Automatica 187 (2026) 112913

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Connections between port-controlled Hamiltonian systems and differential games and their applications to decentralised control of multi-agent systems[☆]

Maria Luisa Scarpa [*], Thulasi Mylvaganam

Department of Aeronautics, Imperial College London, London SW7 2AZ, UK

ARTICLE INFO

Article history:
Received 1 November 2024
Received in revised form 15 August 2025
Accepted 19 January 2026
Available online 12 March 2026

Keywords:
Differential games
Port-controlled Hamiltonian systems
Linear matrix inequalities
Decentralised control
Multi-agent systems

ABSTRACT

Connections between port-controlled Hamiltonian (PCH) systems and differential games are explored in this paper. First, it is shown that a given differential game can be associated with a family of PCH systems. Second, the converse is demonstrated namely that a given PCH system can be associated with a family of differential games. Third, combining these insights, we present a strategy for assigning cost matrices of a differential game, and the free parameters that arise in the context of PCH systems, to achieve a specific closed-loop behaviour, characterised by a desired closed-loop energy function. The construction of the resulting control laws only requires solving linear matrix inequalities. Finally, the latter "integrated approach" is extended to provide a method to design decentralised control laws for a class of networked multi-agent systems.

© 2026 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

1. Introduction

Port-Hamiltonian systems provide a powerful framework for modelling and control of complex physical systems, particularly those characterised by *energy exchanges* taking place between *interconnected components*. Several control design techniques for port-Hamiltonian systems have been developed, such as *energy shaping and power shaping methods, control by interconnection*, and *interconnection and damping passivity-based control* (IDA-PBC) (see, *e.g.* Castaños et al., 2009; Ortega et al., 2008, 2001, 2002; Rodríguez et al., 2001). The aforementioned techniques are aimed at designing a system's control input, such that the closed-loop system is a port-Hamiltonian system, referred to as a *port-controlled* Hamiltonian (PCH) system, and they have in common that they provide meaningful intuition and interpretations in terms of *energy* and related concepts. However, these control techniques often involve free parameters, and selecting them is often neither straightforward nor intuitive (see, *e.g.* Fujimoto & Sugie, 2001; Ortega et al., 2002; Ortega & García-Canseco, 2004; Ortega & Spong, 2000). Differential games, on the other hand, arise in scenarios where multiple decision-makers (referred to as players) influence a dynamical system in a competitive setting. Each player aims to optimise their own objective while accounting for the actions of other players (see, *e.g.* Başar & Olsder, 1999). The so-called Feedback Nash Equilibrium (F-NE) solution is one of the most common solution concepts for dynamic games (see, *e.g.* Başar & Olsder, 1999; Engwerda, 2005). However, obtaining such solutions is generally a challenging task, often requiring one to use iterative algorithms or settle for approximate solutions, even in the context of linear-quadratic (LQ) problems, (see Cappello & Mylvaganam, 2020; Engwerda, 2005; Li & Gajic, 1995; Nortmann et al., 2024). Moreover, similarly to what is the case when selecting the cost functional in the context of optimal control, in practice, the process of *designing* these may be based on trial and error, making it a non-trivial and potentially costly and time-consuming process. Nevertheless, control techniques based on either PCH systems or differential games have gained interest for several applications, including networked systems (see, *e.g.* Arcak, 2007; Cappello & Mylvaganam, 2019, 2022; Gu, 2008; Li & Hu, 2022; Vos, 2015). Yet, the aforementioned challenges associated with these techniques – such as systematically assigning free parameters, constructing solutions of differential games, and designing individual cost functionals for each subsystem that also capture the desired overall behaviour of the networked system – remain unsolved.

With the above in mind, the primary contributions of this paper are twofold. First, we establish novel connections between PCH systems and differential games. These connections can be seen as a generalisation to multi-player settings of the results presented in Vu and Lefèvre (2018) and Ortega et al. (2002) that

☆ The work of M.L. Scarpa has been supported by the EPSRC DTP 2022-24 training grant EP/W524323/1, UK. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Mireille Broucke under the direction of Editor Maria Elena Valcher (System and Control Theory).

* Corresponding author.
E-mail addresses: maria.scarpa18@imperial.ac.uk (M.L. Scarpa), t.mylvaganam@imperial.ac.uk (T. Mylvaganam).

https://doi.org/10.1016/j.automatica.2026.112913
0005-1098/© 2026 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

are specific to single-player settings. While such connections can be drawn for general, nonlinear systems, we focus our attention on linear systems, so that assigning the free parameters and/or constructing the resulting control laws can be done based solely on the solution of linear matrix inequalities (LMIs) that are readily solved. Specifically, we start by showing that a given differential game can be associated with a family of PCH systems. Subsequently, we redirect our focus to the complementary problem, demonstrating that a given PCH system can be associated with a family of differential games. Finally, bringing together these insights, we provide a strategy to assign the cost matrices (in the context of differential games) and the free parameters (in the context of PCH systems) and thereby design a control law that achieves a certain closed-loop behaviour, represented by a desired closed-loop energy function. This "integrated approach" showcases the collective strengths of both methodologies and is a novel contribution that has not been pursued, even in the context of single-player optimal control. The approach also leads us to the second key contribution of the paper. Namely, we specialise the result for application to networked multi-agent systems (MAS), where the inter-agent communication is represented by a static graph. More precisely, leveraging the identified connections between PCH systems and differential games, we propose a technique to design decentralised control laws for each agent via the (offline, centralised) computation of a solution of an LMI. The remainder of the paper is organised as follows. Some preliminaries are recalled in Section 2. The first contribution of the paper is introduced in the following three sections: in Section 3 we demonstrate that a differential game can be related to a family of PCH systems; in Section 4 we demonstrate the converse; in Section 5 we demonstrate that the gained insights can be combined to overcome the challenges associated with the two control methods in isolation. The second key contribution of the paper, namely a method to design decentralised controllers for networked MAS by combining insights from PCH systems and differential games, is presented in Section 6. Finally, in Section 7 we demonstrate the efficacy of the latter control design strategy in the context of a formation control problem involving a MAS made up of a network of mass–spring–damper systems, before some concluding remarks are provided in Section 8.

Notation. The set of real numbers is denoted by $\mathbb{R}$. Given a matrix $A \in \mathbb{R}^{n\times n}$, $A > 0$ ($A \geq 0$) indicates that the matrix is positive definite (positive semi-definite). A block matrix M is denoted by $M = [M^{jk}]$, where M^{jk} is the j−th row, ith column block of M, whereas $M = [M^{j\cdot}]$ and $M = [M^{\cdot j}]$ denote the jth row block and the jth column block of M, respectively. A diagonal matrix with diagonal blocks $A_1, \dots, A_N$ is denoted by blkdiag$\{A_1, \dots, A_N\}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, then $\partial f/\partial x = [\partial f/\partial x_1, \dots, \partial f/\partial x_n]^\top$ denotes the column vector of partial derivatives of f with respect to $x = [x_1, x_2, \dots, x_n]^\top \in \mathbb{R}^n$.

2. Preliminaries and contributions

In what follows we consider systems influenced by $N \geq 0$ control inputs, characterised by the dynamics

$$\dot{x} = Ax + \sum_{i=1}^{N} B_i u_i \,, \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state of the system and $u_i \in \mathbb{R}^{m_i}$ denotes the ith control input, $A \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m_i}$, for all $i \in \mathcal{I}$, where $\mathcal{I} = \{1, \dots, N\}$. Let x_0 denote the initial condition of system (1), *i.e.* $x(0) = x_0$. We focus on two different methods for designing the control inputs u_i, $i \in \mathcal{I}$, based on differential game theory and on PCH systems (IDA-PBC in particular).

2.1. LQ differential games

In the framework of differential games the system (1) is seen as a system influenced by N *players*, via their individual control inputs, also referred to as *strategies*. Namely, u_i represents the strategy of player i. In the following we briefly recall the main components of a differential game. The interested reader is referred to Başar and Olsder (1999), Starr and Ho (1969) for details.

Consider the scenario in which each player $i \in \mathcal{I}$ is associated with a quadratic cost functional

$$J_i(x, u_i) = \frac{1}{2}\int_0^\infty x^\top Q_i x + u_i^\top R_i u_i dt \,, \tag{2}$$

where $R_i = R_i^\top > 0$, $Q_i = Q_i^\top$, which it seeks to minimise via its control input u_i, while taking into consideration the strategies of all other players. The dynamics (1), along with the cost functionals (2), for all $i \in \mathcal{I}$, form a N-player LQ, non-cooperative, infinite horizon, differential game. Considering linear strategies only (as is common in the context of LQ differential games, see, *e.g.* Başar & Olsder, 1999; Engwerda, 2005; Starr & Ho, 1969), a F-NE solution of the game is given by the set of strategies

$$u_i^\star(x) = -R_i^{-1}B_i^\top P_i x \,, \tag{3}$$

for all $i \in \mathcal{I}$, where $P_i = P_i^\top$ constitute a solution to the set of coupled Algebraic Riccati Equations (AREs)

$$\begin{aligned} 0 &= P_iA + A^\top P_i + Q_i + P_iB_iR_i^{-1}B_i^\top P_i \\ &-P_i\left(\sum_{j=1}^{N} B_jR_j^{-1}B_j^\top P_j\right) - \left(\sum_{j=1}^{N} P_jB_jR_j^{-1}B_j^\top\right)P_i \,, \end{aligned} \tag{4}$$

for all $i \in \mathcal{I}$, such that the stability condition

$$\sigma(A_{\mathrm{cl}}^\star) = \sigma(A - \sum_{i=1}^{N} B_iR_i^{-1}B_i^\top P_i) \subset \mathbb{C}^- \tag{5}$$

holds. Obtaining a solution to the coupled AREs is, as discussed in the introduction, challenging, see *e.g.* Cappello and Mylvaganam (2020), Possieri and Sassano (2015). Moreover, the selection of the weighting matrices Q_i and R_i in (2) may, in practice, require costly tuning via a trial-and-error process.

2.2. PCH systems and IDA-PBC

IDA-PBC, instead, is aimed at designing a control law such that the resulting closed-loop system is a PCH system. In the following, we recall the main ideas behind the approach. The interested reader is referred to Ortega et al. (2001), Ortega and García-Canseco (2004), van der Schaft and Jeltsema (2014) for further details. To streamline the presentation, note that (1) can be re-written as

$$\dot{x} = Ax + B_{\mathrm{IDA}}u_{\mathrm{IDA}} \,, \tag{6}$$

where $B_{\mathrm{IDA}} = [B_1, \dots, B_N] \in \mathbb{R}^{n\times m}$, with $m = \sum_{i=1}^N m_i$, and $u_{\mathrm{IDA}} = [u_1^\top, \dots, u_N^\top]^\top \in \mathbb{R}^m$. Loosely speaking, the aim of IDA-PBC is to identify a feedback control law $u_{\mathrm{IDA}}(x)$ so that system (6) in closed loop with $u_{\mathrm{IDA}}(x)$ assumes a desired, closed-loop form given by the PCH system

$$\dot{x} = (J_d(x) - R_d(x))\frac{\partial H_d}{\partial x} \,, \tag{7}$$

for some $J_d(x) = -J_d(x)^\top$ and $R_d(x) = R_d(x)^\top \geq 0$ representing the *desired* interconnection structure and dissipation, respectively, where the closed loop Hamiltonian function $H_d : \mathbb{R}^n \to \mathbb{R}$ represents the total stored energy and has a minimum at the origin (see, *e.g.* Ortega & García-Canseco, 2004; van der Schaft & Jeltsema, 2014). In the following, because (6) is linear, we restrict our analysis to linear control strategies and quadratic closed-loop Hamiltonian functions. Namely, we consider control strategies of the form

$$u_{\mathrm{IDA}}(x) = K_{\mathrm{IDA}}x\,, \tag{8}$$

and quadratic energy functions, *i.e.*

$$H_d(x) = \frac{1}{2}x^\top Q_d x\,, \tag{9}$$

with $Q_d = Q_d^\top > 0$. In this setting, the functions describing the interconnection and dissipation structure are not state-dependent and, thus, the system (7) reduces to

$$\dot{x} = (J_d - R_d)Q_d x\,. \tag{10}$$

We refer to J_d, R_d and Q_d as the interconnection, damping and closed-loop energy matrices, respectively. For linear PCH systems described by (10), the conditions $Q_d > 0$ and $R_d \geq 0$ imply passivity and, therefore, also stability, see van der Schaft and Jeltsema (2014, Chapter 7). The latter can be demonstrated by considering

$$V(x) = H_d(x) > 0 \tag{11}$$

as a candidate Lyapunov function and noting that $\dot{V} = -x^\top Q_d R_d Q_d x \leq 0$. Thus, by LaSalle's invariance principle, the trajectories of the closed-loop system converge to the largest forward invariant set $\mathcal{M}$ contained in

$$\Omega = \{x \in \mathbb{R}^{\bar{n}} \mid \dot{V}(x) = 0\}\,, \tag{12}$$

see *e.g.* Sastry (2013, Chapter 5). Moreover, if $\mathcal{M} = \{0\}$, the closed-loop system (10) is asymptotically stable. Thus, given a solution of the so-called *matching equation*

$$A + B_{\mathrm{IDA}}K_{\mathrm{IDA}} = (J_d - R_d)Q_d\,, \tag{13}$$

the origin of system (6) in closed loop with (8) is stable or asymptotically stable.

While the matching equation (13) can be readily solved, it involves several degrees of freedom *i.e.* the matrices J_d, R_d, Q_d and the solution K_{IDA}, which is generally not unique. *How* to assign these design parameters and/or the target system (10) *systematically* is still largely an open issue. In practice, J_d and R_d are often treated as free variables – subject only to the requirements of skew-symmetry and positive semidefiniteness, respectively – while Q_d may be either partially or fully specified, depending on the problem. (see, *e.g.* Fujimoto & Sugie, 2001; Ortega et al., 2001, 2002; Ortega & García-Canseco, 2004; Ortega & Spong, 2000 for further details).

2.3. Contributions

We have highlighted some inherent challenges associated with designing control laws via differential games or via PCH systems (and, IDA-PBC in particular), *e.g.* in terms of designing the cost functionals (2), determining a F-NE solution or in assigning the free parameters defining the target system (10). In the following sections we leverage insights garnered from both control methods to overcome these challenges.

As detailed in Section 1, the key contributions of the paper are twofold. First, we draw certain connections between differential games and PCH systems: in Lemma 1, we relate the solution of a game to a family of PCH systems, whereas in Proposition 1 we do the converse. Note that the former can be seen as a special case of the results presented in Ortega et al. (2002, Section 4), and while similar results exist for single-player settings (see also Vu & Lefèvre, 2018), they have not yet been established for settings involving multiple players. While interesting *per se*, these insights further lay the foundations for the development of a strategy to design a control law that achieves a certain desired closed-loop behaviour, based only on the solution of an LMI. We stress that the method does not require tuning of free parameters or solutions of any coupled AREs. Moreover, it allows for the inclusion of additional constraints, for instance to "shape" the distributions of individual players' efforts strategically (*e.g.* to account for settings in which players have heterogeneous control capabilities). This represents a novel contribution, which does not have a counterpart even in the single-player setting. Moreover, it is particularly well-suited for applications involving networked systems, which leads to the second key contribution of the paper: a method to design decentralised control laws for networked MAS.

Remark 1. We focus our attention on linear systems, solely because the resulting design processes involve LMIs, for which several efficient solvers exist. Nevertheless, connections between differential games and PCH systems can readily be established for general, nonlinear systems following ideas identical to those presented herein. However, the resulting design processes will involve nonlinear partial differential equations or inequalities in place of LMIs, which are significantly more computationally demanding to solve.

3. On the relation between a differential game and a family of PCH systems

We next demonstrate that under certain conditions, a differential game can be related to a family of PCH systems. Significantly, the solution of a differential game can be used to assign the interconnection and damping matrices, *i.e.* J_d and R_d, of (10) and (13) in a systematic fashion (for a family of closed-loop energy matrices Q_d).

Lemma 1. *Consider the differential game defined by (1), (2), for all $i \in \mathcal{I}$, and suppose it admits a F-NE solution (3), with P_i satisfying (4), for all $i \in \mathcal{I}$, and (5). Then (1) in closed loop with (3), for all $i \in \mathcal{I}$, satisfies the same dynamics as the family of PCH systems with closed-loop form given by (7), with*

$$J_d = \frac{1}{2}(A_{\mathrm{cl}}^\star X - XA_{\mathrm{cl}}^{\star\top})\,, \tag{14}$$

$$R_d = -\frac{1}{2}(A_{\mathrm{cl}}^\star X + XA_{\mathrm{cl}}^{\star\top})\,, \tag{15}$$

$$H_d = \frac{1}{2}x^\top X^{-1} x\,, \tag{16}$$

and with X any symmetric matrix satisfying the LMIs

$$\begin{cases} X > 0 \\ A_{\mathrm{cl}}^\star X + XA_{\mathrm{cl}}^{\star\top} \leq 0\,. \end{cases} \tag{17}$$

Proof. Let x_G denote the state of the closed-loop system (1)–(3) and let x_{PCH} denote the state of the PCH system (7) with J_d, R_d and H_d given by (14), (15), (16) and (17). Substituting (3) into (1) yields $\dot{x}_G = A_{\mathrm{cl}}^\star x_G$. Noting that by (5) a solution X of (17) exists, substituting the corresponding (14), (15) and (16) into (7) yields $\dot{x}_{\mathrm{PCH}} = A_{\mathrm{cl}}^\star x_{\mathrm{PCH}}$, which concludes the proof. □

Remark 2. As noted in Remark 1, results similar to that in Lemma 1 can readily be established for nonlinear systems. In fact, such results are a direct consequence of Ortega et al. (2002, Lemma 1). However, therein the closed-loop energy function H_d is assumed known *a priori* in order to construct J_d and R_d (as reported in Ortega et al. (2002, Section 4)), whereas in Lemma 1 it is obtained from the LMIs (17). If H_d is, instead not known, the result in Ortega et al. (2002, Lemma 1)) would involve solving a PDE, which is significantly more difficult than solving the LMIs (17) that arise in the LQ setting.

Lemma 1 is related, as detailed in Remark 2, to the observation that a system with an asymptotically stable equilibrium point can be represented as a PCH system (7) (Ortega et al., 2002, Section 4), and it provides a link between a solution of a certain differential game, via a solution of the coupled AREs (4) and the corresponding $A_{\rm cl}^{\star}$, and a *family* of PCH systems, parameterised in the solution X of the LMI (17). Namely, the statement essentially provides a systematic procedure to assign the interconnection and damping matrices according to (14) and (15), for a given closed-loop energy matrix X^{-1} via the solution of the differential game. Then, to design an *overall* control input via IDA-PBC the matching equation (13) can be solved for these specific desired interconnection, damping and closed-loop energy matrices.

Remark 3. Whereas the above discussion provides some insights on how a differential game can be utilised to assign desired physical properties of a PCH system (thereby facilitating, *e.g.* control via IDA-PBC), it should be noted that the result in Lemma 1 also allows us to indirectly, via the solution of the coupled AREs (4), $i \in \mathcal{I}$, relate the parameters defining the cost functionals (2), $i \in \mathcal{I}$, to physical properties of the closed-loop system. This sets the inspiration for the constructive control design framework developed in the later sections (Theorem 1 specifically).

4. On the relation between a PCH system and a family of differential games

We now demonstrate that under certain conditions, a given PCH system (10) (*e.g.* controlled via IDA-PBC) can be related to a family of differential games. To streamline the presentation, let $S_i = B_i R_i^{-1} B_i^\top$, $S_{-i} = [S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_N]$ and let $S_{-i_\perp} \in \mathbb{R}^{n\times n}$ be a corresponding left annihilator matrix, *i.e.* such that $S_{-i_\perp} S_{-i} = 0$, for all $i \in \mathcal{I}$.

Proposition 1. *Consider a PCH system (10), such that $\sigma((J_d - R_d)Q_d) \subset \mathbb{C}^-$. Suppose there exists $P_i^{\rm PH} = P_i^{\rm PH^\top}$, such that*

$$\left(\prod_{j\neq i}^{N} S_{-j_\perp}\right) S_i P_i^{\rm PH} = \left(\prod_{j\neq i}^{N} S_{-j_\perp}\right)(A - (J_d - R_d)Q_d)\,, \tag{18}$$

hold, for all $i \in \mathcal{I}$. Let $R_i = R_i^\top > 0$ be given,[1] *and consider the resulting differential game defined by (1), (2), with*

$$\begin{aligned} Q_i &= -P_i^{\rm PH} A - A^\top P_i^{\rm PH} - P_i^{\rm PH} S_i P_i^{\rm PH} \\ &+ P_i^{\rm PH}\left(\sum_{j=1}^{N} S_j P_j^{\rm PH}\right) + \left(\sum_{j=1}^{N} P_j^{\rm PH} S_j\right) P_i^{\rm PH}\,, \end{aligned} \tag{19}$$

for all $i \in \mathcal{I}$. Then, the feedback strategies

$$\hat{u}_i = -R_i^{-1} B_i^\top P_i^{\rm PH} x\,, \tag{20}$$

for all $i \in \mathcal{I}$, constitute a F-NE solution of the differential game, and the system (1) in closed loop with (20), for all $i \in \mathcal{I}$, satisfies the same dynamics as the PCH system (10).

[1] Note that this is without loss of generality, since it is the relative weightings between Q_i and R_i in (2), $i \in \mathcal{I}$, that essentially determine the resulting closed-loop behaviours.

Proof. To begin with note that $P_i = P_i^{\rm PH}$, $i \in \mathcal{I}$, constitute a solution of the coupled AREs (4), with Q_i given by (19), for all $i \in \mathcal{I}$. Moreover, $P_i^{\rm PH}$, $i \in \mathcal{I}$, are such that $(J_d - R_d)Q_d = (A - \sum_{i=1}^{N} S_i P_i^{\rm PH})$. Thus, since, by assumption, the origin of the PCH system (10) is asymptotically stable, (5) holds and it follows that (20), for all $i \in \mathcal{I}$, constitute F-NE strategies for the game defined by (1), (2), with Q_i given by (19), for all $i \in \mathcal{I}$. The final part of the claim can be shown following steps identical to those in the proof of Lemma 1. □

Proposition 1 provides a link between a certain PCH system (*e.g.* controlled using IDA-PBC methods) and a solution, in terms of F-NE, of a differential game. It should, however, be noted that $S_{-j_\perp}$ is generally not unique and that (18) may admit several solutions $P_i^{\rm PH}$. Namely, Proposition 1 entails that a given PCH system may be related to a *family* of differential games, characterised by solutions of (18), for all $i \in \mathcal{I}$. The statement provides a systematic procedure to design the cost functionals (2), $i \in \mathcal{I}$, of a differential game via a given PCH system, with predefined interconnection, damping and closed-loop energy matrices.

Remark 4. The relations (18) and (19) are linear in the variables $P_i^{\rm PH}$ and Q_i, $i \in \mathcal{I}$, respectively. Thus, the conditions are readily satisfied.

Remark 5. The result in Proposition 1 relates the closed-loop dynamics (10) to a differential game and, thus, can be interpreted as an instance of inverse differential games (*i.e.* the multi-player equivalent of inverse optimal control).

5. IDA-PBC and differential games: a combined approach

The results presented in Sections 3 and 4 establish connections between PCH systems and differential games. These connections provide some insights and intuition that can assist in assigning interconnection, damping and closed-loop energy matrices (Lemma 1), or in assigning the cost functionals of a differential game (Proposition 1). However, it should be noted that a solution of the game, and thereby of the AREs (4) is the starting point of Lemma 1, whereas suitable J_d, R_d and Q_d (*i.e.* satisfying the matching equations (13)) are required in Proposition 1. These requirements represent precisely the main bottlenecks associated with the two control methods, as discussed in Section 2. Motivated by this, in this section, we *combine* the insights gained from Lemma 1 and Proposition 1 to overcome these bottlenecks, as detailed in the following statement.

Theorem 1. *Consider the system (1) and its alternative representation (6). Let $R_i \in \mathbb{R}^{m_i\times m_i}$ and $Q_d \in \mathbb{R}^{n\times n}$, such that $R_i = R_i^\top > 0$, for all $i \in \mathcal{I}$, and $Q_d = Q_d^\top > 0$ be given. Let X_i be a solution of the LMIs*

$$\begin{cases} \displaystyle\sum_{i=1}^{N}(S_i X_i + X_i^\top S_i) \geq A Q_d^{-1} + Q_d^{-1} A^\top\,, & \text{(21a)} \\ X_i Q_d = Q_d X_i^\top\,, & \text{(21b)} \end{cases}$$

for all $i \in \mathcal{I}$, and let

$$\hat{P}_i = X_i Q_d\,, \tag{22}$$

be such that $\hat{A}_{\rm cl} = A - \sum_{i=1}^{N} B_i R_i^{-1} B_i^\top \hat{P}_i$ satisfies

$$\sigma(\hat{A}_{\rm cl}) \subset \mathbb{C}^-\,. \tag{23}$$

Then, the feedback control laws

$$\hat{u}_i = -R_i^{-1} B_i^\top \hat{P}_i x\,, \tag{24}$$

for all $i \in \mathcal{I}$, are such that the following statements hold.

(i) The control law $u_{\mathrm{IDA}} = [\hat{u}_1^\top, \ldots, \hat{u}_N^\top]^\top$ *is such that system* (6) *in closed loop with* u_{IDA} *is a passive PCH system with closed-loop dynamics given by* (10)*, where*

$$J_d = \frac{1}{2}(\mathcal{S} - \mathcal{S}^\top), \quad R_d = -\frac{1}{2}(\mathcal{S} + \mathcal{S}^\top), \tag{25}$$

with $\mathcal{S} = A{Q_d}^{-1} - \sum_{i=1}^N S_i X_i$.

(ii) The control laws (24)*, for all* $i \in \mathcal{I}$*, constitute a F-NE solution of the differential game defined by* (1)–(2)*, with* Q_i *given by* (19)*, with* $P_i^{\mathrm{PH}} = \hat{P}_i$.

Proof. Adopting a notation similar to that used in the proof of Lemma 1, let x_{PCH} denote the state of the PCH system (6), and let x_G denote the state of the closed loop system (1)–(24), $i \in \mathcal{I}$. We demonstrate first that statement (i) holds. Namely, substituting $u_{\mathrm{IDA}} = [\hat{u}_1^\top, \ldots, \hat{u}_N^\top]^\top$ into (6) yields

$$\dot{x}_{\mathrm{PH}} = \hat{A}_{\mathrm{cl}} x_{\mathrm{PH}}\,. \tag{26}$$

Note that $\hat{A}_{\mathrm{cl}} = \mathcal{S} Q_d = (J_d - R_d) Q_d$, where the first equality follows from (22) and the second inequality follows from (25). Noting that (21a) implies that $R_d \geq 0$, it follows that (26) is a *passive* PCH system of the form (10), with J_d, R_d and Q_d as specified in the statement. Consider now statement (ii). Substituting (24) into (1) yields $\dot{x}_G = \hat{A}_{\mathrm{cl}} x_G$. Note that (21b) implies that $\hat{P}_i$, $i \in \mathcal{I}$, is symmetric. Consider Q_i given by (19), with $P_i^{\mathrm{PH}} = \hat{P}_i$, for all $i \in \mathcal{I}$, and note that $P_i = \hat{P}_i$, $i \in \mathcal{I}$, constitutes a solution of the corresponding coupled AREs (4), for all $i \in \mathcal{I}$. Noting that x_G and x_{PH} evolve according to the same dynamics, and by (23), the origin of (26) is asymptotically stable, it follows that (24), $i \in \mathcal{I}$, constitute a F-NE solution of the considered game, which concludes the proof. □

Remark 6. Whereas positive semi-definiteness of R_d and positive definiteness of Q_d implies that the closed-loop system (10) is stable, the condition (23) ensures *asymptotic* stability, which is required of any Nash equilibrium solution (see (5)). Given a solution of the LMIs the condition can be readily verified, *e.g.* as done in Section 6 in the context of a formation control problem. Alternatively, the condition (23) can be omitted by imposing instead a more conservative, "sign-definite version" of the condition (21a), thereby ensuring that $R_d > 0$. Since $Q_d > 0$ is assumed, this implies $Q_d R_d Q_d > 0$, which in turn implies asymptotic stability.

Theorem 1 enables the design of control laws for the system (1) through two distinct lenses. Firstly, via PCH considerations, it enables a systematic approach to assign damping and interconnection matrices given a desired closed-loop energy matrix, and to design a feedback control law of the form (8) without requiring a solution of the matching equation (13). Secondly, via differential game considerations, it enables the systematic design of cost functionals (2), thereby avoiding the any need for manual tuning, and enables the construction of F-NE strategies, without requiring the solution of the coupled AREs (4), for all $i \in \mathcal{I}$. Moreover, the dual representation allows us to view the closed-loop dynamics not only in terms of *total energy* – thanks to the PCH interpretation – but also in terms of individual efforts and interactions between multiple players – thanks to the differential game interpretation. This is particularly appealing for applications that involve systems made up of multiple, interacting subsystems, such as networked MAS which are considered in the following section.

Remark 7. Given a solution of the LMIs (21), $V_i(x) = \frac{1}{2} x^\top \hat{P}_i x$ represents the value function for player i. Thus, by definition of value function, the total cost incurred by player i, from any given initial condition x_0, is

$$J_i(x_0, \hat{u}_i) = \frac{1}{2} x_0^\top \hat{P}_i x_0 = \frac{1}{2} x_0^\top X_i Q_d x_0\,. \tag{27}$$

However, for fixed Q_d, which represents the closed-loop energy via (9), the LMIs (21), $i \in \mathcal{I}$, may admit more than one set of solutions X_i, $i \in \mathcal{I}$. The different solutions of these LMIs correspond to different "distributions of costs" amongst the players. That is, via the different solutions of (21), $i \in \mathcal{I}$, Theorem 1 yields a *family* of differential games for which (24), $i \in \mathcal{I}$, constitutes F-NE strategies. The different games correspond to different distributions of *individual* costs for each player, while maintaining the same closed-loop energy function (9).

Remark 8. In light of the observations made in Remark 7, additional convex constraints can be added to the LMIs (21), $i \in \mathcal{I}$, to achieve a certain distribution of costs amongst the players, while preserving the desired energy of the closed-loop system. This is, for instance, advantageous in settings where players have heterogeneous control capabilities. For example, recalling (27), one can easily enforce that player i incurs at least a β_i-fraction of the cost incurred by player j, via the additional LMI constraint

$$x_0^\top (X_i Q_d - \beta_i X_j Q_d) x_0 \geq 0\,, \tag{28}$$

with the parameter $\beta_i > 0$ determining the relative cost allocation between the two players. Since R_i is fixed, this mechanism can be used to capture settings where certain agents are known to have more or less control effort available, effectively distributing the control burden according to each player's capabilities. *E.g.* in the simulations presented in Section 7.2 this mechanism has been used to account for "fast" and "slow" agents.

6. Decentralised control of networked multi-agent systems

We now extend the (centralised) results of the previous section to settings involving networked MAS, *i.e.* MAS that are subject to communication constraints. More precisely, following the observations made in Remarks 7 and 8, additional constraints can also be added to the LMIs (21), $i \in \mathcal{I}$, so that the control law of each agent i, *i.e.* (24), only depends on the information (locally) available to it. This results in a framework to design *decentralised*[2] control laws.

Namely, we consider a system of N agents, each associated with a cost functional of the type (2), $i \in \mathcal{I}$. The communication between the agents is described by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the node set $\mathcal{V} = \{1, \ldots, N\}$ represents the agents, and the edge set $\mathcal{E} = \{1, \ldots, E\}$ represents the communication flow between the agents. That is, if agent i has access to the state of agent j, and the dynamics of agent i may be influenced directly by the control input of agent j, then $(i, j) \in \mathcal{E}$. The incidence matrix $\mathcal{B} = [b_{ik}] \in \mathbb{R}^{N \times E}$ is commonly used to represent the connections between nodes and edges. More precisely, we use the convention that if $(i, j) \in \mathcal{E}$, the edge connecting nodes i and j, which represents the information flow, has a head at node i and a tail at node j (see *e.g.* Mesbahi, 2010). The matrix elements b_{ik} are then given by

$$b_{ik} = \begin{cases} +1 & \text{if node } i \text{ is the head of edge } k\,, \\ -1 & \text{if node } j \text{ is the tail of edge } k\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{29}$$

[2] More precisely, we propose a method to design control laws that, based on a *centralised* (offline) computation, allows the construction of control laws that are decentralised in the sense that they only depend on locally available information.

for $i = 1, \dots, N$ and $k = 1, \dots, E$. In the following, let $\mathcal{N}_i = \{j : (i,j) \in \mathcal{E}\} \cup \{i\}$. Each agent is described by

$$\dot{x}_i = A_i + \sum_{j \in \mathcal{N}_i} B_{ji} u_j \,, \tag{30}$$

where $x_i \in \mathbb{R}^{n_i}$ is the state of agent i and $u_j \in \mathbb{R}^{m_j}$ is the control law of agent j, for $j \in \mathcal{N}_i$, for all $i \in \mathcal{I}$. Similarly to what has been done in Cappello and Mylvaganam (2022), let the *local state* of each agent i be defined as

$$\hat{x}_i = N_i x = [x_{j1}^\top, \dots, x_{j|\mathcal{N}_i|}^\top]^\top \,, \tag{31}$$

where $\hat{x}_i \in \mathbb{R}^{\hat{n}_i}$, $N_i \in \mathbb{R}^{\hat{n}_i \times n}$, $\hat{n}_i = \sum_{j \in \mathcal{N}_i} n_j$ and $j_1, \dots, j_{|\mathcal{N}_i|} \in \mathcal{E}$, where $j_1 = i$. Moreover, given a matrix $X_i \in \mathbb{R}^{n \times n}$, consider the matrix partition $X_i = [X_i^{jk}]$, where $X_i^{jk} \in \mathbb{R}^{n_j \times n_k}$, for $j = 1, \dots, N$ and $k = 1, \dots, N$. In the remainder of this section we consider decentralised control laws, defined as follows.

Definition 1. Consider the MAS (30), for all $i \in \mathcal{I}$, and the communication graph $\mathcal{G}$. A set of feedback control strategies $\{u_1, \dots, u_N\}$ is said to be decentralised if each strategy u_i relies solely on the local information available to agent i, *i.e.* $u_i(x) := \mu_i(\hat{x}_i)$, where $\mu_i : \mathbb{R}^{\hat{n}_i} \to \mathbb{R}_i^n$, for all $i \in \mathcal{I}$.

The result presented in Theorem 1 can be extended to the setting of decentralised control, as detailed in the following statement.

Theorem 2. *Consider the MAS described by the dynamics (30), for all $i \in \mathcal{I}$. Let $R_i \in \mathbb{R}^{m_i \times m_i}$ and let $Q_d = \mathrm{blkdiag}(Q_d^{11}, \dots, Q_d^{NN}) \in \mathbb{R}^{n \times n}$, such that $R_i = R_i^\top > 0$ and $Q_d = Q_d^\top > 0$, be given. Let X_i be a solution of the LMIs (21), with the additional constraints*

$$\begin{cases} X_i Q_d \geq 0 \,, & (32a) \\ X_i^{kk} = 0 \,, \quad \textit{for all } k \notin \mathcal{N}_i \,, & (32b) \end{cases}$$

for all $i \in \mathcal{I}$, and let $\hat{P}_i$ given by (22) be such that $\hat{A}_{\mathrm{cl}} = A - \sum_{i=1}^N R_i^{-1} B_i^\top \hat{P}_i$ satisfies (23). Then, the feedback control laws (24), are decentralised, namely

$$\hat{u}_i = -R_i^{-1} B_i^\top \hat{P}_i \hat{x}_i \,, \tag{33}$$

for all $i \in \mathcal{I}$, and they are such that (i) and (ii) of Theorem 1 hold.

Proof. The proof lies in demonstrating that the control laws (24), are equivalent to the control laws (33), for all $i \in \mathcal{I}$, when X_i is the solution of the LMIs (21), (32). First, note that the left-hand side of (32a) can be re-written as

$$X_i Q_d = \begin{bmatrix} X_i^{11} Q_d^{11} & \cdots & X_i^{1N} Q_d^{NN} \\ \vdots & \ddots & \vdots \\ X_i^{N1} Q_d^{11} & \cdots & X_i^{NN} Q_d^{NN} \end{bmatrix} . \tag{34}$$

Furthermore, by (32b), $X_i^{kk} = 0$ for $k \notin \mathcal{N}_i$. Thus, noting that X_i^{kk}, $k \notin \mathcal{N}_i$, is placed on the diagonal of (34), for the matrix (34) to be positive semi-definite (and thus satisfy (32a)), the corresponding row and column blocks, $X_i^{k\cdot}$ and $X_i^{\cdot k}$, for all $k \notin \mathcal{N}_i$, must be zero (see *e.g.* Dattorro, 2010; Horn & Johnson, 2012). Recalling that $\hat{P}_i$ is given by (22), it follows that $\hat{P}_i x_i = \hat{P}_i \hat{x}_i$, and that (24) is equivalent to the control law (33), for all $i \in \mathcal{I}$. The remainder of the proof follows directly from Theorem 1. □

7. Formation control of networked mechanical systems

In this section we illustrate the efficacy of the results presented in Section 6 on a problem involving the formation control of a networked MAS. In the literature such problems have been studied using a variety of approaches, including differential games (see *e.g.* Cappello & Mylvaganam, 2019, 2022; Gu, 2008; Li & Hu, 2022) and PCH systems theory (see *e.g.* Arcak, 2007; Vos, 2015). This section consists of two parts, the first dedicated to the control design procedure and the second dedicated to numerical simulations.

7.1. Background and control design

We consider a class of MAS that arises in various domains including, for instance, robotics and spacecraft control (see *e.g.* Bullo et al., 2009; Cortes et al., 2004; Mei et al., 2006). Namely, we consider a network of fully actuated mechanical systems, with a communication graph satisfying the following assumption.

Assumption 1. The communication graph $\mathcal{G}$ is a tree graph (see *e.g.* Mesbahi, 2010, Chapter 2), *i.e.* it is connected, acyclic and undirected, namely $(i,j) \in \mathcal{E}$ if and only if $(j,i) \in \mathcal{E}$.

The considered tree graph can be associated with an incidence matrix $\mathcal{B}$ with b_{ik} defined as in (29), by arbitrarily designating one end as the *head*, which receives a positive sign, and the other end as the *tail*, which receives a negative sign. This approach is commonly used in graph theory (see *e.g.* Jafarian et al., 2015; Mesbahi, 2010; Vos et al., 2014). A consequence of Assumption 1 (see, *e.g.* Dharwadker & Pirzada, 2011, Theorem 10.5) is that the incidence matrix satisfies $\ker(\mathcal{B}) = 0$. Then, each agent is described by the dynamics

$$\begin{bmatrix} \dot{q}_i \\ \dot{p}_i \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & M_i^{-1} \\ 0_{n \times n} & -D_i^a M_i^{-1} \end{bmatrix} \begin{bmatrix} q_i \\ p_i \end{bmatrix} + \begin{bmatrix} 0_{n \times n} \\ I_n \end{bmatrix} u_i \,, \tag{35}$$

where $M_i = m_i I_n \in \mathbb{R}^{n \times n}$ corresponds to the inertia matrix and $D_i^a \in \mathbb{R}^{n \times n}$, $D_i^a = D_i^{a\top} \geq 0$ represents the linear viscous Coulomb force acting on the system, and where $u_i \in \mathbb{R}^n$ is the control input associated with each agent i, $i \in \mathcal{I}$. The overall system of N agents is then described by the dynamics

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0_{Nn \times Nn} & M^{-1} \\ 0_{Nn \times Nn} & -D^a M^{-1} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \sum_{i=1}^N \bar{B}_i u_i \,, \tag{36}$$

where $M = \mathrm{blkd}(M_1, \dots, M_N)$, $D^a = \mathrm{blkd}(D_1^a, \dots, D_N^a)$ and

$$\bar{B}_1 = [0_{Nn \times n}, I_n, \dots, 0_n]^\top \,, \dots, \bar{B}_N = [0_{Nn \times n}, 0_n \dots, I_n]^\top .$$

The system (36) constitutes a *mass–spring–damper* system (see, *e.g.* van der Schaft & Jeltsema, 2014, Chapter 12 and Vos, 2015). In what follows we consider the problem of obtaining and maintaining a certain configuration of the agents, as detailed below.

Problem 1. Consider the MAS (35), $i \in \mathcal{I}$. Let $z_k \in \mathbb{R}^n$ denote the relative position between agents i and j, such that

$$z_k = \begin{cases} q_i - q_j \,, & \textit{if node } i \textit{ is the positive end of edge } k \,, \\ q_j - q_i \,, & \textit{if node } j \textit{ is the positive end of edge } k \,, \end{cases} \tag{37}$$

and let $z_k^\star \in \mathbb{R}^n$ denote the *desired* relative position, for $(i,j) \in \mathcal{E}$, $k = 1, \dots, E$. Determine decentralised control laws u_i, for all $i \in \mathcal{I}$, which are such that the agents reach the desired formation, *i.e.* such that $\lim_{t \to \infty} z_k(t) \to z_k^\star$, for all $(i,j) \in \mathcal{E}$, $k = 1, \dots, E$, and $\lim_{t \to \infty} p_i(t) \to 0$, for all $i \in \mathcal{I}$.

By Assumption 1, and from the corresponding definition of the incidence matrix $\mathcal{B}$, the relative displacement between the agents $z = [z_1^\top, \dots, z_E^\top]^\top \in \mathbb{R}^{En}$ is given by $z = (\mathcal{B}^\top \otimes I_n) q$. Considering then the error coordinates $x_1 = (z - z^\star)^\top \in \mathbb{R}^{En}$, $x_2 = p^\top \in \mathbb{R}^{Nn}$, where $z^\star = [z_1^{\star\top}, \dots, z_E^{\star\top}]^\top$, (36) can be re-written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0_{En \times En} & (\mathcal{B}^\top \otimes I_n) M^{-1} \\ 0_{Nn \times En} & -D^a M^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \sum_{i=1}^N \tilde{B}_i u_i \,, \tag{38}$$

where

$$\tilde{B}_1 = [0_{En\times n}, I_n, \dots, 0_n]^\top, \dots, \tilde{B}_N = [0_{En\times n}, 0_n \dots, I_n]^\top.$$

The relative position between the agents z_k, and the desired relative position $z_k^\star$, for $k = 1,\dots,E$, can be analogised to the elongation of a spring and its natural (unstretched) length, respectively (see *e.g.* Jafarian et al., 2015; Vos, 2015; Vos et al., 2014). Based on physical principles, we consider a desired closed-loop energy function of the form $H_d = \frac{1}{2}x_2^\top M^{-1}x_2 + \frac{1}{2}x_1^\top K_c x_1$, where $K_c = \mathrm{blkdiag}\{K_{c_1},\dots,K_{c_E}\}$, with $K_{c_i} > 0$, for $i = 1,\dots,E$, represents the desired stiffness matrix of the springs. In this formulation, the first term in H_d corresponds to the kinetic energy associated with the agents' motion, while the second term represents the potential energy stored in the springs. Recalling (9), it follows that

$$Q_d = \mathrm{blkd}\{K_c, M^{-1}\}. \tag{39}$$

The result in Theorem 2 can then be utilised to design decentralised control laws that solve Problem 1, as detailed in the following statement.

Corollary 1. *Consider the MAS (35), $i \in \mathcal{I}$. Let $R_i = I_n$, for all $i \in \mathcal{I}$, and let Q_d be given by (39). Moreover, let $X_i \in \mathbb{R}^{(En+Nn)\times(En+Nn)}$ be a solution of (21), (32), for all $i \in \mathcal{I}$. Then, the feedback control laws given by (33), with $\hat{P}_i$ given by (22), for all $i \in \mathcal{I}$, solve Problem 1.*

Proof. Under the condition that $\sigma(\hat{A}_{\mathrm{cl}}) \subset \mathbb{C}^-$, the result follows directly from Theorem 2. Thus, the proof lies in demonstrating that the condition indeed holds. To streamline the presentation, recall that by Theorem 2 (and Theorem 1), the overall system represented in error coordinates, *i.e.* (38) with $u_{\mathrm{IDA}} = [\hat{u}_1^\top,\dots,\hat{u}_N^\top]^\top$ and $\hat{u}_i$ given by (24) can be written as (10), with J_d and R_d given by (25), where

$$A = \begin{bmatrix} 0_{En\times En} & (\mathcal{B}^\top\otimes I_n)M^{-1} \\ 0_{Nn\times En} & -D^a M^{-1}\end{bmatrix}, \quad S_i = \tilde{B}_i\tilde{B}_i^\top, \tag{40}$$

for all $i \in \mathcal{I}$. Then, with a slight abuse of notation, consider the partition of X_i given by

$$X_i = \begin{bmatrix} \bar{X}_i^{11} & \bar{X}_i^{12} \\ \bar{X}_i^{21} & \bar{X}_i^{22}\end{bmatrix}, \tag{41}$$

where $\bar{X}_i^{11} \in \mathbb{R}^{En\times En}$, $\bar{X}_i^{12} \in \mathbb{R}^{En\times Nn}$, $\bar{X}_i^{21} \in \mathbb{R}^{Nn\times En}$ and $\bar{X}_i^{22} \in \mathbb{R}^{Nn\times Nn}$. The product term S_iX_i can then be rewritten as

$$S_iX_i = \begin{bmatrix} 0_{En\times En} & 0_{En\times Nn} \\ \Omega_i\bar{X}_i^{21} & \Omega_i\bar{X}_i^{22}\end{bmatrix}, \tag{42}$$

where $\Omega_1 = \mathrm{blkdiag}\{I_N, 0_N,\dots,0_N\}, \dots, \Omega_N = \mathrm{blkdiag}\{0_N,\dots,0_N, I_N\}$. Substituting (39), (40) and (42) into (25) yields

$$R_d = -\frac{1}{2}\left(\begin{bmatrix} 0_{En\times En} & \mathcal{B}^\top\otimes I_n \\ \mathcal{B}\otimes I_n & -2D^a\end{bmatrix} - \begin{bmatrix} 0_{En\times En} & \sum_{i=1}^N \bar{X}_i^{21^\top}\Omega_i \\ \sum_{i=1}^N \Omega_i\bar{X}_i^{21} & \sum_{i=1}^N \Omega_i\bar{X}_i^{22} + \bar{X}_i^{22^\top}\Omega_i\end{bmatrix}\right) \ge 0, \tag{43}$$

where the inequality follows from positive semi-definiteness of R_d (implied by the condition (21a) as established in the proof of Theorem 1). Noting that the first block entry of (43) is the zero matrix, it follows from positive semi-definiteness of R_d (see, *e.g.* Horn & Johnson, 2012, Observation 7.1.10) that

$$\mathcal{B}^\top\otimes I_n = \sum_{i=1}^N \bar{X}_i^{21^\top}\Omega_i. \tag{44}$$

The desired interconnection and damping matrices, J_d and R_d given by (25), can then be written as

$$J_d = \begin{bmatrix} 0_{En\times En} & \mathcal{B}^\top\otimes I_n \\ -(\mathcal{B}\otimes I_n) & \frac{1}{2}\sum_{i=1}^N\left(-\Omega_i\bar{X}_i^{22} + \bar{X}_i^{22^\top}\Omega_i\right)\end{bmatrix}, \tag{45}$$

and

$$R_d = \begin{bmatrix} 0_{En\times En} & 0_{En\times Nn} \\ 0_{En\times En} & D^a + \frac{1}{2}\sum_{i=1}^N\left(\Omega_i\bar{X}_i^{22} + \bar{X}_i^{22^\top}\Omega_i\right)\end{bmatrix}. \tag{46}$$

Consider now the Lyapunov function (11) with $H_d(x)$ given by (9) and Q_d given by (39). It follows from (39) and (46) that the set Ω in (12) is given by $\Omega = \{(x_1^\top, 0)^\top \mid x_1 \in \mathbb{R}^{En}\}$. Substituting $x_2 = 0$ and $\dot{x}_2 = 0$, (39), (45) and (46) into (10) then yields that on the forward invariant set $\mathcal{M}$, $-(\mathcal{B}\otimes I_n)K_cx_1 = 0$. Recalling that (by Assumption 1) $\ker(\mathcal{B}) = 0$, and that $\ker(K_c) = 0$, it follows that $\mathcal{M} = \{0\}$, and thus that $\sigma(\hat{A}_{\mathrm{cl}}) \subset \mathbb{C}^-$. □

7.2. Simulations

We now provide some simulations to demonstrate the efficacy of the result in Corollary 1. To this end, consider a network of $N = 5$ agents whose dynamics are given by (35), with $n = 2$. Each agent i, for $i = 1,\dots,5$, is such that $m_i = 0.150$ kg and $D_i^a = 2I_2$ kg/s. The agents are interconnected via a tree graph with $E = 4$ and

$$\mathcal{B} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1\end{bmatrix}.$$

The desired relative displacement between the agents is set to be $z_k^\star = 0.165I_2$ m, for $k = 1,\dots,4$, whereas the desired stiffness matrix is chosen to be $K_c = \mathrm{blkdiag}\{20I_n, 20I_n, 20I_n, 20I_n\}$. Recall that the corresponding Q_d is given by (39). Considering the error coordinates (38), two sets of simulations – one corresponding to the control design presented in Corollary 1, with additional constraints as suggested in Remark 8 (referred to as Case I), and one corresponding to a direct solution of the matching equation (13) for the given Q_d (referred to as Case II) – are presented. For all simulations we consider $z(0) = [z_k^\star\cos(\frac{\pi}{4}), z_k^\star\sin(\frac{\pi}{4}), z_k^\star\cos(-\frac{\pi}{4}), z_k^\star\sin(-\frac{\pi}{4}), z_k^\star\cos(\frac{\pi}{6}), z_k^\star\sin(\frac{\pi}{6}), z_k^\star\cos(-\frac{\pi}{6}), z_k^\star\sin-(\frac{\pi}{6})]^\top$ and $z^\star = [z_k^\star, 0, z_k^\star, 0, z_k^\star, 0, z_k^\star, 0]^\top$. We suppose that the agents are initially at rest, *i.e.* $p(0) = 0$. This corresponds to the initial conditions $x_1(0) = z(0) - z^\star$ and $x_2(0) = 0$ in the error coordinates. Considering Case I, let X_i, $i = 1,\dots,5$, denote a solution of the LMIs (21), with the additional constraints (as in Remark 8) $x_0^\top(X_1Q_d - 5X_jQ_d)x_0 > 0$, $x_0^\top(X_2Q_d - 5X_jQ_d)x_0 > 0$, for $j = 3,4,5$. These constraints are included to reflect a setting in which the agents have different "control authorities", resulting in a set of "fast" (agents 1 and 2) and "slow" (agents 3, 4, 5) agents. The control laws corresponding to Case I are then given by (33), with $\hat{P}_i$ given by (22). Considering instead Case II, we note that for the same choice of Q_d as in Case I, a possible selection of interconnection and damping matrices is given by

$$J_d = \begin{bmatrix} 0_{En\times En} & \mathcal{B}^\top\otimes I_n \\ -(\mathcal{B}\otimes I_n) & 0_{Nn\times Nn}\end{bmatrix}, R_d = \begin{bmatrix} 0_{En\times En} & 0_{En\times Nn} \\ 0_{En\times En} & D^a + D^d\end{bmatrix},$$

with D^d any symmetric matrix such that $D^a + D^d \ge 0$, and a corresponding control law can be computed by solving (13) for K_{IDA} to obtain the control law (8). In the following we take D^d to be a randomly generated matrix with entries in [0, 1] and such that $D^d = D^{d^\top} > 0$. Note that, since not every selection of J_d and R_d results in a decentralised control law, the (randomly

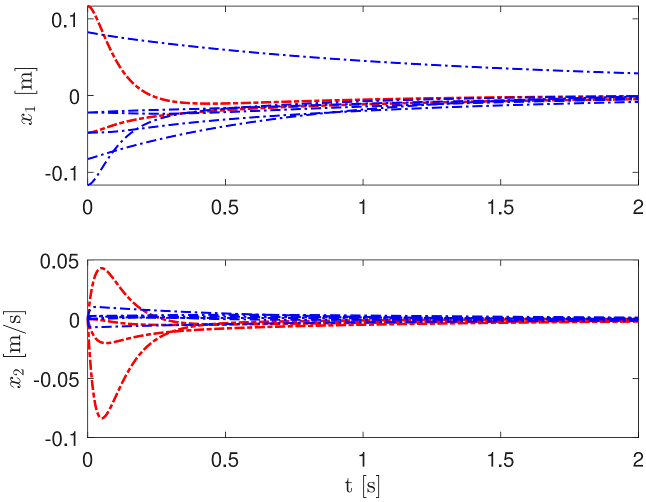

Fig. 1. Time histories of the error coordinates x_1 and x_2 corresponding to Case I.

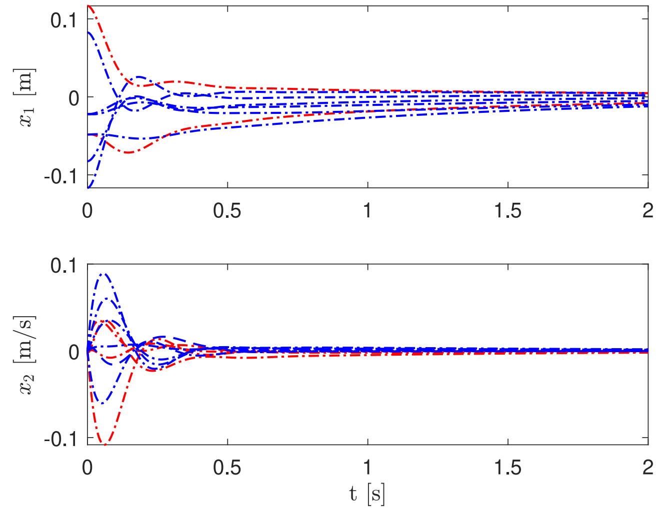

Fig. 2. Time histories of the error coordinates x_1 and x_2 corresponding to Case II.

Table 1
Control efforts corresponding to Case I and Case II.

	$\bar{u}_1$	$\bar{u}_2$	$\bar{u}_3$	$\bar{u}_4$	$\bar{u}_5$
Case I	0.11	0.32	0.015	0.018	0.019
Case II	0.16	0.60	0.44	0.08	0.25

generated) D^d will not generally result in a decentralised control law and, in fact, in the simulations presented herein, they do not. The simulations have been run for $T = 2$ seconds. The time histories of the error coordinates x_1 and x_2 of system (38) in closed loop with the controllers corresponding to Case I and Case II are shown in Figs. 1 and 2, respectively. The error coordinates that are related to the "fast" agents 1 and 2, *i.e.* the first (2D) components of x_1 and the first two (2D) components of x_2 are depicted by red lines, whereas the remaining components of the state, which relate to the "slow" agents, are depicted by blue lines. It can be clearly appreciated that in Case I the fast agents converge quickly with respect to the slow agents. In Case II, the same is *not* observed. Focusing on Case I, the costs (given by (2) with Q_i given by (19), $i = 1, \dots, 5$) at the end of the simulation are $J_1 = 1.77, J_2 = 1.20, J_3 = 0.18, J_4 = 0.15$ and $J_5 = 0.1$, *i.e.* the desired distribution of individual costs is achieved, as dictated by the additional LMI constraints (see Remark 8). Consider next the total control effort of each agent by the end of the simulation, namely $\bar{u}_i = \int_0^T u_i^\top u_i dt$, where $i \in \mathcal{I}$. The values of $\bar{u}_i$ for each agent i are reported in Table 1. In Case I, as expected, the control efforts incurred by players 1 and 2 are higher than those incurred by the remaining agents, whereas the same is not observed for Case II.

The presented example highlights the benefits of the *combined perspective* presented in Theorem 1 and utilised in the MAS setting in Theorem 2 and Corollary 1. Namely, based solely on the solution of LMIs, one can construct decentralised control laws effectively coordinate a team of MAS, accounting for different capabilities amongst the agents. Notably, this is achieved without requiring the direct computation of a solution of the coupled AREs (4), $i \in \mathcal{N}$, associated with the differential game, and also without requiring any tuning of the interconnection and damping matrices associated with IDA-PBC.

8. Conclusion

We have explored connections between PCH systems and differential games. Namely, we have demonstrated that a given differential game can be related to a family of PCH systems and, conversely, that a given PCH system can be related to a family of differential games. Building on these insights, we have provided a strategy to assign the cost matrices (within the framework of differential games) and the interconnection and damping matrices (in the context of PCH systems) to achieve a desired closed-loop behaviour, represented by a specific closed-loop energy function. The corresponding control laws are obtained via the solution of a system of LMIs. Finally, we demonstrate that this "integrated approach" is amenable to decentralised control of networked MAS. The efficacy of the latter result is demonstrated on a formation control problem involving a MAS composed of a network of mass–spring–damper systems.

References

Arcak, M. (2007). Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, *52*(8), 1380–1390.
Başar, T., & Olsder, G. J. (1999). *Dynamic noncooperative game theory, vol. 23* (2. ed. New York, Acad. Press, 1982). Philadelphia, PA: SIAM.
Bullo, F., Cortés, J., & Martinez, S. (2009). Distributed control of robotic networks: a mathematical approach to motion coordination algorithms, vol. 27. Princeton University Press.
Cappello, D., & Mylvaganam, T. (2019). A game theoretic framework for distributed control of multi-agent systems with acyclic communication topologies. In *Proceedings of the 58th IEEE conference on decision and control* (pp. 4477–4482).
Cappello, D., & Mylvaganam, T. (2020). Approximate Nash equilibrium solutions of linear quadratic differential games. *IFAC-PapersOnLine*, *53*(2), 6685–6690.
Cappello, D., & Mylvaganam, T. (2022). Distributed Differential Games for Control of Multi-Agent Systems. *IEEE Transactions on Control of Network Systems*, *9*(2), 635–646.
Castaños, F., Ortega, R., van der Schaft, A., & Astolfi, A. (2009). Asymptotic stabilization via control by interconnection of port-Hamiltonian systems. *Automatica*, *45*(7), 1611–1618.
Cortes, J., Martinez, S., Karatas, T., & Bullo, F. (2004). Coverage control for mobile sensing networks. *IEEE Transactions on Robotics and Automation*, *20*(2), 243–255.
Dattorro, J. (2010). Convex optimization & Euclidean distance geometry. Lulu. com.
Dharwadker, A., & Pirzada, S. (2011). Graph theory. Institute of Mathematics.
Engwerda, J. (2005). LQ dynamic optimization and differential games. John Wiley & Sons.
Fujimoto, K., & Sugie, T. (2001). Canonical transformation and stabilization of generalized Hamiltonian systems. *Systems & Control Letters*, *42*(3), 217–227.
Gu, D. (2008). A differential game approach to formation control. *IEEE Transactions on Control Systems Technology*, *16*(1), 85–93.
Horn, R. A., & Johnson, C. R. (2012). Matrix analysis. Cambridge University Press.

Jafarian, M., Vos, E., De Persis, C., van Der Schaft, A. J., & Scherpen, J. M. (2015). Formation control of a multi-agent system subject to Coulomb friction. *Automatica*, *61*, 253–262.
Li, T., & Gajic, Z. (1995). Lyapunov iterations for solving coupled algebraic Riccati equations of Nash differential games and algebraic Riccati equations of zero-sum games. In *New trends in dynamic games and applications* (pp. 333–351). Springer.
Li, Y., & Hu, X. (2022). A differential game approach to intrinsic formation control. *Automatica*, *136*, Article 110077.
Mei, Y., Lu, Y.-H., Hu, Y. C., & Lee, C. G. (2006). Deployment of mobile robots with energy and timing constraints. *IEEE Transactions on Robotics*, *22*(3), 507–522.
Mesbahi, M. (2010). Graph theoretic methods in multiagent networks. Princeton University Press.
Nortmann, B., Monti, A., Sassano, M., & Mylvaganam, T. (2024). Nash equilibria for linear quadratic discrete-time dynamic games via iterative and data-driven algorithms. *IEEE Transactions on Automatic Control*, *69*(10), 6561–6575.
Ortega, R., van Der Schaft, A., Castanos, F., & Astolfi, A. (2008). Control by interconnection and standard passivity-based control of port-Hamiltonian systems. *IEEE Transactions on Automatic Control*, *53*(11), 2527–2542.
Ortega, R., van Der Schaft, A., Mareels, I., & Maschke, B. (2001). Putting energy back in control. *IEEE Control Systems Magazine*, *21*(2), 18–33.
Ortega, R., van Der Schaft, A., Maschke, B., & Escobar, G. (2002). Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems. *Automatica*, *38*(4), 585–596.
Ortega, R., & García-Canseco, E. (2004). Interconnection and damping assignment passivity-based control: A survey. *European Journal of Control*, *10*(5), 432–450.
Ortega, R., & Spong, M. W. (2000). Stabilization of underactuated mechanical systems via interconnection and damping assignment. *IFAC Proceedings Volumes*, *33*(2), 69–74.
Possieri, C., & Sassano, M. (2015). An algebraic geometry approach for the computation of all linear feedback Nash equilibria in LQ differential games. In *Proceedings of the 54th IEEE conference on decision and control* (pp. 5197–5202).
Rodríguez, H., van der Schaft, A. J., & Ortega, R. (2001). On stabilization of nonlinear distributed parameter port-controlled Hamiltonian systems via energy shaping. *1*, In *Proceedings of the 40th IEEE conference on decision and control* (pp. 131–136). IEEE.
Sastry, S. (2013). Nonlinear systems: analysis, stability, and control, vol. 10. Springer Science & Business Media.
van der Schaft, A., & Jeltsema, D. (2014). Port-Hamiltonian systems theory, vol. 2. (pp. 173–378). Hanover, Massachusetts: now.
Starr, A. W., & Ho, Y. C. (1969). Nonzero-sum differential games. *Journal of Optimization Theory and Applications*, *3*, 184–206.
Vos, E. (2015). *Formation control in the port-Hamiltonian framework* (Ph.D. thesis).
Vos, E., Scherpen, J. M., van der Schaft, A. J., & Postma, A. (2014). Formation control of wheeled robots in the port-Hamiltonian framework. *IFAC Proceedings Volumes*, *47*(3), 6662–6667.
Vu, N. T., & Lefèvre, L. (2018). A connection between optimal control and IDA-PBC design. *IFAC-PapersOnLine*, *51*(3), 205–210.

Maria Luisa Scarpa was born in Turin, Italy in 2000. She received the M.Eng degree in Aeronautics with Spacecraft Engineering from Imperial College London, U.K. in 2022, where she is currently working toward the Ph.D. degree in control theory. Her research interests are focused on optimal control and game theory.

Thulasi Mylvaganam was born in Bergen, Norway, in 1988. She received the M.Eng. degree in Electrical and Electronic Engineering and the Ph.D. degree in Nonlinear Control and Differential Games from Imperial College London, London, U.K., in 2010 and 2014, respectively. From 2014 to 2016, she was a Research Associate with the Department of Electrical and Electronic Engineering, Imperial College London. From 2016 to 2017, she was a Research Fellow with the Department of Aeronautics, Imperial College London, UK, where she is currently Associate Professor. Her research interests include nonlinear control, optimal control, game theory, distributed control and data-driven control. She is Associate Editor of the IEEE Control Systems Letters, of the European Journal of Control, of the IEEE CSS Conference Editorial Board and of the EUCA Conference Editorial Board. She is also Vice Chair of Education for the IFAC Technical Committee 2.4 (Optimal Control) and Member of the UKACC Executive Committee.