The Relaxation Method Applied to the Solution of Problems of Viscous Flow.

A thesis submitted to the University of London

for the degree of Doctor of Philosophy

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ABSTRACT

Abstract of a thesis, 'Relaxation Methods Applied to the Solution of Problems of Viscous Flow', submitted by M. Lister, M.Sc., (Imperial College), to the University of London for the degree of Doctor of Philosophy.

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The flow pattern for the general case of steady laminar motion of an incompressible viscous flow can be obtained from the solution of two non-linear simultaneous partial differential equations. The motion is at so high a speed that neither inertia or viscosity may be neglected. The solution is obtained in terms of a stream function and a vorticity function. The two problems considered are those of viscous flow in the inlet length of a cylindrical pipe of circular cross-section, and viscous flow past a sphere. In both cases there is axial symmetry, and the governing equations are written in terms of cylindrical polar co-ordinates.

For the flow in a pipe the equations of motion are solved for high Reynolds' numbers. The flow patterns change as the Reynolds' number R increases; but an introduction of variables which depend on R makes the change in patterns relatively slow.

Small values of R are taken when the viscous flow past the sphere is considered. An initial conformal transformation changes the independent variables from r and z to α and β, the velocity-potential and the stream function respectively for irrotational flow past a cylinder of circular cross-section.

A numerical solution is obtained to the above equations using relaxation methods. 'A two-diagram technique' in which the values of the stream function and the vorticity function are modified alternately, is used. For the case of the sphere difficulties arise when the boundary condition for points 'at infinity' is expressed in terms of finite differences. By comparing the solution found by relaxation methods with the exact solution for one value of Reynolds' number a satisfactory condition is obtained.

A solution is also obtained to another non-linear partial differential equation. The solution of this equation gives the flow pattern when a plane step shock travels round an infinite plane convex corner and is diffracted. The motion is assumed to be irrotational, and values of the velocity potential are obtained using relaxation techniques.

PREFACE

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Introduction

The derivation of the equations of motion of a viscous incompressible fluid, which is generally accepted today, is given by Lamb (Ref. 9), who states, "The dynamical equations were first obtained by Navier and Poisson on various considerations as to the mutual action of the ultimate molecules of the fluid". This method, adopted by Lamb, "which is free from all hypotheses of this kind, appears to be due in principle to de Saint-Venant and Stokes". It obtains the equations by considering the forces acting on an elementary rectangular parallelopiped. By calculating the change produced in the mass included in this rectangular parallelopiped by the flux across the boundary, we obtain a further kinematical relation between the compponents of velocity and the density. The solution of these equations, together with the boundary conditions gives the flow pattern for any particular problems.

As they are so complex, the equations of viscous flow have only been solved in particular cases, using approximate methods. The relaxation method of solution was suggested by Allen and Southwell (Ref. 3) when the flow patterns for the plane laminar motion past a circular cylinder were computed. It was shown that this method of solution can be applied for Reynolds' number, R, of any magnitude. In

this thesis, relaxation techniques have been used to obtain a solution for certain values of R, to two problems: the flow in the inlet length of a straight pipe of circular section and the plane laminar motion past a sphere of a viscous incompressible fluid.

Before considering in detail some of the methods used to obtain a solution to the two problems stated above, it is necessary to establish a basis for the assumption which is made at the surface of contact of a solid body and a viscous fluid. During the nineteenth century several theories were put forward and finally the condition of 'no slip' (i.e. all the components of velocity zero) on these boundaries, was accepted, for it was found that slip if it did take place, was too small to make any observable difference to the calculations. It was found also that this theory was the one most generally supported by experimental results.

There exists an exact solution to the viscous flow equations in certain cases. The investigation of the steady flow of a liquid through a straight pipe of uniform circular section, when the velocity is assumed parallel to the axis of the pipe, is simple, since the flow pattern can be obtained from the solution of an ordinary second order linear differential equation. This solution is given by Lamb (Ref. 9). In

the inlet length of the pipe the velocity is no longer independent of the distance z, measured along its axis, and so far as is known there is not a complete algebraic solution of the equations of viscous flow in this case. By generalizing Blasius' solution of the two dimensional boundary layer equation, Goldstein (Ref. 6) has obtained an expression for Stokes' stream function ψ.

Before applying relaxation techniques to obtain a numerical solution to the equations of motion for a viscous incompressible fluid in the inlet length of a straight pipe of circular section we first write them in terms of the cylindrical polar co-ordinates r and z and the solution is obtained in terms of ψ, the Stokes' stream function, and ζ, the vorticity. To facilitate the relaxation solution, we change the independent variable z measured along the axis of the cylinder to one which depends on R, since the flow patterns change as R increases. This transformation makes this change in the patterns relatively slow and thereby, for example, the accepted solution for R = 100 is made a good starting assumption for R = 10. We have obtained solutions to the viscous flow equations for the cases R = 10, 100 and $R \rightarrow \infty$. The axial component of velocity was then evaluated and the results compared with those of Goldstein (Ref. 6).

Another problem, which has received attention is that of viscous flow past a sphere. The first attempt to obtain a solution was made by Stokes, who assumed that the inertia terms in the dynamical equations were negligible compared with the viscosity term for a slowly moving fluid. In his method, Stokes expressed his stream function ψ in terms of spherical harmonics. From this solution, the ratio of the neglected inertia terms to the viscosity was found to be $U\rho/\nu$, where U is the velocity of the undisturbed flow, ρ is the distance from the centre of the sphere and ν, the kinematic viscosity. This term increases indefinitely with ρ, however small U may be. Thus in the neighbourhood of the sphere both terms are comparatively small and the character of the motion is unlikely to be seriously affected in these regions, but at a large distance away from the sphere this solution is inaccurate.

This criticism was put forward by Oseen who made an interesting innovation in the treatment of the subject introducing u + U for u, and neglecting terms of the second order in u, v, w only. The inertia terms are thus to some extent taken into account, but it is to be remarked that although the approximation is undoubtedly improved at infinity, where $u = v = w = 0$, it is in some degree impaired at the surface of

thesphere, where now we have $u = -U$. A method of solving these approximate equations for the case of viscous flow past a sphere, is given by Lamb, (Ref. 9). The components of velocity u, v, w are obtained in terms of two functions χ and ϕ:

$$u = -\frac{\partial \phi}{\partial x} + \frac{1}{2k}\frac{\partial \chi}{\partial x} - \chi \ ,$$

$$v = -\frac{\partial \phi}{\partial y} + \frac{1}{2k}\frac{\partial \chi}{\partial y},$$

$$w = -\frac{\partial \phi}{\partial z} + \frac{1}{2k}\frac{\partial \chi}{\partial z},$$

where

$$\nabla^2\phi = 0, \quad \chi = \frac{Ce^{-k(\rho - x)}}{\rho}, \quad \rho^2 = x^2+y^2+z^2, \text{ and } k = U/2\nu.$$

Lamb writes for ϕ an expression, which includes only negative zonal harmonics and obtains values for the constants of integration when $k\rho$ is small. Thus, as in Stokes' method, it will be noticed that the condition for the success of the approximation is that $R = Ua/\nu$ should be small.

Goldstein (Ref. 8) has also obtained expressions for ϕ and χ. These include Legendre and Bessel functions. The condition of no slip on the boundary is written in terms of the radial, u_ρ, and the cross-radial, u_ϕ, components of velocity and is used to obtain the value of the constants of integration. Tomotika and Aoi (Ref. 13) used the u_ρ and u_ϕ given by Goldstein (Ref. 7) to obtain an expression for Stokes' stream function ψ. After detailed numerical calculations they found, for small Reynolds' numbers, the following

approximate expression for ψ:

$$\psi_1 = -\left\{\frac{3}{4}\left(\rho_1 - \frac{1}{\rho_1}\right) - \frac{16+3R}{32}\left(\rho_1^2 - \frac{1}{\rho_1}\right) + \frac{3R}{32}\left(\rho_1^2 - \frac{1}{\rho_1^2}\right)\cos\theta\right\}\sin^2\theta,$$

where $\psi_1 = \psi/Ua^2$ and $\rho_1 = \rho/a$. From this expression for ψ_1 we can see that ψ_1 is not constant over the surface of the sphere.

Numerical values of ψ_1 were computed and the flow patterns round the sphere obtained. It was found that a stationary vortex ring is formed behind the sphere for the case R = 1. Further numerical calculations were carried out for smaller Reynolds' numbers and it was found that even when R assumed very small values, such as 0.1, the stationary vortex ring, though of very weak strength was still formed behind the sphere.

The flow pattern for the case of viscous flow past a sphere for R = 20 has been computed by Mitutosi Kawaguti (Ref. 11). The method of solution used is one described by Thom (Ref. 12), and numerical values are found for Stokes' stream function ψ and the vorticity ς, at the nodal points of a square mesh, which covers the field of flow under consideration. Initially values of ψ and ς are assumed at these nodal points, then using the governing equation ψ and ς are evaluated at the centre of the squares. These latter values are then used to re-calculate values of ψ and ς at the nodal points. This process is then repeated until the values all over the field

have settled. From the streamlines and equi-vorticity lines calculated for the flow round a sphere there was no evidence that there exists a stationary vortex ring behind the sphere.

To reduce the number of independent variables from three to two in the equations of motion for viscous flow past a sphere we write these equations in terms of r and z and Stokes' stream function is introduced. The other independent variable is the vorticity ζ . In order to satisfy the boundary conditions more easily a conformal transformation is used, which changes the independent variables from r and z to α and β , the velocity potential and the stream function for irrotational flow past a circular cylinder Solutions are obtained for the case R = 0, 1, 10, 20. For the case R = 0, there exists the solution obtained by Stokes. The two solutions are compared and certain improvements are suggested to the relaxation method, so that the boundary conditions at infinity may be satisfied more accurately. Finally a method is described by which the equations of viscous flow can be solved for plane laminar motion past a body with axial symmetry.

PART I.

THE RELEVANT THEORY OF FLUID DYNAMICS.

1. Fundamental Equations.

The fundamental equations of this thesis are obtained from the application of the principles of conservation of mass, momentum and energy to the hydrodynamical continuum (Goldstein, Ref. 6).

These may be written

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{q}) = 0, \qquad \text{(conservation of mass)} \qquad \ldots\ldots(1.1)$$

$$\rho \frac{Du}{Dt} = \rho X + \frac{\partial}{\partial x}\hat{xx} + \frac{\partial}{\partial y}\hat{xy} + \frac{\partial}{\partial z}\hat{zx},$$

$$\rho \frac{Dv}{Dt} = \rho Y + \frac{\partial}{\partial x}\hat{xy} + \frac{\partial}{\partial y}\hat{yy} + \frac{\partial}{\partial z}\hat{yz}, \qquad \ldots\ldots(1.2)$$

$$\rho \frac{Dw}{Dt} = \rho Z + \frac{\partial}{\partial x}\hat{zx} + \frac{\partial}{\partial y}\hat{yz} + \frac{\partial}{\partial z}\hat{zz},$$

(conservation of momentum)

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{q}.\text{grad})$

and

$$\frac{\partial E}{\partial t} + (q.\text{grad})E + p\left\{\frac{\partial(\frac{1}{\rho})}{\partial t} + q.\text{grad}\,(\frac{1}{\rho})\right\} = \frac{k}{\rho}\nabla^2 T + \frac{\mu}{\rho}\gamma + Q \qquad \ldots\ldots(1.3)$$

(conservation of energy)

where γ = dissipation function

$$= 2(\frac{\partial u}{\partial x})^2 + 2(\frac{\partial v}{\partial y})^2 + 2(\frac{\partial w}{\partial z})^2 + (\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z})^2 + (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})^2 + (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})^2 - \frac{2}{3}(\operatorname{div} q)^2 \qquad \ldots\ldots(1.4)$$

The equation of state is taken as

$$p = \bar{R}\rho T. \qquad .....(1.5)$$

The following notation is used:-

t = time

(x,y,z) = space coordinates

ρ = density

T = temperature

p = pressure

$\vec{q}$ = velocity vector (u,v,w)

$\vec{F}$ = body force vector (X,Y,Z)

$\hat{rs}$ = stress tensor $\quad (r,s) = (x,y,z)$

$\bar{R}$ = gas constant = $C_p - C_v$

C_p = specific heat at constant pressure (assumed constant)

C_v = specific heat at constant volume

$\gamma = C_p/C_v$

E = internal energy per unit mass = C_vT for polytropic gases

μ = coefficient of viscosity

k = coefficient of heat conduction

Q = heat added.

The assumptions are made that the stresses are linear functions of the rates of strain and the usual conditions of isotropy are used so that the stress in the x-direction on a face perpendicular to this direction can be expressed as

$$xx = -p - 2\mu \operatorname{div}\vec{q}/3 + 2\mu\frac{\partial u}{\partial x} \qquad .....(1.6a)$$

and the shearing forces in the x-direction on the other two faces of the elementary cube are

$$\widehat{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\widehat{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \text{.....(1.6b)}$$

Substituting for the stresses in equations (1.2), we obtain

$$\rho \frac{Du}{Dt} = \rho X - \frac{\partial p}{\partial x} - \frac{2}{3}\frac{\partial}{\partial x}(\mu \operatorname{div} \vec{q}) + 2\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right],$$

$$\rho \frac{Dv}{Dt} = \rho Y - \frac{\partial p}{\partial y} - \frac{2}{3}\frac{\partial}{\partial y}(\mu \operatorname{div} \vec{q}) + 2\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right],$$

$$\rho \frac{Dw}{Dt} = \rho Z - \frac{\partial p}{\partial z} - \frac{2}{3}\frac{\partial}{\partial z}(\mu \operatorname{div} \vec{q}) + 2\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right) + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right], \qquad \text{.....(1.7)}$$

An alternative form of the energy equation is obtained by introducing the entropy S (Courant and Friedrichs, Ref. 4).

The entropy is defined by the equation

$$DE = T\,DS - pD\tau \qquad \text{.....(1.8)}$$

where τ is the volume per unit mass, i.e. $\rho\,\tau = 1$, so that (1.8) can be written

$$DE = T\,Ds - pD\left(\frac{1}{\rho}\right), \qquad \text{.....(1.8)'}$$

and equation (1.3) is then

$$\frac{D}{Dt}(E) + p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \frac{k}{\rho}\nabla^2 T + \frac{\mu}{\rho}\gamma + Q \qquad \text{.....(1.3)'}$$

The left hand side of (1.3)' is $T\frac{DS}{Dt}$ by (1.8)';

therefore $T\frac{DS}{Dt} = \frac{k}{\rho}\nabla^2 T + \frac{\mu}{\rho}\gamma + Q$. .....(1.9)

For polytropic gases the expression for S is

$$S - S_o = C_v \log\left(\frac{p\tau^{\delta}}{\gamma - 1}\right) . \qquad \text{.....(1.10)}$$

2. Axially-Symmetric Viscous Flow.

For steady incompressible flow $\frac{\partial}{\partial t} = 0$ and $\rho = \text{const.}$ so that equations (1.1) and (1.7) take the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \qquad \text{.....(2.1)}$$

and

$$(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})(u,v,w) = -\frac{1}{\rho}[\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}]p + (X,Y,Z) + \nu\,\nabla^2(u,v,w), \qquad \text{.....(2.2)}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity of the fluid. Oz is taken as the axis of the flow and r as the radial coordinate so that

$$r^2 = x^2 + y^2, \qquad \text{.....(2.3)}$$

$$\tan\theta = y/x.$$

Where there is no variation with respect to θ, we denote the radial component of velocity by V and of the body force by T so that

$$\left.\begin{aligned} u &= V\cos\theta, \\ v &= V\sin\theta, \\ X &= T\cos\theta, \\ Y &= T\sin\theta. \end{aligned}\right\} \qquad \text{.....(2.4)}$$

From (2.3) it follows that

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \qquad \text{.....(2.5)}$$

and

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta};$$

hence

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \qquad \text{.....(2.6)}$$

When we substitute (2.6) in (2.1) we obtain the equation of continuity in the form

$$\frac{\partial V}{\partial r} + \frac{V}{r} + \frac{\partial w}{\partial z} = 0$$

therefore $$\frac{\partial}{\partial r}(rV) + \frac{\partial}{\partial z}(rw) = 0.$$

Thus a stream function ψ can be introduced (Lamb, Ref. 9) such that

$$w = \frac{1}{r}\frac{\partial \psi}{\partial r} \qquad \text{.....(2.7)}$$

and $$V = -\frac{1}{r}\frac{\partial \psi}{\partial z}.$$

Again, from (2.4) and (2.5),

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} = V\frac{\partial}{\partial r}. \qquad \text{.....(2.8)}$$

When we use (2.4), (2.5), (2.6), (2.8) the first two of equations (2.2) take the same form:

$$V\frac{\partial V}{\partial r} + w\frac{\partial V}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + T + \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{1}{r}\right)V. \qquad \text{..(2.9)}$$

The third of (2.2) gives

$$V\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + Z + \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)w. \qquad \text{.....(2.10)}$$

It is assumed that a body force potential Ω exists:

$$Z = -\frac{\partial \Omega}{\partial z} \qquad T = -\frac{\partial \Omega}{\partial r}.$$

We now eliminate p and Ω from (2.9) and (2.10) and obtain the following equation:

$$\frac{\partial V}{\partial z}\frac{\partial V}{\partial r} + V\frac{\partial^2 V}{\partial z \partial r} + \frac{\partial w}{\partial z}\frac{\partial V}{\partial z} + w\frac{\partial^2 V}{\partial z^2} - \frac{\partial w}{\partial r}\frac{\partial V}{\partial r} - V\frac{\partial^2 w}{\partial r^2} - \frac{\partial w}{\partial z}\frac{\partial w}{\partial r} - w\frac{\partial^2 w}{\partial z \partial r}$$

$$= \nu\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{1}{r^2}\right]\left[\frac{\partial V}{\partial z} - \frac{\partial w}{\partial r}\right]. \quad \text{.....(2.11)}$$

When we introduce the vorticity ζ, and write

$$\frac{\pi}{r} = \zeta = \frac{\partial V}{\partial z} - \frac{\partial w}{\partial r}, \quad \text{.....(2.12)}$$

the equation (2.11) takes the form

$$\left(\frac{\partial V}{\partial r} + \frac{\partial w}{\partial z}\right)\frac{\pi}{r} + \left(V\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}\right)\frac{\pi}{r} = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{1}{r^2}\right)\frac{\pi}{r},$$

and with the use of (2.7)

$$\nu\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\pi = \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial \pi}{\partial z} - \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial \pi}{\partial r} + \frac{2}{r^2}\frac{\partial \psi}{\partial z}\pi. \quad \text{....(2.13)}$$

Also, when we substitute in (2.12) from (2.7),

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\psi = -\pi. \quad \text{.....(2.14)}$$

In relation to axi-symmetrical flow equations (2.13) and (2.14) have to be satisfied in conjunction with appropriate boundary conditions. The equation of continuity is automatically satisfied by the introduction of ψ.

To facilitate the numerical computations all quantities are made 'non-dimensional'. L is used to denote some representative dimension of length (e.g. the diameter of a sphere or cylinder) and U to denote the velocity of the fluid far upstream. We therefore write

$$r = Lr', \quad z = Lz', \quad V = UV', \quad w = Uw', \quad \psi = UL^2\psi'. \qquad .....(2.15)$$

Then, according to (2.14),

$$\pi = U\pi', \qquad .....(2.16)$$

where $$\pi' = \left(\frac{\partial^2}{\partial r'^2} - \frac{1}{r}\frac{\partial}{\partial r'} + \frac{\partial^2}{\partial z'^2}\right)\psi',$$

and, according to (2.13),

$$\frac{1}{R}\left(\frac{\partial^2}{\partial r'^2} - \frac{1}{r}\frac{\partial}{\partial r'} + \frac{\partial^2}{\partial z'^2}\right)\pi' = \frac{1}{r'}\frac{\partial\psi'}{\partial r'}\frac{\partial\psi'}{\partial z'} - \frac{1}{r'}\frac{\partial\psi'}{\partial z'}\frac{\partial\pi'}{\partial r'} + \frac{2}{r'^2}\frac{\partial\psi'}{\partial z'}\pi'. \qquad ....(2.17)$$

where R (the 'Reynolds' number of the motion) = UL/ν.

II. Viscous Flow in a Straight Pipe of Circular Section.

3. General Equations

We take as the equations which govern the flow

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = -\pi, \qquad \ldots\ldots(3.1)$$

$$\frac{\partial^2\pi}{\partial r^2} - \frac{1}{r}\frac{\partial\pi}{\partial r} + \frac{\partial^2\pi}{\partial z^2} = R\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\pi}{\partial z} - \frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial\pi}{\partial r} + \frac{2}{r^2}\frac{\partial\psi}{\partial z}\pi\right). \qquad \ldots\ldots(3.2)$$

In the equations (3.1) and (3.2) the diameter of the pipe is taken to be the representative dimension of length L, and U is the average velocity of the flow, therefore,

$$R = \frac{Ud}{\nu}.$$

The variable z is replaced by Rx (Allen and Southwell, Ref. 3) so that equations (3.1) and (3.2) take the form

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{R^2}\frac{\partial^2\psi}{\partial x^2} = -\pi, \qquad \ldots\ldots(3.3)$$

$$\frac{\partial^2\pi}{\partial r^2} - \frac{1}{r}\frac{\partial\pi}{\partial r} + \frac{1}{R^2}\frac{\partial^2\pi}{\partial x^2} = \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\pi}{\partial x} - \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial\pi}{\partial r} + \frac{2}{r^2}\frac{\partial\psi}{\partial x}\pi. \qquad \ldots\ldots(3.4)$$

Equations (3.3) and (3.4) are to be solved for ψ and π using relaxation methods. The boundary values of π and ψ have still to be determined.

The velocity at entry to the circular pipe is assumed to be constant over the cross-section. At a sufficient distance from the entry the flow will become uniform and first an approximate solution to equations (3.3) and (3.4) when ψ

and π are independent of z must be obtained. The area of the full solution will then be extended in the x-direction until the values of ψ and π reach these uniform flow values.

4. Uniform Viscous Flow in a Circular Pipe.

When π and ψ are independent of x equations (3.3) and (3.4) are

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = -\pi,$$

$$\frac{\partial^2\pi}{\partial r^2} - \frac{1}{r}\frac{\partial\pi}{\partial r} = 0. \qquad \ldots\ldots(4.1)$$

The diameter of the pipe is taken as 1 so we require a solution for $0 \leq r \leq 1/2$, with boundary conditions that

$$\text{at } r = 0,\ \psi = \pi = 0, \qquad \ldots\ldots(4.2)$$

and at $r = 1/2$, $\partial\psi/\partial r = 0$ and $\psi = 1/8$.

The value $\psi = 1/8$ follows from the fact that if the velocity were independent of r it would be everywhere equal to 1, and then ψ would be $r^2/2$.

The condition $\partial\psi/\partial r = 0$ on $r = 1/2$ is used to provide a boundary value for π. This is given by

$$-\pi = \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r},$$

$$= \frac{\partial^2\psi}{\partial r^2}. \qquad \ldots\ldots(4.3)$$

An exact solution has been obtained to the equation (4.1) (Goldstein, Ref. 6), viz.

$$\psi = -2r^4 + r^2, \qquad \pi = 16r^2. \qquad \ldots\ldots(4.4)$$

In a relaxational treatment equations (4.1) have

to be replaced by approximations in finite differences: in the notation of 'residuals' these are:-

$$F_\psi = \psi_2[1 - \frac{h}{2r_o}] + \psi_4[1 + \frac{h}{2r_o}] - 2\psi_o + h^2\pi_o, \quad .....(4.5)$$

$$F_\pi = \pi_2[1 - \frac{h}{2r_o}] + \pi_4[1 + \frac{h}{2r_o}] - 2\pi_o, \quad .....(4.6)$$

for a typical point O, and mesh length h.

The finite difference approximation for the boundary condition (4.3) is

$$-\pi_o = \frac{\psi_2 + \psi_4 - 2\psi_o}{h^2}. \quad .....(4.7)$$

But $\partial\psi/\partial r = 0$, i.e. $\psi_2 = \psi_4$ so that the fictitious value ψ_2 can be eliminated from (4.7) giving

$$+\pi_o = 2(\psi_o - \psi_4)/h^2. \quad .....(4.8)$$

In the first instance h was taken to be 1/10. Equations (4.5) and (4.6) are then

$$F_\psi = \psi_2[1 - \frac{1}{20r_o}] + \psi_4[1 + \frac{1}{20r_o}] - 2\psi_o + \pi_o/100 \quad .....(4.9)$$

$$F_\pi = \pi_2[1 - \frac{1}{20r_o}] + \pi_4[1 + \frac{1}{20r_o}] - 2\pi_o \quad .....(4.10)$$

for $r_o = 1/10, 1/5, 3/10, 2/5,$

and the boundary condition (4.8) is

$$\pi_o = 200(\psi_o - \psi_4). \quad .....(4.11)$$

We introduce a multiplying factor of 10^3 for both π and ψ so that the boundary conditions (4.2) are

$$r = 0, \qquad \psi = 0 = \pi,$$
$$r = 1/2, \qquad \psi = 125. \qquad \text{.....(4.12)}$$

When we substitute the values of r_0 into (4.10) we obtain the following set of equations

$$\begin{aligned} r = 1/10 = \alpha \ , \ [F_\pi] &= \pi_\beta/2 - 2\pi_\alpha \\ r = 1/5 = \beta \ , \ [F_\pi] &= 3\pi_\gamma/4 + 5\pi_\alpha/4 - 2\pi_\beta \\ r = 3/10 = \gamma \ , \ [F_\pi] &= 5\pi\delta/6 + 7\pi_\beta/6 - 2\pi_\gamma \\ r = 2/5 = \delta \ , \ [F_\pi]_\delta &= 7\pi_\epsilon/8 + 9\pi_\gamma/8 - 2\pi_\delta \end{aligned} \qquad \text{.....(4.13)}$$

where $r = 1/2 = \epsilon$.

It can be seen from (4.13) that when $F_\pi = 0$,

$$\pi_\delta = \frac{16}{25}\pi_\epsilon \ , \ \pi_\gamma = \frac{9}{25}\pi_\epsilon \ , \ \pi_\beta = \frac{4}{25}\pi_\epsilon \ , \ \pi_\alpha = \frac{1}{25}\pi_\epsilon . \qquad (4.14)$$

To obtain a solution to equations (4.9) and (4.10) values of ψ were assumed at the points $r = 1/10, 1/5, 3/10, 2/5$. A value of π_ϵ was then calculated from (4.11). The other values of π were obtained from the relations (4.14). Using (4.9) values for F_ψ were calculated and then the ψ's were altered so as (temporarily) to liquidate the residuals. A new value for π_ϵ was then obtained and the process repeated. From (4.11) it can be seen that large alterations to ψ_δ would make π_ϵ change rapidly and care had to be taken in the ψ-relaxation at the point $r = 2/5$ to prevent the two functions oscillating.

The solutions of (4.9) and (4.10) subject to the boundary conditions (4.11) and (4.12) are given in the first two columns of Table I.

Next, h was taken to be 1/20 and multiplying factors of 10,000 for ψ and 1000 for π were introduced. The residual formulae in this case are

$$F_\psi = \psi_2[1 - \frac{1}{40r_o}] + \psi_4[1 + \frac{1}{40r_o}] - 2\psi_o + \pi_o/40, \qquad .....(4.15)$$

$$F_\pi = \pi_2[1 - \frac{1}{40r_o}] + \pi_4[1 + \frac{1}{40r_o}] - 2\pi_o, \qquad .....(4.16)$$

with boundary conditions

$$\begin{aligned} &r = 0, && \pi = 0 = \psi \\ &r = 1/2, && \psi = 1250, \quad \pi_\epsilon = 80(\psi_o - \psi_4) \end{aligned} \qquad .....(4.17)$$

When the values of r_o are substituted into (4.16) and the equations $F_\pi = 0$ solved for π it can be seen that

$$\pi_\alpha = \frac{\pi_\epsilon}{100} \qquad \pi_\beta = \frac{4\pi_\epsilon}{100} \qquad \text{etc.} \qquad .....(4.18)$$

Thus the same process can be used to solve equations (4.15) and (4.16) subject to the boundary conditions (4.17) as for the case h = 1/10. These values are given in the 3rd and 4th columns of Table I.

Similarly for h = 1/40 we again introduce a multiplying factor of 10,000 for ψ and 1000 for π. The residual formulae in this case are:

$$F_\psi = \psi_2\left[1 - \frac{1}{80r_o}\right] + \psi_4\left[1 + \frac{1}{80r_o}\right] - 2\psi_o + \frac{\pi_o}{160}, \quad \text{.....(4.19)}$$

$$F_\pi = \pi_2\left[1 - \frac{1}{80r_o}\right] + \pi_4\left[1 + \frac{1}{80r_o}\right] - 2\pi_o, \quad \text{.....(4.20)}$$

and on $r = 0 \qquad \pi = 0 = \psi,$

$$r = 1/2 \qquad \psi = 1250 \qquad \pi_\epsilon = 160(\psi_o - \psi_4). \quad \text{.....(4.21)}$$

The corresponding values of ψ and π are given in the fifth and sixth columns of Table I.

In the final two columns of Table I the values of ψ and π obtained from the exact solution to equations (4.1) are given. It can be seen from here that the smaller the values of h the nearer the values of ψ and π approach their exact values.

	RELAXATION SOLUTIONS.						EXACT SOLUTION	
r	$10^3\psi$	$10^3\pi$	$10^4\psi$	$10^3\pi$	$10^4\psi$	$10^3\pi$	$10^4\psi$	$10^3\pi$
1/40					7	10	6	10
1/20			25	39	25	40	25	40
3/40					55	90	56	90
1/10	10	144	98	156	98	160	98	160
1/8					152	250	151	250
3/20			214	351	215	360	215	360
7/40					287	490	287	490
1/5	36	576	366	624	367	640	368	640
9/40					453	810	455	810
1/4			544	975	545	1000	547	1000
11/40					640	1210	642	1210
3/10	72	1296	734	1404	736	1440	738	1440
13/14					831	1690	833	1690
7/20			920	1911	923	1960	925	1960
3/8					1009	2250	1011	2250
2/5	107	2304	1083	2496	1087	2560	1088	2560
17/40					1153	2890	1154	2890
9/20			1201.25	3159	1204	3240	1205	3240
19/40					1237.5	3610	1238	3610

TABLE NO. I

5. Boundary Conditions for Viscous Flow in a circular pipe.

Having obtained the values of ψ and π which satisfy equations (4.1) we can now give completely the boundary conditions for the equations (3.3), (3.4) for all values of R.

On $r = 0$, $\quad \pi = \psi = 0$. .....(5.1)

On $r = 1/2$, $\quad \psi = \frac{1}{8} \quad \frac{\partial \psi}{\partial r} = 0$, ...(5.2)

$$[-\pi] = [\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{R^2}\frac{\partial^2 \psi}{\partial z^2}],$$

$$= [\frac{\partial^2 \psi}{\partial r^2}] = \frac{-2}{h^2}(\psi_0 - \psi_4), \qquad \text{.....(5.3)}$$

since ψ = const. $\frac{\partial \psi}{\partial r} = 0$ on the boundary.

On $x = 0$ it is assumed that the velocity is constant so that

$$\psi = r^2/2, \quad \pi = 0. \qquad \text{.....(5.4)}$$

On $x = a$ (i.e. when the flow becomes uniform) the values of π and ψ are given in Table I.

6. Viscous Flow in a Circular Pipe for large Reynolds' numbers.

Equations (3.3) and (3.4) are now solved for very large R (the terms containing $\frac{1}{R^2}$ may be neglected), subject to the boundary conditions §5. In this case equations (3.3) and (3.4) are

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = -\pi, \qquad \text{.....(6.1)}$$

$$\frac{\partial^2\pi}{\partial r^2} - \frac{1}{r}\frac{\partial\pi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\pi}{\partial x} - \frac{1}{r}\frac{\partial\psi}{\partial x}\frac{\partial\pi}{\partial r} + \frac{2}{r^2}\frac{\partial\psi}{\partial x}\pi. \qquad \text{.....(6.2)}$$

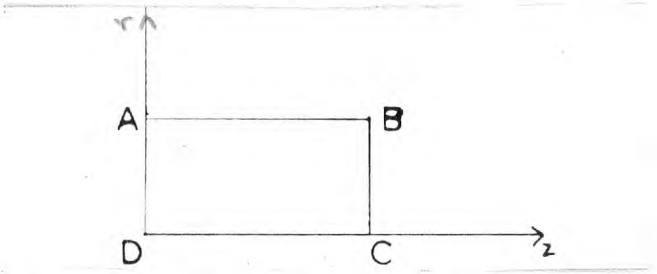


FIG. NO 1.

On AD let the mesh length be h_2 where $h_2 = 1/10 = 10h_1$.

" CD " " " " " " h_1 " $h_1 = 1/100$.

Then the residual formulae corresponding to equations (6.1) and (6.2) are

$$F_\psi = \psi_2[1 - \frac{1}{20r_0}] + \psi_4[1 + \frac{1}{20r_0}] - 2\psi_0 + \pi_0/100. \qquad \text{.....(6.3)}$$

$$F_\pi = \pi_2[1 - \frac{1}{20r_0} + \frac{5}{2r_0}(\psi_1 - \psi_3)] + \pi_4[1 + \frac{1}{20r_0} - \frac{5}{2r_0}(\psi_1 - \psi_3)]$$

$$- 2\pi_0[1 + \frac{1}{2r_0^2}(\psi_1-\psi_3)] - \frac{5}{2r_0}(\psi_2-\psi_4)(\pi_1-\pi_3). \qquad \text{.....(6.4)}$$

We now introduce a multiplying factor of 10^3 for both ψ and π. The boundary conditions given in §5 in this case reduce to

$$r = 0, \quad \psi = \pi = 0, \qquad \text{.....(6.5)}$$

$$r = 1/2, \ \psi = 125, \ \pi = 200(\psi_0 - \psi_4), \qquad \text{.....(6.6)}$$

$$\left.\begin{aligned} &x = 0, \quad \psi = r^2/2, \ \pi = 0, \\ &r = 1/10, \quad \psi = 1/200, \quad 1000\psi = 5, \\ &r = 1/5, \quad \psi = 1/50, \quad 1000\psi = 20, \\ &r = 3/10, \quad \psi = 9/200, \quad 1000\psi = 45, \\ &r = 2/5, \quad \psi = 4/50, \quad 1000\psi = 80. \end{aligned}\right\} \qquad \text{.....(6.7)}$$

As $x \to \infty$, the values of ψ and π are given in Table I. When we substitute for r_0 in (6.4) the residual formulae take the form

$$F_\pi = \pi_2[1 - \tfrac{1}{2n} + \tfrac{1}{40n}(\psi_1 - \psi_3)] + \pi_4[1 + \tfrac{1}{2n} - \tfrac{1}{40n}(\psi_1 - \psi_3)] - \pi_0[2 + \tfrac{1}{10n^2}(\psi_1 - \psi_3)] - \tfrac{1}{40n}(\pi_1 - \pi_3)(\psi_2 - \psi_4) \qquad \text{.....(6.8)}$$

where $n = 1, 2, 3, 4$.

A 'two diagram technique' in which ψ and π were modified alternately was used to obtain a solution to equations (6.3) and (6.8), described by Allen and Southwell (Ref. 3). At the end of a stage of π-relaxation the F_ψ had altered in accordance with the last term of (6.3) and the

ψ's were then altered so as (temporarily) to liquidate the residuals: then, with the altered values of ψ, the boundary values of π changed in accordance with (6.6) and the coefficients of π in equation (6.8) also changed, giving a new set of equations (6.8)' say. The values of π were then modified in accordance with (6.8)' and so on.

To avoid the values of ψ and π oscillating, it was usual to relax completely the ψ residuals with the possible exception of the F_ψ on $r = 2/5$ and to under-relax the π-residuals especially on $r = 2/5$. In practice, it was useful to accumulate π-residuals at points on $r = 2/5$, relaxing almost completely the other F_π and then altering both ψ and π on the line $r = 2/5$ so that $[F_\psi]_{r=2/5}$ were apparently zero. Then the ψ-residuals were recalculated before any further modifications were made to the values of π. The values of ψ were then changed and the process repeated.

The solution of equations (6.3) and (6.8) is given in Fig. 2.

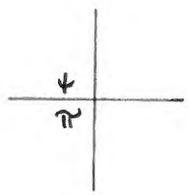
On the line $r = 2/5$ it was necessary to introduce decimal figures into the value of ψ so that the function π on the boundary was smooth.

From Fig. 2 it can be seen that the values of ψ

	125	125	125	125	125	125	125	125	125	125	125	125	125
	4700	4122	3880	3774	3660	3620	3614	3610	3606	3604	3600	3600	3600
80	101.5	104.39	105.6	106.13	106.7	106.9	106.93	106.95	106.97	106.98	107	107	107
0	2177	2394	2386	2376	2336	2313	2309	2307	2306	2305	2304	2304	2304
45	64	68	70	71	71	72	72	72	72	72	72	72	72
0	798	1101	1217	1272	1287	1289	1292	1293	1294	1295	1295	1296	1296
20	30	33	34	35	35	36	36	36	36	36	36	36	36
0	214	371	463	518	547	560	567	571	573	574	575	576	576
5	8	9	9	9	9	10	10	10	10	10	10	10	10
0	32	68	96	115	129	135	139	141	142	143	144	144	144
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0

FIG. NO. 2.

$R \to \infty$, $h_2 = 1/30$.



and π vary most between $x = 0$ and $x = 1/50$. Consequently when a finer net was introduced in the x-direction it was restricted to the range $0 \leq x \leq 1/50$. In the r-direction the mesh-lengths were halved. The network is given in Fig. 4a. There are three different types of residual formulae:

1. in the region $0 \leq x < 1/50$ where the mesh length in the x-direction is 1/400 and is 1/20 in the r-direction.
2. along the line $x = 1/50$.
3. in the region $x > 1/50$ where the mesh length is in the x-direction is 1/100 and is 1/20 in the r-direction.

For 1, 2, 3 the ψ-residuals can be calculated from

$$F_\psi = \psi_2\left[1 - \frac{1}{40r_o}\right] + \psi_2\left[1 + \frac{1}{40r_o}\right] - 2\psi_o + \frac{\pi_o}{40} \cdot \quad \ldots\ldots(6.9)$$

The corresponding π-residuals are obtained below

1. $(0 \leq x < 1/50)$

$$F_\pi = \pi_2\left[1 - \frac{1}{40r_o} + \frac{5}{10^4 r_o}(\psi_1-\psi_3)\right] + \pi_4\left[1 + \frac{1}{40r_o} - \frac{5}{10^4 r_o}(\psi_1-\psi_3)\right]$$
$$- \pi_o\left[2 + \frac{1}{10^4 r_o^2}(\psi_1-\psi_3)\right] - \frac{5}{10^4 r_o}(\psi_2-\psi_4)(\pi_1-\pi_3). \quad \ldots\ldots(6.10)$$

2. $(x = 1/50)$

$h_2 = 1/20$

$h_1 = 1/100$

$h_3 = 1/400$

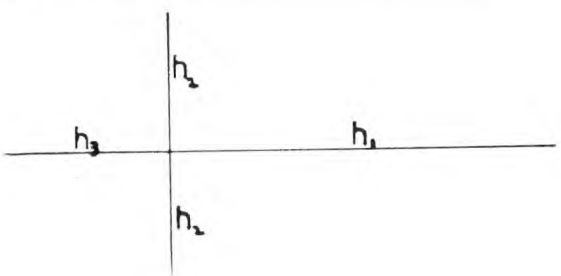


FIG. NO.3

Now,

$$\pi_1 = \pi_0 + h_1(\frac{\partial\pi}{\partial z})_0 + \frac{h_1^2}{2}(\frac{\partial^2\pi}{\partial z^2})_0 + \cdots\cdots$$

$$\pi_3 = \pi_0 - h_3(\frac{\partial\pi}{\partial z})_0 + \frac{h_3^2}{2}(\frac{\partial^2\pi}{\partial z^2})_0 + \cdots\cdots$$

Therefore, when we neglect terms of $O(h^3)$,

$$\pi_1 + 4\pi_3 = 5\pi_0 + \frac{1}{16{,}000}(\frac{\partial^2\pi}{\partial z^2})_0,$$

$$\text{i.e. } (\frac{\partial^2\pi}{\partial z^2})_0 = 16{,}000(\pi_1 + 4\pi_3 - 5\pi_0).$$

$$\pi_1 - 16\pi_3 = -15\pi_0 + \frac{1}{20}(\frac{\partial\pi}{\partial z})_0,$$

$$\text{i.e. } (\frac{\partial\pi}{\partial z})_0 = 20(\pi_1 - 16\pi_3 + 15\pi_0).$$

Hence the finite difference approximation to equation (6.2) is

$$F_\pi = \pi_2[1 - \frac{1}{40r_0} + \frac{1}{20{,}000r_0}(\psi_1-16\psi_3+15\psi_0)]$$
$$+ \pi_4[1 + \frac{1}{40r_0} - \frac{1}{20{,}000r_0}(\psi_1-16\psi_3+15\psi_0)]$$
$$-2\pi_0[1 + \frac{1}{200{,}000r_0^2}(\psi_1-16\psi_3+15\psi_0)]$$
$$- \frac{1}{20{,}000r_0}(\psi_2-\psi_4)(\pi_1-16\pi_3+15\pi_0).$$

3. $(x > 1/50)$

$$F_\pi = \pi_2[1 - \frac{1}{40r_0} + \frac{5}{4.10^4r_0}(\psi_1-\psi_3)]$$
$$+\pi_4[1 + \frac{1}{40r_0} - \frac{5}{4.10^4r_0}(\psi_1-\psi_3)]$$
$$- \pi_0[2 + \frac{1}{4.10^4r_0^2}(\psi_1-\psi_3)] - \frac{5}{4.10^4r_0}(\psi_2-\psi_4)(\pi_1-\pi_3). \quad \cdots\cdots(6.12)$$

The boundary conditions are given by (6.5), (6.7) and the third and fourth columns of Table I.

On $r = 1/2$, $\psi = 1250$, $\pi_0 = 80(\psi_0 - \psi_4)$. $\cdots\cdots(6.6)'$

The method of solving equations (6.9), (6.10), (6.11) and (6.12) for these boundary conditions is the same as the one used previously and the solution is given in Fig. 4a.

From this solution we calculated values of the axial component of velocity and compared the values obtained with those given by Goldstein (Ref. 6). The two graphs are given in Figs. 5a,b and the values of u are recorded in Table II . As can be seen the two sets of results do not agree for $0 \leq x \leq 1/50$. Unfortunately the value of Reynolds number is not given by Goldstein.

To see if we could obtain values of the axial component of velocity which compared more favourably with those of Goldstein the mesh length in the r-direction was taken to be 1/40. The residual formulae in this case are:-

$$F_\psi = \psi_2\left[1 - \frac{1}{80r_o}\right] + \psi_4\left[1 + \frac{1}{80r_o}\right] - 2\psi_o + \pi_o/16. \qquad .....(6.13)$$

1. $(0 \leq x < 1/50)$

$$F_\pi = \pi_2\left[1 - \frac{1}{80r_o} + \frac{1}{4000r_o}(\psi_1-\psi_3)\right] + \pi_4\left[1 + \frac{1}{80r_o} - \frac{1}{4000r_o}(\psi_1-\psi_3)\right]$$

$$-2\pi_o\left[1 + \frac{1}{80,000r_o^2}(\psi_1-\psi_3)\right] - \frac{5}{20,000r_o}(\psi_2-\psi_4)(\pi_1-\pi_3), \quad ..(6.14)$$

	1250	1250	1250	1250	1250	1250	1250	1250
	6808	5360	4888	4640	4480	4380	4280	4200
1012.5	1164.9	1183	1188.9	1192	1194	1195.25	1196.5	1197.5
0	4043	3917	3718	3589	3497	3442	3387	3339
800	992	1031	1046	1055	1061	1065	1069	1071
0	2113	2524	2594	2601	2588	2581	2568	2552
612.5	790	837	859	872	883	890	895	899
0	977	1411	1610	1716	1772	1810	1833	1848
450	592	636	659	674	687	696	702	707
0	398	686	878	1011	1101	1166	1211	1248
312.5	415	450	469	483	495	504	510	516
0	141	290	418	527	612	679	728	776
200	266	290	304	315	324	332	337	341
0	42	106	173	240	300	351	392	429
112.5	150	163	172	179	185	190	194	196
0	10	33	61	95	126	156	182	206
50	66	72	76	80	83	85	87	88
0	1	8	17	29	42	55	67	79
12.5	16	18	19	20	21	21	22	22
0	0	1	3	5	8	11	14	17
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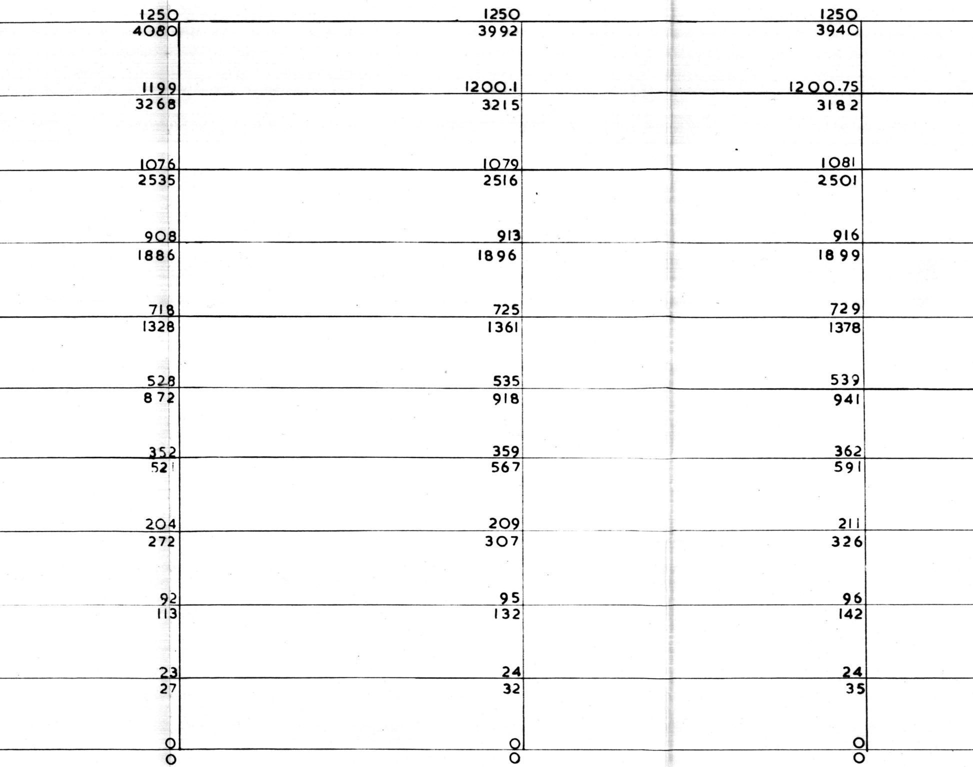
0	0	0	0	0	0	0	0	0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2516
2501
908
913
916
1886
1896
1899
718
725
729
1328
1361
1378
528
535
539
872
918
941
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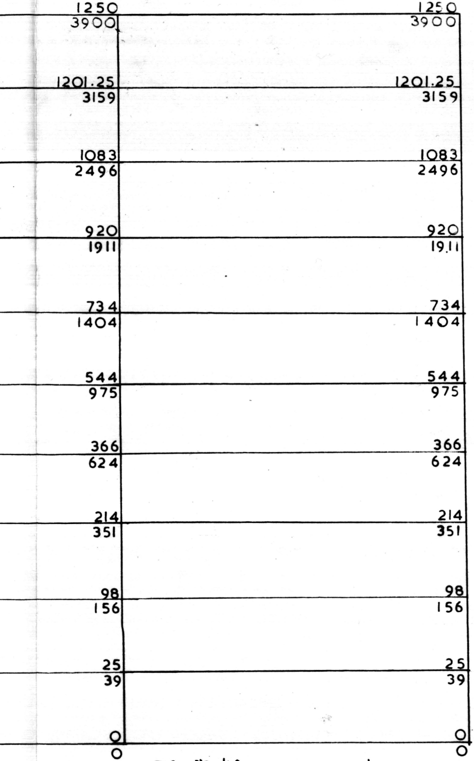
204
209
211
272
307
326
92
95
96
113
132
142
23
24
24
27
32
35
0
0
0
0
0
0

1250	3920	1250	3912	1250	3906
1201	3170	1201.1	3165	1201.175	3162
1082	2497	1082	2496	1083	2496
917	1903	918	1906	919	1908
730	1389	731	1395	732	1398
540	956	541	964	542	968
363	605	364	613	365	617
212	336	213	343	214	346
96	148	97	152	98	153
24	37	25	38	25	38
0	0	0	0	0	0

1250	1250	1250
3904	3902	3900
1201.2	1201.225	1201.25
3161	3160	3159
1083	1083	1083
2496	2496	2496
920	920	920
1909	1910	1911
733	734	734
1400	1402	1403
543	544	544
970	973	974
366	366	366
620	622	623
214	214	214
348	349	350
98	98	98
154	155	156
25	25	25
38	39	39
0	0	0
0	0	0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1911
734
1404
734
1404
544
975
544
975
366
624
366
624
214
351
214
351
98
156
98
156
25
39
25
39
0
0
0
0
ψ
Π

FIG. No. 4a.

$R \to \infty$, $h_2 = 1/20$.

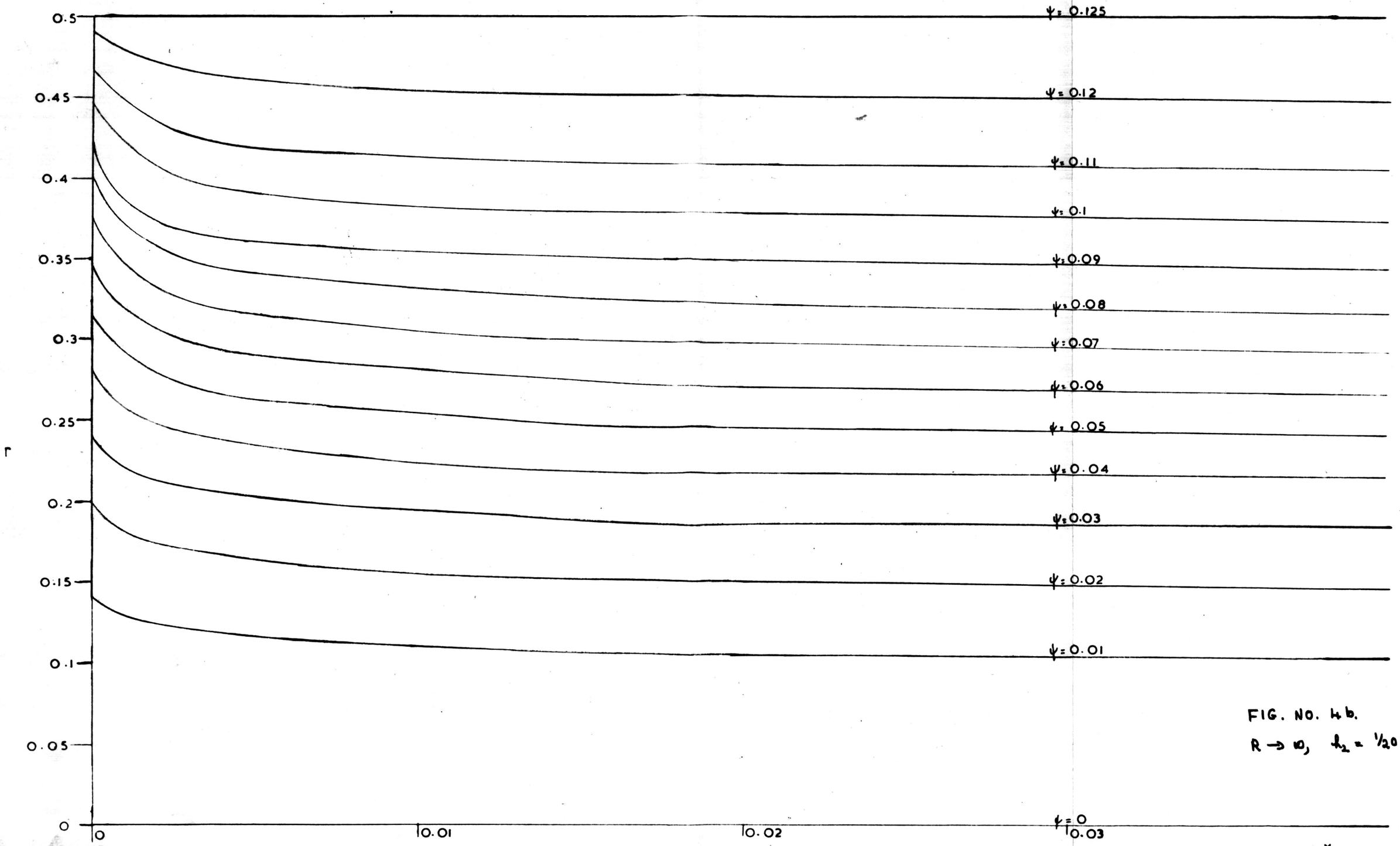
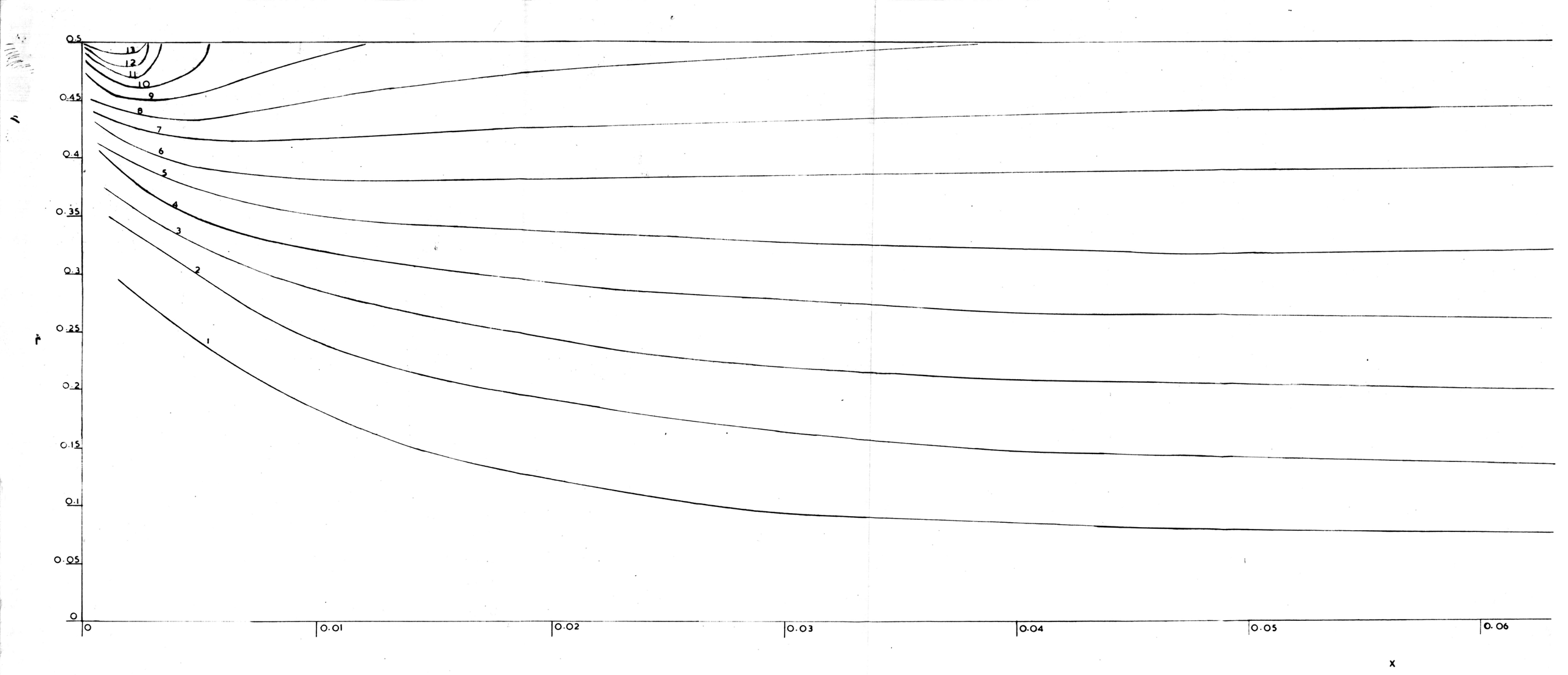


FIG. NO. 4b.

$R \to \infty$, $h_2 = 1/20$.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0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
y
0
0.01
0.02
0.03
0.04
0.05
0.06
x
1
2
3
4
5
6
7
8
9
10
11
12
13

α \ τ	0.10	0.20	0.30	0.35	0.40	0.45
0.0025	1.34	1.33	1.25	1.14	0.94	0.57
0.0050	1.45	1.44	1.29	1.13	0.87	0.50
0.0075	1.53	1.49	1.30	1.11	0.83	0.45
0.0100	1.59	1.52	1.30	1.09	0.80	0.43
0.0125	1.64	1.55	1.29	1.07	0.78	0.42
0.0150	1.69	1.57	1.29	1.05	0.76	0.41
0.0175	1.72	1.58	1.28	1.05	0.75	0.40
0.0200	1.74	1.60	1.28	1.04	0.75	0.40
0.03	1.81	1.62	1.27	1.02	0.73	0.39
0.04	1.85	1.63	1.26	1.01	0.72	0.38
.05	1.87	1.64	1.26	1.01	0.71	0.38
.06	1.88	1.64	1.26	1.01	0.71	0.37
.07	1.88	1.64	1.26	1.00	0.71	0.37
.08	1.89	1.64	1.26	1.00	0.71	0.37
.09	1.89	1.65	1.26	1.00	0.70	0.37
.10	1.89	1.65	1.25	1.00	0.70	0.37
.11	1.89	1.65	1.25	1.00	0.70	0.37

Values of u for $h_2 = 1/20$, $R \to \infty$

TABLE NO. II

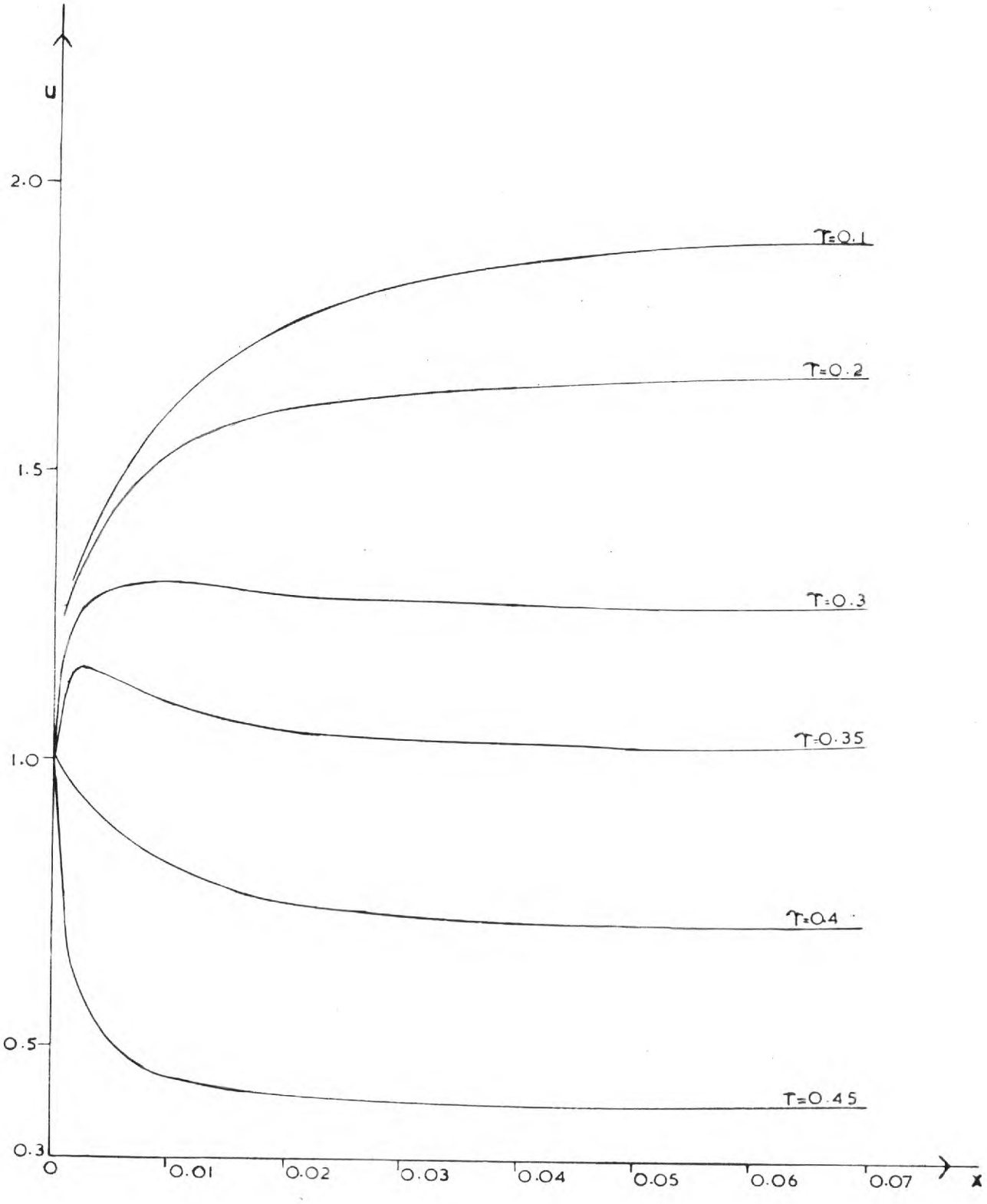


FIG. NO. 5

$R \to \infty$, $h_2 = 1/20$.

2. $(x = 1/50)$

$$F_\pi = \pi_2[1 - \frac{1}{80r_o} + \frac{1}{40,000r_o}(\psi_1-16\psi_3+15\psi_o)]$$
$$+\pi_4[1 + \frac{1}{80r_o} - \frac{1}{40,000r_o}(\psi_1-16\psi_3+15\psi_o)]$$
$$-2\pi_o[1 + \frac{1}{800,000r_o^2}(\psi_1-16\psi_3+15\psi_o) + \frac{15}{2x40,000r_o}(\psi_2-\psi_4)]$$
$$- \frac{1}{40,000r_o}(\pi_1-16\pi_3)(\psi_2-\psi_4), \quad .....(6.15)$$

3. $(x > 1/50)$

$$F_\pi = \pi_2[1 - \frac{1}{80r_o} + \frac{1}{16,000r_o}(\psi_1-\psi_3)]$$
$$+ \pi_4[1 + \frac{1}{80r_o} - \frac{1}{16,000r_o}(\psi_1-\psi_3)]$$
$$-2\pi_o[1 + \frac{1}{320,000r_o^2}(\psi_1-\psi_3)] - \frac{1}{16,000r_o}(\psi_2-\psi_4)(\pi_1-\pi_3),$$
$$.....(6.16)$$

when a multiplying factor of 10^4 is introduced for ψ and 10^2 for π. The multiplying factor of π is reduced in this manner to simplify the π-relaxation. The function ψ is of more interest and we only want the π-contribution to F_ψ. Using this multiplying factor gives it sufficiently accurately. The boundary condition corresponding to (6.6) is

$$r = 1/2, \quad \psi = 1250, \quad \pi = 32(\psi_o - \psi_4). \quad .....(6.6)'$$

The values of ψ and π which satisfy equations (6.13), (6.14), (6.15) and (6.16) are given in Fig. 6.

Comparing the three solutions of ψ and π given by

Figs. 2.4a&b it can be seen that the function ψ is increasing as the mesh length is decreased in the r-direction. The change in ψ from the first network to the second is approximately 6 per cent and the change from the second to the third varies from approximately 3 per cent near the centre of the pipe to 1 per cent (approximately) near the walls. We are therefore approaching the solution to this problem when we take $h_2 = 1/40$. If h_2 is reduced further to 1/80 then the value of ψ obtained will be within 1 per cent of the correct solution.

The changes in π are not so uniform as those for ψ. For $0 \leq r \leq 0.4$ the function π increases. The value of π at the wall is smaller and this has influenced the π-function in the region $0.4 \leq r \leq 0.5$. The decrease in π here is due to the more accurate approximation to the equation $\pi = -\frac{\partial^2 \psi}{\partial r^2}$, from which we obtain this boundary value. When the change in ψ is large then the formulae

$$\frac{\partial \psi}{\partial r} = \frac{\psi_2 - \psi_4}{2h},$$

gives a very crude approximation to the true value. A better approximate value can be obtained by introducing second and third differences. To obtain these differences for ψ at the boundary it is necessary to extrapolate or use graphical methods.

As an alternative method we can expand ψ as a power series in r on this boundary, viz.

$$\psi = \psi_0 + \alpha\left(r - \tfrac{1}{2}\right)^2 + \beta\left(r - \tfrac{1}{2}\right)^3 \qquad \text{.....(16.17)}$$

This satisfies automatically the condition that $\frac{\partial \psi}{\partial r} = 0$n on $r = \frac{1}{2}$. Then

$$\psi_4 = \psi_0 + \alpha h^2 - \beta h^3,$$
$$\psi_{12} = \psi_0 + 4\alpha h^2 - 8\beta h^3. \qquad \text{.....(16.18)}$$

The position of the point 12 is given in Fig. 7.

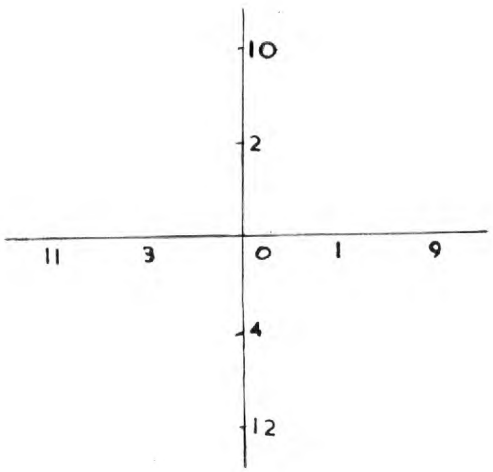


FIG. NO. 7.

Solving the equations (6.18) for α we have

$$\alpha = \frac{1}{4h^2}[8\psi_4 - \psi_{12} - 7\psi_o]. \qquad \ldots\ldots(6.19)$$

Therefore on the boundary

$$\frac{\partial^2\psi}{\partial r^2} = 2\alpha ,$$

$$= \frac{1}{2h^2}(8\psi_4 - \psi_{12} - 7\psi_o]. \qquad \ldots\ldots(6.20)$$

Therefore $\pi_o = \frac{1}{2h^2}[8\psi_4 - \psi_{12} - 7\psi_o]$. $\qquad \ldots\ldots(6.21)$

If we substitute $\psi_4 = 1238.11$ and $\psi_{12} = 1204.88$ in (6.21) then the corresponding value of π_o is 4000. Thus we obtain a value of ψ which is nearer the correct solution. (See Table I). This increase in ψ yields a corresponding increase in the other values of ψ and we obtain a solution which is within 0.5% of the correct one.

The value of the axial component of velocity is also given for the case $h_2 = 1/40$ in Table. III . This is calculated from the formula;

$$u = \frac{\psi_2 - \psi_4}{500 r_o}.$$

A graph of the variation of u along the pipe is given in Fig. 8 . These curves can be compared with those obtained when $h_2 = 1/20$. It can be seen that the maximum value of the velocity on the line r = 0.3 has moved further away from the point of entry, the 0.4 and 0.45 u-curves are almost identical

and that there is a slight increase in the magnitude of the velocity near the centre of the pipe.

This second set of curves therefore differ from those given by Goldstein (Ref. 6). We must assume that the experiment was carried out at a different Reynolds' number from the one used here. In the following sections two further values of R are taken to obtain the variation in u.

x \ r	0.10	0.20	0.30	0.35	0.40	0.45
.0025	1.40	1.37	1.29	1.15	0.94	0.57
.0050	1.54	1.47	1.31	1.14	0.88	0.48
.0075	1.60	1.51	1.31	1.11	0.83	0.46
.0100	1.66	1.53	1.31	1.09	0.81	0.44
.0125	1.70	1.55	1.31	1.08	0.80	0.43
.0150	1.74	1.57	1.30	1.07	0.79	0.42
.0175	1.76	1.59	1.29	1.06	0.78	0.42
.02	1.76	1.61	1.29	1.05	0.77	0.41
.03	1.80	1.64	1.29	1.04	0.75	0.40
.04	1.86	1.65	1.29	1.03	0.73	0.39
.05	1.90	1.66	1.28	1.02	0.73	0.38
.06	1.90	1.66	1.28	1.01	0.73	0.38
.07	1.92	1.66	1.28	1.01	0.72	0.38
.08	1.92	1.66	1.28	1.01	0.72	0.37
.09	1.94	1.66	1.28	1.01	0.72	0.37
.1	1.94	1.66	1.28	1.01	0.72	0.37

Value of u for $R \longrightarrow \infty$, $h_2 = 1/40$

TABLE NO. III

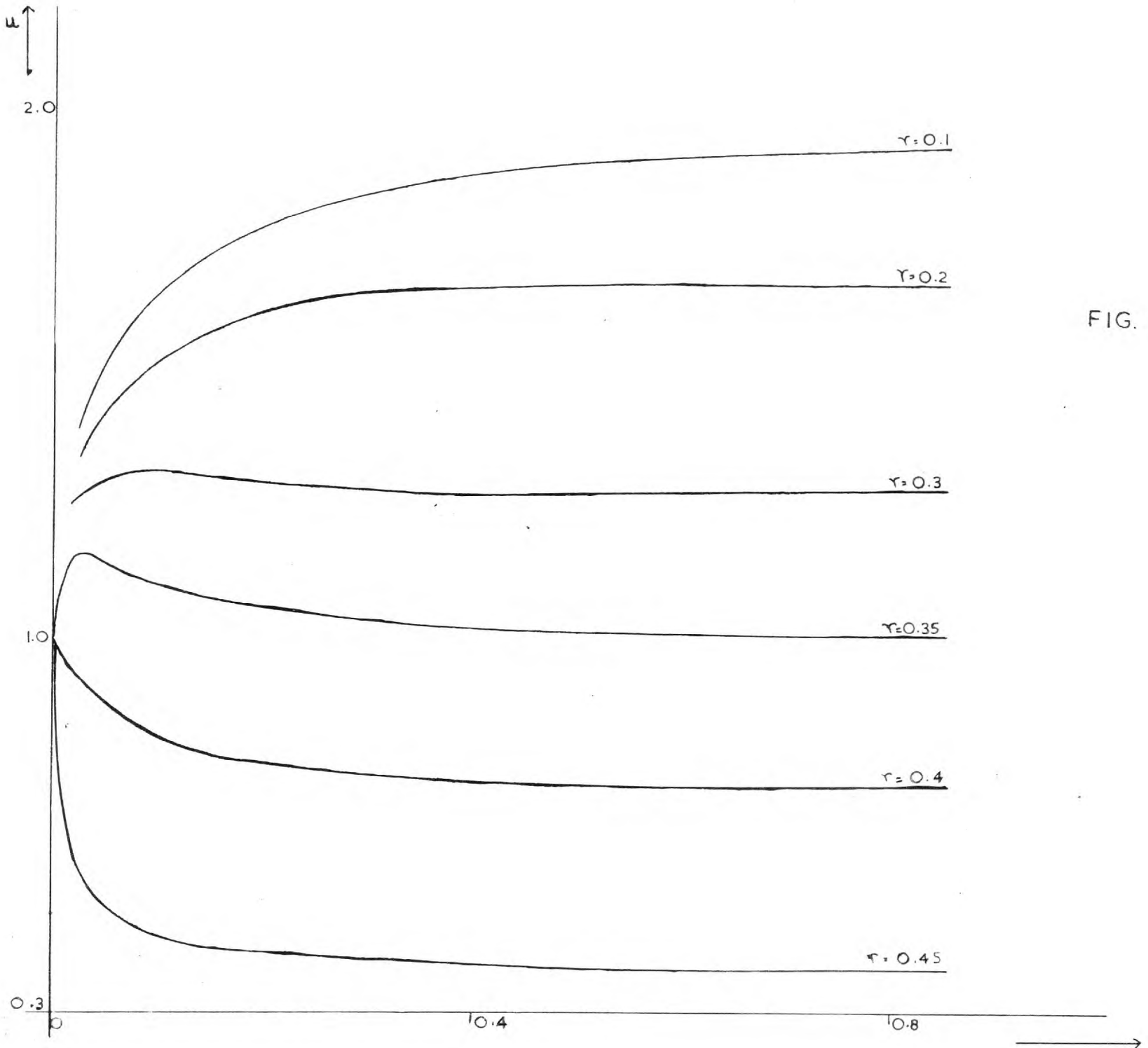


FIG. NO.

7. Viscous Flow in a Circular Pipe for any Reynolds' number

Having obtained one solution to equations (3.3) and (3.4) for one value of the Reynolds' number the other solutions followed quite quickly.

For any value of R the finite difference approximation to (3.3) and (3.4) are

$$F_\psi = \frac{h_2^2}{h_1^2R^2}\psi_1+\psi_2\left[1 - \frac{h_2}{2r_o}\right] + \frac{h_2^2}{h_1^2R^2}\psi_3+\psi_4\left[1 + \frac{h_2}{2r_o}\right] - 2\psi_o\left[1 + \frac{h_2^2}{h_1^2R^2}\right] + \pi_o h_2^2, \qquad \ldots\ldots(7.1)$$

and

$$F_\pi = \pi_1\left[\frac{h_2^2}{h_1^2R^2} - \frac{h_2}{4h_1r_o}(\psi_2-\psi_4)\right]+\pi_2\left[1 - \frac{h_2}{2r_o} + \frac{h_2}{4h_1r_o}(\psi_1-\psi_3)\right]$$
$$+\pi_3\left[\frac{h_2^2}{h_1^2R^2} + \frac{h_2}{4h_1r_o}(\psi_2-\psi_4)\right] + \pi_4\left[1 + \frac{h_2}{2r_o} - \frac{h_2}{4h_1r_o}(\psi_1-\psi_3)\right]$$
$$-\pi_o\left[2 + \frac{2h_2^2}{h_1^2R^2} + \frac{h_2^2}{h_1^2r_o^2}(\psi_1-\psi_3)\right],$$

where h_1 is the mesh length in the x-direction, and h_2 is the mesh length in the r-direction.

The boundary conditions are the same as those given in §5.

As $h_2 = 1/20$ gave a good approximation to the axial velocity for the case of large Reynolds' numbers it was decided to use this value of h_2 again. The values of

h_1 are taken as

(i) $0 \leq x < 1/50$ $h_1 = 1/400$ (ii) $x > 1/50$ $h_1 = 1/100$

When R = 100 and the multiplying factor of 10^4 was introduced for ψ and 10^8 for π, the residual formulae are as follows:

1. $(0 \leq x < 1/50)$

$$F_\psi = 0.04\psi_1 + \psi_2[1 - \frac{1}{40r_o}] + 0.04\psi_3 + \psi_4[1 + \frac{1}{40r_o}] - 2.08\psi_o + \frac{\pi_o}{40}. \quad \ldots\ldots(7.3)$$

$$F_\pi = \pi_1[0.04 - \frac{1}{2000r_o}(\psi_2-\psi_4)] + \pi_2[1 - \frac{1}{40r_o} + \frac{1}{2000r_o}(\psi_1-\psi_3)]$$
$$+\pi_3[0.04 + \frac{1}{2000r_o}(\psi_2-\psi_4)] + \pi_4[1 + \frac{1}{40r_o} - \frac{1}{2000r_o}(\psi_1-\psi_3)]$$
$$-\pi_o[2.08 + \frac{1}{20,000r_o^2}(\psi_1-\psi_3)]. \quad \ldots\ldots(7.4)$$

2. $(x > 1/50)$

$$F_\psi = 0.0025\psi_1 + \psi_2[1 - \frac{1}{40r_o}] + 0.0025\psi_3 + \psi_4[1 + \frac{1}{40r_o}]$$
$$- 2.00025\psi_o + \frac{\pi_o}{40}. \quad \ldots(7.5)$$

$$F_\pi = \pi_1[0.0025 - \frac{1}{8000r_o}(\psi_2-\psi_4) + \pi_2[1 - \frac{1}{40r_o} + \frac{1}{8000r_o}(\psi_1-\psi_3)]$$
$$+\pi_3[0.0025 + \frac{1}{8000r_o}(\psi_2-\psi_4)] + \pi_4[1 + \frac{1}{40r_o} - \frac{1}{8000\ r_o}(\psi_1-\psi_3)]$$
$$-\pi_o[2.005 + \frac{1}{40,000r_o^2}(\psi_1-\psi_3)]. \quad \ldots\ldots(7.6)$$

3. $(x = 1/50)$

As before $\frac{\partial^2 \pi}{\partial z^2} = 16{,}000(\pi_1 + 4\pi_3 - 5\pi_0)$, ...(7.7)

and $\frac{\partial \pi}{\partial z} = 20(\pi_1 - 16\pi_3 + 15\pi_0)$, ...(7.8)

so that the expressions for the residuals are

$$F_\psi = 0.004\psi_1+\psi_2[1 - \frac{1}{40r_0}]+0.016\psi_3+\psi_4[1 + \frac{1}{40r_0}]-2.02\psi_0+\pi_0/40 \quad .....(7.9)$$

and

$$F_\pi = \pi_1[0.004 - \frac{1}{20{,}000r_0}(\psi_2-\psi_4)] + \pi_2[1 - \frac{1}{40r_0} + \frac{1}{20{,}000r_0} (\psi_1-16\psi_3+15\psi_0)]$$

$$+\pi_3[0.016 + \frac{16}{20{,}000r_0}(\psi_2-\psi_4)] + \pi_4[1 + \frac{1}{40r_0} - \frac{1}{20{,}000r_0} (\psi_1-16\psi_3+15\psi_0)]$$

$$-\pi_0[2.2 + \frac{1}{100{,}000r_0^2}(\psi_1-16\psi_3+15\psi_0) + \frac{30}{40{,}000r_0^2}(\psi_2-\psi_4)]. \quad .....(7.10)$$

The only difference between equations (7.3) and (6.9) is a term $0.04(\psi_1+\psi_3-2\psi_0)$. From Fig. 4a it can be seen that the value of $(\psi_1+\psi_3-2\psi_0)$ is small. Consequently the solution obtained for very large Reynolds' numbers is a good approximation to the solution of equations (7.3). For $x > 1/50$ the value of $(\psi_1+\psi_3-2\psi_0)$ is even smaller and consequently there is no variation in the values of ψ.

The value of $0.4(\pi_1+\pi_3-2\pi_0)$, which is the term by which (7.4), (6.10) differ is slightly larger. Consequently the changes in π near the entrance to the pipe are comparatively large. These do not affect the ψ-residuals to a large extent since π must bhange by 40 before F_ψ changes by 1.

The values of ψ and π which satisfy equations (7.3)-(7.10) are recorded in Fig. 9.

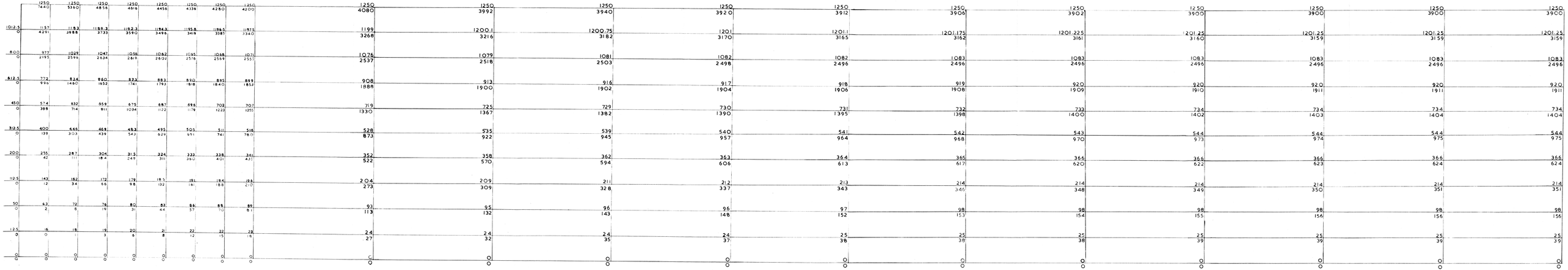
The axial compponent of the velocity was also calculated and the values are given in Table. IV . A graph of u varying with distance is also given and it can be seen that the values of u are approaching those given by Goldstein.

Finally the values of Reynolds' number was taken to be 10. Substituting R = 10, $h_2 = 1/20$, $h_1 = 1/400$, $h_3 = 1/100$, we obtain the following residual formula:-

1. $(0 \leq x < 1/50)$

$$F_\psi = 4\psi_1 + \psi_2\left[1 - \frac{1}{40r_0}\right] + 4\psi_3 + \psi_4\left[1 + \frac{1}{40r_0}\right] - 10\psi_0 + \pi_0/40 \qquad \ldots\ldots(7.11)$$

$$F_\pi = \pi_1\left[4 - \frac{1}{1000r_0}(\psi_2-\psi_4)\right] + \pi_2\left[1 - \frac{1}{40r_0} + \frac{1}{2000r_0}(\psi_1-\psi_3)\right]$$
$$+\pi_3\left[4 + \frac{1}{2000r_0}(\psi_2-\psi_4)\right] + \pi_4\left[1 + \frac{1}{40r_0} - \frac{1}{2000r_0}(\psi_1-\psi_3)\right]$$
$$-\pi_0\left[10 + \frac{1}{100,00r_0^2}(\psi_1-\psi_3)\right]. \qquad \ldots\ldots(7.12)$$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1250 3900
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539 945
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0 0

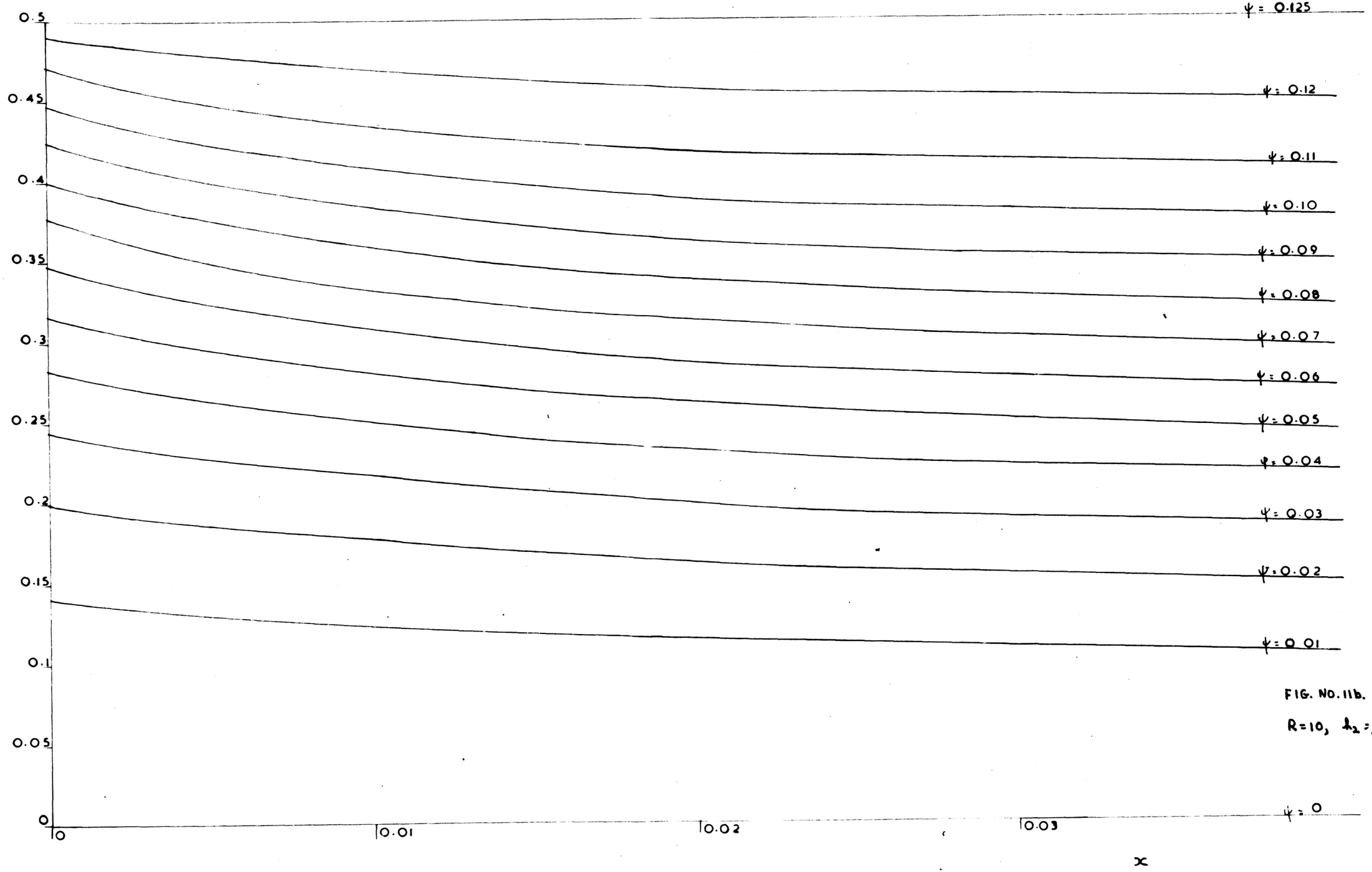


FIG. NO. 11b.

$R = 10, \; k_2 = \frac{1}{\sqrt{10}}$.

x \ r	0.10	0.20	0.30	0.35	0.40	0.45
.0025	1.27	1.29	1.24	1.15	0.96	0.61
.0050	1.44	1.42	1.29	1.13	0.87	0.49
.0075	1.53	1.49	1.30	1.11	0.82	0.45
.0100	1.59	1.52	1.30	1.09	0,80	0.43
.0125	1.64	1.55	1.29	1.07	0.78	0.42
.0150	1.69	1.57	1.28	1.05	0.76	0.41
.0175	1.72	1.59	1.28	1.05	0.75	0.40
.02	1.73	1.60	1.27	1.04	0.75	0.40
.03	1.80	1.62	1.27	1.02	0.73	0.39
.04	1.85	1.63	1.26	1.01	0.72	0.38
.05	1.87	1.64	1.26	1.01	0.71	0.38
.06	1.88	1.64	1.26	1.01	0.71	0.37
.07	1.88	1.64	1.26	1.00	0.71	0.37
.08	1.89	1.64	1.26	1.00	0.71	0.37
.09	1.89	1.65	1.26	1.00	0.70	0.37
.10	1.89	1.65	1.25	1.00	0.70	0.37

Values of u for R = 100, h_2 = 1/20.

TABLE No. IV.

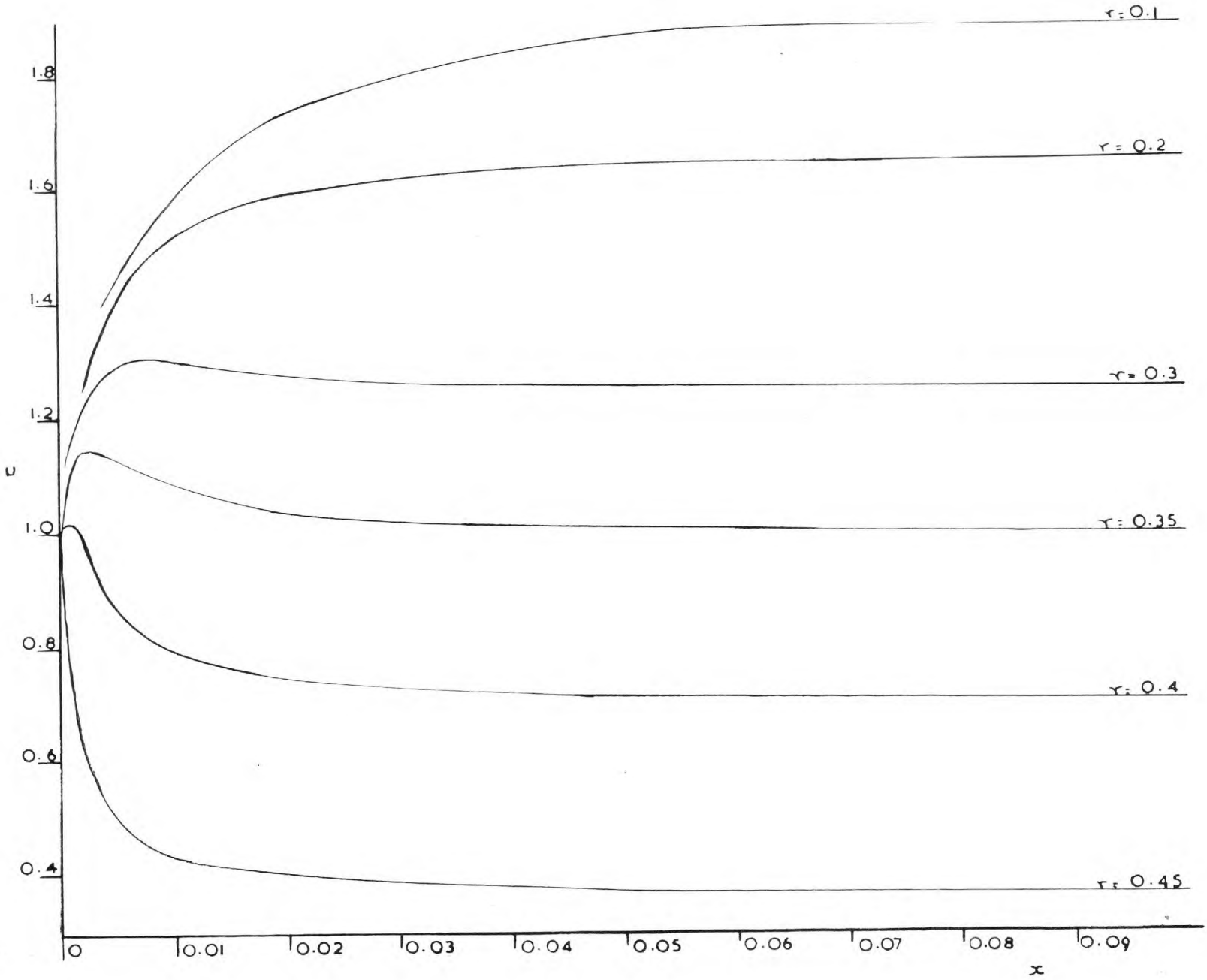


FIG. NO. 10.

$R = 100, \quad k_2 = 1/20$

2. $(x > 1/50)$

$$F_\psi = 0.25\psi_1+\psi_2[1 - \frac{1}{40r_o}]+0.25\psi_3+\psi_4[1 + \frac{1}{40r_o}]-2.5\psi_o+\pi_o/40 \quad .....(7.13)$$

$$F_\pi = \pi_1[0.25 - \frac{1}{8000r_o}(\psi_2-\psi_4)]+\pi_2[1 - \frac{1}{40r_o} + \frac{1}{8000r_o}(\psi_1-\psi_3)]$$
$$+\pi_3[0.25 + \frac{1}{8000r_o}(\psi_2-\psi_4)]+\pi_4[1 + \frac{1}{40r_o} - \frac{1}{8000r_o}(\psi_1-\psi_3)]$$
$$-\pi_o[2.5 + \frac{1}{40,000r_o^2}(\psi_1-\psi_3)]. \quad .....(7.14)$$

3. $(x = 1/50)$

$$F_\psi = 0.4\psi_1+(1 - \frac{1}{40r_o})\psi_2+1.6\psi_3+\psi_4(1 + \frac{1}{40r_o})-4\psi_o + \pi_o/40 \quad .....(7.15)$$

$$F_\pi = \pi_1[0.4 - \frac{1}{20,000r_o}(\psi_2-\psi_4)]$$
$$+\pi_2[1 - \frac{1}{40r_o} + \frac{1}{20,000r_o}(\psi_1-16\psi_3+15\psi_o)$$
$$+\pi_3[1.6 + \frac{16}{20,000r_o}(\psi_2-\psi_4)]$$
$$+\pi_4[1 + \frac{1}{40r_o} - \frac{1}{20,000r_o}(\psi_1-16\psi_3+15\psi_o)]$$
$$-\pi_o[4 + \frac{1}{100,000r_o^2}(\psi_1-16\psi_3+15\psi_o) + \frac{30}{40,000r_o^2}(\psi_2-\psi_4)] \quad .....(7.16)$$

From Fig. 9a it can be seen that, when $0 \leq x < 1/50$ the value of $(\psi_1+\psi_3-2\psi_o)$ is larger than before. However the solution for R = 100 proved quite a good guess for the case R = 10. The ψ-residuals were larger this time, but the relaxation pattern had changed and moving residuals to the

.....

boundary was much easier. It was in this particular problem that the accumulating of residuals on r = 2/5 was found to be of the greatest use. The values of ψ and π are given in Fig. 11a. Again, values of the axial velocity were calculated. These are given in Table V; and graphically in Fig. 11.

These values of u suggest that the Reynolds' number of the motion used in the experimental work lies somewhere between 10 and 100. (The experimental results are shown graphically in Fig. 13).

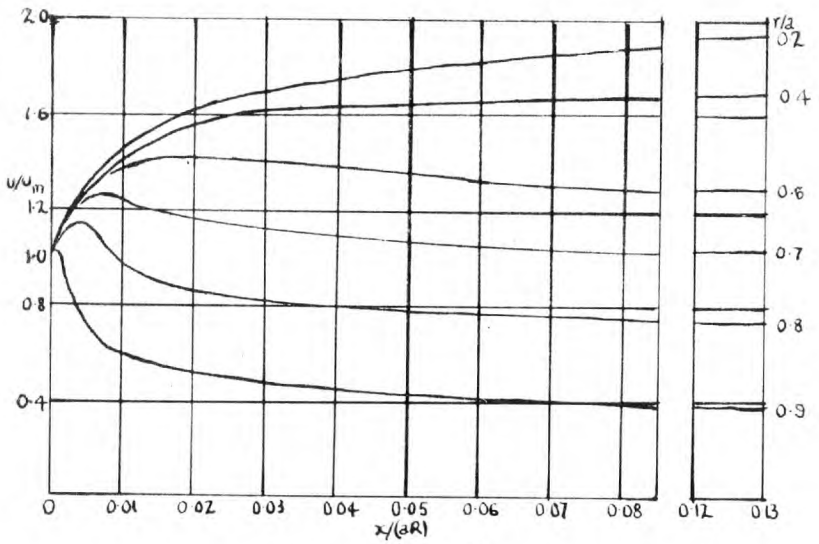


FIG NO 13.

For large values of R, Goldstein (Ref. 6) expanded the stream function ψ in a series. From here he obtained the

axial component of velocity and these results are given in Fig. 14.

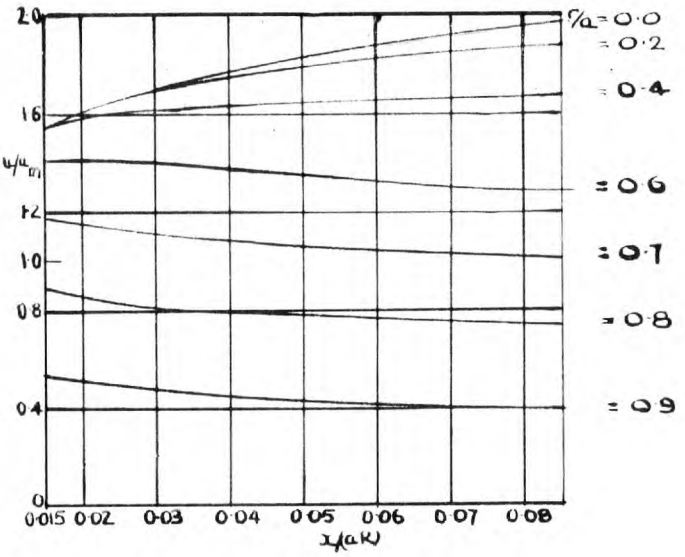
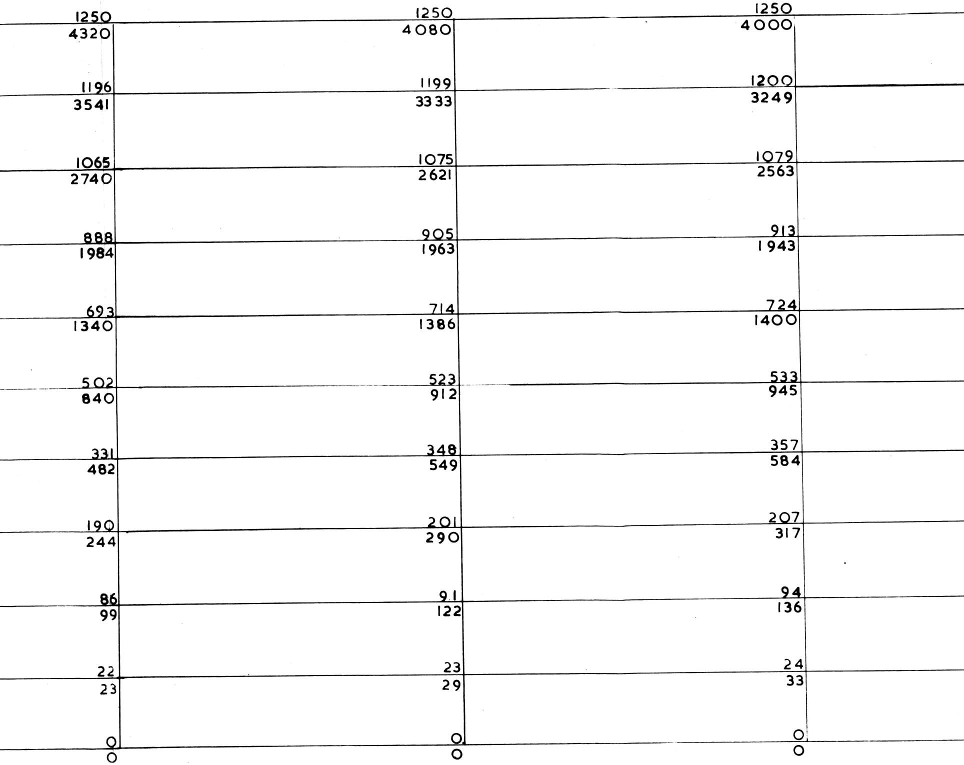


FIG. 14.

The values of u are given in the region $\frac{x}{aR} \geqslant 0.015$ (x is the co-ordinate in the direction of the axis of the pipe and 'a' is the radius of the pipe). For this region $\frac{x}{aR} \geqslant 0.015$ the two sets of results compare favourably. When conditions of uniform flow are established the values of u given by Goldstein are greater than those obtained from the algebraic solution and those obtained by the relaxation method are smaller.

	1250 15080	1250 11944	1250 9632	1250 7984	1250 6832	1250 6024	1250 5448	1250 5032
1012.5 0	1061.5 3312	1100.7 4657	1129.6 5037	1150.2 4985	1164.6 4767	1174.7 4507	1181.9 4260	1187.1 4045
800 0	850 1031	895 1861	934 2415	966 2738	992 2899	1012 2958	1028 2960	1040 2938
612.5 0	654 402	694 806	731 1157	763 1434	791 1639	815 1783	835 1879	851 1942
450 0	482 176	513 370	543 561	570 735	595 886	617 1011	636 1111	652 1191
312.5 0	335 81	357 174	379 272	400 359	419 461	436 544	451 618	465 682
200 0	215 37	229 80	243 128	257 178	269 227	281 275	292 320	302 361
112.5 0	121 16	129 35	137 56	145 79	152 103	159 127	165 150	171 172
50 0	54 5	58 12	61 20	65 29	68 38	71 47	74 57	77 65
12.5 0	13 1	14 2	15 4	16 6	17 8	18 10	19 13	20 15
0 0	0 0	0 0	0 0	0 0	0 0	0 0	0 0	0 0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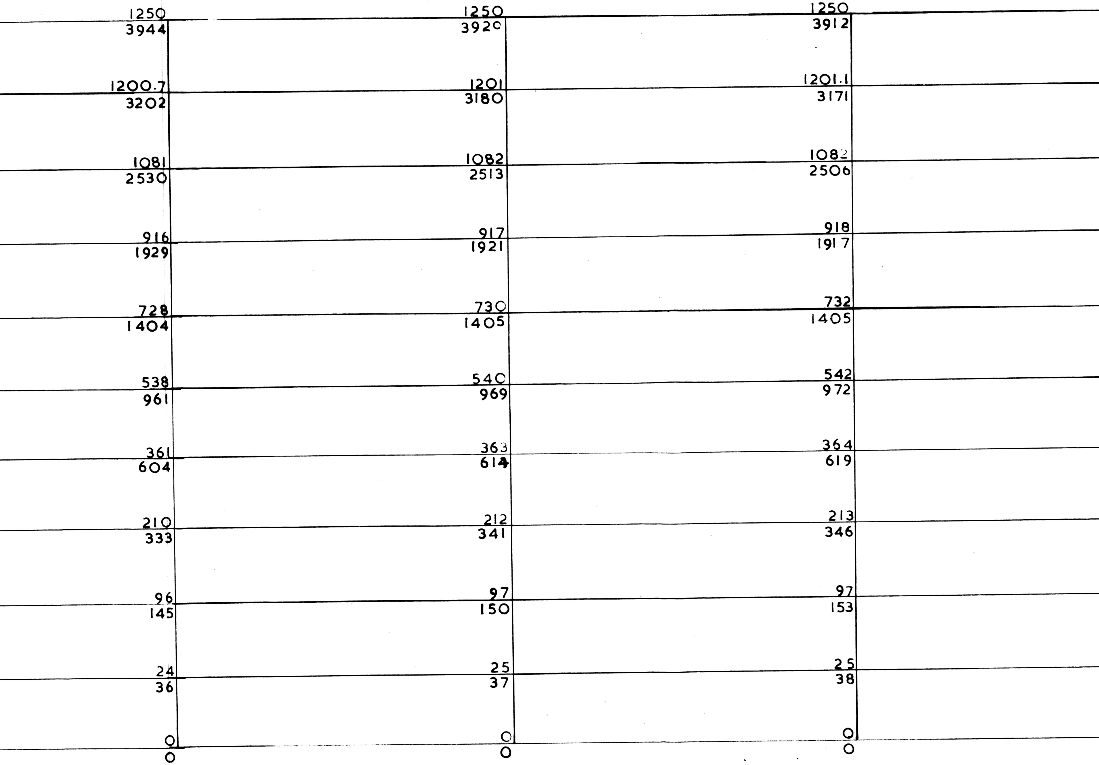
4320
1196
3541
1065
2740
888
1984
693
1340
502
840
331
482
190
244
86
99
22
23
0
0
1250
4080
1199
3333
1075
2621
905
1963
714
1386
523
912
348
549
201
290
91
122
23
29
0
0
1250
4000
1200
3249
1079
2563
913
1943
724
1400
533
945
357
584
207
317
94
136
24
33
0
0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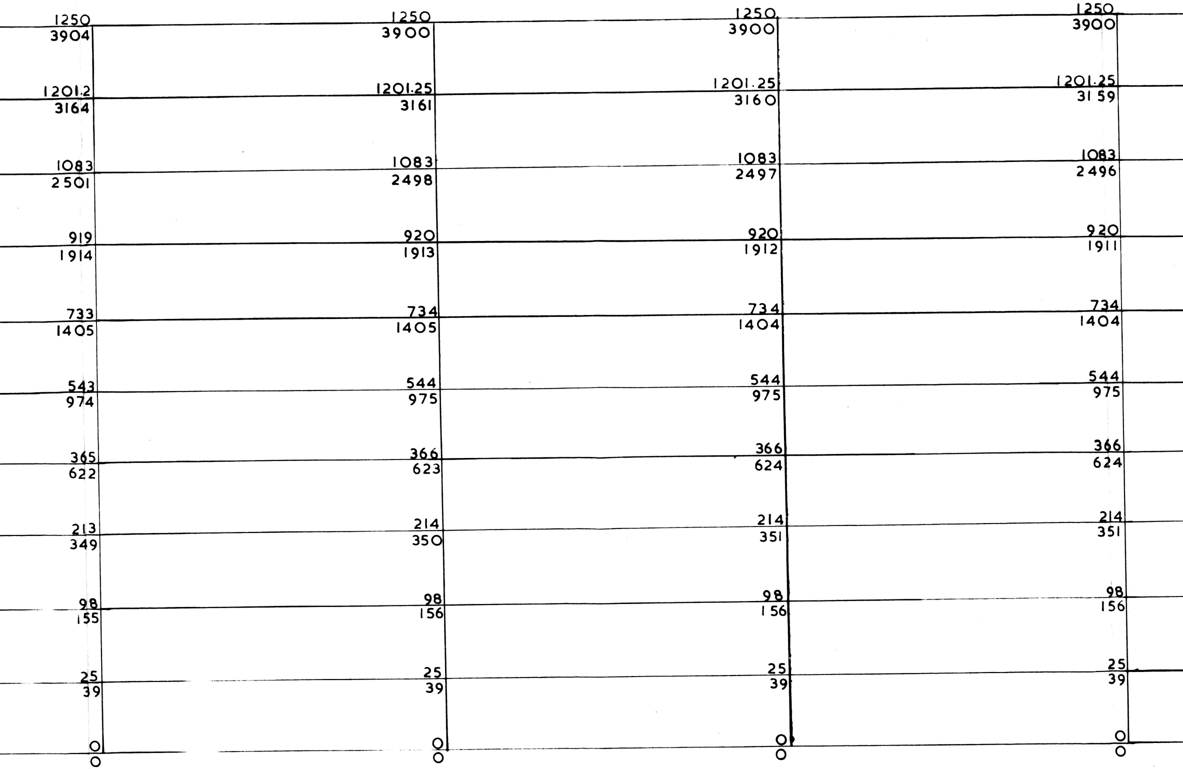
3944
1200.7
3202
1081
2530
916
1929
728
1404
538
961
361
604
210
333
96
145
24
36
0
0
1250
3920
1201
3180
1082
2513
917
1921
730
1405
540
969
363
614
212
341
97
150
25
37
0
0
1250
3912
1201.1
3171
1082
2506
918
1917
732
1405
542
972
364
619
213
346
97
153
25
38
0
0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3904
1201.2
3164
1083
2501
919
1914
733
1405
543
974
365
622
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349
98
155
25
39
0
0
1250
3900
1201.25
3161
1083
2498
920
1913
734
1405
544
975
366
623
214
350
98
156
25
39
0
0
1250
3900
1201.25
3160
1083
2497
920
1912
734
1404
544
975
366
624
214
351
98
156
25
39
0
0
1250
3900
1201.25
3159
1083
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1911
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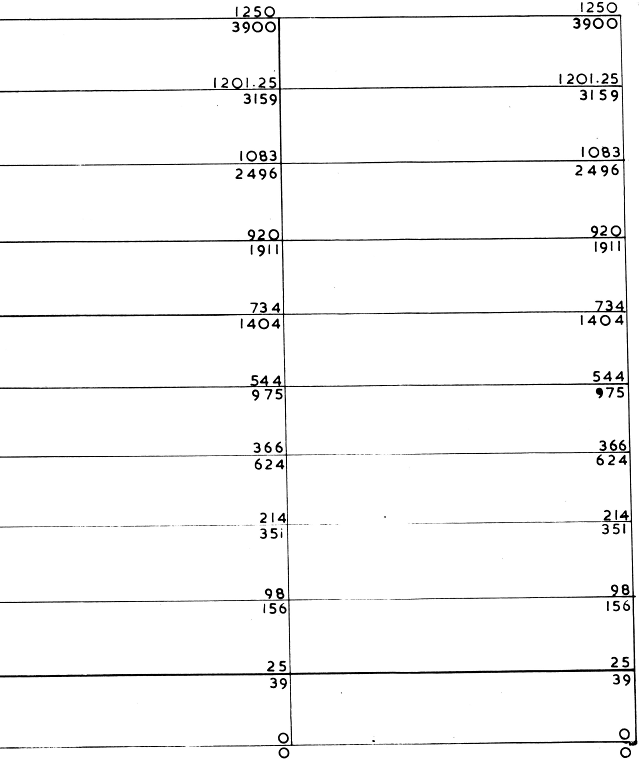
1250
3900
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3159
1201.25
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1083
2496
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920
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1911
734
1404
734
1404
544
975
544
975
366
624
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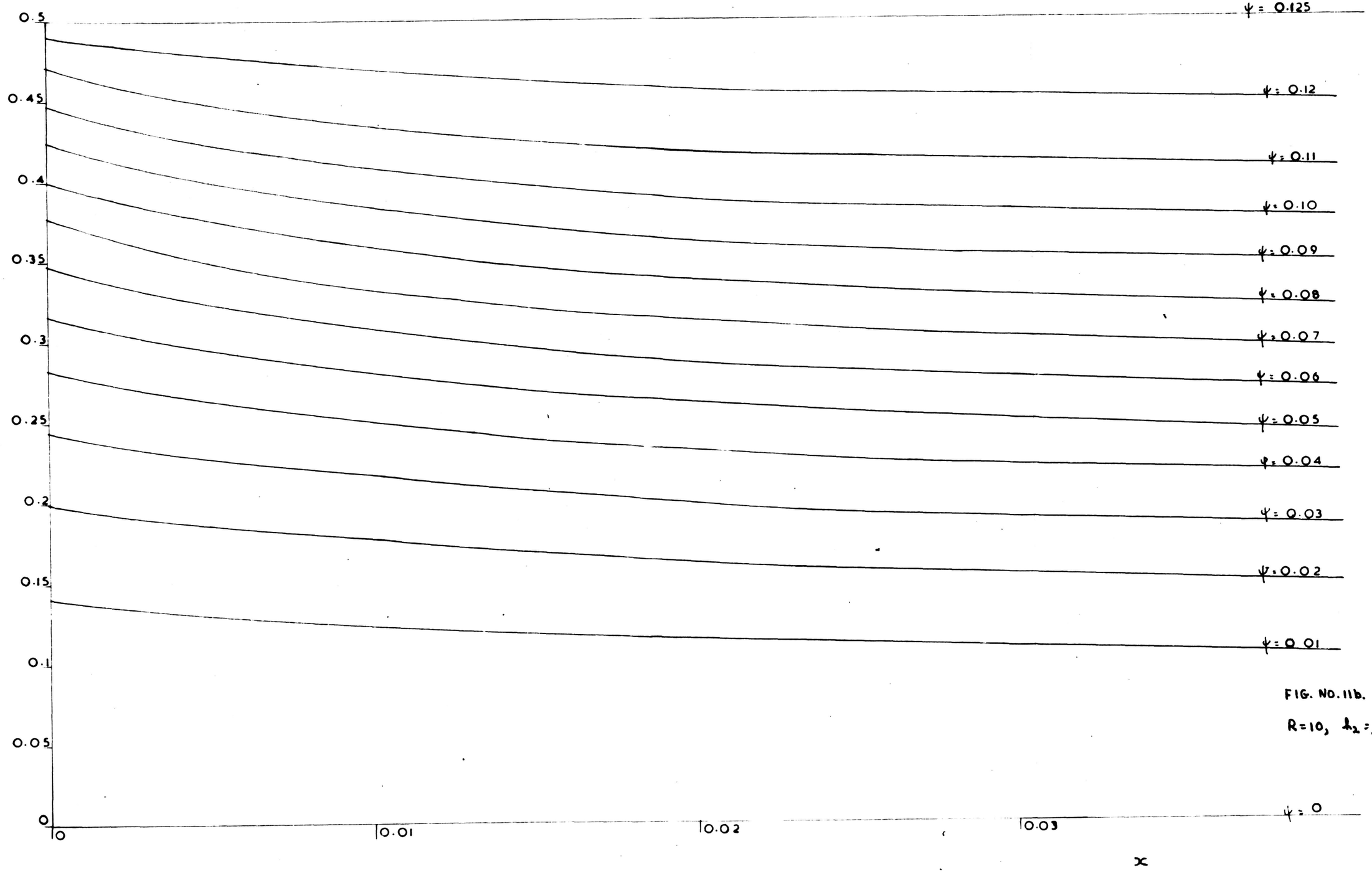
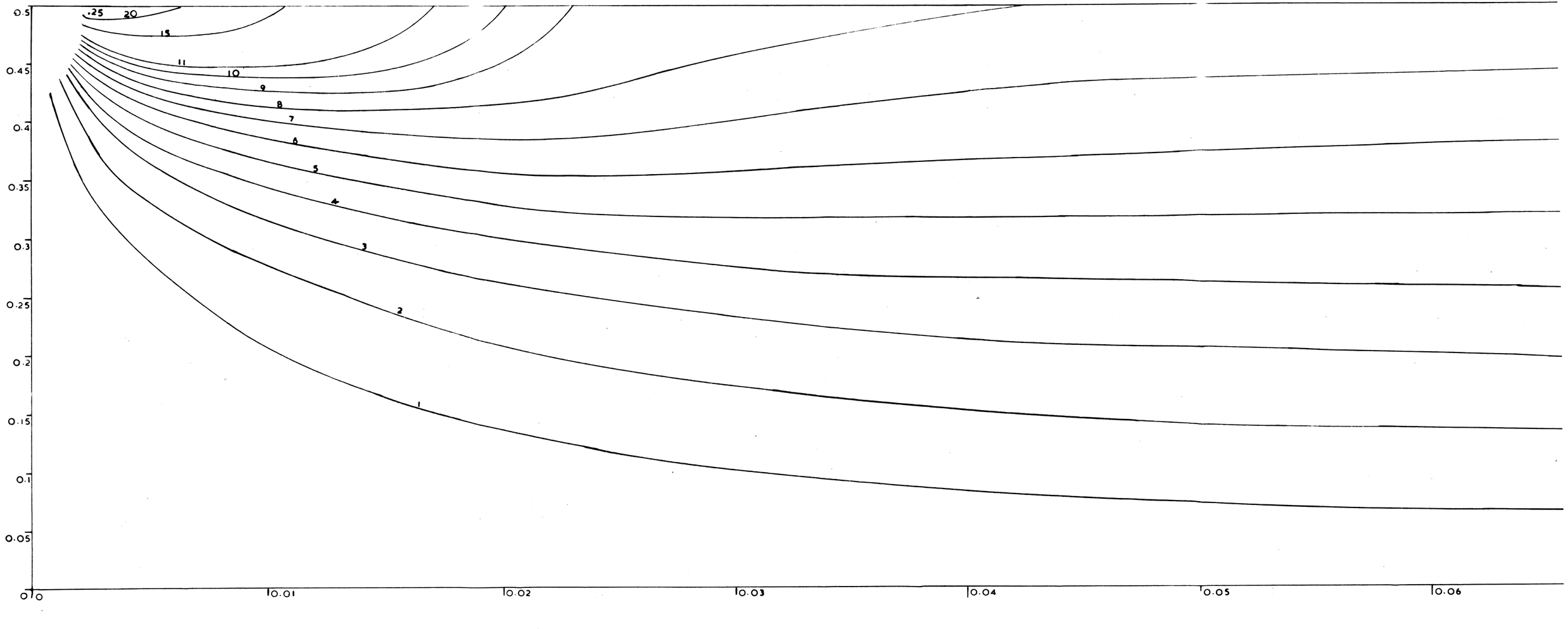


FIG. NO. 11b.

$R = 10, \; k_2 = \frac{1}{\sqrt{10}}$.


0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.01
0.02
0.03
0.04
0.05
0.06
x
.25
20
15
11
10
9
8
7
6
5
4
3
2
1

x \ τ	0.10	0.20	0.30	0.35	0.40	0.45
.0025	1.08	1.07	1.06	1.05	1.02	0.89
.0050	1.15	1.14	1.12	1.09	1.02	0.79
.0075	1.22	1.21	1.17	1.12	1.00	0.70
.0100	1.29	1.28	1.21	1.13	0.97	0.63
.0125	1.35	1.34	1.24	1.13	0.93	0.57
.0150	1.41	1.39	1.26	1.13	0.90	0.53
.0175	1.46	1.43	1.28	1.12	0.87	0.49
.0200	1.51	1.47	1.29	1.11	0.84	0.47
.03	1.68	1.56	1.29	1.06	0.77	0.41
.04	1.78	1.61	1.27	1.03	0.74	0.39
.05	1.83	1.63	1.27	1.01	0.72	0.38
.06	1.86	1.64	1.26	1.01	0.71	0.38
.07	1.87	1.64	1.26	1.01	0.71	0.37
.08	1.88	1.65	1.25	1.00	0.71	0.37
.09	1.88	1.65	1.25	1.00	0.70	0.37
.10	1.89	1.65	1.25	0.98	0.70	0.37
.11	1.89	1.65	1.25	0.98	0.70	0.37

Values of u for R = 10, h_2 = 1/20.

TABLE NO. V

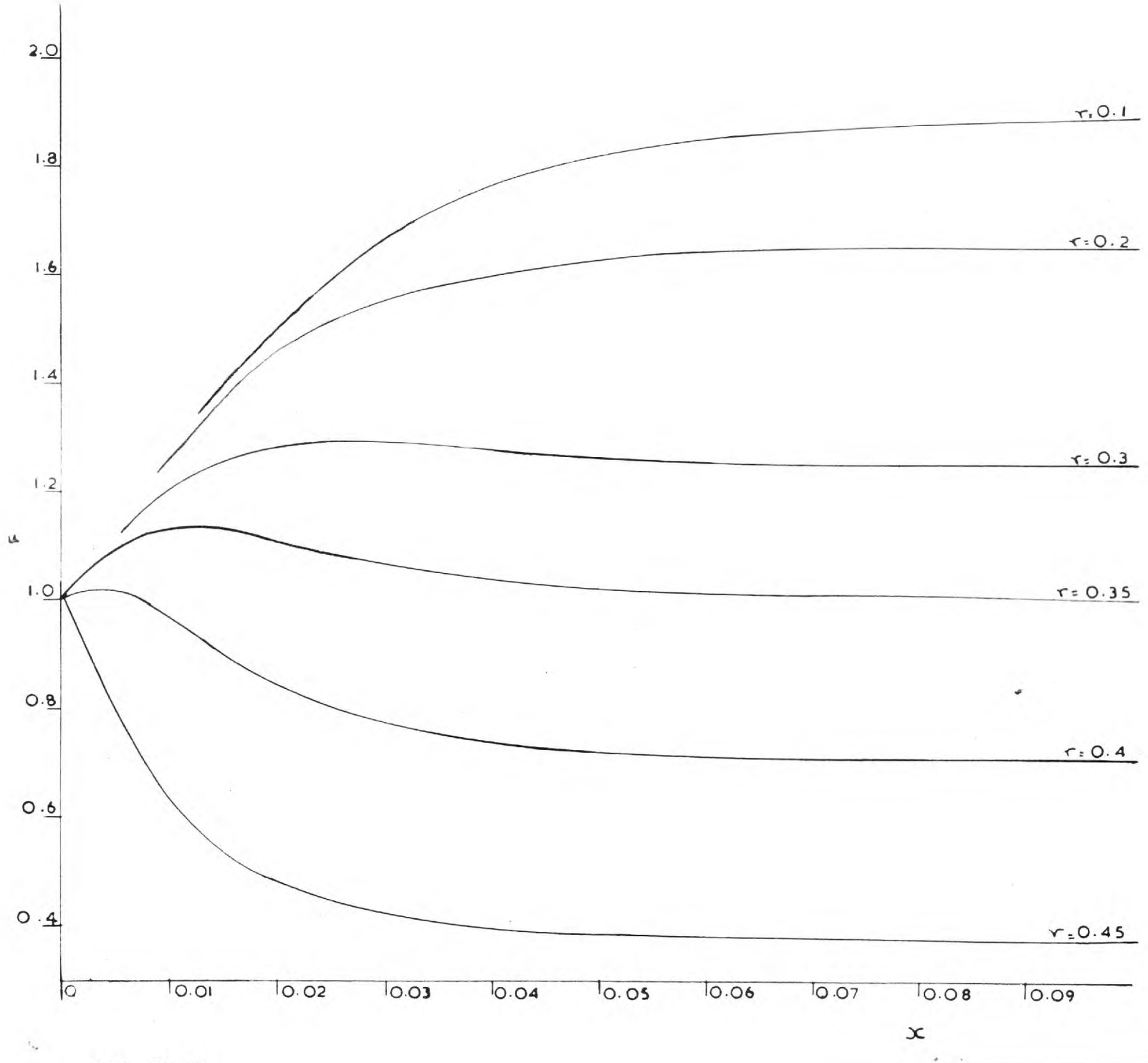


FIG. NO. 12.

$R = 10$, $h_2 = 1/20$.

For the region $\frac{x}{aR} < 0.015$ Goldstein says that his results are inaccurate. Using a relaxation treatment it has been seen that the values of u do not change very much when a finer net is introduced in the r-direction. If a finer net is introduced for $0 \leq x \leq 1/200$, say, the variation of u in this region can be obtained more accurately.

It is, of course, possible to go on and find a solution for the case R = 1 starting with the solution for R = 10. A glance at the residual formulae in this case shows that the coefficients of ψ_1 and ψ_3 have the value 400 and the coefficient of ψ_0 is 802. Working with elements of this size in a relaxation pattern is not easy and in this case it is better to return to the original equations (3.16) and (3.17). As the second problem of this thesis deals with viscous flow past a sphere for small Reynolds' numbers the solutions of the present problem for R = 1, 0 have not been worked out.

PART III

Viscous Flow Past a Sphere

8. General Equations

When we are considering the problem of slow viscous flow (i.e. small values of R) we take the governing equations to be of the form

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = -\pi \ , \qquad \text{.....(8.1)}$$

and

$$\frac{\partial^2 \pi}{\partial r^2} - \frac{1}{r}\frac{\partial \pi}{\partial r} + \frac{\partial^2 \pi}{\partial z^2} = R\left[\frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial \pi}{\partial z} - \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial \pi}{\partial r} + \frac{2}{r}\frac{\partial \psi}{\partial z}\pi\right] \qquad \text{.....(8.2)}$$

These equations (8.1) and (8.2) are to be solved for viscous fluid flowing past a fixed sphere of diameter 1 (in non-dimensional terms).

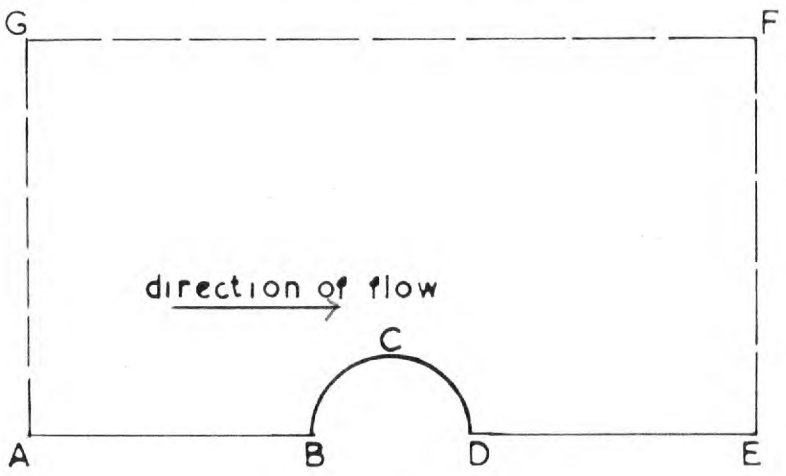


FIG. NO. 15.

The flow is symmetrical about the line AE so that only half of the field of flow is considered.

9. Boundary Conditions

The boundary drawn with a broken line EFGA in Fig.15 is intended to be taken so far from the sphere that the flow is unaffected by it. The stream function ψ then satisfies, on this boundary, the following conditions:-

(1) $\partial\psi/\partial z = 0$ .....(9.1)

(2) $\partial\psi/\partial r = r$ .....(9.2)

The viscosity effects near the boundary EFGA are negligible so, in the first instance, we take π to be zero there. (see P.43.)

The curve ABCDE is a streamline so that ψ has a constant value along it which we take to be zero:

$$\psi = 0. \qquad .....(9.3)$$

Along BCD both the components of velocity are zero, therefore

$$\frac{\partial\psi}{\partial z} = \frac{\partial\psi}{\partial r} = 0 \qquad .....(9.4)$$

On AB and DE, $r = 0$, therefore

$$\pi = r\zeta = 0 \qquad .....(9.5)$$

On BCD it is possible to calculate values for π from (8.1) as ψ is known there. Difficulties arise as this boundary is curved. These are illustrated even if the case $R = 0$ is considered. For the equations (8.1) and (8.2) become then

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = -\pi,$$

$$\frac{\partial^2\pi}{\partial r^2} - \frac{1}{r}\frac{\partial\pi}{\partial r} + \frac{\partial^2\pi}{\partial z^2} = 0. \qquad \text{.....(9.6)}$$

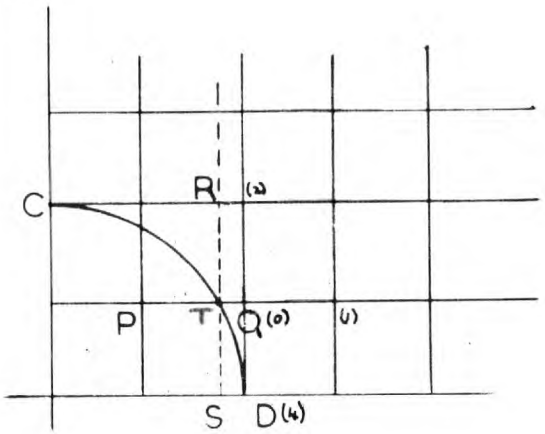


FIG. NO. 16.

Consider the point Q near the boundary point T. We let $QT = \zeta h (0 < \zeta < 1)$ so that

$$\psi_T = \psi_0 - \zeta h\left(\frac{\partial\psi}{\partial z}\right)_Q + \frac{\zeta^2 h^2}{2!}\left(\frac{\partial^2\psi}{\partial z^2}\right)_Q, \qquad \text{.....(9.7)}$$

therefore

$$\left(\frac{\partial\psi}{\partial z}\right)_Q = \frac{\psi_1\zeta^2 - \psi_T}{\zeta h(\zeta+1)},$$

$$\left(\frac{\partial^2\psi}{\partial z^2}\right)_Q = \frac{2(\zeta\psi_1+\psi_T)}{\zeta(\zeta+1)h^2} - \frac{2\psi_0}{h^2\zeta}. \qquad \text{.....(9.8)}$$

Hence the finite difference approximations to equations (9.6) can be obtained and the corresponding residual formulae for the point Q are:-

$$F_\psi = \frac{\psi_1}{\zeta+1} + \frac{\psi_2}{2}[1 - \frac{h}{2r_o}] + \frac{\psi_T}{(\zeta+1)} + \frac{\psi_4}{2}[1 + \frac{h}{2r_o}]$$
$$- 2\psi_o[1 + \frac{1}{\zeta}] + \pi_o h^2/2, \qquad \ldots\ldots(9.9)$$

and

$$F_\pi = \frac{\pi_1}{\zeta+1} + \frac{\pi_2}{2}[1 - \frac{h}{2r_o}] + \frac{\pi_T}{(\zeta+1)} + \frac{\pi_4}{2}[1 + \frac{h}{2r_o}] - 2\pi_o[1 + \frac{1}{\zeta}]. \quad (9.10)$$

The residual F_ψ is easily calculated from (9.9). To obtain a value for F_π at Q it can be seen from (9.10) that a value of π is needed at T. This value can be obtained from the first of equations (9.6).

At T, we have

$$(\frac{\partial^2\psi}{\partial r^2})_T = \frac{\psi_R + \psi_S - 2\psi_T}{h^2}, \quad (\frac{\partial\psi}{\partial r})_T = \frac{\psi_R - \psi_S}{2h}, \qquad \ldots\ldots(9.11)$$

with corresponding expressions for $(\frac{\partial\psi}{\partial z})_T$, $(\frac{\partial^2\psi}{\partial z^2})_T$ in terms of ψ_Q, ψ_P and ζ.

When we substitute these values of $\frac{\partial^2\psi}{\partial r^2}$, $\frac{\partial\psi}{\partial r}$, $\frac{\partial^2\psi}{\partial z^2}$ into (9.6) we obtain an equation of the form:

$$\pi_T = f(\psi_P, \psi_Q, \psi_R, \psi_S, \zeta). \qquad \ldots\ldots(9.12)$$

The quantities ψ_S and ψ_P can be eliminated from (9.12) using the boundary conditions (9.4). The value of ψ at R has still to be interpolated and the obvious scope for inaccuracies to accumulate are such that it was therefore decided to use a transformation for which the half-circle BCD would become a straight line.

10. Change of Independent Variables

Thom (Ref. 12) suggested that the variables r, z are changed to α and β, the velocity potential and stream function for irrotational flow past a sphere or a cylinder with circular cross-section. The boundary of the section is then a stream-line β = const. = 0 (say) i.e. a straight line in the (α, β) plane. Thus the field of computation is transformed into an infinite plane containing a rectilinear slit. The variables α and β are conjugate plane-potential functions, so that $(\alpha + i\beta)$ is a function of $(z + ir)$. Accordingly

$$\frac{\partial \alpha}{\partial z} = \frac{\partial \beta}{\partial r}, \qquad -\frac{\partial \alpha}{\partial r} = \frac{\partial \beta}{\partial z}, \qquad \text{.....(10.1)}$$

and $$\nabla^2 \alpha = \nabla^2 \beta = 0. \qquad \text{.....(10.2)}$$

Then

$$\frac{\partial}{\partial z} = \frac{\partial \alpha}{\partial z}\frac{\partial}{\partial \alpha} + \frac{\partial \beta}{\partial z}\frac{\partial}{\partial \beta}, \quad \frac{\partial}{\partial r} = \frac{\partial \alpha}{\partial r}\frac{\partial}{\partial \alpha} + \frac{\partial \beta}{\partial r}\frac{\partial}{\partial \beta} \qquad \text{.....(10.3)}$$

Therefore

$$\frac{\partial}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

$$= \left[\left(\frac{\partial \alpha}{\partial r}\right)^2 + \left(\frac{\partial \alpha}{\partial z}\right)^2\right]\frac{\partial^2}{\partial \alpha^2} + \left[\left(\frac{\partial \beta}{\partial r}\right)^2 + \left(\frac{\partial \beta}{\partial z}\right)^2\right]\frac{\partial^2}{\partial \beta^2}$$

$$- \frac{1}{r}\frac{\partial \alpha}{\partial r}\frac{\partial}{\partial \alpha} - \frac{1}{r}\frac{\partial \beta}{\partial r}\frac{\partial}{\partial \beta}, \qquad \text{.....(10.4)}$$

and

$$-\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial\pi}{\partial r}+\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\pi}{\partial z}+\frac{2}{r^2}\frac{\partial\psi}{\partial z}\pi$$

$$=-\frac{\partial\psi}{\partial\alpha}\frac{\partial\pi}{\partial\beta}\left[\left(\frac{\partial\alpha}{\partial z}\right)^2+\left(\frac{\partial\alpha}{\partial r}\right)^2\right]+\frac{\partial\psi}{\partial\beta}\frac{\partial\pi}{\partial\alpha}\left[\left(\frac{\partial\alpha}{\partial z}\right)^2+\left(\frac{\partial\alpha}{\partial r}\right)^2\right]$$

$$+\frac{2}{r^2}\pi\left[\frac{\partial\psi}{\partial\alpha}\frac{\partial\alpha}{\partial z}-\frac{\partial\alpha}{\partial r}\frac{\partial\psi}{\partial\beta}\right]. \qquad .....(10.5)$$

Consequently (8.1) and (8.2) are transformed into

$$\frac{-\pi}{H^2}=\frac{\partial^2\psi}{\partial\alpha^2}+\frac{\partial^2\psi}{\partial\beta^2}-\frac{1}{rH^2}\left[\frac{\partial\alpha}{\partial r}\frac{\partial\psi}{\partial\alpha}+\frac{\partial\beta}{\partial r}\frac{\partial\psi}{\partial\beta}\right], \qquad .....(10.6)$$

and

$$\frac{\partial^2\pi}{\partial\alpha^2}+\frac{\partial^2\pi}{\partial\beta^2}-\frac{1}{rH^2}\left[\frac{\partial\alpha}{\partial r}\frac{\partial\pi}{\partial\alpha}+\frac{\partial\beta}{\partial r}\frac{\partial\psi}{\partial\beta}\right]$$

$$=R\left[\frac{\partial\pi}{\partial\alpha}\frac{\partial\psi}{\partial\beta}-\frac{\partial\pi}{\partial\beta}\frac{\partial\psi}{\partial\alpha}+\frac{2\pi}{r^2H^2}\left[\frac{\partial\psi}{\partial\alpha}\frac{\partial\alpha}{\partial z}-\frac{\partial\alpha}{\partial r}\frac{\partial\psi}{\partial\beta}\right]\right], \qquad .....(10.7)$$

where $H^2=\left(\frac{\partial\alpha}{\partial r}\right)^2+\left(\frac{\partial\alpha}{\partial z}\right)^2=\left(\frac{\partial\beta}{\partial r}\right)^2+\left(\frac{\partial\beta}{\partial z}\right)^2$. .....(10.8)

By use of (9.4) and (10.3), the transformed boundary conditions are seen to become:

$\psi=\pi=0$ at all points of the α-axis ($\alpha\geq 1$). .....(10.9)

$\frac{\partial\psi}{\partial\alpha}=\frac{\partial\psi}{\partial\beta}=0$ on that part of the axis ($\beta=0$) which corresponds to the circular cross-section, ($\alpha<1$). .....(10.10)

For a sphere of radius 1/2,

$$\alpha=z\left[1+\frac{1}{16(r^2+z^2)^{1/2}}\right], \qquad .....(10.11)$$

$$\beta=r^2\left[1-\frac{1}{8(r^2+z^2)^{3/2}}\right]/2 \qquad .....(10.12)$$

and for a circular cylinder of radius 1/2,

$$\alpha = z\left[1 + \frac{1}{4(r^2 + z^2)}\right],$$

$$\beta = r\left[1 - \frac{1}{4(r^2 + z^2)}\right]. \qquad \ldots\ldots(10.13)$$

From (10.6) and (10.7) it can be seen that values of H^2, $\frac{\partial \alpha}{\partial z}$, $\frac{\partial \alpha}{\partial r}$ etc. are needed and hence the transformation, from the (r,z) coordinates to the (α , β) coordinates should be as simple as possible. Thus the transformation was taken to be (10.13)

This yields

$$\frac{\partial \alpha}{\partial r} = -\frac{\partial \beta}{\partial z} = \frac{-rz}{2(r^2 + z^2)^2},$$

$$\frac{\partial \alpha}{\partial z} = \frac{\partial \beta}{\partial r} = 1 + \frac{r^2 - z^2}{4(r^2 + z^2)^2}. \qquad \ldots\ldots(10.14)$$

In order to obtain a numerical value for the coefficients in (10.6) and (10.7) values of r and z had to be obtained which satisfied equations (10.13) for different values of α and β . An iterative process was used to do this. Equations (10.13) were written in the form

$$z = \alpha\left[1 + \frac{1}{4(r^2 + z^2)}\right]^{-1} \qquad \ldots\ldots(10.15)$$

$$r = \beta\left[1 - \frac{1}{4(r^2 + z^2)}\right]^{-1} \qquad \ldots\ldots(10.16)$$

Values were assumed for (r,z) and substituted in the right-hand side of equations (10.15) and (10.16). Another set of values for r and z were thus obtained. The initial guesses for r and z were changed in the light of these results. It was found that if the second values of r and z were substituted into the right-hand sides of equations (10.15) and (10.16) in order to obtain a third set of values of r and z, the values obtained did not as a rule converge but oscillated about the true values of r and z. The solutions to equations (10.15) and (10.16) for $\alpha = 0 - 5$, $\beta = 0 - 23/4$ are given in the tables in Appendix I.

11. Non-Viscous Flow past a Sphere

For non-viscous flow, $\mathfrak{F} = 0$ and equations (10.6) and (10.7) reduce to

$$\frac{\partial^2\psi}{\partial\alpha^2} + \frac{\partial^2\psi}{\partial\beta^2} - \frac{\frac{1}{r}\frac{\partial\alpha}{\partial r}}{\left(\frac{\partial\alpha}{\partial r}\right)^2 + \left(\frac{\partial\alpha}{\partial z}\right)^2}\frac{\partial\psi}{\partial\alpha} - \frac{\frac{1}{r}\frac{\partial\beta}{\partial r}}{\left(\frac{\partial\beta}{\partial r}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2}\frac{\partial\psi}{\partial\beta} = 0 \quad \ldots\ldots(11.1)$$

When we substitute for $\frac{\partial\alpha}{\partial r}$, $\frac{\partial\alpha}{\partial z}$ etc. from (10.4) we obtain the following equation:

$$\frac{\partial^2\psi}{\partial\alpha^2} + \frac{\partial^2\psi}{\partial\beta^2} + \frac{z}{2(r^2+z^2)^2 + r^2 - z^2 + 1/8}\frac{\partial\psi}{\partial\alpha} - \frac{(r^2+\lambda^2)^2 + (r^2-z^2)/4}{r\{(r^2+z^2)^2 + (r^2-z^2+1/8)/2\}}\frac{\partial\psi}{\partial\beta} = 0 \quad \ldots\ldots(11.2)$$

We define the symbols A and B as follows;

$$A = \frac{z}{2(r^2+z^2)^2 + r^2 - z^2 + 1/8} = \frac{-\frac{1}{r}\frac{\partial\alpha}{\partial r}}{\left(\frac{\partial\alpha}{\partial r}\right)^2 + \left(\frac{\partial\alpha}{\partial z}\right)^2}, \quad \ldots\ldots(11.3)$$

$$B = \frac{(r^2+z^2)^2 + (r^2-z^2)/4}{r[(r^2+z^2)^2 + (r^2-z^2+1/8)/2]} = \frac{+\frac{1}{r}\frac{\partial\beta}{\partial r}}{\left(\frac{\partial\beta}{\partial r}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2}, \quad (11.4)$$

so that equation (11.2) takes the form

$$\frac{\partial^2\psi}{\partial\alpha^2} + \frac{\partial^2\psi}{\partial\beta^2} + A\frac{\partial\psi}{\partial\alpha} - B\frac{\partial\psi}{\partial\beta} = 0. \quad \ldots\ldots(11.5)$$

Numerical values of A and B for different values of α and β are given in the Tables in Appendix I.

The boundary conditions in this case are:-

On $\alpha = 0$, $\frac{\partial \psi}{\partial \alpha} = 0$ (by symmetry), ...(11.6)

On $\beta = 0$, $\psi = 0$. .....(11.7)

And as $\alpha^2 + \beta^2 \to \infty$, $\frac{\partial \psi}{\partial \alpha} \to \frac{\partial \psi}{\partial z}$, $\frac{\partial \psi}{\partial \beta} \to \frac{\partial \psi}{\partial r}$,

therefore $\frac{\partial \psi}{\partial \alpha} \to 0$, and $\frac{\partial \psi}{\partial \beta} \to r$. .....(11.8)

The finite difference approximation to (11.5) is

$$\psi_1[1 + \tfrac{hA}{2}] + \psi_2[1 - \tfrac{Bh}{2}] + \psi_3[1 - \tfrac{hA}{2}] + \psi_4[1 + \tfrac{Bh}{2}] - 4\psi_0 = 0.$$

The mesh length is taken to be 1/4. We thus have the following expression for F_ψ:-

$$F_\psi = \psi_1[1 + \tfrac{A}{8}] + \psi_2[1 - \tfrac{B}{8}] + \psi_3[1 - \tfrac{A}{8}] + \psi_4[1 + \tfrac{B}{8}] - 4\psi_0. \quad .....(11.9)$$

On $\alpha = 0$ $\psi_1 = \psi_3$ so that the fictitious value ψ_3 can be eliminated from the residual formula (11.9) viz.

$$F_\psi = 2\psi_1 + \psi_2[1 - \tfrac{B}{8}] + \psi_4[1 + \tfrac{B}{8}] - 4\psi_0 \quad .....(11.10)$$

When $\alpha \to \infty$ $\psi_1 = \psi_3$, so the fictitious value ψ_1 can be eliminated from the residual formula (11.9) viz.

$$F_\psi = \psi_2[1 - \tfrac{B}{8}] + 2\psi_3 + \psi_4[1 + \tfrac{B}{8}] - 4\psi_0. \quad .....(11.11)$$

When $\beta \to \infty$ $\frac{\partial \psi}{\partial \rho} = r$, i.e. $\psi_2 - \psi_4 = 2hr$, so that the fictitious value ψ_2 can be eliminated from (11.9) viz.

$$F_\psi = (1 + \tfrac{A}{8})\psi_1 + (1 - \tfrac{A}{8})\psi_3 + 2\psi_4 + r(1 - \tfrac{B}{8})/2 - 4\psi_0 \quad (11.12)$$

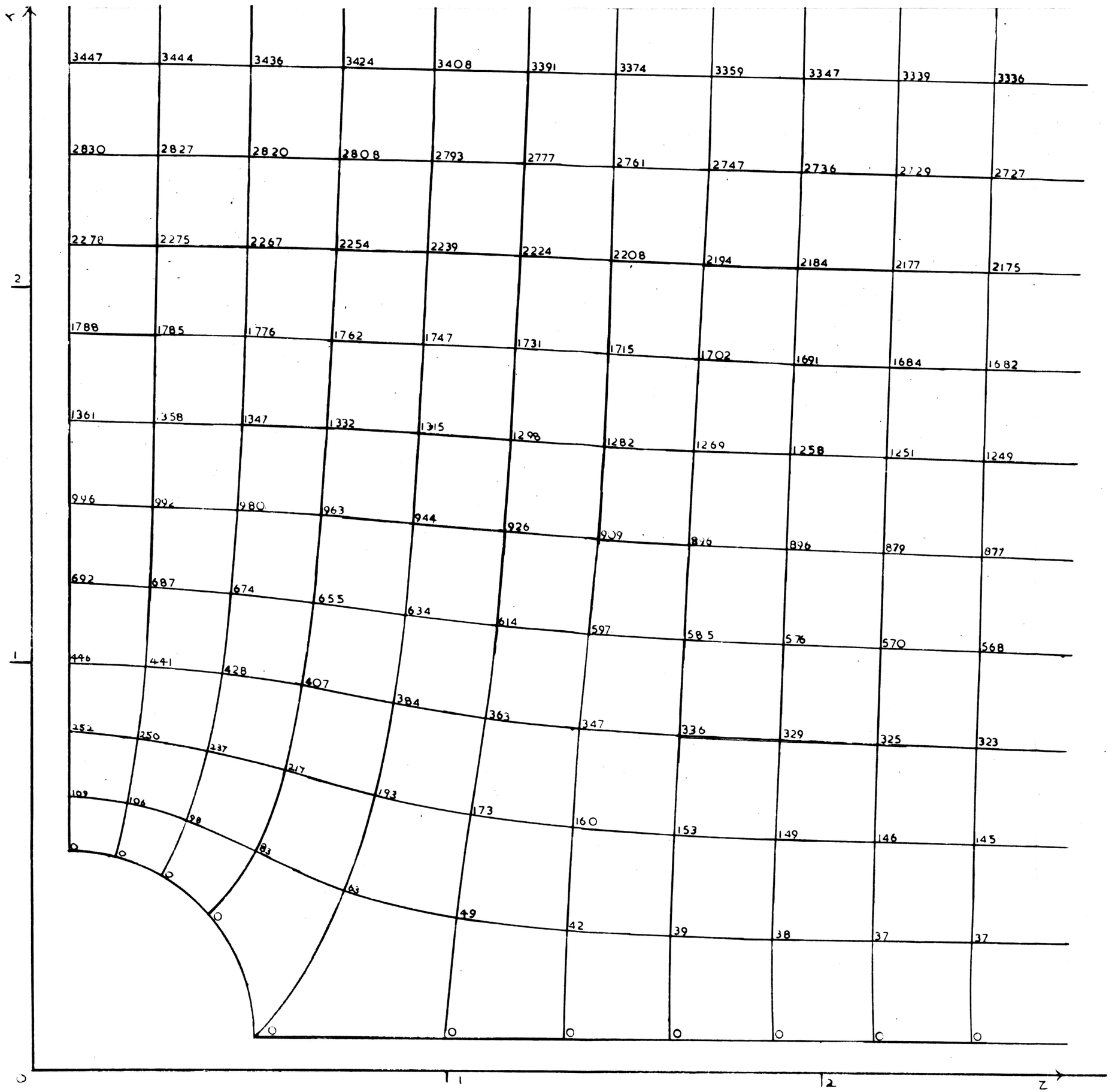
The required values of $r(1 - \frac{B}{8})/2$ are given in Table VI

α \ β	2.5	3	5.75
0	1.238	1.480	2.835
0.25	1.237	1.479	2.835
0.50	1.236	1.479	2.834
0.75	1.234	1.477	2.834
1.00	1.231	1.476	2.834
1.25	1.228	1.474	2.833
1.50	1.225	1.472	2.833
1.75	1.222	1.469	2.833
2.00	1.219	1.467	2.832
2.25	1.216	1.465	2.832
2.50	1.213	1.463	2.831
2.75	1.211	1.461	2.830
3.00	1.208	1.459	2.830
3.25	1.206	1.457	2.829
3.50	1.205	1.455	2.829
3.75	1.203	1.454	2.828
4.00	1.202	1.453	2.827
4.25	1.200	1.452	2.827
4.50	1.199	1.450	2.826
4.75	1.198	1.449	2.826
5.00	1.198	1.448	2.825

Values of $r(1 - \frac{B}{8})/2$

TABLE NO. VI

The solution of equation (11.9) subject to the boundary conditions (11.6), (11.7), (11.8) is given in Fig. 17.



SOLUTION FOR ψ.
IRROTATIONAL MOTION.

FIG. No. 17.

12. General Equation for R = 0

When R = 0 equations (10.6) and (10.7) reduce to

$$\frac{\partial^2\psi}{\partial\alpha^2} + \frac{\partial^2\psi}{\partial\beta^2} + A\frac{\partial\psi}{\partial\alpha} - B\frac{\partial\psi}{\partial\beta} = -\pi/H^2, \qquad \text{.....(12.1)}$$

and

$$\frac{\partial^2\pi}{\partial\alpha^2} + \frac{\partial^2\pi}{\partial\beta^2} + A\frac{\partial\pi}{\partial\alpha} - B\frac{\partial\pi}{\partial\beta} = 0, \qquad \text{.....(12.2)}$$

where

$$H^{-2} = (r^2 + z^2)^2/\{[r^2 + z^2]^2 + (r^2 - z^2 + 1/8)/2\} \qquad \text{.....(12.3)}$$

Values of C/1.6 (= $1/16H^2$) are given in Appendix I.

The boundary conditions for this case are:

on $\alpha = 0$, $\frac{\partial\psi}{\partial\alpha} = \frac{\partial\pi}{\partial\alpha} = 0$, ...(12.4)

since the flow is symmetrical about $\alpha = 0$,

on $\beta = 0$, $0 \leq \alpha < 1$,

$$\psi = 0, \qquad \text{.....(12.5)}$$

and

$$\frac{\partial\psi}{\partial\alpha} = \frac{\partial\psi}{\partial\beta} = \frac{\partial^2\psi}{\partial\alpha^2} = 0, \qquad \text{.....(12.6)}$$

therefore, using (12.1) and (12.6)

$$-C\pi = \frac{\partial^2\psi}{\partial\beta^2}. \qquad \text{.....(12.7)}$$

On $\beta = 0$, $\alpha > 1$, $\psi = 0$, $\pi = r\zeta = 0$, since $r = 0$.

As $\alpha^2 + \beta^2 \to \infty$, $\frac{\partial\psi}{\partial\alpha} = 0$, $\frac{\partial\psi}{\partial\beta} = r$, $\pi \to 0$. .....(12.8)

The finite difference approximations to the equations (12.1) and (12.2) are

$$\psi_1[1 + \tfrac{A}{8}]+\psi_2[1 - \tfrac{B}{8}]+\psi_3[1 - \tfrac{A}{8}] + \psi_4[1 + \tfrac{B}{8}] + \frac{C\pi_o}{16} - 4\psi_o = 0 \quad (12.9)$$

and

$$\pi_1[1 + \tfrac{A}{8}]+\pi_2[1 - \tfrac{B}{8}]+\pi_3[1 - \tfrac{A}{8}]+\pi_4[1 + \tfrac{B}{8}] - 4\pi_o = 0. \quad ....(12.10)$$

We introduce a multiplying factor of 1000 and replace 1000ψ by ψ and 100π by $-\pi$. The residual formulae obtained from equations (12.1) and (12.2) then take the form:

$$F_\psi = \psi_1[1 + \tfrac{A}{8}]+\psi_2[1 - \tfrac{B}{8}]+\psi_3[1 - \tfrac{A}{8}]+\psi_4[1 + \tfrac{B}{8}] - \frac{C\pi_o}{1.6} - 4\psi_o, \quad (12.11)$$

and

$$F_\pi = \pi_1[1 + \tfrac{A}{8}]+\pi_2[1 - \tfrac{B}{8}]+\pi_3[1 - \tfrac{A}{8}]+\pi_4[1 + \tfrac{B}{8}] - 4\pi_o. \quad ....(12.12)$$

The residual formulae for points on $\alpha = 0$ are:-

$$F_\psi = 2\psi_1 + \psi_2[1 - \tfrac{B}{8}] + \psi_4[1 + \tfrac{B}{8}] - \frac{C\pi_o}{1.6} - 4\psi_o, \quad .....(12.13)$$

$$F_\pi = 2\pi_1 + \pi_2[1 - \tfrac{B}{8}] + \pi_4[1 + \tfrac{B}{8}] - 4\pi_o. \quad .....(12.14)$$

The ψ-residual formula for points on the line $\alpha = 17/4$ [x] is

$$F_\psi = \psi_2[1 - \tfrac{B}{8}] + 2\psi_3 + \psi_4[1 + \tfrac{B}{8}] - \frac{C\pi_o}{1.6}. \quad .....(12.15)$$

The ψ-residual formula for points on the line $\beta = 23/4$ [x] is

$$F_\psi = \psi_1[1 + \tfrac{A}{8}] + \psi_3[1 - \tfrac{A}{8}] + 2\psi_4 - 4\psi_o - \frac{C\pi_o}{1.6} + r(1 - \tfrac{B}{8})/2. \quad (12.16)$$

The required values of $r(1 - \frac{B}{8})/2$ for particular values of α are given in Table VI .

The finite difference approximation to (12.7) is

$$-\frac{C\pi_o}{1.6} = +2\psi_2, \text{ i.e. } \pi_o = +\frac{3.2}{C}\psi_2 = 12.8(1 - 4z^2)\psi_2 \quad .....(12.17)$$

x These lines are the external boundaries in the direction $\alpha^2+\beta^2 \to \infty$.

For

$$\alpha = 0,\ z = 0,\ \text{and}\ \pi_0 = +\ 12.8\psi_0.$$
$$\alpha = 1/4,\quad z = 1/8,\ \text{and}\ \pi_0 = +\ 12\psi_0. \qquad \text{.....(12.18)}$$
$$\alpha = 1/2,\quad z = 1/4,\ \text{and}\ \pi_0 = +\ 9.6\psi_0.$$
$$\alpha = 3/4,\ z = 3/8,\ \text{and}\ \pi_0 = +\ 5.6\psi_0.$$

A 'two diagram technique as described in Ref. (3) was employed to obtain a solution to equations (12.11) and (12.12) subject to the boundary conditions (12.13), (12.14), (12.15) (12.16), (12.17) and (12.18).

The first guess was the solution obtained when $\pi = 0$ everywhere. Large positive residuals were obtained at the points $\alpha = 0, 1/4, 1/2, 3/4,\ \beta = 1/4$. These residuals were partly modified (by approximately one quarter of their original value) using the approximate relaxation pattern given in Fig. 18

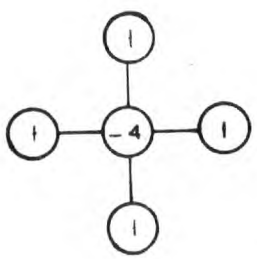


FIG. NO. 18.

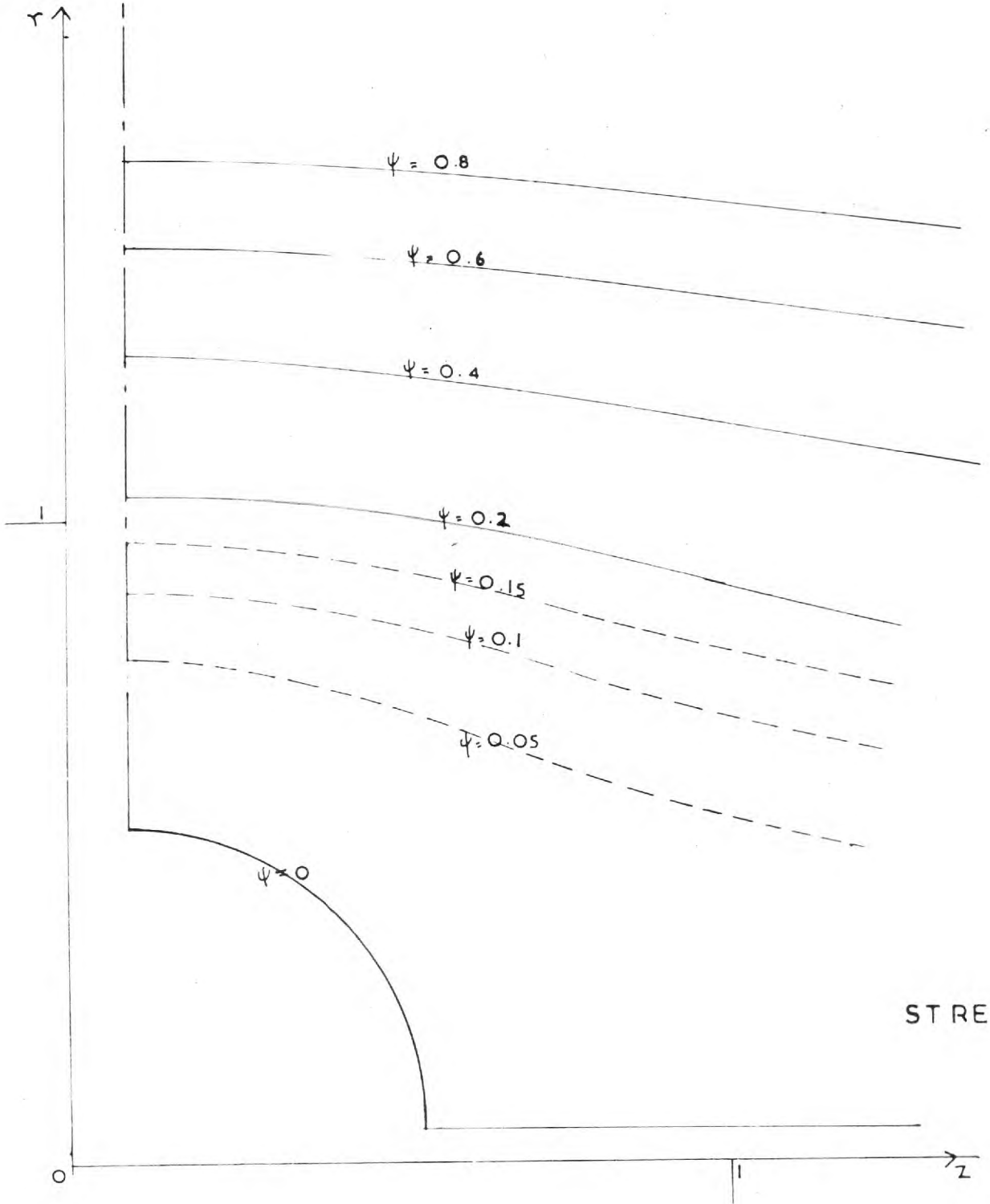
Except at the points $\alpha = 0, 1/4, 1/2, 3/4,\ \beta = 1/4,$

all the π-residuals are 'liquidated'. The ψ-residuals had then altered in accordance with the last term of (12.11). This change was quite large. It should be noticed that if π is changed by 1 then the corresponding ψ residual changes by approximately -0.6. As all the changes in π had been positive all the ψ-residuals were negative, giving a large total negative residual. The only boundary over which this residual could be 'pushed' was $\beta = 0$, $\alpha \geq 1$ since all the other boundaries were subject to a normal gradient condition. The problem of reducing a total negative residual to zero was made easier since a change of 2 in ψ on $\beta = 1/2$ only changed the residual on $\beta = 1/4$ by 1. The easiest way of decreasing the ψ-residuals was by using graded block relaxation. ψ was given constant values along lines $\beta = \text{const.}$ but gradually decreasing in absolute value as β decreased. These changes in ψ were quite large. Consequently the values of ψ at the points $\alpha = 0$, 1/4, 1/2, 3/4, $\beta = 1/4$ changed. As these changed negatively, the value of ψ was less and the corresponding value of π from (12.18) was decreased and the positive π-residual at these 4 points was also decreased. This cycle was then repeated until both the ψ and π-residuals were liquidated. The calculation of residuals in this case was easier than in the solution of the flow through a cylindrical pipe as both equations are linear and the coefficients of π_1, π_2, etc. do not depend on

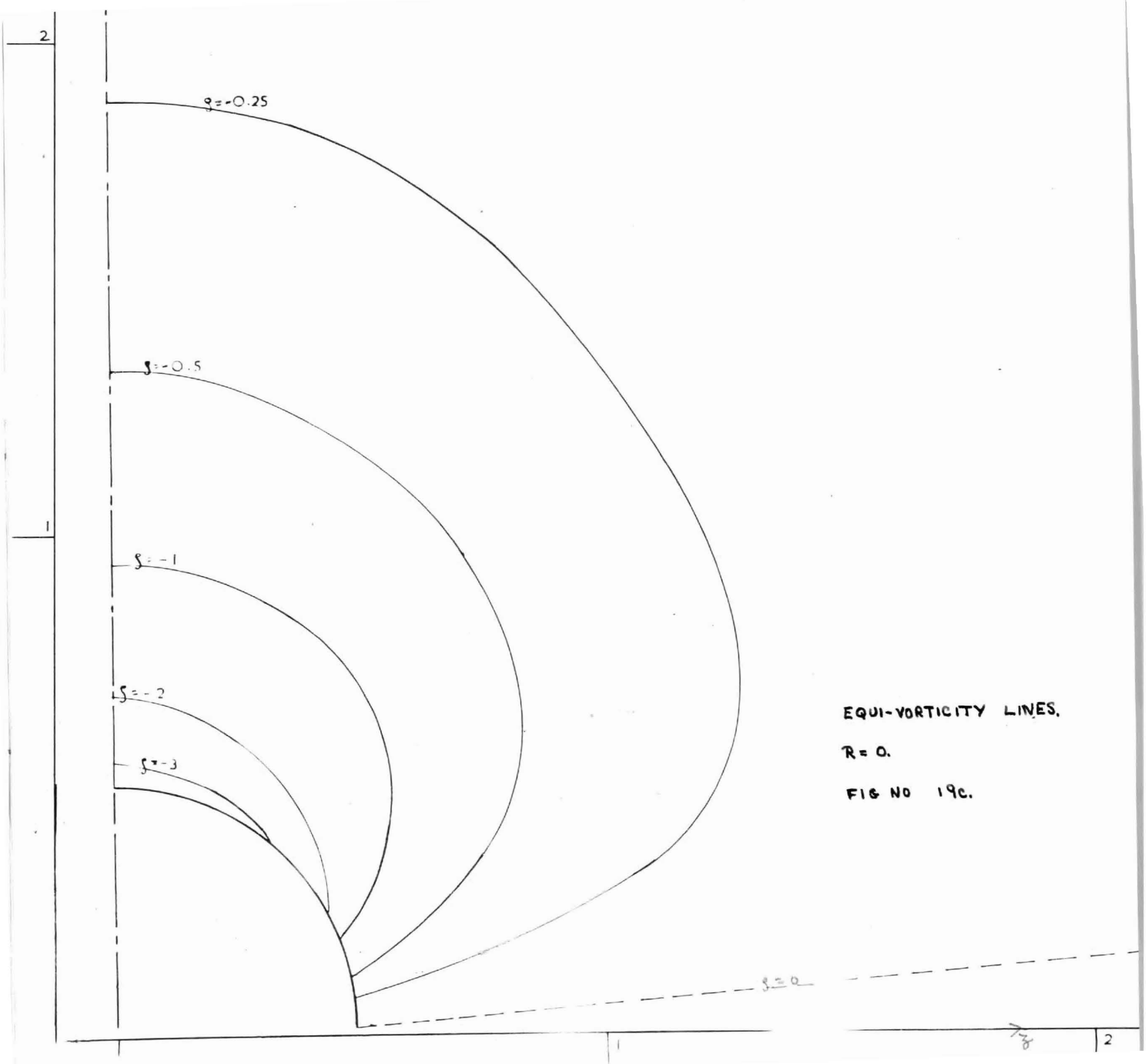
the changes in ψ.

The amount of π-residual to be left at the four points could only be estimated with any degree of accuracy after the cycle had been repeated several times. The tendency was to leave too large π-residuals and to under-relax ψ. It was generally possible to have a total positive ψ-residual but this was not sufficiently large. Though the convergence was slow, by proceeding in this way, the functions ψ and π did not oscillate about their true values.

The solution of the equation (12.11) and (12.12) subject to the boundary conditions is given in Fig. 19a.



STREAMLINES. R=0.
FIG. NO. 19b.



EQUI-VORTICITY LINES.

R = 0.

FIG NO 19c.

13. Finite Difference Equations for flow round the sphere with non-zero R

To obtain the governing equations in a form which can be used to solve the problem in $(\alpha - \beta)$ plane we substitute for $\frac{\partial^2\psi}{\partial r^2}$, $\frac{\partial^2\psi}{\partial z^2}$, etc. from (11.3) and (11.4) in (10.6) and (10.7). This yields

$$\frac{\partial^2\psi}{\partial\alpha^2} + \frac{\partial^2\psi}{\partial\beta^2} + A\frac{\partial\psi}{\partial\alpha} - B\frac{\partial\psi}{\partial\beta} = -C\pi, \qquad \text{.....(13.1)}$$

and

$$\frac{\partial^2\pi}{\partial\alpha^2} + \frac{\partial^2\pi}{\partial\beta^2} + A\frac{\partial\pi}{\partial\alpha} - B\frac{\partial\pi}{\partial\beta} = R\left[\frac{\partial\pi}{\partial\alpha}\frac{\partial\psi}{\partial\beta} - \frac{\partial\pi}{\partial\beta}\frac{\partial\psi}{\partial\alpha} + \frac{2\pi}{r}\left[A\frac{\partial\psi}{\partial\beta} + B\frac{\partial\psi}{\partial\alpha}\right]\right]$$

.....(13.2)

The flow is no longer symmetrical about the axis $\alpha = 0$. Consequently the area of solution is extended to include points for which $\alpha < 0$.

The boundary conditions are thus given by:

(i) as $\alpha \to -\infty$ $\quad \frac{\partial\psi}{\partial\alpha} = 0$, and $\pi \longrightarrow 0$, ...(13.3)

(ii) as $\alpha \to +\infty$ $\quad \frac{\partial\psi}{\partial\alpha} = 0$, and $\pi \longrightarrow 0$, ...(13.4)

(iii) as $\beta \to +\infty$ $\quad \frac{\partial\psi}{\partial\beta} = r$, and $\pi \longrightarrow 0$, ...(13.5)

(iv) on $\beta = 0$, $-1 < \alpha < 1$, $\psi = 0$, and $\frac{\partial\psi}{\partial\beta} = 0$,

$$C\pi = -\frac{\partial^2\psi}{\partial\beta^2} \qquad \text{....(13.6)}$$

(v) $\quad |\alpha| \geq 1 \quad \psi = 0$, and $\pi = 0$.

The finite-difference approximations to equations (13.1) and (13.2) are:-

$$\psi_1[1+\tfrac{A}{8}]+\psi_2[1-\tfrac{B}{8}]+\psi_3[1-\tfrac{A}{8}]+\psi_4[1+\tfrac{B}{8}]-4\psi_o + C\pi_o/16 = 0, \quad (13.7)$$

and

$$\pi_1[1+\tfrac{A}{8}]+\pi_2[1-\tfrac{B}{8}]+\pi_3[1-\tfrac{A}{8}]+\pi_4[1+\tfrac{B}{8}]-4\pi_o - \tfrac{R}{4}(\pi_1-\pi_3)(\psi_2-\psi_4)+\tfrac{R}{4}(\pi_2-\pi_4)(\psi_1-\psi_3) - \tfrac{\pi_o R}{4r}[A(\psi_2-\psi_4)+B(\psi_1-\psi_3)]=0, \quad .....(13.8)$$

and from equation (13.6)

on $\beta = 0$, $-1 < \alpha < 1$,

we find $\pi_o = -128(1-4z^2)\psi_2$. .....(13.9)

We multiply equations (13.7), (13.8) and (13.9) by 1000 and replace 1000ψ by ψ and 100π by $-\pi$. We therefore have the following residual formulae:-

$$F_\psi = \psi_1[1+\tfrac{A}{8}]+\psi_2[1-\tfrac{B}{8}]+\psi_3[1-\tfrac{A}{8}]+\psi_4[1+\tfrac{B}{8}]-4\psi_o-C\pi_o/1.6 \quad .....(13.10)$$

$$F_\pi = \pi_1[1+\tfrac{A}{8}-\tfrac{R}{4000}(\psi_2-\psi_4)]+\pi_2[1-\tfrac{B}{8}+\tfrac{R}{4000}(\psi_1-\psi_3)] +\pi_3[1-\tfrac{A}{8}+\tfrac{R}{4000}(\psi_2-\psi_4)]+\pi_4[1+\tfrac{B}{8}-\tfrac{R}{4000}(\psi_1-\psi_3)] -\pi_o[4+\tfrac{AR}{4000r}(\psi_2-\psi_4)+\tfrac{BR}{4000r}(\psi_1-\psi_3)] \quad .....(13.11)$$

From equation (13.9) we have:

(1) for $\alpha = \pm 3/4$, and $\beta = 0$, $\pi_o = +5.6\psi_2$.

(2) for $\alpha = \pm 1/2$, and $\beta = 0$, $\pi_o = +9.6\psi_2$.

(3) for $\alpha = \pm 1/4$, and $\beta = 0$, $\pi_o = +12\psi_2$. .....(13.12)

(4) for $\alpha = 0$, and $\beta = 0$, $\pi_o = 12.8\psi_o$.

On the boundary $\alpha = 5$ we satisfy the condition (13.4). In terms of finite differences this gives $\psi_1 = \psi_3$. Thus the residual formula for points along this line is

$$F_\psi = 2\psi_3 + \psi_2[1 - \tfrac{B}{8}] + \psi_4[1 + \tfrac{B}{8}] - 4\psi_o - C\pi_o/1.6. \quad .....(13.13)$$

On the line $\beta = 5.75$ we satisfy the conditions (13.5) on this boundary. This gives for the fictitious value of ψ, $\psi_2 = \psi_4 + r/2$. Hence the residual formula for points on this line is

$$F_\psi = \psi_1[1 + \tfrac{A}{8}] + \psi_3[1 - \tfrac{B}{8}] + 2\psi_4 + \tfrac{1}{2}r(1 - \tfrac{B}{8}) - C\pi_o/1.6. \quad ..(13.14)$$

The values of $r(1 - \frac{B}{8})/2$ which are required in the solution of equations (13.10) and (13.11) are given in Table VI.

14. Solution for the case R = 1

The only equation which depends on the value of the Reynolds' number is equation (13.11). When R = 1 it becomes

$$F_\pi = \pi_1[1 + \frac{A}{8} - \frac{1}{4000}(\psi_2-\psi_4)] + \pi_2[1 - \frac{B}{8} + \frac{1}{4000}(\psi_1-\psi_3)]$$
$$+\pi_3[1 - \frac{A}{8} + \frac{1}{4000}(\psi_2-\psi_4)] + \pi_4[1 + \frac{B}{8} - \frac{1}{4000}(\psi_1-\psi_3)]$$
$$-\pi_0[4 + \frac{A}{4000r}(\psi_2-\psi_4) + \frac{B}{4000r}(\psi_1-\psi_3)] \qquad .....(14.1)$$

Initially the values of ψ were taken to be those obtained for the case R = 0. These values were symmetrical about the line $\alpha = 0$. The values of π were modified as it was obvious from equation (14.1) that in this example π would tend to zero more quickly in the direction $\alpha \to -\infty$ than in the direction $\alpha \to +\infty$. In equation (14.1), the difference $\psi_1-\psi_3$ was small at all points compared with 4000. Hence the contributions from the terms π_2, π_4 to F_π would be approximately the same as when R = 0. The difference $(\psi_2-\psi_4)$, on the other hand, varied from about 1000 to 3000 on the lines $\beta = 2$ to $\beta = \frac{23}{4}$. This means that the coefficient of π_1 is smaller and the coefficient of π_3 greater than was the case for R = 0. For R = 0, π was a decreasing function as α increased from $\alpha = 0$. Consequently for $\alpha > 0$, $\pi_1 < \pi_3$. Thus, for $\alpha > 0$, the values of F_π would have been positive here; for $\alpha < 0$, however, $\pi_3 < \pi_1$

and the values of F_π would have been negative, if π had been taken in this case to be the same as had been obtained for $R = 0$. The values of π taken from the solution for $R = 0$ were therefore initially increased at the points for which $\alpha > 1$ and made to tend to zero more slowly. The boundary also was extended from $\alpha = 17/4$ to $\alpha = 5$. At the points for which $0 < \alpha < 1$ the values of π were decreased slightly so that π remained a smooth function. For $\alpha < 0$ the function π was made to decrease rapidly so that it took the value zero when $\alpha = -3$. The values of F_ψ were then calculated and the values of ψ were modified so as to (temporarily) liquidate them F_ψ. The changes in the values of ψ at the points $\alpha = \pm 3/4, \pm 1/2, \pm 1/4, 0$, $\beta = 1/4$ were small. The coefficients $[1 + \frac{A}{8} - \frac{1}{4000}(\psi_2 - \psi_4)]$ etc. in equation (14.1) were then calculated. (A typical set which has been obtained from the final solution is given in Table. VII . Others can be obtained from the author if required.) The values of F_π were then calculated. It was found that there was still a large total positive residual in the region $\alpha > 1$ and a total negative residual in the region $\alpha < 1$. As the total positive residual was larger than the negative, the function was altered positively at first. It soon became apparent that the area of solution would have to be extended in the α-direction.

$\alpha = 3/2$

COEFFICIENTS OF

β	π_1	π_2	π_3	π_4	π_0
0	-	-	-	-	-
1/4	1.01218	0.51056	0.98782	1.48944	4.03455
1/2	0.97892	0.76399	1.02108	1.23601	4.02132
3/4	0.94419	0.84708	1.05581	1.15292	4.01334
1	0.90927	0.88677	1.09073	1.11323	4.01073
5/4	0.87395	0.90948	1.12605	1.09052	4.00683
3/2	0.83799	0.92371	1.16201	1.07629	4.00735
7/4	0.80223	0.93300	1.19777	1.06700	4.00007
2	0.76663	0.93957	1.23337	1.06043	3.99970
9/4	0.73106	0.94386	1.26894	1.05614	3.99935
5/2	0.69616	0.94691	1.30384	1.05309	3.99916
11/4	0.66187	0.94956	1.33813	1.05044	3.99919
3	0.62767	0.95179	1.37233	1.04821	3.99904
13/4	0.59402	0.95375	1.40598	1.04625	3.99918
7/2	0.56091	0.95580	1.43909	1.04420	3.99924
15/4	0.52807	0.95778	1.47193	1.04222	3.99922
4	0.49576	0.95948	1.50424	1.04052	3.99920
17/4	0.46371	0.96122	1.53629	1.03878	3.99960
9/2	0.43217	0.96252	1.56783	1.03748	3.99959
19/4	0.40089	0.96416	1.59911	1.03584	3.99960
5	0.36937	0.96541	1.63063	1.03459	3.99960
21/4	0.33810	0.96680	1.66190	1.03320	3.99961
11/2	0.30708	0.96784	1.69292	1.03216	3.99961
23/4				1.03121	3.99961

TABLE No. VII

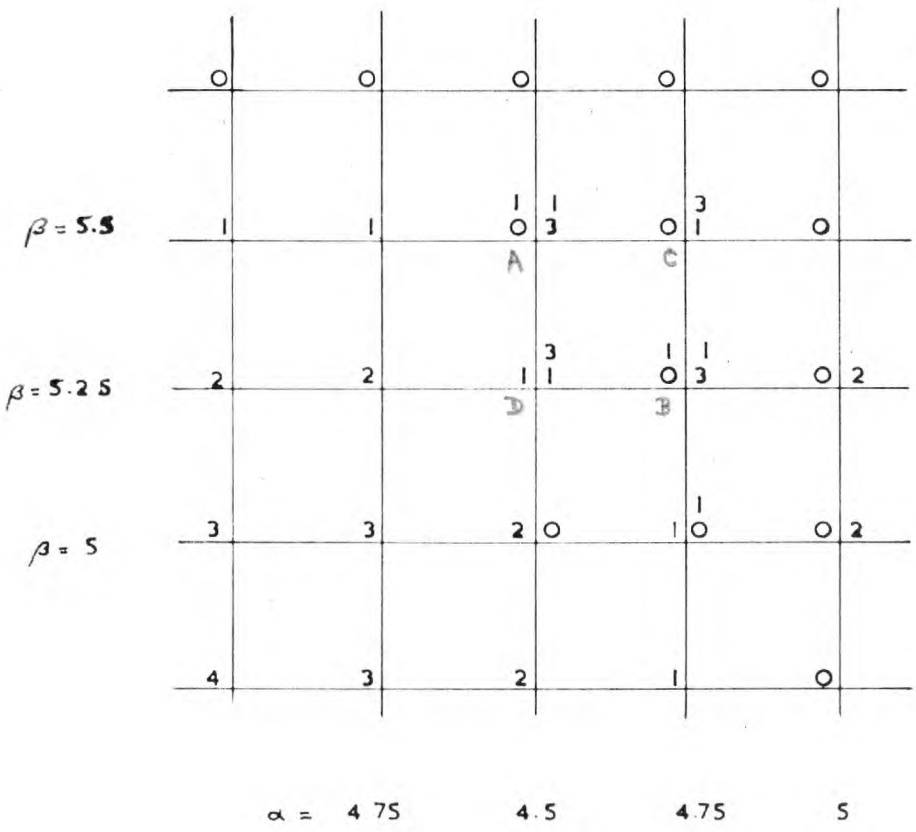
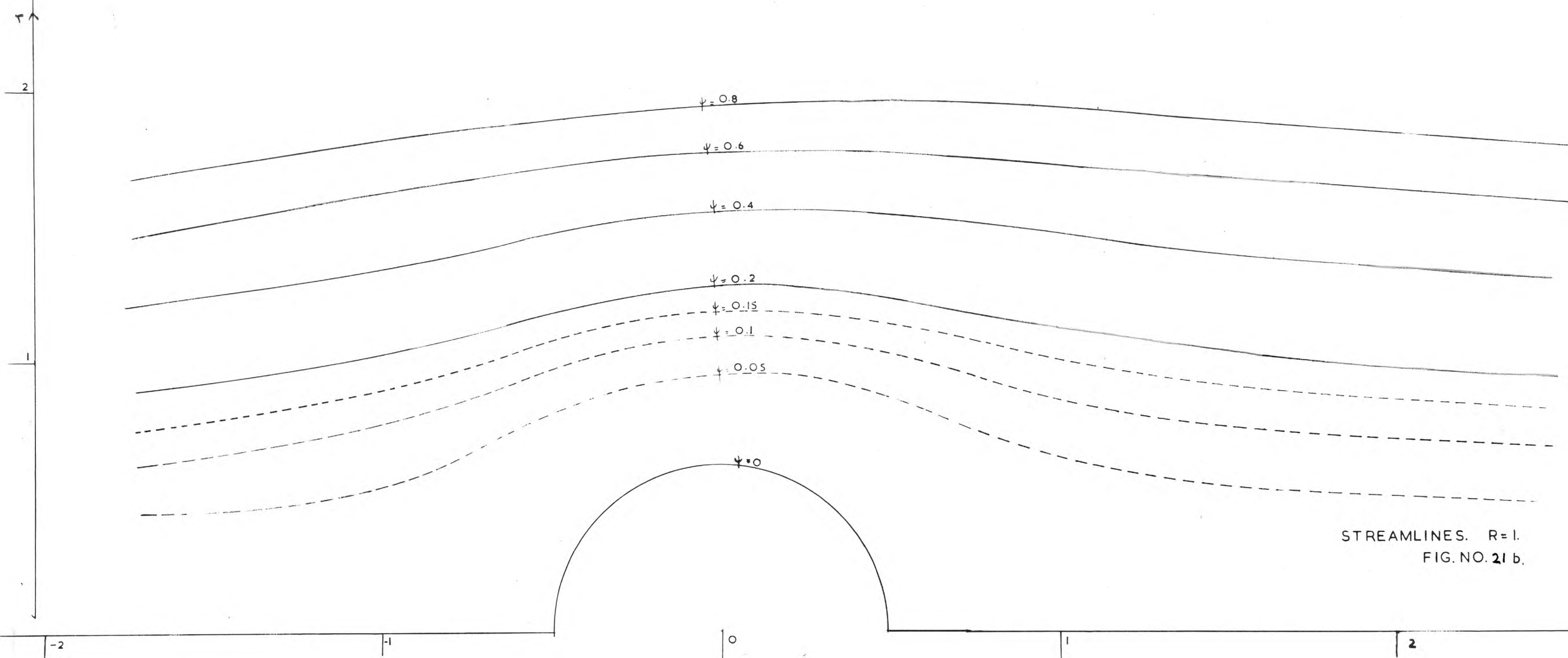


FIG. NO. 20.

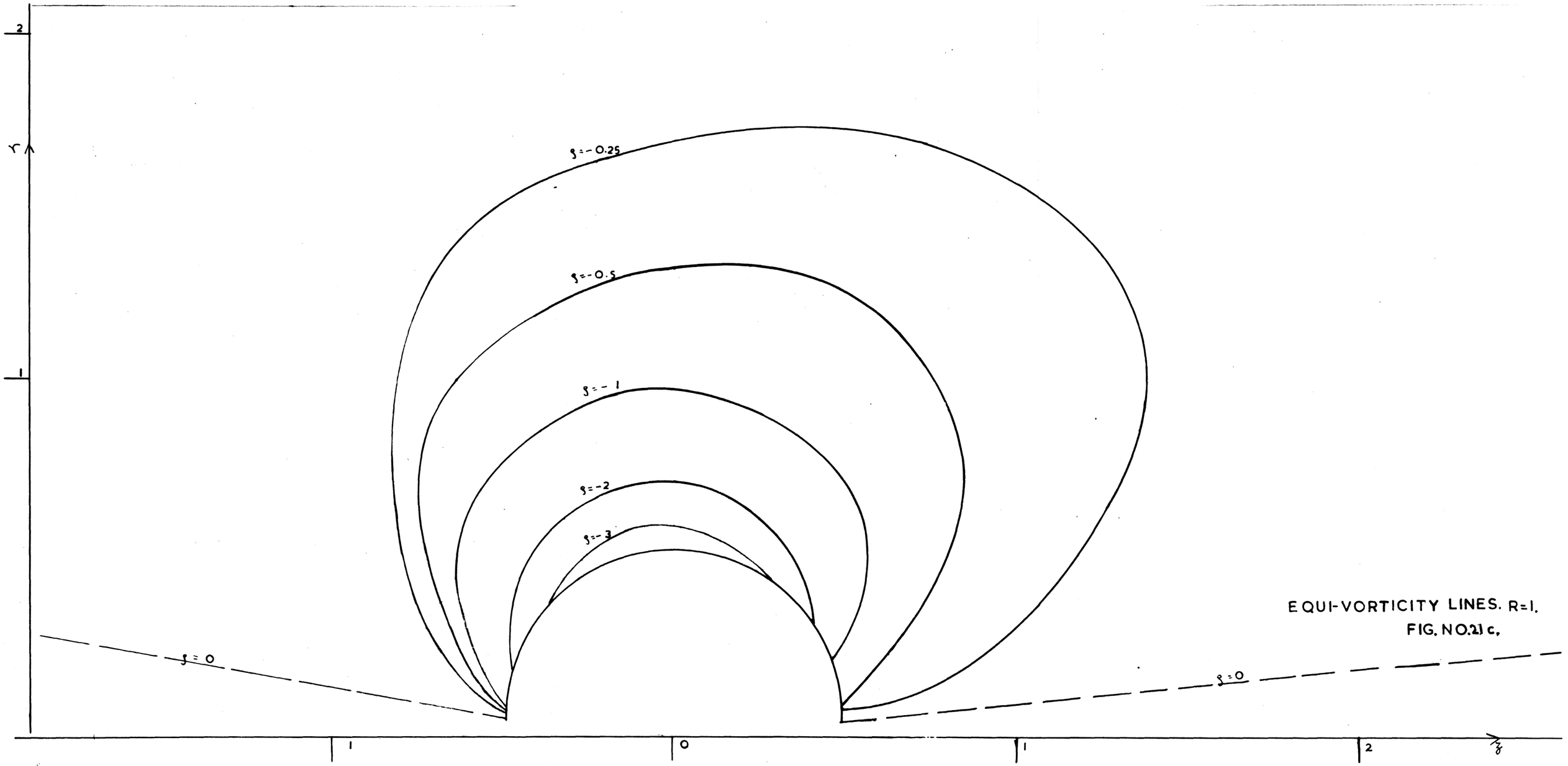
Consider Fig. 20 . The function π is made to tend to zero as $\alpha \longrightarrow 5$, $\beta \rightarrow 23/4$. The residuals are as given. An increment $\psi = 1$ must be made at points A and B, giving at C and D other residuals of 3 (approx.). These in their turn were relaxed and in so doing, it was found that the area had again to be extended. Hence the condition $\frac{\partial \pi}{\partial \alpha} = 0$ on the boundary $\alpha = 5$, was taken as being more

appropriate than $\pi = 0$.

The values of ψ and ζ which were obtained as a solution to equations (13.10) and (14.1) are given in Fig. 21a Figs. 21b and 21c show the stream-lines and the equi-vorticity lines.



STREAMLINES. R = 1.
FIG. NO. 21 b.



EQUI-VORTICITY LINES. R=1.
FIG. NO.21 c.

15. Case R = 10

Equation (13.11) now takes the form

$$F_\pi = \pi_1[1 + \frac{A}{8} - \frac{1}{400}(\psi_2 - \psi_4)] + \pi_2[1 - \frac{B}{8} + \frac{1}{400}(\psi_1 - \psi_3)]$$

$$+\pi_3[1 - \frac{A}{8} + \frac{1}{400}(\psi_2 - \psi_4)] + \pi_4[1 + \frac{B}{8} - \frac{1}{400}(\psi_1 - \psi_3)]$$

$$-\pi_o[4 + \frac{A}{400r}(\psi_2 - \psi_4) + \frac{B}{400r}(\psi_1 - \psi_3)]. \qquad .....(15.1)$$

The values of ψ and π obtained as a solution of equations (13.10) and (14.1) were taken as the initial guess for this case, and the coefficients $[1 + \frac{A}{8} - \frac{1}{400}(\psi_2-\psi_4)]$ etc. from (15.1) were calculated. Here, the difference $(\psi_2-\psi_4)$ was much greater than 400. So, along the lines for which π was an increasing function, in particular in the area $3 \leq \beta \leq 5$, $-3 \leq \alpha \leq 6$, F_π was found initially to be very negative. When these residuals were relaxed, it was found that $\pi \longrightarrow 0$ for $\beta = 7/2$, $\alpha = -2$. The positive residual in the region $0 \leq \beta \leq 2$, $\alpha > 1$ was relaxed before the F_ψ were recalculated. These were then 'liquidated' and the coefficients in equation (15.1) recalculated and the whole process repeated. A typical set of coefficients is given in Table VIII . In this case as in the previous examples it was advisable to build up residuals at the points $\alpha = \pm 3/4, \pm 1/2, \pm 1/4, 0$ $\beta = 1/4$ for both functions and then relax the two together.

The solution appears in Fig. 22a.

$\alpha = 3/2$

β	COEFFICIENTS OF π_1	π_2	π_3	π_4	π_0
1/4	0.83468	0.51931	1.16532	1.48069	4.31104
1/2	0.55942	0.77924	1.44058	1.22076	4.20205
3/4	0.21169	0.85383	1.78831	1.14617	4.12449
1	-0.18073	0.86853	2.18073	1.13147	4.06807
5/4	-0.58130	0.85923	2.58130	1.14077	4.03259
7/2	-0.95572	0.84946	2.95572	1.15054	4.01305
7/4	-1.29702	0.84525	3.29702	1.15475	4.00379
2	-1.61537	0.84582	3.61537	1.15418	3.99932
9/4	-1.92094	0.84711	3.92094	1.15289	3.99700

TABLE No. VIII

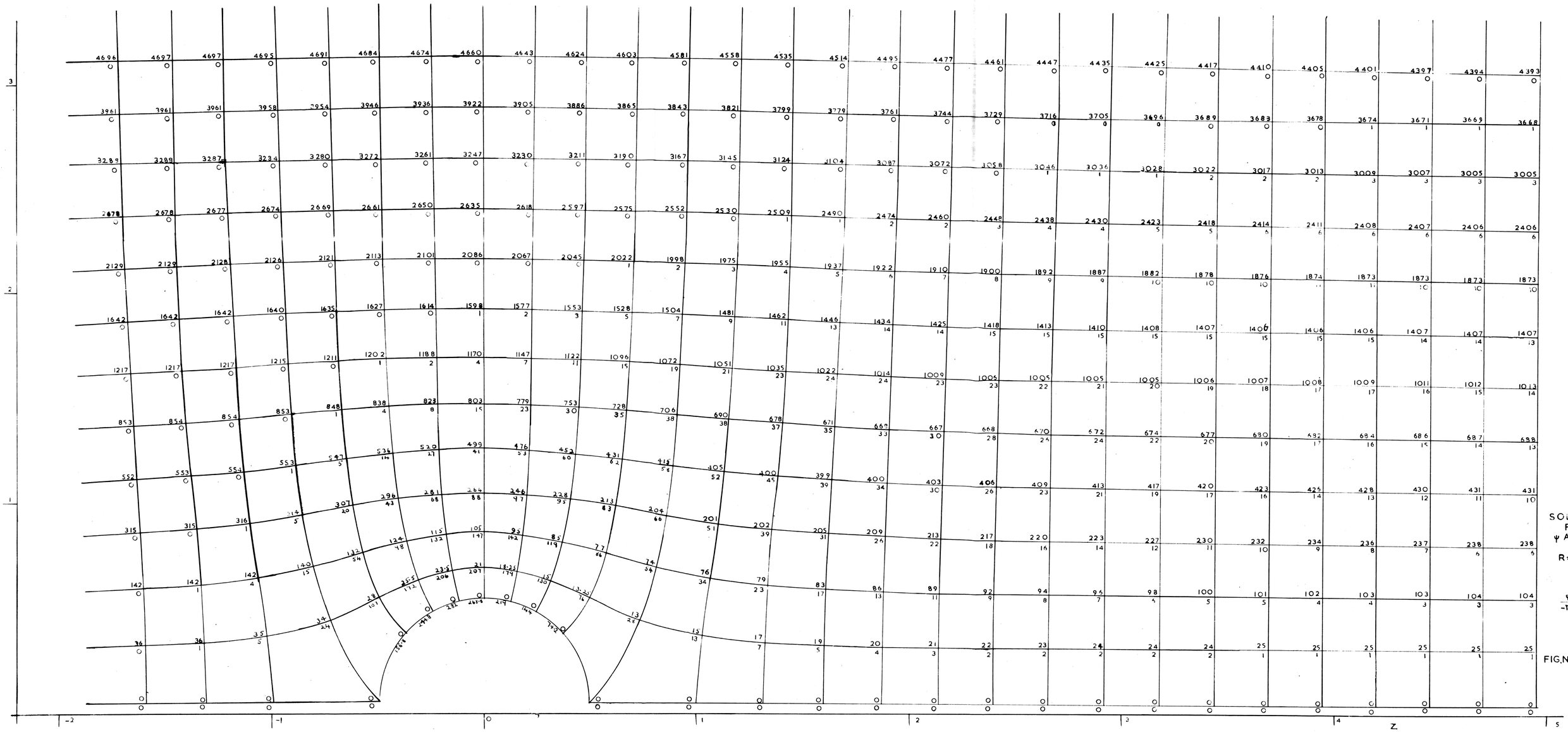
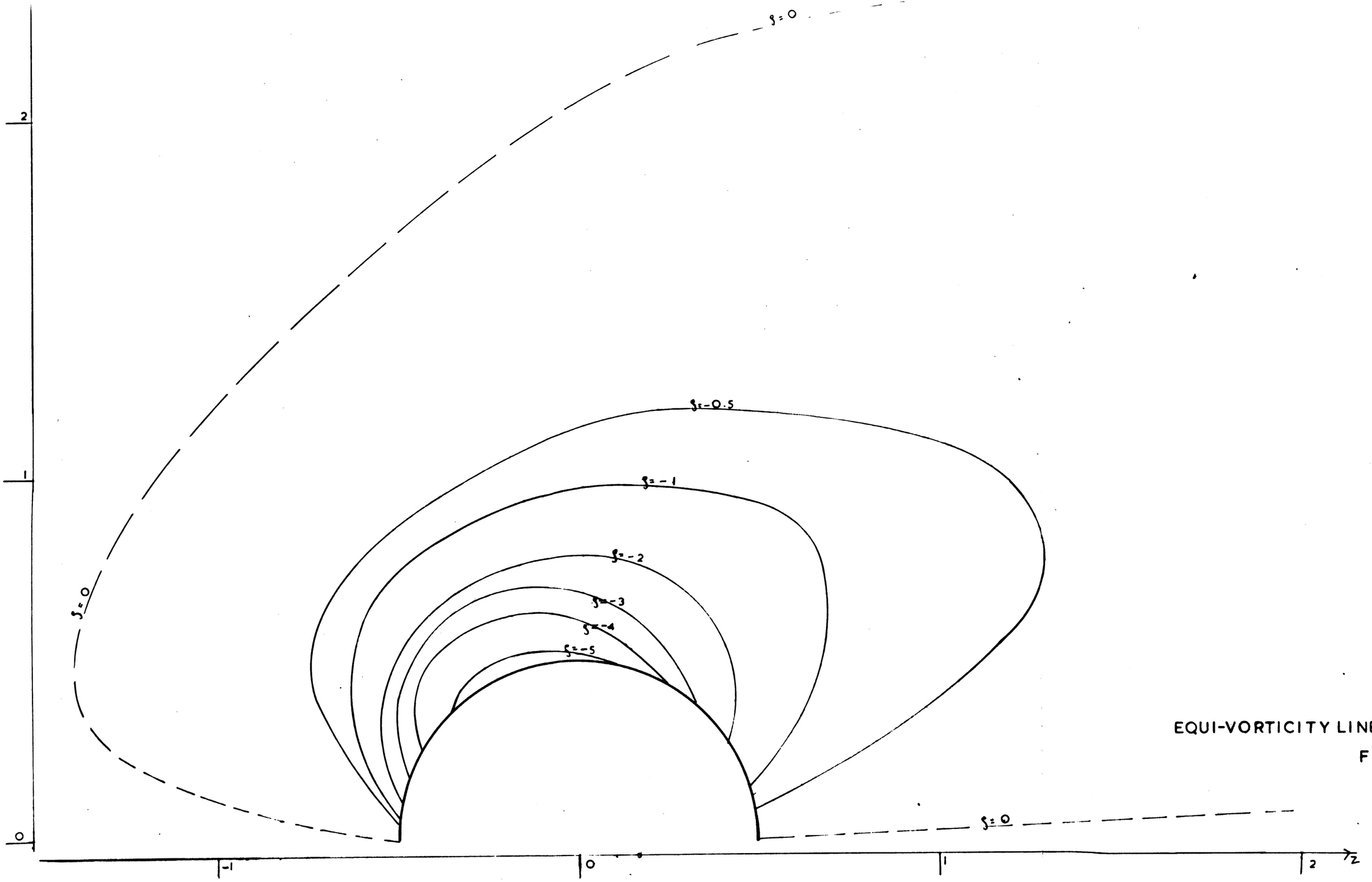


FIG.NO.22a.



EQUI-VORTICITY LINES. R = 10.

FIG. NO. 22c.

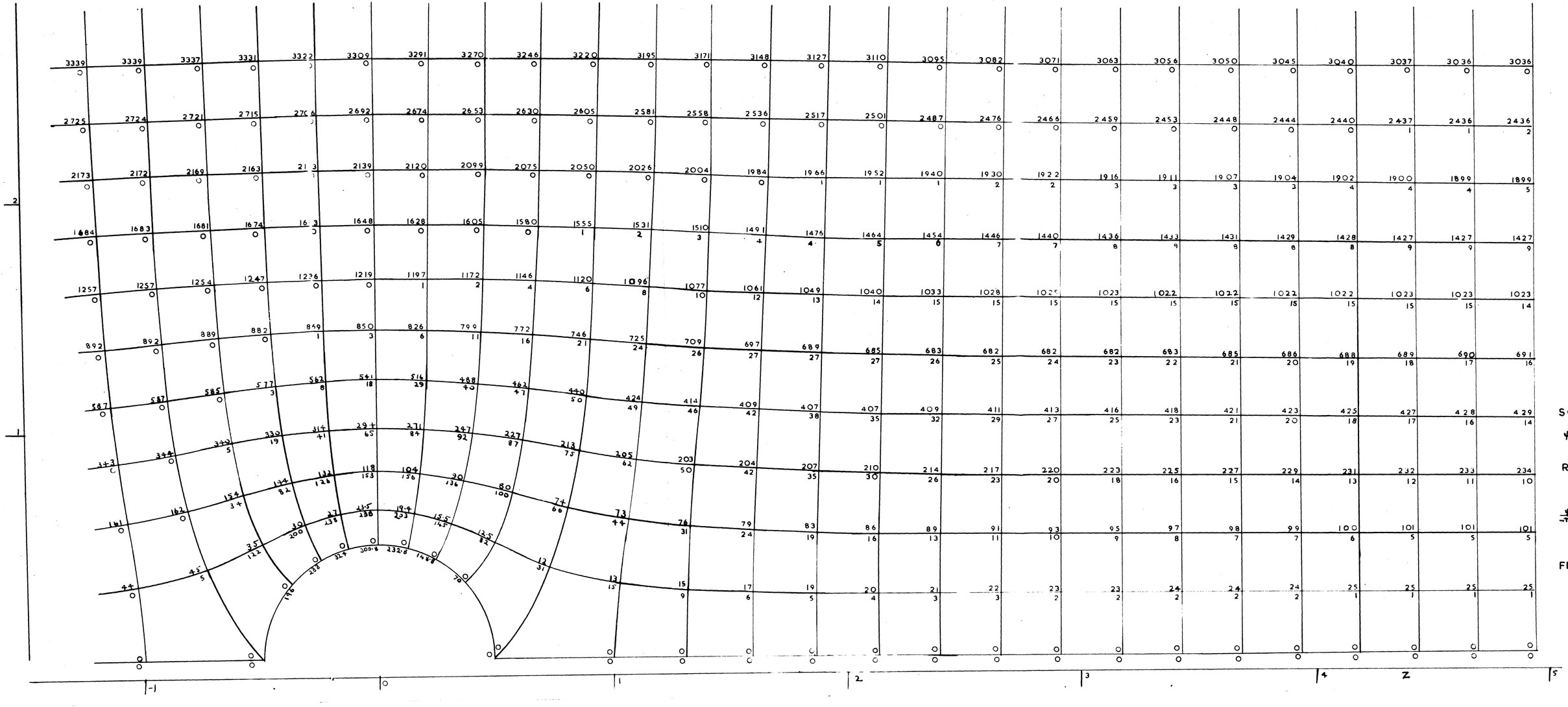
15a. Case R = 20

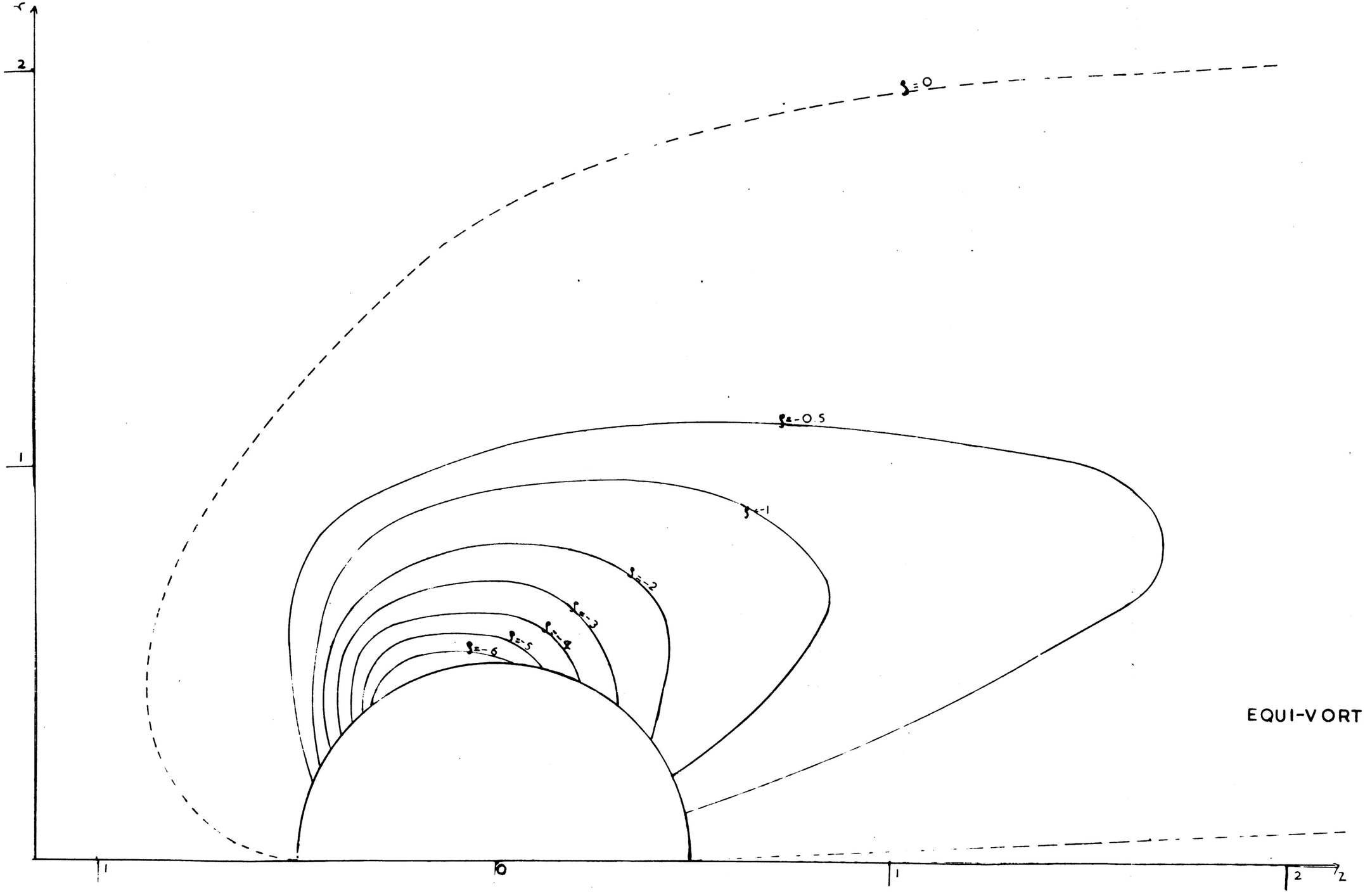
Finally a solution was obtained for R = 20. The values of ψ and π for the case R = 10 were taken as the initial guess. The coefficients $[1 + \frac{A}{8} - \frac{1}{200}(\psi_2 - \psi_4)]$ were calculated. (A typical set of these coefficients is given in Table IX). The values of E_{ψ} were then obtained. These residuals were not very large and a solution was obtained quite quickly. The values of the functions ψ and π which satisfy the equations of viscous flow for R = 20 are given in Fig. 23a. The streamlines and equi-vorticity lines have also been drawn and are given in Figs. 23b and 23c.

$\alpha = 3/2$

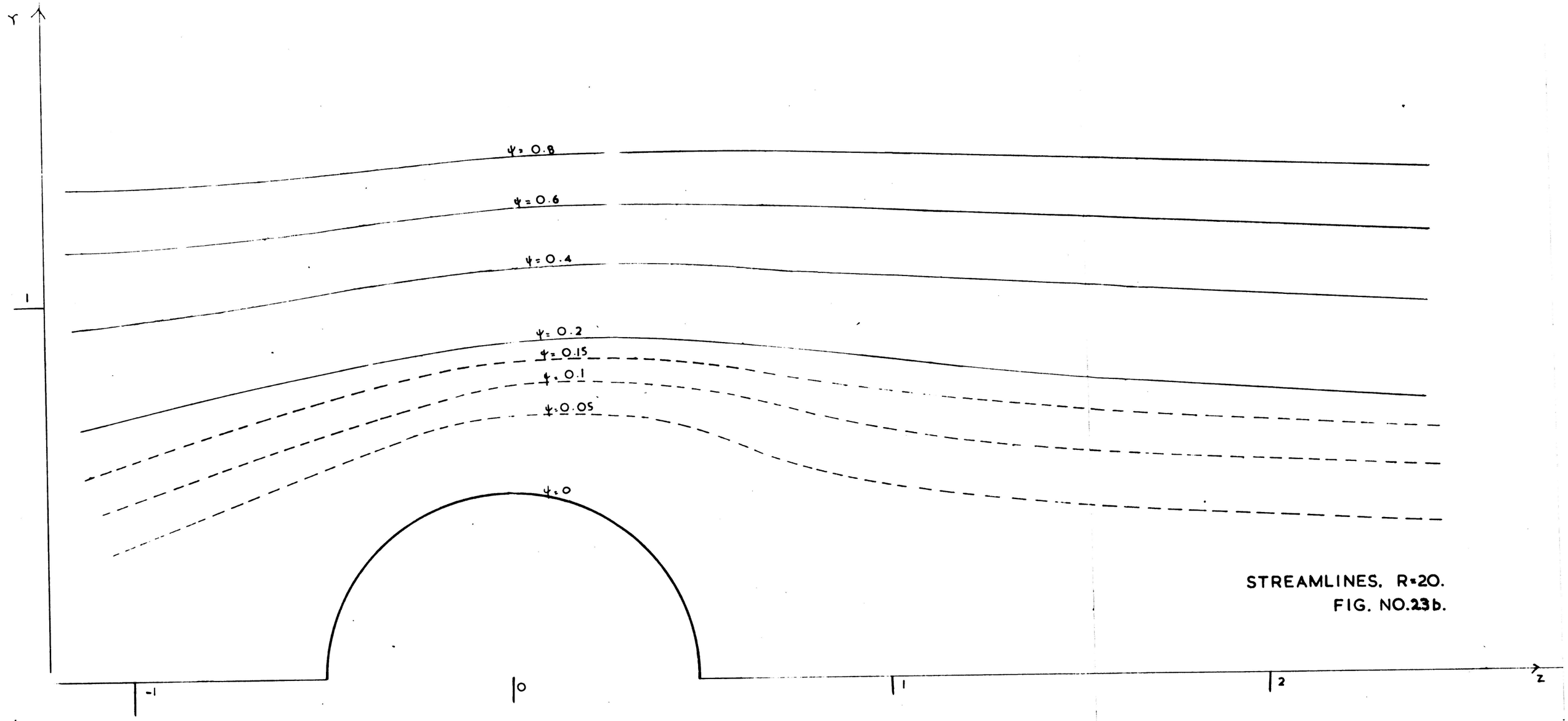
	COEFFICIENTS OF				
β	π_1	π_2	π_3	π_4	π_0
1/4	0.65218	0.52931	1.34782	1.47069	4.60867
1/2	0.08692	0.78674	1.91308	1.21326	4.37336
3/4	-0.67581	0.83883	2.67581	1.16117	4.22334
1	-1.52073	0.80853	3.52073	1.19147	4.10829
5/4	-2.30880	0.77173	4.30880	1.22827	4.04741
3/2	-3.00076	0.74696	5.00076	1.25304	4.01686
7/4	-3.63202	0.73275	5.63202	1.26725	4.00162

TABLE NO. IX


r
z
-1
1
2
3
4
5
SOLUTION FOR ψ AND π.
R=20
ψ/π
FIG. No. 23a.



EQUI-VORTICITY LINES. R=20.
FIG. NO.23c.



STREAMLINES. R=20.
FIG. NO.23b.

16. The algebraic solution for irrotational flow past a sphere

For irrotational flow round a sphere the governing equation is

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0 \qquad \text{.....(16.1)}$$

It can be shown that the function ψ, given by

$$\psi = r^2\left[1 - \frac{1}{8(r^2 + z^2)^{3/2}}\right]/2 \qquad \text{.....(16.2)}$$

satisfies (16.1) and the boundary conditions of §9.

In Figure 24 the values of ψ are given at the intersections of the $\alpha - \beta$ curves for $0 \leq \alpha \leq 11/4$, $0 \leq \beta \leq 11/4$. These numbers can be compared with the values of ψ obtained using relaxation methods given in Fig. 17 It can be seen that in general the values of ψ given in Fig. 17 differ by approximately 2 per cent from their true values. This discrepancy can be accounted for in one of two ways:

(1) either the finite difference approximation to the original differential equation is not sufficiently accurate, or

(2) the finite difference approximation to the boundary conditions is not good.

In Fig. 24 the values of F_ψ, where

$$F_\psi = (1 + \tfrac{A}{8})\psi_1 + (1 - \tfrac{B}{8})\psi_2 + (1 - \tfrac{A}{8})\psi_3 + (1 + \tfrac{B}{8})\psi_4 - 4\psi_0 \qquad \text{.....(16.3)}$$

are also recorded. These residuals are approximately zero and

are reasonably well distributed. Only near the α-axis is there a residual greater than 2. This is due to rounding off errors.

On the curves $\alpha = 5/2$, $\beta = 5/2$, there are three residuals given. The lowest one is obtained when we apply either the condition $\frac{\partial \psi}{\partial r} = r$ or the condition $\frac{\partial \psi}{\partial z} = 0$, to eliminate the values of ψ at the fictitious points. These residuals have a high positive value and are obviously the source of the error obtained in our solution for ψ. The reason is that the finite difference approximation to the boundary conditions, viz.

$$\frac{\partial \psi}{\partial z} = \frac{\psi_1 - \psi_3}{2h} = 0, \qquad \frac{\partial \psi}{\partial r} = \frac{\psi_2 - \psi_4}{2h} = r, \qquad \ldots\ldots(16.4)$$

does not give the correct relation between the values of ψ for uniform flow. This inaccuracy is due probably to the finite difference approximation to the first order derivative. A better value for the derivative can be obtained by using a difference table as suggested by Fox (Ref. 5) or by expanding ψ in a power series, which includes terms of the fourth order as on page 27 .

As an alternative method, we introduce different boundary conditions. These include only second order derivatives. (The finite difference approximation to these derivatives is more accurate than that to the first order

derivatives which is used in equation (16.4)).

In this problem the boundary conditions can be written in the form

$$\frac{\partial^2 \psi}{\partial z^2} = 0, \qquad \text{.....(16.5)}$$

$$\frac{\partial^2 \psi}{\partial r^2} = 1. \qquad \text{.....(16.6)}$$

In terms of the $(\alpha - \beta)$ coordinates, equation (16.6) becomes

$$\left(\frac{\partial \alpha}{\partial r}\right)^2 \left(\frac{\partial^2 \psi}{\partial \alpha^2}\right) + 2\frac{\partial \alpha}{\partial r}\frac{\partial \beta}{\partial r}\frac{\partial^2 \psi}{\partial \alpha \partial \beta} + \left(\frac{\partial \beta}{\partial r}\right)^2 \frac{\partial^2 \psi}{\partial \beta^2} + \frac{\partial^2 \alpha}{\partial r^2}\frac{\partial \psi}{\partial \alpha} + \frac{\partial^2 \beta}{\partial r^2}\frac{\partial \psi}{\partial \beta} = 1.$$

It can be shown that

$$\frac{\partial^2 \alpha}{\partial r^2} = \frac{-z}{2(r^2 + z^2)^2} + \left(\frac{2zr^2}{(r^2 + z^2)^3}\right),$$

and $$\frac{\partial^2 \beta}{\partial r^2} = \frac{r}{2(r^2 + z^2)^2} - \frac{(r^2 - z^2)}{(r^2 + z^2)^3}.$$

The expressions for the other coefficients are given by (10.14) It can be seen that when $r^2 + z^2 \to \infty$ the only non-zero coefficients are $\frac{\partial \alpha}{\partial z}$ and $\frac{\partial \beta}{\partial r}$ and these both have the value 1.

Hence when the flow is uniform

$$\frac{\partial^2 \psi}{\partial \alpha^2} \to \frac{\partial^2 \psi}{\partial z^2}, \quad \frac{\partial^2 \psi}{\partial \beta^2} \to \frac{\partial^2 \psi}{\partial r^2}. \qquad \text{.....(16.7)}$$

Hence the equations (16.5) and (16.6) take the form

$$\frac{\partial^2 \psi}{\partial \alpha^2} = 0, \qquad \text{.....(16.8)}$$

$$\frac{\partial^2 \psi}{\partial \beta^2} = 1. \qquad \text{.....(16.9)}$$

On the boundary $\alpha = 5/2$ the point 1 is fictitious. Hence we use the equation (16.8) to eliminate the value ψ_1 from equation (16.3).

From (16.8)

$$\psi_1 = 2\psi_0 - \psi_3 \qquad \ldots\ldots(16.10)$$

Substituting in (16.3) we have

$$F_\psi = (1 - \tfrac{B}{8})\psi_2 - A\psi_3/4 + \psi_4(1 + \tfrac{B}{8}) - 2\psi_0(1 - \tfrac{A}{8}) \qquad \ldots\ldots(16.11)$$

Hence the values of F_ψ can be calculated at points on $\alpha = 5/2$. The approximate relaxation pattern for points on the line $\alpha = 9/4$ is given by Fig. 25

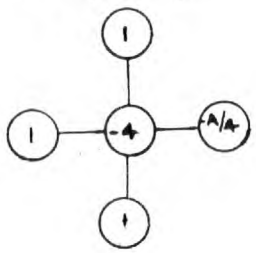


FIG. NO. 25.

The value of A/4 is small so that only small changes are made to the residuals at points on $\alpha = 5/2$ when the values of ψ on $\alpha = 9/4$ are altered, thus making it easier to reduce the total residual. When, as in this problem, there are three boundaries onwhich normal gradient conditions apply then very large changes in the value of the function have to be made before the total residual is reduced, and it is difficult todecide when the best solution to the finite difference equations has been obtained.

When the point 2 is fictitious as on the line $\beta = 5/2$ the boundary condition is taken in the form (16.9). When a multiplying factor of 10^3 is used the finite difference approximation to this is

$$\psi_2 + \psi_4 - 2\psi_0 = 62.5. \qquad \ldots\ldots(16.12)$$

Therefore, when we eliminate ψ_2 between (16.3) and (16.12) we have for F_ψ,

$$F_\psi + (1 + \tfrac{A}{8})\psi_1 + (1 - \tfrac{A}{8})\psi_3 + \tfrac{B}{8}\psi_4 - 2\psi_0(1 + \tfrac{B}{8}) + 62.5(1 - \tfrac{B}{8}). \qquad \ldots\ldots(16.13)$$

Thus for $\alpha = 0$,

$$F_\psi = 2\psi_1 + 0.09285\psi_4 - 2.092840\psi_0 + 59.59875,$$

with similar expressions for the other points on $\beta = 5/2$.

Again, there will be very small changes in the residuals on $\beta = 5/2$, when the values of ψ on $\beta = 9/4$ are altered.

These residual formulae and patterns have not been used in the solution of any example. Using the residual formulae, given by (16.11) and (16.13) on $\alpha = 5/2$ and $\beta = 5/2$ respectively we suggest that the following method be used to obtain the solution for ψ for the case of irrotational flow past a sphere. The values of ψ on the boundary are obtained from the equation

$$\psi = \tfrac{1}{2}r^2,$$

then the values for ψ are guessed at the other points of the area and the residuals are calculated from the exact formulae. Using an approximate relaxation pattern the values of ψ at all internal points are changed, assuming that the values of ψ on the boundary are fixed, until the residuals are 'reduced to zero'. The residuals are then recalculated using the exact formulae and the values of ψ on the boundary altered to satisfy the conditions on $\alpha = 5/2$, $\beta = 5/2$. This process is then repeated. Using this method it is thought that the liquidating of residuals will be easier than was the case when obtaining the solution given in Fig. 17, where we had to satisfy a normal gradient condition on three boundaries.

The values of the residuals obtained when the values of ψ given in Fig. 24 are substituted in the equations (16.11) and (16.13) are given by the top number on the lines $\alpha = 5/2$, $\beta = 5/2$. The middle set of numbers gives the residuals when the value of ψ is known at the points on the curves $\alpha = 11/4$ and $\beta = 11/4$. These two sets of residuals differ very little, and if the formulae (16.11) and (16.13) had been used in the first instance, a much more accurate value for the function ψ would have been obtained.

17. Alternative Methods of solution of the Viscous Flow equations for the case R = 0

When the Reynolds' number is zero there are no product terms in the governing equations (3.1) and (3.2). It is then possible to obtain an algebraic expression for the functions π and ψ. Lamb (Ref. 9).

In terms of cylindrical polar co-ordinates the equations satisfied by ψ and π are

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = -\pi, \qquad \text{.....(17.1)}$$

$$\frac{\partial^2\pi}{\partial r^2} - \frac{1}{r}\frac{\partial\pi}{\partial r} + \frac{\partial^2\pi}{\partial z^2} = 0 \qquad \text{.....(17.2)}$$

If we change to spherical polar coordinates (ρ , ϕ) where

$$r = \rho\sin\phi, \qquad z = \rho\cos\phi, \qquad \text{.....(17.3)}$$

then since

$$\frac{\partial}{\partial r} = \sin\phi\frac{\partial}{\partial\rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi},$$

and

$$\frac{\partial}{\partial z} = \cos\phi\frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi},$$

the equations (3.1) and (3.2) are

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} - \frac{\cot\phi}{\rho^2}\frac{\partial\psi}{\partial\phi} = -\pi, \qquad \text{.....(17.4)}$$

and

$$\frac{\partial^2\pi}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\pi}{\partial\phi^2} - \frac{\cot\phi}{\rho^2}\frac{\partial\pi}{\partial\phi} = 0. \qquad \text{.....(17.5)}$$

Eliminating π between (17.4) and (17.5) we have an equation for ψ, viz.

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} - \frac{\cot\phi}{\rho^2}\frac{\partial}{\partial\phi}\right]^2\psi = 0. \qquad \ldots\ldots(17.6)$$

Equation (17.6) is satisfied by

$$\psi = \sin^2\phi\, f(\rho), \qquad \ldots\ldots(17.7)$$

provided

$$\left[\frac{\partial^2}{\partial\rho^2} - \frac{2}{\rho^2}\right]^2 f(\rho) = 0. \qquad \ldots\ldots(17.8)$$

The solution of this equation (17.8) is

$$f(\rho) = \frac{A}{\rho} + B\rho + C\rho^2 + D\rho^4, \qquad \ldots\ldots(17.9)$$

where A, B, C, D are constant.

In the case of uniform flow at infinity we must have $\psi = \frac{1}{2}U\rho^2\sin^2\phi$ when $\rho \to \infty$, whence

$$D = 0, \qquad C = +\frac{1}{2}U. \qquad \ldots\ldots(17.10)$$

If we denote by u_ρ the velocity along the radius vector, and by u_ϕ the velocity at right angles to the radius, we have

$$u_\rho = \frac{1}{\rho^2\sin\phi}\frac{\partial\psi}{\partial\phi} = U\cos\phi + 2\cos\phi\left(\frac{A}{\rho^3} + \frac{B}{\rho}\right), \qquad \ldots\ldots(17.11)$$

and

$$u_\phi = \frac{-1}{\rho\sin\phi}\frac{\partial\psi}{\partial\rho} = -U\sin\phi + \left(\frac{A}{\rho^3} - \frac{B}{\rho}\right)\sin\phi$$

When there is no slipping on the surface $\rho = a$, we find from (17.11)

$$A = Ua^3/4, \qquad B = -3Ua/4. \qquad \ldots\ldots(17.12)$$

Therefore

$$\psi = \frac{1}{2}U[1 - 3a/2\rho + a^3/2\rho^3]\rho^2\sin^2\phi. \qquad .....(17.13)$$

Hence from (17.4)

$$-\pi = 3aU\sin^2\phi/2\rho. \qquad .....(17.14)$$

In order to compare the solutions obtained for ψ and π by relaxation methods with these analytical solutions we put U = 1, a = 1/2, giving

$$\psi = [1 - 3/4\rho + 1/16\rho^3]\rho^2\sin^2\phi/2, \qquad .....(17.15)$$

$$-\pi = 3\sin^2\phi/4\rho, \qquad .....(17.16)$$

or in terms of cylindrical polar co-ordinates

$$\psi = r^2[1 - 3/4\sqrt{r^2 + z^2} + 1/16(r^2 + z^2)^{3/2}]/2, \qquad ....(17.17)$$

$$-\pi = 3/4(r^2 + z^2)^{1/2}. \qquad .....(17.18)$$

These expressions have been evaluated at the points of intersection of the $(\alpha - \beta)$ curves $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 6$. These values are given in Fig. 24 The values of ψ are multiplied by 10^3 and those of π by -10^2. Comparison of the two solutions shows that there is a considerable error in the relaxation solution.

There are two main reasons for this:-

1) the function π does not tend to zero sufficiently slowly,
2) the boundary condition on $\alpha^2 + \beta^2 \to \infty$ is not given sufficiently accurately by the finite difference approximation.

The first source of error gives a residual of

approximately -6 at all points for which $\beta \geq 3$, and the second source gives a large negative residual at all points on $\alpha = 5$ and $\beta = 23/4$. In eliminating these residuals the values of ψ would tend to their true values.

The numerical values of ψ and π obtained from (17.17) and (17.18) have been substituted in the finite difference approximation to the equations (17.1) and (17.2). The F_ψ obtained are recorded in Fig. 26 . These residuals are mainly due to rounding off errors. For the line $\beta = 23/4$ there are three sets of residuals. The lowest one is the one which is obtained when the value of ψ along $\beta = 6$ is substituted in the residual formulae. The middle one is obtained when the boundary condition $\frac{\partial \psi}{\partial r} = r$ is used to eliminate the fictitious value ψ_2 from the expressions for F_ψ. The top one is the result of eliminating the fictitious ψ_2 using $\frac{\partial^2 \psi}{\partial r^2} = 1$. The residual formulae obtained, in this case, are of the form:

$$F_\psi = (1 + \tfrac{A}{8})\psi_1 + (1 - \tfrac{A}{8})\psi_3 + \tfrac{B}{4}\psi_4 - 2\psi_0(1 + \tfrac{B}{8}) + 62.5(1 - \tfrac{B}{8}) - C\pi_0/1.6 \quad ..(17.19)$$

i.e.

at $\alpha = 0$,

$F_\psi = 2\psi_1 + 0.04284\psi_4 - 2.04284\psi_0 + 61.16125 - 0.61579\pi_0$

with similar expressions at the other points.

These are of similar form to those obtained for

irrotational flow past a sphere. Thus we see that in flow problems of this type, the boundary condition which must be used to eliminate the fictitious value of ψ is

$$\text{either } \frac{\partial^2 \psi}{\partial r^2} = 1, \qquad \text{or } \frac{\partial^2 \psi}{\partial z^2} = 0. \qquad \ldots\ldots(17.20)$$

Within this area, it is not possible to satisfy the condition that $\pi \longrightarrow 0$ on the boundary and hence for further problems boundary conditions for π must be stated, so that the values of π at the fictitious points in the residual formulae can be eliminated. From Fig. 24 it is obvious that such conditions are

$$\frac{\partial^2 \pi}{\partial r^2} = 0, \qquad \text{and} \qquad \frac{\partial^2 \pi}{\partial z^2} = 0. \qquad \ldots\ldots(17.21)$$

Also from Fig. 24 it can be seen that the condition (17.21) can be applied nearer the sphere, viz. on the curve $\beta = 15/4$. Thus reducing the area over which the solution need be obtained, since usually the region which is of greatest interest is round the sphere.

There is also another small source of error. This is again due to the finite difference approximation to the first order differential coefficient.

On the boundary of the sphere the value of π is obtained from

$$-C\pi = \frac{\partial^2 \psi}{\partial \rho^2}, \quad \text{i.e.} \quad -C\pi = \left(\frac{\psi_2 + \psi_4 - 2\psi_o}{h^2}\right), \qquad \ldots\ldots(17.22)$$

and the fictitious value ψ_4 eliminated, using the fact that $\frac{\partial \psi}{\partial \rho} = 0$, i.e. $\psi_2 = \psi_4$ on $\rho = 0$. The values of π obtained in this way are given by the middle set of figures on $\rho = 0$. Again the error is large, and these π are greater than their true values. Thus the values of π at points on $\rho = 1/4$ are greater than their true values. This gives a greater negative contribution to the F_ψ on $\rho = 1/4$ and explains why the two values of ψ on this line are almost the same, when those on $\rho = 1/2$ differ considerably.

To obtain a more accurate value of π from ψ we expand ψ as a power series in ρ, viz.

$$\psi = a\rho^2 + b\rho^3, \qquad \ldots\ldots(17.23)$$

which automatically satisfies the conditions that

$$\psi = \frac{\partial \psi}{\partial \rho} = 0 \text{ on } \rho = 0.$$

From equation (17.22) we have, on $\rho = 0$,

$$\left(\frac{\partial^2 \psi}{\partial \rho^2}\right)_{\rho=0} = 2a. \qquad \ldots\ldots(17.24)$$

Also from (17.23)

$$\psi_2 = ah^2 + bh^3 \qquad \ldots\ldots(17.25)$$

$$\psi_{10} = 4ah^2 + 8bh^3 \qquad \ldots\ldots(17.26)$$

where the points 2 and 10 are as in Fig. 7.

If we eliminate b from (17.25) and (17.26) we have the following expression for 'a':

$$4ah^2 = 8\psi_2 - \psi_{10}. \qquad \text{.....(17.27)}$$

Therefore

$$+\pi = 3.2(8\psi_2 - \psi_{10})(1 - 4z^2), \qquad \text{.....(17.28)}$$

when a multiplying factor of 10^3 is introduced for ψ and -10^2 for π. The values of π obtained using this approximation are given by the lowest set of figures on $\alpha = 0$ Fig. 26. It can be seen that they are much nearer the true values.

When the stream function ψ is found as a solution to a problem, the values of the velocity components are usually calculated. From these the distribution of pressure and stresses can be obtained.

The components of the velocities in terms of r and z are given by

$$u = \frac{1}{r}\frac{\partial\psi}{\partial r}$$

$$= 1 - \frac{3}{4(r^2 + z^2)^{1/2}} + \frac{1}{16(r^2 + z^2)^{3/2}} \qquad \text{.....(17.29)}$$

$$+ r^2\left[\frac{3}{8(r^2 + z^2)^{3/2}} - \frac{3}{32(r^2 + z^2)^{5/2}}\right].$$

$$v = -\frac{1}{r}\frac{\partial\psi}{\partial z},$$

$$= -rz\left[\frac{3}{8(r^2 + z^2)^{3/2}} - \frac{3}{32(r^2 + z^2)^{5/2}}\right]. \qquad \text{.....(17.30)}$$

The values of u, v have been obtained on $\alpha = 0$,

$\alpha = 1/4$, from (17.29) and (17.30) and are given in the last two columns of Tables X and XII.

In order to obtain values of u and v from the relaxation solution u and v must be writtenin a finite difference form

$$u = \frac{1}{r}\frac{\partial \psi}{\partial r},$$

$$= \frac{1}{r}\left[\frac{\partial \psi}{\partial \alpha}\frac{\partial \alpha}{\partial r} + \frac{\partial \psi}{\partial \beta}\frac{\partial \beta}{\partial r}\right], \qquad \text{.....(17.31)}$$

$$= \frac{\partial \alpha}{\partial r}\left[\frac{\psi 1 - \psi 3}{2hr}\right] + \frac{\partial \beta}{\partial r}\left[\frac{\psi 2 - \psi 4}{2hr}\right],$$

where

$$\frac{\partial \alpha}{\partial r} = -\frac{\partial \beta}{\partial z} = \frac{-rz}{2(r^2 + z^2)^2}, \qquad \text{.....(17.32)}$$

and

$$\frac{\partial \beta}{\partial r} = \frac{\partial \alpha}{\partial z} = 1 + \frac{r^2 - z^2}{4(r^2 + z^2)^2}.$$

Similarly

$$v = -\frac{\partial \alpha}{\partial z}\left(\frac{\psi 1 - \psi 3}{2hr}\right) - \frac{\partial \beta}{\partial z}\left(\frac{\psi 2 - \psi 4}{2hr}\right) \qquad \text{.....(17.33)}$$

The values of u and v, obtained from the function ψ given in Fig. 26 are given in Tables X & XII.

It can be seen from these tables that in the region of the sphere the finite difference approximation to the first order differentials is not sufficiently accurate. As the

function ψ varies more in the β-direction it is only necessary to obtain a better value of $\frac{\partial \psi}{\partial \beta}$.

Along any α-curve the function ψ can be expanded as a power series in β, viz.

$$\psi = \psi_0 + A\beta + B\beta^2 + C\beta^3 + D\beta^4. \qquad .....(17.34)$$

$$\psi_2 = \psi_0 + Ah + Bh^2 + Ch^3 + Dh^4.$$

$$\psi_4 = \psi_0 - Ah + Bh^2 - Ch^3 + Dh^4.$$

Therefore

$$\psi_2 - \psi_4 = 2Ah + 2Ch^3$$

$$\psi_{10} = \psi_0 + 2Ah + 4Bh^2 + 8Ch^3 + 16h^4D$$

$$\psi_{12} = \psi_0 - 2Ah + 4Bh^2 - 8Ch^3 + 16h^4D.$$

Therefore

$$\psi_{10} - \psi_{12} = 4Ah + 16Ch^3$$

We thus have the following expression for A:

$$A = \frac{1}{3}[8(\psi_2 - \psi_4) - (\psi_{10} - \psi_{12})]$$

for $h = 1/4$

Therefore $\left(\frac{\partial \psi}{\partial \beta}\right)_{\beta=0} = \frac{8}{3}(\psi_2 - \psi_4) - \frac{1}{3}(\psi_{10} - \psi_{12})$ .....(17.35)

This can be applied at all points except those for the curve $\beta = 1/4$. Here the point 12 is fictitious. In this case we take

$$\psi = \psi_0 + A\beta + B\beta^2 + C\beta^3. \qquad .....(17.36)$$

This gives

$$\left(\frac{\partial \psi}{\partial \beta}\right)_{\beta = 1/4} = \frac{2}{3}(6\psi_2 - 3\psi_0 - \psi_{10}). \qquad \ldots\ldots(17.37)$$

Using the above approximation the value of u at the point $\alpha = 1/4$, $\beta = 1/4$ was found to be 0.2914 and at the point $\alpha = 1/4$, $\beta = 1/2$ to be 0.4750. Both of which compare favourably with the exact values. If ψ had been expanded in a similar manner in the α-direction even more accurate results would be obtained. As the contribution from $\frac{\partial \psi}{\partial \alpha}$ is small, this is not thought to be necessary.

TABLE X

$\alpha = 0$

β	$\frac{1}{r}\frac{\partial \psi}{\partial z}$	$\frac{10^8}{2h}(\psi_2 - \psi_4)$	From Fig. 26. u	From (17.29) u
1/4	2.513479	125	0.31419	0.29543
1/2	1.708189	284.6	0.48615	0.47746
3/4	1.250000	479	0.59875	0.59375
1	0.970557	694	0.67357	0.67157
5/4	0.787889	922	0.72643	0.72612
3/2	0.661063	1160	0.76683	0.76598
7/4	0.568587	1402	0.79716	0.79614
2	0.498448	1644	0.81945	0.81966
9/4	0.443542	1890	0.83829	0.83845
5/2	0.399450	2138	0.85402	0.85378
11/4	0.363286	2384	0.86607	0.86650
3	0.333102	2634	0.87739	0.87722
13/4	0.307535	2884	0.88693	0.88637
7/2	0.285604	3130	0.89394	0.89430
15/4	0.266588	3380	0.90107	0.90115
4	0.249943	3630	0.90729	0.90720
17/4	0.235251	3878	0.91230	0.91256
9/2	0.222190	4130	0.91765	0.91734
19/4	0.210502	4378	0.92158	0.92163
5	0.199981	4628	0.92551	0.92549
21/4	0.190461	4878	0.92907	0.92900
11/2	0.181806	5126	0.93194	0.93219

$\alpha = \frac{1}{4}$

β	$\frac{1}{r}\frac{\partial \psi}{\partial z}$	$-\frac{1}{r}\frac{\partial \psi}{\partial r}$	$\frac{10^8}{2h}(\psi_1-\psi_3)$	$\frac{10^8}{2h}(\psi_2-\psi_4)$
1/4	2.44453	0.450757	125.8	0.6
1/2	1.674967	0.202952	286.4	3.2
3/4	1.234561	0.095718	480.2	8
1	0.963246	0.048227	696	10
5/4	0.784258	0.026010	924	14
3/2	0.659164	0.014933	1160	16
7/4	0.567537	0.009056	1402	18
2	0.497837	0.005757	1646	16
9/4	0.443172	0.003811	1890	18
5/2	0.399215	0.002611	2138	18
11/4	0.363133	0.001844	2386	18
3	0.332999	0.001336	2632	18
13/4	0.307464	0.000990	2882	16
7/2	0.285554	0.000748	3130	16
15/4	0.266551	0.000575	3380	16
4	0.249916	0.000449	3630	14
17/4	0.235232	0.000356	3878	16
9/2	0.222175	0.000285	4128	14
15/4	0.210490	0.000231	4378	14
5	0.199972	0.000190	4628	14
21/4	0.190454	0.000157	4878	12
11/2	0.181800	0.000131	5128	14

TABLE NO XI

$\alpha = \frac{1}{4}$

β	Fig.[26]	Fig.[26]	(17.29)	(17.30)
	u	v	u	v
1/4	0.3073	0.5817	0.28878	0.54527
1/2	0.4791	0.0635	0.47053	0.06228
3/4	0.5921	0.0558	0.58835	0.05450
1	0.6699	0.0432	0.66772	0.04415
5/4	0.7243	0.0350	0.72341	0.03509
3/2	0.7644	0.0279	0.76406	0.02796
7/4	0.7955	0.0229	0.79476	0.02253
2	0.8193	0.0174	0.81864	0.01839
9/4	0.8375	0.0152	0.83768	0.01522
5/2	0.8535	0.0128	0.85319	0.01276
11/4	0.8664	0.0109	0.86604	0.01083
3	0.8764	0.0095	0.87686	0.00929
13/4	0.8861	0.0078	0.88608	0.00809
7/2	0.8838	0.0069	0.89403	0.00703
15/4	0.9009	0.0062	0.90096	0.00619
4	0.9072	0.0051	0.90704	0.00548
17/4	0.9122	0.0051	0.91242	0.00489
9/2	0.9171	0.0043	0.91732	0.00440
19/4	0.9215	0.0040	0.92149	0.00396
5	0.9255	0.0037	0.92540	0.00359
21/4	0.9290	0.0031	0.92892	0.00327
11/2	0.9323	0.0032	0.93212	0.00299

TABLE NO. XII

18. Flow round sphere R = 0

When the boundary conditions (17.20) and (17.21) are used in the solution of a problem the question arises as to how far to extend the area of solution. For the case of a sphere it is possible to transform the governing equations so that a solution can be obtained over the complete field of flow.

The equations satisfied by ψ and π are written in the form given by equations (17.4) and (17.5), viz.

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} - \frac{\cot\phi}{\rho^2}\frac{\partial\psi}{\partial\phi} = -\pi, \qquad \text{.....(18.1)}$$

$$\frac{\partial^2\pi}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\pi}{\partial\phi^2} - \frac{\cot\phi}{\rho^2}\frac{\partial\pi}{\partial\phi} = 0. \qquad \text{.....(18.2)}$$

A further change of co-ordinates is made. We write

$$x = \log 2\rho, \qquad \text{.....(18.3)}$$

so that the circle $\rho = 1/2$ becomes the straight line $x = 0$. From (18.3) we have

$$\frac{\partial}{\partial\rho} = e^{-x}\frac{\partial}{\partial x}, \quad \frac{\partial^2}{\partial\rho^2} = e^{-2x}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right). \qquad \text{.....(18.4)}$$

Thus equations (18.1) and (18.2) now take the form:

$$\frac{\partial^2\psi}{\partial x^2} - \frac{\partial\psi}{\partial x} + \frac{\partial^2\psi}{\partial\phi^2} - \cot\phi\frac{\partial\psi}{\partial\phi} = -\frac{1}{4}e^{2x}\pi \qquad \text{.....(18.5)}$$

$$\frac{\partial^2 \pi}{\partial x^2} - \frac{\partial \pi}{\partial x} + \frac{\partial^2 \pi}{\partial \phi^2} - \cot\phi \frac{\partial \pi}{\partial \phi} = 0. \qquad \text{.....(18.6)}$$

The boundary conditions are:-

(1) on $\phi = 0$, $\psi = 0 = \pi$, ...(18.7)

(2) on $\phi = \pi/2$, $\frac{\partial \psi}{\partial \phi} = \frac{\partial \pi}{\partial \phi} = 0$ (by symmetry), ...(18.8)

(3) on $x = 0$, $\psi = 0 = \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \phi} = \frac{\partial^2 \psi}{\partial \phi^2}$ .....(18.9)

$$\pi = 4e^{-2x}\left[\frac{\partial^2 \psi}{\partial x^2}\right],$$

(4) $x \to \infty$ $\psi \to \frac{1}{8} e^{2x} \sin^2 \phi$, ...(18.10)

$\pi \to 0$.

For even small values of x it can be seen that the function ψ will increase rapidly and as a consequence any finite difference approximation to (18.5) will not be sufficiently accurate. A change is thus made in the dependent variable:-

$$\psi = \bar{\Psi} e^{2x}. \qquad \text{.....(18.11)}$$

Then equation (18.5) becomes

$$\frac{\partial^2 \bar{\Psi}}{\partial x^2} + 3\frac{\partial \bar{\Psi}}{\partial x} + 2\bar{\Psi} + \frac{\partial^2 \bar{\Psi}}{\partial \phi^2} - \cot\phi \frac{\partial \bar{\Psi}}{\partial \phi} + \frac{\pi}{4} = 0 \qquad \text{.....(18.12)}$$

and the boundary conditions are

(1) on $\phi = 0$, $\bar{\Psi} = 0$, ...(18.13)

(2) on $\phi = \pi/2$, $\frac{\partial \bar{\Psi}}{\partial \phi} = 0$, ...(18.14)

(3) on $x = 0$, $\bar{\Psi} = 0$, $\frac{\partial \bar{\Psi}}{\partial x} = 0$, ...(18.15)

$$-\pi = 4e^{-2x}\left[\frac{\partial^2 \Psi}{\partial x^2}\right] = 4\frac{\partial^2 \bar{\Psi}}{\partial x^2}, \qquad .....(18.16)$$

(4) as $x \longrightarrow \infty$, $\bar{\Psi}$ is independent of x. .....(18.17)

Thus we must first obtain an approximate solution by Relaxation Methods to the equation

$$2\bar{\Psi}_0 + \frac{\partial^2 \bar{\Psi}}{\partial \phi^2} - \cot\phi \frac{\partial \bar{\Psi}}{\partial \phi} = 0 \qquad .....(18.18)$$

The finite difference approximation to the equation (18.18) is

$$(1.6211-0.1592\cot\phi_0)\bar{\Psi}_2 + (1.6211+0.1592\cot\phi_0)\bar{\Psi}_4 - 3.1173\bar{\Psi}_0 = 0,$$

with $\bar{\Psi} = 0$ for $\phi = 0$, and for $\phi = \pi/2$, $\bar{\Psi}_2 = \bar{\Psi}_4$. The solution of this equation is given on the line $x = 8.75$ in Fig. 27

We now know either a value of $\bar{\Psi}$ and π on the boundary or have a normal gradient condition, so we can obtain a solution by relaxation methods to equations (18.12) and (18.6).

The finite difference approximation to equation (18.7) is

$$\frac{\bar{\Psi}_1 + \bar{\Psi}_3 - 2\bar{\Psi}_0}{h_1^2} + \frac{3}{2h_1}(\bar{\Psi}_1 - \bar{\Psi}_3) + 2\bar{\Psi}_0 + \frac{\bar{\Psi}_2 + \bar{\Psi}_4 - 2\bar{\Psi}_0}{h_2^2} - \frac{\cot\phi_0}{2h_2}(\bar{\Psi}_2 - \bar{\Psi}_4) + \frac{\pi_0}{4} = 0,$$

where $h_1 = 1/4$, $h_2 = \pi/16$.

The expression for the ψ-residual is therefore

$$F_\psi = 1.375\Psi_1 + (1.6211-0.1592\cot\phi_0)\Psi_2 + 0.625\Psi_3 + (1.6211 + 0.1592\cot\phi_0)\Psi_4 - 5.1173\Psi_0 + \frac{\pi_0}{64}, \quad (18.19)$$

and for the π-residual is

$$F_\pi = 0.875\pi_1 + (1.6211 - 0.1592\cot\phi_0)\pi_2 + 1.125\pi_3 + (1.6211 + 0.1592\cot\phi_0)\pi_4 - 5.2423\pi_0. \quad (18.20)$$

The boundary condition (18.16) must also be expressed in terms of finite differences. The function Ψ can be expressed as a power series in x, viz.

$$\Psi = \alpha x^2 + \beta x^3 + \gamma x^4 \quad .....(18.21)$$

(This automatically satisfies (18.15).)

Hence

$$\Psi_1 = \alpha h^2 + \beta h^3 + \gamma h^4 ,$$
$$\Psi_9 = 4\alpha h^2 + 8\beta h^3 + 16h^4\gamma,$$
$$\Psi_{13} = 9\alpha h^2 + 27h^3\beta + 81h^4\gamma,$$

FIG. NO. 7a.

where h = 1/4. The points 1, 9, 13 are shown in Fig. 7a. When we solve these three equations we obtain the following expression for α :

$$\alpha = \frac{1}{36h^2}[108\Psi_1 - 27\Psi_9 + 4\Psi_{13}],$$

and hence

$$\pi_0 = \frac{1}{18h^2}[108\Psi_1 - 27\Psi_9 + 4\Psi_{13}]. \quad .....(18.22)$$

The solutions of equations (18.19) and (18.20) are given in Fig. 27 The functions Ψ and π have a multiplying

factor of 10^3.

The values of $\bar{\Psi}$ and π have been calculated from equations (17.15) and (17.16) and are given in Fig. 28. These two sets of values compare favourably. The greatest variation in Ψ occurs for large values of x. It is obvious that a much better solution will be obtained using relaxation methods if say, a multiplying factor of 10^5 is used for Ψ. The changes in Ψ along x = const., (for $x \geq 4$) are large. Hence it is also necessary to use a better finite difference approximation to the governing equations. Here it will be advantageous to introduce the second, third and fourth differences as suggested by Fox (5). There will be no need to use graphical methods or extrapolation to obtain values of Ψ for $\phi > \pi/2$ or $\phi < 0$ as Ψ is symmetrical about both these two lines. Also these differences need only be calculated for those lines on which has large variations.

Near the sphere the values of Ψ obtained from the relaxation solution are almost equal to the exact values of Ψ. From these values of Ψ a value of π on the sphere is calculated using (18.22) and it can be seen that these compare favourably with those obtained from the algebraic solution.

It therefore appears that this is a suitable method

FIG. NO. 27. R = 0

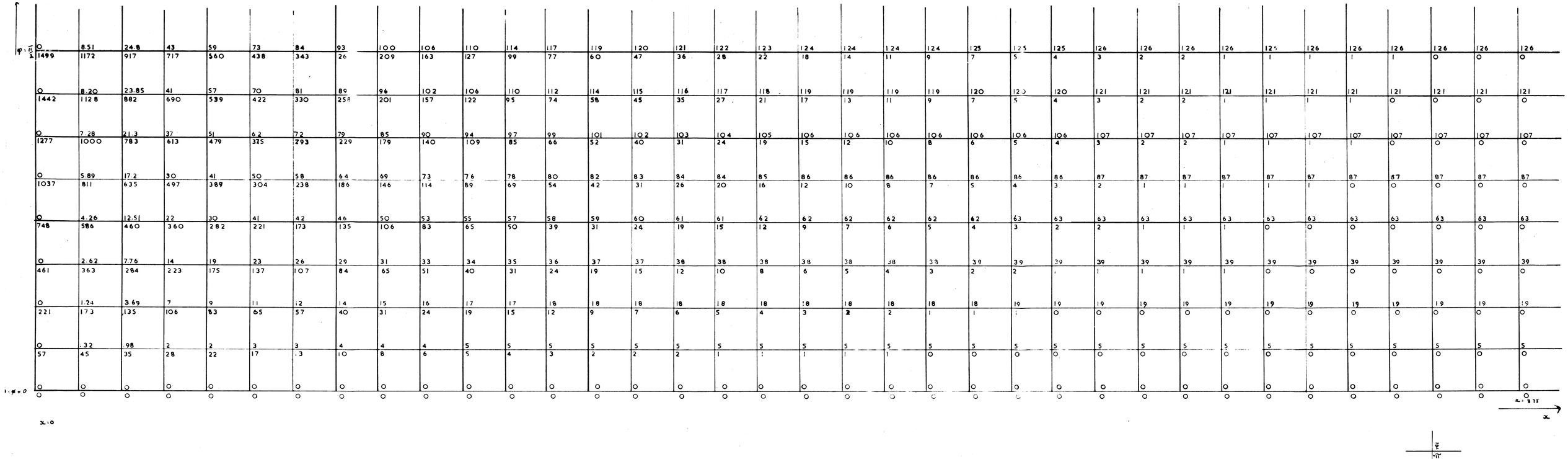
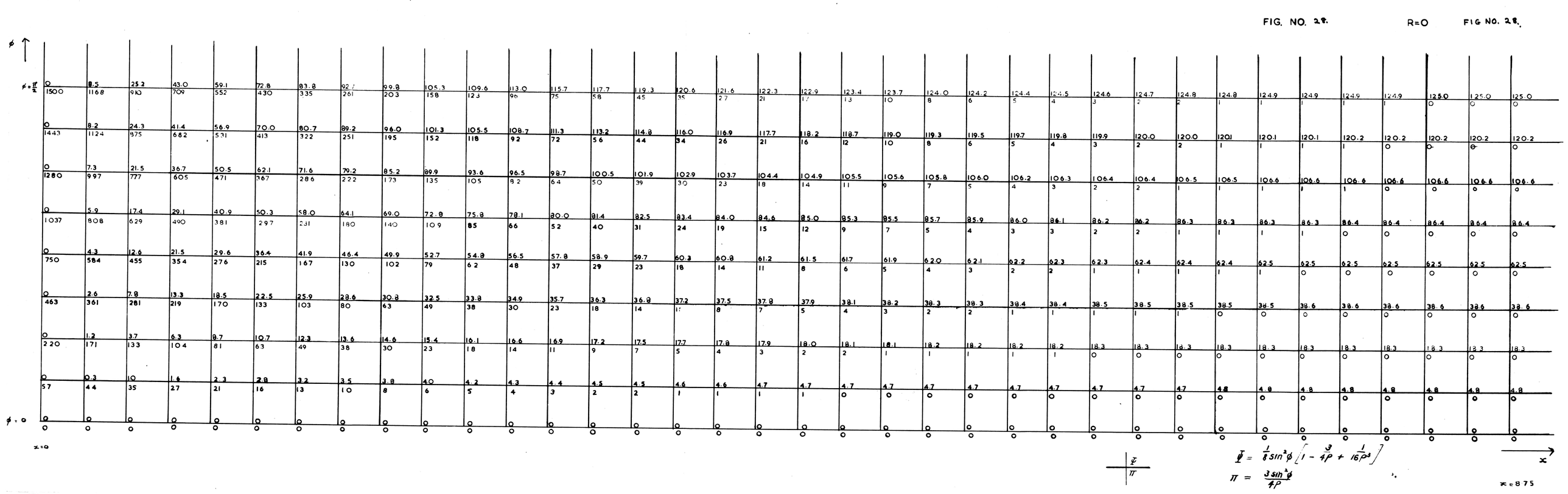

φ = π/2
0 8.51 24.8 43 59 73 84 93 100 106 110 114 117 119 120 121 122 123 124 124 124 124 125 125 125 126 126 126 126 126 126 126 126 126 126 126
1499 1172 917 717 560 438 343 26 209 163 127 99 77 60 47 36 28 22 18 14 11 9 7 5 4 3 2 2 1 1 1 1 1 0 0 0
0 8.20 23.85 41 57 70 81 89 96 102 106 110 112 114 115 116 117 118 119 119 119 119 120 120 120 121 121 121 121 121 121 121 121 121 121 121
1442 1128 882 690 539 422 330 258 201 157 122 95 74 58 45 35 27 21 17 13 11 9 7 5 4 3 2 2 1 1 1 1 0 0 0 0
0 7.28 21.3 37 51 62 72 79 85 90 94 97 99 101 102 103 104 105 106 106 106 106 106 106 106 107 107 107 107 107 107 107 107 107 107 107
1277 1000 783 613 479 375 293 229 179 140 109 85 66 52 40 31 24 19 15 12 10 8 6 5 4 3 2 2 1 1 1 1 0 0 0 0
0 5.89 17.2 30 41 50 58 64 69 73 76 78 80 82 83 84 84 85 86 86 86 86 86 86 86 87 87 87 87 87 87 87 87 87 87 87
1037 811 635 497 389 304 238 186 146 114 89 69 54 42 31 26 20 16 12 10 8 7 5 4 3 2 1 1 1 1 1 0 0 0 0 0
0 4.26 12.51 22 30 41 42 46 50 53 55 57 58 59 60 61 61 62 62 62 62 62 62 63 63 63 63 63 63 63 63 63 63 63 63
748 586 460 360 282 221 173 135 106 83 65 50 39 31 24 19 15 12 9 7 6 5 4 3 2 2 1 1 1 0 0 0 0 0 0 0
0 2.62 7.76 14 19 23 26 29 31 33 34 35 36 37 37 38 38 38 38 38 38 38 38 39 39 39 39 39 39 39 39 39 39 39 39
461 363 284 223 175 137 107 84 65 51 40 31 24 19 15 12 10 8 6 5 4 3 2 2 1 1 1 1 0 0 0 0 0 0 0 0
0 1.24 3.69 7 9 11 12 14 15 16 17 17 18 18 18 18 18 18 18 18 18 18 18 19 19 19 19 19 19 19 19 19 19 19 19
221 173 135 106 83 65 57 40 31 24 19 15 12 9 7 6 5 4 3 2 2 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 .32 98 2 2 3 3 4 4 4 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5
57 45 35 28 22 17 13 10 8 6 5 4 3 2 2 2 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
θ = 0
x = 0
x = 9.75
x

FIG. NO. 28. R=0 FIG NO. 28.



$\frac{x}{\pi}$

$\psi = \frac{1}{8}\sin^2\phi\left[1 - \frac{3}{4P} + \frac{1}{16P^2}\right]$

$\pi = \frac{3\sin^2\phi}{4P}$

x=875

to use to obtain a solution over the whole field for the case of a sphere. The modifications to the finite difference approximation have not been used to obtain a better solution. If we had tried to obtain a solution for Ψ and π for other values of the Reynolds' number then we would have introduced these modifications. These values of R would have been small, since for large R, π tends to zero very quickly and the non-zero π are confined to the boundary layer.

The components of velocity have also been calculated. The values given in Fig. 29 are from the algebraic solution. To obtain values from the relaxation solution we use the equations

$$u_\rho = \frac{4}{\sin\phi}\frac{\partial \Psi}{\partial \phi}, \qquad \text{.....(18.23)}$$

and

$$u_\phi = \frac{-4}{\sin\phi}\left(\frac{\partial \Psi}{\partial x} + 2\Psi\right). \qquad \text{.....(18.24)}$$

To obtain a finite difference approximation to (18.23) we expand Ψ as a power series in ϕ, viz.

$$\Psi = \Psi_0 + a\phi + b\phi^2 + c\phi^3 + d\phi^4.$$

Therefore

$$\left(\frac{\partial \Psi}{\partial \phi}\right)_{\phi=0} = a = \frac{8(\Psi_2 - \Psi_4) - (\Psi_{10} - \Psi_{12})}{12h}$$

where $h = \pi/16$.

Hence

$$u_\rho = \frac{16}{3\pi \sin\phi}\left\{8(\Psi_2 - \Psi_4) - (\Psi_{10} - \Psi_{12})\right\}. \qquad \ldots\ldots(18.25)$$

On $\phi = \pi/2$, $u_\rho = 0$ from symmetry.

On $\phi = \frac{7\pi}{16}$, the point 10 is fictitious, but because of symmetry

$$\Psi_{10} = \Psi_0$$

Therefore, for $\phi = 7\pi/16$

$$[u_\rho] = 16\left\{8(\Psi_2 - \Psi_4) - (\Psi_0 - \Psi_{12})\right\}/3\pi \sin 7\pi/16 \ldots\ldots(18.26)$$

On $\phi = \frac{\pi}{16}$, the point 12 is fictitious. Again Ψ is symmetrical about $\phi = 0$, giving $\Psi_0 = \Psi_{12}$ and hence

$$u = 16\left\{8(\Psi_2 - \Psi_4) - (\Psi_{10} - \Psi_0)\right\}/3\pi \sin \pi/16 . \ldots(18.27)$$

On $\phi = 0$, both $\sin\phi$ and $\frac{\partial \Psi}{\partial \phi}$ are zero. Hence we calculate u_ρ from the equation

$$u_\rho = 4\frac{\partial^2 \Psi}{\partial \phi^2}/\cos\phi \qquad \ldots\ldots(18.28)$$

Since $\Psi_0 = 0$, $\Psi_2 = \Psi_4$, $\Psi_{10} = \Psi_{12}$ on $\phi = 0$, the finite difference approximation to (18.28) is

$$u_\rho = (32\Psi_2 - 2\Psi_{10})/3h^2,$$

where $h = \pi/16$.

In the x-direction the largest changes in Ψ occur near the sphere. Hence we use two different finite difference approximations to (18.24) depending on the difference $(\Psi_1 - \Psi_3)$.

When $(\Psi_1 - \Psi_3)$ is small we write

$$u_\phi = 8(\Psi_1 - \Psi_3 + \Psi_0)/\sin\phi$$

For $(\Psi_1 - \Psi_3)$ large we expand Ψ as a power series in x, viz.

$$\Psi = \Psi_0 + ax + bx^2 + cx^3 + dx^4, \qquad \ldots\ldots(18.29)$$

from which

$$a = [8(\Psi_1 - \Psi_3) - (\Psi_9 - \Psi_{11})]/3.$$

Hence

$$u_\phi = -4[8(\Psi_1 - \Psi_3) - (\Psi_9 - \Psi_{11}) + 6\Psi_0]/3\sin\phi \qquad \ldots\ldots(18.30)$$

For points on $x = 1/4$, the value of Ψ at the point 11 is fictitious. To obtain the value of 'a' in this case we use the four equations

$$\Psi_1 = \Psi_0 + ah + bh^2 + ch^3 + dh^4,$$

$$\Psi_3 = \Psi_0 - ah + bh^2 - ch^3 + dh^4,$$

$$\Psi_9 = \Psi_0 + 2ah + 4bh^2 + 8ch^3 + 16dh^4,$$

and

$$\left(\frac{\partial\Psi}{\partial x}\right)_{x=-h} = 0 = a - 2bh + 3ch^2 - 4dh^3.$$

From the above equations the following value is obtained for 'a':

$$a = [9\Psi_1 - 17\Psi_3 - \Psi_9 + 9\Psi_0]/18h.$$

Since $\Psi_3 = 0$, it follows that

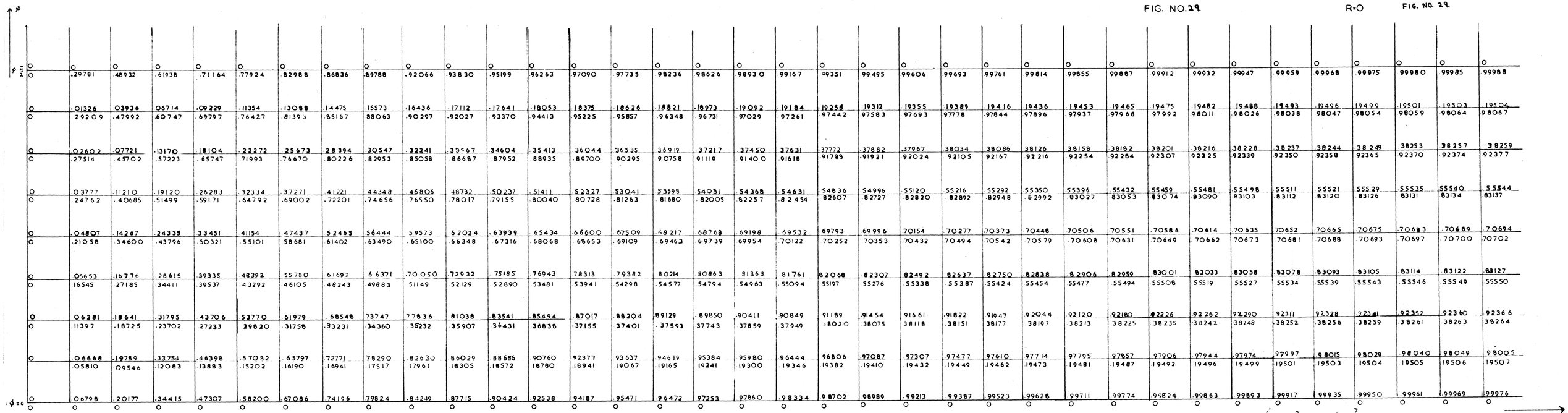
$$u_\phi = \frac{-4}{9\sin\phi}[18\Psi_1 + 36\Psi_0 - 2\Psi_9] \qquad \ldots\ldots(18.31)$$

We calculated values of u_ϕ and u_ρ using equations

(18.31) and (18.30) and these are given in Fig. 30 . The values of Ψ which were used in this calculation were taken from Fig. 28.

The values of u_ϕ and u_ρ obtained from (18.31) and (18.30) agree reasonably well with the values given in Fig. 29. The main disadvantage in using the above formulae (or any other finite difference approximation) to calculate u_ρ and u_ϕ is that the values obtained for the velocities do not, in general, increase smoothly. For this reason it is advisable to work with numbers which have more significant figures. Alternatively the values of u_ρ and u_ϕ can be substituted in the finite difference approximation obtained from equations (2.2) and the errors eliminated in this way. This method is discussed in the [illegible] section 20.

FIG. NO. 29. R=0 FIG. NO. 29.



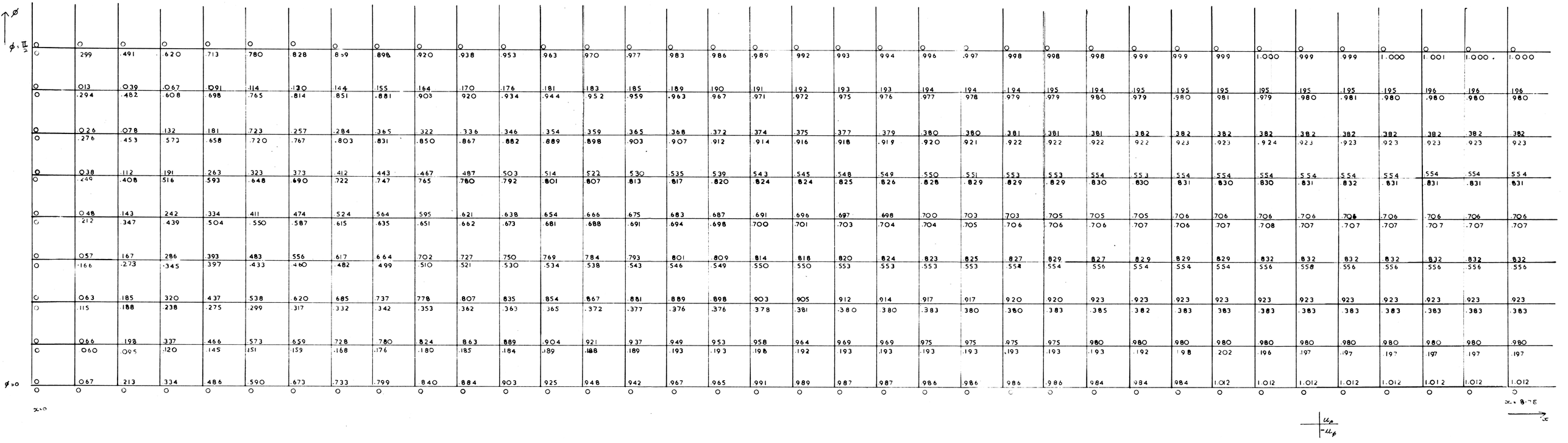
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	.29781	.48932	.61938	.71164	.77924	.82988	.86836	.89788	.92066	.93830	.95199	.96263	.97090	.97735	.98236	.98626	.98930	.99167	.99351	.99495	.99606	.99693	.99761	.99814	.99855	.99887	.99912	.99932	.99947	.99959	.99968	.99975	.99980	.99985	.99988
0	.01326	.03936	.06714	.09229	.11354	.13088	.14475	.15573	.16436	.17112	.17641	.18053	.18375	.18626	.18821	.18973	.19092	.19184	.19256	.19312	.19355	.19389	.19416	.19436	.19453	.19465	.19475	.19482	.19488	.19493	.19496	.19499	.19501	.19503	.19504
0	.29209	.47992	.60747	.69797	.76427	.81393	.85167	.88063	.90297	.92027	.93370	.94413	.95225	.95857	.96348	.96731	.97029	.97261	.97442	.97583	.97693	.97778	.97844	.97896	.97937	.97968	.97992	.98011	.98026	.98038	.98047	.98054	.98059	.98064	.98067
0	.02602	.07721	.13170	.18104	.22272	.25673	.28394	.30547	.32241	.33567	.34604	.35413	.36044	.36535	.36919	.37217	.37450	.37631	.37772	.37882	.37967	.38034	.38086	.38126	.38158	.38182	.38201	.38216	.38228	.38237	.38244	.38249	.38253	.38257	.38259
0	.27514	.45702	.57223	.65747	.71993	.76670	.80226	.82953	.85058	.86687	.87952	.88935	.89700	.90295	.90758	.91119	.91400	.91618	.91789	.91921	.92024	.92105	.92167	.92216	.92254	.92284	.92307	.92325	.92339	.92350	.92358	.92365	.92370	.92374	.92377
0	.03777	.11210	.19120	.26283	.32334	.37271	.41221	.44348	.46806	.48732	.50237	.51411	.52327	.53041	.53599	.54031	.54368	.54631	.54836	.54996	.55120	.55216	.55292	.55350	.55396	.55432	.55459	.55481	.55498	.55511	.55521	.55529	.55535	.55540	.55544
0	.24762	.40685	.51499	.59171	.64792	.69002	.72201	.74656	.76550	.78017	.79155	.80040	.80728	.81263	.81680	.82005	.82257	.82454	.82607	.82727	.82820	.82892	.82948	.82992	.83027	.83053	.83074	.83090	.83103	.83112	.83120	.83126	.83131	.83134	.83137
0	.04807	.14267	.24335	.33451	.41154	.47437	.52465	.56444	.59573	.62024	.63939	.65434	.66600	.67509	.68217	.68768	.69198	.69532	.69793	.69996	.70154	.70277	.70373	.70448	.70506	.70551	.70586	.70614	.70635	.70652	.70665	.70675	.70683	.70689	.70694
0	.21058	.34600	.43796	.50321	.55101	.58681	.61402	.63490	.65100	.66348	.67316	.68068	.68653	.69109	.69463	.69739	.69954	.70122	.70252	.70353	.70432	.70494	.70542	.70579	.70608	.70631	.70649	.70662	.70673	.70681	.70688	.70693	.70697	.70700	.70702
0	.05653	.16776	.28615	.39335	.48392	.55780	.61692	.66371	.70050	.72932	.75185	.76943	.78313	.79382	.80214	.80863	.81363	.81761	.82068	.82307	.82492	.82637	.82750	.82838	.82906	.82959	.83001	.83033	.83058	.83078	.83093	.83105	.83114	.83122	.83127
0	.16545	.27185	.34411	.39537	.43292	.46105	.48243	.49883	.51149	.52129	.52890	.53481	.53941	.54298	.54577	.54794	.54963	.55094	.55197	.55276	.55338	.55387	.55424	.55454	.55477	.55494	.55508	.55519	.55527	.55534	.55539	.55543	.55546	.55549	.55550
0	.06281	.18641	.31795	.43706	.53770	.61979	.68548	.73747	.77836	.81038	.83541	.85494	.87017	.88204	.89129	.89850	.90411	.90849	.91189	.91454	.91661	.91822	.91947	.92044	.92120	.92180	.92226	.92262	.92290	.92311	.92328	.92341	.92352	.92360	.92366
0	.11397	.18725	.23702	.27233	.29820	.31758	.33231	.34360	.35232	.35907	.36431	.36838	.37155	.37401	.37593	.37743	.37859	.37949	.38020	.38075	.38118	.38151	.38177	.38197	.38213	.38225	.38235	.38242	.38248	.38252	.38256	.38259	.38261	.38263	.38264
0	.06668	.19789	.33754	.46398	.57082	.65797	.72771	.78290	.82630	.86029	.88686	.90760	.92377	.93637	.94619	.95384	.95980	.96444	.96806	.97087	.97307	.97477	.97610	.97714	.97795	.97857	.97906	.97944	.97974	.97997	.98015	.98029	.98040	.98049	.98055
	.05810	.09546	.12083	.13883	.15202	.16190	.16941	.17517	.17961	.18305	.18572	.18780	.18941	.19067	.19165	.19241	.19300	.19346	.19382	.19410	.19432	.19449	.19462	.19473	.19481	.19487	.19492	.19496	.19499	.19501	.19503	.19504	.19505	.19506	.19507
0	.06798	.20177	.34415	.47307	.58200	.67086	.74196	.79824	.84249	.87715	.90424	.92538	.94187	.95471	.96472	.97253	.97860	.98334	.98702	.98989	.99213	.99387	.99523	.99628	.99711	.99774	.99824	.99863	.99893	.99917	.99935	.99950	.99961	.99969	.99976
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

u_0
u_1

$u_\rho = \cos\phi \left(1 - \frac{3}{4\rho} + \frac{1}{16\rho^3} \right)$

$u_\phi = \sin\phi \left(-1 + \frac{3}{8\rho} + \frac{1}{32\rho^3} \right)$

FIG. NO. 30. R=0



19. Discussion of the results obtained to the equations of viscous flow for non-zero Reynolds' numbers

As there is no algebraic solution to the equations of viscous flow for non-zero Reynolds' numbers, it is not easy to estimate the accuracy of the solution obtained by relaxation methods. We know that the vorticity function tends to zero more rapidly when R is non-zero. Hence we do not expect a large error in the ψ-function due to the values of π being too small. This applies particularly to the cases when R = 10 and R = 20. For all the solutions, the boundary conditions when $\alpha^2 + \beta^2 \to \infty$ were taken in the form

$$\frac{\partial \psi}{\partial r} = r, \qquad \frac{\partial \psi}{\partial z} = 0. \qquad \text{.....(19.1)}$$

This has been shown to be a source of error for the cases of irrotational flow and slow viscous flow past a sphere. The values of ψ given in Figs. 21a, 22a, and 23a are consequently too high. The pattern of the streamlines in Figs. 21b, 22b, and 23b will not, however, change as the function ψ will be altered by approximately the same amount on each line β = const. The streamlines will be closer together. The vorticity lines in Figs. 21c, 22c, and 23c. are a good approximation.

Comparing the streamlines for the three values of Reynolds' number it can be seen that there is very little

difference between them. As R increases, the turning point on any streamline moves further downstream away from the sphere.

From Fig.21c,22c,23c the equi-vorticity lines are of approximately the same shape for all values of the Reynolds' number. As R increases it can be seen that the effects of viscosity are found more in the region of the sphere.

For the case R = 20, the functions ψ and ζ have been obtained by Mitutosi Kawaguti (Ref. 11) using the method suggested by Thom, which is discussed in the introduction to this thesis. The streamlines and equi-vorticity lines obtained from the solution are given in Figs. 23d and 23e

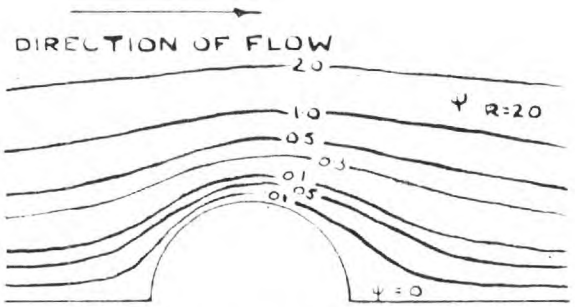


FIG. 23d.

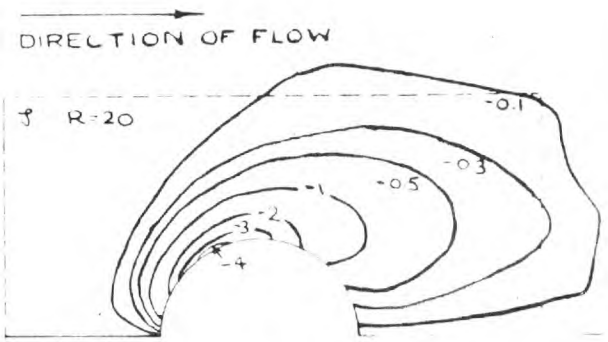


FIG. 23e.

As the values of ψ and ζ are not given at the mesh

points of the area of solution it is difficult to compare the two sets of results. The shape of the streamlines and equi-vorticity lines is approximately the same.

The equations (8.1) and (8.2) have not been solved for any higher values of R, as the coefficients of π_1, π_3 in the expression for F_π increase and the area over which π is non-zero decreases. Hence, before any higher values of the Reynolds' number are taken, it is advisable to make a further change of independent variable. This means that a new set of coefficients in the expressions for the ψ- and π-residuals has to be calculated and an initial guess made for ψ and π.

20. The u, v, p method of solving the equations of viscous flow

An alternative method which can be used to solve the viscous-flow equations for the case R = 0 is given by Allen and Dennis (Ref. !). The velocity components and the pressure are used as variables, and the governing equations are taken to be (1.7) with the cross-product terms neglected, viz.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{\partial p}{\partial x},$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = -\frac{\partial p}{\partial y}, \qquad \text{.....(20.1)}$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = -\frac{\partial p}{\partial z}.$$

To satisfy the equation of continuity (2.1)

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0. \qquad \text{.....(20.2)}$$

These equations can be used in this form on a three-dimensional net to obtain a solution for the flow past the sphere - the advantage being that relaxation patterns are simple and the boundary conditions easy to satisfy. On the other hand the symmetry is not used. We therefore transform to cylindrical polar co-ordinates (r,z). In this case equations (20.1). (20.2) take the form:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\partial p}{\partial r}$$

$$\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial z^2} + \frac{1}{r}\frac{\partial w}{\partial r} = -\frac{\partial p}{\partial z} \qquad \text{.....(20.3)}$$

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} = 0.$$

The boundary conditions in this case are:-

(1) $w = V = 0$ on the sphere.

(2) $\frac{\partial w}{\partial z} = \frac{\partial V}{\partial z} = 0$ (by symmetry) on $z = 0$.

(3) $V \to 0$ and $w \to U$, as $r^2 + z^2 \to \infty$.

(4) $V = 0$, $\frac{\partial w}{\partial r} = 0$ for $r = 0$ $\quad z \geqslant 1/2$.

(5) the pressure p is atmospheric for $r^2 + z^2 \to \infty$ and this can be taken as zero.

(6) $\frac{\partial p}{\partial z} = 0$ (by symmetry) on $z = 0$.

(7) $p = 0$, $r = 0$, $z \gg 1/2$.

(8) the value of the pressure p on the sphere is obtained from the governing equations (20.3) since w and V are both known there. Again there is the difficulty found in section 9 and a further transformation of co-ordinates is needed in order that the circle $r^2 + z^2 = 1/4$ be made a straight line.

If a solution is required over the whole field then the transformation to the (x,ϕ) co-ordinates must be used. In this case the governing equations are

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial \phi^2} + \cot\phi\frac{\partial w}{\partial \phi} = \frac{e^x}{2}(\cos\phi\frac{\partial p}{\partial x} - \sin\phi\frac{\partial p}{\partial \phi})$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial x} + \frac{\partial^2 V}{\partial \phi^2} + \cot\phi\frac{\partial V}{\partial \phi} - \frac{V}{\sin^2\phi} = \frac{e^x}{2}(\sin\phi\frac{\partial p}{\partial x} + \cos\phi\frac{\partial p}{\partial \phi}) \quad ..(20.4)$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial p}{\partial x} + \frac{\partial^2 p}{\partial \phi^2} + \cot\phi\frac{\partial p}{\partial \phi} = 0.$$

The equations are of a similar type to (18.5) and (18.6) except that in this case we have three variables. It seems therefore that for a case of flow which is symmetrical about an axis it is easier to work with the variables ψ and π. Using these latter variables the viscous flow equations for non-zero R can be obtained directly from those for which R = 0. Using the u, v, p equations, however, for non-zero R, alters the governing equation for p. The equation (20.2) is replaced by an equation which includes products of the first and second order derivatives. Though equations of this type are not difficult to solve using relaxation methods, the time taken in the re-calculation of residuals is considerable.

However the variables ψ and π cannot be used for general three-dimensional viscous flow and in this case the u, v, w, p equations must be used, starting with the case R = 0, and using the solution in this case as a good guess for further values of the Reynolds' number.

21. Conclusion

For plane laminar motion past an axi-symmetrical body of a viscous incompressible fluid, it has been shown in the previous sections, that the variables ψ and π can be used to obtain the flow patterns. The computations are related to a sphere, but the method described in sections 10-17 is applicable to any body with axial symmetry. For the case of a sphere, the initial conformal transformation changes the independent variables from r and z to α and β, the velocity-potential and stream-function for irrotational flow past a circular cylinder. This transformation can be used for a cross-section of any shape. If there is not an algebraic expression for α and β, then the values of $\partial\alpha/\partial r$, $\partial\beta/\partial r$ and H^2 can be obtained using relaxation methods (Allen and Evenett, Ref. 2).

A solution to the equations of motion for viscous flow past a sphere was obtained for the Case R = 0. Algebraic expressions for ψ and π exist in this case and it was possible to compare the two sets of results. It was found that the solutions did not agree, and that the reason for this discrepancy was that the conditions of uniform flow at infinity were not satisfied, when the boundary conditions were written in terms of first order differentials. It has been shown that if these boundary conditions are expressed in terms of second order

differentials then the agreement between the two sets of results is good.

By considering the residuals given in Fig. 26 it can be seen that, if the solution had been obtained by relaxation methods, then the largest errors in ψ would have been near the sphere, and in particular on the line $\alpha = 1$. From Fig. 26 we notice that in this area, we have the largest distance between two mesh points. Hence, to obtain a better solution, a more accurate finite difference approximation to the differential equations must be used. If a graded net is introduced then further values of the modulus of the transformation will have to be calculated. Another method is to use difference tables. Finally, it is possible to expand the function under consideration in a power series up to terms of the fourth order and to bring the values of the function at the points 9, 10, 11, 12 into the residual formula. We do not think that it is necessary to introduce these points into the relaxation pattern as the contribution from them is small. The value is used as a correction term in the re-calculation of residuals.

Thus, by using relaxation techniques, we have a method of obtaining a solution to the equations of motion for viscous flow past an axi-symmetric body. The re-calculation of residuals is tedious but the convergence of the two functions is reasonably rapid.

PART V

The diffraction of a plane step shock round an infinite plane convex corner of angle δ

22. Introduction

The problem considered in this section is that of a plane shock, followed by uniform flow, which is moving into still air bounded by a semi-infinite wall perpendicular to the shock. At some line parallel to the shock this plane wall gives place to a second semi-infinite plane wall, inclined to it at an angle δ, and forming a convex corner. As the shock passes this corner it begins to be diffracted, and thereafter will not be a plane, being weaker near the wall than elsewhere. The viscous stresses and heat conduction can be reasonably neglected, since neither will have time to produce substantial effects during the extremely rapid passage of the shock. In consequence the only physical constants defining the problem are the original shock velocity, the pressure and density in still air, and δ. Since these cannot be combined to produce a fundamental length-or-time scale, it follows that while, in this two dimensional problem, each physical quantity is a function of the two space coordinates x,y (with the corner as origin) and the time t, these variables can only occur therein in the combinations x/t, y/t, i.e. the flow is pseudo-stationary.

In a note on this problem, Doris M. Jones, P. Moira E. Martin and C. K. Thornhill (Ref. 8) list the papers where,

when the pseudo-stationary property is assumed, the existing experimental and theoretical knowledge can be found. Lighthill Ref. [10]) has obtained a solution, by assuming that 'δ and hence also the variations in velocity and pressure behind the shock, are small'. As yet, no complete solution for the whole field of flow has been given. It is the purpose of this investigation to see if a solution to this problem can be obtained using relaxation techniques.

23. The Equations of pseudo-stationary compressible flow

In order to have the governing equations in a form suitable for applying relaxation techniques we also assume:-

(1) that the shock is of moderate strength only, and that the shock equations can be replaced by an approximate set of relations, such that the entropy has the same value at all points of the flow,

(2) that the effects of viscosity, heat conduction, etc. can be neglected, so that the flow is irrotational,

(3) that the flow separates at the corner, leaving an area of zero pressure, bounded by a free streamline.

Equation (1.10) can be written as

$$p/\rho^{\gamma} = \text{const.}, \qquad \text{.....(23.1)}$$

using assumption (1).

The equations of motion (1.7) for two dimensional motion are:-

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad \text{.....(23.2)}$$

using assumption (2).

The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0. \qquad \ldots\ldots(23.3)$$

A flow is called pseudo-stationary about the origin of co-ordinates, if in terms of the new variables,

$$\xi = x/t, \qquad \eta = y/t, \qquad \zeta = t, \qquad \ldots\ldots(23.4)$$

it is independent of ζ .

From equation (23.4) we have

$$\frac{\partial}{\partial x} = \frac{1}{t}\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial t} = -\frac{\xi}{t}\frac{\partial}{\partial \xi} - \frac{\eta}{t}\frac{\partial}{\partial \eta}. \qquad \ldots\ldots(23.5)$$

The motion is independent of ζ , so combining (23.2), (23.3) and (23.5) we obtain the following set of equations:-

$$(u - \xi)\frac{\partial \rho}{\partial \xi} + (v - \eta)\frac{\partial \rho}{\partial \eta} + \rho\left(\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta}\right) = 0, \qquad \ldots\ldots(23.6)$$

$$(u - \xi)\frac{\partial u}{\partial \xi} + (v - \eta)\frac{\partial u}{\partial \eta} = -\frac{1}{\rho}\frac{\partial p}{\partial \xi}, \qquad \ldots\ldots(23.7)$$

and $$(u - \xi)\frac{\partial v}{\partial \xi} + (v - \eta)\frac{\partial v}{\partial \eta} = -\frac{1}{\rho}\frac{\partial p}{\partial \eta}. \qquad \ldots\ldots(23.8)$$

We now put $u - \xi = u'$, $v - \eta = v'$, $\ldots\ldots(23.9)$

Thus equations (23.6), (23.7) and (23.8) become

$$u'\frac{\partial u'}{\partial \xi} + v'\frac{\partial u'}{\partial \eta} + u' = -\frac{1}{\rho}\frac{\partial p}{\partial \xi}, \qquad \ldots\ldots(23.10)$$

$$u'\frac{\partial v'}{\partial \xi} + v'\frac{\partial v'}{\partial \eta} + v' = -\frac{1}{\rho}\frac{\partial p}{\partial \eta}, \qquad \ldots\ldots(23.11)$$

$$\frac{\partial u'}{\partial \xi} + \frac{\partial v'}{\partial \eta} + 2 = -\frac{1}{\rho}\left[u'\frac{\partial \rho}{\partial \xi} + v'\frac{\partial \rho}{\partial \eta}\right]. \qquad \ldots\ldots(23.12)$$

The motion is assumed irrotational, so a velocity potential ϕ exists such that

$$u' = \frac{\partial \phi}{\partial \xi}, \qquad v' = \frac{\partial \phi}{\partial \eta}, \qquad \text{.....(23.13)}$$

$$\text{i.e.} \quad \frac{\partial u'}{\partial \eta} = \frac{\partial v'}{\partial \xi}.$$

Equation (23.10) now becomes

$$u'\frac{\partial u'}{\partial \xi} + v'\frac{\partial v'}{\partial \xi} + \frac{\partial \phi}{\partial \xi} = -\frac{1}{\rho}\frac{\partial p}{\partial \xi} = \frac{-1}{\gamma - 1}\frac{\partial a^2}{\partial \xi},$$

where $a^2 = \frac{\gamma p}{\rho}$ is the speed of sound,

This equation can be integrated with respect to ξ, viz.

$$\frac{1}{2}(u'^2 + v'^2) + \phi + \frac{a^2}{\gamma - 1} = \text{const.} = 0 \text{ (say)} \qquad \text{.....(23.14)}$$

Equation (23.11) can be integrated with respect to η, yielding the same result.

Substituting for u' and v' in terms of ϕ, and for ρ in terms of a^2, in (23.12) we obtain the following equation for ϕ,

$$(a^2 - u'^2)\left(\frac{\partial^2 \phi}{\partial \xi^2} + 1\right) + (a^2 - v'^2)\left(\frac{\partial^2 \phi}{\partial \eta^2} + 1\right) - 2u'v'\frac{\partial^2 \phi}{\partial \xi \partial \eta} = 0. \qquad \text{.....(23.15)}$$

Hence equations (23.14) and (23.15) are to be solved, satisfying the boundary conditions, which have still to be stated in order to obtain the flow pattern for this problem.

24. Shock Relations

Certain relations exist between the pressure, density etc. in the regions to the right and left of a plane shock which is moving into still air at a speed $S a_o$. The method used in obtaining these relations is given by Courant and Friedrichs (Ref. 4).

We use the suffix o to denote the quantities ρ , p, a in the region to the right of the plane shock and the suffix 1 to denote the same quantities to the left of the shock.

The on-coming flux of matter must be the same as the departing flux of matter, viz.

$$\rho_1(S a_o - u_1) = \rho_o S a_o. \qquad \ldots\ldots(24.1)$$

where u_1 is the speed at any point behind the shock.

The difference in the pressure thrusts on a small line element $d\ell$ must be equal to the normal flux of momentum through $d\ell$, viz.

$$\begin{aligned} p_1 - p_o &= \rho_o(S a_o)^2 - \rho_1(S a_o - u_1)^2, \\ &= S a_o u_1. \end{aligned} \qquad \ldots\ldots(24.2)$$

The flux of energy is equal to the rate at which work is done by the pressure thrusts, viz.

$$\begin{aligned} p_1(S a_o - u_1) - p_o S a_o = {} & \rho_o S a_o [E_o + \tfrac{1}{2}(S a_o)^2] \\ & - \rho_1(S a_o - u_1)[E_1 + \tfrac{1}{2}(S a_o - u_1)^2], \end{aligned} \qquad \ldots\ldots(24.3)$$

where E is the internal energy $= \frac{1}{\gamma - 1}\frac{p}{\rho}$ from the definition on page 2. Combining (24.1), (24.2) and (24.3) yields

$$\rho_o S a_o \left[\frac{1}{2}\frac{p_1}{\rho_1} \frac{5}{2}\frac{p_o}{\rho_o} + \frac{1}{2}u_1^2\right] = p_1 u_1. \qquad \ldots\ldots(24.4)$$

Equations (24.1), (24.2) and (24.4) can be solved for u_1 and a_1, giving

$$u_1 = a_o(1 - \lambda^2)(S^2 - 1)/S, \qquad \ldots\ldots(24.5)$$

$$a_1^2 = a_o^2[1 + \lambda^2(S^2 - 1)][1 + (1 + \lambda^2)(S^2 - 1)]/S^2, \quad \ldots(24.6)$$

where $\lambda^2 = \frac{\gamma - 1}{\gamma + 1}$.

If we take $\gamma = 1.4$, then for equations (24.5) and (24.6) we have

$$u_1^2/a_1^2 = 25(S^2 - 1)^2/(5 + S^2)(7S^2 - 1) \qquad \ldots\ldots(24.7)$$

and this gives $u_1 < a_1$, if $S < 2.068$ and $u_1 > a_1$ if $S > 2.068$. When $u_1 < a_1$ we have the case of uniform flow behind the undisturbed shock FB in (x,y,t)-space, being sub-sonic. (see Fig. 31).

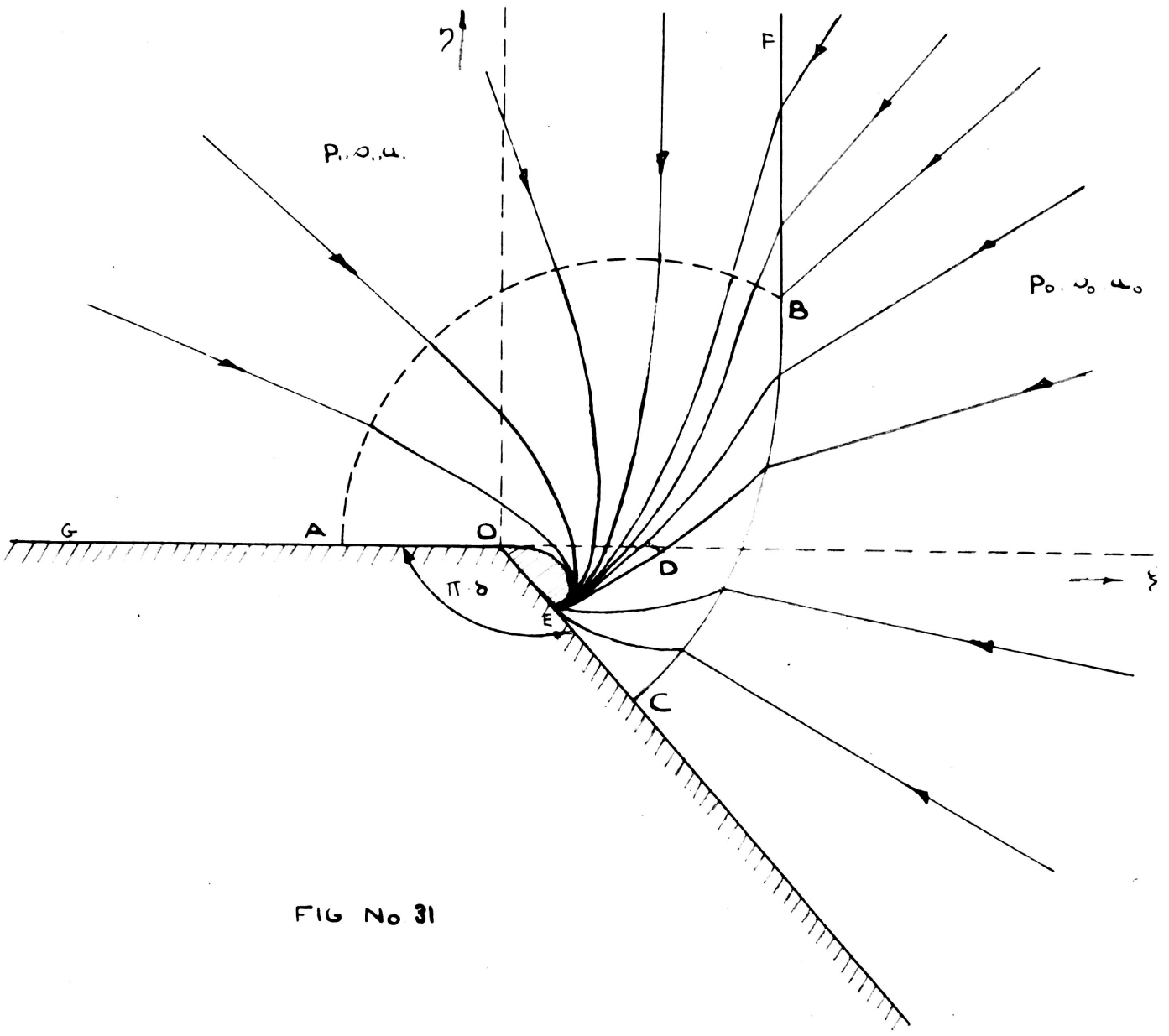


FIG No 31

25. The behaviour of the function ϕ

Fig. 31. illustrates the field of flow. It can be seen from the diagram that the area under consideration can be divided into four parts:

(i) the region ABCEO, where the equations (23.14) and (23.15 govern the flow

(ii) the region to the right of the shock FBC

The flow is isentropic and supersonic throughout the region. Since the flow is undisturbed, the velocity at any point $X(\xi, \eta)$ is directed towards the corner O and is equal in magnitude to $OX = (\xi^2 + \eta^2)^{1/2}$. We therefore have that

$$u' = -\xi,$$
$$v' = -\eta, \qquad \ldots\ldots(25.1)$$
$$a = a_0$$

Therefore, from equation (23.14), we have

$$\phi = -(\xi^2 + \eta^2)/2 - a_0^2/(\gamma-1) \qquad \ldots\ldots(25.2)$$

(iii) the region to the left of the curve GABF

Behind the shock there is an area GABF which is unaware of the existence of the corner O and which corresponds, in (x,y,t)-space to uniform flow behind the undisturbed shock. In this region it can be seen that the velocity at any point $X(\xi, \eta)$ is directed towards the point $D(u_1, o)$ and has mag-

nitude $DX[(\xi - u_1)^2 + \eta^2]^{1/2}$.

We have therefore that

$$u' = -\xi + u_1,$$
$$v' = -\eta, \qquad \text{.....(25.3)}$$
$$a = a_1.$$

Hence from equation (23.14), we obtain the following expression for ϕ:

$$\phi = -[(\xi - u_1)^2 + \eta^2]/2 - a_1^2/(\gamma - 1) \qquad \text{.....(25.4)}$$

(iv) the region between the curve OE and the wall. The behaviour of ϕ is only needed on the boundary OE, to obtain the flow pattern over the area ABCEO.

In order that relaxation techniques can be used to obtain the flow pattern in the region ABCEO we need to know the conditions which ϕ satisfies on those boundaries. The function ϕ is continuous over the curve AB, and when C.K. Thornhill suggested this problem, he gave the following proof that ϕ was continuous across BC.

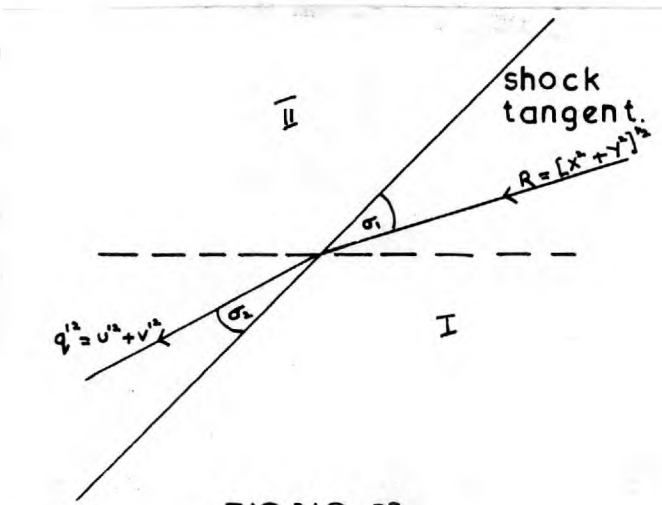


FIG.NO. 31.

Fig. 32 gives the directions of flow at a point on BC.

The shock BC is considered to be at rest. PQ is the shock tangent at a point on BC with co-ordinates (X,Y). The motion to the right of the shock is denoted by suffix 1 and the motion to the left of the shock by suffix 2. σ is the angle the velocity makes with the shock tangent. Similar relations to (24.1), (24.2) and (24.4) exist for flow normal to the shock, viz.

$$q' = \rho_2 \sin\sigma_2 = R\rho_1 \sin\sigma_1, \qquad \text{.....(25.5)}$$

$$\rho_2 q'^2 \sin^2\sigma_2 + p_2 = R\rho_1 \sin^2\sigma_1 + p_1, \qquad \text{.....(25.6)}$$

and

$$q'^2 \sin^2\sigma_2 + \frac{2}{\gamma-1}\frac{p_2}{\rho_2} = R^2 \sin^2\sigma_1 + \frac{2}{\gamma-1}\frac{p_1}{\rho_1}. \qquad \text{.....(25.7)}$$

where $q'^2 = u'^2 + v'^2$, $R^2 = \xi^2 + \eta^2$.

These equations can be solved for q', giving

$$q' \sin\sigma_2 = a_o[1 + \lambda^2(M_1^2 \sin^2\sigma_1 - 1)]/M_1 \sin\sigma_1 \qquad \text{.....(25.8)}$$

with $M_1 = R/a_o$.

The component of the velocity tangential to the shock is the same:

$$q' \cos\sigma_2 = R\cos\sigma_1. \qquad \text{.....(25.9)}$$

Equations (25.5), (25.6), (25.7) can be used to obtain a value for A, the speed of sound just behind the shock at (X,Y), viz.

$$\frac{A^2}{a_o^2} = \frac{[1+\lambda^2(M_1^2\sin^2\sigma_1-1)][1+(1+\lambda^2)[(M_1^2\sin^2\sigma_1-1)]}{M_1^2\sin^2\sigma_1} \quad \text{.....(25.10)}$$

From (25.8), (25.9) and (25.10) it can be shown that

$$\frac{A^2}{\gamma-1} + \frac{1}{2}q'^2 = \frac{1}{2}(X^2 + Y^2) + \frac{a_o^2}{\gamma-1}, \quad \text{.....(25.11)}$$

and that therefore ϕ is continuous across the shock BC. At any point on the shock BC,

$$\phi = -(X^2 + Y^2)/2 - \frac{a_o^2}{\gamma-1} \quad \text{.....(25.12)}$$

from (23.14).

26. Boundary Conditions

The simplest type of flow, compatible with the boundary conditions, is a flow in which the pseudo-stationary streamlines concur at a point E on the wall as illustrated in Fig. 31 . This satisfies the conditions that the flow has no normal component at the wall, and that the flow along the wall is directed away from the corner O just beyond O but towards the corner at the shock. We therefore have to satisfy the following conditions at the point E. Since E lies on the free streamline OE

$$a = 0 \text{ at } E,$$

and

$$\left(\frac{\partial Y}{\partial X}\right)_E = \left(\frac{\phi_\eta}{\phi_\xi}\right)_E. \qquad \text{.....(26.1)}$$

Since also E lies on EC

$$\frac{\partial \phi}{\partial \xi}\sin\delta + \frac{\partial \phi}{\partial \eta}\cos\delta = 0 \text{ at } E. \qquad \text{.....(26.2)}$$

There are two possibilities:

(a) that $\frac{\partial \phi}{\partial \xi}$ and $\frac{\partial \phi}{\partial \eta}$ are both non-zero at E, and

$\frac{\partial \phi}{\partial \xi}\sin\delta + \frac{\partial \phi}{\partial \eta}\cos\delta = 0$ on every streamline ooncurring at E.

i,e. all streamlines concurring at E, including the free stream line OE, must enter the wall tangentially. In this case

$$\frac{\partial \phi}{\partial \xi} = -\sqrt{-2\rho}\cos\delta$$

$$\frac{\partial \phi}{\partial \eta} = \sqrt{-2\rho}\sin\delta \qquad \text{.....(26.3)}$$

Also $\frac{\partial^2 \phi}{\partial \xi^2}\cos^2\delta + \frac{2\partial^2 \phi}{\partial \xi \partial \eta}\sin\delta\cos\delta + \frac{\partial^2 \phi}{\partial \eta^2}\sin^2\delta + 1 = 0$

or (b)

$\frac{\partial \phi}{\partial \xi} = \frac{\partial \phi}{\partial \eta} = 0$ and $a = 0$ at E, and it follows that $\phi = 0$ at E.

.....(26.4)

At the point O, we have $a = 0$, and $\frac{\partial \phi}{\partial \eta} = 0$.

Therefore from equation (23.14)

$$\left(\frac{\partial \phi}{\partial \xi}\right)^2 + 2\phi = 0 \qquad .....(26.5)$$

We have the following conditions on the other boundaries:-

(1) on OA the flow is parallel to the wall, therefore

$\partial\phi/\partial\eta = 0$ .....(26.6)

(2) on the curve AB, ϕ is given by the equation (25.4).

(3) on the curve BC, ϕ is given by the equation (25.12).

(4) on EC, the flow is parallel to the wall,

therefore $v'/u' = -\tan\delta$,

i.e. $\frac{\partial \phi}{\partial \xi}\sin\delta + \frac{\partial \phi}{\partial \eta}\cos\delta = 0.$ .....(26.7)

(5) On OE, $a = 0$ since we have assumed that we have an area of zero pressure near the point O, and the velocity is parallel to the tangent at any point: Hence

$$\frac{\delta Y}{\delta X} = \frac{v'}{u'} = \frac{\partial \phi}{\partial \eta}/\frac{\partial \phi}{\partial \xi} \qquad .....(26.8)$$

where (X,Y) denotes any point on the streamline OE and δX, δY are differentials along the streamline OE.

Hence we have the following two equations satisfied by ϕ_ξ, and ϕ_η :

$$\left(\frac{\partial \phi}{\partial \xi}\right)^2 + \left(\frac{\partial \phi}{\partial \eta}\right)^2 + 2\phi = 0, \qquad \text{.....(26.9)}$$

$$\frac{\partial \phi}{\partial \eta} = \frac{\partial \phi}{\partial \xi} \tan \psi, \qquad \text{.....(26.10}$$

where $\tan\psi = \frac{\delta Y}{\delta x}$.

Therefore from equations (26.9) and (26.10) we have

$$\frac{\partial \phi}{\partial \xi} = \pm \cos\psi \sqrt{-2\phi}, \qquad \text{.....(26.11)}$$

and

$$\frac{\partial \phi}{\partial \eta} = \pm \sin\psi \sqrt{-2\phi} \qquad \text{.....(26.12)}$$

27. The equation of the arc AB

The arc AB marks the outermost limit to which the knowledge of the existence of the corner may be transmitted to the flow in the absence of another shock outside AB.

The equation of the arc is

$$(\xi - u_1)^2 + \eta^2 = a_1^2. \qquad \ldots\ldots(27.1)$$

The co-ordinates of the point A are therefore $[-(a_1-u_1),0]$. The point B is at the intersection of the arc AB and the line $\xi = Sa_0$. Hence the co-ordinates of B are given by

$$\xi = Sa_0, \quad \eta = [a_1^2 - (Sa_0 - u_1)^2]^{1/2}. \qquad \ldots\ldots(27.2)$$

28. The finite difference approximations to the equations of compressible flow

The governing equations in this case are:

$$(a^2 - u^2)\left(\frac{\partial^2 \phi}{\partial \xi^2} + 1\right) + (a^2 - v^2)\left(\frac{\partial^2 \phi}{\partial \eta^2} + 1\right) - 2uv\frac{\partial^2 \phi}{\partial \xi \partial \eta} = 0, \quad (28.1)$$

and

$$\frac{1}{2}(u^2 + v^2) + \phi + \frac{a^2}{\gamma - 1} = 0. \qquad \ldots\ldots(28.2)$$

We write $a^2 - u^2 = C_1$, $a^2 - v^2 = C_2$, $\frac{1}{2}uv = C_3$ .....(28.3)

then equation (28.1) takes the form:

$$C_1\left(\frac{\partial^2 \phi}{\partial \xi^2} + 1\right) + C_2\left(\frac{\partial^2 \phi}{\partial \eta^2} + 1\right) - 4C_3\frac{\partial^2 \phi}{\partial \xi \partial \eta} = 0. \qquad \ldots\ldots(28.4)$$

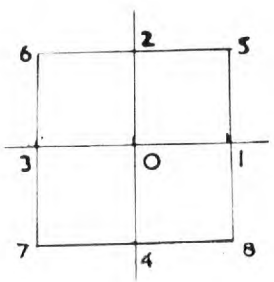


FIG. NO. 7b.

Hence the finite difference approximation to equation (28.4) is

$$C_1\left(\frac{\phi_1 + \phi_3 - 2\phi_0}{h^2} + 1\right) + C_2\left(\frac{\phi_2 + \phi_4 - 2\phi_0}{h^2} + 1\right) - C_3\left(\frac{\phi_5 - \phi_6 + \phi_7 - \phi_8}{h^2}\right) = 0,$$

where the points 5, 6, 7, 8 are shown in Fig. 7b.

Hence

$$F\phi = C_1\phi_1 + C_2\phi_2 + C_1\phi_3 + C_2\phi_4 - 2(C_1 + C_2)\phi_0 + C_3(-\phi_5 + \phi_6 - \phi_7 + \phi_8) + h^2(C_1 + C_2) \quad \ldots\ldots(28.5)$$

29. The Relaxation Treatment

For this particular example the value of S is taken to be 1.34, the angle δ to be 90°, and a_0 to be unity. We therefore have the case of subsonic flow behind the undisturbed shock in (xyt)-space.

Hence

$$a_1 = 1.102832, \qquad u_1 = 0.494776. \qquad \text{.....(29.1)}$$

Therefore A is the point (-0.608056, 0),

B " " " (1.34, 0.708403). .....(29.2)

Hence the boundary AB is the circle with centre (0.494776 , 0) and radius 1.102832.

The function ϕ is continuous over the boundary AB and is given by

$$\phi = -[(\xi - u_1)^2 + \eta^2]/2 - a_1^2/(\gamma - 1) \qquad \text{.....(29.3)}$$

outside AB. Hence we calculated values of ϕ at mesh points outside the boundary.

The boundary BC is not known. The function ϕ is also continuous over this boundary and is given by

$$\phi = -(\xi^2 + \eta^2)/2 - a_0^2/(\gamma - 1) \qquad \text{.....(29.4)}$$

outside BC.

Initially we guessed a position of this curve and calculated values of ϕ at mesh points outside it. These

values of ϕ from (29.3) and (29.4) give a good indication as to the initial guess for ϕ in the region OABC. (We found that this guess was improved if we assumed u = v = 0, a = 1 and relaxed over the whole field using a 4,1,1,1,1, pattern). A curve OE was then introduced. We then obtained values for the velocity components u and v from

$$u = \frac{\partial \phi}{\partial \xi} = \frac{\phi_1 - \phi_3}{2h}, \quad \text{and } v = \frac{\partial \phi}{\partial \eta} = \frac{\phi_2 - \phi_4}{2h}. \qquad \ldots\ldots(29.5)$$

Values of a^2 were then obtained at each mesh point using equation (28.2). Finally the quantities C_1, C_2 and C_3 defined in (28.3) were calculated and the residuals at all points in OABC were obtained. (The residual formulae for points near the boundary will be given in the next section). These residuals were then reduced to zero, using approximate relaxation patterns. The positions of the curves BC and OE were then altered to satisfy the boundary conditions there. New values were obtained for C_1, C_2, C_3 from the new ϕ, and the $F\phi$ were recalculated and again reduced to zero. This process was repeated until the boundary conditions were satisfied on OE and BC and all residuals were zero.

30. The Residual Formulae for points on or near the boundaries

On the wall EC, we have from equation (26.4),

$\partial\phi/\partial\xi = 0$, i.e. $\phi_1 = \phi_3$.

Hence the residual formulae for points on the boundary EC is

$$F_\phi = C_2\phi_2 + 2C_1\phi_3 + C_2\phi_4 - 2(C_1 + C_2)\phi_o + h^2(C_1 + C_2). \quad (30.1)$$

For points near the boundary AB we use equation (28.1) to calculate the residuals. We know the values of the functions ϕ outside AB and that the function ϕ is continuous over it.

On the wall AO, we have to satisfy equation (26.6). This yields $\phi_2 = \phi_4$. Hence the residual formulae for points on AO is

$$F_\phi = C_1\phi_1 + 2C_2\phi_2 + C_1\phi_3 - 2(C_1 + C_2)\phi_o + h^2(C_1 + C_2) \quad (30.2)$$

Next we consider the residual formulae for points near the boundary OE. Initially the curve OE crossed more mesh lines than is shown in Fig. 35 and several nodal points became fictitious. The appropriate one of the boundary conditions (26.11) or (26.12), (or a combination of both) was used, (depending on the direction of the mesh line crossing the boundary) to calculate the values of ϕ at these points. This led to several values of ϕ at each fictitious point and the appropriate one was used in the calculation of the F_ϕ.

The formulae given below are those used in the final solution. The first set give the values of ϕ at the fictitious points O. The second set are the residual formulae for the boundary points near OE. The points affected are shown in Fig. 33

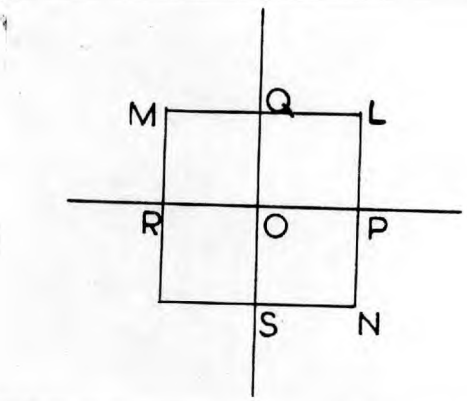


FIG. No. 33.

(1) Point P.

For P the point O is 'fictitious'. We first expand ϕ as a power series in ζ viz.

$$\phi = \phi_0 + A\zeta + B\zeta^2, \qquad \ldots\ldots(30.3)$$

where $A = (\phi_1 - \phi_3)/2h$, $B = (\phi_1 + \phi_3 - 2\phi_0)/2h^2$

From (30.3) we have the following expression for $\partial\phi/\partial\zeta$ at O:

$$\partial\phi/\partial\zeta = A - 2Bh,$$

$$= (-\phi_1 - 3\phi_3 + 4\phi_0)/2h. \qquad \ldots\ldots(30.4)$$

At O, we have to satisfy equation (26.5). Hence we have the following equation from which to obtain a value for ϕ_3

$$(\phi_1 + 3\phi_3 - 4\phi_0)^2 = -8h^2\phi_3, \qquad \ldots\ldots(30.5)$$

when $h = 0.1$, and we introduce a multiplying factor of 100 for ϕ, equation (30.5) is:

$$(\phi_1 + 3\phi_3 - 4\phi_0)^2 = -8\phi_3 \qquad \text{.....(30.6)}$$

(2) Point Q

For Q the point O is fictitious. By expanding ϕ in a power series in η, and satisfying the boundary condition that $\partial\phi/\partial\eta = 0$ at O, we obtain the following equation from which to obtain a value for ϕ_4:

$$\phi_2 + 3\phi_4 - 4\phi_0 = 0. \qquad \text{.....(30.7)}$$

(3) Point R

For the point R we obtain a formula for the value of ϕ at O by interchanging the suffixes 1 and 3 in equation (30.6):

$$(3\phi_1 + \phi_3 - 4\phi_0)^2 = -8\phi_1 \qquad \text{.....(30.8)}$$

(4) Point S

Before we could obtain a value for F_ϕ at S, we had to decide whether ϕ satisfied the equations (26.3) or was zero at the point E. When the a^2 were calculated after the function ϕ had been relaxed for the first time, it was obtious that ϕ satisfied the equations (26.3). By imposing the condition, $\phi = 0$ at E, we found that $\partial\phi/\partial\eta$ was so large, that the values obtained for a^2 from equation (28.2) were negative.

The second of equations (26.3) was used to obtain the value of ϕ at O for the point S. The point E was given co-ordinates $[(\theta h, 0)\ 0 < \theta < 1]$, with the point S as origin and the function ϕ expanded in a power series in η. Using the method

which was applied to obtain the value of ϕ at the first three points we obtained the following equation from which the value of ϕ_2 was calculated.

$$[(1+2\theta)\phi_2-(1-2\theta)\phi_4-4\theta\phi_0]^2 = -4[2(1-\theta^2)\phi_0+\theta(\theta+1)\phi_2+\theta(\theta-1)\phi_4], \quad .....(30.9)$$

since $\delta = 90^\circ$.

When these values of ϕ were known at the point O then values of the quantities u, v were calculated at the points P, Q, R and S from equations (29.5).

The expression for F_ϕ at the points L, M, N includes the value of ϕ at the point O. At P and Q, the finite difference approximation to $\frac{\partial^2\phi}{\partial\xi\partial\eta}$ includes the differences $(\phi_Q-\phi_S)$ and $(\phi_P-\phi_R)$ respectively, and there is a discontinuity of ϕ at O. We therefore obtained other expressions for $\frac{\partial^2\phi}{\partial\xi\partial\eta}$ at these points.

We expand ϕ as a power series in ξ , η :

$$\phi = \phi_0 + A\xi + B\eta + C\xi^2 + D\eta^2 + E\xi\eta, \quad .....(30.10)$$

where

$$A = (\phi_1-\phi_3)/2h, \qquad C = (\phi_1 + \phi_3 - 2\phi_0)2h^2,$$
$$B = (\phi_2-\phi_4)/2h, \text{ and} \qquad D = (\phi_2 + \phi_4 - 2\phi_0)2h^2, \quad ...(30.11)$$

From equation (30.10), we have

$$\begin{aligned} \phi_5 &= -\phi_0 + \phi_1 + \phi_2 + Eh^2, \\ \phi_6 &= -\phi_0 + \phi_2 + \phi_3 - Eh^2, \\ \phi_7 &= -\phi_0 + \phi_3 + \phi_4 + Eh^2, \\ \text{and } \phi_8 &= -\phi_0 + \phi_4 + \phi_1 - Eh^2. \end{aligned} \qquad \ldots\ldots(30.12)$$

Hence, we obtain for $\frac{\partial^2 \phi}{\partial \xi \partial \eta}$ at L, N, P the following expression:

$$\frac{\partial^2 \phi}{\partial \xi \partial \eta} = (\phi_5 - \phi_8 - \phi_2 + \phi_4)/2h^2, \qquad \ldots\ldots(30.13)$$

and at M and Q:

$$\frac{\partial^2 \phi}{\partial \xi \partial \eta} = (\phi_5 - \phi_6 - \phi_1 + \phi_3)/2h^2. \qquad \ldots\ldots(30.14)$$

Thus, the residual formulae for the points L, N, P is:

$$\begin{aligned} F\phi = {} & C_1\phi_1 + C_2\phi_2 + C_1\phi_3 + C_2\phi_4 - 2(C_1+C_2)\phi_0 \\ & + 2C_3(-\phi_5+\phi_8+\phi_2-\phi_4) + h^2(C_1+C_2), \end{aligned} \qquad \ldots\ldots(30.15)$$

and for points M and Q is:

$$\begin{aligned} F\phi = {} & C_1\phi_1+C_2\phi_2+C_1\phi_3+C_2\phi_4 - 2(C_1+C_2)\phi_0 \\ & + 2C_3(-\phi_5+\phi_0+\phi_1-\phi_3) + h^2(C_1+C_2) \end{aligned} \qquad \ldots\ldots(30.16)$$

All the quantities in the above expressions are known and hence we can calculate the residuals at all the points near the boundary OE.

Finally, there is the boundary BC. We have shown that the function ϕ is continuous over BC. The components of velocity u and v are not, since the velocity, normal to the

shock tangent is discontinuous. If ϕ_X denotes the value of ϕ at some point on BC, then we have, at points near the boundary BC, the following expressions for u, v and their derivatives:

$$u = \frac{(\theta^2-1)\phi_0+\phi_X-\theta^2\phi_3}{\theta(\theta+1)h}, \qquad v = \frac{\theta^2\phi_2-\phi_X+(1-\theta^2)\phi_0}{\theta(\theta+1)h}, \tag{30.17}$$

$$\frac{\partial^2\phi}{\partial\xi^2} = \frac{2(\theta\phi_3+\phi_X-(1+\theta)\phi_0)}{h^2\theta(1+\theta)}, \text{ and } \frac{\partial^2\phi}{\partial\eta^2} = \frac{2(\theta\phi_2+\phi_X-(1+\theta)\phi_0}{h^2\theta(1+\theta)},$$

where $0<\theta<1$.

In equations (30.17) all the quantities are known with the exception of ϕ_X and θ. In section 29, we stated that it is necessary to make an initial guess for the curve BC. Thus we know a value for θ and ϕ can be calculated from the equation

$$\phi = -(X^2+Y^2)/2 - a_0^2/(\gamma-1), \qquad .....(30.18)$$

where (X,Y) are the co-ordinates of a point on the curve BC.

A value for $\frac{\partial^2\phi}{\partial\xi\partial\eta}$ is obtained from one of the expressions given in (30.11), (30.12) or (30.13).

31. The equations which determine the position of the curves BC and OE

In section 29 we state that after the residuals have been reduced to zero the positions of the curves BC and OE are altered to satisfy the boundary conditions there.

For the boundary OE we use one of the equations (26.11) or (26.12) to calculate the residuals near OE. The other equation is used to check the position of the curve. In the final solution we have to satisfy the condition $\partial\phi/\partial\xi = 0$ at the point E. At the point with co-ordinates (0, -0.1) the function ϕ is expanded in a power series as in equation (30.10). Relative to the point (0,-0.1) the point E has co-ordinates (0,+θh). Therefore,

$$\left(\frac{\partial\phi}{\partial\xi}\right)_E = (\phi_1 - \phi_3)/2h + \theta(\phi_8 + \phi_0 - \phi_1 - \phi_4)/h = 0.$$

At the point (0,-0.1), the flow is normal to the wall. Hence

$$\phi_1 = \phi_3.$$

Therefore, to satisfy the second boundary condition at E, we must have

$$\phi_0 + \phi_8 = \phi_1 + \phi_4,$$

where the point o has co-ordinates (0,-0.1) relative to 0.

For the boundary BC, we know that the component of velocity parallel to the shock tangent is continuous. Hence

$$\cos\psi\frac{\partial\phi}{\partial\xi} + \sin\psi\frac{\partial\phi}{\partial\eta} = -X\cos\psi - Y\sin\psi, \qquad \ldots\ldots(31.2)$$

where $\tan\psi = \frac{\partial Y}{\partial X}$, and is known for the curve BC.

We can obtain a value to the left hand side of the equation (31.2) using the values of ϕ which lie within the area OABC. This result can be compared with the value of the right-hand side of this equation and the boundary BC adjusted.

The finite difference approximation to equation (31.2) is found by expanding ϕ in a power series in terms of ξ and η;

$$\phi = \phi_0 + A\xi + B\eta + C\xi^2 + D\eta^2 + E\xi\eta, \qquad \ldots\ldots(31.3)$$

where A, B, C, D are given by (30.11), or in the case of an 'irregular star' by (30.17), and E can be obtained from (30.12). Hence the right-hand side of the equation is given by

$$\cos\psi\frac{\partial\phi}{\partial\xi} + \sin\psi\frac{\partial\phi}{\partial\eta} = (A + 2C\xi + E\eta)\cos\psi + (B + 2D\eta + E\xi)\sin\psi, \qquad \ldots(31.4)$$

the quantities ABCDE being evaluated at the mesh point near the boundary.

As an alternative method for checking the position of the boundary BC, we expand ϕ on a line ξ = const in terms of η :

$$\phi = \phi_0 + B\eta + D\eta^2, \qquad \ldots\ldots(31.5)$$

where

$B = (-3\phi_0 - \phi_{10} + 4\phi_2)/2h, \qquad D = (\phi_0 + \phi_{10} - 2\phi_2)/2h^2.$

Therefore if a point on the boundary has co-ordinates $(\theta h, \eta)$,

$$2\phi_X = (\theta+2)(\theta+1)\phi_0 + \theta(\theta+1)\phi_{10} - 2\theta(\theta+2)\phi_2 \qquad .....(31.6)$$

with a similar expression for points on a line η = const.

The value of ϕ obtained from equation (31.6) can be compared with the value obtained from (30.18) and the boundary BC altered.

The position of the curves OE and BC are given in Fig. 35.

											3765	3763	3766							
											X	X	X							
						3768	3724	3689	3665	3650	3645	3651	3667	3692	3728	3773				
						X	X	X	X	X		X	X	X	X	X				
					3782	3757	3663	3618	3583	3558	3543	3539	3544	3560	3586	3622	3668	3724	3789		
					X	X	X										X	X	X		
				3761	3687	3622	3567	3521	3485	3459	3444	3439	3445	3461	3488	3526	3573	3629	3694	3770	
				X	X														X	X	
			3761	3676	3601	3536	3479	3432	3395	3368	3352	3347	3353	3370	3398	3437	3486	3544	3610	3685	3770
			X	X															X	X	
		3780	3686	3601	3525	3458	3400	3352	3313	3385	3269	3263	3269	3287	3316	3356	3406	3465	3534	3610	3695
		X	X																	X
		3715	3622	3536	3459	3390	3330	3280	3239	3210	3193	3187	3193	3212	3242	3283	3334	3395	3465	3544	3660
		X																		X
	3765	3660	3567	3480	3402	3332	3270	3216	3173	3142	3124	3118	3125	3145	3176	3218	3270	3333	3405	3486	3605
	X	X																		X
	3720	3616	3522	3434	3355	3284	3219	3161	3114	3081	3062	3057	3065	3086	3119	3162	3216	3280	3333	3437	3560
	X																			X
	3685	3582	3486	3398	3318	3246	3178	3111	3058	3023	3005	3002	3012	3035	3069	3115	3170	3236	3311	3397	3525
	X																			X
3774	3660	3557	3461	3372	329	322	315	306	300	2964	2951	2951	2965	2991	3028	3076	3134	3201	3278	3365	3500
X	X																		X	X	
3759	3646	3542	3446	3358	328	321	314	299	294	2909	2900	2906	2925	2955	2995	3046	3106	3176	3255	3350	3485
X																				X	X
3754	3641	3537	3441	3354	328	323	314 305	312	293	286	2849	2853	2867	2892	2926	2970	3024	3087	3160	3241	3345
X							254													X	
							269	270	277	2797	2816	2837	2867	2906	2954	3011	3077	3153	3237	3350	
																				X	
							2697	2706	2737	2764	2789	2818	2853	2896	2947	3007	3077	3156	3242	3365	
																				X	
							2703	2710	2729	2753	2780	2812	2851	2897	2951	3014	3087	3169	3265	3390	
																			X	X	
							2716	2721	2736	2758	2786	2820	2861	2909	2966	3032	3107	3191	3300		
																			X		
							2737	2742	2755	2776	2805	2840	2883	2934	2993	3061	3139	3230	3345		
																		X	X		
							2768	2773	2786	2807	2836	2873	2917	2969	3030	3100	3182	3285	3400		
																		X	X		
							2810	2815	2828	2849	2879	2917	2962	3016	3079	3152	3245	3335			
																	X	X			
							2863	2868	2881	2903	2933	2972	3018	3074	3140	3225	3320				
															X	X	X				
							2928	2933	2947	2969	2999	3039	3087	3150	3225						
														X	X						
							3005	3010	3024	3047	3080	3125	3180	3245							
											X	X	X	X							
							3105	3110	3125	3150	3185										
							X	X	X	X	X										

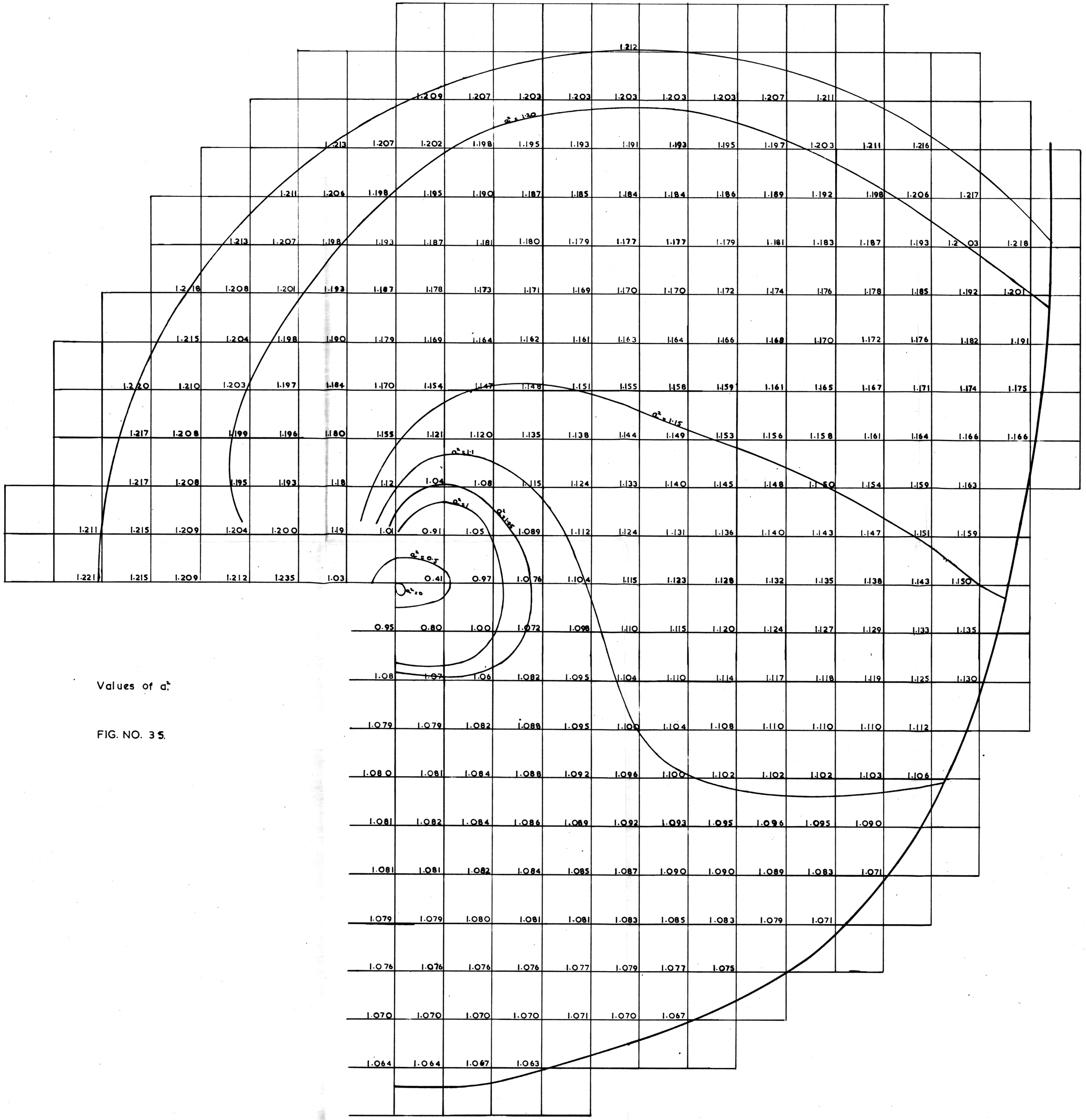
$-\phi$ (X) $\equiv \not\phi$ outside the boundary.

$-\phi$ (X) = relaxation solution for ϕ.

FIG. NO. 34.

Values of a^2

FIG. NO. 35.



32. Conclusions

The solution of equation (28.5) which satisfies the boundary conditions is given in Fig. 34. In the region of the point O, a multiplying factor of 100 has been used in obtaining this solution, elsewhere this factor is 1000. In Fig. 35 the values of a^2 are given, and curves of constant a^2 have been drawn.

The chief difficulties encountered in obtaining this solution were in the region near O, elsewhere the function ϕ converged to the final solution quickly. It was found, that for small changes in ϕ at one of the points P, Q, R, S given in Fig.[33] the coefficients C_1 and C_2 changed considerably and it was very difficult to estimate the effect of changing the value of ϕ at any one of these points. (This is the main reason for the smaller multiplying factor introduced in this region). Also, when calculating the value of ϕ at O from equations (30.6) and (30.8) we obtained two solutions for ϕ. The value of ϕ which made $\partial\phi/\partial\zeta$ positive at O was taken.

From Fig. 35 it can be seen that the value of a^2 at the point (-0.2,0) is too large. The reason is that the values of ϕ at (-0.2,0) and (-0.1,0) must be approximately the same to obtain a reasonable value of ϕ from equation (30.8). This applies at the other mesh points surrounding O. This

difficulty will be partially overcome if a finer net-work is introduced in the area near O, since the values of ϕ will not change so rapidly.

By introducing this finer network in the region near O, it should be possible to obtain a more accurate curve OE. From this investigation we know that the curve is parallel to AO at O, and to OE at E. The position of the point E is (+0, -0.02). C. K. Thornhill has a photograph which shows the shape of OE to be like an 'inverted comma' and OE = 0.04.

In connection with this problem, experiments have been carried out in the Physics Department of Princeton University (Ref. 14). A typical set of results is given in Fig. 36. A diagram comparing the experimental and calculated positions of the shock BC is not given as the two curves obtained are practically indistinguishable. The most our curve differs from the experimental one is by 0.1 of a mesh length.

Curves of constant density are also given in Fig. 36. These correspond to our curves of constant a^2, since

$$\frac{a^2}{a_0^2} = \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = (1 + 0.074n)^{\gamma-1},$$

and it can be seen from the diagrams that they are of the same pattern. Near the point B the value of a^2 has been

calculated as 1.216 from the experimental results and near the point C the value of a^2 is 1.0575. In either case the value of a^2 obtained from the relaxation solution is higher but is within 1 per cent of the experimental values.

The main area of disagreement is near the point O. From Fig. 36 it can be seen that there is no curve for which $a^2 = 0$, and that the smallest value of n is away from the wall OC. It appears from the values of a^2 calculated from the relaxation solution that a may be zero at some point other than at points on the curve OE. For the relaxation treatment it is not necessary to have a boundary condition at O and it would be interesting to see the variation in ϕ along the wall EO in this case.

Thus, we have shown that it is possible to obtain a solution to this problem, using relaxation methods and that over a large part of the area our results agree favourably with the experimental ones.

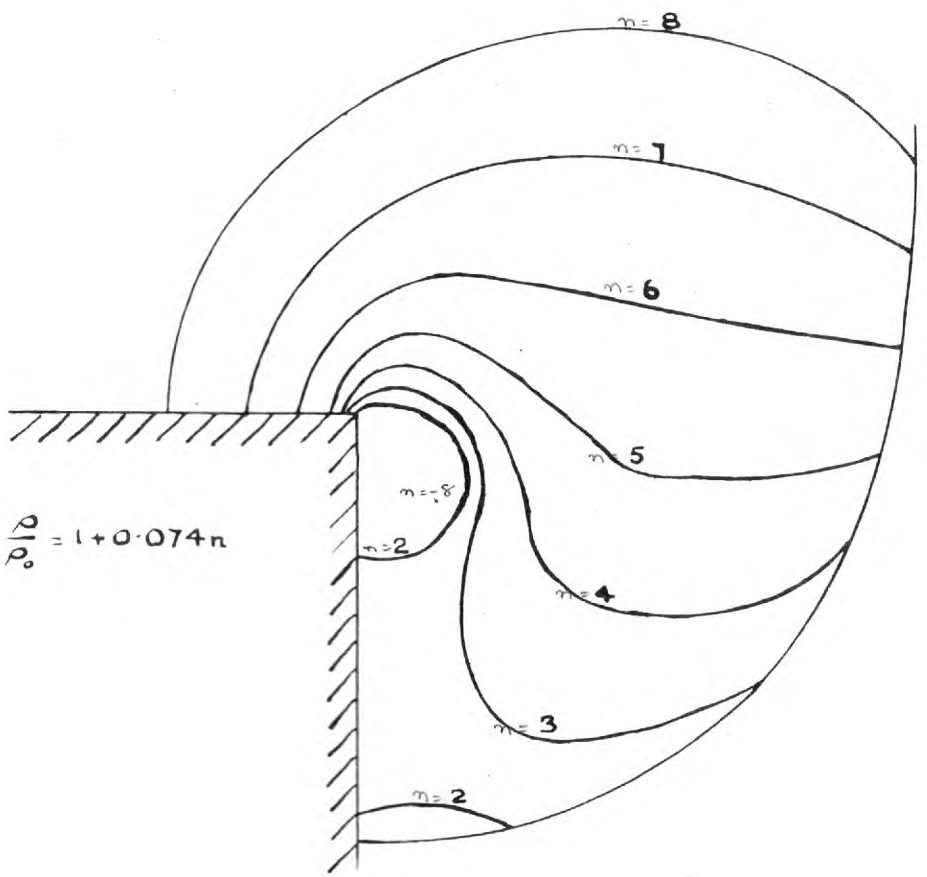


FIG. No 36.

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$\alpha = 0$

β	r	z	A	B	C/1.6
0	0.50000	-	-	1	0.15625
1/4	0.64039	-	-	0.97014	0.24123
1/2	0.80902	-	-	0.89443	0.32726
3/4	1.00000	-	-	0.80000	0.40000
1	1.20711	-	-	0.70711	0.45535
5/4	1.42539	-	-	0.62469	0.49554
3/2	1.65139	-	-	0.55470	0.52444
7/4	1.88278	-	-	0.49614	0.54536
2	2.11803	-	-	0.44721	0.56076
9/4	2.35611	-	-	0.40614	0.57729
5/2	2.59629	-	-	0.37139	0.58110
11/4	2.83809	-	-	0.34174	0.58794
3	3.08114	-	-	0.31623	0.59334
13/4	3.32518	-	-	0.29409	0.59767
7/2	3.57008	-	-	0.27472	0.60118
15/4	3.81552	-	-	0.25766	0.60408
4	4.06155	-	-	0.24254	0.60648
17/4	4.30803	-	-	0.22904	0.60850
9/2	4.55489	-	-	0.21693	0.61021
19/4	4.80206	-	-	0.20601	0.61167
5	5.04951	-	-	0.19612	0.61292
21/4	5.29719	-	-	0.18711	0.61401
11/2	5.54508	-	-	0.17889	0.61496
23/4	5.79315	-	-	0.17134	0.61579

$\alpha = 1/4$

β	r	z	A	B	C/1.6
0	0.48412	0.12500	-	-	-
1/4	0.62597	0.15619	0.18617	1.00965	0.25814
1/2	0.79784	0.18204	0.11200	0.92433	0.34491
3/4	0.99207	0.20096	0.06343	0.81809	0.41416
1	1.20165	0.21408	0.03591	0.71717	0.46533
5/4	1.42163	0.22307	0.02090	0.63022	0.50244
3/2	1.64875	0.22931	0.01264	0.55779	0.52888
7/4	1.88089	0.23373	0.00795	0.49793	0.54834
2	2.11665	0.23694	0.00518	0.44829	0.56279
9/4	2.35507	0.23932	0.00350	0.40681	0.57371
5/2	2.59550	0.24113	0.00243	0.37182	0.58211
11/4	2.83747	0.24252	0.00174	0.34203	0.58867
3	3.08065	0.24362	0.00127	0.31642	0.59389
13/4	3.32479	0.24450	0.00095	0.29422	0.59808
7/2	3.56971	0.24521	0.00072	0.27482	0.60150
15/4	3.81526	0.24580	0.00056	0.25773	0.60432
4	4.06133	0.24628	0.00044	0.24259	0.60667
17/4	4.30784	0.24669	0.00035	0.22908	0.60865
9/2	4.55473	0.24703	0.00028	0.21696	0.61033
19/4	4.80192	0.24733	0.00023	0.20603	0.61177
5	5.04939	0.24758	0.00019	0.19613	0.61301
21/4	5.29709	0.24780	0.00015	0.18713	0.61408
11/2	5.54500	0.24799	0.00013	0.17890	0.61502
23/4	5.79308	0.24815	0.00011	0.17135	0.61584

$\alpha = 1/2$

β	r	z	A	B	C/1.6
0	0.43301	0.25000	-	-	-
1/4	0.58088	0.31855	0.42690	1.16098	0.32268
1/2	0.76454	0.37147	0.22993	1.02602	0.40390
3/4	0.96916	0.40779	0.12200	0.87393	0.45709
1	1.18607	0.43220	0.06728	0.74665	0.49414
5/4	1.41091	0.44881	0.03894	0.64606	0.52119
3/2	1.64121	0.46039	0.02362	0.56661	0.54142
7/4	1.87547	0.46865	0.01495	0.50303	0.55677
2	2.11265	0.47469	0.00982	0.45136	0.56858
9/4	2.35205	0.47921	0.00667	0.40873	0.57777
5/2	2.59318	0.48266	0.00467	0.37306	0.58502
11/4	2.83565	0.48534	0.00335	0.34285	0.59080
3	3.07920	0.48746	0.00246	0.31699	0.59547
13/4	3.32362	0.48916	0.00184	0.29462	0.59928
7/2	3.56875	0.49055	0.00140	0.27510	0.60243
15/4	3.81447	0.49169	0.00109	0.25794	0.60504
4	4.06067	0.49264	0.00085	0.24274	0.60724
17/4	4.30729	0.49344	0.00068	0.22919	0.60911
9/2	4.55426	0.49411	0.00055	0.21705	0.61070
19/4	4.80152	0.49469	0.00045	0.20610	0.61207
5	5.04904	0.49519	0.00037	0.19619	0.61326
21/4	5.29679	0.49562	0.00030	0.18717	0.61429
11/2	5.54473	0.49600	0.00025	0.17893	0.61519
23/4	5.79284	0.49633	0.00021	0.17138	0.61599

$\alpha = 3/4$

β	r	z	A	B	C/1.6
0	0.33072	0.37500	-	-	-
1/4	0.50000	0.50000	0.80000	1.60000	0.50000
1/2	0.71164	0.57808	0.33796	1.23175	0.51639
3/4	0.93460	0.62630	0.16437	0.96731	0.52556
1	1.16286	0.65786	0.08870	0.79215	0.53699
5/4	1.39486	0.67944	0.05148	0.66983	0.54881
3/2	1.62981	0.69467	0.03157	0.57978	0.55971
7/4	1.86716	0.70572	0.02024	0.51070	0.56919
2	2.10645	0.71392	0.01347	0.45602	0.57721
9/4	2.34734	0.72014	0.00926	0.41166	0.58390
5/2	2.58953	0.72494	0.00654	0.37497	0.58946
11/4	2.83278	0.72871	0.00473	0.34414	0.59408
3	3.07690	0.73171	0.00350	0.31787	0.59794
13/4	3.32176	0.73414	0.00264	0.29524	0.60117
7/2	3.56722	0.73613	0.00202	0.27555	0.60389
15/4	3.81320	0.73777	0.00157	0.25827	0.60619
4	4.05961	0.73915	0.00124	0.24298	0.60816
17/4	4.30639	0.74031	0.00099	0.22938	0.60984
9/2	4.55349	0.74129	0.00080	0.21719	0.61130
19/4	4.80086	0.74214	0.00065	0.20621	0.61256
5	5.04847	0.74287	0.00054	0.19627	0.61366
21/4	5.29629	0.74350	0.00045	0.18724	0.61462
11/2	5.54429	0.74406	0.00037	0.17899	0.61547
23/4	5.79246	0.74454	0.00032	0.17142	0.61623

$\alpha = 1$

β	r	z	A	B	C/1.6
0	0	0.50000	-	-	-
1/4	0.39108	0.73489	1.05224	2.80803	0.85957
1/2	0.65012	0.81240	0.36399	1.53381	0.65646
3/4	0.89516	0.86047	0.17419	1.08196	0.60148
1	1.13601	0.89308	0.09564	0.84526	0.58364
5/4	1.37588	0.91618	0.05687	0.69740	0.57928
3/2	1.61603	0.93301	0.03573	0.59521	0.58034
7/4	1.85691	0.94556	0.02341	0.51983	0.58354
2	2.09868	0.95509	0.01588	0.46167	0.58741
9/4	2.34134	0.96245	0.01109	0.41528	0.59130
5/2	2.58482	0.96823	0.00794	0.37737	0.59492
11/4	2.82902	0.97282	0.00581	0.34576	0.59818
3	3.07387	0.97653	0.00434	0.31900	0.60106
13/4	3.31928	0.97955	0.00330	0.29604	0.60358
7/2	3.56518	0.98205	0.00254	0.27613	0.60577
15/4	3.81149	0.98412	0.00199	0.25870	0.60768
4	4.05817	0.98587	0.00158	0.24331	0.60935
17/4	4.30517	0.98735	0.00127	0.22962	0.61081
9/2	4.55244	0.98861	0.00103	0.21738	0.61209
19/4	4.79996	0.98970	0.00084	0.20636	0.61321
5	5.04769	0.99064	0.00070	0.19639	0.61420
21/4	5.29561	0.99146	0.00058	0.18733	0.61508
11/2	5.54370	0.99218	0.00049	0.17906	0.61586
23/4	5.79193	0.99281	0.00041	0.17148	0.61655

$\alpha = 5/4$

β	r	z	A	B	C/1.6
0	0	1.00000	-	-	-
1/4	0.31878	1.02817	0.55430	3.73353	0.90484
1/2	0.59928	1.07235	0.27575	1.78470	0.73201
3/4	0.85912	1.10912	0.15027	1.18307	0.65608
1	1.10987	1.13741	0.08872	0.89440	0.62192
5/4	1.35653	1.15898	0.05548	0.72400	0.60639
3/2	1.60145	1.17553	0.03622	0.61065	0.59982
7/4	1.84575	1.18835	0.02447	0.52926	0.59772
2	2.09002	1.19838	0.01701	0.46766	0.59787
9/4	2.33452	1.20633	0.01213	0.41922	0.59912
5/2	2.57938	1.21268	0.00883	0.38002	0.60084
11/4	2.82463	1.21782	0.00656	0.34760	0.60271
3	3.07029	1.22202	0.00496	0.32030	0.60457
13/4	3.31633	1.22549	0.00380	0.29698	0.60633
7/2	3.56272	1.22838	0.00296	0.27681	0.60796
15/4	3.80942	1.23080	0.00234	0.25921	0.60943
4	4.05642	1.23285	0.00186	0.24369	0.61077
17/4	4.30367	1.23460	0.00150	0.22992	0.61196
9/2	4.55116	1.23611	0.00123	0.21761	0.61303
19/4	4.79885	1.23740	0.00101	0.20654	0.61400
5	5.04672	1.23853	0.00084	0.19653	0.61486
21/4	5.29476	1.23952	0.00070	0.18744	0.61563
11/2	5.54295	1.24039	0.00059	0.17915	0.61632
23/4	5.79127	1.24115	0.00050	0.17156	0.61695

$\alpha = 3/2$

β	r	z	A	B	C/1.6
0	0	1.30902	-	-	-
1/4	0.28970	1.31921	0.25744	3.92550	0.81178
1/2	0.56678	1.34190	0.17539	1.90612	0.73565
3/4	0.83123	1.36646	0.11352	1.24932	0.67959
1	1.08741	1.38840	0.07415	0.93178	0.64570
5/4	1.33875	1.40674	0.04956	0.74613	0.62636
3/2	1.58739	1.42173	0.03396	0.62433	0.61569
7/4	1.83459	1.43388	0.02381	0.53802	0.61011
2	2.08110	1.44374	0.01705	0.47343	0.60750
9/4	2.32733	1.45176	0.01244	0.42312	0.60660
5/2	2.57353	1.45833	0.00925	0.38273	0.60669
11/4	2.81984	1.46375	0.00698	0.34951	0.60732
3	3.06632	1.46824	0.00535	0.32167	0.60822
13/4	3.31302	1.47200	0.00415	0.29798	0.60924
7/2	3.55993	1.47516	0.00327	0.27756	0.61030
15/4	3.80708	1.47785	0.00260	0.25977	0.61134
4	4.05441	1.48014	0.00209	0.24412	0.61232
17/4	4.30195	1.48210	0.00170	0.23025	0.61324
9/2	4.54967	1.48380	0.00139	0.21787	0.61410
19/4	4.79755	1.48528	0.00115	0.20674	0.61488
5	5.04559	1.48657	0.00096	0.19669	0.61560
21/4	5.29377	1.48770	0.00080	0.18767	0.61626
11/2	5.54207	1.48870	0.00068	0.17926	0.61686
23/4	5.79049	1.48958	0.00058	0.17164	0.61741

$\alpha = 7/4$

β	r	z	A	B	C/1.6
0	0	1.59307	-	-	-
1/4	0.27626	1.59810	0.13916	3.97086	0.75304
1/2	0.54727	1.61086	0.11045	1.95538	0.71801
3/4	0.81137	1.62694	0.08140	1.28609	0.68328
1	1.06956	1.64313	0.05843	0.95667	0.65680
5/4	1.32352	1.65790	0.04183	0.76268	0.63875
3/2	1.57469	1.67076	0.03017	0.63540	0.62711
7/4	1.82408	1.68170	0.02201	0.54553	0.61989
2	2.07242	1.69091	0.01627	0.47861	0.61561
9/4	2.32015	1.69864	0.0121$_9$	0.42676	0.61324
5/2	2.56757	1.70513	0.00925	0.38532	0.61208
11/4	2.81486	1.71058	0.00711	0.35138	0.61169
3	3.06214	1.71519	0.00553	0.32305	0.61178
13/4	3.30949	1.71910	0.00435	0.29901	0.61215
7/2	3.55693	1.72243	0.00346	0.27834	0.61268
15/4	3.80450	1.72528	0.00278	0.26036	0.61330
4	4.05220	1.72774	0.00225	0.24458	0.61395
17/4	4.30004	1.72987	0.00184	0.23061	0.61460
9/2	4.54801	1.73172	0.00152	0.21815	0.61523
19/4	4.79610	1.73334	0.00126	0.20697	0.61584
5	5.04432	1.73476	0.00106	0.19687	0.61641
21/4	5.29265	1.73601	0.00089	0.18772	0.61695
11/2	5.54108	1.73712	0.00075	0.17938	0.61745
23/4	5.78961	1.73811	0.00064	0.17174	0.61791

$\kappa = 2$

β	r	z	A	B	C/1.6
0	0	1.866033	-	-	-
1/4	0.26885	1.86895	0.08447	3.98605	0.71809
1/2	0.53512	1.87681	0.07246	1.97646	0.70007
3/4	0.79748	1.88762	0.05815	1.30566	0.67896
1	1.05589	1.89945	0.04497	0.97221	0.66004
5/4	1.31102	1.91105	0.03420	0.77425	0.64522
3/2	1.56369	1.92173	0.02589	0.64381	0.63445
7/4	1.81460	1.93125	0.01964	0.55161	0.62698
2	2.06432	1.93956	0.01499	0.48302	0.62198
9/4	2.31327	1.94676	0.01153	0.42997	0.61876
5/2	2.56172	1.95295	0.00895	0.38768	0.61677
11/4	2.80989	1.95826	0.00701	0.35314	0.61564
3	3.05790	1.96283	0.00554	0.32437	0.61508
13/4	3.30585	1.96677	0.00442	0.30001	0.61491
7/2	3.55381	1.97017	0.00356	0.27910	0.61499
15/4	3.80180	1.97311	0.00289	0.26095	0.61523
4	4.04986	1.97567	0.00236	0.24504	0.61558
17/4	4.29800	1.97791	0.00195	0.23097	0.61597
9/2	4.54622	1.97987	0.00161	0.21844	0.61640
19/4	4.79454	1.98159	0.00135	0.20720	0.61683
5	5.04293	1.98312	0.00114	0.19706	0.61726
21/4	5.29142	1.98447	0.00096	0.18787	0.61768
11/2	5.53999	1.98567	0.00082	0.17950	0.61808
23/4	5.78864	1.98674	0.00070	0.17184	0.61846

$\alpha = 9/4$

β	r	z	A	B	C/1.6
0	0	2.13278	-	-	-
1/4	0.26428	2.13465	0.05552	3.99233	0.69590
1/2	0.52713	2.13985	0.04976	1.98642	0.68570
3/4	0.78764	2.14739	0.04221	1.31629	0.67249
1	1.04552	2.15612	0.03449	0.98175	0.65930
5/4	1.30098	2.16516	0.02756	0.78209	0.64784
3/2	1.55441	2.17390	0.02178	0.64997	0.63868
7/4	1.80628	2.18201	0.01713	0.55634	0.63174
2	2.05698	2.18935	0.01348	0.48662	0.62668
9/4	2.30686	2.19588	0.01064	0.43270	0.62311
5/2	2.55615	2.20164	0.00844	0.38976	0.62066
11/4	2.80505	2.20669	0.00674	0.35473	0.61904
3	3.05371	2.21111	0.00541	0.32559	0.61802
13/4	3.30221	2.21498	0.00438	0.30096	0.61744
7/2	3.55064	2.21836	0.00356	0.27984	0.61715
15/4	3.79904	2.22133	0.00292	0.26154	0.61708
4	4.04744	2.22393	0.00241	0.24550	0.61716
17/4	4.29588	2.22623	0.00201	0.23134	0.61733
9/2	4.54435	2.22825	0.00168	0.21873	0.61756
19/4	4.79288	2.23005	0.00141	0.20744	0.61783
5	5.04146	2.23165	0.00119	0.19726	0.61812
21/4	5.29011	2.23307	0.00102	0.18803	0.61842
11/2	5.53882	2.23434	0.00087	0.17963	0.61872
23/4	5.78759	2.23548	0.00075	0.17195	0.61902

$\alpha = 5/2$

β	r	z	A	B	C/1.6
0	0	2.39564	-	-	-
1/4	0.26123	2.39692	0.03863	3.99541	0.68093
1/2	0.52161	2.40055	0.03558	1.99160	0.67470
3/4	0.78050	2.40599	0.03132	1.32234	0.66613
1	1.03761	2.41255	0.02665	0.98768	0.65689
5/4	1.29295	2.41963	0.02215	0.78735	0.64822
3/2	1.54669	2.42674	0.01814	0.65439	0.64075
7/4	1.79911	2.43357	0.01473	0.55993	0.63469
2	2.05047	2.43995	0.01192	0.48948	0.62996
9/4	2.30101	2.44578	0.00964	0.43497	0.62638
5/2	2.55096	2.45104	0.00781	0.39154	0.62376
11/4	2.80046	2.45575	0.00635	0.35613	0.62187
3	3.04966	2.45994	0.00518	0.32670	0.62056
13/4	3.29865	2.46366	0.00425	0.30183	0.61968
7/2	3.54750	2.46697	0.00351	0.28054	0.61912
15/4	3.79627	2.46990	0.00291	0.26209	0.61880
4	4.04499	2.47250	0.00242	0.24595	0.61865
17/4	4.29371	2.47481	0.00203	0.23170	0.61862
9/2	4.54242	2.47687	0.00171	0.21903	0.61869
19/4	4.79116	2.47870	0.00145	0.20768	0.61881
5	5.03993	2.48035	0.00123	0.19745	0.61898
21/4	5.28874	2.48182	0.00106	0.18819	0.61917
11/2	5.53759	2.48314	0.00091	0.17977	0.61938
23/4	5.78648	2.48434	0.00078	0.17207	0.61959

$\alpha = 11/4$

β	r	z	A	B	C/1.6
0	0	2.65587	-	-	-
1/4	0.25909	2.65679	0.02806	3.99707	0.67032
1/2	0.51763	2.65943	0.02631	1.99452	0.66630
3/4	0.77518	2.66347	0.02377	1.32596	0.66052
1	1.03151	2.66849	0.02084	0.99143	0.65395
5/4	1.28652	2.67408	0.01786	0.79091	0.64742
3/2	1.54030	2.67988	0.01507	0.65754	0.64146
7/4	1.79299	2.68561	0.01257	0.56262	0.63633
2	2.04475	2.69110	0.01043	0.49172	0.63211
9/4	2.29577	2.69625	0.00862	0.43680	0.62875
5/2	2.54620	2.70099	0.00713	0.39303	0.62613
11/4	2.79618	2.70532	0.00590	0.35734	0.62415
3	3.04582	2.70924	0.00489	0.32767	0.62268
13/4	3.29522	2.71277	0.00406	0.30262	0.62161
7/2	3.54444	2.71595	0.00339	0.28117	0.62085
15/4	3.79354	2.71880	0.00284	0.26261	0.62034
4	4.04256	2.72135	0.00239	0.24637	0.62002
17/4	4.29153	2.72364	0.00202	0.23204	0.61983
9/2	4.54047	2.72570	0.00172	0.21931	0.61976
19/4	4.78942	2.72755	0.00147	0.20791	0.61975
5	5.03836	2.72922	0.00126	0.19765	0.61981
21/4	5.28733	2.73072	0.00108	0.18836	0.61990
11/2	5.53631	2.73208	0.00093	0.17990	0.62003
23/4	5.78533	2.73331	0.00081	0.17218	0.62017

$\alpha = 3$

β	r	z	A	B	C/1.6
0	0	2.91421	-	-	-
1/4	0.25752	2.91490	0.02107	3.99803	0.66252
1/2	0.51467	2.91688	0.02000	1.99625	0.65980
3/4	0.77114	2.91996	0.01841	1.32819	0.65577
1	1.02673	2.92388	0.01651	0.99389	0.65101
5/4	1.28135	2.92835	0.01450	0.79334	0.64607
3/2	1.53502	2.93309	0.01253	0.65980	0.64134
7/4	1.78779	2.93790	0.01070	0.56463	0.63709
2	2.03978	2.94262	0.00907	0.49346	0.63343
9/4	2.29110	2.94713	0.00765	0.43828	0.63038
5/2	2.54189	2.95137	0.00644	0.39427	0.62790
11/4	2.79223	2.95530	0.00542	0.35836	0.62593
3	3.04223	2.95893	0.00456	0.32852	0.62440
13/4	3.29196	2.96224	0.00384	0.30332	0.62323
7/2	3.54150	2.96525	0.00324	0.28175	0.62235
15/4	3.79089	2.96799	0.00275	0.26308	0.62171
4	4.04017	2.97047	0.00233	0.24676	0.62125
17/4	4.28937	2.97271	0.00199	0.23237	0.62095
9/2	4.53853	2.97475	0.00170	0.21958	0.62075
19/4	4.78766	2.97659	0.00146	0.20814	0.62064
5	5.03678	2.97825	0.00126	0.19784	0.62060
21/4	5.28589	2.97977	0.00109	0.18851	0.62061
11/2	5.53501	2.98114	0.00095	0.18004	0.62065
23/4	5.78414	2.98239	0.00083	0.17229	0.62073

$\alpha = 13/4$

β	r	z	A	B	C/1.6
0	0	3.17116	-	-	-
1/4	0.25633	3.17169	0.01625	3.99862	0.65660
1/2	0.51240	3.17322	0.01557	1.99736	0.65469
3/4	0.76799	3.17563	0.01453	1.32964	0.65180
1	1.02293	3.17873	0.01326	0.99554	0.64829
5/4	1.27715	3.18234	0.01187	0.79503	0.64451
3/2	1.53062	3.18625	0.01046	0.66144	0.64077
7/4	1.78338	3.19029	0.00911	0.56614	0.63727
2	2.03547	3.19434	0.00787	0.49480	0.63414
9/4	2.28698	3.19828	0.00676	0.43945	0.63144
5/2	2.53801	3.20205	0.00578	0.39528	0.62916
11/4	2.78862	3.20561	0.00494	0.35922	0.62729
3	3.03890	3.20893	0.00421	0.32924	0.62577
13/4	3.28891	3.21200	0.00359	0.30392	0.62456
7/2	3.53870	3.21484	0.00307	0.28226	0.62362
15/4	3.78834	3.21744	0.00263	0.26351	0.62290
4	4.03785	3.21982	0.00225	0.24712	0.62235
17/4	4.28727	3.22199	0.00194	0.23267	0.62195
9/2	4.53662	3.22398	0.00167	0.21983	0.62166
19/4	4.78592	3.22579	0.00145	0.20835	0.62146
5	5.03519	3.22744	0.00125	0.19802	0.62134
21/4	5.28445	3.22895	0.00109	0.18867	0.62128
11/2	5.53370	3.23033	0.00095	0.18017	0.62126
23/4	5.78294	3.23159	0.00083	0.17240	0.62127

$\alpha = 7/2$

ρ	r	z	A	B	C/1.6
0	0	3.42705	-	-	-
1/4	0.25541	3.42746	0.01281	3.99893	0.65200
1/2	0.51062	3.42867	0.01236	1.99809	0.65062
3/4	0.76549	3.43058	0.01166	1.33062	0.64849
1	1.01988	3.43308	0.01078	0.99667	0.64585
5/4	1.27371	3.43603	0.00980	0.79623	0.64293
3/2	1.52696	3.43928	0.00878	0.66263	0.63996
7/4	1.77962	3.44269	0.00778	0.56727	0.63709
2	2.03174	3.44617	0.00683	0.49585	0.63444
9/4	2.28336	3.44961	0.00596	0.44038	0.63208
5/2	2.53454	3.45295	0.00517	0.39610	0.63003
11/4	2.78534	3.45615	0.00447	0.35994	0.62829
3	3.03583	3.45917	0.00387	0.32986	0.62683
13/4	3.28606	3.46201	0.00334	0.30445	0.62564
7/2	3.53607	3.46466	0.00288	0.28271	0.62468
15/4	3.78591	3.46711	0.00249	0.26389	0.62391
4	4.03562	3.46938	0.00216	0.24745	0.62330
17/4	4.28522	3.47147	0.00187	0.23295	0.62283
9/2	4.53475	3.47339	0.00162	0.22007	0.62248
19/4	4.78421	3.47515	0.00141	0.20855	0.62222
5	5.03362	3.47678	0.00124	0.19819	0.62203
21/4	5.28301	3.47827	0.00108	0.18881	0.62190
11/2	5.53238	3.47963	0.00095	0.18029	0.62183
23/4	5.78174	3.48089	0.00083	0.17251	0.62179

$\alpha = 15/4$

β	r	z	A	B	C/1.6
0	0	3.68210	-	-	-
1/4	0.25467	3.68243	0.01029	3.99931	0.64835
1/2	0.50921	3.68340	0.00998	1.99856	0.64732
3/4	0.76348	3.68495	0.00949	1.33129	0.64572
1	1.01739	3.68699	0.00887	0.99746	0.64370
5/4	1.27087	3.68943	0.00817	0.79709	0.64142
3/2	1.52388	3.69214	0.00742	0.66351	0.63904
7/4	1.77642	3.69504	0.00666	0.56813	0.63669
2	2.02851	3.69803	0.00593	0.49665	0.63446
9/4	2.28017	3.70104	0.00524	0.44113	0.63241
5/2	2.53144	3.70399	0.00461	0.39677	0.63059
11/4	2.78238	3.70686	0.00404	0.36053	0.62900
3	3.03302	3.70961	0.00353	0.33045	0.62764
13/4	3.28342	3.71221	0.00308	0.30491	0.62649
7/2	3.53361	3.71467	0.00269	0.28311	0.62554
15/4	3.78362	3.71697	0.00235	0.26423	0.62475
4	4.03350	3.71911	0.00205	0.24774	0.62412
17/4	4.28326	3.72110	0.00179	0.23320	0.62361
9/2	4.53294	3.72295	0.00157	0.22029	0.62321
19/4	4.78254	3,72466	0.00137	0.20874	0.62290
5	5.03209	3.72624	0.00121	0.19835	0.62266
21/4	5.28160	3.72770	0.00106	0.18895	0.62248
11/2	5.53107	3.72905	0.00094	0.18041	0.62236
23/4	5.78053	3.73030	0.00083	0.17261	0.62227

$\alpha = 4$

β	r	z	A	B	C/1.6
0	0	3.93649	-	-	-
1/4	0.25408	3.93676	0.00839	3.99947	0.64540
1/2	0.50806	3.93755	0.00817	1.99890	0.64462
3/4	0.76183	3.93882	0.00783	1.33178	0.64339
1	1.01533	3.94051	0.00738	0.99804	0.64182
5/4	1.26849	3.94254	0.00686	0.79773	0.64002
3/2	1.52128	3.94483	0.00630	0.66417	0.63810
7/4	1.77368	3.94730	0.00573	0.56878	0.63617
2	2.02570	3.94989	0.00516	0.49728	0.63429
9/4	2.27736	3.95251	0.00462	0.44172	0.63253
5/2	2.52869	3.95513	0.00411	0.39731	0.63093
11/4	2.77971	3.95770	0.00364	0.36102	0.62949
3	3.03047	3.96019	0.00322	0.33082	0.62824
13/4	3.28099	3.96257	0.00284	0.30530	0.62715
7/2	3.53132	3.96484	0.00250	0.28345	0.62623
15/4	3.78147	3.96698	0.00220	0.26453	0.62545
4	4.03149	3.96900	0.00194	0.24800	0.62481
17/4	4.28139	3.97089	0.00171	0.23343	0.62428
9/2	4.53119	3.97265	0.00150	0.22048	0.62385
19/4	4.78092	3.97429	0.00133	0.20891	0.62350
5	5.03059	3.97582	0.00117	0.19850	0.62323
21/4	5.28021	3.97725	0.00104	0.18908	0.62301
11/2	5.52979	3.97857	0.00092	0.18052	0.62285
23/4	5.77934	3.97979	0.00082	0.17271	0.62273

$\alpha = 17/4$

β	r	z	A	B	C/1.6
0	0	4.19034	-	-	-
1/4	0.25360	4.19056	0.00694	3.99952	0.64298
1/2	0.50711	4.19121	0.00678	1.99917	0.64237
3/4	0.76047	4.19227	0.00653	1.33212	0.64142
1	1.01361	4.19368	0.00620	0.99846	0.64018
5/4	1.26649	4.19539	0.00581	0.79820	0.63874
3/2	1.51906	4.19734	0.00539	0.66467	0.63719
7/4	1.77132	4.19946	0.00495	0.56929	0.63558
2	2.02326	4.20170	0.00451	0.49778	0.63401
9/4	2.27489	4.20400	0.00407	0.44219	0.63250
5/2	2.52623	4.20632	0.00366	0.39975	0.63109
11/4	2.77731	4.20862	0.00328	0.36143	0.62981
3	3.02814	4.21087	0.00293	0.33119	0.62866
13/4	3.27876	4.21304	0.00260	0.30563	0.62766
7/2	3.52919	4.21513	0.00231	0.28374	0.62678
15/4	3.77946	4.21712	0.00205	0.26480	0.62603
4	4.02960	0.21901	0.00182	0.24823	0.62539
17/4	4.27961	4.22079	0.00162	0.23363	0.62485
9/2	4.52953	4.22247	0.00143	0.22066	0.62440
19/4	4.77937	4.22404	0.00127	0.20907	0.62403
5	5.02914	4.22552	0.00113	0.19864	0.62373
21/4	5.27886	4.22689	0.00101	0.18920	0.62349
11/2	5.52853	4.22818	0.00090	0.18063	0.62330
23/4	5.77817	4.22938	0.00080	0.17281	0.62315

$\alpha = 9/2$

β	r	z	A	B	C/1.6
0	0	4.44374	-	-	-
1/4	0.25319	4.44393	0.00581	3.99974	0.64097
1/2	0.50633	4.44447	0.00569	1.99932	0.64049
3/4	0.75933	4.44536	0.00550	1.33238	0.63974
1	1.01217	4.44655	0.00525	0.99877	0.63875
5/4	1.26479	4.44800	0.00496	0.79855	0.63759
3/2	1.51716	4.44967	0.00464	0.66505	0.63632
7/4	1.76928	4.45150	0.00429	0.56968	0.63499
2	2.02113	4.45345	0.00395	0.49817	0.63365
9/4	2.27271	4.45547	0.00360	0.44257	0.63236
5/2	2.52405	4.45753	0.00327	0.39811	0.63113
11/4	2.77515	4.45959	0.00295	0.36176	0.62999
3	3.02603	4.46162	0.00266	0.33150	0.62896
13/4	3.27672	4.46360	0.00239	0.30591	0.62803
7/2	3.52723	4.46552	0.00214	0.28400	0.62721
15/4	3.77759	4.46737	0.00191	0.26503	0.62649
4	4.02782	4.46913	0.00171	0.24844	0.62587
17/4	4.27793	4.47081	0.00153	0.23382	0.62533
9/2	4.52795	4.47240	0.00136	0.22083	0.62488
19/4	4.77788	4.47389	0.00122	0.20921	0.62450
5	5.02774	4.47531	0.00109	0.19876	0.62418
21/4	5.27755	4.47663	0.00097	0.18932	0.62392
11/2	5.52731	4.47788	0.00087	0.18073	0.62371
23/4	5.77703	4.47904	0.00078	0.17289	0.62353

$\alpha = 19/4$

β	r	z	A	B	C/1.6
0	0	4.69677	-	-	-
1/4	0.25286	4.69693	0.00491	3.99968	0.63928
1/2	0.50566	4.69739	0.00482	1.99948	0.63890
3/4	0.75837	4.69814	0.00468	1.33256	0.63830
1	1.01094	4.69915	0.00449	0.99901	0.63750
5/4	1.26333	4.70040	0.00427	0.79883	0.63655
3/2	1.51553	4.70183	0.00401	0.66534	0.63550
7/4	1.76750	4.70342	0.00375	0.56999	0.63439
2	2.01926	4.70513	0.00347	0.49848	0.63326
9/4	2.27079	4.70691	0.00319	0.44288	0.63215
5/2	2.52210	4.70874	0.00292	0.39841	0.63108
11/4	2.77320	4.71059	0.00266	0.36204	0.63007
3	3.02411	4.71242	0.00241	0.33176	0.62914
13/4	3.27485	4.71423	0.00218	0.30615	0.62829
7/2	3.52542	4.71599	0.00197	0.28422	0.62753
15/4	3.77585	4.71770	0.00177	0.26523	0.62685
4	4.02616	4.71934	0.00160	0.24862	0.62626
17/4	4.27635	4.72091	0.00144	0.23398	0.62574
9/2	4.52645	4.72241	0.00129	0.22097	0.62529
19/4	4.77646	4.72383	0.00116	0.20934	0.62490
5	5.02640	4.72518	0.00104	0.19888	0.62490
21/4	5.27629	4.72645	0.00094	0.18942	0.62430
11/2	5.52612	4.72765	0.00084	0.18082	0.62407
23/4	5.77591	4.72878	0.00076	0.17298	0.62388

$\alpha = 5$

β	r	z	A	B	C/1.6
0	0	4.94949	-	-	-
1/4	0.25257	4.94962	0.00419	3.99979	0.63785
1/2	0.50510	4.95002	0.00412	1.99957	0.63755
3/4	0.75755	4.95066	0.00401	1.33271	0.63705
1	1.00989	4.95153	0.00387	0.99919	0.63640
5/4	1.26208	4.95260	0.00369	0.79904	0.63562
3/2	1.51411	4.95385	0.00349	0.66558	0.63475
7/4	1.76595	4.95523	0.00328	0.57024	0.63382
2	2.01761	4.95673	0.00306	0.49873	0.63286
9/4	2.26908	4.95831	0.00284	0.44312	0.63190
5/2	2.52036	4.95994	0.00261	0.39865	0.63097
11/4	2.77145	4.96160	0.00240	0.36228	0.63008
3	3.02238	4.96326	0.00219	0.33198	0.62924
13/4	3.27314	4.96490	0.00200	0.30636	0.62847
7/2	3.52376	4.96652	0.00181	0.28441	0.62777
15/4	3.77424	4.96809	0.00164	0.26540	0.62713
4	4.02460	4.96962	0.00149	0.24878	0.62657
17/4	4.27486	4.97109	0.00135	0.23412	0.62607
9/2	4.52503	4.97250	0.00122	0.22110	0.62563
19/4	4.77511	4.97384	0.00110	0.20946	0.62525
5	5.02512	4.97513	0.00099	0.19899	0.62492
21/4	5.27508	4.97634	0.00090	0.18951	0.62464
11/2	5.52498	4.97750	0.00081	0.18091	0.62240
23/4	5.77483	4.97859	0.00074	0.17305	0.62420

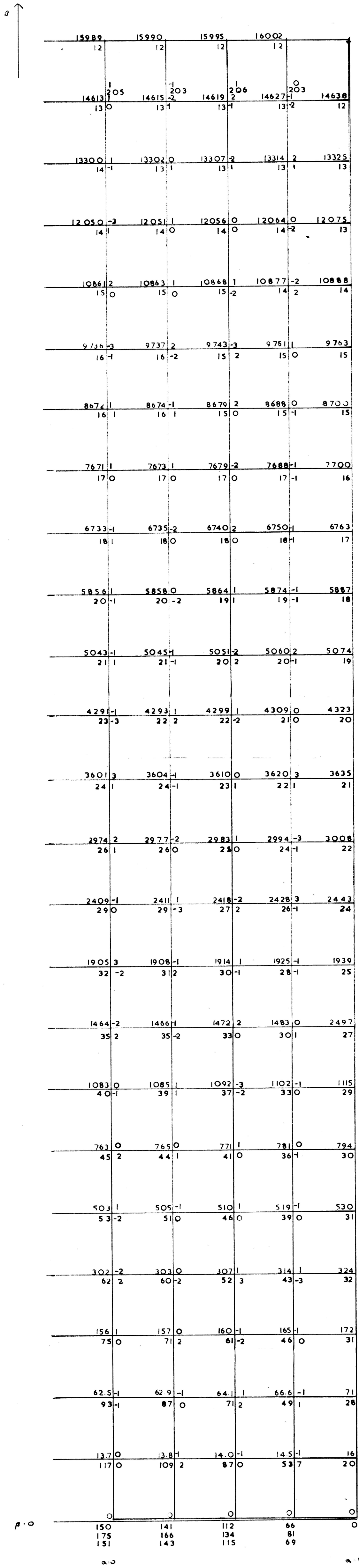

FIG. NO. 26.

Ψ	F_z
π	F_r

$$\Psi = \tfrac{1}{2} r^2 \left[1 - \frac{3}{4\sqrt{r^2+z^2}} + \frac{1}{16\left[r^2+z^2\right]^{2}} \right]$$

$$\pi = \frac{3r^2}{4\left[r^2+z^2\right]^{3/2}}$$

LISTER (M.)

SOLUTION FOR ψ AND π.

FIG. No. 21a.

16332	16333	16336	16340	16346	16352	16360	16368	16375	16382	16389	16396	16401	16406	16410	16412	16413	16413
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14914	14915	14918	14922	14928	14935	14943	14951	14959	14967	14974	14981	14987	14992	14996	14999	15000	15000
1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
13557	13558	13561	13566	13572	13580	13588	13597	13605	13614	13622	13629	13635	13641	13645	13648	13649	13650
2	2	2	2	2	2	1	0	0	0	0	0	0	0	0	0	0	0
12262	12263	12266	12271	12278	12286	12295	12304	12314	12323	12332	12340	12346	12352	12357	12360	12362	12363
3	3	3	3	3	3	2	1	0	0	0	0	0	0	0	0	0	0
11029	11030	11034	11039	11046	11055	11064	11074	11085	11095	11105	11114	11121	11127	11132	11136	11138	11139
4	4	4	4	4	4	3	2	1	0	0	0	0	0	0	0	0	0
9859	9860	9864	9869	9876	9886	9896	9907	9918	9930	9941	9951	9959	9966	9971	9975	9977	9978
5	5	5	5	5	5	4	3	2	1	0	0	0	0	0	0	0	0
8752	8753	8757	8762	8770	8780	8791	8803	8815	8828	8840	8851	8860	8868	8874	8878	8880	8881
6	6	6	6	6	6	5	5	3	2	1	0	0	0	0	0	0	0
7707	7708	7712	7718	7727	7738	7750	7763	7776	7789	7802	7814	7824	7833	7840	7845	7847	7848
7	7	7	7	7	7	6	5	4	3	2	1	0	0	0	0	0	0
6726	6727	6731	6738	6748	6759	6772	6787	6801	6815	6829	6842	6853	6863	6870	6875	6878	6879
9	9	9	9	9	8	7	6	5	4	3	2	1	0	0	0	0	0
5809	5810	5815	5823	5833	5846	5860	5875	5890	5905	5920	5934	5946	5957	5965	5971	5974	5975
12	12	11	11	10	9	8	7	6	5	4	3	2	1	0	0	0	0
4957	4959	4964	4973	4984	4998	5012	5028	5044	5060	5076	5091	5104	5115	5124	5130	5134	5136
15	15	14	13	12	11	10	9	8	7	5	4	3	2	1	0	0	0
4171	4173	4179	4189	4201	4215	4230	4247	4264	4281	4297	4313	4326	4338	4348	4355	4360	4362
18	18	17	16	15	14	12	11	9	8	6	5	4	3	2	1	0	0
3452	3454	3461	3471	3484	3499	3515	3533	3551	3568	3585	3601	3615	3627	3637	3644	3649	3651
22	22	21	20	18	16	14	12	11	9	7	6	5	4	3	2	1	0
2800	2803	2810	2820	2834	2850	2867	2885	2904	2922	2940	2956	2970	2982	2992	2999	3004	3006
27	26	25	23	21	19	16	14	12	10	8	6	5	4	3	2	1	0
2216	2219	2226	2237	2252	2262	2287	2306	2325	2343	2361	2377	2390	2402	2412	2419	2424	2426
33	32	30	27	24	21	18	15	13	10	8	6	5	4	3	2	1	0
1702	1705	1712	1723	1738	1755	1774	1794	1813	1831	1848	1863	1876	1887	1896	1903	1908	1910
39	38	36	32	28	24	20	16	13	10	8	6	5	4	3	2	1	0
1257	1260	1267	1278	1293	1310	1329	1348	1367	1384	1400	1414	1426	1436	1445	1451	1456	1457
46	45	42	37	31	26	21	17	13	10	8	6	5	4	3	2	1	0
883	885	892	903	917	934	952	970	987	1003	1017	1030	1040	1049	1057	1062	1066	1067
54	52	48	42	35	28	22	17	13	10	8	6	5	4	3	2	1	0
579	581	587	597	610	625	642	658	673	686	698	709	718	725	731	735	738	739
63	61	56	46	37	29	22	16	12	9	7	5	4	3	2	1	0	0
344	345	350	358	369	382	396	410	422	432	441	449	456	462	467	470	472	472
19	71	63	51	39	28	20	14	10	7	6	4	3	2	1	0	0	0
175	176	179	185	193	203	213	223	232	239	245	250	255	259	262	264	265	265
89	85	73	55	38	25	16	11	7	5	4	3	2	1	0	0	0	0
68	68	69	72	77	83	90	96	101	104	107	110	113	115	116	117	117	117
109	103	85	59	33	18	11	7	4	3	2	2	1	0	0	0	0	0
13·5	13·5	13·75	14	16	18	21	23	24	25	26	27	28	29	29	29	29	29
136	128	103	65	22	9	4	2	1	1	1	1	0	0	0	0	0	0
172·8	162	132	78·7														

SOLUTION FOR ψ AND π.

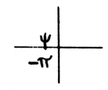

FIG. No. 19a.
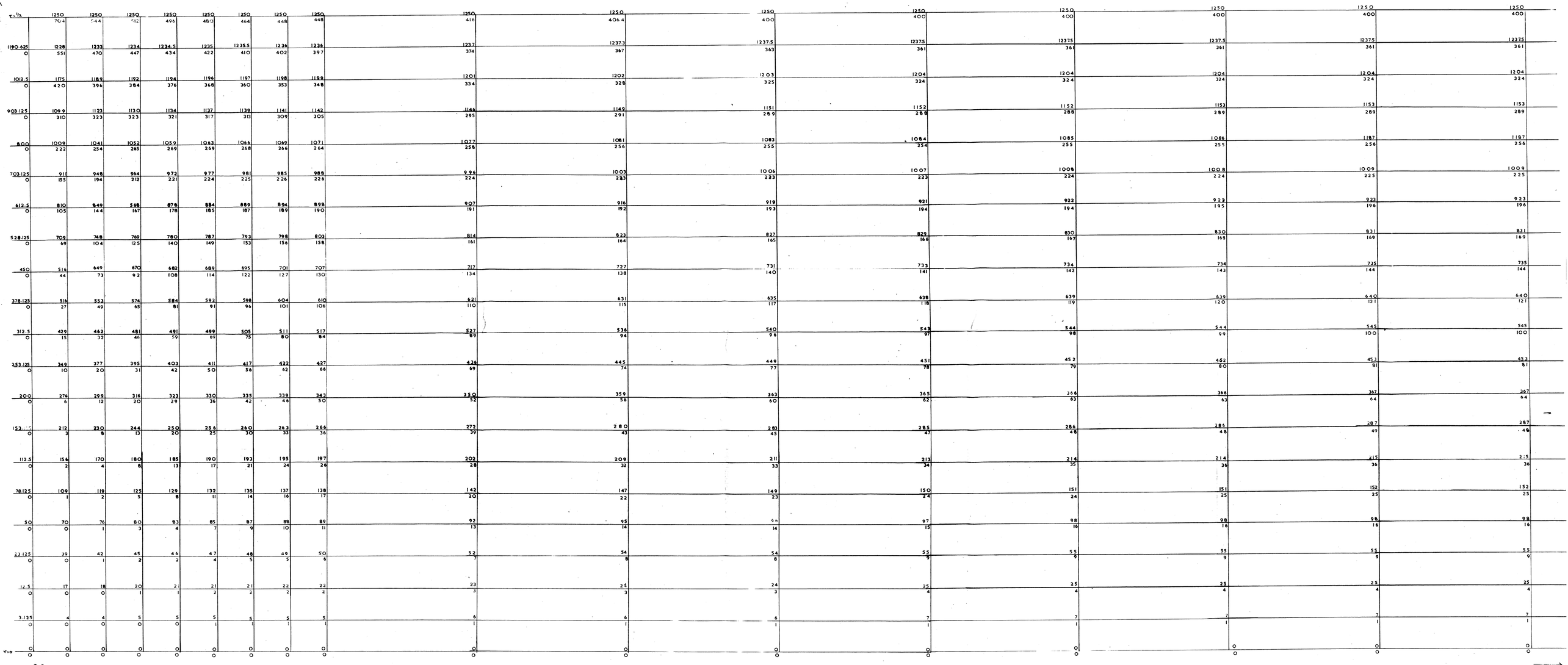
FIG. NO. $R \to \infty$ $\frac{\psi}{\pi}$