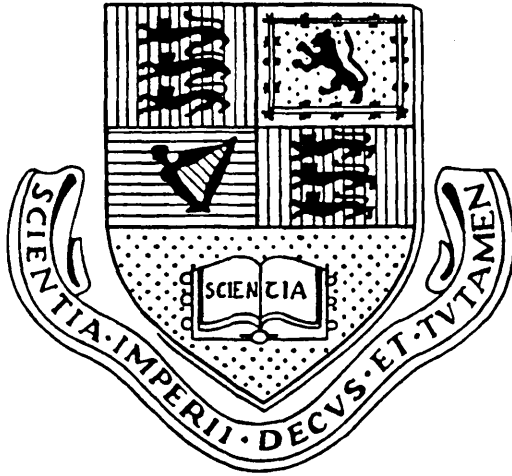




EFFECT OF DYNAMIC RESPONSE ON IMPACT DAMAGE

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ABSTRACT

A theoretical-experimental investigation into the dynamic response of beams subject to transverse, low velocity impact has been conducted, aimed at developing a simple parametric model to aid in the design of structural components under impact conditions.

A uniform *Bernoulli-Euler* beam with an attached impactor mass has formed the basis of a parametric study which has identified the parameters that govern dynamic response. The effect of the individual parameters on the beam's response has been examined by adopting a modal synthesis solution technique. Results of this study have indicated that the maximum bending stress can be predicted using a simple energy formulation.

A comprehensive finite element program has been implemented to validate the modal analysis, and to facilitate the inclusion of shear and rotary inertia effects and the prediction of impact-induced shear stresses, not possible using the modal approach. Contact loadings are also predicted by modelling the multiple bouncing of impactor on the beam. A specially developed high-order *Mindlin* beam element has been found to be ideally suited for dynamic response calculations where complex shear and bending stress variations are encountered.

Drop-weight impact tests have been conducted on a simply-supported, aluminium beam specimen, varying the main governing parameters, namely impactor mass, impactor velocity and beam length. Bending and shear stresses, displacement and contact loading have all been monitored and recorded. Low velocity impact tests have also been conducted on composite laminates to establish threshold energy levels to initiate damage and to monitor the resulting bending stresses and contact forces.

Experimental results show good correlation with theoretical predictions and highlight the effectiveness of the proposed models for estimating impact-induced stresses. The implications for scaling of response and various factors that limit the range of applicability of the proposed models have also been discussed.

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NOTATIONS

a	Arbitrary constant. Defined where used
$\mathbf{a}$	Matrix of constants (a). Defined where used
b	Breadth (width) of beam
c	Wave propagation velocity, $(= \sqrt{\frac{E}{\rho}})$
c_e	Effective wave propagation velocity in composites, $(= \sqrt{\frac{E_e}{\rho_e}})$
d	Depth of beam
$\mathbf{d}$	Matrix of strain-displacement operators
e_0	<i>Belleville</i> disc-washer height
$\mathbf{e}$	Unit vector
f	Arbitrary function. Defined where used
$\mathbf{f}$	Vector of functions (f)
g	Index referring to layers in a composite
h	Arbitrary function. Defined where used
$\mathbf{h}$	Vector of functions (h)
i	Index
j	Index
k	Coefficient of <i>Hertzian</i> contact law
k_e	Effective cross-sectional shear correction coefficient
k_w	Coefficient for the shear variations across the beam width
l	Half length of beam
ℓ	Total length of beam, $(= 2l)$
ℓ_c	Distance from beam support to element centre
m	Impactor mass
n	Number of harmonics
p	Numerically determined exponent of impactor/beam mass ratio
q	Number of element sampling points

r	Nodal displacement
$\mathbf{r}$	Vector of nodal displacements
t	Time
u	Transverse displacement
$\mathbf{u}$	Displacement field
v	Impactor velocity
w	Distributed loading function
w_c	Amplitude of distributed load
$\mathbf{w}$	External nodal load vector
x	Cartesian coordinate along the beam length
y	Cartesian coordinate along the beam depth
z	Cartesian coordinate along the beam width
A	Cross-sectional area
$\mathbf{A}$	Stress smoothing matrix
$\mathbf{B}$	Vector of coefficients. Defined where used
C	Damping coefficient
$\mathbf{C}$	Damping matrix
$\mathbf{D}$	Elasticity matrix
E	Modulus of elasticity
E_e	Effective modulus of elasticity for composites, $(= \sum_{g=1}^r \frac{E_g I_g}{I})$
$\mathbf{F}$	Matrix of functions (f)
G	Shear modulus
H	Arbitrary constant. Defined where used
$\mathbf{H}$	Matrix of constants (H)
I	Second moment of area
$\mathbf{I}$	Iteration matrix
J	Impact energy
K	Stiffness

K^*	Modified stiffness, $(= \frac{1}{2}[K(t) + K(t + \Delta t)])$
$\mathbf{K}$	Stiffness matrix
L	Element length
M	Beam mass
M_a	Additional mass due to displacement flag and protecting channel
$\mathbf{M}$	Mass matrix
N	Number of nodal freedoms
P	Point force
Q	Bending moment
R	Nodal force
$\mathbf{R}$	Nodal force vector
S	Shear force
T	Period of vibration
$T_{\min}$	Period of highest frequency
U	Strain energy
V	Volume
W	Kinetic energy
α	Constant defined as $\frac{\pi \ell_c}{L}$
$\boldsymbol{\alpha}$	Matrix of strain shape functions, $(= \mathbf{d}\omega)$
β	Constant defined as $\frac{\pi \ell}{2L}$
γ	Arbitrary constant. Defined where used
δ	Local indentation due to contact
ϵ	Allowable error
ε	Strain
ε_{xx}	Direct strain
ε_{xy}	Shear strain
$\boldsymbol{\varepsilon}$	Strain matrix
ζ	Dimensionless time interval, $(= \frac{t}{\tau})$
η	<i>Hermitian</i> polynomial

θ	Cross-sectional rotation due to pure bending, $(= \frac{du}{dx})$
κ	Radius of gyration, $(= \sqrt{\frac{I}{A}})$
λ	Beam frequency parameter, where $\lambda^4 = \frac{\Omega^2 \rho A}{EI}$
μ	Constant defined as $\frac{EI}{GA}$
ν	Poisson's ratio
ξ	Local non-dimensional element coordinate, $(= \frac{2x}{L})$
π	Universal constant, $(= 3.1415926536 \dots)$
ρ	Mass density
ρ_e	Effective mass density for composites, $(= \sum_{g=1}^r \frac{\rho_g A_g}{A})$
σ	Stress
σ_{xx}	Direct stress
σ	Stress matrix
τ	Time interval
ϕ	Shear/bending stiffness parameter, $(= \frac{12EI}{GAk_e l^2})$
χ	Arbitrary function. Defined where used
ψ	Cross-sectional rotation due to shear
ω	Shape function
ω	Matrix of element shape functions
Γ	Number of layers in a composite
Λ	Arbitrary function. Defined where used
Φ	Dynamic response function for bending. Defined where used
Ψ	Dynamic response function for shear. Defined where used
Ω	Circular frequency in radians per second

SUBSCRIPTS

e	Effective
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g	Index referring to layers in a composite
i	Index
j	Index
n	Nodal point
q	Sampling point
u	Transverse displacement

B	Bending
C	Damping
E	Exact integration
F	Failure
I	Impactor
K	Stiffness
M	Mass
R	Reduced integration
S	Shear
T	Target

θ	Rotation
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cr	Critical
dyn	Dynamic
min	Minimum
max	Maximum
qst	Quasi-static
sym	Symmetrical

OPERATORS

$[]$	A row or column vector, or a rectangular or square matrix
$\{ \}$	A column vector, when written horizontally to save space
$^{-1}$	Inverse of a square matrix
^{t}	Transpose of a matrix
$\cdot$	Time differentiation, first derivative
$\ddot{}$	Time differentiation, second derivative
—	Virtual quantity
$\sim$	Smoothed quantity
$\frown$	Intermediate value of a quantity during iteration
Δ	Small change in a given quantity
∂	Partial derivative

ABBREVIATIONS

CFRP	Carbon fibre reinforced plastic
CH	Channel
CHIA	Cubic interpolation of inertia algorithm
CT	Composite specimen test case
DC	Direct current
DOF	Degree of freedom
ETB	Engineer's theory of bending
FE	Finite element
MT	Metal specimen test case
SGA	Strain gauge amplifier
TDR	Transient data recorder

UNITS

m (or M)	Metre (unit of length)
cm	Centimetre ($= 10^{-2}$ m)
mm	Millimetre ($= 10^{-3}$ m)
psi	Pound per square inch (unit of pressure $\approx 6895\text{N/m}^2$)
s (or S)	Second (unit of time)
ms (or MS)	Milliseconds ($= 10^{-3}$ s)
C	Celsius (unit of temperature)
Kg	Kilogramme (unit of mass)
Kgf (or KGF)	Kilogramme force (≈ 9.8 N)
N	Newton (unit of force $\approx 0.102\text{Kg m/s}^2$)
MN	Meganewton ($= 10^6$ N)
GN	Giganewton ($= 10^9$ N)
Hz (or HZ)	Hertz (unit of frequency = 1 cycle per second)
KHz	Kilohertz ($= 10^3$ Hz)
MHz	Megahertz ($= 10^6$ Hz)
V	Volt (unit of electrical potential)
Ω^*	Ohm (unit of electical resistance)

Chapter One

INTRODUCTION

1.1 GENERAL OVERVIEW

The past decade has witnessed an increasing activity in aeronautics in the field of impact and impact related phenomena. This newfound interest originates primarily from the increasing importance that has been placed on the use of fibre composites as structural material in aircraft, helicopters, and aero-engines. Although composite materials have been around for quite a long time, their use has been restricted to secondary structural components. Extensive research into various aspects of behaviour and mechanical properties has given confidence to the designer to take advantage of what these non-homogeneous materials have to offer and the aerospace industry has been among the first to appreciate the potential of composite materials as structural components. The economic advantages of such high-strength lightweight materials and the flexibility they offer to designers for "tailor-made" material properties have lead to their increasing use for primary load-carrying structures. In service, such components may be subjected to impact loadings arising from dropping of tools or runway stones which can be classified as low velocity impact and also engine debris and small arms fire which constitute high velocity impact. The impact energy transferred to the target can cause it to deform elastically as well as inelastically. Metals can deform plastically and retain much of their original strength up to the point of failure. Composites, in

comparisons, can absorb energy inelastically only by fracture processes which inherently will lead to a degradation of their mechanical properties. Low velocity impact in particular, can cause sub-surface damage in laminates which can result in reduction of both strength and stiffness without any sign of visible damage.

Many investigators, adopting a variety of different approaches, have contributed to the study of impact but much of this work has been related to ballistic type impact, where wave phenomenon and *Hertzian* contact effects are dominant due to the high velocities involved. These efforts have provided a basis for a design manual as compiled by AVERY (1).

A survey of the literature reveals that the low velocity aspects of impact have received considerably less attention, with much of the interest being directed towards the understanding of the material response of fibre composite targets at the contact zone. The inherent complexity associated with the analysis of composite laminates coupled with complex dynamic effects that form part of the impact process have diverted the research efforts away from the analytical approach in favour of the experimentalists. DOREY and his associates (2-7), among others, have concentrated much of their efforts on studying the degradation of mechanical properties of impacted composite laminates, which is regarded as being of primary importance to the designer. Considering the projectile kinetic energy to be the predominant response parameter, they proceed to perform numerous impact tests on laminate specimens to establish threshold levels of impact energy, at which either the strength or stiffness is significantly reduced or sufficient damage is sustained to affect the life of the component. The various failure mechanisms commonly encountered in impact damaged laminates such as delamination, fibre cracking and matrix crushing have also been investigated with regards to their effect on the threshold energy. DOREY's efforts also entailed examination of the effect of ply orientation, stacking sequence, ply thickness as well as fibre, matrix and interface properties on the promotion of any particular mode of damage thus

affecting established threshold energies. The use of a single quantity, such as energy, as a measure of a components ability to survive damage is a simple concept which has some validity if all of the energy is absorbed by the damage process. Clearly if a sufficient reservoir of elastic energy is available, this approach may not be valid, provided the structure can respond sufficiently quickly to absorb the energy before the local impact force causes damage. This possibility depends on the stiffness of the structure, the mass of both the impactor and the structure, and the impacting velocity. In other words the dynamic response.

The dependency of impact damage on target geometry and flexibility as well as impactor mass, velocity and even impactor geometry has also been examined by a number of researchers (8). These studies have primarily concentrated on the aspects of material response to impact in an effort to provide an insight into the phenomenon of impact damage in fibrous composites and also form a valuable base from which to search for new impact resistant hybrid composites. In addition, the resulting information may provide a basis for a design guideline but falls short of providing a proper understanding of the physical concepts of impact.

Impact phenomenon involves both material response at the contact zone and the structural response in the form of stress waves, and it is the former aspect of impact which has received most of the attention. Although the dynamic structural response of laminates can be neglected for ballistic type impact, the validity of such an assumption, when applied to low velocity impact, is rather doubtful. However, it is evident that in studying the material response aspects of impact, the structural response aspects of the problem have been conveniently undermined, thus casting doubt on some of the findings. The vast majority of this work has involved drop-weight and ball gun impact tests, conducted on coupon sized specimen typically 50 mm long and 25 mm wide for reasons of economy of material. This approach is justifiable only if it can be established that the effects arising from impact loading are localised and that the dynamic response of the entire specimen is

negligible. In the view of the author, it is questionable as to whether the assumption of negligible dynamic response can safely be applied to situations involving low velocity impact. In fact, preliminary work by CANTWELL et al. (9,10) has shown that impact as well as post-impact response of carbon fibre composites is dependent on its geometry, and confirms the authors belief that the coupon tests may be excessively pessimistic. This claim has been further substantiated by HILAIRE and SOULEZELLE (11), who recently embarked on studying the influence of test methods on the damage produced by low velocity impacts on composites. They investigated the impact damage in carbon/epoxy composites as a function of specimen dimension, initial damage energy threshold, specimen deformation and projectile mass, and report that their analysis shows that the free length of specimen has a marked influence on the results. Another assumption subject to criticism is that of regarding the impact energy as being the predominant parameter. Based on their findings, CANTWELL et al. (9) argue that impact energy is not the sole parameter and that impact performance is also affected by projectile mass, momentum and the target response.

The finite element technique forms the basis of much of the analytical effort on the subject of impact. A complete analysis of the impact phenomenon will involve local contact stresses, three-dimensional stress wave propagation, transient structural response, failure theories and even fracture mechanics. The finite element (FE) method is very versatile and well suited for complex analysis, not possible by other analytical methods. It permits detail modelling of the contact zone where the stress state is predominantly three-dimensional. Non-linearities in material properties, resulting from the progression of impact damage and geometric non-linearities arising from large local deformations can also be incorporated. The drawbacks include the vast computer resources necessary to perform such complicated analysis, bearing in mind that the dynamic equations of motion have to be solved iteratively for every timestep and that the complete analysis will require many thousands of timesteps, especially for the case of low velocity impact

where peak stresses which may lead to failure may not occur until sometime after the initial impact. Finally, the results of the analysis have to be carefully interpreted in view of the short-comings of the modelled mesh and that of the element type that has been employed.

GRESZCZUK (12) studied the impact response of both isotropic and composite plates by using the FE computer solution to determine the internal stresses in the target caused by a time-dependant surface pressure distribution under the impactor. This pressure distribution he obtained by analytically combining the dynamic solution of the problem of impact of bodies with the static solution for the pressure between two bodies in contact. Details of various analytical methods for estimating the contact pressure distribution, contact area and duration of contact for both isotropic and composite targets are presented in Reference (13). The internal stress distribution will enable the determination of failure modes and the failure envelope by applying the appropriate failure criteria. Currently static failure criteria are assumed to apply since an experimentally verified dynamic failure criteria does not exist. GRESZCZUK, in collaboration with CHAO (14), later applied this solution technique to investigate the parameters that influence material response and thus control the type and spreading of impact damage. The approach outlined by GRESZCZUK (12) represents a common basis for analytical efforts involving the finite element concept and so the emphasis is put on developing simpler solution techniques, improving the modelling of the contact loading and developing special elements to model the variety of structural effects arising from the non-homogeneity of the target material, in an effort to reduce solution cost and thus render the FE method a viable alternative.

HUMPHREYS (15) attempts to develop an analytical procedure to calculate damage accumulation in composites during low velocity impact. The procedure incorporates damage, as it is predicted, into the displacement solution. Assuming a perfectly plastic impact, the impacting mass was lumped and subjected to an initial velocity. The mesh was modelled by thin shell elements and laminate theory

provided the stress distribution. Predicted damage was found to be far in excess of experimental findings. To improve the predictive capabilities, HUMPHREYS and GOERING (16), employed a shear flexible plate bending element with material bending/extensional coupling facilities and also incorporated a non-linear spring to simulate *Hertzian* contact.

SUN and his associates (17,18) have also applied similar techniques to study laminate response and damage initiation with varying degree of success.

These examples serve to highlight the difficulties in the use of such analytical techniques and although detailed analysis is possible, one has to bear in mind that the complexity of the solution may hide useful information concerning the general underlying physical phenomena of interest to the designer and furthermore the expense involved in the use of such methods may not be justifiable under normal circumstances.

Application of more conventional, analytical techniques to the impact study of laminates is rarely encountered in the literature. BOSTAPH and ELBER (19) have combined fracture mechanics and large deflection plate theory to examine the influence of plate thickness, support condition, and matrix toughness on the onset and propagation of delamination in impacted composites, while CHOW (20) has derived the dynamic equations of orthotropic laminated plates and applied these to study the propagation of flexural waves and the transient response resulting from a normal impact on composite plates. Analytical solutions of this kind have been used extensively in past for studying impact. Works by ERINGEN (21), LEE (22), ZENER and FESHBACH (23) and BARNHARDT and GOLDSMITH (24) on transversely impacted isotropic beams represent some of the earlier contributions to this subject. Their main aim was concerned with developing methodologies to predict contact force, target response and impact induced stresses and the approach most favoured was based on solving the equations of beam vibration, representing dynamic effects, in conjunction with force-indentation laws, representing

contact effects. A detailed account of the various analysis methods applied to the study of impact is contained in GOLDSMITH (25).

Few researchers have attempted to investigate the subject of low velocity impact with the aim to provide an understanding of the underlying physical phenomena. McQUILLEN, GAUSE and LLORENS (26) have reported that low velocity, transverse impact is primarily a structural dynamic response phenomena, at least to 6 m/s which indicates that inplane and transverse wave effects are not important. They found no detectable damage or measurable reduction in residual strength due to impacts below failure level, which is an indication that local *Hertzian* contact effects are of secondary importance. Tests showed that dynamic failure mechanism was identical to the static failure mechanism (i.e. longitudinal tension in the 0° plies) and also that impact energy was not the only factor controlling failure. Based on their observations, McQUILLEN et al. (26) proposed a structural dynamic response model based on a uniform *Bernoulli-Euler* beam and a rigid impactor mass which provided good correlation with their experimental results. In an experimental investigation, ELBER (27) has found that under low velocity impact, the dominant deformation mode of thin composite plates is the first, or static, mode and that the extent of delamination as well as distribution and amount of fibre damage were similar for impact and static tests. These findings are in agreement with those reported by McQUILLEN et al. (26). CAPRINO, CRIVELLI VISCONTI and DI ILIO (28,29) have also observed that the dynamic behaviour of composites at low impact velocities can be predicted by simple static tests. They have considered situations where impactor mass is large in comparison with target mass, so that structural response is not significantly affected by target inertia. As a result, a simple model, based on energy considerations, was proposed for the purpose of predicting the maximum contact force during low velocity impact between an impactor and a composite plate.

1.2 PROJECT OBJECTIVES

The findings reported in References (9,10,26–29) are significant in that these highlight the importance of structural response in situations involving low velocity impact. This aspect of the problem has generally been neglected while attention has been focused on aspects of material response. It is essential to realise that if the results of experimental tests are to be applied to real structures, the effects of structural response have to be considered and included. Identification of the important — hopefully non-dimensional — *parameters* which govern the dynamic response of structural elements to impact by a moving mass would be of valuable assistance to the designer. If these parameters can be summarised then it will enable the result of coupon tests to be extrapolated to other structures with different lengths, thicknesses, densities, rigidities and lay-ups and impacted with different masses and with different velocities. The results of coupon tests — such as residual strength — are usually quoted in terms of projectile energy and it is therefore relevant to be able to quote the separate effects of mass and velocity and how to apply an energy criterion to other structural configurations.

The objective of the present study has been to investigate the structural response aspects of impact with the aim of identifying the governing factors to serve as effective indicators as to the importance of dynamic effects. For this purpose, attention has been focused on the problem of transversely impacted isotropic beam and this has been studied in detail, both numerically and experimentally.

To begin with, a detailed numerical investigation has been carried out on a uniform isotropic beam of rectangular cross-section, simply supported at the ends with a concentrated mass at the centre. The differential equation for the lateral vibration of this particular beam, excluding effects of shear and rotary inertia, has been solved subject to an initial input velocity imparted to the centre mass. In the process a number of variables which affect the lateral vibration of the beam have been identified. These have been further examined to establish the

exact nature of their contribution to the beam's response, thus highlighting the important variables or group of variables. On the basis of this numerical study a theoretical model has been proposed which relates the maximum impact induced stresses to the various governing parameters.

The numerical study has been taken a step further by employing a specially developed finite element package to overcome some of the limitations of the above simplified approach by providing a capability for a more detailed and realistic modelling and analysis. Spring elements have been utilised for the modelling of local contact and estimation of contact loads, including the multiple bouncing of the impactor which has been observed experimentally. High order *Mindlin* beam elements have also been implemented to allow the prediction of transverse shear stresses, for which the modal approach and the standard beam elements have proven unsuitable.

To complement the numerical investigation, a series of dropweight impact experiments have been performed to cover a wide range of test configurations, aimed at providing a sufficiently comprehensive experimental database to facilitate detailed comparisons with numerical findings. In this respect, attention has been focused on highlighting the separate as well as combined effects of the various parameters identified by the numerical study. The impact tests have been conducted on simply-supported aluminium and composite beam specimens, monitoring and recording the contact forces, the bending stresses at various positions along the specimen and, where appropriate, the transverse shear stresses and the centre point deflections.

Finally, the numerical and experimental results have been compared with a view to confirm the relevance of the identified parameters and to establish the applicability of the proposed models for predicting the impact-induced stresses. Implications of the findings, in particular the geometrical factors relating to scaling of the test results to different structural configurations, have been considered and,

furthermore, any limitations that may arise from the dominance of secondary effects have been discussed and put into perspective.

Chapter Two

NUMERICAL ANALYSIS (I) :

Parametric Study

2.1 INTRODUCTION

In the past, investigations relating to the subject of impact behaviour of isotropic targets (25) and more recently of composite targets (9,10,26–29) point to the fact that low velocity impact is a predominantly a structural response phenomenon. It seems appropriate that the parameters that govern the dynamic response be identified and the nature of their contribution be examined. A study of this nature does not appear to have been previously undertaken and would contribute towards a better understanding of the underlying physical concepts of impact.

Research into the structural response aspects of impact has been extensive as regards isotropic targets, and while experimental efforts (35) have considered the influence of variables, such as impactor mass and velocity, on target response, analytical studies have concentrated on devising and improving methods for predicting the contact force, target deflection and induced stresses as a function of time — generally by combining dynamic effects with local contact effects (21–25).

The problem of a simply-supported, isotropic beam of rectangular cross-section, transversely impacted at the centre by a rigid striker has been tackled by numerous authors and forms the basis of the current parametric study.

The most simplified approach to dynamic response is to replace the beam by a massless linear spring with the appropriate stiffness representing the static force required to produce unit transverse beam deflection. The maximum deflection can then be estimated by equating the initial kinetic energy of the impactor to the maximum strain energy stored by the spring at the instant of maximum dynamic deflection. To include the inertia effects of the beam, an equivalent beam mass can be estimated based on the assumption that the dynamic deflection mode of the beam is geometrically similar to the static mode. This approximation can be justified for impact involving a heavy striker on a light mass and leads to the Cox equation (see GOLDSMITH (25)). For impact of light striker on a heavy beam there may be high local curvature even before the maximum displacement is reached and the above simple approach is no longer valid. To realise these more complex local deformations, the higher harmonic responses must also be considered and this is possible when the transverse impact is analysed on the basis of beam vibrations.

For the purpose of the present numerical study, a linear modal synthesis approach has been adopted utilising the *Bernoulli-Euler* equation for the transverse vibration of beams, and so it had to be assumed that the beam remains elastic, and that deflections are small compared to the span.

2.2 LIMITATIONS OF ANALYSIS

An exact analysis is extremely complicated and the complexity of the solution may hide the useful information which the numerical study is designed to extract. It is imperative that simplifications be carefully introduced in order that the problem can be solved in a practical fashion.

In the immediate vicinity of the impact point, the transient stress state is highly three-dimensional in character due to two separate effects, namely stress

waves and contact loading. The propagation and reflection of stress waves will have significant contributions only if the impact velocity (v) is of the order of the wave velocity (c) or if the rise time is of the order of $(\frac{d}{c})$, (d) being the beam depth. In contrast, while the impactor and beam are in contact, there exists a local three-dimensional stress field even under static loading conditions, but as long as the contact stresses are small in comparison with the bending stresses, these contact stresses may be ignored. Thus, subject to low velocity impact conditions, and provided that either the beam remains elastic throughout or any material response is limited to the immediate vicinity of the impact point and damage does not extend across the entire cross-section, then neglecting the three-dimensionality of the transient stress state in the impact region will not detract appreciably from the accuracy of a study of overall beam behaviour.

Away from this localised region, the stress distribution is two-dimensional and the variation of stress is primarily along the length of beam and through its depth as characterised by bending, with the distribution of stress across the width being reasonably uniform. This beam-like action extends over the entire beam, including the impact region, once contact ceases. Approximating the stress distribution within the beam as two-dimensional will reduce the problem to one of plane stress. GOLAND, WICKERSHAM and DENGLER (30) point out that an exact solution of this plane stress beam problem involves both symmetrical and anti-symmetrical cross-sectional stress distribution about the beam mid-height plane. In addition to these fundamental modes, an infinite number of higher mode classes are also present which are characterised by having more than one neutral plane, each class being associated with a certain number of such zero stress planes. By centering attention on the bending action, one may neglect the symmetrical stress system in comparison with the anti-symmetrical counterpart, but an exact theoretical solution, even of this simplified plane-stress problem, poses a great difficulty.

The next order of approximation focuses on the fundamental anti-symmetric mode and ignores all higher order modes, so that the problem is reduced to a one-dimensional analysis, relating to the stress variation along the beam length.

The *Timoshenko* equation of transverse beam vibration has been utilised by many authors (32,33) to study the impact response of beams. It differs from the elementary *Bernoulli-Euler* bending theory by introducing approximate corrections for effects of shear and rotary inertia. GOLAND et al. (30) comment that the *Timoshenko* mechanism permits excellent approximation of the transmission characteristics of the fundamental class of beam modes and affords a reasonable approximation for that of the next highest class of modes, and have confirmed the applicability of this approach by comparing predicted results with experimental data. The conclusion to be drawn is that the higher mode classes do not contribute significantly to the stresses in beams in bending.

The simplifications which have been introduced thus far imply that localised effects in the contact region cannot be properly estimated, but otherwise do not profoundly alter the overall beam response in the velocity range to be investigated.

2.3 SELECTION OF ANALYTICAL APPROACH

The *Timoshenko* method has extensively been applied in the past to the study of transverse impact on isotropic beams and its suitability has been demonstrated. The method solves the equation of forced transverse beam vibration, where the time-dependent forcing function is derived from *Hertzian* type contact law. The equation of beam vibration is derived on the basis of elementary *Bernoulli-Euler* bending theory, but may include secondary effects like the *Timoshenko* equation which introduces shear and rotary inertia effects as approximate corrections. It should be noted in passing that the presence of shear results in a beam cross-section that is not perpendicular to the neutral axis and, in fact, will be warped from the

planar configuration. The *Timoshenko* shear correction, however, is derived on the basis of retaining the fundamental concept of plane sections of the beam remaining plane but relaxing the constraint relating to the section remaining perpendicular to the neutral axis. These kinematics do not permit the warping of the cross-section and thus the shear correction has to be considered as an approximation.

The influence of these second order effects have been investigated by a number of authors, including TRAILL-NASH and COLLAR (31), BOLEY (32) and CHRISTOPHERSON (33). The effects of rotary inertia are generally considered to be negligible. Shear flexibility was shown to have a notable influence on the bending vibration of beam for cases where the sectional shear stiffness was of the same order as the bending stiffness. BOLEY (32) argues that the beam motion will contain high-frequency components in the immediate vicinity of the contact point and that effects of shear and rotary inertia are of great importance in this region. This claim follows from a study of the orders of magnitude of the various terms in the basic equation and indicates that outside a short boundary layer, immediately following the disturbance front these second order effects can be neglected. The length of this layer is of the order of the cross-sectional dimensions of the beam. On this basis, BOLEY (32) proposed an approximate method whereby the *Timoshenko* equations are used within this boundary layer, while for the remainder of the beam the *Bernoulli-Euler* theory should be adequate. The point raised by BOLEY (32) is backed by experimental evidence due to MASON (34), ARNOLD (35) and VIGNESS (36) who indicate the applicability of the *Bernoulli-Euler* theory of bending to lateral impact of beams. It therefore appears that shear and rotary inertia play a somewhat limited role as regards overall beam behaviour, though it should be remembered that the inclusion of these terms is essential to a study of wave propagation, for this elementary bending theory will otherwise predict a physically impossible result of an infinite velocity of propagation for very short waves.

Based on the above discussion, and for this Chapter's parametric study only, the contributions of shear and rotary inertia will henceforth be omitted from the analysis as well as those due to lateral contraction and internal damping to simplify the mathematical solution. The problem is thus reduced to one of obtaining a solution to the *Bernoulli-Euler* equation of forced transverse beam vibration. This is a relatively straightforward task if the time-dependent forcing function is known, but for situations involving a falling mass impacting on a beam complicated interactions arise in which mass of the impactor in relation to the beam mass, vibrations of the impacting mass, and the degree to which the two bodies would stick together due to plastic deformation would all be factors. Under these circumstances, the unknown forcing function has to be either arbitrarily chosen, for example assuming a short-duration impulse loading represented by a *Dirac Delta* function, or determined from conditions relating the displacement of the striker to that of the beam. The latter approach is far more desirable but results in an integral equation which unfortunately can not be solved in closed form, requiring numerical integration and a suitable force-indentation relation.

The solution to this problem has been attempted by numerous authors (21–24), generally employing the *Hertzian* contact law and a small-increment method to extract the contact force variation from the integral equation. The accuracy of the results and computational labour is directly dependent on the number of harmonics that have been included and the fineness of the chosen time interval. ERINGEN (21) and LEE (22) have suggested approximate solution techniques based on simplifying assumptions while BARNHARDT and GOLDSMITH (24) have devised a method to account for the effect of an infinite number of harmonics. The influence of various force-indentation laws have also been investigated and shown to affect the results.

The solution process can be considerably simplified by assuming an analytical form for the contact history. This method has been adopted by some (21–23) and it has been observed that the beam deflection is virtually insensitive to this forcing

function, but as regards the bending stress, being the second derivatives of the displacement function, serious errors may result.

An alternative approach to this forced-vibration problem that brings in some of the mass-beam interaction aspects, avoids the need to solve an integral equation and also does not require the inclusion of some sort of contact law, is to replace the equation of forced vibration by a problem in the free vibration of a beam where impacting mass effects are brought in via added boundary conditions. In other words, the striking mass is lumped at the impact node on the beam by assuming a fully plastic impact — that is striker remains in contact with beam after impact. The modified boundary conditions concern the discontinuity in the beam shear at the contact point which must now equal the inertial effects of the striker and an appropriate initial velocity condition for the beam-impactor system.

This simplified free-vibration problem yields a series solution which implies the existence of an infinite number of bending modes. However, BARNHARDT and GOLDSMITH (24) question the physical significance of the higher modes in view of the fact that the elementary bending theory does not take account of vibration modes other than those due to bending and also excludes internal damping. This theory thus fails to predict a limit to the number of modes which are physically realistic and it is necessary that a limit be imposed in order that a solution to the problem be obtained. BARNHARDT et al. (24) have devised a method to allow the contributions of the higher modes to be included, but in the current investigation the distribution of energy amongst the system's eigenfunctions is used as a guideline for setting a limit on the number of harmonics that should be considered.

It is obvious that whichever approach is adopted certain assumptions and approximations have to be made. With this in mind and realising that the objective of this study is aimed at identifying the parameters that govern the overall beam behaviour, the last approach based on free vibration analysis has been regarded as

being more appropriate and simpler to treat mathematically. It is admitted that the assumption of fully plastic impact is not realised in practice and experimental evidence by MASON (34) and ARNOLD (35) confirm the existence of multiple collision for cases involving a heavy striker on a light beam. However, for the case just mentioned, the response is dominated by the fundamental mode and maximum stress is expected to occur at around maximum deflection. For a light mass striking a heavy beam only a single collision is observed and the excitation of the higher harmonics will mean that the peak stress is likely to be reached much earlier in the response. In Chapter (3), a comparison with the finite element technique, for which a completely numerical integration with respect to time is used, will show this modal prediction approach to be quite accurate. In addition, the results of this parametric study will also be shown subsequently to justify the choice and applicability of the present analysis method.

Lastly, it should be pointed out that contact deformations, vibrations of the striker and the static deflection due to the attachment of the impactor mass are all excluded from the analysis.

2.4 BEAM-IMPACTOR FREE-VIBRATION ANALYSIS

The analysis utilises the *Bernoulli-Euler* equation of lateral beam vibration for reasons outlined in the previous discussion and the derivation of this equation can be found in standard textbooks on dynamics of continuous systems. Subject to the well-known *Bernoulli-Euler* assumptions, the governing differential equation for the free lateral vibration of a beam is given by :

$$\frac{\partial^4 u}{\partial x^4} + \frac{\rho A}{EI} \frac{\partial^2 u}{\partial t^2} = 0 \quad (2.4.1)$$

where E = material elastic modulus

ρ = material density

I = second moment of area of the cross-section

A = cross-sectional area

u = lateral beam displacement

The general solution to equation (2.4.1) gives the lateral response of the beam as :

$$u(x, t) = \sum_{i=1}^{\infty} (a_{1i} \cosh \lambda_i x + a_{2i} \sinh \lambda_i x + a_{3i} \cos \lambda_i x + a_{4i} \sin \lambda_i x) \times (a_{5i} \cos \Omega_i t + a_{6i} \sin \Omega_i t) \quad (2.4.2)$$

$$\text{where } \lambda_i^4 = \Omega_i^2 \frac{\rho A}{EI} \quad (2.4.3)$$

and Ω_i = vibration frequency of the i^{th} harmonic

$(a_{1i}, a_{2i}, a_{3i}, a_{4i})$ are arbitrary constants for the i^{th} mode and are to be determined from the boundary conditions, while constants (a_{5i}) and (a_{6i}) are evaluated from the initial conditions of the problem. The summation extends over all values of ' i ' for which (Ω_i) is a solution of equation (2.4.1). Due to symmetry of the problem, only half the beam will be considered here.

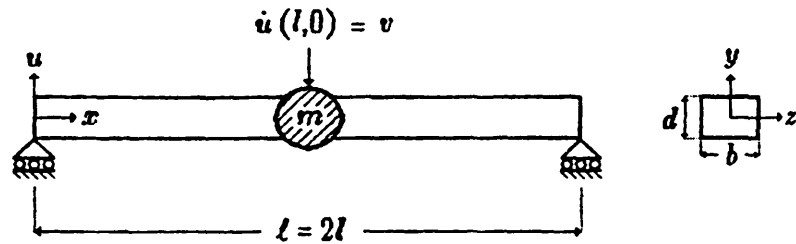


Diagram (2.1) — Simply-supported beam with attached centre mass

Referring to the above sketch, the relevant boundary conditions are as follows :

$$\begin{array}{llll}
 \text{at } x = 0: \text{ displacement} & u(0, t) = 0 & (a) \\
 \text{bending moment} & EI \frac{\partial^2 u}{\partial x^2} \Big|_{(0, t)} = 0 & (b) \\
 \text{at } x = l: \text{ slope} & \frac{\partial u}{\partial x} \Big|_{(l, t)} = 0 & (c) \\
 \text{shear force} & EI \frac{\partial^3 u}{\partial x^3} \Big|_{(l, t)} = -\frac{m}{2} \frac{\partial^2 u}{\partial t^2} \Big|_{(l, t)} & (d)
 \end{array} \quad (2.4.4)$$

where m = total impactor mass

Enforcing the first three boundary conditions (2.4.4a), (2.4.4b) and (2.4.4c) results in :

$$\begin{aligned}
 a_{1i} &= a_{3i} = 0 \\
 a_{4i} &= -a_{2i} \frac{\cosh \lambda_i l}{\cos \lambda_i x}
 \end{aligned} \quad (2.4.5)$$

Applying the remaining boundary condition one obtains

$$\sum_{i=1}^{\infty} m \Omega_i^2 a_{4i} \left(\lambda_i l \cos \lambda_i l \tanh \lambda_i l - \lambda_i l \sin \lambda_i l + \frac{4\rho A l}{m} \cos \lambda_i l \right) = 0 \quad (2.4.6)$$

and for a non-trivial solution (i.e. $a_{4i} \neq 0$), equation (2.4.6) requires that

$$\lambda_i l \cos \lambda_i l \tanh \lambda_i l - \lambda_i l \sin \lambda_i l + \frac{4\rho A l}{m} \cos \lambda_i l = 0$$

It is evident that $(\frac{i\pi}{2})$ does not represent a solution to this equation and it can be rearranged to give

$$\lambda_i l (\tan \lambda_i l - \tanh \lambda_i l) = 2 \frac{(2\rho A l)}{m} = \frac{2M}{m} \quad (2.4.7)$$

The transcendental characteristic equation (2.4.7) must be solved either graphically or numerically to extract the eigenvalues of all the possible modes of lateral vibration for the beam in question.

Combining equations (2.4.2) and (2.4.5) gives

$$u(x, t) = \sum_{i=1}^{\infty} a_{2i} \left(\sinh \lambda_i x - \frac{\cosh \lambda_i l}{\cos \lambda_i l} \sin \lambda_i x \right) (a_{5i} \cos \Omega_i t + a_{6i} \sin \Omega_i t) \quad (2.4.8)$$

The initial conditions to be enforced require that

$$\left. \begin{array}{lll} \text{at } t = 0: \text{ displacement} & u(x, 0) = 0 & (a) \\ \text{velocity} & \left. \begin{array}{l} \frac{\partial u}{\partial t} \Big|_{(x, 0)} = 0 \quad \text{for } x \neq l \\ \frac{\partial u}{\partial t} \Big|_{(l, 0)} = v \end{array} \right\} & (b) \end{array} \right\} \quad (2.4.9)$$

Hence, from (2.4.9a) $a_{5i} = 0$

and equation (2.4.8) reduces to

$$u(x, t) = \sum_{i=1}^{\infty} H_i \left(\sinh \lambda_i x - \frac{\cosh \lambda_i l}{\cos \lambda_i l} \sin \lambda_i x \right) \sin \Omega_i t \quad (2.4.10)$$

Equation (2.4.10) represents the lateral response of a vibrating beam with an attached central lumped mass. The constant ($H_i = a_{2i} \times a_{6i}$) is determined from the second initial condition (2.4.9b) regarding a central velocity imparted to the attached mass. The velocity equation from which the constants (H_i) are determined is given by

$$\frac{\partial u}{\partial t}(x, t) = \sum_{i=1}^{\infty} \Omega_i H_i \left(\sinh \lambda_i x - \frac{\cosh \lambda_i l}{\cos \lambda_i l} \sin \lambda_i x \right) \cos \Omega_i t \quad (2.4.11)$$

The expression for the bending stresses can now be derived by differentiating equation (2.4.10) twice with respect to (w.r.t.) 'x' and using the Engineer's theory of bending (ETB) to relate stresses to bending moments, giving

$$\sigma_B(x, t) = \sum_{i=1}^{\infty} E y \lambda_i^2 H_i \left(\sinh \lambda_i x + \frac{\cosh \lambda_i l}{\cos \lambda_i l} \sin \lambda_i x \right) \sin \Omega_i t \quad (2.4.12)$$

The corresponding shear stress requires the third derivative w.r.t. 'x' of equation (2.4.10) and is given by

$$\sigma_S(x, t) = \sum_{i=1}^{\infty} -\frac{E \kappa^2}{k_e} \lambda_i^3 H_i \left(\cosh \lambda_i x + \frac{\cosh \lambda_i l}{\cos \lambda_i l} \cos \lambda_i x \right) \sin \Omega_i t \quad (2.4.13)$$

where κ = radius of gyration of the cross-section

k_e = shear correction factor for the cross-section

2.5 INTERPRETATION OF EQUATIONS

Identification of the variables that influence the beam stresses, as given by equations (2.4.12) and (2.4.13) is not straightforward and requires that these equations be suitably rearranged.

Concentrating on the bending stresses, equation (2.4.12) can be rewritten as

$$\begin{aligned}\sigma_B(x, t) &= Ey\lambda^2 \sum_{i=1}^{\infty} H_i \left[\left(\frac{\lambda_i}{\lambda} \right)^2 \left(\sinh \lambda_i x + \frac{\cosh \lambda_i l}{\cos \lambda_i l} \sin \lambda_i x \right) \sin \Omega_i t \right] \\ &= Ey\lambda^2 \sum_{i=1}^{\infty} H_i h_i\end{aligned}$$

The above equation can be expressed in matrix form

$$\sigma_B(x, t) = Ey\lambda^2 \mathbf{h} \mathbf{H}$$

where $\mathbf{h}$ is a row matrix = $[h_1 \quad h_2 \quad h_3 \quad \dots \quad h_n]$

$\mathbf{H}$ is a column matrix = $\{H_1 \quad H_2 \quad H_3 \quad \dots \quad H_n\}$

and n = max. no. of harmonics

Substituting for (λ^2) from (2.4.3) results in

$$\sigma_B(x, t) = \sqrt{E\rho} \left(\frac{y}{\kappa} \right) \mathbf{h} \Omega \mathbf{H} \quad (2.5.1)$$

The expression for velocity (2.4.11) can be expressed as

$$\begin{aligned}\frac{\partial u}{\partial t}(x, t) &= \Omega \sum_{i=1}^{\infty} H_i \left(\frac{\Omega_i}{\Omega} \right) \left(\sinh \lambda_i x - \frac{\cosh \lambda_i l}{\cos \lambda_i l} \sin \lambda_i x \right) \cos \Omega_i t \\ &= \Omega \sum_{i=1}^{\infty} H_i f_i\end{aligned}$$

In matrix form
$$\dot{\mathbf{u}}(x, t) = \Omega \mathbf{f} \mathbf{H} \quad (2.5.2)$$

where
$$\mathbf{f}(1 \times n) = [f_1 \quad f_2 \quad f_3 \quad \dots \quad f_n]$$

To solve for the coefficients (H_i), the initial condition given by (2.4.9b) have to be enforced.

Depending on the number of harmonics that are to be included, an equal number of points along the beam length have to be chosen and the initial velocity at these points be specified. The resulting velocity vector for the particular case of central impact would be

$$\dot{\mathbf{u}} = v \{ 1 \quad 0 \quad 0 \quad 0 \quad \dots \quad 0 \} = v \mathbf{e}$$

where (v) is the magnitude of the central impact velocity.

Thus, equation (2.5.2) will become

$$\dot{\mathbf{u}} = \Omega \mathbf{F} \mathbf{H} \quad (2.5.3)$$

where
$$\mathbf{F}(n \times n) = \{ \mathbf{f}_1 \quad \mathbf{f}_2 \quad \mathbf{f}_3 \quad \dots \quad \mathbf{f}_n \}$$

From equation (2.5.3) it follows that

$$\Omega \mathbf{H} = \mathbf{F}^{-1} \dot{\mathbf{u}} = v \mathbf{F}^{-1} \mathbf{e} \quad (2.5.4)$$

Finally, substituting from (2.5.4) into (2.5.1) produces

$$\sigma_B(x, t) = \sqrt{E\rho} \left(\frac{y}{\kappa} \right) v [\mathbf{h} \mathbf{F}^{-1} \mathbf{e}] = \sqrt{E\rho} \left(\frac{y}{\kappa} \right) v \Phi \quad (2.5.5)$$

The factor ($\Phi = \mathbf{h} \mathbf{F}^{-1} \mathbf{e}$) is a complicated scalar function of time which depends on boundary and initial conditions amongst other things and it also contains the

system's eigenvalues and indirectly the impactor/beam mass ratio. Similarly, the shear stresses can be shown to be given by

$$\sigma_S(x, t) = \sqrt{E\rho} \left(\frac{\kappa}{k_e} \right) \frac{v}{l} \Psi \quad (2.5.6)$$

where (Ψ) is a different scalar function but similar in nature to (Φ) .

Apart from material properties, equations (2.5.5) and (2.5.6) indicate that both bending and shear stresses increase linearly with impactor velocity and also depend on the impactor/beam mass ratio. The bending stresses are shown to be inversely proportional and the shear stresses are directly dependent on the radius of gyration. Furthermore, the shear stresses are also affected by the length of the target beam.

Before proceeding to examine the nature of the contribution of the identified variables which control the induced stresses, it would be interesting to compare these findings against the predictions of a simplified approach that is formulated on the assumption that the beam will respond in a static mode and that the total input kinetic energy of the impactor is eventually converted into the strain energy of the beam in this mode.

2.6 QUASI-STATIC APPROACH

The strain energy of a simply-supported beam of length $(2l)$ and loaded by a concentrated force (P) at the centre can be shown to be

$$U = \frac{P^2 l^3}{12EI}$$

The force (P) is related to the bending stress (σ_B) at the centre by ETB

$$\sigma_B = \frac{Pl y}{2I}$$

Assuming that the force acting on the beam is a direct result of a falling weight (m), then the input energy ($\frac{1}{2}mv^2$) of this falling weight will be equal to the work done by the resulting force, which in turn will be absorbed into the strain energy of the beam. Hence equating energies and substituting for the force (P) gives

$$\frac{mv^2}{2} = \frac{\sigma_B^2 Il}{3Ey^2}$$

but letting $I = A\kappa^2$, then

$$\sigma_B^2 = 3E \left(\frac{y}{\kappa} \right)^2 \frac{m}{2Al} v^2 = 3E\rho \left(\frac{y}{\kappa} \right)^2 \left(\frac{m}{2\rho Al} \right) v^2 \quad (2.6.1)$$

$$\text{or} \quad \sigma_B = \sqrt{3}\sqrt{E\rho} \left(\frac{y}{\kappa} \right) \sqrt{\frac{m}{M}} v$$

Equation (2.6.1) can alternatively be expressed in non-dimensional form as

$$\frac{\sigma_B}{E} = \sqrt{3} \left(\frac{y}{\kappa} \right) \sqrt{\frac{m}{M}} \left(\frac{v}{c} \right)$$

$$\text{where} \quad c = \text{wave propagation velocity} = \sqrt{\frac{E}{\rho}}$$

A similar expression can be derived for the average shear stresses (σ_S) as compared to the bending stresses in the beam. Assuming that

$$\sigma_S = \frac{P}{2Ak_e}$$

and substituting for the force (P) from the above relationship, will eventually result in the following expression

$$\sigma_S = \sqrt{3}\sqrt{E\rho} \left(\frac{\kappa}{k_e} \right) \sqrt{\frac{m}{M}} \left(\frac{v}{l} \right) \quad (2.6.2)$$

or, again in non-dimensional form

$$\frac{\sigma_S}{E} = \frac{\sqrt{3}}{k_e} \left(\frac{\kappa}{l} \right) \sqrt{\frac{m}{M}} \left(\frac{v}{c} \right)$$

The expressions in (2.6.1) and (2.6.2) relate to isotropic beams, but similar expressions can be derived for composite beams. Expressed in a general form, the bending and shear stresses in any beam, be it isotropic or composite, can be shown to be given by :

$$\frac{\sigma_{Bg}}{E_g} = \sqrt{3} \left(\frac{y_g}{\kappa} \right) \sqrt{\frac{m}{M}} \left(\frac{v}{c_e} \right) \quad (2.6.3)$$

$$\frac{\sigma_S}{E_e} = \frac{\sqrt{3}}{k_e} \left(\frac{\kappa}{l} \right) \sqrt{\frac{m}{M}} \left(\frac{v}{c_e} \right) \quad (2.6.4)$$

where the subscript 'g' refers to layer 'g', and

$$E_e = \sum_{g=1}^{\Gamma} \frac{E_g I_g}{I} \quad , \quad \rho_e = \sum_{g=1}^{\Gamma} \frac{\rho_g A_g}{A} \quad , \quad c_e = \sqrt{\frac{E_e}{\rho_e}}$$

The observation to be made is that this simple approach identifies the same parameters that were associated with the free vibration analysis with the exception of the variable involving the impactor/beam mass ratio, which in the free vibration analysis is concealed within the complex functions (Φ and Ψ). Bearing in mind that for a large impactor mass the response of the beam is expected to be mainly confined to the fundamental mode it would not appear to be too presumptuous to expect these functions to behave like $(\sqrt{\frac{m}{M}})$ under such circumstances, and then the quasi-static approach would be a fairly good approximation.

In order to examine the exact nature of the above-mentioned complex functions, it will be necessary to perform a series of numerical test cases and these will involve varying all the identified parameters to establish the degree to which these parameters influence the functions (Φ and Ψ) and hence the induced stresses in the beam.

2.7 IMPLEMENTATION OF THE ANALYSIS

The equations that describe the response of the system being analysed are in discretised form and require the superposition of all the harmonics of the system for a complete solution. A numerical solution is called for, and this is best achieved on a computer. The objective of this numerical study is merely to examine the contribution of the parameters that govern the response of the beam-mass system and so the accuracy of the results are only of secondary importance. By limiting the number of harmonics that are included, the analysis can be performed on a microcomputer, thus greatly reducing the solution cost. The *BBC* microcomputer was used for this task, but due to the presence of hyperbolic functions, the equations had to be manipulated to permit inclusion of up to 20 harmonics of the system. This will be shown to be sufficient for the purpose of the current investigation.

The eigenvalues of the system are obtained from equation (2.4.6), which is solved using the standard *Newton-Ralphson* iteration technique and are only dependent on the impactor/beam mass ratio. Table (2.1) lists the first 20 eigenvalues for a range of mass ratios. It can be observed that the mass ratio affects only the first few modes while the eigenvalues ($\lambda_i l$) for the higher modes remain unaffected, tending to almost identical values irrespective of mass ratio. For these higher modes the spacing between the eigenvalues tends to the value of ' π '. This suggests that the higher modes are independent of the boundary conditions.

The bending and shear stresses can be computed once the coefficients (H_i) have been determined and this requires that the velocity distribution along the beam at the start of the motion be known. Depending on the number of harmonics that are to be included an equal number of 'node points', uniformly distributed along the beam, have to be selected and the starting velocities at these points must be specified. For the case of central impact, the resulting velocity vector will be zero at all nodes except for the centre node for which the impactor velocity is

specified. It is worth noting that the initial conditions as stated in (2.4.9b) can not be realised unless an infinite number of harmonics are included. In practice the higher harmonics are truncated and this leads to an approximation to the required initial condition. Improvement to the initial velocity distribution as depicted in Figure (2.1) can be achieved at the expense of including more harmonics. However, there is little point in trying to make $(\dot{u}(x, 0) = 0)$ except at $(x = l)$, since such a discontinuous behaviour would require both shear resistance and rotary inertia near $(x = l)$, and these are excluded anyway. As a consequence, the initial kinetic energy of the system will be in excess of the kinetic energy of the impacting mass by an amount that represents the initial kinetic energy of the beam arising from this approximation of the initial velocity distribution and thus is directly dependent on the number of harmonics being considered and also on the ratio of beam to impactor mass.

The distribution of energy into the various modes is used as a guideline to indicate whether sufficient harmonics have been included. At the start of the motion, the total kinetic energy is the sum of the beam kinetic energy and the kinetic energy of the lumped central mass, giving

$$W = \frac{\rho A l}{2} \int_0^l \left(\frac{\partial u}{\partial t} \right)^2 \Big|_{(x, 0)} dx + \frac{m}{2} \left(\frac{\partial u}{\partial t} \right)^2 \Big|_{(l, 0)}$$

which eventually leads to

$$W = \frac{\rho A l}{2} \sum_{i=1}^n \sum_{j=1}^n H_i H_j \Omega_i \Omega_j \cosh \lambda_i l [\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 + \Lambda_5] \quad (2.7.1)$$

$$\begin{aligned}
\text{where } \Lambda_1 &= \left[\frac{\lambda_j l}{(\lambda_i l)^2 + (\lambda_j l)^2} + \frac{1}{2(\lambda_i l + \lambda_j l)} \right] \cosh \lambda_j l \tanh \lambda_i l \\
\Lambda_2 &= \left[\frac{\lambda_i l}{(\lambda_i l)^2 + (\lambda_j l)^2} + \frac{1}{2(\lambda_i l + \lambda_j l)} - \frac{2}{\lambda_i l} \right] \sinh \lambda_j l \\
\Lambda_3 &= \left[\frac{2}{\lambda_i l} - \frac{\lambda_i l}{(\lambda_i l)^2 + (\lambda_j l)^2} \right] \cosh \lambda_j l \tan \lambda_j l \\
\Lambda_4 &= \left[-\frac{\lambda_j l \tan \lambda_i l}{(\lambda_i l)^2 + (\lambda_j l)^2} - \frac{\sin(\lambda_i l + \lambda_j l)}{2(\lambda_i l + \lambda_j l) \cos \lambda_i l \cos \lambda_j l} \right] \cosh \lambda_j l \\
\Lambda_5 &= \left[\frac{\cosh \lambda_j l \sin(\lambda_i l - \lambda_j l)}{2(\lambda_i l - \lambda_j l) \cos \lambda_i l \cos \lambda_j l} - \frac{\sinh(\lambda_i l - \lambda_j l)}{2(\lambda_i l - \lambda_j l) \cosh \lambda_i l} \right]
\end{aligned}$$

Equation (2.7.1) yields the energy distribution of the system and this has been computed for the various test cases and are presented as a percentage of the total energy in Table (2.2). As can be observed, the energy distribution is primarily a function of impactor/beam mass ratio and while for a large impactor to beam mass the energy is mainly confined to the first mode, as the impactor mass is reduced relative to the beam mass more of the input energy enters the higher modes and the effect of these higher harmonics on the beam response become important and have to be considered. Consequently, either 10, 15 or 20 harmonics were taken into consideration, depending on the impactor/beam mass ratio, when analysing the various numerical test cases. For the range covered, this has meant that the highest harmonic that has been included always contains less than 0.2% of the total energy of the system and is considered as a sufficiently suitable cut-off criterion. It should be realised that this guideline for imposing a limit on the highest mode is quite arbitrary, but its suitability has been checked and demonstrated by analysing one of the test examples, for which a considerable proportion of the total energy is contained in the higher modes, to include 5, 10, 15 and 20 harmonics. The resulting stress histories, which are much more sensitive to the higher harmonics than the displacements, are shown in Figures (2.2–2.5) and it can be observed that no significant difference exists between including 15 or 20 modes and even 10 modes would appear to be sufficient. The differences in peak stress are also within an

acceptable tolerance for the purpose of this investigation bearing in mind that accuracy is of secondary importance.

In all the numerical examples, the analysis was performed up to the quarter period of the fundamental mode of vibration to establish the stress history and obtain the peak stress. In order that the effects of the higher modes be properly taken into account, the stress sampling interval was chosen to be a quarter of the period of the highest harmonic that has been included. Shorter sampling intervals were also considered but the resulting difference in peak stress was found to be less than 1% which is well within the accuracy tolerance of the analysis.

2.8 CONVERGENCE OF STRESSES

A problem that was highlighted during this numerical investigation concerns the transverse shear stresses. It was found that the shear stresses are very dependent on the number of modes that are considered and increase as more harmonics are included and there does not appear to be any sign of convergence, even for the cases where the response is dominated by the fundamental mode. In fact, GOLDSMITH (25) presents a normal mode solution for the problem of free vibration of a beam-mass system for which he shows that the displacement is given by

$$u(x, t) = \frac{4l^2 v}{\sqrt{\frac{EI}{\rho A}}} \sum_{i=1}^{\infty} \frac{1}{(\lambda_i l)^3} \frac{\frac{\sin \lambda_i x}{\cos \lambda_i l} - \frac{\sinh \lambda_i x}{\cosh \lambda_i l}}{\frac{1}{\cos^2 \lambda_i l} - \frac{1}{\cosh^2 \lambda_i l} + \frac{2M}{(\lambda_i l)^2}} \sin \Omega_i t$$

and so the bending and shear stresses at the beam mid-span are respectively derived as

$$\sigma_B(l, t) = -\frac{4yv}{\sqrt{\frac{EI}{\rho A}}} \sum_{i=1}^{\infty} \frac{1}{\lambda_i l} \frac{\tan \lambda_i l + \tanh \lambda_i l}{\frac{1}{\cos^2 \lambda_i l} - \frac{1}{\cosh^2 \lambda_i l} + \frac{2M}{(\lambda_i l)^2}} \sin \Omega_i t$$

$$\sigma_S(l, t) = \frac{4E\kappa^2 v}{k_e l \sqrt{\frac{EI}{\rho A}}} \sum_{i=1}^{\infty} \frac{2}{\frac{1}{\cos^2 \lambda_i l} - \frac{1}{\cosh^2 \lambda_i l} + \frac{2M}{(\lambda_i l)^2}} \sin \Omega_i t$$

It can be observed (see Table (2.1)) that for large values of 'i', the term $(\lambda_i l)$ is of the order of 'i' and all the other factors in the above expressions approach a constant value. Thus, the series solution for the bending stresses is slowly convergent while the shear stresses are represented by a non-converging series. In view of this convergence difficulty the numerical investigation utilising this modal synthesis technique has concentrated on examination of the induced bending stresses and how these are influenced by the variables previously identified.

2.9 NUMERICAL EXAMPLES

The beam-mass free vibration analysis utilising a modal superposition technique has been carried out for a range of test cases in which the beam properties and dimensions as well as impactor mass and velocity have all been varied. Tables (2.3–2.7) list both the configuration of the system and the results for the various test examples that have been analysed. Due to symmetry only half of the system has been used for the analysis and the peak stresses have been calculated at the centre of the beam. Peak stresses at positions away from the centre were obtained but found to be lower than that at the beam mid-span. Peak stresses as predicted by the quasi-static approach are also presented for comparison.

The first series of tests (1–15) (see Table (2.3)) the impactor mass and velocity have been kept constant and only the beam length has been altered, which directly affects the impactor/beam mass ratio. The calculated energy distribution (Table (2.2)) as well as the stress histories (see Figures (2.6–2.20)) clearly indicate the increasing contribution of the higher harmonics with decreasing $(\frac{m}{M})$ and this is also reflected in the dynamic peak stress as compared to that predicted by the quasi-static approach (i.e. $\frac{\sigma_B \text{ dyn}}{\sigma_B \text{ qst}} = 1.3$ for large $\frac{m}{M}$ to $\frac{\sigma_B \text{ dyn}}{\sigma_B \text{ qst}} = 2.0$ for

small $\frac{m}{M}$). The excitation of these higher modes has also meant that while for large impactor/beam mass the peak stress is reached towards the quarter period of the fundamental mode, for small $(\frac{m}{M})$ a high stress level is reached much earlier in the response of the beam. For these tests, both impactor mass and velocity were unaltered, so that the input energy and momentum would be identical in all cases and the findings suggest that the latter two parameters are not the sole or even dominant factors controlling the beam response.

Next the beam length (and hence its mass) and the impactor velocity were held constant while altering the impactor mass to obtain the same $(\frac{m}{M})$ as for test examples (1–15). The results for these tests (16–30) (see Table (2.4)) are found to be identical to those of the previous cases, both from the point of view of peak stresses as well as stress history, though the absolute time scales will differ, and this time the input energy and momentum are different in the various examples. These findings suggest that the impactor/beam mass ratio is the dominant factor controlling the dynamic response of the idealised beam-mass system.

Figure (2.21) represents a logarithmic plot of the variation of the calculated peak stresses with impactor/beam mass ratio, highlighting the linearity of the relationship. The ‘Best Fit’ straight line to these data points indicates that

$$\sigma_{B \max} \propto \left(\frac{m}{M}\right)^{0.3908}$$

The quasi-static approach in comparison gives

$$\sigma_{B \max} \propto \left(\frac{m}{M}\right)^{0.5}$$

which supports the claim that mainly the fundamental mode is influenced by the impactor/beam mass ratio and the boundary conditions while the higher harmonics are much less affected.

Apart from length, the beam mass is also dependent on the cross-sectional dimensions and the density of the beam material. Keeping beam length, impactor mass and velocity constant, the effect of varying the cross-sectional dimensions are considered — see Table (2.5). As expected, the cross-sectional dimensions do not independently influence the beam response but affect it indirectly, by altering the beam mass.

The influence of material properties are considered next (Table (2.6)). From equation (2.5.5) it is evident that density affects the peak stress and indirectly alters the response by altering the target mass. By doubling the density one would expect the response for test case (34) to be identical to those of cases (5 and 20), but the peak stresses would be up by a factor of $\sqrt{2}$, and in fact this is confirmed by the findings.

Test example (35) represents the effect of altering the material elastic modulus and as expected only the peak stresses are modified while the response remains unaltered.

The final set of tests (Table (2.7)) examine the contribution of the impactor velocity. The results reveal that response remains unaffected while peak stresses increase linearly with impactor velocity as suggested by equation (2.5.5) and shown in Figure (2.22).

In conclusion, it is believed that this numerical exercise has successfully identified the parameters that control the global response of beam-type structural components to impact by a rigid body. The importance of each parameter as regards the peak stresses has also been examined in detail as well as the role of the higher harmonics of the system. Furthermore, it was observed that although the response of the beam is markedly different for a light impactor as compared to a heavy one, and the peak stresses may occur anywhere between the initial impact and the quarter period, nevertheless these peak stresses that cause failure

do seem to be related by an expression similar in nature to that derived on basis of quasi-static arguments which can be summarised as

$$\sigma_{B \max} \propto \sqrt{E\rho} \left(\frac{y}{\kappa} \right) \left(\frac{m}{M} \right)^p v \quad (2.9.1)$$

This is highlighted in Figure (2.23) where the stress histories for two extreme cases, namely test example (1) with the highest $\left(\frac{m}{M}\right)$ and test example (15) with the lowest $\left(\frac{m}{M}\right)$, have been scaled in accordance with the above expression (2.9.1) so as to yield the same value of $\left(\sqrt{E\rho} \left(\frac{y}{\kappa}\right) \left(\frac{m}{M}\right)^p v\right)$ in both cases (where $p = 0.39$ as determined earlier). It is clearly evident that although the responses are quite different in the two cases, the peak stresses are more or less identical, irrespective of when during the response these occur, confirming that the correct scaling parameters have been identified.

Chapter Three

NUMERICAL ANALYSIS (II) :

Finite Element Method

3.1 INTRODUCTION

In the previous Chapter, a parametric study was undertaken which identified and investigated in detail, the parameters that influence the low velocity impact response of beams. The analysis was based on a modal synthesis approach, utilising the *Bernoulli-Euler* equations, to provide an inexpensive method for this type of numerical investigation. However, all theoretical analyses are subject to limitations and this is especially true of simplified methods designed to provide a fast, inexpensive approximate solution such as that employed for the aforementioned parametric study. One major shortcoming of the numerical technique of Chapter (2) was that the transverse shear stresses in the impacted beam could not be calculated. Shear deformation is not separately accounted for in the *Bernoulli-Euler* formulation so that the shear stress distribution along the beam can only be estimated from the derivative of the bending moments and this was shown to lead to a solution involving a diverging series. Another uncertainty concerns the lumping of the impacting mass at the impact node requiring the assumption of perfectly plastic impact which is difficult to justify in view of the experimental observations of the author, as well as MASON (34) and ARNOLD (35), with regard to multiple bouncing between impactor and target. As a direct consequence of such an

assumption, the all-important contact force history can not be obtained on the basis of such an analytical technique.

In this Chapter, the finite element (FE) method, which is widely used in engineering applications, will be presented as an alternative analysis technique to overcome the deficiencies of the numerical approach used previously. The finite element method models a complicated continuum by a series of simple elements or components for which the properties and behaviour are/or can readily be defined. Inclusion of shear deformation in the definition of the elements that make up the continuum to be analysed will facilitate the calculation of the transverse shear stress distribution within the system. In addition rotary inertia effects, damping, geometric non-linearities due to large deformations as well as material non-linearities arising from plastic flow or progressive failure can all be modelled, though at a much increased computational cost. The assumption of perfectly plastic impact can also be relaxed since contact between impactor and target can be incorporated into the analysis in a number of ways. The major disadvantage of the FE method is that it requires vast amount of computer resources making the analysis rather costly and this is much more noticeable for dynamic impact analyses, since a closed form solution is not possible, requiring a step-by-step integration of the equations of motion which may, depending on the selected algorithm, have to be iteratively solved for each and every timestep.

The finite element method has been introduced, not to replace either the modal approach or the experimental testing of specimens, but rather to complement these techniques by providing a capability for a more detailed and realistic modelling and analysis. In doing so, it has facilitated comparison between the alternative theoretical approaches to establish the accuracy of the numerical technique employed for the parameter study and to highlight the contributions of shear and rotary inertia on impact response and in particular on the impact induced stresses in the beam target. In addition, the FE analysis has been employed

as an effective tool for checking the correctness of the experimental set-up and verifying the results of the impact tests which form the basis for the validation of the theoretical predictions of the proposed stress models of Chapter (2).

A specially developed finite element package has been implemented and the analysis has been restricted to one dimension, namely variation of stresses along the length of the beam only. Although it is certainly true that in the vicinity of the point of contact the state of stress is three-dimensional — and the FE method could be used to perform such an analysis though prohibitively expensive — the objective of the current research is to consider the global response of the impacted beam target rather than to concentrate on local effects. Facilities for the inclusion of some of the local effects that may have a direct effect on the beam's response (e.g. bouncing of impactor on beam) have been provided in a simple but effective way, in the form of a spring element for which the stiffness variation can be defined in accordance with the non-linearity that is to be modelled. In addition various beam elements have been considered for application to the present impact problem and in assessing the suitability of these elements it has been recognised that the importance of shear, especially where composites are concerned, requires elements that enable accurate estimation of the shear stresses. In existing beam elements the tendency has been to include the effects of shear and rotary inertia as some form of additional correction which, in general, is sufficiently accurate for applications such as calculating the deflection or vibration modes of thicker beam and for the higher modes of any beam. These correction terms are generally obtained by imposing additional constraints on the element (see Section (3.3.2)) and, in the main, this takes the form of enforcing a constant shear along the element. However, where the evaluation of shear stress distribution is the main objective, these constant shear elements may be at a disadvantage, in particular in dynamic impact problems where high order harmonics may be excited that result in complex shear stress variations. In view of this, it has been found necessary to develop high order,

shear-flexible Mindlin beam elements, of which the 4-noded cubic element has been shown to be a very worthy and effective competitor.

3.2 GENERAL OUTLINE OF THE FE METHOD

In the displacement formulation of the finite element method, the displacement field, $\mathbf{u}(x, y, z)$, within the element is uniquely related to the element's nodal displacements ($\mathbf{r}$) by an appropriate set of interpolation functions (ω). Thus

$$\mathbf{u} = \omega \mathbf{r} \quad (3.2.1)$$

Enforcing the necessary and proper compatibility relations, the strain variation (ϵ) within an element will completely be defined in terms of the nodal displacements. This can be expressed as

$$\begin{aligned} \epsilon &= d\mathbf{u} \\ \text{or} \quad \epsilon &= (d\omega)\mathbf{r} = \alpha \mathbf{r} \end{aligned} \quad (3.2.2)$$

Using the constitutive properties of the material ($\mathbf{D}$) the state of stress in the element will be related to its state of strain, giving

$$\sigma = \mathbf{D}\epsilon = \mathbf{D}\alpha \mathbf{r} \quad (3.2.3)$$

Now for a set of external applied forces ($\mathbf{R}$) and internal set of stresses (σ) and a system of displacements ($\mathbf{r}$) and strains (ϵ) the principle of virtual work states that :

$$\bar{\mathbf{r}}^t \mathbf{R} = \int_V \bar{\epsilon}^t \sigma dV \quad (3.2.4)$$

where $\bar{\mathbf{r}}^t$ and $\bar{\epsilon}^t$ denote virtual quantities.

Substituting equations (3.2.2) and (3.2.3) into (3.2.4) yields

$$\bar{\mathbf{r}}^t \mathbf{R} = \int_V \bar{\mathbf{r}}^t \boldsymbol{\alpha}^t \mathbf{D} \boldsymbol{\alpha} r dV \quad (3.2.5)$$

Quantities $\bar{\mathbf{r}}^t$ and $\mathbf{r}$ are independent of the integration variable. In addition, $\bar{\mathbf{r}}^t$ is a virtual quantity and can be arbitrarily chosen. Equation (3.2.5) can therefore be written as

$$\mathbf{R} = \left(\int_V \boldsymbol{\alpha}^t \mathbf{D} \boldsymbol{\alpha} dV \right) \mathbf{r} \quad (3.2.6)$$

The first term in the right-hand side of equation (3.2.6) is the familiar element stiffness matrix, denoted by $\mathbf{K}$.

$$\text{i.e.} \quad \mathbf{K} = \int_V \boldsymbol{\alpha}^t \mathbf{D} \boldsymbol{\alpha} dV \quad (3.2.7)$$

The same interpolation functions can be used to prescribe the velocity variation within the element in terms of its nodal velocities, so that

$$\dot{\mathbf{u}} = \boldsymbol{\omega} \dot{\mathbf{r}} \quad (3.2.8)$$

The total kinetic energy (W) of the element is given by :

$$\begin{aligned} W &= \int_V \left(\frac{1}{2} \dot{\mathbf{u}}^t \rho \dot{\mathbf{u}} \right) dV \\ &= \frac{1}{2} \dot{\mathbf{r}}^t \left(\int_V \boldsymbol{\omega}^t \rho \boldsymbol{\omega} dV \right) \dot{\mathbf{r}} \end{aligned} \quad (3.2.9)$$

It follows that

$$\mathbf{M} = \int_V \boldsymbol{\omega}^t \rho \boldsymbol{\omega} dV \quad (3.2.10)$$

where $\mathbf{M}$ is the element mass matrix.

Element property matrices can be assembled according to element connectivity to form a global property matrix for the entire system, which can be used in conjunction with the correct boundary conditions and knowledge of the external applied loading to solve for the unknown nodal displacements of the system and then obtain the stresses or strains using equations (3.2.3) or (3.2.2) respectively.

3.3 BEAM ELEMENT FORMULATION

The characteristics of an element is greatly influenced by the number of nodes, the nodal variables and the selected interpolation functions and this in turn determines the suitability of the element and the range of its application. Various beam elements that have been implemented for use in the finite element code for the impact problem under investigation will be presented in this Section together with their respective mass and stiffness matrices. For the purpose of this research program, attention has been confined to the one-dimensional case for which the bending action is limited to one principal plane only, resulting in the sole direct strain component, namely ϵ_{xx} (see Diagram (3.1)).

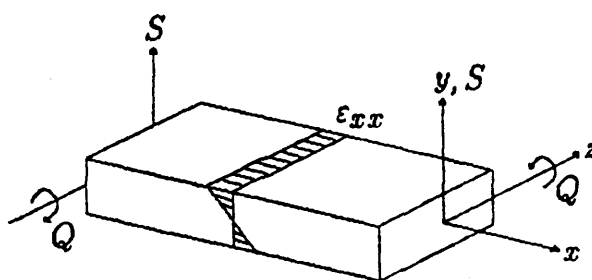


Diagram (3.1) — Beam loading and resulting strain

Transverse motion and rotation in this principal plane are considered to contribute to this strain component while contributions due to axial deformations

are excluded altogether. Furthermore, the beam cross-section has been chosen to be solid and rectangular (thus symmetrical about both principal axes), so that warping effects can be neglected and loading in the principal plane does not result in twisting of the cross-section. In addition, the axis of the beam element has been considered to be straight, and sectional properties do not vary with distance along the axis of the beam in order to appreciably simplify the calculation of the element matrices. Homogeneity of the element material has also been assumed throughout.

Suitability of the presented elements will subsequently be evaluated by means of a series of test cases which should also highlight any problems that may be inherent in the formulation of any particular element — e.g. shear locking of the low-order *Mindlin* element.

3.3.1 The classical *Hermitian* beam element

This element is well-established and applicable to a wide range of beam problems. Detailed derivation of it can be found in many standard textbooks and the intention here is to present an outline of the particular version of this element as implemented in the present code, utilising the displacement approach.

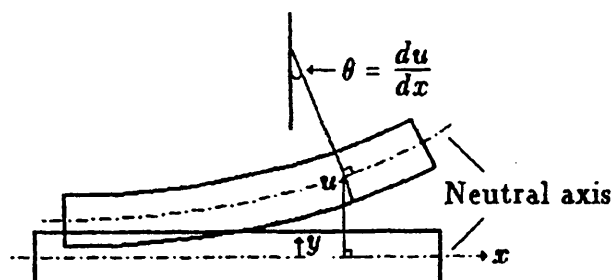


Diagram (3.2) — Deformation of beam section excluding shear effects

The derivation of this beam element is based on the Engineer's theory of bending (ETB) and thus shear effects are excluded from the formulation. The ETB — assuming small deflections — relates the beam's cross-sectional rotation (θ) to its transverse displacement (u) by :

$$\theta = \frac{du}{dx} \quad (3.3.1)$$

so that the strain

$$\epsilon_{xx} = -y \frac{d\theta}{dx} = -y \frac{d^2u}{dx^2} \quad (3.3.2)$$

In order that the integral of equation (3.2.4) can be evaluated, the above conditions impose the c^1 -continuity (in displacement field) requirement upon this element (i.e. both displacements and rotations must be continuous across element interfaces) — see ZIENKIEWICZ (37).

Now, considering a two-noded element with two degrees of freedom per node (1 transverse displacement and 1 rotation, defined on the neutral axis of the element) as shown in Diagram (3.3), a cubic polynomial is necessary to relate the displacement field (u) within the element to the nodal variables (r_u, r_θ). This can be expressed in the following manner

$$u = \omega \mathbf{r} = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 & \omega_4 \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_4 \end{bmatrix} \quad (3.3.3)$$

where the displacement functions of the element are

$$\omega_1 = \frac{1}{4}(1 - \xi)^2(2 + \xi) \quad \omega_2 = \frac{L}{8}(1 - \xi)^2(1 + \xi)$$

$$\omega_3 = \frac{1}{4}(1 + \xi)^2(2 - \xi) \quad \omega_4 = \frac{L}{8}(1 + \xi)^2(\xi - 1)$$

and $\xi = \frac{2x}{L}$



Diagram (3.3) — Two-noded beam element

Using the expression for the strain as defined by equation (3.3.2) and substituting from (3.3.3), it follows that

$$\alpha = \left[-\frac{6y\xi}{L^2} \quad \frac{y(1-3\xi)}{L} \quad \frac{6y\xi}{L^2} \quad -\frac{y(1+3\xi)}{L} \right] \quad (3.3.4)$$

The stress-strain law is simply

$$\sigma_{xx} = E\epsilon_{xx}$$

$$\text{i.e.} \quad \mathbf{D} = [E]$$

Employing equation (3.2.7), the stiffness matrix of this beam can be written as

$$\mathbf{K} = \frac{bL}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{-1}^1 \alpha^t \mathbf{D} \alpha d\xi dy$$

which upon explicit integration results in

$$\mathbf{K} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

The mass matrix for this element, excluding rotary inertia effects, can be obtained directly from (3.2.10) using the interpolation functions given in (3.3.3). Explicit integration gives

$$\mathbf{M} = \frac{\rho AL}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

3.3.2 Timoshenko beam element

In situations where one deals with short, thick beams and in dynamic problems where the higher modes of vibration may be excited, the second order contributions due to shear and rotary inertia become increasingly more important. In order to account for these effects, the formulation of the beam element has to be based on the *Timoshenko* bending theory (38). The cross-section is still assumed to remain plane, but to incorporate the shear deformation, the slope of the deflection curve which represents the total cross-sectional rotation is composed of the cross-sectional rotation (θ) due to pure bending and that due to shear, namely ψ .

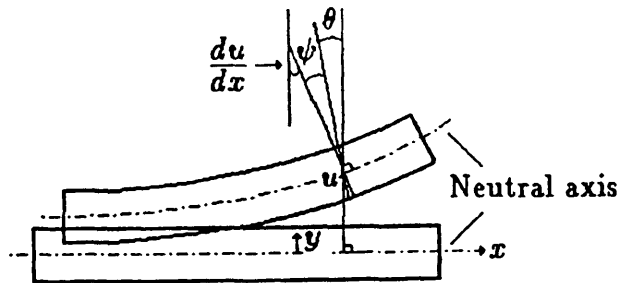


Diagram (3.4) — Beam deformations including shear effects

It is evident from Diagram (3.4) that

$$\frac{du}{dx} = \theta + \psi \quad (3.3.5)$$

Thus the bending strain is given, as before, by

$$\epsilon_{xx} = -y \frac{d\theta}{dx} \quad (3.3.6)$$

and the shear strain is expressed as

$$\epsilon_{xy} = \psi = \frac{du}{dx} - \theta \quad (3.3.7)$$

This shear strain is uniform as a consequence of the assumption of plane deformation. The total strain energy (U) is therefore given by

$$U = \frac{1}{2} \int_0^L EI \left(\frac{d\theta}{dx} \right)^2 dx + \frac{k_e}{2} \int_0^L GA \left(\frac{du}{dx} - \theta \right)^2 dx \quad (3.3.8)$$

where the first term represents the bending energy and the second term expresses the shear energy contribution. The factor (k_e) is referred to as the shear coefficient, dependent on the distribution of shearing stress in the beam cross-section. The choice of an appropriate value for ' k_e ' will be discussed in Section (3.4.1).

A variety of existing beam elements, based on the *Timoshenko* theory outlined above, have been reviewed by DAWE (39) and THOMAS et al. (40) and it appears that the basic *Timoshenko* element is widely favoured for general applications in which shear effects may be of importance. PRZEMIENIECKI (41) and DAVIS et al. (42), among others, have determined the stiffness characteristics of this element by exactly satisfying the differential equations governing the bending of a beam in the presence of shear and unloaded between its ends. This approach permits shear corrections to be conveniently incorporated, while SEVERN (43) has argued that a straightforward application of the more conventional displacement method may result in an erroneous solution. NARAYANASWAMI et al. (44) responded by presenting a displacement formulation for this element, in which the moment equilibrium equations were introduced as a form of constraint in order to couple the bending and shear deformations. Details of this formulation are given by NARAYANASWAMI et al. (44) expressing the resulting coupled displacement field of this two-noded element (see Diagram (3.3)) as :

$$\mathbf{u} = \begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} \\ \omega_{21} & \omega_{22} & \omega_{23} & \omega_{24} \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_4 \end{bmatrix}$$

$$\begin{aligned}
\text{where } \omega_{11} &= \frac{1}{4(1+\phi)} \left((1-\xi)^2(2+\xi) + 2(1-\xi)\phi \right) \\
\omega_{12} &= \frac{L}{8(1+\phi)} \left((1-\xi)^2(1+\xi) + (1+\xi)(1-\xi)\phi \right) \\
\omega_{13} &= \frac{1}{4(1+\phi)} \left((1+\xi)^2(2-\xi) + 2(1+\xi)\phi \right) \\
\omega_{14} &= \frac{L}{8(1+\phi)} \left((1+\xi)^2(\xi-1) - (1-\xi)(1+\xi)\phi \right) \\
\omega_{21} &= \frac{3}{2L(1+\phi)} (\xi-1)(\xi+1) \\
\omega_{22} &= \frac{1}{4(1+\phi)} \left((3\xi+1)(\xi-1) + 2(1-\xi)\phi \right) \\
\omega_{23} &= \frac{3}{2L(1+\phi)} (\xi+1)(1-\xi) \\
\omega_{24} &= \frac{1}{4(1+\phi)} \left((3\xi-1)(\xi+1) + 2(1+\xi)\phi \right)
\end{aligned} \tag{3.3.9}$$

$$\text{and also } \xi = \frac{2x}{L} \quad , \quad \phi = \frac{12EI}{Gk_e AL^2}$$

The shear/bending stiffness parameter (ϕ) can be seen to be of the order of (radius of gyration/beam length)² and so has to be visualised as a small correction to the following terms in the stiffness and mass matrices.

Using the interpolation functions (3.3.9) in conjunction with the appropriate strain definition as given by (3.3.6) and (3.3.7), it can be shown that

$$\alpha = \begin{bmatrix} -\frac{6y\xi}{L^2(1+\phi)} & -\frac{y(3\xi-1-\phi)}{L(1+\phi)} & \frac{6y\xi}{L^2(1+\phi)} & -\frac{y(3\xi+1+\phi)}{L(1+\phi)} \\ -\frac{\phi}{L(1+\phi)} & -\frac{\phi}{2(1+\phi)} & \frac{\phi}{L(1+\phi)} & -\frac{\phi}{2(1+\phi)} \end{bmatrix} \tag{3.3.10}$$

The stress-strain relationship for this element is expressed as

$$\mathbf{D} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix} \tag{3.3.11}$$

Again, the stiffness matrix follows from an explicit integration of equation (3.2.7).

Thus

$$\mathbf{K} = \frac{EI}{L^3(1+\phi)} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & L^2(4+\phi) & -6L & L^2(2-\phi) \\ -12 & -6L & 12 & -6L \\ 6L & L^2(2-\phi) & -6L & L^2(4+\phi) \end{bmatrix}$$

The corresponding mass matrix, including rotary inertia contributions, is determined from the application of equation (3.2.10), using the appropriate interpolation functions given in (3.3.9). It follows that

$$\mathbf{M} = \frac{\rho AL}{840(1+\phi)^2} \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_2 & \gamma_5 & -\gamma_4 & \gamma_6 \\ \gamma_3 & -\gamma_4 & \gamma_1 & -\gamma_2 \\ \gamma_4 & \gamma_6 & -\gamma_2 & \gamma_5 \end{bmatrix} + \frac{\rho AL}{360(1+\phi)^2} \left(\frac{d}{L}\right)^2 \begin{bmatrix} \gamma_7 & \gamma_8 & -\gamma_7 & \gamma_8 \\ \gamma_8 & \gamma_9 & -\gamma_8 & \gamma_{10} \\ -\gamma_7 & -\gamma_8 & \gamma_7 & -\gamma_8 \\ \gamma_8 & \gamma_{10} & -\gamma_8 & \gamma_9 \end{bmatrix}$$

where	$\gamma_1 = 312 + 588\phi + 280\phi^2$	$\gamma_2 = (44 + 77\phi + 35\phi^2)L$
	$\gamma_3 = 108 + 252\phi + 140\phi^2$	$\gamma_4 = -(26 + 63\phi + 35\phi^2)L$
	$\gamma_5 = (8 + 140\phi + 7\phi^2)L^2$	$\gamma_6 = -(6 + 14\phi + 7\phi^2)L^2$
	$\gamma_7 = 36$	$\gamma_8 = (3 - 15\phi)L$
	$\gamma_9 = (4 + 5\phi + 10\phi^2)L^2$	$\gamma_{10} = (-1 - 5\phi + 5\phi^2)L^2$

3.3.3 Mindlin beam elements

An alternative displacement formulation for incorporating the effects of transverse shear and rotary inertia is based on the *Mindlin* concept (45). Accordingly, the transverse displacement (u) and the cross-sectional rotation (θ) are assumed to be unrelated, enabling the independent interpolation of the field variables (u, θ). Utilising the *Timoshenko* beam theory, the element strain fields are defined by enforcing the compatibility relations (3.3.5)–(3.3.7), and equation (3.3.8) expresses

the element strain energy. The present procedure forms the basis for the formulation of a number of beam elements, both low and high order, to be described next.

3.3.3.1 Linear element

The simplest of the shear-flexible beam elements has four degrees of freedom associated with two nodes as shown in Diagram (3.3). The displacements and rotations are defined by independent linear shape functions, presented here in matrix form as

$$\mathbf{u} = \begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} \frac{1-\xi}{2} & 0 & \frac{1+\xi}{2} & 0 \\ 0 & \frac{1-\xi}{2} & 0 & \frac{1+\xi}{2} \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_4 \end{bmatrix} \quad (3.3.12)$$

The element strain field follows from equations (3.3.5) to (3.3.7)

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{xy} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{2y}{L} \frac{d}{d\xi} \\ \frac{2}{L} \frac{d}{d\xi} & -1 \end{bmatrix} \begin{bmatrix} u \\ \theta \end{bmatrix} \quad (3.3.13)$$

to give

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & \frac{y}{L} & 0 & -\frac{y}{L} \\ -\frac{1}{L} & -\frac{(1-\xi)}{2} & \frac{1}{L} & -\frac{(1+\xi)}{2} \end{bmatrix} \quad (3.3.14)$$

Evoking the definition for the stiffness matrix (3.2.7) in conjunction with the stress-strain relationship as expressed by (3.3.11), will, upon explicit exact integration, yield :

$$\mathbf{K}_E = \frac{GA}{12L} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 12\mu + 4L^2 & -6L & -12\mu + 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & -12\mu + 2L^2 & -6L & 12\mu + 4L^2 \end{bmatrix}$$

where

$$\mu = \frac{EI}{GA}$$

As always, explicit integration of equation (3.2.10), on substituting for the shape functions from (3.3.12), will yield the appropriate mass matrix, including rotary inertia effects.

$$\mathbf{M} = \frac{\rho L}{6} \begin{bmatrix} 2A & 0 & A & 0 \\ 0 & 2I & 0 & I \\ A & 0 & 2A & 0 \\ 0 & I & 0 & 2I \end{bmatrix}$$

3.3.3.2 Shear locking and reduced integration

In Section (3.4) it will be demonstrated that the exactly integrated stiffness matrix ($\mathbf{K}_E$) suffers from what is termed as shear locking, tending to produce abnormally stiff elements that lead to worthless results. The existence of this locking phenomenon is widely known and practitioners in the field of finite element have found, mainly by experimenting, ways to remedy the problem, although providing a convincing explanation as to the exact cause of this type of misbehaviour has proven much more challenging. PRATHAP et al. (46) have investigated the problem of shear locking in the linear *Mindlin* element of Section (3.3.3.1) and have demonstrated that the use of exact integration in conjunction with the usual shape functions, result in a spurious shear-related bending-energy term, which arises due to the discretisation process, and which for a finite size element grossly modifies the bending rigidity of the element. Following an extensive investigation into the causes of element misbehaviour, in particular locking and spurious stress oscillations, PRATHAP et al. (47,48,49,50) have proposed a conceptual scheme referred to as the ‘field-consistency’ principle and an operational procedure, called ‘functional reconstitution’, for devising field-consistent (i.e. well-behaved) elements. In general, the problematic elements tend to be associated with the modelling of multi-strain field problems, where the behaviour of the continuum needs to be described by more than one strain-field, and these strain fields are defined in terms of two or more independent nodal field variables (e.g. shear strain field in the *Mindlin* beam elements). Physical behaviour, however, may require that any

number of these strain fields be constrained in particular ways under certain conditions (e.g. shear strain and its associated energy term must vanish in the limit of very thin beams to recover the *Kirchoff* constraint, $\theta = \frac{du}{dx}$ and $U_S = 0$, to give $U = U_B$) and failure to do so leads to an ill- conditioned element that exhibits such characteristics as locking, spurious stress oscillations, etc. and is referred to as field-inconsistent. To ensure field-consistency, all the terms in a strain field expression must have associated with them, coefficients containing contributions from all the independent interpolations of the field variables that appear in the definition of the strain field. Thus, for the linear *Mindlin* element :

$$u = a_{u1} + a_{u2}x \quad , \quad \theta = a_{\theta1} + a_{\theta2}x$$

hence
$$\epsilon_{xx} = -y \frac{d\theta}{dx} = -ya_{\theta1} \quad (a)$$

$$\epsilon_{xy} = \frac{du}{dx} - \theta = (a_{u2} - a_{\theta1}) + a_{\theta2}x \quad (b)$$

The expression in (b) is field-inconsistent, since the coefficient of the term 'x' has only contributions from 'θ' and not 'u'. According to the new concept, the most consistent approach to enforcing the field-consistency requirement, involves the use of appropriate substitute shape functions, which in this case would be ($\theta = a_\theta$) in the definition of 'ε_{xy}' alone to give :

$$\epsilon_{xy} = a_{u2} - a_\theta \quad (c)$$

Alternative techniques for introducing field- consistency have been considered by PRATHAP (47,49), including reduced integration. To be effective, an optimal integration strategy needs to be established for the particular element that is being considered that will integrate all the contributions from the consistent terms and remove all the contributions from the inconsistent terms in the strain field interpolation. Although there may be cases where it may not be possible to achieve true field-consistency by such an integration scheme, HUGHES et al. (51) have

successfully employed reduced integration as a means of alleviating shear locking in the linear *Mindlin* element and PRATHAP et al. (46) have shown that for this particular element, underintegrating the shear contributions has an identical effect to using modified shape functions to achieve field-consistency. Consequently, the two-point Gaussian quadrature, necessary for an exact integration, is replaced by a one-point integration, which still exactly integrates the bending terms while underintegrating the shear contribution. The modified stiffness matrix for the field-consistent, linear element thus becomes

$$\mathbf{K}_R = \frac{GA}{12L} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 12\mu + 3L^2 & -6L & -12\mu + 3L^2 \\ -12 & -6L & 12 & -6L \\ 6L & -12\mu + 3L^2 & -6L & 12\mu + 3L^2 \end{bmatrix}$$

3.3.3.3 Quadratic element

Examination will reveal that the linear element is capable of modelling a linear shear variation along the element length compared to the constant shear offered by the *Timoshenko* element. However, this apparent advantage is offset by the fact that the linear element can only predict a constant bending stress with length, while both the *Hermitian* and the *Timoshenko* elements can represent a linear variation. In order to overcome this shortcoming, it is necessary to opt for a higher order element. This move may also help to mitigate the shear locking problem encountered by the low order element.



Diagram (3.5) — Quadratic beam element model

The quadratic element has an extra centre node as depicted in Diagram (3.5), giving a total of six displacement and rotational freedoms. The field variables (u, θ) , now independently interpolated by second order polynomial functions, can be expressed as follows

$$\mathbf{u} = \begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} -\frac{\xi - \xi^2}{2} & 0 & 1 - \xi^2 & 0 & \frac{\xi + \xi^2}{2} & 0 \\ 0 & -\frac{\xi - \xi^2}{2} & 0 & 1 - \xi^2 & 0 & \frac{\xi + \xi^2}{2} \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_6 \end{bmatrix}$$

The strain field is defined by equation (3.3.13) giving

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & -\frac{y}{L}(2\xi - 1) & 0 & \frac{4y\xi}{L} & 0 & -\frac{y}{L}(2\xi + 1) \\ \frac{(2\xi - 1)}{L} & \frac{\xi - \xi^2}{2} & -\frac{4\xi}{L} & -(1 - \xi^2) & \frac{(2\xi + 1)}{L} & -\frac{\xi + \xi^2}{2} \end{bmatrix} \quad (3.3.15)$$

Substituting from (3.3.11) and (3.3.15) into (3.2.7) and explicitly performing the exact integration will finally yield the element stiffness matrix. This can be expressed in the following format

$$\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2$$

$$\text{where } \mathbf{K}_1 = \frac{GA}{60L} \begin{bmatrix} 140 & 30L & -160 & 40L & 20 & -10L \\ 30L & 140\mu & -40L & -160\mu & 10L & 20\mu \\ -160 & -40L & 320 & 0 & -160 & 40L \\ 40L & -160\mu & 0 & 320\mu & -40L & -160\mu \\ 20 & 10L & -160 & -40L & 140 & -30L \\ -10L & 20\mu & 40L & -160\mu & -30L & 140\mu \end{bmatrix}$$

$$\mathbf{K}_2 = \frac{GA}{60L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8L^2 & 0 & 4L^2 & 0 & -2L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4L^2 & 0 & 32L^2 & 0 & 4L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2L^2 & 0 & 4L^2 & 0 & 8L^2 \end{bmatrix}$$

$$\text{and } \mu = \frac{EI}{GA}$$

Based on the chosen interpolation functions this element can be shown to be field-inconsistent (Section (3.3.3.2)) and thus is expected to be plagued by shear locking. Although far less severe, it is evident from test results (see Section (3.4)) that locking is present and once more reduced integration has been found to alleviate the problem. The two-point Gauss quadrature only affects the stiffness components relating to the shear contribution, so that in the modified matrix ($\mathbf{K}_2$) becomes :

$$\mathbf{K}_2 = \frac{GA}{360L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 40L^2 & 0 & 40L^2 & 0 & -20L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 40L^2 & 0 & 160L^2 & 0 & 40L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -20L^2 & 0 & 40L^2 & 0 & 40L^2 \end{bmatrix}$$

The mass matrix for this element can be derived as before, and is given by :

$$\mathbf{M} = \frac{\rho L}{120} \begin{bmatrix} 16A & 0 & 8A & 0 & -4A & 0 \\ 0 & 16I & 0 & 8I & 0 & -4I \\ 8A & 0 & 64A & 0 & 8A & 0 \\ 0 & 8I & 0 & 64I & 0 & 8I \\ -4A & 0 & 8A & 0 & 16A & 0 \\ 0 & -4I & 0 & 8I & 0 & 16I \end{bmatrix}$$

3.3.3.4 Cubic element

In dynamic problems, especially when the higher harmonics have been excited, the spatial variation of the strain fields will be quite complex and the application of the higher order *Mindlin* type elements appears an attractive proposition. In an article by DAWE (52) on the vibration of plates using *Mindlin* theory, it has been suggested that the use of higher order elements based on independent interpolation of the primary field variables (u, θ) would result in an increase in efficiency.

The cubic element has four equally spaced nodes with two degrees of freedom (u, θ) per node as shown in Diagram (3.6) below

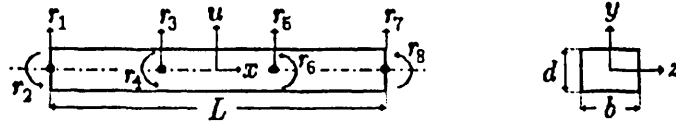


Diagram (3.6) — Four-noded cubic element model

Following the already familiar procedure, cubic shape functions will be used to express the spatial variation of transverse displacement and rotation for this element, so that

$$\mathbf{u} = \begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} \omega_1 & 0 & \omega_2 & 0 & \omega_3 & 0 & \omega_4 & 0 \\ 0 & \omega_1 & 0 & \omega_2 & 0 & \omega_3 & 0 & \omega_4 \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_8 \end{bmatrix}$$

$$\begin{aligned} \text{where } \omega_1 &= -\frac{1}{16}(9\xi^3 - 9\xi^2 - \xi + 1) & \omega_2 &= \frac{9}{16}(3\xi^3 - \xi^2 - 3\xi + 1) \\ \omega_3 &= \frac{9}{16}(-3\xi^3 - \xi^2 + 3\xi + 1) & \omega_4 &= \frac{1}{16}(9\xi^3 + 9\xi^2 - \xi - 1) \end{aligned}$$

The strain field, defined by equation (3.3.13), is simply

$$\boldsymbol{\alpha} = \frac{2}{L} \begin{bmatrix} 0 & -y \frac{d\omega_1}{d\xi} & 0 & -y \frac{d\omega_2}{d\xi} & 0 & -y \frac{d\omega_3}{d\xi} & 0 & -y \frac{d\omega_4}{d\xi} \\ \frac{d\omega_1}{d\xi} & \frac{L\omega_1}{2} & \frac{d\omega_2}{d\xi} & \frac{L\omega_2}{2} & \frac{d\omega_3}{d\xi} & \frac{L\omega_3}{2} & \frac{d\omega_4}{d\xi} & \frac{L\omega_4}{2} \end{bmatrix}$$

Using the stress-strain relation of (3.3.11), the above strain interpolation matrix and equation (3.2.7), the exactly integrated stiffness matrix of this cubic element will be given by :

$$\mathbf{K} = \frac{GA}{3360L} (\mathbf{K}_1 + L^2 \mathbf{K}_2)$$

$$\text{where } \mathbf{K}_1 = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12}^t \\ \mathbf{K}_{12} & \mathbf{K}_{22} \end{bmatrix}$$

$$\mathbf{K}_{11} = \begin{bmatrix} 12432 & 1680L & -15876 & 2394L \\ 1680L & 12432 & -2394L & -15876 \\ -15876 & -2394L & 36288 & 0 \\ 2394L & -15876 & 0 & 36288 \end{bmatrix}$$

$$\mathbf{K}_{12} = \begin{bmatrix} 4536 & 1008L & -24948 & -3402L \\ -1008L & 4536 & 3402L & -24948 \\ -1092 & -294L & 4536 & 1008L \\ 294L & -1092 & -1008L & 4536 \end{bmatrix}$$

$$\mathbf{K}_{22} = \begin{bmatrix} 36288 & 0 & -15876 & 2394L \\ 0 & 36288 & -2394L & -15876 \\ -15876 & -2394L & 12432 & -1680L \\ 2394L & -15876 & -1680L & 12432 \end{bmatrix}$$

$$\mathbf{K}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 256 & 0 & 198 & 0 & -72 & 0 & 38 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 198 & 0 & 1296 & 0 & -162 & 0 & -72 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -72 & 0 & -162 & 0 & 1296 & 0 & 198 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 38 & 0 & -72 & 0 & 198 & 0 & 256 \end{bmatrix}$$

The performance of this cubic element too is improved by 'underintegrating' the shear related stiffness components using a three-point Gauss quadrature, thus ensuring field-consistency (Section (3.3.3.2)). The resulting modified ($\mathbf{K}_2$) matrix becomes

$$\mathbf{K}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 231.7 & 0 & 270.9 & 0 & -144.9 & 0 & 62.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 270.9 & 0 & 1077.3 & 0 & 56.7 & 0 & -144.9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -144.9 & 0 & 56.7 & 0 & 1077.3 & 0 & 270.9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 62.3 & 0 & -144.9 & 0 & 270.9 & 0 & 231.7 \end{bmatrix}$$

The mass matrix for the element is obtained directly from the application of equa-

tion (3.2.10), which upon explicit integration yields :

$$\mathbf{M} = \frac{\rho L}{3360} \begin{bmatrix} 256A & 0 & 198A & 0 & -72A & 0 & 38A & 0 \\ 0 & 256I & 0 & 198I & 0 & -72I & 0 & 38I \\ 198A & 0 & 1296A & 0 & -162A & 0 & -72A & 0 \\ 0 & 198I & 0 & 1296I & 0 & -162I & 0 & -72I \\ -72A & 0 & -162A & 0 & 1296A & 0 & 198A & 0 \\ 0 & -72I & 0 & -162I & 0 & 1296I & 0 & 198I \\ 38A & 0 & -72A & 0 & 198A & 0 & 256A & 0 \\ 0 & 38I & 0 & -72I & 0 & 198I & 0 & 256I \end{bmatrix}$$

3.4 ASSESSMENT OF ELEMENT PERFORMANCE

The various low and high order beam elements presented in the previous Section will next be applied to the analysis of specific numerical examples in order to evaluate and compare the performance of these elements and identify any inherent limitations or misbehaviour that may exist. The ability of the elements to correctly predict displacements and stresses will be judged on the basis of static analysis results of loaded beam systems. A further consideration, in dynamic analysis, is the effectiveness of an element in successfully modelling the harmonics of a system. The accuracy with which the various elements meet this requirement will also be examined by application to the calculation of natural frequencies.

An inherent characteristics of the finite element analysis is that in general using a greater number of smaller elements to model a system can lead to an improved solution accuracy but also an increased solution cost. In static analysis, the general practice is to compromise by refining the mesh (arrangement of elements) in and around regions of expected high stress concentration (e.g. near point loads). In contrast, transient dynamic analysis (e.g. impact) involves the propagation of stress fronts and thus a much more uniform mesh is required to avoid spurious reflections at discontinuous element stiffnesses (see Section (3.6)). In view of the fact that the beam elements are ultimately to be applied to impact analysis, the

numerical tests will be conducted on uniform mesh idealisations. In addition, an equal number of degrees of freedoms will be used in each case, so as to enable a direct comparison of efficiency of the different beam elements.

3.4.1 Shear coefficient

In formulating the beam elements to include shear effects, the shear strain is assumed constant across the cross-section and expressed in terms of the nodal variables (u, θ) , defined along the elements neutral axis (see equation (3.3.7)). At this stage, a correction factor, dependent on the shape of the cross-section needs to be introduced to account for the non-uniform nature of the cross-sectional shear strain distribution. For a rectangular cross-section, the elementary beam theory predicts a through-the-depth parabolic shear distribution, while a more accurate representation (see TIMOSHENKO (53)) includes an additional complementary strain system associated with the variation of shear across the width of the cross-section.

Traditionally, the shear coefficient (k_e) has been regarded as the ratio of the average shear strain on a cross-section to the shear strain at the centroid, and assuming that the basic parabolic distribution is sufficiently accurate, this leads to $(k_e = \frac{2}{3})$ for a rectangular cross-section. A brief survey of earlier work (see COWPER (54)) has indicated that this conventional value of ' k_e ' has frequently led to unsatisfactory results when used in the calculation of the high-frequency spectrum of vibrating beams and consequently arbitrarily adjusted values of ' k_e ' (0.822–0.87) have been employed to give better results.

COWPER (54) has presented a formulae for the shear coefficient that emerges as a by-product of an analysis whereby the *Timoshenko* equations are derived by integration of the three-dimensional elasticity equations. Using the appropriate function to represent the 'complete' shear distribution on a rectangular cross-

section, in conjunction with this formulae, yields :

$$k_e = \frac{10(1 + \nu)}{12 + 11\nu} \quad (3.4.1)$$

giving ($k_e = 0.851$) for ($\nu = 0.33$). In a subsequent paper COWPER (55) has demonstrated that far better accuracy can be achieved using the coefficient as defined in (3.4.1) as compared to the customary value of ($k_e = \frac{2}{3}$) and that discrepancies tend to increase with increasing depth to length ratio of the beam.

Henceforth, the appropriate value for the shear coefficient will be based on COWPER's formulae (3.4.1).

3.4.2 Eigenvalue tests

Finite element idealisations, employing the different beam elements, have been applied to the calculation of the natural frequencies of a simply-supported, uniform beam. Details of the mesh configurations are given in Table (3.1) and one should point out that in conducting the tests, only one half of the beam has been modelled, thus restricting the analysis to the symmetric harmonics of the system. The dimensions and material properties of the beam have been based on that of the specimen to be used in later impact experiments. The parameters used are :

E	$=$	72.0	GN/m ²	ℓ	$=$	1.44	m
G	$=$	27.0	GN/m ²	d	$=$	0.0181	m
ρ	$=$	2720	Kg/m ³	b	$=$	0.0192	m
ν	$=$	0.33					
k_e	$=$	0.851					

With an aspect ratio (length/depth) of nearly 80:1, the beam is very thin and provides an extreme, but realistic, test of shear extraction when the beam response is dominated by bending deformation. Frequencies for the first 10 harmonics have been obtained for each test case and the results compared with values calculated from the solution of the *Timoshenko* beam equations (see TIMOSHENKO (56)),

the exception being the case involving the *Hermitian* element for which the appropriate comparison has been based on values resulting from the *Bernoulli-Euler* equation. The findings, expressed as percentage error in frequencies, are summarised in Tables (3.3,3.4) and where applicable results for both reduced and exactly integrated stiffness matrices are included. A comparison of predicted frequencies based on the *Timoshenko* and *Bernoulli-Euler* theories is also presented in Table (3.5) to highlight the importance of shear and rotary inertia effects. These secondary effects can be seen to mainly influence the higher harmonics, leading to a period elongation.

The results in Table (3.3) clearly indicate that the performance of the shear flexible *Mindlin* elements is significantly improved if the stiffness matrix of the element is appropriately underintegrated to ensure field-consistency, as suggested by PRATHAP (47). In spite of the very noticeable improvement, the predictive capabilities of the linear element, as regards natural frequencies, remains unimpressive, requiring many more elements if better accuracy is to be achieved. In this respect the quadratic elements' performance is much more reassuring, especially in dealing with the lower harmonics, while the cubic element is shown to be undoubtedly the most superior of the *Mindlin* elements.

One common characteristics of the *Mindlin* elements emerges from the graphical comparison of the performance of the different beam elements, presented in Figure (3.1). Despite the use of reduced stiffness matrices, it is evident that for this class of elements the errors increase more rapidly with increasing harmonics than for either the *Timoshenko* or *Hermitian* elements. Thus, although the cubic *Mindlin* element, on the basis of the current findings, is the most accurate in representing the lower vibration modes (see Table (3.3)), the *Timoshenko* element appears to be a better all around contender.

3.4.3 Static tests

A centrally-loaded, simply-supported beam has been analysed using the finite element idealisations employed in the eigenvalue tests and presented in Table (3.1). Dimensions and material properties of the beam are as outlined previously (Section (3.4.2)). The total central load has been taken as ($P = 400$ N) and again only one half of the beam has been considered for analysis purposes. The resulting finite element predictions for the displacements and stresses have been compared with theoretically expected values. These theoretical estimates are based on the elementary beam theory and, where applicable, displacements are corrected to account for shear deflection, evaluated from strain energy arguments. The appropriate expressions are listed in Appendix (A).

Concentrating on displacements first, the findings conclusively demonstrate (see Figure (3.2)) that for the linear *Mindlin* element the exactly integrated stiffness matrix causes severe shear locking. Underintegrating the stiffness matrix is shown to restore credibility to the element, reducing the observed discrepancies to around 0.5%. The 'exact' stiffness matrix for the quadratic element also suffers from shear locking, though to a far lesser extent (see Figure (3.3)), and the application of reduced integration technique can be seen to remedy the problem. For the cubic element, locking is not evident on basis of displacement results and both 'exact' and 'reduced' stiffnesses tend to predict the beam deflection with extreme accuracy, as do the *Timoshenko* and *Hermitian* elements.

As regards stresses, the predicted bending stresses resulting from the application of the cubic *Mindlin*, *Timoshenko* and *Hermitian* elements have been found to be in excellent agreement with theoretical estimates, while the two former elements also correctly represent the expected constant shear stress along the beam length and with similar accuracy. The linear *Mindlin* element, based on exact integration, tends to underestimate the bending stresses as demonstrated in Figure (3.4) and the use of reduced integration is shown to lead to a dramatic improvement

in the performance of this element. As for the discontinuities in bending stresses evident in Figure (3.4), these will be present irrespective of the order of integration employed, since this particular element can only handle a constant bending stress field (see Section (3.3.1)). However, the quadratic *Mindlin* element should be capable of representing a linearly varying bending stress, but as shown in Figure (3.5), if the exact stiffness matrix is used, this leads to stress discontinuity at the element boundaries. Underintegrating the stiffness matrix to remove the mild shear locking of this element can be seen to eliminate the stress discontinuities too.

Another problem associated with the lower order *Mindlin* elements is highlighted in Figures (3.6,3.7). The static analysis is expected to produce a constant shear stress along the length of the beam between the supports and the central load, and as can be observed the nodal shear stresses obtained by using the linear element give rise to large discontinuities at interelement boundaries, oscillating about the expected theoretical value. Although no longer discontinuous, the oscillating nature of the shear stress prediction is also evident in the case of the quadratic element. Reduced integration has been found to be ineffective in overcoming this particular difficulty. Further investigations have indicated that in order to recover the correct shear stress distribution some form of stress smoothing technique is called for. This topic will be considered in the following Section where the cause of this apparent misbehaviour will also be briefly discussed.

3.4.4 Application of smoothing techniques to stress recovery

The fact that the displacement formulation of the finite element technique may deliver a stress field that is discontinuous along the element boundaries is common knowledge. These errors in stress computation may originate from two entirely unrelated sources and it is important to distinguish between the two. The

first type of error is characterised by a discontinuity in stress at element junctures and stems from an inherent feature of the displacement based FE formulation. In this formulation, the assumed function representing the element displacement field satisfies compatibility across elements, thus maintaining displacement continuity across the element interfaces, while the application of virtual work criterion ensures that in general the equations of equilibrium are satisfied only in the mean throughout the element, and point-wise equilibrium is satisfied neither within the element nor along the element periphery. Consequently stress components are not expected to be continuous across element boundaries, although these stress differences decrease as the finite element idealisation is refined and in practice, averaging of element boundary stresses usually give acceptable results. This sort of stress discontinuity serves as a useful indicator in establishing whether refinement of a finite element mesh model is required.

The second and more serious kind of error, encountered by numerous authors, arises due to an embedded characteristic of certain class of elements, resulting in spurious oscillations (e.g. as observed in Figures (3.6,3.7)). An explanation for this misbehaviour by these elements has recently been provided by PRATHAP et al. (49,50), based on the failure of these elements to satisfy the requirements of field-consistency formulation (Section (3.3.3.2)). According to this concept, the problematic elements are associated with the modelling of constrained multi-strain field problems, for which the inconsistently represented constrained strain field can result in oscillatory stress variations. These stress oscillations are considered to represent the mechanism by which the spurious stiffening energy of locking is distributed as a self-equilibrating stress field within the element. Experience has shown that under these circumstances the element nodes appear to be the worst stress sampling points, although there exist positions within the element, usually at the Gauss integration points, where stresses may be correctly sampled. In these situation, the application of smoothing techniques can prove

very effective in extracting useful information provided that the optimal sampling locations can be identified.

The smoothing process can be applied either globally over the entire finite element domain or locally over each individual element and it is the latter approach that has been adopted here. Local smoothing techniques rely on individual element information only and consequently do not lead to unique stresses at nodal points, so that some form of nodal averaging will be required to ensure continuous nodal stress values, but are simpler to implement and less expensive to perform compared to their global counterparts, while generally producing very satisfactory results (see HINTON et al. (57)). The important step in a smoothing scheme is the identification of the correct sampling positions within the element, while the rest of the procedure is essentially a curve-fitting exercise in which an appropriate interpolation function is chosen to relate all the values at the sampling points and subsequently used to extract the nodal point values.

Thus, 'q' sampling points would need a polynomial of order 'q-1' with 'q' number of coefficients. Expressed in matrix notation, the stresses at the sampling points are given by :

$$\sigma_q = a_q B$$

where σ_q represents a column vector ($q \times 1$) of sampling point stresses

B represents a column vector ($q \times 1$) of coefficients

$$a_q = \begin{bmatrix} 1 & a_{q1} & a_{q1}^2 & \dots & a_{q1}^q \\ 1 & a_{q2} & a_{q2}^2 & \dots & a_{q2}^q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{qq} & a_{qq}^2 & \dots & a_{qq}^q \end{bmatrix}$$

and a_{qi} represents the position of the i^{th} sampling point

The nodal stresses can also be expressed in terms of the same polynomials so that :

$$\sigma_n = a_n B$$

where $\mathbf{a}_n$ is a matrix of nodal positions (a_n). Hence, it follows that :

$$\sigma_n = \mathbf{a}_n \mathbf{a}_q^{-1} \sigma_q = \mathbf{A} \sigma_q$$

and the matrix ($\mathbf{A}$) is referred to as the smoothing matrix and is only a function of the sampling positions (a_q) and nodal positions (a_n).

Different schemes for the selection of appropriate sampling points have been proposed by BARLOW (58) and HINTON et al. (57). The procedure adopted by BARLOW (58), imposes on the element a complete polynomial displacement field that is an order higher than the highest complete field the actual element can exactly represent based on the shape functions employed in its derivation. The objective is to identify unique positions within the element at which the stresses have the same degree of accuracy as the nodal displacements. BARLOW has applied this technique to a range of elements and consistently found that the optimal stress points coincide with the Gauss points of the element, which has led numerous authors to conclude that this approach provides justification for sampling element stresses at the Gauss points, as experience had suggested. When this scheme was used in conjunction with the *Mindlin* beam elements, as expected, the reduced Gauss integration points emerged as the optimal sampling locations for the linear and quadratic elements, but this was not the case for the cubic element for which a different set of points resulted (Table (3.6)). HINTON et al. (57) favour an alternative approach, arguing that stress oscillations are a result of the least square nature of the FE displacement method and propose a smoothing technique based on a least square fit approach. To ensure that the smoothed stress function, $\tilde{\sigma}(\xi)$, of order ' q ' is a least square fit to the element stress function, $\sigma(\xi)$, which is an order higher, the latter needs to be sampled at ' q ' locations given by the q -point Gauss rule. This is stated as

$$\int_{-1}^1 (\sigma - \tilde{\sigma}) \xi^{i-1} d\xi = 0 \quad i = 1, 2, \dots, q$$

and is applied in turn to all values of ' i ' up to and including ' q '. When applied to the *Mindlin* beam elements, the sampling points are found, as expected, to be the reduced Gauss integration points for the elements, listed in Table (3.6).

The two schemes outlined above for locating optimal sampling points for use in stress smoothing operation are based on heuristic arguments and it is not surprising that different schemes may predict different sampling points, as observed for the cubic *Mindlin* element. However, according to PRATHAP's (50) field-consistency concept, an element defined in a field-consistent manner (see Section (3.3.3.2)) should not exhibit stress oscillation and thus the application of stress smoothing ought to be unnecessary. Although in deriving the stiffness matrices of the present *Mindlin* elements (Section (3.3.3)), the field-consistency requirement was enforced by employing an appropriate reduced integration scheme, the fact that substitute, field-consistent shape functions were not employed has meant that stress recovery has been based on the field-inconsistent shape functions which consequently have resulted in the spurious stress oscillations that have been observed. In fact, subsequent investigation has revealed that the shape functions necessary to enforce field-consistency in these *Mindlin* elements are identical to those that emerge from the stress smoothing operation based on the reduced Gauss points of the element. It is thus evident that for these particular elements, stress smoothing using the reduced Gauss points, has the effect of enforcing field-consistency at the stress recovery stage.

The stress smoothing matrices for the various elements, as implemented in the present code, are given in Section (c) of Appendix (A) and the technique has been applied to the results of the previous test example. It is evident from Figures (3.6,3.7), that the violent shear stress oscillations observed for the linear and quadratic *Mindlin* elements have been ironed out to recover the theoretically expected stress values, irrespective of whether 'reduced' or 'exact' stiffness matrices are employed. This seems to indicate that shear locking affects only the displacement and bending stress predictions and not the shear stress results. As

regards the remaining elements and the bending stresses in general, these had not exhibited any ill-effects and as expected, were not found to be affected by the smoothing scheme. In fact, it can be shown that for the present beam elements the smoothed bending stress function is identical to the elements bending stress function, thus independent of the choice of sampling points and unaffected by the current smoothing scheme.

3.4.5 Additional static tests

In dynamic problems, the stress variation along the length of a heavy beam will be influenced by the presence of inertia effects and deemed to be rather more complicated than the one analysed in the first test example. Consequently, in this Section, some higher order stress field problems have been analysed, employing the various beam elements, to further examine the ability of these elements from the point of view of future application to dynamic impact problems.

Two test cases have been considered. The first involves a uniformly loaded, simply-supported beam as shown in Diagram (3.7a) in which the loading can be regarded as representing the beams static weight and for which the shear stress is expected to vary linearly and the bending stresses quadratically along the beam length. The magnitude of the loading has been taken as ($w_c = 280 \text{ N/m}$), so as to give a total load of approximately (403 N), similar to that used in the test example of Section (3.4.3). In the second test case shown in Diagram (3.7b), the beam loading is made up of a sinusoidal distribution, which can be regarded as representing the inertia loading of a beam vibrating in its fundamental mode, and a concentrated central load, that can be assumed to be due to the dead weight of an impactor. Again the loading has been chosen such that the two contributions are of similar magnitude — a central load of ($P = 400 \text{ N}$) and a distributed loading ($w_c = 400 \text{ N/m}$), giving a total distributed load of approximately 366 N.

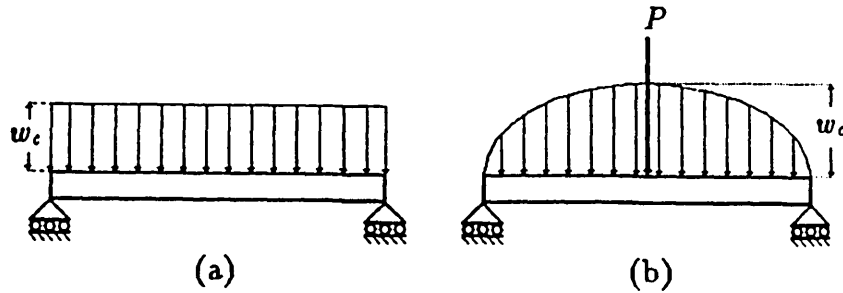


Diagram (3.7) — Schematic representation of test cases

Finite element idealisations, beam dimensions and material properties are as used for the previous tests. The distributed loadings, however, have been modelled in the finite element analysis, by kinematically equivalent nodal forces, which are functions of both element type and mesh idealisation. The appropriate expressions for these forces and resulting loading vectors for the different load cases and the various beam elements are given in Section (b) of Appendix (A). Theoretical estimates of stresses and displacements for comparison with FE predictions, are based on the elementary beam theory and also presented in Appendix (A).

The results of these tests are presented in Figures (3.8–3.31), and the following observations can be made :

a) *Hermitian element* :

Displacements are fractionally underestimated (Figures (3.8,3.9)), but the bending stresses are good (see Figures (3.10) and (3.11)). (Note that theoretical estimates exclude shear effects for this element)

b) *Timoshenko element* :

Displacements are slightly lower than theoretically predicted, as can be seen from Figures (3.8) and (3.9), but the bending stresses (Figures (3.12,3.14)) are correct. This element can only model a constant shear along its length

(see formulation), so that a discontinuity in shear at element interfaces is expected. However, the averaged shear stresses compare well with theory, as is evident from Figures (3.13,3.15).

c) Linear *Mindlin* element :

As previously established, reduced stiffness matrix and stress smoothing are essential for this element. Subject to these restrictions, the displacements are found to be underestimated (see Figures (3.8,3.9)), while averaged bending stresses, presented in Figures (3.16) and (3.18), and the smoothed, averaged shear stresses in Figures (3.17,3.19) are reasonably good. Stress discontinuities at element junctures result from the fact that this element can only represent a constant bending stress along its length and smoothed shear stresses are sampled at the mid-point only and thus are also constant along the element.

d) Quadratic *Mindlin* element :

The 'reduced' stiffness matrix has been used in the present tests and both the displacements (Figures (3.8,3.9)) and the bending stresses in Figures (3.20) and (3.22) are found to be in close agreement with theory. The smoothed shear stresses are also seen to be accurately predicted (Figures (3.21,3.23)).

e) Cubic *Mindlin* element :

The difference in performance based on the use of exact and reduced stiffness matrices has been found to be very small for this element. However, as was explained in Section (3.3.3.2), all of the *Mindlin* elements suffer from locking effects, although the severity of this problem diminishes rapidly with increase in the order of the element, and for the cubic element locking has been found to be extremely mild. Nevertheless, using the reduced stiffness matrix leads to marginally more accurate predictions. The displacement results shown in Figures (3.8,3.9) and the bending stress estimates (Figures (3.24,3.28)) are in excellent agreement with theory, but the shear stresses can be seen to

require smoothing (Figures (3.25,3.29)). As indicated in Table (3.6), different sampling points emerge for the cubic element, depending on which technique is used. Here, both schemes have been considered and it is evident, comparing Figures (3.26,3.27) and (3.30,3.31), that smoothing based on selecting the Gauss points as the sampling points give far superior results, consistent with expectations from PRATHAP's field-consistency concept (see Section (3.4.4)).

3.4.6 Summary of findings

The static and eigenvalue tests have demonstrated that the *Mindlin* beam elements exhibit an erratic behaviour, leading to locking and violent stress oscillations, though the extent of these ill-effects is dependent on the order of the particular element and generally diminishing with an increase in the order of the element. Reasons for this misbehaviour have been briefly outlined in Sections (3.3.3.2) and (3.4.4) and these have been remedied by use of reduced integration in the derivation of the stiffness matrices and smoothing techniques for the recovery of stresses. Such treatments have stabilised and dramatically improved the performance of these elements, with the cubic element, in general, showing closest agreement with theory. The cubic element is also far better suited to the calculation of eigenvalues compared to either the linear or the quadratic elements.

The *Timoshenko* element is on the whole well-behaved and quite accurate, especially in representing the higher frequencies of the system, the only limitation being that it can only model a constant shear stress. Consequently, the accuracy of the shear stress prediction may suffer when applied to problems involving rapidly changing shear stress fields, unless a large number of elements are used.

The *Hermitian* element, as expected performs quite well, the drawback being that it does not model shear and rotary inertia effects.

On the basis of superior performance, the cubic *Mindlin* element is the most suitable for application to the dynamic impact problem. Although the *Timoshenko*

element is better in modelling the higher harmonics of the system, the cubic *Mindlin* element is more accurate in calculating the lower eigenmodes and the displacement field (Figures (3.8,3.9)), while also capable of coping with higher order stress fields, shear as well as bending. Bearing in mind that under low velocity impact conditions, mainly the lower modes will be excited, it is the cubic element that will be the favoured choice.

3.5 SOLUTION OF THE DYNAMIC EQUATIONS

The equation of motion governing the dynamic response of a structural system represents an equilibrium condition at any instant in time (t), between the inertia forces ($\mathbf{R}_M = \mathbf{M}\ddot{\mathbf{r}}$), damping forces ($\mathbf{R}_C = \mathbf{C}\dot{\mathbf{r}}$), elastic stiffness forces ($\mathbf{R}_K = \mathbf{K}\mathbf{r}$) and externally applied loads ($\mathbf{R}(t)$). For a discretised system of finite elements, these set of equilibrium equations are expressed as :

$$\left. \begin{array}{l} \mathbf{R}_M + \mathbf{R}_C + \mathbf{R}_K = \mathbf{R} \\ \text{or} \quad \mathbf{M}\ddot{\mathbf{r}} + \mathbf{C}\dot{\mathbf{r}} + \mathbf{K}\mathbf{r} = \mathbf{R} \end{array} \right\} \quad (3.5.1)$$

It is evident that by ignoring the inertia and damping forces, (3.5.1) reduces to the static equilibrium equations, and the task of solving for the unknown displacements and hence stresses, is greatly simplified. In its present form, (3.5.1) represents a system of second order linear differential equations for which standard solution procedures do exist. However, in general finite element applications a large set of these equations need to be solved and to keep computational cost down it is essential that specific, effective procedures, designed to take advantage of the special properties of the coefficient matrices, are adopted.

BATHE (59) and COOK (60) have reviewed a range of widely popular algorithms, which can be classified into two classes, namely direct integration and mode superposition techniques. The latter approach uses the systems eigenvectors

to transform the equations in (3.5.1) from a finite element displacement basis into a generalised displacement basis, thus effectively de-coupling the set of simultaneous equations in (3.5.1). These can then be solved very efficiently in the new domain and the calculated response will then be transformed back to the original domain for superposition with the other components. Although such modal techniques can be adopted for non-linear analysis (see BATHE (59), COOK (60)), these are most economical when applied to linear analysis of inertial problems in which the response is known to be confined to a few harmonics. The parametric study of the previous Chapter was based on a modal approach, but here, a direct integration scheme will be utilised, since these methods are better suited to shock and wave propagation problems in which a large no. of high-frequency harmonics are excited, and just as effective for vibration problems. Non-linearities can also be incorporated and there is no restriction on the form of the damping matrix (C) as is the case with modal methods.

The direct integration algorithms can be further subdivided into explicit and implicit schemes. In explicit methods, the state at a time $(t + \Delta t)$ is predicted by using the equilibrium conditions at time (t) in conjunction with some form of extrapolation functions, expressing the accelerations and velocities of the system at time (t) in terms of displacements at various times up to and including that at time $(t + \Delta t)$. Such schemes, of which the central difference method is the best known one, usually produce equations that are cheap to solve by avoiding the need to invert (or factorise) the system's stiffness matrix and if damping is excluded and a lumped mass idealisation is adopted, the calculations can be performed at element level without the need to assemble or factorise any global coefficient matrices. Limitations include conditional stability which severely restricts the timestep of integration and the fact that accuracy may suffer in non-linear applications due to the failure to satisfy equilibrium at time $(t + \Delta t)$. Implicit techniques (e.g. *Houbolt*, *Wilson- θ* , *Newmark- β*) enforce the equilibrium conditions at time $(t + \Delta t)$ which is much more accurate, but this inherently involves

the factorisation of the stiffness matrix which increases computational costs. This is even more pronounced in non-linear problems where factorisation needs to be performed continually as the system stiffness properties alter and where in order to satisfy equilibrium at time $(t + \Delta t)$, iterations are required. To limit computational cost, implicit methods can be made unconditionally stable which will allow a much larger integration timestep to be chosen, but this is achieved at the expense of introducing artificial period elongation and response amplitude decay. It is argued that since finite element discretisations do not accurately represent the high harmonics of a system, such numerical effects, if not divergent in nature, will not appreciably affect the systems response especially if this is dominated by the lower modes. NICKELL (61) has compared the performance of the various popular implicit methods listed above and concluded that the *Newmark- β* technique is in general superior, but, as is evident from the brief foregone discussion, the choice of a suitable solution procedure is very much a function of the problem to be solved and the finite element idealisation that is adopted.

The direct integration procedure chosen for the present purposes is based on the cubic interpolation of inertia algorithm (CIIA) originally presented by ARGYRIS and CHAN (62), which although not as widely known as some of the previously mentioned methods, possess some desirable features and represents an attractive alternative. This algorithm can be regarded as implicit in the sense that it uses the equilibrium conditions at time $(t + \Delta t)$, but does not involve the factorisation of the stiffness matrix, thus ideally suited for non-linear problems. In fact, the only matrix to be inverted or factorised is the mass matrix and this is only done once at the start of the solution even for non-linear problems, unless the non-linearities are such as to affect the inertial properties of the system. If a lumped mass idealisation is adopted the solution can be completely performed at element level greatly reducing storage requirements. As its name suggests, inertia forces over each timestep are interpolated by cubic *Hermitian* polynomials so that the truncation errors are at least two orders less than that of the more commonly

used *Newmark- β* method. Applied to a number of shock and vibration problems by ARGYRIS et al. (63), this algorithm has consistently shown far superior accuracy compared with the more commonly used methods. The more accurate, though computationally more expensive, conditionally stable version of this algorithm has been implemented here, but minor alterations to the basic equations can yield an unconditionally stable version as presented by ARGYRIS et al. (63), the penalty being, as before, an artificial period elongation which for this particular algorithm is far less severe than for any of the aforementioned schemes (see ARGYRIS et al. (63)). Solution of the equations of this CHA method is by means of iterations for both linear and non-linear problems and thus for linear problems it may well be computationally less efficient than schemes like the *Newmark- β* method, though for non-linear problems the latter will also require an iterative solution.

3.5.1 Cubic interpolation of inertia algorithm

The algorithm is based on the assumption that the inertia forces ($M\ddot{r}$) vary according to a prescribed function of time, in this case as a cubic function, within a small time interval (τ). To define this variation, the four coefficients of the cubic function are determined by the conditions at the beginning and the end of this time interval and these conditions relate to the inertia forces (R_M) and its first time derivative ($\dot{R}_M = \frac{dR_M}{dt}$). Consequently, the cubic variation of inertia can be written in the form

$$R_M = \eta_1 R_{M0} + \eta_2 \tau \dot{R}_{M0} + \eta_3 R_{M1} + \eta_4 \tau \dot{R}_{M1} \quad (3.5.2)$$

where ' η_i ' are *Hermitian* polynomials expressed in terms of a non-dimensional time variable ($\zeta = \frac{t}{\tau}$) and given as

$$\left. \begin{aligned} \eta_1 &= 1 - 3\zeta^2 + 2\zeta^3 & \eta_2 &= \zeta - 2\zeta^2 + \zeta^3 \\ \eta_3 &= 3\zeta^2 - 2\zeta^3 & \eta_4 &= -\zeta^2 + \zeta^3 \end{aligned} \right\} \quad (3.5.3)$$

so that each produces a unit value for its associated coefficient while zero for all others. Also, the subscripts '0' and '1' in (3.5.2) denote values at the beginning and the end of the time interval when ($\zeta = 0$) and ($\zeta = 1$) respectively.

Integrating equation (3.5.2) with respect to time and substituting for the inertia forces (i.e. $\mathbf{R}_M(t) = \mathbf{M}\ddot{\mathbf{r}}$), it follows that

$$\left. \begin{aligned} \mathbf{M}\dot{\mathbf{r}} &= \mathbf{M}\dot{\mathbf{r}}_0 + \frac{\tau\mathbf{R}_{M0}}{2}(2\zeta - 2\zeta^3 + \zeta^4) + \frac{\tau\mathbf{R}_{M1}}{2}(2\zeta^3 - \zeta^4) \\ &\quad + \frac{\tau^2\dot{\mathbf{R}}_{M0}}{12}(6\zeta^2 - 8\zeta^3 + 3\zeta^4) + \frac{\tau^2\dot{\mathbf{R}}_{M1}}{12}(-4\zeta^3 + 3\zeta^4) \\ \mathbf{M}\mathbf{r} &= \mathbf{M}\mathbf{r}_0 + \tau\zeta\mathbf{M}\dot{\mathbf{r}}_0 + \frac{\tau^2\mathbf{R}_{M0}}{40}(20\zeta^2 - 10\zeta^4 + 4\zeta^5) \\ &\quad + \frac{\tau^2\mathbf{R}_{M1}}{40}(10\zeta^4 - 4\zeta^5) + \frac{\tau^3\dot{\mathbf{R}}_{M0}}{60}(10\zeta^3 - 10\zeta^4 + 3\zeta^5) \\ &\quad + \frac{\tau^3\dot{\mathbf{R}}_{M1}}{60}(-5\zeta^4 + 3\zeta^5) \end{aligned} \right\} \quad (3.5.4)$$

Substituting the value of ($\zeta = 1$) into the equations of (3.5.4) will yield the displacements ($\mathbf{r}_1$) and velocities ($\dot{\mathbf{r}}_1$) at then end of the interval, which are expressed as

$$\left. \begin{aligned} \mathbf{r}_1 &= \mathbf{r}_0 + \tau\dot{\mathbf{r}}_0 + \frac{\tau^2\mathbf{M}^{-1}}{60}(21\mathbf{R}_{M0} + 3\tau\dot{\mathbf{R}}_{M0} + 9\mathbf{R}_{M1} - 2\tau\dot{\mathbf{R}}_{M1}) = \mathbf{r}_0 + \mathbf{d}\mathbf{r} \\ \dot{\mathbf{r}}_1 &= \dot{\mathbf{r}}_0 + \frac{\tau\mathbf{M}^{-1}}{12}(6\mathbf{R}_{M0} + \tau\dot{\mathbf{R}}_{M0} + 6\mathbf{R}_{M1} - \tau\dot{\mathbf{R}}_{M1}) = \dot{\mathbf{r}}_0 + \mathbf{d}\dot{\mathbf{r}} \\ \ddot{\mathbf{r}}_1 &= \mathbf{M}^{-1}\mathbf{R}_{M1} \end{aligned} \right\} \quad (3.5.5)$$

The above set of equations can not be solved explicitly since they are dependent on the inertia force and its time derivative at the end of the interval, namely $\mathbf{R}_{M1}$ and $\dot{\mathbf{R}}_{M1}$, which in turn, according to equation (3.5.1) and its derivative, depend on $\mathbf{r}_1$, $\dot{\mathbf{r}}_1$ and $\ddot{\mathbf{r}}_1$ which themselves are being solved for. The situation may be further complicated by the fact that the stiffness matrix, damping matrix and even the mass matrix may or may not be functions of $\mathbf{r}_1$, $\dot{\mathbf{r}}_1$ and $\ddot{\mathbf{r}}_1$, depending on the problem being solved. Hence, an iterative solution for each timestep will

be necessary. This iterative procedure starts by assuming that at the beginning of a time interval

$$\dot{\mathbf{R}}_{M1} = \dot{\mathbf{R}}_{M0} \quad \text{and} \quad \mathbf{R}_{M1} = \mathbf{R}_{M0} + \tau \dot{\mathbf{R}}_{M0}$$

These first approximations enable $\mathbf{r}_1$ and $\dot{\mathbf{r}}_1$ to be determined using (3.5.5), which in turn are used to obtain a second estimation for the values of $\mathbf{R}_{M1}$ and $\dot{\mathbf{R}}_{M1}$ and this process is repeated until the results converge to within a set tolerance level. The calculation proceeds by using the end values of the current cycle as the initial values for the next time increment.

3.5.2 Stability of the CIIA method

Numerical integration schemes are considered as stable if any perturbations in the solution at a time (t) (e.g. due to round-off or truncation errors, etc.) do not grow due to the integration, thus adversely affecting the results. The integration timestep is generally the critical factor in controlling stability. Schemes for which the stability condition is satisfied independent of the timestep of integration are referred to as unconditionally stable schemes, whereas conditionally stable methods require that the integration timestep be below a critical value to ensure stability.

A systematic procedure for investigating the stability of iteration algorithms is given by BATHE et al. (64). The equations of integration are expressed in a recurrence form, such as

$$\mathbf{R}(t + \Delta t) = \mathbf{I}\mathbf{R}(t)$$

and it becomes clear that in order to ensure that errors in $\mathbf{R}$ do not propagate diversely, the spectral radius of the iteration matrix ($\mathbf{I}$) has to be less than unity. ARGYRIS et al. (63) presented a detailed stability analysis for the CIIA method based on this procedure. By considering the solution of an undamped, linearised

system, the present form of the CIIA technique has been shown to be conditionally stable, subject to the restrictions that $(\Delta t < \frac{1}{2}T_{\min})$, where $T_{\min}$ refers to the period of the highest frequency. The effects of damping and varying stiffness on Δt_{crit} have also been examined and it has been suggested that using a value of $(\Delta t < 0.3T_{\min})$ should ensure stability in the presence of such non-linearities. In the present code, the integration timestep has been chosen as $(\Delta t = 0.2T_{\min})$.

3.5.3 Implementation and practical considerations

The flow-chart in Diagram (3.8) depicts the way in which the CIIA integration scheme has been implemented into the present code. To apply the CIIA method to the solution of non-linear problems (e.g. bouncing (see Section (3.7)), etc.), the elastic forces ($\mathbf{R}_K$) and damping forces ($\mathbf{R}_C$) can not be evaluated directly at each step, but need to be accumulated as the solution proceeds. The non-linear effects can be introduced at two different positions in the iteration cycle, marked (a) and (b) on the flow-chart (Diagram (3.8)). The first is located at a stage where estimates for the new displacements ($\mathbf{r}_1$) and velocities ($\dot{\mathbf{r}}_1$) have been evaluated, which may be used to calculate and undertake any necessary changes to the structural property matrices, before these are used to derive the dynamic force vectors. As an alternative to making changes to the actual matrices, the dynamic force vectors may be modified directly to take account of non-linearities and the necessary alterations can be introduced at the position marked (b).

A criterion for the termination of the iteration cycle has been proposed by ARGYRIS et al. (63) which uses a measure of the vector norm of the change in the $\dot{\mathbf{R}}_M$ vector.

$$\left\| \hat{\mathbf{R}}_{M1} - \dot{\mathbf{R}}_{M1} \right\| = \left[\left(\hat{\mathbf{R}}_{M1} - \dot{\mathbf{R}}_{M1} \right)^t \left(\hat{\mathbf{R}}_{M1} - \dot{\mathbf{R}}_{M1} \right) \right]^{\frac{1}{2}} < \epsilon$$

where ' ϵ ' is a measure of allowable error

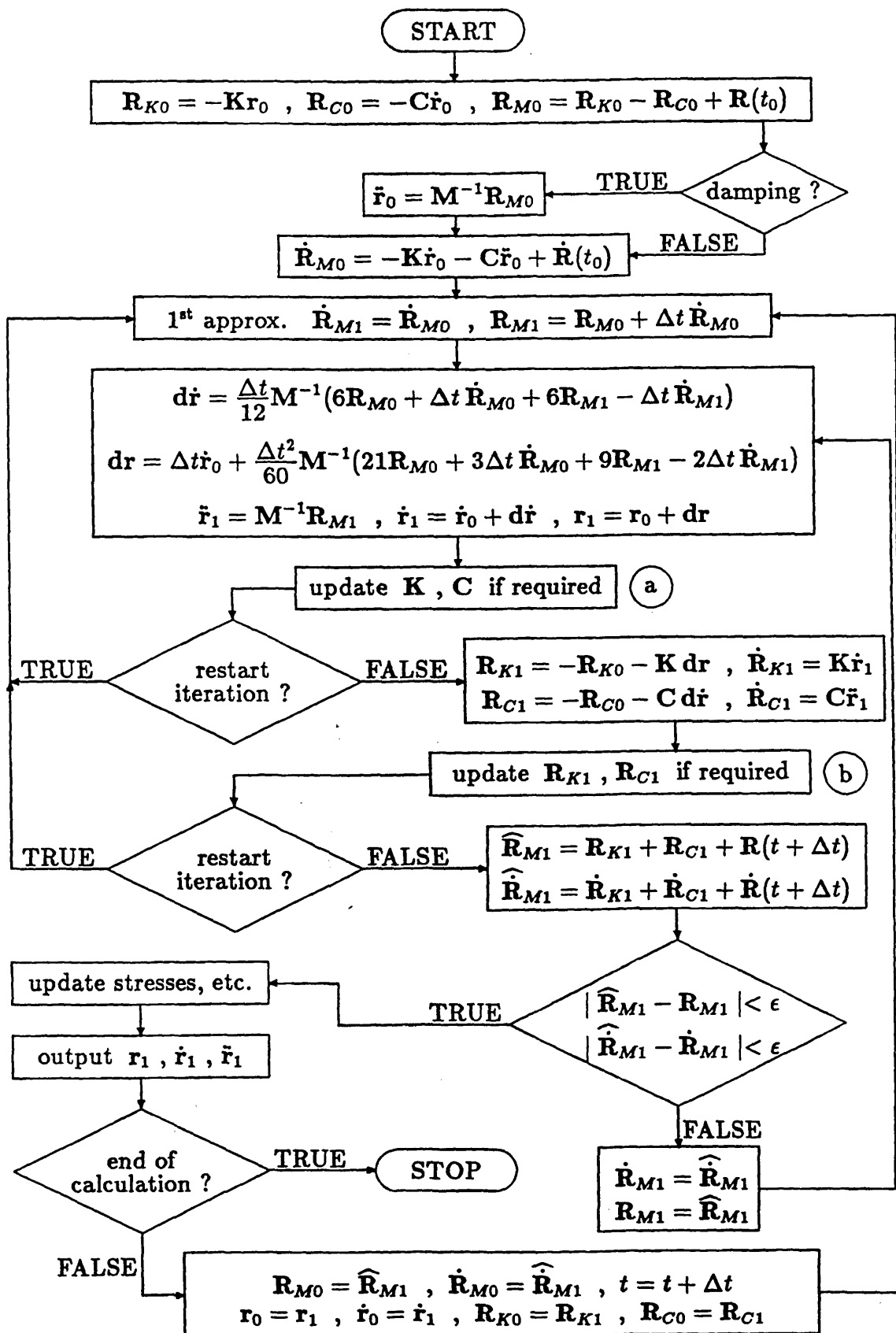


Diagram (3.8) — Flow-chart for the cubic interpolation of inertia algorithm as implemented into the present FE code

This, as shown by ARGYRIS et al. (63), ensures that the out-of-balance forces at the end of the iterative cycle (i.e. $\hat{\mathbf{R}}_{M1} - \mathbf{R}_{M1}$) are much smaller than the tolerated errors in $\dot{\mathbf{R}}_M$, namely by the order of the integration timestep and the accuracy of the estimated velocity and displacement increments is much higher still. To ensure that each individual equation of the set has converged to within the set tolerance, the present implementation checks that the errors in $\dot{\mathbf{R}}_M$ as well as those in $\mathbf{R}_M$ for every equation are within the allowable tolerance. Thus, the present termination criterion can be expressed as

$$\left| \frac{\hat{\mathbf{R}}_{M1} - \dot{\mathbf{R}}_{M1}}{\dot{\mathbf{R}}_{M1}} \right|_i < \epsilon_1$$

$$\left| \frac{\hat{\mathbf{R}}_{M1} - \mathbf{R}_{M1}}{\mathbf{R}_{M1}} \right|_i < \epsilon_1$$

for $i = 1, 2, \dots, N$

A complication that may arise is that either $(\dot{\mathbf{R}}_{M1})_i$ or $(\mathbf{R}_{M1})_i$ may tend to zero (for example if either $\dot{\mathbf{r}}_i$ or $\mathbf{r}_i$ tend to zero). Under these circumstances it may prove very difficult to attain the desired accuracy limit for the particular equation in question and the iteration cycle may repeat indefinitely. To safeguard against such situations a second termination criterion is invoked when necessary, based on the absolute allowable error, so that

$$|\hat{\mathbf{R}}_{M1} - \dot{\mathbf{R}}_{M1}|_i < \epsilon_2$$

$$|\hat{\mathbf{R}}_{M1} - \mathbf{R}_{M1}|_i < \epsilon_2$$

Furthermore, a limit on the number of permissible iteration cycles will automatically terminate the program if all other measures fail. Using a tolerance limit of (0.001%), it has been found that in general linear problems require 2-3 iterations to converge to within the set limit, while non-linear problems may require up to 7 iterative cycles.

A check on the total energy level of the system has also been included which will enable the reliability of the calculation to be verified. Monitoring the energy contents of the system as the calculation proceeds will also help in identifying any problems that may originate from the way a particular non-linear effect (say, bouncing) has been implemented. The kinetic energy ($\frac{1}{2}\dot{\mathbf{r}}^t\mathbf{M}\dot{\mathbf{r}}$) and the strain energy ($\frac{1}{2}\mathbf{r}^t\mathbf{K}\mathbf{r}$) make up the system's total energy which should remain fairly constant unless external work is being done on the system by applied forces or some energy dissipating mechanism, such as damping or plasticity, is present. The strain energy should be accumulated over the time sequence so as to apply to both linear and non-linear problems, using

$$\Delta U = \frac{1}{2}(R_{K0} + R_{K1})\Delta r$$

Finally, the accuracy, reliability and stability aspects of the present implementation of the CIIA integration scheme have been tested and verified on the basis of numerical examples. The undamped vibration of a simple, linear oscillator has been traced for some 150 complete cycles, monitoring the displacement, velocity, acceleration and energy balance of the system, and these were found to remain in excellent agreement with theory. Also considered, was the response of a simply-supported, uniform beam vibrating in its fundamental mode. The analysis followed the beam's response, starting from it being released from an initial deflected shape corresponding to that of its fundamental mode shape, and passing through the undeflected state after a quarter of a period. Once again, the nodal quantities and system energy that were monitored were found to be in close agreement with theoretical expectations.

3.6 FE MODELLING FOR DYNAMIC ANALYSIS

The general approach to finite element discretisation is to employ a fine mesh in regions of expected steeper gradient of field variable with a coarser mesh representing the other remaining areas, in order to optimise solution cost without sacrificing the accuracy in the process. Application of such modelling techniques in dynamic analysis however, is subject to certain restrictions which arise due to the presence of elastic waves, not encountered in static analysis. The topic of elastic waves in solids has been covered in detail by KOLSKY (65) and it suffices to say that when such waves impinge on a boundary between two media, both reflection and refraction take place. Thus, when a homogeneous elastic continuum is modelled by a series of discrete finite elements, any difference in the size of adjacent elements can cause a spurious wave reflection which would not exist in the continuum. The subject of spurious wave reflection in non-uniform finite element meshes has been investigated by BAZANT (66,67) and a number of guidelines for the selection of appropriate finite element meshes for dynamic analysis have emerged. It has been shown that the phenomenon of spurious reflection is significant where there exists a difference in size between adjacent elements even if this difference is very small (less than 10%). An attempt by CELEP et al. (68) to implement a variable mesh by carefully introducing a transition zone with gradually varying elements between two uniform mesh regions has indicated that this approach mitigates the problem to a limited extent, but to eliminate spurious reflections completely requires the use of uniform meshes. The analysis in (66,67,68) also reveal that spurious reflections are only significant for relatively small wavelengths (less than ten times the size of the larger element). Thus, in practice, for transient analyses a uniform mesh should be adopted while for those dynamic analyses where the response is expected to be quasi-static or dominated by a few low harmonics some degree of mesh non-uniformity can be tolerated. In addition, it has further been demonstrated that spurious wave effects are less pronounced for higher order elements and where a consistent mass representation is used.

3.7 CONTACT-SEPARATION MODELLING

Detailed impact analysis of deformable bodies represents an extremely complicated and costly structural problem that requires the modelling of non-linear contact effects and may also involve other geometric and material non-linearities (see HUGHES et al. (69)). Considering that the present task is confined to the case of low velocity impact of a mass on a beam, for which the response is expected to be dominated by the global contribution, such a detailed analysis would be of limited use and can not be justified. A more realistic approach would be to introduce the contributions of some of the aforementioned, local effects through some simplified, approximate means into the global structural response of the system, which although sacrificing the local details, is infinitely more cost-effective, far easier to implement and more appropriate for the present purposes.

Modelling of contact-separation between impactor and target as well as the associated local effects can be accomplished through the use of a spring element as depicted in Diagram (3.9).

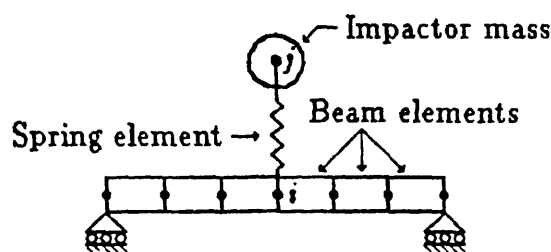


Diagram (3.9) — Idealisation for impact of a mass on a target beam

The impactor can be considered as rigid and idealised as a single lumped mass, while the flexible target beam will be modelled by a series of beam elements. A single connecting spring element should be sufficient for a small contact area, although additional spring elements can be employed as necessary, if for instance

the area of contact between impactor and target is expected to spread to the adjacent nodes. The stiffness of these spring elements can be manipulated and made to vary in any fashion in order to simulate the various local effects that are to be included. Changes to the spring stiffness can be established at every step of the calculation and directly introduced into the assembled global stiffness matrix. An alternative scheme, and one that has been implemented into the present code, introduces the modifications directly into the internal nodal load vector ($\mathbf{R}_K = \mathbf{K}\mathbf{r}$), updating the terms corresponding to the spring contributions at the two nodes 'i' and 'j'. Bearing in mind that the CIIA solution technique is being adopted, the internal nodal load derivatives ($\dot{\mathbf{R}}_K = \mathbf{K}\dot{\mathbf{r}}$), need to be adjusted also.

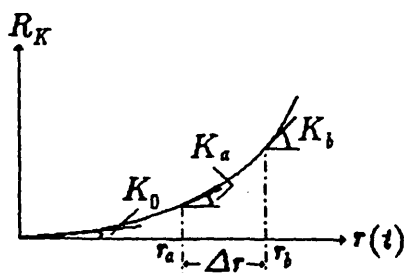


Diagram (3.10) — Typical non-linear force-displacement relationship

Assuming that the spring element is required to follow a force-displacement relationship as shown in Diagram (3.10), a non-linear iteration procedure would be required since the stiffness ' K_b ' depends on the displacement ' r_b ' which in turn is a function of the stiffness variation. The problem can be linearised by assuming the stiffness to remain constant and equal to (K_a) over the interval (Δr) but this is only applicable if (Δr) is small enough (thus a small timestep) for changes in the stiffness to be small over the interval. ARGYRIS et al. (63) have proposed an improved approximation by using $[\frac{1}{2}(K_a + K_b)]$ in place of (K_a) which would

account for second order variations in (R_K) as well. Using this latter scheme, the required modifications take the form of

$$\left. \begin{aligned} R_{Ki}(t + \Delta t) &= R_{Ki}(t + \Delta t) + \left[\frac{1}{2}(K_a + K_b) - K_0 \right] (r_i - r_j) \\ \dot{R}_{Ki}(t + \Delta t) &= \dot{R}_{Ki}(t + \Delta t) + (K_b - K_0)(\dot{r}_i - \dot{r}_j) \\ R_{Kj}(t + \Delta t) &= R_{Kj}(t + \Delta t) - \left[\frac{1}{2}(K_a + K_b) - K_0 \right] (r_i - r_j) \\ \dot{R}_{Kj}(t + \Delta t) &= \dot{R}_{Kj}(t + \Delta t) - (K_b - K_0)(\dot{r}_i - \dot{r}_j) \end{aligned} \right\} \quad (3.7.1)$$

where ' K_0 ' represents the initial spring stiffness as used in the assembled global stiffness matrix. Also note that for the nodal load derivatives it is more consistent and far more accurate to use the actual stiffness (K_b) instead of the approximated value of $[\frac{1}{2}(K_a + K_b)]$.

The specific problem regarding contact-separation modelling can now be dealt with. As long as the spring element remains in compression, this implies contact between impactor and target and the calculation can proceed normally with non-linear effects being introduced if and when required. Tension in the spring signals a state of separation and the equations governing the motion of the two bodies (i.e. impactor and target) have to be de-coupled by merely removing the spring element. All that this requires is to set $(K_a = K_b = 0)$ in equations (3.7.1) and the calculation will thus effectively track the motion of the target and impactor independently until contact is once more established. The transition from a state of contact to that of separation and vice versa requires special care so as to minimise any errors that may inherently be introduced. During the transition interval, the sudden change in the spring stiffness $(K \rightarrow 0 \text{ or } 0 \rightarrow K)$ implies a very rapid variation of ' R_K ', over the interval, which as earlier stated, can only be accommodated by the present approximations if the interval is suitably reduced, otherwise gross errors may be introduced during such a step. Tests were performed to establish the seriousness of such adverse effects and it emerged that by reducing the integration timestep by a factor of '10' during the transition stage any resulting

errors could effectively be contained. This was found to work satisfactorily for the present purposes, but if very high spring stiffnesses are employed it may be necessary to reduce the timestep even further. Once transition is complete, the timestep is reset to its original value. The flow-chart in Diagram (3.11) outlines the way contact-separation has been implemented into the present code.

3.8 COMPARISON OF NUMERICAL TECHNIQUES

The *Bernoulli-Euler* modal technique has been employed for the parametric study in Chapter (2) and it was felt that a brief comparison between this numerical method and the present FE analysis would help to demonstrate any deficiencies that may be inherent in either method and, in addition, would provide an insight into the importance of shear and rotary inertia effects on response, not included in the *Bernoulli-Euler* formulation. The test example chosen for this purpose is that of a long beam ($\ell = 1$ m), simply-supported at either end with an attached, central lumped mass ($m = 0.3$ Kg) and subjected to an initial step-velocity excitation applied at the beam centre (i.e. test case (25) of Chapter (2)). For this particular test case, the ratio of impactor mass to beam mass ($\frac{m}{M} = 0.317$) is quite low and so a number of higher harmonics are expected to be excited (see Table (2.2)) which is desirable if the effects of shear and rotary inertia are to be properly assessed. The FE analysis has been repeated for three different beam elements and details of the various mesh configurations that have been used are listed in Table (3.2). Analysis based on the *Hermitian* element is to enable a direct comparison of the two numerical techniques, while the use of the *Timoshenko* and cubic *Mindlin* elements has been aimed at illustrating the contributions resulting from the inclusion of shear and rotary inertia effects, providing a means of predicting the shear stress history and for comparing the performance of these two shear-flexible elements when applied to dynamic problems.

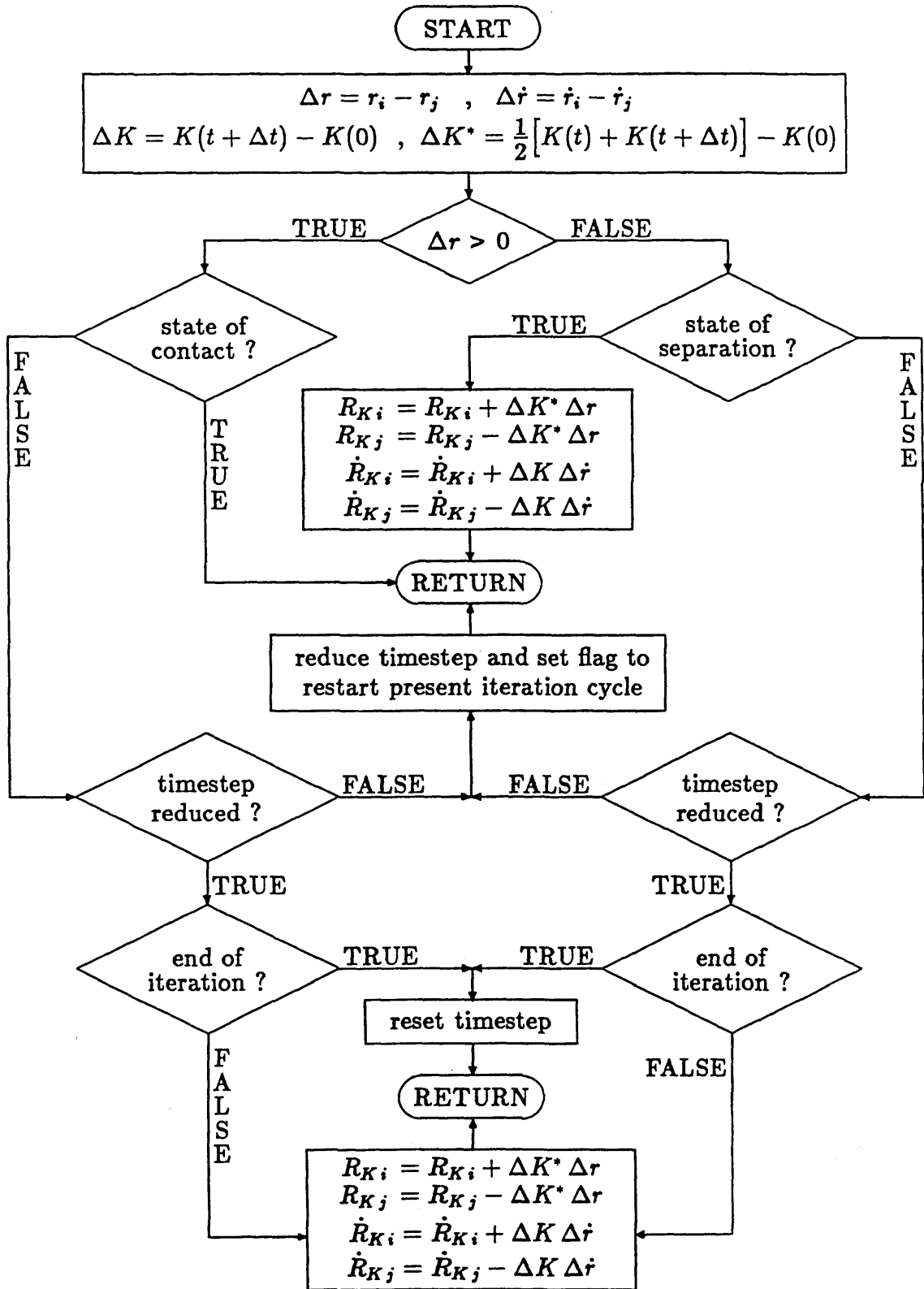


Diagram (3.11) — Flow-chart for implementation of contact-separation into the present FE code

The bending stress history as calculated from the modal approach and the FE analysis using *Hermitian* elements are presented in Figures (3.32) and (3.34) respectively and it is evident that in general the agreement between these two numerical techniques is good. Close inspection does reveal slight local differences in stress amplitude, which represent the combined effects of various sources of error inherent in the nature of the solution methods, but otherwise have an insignificant effect on the overall response. These sources of error include the truncation error resulting from limiting the modal solution to only the ' n ' lowest harmonics, the error due to the chosen FE discretisation (i.e. mesh configuration and type of element) which directly affect the accuracy with which the beam harmonics are modelled, errors relating to the solution algorithm and the fact that sampling the stresses at discrete time intervals may inevitably lead to stress peaks being missed.

The *Timoshenko* and the cubic *Mindlin* elements both take account of shear and rotary inertia effects, and it has already been mentioned in Section (3.4.2) that these secondary effects noticeably reduce the vibration frequency of the higher harmonics relating to beam flexure. However, a comparison of the bending stress responses obtained by the use of these elements, Figures (3.35) and (3.37), with that of the modal approach (Figure (3.32)) and that due to the FE analysis employing the *Hermitian* elements (Figure (3.34)), demonstrates a remarkably close agreement all-around and even the maximum peak stresses can be seen to correlate well despite the aforementioned sources of error. This implies that as regards bending stresses, the effects of shear and rotary inertia are insignificant, even though for this particular case, the modal energy distribution shown in Table (2.2) indicates a significant proportion of the modal energy as being distributed in the harmonics other than the fundamental. On the basis of these findings, it appears that the choice of the *Bernoulli-Euler* formulation for application to the low velocity impact response calculation of Chapter (2) has been appropriate and that shear and rotary inertia effects can safely be assumed to be unimportant as far as bending stresses are concerned.

Regarding the shear stresses, estimation on the basis of using the bending stress derivative in the modal approach was shown to be inappropriate (Section (2.8)), resulting in a non-converging series solution which meant that the number of harmonics that are included in the solution greatly influence the shear stress results. Using the shear-flexible beam elements in a FE analysis will yield an estimate of the shear stresses directly from the displacement field and this was considered to be more suitable. However, both the shear stress histories obtained from the FE analysis (Figures (3.36,3.38)) and that calculated from the modal solution (Figure (3.33)) predict a shear stress response dominated by the high harmonics, leading to extremely rapid fluctuations in shear stress level. In particular, it can be observed, in Figures (3.36,3.38), that the different mesh configurations used in the FE analysis (see Table (3.2)), appear to influence the shear stress response. The finer mesh used in conjunction with the cubic *Mindlin* elements can model more of the higher harmonics and this ability can be seen to be reflected in its prediction of the shear stress variation (Figure (3.38)). The fact that the high harmonics significantly affect the predicted shear stress response, suggests that for the present test example, the solution is either non-converging or convergence is extremely slow, as shown in Section (2.8), and independent of the solution method that is adopted. This is not entirely surprising, considering that a step-velocity excitation applied at the beam centre in effect represents a delta function force excitation at the centre node, which would require a very large number of harmonics to model it satisfactorily. It appears that the present test case is not well suited for studying the shear stress response, due to such severe and unrealistic loading condition that is involved. In practical situations, the finite contact stiffness between the impacting bodies and the fact that the loading will be distributed over the contact area will reduce the severity of the loading, enabling a more realistic estimation of the shear stress response by numerical techniques. This point will be more clearly demonstrated in Chapter (5), where the FE analysis has been employed successfully in predicting the shear stress response in a range of experimental impact

tests. In addition, the inevitable presence of some damping — not included in the present analyses — is also expected to contribute to the reduction of stress peaks, in particular where the stress variations are related to high frequency modes (see Chapter (5)).

Chapter Four

EXPERIMENTAL INVESTIGATIONS

4.1 INTRODUCTION

A number of parameters that control the dynamic response of transversely impacted beams have been identified in Chapter (2), forming the basis of models for the prediction of impact-induced stresses within the target beam under low velocity impact conditions. Detailed numerical investigations that followed examined the effect of the various parameters on the beam's response and the resulting stress variations, thus providing an insight into the phenomenon of impact and confirming the applicability of the proposed bending stress model. The *Bernoulli-Euler* based modal approach used in this investigation proved unsuitable for verifying the shear stress model, requiring the shear prediction capabilities of the FE method. Confidence in the numerical findings was further increased when comparison between the simplified modal approach and the more detailed finite element analysis demonstrated a reasonably good agreement (see Section (3.8)). Numerical examinations of this kind with comparison between different techniques are very useful in providing an understanding of the underlying physical concepts of impact and enhancing the credibility of the theoretically established results, but the validity of such theoretical treatment can best be ascertained by comparison with experimental observation, and since theoretical methods are based on assumptions and approximations, agreement with experimental findings would also justify any simplifications that have been introduced thus promoting a better understanding of

the problem. In addition, there are several factors that only experimentation can determine, namely percentage damping if any, to indicate the corresponding filter needed to apply to the high frequency results of the FE analysis, and also whether static failure threshold for strain are applicable under dynamic loading conditions, or if strain rate effects are important.

Experimental studies on various aspects of transversely impacted metal beams have been conducted by several authors (34–36) in the past and more recently on fibre-composite beams (3,4,7,8,10,26,28,29). However, except for the work of ARNOLD (35), no other experimental effort has been found to have focused attention on the particular ideas that have been numerically approached in Chapter (2), so as to be of any assistance in validating these theoretical results. In view of this fact and the uncertainties surrounding some of ARNOLD's (35) measurements (in particular those that relate to stress and load variations) as a result of using relatively crude instrumentation compared to today's standards, it was considered appropriate to undertake an independent experimental investigation to examine and assess the relevance and accuracy of the theoretical findings, including the validity of the proposed stress models and any limitations that may exist.

The experimental phase of the present study, to be described in this Chapter, is concerned specifically with drop-weight impact testing of centrally impacted simply-supported, isotropic and composite beams. Impact tests have been designed and conducted to cover a wide range of test configurations, by carefully selecting appropriate combinations of the impact velocity (v), impactor mass (m) and beam length (and hence beam mass (M)), which were identified previously as the parameters of primary importance, and in each case the time-dependent impact loads, bending stresses, shear stresses and beam deflection have been monitored and recorded for the entire period of the impact event, resulting in a sufficiently comprehensive experimental database to enable detailed comparisons with numerical findings. Details of the instrumentation, measurement techniques and other practical considerations as well as test configurations and results, will be presented

and discussed qualitatively, but quantitative comparisons between theoretical predictions and experimental observations have been deferred until Chapter (5), where in addition some of the present impact tests have been modelled and analysed by the FE method in an effort to confirm the accuracy of the experimental data as well as illustrating the predictive capabilities of the implemented FE code and also to highlight all-around agreement between the various approaches.

The major portion of the present experimental work has been conducted on an isotropic beam specimen presented in Part (I) of this Chapter, while additional impact tests performed on carbon-fibre beams will be considered in Part (II).

Part (I) — ISOTROPIC TARGET

4.2 DETAILS OF TEST SPECIMEN

A metal beam specimen of reasonably uniform, solid, rectangular cross-section (on average approximately 1.81 cm deep and 1.92 cm wide) and a total initial length of 144 cm has been machined from a piece of high-strength aluminium alloy, designated as '*DTD 5054 TF*' (material property details are listed in Table (4.1)). The drop-weight impact tests have all been conducted on this single specimen, the length of which was varied by gradually cutting the specimen to size as and when it was required, so as to consistently limit the overhang at the simple supports, which can have a marked influence on the beam's response, to a maximum of 2 cm, sufficient to ensure that the specimen does not drop off the supports when deflected. Apart from the obvious economical benefits, the use of a single specimen means that material properties and cross-sectional dimensions are consistent for all impact tests and no erroneous effects are introduced due to slight differences in these parameters, which would be very likely if a number of different specimen were to be used. To survive the multiple impact tests in tact,

the specimen had to be protected against local indentation at the point of impact (see Section (4.5)) and in addition plastic deformation had to be inhibited by ensuring that an adequate margin of safety existed between the maximum expected impact-induced stresses and the material's yield stress. Hence the need to use a high-strength alloy in conjunction with a suitable beam geometry to enable the impactor energy to be safely absorbed, while still reaching a sufficiently high stress level which is desirable for accurate stress measurements.

4.3 IMPACT TEST RIG

An overall view of the test rig, including the monitoring equipment and the mounted test specimen, can be seen in Photo (4.1). The specimen is mounted on a horizontal platform that forms part of the main support framework of heavy gauge steel construction, while the taller box section frame is part of the impact mechanism. The impactor is in the form of a light-weight aluminium carriage, to which weights can be added, and when released, falls under gravity, guided by two tensioned vertical steel rods connected between the two frameworks, and there is a choice of three different carriages depending on the additional mass that is to be carried. The drop-height can be varied up to a maximum of about (1.8 m) and an electrically operated release mechanism has also been provided, though not used in the present experiments.

Photo (4.2) presents a close-up view of the experimental set-up, including the aforementioned drop-weight carriage assembly and the system adopted for supporting the specimen. To simulate simply-supported end conditions, the specimen has been sandwiched between a round-edged wedge shape metal block and a grooved nylon roller (see Photos (4.2,4.4)). This arrangement ensures that the specimen does not lift off its support during impact, nor moves sideways or twists, but still permits axial movement and a limited degree of end rotation. By defi-

nition, true simply-supported end conditions must allow complete rotational freedom at the supports, but simple geometrical arguments will show that this requirement is not exactly satisfied in the present support system. As depicted in Diagram (4.1), during deflection, a contact load (P) will be generated between the beam and the restraining roller of the support system and, in addition, the point of contact will change. The combined effect of this is to produce an opposing end moment that increases with an increase in the end deflection.

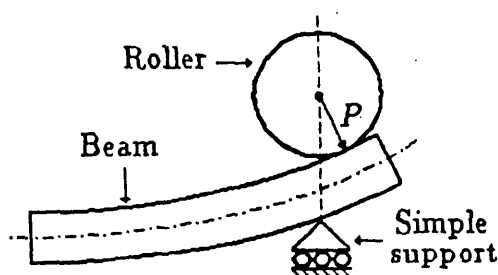


Diagram (4.1) — Local details of the simple-support mechanism

The rotational stiffness that is inevitably being introduced at the supports is mainly a function of the end rotation and also depends on the rigidity of the support assembly. The choice of a nylon roller has been aimed at reducing the latter, while end rotations are generally small for the low harmonics of the beam and it is the lower modes that are expected to dominate the response under low velocity impact conditions. Consequently, any local end-effects due to the support mechanism are not expected to significantly influence the global response of the impacted beam, a fact that is endorsed by the comparisons between experimental results and FE predictions, presented in Chapter (5).

4.4 IMPACT DATA MEASUREMENTS

4.4.1 Measurement of contact force

A load-cell, shown in Photo (4.3), has been used for measuring and monitoring the time-history of the contact force that develops between the impacting mass and the target beam at the point of impact. This load-cell is a single unit, machined from aluminium alloy and embodying a short, thin cylinder reinforced by a circular flange at one end and an end plate at the other (see Diagram (4.2)).

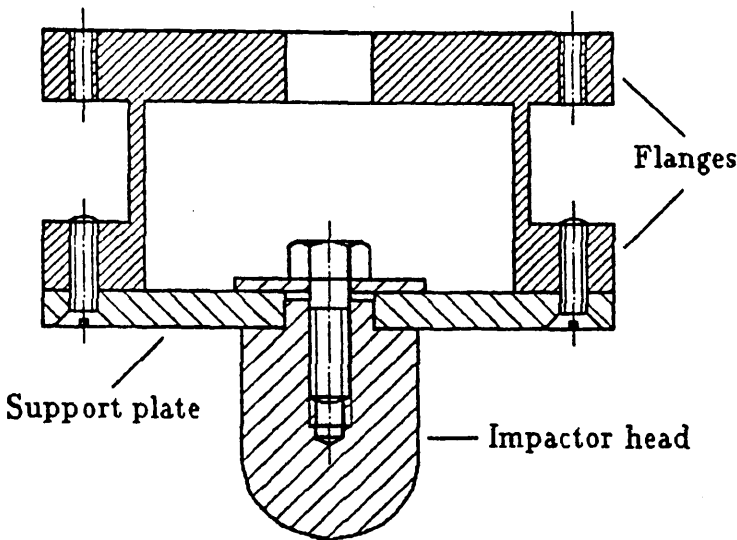


Diagram (4.2) — The load-cell and impactor head assembly

The flanges at either end each have four tapped holes for connecting the load-cell to the drop-weight carriage assembly and for mounting the impactor head assembly, which consists of a solid cylinder with a hemispherical end, some 12.7 mm in diameter, and attached to a flat end plate, both made of hardened steel (also shown in Diagram (4.2)). The applied load is measured in the normal fashion utilising strain gauges of type 'FLA-2' (see Table (4.2)) which have been positioned around

the thin cylindrical cell at regular space intervals (see Photo (4.3) for details). A total of eight gauges sample the strains at the various locations around the cell helping to filter out effects due to local load-cell imperfections and generally giving an improved measure of average strain. Alternate gauges measure the longitudinal strain and the transverse strain, resulting from *Poisson's* ratio, and are connected into a bridge circuitry in such a manner as to increase the effective sensitivity and to eliminate temperature effects.

The proportionality constant between measured strain and applied load has been obtained by calibrating the load-cell assembly in a compression testing machine under static loading conditions. The calibration was repeated for three different configurations of the load-cell assembly :

- a) load-cell without the impactor head
- b) load-cell with small impactor head (6 mm diameter)
- c) load-cell with large impactor head (12.7 mm diameter)

The calibration results are presented in Figure (4.1) and clearly illustrate that the presence and size of the impactor head have a marked effect on the performance of the load-cell. This can be explained by referring to Diagram (4.2), where it can be seen that under an applied load, the impactor head support plate will deform in bending and the extent of this deformation is directly related to the size of the impactor head. The resulting bending moment at the plate support (load-cell flange) will superimpose a bending strain field which is in addition to the direct strain field induced by the applied load. These strain components add on the inside of the cell walls but subtract on the outside, to which the strain gauges are fixed, resulting in a lower strain reading for a given load. This bending action will inevitably reduce the safe maximum load capacity, but for the present load-cell using an impactor head, as is evident from Figure (4.1), has the advantage of leading to a linear calibration curve. The arrangement employed for the present

test purposes uses a large impactor head and appears to behave in a linear fashion over the expected load range with a calibration factor of 8416.5 N/m.

4.4.2 Measurement of impact velocity

Impactor drop-height can be used to get an estimate of impact velocity, but to minimise errors due to frictional losses in the vertical impactor guides, the impact velocity has been measured directly, utilising a digital timer and a small electronic unit, shown in Photo (4.4), which features two light beam triggers separated by a pre-set distance of 5 cm. The unit has been stationed close above the specimen and a small flag fixed to the impactor serves to activate/deactivate the timer as it passes the trigger points. The time taken to travel the distance between the two triggers, recorded to the nearest micro-second, can then be converted to yield the impactor velocity at the half-way position between the triggers which subsequently has to be extrapolated to obtain the actual velocity of the impactor at the point of impact.

Comparisons between measured impact velocities and that calculated directly from impactor drop-heights have revealed discrepancies of up to 3%.

4.4.3 Deflection measurement

The transverse deflection of the beam specimen at the point of impact has been monitored utilising a non-contacting displacement transducer. As illustrated in Photo (4.5), a flag, made of Balsa-wood to minimise the added inertia, has been secured to the beam specimen to reach the transducer which has been mounted directly beneath the point of impact. This is an optical transducer, built by the Aeronautics department, incorporating a light source and photoelectric cells and has a fast response time, a requirement for any dynamic measurement. During specimen deflection, the flag cuts across the light source altering the amount of

incident light reaching the photoelectric cells and the induced power output of these cells gives a direct measure of the deflection. The transducer has been set up and calibrated to ensure a linear relationship between deflection and output voltage over the maximum measuring range of 5 cm (see Figure (4.2)) resulting in a calibration factor of 0.01126 m/V.

4.4.4 Stress measurement

Impact induced stresses have been measured at a number of positions along the test specimen by employing standard strain gauges as shown in Photo (4.3). Bending strains are obtained using pairs of single wire gauges of type 'PL-10' (see Table (4.1)), cemented to the top and bottom surfaces of the specimen at distances of 5, 10, 15, 25 and 35 cm from the point of impact and connected into opposite arms of amplified bridge circuits to eliminate any axial strains, with dummy gauges in the two remaining bridge arms to balance the bridge and provide temperature compensation. For the shear strain measurement, pairs of rosette foil gauges of type 'FCA-6' (see Table (4.1)) have been fixed to either side of the specimen, at distances of 5 and 10 cm from the point of impact, such that the rosette gauges lie along the principal strain direction, at an angle of 45° to the longitudinal beam axis. The gauges of each rosette have been connected into one half of the bridge circuit amplifiers, in adjacent arms, to give a full bridge that eliminates bending and axial strain effects as well as providing temperature compensation.

The strain gauge amplifiers that have been referred to above are purpose designed hybrid low noise low drift linear DC amplifiers specially configured for resistive bridge measurement and having a wide frequency response bandwidth of 400 KHz at unity gain. The particular version used here has a gain of 500, calibrated to produce a one volt output per 1000 micro-strains for a fully active bridge configuration.

Based on the above arrangement, the conversion factors for obtaining the bending and shear stresses from the measured voltage, after accounting for gauge factor and the bridge configuration, can be shown to be as follows :

$$\frac{\sigma_B}{\text{volts}} = \left(\frac{1000 \times 10^{-6}}{0.5} \right) \left(\frac{2.0}{2.1} \right) E = 1904.76 \times 10^{-6} E$$

$$\frac{\sigma_S}{\text{volts}} = \left(\frac{1000 \times 10^{-6}}{1.0} \right) \left(\frac{2.0}{2.1} \right) \frac{Ek_w}{1 + \nu} = 952.38 \times 10^{-6} \left(\frac{Ek_w}{1 + \nu} \right)$$

The correction factor (k_w) has been included to account for the variation of shear across the width of the beam. Bearing in mind that the rosette gauges that measure shear strain are positioned on the side walls, it will be necessary to obtain the shear stress along the beam neutral axis in order to enable a direct comparison between experiment and theory. Elementary theory predicts only a parabolic shear distribution through the depth of the beam, but no variation across the width. TIMOSHENKO (53) has shown that this is applicable in the case of deep, narrow beams, though as the beam width increases, a secondary stress system, resulting in the variation of shear across the width, gradually becomes important. TIMOSHENKO (53) has derived a formulae, based on the theory of elasticity, to account for this effect, and for the rectangular cross-section of the present test specimen, this leads to a correction factor $k_w = 1.178239668$.

4.4.5 Data recording system

A schematic representation of the instrumentation used for monitoring and recording the impact test data is shown in Diagram (4.3). Three dual channel strain gauge amplifiers (SGA) have been used in conjunction with dual channel transient data recorders (TDR) for collecting strain information over a pre-specified time interval. The transient recorders are of type 'DATALAB DL-902', capable of digitising and storing waveforms at a maximum sample rate of 1 MHz per channel. Once triggered, the recorders sample each incoming signal at the

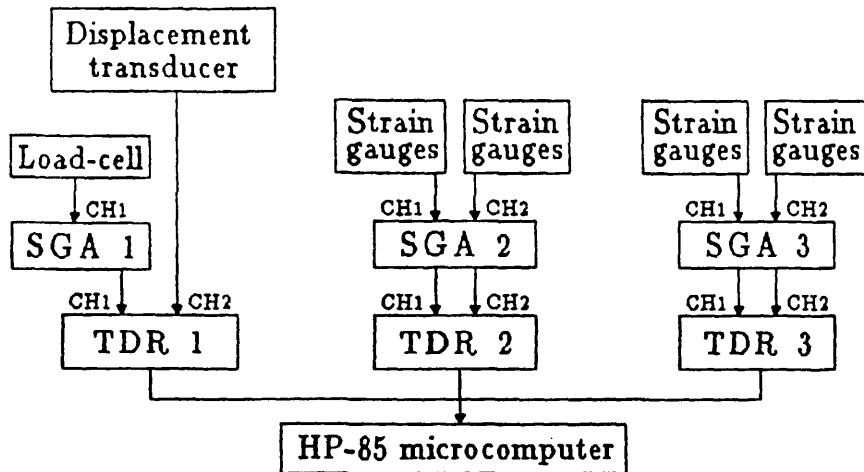


Diagram (4.3) — Configuration of the data recording system

pre-set rate for a total of 2048 record samples. Synchronisation of the transient recorders has ensured simultaneous triggering of all recording channels. Pre-test trials have indicated that internal triggering, based on the load-cell signal, was to be preferred to using the signal from the velocity detector, which activates the electronic timer, to externally trigger the recorders. The rapid increase in load on impact thus provided a reliable trigger signal and using the pre-trigger facility on these recorders has meant that the initial pre-trigger part of the incoming signal would also be captured. The recorded information is then subsequently transferred to a 'HEWLETT-PACKARD 85' microcomputer interfaced with the recorders where it can be viewed and permanently stored on magnetic tape for further processing.

4.4.6 Measurement accuracy

The various equipments that have been used in the measurement of the present impact test data are all subject to limitations and these may adversely affect the

accuracy of the measured data. In this respect, the responsiveness of a measuring system is a critical factor when trying to measure rapidly varying quantities under dynamic conditions. To ensure an accurate measurement, the system's response needs to be fast enough to keep up with the variation in the quantity it is intended to monitor. To provide some means of assessing the measurement accuracy, a rough estimation of the likely frequency content of the signals which are expected to be encountered during the present set of impact tests has been obtained based on the numerical approach of Chapter (2) and this is presented in Table (4.3) for two of the most demanding test cases. The first considers the case of the lowest ($\frac{m}{M}$), which excites the greatest number of harmonics, while the second relates to the shortest beam specimen with the shortest vibration period.

Based on the estimates in Table (4.3), it is evident that the transient recorders (Section (4.4.5)), with a maximum sampling rate of 1 MHz, are capable of capturing signals of up to 250 KHz in frequency, which is far in excess of the expected frequency range shown in Table (4.3). The only source of inaccuracy as regards these recorders is related to the fact that the incoming data signal is stored in digital form, thus imposing a resolution limit of 1 in 256 (i.e. a maximum error of approximately 0.2% of the total scale voltage selected).

The displacement transducer (Section (4.4.3)) has a frequency response range of up to about 10 KHz and this should be quite sufficient bearing in mind that the energy content in the higher harmonics (i.e. those above 10 KHz) of the impacted beams is negligibly small as shown in Table (4.3).

The strain gauge amplifiers used in the measurement of loads and stresses have a frequency bandwidth of 400 KHz at a gain of unity, but the gain of the units employed for the present tests was increased to about 500 which later emerged to have resulted in the lowering of the 3 DB-point of these amplifiers to around 800 Hz. The frequency characteristics of these units were subsequently established and Figures (4.3,4.4) show the variation of signal attenuation and phase shift with

frequency for these units. Using the estimates in Table (4.3) as a guideline, it would appear that some of the high frequency harmonics may be attenuated or even filtered out by the strain gauge amplifiers, but in view of the relatively small percentage of total energy that is expected to be contained in the higher modes it is unlikely for the resulting inaccuracies to significantly modify the measurements.

However, to obtain a more realistic assessment of the effect of the amplifier characteristic on the experimentally measured data, the test results for the two extreme test cases outlined in Table (4.3) have been further processed using *Fast Fourier* numerical technique to compensate for attenuation and phase shift that may be introduced in the amplification stage. The procedure adopted has been to transform the measured time-signal into the frequency domain using *Fast Fourier* transform, correcting the magnitude and phase angle of the individual *Fourier* coefficients in accordance with the amplifier characteristics shown in Figures (4.3,4.4) followed by an inverse transform to yield the corrected time-history for the original signal. One precaution that had to be taken at the correction stage was to realise that the measured input signal had been stored by transient recorders which digitise the signal and thus inevitably, the signal, when transformed, will contain spurious frequency components that originate due to the digitisation process. Thus to avoid erroneously correcting these spurious components, a limit had to be imposed on which frequency components to consider, and since the maximum digitisation error, as mentioned earlier, is approximately 0.2% of the full scale, it was decided to correct only those frequency components for which the *Fourier* coefficient was greater than 0.5% of the highest coefficient in the signal. Of the two test cases considered, the first, as expected, did not indicate any noticeable change in the measured data signal as a result of the amplifier's limitations. For the second test case, relating to the shortest beam, however, even the fundamental mode frequency is sufficiently high for the measured signal to be affected by the amplifiers and this is demonstrated in Figure (4.5) presenting the measured and corrected bending and shear stress histories for this case. It can be

seen that the main effect is a phase shift in the data while the magnitude is still not significantly influenced. Consequently, it would appear that for the present test range, the limitations of the strain gauge amplifiers do not, in general, undermine the accuracy of the measured data, though in the tests on the short beam specimen, a slight underestimation in the absolute magnitude of measured data is anticipated. In addition, while testing, it has been observed that the strain gauge amplifiers register a low level noise signal as shown in Figure (4.6), which appears to be a characteristics of the amplifiers. Fortunately, in the majority of test cases this does not introduce any noticeable error, the exception being the shear stress measurements for the longer specimens due to the low magnitude of the impact induced shear stresses.

Finally, another factor that can affect the measurement accuracy relates to the size of the particular strain gauges employed. These gauges provide an average measure of strain, so that the smaller the gauges, the higher the spatial resolution leading to an increased accuracy. However, gauge size becomes only important when dealing with very short wavelengths and since in the present impact tests only the first few harmonics are expected to be excited, the choice of strain gauge size should not be too critical.

On the whole, however, the possible sources of error listed above do not appear to significantly undermine the accuracy of the experimentally measured data and this has been further confirmed by comparisons with predictions based on the finite element analysis of a number of test cases presented in Chapter (5).

4.5 ADDITIONAL EXPERIMENTAL DETAILS

As mentioned previously, a single beam specimen has been used for conducting the full range of impact tests and as a result it was necessary to protect the specimen against local damage that could arise. An early attempt was to utilise a

small piece of hard rubber that was placed and firmly secured to the beam specimen at the point of impact to dampen the blow from the impactor, protecting both beam and impactor. One major drawback of this arrangement was the complexity of modelling it by the finite element technique, primarily due to the various sources of non-linearities associated with the use of such a rubber damper. These non-linearities arise from a non-linear stress-strain relationship, a non-linearly varying contact area between rubber damper and the hemispherical impactor head and a dependency of the rubber stiffness on strain rate.

Adopting a simplified approach, the rubber damper was initially modelled by a linear spring element and this was followed by implementing into the FE code, experimentally established variations of the rubber stiffness which would combine the effects of the various non-linearities. However, an added complication emerged in relation to contact-separation modelling, namely that the stiffness of the damper on subsequent contacts is a function of the recovery of the compressed rubber damper during separation which is not fully understood. Finite element analysis results however indicated that although the local response near the point of impact, in particular the contact force time-history, is affected to some extent by the modelling of the rubber damper, the displacement and bending stresses are not greatly influenced and compare well with that measured experimentally. This finding is encouraging especially since it implies that the introduction of such a rubber damper as a specimen protection device has not adversely affected the global response of the specimen being investigated.

In order to improve FE predictions of contact forces which have a direct bearing on the induced shear stresses in the specimen would require alleviating the aforementioned modelling difficulties resulting from the non-linear behaviour of the rubber damper. This was achieved by employing an alternative arrangement as shown in Photos (4.2,4.3). The rubber damper has been replaced by a piece of hardened steel in the form of a channel fixed to the specimen for protection against local indentation, while the load-cell carrying the impactor head has been

mounted on a relatively stiff coil spring of known linear stiffness which also helps with reducing the ringing resulting from the impact. The advantage of using such an arrangement is that it greatly simplifies the FE modelling of the impactor assembly, whereby a simple spring-mass system can be employed for approximating the dynamic response of the whole assembly.

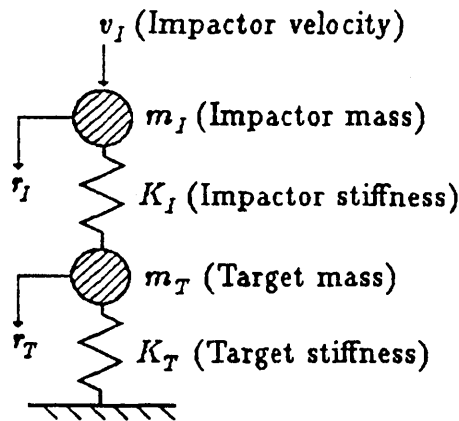


Diagram (4.4) — 2 DOF system to model impact of drop-weight on beam specimen

Such a representation inherently assumes that the spring element dominates the response of the assembly while the dynamic response of the remaining components in the assembly (e.g. load-cell or impactor carriage) is only of secondary importance. The accuracy of the model is thus determined by the extent to which the above requirement is enforced, which in turn depends on the stiffness of the chosen spring in relation to that of the constituent components of the assembly. This would suggest a lowering of the spring stiffness to improve accuracy, but on the other hand, reducing the spring stiffness could affect the impact response of the specimen by elongating the loading period. To ensure impact loading conditions, that is to say a loading period short in comparison with the fundamental vibration period of the impacted specimen, would require increasing the spring stiffness.

These two contrasting requirements impose upper and lower bounds, thus limiting the choice of suitable springs. The appropriate spring stiffness required to ensure a loading period of less than at least $(1/10)^{\text{th}}$ of the fundamental period of the specimen vibration was established for every test case, based on the transient response analysis of a two degree of freedom (DOF) spring-mass system shown in Diagram (4.4) and subject to an initial constant velocity excitation.

It emerged that in order to satisfy the requirement relating to the loading period for every test case configuration, a total of four different springs would suffice. The practical difficulties involved with finding coil springs that have the required stiffness as well as suitable dimensions has meant that only the lowest rated spring was in the form of a coil spring and for the rest a *Belleville* spring assembly that would permit the desired stiffness to be made up from a multitude of *Belleville* disc springs was constructed. The complete *Belleville* spring assembly is shown in Photo (4.5) and a sectioned view is shown in Diagram (4.5).

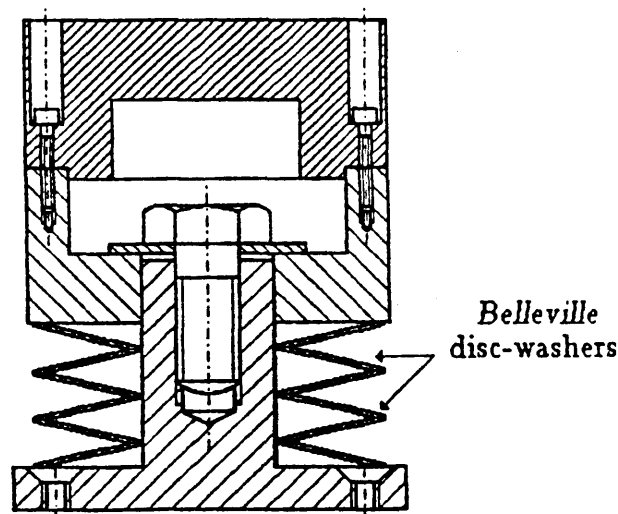


Diagram (4.5) — Sectioned view of *Belleville* spring assembly

Typical stiffness characteristics of disc springs is given in Figure (4.7), which for small deflections is approximately linear. To limit the deflection of the individual disc springs, discs of suitably high stiffness were employed and by stacking a number of these discs in series (see Diagram (4.5)), both linearity and the required stiffness was accomplished. Table (4.4) lists the various configurations that have been employed to achieve the desired stiffness characteristics as determined by the 2 DOF model of Diagram (4.3) and their experimentally determined stiffness for use in later finite element analysis.

4.6 PRESENTATION AND DISCUSSION OF IMPACT TEST RESULTS

A total of 43 drop-weight impact tests have been conducted on the simply-supported aluminium beam specimen to cover a wide range of possible combinations of the previously identified response parameters (v, m, M) giving a total range on the value of the impact parameter $(v\sqrt{\frac{m}{M}})$ of (1.758–4.714). Table (4.5) summarises the various configurations that have been examined. Starting with an initial beam length of 140 cm, this has been gradually reduced in steps of 10 cm, down to a final length of 20 cm, and at every stage a number of different impact tests have been performed, the majority of test configurations having been based around length's of 140 cm, 80 cm and 20 cm, representing long, medium and short beam targets respectively. The impact tests have been carefully designed not only to assess the effects of the parameters of primary importance, namely impactor velocity (v) , impactor mass (m) , and beam length (ℓ) (and hence beam mass (M)), on the beam's response but also to highlight the dependence of maximum stress on just two parameters, the velocity (v) and the mass ratio $(\frac{m}{M})$, as suggested by the numerical findings and to investigate the role, if any, of impactor energy $(\frac{1}{2}mv^2)$ and momentum (mv) as potential controlling parameters in predicting impact response. Quantitative discussion on the effect of the individual parameters (v, m, M)

on the beam's impact response and identification of the main unified controlling parameter, expected to be $(v\sqrt{\frac{m}{M}})$ based on earlier numerical findings, will be presented in Chapter (5), while in this Section the emphasis will be placed on a qualitative discussion of the experimental findings and any consequences that these may entail.

4.6.1 General observations

In conducting the impact tests detailed in Table (4.5), various quantities that include contact force, beam centre-point deflection, bending and shear stresses at a number of positions along the beam, have all been monitored and recorded. A selection of these experimental results are presented in Figures (4.8–4.56) to illustrate some of the typical features that have been observed generally as well as those relating to changes in the parameters (v, m, M) . In all tests, data acquisition was continued for a long enough period to ensure that the dominant response characteristics have been captured. Guided by the displacement measurements, the beam's response has in all cases been monitored from the start of impact past the point of maximum central deflection. A common feature observed in all test cases, except for those involving the very short beam specimen, is the occurrence of multiple impacts between impactor and the target, during what appears, to the naked eye, as a single blow (see Figure (4.8)). The presence of such sub-impacts has also been reported by the likes of MASON (34) and ARNOLD (35), and these sub-impacts appear to be affected by the mass and elasticity of the impactor and that of the target. It is also evident from the experimental data, that the maximum impact load does not necessarily occur on first contact, but that subsequent sub-impacts may give rise to a higher contact force, once again depending on the configuration being tested (compare Figures (4.8) and (4.25)). Comparison between the load and displacement traces clearly show that sub-impacts tend to occur when the beam is about to return to its undeflected position, and the effect these sub-impacts have

on the subsequent response of the target is determined by the target's elasticity and the inertial properties of both impactor and beam.

Presented shear stresses have been measured at a position 5 cm from the point of impact which for the long and medium length beam specimen can be considered to be close enough to the point of impact and the shear variation can be seen to closely follow the variations of the contact load, without being significantly affected by dispersion. Figures (4.22,4.23 and 4.20) compare the shear stresses at 5 cm and 10 cm from the point of impact and the load history for a 40 cm beam specimen and the effect of dispersion is clearly evident, helping to iron out the contributions of the higher frequency components. The effect of dispersion is even more noticeable when it comes to the very short specimen (see Figures (4.37,4.39)) where the measured shear stress variations are found to be quite smooth, no longer reflecting the more rugged nature of the loading pattern. However, the magnitude of the measured shear stresses is as expected, generally small in comparison to the much higher bending stresses, except for the very short beam specimen, and even then their magnitude is about 3 to 4 times smaller compared to the corresponding bending stresses.

Bending stresses have also been recorded at different locations along the beam and the influence of dispersion on the stress peaks as one moves away from the point of impact is clearly demonstrated (see Figures (4.11,4.12,4.13)). A further interesting observation is that the bending stress records for positions distant from the point of impact register an initial negative stress which changes to positive as the impact proceeds (see Figures (4.13,4.19)). This implies that on impact, a portion of the beam specimen at each end bends initially upwards and at this instance the deflected shape of the beam and consequently the bending stress distribution along it are rather complicated and by no means equivalent to the straight line distribution of a centrally-loaded beam. A similar finding has been reported by ARNOLD (35) which is in agreement with the predictions of the FE analysis presented in Section (5.2) where for the cases that have been analysed

the beam deflection for various instances in time have been calculated, exhibiting the upward bending of sections of the beam early on in its response with the beam deflection gradually approaching the fundamental mode response.

4.6.2 Effect of beam length on response

Figures (4.8–4.24) represent the experimental results that have been obtained for test cases (MT5, MT23, MT33) where impactor mass and velocity are identical, the only variable being the specimen length (140 cm, 80 cm and 40 cm respectively). The increase in beam stiffness and the reduction in target mass resulting from a shortening in beam length is seen to reduce both the period of contact and that of separation between successive sub-impacts, which eventually merge to result in a single blow as observed for the very short test specimen in Figure (4.37). The magnitude of the contact force can also be seen to increase with decreasing beam length (see Figures (4.8, 4.14, 4.20)).

As already mentioned, the shear stresses tend to follow the loading pattern, but their magnitude is found to remain relatively small, although a definite increase in peak shear stress with reduction of beam length is clearly evident from Figures (4.10, 4.16, 4.22). It appears that significant shear stress levels can only be expected in very low aspect ratio beam targets.

As regards impact-induced bending stresses, Figures (4.11, 4.17, 4.24) clearly demonstrate that increasing the specimen length results in higher harmonics being excited as was predicted by the numerical study of Chapter (2). While for the short beam the bending stress is dominated by the fundamental mode, the long beam specimen exhibits a significant contribution due to the higher harmonics superimposed on the contributions resulting from its fundamental mode. The presence of the higher harmonics is also evident for the medium size beam target, though far less dominant.

4.6.3 Effect of impactor mass on response

To isolate the effect of impactor mass on the target's response requires that impactor velocity and beam length be kept constant for different drop-weights. A number of different test cases satisfy this requirement and here a few configurations to represent the long (MT1,MT8), the medium (MT22,MT28) and the short specimen (MT36,MT40) have been selected to highlight some of the general features. The recorded time-variations of impact load, displacement, bending and shear stresses for these cases are presented in Figures (4.25–4.40). Comparing the response of the long beam to drop-weights of 6.0 Kg and 1.5 Kg in Figures (4.25–4.30), highlights a substantial increase in the response period of the target impacted by the heavier mass as would be expected on the basis of the earlier numerical investigation of Chapter (2). The contact force history also exhibits a greater number of sub-impacts for the heavier impactor, although the magnitude of the maximum contact force does not appear to have been significantly affected by the impacting mass. It is evident that impactor energy and momentum are transferred to the target in stages and the higher energy and momentum associated with the heavier impactor thus requires more sub-impacts to achieve the transfer. The medium length beam (80 cm) has been impacted by 3.0 Kg and 1.0 Kg weights and the respective responses are compared in Figures (4.31–4.36), showing the expected period elongation associated with the heavier drop-weight. On the basis of the observations relating to the longer specimen, the load history of the 3.0 Kg impactor is expected to exhibit a greater number of sub-impacts. However, as mentioned in Section (4.6.2), the decrease in beam length reduces the separation periods, as a result of which the expected sub-impacts have merged to give more load peaks though fewer actual sub-impacts. For the very short specimen (Figures (4.37–4.40)) period elongation is less noticeable and sub-impacts no longer take place so that impact loadings are similar though now a higher contact force registers due to the higher impactor energy and momentum imparted to the target by the heavier mass.

Shear stress variations are once again seen to reflect the loading pattern, especially for the long and medium size beam targets, with stress levels still remaining generally low. In the case of the very short specimen, the relative position at which shear stresses are sampled is sufficiently far away from the point of impact for dispersion to have a noticeable effect and smooth out any local shear stress variations resulting from the contact load at the point of impact. For this short target, shear stress levels are much higher tending to increase with increase in impactor mass.

The effect of impactor mass on bending stress is clearly highlighted for the long specimen in Figures (4.28), where the lighter mass is seen to promote a response in which the contribution of the higher harmonics is quite significant while the heavier mass results in a response dominated by the fundamental mode, although evidence of the higher harmonics having been excited is still visible. This agrees with findings of the numerical investigation where it was demonstrated that as the ratio of impactor/beam mass is reduced more energy will progressively find its way to the higher response modes. Figures (4.34 and 4.40) also compare the bending stress for different impacting masses relating to the medium and short specimen and it can be observed that the presence of the higher modes diminishes rapidly as the beam is shortened. This is due to the fact that for these cases the ratio of impactor/beam mass is relatively higher than that considered for the long beam and according to the numerical study of Chapter (2), most of the energy should be contained in the fundamental mode as is found to be the case here.

In general, it is clearly observed that specimen deflection and impact-induced stresses tend to increase with increase in impactor mass as would be expected due to higher impactor energy. However, the increase in amplitude is not in direct proportion to the increase in impactor mass (e.g. a four fold increase in mass in the case of the long test specimen has led to only an approximate doubling of the stress level, as seen in Figure (4.28)).

4.6.4 Effect of impactor velocity on response

To examine the role of impact velocity on the behaviour of the beam target, a number of impact tests have been conducted in which impactor mass and beam length were kept constant, only varying impactor drop-height and hence velocity. The resulting variations of contact force, deflection, bending and shear stresses, with time associated with impact tests on the long (MT1,MT3), medium (MT22,MT24) and short specimen (MT36,MT38) are compared in Figures (4.41–4.56). The findings clearly demonstrate that for any given impactor mass and beam length, the time-history of the various measured quantities does not alter as a result of increasing or decreasing the impactor velocity, the only difference being in the magnitude. These observations conclusively confirm the prediction of the numerical investigation, that impactor velocity merely affects the magnitude and not the actual response of the beam target, as long as this is limited to low velocity impact, for which the response can be considered to be quasi-static. Under these conditions, the beam can survive the initial local response due to impact, but if the impact velocity is high enough to cause extensive local damage it is conceivable that the resulting changes in the beam properties can greatly modify the target's response.

Part (II) — FIBRE-COMPOSITE TARGET

4.7 DETAILS OF COMPOSITE TEST SPECIMEN

For the present experimental investigation into the low velocity impact response of composite laminates, attention has been limited to 16-ply (with a stacking sequence of $[(0, 0, +45, -45)_2]_{\text{sym}}$) carbon-fibre reinforced plastic (CFRP) laminated beam specimen based on the *Ciba-Geigy XAS-914C* system. This consists

of surface treated, high tensile *Grafil XA-S* carbon fibres from *Courtlands Ltd.* incorporated in *Fibredux 914* matrix resin system from *Ciba-Geigy*, in the form of epoxy resin pre-impregnated wrap sheet of uni-directional carbon fibres. The chosen material system and lay-up are favoured for possessing characteristics of laminates commonly used in the construction of many aircraft secondary structures. The prepreg sheet is cut and laid up according to the specified stacking sequence to form laminated panels (330 mm × 300 mm). These panels are then cured in an autoclave using the following cure cycle, recommended by *RAE Farnborough* :

- a) Heat panels under vacuum to 140°C in 50 minutes
- b) At 140°C apply 60 psi pressure and vent panel to atmosphere.
Continue heating to 175°C
- c) Cure for 1 hour at 175°C and cool under pressure until 60°C

This process is then followed by post-curing for 4 hours at 190°C in an air-circulating oven to improve machinability and yield optimum high temperature performance. The expected mechanical properties as provided by the manufacturers for this product, designated as *Fibredux 914-XAS-5-34%*, are listed in Table (4.6). The resulting composite laminates have a total thickness of 2 mm with a 60% fibre content by volume.

The manufactured panels were subsequently checked for overall quality using an ultrasonic scanning machine to reveal the presence and location of any small voids or defects, so that these could be avoided in the test specimens. The test specimen were cut from these panels using a diamond tipped slitting wheel. The width of the specimen has been limited to 15 mm, only slightly wider than the impactor head (12.7 mm), in order to minimise plate-type behaviour. Impact tests have been conducted on specimens, simply-supported over a distance of 5, 10, 15, 20 and 25 cm, and in each case the test specimen has been cut to size, only allowing for an overhang of 1 cm at either end.

4.8 APPARATUS AND DATA MEASUREMENT

Drop-weight impact tests on the composite specimens have been conducted utilising essentially the same set-up as was used in the tests on the metal specimen (see Part (I)) except for a few minor difference to be outlined below.

The modified set-up in Photo (4.6) shows the different supports that have been employed in tests involving the composite specimen. This change was necessary since the supports used for the metal beam (Photo (4.2)) were too bulky and could not be placed close enough to support the short composite specimen. However, apart from their physical dimensions, the two support systems are very similar in all other respects, namely preventing the specimen from bouncing off the supports, preventing lateral displacement and twisting, and allowing a limited degree of end rotation as discussed in Section (4.3).

The fact that a number of composite specimens were available for testing purposes has meant that the steel channel, used to protect the metal specimen against local damage was no longer required. In any case, such a protection device could not have been employed due to its substantial inertia compared to the total mass of the laminate. Also absent is the spring assembly employed in tests on the metal beam, which was mounted between the drop-weight carriage and load-cell (see Photo (4.3)), the primary purpose of which had been to aid the FE modelling by allowing the whole carriage assembly to be effectively modelled as a simple spring-mass system of known mass and stiffness (Section (4.5)).

As regards data measurement, only the impact load and the bending stresses have been monitored and recorded, while the impact velocity has been calculated from impactor drop-height information. The impact loads have been measured by means of the load-cell as described in Section (4.4.1), while bending strains were obtained at two positions along the specimen employing single wire gauges of type 'FLA-3' (Table (4.1)) in conjunction with strain gauge amplifiers as discussed in Section (4.4.4). A single wire gauge was placed on the bottom surface of the

specimens directly below the point of impact and a pair of gauges were connected to the top and bottom surfaces of the specimens at a distance of 1 cm away from the point of impact to measure the bending strains. The large strains associated with these composite specimens has also meant that the gain of the strain gauge amplifiers had to be reduced in order to avoid saturation. The gain was thus lowered to 200, producing a 1 volt output per 2500 micro-strains for a fully active bridge configuration. This has the added advantage of increasing the frequency response range of the amplifiers, thus mitigating the problem of signal attenuation as explained in Section (4.4.6), though for the composite specimens employed here, this should not present any problems. The measurement of impact induced shear stresses is impractical in these thin composite specimens and has been omitted, although C-scanning technique has been employed to monitor the specimen and detect any delamination that may initiate as a result of high shear stresses.

4.8.1 Ultrasonic C-scanning technique

The C-scanning method is a non-destructive damage detection technique, widely used in the aerospace industry for inspecting composite materials in order to identify manufacturing faults (e.g. voids, bonding defects) and in-service defects such as delamination. Basically, the technique involves monitoring the signal attenuation resulting from the transmission of high frequency sound waves through the specimen. The ultrasound waves interact with the specimen, its boundaries and any defects that are present and these cause both reflection and refraction of the incident signal leading to an attenuation in signal strength. It is possible to either monitor the attenuation in the transmitted signal or that in the reflected signal and the latter technique, referred to as 'pulse-echo', has been adopted in the present programme, utilising a Meccasonic pulse-echo ultrasonic scanner in conjunction with the arrangement depicted in Diagram (4.6).

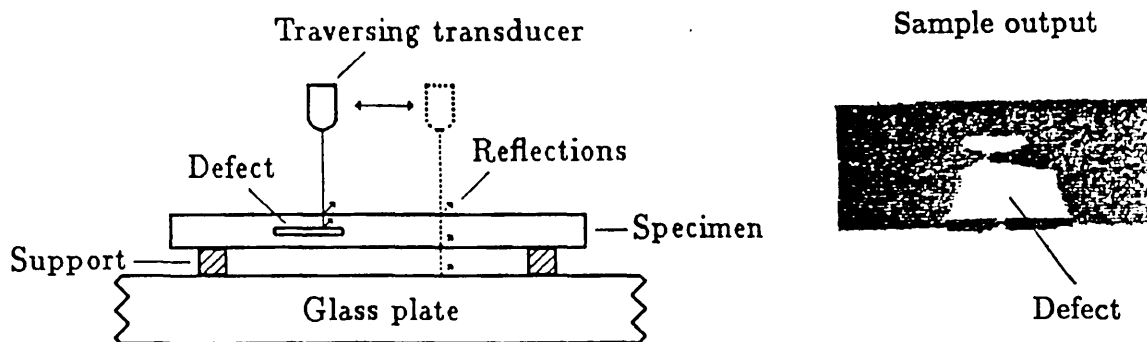


Diagram (4.6) — Typical C-scanning set-up and sample output

The complete set-up is immersed in water which provides a suitable medium for coupling the ultrasound into the specimen, and in addition, as shown, a flat glass plate placed below the specimen acts as a perfect reflector of the sound waves due to the mismatch in the acoustic impedance between the glass and the water. A spherically focused transducer positioned approximately 50 mm above the specimen is used to emit high frequency sound waves (10 MHz) which are reflected off the various boundaries returning to the transducer. The echo of the ultrasonic pulse that passes through the specimen and is reflected off the glass plate to pass through the specimen a second time is generally well separated in time from the other reflections and it is relatively easy to monitor the attenuation in this signal by setting a time gate around it. The amplitude of the gated signal controls an electronic quantiser that generates an appropriate electric current for an electric pen which plots the information on a current sensitive drawing paper, with the magnitude of the current directly affecting the plot density. A complete C-scan map (see Diagram (4.6)) can be generated by traversing the transducer in a rectilinear fashion in a plane parallel to the specimen, yielding a planar view of the composite and highlighting the lateral extent of any region of damage that may be present. The C-scan technique can not differentiate between the various damage mechanisms and neither does it provide any information about the through-thickness

position of any defects. In this programme, the C-scan technique has been employed to assess the overall quality of the test specimens, detecting the presence of any localised impact damage and monitoring the initiation and propagation of delamination.

4.9 IMPACT TEST ON COMPOSITE BEAMS

To complement the experimental efforts of Part (I), a number of additional impact tests on composite laminate beams have been conducted in the low velocity regime to examine the effect of the various parameters on dynamic response, thereby enabling a more complete assessment of the applicability of the impact parameter ($v\sqrt{\frac{m}{M}}$) and that of the proposed models of Chapter (2) for predicting impact-induced stresses. The tests set out to separately illustrate the effect of impactor velocity (v) and target length (and hence target mass (M)) on the impact response, by varying the drop-height as well as the specimen length. The impactor mass has been kept constant ($m = 0.75$ Kg), since investigating the effect of impactor mass (m) on the response of the composite specimen entails complications associated with the fact that these specimens have a very low mass and so a meaningful study of the effect of varying the impactor mass would be beyond the task of the present research programme. Depending on length, the present composite specimens weigh between (2.25–11.25 grams) and in order to highlight excitation of the higher harmonics due to a low impactor mass would require ($\frac{m}{M} \ll 1$), meaning an impactor mass of only a few grams or less, which would need to travel with an impact velocity in excess of 15 m/s to give it sufficient energy for reliable measurements to be possible. CANTWELL (70) has conducted extensive tests on similar composite specimens to study the initiation and development of damage in composite laminates under both low and high velocity impact conditions, and although CANTWELL (70) did not monitor

the impact-induced stresses in the impacted laminates to allow a direct comparison with predictions based on the proposed stress models, some of the findings of CANTWELL (70), particularly those relating to high velocity impact conditions provide useful clues as to the applicability of the stress models under low mass, high velocity conditions and any inherent limitations, which are discussed in detail in Chapter (5).

Two sets of impact tests have been conducted on the present composite specimens and details regarding the test specimens and the apparatus are given in Sections (4.7 and 4.8). The first set of tests were intended to establish the threshold energy to damage initiation in the various test specimens. For each length, a specimen was repeatedly impacted at increasing incident energy levels until damage was observed. After each impact the specimen was C-scanned to check for delamination. Once the energy level to cause damage was obtained, a virgin specimen was impacted at this energy level to verify that repeated impacts had not deteriorated the specimen. In all the tests, no deterioration of the specimen was observed and the initial damage was consistently in the form of top surface cracking running perpendicular to the fibre direction near the edge of the contact area between the impactor head and the composite beam target. The C-scan tests did not detect any delamination prior to this visible surface cracking which suggests that the impact induced shear stress must be low as would be expected according to the predictions of the proposed shear stress model of Chapter (2).

The testing of the long composite beams ($\ell = 20,25$ cm) presented some problems. Due to the large strain to failure of these laminates, the transverse deflections were so large that the simple support mechanism would lock the specimen in the deflected shape, as depicted in Photo (4.7). This was found to result in substantial frictional energy losses and consequently a considerable amount of extra incident energy was required to initiate damage and even then, the damage was found to initiate at the supports. This complication has meant that for these long test

specimens, the threshold energy for damage initiation, which is representative of the energy absorption capabilities of the specimen, could not be determined.

Having determined the energy level to cause damage and confirmed that multiple impacts do not deteriorate the specimen, a second set of impact tests were carried out in which the specimens were fitted with strain gauges, as discussed in Section (4.8), and both the impact force and the impact induced bending stresses were monitored and recorded to enable the effect of impactor velocity and beam length (and hence beam mass) on response of composites to be examined and in addition to assess the accuracy of the proposed stress model in predicting the impact induced bending stresses in composite specimens. A detailed quantitative analysis of the present test results will be deferred until Chapter (5), while here the emphasis will be placed on a qualitative discussion of some of the general findings that have emerged from the various experimental test cases, details of which are listed in Table (4.7).

4.9.1 Effect of impactor velocity on laminate response

The different composite beam specimen have each been impact tested for a range of different impactor drop-heights to assess the role of impactor velocity on loading and bending stress response. To illustrate some of the general observations, the results for the 20 cm and 5 cm specimens (CT5,CT6 and CT18,CT19), which represent the two extremes of long and short specimens in the context of the present set of impact tests, are presented in Figures (4.57–4.60). Although a 25 cm specimen has also been tested, as can be seen from Table (4.7), the severity of the aforementioned locking of the simple support mechanism has meant that there is insufficient data to illustrate the effect of impactor velocity for this longest specimen.

Examination of the measured response in Figures (4.57–4.60), clearly demonstrates that varying the impactor velocity does not alter the composite beam's

response, but rather affects the magnitude, be it loading or bending stresses. This finding is in agreement with that observed for the metal beam specimen in Section (4.6.4) and more importantly with the expectations based on the numerical investigation of Chapter (2).

4.9.2 Effect of specimen length on laminate response

To highlight the way in which the specimen length (and hence its mass (M)) influences the impact response of composite beam specimens, the results for the 25, 15 and 5 cm long specimens have been presented in Figures (4.61–4.66). For these test cases (CT1, CT11 and CT19), although the impactor mass (m) is identical, the impactor velocity (v) is different as can be seen from Table (4.7). However, since impactor velocity has been shown (Section (4.9.1)) only to affect the magnitude of the response and not the response itself, the difference in impactor velocity should not matter in a qualitative assessment of the effect of specimen length.

Inspection of the results presented in Figures (4.61–4.66), indicate that the response of these composite specimens is dominated by the fundamental mode, as would be expected on the basis of the earlier numerical study in situations where the impactor mass (m) is very much larger than the target mass (M) (i.e. $\frac{m}{M} \gg 1$), which is the case in the present tests. In addition, it can also be observed that as the specimen length increases (and hence $\frac{m}{M}$ is reduced), the higher harmonics start to be excited and gradually contribute towards the overall response for both the loading and the bending stresses. Once again, these findings are similar to those for the metal beam specimen presented in Section (4.6.2) and in agreement with the numerical predictions of Chapter (2).

Chapter Five

DISCUSSION AND CONCLUSIONS

5.1 INTRODUCTION

The present research project has been conducted on two fronts, involving detailed numerical studies and an extensive experimental investigation. The numerical work has included a parametric study, presented in Chapter (2), for identifying the various parameters that influence the low-velocity impact response of beam targets and to provide an insight into the nature of their contributions, thus forming the basis of proposed stress models for predicting the impact-induced stresses in transversely impacted beams. In addition, the finite element method, detailed in Chapter (3), has also been utilised as part of this analytical investigation in order to facilitate a more detailed and accurate modelling, to establish the importance of secondary effects (such as shear and rotary inertia) on impact response which had been omitted in the parametric study; to enable a direct estimation of the induced shear stresses and contact forces not possible on the basis of the previous numerical approach and to provide an extremely valuable tool for assessing the accuracy of both the parametric study and that of the experimental investigation.

This numerical effort has been complemented by a detailed experimental impact study involving both isotropic and composite target specimens as outlined in Chapter (4). Experimental findings have been presented and qualitatively discussed (see Chapter (4)), highlighting a number of features that are common to both experimental observations and numerical predictions, with special emphasis

on the role of the various identified parameters in affecting the beam's impact response.

In this concluding Chapter, the results of FE analysis of a number of experimental test cases will be presented and discussed, to verify the suitability of the experimental set-up and the accuracy of the test results and to illustrate the effectiveness of the shear-flexible cubic *Mindlin* beam element for predicting induced shear stresses and for this sort of calculation in general. Furthermore, the different controlling parameters (v , m , M) and the single unified impact parameter ($v\sqrt{\frac{m}{M}}$) that have previously been identified, the proposed stress models of Chapter (2) as well as alternative parameters like momentum and energy will be compared on a quantitative basis with experimental impact test results for both isotropic and composite beams. This will demonstrate the applicability of the parameters and stress models that have emerged from the present work, highlight the inadequacy of energy and momentum as suitable response parameters and illustrate the quasi-static nature of low velocity impact. In view of the findings, the prediction of damage threshold and type of damage in composites, based on the quasi-static approach, will be discussed, with special emphasis on the scaling of coupon test results — relating to the energy to initiate damage and the residual strengths — for application to larger structural configurations. Finally, the range of applicability of the quasi-static approach, the impact parameter ($v\sqrt{\frac{m}{M}}$) and the resulting models for predicting impact-induced stresses, will be reviewed to identify the limitations that are imposed upon their use by various external factors.

5.2 FE PREDICTION OF IMPACT-INDUCED BEAM RESPONSE

The specially developed finite element package, which has been described in detail in Chapter (3), has been employed for predicting the response of centrally

impacted beams and the results compared with experimental measurements to enable an assessment of the accuracy of the experimental set-up, to indicate the effectiveness of the cubic *Mindlin* beam element and to provide an insight into the general dynamic response of transversely impacted beams. In order to minimize computing costs, only a limited number of test cases have been analysed using the finite element method, and these represent the extreme and most demanding cases that have been considered experimentally. The first test case (MT4) considers a simply-supported beam of length (1.4 m), centrally impacted by a 0.5 Kg mass travelling at a velocity of 3 m/s ($v\sqrt{\frac{m}{M}} = 1.7582$), which being the longest beam (and thus the heaviest) and impacted by the lowest mass, is expected to have a significant portion of the incident energy concentrated in the higher harmonics due to its low ($\frac{m}{M}$) (see Table (4.1)). It also represents the case with the lowest value of the impact parameter ($v\sqrt{\frac{m}{M}}$) (see Table (5.1)). The second test case (MT29) is that of a (0.8 m) long, simply-supported specimen that is impacted by a 6.25 Kg mass with an incident velocity of 1.5 m/s (i.e. $v\sqrt{\frac{m}{M}} = 3.7904$), which due to its high ($\frac{m}{M}$) is expected to exhibit a response dominated by the fundamental mode (see Table (4.1)). The final test case (MT38) relates to a simply-supported beam of (0.2 m) length, centrally impacted by a 0.75 Kg mass travelling at a velocity of 2.5 m/s ($v\sqrt{\frac{m}{M}} = 4.0031$), which being the shortest beam tested, has associated with it high frequency low harmonics as well as one of the highest values of the impact parameter ($v\sqrt{\frac{m}{M}}$).

On the basis of preliminary tests presented in Chapter (3), the target beams in all three test cases have been modelled by a series of cubic *Mindlin* beam elements of equal length to yield a uniform mesh which should ensure that the spurious wave reflections of Chapter (3) are eliminated. The choice of element size was further constrained by the need to arrange that either an element node point or Gauss point would coincide with the positions for which the experimentally measured data was available (i.e. require node or Gauss points at 5 cm, 15 cm and 25 cm from the beam centre point). Taking advantage of symmetry, only one half of

the beam has been used in the analyses and the 2 cm overhang at the simple supports (see Chapter (4)) has also been ignored. The resulting meshes employed in modelling the target beams are outlined in Table (5.2). The difficulty arises with modelling the contact between the impacting mass and the beam. As is evident from Photographs (4.2,4.4), the impact carriage assembly is a complicated arrangement comprising the drop-weight, the load-cell and a high stiffness spring to reduce ringing between the impactor and target. As mentioned in Chapter (4), the high stiffness spring was introduced not only to reduce ringing but also to dominate the overall dynamic response of the impactor assembly thus enabling the whole assembly to be simply modelled by a spring-mass system. Adopting this simple model, the load-cell has been considered to be infinitely stiff and the impact carriage to have an effective stiffness identical to that of the spring damper, with the total impacting mass lumped together to form a concentrated point mass (see Diagram (5.1)). Furthermore, a second point mass, attached at the beam centre, is to represent the total mass of the displacement flag and the protecting channel (see Section (4.5) and Photo (4.3)) used in the experiments.

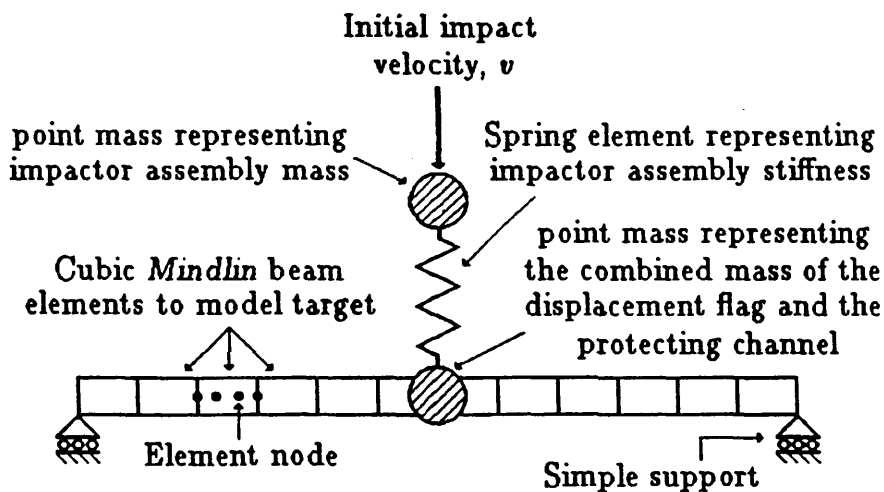


Diagram (5.1) — FE model used in the analyses

The results for the first case (light impactor) are compared in Figures (5.1-5.6). The fact that the impact carriage assembly has been simply modelled by a single spring-mass system has meant that the experimentally observed high frequency transients present in the load history could not be predicted, but only an averaged smooth variation of the general response. Despite this simplification, it appears that the general loading pattern is reasonably well predicted, though the second main impact is seen to occur somewhat later than is observed experimentally and the few minor sub-impacts are also absent. Looking at the centre node displacement in Figure (5.2), it is evident that experimental and FE results show very close similarities, especially for the first 20 milliseconds (i.e. past the quarter period) which is just prior to the second main impact. The fact that the FE analysis predicts a greater centre displacement provides the clue as to why the FE prediction of the second main impact is delayed when compared to experimental observations and why the sub-impacts are altogether absent. The discrepancy in the centre displacement seen in Figure (5.2) indicates that the FE model predicts a greater portion of the energy of the impacting mass being imparted to the target beam on first impact than is observed experimentally, leaving the impacting mass with less kinetic energy than it would otherwise retain in practice. This results in a greater deflection of the target beam with the slowed-down impacting mass unable to catch up with the beam. Consequently, the minor sub-impacts that have been observed experimentally (see Figure (5.1)) fail to occur in the FE model and the second main impact is delayed. A further consequence of this can be seen in Figures (5.4,5.5,5.6) where the measured and estimated bending stresses at positions 5, 15 and 25 cm from the point of impact are compared. Although agreement is seen to be quite good, especially for the initial 20 milliseconds prior to the second main impact, there is a discrepancy in magnitude of the bending stresses, which is directly attributable to the fact that too great a portion of impactor energy is being transferred to the target beam in the FE model. In addition, it can be seen that the period associated with the stress variations is

somewhat longer than that predicted by the FE analysis (see Figure (5.4)) and this is even more noticeable when comparing the results of the shear stresses in Figure (5.3). This is not surprising when one considers that as a result of the minor sub-impacts observed in practice, the impacting mass will effectively be in contact with the target beam for a longer period, which inherently will lead to an elongation in the period of vibration, when compared to the case of the FE model in which these sub-impacts are absent. The presence of damping, ignored in the present FE analyses, may contribute to some of the observed discrepancies, but in general it is believed that damping would have a more notable effect on reducing the stress peaks than to cause the observed phase shift in Figures (5.3–5.6). In fact, additional FE analyses, not included here, have revealed that the inclusion of the additional point mass representing the combined mass of the displacement flag and the protecting channel (see Diagram (5.1)) has a very noticeable effect on reducing the phase shift, supporting the belief that the observed discrepancies are more the result of inaccurate contact modelling rather than due to damping. The present comparison clearly demonstrates that although the FE prediction of impact response is generally in quite good agreement with experimental findings, the accuracy with which the contact loading between impacting mass and beam is modelled can have noticeable consequences, with a far more significant effect on the predicted shear stress history than either the displacement or bending stress responses.

The results for the second test case (heavy impactor) considered are presented in Figures (5.7–5.12). Again the simple spring-mass model for the impactor carriage provides an average smooth variation to the general load response (Figure (5.7)) which except for the excessive peak to trough variation reflects the general loading pattern quite well. As can be seen the high frequency transients rapidly die away which is indicative of the presence of some damping, most probably due to the frictional losses between the *Belleville* disc washers that make up the *Belleville* spring assembly employed for these test cases in the experiments. The

presence of these large load fluctuations can be seen to somewhat affect the prediction of the centre node displacement in Figure (5.8) and the bending stress variations at different positions along the beam presented in Figures (5.10,5.11,5.12). However, the observed discrepancies do not appear to seriously influence the general displacement and bending stress responses which are well predicted. As previously mentioned, the inaccuracies in modelling the loading have a more marked influence on the shear stresses and this is highlighted in Figure (5.9), where although the overall response is very similar to that measured experimentally, the peak to trough fluctuations that were present in the loading response are also evident here, thus significantly over-estimating the magnitude of the shear stresses. To confirm that the discrepancies in the predicted shear stress history are not due to misbehaviour on the part of the cubic *Mindlin* beam element, but rather the result of inaccuracies arising from the modelling of the impact loading, the finite element analysis for this test case was repeated using the well-established, shear flexible *Timoshenko* beam elements (see Chapter (3)). Being lower order beam elements, a finer uniform mesh was employed, so as not to sacrifice accuracy. The results of this analysis when compared to that based on the cubic *Mindlin* beam elements were found to be identical, which indicates that the aforementioned discrepancies are entirely due to modelling inaccuracies in the impact loading resulting from the use of a simple spring-mass system to represent the complex response of the impactor assembly.

Figures (5.13–5.16) compare the theoretical FE estimates with the experimentally measured response for the final test case (largest value of $v\sqrt{\frac{m}{M}}$). As already pointed out in Section (4.6), for the tests conducted on the very short beam specimens (0.2 m long), the frequency of the fundamental mode is sufficiently high to be affected by the attenuation of the strain gauge amplifiers used in measuring the response, so that some error in the experimental data is anticipated. Consequently, in Figures (5.13–5.16), where appropriate, the FE predictions have been modified to account for the behaviour of the SGA, using *Fast Fourier* transform technique,

and both modified as well as original FE results are presented. It is evident from Figure (5.13) that the use of the simple spring-mass model in this case is not sufficient to properly model the contact loading giving a rather poor estimation. However, this is not entirely surprising because for this test case the total response time is of order of 2 milliseconds and if one looks closely at the loading history of the earlier test cases (Figures (5.1) and (5.7)) for a similar period of time it is seen that during this time interval the loading is dominated by the dynamic response of the impactor assembly while the simple spring-mass model can only yield a smooth average approximation to the loading. In earlier test cases the beam's response period is long compared to that of the impactor assembly while in this case the two response periods are of the same order, thus highlighting the deficiencies of the spring-mass model. In contrast, the FE prediction of the central displacement (Figure (5.14)) is extremely close to that measured over the initial response period and although thereafter, due to inaccuracies in modelling the impact loading and damping, the predicted displacement response is seen to start lagging behind the observed response, the displacement variations are still in good agreement. In addition, the modified bending and shear stresses agree reasonably well with the measured data, despite the inaccuracies that were present in modelling the contact loading.

In an effort to improve the modelling of the contact loading, attempts were made to use the statical *Hertzian* contact law, which has been employed by various authors (24,71,72) to model the contact between impactor and target. *Hertzian* contact law relates the contact force to the relative displacement between the two contacting bodies as $P = k\sqrt{\delta^3}$ where ' k ' is a function of material properties and the local geometry of the two bodies and ' δ ' is the relative displacement between them. This characteristics can easily be modelled by means of a non-linear spring, the stiffness of which can be expressed as a function of the relative displacement, hence $K \propto \sqrt{\delta}$. FE analysis including modelling of *Hertzian* contact were conducted on the aforementioned test cases but the results that were obtained were

rather disappointing. Whereas previously the predicted load history generally reflected the measured response in the mean, when *Hertzian* contact was included the results deteriorated and high frequency, high amplitude force variations were predicted that resulted in several short duration micro-impacts over a period for which previously only one single impact had been measured and predicted. The close coupling between loading and the shear stresses meant that the shear stress history now also contained very high frequency transients of large magnitude and consequently the predicted bending stress and displacement variations were also adversely affected, though to a much lesser extent. The reason for this is that contact force is dependent on the relative displacement between impactor and target, which in turn is a function of the dynamics of both impactor and target. Thus, it is obvious that any inaccuracies in modelling of either impactor or target will have a significant effect on the contact loading which in turn will influence the subsequent dynamics of both contacting bodies. In fact, in the past, authors, who have successfully applied this technique, have limited their calculation to a very short initial period of order of 50–100 microseconds and used a rigid mass as impactor. Here the calculation extends over a far longer period so that errors accumulate and most importantly the impactor assembly is a complicated arrangement, the dynamics of which can not easily be modelled, as well as additional complications due to the presence of the protecting channel, completely ignored up to now. Various attempts at modelling the behaviour of the impactor assembly by using a series of spring-mass systems did not succeed either in representing the dominant response characteristics of the assembly and in view of these difficulties it was decided to use the simple spring-mass model and accept the limitations that it has. After all, the main purpose of the FE analysis was to confirm the fact that the experimental instrumentation is capable of correctly measuring the impact response and to assess the suitability of the cubic *Mindlin* beam element for use in dynamic calculations.

In spite of the aforementioned complexities associated with the proper modelling of the actual experimental set-up and the simplifications that have been introduced to approximate the contact loading, the resulting predictions by the FE analysis compare quite well with experimental observations, especially those relating to displacement and bending stresses. Even the variation of the shear stresses are generally reasonable and this in spite of modelling inaccuracies to which the shear stresses seem to be extremely susceptible. Overall however, the present findings clearly indicate that the shear-flexible cubic *Mindlin* beam element is stable and quite suitable for dynamic calculations, especially when determination of the shear stresses is of importance. The foregoing comparison further suggests that the experimental instrumentation is sufficiently accurate for present purposes, though as shown here in Figures (5.13,5.15,5.16) and in Section (4.4.6), the attenuation of the strain gauge amplifiers does appear to lead to slight inaccuracies in the experimentally measured data relating to the very short test specimens. However, as mentioned in Section (4.4.6), the rest of the experimental test cases on longer specimens are not expected to be adversely affected by the amplifier limitations. To further confirm this, the FE predictions for the first two test cases analysed here have also been subjected to a similar treatment as performed for the third test case relating to the very short specimen, namely correcting the FE results, using *Fast Fourier* technique, to account for the amplifier characteristics. As had been expected, and in contrast to the findings of the third test case, the original and corrected FE predictions for the two former examples which relate to longer test specimens were found to be virtually identical. This clearly supports the belief that signal attenuation by the strain gauge amplifiers does not pose any problems in the majority of experimental impact tests conducted as part of the present research work.

Finally, to provide a better insight into the dynamic response of impacted beams, the entire beam deflection and the spatial distribution of bending and shear stresses along the beam for different times after impact are presented in

Figures (5.17–5.22) comparing the responses of a long to that of a short beam. It can clearly be seen that, in both cases, initially the beam's response is very localised which then gradually spreads to the rest of the beam. As a result, the beam experiences an initial reverse curvature at positions away from the point of impact causing an initial reversal of bending stress (Figures (5.18) and (5.21)) consistent with experimental observations (see Figures (5.5,5.6)). This reversal in bending stress is also observed for cases where the response is expected to be dominated by the fundamental mode and is not confined to situations for which a number of higher harmonics are excited (e.g. Figure (4.40)). As regards the bending and shear stress variation along the beam these are shown in Figures (5.18,5.19,5.21,5.22). Comparing the longitudinal distribution of the bending and shear stresses for the two cases presented, it is clearly evident that while in the case of the short beam, the distribution is seen to reflect the dominance of the fundamental response mode, for the long beam target, the distribution of stresses clearly illustrates the noticeable presence of the higher harmonics and this is in complete agreement with expectations based on the findings of the simplified numerical investigation in Chapter (2). It should be noted that the changes in the shear distribution with time are most important for composite laminates in determining whether interlaminar shear failures scale from coupon size test specimens to full scale.

5.3 EVALUATION OF THE IMPACT PARAMETER AND PROPOSED STRESS MODELS AS RESPONSE PREDICTORS

Numerical investigation into the way in which the various parameter (m , v , M , E , ρ , etc.) influence the response of transversely impacted beams, presented in Chapter (2), illustrated the existence of a unified parameter ($v\sqrt{\frac{m}{M}}$), referred to here as the impact parameter, that was numerically demonstrated to govern impact response, especially the development of maximum bending stresses in impacted

beams. In order to confirm the numerical findings and verify the applicability of this impact parameter, a detailed experimental investigation was undertaken to cover a broad range of test configurations and to include both metal and composite specimens. The experimental findings have been presented and qualitatively discussed in Chapter (4) with emphasis on the role of the three main parameters (m, M, v) and the way these affect the impact response of beams. The comparison of numerical and experimental results on a quantitative basis follows next to highlight that the isolated impact parameter ($v\sqrt{\frac{m}{M}}$) is in fact the correct parameter as compared to impactor energy or momentum which have been used by other workers in dynamic response predictions and to validate the suitability of the proposed stress models.

5.3.1 Prediction of impact response for isotropic beams

Knowledge of the maximum stress levels which are likely to be realised under various loading conditions is of primary importance in the design of structural components, this being even more so in the Aircraft industry where weight penalty severely limits the margin of safety that can be tolerated. Although the main consideration is to ensure the integrity of the components under the specified operating conditions, predicting the likely mode of failure once the limiting conditions have been exceeded may also have a direct influence on the design, since certain modes of failures may result in far more devastating consequences than others, and again information regarding levels of stress and its distribution are very necessary. Thus, in the present research, attention has been focused on predicting the maximum stresses that develop in beams as a result of low velocity, transverse impact and how these are affected by various material and structural properties. On the basis of the numerical study of Chapter (2), a model for the prediction of maximum bending stress has been proposed, whereby

$$\sigma_B \propto v\sqrt{\frac{m}{M}} \quad (5.1)$$

The quasi-static arguments that have led to equation (5.1) suggest a similar expression for the prediction of maximum shear stresses, that is

$$\sigma_s \propto \frac{v}{\ell} \sqrt{\frac{m}{M}} \quad (5.2)$$

although this latter model could not be substantiated numerically due to the unsuitability of the chosen numerical technique. Experimental results of Chapter (4), however, provide a sufficiently broad basis for the examination and evaluation of the predictive capabilities of the proposed stress models, presently to be considered.

The experimental investigation regarding an isotropic target has covered a wide range of impact configurations and in each case both the bending and shear stresses at a fixed position, 5 cm away from the point of impact (from now on referred to as station (1)), have been monitored and recorded. This has enabled the maximum bending as well as shear stresses, that develop during the impact period, to be determined and these are tabulated, together with other relevant information in Table (5.1). It is evident that the majority of tests have been carried out at selected beam lengths of 140 cm, 80 cm and 20 cm, representing long, medium and short specimens respectively, although some tests on other intermediate beam lengths have also been performed. For these three selected beam lengths, the experimentally obtained maximum bending stresses and maximum shear stresses at station (1) have been plotted against a variety of parameters, some of which have been used by other workers and include impactor velocity, square-root of impactor mass, momentum, impactor energy and the impact parameter ($v\sqrt{\frac{m}{M}}$). These plots are presented in Figures (5.23–5.37) and where appropriate, a theoretical estimation based on the stress models (equations (2.6.3),(2.6.4)) has also been included, subject to certain assumptions regarding the stress distribution along the beam, to be discussed later.

With reference to equations (5.1) and (5.2), it is evident that both the impact-induced bending and shear stresses are expected to increase linearly with increase in impactor velocity. Figures (5.23–5.25) present the variation of stresses with impactor velocity for different beam sizes but same impactor mass based on the experimental findings of Chapter (4), and these clearly demonstrate the linear dependency of impact-induced stresses on impactor velocity as suggested by both the numerical study of Chapter (2) and the proposed stress models. It can further be observed that the theoretical prediction of bending stresses is quite good and that the prediction of shear stresses, though not accurate for the long specimens, improves rapidly as the specimen size decreases. For the longer specimens, the higher resistance to the impactor mass (m) associated with the increased target inertia (or mass (M)), leads to larger contact forces and consequently higher shear stresses. Inertia forces are of course completely ignored in the quasi-static predictors (equations (5.1) and (5.2)).

Apart from impactor velocity, the impactor mass has also been shown numerically to affect both the bending and shear stresses. In fact, induced stresses are expected to vary linearly with the square-root of impactor mass (see equations (5.1) and (5.2)). Experimental results highlighting the variation of bending and shear stresses with impactor mass for various beam lengths but same impactor velocity are presented in Figures (5.26–5.28). The findings clearly illustrate that, as expected, both bending and shear stresses tend to be linearly dependent on the square-root of impactor mass. As before, the stress models are more accurate in predicting the bending stresses, though the accuracy of the shear stress model improves rapidly as specimen size decreases.

The relationship between induced stresses and the single unified impact parameter ($v\sqrt{\frac{m}{M}}$) can be examined on the basis of experimental results as presented in Figures (5.29–5.31). These fully support the findings of earlier numerical investigation that impact-induced stresses are directly proportional to this impact

parameter as expressed by equations (5.1) and (5.2), forming the basis of the proposed stress models. Once again, the results highlight the accuracy of the bending stress model and the fact that an increase in target inertia leads to a deterioration of the predictive capabilities of the shear stress model.

In contrast, and as expected on the basis of the parameter study in Chapter (2), it is immediately evident that neither the bending stresses nor the shear stresses correlate in any obvious form with the impactor momentum (see Figures (5.32–5.34)). Consequently, impactor momentum — suggested in References (5,7) as a parameter of possible significance — does not appear to be a suitable control parameter in the prediction of impact response.

Impactor energy is a parameter that has in the past been associated with predicting impact response by some researchers. Using the test data of Chapter (4), variations of both bending and shear stress with impactor energy for different beam length have been considered and are presented in Figures (5.35–5.37). At this stage, and based on the results presented, it is not justifiable to discard energy as a governing parameter. However, it should be realised that in the form the data is presented, the effect of beam mass (or length) has not been properly accounted for, since each plot refers to a given length of beam.

In order to verify that the correct parameter, namely $(v\sqrt{\frac{m}{M}})$, as compared to impactor energy, has been identified it will be necessary to compare results of different length of beam, which is only possible if stresses or strains are known at similar relative positions for the different lengths of beam. However, in the tests that have been performed, the strain gauges were fixed at absolute positions along the beam (i.e. 5 cm away from the point of impact). Thus as the beam is cut shorter, the relative position at which the strains are measured is altered, so that direct comparison of results of different beam lengths is not a straightforward matter. Had the strains been measured at the beam centre this problem would

not have arisen, but as the displacement flag (see Chapter (4)) had been attached at this position, strain gauges could no longer be placed there.

To remedy the situation to some extent and permit comparison of stress data for the different length of beam, certain assumptions have to be introduced. For a simply-supported light beam, loaded statically at the centre, the shear force, and hence shear stress, does not vary along the beam length, while the bending stresses will vary linearly with beam length. If one therefore assumes, that since the impact response appears to be a quasi-static phenomenon (see Chapter (2)), that the maximum shear stresses are reasonably constant along the beam and the maximum bending stresses diminish linearly from the beam mid-point to the supports, then a direct comparison for the results of different beam lengths becomes possible. At this point it should be clarified that this assumption does not suggest that shear stress distribution along the beam at any instant of time is constant or that the bending stress distribution at any instant of time varies linearly with distance along the beam (Figures (5.17–5.22)). This, as clearly shown by the FE results of Section (5.2), is not the case and the dynamics of the problem gives rise to more complicated distributions as stress waves traverse the length of the beam. Instead, what is being assumed is that the maximum shear stress measured at different positions along the beam are expected to be equal in magnitude, be it that these occur at different instances in time. Similarly for the bending stresses it will be assumed that the maximum bending stress occurs at the beam mid-point and this decreases linearly down to zero at the beam supports. Thus the maximum bending stresses monitored at different positions along the beam can be extrapolated to yield the expected maximum bending stress at the beam centre.

Based on the above assumptions the shear stress (wherever measured) is expected to vary linearly with the bending stress per distance from the simple support. In order to examine the validity of the above assumptions, one such plot has been presented in Figure (5.38) using the experimentally measured maximum shear and bending stresses at a position 5 cm away from the beam centre (i.e. point of

impact). What the results indicate is that for the shorter beam lengths the above assumption is clearly applicable, though nearer the origin (0,0) where the results correspond to those of longer beam specimens this assumption appears to become less accurate. This is expected since strictly speaking the foregoing assumptions apply to a light beam for which the beam inertia is negligible in comparison to the loading. However, it should be emphasised that the extrapolations are still over a relatively small distance, especially so for the longer specimens.

The above assumptions regarding stress distributions along beams have been used in evaluating the theoretical estimates in earlier Figures (5.23–5.31) and also to enable the data for all the various beam length to be combined on single plots. These are presented in Figures (5.39,5.40), where variations of shear stress with impactor energy per length and $(\frac{v}{\ell}\sqrt{\frac{m}{M}})$ as well as those of centre-point bending stress with impactor energy and $(v\sqrt{\frac{m}{M}})$ have been considered. Concentrating on the results concerning the shear stress (Figure 5.39), it becomes obvious that $(\frac{v}{\ell}\sqrt{\frac{m}{M}})$ as predicted, is indeed the controlling parameter rather than energy/length. there is some scatter near the origin, and indeed the clear linear relationship does not pass through the origin as theory would have it. The reason for this is that points near the (0,0) position correspond to large 'M' (i.e. long and heavy beam) and in these cases the beam inertia forces become influential and the assumption of a light beam with constant shear, on which the derivation of the shear stress model has been based, breaks down. Looking back at earlier plots (Figures (5.23–5.31)) depicting the variation of shear stress with various quantities (v , m and $v\sqrt{\frac{m}{M}}$) for the three different length of beam, it can clearly be seen that for the longest beam the shear stresses are significantly underestimated by the theory by more than a factor of '2', but that these theoretical estimates rapidly improve as the beam length (and hence its inertia) is reduced.

The variation of bending stress with impact parameter $(v\sqrt{\frac{m}{M}})$ and with impactor energy are presented in Figure (5.40) and again quite clearly $(v\sqrt{\frac{m}{M}})$ is the governing parameter as compared to energy. The prediction of the bending

stress model can also be seen to be in good agreement with the experimental findings and unlike the shear stresses, the bending stresses appear to be much less sensitive to beam inertia. This is to be expected since the error in shear is due to the approximation adopted in estimating gradients ($\frac{d}{dx}$) by ($\frac{1}{l}$) which in turn is governed by deformation due to inertia. Referring back to earlier plots in which the variation of bending stress with the different quantities (v , m and $v\sqrt{\frac{m}{M}}$) was considered for the various lengths of beam, it is clearly evident that the theoretical estimates based on the proposed stress model are in very good agreement with experimental results.

5.3.2 Prediction of impact response for composite beams

To further extend the generality of the simple stress predicting models and the use of the impact parameter ($v\sqrt{\frac{m}{M}}$) to include non-isotropic beams, a number of impact tests have been conducted on 16-ply CFRP laminates covering a range of different configurations. Experimental details and a qualitative discussion on the results that were obtained have been presented in Chapter (4), while presently the emphasis will be placed on a quantitative comparison of the results with predictions based on the proposed stress models, in order to establish the applicability of the stress predictors to both isotropic and composite specimens.

The impracticality of measuring shear stresses in these thin composite specimens by conventional means has meant that no data on impact-induced shear stresses in these specimens is available for a direct comparison with the proposed shear stress model. However, on the basis of the proposed models, it can be shown that for the specimen configurations presently employed, the flexural response is far more dominant and no shear failure is likely to occur even if theoretical predictions are inaccurate. This is confirmed by the C-scan results of the tested laminates which did not reveal any delamination prior to visible surface damage indicating that the impact-induced shear stresses are low as expected. A discussion

on the factors that affect the change from flexure to a shear dominated response will follow in Section (5.4).

The maximum bending stresses measured at the mid-span of the various specimens have been plotted against the impactor velocity and are presented in Figures (5.41–5.45), together with theoretical predictions based on the bending stress model. Note that for the 16-ply (0,±45) laminates used in the impact tests, the theoretical estimates are obtained on the basis that $\rho_e = 1500 \text{ Kg/m}^3$ and $E_e = 106.8 \text{ GN/m}^2$. As expected, the variation of bending stress with velocity is linear and in general the agreement between experimental results and theoretical prediction can be considered as extremely good. For the long specimens some experimental points fall below the theoretically predicted line but these points correspond to those test cases where the large frictional forces in the supports kept the specimen in the deflected shape, with the consequence that this energy loss would result in a lowering of the maximum bending stress that is reached (refer to Table (5.3)). The results of the various tests have also been combined into a single plot of bending stress versus impact parameter ($v\sqrt{\frac{m}{M}}$), and are presented in Figure (5.46). The data points corresponding to those test cases where locking was observed have been excluded from this last plot. Once again the expected linear relationship between impact-induced bending stresses and the impact parameter is clearly highlighted and the prediction based on the stress model can be seen to be in excellent agreement with experimental results.

A point of interest also emerges from this test data. In Chapter (4) it was pointed out that for some of the specimens tested (Table (4.7)), impact tests were conducted on the specimen in the presence of an already visible crack (i.e. after a visible crack was noted, the specimens were impacted again from an increased drop-height) and the data points corresponding to these cases are separately highlighted in Figures (5.41–5.46). The results indicate that such surface cracks have not had a serious influence on the impact resistance of the specimens. Thus, as regards prediction of impact-induced bending stresses within the specimen, the results

confirm that a quasi-static argument based on ETB is indeed applicable, and that the impact parameter ($v\sqrt{\frac{m}{M}}$) can be employed even in the presence of limited, localised damage.

5.4 PREDICTION OF DAMAGE THRESHOLD AND TYPE OF DAMAGE

The findings of the present research work have already demonstrated that low velocity impact-induced stresses in target beams can be successfully predicted on the basis of the proposed models (equations (2.6.3),(2.6.4)). However, at present, the common approach, when studying impact resistance behaviour of composites, is to establish energy thresholds for the onset of damage or the reduction in residual strengths and use these measures of the energy absorption capability of the composite to serve as indicators to the effectiveness of the composite in withstanding impact. The fact that the low velocity impact response has been shown to be a quasi-static process and the proposed models have been derived on the basis of energy arguments, means that the absorbed impact energy can be expressed as a function of failure stress, material properties and the target geometry. This can be achieved by simply rearranging equations (2.6.3,2.6.4), relating to the impact-induced stresses, which yield

$$\left. \begin{array}{ll} \text{for flexure} & J = \frac{c_e^2}{6} \left(\frac{\kappa}{y_g} \right)^2 \left(\frac{\sigma_{Bg}}{E_g} \right)^2 M \quad (a) \\ \text{for shear} & J = \frac{c_e^2 k_e^2}{24} \left(\frac{\sigma_s}{E_e} \right)^2 M \left(\frac{\ell}{\kappa} \right)^2 \quad (b) \end{array} \right\} \quad (5.3)$$

The expressions in (5.3) ignore the energy absorbed by the failure process (i.e. fibre fracture or delamination) and thus should predict :

- (i) damage threshold
- (ii) failure if the damage energy is negligible compared to the elastic energy

- (iii) failure if damage is proportional to the elastic energy provided that 'equivalent' stresses are used.

So in order to analytically predict the energy to failure using equations (5.3), it will be necessary to introduce an appropriate value for the respective failure stresses, namely $\sigma_B = F\sigma_B$ and $\sigma_S = F\sigma_S$. The statically measured failure stresses for the present type of laminate are listed in Table (4.6), and concentrating for the time being on flexural type damage, it would appear that the limit ought to be a compressive fibre failure stress of 1350 MN/m^2 . Looking at Figure (5.46) clearly indicates that there is a definite failure threshold, above which damage is observed, although the stress level that defines this threshold can be seen to be substantially higher than the statically expected value. This point is further highlighted in Figure (5.47), where the theoretical prediction, based on the static failure stress of 1350 MN/m^2 , is shown to greatly underestimate the energy absorption capability of the tested composites. However, the theoretical line in Figure (5.47) can be made to coincide with the experimental results, by selecting a value of 1940 MN/m^2 to represent the failure stress.

To examine the reasons for the above discrepancy, it will be necessary to consider the measured bending stress at the mid-span of the tested specimens at the onset of visible damage and these are listed in Table (5.3). It can be observed that the measured stress in the outer fibres is consistently higher than the allowable strength for even the tensile stress (see Table (4.6)), although no damage has been detected on this lower surface in any of the tests conducted as part of the present research work. This suggests that the material may be strongly rate-sensitive and this complicates the analytical prediction of failure threshold. However, in spite of this, the present findings indicate that the energy to failure can be predicted using quasi-static arguments, provided that an appropriate failure stress is chosen which is in agreement with experimental findings (in this case 1940 MN/m^2), rather than the expected static values. Adopting such an approach, it will then be possible to establish an appropriate failure stress value from laboratory tests

on small specimens and use this information in conjunction with equations (5.3) to predict the energy to failure in larger, more realistic composite structures. This is important in as far as both damage threshold and residual strength are directly influenced by the ability of the target to absorb energy elastically, which according to equations (5.3), is expected to be dependent on target geometry as well as failure threshold. A discussion on scaling to account for differences in target geometry and impact conditions is deferred to Section (5.5).

In the above discussion on predicting the energy absorption capability of composites, attention has been focused on situations where the target response will be dominated by flexure and some form of flexural failure ensues (in the present case, this having been a local compression-buckling failure on the top surface close to the point of impact). However, as the specimen size is reduced the targets response is expected to become more shear dominated and failure expected to be in the form of delamination. In fact, equations (5.3) indicate that the transition from a flexure to a shear dominated response is related to the beam's span-to-depth ratio and the ratio of the interlaminar to fibre strength of the material, as suggested by DOREY (2). For the 16-ply $(0,\pm 45)$ laminates used in the present experiments, the transition to a shear dominated response falls outside the range of specimen sizes tested here, so that the validity of equation (5.3b) for predicting the energy to cause a shear failure in these composites can not be directly established. However, in spite of the lack of direct experimental evidence, the fact that delamination was not detected prior to first visible damage for the specimens that have been tested, is in complete agreement with expectations based on equations (5.3). Furthermore, the fact that the quasi-static nature of low velocity impact has been confirmed and, in addition, the impact-induced shear stresses in the tested metal specimen (see Chapter (4)), predicted by the proposed model are in good agreement with test results, would suggest that the shear response of composites ought also be predictable by means of the same model. Consequently, if this is the case, then the energy to initiate a shear failure in composites can be estimated using

equation (5.3b). However, as before, the success of an analytical prediction of damage threshold, this time for shear, may require that a representative shear failure stress be established by conducting suitable impact tests rather than to rely on the static values of the interlaminar shear strength.

5.5 EFFECT OF TARGET GEOMETRY ON DAMAGE THRESHOLD AND RESIDUAL STRENGTH

It is common practice, when studying the impact resistance behaviour of composites, to establish energy thresholds for the onset of damage and the reduction in residual strengths, based on impact tests performed on coupon size targets (usually 50 mm $\times$ 25 mm). Applying the results of such small scale laboratory tests directly to larger structures inherently assumes that the energy absorption capability of the impacted target is independent of target geometry. However, in the present investigation, numerical and experimental studies have confirmed that low velocity impact response is quasi-static in nature, as a result of which the impact response has been shown to be predictable on the basis of the impact parameter ($v\sqrt{\frac{m}{M}}$). Consequently, the target mass (M) and hence target geometry (length (ℓ), depth (d) and width (b)), is expected and has already been shown to affect the impact response. In addition, the energy to initiate damage is also expected to be dependent on target geometry as indicated by equations (5.3), be it a flexural or a shear response.

The influence of target length (ℓ) on the energy absorption capability of composites can be demonstrated using the test data of Chapter (4). For fixed cross-sectional properties, the variation of impact energy to cause damage has been plotted for different length specimen (Figure (5.47)). For the specimens tested, the dominant response is flexure, and it is clearly evident that the ability of the laminates to absorb energy elastically increases linearly with specimen length in accordance with expectations based on equation (5.3a). This finding brings into

question the current practice of conducting impact tests on small coupons to establish the incident energy level to initiate damage and the threshold energy for loss of residual strengths, which are then considered to represent an upper limit, irrespective of the specimen dimensions. In fact, Figure (5.47), illustrates that, at least under low velocity impact conditions, larger specimens can absorb a greater quantity of incident energy in elastic deformation before the onset of damage when compared to shorter ones. Similar results have also been reported by CANTWELL (70) for low velocity impact conditions. Thus damage threshold and residual strengths are directly affected by target geometry and should be scaled accordingly to allow for specimen size, when applying coupon test results to other configurations. In general coupon tests relate energy to residual strength as depicted in Diagram (5.2) below.

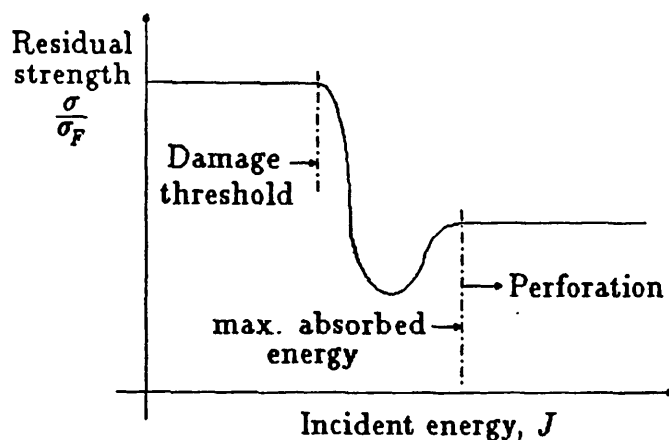


Diagram (5.2) — General pattern of the variation of residual strength with impact energy for composite laminates

Now if damage starts to occur at a certain stress level in a quasi-static fashion, it should be possible to scale the impact energy (J) to allow for specimen size. So if damage is due to flexure (say, fibre fracture), then with reference to equation (5.3a), $J \propto \sigma_B^2 \ell$ for a fixed cross-section, and if the residual strength

in tension is governed by fibre failure, then should be plotted against $(\frac{J}{\ell})$ in order to account for the effect of specimen length. Similarly, if residual strength is governed by delamination (say, the residual compressive strength which depends on the amount of delamination) indicating a shear response, then once again according to equation (5.3b), $J \propto \sigma_S^2 \ell^3$ for a fixed cross-section, and the appropriate factor would appear to be $(\frac{J}{\ell^3})$.

Apart from length, the cross-sectional dimensions of the target are also expected to affect its energy absorbing ability, as indicated by equations (5.3). Changes to cross-sectional property have not been considered as part of the present impact tests presented in Chapter (4) and, to date, little work has been undertaken to provide suitable data to fully assess the influence of cross-sectional property on the ability of laminates to store elastic strain energy. However, as part of a recent experimental study by CANTWELL (70), the effect of target geometry has been investigated and some of the more relevant results — to the present study — that have emerged have been briefly summarised for completeness and are presented in Table (5.4). Concentrating on the effect of laminate thickness in the (± 45) composites, it is clearly evident that for target thicknesses up to and including 2 mm, the energy to initiate damage increases approximately linearly with increasing target thickness, as would be expected from equation (5.3a) for a flexural response. For thicker laminates above 2 mm, the energy to first damage can be seen to reverse trend and actually decrease with any further increase in target thickness. This change in characteristics can be explained by the fact that unlike the thinner laminates, failure no longer initiates in the matrix between the lower surface fibres due to flexure, but is in the form of top surface matrix cracking between fibres and most likely the result of local contact stress field. This clearly suggests that as target thickness increases, local contact forces become more important and thus undermine the basic assumption on which the quasi-static theory is based, namely that flexural response is dominant. Aside from such limitations which will be discussed more fully in Section (5.6), the fact that for thinner laminates, for

which flexure is the dominant mode of response, target thickness has been shown to directly influence damage threshold energy and hence residual strength means that the above argument, regarding scaling with respect to target length, would also apply to scaling with respect to target thickness. In this case the appropriate scaling factors would be $(\frac{J}{d})$ and (Jd) respectively, for a given specimen length and width (see equations (5.3)).

With regards to the effect of laminate width on the energy to damage, the lack of appropriate test data excludes a thorough, conclusive examination. However, the mere fact that low velocity laminate response has already been demonstrated to be a quasi-static phenomenon and that the effects of specimen length and thickness on energy to failure, in general, appear to be in agreement with expectations, suggests a strong possibility that the effect of width on the energy absorption capability of laminates may also be predictable on the basis of equations (5.3). Consequently, provided that laminate width is limited so as to minimize plate-type behaviour, the appropriate scaling factor for impact energy allowing for different specimen width should be $(\frac{J}{b})$ for both shear and flexure according to equations (5.3).

In view of the foregone discussion and in order to simplify matters, it is suggested that scaling of impact energy be generalised with respect to target geometry as a whole rather than the individual components. This can be achieved by using the appropriate scaling factors $(\frac{J}{M})$ and $(\frac{Jd^2}{M\ell^2})$ in Diagram (5.2), depending on the dominant mode of damage (or response) as already explained above.

Finally, although scaling of damage threshold and residual strengths to account for specimen size has been shown to be appropriate for low velocity impact response, certain limitations regarding its application have already been highlighted. The range of applicability of the present proposals will next be discussed in detail to consider the restrictions imposed not only by target geometry, but also by impactor mass and velocity.

5.6 RANGE OF APPLICABILITY OF THE QUASI-STATIC APPROACH

The quasi-static arguments leading to the proposed impact parameter ($v\sqrt{\frac{m}{M}}$) and stress models has been shown to be applicable to beams impacted at low velocities in the range tested (i.e. up to 6 m/s), whether it be metal or composite. In the derivation of the impact parameter, contact effects were considered to be localised and the material assumed to be able to survive the local contact stresses and even if it could not, then the energy dissipated would still be small compared to the total strain energy stored by the whole beam. This assumption should be valid so long as the localised damage does not seriously alter the properties of the beam. In this respect metals present less of a problem than do fibrous composites, which as shown by CANTWELL (70) can be susceptible to contact loading resulting in top surface matrix cracks and leading to planes of delamination that adversely affect the integrity of the laminate and undermine its impact resistance. However, for the present low velocity impact tests on composites the initial damage was in every case caused on the top surface due to local compressive fibre buckling and the stress model successfully predicted the resulting bending stresses. This was even true of specimens tested in the presence of damage. This latter observation confirms the aforementioned assumption that as long as secondary effects or damage are localised, the use of the impact parameter ($v\sqrt{\frac{m}{M}}$) and the quasi-static approach appear to be valid. It should be emphasised that one must differentiate between applying the impact parameter to estimate bending stresses — for which it is truly applicable — and using it to estimate incident energy to cause initial damage as suggested in Sections (5.4,5.5) — for which it can be used (although it may not be strictly applicable) by choosing an appropriate failure stress. The difference, as mentioned earlier, is due to the inherent assumptions of the theory as regards type of failure since strictly speaking, theory assumes that the chosen failure stress relates to a flexural-type damage.

Placing a limit on the range of applicability of the impact parameter and stress models is somewhat difficult. Neither the quasi-static nor the *Bernoulli-Euler* theory of Chapter (2) are capable of predicting this limit, which must certainly depend on the point beyond which the inherent assumptions can no longer be justified. This requires investigation into secondary effects which have up to now been excluded, such as contact and stress wave effects which give rise to a complex three-dimensional stress field in the vicinity of the point of impact, to establish the point at which these effects can no longer be ignored and undoubtedly material considerations will play as important a part in the determination of the relative importance of these secondary effects as do the structural concepts.

To highlight the difficulties associated with imposing an absolute limit it is appropriate to briefly examine some of the findings that have emerged from an extensive experimental study by CANTWELL (70) on the impact response of composite laminates to include both low and high velocity impact regimes. As previously mentioned in Section (5.5), for the (± 45) laminates (see Table (5.4)), the results indicate an initial increase in energy to cause damage with increase in laminate thickness up to and including the 16-ply laminate, followed by a drop in incident energy to damage for the thicker laminates. This change in characteristics has been explained by CANTWELL as being due to a change in the nature of the initial damage from one of lower surface matrix cracking caused by flexure to one of top surface matrix cracking resulting from a local contact stress field. For the $(0, \pm 45)$ laminates, a similar trend is evident. However, in this case the change in the mode of damage is seen not to result in an immediate sharp drop in incident energy to first damage. This suggests that although contact loading may be locally dominant, this has not yet significantly detracted from the flexural response of the laminate. Eventually, with a further increase in thickness up to 32-ply, contact loading is seen to dominate the response, leading to a substantial reduction in the energy needed to initiate damage. The results further indicate that, apart from laminate thickness, the lay-up and ply-orientation may also have

a direct influence on secondary effects such as contact (compare results for the 16-ply (± 45) and (0, ± 45) laminates in Table (5.4)).

Ideally , to estimate the change-over from flexure to a contact dominated response would require a suitable model to express contact-induced stresses in terms of all the governing variables. However, using simplifying assumptions, it is possible to roughly estimate this cross-over point for certain conditions. Contact stresses depend on the contact force for a fixed geometry of stamp face, and for static *Hertzian* contact this force can be expressed in terms of the local material properties and is an upper limit for a half-space. Flexure-induced shear and bending stresses depend on target response, but for static loading or for low velocity impact on laminates, where the combination of a low velocity and a heavy impactor (compared to the weight of the laminate) results in a fundamental mode of response (see Chapter (2)), these stresses can be expressed in terms of the target flexural stiffness which is a function of target geometry and the material properties. Consequently, the point at which contact effects start to dominate the response may be approximated, though this should be considered as an upper bound, since factors such as impactor mass and velocity may lower this cross-over threshold.

Let's consider what limitations will be imposed by the impactor mass and velocity on the range of applicability of the impact parameter ($v\sqrt{\frac{m}{M}}$) and the quasi-static approach and in what way. CANTWELL (70) has observed that for low velocity impact, the energy absorption capacity of laminates increases with an increase in size and appears to be in agreement with the findings of the present study (see Section (5.5)), indicating a quasi-static response. However, the high velocity impact tests have demonstrated that in this velocity range the specimen length does not affect the energy to initiate damage in the laminates, although as discussed in Section (5.5) and according to the numerical findings of Chapter (2), the longer specimen should be capable of absorbing more energy. For a high impact velocity and a low impactor mass, a substantial amount of incident energy will be present in the higher harmonics (see Table (5.5)), but the effect of higher

harmonics was included in the numerical study and shown to result in a maximum stress earlier on in the response, rather than later as is the case when most energy is concentrated in the fundamental mode, and this maximum bending stress was shown to be predictable by a quasi-static energy argument irrespective of when it occurred. Now the specimen used by CANTWELL (70) to investigate the effect of length on energy to initial damage was the 16-ply (0,±45) laminate for which damage, even at low impact velocities, is on the top surface as a result of local contact stresses (see Table (5.4)). At higher impact velocities, these contact effects are expected to be even more dominant and consequently the stress field in the vicinity of impact will be very different to that based on the Engineer's theory of bending (ETB) which is the underlying assumption in the derivation of the impact parameter. Furthermore, the lower impactor mass also contributes to promoting a local contact response by virtue of the fact that reducing impactor mass, as shown, results in higher harmonics being excited which raises the effective flexural stiffness.

In contrast, if one examines CANTWELL's results for tests on 8-ply (0,±45) and 8-ply (±45) laminates, it is found that these laminates exhibit a flexural-type damage for both low and high velocity impact. This indicates that the contact stresses in this case have a much diminished effect due to the target flexibility and reduced target inertia, and one would expect therefore that the impact parameter ($v\sqrt{\frac{m}{M}}$) should be applicable in this case. Indeed, if one examines the energy required to cause first damage in the 8-ply and 16-ply (0,±45) laminates at high velocities, the former is found to exhibit a flexural-type damage as compared to the contact damage found in the latter two specimens, which explains why the 8-ply laminate is capable of absorbing 20% more energy before damage initiation when compared to the other two laminates. This confirms the belief that when contact stresses are of secondary importance, the specimen can respond in flexure, at high as well as low velocities. Actually, the fact that in the 8-ply (0,±45) laminate the initial damage changes from one of lower surface fibre cracking at low

impact velocity to one of lower surface matrix splitting at high impact velocity is of particular interest. Table (5.5) shows that for this 8-ply specimen at higher impact velocities, greater proportion of incident energy is expected to be present in the higher harmonics due to the lower impacting mass. These higher harmonics have a shorter wavelength associated with them, and in fact, the wavelength of the second mode for this specimen is of the order of its width, so that bending stresses will exist locally in both directions. However, the fibres run in only one direction and the weaker matrix material cannot sustain the stresses generated in the cross-fibre direction as a result of which damage in the form of matrix splitting occurs. Thus, although this specimen appears to behave more in accordance with expectations relating to the numerical study of Chapter (2), it is the material properties that impose a limitation on the range of applicability of the impact parameter.

It is apparent that laminate thickness plays a more dominant role in promoting and amplifying contact effects than do impactor mass or velocity, and as the above findings indicate, thin laminates can respond in flexure even at high velocities. In this case, DOREY (2) suggests that the threshold conditions for maximum flexural response be based on when the specimen is deformed at its resonant frequency (of the fundamental mode), resulting in the following expression

$$\frac{v}{c} = \frac{\pi}{\sqrt{27}} \left(\frac{\sigma}{E} \right) \quad (5.4)$$

which, for the present laminates, would yield a transition velocity of approximately 70 m/s. It should be realised that the expression (5.4) does not account for impactor size (and hence incident energy) and nor does it include target geometry (thus no information regarding the energy absorption capacity of the laminates). Consequently, it is possible for target penetration to occur before this transition velocity is reached. In fact, CANTWELL (70) has found that the 4-ply laminate will be penetrated at an incident energy of about 1.2 Joules which yields an impact

velocity of roughly 50 m/s as compared to the expected 70 m/s transition velocity. Thus provided that contact effects are not dominant and the laminate responds in flexure, and that the incident energy is not sufficient to lead to penetration, then the transition velocity based on equation (5.4) may be taken as an upper bound on velocity, beyond which the flexural response and the quasi-static approach will cease to be applicable.

On the other hand, when dealing with thick laminates, in addition to local contact effects, another phenomenon, namely through-thickness stress waves can begin to contribute to the local stress distribution at higher impact velocities, thus leading eventually to tensile (spallation) failure near the lower surface. In this case, a crude analytical measure, for the onset of significant stress waves through the thickness, can be obtained by imagining the impact zone confined to a small cylinder under the impactor. The compressed face — advancing at a velocity (v) — will produce a strain of order $(\frac{v}{c})$ where $c = \sqrt{\frac{E}{\rho}}$ (see JOHNSON (73)). For the present composite laminates, taking the modulus (E) to be that of the matrix and assuming a typical failure strain of the order of 1%, this would lead to an incident velocity of about 25 m/s. Experimental evidence obtained by CANTWELL (70) clearly demonstrates that for the thicker laminates subject to high velocity impact, damage initiates in the form of top surface matrix cracking as a result of local contact stresses, subsequently followed by extensive through-thickness damage and fibre spallation which are indicative of the presence of through-thickness stress wave effects. The impactor velocity associated with this form of ballistic-type damage has been found to range between 25 m/s and 50 m/s.

The foregone discussion demonstrates the difficulties associated with imposing an upper bound on the range of applicability of the quasi-static approach, the impact parameter ($v\sqrt{\frac{m}{M}}$) and the proposed models for predicting impact-induced stresses. As indicated, this is controlled by the relative contributions of secondary effects, such as contact stresses and through-thickness stress waves, which in turn

will depend on target rigidity, inertia, impact velocity, etc.. In addition, material constraints such as non-homogeneity and strain-rate effects may also notably influence local stress distributions and magnify the importance of the secondary effects. It would appear that further research of the kind performed as part of the current project will be necessary to establish similar simplified models which could be used to estimate the contributions of these secondary effects.

5.7 CONCLUSIONS

The present research programme has considered in some detail the dynamics of transversely impacted beams by a low velocity projectile, both theoretically and experimentally, and the main findings are summarised below :

- a) The main parameters that govern the dynamic response of transversely impacted beams have been identified as the impactor velocity (v) and the ratio of impactor mass to target mass ($\frac{m}{M}$). The impactor velocity has been shown to only affect the magnitude of the response and the mass ratio to directly control the distribution of energy in the various harmonics, so that for high ($\frac{m}{M}$) the response will be dominated by the fundamental mode while low ($\frac{m}{M}$) results in the excitation of the higher harmonics.
- b) The maximum impact-induced stresses have been found to be dependent on some unified parameter of the form $v\left(\frac{m}{M}\right)^p$, irrespective of whether the maximum stress occurs early in the response or at a much later stage which is determined by the value of ($\frac{m}{M}$). On the assumption that low velocity impact is mainly a quasi-static process, energy arguments have been used to derive simple models for predicting both the maximum bending and maximum shear stresses. These models are expressed as a function of the impact

parameter $(v\sqrt{\frac{m}{M}})$, so that

$$\sigma_B \propto v\sqrt{\frac{m}{M}}$$

$$\sigma_S \propto \frac{v}{\ell}\sqrt{\frac{m}{M}}$$

The applicability of the above models has been confirmed experimentally for isotropic and composite beams, though the accuracy of the shear stress model has been found to deteriorate somewhat with increase in target inertia. This however, should not present a serious drawback, since in general, the bending stresses are the dominant stress component and shear stresses tend only to become significant in short specimens, for which inertia is normally low anyway.

- c) The present study has confirmed that under low velocity impact conditions, the ability to absorb elastic energy is directly related to target size, as suggested by the proposed models. This is in contrast to current practice in which tests are conducted on small coupons (25 mm × 50 mm) to establish the incident energy to cause initial damage and to relate the residual strengths to impact energy, the results of which are then taken to represent an upper limit, irrespective of specimen dimensions, and applied to different structural configurations. It has been proposed that scaling of test data to incorporate size effects be facilitated in accordance with the findings of the quasi-static approach, by relating the damage threshold and residual strengths to $(\frac{J}{M})$ when dealing with flexural-type failure and to $(\frac{Jd^2}{M\ell^2})$ when considering shear-related damage, rather than just impact energy (J) alone. Also based on the quasi-static approach, the change-over from a flexural to a shear dominated response has been shown to be a function of the span-to-depth ratio and the ratio of interlaminar to fibre strength of the material. It has further been demonstrated that by establishing an appropriate failure stress through tests

on small specimens, the proposed models can be utilised to estimate the energy to first damage, allowing for size differences that may exist between the specimen and the configuration of a real application. However, using the proposed models for the purpose of predicting this damage threshold may not be strictly valid, unless the failure stress relates to a flexural damage, consistent with the inherent assumptions of its derivation.

- d) The range of applicability of the quasi-static approach, the impact parameter and the proposed models have been examined with a view to highlight the various factors that impose limitations on their use. In the main, these restrictions are associated with the relative importance of secondary effects, such as *Hertzian* contact and through-thickness stress-waves, which in turn will be dependent on the structural and material properties of the impacted target. In this respect, target thickness has been found to be very prominent in promoting the dominance of contact effects, so that while the low velocity impact response of thin composites is flexure dominated, the thick laminates exhibit contact-related failure modes. Similarly, high impact velocities tend to introduce additional effects due to through-thickness stress waves in thick laminates resulting in ballistic-type damage, whereas thin composites seem to continue responding in flexure even at elevated impact velocities. However, high impact velocities are generally accompanied by a low impacting mass and this, as shown, leads to the excitation of the higher harmonics which tend to increase contact forces and can also induce transverse stresses that may expose weaknesses in the laminate lay-up configuration (see Section (5.6)). It is apparent that estimating the contributions of these secondary effects is of primary importance in determining the exact range over which the present proposals would be applicable and it is suggested that future work be directed towards establishing simple models for assessing the significance of the various secondary effects.

- e) The dynamic response of beams subject to low velocity impact has also been studied with the aid of the finite element method, permitting a greater flexibility in modelling of the target and impact loading conditions. The initial target response has illustrated a local response near to the point of impact resulting in a reverse curvature away from the contact point. Eventually, the natural modes begin to dominate the response and which modes are excited, as anticipated, will depend on the value of $(\frac{m}{M})$. The effects of shear and rotary inertia on target response have been shown to be negligible unless very high harmonics are excited.
- f) Of the beam elements investigated, the shear-flexible cubic *Mindlin* element has proven to be very suitable for use in dynamic calculations, especially if the shear stress response is of importance (e.g. in composite laminates). In deriving the *Mindlin* family of elements, the presence of shear locking has been dealt with by under-integrating the shear contributions to yield a 'field-consistent' formulation ensuring that the elements are free from spurious effects.
- g) Modelling the impact loading conditions has also been found to be a major obstacle, since any inaccuracies in loading are prominently reflected in the prediction of the shear stress response. However, displacement and bending stress response are not significantly influenced by approximations in the loading conditions.

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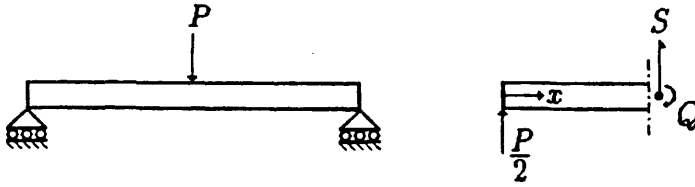
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APPENDIX (A)

A.1 THEORETICAL ESTIMATES

Theoretical estimates, used for comparison with FE predictions, have been derived from the elementary theory of bending, for displacements, bending and shear stresses of loaded beams. A brief outline of the relevant results will be presented here.

A.1.1 Simply-supported, centrally loaded beam



From force equilibrium

$$S = -\frac{P}{2} \quad \text{for } 0 < x < \frac{\ell}{2}$$

so that

$$\sigma_S = -\frac{P}{2Ak_e} \quad (A.1)$$

Moment equilibrium requires that

$$Q = EI \frac{d^2 u_B}{dx^2} = -\frac{Px}{2}$$

giving

$$\sigma_B = \frac{Qy}{I} = -\frac{Pxy}{2I} \quad (A.2)$$

Integration subject to the following boundary conditions

$$\begin{aligned} \text{at } x = 0 : \quad & u_B = 0 \\ \text{and at } x = \frac{\ell}{2} : \quad & \frac{du_B}{dx} = 0 \end{aligned}$$

$$\text{yields} \quad u_B = \frac{P\ell^3}{48EI} \left(-\frac{4x^3}{\ell^3} + \frac{3x}{\ell} \right) \quad (A.3)$$

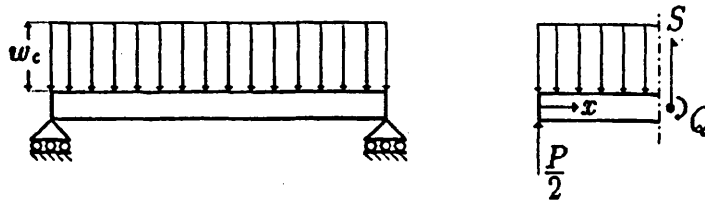
The deflection due to shear can be estimated using the definition for shear angle

$$\psi = \frac{du_S}{dx} = \frac{S}{GAk_e}$$

and the boundary condition that at $x = 0$, $u_S = 0$

$$\text{to give} \quad u_S = -\frac{Px}{2GAk_e} \quad (A.4)$$

A.1.2 Simply-supported, uniformly loaded beam



Force equilibrium requires that for $0 < x < \frac{\ell}{2}$

$$S = -EI \frac{d^3 u_B}{dx^3} = w_c x - \frac{P}{2}$$

$$\text{where} \quad P = \int_0^\ell w_c dx = w_c \ell$$

$$\text{so that} \quad \sigma_S = \frac{S}{Ak_e} = \frac{w_c \ell}{Ak_e} \left(\frac{x}{\ell} - \frac{1}{2} \right) \quad (A.5)$$

Integrating subject to the condition that $\text{at } x = 0, \frac{d^2 u_B}{dx^2} = 0$

gives
$$\sigma_B = \frac{Qy}{I} = Ey \frac{d^2 u_B}{dx^2} = \frac{w_c \ell^2 y}{2I} \left(-\frac{x^2}{\ell^2} + \frac{x}{\ell} \right) \quad (A.6)$$

Further integration using the remaining boundary conditions

$$\begin{aligned} \text{at } x = 0 : \quad & u_B = 0 \\ \text{and at } x = \frac{\ell}{2} : \quad & \frac{du_B}{dx} = 0 \end{aligned}$$

yields
$$u_B = \frac{w_c \ell^4}{48EI} \left(-\frac{x^4}{\ell^4} + \frac{2x^3}{\ell^3} - \frac{x}{\ell} \right) \quad (A.7)$$

As before, using the definition for shear angle

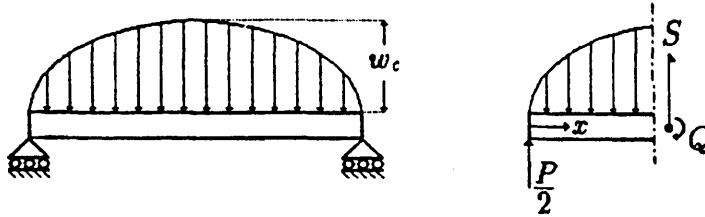
$$\psi = \frac{du_S}{dx} = -\frac{w_c \ell}{GAk_e} \left(\frac{x}{\ell} - \frac{1}{2} \right)$$

in conjunction with the boundary condition that $\text{at } x = 0, u_S = 0$

yields upon integration

$$u_S = -\frac{w_c \ell^2}{2GAk_e} \left(\frac{x^2}{\ell^2} - \frac{x}{\ell} \right) \quad (A.8)$$

A.1.3 Simply-supported, sinusoidally loaded beam



Force equilibrium requires that for $0 < x < \frac{\ell}{2}$

$$S = -EI \frac{d^3 u_B}{dx^3} = \int_0^x w_c \sin \frac{\pi x}{\ell} dx - \frac{P}{2}$$

where
$$P = \int_0^\ell w_c \sin \frac{\pi x}{\ell} dx = \frac{2w_c \ell}{\pi}$$

so that
$$\sigma_S = \frac{S}{Ak_e} = -\frac{w_c \ell}{\pi Ak_e} \cos \frac{\pi x}{\ell} \quad (A.9)$$

Integrating subject to the condition that at $x = 0$, $\frac{d^2 u_B}{dx^2} = 0$

gives
$$\sigma_B = EI \frac{d^2 u_B}{dx^2} = -\frac{w_c \ell^2}{\pi^2 I} \sin \frac{\pi x}{\ell} \quad (A.10)$$

Further integration using the remaining boundary conditions

$$\begin{aligned} \text{at } x = 0 : \quad & u_B = 0 \\ \text{and at } x = \frac{\ell}{2} : \quad & \frac{du_B}{dx} = 0 \end{aligned}$$

yields
$$u_B = \frac{w_c \ell^4}{\pi^4 EI} \sin \frac{\pi x}{\ell} \quad (A.11)$$

Again, using the definition of the shear angle

$$\psi = \frac{du_S}{dx} = -\frac{w_c \ell}{\pi G A k_e} \cos \frac{\pi x}{\ell}$$

and integrating subject to aforementioned boundary conditions yields

$$u_S = -\frac{w_c \ell^2}{\pi^2 G A k_e} \sin \frac{\pi x}{\ell} \quad (A.12)$$

A.2 KINEMATICALLY EQUIVALENT FORCES

In finite element idealisations, distributed loading can only be represented by concentrated forces at the element node points. The distribution of these nodal forces must however be determined in a manner as to correctly reflect the effect of the original distributed loading. A consistent approach for satisfying this requirement is to derive the nodal forces on the basis that the work done by the actual loading over the element is identical to the work done by the equivalent system of nodal forces. The resulting nodal forces are termed 'kinematically equivalent forces' and in this section these will be derived for the various beam elements of Section (3.3) used in the modelling of the test cases outlined in Section (A.1) above.

Thus, for a system of nodal forces ($\mathbf{R}$) and nodal displacements ($\mathbf{r}$) and a set of element displacements ($\mathbf{u}$) and corresponding distributed loading ($\mathbf{w}$) it follows from the application of virtual work that

$$\bar{\mathbf{r}}^t \mathbf{R} = \frac{1}{2} \int_{-1}^1 \bar{\mathbf{u}}^t \mathbf{w} d\xi$$

where $\bar{\mathbf{r}}^t$ and $\bar{\mathbf{u}}^t$ denote virtual quantities and $\xi = \frac{2x}{L}$

Using the element shape functions (ω) the element displacements ($\mathbf{u}$) can be expressed as

$$\mathbf{u} = \omega \mathbf{r}$$

$$\text{and hence} \quad \mathbf{R} = \frac{1}{2} \int_{-1}^1 \omega^t \mathbf{w} d\xi \quad (\text{A.13})$$

A.2.1 Uniform loading

For a uniform distributed loading over an element $w = w_c$

Now using the element shape function in equation (A.13), the equivalent nodal forces for the *Hermitian*, *Timoshenko* and the linear *Mindlin* elements will be given by

$$\mathbf{R} = -\frac{w_c L}{2} \{1 \quad 0 \quad 1 \quad 0\}$$

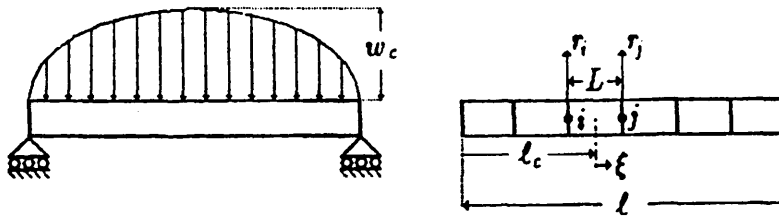
For the quadratic *Mindlin* element, the nodal forces are

$$\mathbf{R} = -\frac{w_c L}{6} \{1 \quad 0 \quad 4 \quad 0 \quad 1 \quad 0\}$$

and for the cubic *Mindlin* element

$$\mathbf{R} = -\frac{w_c L}{8} \{1 \quad 0 \quad 3 \quad 0 \quad 3 \quad 0 \quad 1 \quad 0\}$$

A.2.2 Sinusoidal loading



Here, the loading over an individual element is expressed as

$$w = -w_c (\sin \alpha \cos \beta \xi + \cos \alpha \sin \beta \xi)$$

$$\text{where} \quad \alpha = \frac{\pi \ell_c}{L} \quad \text{and} \quad \beta = \frac{\pi \ell}{2L}$$

The equivalent nodal force vectors for the various beam elements can be shown to be as follows

a) **Hermitian element :**

$$\mathbf{R} = -\frac{w_c L}{\pi} \sin \alpha \sin \beta \{1 \quad 0 \quad 1 \quad 0\} \\ -\frac{w_c L}{\pi} \cos \alpha \left(\cos \beta + \frac{3}{\beta^2} \cos \beta - \frac{3}{\beta^3} \sin \beta \right) \{1 \quad 0 \quad -1 \quad 0\}$$

b) **Timoshenko element :**

$$\mathbf{R} = -\frac{w_c L}{\pi} \sin \alpha \sin \beta \{1 \quad 0 \quad 1 \quad 0\} \\ -\frac{w_c L}{\pi} \chi_1 \cos \alpha \{1 \quad 0 \quad -1 \quad 0\}$$

where
$$\chi_1 = \cos \beta - \frac{\phi \sin \beta}{\beta(1 + \phi)} + \frac{3 \cos \beta}{\beta^2(1 + \phi)} - \frac{3 \sin \beta}{\beta^3(1 + \phi)}$$

c) **Linear Mindlin element :**

$$\mathbf{R} = -\frac{w_c L}{\pi} \sin \alpha \sin \beta \{1 \quad 0 \quad 1 \quad 0\} \\ -\frac{w_c L}{\pi} \cos \alpha \left(\cos \beta - \frac{\sin \beta}{\beta} \right) \{1 \quad 0 \quad -1 \quad 0\}$$

d) **Quadratic Mindlin element :**

$$\mathbf{R} = -\frac{w_c L}{\pi} \sin \alpha \{ \chi_1 \quad 0 \quad -\chi_2 \quad 0 \quad \chi_1 \quad 0 \} \\ -\frac{w_c L}{\pi} \cos \alpha \left(\cos \beta - \frac{\sin \beta}{\beta} \right) \{1 \quad 0 \quad 0 \quad 0 \quad -1 \quad 0\}$$

$$\text{where} \quad \chi_1 = \sin \beta + \frac{2 \cos \beta}{\beta} - \frac{2 \sin \beta}{\beta^2}$$

$$\chi_2 = \frac{4 \cos \beta}{\beta} - \frac{4 \sin \beta}{\beta^2}$$

e) Cubic *Mindlin* element :

$$\mathbf{R} = -\frac{w_c L}{4\pi} \sin \alpha \{ \chi_1 \quad 0 \quad \chi_2 \quad 0 \quad \chi_2 \quad 0 \quad \chi_1 \}$$

$$- \frac{w_c L}{4\pi} \cos \alpha \{ \chi_3 \quad 0 \quad \chi_4 \quad 0 \quad -\chi_4 \quad 0 \quad -\chi_3 \}$$

$$\text{where} \quad \chi_1 = 4 \sin \beta + \frac{9 \cos \beta}{\beta} - \frac{9 \sin \beta}{\beta^2}$$

$$\chi_2 = -\frac{9 \cos \beta}{\beta} + \frac{9 \sin \beta}{\beta^2}$$

$$\chi_3 = 4 \cos \beta - \frac{13 \sin \beta}{\beta} - \frac{27 \cos \beta}{\beta^2} + \frac{27 \sin \beta}{\beta^3}$$

$$\chi_4 = \frac{27 \sin \beta}{\beta} + \frac{81 \cos \beta}{\beta^2} - \frac{81 \sin \beta}{\beta^3}$$

A.3 ELEMENT STRESS SMOOTHING MATRICES

The underlying principle that forms the basis of the stress smoothing technique has been described in Section (3.4.4) and here the resulting smoothing matrices for the various beam elements that have been implemented into the present FE code will be presented.

A.3.1 *Hermitian* beam element

sampling points :

$$\xi = -\frac{1}{\sqrt{3}} \quad , \quad \xi = \frac{1}{\sqrt{3}}$$

smoothing matrix :

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_1 \end{bmatrix}$$

$$\text{where} \quad a_1 = \frac{(1 + \sqrt{3})}{2} \quad , \quad a_2 = \frac{(1 - \sqrt{3})}{2}$$

A.3.2 *Timoshenko* beam element

sampling points :

$$\xi = -\frac{1}{\sqrt{3}} \quad , \quad \xi = \frac{1}{\sqrt{3}}$$

smoothing matrix :

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & a_2 & 0 \\ 0 & a_1 & 0 & a_2 \\ a_2 & 0 & a_1 & 0 \\ 0 & a_2 & 0 & a_1 \end{bmatrix}$$

where
$$a_1 = \frac{(1 + \sqrt{3})}{2} \quad , \quad a_2 = \frac{(1 - \sqrt{3})}{2}$$

A.3.3 Linear *Mindlin* beam element

sampling point :

$$\xi = 0$$

smoothing matrix :

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A.3.4 Quadratic *Mindlin* beam element

sampling points :

$$\xi = -\frac{1}{\sqrt{3}} \quad , \quad \xi = \frac{1}{\sqrt{3}}$$

smoothing matrix :

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & a_2 & 0 \\ 0 & a_1 & 0 & a_2 \\ a_3 & 0 & a_3 & 0 \\ 0 & a_3 & 0 & a_3 \\ a_2 & 0 & a_1 & 0 \\ 0 & a_2 & 0 & a_1 \end{bmatrix}$$

where
$$a_1 = \frac{(1 + \sqrt{3})}{2} \quad , \quad a_2 = \frac{(1 - \sqrt{3})}{2} \quad , \quad a_3 = \frac{1}{2}$$

A.3.5 Cubic *Mindlin* beam element

smoothing matrix :

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & a_2 & 0 & a_3 & 0 \\ 0 & a_1 & 0 & a_2 & 0 & a_3 \\ a_4 & 0 & a_5 & 0 & a_6 & 0 \\ 0 & a_4 & 0 & a_5 & 0 & a_6 \\ a_6 & 0 & a_5 & 0 & a_4 & 0 \\ 0 & a_6 & 0 & a_5 & 0 & a_6 \\ a_3 & 0 & a_2 & 0 & a_1 & 0 \\ 0 & a_3 & 0 & a_2 & 0 & a_1 \end{bmatrix}$$

a) Least square approach :

sampling points :

$$\xi = -\sqrt{\frac{3}{5}} \quad , \quad \xi = 0 \quad , \quad \xi = \sqrt{\frac{3}{5}}$$

and

$$\begin{aligned} a_1 &= \frac{1}{54} \left(45 + 27\sqrt{\frac{5}{3}} \right) & a_2 &= -\frac{2}{3} \\ a_3 &= \frac{1}{54} \left(45 - 27\sqrt{\frac{5}{3}} \right) & a_4 &= \frac{1}{54} \left(5 + 9\sqrt{\frac{5}{3}} \right) \\ a_5 &= \frac{44}{54} & a_6 &= \frac{1}{54} \left(5 - 9\sqrt{\frac{5}{3}} \right) \end{aligned}$$

b) *Barlow's* approach :

sampling points :

$$\xi = -\frac{\sqrt{5}}{3} \quad , \quad \xi = 0 \quad , \quad \xi = \frac{\sqrt{5}}{3}$$

$$\begin{array}{ll}
\text{and} & a_1 = \frac{1}{10} (9 + 3\sqrt{5}) & a_2 = -\frac{8}{10} \\
& a_3 = \frac{1}{10} (9 - 3\sqrt{5}) & a_4 = \frac{1}{10} (1 + \sqrt{5}) \\
& a_5 = \frac{8}{10} & a_6 = \frac{1}{10} (1 - \sqrt{5})
\end{array}$$

TABLES AND FIGURES

— for Chapter (2) —

TABLE (2.1) — Variation of the modal eigenvalues, relating to the beam-mass model, with impactor/beam mass ratio

$\frac{m}{M}$	15.869	7.9344	3.9672	2.2670	1.5869
MODE	EIGENVALUES ($\lambda_i l$)				
1	0.6544	0.7726	0.9059	1.0216	1.0966
2	3.9424	3.9575	3.9861	4.0252	4.0603
3	7.0774	7.0861	7.1028	7.1268	7.1494
4	10.2163	10.2224	10.2342	10.2514	10.2679
5	13.3565	13.3611	13.3703	13.3837	13.3967
6	16.4972	16.5009	16.5084	16.5194	16.5301
7	19.6382	19.6413	19.6476	19.6569	19.6660
8	22.7793	22.7820	22.7875	22.7955	22.8034
9	25.9206	25.9230	25.9278	25.9349	25.9419
10	29.0619	29.0640	29.0683	29.0747	29.0809
11	32.2033	32.2052	32.2091	32.2148	32.2205
12	35.3447	35.3465	35.3500	35.3552	35.3604
13	38.4861	38.4878	38.4910	38.4958	38.5006
14	41.6276	41.6291	41.6321	41.6366	41.6410
15	44.7691	44.7705	44.7733	44.7775	44.7816
16	47.9106	47.9119	47.9145	47.9184	47.9223
17	51.0521	51.0533	51.0558	51.0594	51.0631
18	54.1936	54.1948	54.1971	54.2005	54.2040
19	57.3352	57.3363	57.3384	57.3417	57.3449
20	60.4767	60.4777	60.4798	60.4829	60.4860

TABLE (2.1) — continued

$\frac{m}{M}$	1.0579	0.7934	0.5290	0.3967	0.3174
MODE	EIGENVALUES ($\lambda_i l$)				
1	1.1803	1.2369	1.3102	1.3564	1.3884
2	4.1116	4.1555	4.2266	4.2818	4.3258
3	7.1843	7.2162	7.2721	7.3194	7.3598
4	10.2941	10.3186	10.3632	10.4028	10.4380
5	13.4175	13.4373	13.4742	13.5078	13.5384
6	16.5473	16.5640	16.5953	16.6243	16.6512
7	19.6808	19.6950	19.7222	19.7477	19.7716
8	22.8163	22.8288	22.8528	22.8755	22.8969
9	25.9533	25.9644	25.9859	26.0063	26.0257
10	29.0912	29.1012	29.1206	29.1392	29.1570
11	32.2298	32.2389	32.2566	32.2737	32.2900
12	35.3689	35.3773	35.3936	35.4093	35.4244
13	38.5085	38.5162	38.5312	38.5458	38.5599
14	41.6483	41.6555	41.6695	41.6831	41.6962
15	44.7884	44.7951	44.8082	44.8209	44.8333
16	47.9286	47.9349	47.9472	47.9592	47.9708
17	51.0691	51.0750	51.0866	51.0979	51.1089
18	54.2096	54.2152	54.2261	54.2369	54.2473
19	57.3503	57.3556	57.3660	57.3761	57.3861
20	60.4910	60.4961	60.5059	60.5156	60.5251

TABLE (2.1) — continued

$\frac{m}{M}$	0.2645	0.2267	0.1984	0.1763	0.1587
MODE	EIGENVALUES ($\lambda_i l$)				
1	1.4120	1.4301	1.4445	1.4562	1.4659
2	4.3617	4.3915	4.4167	4.4382	4.4568
3	7.3948	7.4253	7.4521	7.4758	7.4970
4	10.4695	10.4978	10.5234	10.5466	10.5677
5	13.5664	13.5921	13.6157	13.6375	13.6576
6	16.6761	16.6994	16.7210	16.7412	16.7601
7	19.7940	19.8151	19.8349	19.8536	19.8712
8	22.9172	22.9365	22.9547	22.9720	22.9885
9	26.0442	26.0619	26.0787	26.0948	26.1102
10	29.1740	29.1903	29.2059	29.2209	29.2352
11	32.3057	32.3208	32.3354	32.3494	32.3629
12	35.4390	35.4531	35.4667	35.4799	35.4926
13	38.5735	38.5867	38.5995	38.6119	38.6239
14	41.7090	41.7214	41.7335	41.7451	41.7565
15	44.8453	44.8570	44.8684	44.8794	44.8902
16	47.9822	47.9933	48.0040	48.0145	48.0248
17	51.1196	51.1301	51.1404	51.1504	51.1601
18	54.2575	54.2675	54.2773	54.2868	54.2961
19	57.3958	57.4053	57.4146	57.4237	57.4326
20	60.5344	60.5435	60.5524	60.5611	60.5696

**TABLE (2.2) — The variation of the modal energy distribution with
impactor/beam mass ratio**

$\frac{m}{M}$	15.869	7.9344	3.9672	2.2670	1.5869
MODE	PERCENTAGE OF TOTAL ENERGY				
1	97.3587	94.8653	90.2767	84.2448	79.0492
2	1.5714	3.0207	5.5955	8.7811	11.3230
3	0.4934	0.9692	1.8707	3.1068	4.2173
4	0.2358	0.4666	0.9135	1.5488	2.1445
5	0.1352	0.2685	0.5295	0.9074	1.2695
6	0.0849	0.1689	0.3347	0.5772	0.8128
7	0.0553	0.1102	0.2190	0.3795	0.5368
8	0.0356	0.0710	0.1414	0.2460	0.3493
9	0.0209	0.0418	0.0834	0.1457	0.2078
10	0.0089	0.0178	0.0357	0.0626	0.0898

TABLE (2.2) — continued

$\frac{m}{M}$	1.0579	0.7934	0.5290	0.3967	0.3174
MODE	PERCENTAGE OF TOTAL ENERGY				
1	70.6717	64.5809	55.3215	48.6148	43.5441
2	14.2737	16.3871	18.8406	19.9477	20.3482
3	5.7494	7.0481	9.0234	10.3942	11.3448
4	3.0400	3.8355	5.1710	6.2296	7.0702
5	1.8619	2.3865	3.3139	4.1011	4.7688
6	1.2466	1.6137	2.2834	2.8760	3.3997
7	0.8838	1.1517	1.6512	2.1057	2.5188
8	0.6501	0.8513	1.2324	1.5864	1.9147
9	0.4888	0.6426	0.9374	1.2155	1.4776
10	0.3709	0.4892	0.7181	0.9370	1.1459
11	0.2802	0.3706	0.5472	0.7179	0.8827
12	0.2070	0.2747	0.4077	0.5377	0.6647
13	0.1452	0.1933	0.2885	0.3827	0.4757
14	0.0906	0.1210	0.1819	0.2429	0.3041
15	0.0401	0.0539	0.0820	0.1107	0.1400

TABLE (2.2) — continued

$\frac{m}{M}$	0.2645	0.2267	0.1984	0.1763	0.1587
MODE	PERCENTAGE OF TOTAL ENERGY				
1	38.4266	35.5080	32.5732	30.2177	28.4418
2	19.8645	19.6117	18.6640	18.7180	18.3239
3	11.5616	11.8695	12.3428	12.3291	12.4127
4	7.4162	7.9100	8.4393	8.6628	8.8585
5	5.2511	5.5918	6.0318	6.4000	6.5935
6	3.7809	4.0953	4.5513	4.8089	5.0230
7	2.8549	3.1290	3.5379	3.7620	3.9298
8	2.2309	2.4614	2.8002	2.9916	3.1413
9	1.7835	1.9791	2.2553	2.4230	2.5630
10	1.4491	1.6173	1.8409	1.9882	2.1237
11	1.1892	1.3363	1.5150	1.6465	1.7802
12	0.9793	1.1101	1.2525	1.3703	1.5013
13	0.8053	0.9228	1.0363	1.1409	1.2668
14	0.6587	0.7639	0.8547	0.9459	1.0628
15	0.5333	0.6262	0.6982	0.7762	0.8806
16	0.4243	0.5042	0.5595	0.6254	0.7148
17	0.3273	0.3937	0.4331	0.4880	0.5613
18	0.2383	0.2905	0.3154	0.3594	0.4160
19	0.1540	0.1903	0.2038	0.2352	0.2743
20	0.0711	0.0891	0.0949	0.1110	0.1307

TABLE (2.3) — Numerical tests for investigating the effect of varying beam length

Note : for all the following cases

$$\begin{aligned}
 E &= 72.0 \text{ GN/m}^2 \\
 G &= 27.0 \text{ GN/m}^2 \\
 \rho &= 2720.0 \text{ Kg/m}^3 \\
 b &= 0.0192 \text{ m} \\
 d &= 0.0181 \text{ m} \\
 m &= 1.5 \text{ Kg} \\
 v &= 4.43 \text{ m/s}
 \end{aligned}$$

TEST EXAMPLES	(1)	(2)	(3)	(4)	(5)
BEAM LENGTH (m)	0.1000	0.2000	0.4000	0.7000	1.0000
$\frac{m}{M}$	15.869	7.9344	3.9672	2.2670	1.5869
DYNAMIC STRESS (MN/m ²)	956.65	784.04	604.04	543.64	462.59
QUASI-STATIC STRESS (MN/m ²)	740.88	523.88	370.44	280.03	234.29
NO. OF MODES USED	10	10	10	10	10

TABLE (2.3) — continued

TEST EXAMPLES	(6)	(7)	(8)	(9)	(10)
BEAM LENGTH (m)	1.5000	2.0000	3.0000	4.0000	5.0000
$\frac{m}{M}$	1.0579	0.7934	0.5290	0.3967	0.3174
DYNAMIC STRESS (MN/m ²)	344.10	314.66	247.99	225.39	220.44
QUASI-STATIC STRESS (MN/m ²)	191.29	165.67	135.27	117.15	104.78
NO. OF MODES USED	15	15	15	15	15

TEST EXAMPLES	(11)	(12)	(13)	(14)	(15)
BEAM LENGTH (m)	6.0000	7.0000	8.0000	9.0000	10.0000
$\frac{m}{M}$	0.2645	0.2267	0.1984	0.1763	0.1587
DYNAMIC STRESS (MN/m ²)	207.35	202.09	195.12	179.45	165.71
QUASI-STATIC STRESS (MN/m ²)	95.65	88.55	82.83	78.10	74.09
NO. OF MODES USED	20	20	20	20	20

**TABLE (2.4) — Numerical tests for investigating the effect of varying
impactor mass**

Note : for all the following cases

$$\begin{aligned} E &= 72.0 \text{ GN/m}^2 \\ G &= 27.0 \text{ GN/m}^2 \\ \rho &= 2720.0 \text{ Kg/m}^3 \\ b &= 0.0192 \text{ m} \\ d &= 0.0181 \text{ m} \\ \ell &= 1.0 \text{ m} \\ v &= 4.43 \text{ m/s} \end{aligned}$$

TEST EXAMPLES	(16)	(17)	(18)	(19)	(20)
IMPACTOR MASS (Kg)	15.000	7.5000	3.7500	2.1430	1.5000
$\frac{m}{M}$	15.869	7.9344	3.9672	2.2670	1.5869
DYNAMIC STRESS (MN/m ²)	956.65	784.04	604.04	543.64	462.59
QUASI-STATIC STRESS (MN/m ²)	740.88	523.88	370.44	280.03	234.29
NO. OF MODES USED	10	10	10	10	10

TABLE (2.4) — continued

TEST EXAMPLES	(21)	(22)	(23)	(24)	(25)
IMPACTOR MASS (Kg)	1.0000	0.7500	0.5000	0.3750	0.3000
$\frac{m}{M}$	1.0579	0.7934	0.5290	0.3967	0.3174
DYNAMIC STRESS (MN/m ²)	344.10	314.66	247.99	225.39	220.44
QUASI-STATIC STRESS (MN/m ²)	191.29	165.67	135.27	117.15	104.78
NO. OF MODES USED	15	15	15	15	15

TEST EXAMPLES	(26)	(27)	(28)	(29)	(30)
IMPACTOR MASS (Kg)	0.2500	0.2143	0.1875	0.1667	0.1500
$\frac{m}{M}$	0.2645	0.2267	0.1984	0.1763	0.1587
DYNAMIC STRESS (MN/m ²)	207.15	198.99	196.62	177.59	165.96
QUASI-STATIC STRESS (MN/m ²)	95.65	88.55	82.83	78.10	74.09
NO. OF MODES USED	20	20	20	20	20

TABLE (2.5) — Numerical tests for investigating the effect of varying beam cross-sectional dimensions

Note : for all the following cases

$$\begin{aligned} E &= 72.0 \text{ GN/m}^2 \\ G &= 27.0 \text{ GN/m}^2 \\ \rho &= 2720.0 \text{ Kg/m}^3 \\ \ell &= 1.0 \text{ m} \\ m &= 1.5 \text{ Kg} \\ v &= 4.43 \text{ m/s} \end{aligned}$$

TEST EXAMPLES	(31)	(32)	(33)
BEAM WIDTH (m)	0.03840	0.0384	0.0192
BEAM DEPTH (m)	0.00905	0.0181	0.0362
$\frac{m}{M}$	1.5869	0.7934	0.7934
DYNAMIC STRESS (MN/m ²)	462.59	314.69	314.69
QUASI-STATIC STRESS (MN/m ²)	234.29	165.67	165.67
NO. OF MODES USED	10	10	10

TABLE (2.6) — Numerical tests for investigating the effect of varying beam material properties

Note : for all the following cases

$$\begin{aligned} G &= 27.0 \text{ GN/m}^2 \\ b &= 0.0192 \text{ m} \\ d &= 0.0181 \text{ m} \\ \ell &= 1.0 \text{ m} \\ m &= 1.5 \text{ Kg} \\ v &= 4.43 \text{ m/s} \end{aligned}$$

TEST EXAMPLES	(34)	(35)
ELASTIC MODULUS (GN/m ²)	72.0	144.0
DENSITY (Kg/m ³)	5440.0	2720.0
$\frac{m}{M}$	0.7934	1.5869
DYNAMIC STRESS (MN/m ²)	445.00	654.19
QUASI-STATIC STRESS (MN/m ²)	234.29	331.33
NO. OF MODES USED	15	10

TABLE (2.7) — Numerical tests for investigating the effect of varying impact velocity

Note : for all the following cases

$$\begin{aligned}
 E &= 72.0 \text{ GN/m}^2 \\
 G &= 27.0 \text{ GN/m}^2 \\
 \rho &= 2720.0 \text{ Kg/m}^3 \\
 b &= 0.0192 \text{ m} \\
 d &= 0.0181 \text{ m} \\
 \ell &= 1.0 \text{ m} \\
 m &= 1.5 \text{ Kg}
 \end{aligned}$$

TEST EXAMPLES	(36)	(37)	(38)	(39)
IMPACT VELOCITY (m/s)	1.5000	3.0000	6.0000	8.0000
$\frac{m}{M}$	1.5869	1.5869	1.5869	1.5869
DYNAMIC STRESS (MN/m ²)	156.63	313.26	626.53	835.37
QUASI-STATIC STRESS (MN/m ²)	79.33	158.66	317.32	423.09
NO. OF MODES USED	10	10	10	10

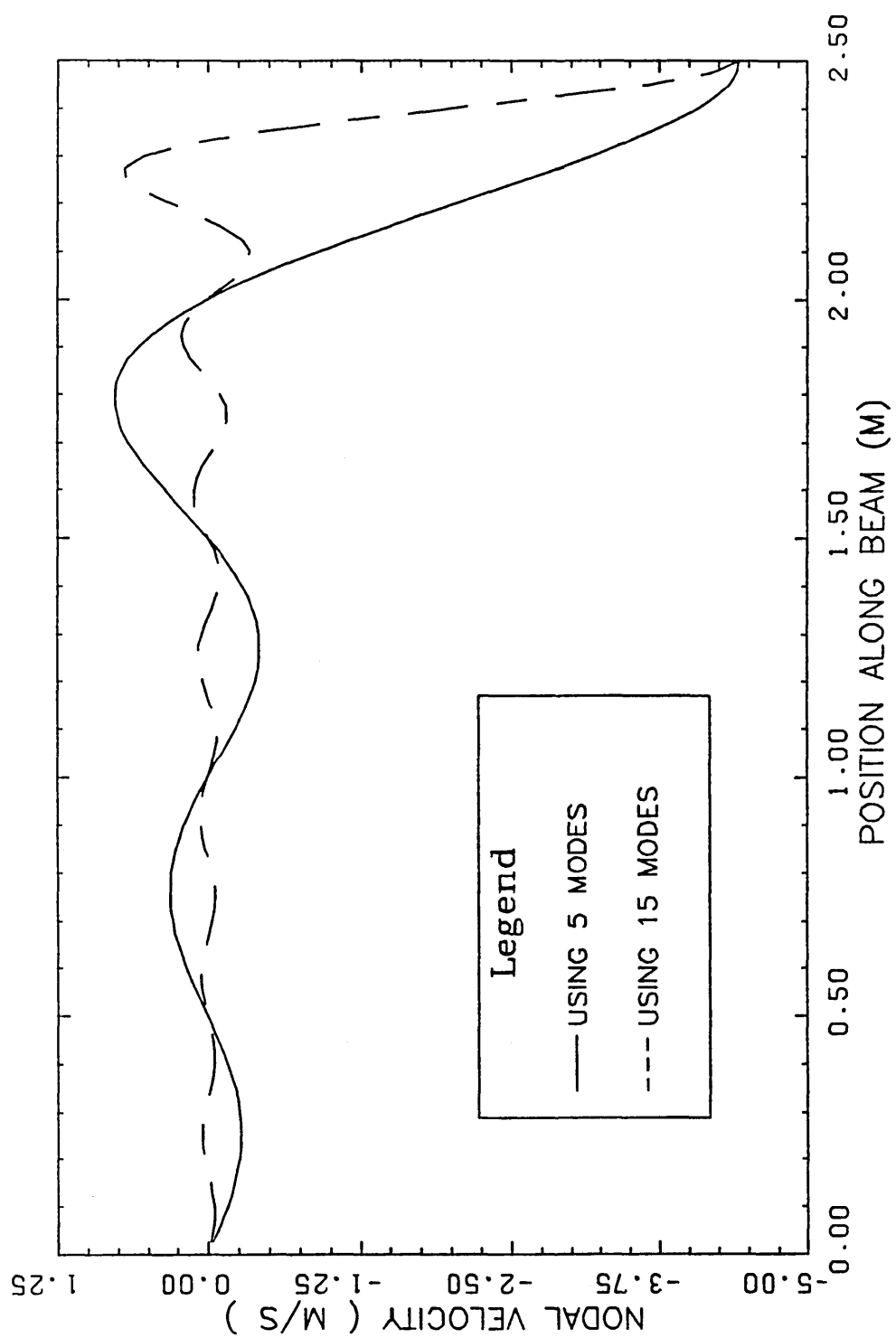


FIG. (2.1) - The initial velocity distribution along a beam that results from the numerical approximation to a step-velocity excitation.

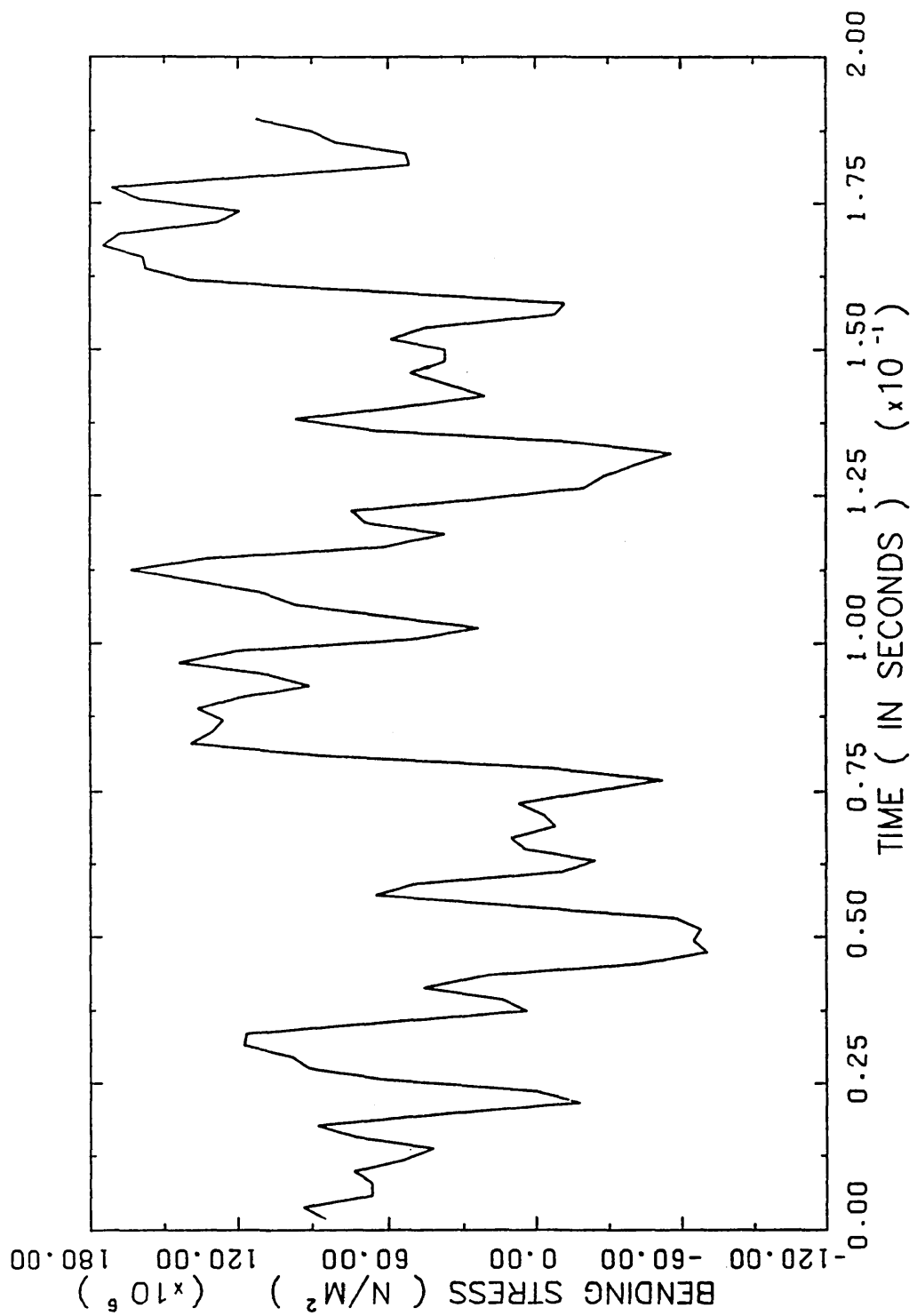


FIG. (2.2) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (10), using the first 5 modes.

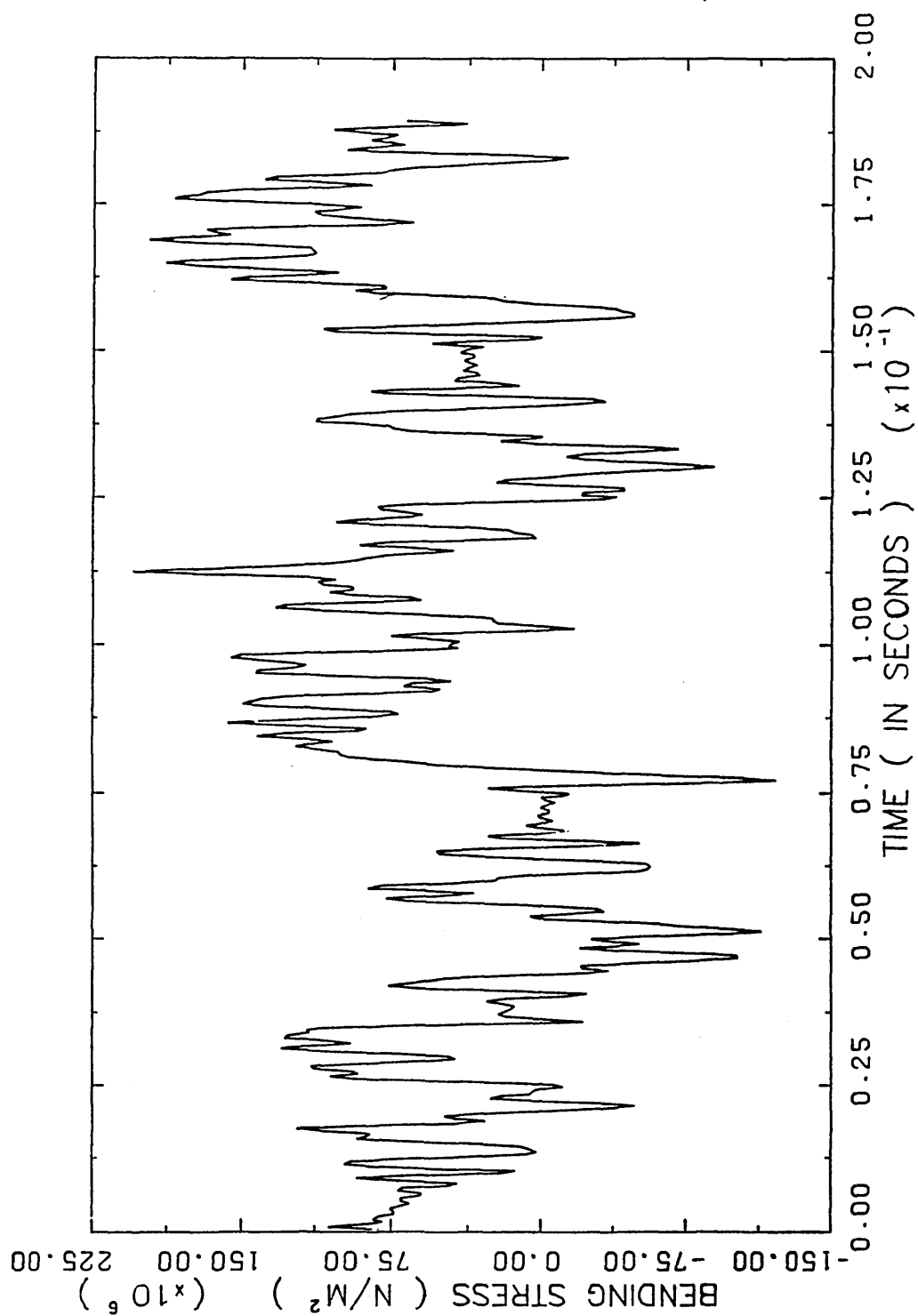


FIG. (2.3) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (10), using the first 10 modes.

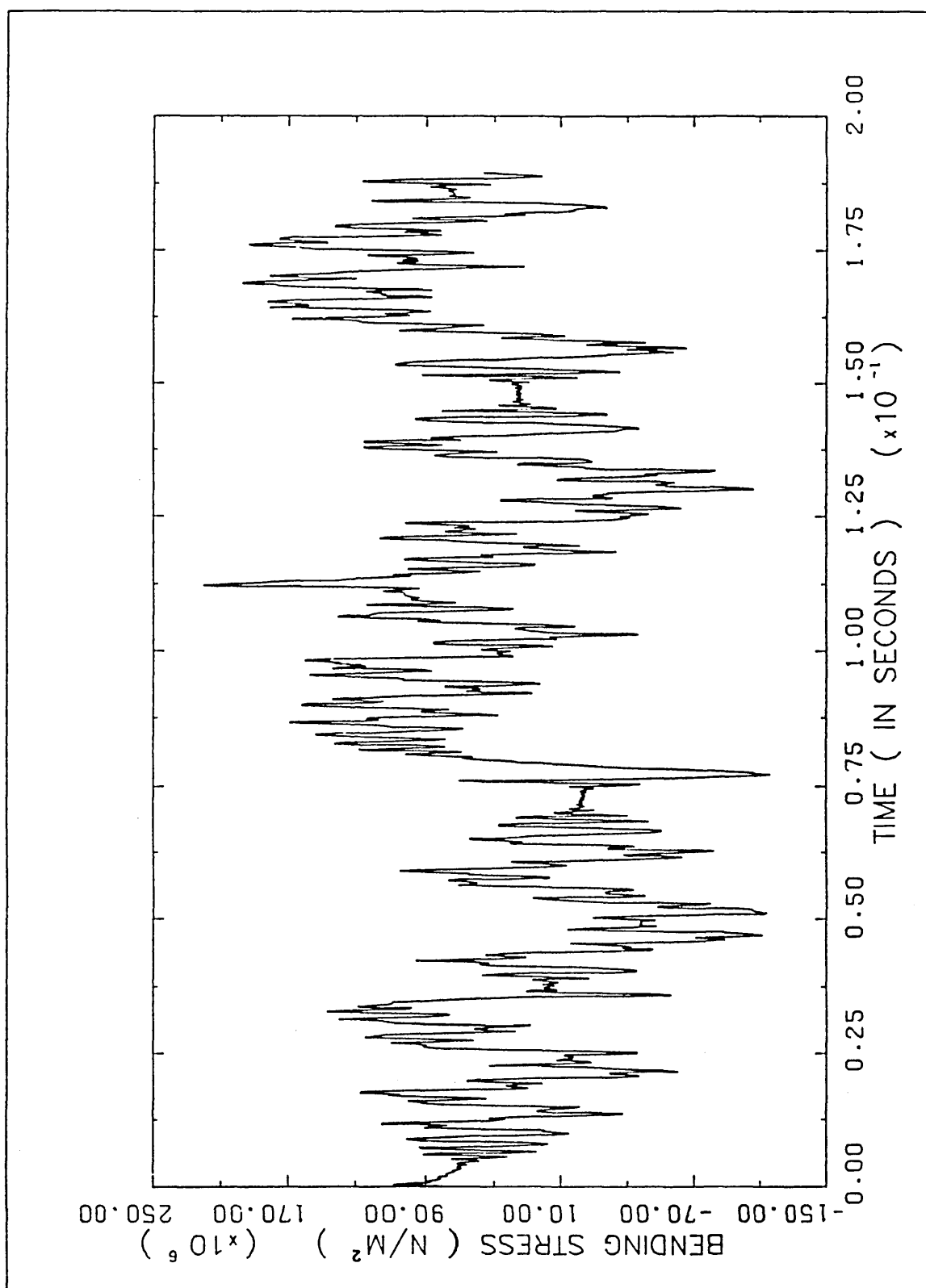


FIG. (2.4) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (10), using the first 15 modes.

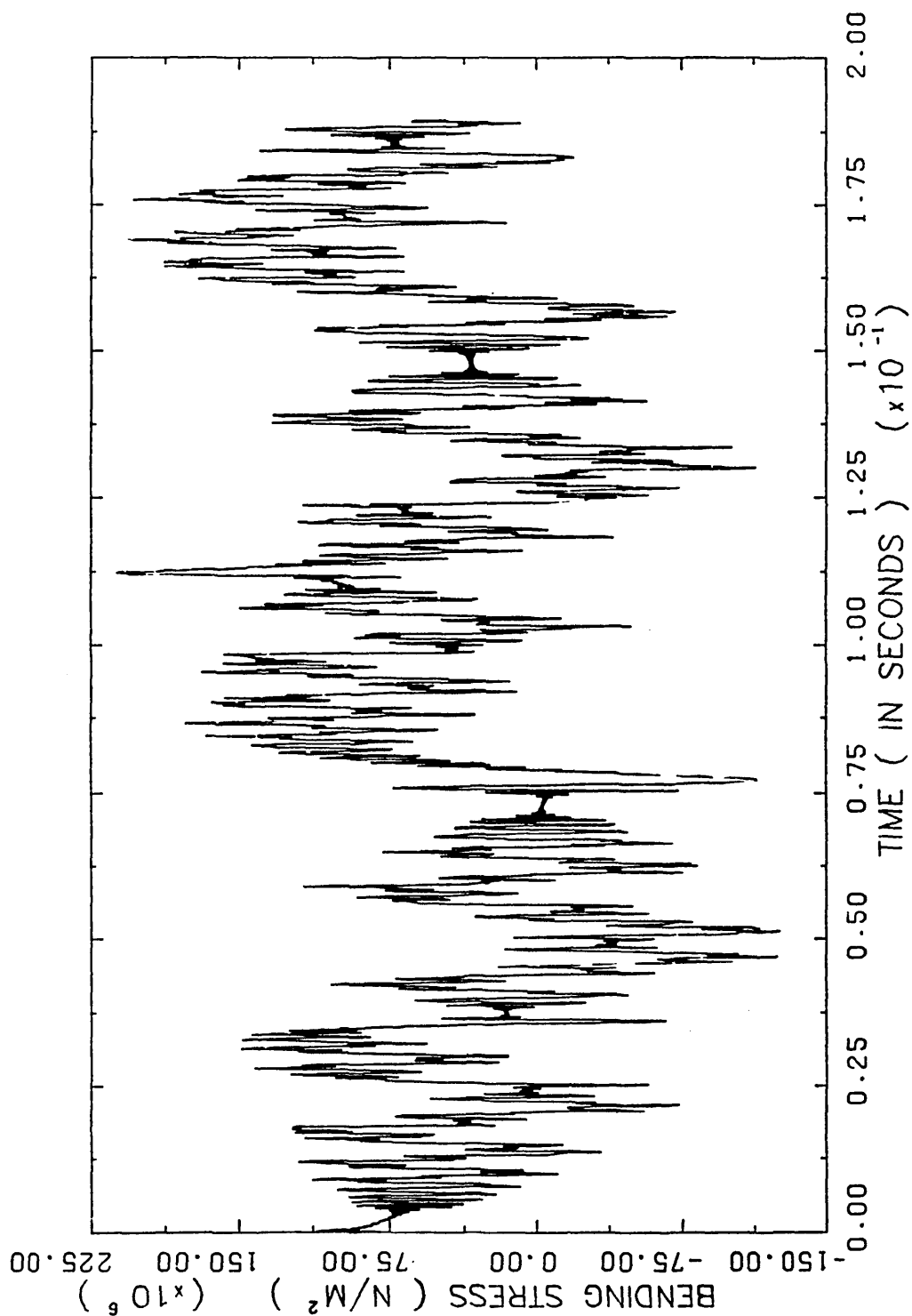


FIG. (2.5) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (10), using the first 20 modes.

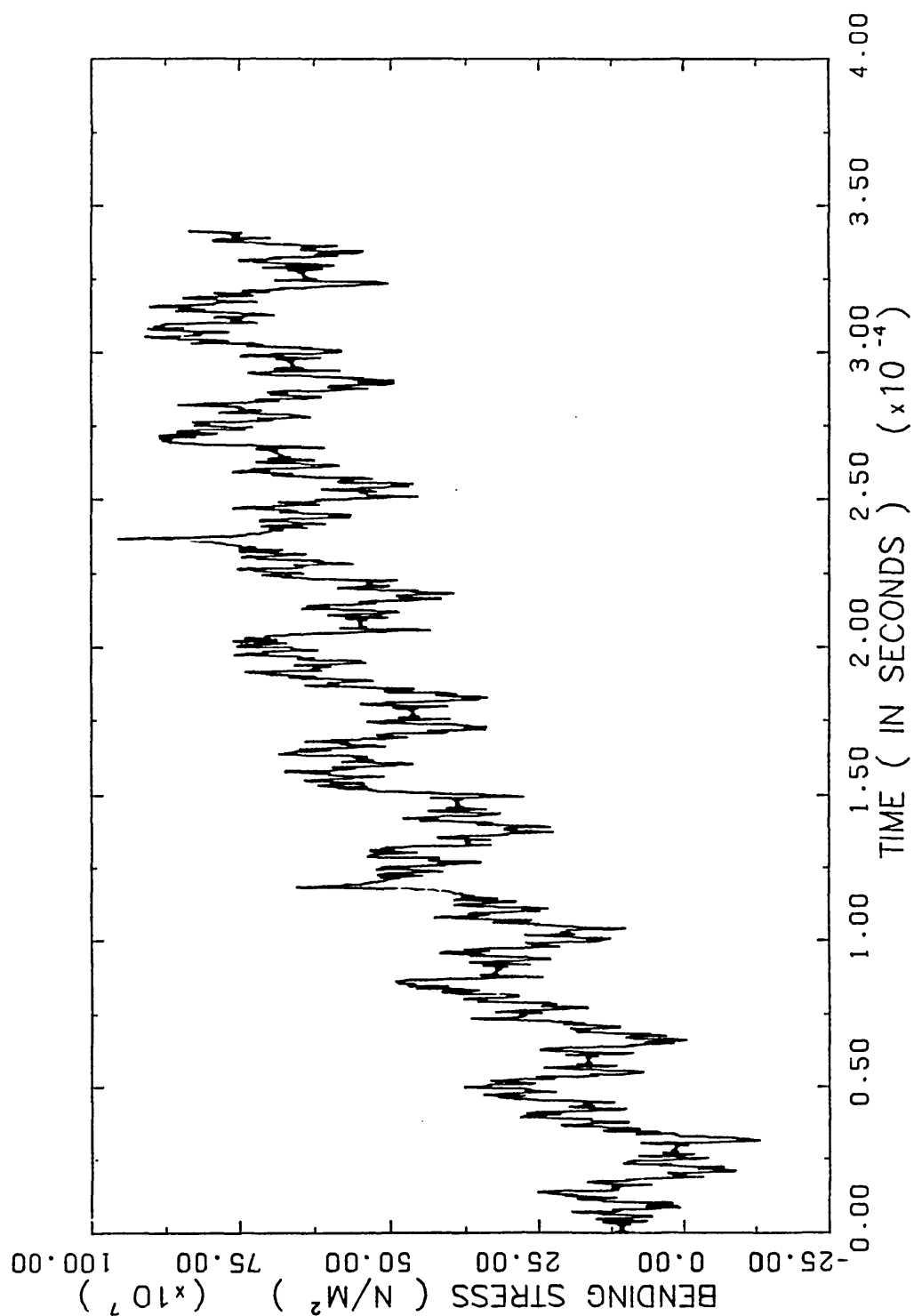


FIG. (2.6) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (1).

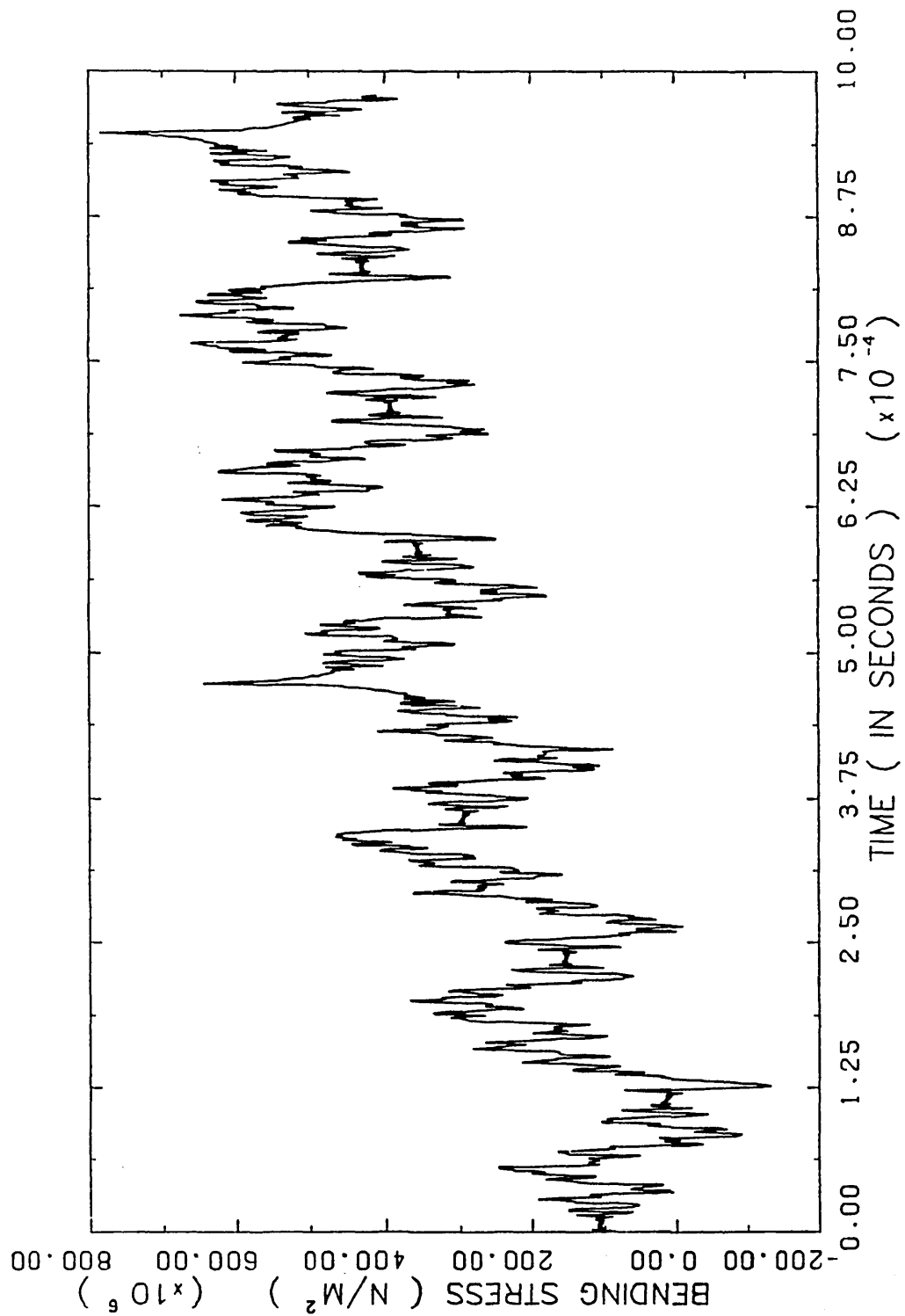


FIG. (2.7) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (2).

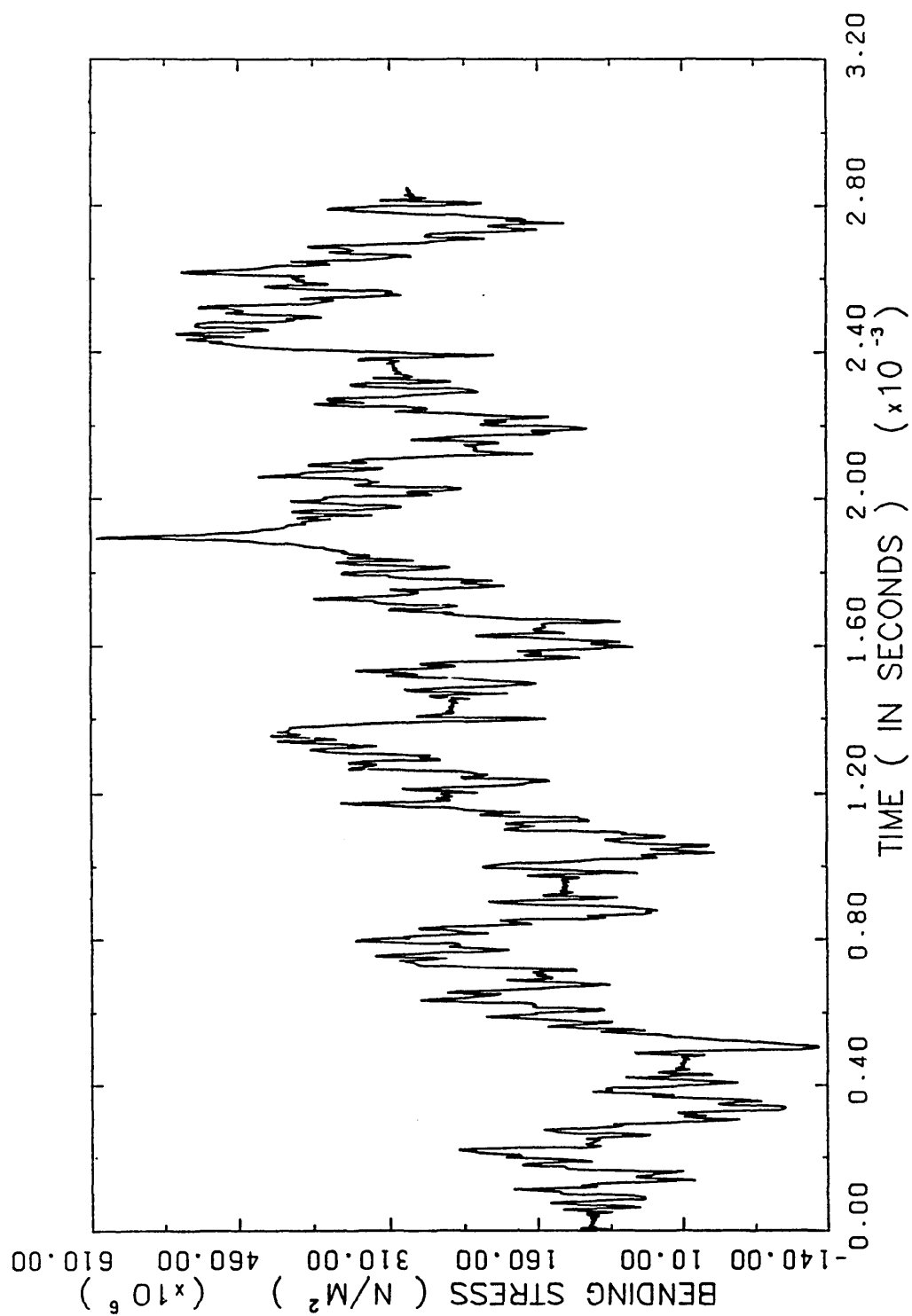


FIG. (2.8) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (3).

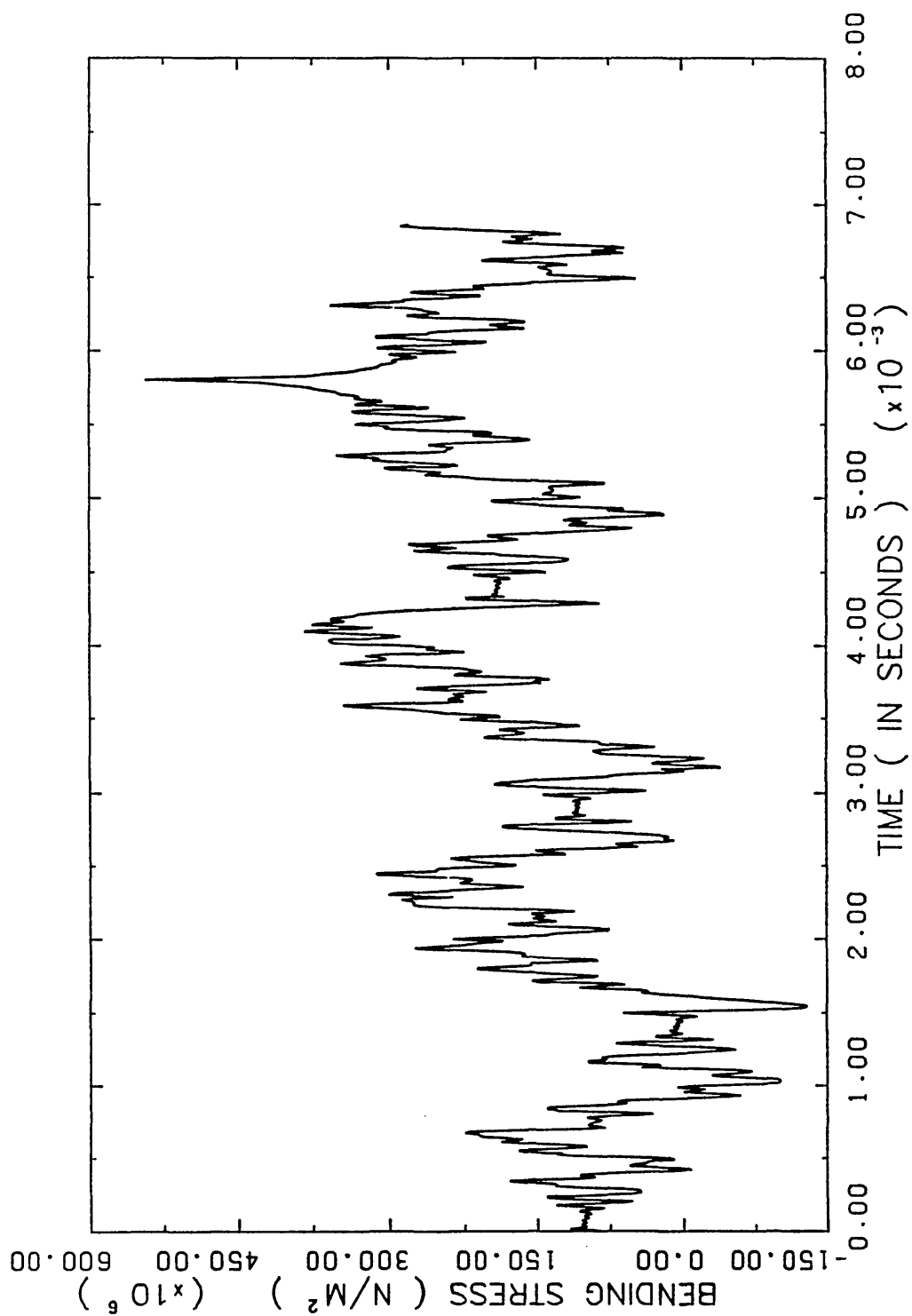


FIG. (2.9) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (4).

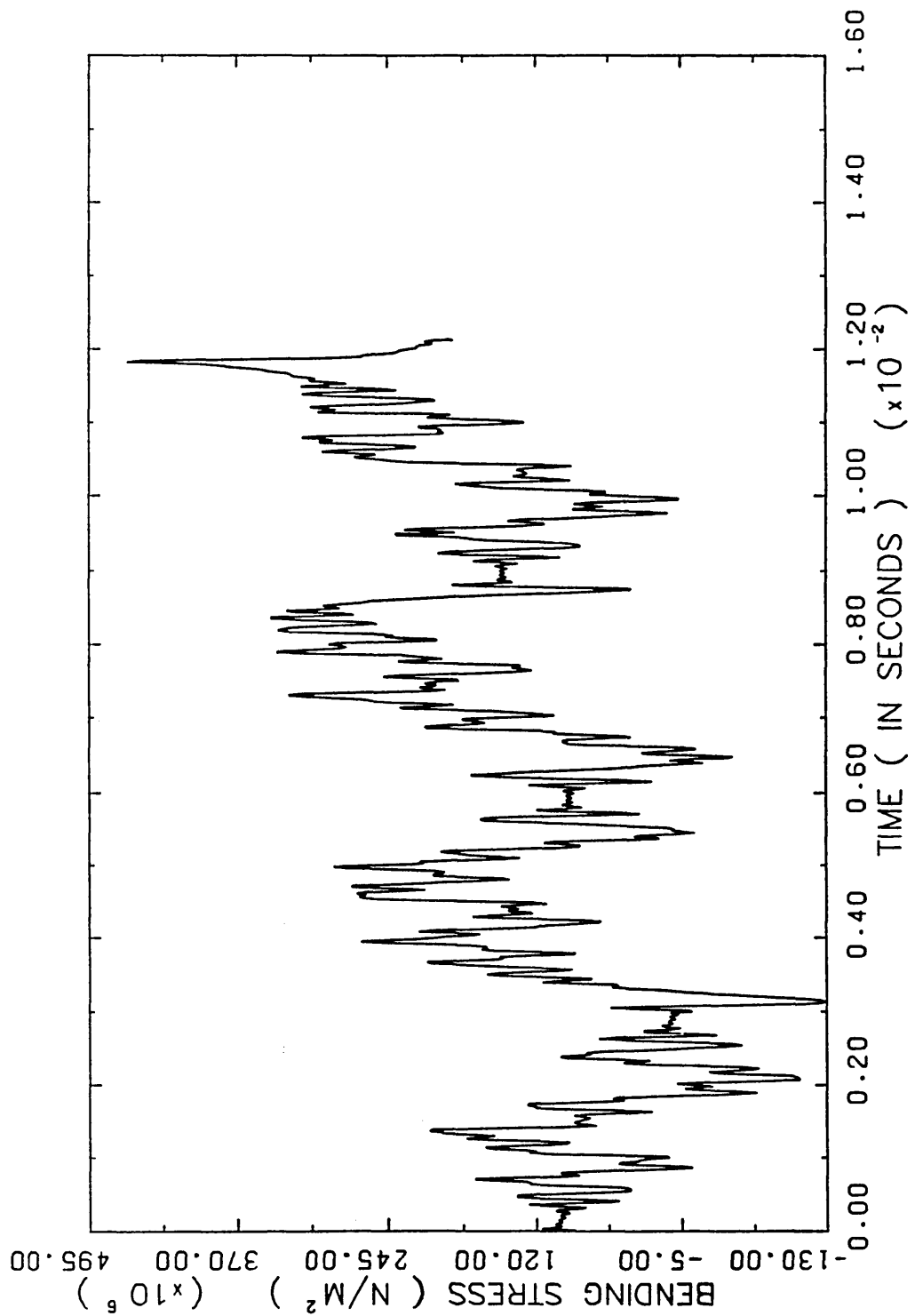


FIG. (2.10) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (5).

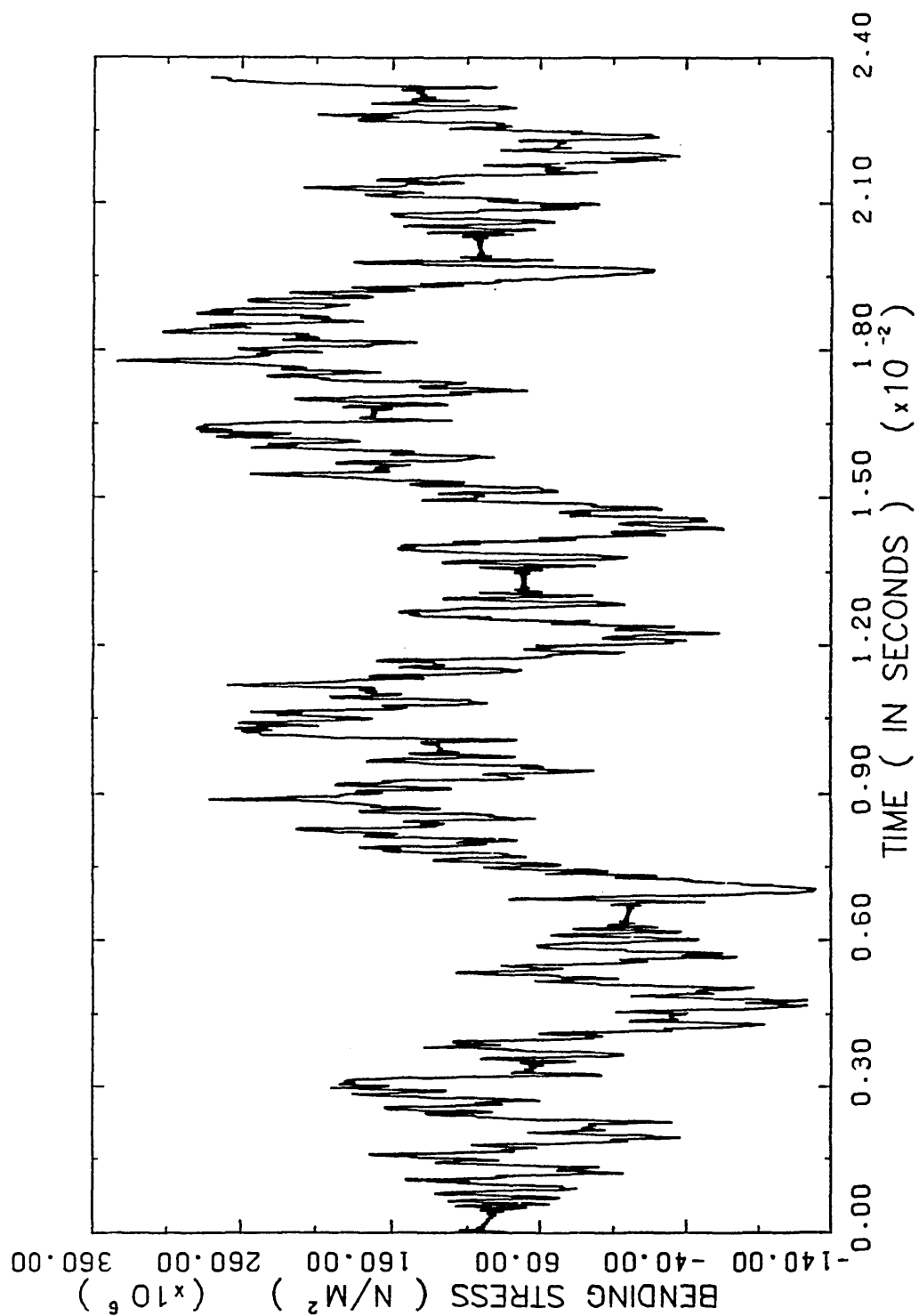


FIG. (2.11) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (6).

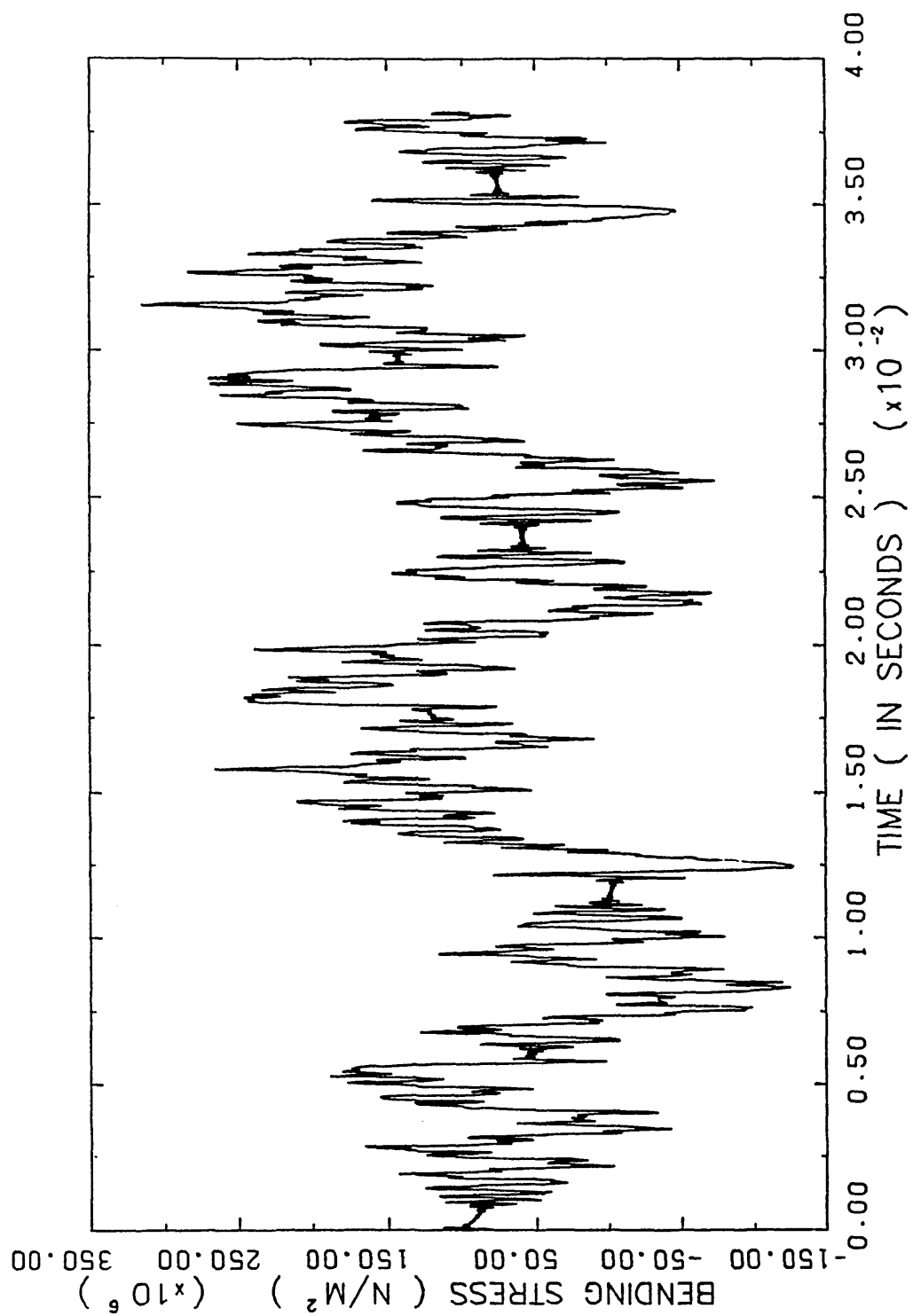
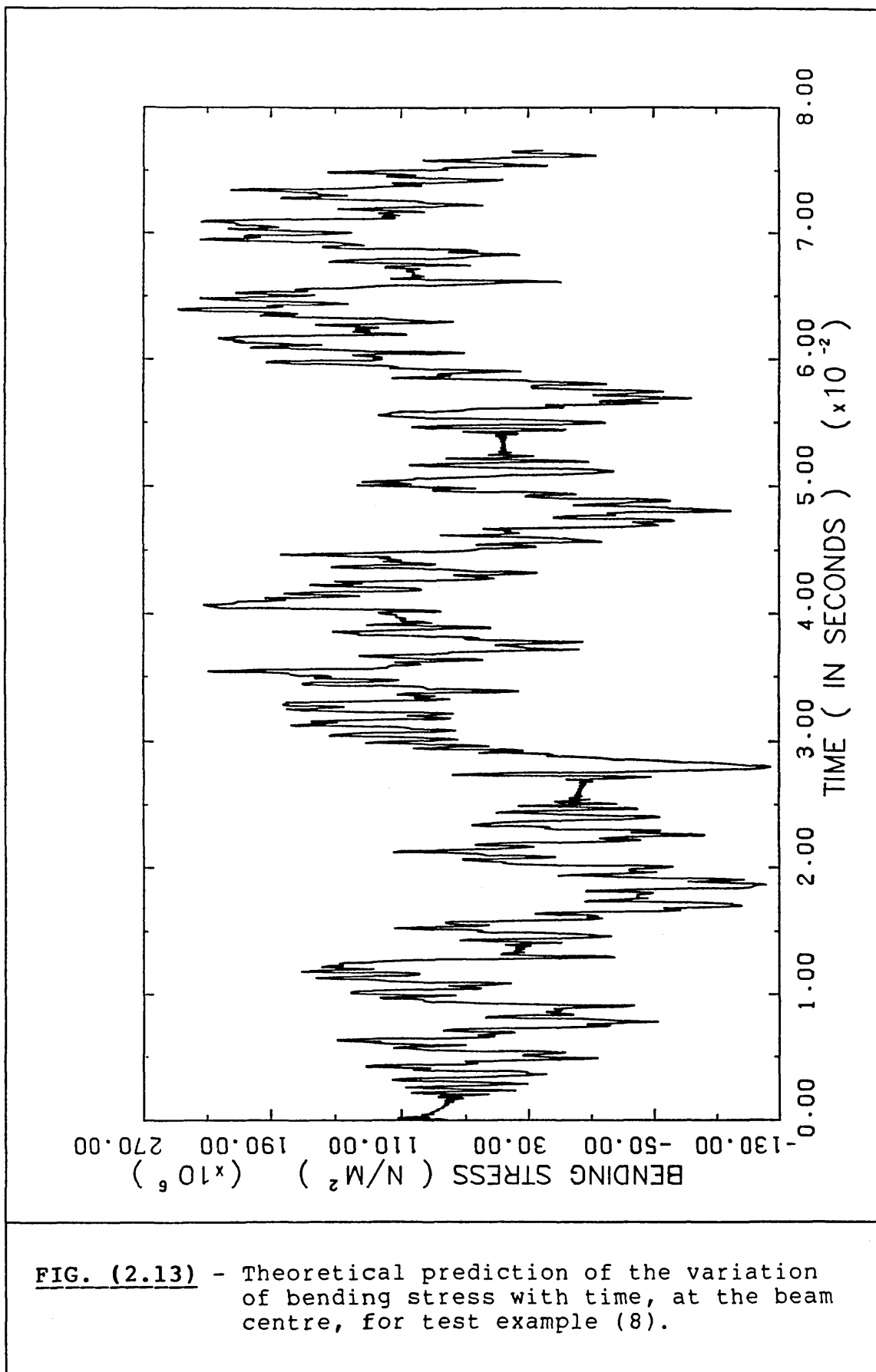


FIG. (2.12) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (7).



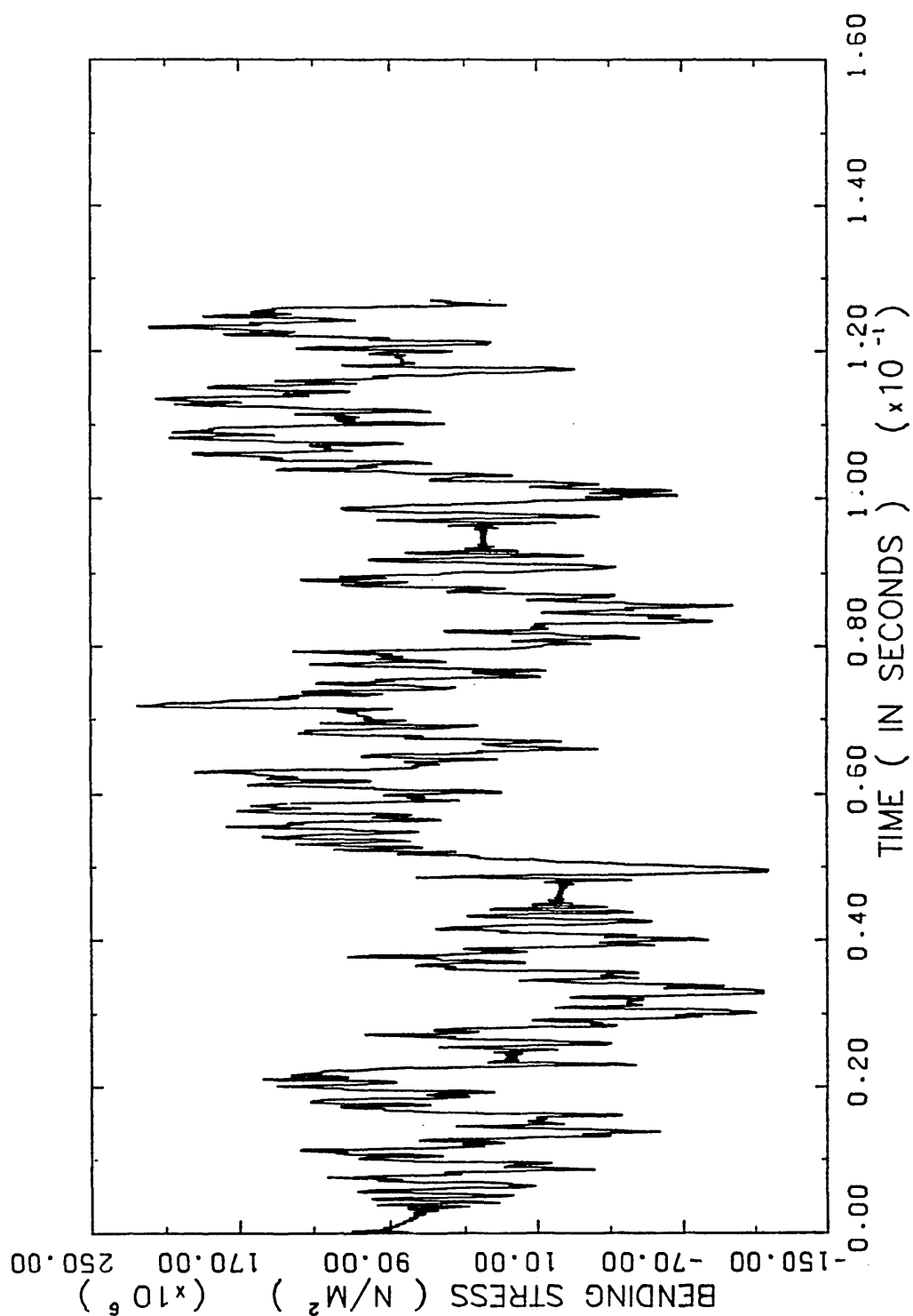


FIG. (2.14) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (9).

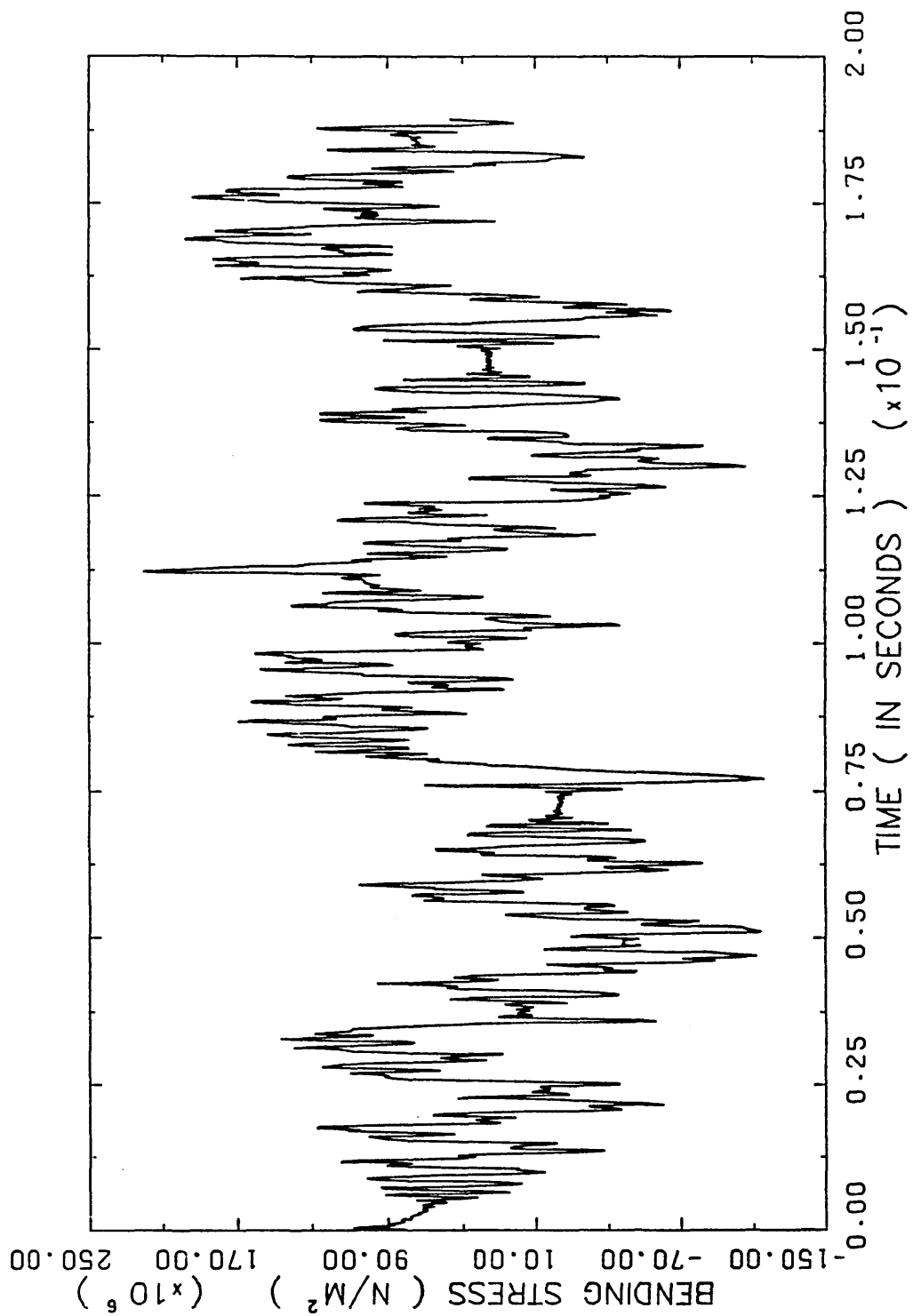


FIG. (2.15) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (10).

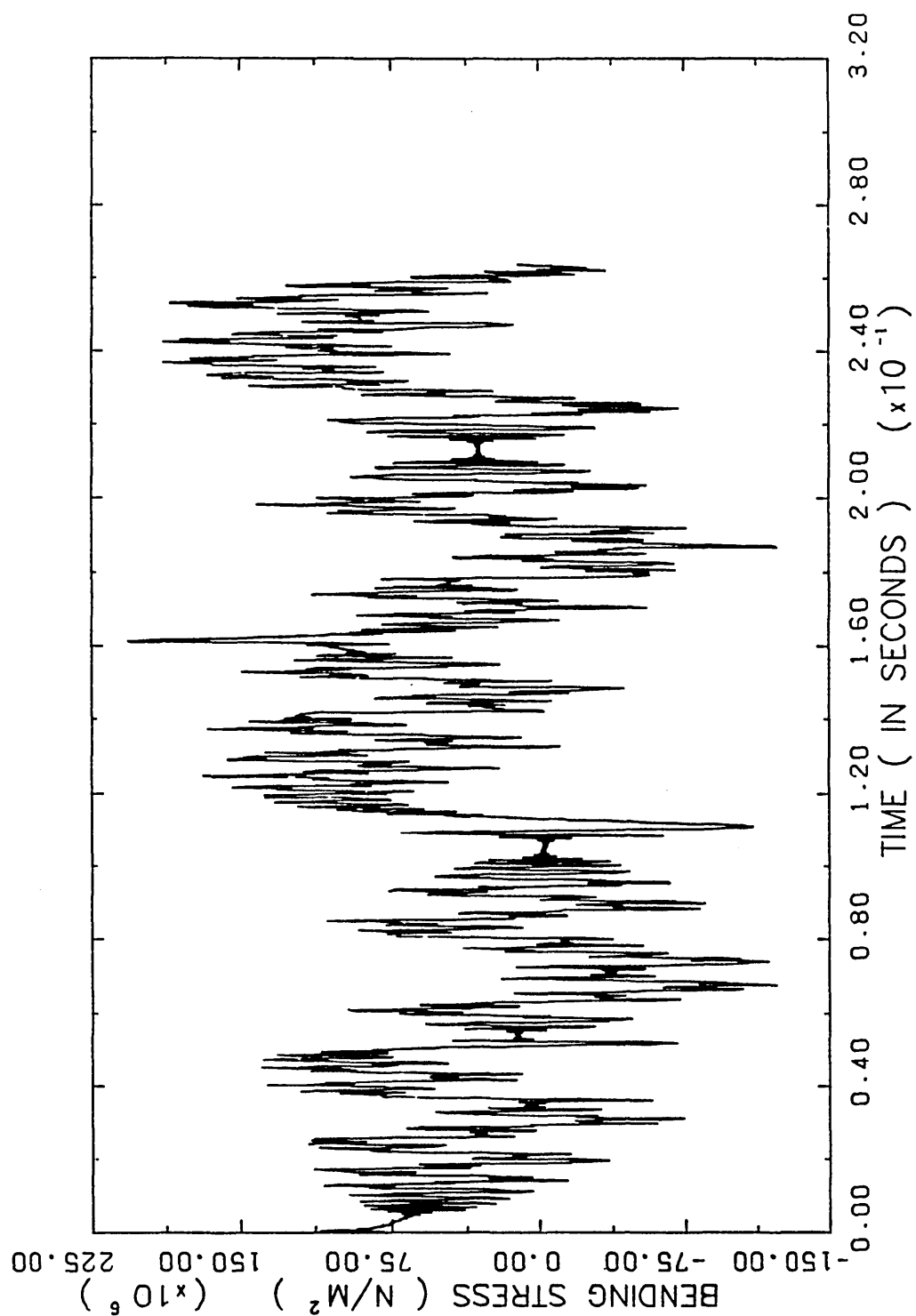


FIG. (2.16) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (11).

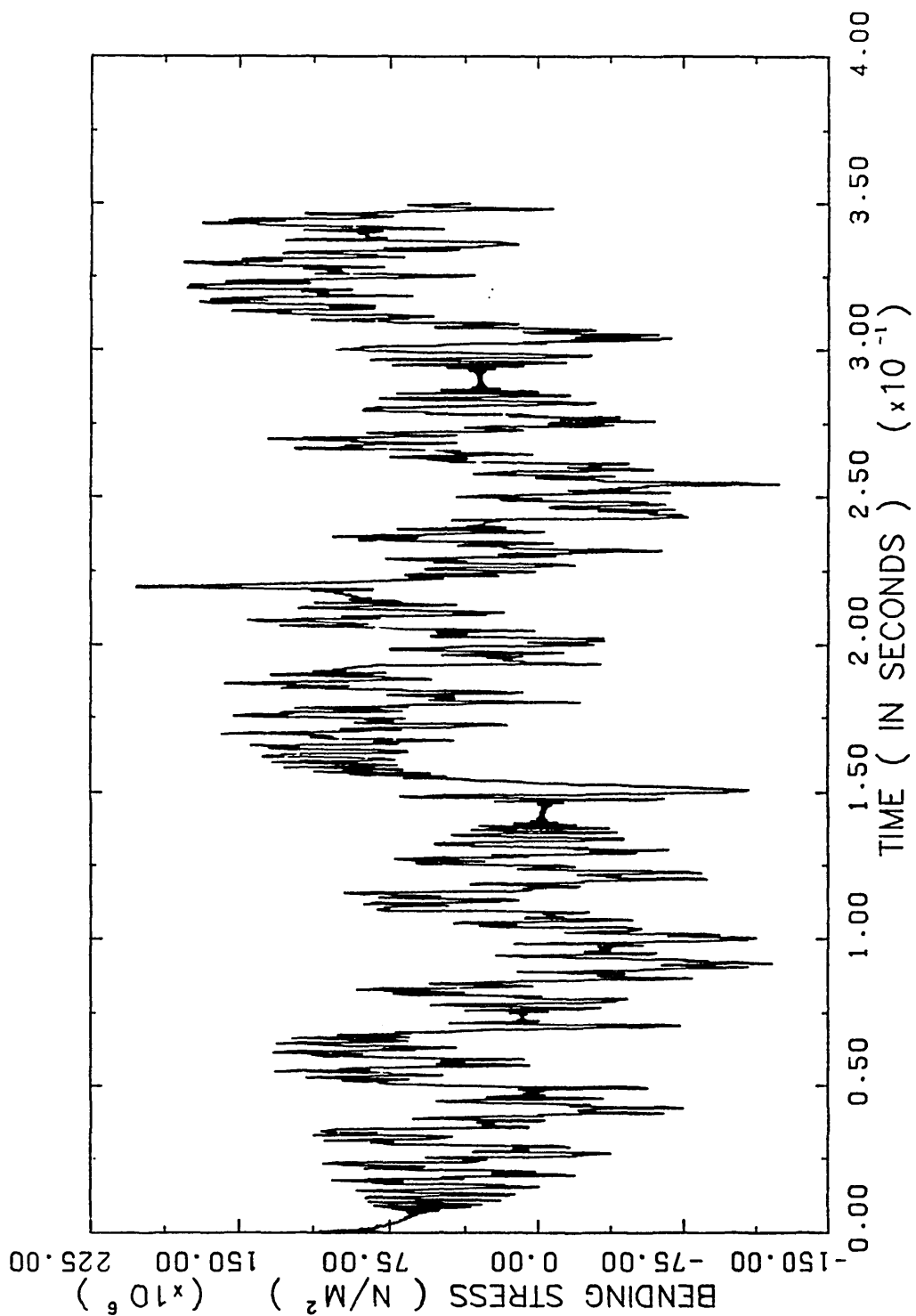


FIG. (2.17) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (12).

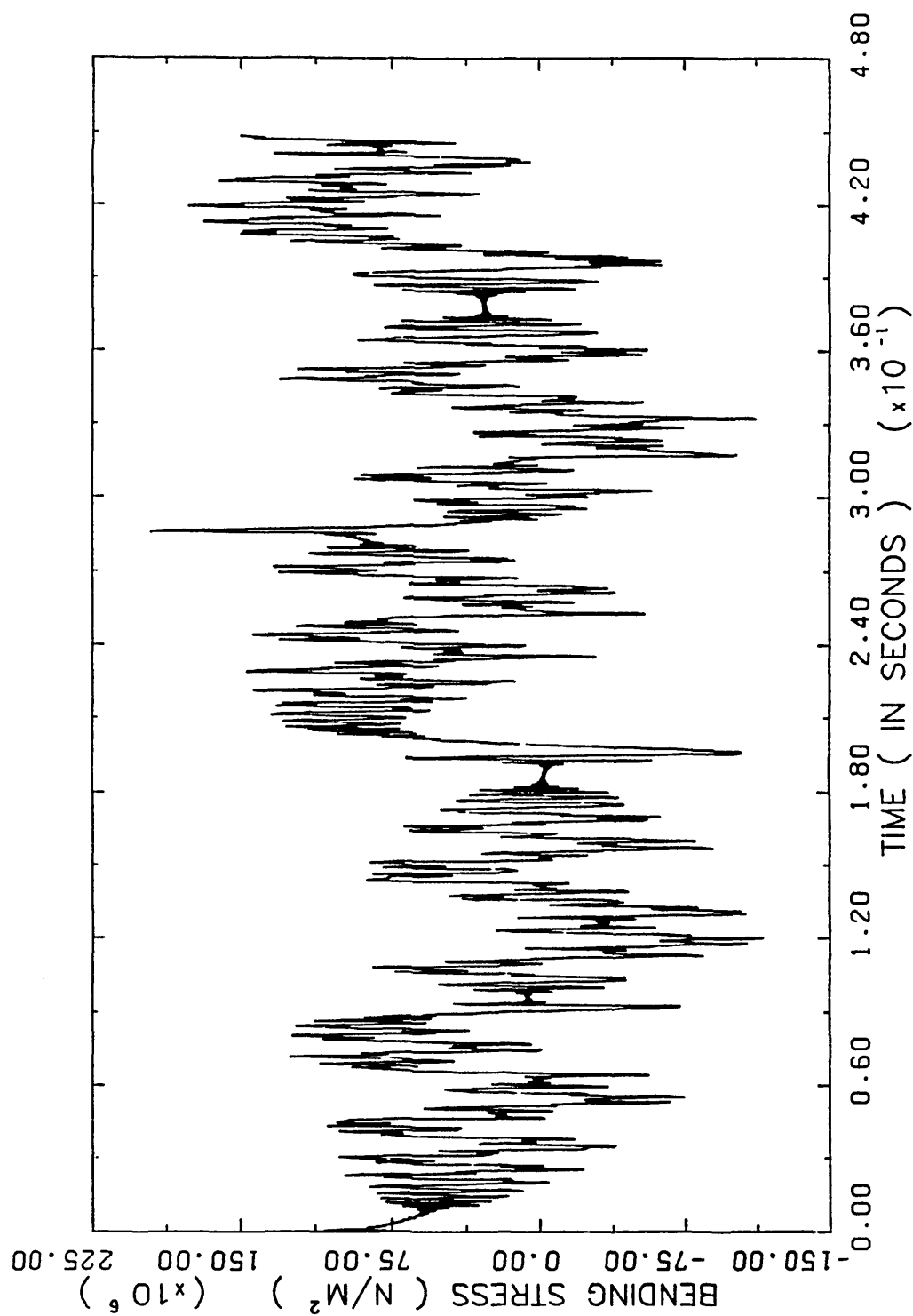


FIG. (2.18) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (13).

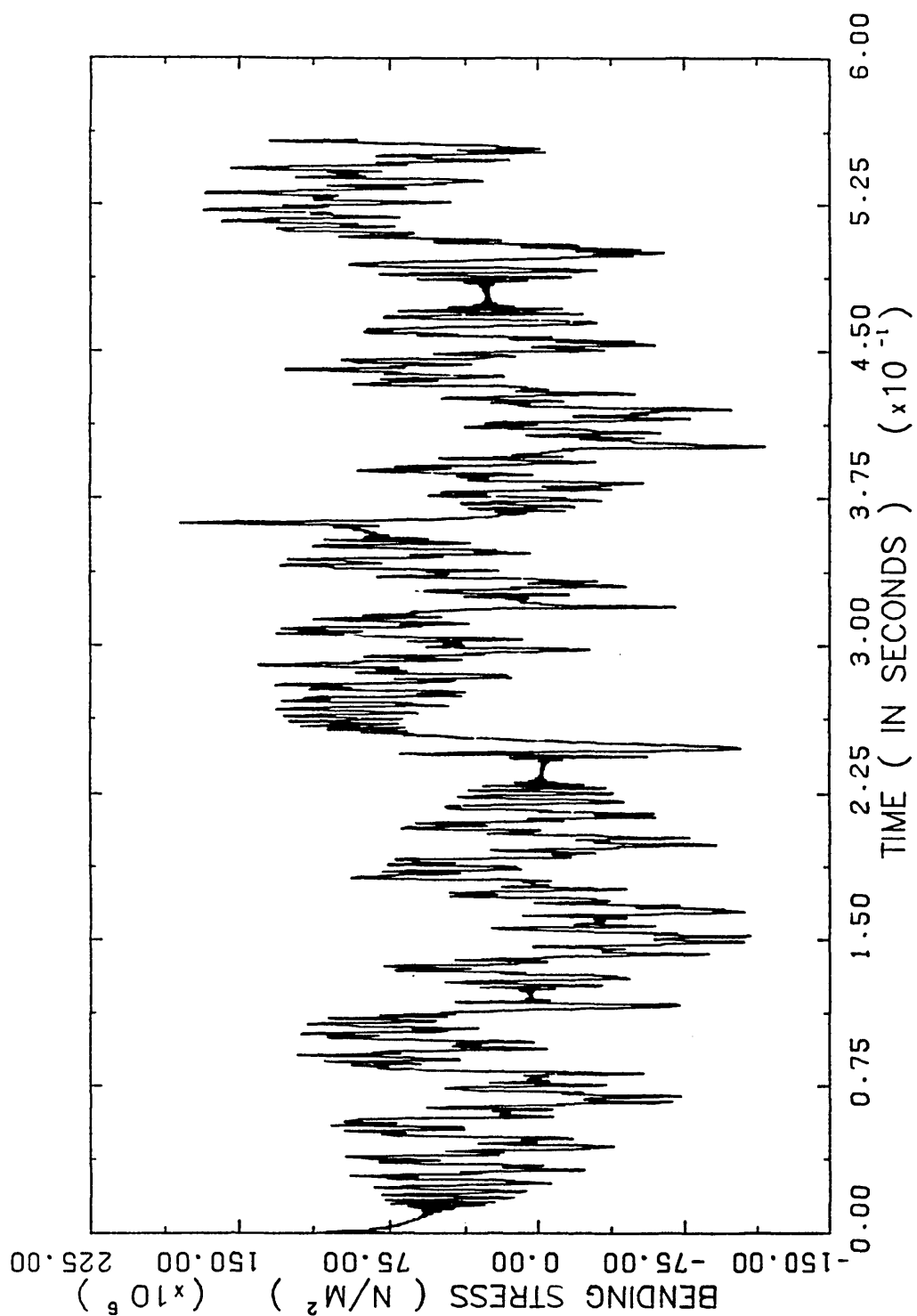


FIG. (2.19) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (14).

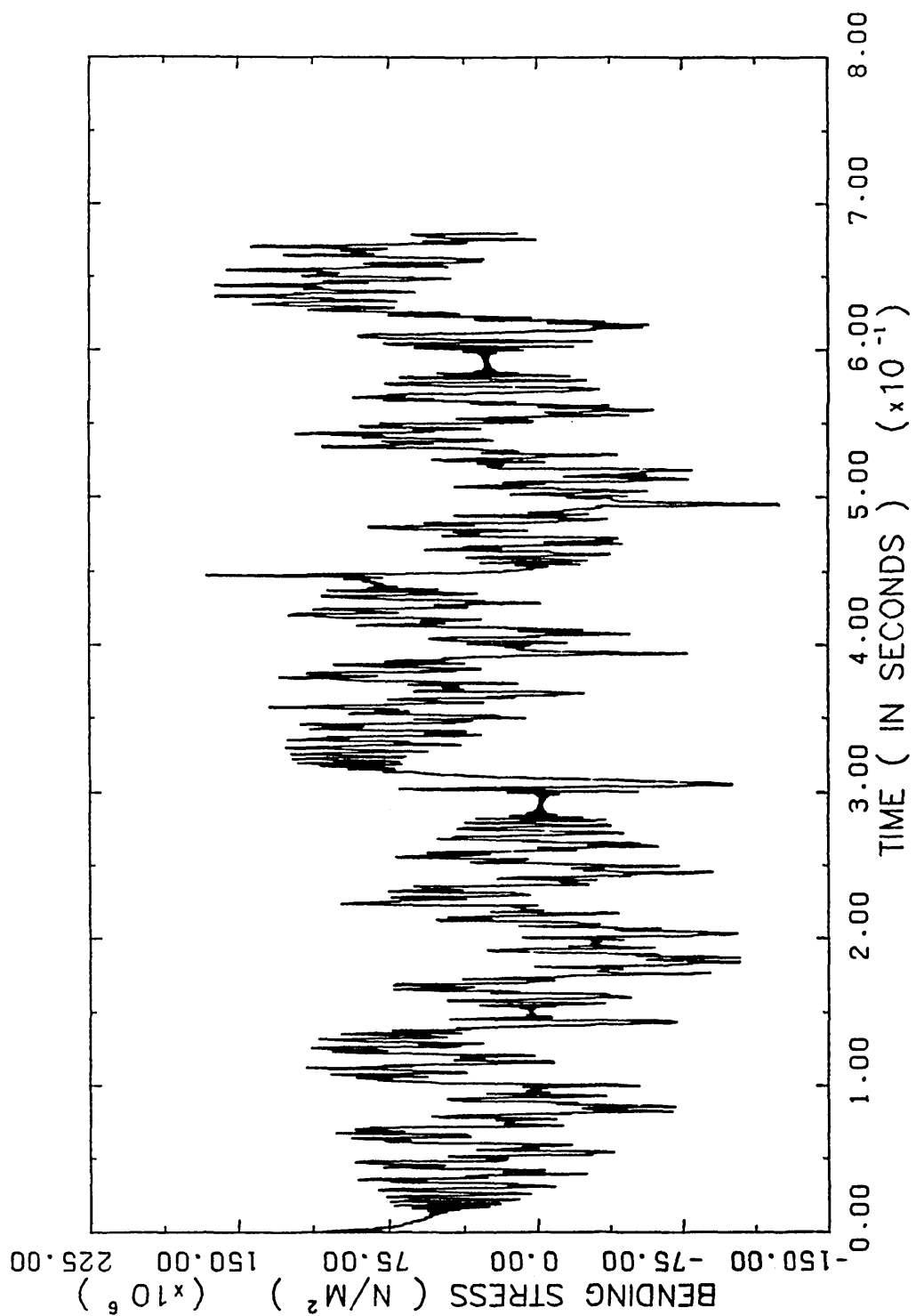


FIG. (2.20) - Theoretical prediction of the variation of bending stress with time, at the beam centre, for test example (15).

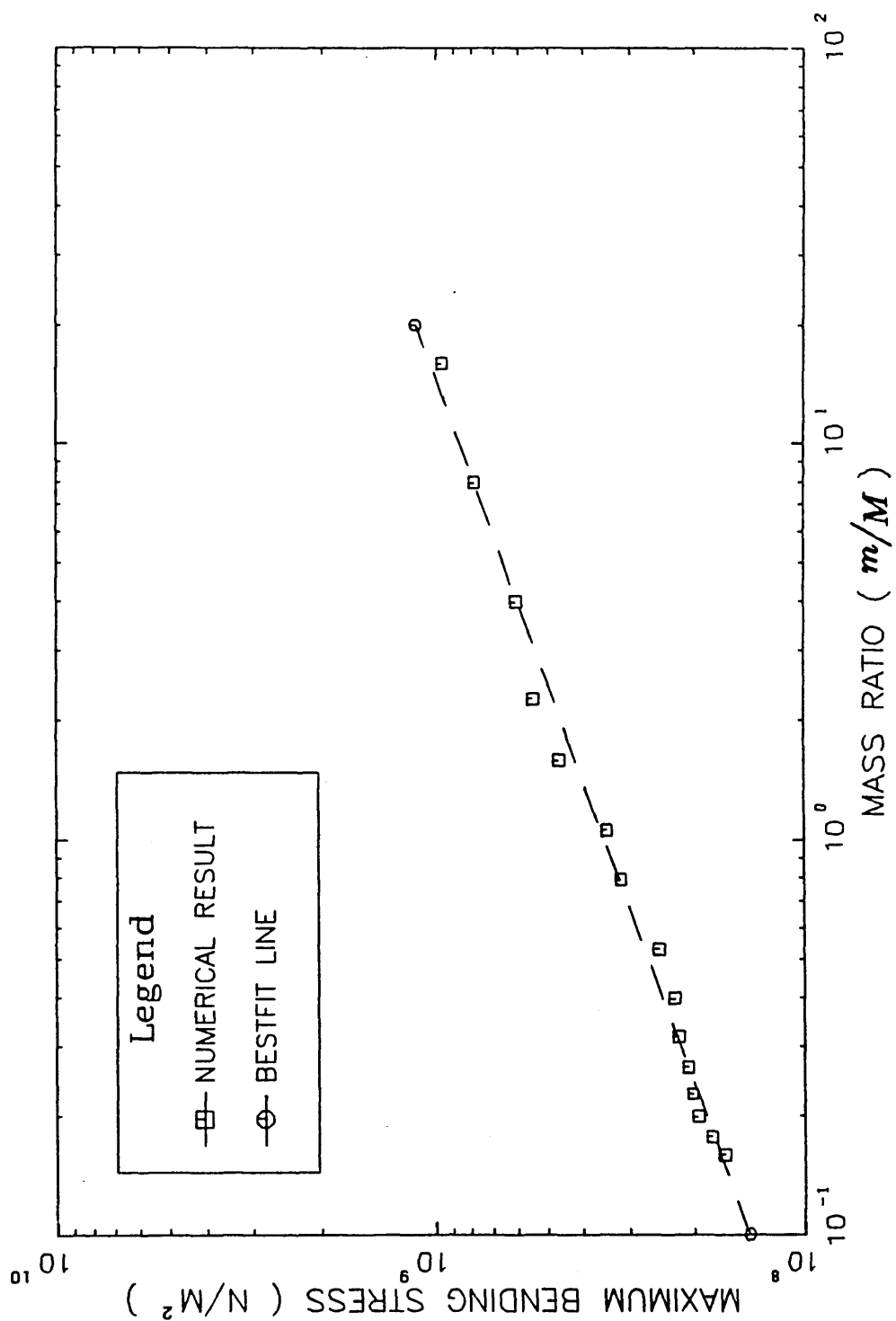


FIG. (2.21) - The variation of the impact-induced peak bending stress with the mass ratio (m/M), based on the results of the numerical investigation.

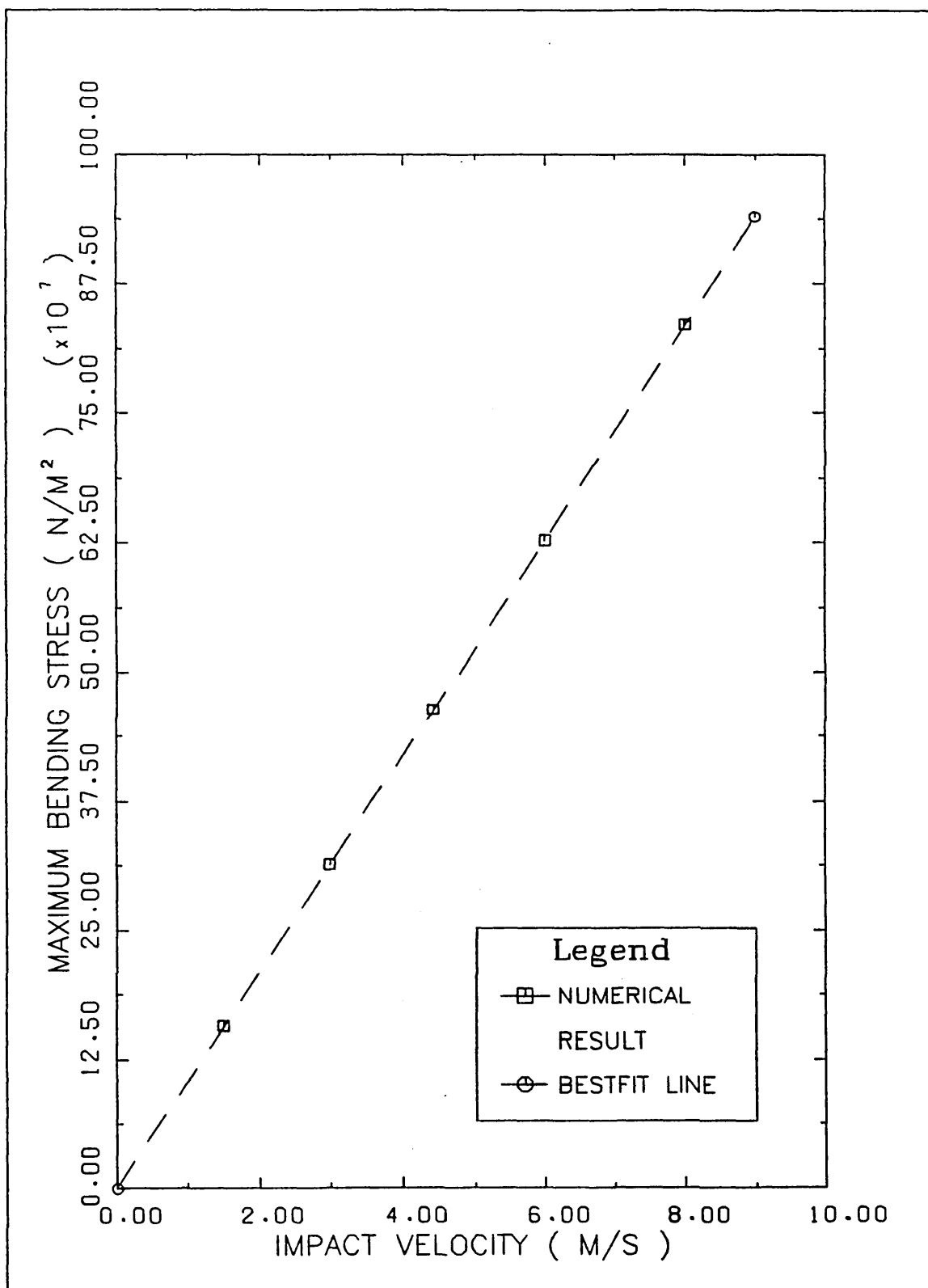
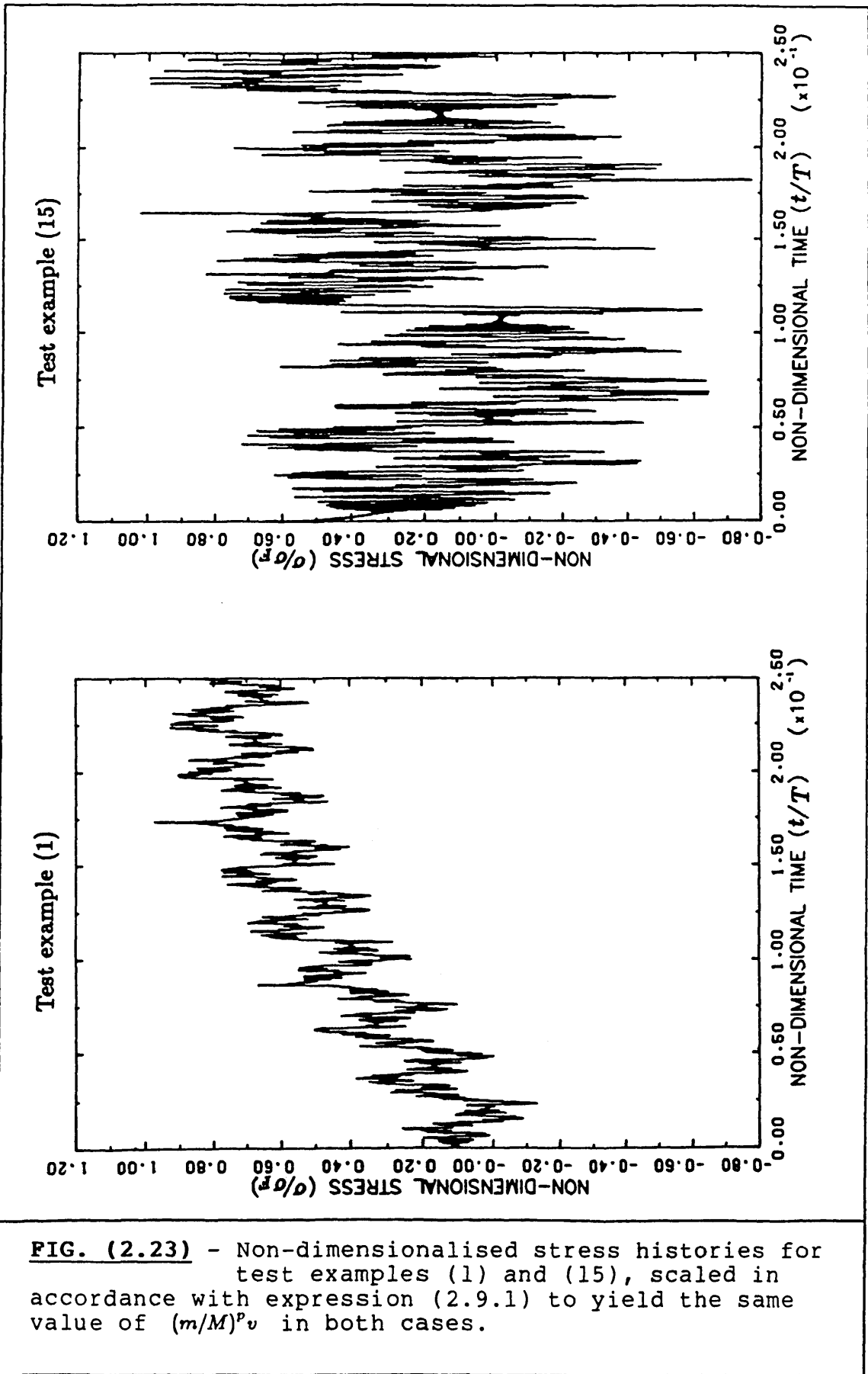


FIG. (2.22) - The variation of the impact-induced peak bending stress with impactor velocity, based on the results of the numerical investigation.



TABLES AND FIGURES

— for Chapter (3) —

TABLE (3.1) — Finite element mesh configurations used in the assessment of element performance

Type of beam element	No. of elements	Element length (m)	No. of freedoms per element	Total no. of freedoms
<i>HERMITIAN</i>	18	0.04	4	38
<i>TIMOSHENKO</i>	18	0.04	4	38
<i>LINEAR MINDLIN</i>	18	0.04	4	38
<i>QUADRATIC MINDLIN</i>	9	0.08	6	38
<i>CUBIC MINDLIN</i>	6	0.12	8	38

TABLE (3.2) — Finite element mesh configurations used in the analysis of a simply-supported beam with an attached central mass subject to an initial step-velocity excitation

Type of beam element	No. of elements	Element length (m)	No. of freedoms per element	Total no. of freedoms
<i>HERMITIAN</i>	20	0.025	4	42
<i>TIMOSHENKO</i>	20	0.025	4	42
<i>CUBIC MINDLIN</i>	8	0.0625	8	50

TABLE (3.3) — Percentage error in predicted frequency compared to the *Timoshenko* theory

MODE	TIMOSHENKO	LINEAR MINDLIN	
		Exact	Reduced
1	1.33×10^{-5}	60.056	0.0952
2	1.36×10^{-3}	60.869	0.8598
3	0.0104	62.496	2.4032
4	0.0392	64.939	4.7545
5	0.1045	68.200	7.9571
6	0.2264	72.274	12.068
7	0.4272	77.146	17.158
8	0.7293	82.776	23.305
9	1.1548	89.088	30.589
10	1.7237	95.939	39.074

TABLE (3.3) — continued

MODE	QUADRATIC <i>MINDLIN</i>		CUBIC <i>MINDLIN</i>	
	Exact	Reduced	Exact	Reduced
1	0.0705	1.87×10^{-4}	1.78×10^{-4}	-7.92×10^{-7}
2	0.6388	0.0154	0.0143	3.41×10^{-4}
3	1.7979	0.1154	0.1081	6.98×10^{-3}
4	3.5903	0.4259	0.4037	0.0492
5	6.0694	1.1094	1.0588	0.2036
6	9.2696	2.3478	2.1797	0.6100
7	13.105	4.3316	4.5971	1.4846
8	16.934	7.2407	7.4986	3.0826
9	16.775	10.982	11.725	5.7362
10	56.415	18.304	17.301	9.8208

TABLE (3.4) — Percentage error in predicted frequency compared to *Bernoulli-Euler* theory

MODE	<i>HERMITIAN</i>
1	-6.48×10^{-3}
2	-0.0581
3	0.1595
4	-0.3073
5	-0.4963
6	-0.7200
7	-0.9706
8	-1.2392
9	-1.5163
10	-1.7919

TABLE (3.5) — Percentage difference in frequency calculated from the *Timoshenko* theory compared to that based on the *Bernoulli-Euler* theory

MODE	% difference
1	0.02685
2	0.24120
3	0.66755
4	1.30130
5	2.13591
6	3.16319
7	4.37371
8	5.75719
9	7.30290
10	8.99994

TABLE (3.6) — Sampling points for stress smoothing based on different techniques for different beam elements

element type	least squares technique	<i>Barlow's</i> method
<i>HERMITIAN</i>	$-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}$
<i>TIMOSHENKO</i>	$-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}$
<i>LINEAR MINDLIN</i>	0	0
<i>QUADRATIC MINDLIN</i>	$-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}$
<i>CUBIC MINDLIN</i>	$-\sqrt{\frac{3}{5}}, 0, \sqrt{\frac{3}{5}}$	$-\frac{\sqrt{5}}{3}, 0, \frac{\sqrt{5}}{3}$

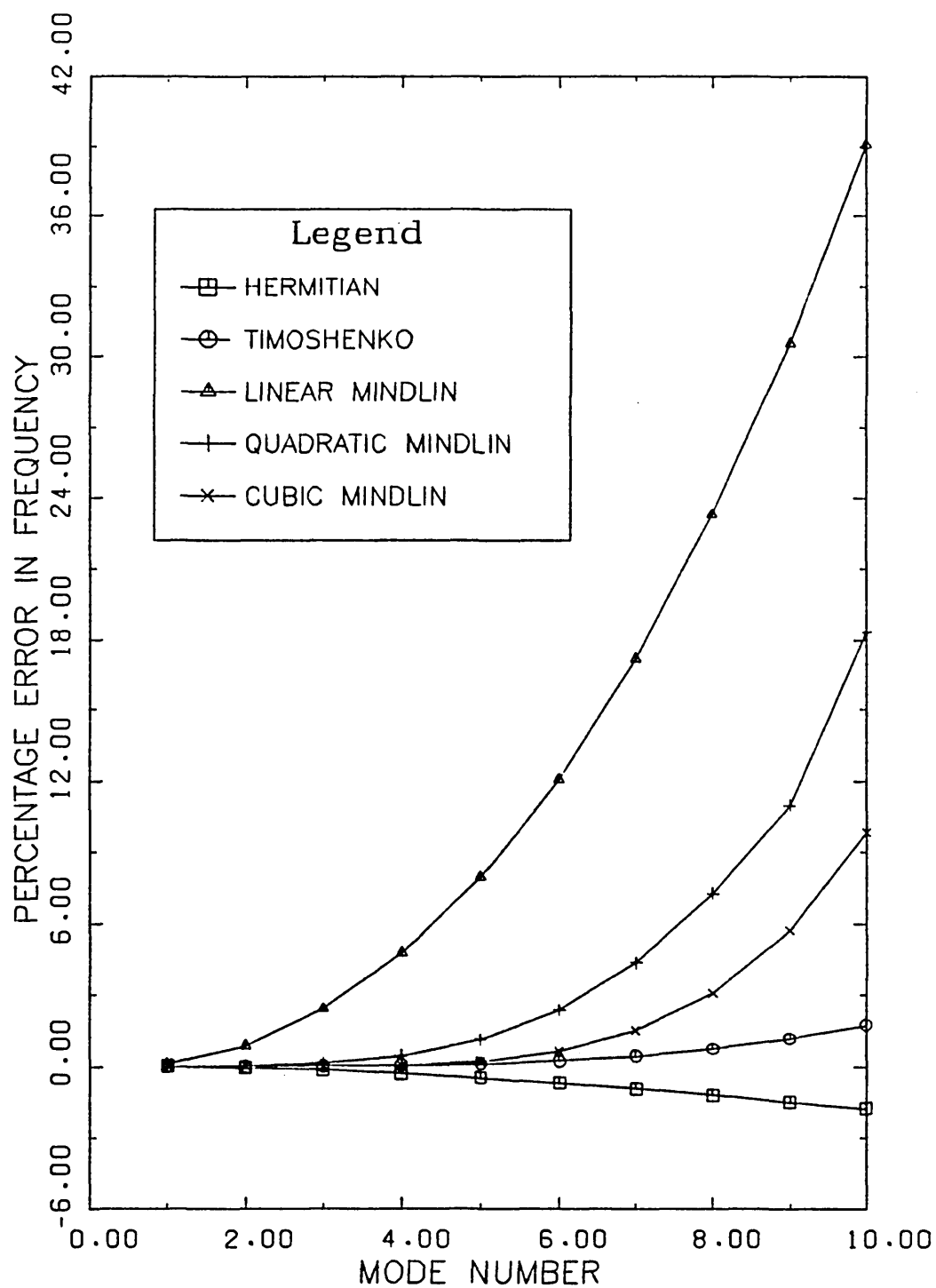


FIG. (3.1) - Percentage error in calculated frequency of a simply-supported beam modelled by different beam elements. The reduced stiffness matrix has been employed for the Mindlin elements.

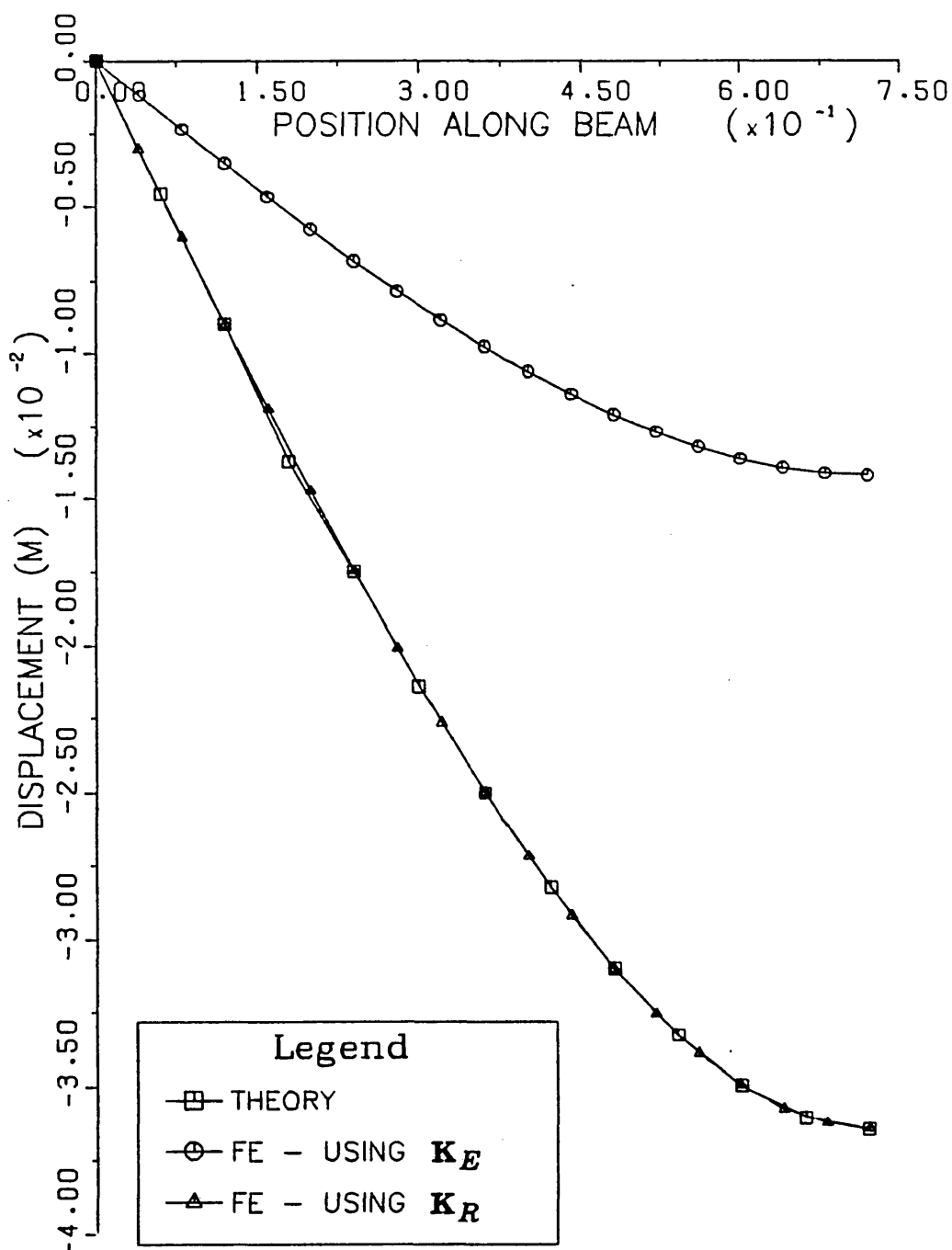


FIG. (3.2) - Calculated deflection of a simply-supported, centrally loaded beam, modelled by linear Mindlin elements, using both exact and reduced stiffness matrices.

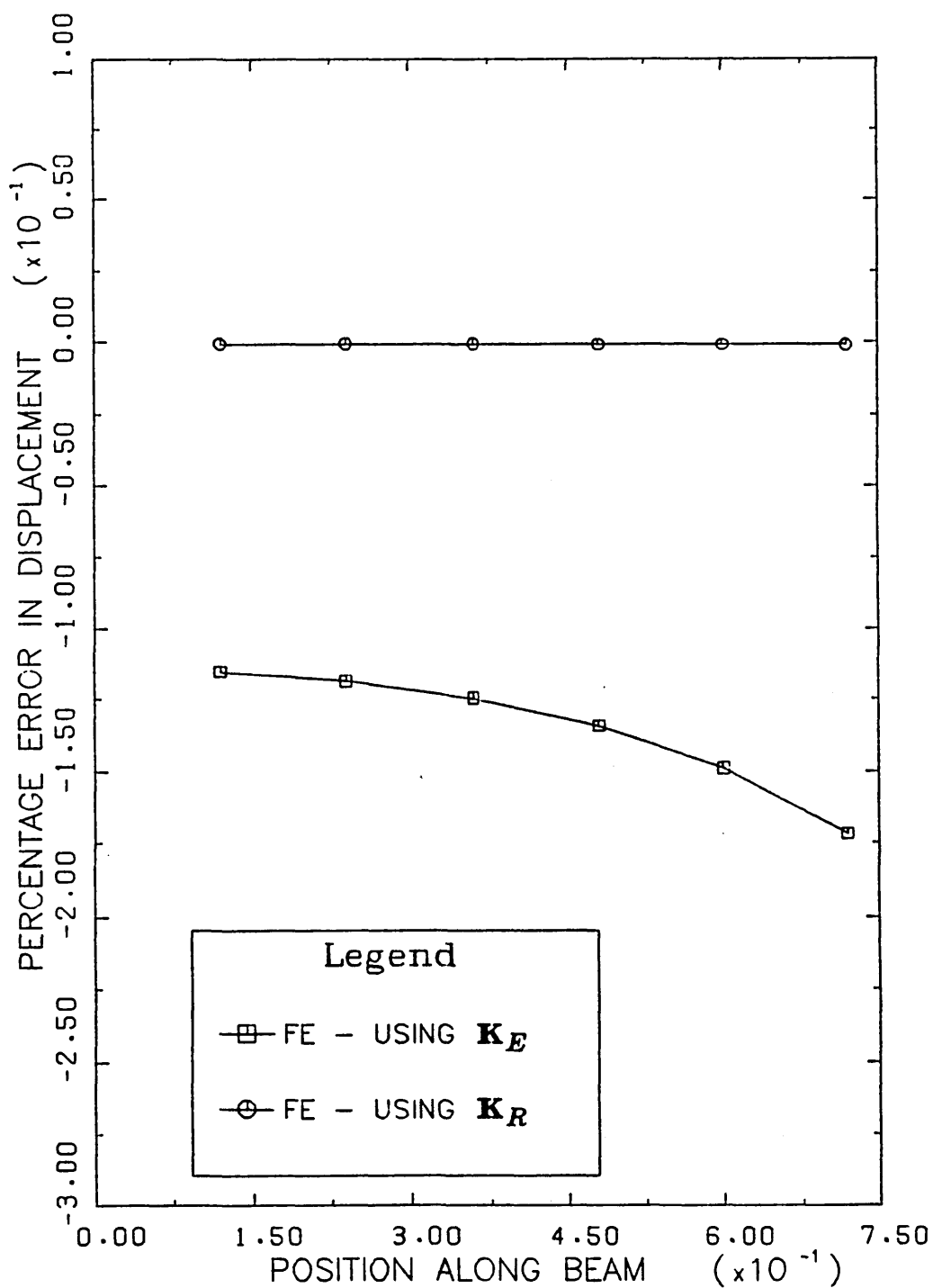


FIG. (3.3) - Percentage error in calculated deflection of a simply-supported, centrally loaded beam, modelled by quadratic Mindlin elements, using both exact and reduced stiffness matrices.

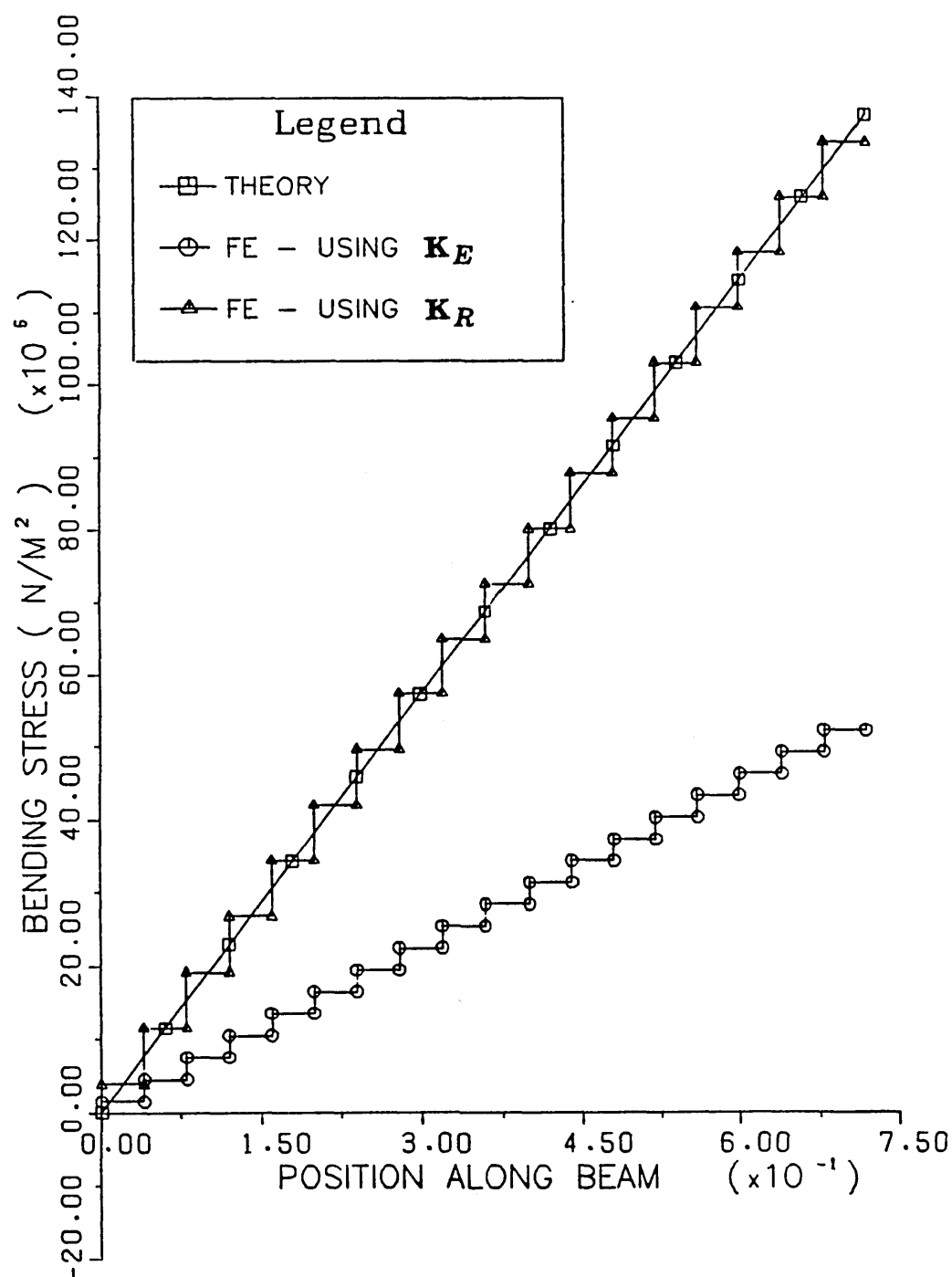


FIG. (3.4) - Predicted bending stresses in a simply-supported, centrally loaded beam, modelled by linear Mindlin elements, using both exact and reduced stiffness matrices.

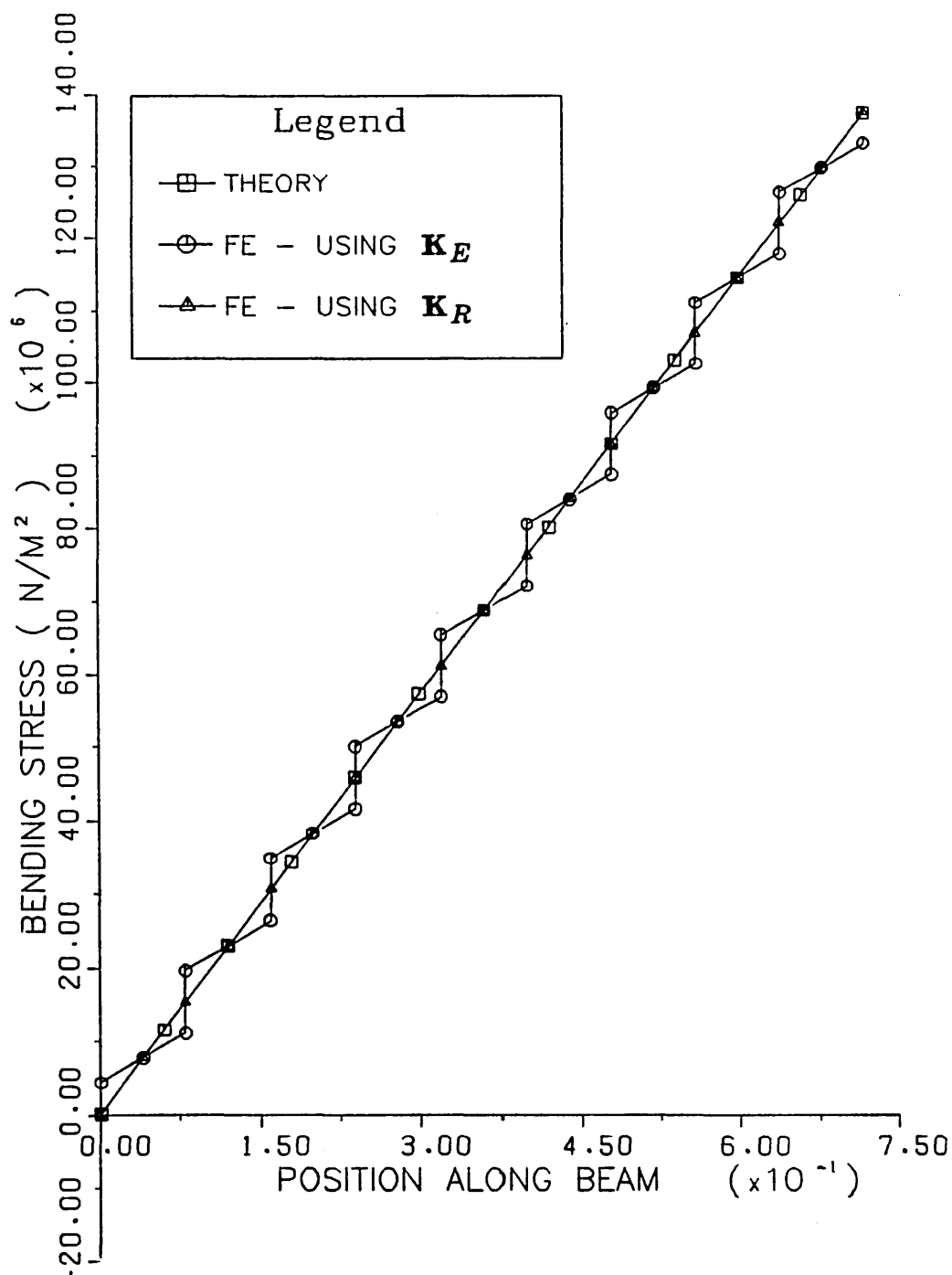


FIG. (3.5) - Predicted bending stresses in a simply-supported, centrally loaded beam, modelled by quadratic Mindlin elements, using both exact and reduced stiffness matrices.

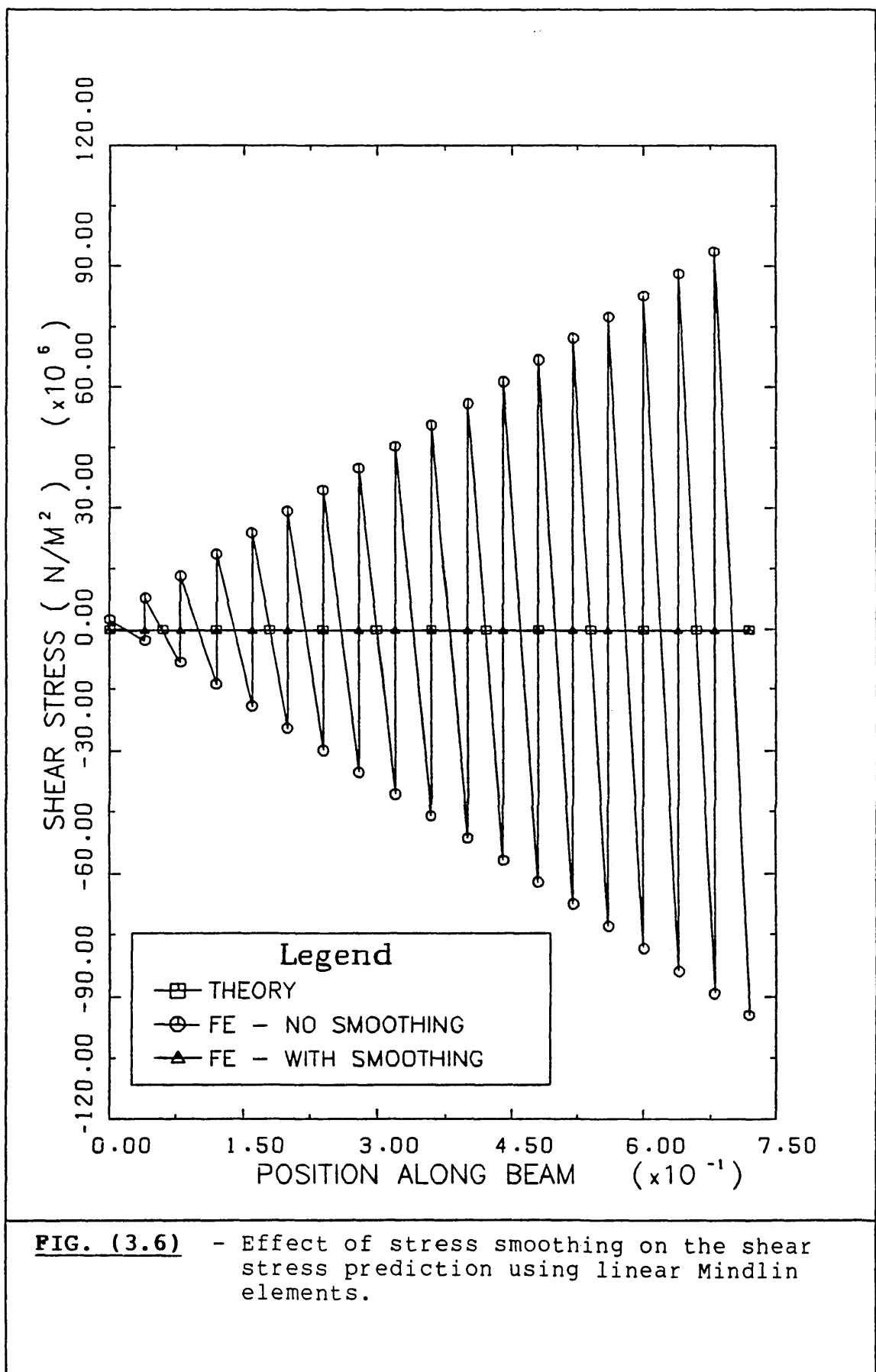


FIG. (3.6) - Effect of stress smoothing on the shear stress prediction using linear Mindlin elements.

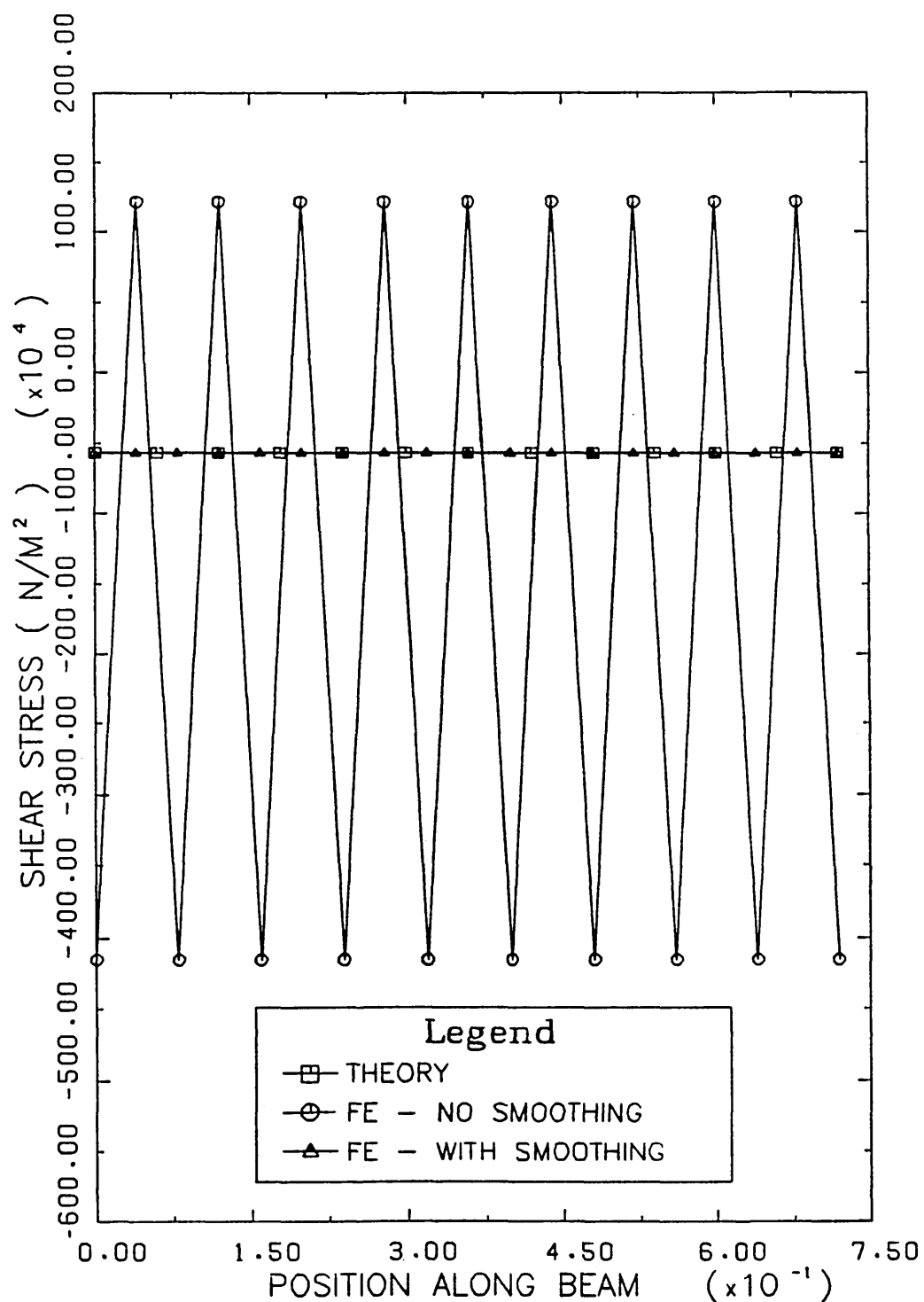


FIG. (3.7) - Effect of stress smoothing on the shear stress prediction using quadratic Mindlin elements.

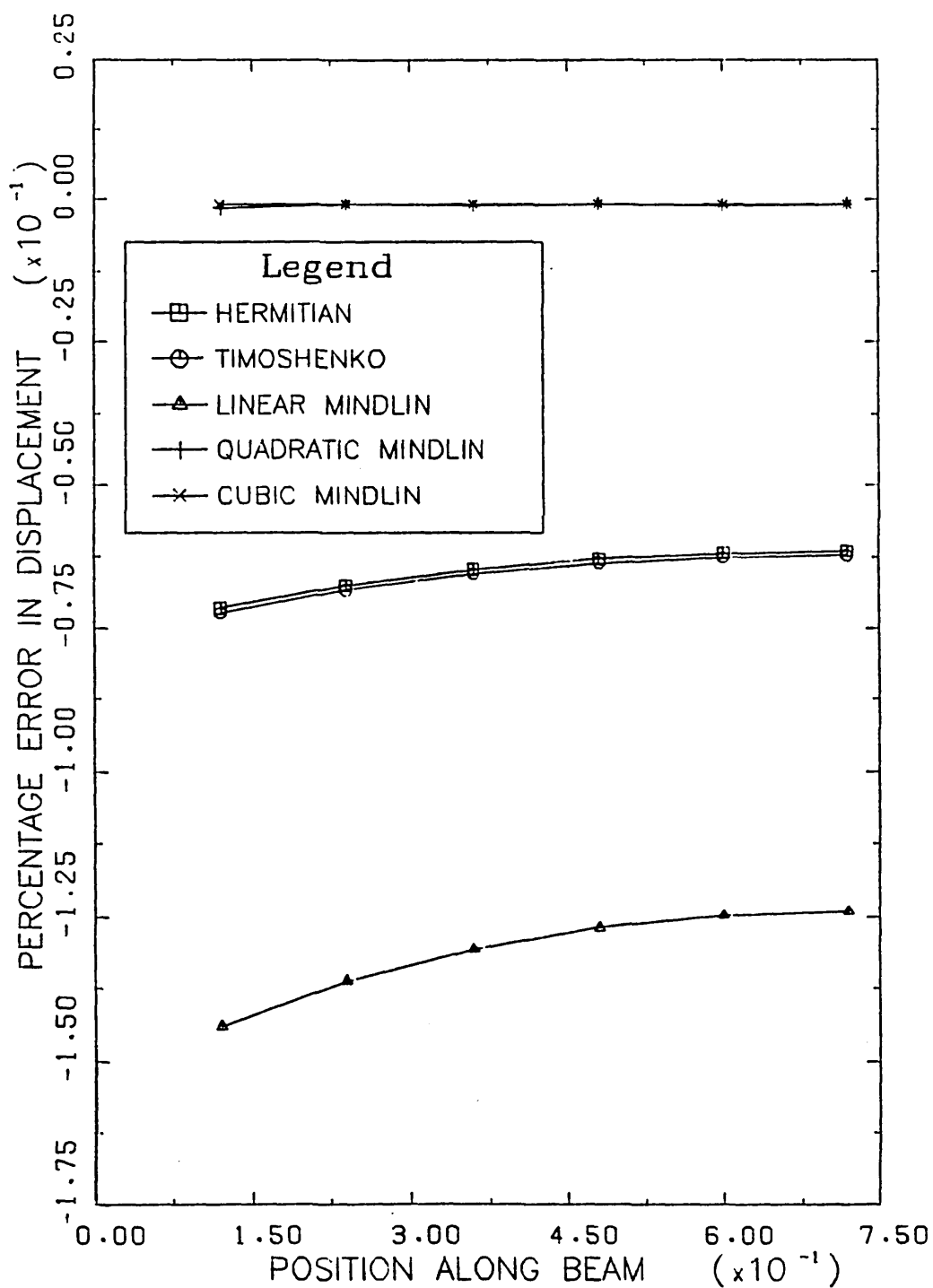


FIG. (3.8) - Percentage error in calculated deflection of a simply-supported, uniformly loaded beam, modelled by different beam elements. The reduced stiffness matrix has been employed for the Mindlin elements.

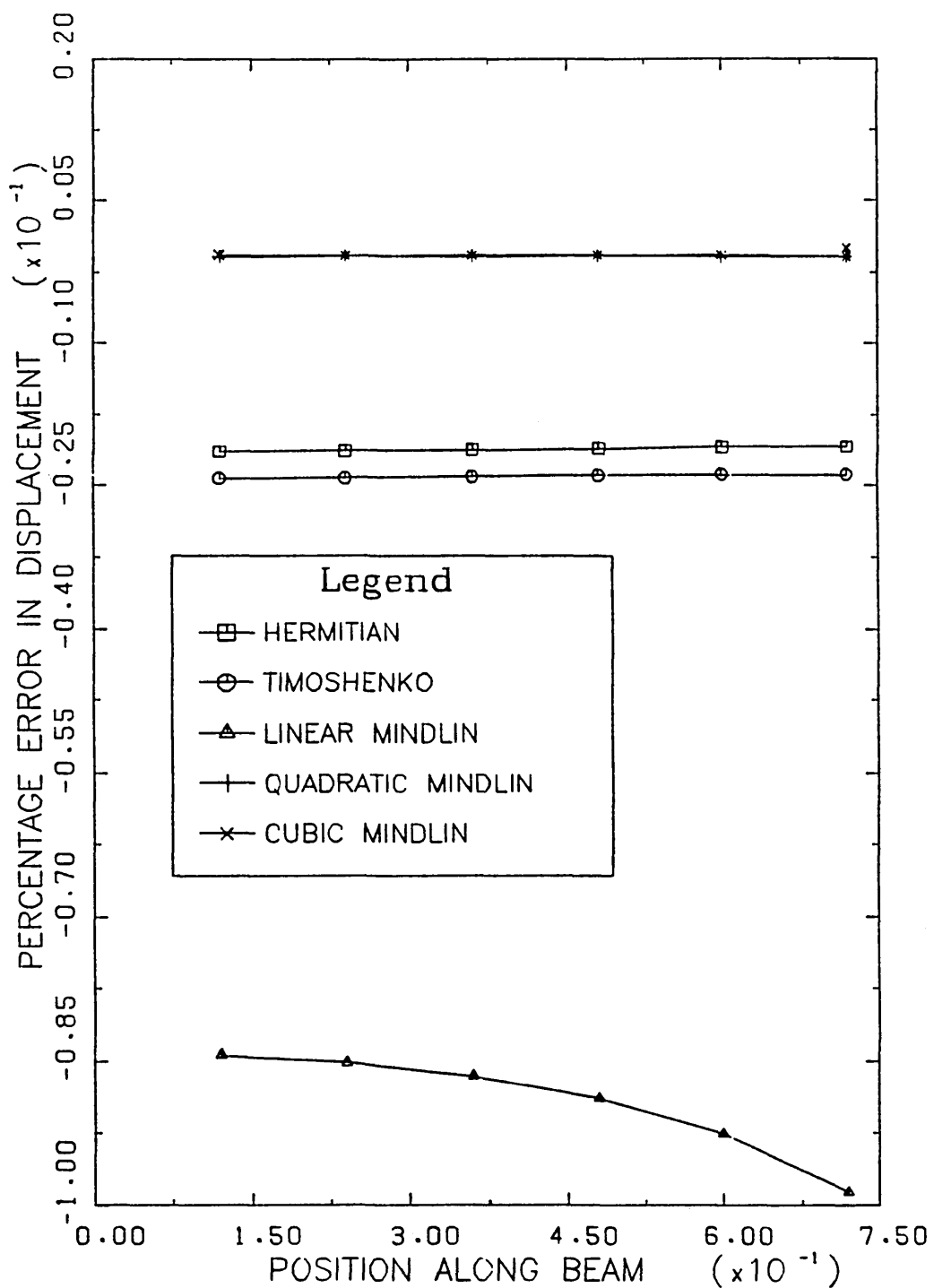


FIG. (3.9) - Percentage error in deflection of a simply-supported beam with a central sinusoidally distributed load, modelled by different beam elements. The reduced stiffness matrix has been employed for the Mindlin elements.

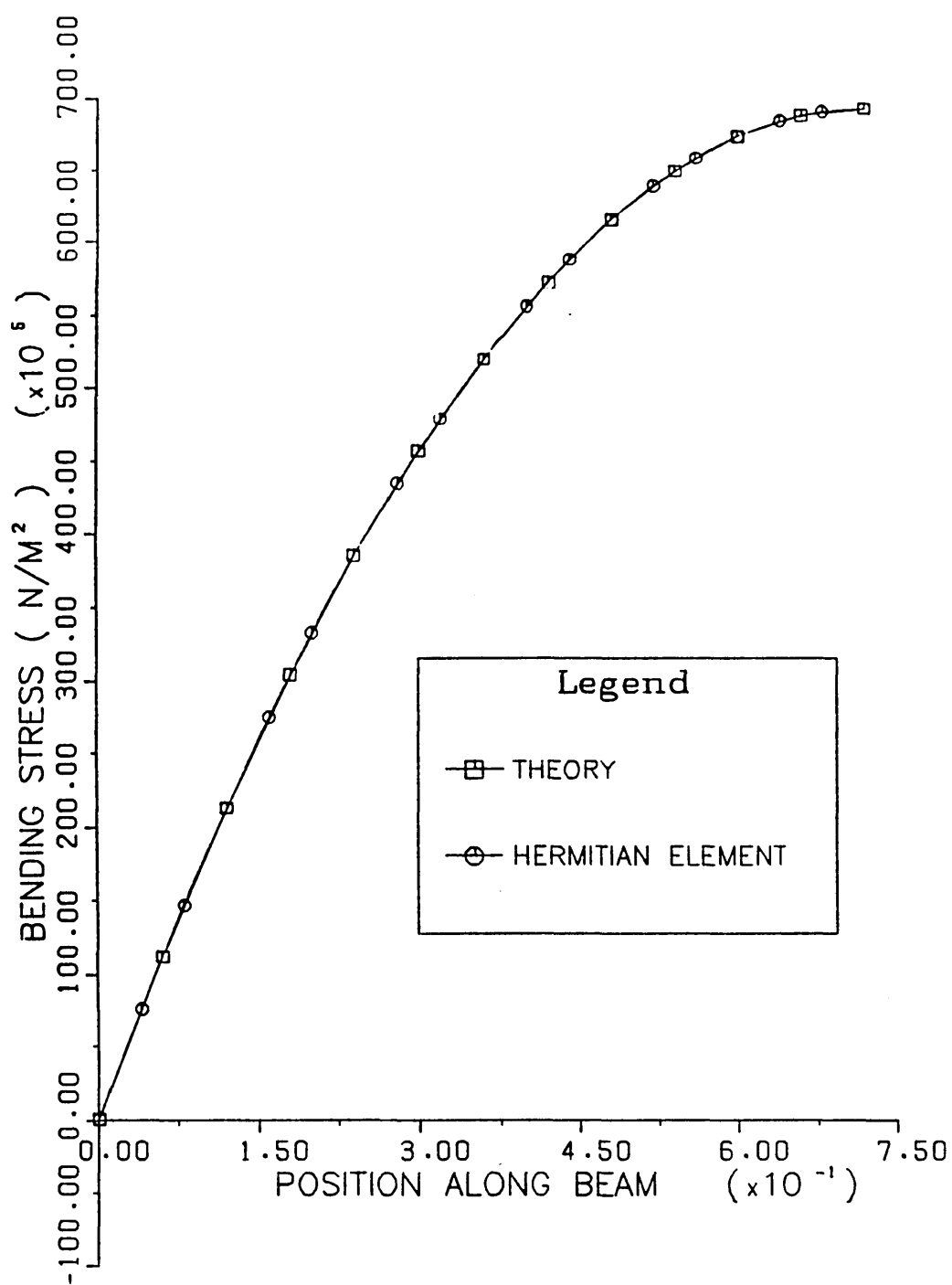


FIG. (3.10) - Bending stresses in a simply-supported, uniformly loaded beam, modelled by Hermitian beam elements.

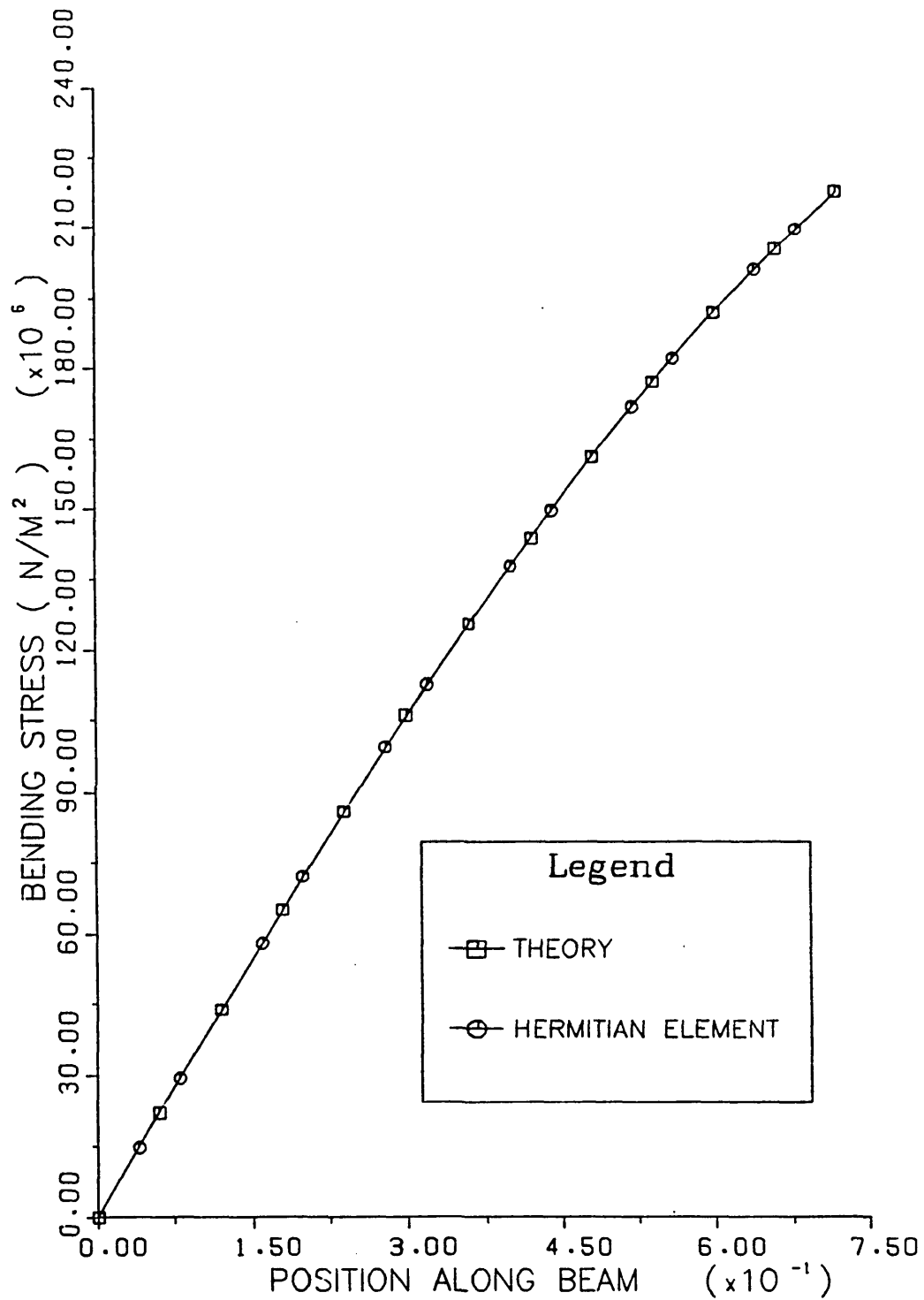


FIG. (3.11) - Bending stresses in a simply-supported beam with a central plus a sinusoidally distributed load, modelled by Hermitian beam elements.

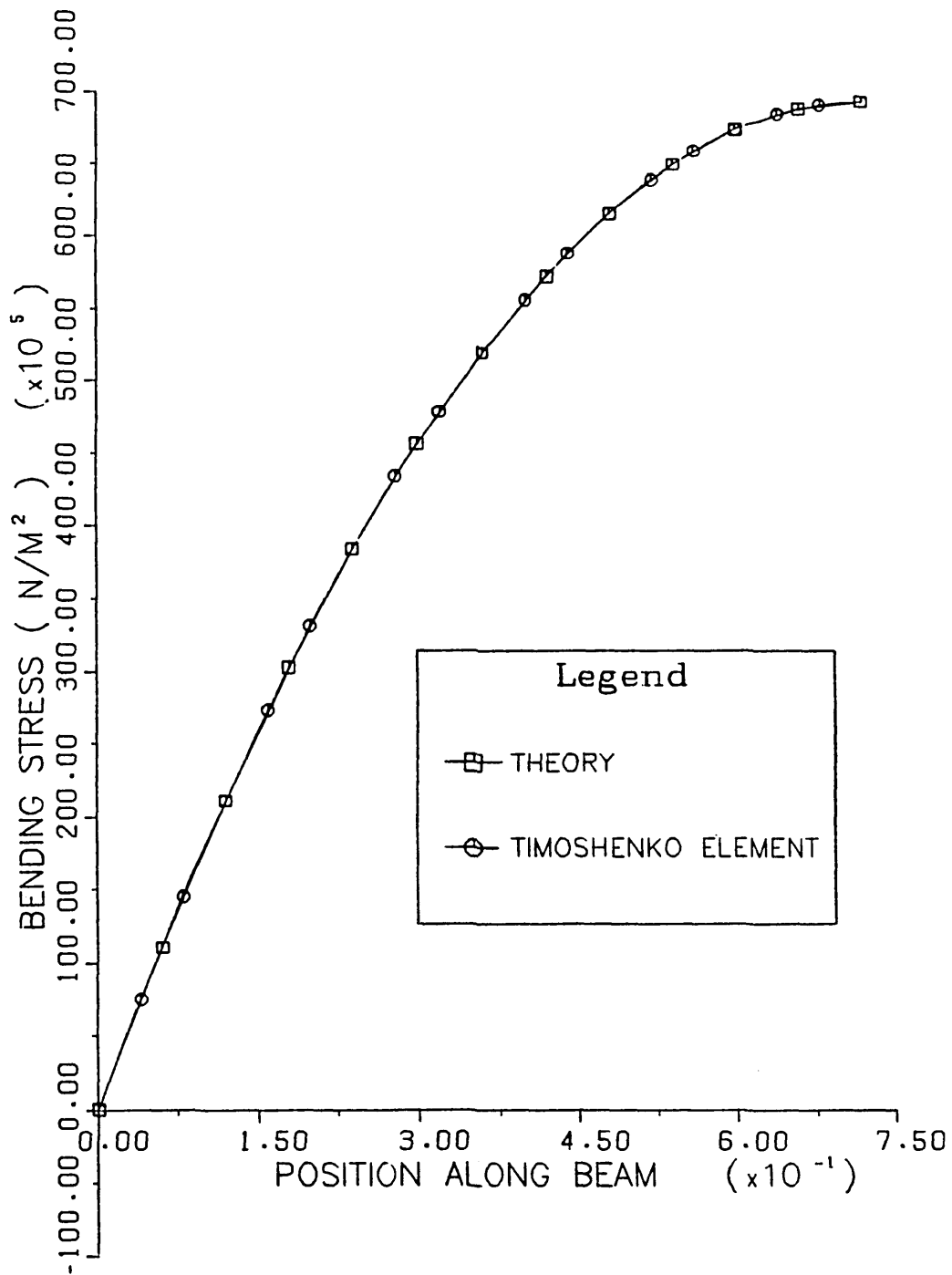


FIG. (3.12) - Bending stresses in a simply-supported, uniformly loaded beam, modelled by Timoshenko beam elements.

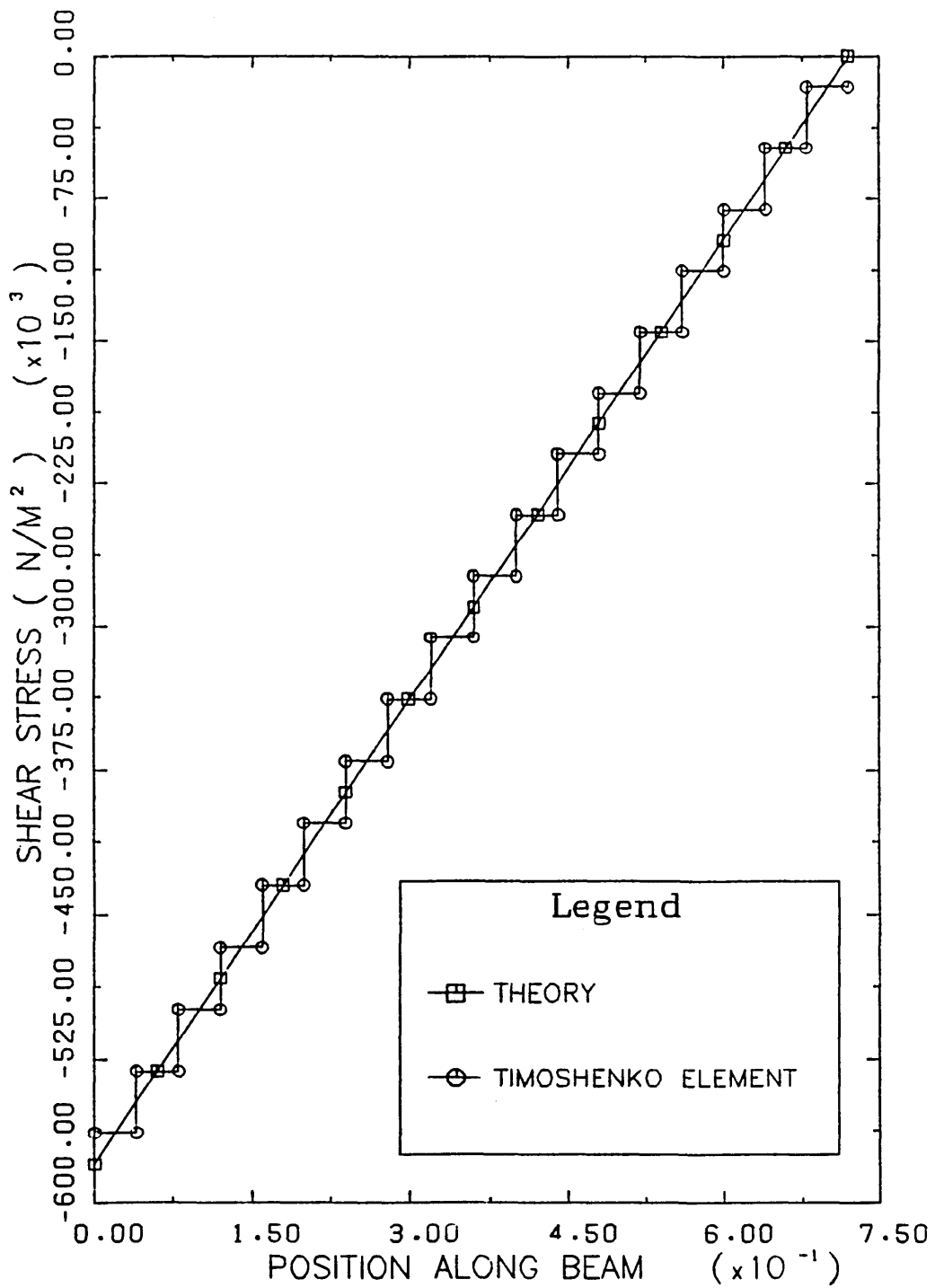


FIG. (3.13) - Shear stresses in a simply-supported, uniformly loaded beam, modelled by Timoshenko beam elements.

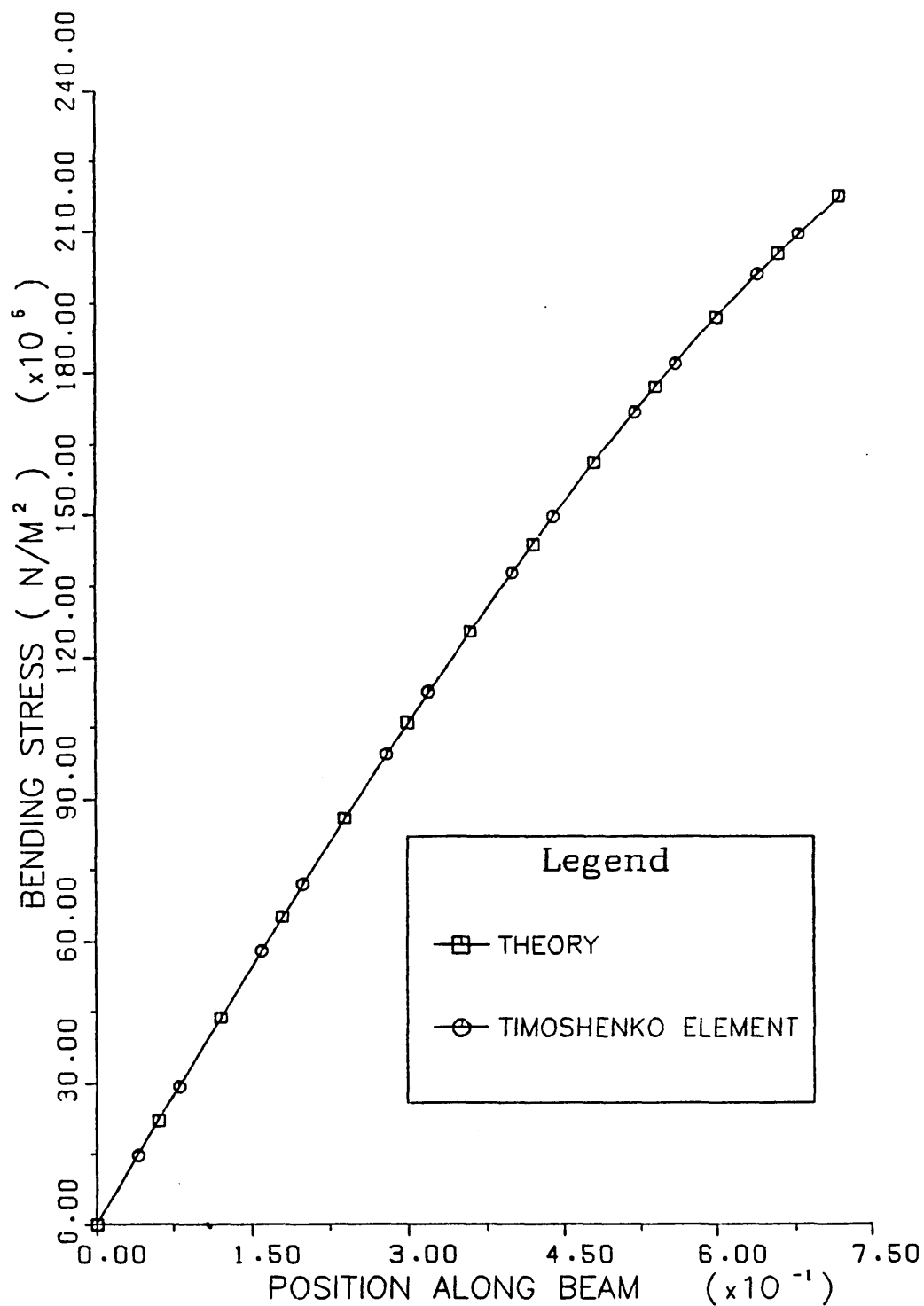


FIG. (3.14) - Bending stresses in a simply-supported beam with a central plus a sinusoidally distributed load, modelled by Timoshenko beam elements.

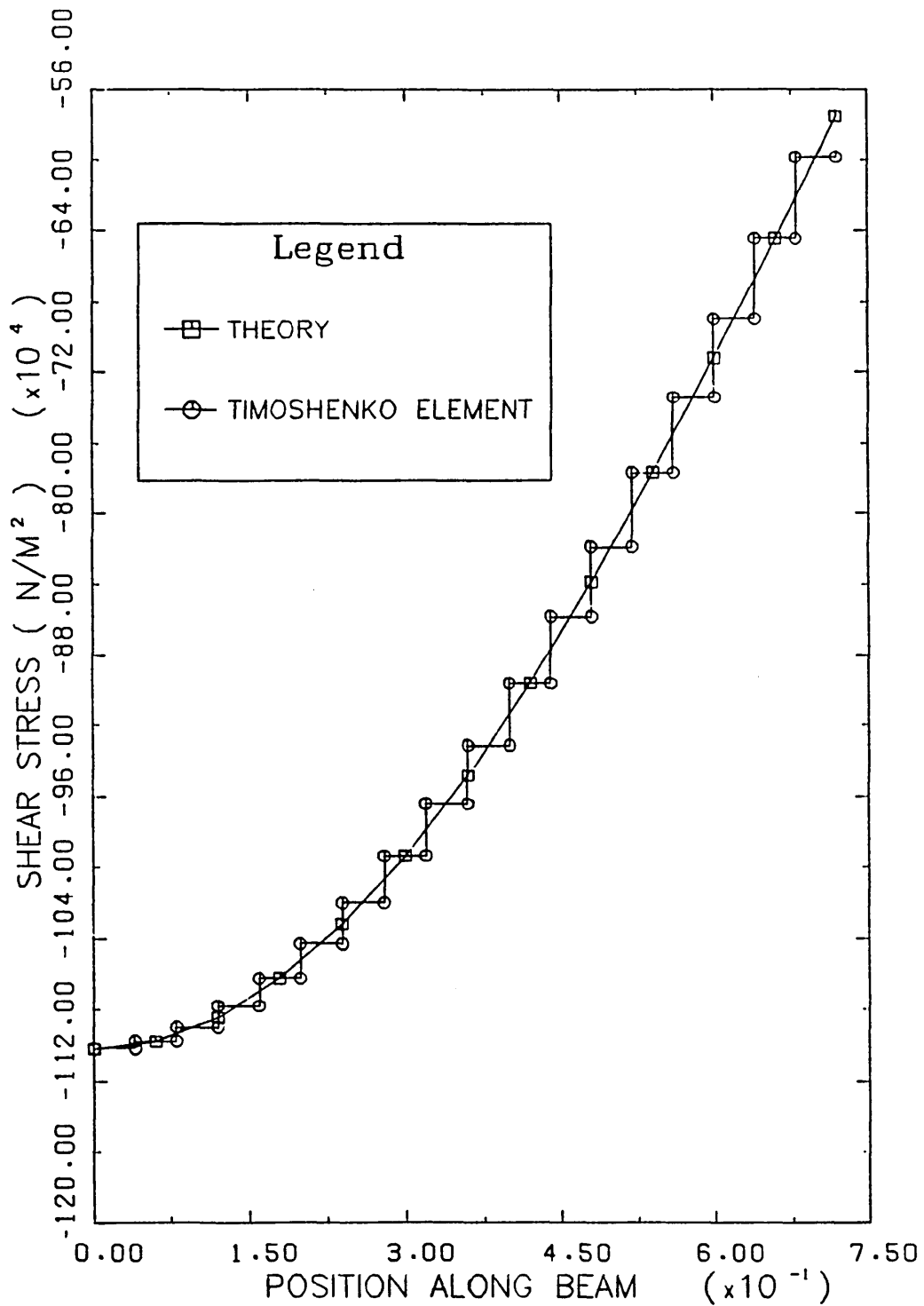
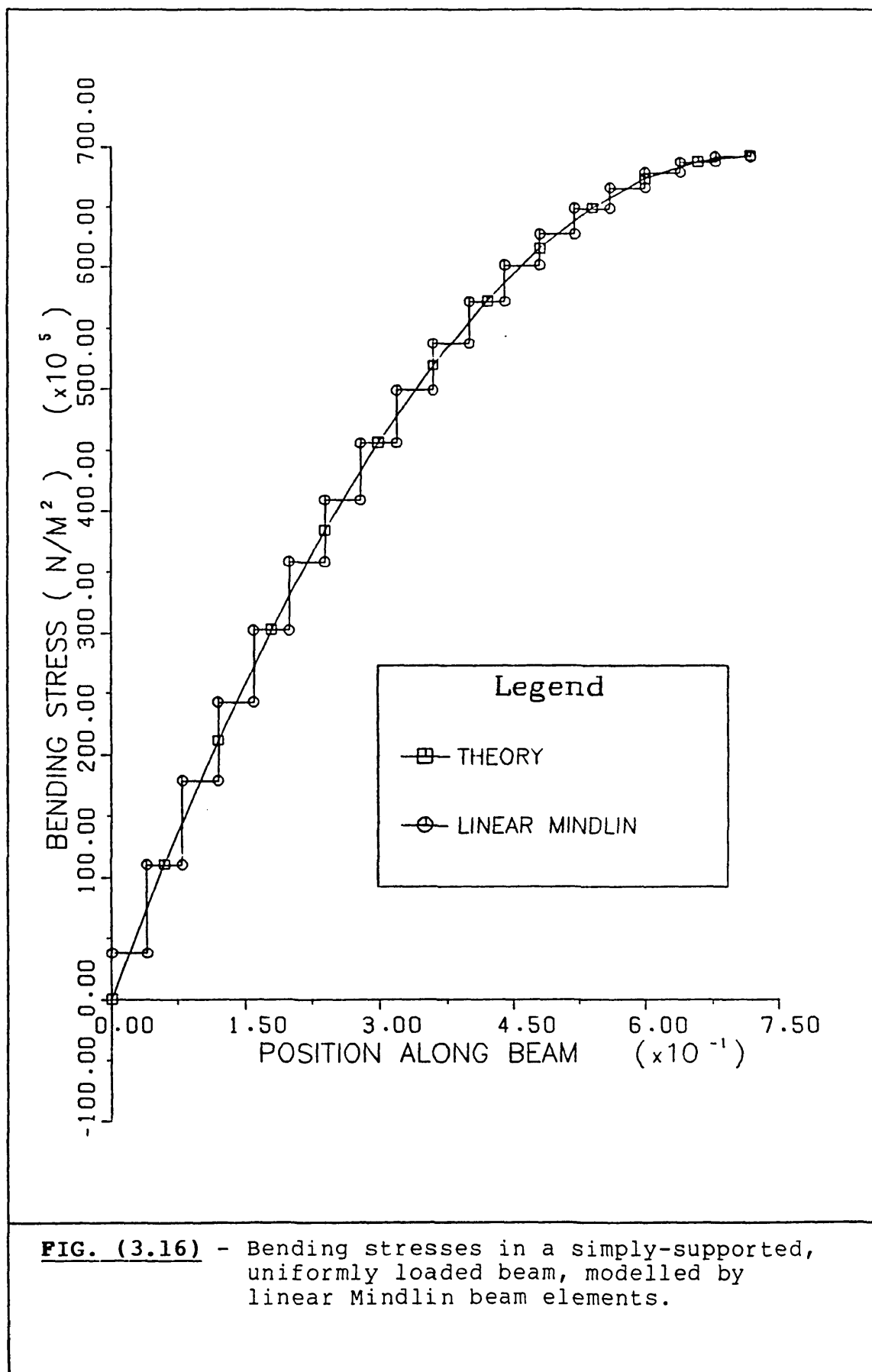


FIG. (3.15) - Shear stresses in a simply-supported beam with a central plus a sinusoidally distributed load, modelled by Timoshenko beam elements.



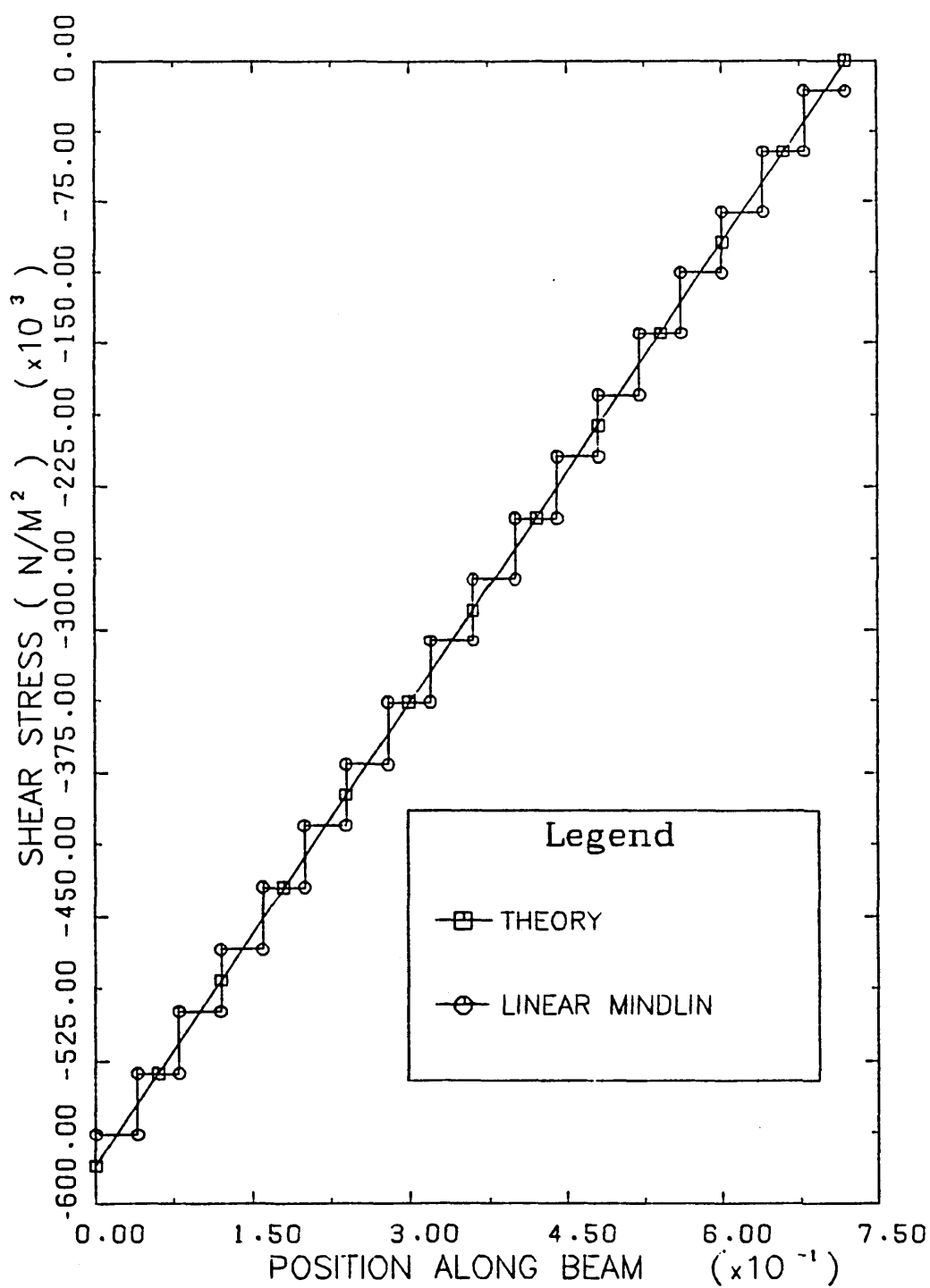


FIG. (3.17) - Shear stresses in a simply-supported, uniformly loaded beam, modelled by linear Mindlin beam elements.

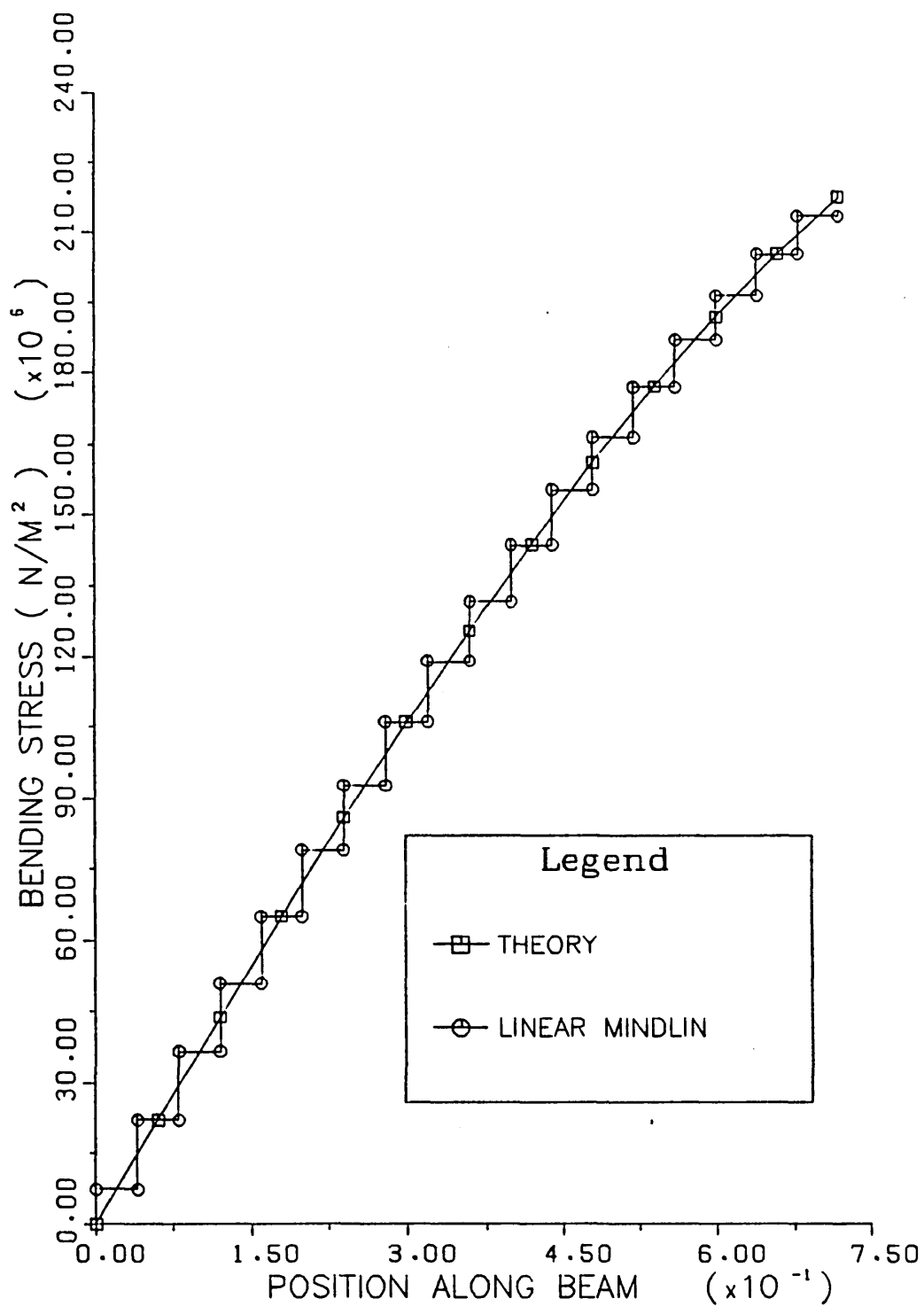


FIG. (3.18) - Bending stresses in a simply-supported beam with a central plus a sinusoidally distributed load, modelled by linear Mindlin beam elements.

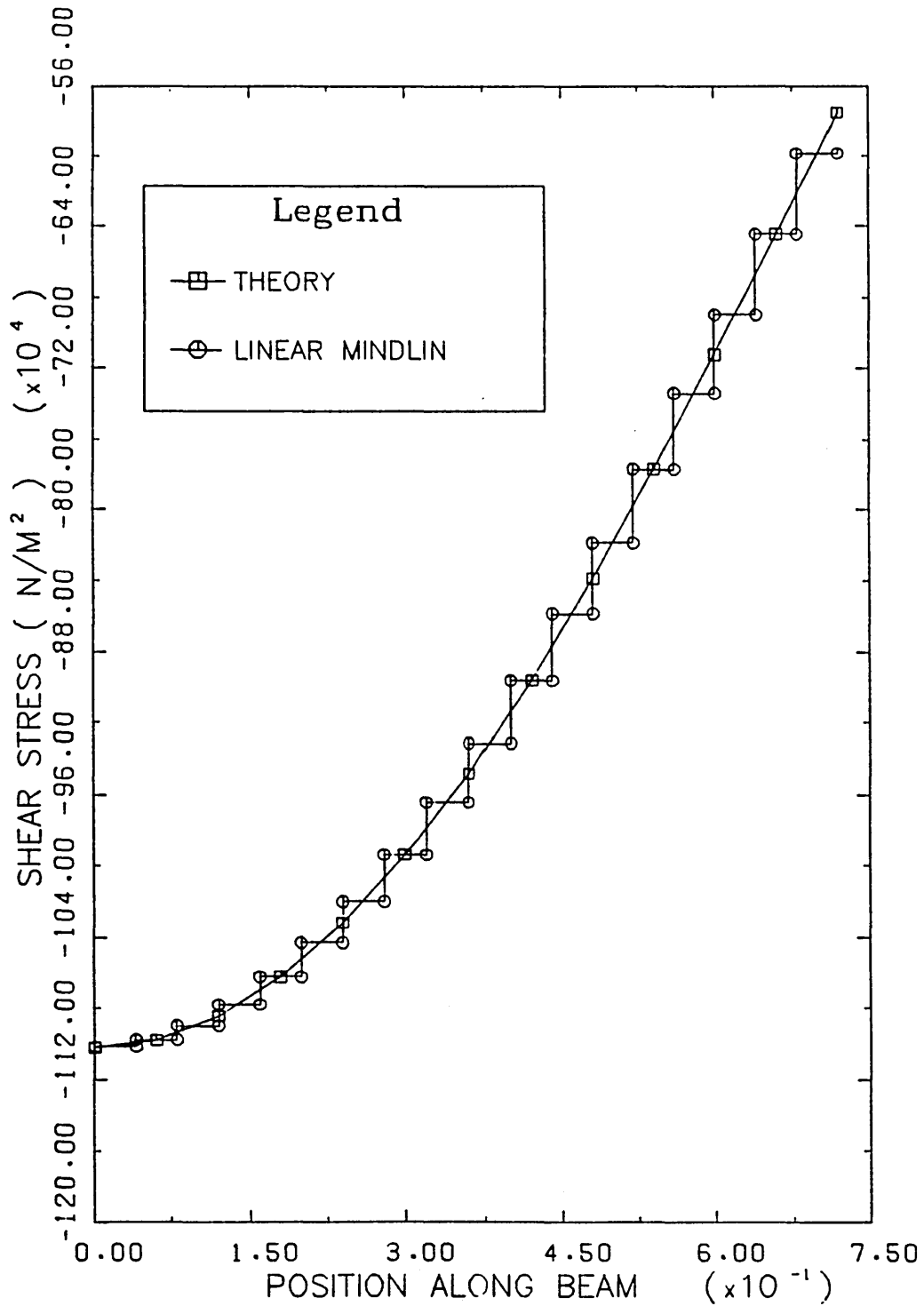


FIG. (3.19) - Shear stresses in a simply-supported beam with a central plus a sinusoidally distributed load, modelled by linear Mindlin beam elements.

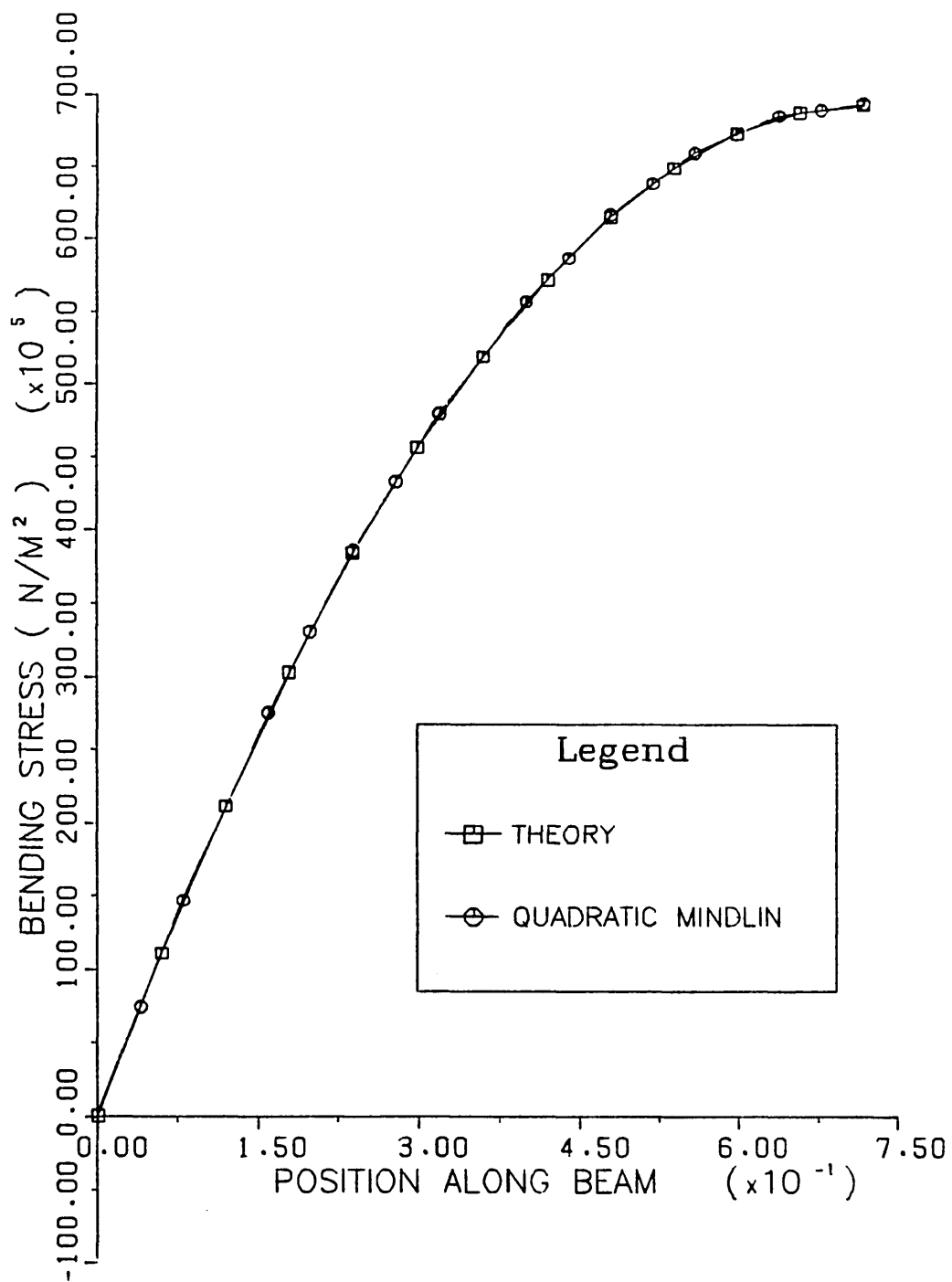


FIG. (3.20) - Bending stresses in a simply-supported, uniformly loaded beam, modelled by quadratic Mindlin beam elements.

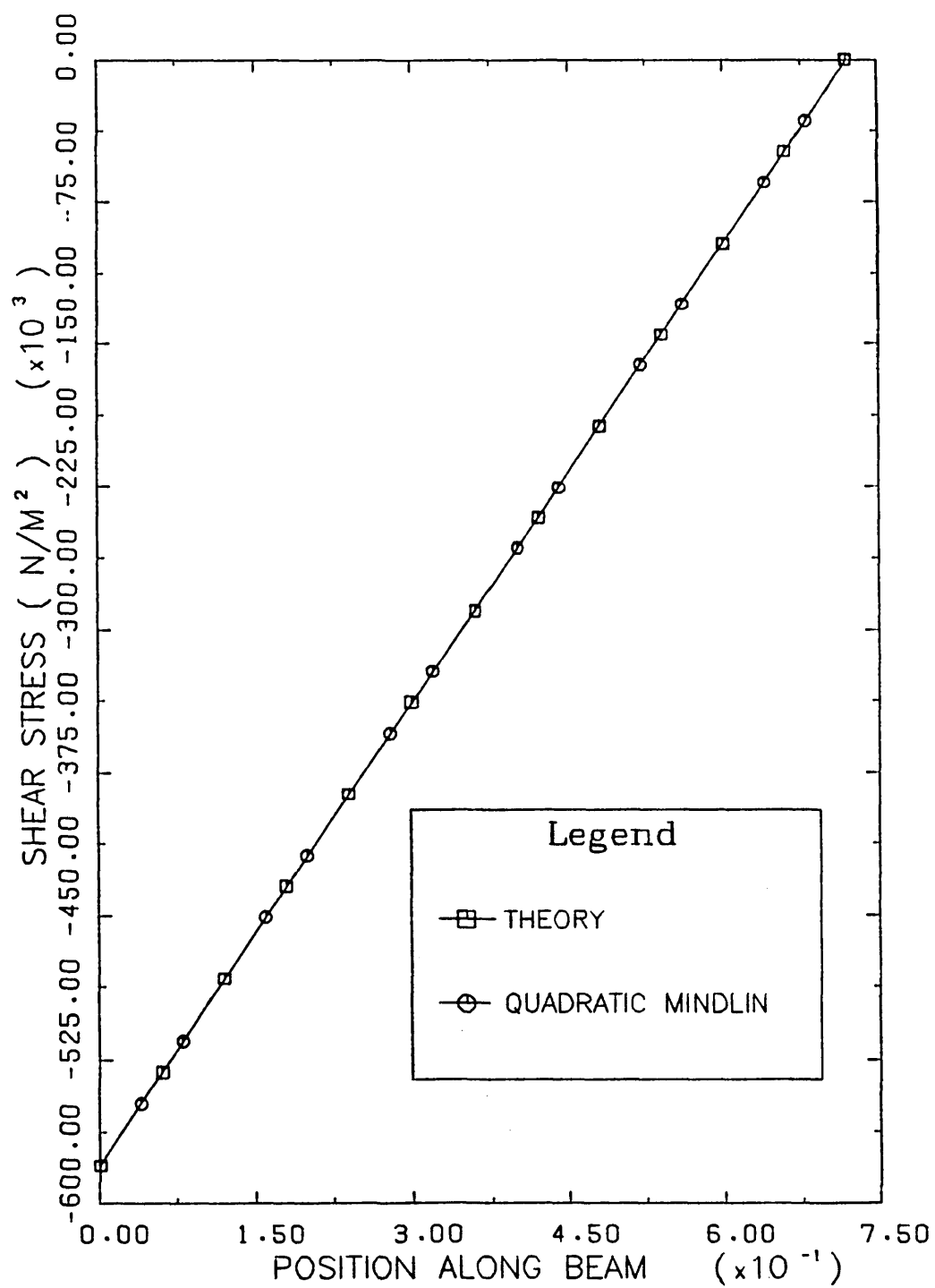


FIG. (3.21) - Shear stresses in a simply-supported, uniformly loaded beam, modelled by quadratic Mindlin beam elements.

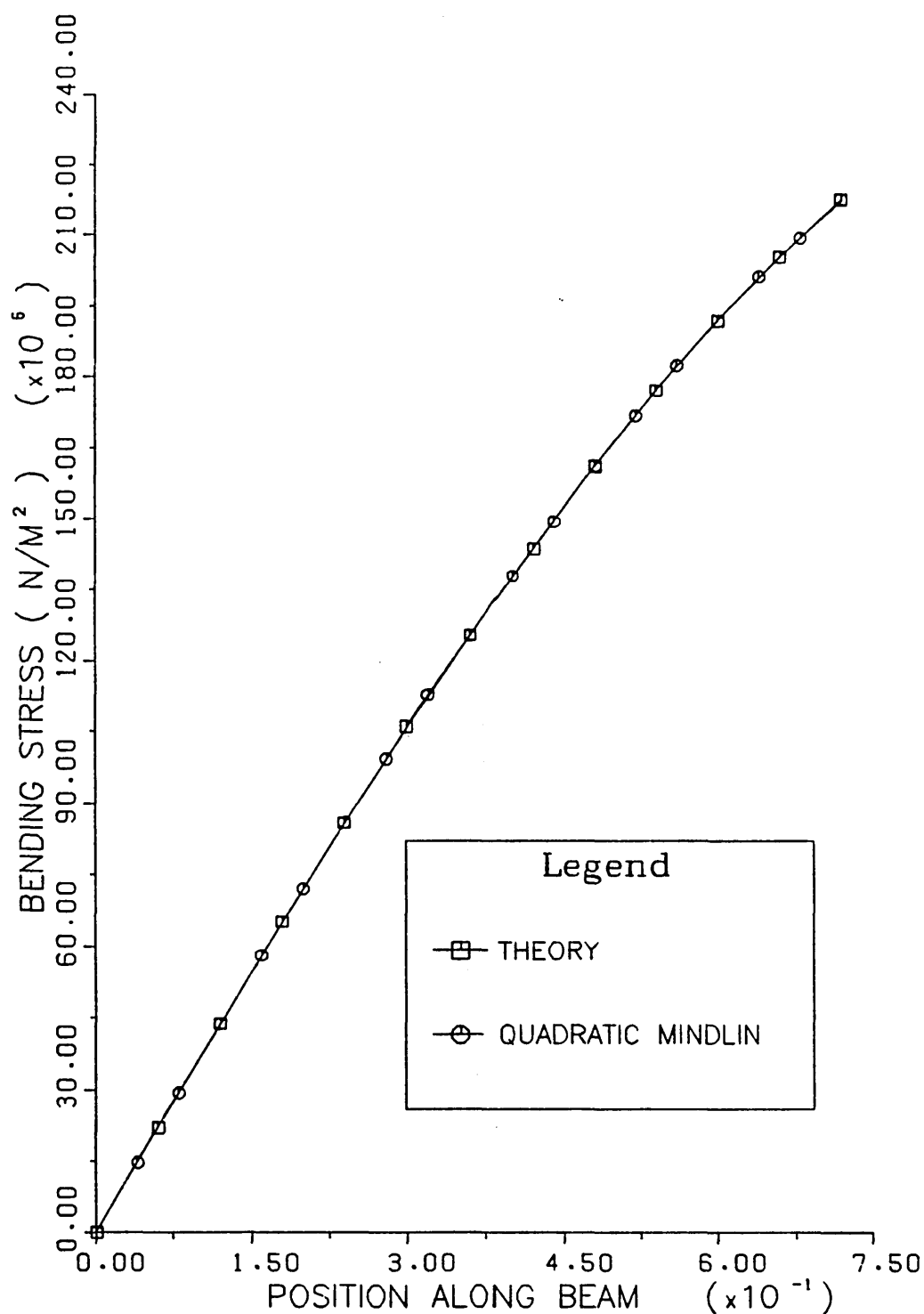


FIG. (3.22) - Bending stresses in a simply-supported beam with a central plus a sinusoidally distributed load, modelled by quadratic Mindlin beam elements.

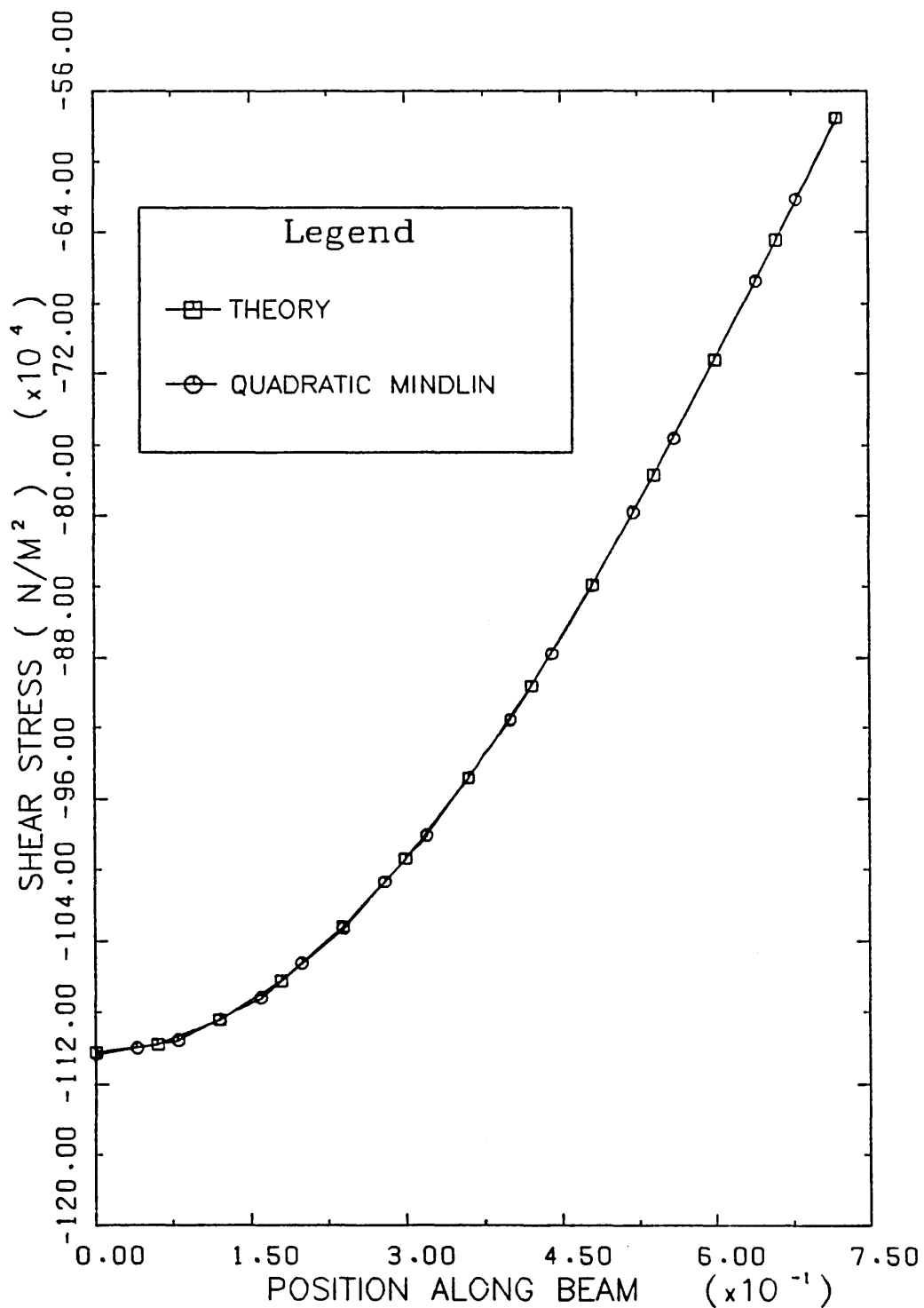


FIG. (3.23) - Shear stresses in a simply-supported beam with a central plus a sinusoidally distributed load, modelled by quadratic Mindlin beam elements.

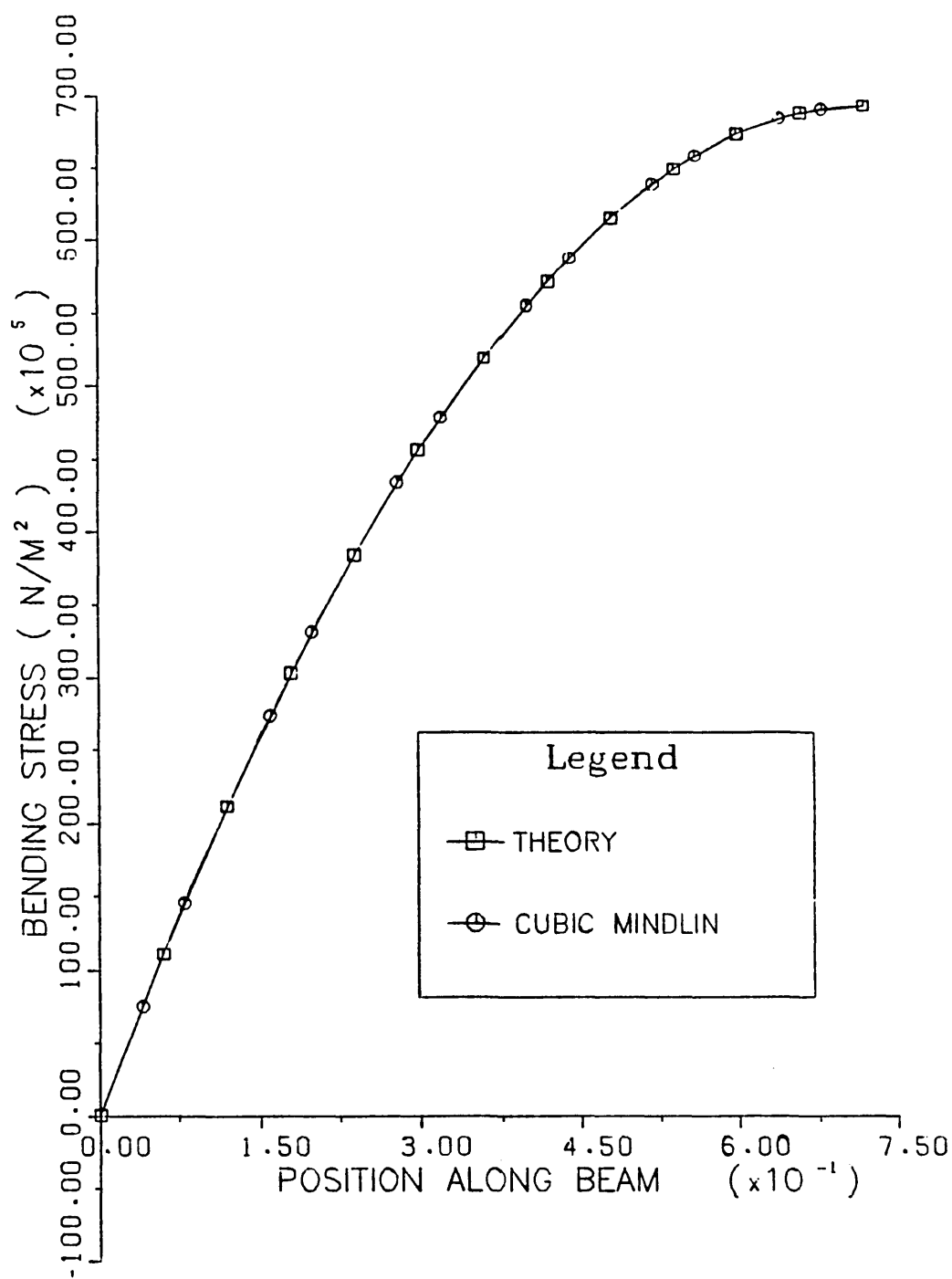


FIG. (3.24) - Bending stresses in a simply-supported, uniformly loaded beam, modelled by cubic Mindlin beam elements.

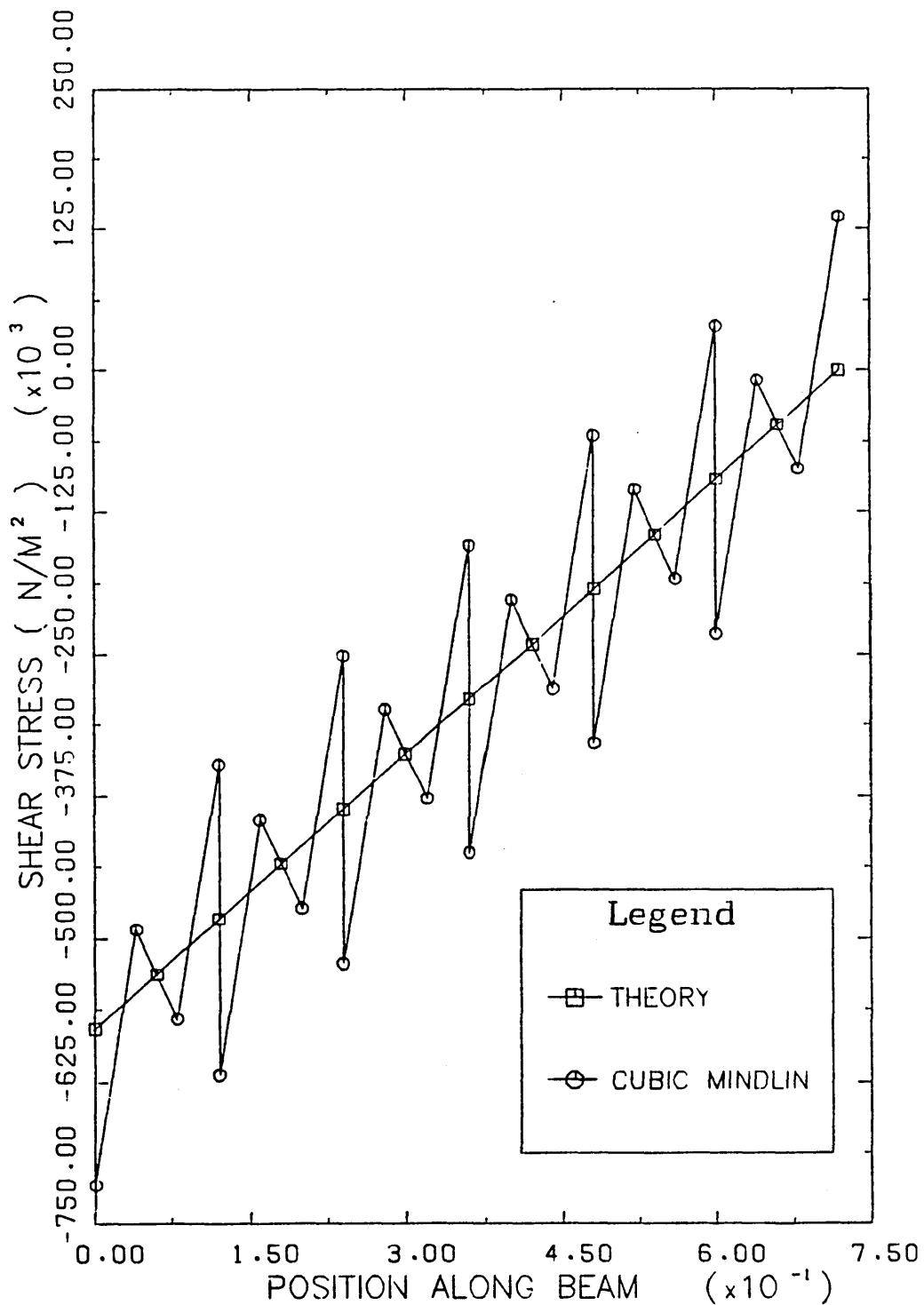


FIG. (3.25) - Shear stresses in a simply-supported, uniformly loaded beam, modelled by cubic Mindlin beam elements without stress smoothing.

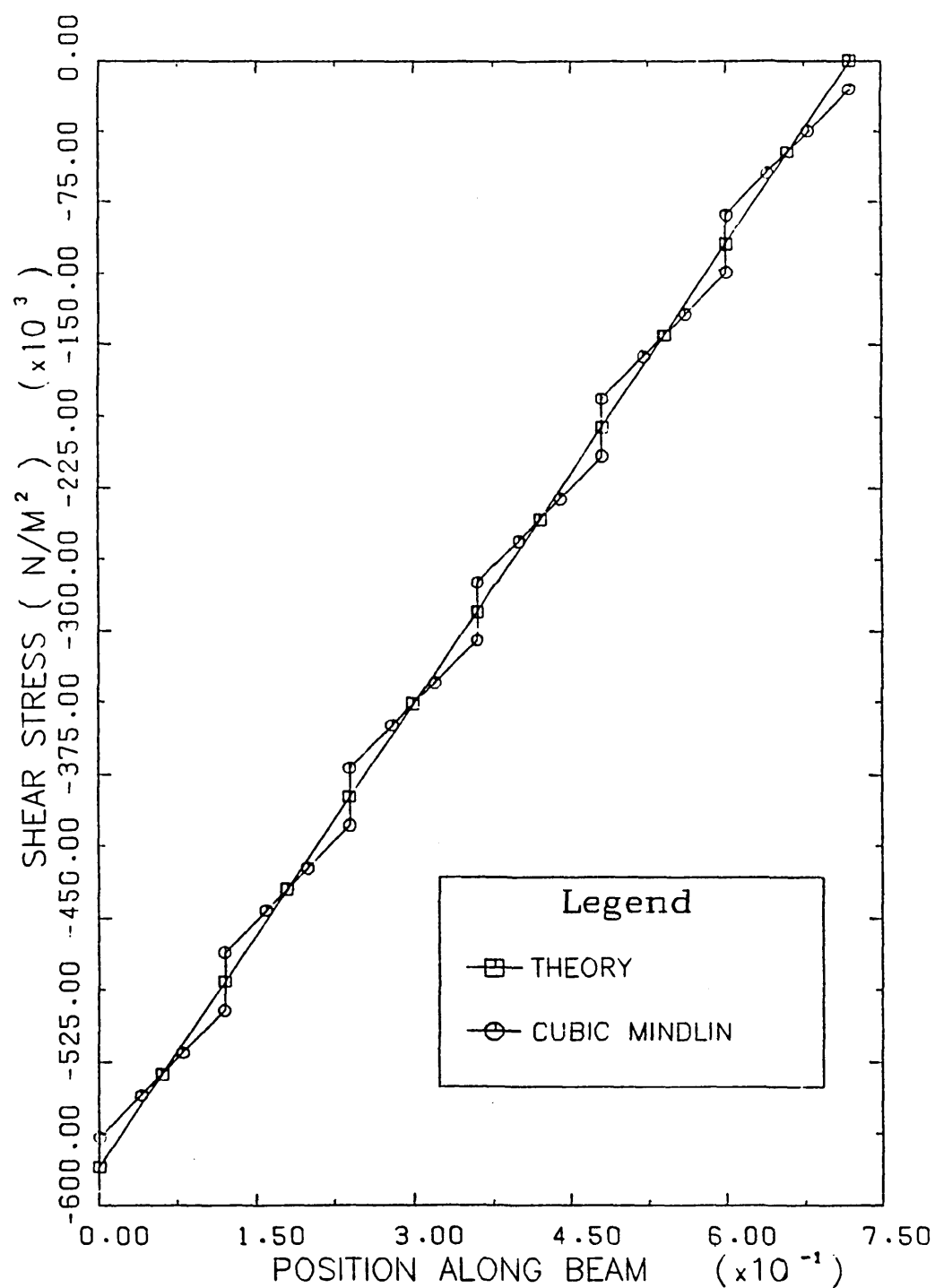


FIG. (3.26) - Shear stresses in a simply-supported, uniformly loaded beam, modelled by cubic Mindlin beam elements with stress smoothing, using sampling points derived from Barlow's technique.

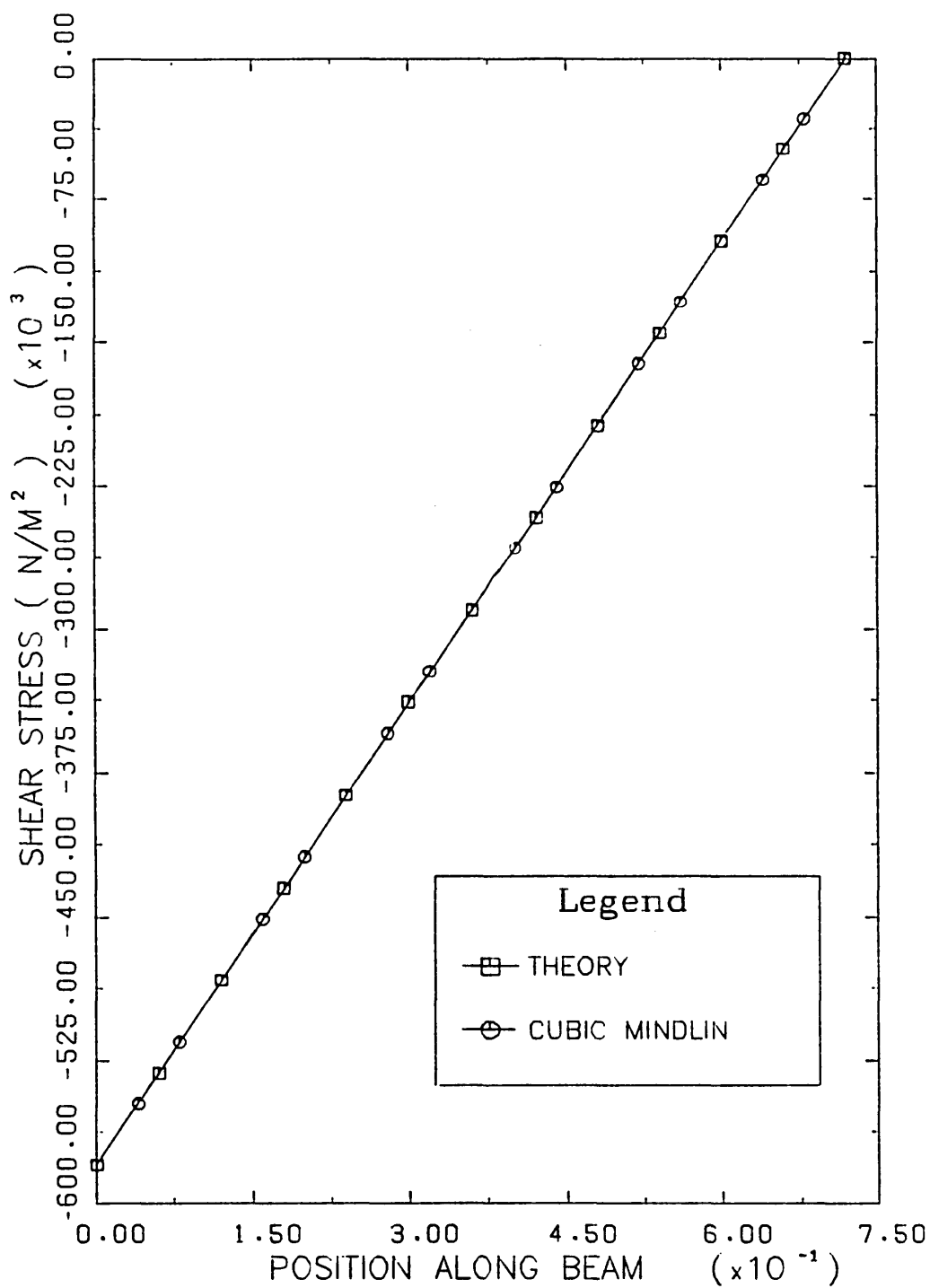


FIG. (3.27) - Shear stresses in a simply-supported, uniformly loaded beam, modelled by cubic Mindlin beam elements with stress smoothing, using the reduced Gauss points as sampling points.

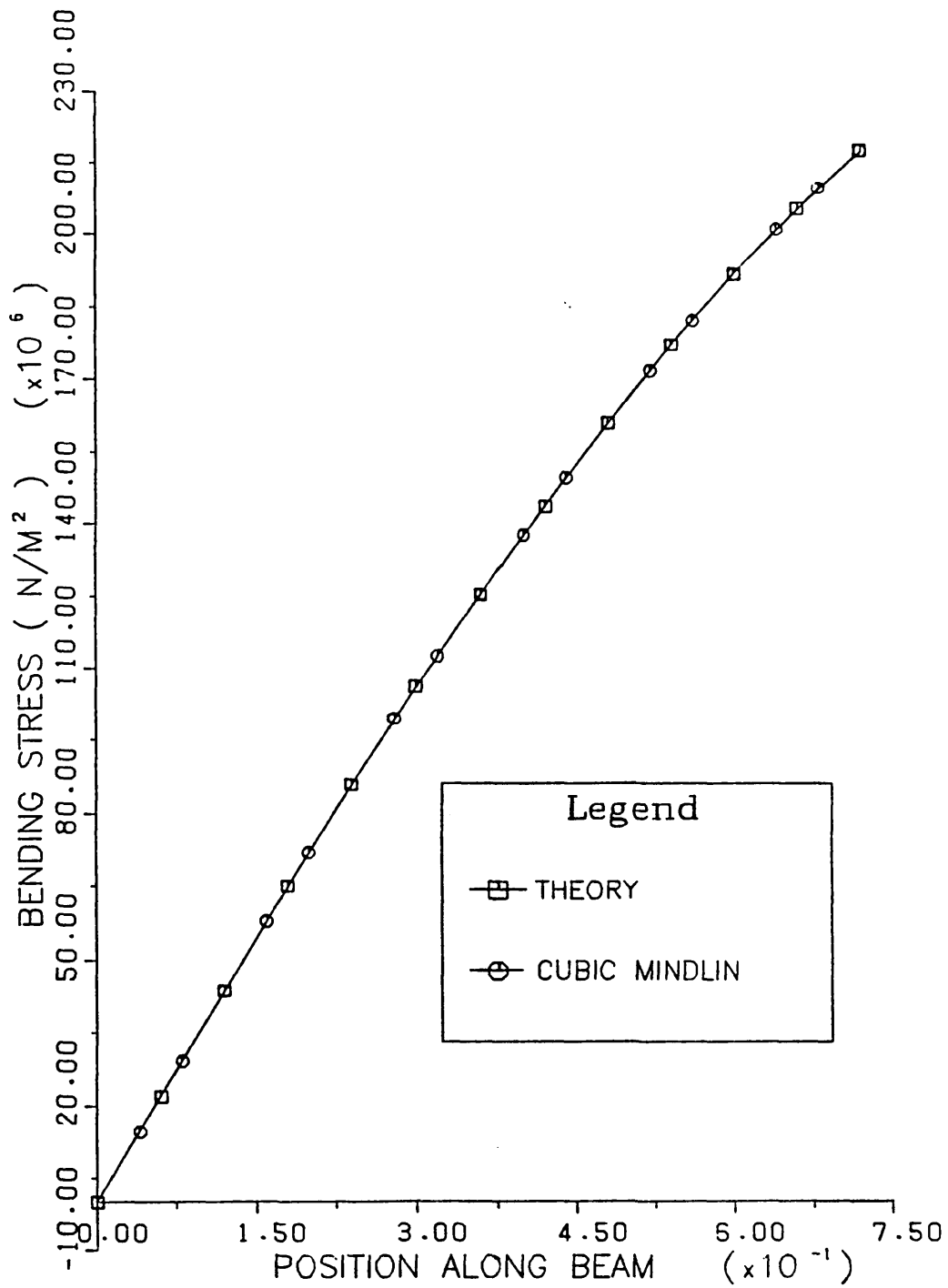


FIG. (3.28) - Bending stresses in a simply-supported beam with a central plus a sinusoidally distributed load, modelled by cubic Mindlin beam elements.

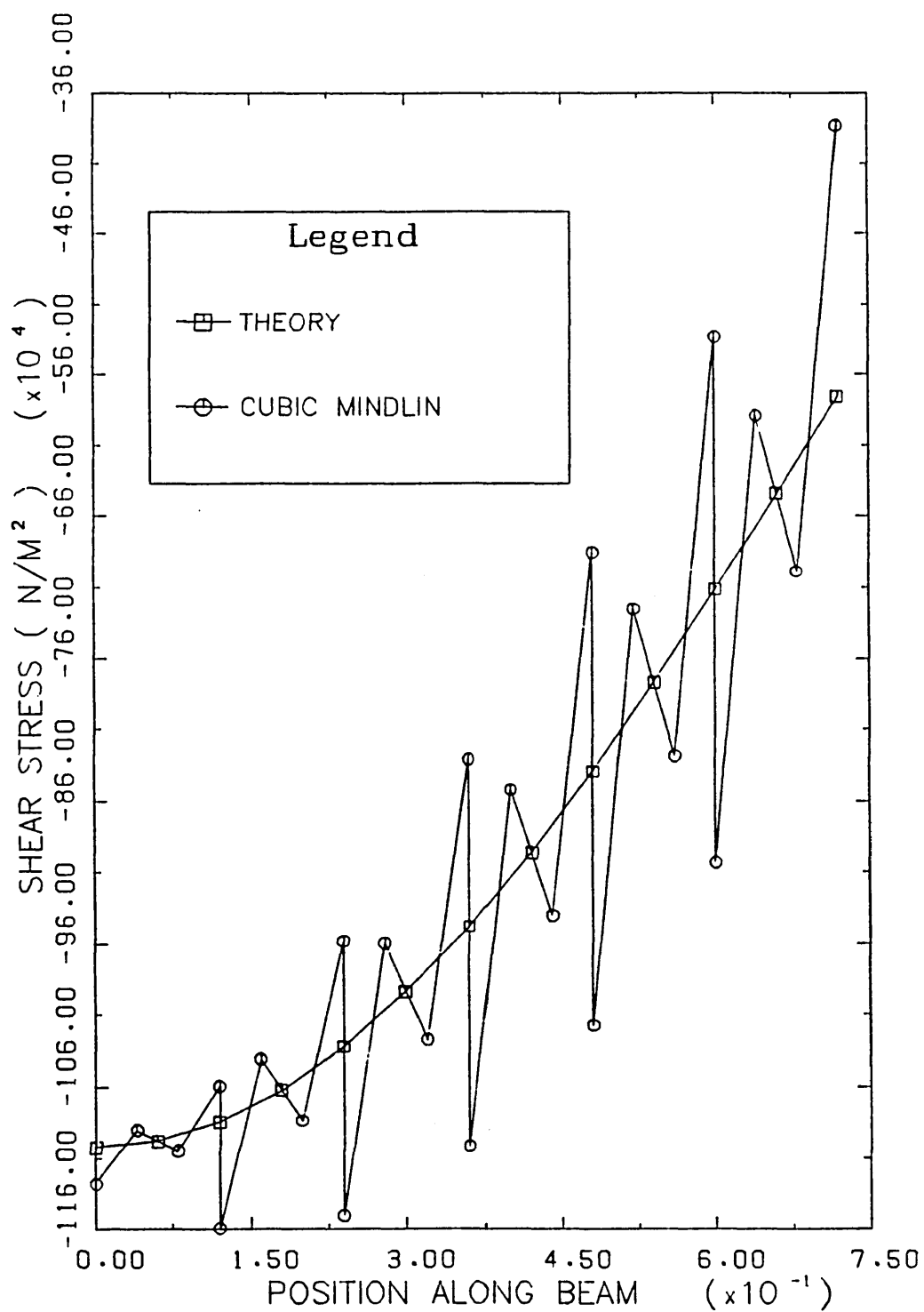


FIG. (3.29) - Shear stresses in a simply-supported beam with a central plus a sinusoidally distributed load, modelled by cubic Mindlin beam elements without stress smoothing.

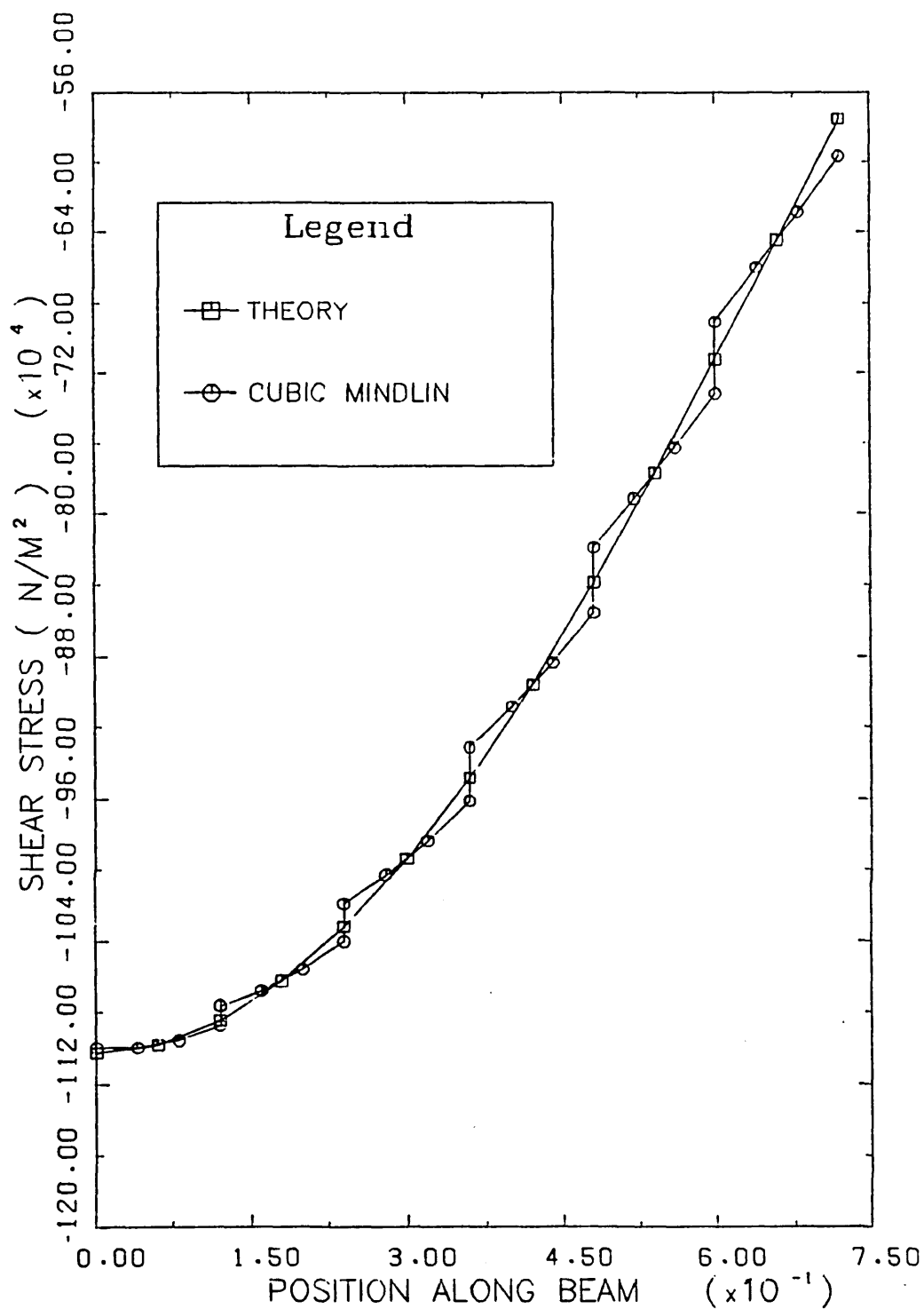


FIG. (3.30) - Shear stresses in a simply-supported beam with a central plus a sinusoidally distributed load, modelled by cubic Mindlin beam elements with stress smoothing, using sampling points derived from Barlow's technique.

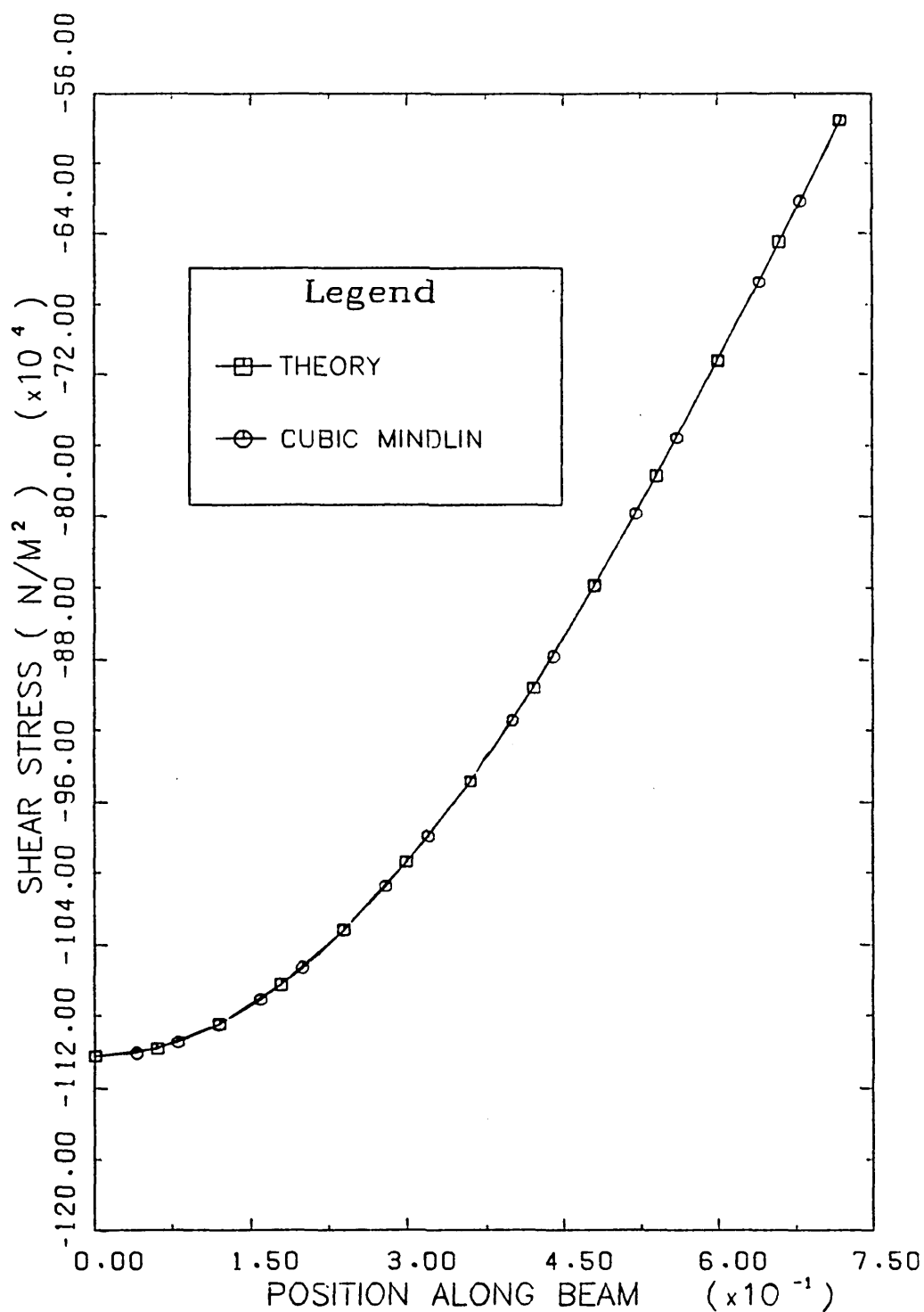


FIG. (3.31) - Shear stresses in a simply-supported beam with a central plus a sinusoidally distributed load, modelled by cubic Mindlin beam elements with stress smoothing, using the reduced Gauss points as sampling points.

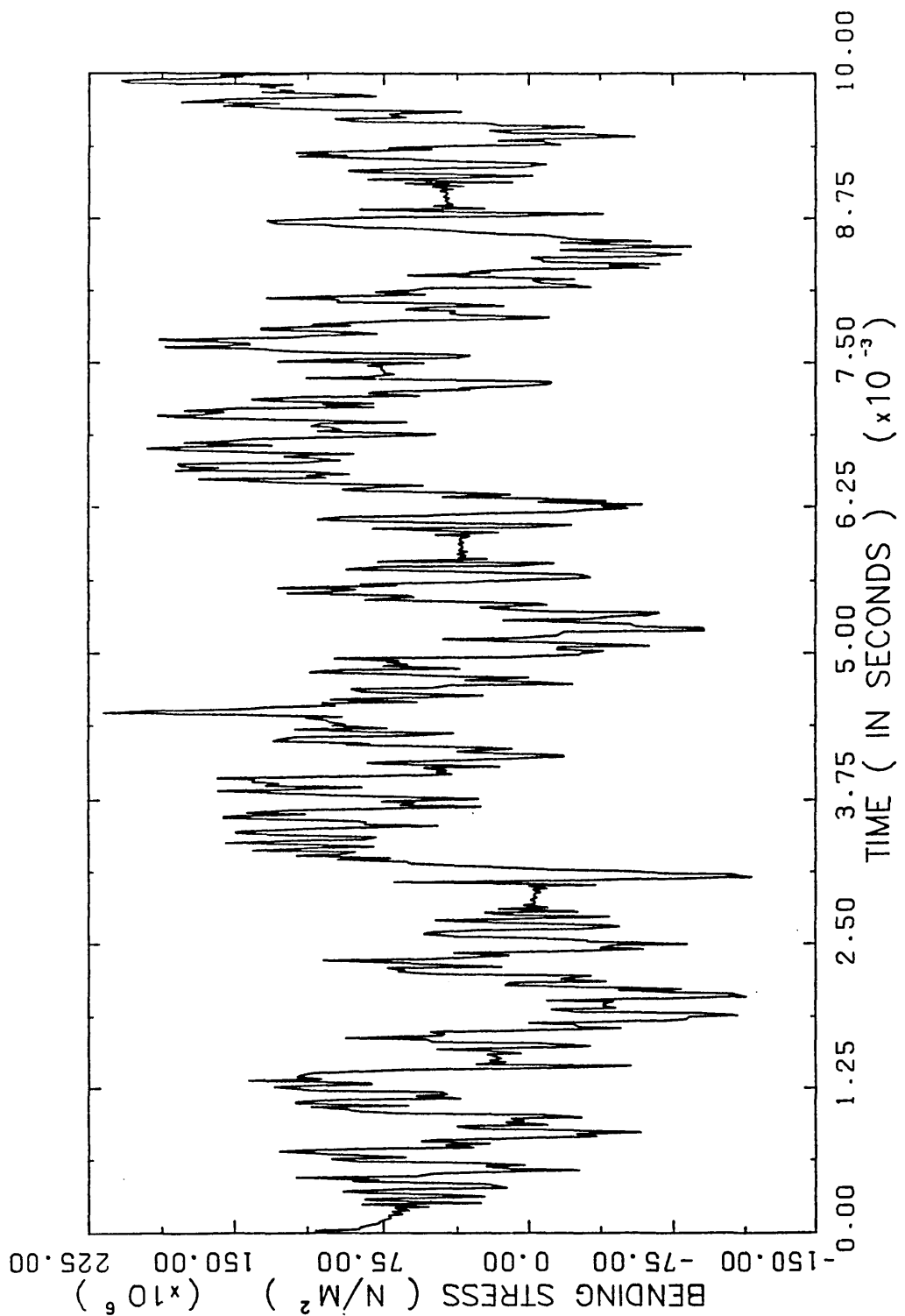


FIG. (3.32) - Centre node bending stress history for a simply-supported beam with an attached centre mass subject to an initial step-velocity excitation, using the Bernoulli-Euler theory and a modal approach.

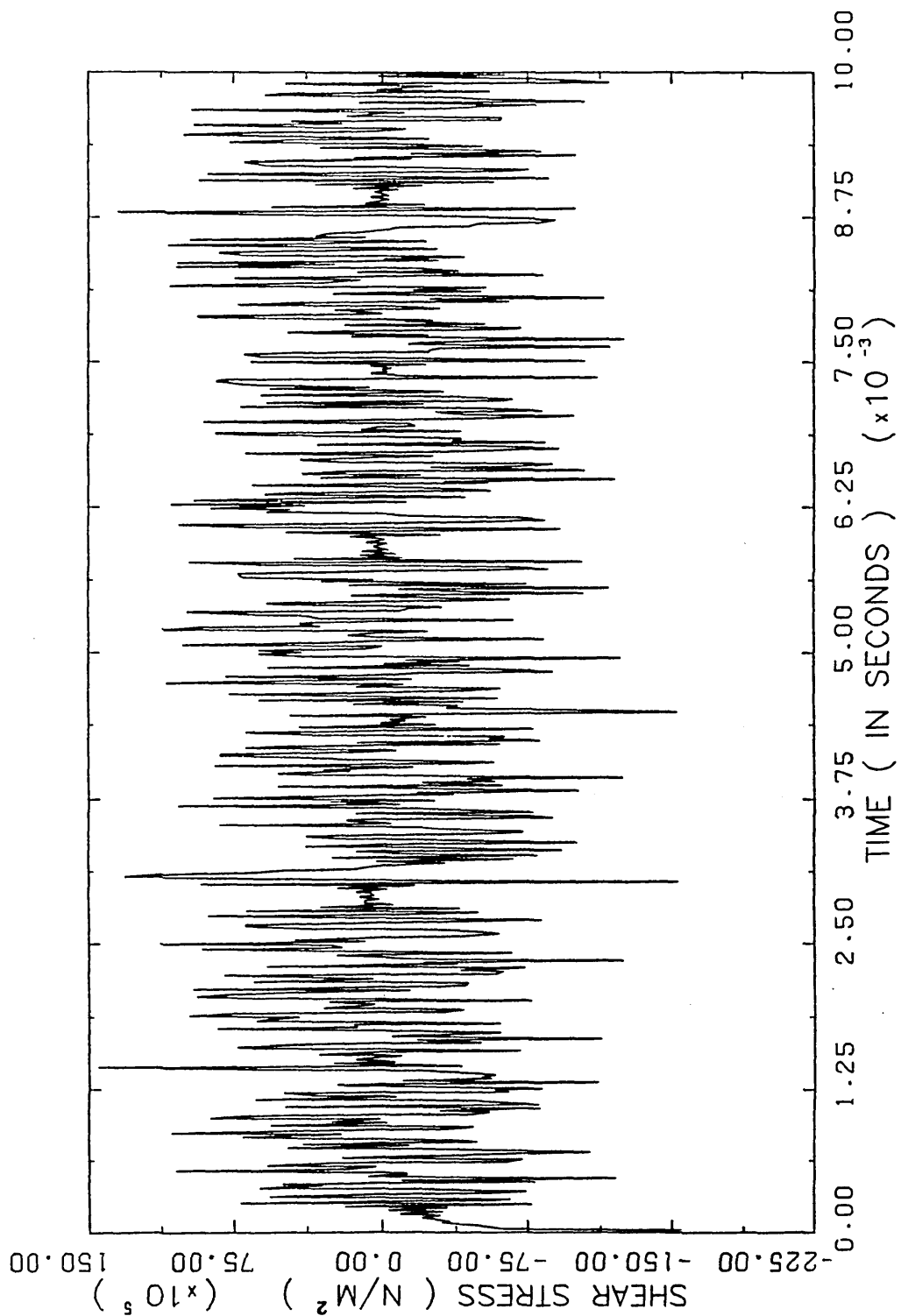


FIG. (3.33) - Centre node shear stress history for a simply-supported beam with an attached centre mass subject to an initial step-velocity excitation, using the Bernoulli-Euler theory and a modal approach.

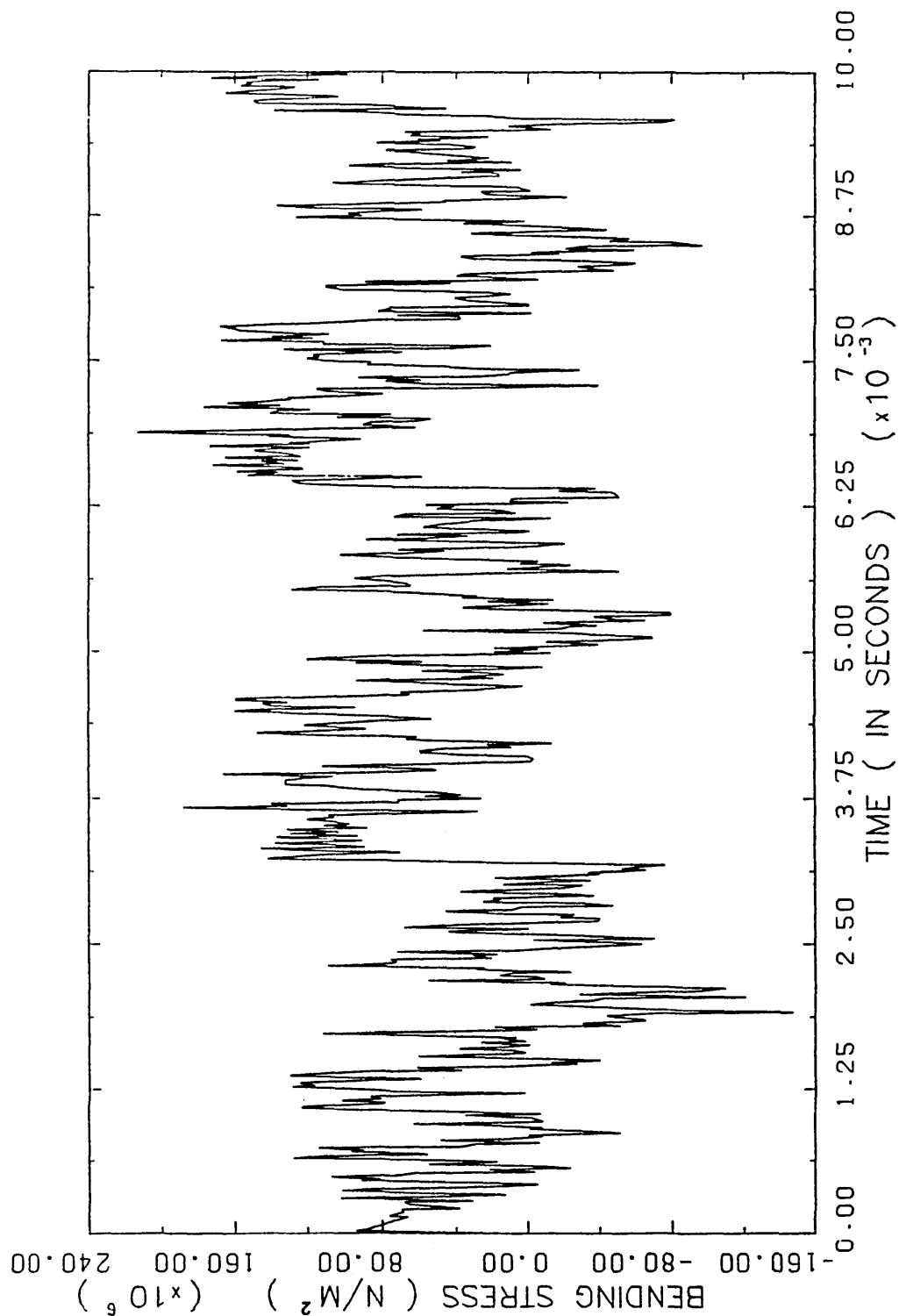


FIG. (3.34) - Centre node bending stress history for a simply-supported beam with an attached centre mass subject to an initial step-velocity excitation, using the FE analysis and Hermitian beam elements.

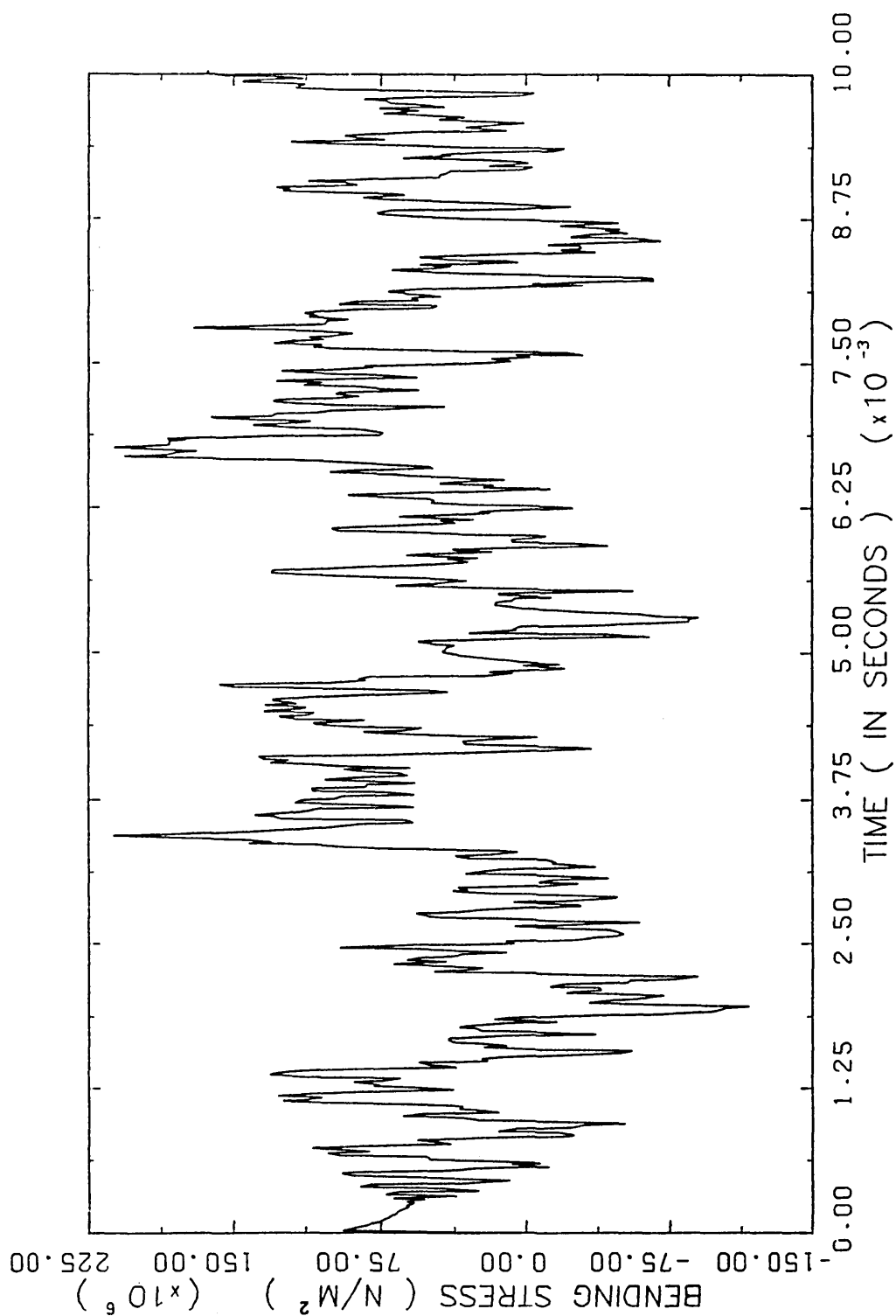


FIG. (3.35) - Centre node bending stress history for a simply-supported beam with an attached centre mass subject to an initial step-velocity excitation, using the FE analysis and Timoshenko beam elements.

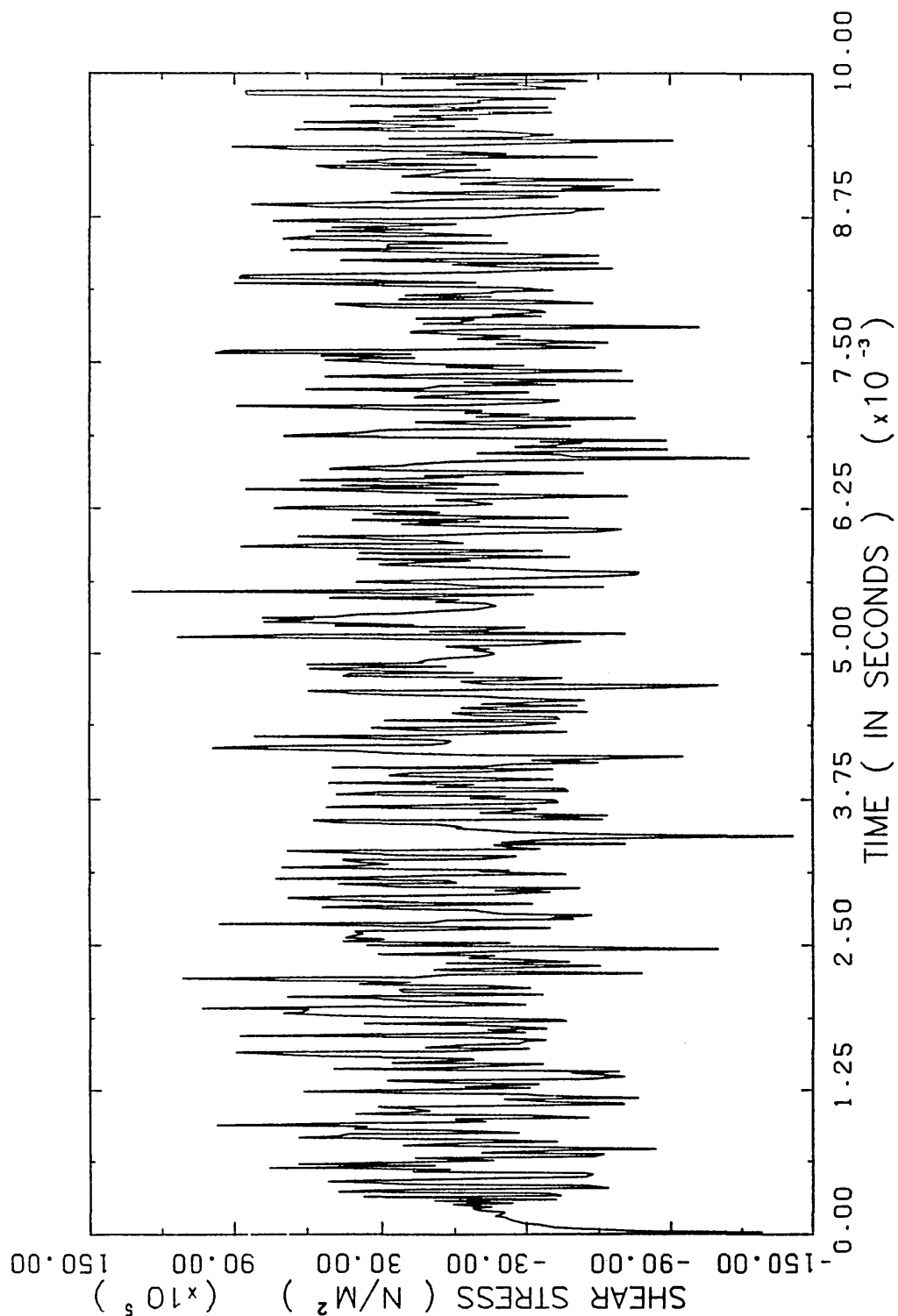


FIG. (3.36) - Centre node shear stress history for a simply-supported beam with an attached centre mass subject to an initial step-velocity excitation, using the FE analysis and Timoshenko beam elements.

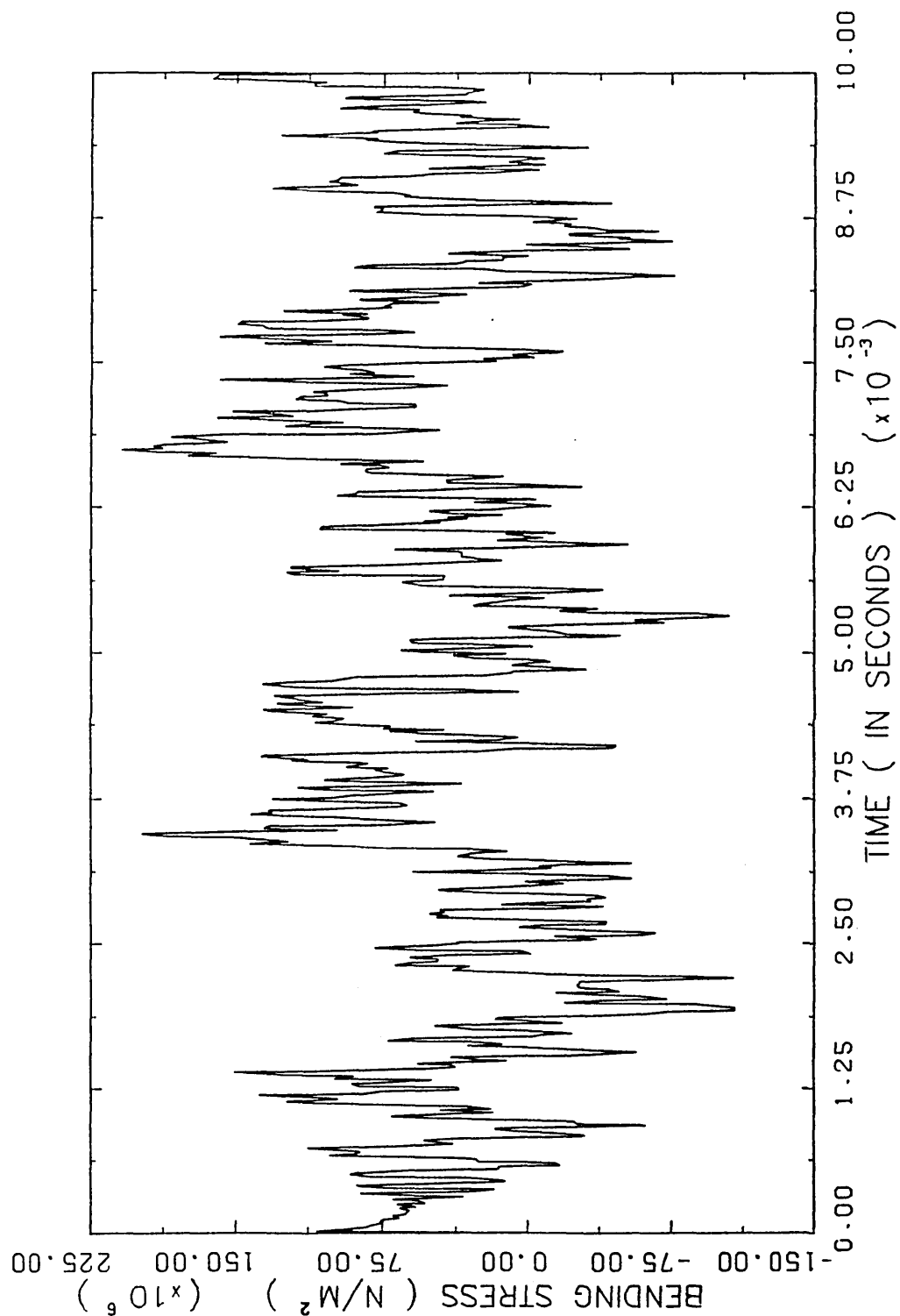


FIG. (3.37) - Centre node bending stress history for a simply-supported beam with an attached centre mass subject to an initial step-velocity excitation, using the FE analysis and cubic Mindlin beam elements.

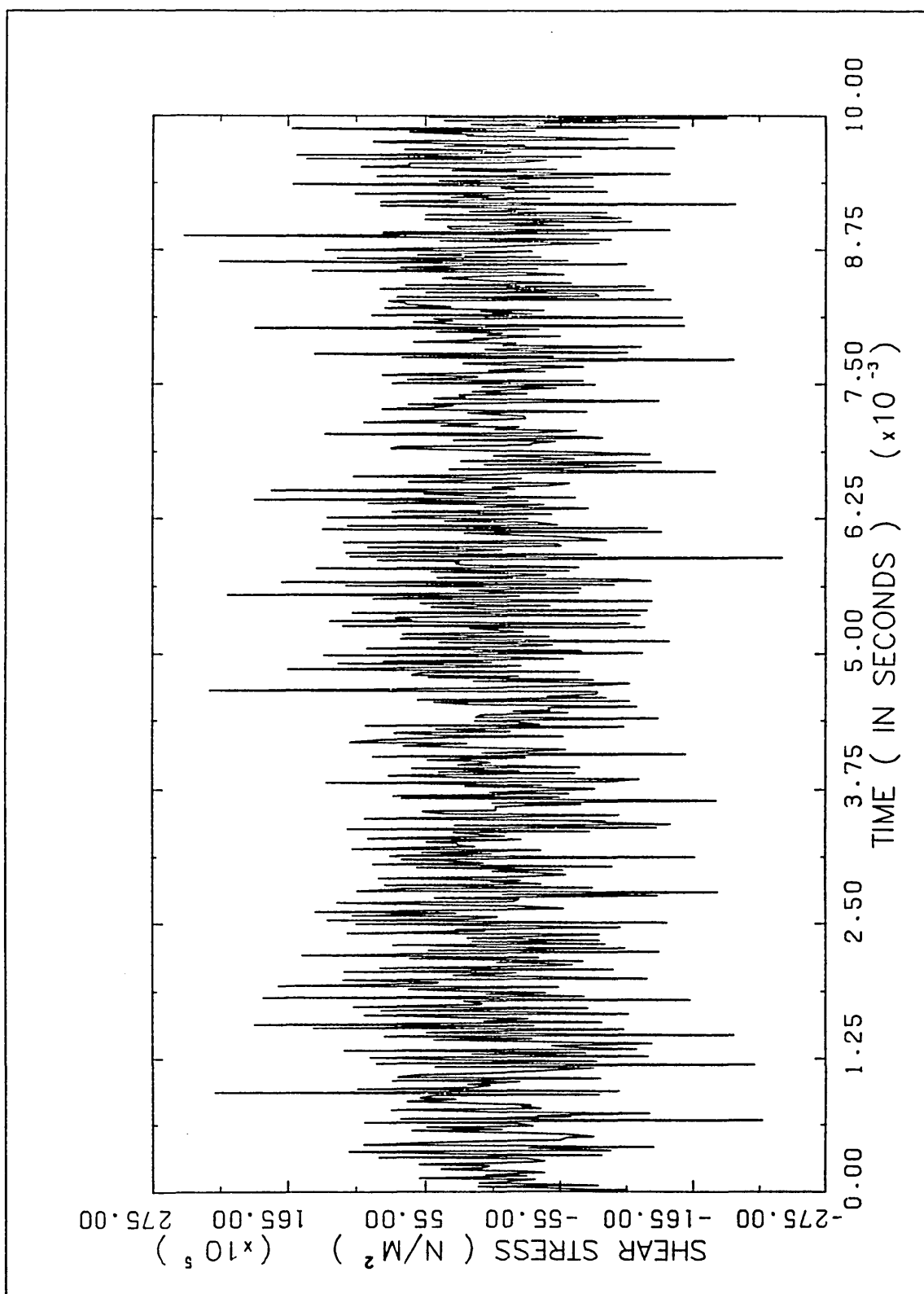


FIG. (3.38) - Centre node shear stress history for a simply-supported beam with an attached centre mass subject to an initial step-velocity excitation, using the FE analysis and cubic Mindlin beam elements.

**TABLES, PHOTOGRAPHS
AND FIGURES**

— for Chapter (4) —

TABLE (4.1) — Material properties of the metal beam specimen

Material : Aluminium alloy DTD-5054-TF (B.A.-703)
Composition : 0.5% Cu , 2.7% Mg , 5.7% Zn , 0.16% Cr
Treatment : Solution treated , aged

Mechanical Properties :

Elastic modulus	:	72.0	GN/m ²
Shear modulus	:	27.0	GN/m ²
Poisson's ratio	:	0.33	
Density	:	2720.0	Kg/m ³
Ultimate tensile strength	:	520.0	MN/m ²
0.1% Proof stress	:	450.0	MN/m ²

TABLE (4.2) — Strain gauge details

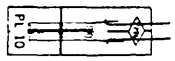
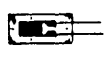
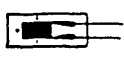
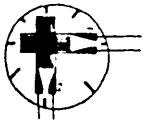
Gauge pattern				
Gauge type	WIRE	FOIL	FOIL	FOIL
Gauge designation	<i>PL-10</i>	<i>FLA-2</i>	<i>FLA-3</i>	<i>FCA-6</i>
Gauge length (mm)	10	2	3	6
Gauge width (mm)	3	1.5	1.8	2.4
Gauge base (mm)	21 × 7	6.5 × 3	9 × 3.5	ø 14
Strain limit (%)	2	3	3	3
Nominal resistance (Ω^*)	120+0.3	120+0.3	120+0.3	120+0.5
Gauge factor (approx.)	2.1	2.1	2.1	2.1

TABLE (4.3) — Frequency and energy contents of two extreme test cases as calculated from modal synthesis

a) Long beam — (Test case MT4) :

MODE	FREQUENCY (Hz)	% ENERGY
1	16.24	49.54
2	160.80	20.93
3	468.88	11.03
4	946.26	6.62
5	1594.66	4.31

b) Short beam — (Test case MT38) :

MODE	FREQUENCY (Hz)	% ENERGY
1	351.15	90.28
2	6798.05	5.59
3	21585.10	1.87
4	44812.43	0.91
5	76483.94	0.53

**TABLE (4.4) — Details of the various impactor assembly springs used
in impact tests on the metal specimen**

a) Die spring :

Material	:	chrome vanadium steel
Wire diameter	:	16.0 mm
Hole diameter	:	32.0 mm
Free length	:	38.1 mm
Max. load	:	4233 N
Spring stiffness	:	371300 N/m

b) Small Belleville disc-washer :

Material	:	alloy steel (50 Cr V4)
Disc diameter	:	50.0 mm
Hole diameter	:	18.3 mm
Disc thickness	:	2.0 mm
Disc height (e_0)	:	1.5 mm
Max. load	:	5594 N

Disc stiffnesses	a)	at $0.25e_0$	:	5106667	N/m
	b)	at $0.50e_0$	:	4516000	N/m
	c)	at $0.75e_0$	:	4056889	N/m
	d)	at $1.00e_0$	:	3729333	N/m

c) Large Belleville disc-washer :

Material	:	alloy steel (50 Cr V4)
Disc diameter	:	70.0 mm
Hole diameter	:	40.5 mm
Disc thickness	:	5.0 mm
Disc height (e_0)	:	1.2 mm
Max. load	:	44471 N

Disc stiffnesses	a)	at $0.25e_0$	:	38460000	N/m
	b)	at $0.50e_0$	:	37860000	N/m
	c)	at $0.75e_0$	:	37392222	N/m
	d)	at $1.00e_0$	:	37059167	N/m

TABLE (4.4) — continued

Note : The compressive stiffness of the various springs listed below were experimentally determined in a compression testing machine under static loading conditions.

CODE	CONFIGURATION	COMPRESSIVE STIFFNESS (N/m)
A	linear die spring	366855
B	6 small <i>Belleville</i> disc-washers stacked in series	721267
C	single small <i>Belleville</i> disc-washer	4462391
D	single large <i>Belleville</i> disc-washer	15936615

TABLE (4.5) — Details of impact test configurations for the metal
beam specimen

TEST CASE	SPRING CODE	BEAM LENGTH (m)	IMPACTOR MASS (Kg)	IMPACTOR VELOCITY (m/s)
MT1	A	1.40	1.5	2.0
MT2	A	1.40	1.5	3.0
MT3	A	1.40	1.5	4.0
MT4	A	1.40	0.5	3.0
MT5	A	1.40	1.0	3.0
MT6	A	1.40	2.0	3.0
MT7	A	1.40	3.0	3.0
MT8	A	1.40	6.0	2.0
MT9	A	1.40	1.0	4.43
MT10	A	1.40	1.375	3.777
MT11	A	1.40	1.375	3.231
MT12	A	1.40	1.75	3.348
MT13	A	1.40	1.75	2.531
MT14	A	1.30	1.0	3.0
MT15	A	1.20	1.0	3.0
MT16	A	1.10	1.0	3.0
MT17	A	1.10	1.375	3.777
MT18	A	1.10	1.375	3.231
MT19	A	1.00	1.0	3.0
MT20	A	1.00	1.75	3.0
MT21	B	0.90	1.0	3.0

TABLE (4.5) — continued

TEST CASE	SPRING CODE	BEAM LENGTH (m)	IMPACTOR MASS (Kg)	IMPACTOR VELOCITY (m/s)
MT22	B	0.80	1.0	2.0
MT23	B	0.80	1.0	3.0
MT24	B	0.80	1.0	4.43
MT25	B	0.80	0.75	3.0
MT26	B	0.80	1.5	3.0
MT27	B	0.80	2.0	2.5
MT28	B	0.80	3.0	2.0
MT29	B	0.80	6.25	1.5
MT30	C	0.70	1.0	3.0
MT31	C	0.60	1.0	3.0
MT32	C	0.50	1.0	3.0
MT33	C	0.40	1.0	3.0
MT34	C	0.40	1.5	2.0
MT35	C	0.40	3.0	1.5
MT36	D	0.20	0.75	1.5
MT37	D	0.20	0.75	2.0
MT38	D	0.20	0.75	2.5
MT39	D	0.20	1.0	1.5
MT40	D	0.20	1.5	1.5
MT41	D	0.20	2.0	1.0
MT42	D	0.20	3.0	1.0
MT43	D	0.20	6.25	0.7

TABLE (4.6) — Material properties of the composite specimens

Material : *Fibredux 914C-XAS-5-34*
Composition : High tensile treated *GRAFIL XA-S* fibres
in *FIBREDUX 914* epoxy resin matrix to give
a 60% fibre content by volume when cured

Mechanical Properties at 20°C :

0° Tensile modulus	:	135.0	GN/m ²
90° Tensile modulus	:	8.0	GN/m ²
0° Flexural modulus	:	125.0	GN/m ²
90° Flexural modulus	:	7.5	GN/m ²
Shear modulus	:	5.9	GN/m ²
Major Poisson's ratio	:	0.31	
Mean density	:	1500.0	Kg/m ³
0° Ultimate tensile strength	:	1650.0	MN/m ²
90° Ultimate tensile strength	:	79.0	MN/m ²
0° Ultimate compression strength	:	1350.0	MN/m ²
90° Ultimate compression strength	:	230.0	MN/m ²
0° Ultimate flexural strength	:	1900.0	MN/m ²
90° Ultimate flexural strength	:	110.0	MN/m ²
Interlaminar shear strength	:	78.0	MN/m ²

Note : The above mechanical properties are affected by the curing cycle employed, the test temperature as well as ageing and humidity conditions. For details of the effect of these variations refer to :

Information Sheet No. FTD819-B, CIBA-GEIGY, (1981)

TABLE (4.7) — Details of impact test configuration for the composite beam specimen (Note : Impactor mass = 0.75 Kg for all the following test cases)

TEST CASE	BEAM LENGTH (m)	DROP HEIGHT (m)	IMPACTOR VELOCITY (m/s)	EXPERIMENTAL OBSERVATIONS
CT1	0.25	0.30	2.426	————
CT2	0.25	0.60	3.431	onset of specimen locking
CT3	0.25	1.00	4.429	locking in deflected position
CT4	0.25	1.50	5.425	locking in deflected position
CT5	0.20	0.24	2.170	————
CT6	0.20	0.50	3.132	————
CT7	0.20	0.80	3.962	onset of specimen locking
CT8	0.20	1.10	4.646	locking in deflected position
CT9	0.20	1.30	5.050	locking in deflected position
CT10	0.15	0.18	1.879	————
CT11	0.15	0.40	2.801	————
CT12	0.15	0.75	3.836	small top surface crack
CT13	0.15	0.90	4.202	top surface crack extended, onset of specimen locking

TABLE (4.7) — continued

TEST CASE	BEAM LENGTH (m)	DROP HEIGHT (m)	IMPACTOR VELOCITY (m/s)	EXPERIMENTAL OBSERVATIONS
CT14	0.10	0.12	1.534	————
CT15	0.10	0.30	2.426	————
CT16	0.10	0.50	3.132	small top surface crack
CT17	0.10	0.65	3.571	top surface crack extended
CT18	0.05	0.06	1.085	————
CT19	0.05	0.15	1.716	————
CT20	0.05	0.25	2.215	small top surface crack
CT21	0.05	0.35	2.621	top surface crack extended

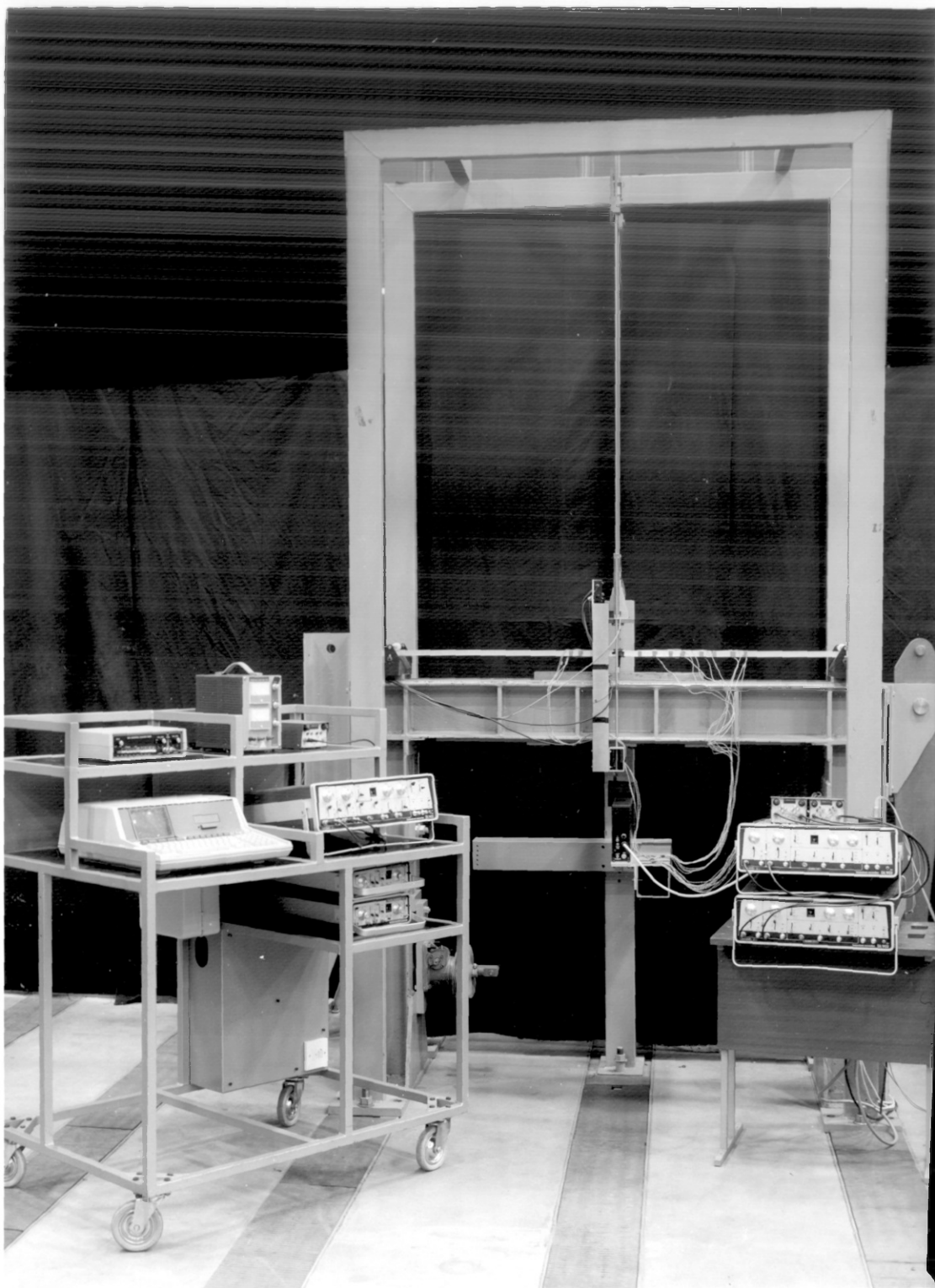


PHOTO. (4.1) - Test rig and instrumentations

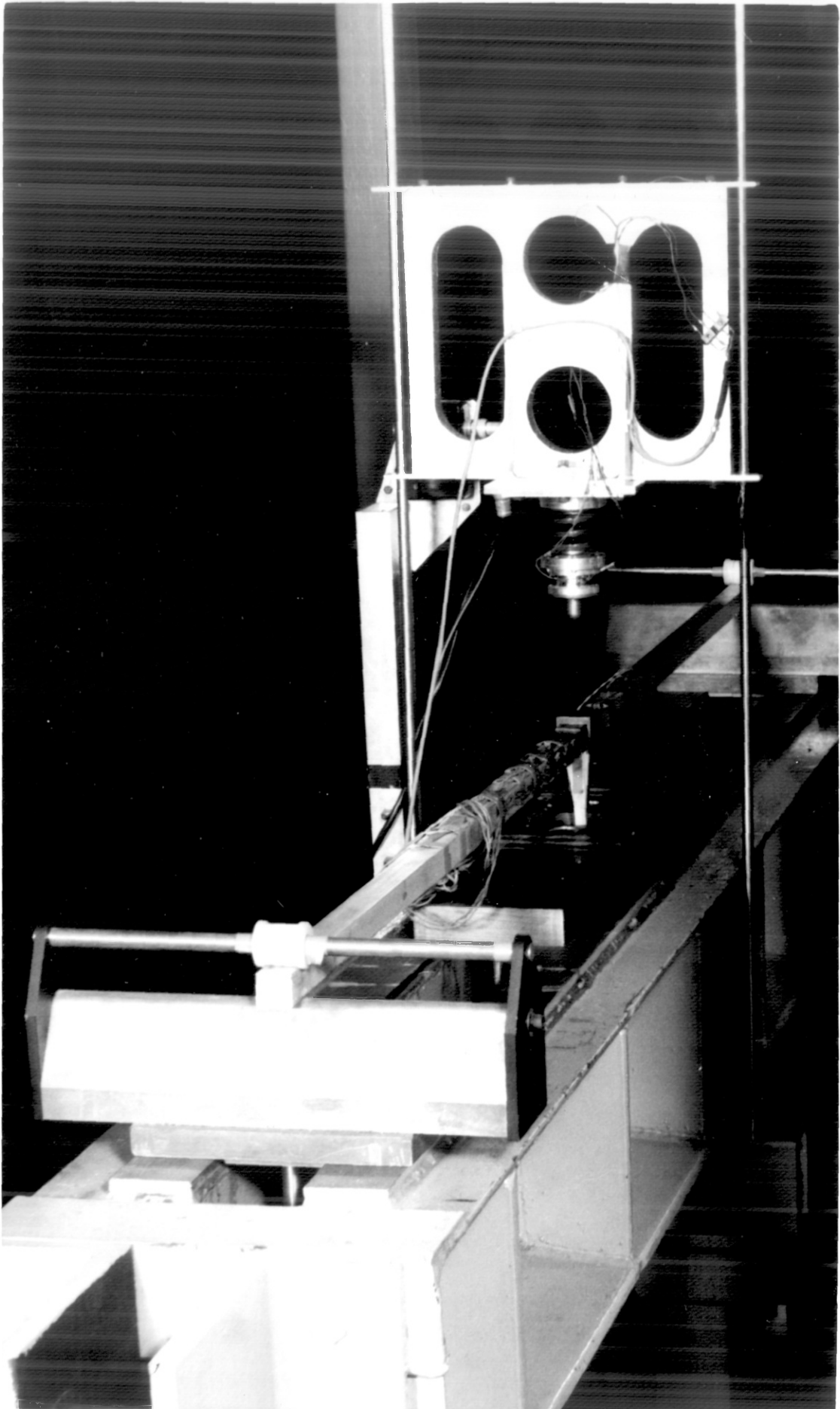


PHOTO. (4.2) - Experimental set-up for the metal specimen

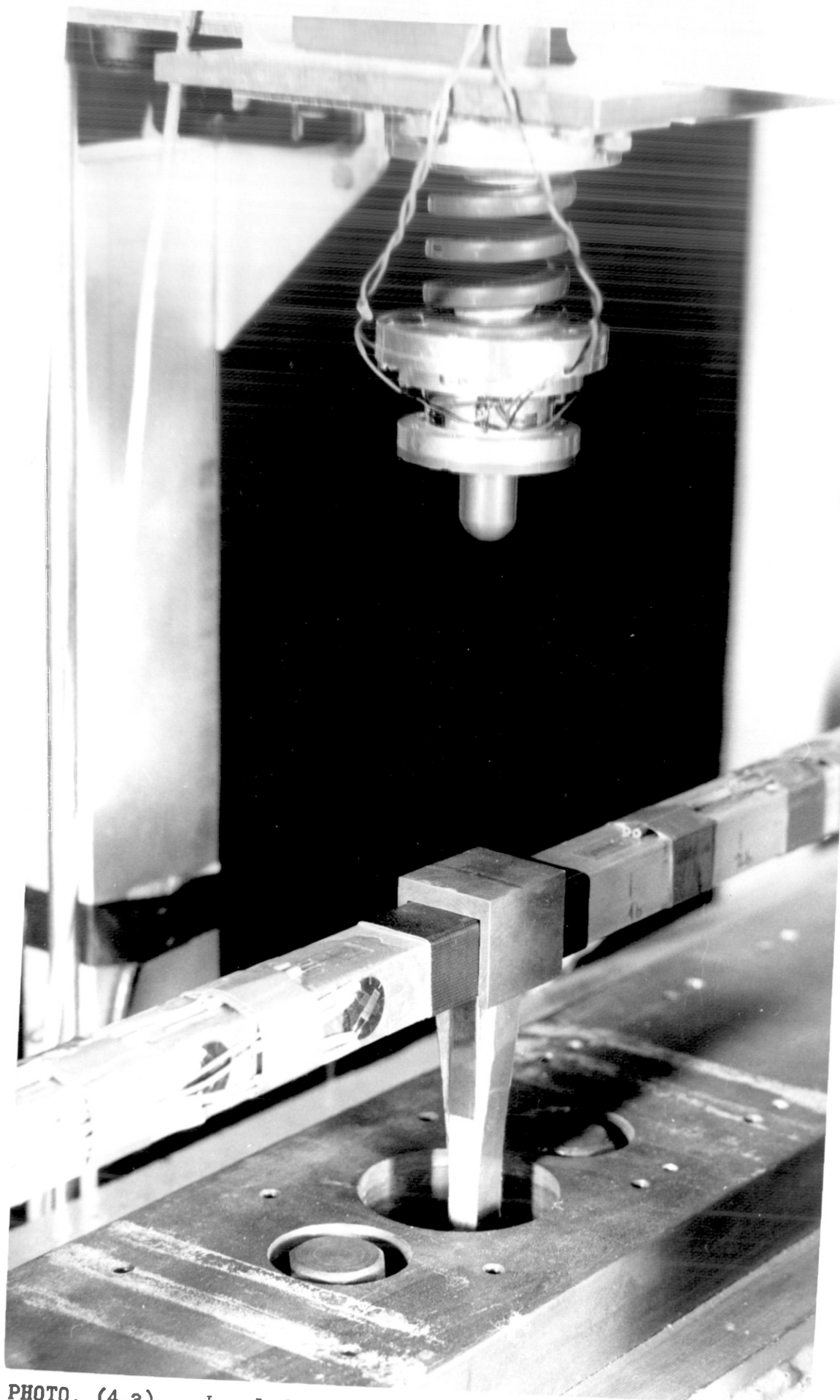


PHOTO. (4.3) - Local details of experimental set-up

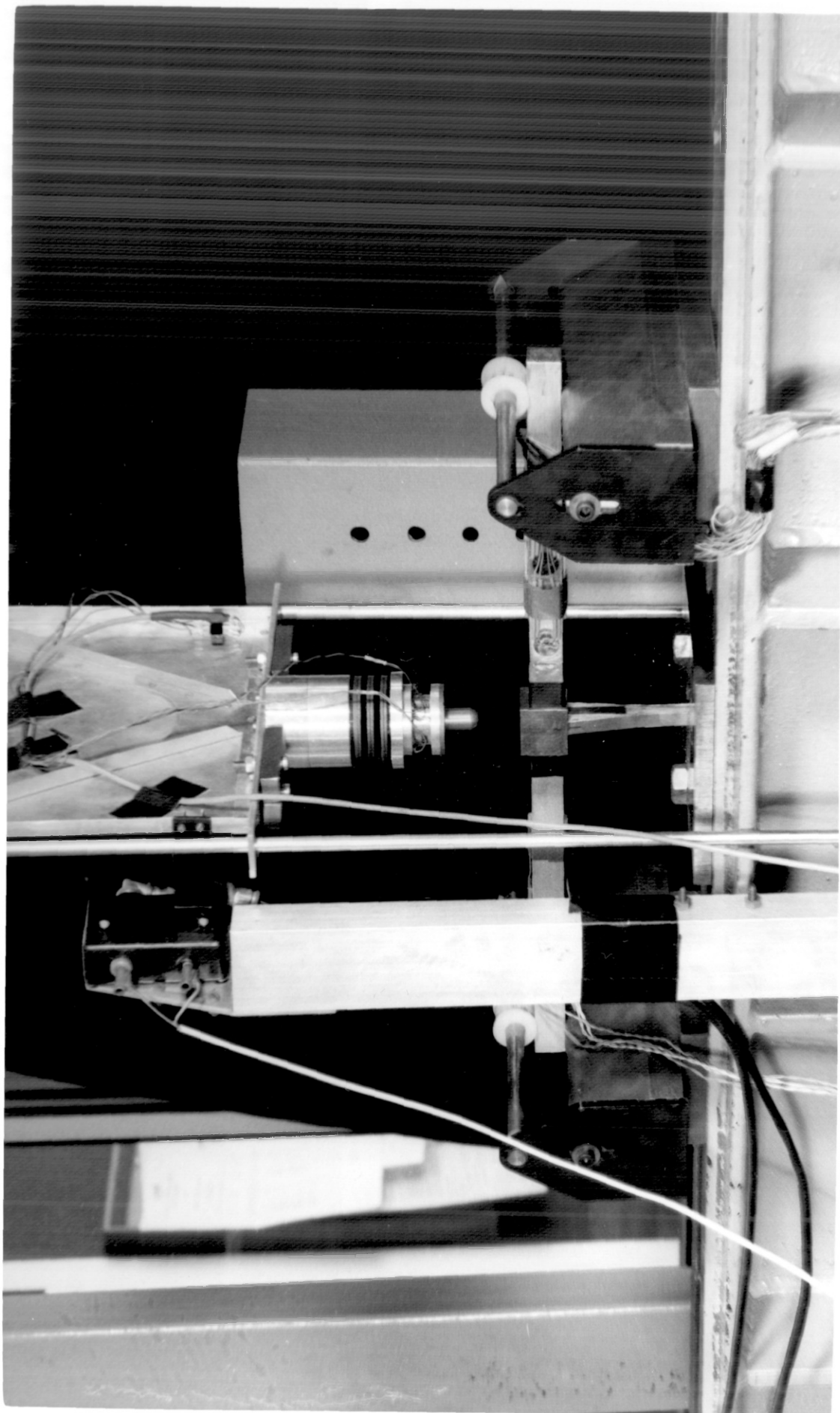


PHOTO. (4.4) - Support mechanism and velocity measuring system

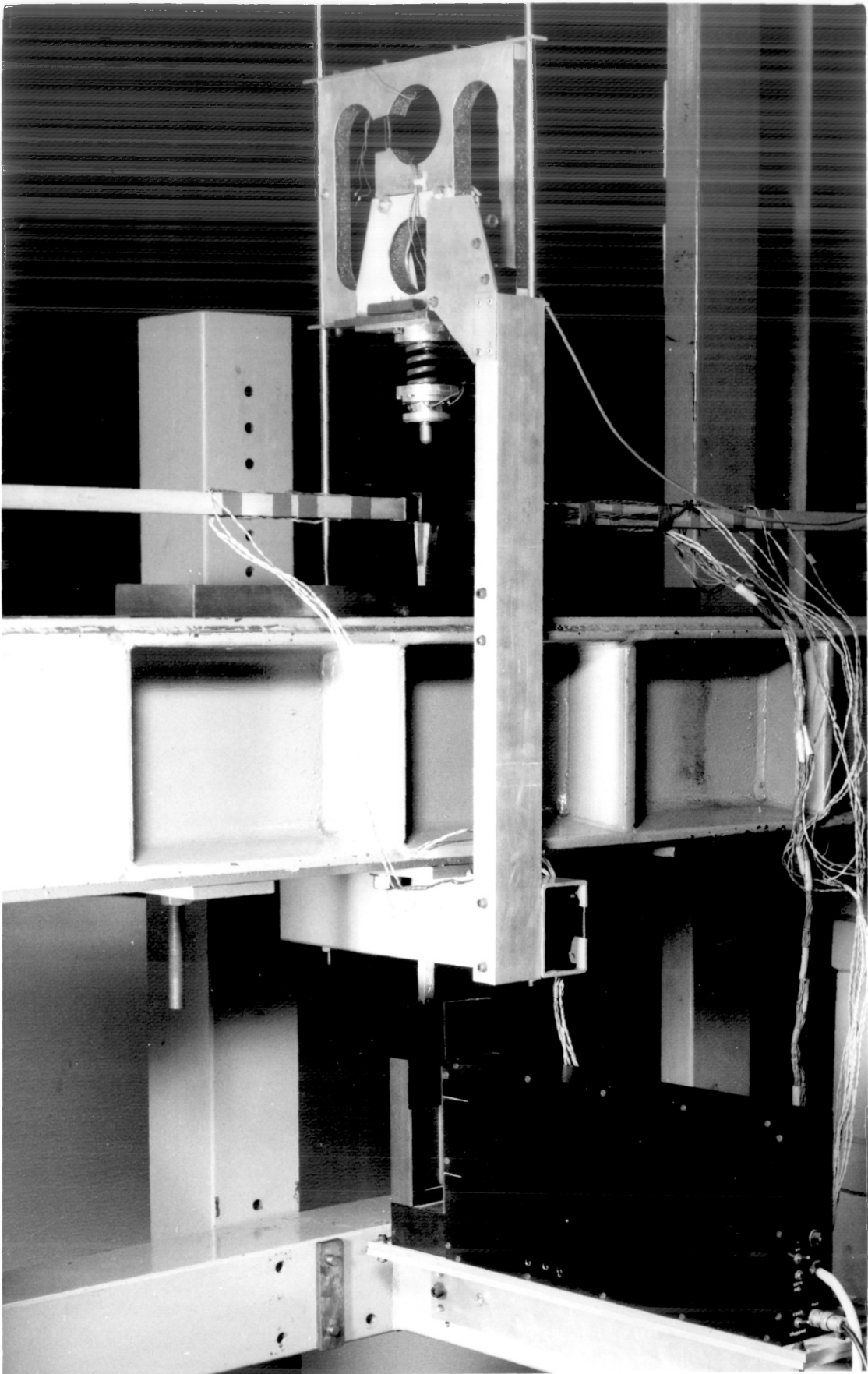


PHOTO. (4.5) - Displacement measuring system

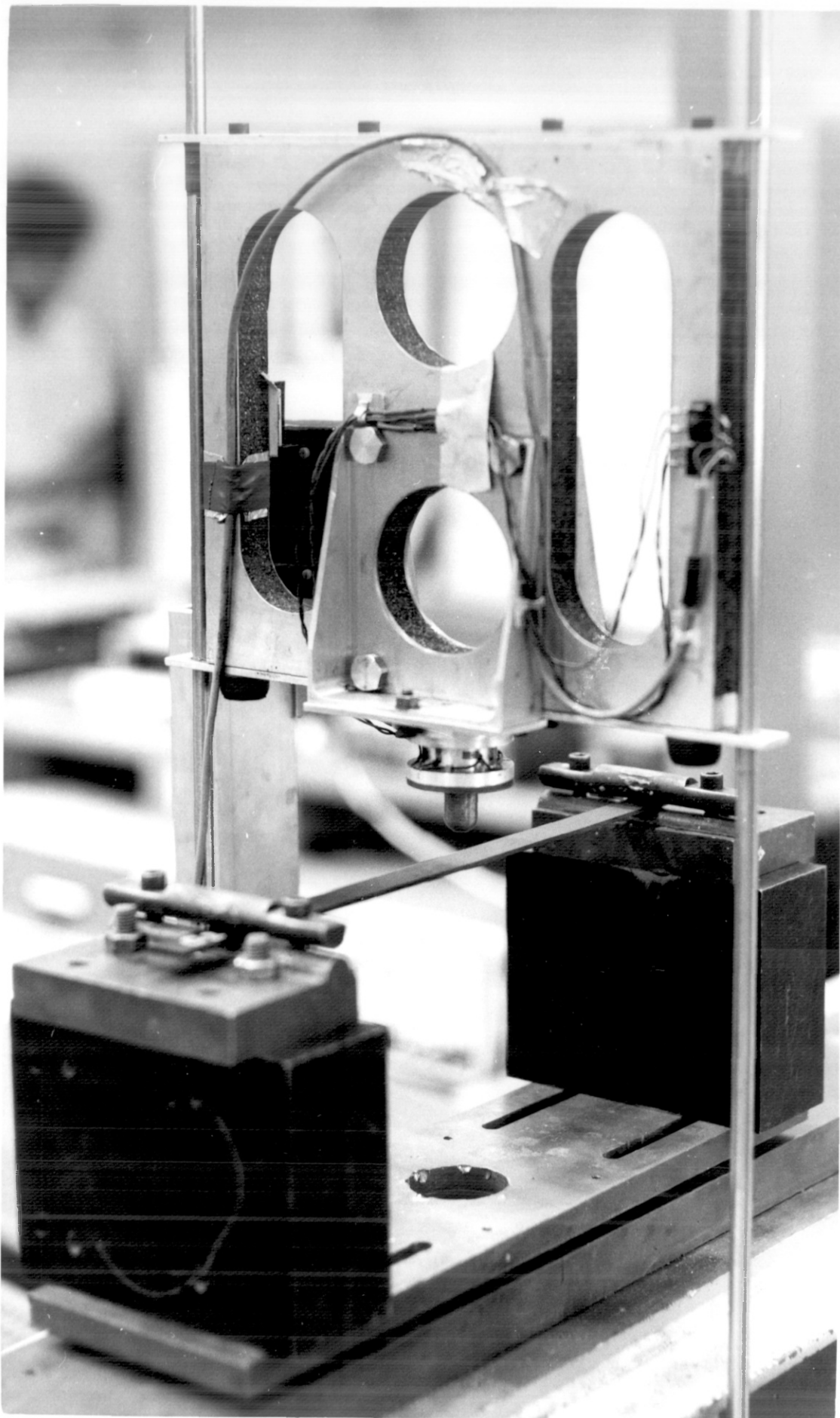


PHOTO. (4.6) - Experimental set-up for composite specimens



PHOTO. (4.7) - Composite beam locked in deflected position

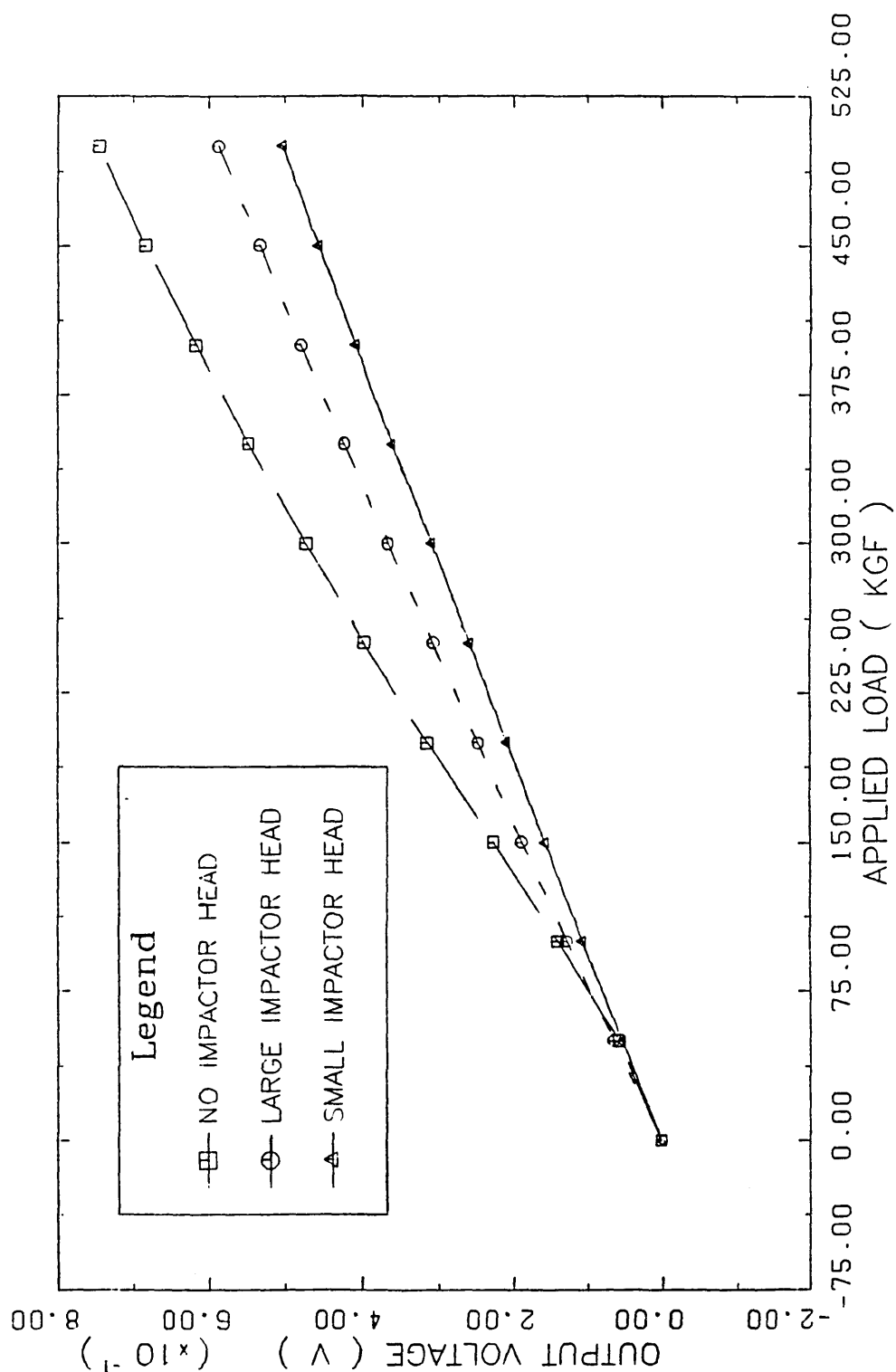


FIG. (4.1) - Load-cell calibration curves for different configurations, experimentally determined in a compression testing machine under static loading conditions.

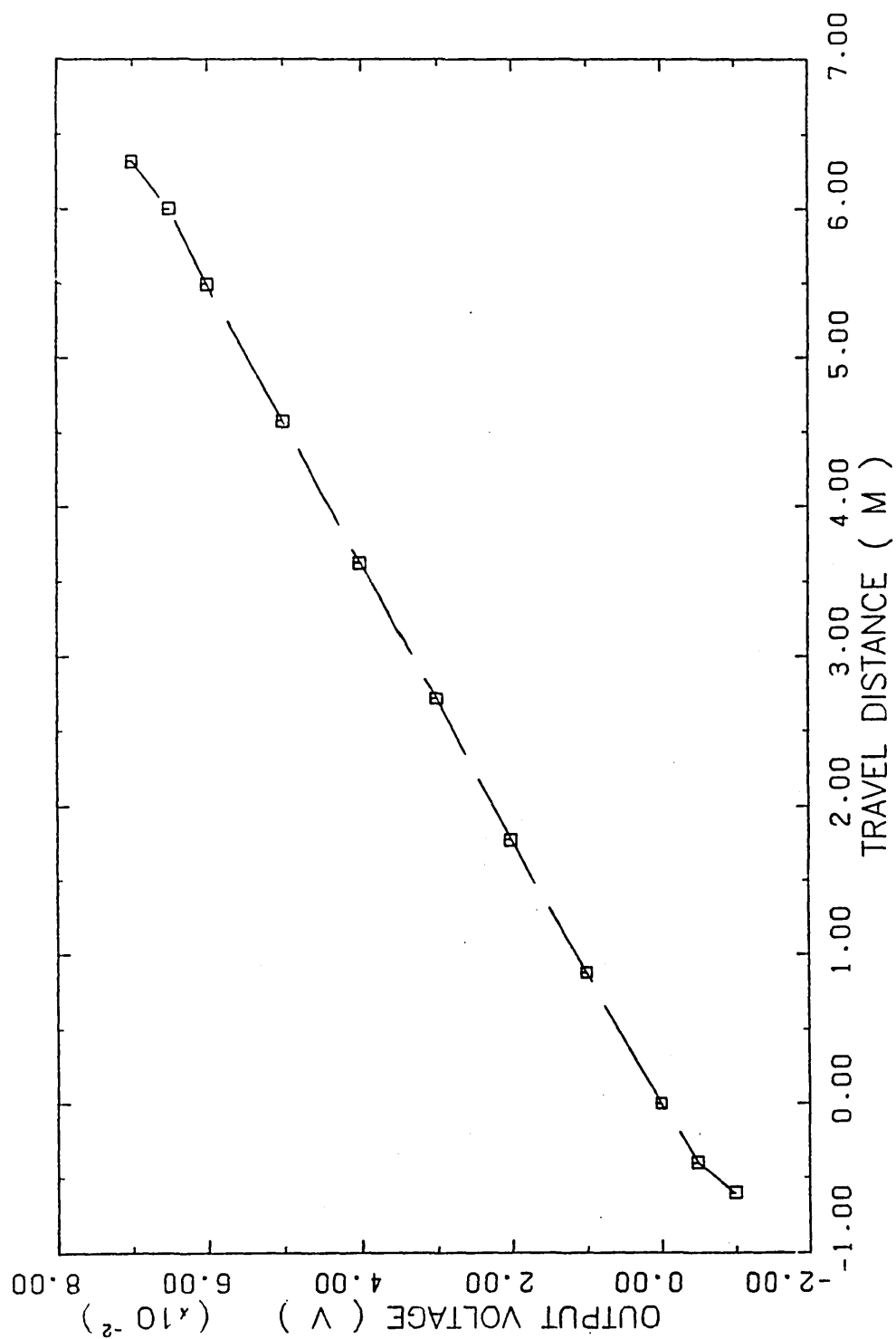


FIG. (4.2) - Calibration curve for the non-contacting displacement transducer.

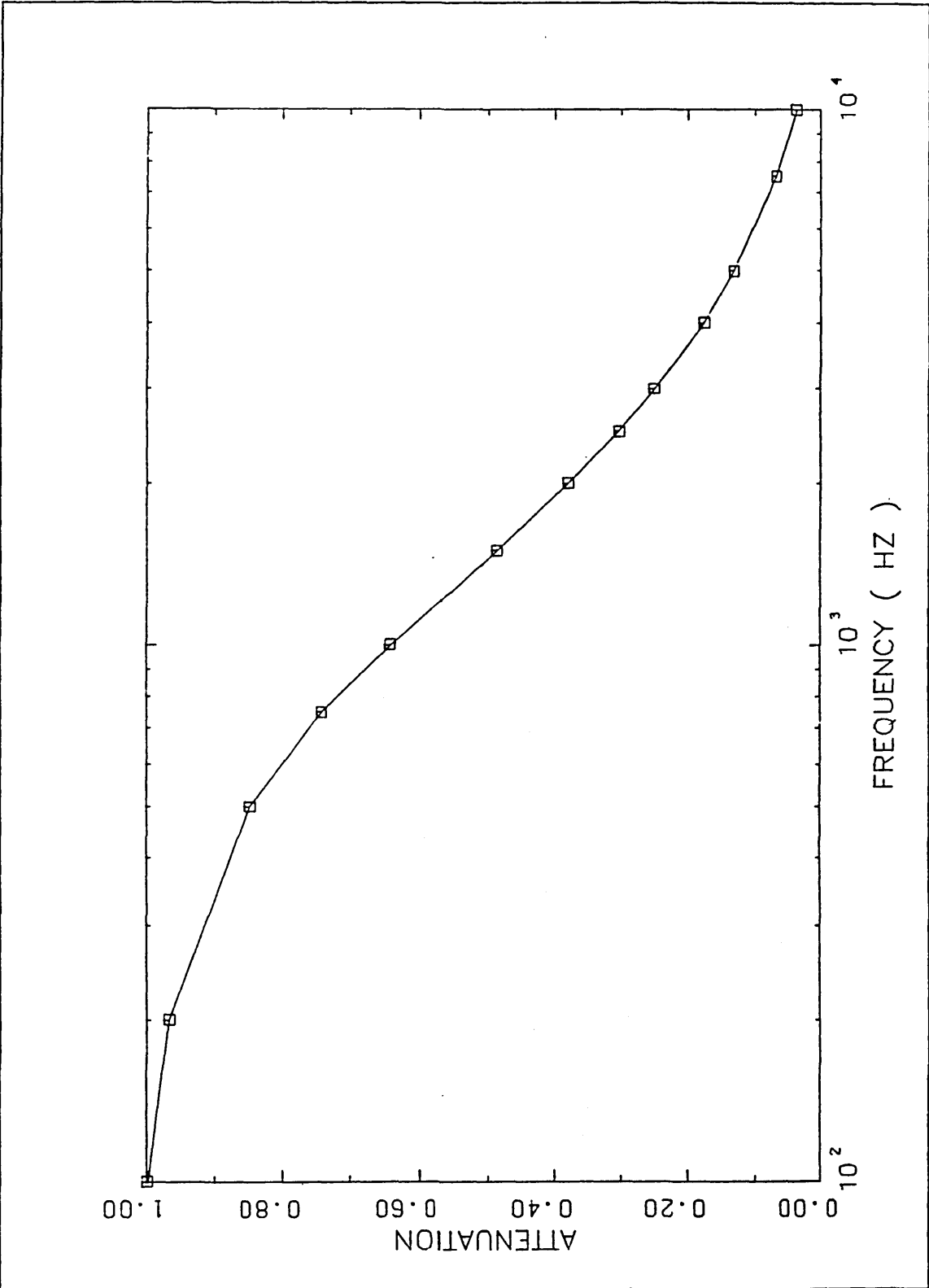


FIG. (4.3) - Attenuation of the input signal with frequency, induced by the strain gauge amplifiers.

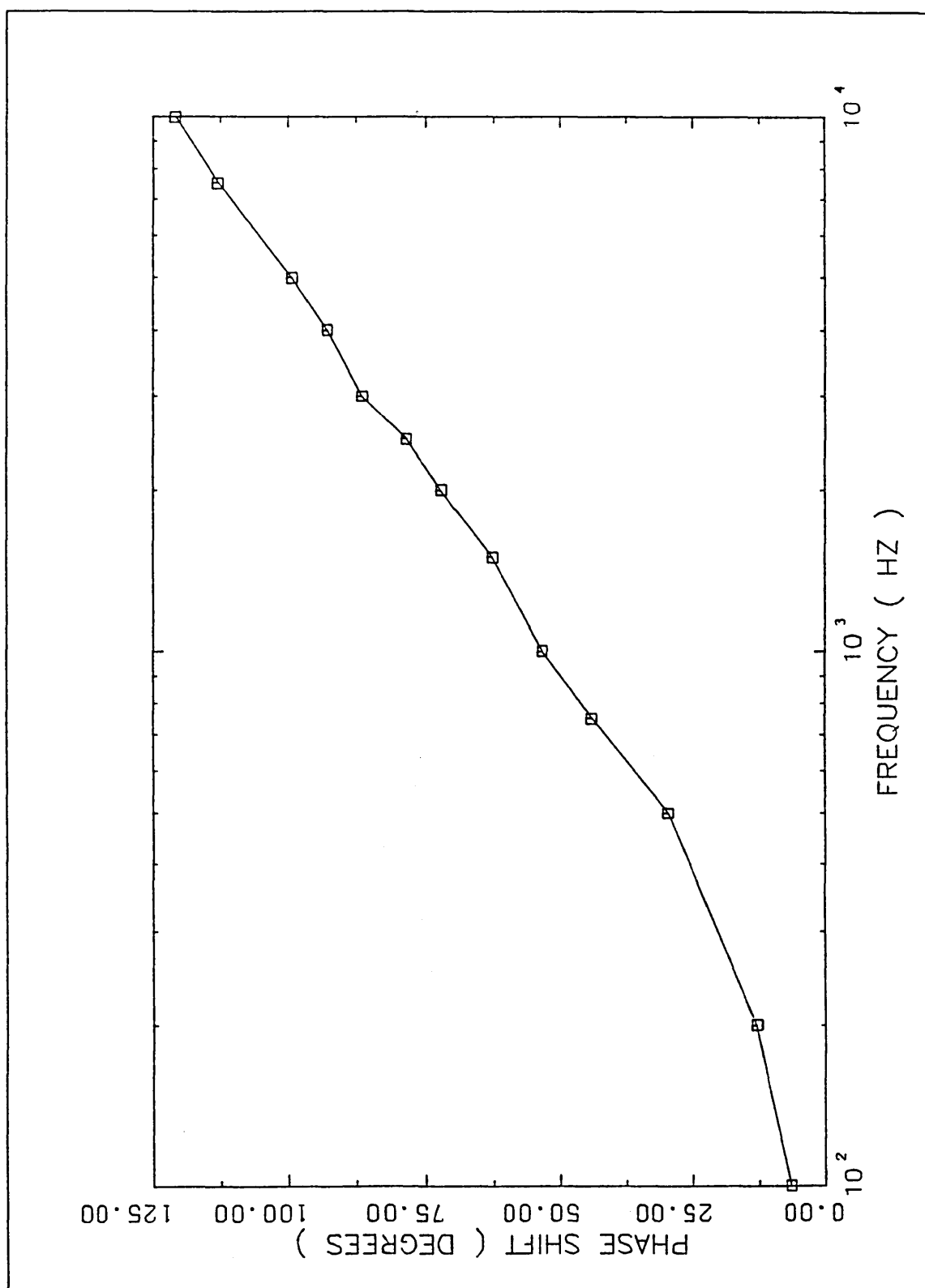


FIG. (4.4) - The change in phase of the input signal with frequency, induced by the strain gauge amplifiers.

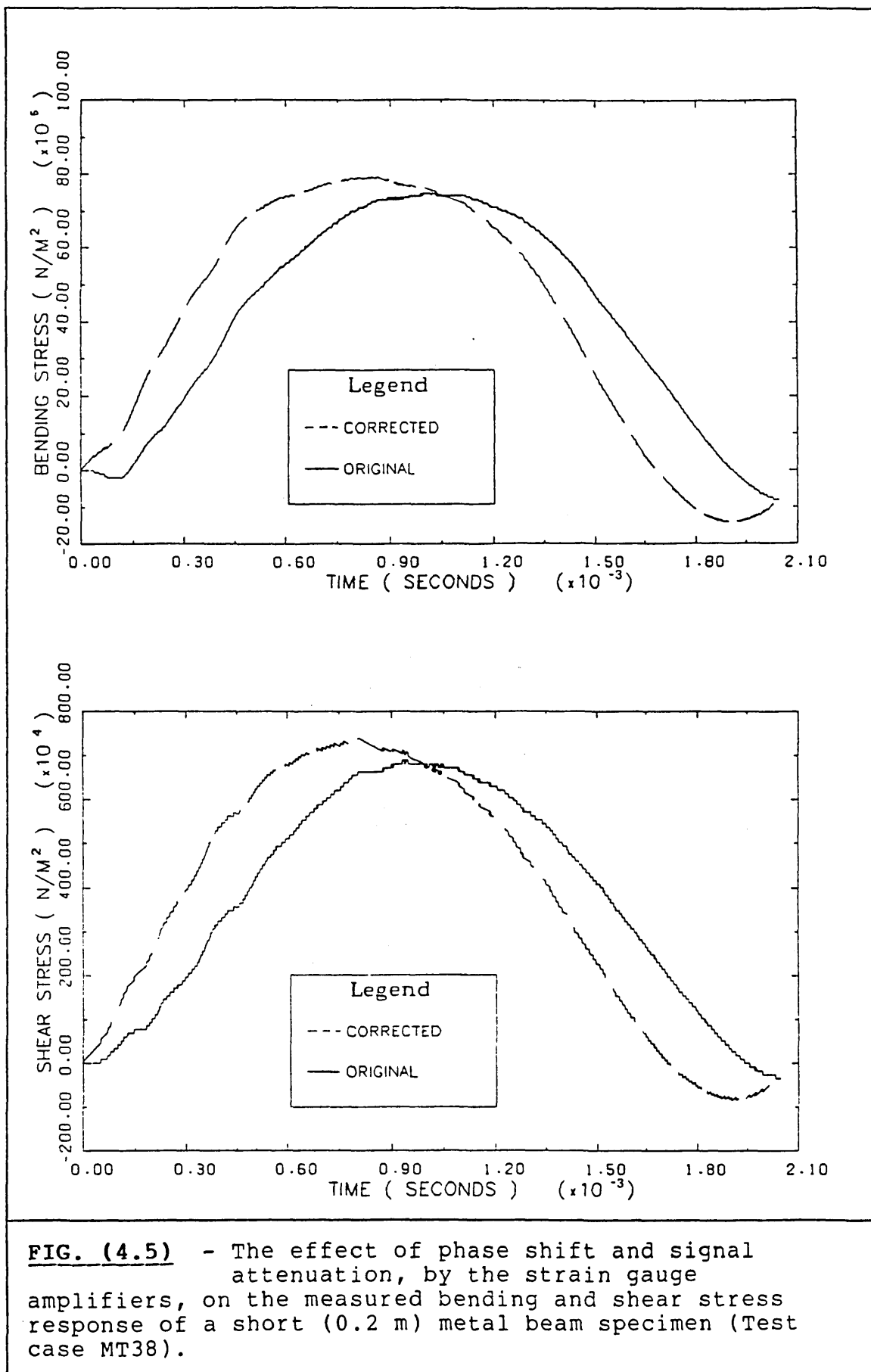


FIG. (4.5) - The effect of phase shift and signal attenuation, by the strain gauge amplifiers, on the measured bending and shear stress response of a short (0.2 m) metal beam specimen (Test case MT38).

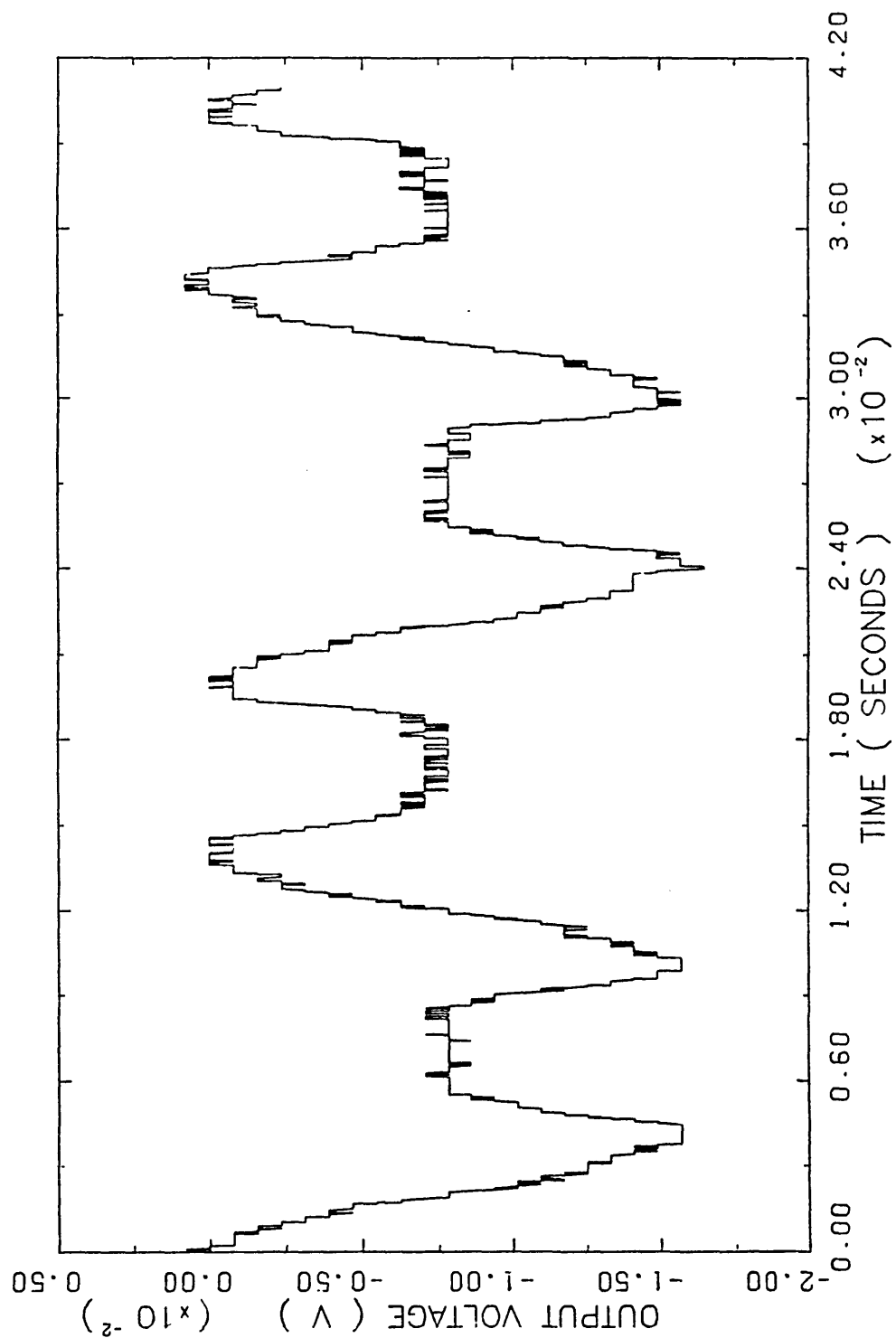


FIG. (4.6) - The characteristic background noise signal of the strain gauge amplifiers.

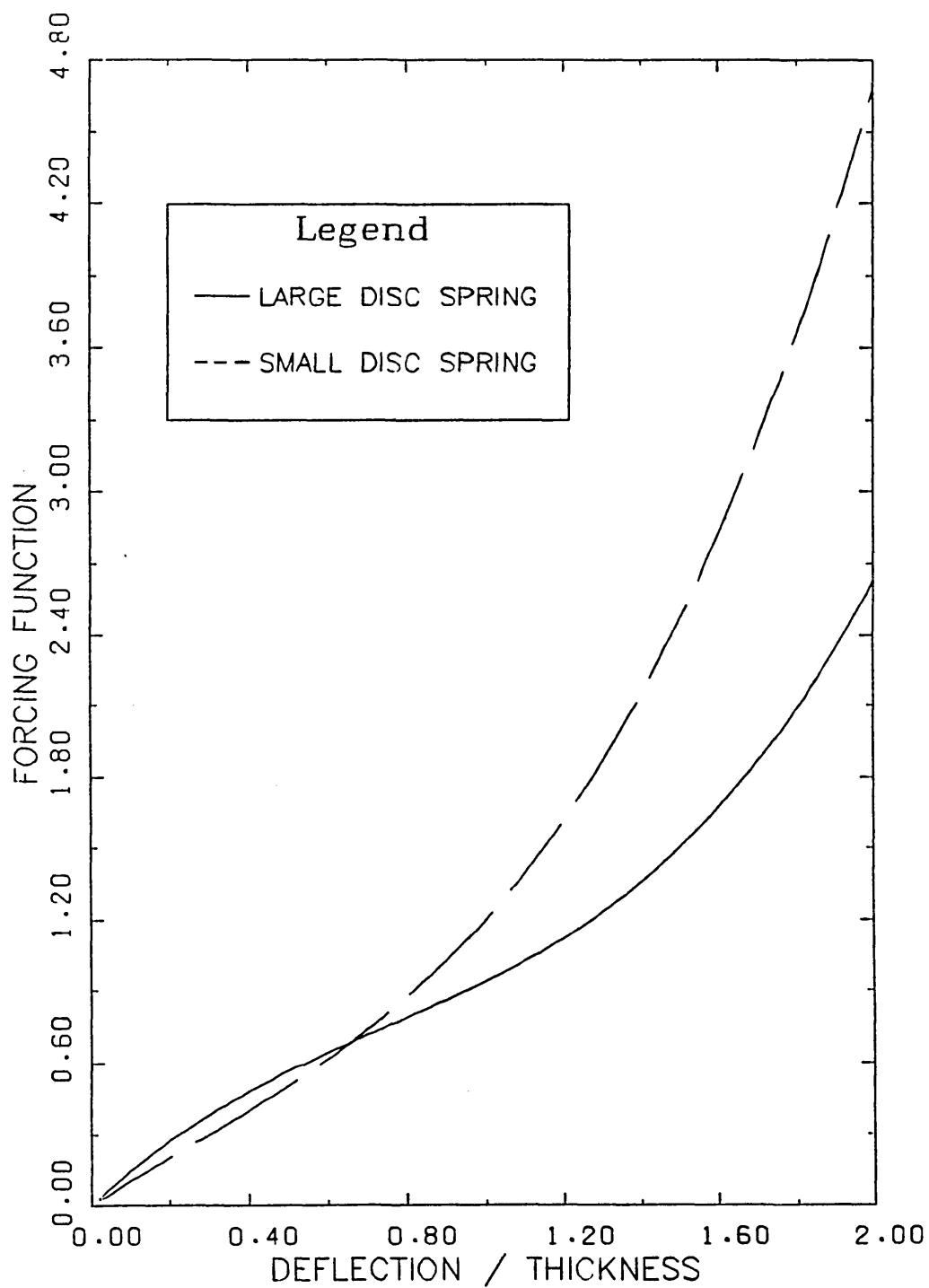


FIG. (4.7) - The force-deflection characteristics of the Belleville disc springs.

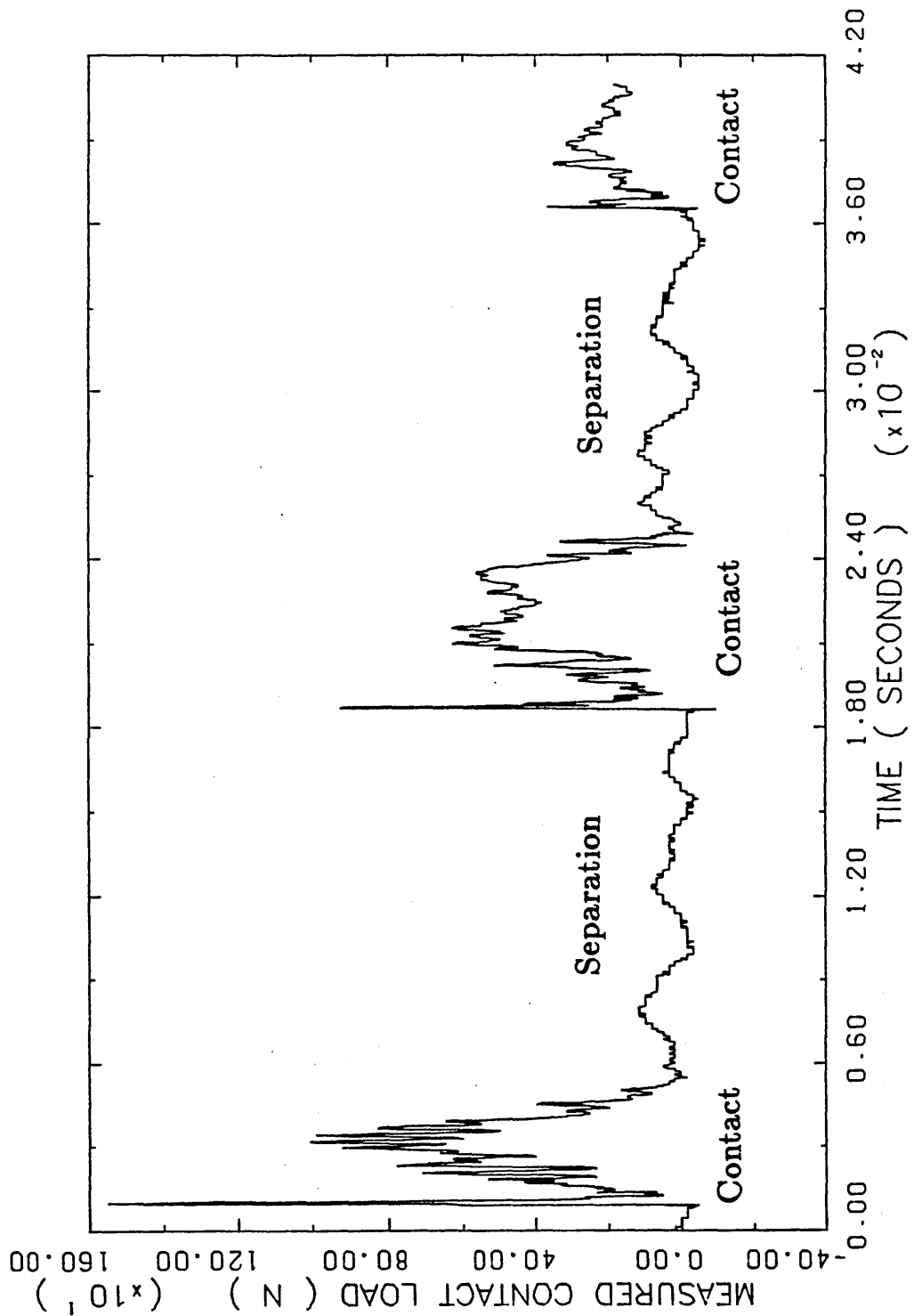


FIG. (4.8) - The experimentally measured variation of contact force with time for test case (MT5), using the long (1.4 m) metal beam specimen.

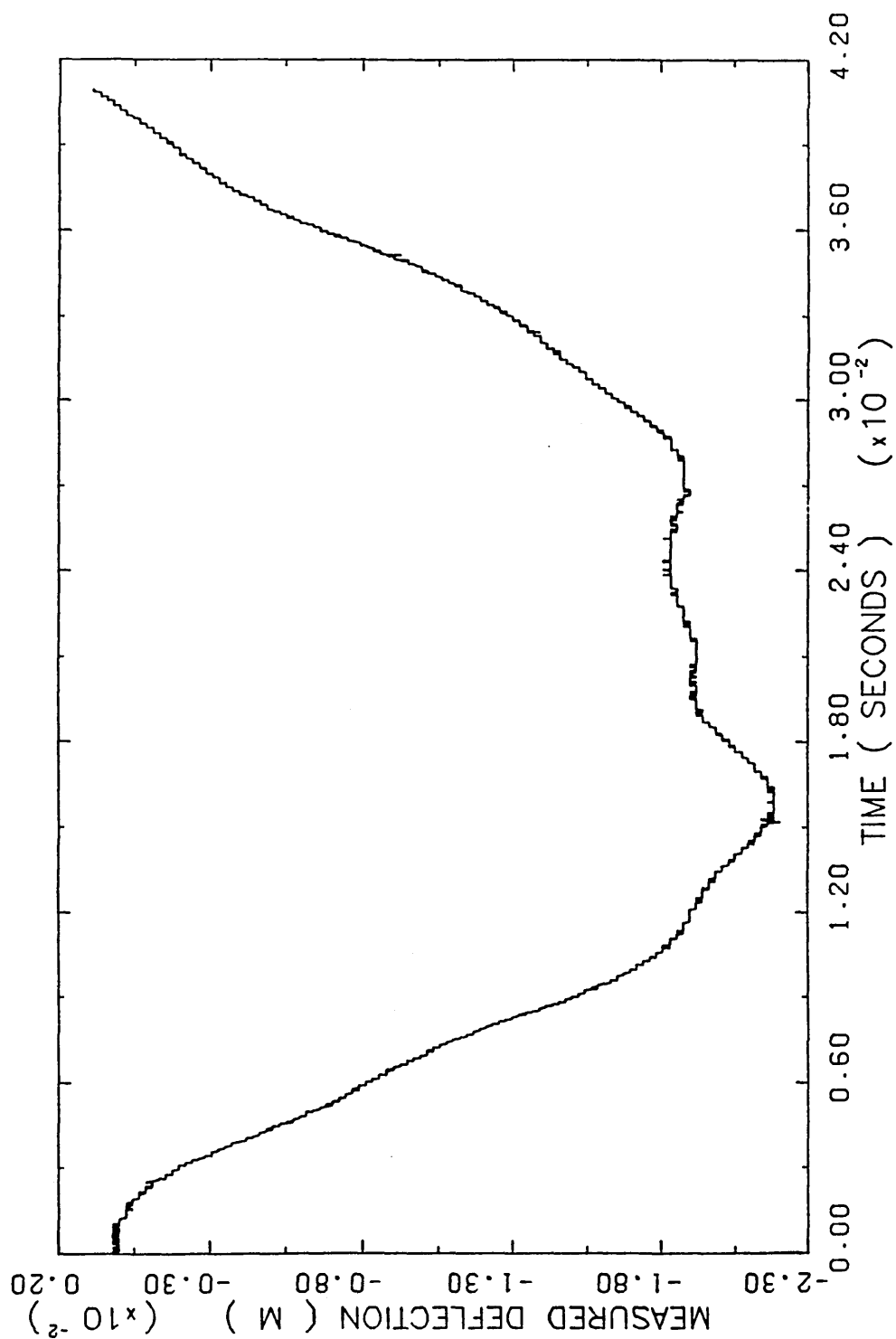


FIG. (4.9) - The experimentally measured variation of displacement with time, at the beam centre, for test case (MT5), using the long (1.4 m) metal beam specimen.

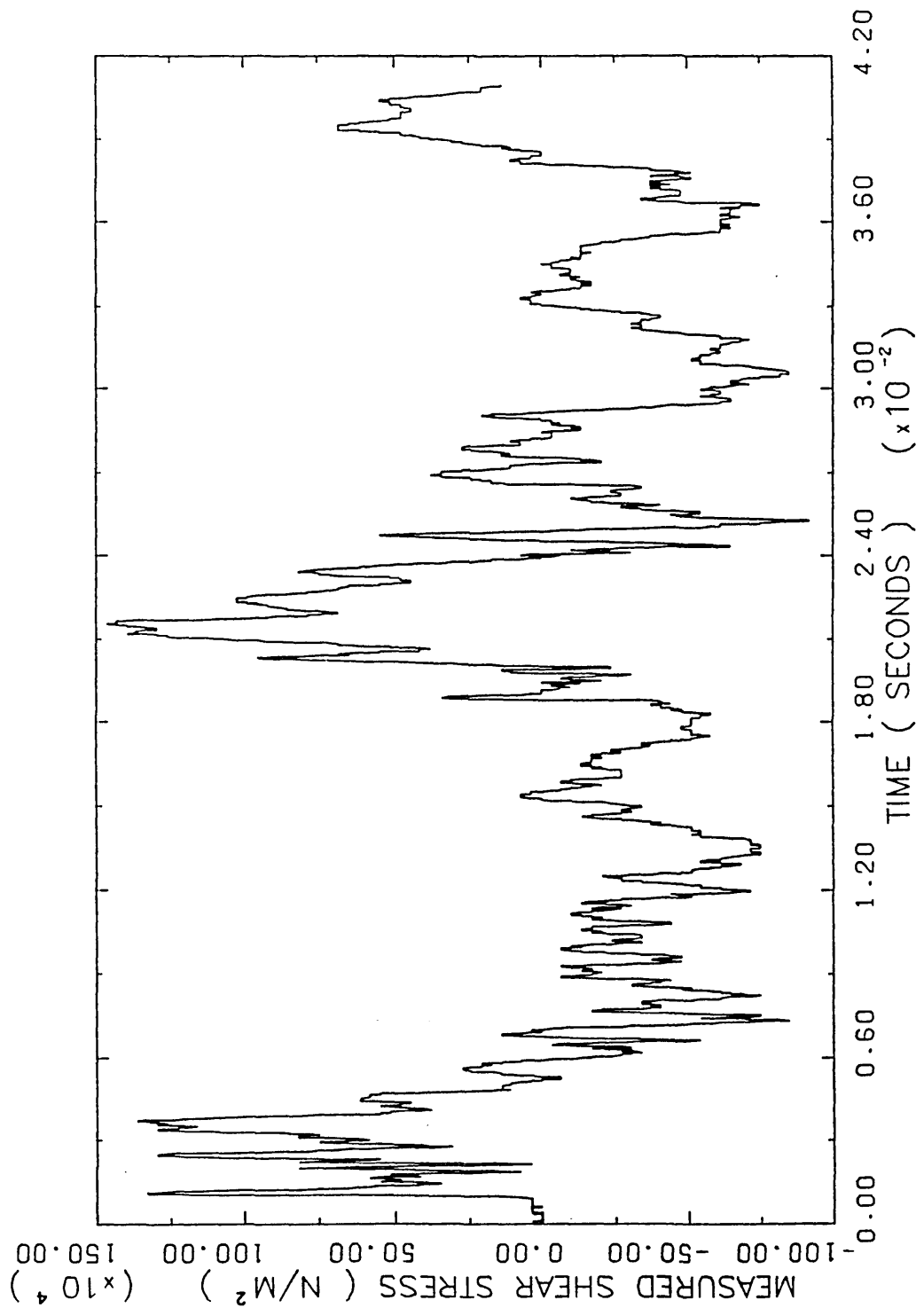


FIG. (4.10) - The experimentally measured variation of shear stress with time, at 5 cm from the point of impact, for test case (MT5), using the long (1.4 m) metal beam specimen.

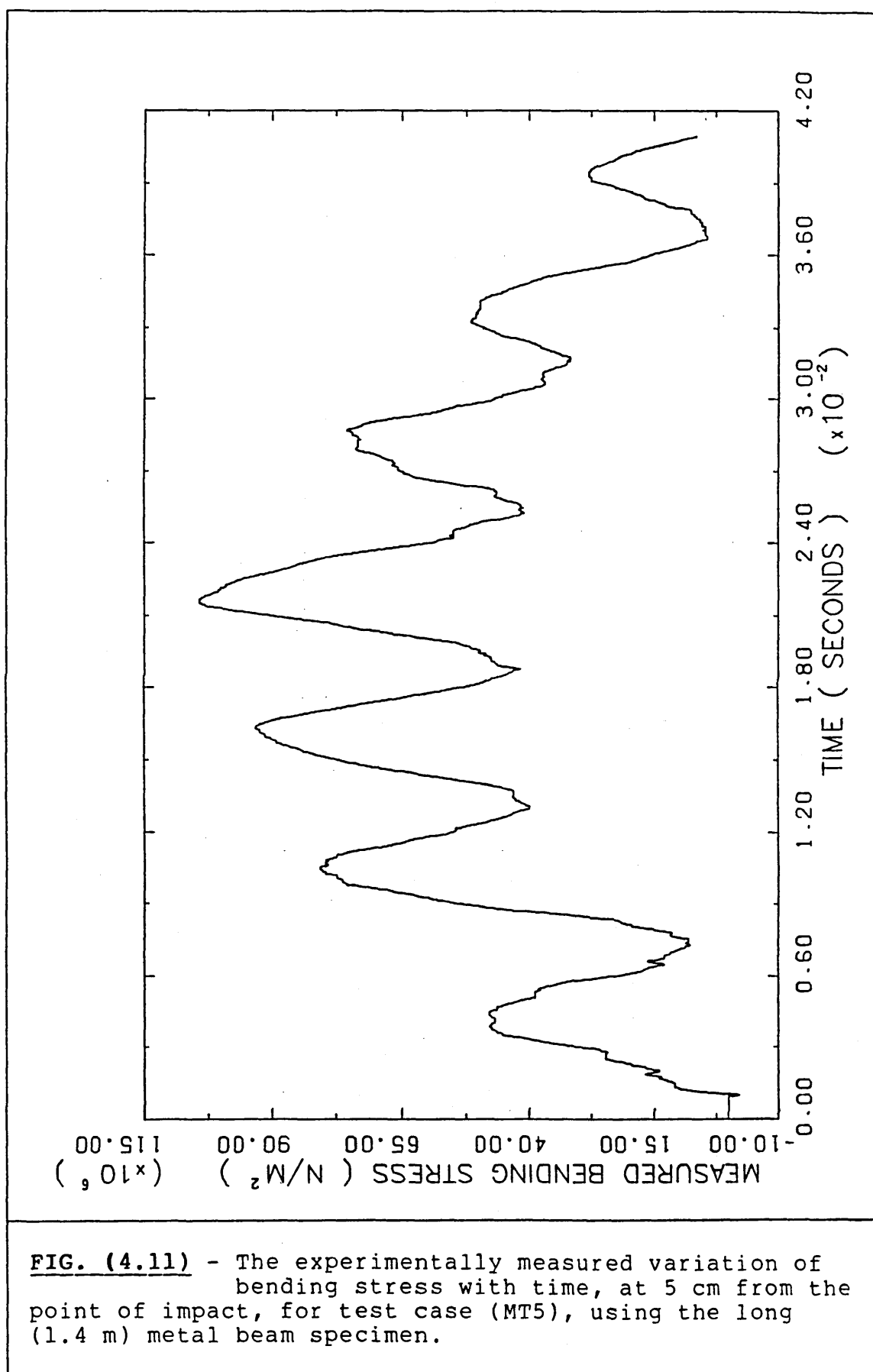


FIG. (4.11) - The experimentally measured variation of bending stress with time, at 5 cm from the point of impact, for test case (MT5), using the long (1.4 m) metal beam specimen.

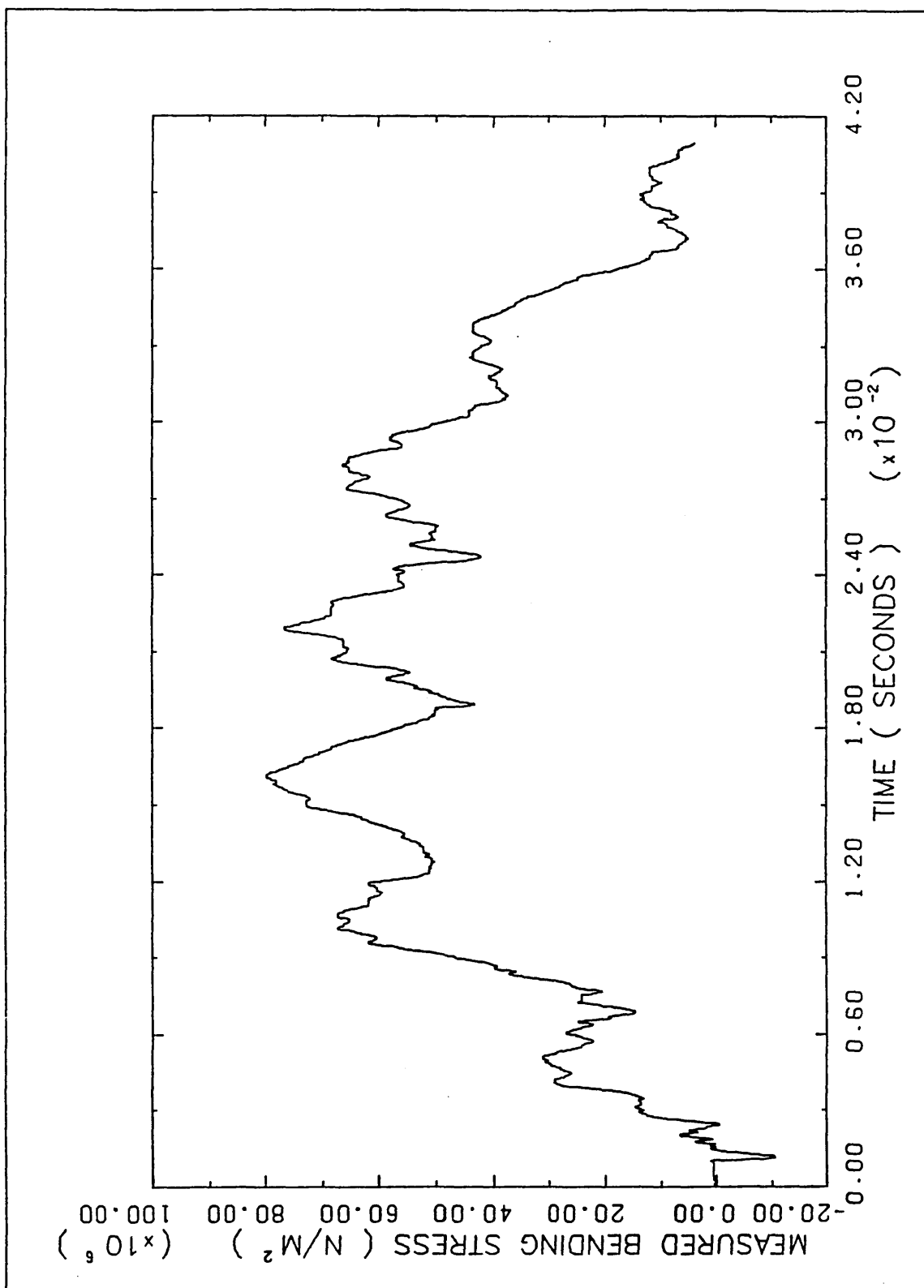


FIG. (4.12) - The experimentally measured variation of bending stress with time, at 15 cm from the point of impact, for test case (MT5), using the long (1.4 m) metal beam specimen.

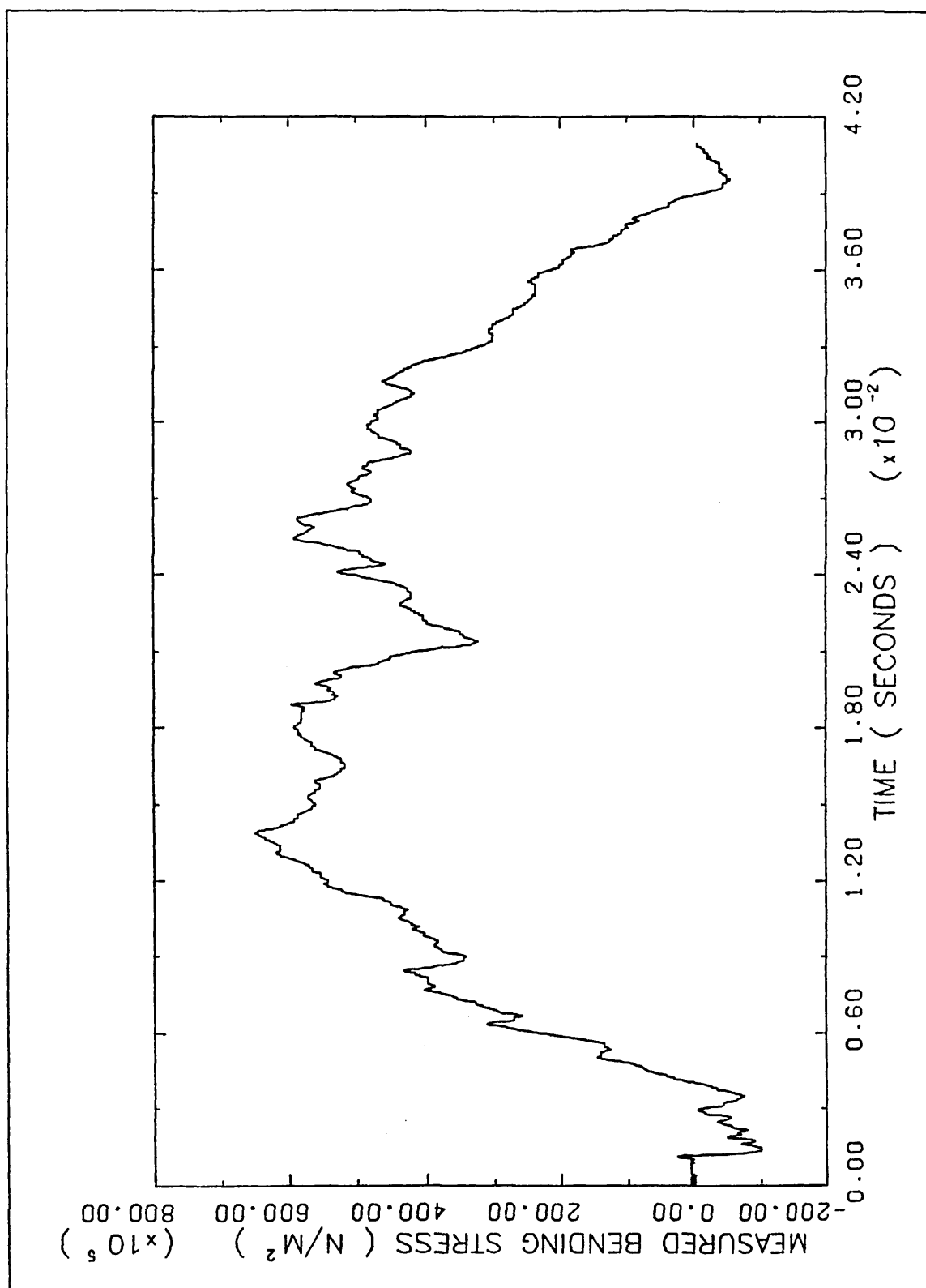


FIG. (4.13) - The experimentally measured variation of bending stress with time, at 25 cm from the point of impact, for test case (MT5), using the long (1.4 m) metal beam specimen.

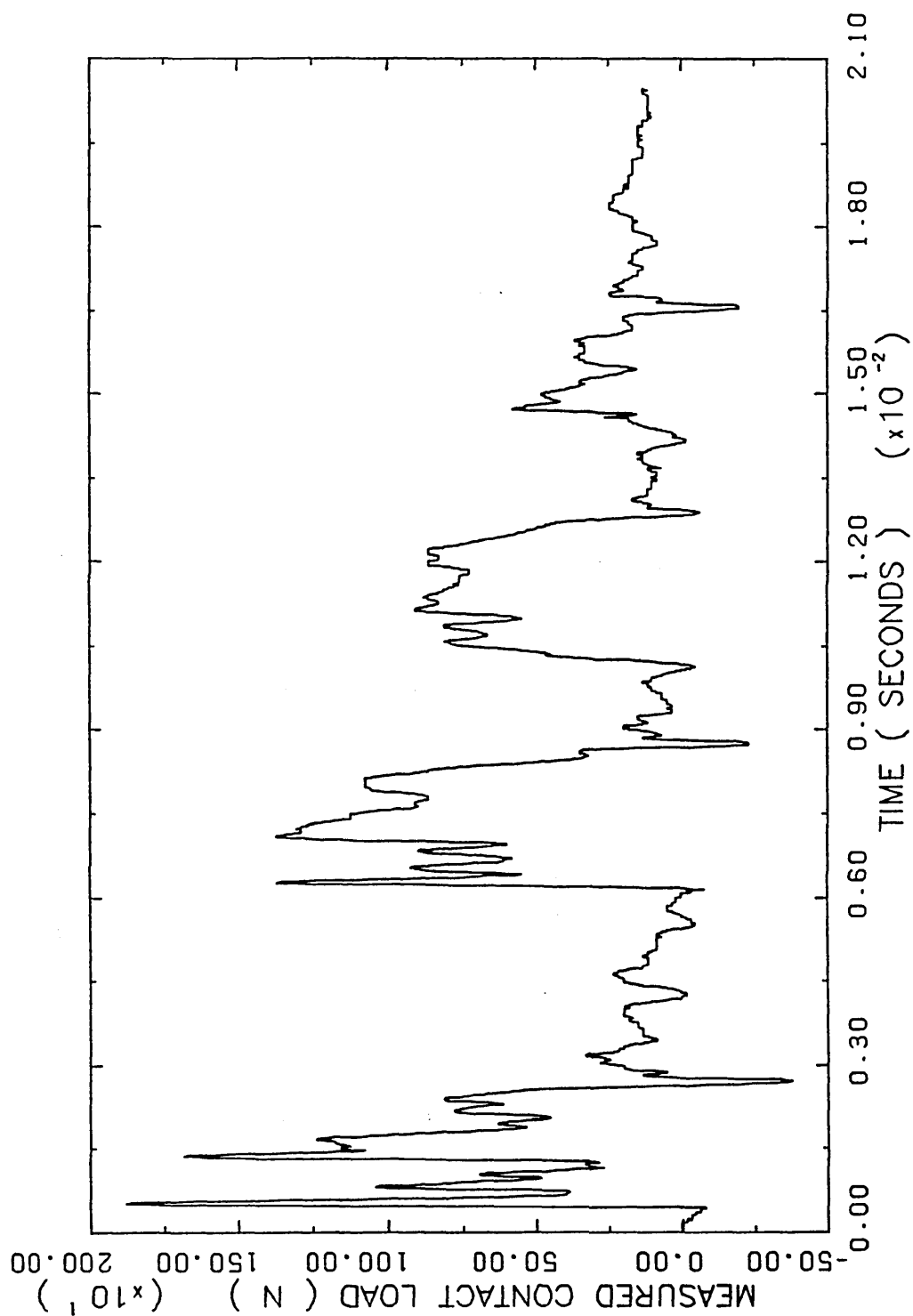


FIG. (4.14) - The experimentally measured variation of contact force with time for test case (MT23), using the medium length (0.8 m) metal beam specimen.

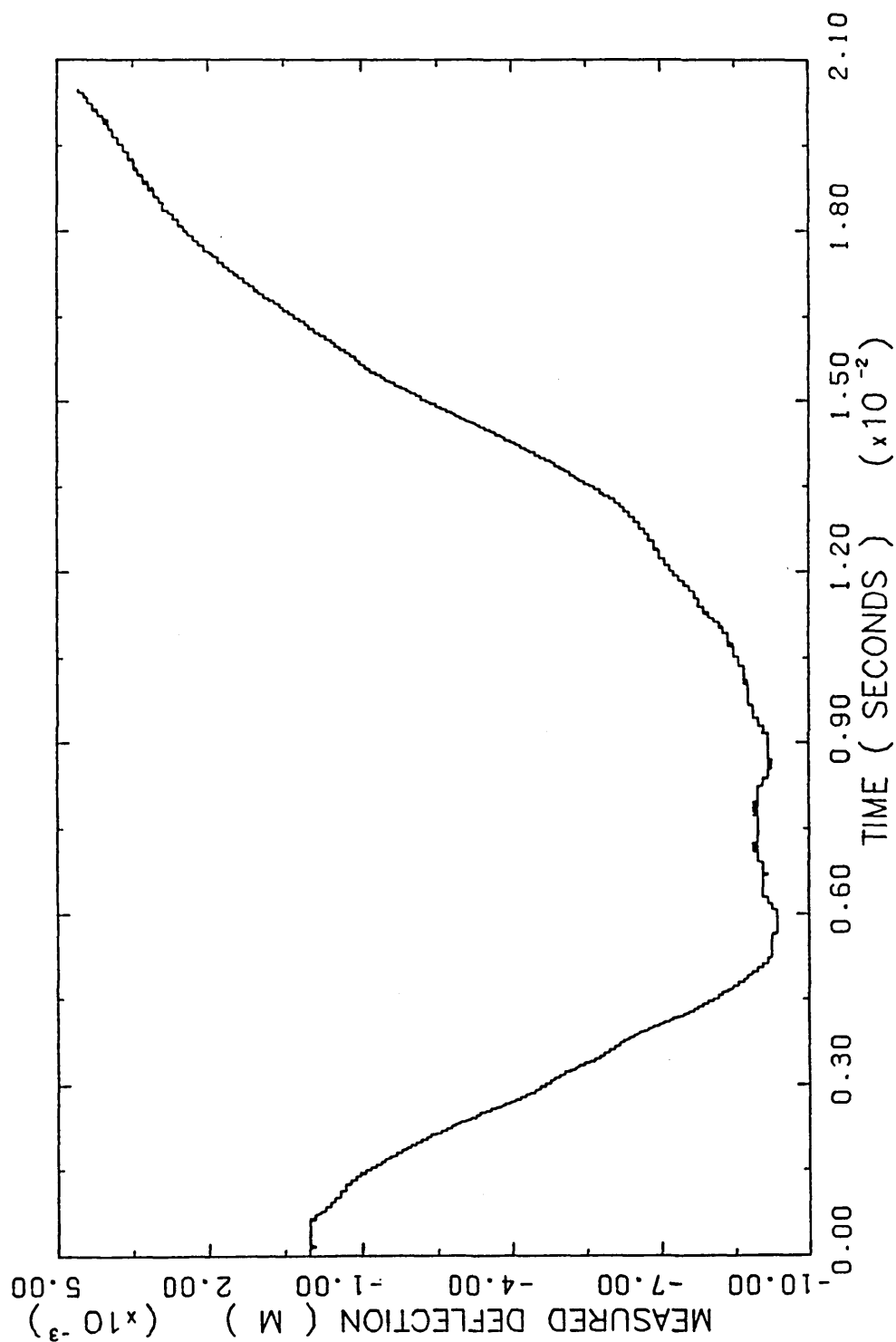


FIG. (4.15) - The experimentally measured variation of displacement with time, at the beam centre, for test case (MT23), using the medium length (0.8 m) metal beam specimen.

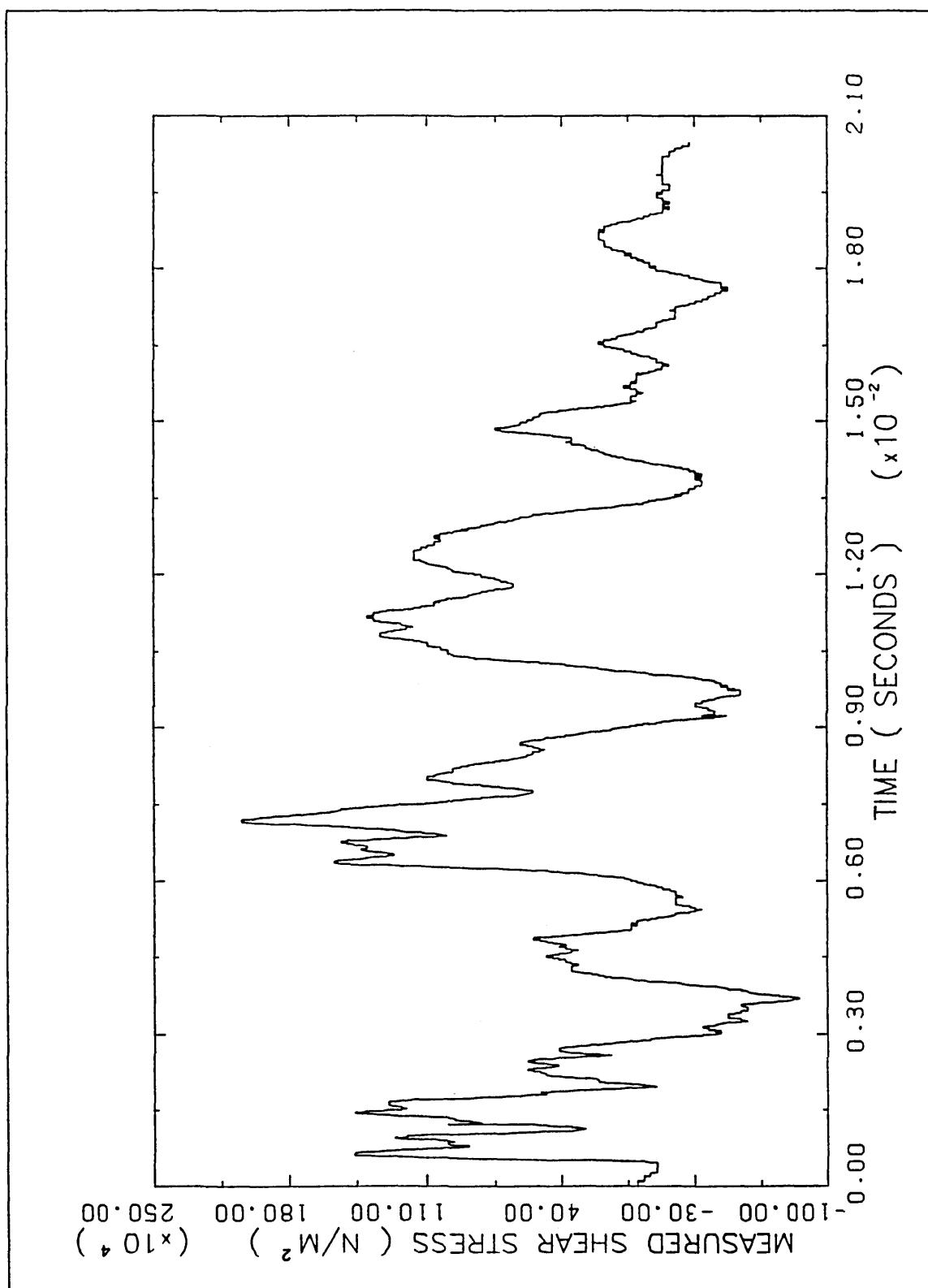


FIG. (4.16) - The experimentally measured variation of shear stress with time, at 5 cm from the point of impact, for test case (MT23), using the medium length (0.8 m) metal beam specimen.

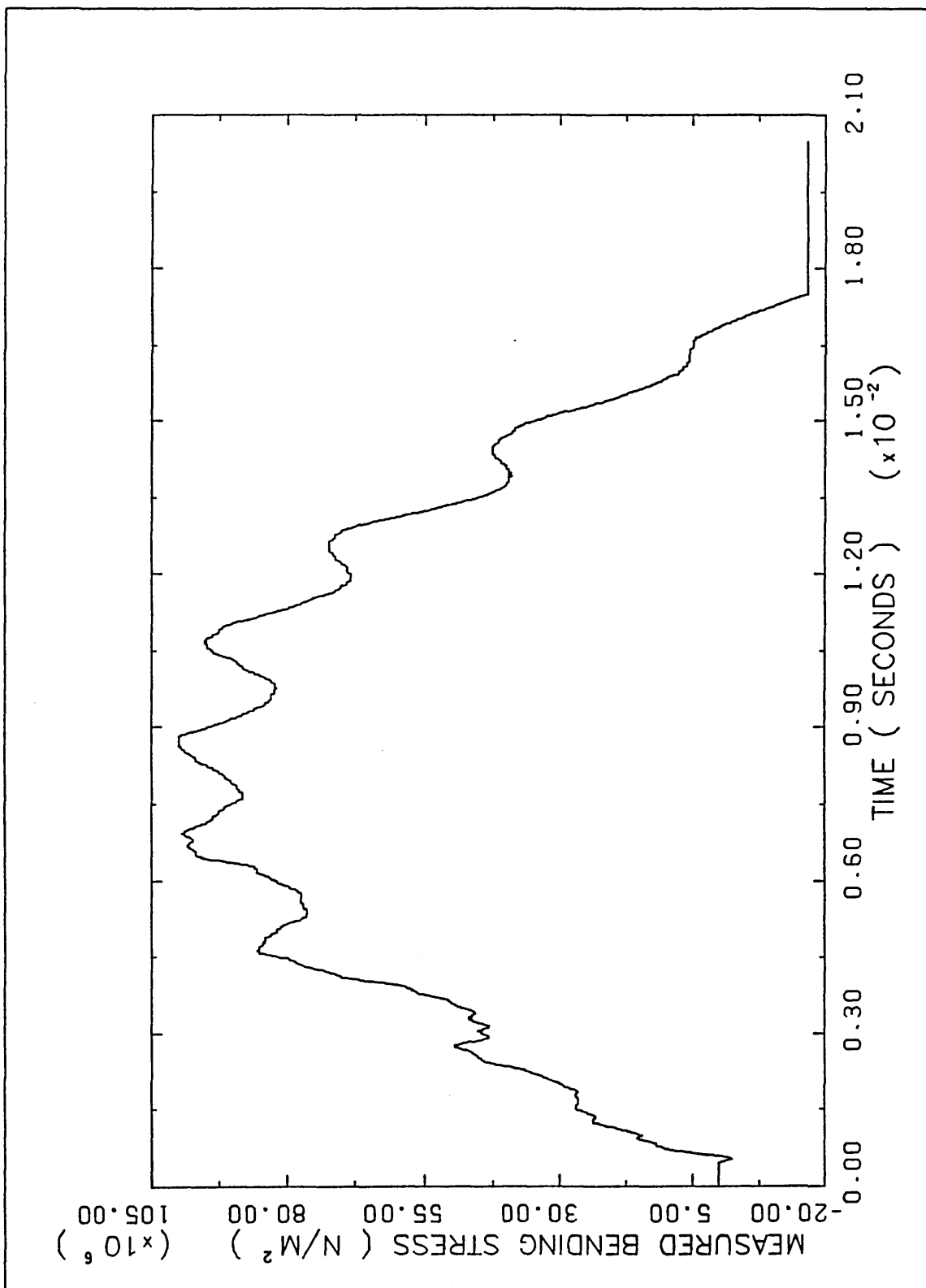


FIG. (4.17) - The experimentally measured variation of bending stress with time, at 5 cm from the point of impact, for test case (MT23), using the medium length (0.8 m) metal beam specimen.

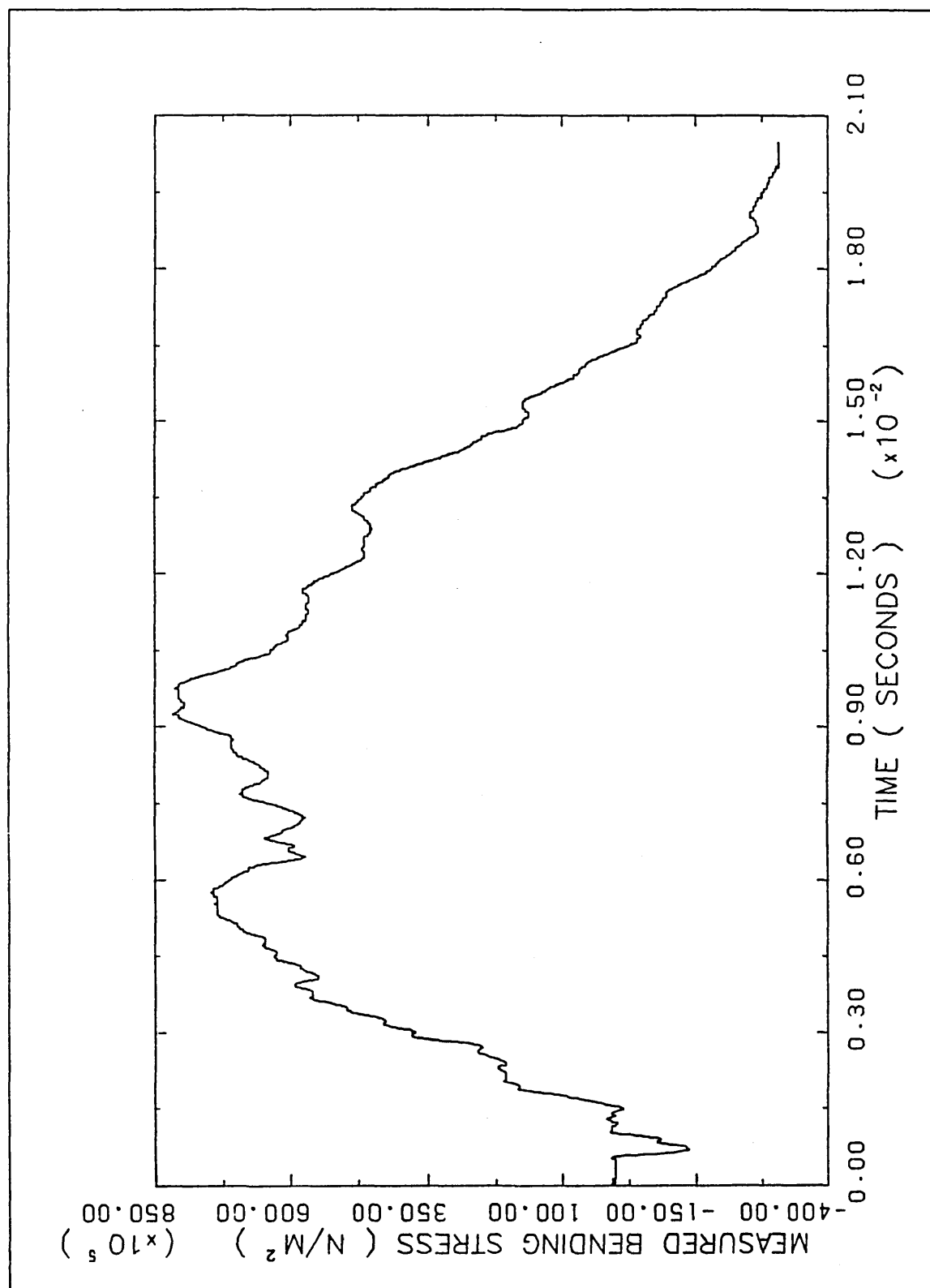


FIG. (4.18) - The experimentally measured variation of bending stress with time, at 15 cm from the point of impact, for test case (MT23), using the medium length (0.8 m) metal beam specimen.

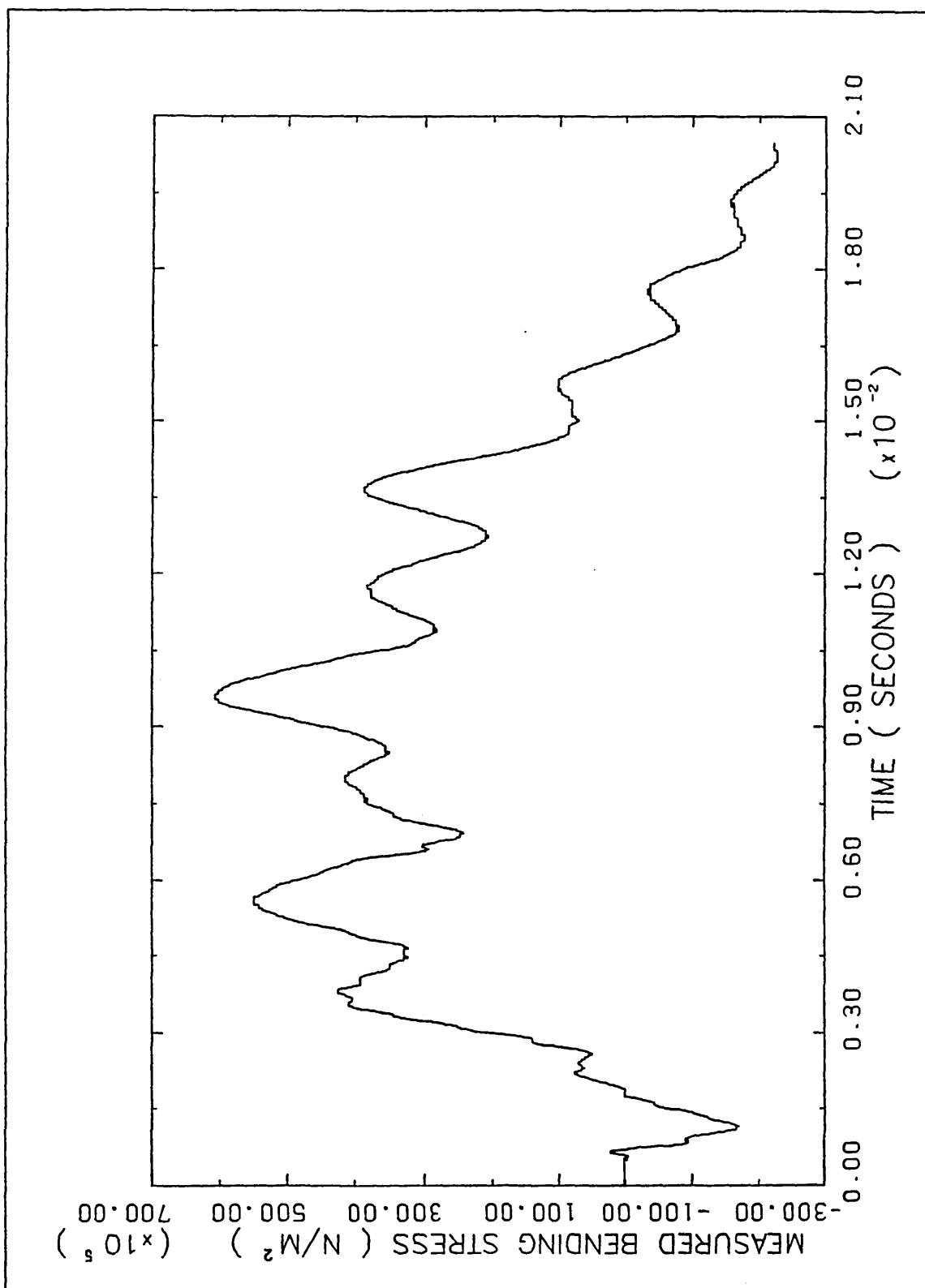


FIG. (4.19) - The experimentally measured variation of bending stress with time, at 25 cm from the point of impact, for test case (MT23), using the medium length (0.8 m) metal beam specimen.

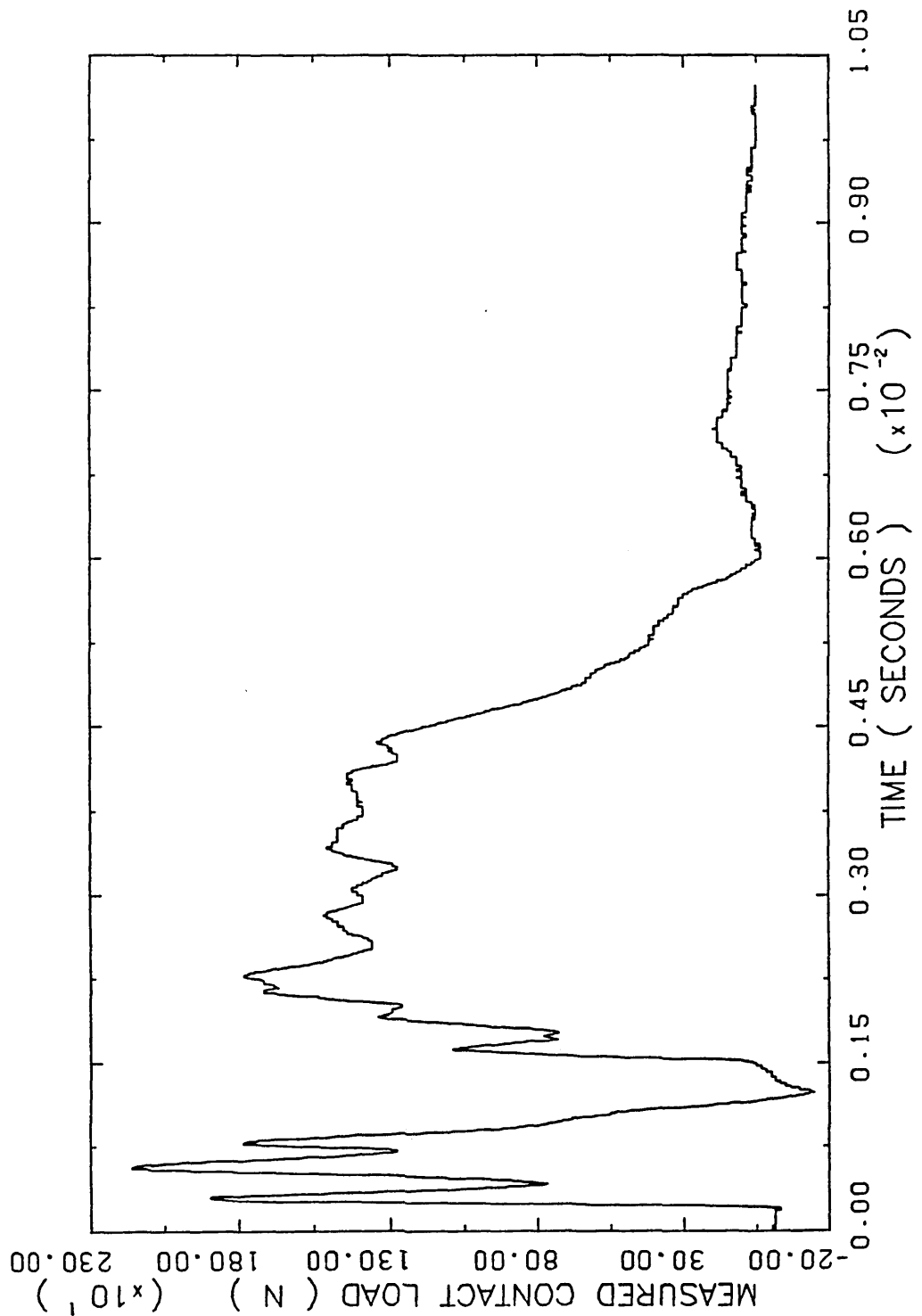


FIG. (4.20) - The experimentally measured variation of contact force with time for test case (MT33), using a metal beam specimen 0.4 m long.

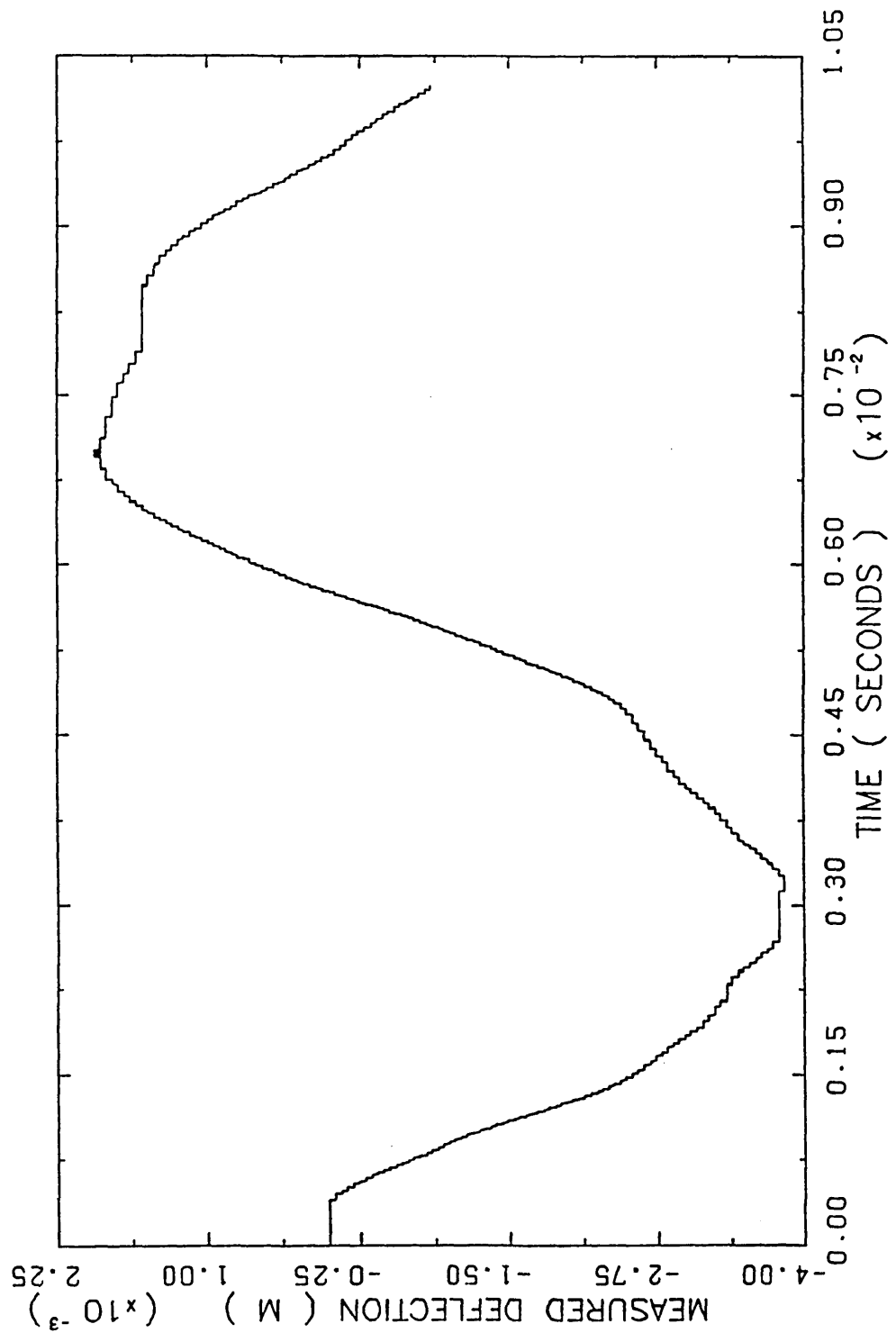


FIG. (4.21) - The experimentally measured variation of displacement with time, at the beam centre, for test case (MT33), using a metal beam specimen 0.4 m long.

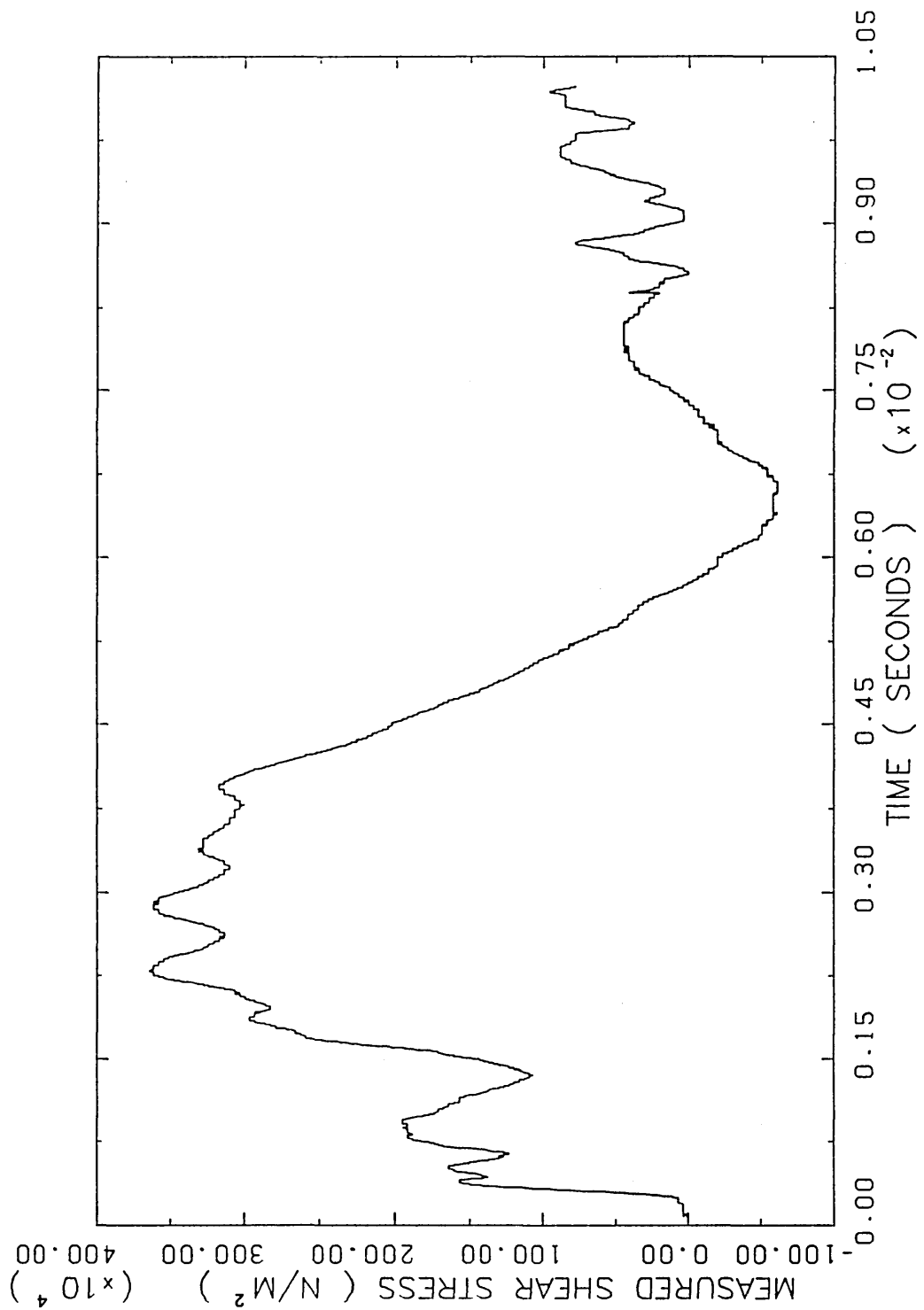


FIG. (4.22) - The experimentally measured variation of shear stress with time, at 5 cm from the point of impact, for test case (MT33), using a metal beam specimen 0.4 m long.

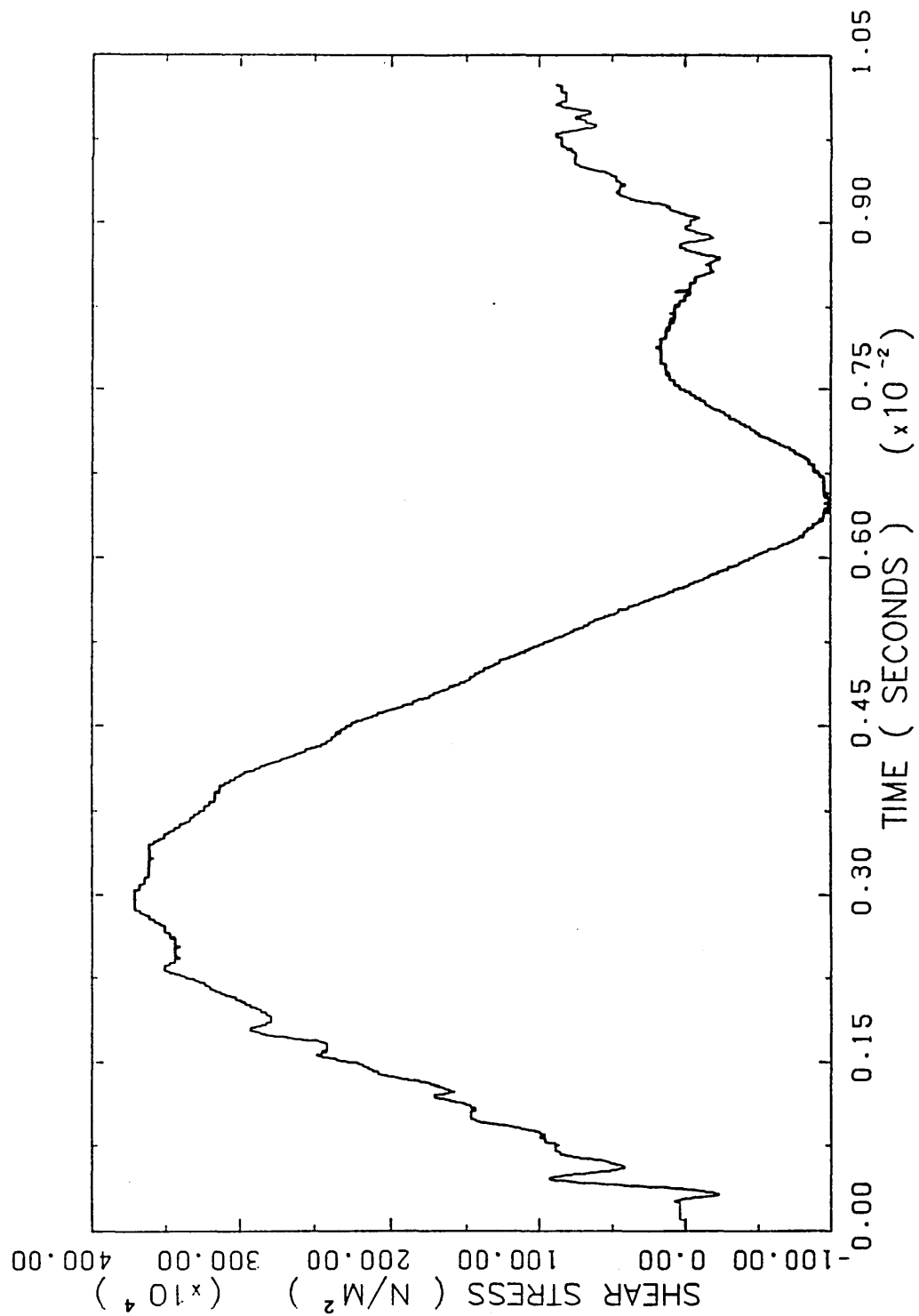
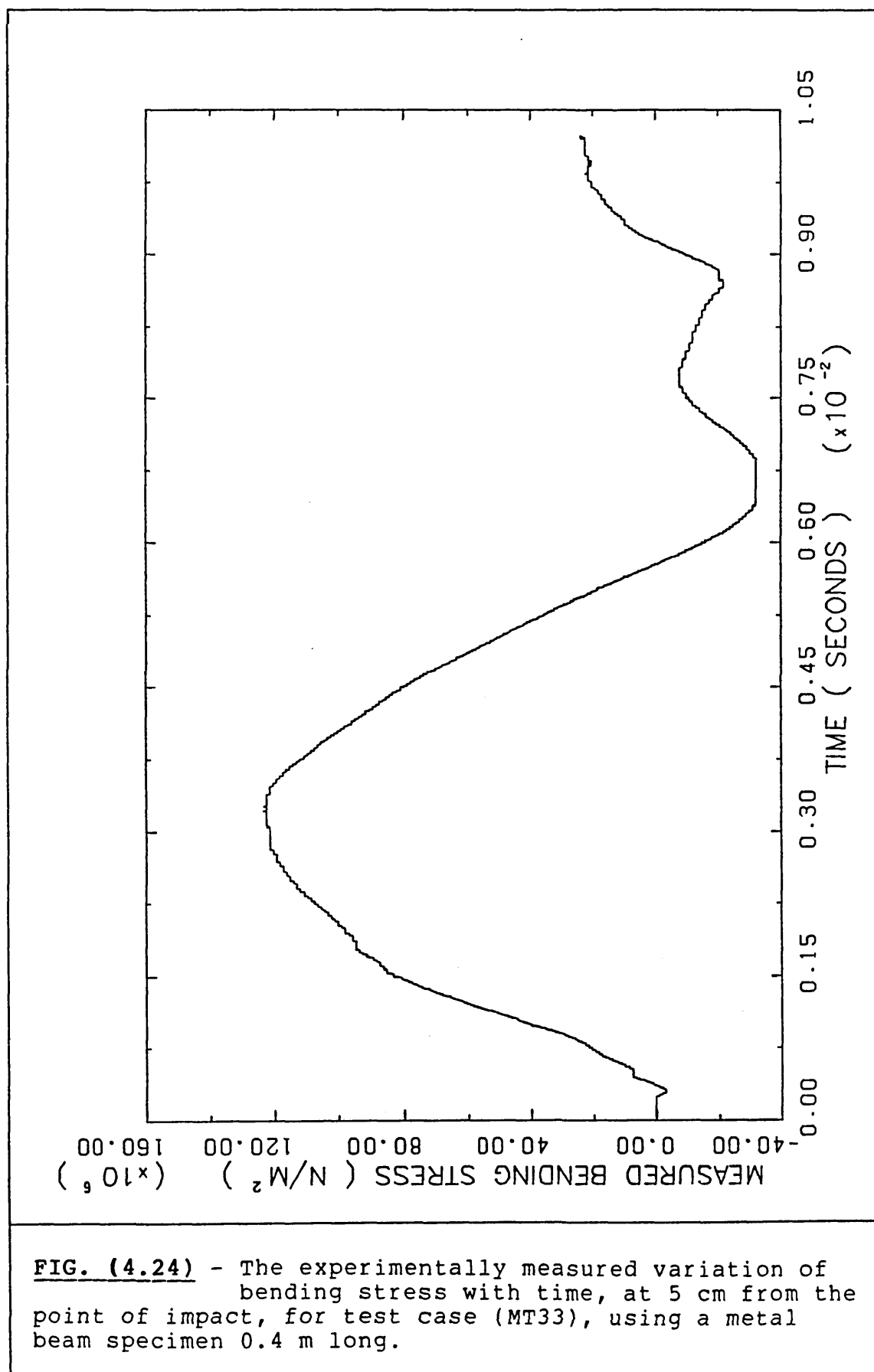


FIG. (4.23) - The experimentally measured variation of shear stress with time, at 10 cm from the point of impact, for test case (MT33), using a metal beam specimen 0.4 m long.



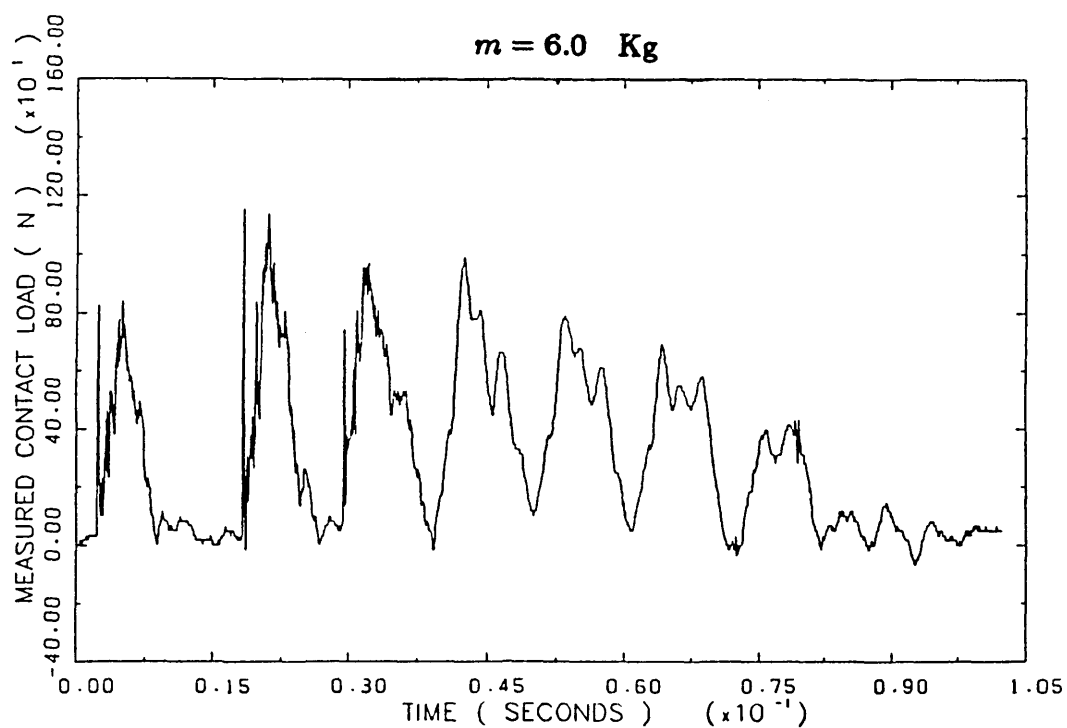
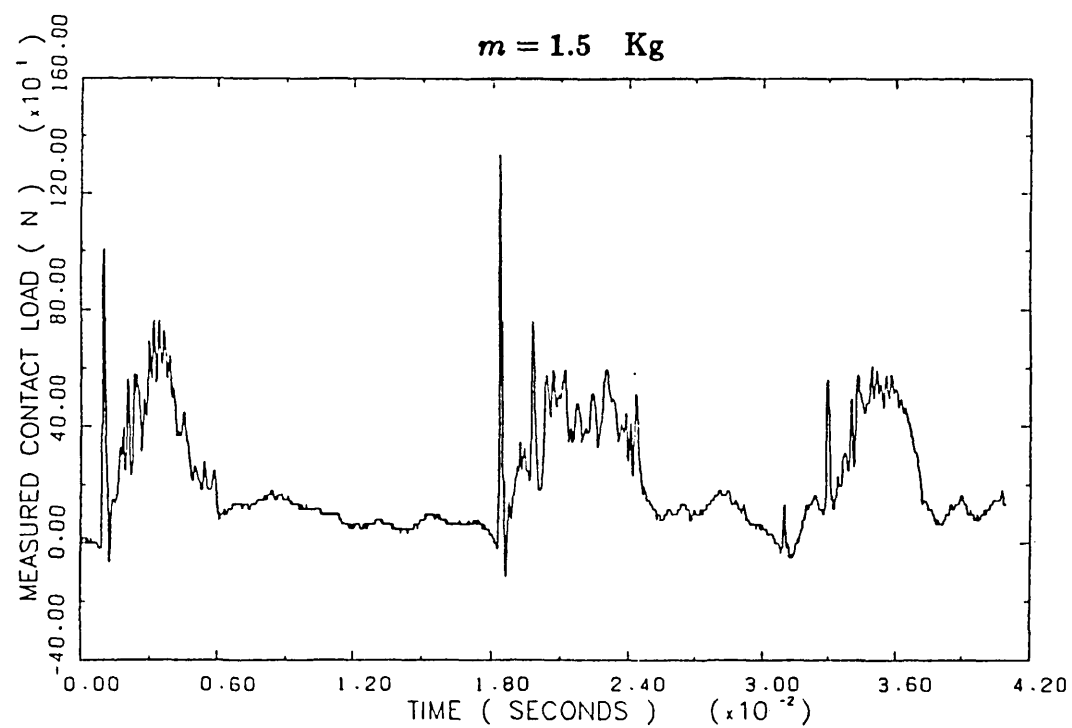


FIG. (4.25) - The experimentally measured variation of contact force with time for test cases (MT1) and (MT8), showing the effect of impactor mass on response for the long (1.4 m) metal beam specimen.

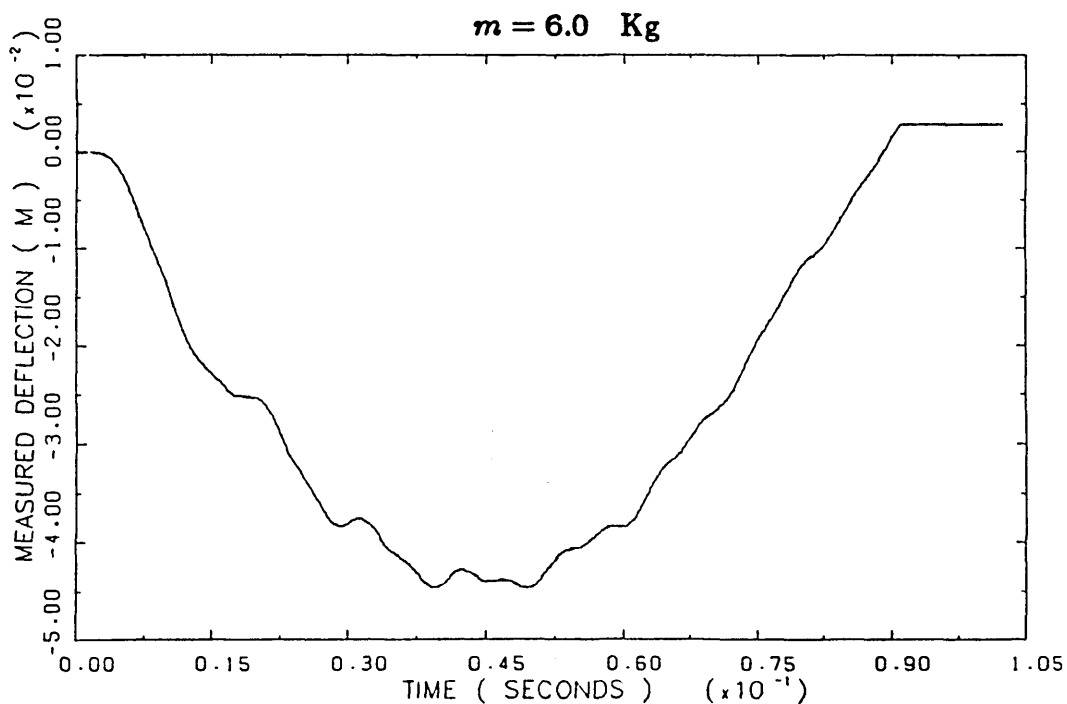
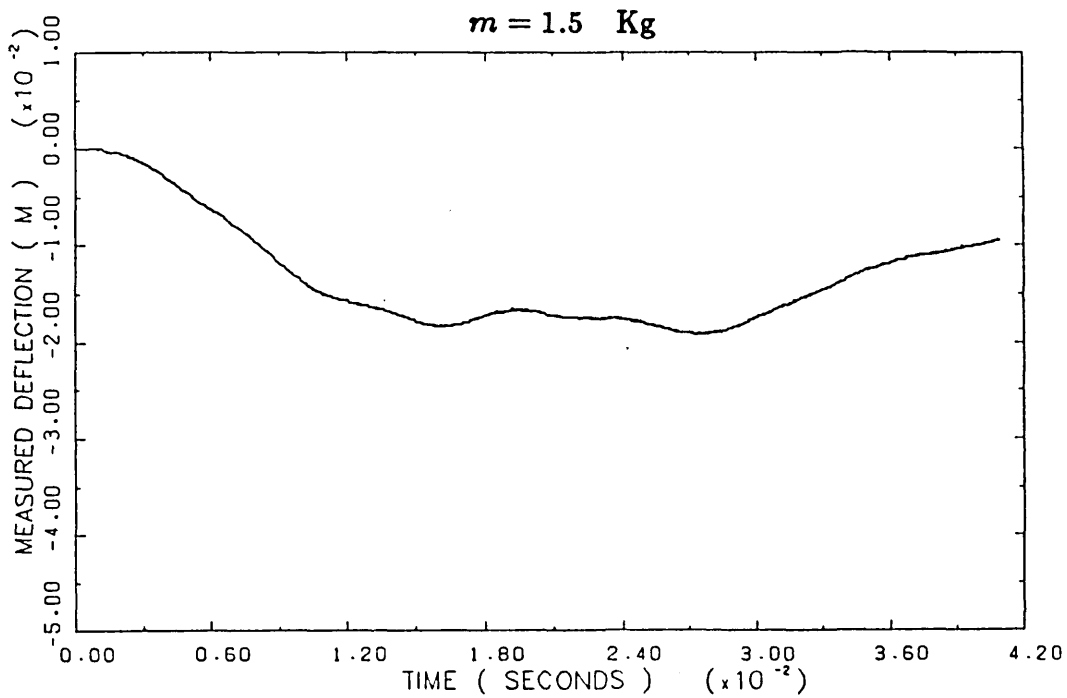


FIG. (4.26) - The experimentally measured variation of displacement with time, at the beam centre, for test cases (MT1) and (MT8), showing the effect of impactor mass on response for the long (1.4 m) metal beam specimen.

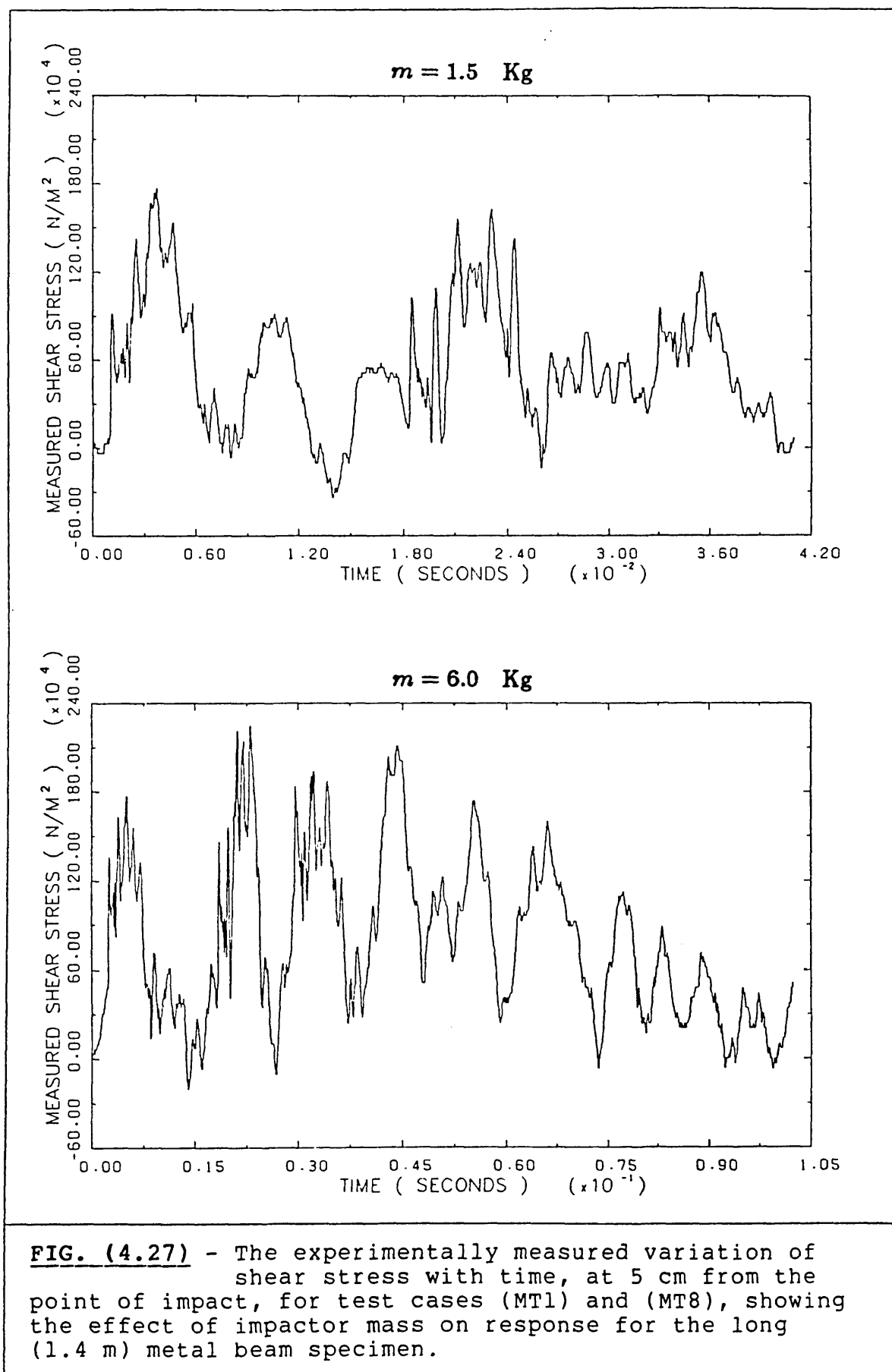


FIG. (4.27) - The experimentally measured variation of shear stress with time, at 5 cm from the point of impact, for test cases (MT1) and (MT8), showing the effect of impactor mass on response for the long (1.4 m) metal beam specimen.

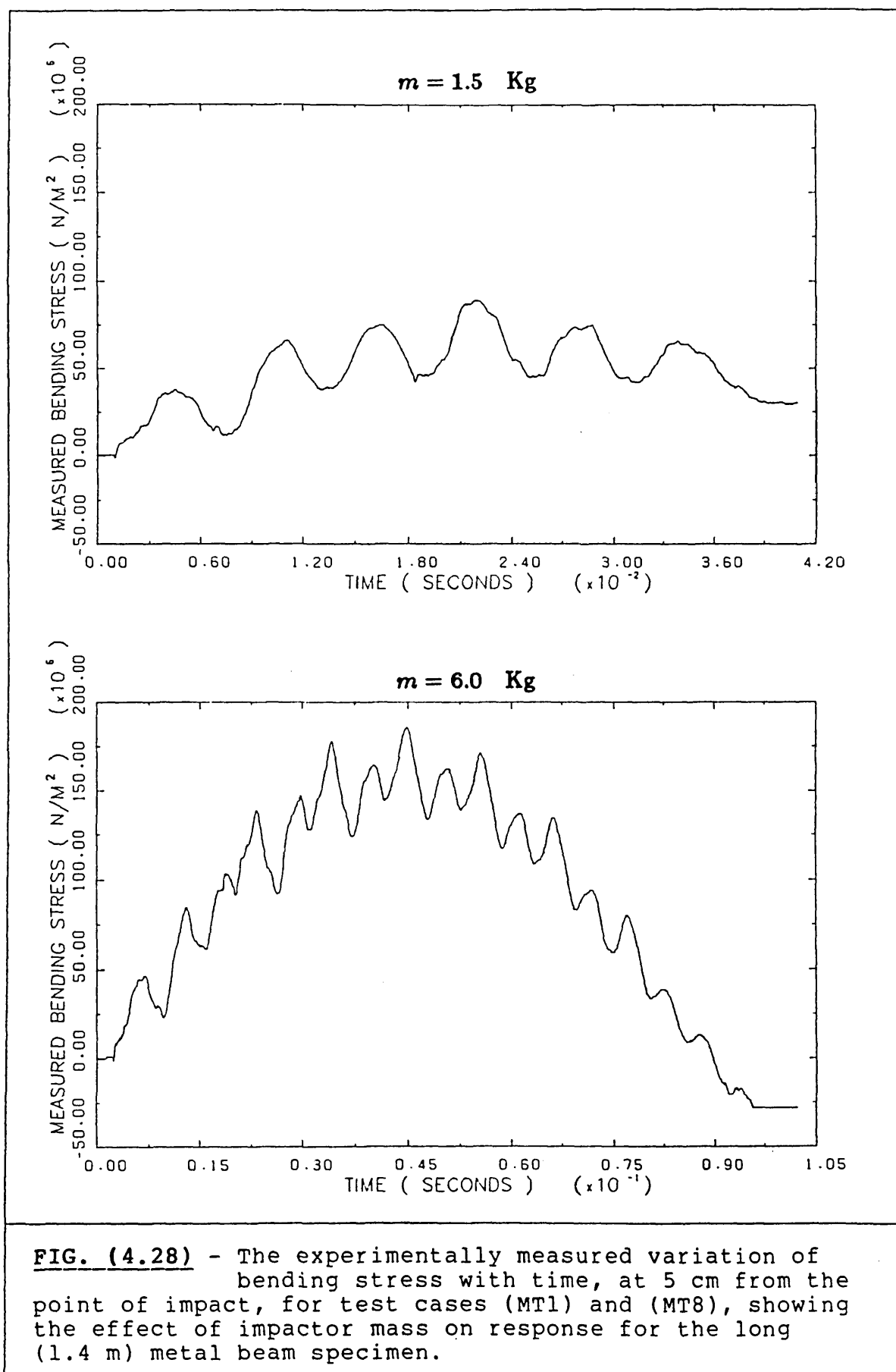


FIG. (4.28) - The experimentally measured variation of bending stress with time, at 5 cm from the point of impact, for test cases (MT1) and (MT8), showing the effect of impactor mass on response for the long (1.4 m) metal beam specimen.

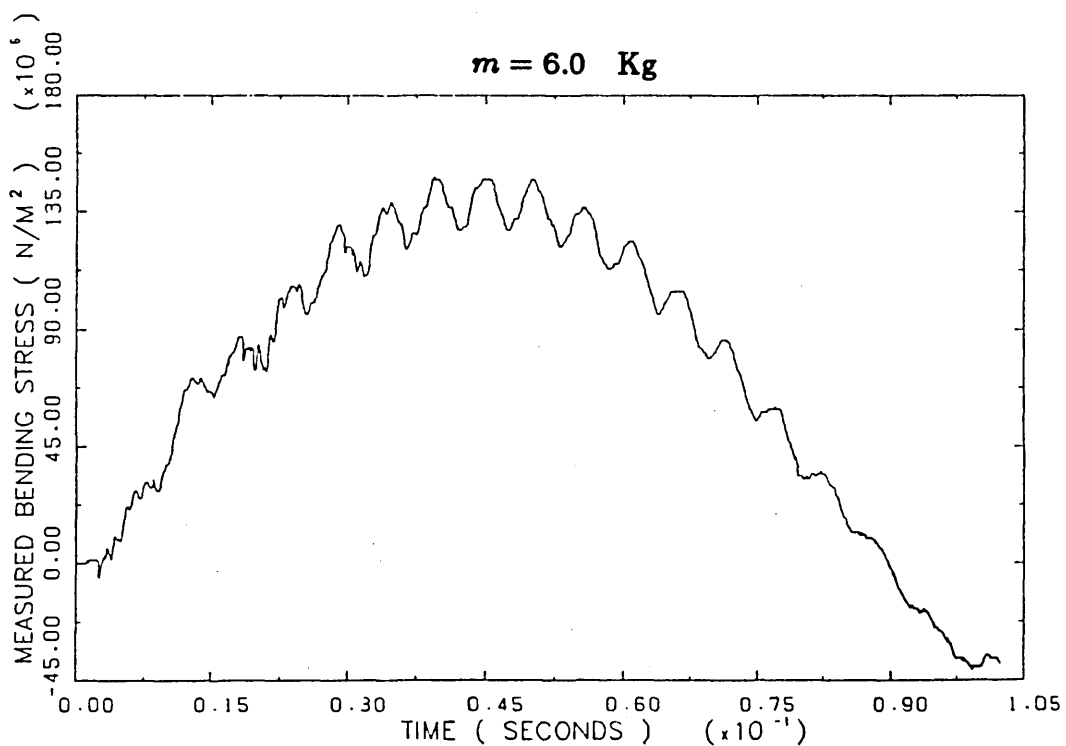
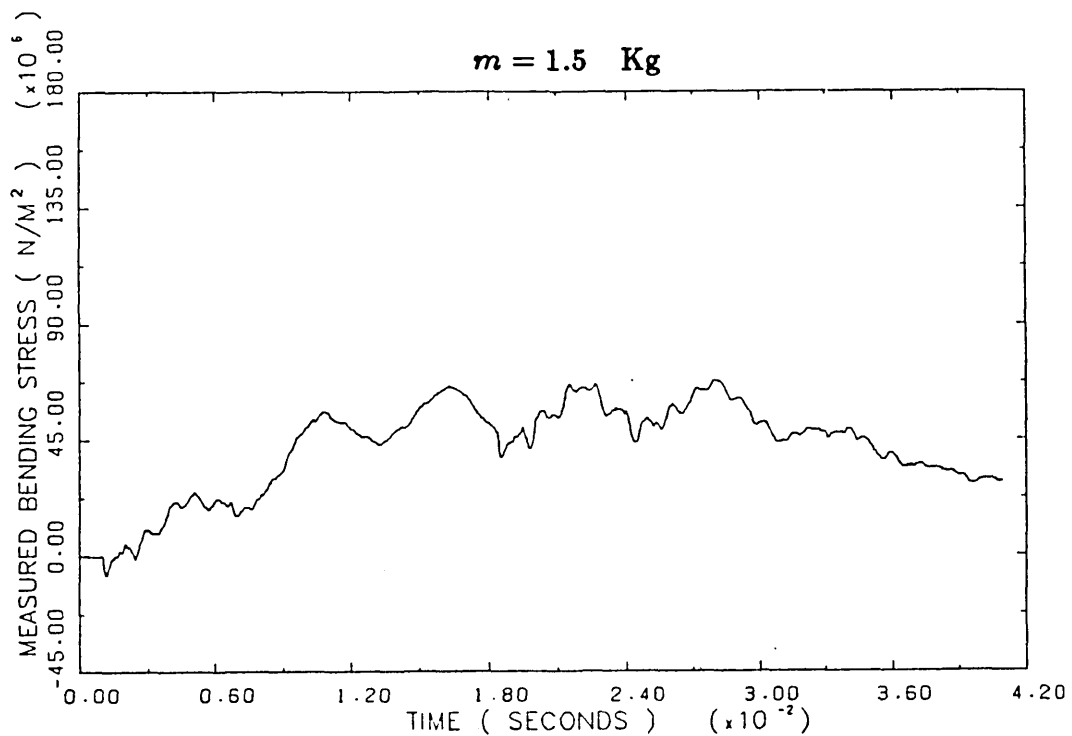


FIG. (4.29) - The experimentally measured variation of bending stress with time, at 15 cm from the point of impact, for test cases (MT1) and (MT8), showing the effect of impactor mass on response for the long (1.4 m) metal beam specimen.

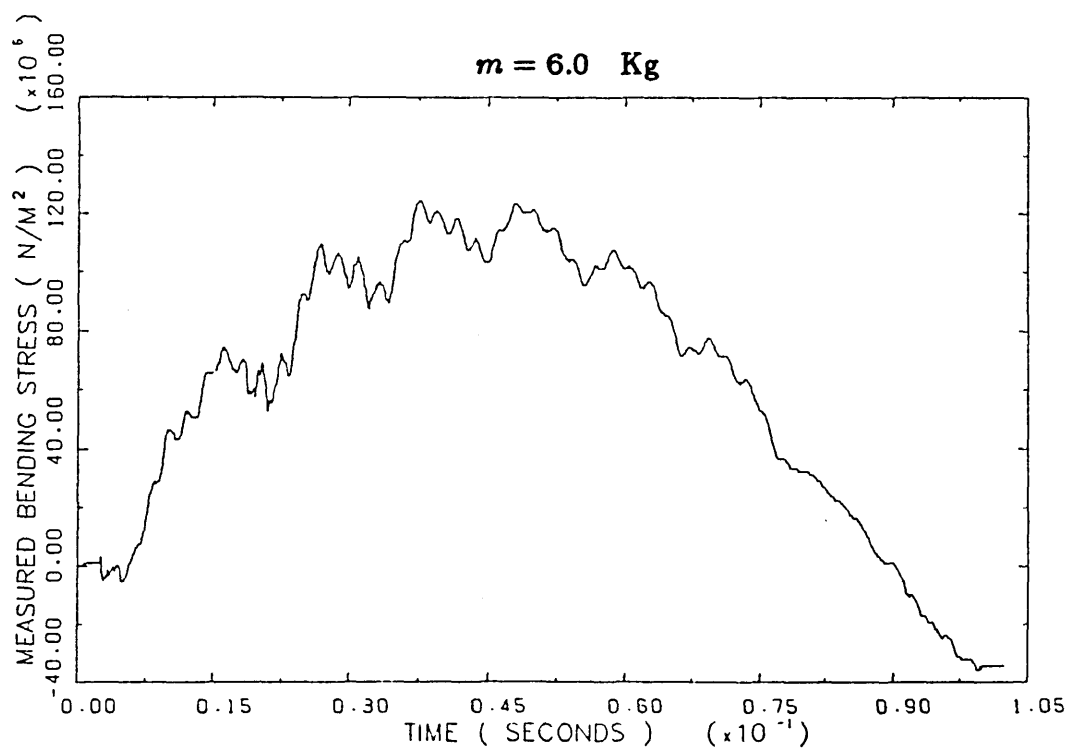
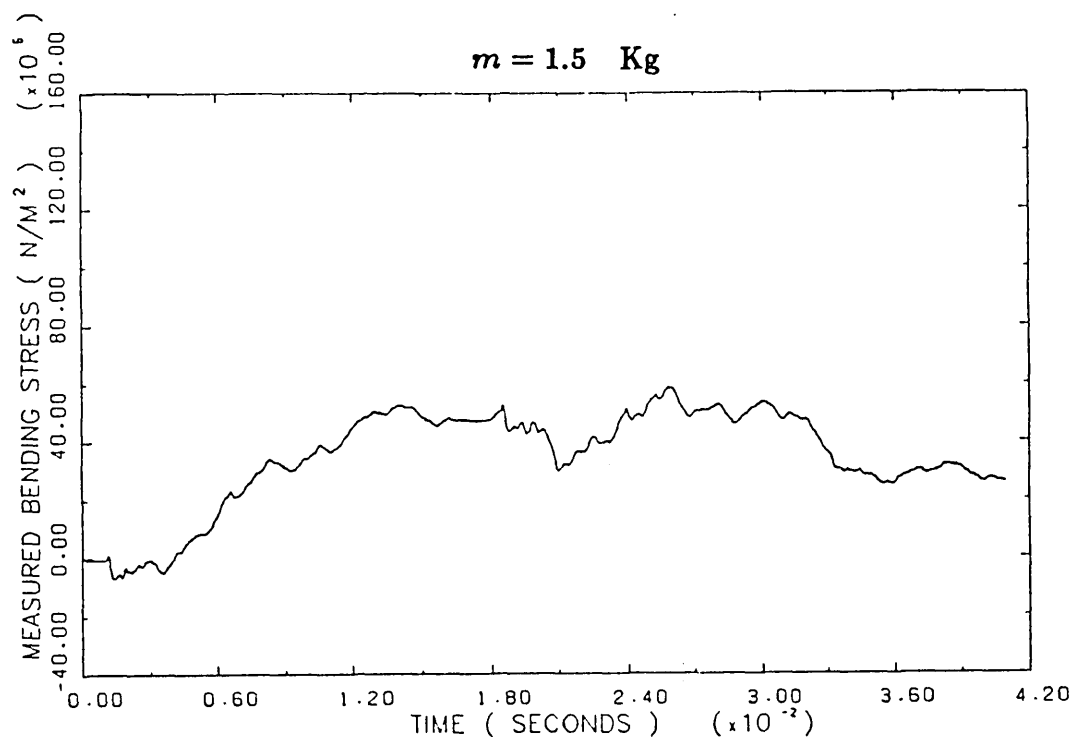


FIG. (4.30) - The experimentally measured variation of bending stress with time, at 25 cm from the point of impact, for test cases (MT1) and (MT8), showing the effect of impactor mass on response for the long (1.4 m) metal beam specimen.

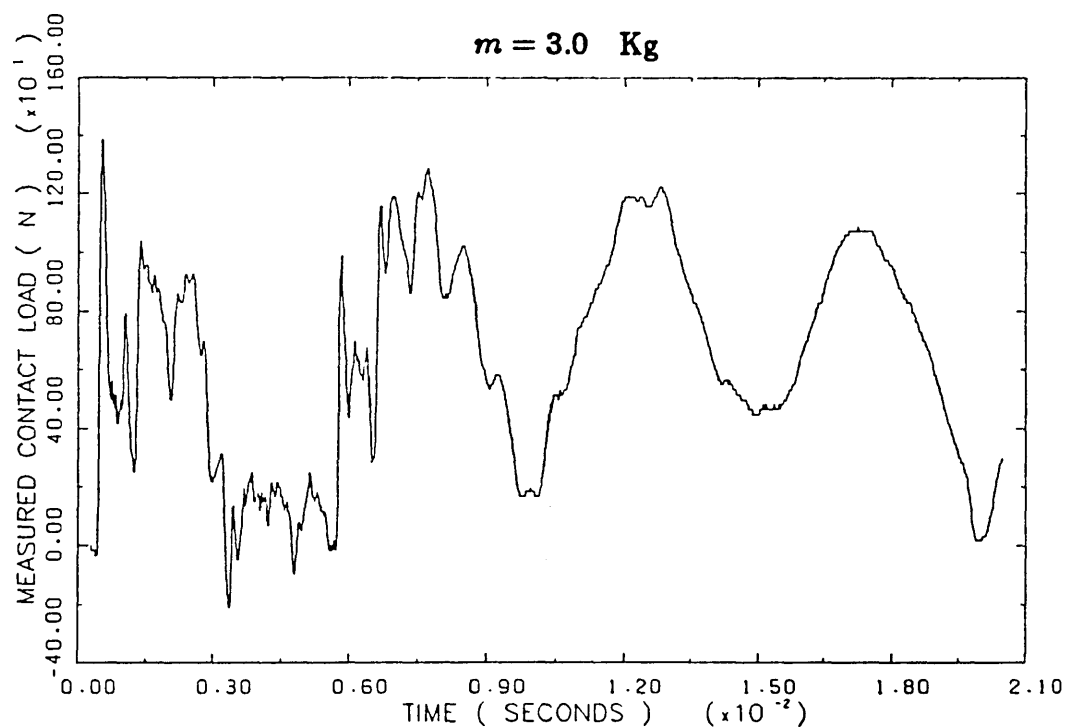
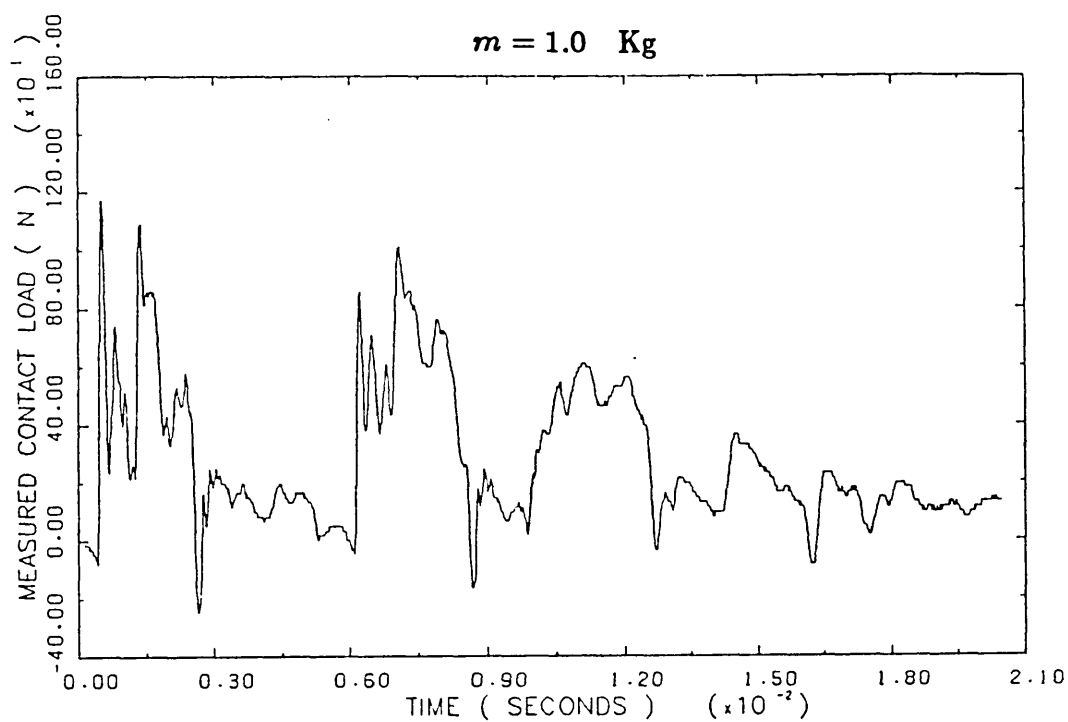


FIG. (4.31) - The experimentally measured variation of contact force with time for test cases (MT22) and (MT28), showing the effect of impactor mass on response for the medium length (0.8 m) metal beam specimen.

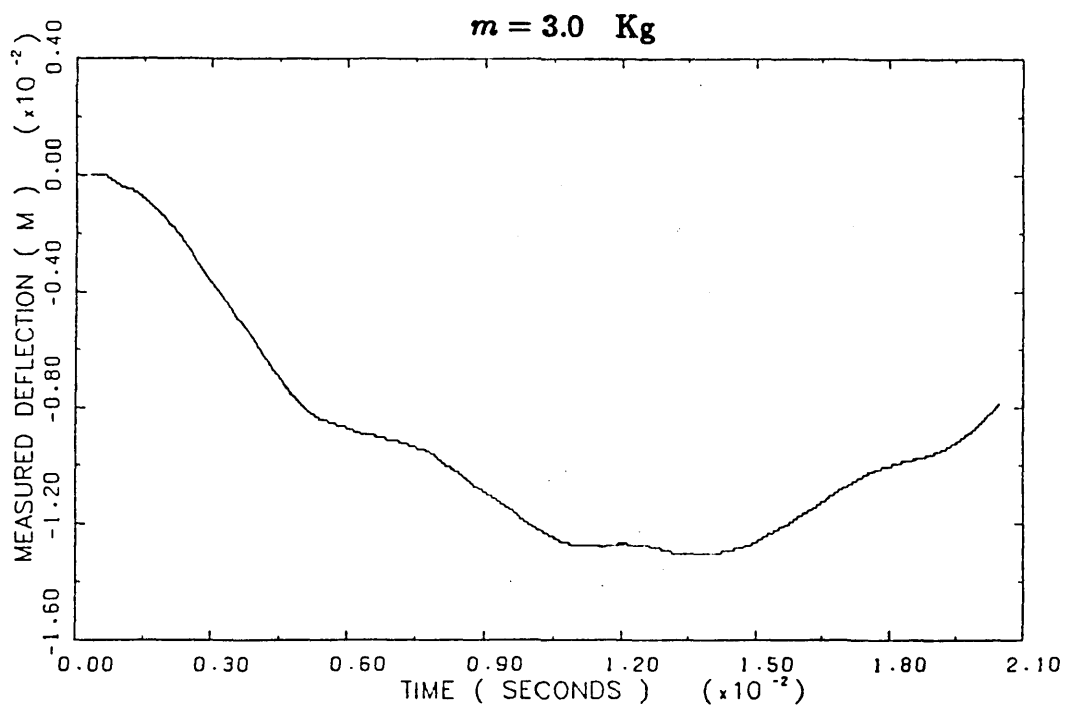
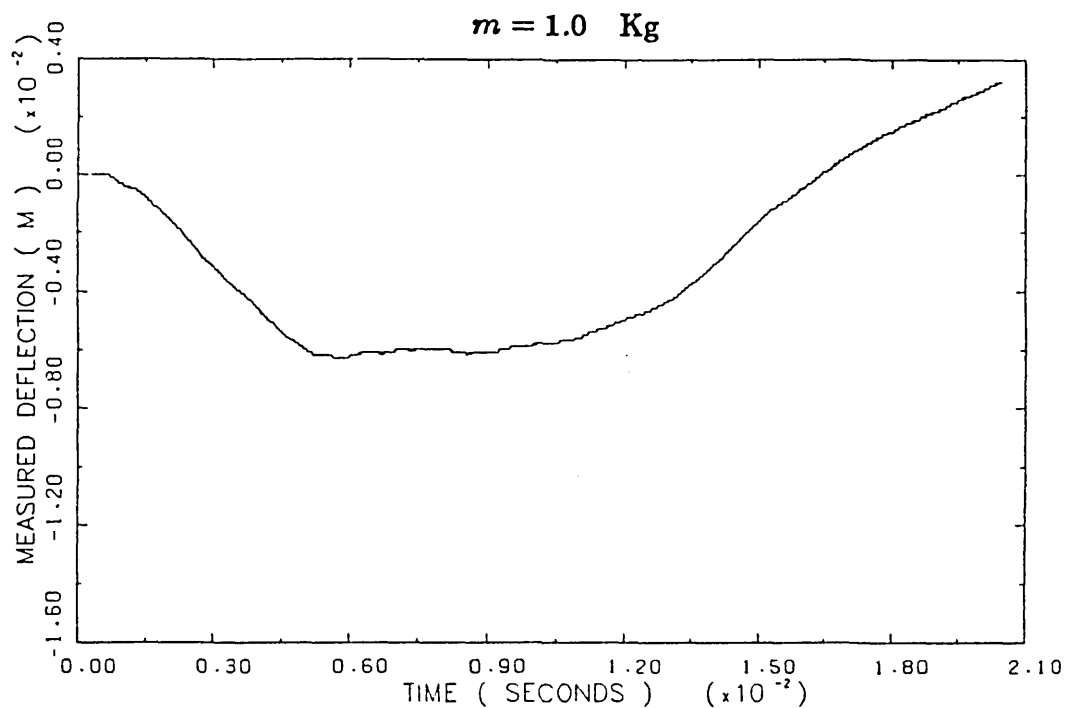


FIG. (4.32) - The experimentally measured variation of displacement with time, at the beam centre, for test cases (MT22) and (MT28), showing the effect of impactor mass on response for the medium length (0.8 m) metal beam specimen.

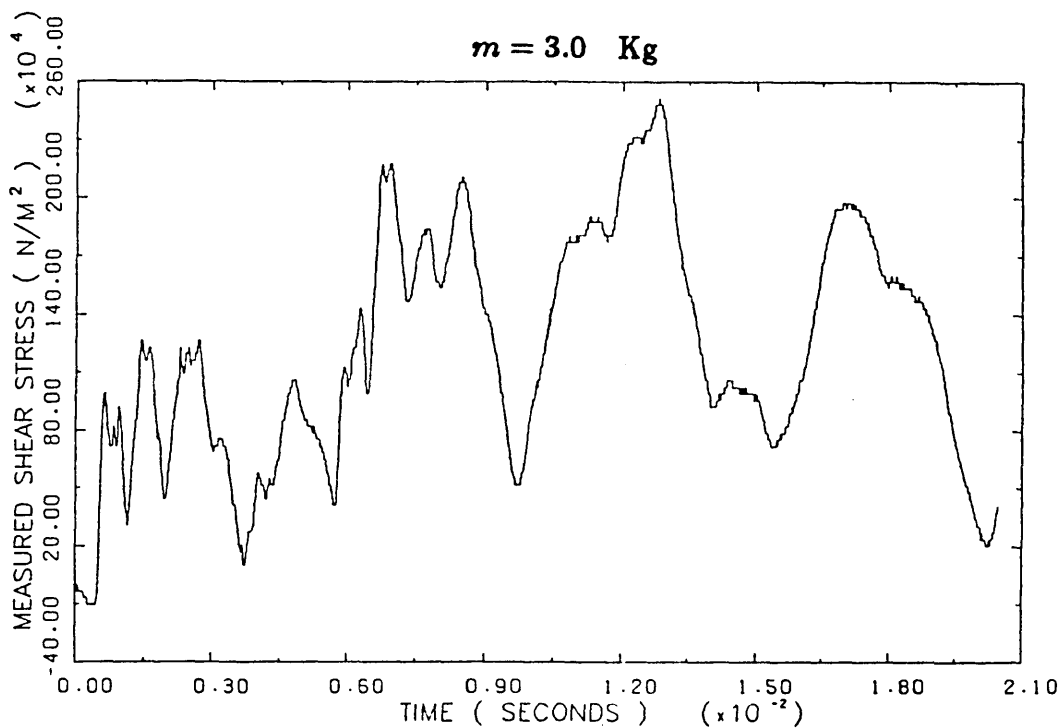
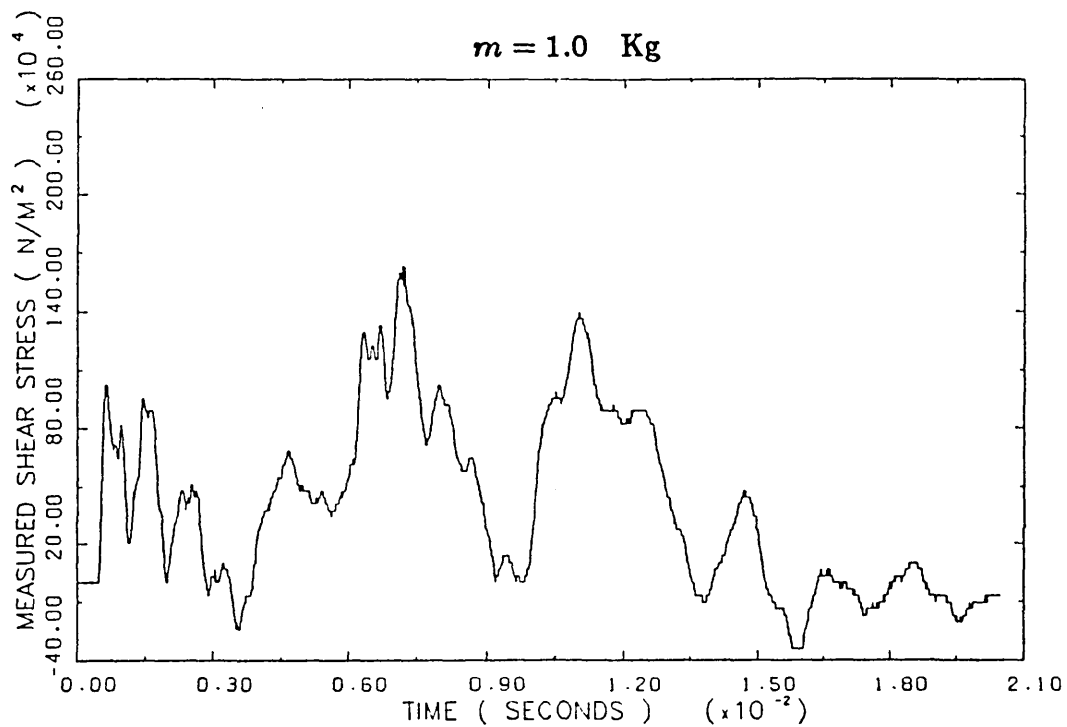


FIG. (4.33) - The experimentally measured variation of shear stress with time, at 5 cm from the point of impact, for test cases (MT22) and (MT28), showing the effect of impactor mass on response for the medium length (0.8 m) metal beam specimen.

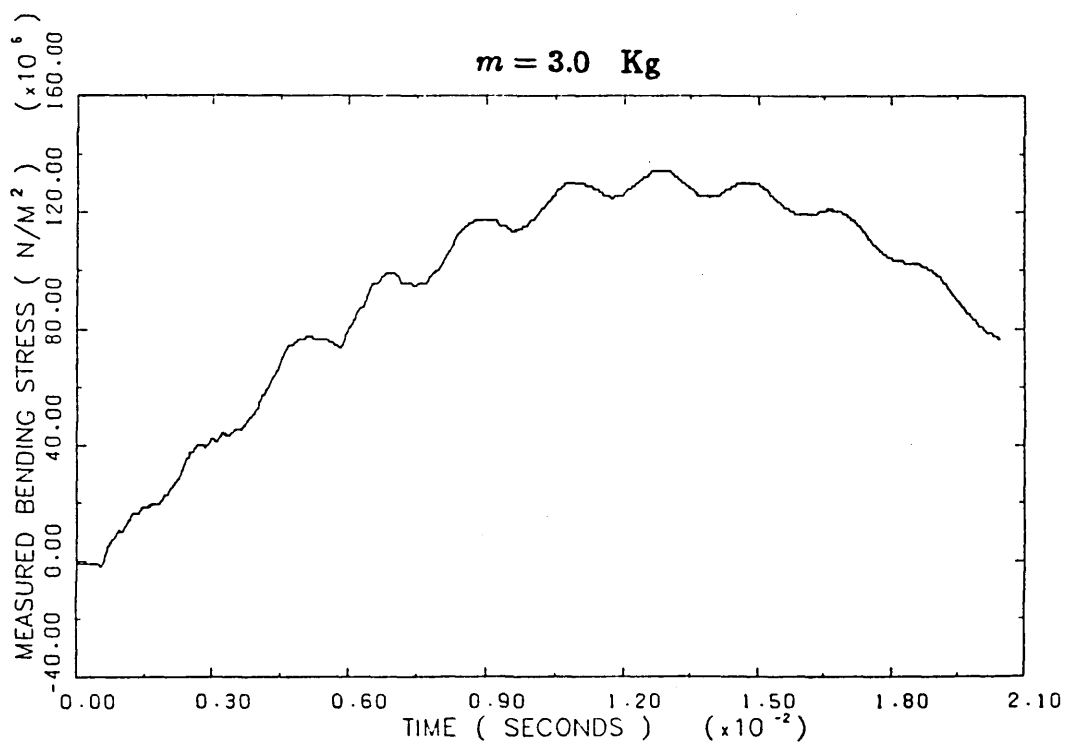
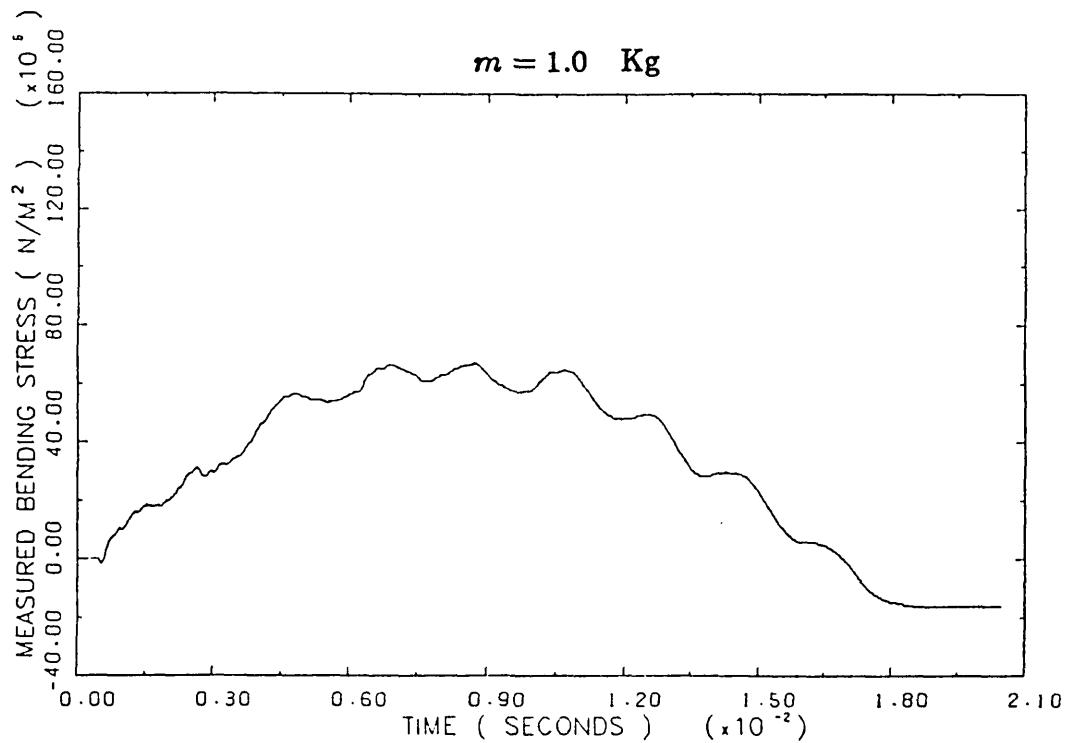


FIG. (4.34) - The experimentally measured variation of bending stress with time, at 5 cm from the point of impact, for test cases (MT22) and (MT28), showing the effect of impactor mass on response for the medium length (0.8 m) metal beam specimen.

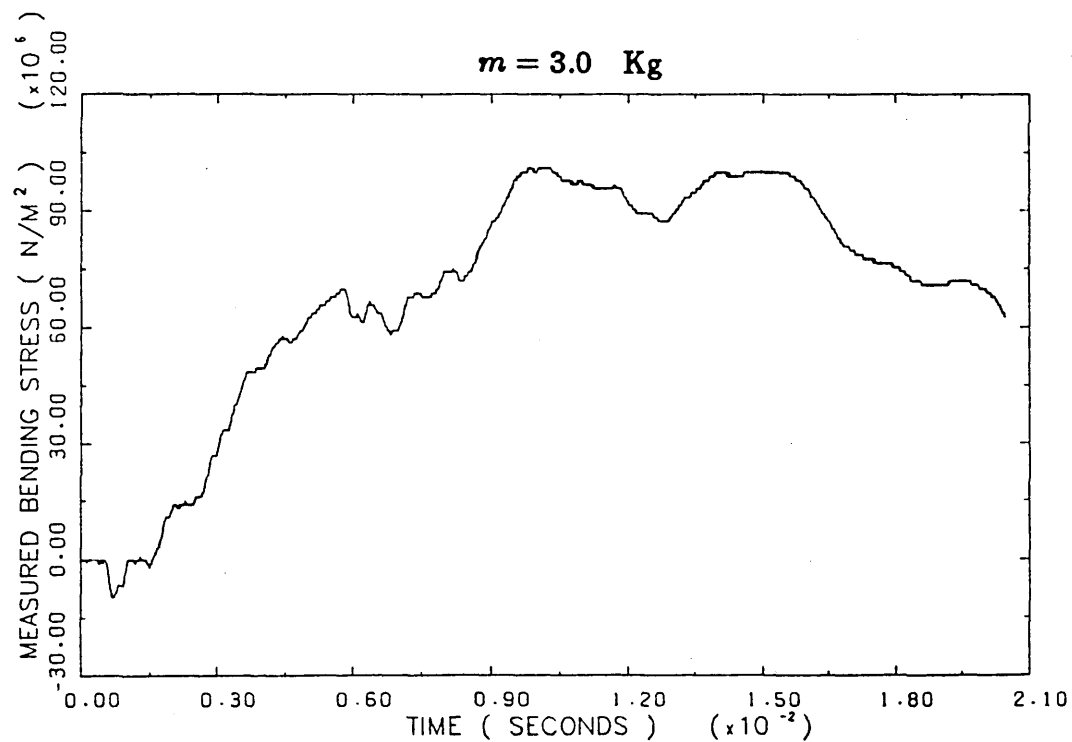
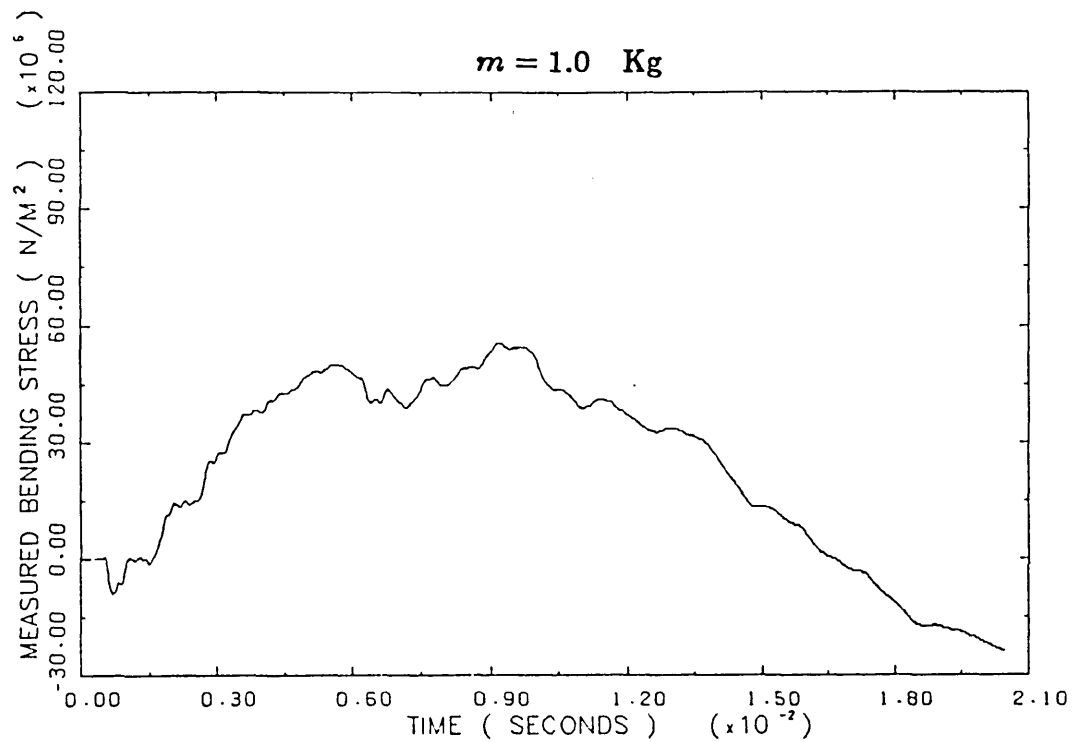


FIG. (4.35) - The experimentally measured variation of bending stress with time, at 15 cm from the point of impact, for test cases (MT22) and (MT28), showing the effect of impactor mass on response for the medium length (0.8 m) metal beam specimen.

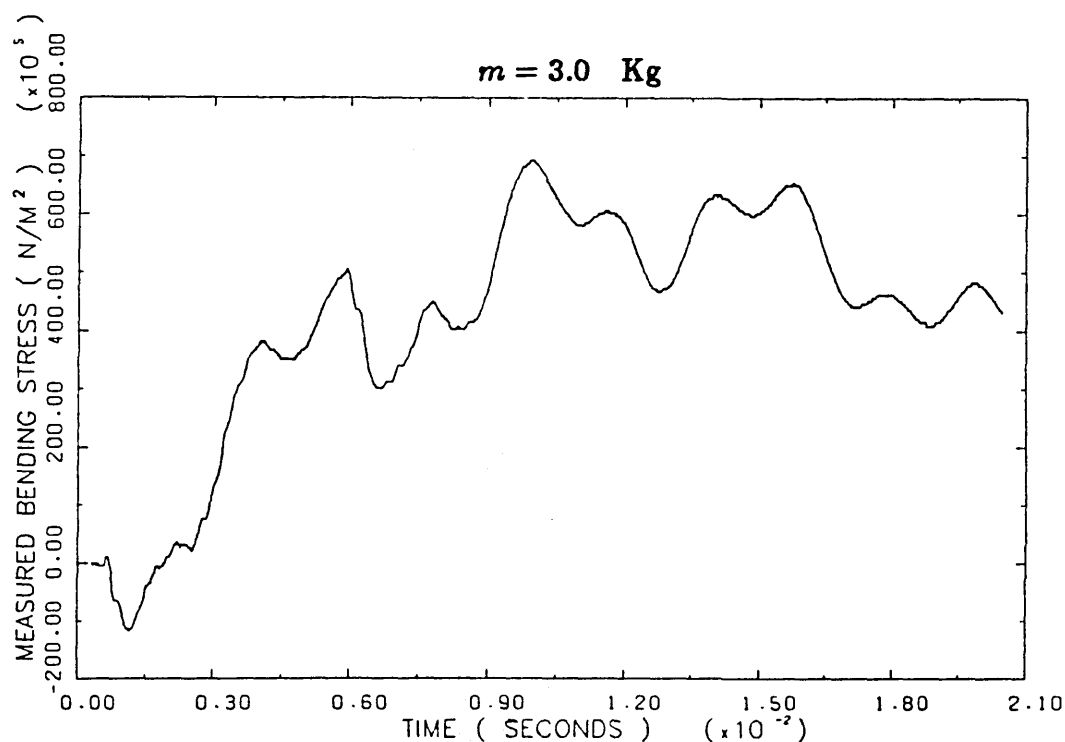
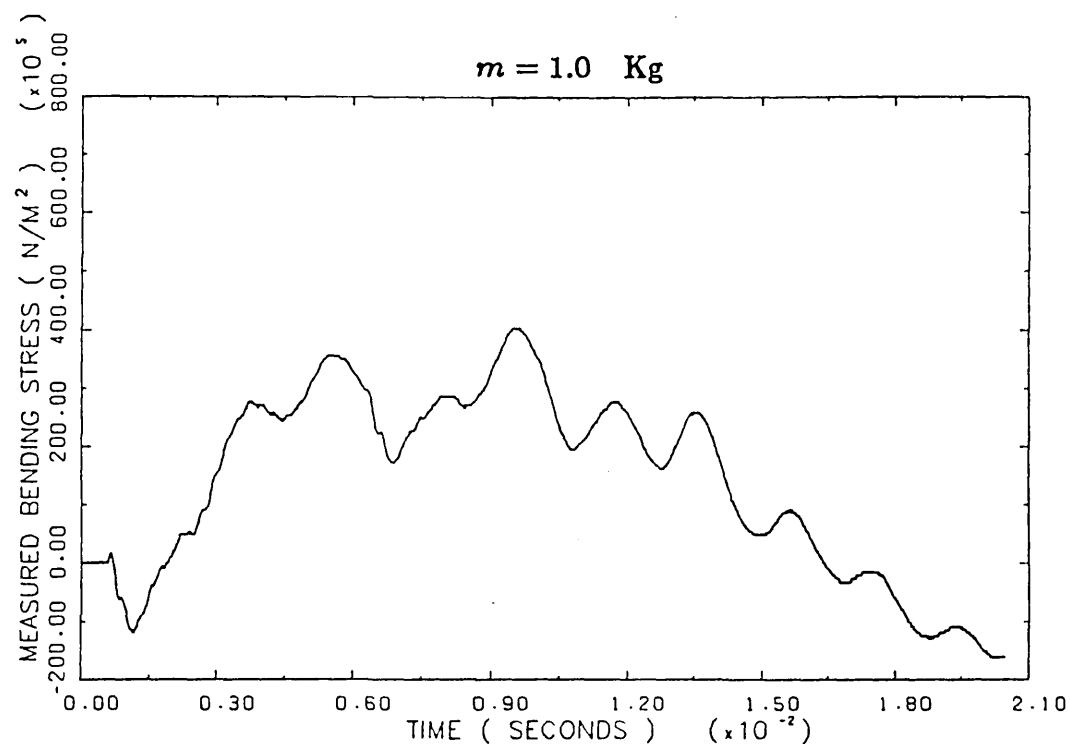


FIG. (4.36) - The experimentally measured variation of bending stress with time, at 25 cm from the point of impact, for test cases (MT22) and (MT28), showing the effect of impactor mass on response for the medium length (0.8 m) metal beam specimen.

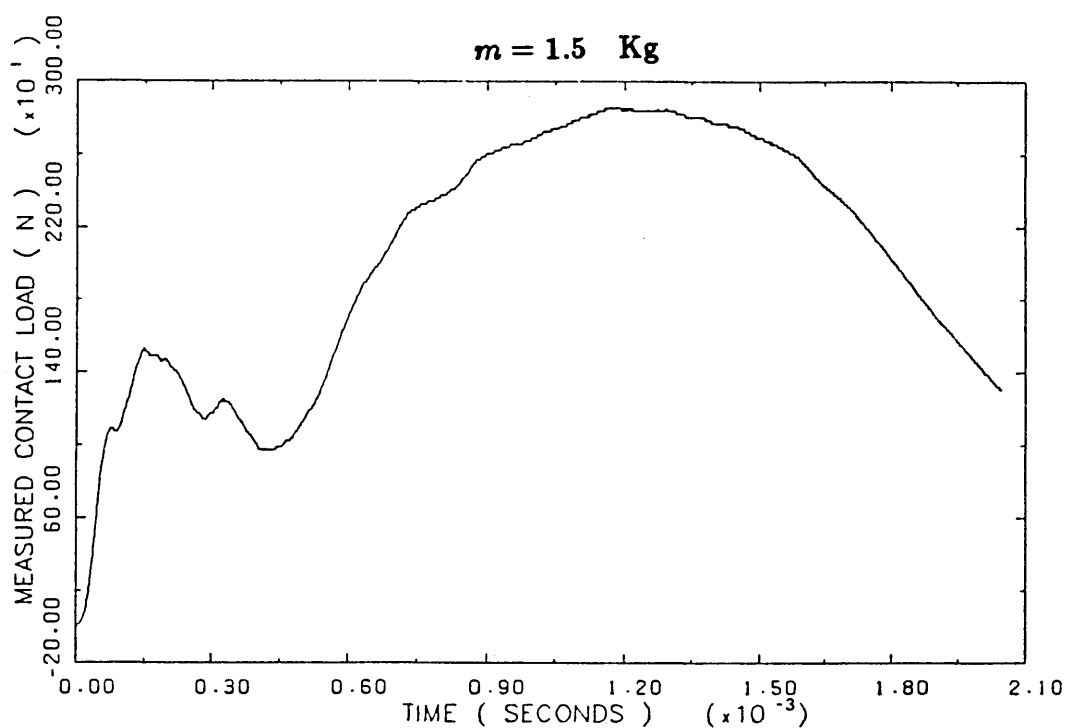
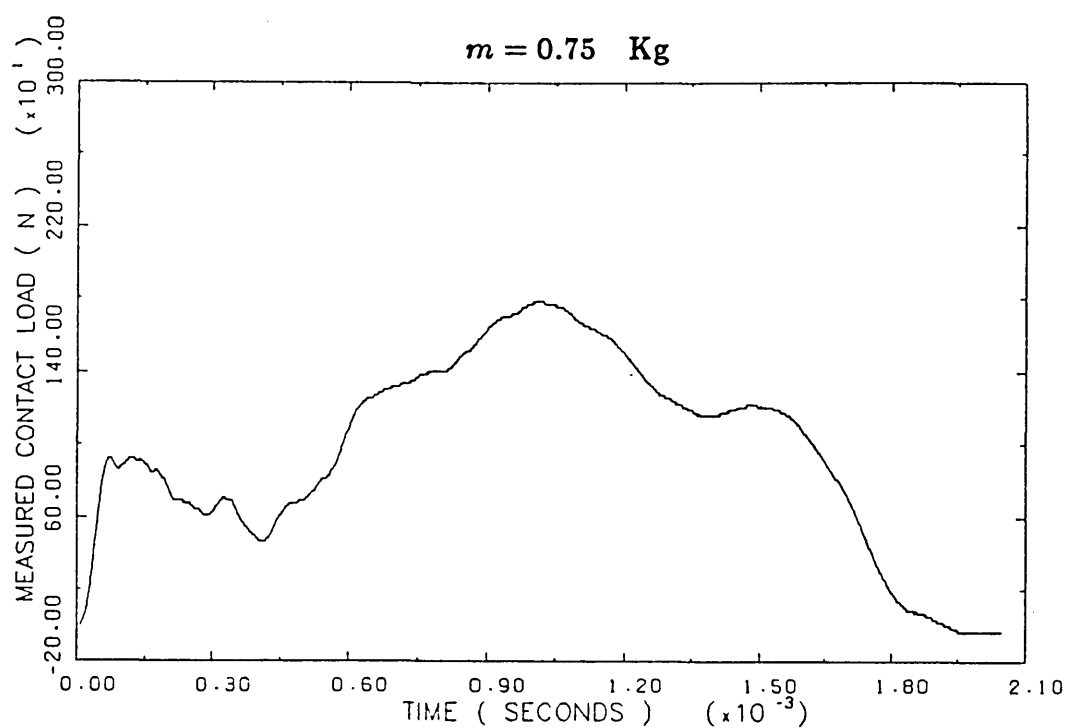


FIG. (4.37) - The experimentally measured variation of contact force with time for test cases (MT36) and (MT40), showing the effect of impactor mass on response for the short (0.2 m) metal beam specimen.

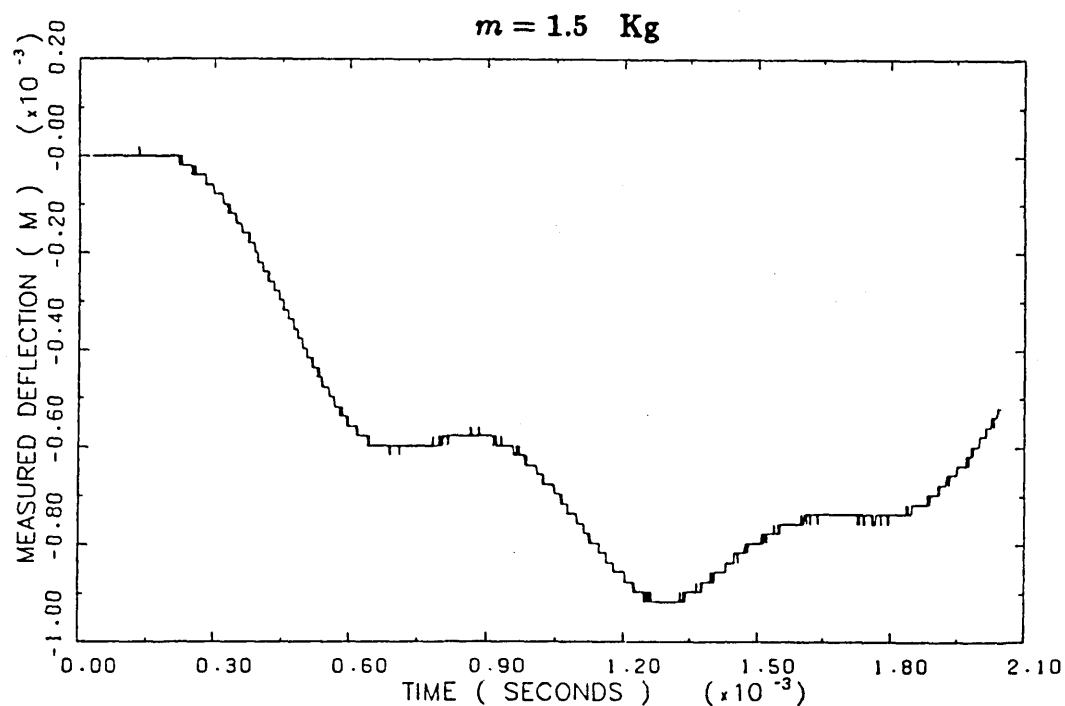
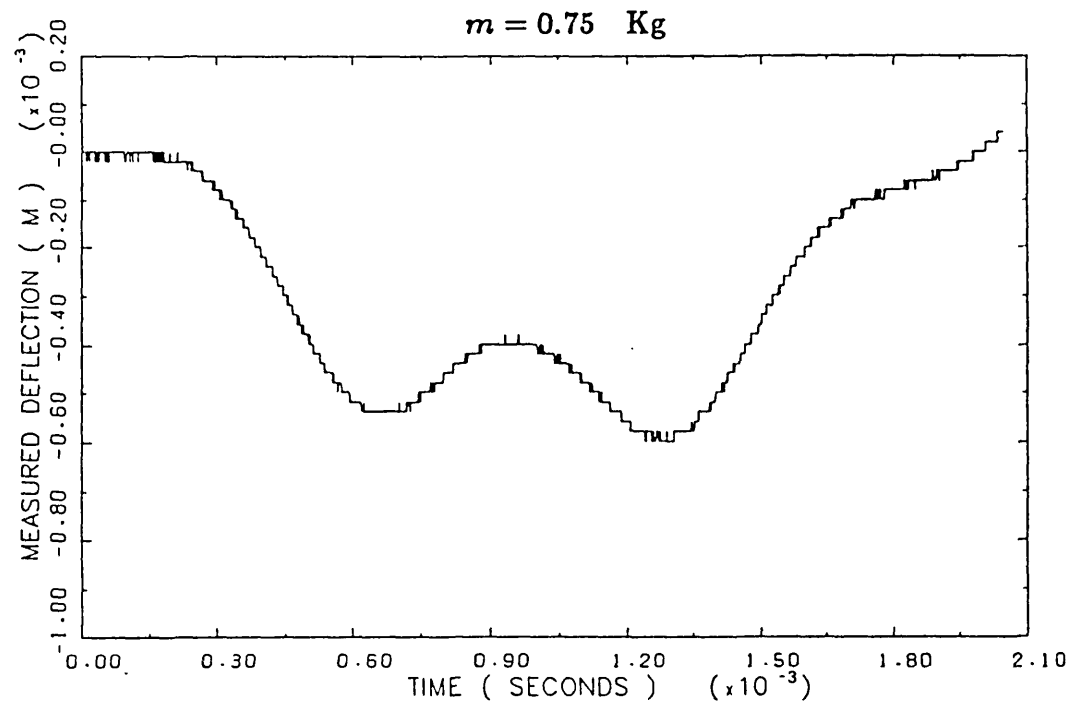


FIG. (4.38) - The experimentally measured variation of displacement with time, at the beam centre, for test cases (MT36) and (MT40), showing the effect of impactor mass on response for the short (0.2 m) metal beam specimen.

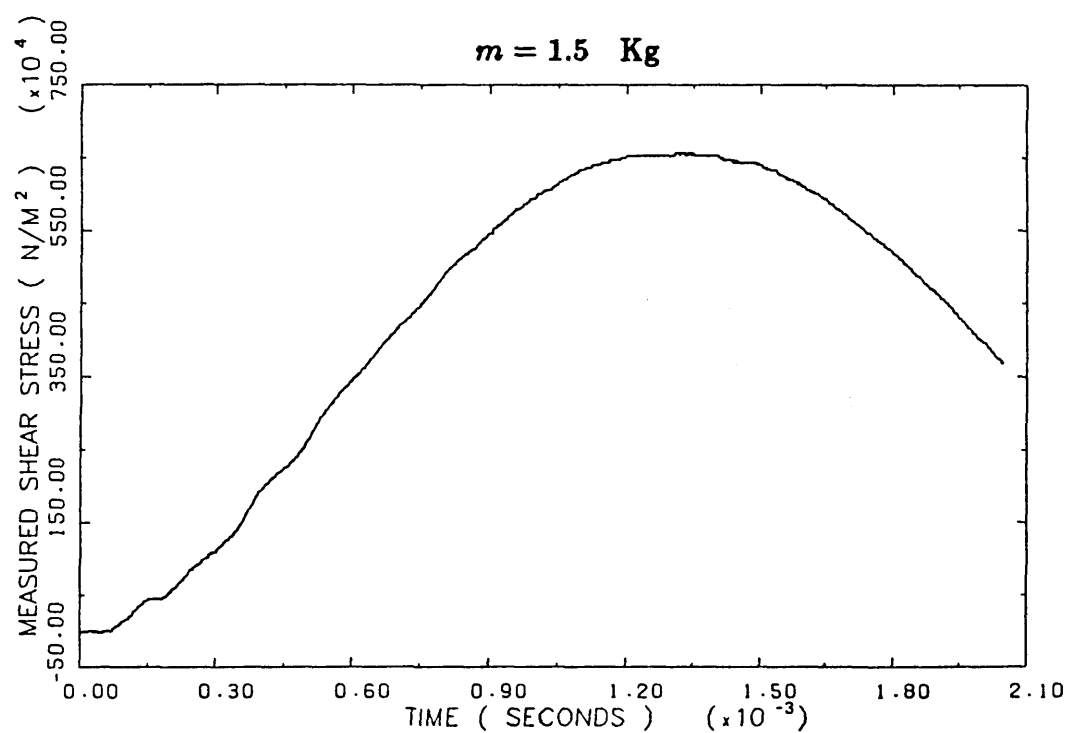
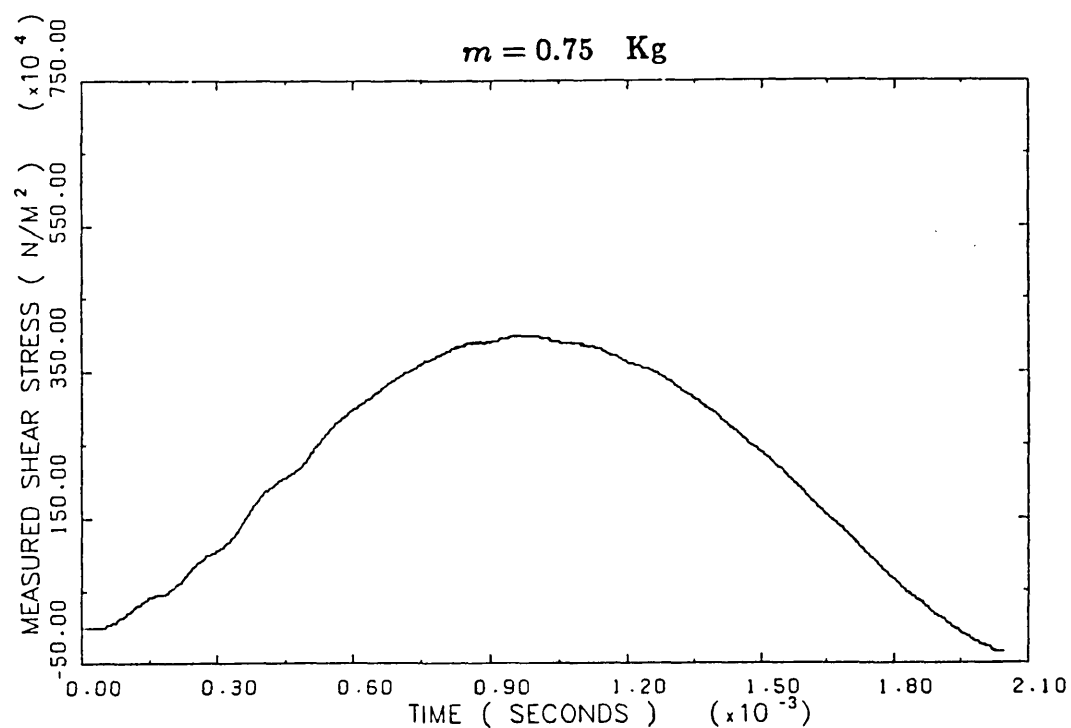


FIG. (4.39) - The experimentally measured variation of shear stress with time, at 5 cm from the point of impact, for test cases (MT36) and (MT40), showing the effect of impactor mass on response for the short (0.2 m) metal beam specimen.

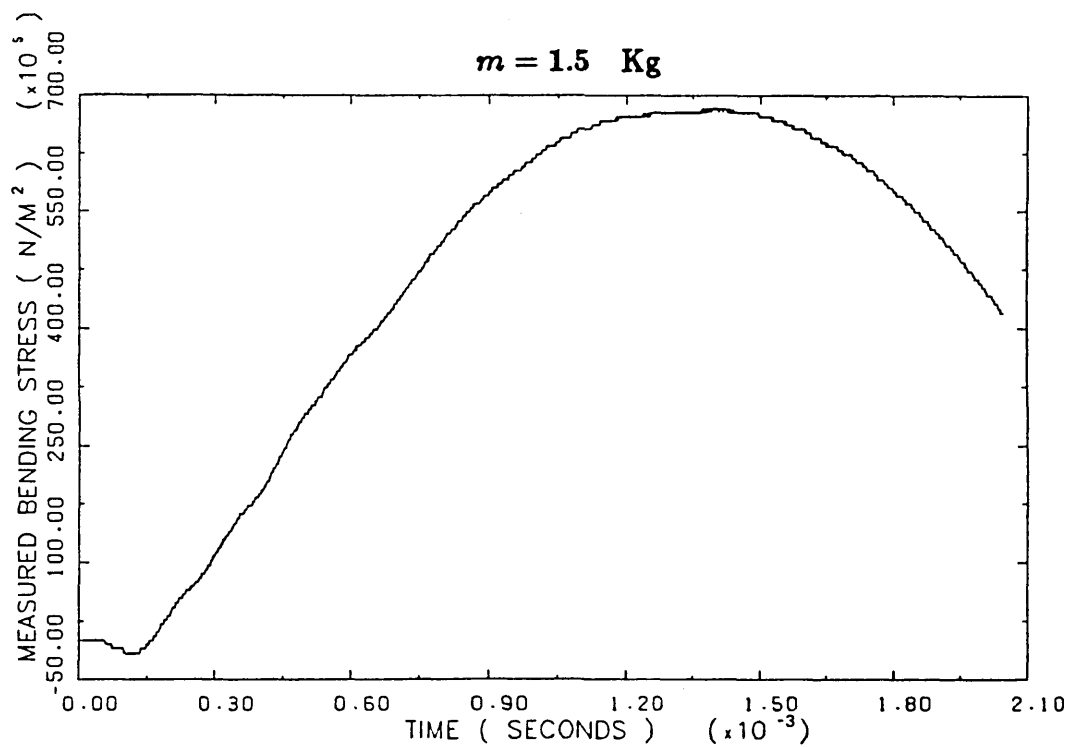
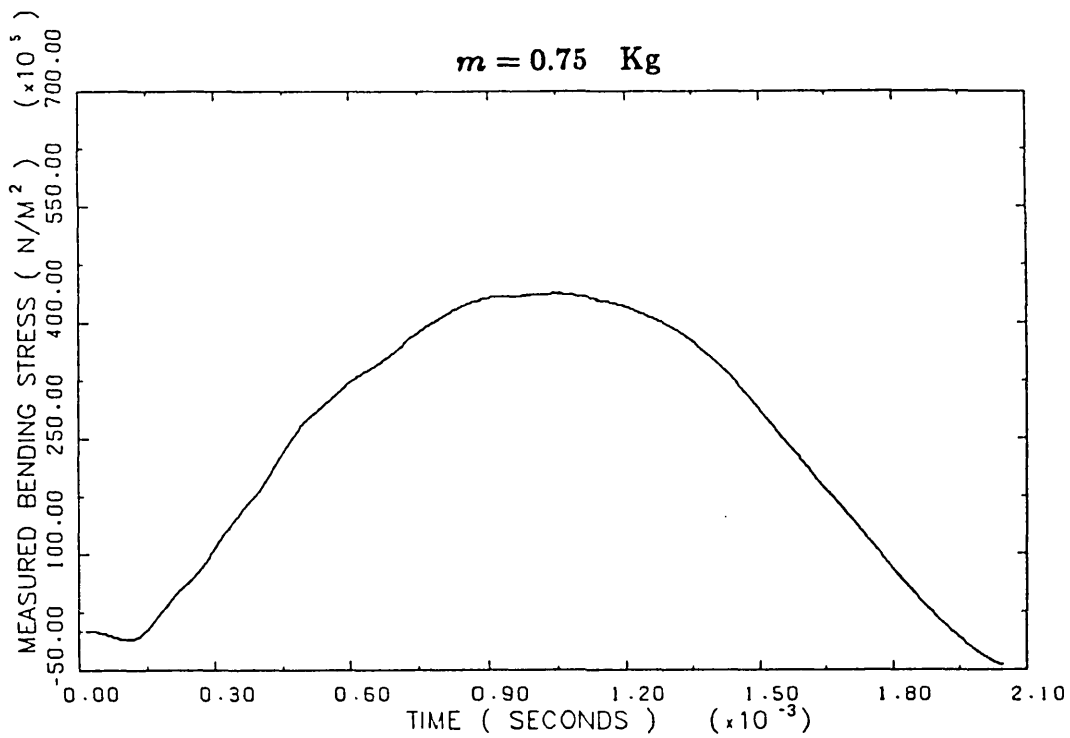
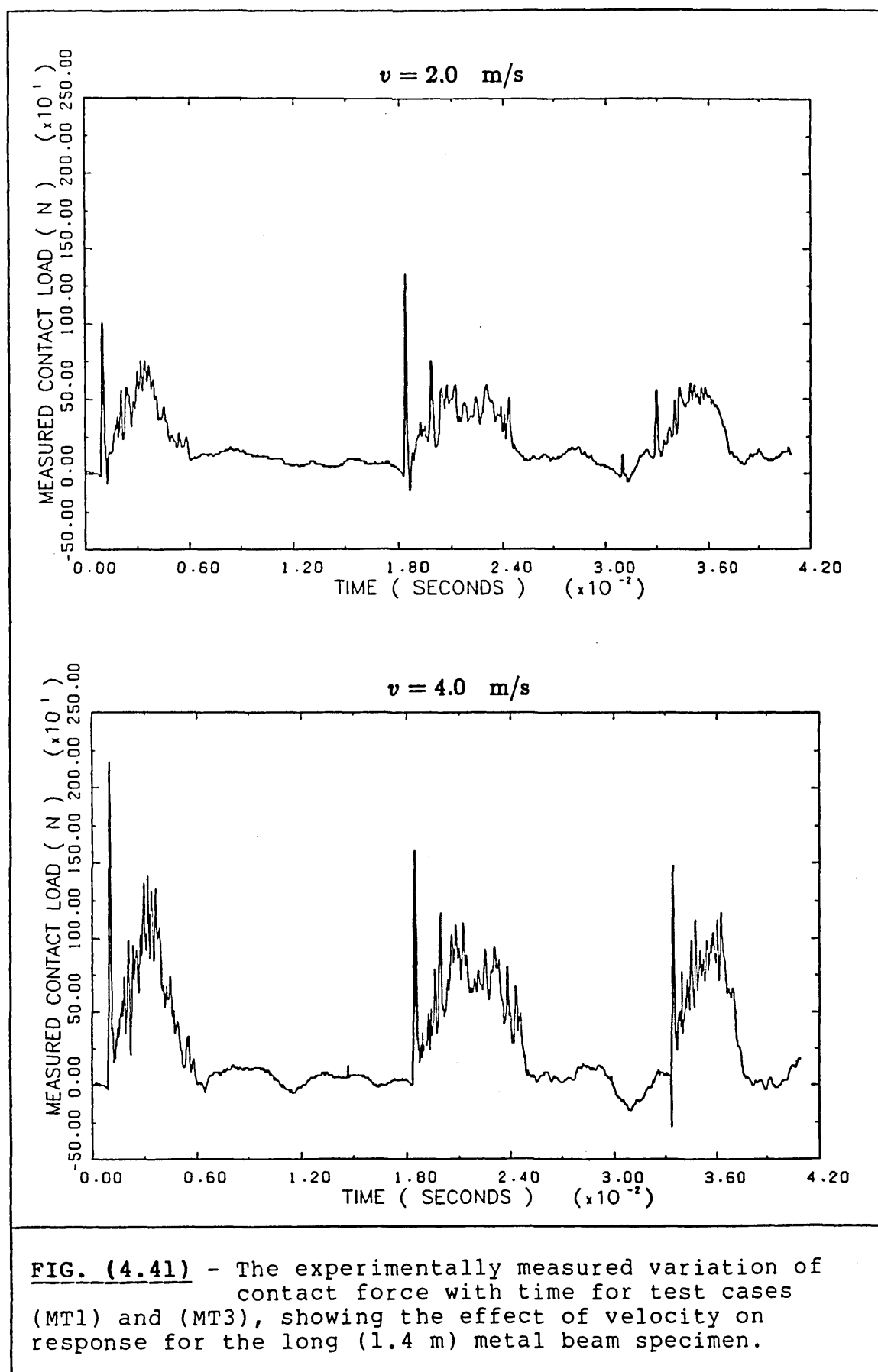


FIG. (4.40) - The experimentally measured variation of bending stress with time, at 5 cm from the point of impact, for test cases (MT36) and (MT40), showing the effect of impactor mass on response for the short (0.2 m) metal beam specimen.



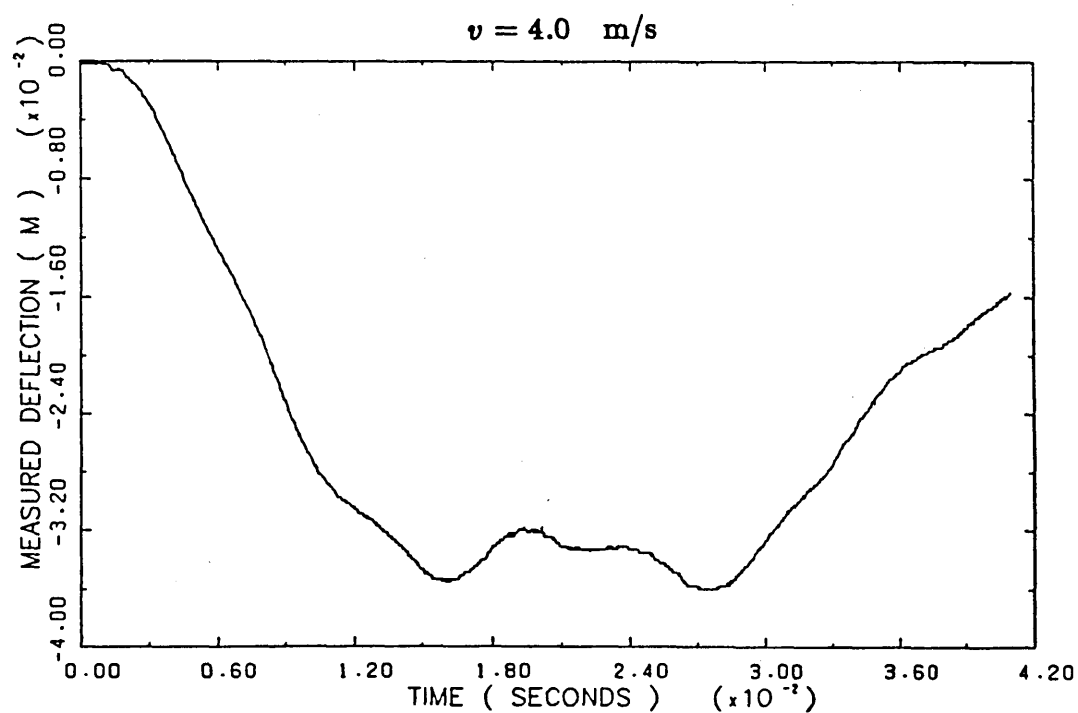
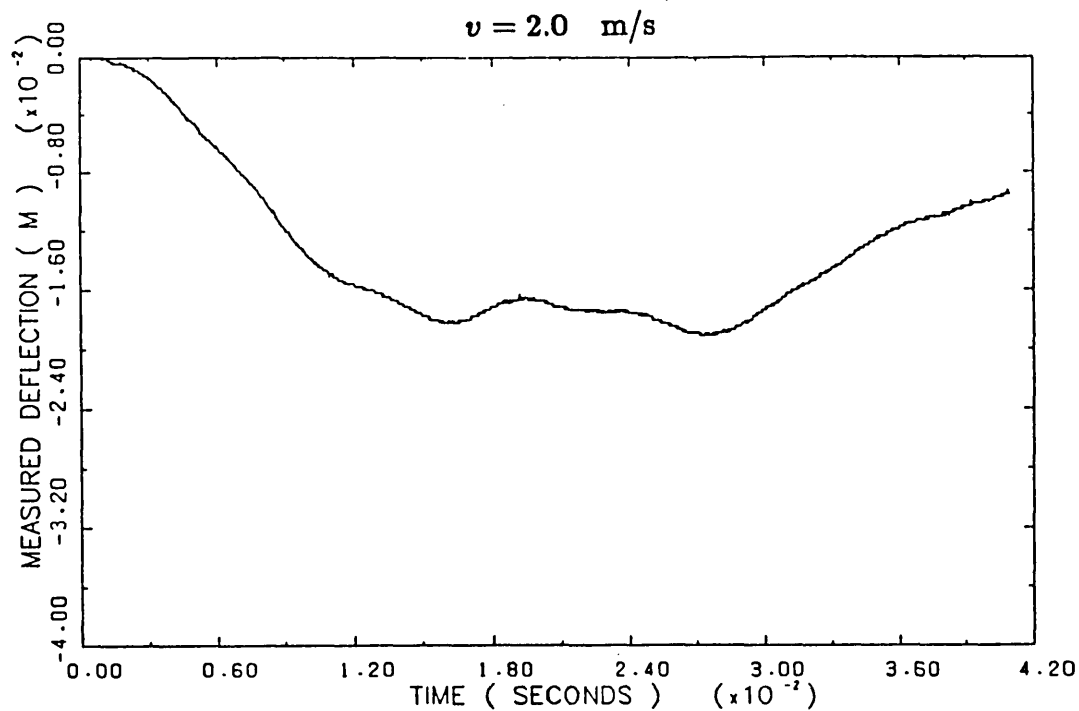


FIG. (4.42) - The experimentally measured variation of displacement with time, at the beam centre, for test cases (MT1) and (MT3), showing the effect of velocity on response for the long (1.4 m) metal beam specimen.

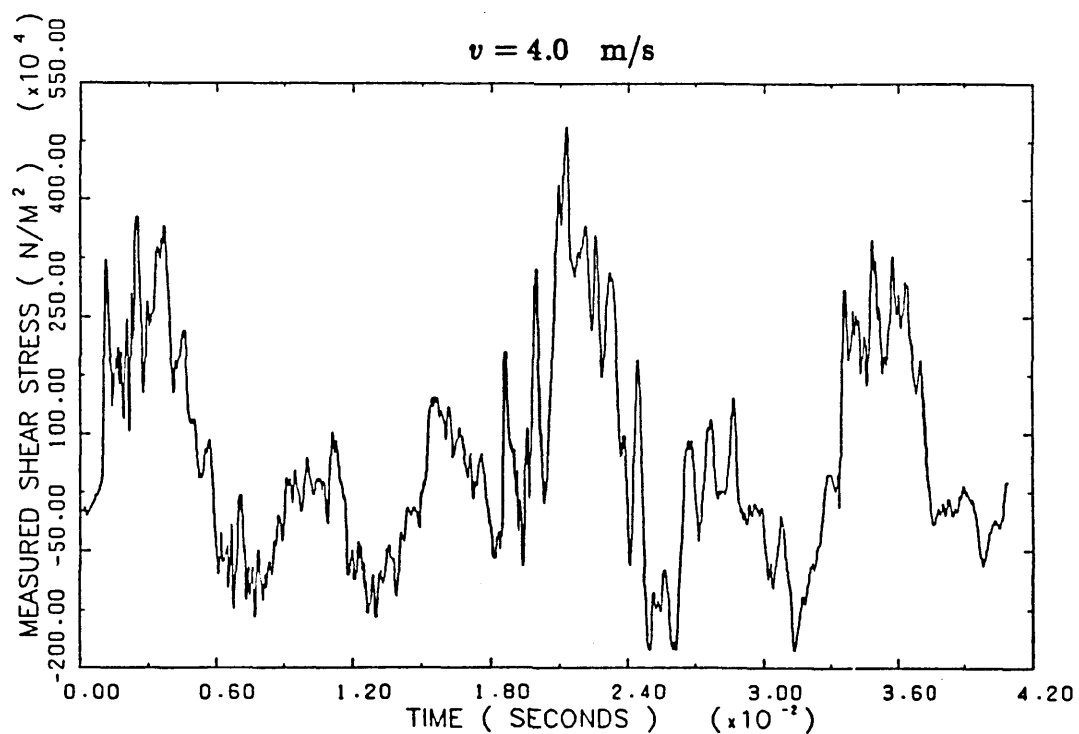
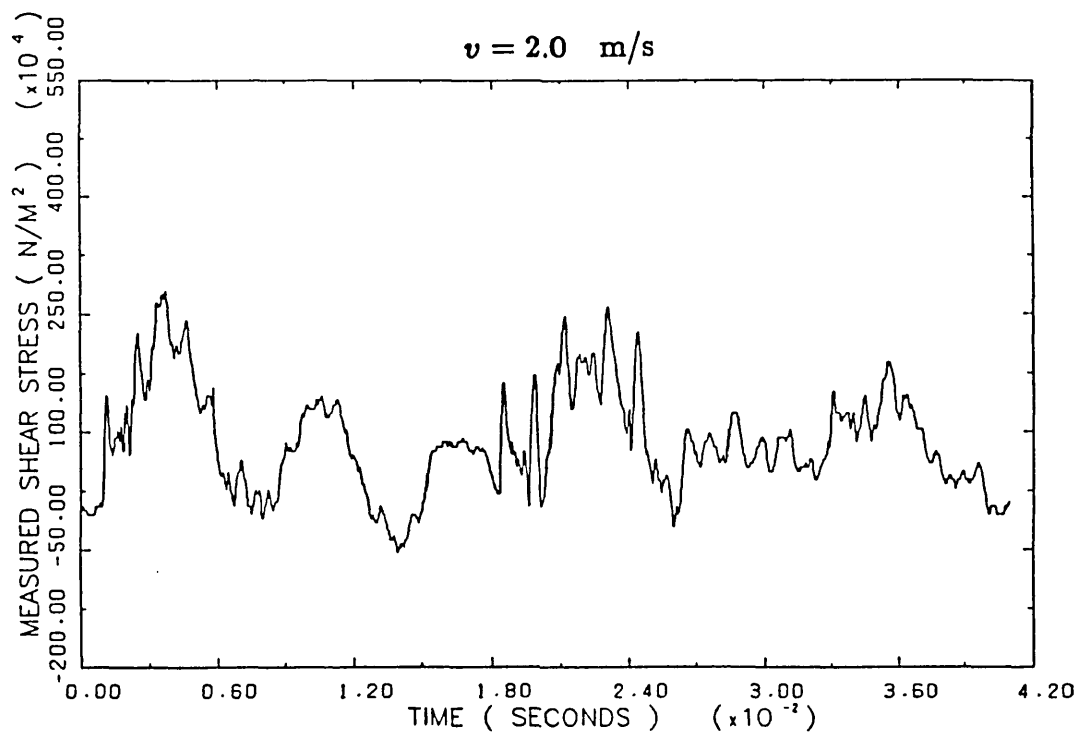


FIG. (4.43) - The experimentally measured variation of shear stress with time, at 5 cm from the point of impact, for test cases (MT1) and (MT3), showing the effect of velocity on response for the long (1.4 m) metal beam specimen.

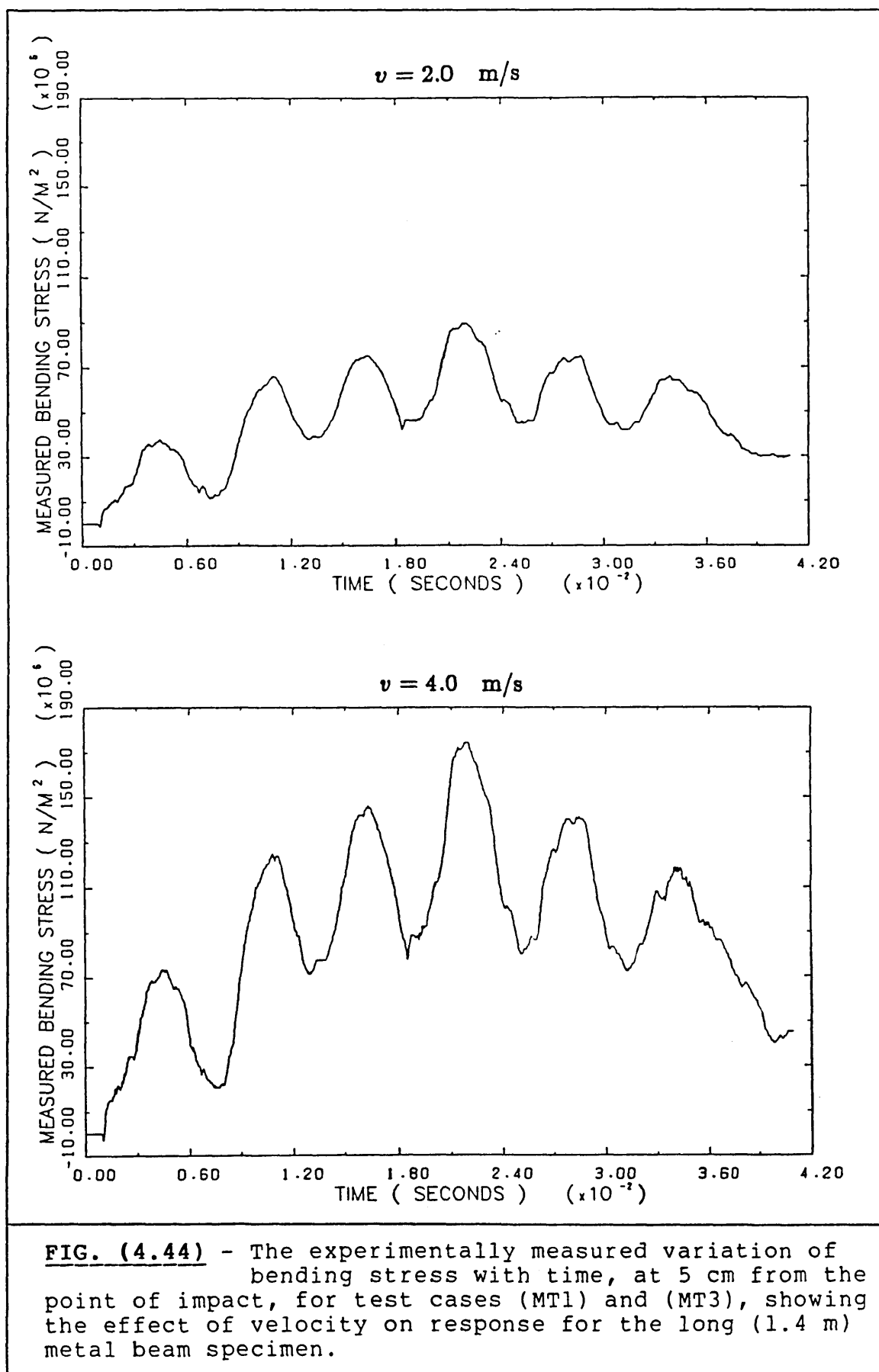


FIG. (4.44) - The experimentally measured variation of bending stress with time, at 5 cm from the point of impact, for test cases (MT1) and (MT3), showing the effect of velocity on response for the long (1.4 m) metal beam specimen.

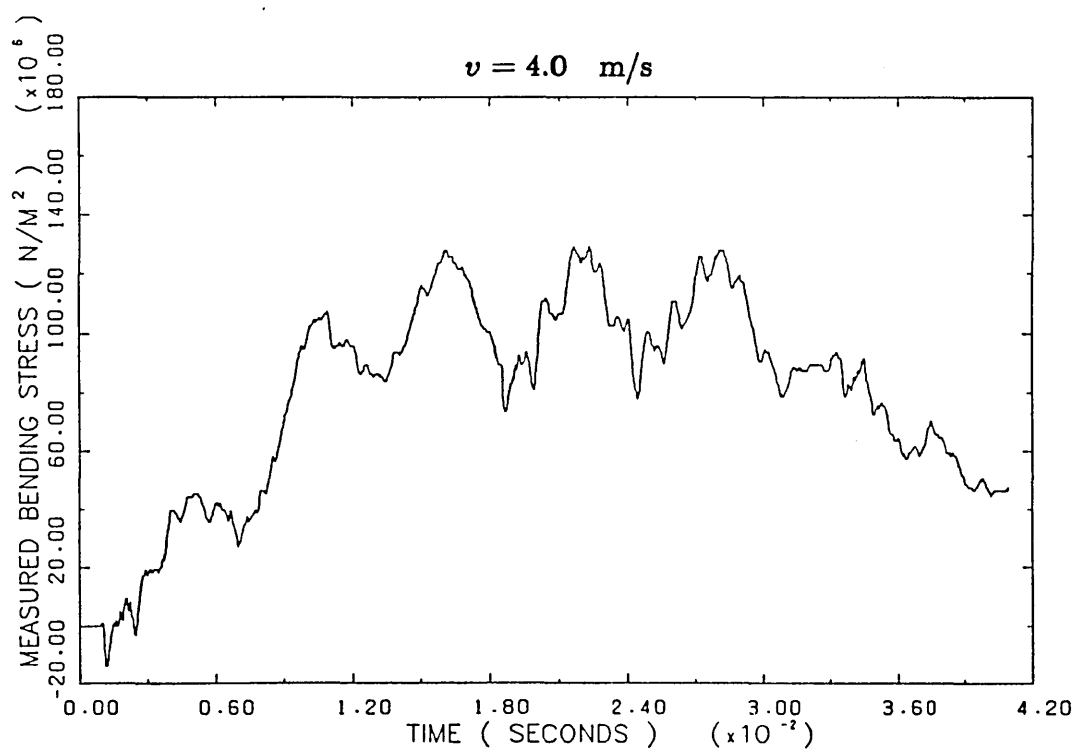
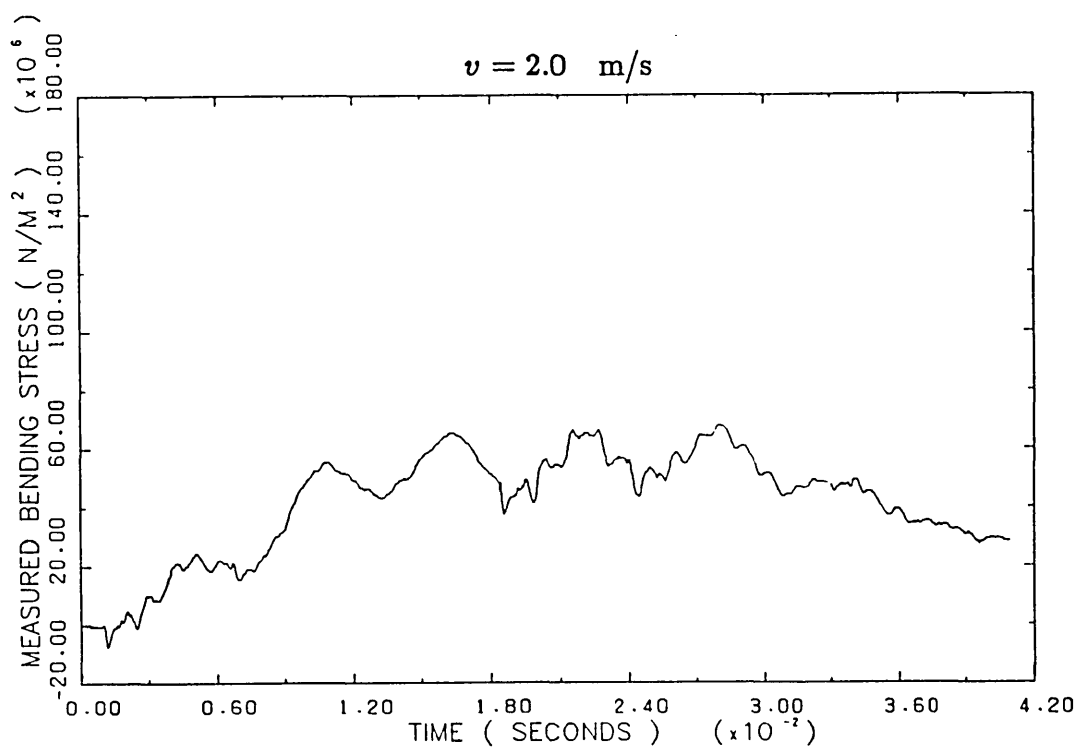


FIG. (4.45) - The experimentally measured variation of bending stress with time, at 15 cm from the point of impact, for test cases (MT1) and (MT3), showing the effect of velocity on response for the long (1.4 m) metal beam specimen.

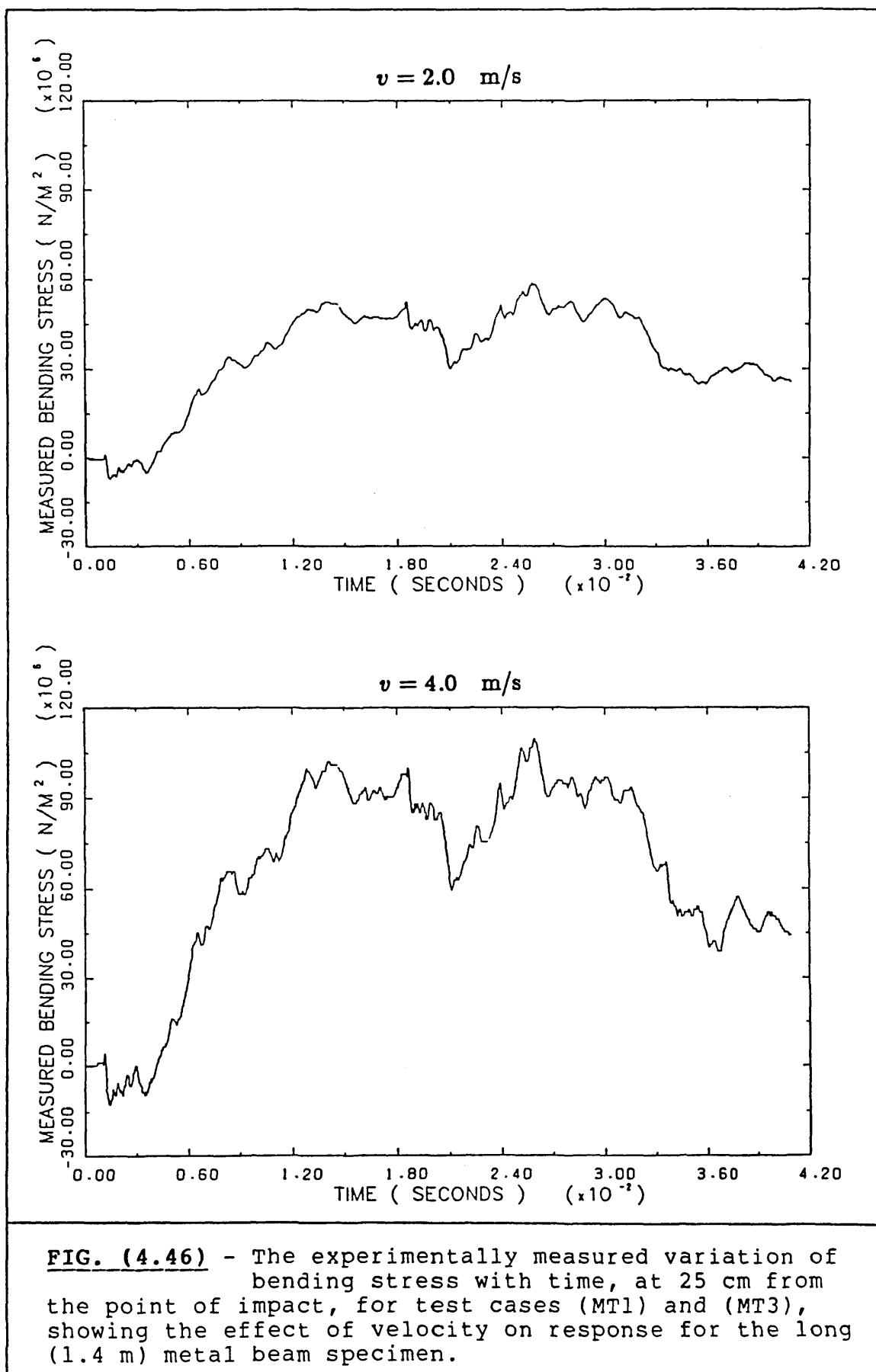
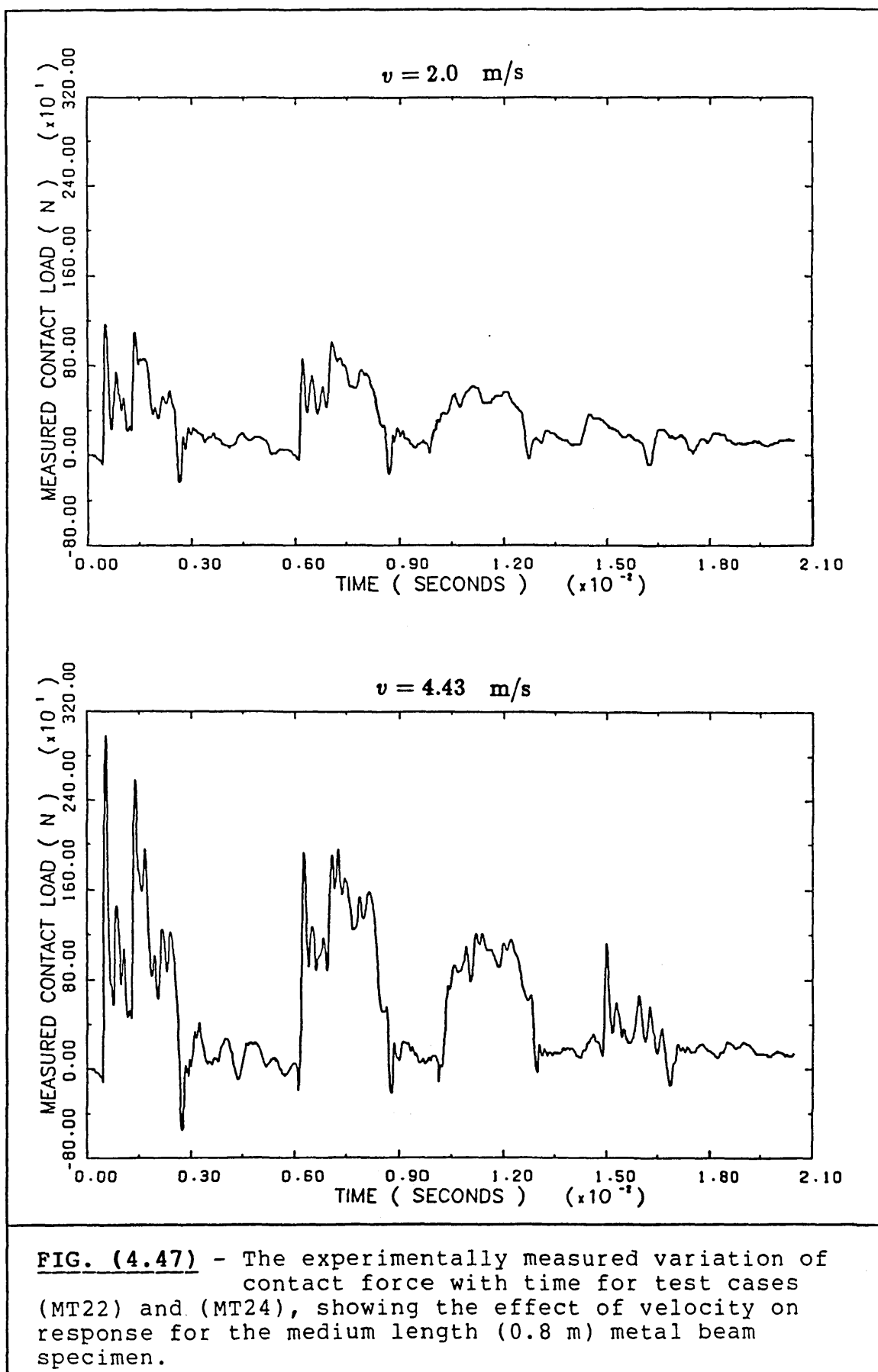
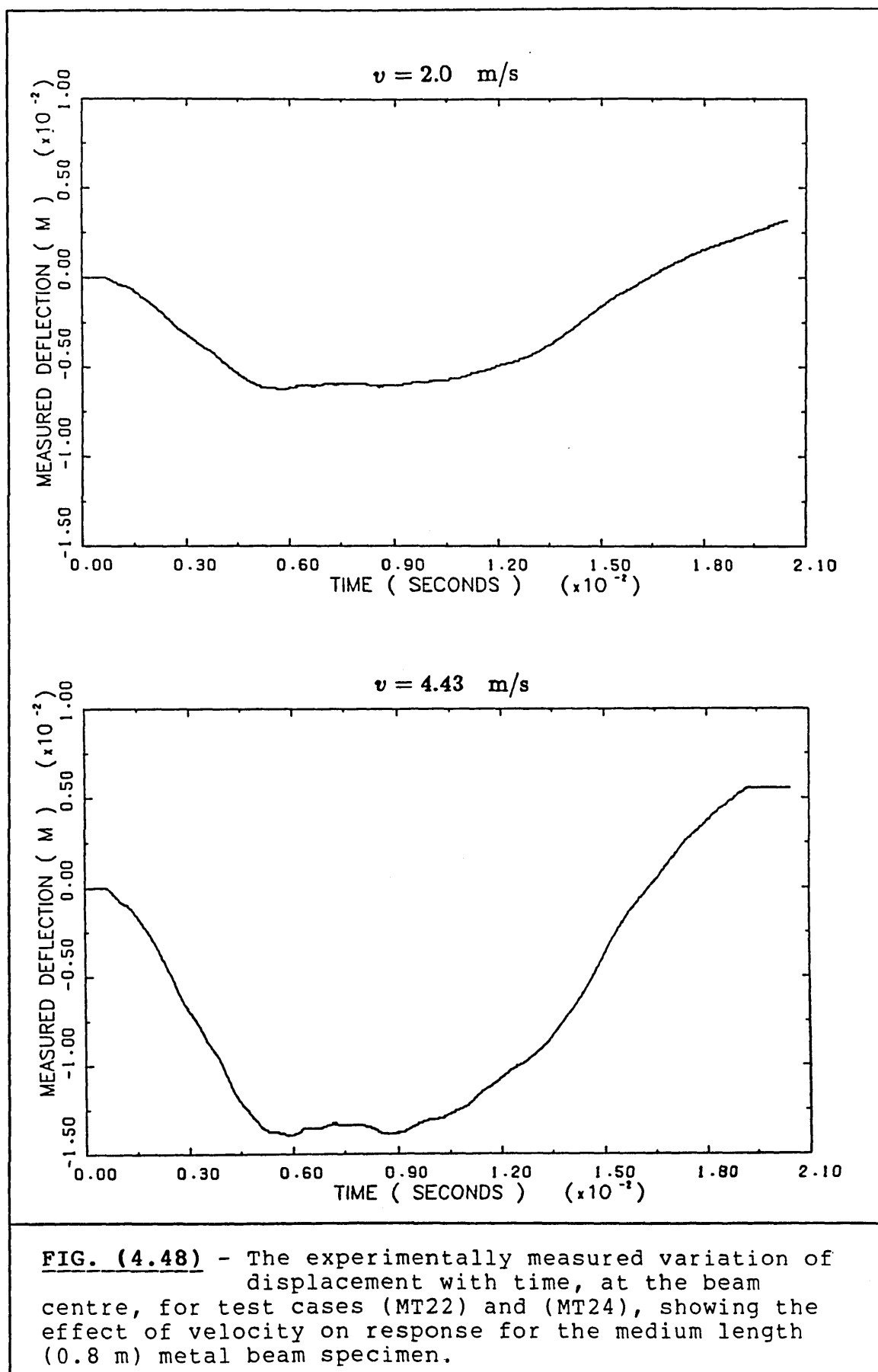


FIG. (4.46) - The experimentally measured variation of bending stress with time, at 25 cm from the point of impact, for test cases (MT1) and (MT3), showing the effect of velocity on response for the long (1.4 m) metal beam specimen.





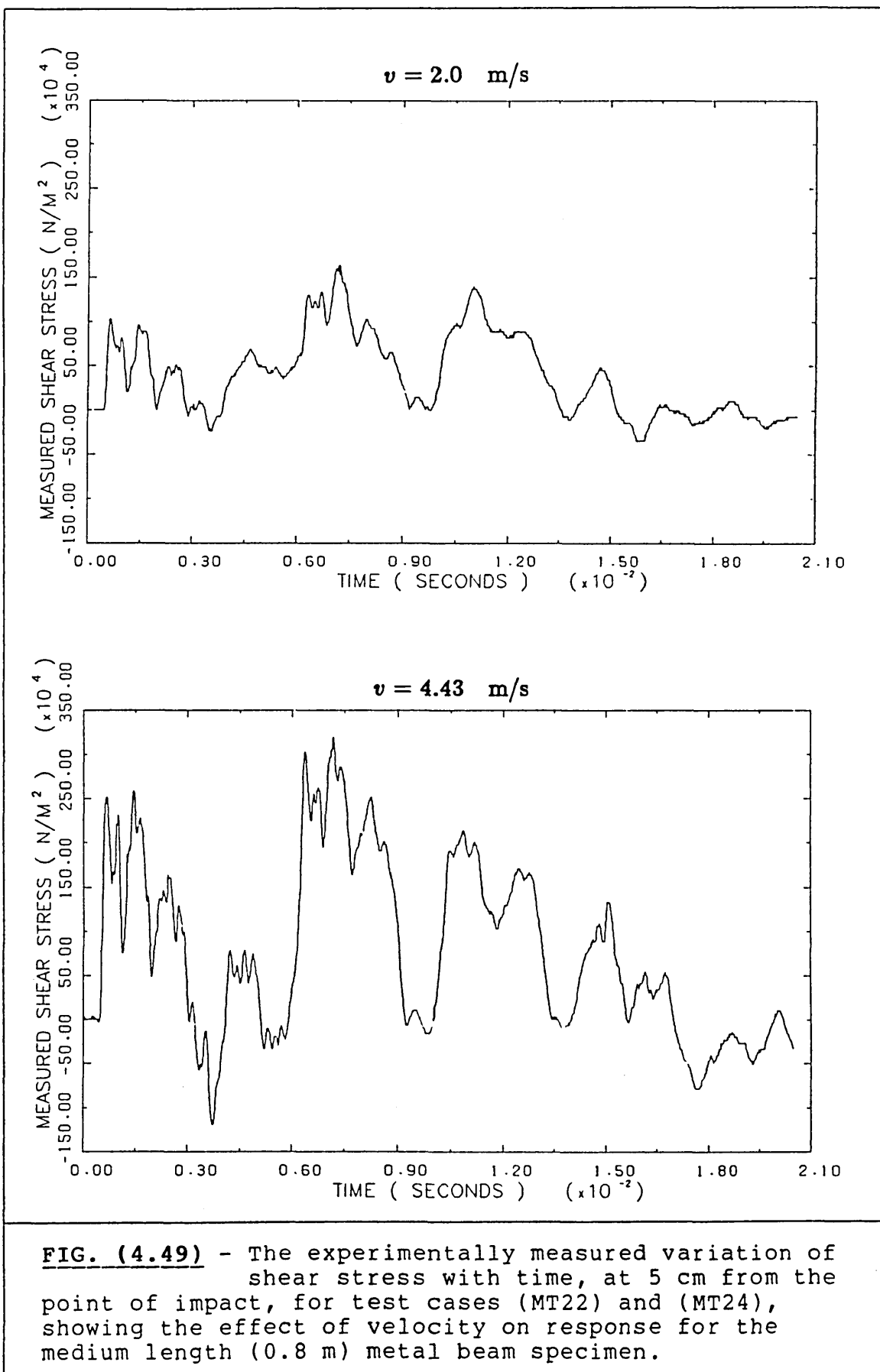


FIG. (4.49) - The experimentally measured variation of shear stress with time, at 5 cm from the point of impact, for test cases (MT22) and (MT24), showing the effect of velocity on response for the medium length (0.8 m) metal beam specimen.

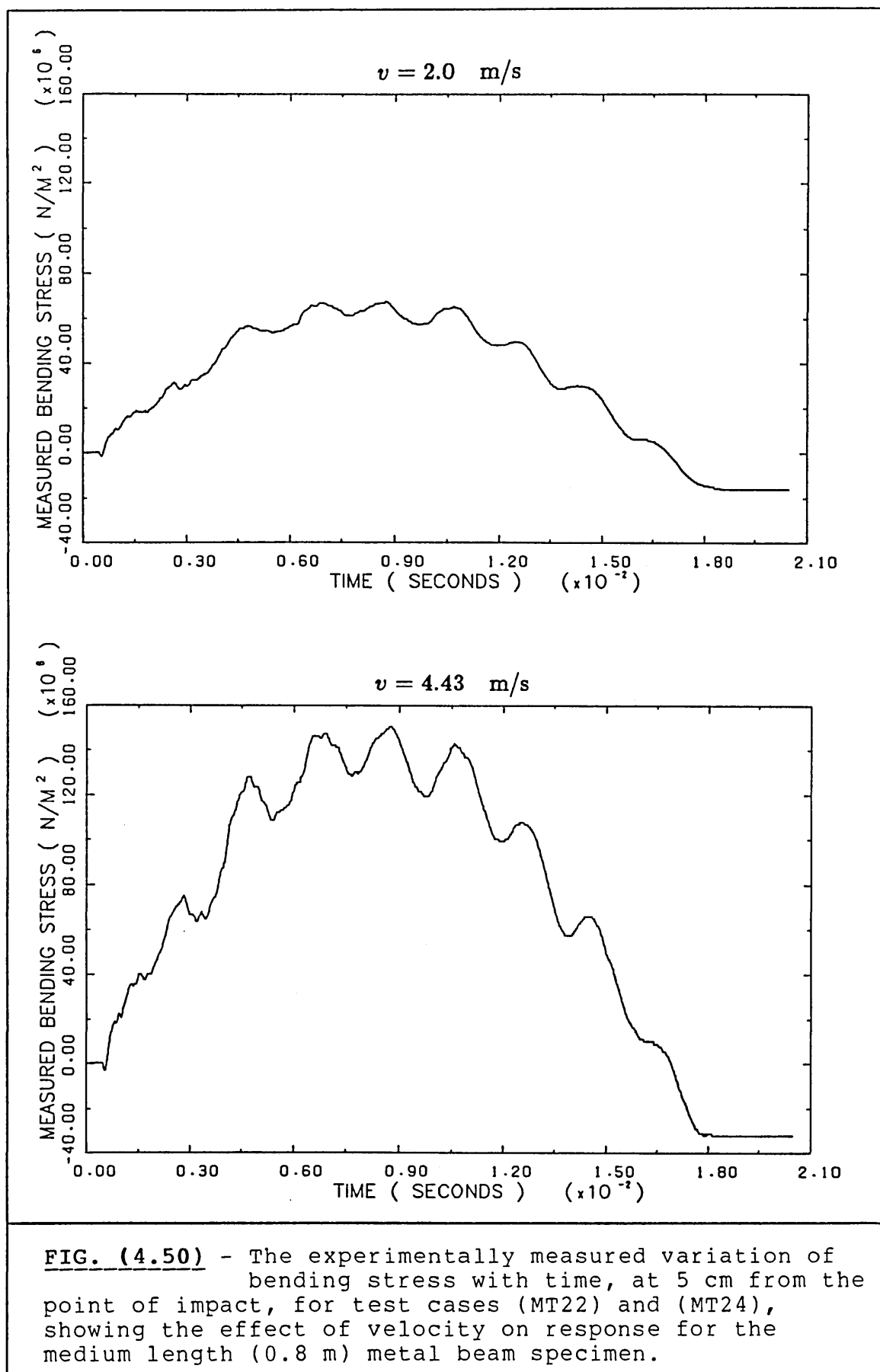


FIG. (4.50) - The experimentally measured variation of bending stress with time, at 5 cm from the point of impact, for test cases (MT22) and (MT24), showing the effect of velocity on response for the medium length (0.8 m) metal beam specimen.

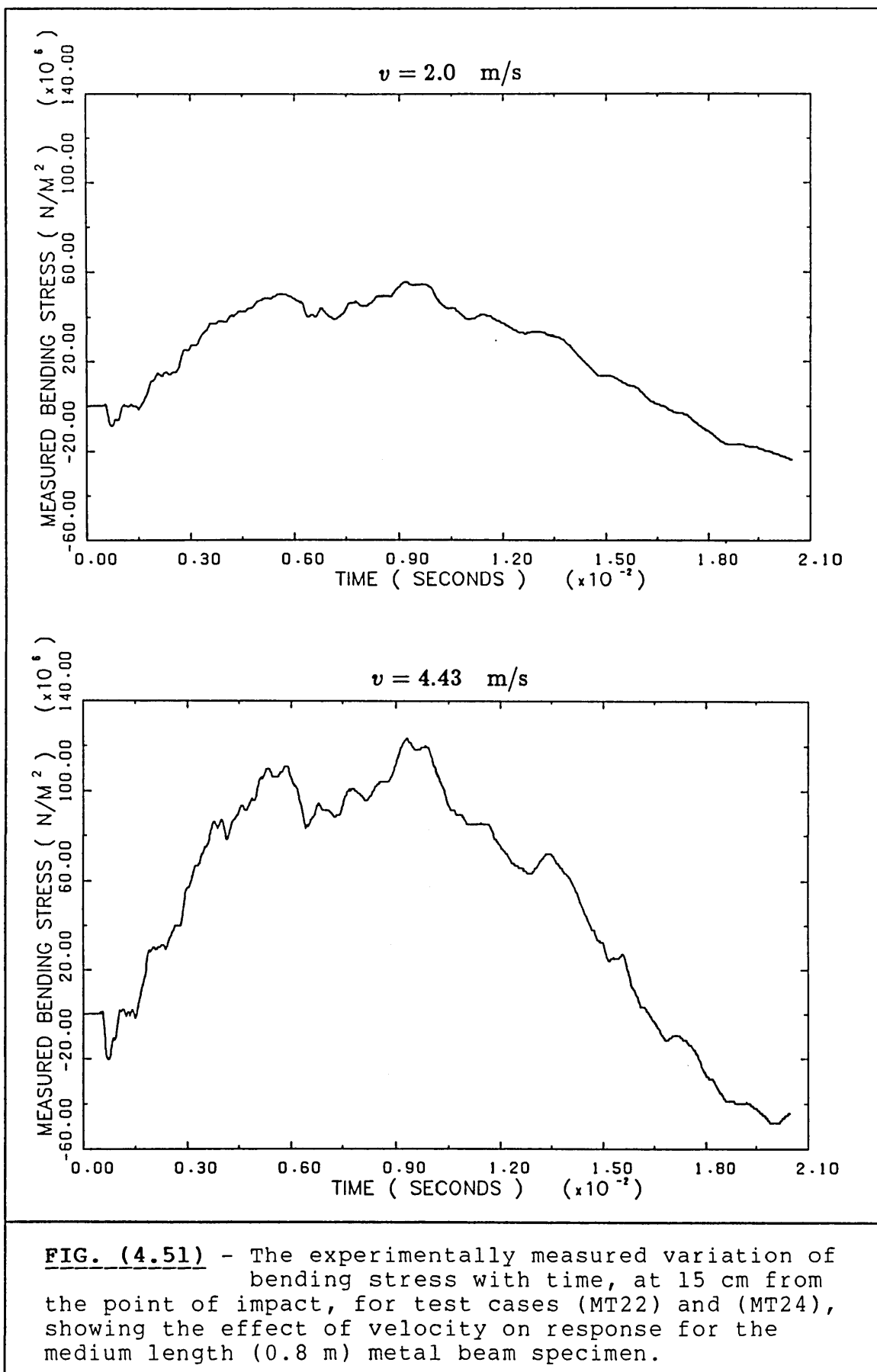


FIG. (4.51) - The experimentally measured variation of bending stress with time, at 15 cm from the point of impact, for test cases (MT22) and (MT24), showing the effect of velocity on response for the medium length (0.8 m) metal beam specimen.

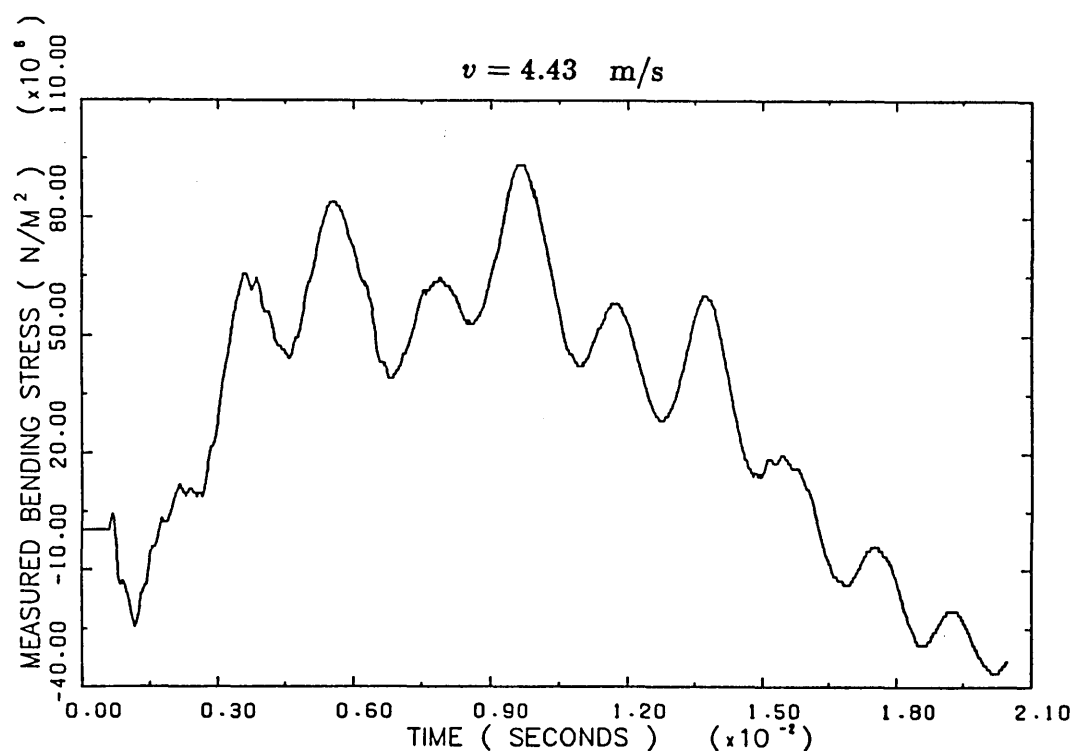
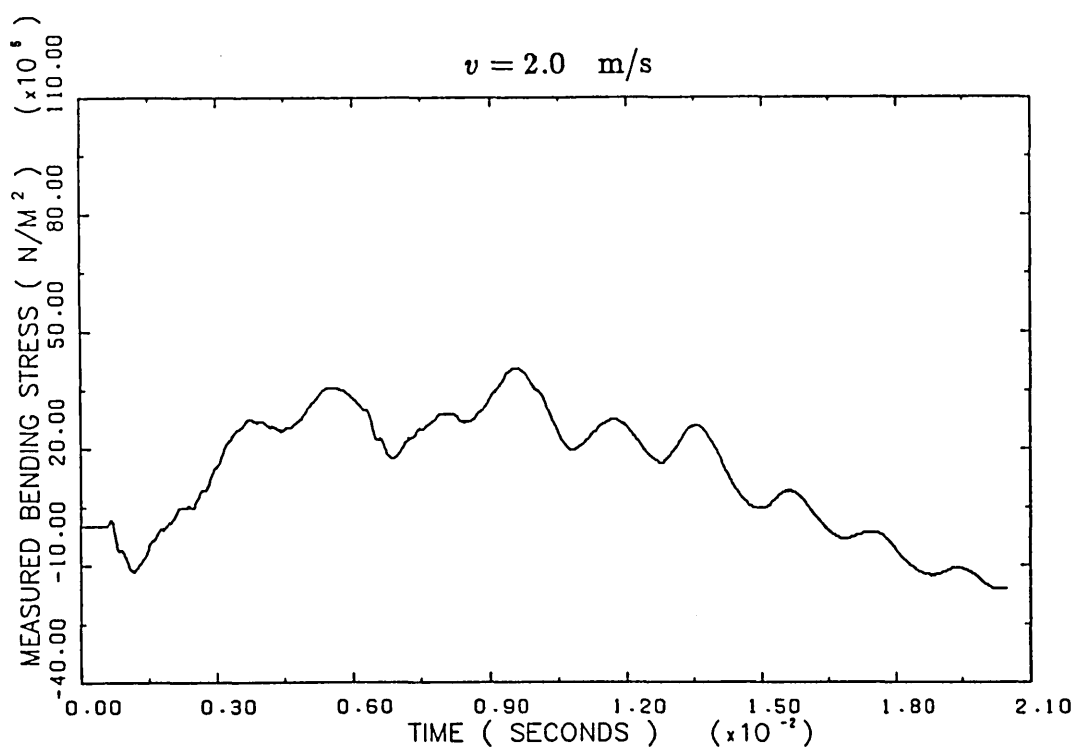


FIG. (4.52) - The experimentally measured variation of bending stress with time, at 25 cm from the point of impact, for test cases (MT22) and (MT24), showing the effect of velocity on response for the medium length (0.8 m) metal beam specimen.

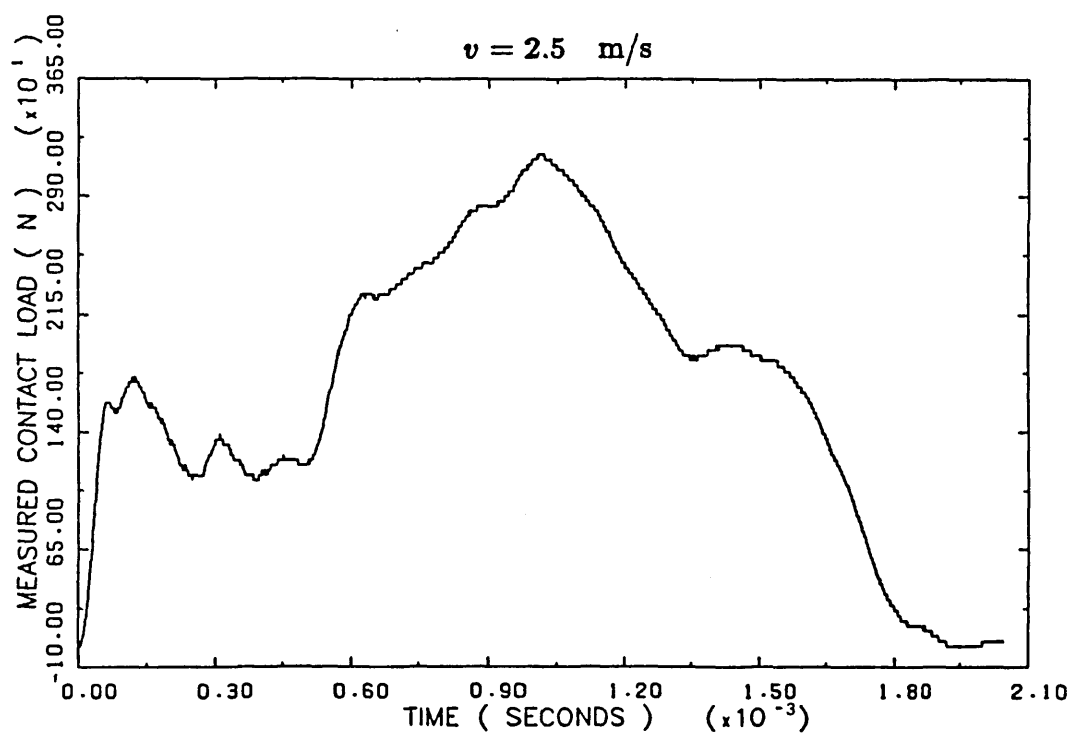
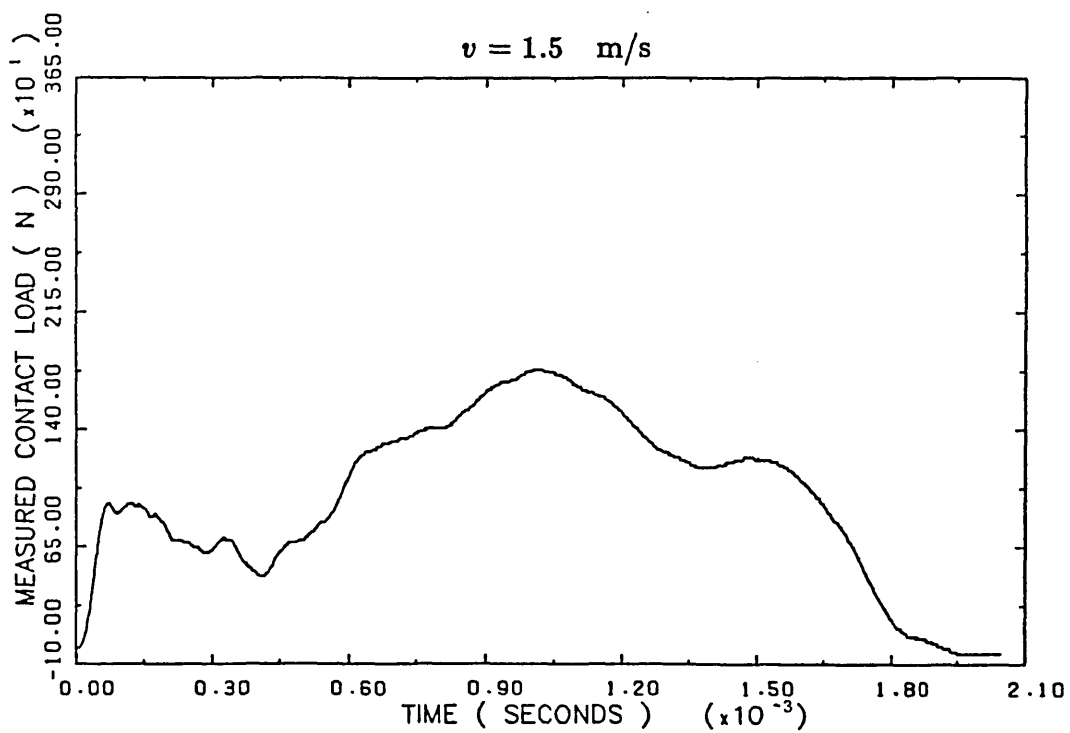


FIG. (4.53) - The experimentally measured variation of contact force with time for test cases (MT36) and (MT38), showing the effect of velocity on response for the short (0.2 m) metal beam specimen.

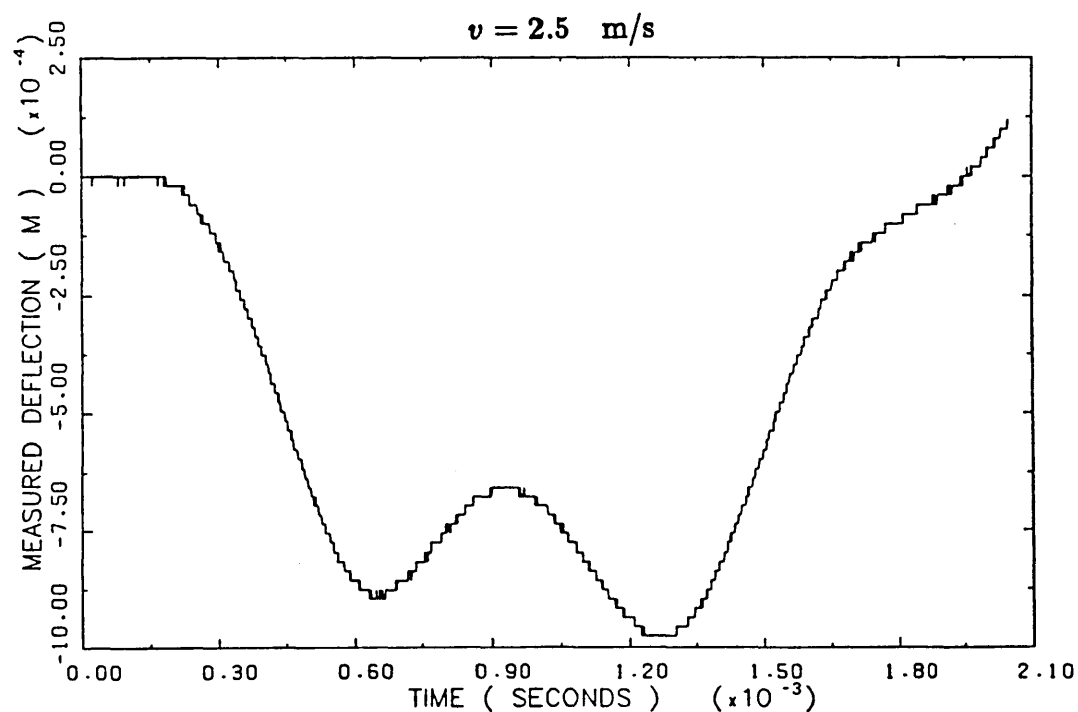
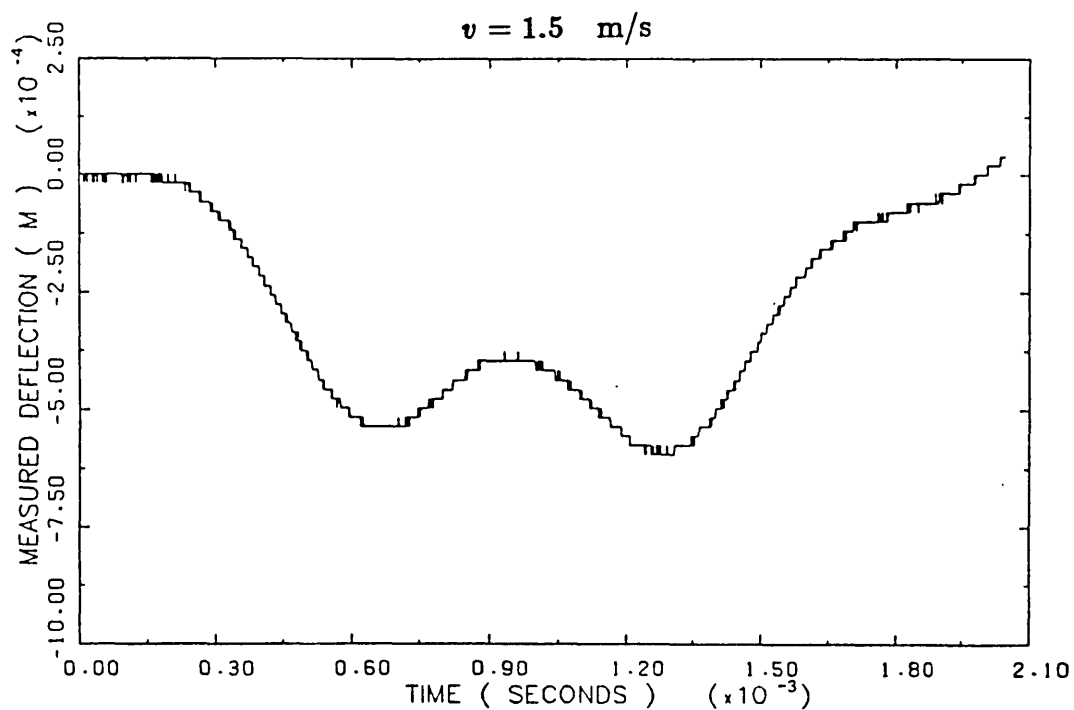


FIG. (4.54) - The experimentally measured variation of displacement with time, at the beam centre, for test cases (MT36) and (MT38), showing the effect of velocity on response for the short (0.2 m) metal beam specimen.

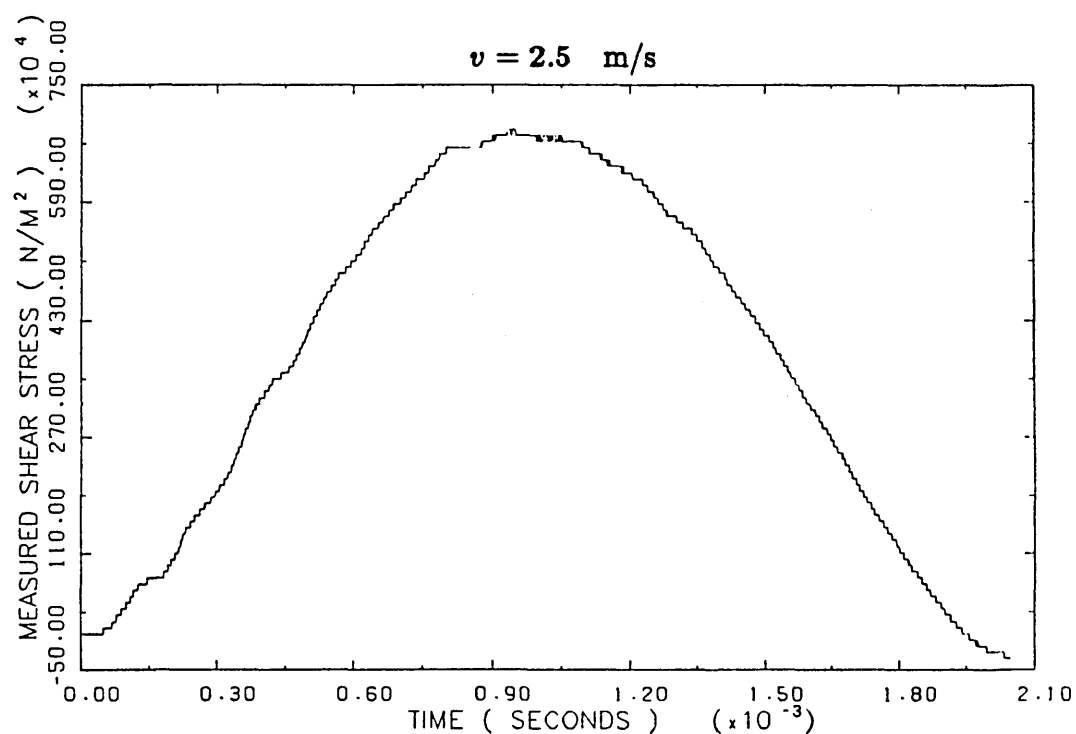
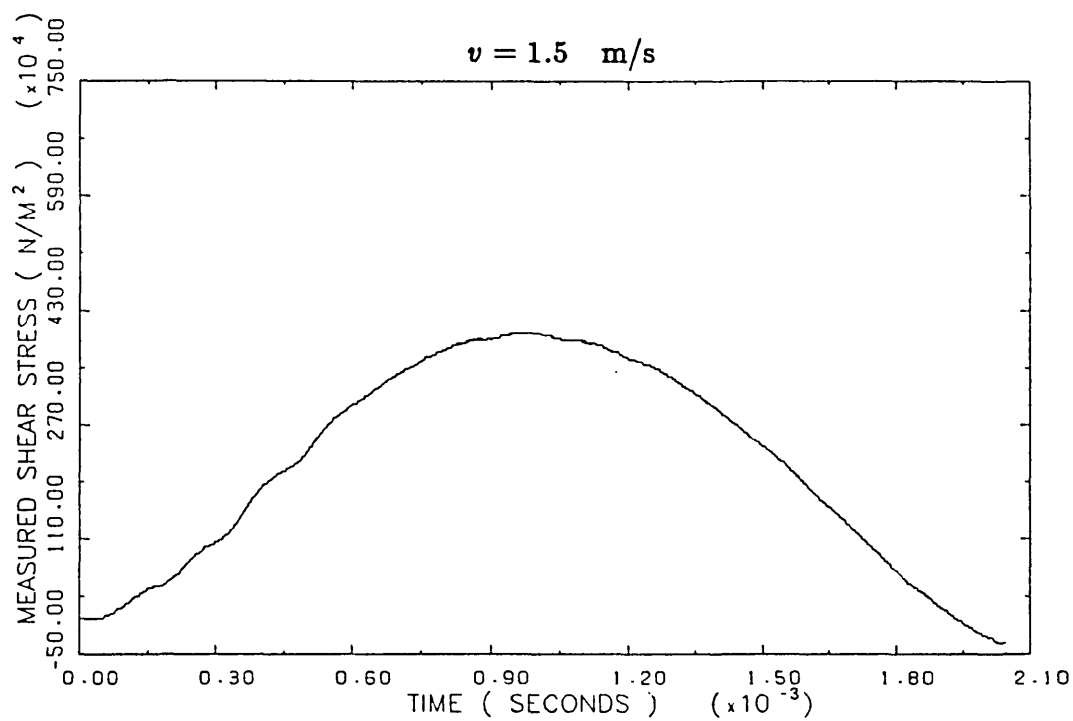


FIG. (4.55) - The experimentally measured variation of shear stress with time, at 5 cm from the point of impact, for test cases (MT36) and (MT38), showing the effect of velocity on response for the short (0.2 m) metal beam specimen.

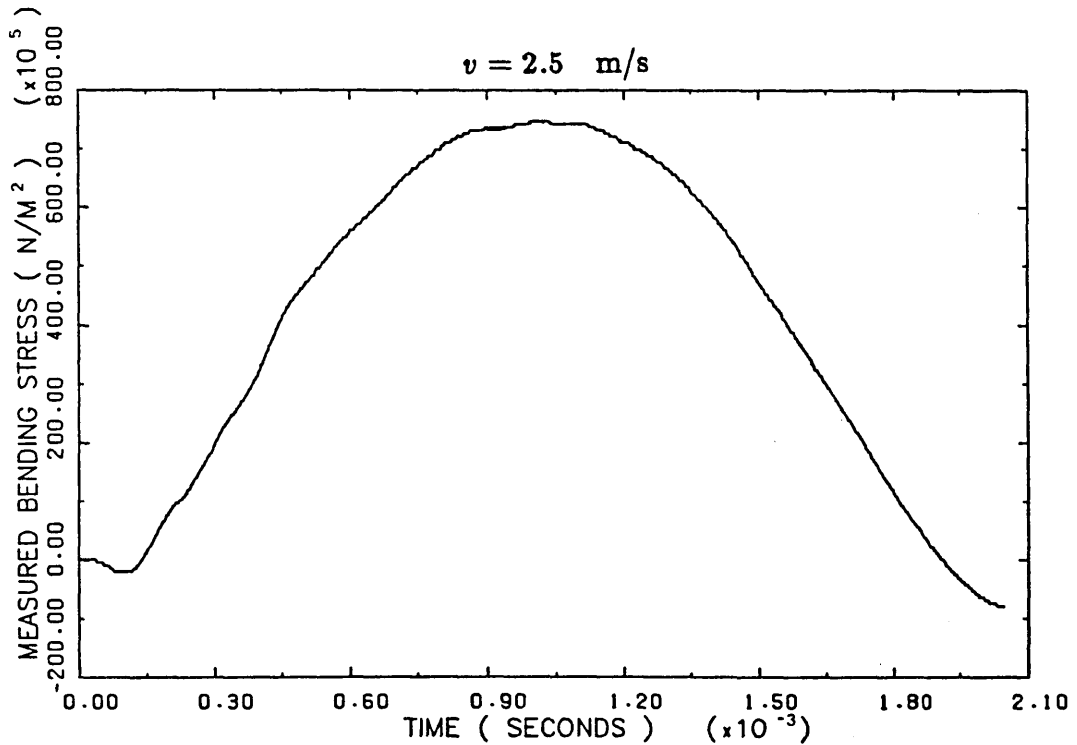
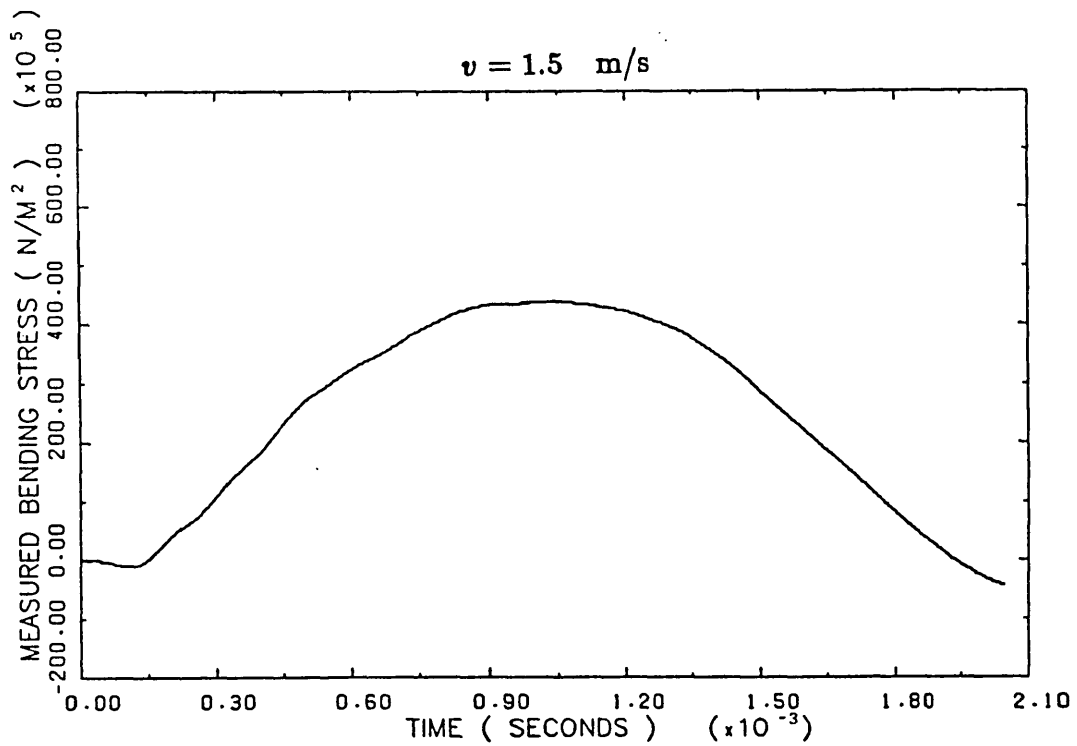


FIG. (4.56) - The experimentally measured variation of bending stress with time, at 5 cm from the point of impact, for test cases (MT36) and (MT38), showing the effect of velocity on response for the short (0.2 m) metal beam specimen.

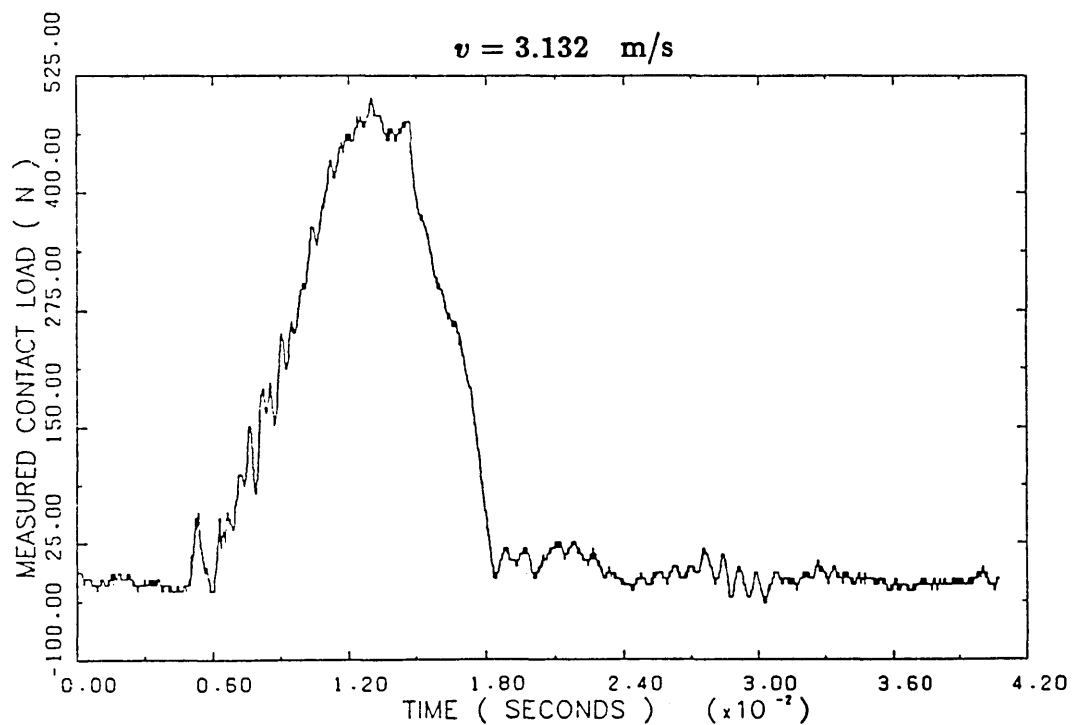
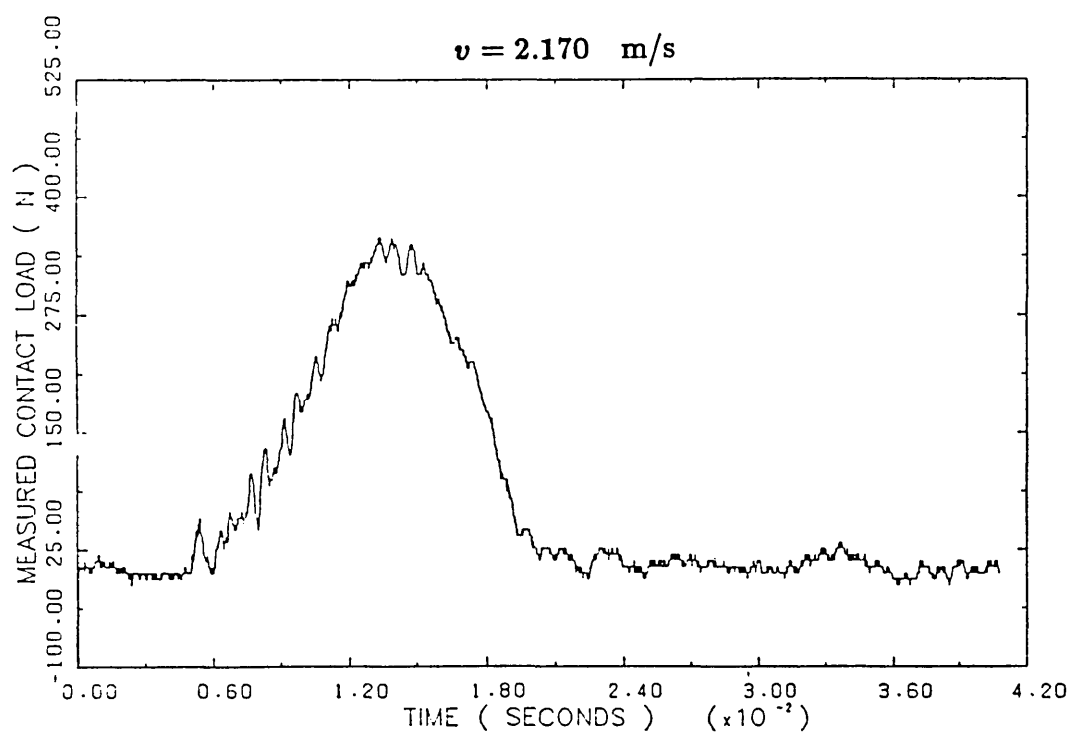


FIG. (4.57) - The experimentally measured variation of contact force with time for test cases (CT5) and (CT6), showing the effect of velocity on response for the 20 cm composite laminate beam specimen.

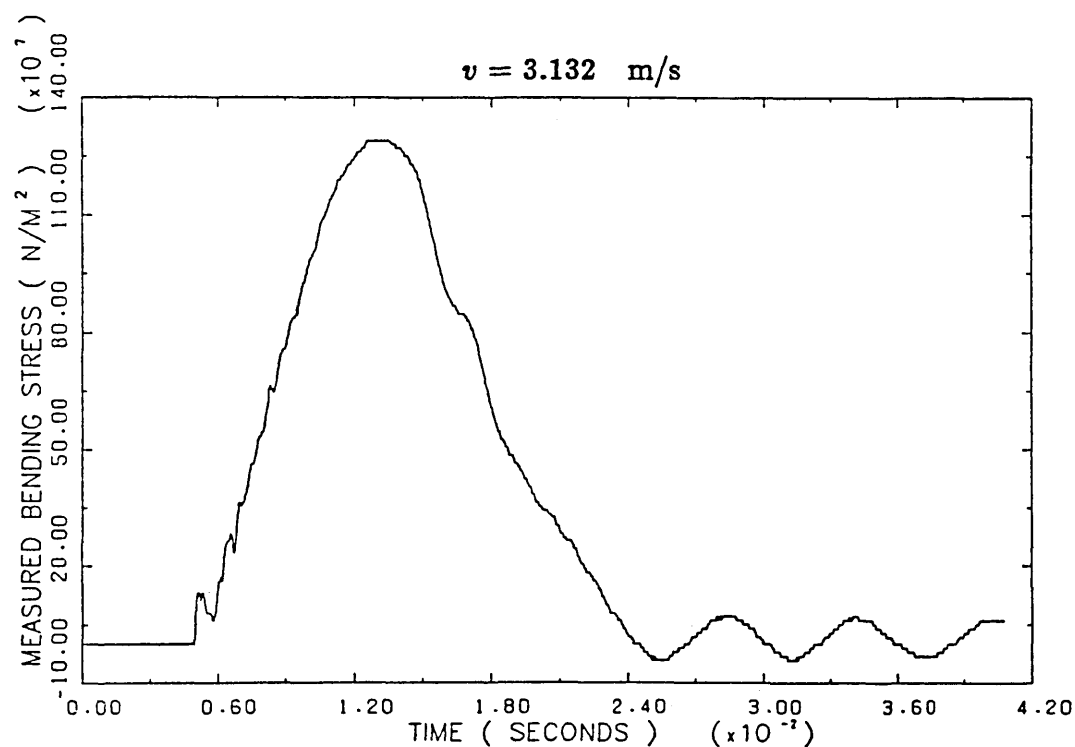
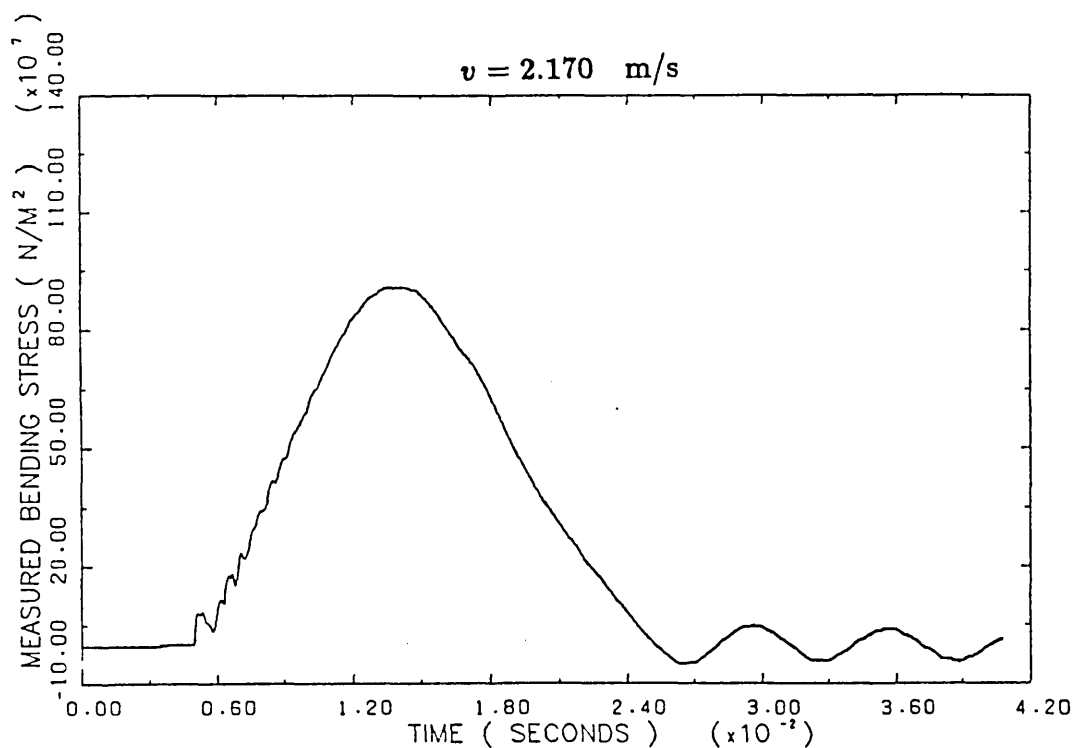


FIG. (4.58) - The experimentally measured variation of bending stress with time, at the beam centre, for test cases (CT5) and (CT6), showing the effect of velocity on response for the 20 cm composite laminate beam specimen.

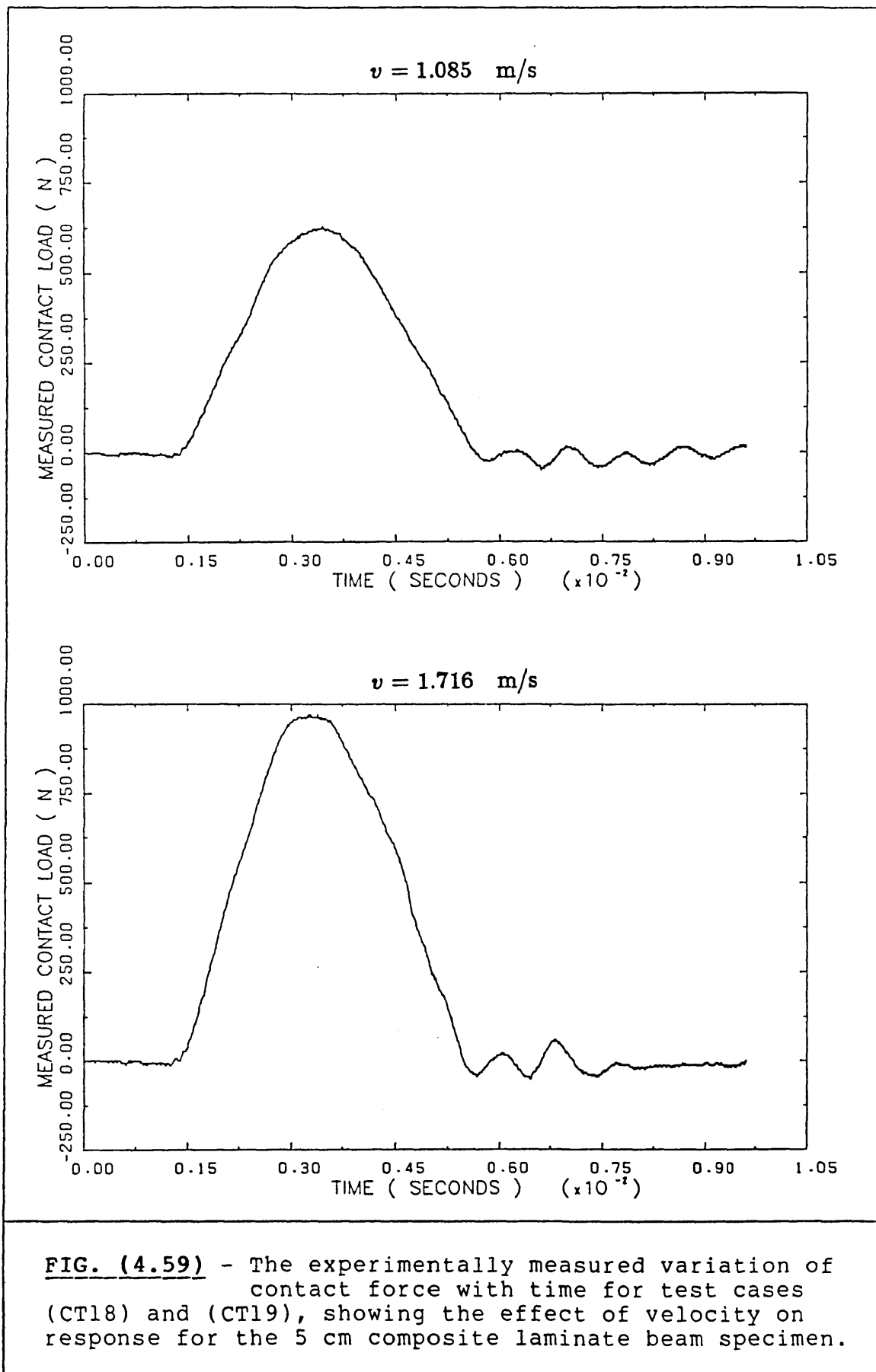
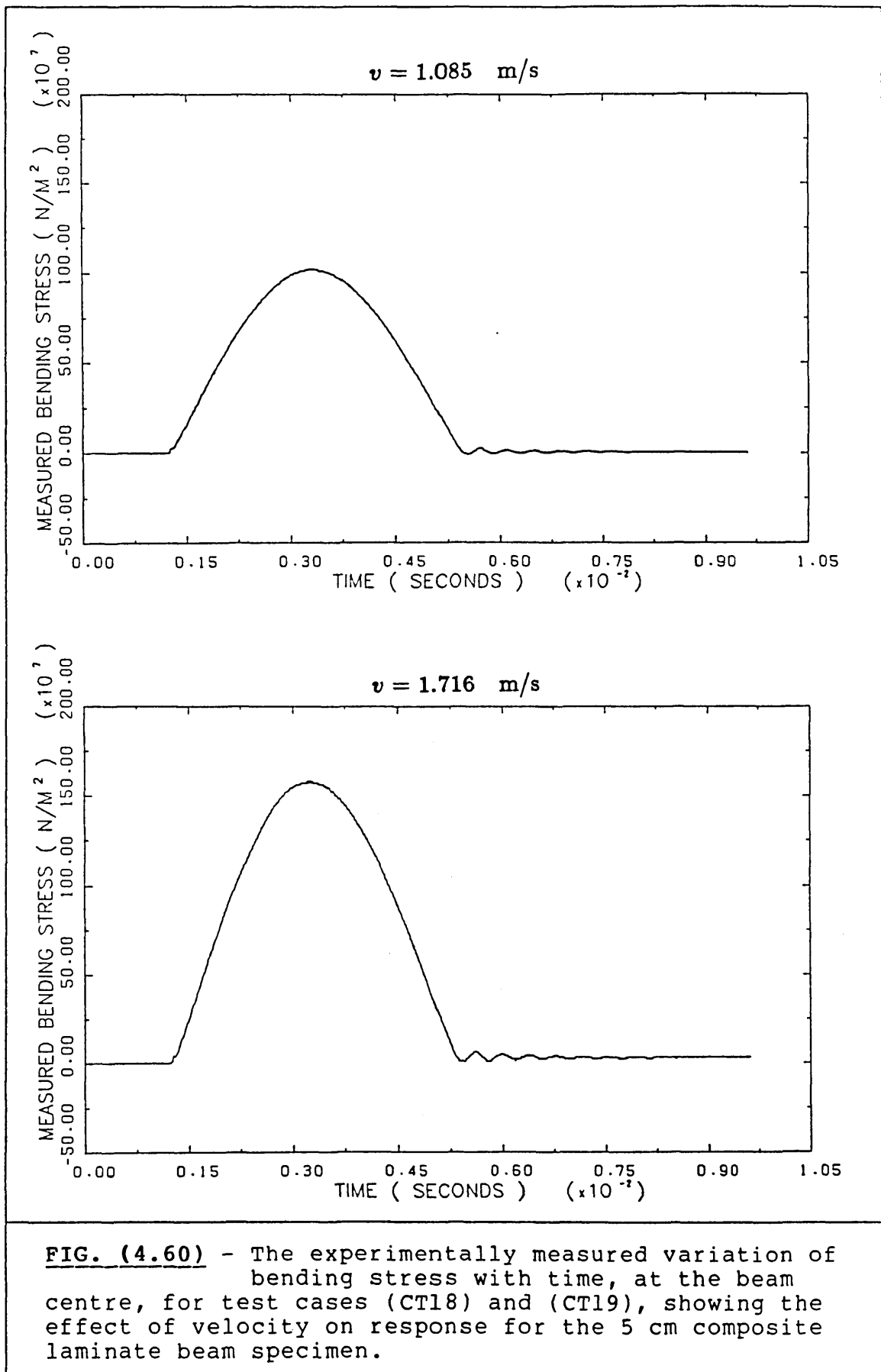


FIG. (4.59) - The experimentally measured variation of contact force with time for test cases (CT18) and (CT19), showing the effect of velocity on response for the 5 cm composite laminate beam specimen.



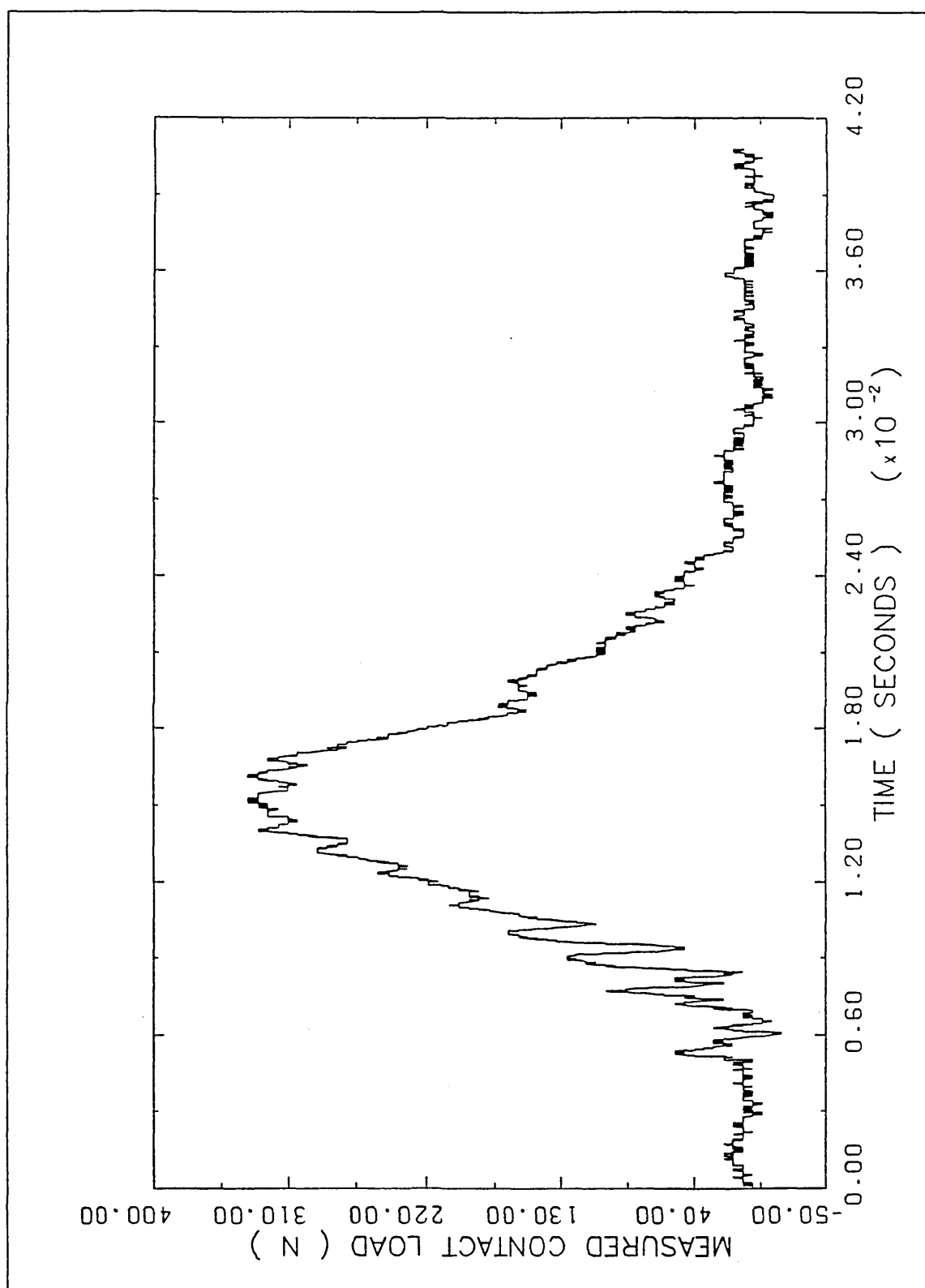


FIG. (4.61) - The experimentally measured variation of contact force with time for test case (CT1), using the 25 cm composite laminate beam specimen.

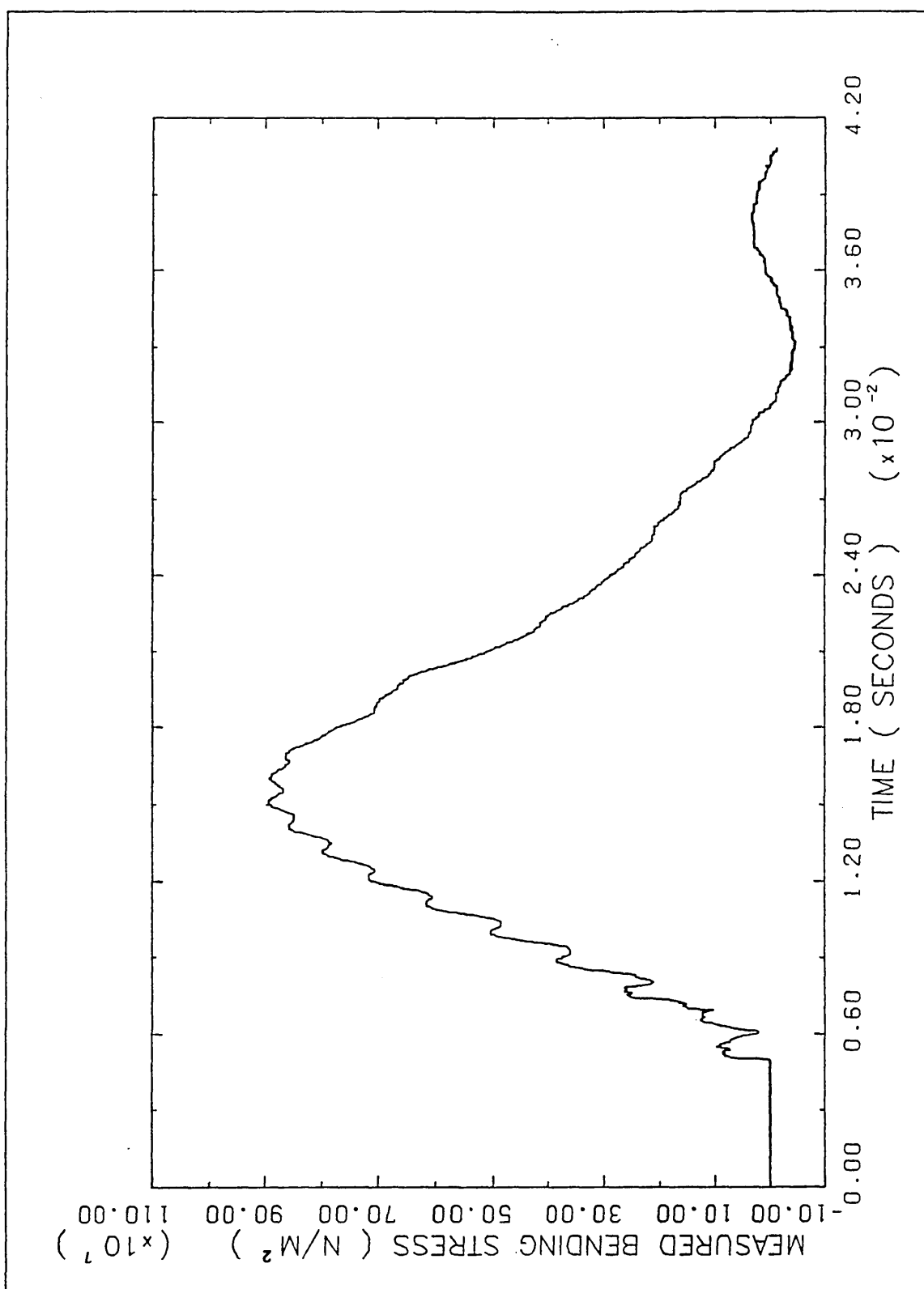


FIG. (4.62) - The experimentally measured variation of bending stress with time, at the beam centre, for test case (CT1), using the 25 cm composite laminate beam specimen.

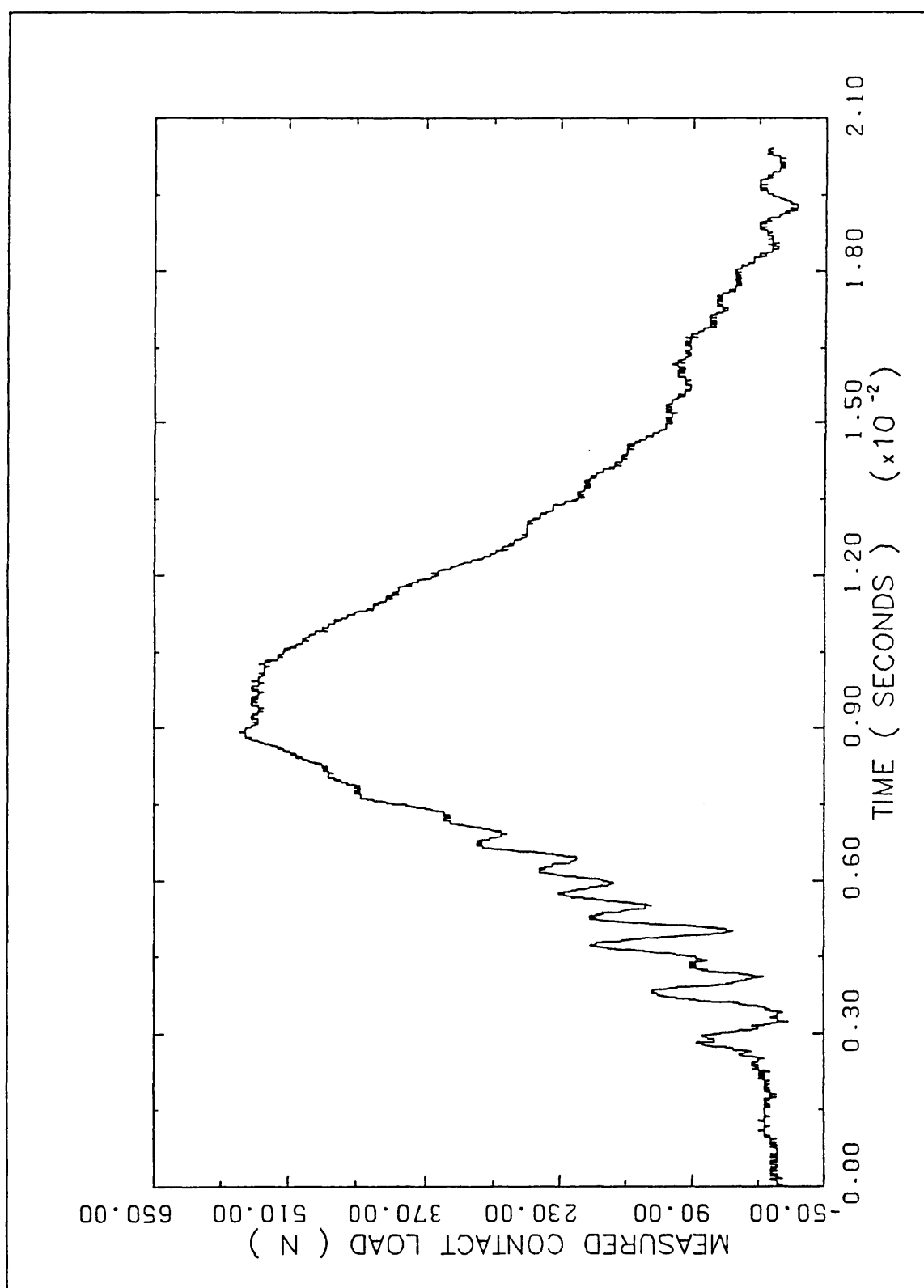


FIG. (4.63) - The experimentally measured variation of contact force with time for test case (CT11), using the 15 cm composite laminate beam specimen.

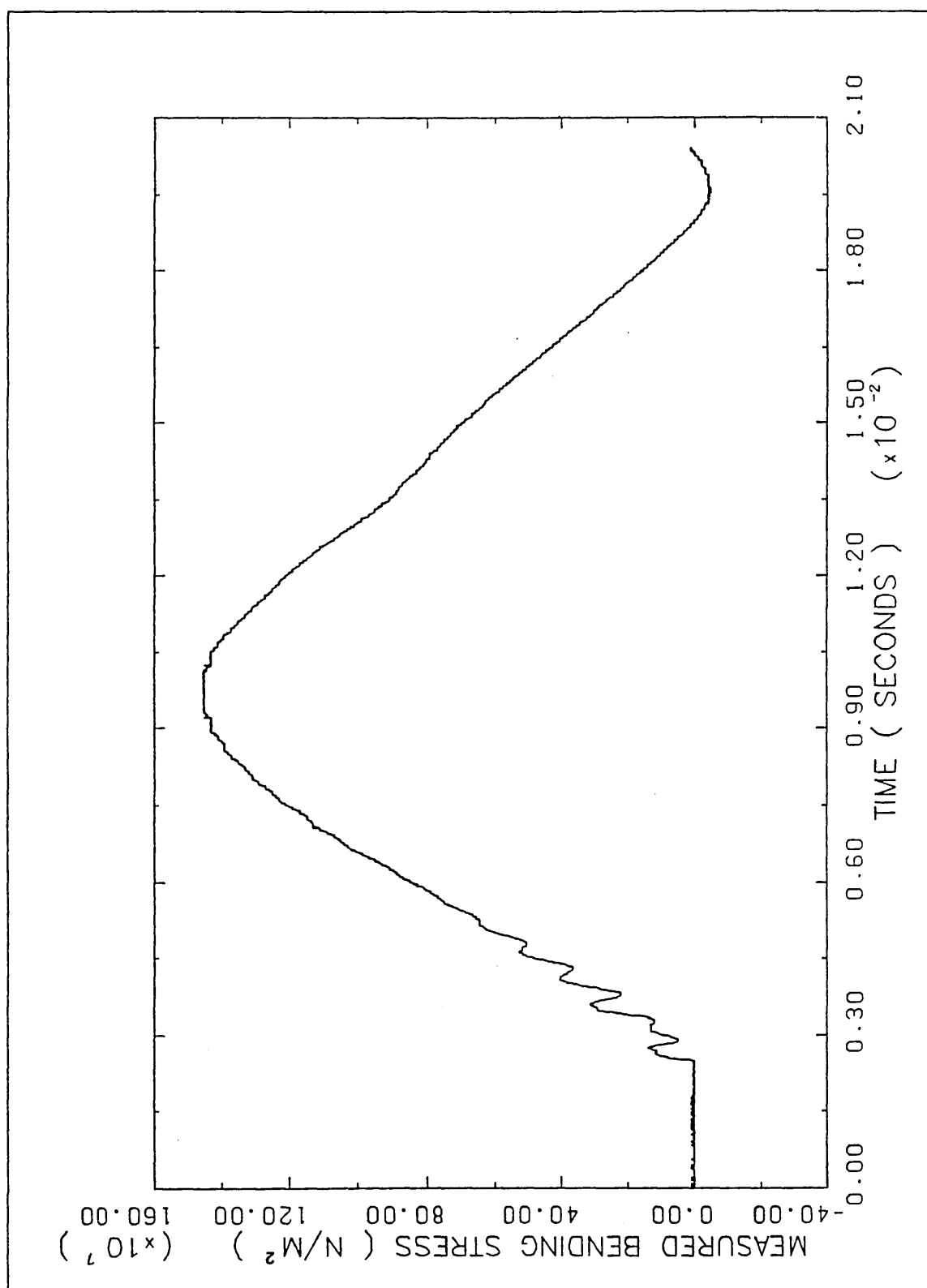


FIG. (4.64) - The experimentally measured variation of bending stress with time, at the beam centre, for test case (CT11), using the 15 cm composite laminate beam specimen.

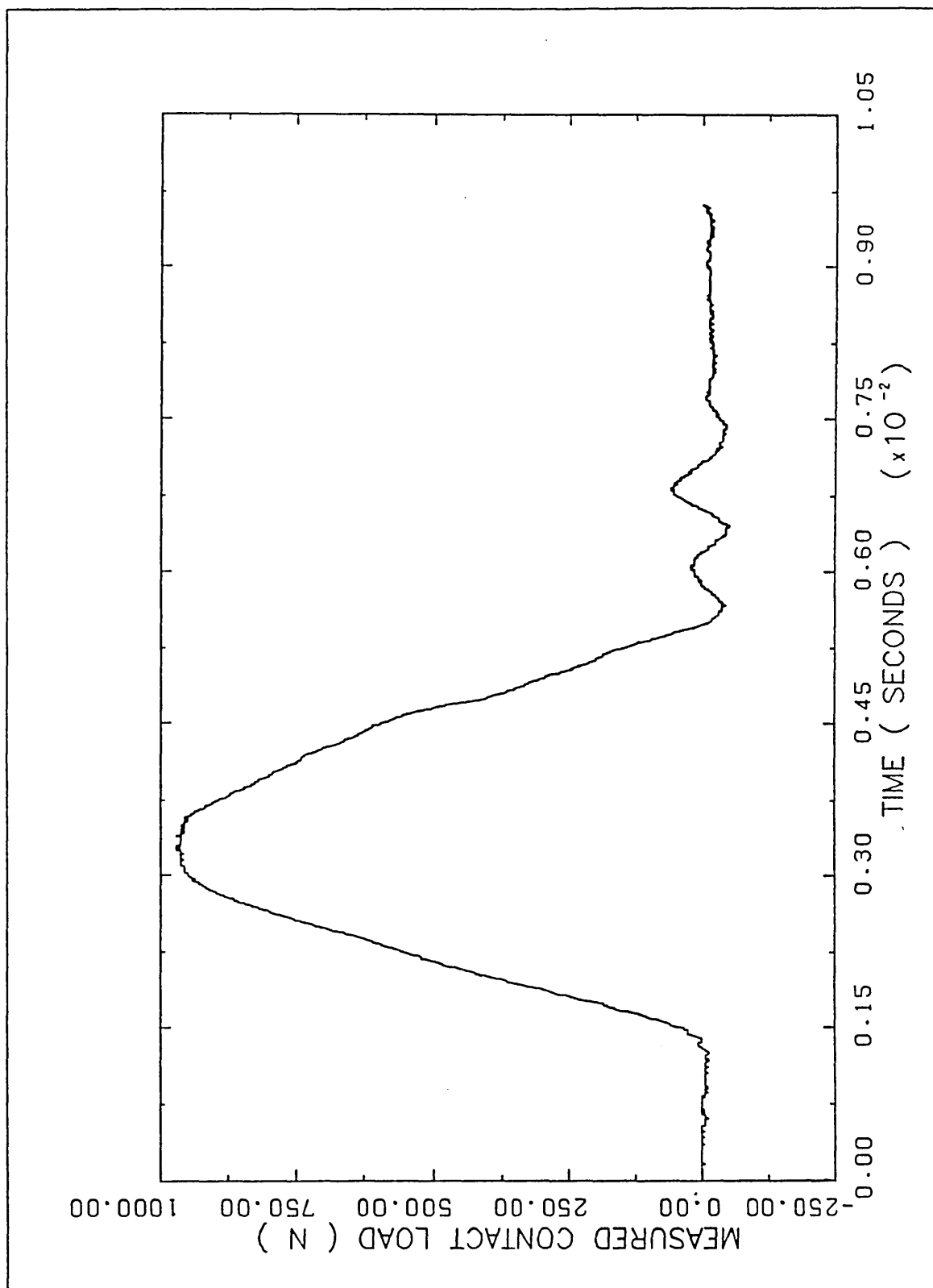


FIG. (4.65) - The experimentally measured variation of contact force with time for test case (CT19), using the 5 cm composite laminate beam specimen.

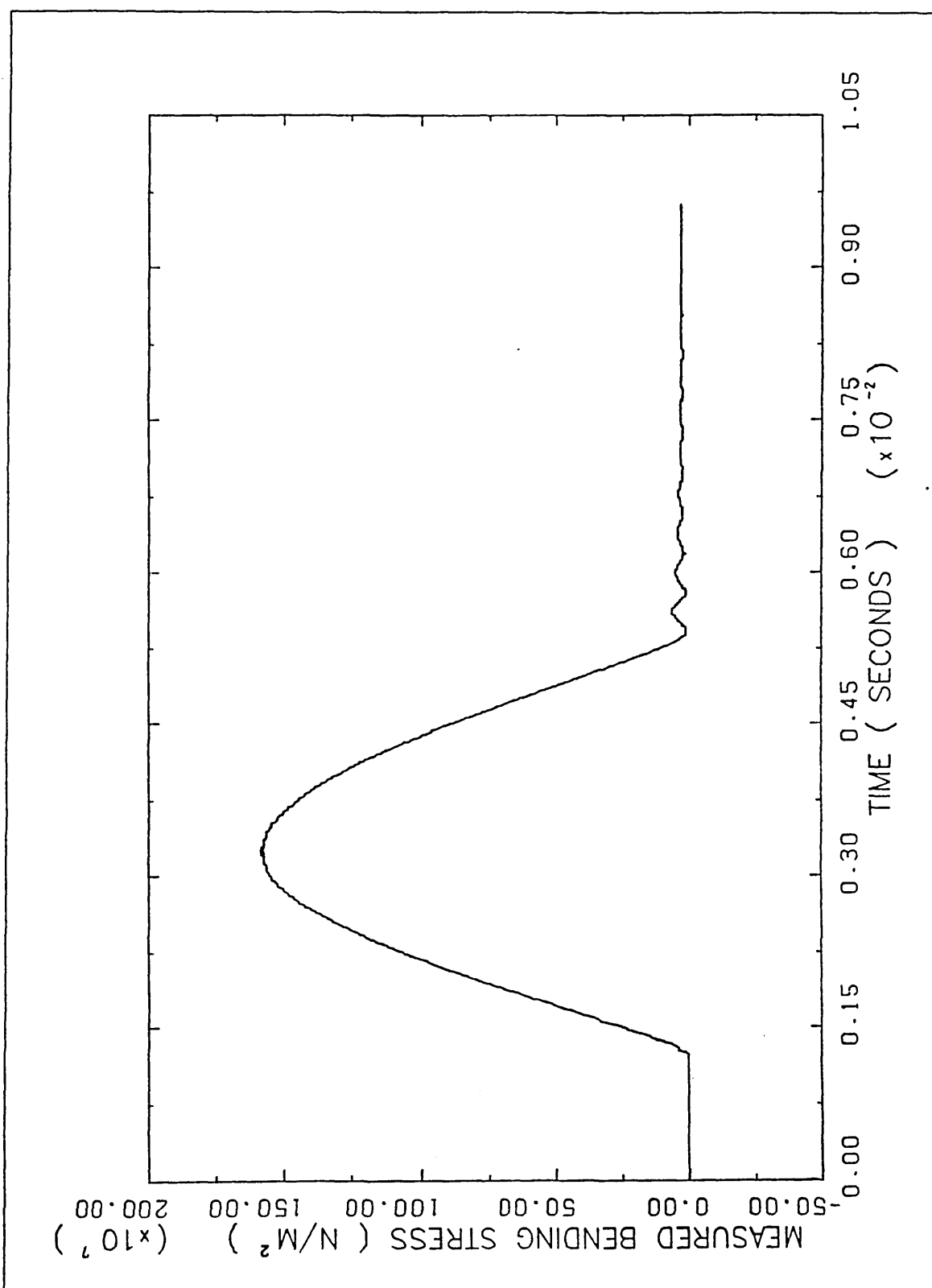


FIG. (4.66) - The experimentally measured variation of bending stress with time, at the beam centre, for test case (CT19), using the 5 cm composite laminate beam specimen.

TABLES AND FIGURES

— for Chapter (5) —

TABLE (5.1) — Details of the experimental results for impact tests on the metal beam specimen

Notes : a) Impact parameter defined as $v\sqrt{\frac{m}{M + M_a}}$

where M_a = mass of displacement flag
+ mass of protecting channel
= 0.086 Kg

(also see Tables (4.1,4.5) and Section (4.4.3))

b) Tabulated values of max. bending and shear stresses are as measured at 5 cm from point of impact

TEST CASE	IMPACTOR VELOCITY (m/s)	IMPACT PARAMETER	MAX. BENDING STRESS (MN/m ²)	MAX. SHEAR STRESS (MN/m ²)
MT1	2.0089	2.0725	89.2779	1.7713
MT2	2.9947	3.0895	133.3774	2.3163
MT3	3.9680	4.0936	174.2511	3.1338
MT4	2.9519	1.7582	62.9248	1.8054
MT5	3.0107	2.5360	104.3369	1.4647
MT6	2.9901	3.5620	150.5874	2.4185
MT7	2.9285	4.2726	179.6292	2.5888
MT8	1.9494	4.0222	186.0830	2.2141
MT9	4.4288	3.7306	150.5874	2.4526
MT10	3.7505	3.7045	157.0411	2.6570

TABLE (5.1) — continued

TEST CASE	IMPACTOR VELOCITY (m/s)	IMPACT PARAMETER	MAX. BENDING STRESS (MN/m ²)	MAX. SHEAR STRESS (MN/m ²)
MT11	3.2436	3.2038	137.6799	2.2482
MT12	3.3521	3.7353	159.1924	2.4526
MT13	2.5494	2.8408	122.6211	1.9416
MT14	2.9966	2.6133	104.3369	1.7372
MT15	2.9978	2.7137	104.8747	2.1120
MT16	3.0039	2.8311	106.4881	1.9076
MT17	3.7617	4.1573	154.8899	2.8614
MT18	3.2320	3.5719	131.4530	2.9635
MT19	2.9535	2.9084	91.9670	1.5669
MT20	2.9845	3.8878	134.4530	2.6229
MT21	2.9445	3.0423	94.1183	2.0098
MT22	1.9804	2.1580	67.2274	1.6351
MT23	2.9272	3.1897	100.0343	2.3163
MT24	4.3263	4.7142	150.5874	3.2020
MT25	2.9276	2.7627	84.4376	1.7713
MT26	2.9971	3.9998	134.4530	2.5548
MT27	2.5158	3.8769	136.6043	2.5548
MT28	2.0186	3.8098	135.5286	2.5548
MT29	1.3914	3.7904	135.5286	2.3845

TABLE (5.1) — continued

TEST CASE	IMAPCTOR VELOCITY (m/s)	IMPACT PARAMETER	MAX. BENDING STRESS (MN/m ²)	MAX. SHEAR STRESS (MN/m ²)
MT30	2.9144	3.3705	105.9503	2.9976
MT31	2.9151	3.6070	110.7907	2.8614
MT32	2.9151	3.9002	118.3186	3.6448
MT33	2.9040	4.2628	123.6968	3.6448
MT34	2.0205	3.6324	110.7907	3.3382
MT35	1.5145	3.8506	118.3202	3.4404
MT36	1.4655	2.4600	43.8322	3.9855
MT37	1.9243	3.2301	59.1601	5.2458
MT38	2.3848	4.0031	74.7568	6.8980
MT39	1.4763	2.8499	52.1684	4.6667
MT40	1.5129	3.5624	68.3030	6.5743
MT41	1.0000	2.7134	53.7819	5.2458
MT42	1.0000	3.3163	58.0844	5.6546
MT43	0.7000	3.3435	77.4459	7.6643

**TABLE (5.2) — Mesh configuration details relating to the FE analysis
of experimental test cases**

Elastic modulus	=	72.0	GN/m ²
Shear modulus	=	27.0	GN/m ²
Density	=	2720.0	Kg/m ³
Shear coefficient	=	0.851	
Element width	=	0.0192	m
Element depth	=	0.0181	m

Note : The mass of the displacement flag and that of the protecting channel have been included by means of a point mass of 0.043 Kg at the centre node.

TEST CASE	MT4	MT29	MT38
IMPACT PARAMETER	1.7582	3.7904	4.0031
BEAM LENGTH (m)	1.40	0.80	0.20
NO. OF BEAM ELEMENTS	7	4	4
LENGTH OF ELEMENTS (m)	0.100	0.100	0.025
IMPACTOR MASS (Kg)	0.250	3.125	0.375
IMPACTOR VELOCITY (m/s)	3.0	1.5	2.5
SPRING STIFFNESS (N/m)	183428	360633	8000000

TABLE (5.3) — Details of the experimental results for impact tests on composite beam specimens

Notes : a) Impact parameter defined as $v\sqrt{\frac{m}{M}}$
 (also see Tables (4.6,4.7))
 b) Tabulated values of max. bending stresses are as measured at the beam centre (i.e. below the point of impact)

TEST CASE	IMPACTOR VELOCITY (m/s)	IMPACT PARAMETER	MAX. BENDING STRESS (MN/m ²)
CT1	2.4261	19.8091	897.4862
CT2	3.4310	28.0143	1078.9865
CT3	4.4294	36.1663	1472.2620
CT4	5.4249	44.2945	1573.1019
CT5	2.1700	19.8091	912.6124
CT6	3.1321	28.5920	1290.7502
CT7	3.9618	36.1663	1552.9339
CT8	4.6456	42.4087	1573.1019
CT9	5.0503	46.1031	1633.6058

TABLE (5.3) — continued

TEST CASE	IMPACTOR VELOCITY (m/s)	IMPACT PARAMETER	MAX. BENDING STRESS (MN/m ²)
CT10	1.8793	19.8091	963.0330
CT11	2.8014	29.5296	1452.0940
CT12	3.8360	40.4351	1714.2777
CT13	4.2021	44.2945	1694.1097
CT14	1.5344	19.8091	993.2853
CT15	2.4261	31.3209	1512.5979
CT16	3.1321	40.4351	1915.9574
CT17	3.5711	46.1031	2036.9652
CT18	1.0850	19.8091	1018.4956
CT19	1.7155	31.3209	1583.1858
CT20	2.2147	40.4351	2218.4770
CT21	2.6205	47.8435	2521.0287

TABLE (5.4) — Summary of the experimental findings presented by CANTWELL (70) relating to impact tests on composite laminates

LOW VELOCITY IMPACT DAMAGE		
COMPOSITE SPECIMEN	ENERGY TO INITIAL DAMAGE (Joules)	TYPE OF DAMAGE
4-ply (± 45)	0.22	lower surface matrix damage
8-ply (± 45)	0.63	lower surface matrix cracking
16-ply (± 45)	1.20	lower surface matrix damage
32-ply (± 45)	0.40	top surface matrix cracking
64-ply (± 45)	0.48	top surface damage
8-ply (0, ± 45)	1.00	lower surface fibre cracking
16-ply (0, ± 45)	1.61	top surface matrix cracking
32-ply (0, ± 45)	0.51	top surface damage

HIGH VELOCITY IMPACT DAMAGE		
COMPOSITE SPECIMEN	ENERGY TO INITIAL DAMAGE (Joules)	TYPE OF DAMAGE
4-ply (± 45)	0.17	————
8-ply (± 45)	0.40	lower surface matrix cracking
16-ply (± 45)	0.61	————
32-ply (± 45)	0.38	top surface matrix cracking
8-ply (0, ± 45)	0.50	top surface matrix splitting
16-ply (0, ± 45)	0.40	top surface matrix cracking

TABLE (5.5) — Estimation of the modal distribution of energy in the composite specimens employed by CANTWELL (70) for high and low velocity impact tests. Calculation based on the numerical technique of Chapter (2)

Laminate material = XAS-914C
Laminate thickness = 2.0 mm
Laminate width = 25.0 mm
Laminate length = 50.0 mm
Effective density = 1500.0 Kg/m³

a) Low velocity impact

impactor mass = 0.75 Kg
impactor velocity = 2.0 m/s

MODE	8-ply	16-ply	32-ply
1	99.89%	99.78%	99.57%
2	0.06%	0.13%	0.26%
3	0.02%	0.04%	0.08%
4	0.01%	0.02%	0.04%
5	—	0.01%	0.02%

TABLE (5.5) — continued

b) High velocity impact

impactor mass = 0.001 Kg
impactor velocity = 20.0 m/s

MODE	8-ply	16-ply	32-ply
1	54.82%	38.57%	24.66%
2	18.12%	20.03%	18.04%
3	8.86%	11.52%	12.26%
4	5.15%	7.48%	9.27%
5	3.29%	5.15%	7.05%

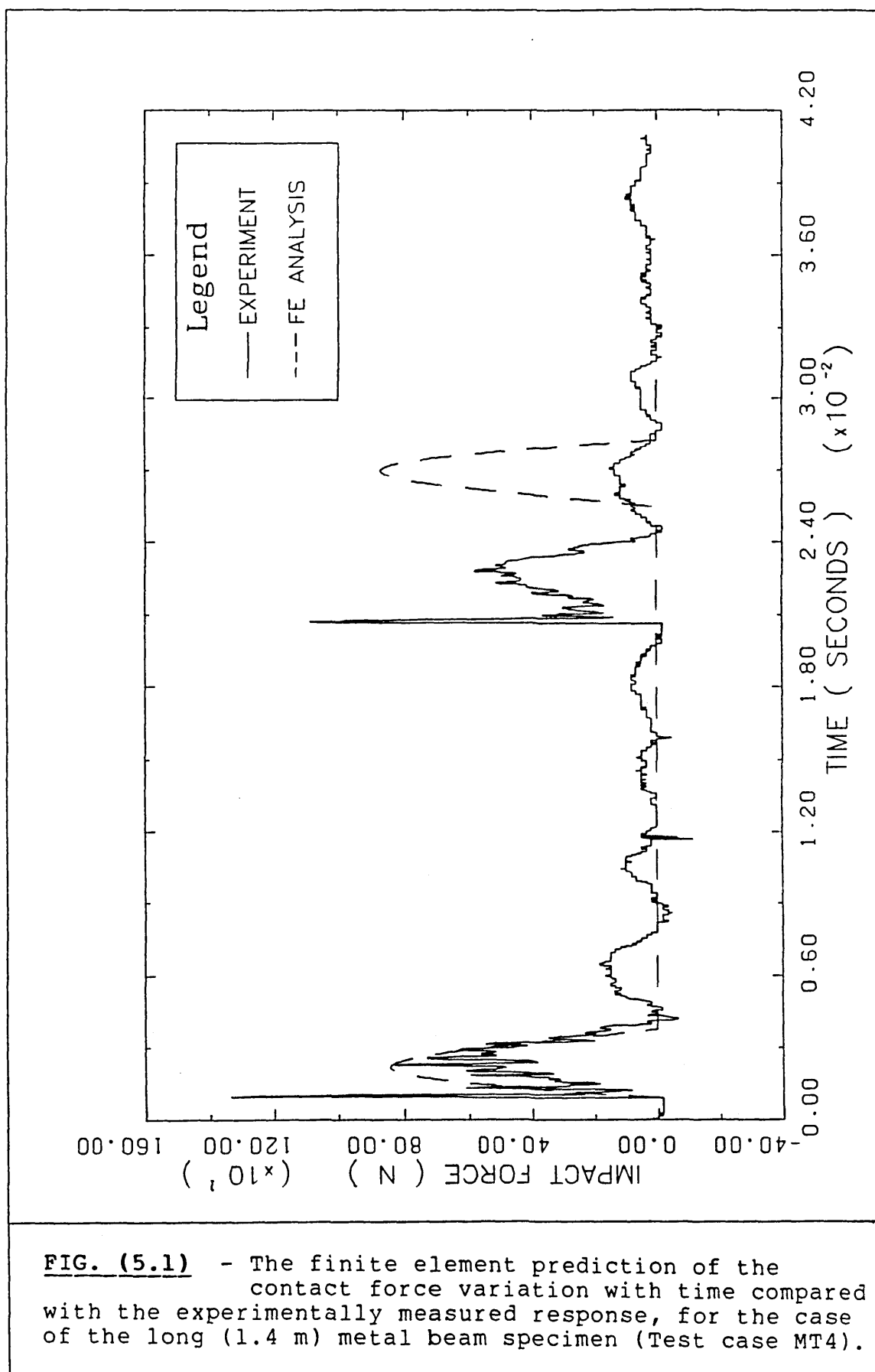


FIG. (5.1) - The finite element prediction of the contact force variation with time compared with the experimentally measured response, for the case of the long (1.4 m) metal beam specimen (Test case MT4).

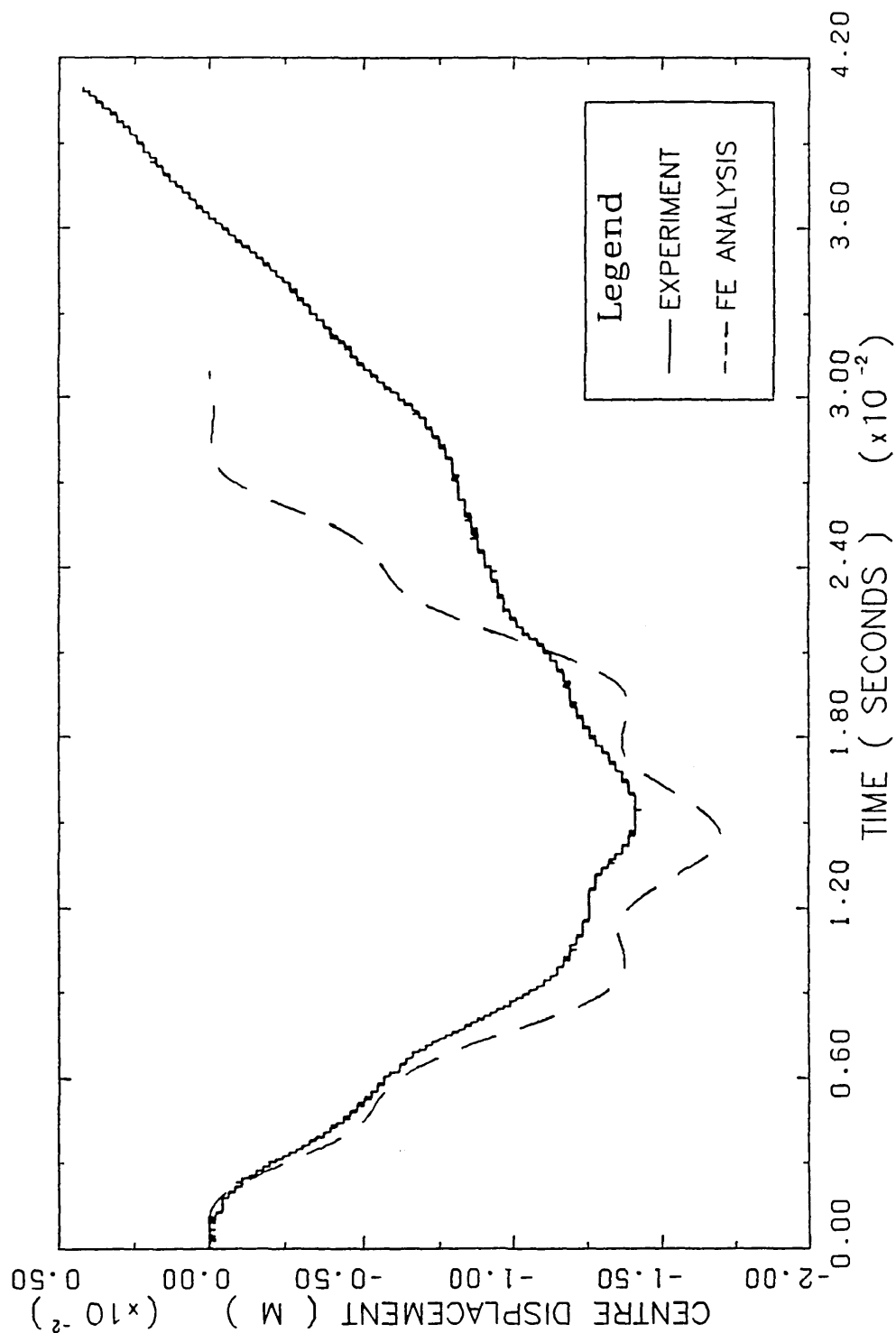
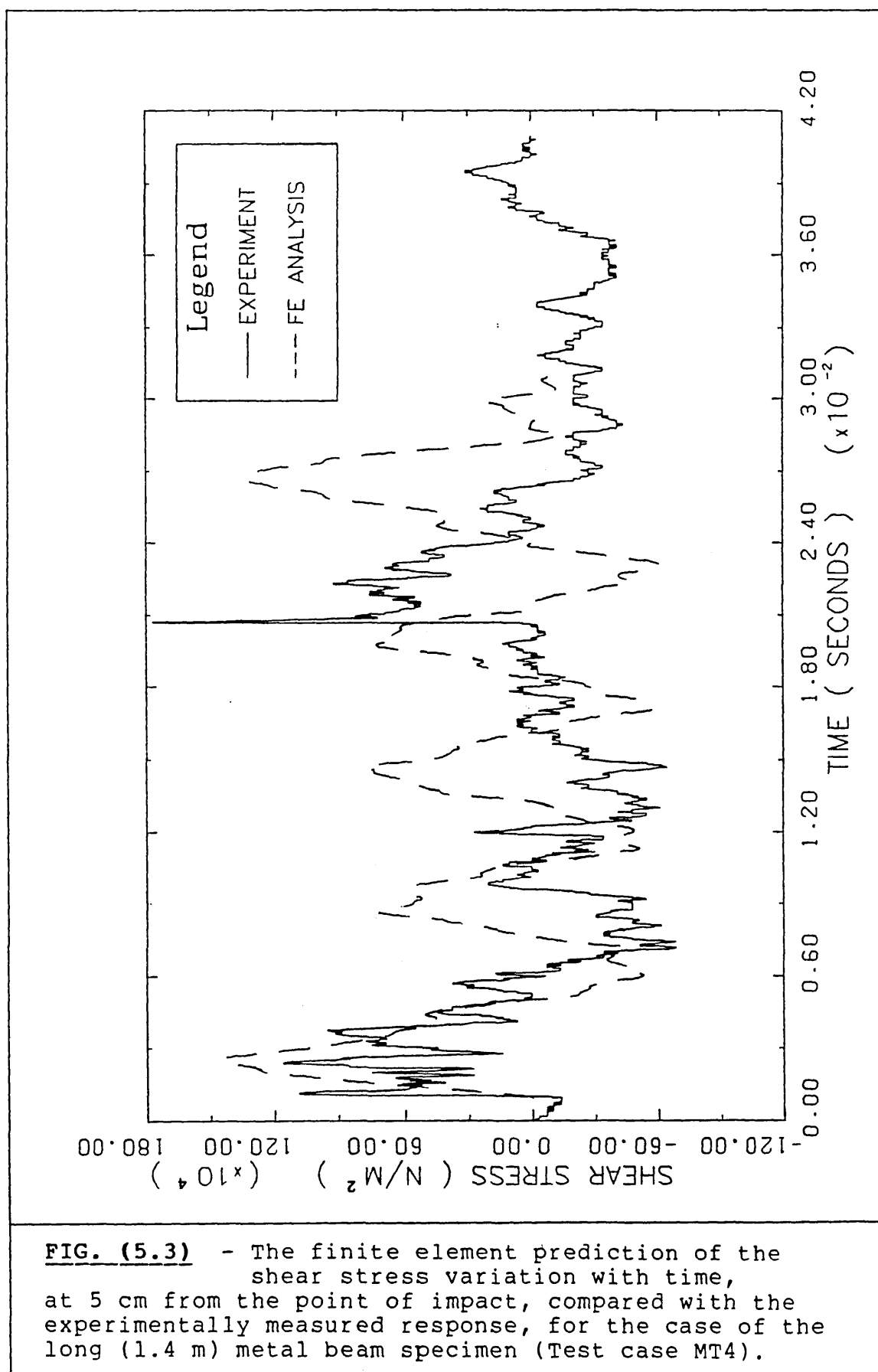


FIG. (5.2) - The finite element prediction of the displacement variation with time, at the beam centre, compared with the experimentally measured response, for the case of the long (1.4 m) metal beam specimen (Test case MT4).



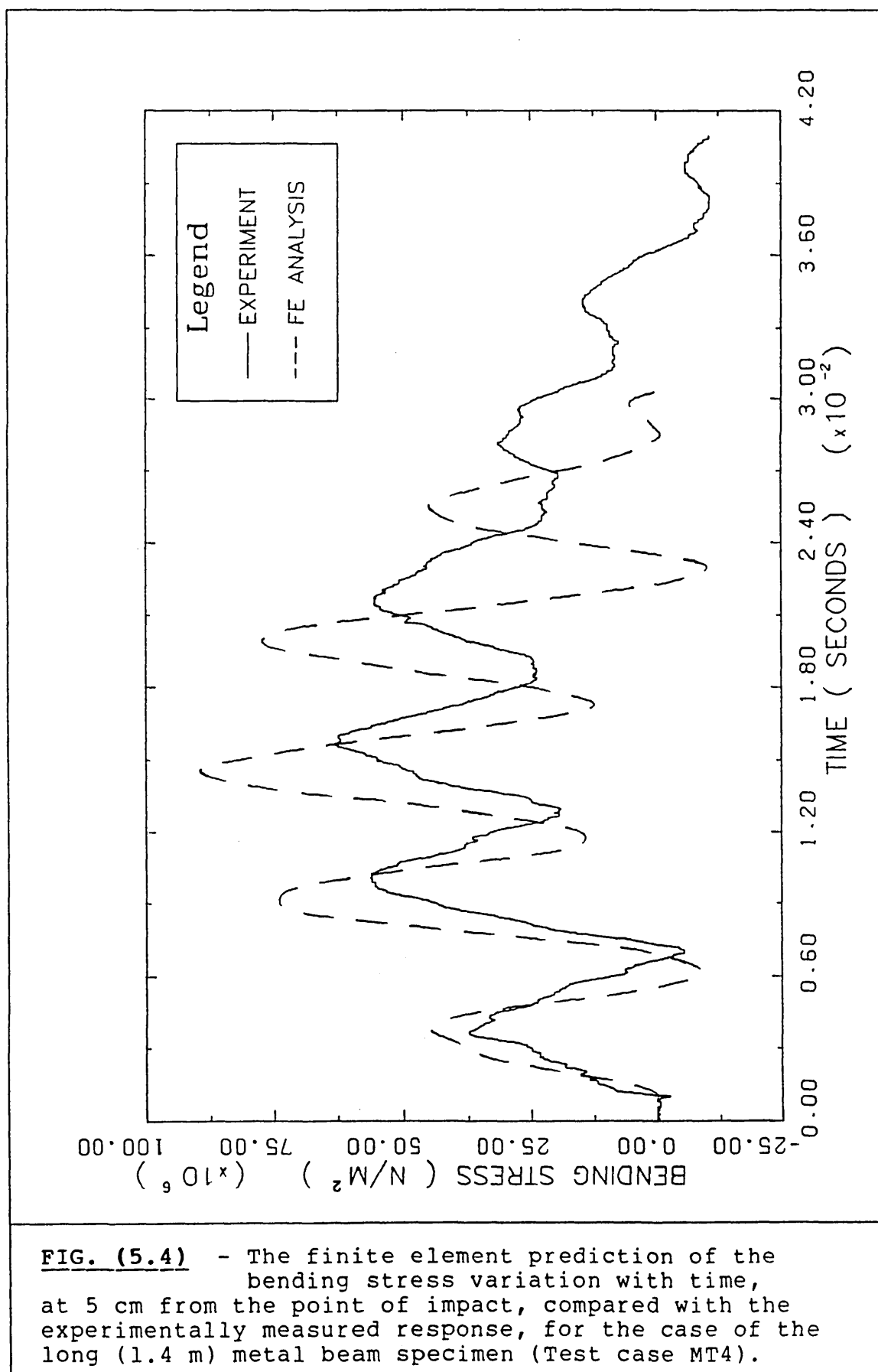
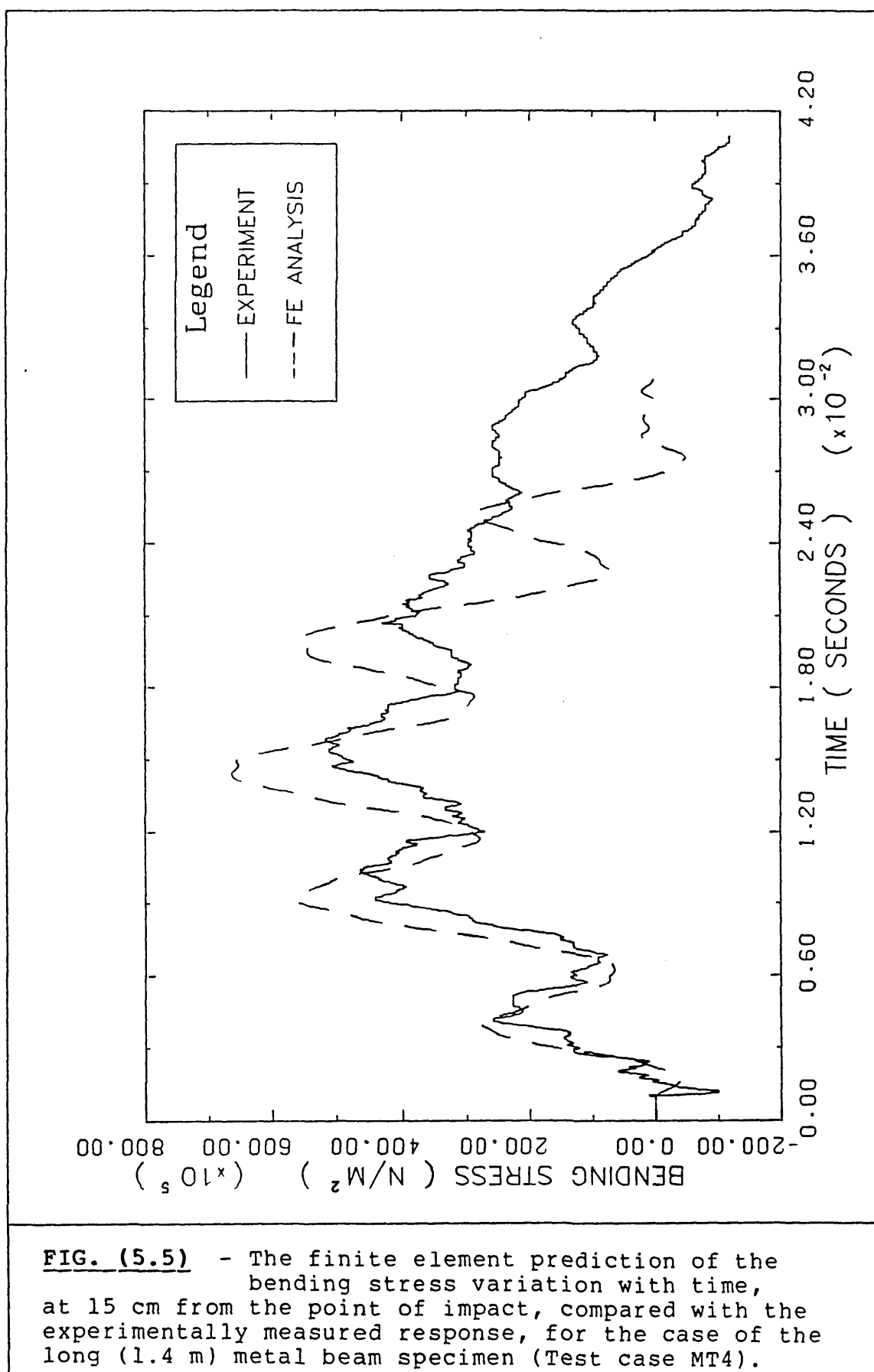


FIG. (5.4) - The finite element prediction of the bending stress variation with time, at 5 cm from the point of impact, compared with the experimentally measured response, for the case of the long (1.4 m) metal beam specimen (Test case MT4).



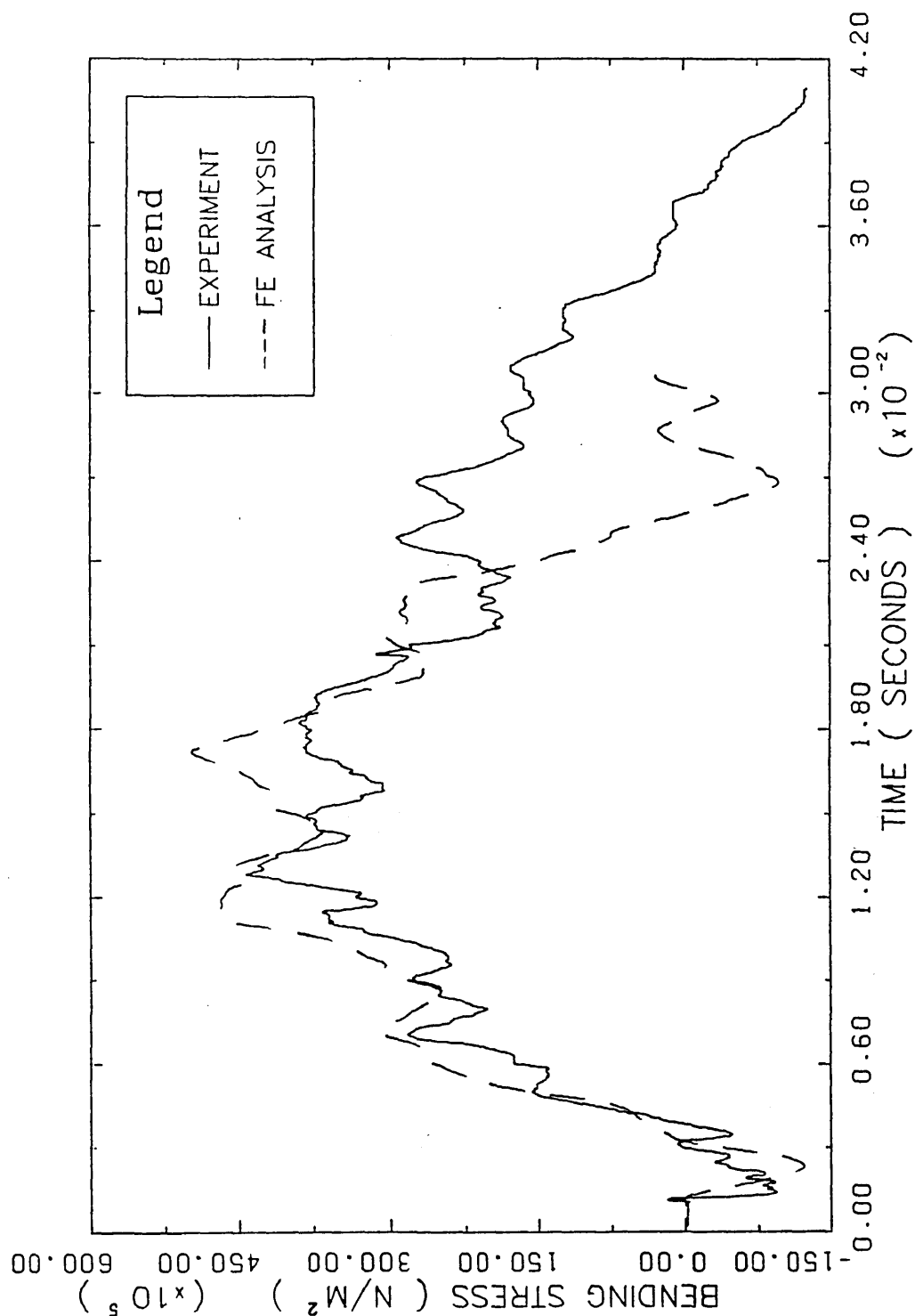


FIG. (5.6) - The finite element prediction of the bending stress variation with time, at 25 cm from the point of impact, compared with the experimentally measured response, for the case of the long (1.4 m) metal beam specimen (Test case MT4).

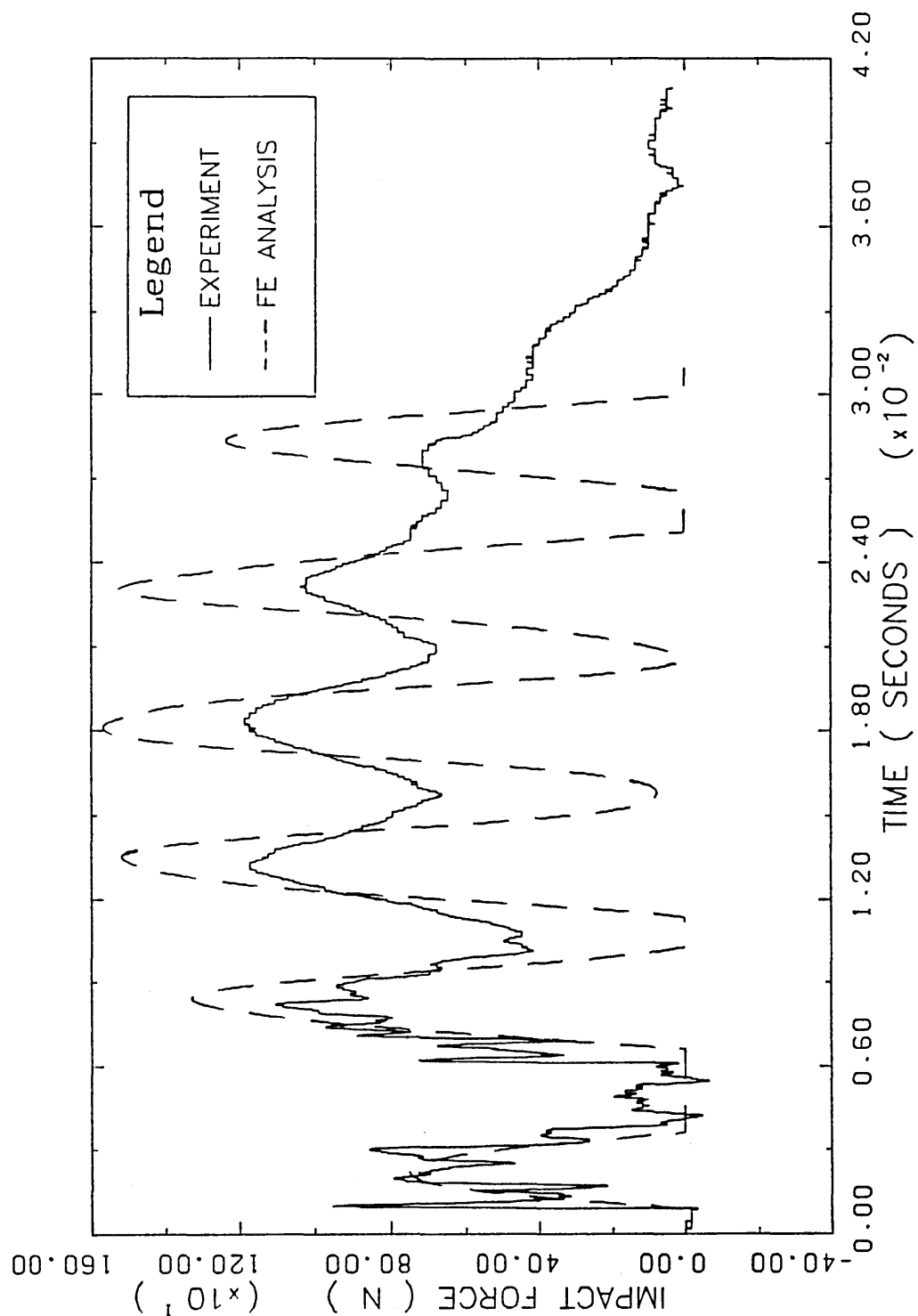


FIG. (5.7) - The finite element prediction of the contact force variation with time compared with the experimentally measured response, for the case of the medium length (0.8 m) metal beam specimen (Test case MT29).

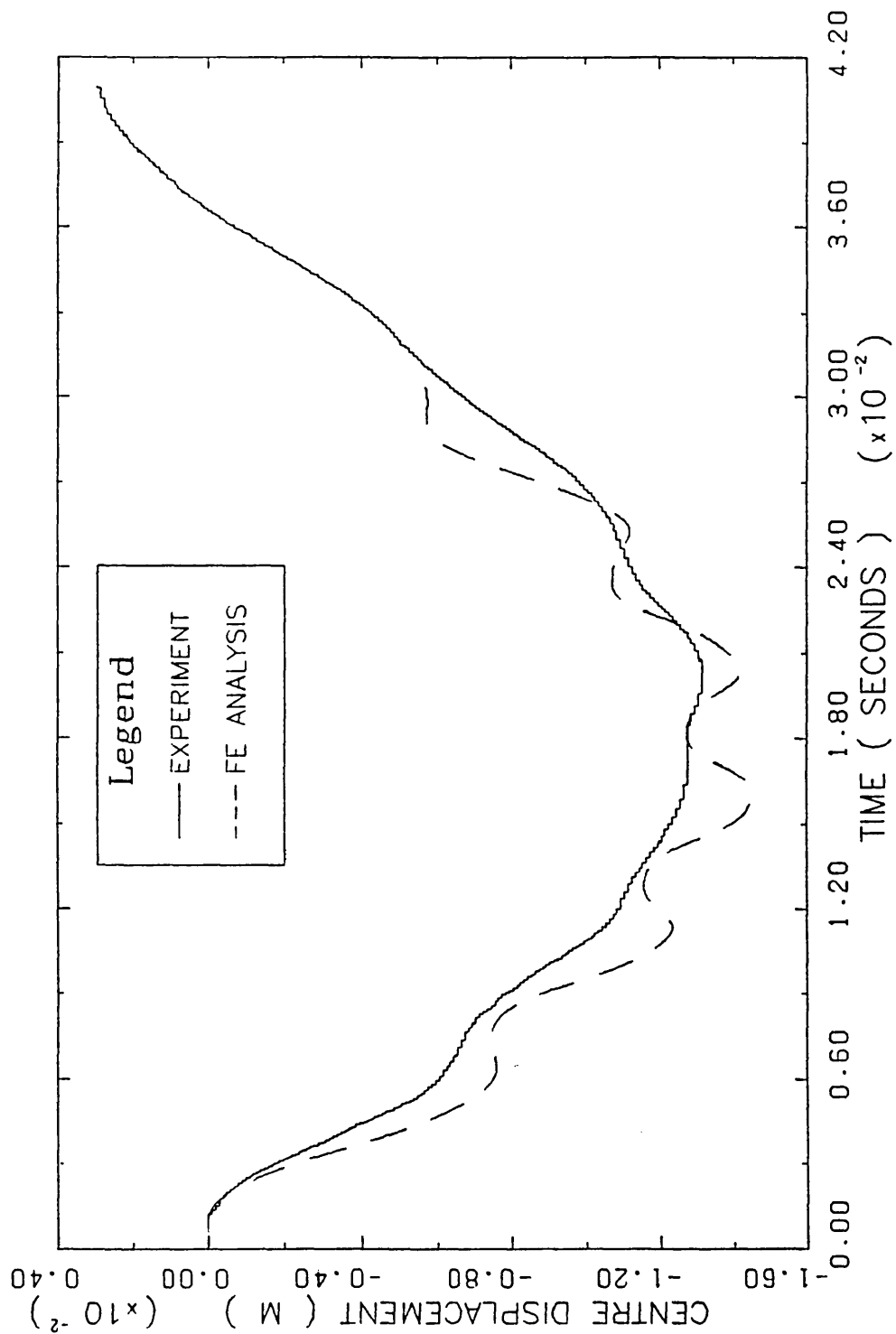


FIG. (5.8) - The finite element prediction of the displacement variation with time, at the beam centre, compared with the experimentally measured response, for the case of the medium length (0.8 m) metal beam specimen (Test case MT29).

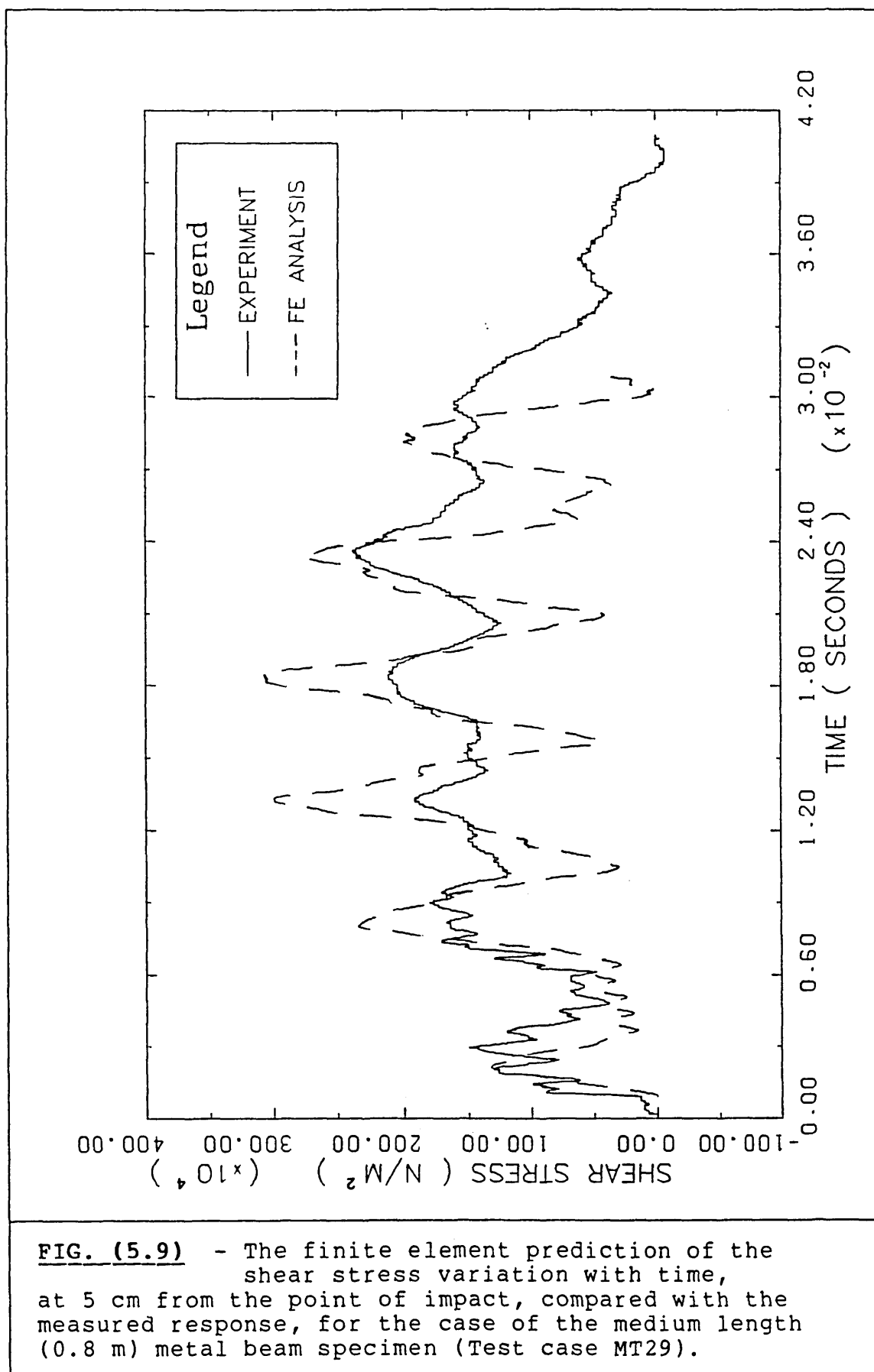


FIG. (5.9) - The finite element prediction of the shear stress variation with time, at 5 cm from the point of impact, compared with the measured response, for the case of the medium length (0.8 m) metal beam specimen (Test case MT29).

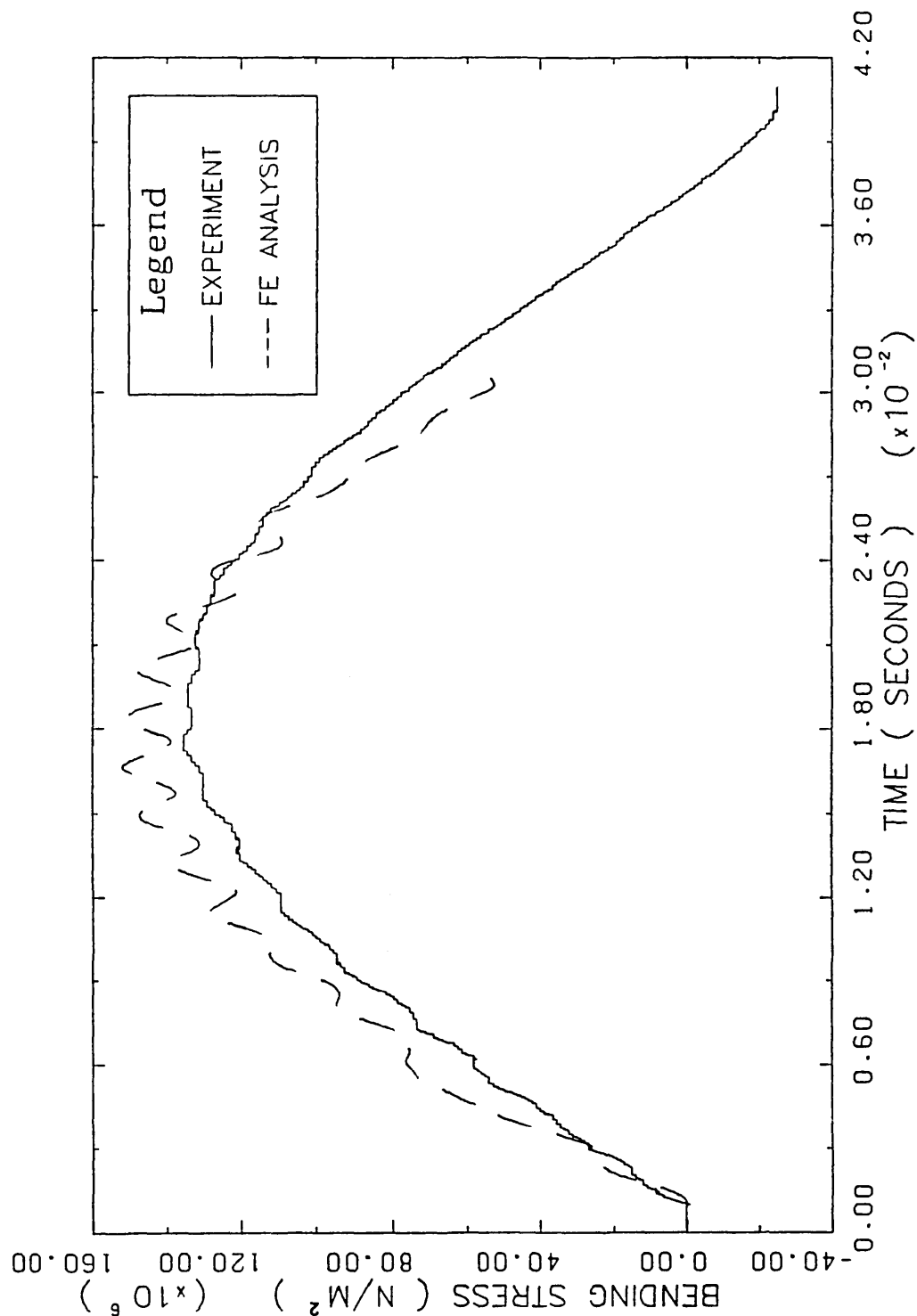
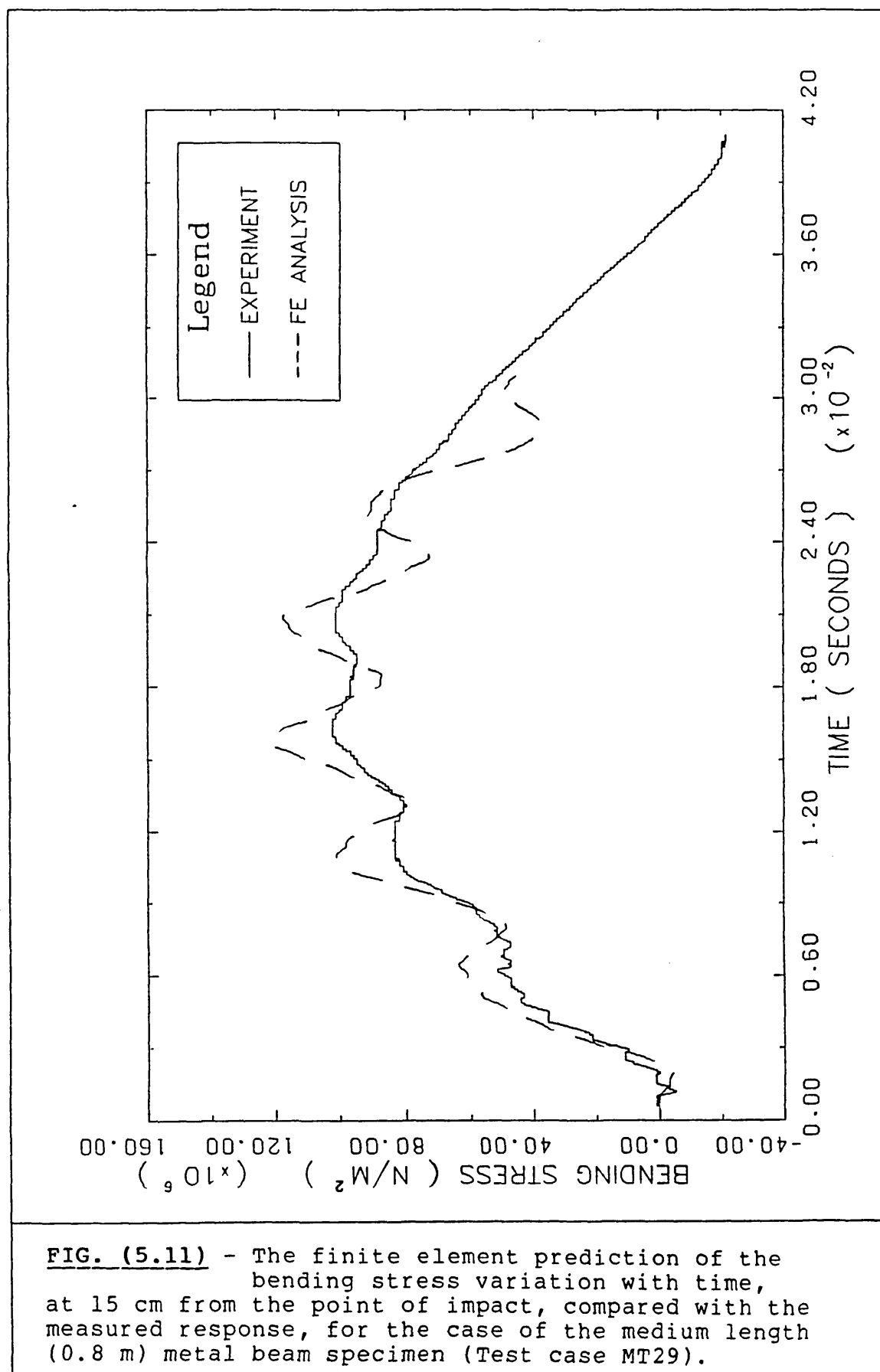


FIG. (5.10) - The finite element prediction of the bending stress variation with time, at 5 cm from the point of impact, compared with the measured response, for the case of the medium length (0.8 m) metal beam specimen (Test case MT29).



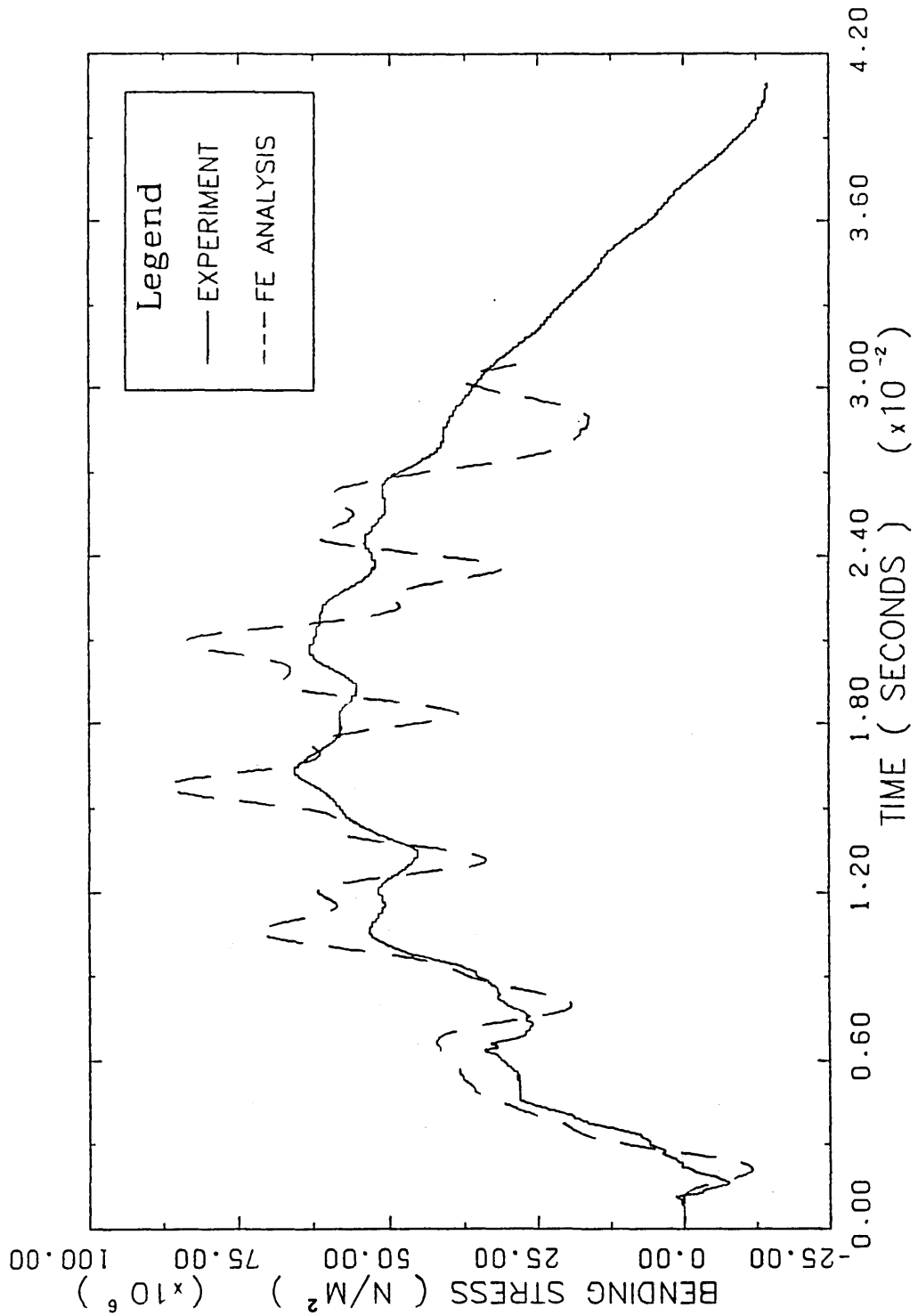
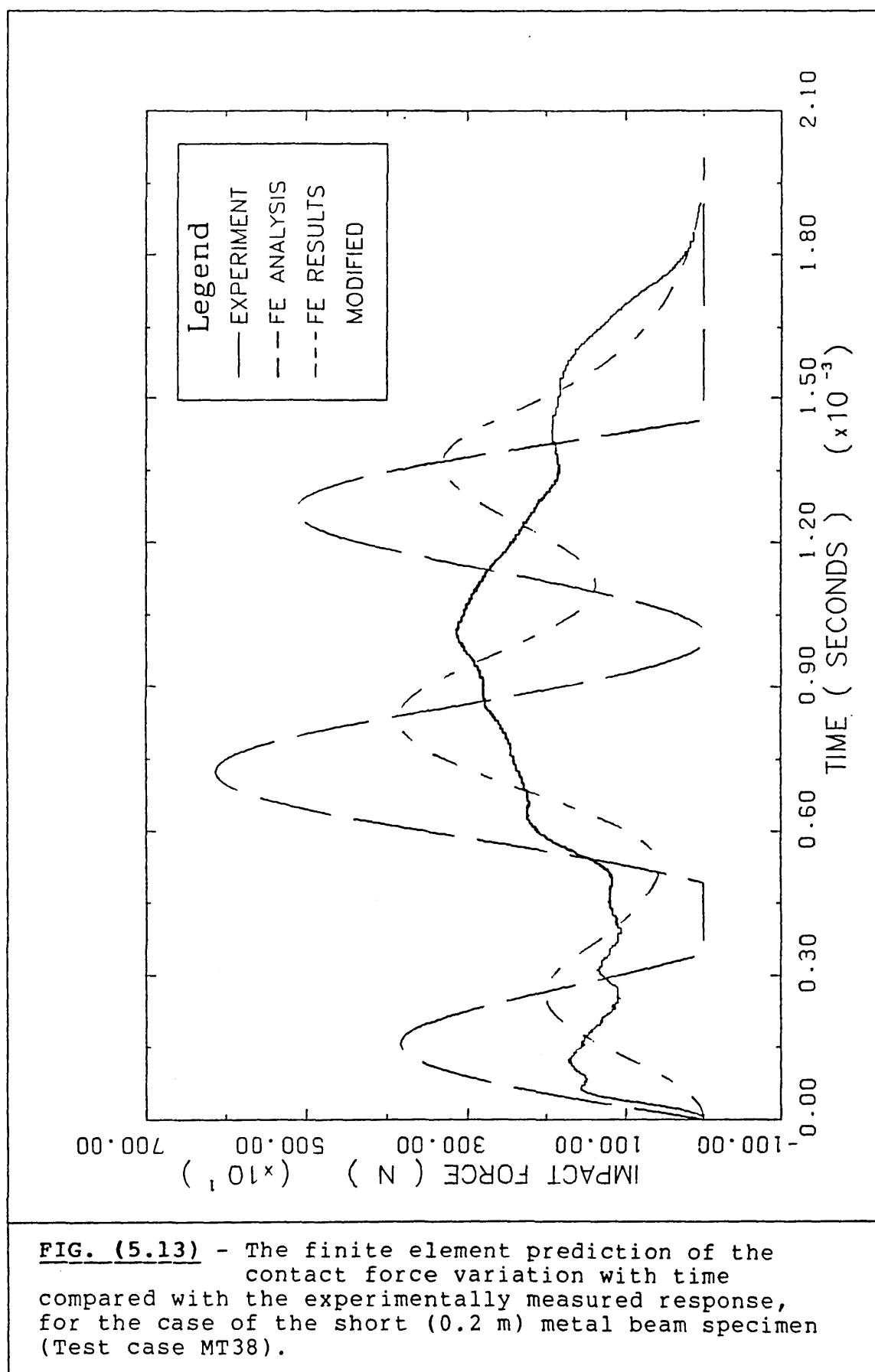


FIG. (5.12) - The finite element prediction of the bending stress variation with time, at 25 cm from the point of impact, compared with the measured response, for the case of the medium length (0.8 m) metal beam specimen (Test case MT29).



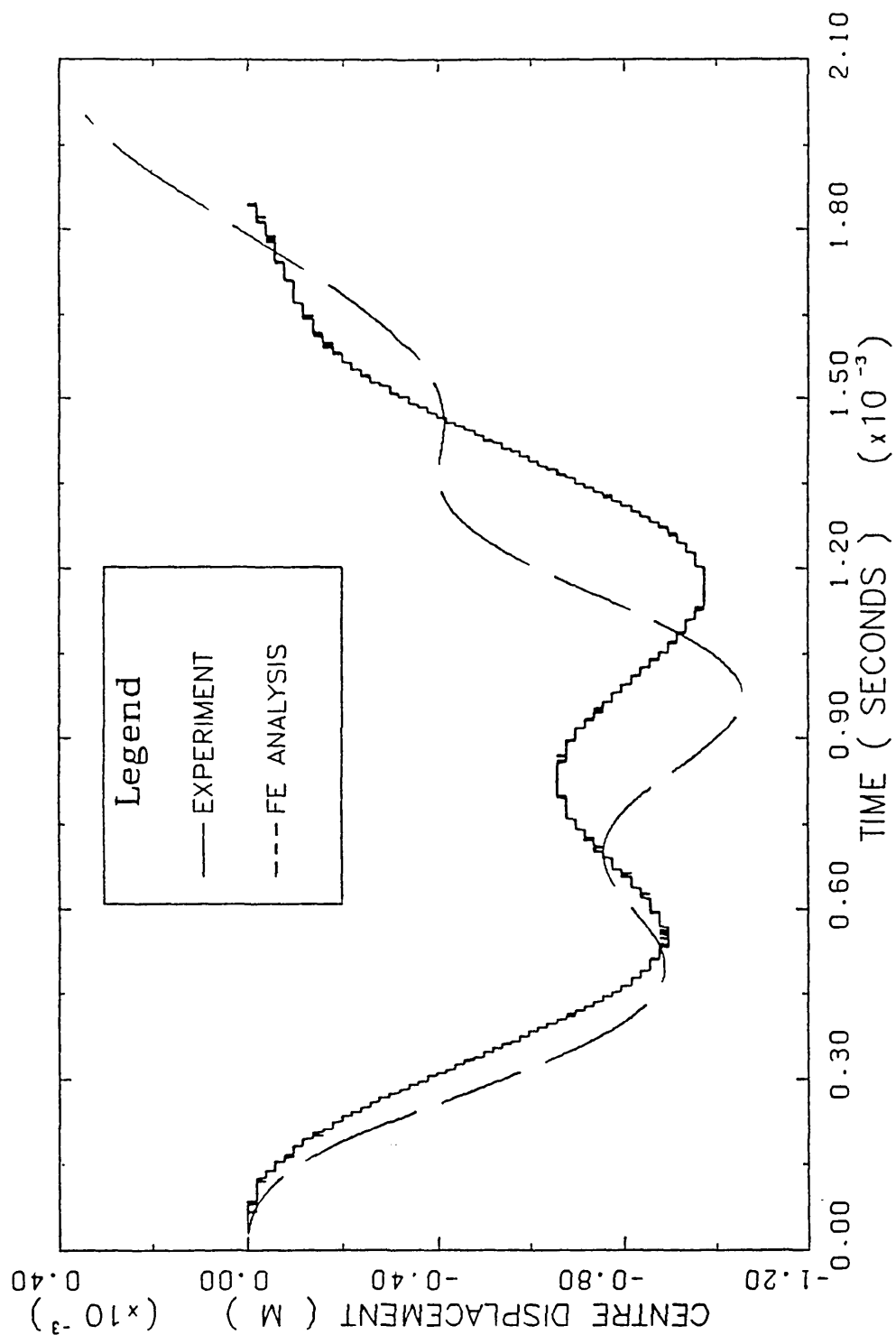


FIG. (5.14) - The finite element prediction of the displacement variation with time, at the beam centre, compared with the experimentally measured response, for the case of the short (0.2 m) metal beam specimen (Test case MT38).

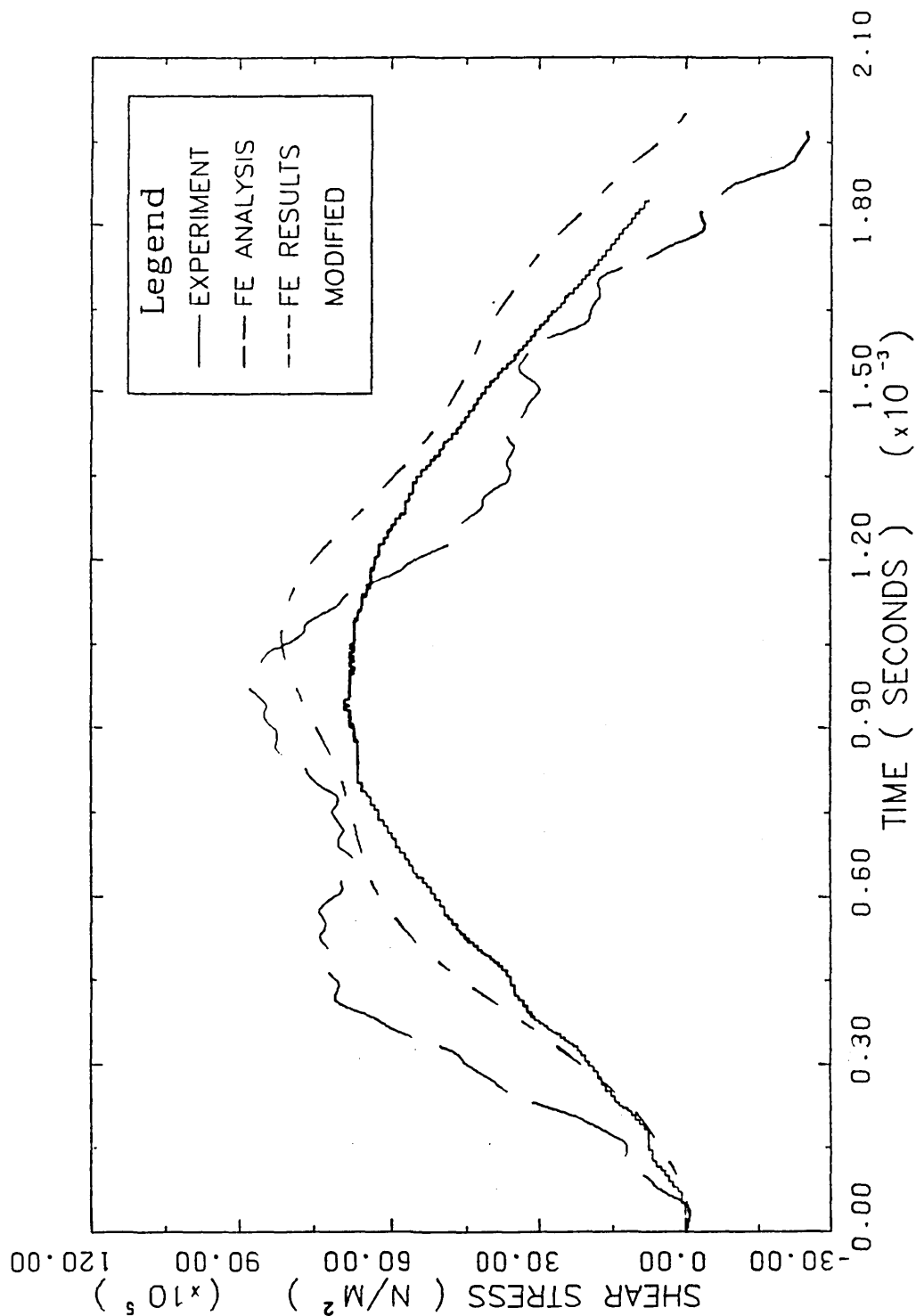


FIG. (5.15) - The finite element prediction of the shear stress variation with time, at 5 cm from the point of impact, compared with the experimentally measured response, for the case of the short (0.2 m) metal beam specimen (Test case MT38).

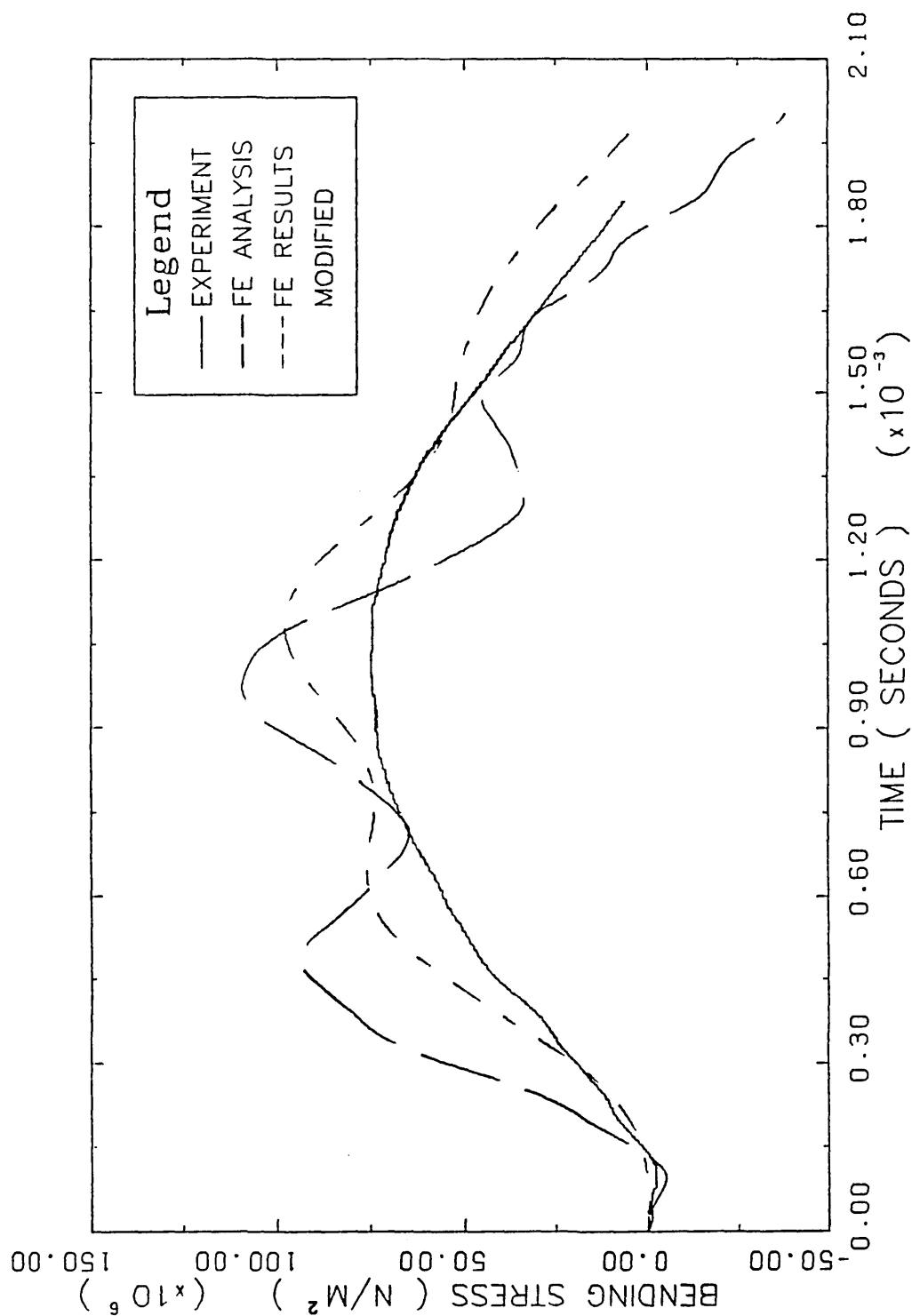


FIG. (5.16) - The finite element prediction of the bending stress variation with time, at 5 cm from the point of impact, compared with the experimentally measured response, for the case of the short (0.2 m) metal beam specimen (Test case MT38).

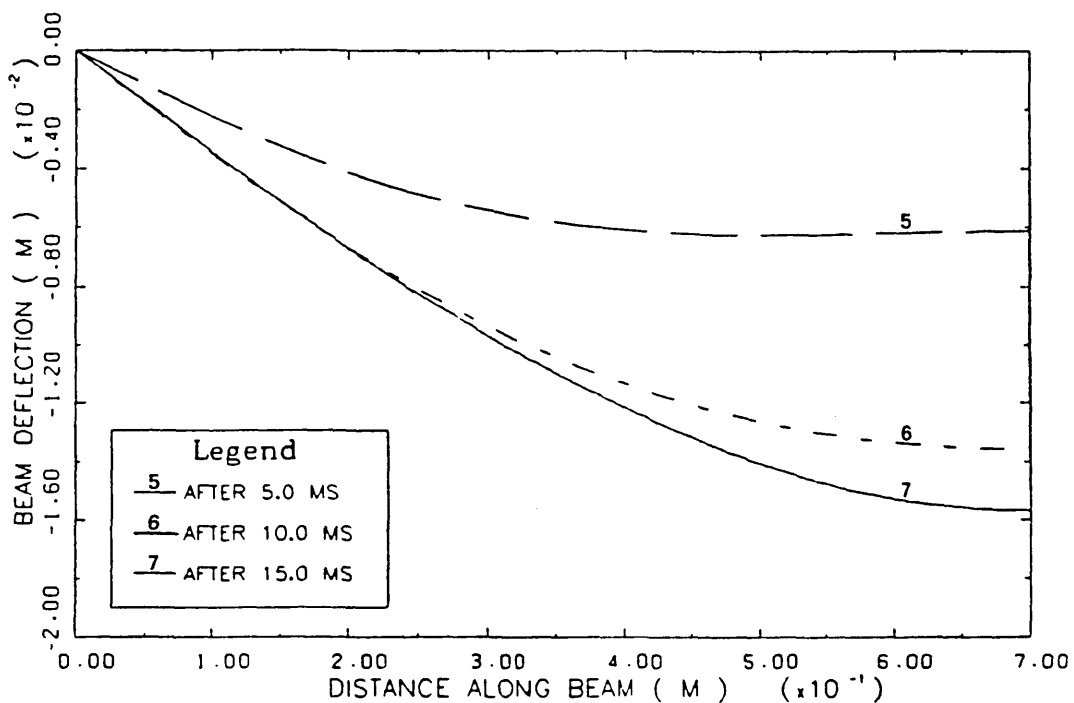
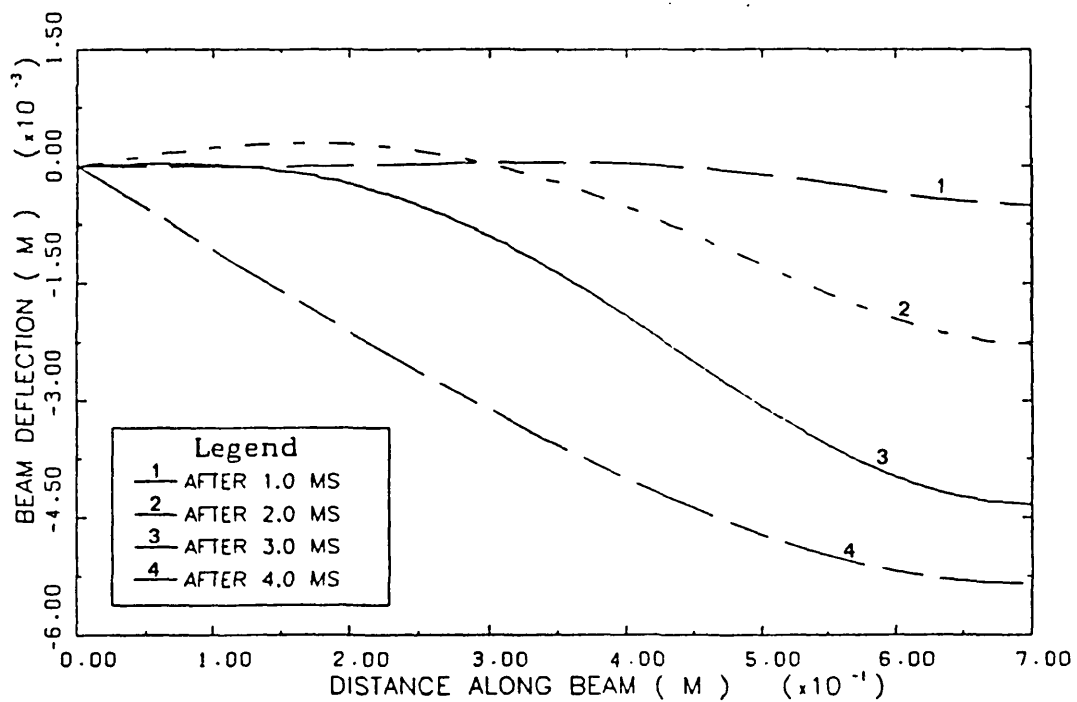


FIG. (5.17) - The finite element prediction for the impact-induced transverse deflection of the long (1.4 m) metal beam specimen (Test case MT4), at various times.

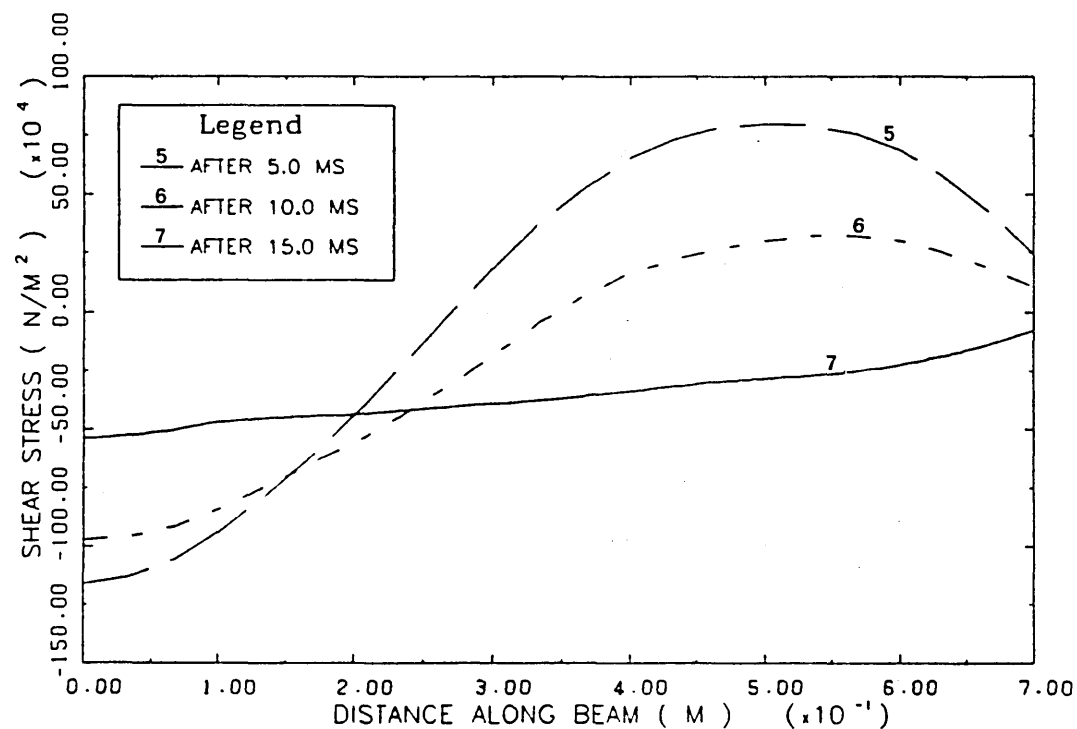
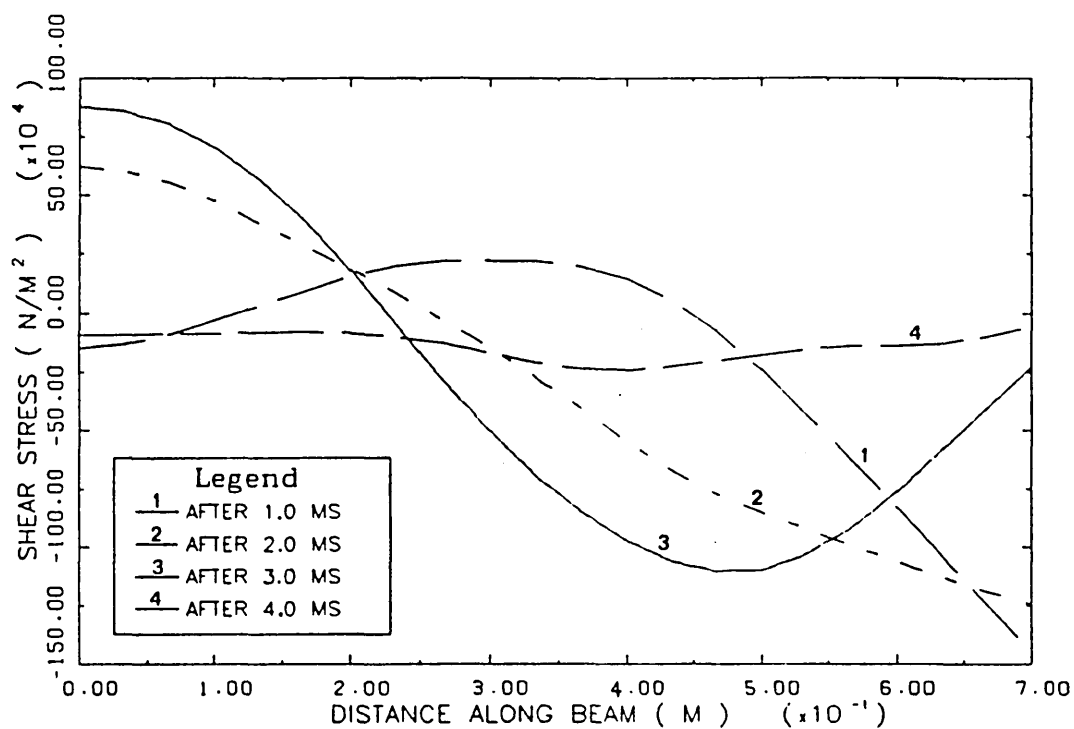


FIG. (5.18) - The finite element prediction of the spatial variation of impact-induced shear stresses in the long (1.4 m) metal beam specimen (Test case MT4), at various times.

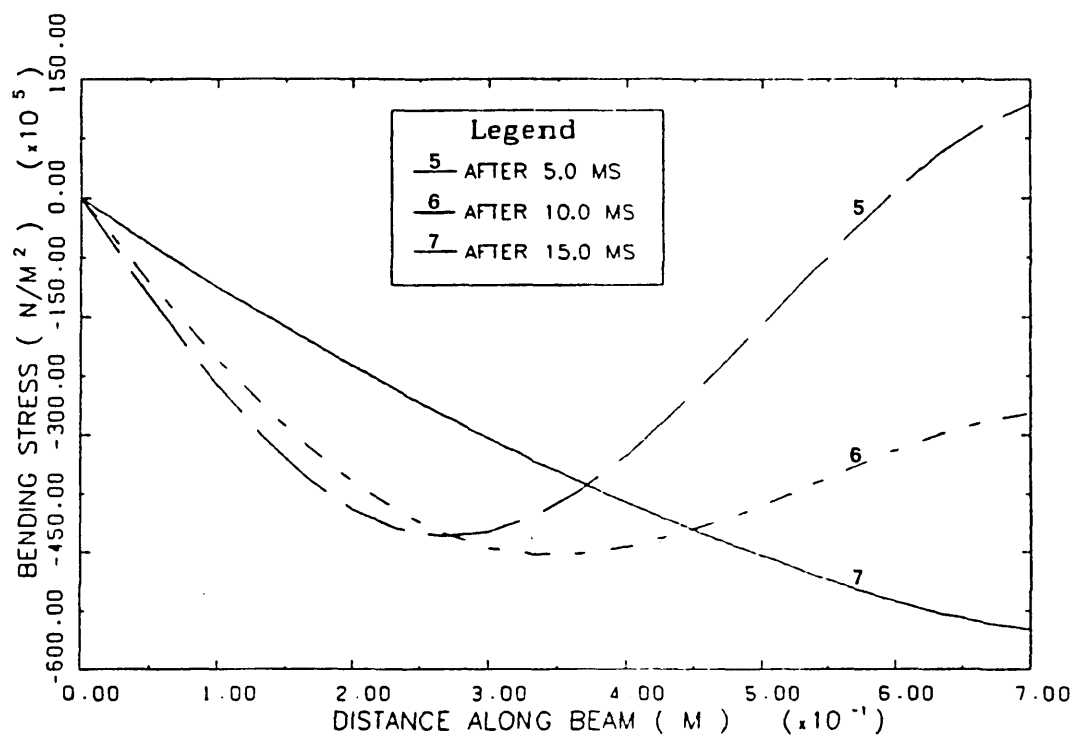
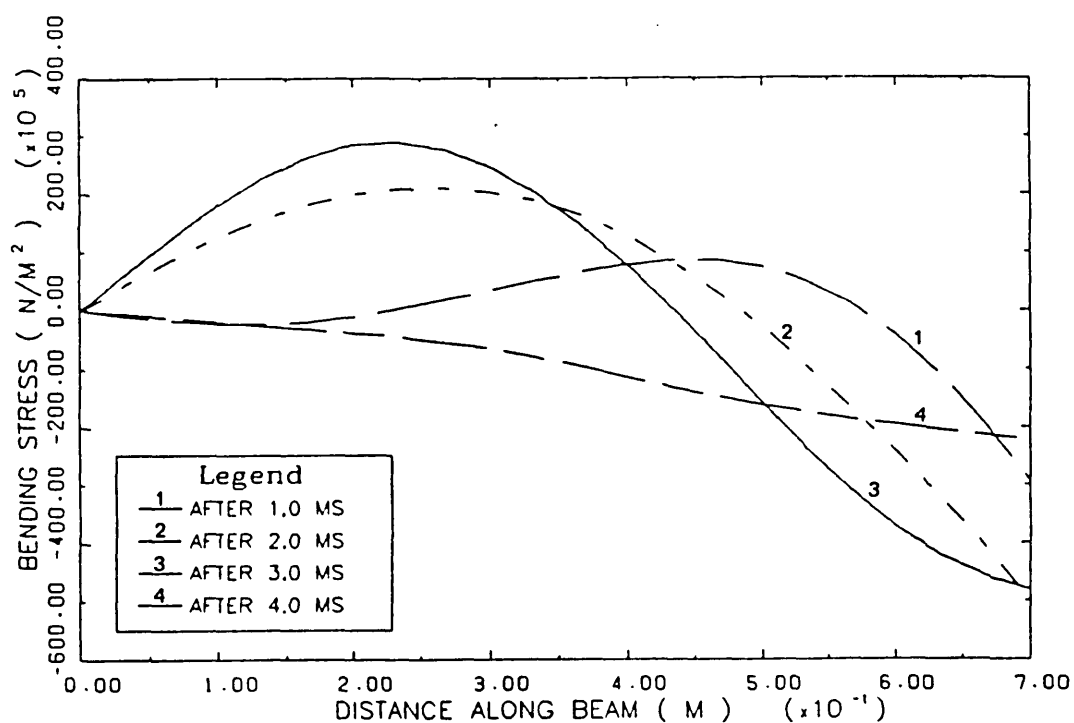


FIG. (5.19) - The finite element prediction of the spatial variation of impact-induced bending stresses in the long (1.4 m) metal beam specimen (Test case MT4), at various times.

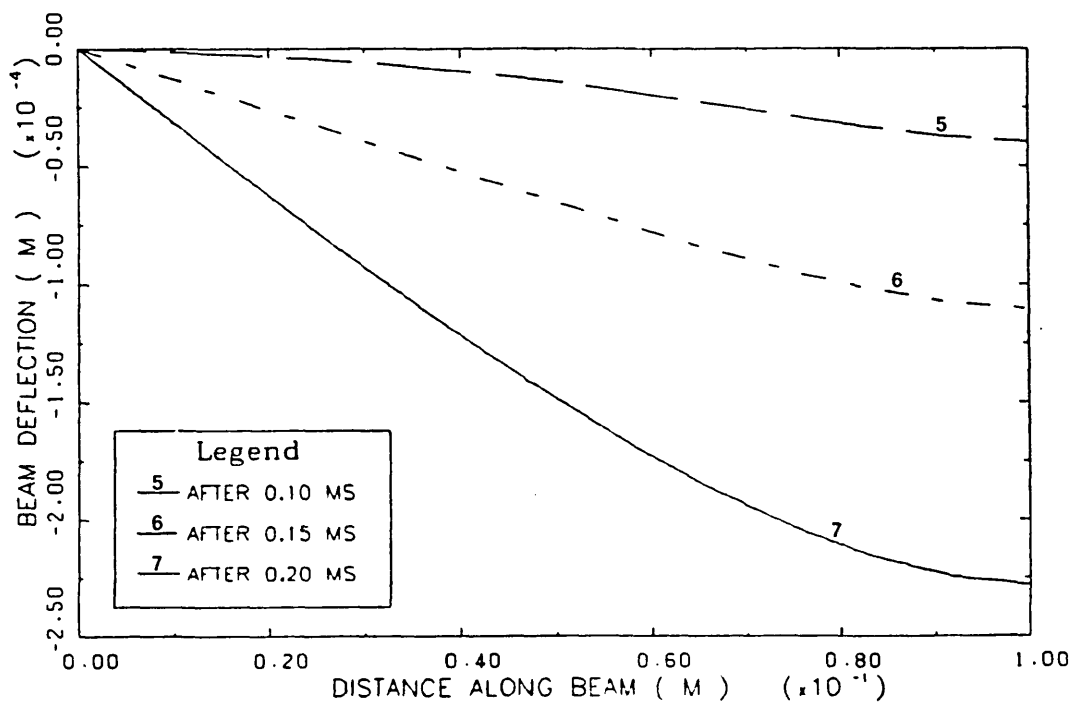
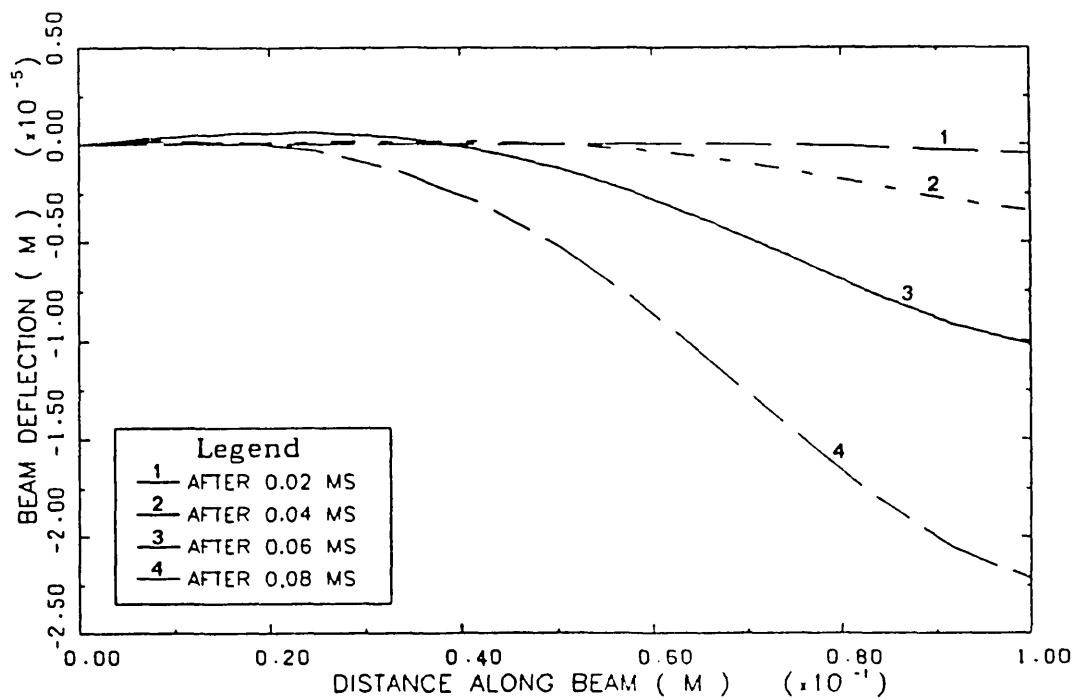


FIG. (5.20) - The finite element prediction for the impact-induced transverse deflection of the short (0.2 m) metal beam specimen (Test case MT38), at various times.

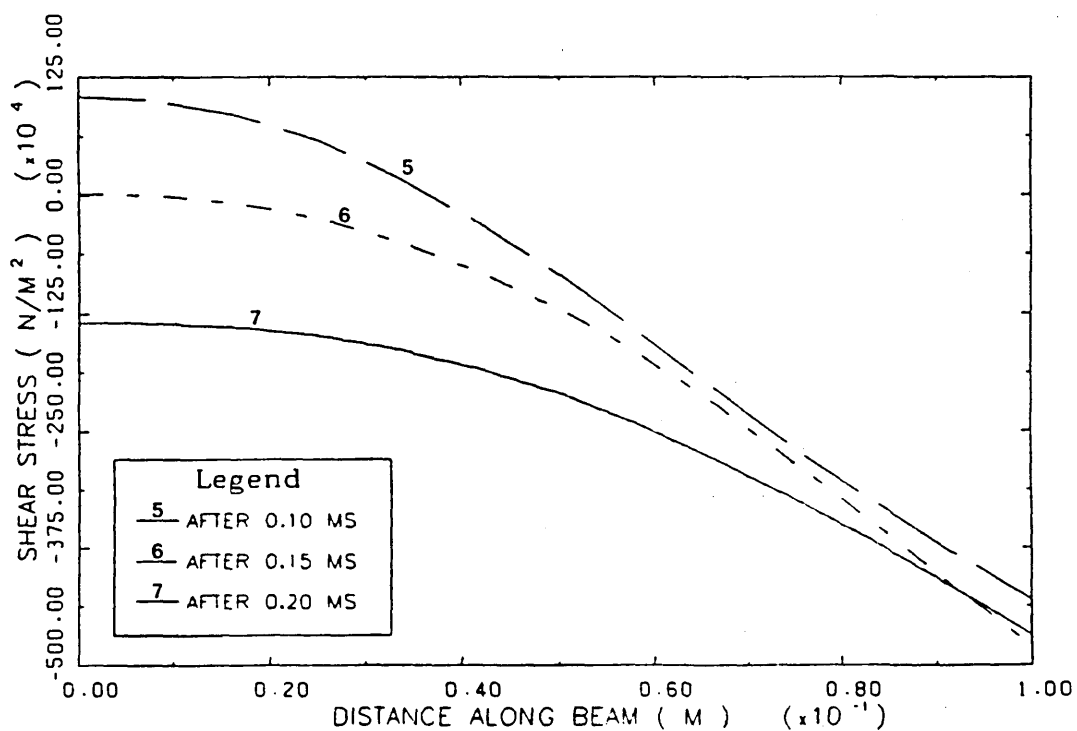
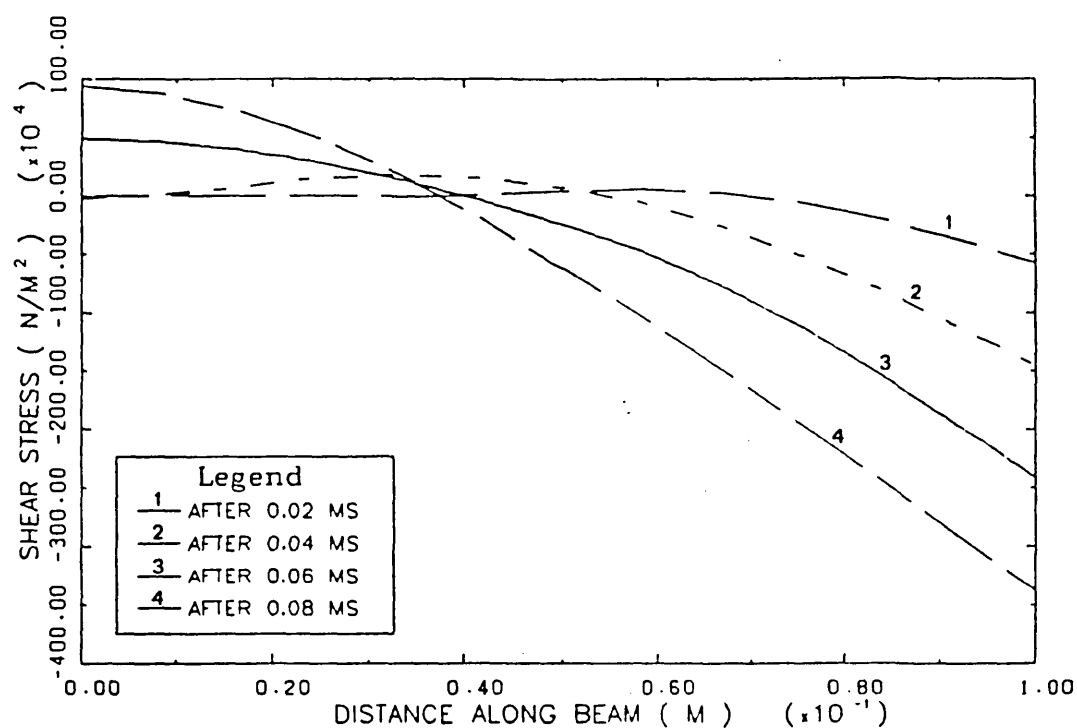


FIG. (5.21) - The finite element prediction of the spatial variation of impact-induced shear stresses in the short (0.2 m) metal beam specimen (Test case MT38), at various times.

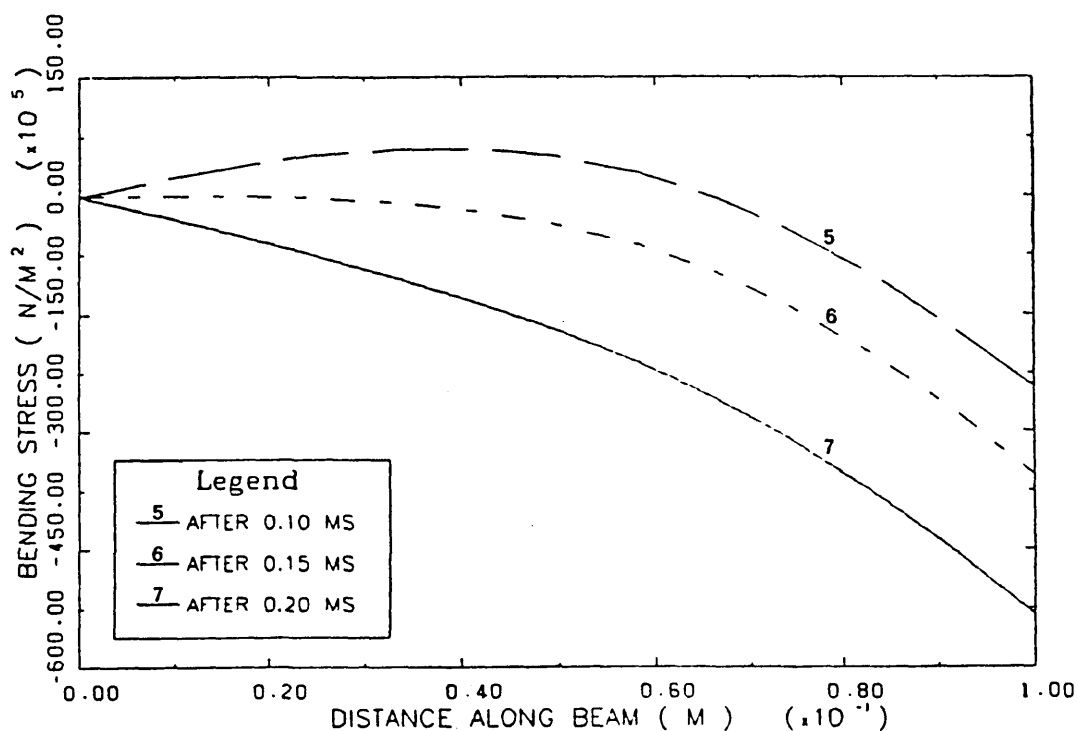
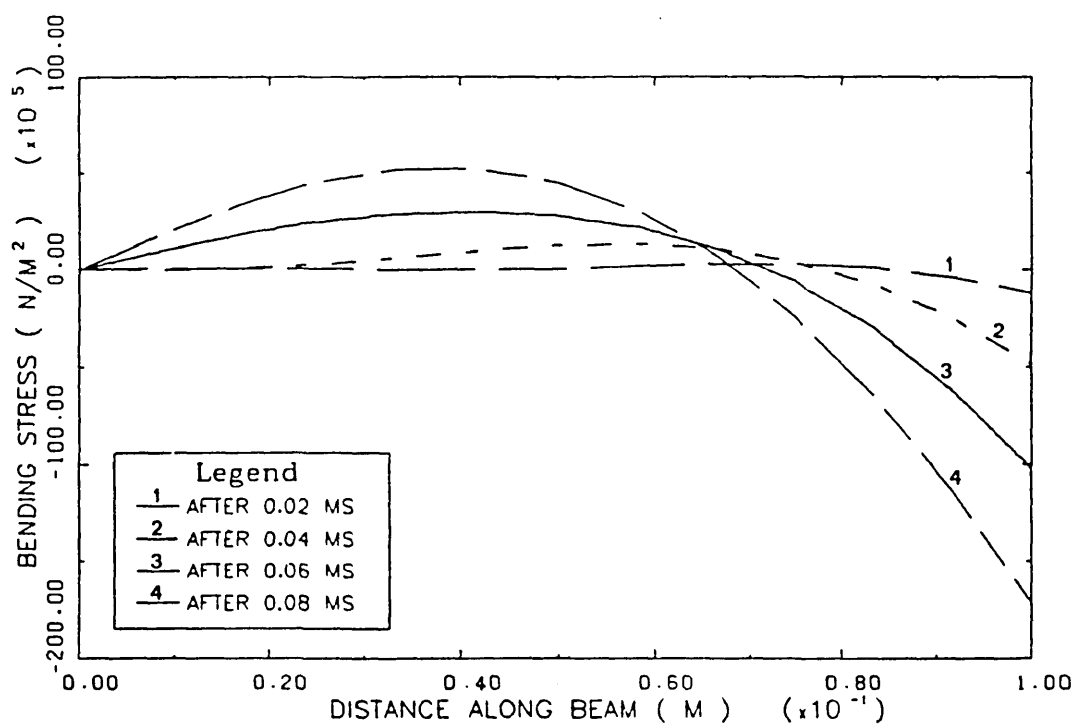
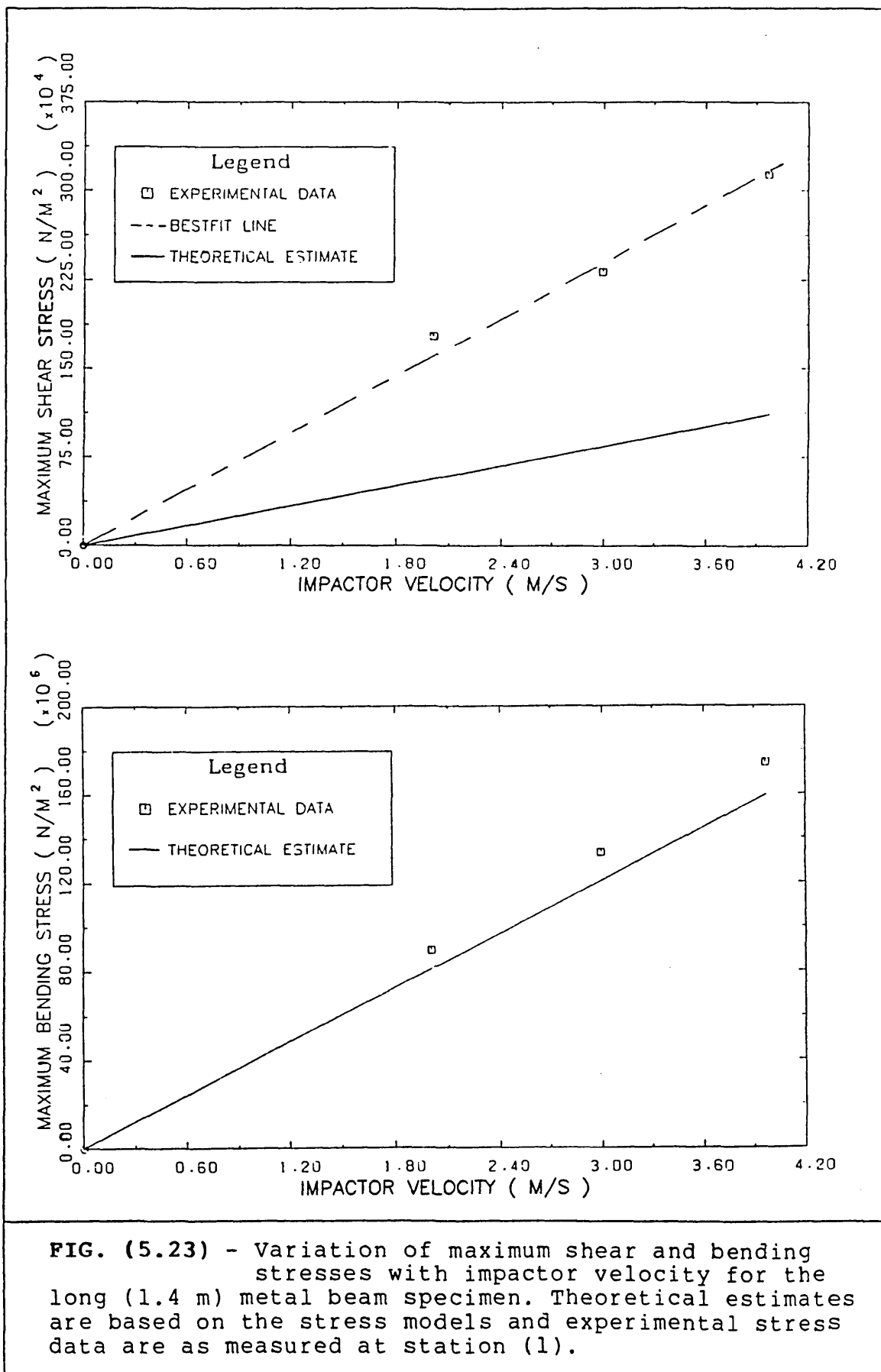
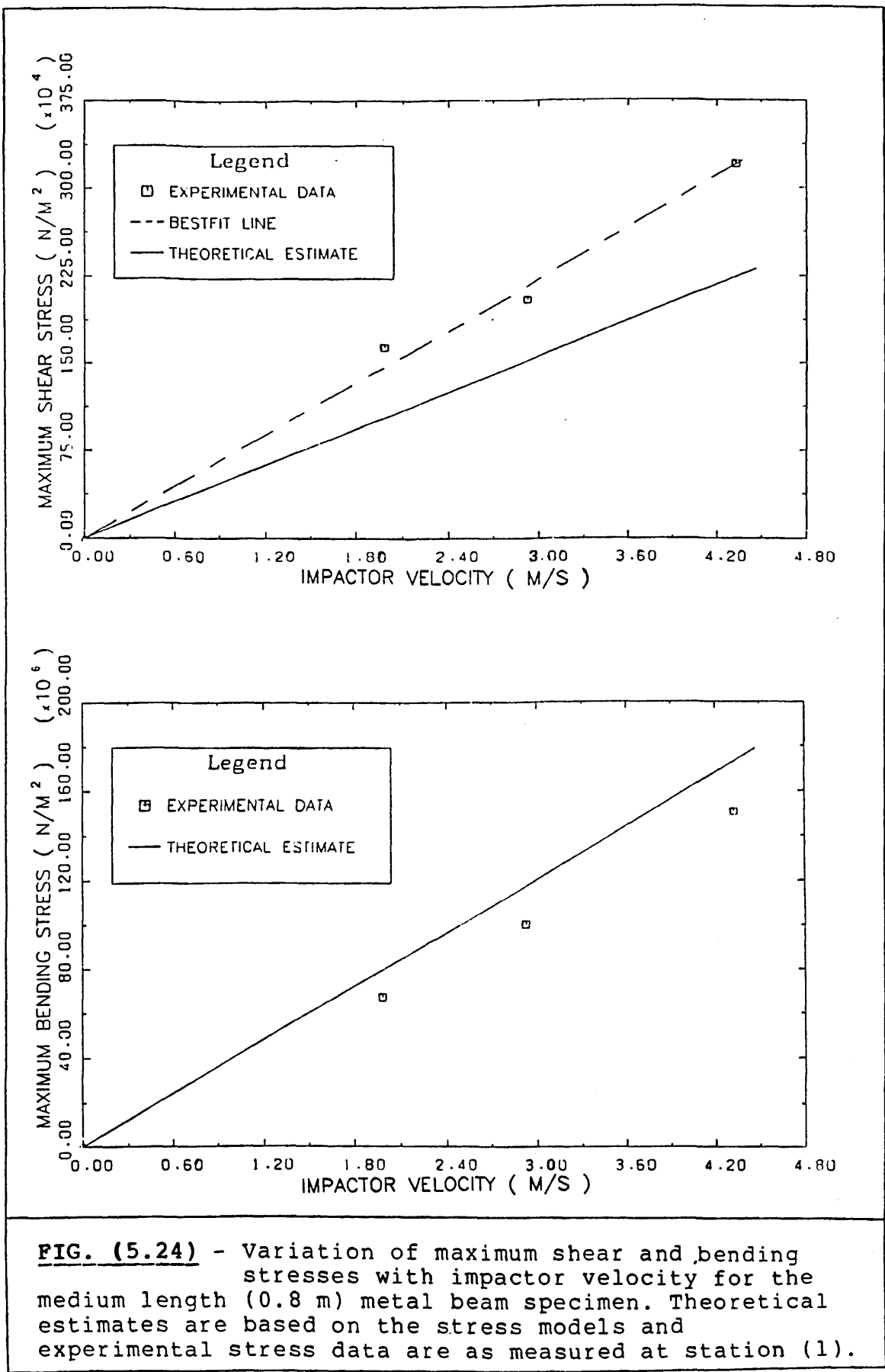
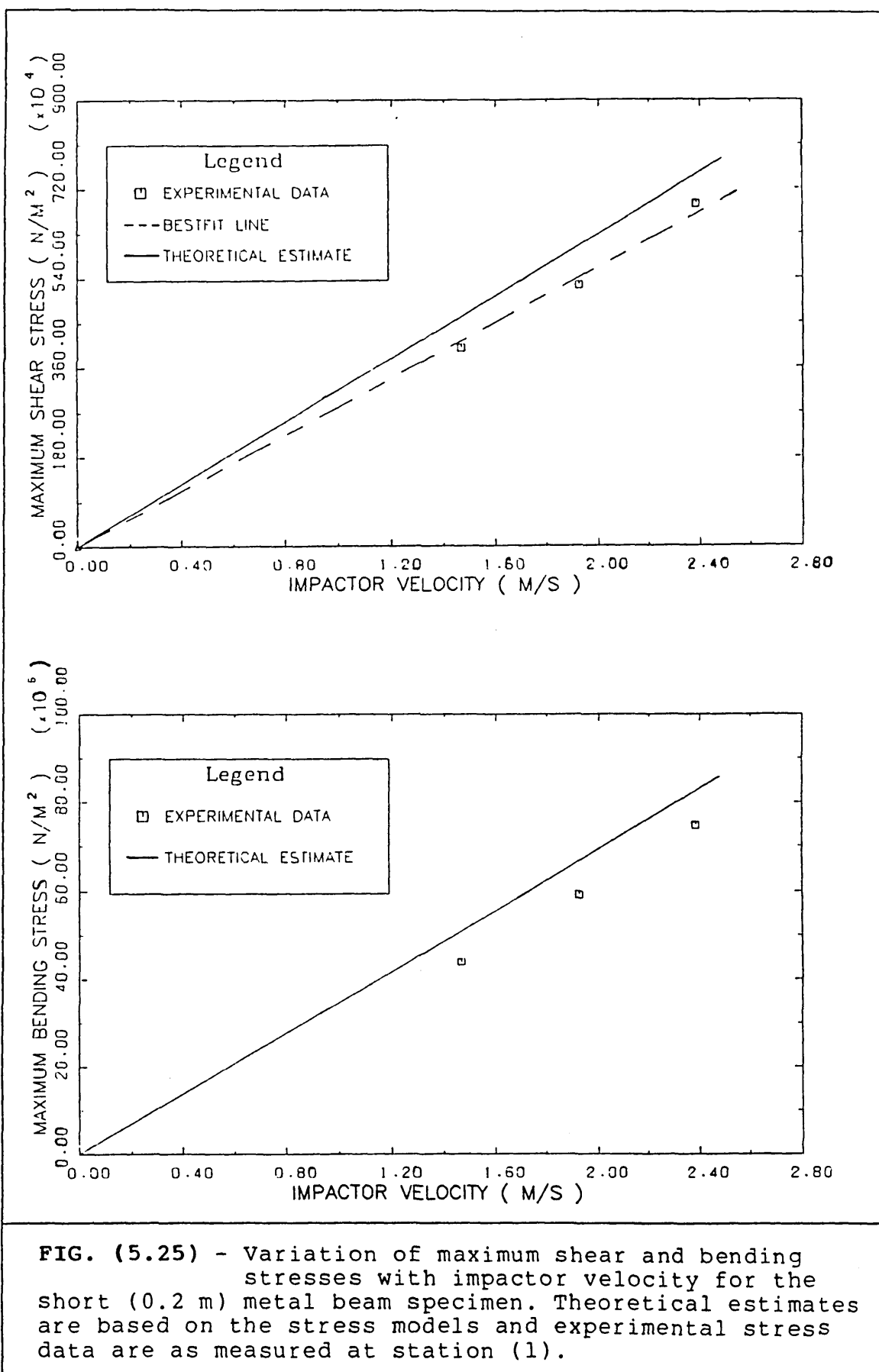
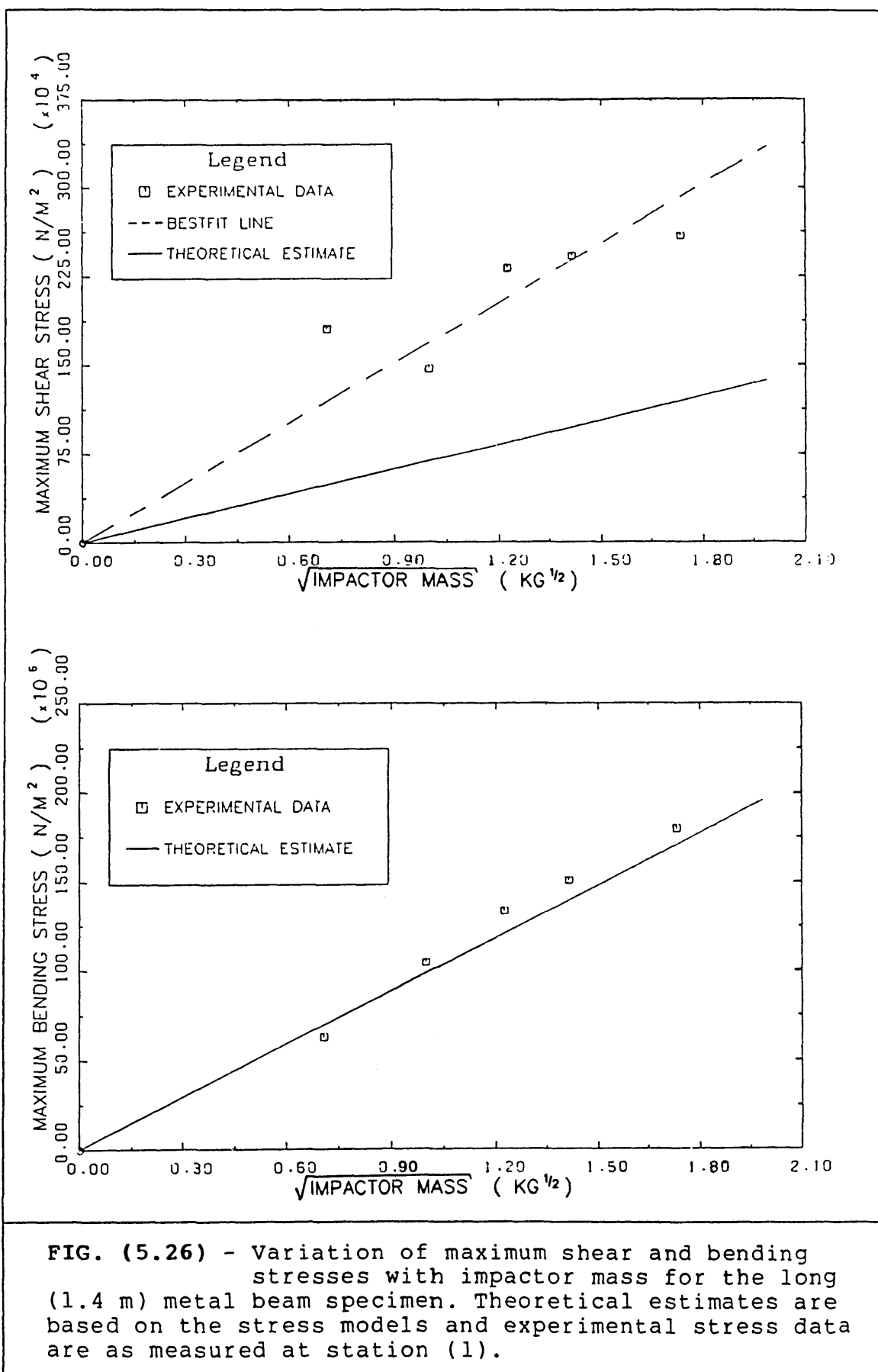


FIG. (5.22) - The finite element prediction of the spatial variation of impact-induced bending stresses in the short (0.2 m) metal beam specimen (Test case MT38), at various times.









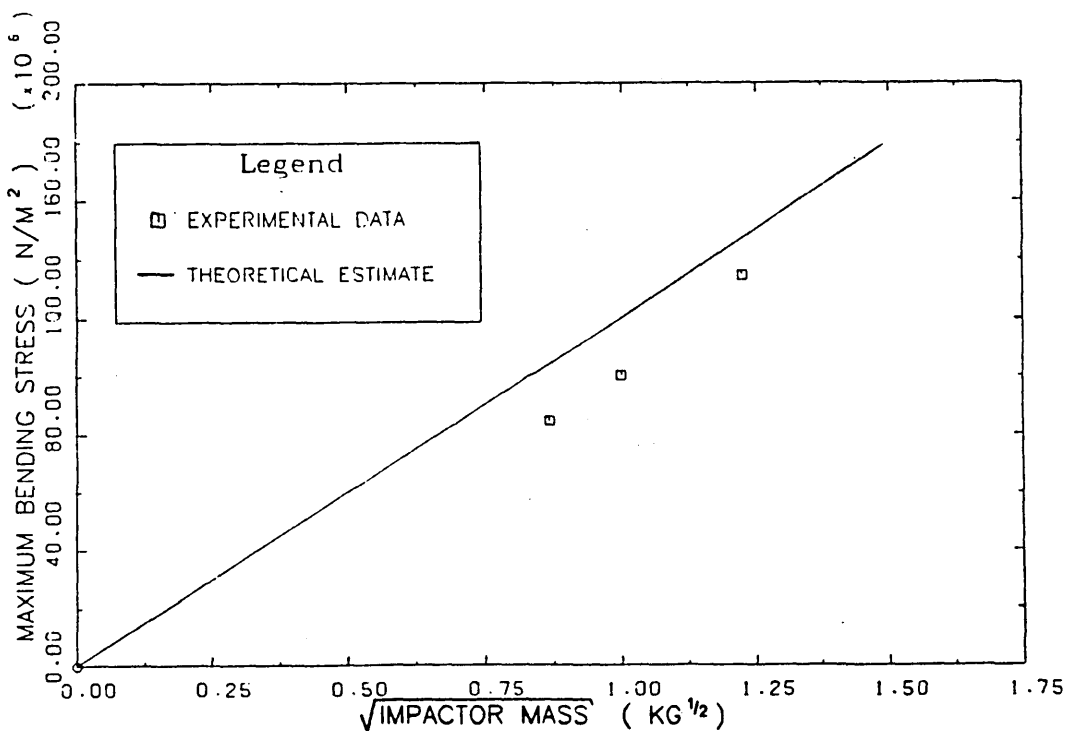
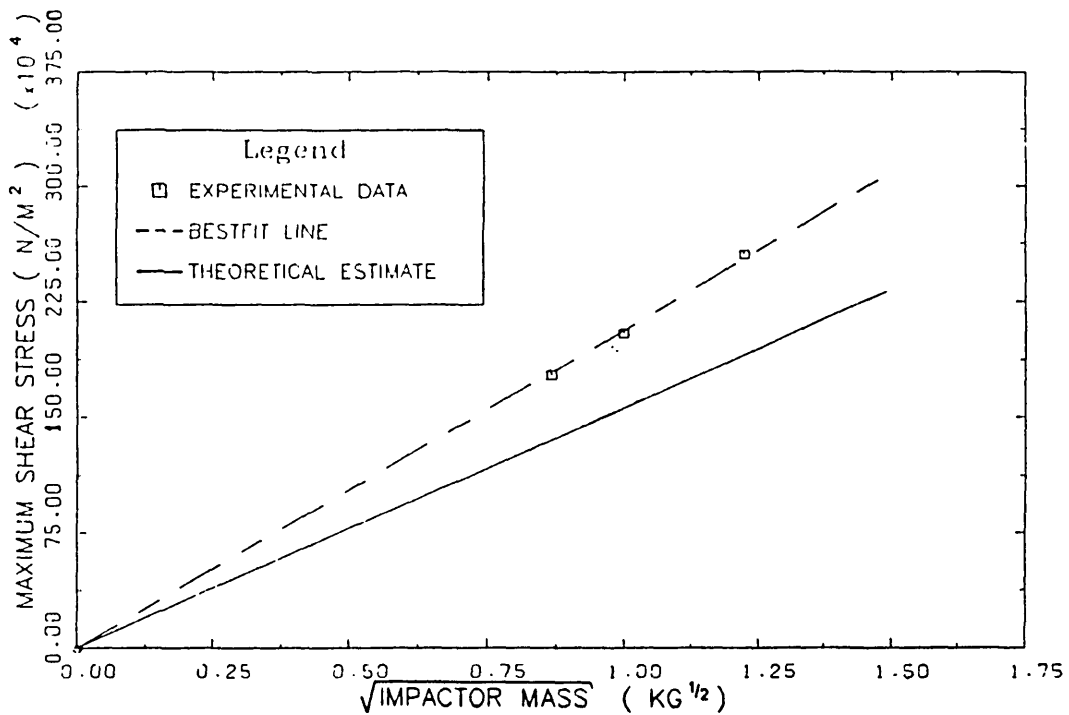


FIG. (5.27) - Variation of maximum shear and bending stresses with impactor mass for the medium length (0.8 m) metal beam specimen. Theoretical estimates are based on the stress models and experimental stress data are as measured at station (1).

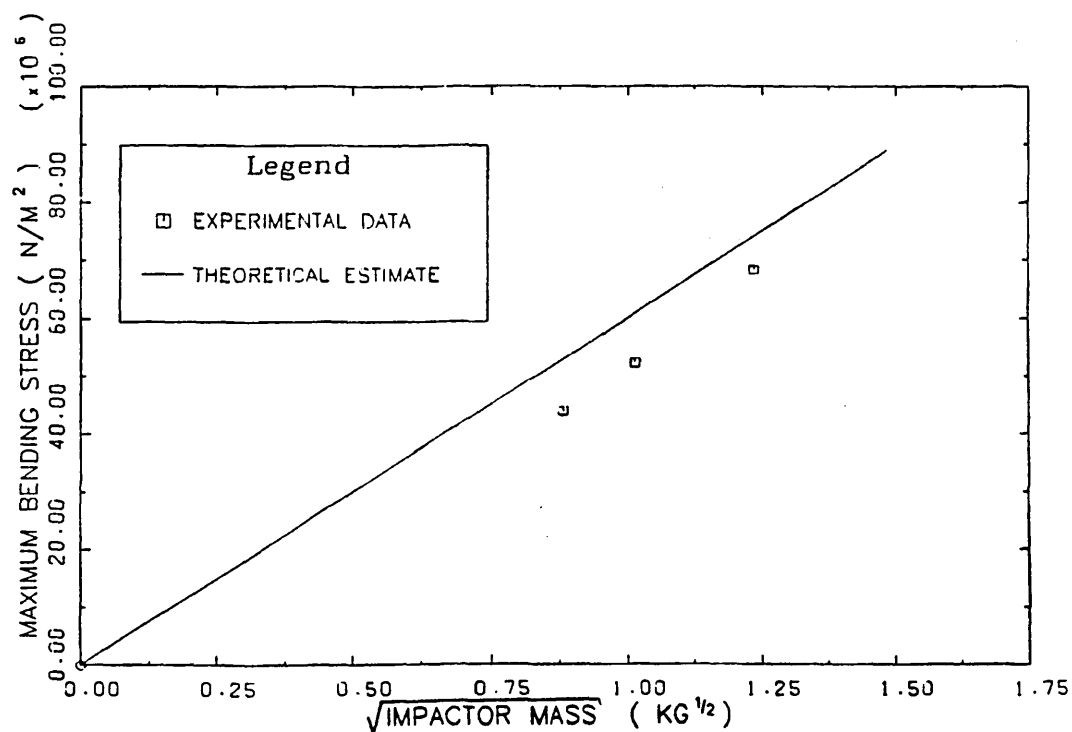
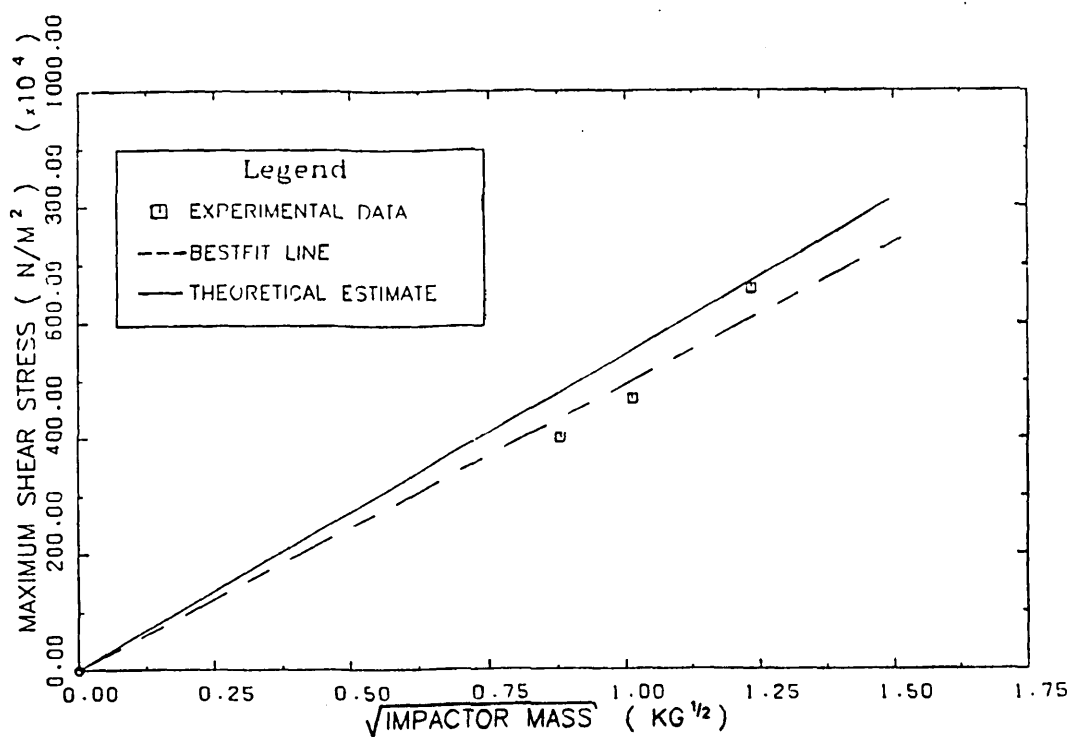


FIG. (5.28) - Variation of maximum shear and bending stresses with impactor mass for the short (0.2 m) metal beam specimen. Theoretical estimates are based on the stress models and experimental stress data are as measured at station (1).

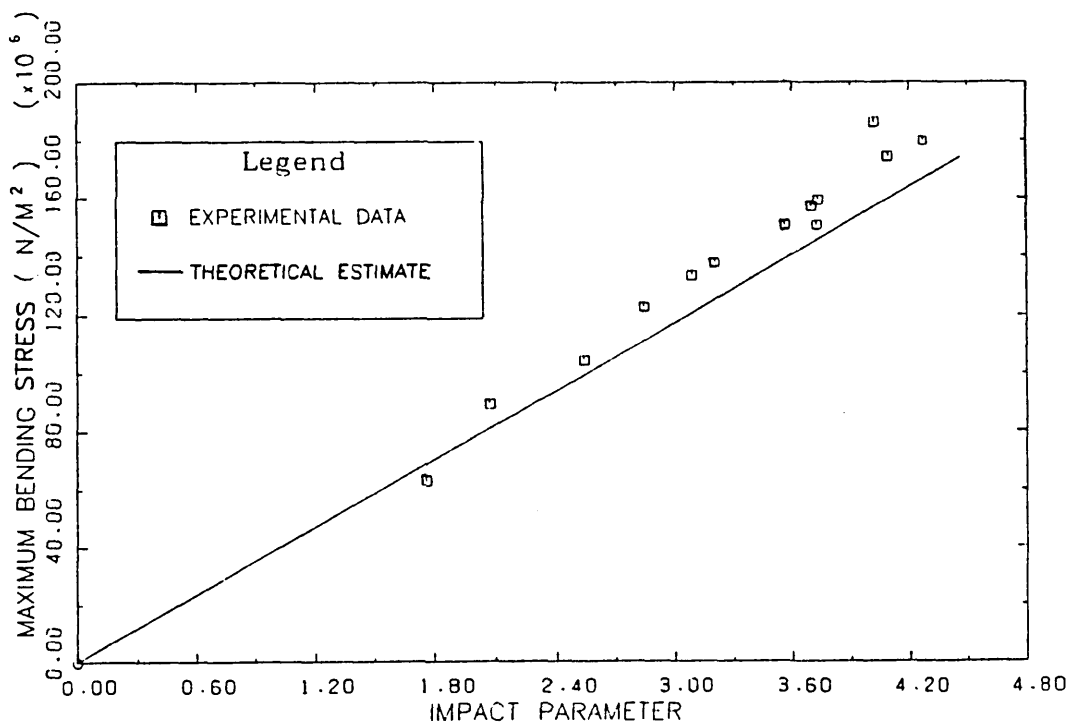
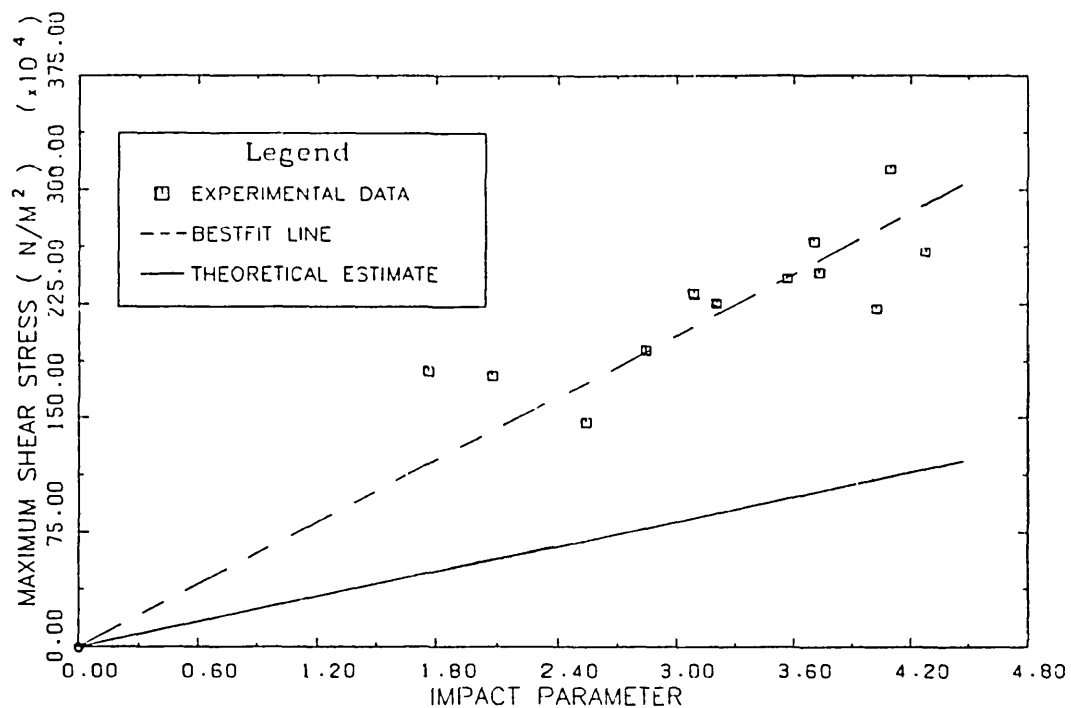


FIG. (5.29) - Variation of maximum shear and bending stresses with impact parameter for the long (1.4 m) metal beam specimen. Theoretical estimates are based on the stress models and experimental stress data are as measured at station (1).

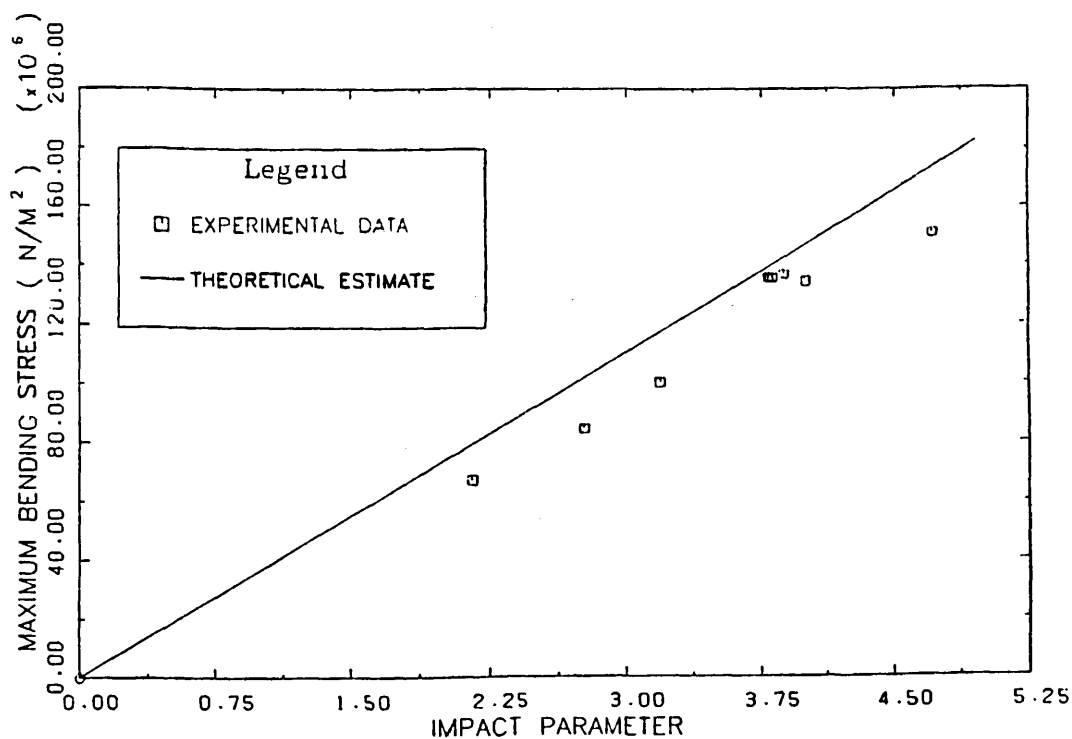
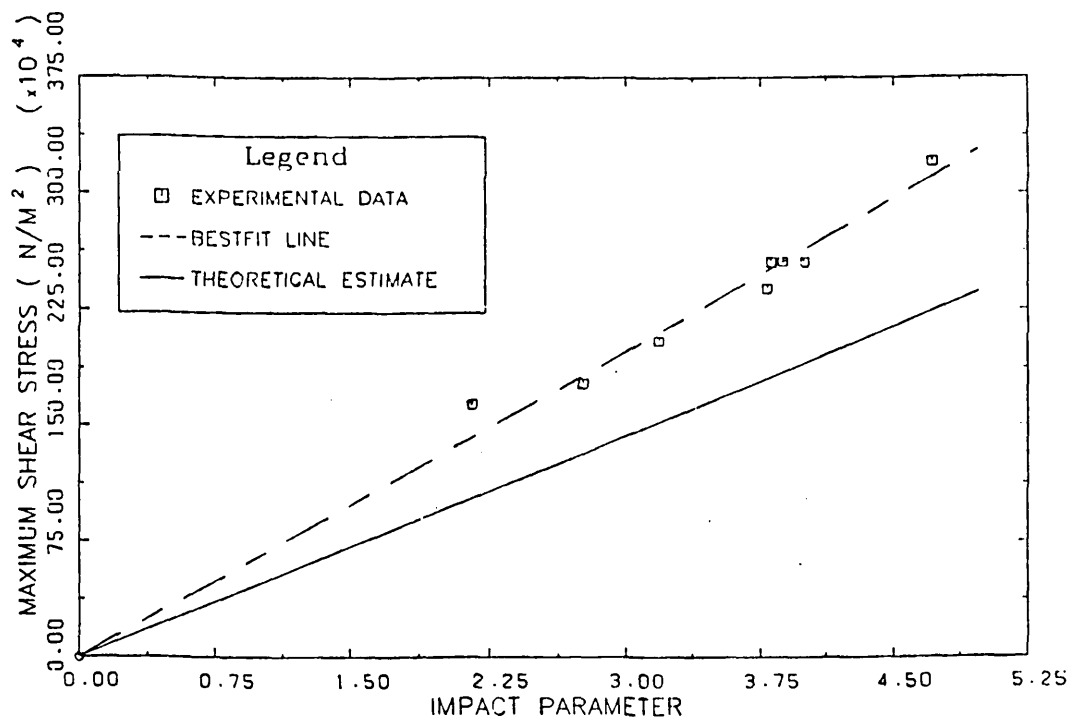
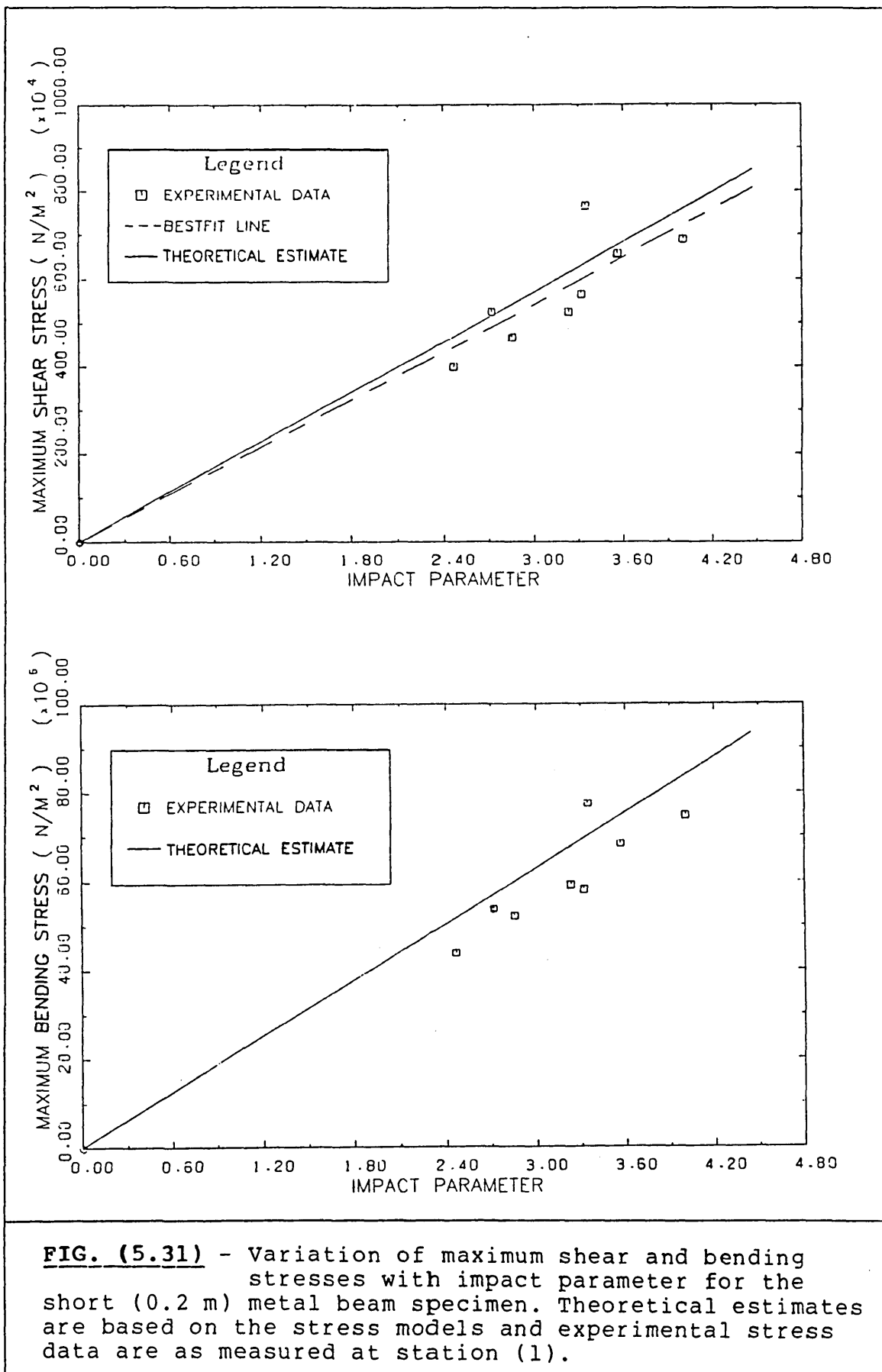


FIG. (5.30) - Variation of maximum shear and bending stresses with impact parameter for the medium length (0.8 m) metal beam specimen. Theoretical estimates are based on the stress models and experimental stress data are as measured at station (1).



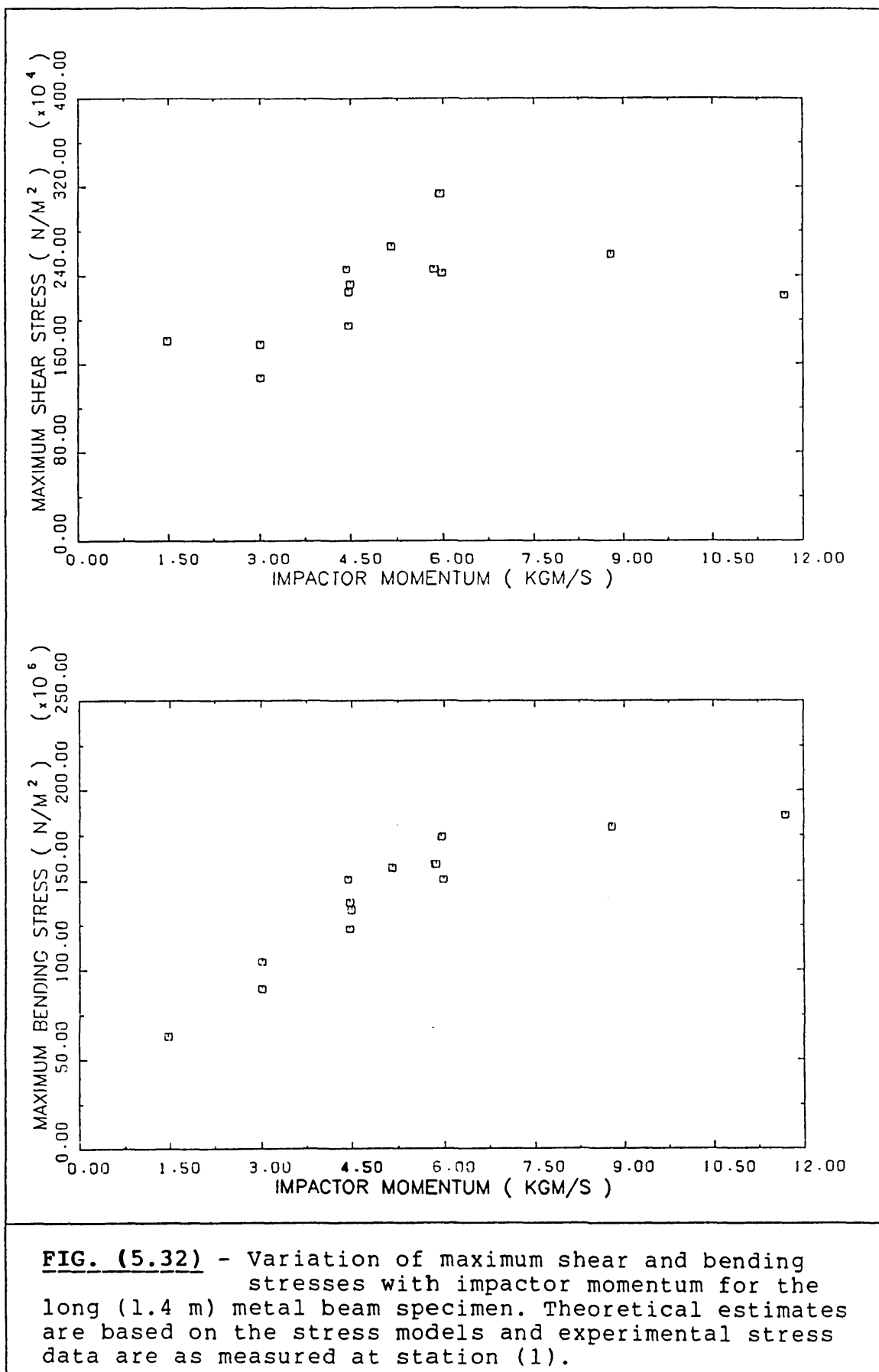
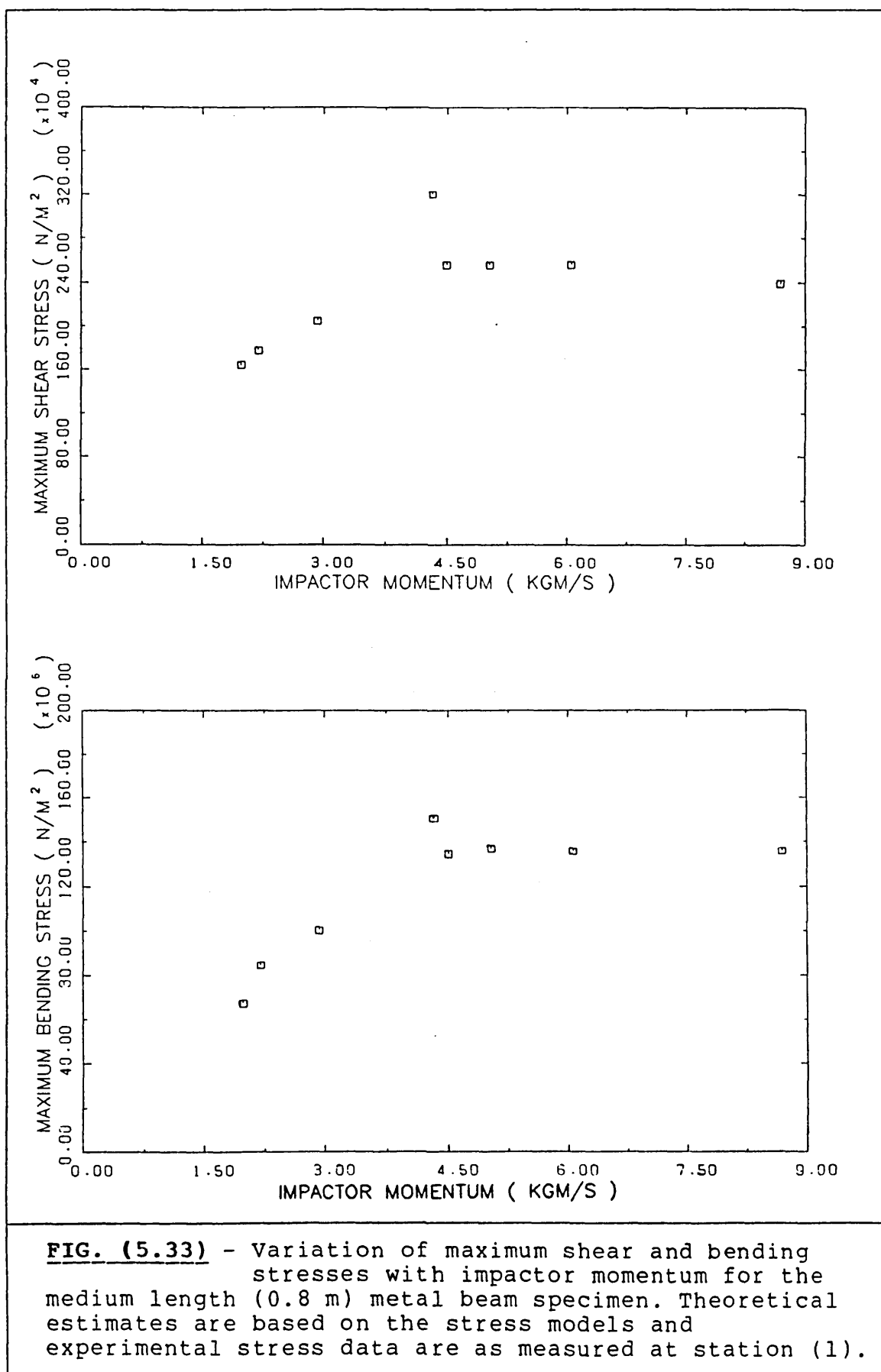


FIG. (5.32) - Variation of maximum shear and bending stresses with impactor momentum for the long (1.4 m) metal beam specimen. Theoretical estimates are based on the stress models and experimental stress data are as measured at station (1).



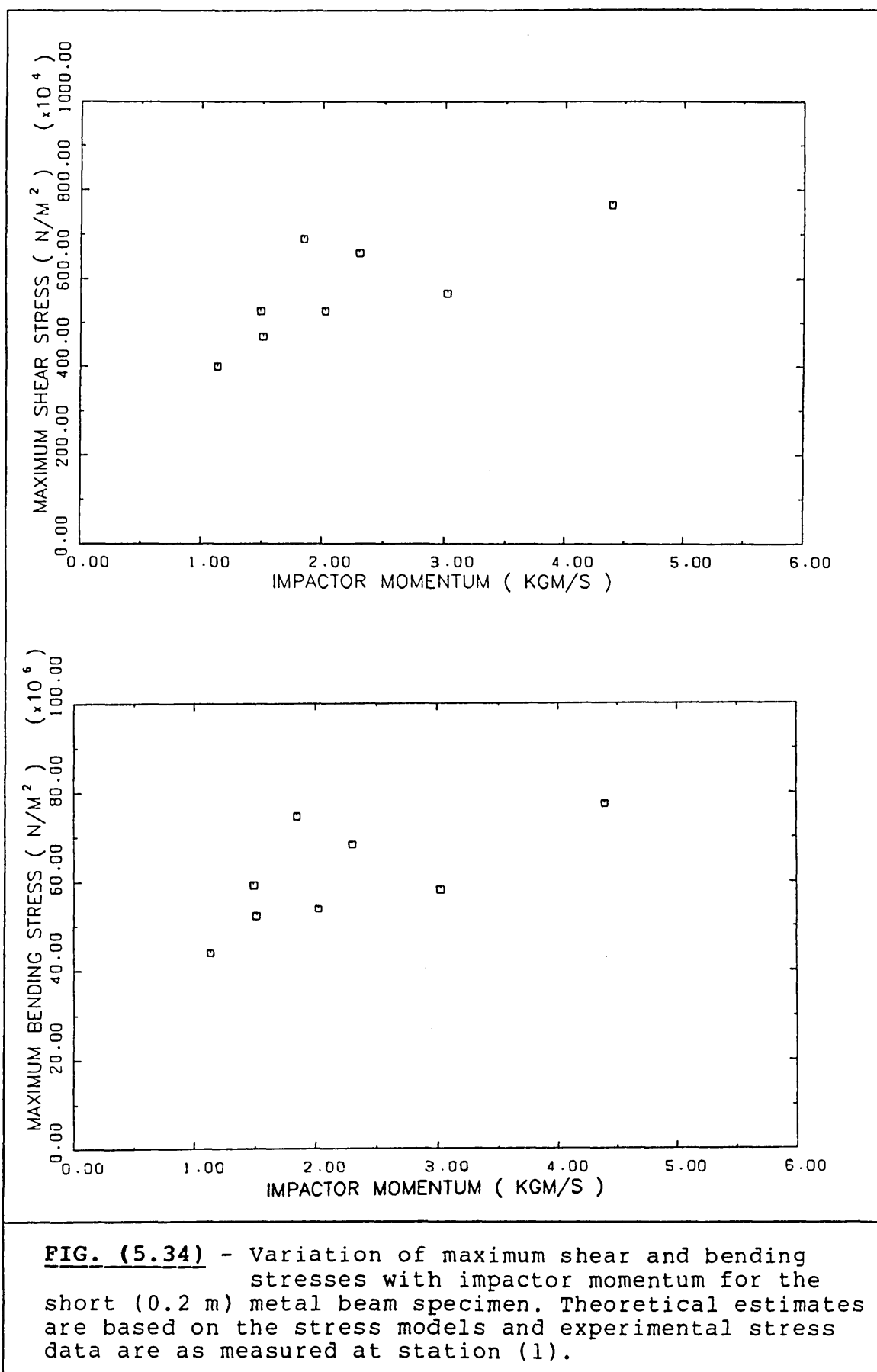


FIG. (5.34) - Variation of maximum shear and bending stresses with impactor momentum for the short (0.2 m) metal beam specimen. Theoretical estimates are based on the stress models and experimental stress data are as measured at station (1).

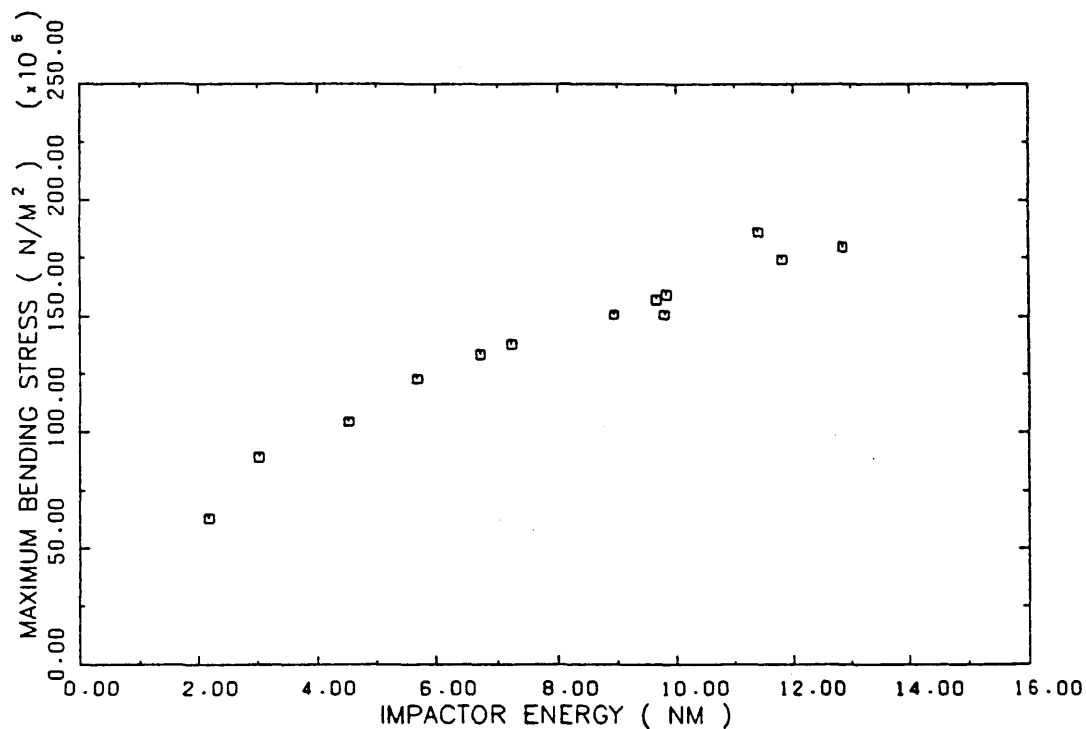
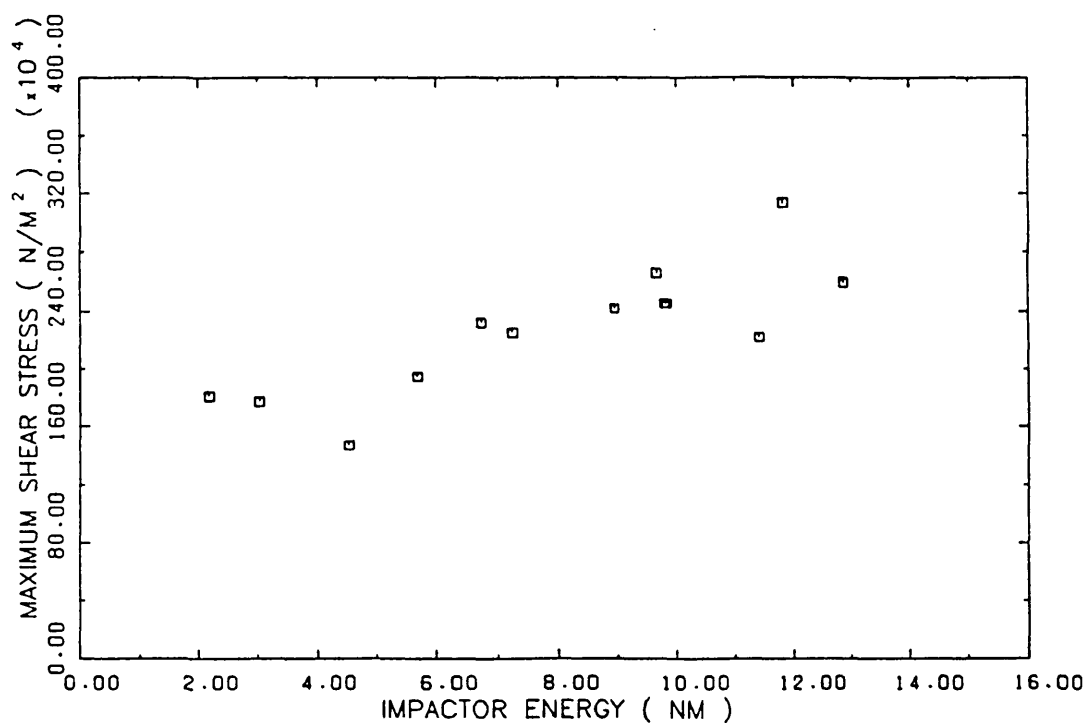
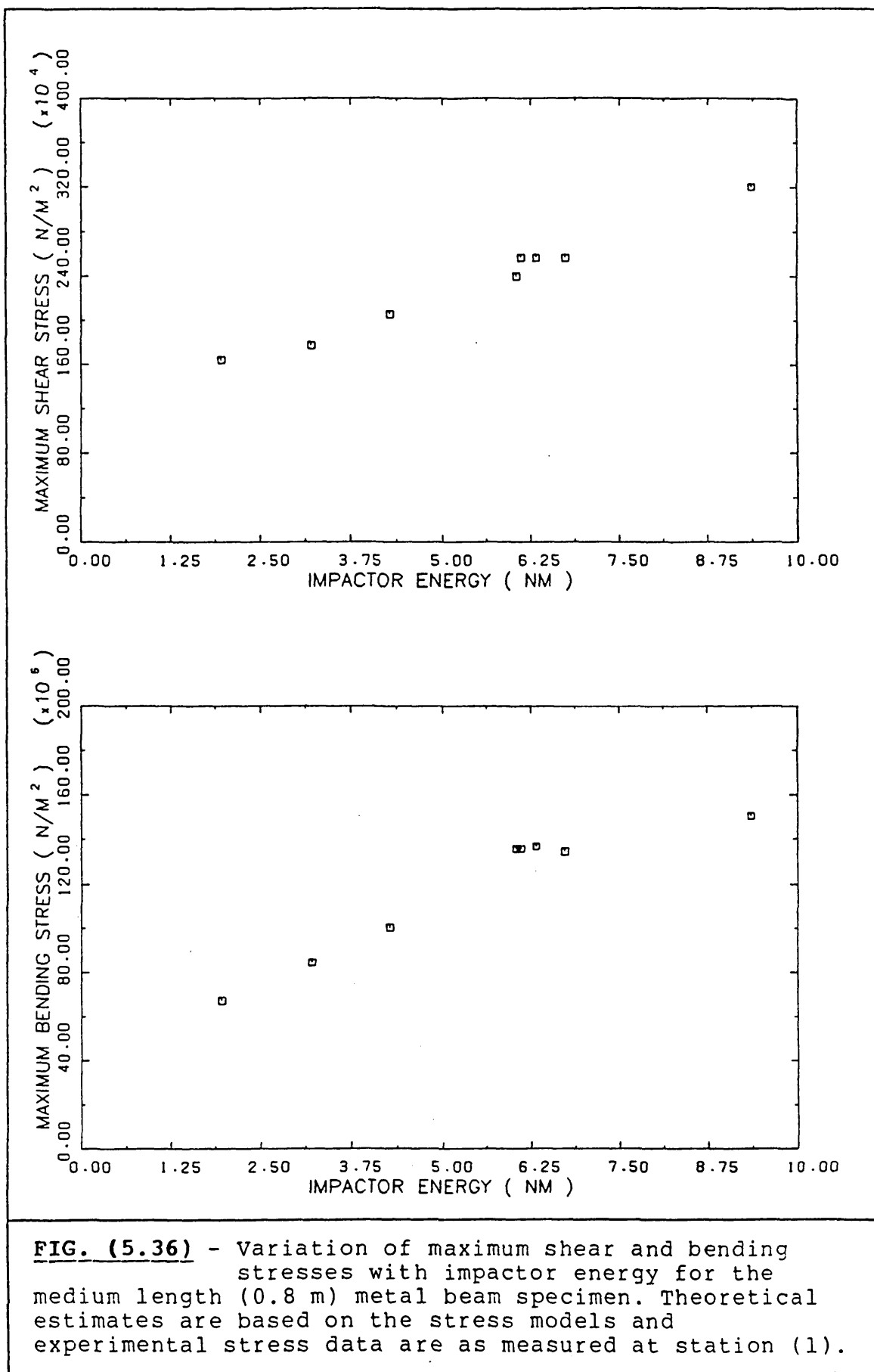
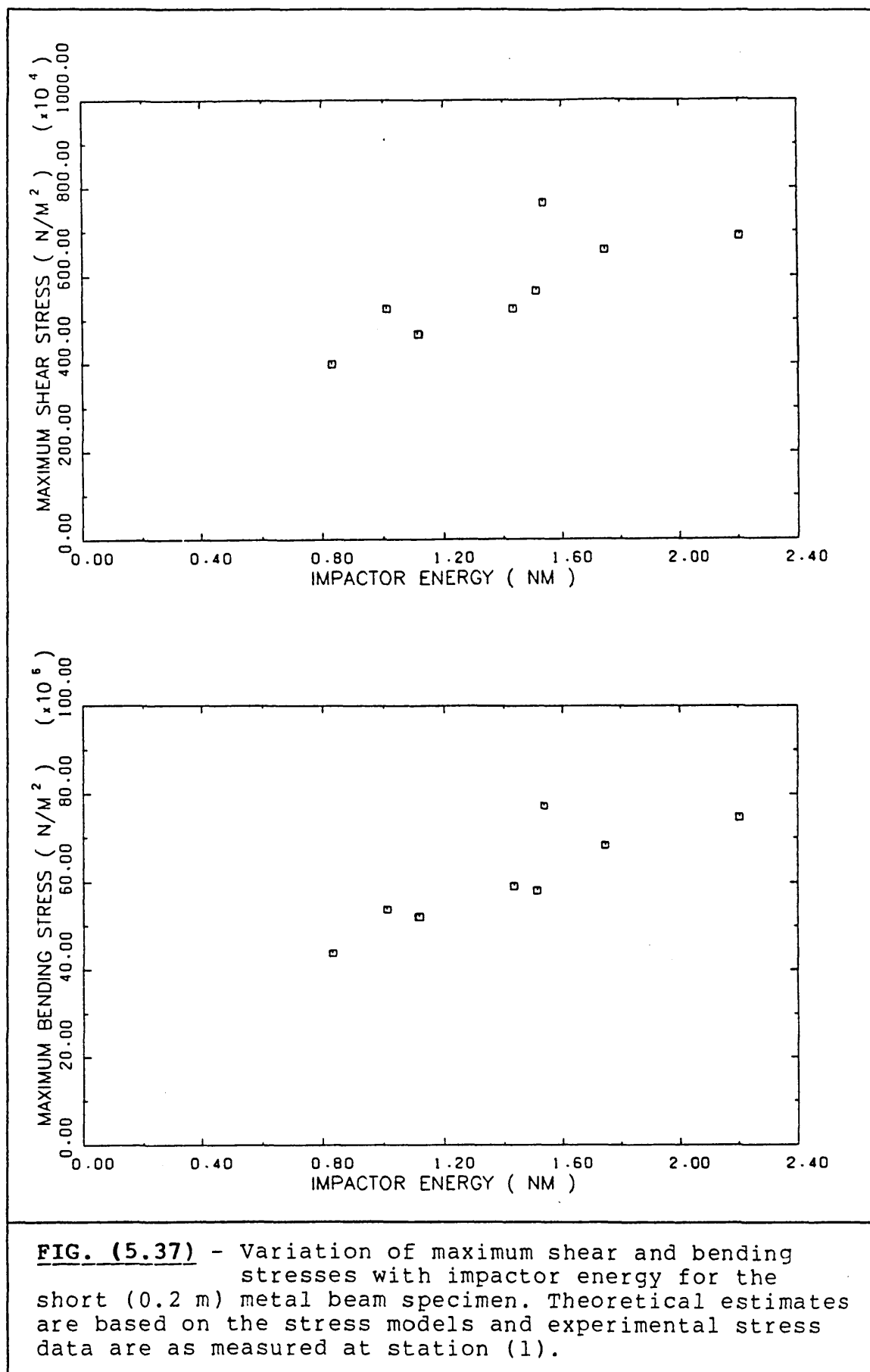
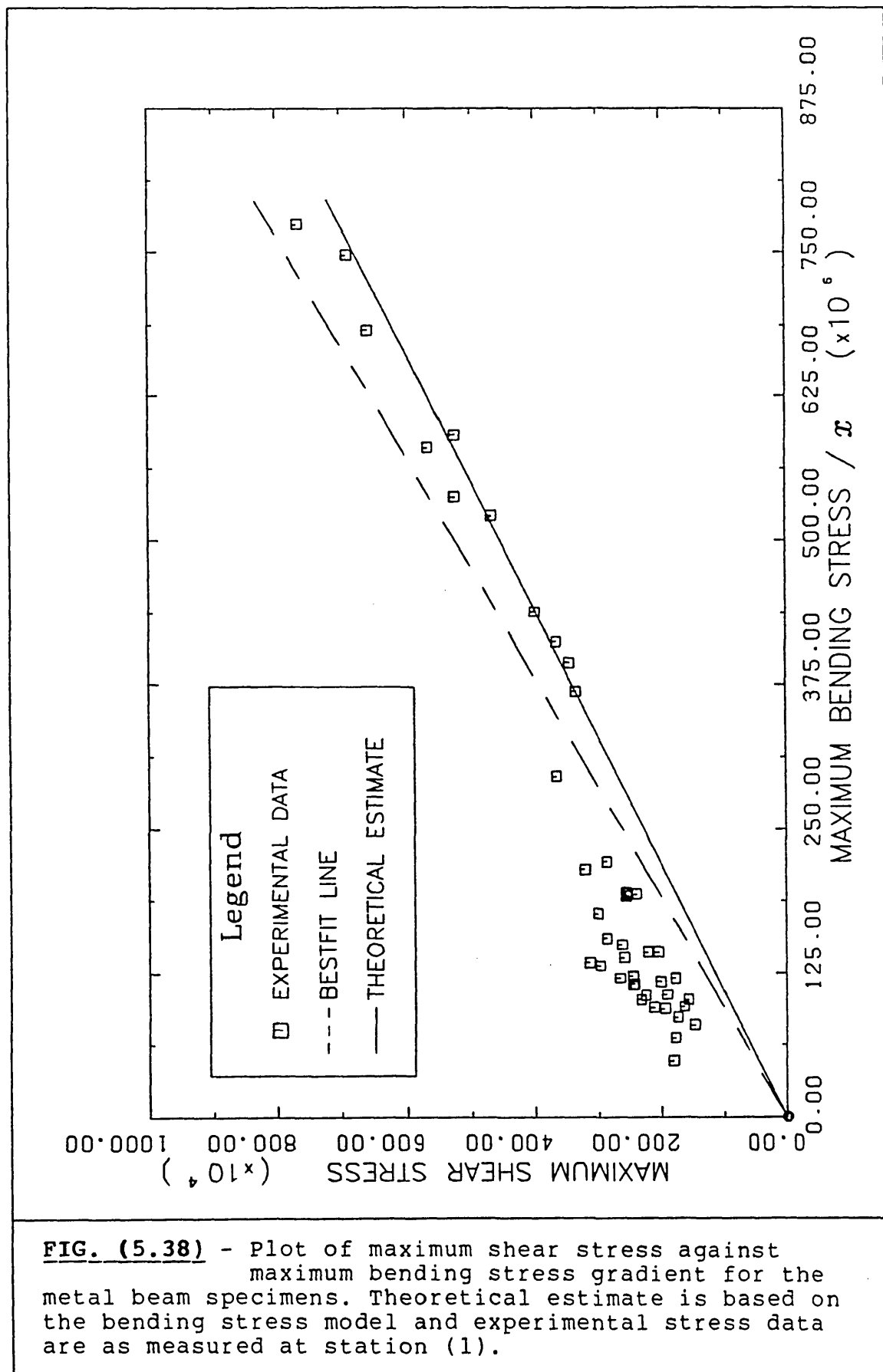


FIG. (5.35) - Variation of maximum shear and bending stresses with impactor energy for the long (1.4 m) metal beam specimen. Theoretical estimates are based on the stress models and experimental stress data are as measured at station (1).







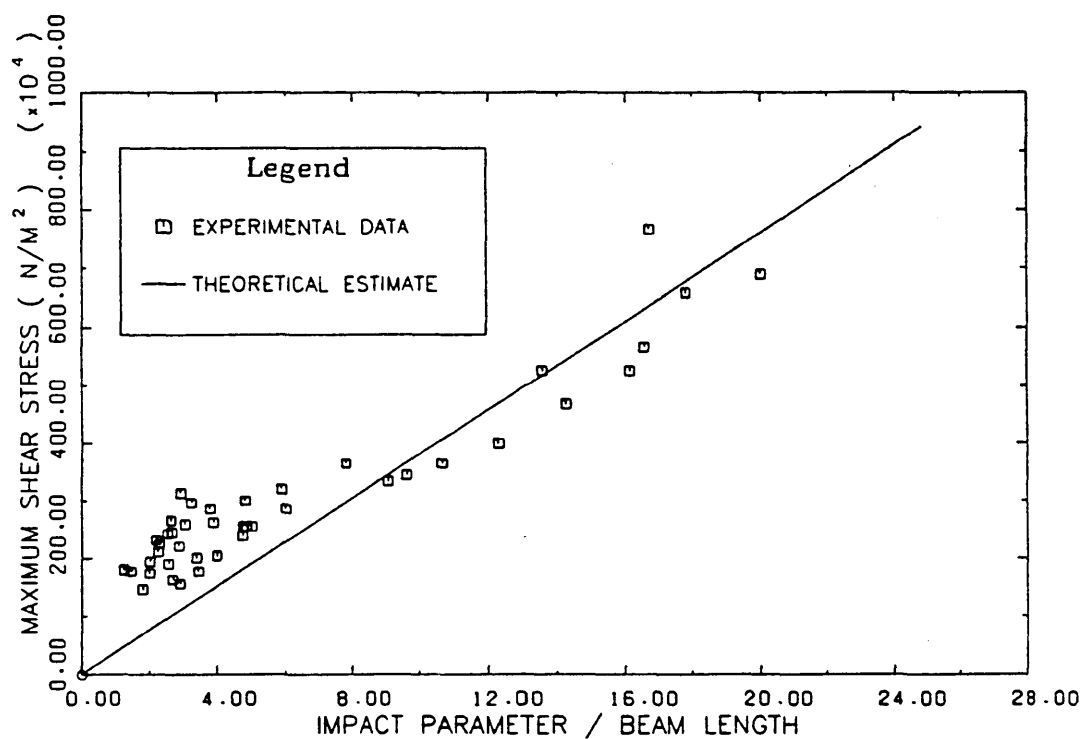
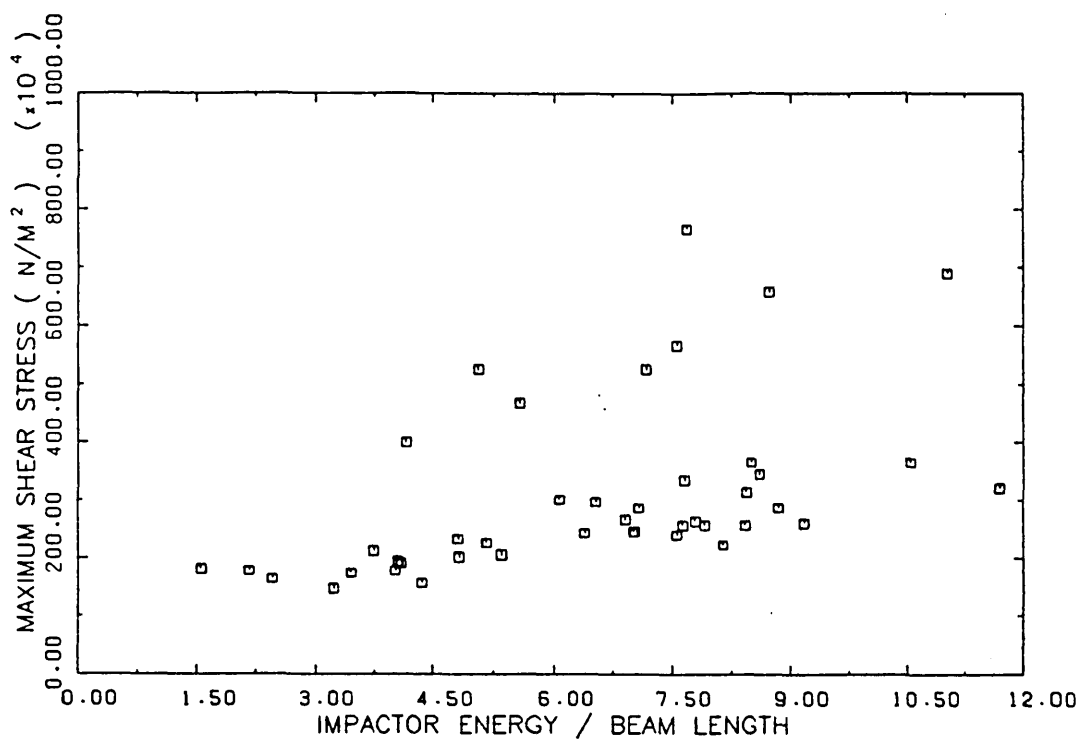


FIG. (5.39) - Variation of maximum shear stress with impactor energy/beam length and impact parameter/beam length for the metal beam specimens. Theoretical estimate is based on the shear stress model and stress data are as measured at station (1).

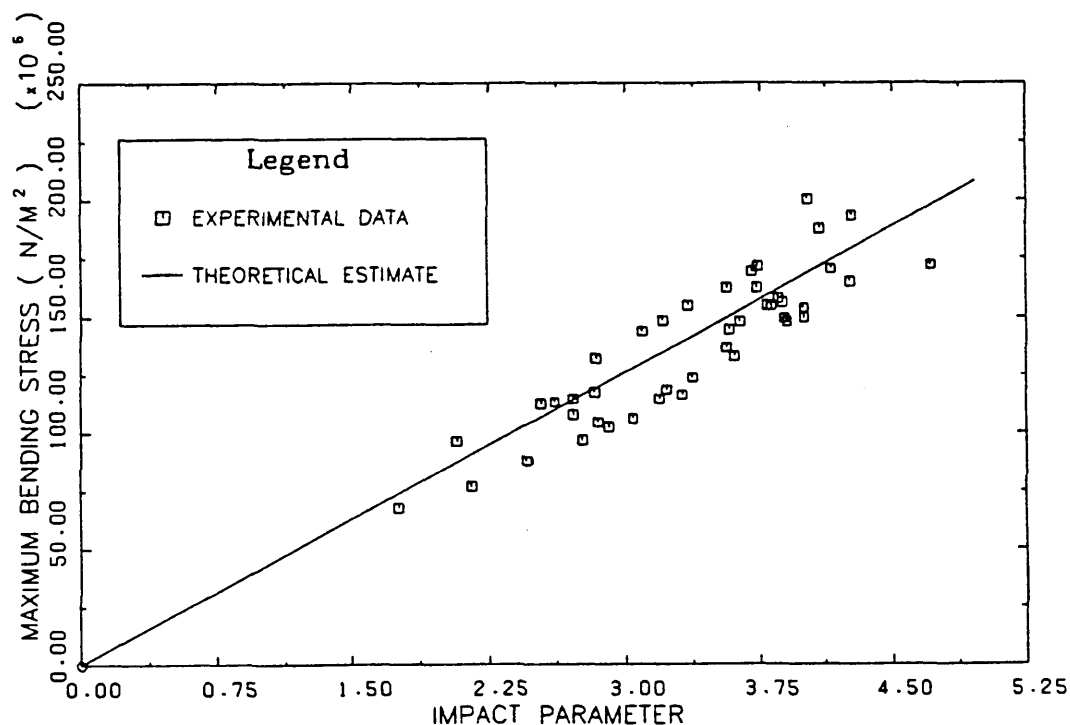
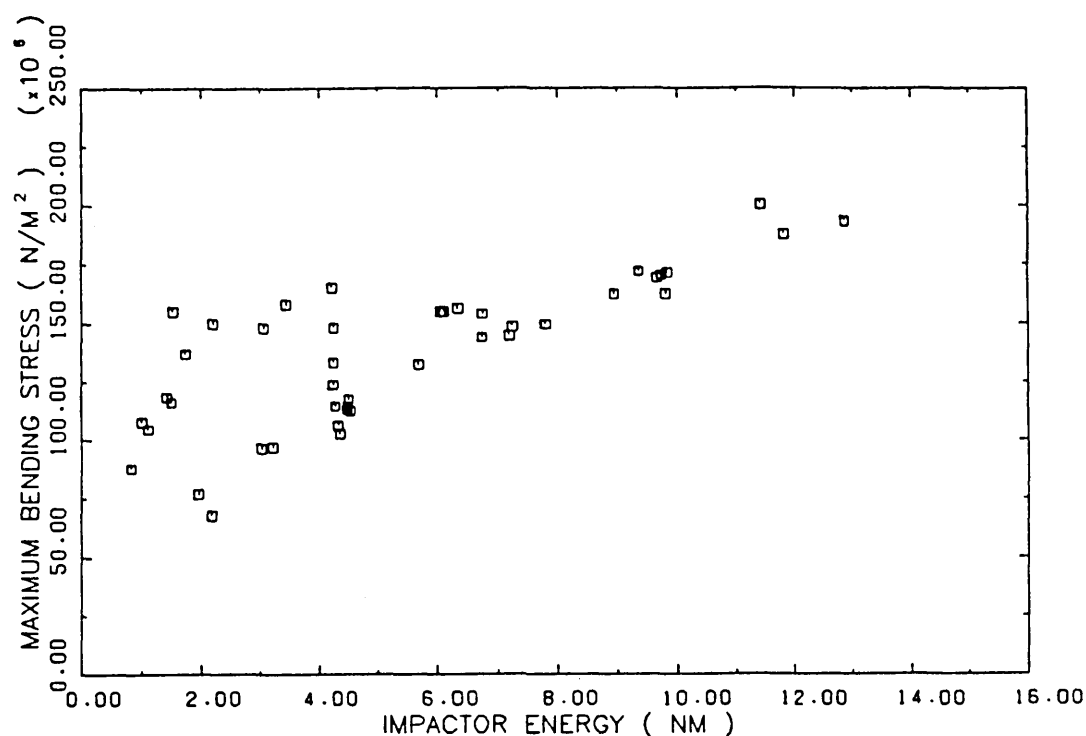
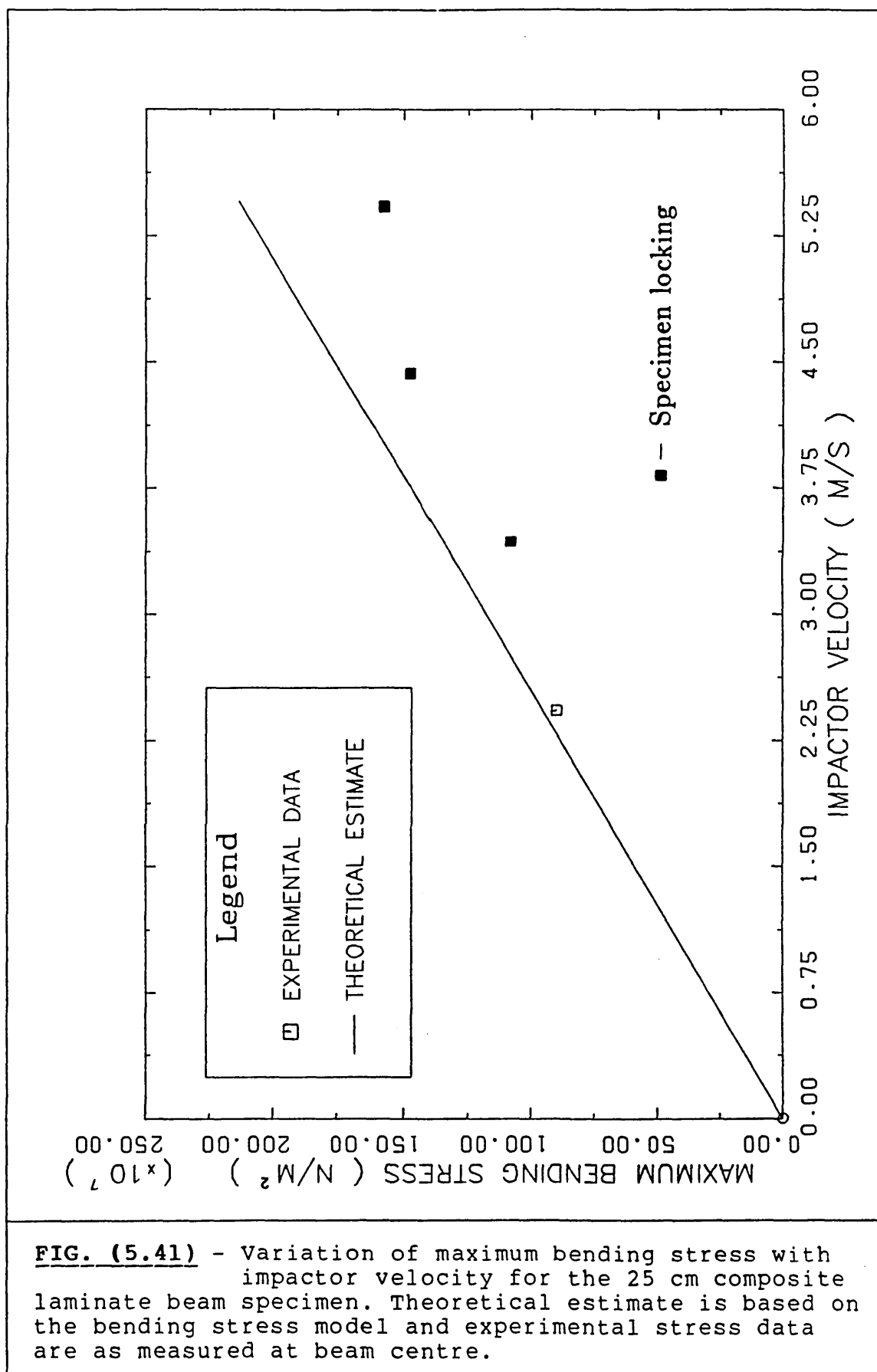


FIG. (5.40) - Variation of maximum bending stress with impactor energy and impact parameter for the metal beam specimens. Measured stress data at station (1) have been extrapolated to yield the stress at the beam centre.



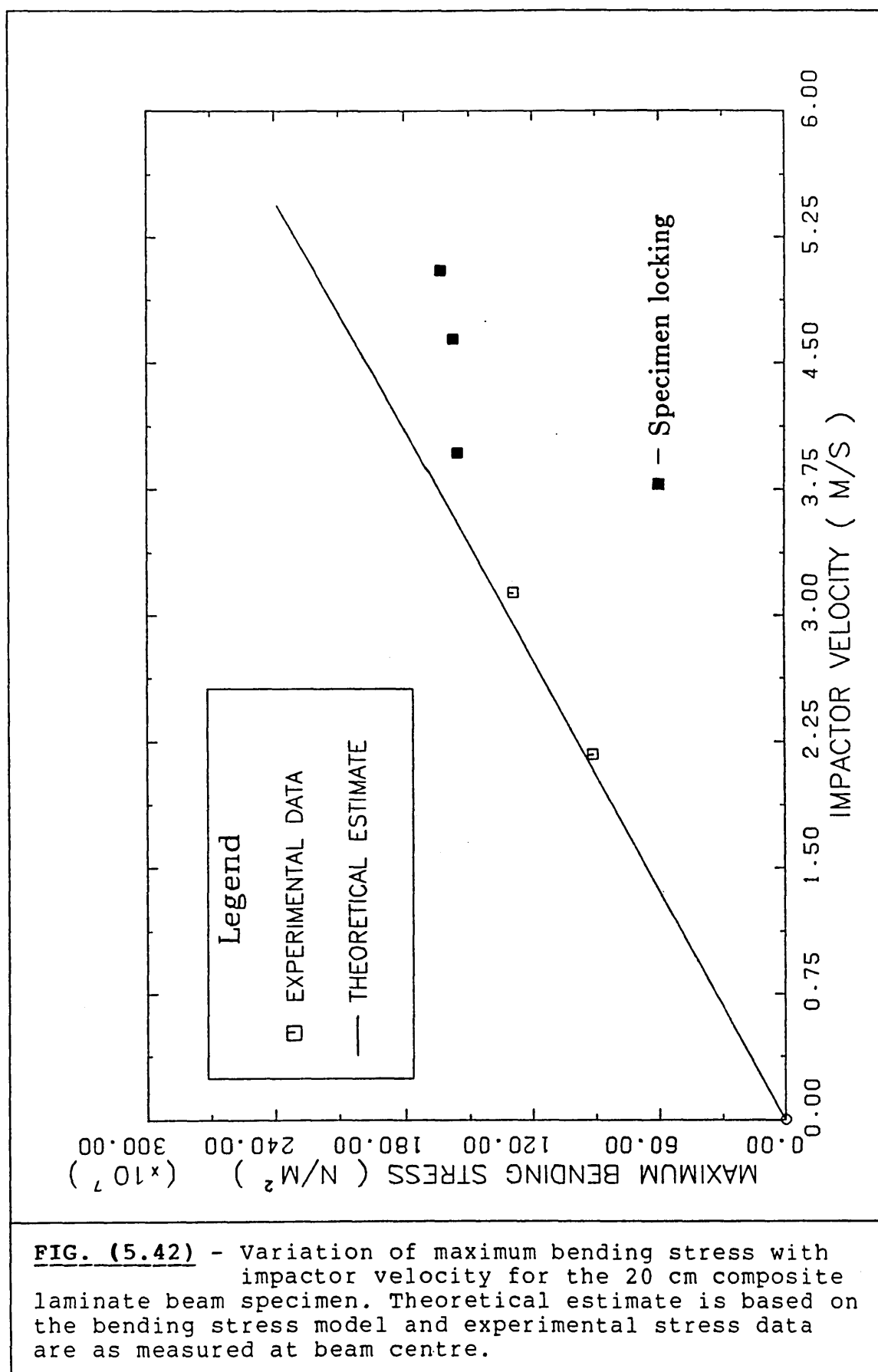


FIG. (5.42) - Variation of maximum bending stress with impactor velocity for the 20 cm composite laminate beam specimen. Theoretical estimate is based on the bending stress model and experimental stress data are as measured at beam centre.

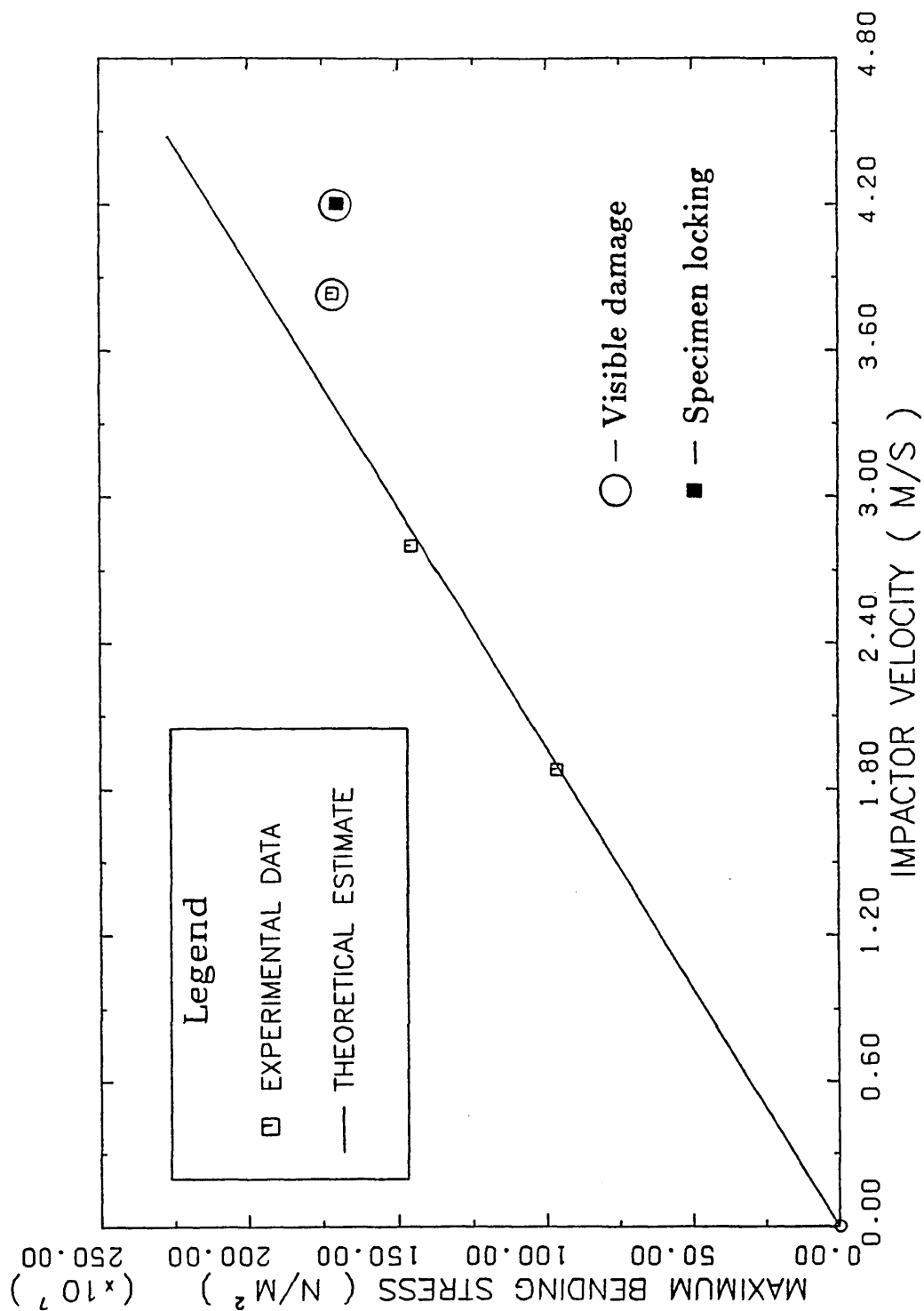
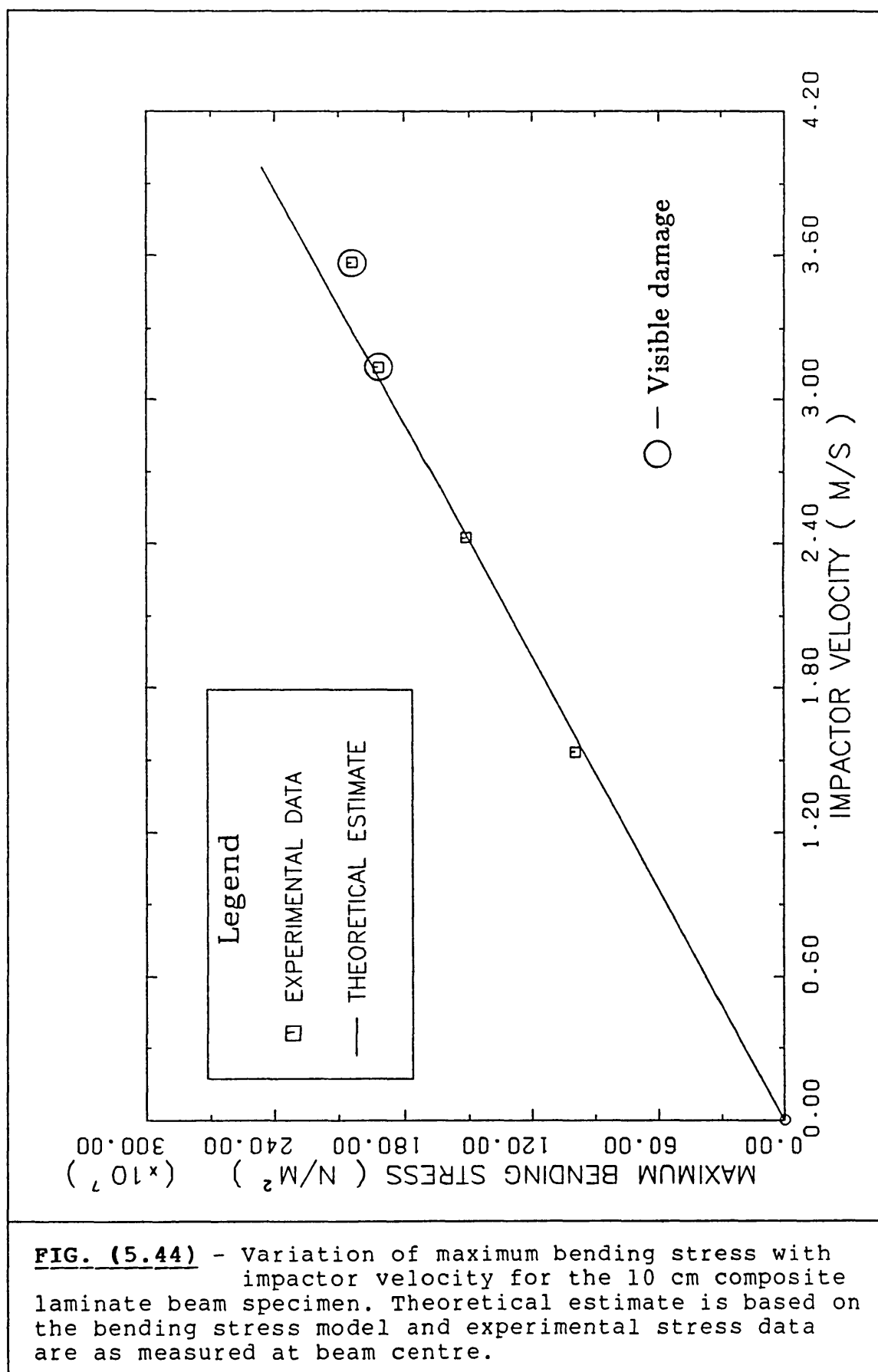


FIG. (5.43) - Variation of maximum bending stress with impactor velocity for the 15 cm composite laminate beam specimen. Theoretical estimate is based on the bending stress model and experimental stress data are as measured at beam centre.



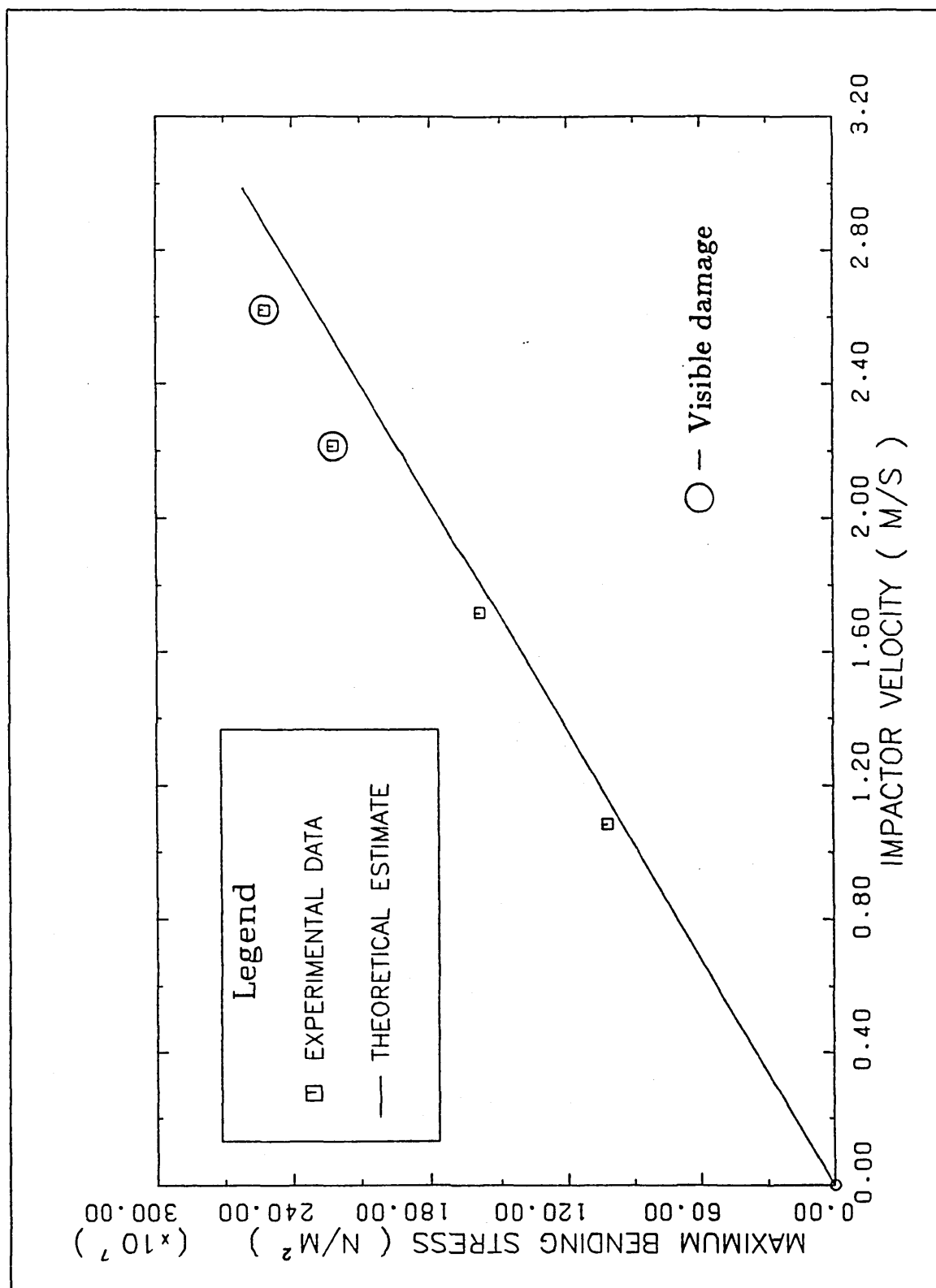
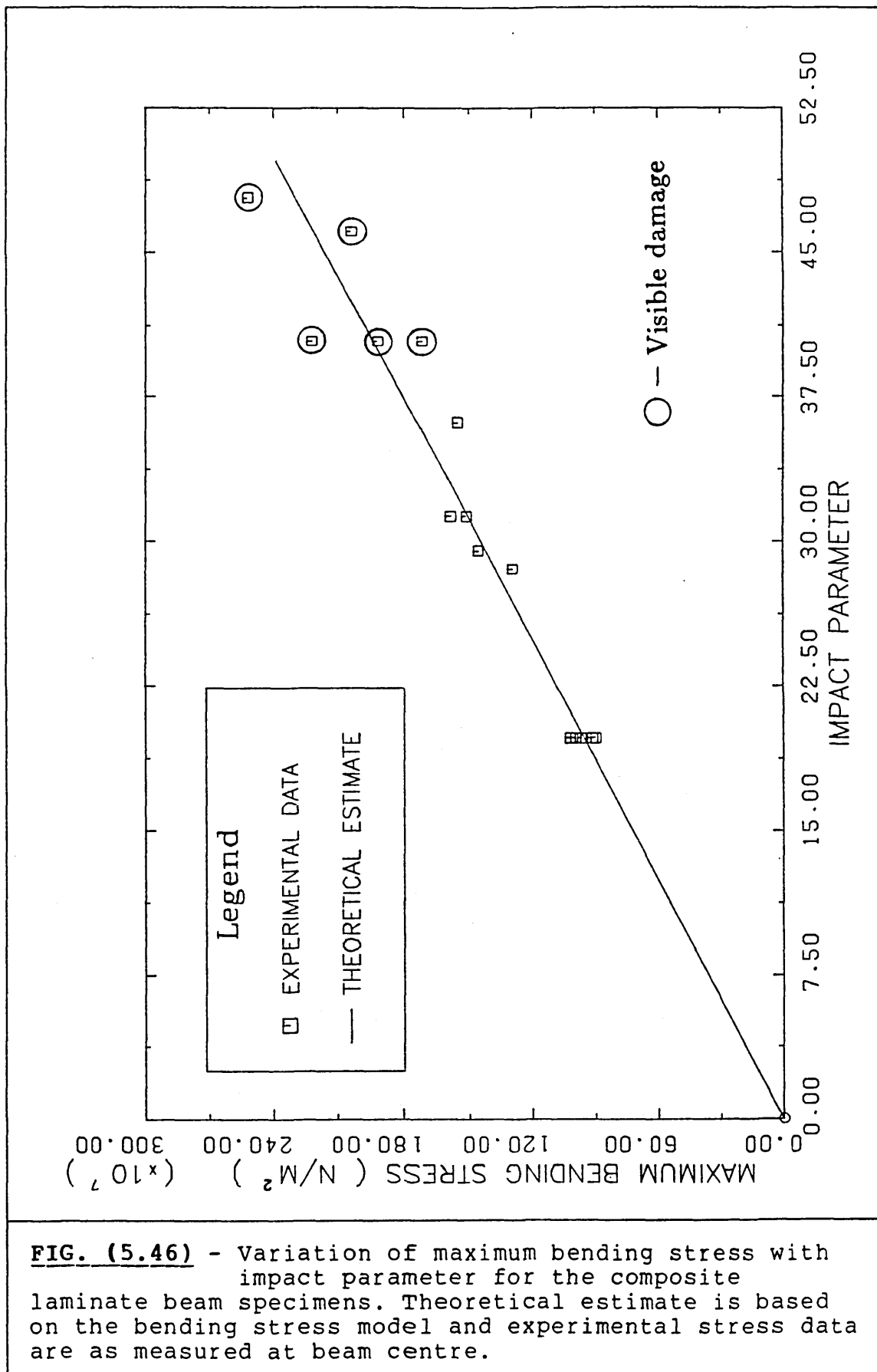


FIG. (5.45) - Variation of maximum bending stress with impactor velocity for the 5 cm composite laminate beam specimen. Theoretical estimate is based on the bending stress model and experimental stress data are as measured at beam centre.



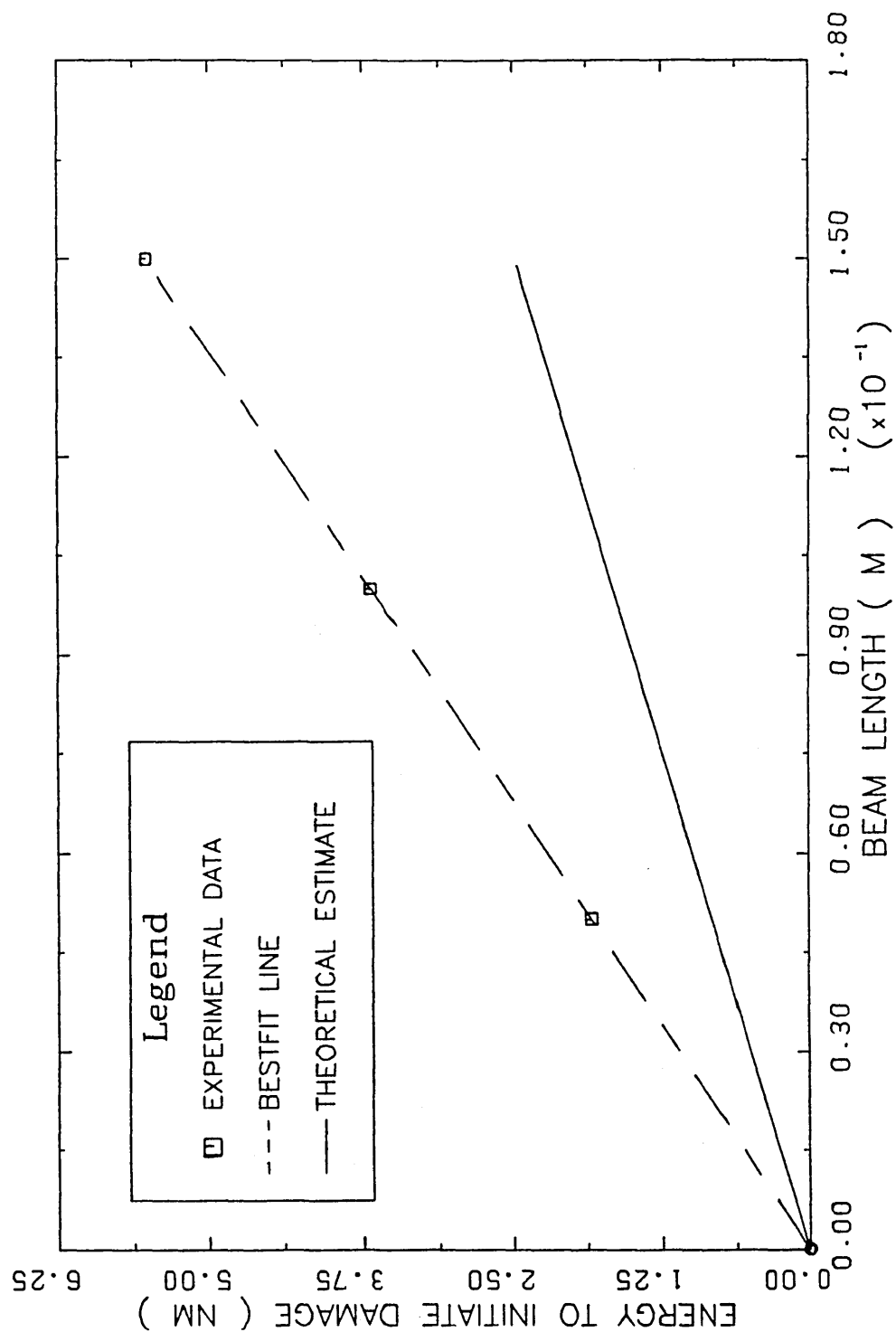


FIG. (5.47) - Energy to initiate damage in the composite laminate beam specimens as a function of beam length. Theoretical estimate is based on the bending stress model.