

A Hybrid Observer for Practical Observability of Linear Stochastic Systems

Zilong Gong and Giordano Scarciootti

Abstract—This paper focuses on the problem of observability and observer design for linear stochastic systems. To introduce our idea, we first construct an *idealistic* observer. This idealistic observer is not causal as it requires perfect knowledge of the Brownian motion. However, after introducing an *a posteriori* method to reconstruct the variations of the Brownian motion in discrete time, we propose a realistic hybrid observer which approximates the idealistic observer. The performance of this hybrid observer can be made arbitrarily close to that of the idealistic observer. Numerical simulations illustrate the results of the paper.

I. INTRODUCTION

The problem of observability and observer design of dynamical systems has been widely studied. It consists in constructing an estimation of the state of the system that converges to the true value of the state. When the dynamical system is described by stochastic differential equations, the problem becomes remarkably more difficult due to the presence of unknown Brownian motion in the dynamics of the system.

Several estimation techniques for stochastic systems have been proposed in the literature. An early approach based on a Lyapunov-like method to solve the stochastic estimation problem was proposed by Tarn and Rasis [1]. Later on, several approaches have been developed for related or generalised classes of stochastic systems, such as discrete-time nonlinear stochastic systems [2], stochastic time-delay systems [3] and networked control systems [4]. In addition to these extensions, several classes of observers have been proposed for stochastic estimation. For instance, in [4]–[8], the observer design problem is changed into an optimization problem where the observer gains are selected by solving linear matrix inequalities (LMI). Other stochastic observers include Luenberger observers [9], reduced-order observers [10] and hybrid observers [11].

In this paper, we focus on the problem of observer design for linear stochastic systems, with the objective of obtaining improved state estimation with respect to the state of the art. We begin by recalling the difficulties of stochastic observability and recalling that perfect state estimation is possible only by using an “idealistic”, but non-causal, stochastic observer which has complete access to the Brownian motion. Of course, this last assumption (knowledge of the Brownian motion) is unrealistic. Thus, our objective is to find an

alternative causal approach that approximates the behaviour of the idealistic observer. We note that although the Brownian motion is not measurable, it is possible to approximate the variations of the Brownian motion in discrete time. To this end, we introduce an *a posteriori* approach that reconstructs the variations of the Brownian motion discretely. By combining this reconstruction with the idealistic observer, a hybrid observer is constructed. The performance of this hybrid observer depends on a sampling period. This sampling period is related to how frequently the variations of the Brownian motion are approximated and, for an idealistic zero sampling period, the behaviour of the idealistic observer is recovered. Although the zero sampling period is not achievable in practical use, by definition of asymptotic stability in probability, the probability of a large estimation error is guaranteed to be small by using a small enough sampling period.

The remainder of the paper is organized as follows. In Section II we first recall the difficulties of stochastic observability and we then formulate the problem. In Section III we construct the hybrid observer and solve our problem. Section IV contains numerical simulations that illustrate the results of the paper. Section V contains some concluding remarks.

Notation: The symbols $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers while the symbol $\mathbb{Z}$ denotes the set of integer numbers. The identity matrix is denoted by I . A^T is the transpose of the matrix A while $A^\dagger$ represents the Moore-Penrose left pseudo-inverse of A . $0_{m \times n}$ represents the $m \times n$ zero matrix. $\|\cdot\|$ denotes the simple absolute value norm. $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space given by the set Ω , the σ -algebra $\mathcal{F}$ defined on Ω and the probability measure $\mathcal{P}$ on the measurable space $(\Omega, \mathcal{F})$. A *stochastic process* with state space $\mathbb{R}^n$ is a family $\{x_t, t \in \mathbb{R}\}$ of $\mathbb{R}^n$ -valued random variables, *i.e.* for every fixed $t \in \mathbb{R}$, $x_t(\cdot)$ is an $\mathbb{R}^n$ -valued random variable and, for every fixed $w \in \Omega$, $x_t(w)$ is an $\mathbb{R}^n$ -valued function of time [12, Section 1.8]. For ease of notation, we often indicate a stochastic process $\{x_t, t \in \mathbb{R}\}$ simply with x_t (this is common in the literature, see *e.g.* [12]). With a slight abuse of notation, any subscript different from the symbol “ t ” indicates the corresponding component of the vector x_t , *e.g.* x_i is the i -th component of the vector x_t . $x(t; x_0, t_0)$ denotes the solution of a stochastic system with zero input and initial value x_0 at t_0 . The solutions of the systems are intended in Itô sense (see, *e.g.*, [13, Definition 3.1.4]).

*This work is partially supported by Shanghai University under the “Foreign Expert Program” of Shanghai University (No. 21WZ0109).

Zilong Gong and Giordano Scarciootti are with the Department of Electrical and Electronic Engineering, Imperial College London, London, SW7 2AZ, UK. {zilong.gong18, g.scarciotti}@imperial.ac.uk

II. PRELIMINARIES

In this section we provide a formulation of the problem that we aim to solve. We first start with the motivation of this paper and explain the potential of a hybrid approach for stochastic observability. To this end, we recall the difficulties involved with stochastic observability which are discussed, for instance, in [14]. Consider a stochastic linear system with two outputs

$$\begin{aligned} dx_t &= (Ax_t + Bu_t)dt + Fx_t d\mathcal{W}_t, \\ y_t^a &= C_a x_t, \quad y_t^b = C_b x_t, \end{aligned} \quad (1)$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}$ is the control input, $y_t^a \in \mathbb{R}$ and $y_t^b \in \mathbb{R}$ are the system outputs, $\mathcal{W}_t$ is the Brownian motion, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $F \in \mathbb{R}^{n \times n}$, $C_a \in \mathbb{R}^{1 \times n}$ and $C_b \in \mathbb{R}^{1 \times n}$. Assume that our objective is to find an observer such that the estimation error of the state converges to zero asymptotically. A first attempt for achieving this is to use a traditional deterministic observer, namely

$$d\hat{x}_t^D = (A\hat{x}_t^D + Bu_t + L(y_t^a - C_a\hat{x}_t^D))dt, \quad (2)$$

where $\hat{x}_t^D$ is the estimation of x_t and L is the observer gain. Interconnecting the deterministic observer (2) with system (1), we obtain the error dynamics

$$de_t^D = (A - LC_a)e_t^D dt + Fx_t d\mathcal{W}_t, \quad (3)$$

where $e_t^D = x_t - \hat{x}_t^D$ is the estimation error. Although (3) has a stable equilibrium point with a suitable L , the stochastic term $Fx_t d\mathcal{W}_t$ may keep exciting the error dynamics. In particular, e_t^D will not converge to zero unless the state x_t converges to zero. In this paper, we focus on the general case of non-zero system state x_t . Therefore, the deterministic observer (3) cannot achieve our estimation objective. To overcome this problem the literature has focused on achieving mean-square stability of the error dynamics by Lyapunov-like methods [1], [5]. In particular, [5] indicates that although perfect observation cannot be achieved by a deterministic observer, the estimation error can be exponentially bounded in the mean-square sense.

In this paper we follow a different approach. To properly introduce this approach let us assume that for now we can construct an *idealistic* stochastic observer of the type

$$\begin{aligned} d\hat{x}_t^I &= (A\hat{x}_t^I + Bu_t + L_1(y_t^a - C_a\hat{x}_t^I))dt \\ &\quad + (F\hat{x}_t^I + L_2(y_t^a - C_a\hat{x}_t^I))d\mathcal{W}_t, \end{aligned} \quad (4)$$

where $\hat{x}_t^I \in \mathbb{R}^n$ is the estimation of the state, L_1 and L_2 are observer gains and $\mathcal{W}_t$ is the Brownian motion. One can see that this observer is unrealistic because $\mathcal{W}_t$ is unknown in practice. By interconnecting (1) and (4), we obtain the error dynamics

$$de_t^I = (A - L_1C_a)e_t^I dt + (F - L_2C_a)e_t^I d\mathcal{W}_t, \quad (5)$$

where $e_t^I = x_t - \hat{x}_t^I$ is the estimation error. Then it is clear that if the gains L_1 and L_2 are chosen such that (5) is asymptotically stable in probability or mean-square, then the

estimation error will go to zero asymptotically in probability or mean-square.

In summary, while the idealistic observer (4) can achieve, with opportune choices of L_1 and L_2 , a zero estimation error in probability, it is unrealistic because it requires knowledge of the Brownian motion. In this paper we propose a new practical observer with the objective of approximating the behaviour of the idealistic observer (4). First of all, consider the quantity $\Delta\mathcal{W}_\varepsilon(k) = \mathcal{W}_{t_k} - \mathcal{W}_{t_{k-1}}$, where $\varepsilon = t_k - t_{k-1}$ is the sampling period. For simplicity, from now on we abuse the notation and indicate with $\mathcal{W}_k$, x_k and so on the quantities $\mathcal{W}_{t_k}$, x_{t_k} and so on. The idea that we want to formulate is the following: if we can construct an approximation $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$ of $\Delta\mathcal{W}_\varepsilon(k)$ and we can find an appropriate concept of convergence, then it could be possible to construct an observer based on the quantity $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$ which has the property of behaving as the idealistic observer (4) when ε goes to zero. To precisely formulate this problem, we introduce the hybrid observer

$$\begin{aligned} d\hat{x}_t^H &= (A\hat{x}_t^H + Bu_t + L_1(y_t^a - C_a\hat{x}_t^H))dt, \\ \hat{x}_k^{H+} &= \hat{x}_k^H + (F\hat{x}_{k-1}^{H+} + L_2(y_{k-1}^a - C_a\hat{x}_{k-1}^{H+}))\Delta\widehat{\mathcal{W}}_\varepsilon(k), \end{aligned} \quad (6)$$

where $\hat{x}_k^{H+}$ indicates the value of $\hat{x}_k^H$ after the jump. Thus, the hybrid dynamics of (6) flow from t_{k-1}^+ to t_k according to the law of the deterministic observer (2). At time t_k there is a jump and the state of the hybrid observer is updated according to the second equation in (6). We note that if $\Delta\widehat{\mathcal{W}}_\varepsilon(k) = \Delta\mathcal{W}_\varepsilon(k)$, then by definition of the Itô integral the limiting solution of (6) as $\varepsilon \rightarrow 0$ is the solution of the idealistic observer. In summary, if we can design an opportune approximation $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$ of $\Delta\mathcal{W}_\varepsilon(k)$ then we can recover the idealistic convergence in the limit. In other words, we want to solve the following problem.

Problem 1. Consider system (1), the hybrid observer (6) and their associated estimation error system with state e_t^H . The problem of ε -approximate stochastic observer design consists in designing the parameters of the hybrid observer (6) such that if $\varepsilon \rightarrow 0$ then the associated estimation error system is asymptotically stable in probability.

Note that from a practical point of view the limiting behaviour is not achievable. However, by definition of asymptotic stability in probability we have the property that for all $\alpha > 0$ and $\gamma > 0$, there exists $\beta > 0$ and $\eta > 0$, such that for all $\varepsilon \leq \beta$ and $\delta \geq \eta$ we have that

$$\mathcal{P} \left\{ \sup_{t \geq \delta} \|e^H(t, e_0, t_0)\| > \gamma \right\} < \alpha.$$

In other words, we can always guarantee that the probability of a large estimation error is small, by picking a small enough ε . In the rest of the paper we design $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$, L_1 and L_2 to solve Problem 1.

III. HYBRID STOCHASTIC OBSERVER

In this section we solve Problem 1. We first focus on the scalar case. This is of course just a toy example (as the

observation problem is trivial in the scalar case) but since this case is simpler, it provides easier to understand insights. Then we solve the general case.

Now, we start by designing $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$. Previous research shows that if one was access to the state, then $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$ can be constructed from the diffusion term of the system dynamics. Consider system (1), assume that there exists $\delta > 0$ such that $|Fx_k| \geq \delta$ almost surely for all $k \in \mathbb{Z}_{\geq 0}$. Then, an approximation of $\Delta\mathcal{W}_\varepsilon(k)$ is given by [15]

$$\Delta\overline{\mathcal{W}}_\varepsilon(k) = (Fx_{k-1})^\dagger [\Delta x_k - (Ax_{k-1} + Bu_{k-1})\varepsilon], \quad (7)$$

where $\Delta x_k = x_k - x_{k-1}$. As discussed in [15], $\Delta\overline{\mathcal{W}}_\varepsilon(k)$ converges to $\Delta\mathcal{W}_\varepsilon(k)$ weakly as $\varepsilon \rightarrow 0$. One can see that this approximation is not an available solution for Problem 1 as the system state is unknown. Note that when $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$ is being computed, all the estimations of the state before time t_k are known. Therefore, a first attempt for constructing $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$ is to approximate (7) by replacing the state with its estimation. However, this is still unrealistic as $\hat{x}_k^{H+}$ is not available at time t_k and, according to the hybrid observer (6), Δx_k cannot be approximated until time t_k^+ when $\hat{x}_k^{H+}$ is provided. For this reason, (7) cannot be used for constructing $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$ of the hybrid observer (6). However, it gives us an idea about how to construct $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$ from the diffusion term.

In our approach, instead of using the diffusion term of the system dynamics, we consider the diffusion term of the output dynamics $dy_t^b = C_b dx_t$. This is motivated by the fact that the quantity $\Delta y_k^b = y_k^b - y_{k-1}^b$, which is the variation of a measurable output y_t^b , can be easily acquired. Similarly to the assumption for (7), the associated diffusion coefficient $C_b F x_{k-1}$ should be non-zero. Considering that this diffusion coefficient is approximated as $C_b F \hat{x}_{k-1}^{H+}$ for constructing $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$, it is reasonable to assume the following.

Assumption 1. Consider system (1), there exists $\delta > 0$ such that $|C_b F \hat{x}_k^{H+}| \geq \delta$ almost surely for all $k \in \mathbb{Z}_{\geq 0}$.

This assumption guarantees that the approximation of the output diffusion coefficient is non-zero at every sampling time. Note that if the system state x_t converges to zero, as discussed in Section II, observation can be simply achieved by a deterministic observer. Therefore, we focus on the case in which the system state x_t and the corresponding state estimation $\hat{x}_k^{H+}$ do not converge to zero. One may also find that if $C_b F = 0_{1 \times n}$, then Assumption 1 does not hold at any sampling time. This simply indicates that the system output contains no information of the Brownian motion and thus cannot be used for constructing $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$. In addition to this, Assumption 1 may still be invalid for some $k \in \mathbb{Z}_{\geq 0}$. From a practical point of view, if $C_b F \hat{x}_{k-1}^{H+} = 0$, then we can skip the sampling at t_k and postpone it to t_{k+1} . With the absence of $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$ at t_k , our hybrid observer would simply degenerate to the deterministic observer at the sampling time t_k . Now let Assumption 1 hold, $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$ is designed as

$$\Delta\widehat{\mathcal{W}}_\varepsilon(k) = (C_b F \hat{x}_{k-1}^{H+})^{-1} [\Delta y_k^b - C_b (A \hat{x}_{k-1}^{H+} + Bu_{k-1})\varepsilon]. \quad (8)$$

It is clear that this approximation is improved as the estimation of the state is improved. In particular, if the perfect observation $\hat{x}_{k-1}^{H+} = x_{k-1}$ is achieved as $\varepsilon \rightarrow 0$, then in (8) we would have

$$\begin{aligned} \Delta\widehat{\mathcal{W}}_\varepsilon(k) &= (C_b F x_{k-1})^{-1} [\Delta y_k^b - C_b (A x_{k-1} + Bu_{k-1})\varepsilon] \\ &= (C_b F x_{k-1})^{-1} C_b [\Delta x_k - (A x_{k-1} + Bu_{k-1})\varepsilon]. \end{aligned} \quad (9)$$

Now recall the forward-Euler discretization of system (1) [16]

$$x_k - x_{k-1} = (A x_{k-1} + Bu_{k-1})\varepsilon + F x_{k-1} \Delta\mathcal{W}_\varepsilon(k) + o(\varepsilon^2), \quad (10)$$

where $o(\varepsilon^2)$ is a higher-order term of ε^2 . Substituting (10) in (9) yields

$$\begin{aligned} \Delta\widehat{\mathcal{W}}_\varepsilon(k) &= (C_b F x_{k-1})^{-1} C_b [F x_{k-1} \Delta\mathcal{W}_\varepsilon(k) + o(\varepsilon^2)] \\ &= \Delta\mathcal{W}_\varepsilon(k) + o(\varepsilon^2), \end{aligned}$$

which implies $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$ converges to $\Delta\mathcal{W}_\varepsilon(k)$ weakly as $\varepsilon \rightarrow 0$. Then as discussed in Section II, the behaviour of the idealistic observer is recovered in the limit.

Now combining (8) with (6), a hybrid stochastic observer is constructed. Note that two different outputs y_t^a and y_t^b are used in this hybrid structure, one for computing $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$ and another one for estimating the state. The need for two outputs is motivated later in Section III B. Now, we focus on the design of L_1 and L_2 for solving Problem 1. For a better understanding of this hybrid observer, we first focus on the scalar case.

A. Scalar Case

First of all, note that in the scalar case the observation problem is trivial because $x_t = y_t^a / C_a$. However, this study of the scalar case is helpful to understand the general case later in this section. Now we introduce a proposition for designing the observer gains L_1 and L_2 .

Proposition 1. Consider system (1) and the hybrid observer (6) in the scalar case. Let Assumption 1 hold. The associated estimation error e_t^H converges to zero asymptotically in probability as $\varepsilon \rightarrow 0$ for any $L_1 C_a > 0$ and $L_2 = 0$.

Proof. For the time being, we focus on the discretization of the estimation error dynamics e_t^H and denote $e_k^{H+} = x_k - \hat{x}_k^{H+}$ (because $x_k = x_k^+$). Note that the forward-Euler discretization of the hybrid observer (6) is given by

$$\begin{aligned} \hat{x}_k^{H+} &= \hat{x}_{k-1}^{H+} + (A \hat{x}_{k-1}^{H+} + Bu_{k-1} + L_1 (y_{k-1}^a - C_a \hat{x}_{k-1}^{H+}))\varepsilon \\ &\quad + (F \hat{x}_{k-1}^{H+} + L_2 (y_{k-1}^a - C_a \hat{x}_{k-1}^{H+}))\Delta\widehat{\mathcal{W}}_\varepsilon(k) + o(\varepsilon^2). \end{aligned} \quad (11)$$

By interconnecting (10) with (11), we have

$$\begin{aligned} e_k^{H+} &= e_{k-1}^{H+} + (A - L_1 C_a) e_{k-1}^{H+} \varepsilon + F x_{k-1} \Delta\mathcal{W}_\varepsilon(k) \\ &\quad - F \hat{x}_{k-1}^{H+} \Delta\widehat{\mathcal{W}}_\varepsilon(k) - L_2 C_a e_{k-1}^{H+} \Delta\widehat{\mathcal{W}}_\varepsilon(k) + o(\varepsilon^2). \end{aligned} \quad (12)$$

To simplify (12), we first focus on the term $F\hat{x}_{k-1}^{H+}\Delta\widehat{\mathcal{W}}_\varepsilon(k)$. Recall the expression of $\Delta\widehat{\mathcal{W}}_\varepsilon(k)$ from (8) and observe that in the scalar case $F\hat{x}_{k-1}^{H+}(C_bF\hat{x}_{k-1}^{H+})^{-1} = C_b^{-1}$, thus

$$\begin{aligned} & F\hat{x}_{k-1}^{H+}\Delta\widehat{\mathcal{W}}_\varepsilon(k) \\ &= C_b^{-1}[(y_k^b - y_{k-1}^b) - C_b(A\hat{x}_{k-1}^{H+} + Bu_{k-1})\varepsilon + o(\varepsilon^2)] \\ &= (x_k - x_{k-1}) - (A\hat{x}_{k-1}^{H+} + Bu_{k-1})\varepsilon + o(\varepsilon^2). \end{aligned} \quad (13)$$

Note that the term $(x_k - x_{k-1})$ is given by the forward-Euler discretization of system (1). Substituting (10) in (13) yields

$$\begin{aligned} F\hat{x}_{k-1}^{H+}\Delta\widehat{\mathcal{W}}_\varepsilon(k) &= (Ax_{k-1} + Bu_{k-1})\varepsilon + Fx_{k-1}\Delta\mathcal{W}_\varepsilon(k) \\ &\quad - (A\hat{x}_{k-1}^{H+} + Bu_{k-1})\varepsilon + o(\varepsilon^2) \\ &= Ae_{k-1}^{H+}\varepsilon + Fx_{k-1}\Delta\mathcal{W}_\varepsilon(k) + o(\varepsilon^2). \end{aligned} \quad (14)$$

Finally, substituting (14) in (12), we have the error dynamics

$$e_k^{H+} = e_{k-1}^{H+} - L_1C_a e_{k-1}^{H+}\varepsilon - L_2C_a e_{k-1}^{H+}\Delta\widehat{\mathcal{W}}_\varepsilon(k) + o(\varepsilon^2). \quad (15)$$

Thus, if $0 < L_1C_a < \frac{2}{\varepsilon}$ and $L_2 = 0$, then e_k^{H+} will converge to zero asymptotically as $\varepsilon \rightarrow 0$. Now we look at what happens between sampling times. The estimation of the state between sampling times is given by the deterministic observer (2). Without loss of generality, consider the time interval (t_{k-1}^+, t_k) , the error dynamics of e_t^H is given by

$$de_t^H = (A - L_1C_a)e_t^H dt + Fx_t d\mathcal{W}_t, \quad (16)$$

where $t \in (t_{k-1}^+, t_k)$. Note that the increments of the Brownian motion follows normal distribution with zero mean and variance $(t - t_{k-1}^+) < (t_k - t_{k-1}^+) = \varepsilon$. Therefore, consider the error dynamics (16), the variations of e_t^H are bounded in probability for $t \in (t_{k-1}^+, t_k)$, which means that for any $\alpha > 0$ and $\gamma > 0$, there exist $\beta > 0$ such that for all $\varepsilon \leq \beta$ we have that

$$\mathcal{P} \left\{ \sup_{t \in (t_{k-1}^+, t_k)} \|e^H(t, e_{k-1}^{H+}, t_{k-1}^+) - e_{k-1}^{H+}\| > \gamma \right\} < \alpha.$$

The probability of large variations in the estimation error is guaranteed to be low by picking a sufficient small sampling period ε . In other words, we can always guarantee that for $t \in (t_{k-1}^+, t_k)$, e_t^H is close enough to e_{k-1}^{H+} with high probability. In conclusion, e_t^H will converge to zero asymptotically in probability as $\varepsilon \rightarrow 0$ with any observer gains satisfying $L_1C_a > 0$ and $L_2 = 0$ and Problem 1 is solved for the scalar case. $\square$

Remark 1. From a practical point of view, ε cannot be exactly zero. For any finite ε , we require $0 < L_1C_a < \frac{2}{\varepsilon}$ as per proof of Proposition 1.

B. General Case

We are now ready to consider the general case. The main difference is that (13) is not valid for the general case and thus the error dynamics of e_k^{H+} cannot be simplified as (15). However, we can still obtain the estimation error dynamics in a similar way.

Proposition 2. Consider the linear stochastic system (1) and the hybrid observer (6). Let Assumption 1 hold. The forward-Euler discretization of the error dynamics e_t^H is given by

$$\begin{aligned} e_k^{H+} &= e_{k-1}^{H+} + [(I - \Psi_{k-1})A - L_1C_a]e_{k-1}^{H+}\varepsilon + o(\varepsilon^2) \\ &\quad + (I - \Psi_{k-1})Fe_{k-1}^{H+}\Delta\mathcal{W}_\varepsilon(k) - L_2C_a e_{k-1}^{H+}\Delta\widehat{\mathcal{W}}_\varepsilon(k), \end{aligned} \quad (17)$$

where $\Psi_k = F\hat{x}_k^{H+}(C_bF\hat{x}_k^{H+})^{-1}C_b$.

System (17) is a discrete time-varying stochastic system. The stability problem of this type of systems is complex. However, a stability criterion is provided, for instance, in [17]. Thus, L_1 and L_2 can be designed by applying [17, Theorem 3.1/4.1]. Once the discrete error dynamics (17) is guaranteed to be stable in probability by using suitable L_1 and L_2 , we again look at what happens between the sampling times. As discussed before, the probability of large variations in the estimation error is low for sufficient small sampling period. Therefore, e_t^H will converge to zero asymptotically in probability as $\varepsilon \rightarrow 0$ and Problem 1 is solved for the general case.

We conclude this section with a series of observations.

Remark 2. If Proposition 2 is applied to the scalar case, we note that

$$\Psi_k = F\hat{x}_k^{H+}(C_bF\hat{x}_k^{H+})^{-1}C_b = 1. \quad (18)$$

Substituting this expression in (17), yields

$$e_k^{H+} = e_{k-1}^{H+} - L_1C_a e_{k-1}^{H+}\varepsilon - L_2C_a e_{k-1}^{H+}\Delta\widehat{\mathcal{W}}_\varepsilon(k) + o(\varepsilon^2),$$

i.e. the error dynamics (15) of the scalar case is retrieved. In other words, the error dynamics (17) can be simplified as (15) in the scalar case and the two results are consistent.

Now based on the error dynamics (17), we can explain the need for the two outputs y_t^a and y_t^b for solving Problem 1. For simplicity, we again set $L_2 = 0_{n \times 1}$.

Proposition 3. If C_a and C_b are linearly dependent, then the dynamics of (17) is not influenced by L_1 .

Proof. If C_a and C_b are linearly dependent, then there exists $\alpha \neq 0$ such that $C_a = \alpha C_b$. We observe that

$$\begin{aligned} C_b(I - \Psi_{k-1}) &= C_b - C_bF\hat{x}_{k-1}^{H+}(C_bF\hat{x}_{k-1}^{H+})^{-1}C_b \\ &= C_b - C_b = 0_{1 \times n} \end{aligned} \quad (19)$$

Thus, multiplying C_b on the both sides of (17) yields

$$\begin{aligned} C_b e_k^{H+} &= C_b e_{k-1}^{H+} + C_b[(I - \Psi_{k-1})A - L_1C_a]e_{k-1}^{H+}\varepsilon \\ &\quad + C_b(I - \Psi_{k-1})Fe_{k-1}^{H+}\Delta\mathcal{W}_\varepsilon(k) + o(\varepsilon^2) \\ &= C_b e_{k-1}^{H+} - C_b L_1 C_a e_{k-1}^{H+}\varepsilon + o(\varepsilon^2) \\ &= C_b e_{k-1}^{H+} - (\alpha C_b L_1) C_b e_{k-1}^{H+}\varepsilon + o(\varepsilon^2). \end{aligned} \quad (20)$$

Denoting $E_k = C_b e_k^{H+}$, the previous scalar system is rewritten as

$$E_k = E_{k-1} - (C_a L_1) E_{k-1} \varepsilon + o(\varepsilon^2). \quad (21)$$

Note that if e_k^{H+} converges to zero, then E_k converges to zero, but the opposite may not hold. Applying the results of Proposition 1, for $C_a L_1 > 0$ and ε sufficiently small we have that E_k converges to zero. Furthermore, since $E_k = 0$ is a stable equilibrium point of the scalar system (21), $\{e_t^H | C_b e_t^H = 0\}$ will be an invariant set as $\varepsilon \rightarrow 0$. In particular, if e_{k-1}^{H+} belongs to this invariant set, then

$$e_k^{H+} = e_{k-1}^{H+} + (I - \Psi_{k-1})Ae_{k-1}^{H+}\varepsilon + (I - \Psi_{k-1})Fe_{k-1}^{H+}\Delta\mathcal{W}_\varepsilon(k) + o(\varepsilon^2), \quad (22)$$

where we have used that $L_1 C_a e_{k-1}^{H+} = \alpha L_1 E_{k-1} \rightarrow 0$. Note that $\Psi_{k-1} = F\hat{x}_{k-1}^H (C_b F \hat{x}_{k-1}^H)^{-1} C_b$. Thus, the error system (22) is completely decided by the estimated state and the sampling period. Tuning the observer gain L_1 is ineffective and the estimation error will either converge to zero or keep increasing inside the invariant set. Therefore, two outputs y_t^a and y_t^b with linearly independent C_a and C_b are required.

Remark 3. Consider Proposition 3 in the scalar case. For any non-zero C_b , $E_k = C_b e_k^{H+} = 0$ is equivalent with $e_k^{H+} = 0$. Thus the result of Proposition 1 is retrieved. This means that one output y_t^a is sufficient for solving the scalar case.

IV. EXAMPLES OF DETERMINISTIC, IDEALISTIC AND HYBRID STOCHASTIC OBSERVER

In this section, we illustrate our results by means of numerical simulations.

Consider a three-dimensional system described by matrices

$$A = \begin{bmatrix} -10 & 7 & 3 \\ 0.2 & -10 & 9 \\ 0 & 1 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad F = 0.5A,$$

$$C_a = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}, \quad C_b = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}.$$

The input and initial values are set as

$$u_t = 10, \quad x_0 = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}^T, \quad \hat{x}_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T.$$

For both idealistic and hybrid observer, the observer gains are set as $L_1 = [-1, 2, 1]^T$ and $L_2 = [0, 0, 0]^T$. The observer gain of the deterministic observer is set as $L = [9, 6, 6]^T$ to guarantee that the estimation error is bounded in the mean-square sense [5]. Two sampling periods $\varepsilon = 10^{-4}$ and $\varepsilon = 10^{-3}$ are tested for the hybrid observer. The simulation step (for the integration of the stochastic differential equations) is set at 10^{-6} . The estimations of the state and of the associated estimation errors of the deterministic, idealistic and hybrid observers are given for $\hat{x}_1$ (Fig. 1), e_1 (Fig. 2), $\hat{x}_2$ (Fig. 3), e_2 (Fig. 4), $\hat{x}_3$ (Fig. 5) and e_3 (Fig. 6).

The figures confirm that the hybrid observers perform well and, as expected, similarly to the idealistic observer. Moreover, they confirm that for our hybrid observer, the estimated state is improved by using a smaller sampling period ε . For instance, the oscillations of the estimation error decrease as the sampling period $\varepsilon \rightarrow 0$.

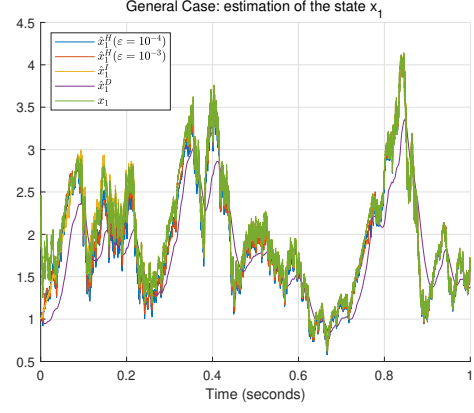


Fig. 1. Time history of the state x_1 (green) and of the corresponding estimated state $\hat{x}_1^D$ (purple), $\hat{x}_1^I$ (yellow) and $\hat{x}_1^H$ (red and blue) for the general case of deterministic, idealistic and hybrid observers, respectively.

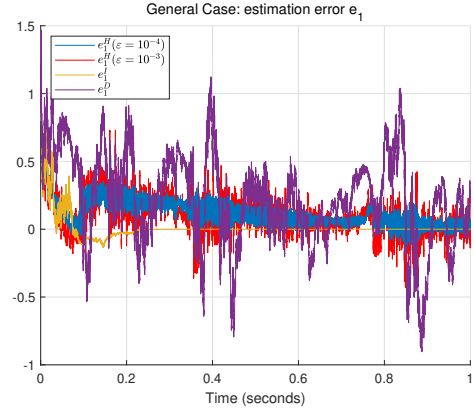


Fig. 2. Time history of the estimation error e_1^D (purple), e_1^I (yellow) and e_1^H (red and blue) for the general case of deterministic, idealistic and hybrid observers, respectively.

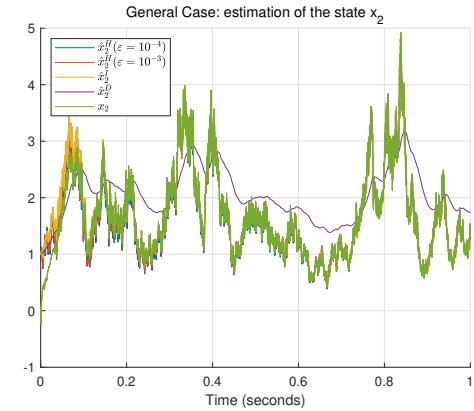


Fig. 3. Time history of the state x_2 (green) and of the corresponding estimated state $\hat{x}_2^D$ (purple), $\hat{x}_2^I$ (yellow) and $\hat{x}_2^H$ (red and blue) for the general case of deterministic, idealistic and hybrid observers, respectively.

V. CONCLUSIONS

In this paper, we proposed a hybrid observer for linear stochastic systems. We first recalled the difficulties arising in the problem of stochastic observability. Then, to properly introduce our approach, we recalled an idealistic observer which assumes that the Brownian motion is known. This idealistic observer is non-causal. Thus, we introduced an *a posteriori* method to approximate the variations of the Brownian motion. Based on this approximation, we constructed a hybrid observer which approximates the behaviour of the idealistic observer. The associated estimation error converges to zero asymptotically in probability as the sampling period $\varepsilon \rightarrow 0$. Although this limiting behaviour cannot be achieved in practice, the probability of a large estimation error can be guaranteed to be small by using small enough ε .

REFERENCES

- [1] T.-J. Tarn and Y. Rasis, "Observers for nonlinear stochastic systems," *IEEE Transactions on Automatic Control*, vol. 21, no. 4, pp. 441–448, 1976.
- [2] E. Yaz and A. Azemi, "Observer design for discrete and continuous non-linear stochastic systems," *International Journal of systems science*, vol. 24, no. 12, pp. 2289–2302, 1993.
- [3] Y. Niu and D. W. Ho, "Robust observer design for itô stochastic time-delay systems via sliding mode control," *Systems & Control Letters*, vol. 55, no. 10, pp. 781–793, 2006.
- [4] X. Fang and J. Wang, "Stochastic observer-based guaranteed cost control for networked control systems with packet dropouts," *IET Control Theory & Applications*, vol. 2, no. 11, pp. 980–989, 2008.
- [5] E. Yaz and A. Azemi, "Lyapunov-based nonlinear observer, design for stochastic systems," in *29th IEEE Conference on Decision and Control (CDC)*, pp. 218–219, IEEE, 1990.
- [6] C. S. Jeong, E. E. Yaz, and Y. I. Yaz, "Stochastically resilient observer design for a class of continuous-time nonlinear systems," *IFAC Proceedings Volumes*, vol. 44, no. 1, pp. 1843–1848, 2011.
- [7] A. P. Dani, S.-J. Chung, and S. Hutchinson, "Observer design for stochastic nonlinear systems using contraction analysis," in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, pp. 6028–6035, IEEE, 2012.
- [8] A. Barbata, M. Zasadzinski, and H. S. Ali, " H_∞ observer design for a class of nonlinear singular stochastic systems," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 3885–3892, 2017.
- [9] S. Bratan and S. Roshchupkin, "Synthesis of lunberger stochastic observer for estimation of the grinding operation state," in *MATEC Web of Conferences*, vol. 224, p. 01133, EDP Sciences, 2018.
- [10] Y.-G. Liu and J.-F. Zhang, "Reduced-order observer-based control design for nonlinear stochastic systems," *Systems & Control Letters*, vol. 52, no. 2, pp. 123–135, 2004.
- [11] X. Hulhoven, R. Hanus, and P. Bogaerts, "Stochastic hybrid observer for bioprocess state estimation," *IFAC Proceedings Volumes*, vol. 37, no. 3, pp. 19–24, 2004.
- [12] L. Arnold, *Stochastic Differential Equations*. A Wiley-Interscience publication, Wiley, 1974.
- [13] B. Øksendal, *Stochastic Differential Equations*. Springer-Verlag, 2003.
- [14] T. Damm, *Rational Matrix Equations in Stochastic Control*. Lecture Notes in Control and Information Sciences, Springer, 2004.
- [15] A. Mellone and G. Scarcioiti, "Output regulation of linear stochastic systems," *IEEE Transactions on Automatic Control*, 2021.
- [16] T. C. Gard, *Introduction to Stochastic Differential Equations*. Monographs and textbooks in pure and applied mathematics, M. Dekker, 1988.
- [17] L. Huang, H. Hjalmarsson, and H. Koepl, "Almost sure stability and stabilization of discrete-time stochastic systems," *Systems & Control Letters*, vol. 82, pp. 26–32, 2015.

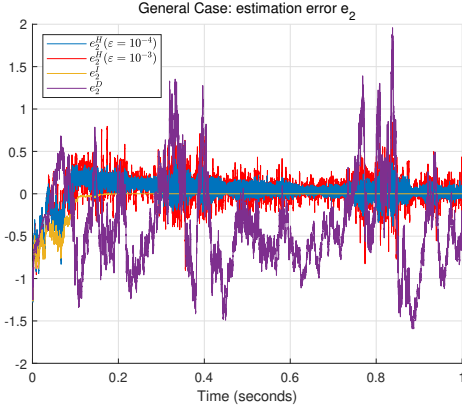


Fig. 4. Time history of the estimation error e_2^D (purple), e_2^I (yellow) and e_2^H (red and blue) for the general case of deterministic, idealistic and hybrid observers, respectively.

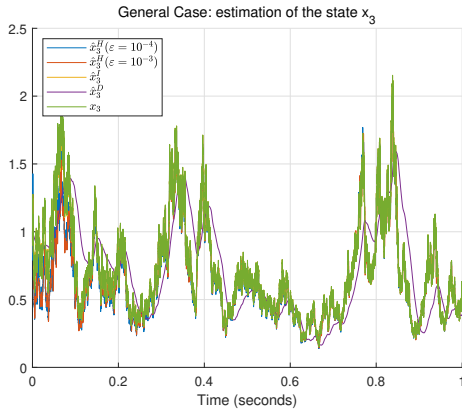


Fig. 5. Time history of the state x_3 (green) and of the corresponding estimated state $\hat{x}_3^D$ (purple), $\hat{x}_3^I$ (yellow) and $\hat{x}_3^H$ (red and blue) for the general case of deterministic, idealistic and hybrid observers, respectively.

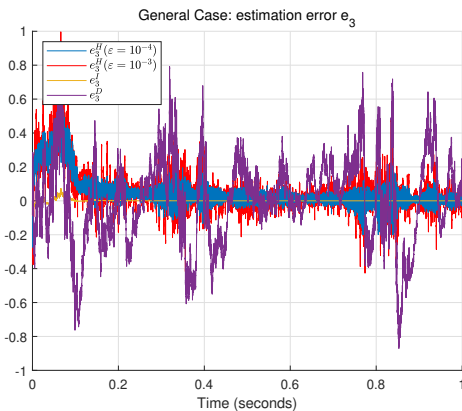


Fig. 6. Time history of the estimation error e_3^D (purple), e_3^I (yellow) and e_3^H (red and blue) for the general case of deterministic, idealistic and hybrid observers, respectively.