of the method to assessing the stability of slopes and underground excavations is illustrated.

ACKNOWLEDGEMENTS

The author wishes to thank John Bray for many hours of stimulating discussion and the insight into mechanics he has inspired, and to John Watson for access to the three dimensional program. He would also like to thank his supervisors, Evert Hoek and Ted Brown, for their encouragement and help, and the staff and students of the Rock Mechanics Section for providing a pleasant working environment.

He appreciates the financial assistance provided to him while a student (1973-1975) by the companies supporting the large underground excavation project at Imperial College. His thanks too, to Jacquetta for typing the thesis.

To Elsie, the author is grateful for all her support throughout this thesis, and to Reg and Sue for their continued interest in his education.

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NOTATION

ENGLISH

<u>Symbol</u>		<u>Represents</u>
a	-	half crack length.
a	-	radius of tunnel.
a^n	-	linear acceleration at time n.
[A]	-	displacement transformation matrix.
B	-	coefficients relating flow to fictitious flows.
$c^{n+\frac{1}{2}}$	-	position of block at time $n+\frac{1}{2}$.
C	-	coefficients relating displacement. to fictitious force.
C	-	stress strain relationships.
D	-	coefficients relating stress component to fictitious force.
E	-	Young's modulus.
f_x	-	applied stress field in x direction.
F^n	-	force on block at time n.
g	-	fictitious flows.
G	-	shear modulus.
I	-	moment of inertia.
k	-	linear stiffness of block assemblage.

<u>Symbol</u>		<u>Represents</u>
k	-	permeability.
$[k]$	-	element stiffness matrix.
k_a	-	angular stiffness of block assemblage.
$[K]$	-	stiffness matrix for entire mesh.
m	-	mass of block.
M	-	$-\frac{\partial V}{\partial x}$
M^n	-	moment on block at time n .
N	-	$-\frac{\partial V}{\partial z}$
p	-	pressure.
p	-	normal fictitious force.
p	-	dipole force.
P	-	$-\frac{\partial \Phi}{\partial x}$
q	-	flow through interface.
q	-	shear fictitious force.
Q	-	$-\frac{\partial \Phi}{\partial z}$
Q	-	flow out of tunnel.
r	-	distance between two points.
r	-	nodal point displacements.
$r^{n+\frac{1}{2}}$	-	linear velocity of block at time $n+\frac{1}{2}$.
R	-	$\frac{\partial T}{\partial z}$
R	-	nodal point loads.
s	-	element corner forces.
S	-	$\frac{\partial T}{\partial x}$
S	-	surface of area S .

SymbolRepresents

t	-	applied tractions.
T	-	tractions due to unit solution.
T_f	-	coefficients relating u and u_f .
u	-	applied displacements.
u	-	pressure at a point.
u_f	-	fictitious displacements.
u_0	-	pressure at centre of tunnel.
U	-	displacements due to unit solution.
U_f	-	coefficients relating t and u_f .
v	-	applied flows.
V	-	$\frac{\partial U}{\partial x}$
V	-	flows due to unit solution.
w^n	-	angular acceleration at time n .
W	-	$\frac{\partial U}{\partial z}$

GREEK

$\omega^{n+\frac{1}{2}}$	-	angular velocity at time $n+\frac{1}{2}$.
β	-	effective porosity.
γ	-	shear strain.
γ_w	-	unit weight of water.

SymbolRepresents

Δ	-	dipole strength /p .
Δt	-	increment of time .
ϵ	-	strain .
$\theta^{n+\frac{1}{2}}$	-	orientation of block at time $n+\frac{1}{2}$.
ν	-	Poisson's ratio .
π	-	pi .
σ	-	normal stress .
Σ	-	summation .
τ	-	shear stress .
ϕ	-	applied potentials .
ϕ	-	friction angle .
Φ	-	potential due to unit solution .

CHAPTER 1: INTRODUCTION

For the design of problems in geotechnics, analysis, whether analytical or numerical, can be of use in aiding engineering judgement and providing an insight into behaviour. A criterion for the analysis of a geotechnical problem as either a continuum or discontinuum must depend on the relative size of the problem (slope, foundation, underground excavation etc.) to the material structure (discontinuity spacing, particle size, etc.). If the problem is large compared with the material structure then it can be effectively treated as a continuum. On the other hand if the problem is small compared to the material structure, then again it can be analysed by continuum methods appropriately modified to take account of the small number of discontinuous features. Between these two extremes the problem is truly discontinuous, that is material structure is moderate in size compared with the problem, and each discrete element in the system requires modelling.

Continuum approach

The popular numerical methods in present usage for analysing continuum problems, the finite difference and the finite element methods, are differential methods. These methods lead to a system of equations whose unknowns concern points lying inside the mass and on the boundary of the problem. Since communication between points or elements is solely via its neighbours, then the system of equations can be written in a banded form. On the other hand, integral methods write the system of equations whose unknowns concern points only lying on the boundary. Thus the problem is reduced by a dimension; however, the system of equations is full and therefore not banded. In many geotechnical problems (infinite and semi infinite media) the ratio of boundary area to volume is small, because the infinite boundary does not require direct modelling, thus these problems are ideally suited to an integral method approach. The reason is that for a differential method points are required throughout the volume, which is terminated in an exterior

boundary some distance removed from the area of required analysis, while integral methods require points only on the boundary itself. The main disadvantage of integral methods is that they are difficult to generalise. However, because of the considerable saving in computational effort and minimal amount of input data required (since only the boundaries require discretization) integral methods warrant attention for the analysis of geotechnical problems.

In the analysis for elastostatics, there exist three different integral methods, the fictitious force, the fictitious displacement and reciprocal methods; the first two are based on the principle of superposition, while the latter on the reciprocal theorem. Integral methods have been used for the analysis of non-homogeneous isotropic media, but have not been extended to non-linear behaviour such as slip and separation on joints and no-tension (material exhibiting zero tensile strength) analysis. Herein is described a two dimensional formulation for the analysis of non-homogeneous media and non-linear analysis as described above.

Integral methods to date have not been applied to the analysis of groundwater seepage problems. A formulation for the analysis of groundwater seepage in non-homogeneous media is given. Iterative methods to cope with the determination of phreatic surface location and transient conditions are described.

Discontinuum Approach

The analysis of discontinua by continuum methods, the finite difference and finite element methods, can be effectively handled provided displacements are small. The reason for this is that the discrete elements of the system are modelled as finite joint elements and continuum elements, thus any relative displacement between two neighbouring continuum elements is modelled by deformation of the joint element. Therefore methods incorporating joint elements have this inherent weakness with respect to modelling large relative displacement or rotation between

discrete neighbouring elements. For this reason a method of analysing large displacements of blocky systems by a rigid block dynamic relaxation technique was investigated. Each discrete element is considered to be rigid and interactions between neighbouring blocks are communicated via block corners. The necessity for blocks to crack was anticipated and the finite element method was incorporated to calculate stresses within the discrete blocks.

In Chapter 2, the integral methods for elastostatics are described and the numerical formulations for heterogeneity, mixed and non-linear problems are presented. The application of the method to elastic geotechnical problems, such as tunnel intersections, an underground power station, a mining complex, inclusions, thin soft seams, slip and separation on joints and no-tension analysis are presented in Chapter 3.

In Chapter 4, the integral method for groundwater seepage is presented and the numerical formulations for heterogeneity, mixed, phreatic surface location for steady and transient conditions and the interface of immiscible fluids are described. In Chapter 5 the method is applied to flow from and into a pressure tunnel and drain near the surface, phreatic surface location of a drain near resevoirs and an underground oil storage scheme.

The numerical formulation for the quasi-rigid block method including the coupled finite element method for stress calculations is described in Chapter 6. In Chapter 7, the method is applied to assessing the stability of slopes and underground excavations.

CHAPTER 2: THE BOUNDARY INTEGRAL METHOD FOR ELASTOSTATICS

1. Review of previous formulations

There exist at present, three different types of boundary integral method for the solution of problems in elastostatics; namely, the fictitious force, the fictitious displacement and the reciprocal method. The reason for using the word fictitious is to imply that the distribution of forces or displacements required to describe the problem have no physical significance as regards the actual problem. It is important to briefly discuss each method in turn in order to illustrate the advantages and/or disadvantages of each method for certain applications.

The fictitious force method has been described by Massonet (1965), who used the solution of a line load on a plane boundary as the unit solution, whereas other workers, Oliveira (1968) and Bray (1976a) used the solution of a line load in an infinite mass. Irrespective of what type of unit solution has been used, the basic formulation of the method given by all three of these workers is the same, i.e. to find a distribution of fictitious forces applied at the boundary of the problem in order to achieve the necessary boundary conditions. The most sophisticated two dimensional element chosen to represent this distribution of fictitious forces is the one of Oliveira, who used a straight line element with linear variation of the unknown (i.e. fictitious forces). The method is equally applicable to three dimensions, Clarke and Thompson (1976); however, sophisticated elements (e.g. quadratic variation of unknowns) have not at present

been incorporated in the method. Formulations used to date have written the equations for the midpoint of each segment of fictitious force distribution, thus alleviating the problem of singularities. A more complete description of the method is given in section 2.2.

The fictitious displacement method has been described by Crouch (1976), who called it the displacement discontinuity method. The basic formulation is to find a distribution of fictitious displacements applied at the boundary of the problem in order to achieve the necessary boundary conditions. Crouch used straight line elements with constant unknown over the element, and he wrote the system of equations for the midpoint of the element. The method can be written in matrix notation as:

$$[t] = [U_f] [u_f] \quad \dots (2.1.1.)$$

where t are the applied tractions,

U_f coefficients relating t and u_f ,

u_f fictitious displacements (displacement discontinuities),

$$[u] = [T_f] [u_f] \quad \dots (2.1.2.)$$

where u are the boundary displacements, and

T_f coefficients relating u and u_f .

From the above equations, a relationship between t and u can be formulated.

The reciprocal method, which is described in detail in section 2.3, relates boundary displacements to applied tractions. Mixed problems (i.e. combination of traction and displacement controlled boundary conditions) are most easily analysed using this method, in comparison to the other two. Rizzo (1967) was the first to describe the method applied to elastostatics. Earlier work by Jaswon and Ponter (1963), Jaswon (1963) and Symm (1963) described the method for problems of the potential. Cruse (1969) presented a three dimensional representation of the reciprocal method for elastostatics. Cruse used triangular boundary elements with constant unknown over the element in this attempt; however, he later refined these elements by incorporating linear variation of the unknown (Cruse (1974)). Cruse demonstrated the use of the method for only very simple problems. Lachat and Watson (1975) departed from previous workers, who generally used analytical solutions for their distribution of unknown, by using numerical integration. They incorporated the analogous advantages made in the sophistication of finite elements to boundary elements i.e. isoparametric elements. They found that parabolic shaped elements adequately modelled geometry and that parabolic variation of unknown using Gaussian integration, was in general satisfactory for calculation of the relevant integrals (see section 2.3). They also introduced the concept of subregions, which have a number of advantages; namely, introduction of regions of different elastic parameters and the formation of a banded matrix. The calculation of the free term (which is one half if the boundary is smooth at the load point) and the principal value (which can be calculated directly for cases of distributed load) of a concentrated load can be found implicitly by rigid body motion (Cruse (1974)).

2. Fictitious force (boundary element) method

The fictitious force method, herein after designated as the the boundary element method, used in this study is the one by Bray (1976a). It is at this stage worth while describing this method in detail, since it is called upon in the analysis of a no-tension material (section 2.7) and infinitesimal slip on discontinuities (section 2.8).

The problem posed is to find the stress distribution around an excavation or within a body due to applied tractions. For the purpose of explanation, consider the problem of determining the state of stress around a long excavation of a specified cross-section (Figure 2.2.1) given the field stresses f_x (horizontal) applied at infinity. This problem is the sum of the solutions to (a) the excavation with applied tractions to balance the field stresses, Figure 2.2.2, i.e. a homogeneous stress field within the mass and (b) the excavation with applied tractions on its boundary as shown, Figure 2.2.3, i.e. with zero field stresses. Therefore the essential problem to be solved is that of Figure 2.2.3.

To solve the problem shown in Figure 2.2.3, fictitious forces are distributed within an infinite mass, without any holes, along where the excavation boundary would be if present. These fictitious forces are uniformly distributed over straight line elements, which constitute the imaginary boundary. The procedure is now to adjust these fictitious forces so that the normal and shear components of stress at the centre of each element are equal to the corresponding normal and shear applied tractions. This adjustment of the fictitious forces is carried out by an iterative procedure as follows:- starting with the first element,

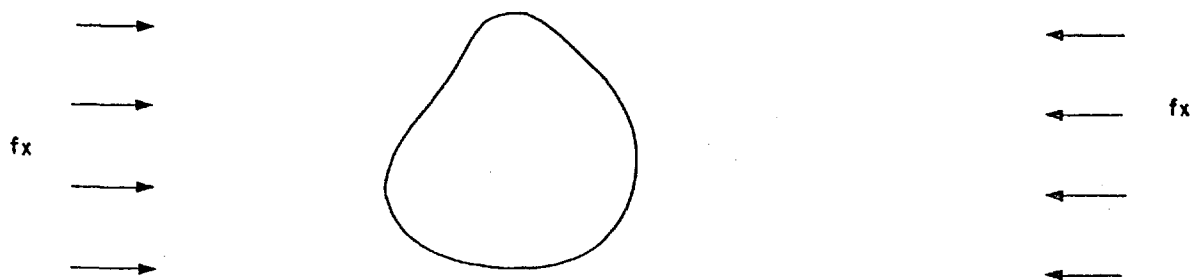


FIGURE 2.2.1 HOLE IN AN INFINITE MAISS WITH HORIZONTAL FIELD STRESS f_x

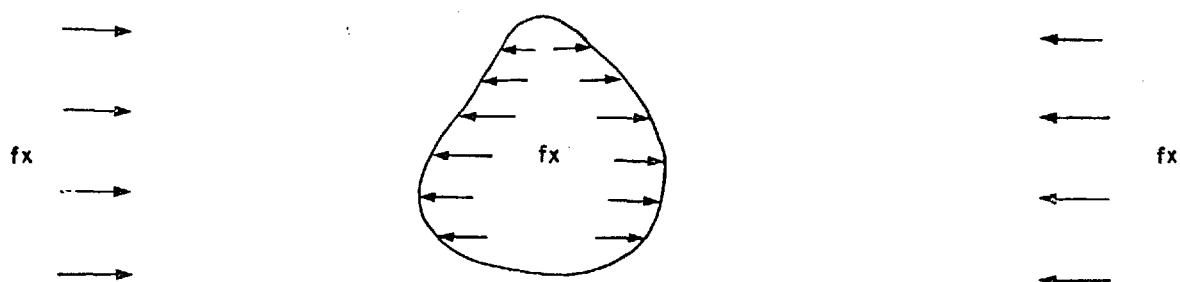


FIGURE 2.2.2 HOMOGENEOUS STRESS FIELD f_x

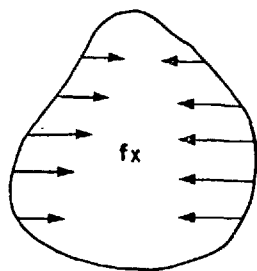


FIGURE 2.2.3 APPLIED TRACTIONS ON BOUNDARY OF HOLE

the normal and shear fictitious forces are set to yield the required result; we then pass on to the remainder of the elements carrying out similar adjustments in turn. In correcting the values of normal and shear components of stress for any given element, we disturb the stresses at all other elements and hence the procedure must be continued for a series of cycles around the "boundary" until no further adjustment is considered necessary.

The equations required for the formulation of the boundary element method are now presented. For the case of a line load applied in an infinite mass, the stresses and displacements for any point are given in Appendix A 2.2.

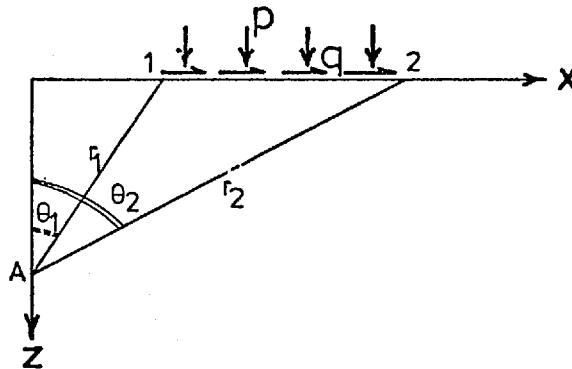


Figure 2.2.4 Stresses and displacements at a point A due to strip fictitious forces - plane strain.

$$\begin{aligned}\sigma_x &= \frac{1}{4\pi(1-\nu)} \left[p \left\{ 2\nu\theta - \frac{xz}{r^2} \right\} + q \left\{ (3-2\nu) \ln r + \frac{z^2}{r^2} \right\} \right]_1^2 \\ \sigma_z &= \frac{1}{4\pi(1-\nu)} \left[p \left\{ 2(1-\nu)\theta + \frac{xz}{r^2} \right\} + q \left\{ -\frac{z^2}{r^2} - (1-2\nu) \ln r \right\} \right]_1^2 \\ \tau_{zx} &= \frac{1}{4\pi(1-\nu)} \left[p \left\{ (1-2\nu) \ln r - \frac{z^2}{r^2} \right\} + q \left\{ 2(1-\nu)\theta - \frac{xz}{r^2} \right\} \right]_1^2 \quad \dots (2.2.1.)\end{aligned}$$

$$u_x = \frac{1}{8\pi(1-\nu)G} \left[p \left\{ z \ln r \right\} - q \left\{ (3-4\nu) (x \ln r - \dot{x} + z\theta) + z\theta \right\} \right]_1^2$$

$$u_z = \frac{1}{8\pi(1-\nu)G} \left[q \left\{ z \ln r \right\} - p \left\{ (3-4\nu) (x \ln r - x + z\theta) + x - z\theta \right\} \right]_1^2$$

The problem can be written for an element i as:-

$$t_{n_i} = \sum_{j=1}^n T_{nn_{ij}} p_j + T_{nm_{ij}} q_j \quad \dots (2.2.2.)$$

$$t_{m_i} = \sum_{j=1}^n T_{mn_{ij}} p_j + T_{mm_{ij}} q_j$$

where j is an element from 1 to n elements, including i ,

p and q are the normal (n direction) and shear (m direction) fictitious forces,

T_{nm} is the normal component of stress due to a shear fictitious force q ,

T_{nn} , T_{mn} , and T_{mm} in similar notation, and t_n , t_m are the normal and shear applied tractions respectively.

Thus the problem can be given the following matrix identity:

$$\begin{bmatrix} T_{nn} & T_{nm} \\ T_{mn} & T_{mm} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} t_n \\ t_m \end{bmatrix} \quad \dots (2.2.3.)$$

Therefore by inversion or using an iterative scheme the fictitious forces can be found. Once the fictitious forces have been calculated, then the stresses parallel to the boundary can be computed by summing the contributions of the various fictitious forces, equation (2.2.4.). The stress parallel to the boundary at the midpoint of element i is:

$$\sigma_{m_i} = \sum_{j=1}^n D_{mn_{ij}} p_j + D_{mm_{ij}} q_j \quad \dots (2.2.4.)$$

where σ_{m_i} is the stress parallel (m direction) to the boundary at the midpoint of element i .

j is an element from 1 to n elements, including i ,

and D_{mn} , D_{mm} are the normal components (m direction) of stress due to p and q respectively. Similarly stresses can be computed for any internal point, k . In similar notation as above, we have:

$$\sigma_{x_k} = \sum_{j=1}^n D_{xn_{kj}} p_j + D_{xm_{kj}} q_j$$

$$\sigma_{z_k} = \sum_{j=1}^n D_{zn_{kj}} p_j + D_{zm_{kj}} q_j \quad \dots (2.2.5.)$$

$$\tau_{zx_k} = \sum_{j=1}^n D_{zxn_{kj}} p_j + D_{zxm_{kj}} q_j$$

where σ_x , σ_z and τ_{zx} are the stress components at the point,

D_{zxn} , D_{zxm} are the shear components of stress due to p and q respectively,

and D_{xn} , D_{zn} are the normal components (x and z directions) of stress due to p , likewise D_{xm} and D_{zm} for q .

Boundary displacements or displacements at interior points can be found by summing the effects of the fictitious forces (Hocking (1976a)). For example at a point k , we have:

$$u_{x_k} = \sum_{j=1}^n C_{xn_{kj}} p_j + C_{xm_{kj}} q_j \quad \dots (2.2.6.)$$

$$u_{z_k} = \sum_{j=1}^n C_{zn_{kj}} p_j + C_{zm_{kj}} q_j$$

where C_{xn} and C_{xm} are the x components of displacement due to fictitious forces p and q respectively, and similarly the z components C_{zn} and C_{zm} .

If the problem possesses symmetry with respect to both geometry and loading, then there is a reduction in the number of equations required to describe the problem. However, the number of coefficients of $[T]$, see equation (2.2.3.), needed to be computed will not change (in other words, the system of equations must be integrated around the entire boundary irrespective of symmetry). In general, the iteration procedure converges rapidly, however, for strongly coupled problems, e.g. long thin excavations, inversion of $[T]$ in equation (2.2.3.) may be necessary.

3. Reciprocal (boundary integral) method:

The reciprocal method, here after designated as the boundary integral method, used in this study is the one described by Rizzo (1967). The problem posed is the linear elastostatic one, which is the determination of stresses and/or displacements at any point in or on the body due to imposed boundary conditions (whether traction, displacement or mixed).

Consider the finite, homogeneous, isotropic body of surface S , Figure 2.3.1, subjected to applied tractions and displacements, t and u respectively. In the same body, but with t and u removed, Figure 2.3.2, apply a unit force in the x direction at the interior point i . The tractions (T_{xx} , T_{xy} , T_{xz}) and displacements (U_{xx} , U_{xy} , U_{xz}) at a point j on the boundary of the body, due to the unit force at i , can be calculated from Kelvin's solution (Love (1945)). By applying Betti's reciprocal work theorem to the cases shown in Figures 2.3.1 and 2.3.2, the following is obtained:

$$u_{x_i} + \int (T_{xx}u_x + T_{xy}u_y + T_{xz}u_z)ds = \int (U_{xx}t_x + U_{xy}t_y + U_{xz}t_z)ds \quad \dots (2.3.1.)$$

The above formula is known as Somigliana's integrals (Love (1945)).

where U_{xx} , U_{xy} , U_{xz} are the displacement at a point in the x , y and z directions respectively, due to the point load applied at i in the x direction, (calculated from Kelvin's solution) T_{xx} , T_{xy} , T_{xz} are the tractions (from Kelvin's solution) in the x , y and z direction at a point, due to the point load applied at i in the x direction.

u_x , u_y , and u_z are the boundary displacements in the x , y and z directions respectively. t_x , t_y and t_z are the applied tractions in the x , y and z directions, and u_{x_i} is the x component of displacement in the loaded body, Figure 2.3.1, at the interior point i .

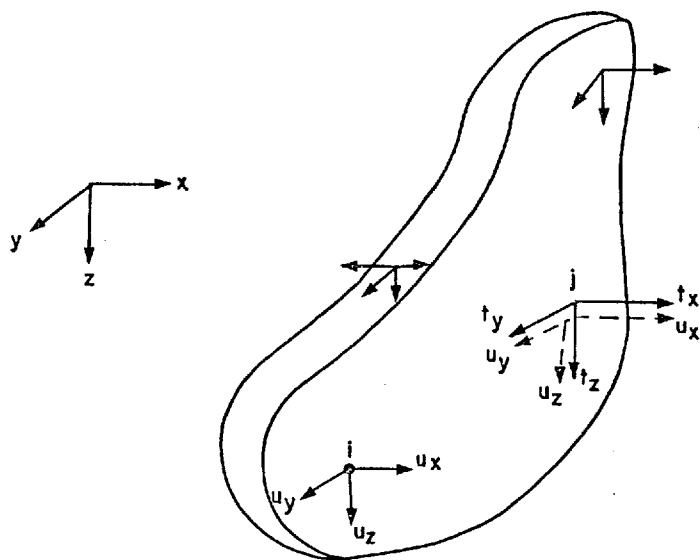


FIGURE 2.3.1 APPLIED TRACTIONS AND DISPLACEMENTS ON BOUNDARY OF BODY OF SURFACE S , WITH ASSOCIATED DISPLACEMENTS AT INTERIOR POINT i

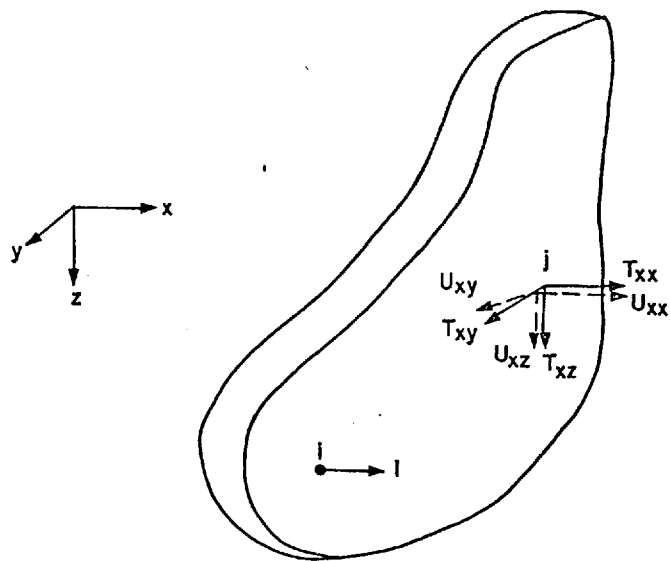


FIGURE 2.3.2 UNIT FORCE IN x DIRECTION APPLIED AT INTERIOR POINT, WITH ASSOCIATED TRACTIONS AND DISPLACEMENTS ON THE BOUNDARY

Equation (2.3.1.) is simply, the tractions of one system times the displacement of a second, equals the tractions of the second system times the displacements of the first.

Now the essence of the boundary integral method is to move the point i to the boundary. If the boundary is smooth (continuous tangent at the point) at this point, then the unit force will be equally divided between the interior and exterior regions. Then equation 2.3.1. becomes:

$$\frac{1}{2}u_{x_i} + \int (T_{xx}u_x + T_{xy}u_y + T_{xz}u_z) dS = \int (U_{xx}t_x + U_{xy}t_y + U_{xz}t_z) dS \quad \dots (2.3.2.)$$

Equation (2.3.2.) is a boundary constraint equation, i.e. it relates tractions to boundary displacements. By applying unit point forces in the y and z directions at point i on the boundary in turn, and then in succession to all other points on the boundary, the complete boundary constraint equation can be formulated. For example, in matrix notation:

$$\begin{bmatrix} T_{xx_{11}} & T_{xy_{11}} & \cdot & \cdot & \cdot & \cdot \\ T_{yx_{11}} & T_{yy_{11}} & \cdot & \cdot & \cdot & \cdot \\ T_{zx_{11}} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & T_{zy_{nn}} & T_{zz_{nn}} \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{y_1} \\ \vdots \\ u_{z_n} \end{bmatrix} = \begin{bmatrix} U_{xx_{11}} & U_{xy_{11}} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & U_{zz_{nn}} & \cdot \end{bmatrix} \begin{bmatrix} t_{x_1} \\ t_{y_1} \\ \vdots \\ t_{z_n} \end{bmatrix} \quad \dots (2.3.3.)$$

Thus all the knowns are assembled on the right hand side and all the unknowns on the left hand side. The system of equations can now be reduced, either by inversion or by an iterative procedure.

As mentioned in section 2.2, any symmetry the problem possesses will reduce the number of equations required to describe the problem, thus saving computational effort.

For the case of multi-connected bodies (finite or infinite mass with holes), identity (2.3.3.) still holds, however for proof of this, the body must be cut into simply connected regions (Rizzo (1967)). For the exterior problem (i.e. infinite mass), the identity (2.3.4.) is applicable, provided that $[T][u]$ tends to zero as infinity is approached.

4. Numerical formulation of the plane problem

A two dimensional boundary integral program was written for solving problems in elastostatics. The boundary of the body was divided up into straight line segments and the functions U and T were taken to be constant over each element. The stresses and displacements at any point for strip normal and shear forces are given by the equations (2.2.1.); thus the matrices $[T]$ and $[U]$ can be constructed. The system of equations is written for the midpoints of the elements, thus avoiding singularities at the ends of the elements. To aid explanation, consider the problem shown in Figure 2.4.1 in which the problem is represented by only two (2) boundary elements.

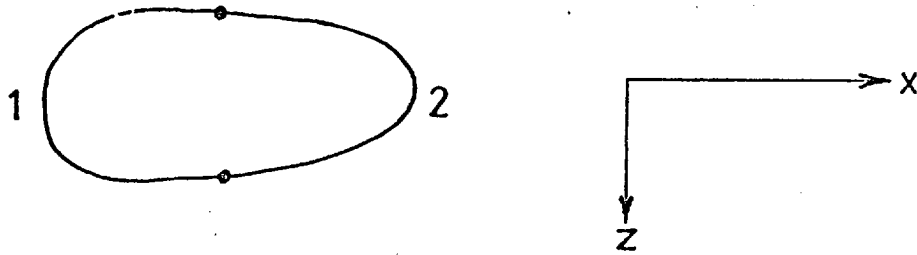


Figure 2.4.1 Problem represented by two boundary elements.

Then in the same notation as used in section 2.3, the basic equations for the problem given in Figure 2.4.1 are:

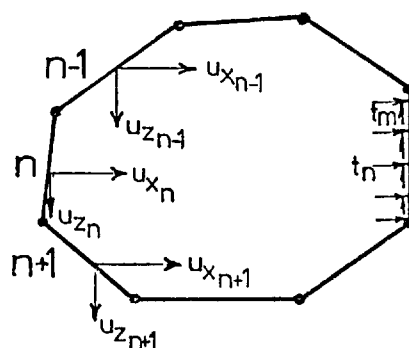
$$\begin{bmatrix} T_{xx_{11}} & T_{xz_{11}} & T_{xx_{12}} & T_{xz_{12}} \\ T_{zx_{11}} & T_{zz_{11}} & T_{zx_{12}} & T_{zz_{12}} \\ T_{xx_{21}} & T_{xz_{21}} & T_{xx_{22}} & T_{xz_{22}} \\ T_{zx_{21}} & T_{zz_{21}} & T_{zx_{22}} & T_{zz_{22}} \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{z_1} \\ u_{x_2} \\ u_{z_2} \end{bmatrix} = \begin{bmatrix} U_{xx_{11}} & U_{xz_{11}} & U_{xx_{12}} & U_{xz_{12}} \\ U_{zx_{11}} & U_{zz_{11}} & U_{zx_{12}} & U_{zz_{12}} \\ U_{xx_{21}} & U_{xz_{21}} & U_{xx_{22}} & U_{xz_{22}} \\ U_{zx_{21}} & U_{zz_{21}} & U_{zx_{22}} & U_{zz_{22}} \end{bmatrix} \begin{bmatrix} t_{x_1} \\ t_{z_1} \\ t_{x_2} \\ t_{z_2} \end{bmatrix}$$

... (2.4.1.)

When the identity (2.4.1.) is being assembled, the coefficients are scaled to obtain a non-dimensional and thus numerically stable system (Lachat and Watson (1975)). The coefficients of $[U]$ are multiplied by the Young's modulus (E) and divided by the maximum distance between two elements ($r_{\max}$); given values of displacement are divided by $r_{\max}$ and given values of traction are divided by E . The unknowns are arranged to be placed in $[u]$, the knowns in $[t]$ and the columns of $[U]$ and $[T]$ are interchanged where necessary, with due account being taken of signs (see section 2.3 for the formulation of the mixed problem). As the system of equations is to be solved by inversion, then they are written in global coordinates as suggested in the matrix identity, equation (2.4.1.) The right hand sides, i.e. $[U]$ $[t]$ are now multiplied and held in core, while the matrix $[T]$ is read to a mass storage random access file for eventual reduction by a block solver (Lachat (1975)). After solution the calculated values of u and t are multiplied by $r_{\max}$ and E respectively.

For the exterior problem (i.e. infinite mass with holes) with stress fields applied at infinity, the problem is subdivided into two parts, (a) and (b) as mentioned in section 2.2, i.e. find the solution of (b) with eventual superposition of (a).

Upon solution, boundary stresses are calculated from surface traction and tangential strain (Cruse (1974)). For example say the respective boundary displacements and tractions are known for three elements ($n-1$, n and $n+1$), Figure 2.4.2.



t_n - normal traction on element.
 t_m - shear traction on element.

Figure 2.4.2 After solution known boundary displacements and tractions.

Simply by averaging the displacements of neighbouring elements ($n + 1$ and $n - 1$) towards or away from the element in question (element n) then the tangential strain (ϵ_m) can be found. In the case of plane strain, the tangential stress (σ_m) is given by:-

$$\sigma_m = \frac{E}{(1-\nu^2)} \epsilon_m + \frac{\nu}{(1-\nu)} t_n \quad \dots (2.4.2.)$$

where σ_m is the tangential stress

t_n is the normal traction

ϵ_m is the tangential strain

and E and ν are Young's modulus and Poisson's ratio respectively.

Stresses and displacements inside the body are calculated as discussed in section 2.6.

The program has a number of generation routines for the construction of boundary elements, in order to reduce input data and errors in construction of the problem. Also any symmetry the problem possesses is taken into account in order to reduce the system of equations required to describe the problem (Bray (1976a)). For example, suppose the problem to be solved has symmetry with respect to both co-ordinate axes, Figure 2.4.3, and is represented by only two boundary elements.

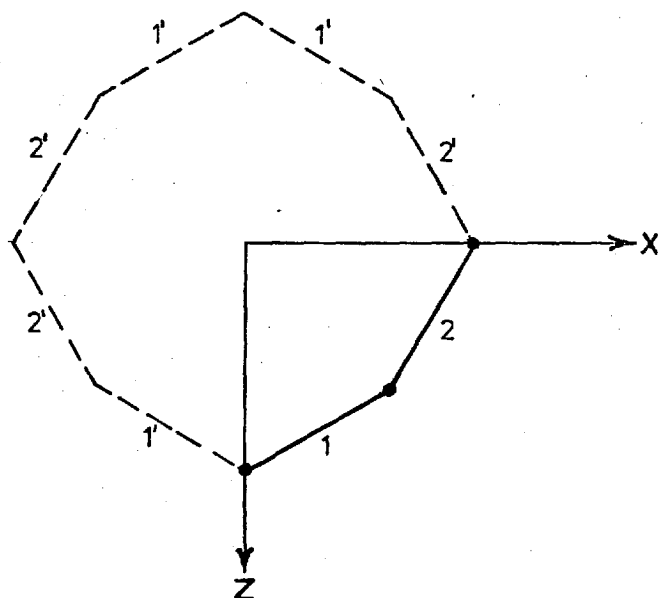


Figure 2.4.3 Problem with symmetry with respect to the x and z coordinate axes.

The problem, Figure 2.4.3, represented by two boundary elements (1 and 2) is shown with its entire surface, which include mirror images of elements 1 and 2 (shown 1', 2' etc.) First element 1 is loaded in the normal and shear directions and response at the midpoint of elements 1 and 2 noted. Then the mirror images of element 1 (shown 1') are loaded in turn, and the response at the midpoints of elements 1 and 2 noted. Finally the remainder of boundary elements, designated as 2 and 2' are loaded and response recorded. Thus the procedure is to integrate around the entire boundary, but lump all effects on only elements 1 and 2 for the problem given in Figure 2.4.3.

5. Subregions

Lachat and Watson (1975) describe the use of subregions in order to reduce the size of the system of equations to be constructed and also handle non-homogeneous problems.

For the sake of explanation, consider the simple problem shown in Figure 2.5.1. This problem consists of only two subregions designated (1) and (2), and is represented by three elements, 1, 2 and 3. Element 2 is shared by both subregions (1) and (2).

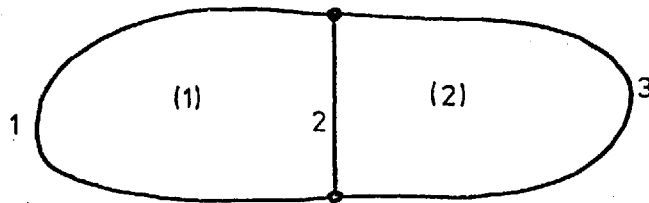


Figure 2.5.1 Finite body subdivided into two subregions represented by three boundary elements.

As shown above for a problem with subregions, elements must be defined along the boundary of the subregions, i.e. around the boundary of the body and in the body to delineate the subregions. To construct the system of equations, first a boundary constraint equation, i.e. equation (2.3.3.) is written for each subregion in turn. For the problem shown in Figure 2.5.1, consider each subregion separately as illustrated in Figure 2.5.2.

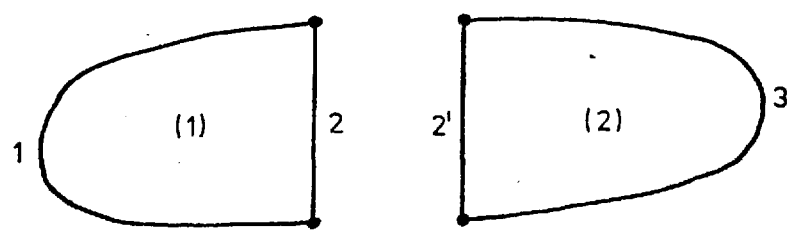


Figure 2.5.2 Subregions (1) and (2) considered separately.

Then along the boundary between the two subregions, designate 2 and 2' as the elements, which lie in subregion (1) and (2) respectively. The boundary constraint equation can now be written for each subregion:

$$\begin{bmatrix} T_{xx_{11}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{z_1} \\ u_{x_2} \\ u_{z_2} \\ u_{x_{2^1}} \\ u_{z_{2^1}} \\ u_{x_3} \\ u_{z_3} \end{bmatrix} =$$

$$\begin{bmatrix} U_{xx_{11}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} t_{x_1} \\ t_{z_1} \\ t_{x_2} \\ t_{z_2} \\ t_{x_{2^1}} \\ t_{z_{2^1}} \\ t_{x_3} \\ t_{z_3} \end{bmatrix}$$

... (2.5.1.)

Between subregions (1) and (2), i.e. along element 2 in this case, the following conditions apply

$$\begin{aligned} t_{x_2} &= -t_{x_2^1} \\ t_{z_2} &= -t_{z_2^1} \end{aligned} \quad \dots (2.5.2.)$$

and

$$\begin{aligned} u_{x_2} &= u_{x_2^1} \\ u_{z_2} &= u_{z_2^1} \end{aligned} \quad \dots (2.5.3.)$$

where t_x , t_z are the tractions on an element of subscript given and u_x , u_z are the boundary displacements.

For elements which are shared by two subregions, in this case element 2, the tractions t and displacements u for these elements are unknown. Therefore by substituting all unknowns on element 2 for the unknowns on element 2 into equation (2.5.1.), the system of equations can be rewritten (adding and subtracting the relevant equations) to solve for both u and t on element 2.

Consider the problem given in Figure 2.5.1, and undertake the relevant manipulation for placing unknowns on the left hand side and knowns on the right. The resultant equations are of the following form:-

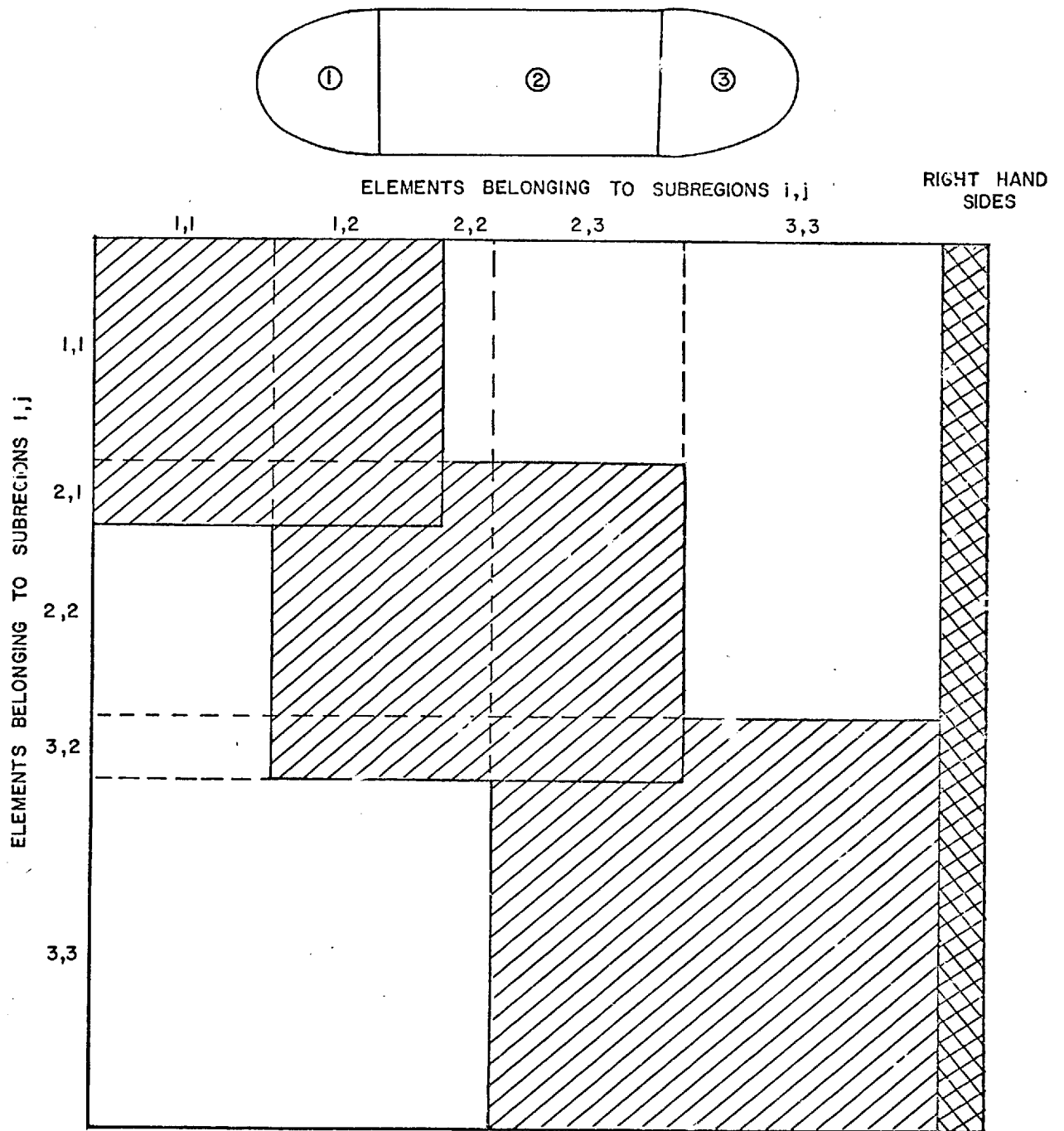


FIGURE 2.5.3 CONSTRUCTION OF $[T]$ FOR SUBREGION PROBLEM SHOWN ABOVE

6. Calculation of stresses and displacements at interior points.

Consider the loaded homogeneous, isotropic finite body shown in Figure 2.6.1, in which the interior point p has displacement components u_x^p and u_z^p due to the applied tractions (t_x and t_z) and displacements (u_x and u_z) acting on the boundary of the body. In the same body, but unloaded, Figure 2.6.2, apply a unit force in the x direction at p and calculate the relevant tractions (T_{xx} and T_{xz}) and displacements (U_{xx} and U_{xz}) for points on the boundary from equation (2.6.1) to maintain the body in equilibrium. These tractions (T_{xx} and T_{xz}) and displacements (U_{xx} and U_{xz}) can be found from the solution of a line load in an infinite mass (Love (1945)), which is given below, equation (2.6.1.).

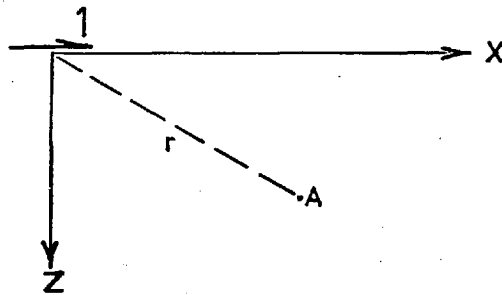


Figure 2.6.3 Stresses and displacements at a point A , due to a unit line load in an infinite mass - plane strain case.

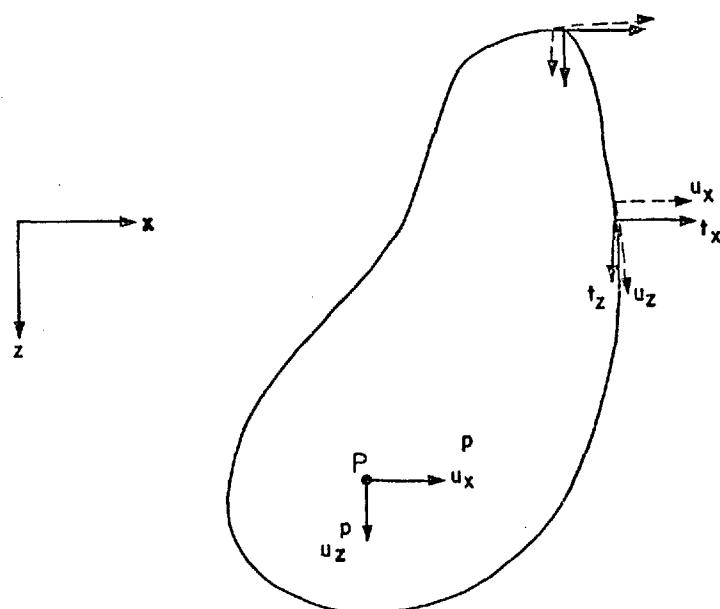


FIGURE 2.6.1 LOADED BODY WITH APPLIED TRACTIONS AND DISPLACEMENTS ON THE BOUNDARY, WITH ASSOCIATED DISPLACEMENTS AT INTERIOR POINT P

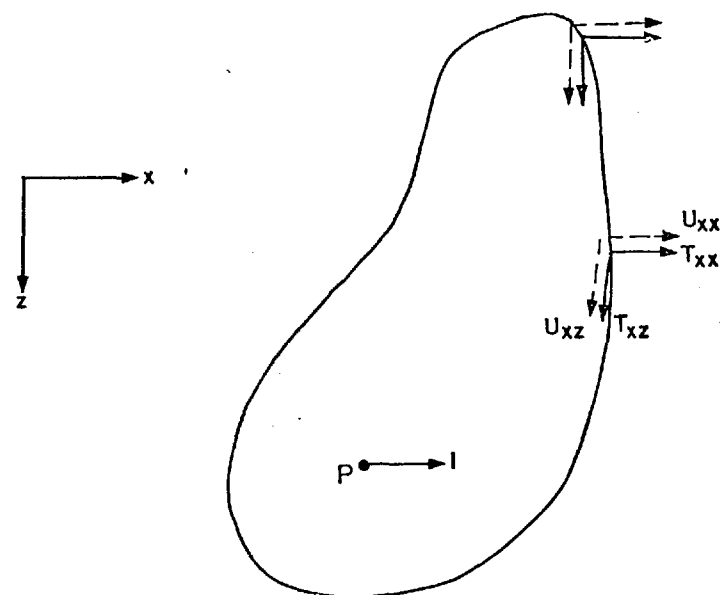


FIGURE 2.6.2 UNIT FORCE APPLIED IN x DIRECTION AT INTERIOR POINT P , WITH ASSOCIATED TRACTIONS AND DISPLACEMENTS ON THE BOUNDARY

$$\begin{aligned}
u_x &= \frac{-1}{8\pi(1-\nu)G} \left\{ (3-4\nu) \ln r + \frac{z^2}{r^2} \right\} \\
u_z &= \frac{1}{8\pi(1-\nu)G} \left\{ \frac{xz}{r^2} \right\} \\
\sigma_x &= \frac{1}{4\pi(1-\nu)} \left\{ (3-2\nu) \frac{x}{r^2} - \frac{2xz^2}{r^4} \right\} \quad \dots (2.6.1.) \\
\sigma_z &= \frac{1}{4\pi(1-\nu)} \left\{ \frac{2xz^2}{r^4} - (1-2\nu) \frac{x}{r^2} \right\} \\
\tau_{zx} &= \frac{1}{4\pi(1-\nu)} \left\{ (1-2\nu) \frac{z}{r^2} + \frac{2zx^2}{r^4} \right\}
\end{aligned}$$

By applying Betti's reciprocal work Theorem (similar to section 2.3) for the cases shown in Figures 2.6.1 and 2.6.2, we obtain:

$$u_x^p + \int (T_{xx}u_x + T_{xz}u_z) dS = \int (U_{xx}t_x + U_{xz}t_z) dS \quad \dots (2.6.2.)$$

where U_{xx} , U_{xz} and T_{xx} , T_{xz} are the displacements and tractions respectively, on the boundary of the body due to the unit line load at p, see Figure 2.6.2. The first subscript refers to the direction of the unit force applied at p, and the second to the direction of the vectors U and T. And u_x , u_z and t_x , t_z are the displacements and tractions respectively on the boundary of the loaded body, Figure 2.6.1; likewise u_z^p can be found.

To obtain stresses at p, we must first calculate the strains at p, i.e.

$$\epsilon_{xx} = \frac{\partial u_x^p}{\partial x} \quad \dots (2.6.3.)$$

$$\epsilon_{zz} = \frac{\partial u_z^p}{\partial z}$$

$$\text{and } \epsilon_{zx} = \frac{\partial u_x^p}{\partial z} + \frac{\partial u_z^p}{\partial x}$$

and then from Hooke's law deduce the stresses. The partial differentials of the left hand sides in equation (2.6.1.) with respect to x and z are given in Appendix A 2.6.

To simplify notation, equate the following:-

$$\begin{aligned} V &= \frac{\partial U}{\partial x} & W &= \frac{\partial U}{\partial z} \\ S &= \frac{\partial T}{\partial x} & R &= \frac{\partial T}{\partial z} \end{aligned} \quad \dots (2.6.4.)$$

Then the displacements and strains at the interior point p are:

$$\begin{aligned} u_x^p &= \int (U_{xx} t_x + U_{xz} t_z) dS - \int (T_{xx} u_x + T_{xz} u_z) dS \\ u_z^p &= \int (U_{zx} t_x + U_{zz} t_z) dS - \int (T_{zx} u_x + T_{zz} u_z) dS \\ \epsilon_{xx}^p &= \int (V_{xx} t_x + V_{xz} t_z) dS - \int (S_{xx} u_x + S_{xz} u_z) dS \\ \epsilon_{zz}^p &= \int (W_{zx} t_x + W_{zz} t_z) dS - \int (R_{zx} u_x + R_{zz} u_z) dS \\ \epsilon_{zx}^p &= \int ((W_{xx} + V_{zx}) t_x + (W_{xz} + V_{zz}) t_z) dS \\ &\quad - \int ((R_{xx} + S_{zx}) u_x + (R_{xz} + S_{zz}) u_z) dS \end{aligned} \quad \dots (2.6.5.)$$

Therefore by simply integrating the relevant terms around the boundary the strains and displacements at an interior point can be calculated. For the case of plane strain, the stresses can be found from the strains by the following equations:

$$\gamma_{zx}^p = \frac{1}{2} \epsilon_{zx}^p \quad \dots (2.6.6.)$$

$$\sigma_z = \frac{(1 - \nu) E}{(1 + \nu)(1 - 2\nu)} \left\{ \frac{\nu}{1 - \nu} \epsilon_x + \epsilon_z \right\}$$

$$\sigma_x = \frac{(1 - \nu) E}{(1 + \nu)(1 - 2\nu)} \left\{ \frac{\nu}{1 - \nu} \epsilon_z + \epsilon_x \right\} \quad \dots (2.6.7.)$$

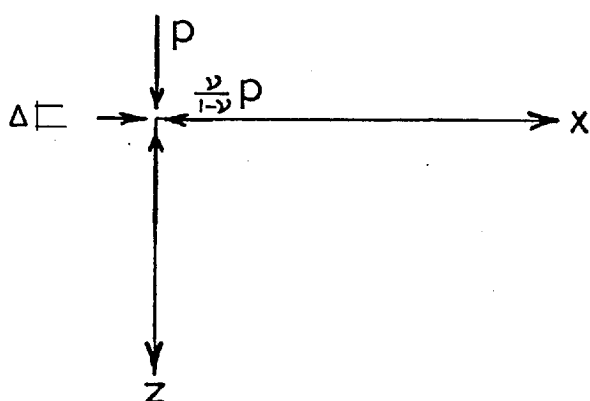
$$\tau_{zx} = G\gamma_{zx} \quad -$$

7. No-tension analysis

First the problem in question is solved in the normal manner, i.e. either by the boundary element (section 2.2) or the boundary integral (section 2.3) method. In this section only the boundary element method will be considered, because due to its iterative procedure it is by far the more easily adaptable for the no-tension analysis. After the problem has been solved for the linear elastic case, then a search is conducted of boundary elements to determine whether tensile stresses are present around the boundary. If so, then at the boundary element in question, crack elements (to be described shortly) are inserted in the direction of the maximum principal stress at points inside the mass, to relieve any tensile stresses present. The effect of relieving these tensile stresses on all boundary elements is recorded. The procedure of relieving tensile stresses at points inside the mass is continued for all other boundary elements exhibiting a low tangential stress. Throughout the entire procedure the accumulated sum of the effect of relief on all boundary elements is recorded. The fictitious forces (see section 2.2) are now readjusted to remove the normal and shear tractions present on boundary elements, due to tensile stress relief.

The search for tensile stresses is again carried out, and the entire relief procedure repeated until tensile stresses have been reduced to zero.

The type of crack element used in the tensile stress relief was suggested by Bray (1976b). The crack element consists of a "compensated" dipole, Figure 2.7.1, integrated over the length of the element in an approximately elliptical distribution as suggested by Brady (1977). The solutions of the unit compensated dipole, Figure 2.7.1, and the constant compensated dipole are given in Appendix A 2.7.



Strength of dipole
= $p\Delta = 1$

Figure 2.7.1 Unit compensated dipole

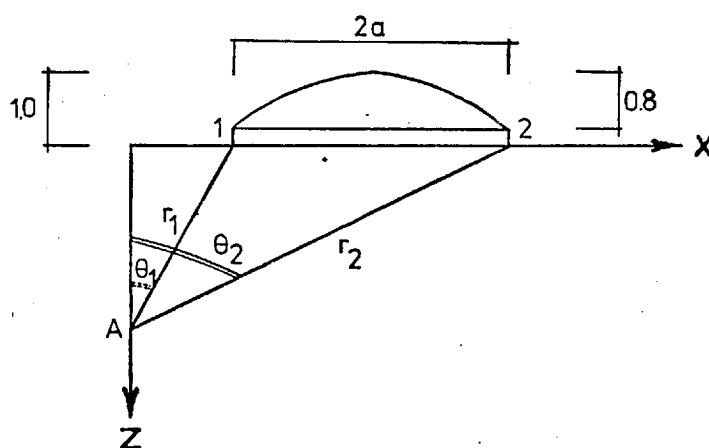


Figure 2.7.2 Approximate elliptical distribution of the compensated dipole over the element - plane strain.

The equations relating to the above distribution of the compensated dipole, Figure 2.7.2, for a point A are:

$$\begin{aligned}
 \sigma_z &= \frac{-(1-2\nu)}{4\pi(1-\nu)^2} \left\{ \left[\frac{x}{r^2} + \frac{2z^2x}{r^4} \right] + \frac{0.8}{a^2} \left\{ (x_1-a)^2 \left(\frac{-x}{r^2} - \frac{2z^2x}{r^4} \right) \right. \right. \\
 &\quad \left. \left. - 2(x_1-a) \left(\frac{z^2}{2r^2} + \frac{x^2}{2r^2} + \ln r - \frac{2x^2z^2}{r^4} \right) + x - \frac{3z^2x}{r^2} + \frac{2z^4x}{r^4} \right\} \right\}_1^2 \\
 \sigma_x &= \frac{-(1-2\nu)}{4\pi(1-\nu)^2} \left\{ \left[\frac{x}{r^2} - \frac{2z^2x}{r^4} \right] + \frac{0.8}{a^2} \left\{ (x_1-a)^2 \left(\frac{-x}{r^2} + \frac{2z^2x}{r^4} \right) \right. \right. \\
 &\quad \left. \left. - 2(x_1-a) \left(\frac{z^2}{2r^2} - \frac{3x^2}{2r^2} + \ln r + \frac{2x^2z^2}{r^4} \right) + x - 4z\theta + \frac{5z^2x}{r^2} \right. \right. \\
 &\quad \left. \left. - \frac{2z^4x}{r^4} \right\} \right\}_1^2 \\
 \tau_{zx} &= \frac{-(1-2\nu)}{4\pi(1-\nu)^2} \left\{ \left[\frac{z}{r^2} - \frac{2z^3}{r^4} \right] + \frac{0.8}{a^2} \left\{ (x_1-a)^2 \left(\frac{-z}{r^2} + \frac{2z^3}{r^4} \right) \right. \right. \\
 &\quad \left. \left. - 2(x_1-a) \left(\frac{-2xz}{r^2} + \frac{2xz^3}{r^4} \right) + 2z \ln r + \frac{z^3}{r^2} - \frac{2x^4z}{r^4} \right\} \right\}_1^2 \quad \dots(2.7.1.) \\
 u_x &= \frac{(1-2\nu)}{8\pi(1-\nu)^2 G} \left\{ \left[(1-2\nu) \ln r + \frac{z^2}{r^2} \right] - \frac{0.8}{a^2} \left\{ (x_1-a)^2 \left((1-2\nu) \ln r + \frac{z^2}{r^2} \right) \right. \right. \\
 &\quad \left. \left. - 2(x_1-a) \left((1-2\nu) (x-z\theta) - \frac{x^3}{r^2} \right) + \frac{(1-2\nu)}{2} (x^2 + z^2 \ln r) + \frac{x^2 z^2}{r^2} \right. \right. \\
 &\quad \left. \left. - 2z^2 \ln r \right\} \right\}_1^2 \\
 u_z &= \frac{(1-2\nu)}{8\pi(1-\nu)^2 G} \left\{ \left[(1-2\nu) \theta - \frac{z^2}{r^2} \right] - \frac{0.8}{a^2} \left\{ (x_1-a)^2 \left((1-2\nu) \theta - \frac{z^2}{r^2} \right) \right. \right. \\
 &\quad \left. \left. - 2(x_1-a) \left((1-2\nu) z \ln r - \frac{z^2x}{r^2} + z\theta \right) + (1-2\nu) z (x-z\theta) \right. \right. \\
 &\quad \left. \left. - \frac{x^2 z^2}{r^2} + 2z^2 \ln r \right\} \right\}_1^2
 \end{aligned}$$

The reason for using such a distribution as shown in Figure 2.7.2, is to achieve good agreement with the analytical solution (Muskhelishvili (1953)). The close agreement is illustrated in Figure 2.7.3 where the stress distributions for the approximately elliptical and constant distribution of the unit compensated dipole are compared with the analytical solution.

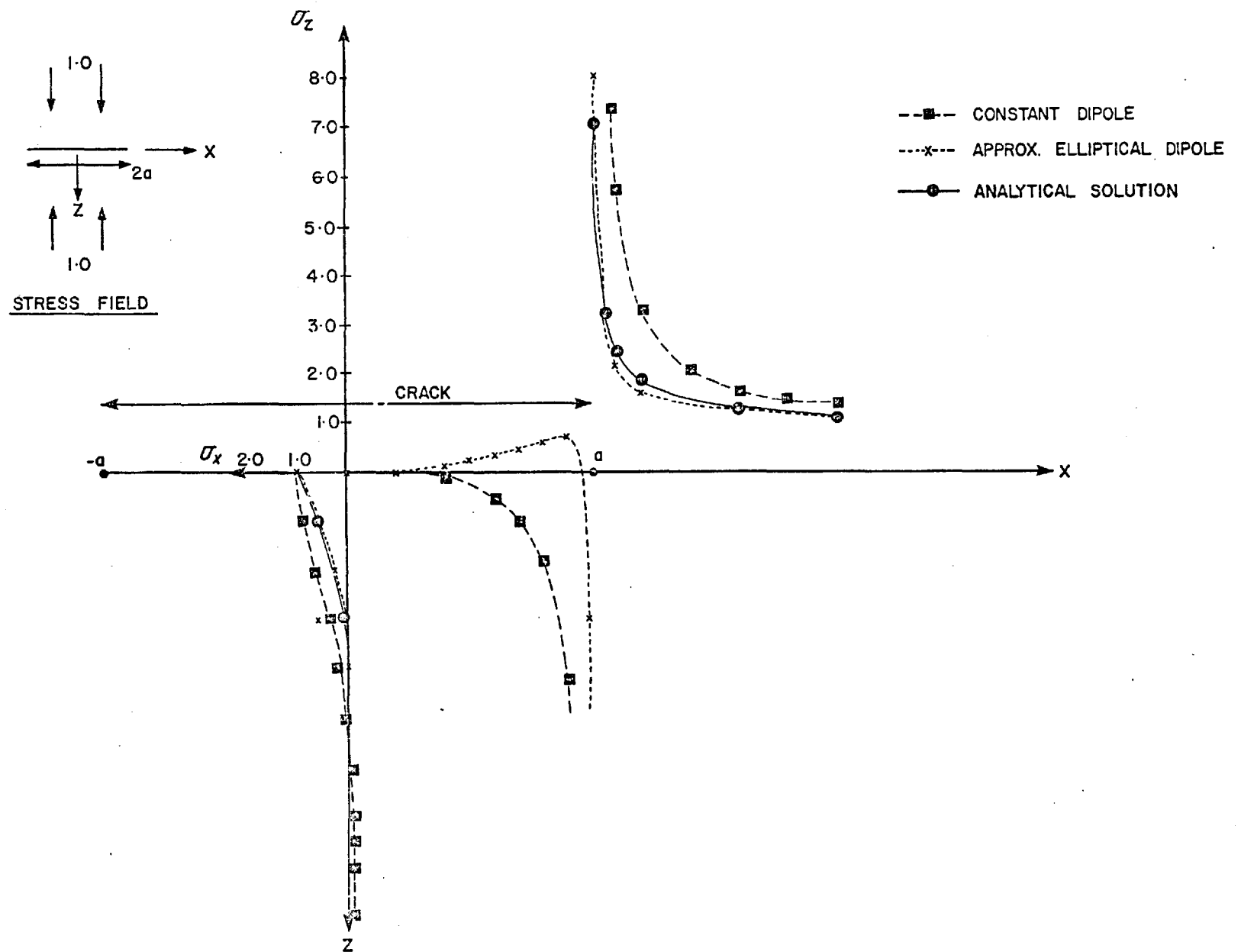


FIGURE 2.7.3 STRESS DISTRIBUTION ALONG THE X AND Z AXES NEAR A CRACK IN A UNIT FIELD STRESS P_z

8. Infinitesimal slip on discontinuities

Initially the elastic analysis, with no slip, is computed using either the boundary element (section 2.2) or the boundary integral (section 2.3) method. Only the boundary element method will be considered here, for the same reason stated in section 2.7, i.e. the iterative procedure allows easy adaptation of the program to this particular problem.

After solution of the elastic analysis, a search is conducted along the discontinuity for the possibility of slip along or separation of the discontinuity. If the normal stress across the discontinuity is tensile(i.e. separation) then this tensile stress is relieved as described in section 2.7. On the other hand if the shear component of stress along the discontinuity exceeds the shear strength, then the excess shear stress is relieved by a "shear" element (to be described shortly). The effect of relieving this shear stress on all boundary elements is recorded, and the procedure of relieving excess shear stresses is continued for all other points on the discontinuity. Throughout the entire procedure the accumulated sum of the effect of relief on all boundary elements is stored. The fictitious forces (section 2.2) are now adjusted and the entire procedure repeated (similar to section 2.7) until the discontinuity is stable.

Slip in this analysis is of the order of elastic displacements, i.e. does not imply major movement, but simply a redistribution of stresses.

The "shear" element consists of the unit solution illustrated in Figure 2.8.1, integrated over the length of the crack in an approximately elliptical distribution. The solutions for the unit shear dipole, Figure 2.8.1, and constant distributed shear dipole are given in Appendix A 2.8.

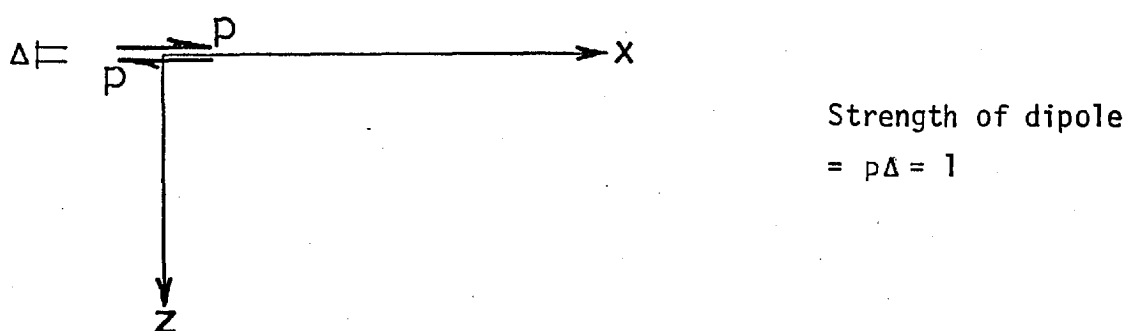


Figure 2.8.1 Unit shear dipole

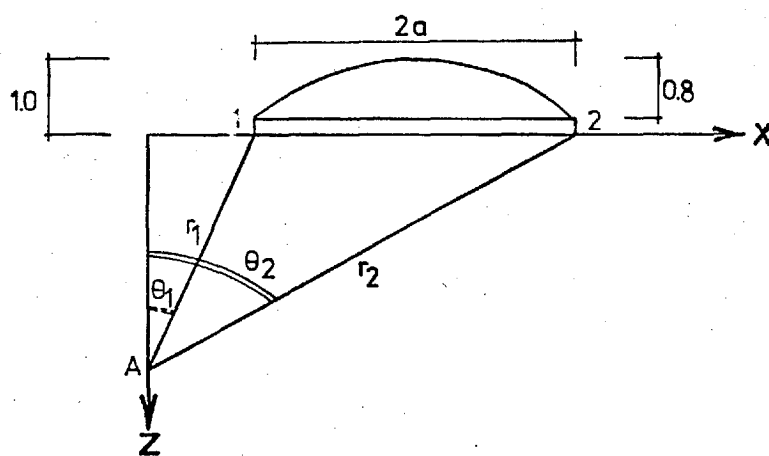


Figure 2.8.2 Approximate elliptical distribution of unit shear dipole over element - plane strain.

The equations for the stresses and displacements at a point A, due to the above distribution, Figure 2.8.2, of a unit shear dipole are:

$$\begin{aligned}
 \sigma_z &= \frac{-1}{4\pi(1-\nu)} \left[\left\{ \frac{-(3-2\nu)z}{r^2} + \frac{2z^3}{r^4} \right\} - \frac{0.8}{a^2} \left\{ (x_1-a)^2 \left(\frac{-(3-2\nu)z}{r^2} + \frac{2z^3}{r^4} \right) \right. \right. \\
 &\quad \left. \left. - 2(x_1-a) \left(\frac{2(1-\nu)\theta - 2(2-\nu)xz}{r^2} + \frac{2xz^3}{r^4} \right) + 2(3-2\nu)z \ln r \right. \right. \\
 &\quad \left. \left. + \frac{z^3}{r^2} + \frac{2x^2z^3}{r^4} - 2(2-\nu) \frac{x^2z}{r^2} \right\} \right]_1^2 \\
 \sigma_x &= \frac{-1}{4\pi(1-\nu)} \left[\left\{ \frac{(5-2\nu)z}{r^2} - \frac{2z^3}{r^4} \right\} - \frac{0.8}{a^2} \left\{ (x_1-a)^2 \left(\frac{(5-2\nu)z}{r^2} - \frac{2z^3}{r^4} \right) \right. \right. \\
 &\quad \left. \left. - 2(x_1-a) \left(\frac{2(3-\nu)xz}{r^2} - 2(2-\nu)\theta - \frac{2xz^3}{r^4} \right) + 2(3-\nu) \frac{x^2z}{r^2} \right. \right. \\
 &\quad \left. \left. - 2(5-2\nu)z \ln r - \frac{z^3}{r^2} - \frac{2x^2z^3}{r^4} \right\} \right]_1^2 \quad \dots (2.8.1.) \\
 \tau_{zx} &= \frac{-1}{4\pi(1-\nu)} \left[\left\{ \frac{-(3-2\nu)x}{r^2} + \frac{2xz^2}{r^4} \right\} - \frac{0.8}{a^2} \left\{ (x_1-a)^2 \left(\frac{-(3-2\nu)x}{r^2} \right. \right. \right. \\
 &\quad \left. \left. + \frac{2xz^2}{r^4} \right) - 2(x_1-a) \left(\frac{(3-2\nu) \ln r + 2(1-\nu)z^2}{r^2} - \frac{2x^2}{r^2} + \frac{2x^2z^2}{r^4} \right) \right. \\
 &\quad \left. \left. - 4(2-\nu)z\theta + (3-2\nu)x + \frac{(5-2\nu)xz^2}{r^2} + \frac{2x^3z^2}{r^4} \right\} \right]_1^2 \\
 u_x &= \frac{1}{8\pi(1-\nu)G} \left[\left\{ \frac{4(1-\nu)\theta}{r^2} - \frac{xz}{r^2} \right\} - \frac{0.8}{a^2} \left\{ (x_1-a)^2 \left(\frac{4(1-\nu)\theta}{r^2} - \frac{xz}{r^2} \right) \right. \right. \\
 &\quad \left. \left. - 2(x_1-a) \left(\frac{(5-4\nu)z \ln r + \frac{z^3}{r^2}}{r^2} + \frac{(5-4\nu)z(x-z\theta)}{r^2} + \frac{xz^3}{r^2} \right. \right. \right. \\
 &\quad \left. \left. \left. - z^2\theta \right\} \right]_1^2
 \end{aligned}$$

$$u_z = \frac{1}{8\pi(1-\nu)G} \left[\left\{ -\ln r - \frac{z^2}{r^2} \right\} - \frac{0.8}{a^2} \left\{ (x_1-a)^2 \left(-\ln r - \frac{z^2}{r^2} \right) \right. \right. \\ \left. \left. - 2(x_1-a) \left(x - 2z\theta + \frac{xz^2}{r^2} \right) + \frac{x^2}{2} - \frac{3}{2} z^2 \ln r + \frac{x^2 z^2}{r^2} \right\} \right]_1^2$$

The stress distribution around a crack in a unit shear stress field, given by the analytical solution, the approximately elliptical and the constant distribution of unit shear dipole are plotted in Figure 2.8.3. Again the choice for using a pseudo-elliptical distribution is borne out, i.e. because of its close agreement with the analytical solution. It is interesting to note the similar form of the equations (2.7.1.) and (2.8.1.).

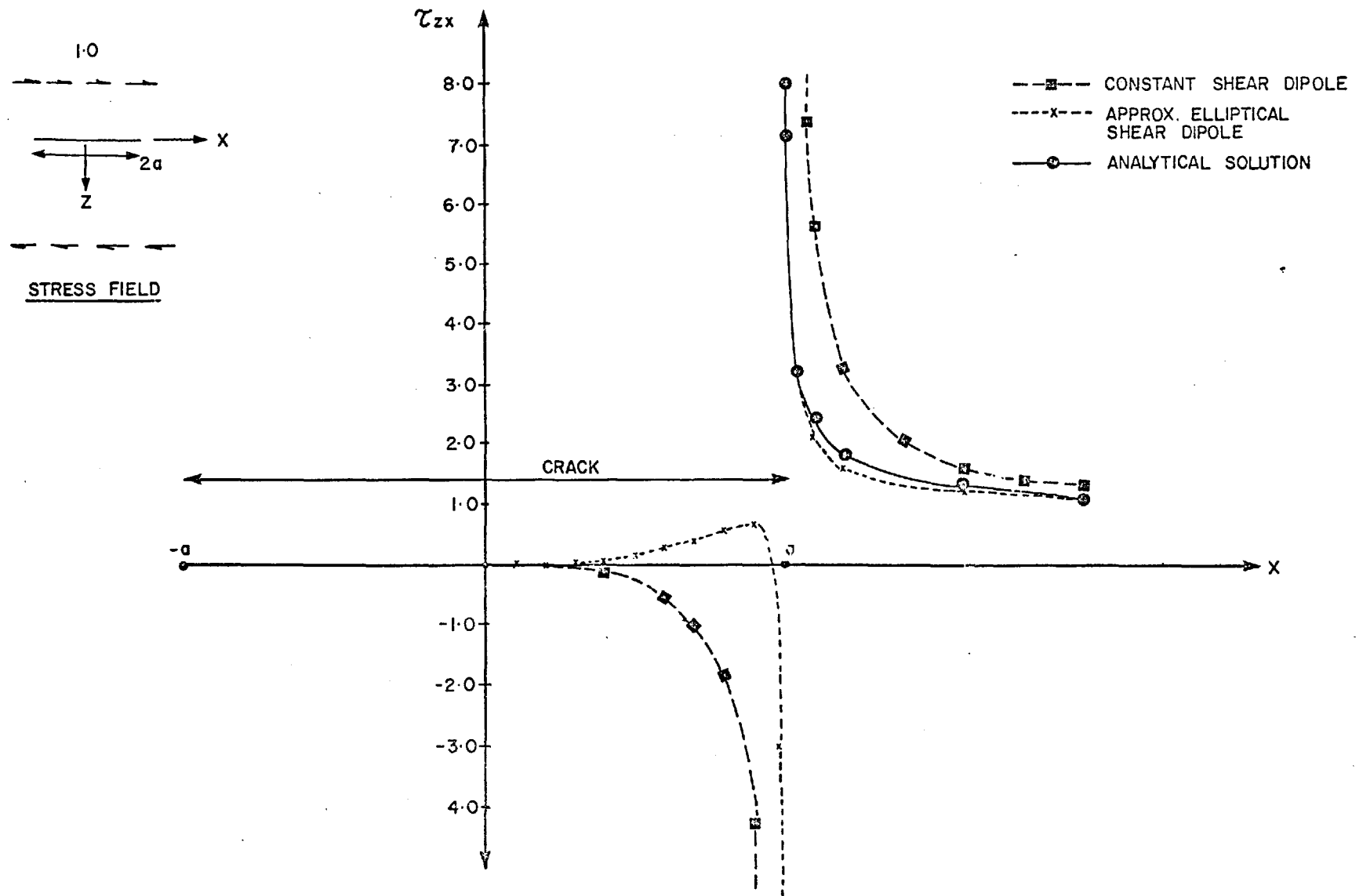


FIGURE 2.8.3 STRESS DISTRIBUTION ALONG THE X AXIS NEAR A CRACK IN A UNIT SHEAR FIELD STRESS

CHAPTER 3: APPLICATION OF THE METHOD TO STRESS ANALYSIS PROBLEMS

1. Check on accuracy of the two and three dimensional programs.

For a new numerical method, such as the boundary integral method (B.I.M.), a comprehensive check is worthwhile for a number of reasons; e.g. (a) to be confident of the method when applied to problems when no alternative solution is available, (b) to aid insight into the accuracy of the method and (c) to possibly highlight areas that may require improvement. For the above reasons, this section is entirely devoted to checking of the method, and in particular the two dimensional program.

In Figure 3.1.1, the boundary stresses and displacements around a circular tunnel from the B.I.M. with ten elements and the analytical solution (Jaeger & Cook (1971)) are given. There is excellent agreement between the analytical solution and B.I.M. analysis. The circular tunnel is not a severe test, therefore the comparison was made between the B.I.M. with fifteen elements and the analytical solution for a 3:1 elliptical tunnel, Figure 3.1.2. In this case the boundary stress gradients near the tip of the excavation are high; however, good agreement is present. The B.I.M. and the analytical solution are almost identical for boundary displacements. A more extreme test is of a 6:1 elliptical tunnel, Figure 3.1.3, where excellent agreement exists between the B.I.M. with twenty elements and the analytical solution for boundary stresses. Further still, for a 9:1 elliptical tunnel, excellent agreement of boundary stresses occurs for the case of twenty boundary elements, Figure 3.1.4.

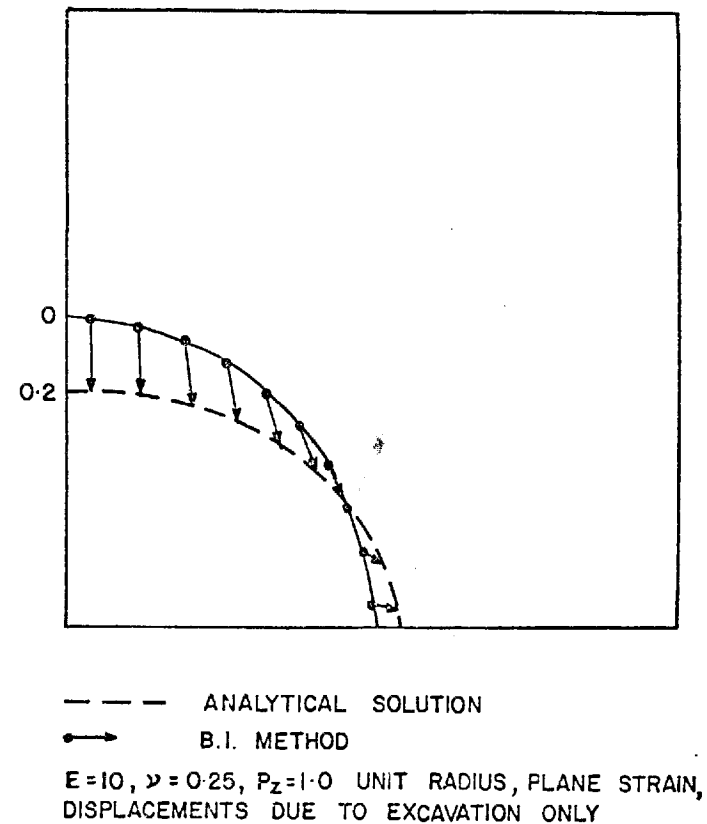
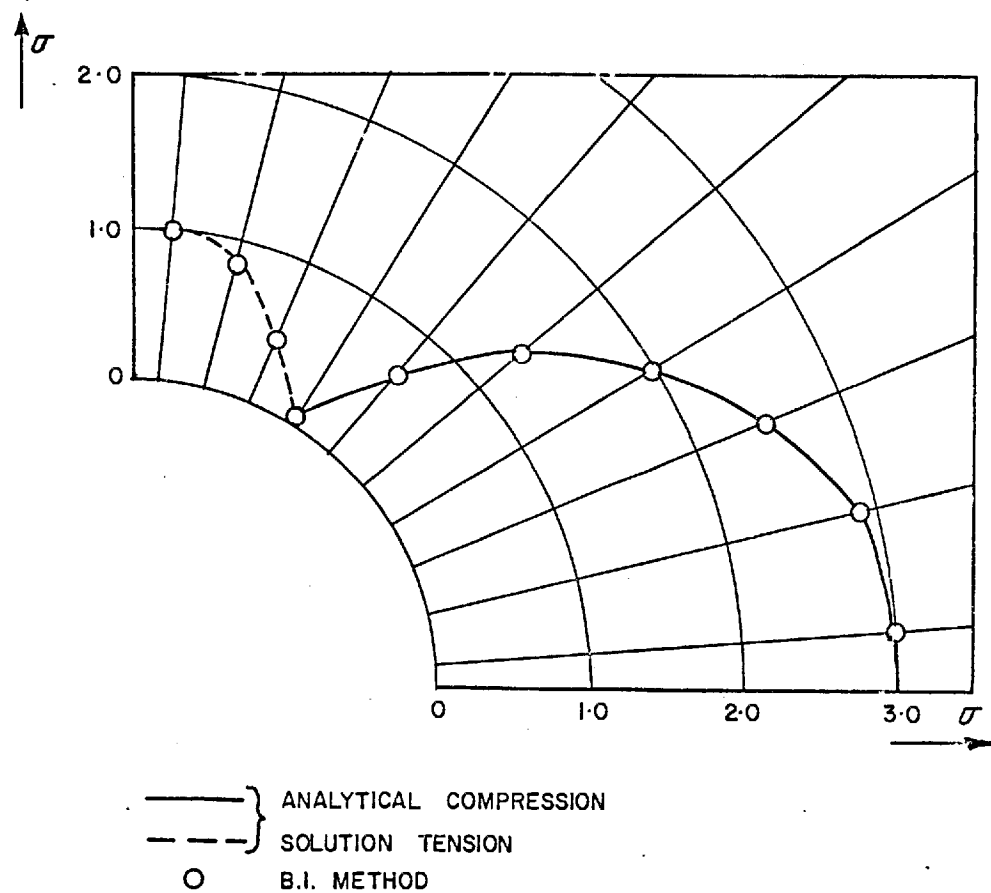
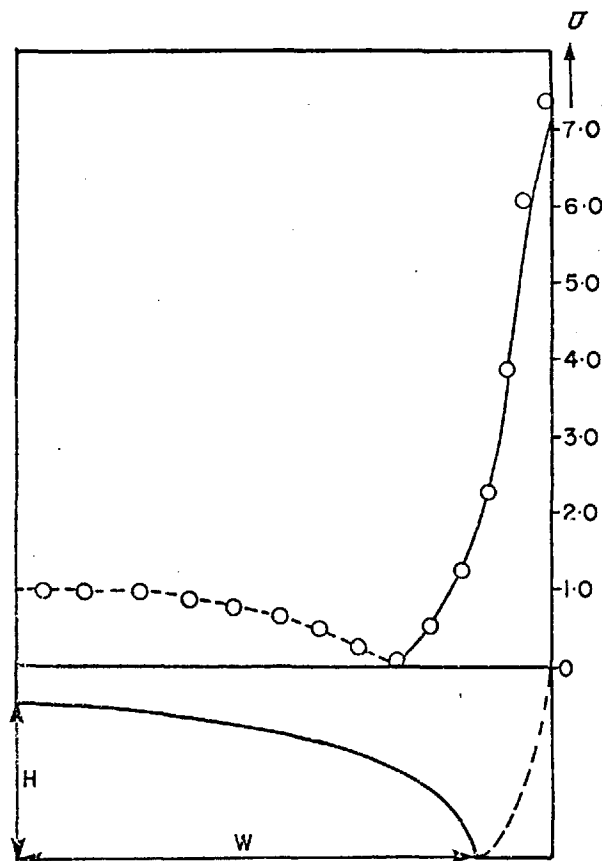
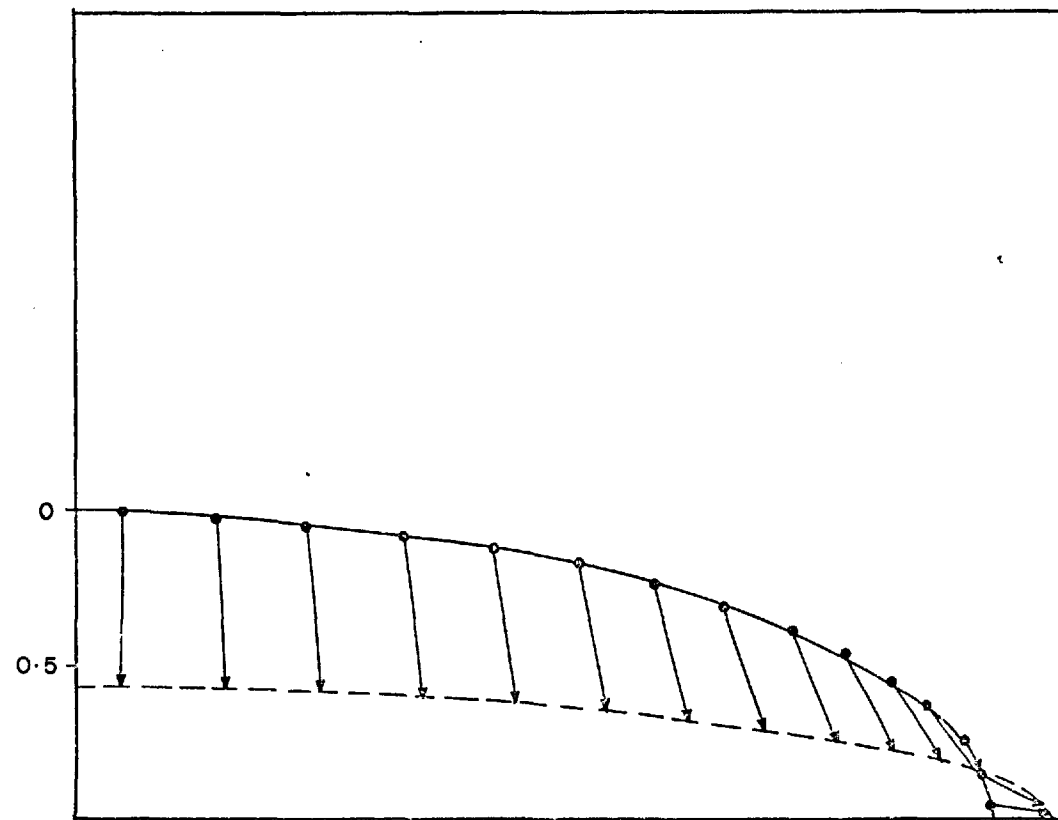


FIGURE 3.1.1 STRESSES AND DISPLACEMENTS OF A CIRCULAR TUNNEL
SUBJECTED TO A UNIT VERTICAL STRESS FIELD



— } ANALYTICAL COMPRESSION
 - - - } SOLUTION TENSION
 ○ B.I. METHOD
 $W/H = 3$



- - - ANALYTICAL SOLUTION
 ○ — B.I. METHOD
 $E = 10, \nu = 0.25, P_z = 1.0, H = 1.0$ PLANE STRAIN, DISPLACEMENTS DUE TO EXCAVATION ONLY

FIGURE 3.1.2 STRESSES AND DISPLACEMENTS OF A ELLIPTICAL TUNNEL ($W/H = 3$)
SUBJECTED TO A UNIT VERTICAL STRESS FIELD

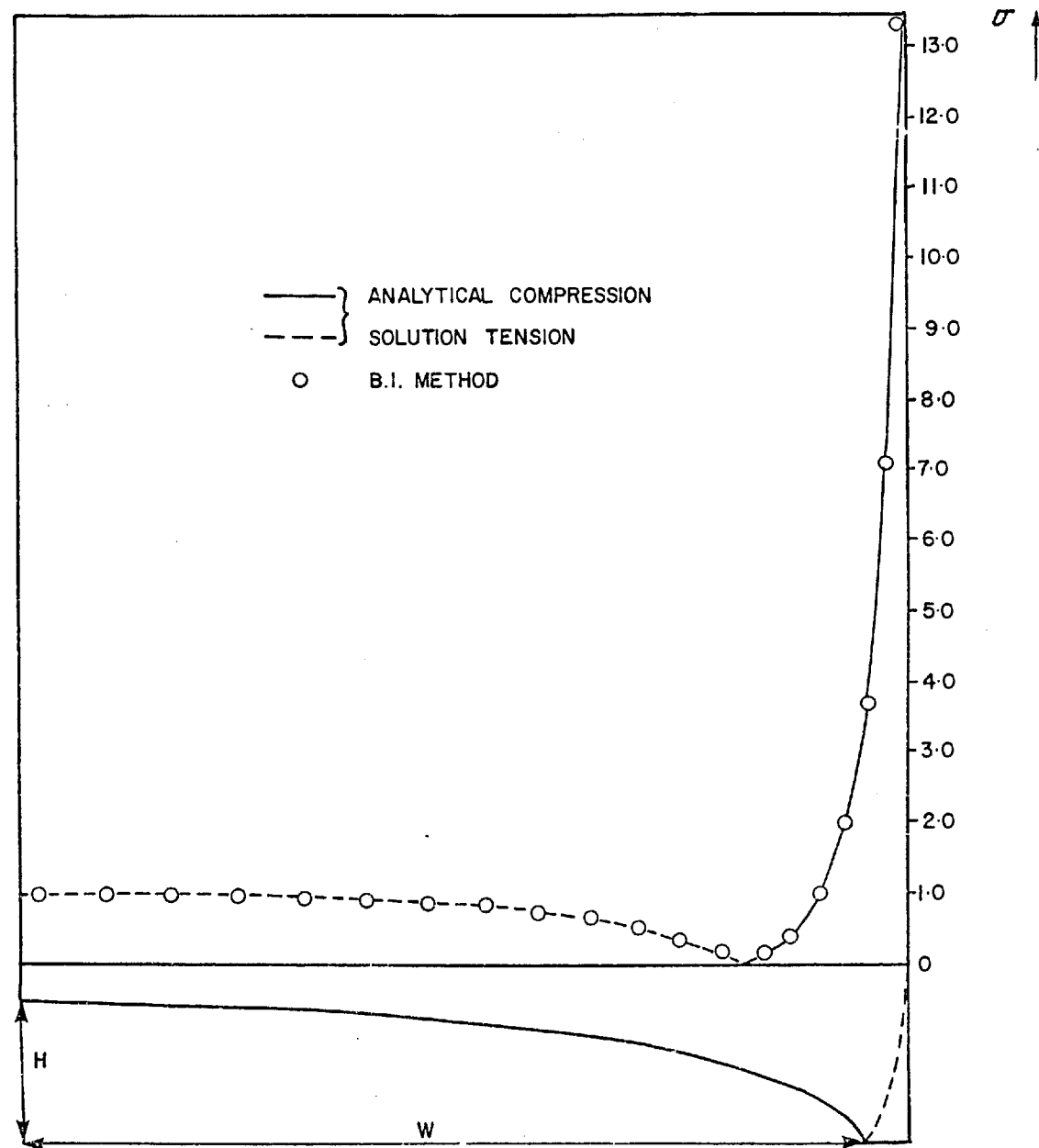


FIGURE 3.1.3 BOUNDARY STRESSES OF AN ELLIPTICAL TUNNEL ($W/H = 6$)
SUBJECTED TO A UNIT VERTICAL STRESS FIELD

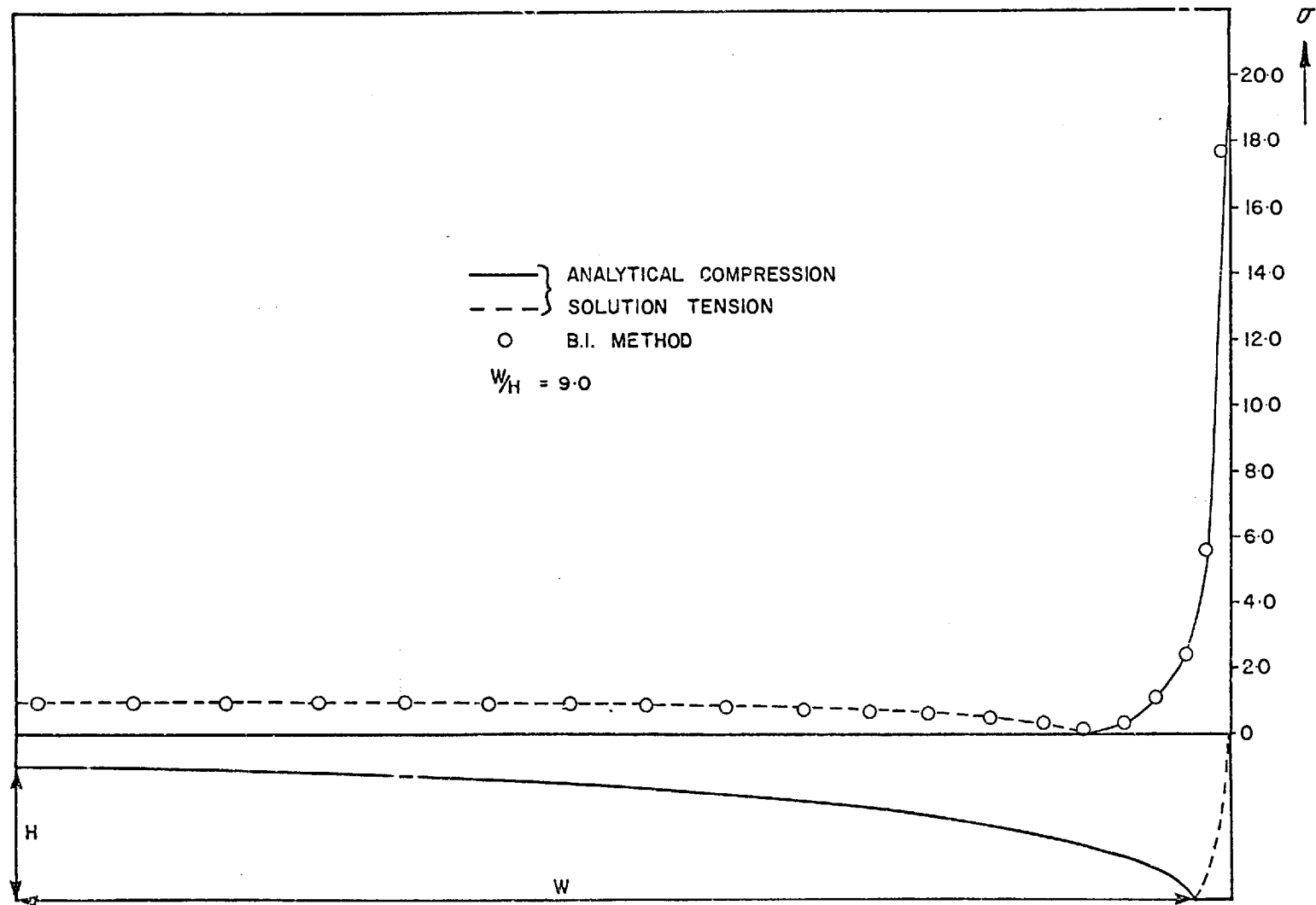


FIGURE 3.1.4 BOUNDARY STRESSES AROUND AN ELLIPTICAL TUNNEL ($W/H = 9$)
 SUBJECTED TO A UNIT VERTICAL STRESS FIELD

In order to compare numerical values from the B.I.M. and the analytical solution, the boundary stresses and displacements at points A and B on the surface of an elliptical tunnel, Figure 3.1.5, are tabulated for a number of width to height ratios, Tables 3.1.1. and 3.1.2. The location of boundary stresses and displacements for the B.I.M. in Tables 3.1.1. and 3.1.2. are not exactly at the points A and B but located at the centre of the nearest boundary element. In all cases, where a reasonable number of boundary elements have been considered, good to excellent agreement is present. In the cases where only a small number of elements have been incorporated in the analysis, the errors involved will be due to

- (a) poor modelling of geometry and function and
- (b) large distance between the centre of the nearest element and the points concerned.

The boundary stresses around two parallel circular tunnels in a unit vertical stress field are illustrated in Figure 3.1.6. At the springline level on the pillar side, i.e. between the two tunnels, the boundary stresses are 3.1, 3.4 and 4.1 for pillar dimensions of 1.0, 0.5 and 0.25 of a diameter, respectively. The boundary stresses for points A and B are compared with the analytical solution (Savin (1961)) Table 3.1.3. Good agreement exists between the B.I.M., with twenty boundary elements modelling one half of the circular tunnel, and the analytical solution. The boundary displacements for points A, B and C are quoted in Table 3.1.4. It is interesting to compare the x displacement of points A and B. In the case of the pillar width equal to one diameter, the displacement of point B is into the pillar, while with reduced pillar width the point B moves into the opening under this loading condition.

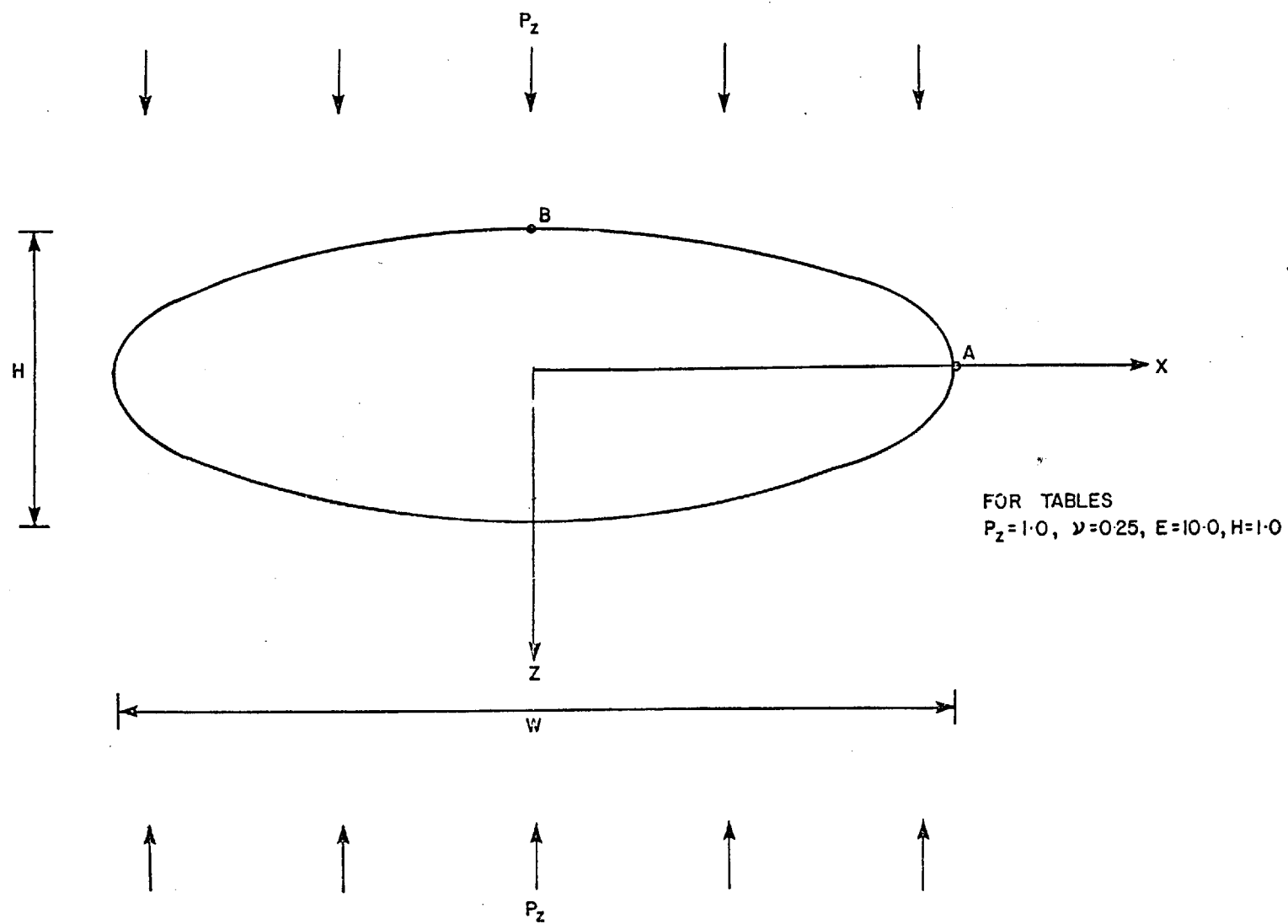


FIGURE 3.1.5 ELLIPTICAL TUNNEL WITH FIELD STRESS FIELD P_z

TABLE 3.1.1. BOUNDARY STRESSES FOR CIRCULAR AND ELLIPTICAL TUNNELS AS IN FIGURE 3.1.5.

$\frac{W}{H}$	B.I. Method		Analytical		Number of Elements
	A	B	A	B	
1.0	2.94	-0.86	3.0	-1.0	5
	3.06	-1.02			10
3.0	6.46	-0.96	7.0	-1.0	10
	7.36	-0.99			15
6.0	10.60	-0.98	13.0	-1.0	15
	13.05	-0.98			20
	13.38	-0.99			40
9.0	8.80	-.098	19.0	-1.0	15
	14.25	-.098			20
	17.74	-.099			40
12.0	4.39	-0.98	25.0	-1.0	15
	11.31	-0.98			20
	29.21	-0.99			40
15.0	-	-0.97	31.0	-1.0	15
	5.96	-0.98			20
	32.14	-0.99			40

TABLE 3.1.2. BOUNDARY DISPLACEMENTS FOR CIRCULAR
AND ELLIPTICAL TUNNELS AS IN FIGURE 3.1.5.

$\frac{W}{H}$	B.I. Method		Analytical		Number of Elements
	A	B	A	B	
1.0	0.062	-0.193	0.063	-0.188	5
	0.062	-0.191			10
3.0	0.199	-0.571	0.188	-0.563	10
	0.195	-0.567			15
6.0	0.412	-1.135	0.377	-1.125	15
	0.402	-1.133			20
	0.389	-1.129			40
9.0	0.650	-1.700	0.565	-1.688	15
	0.627	-1.697			20
	0.594	-1.692			40
12.0	0.908	-2.226	0.754	-2.250	15
	0.868	-2.258			20
	0.807	-2.254			40
15.0	1.184	-2.824	0.942	-2.813	15
	1.126	-2.818			20
	1.029	-2.816			40

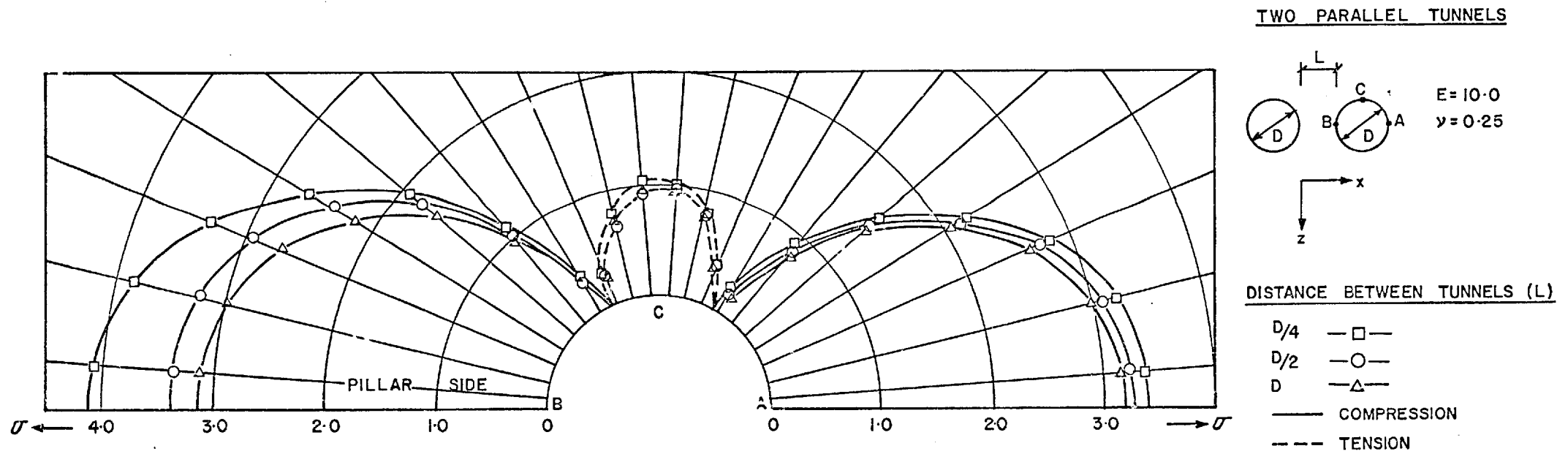


FIGURE 3.1.6 BOUNDARY STRESSES AROUND TWO PARALLEL CIRCULAR TUNNELS FOR A UNIT VERTICAL STRESS FIELD

TABLE 3.1.3. BOUNDARY STRESSES FOR TWO PARALLEL CIRCULAR TUNNELS AS IN FIGURE 3.1.6.

Distance between tunnels (L)	B.I. Method		Analytical	
	A	B	A	B
D	3.14	3.10	3.07	3.02
D/2	3.23	3.37	3.15	3.26
D/4	3.35	4.08	3.34	-

TABLE 3.1.4. BOUNDARY DISPLACEMENTS FOR TWO PARALLEL CIRCULAR TUNNELS AS IN FIGURE 3.1.6.

Distance between tunnels (L)	B.I. Method			
	A	B	C	
	U_x	U_x	U_x	U_z
D	0.075	-0.011	0.020	0.198
D/2	0.077	0.007	0.021	0.211
D/4	0.081	0.015	0.023	0.231

Stresses at interior points near a circular tunnel under a unit vertical stress field are given for the B.I.M. and the analytical solution (Jaeger & Cook (1971)) in Figure 3.1.7. In Table 3.1.5, the displacements at the same interior points are quoted. In both cases good agreement between the B.I.M. and the analytical solution exists. Similarly for interior points near a 3:1 elliptical tunnel, stresses are illustrated in Figure 3.1.8. and displacements given in Table 3.1.6. Here again the B.I.M. agrees very closely to the analytical solution, Bray (1976a). If the interior points lie between the centres of two elements (i.e. the normal projection from the surface) then slight inaccuracies will occur. The reason for this is the large contribution the nearest element has to the calculation of stress at interior points. This is not the case with displacements, because displacements vary as $\frac{1}{r}$ whilst stresses vary as $\frac{1}{r^2}$ in two dimensions.

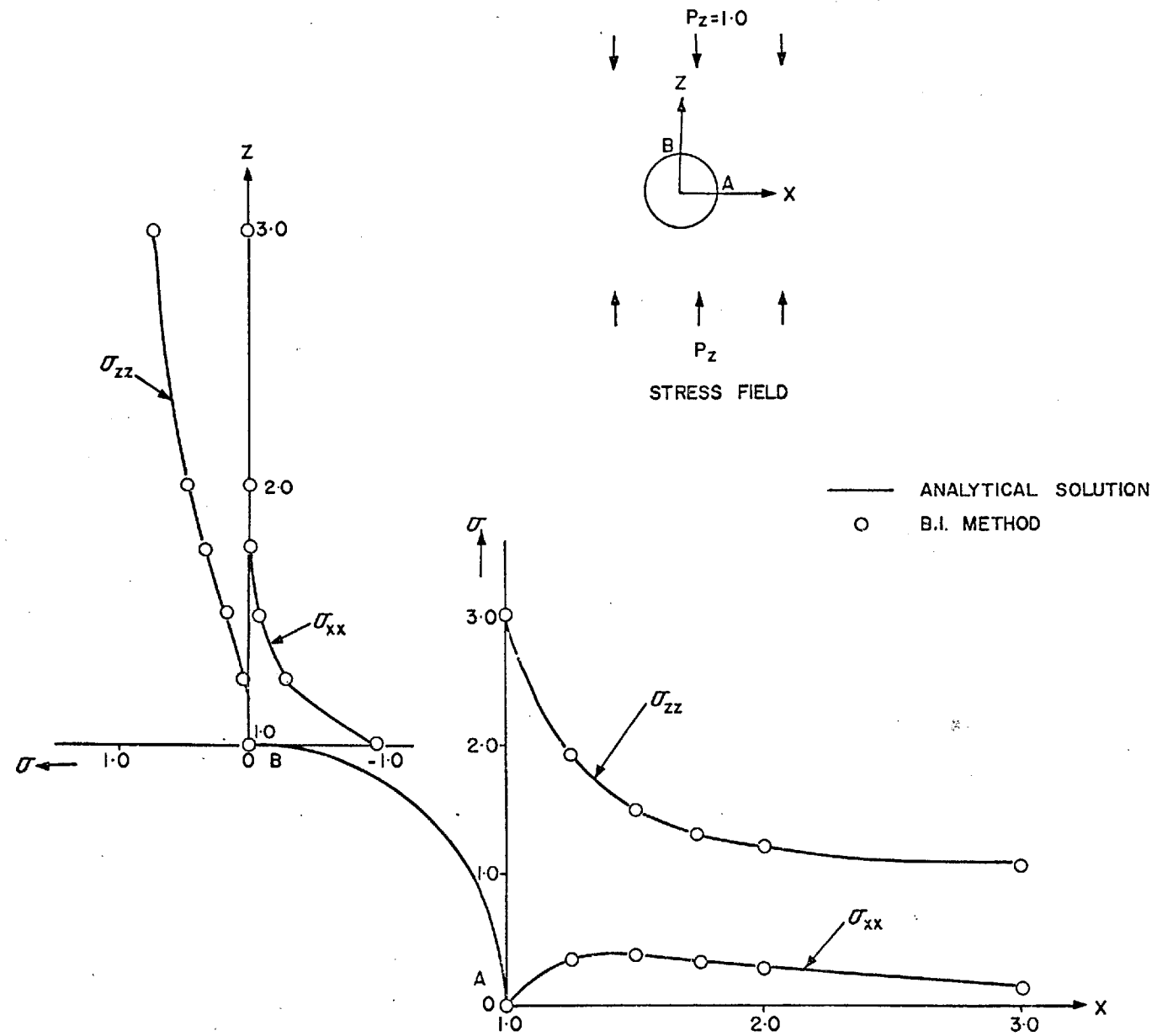


FIGURE 3.1.7 STRESSES AT INTERIOR POINTS NEAR A CIRCULAR TUNNEL UNDER UNIT VERTICAL STRESS FIELD

TABLE 3.1.5. DISPLACEMENTS AT INTERIOR POINTS
AROUND A CIRCULAR TUNNEL ($W/H = 1$)
AS IN FIGURE 3.1.5.

Location		B.I. Method		Analytical	
x	z	U_x	U_z	U_x	U_z
1.25	0.0	0.069	o	0.068	o
1.5	0.0.	0.066	o	0.065	o
1.75	0.0.	0.061	o	0.060	o
2.0	0.0	0.056	o	0. 055	o
3.0	0.0.	0.040	o	0.039	o
0.0.	1.25	o	-0.171	o	-0.168
0.0.	1.5	o	-0.151	o	-0.148
0.0	1.75	o	-0.134	o	-0.131
0.0	2.0	o	-0.120	o	-0.117
0.0	3.0	o	-0.083	o	-0.081

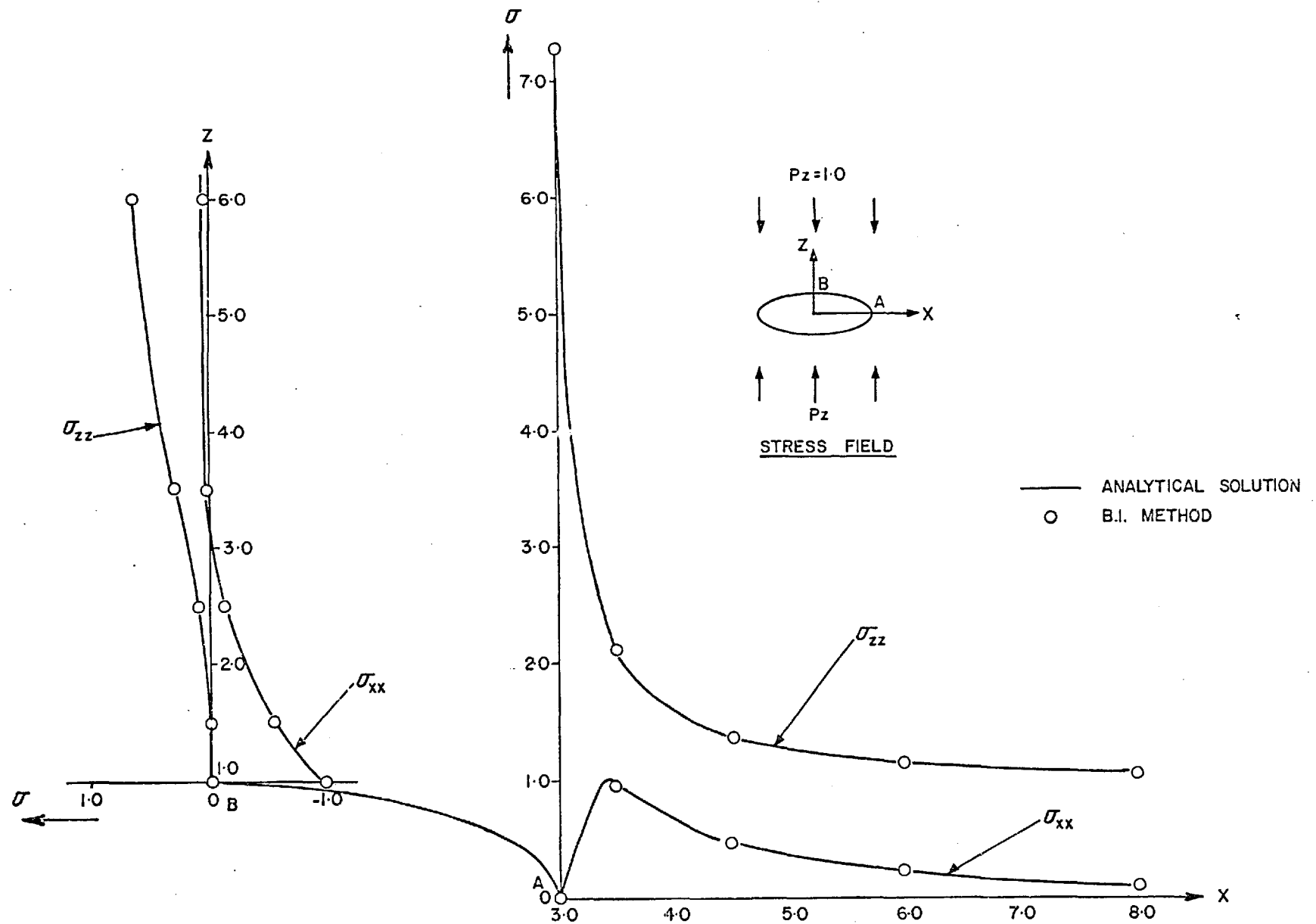


FIGURE 3.1.8 STRESSES AT INTERIOR POINTS NEAR AN ELLIPTICAL TUNNEL UNDER UNIT VERTICAL STRESS FIELD

TABLE 3.1:6.

DISPLACEMENTS AT INTERIOR POINTS
AROUND AN ELLIPTICAL TUNNEL ($W/H = 3$)
AS IN FIGURE 3.1.5.

Location		B.I. Method		Analytical	
x	z	U_x	U_z	U_x	U_z
3.5	0.0	0.185	o	0.179	o
4.5	0.0.	0.137	o	0.134	o
6.0	0.0.	0.099	o	0.027	o
8.0	0.0.	0.073	o	0.072	o
0.0	1.5	o	-0.534	o	-0.527
0.0	2.5	o	-0.453	o	-0.453
0.0	3.5	o	-0.381	o	-0.375
0.0	6.0	o	-0.259	o	-0.254

The boundary integral program of Lachat and Watson (1975) has been modified by Watson (1976) for the exterior problem. This three dimensional program was checked by solving the problems of a spherical cavity in a uniaxial stress field and a spheroidal cavity in a triaxial stress field. The boundary element configurations for these two cavities, (a) spherical and (b) penny shaped are illustrated in Figure 3.1.9. For the spherical cavity a unit stress field was applied in the z direction with Poisson's ratio of $\nu = 0.3$. The tangential stresses computed at points A and B were -0.67 and 2.04 respectively; those given by the analytical solution (Terzaghi & Richart (1952)) are -0.682 and 2.045. The spheroidal cavity, in which OB is five times the length of OA, was subjected to a unit stress field in the z direction and a horizontal stress field of 0.25 (i.e. in the xy plane). The tangential stresses computed at points A and B were -0.32 and 7.0, respectively; those given by the analytical solution are -0.323 and 7.059. The high degree of curvature of the boundary near point B makes this a relatively severe test. Other checks of this program have been mentioned by Lachat and Watson (1975), Lachat (1975), Hocking (1976b) and Hocking et al (1976).

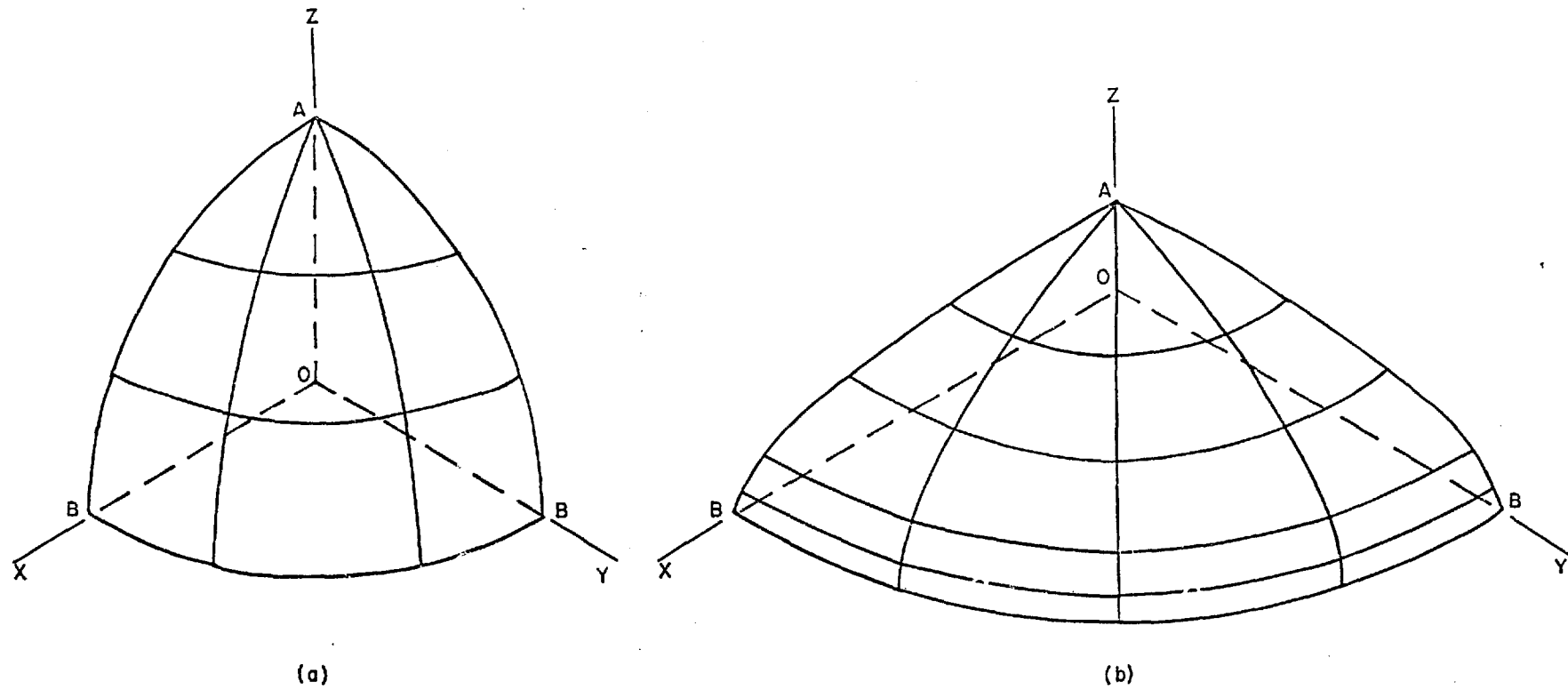


FIGURE 3.1.9 BOUNDARY ELEMENT CONFIGURATION OF a) SPHERICAL AND b) PENNY SHAPED CAVITIES

2. Stresses around tunnel intersections

The three dimensional stress distribution around five geometrically different junctions of circular tunnels, the cross +, the tee T and three bifurcations (15° , 30° and 45° Y intersections), have been computed using the three dimensional boundary integral program of Watson (1976). The results presented give a clear indication of the stress concentrations and stress gradients around the junctions and the zones of influence of the intersections.

+ - junction

The + - junction, illustrated in Figure 3.2.1, is the intersection of two circular tunnels of unit radius. Field stresses of $p_z = 1.0$ and $p_x = p_y = Np_z$ were applied at infinity in directions parallel to the co-ordinate axes, see Figure 3.2.1. Calculations were carried out for $N = 0, 0.33, 1.0$ and 2.0 with a Poisson's ratio of $\nu = 0.25$. Since the problem is symmetric with respect to the three co-ordinate axes, x , y and z , then only one eighth of the problem was modelled. The boundary element configuration for the + - junction is shown in Figure 3.2.2. Figure 3.2.3 shows the distribution of boundary stresses at sections AA, BB and CC as defined in Figure 3.2.1.

The boundary stresses σ_{xx} in the crown of the tunnels, Figure 3.2.3 (a), are approximately equal to those given by the plane strain solution one diameter (i.e. $x = 2.0$) from the centre of the intersection. These stresses are not greatly different over the centre of the intersection (i.e. $x = 0$) than those removed from the intersection. In Figure 3.2.3 (b), (i.e. section BB), it is important to highlight the high compressive stress at the springline level, where the boundary stress, σ_{zz} , is 6.5, 5.7, 4.2 and 1.9 for $N = 0, 0.33, 1.0$ and 2.0 respectively. The plane strain solution for a circular tunnel

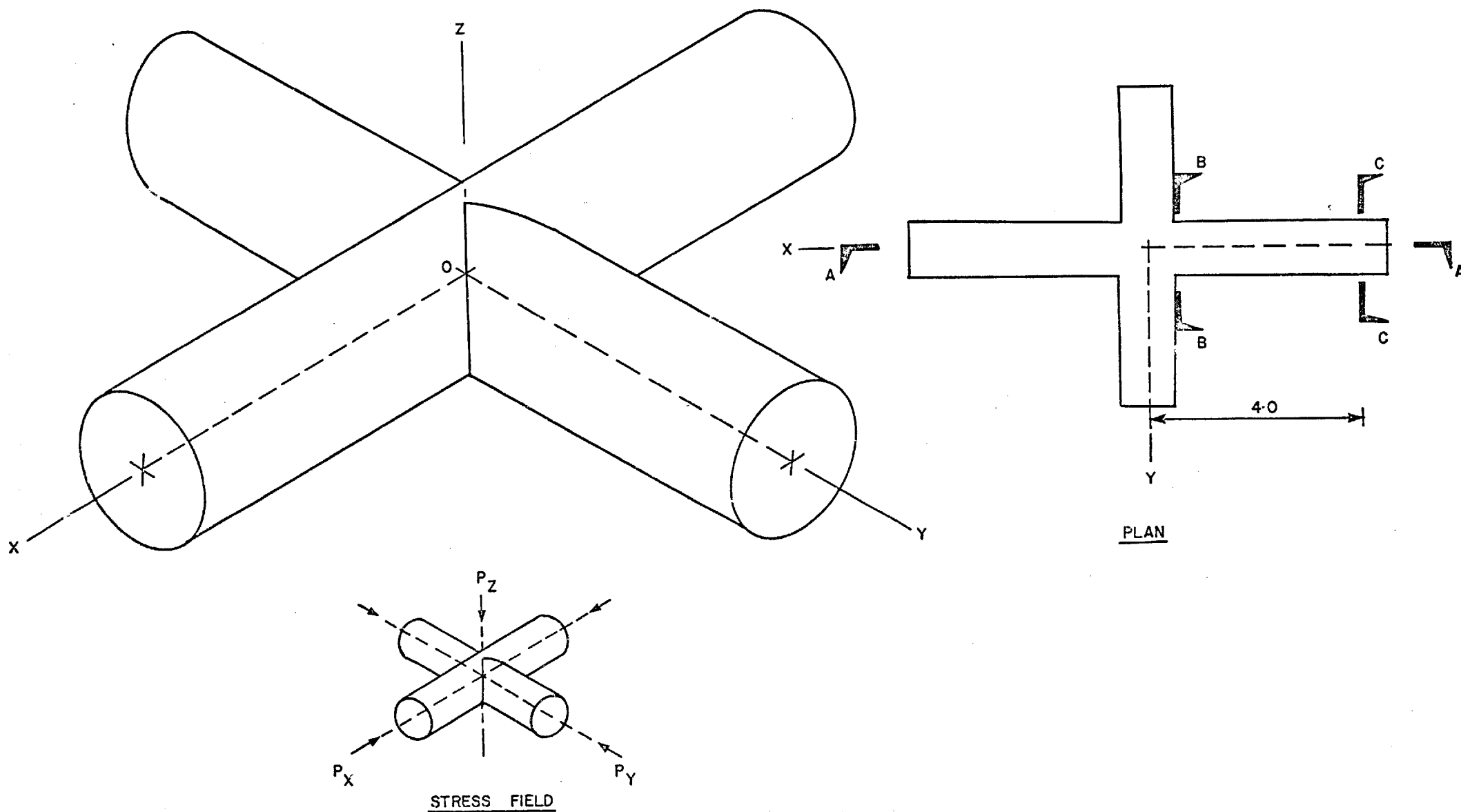


FIGURE 3.2.1 GEOMETRY OF THE CROSS JUNCTION

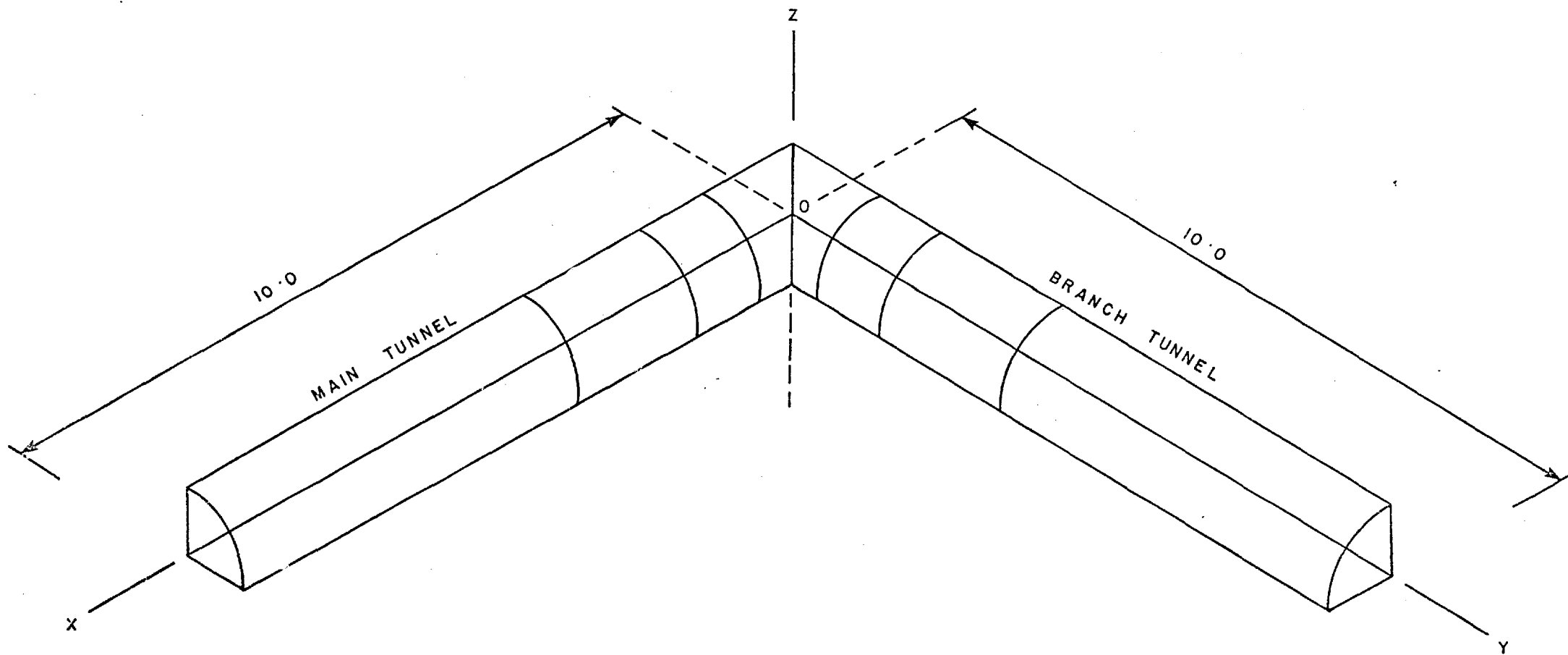
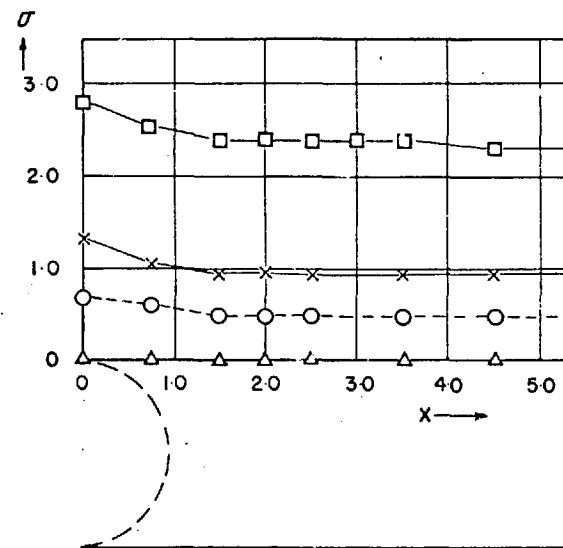
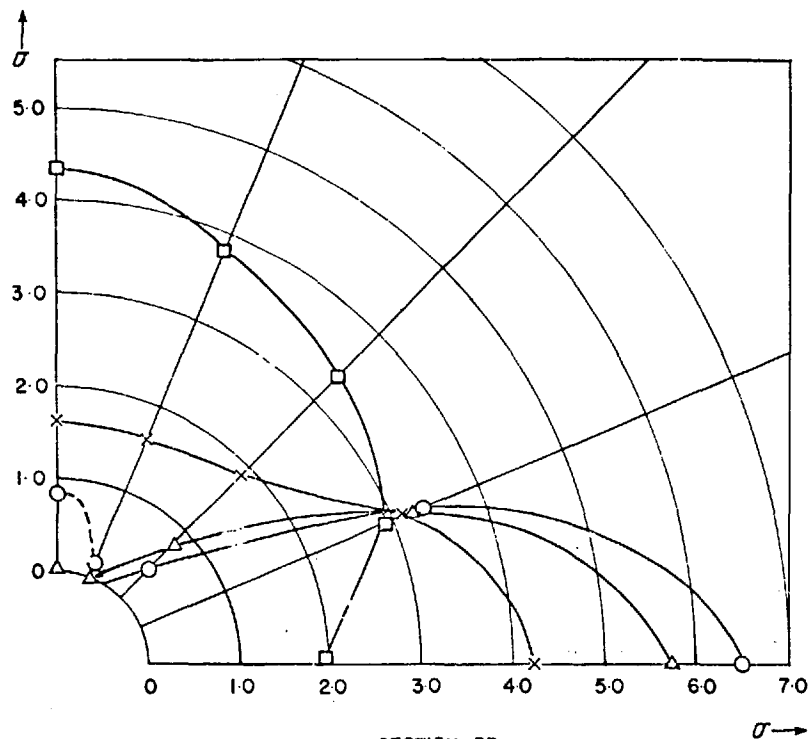


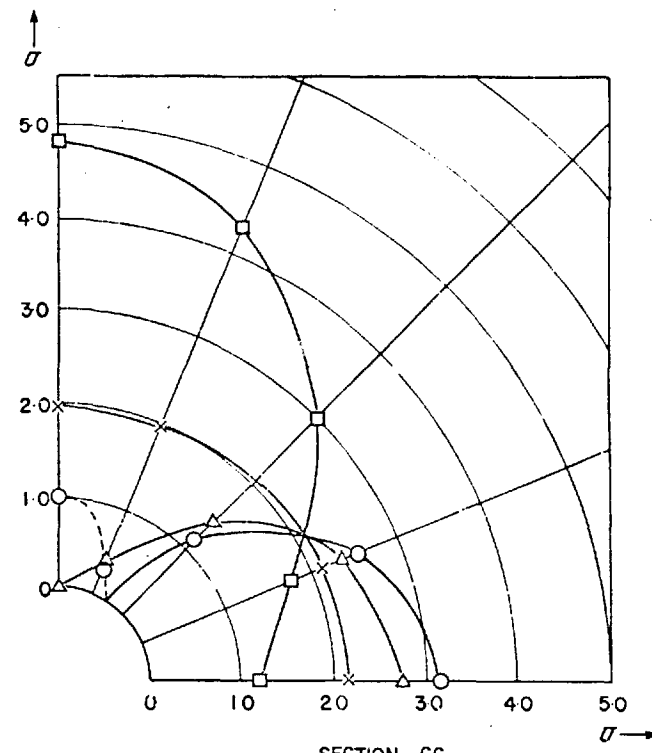
FIGURE 3.2.2 BOUNDARY ELEMENT CONFIGURATION FOR CROSS JUNCTION



SECTION AA
(a)



SECTION BB
(b)



SECTION CC
(c)

FIGURE 3.2.3 BOUNDARY STRESSES FOR CROSS JUNCTION a) SECTION AA b) SECTION BB AND c) SECTION CC

yields boundary stresses at the springline level of 3.0, 2.67, 2.0 and 1.0 for $N = 0, 0.33, 1.0$ and 2.0 respectively. Therefore for the case of a uniaxial stress field ($N = 0$) the stress concentration for a + - junction is approximately twice that of a long circular tunnel. It is interesting to note that at two diameters removed from the centre of the intersection (i.e. at $x = 4.0$), the boundary stresses are approximately the same as those in the plane strain case, shown in Figure 3.2.3(c). For the field stresses considered here, the zone of influence of the + - junction intersection is twice the diameter of the tunnels. The zone of influence was here defined as the zone in which the boundary stresses departed from the plane strain solution by more than 5% of the maximum stress concentration present in the plane strain case.

T - Junction

The T - Junction illustrated in Figure 3.2.4., consists of the intersection of a main tunnel and a branch tunnel (perpendicular to the main), both of unit radius. Field stresses of $p_z = 1.0$ and $p_y = p_x = Np_z$ were applied at infinity in directions parallel to the co-ordinate axes, see Figure 3.2.4. Calculations were carried out for $N = 0, 0.33, 1.0$ and 2.0 with a Poisson's ratio of $\nu = 0.25$. Since the problem is symmetric with respect to the x and z co-ordinate axes, then only one quarter of the problem was modelled. The boundary element configuration for the T - junction is shown in Figure 3.2.5. The distribution of boundary stresses at selected sections, as defined in Figure 3.2.4, are plotted in Figure 3.2.6.

Over the centre of the intersection, Figure 3.2.6 (a) i.e. section AA, the tensile stress (σ_{yy}) is less than 1.0, but tends to the plane strain solution approximately two diameters from the centre of the intersection (i.e. at $y = 4.0$). The compressive stresses at springline level in this section for $N = 0$ and 0.33 are only slightly greater than those for the plane strain case. For the cases of $N = 1.0$ and 2.0 , the compressive stresses at this point depart

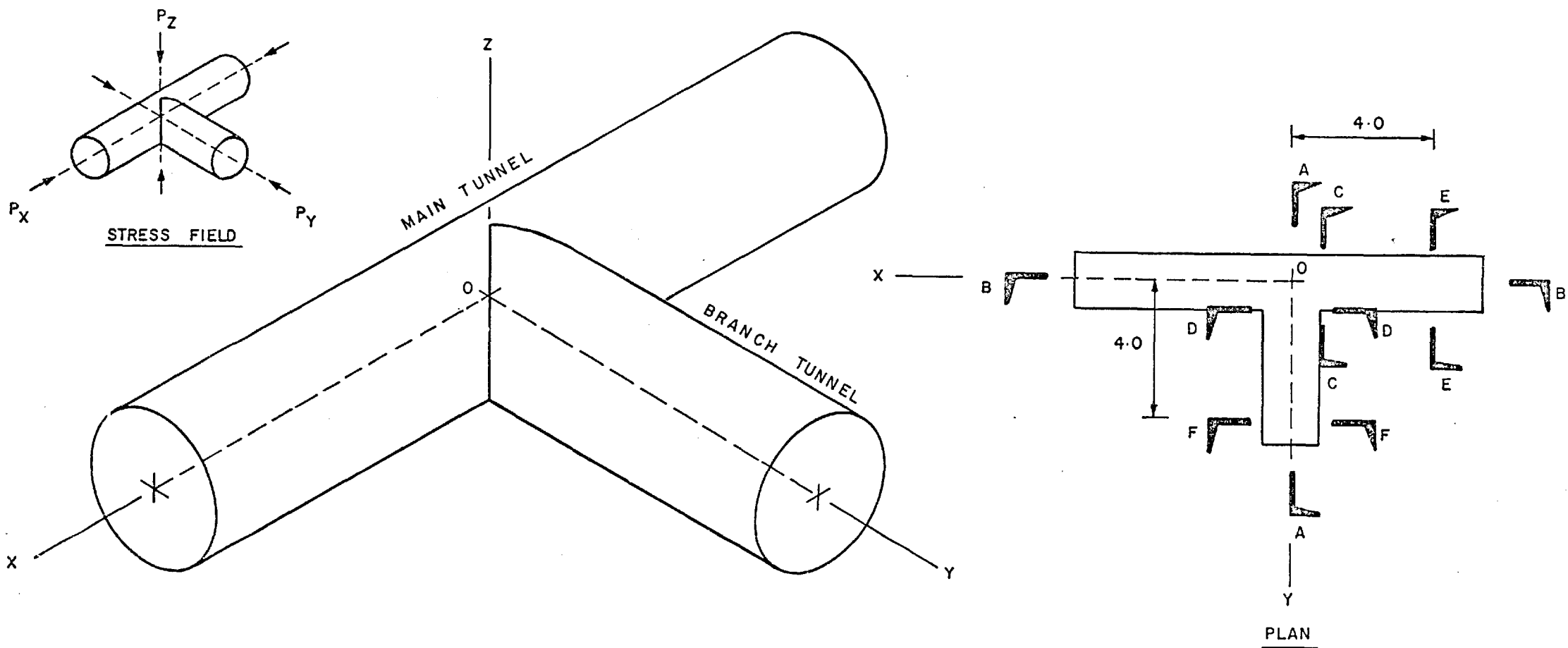


FIGURE 3.2.4 GEOMETRY OF THE T JUNCTION

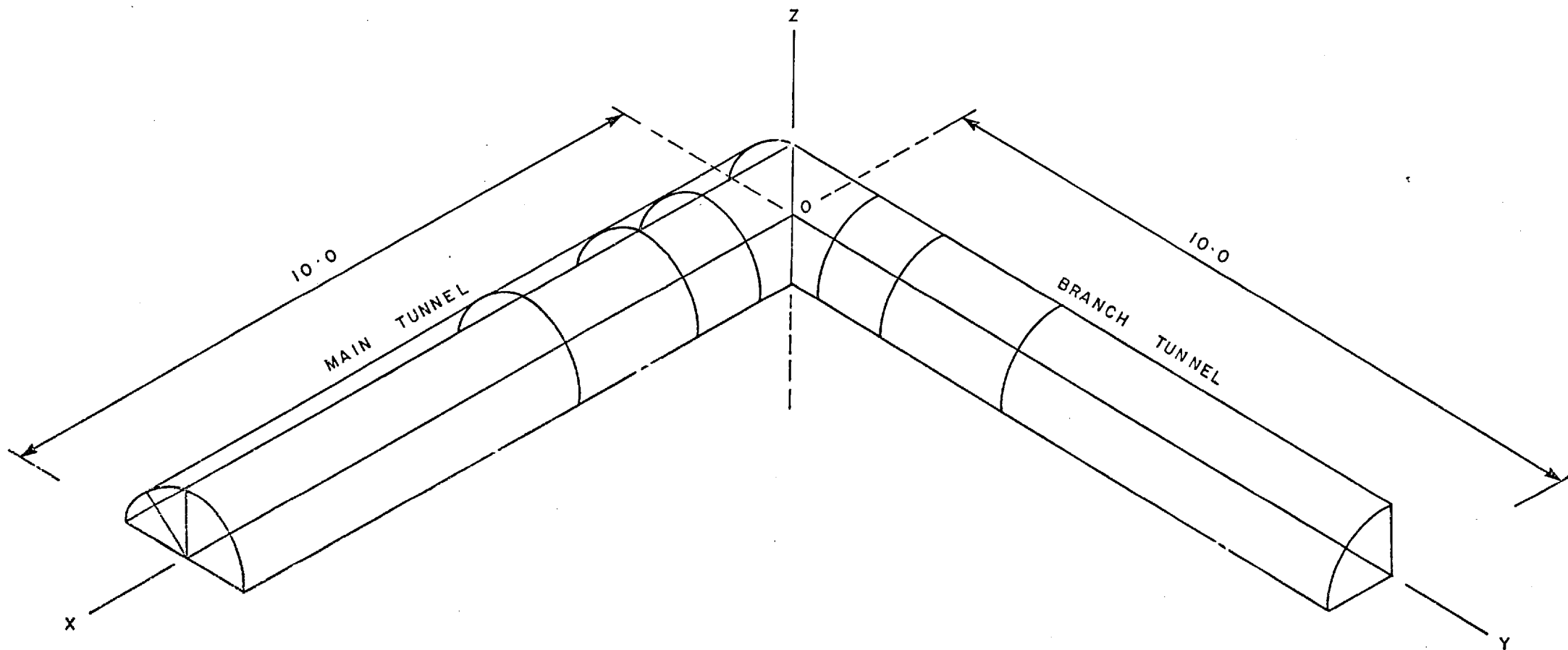


FIGURE 3.2.5 BOUNDARY ELEMENT CONFIGURATION FOR T JUNCTION

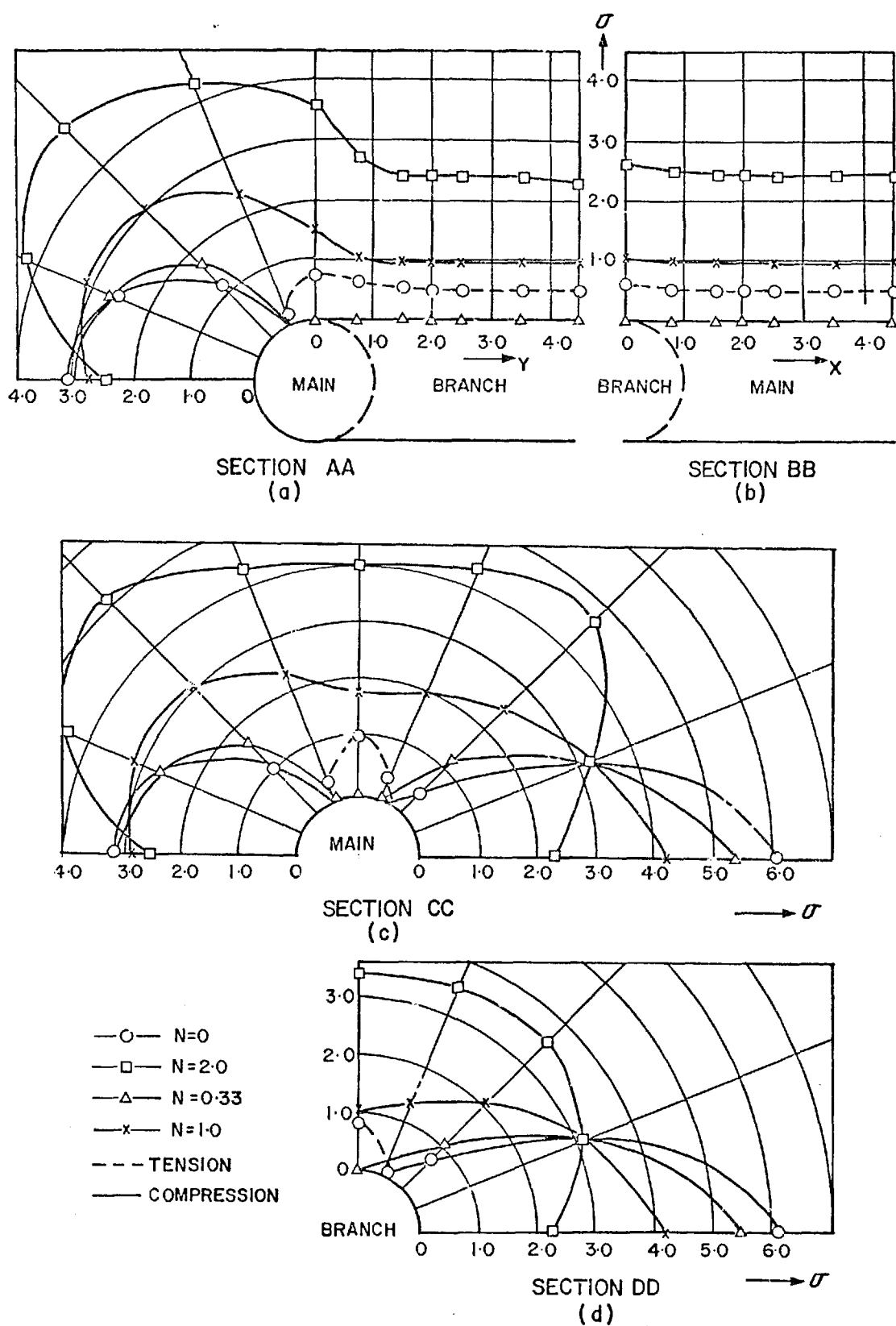


FIGURE 3.2.6 BOUNDARY STRESSES FOR T JUNCTION AT a) SECTION AA
b) SECTION BB c) SECTION CC AND d) SECTION DD

greatly from the plane strain solution, as would be expected due to the presence of the branch tunnel. The boundary stresses along the crown, section BB, Figure 3.2.6 (b), differ only slightly over the centre of the intersection than those removed from the intersection. By far the most important effect is the high compressive stresses at springline level for the cases of $N = 0, 0.33$ and 1.0 as shown in sections CC and DD, Figures 3.2.6 (c) and (d) respectively. At this point the boundary stresses are 6.1, 5.4, 4.2 and 2.3 for the cases of $N = 0, 0.33, 1.0$ and 2.0 respectively. For the field stresses considered the zone of influence (see definition in the + - junction case) of the T - junction is twice the diameter of the tunnels.

Y - junctions

The geometry of the three bifurcations (Y-junctions) is illustrated in Figure 3.2.7. The three cases considered were the intersection of circular tunnels of unit radii with bifurcation angles, α , of 15° , 30° and 45° respectively. The stress distribution around each Y - junction was computed for field stresses of $p_z = 1.0$ and $p_x = p_y = Np_z$ applied at infinity in directions parallel to the coordinate axes, see Figure 3.2.7. Calculations were carried out for four cases of $N = 0, 0.33, 1.0$ and 2.0 with a Poisson's ratio of $\nu = 0.25$. Since the Y - junction were all symmetric with respect to the y and z axes, only one quarter of the problem was modelled. The configuration of boundary elements required to represent the 15° Y - junction is given in Figure 3.2.8. Boundary stresses for selected sections, as shown in Figure 3.2.7, are given in Figures 3.2.9, 3.2.10 and 3.2.11, for the 15° , 30° and 45° bifurcations respectively.

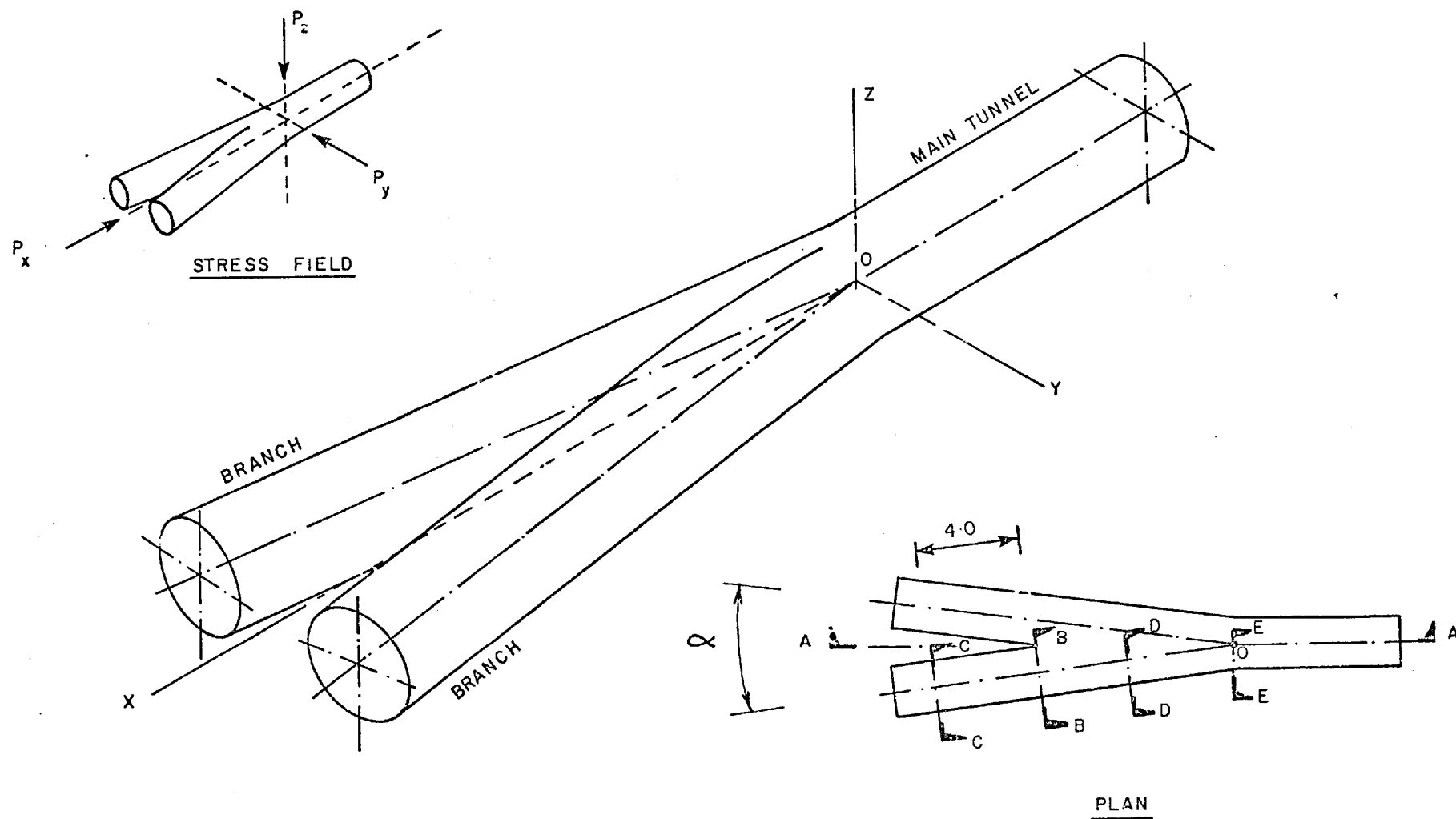


FIGURE 3.2.7 GEOMETRY OF THE BIFURCATION JUNCTION

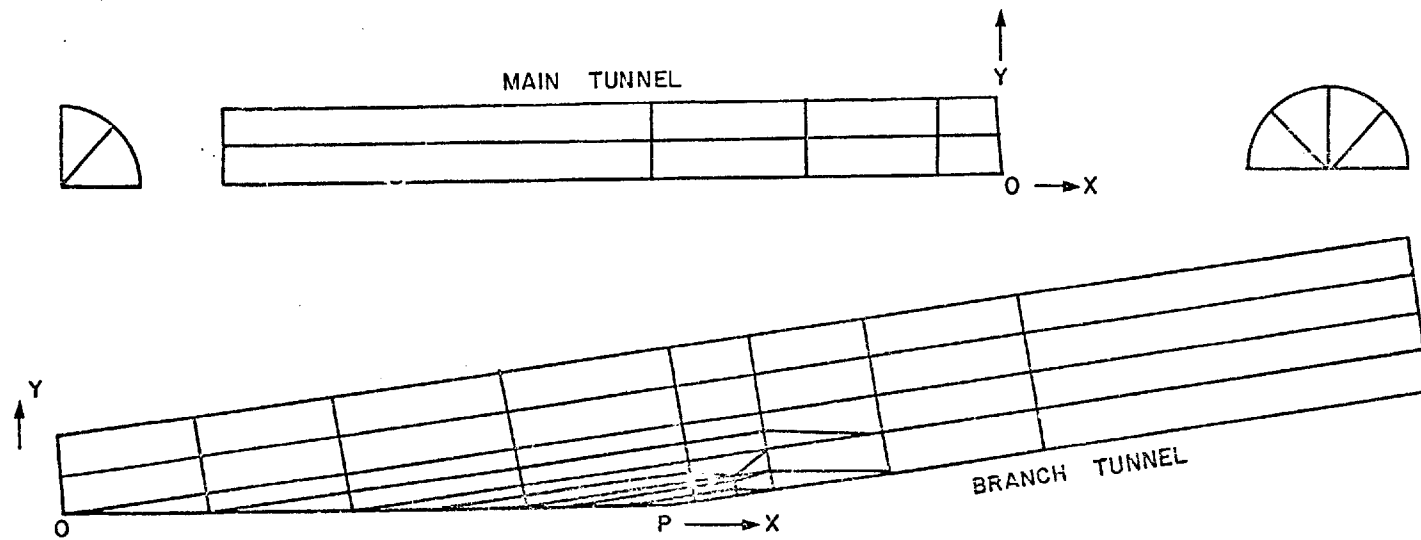


FIGURE 3.2.8 BOUNDARY ELEMENT CONFIGURATION FOR BIFURCATION JUNCTION

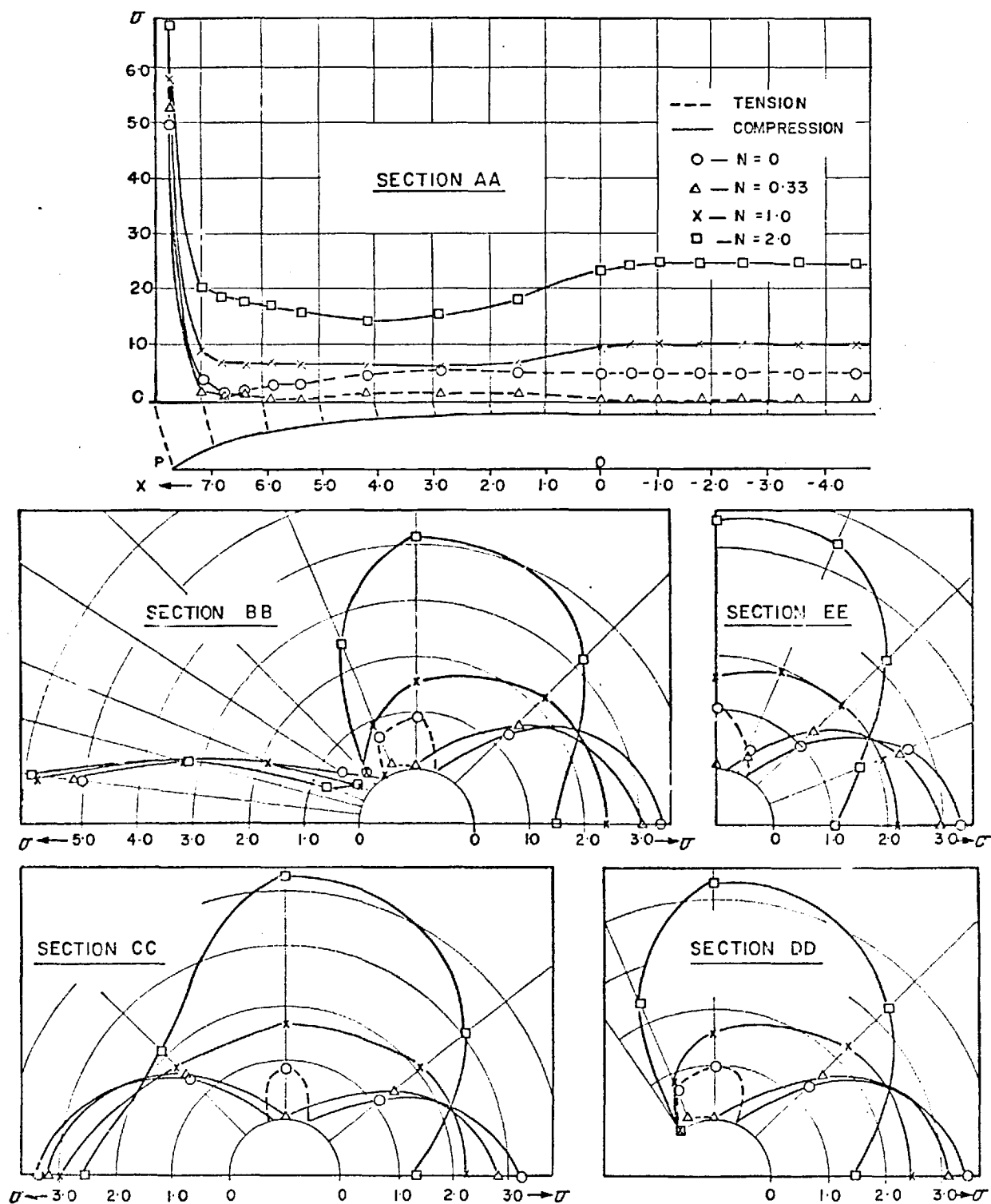


FIGURE 3.2.3 BOUNDARY STRESSES AT SELECTED SECTIONS FOR 15° BIFURCATION

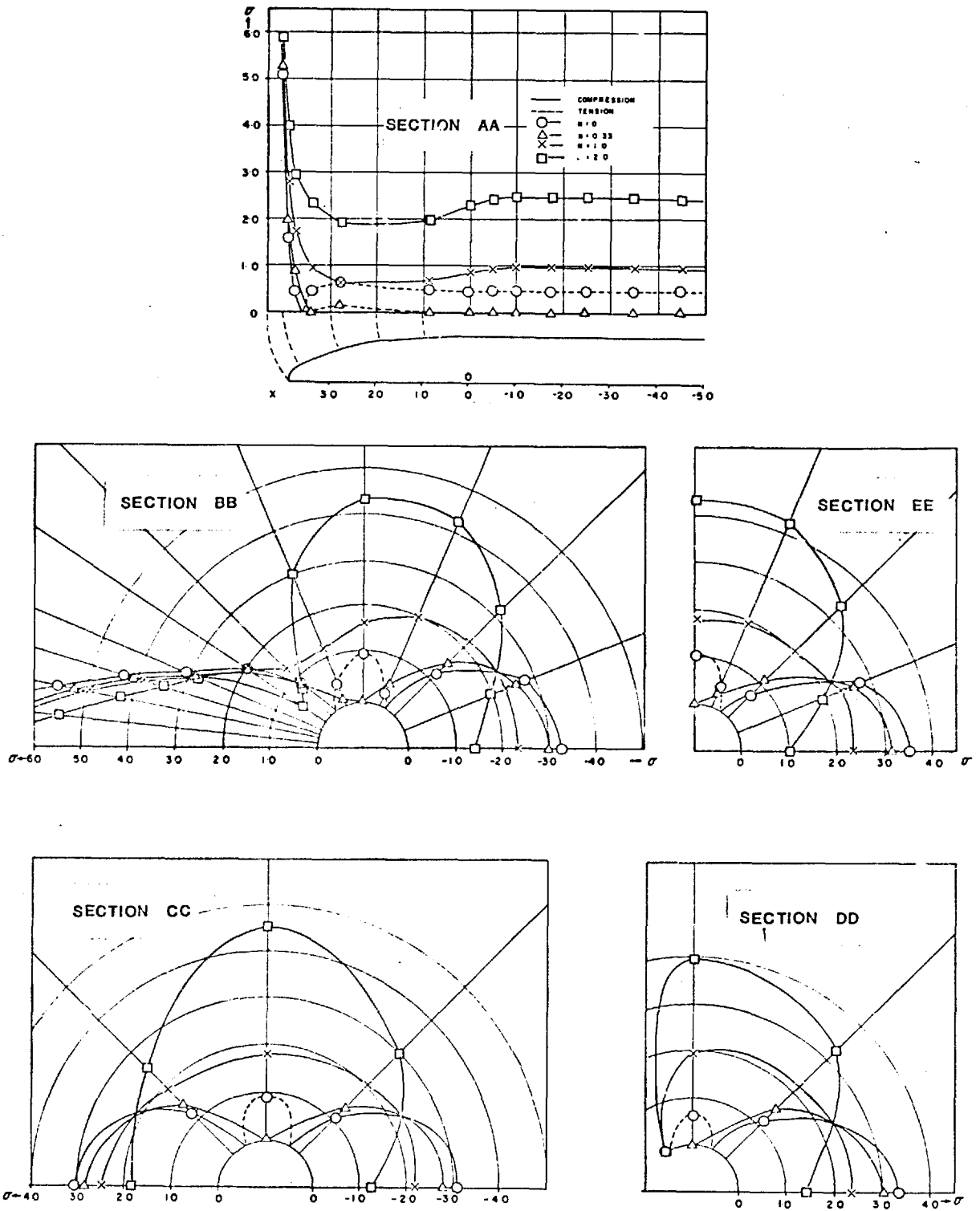


FIGURE 3.2.10 BOUNDARY STRESSES AT SELECTED SECTIONS FOR 30° BIFURCATION

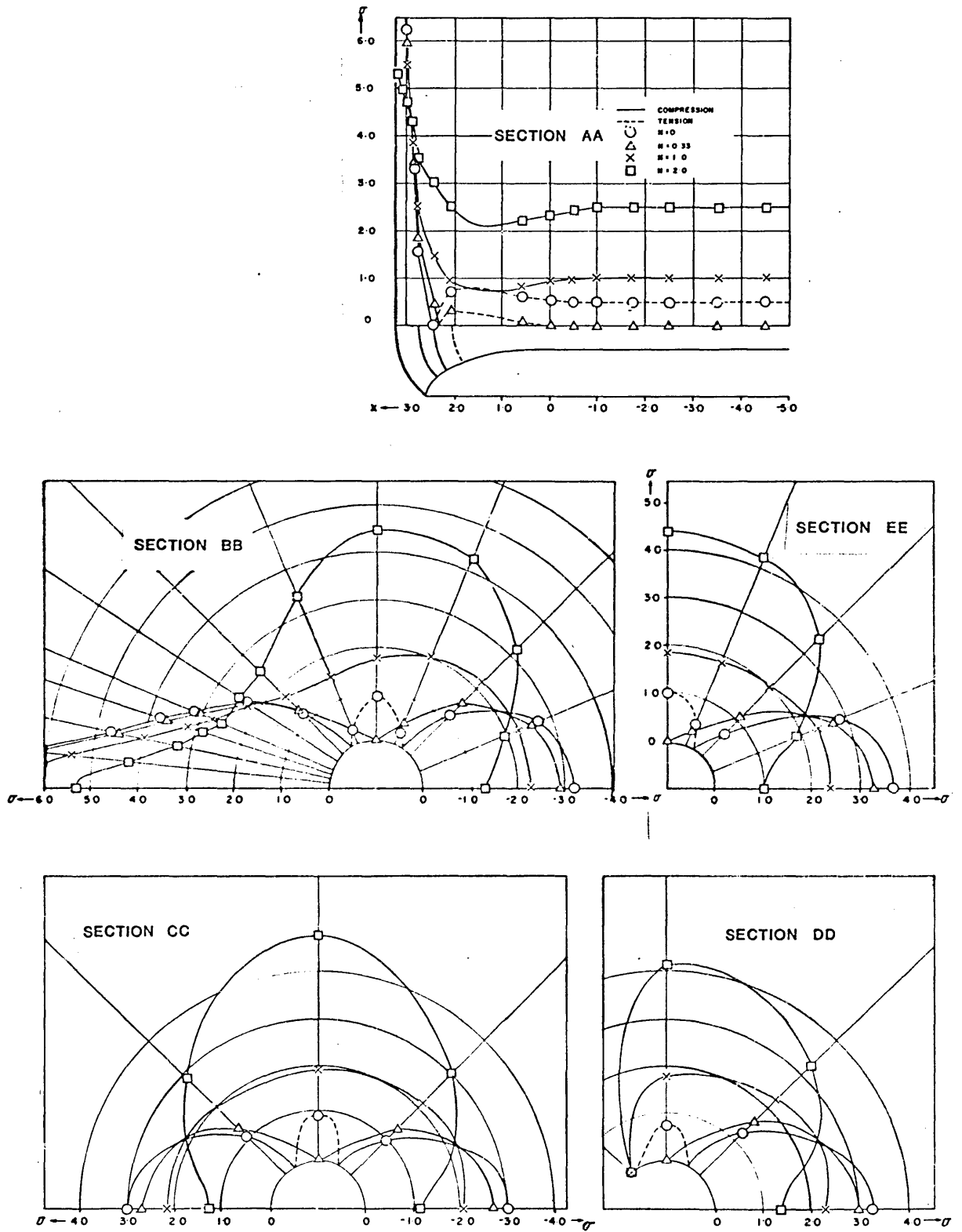


FIGURE 3.2.11 BOUNDARY STRESSES AT SELECTED SECTIONS FOR 45° BIFURCATION

In the case of the 15° bifurcation, the boundary stresses plotted for section AA, Figure 3.2.9 (a), over the centre of the intersection (point O) differ only slightly from those at a point well removed from the intersection (i.e. $x = \text{large negative}$). On the other hand, at the springline level, i.e. point P, Figure 3.2.9 (a), there exist high compressive stress concentrations for all values of N considered. For the cases of $N = 0, 0.33, 1.0$ and 2.0 , the boundary stresses σ_{zz} , at point P are 35.7, 32.3, 28.6 and 22.7 respectively. Obviously these stress concentrations are very limited in areal extent and the stress gradients near the point P are extremely high. At section CC, Figure 3.2.9 (c), the boundary stresses are very close to those in the case of two long parallel circular tunnels. This point highlights the areal influence that the intersection has on the remainder of the excavation. The zone of influence of the 15° Y-junction intersection for the stress fields considered is three times the diameter of the circular tunnels. However, it must be borne in mind that the two branch tunnels, Figure 3.2.7, still affect each other some distance removed from the intersection.

Boundary stresses plotted for the 30° bifurcation are shown in Figure 3.2.10. For section AA, Figure 3.2.10 (a), the boundary stresses are similar to the 15° bifurcation except that at point P, σ_{zz} has values of 17.2, 15.9, 13.1 and 9.0 for the cases of $N = 0, 0.33, 1.0$ and 2.0 respectively. Two diameters removed from point P, i.e. section CC, Figure 3.2.10 (c), the boundary stresses around each branch tunnel for $N = 0$ and 0.33 are approximately the same as those for the case of a single long circular tunnel. However, for the cases of $N = 1.0$ and 2.0 , the boundary stresses depart further from the plane strain isolated long circular tunnel solution, primarily because of the shadowing effect of the neighbouring tunnel to horizontal field stresses. This is apparent when comparing the boundary stresses, σ_{zz} , at springline level in Figure 3.2.10 (c); i.e. on the left σ_{zz} is 2.5 and 1.8 for $N = 1.0$ and 2.0 , while on

the right σ_{zz} is 2.2 and 1.2 for $N = 1.0$ and 2.0 respectively. The zone of influence of the intersection is two and a half times the diameter of the circular tunnels. However as noted previously, the two branch tunnels will affect each other some distance removed from the intersection.

For the 45° bifurcation considered, the boundary stresses are plotted at selected sections in Figure 3.2.11. For section AA, Figure 3.2.11 (a), the boundary stresses are similar to those for the previous two bifurcations, except that at point P, σ_{zz} has values of 11.6, 10.6, 8.5 and 5.3 for the four cases of field stresses applied, $N = 0, 0.33, 1.0$ and 2.0 respectively. At section CC, Figure 3.2.11 (c) two tunnel diameters removed from the point P, the boundary stresses for $N = 0$ and 0.33 are the same as those of an isolated circular tunnel under plane strain conditions. For the cases of $N = 1.0$ and 2.0 , the departure from plane strain is very slight, within 5% of the maximum stress concentration under plane strain conditions.

An interesting point is the minimal effect the neighbouring branch tunnel has on the boundary stresses, two tunnel diameters from point P. (e.g. compare boundary stresses on the left and right sides of the tunnel at section CC, Figure 3.2.11 (c)). The zone of influence of the 45° Y - junction is twice the diameter of the circular tunnels.

The vertical stresses at springline level within the pillar, i.e. between the two branch tunnels, are plotted for the three bifurcations in Figure 3.2.12. The most important feature of all three bifurcations is the high gradient of σ_{zz} near the point P. From these curves and the relevant sections, the volume of the pillar subjected to a certain level of vertical stress and greater can be assessed. It is worth emphasising that it is the extent of a stress concentration that is important in the design of excavations in rock and not merely the magnitude of the stress concentration.

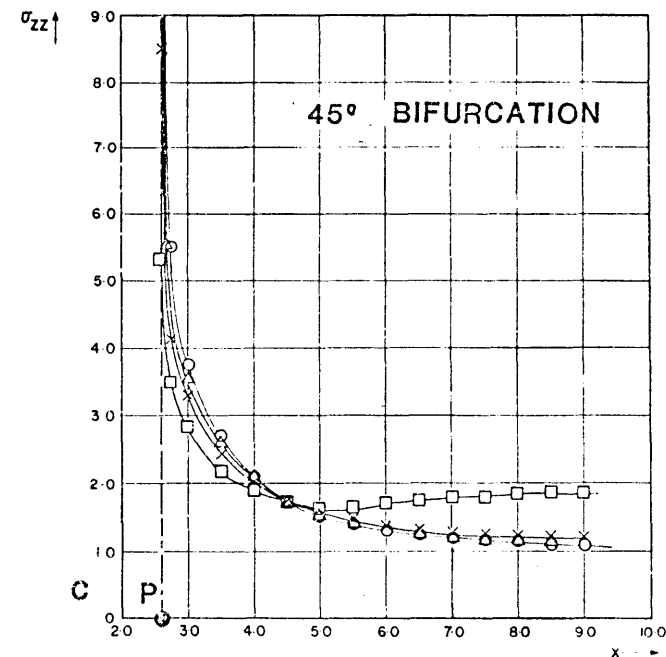
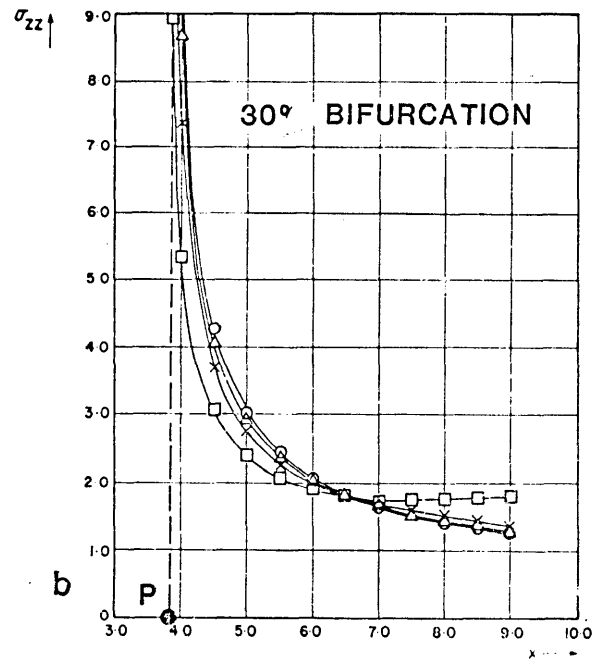
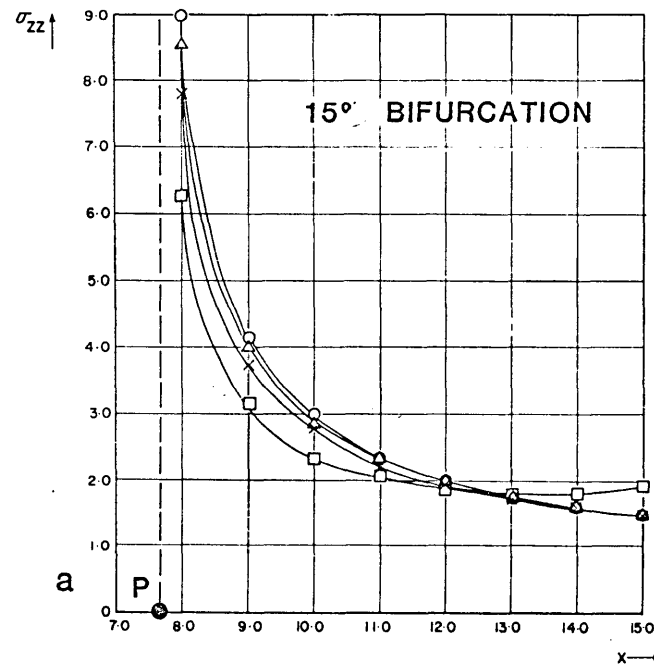


FIGURE 3.2.12 VERTICAL STRESSES AT MID-HEIGHT ALONG PILLAR CENTRELINE FOR a) 15° BIFURCATION
b) 30° BIFURCATION c) 45° BIFURCATION

Comparison of the + and T Junctions

To facilitate a comparison between these two types of intersection of circular tunnels, the boundary stresses at selected points for the two junctions are shown in Figures 3.2.13 and 3.2.14 for $N = 0$ and $\nu = 0.25$.

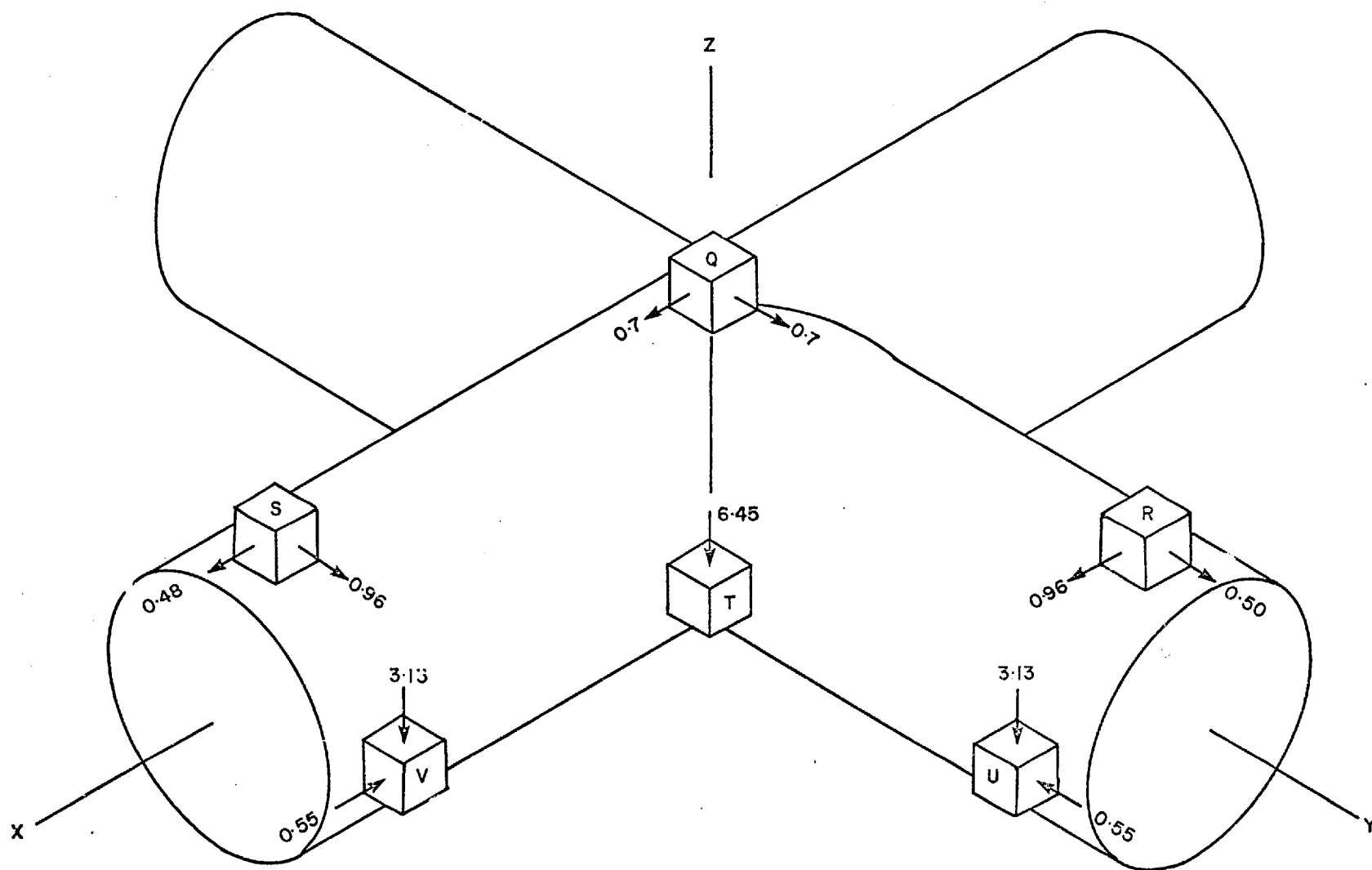


FIGURE 3.2.13 BOUNDARY STRESSES AT SELECTED POINTS FOR CROSS JUNCTION

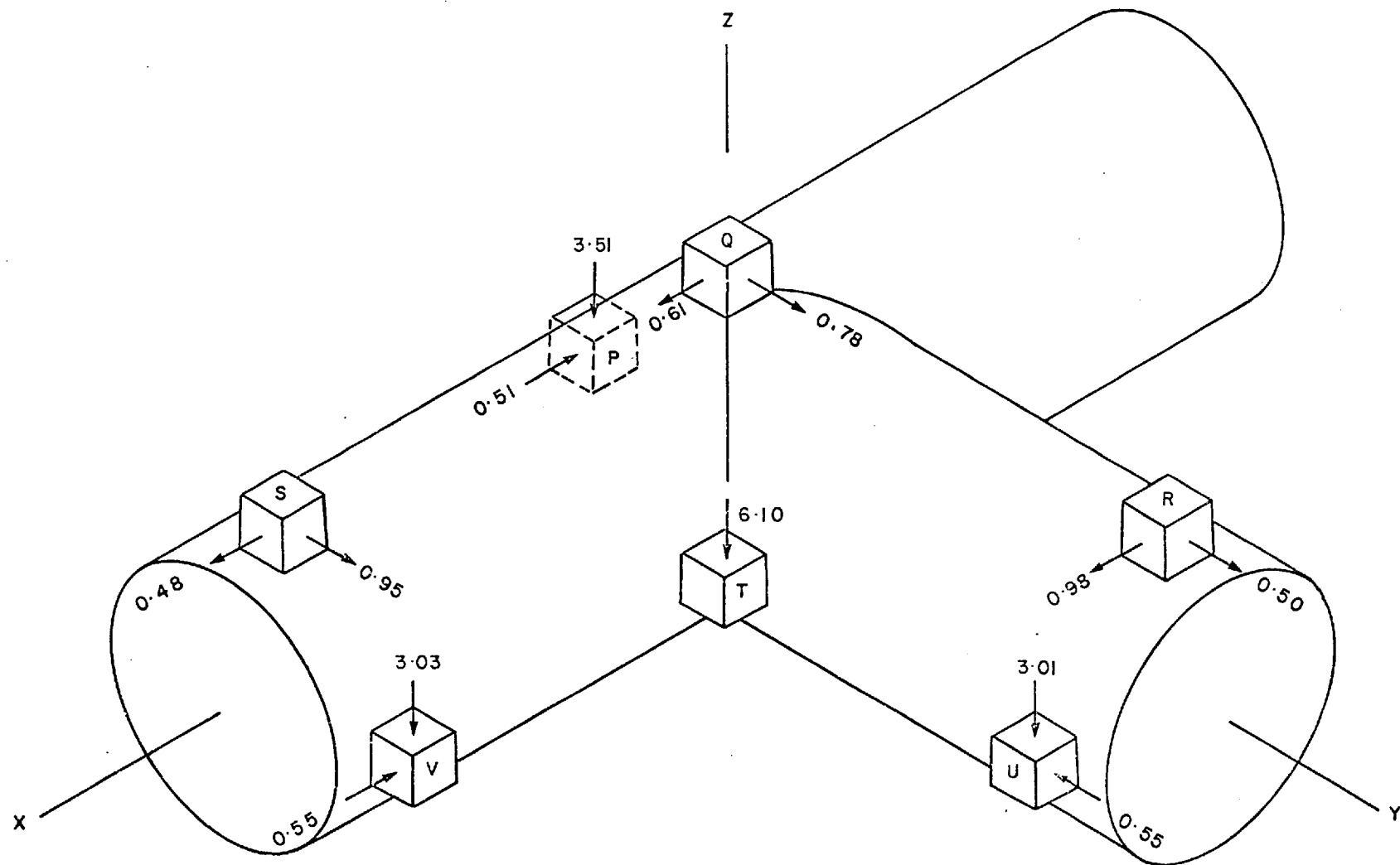


FIGURE 3.2.14 BOUNDARY STRESSES AT SELECTED POINTS FOR T JUNCTION

3. The 1100 Orebody at Mount Isa.

Introduction

The Mount Isa Mine, at Mount Isa, Queensland, Australia, is a major producer of copper, lead, zinc and silver, obtained primarily by underground mining methods. A major proportion of the copper ore is extracted from the 1100 orebody, which is shown in plan, Figure 3.3.1, and longitudinal section, Figure 3.3.2 (Morris and Blair (1976)), (Bridges (1976)). The ore rocks consist of siliceous and dolomitic breccias within a pyritic shale sequence and have unconfined compressive strengths from 150 - 250 MPa. The ore zone is traversed by several major fault sets. Sub-level open stoping with subsequent pillar recovery between cemented rock filled stopes has been selected as the mining method. At present the orebody has main transverse (East-West) pillars at 80 metre centres linked to rib (North-South) pillars, Figure 3.3.1. The maximum vertical extent of the stopes is approximately 300 metres.

The reason for undertaking two and three dimensional stress analysis of this orebody was primarily to determine the applicability of two dimensional analyses. Therefore it was not necessary to model the 1100 orebody exactly but rather study a simplified geometry in order to help interpretation of the analyses and thus provide an insight into the applicability of two dimensional analysis.

The problem posed

The 1100 orebody as shown in Figures 3.3.1 and 3.3.2 was simplified by removing the rib pillars to that illustrated in Figure 3.3.3. Only one eighth of the problem is shown since it possesses symmetry with respect to all three axes. The main transverse pillars are shaded

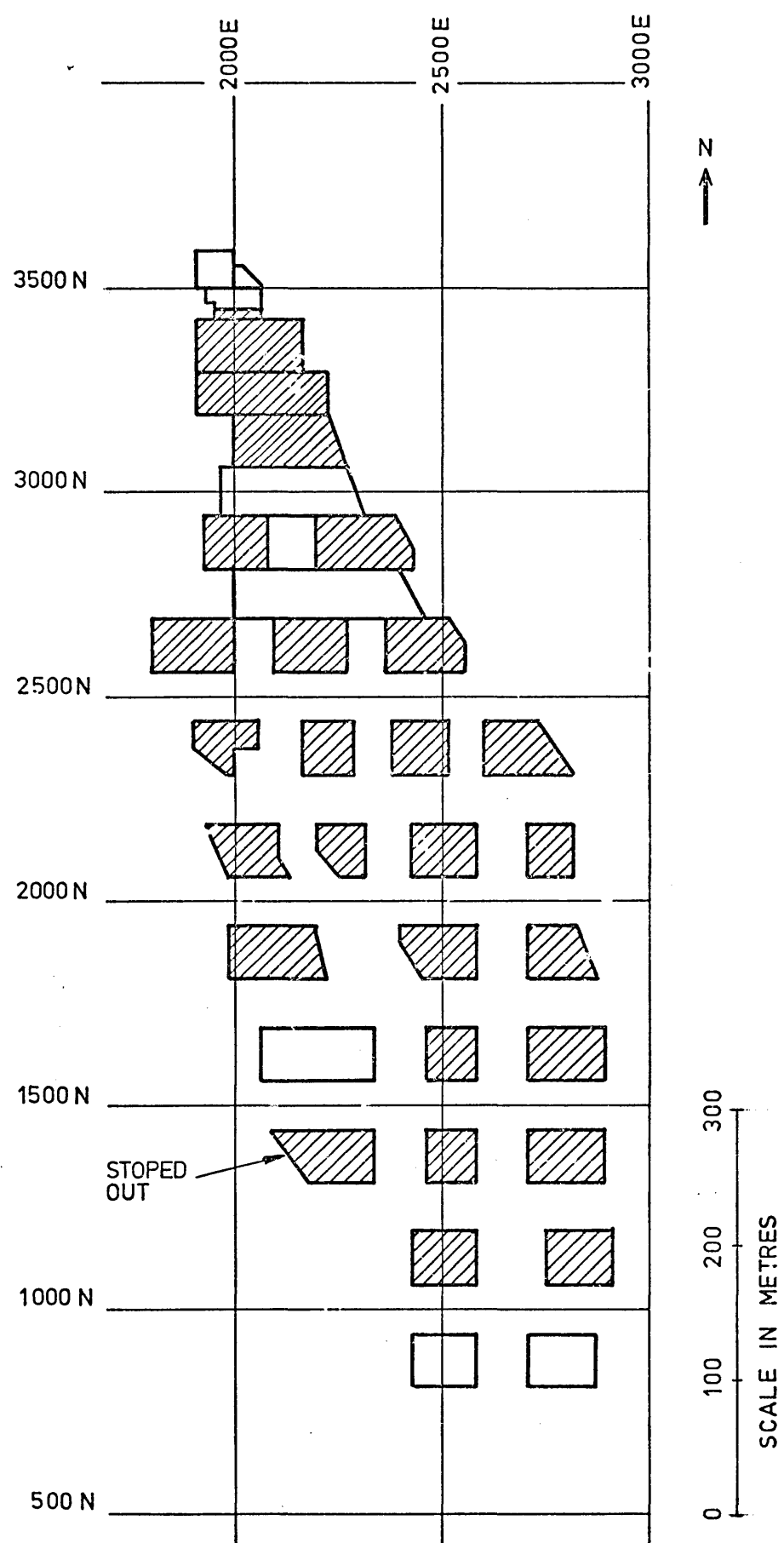


FIGURE 3.3.1A GENERALISED PLAN OF 1100 OREBODY OPEN STOPES AT ABOUT THEIR MID-HEIGHT. (BRIDGES (1976)).

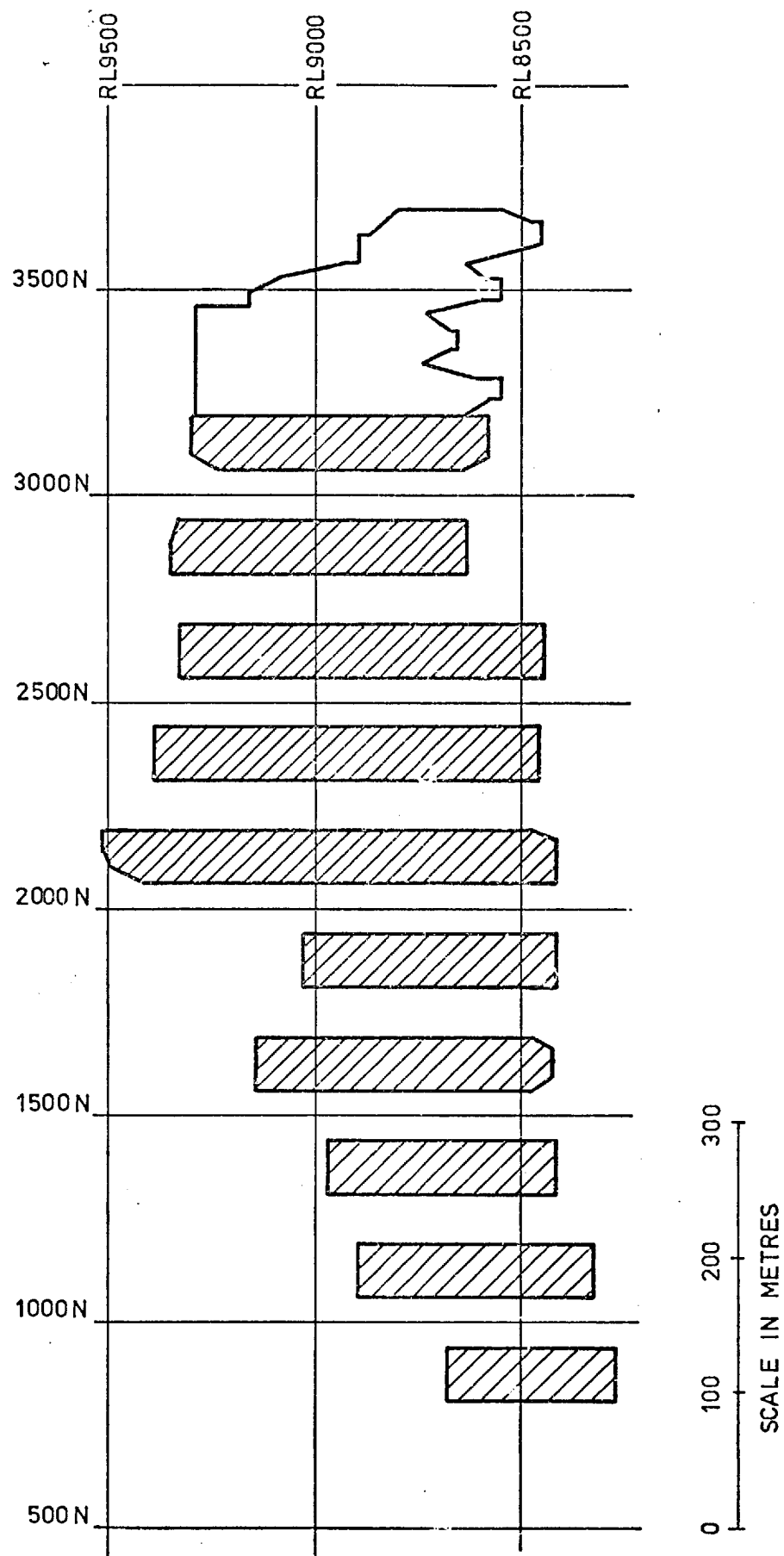


FIGURE 3.3.2 A COMPOSITE LONGSECTION OF 1100 OREBODY OPEN STOPES. (BRIDGES (1976)).

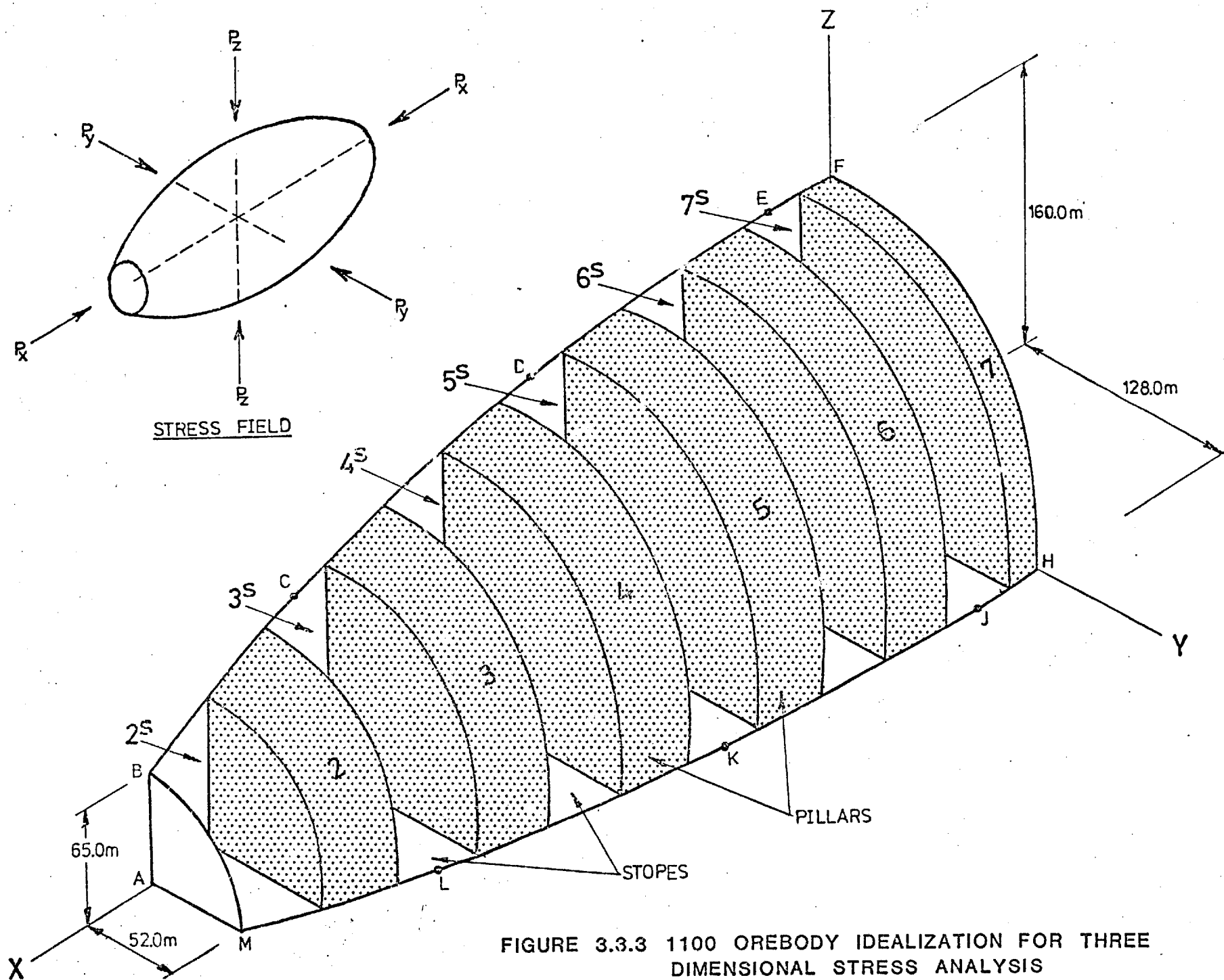


FIGURE 3.3.3 1100 OREBODY IDEALIZATION FOR THREE DIMENSIONAL STRESS ANALYSIS

and numbered 2 to 7 accompanied by stopes 2^S to 7^S . The stopes and pillars are all 40 metres in the X direction, and the overall shape of the problem looks like a fat cigar with maximum dimensions of 920 metres, 256 metres and 320 metres in the X, Y and Z directions respectively. The problem was sub-divided into 7 regions, the first being the infinite region containing a cigar shaped cavity and regions 2 to 7 corresponding to the pillars. The boundary element configuration for subregion one, and subregions two to seven are shown in Figures 3.3.4 and 3.3.5 respectively.

The stress field in the area has been determined previously, (Hoskins (1967), Brady et al (1976)), as $P_x = 13.8$ MPa, $P_y = 34.5$ MPa and $P_z = 20.7$ MPa. Young's modulus E and Poisson's ratio ν , for the rock mass were 69000 MPa and 0.3 respectively.

The pillars were reduced in stiffness by scaling down E and ν for the pillar by the same amount. This was to take into account the effect of geological weaknesses present in the pillars. The cases analysed are listed in Table 3.3.1.

TABLE 3.3.1. THE SIX CASES ANALYSED

A. Pillar 2 and stope 2^S were not modelled

- A1 - Pillars 3, 4, 5, 6 and 7 same properties as host rock (E, ν)
- A2 - Pillars 3 and 4 = E and ν , Pillars 5, 6, and 7 = $0.5E$ & 0.5ν
- A3 - Pillars 3 and 4 = E and ν , Pillars 5, 6 and 7 = $0.2E$ & 0.2ν
- A4 - Pillars 3 and 4 = E and ν , Pillars 5, 6 and 7 removed.

B. Pillars 5, 6 and 7 removed, pillar 2 and stope 2^S modelled

- B1 - Pillars 2, 3 and 4 same properties as host rock (E, ν)

C. Pillars 2, 3, 4, 5, 6 and 7 removed

- C1 - All pillars removed.

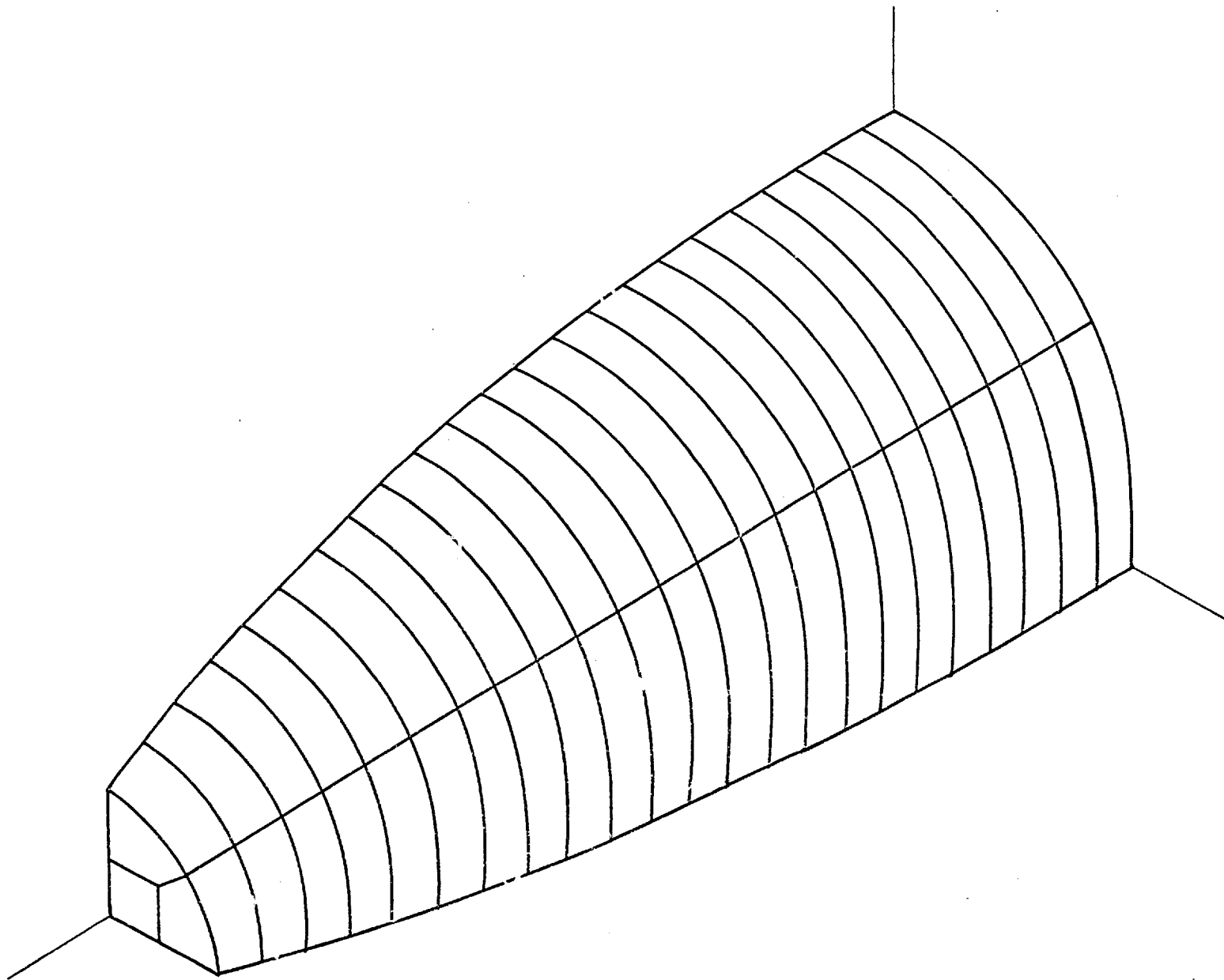
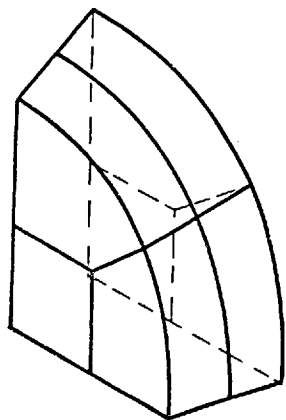
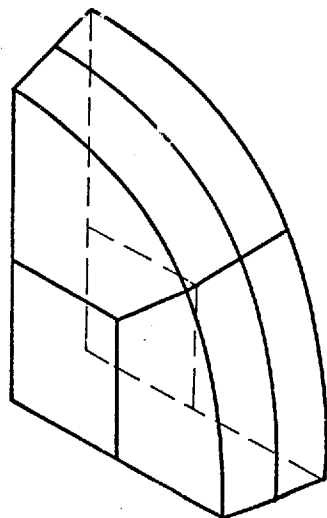


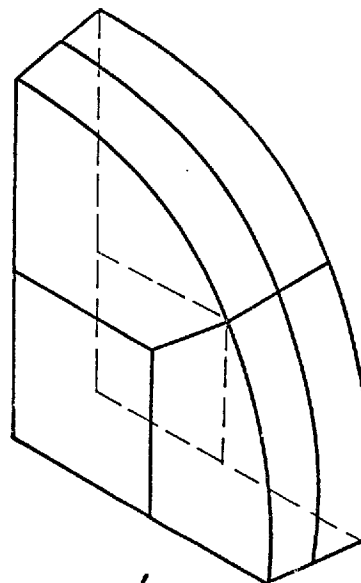
FIGURE 3.3.4 BOUNDARY ELEMENT CONFIGURATION FOR SUBREGION 1



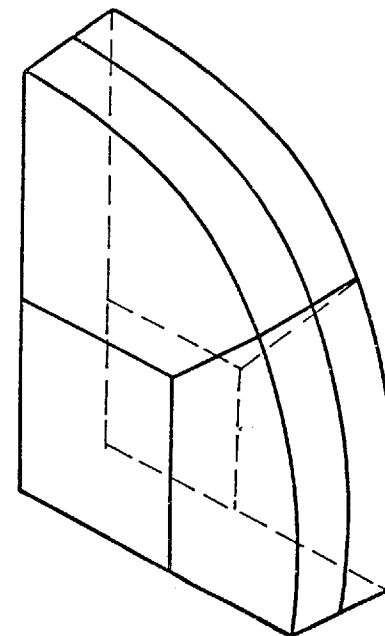
2



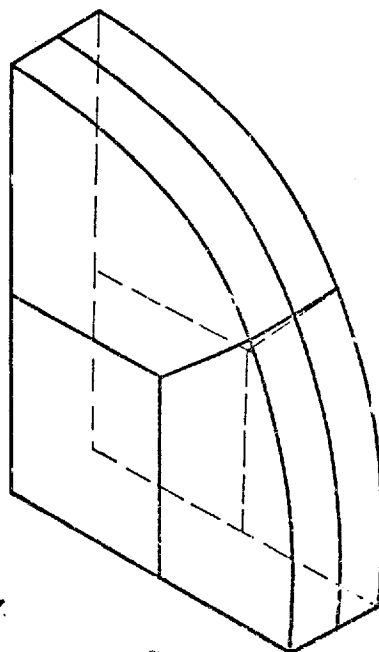
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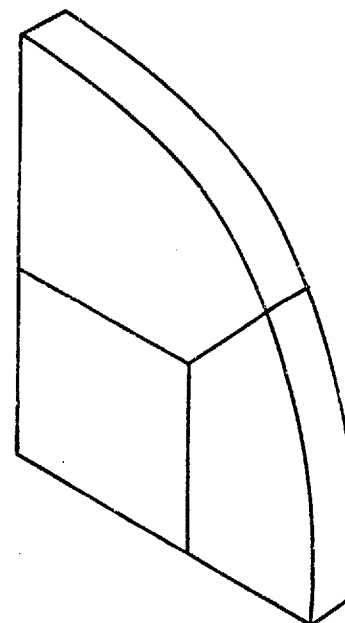
4



5



6



7

FIGURE 3.3.5 BOUNDARY ELEMENT

CONFIGURATION FOR SUBREGIONS 2-7.

In all cases computed the pillars were each subjected to approximately constant biaxial compression. Listed in Table 3.3.2 are the stresses in MPa at the centres of the pillars for the three dimensional analyses and in Table 3.3.3 for the two dimensional analyses.

For the cases listed in Table 3.3.2 the pillar with the maximum variation in compressive stresses across the pillar was pillar 4. In pillar 4, the variation along the X axis, travelling from one face to the other in a positive X direction was σ_{yy} from 60 to 50 MPa and σ_{zz} from 31 to 25 MPa. All the pillars were each under approximately constant biaxial compression for at least 80% of their respective volumes. The close agreement between the two and three dimensional analyses for case A1 (Tables 3.3.2 and 3.3.3), indicates that two dimensional analyses appear applicable for up to 50% extraction of the orebody. However, for case B1, the two and three dimensional analyses are not in good agreement, because of the major effect the three dimensional shape of the orebody has on the resulting stress distributions. For all other cases, A2 - A4, the level of agreement between two and three dimensional analyses, lies between those of cases A1 and B1.

Boundary stresses at cross sections in the YZ plane, through the centres of Stopes 3^S, 5^S, and 7^S are illustrated in Figure 3.3.6. The boundary stresses are those acting in the plane of the figure and have been plotted perpendicular to the stope walls. For all three cross-sections, the boundary stresses are identical for cases A1 and C1. It is worth highlighting the small effect that removing pillars 5, 6 and 7 has on the boundary stresses plotted in Figure 3.3.6(a), and also the small effect on the boundary stresses in Figure 3.3.6(c) to the removal of pillars 2, 3 and 4. Under plane strain conditions the boundary stresses for an elliptical hole with a cross-section as in Figure 3.3.6 are (Terzaghi and Richart (1952)) 102 MPa in the crown and 39 MPa in the sidewall i.e. at points C,D and E and points I, K and J respectively.

TABLE 3.3.2. PILLAR STRESSES CALCULATED BY THREE DIMENSIONAL
PROGRAM

Case Computed	Stress	Pillar					
		2	3	4	5	6	7
A1	σ_{yy}		45.7	46.5	46.8	46.9	46.9
	σ_{zz}		24.3	24.6	24.8	24.9	24.8
A2	σ_{yy}		47.0	50.1	27.5	28.6	28.7
	σ_{zz}		24.8	26.3	11.6	12.1	12.1
A3	σ_{yy}		48.0	52.9	12.6	13.4	13.5
	σ_{zz}		25.0	27.3	4.2	4.4	4.4
A4	σ_{yy}		49.0	55.4			
	σ_{zz}		25.1	27.8			
B1	σ_{yy}	47.7	49.9	55.7			
	σ_{zz}	24.6	25.5	27.9			

TABLE 3.3.3. PILLAR STRESSES CALCULATED BY TWO DIMENSIONAL
PROGRAM

Case computed	Plane strain with respect to	Stress	Pillar					
			2	3	4	5	6	7
A1	Z	σ_{yy}		50.7	52.3	52.7	53.0	53.2
	Y	σ_{zz}		27.6	28.9	29.3	29.5	29.5
B1	Z	σ_{yy}	65.6	73.0	88.9			
	Y	σ_{zz}	36.8	39.8	47.7			

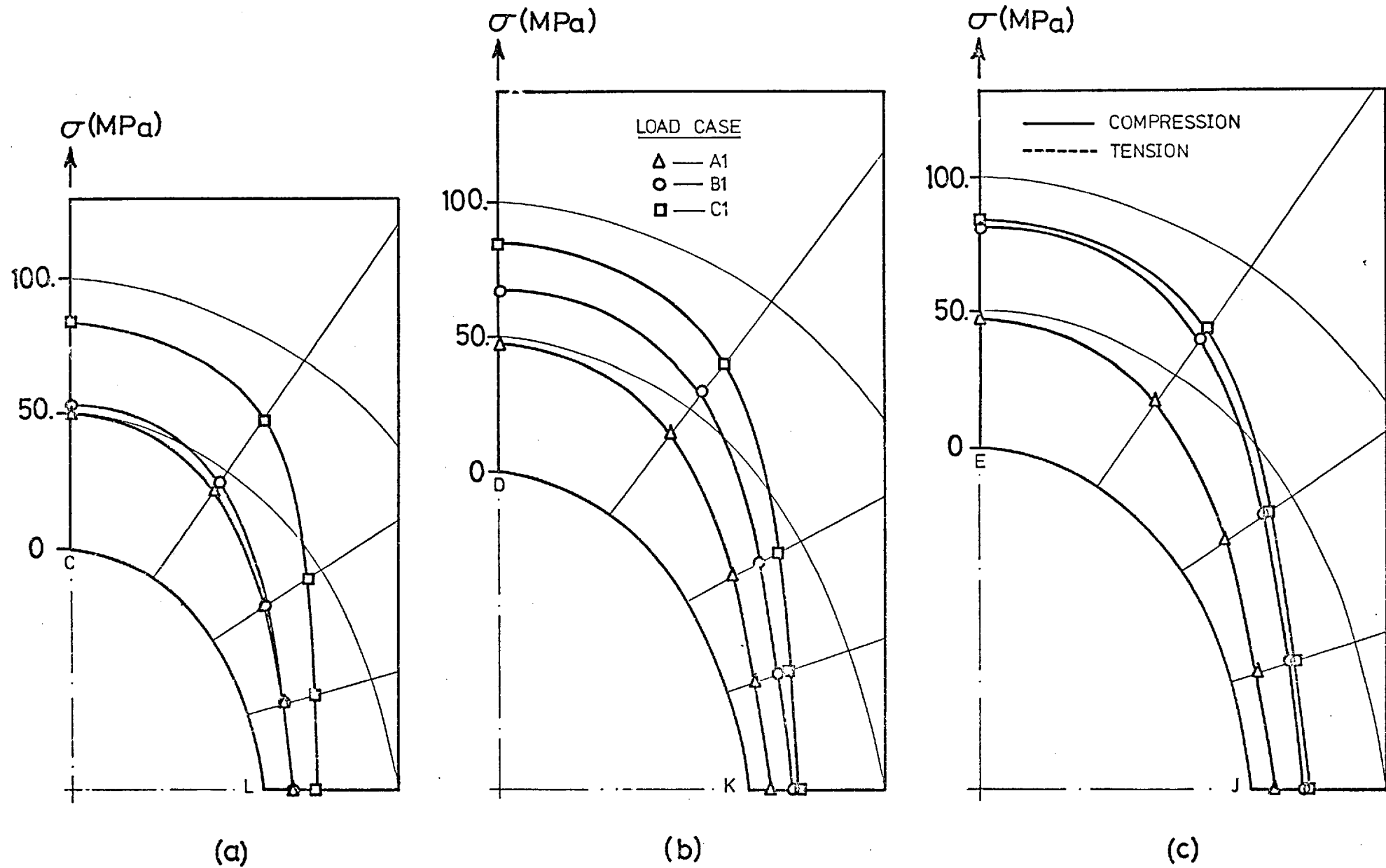


FIGURE 3.3.6 BOUNDARY STRESSES IN THE YZ PLANE
AT CENTRE OF (a) 3° STOPE, (b) 5° STOPE AND (c) 7° STOPE

In Figure 3.3.7 are plotted the boundary stresses for the cross-section in the XZ plane at $Y = 0$ for case A1. In the abutment area, i.e. at X co-ordinate of 380 metres, the two and three dimensional analyses are in good agreement. Over the stope "roof", the two dimensional results have not been plotted; however, they are generally approximately the same as those given by the three dimensional analysis. For a section in the same position but for case B1, boundary stresses are illustrated in Figure 3.3.8. Here the boundary stresses at the abutment for the two dimensional analysis are greater than those computed from the three dimensional analysis. This increase is greater than that shown in Figure 3.3.7 because the effect of the Y dimension of the orebody has a greater impact for this case. This is also demonstrated by the boundary stresses in the roof of the large stoped out area. The extreme case to demonstrate this effect is given in Figure 3.3.9 where boundary stresses are plotted at the same section for case C1. The boundary stresses at the abutment i.e. $X = 420, Z = 0$, are 28 MPa given by the three dimensional analysis compared with 47 MPa from the two dimensional analysis. This difference is also borne out by a comparison of the boundary stresses in the roof of the stoped area for the two analyses.

Boundary stresses for a cross-section in the XY plane at $Z=0$ for case A1 are shown in Figure 3.3.10. In the abutment area, i.e. at X co-ordinate of 380 metres, the two and three dimensional analyses are in good agreement. In the stope roofs, the boundary stresses have not been plotted; however, they differ only slightly from those computed by the three dimensional analysis. At the same section for case B1, the boundary stresses are illustrated in Figure 3.3.11.

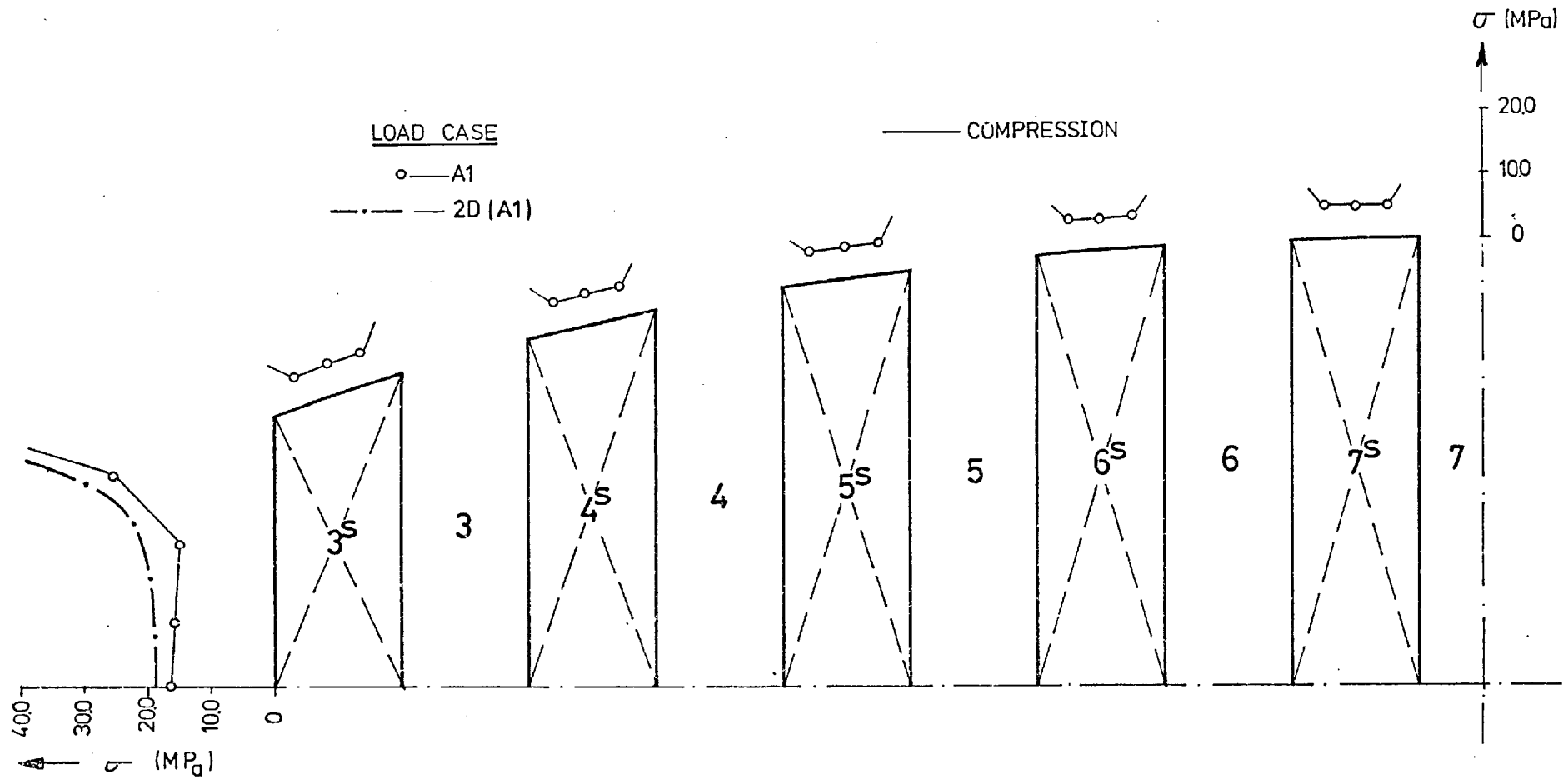


FIGURE 3.3.7 BOUNDARY STRESSES IN THE XZ PLANE AT $Y = 0$
FOR LOAD CASE A1

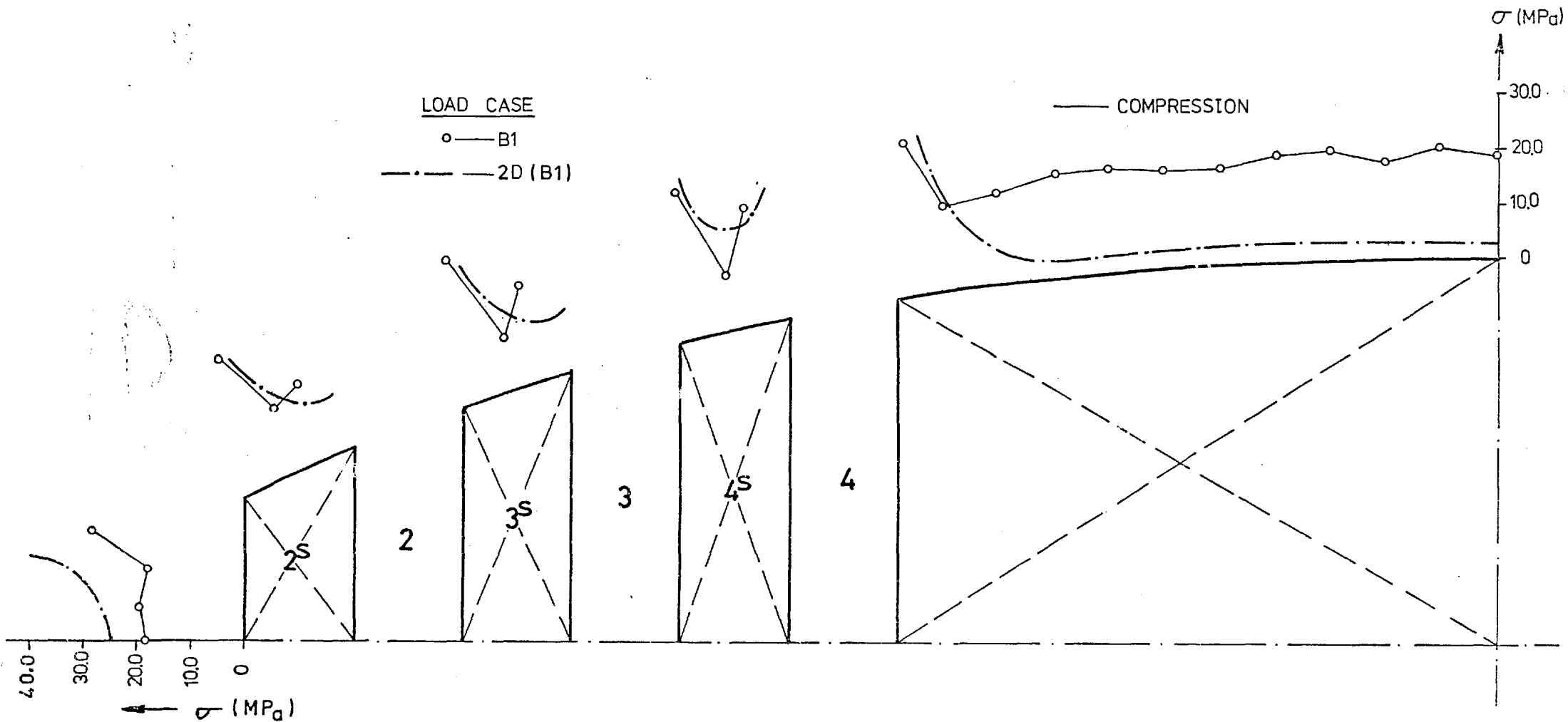


FIGURE 3.3.8 BOUNDARY STRESSES IN THE XZ PLANE
AT $Y = 0$ FOR LOAD CASE B1

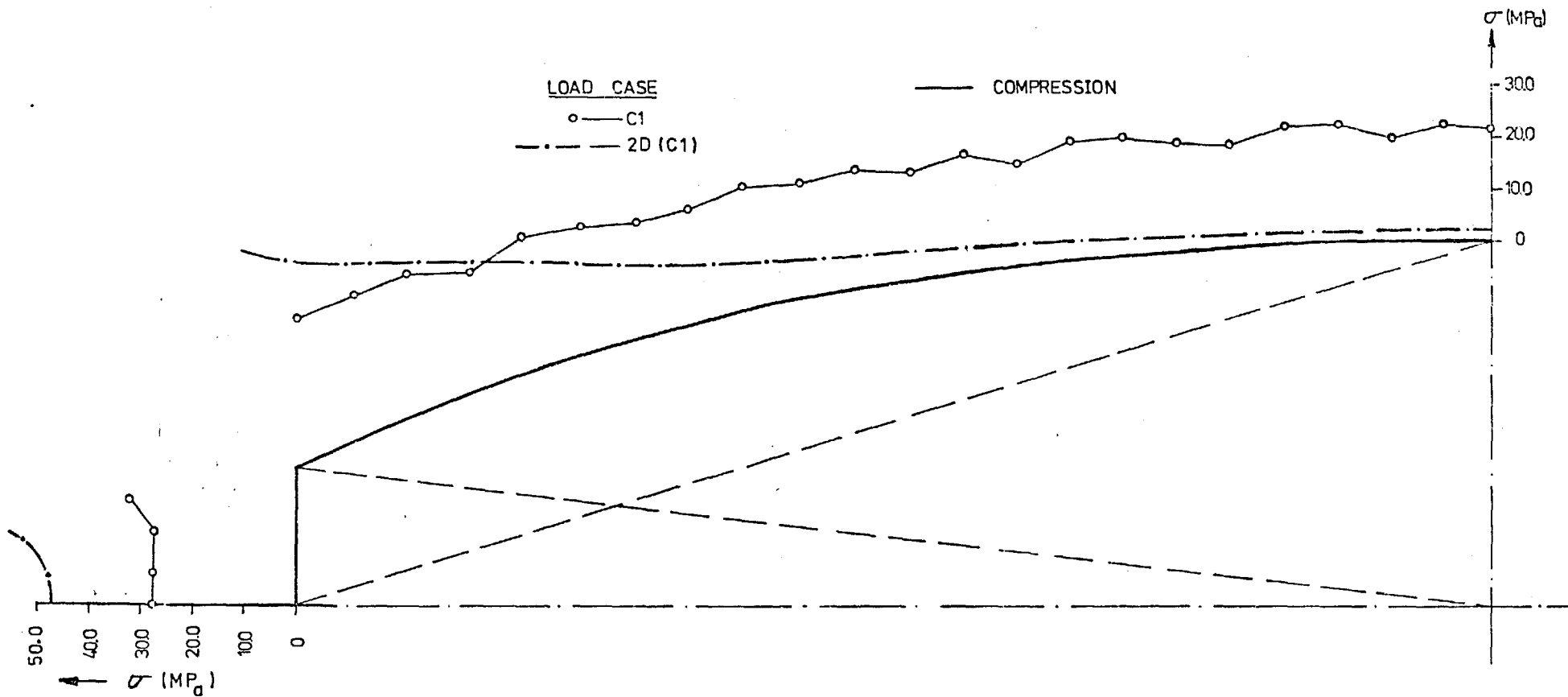


FIGURE 3.3.9 BOUNDARY STRESSES IN THE XZ PLANE
AT $Y = 0$ FOR LOAD CASE C1.

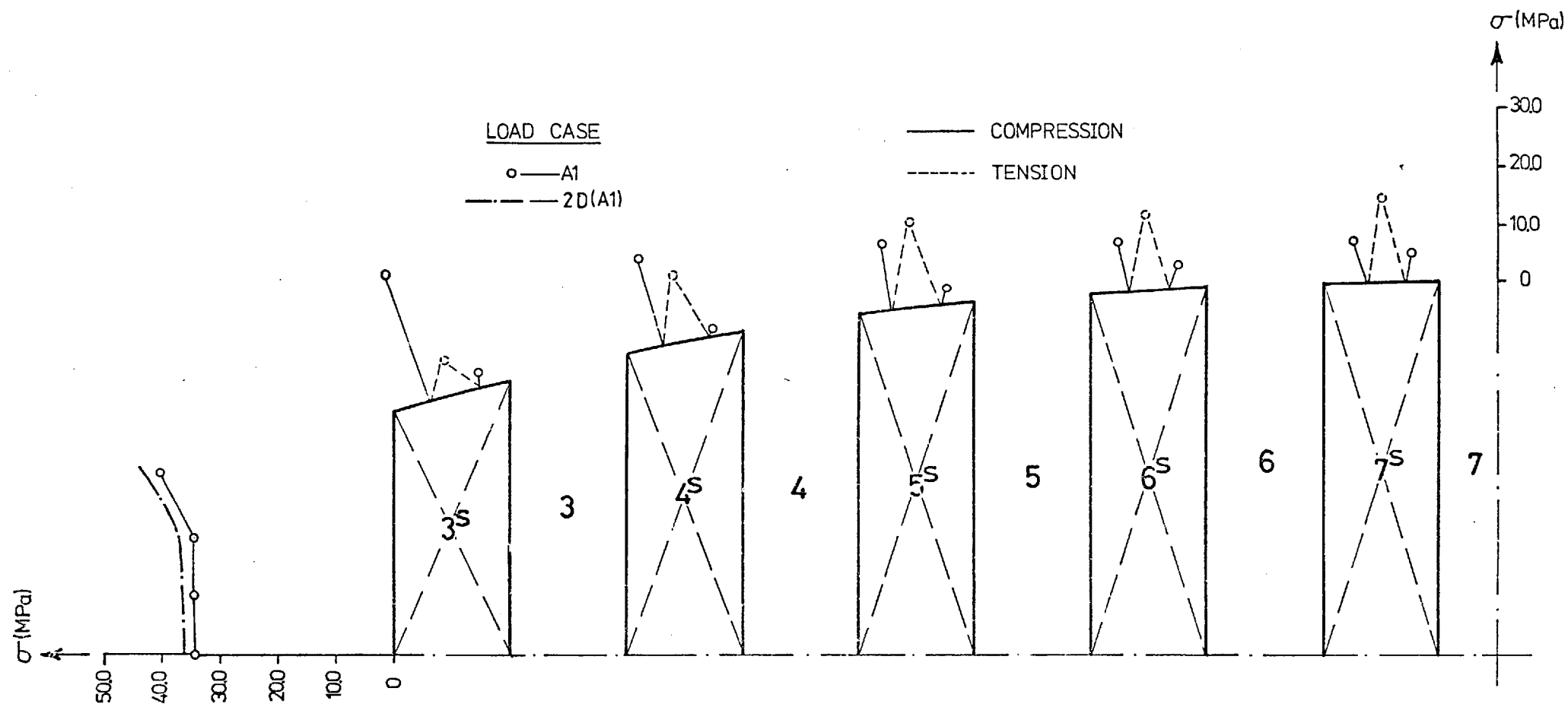


FIGURE 3.3.10 BOUNDARY STRESSES IN THE XY PLANE
 AT $Z = 0$ FOR LOAD CASE A1.

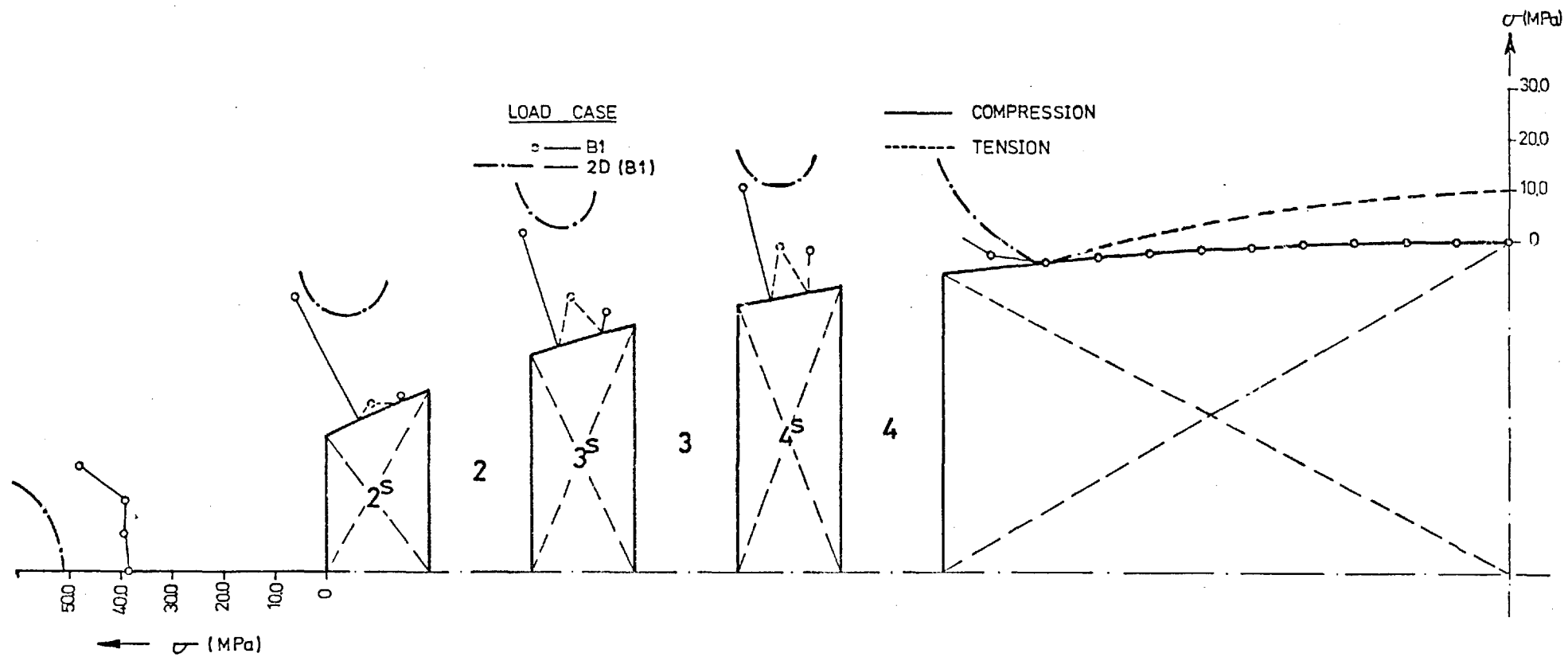


FIGURE 3.3.11 BOUNDARY STRESSES IN THE XY PLANE
AT $Z = 0$ FOR LOAD CASE B1

Results from the two and three dimensional analyses do differ slightly in the abutment area; however, the major point is that two dimensional analysis yields tensile stresses in the roof of the stoped out area. Boundary stresses for case C1 at the same section are shown in Figure 3.3.12. Results from the two and three dimensional analyses differ greatly in the abutment area, e.g. 100 MPa (from two dimensional analysis) compared with 58 MPa (from three dimensional analysis).

Analytical solutions

It is generally more rewarding to look at solutions to simple problems before a complex thorough analysis is undertaken. Firstly a greater insight into the behaviour of the problem is achieved, and secondly simple, quick checks of the more complete analysis are readily available. In Figure 3.3.13 compiled from Obert and Duvall (1967), the variation of stresses at two points on the surface of an ellipsoidal cavity with cavity length are illustrated. The ratio of axes c to b are approximately the same as those of the simplified 1100 orebody, Figure 3.3.3. An important point to note is that the boundary stresses at B, approach those for plane strain conditions when a equals approximately five times b .

Bray (1976b), assuming that the pillars could be approximated by an inclusion material which is in a plane stress condition, solved two dimensionally the 1100 orebody problem, Figure 3.3.3 for case A1. By replacing the inclusion with discrete pillars, assuming the pillars exerted a normal pressure on the cavity walls, yielded the results given in Figure 3.3.14. In Figure 3.3.14 (a) the pillar

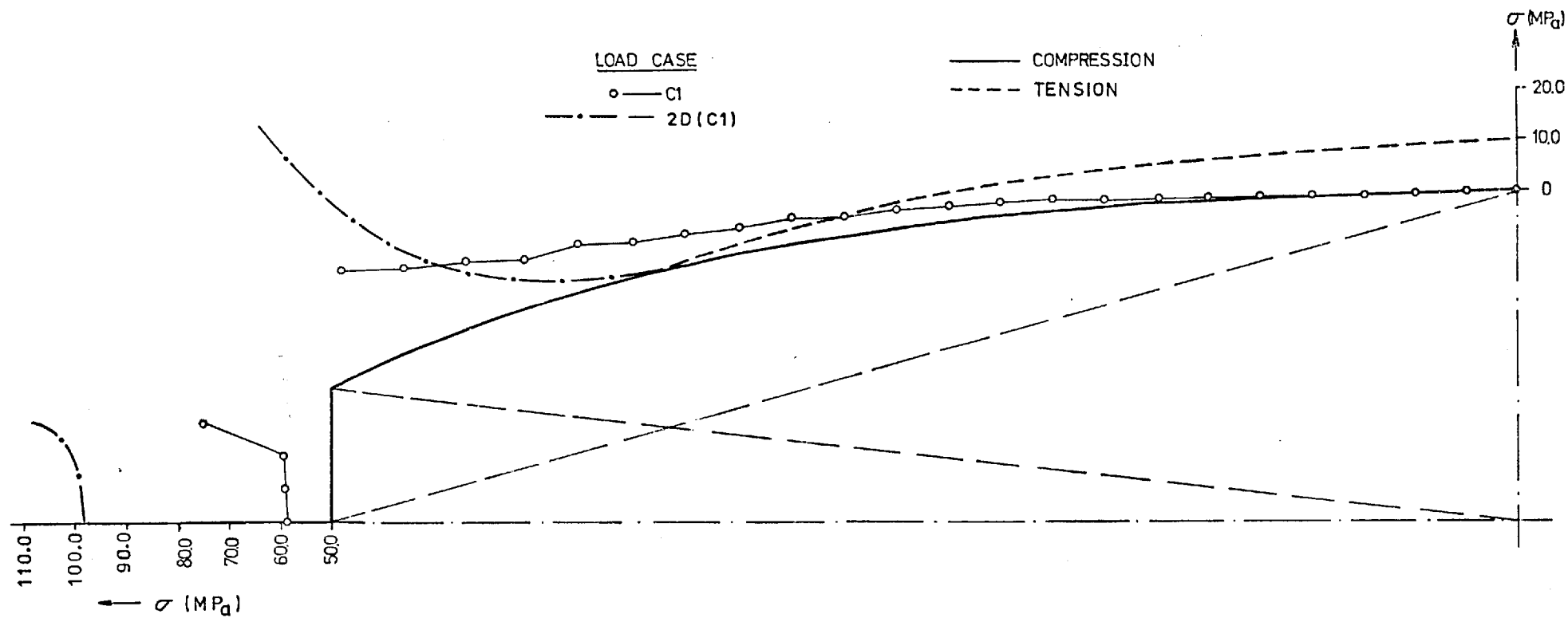


FIGURE 3.3.12 BOUNDARY STRESSES IN THE XY PLANE
AT $Z = 0$ FOR LOAD CASE C1

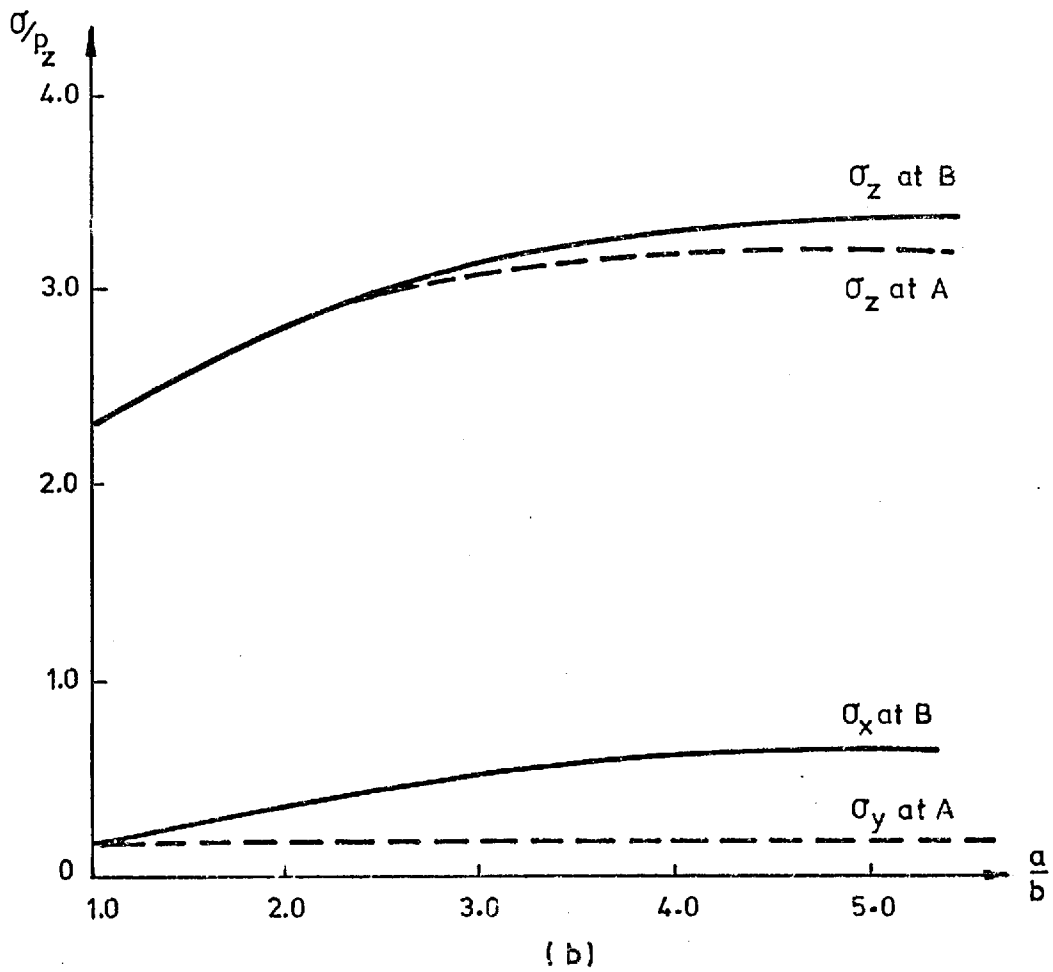
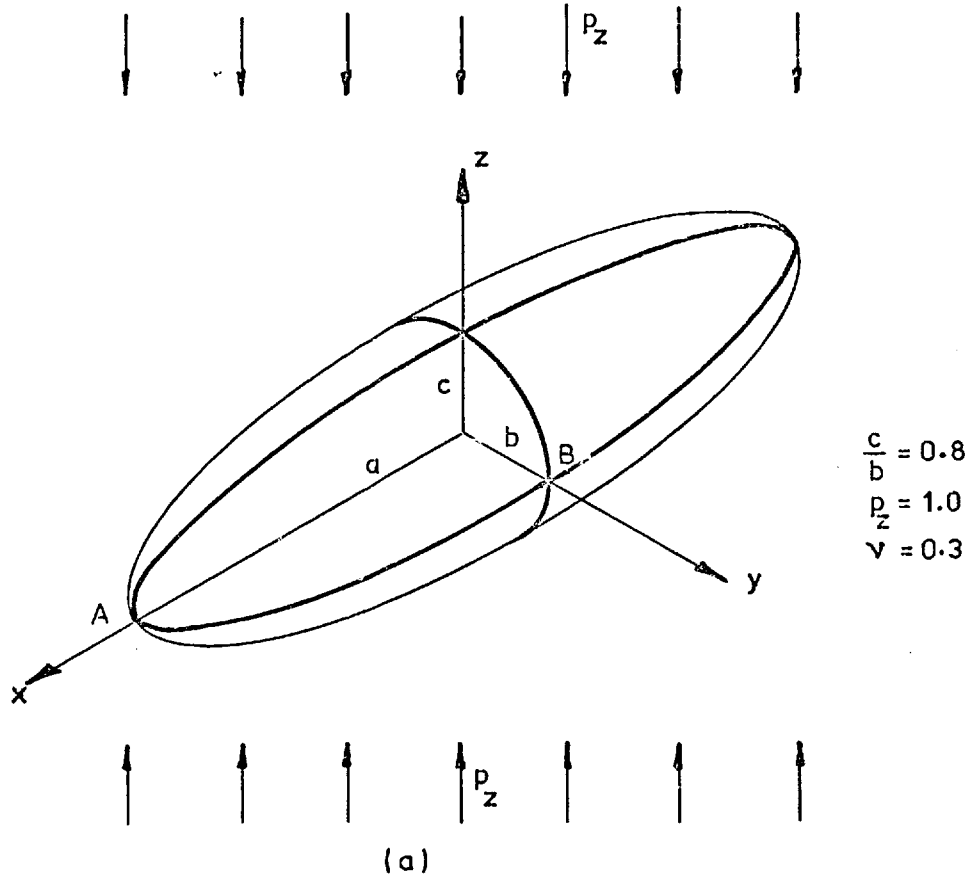
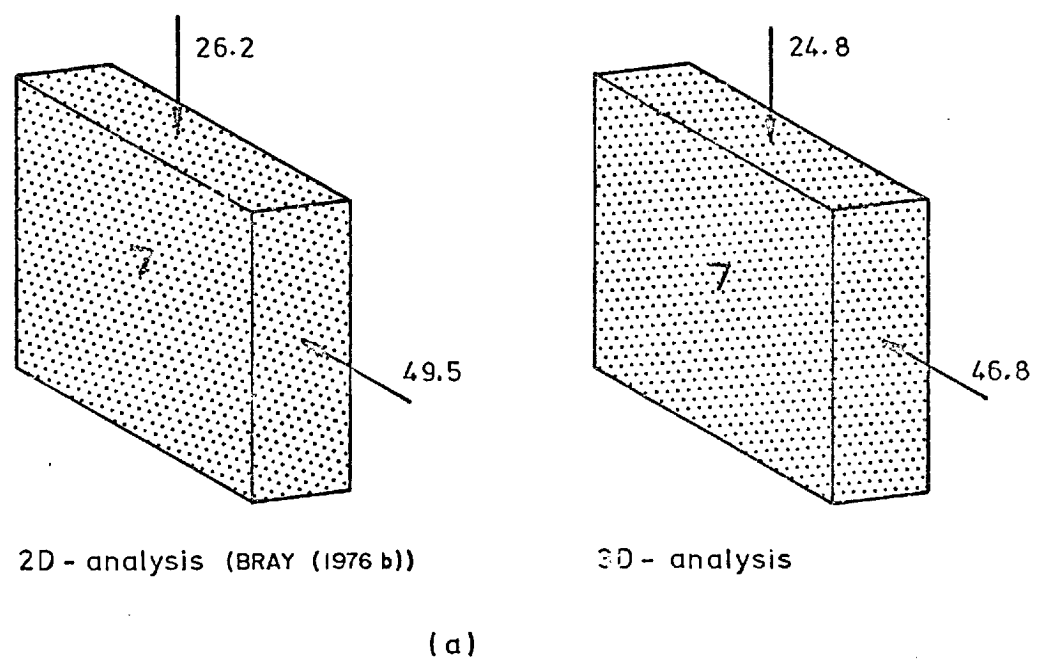


FIGURE 3.3.13 a) ELLIPSOIDAL CAVITY IN UNIAXIAL STRESS FIELD
 b) BOUNDARY STRESSES AT A AND B.



TENSILE STRESSES NEGATIVE

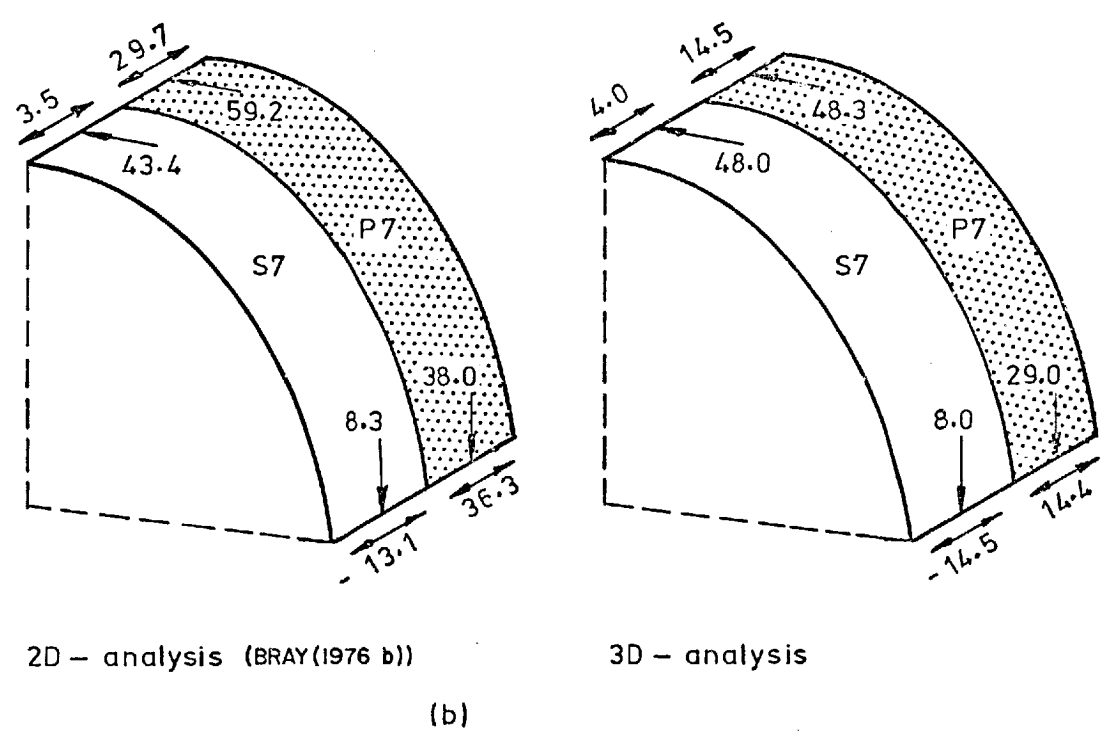


FIGURE 3.3.14 a) STRESSES IN PILLAR 7 b) STRESSES IN PILLAR 7 AND SLOPE 7s AT JUNCTION WITH SUBREGION 1

stresses given by the three dimensional and the two dimensional analysis of Bray are shown to be in good agreement. The stresses, Figure 3.3.14 (b) in the walls and roof of stope 7^S and in pillar 7 at the junction with subregion 1 are not in good agreement. Possible reasons for this discrepancy are (a) the boundary stresses quoted for the two dimensional analysis are the average values over the stope and pillar, while those for the three dimensional analysis are for the central position of stope 7^S and pillar 7, and (b) the pillar is an integral unit of the rock mass (subregion 1) and this fact was not modelled by the two dimensional analysis.

4. The Dinorwic Underground Power Station

The Dinorwic Pumped Storage Scheme (Douglas et al (1977)), is situated in the Snowdonia area of North Wales. The hydraulic head of the scheme is approximately 550 feet and the maximum power output is 1800 MW. The machine and transformer underground chambers lie in the main Cambrian Slate Belt, Anderson et al (1977), exploited in the past on a vast scale and still worked in the nearby Penrhyn Quarries. The state of stress has been determined in the area of the underground complex by C.S.I.R. and Talbott cells as 17.5 MPa in dip direction/dip of $300^{\circ}/26^{\circ}$, 8.0 MPa in $198^{\circ}/26^{\circ}$ and 6.5 MPa in $074^{\circ}/52^{\circ}$ (Anderson et al (1977), Douglas et al (1977)).

In the early stages of the investigation, the stress and displacement distributions around the machine hall test enlargement, Figure 3.4.1, were required for comparison with field measurements. At this time preliminary stress field measurements indicated the probable stress fields shown in Figure 3.4.1. Three dimensional stress analyses of this excavation were undertaken using Watson's program and two dimensional analyses using Bray's boundary element program.

The boundary stresses at particular sections are shown in Figure 3.4.2 for cases 1 and 2. The magnitudes of the boundary stresses have been plotted normal to the excavation, compressive stress plotted inside the medium, tensile inside the excavation. There is close agreement between the two and three dimensional analyses in the crown of the excavation, section AA; however, at the sidewalls they tend to

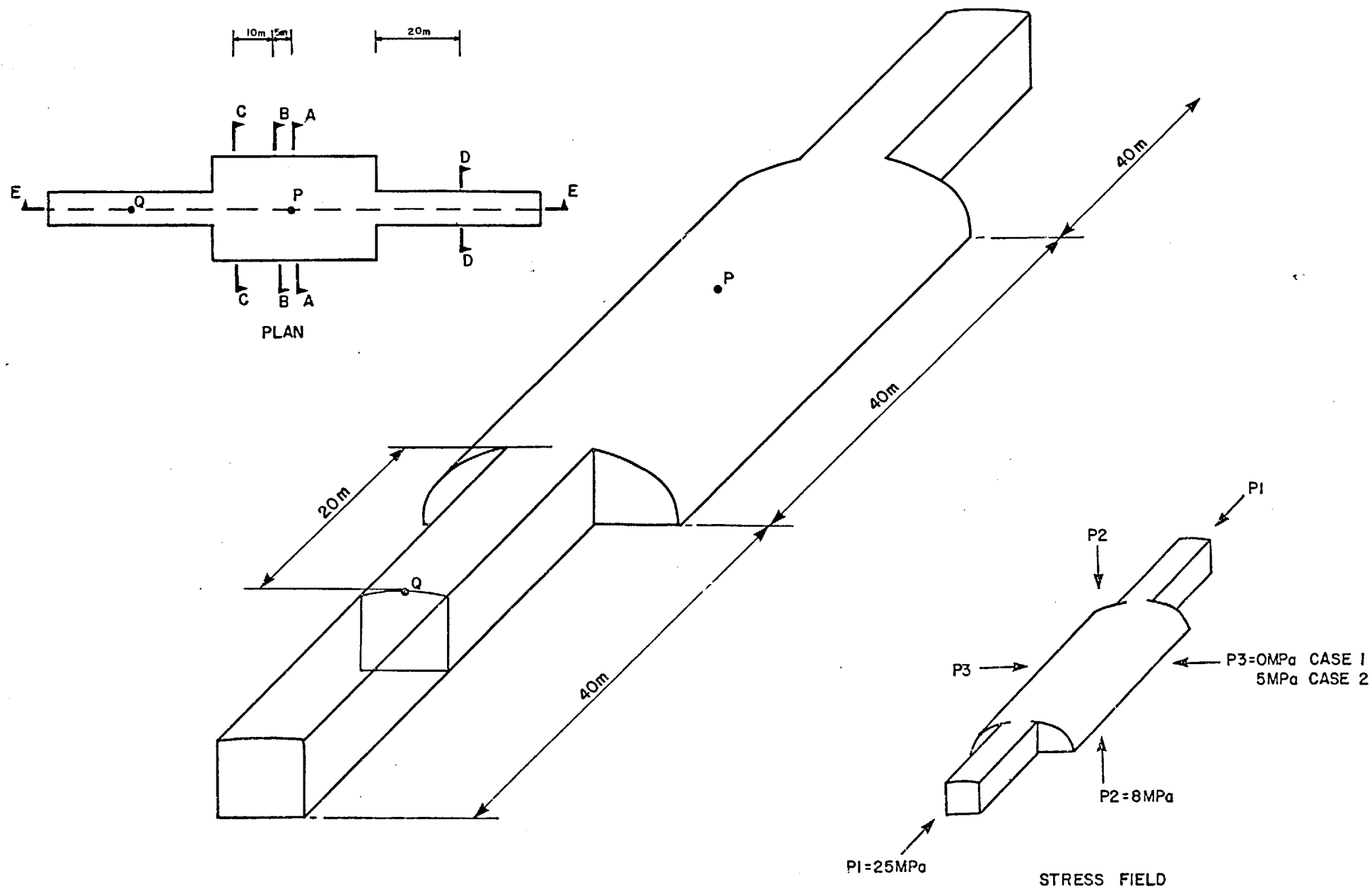
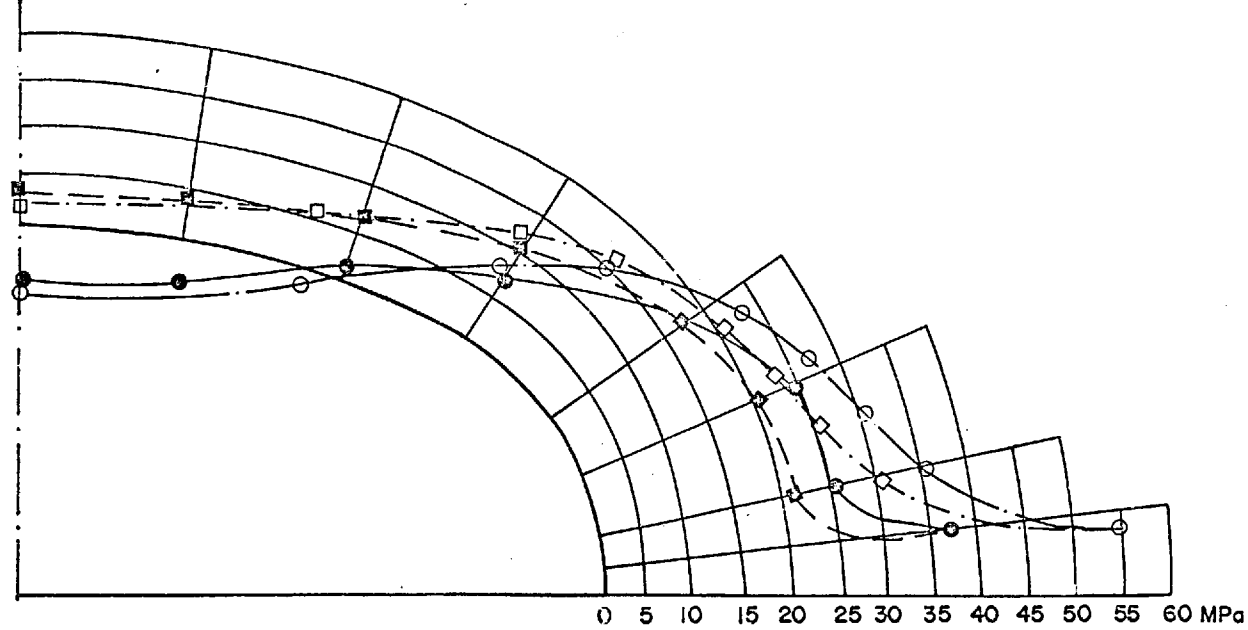
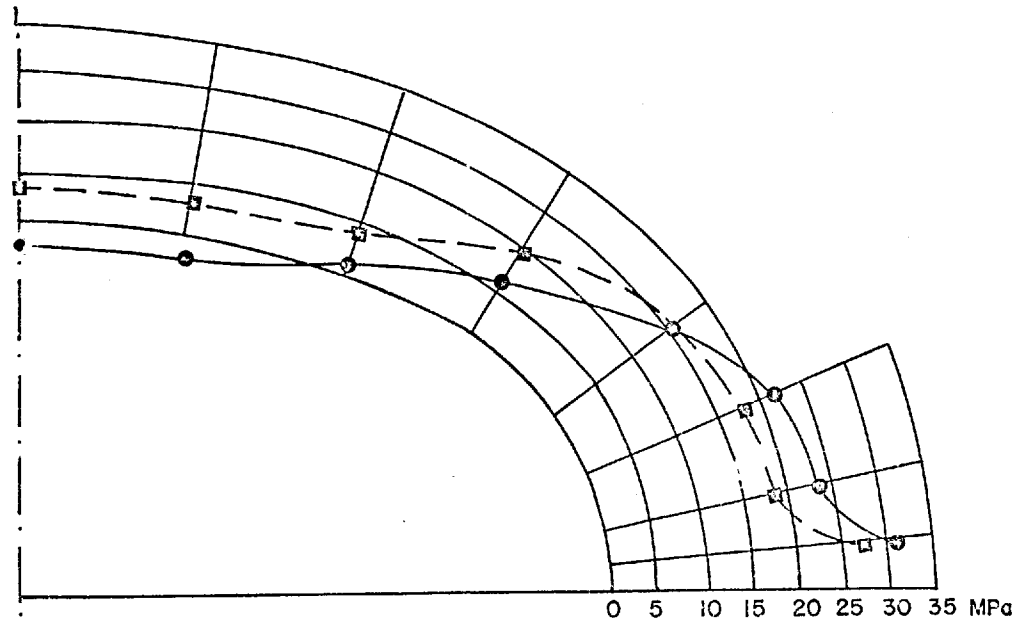


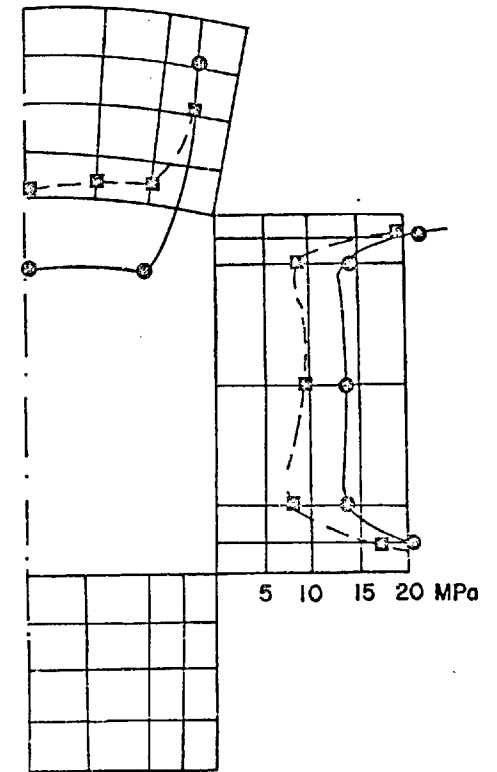
FIGURE 3.4.1 GEOMETRY OF THE MACHINE HALL TEST ENLARGEMENT



(a) SECTION AA



(b) SECTION CC



(c) SECTION DD

- 2D CASE 1
- 3D CASE 1
- -□- - 2D CASE 2
- -□- - 3D CASE 2

FIGURE 3.4.2 BOUNDARY STRESSES AT a) SECTION AA b) SECTION CC AND c) SECTION DD FOR THE MACHINE HALL TEST ENLARGEMENT

differ slightly because of the 3-D geometry of the excavation, Figure 3.4.1. At section DD, the two dimensional results have not been plotted; however, they are almost exactly the same as the three dimensional results shown. The boundary displacements for the machine hall test enlargement are illustrated in Figure 3.4.3. Only the three dimensional results are presented. Section EE is interesting as it illustrates the zone of influence of the machine hall test enlargement on the access heading as regards displacement. Half way between points P and Q, the displacement is the same as at point Q, removed from the large excavation.

The three dimensional stress analysis of the intersection of the machine hall and busbar excavations was analysed with updated stress fields as in Figure 3.4.4. Only a single busbar excavation was considered, in order to simplify the analysis. The location of sections for presentation of results are shown in Figure 3.4.5. Boundary stresses and displacements for sections 1, 2, 3, 4 and 5 are illustrated in Figures 3.4.6, 3.4.7, 3.4.8, 3.4.9 and 3.4.10 respectively. The most important feature of the above results is that the boundary stresses in the near vicinity of the intersection are not significantly different from those slightly removed from the intersection. For example, compare section 1 with section 2, and section 4 with section 5. Since sections 1 and 5 could be analysed two dimensionally, (see zone of influence of machine hall test enlargement) then, for this problem a two dimensional analysis may be sufficient for stress analysis purposes. Obviously the three dimensional analysis was required to arrive at this decision.

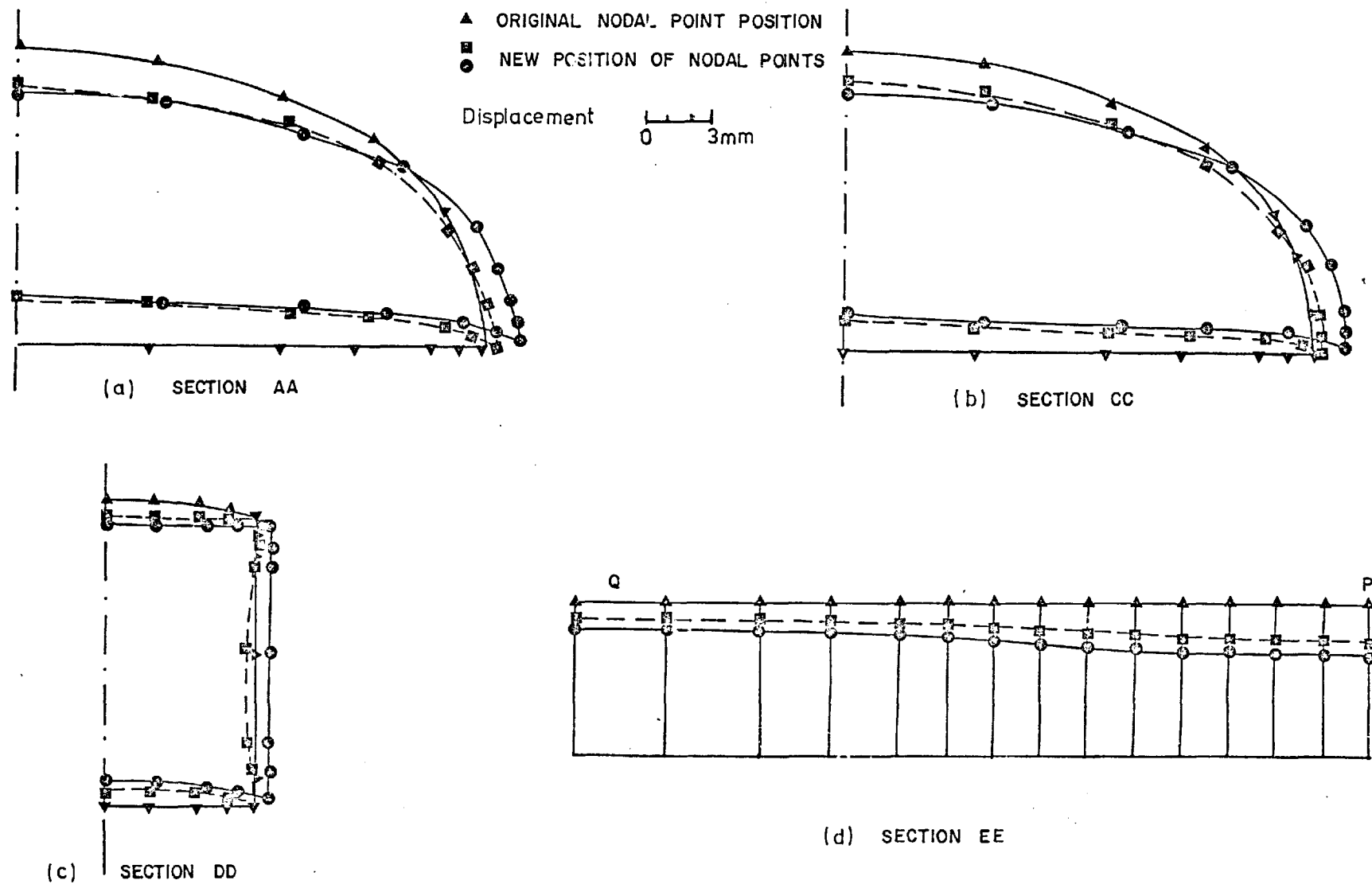


FIGURE 3.4.3 BOUNDARY DISPLACEMENTS FOR a) SECTION AA b) SECTION CC c) SECTION DD AND d) SECTION EE FOR MACHINE HALL TEST ENLARGEMENT

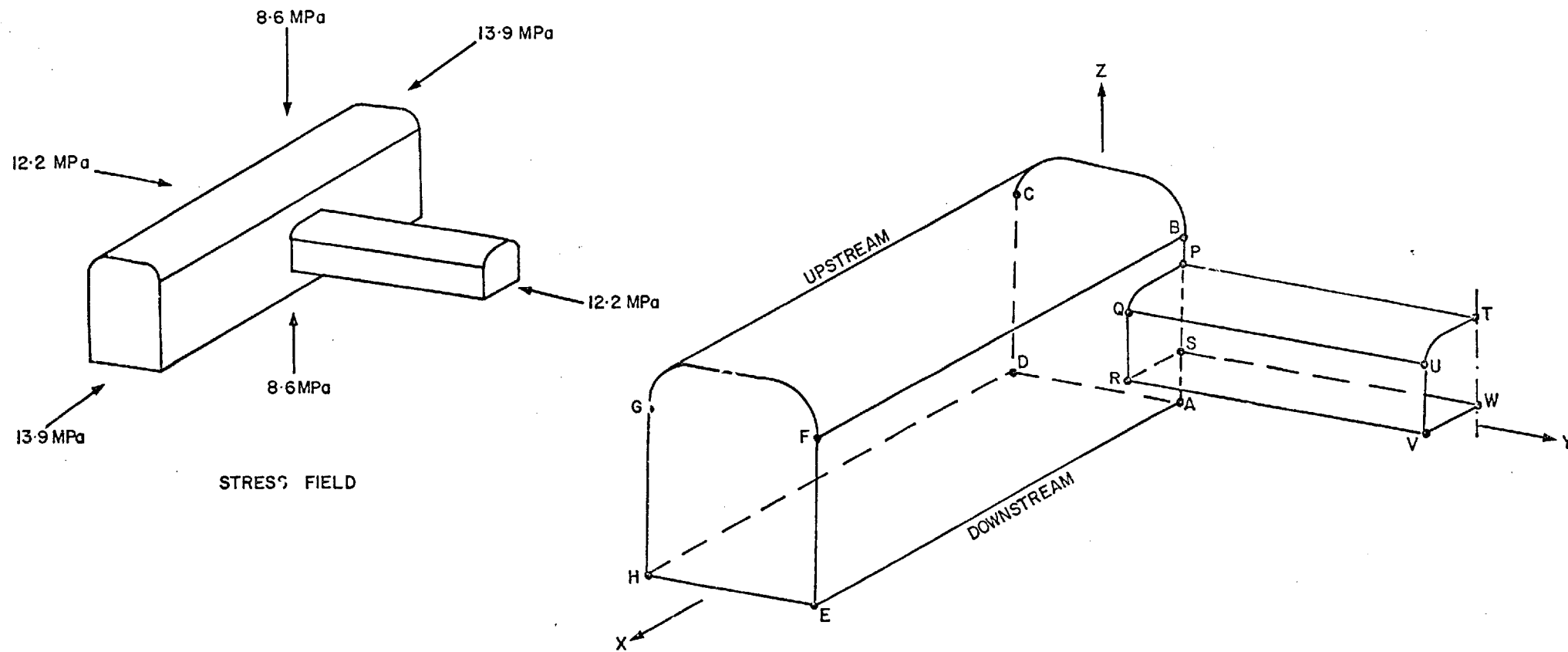


FIGURE 3.4.4 GEOMETRY OF MACHINE HALL - BUSBAR INTERSECTION

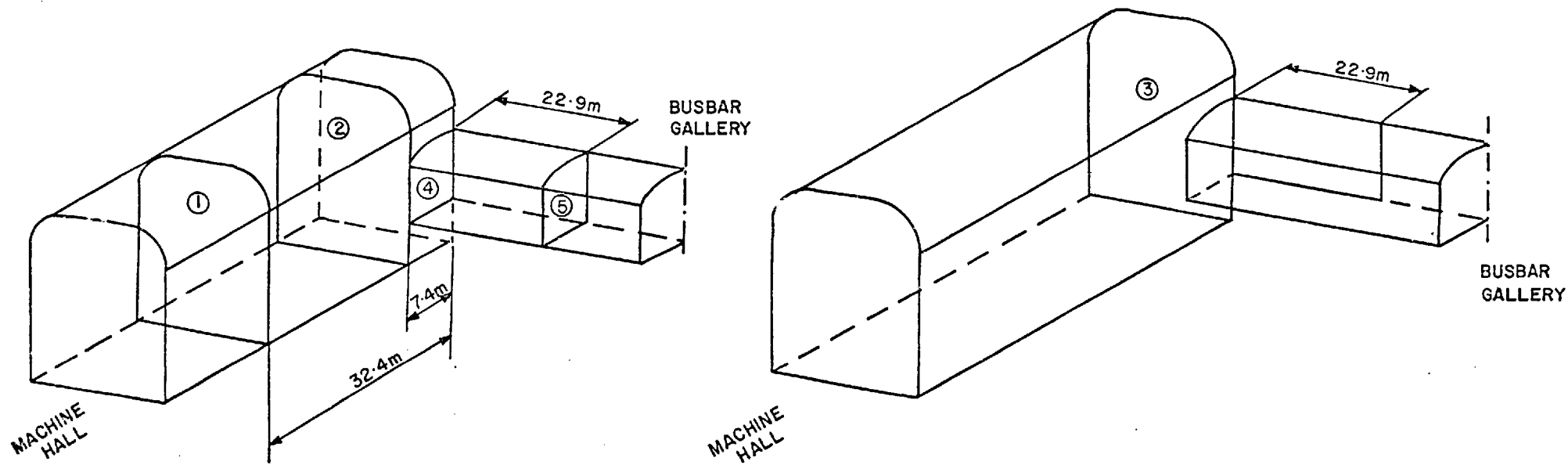
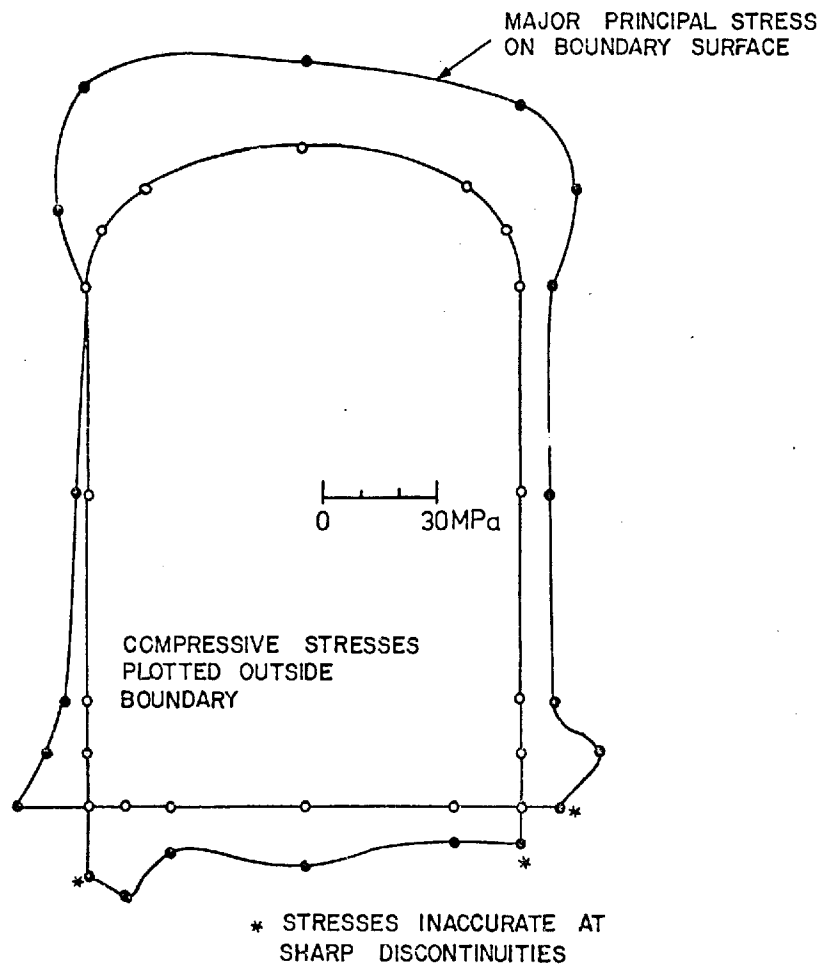
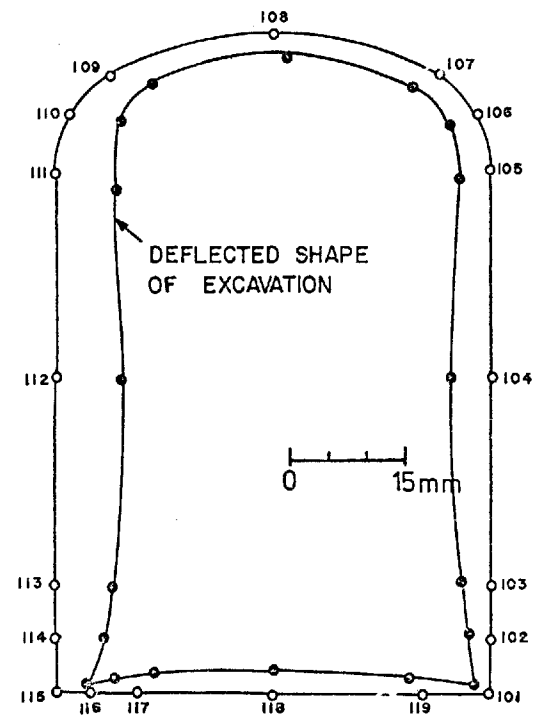


FIGURE 3.4.5 LOCATION OF SECTIONS FOR MACHINE HALL - BUSBAR INTERSECTION

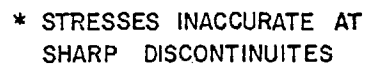


BOUNDARY STRESSES



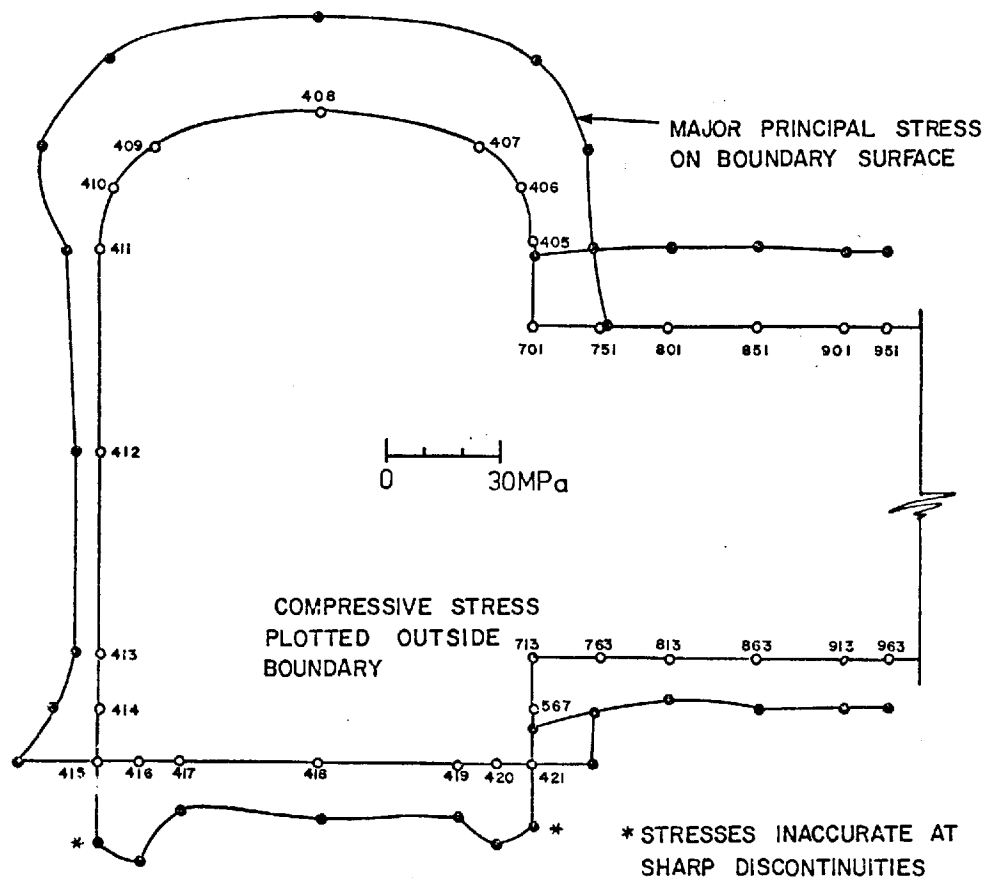
BOUNDARY DISPLACEMENTS

FIGURE 3.4.6 BOUNDARY STRESSES AND DISPLACEMENTS FOR SECTION 1

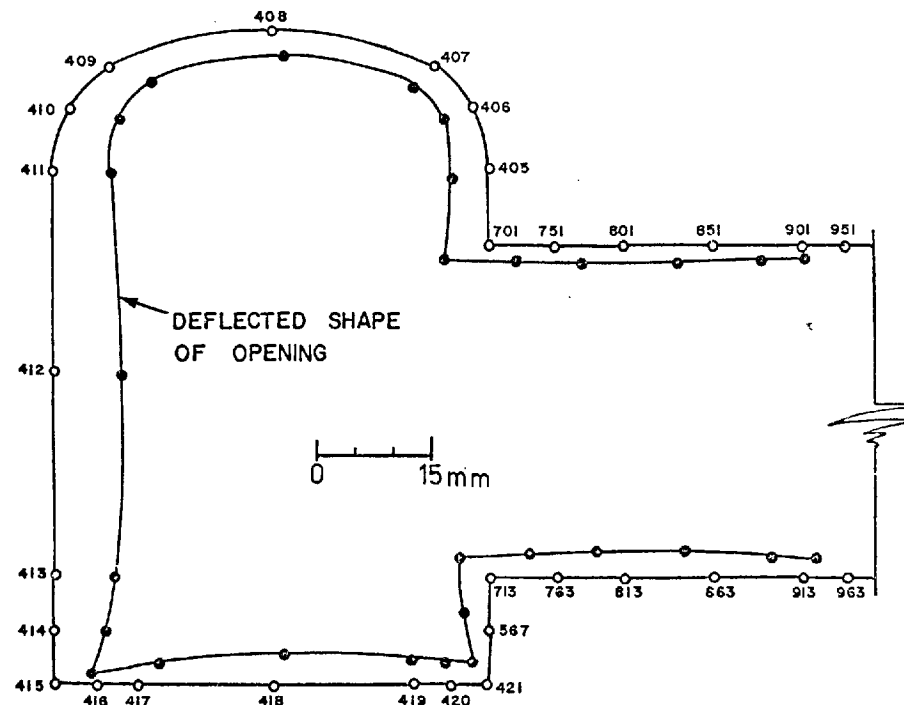


BOUNDARY DISPLACEMENTS

FIGURE 3.4.7 BOUNDARY STRESSES AND DISPLACEMENTS FOR SECTION 2



BOUNDARY STRESSES



BOUNDARY DISPLACEMENTS

FIGURE 3.4.8 BOUNDARY STRESSES AND DISPLACEMENTS FOR SECTION 3

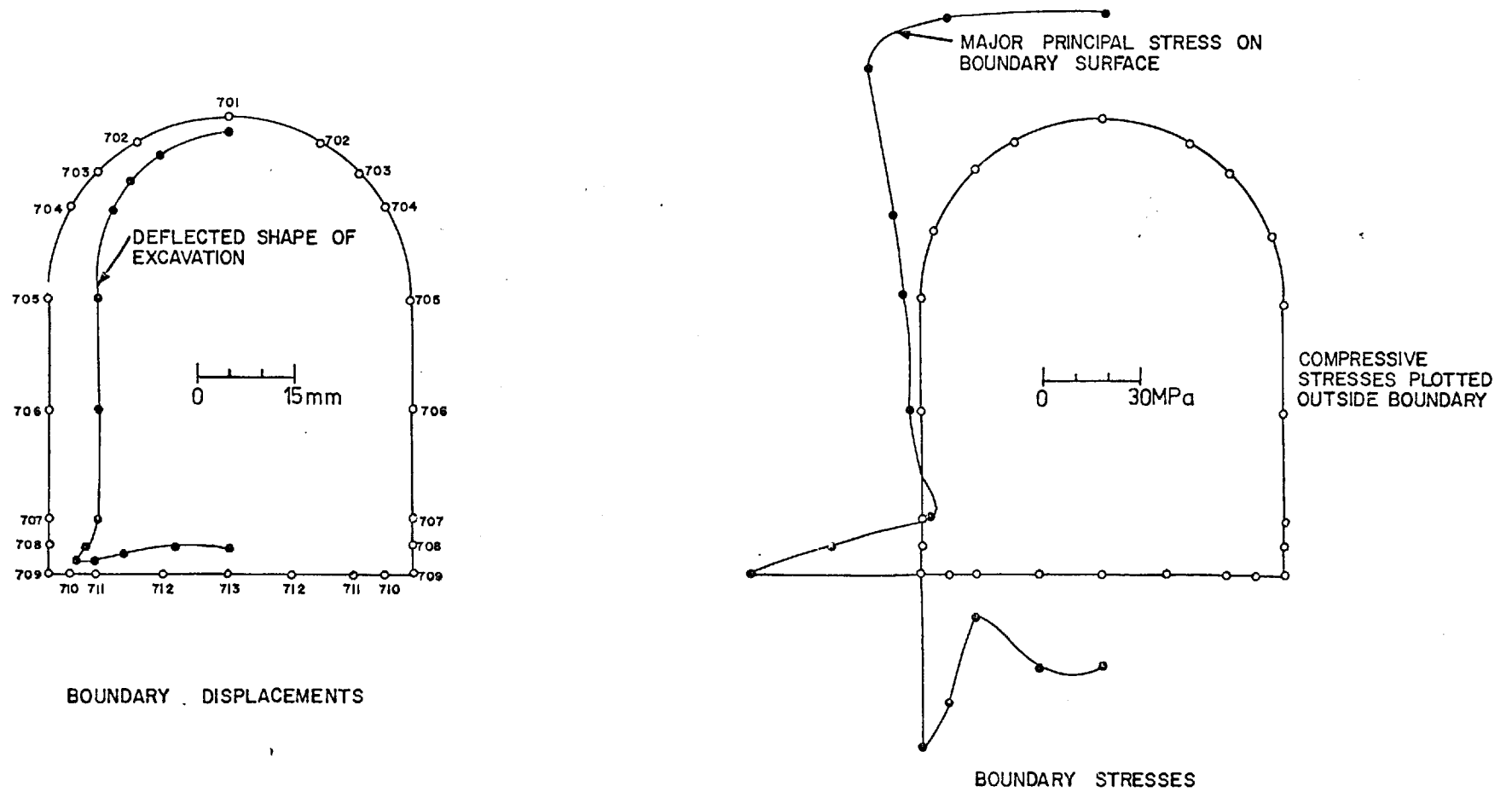
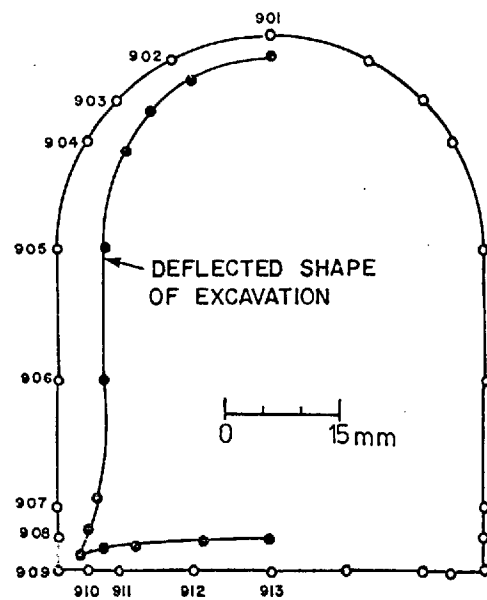
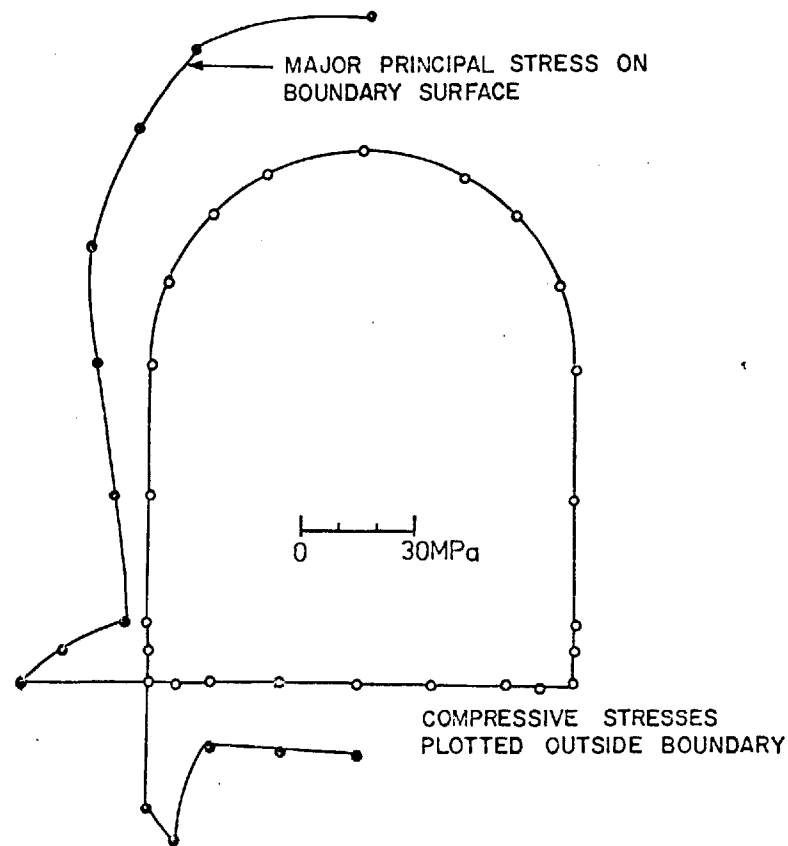


FIGURE 3.4.9 BOUNDARY STRESSES AND DISPLACEMENTS FOR SECTION 4



BOUNDARY DISPLACEMENTS



BOUNDARY STRESSES

FIGURE 3.4.10 BOUNDARY STRESSES AND DISPLACEMENTS FOR SECTION 5

The proposed complete underground complex is shown in Figure 3.4.11. The machine hall test enlargement is illustrated on the right. The geometrical complexity of this project hardly requires comment; however, it emphasises the importance of knowing the zone of influence of excavations in order to simplify the geometry for stress analysis purposes.

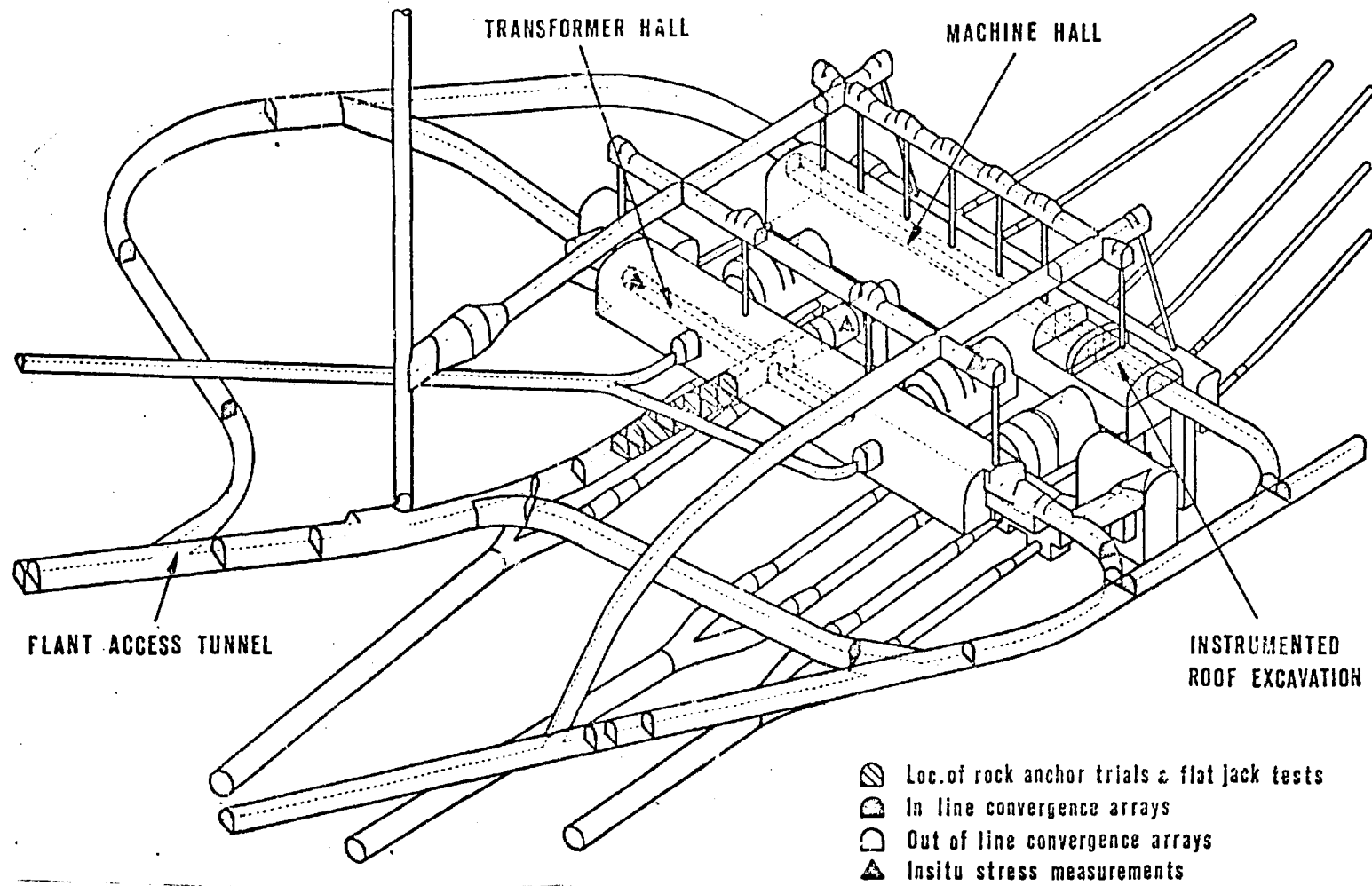


FIGURE 3.4.11 ISOMETRIC VIEW OF THE DINORWIC MAIN UNDERGROUND COMPLEX

(ANDERSON ET AL (1977))

5. Heterogeneous problems - inclusions and thin soft seams

Inclusions

In order to check the two dimensional program for subregions with different elastic parameters, the stress distribution in and around a circular inclusion was computed. In Table 3.5.1, the stresses within the inclusion calculated from the B.I.M. and the analytical solution, Jaeger & Cook (1971), are presented. There is good agreement between the results quoted. In Table 3.5.2 the stresses within two elliptical inclusions, one with its major axis perpendicular to stress field direction, $W/H = 3$, and the other parallel, $W/H = 0.33$, are given. The feature of homogeneous stress field within the inclusion material has been used to advantage in the design of stress field determination equipment, Roberts (1968). In Table 3.5.3, the stresses in the host medium at the inclusion interface for a circular inclusion are given for both the B.I.M. and analytical solution. Good to excellent agreement exists between the results from the B.I.M. and the analytical solution. In Table 3.5.4, the host medium stresses at the inclusion interface are presented for two elliptical inclusions, as mentioned above.

The case of two neighbouring inclusions, in this case two parallel circular inclusions as shown in Figure 3.5.2, was studied in order to determine whether a homogeneous stress field was present within the inclusions. Stresses within the inclusion for circular inclusions at various spacings are presented in Table 3.5.5 for points A, B and C, Figure 3.5.2. From these results it is clearly

TABLE 3.5.1. STRESSES WITHIN CIRCULAR INCLUSION
AS IN FIGURE 3.5.1.

$\frac{E_I}{E}$	B.I. Method		Analytical	
	σ_{xx}	σ_{zz}	σ_{xx}	σ_{zz}
10.0	0.29	1.54	0.26	1.55
2.0	0.12	1.25	0.10	1.24
0.5	-0.05	0.75	-0.07	0.73
0.1	-0.06	0.26	-0.06	0.24

TABLE 3.5.2. STRESSES WITHIN ELLIPTICAL INCLUSIONS
AS IN FIGURE 3.5.1.

$\frac{W}{H}$	$\frac{E_I}{E}$	B.I. Method	
		σ_{xx}	σ_{zz}
3	10.0	0.27	1.22
	2.0	0.10	1.06
	0.5	-0.04	0.89
	0.1	-0.05	0.45
0.33	10.0	0.26	2.42
	2.0	0.11	1.50
	0.5	-0.03	0.63
	0.1	-0.05	0.16

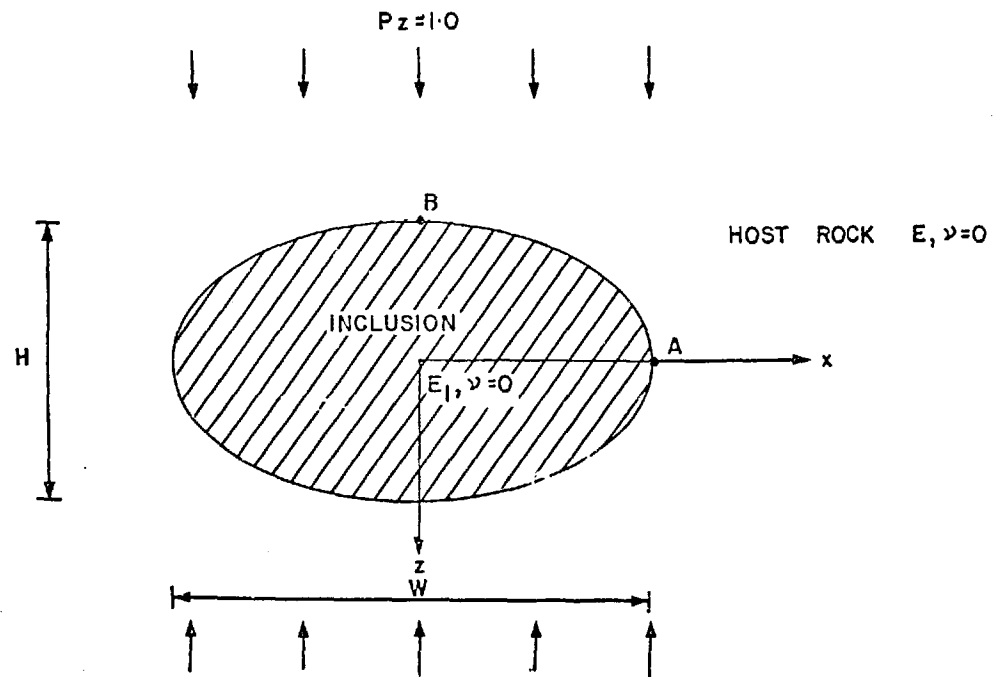


FIGURE 3.5.1 ELLIPTICAL INCLUSION SUBJECTED TO A UNIT VERTICAL STRESS FIELD

TABLE 3.5.3. STRESSES IN HOST ROCK AT INCLUSION
INTERFACE AROUND CIRCULAR INCLUSION
AS IN FIGURE 3.5.1.

$\frac{E_I}{E}$	B.I. Method				Analytical			
	A		B		A		B	
	σ_{xx}	σ_{zz}	σ_{xx}	σ_{zz}	σ_{xx}	σ_{zz}	σ_{xx}	σ_{zz}
10.0	0.31	0.14	0.03	1.58	0.26	0.16	0.03	1.55
2.0	0.12	0.63	0.06	1.26	0.10	0.62	0.05	1.24
0.5	-0.05	1.51	-0.12	0.75	-0.07	1.47	-0.13	0.73
0.1	-0.05	2.55	-0.61	0.26	-0.06	2.45	-0.63	0.24

TABLE 3.5.4. STRESSES IN HOST ROCK AT INCLUSION
INTERFACE AROUND ELLIPTICAL INCLUSIONS
AS IN FIGURE 3.5.1.

$\frac{W}{H}$	$\frac{E_I}{E}$	B.I. Method			
		A		B	
		σ_{xx}	σ_{zz}	σ_{xx}	σ_{zz}
3	10.0	0.33	0.07	0.03	1.22
	2.0	0.08	0.56	0.05	1.11
	0.5	-0.04	1.73	-0.09	0.89
	0.1	-0.07	4.72	-0.52	0.45
0.33	10.0	0.28	0.22	0.09	2.41
	2.0	0.11	0.75	0.15	1.48
	0.5	-0.02	1.25	-0.26	0.67
	0.1	-0.05	1.57	-1.23	0.19

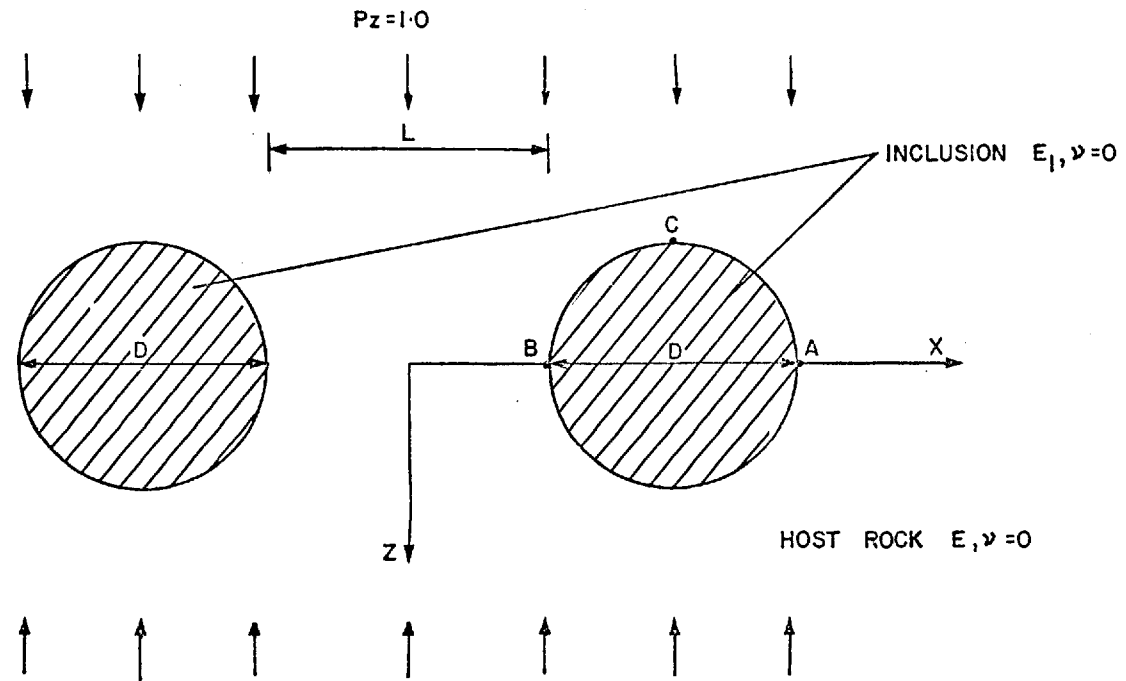


FIGURE 3.5.2 TWO PARALLEL CIRCULAR INCLUSIONS SUBJECTED TO A UNIT VERTICAL STRESS FIELD

TABLE 3.5.5. STRESSES WITHIN TWO PARALLEL CIRCULAR INCLUSIONS AS IN FIGURE 3.5.2.

L	$\frac{E_I}{E}$	A		B		C	
		σ_{xx}	σ_{zz}	σ_{xx}	σ_{zz}	σ_{xx}	σ_{zz}
D	10	0.28	1.48	0.30	1.53	0.25	1.36
	2.0	0.13	1.20	0.12	1.24	0.12	1.18
	0.5	-0.03	0.75	-0.06	0.77	-0.02	0.76
	0.1	-0.05	0.26	-0.06	0.27	-0.04	0.27
D/2	10.0	0.27	1.48	0.31	1.50	0.22	1.22
	2.0	0.12	1.20	0.12	1.23	0.11	1.14
	0.5	-0.03	0.76	-0.06	0.78	0.0	0.79
	0.1	-0.05	0.27	-0.06	0.28	-0.03	0.29
D/4	10.0	0.27	1.48	0.33	1.49	0.21	1.03
	2.0	0.12	1.20	0.12	1.23	0.12	1.07
	0.5	-0.03	0.76	-0.06	0.79	0.0	0.84
	0.1	-0.04	0.28	-0.06	0.29	-0.02	0.34

apparent that the stress within the inclusion is not homogeneous and varies greatly as the spacing L is decreased. In Table 3.5.6 the stresses in the host medium for two parallel circular inclusions are presented.

The single inclusions analysed were all convex as regards the inclusion material; therefore, it was considered worthwhile analysing an inclusion with a general shape. For this purpose, the shape of a dog bone was used, see Figure 3.5.3. Stresses within the dog bone inclusion are quoted in Table 3.5.7. As can be seen from these results the stresses within the dog-bone inclusion are not uniform.

TABLE 3.5.6. STRESSES IN HOST ROCK AROUND TWO
PARALLEL CIRCULAR INCLUSION AS IN FIGURE 3.5.2.

L	$\frac{E_I}{E}$	A		B		C	
		σ_{xx}	σ_{zz}	σ_{xx}	σ_{zz}	σ_{xx}	σ_{zz}
D	10.0	0.34	0.09	0.02	1.53	0.31	0.08
	2.0	0.12	0.61	0.05	1.24	0.11	0.61
	0.5	-0.03	1.50	-0.11	0.77	-0.02	1.52
	0.1	-0.05	2.62	-0.59	0.27	-0.04	2.65
D/2	10.0	0.33	0.09	0.02	1.50	0.27	0.07
	2.0	0.11	0.61	0.05	1.23	0.10	0.58
	0.5	-0.03	1.51	-0.11	0.78	0.0	1.57
	0.1	-0.04	2.69	-0.60	0.28	-0.02	2.86
D/4	10.0	0.33	0.09	0.03	1.49	0.24	0.08
	2.0	0.11	0.61	0.06	1.23	0.12	0.55
	0.5	-0.02	1.52	-0.11	0.79	0.0	1.67
	0.1	-0.04	2.76	-0.61	0.29	-0.02	3.35

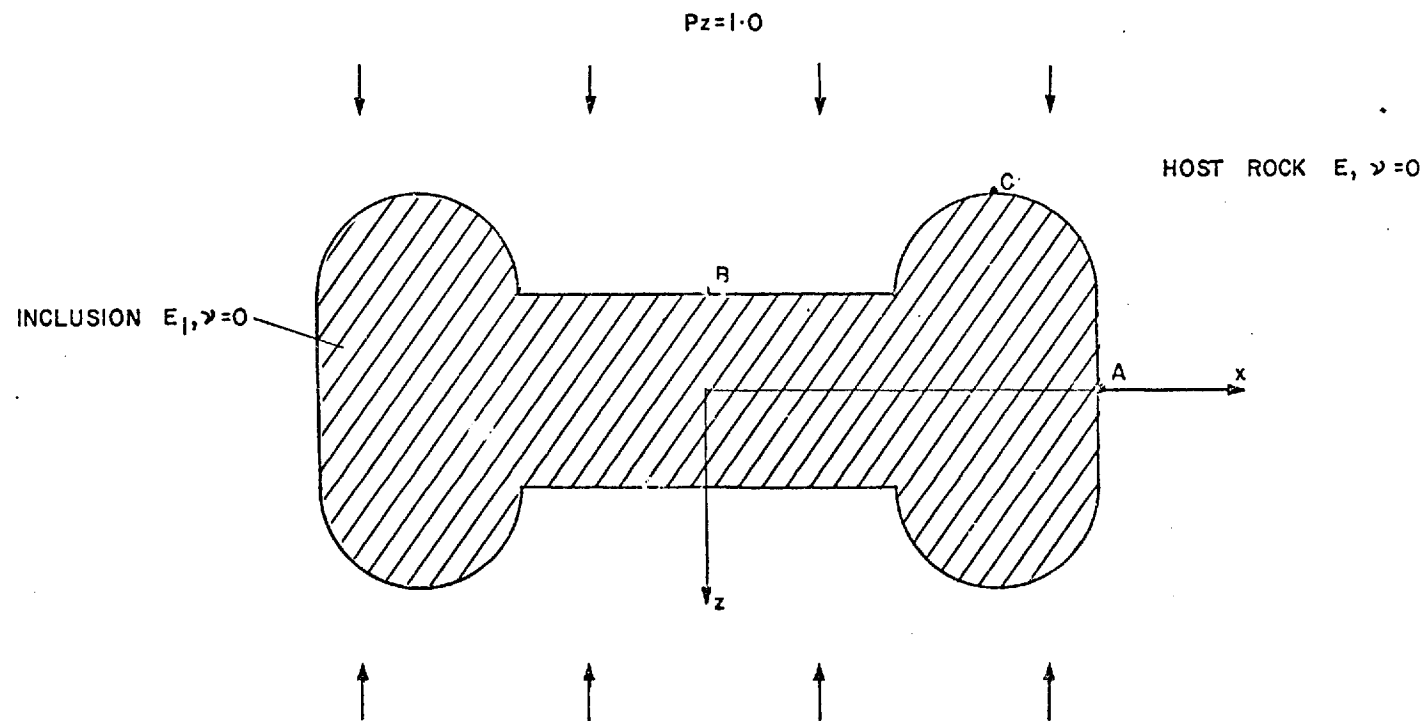


FIGURE 3.5.3 DOG BONE INCLUSION SUBJECTED TO UNIT VERTICAL STRESS FIELD

TABLE 3.5.7. STRESSES WITHIN A DOG BONE INCLUSION
AS IN FIGURE 3.5.3

$\frac{E_I}{E}$	A		B		C	
	σ_{xx}	σ_{zz}	σ_{xx}	σ_{zz}	σ_{xx}	σ_{zz}
0.5	-0.15	0.51	-0.07	0.90	-0.03	0.92
0.1	-0.01	0.09	-0.08	0.34	-0.05	0.50

Thin soft seams

In Figure 3.5.4 seven configurations of thin horizontal seams near a circular tunnel are given. These seven cases were analysed with various elastic parameters of host medium and seam. The stresses at points A, B and C are quoted in Table 3.5.8 for all seven cases under a unit vertical stress field.

For case I, when the seam is very soft, the host medium displaces greatly at point A, thus reducing the compression at A and increasing the tension at B. It is interesting to note the only minor variation in stress at B for modulus ratios ranging from 1 to 100, and the minor change in compressive stress at A for modulus ratios of 100 to 1000. In case II, the fact that the compressive stress at A does not change with increasing softness of the seam is interesting. The tensile stress at point B, case III, obviously reduces as the modulus ratio varies from 1 to 100, since it is located within the soft seam. The large increase in compressive stress at A for this case is due to large change in direction of the major principal stress per length for this case (i.e. when the seam is very soft it is analogous to the loaded boundary being located at the seam). This fact is also illustrated for case IV, for both points A and B. The remaining three cases give similar results to their symmetrical counterparts, i.e. case V and case II, except that the effect at point B is in all cases greater than the change at C.

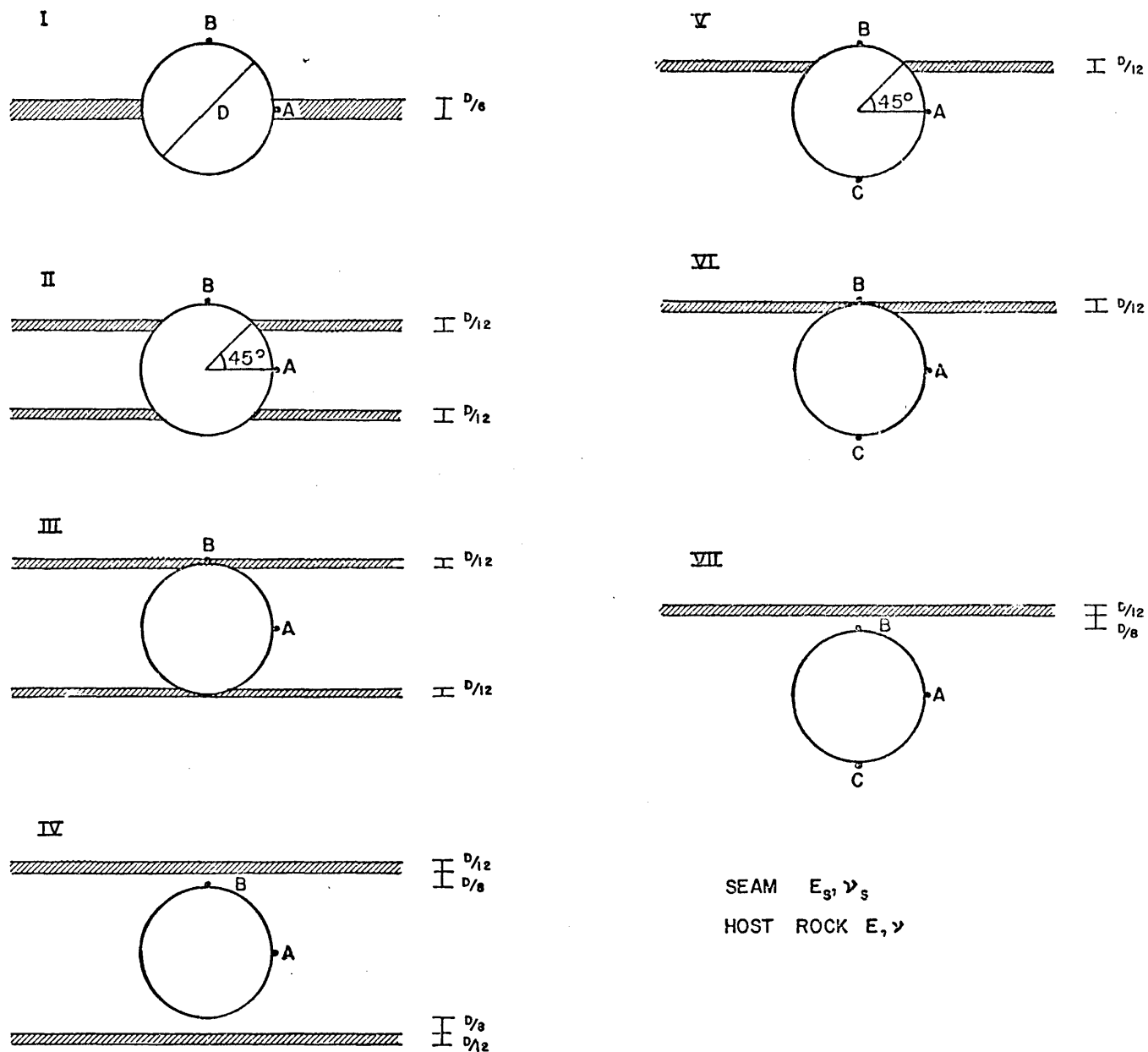


FIGURE 3.5.4 THIN HORIZONTAL SEAMS NEAR A CIRCULAR TUNNEL
SUBJECTED TO A UNIT VERTICAL STRESS FIELD

TABLE 3.5.8. BOUNDARY STRESSES AT SELECTED POINTS
AROUND A CIRCULAR TUNNEL WITH THIN
HORIZONTAL SEAMS AS IN FIGURE 3.5.4.

Case	$\frac{E_s}{E}$	ν	ν_s	A	B	C
I	1	0.25	0.25	2.98	-0.97	-
	0.75	"	"	2.83	-0.98	-
	0.5	"	"	2.61	-1.00	-
	0.25	"	"	2.26	-1.04	-
	0.1	"	"	1.85	-1.11	-
	0.01	"	0.49	1.20	-1.15	-
	0.001	"	0.49	1.15	-1.33	-
II	1	0.25	0.25	3.00	-1.00	-
	0.5	"	"	3.02	-1.04	-
	0.1	"	"	3.06	-1.17	-
	0.01	"	"	3.02	-1.36	-
III	1	0.25	0.25	3.03	-1.03	-
	0.5	"	"	3.07	-0.69	-
	0.1	"	"	3.29	-0.32	-
	0.01	"	"	4.47	-0.25	-
IV	1	0.25	0.25	3.05	-0.97	-
	0.5	"	"	3.09	-0.96	-
	0.1	"	"	3.35	-1.05	-
	0.01	"	"	4.69	-2.41	-

TABLE 3.5.8.(continued)

Case	$\frac{E_s}{E}$	ν	ν_s	A	B	C
V	1	0.25	0.25	3.06	-1.01	-1.01
	0.5	"	"	3.07	-1.05	-1.02
	0.1	"	"	3.09	-1.17	-1.03
	0.01	"	"	3.09	-1.36	-1.12
VI	1	0.25	0.25	3.05	-1.02	-1.02
	0.5	"	"	3.07	-0.68	-1.03
	0.1	"	"	3.19	-0.31	-1.06
	0.01	"	"	3.86	-0.24	-1.13
VII	1	0.25	0.25	3.05	-0.96	-1.02
	0.5	"	"	3.07	-0.95	-1.02
	0.1	"	"	3.21	-0.98	-1.07
	0.01	"	"	3.88	-1.68	-1.29

6. No-tension analysis of a circular tunnel

A no tension analysis has been incorporated in the finite element method F.E.M., Zienkiewicz et al (1968).

Unfortunately no simple solution, e.g. circular tunnel in a unit stress field, by the F.E.M. is available in the literature for a quantitative check of the results presented here. The no tension analysis considered, is of a material which has zero tensile strength; obviously the problem is non-linear and therefore path dependent. The analysis is carried out as mentioned in section 2.7 i.e. elastic stresses around the circular tunnel are computed and a step-wise linear path to the solution is followed. Path dependence has not been considered.

The boundary stresses around a circular tunnel under a unit vertical stress field for a tension and a no-tension material are shown in Figure 3.6.1. All tensile stresses in the crown have been eliminated with subsequent reduction in the compressive stress at springline level. This reduction is small from 3.0 to 2.8. In order to approximately check this solution, the principal stress trajectories and the 0.1 tensile stress contour are drawn for the elastic case, Figure 3.6.2. The material will form cracks along the major principal stress trajectories inside the tensile region. For this example consider only the 0.1 tensile stress region. Thus in this region, compressive columns, delineated by the major principal stress trajectories will be formed. No load will be transferred between these columns in the tensile region.

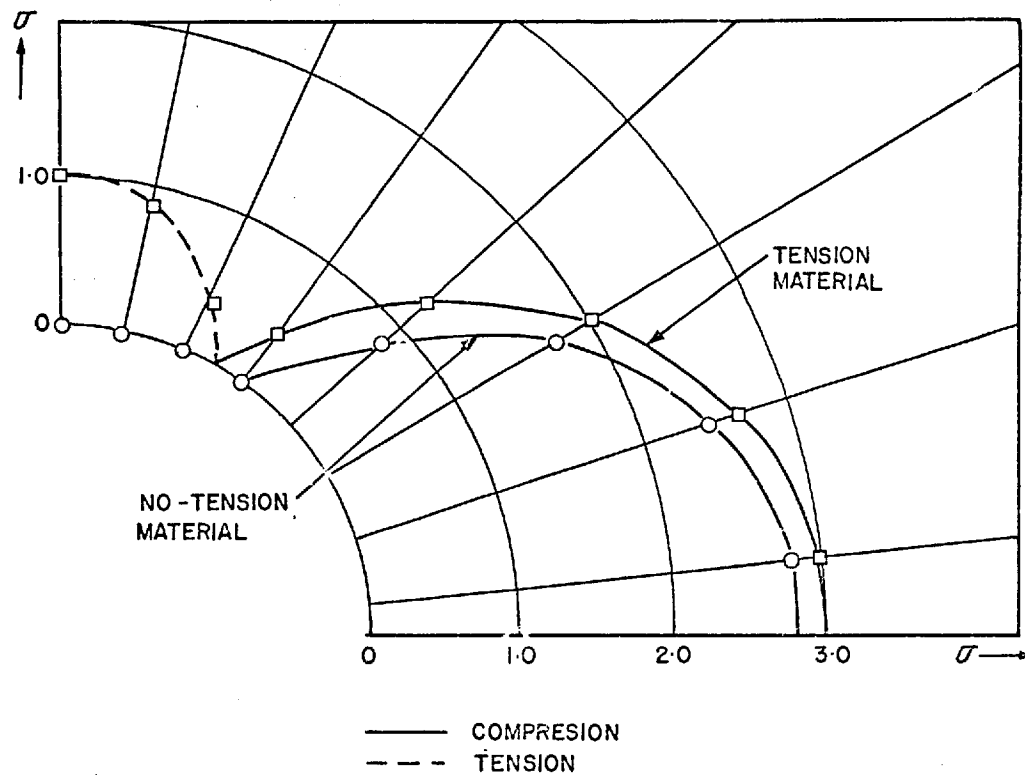


FIGURE 3.6.1 BOUNDARY STRESSES AROUND A CIRCULAR TUNNEL UNDER A UNIT VERTICAL STRESS FIELD FOR A NO - TENSION MATERIAL

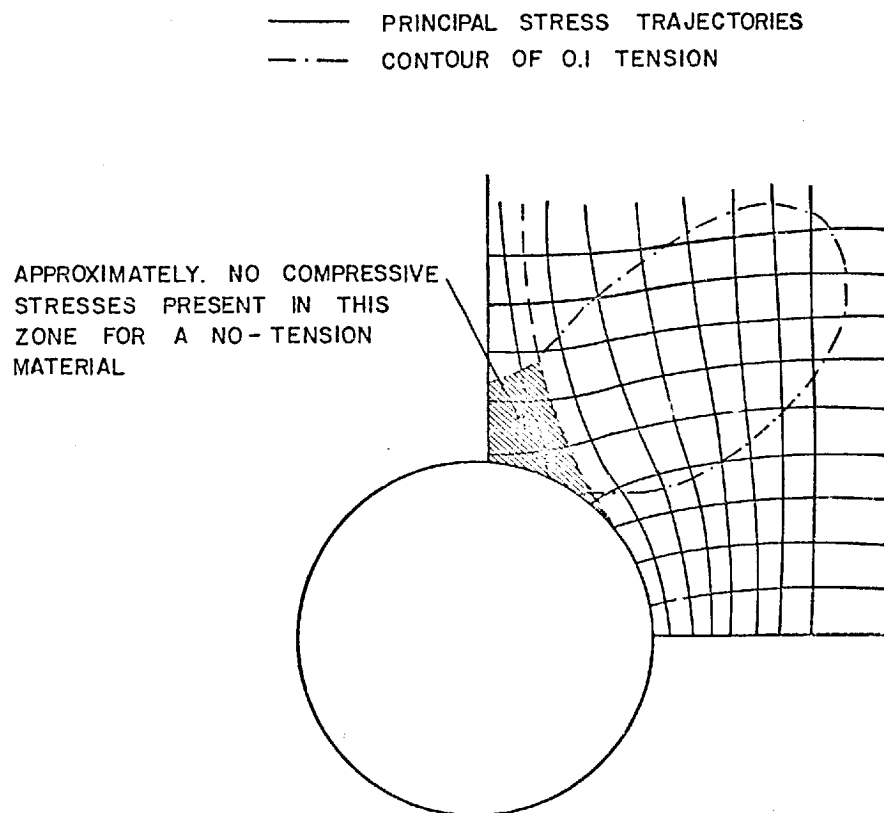


FIGURE 3.6.2 PRINCIPAL STRESS TRAJECTORIES AND 0.1 TENSILE STRESS CONTOUR FOR A CIRCULAR TUNNEL IN A UNIT VERTICAL STRESS FIELD

Therefore the columns in the shaded portion, Figure 3.6.2, will not carry any load, since the tractions on the boundary are zero. As a rough check to the no-tension analysis, the case of a zero load carrying region was analysed as an elliptical tunnel. For this case the compressive stress at the springline level is 2.5, which approximately agrees with the no-tension solution.

For the case of a intact material with tensile strength, the above mentioned crack pattern is not observed; in fact, a single crack propagates from the roof and floor of the circular tunnel and stops approximately half a radius from the excavation (Hoek (1965)). Stresses at the tip of a constant width elliptical crack of varying length in the roof and floor of a circular tunnel under a unit vertical stress field are shown in Figure 3.6.3. The stress at the tip is tensile for a crack length less than 0.4, and compressive when the crack length is greater than 0.4. For a crack of length 0.4, the compressive stress at springline level is approximately the same as in the case where no crack is present.

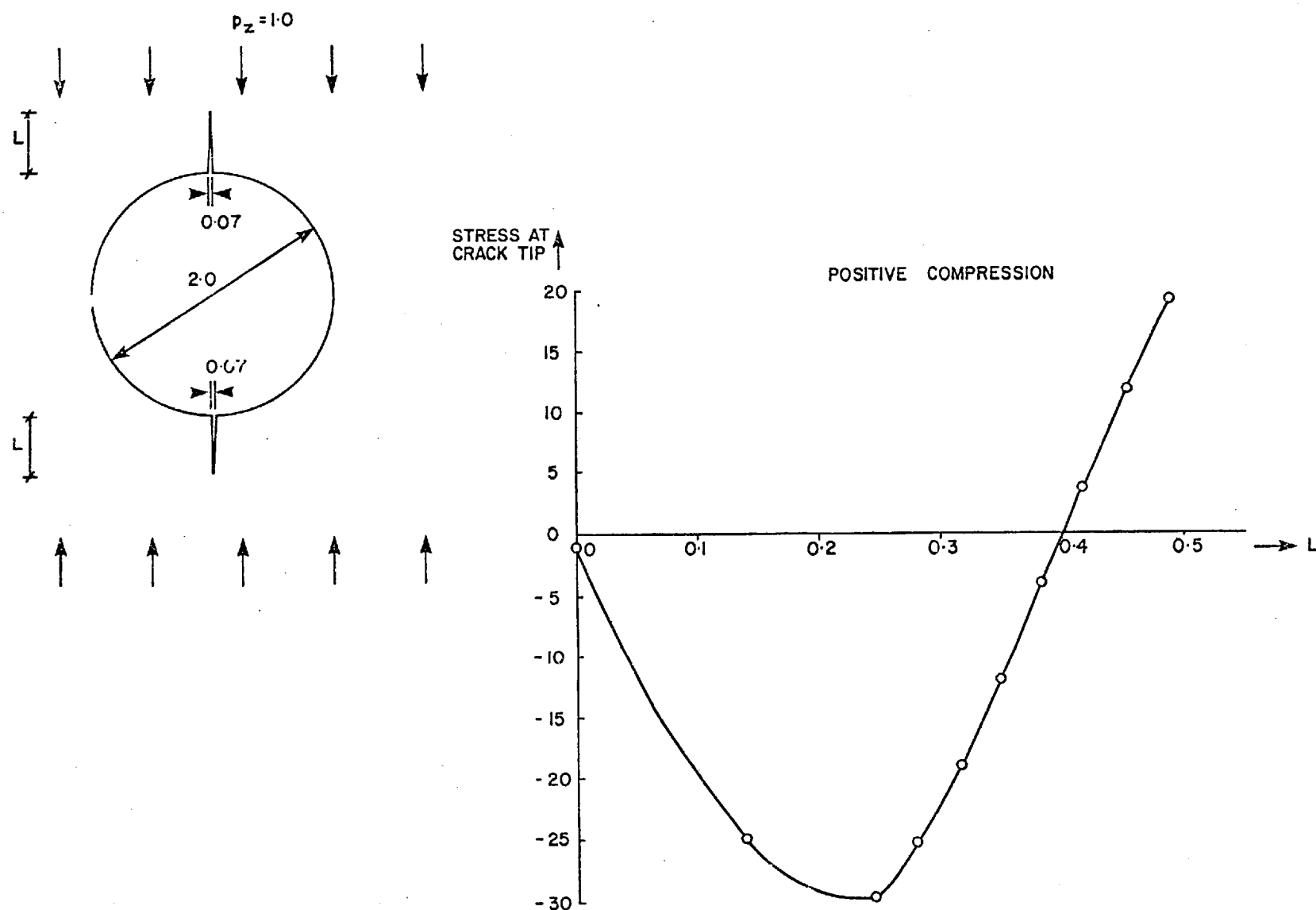


FIGURE 3.6.3 STRESSES AT THE TIP OF AN ELLIPTICAL CRACK OF VARYING LENGTH IN THE ROOF AND FLOOR OF A CIRCULAR TUNNEL UNDER UNIT VERTICAL STRESS FIELD

7. Slip on discontinuities

The F.E.M. can analysis slip on discontinuities by joint elements (Goodman et al (1968)), however, no simple solution, e.g. circular tunnel in a uniaxial stress field, is available for a quantitative check of the results presented here. Four different cases of discontinuity location around a circular tunnel were considered, Figure 3.7.1. Each case was analysed for two values of friction angle, 15 and 20 degrees, with zero cohesion. The analysis is carried out as described in section 2.8, i.e. elastic stresses around the circular tunnel are computed and a step-wise linear path to the solution is followed. The problem is obviously non-linear and therefore path dependent; however, path dependence was not considered here. The stresses at points A, B and C are quoted in Table 3.7.1 for all cases considered, under a unit vertical stress field.

TABLE 3.7.1. BOUNDARY STRESSES AT POINTS
A, B AND C AS IN FIGURE 3.7.1.

Case	ϕ	A	B	C
I	15 ⁰	2.80	-0.93	-
	20 ⁰	2.84	-0.95	-
II	15 ⁰	2.29	-0.55	-
	20 ⁰	2.69	-0.65	-
III	15 ⁰	2.90	-0.94	-0.95
	20 ⁰	2.91	-0.95	-0.96
IV	15 ⁰	2.51	-0.58	-0.68
	20 ⁰	2.80	-0.69	-0.80

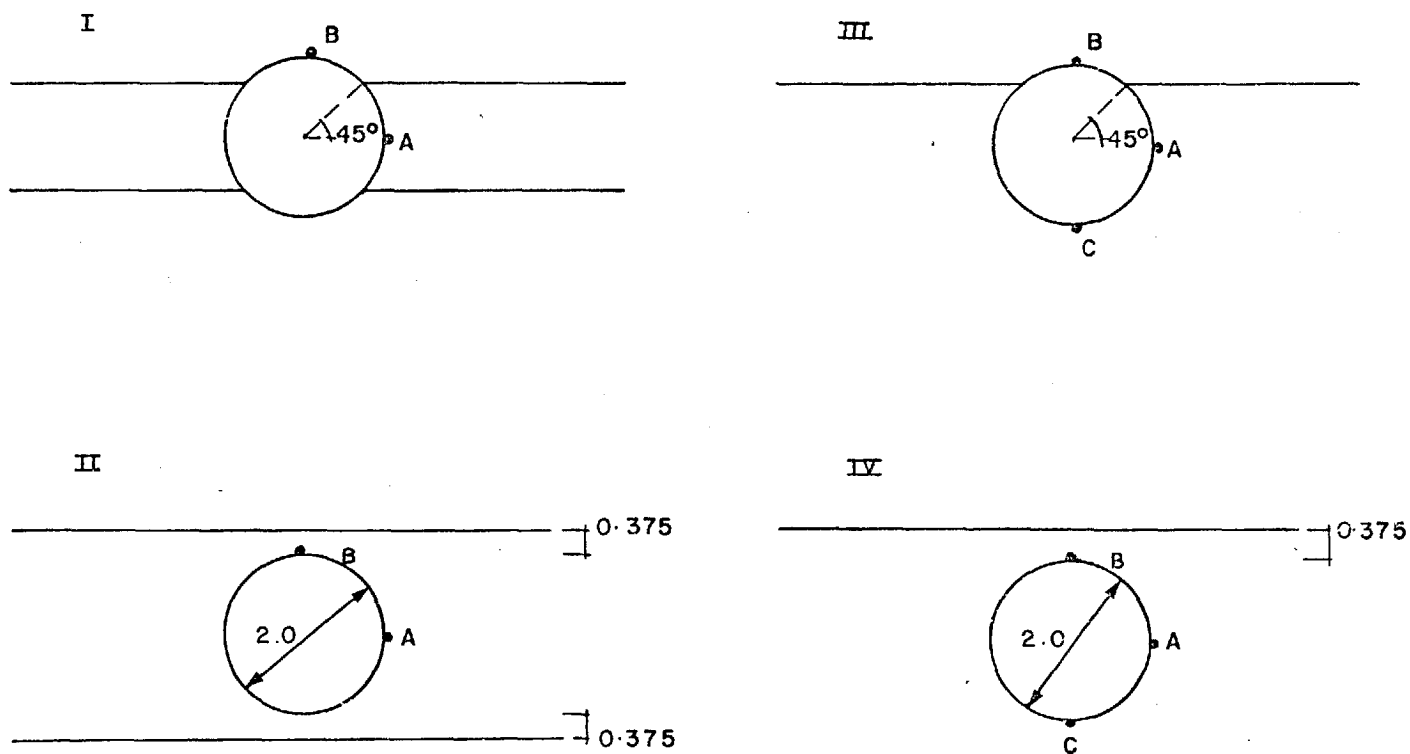


FIGURE 3.7.1 HORIZONTAL DISCONTINUITIES NEAR A
CIRCULAR TUNNEL SUBJECTED TO A UNIT VERTICAL STRESS FIELD

All cases quoted in Table 3.7.1 converged to a stable value; however, the results do not appear to be compatible with the relative slip that occurs. From the slip to be expected in say case III, the top section would move outwards at the discontinuity while the bottom section would move inwards, relative to the top section. This relative motion would result in boundary stresses at B becoming more tensile, and at C more compressive. Therefore the results in Table 3.7.1 should be viewed with suspicion until validation is obtained.

8. Computational efficiency

An important point is the efficiency, as regards cost, of a particular numerical method over any other. In the case of underground openings, the B.I.M. has huge savings in data preparation compared with the F.E.M. even when elaborate mesh generation routines are available. Computational cost is difficult to compare, because the programs need to be of similar sophistication for each method under consideration. It is for this reason that computational costs of the two dimensional B.I.M. program are given, Figure 3.8.1, without any comparison with other techniques. The reason why computation costs of the three dimensional B.I.M. program of Watson are not included, is because the C.P. time is not simply dependent on the number of elements, but also on the number of Gauss points chosen. The number of Gauss points chosen is problem dependent. The 2D B.I.M. program (ELAS 2D) requires 25k (decimal) small core memory for 150 elements. Input/output time never exceeded 5% of the C.P. time.

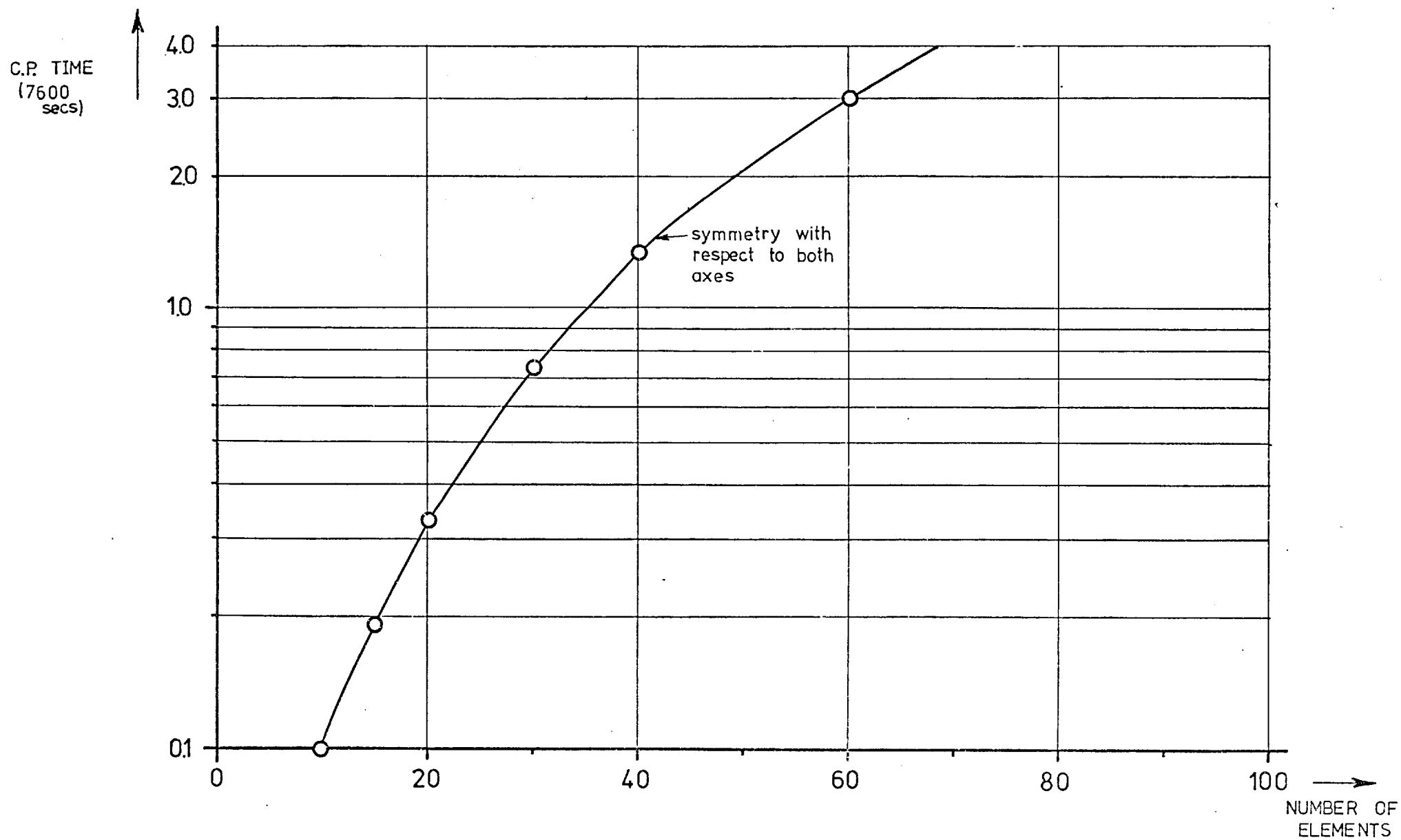


FIGURE 3.8.1 CENTRAL PROCESSOR TIME FOR ELAS 2D

CHAPTER 4: THE BOUNDARY INTEGRAL METHOD FOR GROUNDWATER SEEPAGE

1. - Review of previous formulations

As discussed earlier in section 2.1, Jaswon and Porter (1963) Jaswon (1963) and Symm(1963) described the boundary integral method applied to problems of the potential. However, it appears that application of this method to groundwater seepage problems has not been recorded as yet. Although formulations based on the boundary element (fictitious flow) method Grodtkjaer (1973) and Hocking (1976 c) have been described, these only relate to irrotational fluid flow around rigid bodies, and do not consider the all important mixed problem so common in practical groundwater seepage problems. However it is worthwhile at this stage to briefly discuss the above mentioned formulation in order to appreciate the similarities with the boundary element method used in elastostatics.

The boundary element (fictitious flow) method described here is the one by Hocking (1976c). The problem posed is the behaviour of irrotational flow about rigid bodies, e.g. Figure 4.1.1. The solution to the problem, Figure 4.1.1. is the sum of the solutions of (a) a uniform flow field without any obstructions to flow, Figure 4.1.2, and (b) the bodies with applied normal boundary velocities as shown and zero field flow, Figure 4.1.3. Therefore the essential problem to be solved is that of part (b) Figure 4.1.3. To solve this problem, sources are distributed within an infinite medium without any bodies, along where the boundaries of the bodies would be if present. These fictitious flows are uniformly distributed over straight line elements, which constitute the imaginary boundary. The procedure is now to adjust these fictitious flows so that the normal boundary velocities at the centre of each element are equal to the corresponding applied normal

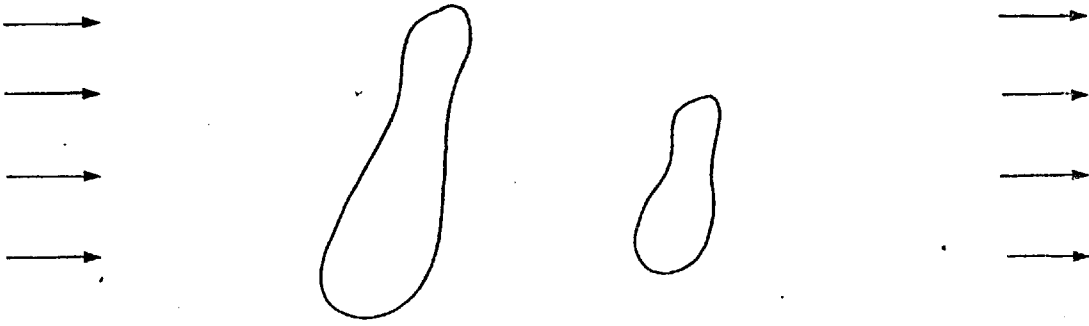


FIGURE 4.1.1 IRROTATIONAL FLOW ABOUT RIGID BODIES

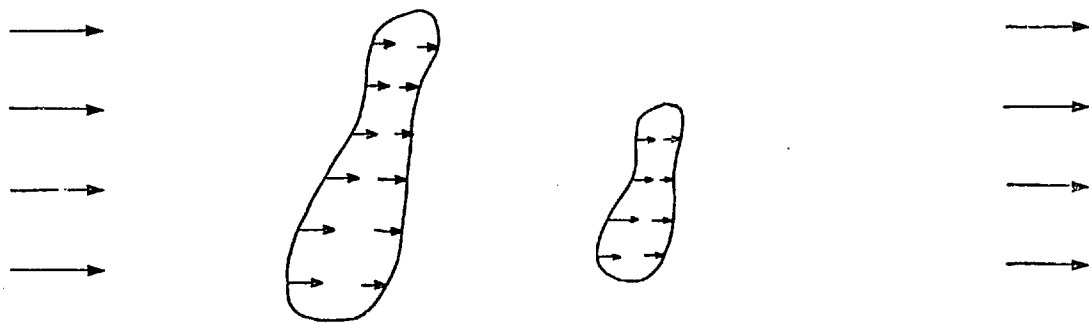


FIGURE 4.1.2 UNIFORM HOMOGENEOUS FLOW FIELD

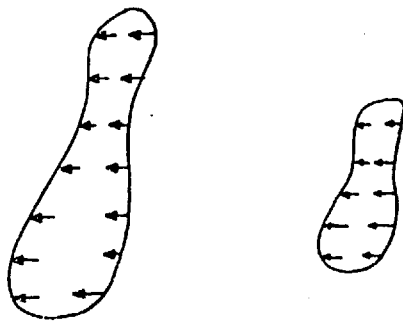


FIGURE 4.1.3 APPLIED NORMAL BOUNDARY VELOCITIES ON RIGID BODIES

boundary velocities shown in Figure 4.1.3. This adjustment is carried out by an iterative procedure, similar to that described in section 2.2. The problem can be written for an element i as:

$$v_i = \sum_{j=1}^n B_{n_{ij}} g_j \quad \dots (4.1.1.)$$

where g_j is the fictitious normal boundary velocity for element j , $B_{n_{ij}}$ is the normal boundary velocity at element i due to the unit source at element j , n is the number of elements, and v_i is the normal boundary velocity at element i .

Thus the problem can be given the following matrix identity

$$\begin{bmatrix} B_n \end{bmatrix} \begin{bmatrix} g \end{bmatrix} = \begin{bmatrix} v \end{bmatrix} \quad \dots (4.1.2.)$$

where $\begin{bmatrix} B_n \end{bmatrix}$ is $n \times n$, and $\begin{bmatrix} g \end{bmatrix}$ and $\begin{bmatrix} v \end{bmatrix}$ are $n \times 1$.

2. Green's equation (boundary integral) method

For solving groundwater seepage problems, the reciprocal theorem (in this case Green's equation) is again drawn upon to set up the system of equations to describe the problem. The problem posed is percolation through a porous medium, in which inertia forces are considered negligible. This is a potential problem which can be described by the Laplace equation.

$$\nabla^2 \phi = 0 \quad \dots (4.2.1.)$$

where ϕ is the potential (in this case total head).

Consider the finite, homogeneous two dimensional body of surface S , Figure 4.2.1, subjected to total head and normal (to the boundary) components of velocity, ϕ and v respectively, on the boundary of the body. This problem, Figure 4.2.1, is the one posed, i.e. either ϕ or v are known at any point and we wish to find the unknowns.

In the same body as illustrated in Figure 4.2.1, but with ϕ and v removed, place a line source of unit flow at the interior point i , Figure 4.2.2. The velocity component normal to the boundary V and the potential ϕ for any point j on the boundary due to the point source at i can be found (Batchelor (1974)). By applying the reciprocal theorem (i.e. Green's equation) the following is obtained:

$$\phi_i + \int V_{ij} \phi_j \, dS = \int \phi_{ij} v_j \, dS \quad \dots (4.2.2.)$$

where V_{ij} and ϕ_{ij} are the normal (to the boundary) component of flow and the total head (potential) respectively at the point j on the boundary, due to the source at the interior point i . The first subscript refers to the location of the source, and the second to the location of the point on the boundary, Figure 4.2.2.

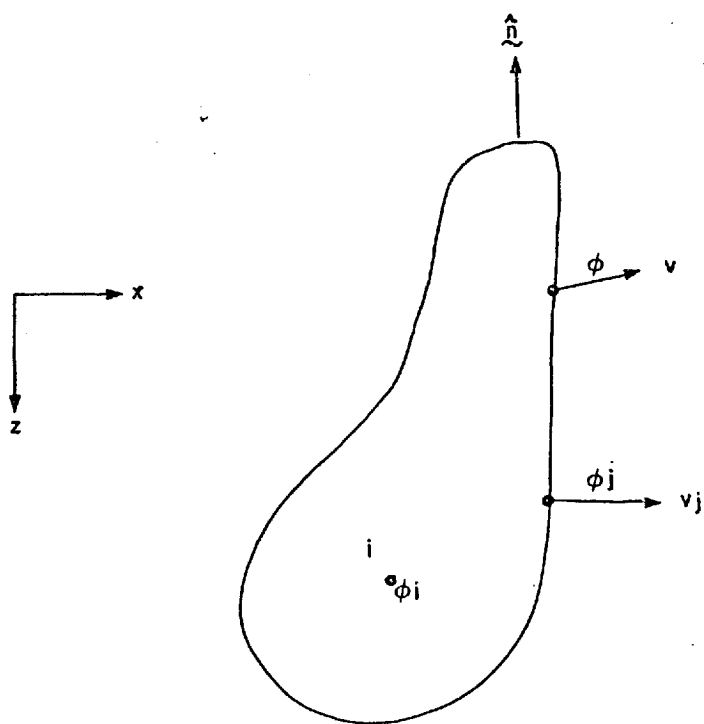


FIGURE 4.2.1 APPLIED POTENTIALS AND NORMAL BOUNDARY VELOCITIES OF FLOW ON THE BODY WITH SURFACE S

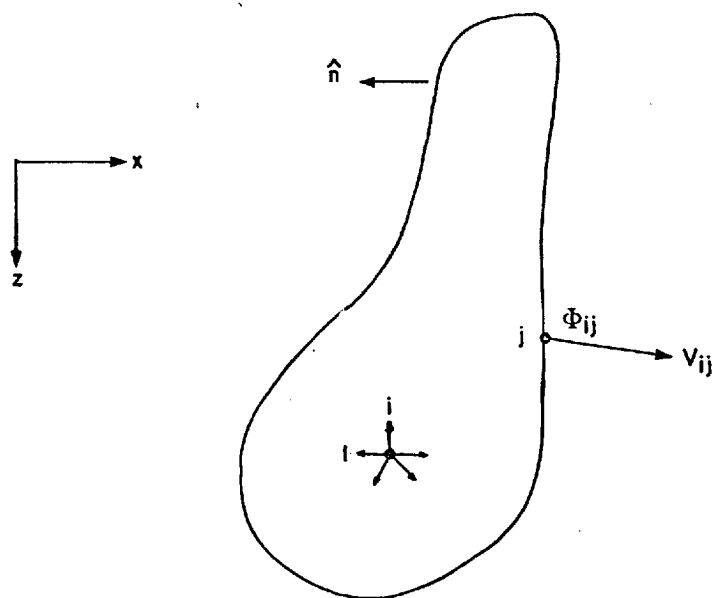


FIGURE 4.2.2 SOURCE OF UNIT FLOW AT INTERIOR POINT i , WITH ASSOCIATED POTENTIALS AND NORMAL BOUNDARY VELOCITIES OF FLOW ON THE BODY

v_j and ϕ_j are the normal (to the boundary) component of flow and the total head (potential) boundary conditions applied to the problem, Figure 4.2.1 and ϕ_i is the total head at the interior point i in the problem shown in Figure 4.2.1.

Equation (4.2.2) is simply, the normal (to the boundary) component of velocities of one system times the potentials (total heads) of a second, equals the normal boundary velocities of the second system times the potentials of the first.

Now similar to section 2.3, the point i is moved to the boundary of the body. If the boundary at this point is smooth, then the flow due to the source will be equally divided between the interior and exterior regions. Then equation (4.2.2) becomes:-

$$\frac{1}{2}\phi_i + \int v_{ij} \phi_j dS = \int \phi_{ij} v_j dS \quad \dots (4.2.3.)$$

Equation (4.2.3) is analogous to the boundary constraint equation (see section 2.3, equation (2.3.2)). By applying unit sources at all other points on the boundary in succession, the complete relationship between ϕ and v can be formulated. For example in matrix notation:-

$$\begin{bmatrix} v_{11} & v_{12} & \dots & \dots & \dots & \dots & v_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{n1} & v_{n2} & \dots & \dots & \dots & \dots & v_{nn} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \phi_n \end{bmatrix} =$$

$$\begin{bmatrix}
 \phi_{11} & \phi_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \phi_{1n} \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \phi_{n1} & \phi_{n2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \phi_{nn}
 \end{bmatrix}
 \begin{bmatrix}
 v_1 \\
 v_2 \\
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot \\
 v_n
 \end{bmatrix}$$

... (4.2.4.)

Where V_{mn} is the normal boundary velocity integrated over the element at the n th element, due to a source located at the m th element, and n is the number of elements the boundary has been subdivided into. The matrix identity, equation (4.2.4.) can be simply written as:-

$$[V] [\phi] = [\Phi] [v] \quad \dots (4.2.5.)$$

where $[V]$ and $[\phi]$ are the normal boundary velocities and potentials integrated over the element from the unit sources and $[\Phi]$ and $[v]$ are the potentials and normal boundary velocities imposed on the body.

If the problem consists of mixed boundary conditions, i.e. some potential and some flow controlled, then the relevant equations are interchanged where necessary, taking due account of signs. For example, in the identity equation (4.2.4.), say that all normal boundary velocities are known, except for element 2, i.e. v_2 . That is, the potentials of all elements, except element 2, are unknown; therefore, for this case the system of equations is:-

$$\begin{bmatrix} V_{11} & -\phi_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & V_{1n} \\ \cdot & -\phi_{22} & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ V_{n1} & -\phi_{n2} & \cdot & \cdot & \cdot & \cdot & \cdot & V_{nn} \end{bmatrix} \begin{bmatrix} \phi_1 \\ v_2 \\ \phi_3 \\ \cdot \\ \cdot \\ \cdot \\ \phi_n \end{bmatrix} =$$

$$\begin{bmatrix} \phi_{11} & -v_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \phi_{1n} \\ \cdot & -v_{22} & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \phi_{n1} & -v_{n2} & \cdot & \cdot & \cdot & \cdot & \cdot & \phi_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ \phi_2 \\ v_3 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix}$$

... (4.2.6.)

The system of equations can now be reduced, either by inversion or an iterative procedure.

As described in sections 2.2 and 2.3 on elastostatics, any symmetry the problem possesses will reduce the number of equations required to define the problem. For the case of multi-connected bodies (infinite or finite mass with holes), the identity (4.2.4.) still holds, however for proof of this, the body must be cut into simply connected regions.

It is interesting to note the analogous rigid body shift for this case, i.e. a potential shift. In equation (4.2.3.), say normal boundary velocities are equal to zero, and give all potentials a value of one, then equation (4.2.3.) becomes:-

$$\frac{1}{2} + \int V_{ij} dS = 0 \quad \dots (4.2.7.)$$

Therefore the free term and principal value can be obtained implicitly from (4.2.7.), which is analogous to that mentioned by Lachat and Watson (1975) for elastostatics. This fact is an important point when considering numerical integration using a point or line source (singularity) to determine the coefficients for $[V]$ and $[\Phi]$.

3. Numerical formulation

In the program, the boundary of the body was divided into straight line segments and the functions V and ϕ were taken to be constant over each element (similar to section 2.4). The equations for the potential and velocities at a point A, Figure 4.3.1, due to a strip source are given below. For the case of a single line source, the relevant equations are given in Appendix A 4.3.

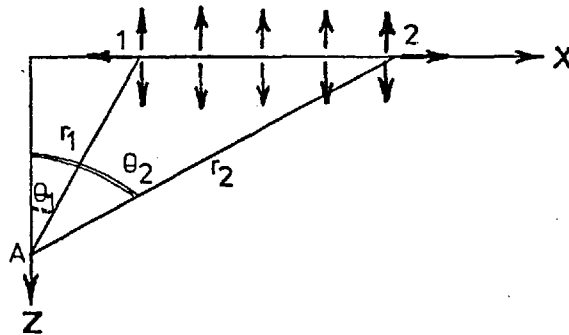


Figure 4.3.1 Potential and velocities at point A due to strip source.

$$\begin{aligned}\phi &= \frac{m}{2\pi} \left[x \ln r - x + z\theta \right]_1^2 \\ -\frac{\partial \phi}{\partial x} &= -\frac{m}{2\pi} \left[\ln r \right]_1^2 \\ -\frac{\partial \phi}{\partial z} &= -\frac{m}{2\pi} \left[\theta \right]_1^2\end{aligned}\quad \dots (4.3.1.)$$

$$v_x = -k_{xx} \frac{\partial \phi}{\partial x} - k_{xz} \frac{\partial \phi}{\partial z}$$

$$v_z = -k_{zx} \frac{\partial \phi}{\partial x} - k_{zz} \frac{\partial \phi}{\partial z}$$

where ϕ is the total head (potential) at the point A due to the strip source,

k_{xx} , $k_{xz} = k_{zx}$, k_{zz} are the coefficients of permeability for anisotropic media, and v_x and v_z are the velocities at point A in the x and z directions respectively.

The system of equations is written for the midpoints of the elements, thus avoiding the singularities at the ends of the elements. To aid explanation, consider the problem shown in Figure 4.3.2, in which the problem is represented by only three boundary elements.

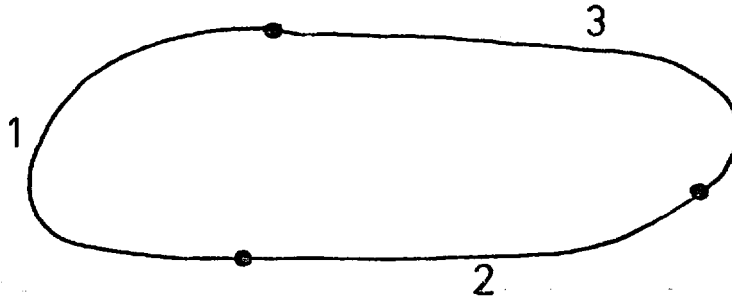


Figure 4.3.2 Problem represented by three boundary elements.

Then the system of equations for this problem:-

$$\begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \dots (4.3.2.)$$

When the identity (4.2.4.) is being assembled, the coefficients are scaled to obtain a non-dimensional and thus numerically stable system. The coefficients of $[V]$ are divided by the average permeability (k), the coefficients of $[\phi]$ are divided by the maximum distance between two elements ($r_{\max}$), given values of velocity are divided by k , and given values of potential are divided by $r_{\max}$. The unknowns are arranged to be placed in $[\phi]$ the knowns in $[v]$ and the columns of $[V]$ and $[\phi]$ are interchanged where necessary, with due account being taken of signs. The right hand sides, i.e. $[\phi]$ $[v]$ are now multiplied and held in core, while the matrix $[V]$ is read to a mass storage random access file for eventual reduction by a block solver (as in section 2.4).

Potentials and velocities at interior points (i.e. inside the body) are calculated as described in section 4.5. The program has a number of generation routines in order to simplify input. Any symmetry the problem possesses is taken into account in order to reduce the system of equations required to describe the problem.

4. Subregions

A similar procedure to that described in section 2.5 for elastostatics is adopted for the analysis of non-homogeneous media for ground-water seepage problems. To construct the system of equations, first the identity, equation (4.2.4.), is written for each homogeneous region in turn, then by adding and subtracting the relevant equations the complete (for the heterogeneous problem) system of equations is found. For ease of explanation, consider the problem illustrated in Figure 4.4.1, which consists of two subregions with the entire problem modelled by three boundary elements.

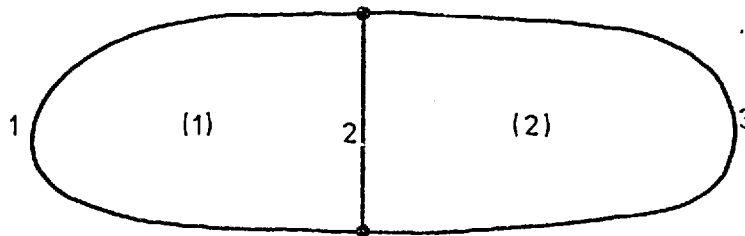


Figure 4.4.1 Body consisting of two subregions modelled by three boundary elements.

The identity (equation 4.2.4.) can now be written for each subregion, i.e.

$$\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \\ & v_{2^1 2^1} & v_{2^1 3} \\ & v_{32^1} & v_{33} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_{2^1} \\ \phi_3 \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \\ & \phi_{2^1 2^1} & \phi_{2^1 3} \\ & \phi_{32^1} & \phi_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_{2^1} \\ v_3 \end{bmatrix} \quad \dots (4.4.1.)$$

where elements 1 and 2 lie in subregion (1) and elements 2¹ and 3 lie in subregion (2), as illustrated in Figure 4.4.2.

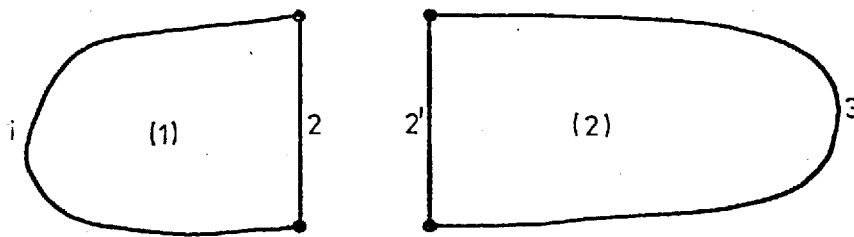


Figure 4.4.2 Body considered as two separate subregions.

Now along the boundary between the two subregions, i.e. in this case elements 2 and 2¹, the following conditions apply:

$$v_2 = -v_{2^1}$$

and $\phi_2 = \phi_{2^1}$

... (4.4.2.)

where v_2 and v_{2^1} are the normal boundary velocities of flow for elements 2 and 2^1 , which lie on the boundary of subregions (1) and (2), Figure 4.4.2,

and ϕ_2 and ϕ_{2^1} are the potentials for elements 2 and 2^1 in Figure 4.4.2.

Therefore from equations (4.4.1.) and (4.4.2.) the system of equations for the problem, Figure 4.4.1, can be written as:-

$$\begin{bmatrix} v_{11} & \cdot & \cdot \\ v_{21} & \cdot & \cdot \\ v_{21} & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \phi_1 \\ v_2 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{21} \\ \cdot \end{bmatrix} \begin{bmatrix} v_1 \\ 0 \\ 0 \\ v_3 \end{bmatrix} \quad \dots (4.4.3.)$$

Obviously for elements which are shared by two subregions, the normal boundary velocity and potential are both unknown, thus for the problem shown in Figure 4.4.1 we have four unknowns. That is, for each element shared by two subregions the number of equations is

doubled; however the reduction in the total size of the problem (e.g. banded matrix) and the ability to analyse heterogeneous problems far outweigh this disadvantage.

In Figure 4.4.3, the assemblage of the matrix $[V]$ is given for a more general problem. The reason why velocities for each shared element are written first is because velocities do not uniquely define potential, i.e. for a subregion without potential controlled boundary conditions, a shift in potential is required for the complete solution (c.f. rigid body displacement).

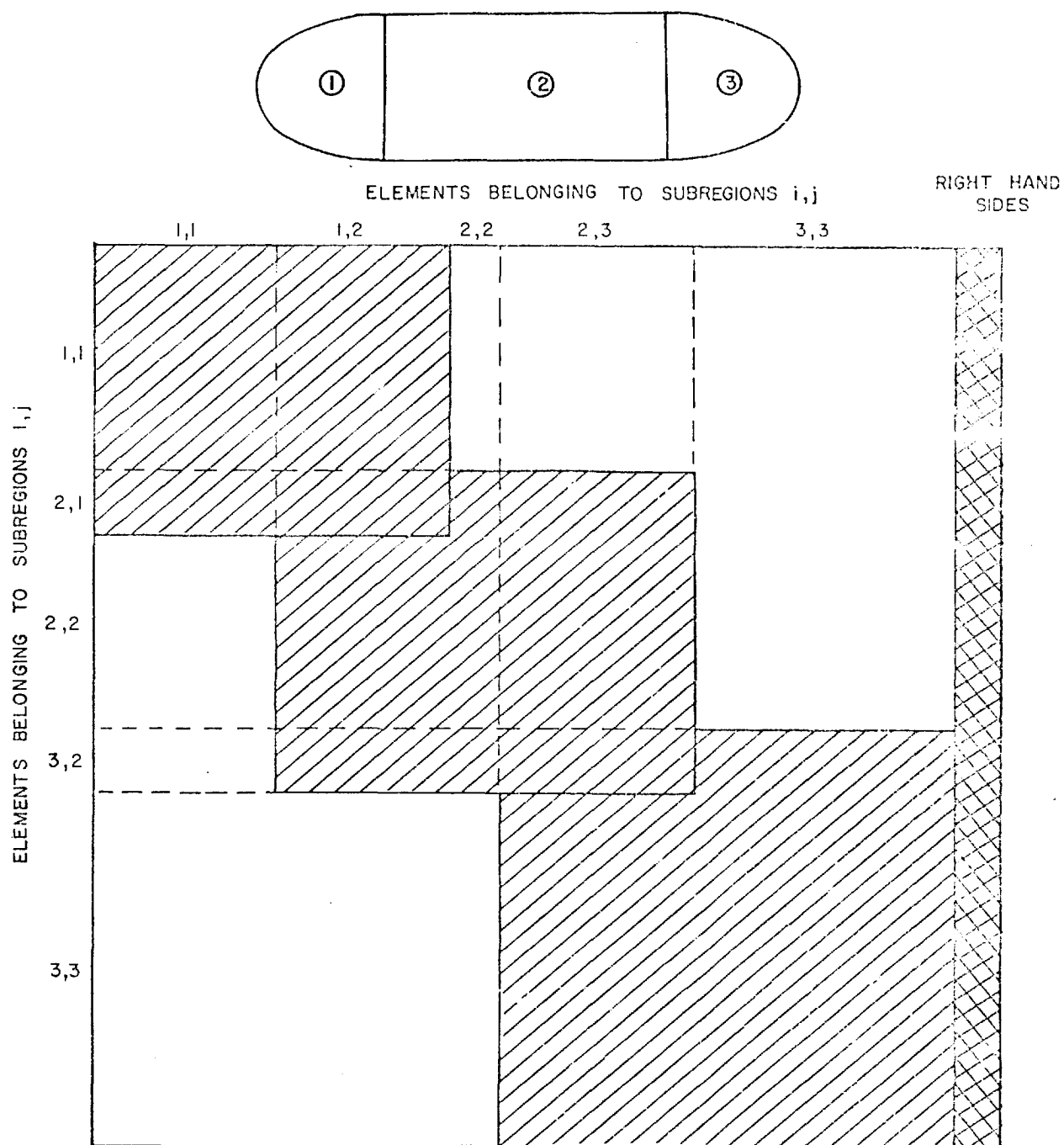


FIGURE 4.4.3 CONSTRUCTION OF $[V]$ FOR SUBREGION PROBLEM SHOWN ABOVE

5. Calculation of potentials and velocities at interior points

Consider the homogeneous, anisotropic finite body, Figure 4.5.1, in which an interior point p has potential ϕ_p and velocity components v_{xp} and v_{zp} , due to applied potentials (ϕ) and normal (to the boundary) components of velocity of flow (v) acting on the boundary of the body. In the same body, but with the boundary conditions ϕ and v removed, Figure 4.5.2, place a source of unit flow at p and calculate the relevant velocities (V) and potentials (ϕ) for boundary points from equation (4.5.1.) to maintain the body in equilibrium. These velocities (V) and potentials (ϕ) can be found from the solution of a line source in an infinite mass (Batchelor (1974)).

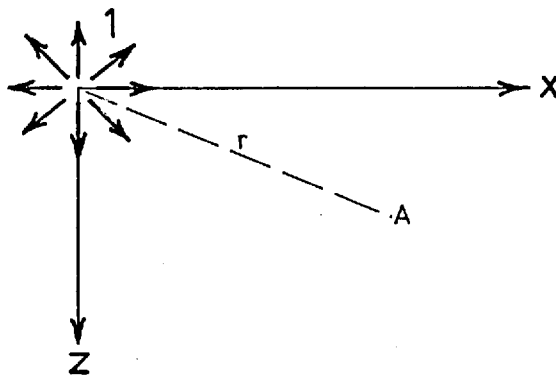


Figure 4.5.3 Velocities and potential at a point A, due to a unit line source in an infinite mass.

$$\begin{aligned}
 \phi &= \frac{1}{2\pi} \ln r \\
 v_x &= -k_{xx} \frac{x}{2\pi r^2} - k_{zz} \frac{z}{2\pi r^2} \quad \dots (4.5.1.) \\
 v_z &= -k_{zx} \frac{x}{2\pi r^2} - k_{zz} \frac{z}{2\pi r^2}
 \end{aligned}$$

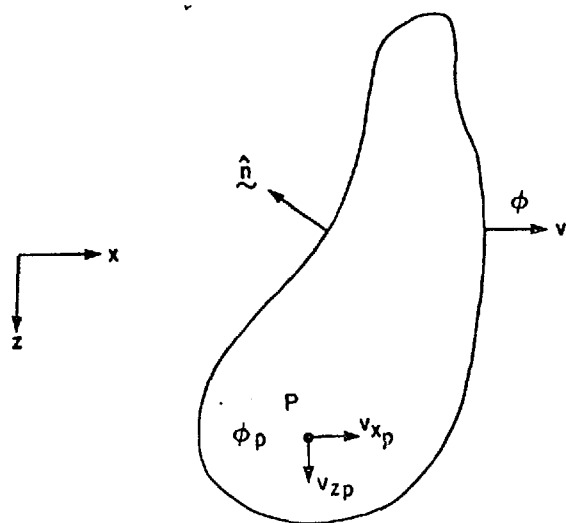


FIGURE 4.5.1 APPLIED POTENTIALS AND NORMAL BOUNDARY VELOCITIES OF FLOW ON BODY OF SURFACE S , WITH ASSOCIATED POTENTIAL AND VELOCITY COMPONENTS AT INTERIOR POINT P

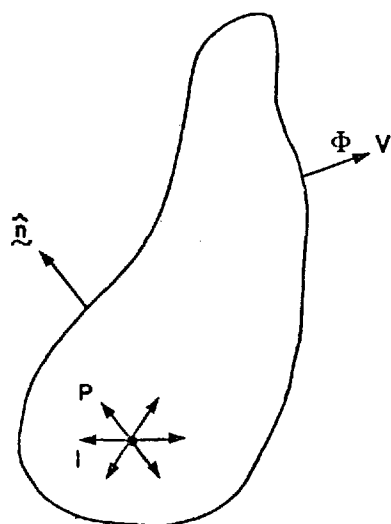


FIGURE 4.5.2 SOURCE OF UNIT FLOW AT INTERIOR POINT P , WITH ASSOCIATED POTENTIALS AND NORMAL BOUNDARY VELOCITIES OF FLOW ON THE BODY

By applying Green's equation (similar to section 4.2) we obtain:

$$\phi_p + \int V \phi dS = \int \phi v dS \quad \dots (4.5.2.)$$

where V and ϕ are the normal boundary velocities and potentials on the boundary of the body, Figure 4.5.2, due to the unit line source at the interior point p ,

v and ϕ are the normal boundary velocities and potentials on the boundary of the body shown in Figure 4.5.1, and ϕ_p is the potential at the interior point p in the body, Figure 4.5.1, with boundary conditions of ϕ and v .

To obtain velocity components at p , we must differentiate the potential at p with respect to the velocity direction. In order to simplify notation, equate the following:-

$$\begin{aligned} M &= - \frac{\partial V}{\partial x} & N &= - \frac{\partial V}{\partial z} & \dots (4.5.3.) \\ P &= - \frac{\partial \phi}{\partial x} & Q &= - \frac{\partial \phi}{\partial z} \end{aligned}$$

The above partial derivatives (i.e. in equation (4.5.3.)) for the unit line source are given in Appendix A 4.3.

Therefore the velocity components (x and z) at the interior point p are given by:

$$\frac{-\partial\phi_p}{\partial x} = \int P_v dS - \int M_\phi dS$$

$$-\frac{\partial\phi_p}{\partial z} = \int Q_v dS - \int N_\phi dS$$

... (4.5.4.)

$$v_{xp} = -k_{xx} \frac{\partial\phi_p}{\partial x} - k_{xz} \frac{\partial\phi_p}{\partial z}$$

$$v_{zp} = -k_{zx} \frac{\partial\phi_p}{\partial x} - k_{zz} \frac{\partial\phi_p}{\partial z}$$

where v_{xp} and v_{zp} are the x and z components of velocity of flow at the interior point p, Figure 4.5.i, with boundary conditions of ϕ and v .

Thus by simply integrating the relevant terms (in this case considered constant over each element) around the boundary the potential and velocity components at an interior point can be found. Obviously if the body consists of a number of subregions, then the integrals in equations (4.5.2.) and (4.5.4.) are only considered for the elements defining the subregion in which the point lies.

6. Steady flow with phreatic surface

In section 4.2, only confined flow problems were considered, i.e. boundaries were fixed in space and had prescribed values of either potential ϕ or normal boundary velocity v . In this section, the method is extended to include the analysis of unconfined flow problems, i.e. the location of a phreatic surface is unknown.

The potential along the phreatic surface is equal to the elevation head, since the pressure at the free surface is zero (atmospheric in most cases). The case of a seepage surface must also be considered; however, this is simply done by prescribing a potential boundary condition with flow being unknown. The following iteration scheme is used to determine the phreatic surface location.

- (1) The position of the phreatic surface is estimated.
- (2) The problem is then solved in the usual manner (see section 4.2) with zero (if no precipitation) flow boundary condition for the free surface.
- (3) The pressure along the free surface now determines the new position of the free surface. If the pressure is zero at all points on the free surface then the problem is solved, otherwise a new position of the free surface is calculated and the problem is looped to step (2) for the next iteration cycle.

7. Unsteady flow with phreatic surface

In this section unsteady (transient) flow will be considered. Unsteady flow resulting from drainage of the media (i.e. fluid stored in the pores of the media) will only be considered, thus storage of fluid due to volumetric deformation of fluid and media is ignored. The problem is the same as confined flow (section 4.2) except that time dependency is introduced through one or more of the boundary conditions being time dependent.

Suppose we consider the problem illustrated in Figure 4.7.1, where at some time t , the free surface is located at y^t and at time $t + 1$, the location of the free surface is y^{t+1} (shown dotted). The volume of fluid removed from the shaded area in an increment of time $(t + 1 - t)$ is given by,

$$W = A\beta \dots (4.7.1.)$$

where A is the area of the shaded portion in Figure 4.7.1, β is the effective porosity (the partial volume of the pore space which will be free to move per unit volume of the media) and W is the volume of fluid removed from the shaded portion, as shown in Figure 4.7.1.

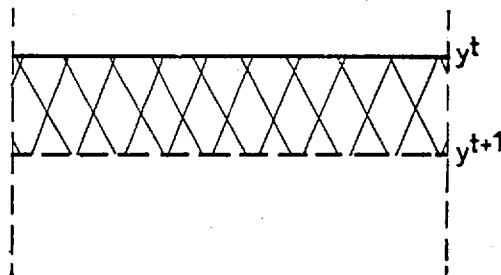


Figure 4.7.1 Unsteady flow, phreatic surface position at times t and $t + 1$

The following iteration scheme is used to find the position of the free surface (or surfaces) with respect to time.

- (1) The position of the free surface is known at time $t = 0$.
- (2) The problem is then solved in the usual manner (see section 4.2) with pressure equal to zero (i.e. potential equal to elevation head) on the free surface.
- (3) The free surface is now moved from position at time t , i.e. y^t to position at time $t + 1$, i.e. y^{t+1} (see Figure 4.7.1) so that for each element equation (4.7.2.) holds:

$$A_i^t = \frac{q_i^t \Delta t}{\beta} \quad \dots (4.7.2.)$$

where A_i^t is the area of the shaded portion, Figure 4.7.1 for element i at time t . and q_i^t is the flow of fluid across the element i at time t found from the solution of the problem described in step (2). If any precipitation occurs (e.g. evaporation or rainfall) then the flow of fluid due to precipitation at time t is taken from q_i^t , and Δt is the increment of the time step ($t + 1 - t$).

- (4) If there is no change (or negligible change) in the position of the free surface between time $t + 1$ and time t , then the problem has reached a "steady" flow condition. Otherwise, with the new position of the free surface at $t + 1$, the program loops to step (2) in the iteration cycle.

The time step can be preset to a constant value, used in problems where geometry changes are similar from one step to another (e.g. position of the phreatic surface in an embankment dam due to fluctuation of the reservoir level); or calculated automatically in the program, from a preset maximum movement of the free surface for each cycle.

8. Interface of immiscible fluids

The interface condition between two immiscible fluids of different densities bears a close resemblance to that of a free surface. In the case of oil and water, there may well be an abrupt interface; however, in the case of fresh and salt water, there will exist a transition zone, in which mixing will occur. As a first approximation it may be worthwhile to simplify the problem by assuming an abrupt interface. An interface of two immiscible fluids of different densities is shown in Figure 4.8.1. The conditions existing at the interface are given in equation (4.8.1.)

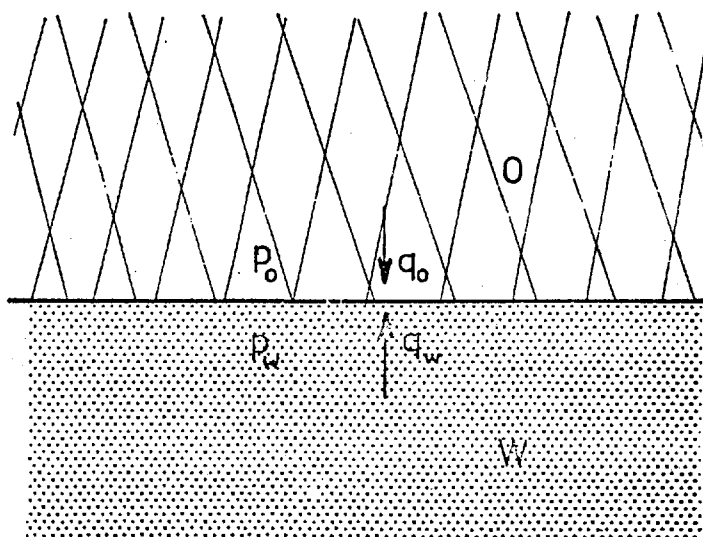


Figure 4.8.1

Interface of two immiscible
fluids of different densities.

$$p_o = p_w$$

... (4.8.1.)

$$q_o = q_w = 0$$

where p_o , p_w are the pressures at the interface in the fluids o and w respectively,

and q_o , q_w are the flows out of the fluids o and w respectively through the interface.

Thus the pressure on either side of the interface must be equal and there exists no flow through the interface.

Therefore this case is analogous to the steady state phreatic surface problem, section 4.6. The problem is begun either at rest or unbalanced and the interface is found by successive iterations as stated in section 4.6, until an equilibrium position for the interface is found.

CHAPTER 5: APPLICATION OF THE METHOD
 TO GROUNDWATER SEEPAGE PROBLEMS

1. Check on accuracy of the numerical formulation

Bray (1976 b) has determined the analytical solution for seepage from a pressure tunnel to a free surface in a homogeneous isotropic medium, see Figure 5.1.1. This solution is based on the fact that the flow lines and isopotentials consist of families of circles for a source sink combination, Figure 5.1.2. Equation (5.1.1.) is the flow from the pressure tunnel, Figure 5.1.1., and equation (5.1.2.) is the pressure at any point in the medium. The solution is correct when the potential around the tunnel wall is a constant, thus the analysis is approximate when the tunnel is acting as a drain and runs only partly full.

$$Q = 2\pi k \left(\frac{u_0}{\gamma_w} - z_0 \right) / \ln \left(\frac{d + z_0}{a} \right) \quad \dots (5.1.1.)$$

where $d = \sqrt{z_0^2 - a^2}$

u_0 = pressure at centre of tunnel

γ_w = unit weight of water

k = permeability

and Q = flow out of tunnel

$$u = \gamma_w \left\{ \frac{Q}{2\pi k} \ln \left(\frac{x^2 + (z + d)^2}{x^2 + (z - d)^2} \right)^{\frac{1}{2}} + z \right\} \quad \dots (5.1.2.)$$

where u is the pressure at point (x, z)

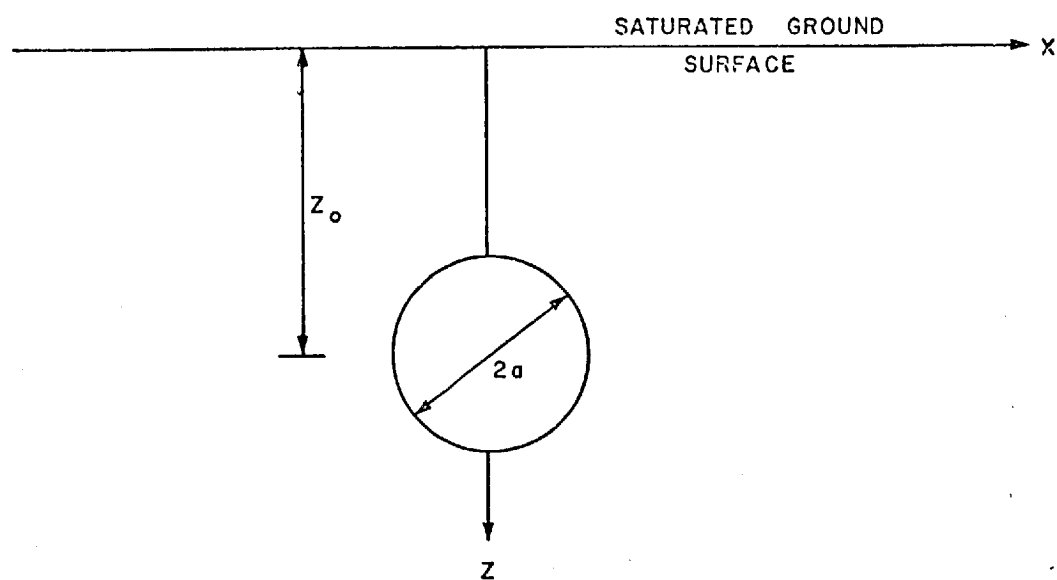


FIGURE 5.1.1 CIRCULAR PRESSURE TUNNEL (OR DRAIN)
IN SATURATED GROUND

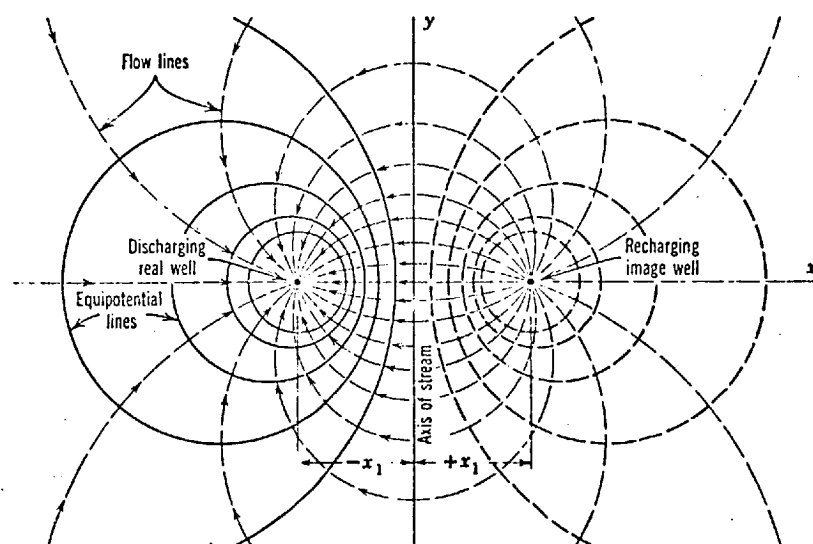


FIGURE 5.1.2 EQUIPOTENTIAL AND FLOW LINES FOR A
SOURCE AND SINK SYSTEM (TODD (1967))

A number of problems, as shown in Figure 5.1.1 were analysed by the 2-D B.I.M. program and the results are compared with the analytical solution of Bray in Table 5.1.1.

TABLE 5.1.1. FLOW FROM A CIRCULAR PRESSURE TUNNEL
(OR DRAIN) IN SATURATED GROUND ($k = 1.0$)

$\frac{z_0}{2a}$	Pressure Tunnel $\frac{u_0}{\gamma_w} = \frac{z_0}{2a} + 1$		Drain $\frac{u_0}{\gamma_w} = 0$	
	B.I.M.	Analytical	B.I.M.	Analytical Approx.
8.0	1.76	1.81	-14.07	-14.51
6.0	1.93	1.98	-11.51	-11.87
5.0	2.05	2.10	-10.18	-10.50
4.0	2.21	2.27	-8.80	-9.08
3.0	2.47	2.54	-7.33	-7.61
2.0	2.97	3.05	-5.77	-6.09
1.5	3.47	3.56	-4.93	-5.35
1.0	4.57	4.77	-4.04	-4.77

All the cases quoted in Table 5.1.1., had fifteen boundary elements to model one half of the tunnel, and surface elements immediately above the tunnel were five times longer than the tunnel elements. There is reasonable agreement between the B.I.M. and the analytical solution for the whole range of tunnel depths considered. The analytical solution for the drain case computed inflows 20% high, for a drain one diameter below the surface. At depth there exists good agreement between the approximate analytical solution and the B.I.M. for the drain case.

The pressure at points inside the medium near a circular pressure tunnel are plotted in Figure 5.1.3 for both the analytical solution and the B.I.M. program. Excellent agreement exists for points along both axes. The components of flow velocity for these interior points are given in Figure 5.1.4. There is good agreement between: results calculated from the B.I. Method and those from the analytical solution.

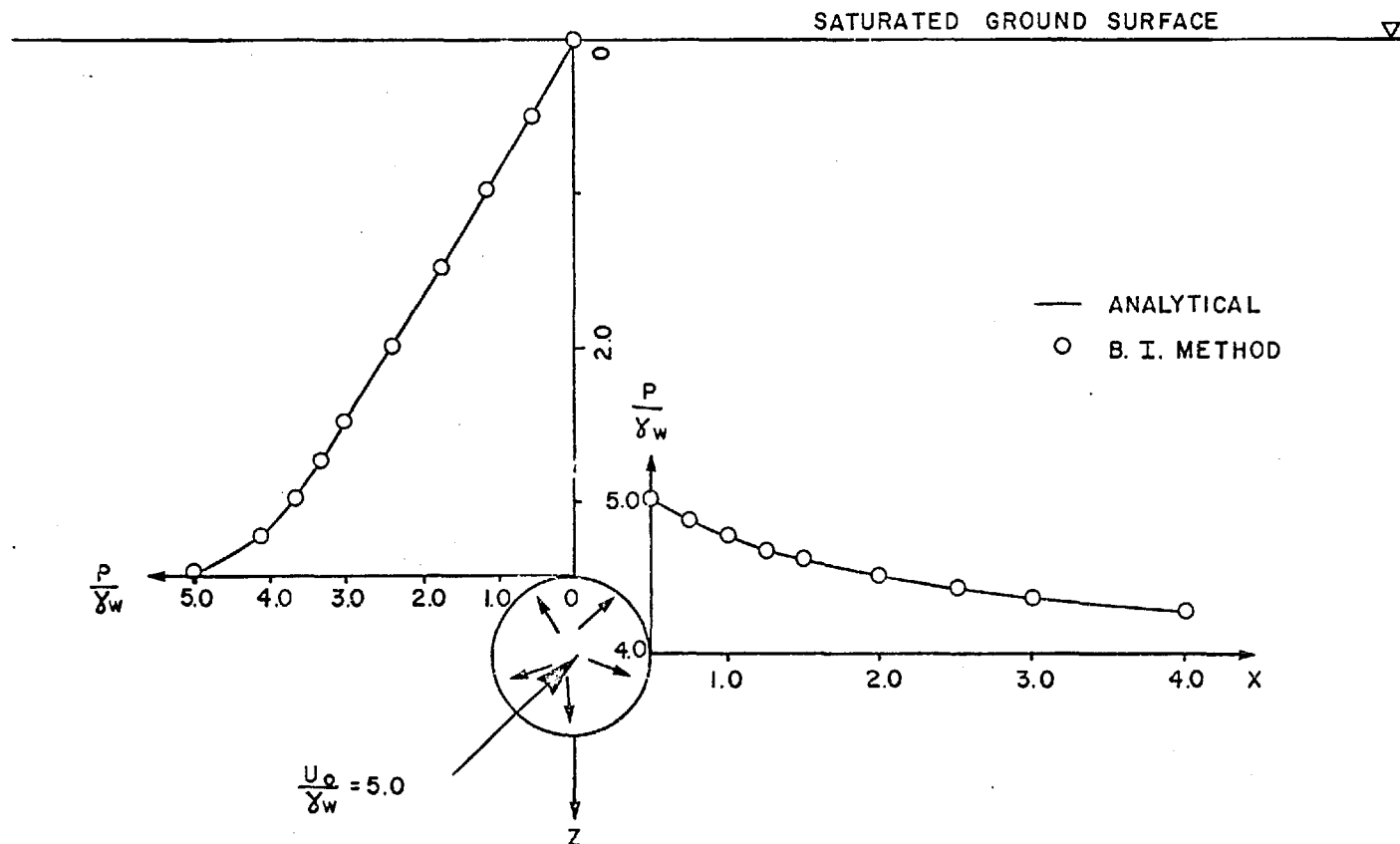


FIGURE 5.1.3 PRESSURE HEAD AT INTERIOR POINTS NEAR A CIRCULAR
PRESSURE TUNNEL AND FREE SURFACE

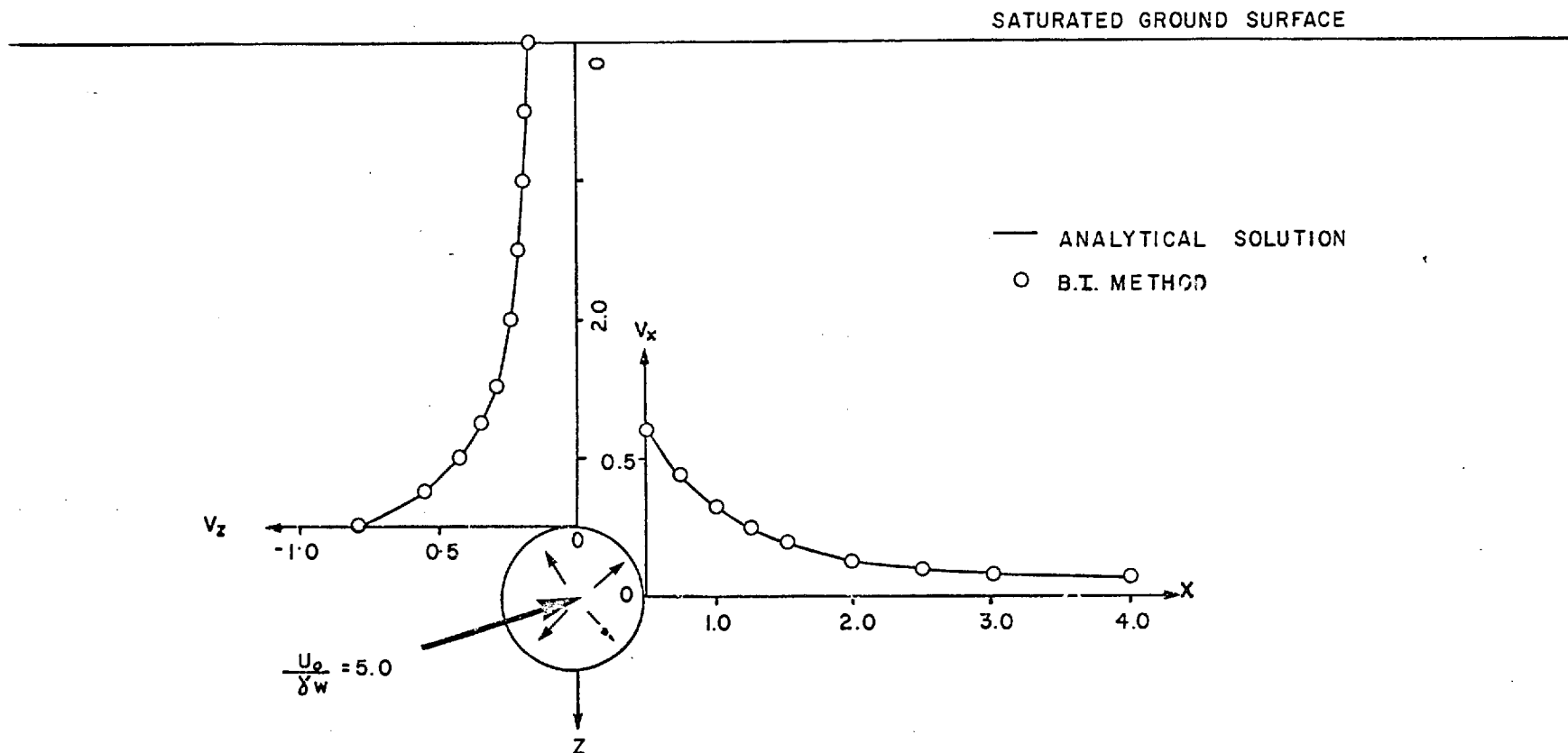


FIGURE 5.1.4 SEEPAGE VELOCITIES AT INTERIOR POINTS NEAR A CIRCULAR
PRESSURE TUNNEL AND FREE SURFACE

2. Pressure tunnel and drain near a free surface

The boundary flow velocities for the pressure tunnel in a saturated medium four diameters below the ground surface are given in Figures 5.2.1 and 5.2.2. The pressure head at the centre of the tunnel is 4.0. The velocities of flow out of the ground, Figure 5.2.1, do not differ greatly for the three anisotropic permeabilities considered (i.e. principal permeabilities of 10 and 1, (a) maximum in x direction (b) maximum 45° to x axis and (c) maximum in z direction). If the tunnel was closer to the ground surface then the differences would be greater. In this case the flow out of the pressure tunnel was 2.21 for isotropic permeability of 1.0, 12.1 for $k_x = 10.0$ and $k_z = 1.0$, 12.2 for $k_{\max} 45^\circ$ to x axis, and 12.2 for $k_z = 10.0$ and $k_x = 1.0$. The distribution of flow velocities out of the pressure tunnel, Figure 5.2.2, are interesting, since as is to be expected, they are elongated in the direction of least resistance to flow.

The flow velocity distributions out of surfaces for the case of a drained tunnel four diameters below a free surface are illustrated in Figures 5.2.3 and 5.2.4. In Figure 5.2.3, the flow velocities for the three anisotropic permeability cases are very similar. The flows into the drained tunnel are 8.8 for isotropic permeability of 1.0, 48.4 for $k_x = 10.0$ $k_z = 1.0$, 48.4 for $k_{\max} 45^\circ$ to x axis and 48.4 for $k_z = 10.0$ and $k_x = 1.0$. The similar inflows between the different permeability ratios considered is an interesting point, especially since this same feature was present in the pressure tunnel case.

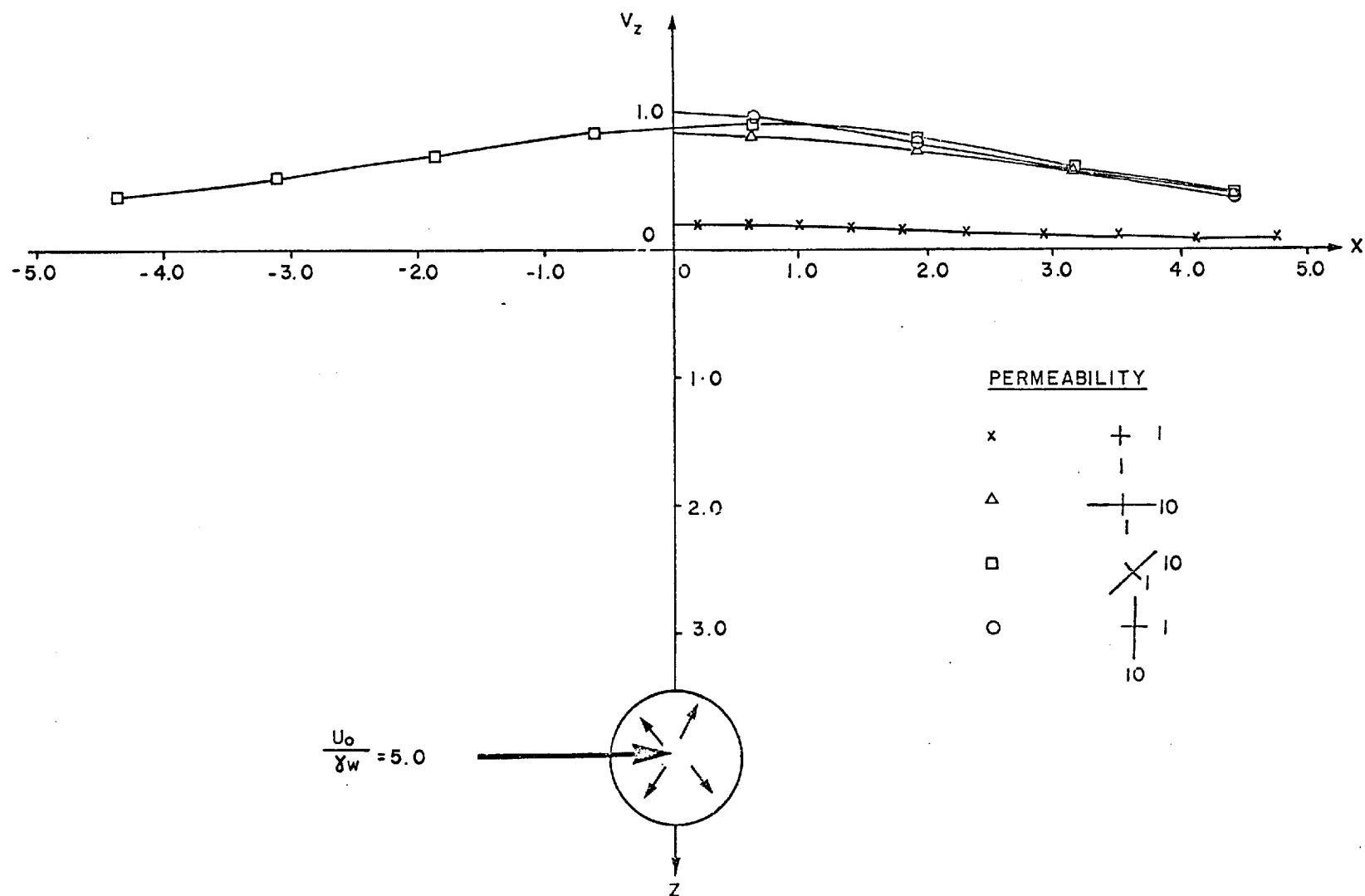


FIGURE 5.2.1 FLOW VELOCITY OUT OF GROUND SURFACE NEAR A PRESSURE TUNNEL

PERMEABILITY

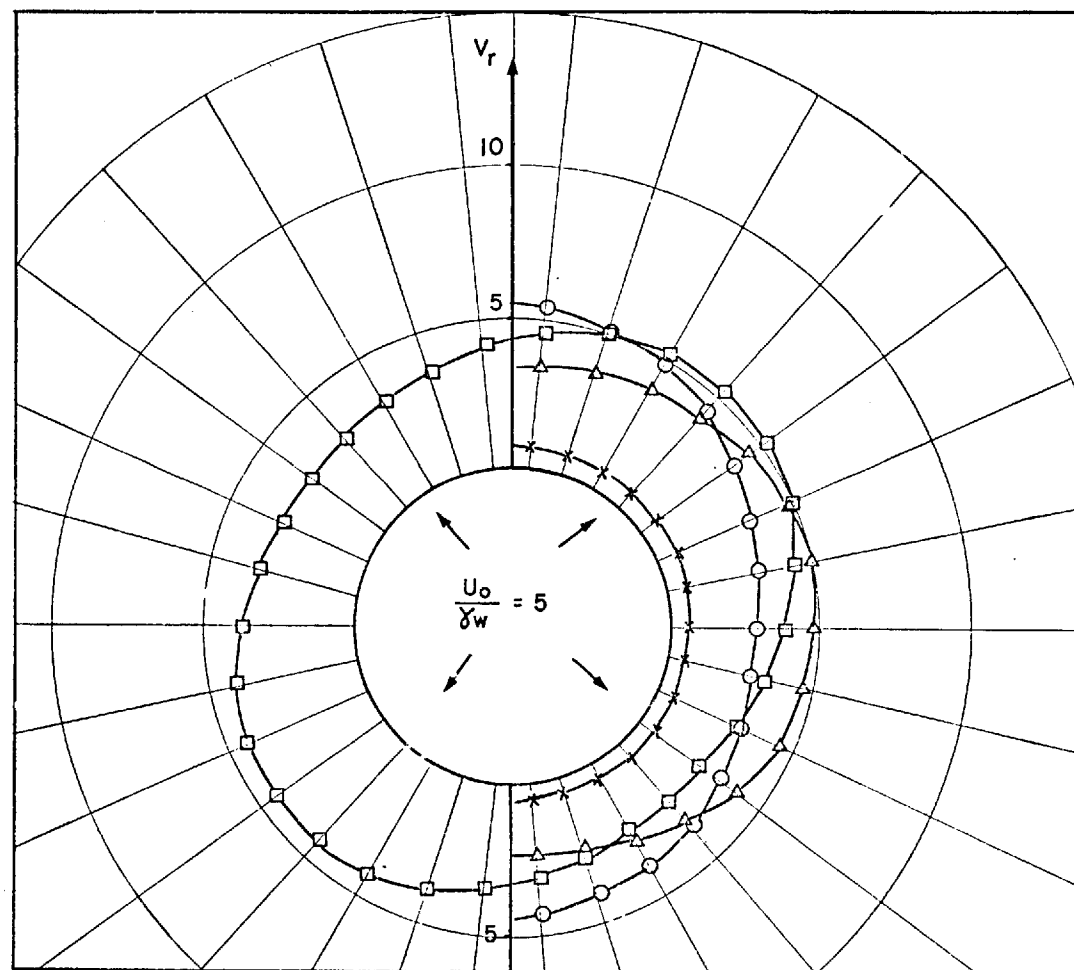
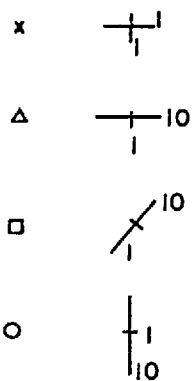


FIGURE 5.2.2 FLOW VELOCITY OUT OF A PRESSURE TUNNEL NEAR THE GROUND SURFACE

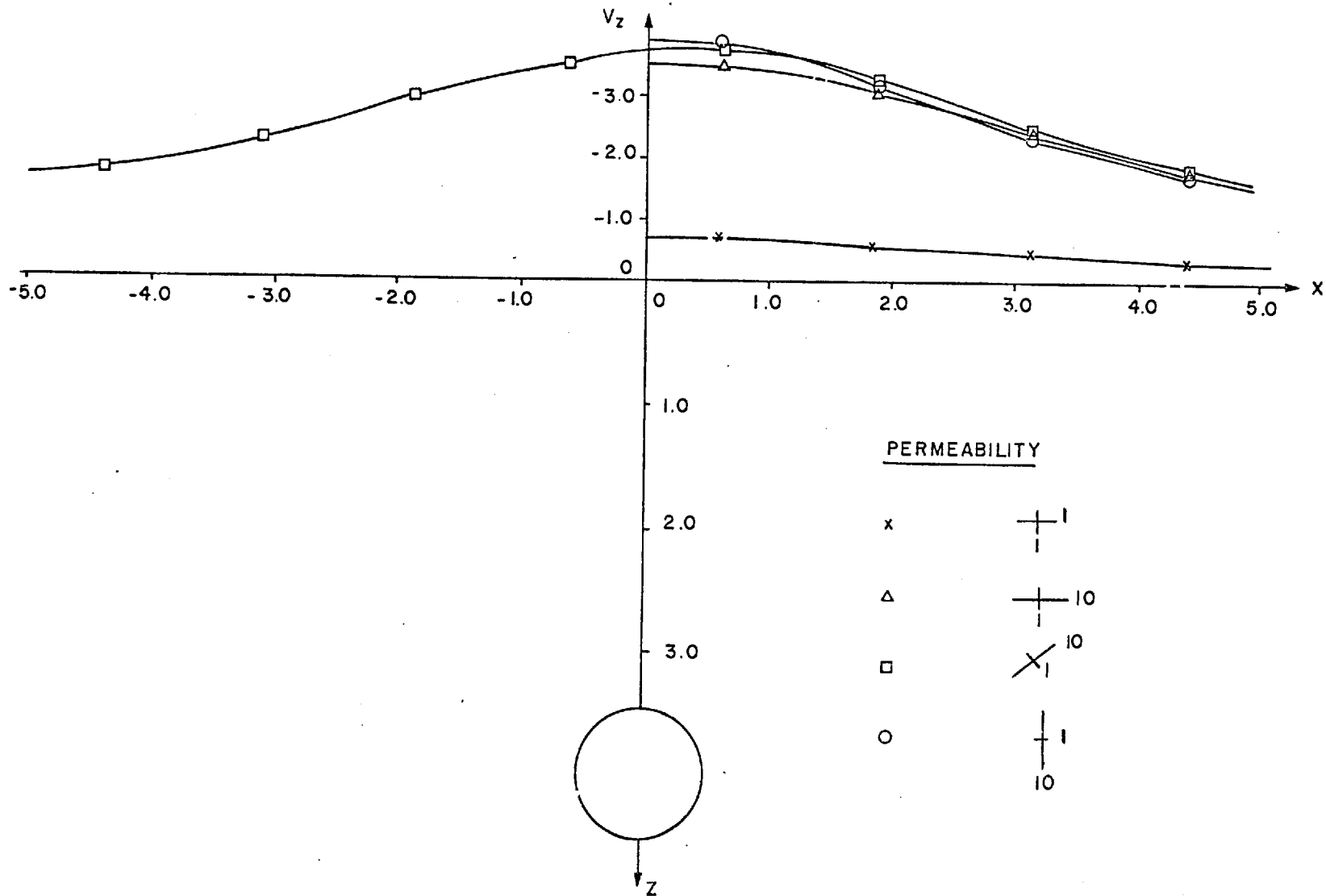


FIGURE 5.2.3 FLOW VELOCITY INTO GROUND NEAR A DRAINED TUNNEL

PERMEABILITY

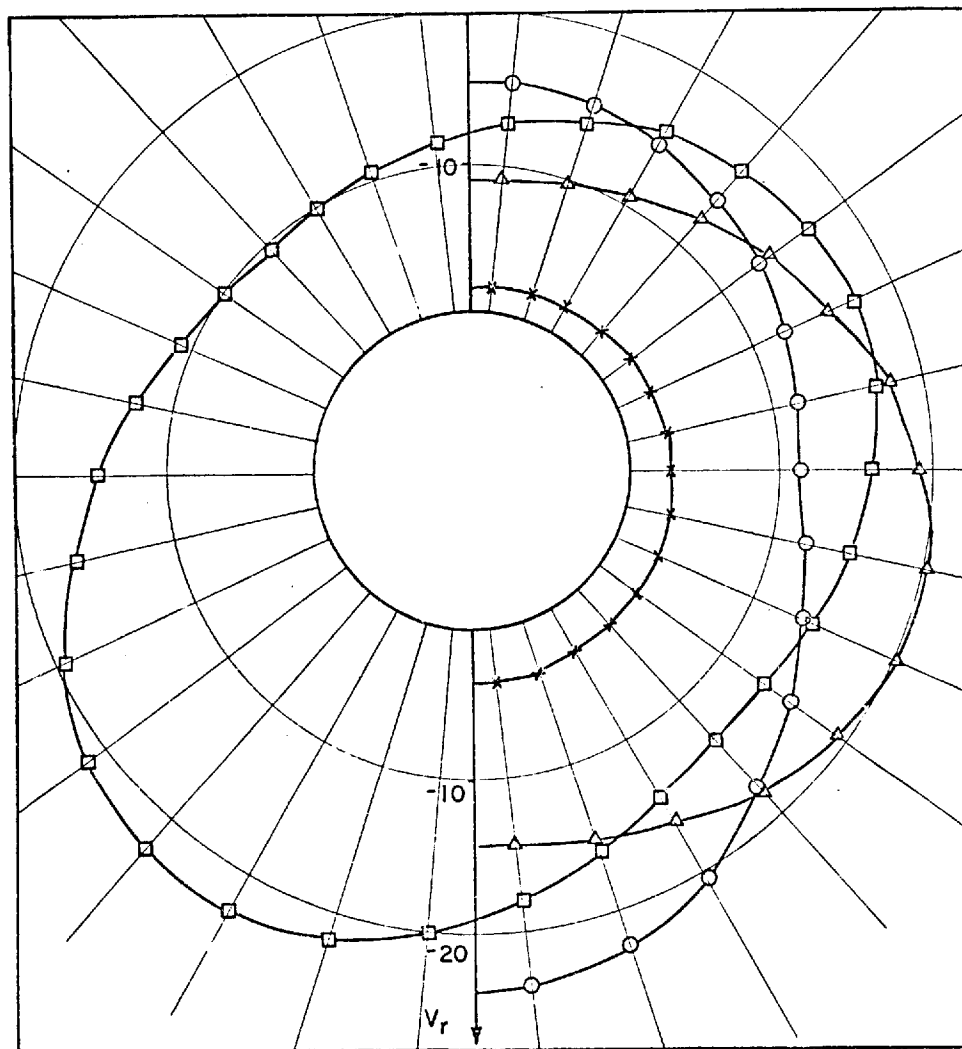
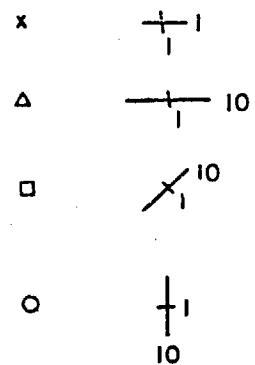


FIGURE 5.2.4 FLOW VELOCITY INTO A DRAINED TUNNEL NEAR THE GROUND SURFACE

3. Steady state flow with phreatic surface

A drained tunnel located four diameters from infinitely deep reservoirs was analysed by the B.I.M., Figure 5.3.1. The analysis was begun with an upper bound of the phreatic surface, in this case horizontal at $z = 0$. Then by iteration as described in section 4.6, the final solution obtained. An electrical analogue (Sharp (1970)) was used to solve the same problem and there was good agreement between the two methods, Figure 5.3.1. The solution from the electrical analogue cannot be considered precise for this case, since a lower impermeable boundary was installed because of mesh size limitations, and also the problem was modelled with a coarse mesh. However, due to the geometry of the problem these effects did not significantly control the position of the phreatic surface, even though they did alter the quantity of flow into the tunnel, and therefore the electrical analogue was considered to provide a worthwhile check.

4. Unsteady state flow with phreatic surface.

A drained tunnel, four diameters below the ground surface, was analysed with respect to time, Figure 5.4.1. The problem commenced with the ground being totally saturated and the circular drain installed. Then the solution with respect to time was calculated as described in section 4.7. Each drawdown curve shown in Figure 5.4.1, denoted t_0, t_1 etc. was calculated for a variable time step. The time step was automatically calculated within the program, so that the maximum drawdown for each increment was the same. Therefore to gain insight into the drawdown curves plotted, the respective time steps are important. For example, $t_1 - t_0$ was 0.076, $t_2 - t_1$ was 0.077, and $t_3 - t_2$ was 0.083. The rapid increase of the time step is significant and indicates the large time required for the phreatic surface to intersect the drain. The final drawdown curve as $t \rightarrow \infty$ would eventually approach a horizontal line, coinciding with the invert level of the drain.

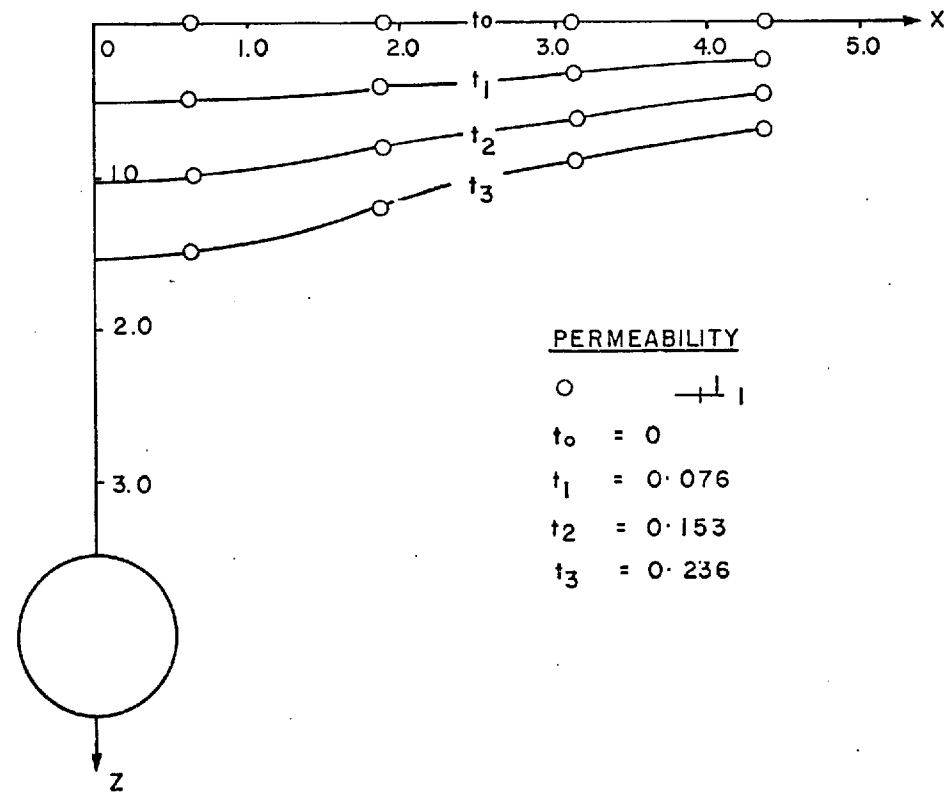


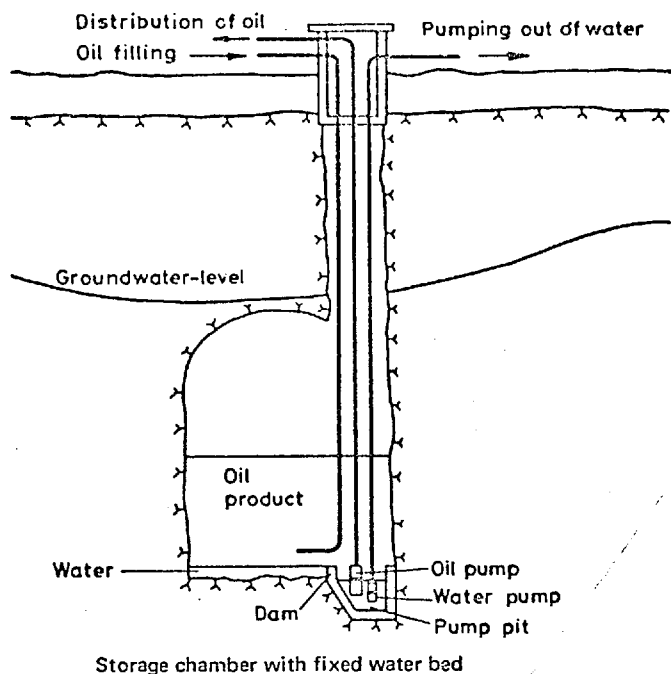
FIGURE 5.4.1 POSITION OF PHREATIC SURFACE WITH TIME FOR A DRAINED TUNNEL

5. Underground oil storage complexes

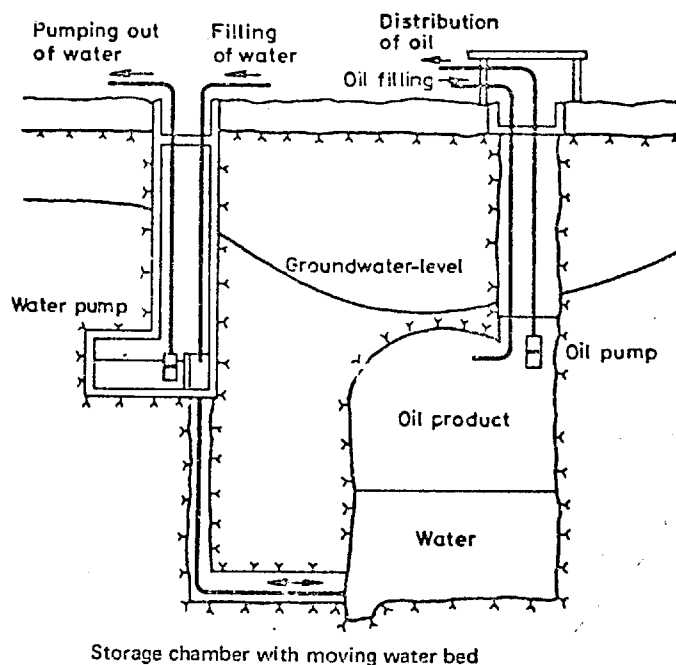
Underground oil storage schemes have been built in various parts of Scandinavia (Morfeldt (1972), Johansson & Lahtinen (1976)). The chambers are generally unlined and consist of - natural caverns, abandoned mines, leached caverns in salt domes, and excavated chambers in rock. For unlined caverns the basic requirements for planning are: the bedrock be competent, near the surface, able to stand virtually unsupported when excavated, the ground water table be sufficiently above the proposed caverns, and the oils to be stored should be lighter than water and insoluble in it.

The basic principle of operation is that the ground water pressure at any point around the chamber is greater than the stored oil pressure. Therefore flow of ground water towards the caverns takes place and thus the oil is prevented from escaping into the bedrock. The flow of water into the chambers depends on the permeability of the bedrock. Grouting may be required to seal any fissures present and special care given to shattered zones.

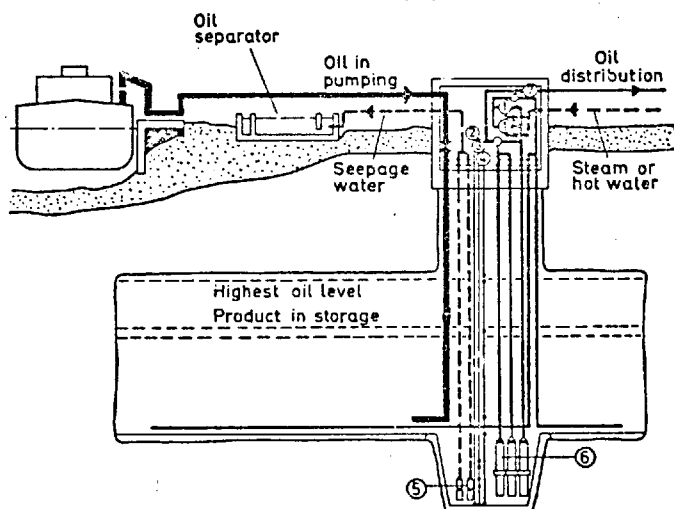
There are three main types of underground oil storage; namely, fixed water bed, moving water bed and dry bottom, Figure 5.5.1. In the case of the fixed water bed, the stored oil lies on a bed of water approximately 0.5m. deep. Ground-water inflow is pumped out to maintain the water bed oil boundary at a fixed level. Oil and water pumps are situated in a pit in the base of the chamber. Heating of the oil is effected by heating the water bed.



(a)



(b)



(c)

FIGURE 5.5.1 THREE UNDERGROUND STORAGE SYSTEMS

a) FIXED WATER BED b) MOVING WATER BED AND c) DRY BOTTOM

(JOHANSSON & LAHTINEN (1976))

The moving water bed storage system, Figure 5.5.1, consists of maintaining a constant level of the upper surface of the stored oil. This is carried out by pumping out ground-water inflow to maintain a constant elevation of the upper surface. In filling the store, a corresponding amount of water is pumped out of the water bed, and similarly when emptying.

The dry bottom storage, Figure 5.5.1, is generally constructed with a concrete floor with trenches to enable groundwater inflow to be removed. Heating of the oil is carried out directly.

Each type of storage has certain advantages, which can be summarized as follows (Johansson and Lahtinen (1976)). A moving water bed is safer in dealing with light oil products with high vapour pressure because the risk of internal fire, as well as overall evaporation losses, can be reduced more readily. A fixed water bed is mostly applied to middle distillates, fuel oils, crude, etc. Its great advantage lies in its much lower need for water treatment and purification. Consequently, in a fixed water bed it is easier to use high filling and outpumping capacities. The use of dry-bottom storage arises for the heaviest fuel oils, when the risk of contamination between water and oil exists. When technically acceptable, a fixed water bed gives the best solution economically for underground storage facilities in rock.

To illustrate the application of the B.I.M. to flow problems a fixed water bed storage system, similar to the Sentab scheme outlined by Morfeldt (1972), was analysed. The cross section of the problem is illustrated in Figure 5.5.2. The problem is in fact three dimensional; however as the caverns are long compared to their cross-sectional dimensions and depth below the phreatic surface, a two dimensional analysis is a worthwhile first approximation. The problem was analysed initially as a balanced system, i.e. both chambers filled to the same level, Figure 5.5.2. A water filled tunnel was placed between and above the two storage caverns, with a pressure head at the centre of the tunnel of 5 metres. The permeability of the bedrock was taken as isotropic with $k = 1 \times 10^{-8}$ m/sec. The vapour pressure was inputted as equivalent to 2.5 metres head of water. With zero precipitation and effective porosity of 1%, the phreatic surface locations with time are illustrated in Figure 5.5.2. The respective times in seconds were 0.0, 3.65×10^6 , 7.38×10^6 and 9.34×10^6 for t_0 , t_1 , t_2 and t_3 . The flows (m^3/sec) into each storage chamber were 8.49×10^{-7} , 8.28×10^{-7} and 7.50×10^{-7} for times t_0 , t_1 , and t_2 respectively. The flows (m^3/sec) out of the water filled tunnel were -9.63×10^{-9} , 5.48×10^{-9} and 1.11×10^{-8} for t_0 , t_1 and t_2 . The flow velocity distributions into or out of the storage chambers and the water filled tunnel are given in Figure 5.5.3.

The same problem was analysed for an unbalanced system, Figure 5.5.4. In this case oil has been removed from the right hand chamber. The positions of the phreatic surface are shown for times (secs) of 0.0, 3.52×10^6 and 7.10×10^6 noted on the diagram as t_0 , t_1 and t_2 respectively.

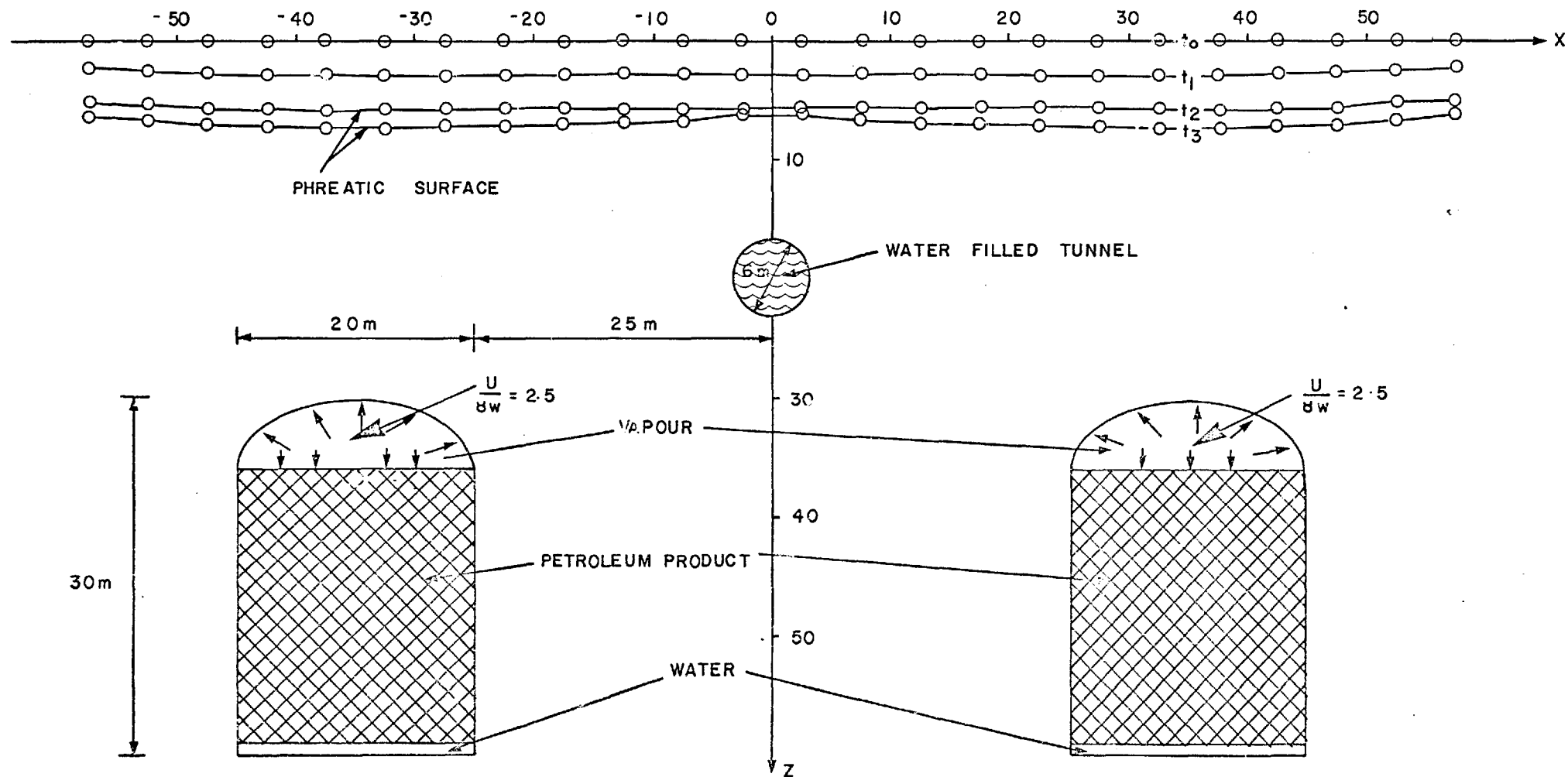
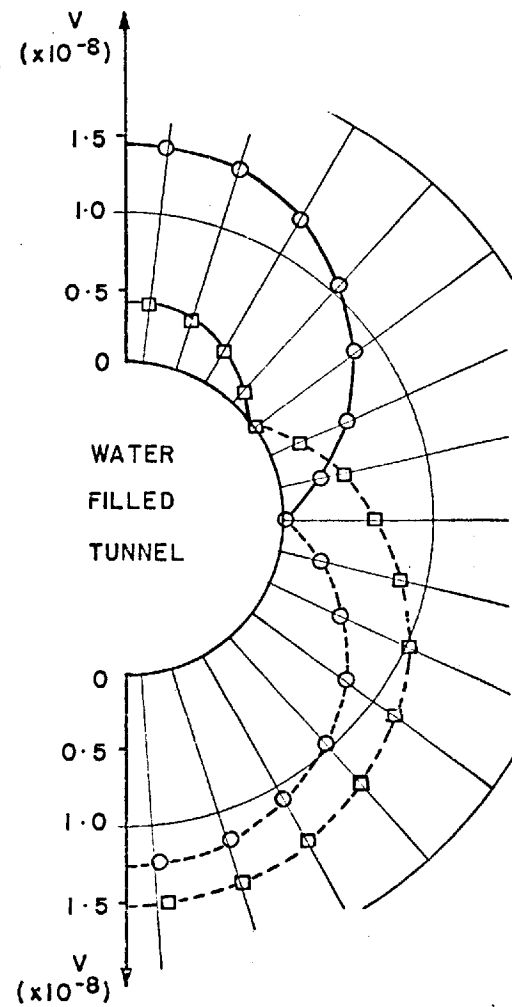


FIGURE 5.5.2 PHREATIC SURFACE LOCATION WITH TIME FOR BALANCED STORAGE CHAMBERS



NOTE:

- - t_0
- - t_3
- FLOW INTO CAVITY
- - - FLOW OUT OF CAVITY

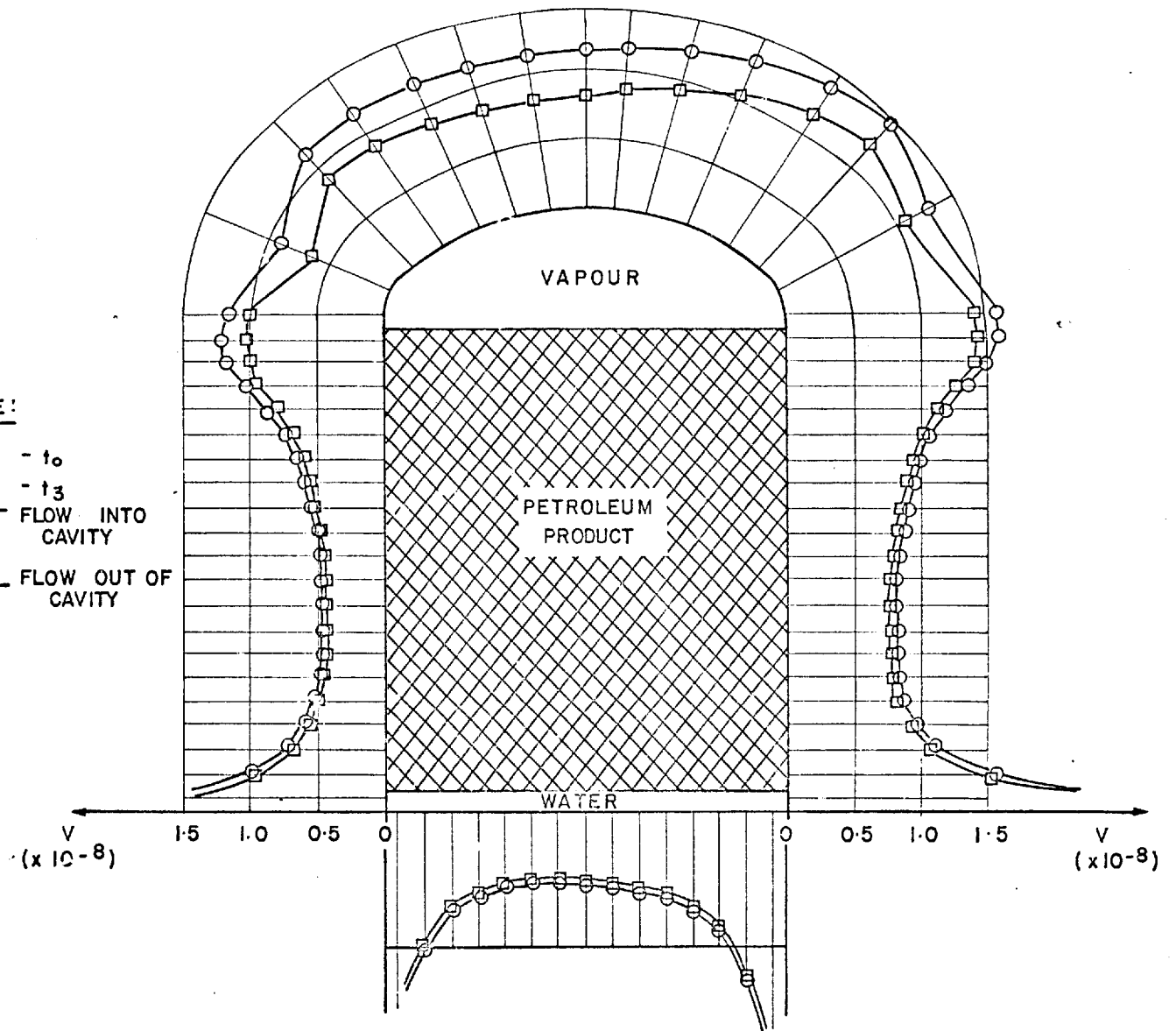


FIGURE 5.5.3 FLOW VELOCITY INTO CHAMBERS FOR BALANCED SYSTEM

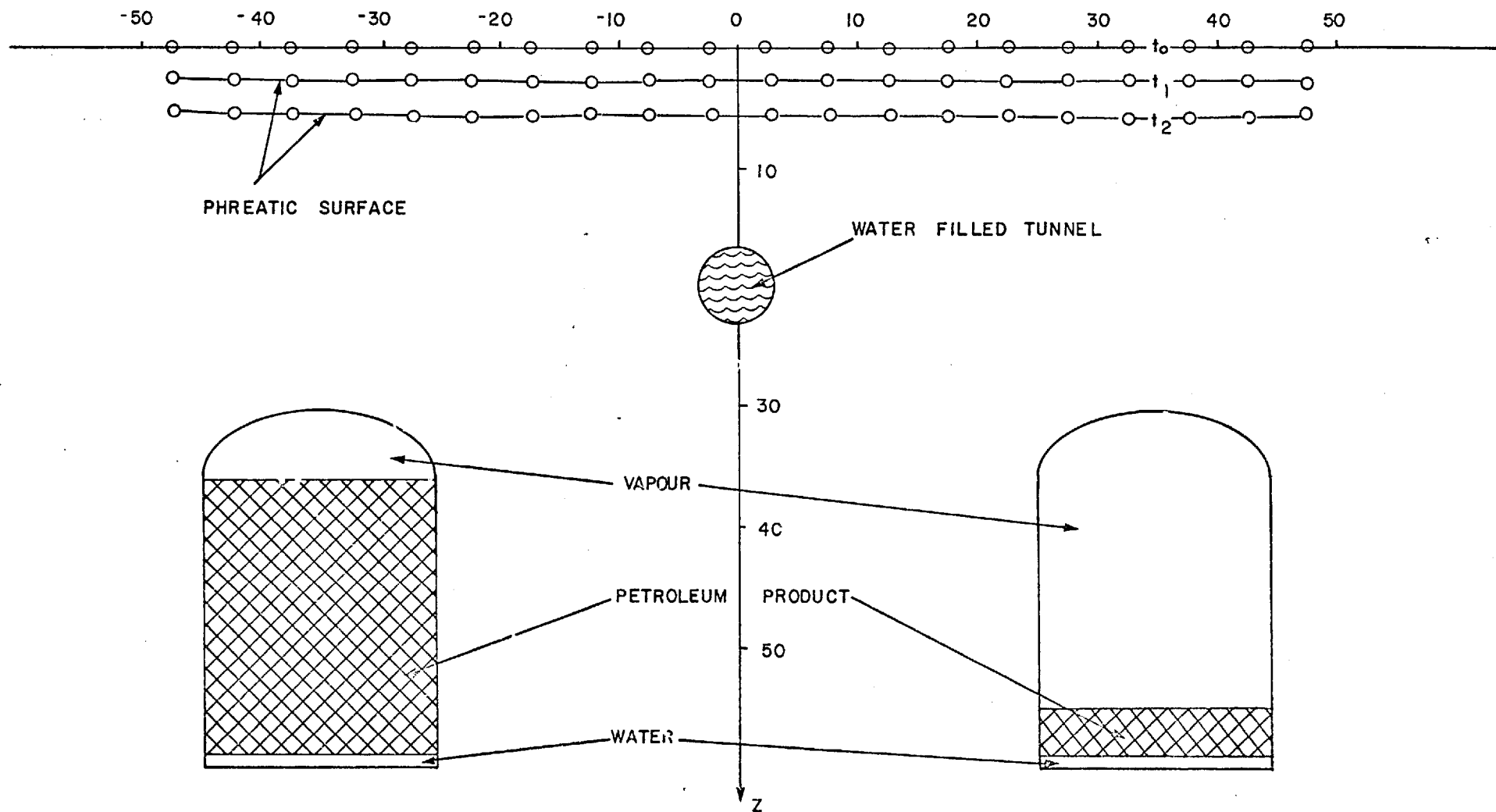


FIGURE 5.5.4 PHREATIC SURFACE LOCATION WITH TIME FOR UNBALANCED STORAGE CHAMBERS

As is to be expected the above times are less than the corresponding ones for the balanced system (the time step was calculated for a particular maximum drop in the phreatic surface). The flows (m^3/sec) into the storage chambers were 1.89×10^{-6} , 1.84×10^{-6} and 1.61×10^{-6} for t_0 , t_1 and t_2 respectively. The flows (m^3/sec) out of the water filled tunnel were 2.32×10^{-8} , 5.51×10^{-8} and 1.88×10^{-7} for times t_0 , t_1 and t_2 . The flow velocity distributions into or out of the storage chambers and the water filled tunnel are given in Figures 5.5.5 and 5.5.6. In Figure 5.5.6 the flow velocity into the full storage chamber is low on the lower right hand side. In order to avoid migration of petroleum product from one chamber to another, it may be prudent to install another water filled tunnel between the two chambers. An optimum location of this additional water filled tunnel was aimed for by considering two cases, Figure 5.5.7. The flow velocities into the full storage chamber for both case I and II are illustrated in Figure 5.5.8. It is clearly obvious that case II, the lower position of the water filled tunnel, gives the largest increase in flow velocity along the right hand side of the chamber. The flows (m^3/sec) into the storage chambers for case I were 2.08×10^{-6} , 2.03×10^{-6} and 1.79×10^{-6} for times t_0 , t_1 and t_2 respectively. The times in seconds were 0.0, 3.60×10^6 and 7.25×10^6 for t_0 , t_1 and t_2 . For case II the flows (m^3/sec) into the chambers were 2.19×10^{-6} at time 0.0, 2.14×10^{-6} at time of 3.59×10^6 secs and 1.91×10^{-6} at 7.24×10^6 secs. The difference between flow out of the water filled tunnels for cases I and II at the same time t_2 was $1.2 \times 10^{-7} \text{ m}^3/\text{sec}$.

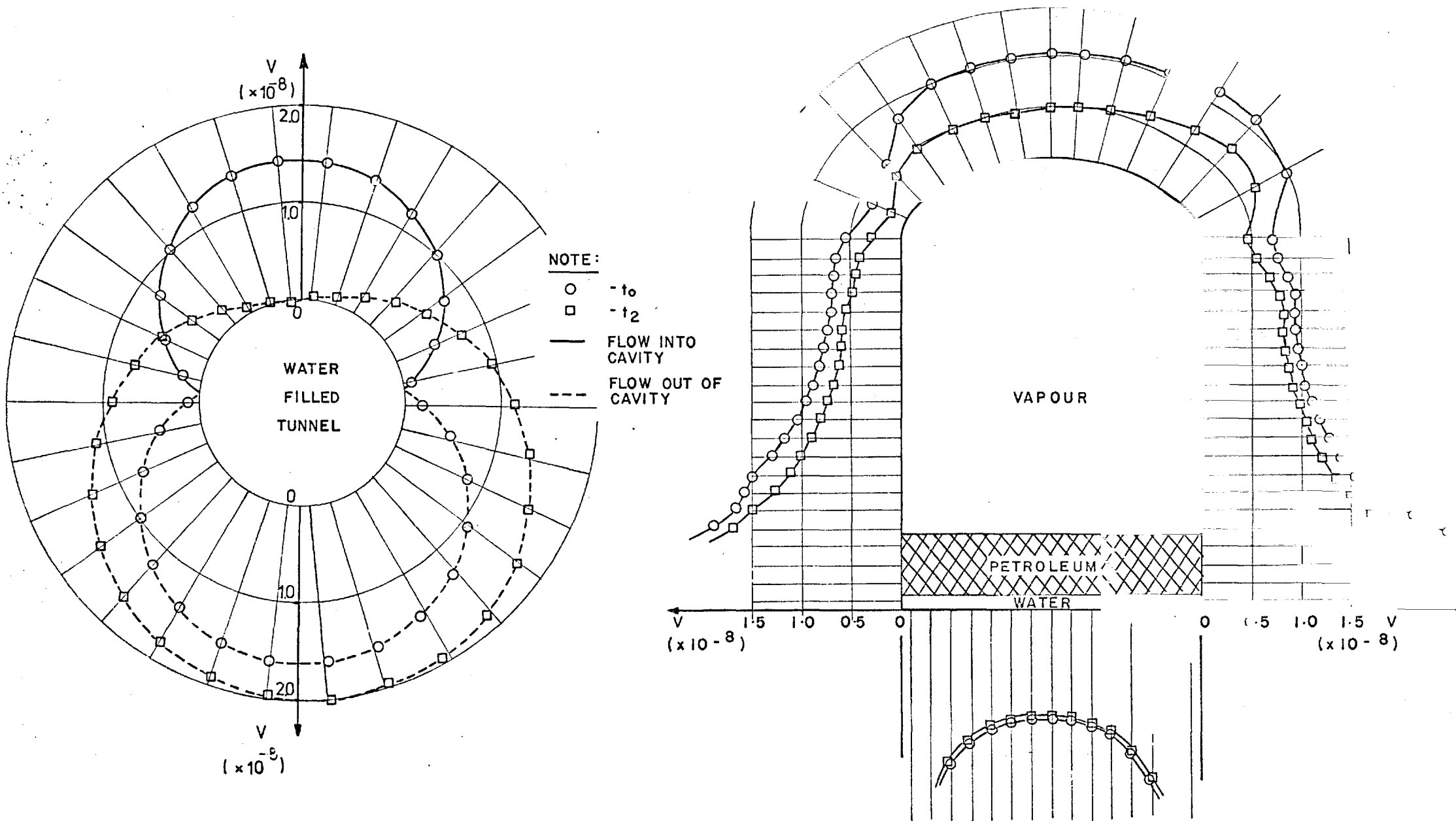


FIGURE 5.5.5 FLOW VELOCITY INTO CHAMBERS FOR UNBALANCED SYSTEM

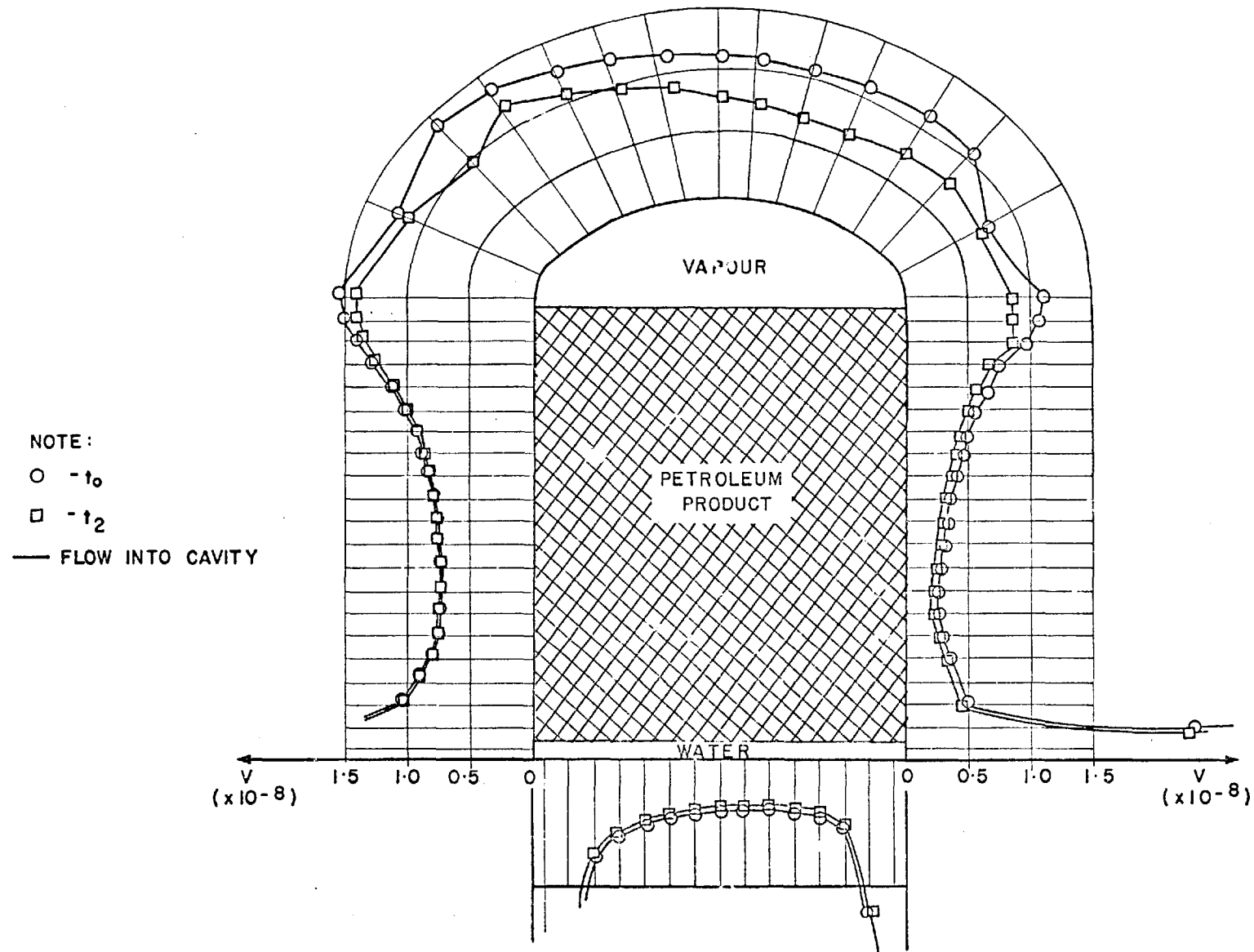


FIGURE 5.5.6 FLOW VELOCITY INTO FULL STORAGE CHAMBER FOR UNBALANCED SYSTEM

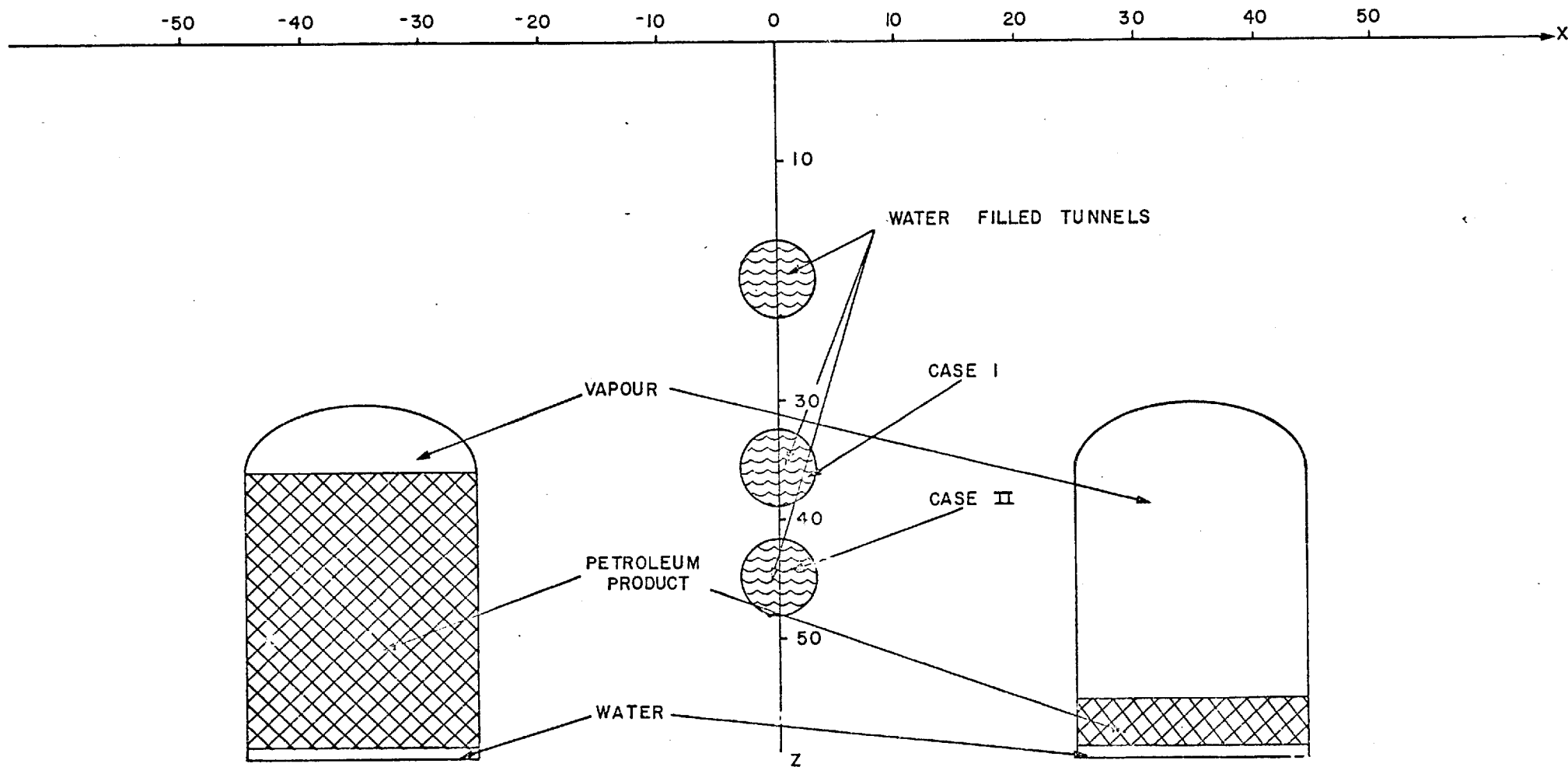
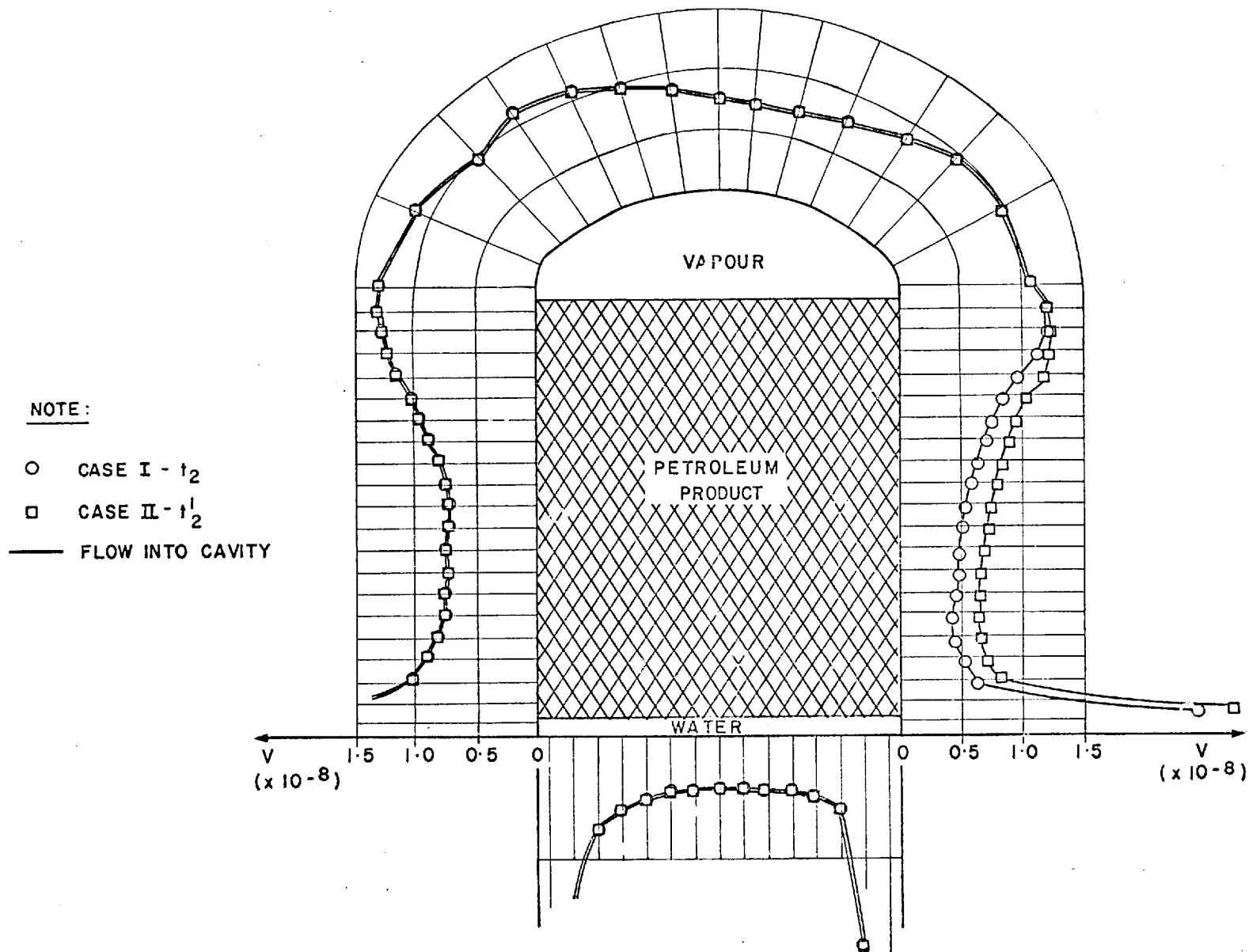


FIGURE 5.5.7 UNBALANCED CASE WITH ADDITIONAL WATER FILLED TUNNELS I AND II



5.5.8 FLOW VELOCITY INTO FULL STORAGE CHAMBER FOR ADDITIONAL WATER TUNNELS I AND II

For an underground storage system of approximately 200,000 m³, which would be two storage chambers, as shown in Figures 5.5.2, 5.5.4 and 5.5.7, of about 150 metres long, the maximum steady inflow into the chambers for the conditions analysed was 25m³/day. The mean value of tolerable inflows into a storage system of this size is approximately 20 - 40m³/day (Morfeldt (1972)).

6. Computational efficiency

The F.E.M. is ideal for handling flow problems; however, the B.I.M. presented here has distinct advantages; namely simple data preparation, reduced computational time for large volume to surface area problems and simplicity of redefining the element configuration for an iterative procedure. The C.P. time is plotted against number of boundary elements in Figure 5.6.1, for the two dimensional B.I.M. program, FLOW 2D. This program required 20k (decimal) of small core memory for 300 elements. Input/output time never exceeded 5% of the C.P. time.

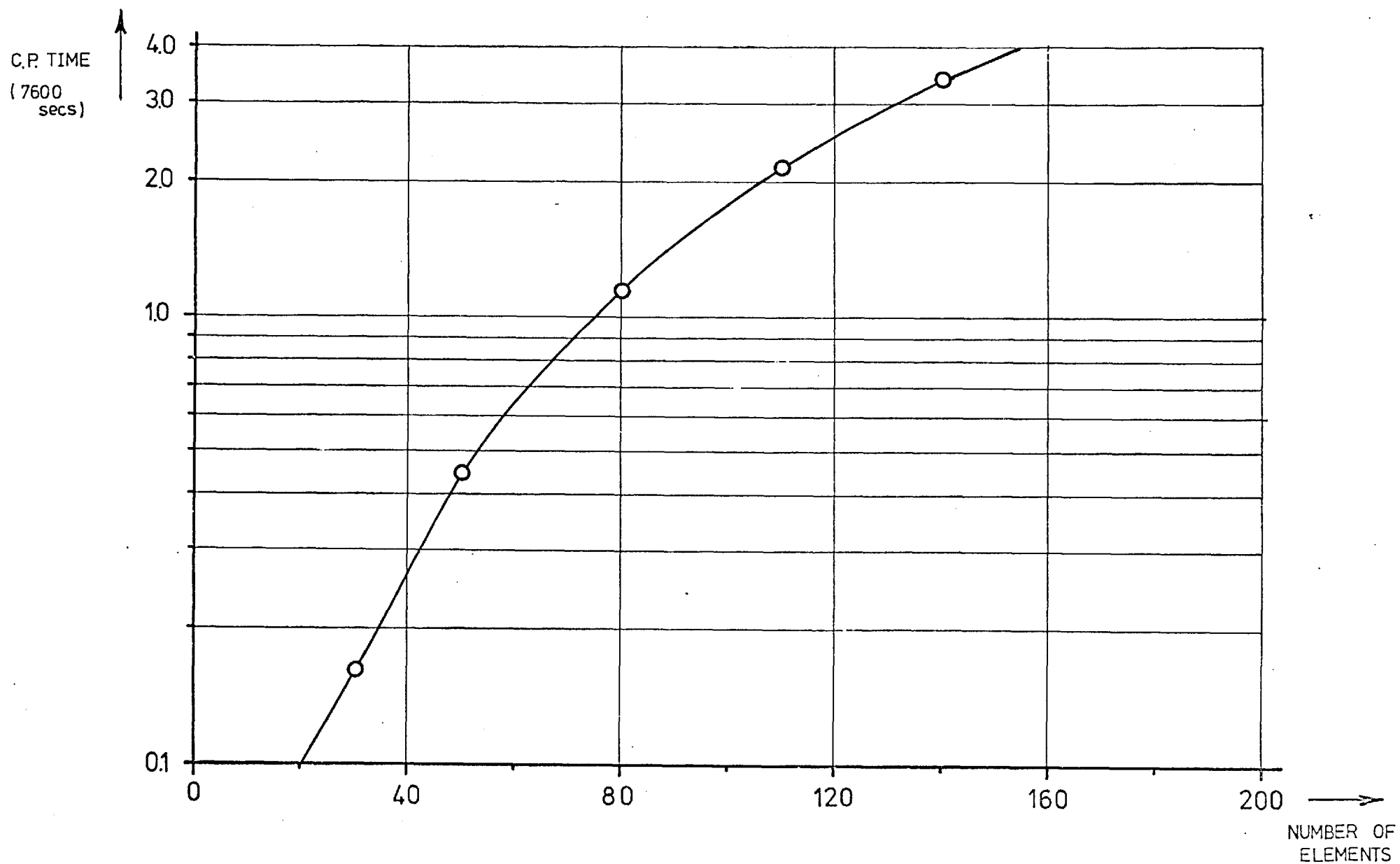


FIGURE 5.6.1 CENTRAL PROCESSOR TIME FOR FLOW 2D

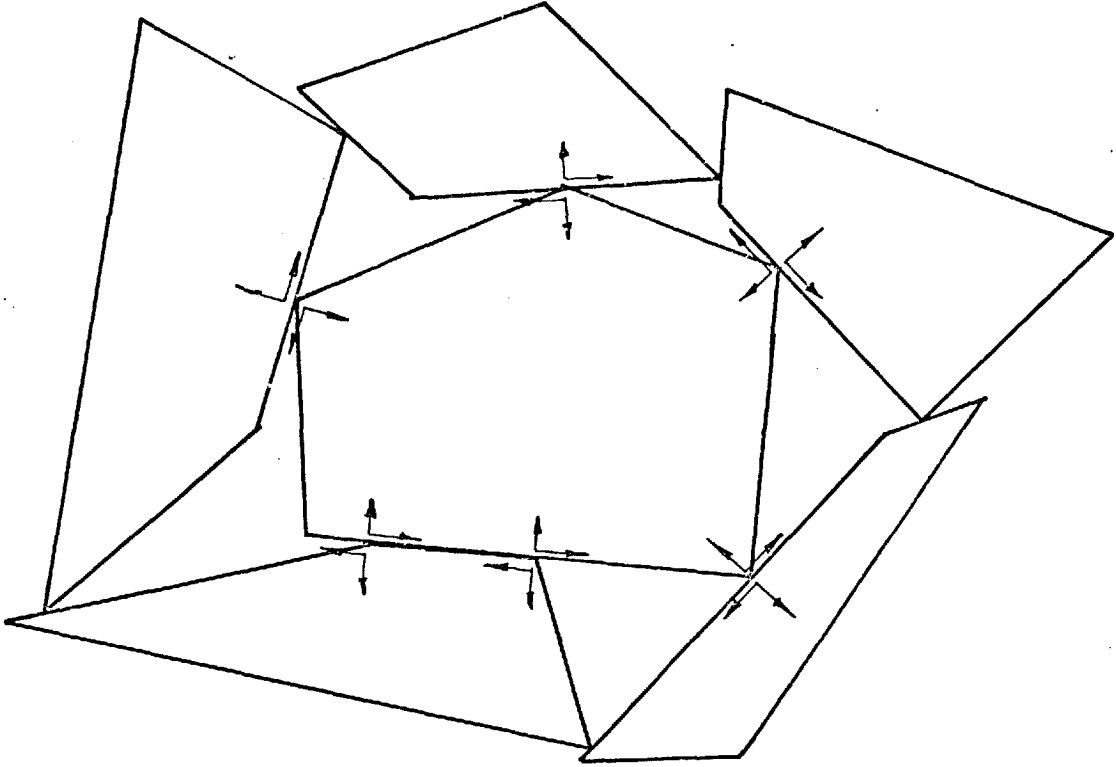
CHAPTER 6: THE QUASI-RIGID BLOCK METHOD

1. Review of previous formulations

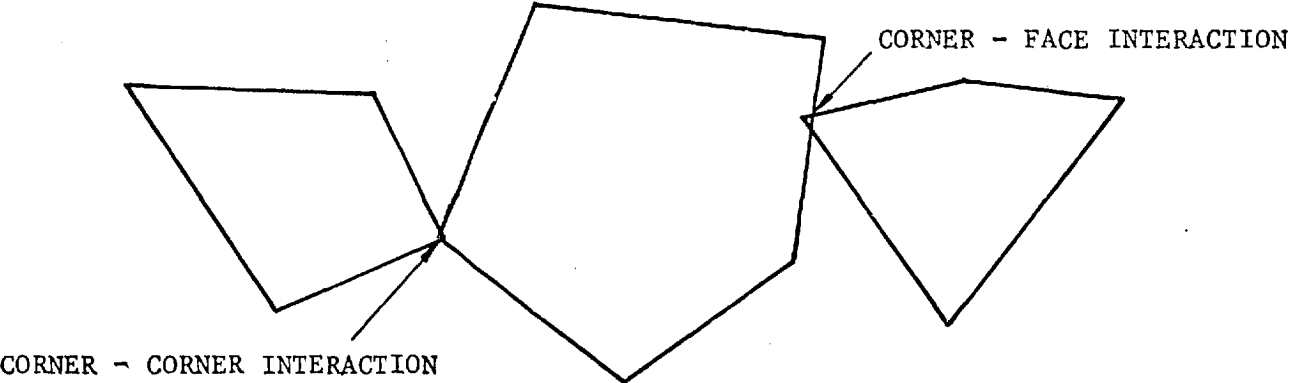
The technique of modelling large movements of a blocky rock system by discrete rigid blocks was devised by Cundall (1971 a) and b)). The method consists of approximating all deformations that occur within a rock mass, at the surfaces of discrete rock blocks. (The deformation properties of both the intact material and the joint are lumped together). Interactions between blocks are communicated via the corners, Figure 6.1.1. Two possible types of interaction are possible; corner-face interaction and corner-corner interaction.

The interaction between blocks can be subdivided into normal interaction, Figure 6.1.2, and shear interaction, Figure 6.1.3. Cundall(1971 a) and b), 1974) modelled the normal interaction as a spring dashpot system, Figure 6.1.2. The contraction of this system was simulated by overlapping of neighbouring blocks.

The shear interaction was also modelled as a spring dashpot system, Figure 6.1.3, except that once slip occurred the shear damping was removed. A simple shear force-displacement curve is illustrated in Figure 6.1.4; however, complex curves can be handled with ease (Cundall (1974)).



BLOCKS ONLY INTERACT THROUGH THEIR CORNERS



TWO POSSIBLE TYPES OF INTERACTION

FIGURE 6.1.1 BLOCK INTERACTION

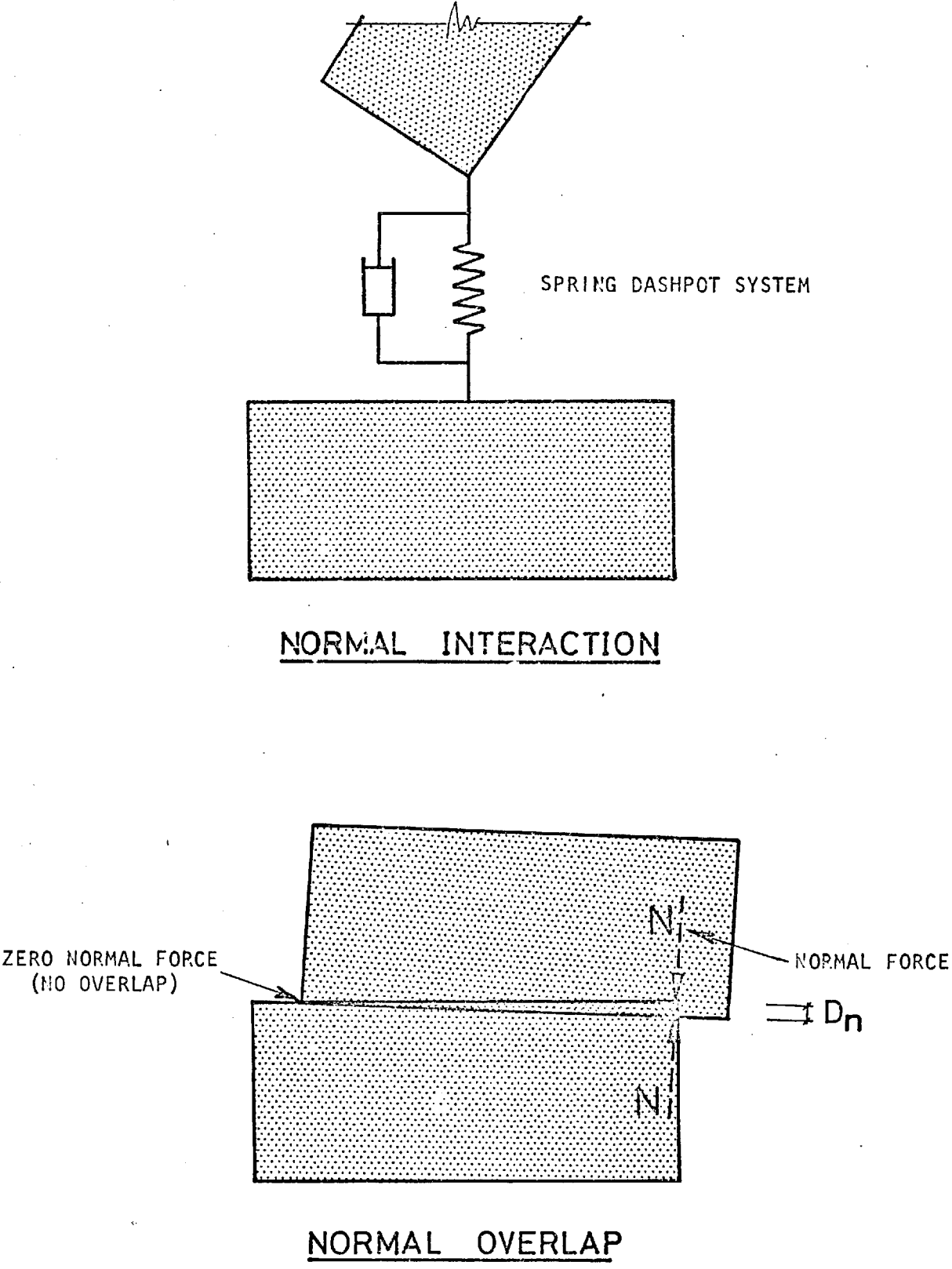
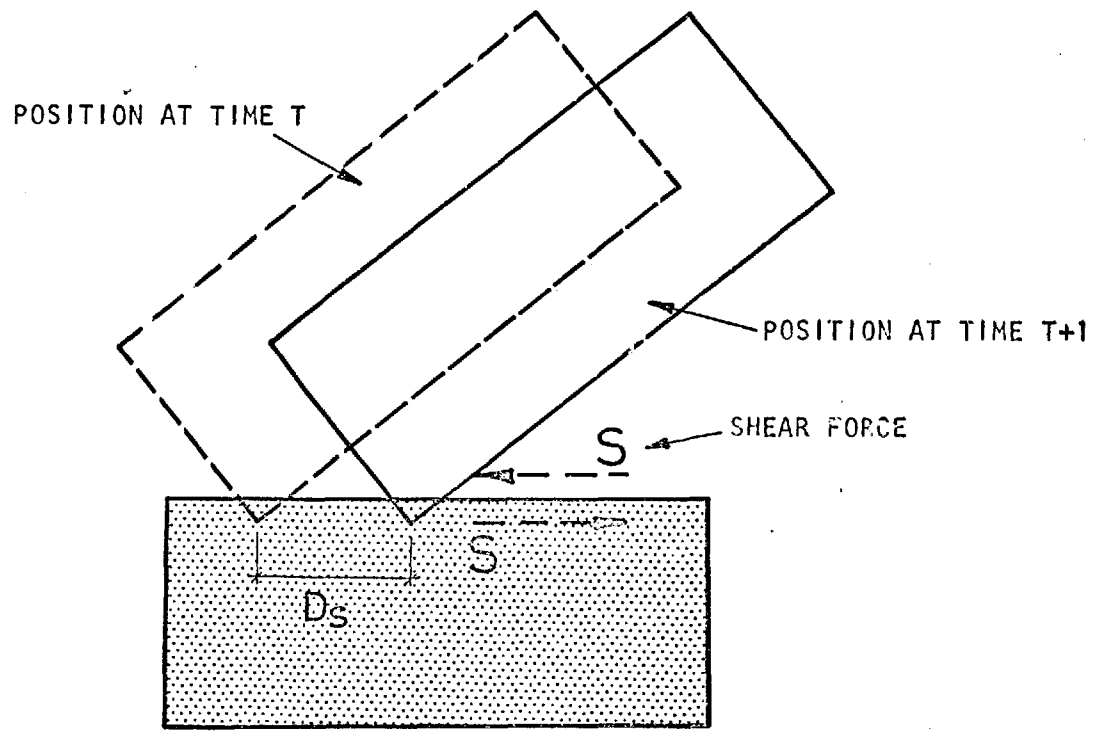
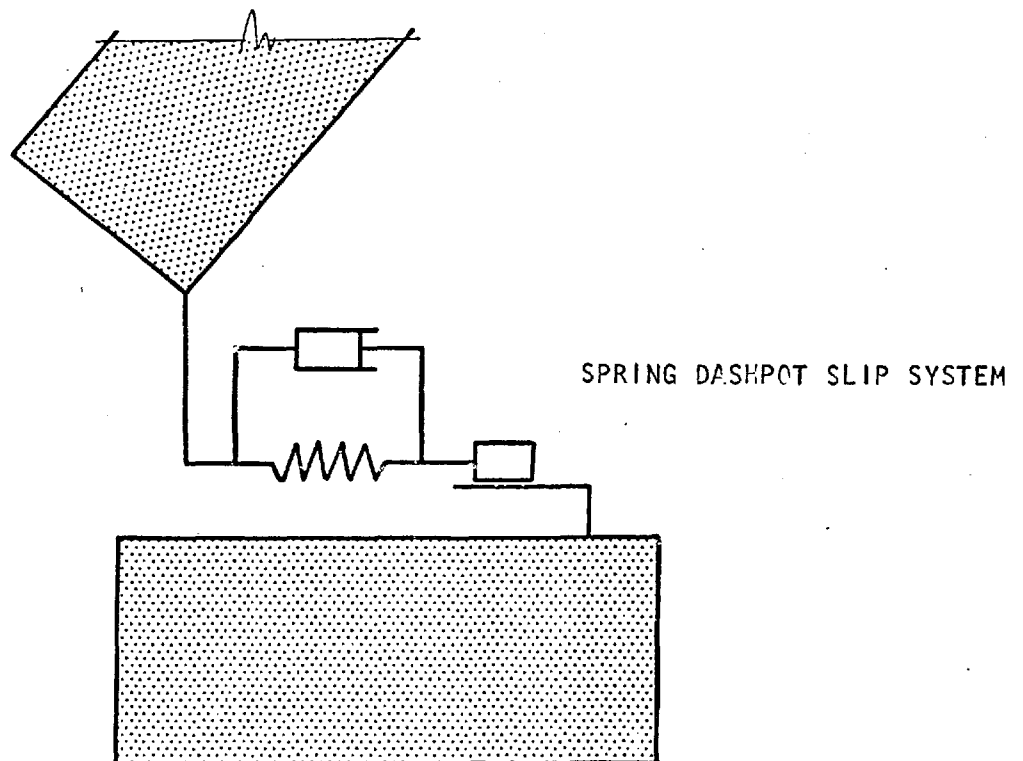


FIGURE 6.1.2 NORMAL INTERACTION



SHEAR DEFORMATION



SHEAR INTERACTION

FIGURE 6.1.3 SHEAR INTERACTION

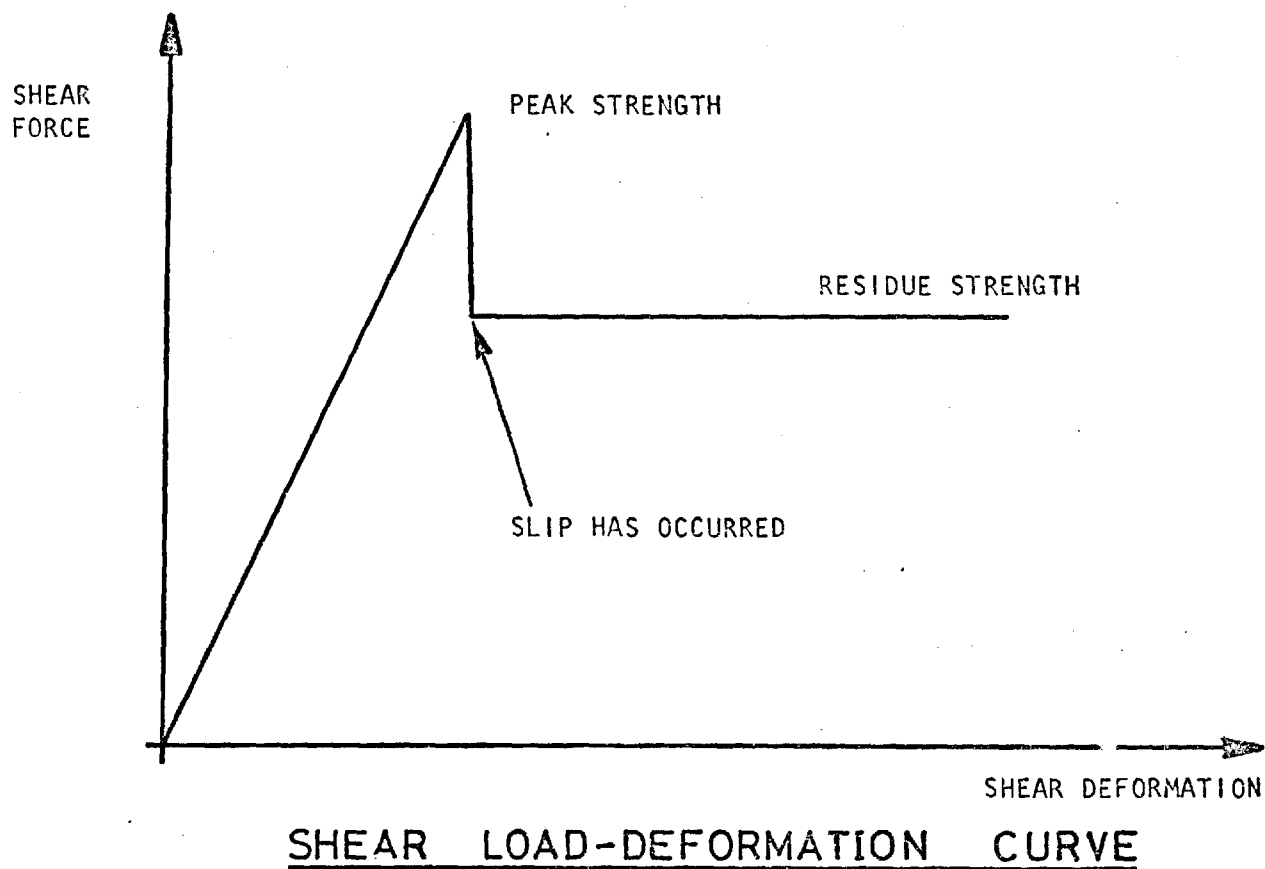


FIGURE 6.1.4 SIMPLE SHEAR STRENGTH DEFORMATION RELATIONSHIP

The method is based on a dynamic relaxation approach (Otter et al (1966)), where the calculation cycle is based on a stable time increment (i.e. calculation of interactions between blocks is faster than the physical interaction). The problem is begun at some position in time, say, 0.0; then from the location of blocks, interacting forces between blocks are computed as shown in Figures 6.1.2. and 6.1.3. The forces and moments acting on any particular block for this time step are added to any existing forces or moments acting on the block. From Newton's second law of motion, the linear and angular accelerations of a block are found from the forces and moments acting. These accelerations acting over the time step give rise to velocities and hence displacement of the block position. Now the whole cycle is repeated for the new block position. This iteration loop is illustrated diagrammatically in Figure 6.1.5. For completeness, the equations describing the above mentioned behaviour are:

$$a^n = \frac{F^n}{m} \quad \dots (6.1.1.)$$

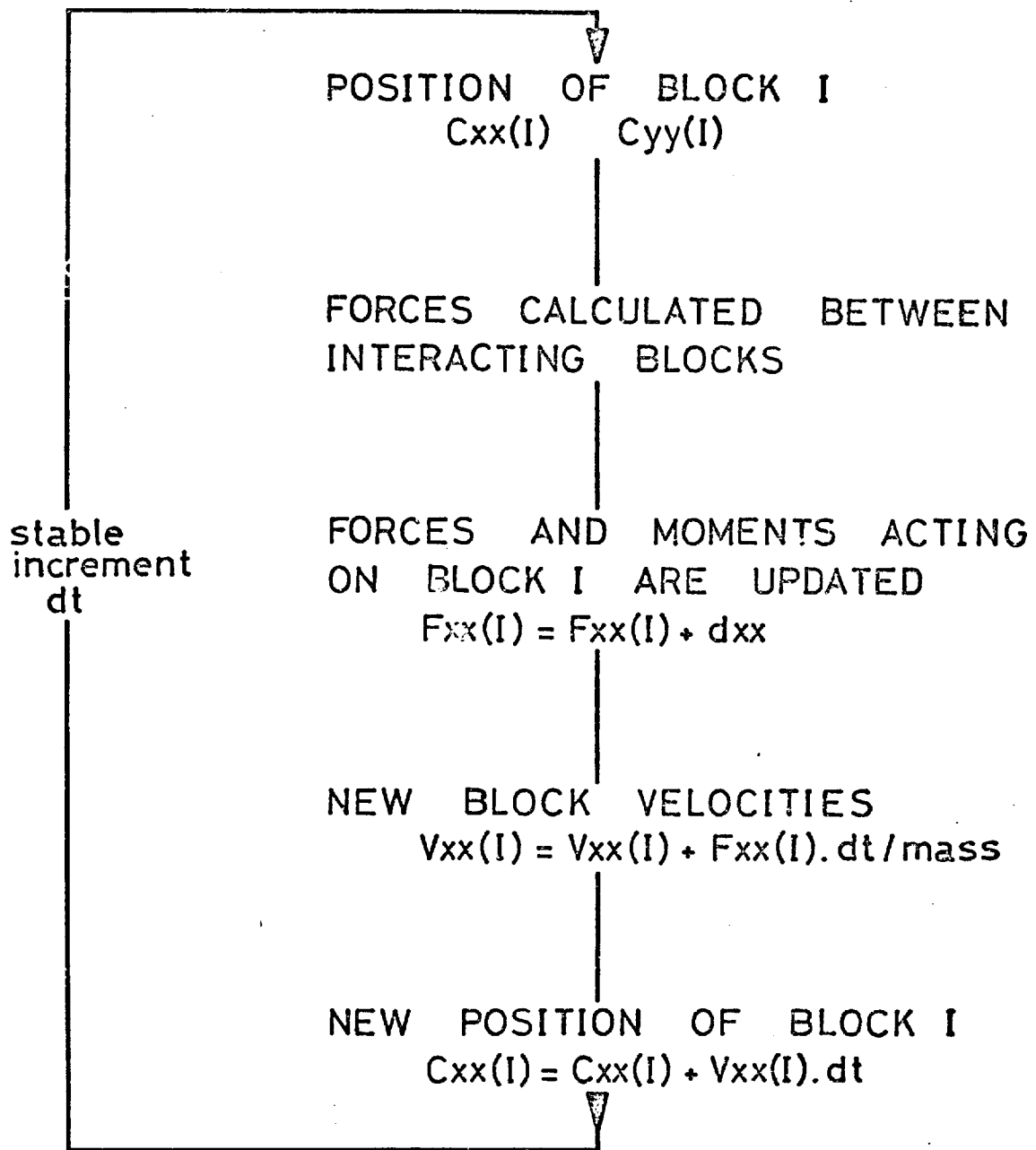
$$v^{n+\frac{1}{2}} = v^{n-\frac{1}{2}} + a^n \Delta t$$

$$c^{n+\frac{1}{2}} = c^{n-\frac{1}{2}} + v^{n+\frac{1}{2}} \Delta t$$

$$w^n = \frac{M^n}{I}$$

$$\alpha^{n+\frac{1}{2}} = \alpha^{n-\frac{1}{2}} + w^n \Delta t \quad \dots (6.1.2.)$$

$$\theta^{n+\frac{1}{2}} = \theta^{n-\frac{1}{2}} + \alpha^{n+\frac{1}{2}} \Delta t$$



ITERATION LOOP

FIGURE 6.1.5 MAIN ITERATION LOOP

where Δt = time increment.

a^n = linear acceleration (x and y components) at time n.

F^n = accumulative force acting at time n (x and y components).

m = block mass.

$v^{n-\frac{1}{2}}$ = new linear velocity (x and y components).

$v^{n-\frac{1}{2}}$ = previous linear velocity.

$c^{n+\frac{1}{2}}$ = new block position (x and y components).

$c^{n-\frac{1}{2}}$ = old block position.

w^n = angular acceleration at time n.

M^n = accumulative moment acting at time n.

I = moment of inertia.

$\alpha^{n+\frac{1}{2}}$ = new angular velocity.

$\alpha^{n-\frac{1}{2}}$ = previous angular velocity.

$\theta^{n+\frac{1}{2}}$ = new block orientation.

$\theta^{n-\frac{1}{2}}$ = previous block orientation.

As regards the value of the time step increment, Cundall (1971a) demonstrated that for a problem with one degree of freedom the following equations (6.1.3.) and (6.1.4.) apply for a stable Δt

$$\Delta t < 2\sqrt{\frac{m}{k}} \quad \dots (6.1.3.)$$

where k is the linear stiffness of the system
and

$$\Delta t < 2\sqrt{\frac{I}{k_a}} \quad \dots (6.1.4.)$$

where k_a is the angular stiffness of the system.

In order to ensure stability, Δt must be much less than given by the above equations (Cundall (1974)).

In the numerical formulations of Cundall (1974), general shaped blocks could be modelled. A linked list system of storage was also incorporated to conserve computer storage.

2. Numerical formulation

A Fortran program was written to handle general shaped blocks, but in particular model efficiently large number of parallelogram shaped blocks. The reason for this is because many rock mechanics problems consist of sets of major and minor joints, e.g. bedding and cross joints in sedimentary rocks, cleavage and joints in slate, etc. The same logic used by Cundall was incorporated (as in section 6.1); however, there exist a number of features additional to his work, which will be described shortly. The program has been run interactively, using graphics; however, except for trivial problems the use of the program is recommended in batch mode. The excellent graphic facilities at Imperial College, Tektronic terminals plus microfilm, have been utilized to the full.

In order to simplify description of the program, each operation will be described in its correct order of being called. The title of the operation includes the name of the subroutine, which carries out the respective operations.

COORDS Coordinate generation

Subroutine GEN, which is block generation routine is called by COORDS. COORDS then finds the masses, moments of inertia etc.

NEIGHB Location of neighbouring blocks

In order to save computational time, the location of the nearest neighbours to any particular block are stored. NEIGHB searches the complete assemblage of blocks with a coarse mesh; then once the blocks within this coarse mesh are found, they are searched for nearest neighbours and stored as illustrated diagrammatically in Figure 6.2.1.

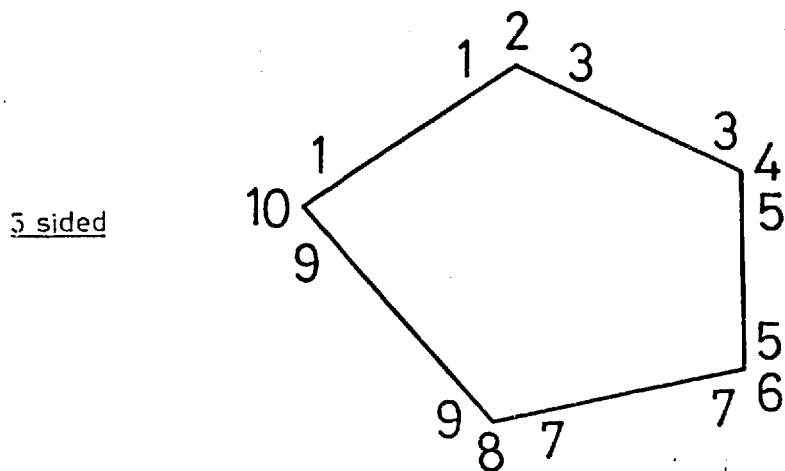
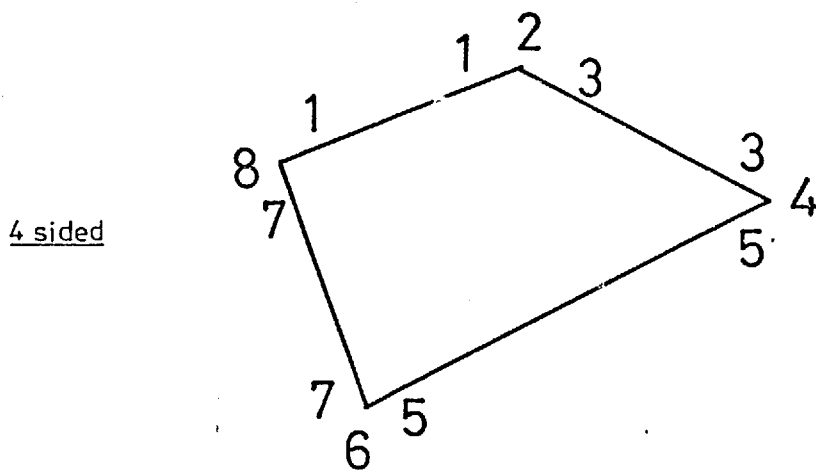
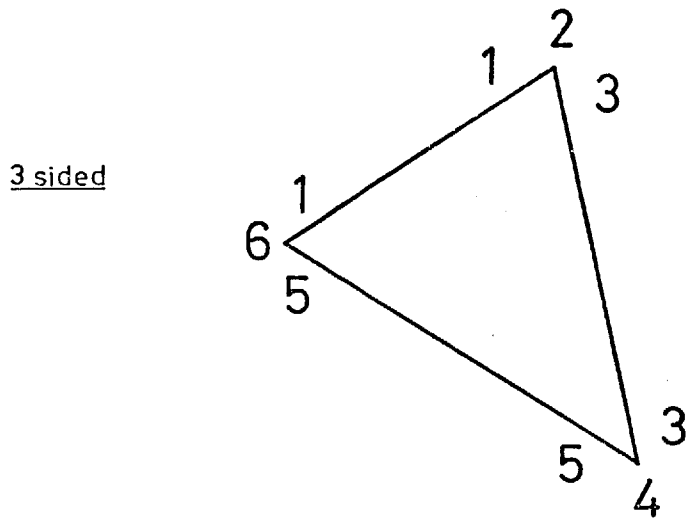


FIGURE 6.2.1 STORAGE LOCATIONS OF NEIGHBOURING BLOCKS

MAINIteration Hoop

The main program, MAIN, finds the corner interaction, calculates the relevant forces and moments and updates the velocities and positions of the blocks. The normal and shear interaction incorporated are those mentioned in section 6.1.

CRACKCracking of Blocks

The process of blocks cracking has not been incorporated in the program but suggestions are discussed in section 6.3. However, the main body of work leading up to cracking i.e. calculation of stresses within blocks, is included. The calculation of stresses within the blocks is carried out by the F.E.M. (Zienkiewicz (1971)). Parts of the program of Wilson (1963) were used for these relevant calculations. Automatic mesh generators are included and consist of two different groups, those for parallelogram shaped blocks and those for general shaped blocks. The finite element mesh for parallelogram shaped blocks is shown in Figure 6.2.2. The mesh for general shaped blocks is subdivided into a fine and coarse mesh. Each block is subdivided, in triangles, as illustrated in Figure 5.2.3., and each of these triangles is subdivided as shown in Figure 6.2.4. All parallelogram shaped blocks use the same stiffness matrix, which is therefore only called once. Then for the calculation of stresses the stiffness matrix is reduced with the relevant nodal forces, and the stresses found from the nodal displacements. This feature saves considerable computational effort when a large number of parallelogram blocks are present. For the general shaped blocks the complete procedure i.e. calculation of stiffness matrix and subsequent reduction with the relevant nodal forces, is required for each block. The following equations in simplified form indicate the procedure of the F.E.M. used here.

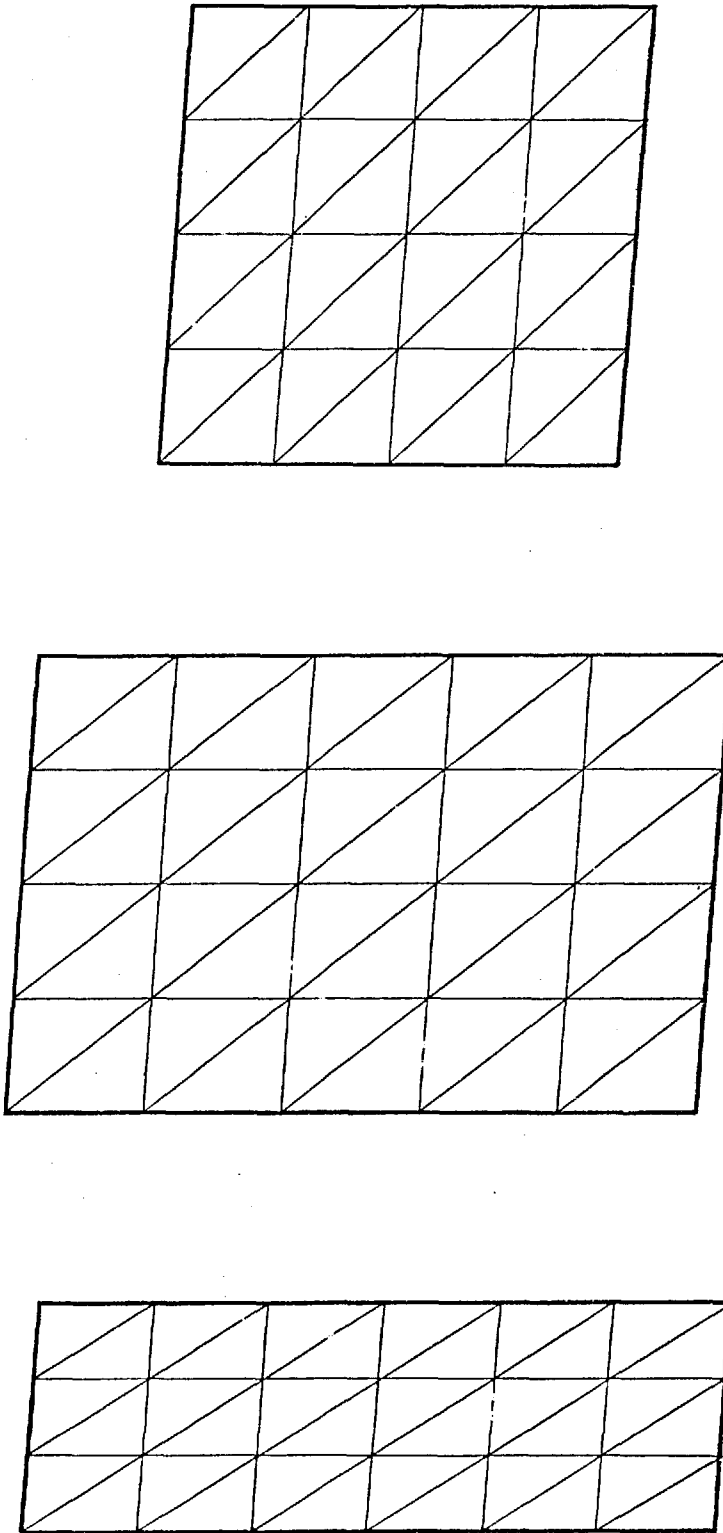
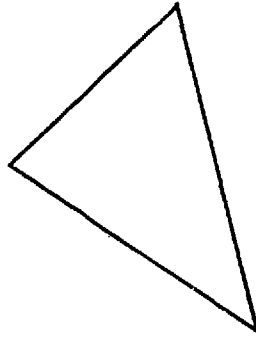
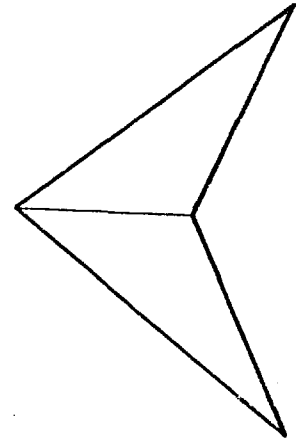
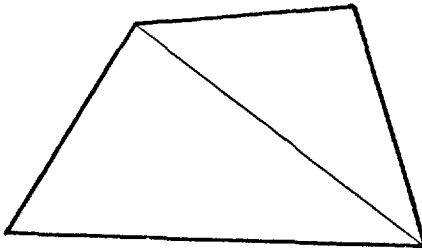


FIGURE 6.2.2 FINITE ELEMENT MESH FOR PARALLELOGRAM BLOCKS

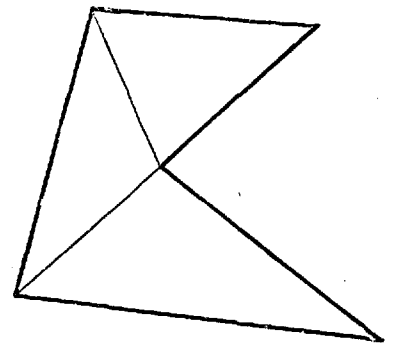
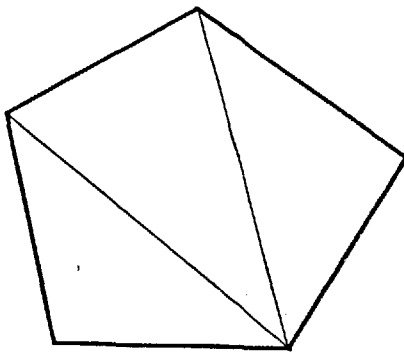
3 sided



4 sided



5 sided



6 sided

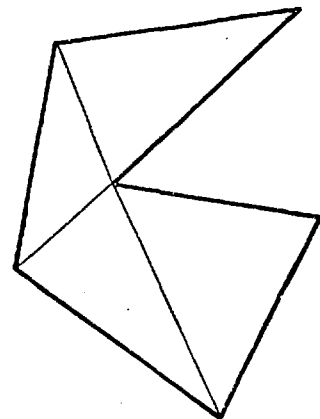
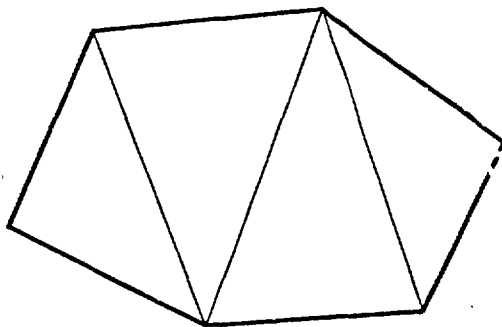


FIGURE 6.2.3 COARSE SUBDIVISION FOR GENERAL BLOCKS

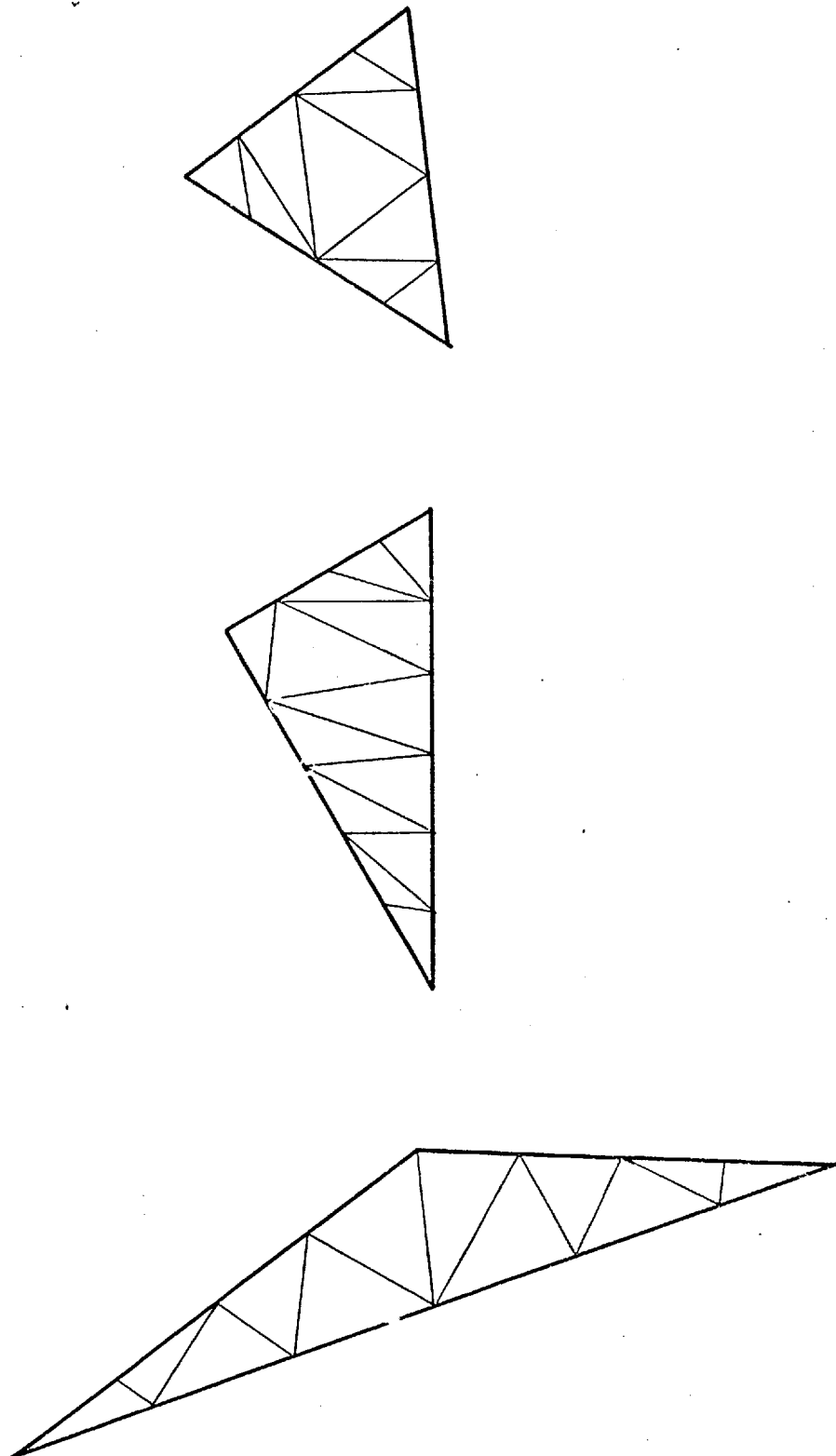


FIGURE 6.2.4 FINITE ELEMENT MESH FOR COARSE SUBDIVISION

$$[\epsilon] = [A] [r] \quad \dots (6.2.1.)$$

where $[\epsilon]$ are the element strains.
 $[A]$ the displacement transformation matrix.
 $[r]$ the nodal point displacements.

$$[\sigma] = [C] [\epsilon] \quad \dots (6.2.2.)$$

where $[\sigma]$ are the element stresses.
 $[C]$ the stress strain relationship.

$$[s] = [k] [r] \quad \dots (6.2.3.)$$

where $[s]$ are the element corner forces.
 $[k]$ the element stiffness matrix.

$$[R] = [K] [r] \quad \dots (6.2.4.)$$

where $[R]$ are the nodal point loads.
 $[K]$ the stiffness of the complete structure.

The present program solves equation (6.2.4.) by an over-relaxation Gauss - Seidel iteration procedure. Whether it is more efficient to invert $[K]$ requires consideration and will depend greatly on the number of blocks. Since the blocks are simply connected bodies (Muskhelishvili (1953)), then the stresses within the blocks are independent of the elastic constants.

Subroutines called by CRACK are:

NODALF	generates the finite element mesh for parallelogram blocks and calculates nodal forces from block interactions and body forces.
NODALG	generates the finite element meshes for general shaped blocks and calculates nodal forces from block interactions and body forces.
STIFF	computes the stiffness matrix $[K]$ for each finite element mesh.
STRESS	solves equation (6.2.4.) by Gauss - Seidel iteration and then calculates stresses at the centre of all elements.

Graphics output

BLOCK	draw all blocks.
VELOC	plots linear velocities of each block by an arrow.
TENSOR	plots stresses within a block as tensors.
MESH	draws the finite element mesh for each block.
VECTOR	plots vectors as arrows.

3. Suggested improvements

The most important improvements required to enable the program to effectively model blocky rock masses are more realistic interaction relationships and the cracking of blocks. The normal and shear block interaction relationships, especially normal interaction, should be modified to correspond more closely with true physical behaviour. In Figure 6.3.1 the normal interaction behaviour of two rock blocks are shown (Goodman (1975)). The most important feature in Figure 6.3.1 is the marked non-linear relationship of this interaction (i.e. the high increase in resistance to further penetration). A normal interaction relationship, similar to that shown in Figure 6.3.1, would include empirical modelling of corners crushing. Realistic shear strength parameters, such as complete shear strength curves including dilatant behaviour should also be introduced.

It is suggested that a criterion for cracking of blocks be best kept simple, e.g. Mohr-Coulomb with tension cut-offs (Jaeger and Cook (1971)). Checks on block fragmentation can be carried out by comparison with physical models under controlled loading conditions, e.g. Brown and Hudson (1972).

(a) Idealization of a joint.
 (b) Behavior of a joint in compression.
 ξ is the seating load. (c) Normal compression of an extension fracture in a granodiorite specimen.

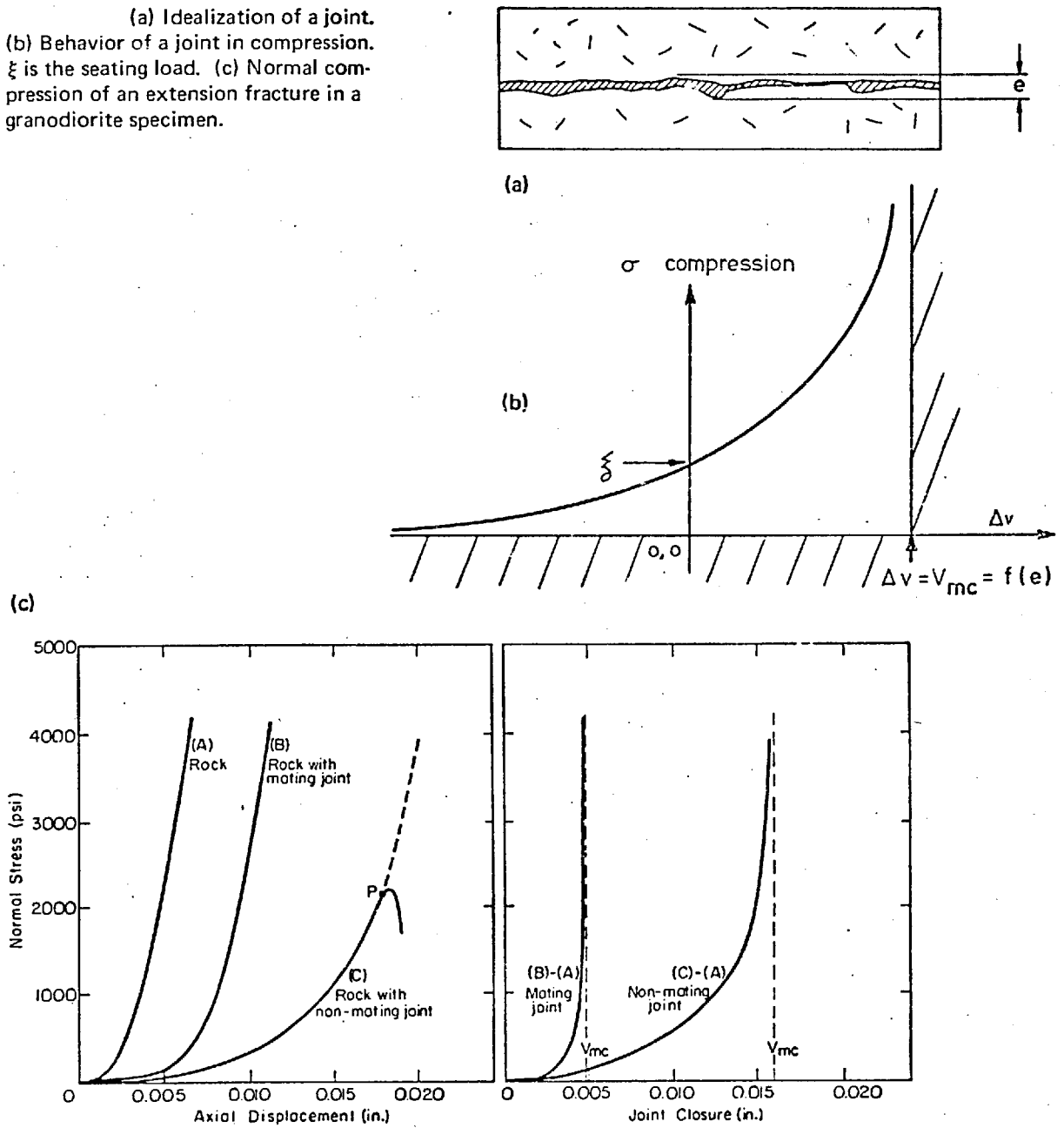


FIGURE 6.3.1 BEHAVIOUR OF A JOINT IN COMPRESSION (GOODMAN (1975))

CHAPTER 7: APPLICATION OF THE RIGID BLOCK METHOD

1. Check on accuracy

The program has been checked for the cases of simple sliding and toppling of individual blocks. More elaborate checks have been undertaken by Ozgenoglu (1977) such as toppling and sliding of two or more blocks, Figure 7.1.1., in which could agreement exists with the limit equilibrium solution of Goodman and Bray (1976). The mode of failure of toppling on a stepped base from Goodman and Bray (1976) is shown in Figure 7.1.2.

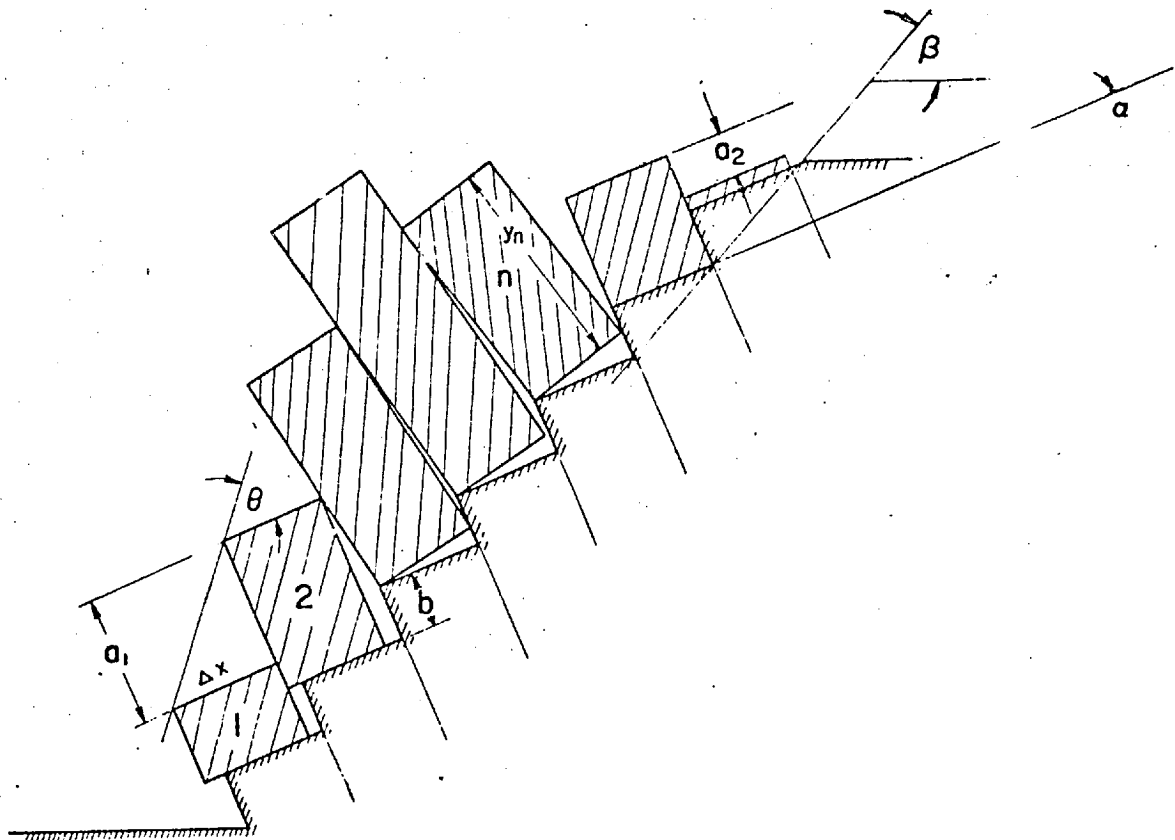
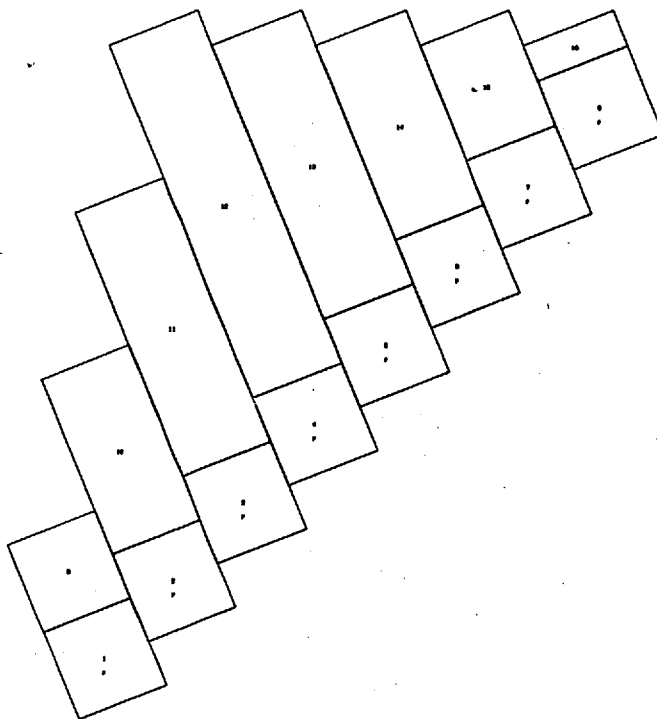


Figure 7.1.2. Toppling on a stepped base (Goodman and Bray (1976)).

TOP. ANAL. STEPPED BASE FI=3B D. TETA=21.8 D. DT=1E-4 SN=2E+7
 BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.80 CYCLE NUMBER = 0



TDP. ANAL. STEPPED BASE FI=3B D. TETA=21.8 D. DT=1E-4 SN=2E+7
 BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.80 CYCLE NUMBER = 1400

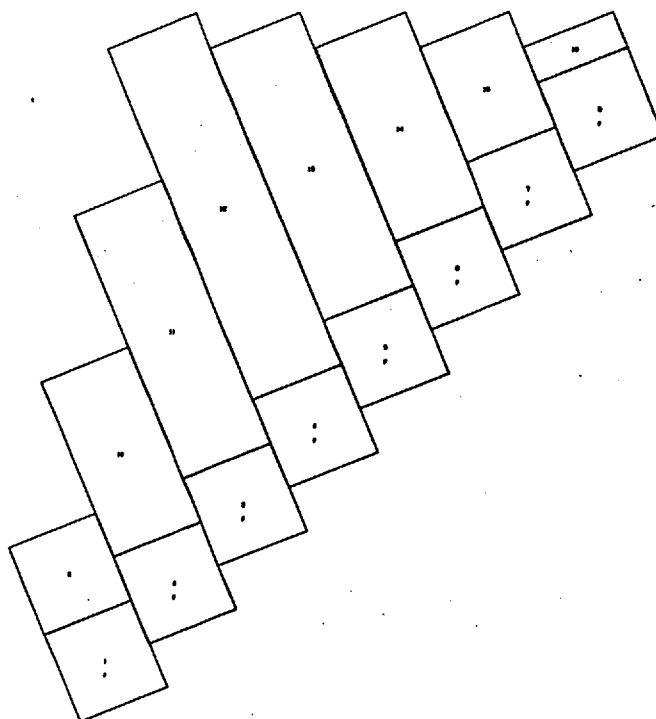
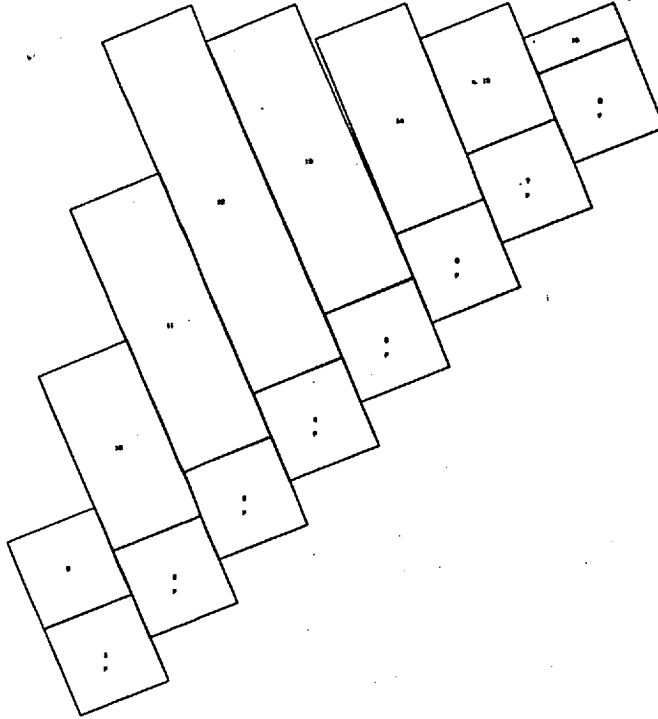


FIGURE 7.1.1 TOPPLING ON A STEPPED BASE

TOP. ANAL. STEPPED BASE $F1=38$ D. $TETA=21.8$ D. $DT=1E-4$ $SN=2E+7$
BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.80 CYCLE NUMBER = 2800



TOP. ANAL. STEPPED BASE $F1=38$ D. $TETA=21.8$ D. $DT=1E-4$ $SN=2E+7$
BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.80 CYCLE NUMBER = 4200

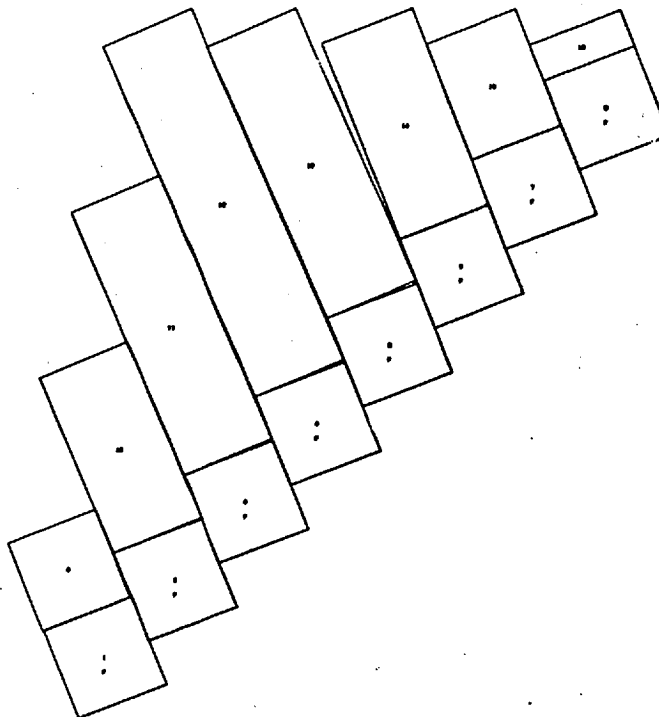
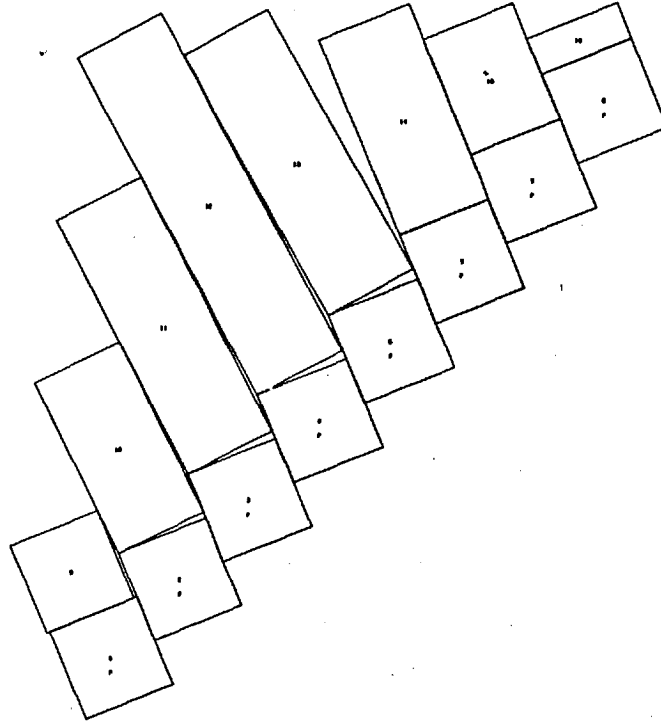


FIGURE 7.1.1 CONTINUED

TOP. ANAL. STEPPED BASE F1=38 D. TETA=21.8 D. DT=1E-4 SN=2E+7
BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.80 CYCLE NUMBER = 7600



TOP. ANAL. STEPPED BASE F1=38 D. TETA=21.8 D. DT=1E-4 SN=2E+7
BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.80 CYCLE NUMBER = 12600

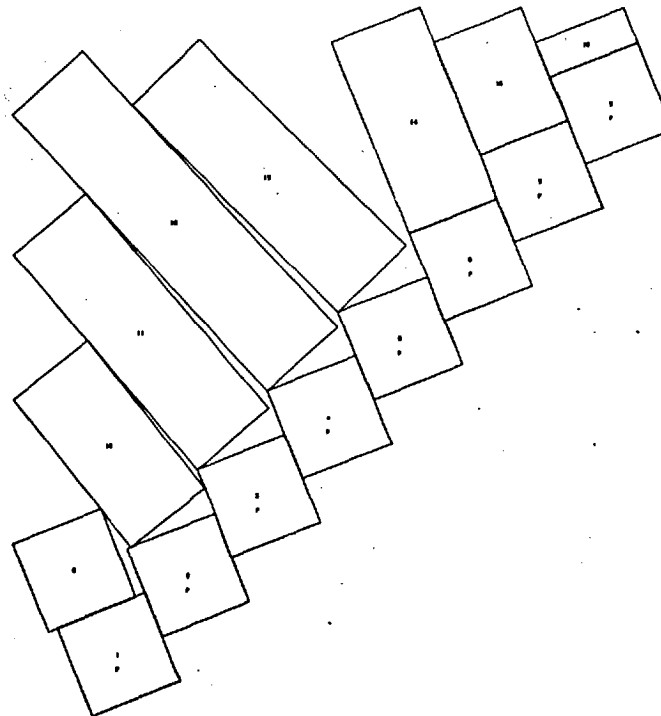


FIGURE 7.1.1 CONTINUED

2. Slope stability

In Figure 7.2.1 the behaviour of a steep slope with a major discontinuity set dipping into the excavation and with offset secondary joints is illustrated. The dip of the major joints is 23° while the friction angle (peak and residual) is 20° . The blocks were exactly preconsolidated to overburden load by an included subroutine. The blocks marked with an F are fixed throughout the analysis. At iteration cycle number 200, the blocks do not appear to have moved (see block plot), however their velocities (see velocity plot) indicate motion. The maximum velocity vector has a magnitude of 0.07m/sec. By the time of 2200 cycles, i.e. real time of 0.44 secs, the blocks have moved considerably. The toe block (24) is sliding, while immediately above the toe columns of blocks are toppling with no contact at the top of the column. Above this zone blocks are rotating but maintaining their original form, and the uppermost zone consists of block columns with contact only at the top of the columns. The velocities at cycle 2200 are greatly increased from cycle 200 and indicate a probable potential failure. The magnitude of the maximum velocity vector is 0.59 m/sec. At cycle 2800, the blocks have moved further and are in fact in an unstable situation (maximum velocity is 0.78 m/sec.).

A shallower slope to the one illustrated in Figure 7.2.1. but of similar joint orientation and spacing is shown in Figure 7.2.2. The friction angle is again 20° and the dip of the major joints 23° . At cycle 200 the blocks do not appear to have moved, the maximum velocity is 0.06 m/sec which is less than the corresponding velocity for the previous case. At cycle 2200 the blocks have moved considerably and in general appear to be acting as double block columns.

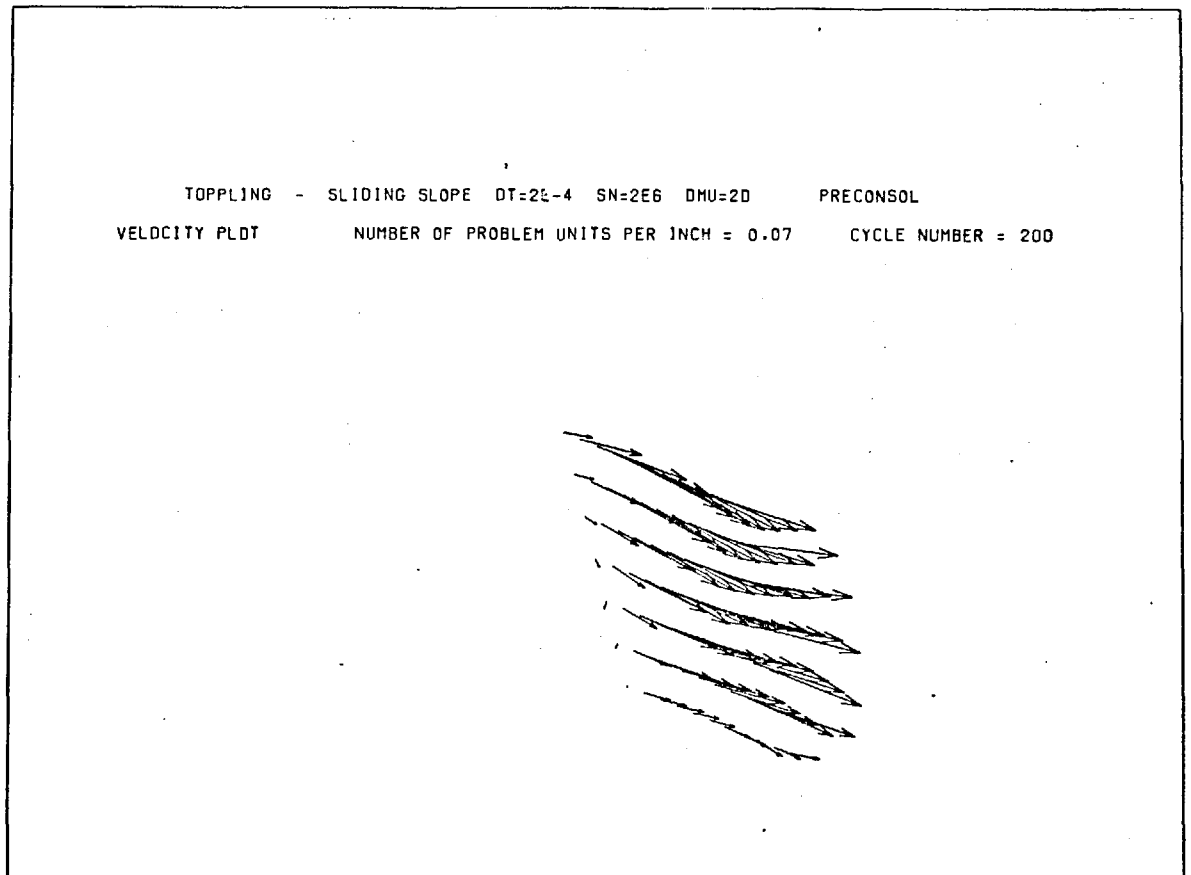
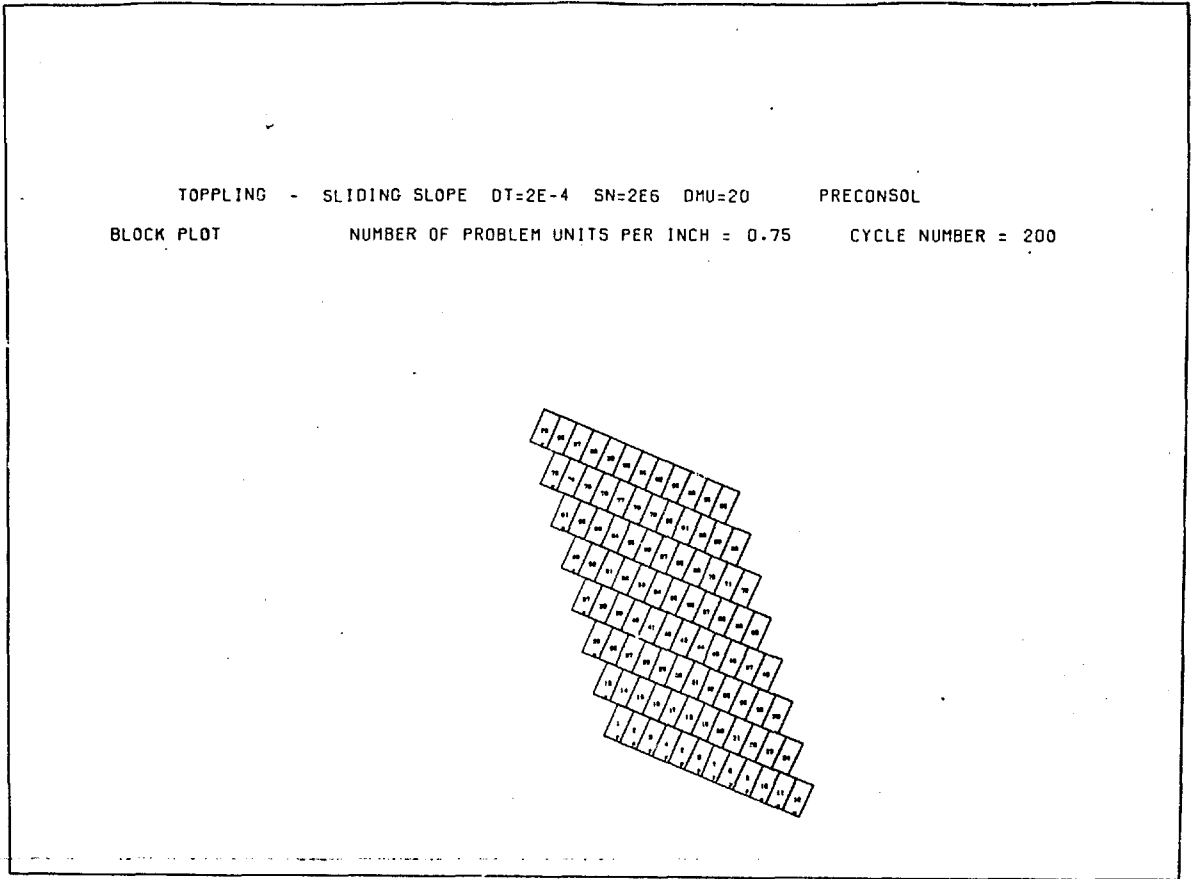
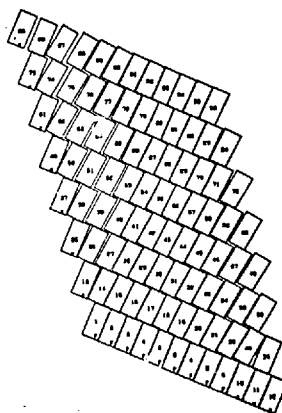


FIGURE 7.2.1 TOPPLING SLIDING FAILURE OF A STEEP SLOPE

TOPPLING - SLIDING SLOPE DT=2E-4 SN=2E6 DMU=20 PRECONSOL
BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 0.75 CYCLE NUMBER = 2200



TOPPLING - SLIDING SLOPE DT=2E-4 SN=2E6 DMU=20 PRECONSOL
VELOCITY PLOT NUMBER OF PROBLEM UNITS PER INCH = 0.59 CYCLE NUMBER = 2200

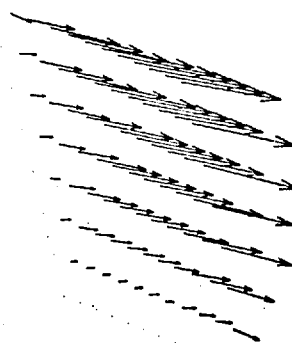
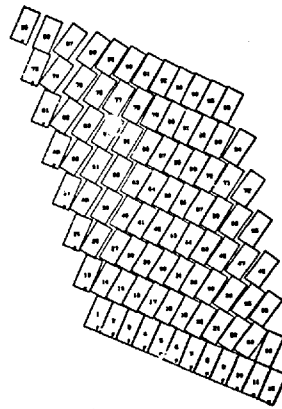


FIGURE 7.2.1 CONTINUED

TOPPLING - SLIDING SLOPE DT=2E-4 SN=2E6 DMU=20 PRECONSOL
BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 0.75 CYCLE NUMBER = 2800



TOPPLING - SLIDING SLOPE DT=2E-4 SN=2E6 DMU=20 PRECONSOL
VELOCITY PLOT NUMBER OF PROBLEM UNITS PER INCH = 0.78 CYCLE NUMBER = 2800

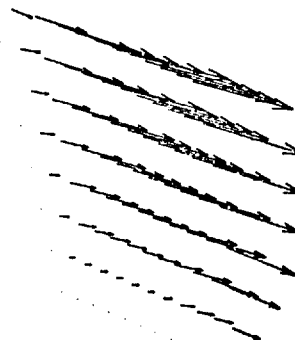


FIGURE 7.2.1 CONTINUED

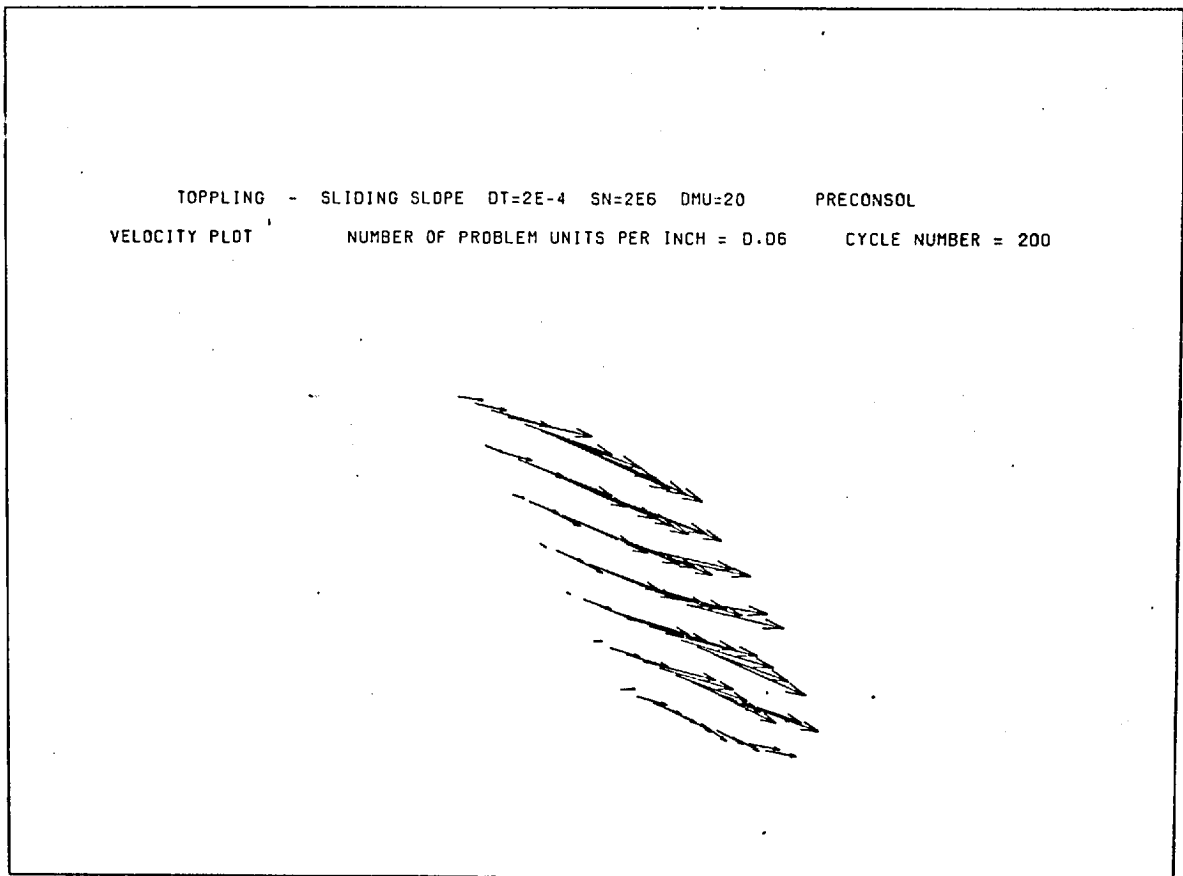
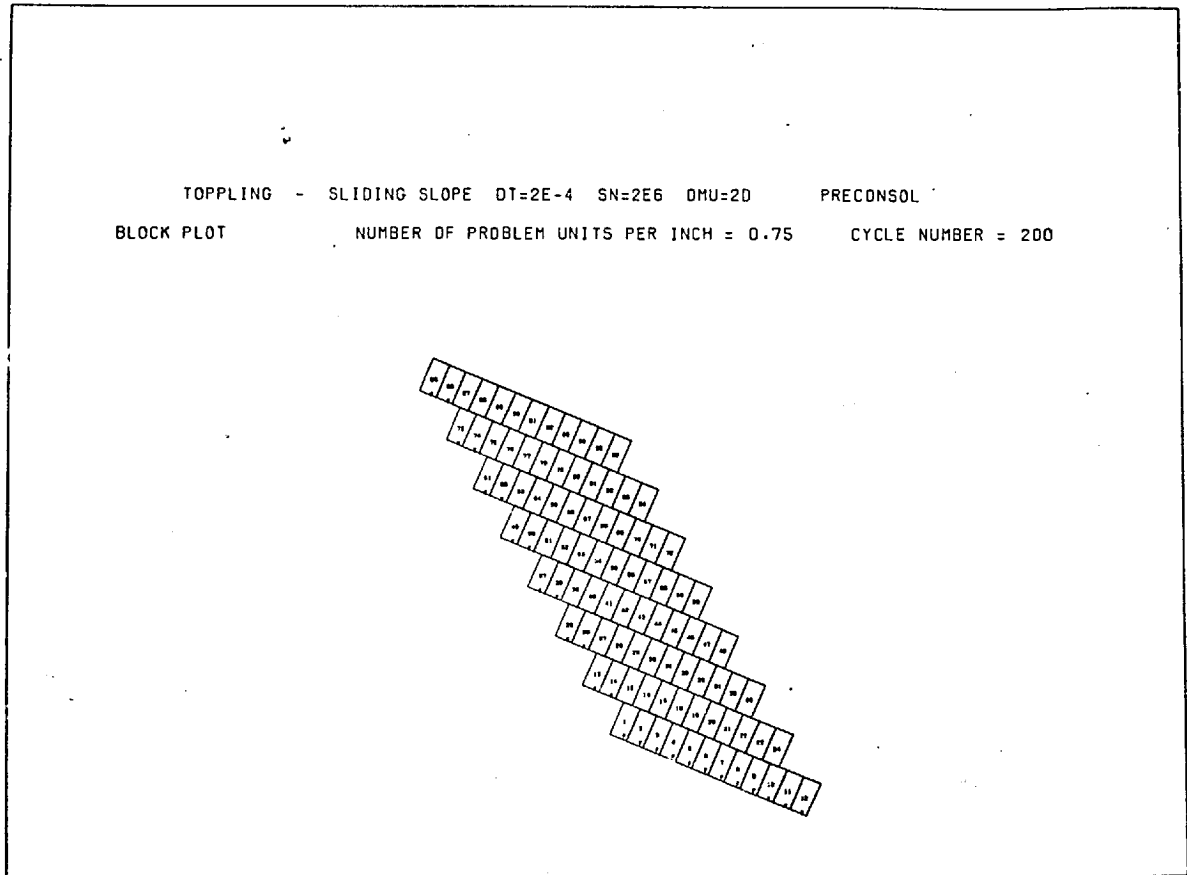
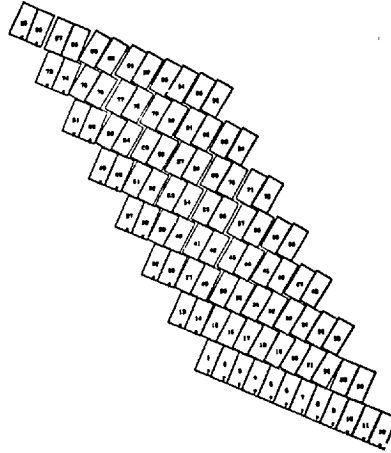


FIGURE 7.2.2 TOPPLING SLIDING FAILURE OF A FLATTER SLOPE

TOPPLING - SLIDING SLOPE DT=2E-4 SN=2E6 DMU=20 PRECONSOL
BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 0.75 CYCLE NUMBER = 2200



TOPPLING - SLIDING SLOPE DT=2E-4 SN=2E6 DMU=20 PRECONSOL
VELOCITY PLOT NUMBER OF PROBLEM UNITS PER INCH = 0.55 CYCLE NUMBER = 2200

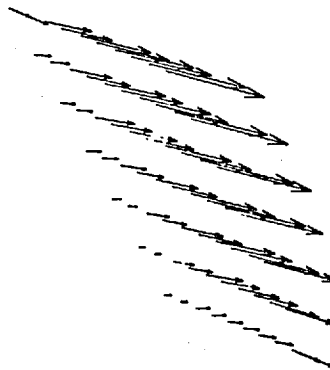
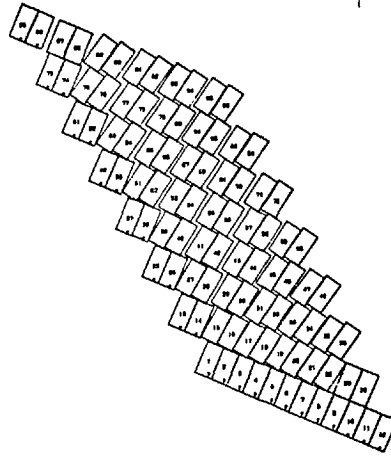


FIGURE 7.2.2 CONTINUED

TOPPLING - SLIDING SLOPE DT=2E-4 SN=2E6 DMU=20 PRECONSOL
BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 0.75 CYCLE NUMBER = 2800



TOPPLING - SLIDING SLOPE DT=2E-4 SN=2E6 DMU=20 PRECONSOL
VELOCITY PLOT NUMBER OF PROBLEM UNITS PER INCH = 0.68 CYCLE NUMBER = 2800



FIGURE 7.2.2 CONTINUED

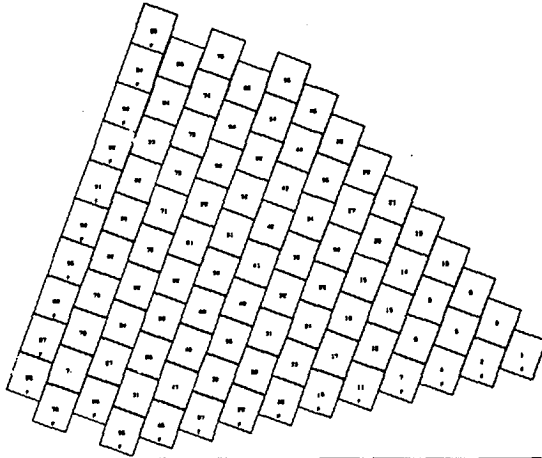
The maximum velocity at this cycle is 0.55 m/sec, which is again less than the previous case. At cycle 2800 large movements have occurred and the blocks are in an unstable mode.

A possible toppling failure block geometry shown in Figure 7.2.3 with a major discontinuity set dipping 67° into the slope and perpendicular cross joints was analysed with a friction angle of 25° . The blocks were preconsolidated to overburden load. At cycle number 200 the maximum velocity is high 1.43 m/sec but blocks do not appear to have moved. At cycle 400 the maximum velocity has dropped to 1.01 m/sec while at cycle 700 it has dropped further to 0.04 m/sec. This behaviour indicates stability and the motion of the blocks is simply a bending over of the columns as would be expected.

In Figure 7.2.4 is illustrated an almost identical problem to that in Figure 7.2.3 except that the toe block has been removed. At cycle 300 the maximum velocity is high (2.24 m/sec) and the velocity vectors have an interesting pattern completely different from those in the previous case. Block number 3 appears to have moved slightly. At cycle 800 the blocks have all bent over but also the uppermost blocks 3, 6, 10, 15, 21, 28, 36, 45 and 55 have all slid down the secondary joint set. The velocity vectors highlight this point, all blocks except the uppermost blocks are approximately stationary; while the uppermost blocks appear to be all moving at approximately 1.64 m/sec. Illustrated at cycles 1000 and 1200 are the positions of the moving uppermost blocks. At cycle 1000 the blocks are sliding, while at cycle 1200 the lower blocks have rotated.

In Figure 7.2.5 is a geometrically similar problem to Figure 7.2.4, except that the friction angle is 30° . In this case the complete assemblage of blocks is stable.

TOPPLING SLOPE 70DEG DMU=25 DT=1E-3 SN=2E6 PRECONSOL
 BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.50 CYCLE NUMBER = 200



TOPPLING SLOPE 70DEG DMU=25 DT=1E-3 SN=2E6 PRECONSOL
 VELOCITY PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.43 CYCLE NUMBER = 200

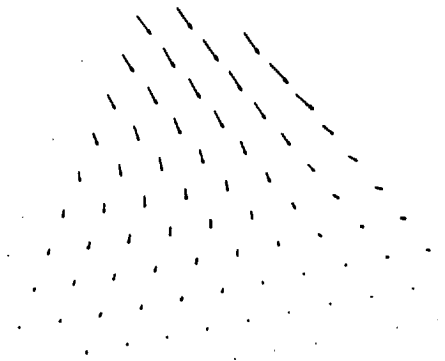
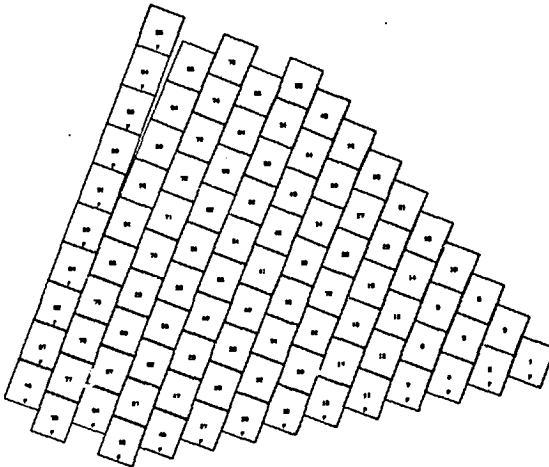


FIGURE 7.2.3 TOPPLING SLOPE WITH TOE

TOPPLING SLOPE 70DEG DMU=25 DT=1E-3 SN=2E6 PRECONSOL
BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.50 CYCLE NUMBER = 400



TOPPLING SLOPE 70DEG DMU=25 DT=1E-3 SN=2E6 PRECONSOL
VELOCITY PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.01 CYCLE NUMBER = 400

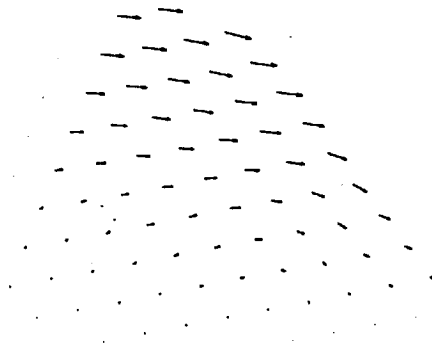
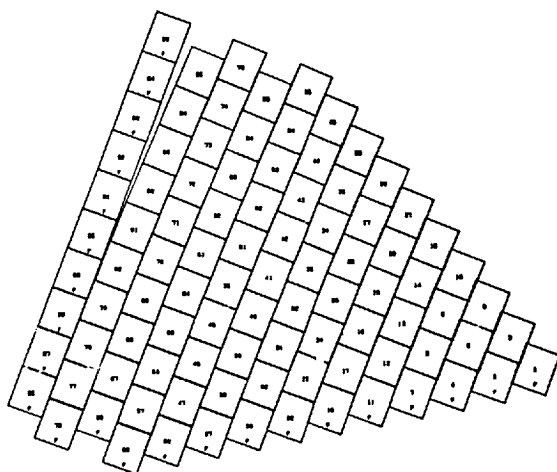


FIGURE 7.2.3 CONTINUED

TOPPLING SLOPE 700EG DMU=25 DT=1E-3 SN=2E6 PRECONSOL
BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.50 CYCLE NUMBER = 700



TOPPLING SLOPE 700EG DMU=25 DT=1E-3 SN=2E6 PRECONSOL
VELOCITY PLOT NUMBER OF PROBLEM UNITS PER INCH = 0.04 CYCLE NUMBER = 700

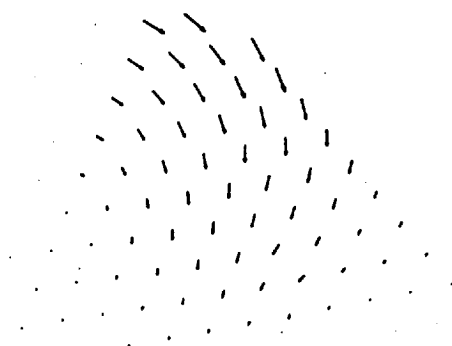
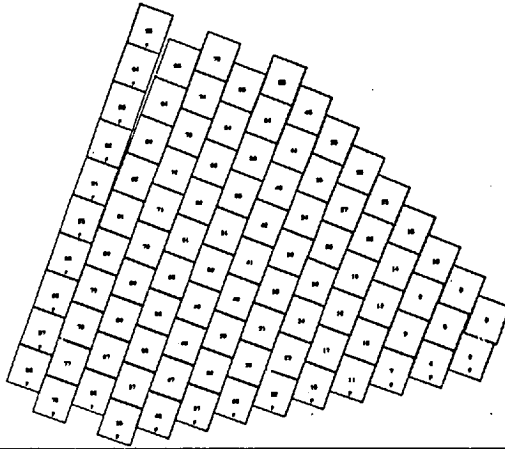


FIGURE 7.2.3 CONTINUED

TOPPLING SLOPE 70DEG DMU=25 DT=1E-3 SN=2E6 PRECONSOL-TOE REMOVED
 BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.50 CYCLE NUMBER = 300



TOPPLING SLOPE 70DEG DMU=25 DT=1E-3 SN=2E6 PRECONSOL-TOE REMOVED
 VELOCITY PLOT NUMBER OF PROBLEM UNITS PER INCH = 2.24 CYCLE NUMBER = 300

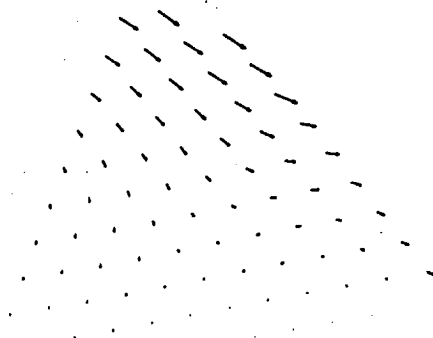
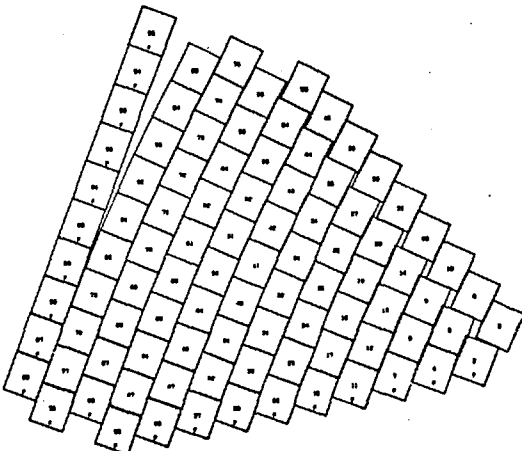


FIGURE 7.2.4 TOPPLING SLOPE WITHOUT TOE $\phi = 25^\circ$

TOPPLING SLOPE 70DEG DMU=25 DT=1E-3 SN=2E6 PRECONSOL-TOE REMOVED
 BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.50 CYCLE NUMBER = 800



TOPPLING SLOPE 70DEG DMU=25 DT=1E-3 SN=2E6 PRECONSOL-TOE REMOVED
 VELOCITY PLOT, NUMBER OF PROBLEM UNITS PER INCH = 1.64 CYCLE NUMBER = 800

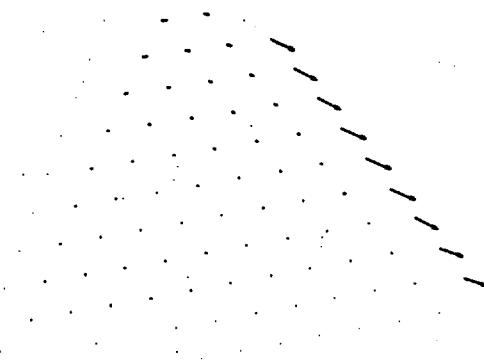
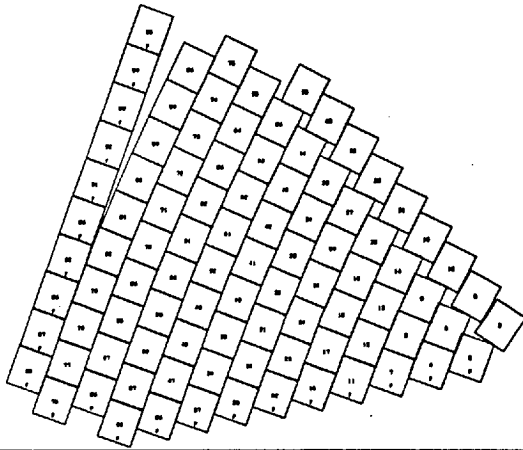


FIGURE 7.2.4 CONTINUED

TOPPLING SLOPE 70DEG DMU=25 DT=1E-3 SN=2E6 PRECONSOL-TOE REMOVED
BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.50 CYCLE NUMBER = 1000



TOPPLING SLOPE 70DEG DMU=25 DT=1E-3 SN=2E6 PRECONSOL-TOE REMOVED
BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.50 CYCLE NUMBER = 1200

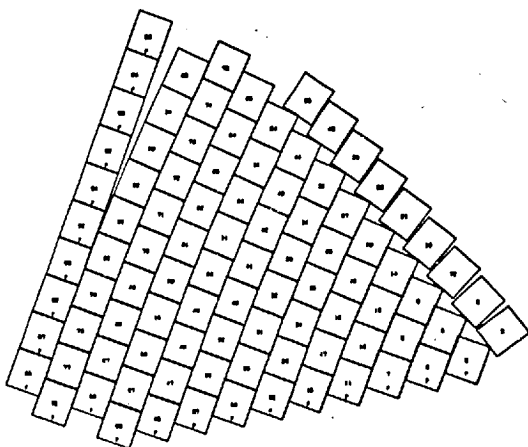
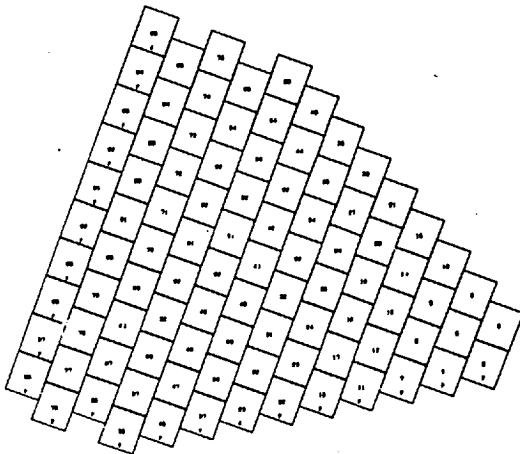


FIGURE 7.2.4 CONTINUED

TOPPLING SLOPE 70DEG DMU=30 DT=1E-3 SN=2E6 PRECONSOL-TOE REMOVED
 BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.50 CYCLE NUMBER = 100



TOPPLING SLOPE 70DEG DMU=30 DT=1E-3 SN=2E6 PRECONSOL-TOE REMOVED
 VELOCITY PLOT NUMBER OF PROBLEM UNITS PER INCH = 0.85 CYCLE NUMBER = 100

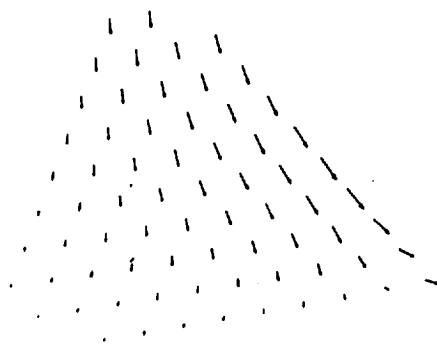
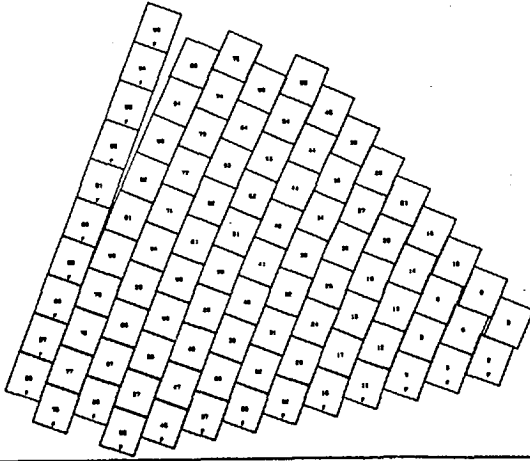


FIGURE 7.2.5 TOPPLING SLOPE WITHOUT TOE $\phi = 30^\circ$

TOPPLING SLOPE 70DEG DMU=3D DT=1E-3 SN=2E6 PRECDNSCL-TOE REMOVED
 BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.50 CYCLE NUMBER = 600



TOPPLING SLOPE 70DEG DMU=3D DT=1E-3 SN=2E6 PRECDNSCL-TOE REMOVED
 VELOCITY PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.23 CYCLE NUMBER = 600

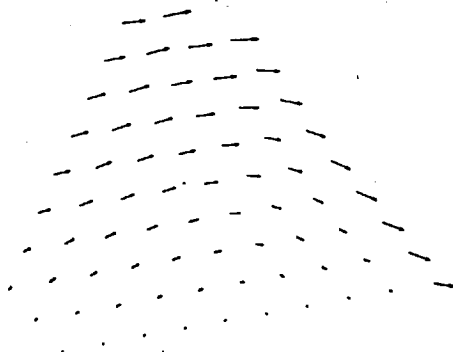


FIGURE 7.2.5 CONTINUED

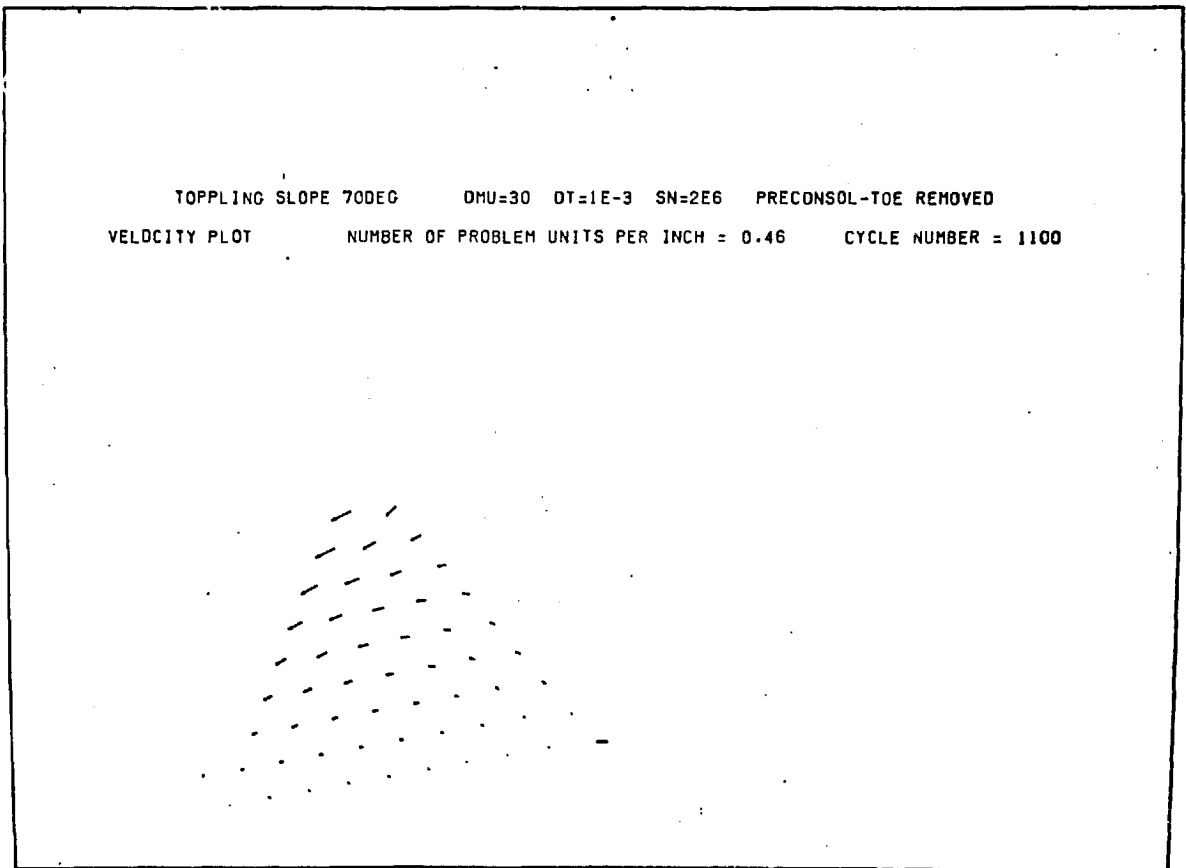
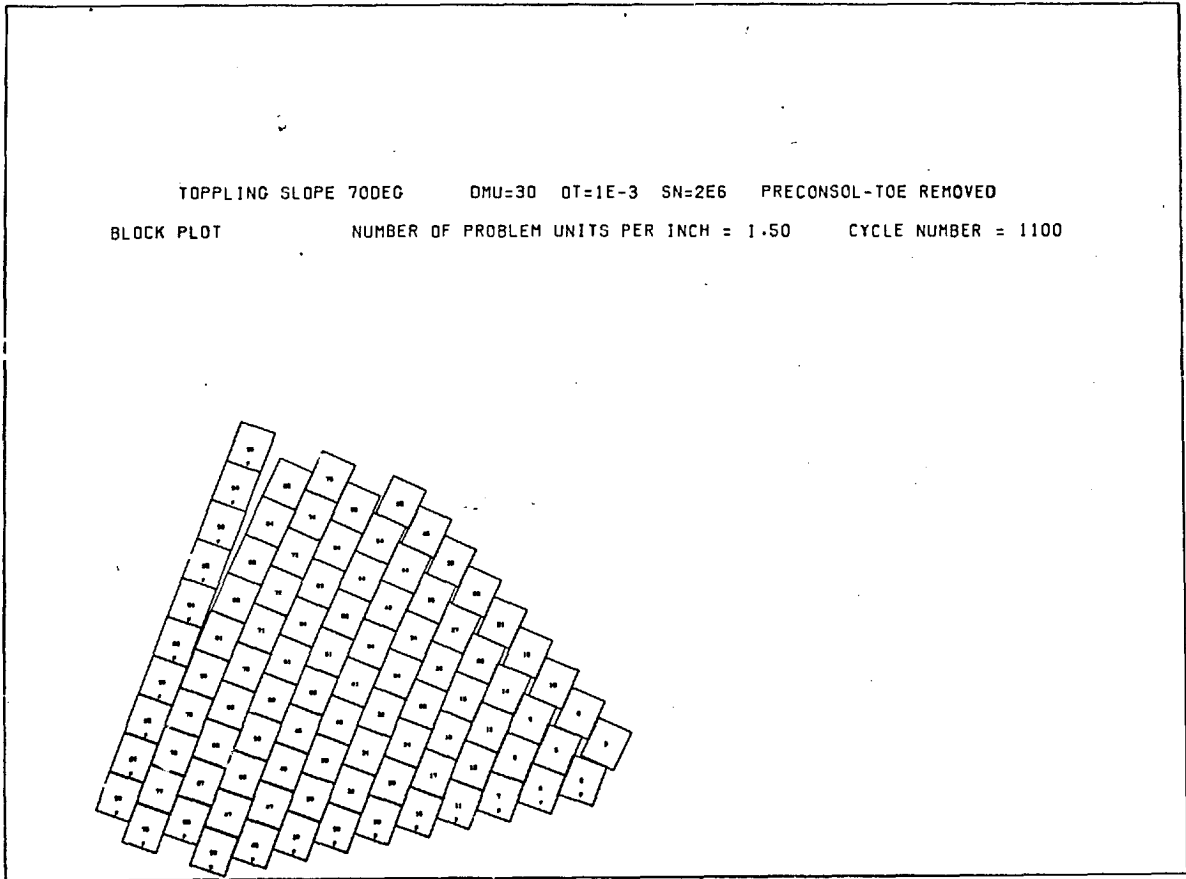


FIGURE 7.2.5 CONTINUED

3. Horizontal seam

Illustrated in Figure 7.3.1 is a horizontal seam at very shallow depth with five blocks extracted. The problem was preconsolidated to overburden load and blocks marked with an F are fixed. From Figure 7.3.1 it is clear that the assemblage of blocks is unstable. The friction angle, both peak and residual, was 25° . The almost final collapsed form of the blocks is illustrated at cycle 800. The large overlap between some blocks is due to the low stiffness used. This could have been alleviated by increasing the stiffness or more realistically improving the interaction relationships as described in section 6.3. The blocks 13, 18, 27, 28, 32, 33 etc. could well have fractured due to tensile stresses in many rock types; however, in this case using infinitely strong blocks did not appear to prevent total collapse.

In Figure 7.3.2. a horizontal seam at shallow depth with extracted blocks was analysed. The assemblage of blocks was preconsolidated to overburden load and the friction angle, both peak and residual, was 25° . The problem is clearly unstable as illustrated in Figure 7.3.2. The approximately final collapse form of the blocks is shown at cycle 900. The velocity vectors at this cycle are interesting since they indicate the zone of moving blocks. The mode of failure is similar to that shown in the previous example, Figure 7.3.1. The comments made earlier as regards blocks overlapping are applicable to this case.

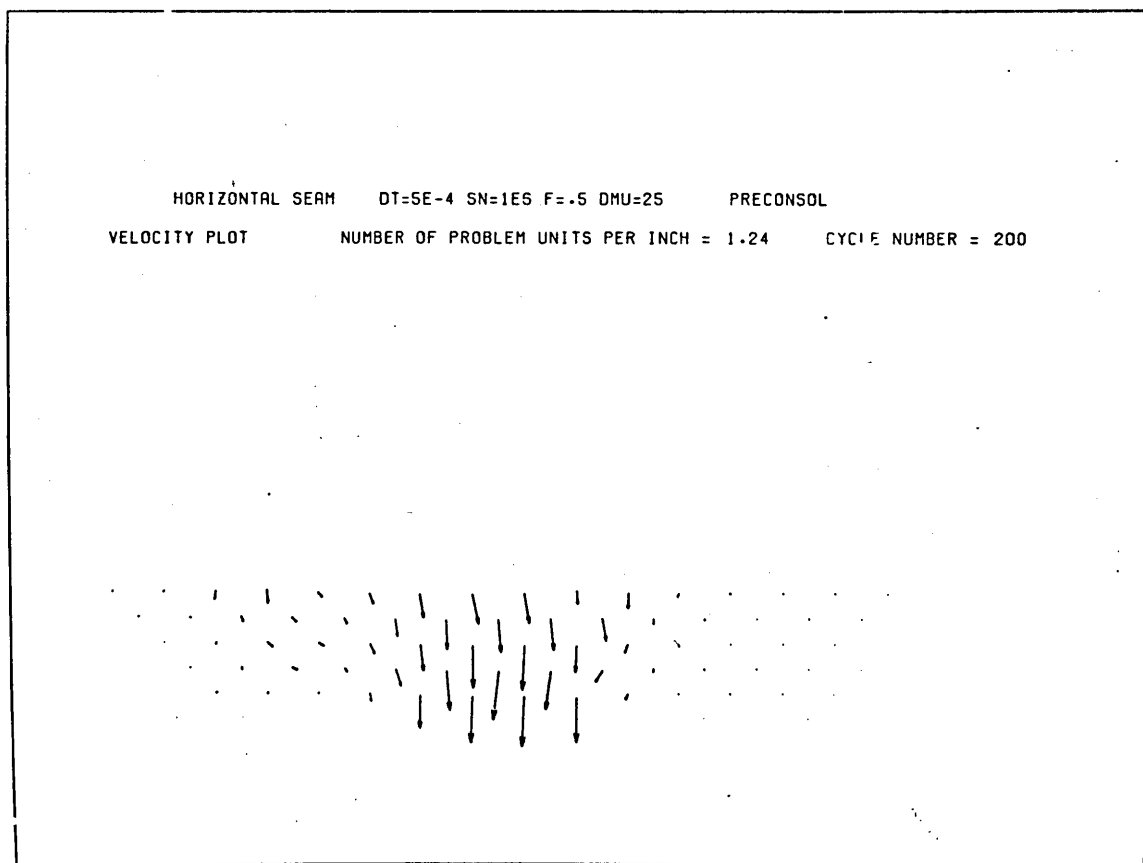
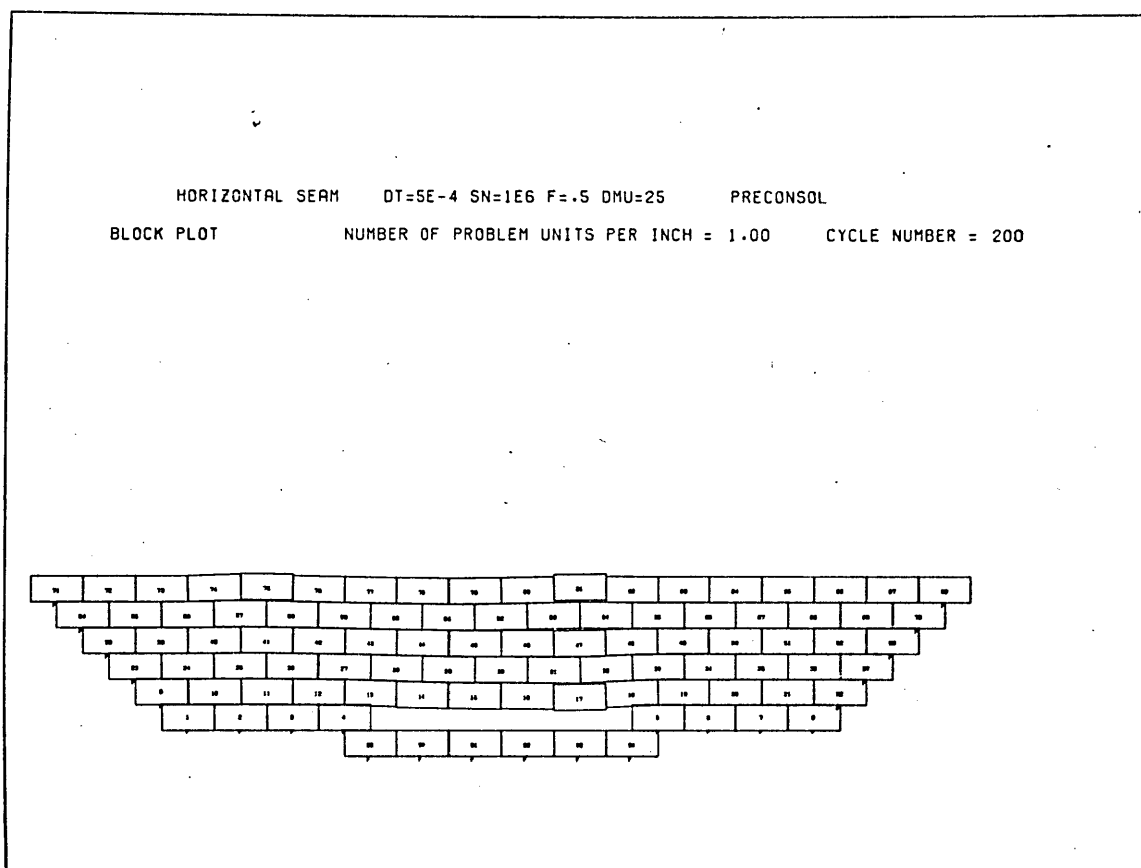
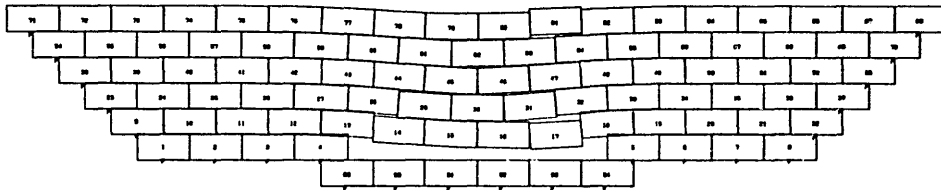


FIGURE 7.3.1 EXCAVATED HORIZONTAL SEAM AT VERY SHALLOW DEPTH

HORIZONTAL SEAM DT=5E-4 SN=1E6 F=.5 DMU=25 PRECONSOL
BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.00 CYCLE NUMBER = 400

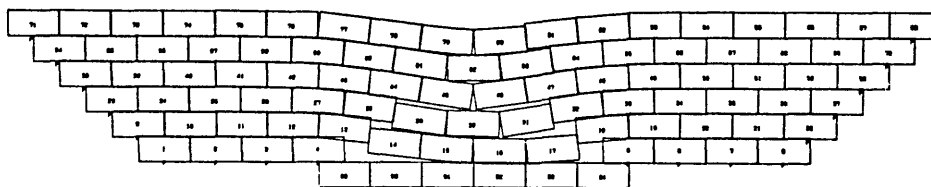


HORIZONTAL SEAM DT=5E-4 SN=1E6 F=.5 DMU=25 PRECONSOL
VELOCITY PLOT NUMBER OF PROBLEM UNITS PER INCH = 2.24 CYCLE NUMBER = 400



FIGURE 7.3.1 CONTINUED

HORIZONTAL SEAM DT=5E-4 SN=1E6 F=.5 DMU=25 PRECONSOL
BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.00 CYCLE NUMBER = 600



HORIZONTAL SEAM DT=5E-4 SN=1E6 F=.5 DMU=25 PRECONSOL
VELOCITY PLOT NUMBER OF PROBLEM UNITS PER INCH = 2.70 CYCLE NUMBER = 600



FIGURE 7.3.1 CONTINUED

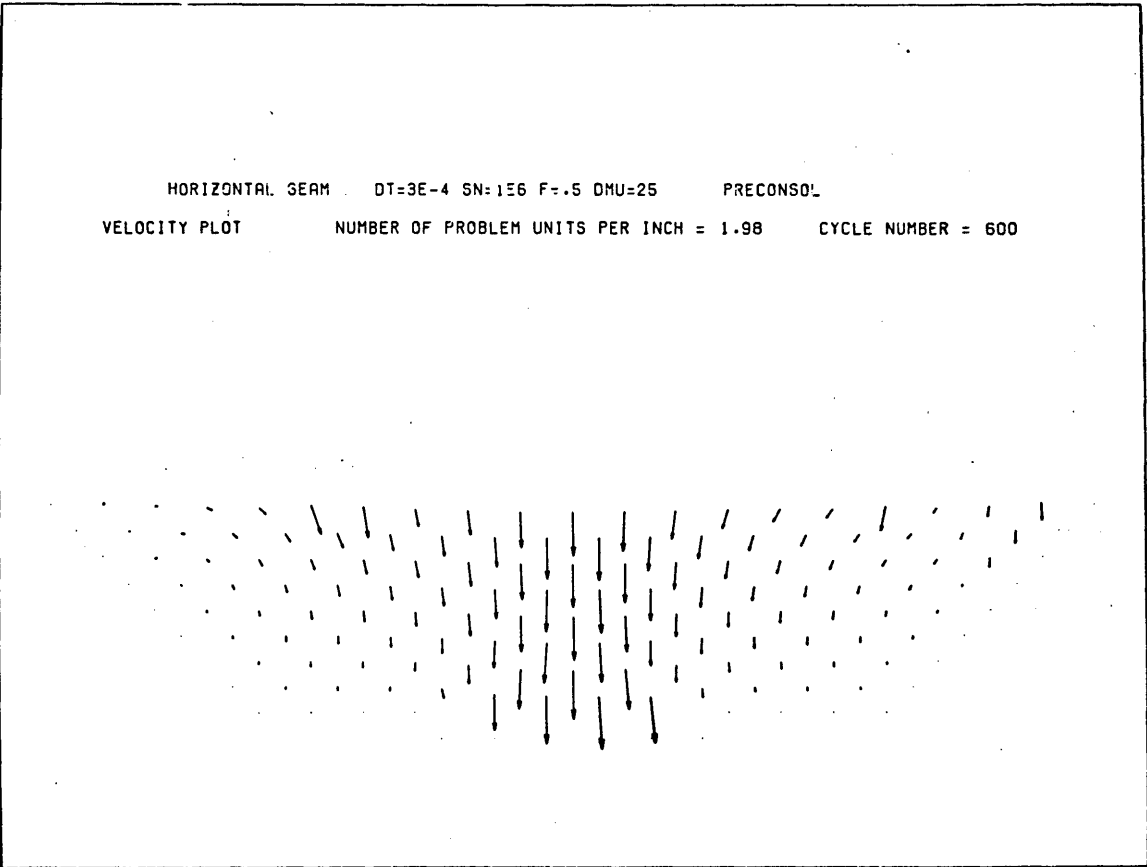
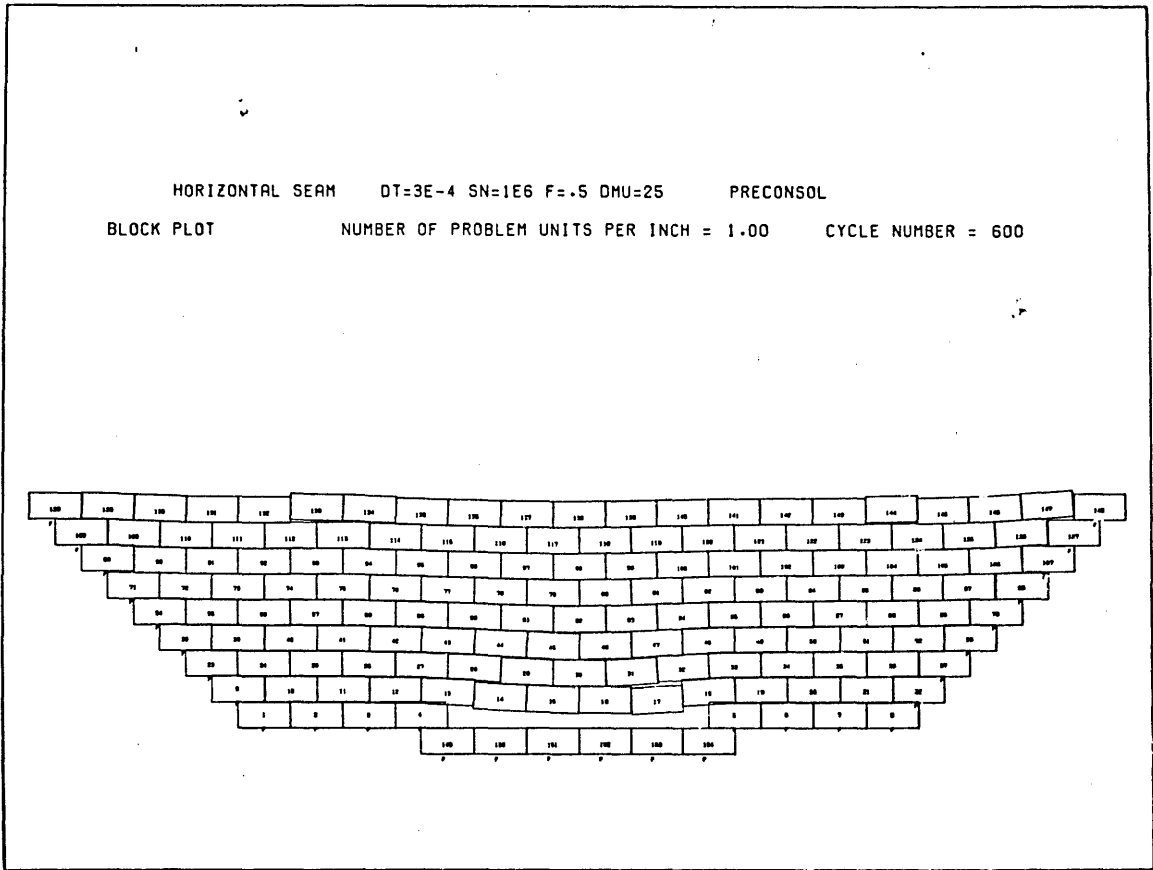


FIGURE 7.3.2 EXCAVATED HORIZONTAL SEAM AT SHALLOW DEPTH

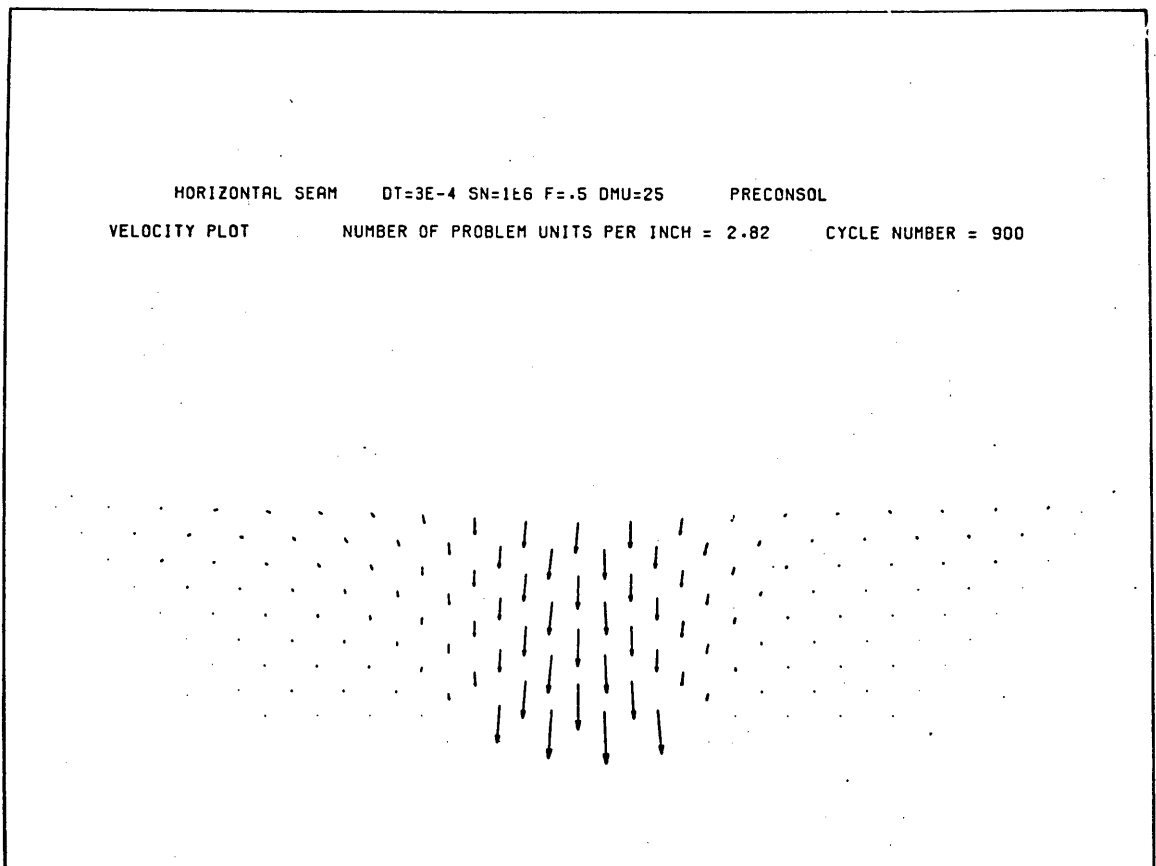
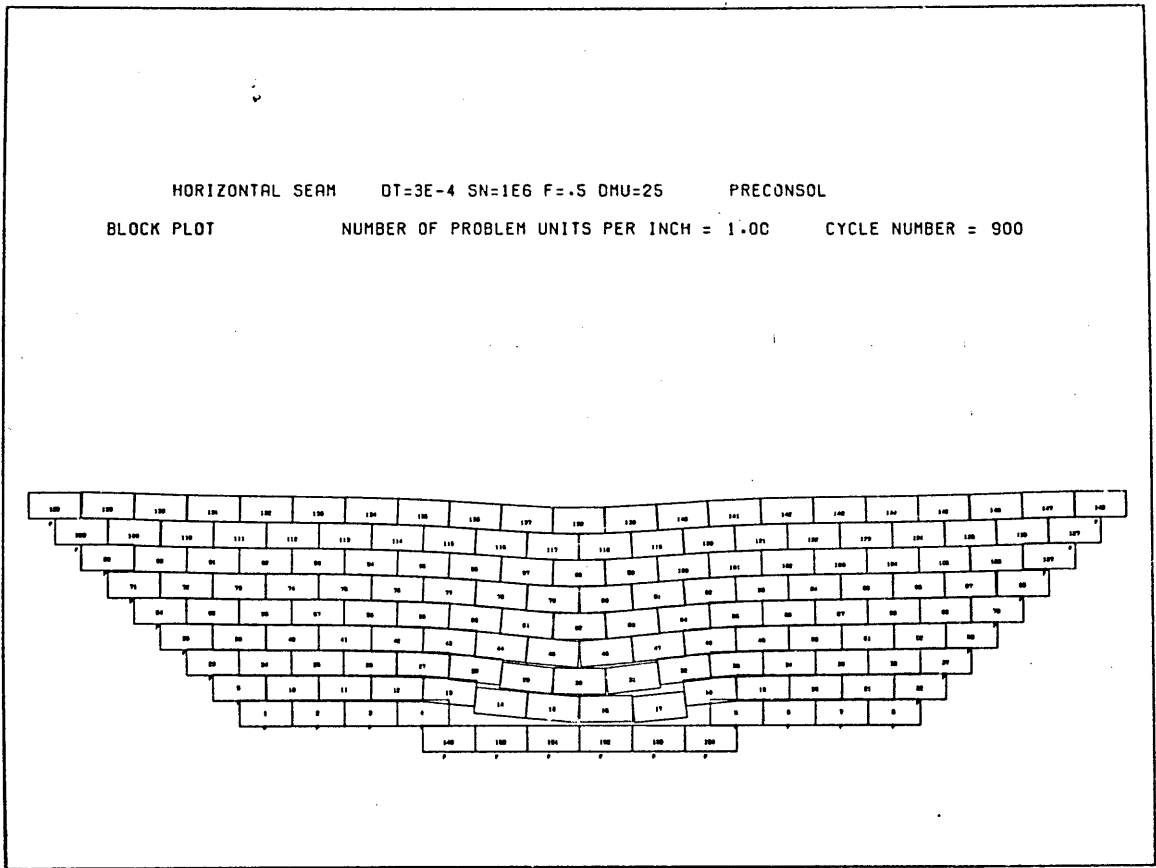


FIGURE 7.3.2 CONTINUED

4. Inclined orebody

An inclined orebody with major continuous joints dipping at 35° and offset secondary joints dipping at 75° is illustrated in Figure 7.4.1. Blocks marked with an F are fixed and the entire assemblage was preconsolidated to overburden load. The friction angle both peak and residual was 25° . The geometry of the problem before any motion is illustrated at cycle 0. The first signs of blocks moving is shown at cycle 300. At cycle 600 there have been large movements of blocks and the velocity vectors clearly indicate the motion of blocks. The motion of blocks up dip, towards a central "draw" zone and the rapid decrease in velocity down dip of the excavated region are features to be expected. At cycle 1000 the assemblage of blocks is almost in a final collapse form. At cycle 1400 the blocks near the excavated area are still and the structure of the blocks has clearly a bent beam appearance. The large overlap of some blocks is due to the low stiffness used.

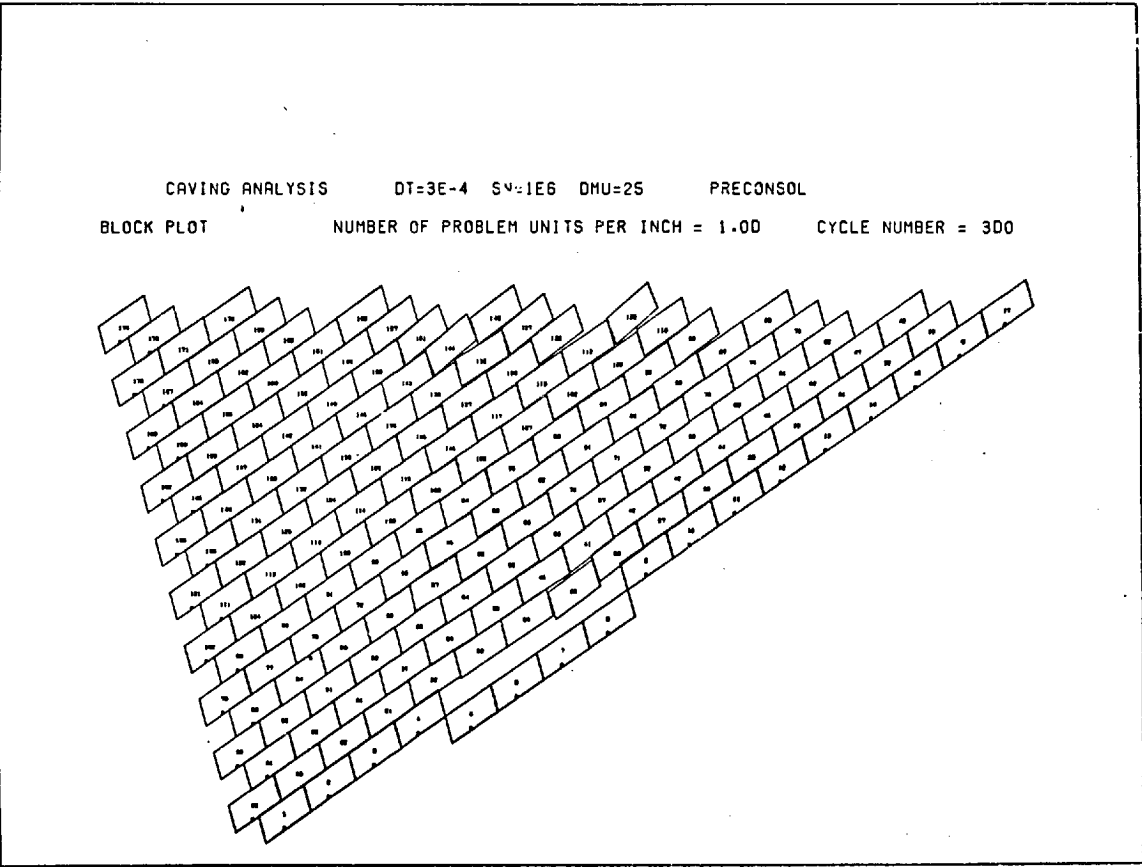
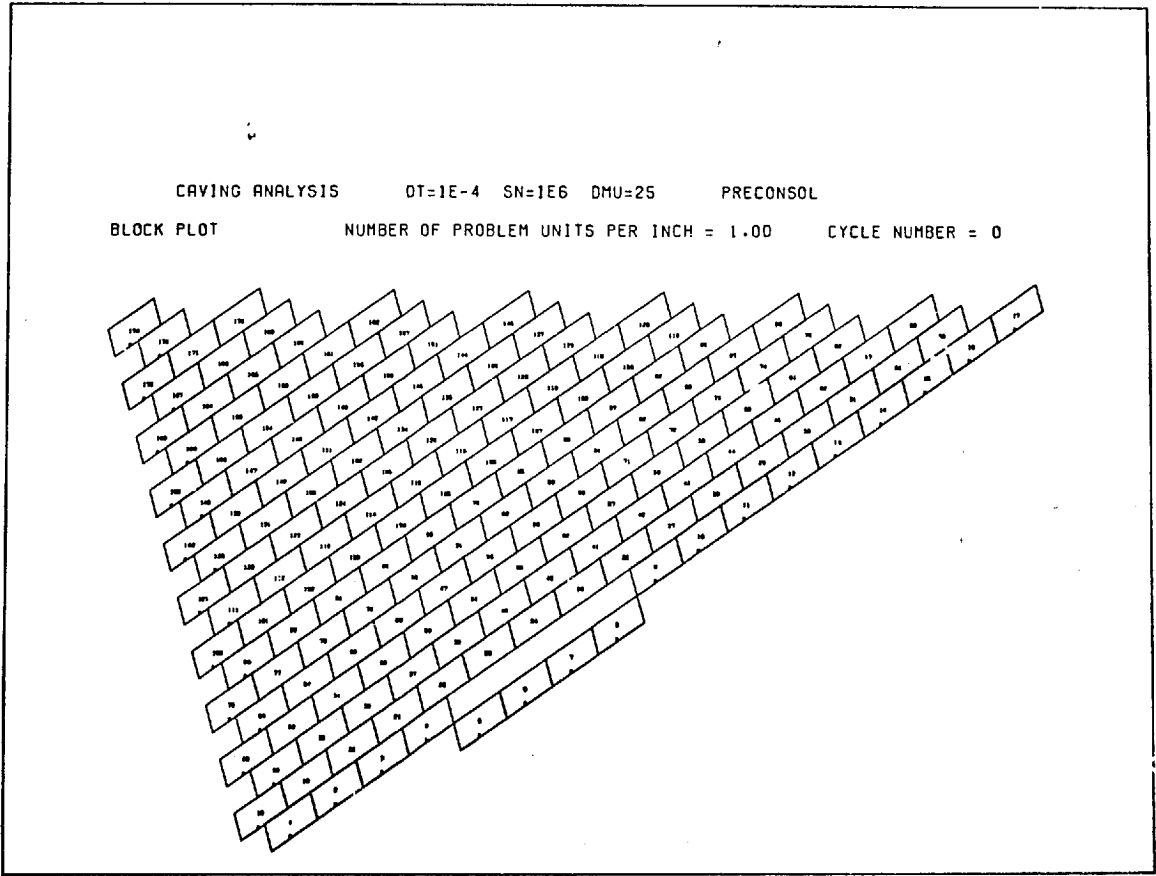


FIGURE 7.4.1 EXCAVATED INCLINED OREBODY

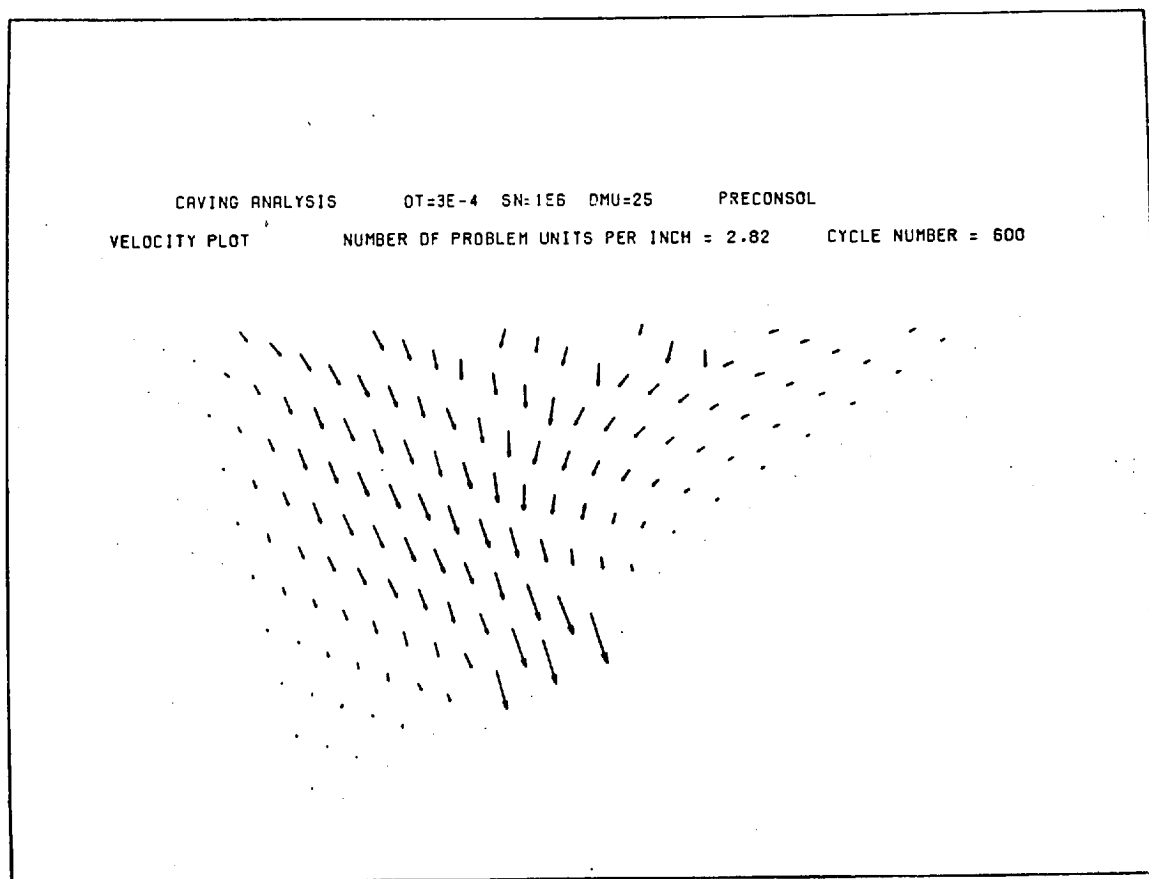
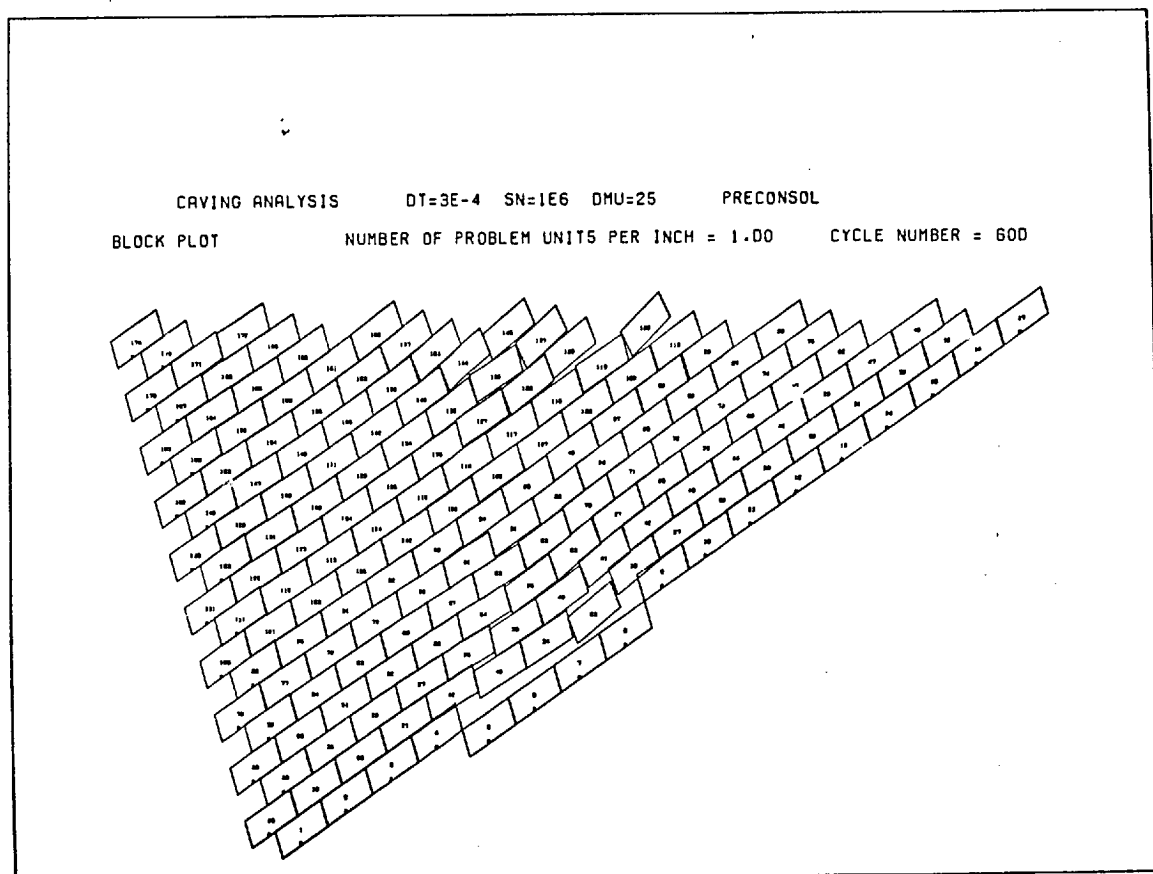


FIGURE 7.4.1 CONTINUED

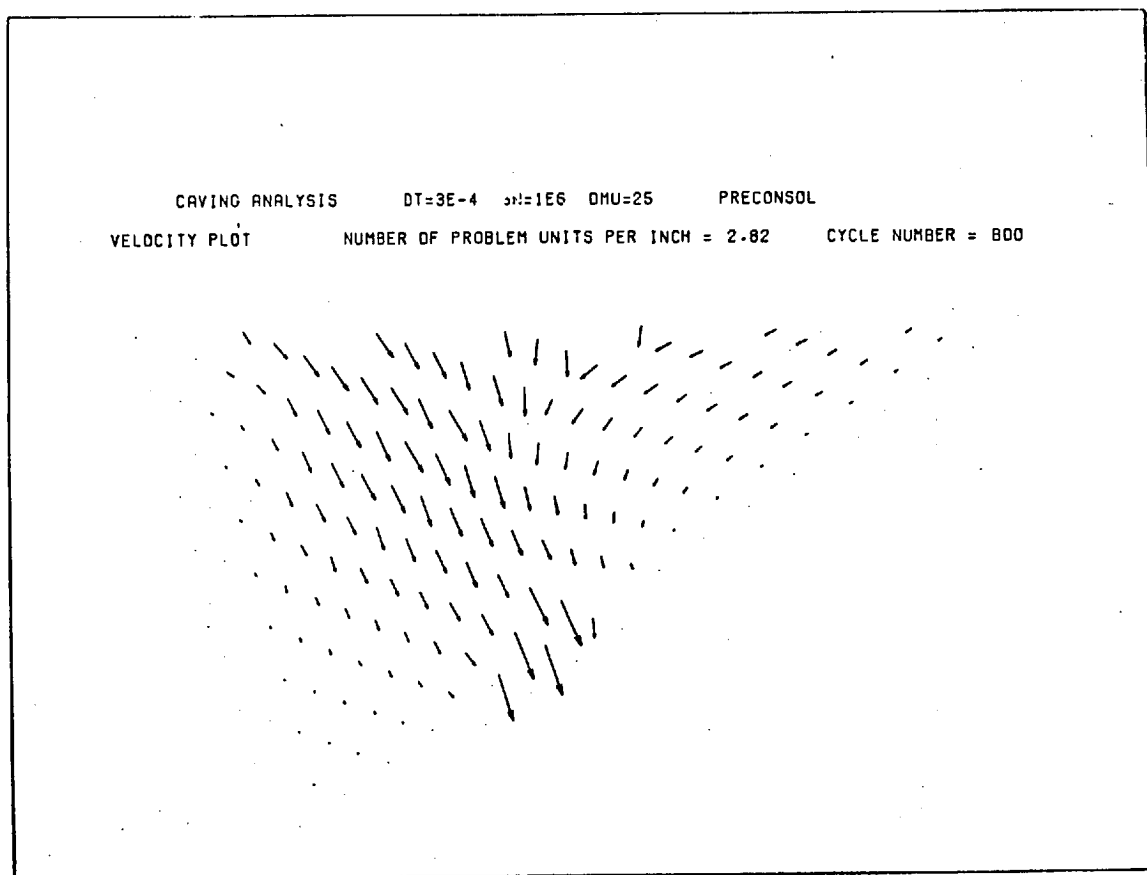
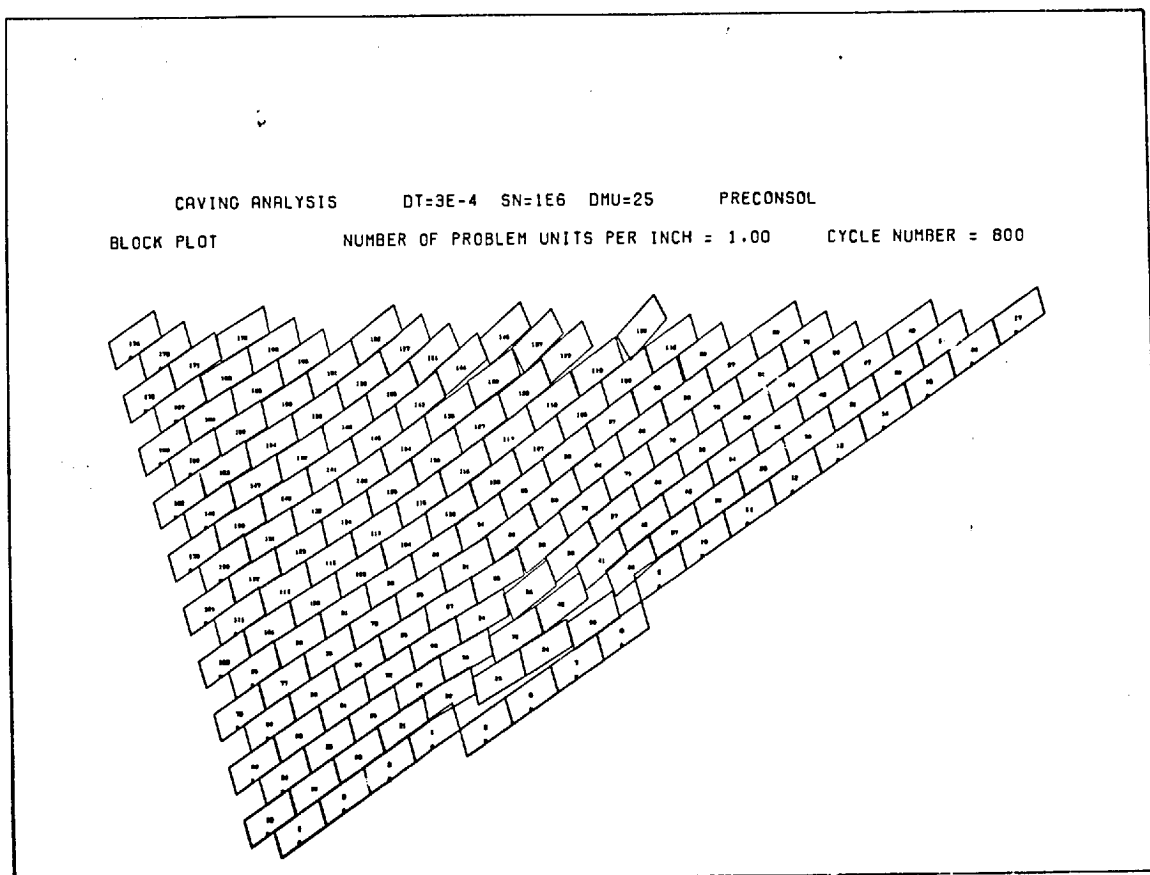
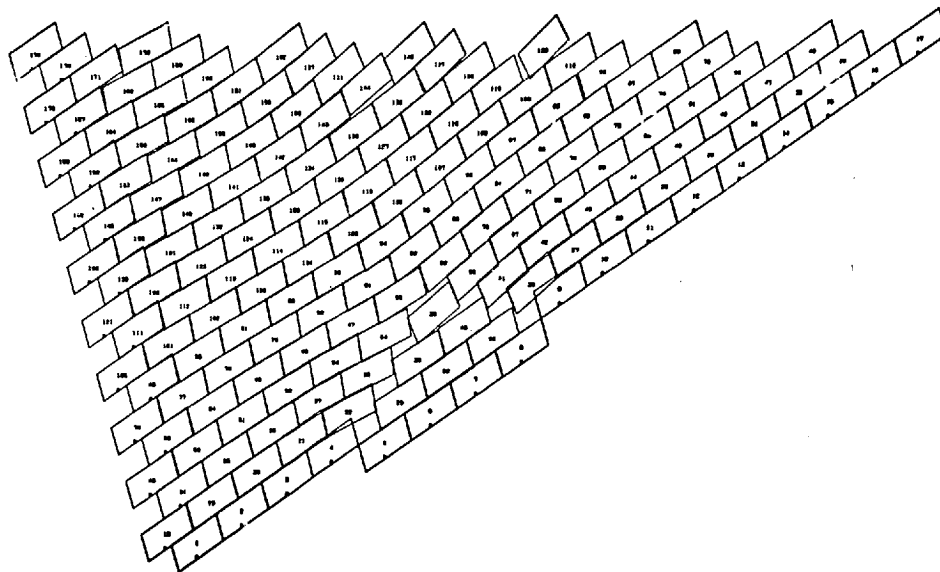


FIGURE 7.4.1 CONTINUED

CAVING ANALYSIS DT=3E-4 SN=1E6 DMU=25 PRECONSOL
 BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.00 CYCLE NUMBER = 1000



CAVING ANALYSIS DT=3E-4 SN=1E6 DMU=25 PRECONSOL
 VELOCITY PLOT NUMBER OF PROBLEM UNITS PER INCH = 2.17 CYCLE NUMBER = 1000

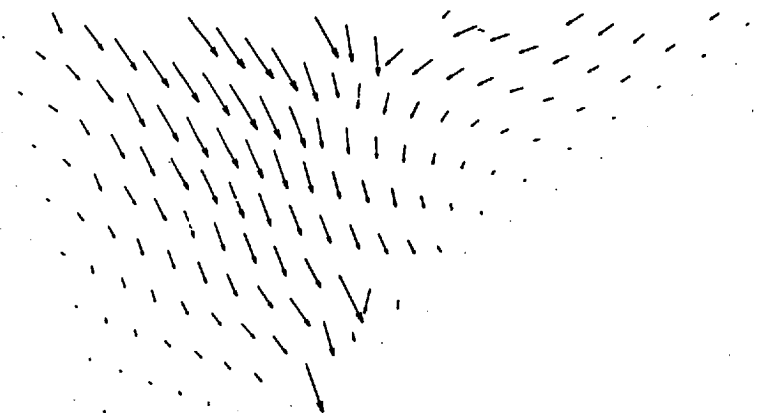


FIGURE 7.4.1 CONTINUED

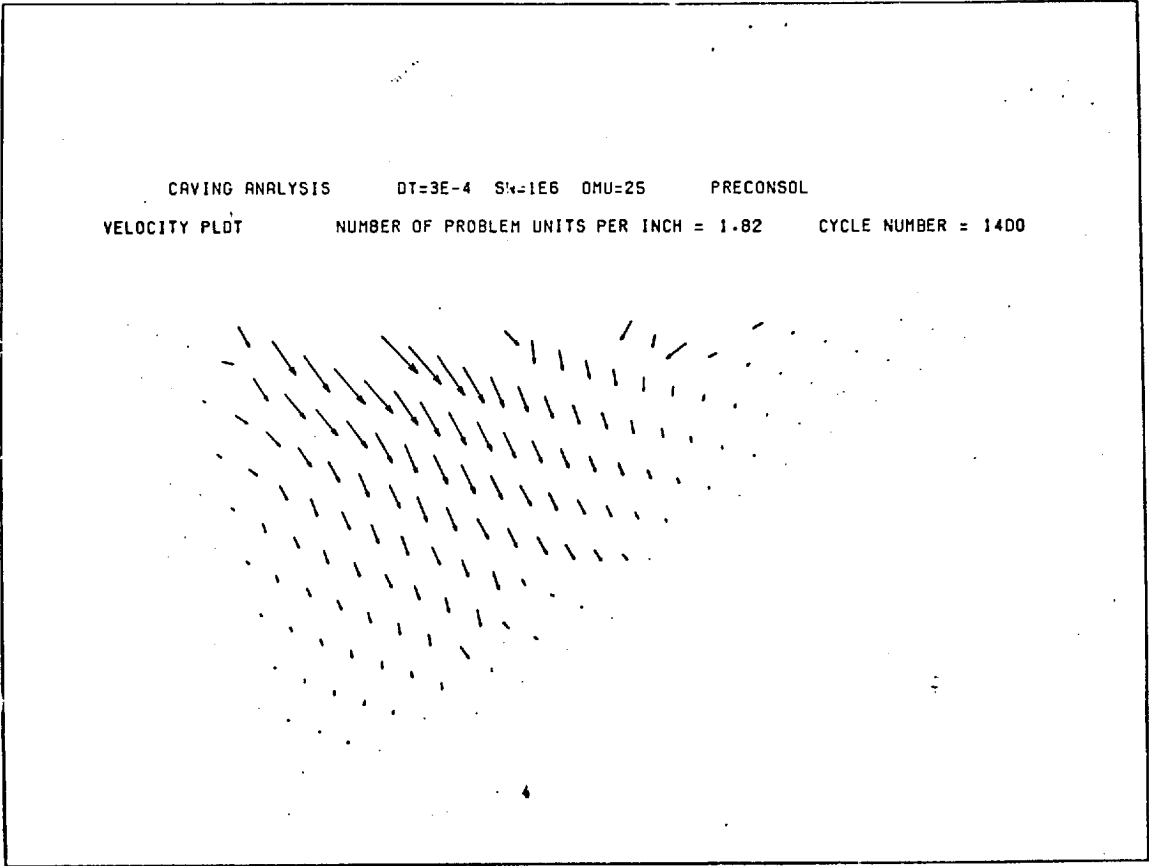
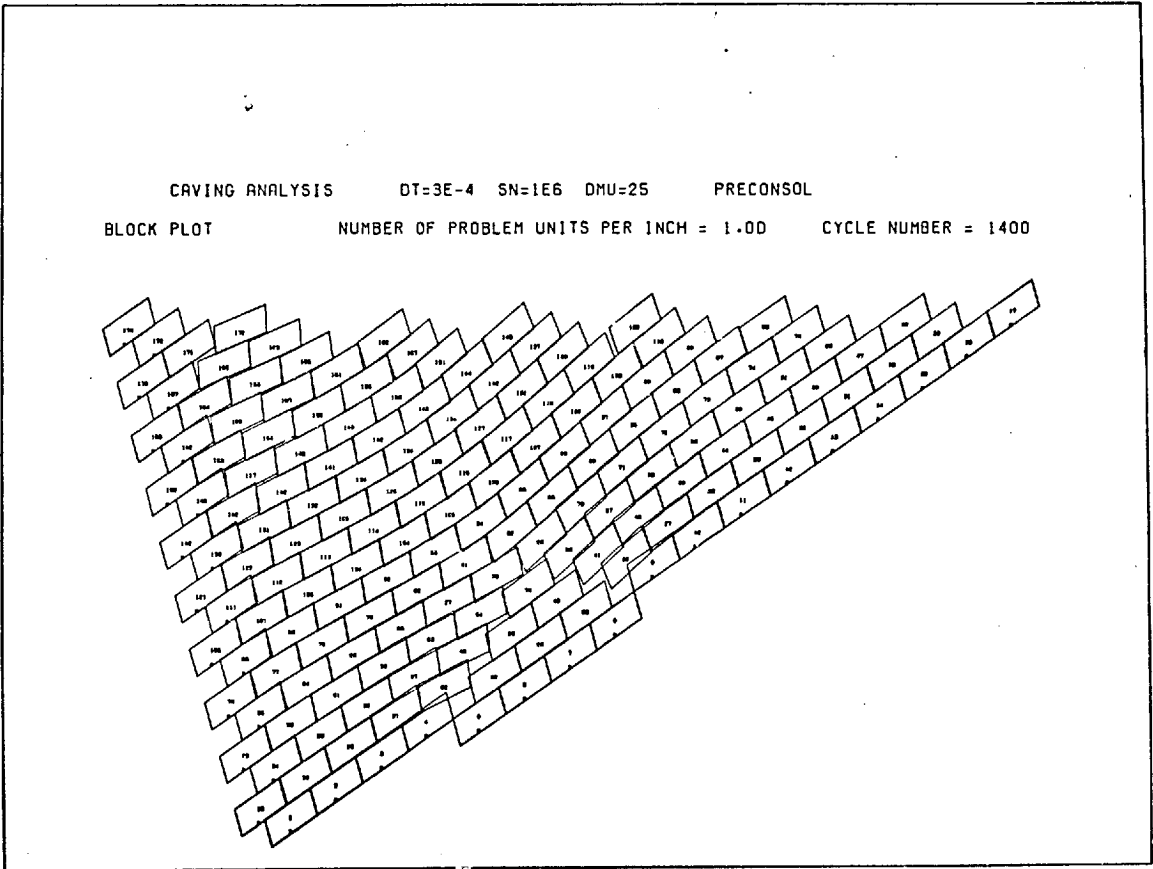
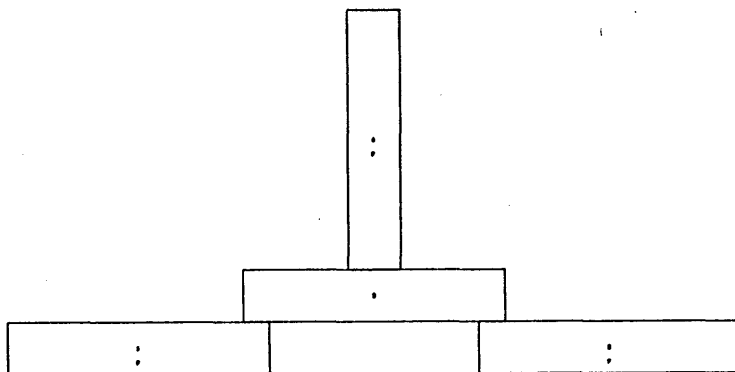


FIGURE 7.4.1 CONTINUED

5. Stresses within blocks

In order to illustrate the coupling of the finite element method to the quasi-rigid block program a simple block beam shown in Figure 7.5.1 was analysed. The blocks marked with an F are fixed which therefore leaves only block 3 free. Block number 4 weighs ten times that of the other three blocks. Therefore block 3 is subjected to a load equivalent to ten blocks applied at midspan and its' self weight. Support for block 3 is provided by blocks 1 and 2. The block program was run for only a single cycle, and corner forces of respective blocks fed into the coupled finite element program. The finite element mesh for block 3 is shown in Figure 7.5.1 along with a tensor plot of stresses at the centre of each finite element. Tensors with arrows indicate tensile stresses.

STRESSES IN BLOCK - BEAM DT=1E-4 SN=1E7 DMU=200
BLOCK PLOT NUMBER OF PROBLEM UNITS PER INCH = 1.00 CYCLE NUMBER = 0



STRESSES IN BLOCK - BEAM DT=1E-4 SN=1E7 DMU=200
MESH PLOT NUMBER OF PROBLEM UNITS PER INCH = 0.50 CYCLE NUMBER = 1

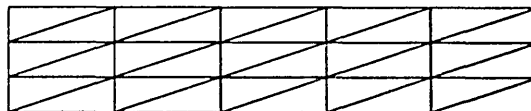


FIGURE 7.5.1 STRESSES IN BLOCK BEAM

STRESSES IN BLOCK - BEAM DT=1E-4 SN=1E7 OMU=200
TENSOR PLOT NUMBER OF PROBLEM UNITS PER INCH = 1000000.00 CYCLE NUMBER = 1

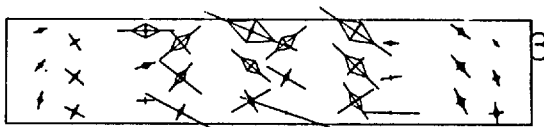


FIGURE 7.5.1 CONTINUED

6. Computational Efficiency

The program BLOCK could handle 120 blocks for 30k decimal storage on a CDC 6400 computer. Storage for buffers is included in this figure while in previous cases, sections 3.8 and 5.6, they did not (the reason being that the 7600 machine allocates buffer storage in L.C.M.). The block program could be altered to handle approximately 1000 blocks for the same storage requirements as above with a minimum of input output time if random access files were used. The central processor time for a single cycle versus number of blocks is illustrated in Figure 7.6.1.

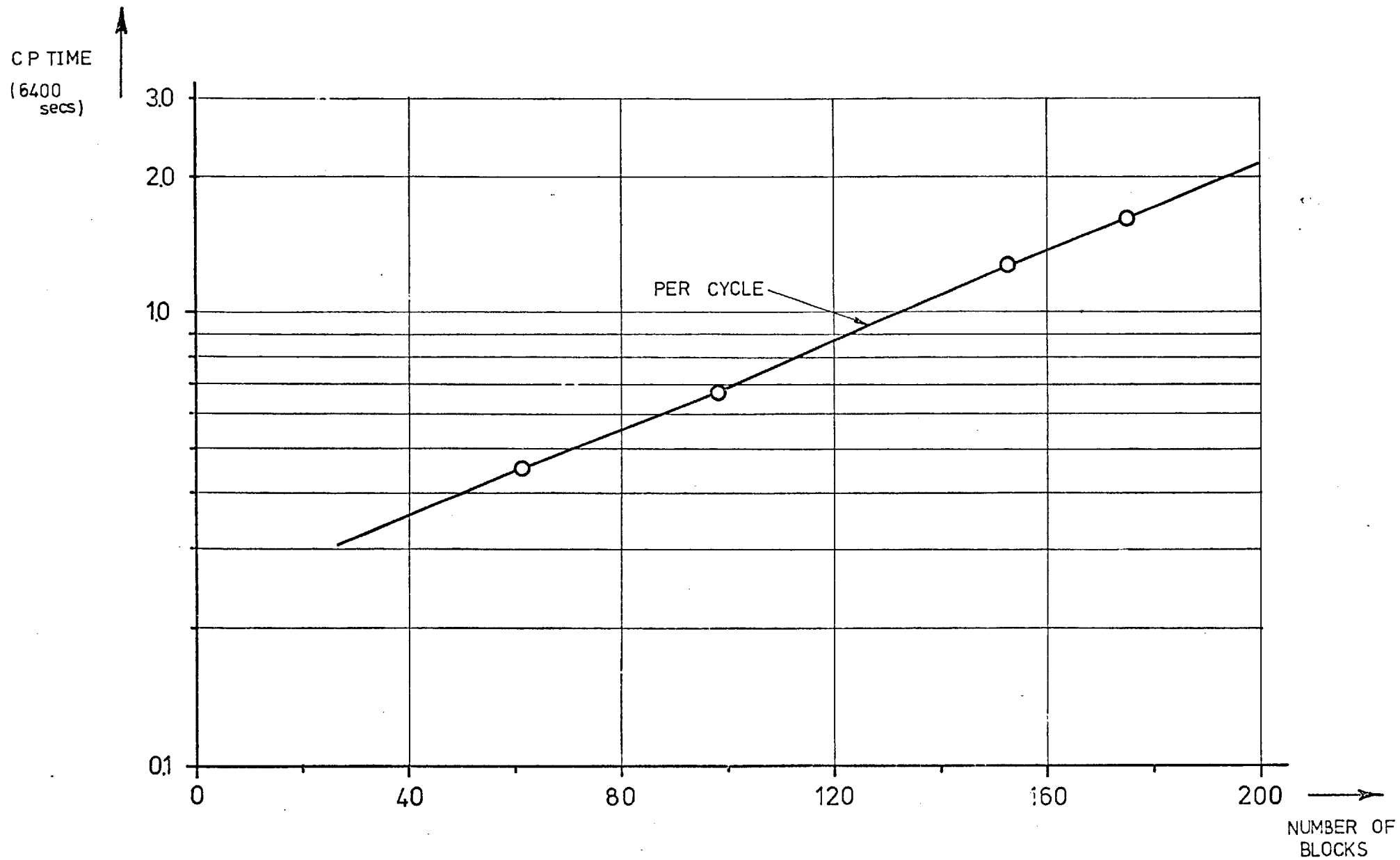


FIGURE 7.6.1 CENTRAL PROCESSOR TIME FOR BLOCK

CHAPTER 8: CONCLUSIONS

An efficient two dimensional formulation of the boundary integral method for the analysis of elastostatics in non-homogeneous media has been presented. The integral methods have been shown to be applicable to non-linear problems, slip and separation on joints and no-tension analysis, by following a step-wise linear path to the solution. The boundary integral method has been successfully applied to; tunnel intersections, five geometrically different circular tunnel intersections the cross, the tee and three bifurcations (15° , 30° , and 45°), an underground mining complex, an underground power station, inclusions, thin soft seams near a circular tunnel, slip and separation of joints near a circular tunnel and the analysis of a circular tunnel in a no-tension material.

The analysis of groundwater seepage in non-homogeneous media by the boundary integral method has been described. A two dimensional formulation of the method has been successfully applied to a number of steady state problems; flow to and from a pressure tunnel and drain near the surface of a saturated medium. An iterative technique for the calculation of the phreatic surface location under steady state conditions has been presented and illustrated to converge satisfactorily by application to the problem of a drain near infinite resevoirs. The analysis of transient groundwater flow (time dependency due to a change in boundary conditions and locations) has been described and applied to a circular drain near the surface of a saturated medium. The method has also been applied to the analysis of groundwater flow around two underground oil storage chambers and for the determination of the optimum location of nearby water filled tunnels.

The boundary integral methods for elastostatics and groundwater seepage have been shown to be efficient and practical tools in the analysis of geotechnical problems. Therefore the methods do warrant further attention in order to broaden their scope and improve their efficiency. Recommendations for future work are the inclusion of numerical integration and curved boundary elements, a formulation to solve transient heat flow problems by a suitable transformation, the application to dynamic problems and extension of the method to other non-linear behaviour.

The quasi-rigid block method was shown to model the behaviour of blocky mass systems effectively. The method was applied to assessing the stability of slopes and underground excavations for which problems it is an ideal technique. The coupling of the finite element method for stress calculations was demonstrated by a long thin block under simple loading conditions. Recommended for future work are the inclusion of a cracking criterion and more realistic interaction relationships. The method is suitable for the analysis of the cavability of a hangingwall and drawing of broken ore.

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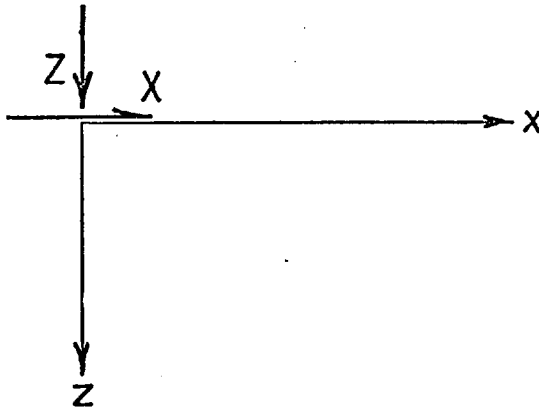
APPENDIX A.2A 2.2 - Stresses and displacements due to line forces in infinite isotropic mass.

Figure A 2.2.1 Line forces in infinite mass.

$$\sigma_x = \frac{1}{4\pi r(1-\nu)} \left\{ X \left[(3-2\nu) \frac{x}{r} - \frac{2xz^2}{r^3} \right] + Z \left[\frac{2zx^2}{r^3} - (1-2\nu) \frac{z}{r} \right] \right\}$$

$$\sigma_z = \frac{1}{4\pi r(1-\nu)} \left\{ X \left[\frac{2xz^2}{r^3} - (1-2\nu) \frac{x}{r} \right] + Z \left[\frac{2zx^2}{r^3} - (1-2\nu) \frac{z}{r} \right] \right\}$$

$$\tau_{zx} = \frac{1}{4\pi r(1-\nu)} \left\{ X \left[(1-2\nu) \frac{z}{r} + \frac{2zx^2}{r^3} \right] + Z \left[(1-2\nu) \frac{x}{r} + \frac{2xz^2}{r^3} \right] \right\}$$

... (A 2.2.1.)

$$u_x = \frac{1}{8\pi(1-\nu)G} \left\{ -X \left[(3-4\nu) \ln r + \frac{z^2}{r^2} \right] + Z \left[\frac{xz}{r^2} \right] \right\}$$

$$u_z = \frac{1}{8\pi(1-\nu)G} \left\{ X \left[\frac{xz}{r^2} \right] - Z \left[(3-4\nu) \ln r + \frac{x^2}{r^2} \right] \right\}$$

A. 2.6 - Equations for the calculation of stresses and displacements at interior points.

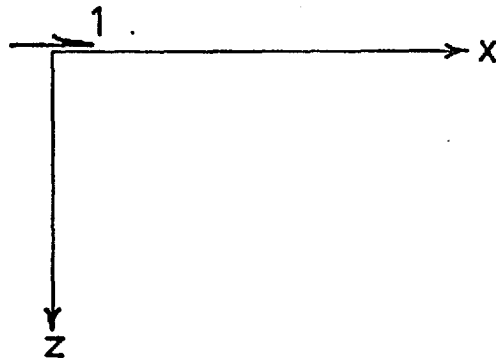


Figure A 2.6.1 Unit line force in the positive x direction.

$$\sigma_x = \frac{1}{4\pi(1-\nu)} \left\{ \frac{(3-2\nu)x}{r^2} - \frac{2xz^2}{r^4} \right\}$$

$$\sigma_z = \frac{1}{4\pi(1-\nu)} \left\{ \frac{2xz^2}{r^4} - \frac{(1-2\nu)x}{r^2} \right\}$$

$$\tau_{zx} = \frac{1}{4\pi(1-\nu)} \left\{ \frac{(1-2\nu)z}{r^2} + \frac{2zx^2}{r^4} \right\}$$

.... (A 2.6.1.)

$$u_x = \frac{-1}{8\pi(1-\nu) G} \left\{ (3-4\nu) \ln r + \frac{z^2}{r^2} \right\}$$

$$u_z = \frac{1}{8\pi(1-\nu) G} \left\{ \frac{xz}{r^2} \right\}$$

$$\frac{\partial \sigma_x}{\partial x} = \frac{1}{4\pi(1-\nu)} \left\{ \frac{(3-2\nu)\underline{1}}{r^2} - \frac{2(3-2\nu)\underline{x}^2}{r^4} - \frac{2\underline{z}^2}{r^4} + \frac{8\underline{x}^2\underline{z}^2}{r^6} \right\}$$

$$\frac{\partial \sigma_x}{\partial x} = \frac{1}{4\pi(1-\nu)} \left\{ \frac{2\underline{z}^2}{r^4} - \frac{8\underline{x}^2\underline{z}^2}{r^6} - (1-2\nu) \frac{\underline{1}}{r^2} + 2(1-2\nu) \frac{\underline{x}^2}{r^4} \right\}$$

$$\frac{\partial \tau_{zx}}{\partial x} = \frac{1}{4\pi(1-\nu)} \left\{ \frac{-2(1-2\nu)\underline{x}\underline{z}}{r^4} + \frac{4\underline{x}\underline{z}}{r^4} - \frac{8\underline{x}^3\underline{z}}{r^6} \right\}$$

... (A 2.6.2.)

$$\frac{\partial u_x}{\partial x} = \frac{-1}{8\pi(1-\nu)G} \left\{ \frac{(3-4\nu)\underline{x}}{r^2} - \frac{2\underline{x}\underline{z}^2}{r^4} \right\}$$

$$\frac{\partial u_z}{\partial x} = \frac{1}{8\pi(1-\nu)G} \left\{ \frac{\underline{z}}{r^2} - \frac{2\underline{x}^2\underline{z}}{r^4} \right\}$$

$$\frac{\partial \sigma_x}{\partial z} = \frac{1}{4\pi(1-\nu)} \left\{ \frac{-2(3-2\nu)\underline{x}\underline{z}}{r^4} - \frac{4\underline{x}\underline{z}}{r^4} + \frac{8\underline{x}\underline{z}^3}{r^6} \right\}$$

$$\frac{\partial \sigma_z}{\partial z} = \frac{1}{4\pi(1-\nu)} \left\{ \frac{4\underline{x}\underline{z}}{r^4} - \frac{8\underline{x}\underline{z}^3}{r^6} + 2(1-2\nu) \frac{\underline{x}\underline{z}}{r^4} \right\}$$

$$\frac{\partial \tau_{zx}}{\partial z} = \frac{1}{4\pi(1-\nu)} \left\{ \frac{(1-2\nu)\underline{1}}{r^2} - \frac{2(1-2\nu)\underline{z}^2}{r^4} + \frac{2\underline{x}^2}{r^4} - \frac{8\underline{z}^2\underline{x}^2}{r^6} \right\}$$

... (A 2.6.3.)

$$\frac{\partial u_x}{\partial z} = \frac{-1}{8\pi(1-\nu)G} \left\{ \frac{(3-4\nu)\underline{z}}{r^2} + \frac{2\underline{z}}{r^2} - \frac{2\underline{z}^3}{r^4} \right\}$$

$$\frac{\partial u_z}{\partial z} = \frac{1}{8\pi(1-\nu)G} \left\{ \frac{\underline{x}}{r^2} - \frac{2\underline{x}\underline{z}^2}{r^4} \right\}$$

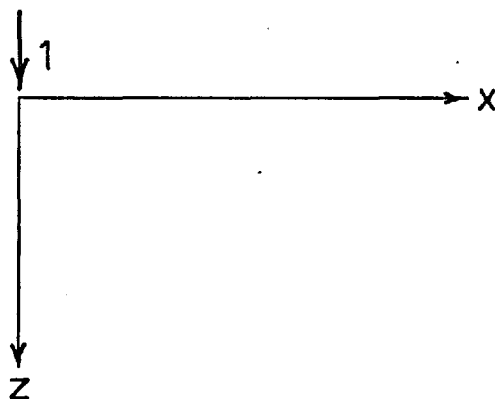


Figure A 2.6.2 Unit line force in the positive z direction.

$$\sigma_x = \frac{1}{4\pi(1-\nu)} \left\{ \frac{2x^2z}{r^4} - (1-2\nu) \frac{z}{r^2} \right\}$$

$$\sigma_z = \frac{1}{4\pi(1-\nu)} \left\{ (3-2\nu) \frac{z}{r^2} - \frac{2xz^2}{r^4} \right\}$$

$$\tau_{zx} = \frac{1}{4\pi(1-\nu)} \left\{ (1-2\nu) \frac{x}{r^2} + \frac{2xz^2}{r^4} \right\} \quad \dots (A 2.6.4.)$$

$$u_x = \frac{1}{8\pi(1-\nu)G} \left\{ \frac{xz}{r^2} \right\}$$

$$u_z = \frac{1}{8\pi(1-\nu)G} \left\{ (3-4\nu) \ln r + \frac{x^2}{r^2} \right\}$$

$$\frac{\partial u_x}{\partial z} = \frac{1}{4\pi(1-\nu)} \left\{ \frac{2x^2}{r^4} - \frac{8z^2x^2}{r^6} - (1-2\nu) \frac{1}{r^2} + 2(1-2\nu) \frac{z^2}{r^4} \right\}$$

$$\frac{\partial u_z}{\partial z} = \frac{1}{4\pi(1-\nu)} \left\{ (3-2\nu) \frac{1}{r^2} - 2(3-2\nu) \frac{z^2}{r^4} - \frac{2xz^2}{r^4} + \frac{8z^2x^2}{r^6} \right\}$$

$$\frac{\partial \tau_{zx}}{\partial z} = \frac{1}{4\pi(1-\nu)} \left\{ -2(1-2\nu) \frac{xz}{r^4} + \frac{4xz}{r^4} - \frac{8xz^3}{r^6} \right\}$$

... (A 2.6.5.)

$$\frac{\partial u_x}{\partial z} = \frac{1}{8\pi(1-\nu)G} \left\{ \frac{x}{r^2} - \frac{2xz^2}{r^4} \right\}$$

$$\frac{\partial u_z}{\partial z} = \frac{-1}{8\pi(1-\nu)G} \left\{ \frac{(3-4\nu)z}{r^2} - \frac{2x^2z}{r^4} \right\}$$

$$\frac{\partial \sigma_x}{\partial x} = \frac{1}{4\pi(1-\nu)} \left\{ \frac{4xz}{r^4} - \frac{8x^3z}{r^6} + \frac{2(1-2\nu)xz}{r^4} \right\}$$

$$\frac{\partial \sigma_z}{\partial x} = \frac{1}{4\pi(1-\nu)} \left\{ -\frac{2(3-2\nu)xz}{r^4} - \frac{4xz}{r^4} + \frac{8x^3z}{r^6} \right\}$$

$$\frac{\partial \tau_{zx}}{\partial x} = \frac{1}{4\pi(1-\nu)} \left\{ \frac{(1-2\nu)1}{r^2} - \frac{2(1-2\nu)x^2}{r^4} + \frac{2z^2}{r^4} - \frac{8x^2z^2}{r^6} \right\}$$

... (A 2.6.6.)

$$\frac{\partial u_x}{\partial x} = \frac{1}{8\pi(1-\nu)G} \left\{ \frac{z}{r^2} - \frac{2x^2z}{r^4} \right\}$$

$$\frac{\partial u_z}{\partial x} = \frac{-1}{8\pi(1-\nu)G} \left\{ \frac{(3-4\nu)x}{r^2} - \frac{2x^3}{r^2} - \frac{2x^3}{r^4} \right\}$$

$$\sigma_x = \frac{-(1-2\nu)}{4\pi(1-\nu)^2} \left[\frac{x}{r^2} - \frac{2z^2x}{r^4} \right]_1^2$$

$$\sigma_z = \frac{-(1-2\nu)}{4\pi(1-\nu)^2} \left[\frac{x}{r^2} + \frac{2z^2x}{r^4} \right]_1^2$$

$$\tau_{zx} = \frac{-(1-2\nu)}{4\pi(1-\nu)^2} \left[\frac{z}{r^2} - \frac{2z^3}{r^4} \right]_1^2$$

... (A 2.7.2.)

$$u_x = \frac{(1-2\nu)}{8\pi(1-\nu)^2G} \left[(1-2\nu) \ln r + \frac{z^2}{r^2} \right]_1^2$$

$$u_z = \frac{(1-2\nu)}{8\pi(1-\nu)^2G} \left[(1-2\nu)\theta - \frac{z^2}{r^2} \right]_1^2$$

A 2.8 - Equations for infinitesimal slip analysis

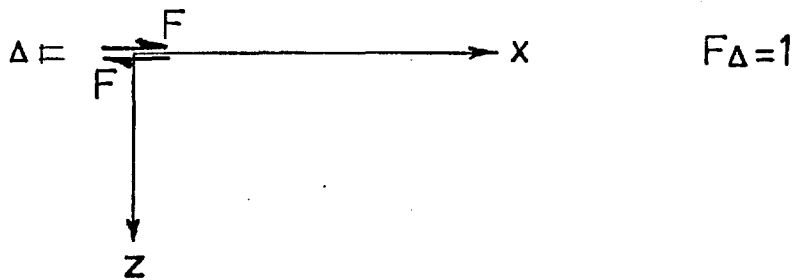


Figure A 2.8.1 Unit shear dipole.

$$\sigma_x = \frac{-1}{4\pi(1-\nu)} \left\{ -2(5-2\nu) \frac{xz}{r^4} + \frac{8xz^3}{r^6} \right\}$$

$$\sigma_z = \frac{-1}{4\pi(1-\nu)} \left\{ 2(3-2\nu) \frac{xz}{r^4} - \frac{8xz^3}{r^6} \right\}$$

$$\tau_{zx} = \frac{-1}{4\pi(1-\nu)} \left\{ (1-2\nu) \frac{1}{r^2} - 2(1-2\nu) \frac{z^2}{r^4} - \frac{8x^2z^2}{r^6} + \frac{2x^2}{r^4} \right\}$$

$$u_x = \frac{1}{8\pi(1-\nu)G} \left\{ (5-4\nu) \frac{z}{r^2} - \frac{2z^3}{r^4} \right\} \quad \dots \text{ (A 2.8.1.)}$$

$$u_z = \frac{-1}{8\pi(1-\nu)G} \left\{ \frac{x}{r^2} - \frac{2xz^2}{r^4} \right\}$$

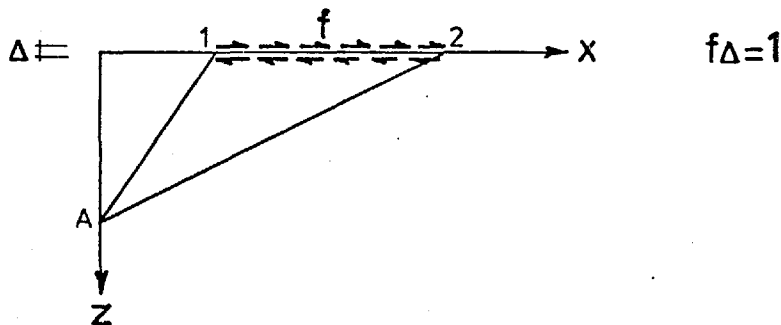


Figure A 2.8.2 Stresses and displacements at point A, due to unit constant shear dipole.

$$\sigma_x = \frac{-1}{4\pi(1-\nu)} \left[(5-2\nu) \frac{z}{r^2} - \frac{2z^3}{r^4} \right]_1^2$$

$$\sigma_z = \frac{-1}{4\pi(1-\nu)} \left[-(3-2\nu) \frac{z}{r^2} + \frac{2z^3}{r^4} \right]_1^2$$

$$\tau_{zx} = \frac{-1}{4\pi(1-\nu)} \left[-(3-2\nu) \frac{x}{r^2} + \frac{2xz^2}{r^4} \right]_1^2$$

... (A 2.8.2.)

$$u_x = \frac{1}{8\pi(1-\nu)G} \left[4(1-\nu)\theta - \frac{xz}{r^2} \right]_1^2$$

$$u_z = \frac{-1}{8\pi(1-\nu)G} \left[\ln r + \frac{z^2}{r^2} \right]_1^2$$

APPENDIX A.4

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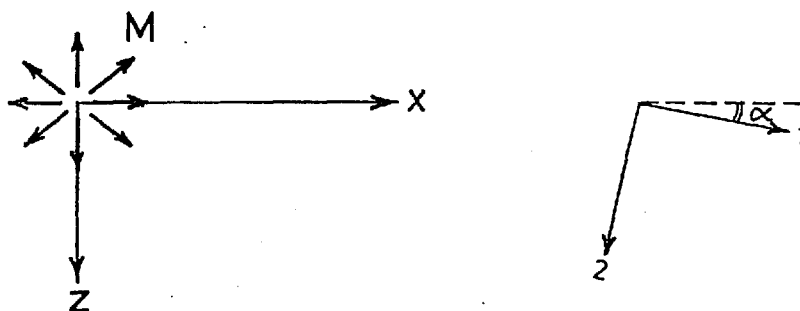
A 4.3 - Equations for a line source in an infinite medium

Figure A 4.3.1 Line source in an infinite medium

$$\phi = \frac{M}{2\pi} \ln r$$

$$\frac{-\partial\phi}{\partial x} = -\frac{M}{2\pi} \frac{x}{r^2} \quad \dots (A 4.3.1.)$$

$$\frac{-\partial\phi}{\partial z} = -\frac{M}{2\pi} \frac{z}{r^2}$$

$$\begin{aligned} k_{xx} &= k_1 \cos^2 \alpha + k_2 \sin^2 \alpha \\ k_{zz} &= k_1 \sin^2 \alpha + k_2 \cos^2 \alpha \\ k_{zx} &= k_{xz} = (k_1 - k_2) \sin \alpha \cos \alpha \end{aligned} \quad \dots (A 4.3.2.)$$

$$v_x = -k_{xx} \frac{\partial\phi}{\partial x} - k_{xz} \frac{\partial\phi}{\partial z} \quad \dots (A 4.3.3.)$$

$$v_z = -k_{zx} \frac{\partial\phi}{\partial x} - k_{zz} \frac{\partial\phi}{\partial z}$$

Where 1 and 2 are principal permeability directions with k_1 and k_2 permeabilities respectively.