SUM RULES IN PARTICLE PHYSICS.

by

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PREFACE.

The work presented in this thesis was carried out in the Department of Theoretical Physics, Imperial College, London, between October 1965 and May 1968 under the Supervision of Professor P.T. Matthews. The author wishes to thank him, Dr. K.J. Barnes, Dr. H.F. Jones, and Dr. M.D. Scadron for their continued guidance and encouragement, without which this thesis could never have been written.

The thesis is based in part on three papers by the author and falls into two distinct parts: (i) The general M-function with applications to superconvergence which occupies the main body of the thesis, and (ii) a Current Algebra calculation on meson decays which, as it is distinct from the rest of the thesis, has been presented in the form of a reprint of the author's paper on the calculation.

The work described in the text and the reprints is, except where explicitly stated, original and has not been submitted to this, or any other University for any other degree.

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ABSTRACT.

A general method by which superconvergence sum rules may be obtained for any scattering process is developed. Working in the covariant four-component spinor formalism, the M-function expansion in terms of kinematic covariants and kinematic singularity free amplitudes is investigated. The form of these singularity free amplitudes, together with the relevant equivalence theorems and crossing matrices, is given in detail for examples of fermion-fermion and abnormal boson-boson scattering.

The procedure for obtaining partial wave expansions of invariant amplitudes is summarized and used, in conjunction with the Regge pole model, in order to obtain the asymptotic behaviour of the invariant amplitudes. Superconvergence sum rules are derived, by use of these results, for $N\Sigma \rightarrow N\Sigma$ and $\pi\rho \rightarrow \rho\omega$ scattering. The generalization of these sum rules to other scattering processes with the same spin-parity assignment is discussed, and the results compared with the predictions of the U(6,6) symmetry scheme.

CONTENTS.

INTRODUCTION.

In elementary particle physics, the idea of "Sum Rules" has received considerable attention lately. The approximation of a physical system by a sum over simple states is a widely used technique of physics, and sum rules of various kinds have been known for many years. In particular, the development of elementary particle physics has seen the introduction of sums over elementary particle states, which has resulted in the derivation of sum rules for many processes in both weak and strong interaction physics.

Two particularly important general techniques used for the derivation of sum rules in recent years have been respectively the algebra of currents and superconvergence. These names are to some extent self explanatory. The algebra of currents is based on the assumption that the commutation rules of charges, or the currents of which they are the spatial integrals, are those predicted by some simple model for the currents, such as the quark model(1). Superconvergence, on the other hand, arises on parts of the scattering amplitude which vanish sufficiently rapidly asymptotically for unsubtracted dispersion relations to be constructed, both for the particular portion of the scattering amplitude and its first moment. The asymptotic behaviour is generally obtained from some model such as the Regge pole model(2). The summation in current algebra is generally over a set of intermediate states inserted between the two components of a current commutator, while in superconvergence, the sum is over an elementary particle pole approximation to a dispersion type integral.

Following the original idea of Gell-Mann[3], a large number of papers appeared in which current algebras were utilised for the construction of sum rules for a wide variety of interactions[4]. Many of these applications using the partial conservation of the axial current hypothesis[5] and the soft pion and infinite momentum[6] limits have proved remarkably successful. One of the developments of the current algebra technique was to apply the dispersion theory of broken symmetries[7] to current commutators[8,9,10]. This soon led to a paper by de Alfaro, Fubini, Rossetti, and Furlan[11] in which superconvergence relations were obtained. The possibility of superconvergence was also noted independently by Soloviev[12] at about the same time. The work of de Alfaro et al.[11] was quite firmly based in the current algebra scheme and some of the subsequent work on superconvergence relations has retained this explicit basis in current algebras[13-18]. However, the fact that the possibility of superconvergence can be seen in a more general framework was soon utilised and the connections with current algebras overshadowed[19-43]. It has been suggested, nevertheless, that the connection between the results of superconvergence sum rules and the predictions of higher symmetries may actually be from current algebras[23].

As was mentioned above, superconvergence arises when parts of the scattering amplitude, or the T matrix, have suitable asymptotic behaviour. Some decision has therefore to be made about the way in which the T matrix will be separated into simpler elements. Two main ways of dealing with the scattering matrix for processes involving particles with spin have been developed. These are the helicity amplitude[44] and invariant amplitude[45] approaches. In the helicity amplitude approach, the concept of "helicity" as the component of a particle's spin in the direction of its momontumis called, is paramount. This approach has the advantage that general formulae for any process are readily obtained and that the physical cross-section is proportional to a direct sum of the squares of the helicity amplitudes which are incoherent. On the other hand, in the invariant amplitude approach, firstly a covariant M-function is introduced[46,47]. This is then decomposed into a sum of kinematic covariants multiplied by scalar (invariant) amplitudes. Invariant amplitudes have the advantage of simpler analytic and crossing properties at the expense of a more complex relation to physical cross-sections than that of the helicity amplitudes.

The choice between these alternative approaches has, on the whole, tended to favour helicity amplitudes. This is partly because of the relative ease of applying the Regge pole model to helicity amplitudes[48,49]. Although the method of Reggeizing invariant amplitudes by analytically continuing the partial wave expansion obtained from the consideration of the exchange of a high spin isobar[50] has been known for a long time, the difficulty of constructing the general high spin projection operators necessary for the partial wave expansion was not adequately resolved until recently[51-53]. Since the Jacob and Wick[44] helicity states are essentially angular momentum states, their partial wave analysis is immediate and the Reggeization procedure relatively straightforward. Consequently, soon after the rise in interest in superconvergence, Trueman[19] proposed a general method for superconvergence based on helicity amplitude techniques. However, much of the work on superconvergence sum rules has been done in the invariant amplitude formalism[21-24]. It is possible, in fact, that optimum convergence properties may be obtained from a combination of the two methods[54].

Since superconvergence depends on the analytic properties of the scattering amplitude, care must be taken that in the decomposition of the T matrix into helicity amplitudes or the M-function and invariant amplitudes, spurious kinematical singularities are not introduced. This is possibly the greatest single problem of either approach. The problem of constructing kinematic singularity free (KSF) helicity amplitudes has been essentially solved by Hara[55] and Wang[56] using the crossing relations for the helicity amplitudes[57,58]. The decomposition of the covariant M-function into KSF invariant amplitudes has been treated in a general way recently[59,60].

The idea is basically that the complete set of kinematic covariants as obtained from perturbation theory does not contain any kinematical singularities[45]. Therefore, as long as no explicit singularities are introduced in the reduction from the complete set to the independent set, the final independent invariant amplitudes will be free of kinematical singularities and zeros in any kinematic variable.

The invariant amplitude approach to superconvergence problems has been followed in this thesis for several reasons. Probably the most important of these is the simplicity of the analytic properties of the invariant amplitudes. Although it is possible to construct KSF helicity amplitudes, singularities in only one variable may be removed at any one time and the helicity amplitudes must satisfy certain constraint equations at thresholds, pseudothresholds[61-63], and in the forward direction. Also the helicity amplitudes appropriate to the different channels of a scattering process are not simply related and the crossing matrices are complicated functions which involve kinematical variables[57,58]. Further, the Lorentz transformation properties of the T matrix for particles with spin are complicated by the "reshuffling" of the helicity components by the Wigner rotations[66]. On the other hand, although the initial difficulty in the M-function approach of choosing the independent set of kinematic covariants such that the invariant amplitudes will be KSF may be great (since, unlike in the helicity formalism, there is no general formala for obtaining the correct set), once a KSF set of invariant amplitudes has been found, the problem of kinematic singularities and zeros has been solved completely. Also, the correct set of invariant amplitudes are KSF in both the energy and angle variables simultaneously; they are uncorrelated at thresholds, and the Lorentz transformation properties are simple since the M-function is an explicit Lorentz covariant, while the invariant amplitudes are, as the name implies, Lorentz scalars. Crossing properties are also much simpler; the invariant amplitudes cross one to one, and the crossing matrices relating the amplitudes from the M-function expansion in different channels can be made free of kinematic variables if the appropriate amplitudes are chosen[60]. This latter possibility is of great importance, both in guiding the choice of amplitudes in fermion-fermion scattering, and in the proof that the invariant amplitudes correctly chosen are free of kinematic singularities in both the s and t variables simultaneously.

In this thesis, kinematic singularity free amplitudes are obtained for both normal and abnormal parity $\frac{1}{2}+\frac{1}{2} \to \frac{1}{2}+\frac{1}{2}$ scattering as well as for the abnormal parity processes $\frac{1}{2}+\frac{1}{2} \to \frac{1}{2}+\frac{3}{2}$ and $0+1 \to 1+1$. Superconvergence relations are obtained for general $\frac{1}{2}^+ + \frac{1}{2}^+ \to \frac{1}{2}^+ + \frac{1}{2}^+$ and $0^- + 1^- \to 1^- + 1^-$ processes and an attempt is made to saturate these with low lying meson states. The resulting sum rules are compared with the predictions of the U(6,6) symmetry scheme[67–73] and some degree of correspondence is found.

The first three chapters are devoted to a general investigation of the covariant M-function and invariant amplitude approach to the scattering of particles with spin. General kinematical considerations and a review of the work of Scadron and Jones[59] are given in chapter 1, while in chapters 2 and 3 a detailed investigation of the problem in fermion-fermion scattering and a particular boson-boson scattering process is made. The method of obtaining the partial wave expansion for invariant amplitudes by considering general high spin projection operators[51] is reviewed in chapter 4. The procedure for obtaining the asymptotic behaviour of the invariant amplitudes by Reggeizing this expansion[52,53] is also discussed, with the results necessary for the superconvergence problems dealt with in chapter 5 obtained as examples.

There are several ways of obtaining physical consequences from superconvergence relations, and these are discussed in chapter 5. In the two particular superconvergence problems considered here, the results obtained by saturating the superconvergence relations with low lying meson states are compared with the predictions of the U(6,6) symmetry scheme for the coupling constants. The agreement is perhaps not as striking as earlier work may have suggested[23], but possible extensions of these results are discussed.

In two appendices, the general conventions used for the metric and γ matrices are given; useful identities stated, and coupling constants as predicted by the U(6,6) currents are given.

Finally, reprints of two of the author's papers are enclosed in the back pocket. The first of these, "A Limitation on the Use of Current Algebra Techniques for Meson Decays", shows that the conventional current algebra method for obtaining meson decay branching ratios is not immediately applicable to abnormal decays involving 0^- and 1^- mesons. Since this work is not directly related to the investigation of superconvergence relations in the main body of the thesis, it has not been rewritten in the text.

CHAPTER 1.

THE M-FUNCTION.

1.1 Kinematics.

It is well known that we can consider any two particle scattering process in three independent "channels", labelled by the Mandelstam invariants s,t,or u. The label indicates which of the three invariants represents the square of the centre of mass energy in a particular channel. In quantum electrodynamics, or in any scattering process where the interaction potential is known, it can be shown that the three channels are indeed equivalent and can each be described by the same analytic function. Passing from one channel to another, known as "crossing", may be effected by analytically continuing the scattering matrix elements of one channel into the kinematic domain of the crossed channel.

It is a fundamental assumption of strong interaction physics that, although it cannot be proved rigorously in a theory without a known potential, "crossing" is valid. In other words, we assume that the scattering of strongly interacting particles is described by the same analytic function in the s,t,and u channels. For this reason we will define kinematic quantities in each channel in a similar way. The analysis of any given problem may be done completely in terms of the set of kinematical quantities of just one channel, but in the following work it is found to be convenient to deal with different aspects of one scattering process in terms of different channels. Great simplification of the necessary crossing relations is achieved if the three channels are treated in such a way that they manifestly "look alike". This "look alike" principle guides us in the following work, and is found to be of great significance, especially in the analysis of fermion-fermion scattering.

Throughout this work we will use the metric $g_{00} = -g_{11} = -g_{22} = -g_{33} = +1$ and our convention for γ matrices will be that $\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}$. We define

$$\sigma_{\mu\nu} = \frac{i}{2}\,[\gamma_\mu,\gamma_\nu], \text{ and } \quad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3, \quad \gamma_5^2 = -1.$$

With the hermitian adjoint defined by $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$, it is clear that γ_0 and σ_{ij}, where $i,j = 1,2,3$, will be hermitian, while γ_5, γ_i, and σ_{0i} will be antihermitian. Several useful identities between the γ matrices are given in appendix A.

The s-channel for the scattering of particles of any spin is taken to be

$$s_1 + s_2 \rightarrow s_1' + s_2', \tag{1.1}$$

where s_1, s_2 etc. refer to the spins of the particles, either integral or half odd-integral. We describe the particles by Rarita-Schwinger type(74-77) wave functions and the structure of these wave functions is investigated in section 1.2. We have chosen to work with the Rarita-Schwinger formalism rather than the alternative SL(2,C) approach of Hepp[47] because of the extremely simple way in which discrete symmetries such as parity invariance can be incorporated in the former.

We identify the spin, helicity, momentum, mass, and possible Dirac spinor indices for the s-channel as shown in Fig. 1, where the labels are written in the order stated.

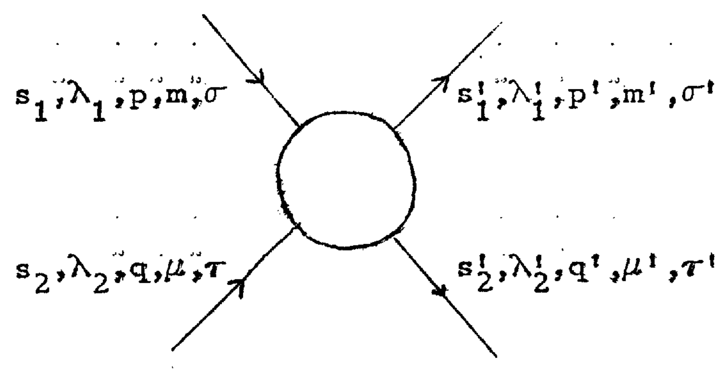

Fig.1. The s-channel.

With this convention for the momentum labels, we define the Mandelstam invariants s,t, and u by

$$s = (p + q)^2, \quad t = (p' - p)^2, \text{ and } \quad u = (p - q')^2.$$

If all four of the particles are fermions, some convention must be given for the ordering of the associated spinors. There is a "natural" way in which to do this because in many applications we wish to consider the exchange of single particle poles in the different channels. The calculation of such pole graphs involves the vertex functions connecting the external particles to the intermediate pole. These vertex functions are discussed further in chapter 4 and it is sufficient to say that the most convenient ordering of the spinors is that which defines the γ matrices in a way appropriate for the vertices in each channel. It is obvious that the "natural" definition of γ matrices for the s-channel is given by the following ordering of the spinors, suppressing any possible vector indices,

$$\bar{u}^{\sigma'}(p')(\gamma' C)_{\sigma'\tau'}\{\bar{u}^{\tau'}(q')\}^{T}\{u_{\tau}(q)\}^{T}(C^{-1}\gamma)^{\tau\sigma}u_{\sigma}(p), \qquad (1.2)$$

where T indicates the transpose, and C is the Dirac matrix such that $C^T = -C$ and $C\gamma_\mu^T C^{-1} = -\gamma_\mu$. In order that we may distinguish between the two independent sets of γ matrices without writing out the spinor indices explicitly, we have primed γ matrices defined between the final state spinors in the s-channel and left unprimed those defined between the initial spinors. The matrix $C_{\sigma\tau}$ acts as a lowering operator for spinor indices, and its inverse, $(C^{-1})^{\sigma\tau}$, acts as the corresponding raising operator. Using these matrices we can define the antiparticle wave functions with relations like

$$v_{\sigma}(p) = C_{\sigma\tau}\ \{\bar{u}^{\tau}(p)\}^{T}. \qquad (1.3)$$

As well as natural γ matrices for each channel, it is convenient to define natural momenta. These will be combinations of the separate particle momenta appropriate for the vertex functions mentioned above. For the s-channel we define the following natural momenta

$$\Lambda = \tfrac{1}{2}(p - q), \ \Lambda' = \tfrac{1}{2}(p' - q'), \quad K = p + q = p' + q'. \qquad (1.4)$$

The square of the centre of mass energy is then $K^2 = s$.

In accordance with our principle of making the channels look alike, we define natural momenta and γ matrices in the t-channel in a similar way. With reference to the s-channel defined in eqn. (1.1), we define the crossed t-channel as

$$s_1 + \bar{s}_1' \to \bar{s}_2 + s_2' \;, \tag{1.5}$$

where the bar indicates the antiparticle. The various labels are now identified as in Fig. 2.

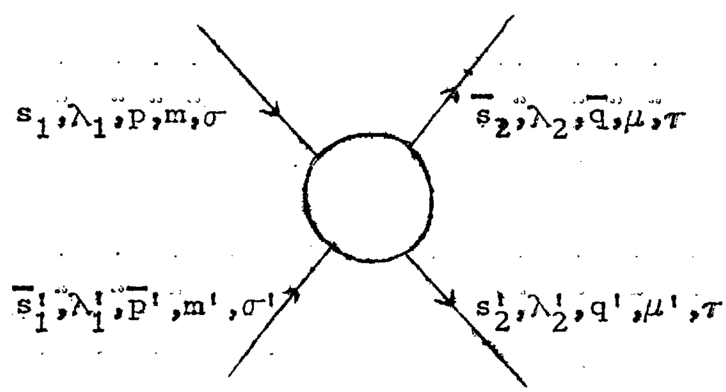

Fig. 2. The t-channel.

The natural definition of γ matrices for the t-channel is obtained by reordering the s-channel spinors of eqn.(1.2) according to

$$\bar{u}^{\tau'}(q')(\gamma^{(2)})_{\tau'}{}^{\tau}\, u_{\tau}(q)\; \bar{u}^{\sigma'}(p')(\gamma^{(1)})_{\sigma'}{}^{\sigma}\, u_{\sigma}(p), \tag{1.6}$$

where we have used superscripts (1) and (2) to distinguish these γ matrices from the s-channel set when spinor indices are suppressed. From (1.6) it is clear that the (1) refers to the γ matrices defined between initial state spinors and the (2) refers to the final state set.

The natural momenta for the t-channel corresponding to the set (1.4) are again defined as combinations of initial and final particle momenta:

$$P = \tfrac{1}{2}(p + p'), \quad Q = \tfrac{1}{2}(q + q'), \quad \Delta = p' - p = q - q', \tag{1.7}$$

with the square of the centre of mass energy in the t-channel now $\Delta^2 = t$.

For the u-channel, we can again define natural momenta and γ matrices. Taking the u-channel to be

$$s_1 + \bar{s}_2' \to s_1' + \bar{s}_2 , \tag{1.8}$$

we identify the various labels as in Fig.3

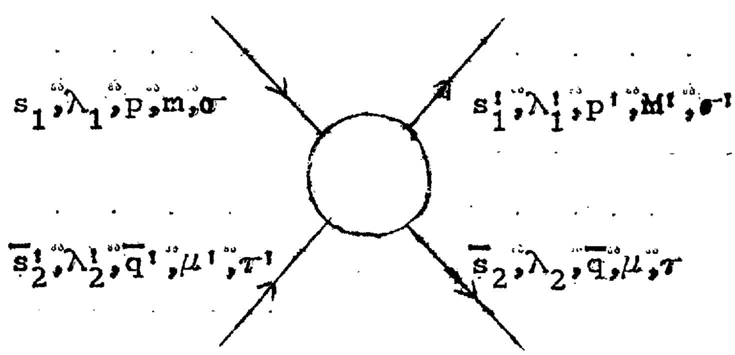

Fig.3. The u-channel.

We again reorder s-channel spinors for the definition of natural u-channel γ matrices in four fermion scattering problems. We distinguish the independent sets of γ matrices, defined between the initial and final spinors, by the superscripts (a) and (b) respectively according to

$$\bar{u}^{\sigma'}(p')(\gamma^{(b)})_{\sigma'\tau}\, u_{\tau}(q)\bar{u}^{\tau'}(q')(\gamma^{(a)})_{\tau'\sigma}\, u_{\sigma}(p). \qquad (1.9)$$

Finally, the corresponding u-channel natural momenta are defined as the combinations

$$\bar{\Lambda} = \tfrac{1}{2}(p+q'),\ \bar{\Lambda}' = \tfrac{1}{2}(p'+q),\quad \bar{K} = p-q' = p'-q\ . \qquad (1.10)$$

The square of the centre of mass energy in the u-channel is now $\bar{K}^2 = u$.

It is to be noted that any one of these three sets of natural momenta and γ matrices is sufficient for the complete determination of the kinematics of a given scattering process. The various quantities defined are not independent and in particular we have the relation

$$s + t + u = m^2 + m'^2 + \mu^2 + \mu'^2. \qquad (1.11)$$

Also, there are equations expressing the natural momenta of any channel in terms of those for the other channels. For example, we can see that

$$\begin{aligned} \Lambda &= \tfrac{1}{2}(P - Q - \Delta) &&= \tfrac{1}{2}(\bar{\Lambda} - \bar{\Lambda}' + \bar{K}), \\ \Lambda' &= \tfrac{1}{2}(P - Q + \Delta) &&= \tfrac{1}{2}(\bar{\Lambda}' - \bar{\Lambda} + \bar{K}), \\ K &= P + Q &&= \bar{\Lambda} + \bar{\Lambda}'\ . \end{aligned} \qquad (1.12)$$

Various products of the natural momenta in each channel, being scalars, are simply combinations of the masses and the Mandelstam invariants. In the s-channel we have

$$\Lambda.\Lambda' \equiv \tilde{\nu} = \tfrac{1}{4}(t - u),$$
$$\Lambda.K = \tfrac{1}{2}(m^2 - \mu^2), \qquad \Lambda'.K = \tfrac{1}{2}(m'^2 - \mu'^2), \tag{1.13}$$
$$\Lambda^2 = \tfrac{1}{2}(m^2 + \mu^2) - \tfrac{1}{4}s, \; \Lambda'^2 = \tfrac{1}{2}(m'^2 + \mu'^2) - \tfrac{1}{4}s.$$

For the t-channel momenta we have

$$P.Q \equiv \nu = \tfrac{1}{4}(s - u),$$
$$P.\Delta = \tfrac{1}{2}(m'^2 - m^2), \qquad Q.\Delta = \tfrac{1}{2}(\mu^2 - \mu'^2), \tag{1.14}$$
$$P^2 = \tfrac{1}{2}(m'^2 + m^2) - \tfrac{1}{4}t, \quad Q^2 = \tfrac{1}{2}(\mu'^2 + \mu^2) - \tfrac{1}{4}t,$$

and similarly for the u-channel,

$$\bar{\Lambda}.\bar{\Lambda}' \equiv \bar{\nu} = \tfrac{1}{4}(s - t),$$
$$\bar{\Lambda}.\bar{K} = \tfrac{1}{2}(m^2 - \mu'^2), \qquad \bar{\Lambda}'.\bar{K} = \tfrac{1}{2}(m'^2 - \mu^2), \tag{1.15}$$
$$\bar{\Lambda}^2 = \tfrac{1}{2}(m^2 + \mu'^2) - \tfrac{1}{4}u, \quad \bar{\Lambda}'^2 = \tfrac{1}{2}(m'^2 + \mu^2) - \tfrac{1}{4}u.$$

Finally we define the normal vector N_μ as

$$N_\mu = \varepsilon_\mu(PQ\Delta), \tag{1.16}$$

where we have used the abbreviation $\varepsilon_\mu(PQ\Delta) = \varepsilon_{\mu\nu\lambda\rho}P^\nu Q^\lambda \Delta^\rho$, and $\varepsilon_{\mu\nu\lambda\rho}$ is the four-dimensional Levi-Civita completely antisymmetric tensor defined in appendix A. We can express N_μ in terms of the natural momenta of the other channels as

$$N_\mu = -\varepsilon_\mu(\Lambda'K\Lambda) = \varepsilon_\mu(\overline{\Lambda}'\overline{K\Lambda}) = -\varepsilon_\mu(p'Qp). \tag{1.17}$$

One important feature of N_μ is that $-N^2$ is the determinant of Kibble[78] and $N^2 = 0$ determines the boundary of the physical region. The Kibble determinant is related to the centre of mass scattering angles by

$$-\frac{N^2}{s} = p_s^2\, p_s'^2 \sin^2\theta_s , \tag{1.18}$$

and similarly for the t and u channels.

1.2 Definition of the M-function.

As was mentioned in the previous section, we shall use the Rarita-Schwinger[74] formalism to describe particles of any spin. In this formalism, the wave function for any particle of integral or half odd-integral spin is made up from elementary spin 1 and spin $\frac{1}{2}$ wave-functions. For a boson of spin $s = J$, the covariant wave-function is given by

$$\varepsilon^{(\lambda)}_{\alpha_1\ldots\alpha_J}(p) = \sum_{\lambda_1\ldots\lambda_J} \langle\lambda_1\ldots\lambda_J|J,\lambda\rangle \varepsilon^{(\lambda_1)}_{\alpha}(p)\ldots\varepsilon^{(\lambda_J)}_{\alpha_J}(p), \tag{1.19}$$

where $\langle\lambda_1\ldots\lambda_J|J,\lambda\rangle$ is the "parallel coupling coefficient" defined by[79,80]

$$\langle\lambda_1\ldots\lambda_J|J,\lambda\rangle = \left[2^{J-\bar{\lambda}}\,\frac{(J+\lambda)!\,(J-\lambda)!}{(2J)!}\right]^{\frac{1}{2}} \delta_{\bar{\lambda},\lambda}\ , \quad (1.20)$$

with $\bar{\lambda} = \Sigma|\lambda_i|$. In these formulae, as before, λ is the helicity label for the wave-functions.

For a fermion, we have a residual $\frac{1}{2}$-integral spin which we will describe by a Dirac four-component spinor $u^{(\lambda_{\frac{1}{2}})}(p)$. The general fermion wave-function can then be written(77)

$$u^{(\lambda)\sigma}_{\alpha_1\ldots\alpha_J}(p) = \sum_{\lambda_1\ldots\lambda_J\lambda_{\frac{1}{2}}} \langle\lambda_1\ldots\lambda_J\lambda_{\frac{1}{2}}|J+\tfrac{1}{2},\lambda\rangle \times$$

$$\times\ \varepsilon^{(\lambda_1)}_{\alpha_1}(p)\ldots\varepsilon^{(\lambda_J)}_{\alpha_J}(p)\ u^{(\lambda_{\frac{1}{2}})\sigma}(p), \quad (1.21)$$

where J is now the integral part of s.

The phase convention of Jacob and Wick(44) is followed so that the antiparticle wave-functions are given by

$$\begin{aligned} \varepsilon^{*(\lambda)}_{\alpha}(p) &= -\ \xi_\lambda\ \varepsilon_\alpha^{(-\lambda)}(p), \\ v^{(\lambda)}(p) &= C\ \bar{u}^{T(\lambda)}(p) \qquad\qquad (1.22) \\ &= i\gamma_5\xi_\lambda\ u^{(-\lambda)}(p), \end{aligned}$$

where $\xi_\lambda = (-)^{s-\lambda}$.

It is easily seen that the wave-functions (1.19) and (1.21) are symmetric. The required tracelessness condition follows from the form of the coupling coefficient (1.20). Also, the construction of these expressions from elementary wave functions which separately satisfy the subsidiary conditions

$$p.\varepsilon(p) = (p.\gamma - m)u(p) = \gamma.u(p) = 0,$$

ensures that the wave functions (1.19) and (1.21) satisfy the appropriate conditions.

If we now define the usual T matrix in terms of the scattering matrix S by

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \, \delta^{(4)}(p_f - p_i) \, T_{fi}, \qquad (1.23)$$

where i and f refer to the initial and final states respectively, we can define the M-function from the T matrix by

$$< s_1^{\prime}s_2^{\prime};\lambda_1^{\prime}\lambda_2^{\prime} \mid T \mid s_1s_2;\lambda_1\lambda_2> \equiv \bar{\Psi}^{(\lambda_1^{\prime})}\bar{\Psi}^{(\lambda_2^{\prime})}.\mathcal{M}.\Psi^{(\lambda_1)}\Psi^{(\lambda_2)}. \qquad (1.24)$$

In eqn. (1.24) the dots indicate contraction over all vector and spinor indices. Throughout this work we have used the covariant normalization for particle states so that

$$< \underline{p}'\alpha' \mid \underline{p}\alpha> = 2E(2\pi)^3\delta^{(3)}(\underline{p}' - \underline{p}) \, \delta_{\alpha'\alpha},$$

and $\bar{u}(\underline{p})\; u(\underline{p}) = 2m$, where $\underline{p}$ is the three-momentum vector.

The S matrix is known to be invariant under transformations of the inhomogeneous Lorentz group while the T matrix defined in (1.23) is invariant under the homogeneous Lorentz group. The advantage of defining an M-function as in eqn.(1.24) is that it is a covariant function transforming like a tensor-spinor under Lorentz transformations. We could have obtained a similarly simple function by following the SL(2,C) formalism of Hepp[47], but we have chosen to work in the four-component Rarita-Schwinger formalism because of the simplicity of incorporating discrete symmetries and the relatively greater familiarity with applying Rarita-Schwinger wave functions to physical particles. This does lead to some loss of elegance in that we must have subsidiary conditions in order to eliminate the redundant components of the formalism, but we prefer to use the more familiar wave-functions with covariant vectors and Dirac spinors.

Our approach to the covariant M-function defined in eqn. (1.24) is determined by the possibility of defining a set of scalar (invariant) amplitudes. The Lorentz transformation properties of $\mathcal{M}$ must be the same as those of appropriate combinations of momentum vectors and γ matrices. In general, there will be as many such independent combinations of momenta and γ matrices as there are helicity amplitudes for the process, namely $\Pi(2s_i + 1)$, where s_i are the spins of the external particles. Discrete symmetries, such as parity invariance, will reduce this number for any given process, but there will always be a well-defined number of <u>independent</u> kinematic covariants. We construct these covariants from the natural momenta and γ matrices defined in section 1.1 and, as may have been expected, there are three ways of obtaining an independent set, one way for the natural kinematics of each channel. It must be remembered that the three sets of kinematic covariants are equivalent, and we introduce covariants natural to each channel for convenience only.

We can, therefore, expand the M-function as a linear combination of these kinematic covariants. For example, in the t-channel

$$\mathcal{M}_{\beta;\alpha} = \sum_{\kappa} A_{\kappa}^{(t)}(s,t)\, \mathcal{K}^{\kappa}_{\beta;\alpha}(P,Q) \,, \qquad (1.25)$$

where α,β refer to the vector and spinor indices of the initial and final states respectively. The invariant amplitudes $A_{\kappa}^{(t)}(s,t)$ have a superscript (t) to indicate that they are defined as the coefficients of natural t-channel covariants and they are scalar functions of the Mandelstam variables s and t only. The alternative helicity amplitudes can also be expressed as linear combinations of the invariant amplitudes $A_{\kappa}^{(t)}$, but it is of great importance to notice that the corresponding helicity amplitudes in different channels are given by <u>different</u> combinations of the $A_{\kappa}^{(t)}$. The crossing properties of the helicity amplitudes are, therefore, complicated, but the invariant amplitudes cross one to one. In the different channels we merely have different ranges for the arguments s and t.

The M-function is, however, channel independent, and we can also expand it in terms of the natural s-channel covariants :

$$\mathcal{M}_{\beta;\alpha} = \sum_{\kappa} A_{\kappa}^{(s)}(s,t)\, \mathcal{K}^{\kappa}_{\beta;\alpha}(\Lambda,\Lambda'), \qquad (1.26)$$

and similarly for the u-channel. The above remarks about the t-channel amplitudes apply equally to those for the s and u channels as the expansions are completely equivalent. As we stated previously, a guiding principle in our formulation of a problem in the various channels is to make them "look alike" and this principle leads to simple relations between the covariants and amplitudes for different channels. We express these relations which give the amplitudes of one channel in terms of the amplitudes of the other channels as crossing matrices. These crossing matrices are distinct from the crossing properties of the amplitudes in the sense that they arise only because, for convenience, we introduce alternative expansions of our M-function. The amplitudes have crossing properties independent of the channel in which they are defined and of whether amplitudes are defined for expansions in other channels.

At this point we can introduce a third concept related to the crossing properties of the invariant amplitudes. This is crossing symmetry which arises when two of the particles are identical. Crossing symmetry is then a consequence of our principle of making the different channels look alike as far as is possible. From the definitions of the three channels it is clear that if, for example, the t-channel final state consists of two identical self-conjugate bosons, then the s and u channels are identical and crossing $s \leftrightarrow u$ exhibits a simple odd or even property of the t-channel invariant amplitudes. Crossing symmetry will be discussed in greater detail later.

Finally, we come to the problem of kinematic singularities. This is, perhaps, the greatest single problem in any discussion of the kinematics of the T matrix. It is known that the T matrix is directly related to the physical cross-section for a process and that such cross-sections are finite and smooth. The singularities of T are solely of dynamical origin and we must be very careful to ensure that in our discussion of the T matrix in terms of an invariant amplitude expansion of the M-function, we do not introduce spurious kinematical singularities. Extracting the external particle wave-functions in the definition (1.24) does not introduce singularities, but the expansions (1.25) and (1.26) may. For example, a kinematic covariant could go to zero and the corresponding amplitude go to infinity in such a way that the product is finite and smooth. Such a situation is undesirable and must be avoided.

Much of the work in this thesis is devoted to the problem of choosing an independent set of kinematic covariants such that the invariant amplitudes are free of kinematical singularities (KSF). Because we use the Rarita-Schwinger formalism, our wave-functions have redundant components and satisy subsidiary conditions. This in turn implies that in general it is possible to construct more covariants than the known number in the independent set. The subsidiary conditions imply various relations between these covariants known as equivalence theorems. In choosing our independent set from the complete set, we must ensure that no spurious kinematical singularities are introduced. The general method for doing this is discussed in section 1.4 and it is shown in 1.5 that if the procedure is correctly followed, the resulting invariant amplitudes are KSF in both the s and t variables simultaneously.

1.3 Symmetry Transformations on M-Functions.

Before beginning the investigation of the general method for obtaining KSF invariant amplitudes, we shall consider some general properties that must be shown by the kinematic covariants. For any given scattering process, discrete symmetries such as parity, charge conjugation, and time reversal invariance will restrict the possible forms which covariants can take. The conditions which arise from these discrete symmetry operations give us a guide for obtaining the complete set from which the independent set is chosen. We also investigate the consequences of the condition that the S matrix be unitary.

The discrete symmetry most obvious in its effects is parity. Each particle in a scattering process has an intrinsic parity and it has been found convenient to define the concept of "normality" in order to describe the relation between spin and parity for a particle. For a particle of spin J or $J + \frac{1}{2}$, we define the normality as $n = (-)^J$ times the intrinsic parity[51,81]. A normal particle is then one which has $n = 1$. This is ensured if the particle has intrinsic parity $(-)^J$ so that, for bosons, the normal particles have the spin-parity $J^P = 0^+, 1^-, 2^+$, etc and normal fermions have $J^P = \frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+$, etc. Abnormal particles have $n = -1$ and consequently for the same spin, they have the opposite intrinsic parity to the normal particles. Abnormal bosons come in the series $J^P = 0^-, 1^+, 2^-$, etc and abnormal fermions in $J^P = \frac{1}{2}^-, \frac{3}{2}^+, \frac{5}{2}^-$, etc.

We can extend the concept of normality to three and four point functions by defining the normality of a vertex or scattering process to be the product of the normalities of the individual particles. A normal vertex then has $n_v = n_1 n_2 n_3 = 1$, while an abnormal vertex is one with $n_v = -1$. Similarly, a normal scattering process has $n_s = n_1 n_2 n_3 n_4 = 1$ and an abnormal process has $n_s = -1$.

Under the action of the parity operator, the helicity states of individual particles transform as

$$|\underline{p}\lambda> \rightarrow \eta_p \xi_\lambda \, |-\underline{p},-\lambda>$$

where η_p is the intrinsic parity; $\xi_\lambda = (-)^{s-\lambda}$ as before, and $\underline{p}$ is the momentum which is taken, for convenience, to lie in the 1-3 plane. The statement of parity invariance for helicity amplitudes is, therefore,

$$<\underline{p}_1',\lambda_1';\underline{p}_2',\lambda_2' \,|\, T \,|\, \underline{p}_1,\lambda_1;\underline{p}_2,\lambda_2> \;=$$
$$= n_s(-)^{\lambda' - \lambda} <-\underline{p}_1',-\lambda_1';-\underline{p}_2',-\lambda_2' \,|\, T \,|\, -\underline{p}_1,-\lambda_1;-\underline{p}_2,-\lambda_2>, \quad (1.27)$$

where $\lambda = \lambda_1 - \lambda_2$ and $\lambda' = \lambda_1' - \lambda_2'$, and n_s is the overall normality for the scattering process.

If we now relate the parity reversed wave-functions to the original wave-functions by

$$\varepsilon_{\alpha}^{(-\lambda)}(-\underline{p}) = -\xi_{\lambda}\, g_{(\alpha)}\varepsilon_{\alpha}^{(\lambda)}(\underline{p}),$$
$$u^{(-\lambda)}(-\underline{p}) = \xi_{\lambda}\, \gamma_0 u^{(\lambda)}(\underline{p}), \tag{1.28}$$

where $g_{(\alpha)} = (1,-1,-1,-1)$, we can rewrite (1.27) as a parity statement on the M-function:

$$\mathcal{M}_{\beta\alpha}(f,i) = n_s g_{(\alpha)} g_{(\beta)} \gamma_0 \mathcal{M}_{\beta\alpha}(Pf,Pi)\gamma_0. \tag{1.29}$$

From (1.29) we can see that if the reaction is normal, $\mathcal{M}$ should be expanded in terms of a set of proper tensors which we shall call $\mathcal{K}^+$, whereas for an abnormal reaction, $\mathcal{M}$ must be expanded in terms of a set of pseudotensors $\mathcal{K}^-$. Pseudotensors contain one overall γ_5 or one overall Levi-Civita tensor $\varepsilon_{\mu\nu\lambda\rho}$.

Because of this division of a reaction with given external spins into two distinct categories by the condition of parity invariance, the number of independent covariants for a particular reaction is much smaller than the possible $N = \Pi(2s_i + 1)$. For boson-fermion (BF) or fermion-fermion (FF) reactions, $N^{\pm} = \frac{1}{2}N$, where N^+ is the number of independent covariants in the normal case and N^- the corresponding number for the abnormal case. For boson-boson (BB) reactions we find that $N^+ = \frac{1}{2}(N + 1)$ and $N^- = \frac{1}{2}(N - 1)$. This counting agrees with the number of independent helicity amplitudes under parity conservation in the various cases.

In general, the conditions of time-reversal invariance and charge conjugation invariance do not reduce the number of covariants required to describe the process; they merely relate one process to another. However, when we have elastic scattering, $s + s' \to s + s'$, or superelastic scattering, $s + s \to s + s$, where all four particles are identical, these conditions do, in fact, reduce the number of covariants. We shall discuss this effect in more detail in our investigation of normal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ scattering in section 2.1.

By considering the time-reversal invariance condition on helicity amplitudes, we find for the M-function the corresponding condition:

$$\mathcal{M}_{\beta\alpha}(f,i) = \eta_T g_{(\beta)} g_{(\alpha)}\; T^{-1} \mathcal{M}^T_{\alpha\beta}(Ti,Tf)\; T, \qquad (1.30)$$

where η_T is the product of the time-reversal phases of the four particles, the superscript T indicates the transpose, and the matrix T is the Dirac matrix such that $T\gamma_\mu T^{-1} = g_{(\mu)}\gamma_\mu^T$. It is clear from this definition that T is related to the charge conjugation matrix by $T = i\gamma_0\gamma_5 C^{-1}$.

We can now combine the parity and time-reversal conditions to obtain the condition for invariance under the combined transformation PT:

$$\mathcal{M}_{\beta\alpha}(f,i) = n_s\, \eta_T\, B^{-1}\, \mathcal{M}^T_{\alpha\beta}(i,f)\, B, \qquad (1.31)$$

where B is the Dirac matrix such that $B\gamma_\mu B^{-1} = \gamma_\mu^T$, and thus $B = i\gamma_5 C^{-1}$.

Using relations like (1.3), we obtain the condition for charge conjugation invariance for the M-function:

$$\mathcal{M}_{\beta\alpha}(f,i) = \eta_C(-C)\,\mathcal{M}^T_{\beta\alpha}(Cf,Ci)\;C^{-1}, \qquad (1.32)$$

where η_C is the product of the four charge conjugation phases. Combining eqns. (1.31) and (1.32), we get the relation for the CPT transformation:

$$\mathcal{M}_{\beta\alpha}(f,i) = n_s\eta_C\eta_T(-i\gamma_5)\;\mathcal{M}_{\alpha\beta}(Ci,Cf)(i\gamma_5). \qquad (1.33)$$

Comparing (1.33) with the result of crossing all four particles[59], we get the result that the overall C,P,T phases are constrained to satisfy $\eta_P\eta_C\eta_T = +1$.

We now consider the unitarity condition in relation to invariant amplitudes. If the S matrix is unitary, ie. $SS^\dagger = 1$, it is well known that we get a discontinuity condition for the T matrix defined in eqn. (1.23) which reads

$$-\tfrac{1}{2}i(T_{fi} - T^*_{if}) = \tfrac{1}{2}\sum_n T^*_{nf}\,T_{ni}\,(2\pi)^4\delta^{(4)}(p_f - p_n), \quad (1.34)$$

where the sum is over all intermediate states n. The left-hand side of (1.34) is the imaginary part of the T matrix, ImT, and corresponds to the discontinuity across the unitarity cut. This is an analytic condition independent of the kinematics of the problem so we would like to be able to chose our kinematic covariants in the M-function expansion so that this discontinuity corresponds directly to the discontinuities of the invariant amplitudes.

The condition (1.34) becomes, in terms of the M-function

$$-\tfrac{1}{2}i\{\mathcal{M}_{\beta\alpha}(f,i) - \overline{\mathcal{M}}_{\alpha\beta}(i,f)\} =$$

$$= \tfrac{1}{2}\sum_n \overline{\mathcal{M}}_{\beta}(n,f)\,\mathcal{P}_n\,\mathcal{M}_{\alpha}(n,i)\;(2\pi)^4\delta^{(4)}(p_f - p_n), \qquad (1.34a)$$

where $\mathcal{P}_n$ is the product of the spin projection operators for the intermediate state and $\overline{\mathcal{M}} = \gamma_0\mathcal{M}^\dagger\gamma_0$. Using the PT invariance relation (1.31) and the result $\eta_P\eta_C\eta_T = 1$, we find that

$$\{\mathcal{M}_{\beta\alpha}(f,i) - \overline{\mathcal{M}}_{\alpha\beta}(i,f)\} = \{\mathcal{M}_{\beta\alpha}(f,i) - \eta_C B^{-1}\,\overline{\mathcal{M}}^T_{\beta\alpha}(f,i)B(-)^J\}, \qquad (1.35)$$

where J is the total number of vector indices on the M-function.

It follows directly from eqns. (1.34) to (1.35) that if we always choose our kinematic covariants to satisfy

$$\overline{\mathcal{K}}_{\beta\alpha}(P,Q) = \eta_C\,B^{-1}\mathcal{K}^T_{\beta\alpha}(P,Q)\,B \times (-)^J, \qquad (1.36)$$

the unitarity relation can be cast in the form

$$\sum_\kappa(\mathrm{Im}\,A_\kappa)\,\mathcal{K}_{\beta\alpha}(f,i) = \tfrac{1}{2}\sum_n \overline{\mathcal{M}}_{\beta}(n,f)\,\mathcal{P}_n\mathcal{M}_{\alpha}(n,i)(2\pi)^4\delta^{(4)}(p_f-p_n) \qquad (1.37)$$

which has the discontinuity entirely in the invariant amplitudes as desired. Equation (1.36) is called the discontinuity condition[59].

1.4 Obtaining KSF Invariant Amplitudes.

We now return to the central problem: how do we find a set of kinematic covariants so that the invariant amplitudes are kinematic singularity free? The discrete symmetry conditions of section 1.3 guide us in our choice of possible covariants, but because of the inherent redundancy of components in the four-component formalism, we can, in general, write down more covariants than the number known to be independent. Some obvious reductions in the number of covaraints follow immediately from the subsidiary conditions on the Rarita-Schwinger wave-functions. For instance, the tracelessness condition excludes metric tensors contracting two indices on the same wave-function, and from the condition

$$p_\mu \, \varepsilon^\mu(p) = 0, \tag{1.38}$$

we get immediately that $\Delta_\mu = 2P_\mu$. Such reductions obviously do not introduce kinematical singularities and since they are so trivial, they will not be mentioned further.

Again, we can often eliminate further covariants by conditions that Scadron and Jones[59] call "abnormal reductions". Abnormal reductions are simply identities arising, not from the subsidiary conditions directly, but from redundancies of the sort contained in $\gamma_5 N_\mu$ or $N_\mu N_\nu$ where we multiply two abnormal-type covariants together to form a possible normal covariant. The value of such abnormal reductions is that they are reasonably obvious and there is no doubt as to which covariant must be eliminated by the identity - it is always the one most complicated in the sense of containing the largest number of momenta, and expressions like $\gamma_5 \varepsilon_{\mu\nu\lambda\rho}$.

However, even after we have used the subsidiary conditions in all obvious ways and employed the abnormal reductions, we may still have redundant covariants. These must be eliminated by the "equivalence theorems"[59]. Equivalence theorems depend in more complicated ways on the subsidiary conditions and there is no gendral formula for finding them for all problems. The relevant equivalence theorems must be found separately for each new problem, although the work is often simplified by noticing features of simpler problems which may be carried across into more difficult situations. Equivalence theorems relate covariants for which there is no a priori reason for choosing one in preference to any other. This makes the choice of a set of kinematic covariants which will give KSF amplitudes difficult, but there are certain procedures which, if followed, make the correct choice possible. Equivalence theorems will involve kinematic variables such as ν and t as well as masses, and it is clear that if we use an equivalence theorem to eliminate a covariant which is multiplied by such a variable in the theorem, we must divide by a factor which will give rise to singularities. It is, therefore, taken as a general principle that only those covariants in the equation which are multiplied by constants or constant mass factors may be eliminated.

Another general principle which guides the choice of the correct set of covariants is that the crossing matrices should not involve kinematic variables such as ν and t. This principle is of great importance in high spin FF scattering processes where there are many equivalence theorems and the crossing matrices are very complicated. We investigate this problem in detail in chapter 3.

In chapters 2 and 3 we also require relations which are obtained in ways similar to the equivalence theorems, but these relations do not involve kinematic variables. We could, therefore, eliminate any of the covariants occurring in them except that care must be taken not to divide by mass differences where the masses may become equal. We call these relations "spinor identities" because they usually arise from applications of the Dirac equation. They are important in FF scattering, but do not cause ambiguities because the condition of constant crossing matrices is sufficiently stringent to make the choice of the correct covariants reasonable obvious.

With these conventions for the elimination of redundant covariants, Scadron and Jones[59] classify M-functions according to four types. M-functions of the first kind are those for the basic reactions (in spin space): both normal and abnormal $0 + 0 \rightarrow 0 + s'_B$ scattering, and both normal and abnormal $\frac{1}{2} + 0 \rightarrow s'_F + 0$ scattering, where s'_B and s'_F can take any integer or half odd-integer values respectively. For such reactions, using the trivial constraints from the subsidiary conditions, one can write down only the correct number of covariants; there are no redundancies; and no equivalence theorems are needed. For example, normal $0 + 0 \to 0 + 1$ has the covariants P_μ and Q_μ, while the abnormal case has only N_μ. Similarly, $\frac{1}{2} + 0 \to \frac{1}{2} + 0$ has the normal covariants 1 and $Q.\gamma$, $P.\gamma$ being trivially eliminated by the Dirac equation. The abnormal covariants for this reaction are simply γ_5 and $\gamma_5 Q.\gamma$. The possible normal covariant $\gamma_5 N.\gamma$ is eliminated by the abnormal reduction

$$\gamma_5 N.\gamma = 2(P^2 - \tfrac{1}{4}(m-m')^2)Q.\gamma - (m+m')\nu - \tfrac{1}{2}(m-m')Q.\Delta, \quad (1.39)$$

which is obtained by saturating eqn. (A.5) with P, Q, Δ, and γ.

M-functions of the second kind are those for the basic reactions: both normal and abnormal $0 + 1 \to 0 + s_B'$, and both normal and abnormal $\frac{1}{2} + 0 \to \frac{1}{2} + s_B'$, obtained essentially by compounding reactions with M-functions of the first kind. For these reactions we require, at most, abnormal reductions in order to get the independent set of covariants. For example, the normal reaction $0 + 1 \to 0 + 1$ has covariants $P_\mu P_\nu$, $Q_\mu Q_\nu$, $P_\nu Q_\mu$, $P_\mu Q_\nu$, and $g_{\mu\nu}$, where the possible $N_\mu N_\nu$ is eliminated by the abnormal reduction

$$N_\mu N_\nu = N^2 g_{\mu\nu} + (tQ^2 - (Q.\Delta)^2) P_\mu P_\nu + (P.\Delta Q.\Delta - t\nu)\{P,Q\}_{\mu\nu}$$
$$- 2(\nu Q.\Delta - Q^2 P.\Delta)[P,Q]_{\mu\nu} + \{4(\nu^2 - P^2Q^2) + tP^2 - (P.\Delta)^2\}Q_\mu Q_\nu. \quad (1.40)$$

Similarly, for normal $\frac{1}{2} + 0 \to \frac{1}{2} + 1$, we have the six covariants P_μ, Q_μ, $P_\mu Q.\gamma$, $Q_\mu Q.\gamma$, γ_μ, and $i\sigma_{\mu\nu}Q^\nu$. Further examples of M-functions of the first and second kinds are given, with the appropriate abnormal reductions, by Scadron and Jones[59].

M-functions of the third and fourth kinds are those for which redundant covariants remain after all trivial relations and abnormal reductions are employed. Thus we require equivalence theorems in order to obtain the independent set of covariants. The distinction between M-functions of the third and fourth kinds is that for the fourth kind, the number of equivalence theorems becomes of the same order of magnitude as the number of independent covariants while for the third kind, the number of equivalence theorems is small. M-functions of the third kind arise for reactions like $\frac{3}{2} + 0 \to \frac{3}{2} + 0$, $\frac{1}{2} + 0 \to \frac{3}{2} + 1$, $\frac{1}{2} + 1 \to \frac{1}{2} + 1$, and $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ in both the normal and abnormal parity cases. Normal BB reactions such as $0 + 2 \to 0 + 2$, $0 + 1 \to 1 + 2$, and $1 + 1 \to 1 + 1$ also have M-functions of the third kind. Scadron and Jones have given a general treatment of the equivalence theorems for M-functions of the third kind and several examples occur in the literature[21-23,28-30]. We investigate both normal and abnormal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ scattering in chapter 2.

Higher spin BF and FF reactions in general have M-functions of the fourth kind, as do abnormal BB processes like $0 + 1 \to 1 + 1$ [24] investigated in chapter 3. We also investigate the M-function of the fourth kind for abnormal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{3}{2}$ [60] in chapter 3.

2.5 Proof of Freedom from Kinematic Singularities.

In section 1.4 the general procedure for obtaining KSF invariant amplitudes was outlined. It could be argued that a disadvantage of the approach followed is that no general formula can be found which will give the correct covariants for all processes with spin; the choice has to be made by reference to the equivalence theorems which must be derived separately for each process. It has not, indeed been proved rigorously that it is possible, for any general process, to make a choice of covariants such that the amplitudes are KSF. However, in the work of Scadron and Jones, and that contained in this thesis, general techniques are employed to obtain KSF amplitudes for a wide range of processes, including the most difficult type of process with M-functions of the fourth kind. For a further range of processes the KSF amplitudes are trivially obtained by simple generalizations of this work, so the conjecture that it is always possible to choose KSF amplitudes appears to be reasonable.

The claim that the amplitudes obtained by the techniques outlined in the previous section are KSF in both s and t simultaneously, is based, to some extent, on the work of Hearn[45]. It is considered that the <u>complete</u> set of covariants which would arise from perturbation theory have KSF amplitudes. Consequently, if no explicit singularities are introduced when we reduce this complete set to an <u>independent</u> set, the independent set of amplitudes will also be KSF. Thus, the procedure which allows only those covariants which appear in the equivalence theorems multiplied by constants (or constant mass factors) to be eliminated, manifestly does not introduce kinematic singularities. Some effort may be required in order to cast the equivalence theorems into a form in which this procedure may be followed. For example, in normal $1 + 1 \rightarrow 1 + 1$ scattering considered by Scadron and Jones, we need to take rather obscure combinations of simple covariants in order to get two independent combinations multiplied by masses only.

We could also show the KSF structure of invariant amplitudes by relating them to the helicity amplitudes. According to Hara[55] and Wang[56], the singularities of $T^{(t)}_{\lambda'\lambda}$ in s are entirely contained in the factors

$$(\cos \tfrac{1}{2}\theta_t)^{|\lambda' + \lambda|} (\sin \tfrac{1}{2}\theta_t)^{|\lambda'-\lambda|} ,$$

so that the reduced helicity amplitude

$$\bar{T}^{(t)}_{\lambda'\lambda} = (\cos \tfrac{1}{2}\theta_t)^{-|\lambda'+\lambda|} (\sin \tfrac{1}{2}\theta_t)^{-|\lambda'-\lambda|} \; T^{(t)}_{\lambda'\lambda}$$

is KSF in s.

We can now find the relation between the helicity amplitudes and the invariant amplitudes by considering the expansion of the M-function in terms of the t-channel covariants and forming the helicity amplitudes according to

$$T^{(t)}_{\lambda'\lambda}(s,t) = \sum_{\kappa} A^{(t)}_{\kappa}(s,t) \langle\lambda'|\mathcal{K}_{\kappa}(P,Q)|\lambda\rangle \ . \tag{1.41}$$

If we now consider the t-channel centre of mass frame, we find the conditions

$$\varepsilon^{(\pm)}(q').Q = \varepsilon^{(\pm)}(q).Q = \varepsilon^{(\pm)}(p').P = \varepsilon^{(\pm)}(p).P = 0;$$

and $\bar{v}^{(\pm)}(p')\ u^{(\pm)}(p) = 0,$

from which we find that the helicity amplitude for maximum helicity-flip, $\bar{T}_{(\Delta\lambda)_{max}}$, is proportional to just one invariant amplitude. It has the correct angular factors $(\cos\frac{1}{2}\theta_t)^{|\lambda'+\lambda|}(\sin\frac{1}{2}\theta_t)^{|\lambda'-\lambda|}$ so that the invariant amplitude is KSF in s. Similarly, the next reduced helicity amplitude will be proportional to a linear combination of the first invariant amplitude and another one, from which we conclude that the second invariant amplitude is also KSF in s. This process may be continued and the "triangular" structure of the relation between helicity and invariant amplitudes implies that the invariant amplitudes are all KSF in s.

We can then carry out the entire procedure equivalently for the s-channel reduced helicity amplitudes and the s-channel M-function expansion. The same triangular structure shows that the invariant amplitudes are KSF in t. Finally, the s and t channel invariant amplitudes are related by the crossing matrices. But we have made it a condition that this crossing matrix does not contain kinematic variables, so crossing does not introduce singularities.

The conclusion is that both $\Lambda_{\kappa}^{(s)}$ and $\Lambda_{\kappa}^{(t)}$ are KSF in both the s and t variables.

This completes our proof that the invariant amplitudes are KSF and we also note that they are free of kinematic zeros. These properties, together with the fact that they are independent at thresholds and pseudothresholds, encourage us in the belief that the invariant amplitude approach to scattering processes is useful.

CHAPTER 2.

M-FUNCTIONS OF THE THIRD KIND.

We recall that M-functions of the third kind are more complicated than those of the first and second kinds since, in order to eliminate redundant covariants, we require equivalence theorems. The number of such relations between covariants is, however, small compared with the number of independent covariants required for the process. In this chapter we will investigate in detail the M-function expansion for both normal and abnormal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ scattering. The simple counting formulae given in chapter 1 show that there are eight independent invariant amplitudes for each process, although for normal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ it is possible to have elastic or superelastic scattering which reduces the number of independent amplitudes. We consider the transition from inelastic, through elastic, to superelastic scattering for this case in some detail, and express our results in a form such that this transition is manifest and smooth.

2.1 Normal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ Scattering.

(i) Covariants.

Nucleon-nucleon scattering is undoubtedly the best known normal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ scattering process and it has been discussed in detail by several authors(82-84). It is, however, a superelastic process and consequently the M-function must be invariant under time reversal and also under the interchange of the identical particles in the initial and final states. As was mentioned in section I.3, such special conditions reduce the number of independent amplitudes, and for NN scattering, it is well known that there are five amplitudes only. For a process such as $N\Sigma \rightarrow N\Sigma$, the scattering is elastic and there are six independent amplitudes, the condition of invariance under the interchange of particles in the initial or final states no longer obtaining. In this, and the following work, we will choose the s-channel to be the direct particle-particle scattering channel with the crossed t and u channels as particle-antiparticle scattering channels and use the conventions of eqns. (1.2), (1.6), and (1.9) for the ordering of the four spinors and labelling of the independent sets of γ matrices.

The eight kinematic covaraints for the t-channel which give KSF invariant amplitudes for inelastic scattering are

$$\mathcal{K}_1^{(t)} = 1^{(2)}1^{(1)}, \qquad \mathcal{K}_2^{(t)} = \gamma^{(2)}.\gamma^{(1)},$$

$$\mathcal{K}_3^{(t)} = \gamma_5^{(2)}\gamma^{(2)}.\gamma_5^{(1)}\gamma^{(1)}, \qquad \mathcal{K}_4^{(t)} = \tfrac{1}{2}\sigma_{\mu\nu}^{(2)}\sigma_{\mu\nu}^{(1)} = -\sum_{\mu<\nu}\gamma_\mu^{(2)}\gamma_\nu^{(2)}\gamma_\mu^{(1)}\gamma_\nu^{(1)},$$

$$\mathcal{K}_5^{(t)} = \gamma_5^{(2)}\gamma_5^{(1)}, \qquad \mathcal{K}_6^{(t)} = (P.\gamma^{(2)}1^{(1)} - 1^{(2)}Q.\gamma^{(1)}),$$

$$\mathcal{K}_7^{(t)} = \gamma_5^{(2)}P.\gamma^{(2)}\gamma_5^{(1)}, \qquad \mathcal{K}_8^{(t)} = \gamma_5^{(2)}\gamma_5^{(1)}Q.\gamma^{(1)}. \qquad (2.1)$$

This choice is well known from the literature. For elastic scattering, the condition of time reversal invariance in the direct channel requires that in the crossed t-channel the covariants satisfy

$$\mathcal{K}(P,Q) = C^{(1)}C^{(2)}\mathcal{K}(-P,-Q)\{C^{(1)}\}^{-1}\{C^{(2)}\}^{-1}, \qquad (2.2)$$

where $C^{(1)}$ and $C^{(2)}$ are the Dirac charge conjugation matrices appropriate to the (1) and (2) sets of spinors respectively. We find that the condition (2.2) excludes both $\mathcal{K}_7^{(t)}$ and $\mathcal{K}_8^{(t)}$ from the set (2.1), leaving us with six covariants for elastic scattering as desired. For superelastic scattering our covariants must remain unchanged under the interchange of identical particles, the condition for which takes the form of the simple relabelling constraint

$$\mathcal{K}(P,Q) = \mathcal{K}(Q,P), \qquad \gamma^{(1)} \leftrightarrow \gamma^{(2)}. \qquad (2.3)$$

Condition (2.3) further excludes $\mathcal{K}_6^{(t)}$ from the set (2.1) and we are left with the familiar GGMW[84] set of five "Beta-decay invariants" for superelastic scattering.

Instead of defining our s-channel covariants as the linear combinations of the set (2.1) given by the $t \to s$ crossing matrix as in ALV[83], we follow our "look alike" principle and go straight to the natural s-channel covariants. The form of these covariants may be obtained from (2.1) by the simple substitutions $\gamma^{(1)} \to \gamma$, $\gamma^{(2)} \to \gamma'$, $P \to \Lambda$, and $Q \to \Lambda'$, giving

$$\mathcal{K}_1^{(s)} = 1'1, \qquad \mathcal{K}_2^{(s)} = \gamma'.\gamma,$$

$$\mathcal{K}_3^{(s)} = \gamma_5'\gamma'.\gamma_5\gamma, \qquad \mathcal{K}_4^{(s)} = \tfrac{1}{2}\sigma'_{\mu\nu}\sigma_{\mu\nu},$$

$$\mathcal{K}_5^{(s)} = \gamma_5'\gamma_5, \qquad \mathcal{K}_6^{(s)} = (\Lambda.\gamma'1 - 1'\Lambda'.\gamma), \tag{2.4}$$

$$\mathcal{K}_7^{(s)} = \gamma_5'\Lambda.\gamma'\gamma_5, \qquad \mathcal{K}_8^{(s)} = \gamma_5'\gamma_5\Lambda'.\gamma \ .$$

Unfortunately, the set (2.4) does not simply satisfy the conditions of time reversal invariance and identical particles. Under the time reversal invariance condition for the s-channel,

$$\mathcal{K}(\Lambda,\Lambda') = \mathcal{K}(-\Lambda',-\Lambda), \qquad (\gamma'C) \leftrightarrow (C^{-1}\gamma)^T, \tag{2.5}$$

we find that $\mathcal{K}_6^{(s)}$ is excluded while $\mathcal{K}_7^{(s)} \leftrightarrow -\mathcal{K}_8^{(s)}$. Similarly, the condition for superelastic scattering

$$\mathcal{K}(\Lambda,\Lambda') = CC'\,\mathcal{K}(-\Lambda,-\Lambda')\; C^{-1}(C')^{-1}, \tag{2.6}$$

allows $\mathcal{K}_6^{(s)}$ to remain and excludes $\mathcal{K}_7^{(s)}$ and $\mathcal{K}_8^{(s)}$. Obviously the covariants $\mathcal{K}_6^{(s)}$ to $\mathcal{K}_8^{(s)}$ as given in (2.4) do not show a smooth transition through elastic to superelastic scattering. However, if we take the linear combinations

$$\mathcal{K}_6'^{(s)} = (\gamma_5'\Lambda.\gamma'\gamma_5 - \gamma_5'\gamma_5\Lambda'.\gamma),$$

$$\mathcal{K}_7'^{(s)} = (\gamma_5'\Lambda.\gamma'\gamma_5 + \gamma_5'\gamma_5\Lambda'.\gamma), \tag{2.7}$$

$$\mathcal{K}_8'^{(s)} = (\Lambda.\gamma'1 - 1'\Lambda'.\gamma),$$

it is clear that condition (2.5) excludes $\mathcal{K}_7'^{(s)}$ and $\mathcal{K}_8'^{(s)}$

while condition (2.6) further excludes $\mathcal{K}_6'^{(s)}$ and we have reproduced the pattern shown by the t-channel covariants (2.1).

A similar situation arises with the natural u-channel variables. We can go straight to the natural u-channel set by the substitutions $\gamma^{(1)} \to \gamma^{(a)}$, $\gamma^{(2)} \to \gamma^{(b)}$, $P \to \bar{\Lambda}$, $Q \to \bar{\Lambda}'$, in (2.1) and obtain

$$\mathcal{K}_1^{(u)} = 1^{(b)}1^{(a)}, \qquad \mathcal{K}_2^{(u)} = \gamma^{(b)}.\gamma^{(a)},$$
$$\mathcal{K}_3^{(u)} = \gamma_5^{(b)}\gamma^{(b)}.\gamma_5^{(a)}\gamma^{(a)}, \qquad \mathcal{K}_4^{(u)} = \tfrac{1}{2}\sigma_{\mu\nu}^{(b)}\sigma_{\mu\nu}^{(a)},$$
$$\mathcal{K}_5^{(u)} = \gamma_5^{(b)}\gamma_5^{(a)}, \qquad \mathcal{K}_6^{(u)} = (\bar{\Lambda}.\gamma^{(b)}1^{(a)} + 1^{(b)}\bar{\Lambda}'.\gamma^{(a)}),$$
$$\mathcal{K}_7^{(u)} = \gamma_5^{(b)}\bar{\Lambda}.\gamma^{(b)}\gamma_5^{(a)}, \qquad \mathcal{K}_8^{(u)} = \gamma_5^{(b)}\gamma_5^{(a)}\bar{\Lambda}'.\gamma^{(a)}. \tag{2.8}$$

In the natural set (2.8), $\mathcal{K}_1^{(u)}$ to $\mathcal{K}_5^{(u)}$ are, as before, the appropriate β-decay invariants, but as for the s-channel covariants, we find that we need the combinations

$$\mathcal{K}_6'^{(u)} = (\gamma_5^{(b)}\bar{\Lambda}.\gamma^{(b)}\gamma_5^{(a)} - \gamma_5^{(b)}\gamma_5^{(a)}\bar{\Lambda}'.\gamma^{(a)}),$$
$$\mathcal{K}_7'^{(u)} = (\gamma_5^{(b)}\bar{\Lambda}.\gamma^{(b)}\gamma_5^{(a)} + \gamma_5^{(b)}\gamma_5^{(a)}\bar{\Lambda}'.\gamma^{(a)}), \tag{2.9}$$
$$\mathcal{K}_8'^{(u)} = (\bar{\Lambda}.\gamma^{(b)}\,1^{(a)} - 1^{(b)}\,\bar{\Lambda}'.\gamma^{(a)}).$$

Now the time reversal invariance condition for the u-channel

$$\mathcal{K}(\bar{\Lambda},\bar{\Lambda}') = C^{(a)}C^{(b)}\mathcal{K}(-\bar{\Lambda}',-\bar{\Lambda})\{C^{(a)}\}^{-1}\{C^{(b)}\}^{-1}, \quad \gamma^{(a)} \leftrightarrow \gamma^{(b)}, \tag{2.10}$$

excludes $\mathcal{K}_7'^{(u)}$ and $\mathcal{K}_8'^{(u)}$, while the identical particle condition

$$\mathcal{K}(\bar{\Lambda},\bar{\Lambda}') = \mathcal{K}(\bar{\Lambda}',\bar{\Lambda}), \qquad \gamma^{(a)} \leftrightarrow \gamma^{(b)}, \tag{2.11}$$

excludes $\mathcal{K}_6'^{(u)}$ as required.

It is in some ways unfortunate that the combinations (2.7) and (2.9) of the natural s and u channel covariants are required for a smooth transition from inelastic to elastic and superelastic scattering as taking them destroys the symmetry which we could see when, in the inelastic case, either the initial two, or the final two particles of the s-channel are identical. From eqns. (1.1), (1.5), and (1.8) it is clear that if, for example, the particles with spins s_1' and s_2' are identical, then the t and u channels are essentially the same. Crossing $t \leftrightarrow u$ then simply corresponds to interchanging the identical particles in the s-channel final state and we would expect the s-channel covariants to show some simple properties under $t \leftrightarrow u$ crossing. These are the crossing symmetry relations mentioned in section 1.2 and they arise because, in normal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ scattering, the interchange of identical final state particles in the s-channel makes $t \leftrightarrow u$ crossing simply a matter of fermi statistics. The s-channel covariants have, therefore, the property that under $t \leftrightarrow u$ crossing

$$\mathcal{K}_i(\Lambda,\Lambda') \to \mathcal{K}_i(\Lambda,-\Lambda'), \quad (O_s'C) \to (O_s'C)^T, \tag{2.12}$$

where $\mathcal{K}_i(\Lambda,\Lambda')$ are the set of natural covariants (2.4).

This crossing symmetry is reflected in the invariant amplitudes so that for the natural s-channel amplitudes, we have under $t \leftrightarrow u$ crossing

$$A_i^{(s)}(\tilde{\nu},s) = -(-)^i A_i^{(s)}(-\tilde{\nu},s), \quad i = 1,\ldots,8. \tag{2.13}$$

The extra minus sign in (2.13) arises from the fact that the M-function must have an overall sign change under $t \leftrightarrow u$ crossing since this involves the interchange of two fermions.

This crossing symmetry could be seen in an inelastic process such as $N\Xi \to \Sigma\Sigma$, but as such processes are comparatively rare, the form in which we have given the covariants in eqns. (2.1),(2.7), and (2.9) is probably more useful. It is well known, though, that we observe this same crossing symmetry in superelastic scattering where the amplitudes A_6 to A_8 are not present. For this case, eqn. (2.13) (with $i = 1,\ldots,5$) reproduces the familiar symmetry of the s-channel amplitudes for NN elastic scattering under $t \leftrightarrow u$ crossing.

(ii) The Equivalence Theorems.

We now come to a consideration of the equivalence theorems and spinor identities by which such possible covariants as $P.\gamma^{(2)}Q.\gamma^{(1)}$, $(P.\gamma^{(2)}1^{(1)} + 1^{(2)}Q.\gamma^{(1)})$, and $Q_\mu\gamma_\nu^{(2)}i\sigma_{\mu\nu}^{(1)}$, etc. are eliminated to leave the eight independent covariants discussed in the previous section. These relations are well known from the literature, but we will reproduce them here for completeness. They are most easily derived from the expansion of forms such as

$\varepsilon_{\mu\nu}(Q\gamma^{(2)})\varepsilon^{\mu\nu}(P\gamma^{(1)})$, $\varepsilon_{\mu\nu\alpha\beta}\Delta^{\alpha}{}_{\sigma}{}^{(2)\beta\lambda}\varepsilon^{\mu\nu\gamma\delta}\Delta_{\gamma}\sigma_{\delta\lambda}^{(1)}$, etc., and we find three equivalence theorems and four spinor identities.

If we compare the expansion of $\varepsilon_{\mu\nu}(\Delta\sigma^{(2)\lambda})\varepsilon^{\mu\nu}(\Delta\sigma_{\lambda}^{(1)})$ according to eqn. (A.1) with the expansion obtained from the use of eqn. (A.7) for each ε-symbol, we obtain the equivalence theorem

$$\tfrac{1}{4}(m+m'+\mu+\mu')\{P.\gamma^{(2)}1^{(1)} + 1^{(2)}Q.\gamma^{(1)}\} = \nu\mathcal{K}_1^{(t)} + \\ + \tfrac{1}{4}(m+m')(\mu+\mu')\mathcal{K}_2^{(t)} - \tfrac{1}{4}(m-m')(\mu-\mu')\mathcal{K}_3^{(t)} - \tfrac{t}{4}\mathcal{K}_4^{(t)} - \\ - \nu\mathcal{K}_5^{(t)} - \tfrac{1}{4}(m-m'-\mu+\mu')\mathcal{K}_6^{(t)} + \tfrac{1}{2}(\mu-\mu')\mathcal{K}_7^{(t)} + \\ + \tfrac{1}{2}(m-m')\mathcal{K}_8^{(t)} . \qquad (2.14)$$

Similarly, by considering the form $\varepsilon_{\mu\nu}(Q\gamma^{(2)})\varepsilon^{\mu\nu}(P\gamma^{(1)})$, we obtain the second equivalence theorem

$$P.\gamma^{(2)}Q.\gamma^{(1)} = \nu\mathcal{K}_2^{(t)} + \tfrac{1}{4}\{(\mu-\mu')^2 + (m-m')^2 - t\}\mathcal{K}_3^{(t)} + \\ + \tfrac{1}{4}(m-m')(\mu-\mu')\mathcal{K}_4^{(t)} - \tfrac{1}{4}(m+m')(\mu+\mu')\mathcal{K}_5^{(t)} - \\ - \tfrac{1}{2}(m-m')\mathcal{K}_7^{(t)} - \tfrac{1}{2}(\mu-\mu')\mathcal{K}_8^{(t)}. \qquad (2.15)$$

The third equivalence theorem is obtained by multiplying (2.15) by $\gamma_5^{(2)}\gamma_5^{(1)}$ which changes the signs of the masses m' and μ' as well as shuffling the covariants in an obvious manner. It is

$$\gamma_5^{(2)}P.\gamma^{(2)}\gamma_5^{(1)}Q.\gamma^{(1)} = -\tfrac{1}{4}(m-m')(\mu-\mu')\mathcal{K}_1^{(t)} + \\ + \tfrac{1}{4}\{(\mu+\mu')^2 + (m+m')^2 - t\}\mathcal{K}_2^{(t)} + \nu\mathcal{K}_3^{(t)} - \tfrac{1}{4}(m+m')(\mu+\mu')\mathcal{K}_4^{(t)} - \\ - \tfrac{1}{2}(m+m')P.\gamma^{(2)}1^{(1)} - \tfrac{1}{2}(\mu+\mu')1^{(2)}Q.\gamma^{(1)}. \qquad (2.16)$$

The spinor identities may be obtained by considering the expressions $\Delta_{\mu}\gamma_{\nu}^{(2)}\sigma^{(1)\mu\nu}$ and $\sigma_{\mu\nu}^{(2)}\Delta^{\mu}\gamma^{(1)\nu}$ and the fact that we can multiply them by $1 = (\gamma_5^{(2)})^2(\gamma_5^{(1)})^2$. Simple applications of the Dirac equation leads to

$$\gamma_5^{(2)} Q_{\mu}\gamma_{\nu}^{(2)}\gamma_5^{(1)} i\sigma_{\mu\nu}^{(1)} = -\tfrac{1}{2}(m+m')\mathcal{K}_2^{(t)} + \tfrac{1}{2}(\mu+\mu')\mathcal{K}_4^{(t)} + + P.\gamma^{(2)}1^{(1)}, \tag{2.17}$$

and

$$\gamma_5^{(2)} i\sigma_{\mu\nu}^{(2)} P_{\mu}\gamma_5^{(1)}\gamma_{\nu}^{(1)} = -\tfrac{1}{2}(\mu+\mu')\mathcal{K}_2^{(t)} + \tfrac{1}{2}(m+m')\mathcal{K}_4^{(t)} + + 1^{(2)}Q.\gamma^{(1)}. \tag{2.18}$$

Multiplying (2.17) and (2.18) by $\gamma_5^{(2)}\gamma_5^{(1)}$ leads to two further identities

$$Q_{\mu}\gamma_{\nu}^{(2)} i\sigma_{\mu\nu}^{(1)} = -\tfrac{1}{2}(m-m')\mathcal{K}_3^{(t)} - \tfrac{1}{2}(\mu-\mu')\mathcal{K}_4^{(t)} + \mathcal{K}_7^{(t)}, \tag{2.19}$$

and

$$i\sigma_{\mu\nu}^{(2)} P_{\mu}\gamma_{\nu}^{(1)} = -\tfrac{1}{2}(\mu-\mu')\mathcal{K}_3^{(t)} - \tfrac{1}{2}(m-m')\mathcal{K}_4^{(t)} + \mathcal{K}_8^{(t)}. \tag{2.20}$$

We can readily see that the three equivalence theorems (2.14-2.16) and the four identities (2.17-2.20) satisfy the conditions of section 1.5 for the reduction of the complete set of covariants to the independent set giving KSF amplitudes.

In the above work we have given the equivalence theorems in terms of the natural t-channel covariants because this is their customary form, but because we are treating these problems in a symmetric way for all three channels, it is necessary that eqns. (2.14) to (2.20) be written in terms of the s and u channel covariants as well. We can obtain the s-channel set by the simple substitutions

$$m \to m, \quad \mu \to \mu', \quad m' \to \mu, \quad \mu' \to m', \quad \nu \to \tilde{\nu}, \quad t \to s,$$
$$P \to \Lambda, \quad Q \to \Lambda', \quad \gamma^{(2)} \to \gamma', \text{ and } \gamma^{(1)} \to \gamma, \tag{2.21}$$

in equations (2.14) to (2.20). For example, eqn. (2.14) becomes for the s-channel covariants

$$\tfrac{1}{4}(m+\mu+m'+\mu')\{\Lambda.\gamma' 1 + 1'\Lambda'.\gamma\} = \tilde{\nu}\mathcal{K}_1^{(s)} + \tfrac{1}{4}(m+\mu)(m'+\mu')\mathcal{K}_2^{(s)} +$$
$$+ \tfrac{1}{4}(m-\mu)(m'-\mu')\mathcal{K}_3^{(s)} - \tfrac{s}{4}\mathcal{K}_4^{(s)} - \tilde{\nu}\mathcal{K}_5^{(s)} + \tfrac{1}{4}(m+\mu-m'-\mu')\mathcal{K}_6^{(s)} +$$
$$+ \tfrac{1}{2}(m'-\mu')\mathcal{K}_7^{(s)} - \tfrac{1}{2}(m-\mu)\mathcal{K}_8^{(s)}. \tag{2.22}$$

Notice that $\mathcal{K}_6^{(s)}$, $\mathcal{K}_7^{(s)}$, and $\mathcal{K}_8^{(s)}$ are here those of eqn. (2.4) and not the combinations of (2.7).

Similarly, for the equivalence theorems in the u-channel, we make the substitutions

$$m \to m, \quad \mu \to \mu, \quad m' \to \mu', \quad \mu' \to m', \quad \nu \to \overline{\nu}, \quad t \to u,$$
$$P \to \overline{\Lambda}, \quad Q \to \overline{\Lambda}', \quad \gamma^{(2)} \to \gamma^{(b)}, \text{ and } \gamma^{(1)} \to \gamma^{(a)}, \tag{2.23}$$

in equations (2.14) to (2.16) and obtain the u-channel spinor identities by the same substitutions in eqns. (2.17) to (2.20).

(iii) The Crossing Matrices.

The crossing matrices relate the natural covariants and invariant amplitudes of the M-function expansion in one channel to those of the expansion in another channel. For many processes, these crossing matrices are almost trivial, arising solely from the relations between the natural momenta (1.12) with the addition of the subsidiary conditions like (1.38). For four fermion scattering, however, we have the additional complication that the spinors must be reordered as we cross from one channel to another. The matrices expressing this reordering are known as the Fierz transformation matrices and we must calculate these in order to obtain the normal $\frac{1}{2} + \frac{1}{2} \rightarrow \frac{1}{2} + \frac{1}{2}$ crossing matrices. Once this Fierz matrix is known, it is a reasonably straightforward calculation involving the use of eqns. (1.12) and the Dirac equation to obtain the full crossing matrices.

Our convention for labelling the crossing matrices is given by the equation

$$\mathcal{K}^{(s)} = \Gamma_{[st]} \mathcal{K}^{(t)}, \tag{2.24}$$

which means that the 8×8 matrix $\Gamma_{[st]}$ gives the linear combinations of the t-channel covariants equivalent to the s-channel covariants. The suffices [st] merely label the matrix and are not tensor-like indices so that $\Gamma^T_{[st]} \neq \Gamma_{[ts]}$. We assume that the eight covariants, expressed symbolically in (2.24) as the column vector $\mathcal{K}$, are ordered according to equations (2.1) and the combinations (2.7) and (2.9) are used for the s and u channel sets. The eight component column vector of the invariant amplitudes, similarly ordered, consequently cross according to

$$A^{(s)} = \Gamma^T_{[ts]} A^{(t)}. \tag{2.25}$$

We also have the obvious combination properties for the crossing matrices:

$$\Gamma_{[st]}\,\Gamma_{[ts]} = \Gamma_{[ts]}\,\Gamma_{[st]} = \Gamma_{[tu]}\,\Gamma_{[ut]} = \\ = \Gamma_{[su]}\,\Gamma_{[us]} = \text{etc.} = 1, \tag{2.26}$$

and

$$\Gamma_{[st]} = \Gamma_{[su]}\,\Gamma_{[ut]}, \quad \Gamma_{[ts]} = \Gamma_{[tu]}\,\Gamma_{[us]}, \text{ etc.} \tag{2.27}$$

We find that each of the six crossing matrices has the block form

$$\Gamma = \tfrac{1}{4}\left[\begin{array}{c|c} \Gamma^{a} & 0 \\ \hline \Gamma^{c} & \Gamma^{b} \end{array}\right],$$

with Γ^a a 5 x 5 matrix of numbers only which crosses the five β-decay invariants into each other, Γ^b a 3 x 3 matrix of numbers only, and Γ^c is a 5 x 3 matrix involving the masses.

There are several ways of calculating the Fierz transformation matrices. For instance, one could calculate appropriate traces, but in this work we will base our approach on the completeness relation for the Dirac γ matrices

$$\delta_{\sigma}^{\ \tau'}\delta_{\tau}^{\ \sigma'} = \tfrac{1}{4}\sum_{R}(\gamma_R)_{\sigma}^{\ \sigma'}(\gamma_R)_{\tau}^{\ \tau'}, \tag{2.28}$$

where the summation extends over the 16 γ matrices and our convention for these matrices is given in detail in appendix A.

Contracting (2.28) with $(O_s^{\prime}C)_{\omega'\tau'}(C^{-1}O_s)^{\tau\omega}$, where $O_s^{\prime}$ and O_s are the appropriate combinations of momenta and γ matrices to give the natural s-channel covariants, we obtain, after relabelling,

$$(O_s^{\prime}C)_{\sigma'\tau'}(C^{-1}O_s)^{\tau\sigma} = \tfrac{1}{4}\sum_R(\gamma_R)_{\tau'}{}^{\tau}(O_s^{\prime}C\gamma_R^{T}C^{-1}O_s)_{\sigma'}{}^{\sigma}, \qquad (2.29)$$

which expresses the s-channel covariants in terms of those of the t-channel. Use of eqns. (2.29), (1.12), and the Dirac equation then leads to the following results for the crossing matrix $\Gamma_{[st]}$:

$$\Gamma^{a}_{[st]} = \begin{bmatrix} 1 & -1 & 1 & -1 & -1 \\ 4 & 2 & 2 & 0 & 4 \\ 4 & -2 & -2 & 0 & 4 \\ 6 & 0 & 0 & 2 & -6 \\ -1 & -1 & 1 & 1 & 1 \end{bmatrix}, \qquad (2.30)$$

which gives the expansion of the five s-channel covariants $\mathcal{K}_1^{(s)},\ldots,\mathcal{K}_5^{(s)}$ in terms of the five superelastic t-channel covariants. This matrix has been given by ALV in their definition of the s-channel covariants in terms of the t-channel set in nucleon-nucleon scattering. But we have also

$$\Gamma^{b}_{[st]} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & -4 & -4 \end{bmatrix}, \qquad (2.31)$$

and

$$\Gamma^{c}_{[st]} = \tfrac{1}{2}\begin{bmatrix} M & -3M & -M & -M & 3M \\ 3M' & -M' & -3M' & M' & M' \\ 3M'' & M'' & 3M'' & M'' & M'' \end{bmatrix}, \qquad (2.32)$$

where we define $M \equiv (m-\mu+m'-\mu')$, $M' \equiv (m-\mu-m'+\mu')$, and $M'' \equiv (m+\mu-m'-\mu')$.

From equations (2.31) and (2.32) we can see why it was desirable to take linear combinations of the s-channel covariants. The crossing matrices in the form given above show clearly the "smooth" transition from inelastic to elastic and superelastic scattering. For elastic scattering, $m = m'$ and $\mu = \mu'$, so that $M' = M'' = 0$ and the last two rows of $\Gamma^c_{[st]}$ vanish. Under these circumstances $\mathcal{K}'_7$ and $\mathcal{K}'_8$ transform completely independently of the other six covariants and they can be dropped without disrupting the crossing of the other six. Similarly for superelastic scattering, when all the masses become equal, $\Gamma^c_{[st]}$ vanishes completely so that we can drop $\mathcal{K}'_6$, $\mathcal{K}'_7$, and $\mathcal{K}'_8$ without upsetting the transformations of the remaining five covariants. We find that this pattern is followed in all six crossing matrices, provided always that we employ the combinations of covariants (2.7) and (2.9).

The inverse crossing matrix, $\Gamma_{[ts]}$, may be obtained by similar use of the Fierz transformation formula

$$(O_t^{(2)})_{\tau'}{}^{\tau}(O_t^{(1)})_{\sigma'}{}^{\sigma} = \tfrac{1}{4}\sum_R (\gamma_R C O_t^{(2)T})_{\sigma'\tau'}(C^{-1}\gamma_R O_t^{(1)})^{\tau\sigma}, \quad (2.33)$$

and the relations (1.12). It turns out that the elements of $\Gamma^a_{[ts]}$ have the same absolute values as the elements of $\Gamma^a_{[st]}$. It is not necessary, therefore, to write out $\Gamma^a_{[ts]}$ in detail, as it can readily be obtained from $\Gamma^a_{[st]}$ by inspection since it is the inverse of the matrix (2.30). The rest of $\Gamma_{[ts]}$ is given by

$$\Gamma^{b}_{[ts]} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & -2 \end{bmatrix}, \tag{2.34}$$

and

$$\Gamma^{c}_{[ts]} = \tfrac{1}{2} \times$$
$$\begin{bmatrix} -3M & M & -3M & M & -M \\ (2m-\mu-2m'+\mu') & (2m+\mu-2m'-\mu')-(2m-\mu-2m'+\mu') & (\mu-\mu') & (2m+\mu-2m'-\mu') \\ -(m-2\mu-m'+2\mu') & (m+2\mu-m'-2\mu') & (m-2\mu-m'+2\mu') & (m-m') & (m+2\mu-m'-2\mu') \end{bmatrix}. \tag{2.35}$$

The crossing matrix $\Gamma_{[su]}$ is obtained in a similar way. The appropriate Fierz transformation formula is

$$(O_s'C)_{\sigma'\tau'}(C^{-1}O_s)^{\tau\sigma} = \tfrac{1}{4}\sum_R (O_s'C\gamma_R^T(C^{-1}O_s)^T)_{\sigma'}{}^{\tau}(\gamma_R)_{\tau'}{}^{\sigma}. \tag{2.36}$$

Once more it is not necessary to write out $\Gamma^a_{[su]}$ since it can be obtained directly from $\Gamma^a_{[st]}$ by changing the signs of the odd numbered rows, leaving rows 2 and 4 as they are. This possibility reflects the fact that we have crossing symmetry on these five covariants if the t and u channels are identical.

Similarly, $\Gamma_{[us]}$ can be obtained with the use of the Fierz transformation

$$(O_u^{(b)})_{\sigma'}{}^{\tau}(O_u^{(a)})_{\tau'}{}^{\sigma} = \tfrac{1}{4}\sum_R (O_u^{(b)}\gamma_R C O_u^{(a)T})_{\sigma'\tau'}(C^{-1}\gamma_R)^{\sigma\tau}, \tag{2.37}$$

and again $\Gamma^a_{[us]}$ has elements of the same absolute value as those of $\Gamma^a_{[st]}$. It can be obtained directly from $\Gamma^a_{[ts]}$ by changing the signs of the odd-numbered columns which can also be seen as a manifestation of possible crossing symmetry. It turns out, in fact, that $\Gamma^a_{[su]} = \Gamma^a_{[us]}$ and this matrix agrees with that given by GGMW.

The remaining portions of $\Gamma_{[su]}$ and $\Gamma_{[us]}$ are found to be

$$\Gamma^{b}_{[su]} = \Gamma^{b}_{[us]} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} , \qquad (2.38)$$

$$\Gamma^{c}_{[su]} = \tfrac{1}{2} \begin{bmatrix} -3M & M & 3M & -M & -M \\ -M' & 3M' & M' & M' & -3M' \\ 3M'' & M'' & 3M'' & M'' & M'' \end{bmatrix} , \qquad (2.39)$$

and

$$\Gamma^{c}_{[us]} = \tfrac{1}{2} \begin{bmatrix} 3M & -M & -3M & M & M \\ 3M' & M' & 3M' & M' & M' \\ -M'' & 3M'' & M'' & M'' & -3M'' \end{bmatrix} . \qquad (2.40)$$

The crossing matrices $\Gamma_{[tu]}$ and $\Gamma_{[ut]}$ can be obtained from the Fierz transformation formulae

$$(O^{(2)}_{t})_{\tau'}{}^{\tau}(O^{(1)}_{t})_{\sigma'}{}^{\sigma} = \tfrac{1}{4} \sum_{R}(\gamma_{R}O^{(2)}_{t})_{\sigma'}{}^{\tau}(\gamma_{R}O^{(1)}_{t})_{\tau'}{}^{\sigma} \qquad (2.41)$$

and

$$(O^{(b)}_{u})_{\sigma'}{}^{\tau}(O^{(a)}_{u})_{\tau'}{}^{\sigma} = \tfrac{1}{4} \sum_{R}(\gamma_{R}O^{(b)}_{u})_{\tau'}{}^{\tau}(\gamma_{R}O^{(a)}_{u})_{\sigma'}{}^{\sigma}, \qquad (2.42)$$

respectively, or from the relations (2.27). They are given by

$$\Gamma^{a}_{[tu]} = \Gamma^{a}_{[ut]} = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 4 & -2 & 2 & 0 & 4 \\ 4 & 2 & -2 & 0 & 4 \\ 6 & 0 & 0 & -2 & -6 \\ -1 & 1 & 1 & -1 & 1 \end{bmatrix} , \qquad (2.43)$$

which again agrees with the matrix given by GGMW,

$$\Gamma^{b}_{[tu]} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & 2 \end{bmatrix} , \tag{2.44}$$

$$\Gamma^{b}_{[ut]} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & 4 & 4 \end{bmatrix} , \tag{2.45}$$

$$\Gamma^{c}_{[tu]} = \tfrac{1}{2} \times$$

$$\begin{bmatrix} 3M & -M & 3M & -M & M \\ (2m+\mu-2m'-\mu') & (2m-\mu-2m'+\mu') & -(2m+\mu-2m'-\mu') & -(\mu-\mu') & (2m-\mu-2m'+\mu') \\ (m+2\mu-m'-2\mu') & -(m-2\mu-m+2\mu') & -(m+2\mu-m'-2\mu') & -(m-m') & -(m-2\mu-m'+2\mu') \end{bmatrix} \tag{2.46}$$

and

$$\Gamma^{c}_{[ut]} = \tfrac{1}{2} \begin{bmatrix} -M & -3M & M & -M & -3M \\ 3M' & -M' & 3M' & -M' & M' \\ 3M'' & M'' & -3M'' & -M'' & M'' \end{bmatrix} . \tag{2.47}$$

It will be noted that these six 8 × 8 crossing matrices do not contain kinematic variables and thus do not introduce kinematic singularities. The independent set of kinematic covariants given in section (i) and obtained from the complete set by the equivalence theorems of section (ii) will, therefore, lead to KSF invariant amplitudes.

2.2 Abnormal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ Scattering.

(i) Covariants.

This process is simply the abnormal equivalent of the normal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ scattering treated in section 2.1. Many of the features of the abnormal case can be obtained directly from the previous results by simple chiral γ_5 transformations. We will assume that the opposite parity particle is always in the final state for the s-channel process and it is clear that there is no possibility of elastic or superelastic scattering. There are, therefore, always the full eight covariants and we do not need to take linear combinations of the natural covariants in the s and u channels.

The eight t-channel covariants are obtained by multiplying the set (2.1) by $\gamma_5^{(1)}$ and $\gamma_5^{(2)}$ and we choose

$$
\begin{aligned}
\mathcal{K}_1^{(t)} &= \gamma_5^{(2)} 1^{(1)}, & \mathcal{K}_2^{(t)} &= \gamma_5^{(2)} \gamma^{(2)} . \gamma^{(1)}, \\
\mathcal{K}_3^{(t)} &= \gamma^{(2)} . \gamma_5^{(1)} \gamma^{(1)}, & \mathcal{K}_4^{(t)} &= \tfrac{1}{4}\varepsilon(\sigma^{(2)}\sigma^{(1)}), \\
\mathcal{K}_5^{(t)} &= 1^{(2)} \gamma_5^{(1)}, & \mathcal{K}_6^{(t)} &= (P.\gamma^{(2)}\gamma_5^{(1)} - \gamma_5^{(2)} Q.\gamma^{(1)}), \\
\mathcal{K}_7^{(t)} &= 1^{(2)} \gamma_5^{(1)} Q.\gamma^{(1)}, & \mathcal{K}_8^{(t)} &= \gamma_5^{(2)} P.\gamma^{(2)} 1^{(1)}.
\end{aligned}
\tag{2.48}
$$

Similarly, the set of s-channel covariants can be obtained from eqn. (2.4) or directly from (2.48) by the substitutions

$$
\gamma^{(2)} \to \gamma', \quad \gamma^{(1)} \to \gamma, \quad P \to \Lambda, \quad Q \to \Lambda'. \tag{2.49}
$$

The s-channel covariants then are

$$\mathcal{K}_1^{(s)} = \gamma_5' 1, \qquad \mathcal{K}_2^{(s)} = \gamma_5'\gamma'\cdot\gamma,$$
$$\mathcal{K}_3^{(s)} = \gamma'\cdot\gamma_5\gamma, \qquad \mathcal{K}_4^{(s)} = \tfrac{1}{4}\varepsilon(\sigma'\sigma),$$
$$\mathcal{K}_5^{(s)} = 1'\gamma_5, \qquad \mathcal{K}_6^{(s)} = (\Lambda\cdot\gamma'\gamma_5 - \gamma_5'\Lambda'\cdot\gamma), \tag{2.50}$$
$$\mathcal{K}_7^{(s)} = 1'\gamma_5\Lambda'\cdot\gamma, \qquad \mathcal{K}_8^{(s)} = \gamma_5' \Lambda\cdot\gamma' 1,$$

which covariants conform with our "look alike" principle.

A possible set of u-channel covariants is obtained from the set (2.48) by the substitutions

$$\gamma^{(2)} \to \gamma^{(b)}, \quad \gamma^{(1)} \to \gamma^{(a)}, \quad P \to \overline{\Lambda}, \quad Q \to \overline{\Lambda}', \tag{2.51}$$

but this set is not the most convenient. Crossing symmetry in abnormal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ will be observed when the two initial s-channel particles are identical. From eqns. (1.1), (1.5), and (1.8) we see that if the particles with spins s_1 and s_2 are identical, the t and u channels are alike, one in fact being the TCP transform of the other. The necessity for this TCP transform masks to some extent the possible crossing symmetry and as the covariants obtained by the substitutions (2.51) are the natural set for the u-channel as given in eqn. (1.8), they do not make the crossing symmetry manifest. The crossing symmetry can be made explicit either by redefining the u-channel as the TCP transform of (1.8) or by reordering the u-channel covariants obtained from (2.48) by the substitutions (2.51). We shall follow the latter approach and the reordered set can be obtained directly from (2.48) by the substitutions

$$\gamma^{(2)} \to \gamma^{(a)}, \quad \gamma^{(1)} \to \gamma^{(b)}, \quad P \to \bar{\Lambda}', \quad Q \to \bar{\Lambda}, \qquad (2.52)$$

Since this is still a simple relabelling transformation, the set of u-channel covariants obtained conforms to our principle of defining the covariants in the separate channels in a similar way.

Performing the substitutions (2.52) in the set (2.48), we obtain our u-channel covariants

$$\begin{aligned}
\mathcal{K}_1^{(u)} &= 1^{(b)}\gamma_5^{(a)}, & \mathcal{K}_2^{(u)} &= \gamma^{(b)}.\gamma_5^{(a)}\gamma^{(a)}, \\
\mathcal{K}_3^{(u)} &= \gamma_5^{(b)}\gamma^{(b)}.\gamma^{(a)}, & \mathcal{K}_4^{(u)} &= \tfrac{1}{4}\varepsilon(\sigma^{(b)}\sigma^{(a)}), \\
\mathcal{K}_5^{(u)} &= \gamma_5^{(b)}\ 1^{(a)}, & \mathcal{K}_6^{(u)} &= (\gamma_5^{(b)}\bar{\Lambda}'.\gamma^{(a)} - \bar{\Lambda}.\gamma^{(b)}\gamma_5^{(a)}), \\
\mathcal{K}_7^{(u)} &= \gamma_5^{(b)}\bar{\Lambda}.\gamma^{(b)}1^{(a)}, & \mathcal{K}_8^{(u)} &= 1^{(b)}\gamma_5^{(a)}\bar{\Lambda}'.\gamma^{(a)}.
\end{aligned} \qquad (2.53)$$

With the covariants as chosen in (2.48), (2.50), and (2.53), we obtain explicit crossing symmetry on the s-channel invariant amplitudes for identical initial s-channel particles. The derivation of the crossing symmetry relations is similar to that given in the previous section for the normal case. The result is that the invariant amplitudes for abnormal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ scattering behave as

$$A_i^{(s)}(\tilde{\nu},s) = -(-)^i\, A_i^{(s)}(-\tilde{\nu},s), \quad i = 1,\ldots,8, \qquad (2.54)$$

under $t \leftrightarrow u$ crossing.

<u>(ii) The Equivalence Theorems.</u>

The equivalence theorems for abnormal $\frac{1}{2}+\frac{1}{2}\rightarrow\frac{1}{2}+\frac{1}{2}$ are obtained directly from those for the normal scattering by multiplying by $\gamma_5^{(1)}$ or $\gamma_5^{(2)}$ in a similar way to that in which we obtained the abnormal covariants. The effect of the $\gamma_5^{(1)}$ transformation is to change the sign of the mass m' and $\gamma_5^{(2)}$ changes the sign of μ'. The normal $\frac{1}{2}+\frac{1}{2}\rightarrow\frac{1}{2}+\frac{1}{2}$ covariants change in an obvious manner into their abnormal counterparts and we obtain the three equivalence theorems:

$$\begin{aligned}\tfrac{1}{4}(m+\mu+m'+\mu')\{P.\gamma^{(2)}\gamma_5^{(1)}+\gamma_5^{(2)}Q.\gamma^{(1)}\} &= \nu\mathcal{K}_1^{(t)} + \\ &+\tfrac{1}{4}(m+m')(\mu-\mu')\mathcal{K}_2^{(t)}+\tfrac{1}{4}(m-m')(\mu+\mu')\mathcal{K}_3^{(t)}-\tfrac{t}{4}\mathcal{K}_4^{(t)}+ \\ &+\nu\mathcal{K}_5^{(t)}+\tfrac{1}{4}(m-\mu+m'-\mu')\mathcal{K}_6^{(t)}-\tfrac{1}{2}(m-m')\mathcal{K}_7^{(t)}-\tfrac{1}{2}(\mu-\mu')\mathcal{K}_8^{(t)},\end{aligned} \tag{2.55}$$

$$\begin{aligned}\gamma_5^{(2)}P.\gamma^{(2)}Q.\gamma^{(1)} &= \nu\mathcal{K}_2^{(t)}-\tfrac{1}{4}\{(\mu+\mu')^2+(m-m')^2-t\}\mathcal{K}_3^{(t)}+ \\ &+\tfrac{1}{4}(m-m')(\mu+\mu')\mathcal{K}_4^{(t)}+\tfrac{1}{4}(m+m')(\mu-\mu')\mathcal{K}_5^{(t)}+ \\ &+\tfrac{1}{2}(\mu+\mu')\mathcal{K}_7^{(t)}+\tfrac{1}{2}(m-m')P.\gamma^{(2)}\gamma_5^{(1)},\end{aligned} \tag{2.56}$$

and

$$\begin{aligned}P.\gamma^{(2)}\gamma_5^{(1)}Q.\gamma^{(1)} &= \tfrac{1}{4}(m-m')(\mu+\mu')\mathcal{K}_1^{(t)}-\tfrac{1}{4}\{(\mu-\mu')^2+(m+m')^2-t\}\mathcal{K}_2^{(t)}+ \\ &+\nu\mathcal{K}_3^{(t)}+\tfrac{1}{4}(m+m')(\mu-\mu')\mathcal{K}_4^{(t)}+\tfrac{1}{2}(m+m')\mathcal{K}_8^{(t)}+ \\ &+\tfrac{1}{2}(\mu-\mu')\gamma_5^{(2)}Q.\gamma^{(1)}.\end{aligned} \tag{2.57}$$

We obtain two abnormal spinor identities in a similar way from eqns. (2.19) and (2.20). They are

$$\gamma_5^{(2)} Q_\mu \gamma_\nu^{(2)} i\sigma_{\mu\nu}^{(1)} = \tfrac{1}{2}(m-m')\mathcal{K}_3^{(t)} - \tfrac{1}{2}(\mu+\mu')\mathcal{K}_4^{(t)} - P.\gamma^{(2)}\gamma_5^{(1)}, \tag{2.58}$$

and

$$i\sigma_{\mu\nu}^{(2)} P_\mu \gamma_5^{(1)} \gamma_\nu^{(1)} = \tfrac{1}{2}(\mu-\mu')\mathcal{K}_2^{(t)} - \tfrac{1}{2}(m+m')\mathcal{K}_4^{(t)} - \gamma_5^{(2)} Q.\gamma^{(1)}. \tag{2.59}$$

The two further identities for $Q_\mu \gamma_\nu^{(2)} \gamma_5^{(1)} i\sigma_{\mu\nu}^{(1)}$ and $\gamma_5^{(2)} i\sigma_{\mu\nu}^{(2)} P_\mu \gamma_\nu^{(1)}$ may also be obtained, but they are trivially related to (2.58) and (2.59) above and will not be written out here.

The equations for the s-channel are obtained from these by the substitutions (2.21) in eqns. (2.55) to (2.59), but since the u-channel is now treated differently, we obtain the u-channel relations by the substitutions

$$m \to \mu, \quad \mu \to m, \quad m' \to m', \quad \mu' \to \mu', \quad t \to u,$$
$$\nu \to \bar{\nu}, \quad P \to \bar{\Lambda}', \quad Q \to \bar{\Lambda}, \quad \gamma^{(2)} \to \gamma^{(a)}, \text{ and } \gamma^{(1)} \to \gamma^{(b)}, \tag{2.60}$$

in equations (2.55) to (2.59).

(iii) The Crossing Matrices.

The crossing matrices for abnormal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ scattering are obtained from equations (1.12) and (2.29) etc. We will use the same notation as in section 2.1 and we again find that the six crossing matrices have the block form

$$\Gamma \;=\; \tfrac{1}{4}\left[\begin{array}{c|c} \Gamma^{a} & 0 \\ \hline \Gamma^{c} & \Gamma^{b} \end{array}\right] ,$$

where Γ^{a} is a 5×5 matrix of numbers whose elements have the same absolute values in all six matrices, Γ^{b} is a 3×3 matrix which has also purely numerical elements of the same magnitude in all cases, while Γ^{c} is a 5×3 matrix of the masses.

It will not be necessary to give details of the calculations as they involve only simple algebra along the lines described in section 2.1. The matrix $\Gamma_{[st]}$ is given by

$$\Gamma^{a}_{[st]} \;=\; \begin{bmatrix} 1 & -1 & -1 & -1 & 1 \\ -4 & -2 & 2 & 0 & 4 \\ 4 & -2 & 2 & 0 & -4 \\ 6 & 0 & 0 & 2 & 6 \\ 1 & 1 & 1 & -1 & 1 \end{bmatrix} , \tag{2.61}$$

$$\Gamma^{b}_{[st]} \;=\; \begin{bmatrix} 0 & 4 & 4 \\ 2 & -2 & 2 \\ -2 & -2 & 2 \end{bmatrix} , \tag{2.62}$$

and $\Gamma^{c}_{[st]} \;=\; \tfrac{1}{2} \times$

$$\begin{bmatrix} (3m-\mu-3m'+\mu') & -(3m-\mu+3m'-\mu') & (m-3\mu+m'-3\mu') & (m+\mu-m'+\mu') & -(m-3\mu-m'+3\mu') \\ (-m+\mu+2m'+2\mu') & -(m-\mu+2m'+2\mu') & (-m+\mu+2m'+2\mu') & -(m-\mu) & -(m-\mu+2m'+2\mu') \\ -(2m+2\mu-m'+\mu') & -(2m+2\mu+m'-\mu') & (2m+2\mu-m'+\mu') & (m'-\mu') & (2m+2\mu+m'-\mu') \end{bmatrix}$$

(2.63)

The inverse matrix $\Gamma_{[ts]}$ will not be written out in detail. As before, $\Gamma^a_{[ts]}$ can be obtained easily by inspection from $\Gamma^a_{[st]}$ and the same is true for $\Gamma^b_{[ts]}$ from $\Gamma^b_{[st]}$. Instead of writing out $\Gamma^c_{[ts]}$ in full, we give an ad hoc prescription for obtaining it from $\Gamma^c_{[st]}$. If we interchange μ and m' and simultaneously change the signs of the first and second rows and the third and fourth columns in $\Gamma^c_{[st]}$ of eqn. (2.63), we obtain $\Gamma^c_{[ts]}$. No deep significance would seem to attatch to this prescription.

The crossing matrix $\Gamma_{[su]}$ may be obtained from the Fierz transformation (2.36) and the various momentum relations. However, it is not necessary to write this matrix out since it reflects the $t \leftrightarrow u$ crossing symmetry of (2.54). Remembering that $t \leftrightarrow u$ crossing involves the interchange of p and q so that for $\Gamma^c_{[su]}$, m and μ are interchanged in $\Gamma^c_{[st]}$, we obtain $\Gamma_{[su]}$ from $\Gamma_{[st]}$ by changing the signs of the odd-numbered rows. We will use the notation $\Gamma^c_{[su]} = \Gamma^c_{[st]}(m \leftrightarrow \mu)$ to denote this interchange of masses in future, remembering that there may also be some sign changes which will be mentioned explicitly whenever the abbreviation is employed. The crossing symmetry is also reflected in $\Gamma_{[us]}$ and enables us to write it down from $\Gamma_{[ts]}$ by changing the signs of the odd-numbered columns and remembering $\Gamma^c_{[us]} = \Gamma^c_{[ts]}(m \leftrightarrow \mu)$.

Finally we find that

$$\Gamma^a_{[ut]} = \Gamma^a_{[tu]}, \quad \Gamma^b_{[ut]} = \Gamma^b_{[tu]}, \quad \Gamma^c_{[ut]} = \Gamma^c_{[tu]}(m \leftrightarrow \mu), \tag{2.64}$$

where these matrices are given by

$$\Gamma^{a}_{[ut]} = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 4 & -2 & -2 & 0 & -4 \\ -4 & -2 & -2 & 0 & 4 \\ 6 & 0 & 0 & -2 & 6 \\ 1 & -1 & 1 & 1 & 1 \end{bmatrix}, \tag{2.65}$$

$$\Gamma^{b}_{[ut]} = \begin{bmatrix} 0 & 4 & -4 \\ 2 & 2 & 2 \\ -2 & 2 & 2 \end{bmatrix}, \tag{2.66}$$

and

$$\Gamma^{c}_{[ut]} = \tfrac{1}{2} \begin{bmatrix} -(3m+\mu-3m'-\mu') & (3m-\mu+3m'-\mu') & (m-3\mu+m'-3\mu') & (m-\mu-m'+\mu') & (m+3\mu-m'-3\mu') \\ -(2m+\mu-m'+2\mu') & -(2m-\mu+m'+2\mu') & -(2m+\mu-m'+2\mu') & (\mu-m') & (2m-\mu+m'+2\mu') \\ -(m-2\mu-2m'-\mu') & -(m+2\mu+2m'-\mu') & (m-2\mu-2m'-\mu') & (m-\mu') & -(m+2\mu+2m'-\mu') \end{bmatrix}. \tag{2.67}$$

Of course, in the crossing matrices given above, the covariants are always ordered as in eqns. (2.48), (2.50), and (2.53) and we see that these covariants do cross simply without the introduction of kinematic variables.

CHAPTER 3.

M-FUNCTIONS OF THE FOURTH KIND.

According to the way in which we classified M-functions for processes of any spin in section 1.4, we expect M-functions of the fourth kind to show some features in common with those of the third kind, except that the number of equivalence theorems is greatly increased for processes with M-functions of the fourth kind. In this chapter we examine in detail the M-function decomposition into invariant amplitudes for two abnormal scattering processes: the abnormal BB scattering $0 + 1 \rightarrow 1 + 1$ (for example $\pi\rho \rightarrow \rho\omega$) and the abnormal FF scattering $\frac{1}{2} + \frac{1}{2} \rightarrow \frac{1}{2} + \frac{3}{2}$ (for example $NN \rightarrow NN^*$).

For the BB example, the usual counting procedures show that we must have thirteen independent invariant amplitudes, but we can construct sixty possible covariants. This number is reduced to thirty-six by trivial applications of the subsidiary conditions, but we still require eleven equivalence theorems and twelve abnormal reductions in order to obtain our independent set of thirteen.

For abnormal $\frac{1}{2} + \frac{1}{2} \rightarrow \frac{1}{2} + \frac{3}{2}$, it is easily seen that we require sixteen independent amplitudes. In this case we find ten equivalence theorems and seven spinor identities. From the multiplicity of possible covariants and the large numbers of equivalence theorems required, it is clear that both of the processes considered in this chapter have

M-functions of the fourth kind, but we show that by following the procedures outlined above, it is possible to choose an independent set of kinematic covariants such that the invariant amplitudes are KSF.

It is interesting to note that, although the crossing matrices for abnormal $0 + 1 \to 1 + 1$ scattering are trivial compared with those for abnormal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{3}{2}$, in both cases the requirement of simple behaviour of the covariants under crossing, in the sense discussed above, is an essential factor in determining the choice of the independent set of covariants.

3.1 Abnormal $0 + 1 \to 1 + 1$ Scattering.

(i) Covariants.

This problem, for the first time in our detailed discussion of particular scattering processes, introduces vector indices, and our convention for the assignment of these indices to the particular vector particles is given in Fig. 4.(*) The s-channel is the direct channel as

(*) Notice that in order to bring the notation into line with that used in the rest of the thesis, this labelling differs somewhat from that used in the author's paper "The General Scattering PV → VV and Superconvergence", Nuovo Cimento 53, 625 (1968).

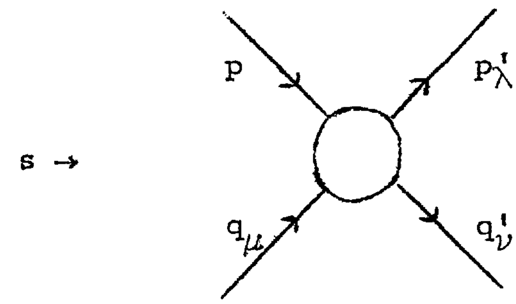

Fig. 4. Abnormal $0 + 1 \to 1 + 1$ Scattering.

shown, and q_μ, etc. symbolize the wave-functions for spin one particles with vector indices μ etc. The spin zero particle is associated with the momentum p.

Our thirteen independent covariants must be chosen from the twenty-seven of the form $\varepsilon_{\nu\lambda}(AB)C_\mu$, $\varepsilon_{\mu\lambda}(AB)C_\nu$, $\varepsilon_{\mu\nu}(AB)C_\lambda$, the twenty-seven $N_\mu A_\nu B_\lambda$, $N_\nu A_\lambda B_\mu$, $N_\lambda A_\mu B_\nu$, the three $N_\mu g_{\nu\lambda}$, $N_\nu g_{\mu\lambda}$, $N_\lambda g_{\mu\nu}$, and finally, the three of the form $\varepsilon_{\mu\nu\lambda}(A)$, where in these expressions A,B, and C can be any of the momenta P, Q, or Δ. Our set of natural covariants in the t-channel is chosen to be

$$\mathcal{K}_1 = \{\varepsilon_{\mu\lambda}(PQ)P_\nu + \varepsilon_{\nu\lambda}(PQ)P_\mu\},\quad \mathcal{K}_2 = \{\varepsilon_{\mu\lambda}(PQ)P_\nu - \varepsilon_{\nu\lambda}(PQ)P_\mu\},$$

$$\mathcal{K}_3 = \{\varepsilon_{\mu\lambda}(P\Delta)P_\nu + \varepsilon_{\nu\lambda}(P\Delta)P_\mu\},\quad \mathcal{K}_4 = \{\varepsilon_{\mu\lambda}(P\Delta)P_\nu - \varepsilon_{\nu\lambda}(P\Delta)P_\mu\},$$

$$\mathcal{K}_5 = \{\varepsilon_{\mu\lambda}(Q\Delta)P_\nu + \varepsilon_{\nu\lambda}(Q\Delta)P_\mu\},\quad \mathcal{K}_6 = \{\varepsilon_{\mu\lambda}(Q\Delta)P_\nu - \varepsilon_{\nu\lambda}(Q\Delta)P_\mu\},$$

$$\mathcal{K}_7 = \varepsilon_{\mu\nu}(PQ)Q_\lambda,\quad \mathcal{K}_8 = \varepsilon_{\mu\nu}(P\Delta)Q_\lambda,\quad \mathcal{K}_9 = \varepsilon_{\mu\nu}(Q\Delta)Q_\lambda,$$

$$\mathcal{K}_{10} = \varepsilon_{\mu\nu\lambda}(P),\quad \mathcal{K}_{11} = \varepsilon_{\mu\nu\lambda}(Q),\quad \mathcal{K}_{12} = \varepsilon_{\mu\nu\lambda}(\Delta),$$

$$\mathcal{K}_{13} = N_\lambda g_{\mu\nu}. \tag{3.1}$$

The sums and differences have been chosen in order that the covariants will exhibit simple crossing properties.

It would be possible to write down similar sets of kinematic covariants in terms of the s and u channel natural momenta by making simple substitutions in the set (3.1), but in general we do not find this necessary for BB scattering processes. Since we do not have the complication of different orderings of four spinors and two independent sets of γ matrices for each channel, the crossing matrices relating the covariants expressed in terms of the natural momenta of different channels are obtained from the use of eqns. (1.12) only. Such crossing matrices are reasonably trivial, but it is usually easier to perform calculations in the s and u channels directly in terms of the t-channel covariants (3.1) than it is to use the natural momenta of these channels and calculate the necessary crossing matrices. Such a procedure using only the covariants (3.1) means that the crossing matrices are used <u>implicitly</u> before any calculation in the s or u channels is performed, rather than <u>explicitly</u> after the calculation is completed. Our basic principle of treating all channels in a manifestly similar manner is not in any way disturbed by this use of just one set of covariants; it is, if anything, made stronger since we use the same covariants in each channel.

Crossing symmetry in abnormal $0 + 1 \rightarrow 1 + 1$ takes a slightly different form than it did for the processes considered in chapter 2. If, in the s-channel of any scattering process, one initial and one final state particle are identical self-conjugate bosons, then it is easy to see from eqns. (1.1), (1.5), and (1.8) that the s and u channels are identical. Under these circumstances crossing symmetry is observed for the t-channel amplitudes under $s \leftrightarrow u$ crossing. For the problem we are considering, this will occur if the particles symbolized by q_μ and q'_ν are identical self-conjugate bosons. The effect of $s \leftrightarrow u$ crossing is to interchange and charge conjugate these particles simultaneously. The kinematic covariants then behave as

$$\mathcal{K}_{\mu\nu\lambda}(P,Q) \rightarrow \mathcal{K}_{\nu\mu\lambda}(P,-Q), \tag{3.2}$$

and since the M-function is invariant under the crossing operation, we must obtain the crossing property of the t-channel invariant amplitudes under $s \leftrightarrow u$ crossing:

$$A_\kappa^{(t)}(\nu,t) = \xi_\kappa A_\kappa^{(t)}(-\nu,t), \tag{3.3}$$

where κ is the summation index in the M-function expansion and $\xi_\kappa = \pm 1$. For our amplitudes $A_1(\nu,t)$ to $A_{13}(\nu,t)$ for abnormal $0 + 1 \rightarrow 1 + 1$ scattering, it is easily seen that

$$\xi_1 = \xi_4 = \xi_5 = \xi_7 = \xi_9 = \xi_{10} = \xi_{12} = \xi_{13} = -1, \tag{3.4}$$

while

$$\xi_2 = \xi_3 = \xi_6 = \xi_8 = \xi_{11} = +1. \tag{3.5}$$

(ii) The Equivalence Theorems.

We mentioned in the introduction to this chapter that for abnormal $0 + 1 \to 1 + 1$ scattering we can write down sixty possible covariants. This list is trivially reduced to thirty-six covariants by use of the subsidiary conditions which in our formalism are

$$\Delta_\mu \to -2Q_\mu, \quad \Delta_\nu \to 2Q_\nu, \quad \Delta_\lambda \to -2P_\lambda. \tag{3.6}$$

The remaining twentythree equivalence theorems and abnormal reductions are all derived from the basic identity

$$\varepsilon_{\mu\nu\lambda\rho}g_{\alpha\beta} = \varepsilon_{\alpha\nu\lambda\rho}g_{\mu\beta} + \varepsilon_{\mu\alpha\lambda\rho}g_{\nu\beta} + \varepsilon_{\mu\nu\alpha\rho}g_{\lambda\beta} + \varepsilon_{\mu\nu\lambda\alpha}g_{\rho\beta} \tag{3.7}$$

by saturating with various momenta. We first contract eqn. (3.7) against $A^\lambda B^\rho C^\beta$ and use of the conditions (3.6) leads to nine equations between twentyone of the covariants. These equations do not, however, make it immediately obvious which of the covariants should be eliminated, as only the three covariants $\mathcal{K}_{10}$, $\mathcal{K}_{11}$, and $\mathcal{K}_{12}$ are multiplied by kinematic factors. But taking linear combinations of these equations makes it clear that the nine covariants on the left-hand-sides of the equivalence theorems (3.8) to (3.16) may be eliminated.

$$\varepsilon_{\mu\nu}(PQ)P_\lambda = \mathcal{K}_2 - \nu\mathcal{K}_{10} + P^2\mathcal{K}_{11}, \tag{3.8}$$

$$\varepsilon_{\mu\nu}(P\Delta)P_\lambda = \mathcal{K}_4 - P.\Delta\mathcal{K}_{10} + P^2\mathcal{K}_{12}, \tag{3.9}$$

$$\varepsilon_{\mu\nu}(Q\Delta)P_\lambda = \mathcal{K}_6 - P.\Delta\mathcal{K}_{11} + \nu\mathcal{K}_{12}, \tag{3.10}$$

$$2\varepsilon_{\mu\lambda}(PQ)Q_\nu = -\mathcal{K}_2 + \mathcal{K}_7 + (Q^2+\nu+\tfrac{1}{2}Q.\Delta)\mathcal{K}_{10} - (P^2+\nu+\tfrac{1}{2}P.\Delta)\mathcal{K}_{11}, \tag{3.11}$$

$$2\varepsilon_{\mu\lambda}(P\Delta)Q_\nu = -\mathcal{K}_4 + \mathcal{K}_8 + (P.\Delta + Q.\Delta + \tfrac{1}{2}\Delta^2)\mathcal{K}_{10} - $$
$$- (P^2 + \nu + \tfrac{1}{2}P.\Delta)\mathcal{K}_{12}, \tag{3.12}$$

$$2\varepsilon_{\mu\lambda}(Q\Delta)Q_\nu = -\mathcal{K}_6 + \mathcal{K}_9 + (P.\Delta + Q.\Delta + \tfrac{1}{2}\Delta^2)\mathcal{K}_{11} - $$
$$- (Q^2 + \nu + \tfrac{1}{2}Q.\Delta)\mathcal{K}_{12}, \tag{3.13}$$

$$2\varepsilon_{\nu\lambda}(PQ)Q_\mu = -\mathcal{K}_2 - \mathcal{K}_7 - (Q^2 - \nu - \tfrac{1}{2}Q.\Delta)\mathcal{K}_{10} - $$
$$- (P^2 - \nu + \tfrac{1}{2}P.\Delta)\mathcal{K}_{11}, \tag{3.14}$$

$$2\varepsilon_{\nu\lambda}(P\Delta)Q_\mu = -\mathcal{K}_4 - \mathcal{K}_8 + (P.\Delta - Q.\Delta + \tfrac{1}{2}\Delta^2)\mathcal{K}_{10} - $$
$$- (P^2 - \nu + \tfrac{1}{2}P.\Delta)\mathcal{K}_{12}, \tag{3.15}$$

$$2\varepsilon_{\nu\lambda}(Q\Delta)Q_\mu = -\mathcal{K}_6 - \mathcal{K}_9 + (P.\Delta - Q.\Delta + \tfrac{1}{2}\Delta^2)\mathcal{K}_{11} + $$
$$+ (Q^2 - \nu - \tfrac{1}{2}Q.\Delta)\mathcal{K}_{12}. \tag{3.16}$$

Equation (3.7) may be contracted against all possible combinations of four of the P, Q, and Δ and another nine nontrivial equations are obtained. Taking linear combinations of these equations multiplied by P or Q as the case may be, and using eqns. (3.8) to (3.16), we obtain twelve abnormal reductions eliminating covariants of the form of N multiplied by two other momenta. We distinguish these relations from the equivalence theorems because the covariants eliminated are of a different type from the set (3.1).

$$2N_{\mu}Q_{\nu}Q_{\lambda} = (Q.\Delta + \tfrac{1}{2}\Delta^2)\mathcal{K}_7 + (\nu + \tfrac{1}{2}P.\Delta)\mathcal{K}_9 - (Q^2 + \tfrac{1}{2}Q.\Delta)\mathcal{K}_8, \tag{3.17}$$

$$\begin{aligned} 2N_{\mu}Q_{\nu}P_{\lambda} = {} & (Q.\Delta + \tfrac{1}{2}\Delta^2)\mathcal{K}_2 - (Q^2 + \tfrac{1}{2}Q.\Delta)\mathcal{K}_4 + (\nu + \tfrac{1}{2}P.\Delta)\mathcal{K}_6, + \\ & + (Q^2P.\Delta - \nu Q.\Delta + \tfrac{1}{2}P.\Delta Q.\Delta - \tfrac{1}{2}\nu\Delta^2)\mathcal{K}_{10} + \\ & + (P^2Q.\Delta - \nu P.\Delta - \tfrac{1}{2}(P.\Delta)^2 + \tfrac{1}{2}P^2\Delta^2)\mathcal{K}_{11} + \\ & + (\nu^2 - P^2Q^2 + \tfrac{1}{2}\nu P.\Delta - \tfrac{1}{2}P^2Q.\Delta)\mathcal{K}_{12}, \end{aligned} \tag{3.18}$$

$$\begin{aligned} 2N_{\mu}P_{\nu}Q_{\lambda} = {} & Q.\Delta\mathcal{K}_1 + (P.\Delta + \tfrac{1}{2}\Delta^2)\mathcal{K}_2 - Q^2\mathcal{K}_3 - (\nu + \tfrac{1}{2}Q.\Delta)\mathcal{K}_4 + \\ & + \nu\mathcal{K}_5 + (P^2 + \tfrac{1}{2}P.\Delta)\mathcal{K}_6 + P.\Delta\mathcal{K}_7 - \nu\mathcal{K}_8 + P^2\mathcal{K}_9, \end{aligned} \tag{3.19}$$

$$\begin{aligned} 2N_{\mu}P_{\nu}P_{\lambda} = {} & -\tfrac{1}{2}\Delta^2\mathcal{K}_1 + (P.\Delta - Q.\Delta)\mathcal{K}_2 + \tfrac{1}{2}Q.\Delta\mathcal{K}_3 + (Q^2 - \nu)\mathcal{K}_4 - \\ & - \tfrac{1}{2}P.\Delta\mathcal{K}_5 + (P^2 - \nu)\mathcal{K}_6 + P.\Delta\mathcal{K}_7 - \nu\mathcal{K}_8 + P^2\mathcal{K}_9, \end{aligned} \tag{3.20}$$

$$2N_{\nu}Q_{\mu}Q_{\lambda} = -(Q.\Delta - \tfrac{1}{2}\Delta^2)\mathcal{K}_7 + (Q^2 - \tfrac{1}{2}Q.\Delta)\mathcal{K}_8 - (\nu - \tfrac{1}{2}P.\Delta)\mathcal{K}_9, \tag{3.21}$$

$$\begin{aligned} 2N_{\nu}Q_{\mu}P_{\lambda} = {} & -(Q.\Delta - \tfrac{1}{2}\Delta^2)\mathcal{K}_2 + (Q^2 - \tfrac{1}{2}Q.\Delta)\mathcal{K}_4 - (\nu - \tfrac{1}{2}P.\Delta)\mathcal{K}_6 + \\ & + (\nu Q.\Delta - Q^2P.\Delta + \tfrac{1}{2}P.\Delta Q.\Delta - \tfrac{1}{2}\nu\Delta^2)\mathcal{K}_{10} + \\ & + (\nu P.\Delta - P^2Q.\Delta - \tfrac{1}{2}(P.\Delta)^2 + \tfrac{1}{2}P^2\Delta^2)\mathcal{K}_{11} + \\ & + (P^2Q^2 - \nu^2 + \tfrac{1}{2}\nu P.\Delta - \tfrac{1}{2}P^2Q.\Delta)\mathcal{K}_{12}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} 2N_{\nu}P_{\mu}Q_{\lambda} = {} & Q.\Delta\mathcal{K}_1 + (P.\Delta + \tfrac{1}{2}\Delta^2)\mathcal{K}_2 - Q^2\mathcal{K}_3 - (\nu + \tfrac{1}{2}Q.\Delta)\mathcal{K}_4 + \\ & + \nu\mathcal{K}_5 + (P^2 + \tfrac{1}{2}P.\Delta)\mathcal{K}_6 - P.\Delta\mathcal{K}_7 + \nu\mathcal{K}_8 - P^2\mathcal{K}_9. \end{aligned} \tag{3.23}$$

$$2N_\nu P_\mu P_\lambda = -\tfrac{1}{2}\Delta^2\mathcal{K}_1 - (P.\Delta + Q.\Delta)\mathcal{K}_2 + \tfrac{1}{2}Q.\Delta\mathcal{K}_3 + (Q^2 + \nu)\mathcal{K}_4 -$$
$$- \tfrac{1}{2}P.\Delta\mathcal{K}_5 - (P^2 + \nu)\mathcal{K}_6 + P.\Delta\mathcal{K}_7 - \nu\mathcal{K}_8 + P^2\mathcal{K}_9, \tag{3.24}$$

$$2N_\lambda Q_\mu Q_\nu = Q.\Delta\mathcal{K}_2 - Q^2\mathcal{K}_4 + \nu\mathcal{K}_6 + \tfrac{1}{2}\Delta^2\mathcal{K}_7 - \tfrac{1}{2}Q.\Delta\mathcal{K}_8 +$$
$$+ \tfrac{1}{2}P.\Delta\mathcal{K}_9 + (Q^2P.\Delta - \nu Q.\Delta - \tfrac{1}{2}(Q.\Delta)^2 + \tfrac{1}{2}Q^2\Delta^2)\mathcal{K}_{10} +$$
$$+ (P^2Q.\Delta - \nu P.\Delta + \tfrac{1}{2}P.\Delta Q.\Delta - \tfrac{1}{2}\nu\Delta^2)\mathcal{K}_{11} +$$
$$+ (\nu^2 - P^2Q^2 - \tfrac{1}{2}Q^2P.\Delta + \tfrac{1}{2}\nu Q.\Delta)\mathcal{K}_{12}, \tag{3.25}$$

$$2N_\lambda Q_\mu P_\nu = (P.\Delta - Q.\Delta + \tfrac{1}{2}\Delta^2)\mathcal{K}_2 - (\nu - Q^2 + \tfrac{1}{2}Q.\Delta)\mathcal{K}_4 -$$
$$- (\nu - P^2 - \tfrac{1}{2}P.\Delta)\mathcal{K}_6 + P.\Delta\mathcal{K}_7 - \nu\mathcal{K}_8 + P^2\mathcal{K}_9, \tag{3.26}$$

$$2N_\lambda P_\mu Q_\nu = (P.\Delta + Q.\Delta + \tfrac{1}{2}\Delta^2)\mathcal{K}_2 - (\nu + Q^2 + \tfrac{1}{2}Q.\Delta)\mathcal{K}_4 +$$
$$+ (\nu + P^2 + \tfrac{1}{2}P.\Delta)\mathcal{K}_6 - P.\Delta\mathcal{K}_7 + \nu\mathcal{K}_8 - P^2\mathcal{K}_9, \tag{3.27}$$

$$2N_\lambda P_\mu P_\nu = -(P.\Delta + \tfrac{1}{2}\Delta^2)\mathcal{K}_1 - Q.\Delta\mathcal{K}_2 + (\nu + \tfrac{1}{2}Q.\Delta)\mathcal{K}_3 + Q^2\mathcal{K}_4 -$$
$$- (P^2 + \tfrac{1}{2}P.\Delta)\mathcal{K}_5 - \nu\mathcal{K}_6 + P.\Delta\mathcal{K}_7 - \nu\mathcal{K}_8 + P^2\mathcal{K}_9. \tag{3.28}$$

Equations (3.8) to (3.28) give us twentyone of our twentythree equations and the remaining two equivalence theorems must be used to eliminate two of the remaining three covariants $N_\mu g_{\nu\lambda}$, $N_\nu g_{\lambda\mu}$, and $N_\lambda g_{\mu\nu}$. We choose to retain $N_\lambda g_{\mu\nu}$ as our thirteenth covariant because from eqn. (3.2) we see that s ↔ u crossing involves interchanging the μ and ν indices and $N_\lambda g_{\mu\nu}$ has simple crossing properties while the other two covariants cross into each other. We have chosen the linear combinations of eqns. (3.1) for $\mathcal{K}_1$ to $\mathcal{K}_6$ for similar simplicity under s ↔ u crossing.

By an appropriate contraction of (3.7), we obtain

$$g_{\alpha\beta}N_\gamma - g_{\alpha\gamma}N_\beta = -\varepsilon_{\beta\gamma}(PQ)\Delta_\alpha + \varepsilon_{\beta\gamma}(P\Delta)Q_\alpha - \varepsilon_{\beta\gamma}(Q\Delta)P_\alpha, \quad (3.29)$$

from which, with appropriate choices of the indices μ, ν, λ for α, β, γ, and use of eqn. (3.6) and the other equivalence theorems, we get our two remaining equivalence theorems:

$$\begin{aligned} 2N_\mu g_{\nu\lambda} = {} & -2\mathcal{K}_2 + \mathcal{K}_4 + \mathcal{K}_5 + \mathcal{K}_6 + 2\mathcal{K}_7 - \mathcal{K}_8 + \\ & + (2Q^2 + 2\nu - P.\Delta - \tfrac{1}{2}\Delta^2)\mathcal{K}_{10} - (2P^2 + 2\nu + P.\Delta)\mathcal{K}_{11} + \\ & + (P^2 + \nu + \tfrac{1}{2}P.\Delta)\mathcal{K}_{12} + 2\mathcal{K}_{13}, \end{aligned} \quad (3.30)$$

and

$$\begin{aligned} 2N_\nu g_{\mu\lambda} = {} & 2\mathcal{K}_2 + \mathcal{K}_4 + \mathcal{K}_5 - \mathcal{K}_6 + 2\mathcal{K}_7 + \mathcal{K}_8 + \\ & + (2Q^2 - 2\nu - P.\Delta - \tfrac{1}{2}\Delta^2)\mathcal{K}_{10} + (2P^2 - 2\nu + P.\Delta)\mathcal{K}_{11} + \\ & + (P^2 - \nu + \tfrac{1}{2}P.\Delta)\mathcal{K}_{12} + 2\mathcal{K}_{13}. \end{aligned} \quad (3.31)$$

It is clear from the above equations that we have succeeded in reducing the complete set of covariants to the independent set of thirteen (3.1) without introducing kinematic singularities and thus, according to our general philosophy, the corresponding invariant amplitudes will be KSF.

3.2 Abnormal $\frac{1}{2}+\frac{1}{2}\to\frac{1}{2}+\frac{3}{2}$ Scattering.

(i) Covariants.

The second process which we will consider with an M-function of the fourth kind is abnormal $\frac{1}{2}+\frac{1}{2}\to\frac{1}{2}+\frac{3}{2}$ scattering. The physically interesting example of this type of process is, of course, $NN \to NN^*$. As this is an abnormal FF process, we expect it to have features in common with abnormal $\frac{1}{2}+\frac{1}{2}\to\frac{1}{2}+\frac{1}{2}$ considered in section 2.2. This is indeed the case, and quite a lot of the work of section 2.2 is easily generalized for the present problem. We will assume that our spin $\frac{3}{2}$ particle is associated with the momentum q' and is described by the Rarita-Schwinger type wave-function $u_\mu^{\tau'}(q')$ where μ is the vector index and τ' the spinor index.

We obtain sixteen possible covariants for abnormal $\frac{1}{2}+\frac{1}{2}\to\frac{1}{2}+\frac{3}{2}$ by multiplying those of eqn. (2.48) by P_μ or Q_μ. However, there is also the possibility of some covariants of an entirely new type and associated new equivalence theorems. We require a total of sixteen independent covariants to be chosen from the sixteen mentioned above and new covariants like $\gamma_5^{(2)}\gamma_\mu^{(1)}$, $1^{(2)}\gamma_5^{(1)}\gamma_\mu^{(1)}$, etc. Note that $\gamma_\mu^{(2)}$ can never appear because of the subsidiary condition

$$\gamma_\mu^{(2)}u^\mu(q') = 0. \tag{3.32}$$

The t-channel set which we choose is

$$\mathcal{K}_1^{(t)} = Q_\mu 1^{(2)}\gamma_5^{(1)}, \qquad \mathcal{K}_2^{(t)} = P_\mu 1^{(2)}\gamma_5^{(1)},$$

$$\mathcal{K}_3^{(t)} = Q_\mu\gamma_5^{(2)}1^{(1)}, \qquad \mathcal{K}_4^{(t)} = P_\mu\gamma_5^{(2)}1^{(1)},$$

$$\mathcal{K}_5^{(t)} = Q_\mu\gamma^{(2)}.\gamma_5^{(1)}\gamma^{(1)}, \quad \mathcal{K}_6^{(t)} = P_\mu\gamma^{(2)}.\gamma_5^{(1)}\gamma^{(1)},$$

$$\mathcal{K}_7^{(t)} = P_\mu\gamma_5^{(2)}\gamma^{(2)}.\gamma^{(1)}, \qquad \mathcal{K}_8^{(t)} = Q_\mu\gamma_5^{(2)}\gamma^{(2)}.\gamma^{(1)},$$

$$\mathcal{K}_9^{(t)} = \tfrac{1}{4}P_\mu\varepsilon(\sigma^{(2)}\sigma^{(1)}), \qquad \mathcal{K}_{10}^{(t)} = \tfrac{1}{4}Q_\mu\varepsilon(\sigma^{(2)}\sigma^{(1)}), \tag{3.33}$$

$$\mathcal{K}_{11}^{(t)} = \{P_\mu\gamma_5^{(2)}P.\gamma^{(2)}1^{(1)} - 3Q_\mu 1^{(2)}\gamma_5^{(1)}Q.\gamma^{(1)}\},$$

$$\mathcal{K}_{12}^{(t)} = \{P_\mu 1^{(2)}\gamma_5^{(1)}Q.\gamma^{(1)} - 3Q_\mu\gamma_5^{(2)}P.\gamma^{(2)}1^{(1)}\},$$

$$\mathcal{K}_{13}^{(t)} = P_\mu\{P.\gamma^{(2)}\gamma_5^{(1)} - \gamma_5^{(2)}Q.\gamma^{(1)}\}, \qquad \mathcal{K}_{14}^{(t)} = \gamma_5^{(2)}\gamma_\mu^{(1)},$$

$$\mathcal{K}_{15}^{(t)} = 1^{(2)}\gamma_5^{(1)}\gamma_\mu^{(1)}, \qquad \mathcal{K}_{16}^{(t)} = \gamma_5^{(2)}\gamma^{(2)\nu}i\sigma_{\mu\nu}^{(1)} .$$

The combinations in $\mathcal{K}_{11}$, $\mathcal{K}_{12}$, and $\mathcal{K}_{13}$ were chosen so that the crossing matrices would be free of kinematic variables. The requirements of simple crossing properties and crossing matrices free of kinematic variables impose stringent conditions on the possible form of the covariants in this case. We find that we have to choose the three covariants from the six $P_\mu\gamma_5^{(2)}P.\gamma^{(2)}1^{(1)}$, $P_\mu 1^{(2)}\gamma_5^{(1)}Q.\gamma^{(1)}$, $Q_\mu\gamma_5^{(2)}P.\gamma^{(2)}1^{(1)}$, $Q_\mu 1^{(2)}\gamma_5^{(1)}Q.\gamma^{(1)}$, $P_\mu\{P.\gamma^{(2)}\gamma_5^{(1)} - \gamma_5^{(2)}Q.\gamma^{(1)}\}$, and $Q_\mu\{P.\gamma^{(2)}\gamma_5^{(1)} - \gamma_5^{(2)}Q.\gamma^{(1)}\}$. We shall see in the next sectionthat these covariants are all multiplied by constants in the equivalence theorems so we can eliminate any of them at will. In order to make our choice, we must consider the form of the crossing matrices, bearing in mind our basic principle that the covariants in different channels must look alike. The factor of -3 in $\mathcal{K}_{11}$ and $\mathcal{K}_{12}$ comes from eqns. (1.12) and the subsidiary condition $q'_\mu u^\mu(q') = 0$, which together mean that the momenta are related by

$$\Lambda_\mu = \tfrac{1}{2}(P_\mu - 3Q_\mu) = \tfrac{1}{2}(3\bar{\Lambda}_\mu - \bar{\Lambda}'_\mu),$$

$$\Lambda'_\mu = \tfrac{1}{2}(P_\mu + Q_\mu) = \tfrac{1}{2}(\bar{\Lambda}_\mu + \bar{\Lambda}'_\mu).$$

Similarly, for P_μ and Q_μ in terms of Λ_μ, Λ'_μ, $\bar{\Lambda}_\mu$, and $\bar{\Lambda}'_\mu$, and $\bar{\Lambda}_\mu$ and $\bar{\Lambda}'_\mu$ in terms of P_μ, Q_μ, Λ_μ, and Λ'_μ, the factors -3 and 1 occur , making the choice of covariants (3.33) consistent with our principles.

The s-channel natural covariants are obtained from the set (3.33) by the substitutions (2.49). They are

$$\begin{aligned}
&\mathcal{K}_1^{(s)} = \Lambda'_\mu 1'\gamma_5, && \mathcal{K}_2^{(s)} = \Lambda_\mu 1'\gamma_5,\\
&\mathcal{K}_3^{(s)} = \Lambda'_\mu \gamma'_5 1, && \mathcal{K}_4^{(s)} = \Lambda_\mu \gamma'_5 1,\\
&\mathcal{K}_5^{(s)} = \Lambda'_\mu \gamma'\cdot\gamma_5\gamma, && \mathcal{K}_6^{(s)} = \Lambda_\mu \gamma'\cdot\gamma_5\gamma,\\
&\mathcal{K}_7^{(s)} = \Lambda_\mu \gamma'_5\gamma'\cdot\gamma, && \mathcal{K}_8^{(s)} = \Lambda'_\mu \gamma'_5\gamma'\cdot\gamma,\\
&\mathcal{K}_9^{(s)} = \tfrac{1}{4}\Lambda_\mu \varepsilon(\sigma'\sigma), && \mathcal{K}_{10}^{(s)} = \tfrac{1}{4}\Lambda'_\mu \varepsilon(\sigma'\sigma),\\
&\mathcal{K}_{11}^{(s)} = \{\Lambda_\mu \gamma'_5\Lambda.\gamma' 1 - 3\Lambda'_\mu 1'\gamma_5\Lambda'.\gamma\}, &&\\
&\mathcal{K}_{12}^{(s)} = \{\Lambda_\mu 1'\gamma_5\Lambda'.\gamma - 3\Lambda'_\mu\gamma'_5\Lambda.\gamma' 1\}, &&\\
&\mathcal{K}_{13}^{(s)} = \Lambda_\mu\{\Lambda.\gamma'\gamma_5 - \gamma'_5\Lambda'.\gamma\}, && \mathcal{K}_{14}^{(s)} = \gamma'_5\gamma_\mu,\\
&\mathcal{K}_{15}^{(s)} = 1'\gamma_5\gamma_\mu, && \mathcal{K}_{16}^{(s)} = \gamma'_5\gamma'^\nu i\sigma_{\mu\nu}.
\end{aligned} \tag{3.34}$$

For the u-channel covariants, a similar situation arises as in abnormal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ scattering. Crossing symmetry is shown for identical <u>initial</u> s-channel particles and we again require the TCP of the u-channel defined in eqn. (1.8). We take account of this, as before, by re-ordering the natural u-channel covariants and once again, obtain the reordered set from (3.33) by the substitutions (2.52). The set of u-channel covariants which we shall use is then

$$
\begin{aligned}
&\mathcal{K}_1^{(u)} = \bar{\Lambda}_\mu \gamma_5^{(b)} 1^{(a)}, && \mathcal{K}_2^{(u)} = \bar{\Lambda}'_\mu \gamma_5^{(b)} 1^{(a)}, \\
&\mathcal{K}_3^{(u)} = \bar{\Lambda}_\mu 1^{(b)} \gamma_5^{(a)}, && \mathcal{K}_4^{(u)} = \bar{\Lambda}'_\mu 1^{(b)} \gamma_5^{(a)}, \\
&\mathcal{K}_5^{(u)} = \bar{\Lambda}_\mu \gamma_5^{(b)} \gamma^{(b)} \cdot \gamma^{(a)}, && \mathcal{K}_6^{(u)} = \bar{\Lambda}'_\mu \gamma_5^{(b)} \gamma^{(b)} \cdot \gamma^{(a)}, \\
&\mathcal{K}_7^{(u)} = \bar{\Lambda}'_\mu \gamma^{(b)} \cdot \gamma_5^{(a)} \gamma^{(a)}, && \mathcal{K}_8^{(u)} = \bar{\Lambda}_\mu \gamma^{(b)} \cdot \gamma_5^{(a)} \gamma^{(a)}, \\
&\mathcal{K}_9^{(u)} = \tfrac{1}{4}\bar{\Lambda}'_\mu \varepsilon(\sigma^{(b)}\sigma^{(a)}), && \mathcal{K}_{10}^{(u)} = \tfrac{1}{4}\bar{\Lambda}_\mu \varepsilon(\sigma^{(b)}\sigma^{(a)}), \qquad (3.35) \\
&\mathcal{K}_{11}^{(u)} = \{\bar{\Lambda}'_\mu 1^{(b)} \gamma_5^{(a)} \bar{\Lambda}' \cdot \gamma^{(a)} - 3\bar{\Lambda}_\mu \gamma_5^{(b)} \bar{\Lambda} \cdot \gamma^{(b)} 1^{(a)}\}, \\
&\mathcal{K}_{12}^{(u)} = \{\bar{\Lambda}'_\mu \gamma_5^{(b)} \bar{\Lambda} \cdot \gamma^{(b)} 1^{(a)} - 3\bar{\Lambda}_\mu 1^{(b)} \gamma_5^{(a)} \bar{\Lambda}' \cdot \gamma^{(a)}\}, \\
&\mathcal{K}_{13}^{(u)} = \bar{\Lambda}'_\mu \{\gamma_5^{(b)} \bar{\Lambda}' \cdot \gamma^{(a)} - \bar{\Lambda} \cdot \gamma^{(b)} \gamma_5^{(a)}\}, && \mathcal{K}_{14}^{(u)} = \gamma_\mu^{(b)} \gamma_5^{(a)}, \\
&\mathcal{K}_{15}^{(u)} = \gamma_5^{(b)} \gamma_\mu^{(b)} 1^{(a)}, && \mathcal{K}_{16}^{(u)} = i\sigma_{\mu\nu}^{(b)} \gamma_5^{(a)} \gamma^{(a)\nu}.
\end{aligned}
$$

With this choice of u-channel covariants, the crossing symmetry relation for the s-channel invariant amplitudes under $t \leftrightarrow u$ crossing is obtained in the same way as the previous crossing symmetry relations, and it is

$$A_i^{(s)}(\tilde{\nu}, s) = -(-)^i A_i^{(s)}(-\tilde{\nu}, s), \quad i = 1,\ldots,16. \qquad (3.36)$$

(ii) The Equivalence Theorems.

Just as we obtained several of the covariants for abnormal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{3}{2}$ scattering by a simple generalization of the work of section (2.2), it is clear that we obtain six equivalence theorems by multiplying eqns. (2.55), (2.56), and (2.57) by P_μ and Q_μ. These theorems eliminate $[\{P.\gamma^{(2)}\gamma_5^{(1)} + \gamma_5^{(2)}Q.\gamma^{(1)}\},\ \gamma_5^{(2)}P.\gamma^{(2)}Q.\gamma^{(1)},\ P.\gamma^{(2)}\gamma_5^{(1)}Q.\gamma^{(1)}]$ $\times[P_\mu,\ Q_\mu]$ where the notation signifies that each of the covariants in the first bracket is multiplied by P_μ and Q_μ to give two covariants for the higher spin process.

In exactly the same way, multiplying eqns. (2.58) and (2.59) by P_μ and Q_μ gives us four spinor identities for abnormal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{3}{2}$. These identities eliminate $[\gamma_5^{(2)}Q_\lambda\gamma_\nu^{(2)}i\sigma_{\lambda\nu}^{(1)},\ i\sigma_{\lambda\nu}^{(2)}P_\lambda\gamma_5^{(1)}\gamma_\nu^{(1)}]\times[P_\mu,\ Q_\mu]$. Since these ten equations are obtained in such a trivial manner from the previous work, they will not be written out here. It is easy to see from eqns. (2.48) and (3.33) how the covariants of eqn. (2.48) change under these multiplications.

The possibility of the entirely new type of covariant involving $\gamma_\mu^{(1)}$ mentioned above does, however, mean that we must have new equivalence theorems. These are derived from the consideration of double epsilon forms as before and we find four new equivalence theorems and three spinor identities. First we consider the form $\varepsilon^\nu(Q\gamma^{(2)}\gamma^{(2)})\gamma_5^{(1)}\varepsilon_{\mu\nu}(P\gamma^{(1)})$ and expand it in two ways as before. This leads to the equivalence theorem

$$P_\mu 1^{(2)}\gamma_5^{(1)}Q.\gamma^{(1)} = \tfrac{1}{2}(m-m')\mathcal{K}_3^{(t)} + \tfrac{1}{2}(m+m')\mathcal{K}_4^{(t)} + \mu'\mathcal{K}_8^{(t)} +$$

$$+ \tfrac{1}{2}(m+m')\mathcal{K}_{10}^{(t)} - \tfrac{1}{4}\{(\mu+\mu')^2 + (m+m')^2 - t\}\mathcal{K}_{14}^{(t)} +$$

$$+ \nu\mathcal{K}_{15}^{(t)} + \tfrac{1}{4}(m+m')(\mu+\mu')\mathcal{K}_{16}^{(t)} + Q_\mu\gamma_5^{(2)}Q.\gamma^{(1)}. \quad (3.37)$$

Multiplying eqn. (3.37) by $\gamma_5^{(2)}\gamma_5^{(1)}$ leads to

$$Q_\mu 1^{(2)}\gamma_5^{(1)}Q.\gamma^{(1)} = -\tfrac{1}{2}(m+m')\mathcal{K}_1^{(t)} - \tfrac{1}{2}(m-m')\mathcal{K}_2^{(t)} + \mu'\mathcal{K}_5^{(t)} -$$

$$- \tfrac{1}{2}(m-m')\mathcal{K}_{10}^{(t)} - \nu\mathcal{K}_{14}^{(t)} + \tfrac{1}{4}\{(m-m')^2 + (\mu-\mu')^2 - t\}\mathcal{K}_{15}^{(t)} -$$

$$- \tfrac{1}{4}(m-m')(\mu-\mu')\mathcal{K}_{16}^{(t)} + P_\mu\gamma_5^{(2)}Q.\gamma^{(1)}. \quad (3.38)$$

The expansion of $\gamma_5^{(2)}\varepsilon^\nu(P\gamma^{(2)}\gamma^{(2)})\gamma^{(2)\lambda}\varepsilon_{\mu\nu}(\Delta\gamma^{(1)})\gamma_\lambda^{(1)}$ leads to

$$Q_\mu\gamma_5^{(2)}P.\gamma^{(2)}1^{(1)} = \tfrac{1}{2}(\mu-\mu')\mathcal{K}_2^{(t)} + \tfrac{1}{2}(\mu+\mu')\mathcal{K}_4^{(t)} +$$

$$+ \tfrac{1}{2}(m-m')\mathcal{K}_5^{(t)} + \tfrac{1}{2}(m+m')\mathcal{K}_8^{(t)} - \tfrac{1}{4}(m+m')(\mu+\mu')\mathcal{K}_{14}^{(t)} -$$

$$- \tfrac{1}{4}(m-m')(\mu-\mu')\mathcal{K}_{15}^{(t)} + \tfrac{t}{4}\mathcal{K}_{16}^{(t)} - Q_\mu P.\gamma^{(2)}\gamma_5^{(1)}, \quad (3.39)$$

and the fourth equivalence theorem is obtained by considering the expansion of $\gamma_5^{(2)}\varepsilon_{\mu\nu}(Q\gamma^{(2)})\varepsilon^\nu(P\gamma^{(1)}\gamma^{(1)})$:

$$\gamma_5^{(2)}P.\gamma^{(2)}Q_\nu i\sigma_{\mu\nu}^{(1)} = \mu'\mathcal{K}_1^{(t)} - \tfrac{1}{2}(m+m')\mathcal{K}_5^{(t)} - \tfrac{1}{2}(m-m')\mathcal{K}_6^{(t)} +$$

$$+ \tfrac{1}{2}(\mu+\mu')\mathcal{K}_9^{(t)} + \tfrac{1}{4}(m+m')(\mu+\mu')\mathcal{K}_{15}^{(t)} + \nu\mathcal{K}_{16}^{(t)} +$$

$$+ P_\mu P.\gamma^{(2)}\gamma_5^{(1)}. \quad (3.40)$$

The three spinor identities are similarly obtained: first by considering $\varepsilon_{\mu\nu\lambda}(\gamma^{(2)})\gamma_5^{(1)}\varepsilon^{\nu\lambda}(P\gamma^{(1)})$, which gives

$$P.\gamma^{(2)}\gamma_5^{(1)}\gamma_\mu^{(1)} = \mathcal{K}_6^{(t)} - \mathcal{K}_8^{(t)} + \tfrac{1}{2}(\mu+\mu')\mathcal{K}_{14}^{(t)} - $$

$$- \tfrac{1}{2}(m+m')\mathcal{K}_{16}^{(t)}, \qquad (3.41)$$

and multiplying (3.41) by $\gamma_5^{(2)}\gamma_5^{(1)}$ leads to the second identity

$$\gamma_5^{(2)}P.\gamma^{(2)}\gamma_\mu^{(1)} = -\mathcal{K}_5^{(t)} + \mathcal{K}_7^{(t)} + \tfrac{1}{2}(\mu-\mu')\mathcal{K}_{15}^{(t)} - $$

$$- \tfrac{1}{2}(m-m')\mathcal{K}_{16}^{(t)}, \qquad (3.42)$$

while the expansion of $\varepsilon^{\nu}(\Delta\gamma^{(2)}\gamma^{(2)})\gamma_5^{(1)}\varepsilon_{\mu\nu}(P\gamma^{(1)})$ leads to the third,

$$\gamma_5^{(2)}Q_\nu i\sigma_{\mu\nu}^{(1)} = \mathcal{K}_2^{(t)} + \mathcal{K}_{10}^{(t)} - \tfrac{1}{2}(m-m')\mathcal{K}_{15}^{(t)} + \tfrac{1}{2}(\mu-\mu')\mathcal{K}_{16}^{(t)}. \qquad (3.43)$$

There are, of course, other equivalent ways of obtaining these equations and also further relations for covariants involving the Levi-Civita tensor $\varepsilon_{\mu\nu\lambda\rho}$ rather than γ_5, but these are easily derivable from the equations above and the γ matrix formulae of appendix A.

The corresponding equations for the s and u channels can be obtained as before by the substitutions (2.21) and (2.60) respectively in the above equations.

(iii) The Crossing Matrices.

Abnormal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{3}{2}$ is a four spinor scattering process and consequently the crossing matrices are non-trivial. Our notation is the same as that outlined in chapter 2 and as stated above, the calculation of these crossing matrices is an essential part of the procedure for finding KSF invariant amplitudes. We must use the Fierz transformations (2.29) etc, but because of the similarity between abnormal $\frac{1}{2} + \frac{1}{2} \rightarrow \frac{1}{2} + \frac{3}{2}$ and abnormal $\frac{1}{2} + \frac{1}{2} \rightarrow \frac{1}{2} + \frac{1}{2}$, we have essentially already calculated the appropriate Fierz matrices for the first thirteen covariants. The three new covariants $\mathcal{K}_{14}$, $\mathcal{K}_{15}$, and $\mathcal{K}_{16}$ in fact transform independently of the other covariants and the necessary 3 × 3 matrices (Γ^c below) are easily calculated. We have, however, the additional new feature of a vector index and the related momentum subsidiary conditions. This leads to the relations

$$\Delta_\mu \rightarrow 2Q_\mu, \quad K_\mu \rightarrow 2\Lambda'_\mu, \quad \overline{K}_\mu \rightarrow 2\overline{\Lambda}_\mu, \tag{3.44}$$

which are also required for the calculation of the full 16 × 16 crossing matrices.

As before, we find that the six crossing matrices may conveniently be split into the block form:

$$\Gamma = \frac{1}{8}\left[\begin{array}{c|c|c} \Gamma^a & \multicolumn{2}{c}{0} \\ \hline \Gamma^d & \Gamma^b & 0 \\ \hline 0 & 0 & \Gamma^c \end{array}\right],$$

where Γ^a is a 10×10, and Γ^b and Γ^c are 3×3, matrices of numbers only, while Γ^d is a 10×3 matrix of the masses. The elements of Γ^a, Γ^b, and Γ^c all have the same absolute values in each of the six matrices.

The result for $\Gamma_{[st]}$ is given by:

$$\Gamma^{a}_{[st]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -3 & 1 & -3 & 1 & -3 & 1 & 1 & -3 & -1 & 3 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -3 & 1 & -3 & 1 & 3 & -1 & -1 & 3 & -1 & 3 \\ -4 & -4 & 4 & 4 & 2 & 2 & -2 & -2 & 0 & 0 \\ 12 & -4 & -12 & 4 & -6 & 2 & -2 & 6 & 0 & 0 \\ -12 & 4 & 12 & -4 & -6 & 2 & -2 & 6 & 0 & 0 \\ 4 & 4 & -4 & -4 & 2 & 2 & -2 & -2 & 0 & 0 \\ -18 & 6 & -18 & 6 & 0 & 0 & 0 & 0 & 2 & -6 \\ 6 & 6 & 6 & 6 & 0 & 0 & 0 & 0 & 2 & 2 \end{bmatrix}, \qquad (3.45)$$

$$\Gamma^{b}_{[st]} = \begin{bmatrix} -4 & 4 & -8 \\ -4 & 4 & 8 \\ 4 & 4 & 0 \end{bmatrix}, \qquad (3.46)$$

$$\Gamma^{c}_{[st]} = \begin{bmatrix} 4 & 4 & -4 \\ 4 & 4 & 4 \\ -8 & 8 & 0 \end{bmatrix}, \qquad (3.47)$$

and

$$\Gamma^{d}_{[st]} = \frac{1}{2}\begin{bmatrix} -3(m+3\mu-m'-3\mu') & (5m-\mu+7m'+5\mu') & 3(3m+\mu-3m'-\mu') & (m-5\mu-5m'-7\mu') & -3(m+3\mu+m'+3\mu') & (5m-\mu-7m'-5\mu') & (m-5\mu+5m'+7\mu') & 3(3m+\mu+3m'+\mu') & (3m-3\mu+m'-\mu') & 3(m-\mu-m'+\mu') \\ -3(m+3\mu-m'-3\mu') & -(7m+5\mu+5m'-\mu') & 3(3m+\mu-3m'-\mu') & (5m+7\mu-m'+5\mu') & -3(m+3\mu+m'+3\mu') & -(7m+5\mu-5m'+\mu') & (5m+7\mu+m'-5\mu') & 3(3m+\mu+3m'+\mu') & -(m-\mu+3m'-3\mu') & 3(m-\mu-m'+\mu') \\ 3(m-3\mu-m'+3\mu') & -(m-3\mu-m'+3\mu') & -3(3m-\mu-3m'+\mu') & (3m-\mu-3m'+\mu') & -3(m-3\mu+m'-3\mu') & (m-3\mu+m'-3\mu') & -(3m-\mu+3m'-\mu') & 3(3m-\mu+3m'-\mu') & (m+\mu-m'-\mu') & -3(m+\mu-m'-\mu') \end{bmatrix} . \quad (3.48)$$

The inverse matrix $\Gamma_{[ts]}$ may be obtained by a prescription similar to that given in section 2.2. Inspection of $\Gamma^{a}_{[st]}$, $\Gamma^{b}_{[st]}$, and $\Gamma^{c}_{[st]}$ leads directly to their inverses $\Gamma^{a}_{[ts]}$, $\Gamma^{b}_{[ts]}$, and $\Gamma^{c}_{[ts]}$ since their respective elements have the same absolute values. We again give an <u>ad hoc</u> prescription for obtaining $\Gamma^{d}_{[ts]}$ from $\Gamma^{d}_{[st]}$. $\Gamma^{d}_{[ts]} = \Gamma^{d}_{[st]}(\mu \leftrightarrow m')$ with the simultaneous changing of the signs of the first row and the second, fourth, fifth, seventh, and ninth columns.

The crossing matrix $\Gamma_{[su]}$ is obtained in a similar way from the Fierz transformation (2.36) and the various momentum relations. Once again it is not necessary to write this matrix out explicitly since it reflects the $t \leftrightarrow u$ crossing symmetry (3.36). We obtain $\Gamma_{[su]}$ from $\Gamma_{[st]}$ by changing the signs of the odd-numbered rows with $\Gamma^{d}_{[su]} = \Gamma^{d}_{[st]}(m \leftrightarrow \mu)$. Similarly, the inverse matrix $\Gamma_{[us]}$

is obtained from $\Gamma_{[ts]}$ by changing the signs of the odd-numbered columns with $\Gamma^d_{[us]} = \Gamma^d_{[ts]}(m \leftrightarrow \mu)$.

Finally, we find that

$$\Gamma^a_{[ut]} = \Gamma^a_{[tu]}, \quad \Gamma^b_{[ut]} = \Gamma^b_{[tu]}, \quad \Gamma^c_{[ut]} = \Gamma^c_{[tu]},$$
$$\Gamma^d_{[ut]} = \Gamma^d_{[tu]}(m \leftrightarrow \mu), \tag{3.49}$$

and these matrices are given by

$$\Gamma^a_{[ut]} = \begin{bmatrix} -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 3 & 1 & 3 & 1 & 3 & 1 & -1 & -3 & 1 & 3 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 3 & 1 & 3 & 1 & -3 & -1 & 1 & 3 & 1 & 3 \\ -4 & 4 & 4 & -4 & 2 & -2 & -2 & 2 & 0 & 0 \\ 12 & 4 & -12 & -4 & -6 & -2 & -2 & -6 & 0 & 0 \\ -12 & -4 & 12 & 4 & -6 & -2 & -2 & -6 & 0 & 0 \\ 4 & -4 & -4 & 4 & 2 & -2 & -2 & 2 & 0 & 0 \\ 18 & 6 & 18 & 6 & 0 & 0 & 0 & 0 & -2 & -6 \\ -6 & 6 & -6 & 6 & 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix}, \tag{3.50}$$

$$\Gamma^b_{[ut]} = \begin{bmatrix} -4 & -4 & -8 \\ -4 & -4 & 8 \\ -4 & 4 & 0 \end{bmatrix}, \tag{3.51}$$

$$\Gamma^c_{[ut]} = \begin{bmatrix} 4 & -4 & -4 \\ -4 & 4 & -4 \\ -8 & -8 & 0 \end{bmatrix}, \tag{3.52}$$

and

$$\Gamma^{d}_{[ut]} = \frac{1}{2}\left[\begin{array}{cccccccccc} 3(m-3\mu-m'+3\mu') & -(7m-\mu+5m'+5\mu') & -3(3m-\mu-3m'+\mu') & (5m+5\mu-m'+7\mu') & -3(m+3\mu+m'+3\mu') & (7m+\mu-5m'+5\mu') & (5m-5\mu+m'+7\mu') & -3(3m+\mu+3m'+\mu') & (m-3\mu+3m'-\mu') & 3(m+\mu-m'-\mu') \\ 3(m-3\mu-m'+3\mu') & (5m+5\mu+7m'-\mu') & -3(3m-\mu-3m'+\mu') & (m-7\mu-5m'-5\mu') & -3(m+3\mu+m'+3\mu') & -(5m-5\mu-7m'-\mu') & (m+7\mu+5m'-5\mu') & -3(3m+\mu+3m'+\mu') & -(3m-\mu+m'-3\mu') & 3(m+\mu-m'-\mu') \\ 3(m+3\mu-m'-3\mu') & (m+3\mu-m'-3\mu') & -3(3m+\mu-3m'-\mu') & -(3m+\mu-3m'-\mu') & 3(m-3\mu+m'-3\mu') & (m-3\mu+m'-3\mu') & (3m-\mu+3m'-\mu') & 3(3m-\mu+3m'-\mu') & (m-\mu-m'+\mu') & 3(m-\mu-m'+\mu') \end{array}\right] . \qquad (3.53)$$

The work in this chapter shows clearly that it is possible to expand the M-functions for the processes considered in terms of invariant amplitudes which are KSF. All our conditions have been fulfilled. We have treated each of the three channels in a manifestly similar way and have shown that the crossing properties of the amplitudes are simple and that the crossing matrices do not involve kinematic variables. The way in which we were able to use many of the results for abnormal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ in our investigation of abnormal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{3}{2}$, indicates the way in which the M-function expansion for higher spin processes may be obtained by generalizing the results for basic low spin processes. Although this does not constitute a proof that any process with spin is amenable to this approach of expanding the M-function in terms of KSF invariant amplitudes, it does make it a reasonable assumption.

CHAPTER 4.

REGGEIZATION OF INVARIANT AMPLITUDES.

The M-function approach to processes with spin outlined in the previous three chapters is well suited to the calculation of dispersion integrals because of the simple analytic and constraint free properties of the invariant amplitudes. In the applications of the M-function technique which we shall investigate, we are primarily interested in fixed momentum transfer dispersion integrals of the type known as superconvergence relations. Superconvergence, as the name implies, occurs for those parts of the scattering amplitude for a process which fall off more rapidly asymptotically than does the whole amplitude. It is clear, then, that if we can discover the asymptotic behaviour of the various invariant amplitudes for a process, we will be in a position to form super-convergence integrals on those amplitudes which have an appropriate asymptotic behaviour.

We shall determine these asymptotic behaviours by "Reggeizing" the invariant amplitudes. The necessary technique has been developed by Jones and Scadron[52,53] and it depends on obtaining a partial wave expansion for the invariant amplitudes. From this partial wave expansion we can perform a Sommerfeld-Watson transform in order to analytically continue the angular momentum J.

The well-known result is that J becomes $\alpha(t)$, the complex angular momentum, which is a function of the momentum transfer and is known as the trajectory of the Regge pole. The process of Reggeizing the partial wave expansion is not without difficulties, especially in connection with singularities at $t = 0$ [85-88], but as we will be interested in the behaviour of the leading term in the asymptotic limit only, we will not discuss these problems here and refer the reader to the literature[52,53,89,90].

The covariant partial wave expansion of the invariant amplitudes is obtained by a method developed by Scadron[51]. We consider the projection operators for arbitrary spin intermediate states coupled to general covariant vertex functions. By comparing this with the expansion of the M-function in terms of invariant amplitudes, we obtain a partial wave expansion in terms of "solid" Legendre functions which correspond to the usual $d^J_{\lambda'\lambda}(\theta)$ angular functions. To perform this calculation, it is necessary to construct general high spin projection operators and vertex functions and, before proceeding and further, we will give a brief summary of the theory of these entities.

4.1 Contracted Projection Operators.

The high spin projection operator in the s-channel is defined as

$$\mathcal{P}^{s}_{\mu_1\ldots\mu_J;\nu_1\ldots\nu_J}(K) = \sum_{\lambda} \Psi^{(\lambda)}_{\mu_1\ldots\mu_J}(K)\overline{\Psi}^{(\lambda)}_{\nu_1\ldots\nu_J}(K), \qquad (4.1)$$

where Ψ and $\overline{\Psi}$ are the Rarita-Schwinger type wave-functions for either bosons (1.19) or fermions (1.21), and K is the exchange momentum. We contract $\mathcal{P}^{s}_{\mu_1\ldots\mu_J;\nu_1\ldots\nu_J}(K)$ with with initial momenta p^{ν} and final momenta p'^{μ}, which produces the contracted projection operator $\mathcal{P}^{s}(p',p;K)$, with s=J or J+½. If we now take a virtual boson of spin J to its rest frame, we find $\varepsilon^{(\lambda)}(M).p = p_{\lambda}$, where the helicity $\lambda = 1,0,-1$. Then by aligning $\underline{p}$ along the third direction so that in equation (1.20), $\lambda = 0$, one can show that

$$\mathcal{P}^{J}(p',p;M) = c_J\, \mathcal{P}_J(\underline{p}'.\underline{p}), \qquad (4.2)$$

where $\mathcal{P}_J(\underline{p}'.\underline{p})$ is the "solid" Legendre polynomial

$$\mathcal{P}_J(\underline{p}'.\underline{p}) = |\underline{p}'|^{J}|\underline{p}|^{J}\, \mathcal{P}_J(\hat{p}'.\hat{p}), \qquad (4.3)$$

and

$$c_J = \frac{2^J\, J!\, J!}{(2J)!} \quad . \qquad (4.4)$$

We can now boost this result up to momentum K, providing $K^2 = M^2$, ie. the particle must be on-shell. The boost prescriptions are

$$\delta_{ij} \rightarrow -\left(g_{\mu\nu} - \frac{K_{\mu}K_{\nu}}{M^2}\right) \equiv -g_{\mu\nu}(K), \qquad (4.5)$$

and

$$p_i \rightarrow -\left(p_{\mu} - \frac{p.K}{M^2}K_{\mu}\right) \equiv -p_{\mu}(K), \qquad (4.6)$$

so that

$$\underline{p}'.\underline{p} \rightarrow -(p'.p - \frac{p'.K\ p.K}{M^2}) = -p'(K).p(K). \qquad (4.7)$$

Thus the covariant on-shell result for the contracted projection operator is $c_J \mathcal{P}_J$ which completely specifies the spin J boson projection operator coupled to spin zero particles. We suppress the dependence of the solid harmonic $\mathcal{P}_J$ on its invariant argument $-p'(K).p(K)$ for convenience.

When the particles to which the projection operators must couple have spin other than zero, it is clear that the projection operator must have free indices. We obtain these by "freeing" indices from $\mathcal{P}^J(p',p;K)$. The solid harmonic $\mathcal{P}_J$ is a homogeneous polynomial of degree J in either p' or p and we can free indices by a covariant version of Zemach's O(3) differential technique[91]. In this way we can free indices as required for either the initial or the final state. The details of this procedure will not be reproduced here; the reader is referred to Scadron[51] for a full discussion. We are content to quote the results for the first few projection operators.

Where α is a general initial particle index and β a final particle index, the boson projection operator formulae are

$$\mathcal{P}^J(p',p;K) = c_J \mathcal{P}_J, \qquad (4.8)$$

$$\mathcal{P}^J_{;\alpha}(p',p;K) = -\frac{c_J}{J}[p'_\alpha(K)\mathcal{P}'_J + p'^2(K)p_\alpha(K)\mathcal{P}'_{J-1}], \qquad (4.9)$$

$$\mathcal{P}^J_{\beta;}(p',p;K) = -\frac{c_J}{J}[p_\beta(K)\mathcal{P}'_J + p^2(K)p'_\beta(K)\mathcal{P}'_{J-1}], \qquad (4.10)$$

$$\mathcal{P}^J_{\beta;\alpha}(p',p;K) = \frac{c_J}{J^2}\,[\{p'_\beta(K)p_\alpha(K) + p_\beta(K)p'_\alpha(K)\}\mathcal{P}''_J + $$
$$+\{p^2(K)p'_\beta(K)p'_\alpha(K) + p'^2(K)p_\beta(K)p_\alpha(K)\}\mathcal{P}''_{J-1} -$$
$$- g_{\beta\alpha}(K)\mathcal{P}'_J - (2J+1)p'_\beta(K)p_\alpha(K)\mathcal{P}'_{J-1}\ ], \qquad (4.11)$$

$$\mathcal{P}^J_{;\alpha_1\alpha_2}(p',p;K) = \frac{c_J}{J(J-1)}\,[\ p'_{\alpha_1}(K)p'_{\alpha_2}(K)\mathcal{P}''_J +$$
$$+p'^2(K)\{p'_{\alpha_1}(K)p_{\alpha_2}(K) + p_{\alpha_1}(K)p'_{\alpha_2}(K)\}\mathcal{P}''_{J-1} +$$
$$+p'^4(K)p_{\alpha_1}(K)p_{\alpha_2}(K)\mathcal{P}''_{J-2} -$$
$$- p'^2(K)g_{\alpha_1\alpha_2}(K)\mathcal{P}'_{J-1}\ ], \qquad (4.12)$$

$$\mathcal{P}^J_{\beta;\alpha_1\alpha_2}(p',p;K) = -\frac{c_J}{J^2(J-1)}\,[\{p'p'p\}_{\beta\alpha_1\alpha_2}\mathcal{P}'''_J +$$
$$+\{p'p'p';p^2\}_{\beta\alpha_1\alpha_2}\mathcal{P}'''_{J-1} +$$
$$+ p'^4(K)p_\beta(K)p_{\alpha_1}(K)p_{\alpha_2}(K)\mathcal{P}'''_{J-2} - \{gp'\}_{\beta\alpha_1\alpha_2}\mathcal{P}''_J -$$
$$- [(2J+1)p'_\beta(K)\{p'p\}_{\alpha_1\alpha_2} + p'^2(K)\{gp\}_{\beta\alpha_1\alpha_2}]\mathcal{P}''_{J-1} -$$
$$- (2J+1)p'^2(K)p'_\beta(K)p_{\alpha_1}(K)p_{\alpha_2}(K)\mathcal{P}''_{J-2} +$$
$$+ (2J+1)p'_\beta(K)g_{\alpha_1\alpha_2}(K)\mathcal{P}'_{J-1}\ ], \qquad (4.13)$$

and so on, with $\mathcal{P}^J_{;\alpha_1\alpha_2} \to \mathcal{P}^J_{\beta_1\beta_2;}$ and $\mathcal{P}^J_{\beta;\alpha_1\alpha_2} \to \mathcal{P}^J_{\beta_1\beta_2;\alpha}$ when $\alpha\leftrightarrow\beta$ and $p\leftrightarrow p'$. In eqn. (4.13), the brackets $\{\quad\}_{\dots}$ indicate symmetric combinations such as

$$\{p'p'p\}_{\beta\alpha_1\alpha_2} = p'_\beta(K)p'_{\alpha_1}(K)p_{\alpha_2}(K) + p'_\beta(K)p_{\alpha_1}(K)p'_{\alpha_2}(K) +$$
$$+ p_\beta(K)p'_{\alpha_1}(K)p'_{\alpha_2}(K),$$

and

$$\{p'p'p';p^2\}_{\beta\alpha_1\alpha_2} = p^2(K)p'_\beta(K)p'_{\alpha_1}(K)p'_{\alpha_2}(K) +$$

$$+ p'^2(K)[p_\beta(K)p'_{\alpha_1}(K)p_{\alpha_2}(K) + p_\beta(K)p_{\alpha_1}(K)p'_{\alpha_2}(K) +$$

$$+ p'_\beta(K)p_{\alpha_1}(K)p_{\alpha_2}(K)].$$

For the interaction $s_1 + s_2 \to J \to s'_1 + s'_2$ one needs to consider the above boson projection operator formulae with at most $s_1 + s_2$ free α labels and $s'_1 + s'_2$ free β labels. Also, the subsidiary conditions mean that we can replace p by Λ and p' by Λ' in the above formulae in order to get the projection operators in terms of the natural s-channel momenta. From the s-channel form we can cross directly to the t-channel where $p' \leftrightarrow -q$. Thus the s-channel variables cross according to the substitutions $\Lambda \to P$, $\Lambda' \to -Q$, $K \to -\Delta$, or we can obtain the t-channel forms by the substitutions $p \to P$, $p' \to -Q$, and $K \to -\Delta$ in eqns. (4.8) to (4.13), noting that $\underline{p}'.\underline{p} \to -P.Q$.

In the applications which we shall investigate, we reggeize in boson channels, so that we will require only boson projection operators. It is possible to extend the above general technique to the calculation of arbitrary spin fermion projection operators, but once again we do not go into this here and refer the reader to reference (51). This restriction to bosons also introduces a simplification in the next section where we need consider covariant vertex functions coupling only external BB or FF states to high spin bosons. The couplings of external BF states to high spin fermions is not considered.

4.2 Covariant Vertex Functions.

In this section we shall investigate the coupling of a high spin J particle to other particles. For each vertex between three particles of any given spin there will be two distinct types of coupling as well as several independent coupling constants. The distinction between the two types is based on the notion of normality discussed in section 1.3. The couplings of three particles of given spin with parity assignments such that the vertex is normal are entirely disjoint from the couplings for the same spins but abnormal overall parity.

It is found that the number of independent couplings at a given vertex depends on the two lowest spins only, so that the third spin may become arbitrarily large without increasing the number of coupling functions required. This is obviously of great significance for the problems which we shall consider where we wish to investigate the coupling of two fixed (usually small) spin particles to an arbitrary high spin state in the derivation of the partial wave expansion of the invariant amplitudes of a scattering process. Consider the general coupling $s_1 + s_2 \rightarrow s_3$. We take s_1 and s_2 to be the lowest of the three spins and combine them to form $\underline{s} = \underline{s}_1 + \underline{s}_2$ so that the problem is reduced to couplings of the form $0 + s \rightarrow s_3$ when $|s_1 - s_2| < s < s_1 + s_2$. It is easy to see, by going to the rest frame of one of the particles, that in this form there are $2s_m + 1$ couplings for on-shell (physical) particles for each value of

$s_m = \min(s, s_3)$. Hence, the total number of reduced couplings for $s_1 + s_2 \rightarrow s_3$ is $\sum_{s_m} (2s_m + 1)$. It is then easy to show that if $s_1 + s_2 \leqslant s_3$, there are

$$N^{\pm} = \tfrac{1}{2}(2s_1 + 1)(2s_2 + 1) \tag{4.14}$$

independent normal (N^+) or abnormal (N^-) couplings for FFB interactions, and

$$N^{\pm} = \tfrac{1}{2}[(2s_1 + 1)(2s_2 + 1) \pm 1] \tag{4.15}$$

independent normal (N^+) or abnormal (N^-) couplings for BBB interactions[92].

On the other hand, if $s_1 + s_2 > s_3$ but s_1 and s_2 are still both less than s_3, there are

$$N^{\pm} = \tfrac{1}{2}S(S + 1) \tag{4.16}$$

normal or abnormal couplings for FFB or BBB interactions where $S \equiv s_1 + s_2 - s_3$. These counting rules will apply to covariant three point functions provided that all three particles are on shell with $p_i^2 = m_i^2$. Photon couplings require a somewhat different treatment and as we will not be considering processes involving photons, they will not be discussed here.

In order to study the vertex functions, we consider effective Lagrangian interactions in momentum space:

$$\mathcal{L} = \bar{\Psi}^{\alpha_1 \cdots \alpha_{J_3}}(K)\, \mathcal{C}_{\alpha_1 \ldots \alpha_{J_3}; \mu_1 \ldots \mu_{J_1}; \nu_1 \ldots \nu_{J_2}} \times$$

$$\times \; \Psi^{\mu_1 \cdots \mu_{J_3}}(p)\, \Psi^{\nu_1 \cdots \nu_{J_2}}(q) + \text{h.c.}, \tag{4.17}$$

for the interaction $s_1(p) + s_2(q) \rightarrow s_3(K)$, $K = p + q$ as usual. In eqn. (4.17), $\mathcal{G}$ is the function which we shall call the covariant coupling function or the "$\mathcal{G}$-function". It will depend upon various combinations of momenta, metric tensors, and γ matrices; the kinematic covariants of section 1.1. It is clear that the $\mathcal{G}$-functions defined by (4.17) correspond to the coupling of the initial to the intermediate state of our s-channel, so we will use the natural momenta of (1.4) and natural γ matrices as in (1.2). These $\mathcal{G}$-functions can, of course, be written in an equivalent form for the other channels. However, as well as the kinematic dependence, the $\mathcal{G}$-functions depend on the independent on-shell coupling constants which are, in fact, the vertex form factors evaluated for all particles on their respective mass shells.

In an obvious notation, we will write the normal vertex functions as $\mathcal{G}^+$ and those for abnormal vertices as $\mathcal{G}^-$. The various indices and spins are written according to $\mathcal{G}(s_1, s_2; s_3)$ for the interaction $s_1(p)_\mu + s_2(q)_\nu \rightarrow s_3(K)_\alpha$. In this notation, the first few vertex functions for the coupling of two low spin bosons to an arbitrary high spin boson are:

$$\mathcal{G}^+(0,0;J) = g\Lambda_{\alpha_1} \ldots \Lambda_{\alpha_J}, \tag{4.18}$$

$$\mathcal{G}^-(0,0;J) = 0, \tag{4.19}$$

$$\mathcal{G}^+(1,0;J) = \Lambda_{\alpha_2} \ldots \Lambda_{\alpha_J} \{g_1 g_{\alpha_1\mu} + g_2 \Lambda_{\alpha_1} \Lambda_\mu\}, \tag{4.20}$$

$$\mathcal{G}^-(1,0;J) = \Lambda_{\alpha_2}\cdots\Lambda_{\alpha_J}\ g\{\varepsilon_{\alpha_1\mu}(K\Lambda)\},\tag{4.21}$$

$$\mathcal{G}^+(2,0;J) = \Lambda_{\alpha_3}\cdots\Lambda_{\alpha_J}\{g_1 g_{\alpha_1\mu_1} g_{\alpha_2\mu_2} + g_2 g_{\alpha_1\mu_1}\Lambda_{\alpha_2}\Lambda_{\mu_2} + + g_3\Lambda_{\alpha_1}\Lambda_{\alpha_2}\Lambda_{\mu_1}\Lambda_{\mu_2}\},\tag{4.22}$$

$$\mathcal{G}^-(2,0;J) = \Lambda_{\alpha_3}\cdots\Lambda_{\alpha_J}\varepsilon_{\alpha_1\mu_1}(K\Lambda)\{g_1 g_{\alpha_2\mu_2} + g_2\Lambda_{\alpha_2}\Lambda_{\mu_2}\},\tag{4.23}$$

$$\mathcal{G}^+(1,1;J) = \Lambda_{\alpha_3}\cdots\Lambda_{\alpha_J}[g_1 g_{\alpha_1\mu} g_{\alpha_2\nu} + g_2 g_{\alpha_1\mu}\Lambda_{\alpha_2}\Lambda_\nu + + g_3 g_{\alpha_1\nu}\Lambda_{\alpha_2}\Lambda_\mu + g_4 g_{\mu\nu}\Lambda_{\alpha_1}\Lambda_{\alpha_2} + g_5\Lambda_{\alpha_1}\Lambda_{\alpha_2}\Lambda_\mu\Lambda_\nu\ ],\tag{4.24}$$

$$\mathcal{G}^-(1,1;J) = \Lambda_{\alpha_3}\cdots\Lambda_{\alpha_J}[g_1\{\varepsilon_{\alpha_1\mu}(K\Lambda)g_{\alpha_2\nu} + \varepsilon_{\alpha_1\nu}(K\Lambda)g_{\alpha_2\mu}\} + + g_2\varepsilon_{\mu\nu\alpha_1}(K)\Lambda_{\alpha_2} + g_3\varepsilon_{\mu\nu\alpha_1}(\Lambda)\Lambda_{\alpha_2} + + g_4\varepsilon_{\mu\nu}(K\Lambda)\Lambda_{\alpha_1}\Lambda_{\alpha_2}\ ],\tag{4.25}$$

and so on. As the wave-functions in eqn. (4.17) are symmetric in their indices, the $\mathcal{G}$-functions need not be symmetric in the μ, ν, or α labels.

The $\mathcal{G}$-functions coupling two low spin fermions to a high spin boson are quite similar to those given above except that now we have the additional covariant γ_ρ, where ρ is a boson label. It is important that the spinor indices of these γ matrices be contracted with the indices of the two spinors involved in the vertex function in which they occur. It is for this reason that we defined our natural γ matrices for four fermion processes as in section 1.1. We need consider only normal coupling functions $\mathcal{C}^{+}$ as $\mathcal{C}^{+} \to \mathcal{C}^{-}$ when $1 \to \gamma_5$ in the Dirac spin space. Also, covariants like $\gamma_\mu \gamma_\nu$, $\varepsilon_{\mu\nu\alpha\beta}$, and $\gamma_5 \varepsilon_{\mu\nu\alpha\beta}$ are never necessary in on-shell couplings because they can be related to the other forms by the identities of appendix A. We write out the first few FFB interactions with high spin bosons:

$$\mathcal{C}^{+}(\tfrac{1}{2},\tfrac{1}{2};J) = \Lambda_{\alpha_2} \cdots \Lambda_{\alpha_J} \{ g_1 \gamma_{\alpha_1} + g_2 \Lambda_{\alpha_1} \}, \tag{4.26}$$

$$\begin{aligned} \mathcal{C}^{+}(\tfrac{1}{2},\tfrac{3}{2};J) = \Lambda_{\alpha_3} \cdots \Lambda_{\alpha_J} \{ & g_1 \gamma_{\alpha_2} g_{\alpha_1\mu} + g_2 \Lambda_{\alpha_2} g_{\alpha_1\mu} + \\ & + g_3 \gamma_{\alpha_1} \Lambda_{\alpha_2} \Lambda_\mu + g_4 \Lambda_{\alpha_1} \Lambda_{\alpha_2} \Lambda_\mu \}, \end{aligned} \tag{4.27}$$

$$\begin{aligned} \mathcal{C}^{+}(\tfrac{3}{2},\tfrac{3}{2};J) = \Lambda_{\alpha_4} \cdots \Lambda_{\alpha_J} \{ & g_1 g_{\alpha_1\mu} g_{\alpha_2\nu} \gamma_{\alpha_3} + g_2 g_{\alpha_1\mu} g_{\alpha_2\nu} \Lambda_{\alpha_3} + \\ & + g_3 g_{\alpha_1\mu} \gamma_{\alpha_2} \Lambda_{\alpha_3} \Lambda_\nu + g_4 g_{\mu\nu} \gamma_{\alpha_1} \Lambda_{\alpha_2} \Lambda_{\alpha_3} + \\ & + g_5 g_{\alpha_1\mu} \Lambda_{\alpha_2} \Lambda_{\alpha_3} \Lambda_\nu + g_6 g_{\mu\nu} \Lambda_{\alpha_1} \Lambda_{\alpha_2} \Lambda_{\alpha_3} + \\ & + g_7 \gamma_{\alpha_1} \Lambda_{\alpha_2} \Lambda_{\alpha_3} \Lambda_\mu \Lambda_\nu + g_8 \Lambda_{\alpha_1} \Lambda_{\alpha_2} \Lambda_{\alpha_3} \Lambda_\mu \Lambda_\nu \}. \end{aligned} \tag{4.28}$$

It should be noted that if J is less than $s_1 + s_2$, some of the couplings in eqns. (4.18–4.28) vanish in accordance with the counting rule (4.16). Further restrictions on the couplings arise when we impose invariance under the discrete symmetry operations on the vertex functions. Time reversal invariance ensures the reality of the appropriately chosen coupling constants and charge conjugation invariance on self-conjugate states $|s_1, s_2 = \bar{s}_1\rangle$ implies relations between some couplings. Charge conjugation invariance further eliminates half the abnormal couplings for FFB vertices because $\gamma_5(\gamma_\alpha,\Lambda_\alpha) \rightarrow \gamma_5(\gamma_\alpha,-\Lambda_\alpha)$, whereas $(\gamma_\alpha,\Lambda_\alpha) \rightarrow (-\gamma_\alpha,-\Lambda_\alpha)$. It is also clear from the way in which we have written the vertex functions that the coupling constants we use are not dimensionless. The power of an arbitrary mass factor necessary to make the couplings dimensionless is, however, easily obtained by counting the number of relevant momentum factors multiplying each constant.

4.3 Asymptotic Behaviour.

In applying fixed momentum transfer dispersion relations to scattering processes, one hopes to obtain sum rules between coupling constants by saturating the superconvergence sum rules with known elementary particle poles. If there is crossing symmetry between two of the channels for the reaction, the same single particle poles will be used in this saturation for each of the symmetric channels. It is, therefore, obviously desirable to form the superconvergence integral in such a way that we can make use of this simplicity. To make clear how we in fact do this, consider the case of BB scattering. In section 3.1 we saw that if, in the t-channel of a BB scattering process, the two final state particles were identical, then we have crossing symmetry on the t-channel invariant amplitudes under $s \leftrightarrow u$ crossing. Under these circumstances, the s and u channels are saturated with the same single particle poles, although the contributions of these poles in the direct s-channel and the crossed u-channel to the crossed t-channel amplitudes may differ in sign. Consequently, the simplest approach is to form superconvergence integrals on the t-channel amplitudes and then approximate these integrals by the appropriate contributions from the s and u channel single particle poles.

A similar situation arises for FF scattering processes where, as we have pointed out in the examples of FF processes considered in chapters 2 and 3, crossing symmetry may be seen for the s-channel amplitudes under $t \leftrightarrow u$ crossing. The s-channel of FF scattering as we have defined it will be the baryon number 2 channel for baryon-baryon scattering, while the t and u channels have baryon number zero and are thus saturated with meson states. Therefore we reggeize the s-channel amplitudes in FF scattering and form superconvergence integrals on these.

We must first obtain the appropriate partial wave expansion using the formulae of sections 4.1 and 4.2. Since we have written these in terms of s-channel variables, we will, in the first instance, form the s-channel partial wave expansion. It is clear from the previous work that the contribution to the amplitude of spin J exchange has the form

$$\mathcal{M}^{(J)} = \bar{\mathcal{L}}(s_1', s_2'; J) : \mathcal{P}^J(\Lambda, \Lambda'; K) : \mathcal{L}(s_1, s_2; J). \qquad (4.29)$$

As an example of this technique, we shall consider the partial wave expansions of the five amplitudes for nucleon-nucleon scattering, ie. normal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ with four identical particles. Since the s-channel exchange can be either normal or abnormal, and the vertex functions are distinct for either case, we will get two distinct contributions to $\mathcal{M}$, one from normal exchange ($\mathcal{M}_+^{(J)}$) and the other from abnormal exchange ($\mathcal{M}_-^{(J)}$). Using the $\mathfrak{C}$-function (4.26), remembering that in the final state coupling we have $\bar{\mathfrak{C}}$ rather than $\mathfrak{C}$, we obtain for normal exchange

$$\mathcal{M}_+^{(J)} = g_1^2 \Upsilon'_\beta \mathcal{P}^J_{\beta;\alpha}\Upsilon_\alpha + g_1 g_2 \{\Upsilon'_\beta \mathcal{P}^J_{\beta;} + \mathcal{P}^J_{;\alpha}\Upsilon_\alpha\} + g_2^2 \mathcal{P}^J . \quad (4.30)$$

Using the projection operator formulae (4.8) to (4.11), the Dirac equation and the equivalence theorems (2.14) to (2.16), we obtain the following partial wave expansions for the invariant amplitudes:

$$A_1^{(J)} = \frac{c_J}{J^2} g_1^2 \{m^2 \mathcal{P}''_J + \tilde{\nu}\Lambda^2 \mathcal{P}''_{J-1} - (2J+1)m^2 \mathcal{P}'_{J-1}\} + $$
$$+ \frac{c_J}{J}\frac{g_1 g_2}{m}\{\tilde{\nu}\mathcal{P}'_J + 2m^2\Lambda^2 \mathcal{P}''_{J-1}\} + c_J\, g_2^2\, \mathcal{P}_J , \quad (4.31)$$

$$A_2^{(J)} = \frac{c_J}{J^2} g_1^2 \{\tilde{\nu}\mathcal{P}''_J + m^2\Lambda^2 \mathcal{P}''_{J-1} - \mathcal{P}'_J\} + \frac{c_J}{J} g_1 g_2\, m\, \mathcal{P}'_J , \quad (4.32)$$

$$A_3^{(J)} = \frac{c_J}{J^2} g_1^2 \frac{s}{4} \mathcal{P}''_J , \quad (4.33)$$

$$A_4^{(J)} = \frac{c_J}{J^2} g_1^2 \frac{s}{4} \Lambda^2 \mathcal{P}''_{J-1} + \frac{c_J}{J} g_1 g_2 \frac{s}{4m} \mathcal{P}'_J , \quad (4.34)$$

$$A_5^{(J)} = \frac{c_J}{J^2} g_1^2 \{m^2 \mathcal{P}''_J + \tilde{\nu}\Lambda^2 \mathcal{P}''_{J-1}\} + \frac{c_J}{J}\frac{g_1 g_2}{m}\tilde{\nu}\mathcal{P}'_J . \quad (4.35)$$

A similar partial wave expansion may be obtained for abnormal exchange, but as the couplings at either end differ only by a factor of γ_5 from those used in (4.30), the result is not greatly different and will not be written out in detail here.

Since, in general as in eqns. (4.31) to (4.35), the partial wave expansions of the invariant amplitudes involve just Legendre polynomials and their derivatives, we can, using the well-known orthogonality properties of these functions, invert the expansion to give Froissart-Gribov expressions for the partial wave amplitudes. According to the prescriptions of Frautschi, Gell-Mann, and Zachariasen[50], therefore, we can perform a Sommerfeld-Watson transform to obtain the form of the Regge pole term:

$$\mathcal{M}^{(\alpha)} = \mathcal{G}(s_1', s_2'; \alpha(s)) : \mathcal{P}^{\alpha}(\Lambda, \Lambda'; K) : \mathcal{G}(s_1, s_2; \alpha(s)). \quad (4.36)$$

In eqn. (4.36), the coupling constants g_i of $\mathcal{G}(J)$ become the Regge residue functions $g_n(s)$ of $\mathcal{G}(\alpha(s))$. The Legendre functions $\mathcal{P}_J(z_s)$ occurring in $\mathcal{P}^J(\Lambda, \Lambda'; K)$ are replaced by

$$\mathcal{P}_\alpha(-z_s) \times \tfrac{1}{2}(1 \pm e^{i\pi\alpha}) \Big/ \sin \pi\alpha \; ,$$

giving the appropriate Regge form.

With the Reggeization of eqn. (4.36), we should notice that all the covariant residue functions are factorized immediately because of the way in which we obtained our partial wave expansion with covariant vertex functions. Also, because the projection operators contain only "solid" spherical harmonics, the correct threshold factors $p^\alpha(s)$, $q^\alpha(s)$ appear automatically. The problems of singularities at s=0 which may arise for the unequal mass case when we

Reggeize in the s-channel as above, or at t=0 if we Reggeize in the t-channel as in $B\bar{B}$ scattering, will not be discussed further here since we are interested only in the asymptotic behaviour which is dominated by the leading term in the expansion.

For normal Regge exchange in superelastic normal $\frac{1}{2} + \frac{1}{2} \rightarrow \frac{1}{2} + \frac{1}{2}$ scattering, it is clear from eqns. (4.31) to (4.35), that as $\tilde{\nu} \rightarrow \infty$, the five amplitudes have the behaviour

$$A_1 \sim \tilde{\nu}^{\alpha},\ A_2 \sim \tilde{\nu}^{\alpha-1},\ A_3 \sim \tilde{\nu}^{\alpha-2},\ A_4 \sim \tilde{\nu}^{\alpha-1},\ A_5 \sim \tilde{\nu}^{\alpha}, \qquad (4.37)$$

since $c_{\alpha} P_{\alpha} \sim \tilde{\nu}^{\alpha}$, $\frac{c_{\alpha}}{\alpha} P'_{\alpha} \sim \tilde{\nu}^{\alpha-1}$, etc. as $\tilde{\nu} \rightarrow \infty$.

This behaviour is slightly changed in the unequal mass inelastic case, and the resulting asymptotic forms for normal Regge exchange are

$$\begin{array}{ll} A_1 \sim \tilde{\nu}^{\alpha}\ , & A_2 \sim \tilde{\nu}^{\alpha-1}, \\ A_3 \sim \tilde{\nu}^{\alpha-1}, & A_4 \sim \tilde{\nu}^{\alpha-1}, \\ A_5 \sim \tilde{\nu}^{\alpha}\ , & A_6 \sim \tilde{\nu}^{\alpha-1}, \\ A_7 \sim \tilde{\nu}^{\alpha-1}, & A_8 \sim \tilde{\nu}^{\alpha-1}, \end{array} \qquad (4.38)$$

Similarly, we may obtain the asymptotic behaviours for abnormal Regge exchange:

$$\begin{array}{ll} A_1 \sim \tilde{\nu}^{\alpha-1}, & A_2 \sim \tilde{\nu}^{\alpha-2}, \\ A_3 \sim \tilde{\nu}^{\alpha-2}, & A_4 \sim \tilde{\nu}^{\alpha-2}, \\ A_5 \sim \tilde{\nu}^{\alpha}\ , & A_6 \sim \tilde{\nu}^{\alpha-2}, \\ A_7 \sim \tilde{\nu}^{\alpha-2}, & A_8 \sim \tilde{\nu}^{\alpha-2}, \end{array} \qquad (4.39)$$

The differences between eqns. (4.38) and (4.39) are largely due to the fact mentioned previously that charge conjugation invariance halves the number of abnormal couplings. Since both normal and abnormal trajectories may contribute, the final asymptotic behaviour must be the "aggragate" of the behaviours of (4.38) and (4.39). Which will in fact dominate in any particular case will depend on the particular value of α for the leading Regge trajectory which can contribute.

We now consider the application of this Reggeization procedure to abnormal $0 + 1 \rightarrow 1 + 1$ scattering. For the reasons discussed above, we Reggeize the t-channel of this process and must recast our formulae in terms of the t-channel variables. The partial wave expansion for normal Regge exchange is obtained from

$$\mathcal{M}_{+}^{(J)} = \mathcal{G}^{+}(1,1;J) : \mathcal{P}^{J}(P,-Q;-\Delta) : \mathcal{F}^{-}(1,0;J). \tag{4.40}$$

Using eqns. (4.21) and (4.24), we obtain

$$\begin{aligned}\mathcal{M}_{+}^{(J)} &= g_1 f\,\mathcal{P}^{J}_{\mu\nu;\alpha}\varepsilon_{\alpha\lambda}(P\Delta) + g_2 f\,\mathcal{P}^{J}_{\mu;\alpha}\varepsilon_{\alpha\lambda}(P\Delta)Q_{\nu} + \\ &+ g_3 f\,\mathcal{P}^{J}_{\nu;\alpha}\varepsilon_{\alpha\lambda}(P\Delta)Q_{\mu} + g_4 f\,\mathcal{P}^{J}_{;\alpha}\varepsilon_{\alpha\lambda}(P\Delta)g_{\mu\nu} + \\ &+ g_5 f\,\mathcal{P}^{J}_{;\alpha}\varepsilon_{\alpha\lambda}(P\Delta)Q_{\mu}Q_{\nu},\end{aligned} \tag{4.41}$$

where $g_1,\ldots,g_5$, and f are independent coupling constants. Using eqns. (4.9) to (4.13) and the equivalence relations (3.8) to (3.31), we obtain the partial wave expansion for each of the invariant amplitudes. The resulting thirteen **expressions** are too long to reproduce in detail here and we give only the asymptotic behaviour resulting from following the Reggeization procedure outlined above.

$$\begin{array}{ll} A_1 \sim \nu^{\alpha-3}, & A_2 \sim \nu^{\alpha-2}, \\ A_3 \sim \nu^{\alpha-3}, & A_4 \sim \nu^{\alpha}, \\ A_5 \sim \nu^{\alpha-2}, & A_6 \sim \nu^{\alpha-1}, \\ A_7 \sim \nu^{\alpha-1}, & A_8 \sim \nu^{\alpha-1}, \\ A_9 \sim \nu^{\alpha-1}, & A_{10} \sim \nu^{\alpha}, \\ A_{11} \sim \nu^{\alpha-1}, & A_{12} \sim \nu^{\alpha+1}, \\ A_{13} \sim \nu^{\alpha-1}, & \end{array} \tag{4.42}$$

For abnormal Regge exchange we must consider

$$\mathcal{M}_-^{(J)} = \mathcal{L}^-(1,1;J) : \mathcal{P}^J(P,-Q;-\Delta) : \mathcal{L}^+(1,0;J). \tag{4.43}$$

Using eqns. (4.20) and (4.25) in the usual way, we obtain

$$\begin{aligned} \mathcal{M}_-^{(J)} = {} & g_1' f_1 \{ \mathcal{P}^J_{\alpha\mu;\lambda} \varepsilon_{\nu\alpha}(Q\Delta) + \mathcal{P}^J_{\alpha\nu;\lambda} \varepsilon_{\mu\alpha}(Q\Delta) \} + \\ & + g'_2 f_1 \mathcal{P}^J_{\alpha;\lambda} \varepsilon_{\mu\nu\alpha}(\Delta) + g_3' f_1 \mathcal{P}^J_{\alpha;\lambda} \varepsilon_{\mu\nu\alpha}(Q) + \\ & + g_4' f_1 \mathcal{P}^J_{;\lambda} \varepsilon_{\mu\nu}(Q\Delta) + g_1' f_2 \{ \mathcal{P}^J_{\alpha\nu;} \varepsilon_{\mu\alpha}(Q\Delta) + \mathcal{P}^J_{\alpha\mu;} \varepsilon_{\nu\alpha}(Q\Delta) \} P_\lambda + \\ & + g_2' f_2 \mathcal{P}^J_{\alpha;} \varepsilon_{\mu\nu\alpha}(\Delta) P_\lambda + g_3' f_2 \mathcal{P}^J_{\alpha;} \varepsilon_{\mu\nu\alpha}(Q) P_\lambda + \\ & + g_4' f_2 \mathcal{P}^J \varepsilon_{\mu\nu}(Q\Delta) P_\lambda, \end{aligned} \tag{4.44}$$

where $g_1', \ldots, g_4'$, f_1, and f_2 are a different set of independent coupling constants. The resulting asymptotic behaviours for abnormal exchange are

$$\begin{array}{ll} A_1 \sim \nu^{\alpha-2}, & A_2 \sim \nu^{\alpha-1}, \\ A_3 \sim \nu^{\alpha-2}, & A_4 \sim \nu^{\alpha}, \\ A_5 \sim \nu^{\alpha-3}, & A_6 \sim \nu^{\alpha-1}, \\ A_7 \sim \nu^{\alpha-2}, & A_8 \sim \nu^{\alpha-1}, \\ A_9 \sim \nu^{\alpha-1}, & A_{10} \sim \nu^{\alpha}, \\ A_{11} \sim \nu^{\alpha}, & A_{12} \sim \nu^{\alpha+1}, \\ A_{13} \sim \nu^{\alpha-2}. & \end{array} \tag{4.45}$$

Once again, whether a given amplitude will superconverge depends on the aggragate behaviour of (4.42) and (4.45) and the particular value of α for the dominant Regge trajectory.

The method given here for obtaining partial wave expansions of invariant amplitudes and Reggeizing these in order to obtain asymptotic behaviours is quite general and may be applied to any process. When the partial wave expansions are written out in detail, it is sometimes possible to discover linear combinations of invariant amplitudes which fall off more rapidly asymptotically than do the individual amplitudes. It happens that such combinations do not arise for the examples which we have considered. However, in $\pi N \rightarrow \rho N$ scattering, for instance, the partial wave expansions of the invariant amplitudes obtained as above lead to the same result as the partial wave expansions of the helicity amplitudes. Superconvergence for $\pi N \rightarrow \rho N$ in the helicity formalism has been considered by Graham and Huq[27] and their results agree exactly with those that would be obtained from the present approach.

CHAPTER 5.

APPLICATIONS TO SUPERCONVERGENCE.

We have already introduced the idea of superconvergence in the previous chapter in order to explain the motive behind Reggeizing the invariant amplitudes to obtain their asymptotic behaviour. The idea of superconvergence arises from ordinary dispersion theory. It is well-known that we can write a dispersion integral for a t-channel invariant amplitude if, as $s \to \infty$, the amplitude tends to zero. For convenience we write the dispersion relation in terms of the variable $\nu = \frac{1}{4}(s - u)$ rather than s. Such a convention leads to simple crossing properties as we shall see later. The dispersion relation is then

$$A(\nu, t) = \frac{1}{\pi}\int_{\nu_0}^{\infty} \frac{\text{Im } A(\nu', t) \, d\nu'}{\nu' - \nu} , \tag{5.1}$$

where the integral extends over the physical region of the right-hand cut, ν_0 being the threshold energy and Im $A(\nu', t)$ the discontinuity across the cut. We saw in section 1.3 that the discontinuity of the T-matrix is entirely contained in the discontinuities of the invariant amplitudes provided the kinematic covariants satisfy the discontinuity condition (1.36). Our covariants are always chosen to satisfy this condition so we can always attatch physical meaning to the dispersion integral (5.1).

However, if the amplitude asymptotically behaves as $\nu^{-1-\varepsilon}$ where ε is some small positive number, we can also write a dispersion relation for $\nu A(\nu,t)$:

$$\nu A(\nu,t) = \frac{1}{\pi}\int_{\nu_0}^{\infty}\frac{\nu' \mathrm{Im}\, A(\nu',t)\, d\nu'}{\nu' - \nu} , \tag{5.2}$$

It follows immediately that eqn. (5.2) minus ν times eqn. (5.1) gives

$$\int_{\nu_0}^{\infty} \mathrm{Im}\, A(\nu',t)\, d\nu' = 0. \tag{5.3}$$

Equation (5.3) is known as a superconvergence relation.

It is obvious that if the asymptotic behaviour is $\nu^{-2-\varepsilon}$, this process may be repeated, leading to

$$\int_{\nu_0}^{\infty} \nu\, \mathrm{Im}\, A(\nu,t)\, d\nu = 0, \tag{5.4}$$

or in general, if the amplitude falls off asymptotically as $\nu^{-n-\varepsilon}$, we obtain the superconvergence relation

$$\int_{\nu_0}^{\infty} \nu^{n-1}\, \mathrm{Im}\, A(\nu,t)\, d\nu = 0. \tag{5.5}$$

We can obviously disperse in the variable $\tilde{\nu}$ and obtain superconvergence relations for the s-channel amplitudes in a similar manner if these are desired.

There are several possible approaches to superconvergence relations if one wishes to use them to obtain physical predictions. A recent idea[31-35] is to subtract off the higher Regge behaviour obtained as in the previous chapter and from the much faster convergence of the residue, obtain sum rules which hold for a range of values of t, but we will follow the original approach to superconvergence and consider those invariant amplitudes for which the Regge behaviour allows superconvergence at a particular value of the momentum transfer. Since for any given Regge pole, the value of $\alpha(t)$ tends to decrease as $t \to 0$, in order to obtain the optimum value, it is usual to take $t = 0$ in eqn. (5.5) and saturate the superconvergence relation in the forward direction. The $t = 0$ intercepts of the Regge trajectories must, therefore, be known, and care must be taken to ensure that the dominant Regge poles in both the normal and abnormal exchange cases are considered in deciding whether a particular invariant amplitude will superconverge. Care must also be taken that the presence of Regge cuts does not obviate the asymptotic behaviour indicated by the exchange of only single Regge poles[93,94].

Once superconvergence relations of the form (5.5) with $t = 0$ have been obtained, we get our physical sum rules by saturating these relations with direct s and crossed u channel elementary particle poles. These poles may actually occur below the physical threshold for the scattering process, but we assume that, in fact, they make a significant contribution to the amplitude. If the saturation is to be complete, we must, of course, include all possible resonances and multiparticle states, but the assumption is made that the amplitude is reasonably well approximated by just the well-known low lying elementary particle states. Several authors have, however, attempted to include these higher resonances[20,25,26,29-40].

Frampton and Taylor[25] found that for $\rho\pi$ scattering, the higher contributions were not necessarily small, but in 2^+0^- scattering, Rivers[29] found that their inclusion did not necessarily remove all inconsistencies between the sum rules without the use of further assumptions. More recently it has been suggested that the non-resonant contributions may be important[37], or that some Bootstrap-like mechanism can be used[36]. We do not investigate these possibilities here but assume that the higher contributions are small for at least some of the sum rules if not for all.

It has been proposed[23] that a possible criterion for selecting those sum rules which are reasonably well saturated by just low lying states is obtained by comparing the sum rules with the predictions of the U(6,6) symmetry scheme[67-73]. If the sum rules are satisfied by the U(6,6) values of the coupling constants, it is hoped that by inserting the physical masses, a means of obtaining a form of mass broken U(6,6) would have been achieved. It was also postulated that superconvergence using only isospin invariance and saturating with the low lying SU(6) multiplets, might have some direct connection with the U(6,6) symmetry. We test this hypothesis in the particular superconvergence sum rules obtained in the following sections.

5.1 Sum Rules for Normal $\frac{1}{2} + \frac{1}{2} \rightarrow \frac{1}{2} + \frac{1}{2}$.

There are several baryon-baryon scattering processes which are of the normal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ type. The most obvious is of course $NN \to NN$ for which superconvergence has been considered by D'Auria and de Alfaro[21]. In the general case, however, we should also consider the processes $N\Sigma \to N\Sigma$, $N\Sigma \to N\Lambda$, $\Sigma\Sigma \to \Sigma\Sigma$, $N\Xi \to N\Xi$, $\Sigma\Xi \to \Sigma\Xi$, $\Xi\Xi \to \Xi\Xi$, $N\Xi \to \Sigma\Sigma$, etc. It is to be expected that these various scattering processes will be similar in many respects, and it is found that, once the sum rules for one of the processes are found, it is a simple matter to generalize them so that they obtain for the other processes as well. We will, therefore, investigate only the reaction $N\Sigma \to N\Sigma$ in detail, and merely state the corresponding results for the other processes.

We investigate superconvergence in $N\Sigma$ elastic scattering in the formalism of the previous chapters but, in addition, we require the isotopic spin crossing matrices. It is in these isospin matrices that the main differences between the various processes above arise as the various particles carry different isospins. The relevant crossing matrices have been calculated by various authors[95] and here we quote the necessary matrices for $\frac{1}{2} + 1 \to \frac{1}{2} + 1$ in isospin space.

$$\begin{bmatrix} A_{\frac{1}{2}}^{(s)} \\ A_{\frac{3}{2}}^{(s)} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} A_0^{(t)} \\ A_1^{(t)} \end{bmatrix} , \qquad (5.6)$$

and

$$\begin{bmatrix} A_{\frac{1}{2}}^{(s)} \\ A_{\frac{3}{2}}^{(s)} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{9} \\ 2 & \frac{1}{3} \end{bmatrix} . \begin{bmatrix} A_{\frac{1}{2}}^{(u)} \\ A_{\frac{3}{2}}^{(u)} \end{bmatrix} , \qquad (5.7)$$

where in an obvious notation, $A_I^{(s)}$ indicates the s-channel amplitude corresponding to isospin I, etc.

There are six invariant amplitudes for $N\Sigma$ elastic scattering and their asymptotic behaviour as $t \to \infty$ is given by eqns. (4.38) and (4.39) for normal and abnormal exchange respectively. The appropriate values for the parameters $\alpha(s)$ are given by the Regge exchange trajectories in the s-channel. The s-channel has baryon number two, so we expect the dominant Regge poles in both the normal and abnormal cases to be some $N\Sigma$ resonances. Since baryons have, in general, the momentum transfer zero intercepts of their Regge trajectories less than zero, it is reasonable to assume that $\alpha(s=0)$, for both the normal and abnormal resonances of isospin $\frac{1}{2}$ and $\frac{3}{2}$, is less than zero in this case. From eqns. (4.38), (4.39), and the foregoing discussion, it is clear that we can have superconvergence relations of the form

$$\int \mathrm{Im}\, A_i^{(s)}(\bar{\nu},0)\, d\bar{\nu} = 0, \tag{5.8}$$

for i = 2,3,4, and 6.

We now wish to saturate eqns. (5.8) with single particle poles in the t and u channels. Both of these channels have baryon number zero so we saturate with 0^- and 1^- mesons as the lowest meson states likely to dominate the sum rules. The t-channel has isospin 0 or 1, so we saturate with the π, η, X^o, ρ, ω, and φ poles, while the u-channel has isospin $\frac{1}{2}$ or $\frac{3}{2}$ so we attempt to saturate this channel with the K and K^* poles.

The method of saturation is to consider effective interaction Lagrangians for each vertex of a pole graph, and with the appropriate spin and isospin propagators, calculate the contributions to the various invariant amplitudes in each channel. We obtain expressions for the t-channel amplitudes in terms of the masses and coupling constants for the π, ρ, etc. poles, and the u-channel amplitudes in terms of masses and coupling constants for the K and K^* poles. We then use the crossing matrices of section 2.1 (iii) and (5.6) and (5.7), in order to find the contributions of these poles to the s-channel invariant amplitudes of eqn. (5.8).

The effective interaction Lagrangian which we use, omitting the isospin couplings for the moment, are: for 0^- (P) coupling to two $\frac{1}{2}^+$ baryons (B)

$$\mathcal{L}_{PBB} = g_{PBB}\,\overline{B}\gamma_5 PB, \tag{5.9}$$

and for 1^- (V) coupling to two baryons

$$\mathcal{L}_{VBB} = \overline{B}\{\, g_{VBB}\gamma_\mu + \frac{1}{\mu}\, g'_{VBB}P_\mu \}\, V^\mu B, \tag{5.10}$$

where μ is a mass scale factor which makes g' of the same dimensions as g, and for the u-channel, the momentum P_μ is replaced by $\overline{\Lambda}_\mu$. These effective interaction Lagrangians are related to the $\mathcal{G}$-functions of section 4.2 by eqn. (4.17).

Since we wish to compare our sum rules with the predictions of U(6,6), we will absorb the φ coupling into the ω coupling and take the U(6,6) equal mass limit, with m the baryon mass and μ the meson mass. From eqn (5.8) we see that we have four sum rules for s-channel isospin $\frac{1}{2}$ and four for isospin $\frac{3}{2}$. For isospin $\frac{1}{2}$, the sum rule on $A_2^{(s)}$ gives

$$
\begin{aligned}
&(2g_{\pi NN}g_{\pi\Sigma\Sigma} + g_{\eta NN}g_{\eta\Sigma\Sigma} + g_{X^oNN}g_{X^o\Sigma\Sigma} - g^2_{KN\Sigma}) - \\
&- (1 - \frac{\mu^2}{4m^2})(2g'_{\rho NN}g'_{\rho\Sigma\Sigma} + g'_{\omega NN}g'_{\omega\Sigma\Sigma} - g'^2_{K^*N\Sigma}) + \\
&+ 2(2g_{NN\rho}g_{\rho\Sigma\Sigma} + g_{\omega NN}g_{\omega\Sigma\Sigma} - g^2_{K^*N\Sigma}) + (2g_{\rho NN}g'_{\rho\Sigma\Sigma} + 2g'_{\rho NN}g_{\rho\Sigma\Sigma} + \\
&+ g_{\omega NN}g'_{\omega\Sigma\Sigma} + g'_{\omega NN}g_{\omega\Sigma\Sigma} - 2g_{K^*N\Sigma}g'_{K^*N\Sigma}) = 0. \qquad (5.11)
\end{aligned}
$$

The sum rule on $A_3^{(s)}$ for $I = \frac{1}{2}$ gives

$$
\begin{aligned}
&(2g_{\pi NN}g_{\pi\Sigma\Sigma} + g_{\eta NN}g_{\eta\Sigma\Sigma} + g_{X^oNN}g_{X^o\Sigma\Sigma} + g^2_{KN\Sigma}) - \\
&- (1 - \frac{\mu^2}{4m^2})(2g'_{\rho NN}g'_{\rho\Sigma\Sigma} + g'_{\omega NN}g'_{\omega\Sigma\Sigma} + g'^2_{K^*N\Sigma}) - \\
&- 2(2g_{\rho NN}g_{\rho\Sigma\Sigma} + g_{\omega NN}g_{\omega\Sigma\Sigma} + g^2_{K^*N\Sigma}) - (2g_{\rho NN}g'_{\rho\Sigma\Sigma} + 2g'_{\rho NN}g_{\rho\Sigma\Sigma} + \\
&+ g_{\omega NN}g'_{\omega\Sigma\Sigma} + g'_{\omega NN}g_{\omega\Sigma\Sigma} - 2g_{K^*N\Sigma}g'_{K^*N\Sigma}) = 0. \qquad (5.12)
\end{aligned}
$$

The sum rule for $A_4^{(s)}$ with $I = \frac{1}{2}$ gives

$$
\begin{aligned}
&(2g_{\pi NN}g_{\pi\Sigma\Sigma} + g_{\eta NN}g_{\eta\Sigma\Sigma} + g_{X^oNN}g_{X^o\Sigma\Sigma} - g^2_{KN\Sigma}) + \\
&+ (1 - \frac{\mu^2}{4m^2})(2g'_{\rho NN}g'_{\rho\Sigma\Sigma} + g'_{\omega NN}g'_{\omega\Sigma\Sigma} - g'^2_{K^*N\Sigma}) + \\
&+ (2g_{\rho NN}g'_{\rho\Sigma\Sigma} + 2g'_{\rho NN}g_{\rho\Sigma\Sigma} + g_{\omega NN}g'_{\omega\Sigma\Sigma} + g'_{\omega NN}g_{\omega\Sigma\Sigma} - 2g_{K^*N\Sigma}g'_{K^*N\Sigma}) \\
&= 0. \qquad (5.13)
\end{aligned}
$$

The sum rule on $A_6^{(s)}$ for $I = \frac{1}{2}$ gives

$$
2g_{\rho NN}g'_{\rho\Sigma\Sigma} - 2g'_{\rho NN}g_{\rho\Sigma\Sigma} + g_{\omega NN}g'_{\omega\Sigma\Sigma} - g'_{\omega NN}g_{\omega\Sigma\Sigma} = 0. \qquad (5.14)
$$

Similarly, the sum rules for isospin $\frac{3}{2}$ in the s-channel may be obtained. They differ from the above only in the isospin crossing matrix factors and given in the same order, they are

$$\begin{aligned}&(-g_{\pi NN}g_{\pi\Sigma\Sigma} + g_{\eta NN}g_{\eta\Sigma\Sigma} + g_{X^oNN}g_{X^o\Sigma\Sigma} + 2g^2_{KN\Sigma}) + \\ &+ (1-\frac{\mu^2}{4m^2})(g'_{\rho NN}g'_{\rho\Sigma\Sigma} - g'_{\omega NN}g'_{\omega\Sigma\Sigma} - 2g'^*_{K}{}_{N\Sigma}) - \\ &- 2(g_{\rho NN}g_{\rho\Sigma\Sigma} - g_{\omega NN}g_{\omega\Sigma\Sigma} - 2g^{2*}_{K}{}_{N\Sigma}) - (g_{\rho NN}g'_{\rho\Sigma\Sigma} + g'_{\rho NN}g_{\rho\Sigma\Sigma} - \\ &-g_{\omega NN}g'_{\omega\Sigma\Sigma} - g'_{\omega NN}g_{\omega\Sigma\Sigma} - 4g_{K^*N\Sigma}g'_{K^*N\Sigma}) = 0, \qquad (5.15)\end{aligned}$$

$$\begin{aligned}&-(g_{\pi NN}g_{\pi\Sigma\Sigma} - g_{\eta NN}g_{\eta\Sigma\Sigma} - g_{X^oNN}g_{X^o\Sigma\Sigma} + 2g^2_{KN\Sigma}) + \\ &+ (1 - \frac{\mu^2}{4m^2})(g'_{\rho NN}g'_{\rho\Sigma\Sigma} - g'_{\omega NN}g'_{\omega\Sigma\Sigma} + 2g'^{2}_{K^*N\Sigma}) - \\ &- 2(-g_{\rho NN}g_{\rho\Sigma\Sigma} + g_{\omega NN}g_{\omega\Sigma\Sigma} - 2g^{2}_{K^*N\Sigma}) + (g_{\rho NN}g'_{\rho\Sigma\Sigma} + g'_{\rho NN}g_{\rho\Sigma\Sigma} - \\ &- g_{\omega NN}g'_{\omega\Sigma\Sigma} - g'_{\omega NN}g_{\omega\Sigma\Sigma} + 4g_{K^*N\Sigma}g'_{K^*N\Sigma}) = 0, \qquad (5.16)\end{aligned}$$

$$\begin{aligned}&(g_{\pi NN}g_{\pi\Sigma\Sigma} - g_{\eta NN}g_{\eta\Sigma\Sigma} - g_{X^oNN}g_{X^o\Sigma\Sigma} - 2g^2_{KN\Sigma}) + \\ &+ (1 - \frac{\mu^2}{4m^2})(g'_{\rho NN}g'_{\rho\Sigma\Sigma} - g'_{\omega NN}g'_{\omega\Sigma\Sigma} - 2g'^{2}_{K^*N\Sigma}) + \\ &+ (g_{\rho NN}g'_{\rho\Sigma\Sigma} + g'_{\rho NN}g_{\rho\Sigma\Sigma} - g_{\omega NN}g'_{\omega\Sigma\Sigma} - g'_{\omega NN}g_{\omega\Sigma\Sigma} - 4g_{K^*N\Sigma}g'_{K^*N\Sigma}) = \\ &= 0, \qquad (5.17)\end{aligned}$$

and

$$g'_{\rho NN}g_{\rho\Sigma\Sigma} - g_{\rho NN}g'_{\rho\Sigma\Sigma} - g'_{\omega NN}g_{\omega\Sigma\Sigma} + g_{\omega NN}g'_{\omega\Sigma\Sigma} = 0. \qquad (5.18)$$

The similarity between these sum rules and those given in ref. (21) is obvious and it is clear that the only differences encountered in writing out the sum rules for the various other scattering processes will be in the iso-spin crossing matrix elements and the actual couplings involved. In processes where there is crossing symmetry, it may be the case that some of the sum rules are identically satisfied by crossing. In obtaining the relative signs of the contributions from the t and u channels to eqns. (5.11) to (5.18), we must remember that as well as there being different crossing matrices involved, there is the overall sign change from Fermi statistics mentioned earlier and we also get an additional minus sign from taking the imaginary part of the amplitude when we cross $t \leftrightarrow u$.

Because of the large number of coupling constants needed in the above sum rules, their predictive value is rather small. We could follow the procedure of D'Auria and de Alfaro[21] and make physical assumptions about the relative sizes of the couplings and then investigate the degree to which the sum rules are satisfied by the physical values of the couplings. We will, however, investigate the consistency of our sum rules with the predictions of the U(6,6) symmetry scheme. Using the values for the coupling constants obtained from the U(6,6) current as in appendix B, we find that none of the eight sum rules for $N\Sigma$ elastic scattering are in fact consistent with these values. However, if we take out the factor $g^2(1 + \frac{2m}{\mu})^2(1 - \frac{\mu^2}{4m^2})$ in each case, where g is the U(6,6) coupling constant, we find that integral multiples of this factor always cancel, and we are left with only $\frac{\mu}{m}$ or $(\frac{\mu}{m})^2$ times this factor. If we absorb the factor $(1 + \frac{2m}{\mu})^2$ into g^2 as some sort of renormalization factor, we obtain consistency in the limit as $(\frac{\mu}{m}) \to 0$. It is not clear just what significance this procedure has, but consistency with U(6,6) predictions in this limit has been observed in other cases[20,23,26,31]. It would seem to be some generalization of the notion of soft pions in that we are taking all meson masses to zero in a limiting theory which has all meson masses equal in any case.

The difficulty in obtaining consistency with U(6,6) couplings appears to arise in the terms linear in $\frac{\mu}{m}$ in the g'_{VBB} couplings (see appendix B). In taking $\frac{\mu}{m} \to 0$, we are essentially simplifying these couplings in the U(6,6) theory. It is interesting to notice that in this limit, not only do the eight sum rules (5.11) to (5.18) agree exactly with the U(6,6) predictions, but also all the sum rules similarly obtained for all the other normal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ scattering processes agree with the U(6,6) predictions, whereas none agree before this limit is taken.

It would seem, therefore, that there is some general consistency in the normal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ sum rules and that the saturation with only 0^- and 1^- meson states is a reasonable approximation. The contribution from higher spin meson states cannot be compared with the contributions given above in the U(6,6) limit, because they are coupled to the baryons with independent coupling constants in the U(6,6) theory. The assumption that these higher spin contributions are small would seem, however, to be reasonable in view of the consistency which we have already achieved.

5.2 Sum Rules for Abnormal $0 + 1 \to 1 + 1$.

As we have mentioned before, in order to make use of the possible $s \leftrightarrow u$ crossing symmetry in abnormal $0+1 \to 1+1$ scattering, we form our superconvergence integrals on the t-channel invariant amplitudes. The asymptotic behaviour for these as $s \to \infty$ has been given in eqns. (4.42) and (4.45). As we are now dealing with processes like $\pi\rho \to \rho\omega$, $\pi K^* \to \rho K^*$, $\eta\rho \to \rho\rho$, $K\rho \to K^*\omega$, $K\rho \to K^*\rho$, $\pi K^* \to \omega K^*$, $KK^* \to K^*K^*$, etc, we will have meson Regge trajectories and we must consider the selection rules arising from G-parity as well as spin parity and isotopic spin. In the t-channel, the dominant normal trajectories of positive G-parity are the ρ for isospin 1 and the Pomeron for isospin 0. For negative G-parity, the A_2 dominates the $I = 1$ channel and the ω dominates for $I = 0$. The dominant abnormal trajectories of positive G-parity are the B meson for $I = 1$ and the η for $I = 0$. For negative G-parity, the A_1 dominates the isotopic spin 1 channel and there is no known trajectory for $I = 0$. In the following we will assume that the $t = 0$ intercepts of the ρ, ω, A_1, A_2, and B trajectories lie between 0 and 1, the Pomeron trajectory passes through 1 at $t = 0$, while the trajectories for the η and the case where no particle is known, are less than zero at $t = 0$.

From the thirteen amplitudes (3.1) and the asymptotic behaviours of (4.42) and (4.45) with the above assumptions about $\alpha(t=0)$, we find the following sum rules for processes with positive G-parity Regge exchange:

$$\int \mathrm{Im}\, A_1^{(1)}(\nu,0)\, d\nu = 0, \tag{5.19}$$

$$\int \mathrm{Im}\, A_1^{(0)}(\nu,0)\, d\nu = 0, \tag{5.20}$$

$$\int \mathrm{Im}\, A_3^{(1)}(\nu,0)\, d\nu = 0, \tag{5.21}$$

$$\int \mathrm{Im}\, A_3^{(0)}(\nu,0)\, d\nu = 0, \tag{5.22}$$

$$\int \mathrm{Im}\, A_5^{(1)}(\nu,0)\, d\nu = 0, \tag{5.23}$$

where the bracketed superscript indicates the t-channel isospin and the subscript labels the amplitude corresponding to the covariants as in eqn. (3.1). For negative G-parity Regge exchange, we get these same five sum rules plus

$$\int \mathrm{Im}\, A_2^{(0)}(\nu,0)\, d\nu = 0, \tag{5.24}$$

$$\int \mathrm{Im}\, A_5^{(0)}(\nu,0)\, d\nu = 0, \tag{5.25}$$

$$\int \nu \mathrm{Im}\, A_1^{(0)}(\nu,0)\, d\nu = 0, \tag{5.26}$$

$$\int \nu \mathrm{Im}\, A_3^{(0)}(\nu,0)\, d\nu = 0, \tag{5.27}$$

Once again the isotopic spin analysis will differ for the different processes considered and we refer to Carruthers[95] for details of the appropriate crossing matrices. We also get a relative minus sign between the s and u channel conributions arising from the fact that we have taken the imaginary part of the amplitude, but with bosons we do not get an overall sign change from statistics.

We will attempt to saturate the sum rules (5.19) to (5.27) with 0^- and 1^- particles in the s and u channels. As in the previous case, the analysis for the different abnormal

$0 + 1 \to 1 + 1$ processes is very similar and we will consider only $\pi\rho \to \rho\omega$ scattering in detail. In this reaction we can have the pion pole in the s-channel and the rho pole in the crossed u-channel. We perform our saturation as before, using the effective interaction Lagrangians

$$\mathcal{L}_{\rho\pi\pi} = g_1 \; i\varepsilon_{abc}\rho^{\mu a}\pi^b\pi^c\Lambda_\mu , \tag{5.28}$$

$$\mathcal{L}_{\rho\omega\pi} = g_2\omega^\nu\rho^{\lambda a}\pi_a\varepsilon_{\nu\lambda}(\Lambda' K), \tag{5.29}$$

$$\mathcal{L}_{\rho\rho\rho} = i\varepsilon_{abc}\rho^{\mu a}\rho^{\lambda b}\rho^{\sigma c}\{g_3 g_{\lambda\sigma}\bar{\Lambda}_\mu + g_3' g_{\mu\sigma}\bar{\Lambda}_\lambda + \\ + g_3'' g_{\mu\lambda}\bar{\Lambda}_\sigma + g_3''' \bar{\Lambda}_\mu\bar{\Lambda}_\lambda\bar{\Lambda}_\sigma\}, \tag{5.30}$$

where a, b, c = 1, 2, 3 are isotopic spin indices and the momenta for the s and u channels are defined as in sec. 1.1.

Since in $\pi\rho \to \rho\omega$ scattering we can have only isospin 1 in each channel, the crossing matrices are trivial and of the nine possible sum rules, the only ones to apply are (5.19), (5.21), and (5.23). Evaluating these in the U(6,6) equal mass limit, we obtain from eqn. (5,19)

$$(g_1 + g_3)g_2 = 0, \tag{5.31}$$

from eqn. (5.21)

$$(g_1 - g_3 + 2g_3'' - \mu^2 g_3''')g_2 = 0, \tag{5.32}$$

and from eqn. (5.23)

$$(g_1 + g_3 + 2\mu^2 g_3''')g_2 = 0. \tag{5.33}$$

Comparing these sum rules with the predictions of the U(6,6) current given in appendix B, we see that eqn. (5.31) gives the U(6,6) result $g_1 = -g_3$, while (5.32) and (5.33) are inconsistent with the U(6,6) predictions. The sum rules for the other possible process are similar to those given here for $\pi\rho \to \rho\omega$. For example, in $\pi K^* \to \rho K^*$ we find that all nine sum rules (5.19) to (5.27) are possible, but (5.19), (5.21), and (5.24)-(5.27) are identically satisfied by crossing. None of the three non-trivial sum rules agree with the U(6,6) predictions. Similarly for the other possible scattering processes we find that only the sum rule (5.19) ever gives the U(6,6) result and then it is sometimes satisfied identically by crossing. There is, therefore, no general consistency with U(6,6) in these results as there was for superconvergence in normal $\frac{1}{2} + \frac{1}{2} \to \frac{1}{2} + \frac{1}{2}$ scattering. The $\mu \to 0$ limit does not help us in this case. But eqn. (5.19) does always give the U(6,6) value or is identically satisfied, so one would hope that this sum rule at least may be reasonably approximated by just 0^- and 1^- meson poles. The higher resonance contributions to the other sum rules are probably not small. (cf. Frampton and Taylor[25]).

Once again we do not calculate these higher contributions because they cannot be related to the 0^- and 1^- contributions in U(6,6) theory where they introduce a further arbitrary coupling constant. Also, the multiplicity of coupling constants introduced by including further contributions reduces the predictive power of the sum rules.

5.3 Discussion.

The formalism which we have established in the first four chapters has been applied by various authors to obtain superconvergence sum rules for a large variety of processes. It is clear that by using this completely general technique, one could, in principle, determine the superconvergence relations for all possible scattering processes. The formalism is not, however, of value for superconvergence relations only. The M-function approach gives a possible method for the analysis of any scattering process of whatever spin, which has many attractive features not shared by the alternative helicity amplitude approach. We have shown that it is possible to obtain a set of kinematic singularity free invariant amplitudes for two processes of the most complicated type; those which have M-functions of the fourth kind. The conjecture that it is always possible to find such a KSF set of invariant amplitudes, although not proven, seems likely to be true, so that the method provides a powerful tool for handling the problems of kinematic singularities and zeros in processes with spin.

The partial wave expansions of the invariant amplitudes obtained covariantly by the method of contracted projection operators and covariant vertex functions in chapter 4 is also of obviously greater general value than just for finding asymptotic behaviours. It provides a possible approach to the general problems of Reggeism, the occurrence of fixed poles, daughters and conspirators[52, 53,85], as well as the general partial wave analysis of scattering processes.

In the present work, however, we have developed the formalism purely for application to the problem of superconvergence and have not considered the more general aspects of the theory in any detail. We have limited our investigation of the superconvergence sum rules to the attempt to find if there is any consistency with the predictions of the U(6,6) symmetry scheme in the results of superconvergence calculations based on isospin invariance alone. Earlier work seemed to indicate that there was some such relation between the two theories, and we find that there is some degree of consistency in the two general cases of superconvergence considered in sections 5.1 and 5.2. All the sum rules for normal $\frac{1}{2} + \frac{1}{2} \rightarrow \frac{1}{2} + \frac{1}{2}$ scattering agree in the $(\frac{\mu}{m}) \rightarrow 0$ limit and there is one sum rule of nine in abnormal $0 + 1 \rightarrow 1 + 1$ scattering which either gives the predicted U(6,6) values for the coupling constants or is identically satisfied. One would hope that in some extension of the superconvergence sum rules obtained here, those sum rules which we have found to be consistent with U(6,6) will in general be the most readily amenable for obtaining reasonable physical predictions. Such an investigation has not, however, come within the scope of the work done for this thesis.

The analysis of abnormal $\frac{1}{2} + \frac{1}{2} \rightarrow \frac{1}{2} + \frac{3}{2}$ scattering in section 3.2 could be used immediately to investigate superconvergence sum rules for this process. In this way we could go on to consider all possible processes and perhaps some pattern would emerge to enable us to understand more clearly the physical implications of the phenomenon of superconvergence. Consistency with the predictions of the U(6,6) symmetry scheme is one possible general pattern.

APPENDIX A.

We use the metric $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$ and γ matrices such that $\gamma_\mu = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ and $\gamma^\mu = (\gamma_0, -\gamma_1, -\gamma_2, -\gamma_3)$. Then we have the anticommutator

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu},$$

and γ_5 defined by

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = -\frac{1}{4!}\,\varepsilon_{\mu\nu\lambda\rho}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho,$$

with $\gamma_5^2 = -1$.

We define

$$\sigma_{\mu\nu} = \frac{i}{2}\,[\gamma_\mu, \gamma_\nu],$$

and the Levi-Civita tensor $\varepsilon_{\mu\nu\lambda\rho}$ such that $\varepsilon_{0123} = +1$ and

$$\varepsilon_{\mu\nu\lambda\rho}\varepsilon_{\alpha\beta\gamma\delta} = -\begin{vmatrix} g_{\mu\alpha} & g_{\mu\beta} & g_{\mu\gamma} & g_{\mu\delta} \\ g_{\nu\alpha} & g_{\nu\beta} & g_{\nu\gamma} & g_{\nu\delta} \\ g_{\lambda\alpha} & g_{\lambda\beta} & g_{\lambda\gamma} & g_{\lambda\delta} \\ g_{\rho\alpha} & g_{\rho\beta} & g_{\rho\gamma} & g_{\rho\delta} \end{vmatrix}. \tag{A.1}$$

With the totally antisymmetric Levi-Civita tensor we often use the abbreviated notation

$$\varepsilon(ABCD) = \varepsilon_{\mu\nu\lambda\rho}A^\mu B^\nu C^\lambda D^\rho,$$
$$\varepsilon_\mu(ABC) = \varepsilon_{\mu\nu\lambda\rho}A^\nu B^\lambda C^\rho,$$

etc. With this tensor we have the useful identity

$$g_{\alpha\beta}\varepsilon_{\mu\nu\lambda\rho} = g_{\mu\beta}\varepsilon_{\alpha\nu\lambda\rho} + g_{\nu\beta}\varepsilon_{\mu\alpha\lambda\rho} + g_{\lambda\beta}\varepsilon_{\mu\nu\alpha\rho} + g_{\rho\beta}\varepsilon_{\mu\nu\lambda\alpha}. \tag{A.2}$$

Products of the sixteen Dirac γ matrices are decomposed according to the following identities:

$$\gamma_\lambda\sigma_{\mu\nu} = i(g_{\lambda\mu}\gamma_\nu - g_{\lambda\nu}\gamma_\mu) - i\varepsilon_{\lambda\mu\nu}(\gamma_5\gamma), \tag{A.3}$$

$$\gamma_5\sigma_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu}(\sigma), \tag{A.4}$$

$$\begin{aligned}\sigma_{\mu\nu}\sigma_{\lambda\rho} = {} & i(g_{\mu\rho}\sigma_{\nu\lambda} + g_{\nu\lambda}\sigma_{\mu\rho} - g_{\mu\lambda}\sigma_{\nu\rho} - g_{\nu\rho}\sigma_{\mu\lambda}) \\ & (g_{\mu\lambda}g_{\nu\rho} - g_{\nu\lambda}g_{\mu\rho} - \varepsilon_{\mu\nu\lambda\rho}\gamma_5),\end{aligned} \tag{A.5}$$

and

$$\gamma_5\gamma_\mu\gamma_5\gamma_\nu = g_{\mu\nu} - i\sigma_{\mu\nu}. \tag{A.6}$$

From these it follows that

$$\begin{aligned}\gamma_5\varepsilon_{\mu\nu}(A\gamma) &= -i\sigma_{\mu\nu}\gamma.A + A_\mu\gamma_\nu - A_\nu\gamma_\mu \\ &= -\gamma.Ai\sigma_{\mu\nu} - A_\mu\gamma_\nu + A_\nu\gamma_\mu \;.\end{aligned} \tag{A.7}$$

The completeness relation for the sixteen γ matrices is written

$$\delta_{\sigma'}{}^{\tau}\delta_{\tau'}{}^{\sigma} = \tfrac{1}{4}\sum_R (\gamma_R)_{\sigma'}{}^{\sigma}(\gamma_R)_{\tau'}{}^{\tau}$$

$$= \tfrac{1}{4}[\,(1)_{\sigma'}{}^{\sigma}(1)_{\tau'}{}^{\tau} + (\gamma_\mu)_{\sigma'}{}^{\sigma}(\gamma^\mu)_{\tau'}{}^{\tau} + (\gamma_5\gamma_\mu)_{\sigma'}{}^{\sigma}(\gamma_5\gamma^\mu)_{\tau'}{}^{\tau} +$$

$$+ \tfrac{1}{2}(\sigma_{\mu\nu})_{\sigma'}{}^{\sigma}(\sigma^{\mu\nu})_{\tau'}{}^{\tau} - (\gamma_5)_{\sigma'}{}^{\sigma}(\gamma_5)_{\tau'}{}^{\tau}]. \tag{A.8}$$

APPENDIX B.

The U(6,6) currents from which the symmetry predictions for the coupling constants are obtained are[67-73], for the total pseudoscalar current

$$j_5 = g(1 + \frac{2m}{\mu})[\ \frac{P^2}{4m^2}\{(\bar{N}\gamma_5 N)_{D+\frac{2}{3}F-S} + 3\bar{D}_\lambda\gamma_5 D^\lambda\} + + \frac{3}{2m^2}q_\lambda\bar{D}^\lambda\gamma_5 q_\kappa D^\kappa + \frac{q_\lambda\bar{D}^\lambda}{m} N + h.c.] - - \frac{3ih}{2\mu}\ \varepsilon_{\mu\nu\rho\lambda}q^\mu P^\nu(\bar{\varphi}^\rho\varphi^\lambda)_D - 3h\ q^\lambda(\bar{\varphi}_\lambda\varphi_5 - \bar{\varphi}_5\varphi_\lambda)_F, \quad (B.1)$$

and for the total vector current

$$j_\mu = g\ \frac{P_\mu}{2m}(1 + \frac{\mu}{2m})(\bar{N}N)_{F+3S} + g(1 + \frac{2m}{\mu})(\bar{N}\ \frac{r_\mu}{4m^2}\ N)_{D+\frac{2}{3}F-S} + + \frac{3gP^2}{4m^2}\ \bar{D}^\lambda[(1 + \frac{2m}{\mu})\gamma_\mu - \frac{P_\mu}{\mu}]D_\lambda + \frac{3g}{2m^2}\ q^\lambda\bar{D}_\lambda[(1 + \frac{2m}{\mu})\gamma_\mu - \frac{P_\mu}{\mu}]q^\rho D_\rho + + \frac{g}{2m^2}(1 + \frac{2m}{\mu})\varepsilon_{\mu\nu\rho\lambda}P^\nu q^\rho\bar{D}^\lambda N + h.c. + + h[\frac{3}{2}P_\mu(\bar{\varphi}_5\varphi_5 - \bar{\varphi}^\lambda\varphi_\lambda)_F + \frac{P_\mu q_\nu q_\lambda}{\mu^2}(\bar{\varphi}^\nu\varphi^\lambda)_F + 3q^\nu(\bar{\varphi}_\nu\varphi_\mu - \varphi_\nu\bar{\varphi}_\mu)_F] + \frac{ih}{4\mu}\ \varepsilon_{\mu\nu\rho\lambda}P^\rho q^\lambda(\bar{\varphi}^\nu\varphi_5 + \bar{\varphi}_5\varphi^\nu)_D. \quad (B.2)$$

In eqns. (B.1) and (B.2), the momenta are defined by $q = p - p'$, $P = p + p'$, $r_\mu = \varepsilon_{\mu\nu\rho\lambda}P^\nu q^\rho\gamma^\lambda\gamma_5$, where p, p' denote incoming and outgoing particle momenta respectively. The symbols N, D_μ, φ_μ, and φ_5 denote the baryon octet, decuplet, and vector and pseudoscalar meson octets respectively. The F, D, and S couplings are defined by

$$(\bar{N}N)_F = (N\bar{N} - \bar{N}N)$$
$$(\bar{N}N)_D = (N\bar{N} + \bar{N}N)$$
$$(\bar{N}N)_S = \mathrm{Tr}(\bar{N}N).$$

The mixing angles for the isoscalar component of the vector and pseudoscalar SU(3) octets are defined by the matrices

$$\varphi_\mu = \begin{bmatrix} \frac{\rho^o}{\sqrt{2}}+\frac{\omega}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & -\frac{\rho}{\sqrt{2}}+\frac{\omega}{\sqrt{2}} & K^{*o} \\ K^{*-} & \bar{K}^{*o} & \varphi \end{bmatrix}, \tag{B.3}$$

and

$$\varphi_5 = \begin{bmatrix} \frac{\pi^o}{\sqrt{2}}+\frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^o}{\sqrt{2}}+\frac{\eta}{\sqrt{6}} & K^o \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{bmatrix} + \frac{i}{\sqrt{3}}\,X^o. \tag{B,4}$$

For the comparison of the superconvergence sum rules obtained in chapter 5 with the predictions of U(6,6), we require the couplings $\bar{N}N\varphi_5$, $\bar{N}N\varphi_\mu$, $\varphi_\mu\varphi_\nu\varphi_5$, and $\varphi_\mu\varphi_\nu\varphi_\lambda$. With the notation for couplings used in chapter 5, we find that

$$g_{\pi NN} = \frac{5g}{3\sqrt{2}}\left(1 + \frac{2m}{\mu}\right)\left(1 - \frac{\mu^2}{4m^2}\right), \tag{B.5}$$

and in terms of this

$$\begin{aligned} g_{\pi NN} &= \frac{5}{\sqrt{3}}g_{\eta NN} = \frac{5}{\sqrt{6}}g_{X^oNN} = \frac{5}{4}g_{\pi\Sigma\Sigma} = \frac{5}{2\sqrt{3}}g_{\eta\Sigma\Sigma} = \frac{5}{\sqrt{6}}g_{X^o\Sigma\Sigma} = \\ &= 5g_{KN\Sigma} = \frac{5}{2\sqrt{3}}g_{\pi\Sigma\Lambda} = -\frac{5}{3\sqrt{3}}g_{KN\Lambda} = -5g_{\pi\Xi\Xi} = -\frac{5}{3\sqrt{3}}g_{\eta\Xi\Xi} = \\ &= \frac{5}{\sqrt{6}}g_{X^o\Xi\Xi} = g_{K\Sigma\Xi} = \frac{5}{\sqrt{3}}g_{K\Xi\Lambda} = -\frac{5}{2\sqrt{3}}g_{\eta\Lambda\Lambda} = \frac{5}{\sqrt{6}}g_{X^o\Lambda\Lambda}. \end{aligned} \tag{B.6}$$

There are two couplings for vector mesons and the octet of baryons, defined by eqn. (5.10) and we find that

$$g_{\rho NN} = g_{\pi NN} = \frac{5g}{3\sqrt{2}}(1 + \frac{2m}{\mu})(1 - \frac{\mu^2}{4m^2}) =$$

$$= \frac{5}{3}g_{\omega NN} = \frac{5}{4}g_{\rho\Sigma\Sigma} = \frac{5}{4}g_{\omega\Sigma\Sigma} = -\frac{5}{\sqrt{2}}g_{\varphi\Sigma\Sigma} = 5g_{K^*N\Sigma} = \frac{5}{2\sqrt{3}}\, g_{\rho\Sigma\Lambda} =$$

$$= -\frac{5}{3\sqrt{3}}\, g_{K^*N\Lambda} = -5g_{\rho\Xi\Xi} = -5g_{\omega\Xi\Xi} = \frac{5}{4\sqrt{2}}\, g_{\varphi\Sigma\Sigma} = g_{K^*\Sigma\Xi} =$$

$$= \frac{5}{\sqrt{3}}g_{K^*\Xi\Lambda} = \frac{5}{3\sqrt{2}}\, g_{\varphi\Lambda\Lambda} \,,$$

and

$$g_{\varphi NN} = g_{\omega\Lambda\Lambda} = 0. \tag{B.7}$$

For g'_{VBB}, we find that each coupling contains the factor $g(1 + \frac{2m}{\mu}) = G$, so that

$$g'_{\rho NN} = \frac{1}{3\sqrt{2}}\, G(\frac{3\mu}{2m} - 5), \quad g'_{\omega NN} = \frac{1}{\sqrt{2}}\, G(\frac{3\mu}{2m} - 1), \quad g'_{\varphi NN} = 0,$$

$$g'_{\rho\Sigma\Sigma} = \frac{1}{3\sqrt{2}}\, G(\frac{3\mu}{m} - 4) = g'_{\omega\Sigma\Sigma}, \quad g'_{\varphi\Sigma\Sigma} = \frac{1}{3}\, G(\frac{3\mu}{2m} + 1) = -\sqrt{2}g'_{K^*N\Sigma},$$

$$g'_{\rho\Sigma\Lambda} = -\frac{2}{\sqrt{6}}\, G, \quad g'_{K^*N\Lambda} = -\frac{3}{\sqrt{6}}\, G(\frac{\mu}{2m} - 1), \quad g'_{\rho\Xi\Xi} = \frac{1}{3\sqrt{2}}\, G\,(\frac{3\mu}{2m} + 1) =$$

$$= g'_{\omega\Xi\Xi}, \quad g'_{\varphi\Xi\Xi} = \frac{1}{3}\, G(\frac{3\mu}{m} - 4), \quad g'_{K^*\Sigma\Xi} = \frac{1}{3\sqrt{2}}\, G(\frac{3\mu}{2m} - 5),$$

$$g'_{K^*\Xi\Lambda} = -\frac{1}{\sqrt{6}}\, G(\frac{3\mu}{2m} + 1), \quad g'_{\omega\Lambda\Lambda} = \frac{1}{\sqrt{2}}\, G\,\frac{\mu}{m}, \quad g'_{\varphi\Lambda\Lambda} = G(\frac{\mu}{2m} - 1). \tag{B.8}$$

The three meson couplings are somewhat simpler, and we find that, in the notation of eqns. (5.28), (5.29), and (5.30),

$$g_{\rho\pi\pi} = \frac{3}{\sqrt{2}}\, h = 2g_{\rho KK} = 2g_{K^*K\pi} = -\frac{2}{\sqrt{3}}\, g_{K^*K\eta} = -2g_{\omega KK} = \sqrt{2}g_{\varphi KK}. \tag{B.9}$$

$$g_{\rho\omega\pi} = \frac{3\sqrt{2}}{\mu}\, h = 2g_{K^*K^*\pi} = 2g_{\rho K^*K} = 2g_{\omega K^*K} = \sqrt{2}g_{\varphi K^*K} =$$

$$= 2\sqrt{3} g_{\rho\rho\eta} = 2\sqrt{3} g_{\omega\omega\eta} = -\sqrt{3} g_{\varphi\varphi\eta} = -2\sqrt{3} g_{K^* K^* \eta} = \sqrt{6} g_{\rho\rho X^o} =$$

$$= \sqrt{6} g_{\omega\omega X^o} = \sqrt{6} g_{\varphi\varphi X^o} = \frac{\sqrt{6}}{2} g_{K^* K^* X^o}. \qquad (B.10)$$

The three vector couplings follow immediately from eqns. (B.2) and (B.9). For example

$$g_{\rho\pi\pi} = - g_{\rho\rho\rho} = g'_{\rho\rho\rho} = - g''_{\rho\rho\rho} = \frac{3\mu^2}{2} g'''_{\rho\rho\rho} . \qquad (B.11)$$

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A Limitation on the Use of Current Algebra Techniques for Meson Decays (*).

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Summary. — It is shown that for decays involving only 1^- particles and an odd number of 0^- particles, the current algebra approach is not useful.

1. – Introduction.

Several papers have been published recently [1-3] using current algebra techniques to derive results for various meson decay modes. In this paper we shall examine the application of current algebra techniques to a particular type of meson decay. Our discussion will relate only to those decays which satisfy the following two conditions:

a) only 1^- or 0^- particles may be involved in the interaction,

b) the over-all number of 0^- particles must be odd.

Our aim is to show that for decays satisfying these two conditions, the application of current algebra techniques does not lead to physically useful results.

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Our starting point is the equal-time commutation relations

$$[A_0^a(x),\, A_\mu^b(0)]\delta(x_0) = i\varepsilon_{abc}V_\mu^c(x)\,\delta^{(4)}(x)\,, \tag{1}$$

$$[A_0^a(x),\, V_\mu^b(0)]\delta(x_0) = i\varepsilon_{abc}A_\mu^c(x)\,\delta^{(4)}(x)\,, \tag{2}$$

$$[V_0^a(x),\, V_\mu^b(0)]\delta(x_0) = i\varepsilon_{abc}V_\mu^c(x)\,\delta^{(4)}(x)\,, \tag{3}$$

where $A_\mu^a(x)$ and $V_\nu^b(x)$ are respectively the axial-vector and vector currents and $(a, b, c) = (1, 2, 3)$ are isotopic-spin indices. Following the method of Alessandrini, Bég, and Brown (4), for the commutator (1) we write down the amplitude

$$T_{\mu\nu}^1 = i\int \mathrm{d}^4x \exp[ik^a\cdot x]\theta(x_0)\,\langle\alpha|[A_\mu^a(x),\, A_\nu^b(0)]|\beta\rangle\,, \tag{4}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are some meson states, as yet unspecified. From (4) by partial integration we obtain the equation

$$\begin{aligned} k_\mu^a k_\nu^b T_{\mu\nu}^1 + i\varepsilon_{abc}k_\nu^b\langle\alpha|V_\nu^c(0)|\beta\rangle = \\ = i\int \mathrm{d}^4x \exp[ik^a\cdot x]\delta(x_0)\,\langle\alpha|\partial_\mu A_\mu^a(x),\, A_0^b(0)]|\beta\rangle + \\ + i\int \mathrm{d}^4x \exp[ik^a\cdot x]\theta(x_0)\,\langle\alpha|[\partial_\mu A_\mu^a(x),\, \partial_\nu A_\nu^b(0)]|\beta\rangle\,, \end{aligned} \tag{5}$$

where we have used eq. (1).

Similarly for (2) we form

$$T_{\mu\nu}^2 = i\int \mathrm{d}^4x \exp[ik^a\cdot x]\theta(x_0)\,\langle\beta|[A_\mu^a(x),\, V_\nu^b(0)]|\beta\rangle\,, \tag{6}$$

and from this we obtain the equation

$$k_\mu^a T_{\mu\nu}^2 + i\varepsilon_{abc}\langle\alpha|A_\nu^c(0)|\beta\rangle = -\int \mathrm{d}^4x \exp[ik^a\cdot x]\theta(x_0)\langle\alpha|[\partial_\mu A_\mu^a(x),\, V_\nu^b(0)]|\beta\rangle\,. \tag{7}$$

Again for (3) we form

$$T_{\mu\nu}^3 = i\int \mathrm{d}^4x \exp[ik^a\cdot x]\theta(x_0)\,\langle\alpha|[V_\mu^a(x),\, V_\nu^b(0)]|\beta\rangle\,, \tag{8}$$

and obtain

$$k_\mu^a T_{\mu\nu}^3 + i\varepsilon_{abc}\langle\alpha|V_\nu{}^c(0)|\beta\rangle = 0\,, \tag{9}$$

(4) V. A. Alessandrini, M. A. Bég and D. S. Brown: *Phys. Rev.*, **144**, 1137 (1966).

since we assume that vector currents are conserved. Equations (5), (7) and (9) will form the basis of our discussion.

2. – Meson interactions.

If we assume the PCAC relation

$$\partial_\mu A^a_\mu(x) = f_\pi \varphi^a_\pi(x)\,, \tag{10}$$

where

$$f_\pi = \frac{M m_\pi^2 g_A}{g_r K_{NN\pi}(0)}\,,$$

$\varphi^a_\pi(x)$ is the renormalized pion field operator, and the other constants are defined as usual [5], and put α and β equal to various 1^- and 0^- particle states or the vacuum state in equations (5), (7) and (9) we would hope to be able to get relations between various physical decay processes. In particular we are interested in interactions involving an odd number of pseudoscalar mesons and we find that there are seven ways not necessarily all different, of identifying α and β so that our conditions are satisfied.

i) In (5) put a 0^- particle for α and a 1^- for β.

ii) In (7) put a 1^- particle for α and a 1^- for β.

iii) In (9) put a 0^- particle for α and a 1^- for β.

iv) In (5) put a 0^- particle for β and the vacuum for α.

v) In (7) put the vacuum for α and a 1^- particle for β.

vi) In (7) put a 0^- particle for α and a 0^- for β.

vii) In (9) put the vacuum for α and a 0^- particle for β.

Because we have the particular situation of processes involving an odd number of pseudoscalar mesons, parity considerations necessitate the use of the Levi-Civita tensor $\varepsilon_{\mu\nu\lambda\sigma}$ in exhibiting the explicit momentum dependence of each term of the resulting equations. In each of the seven cases above we have only a limited number of momentum and polarization vectors available for contracting with the indices on $\varepsilon_{\mu\nu\lambda\sigma}$ and it turns out that if we take the limit as some momentum or momenta go to zero in order to get rid of physically uninteresting terms, the physically interesting terms also go to zero. We will

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(5) See, for example, S. A. ADLER: *Phys. Rev.*, **137**, B 1022 (1965).

illustrate this point by considering the second example discussed by KAWARABAYASHI and SUZUKI in which they attempt to relate the coupling constants for the decays $\omega \to \pi^+\pi^-\pi^0$ and $\omega \to \pi^0\gamma$. This is of type i) in the above classification where we put $\alpha = \pi^0$ and $\beta = \omega$ in eq. (5). The equation then reads

$$(11)\quad ik_\mu^+ k_\nu^- \int d^4x \exp[ik^+\cdot x]\theta(x_0)\langle\pi^0|[A_\mu^+(x), A_\nu^-(0)]|\omega\rangle + k_\nu^-\langle\pi^0|V_\nu^3(0)|\omega\rangle = i\int d^4x \exp[ik^+\cdot x]\delta(x_0)\langle\pi^0|[\partial_\mu A_\mu^+(x), A_0^-(0)]|\omega\rangle + i\int d^4x \exp[ik^+\cdot x]\theta(x_0)\langle\pi^0|[\partial_\mu A_\mu^+(x), \partial_\nu A_\nu^-(0)]|\omega\rangle,$$

where we have used the combinations

$$\frac{1}{\sqrt{2}}(A_\mu^1 \pm A_\mu^2) = A_\mu^\pm.$$

It is clear that we can relate the second term on the left-hand side of (11) to the decay $\omega \to \pi^0\gamma$ and the second term on the right-hand side to $\omega \to \pi^+\pi^-\pi^0$, so it is tempting to take the limit of eq. (11) as both k^+ and $k^- \to 0$ in order to get rid of the other two terms which cannot easily be related to known physical processes. That this does not, however, work can be seen from a closer examination of the dependence of each term of (11) on k^+ and k^-. We have only four possible vectors available for contracting with the indices on $\varepsilon_{\mu\nu\lambda\sigma}$, they are k_μ^+, k_ν^-, the momenta associated with the axial-vector currents, and e_λ and p_σ, respectively the polarization and momentum vectors of the ω meson. Consequently each term must involve both k^+ and k^- in the same way and so all terms will fall off at the same rate as k^+ and $k^- \to 0$.

For the first term of (11) we can write

$$g\varepsilon_{\mu\nu\lambda\sigma}k_\mu^+ k_\nu^- e_\lambda p_\sigma,$$

where g is some coupling constant. Similarly for the second term which we relate to the $\omega \to \pi^0\gamma$ decay we write

$$g_{\omega\pi^0\gamma}\varepsilon_{\mu\nu\lambda\sigma}k_\mu^+ k_\nu^- e_\lambda p_\sigma.$$

By using PCAC and taking the limit of zero 4-momentum transfer squared for the pion current, the third term becomes

$$\frac{f_\pi}{m_\pi^2} g'\varepsilon_{\mu\nu\lambda\sigma}k_\mu^+ k_\nu^- e_\lambda p_\sigma,$$

where g' is again some constant defined by this expression. Similarly for the fourth term, this time using PCAC twice, we obtain

$$\frac{f_\pi^2}{m_\pi^4}\, g_{\omega\pi^+\pi^-\pi^0}\varepsilon_{\mu\nu\lambda\sigma}k_\mu^+ k_\nu^- e_\lambda p_\sigma\,.$$

From these expressions it is immediately obvious that each of the terms falls off at the same rate if we take the limit as k^+, $k^- \to 0$ and we cannot in this way obtain a useful physical result because our equation contains the unknowns g and g'. If, however, some other physical effect depresses these unwanted terms or if they can be simply estimated, it would be possible to obtain a relation between the interesting quantities. Some such effect as this probably explains the good results obtained from the use of this current algebra approach for these particular problems, but the result does not come from a current-algebra-type argument alone.

It can easily be seen that exactly the same situation arises for Kawarabayashi and Suzuki's attempt to relate the $\gamma \to \pi^+\pi^-\pi^0$ vertex and the $\pi^0 \to \gamma\gamma$ decay and for Wada's attempt to relate the $\eta \to \pi^+\pi^-\gamma$ and $\eta \to \gamma\gamma$ decays because these are also of type i) in the present classification. We would run into the same difficulty if we attempted to take the limit of some momentum going to zero in any of the other six cases listed above although some of these are of academic interest only since at most one amplitude in the equation can be related to a known physical process. Cases ii) and iii) are of this type.

However, TAYLOR (3) has used a different technique from that outlined above to obtain his sum rule which he claims contradicts ρ-dominance. His method is applicable to types iv)-vii) in the above classification and depends on the fact that in these cases the term obtained from putting states around the right-hand side of the equal-time commutators does not contribute anything because from the momentum and polarization vectors available we are not able to construct a quantity with the appropriate Lorentz transformation properties. Our general thesis that one cannot obtain physically useful sum rules from current algebras applied to any of the seven types listed above is not, however, upset because the method involves taking the pions to be at rest and under such circumstances we cannot apply the usual prescription of taking zero-mass pions to avoid the pole introduced by application of PCAC.

We can reproduce Taylor's argument if we take

$$T_{\mu\nu} = i\int \mathrm{d}^4x \exp\left[-ik\cdot x\right]\langle 0|\left[A_\mu^a(x),\ V_\nu^b(0)\right]|\omega\rangle\,, \tag{12}$$

which is of type v) with the ω meson for the vector particle. Carrying out the usual partial integrations on (12) leads to

$$k_\mu T_{\mu\nu} = \int \mathrm{d}^4x \exp[-ik\cdot x]\langle 0|[\partial_\mu A_\mu^a(x),\ V_\nu^b(0)]|\omega\rangle . \tag{13}$$

The important feature of Taylor's method is to now integrate (13) over $\int(\mathrm{d}^4k/k_0)\delta^{(3)}(\boldsymbol{k})$ which is equivalent to choosing the frame in which the 3-momentum $\boldsymbol{k}$ is zero. In doing this, however, we must be careful to remember that the 4-momentum k must have a finite length; it cannot be the momentum of a zero-mass pion since we cannot have a zero-mass particle at rest. Carrying out this integration on (13), the left-hand side gives

$$2\pi i\int \mathrm{d}^4x\delta(x_0)\langle 0|[A_0^a(x),\ V_\nu^b(0)]|\omega\rangle = -2\pi\varepsilon_{abc}\langle 0|A_\nu^c(0)|\omega\rangle = 0\,, \tag{14}$$

where we have used eq. (2). Using PCAC, the right-hand side gives

$$\int\frac{\mathrm{d}^4k}{k_0}\delta^3(k)f_\pi\int\frac{\mathrm{d}^4x\exp[-ik\cdot x]}{k^2-m_\pi^2}\langle 0|[j_\pi^a(x),\ V_\nu^b(0)]|\omega\rangle = \tag{15}$$

$$= f_\pi\int\frac{\mathrm{d}k_0F^{ab}(\nu,\ k_0^2)}{k_0^2-m_\pi^2}\,\varepsilon_{0\nu\lambda\sigma}e_\lambda p_\sigma\,,$$

where e_λ and p_σ are respectively the polarization and momentum 4-vectors of the ω meson, $F^{ab}(\nu,\ k^2)$ is some function involving coupling constants and form factors, and ν is defined by

$$\nu = p\cdot k\,.$$

Equations (14) and (15) together imply that

$$\int\frac{\mathrm{d}k_0F^{ab}(\nu,\ k_0^2)}{k_0^2-m_\pi^2} = 0\,. \tag{16}$$

Equation (16) as it stands is not obviously inconsistent with ρ-dominance and dispersion theory and does not have any immediate physical relevance. We cannot get rid of the denominator by the usual trick of taking a zero-mass pion because this would imply integration over a zero interval. Without the denominator, (16) would contradict ρ-dominance by implying that

$$tF^{ab}(t)\to 0 \qquad \text{as} \qquad t\to\infty\,,$$

when t is some momentum transfer, but for the case considered above, current algebras do not contradict ρ-dominance.

3. – Conclusion.

There are many other meson interactions which can be fitted into one or another of our seven categories. For instance under v) we could consider decays like $\omega \to \pi^0\gamma$, $\varphi \to \pi^0\gamma$, $\rho \to \pi\gamma$ and under vi) we could consider $\eta \to \pi^+\pi^-\gamma$, $K^\pm \to \pi^\pm\pi^0\gamma$, and so on. We have shown, however, that in general one gets unwanted terms in the equations obtained from the usual current algebra methods when they are applied to interactions involving only 1^- and an odd number of 0^- particles. It is not possible to get rid of these terms by taking the limit as some momentum or momenta go to zero because if one does this one finds that all the terms in the equation fall off at the same rate, not allowing any conclusions to be reached about the interesting terms. On the other hand, TAYLOR has given an interesting new prescription for eliminating unphysical terms for interactions in which some of the terms cannot be constructed for some reason. But it has been seen that this method, although it eliminates the unwanted terms, reduces the physically interesting part of the equation to a form in which it is no longer as useful.

These considerations would also apply to the decays of 0^+ and 1^+ mesons if they exist. Any decay of these for which the Lorentz transformation properties necessitate the use of the Levi-Civita tensor would, if we used current algebra techniques, run into similar difficulties.

The conclusion would seem to be that current algebras are not a useful tool for dealing with meson inteactions of the type considered.

* * *

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Note added in proofs.

After this paper was completed two recent papers [6,7] in which the branching ratio $(\eta \to 2\pi\gamma)/(\eta \to 2\gamma)$ was calculated by current-algebra methods came to the author's notice. This calculation is of type i) and neither of these two recent papers avoids the difficulties outlined above.

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(7) J. PASUPATHY and R. E. MARSHAK: *Phys. Lev. Lett.*, **17**, 888 (1966).

RIASSUNTO (*)

Si dimostra che per decadimenti che coinvolgono solo particelle 1^- e un numero dispari di particelle 0^-, il metodo dell'algebra delle correnti non è adatto.

(*) *Traduzione a cura della Redazione.*

Ограничение на использование техники алгебры токов для распадов мезонов.

Резюме (*). — Показано, что для распадов, включающих только 1^- частицы и четное число 0^- частиц приближение алгебры токов не является пригодным.

(*) *Переведено редакцией.*

B. H. Kellett
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The General Scattering Process $PV \to VV$ and Superconvergence (*).

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Summary. — The general scattering process $PV \to VV$ is analysed in detail and nine superconvergent sum rules are found. These are saturated with 0^- and 1^- particles for the various possible reactions and the results are compared with the predictions of $U_{6,6}$. It is found that one of the sum rules gives, in every case, either the $U_{6,6}$ result or is identically satisfied.

1. – Introduction.

There has been a great deal of interest recently in the idea of superconvergent sum rules. Since De Alfaro, Fubini, Furlan, and Rossetti [1] first derived superconvergent sum rules for $\rho\pi$ scattering, a number of papers have appeared [2-5] giving sum rules for many other processes. Superconvergence

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(1) V. de Alfaro, S. Fubini, G. Furlan and G. Rossetti: *Phys. Lett.*, **21** 576 (1966).

(2) H. F. Jones and M. D. Scadron: *Nuovo Cimento*, **48** A, 545 (1967); Imperial College Preprint ICTP/67/17 (to be published in *Nuovo Cimento*). See these papers for further references.

(3) R. J. Rivers: Imperial College Preprints ICTP/67/11 and ICTP/67/18.

(4) R. H. Graham and M. Huq: Syracuse Preprint.

(5) G. Venturi: Enrico Fermi Institute Preprint.

occurs when we apply fixed-momentum-transfer dispersion relations to those parts of the scattering amplitude for which Regge behaviour in the crossed channel predicts suitable asymptotic behaviour. The sum rules so obtained may be saturated with single-particle poles in the direct channel and the resulting relations compared with group theory or experiment.

Obviously, in the study of superconvergence, the Reggeization procedure is very important and several ways of doing this have been suggested. The most common method is to consider the Regge behaviour of the helicity amplitudes, but recently SCADRON (6) has given a method by which one can obtain the Regge behaviour of the invariant amplitudes directly. This method is particularly useful in the consideration of scattering processes in which more than two of the external particles have spin because in this case the relation between helicity amplitudes and invariant amplitudes is more complicated than in the case of processes involving only two particles with spin. Because helicity amplitudes have to satisfy certain conditions which are automatically satisfied by the invariant amplitudes, it would seem that Scadron's more direct approach may be useful in considering superconvergence. Also, the fact that when we have more than two particles with spin, both normal and abnormal (6) Regge poles can be exchanged in the crossed channel, introduces complications not present when the Regge pole has to couple to two spin-0 particles. Scadron's method makes clear the differences in asymptotic behaviour arising from the normal and abnormal pole exchange. For these reasons in the examination of the general scattering process PV $\to$ VV in this paper we will follow Scadron's prescription for Reggeization. It would perhaps be useful to point out that we have done the calculation of the asymptotic behaviour of the invariant amplitudes for the process $\pi\mathcal{N} \to \rho\mathcal{N}$ by this method and that we obtained the same sum rules as were obtained by GRAHAM and HUQ (4) who used the helicity amplitude approach. The PV $\to$ VV problem has also been discussed recently by VENTURI (5), but his approach is quite different from that used in this paper.

In Sect. **2** we consider in detail the M-function for a general PV $\to$ VV scattering process, derive the numerous equivalence theorems required, and indicate the nature of s-to-u crossing. Section **3** deals with the Reggeization procedure and the derivation of the sum rules, while in Sect. **4** we consider the saturation of these sum rules in particular cases of PV $\to$ VV scattering and compare the results with the predictions of $U_{6,6}$. The sum rules for any particular case are, in general, inconsistent, but there is one sum rule which in every case either gives the $U_{6,6}$ result or is identically satisfied.

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(6) M. D. SCADRON: Imperial College Preprint ICTP/67/23 (submitted to *Phys. Rev.*).

2. – The M-function.

We take the momenta as shown in Fig. 1, where μ, ν, and λ are the spin labels of the vector particles, and we define

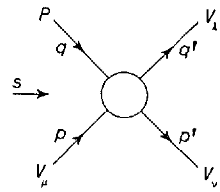

Fig. 1.

$$P=\tfrac{1}{2}(p+p')\,,\qquad Q=\tfrac{1}{2}(q+q')\,,$$
$$\Delta=\tfrac{1}{2}(p'-p)\,,\qquad N_\mu=\varepsilon_{\mu\nu\lambda\sigma}P^\nu Q^\lambda \Delta^\sigma\,,$$

and the Mandelstam invariants

$$s=(p+q)^2\,,\qquad t=(p'-p)^2\,,\qquad u=(p-q')^2\,,$$

so that

$$P\cdot Q\equiv\nu=\tfrac{1}{4}(s-u)\qquad\text{and}\qquad \Delta^2=\tfrac{1}{4}t\,.$$

The amplitude for the process PV → VV is written as

$$T=\varepsilon^{\nu *}(p')\,\varepsilon^{\lambda *}(q')\,M_{\mu\nu\lambda}\varepsilon^\mu(p) \tag{1}$$

where $\varepsilon^\mu(p)$, $\varepsilon^\nu(p')$, and $\varepsilon^\lambda(q')$ are the vector wave functions for the general vector particles. The covariant M-function can be expressed in terms of 13 invariant amplitudes as

$$\begin{aligned}M_{\mu\nu\lambda}={}&A_1(\varepsilon_{\nu\lambda}PQQ_\mu+\varepsilon_{\mu\lambda}PQQ_\nu)+A_2(\varepsilon_{\nu\lambda}PQQ_\mu-\varepsilon_{\mu\lambda}PQQ_\nu)+\\&+A_3(\varepsilon_{\nu\lambda}P\Delta Q_\mu+\varepsilon_{\mu\lambda}P\Delta Q_\nu)+A_4(\varepsilon_{\nu\lambda}P\Delta Q_\mu-\varepsilon_{\mu\lambda}P\Delta Q_\nu)+\\&+A_5(\varepsilon_{\nu\lambda}Q\Delta Q_\mu+\varepsilon_{\mu\lambda}Q\Delta Q_\nu)+A_6(\varepsilon_{\nu\lambda}Q\Delta Q_\mu-\varepsilon_{\mu\lambda}Q\Delta Q_\nu)+\\&+A_7\,\varepsilon_{\mu\nu}PQP_\lambda+A_8\varepsilon_{\mu\nu}P\Delta P_\lambda+A_9\varepsilon_{\mu\nu}Q\Delta P_\nu+\\&\qquad+A_{10}\varepsilon_{\mu\nu\lambda}P+A_{11}\varepsilon_{\mu\nu\lambda}Q+A_{12}\varepsilon_{\mu\nu\lambda}\Delta+A_{13}N_\lambda g_{\mu\nu}\,,\end{aligned}\tag{2}$$

where we have used the abbreviations $\varepsilon_{\alpha\beta}AB\equiv\varepsilon_{\alpha\beta\gamma\delta}A^\gamma B^\delta$ and $\varepsilon_{\alpha\beta\gamma}A\equiv\varepsilon_{\alpha\beta\gamma\delta}A^\delta$. The reason for this choice of covariants will become clear when we consider the equivalence theorems by which we can express any other possible covariant in terms of these 13. Because the process is, over-all, abnormal, we have to select our 13 covariants from the 27 $\varepsilon_{\nu\lambda}ABC_\mu$, $\varepsilon_{\mu\lambda}ABC_\nu$, $\varepsilon_{\mu\nu}ABC_\lambda$, the 27 $N_\mu A_\nu B_\lambda$, $N_\nu A_\lambda B_\mu$, $N_\lambda A_\mu B_\nu$, the 3 $N_\mu g_{\nu\lambda}$, $N_\nu g_{\lambda\mu}$, $N_\lambda g_{\mu\nu}$, and the 3 $\varepsilon_{\mu\nu\lambda}A$, where A, B, C can be any of the P, Q, Δ.

We can reduce this list considerably by applying the subsidiary conditions $p^\mu\varepsilon_\mu(p)=0$, whence

$$\Delta_\mu\to P_\mu\,,\qquad \Delta_\nu\to -P_\nu\,,\qquad \Delta_\lambda\to Q_\lambda\,. \tag{3}$$

This leaves us with 36 possible covariants and as we require only 13, we must have 23 equivalence theorems. We obtain these by saturating the basic identity

$$\varepsilon_{\mu\nu\sigma\tau} g_{\alpha\beta} = \varepsilon_{\alpha\nu\sigma\tau} g_{\mu\beta} + \varepsilon_{\mu\alpha\sigma\tau} g_{\nu\beta} + \varepsilon_{\mu\nu\alpha\tau} g_{\sigma\beta} + \varepsilon_{\mu\nu\sigma\alpha} g_{\tau\beta} \tag{4}$$

in various ways.

First we contract eq. (4) against $A^\sigma B^\tau C^\beta$, use the conditions (3), and from the resulting 9 equations between 21 of the covariants obtain, after a little algebra

$$\varepsilon_{\mu\nu} PQQ_\nu = -(\varepsilon_{\nu\lambda} PQQ_\mu - \varepsilon_{\mu\lambda} PQQ_\nu) - Q^2 \varepsilon_{\mu\nu\lambda} P + P\cdot Q \varepsilon_{\mu\nu\lambda} Q\,, \tag{5}$$

and two similar equations for $\varepsilon_{\mu\nu} P\varDelta Q_\lambda$ and $\varepsilon_{\mu\nu} Q\varDelta Q_\lambda$;

$$\begin{aligned} 2\varepsilon_{\mu\lambda} PQP_\nu = {} & (\varepsilon_{\nu\lambda} PQQ_\mu - \varepsilon_{\mu\lambda} PQQ_\nu) + \varepsilon_{\mu\nu} PQP_\lambda + \\ & + (Q^2 + P\cdot Q - Q\cdot\varDelta)\varepsilon_{\mu\nu\lambda} P - (P^2 + P\cdot Q)\varepsilon_{\mu\nu\lambda} Q\,, \end{aligned} \tag{6}$$

and similar equations for $\varepsilon_{\mu\lambda} P\varDelta P_\nu$ and $\varepsilon_{\mu\lambda} Q\varDelta P_\nu$;

$$\begin{aligned} 2\varepsilon_{\nu\lambda} PQP_\mu = {} & (\varepsilon_{\nu\lambda} PQQ_\mu - \varepsilon_{\mu\lambda} PQQ_\nu) - \varepsilon_{\mu\nu} PQP_\lambda + \\ & + (Q^2 - P\cdot Q - Q\cdot\varDelta)\varepsilon_{\mu\nu\lambda} P + (P^2 - P\cdot Q)\varepsilon_{\mu\nu\lambda} Q\,, \end{aligned} \tag{7}$$

and similar equations for $\varepsilon_{\nu\lambda} P\varDelta P_\mu$ and $\varepsilon_{\nu\lambda} Q\varDelta P_\mu$. In these equations, and in the subsequent work, we assume that the particles with momenta p and p' have equal mass so that $P\cdot\varDelta = 0$. The combinations $(\varepsilon_{\nu\lambda} PQQ_\mu \pm \varepsilon_{\mu\lambda} PQQ_\nu)$ etc. have been chosen for simplicity in crossing as will be seen later.

Next we contract eq. (4) against all possible combinations of 4 of the P, Q, $\varDelta$ and, from the 9 nontrivial equations obtained, after some algebra and the use of eqs. (3), (5), (6), (7), we get expressions for covariants of the type $N_\alpha A_\beta B_\gamma$ in terms of our chosen set. The resulting equations are quite complicated and we write out here explicitly only 3 of the 12:

$$2N_\mu P_\nu P_\lambda = -\varDelta^2 \varepsilon_{\mu\nu} PQP_\lambda + (Q\cdot\varDelta - P\cdot Q)\varepsilon_{\mu\nu} P\varDelta P_\lambda + P^2 \varepsilon_{\mu\nu} Q\varDelta P_\lambda\,, \tag{8}$$

$$2N_\nu P_\mu P_\lambda = -\varDelta^2 \varepsilon_{\mu\nu} PQP_\lambda + (Q\cdot\varDelta + P\cdot Q)\varepsilon_{\mu\nu} P\varDelta P_\lambda - P^2 \varepsilon_{\mu\nu} Q\varDelta P_\lambda\,, \tag{9}$$

$$\begin{aligned} 2N_\lambda P_\mu P_\nu = {} & P\cdot Q(\varepsilon_{\nu\lambda} P\varDelta Q_\mu - \varepsilon_{\mu\lambda} P\varDelta Q_\nu) - P^2(\varepsilon_{\nu\lambda} Q\varDelta Q_\mu - \varepsilon_{\mu\lambda} Q\varDelta Q_\nu) - \\ & - \varDelta^2 \varepsilon_{\mu\nu} PQP_\lambda + Q\cdot\varDelta \varepsilon_{\mu\nu} P\varDelta P_\lambda + P\cdot Q(Q\cdot\varDelta - \varDelta^2)\varepsilon_{\mu\nu\lambda} P - \\ & - P^2(Q\cdot\varDelta - \varDelta^2)\varepsilon_{\mu\nu\lambda} Q + \left(P^2Q^2 - (P\cdot Q)^2 - P^2 Q\cdot\varDelta\right)\varepsilon_{\mu\nu\lambda} \varDelta\,. \end{aligned} \tag{10}$$

There is no difficulty in principle in deriving similar expressions for $N_\mu P_\nu Q_\lambda$, $N_\mu Q_\nu P_\lambda$, $N_\nu Q_\mu Q_\lambda$, $N_\nu Q_\mu Q_\lambda$, $N_\nu Q_\mu P_\lambda$, $N_\nu P_\mu Q_\lambda$, $N_\lambda P_\mu Q_\nu$, $N_\lambda Q_\mu P_\nu$, and $N_\lambda Q_\mu Q_\nu$.

It is clear now that we have chosen the first 12 of the covariants used in the expansion (2) of the M-function in such a way as to enable us to express all the other possible covariants in terms of these without introducing kinematic singularities. For our thirteenth covariant we have the choice of $N_\mu g_{\nu\lambda}$, $N_\nu g_{\lambda\mu}$, or $N_\lambda g_{\mu\nu}$. By an appropriate contraction of (4) we obtain

$$g_{\alpha\beta}N_\gamma - g_{\alpha\gamma}N_\beta = -\varepsilon_{\beta\gamma}PQ\Delta_\alpha + \varepsilon_{\beta\gamma}P\Delta Q_\alpha - \varepsilon_{\beta\gamma}Q\Delta P_\alpha\,, \tag{11}$$

from which, with appropriate choices of the indices μ, ν, λ for α, β, γ, we get relations between these three and the other covariants, making use of the equivalence theorems given above. No kinematic singularities are introduced in these equations so we could choose any one of the three, but for simplicity in our crossing relations we have chosen $N_\lambda g_{\mu\nu}$. The equations relating $N_\mu g_{\nu\lambda}$ and $N_\nu g_{\mu\lambda}$ to $N_\lambda g_{\mu\nu}$ and the twelve other covariants then give us our remaining two equivalence theorems.

One of the effects of performing s-to-u crossing on our M-function is to interchange the μ and ν space-time indices. It is for this reason that we have, where necessary, chosen linear combinations of covariants so that they show a definite symmetry or antisymmetry under the interchange of μ and ν. With the invariant amplitudes as shown in eq. (2), therefore, A_1, A_3, A_6, A_7, A_8, A_{11}, A_{12}, and A_{13} are all odd under s-to-u crossing, while A_2, A_4, A_5, A_9 and A_{10} are even.

3. – Reggeization and the sum rules.

The essence of Scadron's prescription for Reggeization is to consider the exchange of a particle pole of arbitrary spin. In this way we obtain what is essentially a partial-wave expansion of each of the invariant amplitudes and from this, by letting the angular momentum of the exchanged particle become complex, we obtain the asymptotic behaviour of each invariant amplitude. This partial-wave expansion must agree with that obtained by considering the expansion of the helicity amplitudes in the normal way, but with this method we can see directly which combinations of invariant amplitudes will have the best asymptotic behaviour.

We will, therefore, consider the exchange of a particle of arbitrary spin J in the t-channel of our process PV → VV. The exchanged particle may be either normal or abnormal and as the normality of the particles determines the nature of the couplings, we split our M-function into two parts and consider these separately. We will call M^+ the contribution to M from the exchange of normal particles and M^- the contribution from abnormal particles. Then $M = M^+ + M^-$.

Consider M^+ first. This has abnormal coupling at the PV vertex and normal coupling at the VV vertex so, following Scadron's terminology, we write

$$M^+ = \mathscr{C}^+(1, 1, J)\mathscr{P}^J\mathscr{C}^-(1, 0, J)\,. \tag{12}$$

Using the expressions of ref. [6] for the couplings we obtain

$$\begin{aligned} M^+ = {} & g_1 f \mathscr{P}^J_{\mu\nu;\alpha}\varepsilon_{\alpha\lambda}Q\Delta + g_2 f \mathscr{P}^J_{\mu;\alpha}\varepsilon_{\alpha\lambda}Q\Delta P_\nu + \\ & + g_3 f \mathscr{P}^J_{\nu;\alpha}\varepsilon_{\alpha\lambda}Q\Delta P_\mu + g_4 f \mathscr{P}^J_{;\alpha}\varepsilon_{\alpha\lambda}Q\Delta g_{\mu\nu} + g_5 f \mathscr{P}^J_{;\alpha}\varepsilon_{\alpha\lambda}Q\Delta P_\mu P_\nu\,, \end{aligned} \tag{13}$$

where g_i and f are independent coupling constants. From eq. (13), using the expressions of ref. [6] for the propagators and the equivalence theorems derived in Sect. 2, after a rather tedious calculation, we obtain the desired expansions for the invariant amplitudes. These expressions are too long to write out explicitly here and we give only the asymptotic behaviour of the invariant amplitudes obtained from the Reggeization procedure in terms of the variable ν.

$$\left\{\begin{array}{lll} A_1 \sim \nu^{\alpha-3}\,, & A_2 \sim \nu^{\alpha-2}\,, & A_3 \sim \nu^{\alpha-3}\,, \\ A_4 \sim \nu^{\alpha}\,, & A_5 \sim \nu^{\alpha-2}\,, & A_6 \sim \nu^{\alpha-1}\,, \\ A_7 \sim \nu^{\alpha-1}\,, & A_8 \sim \nu^{\alpha-1}\,, & A_9 \sim \nu^{\alpha-1}\,, \\ A_{10} \sim \nu^{\alpha}\,, & A_{11} \sim \nu^{\alpha-1}\,, & A_{12} \sim \nu^{\alpha+1}\,, \\ A_{13} \sim \nu^{\alpha-1}\,. & & \end{array}\right. \tag{14}$$

Which Regge trajectory will dominate this behaviour depends on the G-parity of the external particles, the isospin in the t-channel, and the fact that for M^+ only normal particles are exchanged. The dominant normal trajectories of positive G-parity are the ϱ for $I = 1$ and the Pomereon for $I = 0$. For negative G-parity, the A_2 dominates the $I = 1$ channel and the ω dominates for $I = 0$. We take the $t = 0$ intercepts of the ϱ, ω, and A_2 trajectories to lie between 0 and 1 and the Pomereon trajectory to pass through 1 at $t = 0$.

We must now consider the contributions to M through M^- and abnormal particle exchange. As before we can write

$$\begin{aligned} M^- = {} & \mathscr{C}^-(1, 1, J)\mathscr{P}^J\mathscr{C}^+(1, 0, J) = \\ = {} & g_1' f_1 \{\mathscr{P}^J_{\alpha\mu;\lambda}\varepsilon_{\nu\alpha}P\Delta + \mathscr{P}^J_{\alpha\nu;\lambda}\varepsilon_{\mu\alpha}P\Delta\} + g_2' f_1 \mathscr{P}^J_{\alpha;\lambda}\varepsilon_{\mu\nu\alpha}\Delta + \\ & + g_3' f_1 \mathscr{P}^J_{\alpha;\lambda}\varepsilon_{\mu\nu\alpha}P - g_4' f_1 \mathscr{P}^J_{;\lambda}\varepsilon_{\mu\nu}P\Delta - \\ & - g_1' f_2 \{\mathscr{P}^J_{\alpha\nu;}\varepsilon_{\mu\alpha}P\Delta Q_\lambda + \mathscr{P}^J_{\alpha\mu;}\varepsilon_{\nu\alpha}P\Delta Q_\lambda\} - g_2' f_2 \mathscr{P}^J_{\alpha;}\varepsilon_{\mu\nu\alpha}\Delta Q_\lambda - \\ & - g_3' f_2 \mathscr{P}^J_{\alpha;}\varepsilon_{\mu\nu\alpha}PQ_\lambda + g_4' f_2 \mathscr{P}^J\varepsilon_{\mu\nu}P\Delta Q_\lambda\,, \end{aligned} \tag{15}$$

where g_i', f_1, and f_2 are independent coupling constants, independent of those used in eq. (13). Once again, using the propagators and equivalence theorems, we obtain the expansions of the invariant amplitudes and here write out the resulting asymptotic behaviour:

$$(16)\qquad \left|\begin{array}{lll} A_1 \sim \nu^{\alpha-2}, & A_2 \sim \nu^{\alpha-1}, & A_3 \sim \nu^{\alpha-2}, \\ A_4 \sim \nu^{\alpha}\;, & A_5 \sim \nu^{\alpha-3}, & A_6 \sim \nu^{\alpha-1}, \\ A_7 \sim \nu^{\alpha-2}, & A_8 \sim \nu^{\alpha-1}, & A_9 \sim \nu^{\alpha-1}, \\ A_{10} \sim \nu^{\alpha}\;, & A_{11} \sim \nu^{\alpha}\;, & A_{12} \sim \nu^{\alpha+1}, \\ A_{13} \sim \nu^{\alpha-2}. & & \end{array}\right.$$

Which abnormal Regge trajectory dominates for M^- depends on G-parity as before. In this case the dominant trajectories of positive G-parity are the B for $I = 1$ and the η for $I = 0$. For negative G-parity, the A_1 dominates $I = 1$ and there is no known trajectory for $I = 0$. We assume that the $t = 0$ intercepts of the B and A_1 are between 0 and 1 and that for the η and the case where no particle is known, the $t = 0$ intercepts are less than zero.

When we are considering which of the invariant amplitudes or combinations of them will give superconvergent sum rules, we must remember that for any amplitude the asymptotic behaviour will be determined by both M^+ and M^-, whichever falls off least rapidly. Therefore, from the asymptotic behaviours given in eqs. (14) and (16), we find the following sum rules for positive G-parity Regge exchange and t close to zero:

$$(17)\qquad \int \mathrm{d}\nu \, \mathrm{Im}\, A_1^{(1)}(\nu, t) = 0\,,$$

$$(18)\qquad \int \mathrm{d}\nu \, \mathrm{Im}\, A_1^{(0)}(\nu, t) = 0\,,$$

$$(19)\qquad \int \mathrm{d}\nu \, \mathrm{Im}\, A_3^{(1)}(\nu, t) = 0\,,$$

$$(20)\qquad \int \mathrm{d}\nu \, \mathrm{Im}\, A_3^{(0)}(\nu, t) = 0\,,$$

$$(21)\qquad \int \mathrm{d}\nu \, \mathrm{Im}\, A_5^{(1)}(\nu, t) = 0\,.$$

For negative G-parity Regge exchange we get these same sum rules plus

$$(22)\qquad \int \mathrm{d}\nu \, \mathrm{Im}\, A_2^{(0)}(\nu, t) = 0\,,$$

$$\int d\nu \,\mathrm{Im}\, A_5^{(0)}(\nu, t) = 0\,, \tag{23}$$

$$\int d\nu\, \nu \mathrm{Im}\, A_1^{(0)}(\nu, t) = 0\,, \tag{24}$$

$$\int d\nu\, \nu \mathrm{Im}\, A_3^{(0)}(\nu, t) = 0\,. \tag{25}$$

It might appear that several of these sum rules would be trivially satisfied from the crossing relations given in Sect. 2, but this is not necessarily the case, because in some examples of PV → VV scattering different particles are exchanged in the direct and crossed channels so there can be no possibility of their contributions cancelling.

4. – Saturation of the sum rules.

We now attempt to saturate our sum rules (17)-(25) with 0^- and 1^- particles only for the various possible PV → VV reactions. As all these reactions will obviously have the same spin analysis, the only differences being in isotopic spin and G-parity, we will consider only the reaction $\pi\rho \to \rho\omega$ in detail. In this reaction we can have a pion pole in the direct channel and the ρ in the crossed channel. In the t-channel the isotopic spin is 1 and G-parity is negative. Using the momenta shown in Fig. 2, we define the effective interaction Lagrangians in momentum space as

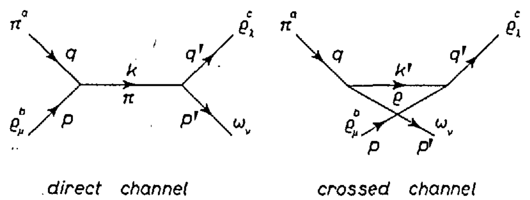

Fig. 2.

$$\begin{aligned}
\mathscr{L}_{\rho\pi\pi} &= g_1 i\varepsilon_{abc}\varrho^a_\mu \pi^b \pi^c P_\mu\,,\\
\mathscr{L}_{\rho\omega\pi} &= g_2 \omega_\nu \varrho^a_\lambda \pi_a \varepsilon_{\nu\lambda\alpha\beta} P_\alpha q_\beta\,,\\
\mathscr{L}_{\rho\rho\rho} &= i\varepsilon_{abc}\varrho^a_\mu \varrho^b_\lambda \varrho^c_\sigma \{g_3 g_{\lambda\sigma} P_\mu + g_3' g_{\mu\sigma} q_\lambda + g_3'' P_\mu q_\lambda q_\sigma + g_3''' g_{\mu\lambda} q_\sigma\}\,,
\end{aligned}$$

where a, b, c are isotopic spin indices and P is half the sum of the incoming and outgoing momenta and q is the outgoing momentum minus the incoming. In this context we take for our incoming and outgoing particles those with momenta q, q', k, or k'. We define our Lagrangians in this way in order to facilitate the comparison of our results with the predictions of the $U_{6,6}$ current [7]. Of our nine sum rules the only ones to apply to $\pi\rho \to \rho\omega$ scattering are (17), (19) and (21). Evaluating these at $t = 0$ in the $U_{6,6}$ equal-mass limit, we obtain from eq. (17)

$$(g_1 + g_3)g_2 = 0\,, \tag{26}$$

from (19)

$$(g_1 - g_3 - m^2 g_3'' + 2g_3''')g_2 = 0\,, \tag{27}$$

and from (21)

$$(g_1 + g_3 + 2m^2 g_3'')g_2 = 0\,. \tag{28}$$

Comparing these equations with the predictions of the $U_{6,6}$ current in ref. (7) we see that eq. (26) gives the $U_{6,6}$ result $g_1 = -g_3$, while (27) and (28) are inconsistent with the $U_{6,6}$ predictions.

If we consider the process $\pi K^* \to \rho K^*$ when all nine of the sum rules are possible we do not find any which give the $U_{6,6}$ results but sum rules (17), (19), (22)-(25) are all identically satisfied by crossing. Similarly, in the other possible processes $\eta\rho \to \rho\rho$, $K\rho \to K^*\rho$, $K\rho \to K^*\omega$, $\pi K^* \to \omega K^*$, and $K\bar{K}^* \to \bar{K}^*K^*$, we find that the sum rule (17) either gives the $U_{6,6}$ result or is identically satisfied, while no other sum rule gives $U_{6,6}$, although some may sometimes be identically satisfied by crossing. It would seem, then, that the idea of JONES and SCADRON (2) that there is some close correspondence between the results of superconvergence using only isospin invariance and the predictions of $U_{6,6}$ is not verified when we have more particles with spin. For the case of PV → VV scattering we find only one sum rule out of nine which gives the results of $U_{6,6}$. We could, of course, include higher-spin resonances in our saturation of the sum rules, but because of the multiplicity of coupling constants at each vertex, the predictive power of such a saturation is small.

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The author should like to thank Drs. JONES and SCADRON for many helpful discussions during the completion of this work. He should also like to acknowledge financial support from the New Zealand University Grants Committee.

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(7) R. DELBOURGO, M. A. RASHID, A. SALAM and J. STRATHDEE: *Trieste Lectures in High Energy Physics and Elementary Particles* (1965). Note that there is a misprint in the meson part of the vector current on p. 498 of these lectures. This part of the current should read

$$\left[\left(1 + \frac{q^2}{2\mu^2}\right) P_\mu(\bar{\varphi}_5\varphi_5 - \bar{\varphi}_\lambda\varphi_\lambda)_F + \frac{1}{\mu^2} P_\mu q_\nu q_\lambda(\bar{\varphi}_\nu\varphi_\lambda)_F + 3q_\nu(\bar{\varphi}_\nu\varphi_\mu - \varphi_\nu\,\bar{\varphi}_\mu)_F\right].$$

RIASSUNTO (*)

Si analizza in dettaglio lo scattering generico $PV \to VV$ e si trovano dodici regole di somma superconvergenti. Queste sono saturate con particelle di 0^- e 1^- per le diverse reazioni possibili e si confrontano i risultati con le predizioni di $U_{6,6}$. Si trova che in ogni caso una delle regole di somma fornisce o il risultato di $U_{6,6}$ oppure è identicamente soddisfatta.

(*) *Traduzione a cura della Redazione.*

Общий процесс рассеяния $PV \to VV$ и сверхсходимость.

Резюме (*). — Подробно анализируется общий процесс рассеяния $PV \to VV$ и выводятся девять сверхсходящихся правил сумм. Они насыщаются с помощью 0^- и 1^- частиц для различных возможных реакций, и результаты сравниваются с предсказаниями $U_{6,6}$. Обнаружено, что одно из этих правил сумм приводит, каждый раз, либо к результату $U_{6,6}$; либо тождественно выполняется.

(*) *Переведено редакцией.*