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Department of Physics

**Microwave Probing of Topological Insulator-based
Josephson Quantum Circuits**

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Statement of Originality

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Abstract

Majorana zero modes (MZMs) are exotic quasiparticles predicted to emerge in specific condensed matter systems, such as superconductor-topological insulator hybrids. Their anticipated non-Abelian exchange statistics could lead to robust quantum computing, mitigating issues of environmental noise that challenge existing quantum platforms like superconducting qubits. However, definitive evidence of MZMs remains elusive. This thesis explores the integration of three-dimensional topological insulators (3DTIs) with superconducting qubits within circuit quantum electrodynamics (cQED) to leverage 3DTI properties for MZM detection. Through the fabrication, characterization, and analysis of fixed-frequency and flux-tunable TImon devices, this work establishes a foundation for future experimental advancements in MZM detection using transmon qubits.

Chapter 4 confirms the compatibility of topological insulator Josephson junctions (TI JJs) with cQED systems, demonstrating effective control and coherence in a fixed-frequency TI-based transmon (TImon) device. The primarily diffusive nature of TI JJs was inferred from qubit anharmonicity and flux-tuning patterns observed in a separate flux-tunable TImon, indicating minimal ballistic contributions.

Chapter 5 tests the feasibility of using the TImon platform for MZM detection by subjecting the fixed-frequency TImon in Chapter 4 to an in-plane magnetic field of 900 mT. Although the device demonstrated field compatibility, definitive MZM signatures were not observed. Neither were exotic magnetic effects of 3DTI likely due to interference effects in the diffusive TI JJs.

Chapter 6 addresses the challenges posed by the diffusive nature of TI JJs through microwave spectroscopy and DC transport measurements. These experiments further confirmed the diffusive behavior and identified design limitations, providing guidance for future device modifications. The next steps include redesigning the qubit frequency to 15-20 GHz range for compatibility with higher temperatures around 1.6 K.

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Chapter 1

Introduction

1.1 Quantum Information Processing

As early as 1960s, Gordon Moore noticed the complexity of integrated circuits increasing exponentially [2]. Despite giving his famous prediction that the trend should go on in the foreseeable future, he also foretold a conflict between electronics' performance and size imposed by quantum phenomena [3]. More than forty years have passed and we are already standing at the boundary between classical and quantum physics, when the industry is massively producing chips at a few nanometers [4]. Facing the challenge, researchers start to look for alternative computing technologies. In 1981, Richard Feynman proposed to process information using the quantum effects that are unwanted in classical information processing. In his proposal, quantum information processing based on familiar and controllable quantum systems can be specifically useful for studying the inherent quantum effects in more complex quantum systems [5]. Later different quantum algorithms are developed that showcase computational advantages of quantum computing in more universal problems [6, 7] than just simulating quantum physics.

The smallest unit that stores information in quantum computing is called a quantum bit or a qubit. A qubit is a two-level quantum system. Like a classical bit, it also has two states $|0\rangle$ and $|1\rangle$, formed from the ground and excited states of the quantum system. However, quantum properties allow information to be stored in more than $|0\rangle$ and $|1\rangle$ states, but any superpositions of the two states, which can be represented as $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are complex numbers that describe the probability amplitudes of the qubit's state. Single-qubit states can

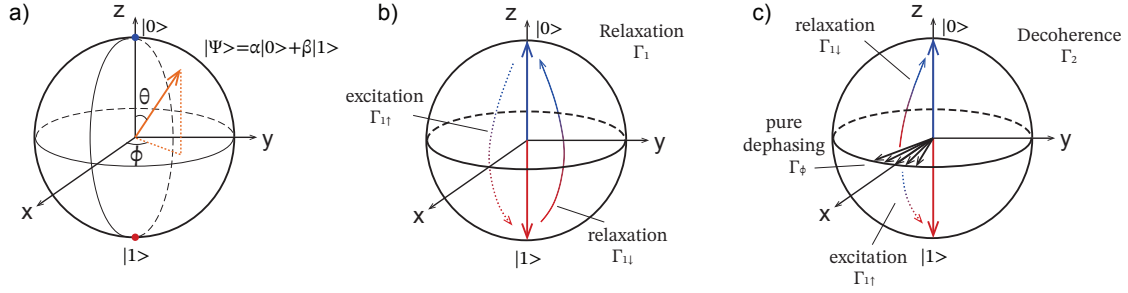


Figure 1.1: Qubit represented on the Bloch sphere. (a) Bloch sphere representation of an arbitrary quantum state $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|0\rangle$ and $|1\rangle$ are the ground and excited states of a qubit located at the two poles along z-axis on the Bloch sphere. (b) Qubit longitudinal relaxation Γ_1 that results in transitions $|0\rangle \leftrightarrow |1\rangle$. (c) Transverse relaxation Γ_2 as an effect of longitudinal relaxation and pure dephasing that causes qubit decoherence. Figures adapted from [8].

be represented as vectors on a Bloch sphere as shown in Figure 1.1a. In the Bloch sphere representation, an arbitrary qubit state is characterized by the polar and azimuthal angles θ and ϕ . The $|0\rangle$ and $|1\rangle$ are at the two poles on the longitudinal axis (z-direction) with $\phi = 0$ and $\theta = 0, \pi$. With information stored in qubit states, computations are then given by a series of unitary transformations on the qubit states that move them around on the Bloch sphere, after which the resulting state can be read out.

Despite the computational advantages of quantum information processing, qubits are prone to decoherence from environmental noise during computations, causing qubit relaxations that lead to a loss of quantum information and therefore high infidelity in information processing. Figure 1.1b-c demonstrate two kinds of qubit relaxations. The first is the longitudinal relaxation (Γ_1). Intuitively, it can be thought of as flips between the ground and excited states of a qubit [Figure 1.1b]. In general, it has an effect of depolarization that blurs the polar angle θ . Such relaxation events are caused by resonant transverse noises in the xy plane that mix the ground and excited states ($|0\rangle$ and $|1\rangle$) with off-diagonal interaction terms. The second relaxation is called transverse relaxation (Γ_2), which describes the qubit in a superposition state losing its coherence. Therefore, it is also called qubit decoherence. Consider a superposition state along x-axis on the equator of the Bloch sphere [Figure 1.1c], blurring either θ or ϕ leads to decoherence of the qubit state. The former, as mentioned before, can be caused by longitudinal relaxation while the latter is an effect of pure dephasing (Γ_ϕ). Pure dephasing

is caused by longitudinal noises that cause the qubit frequency f_{01} to oscillate such that the superposition state starts to precess that blurs the angle ϕ . Therefore, qubit decoherence is a combined event of longitudinal relaxation and pure dephasing.

Multiple systems can act as platforms for quantum processors such as trapped ions, ultracold atoms, polarized photons and electron spins in quantum dots. Unlike the aforementioned natural microscopic quantum systems, superconducting qubits are a solid-state approach to realize qubits. They are artificial quantum two-level systems arising from a nonlinear energy spectrum of macroscopic superconducting circuits incorporating Josephson junctions (JJs). Compared with other physical realizations of qubits, superconducting qubits are advantageous because of their high scalability and modifiability, exploiting advanced nanofabrication techniques. On the other side, the macroscopic elements of superconducting qubits make them more sensitive to noise in the environment, leading to faster qubit decoherence and higher infidelity through processes mentioned previously. To correct the error caused by decoherence, logical qubits formed from arrays of physical superconducting qubits are used in practical information processing [9]. Depending on the specific error correction scheme, the actual number of superconducting qubits needed for every single logical qubit varies, but always imposes great challenges in terms of scalability. An alternative way to mitigate the effect of decoherence is to use states of matter with topological order to protect quantum information. Majorana zero modes (MZMs), predicted to exist in certain topological superconductors, are an example of states that are theoretically expected to provide such protection.

1.2 Majorana Zero Modes

Majoranas are exotic fermionic quasiparticles that emerge in certain condensed matter systems, particularly in topological superconductors. Unlike conventional fermionic particles, Majoranas are their own antiparticles, which means that they are their own conjugates, described by the mathematical expression $\gamma = \gamma^\dagger$ and $\gamma^2 = 1$, where γ is a Majorana operator. Majoranas have to come in pairs, and they are related to fermionic modes in the sense that a pair of Majorana modes (γ_1, γ_2) forms one fermionic mode either being empty or occupied, given by:

$$c^\dagger = \frac{1}{\sqrt{2}}(\gamma_1 + i\gamma_2), \quad c = \frac{1}{\sqrt{2}}(\gamma_1 - i\gamma_2), \quad c^\dagger c = \frac{1}{2}(1 - i\gamma_1\gamma_2) \quad (1.1)$$

where $c^\dagger$ and c are the creation and annihilation operators for a single fermionic mode [10].

Given the relation in Eq 1.1, any electronic system can be rewritten in terms of even Majorana operators, according to a trivial or topological pairing of Majoranas, which depends on the system properties. A one-dimensional lattice model proposed by Alexei Kitaev concisely demonstrates the difference between the two pairings [10]. Figure 1.2 shows the Kitaev model where every single fermionic mode is represented as a site block that hosts two Majorana modes. In a trivial Kitaev chain [Figure 1.2a], Majoranas of the same site are paired so that every site is either empty or filled by one fermionic quasiparticle. Alternatively, Majoranas from different sites can be paired together such that two unpaired Majorana modes at both ends are isolated from the bulk region with fermionic excitations [Figure 1.2b]. This specific pairing of Majoranas results in a topological Kitaev chain and the two unpaired Majorana modes bounded at the edge, also called as zero-energy edge states or Majorana zero modes (MZMs), are topologically protected by the bulk gap. Note that the MZMs are free to move if there exist neighboring sites that allow inter-site pairing. Therefore, a domain wall is needed to maintain the edge states and keep them localized in space. Such a domain wall can be created by connecting a topological Kitaev chain with a trivial one as shown in Figure 1.2c, and presents a method to manipulate MZMs. It is important to note that in the discussion above, the spin of the fermionic modes is completely ignored, which is only possible in spinless systems with only one active spin species. We shall see later how p-wave superconductivity leads to effectively spinless systems.

Given the topological protection, storing information in the MZMs results in topological protection of information from local noise, which is the main resource of qubit decoherence. A pair of Majoranas being brought together forms either $|0\rangle$ or $|1\rangle$ state for a fermionic mode. Therefore, in a topological system with a pair of MZMs [Eq 1.1], there are two different configurations for the isolated pair of MZMs characterized by the state of the fermionic mode they form when no longer isolated. Given that they are zero-energy states, the choice of configura-

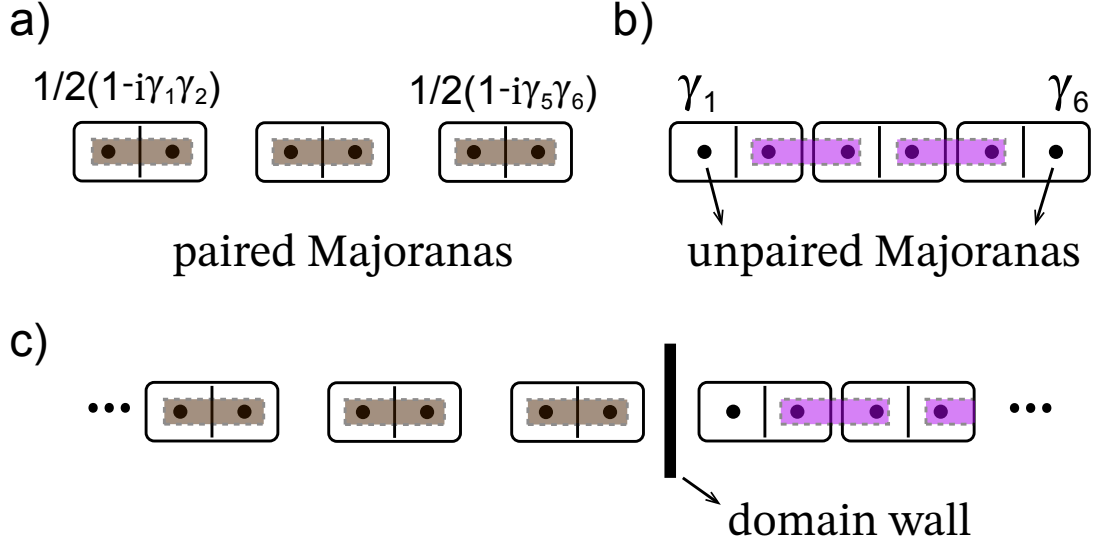


Figure 1.2: Trivial and topological Kitaev chain. (a) Trivial Kitaev chain formed by pairing Majoranas of the same sites together. (b) Topological Kitaev chain formed by pairing Majoranas of different sites together that leaves two unpaired Majorana modes at two ends. (c) Majorana localized on the domain wall by bringing a trivial and topological Kitaev chains together. Figures adapted from [11].

tions does not affect the ground-state energy of the system. As a result, the topological system has two degenerate ground states that form a two-level system required to store information in qubit processing. The two ground states can be characterized by the configurations of the MZMs ($|0\rangle$ or $|1\rangle$ fermionic states). Since the parity of the number of electrons is fixed in the bulk region, the two ground states separately render an odd or even parity. Therefore, the two ground states are equivalently characterized by the total parity of the system. Therefore, the state of an arbitrary qubit as a superposition of the two ground states can be read out by measuring the parity of the system after physically bringing the two MZMs close together.

On top of the topological protection of information and information readout, another interesting property of MZMs leads to the completion of topological quantum computing. Normally, physical particles can be categorized into fermions and bosons, which follow certain exchange statistics accordingly to their type. In a two-particle system, an exchange of particles simply adds a sign of ± 1 in front of the wavefunction. Alternatively, the exchange statistics can be described as particle exchanges introduce a phase shift $e^{i\phi}$ to the wavefunction, with $\phi = 0, \pi$ depending on the particle type. However, MZMs do not follow this statistics. The exchange of a pair of MZMs results in a phase shift with an angle different from 0 and π , as well as a transition

between the ground states. In general, every exchange can be described by a unitary transformation, and the order of MZMs exchange matters since the exchange operations are nonabelian (non-commutative) [12, 13]. Therefore, MZMs are also called nonabelian anyons, and a series of MZMs exchange is called braiding. Since braiding is a unitary transformation, we can use it to perform the computational task in topological quantum computing. Figure 1.3 shows a braiding example that transforms a two-qubit state $|00\rangle$ to an entangled state $(|00\rangle + |11\rangle)/\sqrt{2}$.

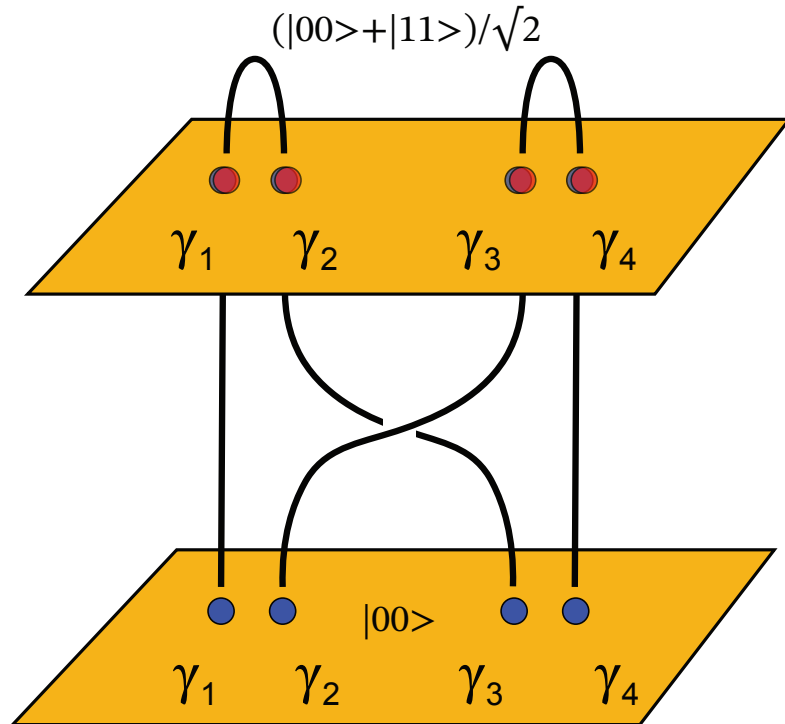


Figure 1.3: Topological computation through Majorana braiding. A simple operation in which two unpaired Majoranas are exchanged. The (isolated) Majorana pairs 12 and 34 are created in the vacuum state $|00\rangle$. When they are brought back together they are in an entangled superposition of 0 and 1 quasiparticle states $(|00\rangle + |11\rangle)/\sqrt{2}$. Figures adapted from [14].

1.3 Detection of MZMs

MZMs possess great potential to implement reliable quantum computing. However, physical realization of MZMs in solid state systems remains elusive due to the lack of unambiguous evidence of the presence of MZMs, despite great effort dedicated to it. Theoretically, MZMs are

predicted to emerge in superconductors with a novel pairing symmetry (p-wave triplet pairing) that can be engineered in some normal materials with proximity-induced s-wave superconductivity. Such superconductors that support MZMs are called topological superconductors. Commonly used materials to implement topological superconductors are semiconducting materials with strong spin-orbit coupling. Among them, topological insulators (TIs) are a special class of materials. Topological insulators, as suggested by their name, have an insulating bulk. However, due to strong spin-orbit coupling that causes band inversion, TIs also have gapless conducting states that are protected by the inherent time-reversal symmetry of TIs and exhibit a spin-momentum locking such that the electron spin of the surface states aligns perpendicularly to their momentum vector. In three-dimensional topological insulators (3DTIs), these conducting states are two-dimensional and become surface states that can be described by a single gapless Dirac cone [15]. When proximity-induced superconductivity is introduced into the system, gapless surface states can behave in the same way as MZMs and be described by Majorana operators [16, 17, 18]. Normally, the pairing of these surface states is trivial. Under certain conditions such as the presence of a $h/2e$ vortex, topological pairing can be induced [19], leading to topological superconductors.

In heterostructures of topological superconductors in contact with normal materials, the domain wall(s) as in Figure 1.2c is formed at the interface(s) between the two types of matter, leading to MZMs localized at the interface(s). Such topological heterostructures can be easily created by proximitizing a section of a long nanowire/nanoribbon made of normal materials that support Majoranas to s-wave superconductors. The resulting superconductor-normal heterostructures could theoretically host a single localized MZM at the SN interface. Electron tunneling via the coupling between the MZM and the fermionic modes in the normal region results in a quantized zero-bias conductance peak (ZBCP) [20, 21]. However, these ZBCPs are not exclusive signatures of the MZMs. Due to limitations in the experimental setup and material systems, it is possible to have trivial ZBCPs produced by other effects such as the presence of trivial bound states near zero energy [22, 23, 24, 25]. Therefore, despite the observations of ZBCPs in semiconducting materials [26, 27] and in TI systems [28], it is hard to conclude that MZMs indeed exist in these solid-state systems.

Alternatively, more complicated heterostructures, such as S-N-S junctions, can be used to realize MZMs. Similarly to S-N heterostructures, S-N-S junctions can be fabricated by proximitizing s-wave superconductors to the two ends of the nanowire/nanoribbon with the same normal materials, leaving a thin non-proximitized normal region in the middle. Two MZMs can reside in the S-N interfaces on the two sides of the normal region. The coupling between the two MZMs leads to single-electron tunneling and 4π -periodic fractional Josephson effect, different from the 2π -periodic supercurrent transport that is expected in S-N-S heterostructures without topological superconductivity [29, 30, 21]. The fractional Josephson effect is considered stronger evidence for the existence of MZMs and has been observed in experiments on the S-N-S junctions [31, 32, 33]. However, the fractional Josephson effect can still arise out of trivial systems without the presence of MZMs [34, 35]. Due to the challenges in obtaining unambiguous evidence for the existence of MZMs, newer and smarter detection schemes are needed.

In response to the need for novel detection schemes, this thesis explores the practicalities of integrating 3DTI SNS Josephson junctions into superconducting qubits with the aim of observing the parity-mixing effect as a signature of MZMs in qubit spectroscopy measurements [36]. Although this signature, like the previous signature, can also be caused by trivial phenomena, this new detection scheme involves qubit measurements, different from transport measurements, which might circumvent some experimental limitations in previous detection schemes such as resolutions and phase slips. Moreover, embedding the Majorana transmon circuit in cQED for detection enables later exploitation as a topological qubit [37, 38].

Chapter 2

Theoretical background

2.1 Superconductivity

2.1.1 An Overview of Superconductivity

In the beginning of twentieth century, as a result of developments in achieving low temperature environment, the phenomenon of superconductivity was first discovered by Kamerlingh Onnes of the University of Leiden as his team noticed the resistance of Mercury vanishing at 4.2K. Two hallmarks of superconductivity are perfect current transfer with zero resistance and the repulsion of the magnetic field (Meissner effect). These phenomena were found to disappear at high temperatures and strong magnetic fields. The disappearances of superconductivity are characterized by two critical values — critical temperature (T_c) and critical field (H_c), above which the superconductivity is destroyed. In order to describe the superconducting phenomena, phenomenological theories with characteristic length scales corresponding to the superconducting properties were developed. For example, regarding the Meissner effect, λ is defined in the London theory [39] as the penetration depth for the exponential screening of a magnetic field from the interior of a superconductor. According to the London theory, $\lambda \propto n_s^{-\frac{1}{2}}$, where n_s is the number density of superconducting electrons.

While the London theory assumes a constant n_s in space, Ginzburg-Landau (GL) theory uses a pseudo-wavefunction $\psi(r) \propto \Delta(r)$ to describe superconducting systems with $n_s = |\psi(r)|^2$ that varies in space. The superconducting pseudowavefunction decays in space with a coherence length ξ when a disturbance to the ψ is present. It is postulated in GL theory that the

interplay of the penetration depth (λ) and coherence length (ξ) leads to two types of superconductors with respect to the penetration of the magnetic field, depending on the ratio between the two characteristic lengths ($\kappa = \lambda/\xi$) [40]. Type-I superconductors with $\kappa \ll 1$, as expected from the London theory, completely repel magnetic field below a critical magnetic field H_c until the superconductivity is destroyed when the field exceeds H_c . Type-I superconductors include Aluminum (Al), Mercury (Hg) and Indium (In).

On the other side, Type-II superconductors with $\kappa \gg 1$ have two critical fields H_{c1} and H_{c2} . Below H_{c1} , Type-II superconductors are said to be in the Meissner phase, behaving similarly as Type-I superconductors. However, once the field exceeds H_{c1} , Type II superconductors are in an intermediate state, where it is energetically favorable to allow partial penetration of the magnetic field in the form of quantized $h/2e$ flux vortices. The vortices have a small non-superconducting core that allows penetration of quantized flux lines with a concentrated magnetic field, surrounded by circulation of superconducting current around the vortex core that cancels the external field within the superconductor, except along the vortex lines. Therefore, the surrounding area of the vortices remain superconducting, until the field exceeds the second critical field H_{c2} such that the external field within the superconductors becomes too strong to be canceled by the circulating supercurrent and the superconductivity is completely destroyed. Common type-II superconducting materials include Niobium (Nb), which is used for the measured devices in the future chapters.

Given the form of the flux vortices, it is intuitive to see that for thin superconducting film, the vortices with flux lines perpendicular to the thin film are easier to form. Therefore, when a field is applied parallel to the thin film, the penetration of the field is suppressed due to the geometry restriction, leading to a higher critical field limit $H_{c\parallel}$. According to GL theory, the presence of a magnetic field H effectively reduces the density of superconducting electrons [40].

$$n_s(H) \equiv |\psi|^2 \propto 1 - \frac{H^2}{H_{c\parallel}^2}. \quad (2.1)$$

2.1.2 BCS Theory

As we can see previously, the phenomenological theories of superconductivity are useful when describing the macroscopic properties of superconductors. However, they did not provide any explanation for the mechanism behind the emergence of superconductivity. It was until the discovery of Bardeen, Cooper, and Schrieffer (BCS) that finally unveiled the microscopic mechanism behind superconductivity — bound pairs of electrons (Cooper pair). There are two significant values in the BCS theory, the complex pairing potential Δ between the bounded electrons and the coherence length of Cooper pairs $\xi_0 = \frac{\hbar v_F}{\pi \Delta(0)}$, where v_F is the Fermi velocity of electrons and $\Delta(0)$ is the pairing gap (magnitude of Δ) at absolute zero temperature. In relation to the GL theory, the pseudowavefunction for superconductors $\psi \propto \Delta$ and its coherence length $\xi \approx \xi_0$ for pure superconductors well below T_c [40].

According to BCS theory, which was formulated to describe conventional superconductors, the pairing potential arises from a spatially isotropic weak phonon-mediated attractive interaction between electrons, leading to the formation of bound pairs of electrons (Cooper pairs) [41, 42]. We can write the attractive interaction between these electrons in a quadratic form:

$$-V \int dx \Psi_s^\dagger(x) \Psi_{-s}^\dagger(x) \Psi_{-s}(x) \Psi_s(x), \quad (2.2)$$

where $s = \uparrow, \downarrow$ is the spin of the electrons ($-s$ represents the opposite spin), and $\Psi_s(x)$ is the electron wavefunction in position space. Due to Pauli exclusion principle, this interaction is between electrons with opposite spins. The pairing potential $\Delta = \Delta e^{i\varphi} = -V \langle \Psi_s(x) \Psi_{-s}(x) \rangle$ is defined when the quadratic form is rewritten into a bilinear one with mean field approximation, where φ is the superconducting phase. The bilinear form is given in the following:

$$\int dx \Delta e^{i\varphi} \Psi_s^\dagger(x) \Psi_{-s}^\dagger(x) + \Delta e^{-i\varphi} \Psi_{-s}(x) \Psi_s(x). \quad (2.3)$$

Using Bloch theorem, we can rewrite Eq 2.3 in the momentum space, such that:

$$\sum_{k,s} \Delta e^{i\varphi} c_{k,s}^\dagger c_{-k,-s}^\dagger + H.c. \quad (2.4)$$

Note that electrons with opposite momentum and spin are paired together in the k-space and Δ is the same for all k as

$$\Delta e^{i\varphi} = - \sum_l V \langle c_{-l,-s} c_{l,s} \rangle. \quad (2.5)$$

This specific singlet pairing order of electrons, called s-wave pairing, prevails in conventional superconductors. The conventional superconductors, therefore, are also called s-wave superconductors.

2.1.3 Bogoliubov-de Gennes Hamiltonian for S-wave Superconductors

With the pairing term given by Eq 2.4, a general BCS hamiltonian that includes the electron excitations can be written as follow,

$$H_{\text{BCS}} = \sum_{k,s} \mathcal{E}_k c_{k,s}^\dagger c_{k,s} + \Delta e^{i\varphi} c_{k,s}^\dagger c_{-k,-s}^\dagger + H.c. , \quad (2.6)$$

where $\mathcal{E}_k = \frac{\hbar^2 k^2}{2m_e} - \mu$, and $\mu = E_F$ is the chemical potential. Since only electrons of opposite spins are paired together, we can reduce the spin degree of freedom by writing $c_k = c_{k,s}$, and defining a hole operator $h_k = c_{-k,-s}^\dagger$ as a counterpart of electrons with equal and opposite momentum (k), spin (s), and energy ($\mathcal{E}_k$). With c_k and h_k , Eq 2.6 becomes

$$H_{\text{BCS}} = \sum_k \mathcal{E}_k c_k^\dagger c_k + h_k^\dagger h_k + \Delta e^{i\varphi} c_k^\dagger h_k + \Delta e^{-i\varphi} h_k^\dagger c_k. \quad (2.7)$$

Eq 2.7 is further simplified using the Bogoliubov-De Gennes (BdG) transformation by the introduction of electron-hole spinor $\Psi_k = (c_k, h_k)^T$, such that:

$$H_{\text{BCS}} = \frac{1}{2} \sum_k \Psi_k^\dagger H_k \Psi_k, \quad (2.8)$$

with

$$H_k = \begin{pmatrix} \mathcal{E}_k & \Delta e^{i\varphi} \\ \Delta e^{-i\varphi} & -\mathcal{E}_k \end{pmatrix} = \mathcal{E}_k \tau_z + \Delta (\tau_x \cos \varphi + \tau_y \sin \varphi), \quad (2.9)$$

where $\tau_{x,y,z}$ are Pauli matrices acting on the electron-hole space.

We can diagonalize H_k easily with a unitary transformation $U_k = \begin{pmatrix} u_k^* & v_k^* \\ -v_k & u_k \end{pmatrix}$ such that the BCS becomes diagonal with new quasiparticle operators $(\gamma_{k,+}, \gamma_{k,-})^T = U_k \Psi_k$:

$$H_{\text{BCS}} = \sum_k \epsilon_k (\gamma_{k,+}^\dagger \gamma_{k,+} - \gamma_{k,-}^\dagger \gamma_{k,-}), \quad (2.10)$$

where $\pm \epsilon_k = \pm \sqrt{\mathcal{E}_k^2 + \Delta^2}$. $\gamma_{k,\pm}$ are called annihilation operators for Bogoliubov quasiparticles [43]. As seen from U_k , the Bogoliubon operators are coherent pairings of electron and hole operators. The coherence factors (u_k, v_k) are given by

$$|v_k|^2 = 1 - |u_k|^2 = \frac{1}{2} \left(1 - \frac{\mathcal{E}_k}{\epsilon_k} \right). \quad (2.11)$$

Assuming $u_k = |u_k|$ is real, then v_k carries the same superconducting phase φ as the pairing potential such that $v_k = |v_k|e^{i\varphi}$.

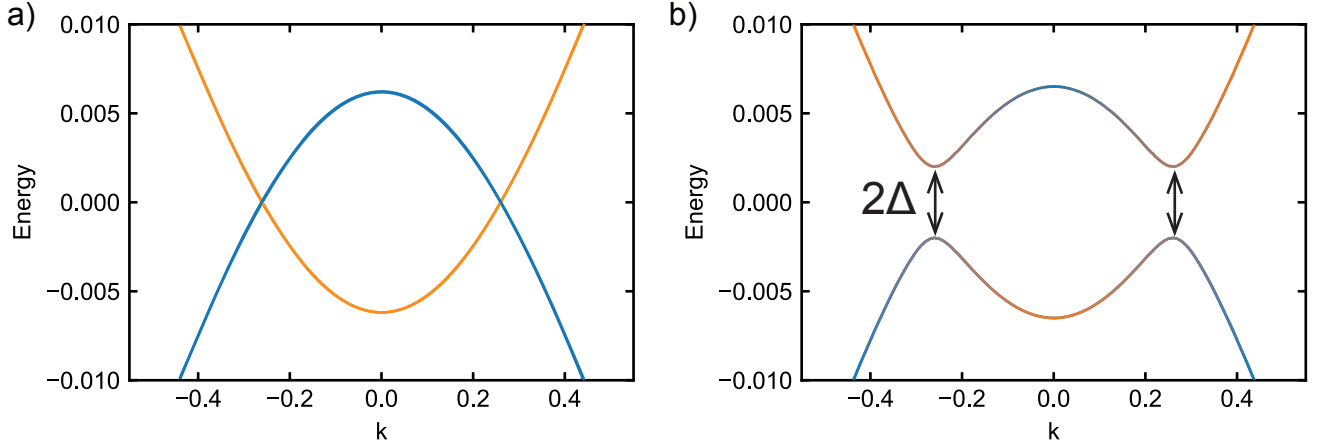


Figure 2.1: Gap opening due to superconducting pairing potential. (a) Quasiparticle excitation band structure that describes one of the bands as holes results in a flip of momentum as well as energy. (b) With superconducting pairing potential Δ added, an avoided crossing at k_F opens up a gap of 2Δ between the two bands, with the new eigenstates having mixed electron-hole character.

Figure 2.1 shows the band structure of a 1D system of free propagating electrons with and without pairing potential. We shall see that a nonzero superconducting pairing potential Δ opens up a gap at Fermi level in the band structure. In Figure 2.1a, when the pairing potential $|\Delta| = 0$, we have a typical parabolic electron [orange] band described by $\mathcal{E}_k$. Due to particle-

hole symmetry, we can set $k \rightarrow -k$, and $\mathcal{E}_k \rightarrow -\mathcal{E}_k$ and end up with a hole [blue] band. The two bands cross at $\pm k_F$, where $\mathcal{E}_k - \mu = 0$. However, when the pairing potential is applied [Figure 2.1b], avoided crossings emerge at k_F due to the mixing of electron and hole bands. The bands are no longer purely electrons or holes but Bogoliubov quasiparticles. The energies of these bands are given by $\epsilon_k = \pm \sqrt{\mathcal{E}_k^2 + \Delta^2}$. As suggested by ϵ_k , the gap opening at k_F is 2Δ and every sub-gap quasiparticle (with $|\mathcal{E}_k| < \Delta$) will be pushed outside of the gap. Another prominent feature of superconducting band structures is that the bands come in pairs, which shows that particle-hole symmetry is preserved such that

$$\gamma(E) = \gamma^\dagger(-E) \quad (2.12)$$

which means annihilating a band with energy E is equivalent of creating a band with energy $-E$, or vice versa.

2.1.4 More on BCS Theory, Temperature Dependence and Field Parity Breaking Effect

In the following, we use Eq 2.10 to find the ground state of a superconductor and the temperature dependence of Δ . The ground state of a system is the lowest energy state. Therefore, given the BCS spectrum given by $\pm\epsilon_k$, the ground state of BCS hamiltonian forms from condensation of all quasiparticle excitations with all of the negative energy states populated. This state can be created from acting on the vacuum state $|V\rangle$ with all the negative energy Bogoliubons:

$$|g_\varphi\rangle = \prod_k \gamma_{k,-}^\dagger |V\rangle \rightarrow \prod_k (u_k - v_k e^{i\varphi} c_k^\dagger h_k) |0\rangle \quad (2.13)$$

We can apply a fourier transform to $|g_\varphi\rangle$ and redefine the ground state in the N -space where N represents number of Cooper pairs, such that

$$|g_N\rangle = \int_0^{2\pi} d\varphi e^{-iN\varphi} |g_\varphi\rangle \quad (2.14)$$

Define $N = -i \frac{\partial}{\partial \varphi}$, where φ is now consider an operator, $|g_N\rangle$ is then an eigenstate of N with eigenvalue N .

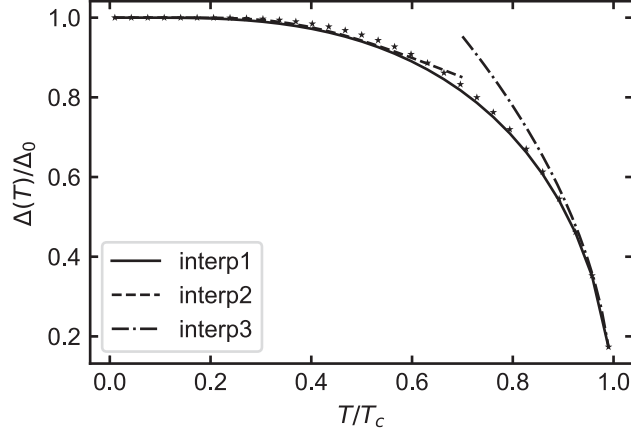


Figure 2.2: Temperature dependence of $\Delta(T)$. The simulated $\Delta(T)$ as a function of Δ_0 is plotted against T/T_c (*). Three interpolation models are fitted to the temperature dependence of $\Delta(T)$.

Next we substitute Eq 2.10 and Eq 2.11 back into Eq 2.5, and get

$$\Delta e^{i\varphi} = - \sum_l V u_l^* v_l \langle 1 - \gamma_{l,+}^\dagger \gamma_{l,+} - \gamma_{l,-}^\dagger \gamma_{l,-} \rangle \quad (2.15)$$

The probability of $\gamma_{k,\pm}$ being occupied can be calculated using the usual Fermi function

$$f(\epsilon_k) = (e^{\beta\epsilon_k} + 1)^{-1}, \quad (2.16)$$

where $\beta = 1/kT$.

As a result, the zero-temperature superconducting pairing potential is found to be $\Delta(0) = 1.764kT_c$, and its temperature dependence of $\Delta(T)$ can be numerically simulated [44]. In Figure 2.2, the simulated temperature dependence is fitted to three interpolation models in Table 2.1.4. The first model fits the temperature dependence quite well over the whole range.

	Math Expression
Interp1	$\tanh 1.74\sqrt{1/t} - 1$
Interp2	$\tanh \frac{1.76}{2t}$
Interp3	$1.74\sqrt{1-t}$

Table 2.1: Interpolation models for the temperature dependence of the superconducting pairing potential. In the mathematical expressions, $t = \frac{T}{T_c}$.

As seen from the derivation of $\Delta(T)$, the reduction of the superconducting pairing potential stems from the excitation of Bogoliubons from the breaking of Cooper pairs due to the thermal effect. Next, we will explore the nature of decreasing n_s with an applied magnetic field [Eq 2.1]. Unlike the thermal effect, if a magnetic field is present, the time-reversal symmetry is broken. There is now a difference between the k and $(-k)$ terms as a result of interactions with the field, leading to Cooper pair breaking. The pair-breaking strength is defined as $2\alpha = |E(-k) - E(k)|$ [45]. Let us take the magnetic perturbation coupled to the momentum as an example that causes Cooper pair breaking. The energy difference between $\pm k$ states is given by $(2e/mc)\mathbf{p}_k \cdot \mathbf{A}$. This energy difference leads to a phase change with time such that

$$\frac{d\varphi}{dt} = \left(\frac{2e}{\hbar c}\right) \mathbf{v}_k \cdot \mathbf{A} \quad (2.17)$$

When bound pairs of electrons ($c_{k,s}$, $c_{-k,-s}$) lose their coherence due to the phase change, the Cooper pairs they form break. Assuming the scattering time τ between collisions is short compared to $\frac{d\varphi}{dt}$, $\frac{d\varphi}{dt}$ is integrated over the whole trajectory to find the overall phase change. The time τ_k required for a Cooper pair to decohere is then given by

$$\frac{1}{\tau_k} = D \left(\frac{2\pi}{\Phi_0}\right)^2 \langle A^2 \rangle, \quad (2.18)$$

where D is the electronic diffusion constant, and $\Phi_0 = h/2e$ is the superconducting flux quantum.

Considering a thin-film superconductor with thickness d subject to parallel field H , α is modified

$$\alpha = \frac{1}{6} \frac{De^2 H^2 d^2}{\hbar c^2} \quad (2.19)$$

For weak pair breaking, the reduced critical temperature $T_c(\alpha)$ and the reduced superconducting gap at zero temperature $\Delta_0(\alpha)$ are linear in α , such that:

$$k(T_{c0} - T_c) = 1.76(\Delta_{00} - \Delta_0(\alpha)) = \frac{\pi\alpha}{4} \quad (2.20)$$

where $T_{c0} = T_c(\alpha = 0)$ and $\Delta_{00} = \Delta_0(\alpha = 0)$. Combining Eq 2.19-2.20 into Eq 2.1 with the

limit $H \ll H_{c\parallel}$, we have

$$\frac{H^2}{H_{c\parallel}^2} \approx \frac{1}{6} \frac{De^2 H^2 d^2}{\hbar c^2}. \quad (2.21)$$

2.1.5 Majorana Zero Modes in 3D Topological Insulators

Recall that within superconductors, the particle-hole symmetry leads to $\gamma(E) = \gamma^\dagger(-E)$, where γ and $\gamma^\dagger$ are creation and annihilation operators that act on eigenstates of superconducting systems. These operators would meet the Majorana condition ($\gamma_M = \gamma_M^\dagger$) when there are zero-energy states present. However, as shown in Section 2.1.4, subgap quasiparticle states are prohibited in superconductors, as shown in the quasiparticle band structure in Figure 2.1b. Additionally, we have shown previously that the quasiparticle excitations in superconductors are coherent superpositions of electron and hole states, with an inherent spin dependence determined by the pairing order of the superconductors. For s-wave pairing, which is assumed in our previous discussion of the BCS theory, the Bogoliubov quasiparticles have a spin texture as shown in the generic expression ($\gamma = uc_\uparrow^\dagger + vc_\downarrow$), that forces $\gamma^\dagger \neq \gamma$, which violates the Majorana condition. With a different pairing order such that electrons with the same spin textures are paired together, the Bogoliubov operators can meet the Majorana condition. However, with a nonzero pairing potential, the subgap quasiparticle states are still prohibited. We shall see that with the p-wave superconductivity supported by 3DTI materials, zero-energy states (MZMs) can emerge with a magnetic field applied.

3DTIs are materials that have an inherent time-reversal symmetry and large spin-orbit couplings. The latter induces gapless conducting surface states that connect the bulk states while the former equips the surface states with spin-momentum locking. The effective low-energy hamiltonian for 3DTIs with z as the direction of uniaxial anisotropy and $\mathbf{k} \equiv (k_x, k_y, k_z)$ is given by the following four-orbital hamiltonian in the spin and orbital subspace [46, 47, 48, 49]:

$$\begin{aligned} [\hat{H}_{3\text{DTI}}(\mathbf{k}) &\equiv [\epsilon(\mathbf{k}) - \mu] + \tau_z M(\mathbf{k}) + \tau_x A_\perp (k_y \sigma_x - k_x \sigma_y) + \tau_y A_z k_z \\ \epsilon(\mathbf{k}) &\equiv C_0 - C_\perp (k_x^2 + k_y^2) - C_z k_z^2, \\ M(\mathbf{k}) &\equiv M_0 - M_\perp (k_x^2 + k_y^2) - M_z k_z^2, \end{aligned} \quad (2.22)$$

where σ and τ are Pauli matrices acting on the spin and orbital subspaces. The parameters C_0 , $C_\perp$, C_z , M_0 , M_z , $M_\perp$, A_z , $A_\perp$ reflect the specific material properties of different 3DTI materials. $C_{0,\perp,z}$ together set the Fermi level (C_0) with a slight asymmetry between the electron-hole bands of surface states ($C_{\perp,z}$). M_0 ($M_{\perp,z}$) describes the band-inversion of the bulk (the surface states). $A_{\perp,z}$ determine the group velocity of the surface states. Different $C_{\perp,z}$, $M_{\perp,z}$ and $A_{\perp,z}$ reflect the uniaxial anisotropy. The spin-momentum locking ($k_y\sigma_x - k_x\sigma_y$) of the gapless surface states protect them from backscattering since reversing their momenta guarantees a flip of their spins [50]. Meanwhile, this spin-momentum locking is found to modify the pairing symmetry of conventional superconductors and leads to a p-wave pairing for low-energy surface states [16, 51, 52]. The actual derivation is not given here, but the key is a definition of spinless momentum-dependent electron and hole operators $c_{\mathbf{k}} = (\psi_{\uparrow\mathbf{k}} + e^{i\theta_{\mathbf{k}}}\psi_{\downarrow\mathbf{k}})/\sqrt{2}$ for $\mathbf{k} = k_0(\cos\theta_k, \sin\theta_k)$ and $\nu k_0 \sim \mu$. With these spinless zero-energy electrons and holes, the hamiltonian of a 3DTI system with induced s-wave superconductivity can be rewritten into a form that resembles $p_x + ip_y$ superconductors.

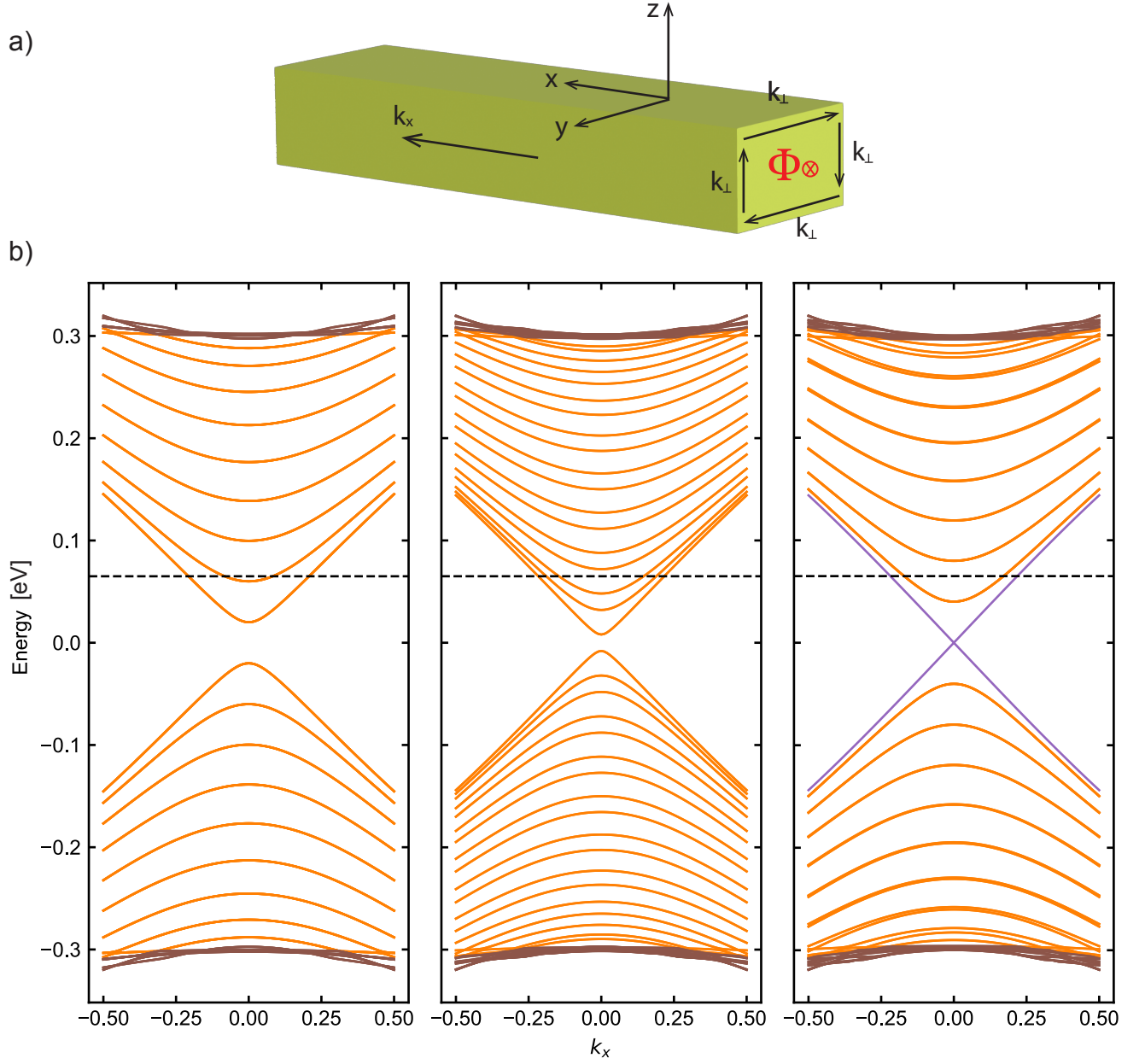


Figure 2.3: (a) Schematic of a long 3DTI nanoribbon along the x-direction. The nanoribbon has a confined cross-sectional area in the y-z plane, leading to quantized $k_\perp$. A magnetic field applied along the x-direction threads a flux Φ through the confined cross-sectional area. (b) Band structures of the 3DTI nanoribbon when $\Phi = 0, 0.6\Phi_0, \Phi_0$. The brown bands represent the bulk states of our 3DTI. Surface states (orange) exist within the band gap between the valence and conduction bands. At $\Phi = \Phi_0$, gapless surface states (purple) emerge, which can host unpaired MZMs.

Now, consider a long 3DTI nanoribbon with the longitudinal axis along the x-direction, as shown in Figure 2.3a. The TI nanoribbon is confined in the y-z (transverse) plane, leading to the quantization of $k_{\perp}$ given by $2\pi l/C$, where $l = 0, \pm 1, \pm 2, \dots$ is the quantum number for the angular momentum and C is the circumference of the path surrounding the transverse surface. Considering other effects such as the Berry's phase and the Aharonov-Bohm effect due to an applied magnetic field, the total quantization of $k_{\perp}$ is given by [17, 18]:

$$k_{\perp} = \frac{2\pi}{C} \left(l + 0.5 - \frac{\Phi}{2\Phi_0} \right), \quad (2.23)$$

where Φ is the flux threading through the nanoribbon due to an external magnetic field and $\Phi_0 = h/2e$. In the equation above, the first term is attributed to the angular momentum. The second term is attributed to the Berry's phase, while the third term is attributed to the Aharonov-Bohm effect. Since the nanoribbon is long along the longitudinal axis (x-axis in Figure 2.3a), we can assume a continuum in k_x . The energy band is given by

$$E_l(k_x) = \pm \hbar v_F \sqrt{k_x^2 + k_{\perp}^2}. \quad (2.24)$$

As Eqs 2.23 and 2.24 show, the band structures of a long 3DTI nanoribbon, depending on the values of Φ , are separated into three cases:

1. Due to the Berry's phase, at $\Phi = 0$, every energy band is gapped and doubly degenerate ($E(l) = E(-l - 1)$).
2. The degeneracy of the band structure is lifted with an arbitrary $\Phi \neq \Phi_0$. The bands remain gapped.
3. When $\Phi = (2n + 1)/\Phi_0$, the degeneracy of most energy bands is recovered except from the $l = 0$ state. Moreover, the $l = 0$ state is gapless.

Using a tight-binding formulation of the effective 3D Hamiltonian on a cubic lattice with artificial lattice constant a , and imposing a geometry of an infinitely long nanoribbon with translational symmetry along the longitudinal axis of the nanoribbon, the band structure of the nanoribbon is simulated with Kwant [53] and plotted in Figure 2.3b. The parameters used in the simulation are given in Supplementary D.1. The simulated band structures confirm the

previous discussions of the three flux-dependent situations. As shown in Figure 2.3b, without any flux piercing the nanoribbon, all surface states (orange) are gapped and degenerate [Left of Figure 2.3b]. Due to the gap opening, these surface states are sensitive to non-magnetic disorder, such as chemical potential disorder that can lead to localization and interband backscattering [54, 55], different from the gapless surface states that are perfectly conducting in ideal 3DTIs independent of the Fermi level. With an arbitrary pierced flux $\Phi \neq \Phi_0$, the time-reversal symmetry is broken and the gapped surface states (orange) become non-degenerate while remaining gapped [Middle of Figure 2.3b].

With $\Phi = \Phi_0$ [Right of Figure 2.3b], time-reversal symmetry is restored and the once again degenerate gapped surface states (orange) have opposite ls . Assuming the Fermi level is at an arbitrary μ represented by the dashed line in Figure 2.3b, the degenerate zero-energy surface states (degenerate bands that cross μ) can be paired by induced superconductivity that pushes the surface states out of the induced gap. On the other hand, the isolated energy band (purple) that represents gapless surface states with $l = 0$ does not have a companion band to be paired with and therefore remains a zero-energy state even with the presence of a pairing potential. This gapless surface state, different from incident subgap states, is topologically protected by time-reversal symmetry and hosts two unpaired Majorana zero modes. Without any additional domain walls such as magnetic defects inside the 3DTI nanoribbon superconductor, it is posited that the two MZMs are located at the two ends of the nanoribbon, where the induced superconductivity is off. As a result, the two MZMs can be described by the following wavefunctions

$$\gamma_{1,2} \equiv \frac{1}{\sqrt{2}} \int dx e^{-i\varphi/2} \xi_{1,2}(x) c^\dagger(x) + e^{i\varphi/2} \xi_{1,2}^*(x) c(x). \quad (2.25)$$

Here $\sqrt{2}$ is a normalization factor, φ is the superconducting phase, and $\xi_{1,2}(x)$ are bound state wave functions such that:

$$\xi_{1,2}(x) = e^{-(x-x_{1,2})/|\xi_{1,2}|} e^{i\theta_{1,2}}, \quad (2.26)$$

and

$$\theta_1 - \theta_2 = \pi/2, \quad (2.27)$$

Note that Eq 2.27 is calculated to ensure that when MZMs are brought together, they form a single Fermionic mode. The origin of it is the $\mathbf{k}$ -dependent pairing due to spin-momentum locking.

Apart from the gapped and gapless surface states, our long 3DTI nanoribbon also hosts bulk states (brown), as shown in Figure 2.3b. These bulk states stay insulating when the Fermi level is below the conduction band. However, it has been found that the Fermi level is usually well above the Dirac points in 3DTI materials (Bi_2Se_3 and Bi_2Te_3) [56, 57]. If the Fermi level is high enough to cross through the conduction band, the bulk states become conducting. These conducting bulk states, although considered to be a small contributing part, can complicate the electronic properties of 3DTI systems.

2.2 Josephson Junctions

After introducing the basics of superconductivity, we now start to look at properties of superconducting heterostructures, in particular, Josephson junctions (JJs) — two superconducting leads separated by a weak link. Depending on the type of the weak link, Josephson junctions can be categorized into superconductor-insulator-superconductor (SIS), superconductor-normal-superconductor (SNS) junctions and more. It was first predicted for SIS junctions, that supercurrent as a function of the superconducting phase difference between the two superconducting leads can flow through the insulating region from one lead to another [58, 59]. It was later found that the phenomenon is general for all kinds of JJs although the specific current-phase (or energy-phase) relation varies. This phenomenon is rooted from the extension of Cooper pairs coherence into the weak link region. Measuring the phase-dependence can reveal the electronic properties of the material systems involved in the junctions. Moreover, non-linear inductance of Josephson junctions is especially useful for creating a non-linear oscillator energy spectrum.

2.2.1 SIS Junction

SIS junctions are also called tunnel junctions because the insulating region acts as a potential barrier for Cooper pairs to tunnel through. For a short insulating region with respect to the cooper pair coherence length, the transport of supercurrent in these junctions follows Josephson effect:

$$I = I_c \sin \varphi \quad (2.28)$$

$$V = \frac{2\pi}{\Phi_0} \frac{d\varphi}{dt} \quad (2.29)$$

where φ is now defined as the phase difference between the superconducting leads $\varphi_2 - \varphi_1$. Josephson effect implies that with zero potential, there is a sinusoidal current-phase relation, and with nonzero potential, current oscillates in time with a frequency given by $V\Phi_0/(2\pi)$.

We can easily use the Ginzburg-Landau approach to verify the Josephson effect by construct-

ing a wavefunction $(\psi_1, \psi_2)^T$, where $\psi_1 = \sqrt{n_1}e^{i\varphi_1}$ and $\psi_2 = \sqrt{n_2}e^{i\varphi_2}$ are the pseudowavefunctions used in the Ginzburg-Landau theory [Section 2.1.1]. Given an SIS junction as shown in Figure 2.4, the system Hamiltonian can be written as:

$$H = \begin{pmatrix} eV & T \\ T & -eV \end{pmatrix} \quad (2.30)$$

where T describes the interaction between the two superconducting leads that causes the transport between them.

Solving the Schrödinger equation, we end up with

$$\dot{n}_A = \frac{2T\sqrt{n_A n_B}}{\hbar} \sin \varphi \quad (2.31)$$

$$\dot{\varphi} = \dot{\varphi}_2 - \dot{\varphi}_1 = \frac{2eV}{\hbar} \quad (2.32)$$

Assuming $n_A \approx n_B = n_s$, the solution (2.31) transforms to the Josephson effect with $I_c \propto n_s$, and therefore $I_c \propto |\Delta|^2$, where Δ is the gap parameter assuming to be a constant in the bulk of the two leads, and can be estimated from the BCS pairing potential.

The Josephson potential can be evaluated by combining Eq 2.28 and Eq 2.29:

$$H_J = \int_1^2 IV dt = \int_0^\varphi \frac{2\pi I_c}{\Phi_0} \sin \varphi = -\frac{2\pi I_c}{\Phi_0} \cos \varphi \quad (2.33)$$

The Josephson inductance L_J is defined as $V = L_J \frac{dI}{dt}$ and is given by the following:

$$L_J = \frac{\Phi_0}{2\pi I_c \cos \varphi} \quad (2.34)$$

As we can see from the derivation of Josephson effect above, the nature behind the supercurrent transport in tunnel junctions is the interaction of superconducting wavefunctions despite the insulating region. Such an interaction is allowed because the superconducting wavefunctions vary in space and have a coherence length $\xi_{1,2}$ that can extend into the insulating region. As the separation between the two superconducting leads lengths, intuitively, we would expect a decrease in the transported supercurrent, which is the iconic exponential decay of critical

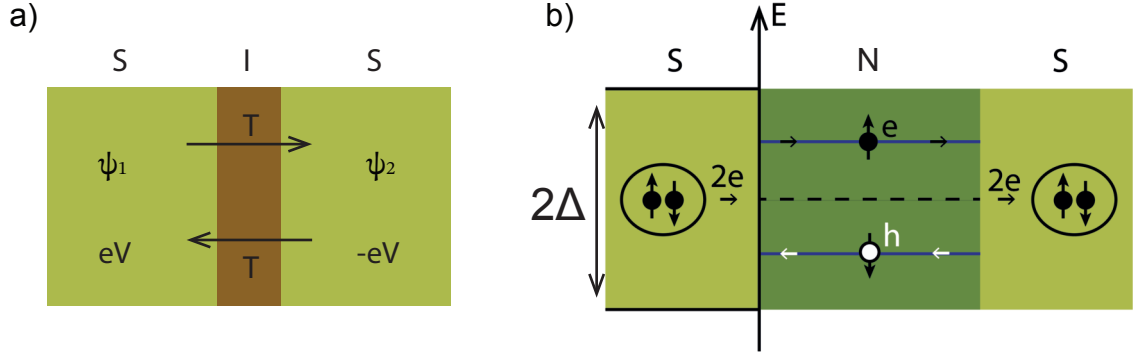


Figure 2.4: SIS and SNS Junctions (a) SIS junctions where two superconducting [light green] leads are separated by a short insulating [brown] region. The two leads are described by pseudowavefunctions $\psi_{1,2} = \sqrt{n_{1,2}}e^{i\varphi_{1,2}}$, where $n_{1,2}$ is the number of Cooper pairs and $\varphi_{1,2}$ is the superconducting phase. The transport from one lead to another is characterized by T in the unit of energy. (b) SNS junctions where two superconducting [light green] lead are separated by a short normal [dark green] weak link. The supercurrent transport is carried by Andreev bound states formed from constructive interferences of multiple Andreev reflections, where an incoming electron (hole) gets reflected as a hole (electron) with opposite momentum and spin.

current in long SIS junctions.

2.2.2 SNS Junction and Andreev Bound States

Now, if we have SNS junctions, supercurrent transport is facilitated by Andreev reflections, rather than the tunneling of Cooper pairs, which means that an electron gets reflected into a hole with the opposite spin and momentum (and vice versa, for an incoming hole) at an S-N interface. In general, a charge quasiparticle with a coherent superposition of pure electron and hole states gets reflected into its particle-hole symmetric partner state. For the sake of simplicity, the following discussion uses pure electron and hole states as examples.

Given an SNS junction as shown in Figure 2.4, let us consider a subgap electron state with positive energy that travels inside the normal region towards the right superconducting lead. Since subgap states are not allowed inside superconductors as shown in Section 2.1.4, the incoming electron has to condense into a Cooper pair. Such condensation happens when the right superconducting lead ejects an outgoing hole with the opposite energy and spin as the incoming electron [60, 20]. The annihilation of a hole inside the superconductor is equivalent to adding an electron state inside that can be paired with the incoming electron to form Cooper

pairs, when the phase difference between the two match the superconducting phase of the lead. To allow the condensation, the phase difference between the two electrons must match the superconducting phase of the lead φ_2 . Assuming the incoming electron has a zero phase, the phase of the other electron arising out of the removal of a hole is given by $e^{i\varphi_2}$. Therefore, the reflected hole picks up the same phase $e^{i\varphi_2}$. The reflected hole, once arriving at the other S-N interface between the normal region and the left superconducting leads, is reflected back into electron state that is in the phase coherence with the original electron, which leaves a hole state inside the right superconductor. The resulting hole state is bound with the incoming hole, with a phase difference $e^{-i\varphi_1}$, equivalent to destroying a Cooper pair. Therefore during this Andreev reflection, the reflected electron state again picks up an additional phase $e^{-i\varphi}$.

Combining the effect of two Andreev reflections at both interfaces, we see that a Cooper pair is transported from the left lead to the right lead via couplings to the electron and hole states in the normal region. We also see that the overall phase accumulation due to Andreev reflection instead of trivial phase decoherence events due to noises inside the normal region, is given by $e^{i(\varphi_2-\varphi_1)} = e^{i\varphi}$, where φ is the superconducting phase difference between the two leads. The electron and hole states can undergo multiple Andreev reflections more than twice. Andreev bound states (ABSs) are formed due to interference between all the reflected electrons and holes. Restricting to only two Andreev reflections and assuming a perfect transmission inside the normal region without any phase decoherences, as the accumulated phase $e^{i\varphi}$ suggests, ABSs are 2π -periodic, with constructive interference at $\varphi = 2n\pi$ and completely destructive interference at $\varphi = (2n + 1)\pi$, where $n = 0, 1, 2, \dots$. With imperfect transmissions considered (for example due to disordered normal region and long junction limit), the analytic expression for the ABS is given by the following:

$$E_{ABS} = \pm \Delta \sqrt{1 - T \sin^2(\varphi/2)}, \quad (2.35)$$

where T is the transmission coefficient of the ABS [61]. An ABS with high T is said to be ballistic, and otherwise diffusive. Using the equation above, the energy-phase relation of ABSs is shown in the left panel of Figure 2.5a. The sign of E_{ABS} depends on the direction of supercurrent. Using the definition $I \propto \frac{dE}{d\varphi}$, the phase dependence of the supercurrent carried by

ABSs is shown in the right panel of Figure 2.5a.

Within an SNS junction, there could be multiple ABSs, each with a characteristic transmission coefficient (T_i), the SNS junction energy due to the supercurrent transport is then the sum of all the ABSs (with the same direction) inside:

$$H_J = \sum_i E_{i,ABS} = -\Delta^* \sum_i \sqrt{1 - T_i \sin^2(\varphi/2)}. \quad (2.36)$$

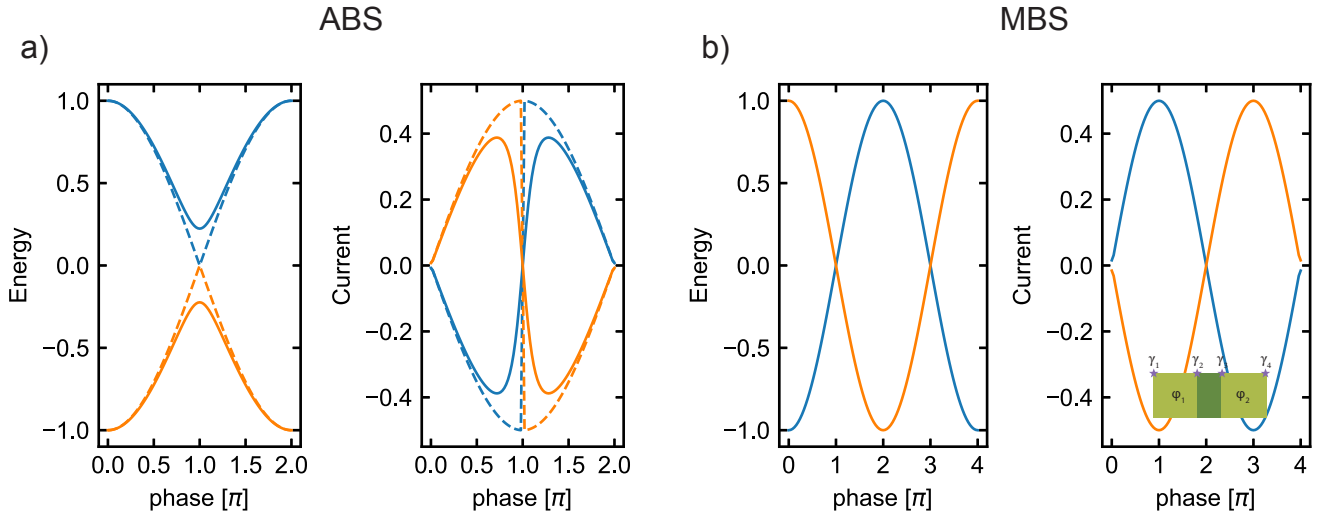


Figure 2.5: Energy-phase and current-phase relations of Andreev and Majorana bound states. (a) The energy-phase and current-phase relation of two Andreev bound states (with opposite supercurrent direction) formed inside an SNS junction with $T = 1$ (dashed line) and $T = 0.95$ (solid line). (b) 4π -periodic energy-phase and current-phase relation of two Majorana bound states (with opposite supercurrent direction) that can be formed in an STI-N-STI junction as shown in the inset of the right panel in (b).

2.2.3 3DTI Junctions

We can build an SNS junction using a long 3DTI nanoribbon. By proximitizing conventional superconducting materials on top of the left and right regions of the nanoribbon, separated by a short normal unproximitized region, the heterostructure becomes an STI-N-STI junction. Effectively, the two ends of the nanoribbon with induced superconductivity turn into p-wave superconductors [Section 2.1.5]. Each lead can potentially host two MZMs at its boundaries, where the induced superconductivity turns zero. As a result, there are in total four MZMs ($\gamma_1, \gamma_2, \gamma_3, \gamma_4$) inside the STI-N-STI junction, as shown in the inset of Figure 2.5b.

Since MZMs are zero energy states, if they stay isolated, we would have $H_M = 0$. However, if their wavefunctions overlap, they form a fermionic mode that has a finite energy depending on its occupancy. Assuming that the superconducting leads have lengths ($L_{SC} \gg \xi$), where ξ is the coherence length of the MZMs, the couplings between the two pairs of MZMs (γ_1, γ_2) and (γ_3, γ_4) can be safely ignored. Therefore, the only non-zero Majorana energy term is due to the coupling between γ_2 and γ_3 . Using the MZMs wavefunctions [Eq 2.25], we end up with

$$H_M = iE_M\gamma_2\gamma_3 \cos \varphi/2, \quad (2.37)$$

where φ is the superconducting phase difference between the two superconducting leads, and $E_M \propto e^{-L/\xi}$. The Majorana coupling between γ_2 and γ_3 allows a free fermionic mode to transport from one lead to another coherently. This coherent single-electron transport will also lead to the formation of bound states similarly to the emergence of ABSs. We call these bound states that are originated from Majorana coupling Majorana bound states (MBSs). The energy-phase and current-phase relations of the MBSs are plotted in Figure 2.5b. Notice that MBSs have a 4π periodicity as opposed to the 2π periodicity of ABSs and the SIS Josephson effect. This difference stems from the fact that MBSs support single-electron transport instead of supercurrent transport.

As discussed in Section 2.1.5, MZMs in confined 3DTI nanoribbons are hosted by gapless non-degenerate surface states. As shown in Figures 2.3 and 2.6a, these gapless non-degenerate surface states (purple) emerge when an odd integer of quantum flux is threaded through the nanoribbon. Meanwhile, as the figures show, gapped surface states (orange) and insulating bulk states (brown) coexist with the gapless surface states that host MBSs. The gapped surface states follows the same mechanism as described in the previous section [Section 2.2.2] and form ABSs that facilitate supercurrent ($2e$) transport [62]. These gapped surface states are subject to backscattering due to weaker topological protection compared to that of gapless surface states [54, 55] [Section 2.1.5], resulting in imperfect supercurrent transmission ($T < 1$). Whether the supercurrent transport is ballistic or diffusive depends on the gap opening of the surface states as well as the intensity of disorders that cause the backscattering. Figure 2.6b

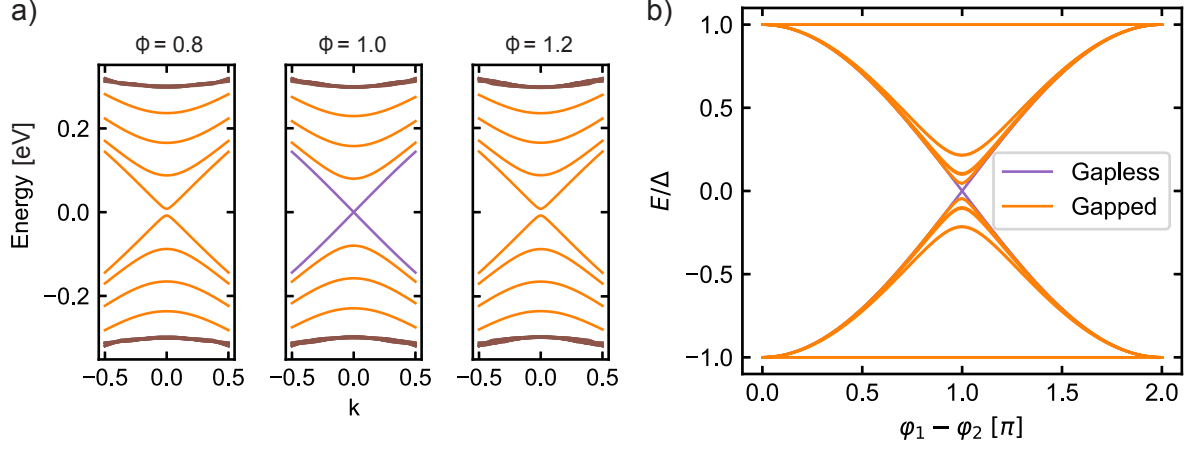


Figure 2.6: Charge transport carried by surface states in TI hybrid junctions. (a) Band structures of an STI lead with threaded flux $\Phi = 0.8\Phi_0$, Φ_0 , $1.2\Phi_0$. When $\Phi \neq \Phi_0$, the charge transport can be carried by surface states (orange) and bulk states (brown). When $\Phi = \Phi_0$, a gapless non-degenerate surface state (purple) can host Majorana zero modes and facilitate charge transport with other surface states and bulk states. (b) The simulated energy-phase relations of the MBS and ABS charge transport carried by gapless surface states (purple) and gapped surface states (orange).

plots the simulated energy-phase relations of an MBS formed by gapless surface states and ABSs formed by gapped surface states. The simulation is based on the same tight-binding microscopic model of a 3DTI nanoribbon as the band structure simulation in Section 2.1.5 with $\Phi = \Phi_0$. To induce backscattering among the gapped surface states, a potential mismatch between the two superconducting leads and a chemical potential disorder profile are deliberately introduced into the 3DTI nanoribbon [Supplementary D.1]. As Figure 2.6b shows, the energy-phase traces of all the ABSs formed by gapped surface states (orange) have non-zero gaps at the π -point. According to Eq 2.35, they have imperfect transmissions in the range of $(0, 1)$. The ABSs with almost zero gap opening have $T \rightarrow 1$ and the ABSs with flat energy-phase relation have $T \rightarrow 0$. On the other hand, despite the potential mismatch and disorders, the MBS remains 4π -periodic.

Additionally, if the bulk states are partially conducting, for example, when the Fermi level crosses the conduction band [56, 57] [Section 2.1.5], they can also form ABSs and contribute to supercurrent transport in TI JJs. Since these conducting bulk states are normal states, they are exempted from topological protection and are more inclined to be diffusive. In general, the TI JJ can be considered as an SNS junction in the trivial regime where MBSs are absent. However, in the topological regime when MBSs emerge with odd number of flux quantum

pierced through, the total TI JJ Hamiltonian can be written as:

$$H_{\text{TIJJ}} = H_{\text{ABS}} + H_M. \quad (2.38)$$

2.2.4 SQUIDS

We have shown different types of JJs, all dependent on the superconducting phase difference (φ). However, without additional control, φ is usually fixed, resulting in fixed JJ energies. In this section, we shall see how a superconducting quantum interference device (SQUID) can be created from two JJs in a loop [Figure 2.7], leading to flux-dependent JJ energies.

Consider two JJs joined together by superconducting links to form a mostly superconducting loop as shown in Figure 2.7. If a magnetic field is applied perpendicular to the enclosed area of the loop, we have [63, 64]

$$\varphi_1 + \varphi_2 - \frac{2\pi\Phi}{\Phi_0} = 2\pi n, \quad (2.39)$$

where Φ is the magnetic flux threaded through the loop area, and φ_1 and φ_2 are the phase differences across superconducting leads of the two JJs. As Eq 2.39 shows, φ_1 and φ_2 will change with Φ periodically. We can then set φ_1 as an independent variable and look at the flux dependence of the SQUID energy (H_{tot}). For purely SIS junctions and SNS junctions (with ABSs), H_{tot} simply sums over the energies of the two JJs, such that

$$H_{\text{tot}} = H_{J,1}(\varphi_1) + H_{J,2}\left(\frac{2\pi\Phi}{\Phi_0} + 2\pi n - \varphi_1\right), \quad (2.40)$$

with H_J defined in Eqs 2.33 and 2.36. If the two junctions are identical such that they have the same Josephson energy (E_J) or equivalently, they have the same junction energy when $\varphi_1 = \varphi_2$ for junctions that do not have well-defined E_J , we say the SQUID loop they form are symmetric. We can then define an asymmetry factor d with $d = \frac{E_{J,1} - E_{J,2}}{E_{J,1} + E_{J,2}}$ for SIS junctions and $d = \frac{\Delta_1 - \Delta_2}{\Delta_1 + \Delta_2}$ for SNS junctions. Figure 2.8 shows the simulated energy-phase relations of a SIS SQUID and a SNS SQUID at $\Phi = 0, 0.25\Phi_0, 0.5\Phi_0, 0.75\Phi_0, 1\Phi_0$. During the simulation, a slight asymmetry ($d = 0.2$) between the two JJs in the SQUID is assumed. Details about other simulation parameters are given in the Supplementary D.2.1.

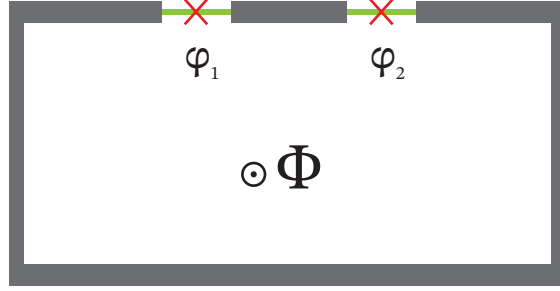


Figure 2.7: Josephson junctions in a SQUID loop. Two JJs (red crosses) are joined by superconductors (gray) and form a superconducting loop. The two weak link regions of the JJs are depicted as two green short lines. The phase differences across the two weak links are φ_1 and φ_2 . A magnetic field applied perpendicularly to the loop area can thread a flux of Φ_0 .

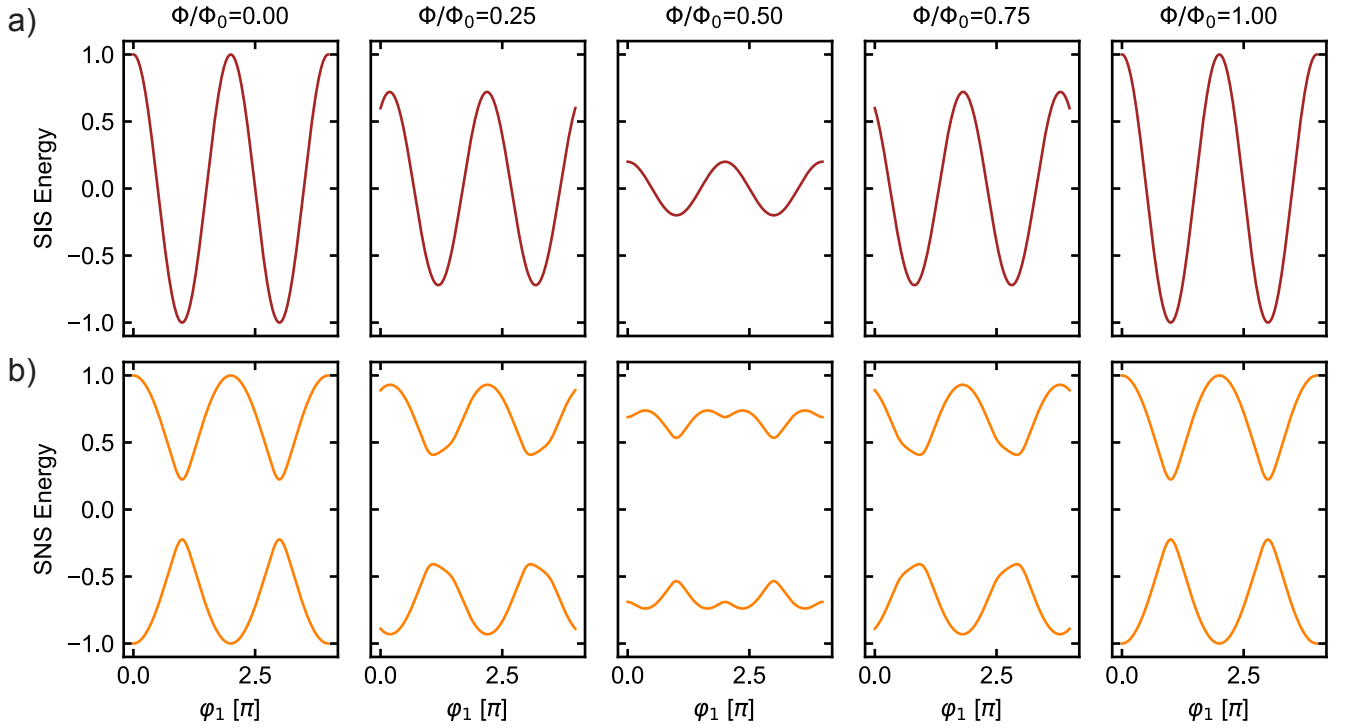


Figure 2.8: Periodicity with external flux in the energy-phase relation of (a) SIS junction, (b) Andreev bound states with different external flux values $\Phi/\Phi_0 = 0, 0.25, 0.5, 0.75, 1$.

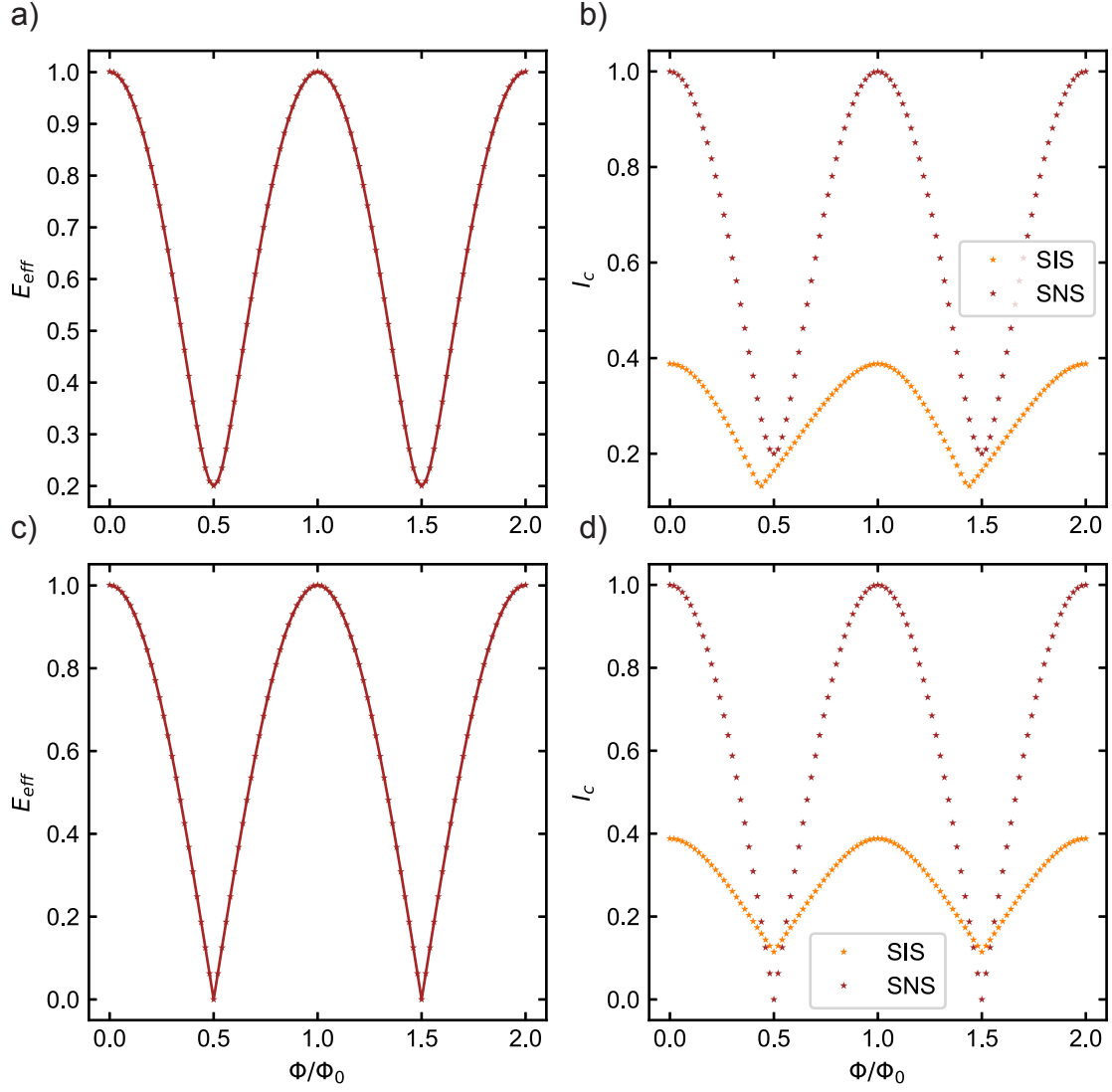


Figure 2.9: (a) Flux dependence of E_{eff} of an SIS SQUID with a slight asymmetry $d = 0.2$. (b) Flux dependence of the critical currents (I_c) for an SIS SQUID and an SNS SQUID when $d = 0.2$. (c) Flux dependence of E_{eff} of a perfectly balanced SIS SQUID with $d = 0$. (d) Flux dependence of the critical currents (I_c) for a symmetric SIS SQUID and a symmetric SNS SQUID.

As suggested from the term $\frac{2\pi\Phi}{\Phi_0}$ in Eq 2.39, we would expect a Φ_0 -periodicity in the flux dependence of H_{tot} for H_J with 2π -periodic phase dependence. Figure 2.8 shows this expected Φ_0 -periodicity for both SIS and SNS SQUIDS. Figure 2.8 also shows that the flux dependence of an SIS SQUID has a more regular pattern, which can be described by a phase shift as well as an amplitude change in its SIS-like energy-phase relation. Since we do not care about an arbitrary phase shift in the energy-phase relation, we can treat an SIS SQUID as an effective SIS JJ with its E_J tunable by a magnetic flux Φ_0 . In fact, by defining a new phase $\varphi' = (\varphi_1 + \varphi_2)/2$, H_{tot} in Eq 2.40 with SIS Josephson energy can be written in the following analytic form:

$$H_{\text{tot}}(\Phi) = E_{\text{eff}}(\Phi) \cos(\varphi' - \varphi_0), \quad (2.41)$$

with

$$E_{\text{eff}}(\Phi) = E_{J,\text{all}} \sqrt{\cos^2\left(\frac{\pi\Phi}{\Phi_0}\right) + d^2 \sin^2\left(\frac{\pi\Phi}{\Phi_0}\right)}, \quad (2.42)$$

where $E_{J,\text{all}} = E_{J,1} + E_{J,2}$, and $\varphi_0 = d \tan \pi\Phi/\Phi_0$. The agreement between numerical simulations and the analytical expression above is shown in Figures 2.9a and 2.9c. The minimum $E_{J,\text{all}}$ appears at the half-flux point. When the SQUID is perfectly symmetric ($d = 0$), $E_{J,\text{all}}$ goes to zero at its minimum [Figure 2.9c]. As d increases, the minimum $E_{J,\text{all}}$ increases and reduces the tuning range of the effective Josephson energy.

On the other hand, the flux dependence of an SNS SQUID, despite having the same Φ_0 -periodicity, has a much more complicated form. Since the ABS energy [Eq 2.35] allows for higher harmonics, the energy-phase relation with a nonzero Φ_0 can become complex as shown in Figure 2.8. Due to the change in the functional form, we cannot use a single variable to capture the whole flux-dependence of an SNS SQUID. However, the critical current (I_c) of an SNS SQUID, which measures the maximum current that can be carried through the loop, is well defined because it ignores the detailed phase dependence. The critical currents of an SIS and an SNS SQUID are shown in Figures 2.9b and 2.9d. As we can see from Figure 2.9b, with $d \neq 0$, the SNS current-phase relation, with an apparent skewedness, differs from that of the SIS. However, for perfectly-balanced SNS SQUIDS ($d = 0$), the current-phase relation becomes symmetric with the minimum I_c at the half-flux point like SIS SQUIDS.

2.3 Transmon Qubit System

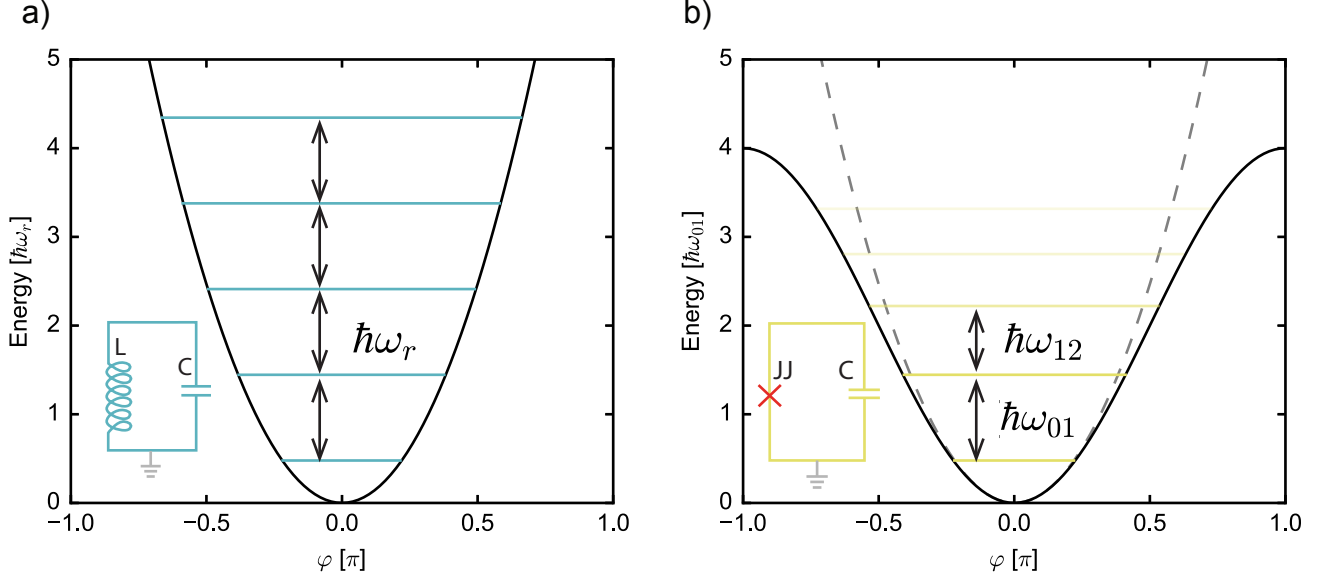


Figure 2.10: Energy spectra of a quantum simple harmonic oscillator (SHO) and a superconducting qubit. The inset of (a) Circuit for a parallel LC-resonator. The superconducting phase on the island is denoted ϕ , referencing ground as zero. (a) Linear energy spectrum of the LC-resonator with quadratic potential, characterized by a uniform energy splitting of $\hbar\omega_r$. The inset of (b) Superconducting qubit circuit with a nonlinear element Josephson junction (JJ) in parallel with a capacitor. (b) The Josephson inductance reshapes the quadratic energy potential (dashed) into sinusoidal (solid) which yields a nonlinear energy spectrum. This allows us to isolate the two lowest energy levels $|0\rangle$ and $|1\rangle$, forming a computational subspace with an energy separation $\hbar\omega_{01}$, which is different than $\hbar\omega_{12}$. Figures adapted from [8].

In the previous section, we have seen that the energy-phase relations of different types of Josephson junctions (JJ) that deviate from a simple quadratic form. This deviation leads to rich energy spectra when embedding JJs in superconducting electrical circuits [Figure 2.10 b], such as the Cooper pair box circuits [65, 66]. From the non-linearity, the lowest two-level subspace with transition energy $\hbar\omega_{01}$ can be selected to form a qubit basis. Qubits implemented via superconducting circuits are called superconducting (SC) qubits. Charge-insensitive SC (transmon) qubits can be created from Cooper-pair box circuits by carefully choosing the values of the key electronic components [64].

To perform qubit measurements on SC qubits, the most common scheme is to create a dispersive coupling between a SC qubit and a superconducting resonator [67, 68], which is a quantum harmonic oscillator arising from superconducting circuits with linear components

only [Figure 2.10 a]. The qubit-resonator system resembles an atom inside the cavity, where the cavity resonances change according to the state of the atom [69]. As a result, we can read out the SC qubit state via measuring the change in the superconducting resonator frequency. It is important to study the complete SC qubit system which includes not only the qubit itself, but also the superconducting resonator that the qubit is coupled to and the interaction between them.

2.3.1 Superconducting Resonators

Superconducting resonators are essentially quantized LC circuits built from superconducting materials, as shown in the inset of Figure 2.10a. Therefore we start our discussion of superconducting resonators from LC oscillators. An LC circuit simply contains an inductor (with inductance L) and a capacitor (with capacitance C). We then have the current through the inductor $I = \Phi/L$ and the voltage across the capacitor $V = Q/C$. The linear relationships between the two pairs (I, Φ) and (V, Q) characterize normal inductors and capacitors as linear circuit elements, which determines the harmonicity of the system. We can then construct a Hamiltonian based on the linear relationships. $H = \Phi^2/2L + Q^2/2C$ in quadratic form, which resembles the Hamiltonian of mechanical oscillators. The bare resonance of the LC oscillator is given by $\omega_r = 1/(2\pi\sqrt{LC})$. At room temperature, due to thermal energy, the energy of an LC oscillator is a continuous quantity. Moreover, dissipation in normal currents imply that there is always a resistance that adds damping to the oscillator. If we reduce the temperature such that the environmental thermal energy is small compared to $\hbar\omega_r$, the quantized energy spectrum of the LC oscillator becomes detectable. Additionally, switching to superconductors in place of normal conductors, we can construct a dissipationless LC circuit.

In such a quantized LC oscillator, where $Q \rightarrow \hat{Q}$ and $\Phi \rightarrow \hat{\Phi}$, we can define lowering and raising operators $\hat{a}$ and $\hat{a}^\dagger$ in similar fashions as quantum mechanical oscillators

$$\hat{H}_{LC} = \frac{\hat{Q}^2}{2C} + \frac{1}{2}C\omega_r^2\hat{\Phi}^2 = \hbar\omega_r(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \quad (2.43)$$

$$\hat{\Phi} = \Phi_{\text{zpf}}(\hat{a}^\dagger + \hat{a}), \quad \hat{Q} = iQ_{\text{zpf}}(\hat{a}^\dagger - \hat{a}) \quad (2.44)$$

where $\Phi_{\text{zpf}} = \sqrt{\hbar/2\omega_r C}$ and $Q_{\text{zpf}} = \sqrt{\hbar\omega_r C/2}$. Therefore, the energy spectrum consists of evenly spaced levels as shown in Figure 2.10a.

Coplanar Waveguide Resonator

Superconducting (SC) coplanar waveguide (CPW) is an easy and highly controllable way to implement a superconducting resonator. Consider a coplanar waveguide of finite length l formed by a superconducting center conductor of width w and thickness t , separated on both sides by a distance s from a superconducting ground plane of the same thickness on top of a substrate with a dielectric constant ϵ_{eff} [Figure 2.11a]. For an infinitesimally short section of the CPW, as Figure 2.11b shows, the electromagnetic field is confined to a small volume between the center conductor and the ground of the CPW, such that each unit length is equivalently a LC circuit with a unit length inductance L_0 and capacitance C_0 . Using the conformal mapping technique, L_0 and C_0 are given by [70]:

$$\begin{aligned} L_0 &= L_g + L_k, \\ L_g &= \frac{\mu_0}{4} \frac{K(k'_0)}{K(k_0)}, \\ L_k &= \mu_0 \frac{\lambda^2}{wt} g(s, w, t), \end{aligned} \quad (2.45)$$

and

$$C_0 = 4\epsilon_0\epsilon_{\text{eff}} \frac{K(k_0)}{K(k'_0)}. \quad (2.46)$$

Here, K denotes the complete elliptic integral of the first kind with

$$k_0 = \frac{w}{w + 2s}, \quad (2.47)$$

$$k'_0 = \sqrt{1 - k_0^2}, \quad (2.48)$$

and $g(s, w, t)$ is a geometrical factor, that has the strongest dependence on w [70]. λ is the penetration depth of the superconductor used in the CPW. The unit length inductance (L_0) consists of a geometric component (L_g) arising from the geometry of the CPW and a kinetic

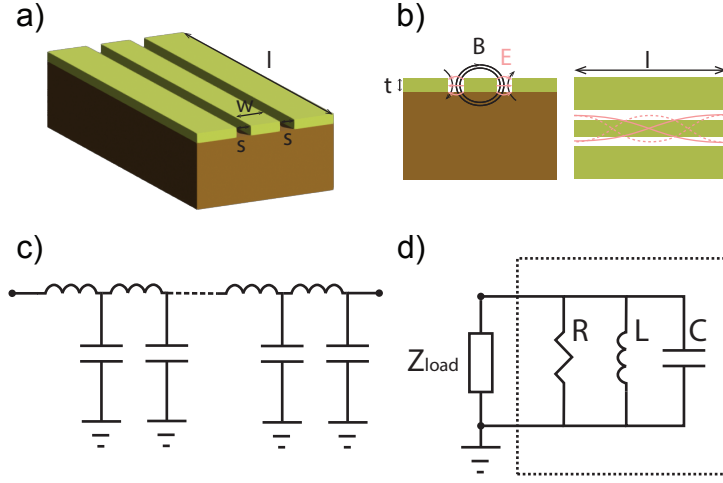


Figure 2.11: Superconducting coplanar waveguide (CPW) and equivalent circuit diagram. (a) A 3d view of a coplanar waveguide. Center conductor and lateral ground metallization are superconducting [light green]. The width of the center conductor is w and the gap between it and lateral superconducting ground is s . The dielectric constant of the substrate [brown] is ϵ_{eff} . (b) Left: cross-sectional cut of the CPW as well as schematic representations of the E and B field distributions. Right: top-down view of a CPW with short-circuited boundary condition (solid line) and open-circuited boundary condition (dashed) imposed, leading to quarter-wave and half-wave resonators. Figure adapted from [67]. (c) Equivalent circuit diagram of the CPW. (d) Equivalent circuit diagram of a CPW with the boundary conditions.

component (L_k) due to the inertia of the Cooper pairs inside the superconductors.

Given L_0 and C_0 , the circuit diagram of the entire CPW is shown in Figure 2.11c. A CPW resonator is formed with a boundary condition imposed on the CPW. Typical boundary conditions are shown in the top-down view of the CPW in Figure 2.11b. With the two ends of the CPW open, the CPW results in a half-wave resonator such that its fundamental frequency f_r is given by $1/(2l\sqrt{L_0C_0})$. On the other end, if one end of the CPW is grounded, a quarter-wave resonator with $f_r = 1/(4l\sqrt{L_0C_0})$ can be created. Given the length dependence of f_r for CPW resonators, setting the right length sets the resonator frequency. For my research purpose, the desired resonator frequency is in the microwave range, usually ranging from 4-8 GHz. The equivalent circuit diagram of a CPW resonator is a parallel RLC circuit as shown in Figure 2.11d. The inclusion of a nonzero resistance (R) reflects the presence of noises in actual physical system.

Resonator Microwave Losses and Two Level Systems

We have assumed that superconducting resonators have zero resistance. However, a physical resonator system, such as a CPW resonator, is not made solely of superconductors and it is not completely isolated from the outer environment. Ignoring the outer environment that can be characterized by an overall impedance Z_{load} , we have the superconductors in contact with dielectrics and exposed to air inside the resonator system. Therefore we have to consider the loss mechanisms involved in the resonator system. In an equivalent circuit diagram, the loss mechanisms are in general represented by an impedance. We can isolate the imaginary part, and end up with modified C and L . The real part left is an effective resistance. A realistic equivalent microwave resonator circuit is shown in Figure 2.11b. Internal quality factor $Q_i = \omega_r L / R$ is then a good measure of the magnitude of losses inside the resonators, which can be found from the absorption (emission) spectrum of the superconducting resonator, by applying an EM drive around the resonator frequency. However, it is important to note that in quality factor measurements, the resonator is unavoidably coupled to the environment, including the readout circuitry. Therefore, we have the total quality factor $Q_{\text{tot}} = \omega_r / \text{FWHM}$ related to Q_i given by the following equation

$$\frac{1}{Q_{\text{tot}}} = \frac{1}{Q_{\text{ext}}} + \frac{1}{Q_i}, \quad (2.49)$$

where Q_{ext} is dependent on the resonator circuit and Z_{load} . By simplifying the total circuits with a few effective component values, we can extract ω_r and Q_i by fitting the microwave absorption (emission) data around the resonance frequency. This procedure is a typical resonator measurement and as we shall see is also the key in qubit measurements.

Among different kinds of losses that affect Q_i , losses due to two level systems (TLSs) that largely reside in amorphous dielectric layers including metal-air and metal-substrate interfaces are generally considered to be the dominant mechanism. Although the actual physical entity of the TLSs is still unknown, a commonly used microscopic model attributes the emergence of TLSs to the electric dipole moment in polar molecules creating an energy splitting with a presence of electric field. The presence of TLSs modifies the dielectric constant of the materials in the resonator system $\epsilon_{\text{eff}} \rightarrow \tilde{\epsilon}_{\text{eff}}$, so that $\tilde{\epsilon}_{\text{eff}}$ is now a complex number. The TLSs can be categorized into two kinds depending on their frequencies: 1) on-resonant ones with transition

frequencies close to the drive frequency and ω_r ; 2) highly detuned ones from the drive frequency and ω_r . It is found that 1) contributes more to the imaginary part of $\tilde{\epsilon}_{\text{eff}}$. According to Eq 2.46, an imaginary ϵ_{eff} leads to an imaginary C_l , which is equivalent to a frequency-dependent resistance R that changes the Q_i . Since these TLSs are on-resonant with the drive, they can be saturated at high power. Therefore, their effect on the Q_i is power-dependent. Overall, losses due to TLSs are captured by the following equation:

$$Q_i^{-1} = \frac{F \tan \delta_0}{(1 + \langle n \rangle / n_c)^\alpha} + Q_{i,0}^{-1} \quad [71], \quad (2.50)$$

where F is the filling factor describing the ratio of E-field in the TLS host volume to the total volume. $\tan \delta_0 = 1/Q_{\text{TLS}}$ is the TLS loss tangent, n_c is the critical number of photons within the resonator to generate the E-field required to saturate one TLS. $Q_{i,0}^{-1}$ is the quality factors of non-TLS loss mechanisms, which generally dominate the losses at high microwave drives. On the other hand, 2) mainly contributes to the real part of $\tilde{\epsilon}_{\text{eff}}$, which according to Eq 2.45 and $f_r = 1/(4l\sqrt{L_0 C_0})$ leads to the shift of f_r or ω_r , given by the following equation:

$$\frac{f_r(T) - f_r(T_0)}{f_r(T_0)} = F \tan \delta_i \left[\text{Re} \left(\Psi \left(\frac{1}{2} + \frac{hf_r(T_0)}{2\pi i k_B T} \right) + \left(\frac{1}{2} + \frac{hf_r(T_0)}{2\pi i k_B T_0} \right) - \ln \frac{T}{T_0} \right) \right] \quad [72]. \quad (2.51)$$

2.3.2 Transmon Qubit

Transmon qubits are a derivative of superconducting charge qubit that have a reduced sensitivity to charge fluctuations. The transmon qubit circuit stems from the Cooper pair box circuit, which involves a realistic Josephson junction. A realistic JJ system not only includes a non-linear Josephson inductance given by the Josephson effect [section 2.2.1] but also a parallel capacitance C_J from the two superconducting island. Therefore, the Hamiltonian of a realistic JJ (or Cooper pair box) includes a charging term:

$$\hat{H}_{\text{CPB}} = H_C + H_J = 4E_C (\hat{n} - n_g)^2 + H_J(\hat{\varphi}) \quad (2.52)$$

where $E_C = e^2/2C_J$. $\hat{n}$ is the charge operator that measures the number of Cooper pairs across the islands. $\hat{\varphi}$ is the phase operator that measures the phase difference between the two islands.

How they are defined as operators is briefly covered in the previous section 2.1.4. n_g accounts for charge fluctuations and $H_J(\hat{\varphi})$ is a general term for the JJ energy to include different types of junction.

Anharmonicity of Transmon Qubits

It is evident from Eq 2.52 that the energy spectrum of $\hat{H}_{\text{CPB}}$ will be non-harmonic given $H_J(\hat{\varphi})$ deviates from a quadratic potential [demonstrated in Figure 2.10b]. Differences in the transition energies between two neighboring levels are used to quantify the anharmonicity (α) of the resulting spectrum. Because of the anharmonicity, the CPB is used as a superconducting charge qubit with information stored in the subspace spanned by the two lowest energy levels. However, due to the small C_J that is limited by the size of the superconducting islands, charge fluctuations lead to a high variation in the total energy of the system, making the system sensitive to charge noise. Adding a shunt capacitor (C_S) to increase the overall capacitance of the qubit system would alleviate this issue and it was found that the right E_J/E_C ratio allows for a charge-insensitive qubit spectrum while maintaining sensible anharmonicity [64]. Such qubits are called transmon qubits. Their Hamiltonian is still described by Eq 2.52 apart from a small change in the E_C

$$E_c = \frac{e^2}{2C_\Sigma} = \frac{e^2}{2(C_J + C_S)}. \quad (2.53)$$

We can analytically evaluate the resulting anharmonicity of a transmon qubit with a sinusoidal SIS junction such that

$$H_{\text{J,SIS}}(\hat{\varphi}) = -E_J \cos \hat{\varphi} \quad (2.54)$$

$$= +\frac{1}{2}E_J \hat{\varphi}^2 - \frac{1}{4!}E_J \hat{\varphi}^4 + \mathcal{O}(\hat{\varphi}^6) \quad (2.55)$$

where in the end, H_J is expanded into a power series. As can be seen from the power series, the leading quadratic term creates a harmonic spectrum but the higher order terms disrupt the harmonicity. We can then define ladder operators ($\hat{b}$, $\hat{b}^\dagger$) for the qubit

$$\hat{\varphi} = \left(\frac{2E_C}{E_J}\right)^{1/4} (\hat{b}^\dagger + \hat{b}), \quad \hat{n} = \frac{i}{2} \left(\frac{E_J}{2E_C}\right)^{1/4} (\hat{b}^\dagger - \hat{b}). \quad (2.56)$$

We then rewrite the qubit Hamiltonian in terms of $\hat{b}$, $\hat{b}^\dagger$ up to the quartic order such that

$$\begin{aligned}\hat{H}_q &= \sqrt{8E_c E_J} \hat{b}^\dagger \hat{b} - \frac{E_C}{12} (\hat{b}^\dagger + \hat{b})^4 \\ &\approx \hbar\omega_q \hat{b}^\dagger \hat{b} - \frac{E_c}{2} \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b}\end{aligned}\tag{2.57}$$

In the equation, any terms with an unequal number of creation and annihilation operators oscillate in the rotating frame at ω_q . In the transmon regime, we can safely ignore those rotating terms given that charge fluctuations are quickly averaged out. Eq 2.57 then becomes

$$\hat{H}_q = \hbar \left(\omega_q - E_c(\hat{b}^\dagger \hat{b} - 1)/2\hbar \right) \hat{b}^\dagger \hat{b}\tag{2.58}$$

where $\hbar\omega_q = \sqrt{8E_c E_J} - E_C \approx \sqrt{8E_C E_J}$. Therefore the anharmonicity of the transmon with a SIS junction is given by $\alpha = -E_C$.

We can also evaluate the anharmonicity of a transmon qubit with a SNS junction [73] by expanding the SNS Josephson potential introduced in section 2.2.2

$$H_{J,\text{SNS}}(\hat{\varphi}) = -\Delta \sum_i \sqrt{1 - T_i \sin \hat{\varphi}/2} \tag{2.59}$$

$$= \frac{1}{2} E_J \hat{\varphi}^2 - E_J \left(1 - \frac{3 \sum T_i^2}{4 \sum T_i} \right) \frac{\hat{\varphi}^4}{24} + \mathcal{O}(\hat{\varphi}^6) \tag{2.60}$$

where Δ is the superconducting gap in the leads, T_i is the transmission coefficient for the i -th Andreev bound state (ABS), and $E_J = \frac{\Delta}{4} \sum T_i$ [64, 74]. Again, we ignore terms with orders higher than the power 4. By treating the quartic term as a perturbation to the harmonic system, the anharmonicity can be found from the perturbation matrix elements. As a result, we have

$$\alpha \approx -E_C \left(1 - \frac{3 \sum T_i^2}{4 \sum T_i} \right). \tag{2.61}$$

Assuming T_i is uniform, in the limit of $T_i \rightarrow 0$, $\alpha \approx -E_C$ as expected, since diffusive ABSs conform to the SIS sinusoidal Josephson effect. For $T_i = 1$, $\alpha \approx -E_C/4$, reducing the qubit nonlinearity.

In general, given the complexity of Eq 2.60, it is hard to analytically find the transmon

energy spectrum and the anharmonicity. However we can use a numerical method to solve the qubit Hamiltonian for any arbitrary combinations of T_i . QuTiP [75] is a particularly useful Python package that conveniently solves superconducting circuits. In this thesis, all simulations of the qubit spectrum are done via QuTiP.

Flux-tunable Transmon

As shown in section 2.2.4, the Josephson energy can be tuned by external flux with SQUIDs. Incorporating the flux-tunable SQUIDs into the qubit circuits allows the qubit frequency to be tuned. By inserting the flux-tunable Josephson potential [Eq 2.40] into the qubit Hamiltonian [Eq 2.52], the simulated flux-tuning patterns of transmon qubits with SIS and SNS SQUIDs are shown in Figure 2.12 [Supplementary D.2.2]. As Figure 2.12 shows, the flux-tuning patterns of flux-tunable transmons resemble those of Josephson energy shown in Figure 2.9, such that:

1. The qubit flux-tuning pattern is Φ_0 -periodic,
2. The minimum f_q appears at the half-flux point,
3. The minimum f_q goes to zero when $d = 0$.

Both the SIS and SNS flux-tunable qubits exhibit such flux-tuning patterns. In Section 2.2.4, SNS SQUIDs can be distinguished from SIS SQUIDs by the skewedness in the flux dependence of I_c . Here, flux-tunable qubits do not exhibit a skewedness in their flux dependence regardless of their junction types, as opposed to the critical currents [Figure 2.9]. However, it is still obvious that the two kinds of qubits have distinctive flux-tuning patterns. SNS flux-tunable qubits have flatter caps and more pointy dips given the same asymmetry factor d as shown in Figure 2.12.

For SIS flux-tunable qubits, as long as the transmon limit stays true, we can analytically calculate the qubit frequency by inserting Eq. 2.42 into $\hbar\omega_q \approx \sqrt{8E_C E_J}$. On the other hand, since there are no analytical expressions for SNS SQUIDs, the data in Figure 2.9c-d are obtained from numerical simulations. During the simulations, the energies of the SQUIDs at each flux point as a function of phase are calculated, Fourier transformed to retrieve the functional

form of their energy-phase relations. Lastly, the exact functional form is included in the qubit Hamiltonians to allow for the calculation of qubit frequencies at every applied flux. The parameters used in the numerical simulation are given in Supplementary D.2.2.

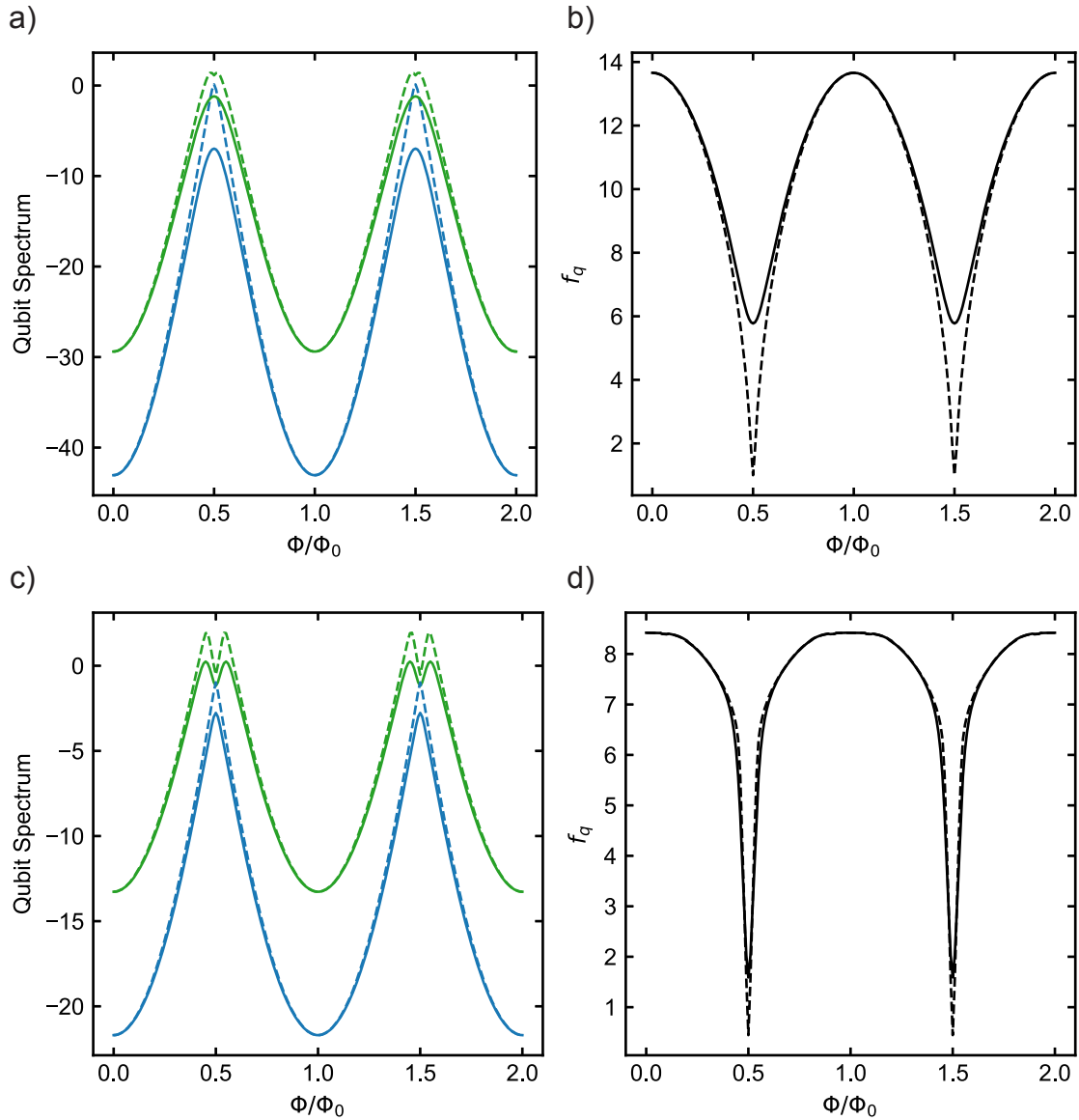


Figure 2.12: Flux-tunable transmon with SIS and SNS Junctions. (a) Simulated energy spectrum of a flux-tunable transmon qubit with an SIS SQUID as a function of applied flux. (b) Extracted qubit transition frequency $f_q = f_{01}$ from the spectrum in (a) as a function of applied flux. (c) and (d) Simulated energy spectrum and the qubit frequency of a flux-tunable transmon qubit with an SNS SQUID. In all the figures, $d = 0.2$ for solid lines and $d \rightarrow 0$ for dashed lines.

Majorana-transmon Qubit

We have introduced the CPB and transmon qubits based on their Hamiltonians. Let us now develop an intuitive picture of these qubits, starting from a CPB qubit that is compatible with the emergence of MZMs. In the CPB Hamiltonian, H_C renders identical parabolic curves that are shifted in the charge space by their charge offsets, which is depicted in Figure 2.13a. Considering three neighboring curves with charge offsets $(n_g, n_g + 1, n_g + 2)$, the bands of n_g and $n_g + 2$ have a degeneracy point at $n = n_g + 1$, and the bands of n_g and $n_g + 1$ have a degeneracy point at $n = n_g + 0.5$. Note that n is now the number of electrons instead of Cooper pairs on the superconducting islands and the charge offset is also expressed in terms of single electron (e). H_J that arises from the coherent Cooper pair transition, connects the bands of n_g and $n_g + 2$ and opens up a gap ($\sim E_J$) at the degeneracy point between the two bands, leading to the gapped qubit spectrum. With a large E_J/E_C ratio, the charge dispersion determined by E_C is small compared to the gap opening, resulting in flat bands in the qubit spectrum.

Notice that even H_J hybridizes the well defined charge states of H_C , the parity state of the qubit is preserved. However, with the presence of MZMs in the junction and a coupling between the MZMs, a coherent single-electron transport connects the bands of n_g and $n_g + 1$ with a secondary gap opening ($\sim E_M$) at their degeneracy point, and mixes the parity states of the qubit. The parity mixing effect leads to energy splittings in the qubit spectrum as shown in Figure 2.13b, that are simulated using the Majorana-transmon qubit Hamiltonian:

$$\hat{H}_q = H_C + H_J + H_M, \quad (2.62)$$

where $H_M = iE_M\gamma_1\gamma_2\cos(\hat{\varphi}/2)$ is defined previously in Section 2.2.3, and $\gamma_{1,2}$ are the unpaired MZMs on two sides of the TI junction normal region.

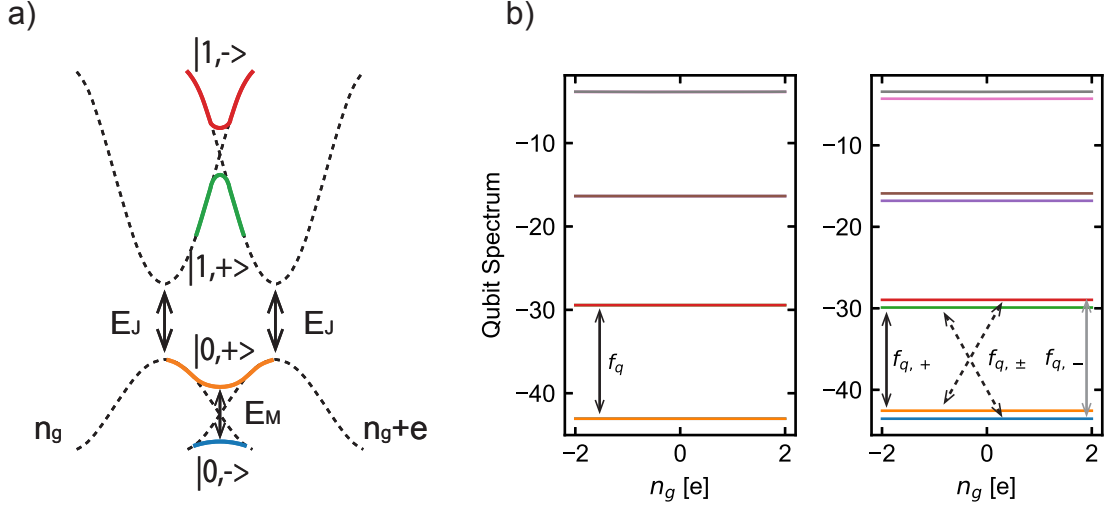


Figure 2.13: Parity mixing of transmon qubit spectrum with the presence of Majorana coupling E_M . (a) When the unpaired Majorana Bound States (MBSs) across the TI junction overlap and create a Majorana coupling E_M , coherent single-electron tunneling mixes the two parity states of the total unpaired charge n_g of the transmon qubit. This effect is equivalent to hybridizing the spectra of $H_T[n_g]$ and $H_T[n_g + 1]$, resulting in anti-crossings at the degeneracy point $n_g = 1/2$. Note that n_g is expressed in terms of single electron. To better illustrate the effect of mixing parity, the qubit spectra ($H_T[n_g]$ and $H_T[n_g + 1]$) are plotted with apparent charge dispersion. (b) Simulated hybridization of qubit spectrum in the transmon regime, where charge dispersion is suppressed. In the simulation, the junction in the qubit model is considered to be SIS-like junction with $E_J = 50$, and $E_C = E_M = 0.5$.

2.3.3 Circuit-QED and Dispersive Qubit Readout

We have discussed the inherent nature of different transmon qubits determined by their Hamiltonians. However, in reality, we need to conduct measurements to read out the qubit properties. We can directly couple an electromagnetic signal to the qubit such that the transition frequency of the qubit can be measured from the absorption spectrum. However, a direct coupling would not allow for a non-demolition readout of the qubit state. A simple non-demolition readout scheme of the qubit can be implemented using a dispersive coupling between a qubit and a superconductor resonator, which resembles an iconic atom-cavity system [69, 76]. The superconducting resonator coupled to the qubit is therefore often called a cavity and the interaction between the qubit and the cavity is described by Circuit-QED.

The desired coupling between a qubit and a resonator is usually capacitively induced by adding an electrostatic gate between the qubit and the resonator. In this situation, the resonator acts as a voltage source with a dynamical gate voltage such that n_g in the Eq 2.52 can be

replaced by $\hat{n}_r \sim \hat{Q}_r$, representing the effective offset charge term of the transmon due to the quantum electric field operator of the resonator. The Hamiltonian of the qubit-cavity system is then given by

$$\hat{H}_{\text{tot}} = 4E_C(\hat{n} + \hat{n}_r)^2 + \hat{H}_J + \hbar_r \hat{a}^\dagger \hat{a} \quad (2.63)$$

We can expand the first term into:

$$\hat{H}_C = 4E_C \hat{n}^2 + 8E_C \hat{n} \hat{n}_r + 4E_C \hat{n}_r^2, \quad (2.64)$$

in which the last term can be bundled with the resonator's Hamiltonian and renormalize the cavity frequency. Replacing the $\hat{n}$ and $\hat{n}_r$ with the previously defined raising and lowering operators for resonators ($\hat{a}^\dagger, \hat{a}$) and qubits ($\hat{b}^\dagger, \hat{b}$), the total Hamiltonian of a qubit-cavity system can be rewritten as:

$$\begin{aligned} \hat{H}_{\text{tot}} &= \hat{H}_q + \hat{H}_r + \hat{H}_{\text{int}}, \\ \hat{H}_{\text{int}} &= -\hbar g (\hat{b}^\dagger - \hat{b})(\hat{a}^\dagger - \hat{a}), \end{aligned} \quad (2.65)$$

where g as the coupling strength between the qubit and its cavity is given by

$$g = \omega_r \frac{C_g}{C_\Sigma} \left(\frac{E_J}{2E_C} \right)^{1/4} \sqrt{\frac{\pi Z_r}{R_K}}. \quad (2.66)$$

Here Z_r is the characteristic impedance of the cavity resonator and $R_K = h/e^2 \sim 25.8\text{k}\Omega$. C_g is the gate capacitance and C_Σ is the total capacitance of the qubit.

Restricting the qubit Hamiltonian to its lowest two-level subspace such that $\hat{b}^\dagger \rightarrow \sigma^+ = |e\rangle\langle g|$ and $\hat{b} \rightarrow \sigma^- = |g\rangle\langle e|$, the effective Hamiltonian of Eq 2.65 is given by:

$$\hat{H}_{\text{tot}} = \frac{\hbar\omega_q}{2} \sigma_z + \hbar\omega_r (\hat{a}^\dagger \hat{a} + \frac{1}{2}) + \hbar g (\hat{a}^\dagger \sigma^- - \hat{a} \sigma^+), \quad (2.67)$$

where σ_z is by convention ($|e\rangle\langle e| - |g\rangle\langle g|$). The equation above has the exact same form as the Jaynes-Cumming (JCM) Hamiltonian in atom-cavity interaction, which can be exactly diagonalized, which yields the eigenenergies

$$E_{\pm, n} = (n+1)\hbar\omega_r \pm \frac{\hbar}{2} \sqrt{4g^2(n+1) + \Delta^2}, \quad (2.68)$$

and the ground state energy

$$E_{\downarrow,0} = -\frac{\hbar\Delta}{2}, \quad (2.69)$$

with $\Delta \equiv \omega_q - \omega_r$ as the qubit-cavity detuning. From Eq 2.68, we find that the dressed cavity frequency gains a power-dependent pull due to the qubit-cavity coupling.

In the dispersive regime, where $g \ll \omega_q, \omega_r, \Delta$, expansion of Eq 2.68 gives

$$\tilde{\omega}_r = E_{\pm,n+1} - E_{\pm,n} = \hbar\omega_r + \frac{g^2}{\Delta}, \quad (2.70)$$

and the difference in the dressed cavity frequency dependent on the qubit state is given by

$$\Delta\tilde{\omega}_r = E_{+,n} - E_{-,n} = 2\frac{g^2}{\Delta}. \quad (2.71)$$

It is important to note that the findings above are valid under two conditions

- The qubit-cavity coupling is in the dispersive regime,
- The cavity photon number is small compared to Δ^2 such that $\frac{g^2(n+1)}{\Delta^2} \ll 1$.

A unitary transformation on the JCM Hamiltonian with

$$U = \exp\left[\frac{g}{\Delta}(\hat{a}\sigma^+ - \hat{a}^\dagger\sigma^-)\right] \quad (2.72)$$

invoking the rotating wave approximation and expanding to second order in g , the following Hamiltonian is obtained:

$$UHU^\dagger \approx \hbar\left[\omega_r + \frac{g^2}{\Delta}\sigma_z\right]\hat{a}^\dagger\hat{a} + \frac{\hbar}{2}\left[\omega_q + \frac{g^2}{\Delta}\right]\sigma_z. \quad (2.73)$$

The first term of Eq 2.73 again shows the qubit-state dependent push on the cavity frequency as in Eq 2.71. We can utilize this state-dependent push to implement a qubit readout scheme by determining the qubit state from the measured cavity frequency. This readout scheme is non-demolition because the large detuning (Δ) in the dispersive regime suppresses direct photon exchange between the qubit and the cavity, such that the qubit state is hardly perturbed by driving the cavity. However, as we shall see, the qubit-cavity coupling exerts a push on the

qubit frequency.

Rearranging Eq 2.73, we have

$$\hbar\omega_r\hat{a}^\dagger\hat{a} + \left[\frac{\hbar}{2}\omega_q + \frac{g^2}{\Delta}(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \right] \sigma_z. \quad (2.74)$$

As is clear from this expression, the qubit transition is ac Stark shifted by $g^2/\Delta(n + \frac{1}{2})$.

In the treatment above, we have treated the qubit in the qubit-cavity system as a two-level system reduced from a CPB qubit spectrum and completely ignored the effect of higher qubit energy levels. To account for the transmon qubit regime, the qubit-state dependent push on the cavity frequency is modified to [64, 67]:

$$-g^2 \frac{E_C/\hbar}{\Delta(\Delta - E_C/\hbar)}. \quad (2.75)$$

If $\hbar\Delta \gg E_C$, this term reduces to the form (g^2/Δ) we see in Eq 2.73.

2.3.4 Qubit Lifetime

In applications of qubits, since the information is encoded in the qubit state, it is important for the qubits to have a long coherence time, which is at risk from the couplings between the qubits and the environment. Although it is tempting to isolate the qubits in order to protect its coherence, as shown previously, some couplings are inevitable for the qubit operations. Therefore, it is important to understand the mechanisms behind qubit lifetime and the sources of noise to optimize the setups for qubit applications.

The qubit lifetime (T) as a time measure of qubit coherence is defined to be the inverse of the qubit decoherence rate (Γ) such that $T = 1/\Gamma$. The decoherence of qubit states can be categorized into three kinds. The first is the longitudinal relaxation (Γ_1) that describes the depolarization along the longitudinal axis (along z-axis in Figure 1.1), which can be understood as obscuring the polar angle θ due to the excitation of $|0\rangle$ (denoted as $\Gamma_{1,\uparrow}$) or relaxation of $|1\rangle$ ($\Gamma_{1,\downarrow}$). Such depolarization is caused by resonant transversal noises on the equator (x-y plane), resulting in off-diagonal interaction terms that connect and drive transitions between

states $|0\rangle$ and $|1\rangle$. The total longitudinal relaxation rate is given by $\Gamma_1 = \Gamma_{1,\uparrow} + \Gamma_{1,\downarrow}$. A characteristic time scale $T_1 = 1/\Gamma_1$ is defined over which qubit population will relax to its steady state value. Although both excitation and relaxation events can happen, in a cold environment — the typical operation environment for qubit, the excitation rate is suppressed given $\Gamma_{1\uparrow} = \exp(-\hbar\omega_q/k_B T)\Gamma_{1\downarrow}$, where T is the temperature and k_B is the Boltzmann constant [8]. As the Boltzmann term suggests that a major noise for qubit excitation is thermal noise. With the thermal noise suppressed, dissipations of qubit energy to the environment dominate the lifetime of the qubit T_1 .

Due to the qubit-cavity coupling, the cavity is an inevitable channel for qubit relaxations. However, by a large detuning between the qubit and the cavity, the spontaneous emission rate can be reduced to minimize these relaxations. Another major source for qubit relaxations is the dielectric losses due to TLSs in the substrate [77], which is also the dominant loss mechanism of the superconducting resonator. Assuming an even distribution of TLSs in the substrate, we can limit the surface area of the superconducting qubits to mitigate the substrate losses. We can also choose to use materials with a lower density of TLSs to reduce the substrate losses. It is found that trapped quasiparticles in the superconducting film can act as TLSs, that provide additional channels for the qubit relaxations [78]. The density of trapped quasiparticles depends on the spatial variation of the superconducting gap. Therefore, having a clean superconducting interface is important.

The second kind of qubit decoherence is called pure dephasing (Γ_ϕ), which describes depolarization in the equator (x-y plane) of the Bloch sphere. It arises from longitudinal noise that causes the qubit frequency to fluctuate, leading to the precession of qubit state in the x-y plane. The most common noise for dephasing are the charge and flux noise. Thermal quasiparticles can lead to pure dephasing as well. Since the break of Cooper pairs leads to the change in the superconducting gap, it causes a change in the qubit frequency via the Josephson term. Since the pure dephasing is not a resonant phenomenon, longitudinal noise at any frequency can modify the qubit frequency and cause dephasing. Luckily, pure dephasing is in principle reversible as it only adds a phase to the qubit state. Therefore, pure dephasing due to semi-

static noise or known single-shot noise can be cancelled by unitary transformation.

The transverse relaxation (Γ_2) describes the general qubit decoherence, which is the loss of coherence of a superposition state, and is generally related to Γ_1 and Γ_ϕ through the relationship:

$$\Gamma_2 = \Gamma_1/2 + \Gamma_\phi \tag{2.76}$$

The characteristic time scale T_2 is defined as the $1/\Gamma_2$.

Chapter 3

Experimental Setup and Techniques

The fundamental principles of a superconducting qubit, as introduced in the previous chapter, are (1) superconductivity and (2) selection of the lowest two levels, $|0\rangle$ and $|1\rangle$, from an anharmonic energy spectrum created by incorporating JJs — a superconducting nonlinear electrical component. Both principles impose a requirement on the operating temperature of superconducting qubits. This temperature has to stay well below the critical temperature of the superconducting materials used in qubit devices to prevent the circuit from exhibiting losses. It ranges from 1 K to 10 K for commonly-used materials such as Al and Nb. On top of this, for qubits to reside in the sub-two-level system, thermal noise must be considerably lower than the transition energy from $|0\rangle$ to $|1\rangle$, which sets another constraint on the temperature. Given that all the qubit devices in this thesis have f_{01} in the GHz range, the temperature requirement is further reduced to below 100 mK. To meet the requirement, we use a cryogenic dilution fridge that can go down to sub-10 mK. During measurements, qubit devices are mounted inside the fridge with enough thermalisation to ensure effective cooling.

Measurements of superconducting qubits utilize the qubit state dependent cavity shift arising from the qubit-cavity interaction in the dispersive regime as described in section 2.3.3. With qubit-cavity coupling present in the dispersive regime, the cavity resonance frequency is shifted depending on the qubit state. We can then infer the qubit state by measuring the power response of the cavity [76]. Spectroscopic measurements allow us to identify transition frequency, while time-domain measurements allow for direct qubit manipulation and characterisations of qubit lifetimes. Regardless of the types of measurements, reading the qubit state via the cavity response is essential. To obtain the cavity power response, we need to send signals

through the cavity (signal in) and analyze the signals back (signal out) using various microwave instruments. These instruments operate at room temperature, while the qubit devices operate at ultra-low temperatures inside the dilution fridge. Therefore, we add RF and DC lines in the dilution fridge to join the room temperature instruments with the devices.

In the first section, I shall introduce the basics of a dilution fridge and the typical wiring scheme inside the fridge for qubit measurements. In the second and third sections, I will go through the measurement setups for spectroscopic and time-domain measurements separately. Data from measurements on a flux-tunable test qubit device are used as examples.

3.1 Cryogenics and Dilution Fridge

It was found in the early 1950s that a spontaneous phase separation in mixtures of ^3He and ^4He is able to bring the temperature down to the sub-100 mK temperature required by superconducting qubit measurements [79]. The phase separation in the helium mixtures emerges when they are cooled below a temperature generally at 1K but subject to change depending on the pressure. From the temperature downward to absolute zero, the mixtures of ^3He and ^4He are separated into the concentrated phase (practically 100% ^3He) and the dilute phase (about 6.6% ^3He and 93.4% ^4He). The concentrate phase floats above the denser dilute phase and a phase boundary forms. Other concentration levels of ^3He in the mixtures are forbidden. Therefore, pure ^3He liquids undergo a spontaneous dilution process when being injected into the lower region of the mixtures in the dilute phase. This dilution process is endothermic and offers a continuous cooling effect as long as the phase separation is present. A dilution fridge utilises this cooling effect and includes a circulation unit which injects ^3He into the dilute phase and then collects the extra ^3He after dilution process which can be reused for the next cycle.

The inside of a dilution fridge (Oxford Instruments Proteox) is shown in Figure 3.1a. The fridge has six plates fixed at different temperatures and thermally separated from each other when in operation. We installed RF coaxial cables as well as DC looms in between the plates as an integral part of measurement circuits. With them installed, input and output signals used in measurements can go through the whole fridge from the ports exposed outside at the top of the fridge (Figure 3.1b) to the ones on the bottom-most plate. For qubit measurements, we mainly

use the RF lines in the measurement circuits. The DC looms are used to power active components in the circuits only. However, in DC transport measurements, the DC looms become the main carriers of signals. We normally use copper for DC looms because of the material's low resistivity. However, when we need to run large currents, we would use looms made of superconducting materials such as Nb to avoid thermal dissipation that could potentially heat up the fridge. As for the RF lines, in the form of metallic coaxial cables they directly join the plates together. Therefore we have to consider materials' thermal conductivity besides their electrical conductivity. Common materials for RF lines are NbTi (Niobium Titanium) and stainless steel. Both materials have lower thermal conductivity than Copper to ensure thermal separation between plates joined by the RF cables.

As shown in Figure 3.1c, we use compatible brackets to mount sample boxes directly onto the bottom-most plate (mixing chamber plate). This plate is in physical contact with a bath of ^3He and ^4He mixtures where dilution process happens. Therefore it can reach sub-10 mK at coldest. In the figure, a copper L-shaped bracket is used to mount a sample box made of brass. We chose these materials to ensure a good thermal contact with the plate. In terms of signal connectivity, two RF cables on one end extend from the fridge wiring and go into the sample box on the other end through two mini-smp ports on a PCB (Printed Circuit Board) that is screwed inside the sample box. Our device to be measured is glued on the PCB using thermal paste for thermal connection and wired into the measurement circuit via aluminum bond wires (Figure 3.1d-e).

Figure 3.1f shows the overall circuit diagram inside the fridge. The top circuit in black demonstrates the fridge wiring. The section on the left is the input line where source signals generated outside of the fridge flow into the device. This section is mainly made of stainless-steel coaxial cables. Because we are relatively free to choose the strength of injected signals, signal losses due to using non-superconducting materials are not an issue. Attenuators are added at each plate to thermalise the inner conductor of the coaxial lines and reduce thermal noise from higher temperature stages. Besides the attenuators, a 10GHz low-pass filter (K&L 10GHz LP filter) is present in the input line. The common frequency range for qubit measurements is 4-

8GHz. The low-pass filter gets rid of unwanted signals of higher frequencies. After attenuation and filtering, input signals flow into the device via a long feedline (drawn as a black inductor in Figure 3.1f) and eventually enter our qubit-cavity system. Figure 3.1f only shows one such system in colors. After signal absorption by the qubit system, the transmitted signal from the device passes through the right section in black, which is the output line. Transmitted signals carry the information about our device and we do not have direct control on their intensity. Therefore we want to minimize signal losses as much as possible. As a result, we choose superconductors NbTi as the main material for the whole section. We again add the 10GHz low-pass filter for high frequency filtering. Moreover, we add in a cryogenic isolator (ISC4_12A) with an operating frequency range of 4-8GHz. An isolator is a passive unidirectional component. In our case, we use it to strongly suppress noise that are flowing down the output line to the device from sources above the mixing chamber plate and those out of the fridge. At 4K plate, we also add in a low-noise cryogenic amplifier (LNC4.8C), which can amplify signals in 4-8GHz. The amplifier is an active component in the sense that it needs to be powered. We wire one DC loom to run DC voltage and power the amplifier.

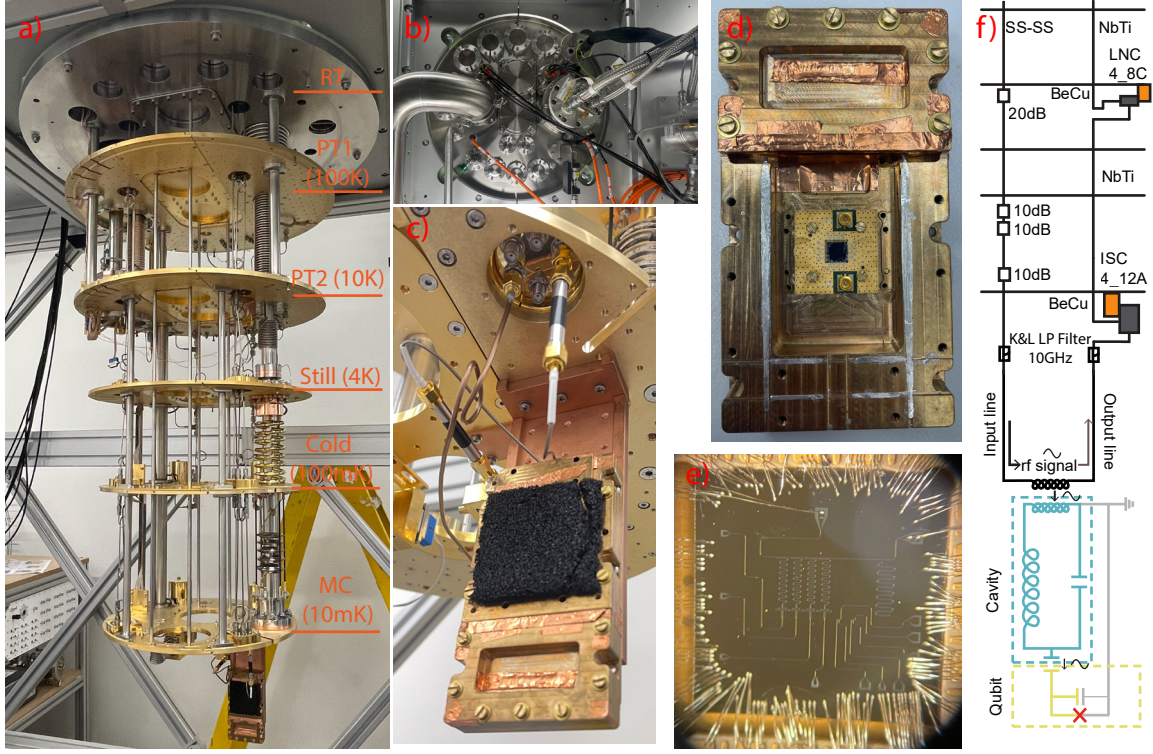


Figure 3.1: Inside of dilution fridge with a sample mounted. (a) A dilution fridge of six thermally isolated plates (RT, PT1, PT2, Still Plate, Cold Plate and Mixing Chamber Plate) with characteristic temperatures at room temperature, 100K, 10K, 4K, 100mK and 10mK separately. Several RF lines made of NbTi and stainless steel as well as three DC looms run through each plate, all of which can be connected to measuring equipment outside the dilution fridge via the connectors on top of the RT plate, as shown in (b). (c) A brass sample box is mounted on the mixing chamber plate using an L-shaped copper bracket. (d) Inside the sample box, a device is glued with GE Varnish to a PCB. (e) Multiple bond wires connect the sample ground to the common ground of the PCB and therefore the fridge. They also connect the feedline of the device to the two mini-smp ports on the PCB, which allows the device to be wired into the fridge circuit by plugging the RF lines in the fridge into the ports on the PCB. With two specific RF lines plugged in, one for RF signals in and another one for signals out of the device, the effective circuit diagram inside the fridge plus the device is given in (f).

3.2 Spectroscopic Measurements

The dilution fridges and the wiring schemes as introduced in the section above provide a suitable operation environment and a signal path through the device. However, to complete a full measurement setup, we need to include instruments at room temperature that can send input signals and analyze the corresponding response for different types of qubit measurements. We generally categorize qubit measurements into spectroscopic and time-domain measurements. In the former, we vary input signal frequencies and analyze the device response as a function of signal frequencies. Since our devices have discrete energy spectra, they will absorb (or reflect

depending on the type of devices) signals only at certain frequencies that are on resonance. Therefore, absorption spectroscopy allows us to identify the characteristic frequencies of our devices. In practice, we can add extra experimental variables on top of the frequency variation and unlock more information about our devices by studying the dependence of devices' characteristic frequencies on the extra variables. Some common choices for these extra variables are the power of input signals and the environmental temperature. For tunable devices, we can also vary their bias sources. For example, for flux tunable qubits, we can sweep a bias current source that tunes the applied flux, while doing spectroscopic measurements to demonstrate flux tunability.

In spectroscopic measurements, for the sake of simplicity, we normally just use RF signals in continuous wave (CW) mode, which can be easily generated by RF signal sources. Beside signal sources, another frequently-used instruments in spectroscopic qubit measurements are VNAs (Vector Network Analyzers). VNAs have at least two ports, one for input and another one for output. They can send out CW signals at different frequencies with tunable power via the output port. Received signals through the input port will be locked at the same frequencies and analyzed with respect to the generated signals. Throughout the process, VNAs can extract signal reflections and transmissions at specific frequencies, which is perfect for reading out cavity response to RF signals in the frequency domain. However, due to the usage of CW signals, VNAs rarely take single-shot measurement and do not provide accurate timestamps for each measurement. Due to the lack of time information, we cannot use VNAs to record how the cavity response vary with time. In Figure 3.2, a typical setup for spectroscopic measurements is shown in solid lines.

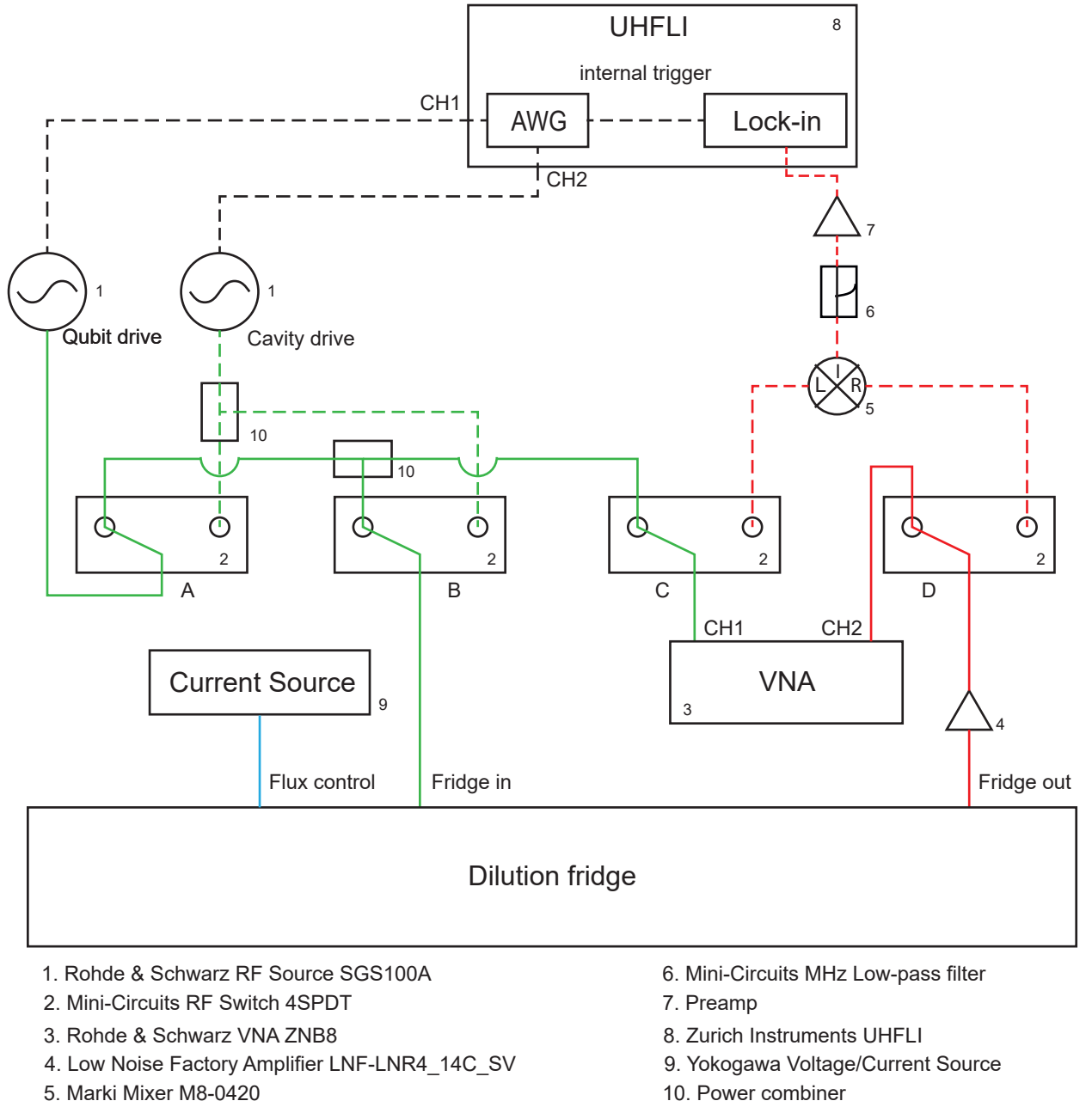


Figure 3.2: Instrument setup outside of dilution fridge. Instrument such as RF source, VNA, AWG, Lock-in amplifier are connected to four interconnected RF switches. Some of them contribute to the signal going into the fridge and the feedline of measured device (Green). Some are used to measure the signal coming out of the device (Red). With certain device, additional instrument such as a current source is used to generate a bias that tune the qubits (Blue). When all the RF switches are switched to the left side, this figure shows the instrument setup for continuous wave spectroscopic measurements (Solid line). When the switches are switched to the right side, this figure shows the setup for pulsed time domain measurements.

3.2.1 Single-tone Spectroscopic Measurements

Given the design of my qubit devices, where every single qubit is coupled to a readout resonator, each device has two characteristic frequencies corresponding to the energy spectra of

the cavity and the qubit separately. Before harnessing the qubit, we first only use one CW (continuous wave) input signal sweeping in the suspected cavity frequency range to identify the exact drive frequency that can excite our readout resonator. These measurements are called single-tone spectroscopic measurements and the RF tone is referred to as the cavity drive. We can easily use VNAs to carry out such measurements. VNAs themselves already provide the CW signal and can automatically render absorption spectra as shown in Figure 3.3.

The example device being measured in Figure 3.3 is a flux-tunable qubit made of AlO_x JJs coupled to a half-wave resonator borrowed from Prof. Paul Warburton at UCL and fabricated by Lincoln Laboratory MIT. We measured this device to test the measurement setup in Section 4.3. Throughout the rest of this chapter, the characterization data of this example device are used to demonstrate typical results of qubit measurements. On the example device, the resonator being half-wave leads to a maximum power transfer when the cavity drive is on resonance. This appears as a sharp peak in the absorption spectrum at fixed drive power, just like Figure 3.3b. However, for other types of resonators, such as quarter-wave resonators, we can instead have dips at the resonance. Using appropriate fitting routines allows us to find the resonance frequency from the peaks/dips.

We can repeat the single-tone spectroscopic measurements while varying the cavity drive power. The power-dependent data obtained can be presented as color maps as shown in Figure 3.3a. In the figure, we can clearly see that the sharp resonance peaks of the half-wave resonator in our example device at different powers link up and form a curved bright trace, marking the resonator frequency at different drive power. The fact that the trace is curved instead of being a straight line is a possible indicator of an active qubit existing. Due to qubit-cavity coupling, at low drive power which has low photon numbers, the resonator would be dressed by the qubit at its ground state with a resonance frequency $\tilde{f}_r$ shifted from its bare resonance f_r . As we increase the drive power, the resonator would become strongly-driven and restore its bare resonance f_r [Section 2.3.3]. As a result, we would see a power-dependent frequency shift if the qubit-cavity coupling is present. Given a dispersive coupling, the resonator frequency shift $\chi = f_r - \tilde{f}_r$ can be expressed as g^2/Δ , where g is the coupling strength and $\Delta = f_q - f_r$

is the detuning between the qubit and the resonator frequencies. Using the equation, we can estimate the qubit frequency f_q from the power-dependent single-tone spectrum if we know the coupling strength g from the device design, while f_r and χ can normally be identified from the single-tone spectrum. However, in the example (Figure 3.3a), we only see a shift without having f_r recovered at high power, which is normally demonstrated as a straight tail at high power in the spectrum. Possible explanations for the absence of the high power tail are 1) The power where f_r starts to recover is beyond the power range we used in the measurement and 2) The cavity is driven way too strongly such that the quality factor of its resonance becomes too low to be picked up from the background. Regardless of the explanations, Figure 3.3a as an example really challenges the reliability of power-dependent single-tone spectroscopy in terms of extracting qubit information. An even bigger flaw of it is that not all dispersive shifts of the cavity resonances in the power-dependent spectrum are caused by qubits. Unwanted two-level systems in the substrate that resemble qubits, for example, can lead to similar dispersive shifts.

Since the example device being used is flux-tunable, we also measured the cavity frequency as a function of applied flux, which could provide stronger evidence of active qubits. For this specific device, the applied flux is generated by currents passing through a coil mounted close to the sample. We used a Yokogawa current source (GS210) to precisely control the currents. In Figure 3.2, this flux control module is presented in color blue. When measuring the example device, we repetitively performed single-tone spectroscopy fixed at -115dBm with the bias current set differently in the range from -5 mA to 5 mA. The results are plotted in Figure 3.3c. From the figure, we can see a bright curve that periodically changes with the applied flux. Since we were measuring at low power, the bright curve marks the dressed resonator frequency $\tilde{f}_r$, which depends on the qubit frequency if there is a qubit present. Given the flux tunability of the qubit (Section 2.3.2), we expect to see a periodic flux tuning pattern of $\tilde{f}_r$. Within the flux tuning pattern in Figure 3.3c, we can also see anti-crossing features at $\Phi \approx \pm 0.4$. These features appear when the tuning range of the qubit frequency f_q crosses the bare resonator frequency f_r . On one side of the anti-crossing features, f_q is just above f_r and results in a large negative χ that pushes $\tilde{f}_r$ down strongly from f_r . On the other side, f_q has just come below f_r and instead pushes $\tilde{f}_r$ up highly. Since flux tuning is specific to the qubit only, and

then indirectly to the cavity, these flux tuning patterns seen in Figure 3.3c are indeed stronger evidence of the presence of qubits compared to the dispersive frequency shift in Figure 3.3a. However, to better extract qubit information, we still need next-step measurements.

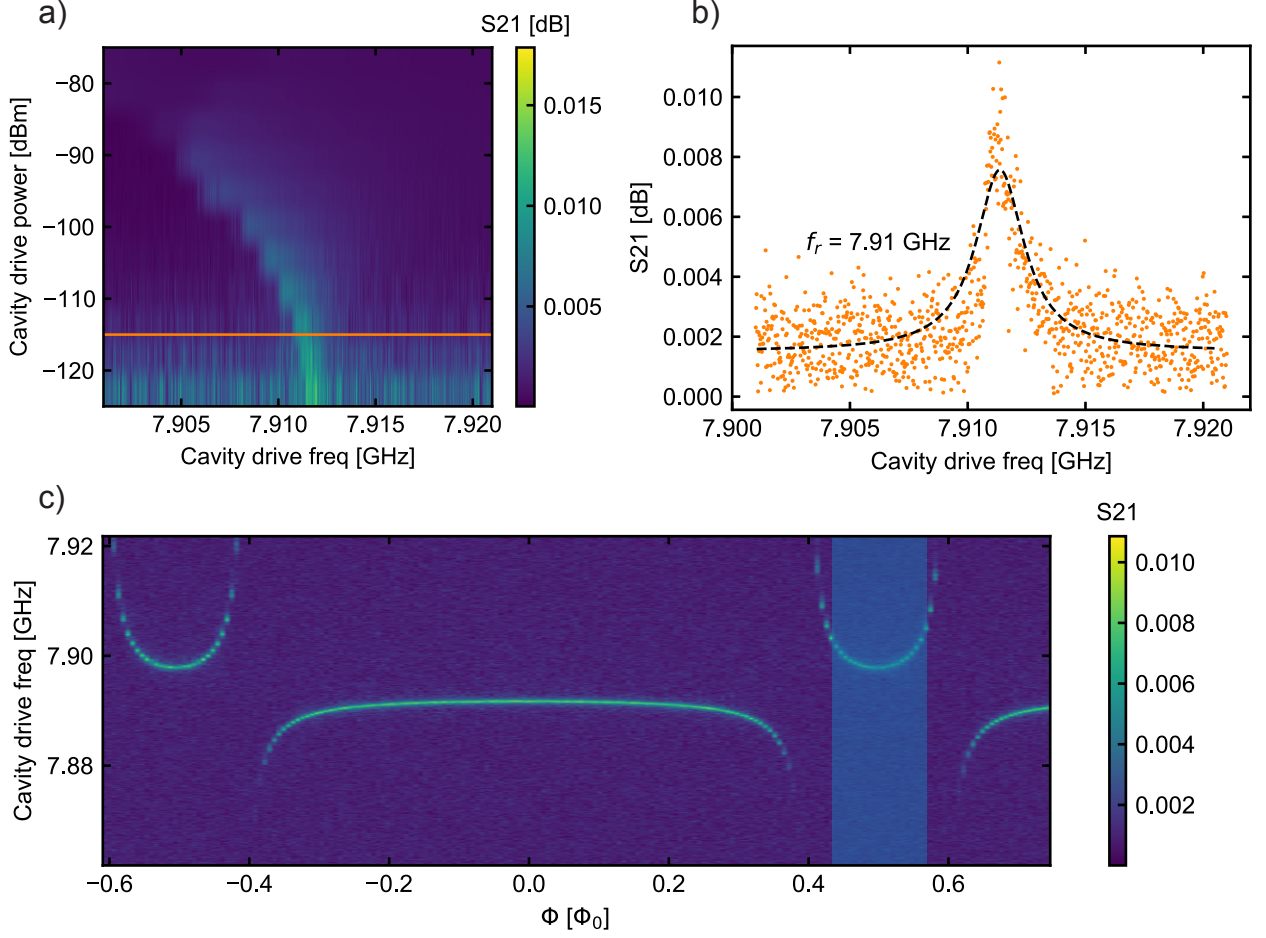


Figure 3.3: Single-tone spectroscopy measured on a flux-tunable AlO_2 junction qubit device. (a) Cavity response to a RF signal at different powers. The orange line marks where the signal power is at -115dBm, and the line cut is plotted in (b). In (a) and (b), the bias current was fixed at 2.58mA (corresponding to $\Phi \approx 0.42$). (c) Cavity response to the RF signal at -115dBm while sweeping the bias current in the range from -5mA to 5mA. The x-axis in the figure displays applied flux calibrated from applied currents. The highlighted area from $\Phi \approx 0.43$ to $\Phi \approx 0.57$ shows the flux range where two-tone spectroscopy in the next section was carried out.

3.2.2 Two-tone Spectroscopic Measurements

As shown previously, single-tone spectroscopic measurements that only have a cavity drive are not technically qubit measurements but characterizations of readout resonators. We shall see that they are indeed just prerequisites to actual qubit measurements. Qubit measurements consist of two parts. One is to drive qubit transitions and another is to read out the current qubit states. Driving qubit transitions is pretty straightforward and only involves an RF signal

around the qubit frequency, which is referred to as the qubit drive tone. However, reading out the qubit states is a bit complicated. We adopt an indirect readout scheme that allows us to extract the qubit states from the reflected/transmitted photons of cavities, hence the name readout cavities. The principle behind the scheme is once again the qubit-cavity coupling, with which the cavity resonances are sensitive to the qubit states. Qubits in the ground/excited state will push the cavity resonances in opposite directions, leading to dressed frequencies $\tilde{f}_r = f_r \mp g^2/\Delta$. If we irradiate the cavity at one of the pushed frequencies $\tilde{f}_r = f_r \mp g^2/\Delta$ by applying a fixed-frequency cavity drive, the transmission of the cavity will be close to unity for one state of the qubit and close to zero for the other. Therefore the qubit state information is entangled with and can be extracted from the reflection/transmission of the readout cavity [68]. In summary, qubit measurements involve applying a qubit drive tone for preparations and manipulations of the qubit state, followed by a cavity drive tone that allows for reading out the qubit state through probing the reflected/transmitted photons of the cavity at $\tilde{f}_r$. From single-tone spectroscopic measurements, $\tilde{f}_r$ can be found. To find what frequency the qubit drive tone should be, we need two-tone spectroscopic measurements.

The working principle of two-tone spectroscopic measurements is very simple. Basically, we continue monitoring the qubit state in these measurements using the indirect readout scheme explained above, which involves a fixed-frequency cavity drive. Meanwhile, we apply a second drive tone and sweep its frequency in the range close to the suspected qubit frequency. When the second drive tone is off-resonance with f_q , we expect to see the qubit remain in the ground state. Yet, when the drive tone is on-resonance, the qubit would get excited. Therefore, the qubit frequency can be found at the place where the qubit changes its state. The easiest version of such measurements uses signals in continuous wave mode for both drive tones. Such an implementation is given in Figure 3.2. In the setup, we use a VNA and an additional RF source (SGS100A) to provide the two drive tones. The sweep mode of the VNA is turned off and set to output the fixed cavity drive, while the output signal of the SGS100A is swept as the qubit drive. The two drives are combined by a power combiner before entering the fridge through the fridge input line. The cavity response to the input signals comes out via the fridge output and goes back to the VNA. Then it is analyzed at the cavity drive frequency to extract

the cavity's transmission, which will be recorded as a function of the qubit drive frequency. Just like single-tone spectroscopy, two-tone spectroscopic measurements can be more complicated by introducing more independent variables such as the qubit drive power, the cavity drive power, the bias source for tunable qubits, etc., in addition to the qubit drive frequency. Each added independent variable provides additional information about the qubit being measured. Increasing the qubit drive power allows for higher transitions so that we might be able to see their resonances in two-tone spectroscopy. Increasing the cavity drive power, we expect to see the qubit resonances shift due to the ac stark effect, which is basically a backaction of the cavity on the qubit since their coupling is mutual. This measurement allows us to distinguish random peaks/dips in basic two-tone spectroscopy from actual resonances since those random peaks/dips are fixed in frequency regardless of the cavity drive power. However, we cannot attribute every resonance that shifts with cavity drive power to qubit transitions because it is possible to observe resonances not of qubits but other quantum systems that are also coupled to the readout cavity. For tunable qubits, varying the bias source would help us identify the actual qubit transitions among all kinds of resonances because only the qubit would be tuned accordingly by the bias source.

Figure 3.4 plots the results of two-tone spectroscopy on the same example device. Figure 3.4b shows what a basic two-tone spectrum would look like. Given the design of our example device, that the cavity resonance appears as a peak in the single-tone spectrum, the qubit resonance in two-tone spectrum would appear as a dip instead. In Figure 3.4b, there are some features that look like dips. However, due to the excess noise in the data, it is hard to conclude whether those dips correspond to the qubit transition resonances. In each situation, the flux tunability is useful for the real qubit dips are tuned by external flux as shown in Figure 3.4a.

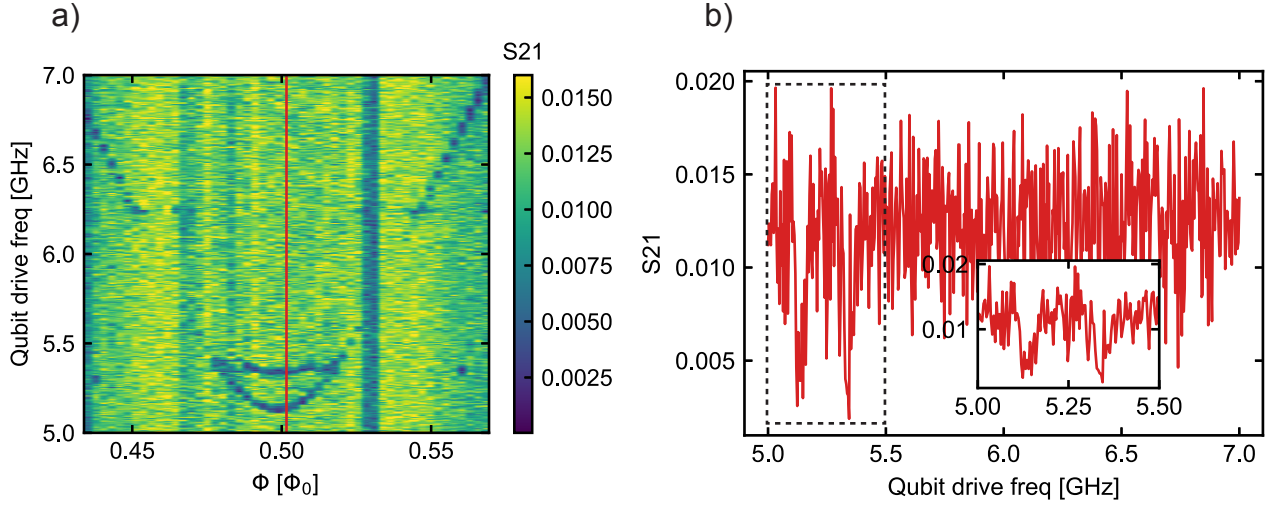


Figure 3.4: Two-tone spectroscopy measured on the same example device as in Figure 3.3. (a) Averaged cavity response measured as we swept the qubit drive frequency at different bias currents. The applied bias currents range from 2.7mA to 3.7mA, corresponding to $\Phi \approx 0.42 - 0.57$ after calibration. The red line in (a) indicates a line cut at $\Phi \approx 0.50$, which is plotted in (b). A section of the line cut enclosed by a dashed rectangle is plotted as an inset in (b).

3.3 Time-domain Measurements

Spectroscopic measurements are useful for frequency characterization. However, the lack of control over time during spectroscopic measurements prevents fine control over qubit states and the characterization of time-related properties of qubits, such as their lifetimes. Time-domain measurements that involve the application of precisely controlled pulse sequences complement spectroscopic measurements.

The process of time domain measurements begins with the generation of control pulses using an arbitrary waveform generator (AWG). We carefully vary the pulse duration and amplitude to implement specific qubit operations. For instance, a typical experiment might involve applying a sequence of π and $\pi/2$ pulses to rotate the qubit state on the Bloch sphere. Once the control pulses are generated, they are up-converted to the qubit drive frequency using IQ mixers and CW signals from microwave generators. These microwave pulses are then applied to the qubit, causing transitions between its energy states. Shortly after the qubit pulses, a readout pulse is generated and also up-converted to the cavity drive frequency. Information on the qubit state is captured by measuring the cavity response. This response signal is down-converted back to low-frequency IQ signals using another IQ mixer, which is then read by a lock-in amplifier.

With high-resolution triggering between the AWG and the lock-in amplifier, the delay between the qubit readout and the qubit drive can be precisely timed, which, as we shall see, is essential for lifetime measurements. The dashed lines in Figure 3.2 show the complete setup we used for time domain measurements.

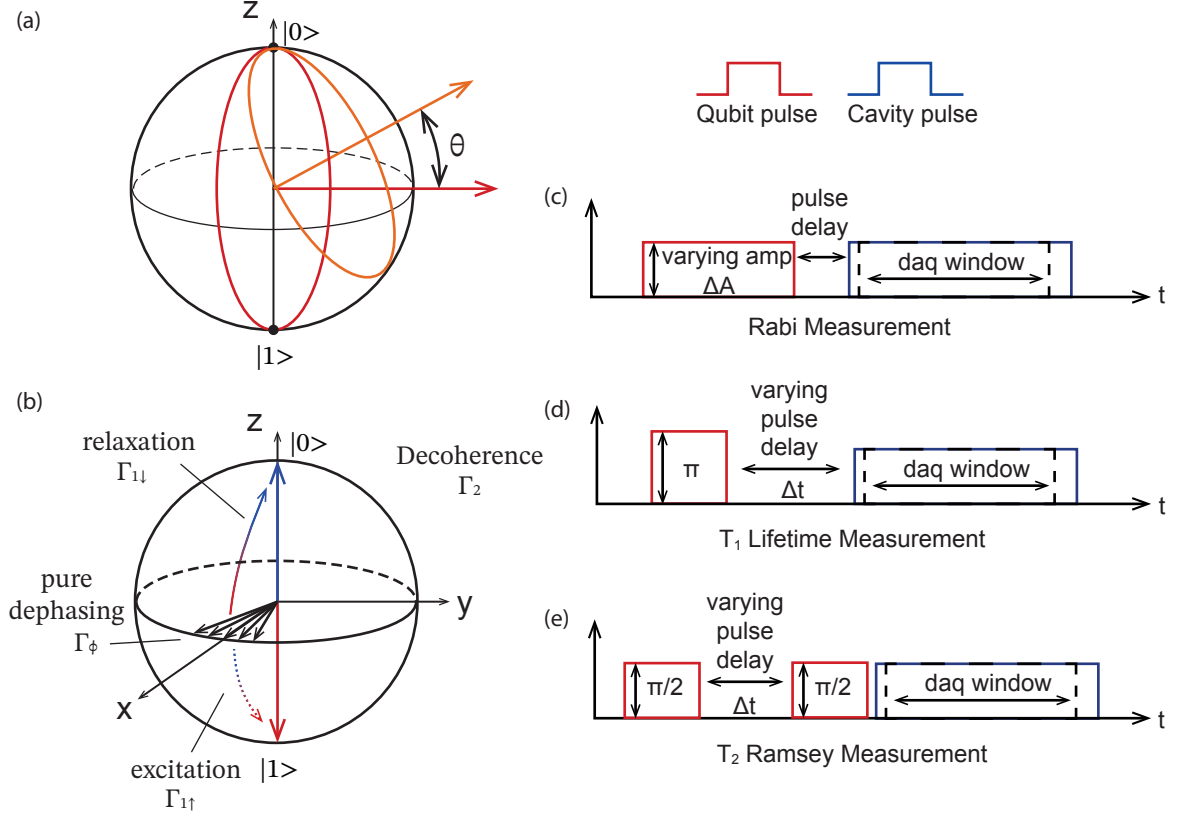


Figure 3.5: Principles of time domain measurements. (a) Eigenstates on the Bloch sphere for a driven qubit. The eigenstates for a detuned driven qubit cut through the Bloch sphere with an angle θ from the equator. (b) Decoherence of a qubit as an combined effect of longitudinal relaxation and pure dephasing that causes qubit decoherence. (c)-(e) Pulse sequences for time-domain measurements.

3.3.1 Rabi Oscillation

The key principle of time domain measurements involve carefully shaped and timed pulses for qubit manipulation. The pulses are found from Rabi oscillation, which is the coherent oscillation of the qubit state when the qubit is driven by a signal with varying amplitude (or duration). The interaction between an effectively two-level qubit and a classical signal, in the

semi-classical approach, is given by [80]:

$$H_{\text{semi-classic}} = -\frac{1}{2}\omega_q\sigma_z - A\cos(\omega_d t)\sigma_x, \quad (3.1)$$

where $A = Ed$ quantifies the interaction between a classical signal $E(t) = E\cos(\omega_d t)$ and the dipole moment of the qubit d . $\sigma_{z,x}$ are Pauli matrices acting on the qubit subspace.

By going to a rotating frame of drive with $U(t) = e^{-i\frac{\omega_d}{2}\sigma_z t}$, the Hamiltonian (Eq 3.1) becomes:

$$\hat{H} = -\frac{1}{2}\Delta_d\sigma_z - \frac{Ed}{2}\sigma_x, \quad (3.2)$$

where $\Delta_d = \omega_q - \omega_d$ is the detuning between the qubit frequency and the drive frequency.

Diagonalizing the rotating frame Hamiltonian above gives the eigenstates,

$$|V_-\rangle = \cos(\theta)|e\rangle - \sin(\theta)|g\rangle, \quad (3.3)$$

$$|V_+\rangle = \sin(\theta)|e\rangle + \cos(\theta)|g\rangle, \quad (3.4)$$

where $\theta = \tan^{-1}(\frac{A}{\sqrt{A^2 + \Delta_d^2}})$, and the eigenvalues of $|V_\pm\rangle$ are given by $E_\pm = \pm\frac{1}{2}\sqrt{A^2 + \Delta_d^2}$.

The evolution of the system can then be described by

$$|\varphi(t)\rangle = [C_+e^{-iE_+t}\sin(\theta) + C_-e^{-iE_-t}\cos(\theta)]|e\rangle \quad (3.5)$$

$$+ [C_+e^{-iE_+t}\sin(\theta) - C_-e^{-iE_-t}\cos(\theta)]|g\rangle, \quad (3.6)$$

where $C_\pm$ are constants determined by the initial condition. For a qubit starting in the ground state, the time evolution of the qubit state would be:

$$|\varphi(t)\rangle = -i\sin(E_+t)\sin(2\theta)|e\rangle + \cos(E_+t)\sin(2\theta)|g\rangle, \quad (3.7)$$

with the probability of being in the excited state given by:

$$P_e = \sin^2(2\theta)\sin^2(E_+t) = \frac{A^2}{A^2 + \Delta_d^2} \sin^2\left(\frac{\sqrt{A^2 + \Delta_d^2}t}{2}\right). \quad (3.8)$$

As Eq 3.8 shows, a qubit when driven by a drive at exactly the qubit frequency, oscillates between its ground and excited states with a frequency determined by the interaction strength between the qubit and the drive signal (A). As a result, we can choose the duration (t_0) for a qubit drive pulse depending on the desired qubit state $P_e(t_0)$. However, not all desired qubit states are available. With a non-zero detuning present, the qubit can never be completely excited, as suggested by $\frac{A^2}{A^2 + \Delta_d^2} < 1$ and demonstrated in Figure 3.5a.

Figures 3.6a and b show the measured Rabi oscillations of the example device with a pulse sequence defined in Figure 3.5c. According to Eq 3.8, varying E would also create a Rabi oscillation. When Figures 3.6a and b were measured, the drive amplitude (E) was changed, instead of the pulse duration (t). This choice is made to avoid any trivial errors when creating the arbitrary waveforms of different durations. However, this choice leads to a more complicated form of Rabi oscillations with varying prefactors.

3.3.2 T_1 Lifetime and T_2 Lifetime Measurements

After measuring the Rabi oscillations and fitting the data to Eq 3.8, the pulses required to drive the qubit into different states with a certain amplitude (or duration) can be found. For example, according to Figure 3.6b, a pulse amplitude of 0.4 V should correspond to a π -pulse that drives a qubit in the ground state to the excited state. With π - and $\pi/2$ -pulses found, lifetime measurements can be carried out using the pulses. The principle of lifetime measurements involves varying the wait time between the qubit drive and the qubit readout, which allows the qubit to decohere. The characteristic decoherence time scales are found by measuring the decay time of the qubit state from an initialized state to its steady state. In the following, we will discuss the measurements designed to find the T_1 and T_2^* qubit lifetimes.

Given by the definition of T_1 lifetime in Section 2.3.4, the qubit is driven into its excited state by a π -pulse, by varying the delay in reading out the qubit state, we can find the qubit state as a function of the delay. With each delay, the measurement is repeated many times to find the averaged qubit state. The characteristic decoherence time T_1 can be found by fitting

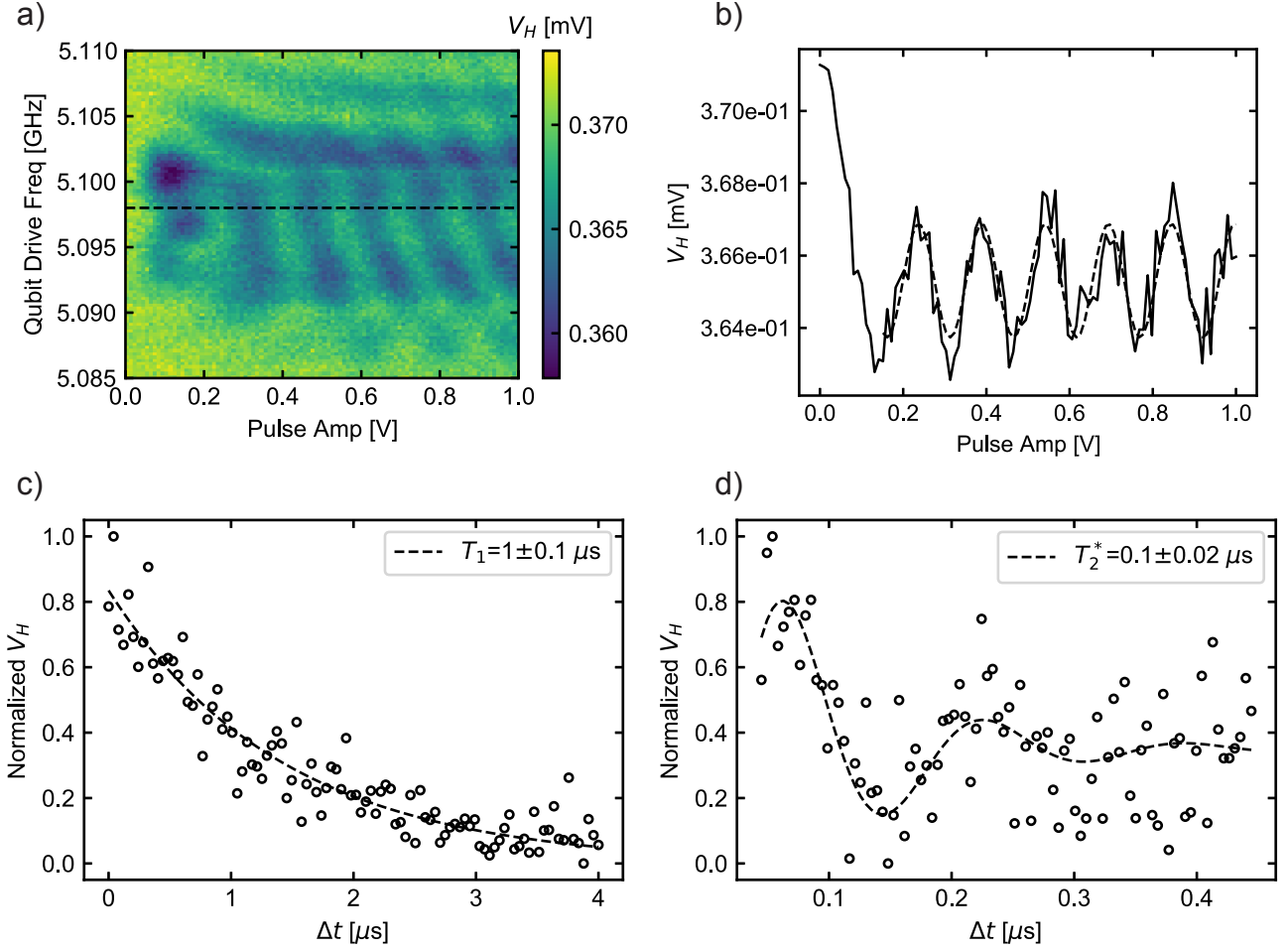


Figure 3.6: Time-domain measurements on the example device. (a) Rabi oscillations with a varying qubit drive as a function of the drive amplitude and the drive frequency. (b) A linecut of the Rabi oscillations in (a) with the qubit drive frequency set at 5.098 GHz. (c) Normalized heterodyne voltage as a function of delay time Δt between the qubit drive pulse and cavity readout pulse in T_1 lifetime measurement as demonstrated in Figure 3.5d. (d) Normalized heterodyne voltage as a function of delay time Δt between the qubit drive pulse and cavity readout pulse in T_2 lifetime measurement as demonstrated in Figure 3.5e.

an exponential decay to the averaged data. An example of T_1 lifetime measurement data T_1 is shown in Figure 3.6c. In the example, the qubit has a T_1 lifetime of 1 μs .

Since T_2^* lifetime describes the decoherence time of qubits in a superposition state [Section 2.3.4], a $\pi/2$ -pulse is used to prepare the qubit in a superposition state. Similarly to T_1 lifetime measurements, the qubit state is read out with a varying delay after the qubit drive. The decay in the probability of measuring the qubit in the original superposition state gives T_2^* . The star sign of T_2^* is to show that it is subject to pure dephasing, with no extra step in the measurement to mitigate the pure dephasing. In T_2^* measurements, we normally use a qubit drive with an

intentional detuning Δ_d to create an oscillation on top of the T_2^* exponential decay, leading to Ramsey fringes. This allows a more accurate lifetime extraction. This technique is not applicable to T_1 lifetime measurement because a nonzero detuning prohibits the qubit to be driven into its excited state, which is the prerequisite for T_1 measurement. Data of the T_2^* lifetime measurement on the example device are plotted in Figure 3.6d. Due to an excess of noise, we can barely see Ramsey fringes. With zero detuning, it would be even harder to extract T_2^* .

Chapter 4

Microwave Characterization of Fixed-frequency and Flux-tunable TImon qubits

The integration of semiconductor Josephson Junctions (JJs) in superconducting quantum circuits provides a versatile platform for hybrid qubits and offers a powerful way to probe exotic quasiparticle excitations. Recent proposals for using circuit Quantum Electrodynamics (cQED) to detect topological superconductivity motivate the integration of novel topological materials in such circuits [81, 82, 36]. As shown in Section 2.1.5, Majorana zero modes (MZMs) are predicted to emerge in hybrid TI structures. Embedding S-TI-S JJs into superconducting qubit circuits allows for detecting MZMs from the qubit spectrum. Figure 2.13 shows simulated signatures of MZMs in a TI-based transmon (TImon) device. In reality, the experimental feasibility of this detection scheme depends on the properties of the qubit. Consistent control of the properties and the ability to engineer them are essential. In this chapter, I will present data from measurements on two TImon devices serving as a proof-of-principle for the integration of TIs into superconducting circuits and as a preliminary investigation for the physics behind complex TI-based cQED devices.

4.1 Qubit Design and Fabrication

Transport measurements on S-TI-S JJs [83, 84, 85] have been reported including the work of our collaborators at Forschungszentrum Jülich [1]. Integrating TIs into hybrid cQED circuits,

however, creates additional demands on the microwave compatibility because the environment needs to have low dielectric losses [See Section 2.3.1]. BST ($(\text{Bi,Sb})_2\text{Te}_3$), a 3DTI material that can be grown by molecular beam epitaxy on low-loss silicon substrates [86], has a large helical gap of up to 300 meV [46, 87, 88]. These material properties allow for a reliable realization of S-TI nanostructure networks with transparent interfaces on silicon [1]. However, BST's sensitivity to ambient conditions imposes challenges on the fabrication. Facing the challenges, our collaborators developed a complete ultra-high vacuum (UHV) fabrication process, based on monolithically integrated dielectric masks for SAG (selective area growth) and stencil metallization to achieve a scalable and deterministic patterning of TI nanoribbon-superconductor networks [1, 89]. Parasitic two-level systems (TLSs) that can be hosted in the global SAG dielectric layers are identified as a ubiquitous loss mechanism in superconducting circuits [Section 2.3.1]. Characterizing microwave losses from the bulk of the stack materials is essential to validate the microwave compatibility of the fabrication process.

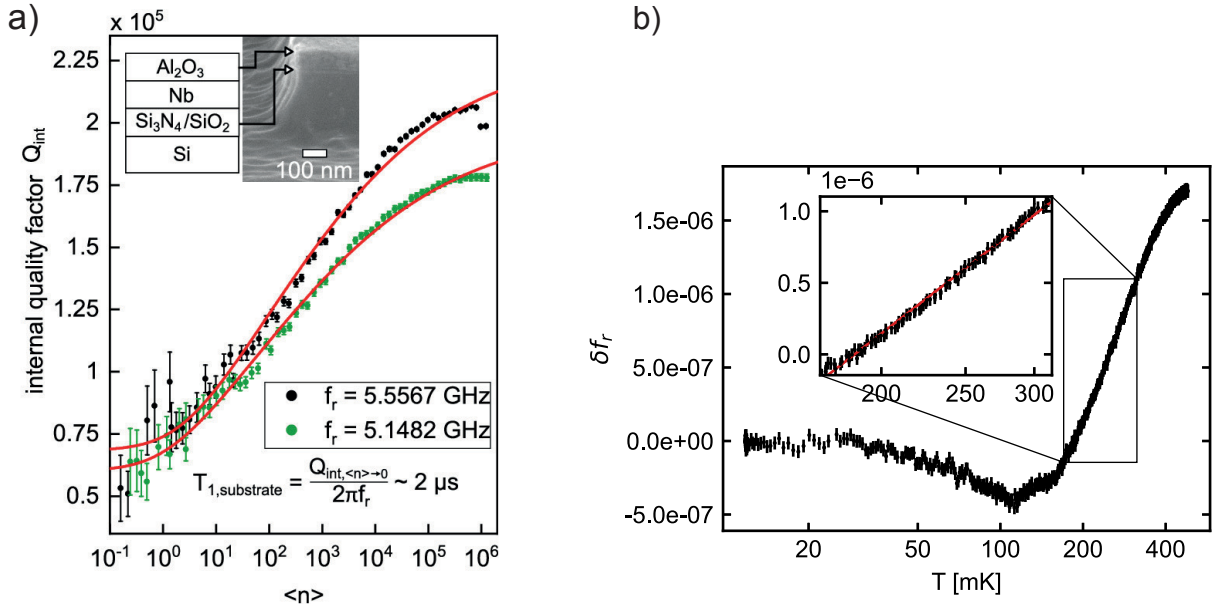


Figure 4.1: Evaluation of substrate losses via measurements on lumped-element resonators. (a) Internal quality factor of two resonators as a function of average photon number $\langle n \rangle$ in the resonator. This is fit to a model of TLS induced losses. Inset: SEM micrograph showing a cross-sectional view of the device (including trenching of the Si substrate). Adapted with permission from [89]. (b) Normalized cavity frequency shift δf_r of one of the resonator (~ 5.56 GHz) as a function of fridge temperature. Inset: Linear fit based on TLS model to data in the range of where TLS dominates the frequency shift.

For the purpose of testing the qubit fabrication process, I measured two Nb lumped element

resonators that were fabricated using the same stack materials [Figure 4.1a inset] following a similar fabrication process that excludes prior processing of a local stencil mask. Although the excluded step is essential in TImon qubit fabrications, local stencil lithography mitigates possible dielectric losses from the stencil mask compared to the global stack materials. Via microwave transmission measurements, I first measured the internal quality factors Q_{int} of the two resonators as a function of their average numbers of photons $\langle n \rangle$ [Figure 4.1a]. At single-photon limit, the resonators show an average Q_{int} of $6.6 \pm 0.3 \times 10^4$ determined using a robust and traceable fit routine [90]. Fitting Q_{int} in the whole range of $\langle n \rangle$ to a widely-used TLS model [Eq 2.50], I found the average loss tangent of the resonators due to on-resonant TLSs in a narrow window around the frequencies of the resonators $F\delta_{\text{TLS}}^0 = 1.1 \times 10^{-5}$. This value is an order of magnitude worse than mature technologies of Al resonators on high-resistivity silicon [91]. Assuming that the energy relaxation time (T_1) of our TImon qubit is dominantly limited by the same TLS losses in global materials, we can use the average Q_{int} to estimate T_1 using the formula $T_1 = Q/2\pi f \approx 2\mu\text{s}$ [92]. The validity of the formula can be understood from the energy-time uncertainty principle, which relates the qubit resonance linewidth to its lifetime.

Next, I chose one resonator and measured its resonance frequency at different temperatures. Normalized frequency shift of the resonator $\delta f_r = \frac{f_r(T) - f_r(T_0)}{f_r(T_0)}$ as a function of temperature is shown in Figure 4.1b. When the temperature is far below the critical temperature of Nb, the frequency shift is dominated by the contribution from highly detuned TLSs. Fitting the linear region of the data to Eq 2.51, the loss tangent of the resonator due to off-resonant TLSs $F\delta_{\text{TLS}}^i$ is found to be 2.05×10^{-6} . The fact that $F\delta_{\text{TLS}}^0$ is an order of magnitude higher than $F\delta_{\text{TLS}}^i$ suggests that the on-resonant TLSs have a higher density of states than the highly-detuned TLSs, and dominate the dielectric losses due to TLSs.

With the feasibility of the TImon fabrication process supported by the resonator test, our collaborators fabricated several TImon chips. Two of them are featured below. The first device chip, labeled BST-1959, hosts four distinct qubit-cavity systems [Figure 4.2a]. Its circuitry design features a signal feedline positioned on top, with four microwave quarter-wave CPW resonators below serving as readout cavities, inductively coupled to the feedline. To tune the

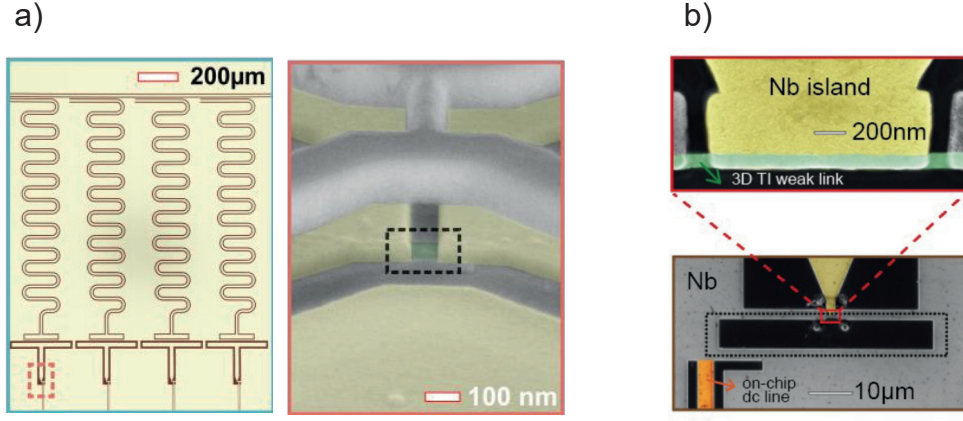


Figure 4.2: Integration of topological insulator Josephson junctions in superconducting quantum circuits. (a) Left: false-colored SEM of the four qubits and readout cavities on fixed-frequency TImon chip BST-1960. Right: zoomed-in false-colored SEM of one of the JJs shows the BST nanoribbon (green) and the two Nb leads (yellow) under the stencil mask used to shadow a part of the BST nanoribbon from the Nb. Adapted with permission from [89]. (b) Bottom: false-colored SEM of two TI JJs forming a SQUID loop on a flux-tunable TImon chip BST-2162 equipped with an on-chip DC line (orange) used to tune the flux through the loop. Top: zoomed-in image on the two JJs formed from one long BST nanoribbon (green) and three Nb islands.

bare resonance frequencies of these resonators apart, we slightly varied their lengths [Section 2.3.1]. Designed within the 4-8 GHz frequency range typical for qubit measurements, we utilized a serpentine shape in the resonator design to optimize length while conserving space. Each resonator terminates in a T-shaped island, below which four charge qubits are individually coupled. The qubits also incorporate T-islands, acting as shunt capacitors to adjust the charging energy E_C into the transmon regime [Section 2.3.2]. Moreover, the proximity of the qubit and resonator T-islands forms parallel plate capacitors and generates a capacitive coupling between the qubits and their readout cavities, which is essential for qubit readout measurements [Section 2.3.3].

To validate the circuitry design of chip BST-1959, electrostatic simulations were conducted using Comsol software [Appendix B]. The simulated value of E_C was found to be approximately 240 MHz, ensuring the qubits with $f_{01} \sim$ GHz operate in the transmon regime. Furthermore, by assuming a qubit frequency $f_{01} = 6$ GHz, the designed qubit-cavity coupling strength $g/2\pi$ was calculated to be around 90 MHz. This coupling strength enables a dispersive readout scheme. Satisfied with these values, we adopted this circuitry design for all TImon chips in this thesis, maintaining consistency in T-island design across chips, with occasional adjustments in

qubit-cavity system numbers. Changes in qubit-cavity system necessitate corresponding modifications to specific cavity resonances. Apart from these minor alterations, primary design variations among chips typically reside in qubit junction configurations.

Chip BST-1959 exclusively hosts fixed-frequency qubits, each comprising a single TI Josephson junction (JJ). Fabrication involves sputtering Type-II superconducting materials Nb onto a BST nanoribbon via a shadow mask, creating a pure TI middle section connected to Nb-coated TI leads [Figure 4.2a, right panel]. Due to the proximity effect, the two TI leads are effectively superconducting with a superconducting gap Δ^* that depends on Δ_{Nb} and the transparency of the Nb/TI interface D . These superconducting leads, together with the unproximitized middle region, form S-TI-S junctions. Based on past measurements on junction-only devices by our collaborators [1], all JJs on this chip were designed to be 100 nm wide and 150 nm long, ensuring qubit frequencies f_{01} are within the dispersive readout regime. However, fabrication uncertainties led to slight deviations in junction dimensions, with lengths (L) ranging from approximately 90-150 nm and widths (W) from around 110-140 nm. To evaluate fabrication quality and troubleshoot potential issues, a dedicated check resonator, uncoupled to any qubits, was also included in the chip design, providing insight into film cleanliness within the material stack.

The second qubit device chip, BST-2162, shares a similar circuitry design with BST-1959 but incorporates six flux-tunable TImons instead of fixed-frequency ones. Each flux-tunable TImon is coupled to its readout cavity via T-shaped islands, maintaining consistency with BST-1959 in T-island design. Therefore, we expected the same E_c value as BST-1959. However, variations in qubit designs can result in different values of the coupling strength g , which is dependent on E_J and f_{01} . Flux-tunability in BST-2162 is achieved using pairs of TI JJs forming SQUID loops [Figure 4.2b], enabling E_J and f_{01} to be tuned via external flux Φ_{loop} [Section 2.3.2]. To provide the required external flux, we put an on-chip DC line next to every SQUID loop. When currents pass through the DC line, the induced magnetic field of the currents creates a flux that penetrates the loop such that $\Phi_{\text{loop}} = B_{\text{ind}}A$, where A is the surface area of the loop. As the equation shows, the value of A dictates the sensitivity of the flux-tunable qubits

to magnetic fields. For BST-2162, A was designed to be $318.5 \mu\text{m}^2$ for every qubit with a width of $6.5 \mu\text{m}$ and a length of $49 \mu\text{m}$. Within each qubit, the two JJs in the SQUID loop were designed to have identical geometries to maximize the qubit tuning range, although fabrication errors normally occur and result in slight asymmetry between the JJs, quantified by parameter d . Among qubits, we intentionally varied their junction designs particularly in junction length. SEM analysis revealed junction widths of approximately 100 nm for all qubits. Three qubits with a shorter junction design have junction lengths of approximately 90 nm , while the other three have junction lengths of approximately 110 nm . Similar to BST-1959, BST-2162 includes a stand-alone check resonator for diagnostic purposes.

4.2 Fixed-frequency TImon

This section presents data collected by our collaborators from spectroscopic and time-domain qubit measurements on BST-1959 and published in [89]. For more details, please see the acknowledgements 4.5. The experimental results demonstrate successful qubit characterization and manipulation of a TImon qubit.

4.2.1 Qubit Spectroscopy

We used single-tone and two-tone spectroscopic measurements to search for qubit transitions. Before measuring the qubit, its readout cavity was first characterized. Figure 4.3a shows the measured power-dependent spectrum of the cavity on BST-1959 with a bare resonance frequency f_r at around 7.133 GHz . From the figure, we can observe a power-dependent shift $\chi \sim -11.7 \text{ MHz}$ in the cavity resonance frequency with its dressed resonance frequency $\tilde{f}_r \sim 7.145 \text{ GHz}$ when the cavity drive power is below -100 dBm . The presence of such a dispersive shift is normally an indicator of qubit-cavity coupling. The fact that the dressed cavity resonance is pushed up by the qubit at low power suggests that the qubit frequency is below the cavity frequency. Given $\chi \approx g^2/\Delta$, where the simulated g is around 90 MHz , and Δ is the detuning between the qubit and cavity frequency, we estimated that the qubit coupled to the cavity has a frequency of roughly 6.39 GHz . When measuring all cavities on BST-1959, one other cavity also shows an evident dispersive shift in its resonance frequency. However, this

shift is in the opposite direction, suggesting that the qubit to which it is coupled has a higher frequency. The variations in qubit frequency are expected from the dimension variations in the TI JJs. Extra knobs are needed to fine-control the qubit frequency.

Next we started to locate the qubit frequency using two-tone spectroscopy by adding a dedicated RF tone to directly harness the qubit while keeping the cavity drive at its dressed resonance frequency and a low power [Section 3.2.2]. Figure 4.3b shows the cavity response in the expected frequency range of the qubit while sweeping the qubit drive power. At low qubit drive power below -90dBm, we see only one peak around 6.30 GHz. We attribute this peak to the $|0\rangle \rightarrow |1\rangle$ transition, and establish that the qubit-cavity detuning Δ is approximately 850 MHz with the qubit frequency $f_{01} \sim 6.30$ GHz. With Δ and χ known, the coupling strength $g/2\pi$ was calculated to be 93 ± 9.3 MHz, which is consistent with the simulated value. When the drive power increases, more peaks start to occur, corresponding to higher level transitions of $|0\rangle \rightarrow |2\rangle$ and $|0\rangle \rightarrow |3\rangle$. We attribute one of the peaks, which starts to appear at around -90dBm, to the two-photon process of $|0\rangle \rightarrow |2\rangle$ transition. With $f_{02}/2 \approx 6.15$ GHz, the anharmonicity of the qubit from $|\alpha/h| = |2(f_{02}/2 - f_{01})|$ is estimated to be approximately 300 MHz. Compared to hybrid transmons with only a few highly transmitting channels [93], which typically show $|\alpha| < E_C$, our device is closer to the metal-oxide limit ($|\alpha| \approx E_C$) [Section 2.3.2]. The metal-oxide limit suggests that most transport channels in our TImon qubit resemble the sinusoidal tunneling supercurrent. In Section 2.2.3, we identified two supercurrent carriers in confined TI JJs — gapped surface states and bulk states, both in the form of Andreev bound states (ABSs). While both are susceptible to impurities or disorders leading to reduced transmission coefficients, the bulk states lacking the topological protection of time-reversal symmetry, are usually more likely to be diffusive [Sections 2.1.5 and 2.2.3]. As we shall see in Section 4.3.2, diffusive ABSs with low transmission coefficients resemble sinusoidal Josephson supercurrent. Therefore the metal-oxide limit of α suggests that the supercurrent in our TImon qubit is dominantly diffusive.

To eliminate the possibility that the observed peaks are merely background noise, further two-tone spectroscopic measurements by sweeping the cavity drive power to detect any AC

Stark shifts. Since the cavity number operator acts on the dressed qubit states in the effective Jaynes-Cummings Hamiltonian equation [Section 2.3.3], the readout cavity exerts a push on the qubit frequency with a strength dependent on the photon number $\langle n \rangle$ of the cavity. Fixing the qubit drive power at -101 dBm, where we only see the qubit ($|0\rangle \rightarrow |1\rangle$) peak, we indeed observed an ac Stark shift in the peak frequency as shown in Figure 4.3c while varying the cavity drive power. Figure 4.3d shows two traces of the qubit peak at $p_r = -89$ dBm and $p_r = -106$ dBm. The trace at higher power has a much better signal-to-noise ratio. Therefore, we used this readout power in the following time-domain measurements to demonstrate coherent control of the TImon qubit.

4.2.2 Qubit Lifetime

After characterizing the spectroscopic properties of our TImon qubit, we moved on to time-domain measurements that use controlled pulse sequences to demonstrate coherent qubit control and unveil more information about our device such as its T_1 and T_2 lifetimes.

First, we show coherent qubit control via Rabi oscillations. During the measurement, we used a qubit drive pulse with a frequency around f_{01} to drive the qubit transition. As demonstrated in Figure 4.4a, a readout pulse fixed at $\tilde{f}_r$ follows right after the first pulse to read the qubit state from cavity response. By varying the duration of the qubit drive pulse, we saw the qubit state oscillates between $|0\rangle$ and $|1\rangle$ in Figure 4.4a. The prominence and period of these oscillations depend on both the qubit drive frequency f_d and the drive power p_d . When $f_d = f_{01}$, Rabi oscillations have the longest period and are the most prominent. While with f_d fixed, higher drive power p_d renders more prominent and faster Rabi oscillations. Fixing f_d and p_d , we can find the pulse durations that correspond to $\pi/2$ and π pulses from the Rabi oscillations, and use them for qubit lifetime measurements.

After calibrating the $\pi/2$ and π pulses, we performed a T_1 lifetime measurement that gives the relaxation time of our TImon qubit. The qubit was first excited from $|0\rangle$ to $|1\rangle$ by a π pulse. A wait time of τ_{T_1} was applied before reading out the qubit state after the π pulse. As

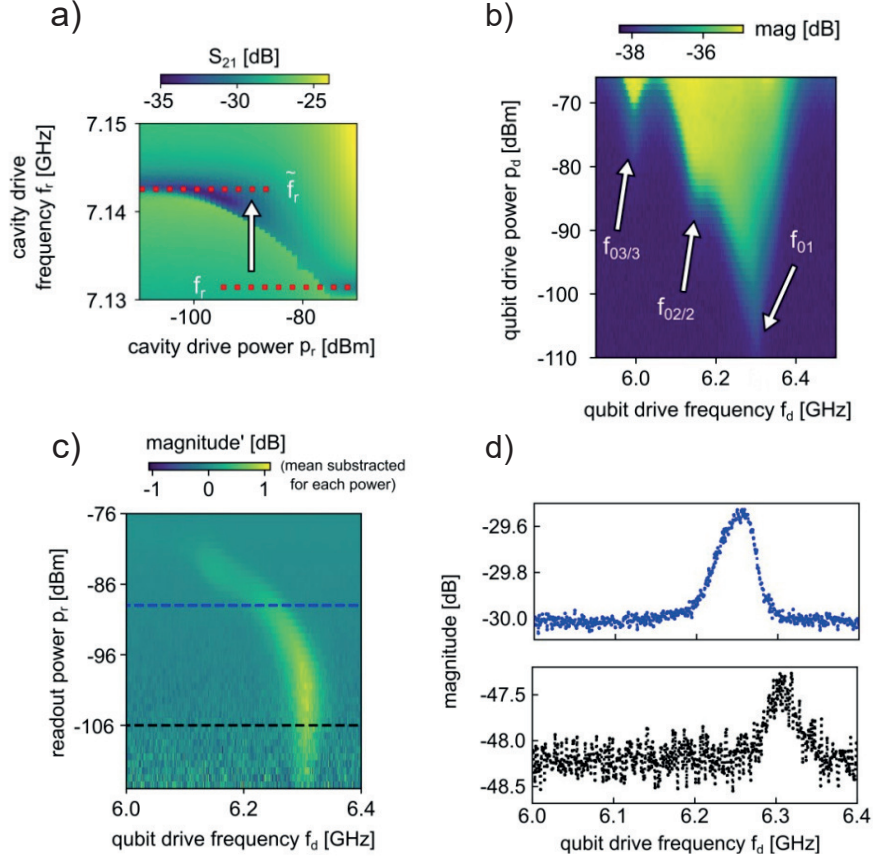


Figure 4.3: Qubit spectroscopic measurements. (a) S_{21} response of one of the four cavities as a function of cavity drive frequency f_r and drive power p_r in single-tone spectroscopic measurements showing a dispersive shift. (b) Cavity S_{21} response as a function of qubit drive frequency f_d and qubit drive power p_d while fixing cavity drive in two-tone spectroscopic measurements showing peaks at frequencies corresponding to the f_{01} , $f_{02/2}$, $f_{03/3}$ qubit transitions. (c) Cavity S_{21} response as a function of qubit drive frequency f_d and cavity drive power p_r with the qubit drive fixing at $p_q = -101$ dBm demonstrating ac Stark shift where the qubit frequency shifts with readout power. The mean value at each readout power has been subtracted for visibility. (d) Two single two-tone spectroscopy measurements at readout power $p_r = -89$ dBm (blue) and $p_r = -106$ dBm (black) without mean value subtraction. Adapted with permission from [89].

τ_{T_1} increases, the qubit is more likely to decay back to the ground state, and the probability of finding the qubit in the excited state decreases exponentially. In Figure 4.4 c, each point in the main plot is an averaged value of normalized cavity response over multiple repetitions that indicates the probability of finding the qubit in the excited state. Fitting an exponential decay model to the data allows $T_1 = 28$ ns to be extracted. $T_1 = 28$ ns is much smaller than the potential relaxation time estimated from the test resonator results $\sim \mu\text{s}$. While the test resonator does not capture the impact of all fabrication steps of the qubit chip, the discrepancy between the test resonator results and the measured lifetime implies that our TImon qubits

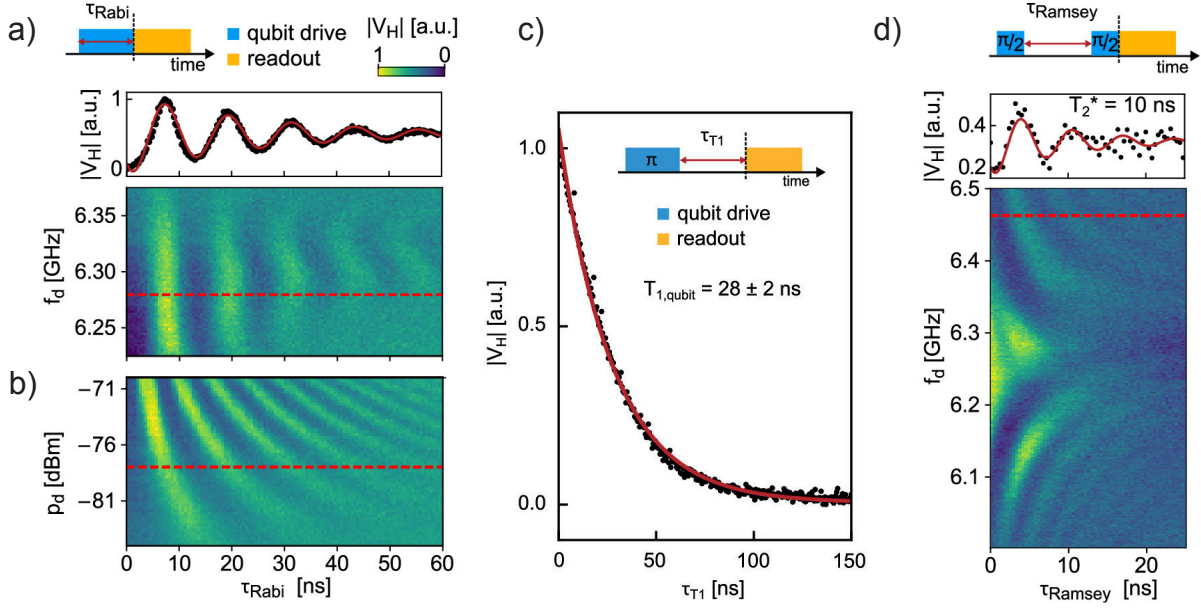


Figure 4.4: Qubit control and lifetime measurements. (a) Rabi oscillations of a TI transmon qubit measured with the pulse sequence shown at the top. The single line plot shows an average of 10 Rabi measurements taken at a drive frequency $f_d = 6.28$ GHz and power $p_d = -78$ dBm with the solid red line being a fitted damped sinusoid. The color plot shows the dependence of the Rabi oscillations on the drive frequency f_d . (b) Power dependence of the Rabi oscillations at a drive frequency $f_d = 6.28$ GHz. (c) Heterodyne voltage as a function of measurement delay time τ_{T1} using the pulse sequence shown in the inset. Data are averaged over a total of 150 measurement runs with a small temporal variance of ± 2 ns. By fitting an exponential decay to the data (red), we extract an energy relaxation time of the qubit $T_1 = 28$ ns. (d) Heterodyne voltage as a function of qubit drive frequency f_d and the delay time between two $\pi/2$ pulses τ_{Ramsey} as in the Ramsey pulse sequence shown in the top scheme. Line cut taken at $f_d = 6.462$ GHz (red dashed line) is plotted and shows the Ramsey oscillations with a finite detuning from f_{01} . $T_2^* = 10$ ns is extracted from a damped sinusoid fit (solid red line). Adapted with permission from [89].

are currently not limited by TLS losses in the bulk materials of the SAG mask. The additional losses could come from imperfect selective growth of TI nanoribbon that resulted in TI crystals grown on the SAG mask [See Appendix A]. Additionally, trapped quasiparticles in the JJ [78], a soft induced superconducting gap Δ_{ind} and low Nb/TI interface transparency D [94] can all contribute to short coherence times.

Next we performed a Ramsey interference measurement using $\pi/2$ qubit drive pulse instead to find the decoherence time T_2^* of our TImon qubit. Different from T_1 lifetime measurement, Ramsey measurements use two $\pi/2$ qubit drive pulses to complete a full π -rotation from $|0\rangle$ to $|1\rangle$. A variable period of time τ_{Ramsey} is added between the two pulses $\pi/2$ to allow the qubit to evolve starting in the transverse plane. During the time, the qubit loses coherence

due to qubit relaxation and dephasing. Likewise this results in an exponential decay in the probability of finding the qubit in the excited state characterized by the decoherence time T_2^* . When measuring, we intentionally added a detuning between f_d and f_{01} to create an oscillation on top of the exponential decay to better extract the characteristic decay time T_2^* as shown by the Ramsey fringes in Figure 4.4d. A fit to the Ramsey fringe estimated $T_2^* = 10$ ns. When decoherence is dominated by qubit relaxation events, we expect to see $T_2^* = 2T_1$. The fact that $T_2^* < 2T_1$ indicates that the main contribution to the decoherence is pure dephasing in the transverse plane, which could be caused by 1/f-type noise such as critical current noise that leads to fluctuations of E_J caused by excess quasiparticles [95] and capacitance noise that leads to fluctuations of E_C caused by interacting-TLS [96, 97].

4.3 Flux-tunable TImon

In the previous section, the characterization and coherence control of a fixed-frequency TImon demonstrate a successful integration of locally grown TI nanoribbon with cQED compatible circuits. However, it is still immature to use the fixed-frequency TImon qubit in quantum information processing. We lack fine control over the qubit frequency. Without an extra tuning method, there are unavoidable discrepancies between the desired and the actual qubit frequencies due to fabrication uncertainty. Moreover, compared to the current stateplay of mature metal-oxide transmon qubits used in quantum information processing, the coherence time of our TImon qubit is too short. On the other hand, there are also questions about using TImon qubits for the detection of topological superconductivity. What is the nature of the conducting channels inside a TI JJ? Will they mask the feature of topological superconductivity? Facing the aforementioned issues and questions, a device of next generation BST-2162 was fabricated, which adopts a more complicated qubit design using a SQUID loop that allows the qubit frequency to be tuned [Figure 4.2b]. This section presents experimental data and corresponding theoretical study on this device.

4.3.1 Experimental Characterization

Following the typical dispersive readout scheme, single-tone spectroscopy was performed on the new device. The cavity response S_{21} to a low power drive tone was recorded when the frequency of the drive tone and the applied current through the DC line of the device were swept [Figure 4.5a]. At low drive power, the cavity undergoes a dispersive frequency push from the coupled qubit. Therefore, the measured resonance frequency of the cavity $\tilde{f}_r$ depends on the qubit frequency f_q . The dressed cavity frequency $\tilde{f}_r$ was extracted from S_{21} data in Figure 4.5a and plotted in Figure 4.5b. Since the qubit frequency oscillates with the external flux through the SQUID loop that varies when the applied current is swept [See section 2.3.2], the dressed cavity frequency is expected to be modulated along with f_q . Figure 4.5 displays such an oscillation that demonstrates the flux-tuneability of the SQUID transmon device.

As discussed in Section 2.3.2, the flux-tuning pattern of f_q has a periodicity of Φ_0 , and reaches its maximum at zero flux and its minimum at half flux point regardless of the type of SQUIDs [See Figure 4.6b]. The applied flux in Figure 4.5 was calibrated using the periodicity in applied current $P = 548.5 \mu\text{A}$ and the current value $I_{\text{max}} = 90.2 \mu\text{A}$ that gives the maximum $\tilde{f}_r$. These values were found from the first modulation cycle with negative bias currents. The drift of I_{max} from zero indicates the existence of a background flux noise in the environment at a level of $\Phi_{\text{bg}} = 0.16\Phi_0$. After calibration, the value of $\tilde{f}_r$ in Figure 4.5a-b is shown to decrease monotonically between zero and half flux point. The decreasing trend indicates $f_q > f_r$. The absence of avoided crossings indicates that f_q never crosses the bare resonance of the cavity f_r when the value of f_q continues to drop in the flux range. Therefore, f_q stays above f_r in the whole flux range.

The tunable range of f_q depends on the total Josephson energy $E_{J,\text{tot}} = E_{J,1} + E_{J,2}$ and the asymmetry factor $d = |E_{J,1} - E_{J,2}|/E_{J,\text{tot}}$. $E_{J,\text{tot}}$ sets the upper bound for tunable f_q , while d and $E_{J,\text{tot}}$ together determine the lower bound. With perfect symmetry $d = 0$, f_q can go to zero. The leftmost modulation cycle of $\tilde{f}_r$ data in Figure 4.5b was fitted to a model based on SIS SQUID and dispersive qubit-cavity coupling. In the fit model, some geometry-dependent parameters such as E_C are decided from COMSOL simulation. From the fit, $E_{J,\text{tot}} = 215.8$

GHz and $d = 0.27$ were extracted. Therefore, $E_{J,1;2} = 137.0; 78.8$ GHz. Given $E_{J,tot}$ and d , f_q versus applied flux was also calculated, ranging from 10.4 to 19.9 GHz [See Figure 4.5c]. Consequently, the coupling strength g ranges from 88 to 122 MHz. From the extracted parameters, we can again clearly see discrepancies between the design and the actual device resulted from fabrication uncertainty. Even with identical designs for the two JJs in the SQUID, we still ended up with a non-negligible d that significantly reduces the tuning range of our qubit. Moreover, the JJs on BST-2152 are designed to be narrower than the ones on BST-1959 and are expected to have lower Josephson energy. Instead, extracted $E_{J,1;2}$ are both higher than the Josephson energy of the fixed-frequency TImon. As a result, the highest qubit frequency which corresponds to $E_{J,tot}$ are much higher than the qubit.

Although the pure SIS model fits the data quite well as seen in Figure 4.5b, our TI JJs being SNS junctions are expected to deviate from a typical sinusoidal energy-phase relation, and therefore deviate from the SIS model. We defined a quantity to measure the deviation as the following: $(\sum^n |\frac{f_r - f_{r,sim}}{f_r}|)/n$, where f_r is the extracted resonance frequency from spectroscopic measurements, $f_{r,sim}$ is the simulated resonance frequency from the SIS model fit at each flux point, and n is the total number of data points. Getting rid of the data points that have error bars larger than 0.1 MHz (roughly 0.0015% of the data values), the deviation calculated from our single cycle fit is 0.00022%. To understand the implication of this deviation value for our TI JJs, I conducted a theoretical study on flux-tunable qubits that incorporate SNS JJs, which resemble our flux-tunable TImon.

4.3.2 Theoretical Simulation

To theoretically study flux-tunable TImons, I first need to build a model that captures the essence of our devices — supercurrent transport in the TI JJs. As discussed in Sections 2.2.2 and 2.2.3, the major supercurrent contribution in our TI JJs is from ABS formed from helical surface states facilitated by multiple Andreev reflections. When the temperature is sufficiently low, the Josephson potential of junction k ($k = 1, 2$) $U_{J,k}$ is given by adding the energies of those Andreev bound states, mediated by a number ($i = 1, 2, \dots$) of Andreev bound states,

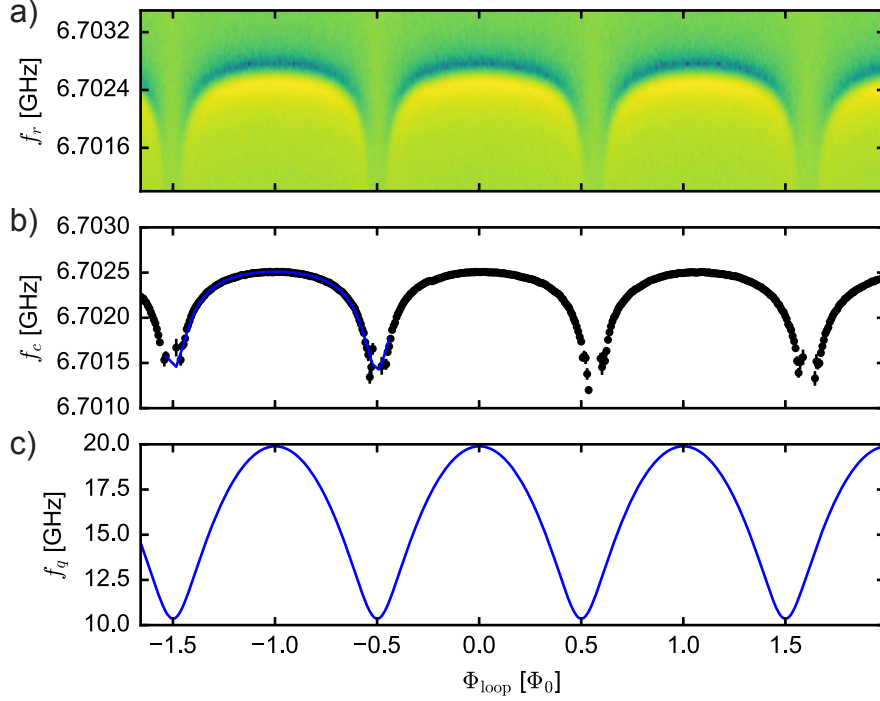


Figure 4.5: Flux tuning via SQUID junctions. (a) Cavity S21 response as a function of cavity drive frequency f_r and applied flux with the drive power fixed at -110 dBm. The inset shows cavity transmission at effective zero-flux point. (b) Cavity frequency f_c versus applied flux extracted from transmission data in (a) at each flux point. (c) Qubit frequency f_q versus applied flux deduced from the cavity frequency in (b) using Jaynes-Cummings model. The applied flux is calibrated from the current applied through the on-chip flux line.

each characterized by a transmission coefficient, $T_i^{(k)}$. $U_{J,k}$ can be analytically described in the following equation:

$$U_{J,k} = \sum_i V_i^{(k)}(\varphi_k) = -\Delta^* \sum_i \sqrt{1 - T_i^{(k)} \sin^2(\varphi_k/2)}, \quad (4.1)$$

where Δ^* is the induced superconducting gap of the superconducting leads in the JJs, and φ_k is the phase difference across junction k. With an external flux Φ_{loop} , we can set $\varphi_1 = \varphi$ such that $\varphi_2 = \varphi - 2\pi\Phi_{\text{loop}}/\Phi_0$.

Before studying the total Josephson energy, I investigated the effect of $T_i^{(k)}$ for a single junction. Although surface states are normally expected and measured to be ballistic with high supercurrent transmissions [98, 99, 100], gapped surface states could have low transmissions induced by disorders [Section 2.2.3], as suggested in the previous discussion on the anharmonicity of our fixed frequency TImon [101]. Figure 4.6a shows the energy-phase relation for each ABS with different transmission coefficients T . In the ballistic limit of $T \rightarrow 1$, the ABS energy spec-

trum resembles a quadratic potential within a 2π phase range. As T moves away from perfect transmission, the ABS energy spectrum becomes more sinusoidal. As a result, I isolated the high-transmission channels from slightly more diffusive channels. For simplicity, I treated our TI JJs as hybrid junctions with supercurrent composed of one high-transmitting ABS channel ($T = 0.9$) and one diffusive channel. In the effective model, both channels now have prefactors that physically represent the number of channels of similar type. Moreover, when describing the diffusive channel, I used a sinusoidal SIS junction model. The effective Hamiltonian of such hybrid junctions is then given by:

$$H_{J,k} = H_{J,\text{SNS}}^{(k)}(\hat{\varphi}_k) + H_{J,\text{SIS}}^{(k)}(\hat{\varphi}_k). \quad (4.2)$$

Given the hybrid junction model, I varied the prefactors in front of the two distinctive channels and plotted the energy spectrum of a single hybrid junction with different combinations of the prefactors in Figure 4.6b. These different scenarios are characterized by a quantity $r = E_{J,\text{SNS}}/E_{J,\text{SIS}}$ that measures the ratio between the contributions of the two channels to the total Josephson energy of the junction, where $E_{J,\text{SNS/SIS}}$ are the quantities that parameterize their corresponding Hamiltonians. For SIS, $E_{J,\text{SIS}}$ is well-defined as the constant in front of the sinusoidal function. For SNS, since its functional form is more complicated, it is hard to adopt a single parameter to parameterize the whole Hamiltonian. By convention [93], we define $E_{J,\text{SNS}}$ as related to the potential in the harmonic-oscillator Hamiltonian such that when expanding $H_{J,\text{SNS}}$ in terms of $\hat{\varphi}$,

$$H_{J,\text{SNS}}(\hat{\varphi}) = \frac{E_{J,\text{SNS}}}{2}\hat{\varphi}^2 + \text{higher-order terms}. \quad (4.3)$$

Figure 4.6b shows how the increase in r shifts the energy spectrum of a hybrid junction away (dashed lines) from one of a simple SIS junction (solid line).

After the investigation of a single hybrid junction, the total Josephson potential for a SQUID

formed by two hybrid junctions is considered:

$$H_J(\Phi_{\text{loop}}) = H_J(\hat{\varphi}) + H_J(\hat{\varphi} - 2\pi\Phi_{\text{loop}}/\Phi_0). \quad (4.4)$$

Like realistic junctions, asymmetry between the two junctions is allowed. I assumed the same asymmetry factor d applies to both types of channels in the two junctions such that $d = |E_{J,\text{SIS};\text{SNS}}^1 - E_{J,\text{SIS};\text{SNS}}^2|/(E_{J,\text{SIS};\text{SNS}}^1 + E_{J,\text{SIS};\text{SNS}}^2)$, and the same r for both junctions such that $r = E_{J,\text{SNS}}^{1;2}/E_{J,\text{SIS}}^{1;2}$. In order to have the theoretical SQUID emulate our actual flux-tunable device as much as possible, I fixed d at 0.27 which is found from the experimental data, and allowed r to be a free parameter. Based on the assumptions above, inserting Eq 4.4 into the qubit Hamiltonian such that $H_q = H_C + H_J$, we can calculate the qubit frequency for any given Φ_{loop} . Choices of other simulation parameters are given in Supplementary D.2.3. Figure 4.6 c shows the calculated qubit frequency as a function of Φ_{loop} for two different values of r . The same flux-tuning model based on SIS junctions that we used to fit our experimental data was used to fit the simulated qubit frequency. As expected, the model fits our $r = 0$ simulation results perfectly. On the other hand, we saw a slight deviation when $r = 0.45$.

To better compare theories with experimental data that measure f_r only, I further calculated f_r from the simulated qubit frequency as a function of Φ_{loop} for different r ranging from 0 to 1. The same SIS flux tuning model was applied to the simulated f_r , and the deviations between the SIS model and the theoretical hybrid junction model are plotted in Figure 4.6d. As shown in the figure, at $r = 0.45$, the deviation is around 0.00049%. However, it is hard to see the deviation with the eyes if the simulated data in the inset of Figure 4.6d are slightly obscured by noise. The deviation found from flux-tunable TImon measurements is 0.00022%. Given enough data points, this value sets an upper bound for the actual deviation due to the failure of the SIS model describing more complicated hybrid junctions, taking into account the uncertainty that might boost the deviation. According to the theoretical simulation results, this deviation value suggests that a ballistic contribution with $r < 0.3$ might exist in our flux-tunable TImon. However, it is important to note that when calculating the deviation from experimental data, data points with high uncertainties are taken out. Moreover, there are limitations within the theoretical simulation. Assumptions are made during the simulation. The

most significant one is that the analytical expressions of the hybrid junctions assume a short junction limit for both ballistic and SIS-like diffusive supercurrent.

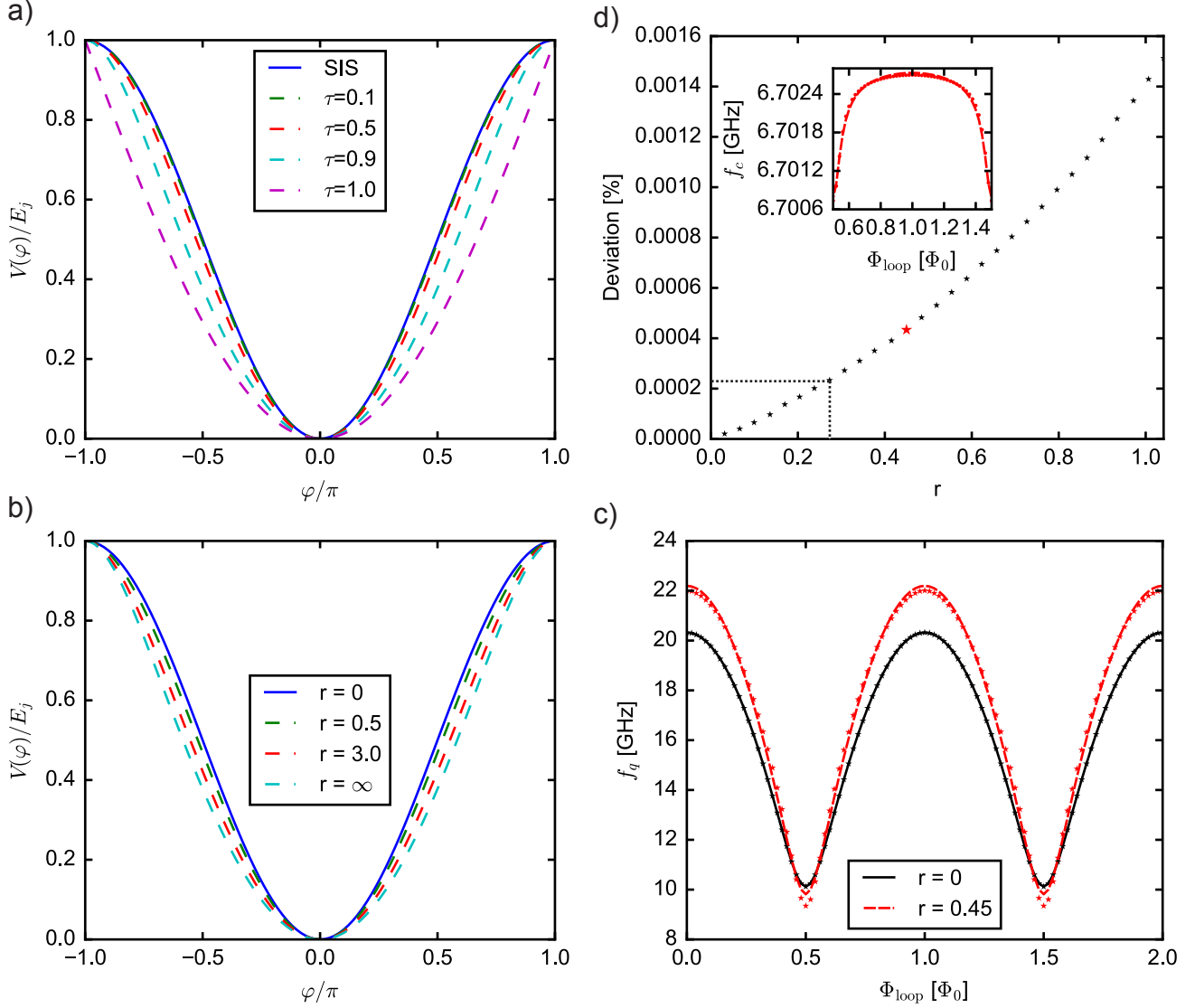


Figure 4.6: (a) Normalized Josephson potential $V(\phi)$ of single-channel SNS Josephson junctions with different transmission coefficients τ [dashed lines] ranging from 0.1 to 1 (perfect transmission) as a function of phase difference between superconducting islands ϕ . For comparison, a sinusoidal potential of SIS Josephson junctions is also shown [blue solid line]. (b) Normalized Josephson potential $V(\phi)$ of a hybrid junction with contributions from both ballistic modes ($\tau = 0.9$) and diffusive modes as a function of ϕ . With different ratios r between the ballistic and diffusive contributions, the corresponding Josephson potentials are plotted in different colors. A pure SIS junction potential ($r = 0$) is once again plotted in blue solid line. (c) Simulated qubit frequency f_q of a flux-tunable qubit modeled from our actual device as a function of applied flux Φ_{loop} through the SQUID at $r = 0$ [black] and $r = 0.45$ [red]. The dashed lines are fits of the SIS model to f_q . (d) Deviation between hybrid model simulations of readout cavity resonance f_c and SIS model fits to f_c versus r . The inset gives the simulated f_c and its SIS model fit as a function of Φ_{loop} at $r = 0.45$ [marked by the red star]. The deviation calculated from the experimental data (0.00022%) is marked by dashed lines.

4.4 Conclusion

In conclusion, we have demonstrated the compatibility of ultrahigh vacuum fabricated topological insulator Josephson junctions with cQED. By measuring a fixed-frequency TImon on BST-1959, we showed that our device fabrication based on selective area epitaxy and local stencil lithography allows for designing Josephson energies relatively reliably on a cQED compatible substrate and demonstrated qubit control and coherence. Via successful tuning of qubit frequency on a flux-tunable TImon on BST-2162, we demonstrated that the slight uncertainty in getting the designed Josephson energy can be mediated by adopting SQUIDs in the qubit circuit. This proof-of-concept work confirms the presented technique as a flexible and scalable approach to integrate TIs into superconducting circuits which could be used to study topological Josephson junctions and explore novel TI-based superconducting qubits, such as gatemons, Majorana transmons [102], or top transmons [37]. Furthermore, it paves the way for reading out of topological qubits in TI-superconductor networks.

Through analyses on the results of TImon characterizations, we also gained insights about the nature of electron transport in our devices. The large anharmonicity measured in fixed-frequency TImon together with the SIS model fit to the flux tunable TImon suggests that TI JJs in our devices are dominated by diffusive supercurrent. The low lifetimes of the fixed-frequency TImon, especially compared to the high quality factors of the test resonators on the same substrate, imply that the limiting loss mechanism is within the small junction region. Regarding these findings, we ask further questions. For example, what exactly leads to the low qubit lifetimes and can they be improved in order to meet the requirement of quantum information processing? To answer these questions, further studies need to be done. For example, the reason behind the low coherence times of TI transmon qubits can be further explored using a thorough analysis of the induced superconducting gap in TI nanoribbons on the chip or a direct probe of TLSs of a qubit device [103]. Another key question for the purpose of detecting topological superconductivity is whether we can still see signs of MZMs when our TI JJs are dominantly populated by diffusive supercurrent? In the following chapters, I continued my study on TImon devices with the aim of answering this question.

4.5 Acknowledgements

Both devices (BST-1959 and BST-1960) were fabricated by our collaborators (Tobias Schmitt, et al.) at Forschungszentrum Jülich, Germany. I conducted the characterization of the two Nb lumped element resonators at the National Physical Laboratory, Teddington with the help of Tobias Lindström and Sebastian E. de Graaf. The characterization of the fixed-frequency TImon (BST-1959) was carried out by Tobias Schmitt [89]. I measured the flux-tunable TImon at UCL mainly with the help of Oscar Kennedy. Louis Fry-Bouriaux and Katie Porsch also provided valuable advice regarding the experimental setup for the flux-tunable TImon measurements.

Chapter 5

TImon Qubit in Magnetic Field

Building on the foundational work presented in the previous chapter, where we established the integration of topological insulator (TI) Josephson junctions (JJs) within circuit quantum electrodynamics (cQED) frameworks, this chapter aims to further investigate the capability of this robust platform for the detection of topological superconductivity. Specifically, I tested the magnetic field compatibility of our TImon device and conducted a preliminary measurement looking for signatures of the Majorana zero modes (MZMs) in TImon qubit spectroscopy.

As discussed in Section 2.1.5 and 2.3.2, MZMs could emerge in TI junctions when an axial magnetic field threads an odd integer number of flux quantum through the TI nanoribbon. The coupling between MZMs on the two ends of the junction facilitates coherent single-electron transport across the junction, which generates a parity mixing effect that hybridizes the qubit transition frequency [Section 2.3.2]. According to the dimensions of the TI JJ in the fixed-frequency TImon device (BST-1959) in Chapter 4.2 ($W \approx 110$ nm, $L \approx 85$ nm and $t \approx 20$ nm), the smallest field strength that allows for the emergence of MZMs in the TImon device is given by $h/(2eWt) \approx 940$ mT. To demonstrate the field compatibility of our TImon device with the required field strength, I measured the TImon device in a dilution fridge equipped with 2-2-2 T vector magnet capable of applying 2 T field in every direction. Adjusting the supply currents for the vector magnet during the qubit measurements, an in-plane magnetic field was applied in the x-y plane that went up to 900 mT along both the x- and the y-directions, which correspond to the transverse and axial directions separately. The schematic is shown in Figure 5.1a. The application of a field in the x-direction serves as a control experiment.

In addition to the effects of magnetic fields that are intrinsic to the TI nanoribbon, the fields can penetrate the superconducting leads, which leads to Cooper pair breaking and a reduction in the superconducting gap. Since Nb is a type-II superconductor, when the applied field is small, the leads remain in the Meissner phase [Section 2.1.1]. As the field strength increases, the superconducting leads start to trap flux and lead to magnetic hysteresis originating from the magnetization of the superconducting leads [104]. During the measurements to be presented below, the in-plane fields were swept up and down, covering both positive and negative values to evaluate the magnetic hysteresis. The field sweeping order is illustrated in Figure 5.1b. Starting from zero field, we first swept B_y in the positive direction up to 900 mT. Then B_y was swept all the way down to -900 mT in the negative direction. After that, we went back to zero field and repeated the same field sweep for B_x . Before the initial sweep, a field alignment was performed.

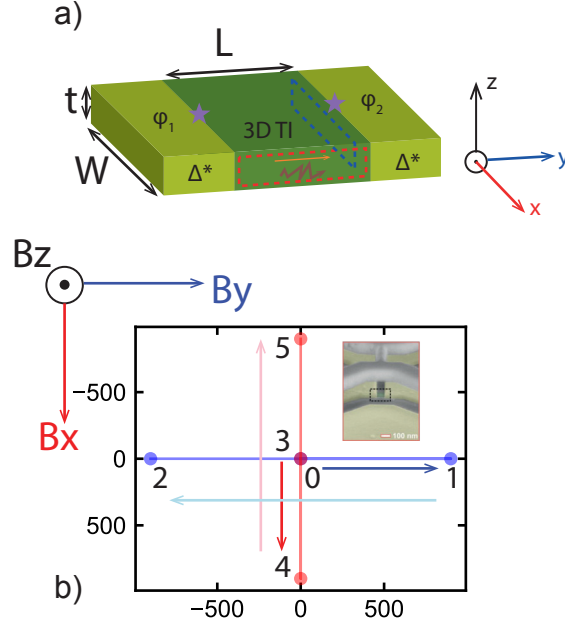


Figure 5.1: Theoretical and experimental schematics of planar TI junctions with in-plane fields. (a) Schematic of a planar 3DTI-JJ subject to magnetic fields. The two ends of the JJ (proximitized to superconductors) are presented in lighter green as two superconducting leads with an induced gap Δ^* and phases $\varphi_{1,2}$. The middle normal (non-proximitized) region is shown in darker green. The orange and brown arrowed lines indicate two types of transport modes — ballistic (orange) and diffusive (brown) in the normal region. The colored axes (x in red and y in blue) mark the plane of the planar JJ, and z -axis is the direction perpendicular to the superconducting film. Fields along x - and y - directions are transverse and axial fields. Correspondingly, two dashed rectangles in red and blue show the surface area through which the magnetic fields penetrate. When the threaded flux of the axial field $\Phi_y = h/2e$, MZMs represented by the two purple stars are expected to emerge. (b) 5-step field sweep. The field sweep started from zero field, with the vector magnet aligned along the y -direction (point 0). Then the field was swept up to 900 mT, reaching point 1. Afterwards, the field was swept in the opposite direction, all the way down to point 2 at -900 mT. After returning to the zero field, the field was rotated 90 degrees in the x - y plane so that it is now along the x -direction (point 3). The field was once again swept up to 900 mT (point 4) and finally swept down to -900 mT (point 5).

5.1 Field Compatibility and Qubit Noises

In each field point measurement, I applied the same qubit characterization protocol used in the previous chapter to assess the fixed-frequency TImon qubit device. The readout cavity was first measured twice with the readout drive power set at -55 dBm and -110 dBm. At low power, I measured the dressed cavity resonance frequency ($\tilde{f}_r$), while at high power, the bare cavity resonance frequency (f_r) is measured. Following the cavity measurements, the qubit frequency ($f_q=f_{01}$) was measured in a frequency range estimated from the power-dependent frequency difference. Afterwards, Rabi and T_1 lifetime measurements were conducted to find the qubit relaxation time. The resulting raw data of bare cavity resonance traces and qubit transition peaks at various B_x and B_y settings are depicted in Figure 5.2. It is clear from the data that both the cavity and the qubit remain operational within the specified range of magnetic fields in both the x- and y-directions. However, as field strength increases, the linewidths of the cavity resonances and the qubit transitions broaden. This broadening is anticipated due to the increased density of quasiparticles from Cooper pair breaking, resulting in enhanced losses in both the resonator and the qubit.

Further analysis is presented in Figure 5.3, which details both low- and high-power cavity transmission dips (measured at readout drive powers of -110 dBm and -55 dBm, respectively), the qubit transition peaks (measured at a qubit drive power of -80 dBm), and the T_1 decay curves of the qubit at specific field points along the y-direction ($B_y = 0$ and 900 mT). At these points, increased field strength visibly broadens and adds noise to the cavity and qubit traces, adversely affecting the precision of T_1 lifetime measurements. Notably, at $B_y = 900$ mT, the qubit's peak width at half maximum (FWHM) approximates 100 MHz, setting a lower threshold for E_M , in order for it to be detected through qubit spectroscopy. Interestingly, the qubit at zero field exhibits a longer T_1 lifetime (84 ± 1 ns) compared to the previous measurement ($T_1 \sim 28 \pm 2$ ns), aligning with the hypothesis that T_1 lifetime in our TImon qubit is limited by quasiparticles trapped in the TI JJ acting as two-level systems (qTLS) [78]. Going through thermal cycling between experimental runs, trapped quasiparticles are released and can be re-trapped with a different distribution, leading to a prolonged T_1 lifetime.

Returning to Figure 5.2, both f_q and f_c exhibit a general monotonically decreasing trend. However, within the data there are some notable discontinuities that do not conform to the general trend. For instance, in Figure 5.2a, f_c shows a one-time increase at the initial non-zero field strength throughout the whole 5-step field sweep process. This jump is likely caused by vortices initially trapped during the alignment process being expelled, and since it occurs only once, it is considered an anomaly rather than a trend. Another anomaly of f_c occurs in the field range of B_x between 600-700 mT, as shown in Figure 5.2d. The observed discontinuity is attributed to the readout cavity falling out of the readout drive frequency range (f_r). Another cavity resonator on the same chip, resonating at a slightly higher frequency, was mistakenly tracked in its place, leading to erroneous readings for this segment of the sweep. This error caused subsequent qubit measurements to be set to incorrect drive frequencies, resulting in a false disappearance of the qubit transitions, as shown in Figure 5.2e.

With the tracking error resolved during the downward sweep of the magnetic field along the x-direction (from 4 $\rightarrow$ 5 step), f_q shows a smooth dependence on B_x , as depicted in the inset of Figure 5.2e. In contrast, applying the field along the y-direction reveals several distinct discontinuities in f_q , many of which resemble qubit hybridization, as indicated in Figure 5.2b-c. These discontinuities, unlike earlier anomalies, are recurring features that consistently occur during the experiment run at similar field strengths ($B_y \approx 240$ and $640 - 680$ mT), irrespective of the sweep direction or the sign of the field strength. The higher field, where the discontinuities occur, approximately triples the lower field, suggesting that these are periodic hybridization events of the qubit transition. Interestingly, this periodicity aligns with the expected emergence of Majorana zero modes (MZMs), which occur when the threaded flux of an axial field B_y equals multiples of $\frac{h}{2e}, \frac{3h}{2e}, \frac{5h}{2e}, \dots$. Despite this alignment, the presence of MZMs is unlikely to account for the discontinuities, as the field strength where the discontinuities first occur ($B_y \sim 240$ mT) is too low to induce a flux quantum through the TI nanoribbon based on the calculated field strength of approximately 940 mT required by the device geometry.

Due to the mismatch in the field strength where the discontinuities occur, the explanations of the discontinuities remain elusive. To find the plausible mechanisms that could explain the

discontinuities, it is helpful to summarize their properties, mainly being (1) the discontinuities resemble qubit hybridization, and (2) they only occur under y-direction field. To satisfy the properties above, the discontinuities should be results of mechanisms local to the qubit and sensitive to the field direction. As we shall see in the next section, there exists a native magnetic anisotropy in the field direction due to field misalignment such that the same magnitude in the x- and y-directions gives different out-of-plane components. Therefore, the mechanisms could be either inherently sensitive to the field direction or just sensitive to the field strength.

The ambiguous nature of the observed discontinuities presents an issue with the detection of MZMs using a fixed-frequency TImon device, plainly from the observation of periodic qubit hybridizations. To robustly detect MZMs, additional control mechanisms that can tune and distinguish the signatures of MZMs are necessary. This requirement underscores the need for a more sophisticated experimental setup or device architecture that can definitively isolate and identify the unique properties of MZMs.

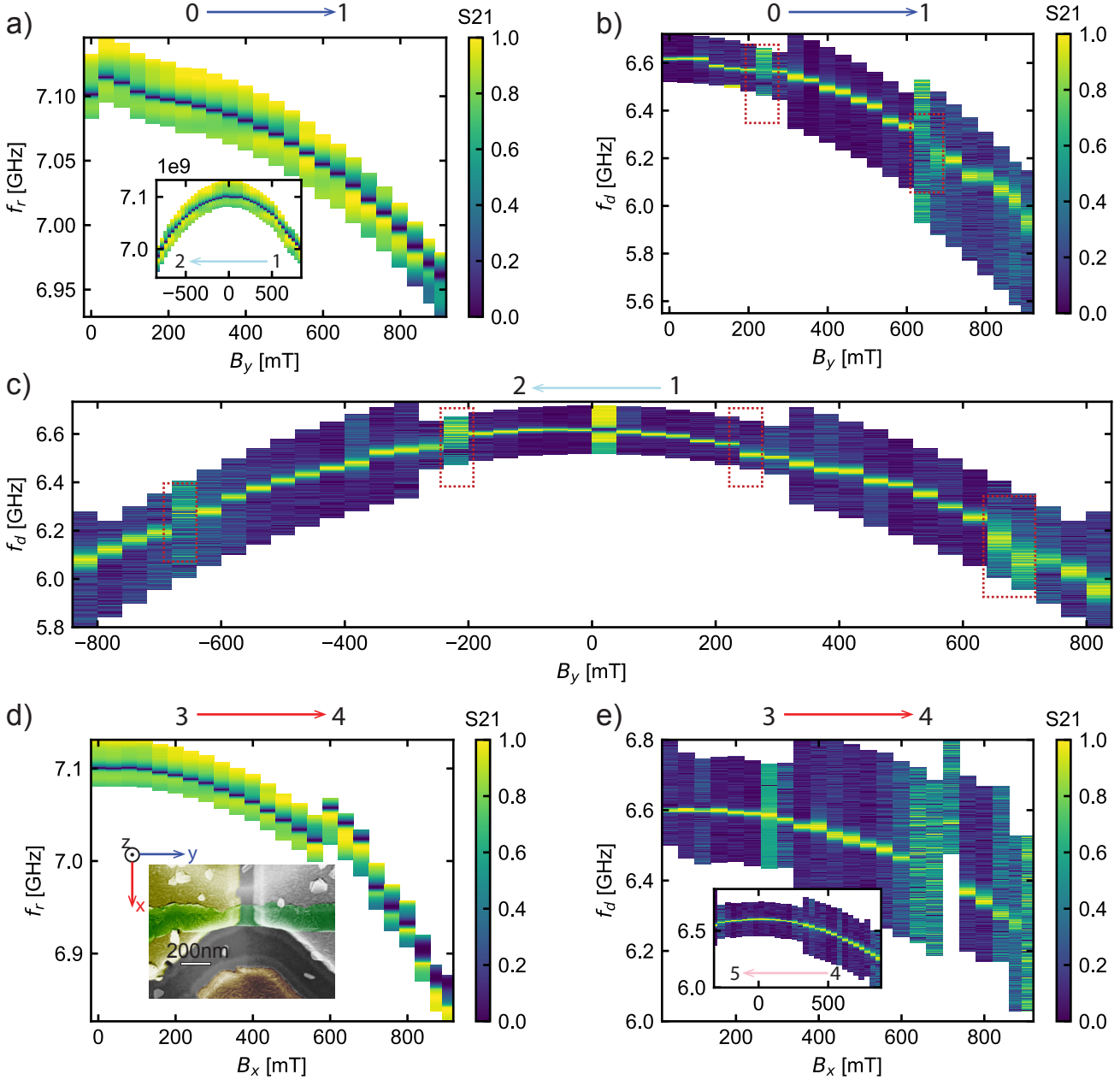


Figure 5.2: Qubit and cavity in parallel axial magnetic field. (a) S21 response of one readout cavity on our chip as a function of the cavity drive frequency f_r and the strength of the applied parallel axial magnetic field B_y in single-tone measurements. The inset of (a) S21 response of the readout cavity when B_y is swept in the opposite direction. The readout drive power p_r is set at -55 dBm high enough to recover the bare cavity resonance. (b) and (c) S21 response of the same cavity in two-tone measurements as a function of qubit drive frequency f_d and B_y . (d) Single-tone S21 response of the same readout cavity as a function of readout drive frequency f_r and the strength of a parallel transversal magnetic field B_x . Again, $p_r = -55$ dBm. The inset in (d) includes an SEM image of the device chip showing a 3DTI-JJ and the corresponding direction of B_y is marked in the coordinate system in color blue with respect to the device. (e) Two-tone S21 response as a function of f_d and B_x . The inset of (e) Two-tone S21 response as a function of f_d and B_x when B_x is swept in the opposite direction. In all of the figures, the sweep directions of the parallel field during measurements are indicated by arrows and marked by the start point and the end point of each sweep defined in Figure 5.1b.

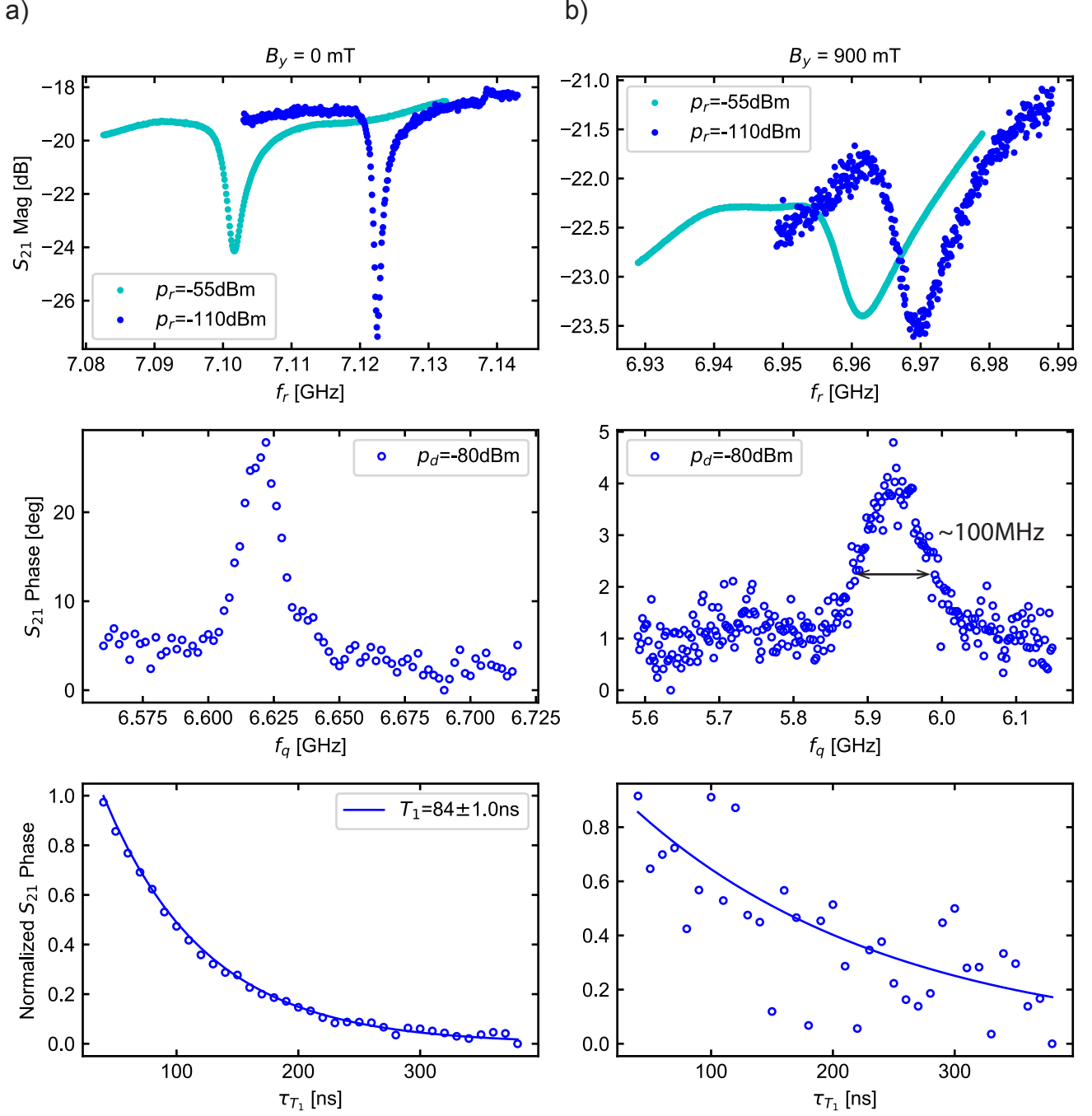


Figure 5.3: (a) Cavity resonance, qubit transition peak and T_1 lifetime at zero field and (b) at $B_y = 900$ mT.

5.2 Magnetic Anisotropy and Hysteresis

The raw data in Figure 5.2 have demonstrated the field compatibility of our TImon device. Although convincing signatures of MZMs were not observed, examining the field dependence of qubit frequency in detail, can reveal properties of our TI JJ. Although the effects of magnetic fields on TI JJs are complicated, they can be categorized into two kinds. One acts on the superconducting leads, modulating the induced gap Δ^* and the other acts on the gauge-invariant phase difference of each transport path in the normal region of TI JJs. The latter, which induces interferences, is evaluated differently depending on the type of transport being either ballistic or diffusive. The former includes a trivial magnetic effect that reduces the superconducting gap of the niobium superconducting film on top of the TI nanoribbon due to Cooper pair breaking, and other exotic effects that modify the electronic structure of the TI nanoribbon such as Aharonov-Bohm effect [105, 106], spin-orbit coupling and Zeeman effect [106, 107, 108, 109], which in turn affects the induced superconductivity.

To reveal more details in the field dependence of the qubit frequency, the qubit frequency (f_q) is extracted from the raw resonances measured in the upward sweep of B_x and B_y (corresponding to the field sweep steps $0 \rightarrow 1$ and $3 \rightarrow 4$) [Figure 5.4 a]. Since the trivial effect of field on the global niobium superconducting film would affect the cavity frequency, the cavity frequency (f_c) as a function of B_x and B_y is also extracted and serves as a control for the trivial effect. Differences in the field dependences of f_q and f_c are attributed to interference effects due to the magnetic field. As shown in Figure 5.2a, the field dependences of f_c along the x- and y-directions both follow the conventional quadratic trend expected from the trivial magnetic effect [110]. However, there is an anisotropy in the decreasing rate between the two directions. This anisotropy can be explained by misalignment of the applied field. Due to misalignment, the field applied in the x-direction might have a stronger out-of-plane component that leads to a faster decrease in f_c . The qubit frequency, within the measurement range, also exhibits a monotonically decreasing trend that is quadratic-like. Yet, the directional anisotropy of f_q is the opposite of f_c , which is demonstrated more explicitly in Figure 5.4b using an expression $\frac{f_{c/q,y}(B) - f_{c/q,x}(B)}{f_{c/q,x}(B)}$ to quantify the anisotropy, where $f_{c/q,x/y}(B)$ is the cavity/qubit frequency measured at a field strength of B with the field applied in the x/y-direction.

To ensure that the opposite anisotropy of f_c and f_q is real and replicable, the qubit and cavity frequency measured during all the field sweep steps were extracted and plotted in Figure 5.5. From Figure 5.5a, we can see that in both the upward and downward sweeps of the magnetic field, the cavity frequency is always dropping faster with the field along the x-direction. Aligning with the anisotropy of the decreasing rate, the hysteresis loop is more pronounced when the field is applied along the x-direction. As the magnetic hysteresis is normally attributed to the effect of trapped flux inside superconductors and stronger out-of-plane component induces more trapped flux inside the niobium superconducting film, the explanation for the anisotropy of f_c that B_x has a larger out-of-plane component due to a field misalignment can also account for the anisotropy observed in the magnetic hysteresis of f_c . The extracted field dependence of the cavity internal quality factor (Q_{int}) further corroborates the explanation. The non-smoothness observed in the trend of Q_{int} [the inset of Figure 5.5a] is mostly due to the entrance and exit of trapped flux. We can see that jumps in the x-direction always occur at a lower field strength than the jumps in the y-direction.

As shown in Figure 5.5b, the previously observed anisotropy in the qubit frequency that opposes the anisotropy of the cavity frequency is also present in both the upward and downward field sweeps, suggesting an existence of nontrivial magnetic effects on the TI JJ, which could stem from anisotropic interference effects or exotic effects that are specific to the TI materials. Additionally, the magnetic hysteresis loop of f_q is more prominent with the field applied along the y-direction, which is anisotropic with a trend opposing that of f_c . Since the interference effects do not induce magnetic hysteresis, the anisotropy of the magnetic hysteresis in f_q suggests the presence of exotic magnetic effects.

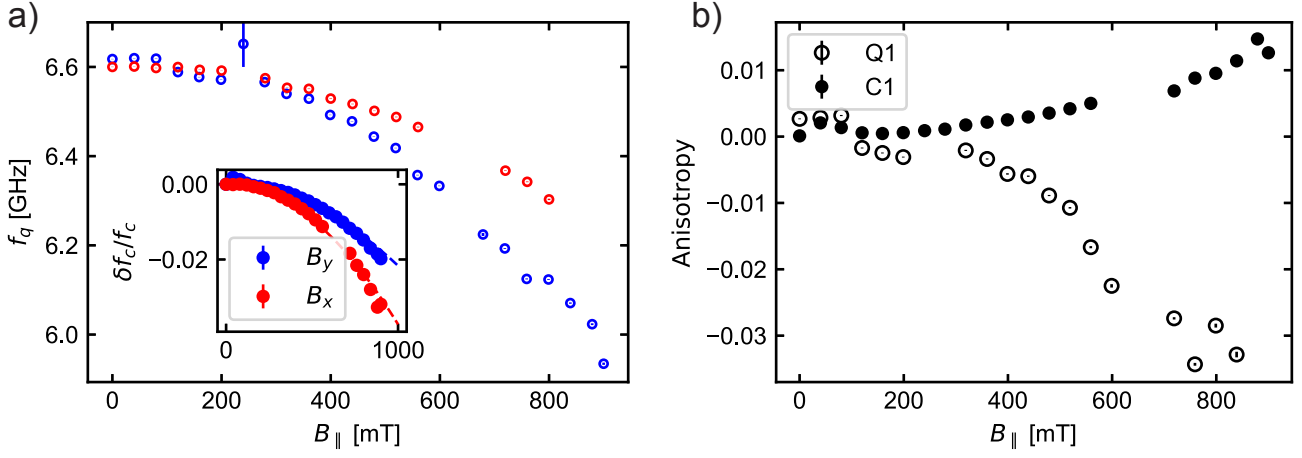


Figure 5.4: Anisotropy of qubit and cavity in parallel magnetic fields. (a) Extracted qubit frequency f_q as a function of the parallel axial field B_y [blue] and the transverse field B_x [red] being upwardly swept. The inset correspondingly plots the normalised extracted cavity resonance frequency f_c as a function of B_y and B_x . (b) Anisotropy in f_q [empty circle] and f_c [filled circle] when changing field directions between x and y as a function of the applied parallel field strength $B_{\parallel}$.

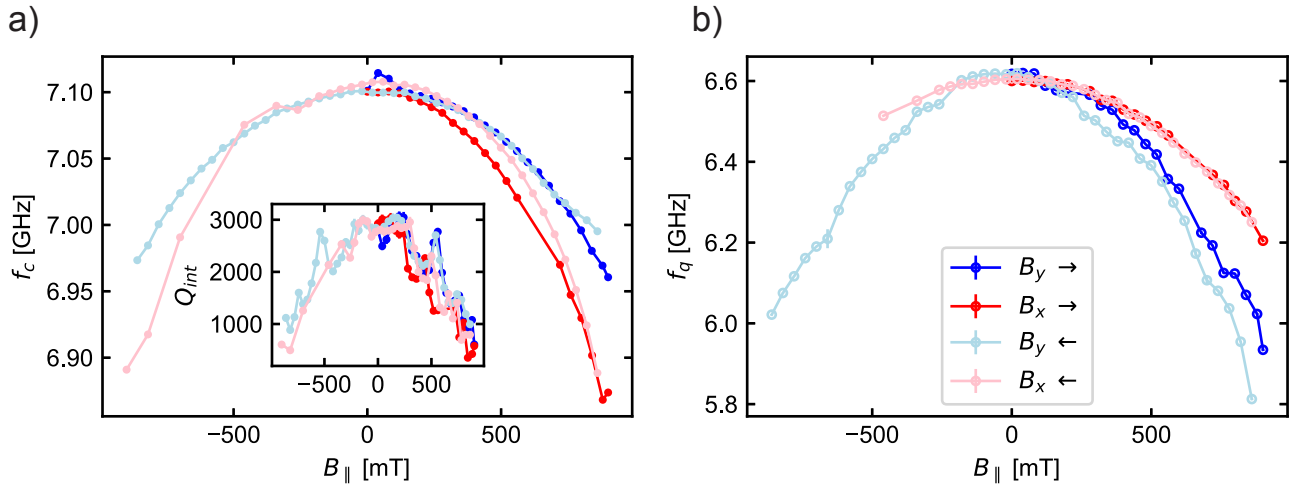


Figure 5.5: Hysteresis of qubit and cavity in parallel magnetic fields. (a) Extracted cavity frequency f_c as a function of the parallel axial field B_y [blue] and the transverse field B_x [red] during the whole sweep process. Inset of (a) Extracted cavity internal quality factor Q_i as a function of the applied in-plane field $B_{\parallel}$. (b) Extracted qubit frequency f_q as a function of $B_{\parallel}$.

5.3 Simulations on Anisotropic Interference Effects

While the anisotropy in hysteresis suggests the existence of exotic field effects, the magnitudes of these effects such as spin-orbit couplings and Zeeman effects are usually small with fields under 1T, compared to the trivial and the interference effects. This section shows that the anisotropy in the qubit frequency can be explained solely by anisotropic interference effects of magnetic fields on the supercurrent transport of TI JJs.

Figure 5.6a shows a schematic of our TI JJ. The types of supercurrent transport include ballistic transport facilitated by surface states as shown by the illustrative paths $1 \rightarrow 1'$ and $2 \rightarrow 2'$, which are localised in the top and bottom surfaces of the TI nanoribbon. Applying a transverse field (B_x) that is perpendicular to the ballistic transport paths, there exists an interference effect between the supercurrent on the top and bottom surfaces that resembles the interference effect of a SQUID device [111, 112], which is periodic with threaded flux $\Phi_x = B_x S_x$, where S_x is the surface area of our TI JJ in the y-z plane. Assuming the top- and bottom-surface supercurrents are symmetric, the interference pattern is given by [110, 113]:

$$E_J(\Phi_x) \propto \left| \cos \frac{\pi \Phi_x}{\Phi_0} \right|. \quad (5.1)$$

However, with an axial field (B_y), we expect no interference effect. As a result, an anisotropy as shown in Figure 5.6 b would be present. When simulating the field dependences in the figure, an isotropic trivial effect [Eq 2.1] is added on top of the interference pattern with the critical field (B_c) set at 3 T. More details about the simulation are given in Supplementary D.3

Additionally, TI JJs contain diffusive supercurrent transport that stems from surface states scattered by disorder inside the junctions and bulk states [Section 2.2.3]. The path $1 - 2$ in Figure 5.6a is an example of diffusive supercurrent transport. As we can see that the diffusive path gives a non-zero accumulated phase $\oint \mathbf{A} \cdot d\mathbf{l}$ for a magnetic field applied along both the x- and y-directions, where $\mathbf{A}$ is the magnetic vector potential with an arbitrary gauge choice, and is integrated along a closed contour of $d\mathbf{l}$ (such as $1 - 1' - 2 - 1$). As a result, every diffusive path with a distinct phase difference interferes with each other, which resembles Fraunhofer

diffraction.

The accumulated phase difference for each diffusive path can be rewritten in a gauge-invariant form such that $\oint \mathbf{A} \cdot d\mathbf{l} = B_x S_{y-z}$ for transverse field or $B_y S_{x-z}$ for axial field, where S_{y-z} (S_{x-z}) is the surface area enclosed by the closed path projected onto the $y-z$ ($x-z$) plane. Although the evaluation of the diffraction pattern requires extensive calculation that averages the phase difference of all diffusive paths, the maximum accumulated phase differences with B_x (B_y) are proportional to S_x (S_y)—the cross-sectional area of the normal region in the $y-z$ ($x-z$) plane. It is intuitive to foresee a magnetic anisotropy given $S_x \neq S_y$. Although I did not calculate the exact functional form for the diffraction pattern, to visually demonstrate the anisotropy that would arise out of the interference effect of diffusive supercurrent, the following analytic expressions based on Fraunhofer models for short and long junctions were used to plot the field dependence of the qubit frequency in Figure 5.6c:

$$E_J(\Phi) \propto \left| \frac{\sin \frac{\pi\Phi}{\Phi_0}}{\frac{\pi\Phi}{\Phi_0}} \right| \quad (\text{short limit [110]}), \quad E_J(\Phi) \propto \frac{\frac{\pi\Phi}{\sqrt{3}\Phi_0}}{\sinh \frac{\pi\Phi}{\sqrt{3}\Phi_0}} \quad (\text{long limit [114]}) \quad (5.2)$$

$\Phi = B_{x,y} S_{x,y}$ and $S_{x,y}$ are set to be the the geometric surface areas of the TI JJ in our qubit device. Again an isotropic trivial magnetic effect with $B_c = 3$ T is included in the simulation [Supplementary D.3]. As Figure 5.6c shows, in the long junction limit, the oscillating diffraction pattern becomes a monotonically decreasing trace.

As we can see from Figure 5.6b and c, the anisotropy is reversed when the JJ goes from the ballistic limit to the diffusive limit. For Josephson junctions that include both ballistic and diffusive contributions, the overall anisotropy depends on the exact ratio between the two contributions. If the overall anisotropy agrees with the diffusive limit, and the ballistic contribution is non-negligible, a kink is expected in the field dependence of the anisotropy with the turning point at the antinode of the ballistic interference pattern. Comparing the experimental data with the simulation results, where we do not observe oscillatory patterns or kinks, suggests that the TI JJ in our qubit device is in the long junction limit with predominantly diffusive supercurrent. However, it is important to note that the analysis above neglects other exotic magnetic effects. Moreover, as the more realistic schematic in Figure 5.6a shows, there is a

height difference ($d = 35\text{nm}$) between the centers of the nanoribbon and of the superconducting leads. An additional phase arises due to the magnetic flux penetrating the area dW formed between the leads and the TI nanoribbon due to this height difference, where W is the width of the TI junction [107]. The interference effect due to this additional phase is also ignored in the analysis above.

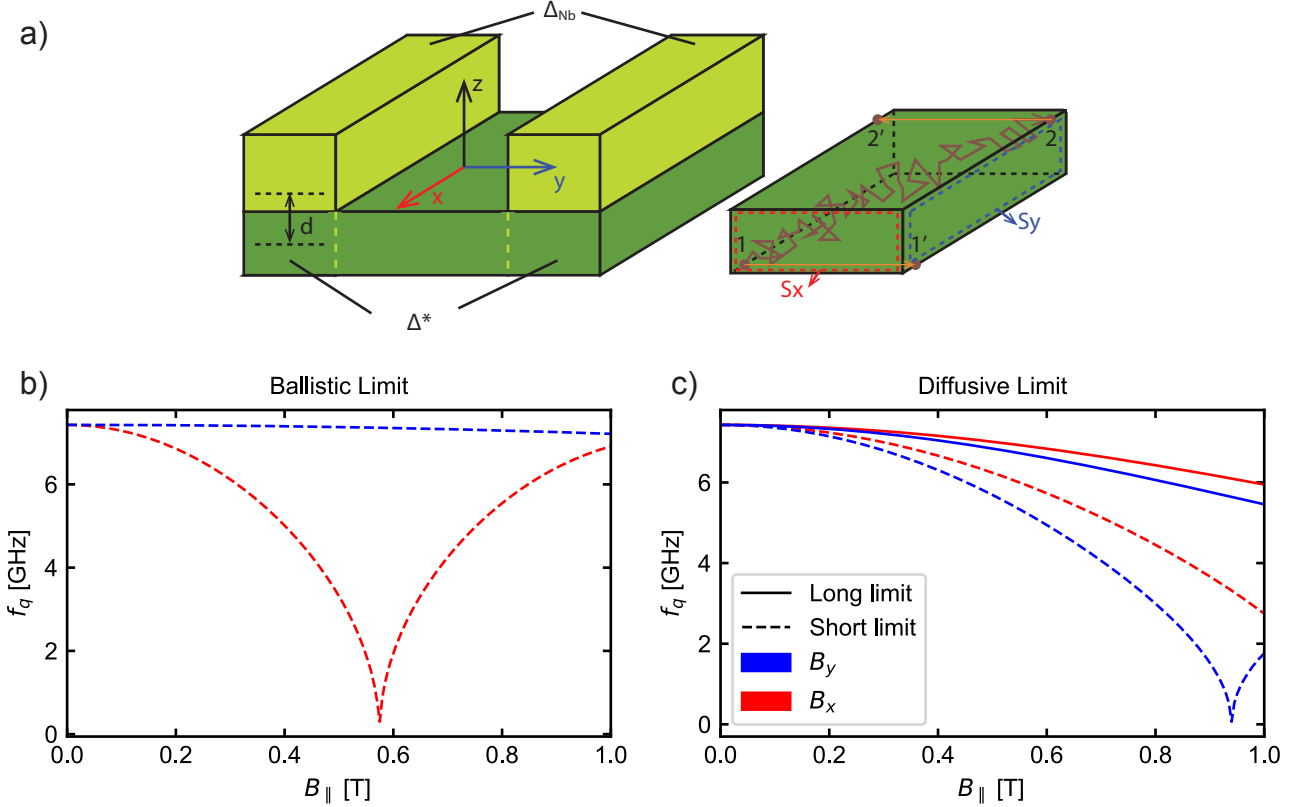


Figure 5.6: Theoretical simulation of the parallel magnetic field dependence in TImon qubits. The inset of (a) gives a visualization of the model used in the simulation and marks the directions of applied magnetic fields x [red] and y [blue]. (a)-(c) Simulated qubit frequency f_q as a function of B_x [red] and B_y [blue] in the long limit [solid line] and short limit [dashed line] with different ratios between ballistic transport and diffusive transport $r = 0, 0.16, 0.47$. The inset of (b) plots f_q in a zoomed-in field range enclosed by a black rectangle in (b) where the threaded magnetic flux Φ_y by B_y is close to $\Phi_0 = h/2e$.

5.4 Conclusion

In this chapter, we put our fixed-frequency TImon device under an in-plane field. The device is operational with a field strength close to the required value of the emergence of MZMs. Under this field strength, the device as a detector for Majorana couplings has a resolution around

100MHz. Although we did not see convincing signatures of MZMs in the measured qubit spectrum, a non-trivial field anisotropy was observed despite the presence of field misalignments favoring a trivial anisotropy with the opposite trend. A long diffusive junction with a geometrically dependent interference effect could account for the observed anisotropy. The results in the previous chapter that show the anharmonicity of this device $\alpha \sim E_C$ and the SIS-like flux tuning pattern of another flux-tunable TImon device support the diffusive model. Although exotic magnetic field effects could exist, it is hard to isolate them with complex interference effects present.

To optimize the measurement setting for future experiments that aim to investigate the novel properties of TI materials and to detect the signatures of MZMs via the platform of TImon qubit, we can

- conduct better field alignment to minimize the anisotropy due to trivial magnetic effects,
- include lithographically defined artificial defect sites in the niobium superconducting film to fix the Abrikosov vortices in place to minimize noises due to trapped flux and the magnetic hysteresis [115],
- enhance the ballistic contribution to simplify the interference effects, allowing the intricate natures of exotic magnetic field effects to be revealed,
- use flux-tunable or gate-tunable TImon device to include a tunability of MZMs signatures to distinguish them from trivial qubit hybridization events. Note that we might need to intentionally increase the sensitivity to charge dispersion such that the qubit is no longer in the transmon regime in order to tune the MZMs signatures via electrostatic gates on the superconducting leads [36].

Chapter 6

Temperature Dependence of TImon Qubit

In the previous chapters, we have demonstrated the feasibility of our TImon qubit platform for studying novel properties of helical surface states and detecting topological superconductivity. We failed to see convincing signs of MZMs in our first experimental attempt. However, the field anisotropy observed in the attempt plus previous qubit characterization results suggest that the TI JJs in our TImon devices are dominated by diffusive states in the long junction limit. The diffusive states hide the intricate properties of the TI nanoribbon inside the qubit.

Previous studies on DC devices with 3DTI JJs have used temperature-dependent measurements to identify types of transport modes (long/short, diffusive/ballistic) and demonstrated that the diffusive states can be tuned out via temperature since they have a lower critical temperature compared to the ballistic states [1, 85, 116]. Specifically, in [1], a 3DTI JJ was fabricated and measured by our collaborator using the same materials and fabrication process as those of the 3DTI JJs in our TImon qubits. Without embedding the 3DTI JJ inside a qubit circuitry, our collaborator identified both ballistic and diffusive supercurrent channels inside the junction, and characterized their properties including their superconducting coherence lengths (ξ), their critical temperatures (T_c) and the elastic mean free path of the diffusive channels (l_{diff}). Table 6.1 summarizes these transport properties. Inspired by the previous DC studies, in this chapter we measured another fixed-frequency TImon chip (BST-1960) at different temperatures to verify the long junction limit of our devices suggested by the previous chapters,

Length	1000 nm
Width	30 nm
I_c	14 μ A
$T_{c, \text{ball}}$	2 K
$T_{c, \text{diff}}$	4 K
ξ_{ball}	110 nm
ξ_{diff}	24.6 nm
l_{diff}	7.6 nm

Table 6.1: DC Characterization of a Stand-alone 3DTI JJ [1]

and to demonstrate the feasibility of using temperature to experimentally control the ratio between diffusive and ballistic modes in cQED measurements.

The TImon chip that we measured in this chapter, BST-1960, has the same design as BST-1959, featuring four qubit devices. We observed power-dependent resonance shifts in the readout cavities for all four qubits. Unfortunately, we failed to measure qubit frequency directly from two-tone spectroscopy but we can use the observed resonance shifts to estimate the qubit frequency. During the experiment, we repeated power-dependent single-tone spectroscopic measurements on all four readout cavities at fixed temperatures, starting from the base temperature around 10mK up to 4K. As an example of the raw data obtained, Figure 6.1 shows the drive power-dependent S21 spectra of cavity 1 measured at the base temperature [Figure 6.1a-b] and 1K [Figure 6.1c-d]. After applying a fitting routine to the low-power and high-power S21 traces [as shown in Figure 6.1b and d], we can extract the resonance frequency as well as the quality factor to study their temperature dependence. Using the simulated $g \sim 90\text{MHz}$, we can also estimate the qubit frequency and investigate how it varies with temperature.

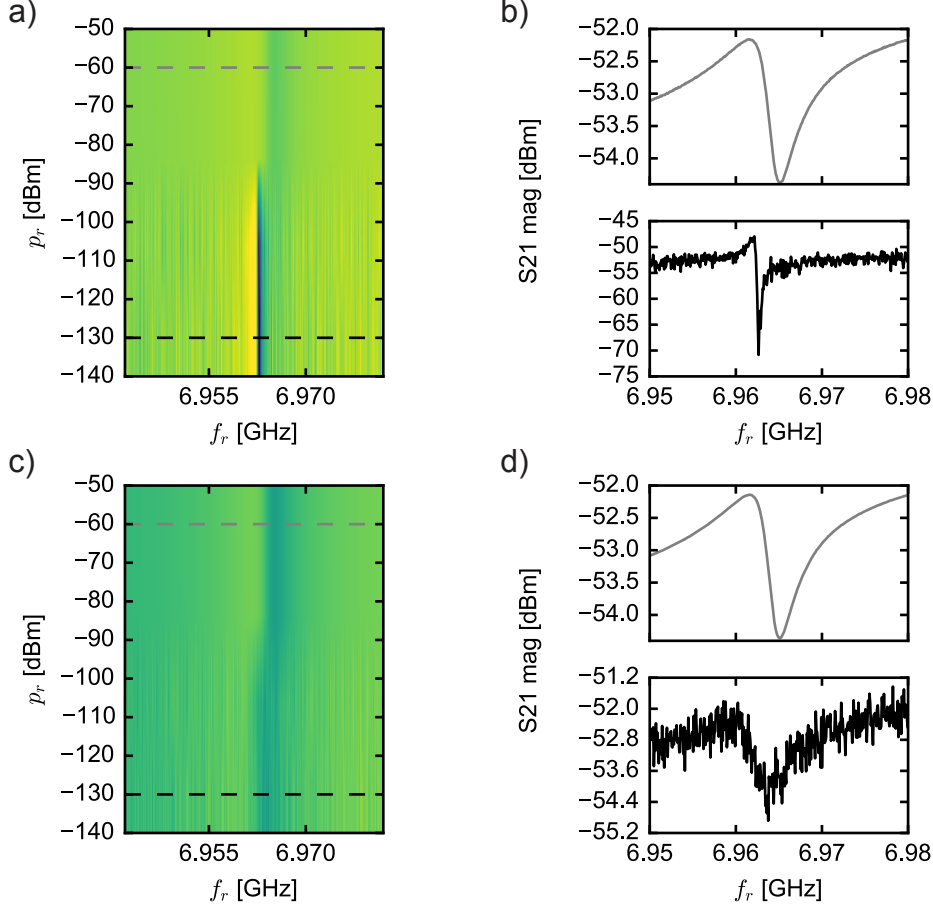


Figure 6.1: (a) S21 response at base temperature in single-tone measurements as a function of cavity drive frequency f_r and cavity drive power p_r . (b) Line cuts of (a) at $p_r = -60$ dBm [gray] and $p_r = -130$ dBm [black]. (c)-(d) S21 response at temperature $T=1$ K plotted in the same way.

6.1 Temperature Dependence of Readout Cavities

Before looking into the temperature dependence of the qubit frequency, we first checked how the readout cavities react to the change in temperature by looking at their bare resonances measured at high drive power. Figures 6.2a-b show the resonances of the extracted cavity and the quality factors as a function of the fridge temperature.

In Figure 6.2a and its inset, we see that normalized cavity frequencies $\delta f_{r,bare}$ first increase with temperature below 500 mK, and then start to drop with temperature all the way up to 4 K. The two distinctive behaviors demonstrate two effects on the cavity resonances. Firstly, as temperature increases, we start to have a higher density of quasiparticles in the superconducting film due to thermal effect as well as Cooper pair breaking, which leads to a change

in the surface impedance that reduces both the cavity resonances $f_{c,bare}$ and the quality factors Q_{int} [117]. According to the mechanism, the changing rate of $\delta f_{c,bare}$ depends on the ratio $\alpha = L_k/(L_k + L_m)$, where L_k is the kinetic inductance that depends on the film properties and varies with T, and L_m is the geometric inductance that depends on the cross-sectional geometry of our resonators [70, 117, 118]. Assuming an even superconducting film on chip BST-1960 and given the designs of our readout cavities only vary in the longitudinal lengths, we expect the temperature dependence of $\delta f_{c,bare}$ converging for all cavities as shown in Figure 6.2. Moreover, since the quasiparticle generation is negligible at very low temperature, the change in $f_{c,bare}$ and Q_{int} should also be negligible, displaying as flat plateaus. However, if we zoom in on the low-temperature range, we can observe the dominating effect of two-level-systems (TLSs) acting upon the cavity resonances. As the temperature increases, TLSs get saturated and modify $f_{c,bare}$ according to Eq 2.51. Therefore, we see a similar linearly increasing trend in the inset of Figure 6.2a as in Figure 4.1b. Overall, we see that the readout cavities on BST-1960 react to different temperatures as expected. Given their resonances and quality factors are relatively stable till 1.6K, they are able to support cQED measurements stably below 1.6K.

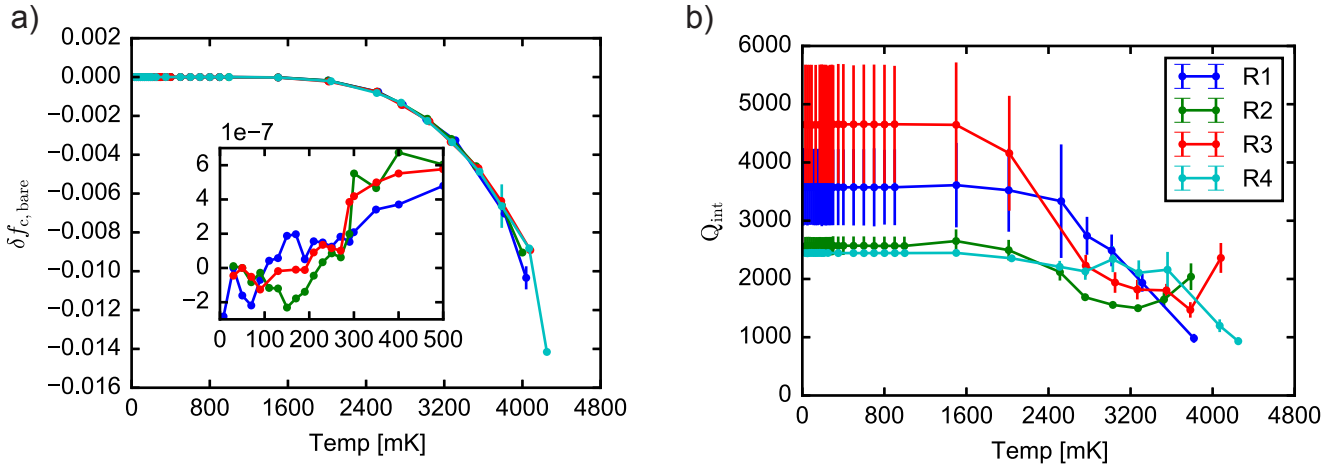


Figure 6.2: Temperature dependence of readout cavities at high power. (a) Changes in cavity resonance frequencies (in reference to the base temperature value) for all readout cavities measured at high drive power as a function of fridge temperature. The inset shows a zoomed-in plot on the low temperature range. (b) Internal quality factors Q_{int} for all readout cavities at high drive power as a function of fridge temperature.

6.2 Disappearance of Power-dependent Cavity Shift

Next, we shifted our attention to the qubit information that is hidden behind the power-dependent shifts (χ) between the high-power bare cavity resonances and the low-power dressed cavity resonances. Given $\chi \approx g^2/\Delta = g^2/(f_q - f_c)$, we can detect the temperature dependence of f_q from the temperature dependence of χ , which is plotted in Figure 6.3a. As we can see from the figure, three cavities (C1, C2 and C4) show positive χ values and one (C3) has negative χ . These indicate that $f_q^{(1,2,4)} > f_c^{1,2,4}$ and $f_q^{(3)} < f_c^{(3)}$. Using the χ -values measured at the base temperature and the $g \approx 90$ MHz found before, $f_q^{(1,2,3,4)}$ are estimated to be around 12.5, 10, 4.3, 8 GHz.

As the environmental temperature increases, we expect that the qubit frequency drops due to the suppression of the Nb superconducting gap, although the specific functional form depends on the details of the junctions. As a result, the magnitude of χ should increase for C1, C2 and C4 since Δ is now smaller with f_q moving down and closer to f_c . As for C3, χ should change in the opposite direction. In Figure 6.3a, the most prominent trend shared by all cavities is $\chi \rightarrow 0$ at certain temperatures, denoted by T_c . T_c for C3 and C4 are low such that it is hard to tell the trend of χ before its disappearance. However, C1 and C2 display the increasing trend in the magnitude of χ as what we expected, prior to $\chi \rightarrow 0$. The low-power quality factors Q_{int} of C1-4 also show a similar global trend in their temperature dependence that Q_{int} would drop with temperature and recover their high power quality factors at temperatures slightly lower than T_c [Figure 6.3b].

Figure 6.3c plots $k_B T_c$ against hf_q for qubits 1-4 (Q1-4, corresponding to C1-4). The data points display a linear trend and a linear regression fit gives a slope of 2.35. This linear correspondence suggests that the disappearance of χ is related to the excitation rates of the qubits given by $\Gamma_{\uparrow} = \exp(-hf_q/k_B T)\Gamma_{\downarrow}$, where $\Gamma_{\downarrow}$ is the decay rate of the qubits. To verify the reason behind the disappearance of χ , I built a simple qubit-cavity model using QuTiP [Supplementary D.2.4] such that the qubit is an effective two-level system with $f_q = 8$ GHz, $f_c = 7$ GHz, and $g = 100$ MHz. The cavity dissipation rate $\kappa_{\downarrow}$ is set to be 0.005, rendering a base-temp $Q_{int} \approx 1256$, and $\Gamma_{\downarrow} = 0.05$. Preliminary simulations show that varying $\kappa_{\downarrow}$ and $\Gamma_{\downarrow}$ does not

affect χ much. Therefore, in the simulation, they are considered to be constants. The only temperature dependence appears in $\Gamma_{\uparrow}$. Figure 6.3d shows χ and Q as a function of $k_B T/hf_q$ extracted from the simulated spectra plotted in the inset. The simulation replicates our observations well, and agrees with the linear regression fit result in Figure 6.3c within an acceptable margin of error. Due to the disappearance of χ , the dispersive readout scheme would fail at $T > T_c$. Therefore, if we want to measure our TImon qubits at high temperature, we can utilize the correspondence between T_c and f_q , and intentionally design qubits with higher frequencies.

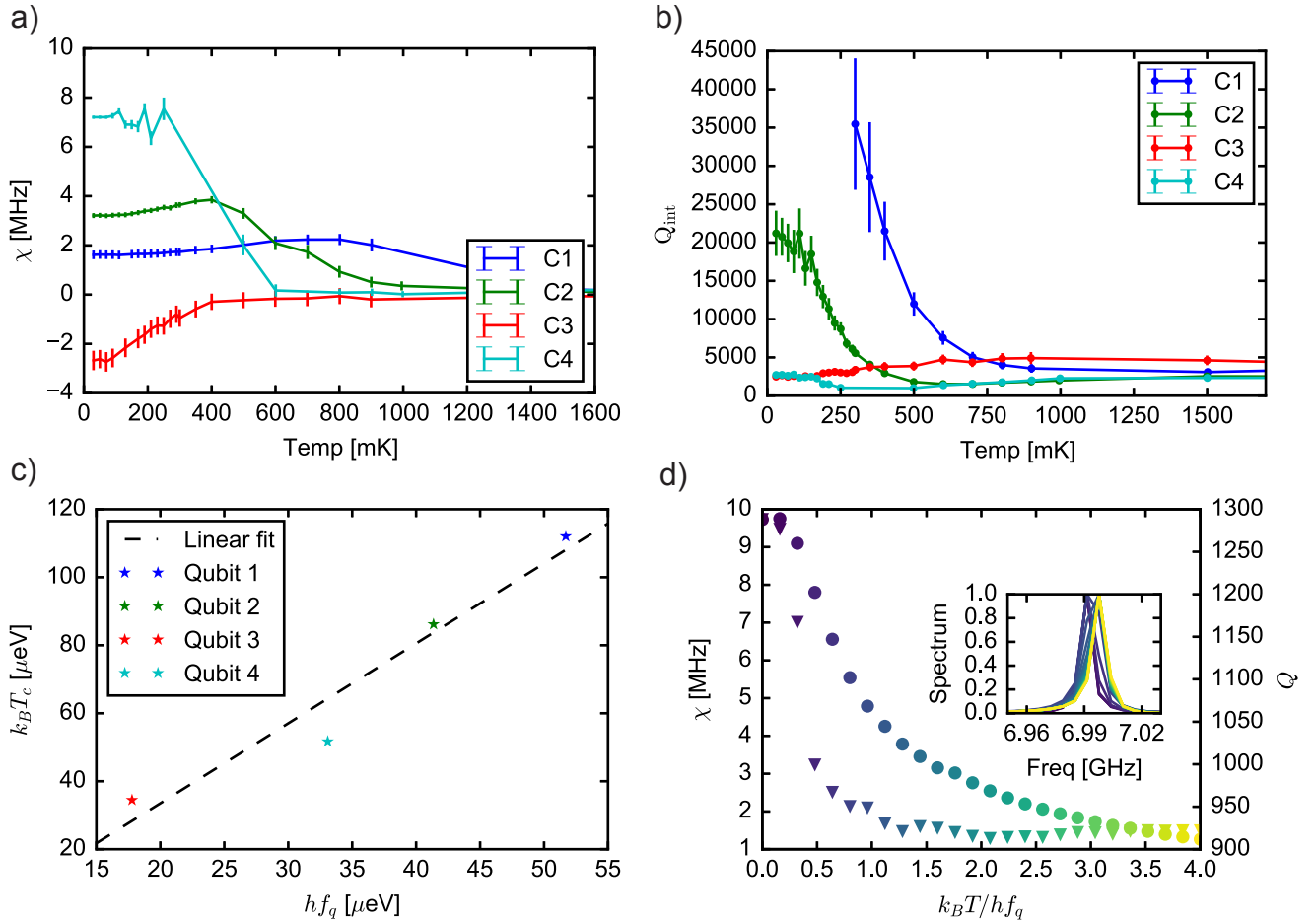


Figure 6.3: Disappearance of power-dependent cavity shift. (a) Power-dependent cavity shift χ as a function of fridge temperature for all readout cavities. (b) Q_{int} for all readout cavities at single-photon limit as a function of fridge temperature. (c) Critical temperature T_0 where $\chi \rightarrow 0$ versus qubit frequency for all qubits coupled to the readout cavities. Dashed line shows a linear fit to the data. (d) χ [circle] and quality factor Q [triangle] extracted from the simulated cavity spectrums in the inset as a function of normalised fridge temperature.

6.3 Temperature Dependence of Qubit Frequency

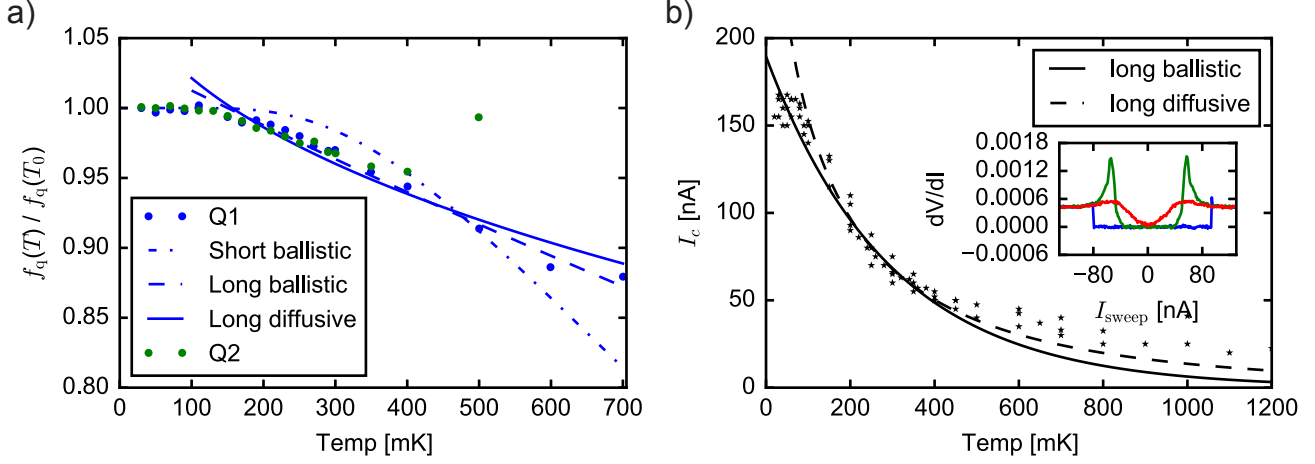


Figure 6.4: Temperature dependence of TI-mon qubit compared to TI-JJ. (a) Normalised qubit frequency of Q1 [blue] and Q2 [green] extracted from the power-dependent cavity shift of their readout cavities as a function of fridge temperature. Models in the short ballistic [dash-dotted line], long ballistic [dashed line] and long diffusive limits [solid line] are fitted to the data and plotted in different styles. (b) Critical currents I_c of a TI-JJ versus fridge temperature [black dots]. I_c is fitted to long ballistic and long diffusive junction models separately and plotted in dashed and solid lines. The inset plots the differential conductance of the TI-JJ as a function of current through the JJ from which I_c is extracted at three characteristic temperatures $T = 200\text{mK}$ [blue], $T = 400\text{mK}$ [green] and $T = 750\text{mK}$ [red].

Lastly, we looked into the temperature dependence of qubit frequency to investigate the junction properties by studying the detailed functional forms extracted from χ before it starts to disappear. Since Cavities 1 and 2 have higher T_c , therefore, we chose to extract qubit frequency for Q1 and Q2 in temperature range before T_c as shown in Figure 6.4a. We can see from the figure that after a saturating plateau extending to 150mK, both Q1 and Q2 start to decrease with temperature following a similar trend until the extraction of f_q fails for Q2 above 500mK due to the disappearance of χ . Using the qubit frequency $f_q \approx \sqrt{8E_C E_J}$ and different models of E_J , the Q1 data were fitted to junction models in the short ballistic, long ballistic and long diffusive limits. In the short ballistic junction model, the Josephson energy is the sum of high-transmitting Andreev Bound States (ABS) described by Eq 4.1 such that the temperature dependence mainly resides in the superconducting gap: $E_J(T) = E_{J, SNS}(\tau \sim 1, T) \propto \Delta^*(T)$. Having $\Delta^*(T) \propto \Delta_{\text{Nb}}(T) \propto \tanh(\Delta_{\text{Nb},0}/k_B T)$ with the last relation given by BCS theory [110], we have $f_q(T)/f_{q,0} \propto \sqrt{\tanh(\Delta_{\text{Nb},0}/k_B T)}$ in the short ballistic limit. For models in the long ballistic and diffusive limits, we use $E_J \propto I_c$ and $I_c \propto e^{-\alpha T}$ or $e^{-\beta\sqrt{T}}$ for junctions in the bal-

listic or diffusive limits [110, 119]. We fitted the data after the saturating plateau to the three models separately and found that both models in the long limit describe our experimental data much better.

My colleague, Declan Burke, also measured the critical current I_c of a DC device with TI SQUIDS of dimensions similar to those in the flux-tunable TImon qubit (BST-2162) [Figure 6.4b]. I_c as the maximum current that can flow through the junction without inducing a voltage through it, was extracted from differential resistance in the DC transport measurements. The inset of Figure 6.4b shows three differential resistance traces measured at distinct temperatures $T = 200, 400, 750\text{mK}$. By definition, I_c refers to the current where the differential resistance turns zero. However, as the temperature increases, the phase shift into supercurrent is less abrupt and less well-defined. For the sake of uniformity in extraction, we chose I_c to be the current right before the phase shift that gives the maximum differential resistance. Figure 6.4b plots the extracted I_c . Like the temperature curve of f_q in Figure 6.4a, we also observe a saturating plateau for temperatures below 150mK. The saturated I_c at low temperature is around 160nA. With $E_J \sim \hbar I_c / 2e$, if the dc device were integrated into a transmon circuit, we would end up with a qubit device with $f_q \approx 12\text{GHz}$, comparable to the qubit frequencies of Q1 and Q2 (on BST-1960). Although I_c was measured up to 1.2K, it is important to note that the high-temperature values might not be accurate due to the softer superconducting phase transition at higher temperatures.

The long-junction-limit models used to fit the qubit frequency were also applied to the DC data and both fit the data below 400mK quite well. A previous study done by our collaborators measured the superconducting coherence length $\xi = 110\text{nm}$ for larger TI JJs that have $I_c \sim 14.1\mu\text{A}$ [1] [Table 6.1]. Therefore, the coherence length of our nanoscale JJs could be much smaller such that they are indeed in the long junction limit given the junction lengths $L \sim 110\text{nm}$ [Table 6.1]. On the other hand, we noticed a deviation between the high temperature data and the long-junction-limit models. Similar deviations were also observed in previous studies [85, 1, 116], often showing as a sharp kink. Their explanation for the deviations is a change in the junction type due to the diffusive contribution gradually phasing out at a lower

temperature, leaving the ballistic contribution dominant in I_c at high temperature. Although our data do not show a sharp kink, our junctions should undergo a similar transition if both diffusive and ballistic contributions exist. Assuming that I_c is dominated by ballistic contributions $I_{c,\text{ball}}$ in the high temperature tail and $I_{c,\text{ball}}$ is relatively flat at small T , the ratio $r = I_{c,\text{ball}}/I_{c,\text{diff}} = 40/(160 - 40) \approx 0.33$ at base temperature is found. It is important to note that this is just a simple estimate that does not consider the temperature dependence of the ballistic contribution, the errors in extracting I_c at high temperatures and ignores the residual of diffusive contributions.

Despite the temperature dependences of both f_q and I_c supporting the argument of the long-junction limit, we cannot distinguish the ballistic and diffusive models solely from the temperature dependences. Therefore, we ask whether our junctions are (dominantly) ballistic or diffusive at low temperature. To answer the question, we have qubit measurements in the previous chapters supporting the TI JJs being dominantly diffusive. Moreover, the complementary flux-tuning data of the DC device measured at low temperature in Figure 6.5a. The DC flux tuning pattern is sinusoidal without apparent skewedness. As the temperature increases, we still cannot observe an apparent skewness in the high-temperature flux-tuning pattern [Figure 6.5b]. The most prominent temperature-dependent feature is the reduction of the maximum I_c with temperature. The lack of skewedness seems to suggest that we have little ballistic contribution on our TI JJs. However, the raw data in Figure 6.5a shows that I_c can be tuned to nearly zero. This indicates a very low asymmetry factor, which leads to non-skewedness despite the existence of ballistic contribution.

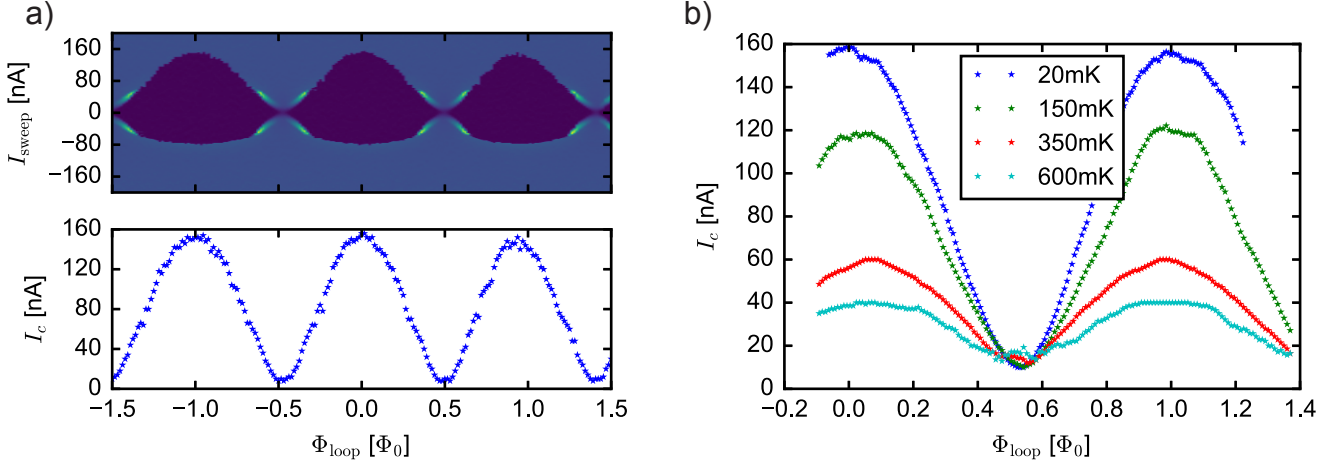


Figure 6.5: (a) Flux tuning pattern of I_c measured at base temperature. The top shows the raw data while the bottom plots the extracted I_c from the raw data as a function of calibrated external flux through the SQUID loop Φ_{loop} . (b) Flux tuning pattern of I_c at different temperatures. The x-axis is calibrated from the data at 20mK.

6.4 Conclusion

In this chapter, it is verified that our TI JJs are in the long junction limit, initially suggested in the field-dependent data in the previous chapter, from the temperature dependences obtained in the TImon and TI DC device measurements. Attempting to demonstrate the temperature tunability of the ratio between the ballistic and diffusive supercurrent, the dispersive readout scheme of qubit was found to be incompatible with high temperature. In DC measurements, without an apparent kink in the measured data that marks the transition between the diffusive and ballistic limits, and the lack of skewedness in the flux tuning pattern with a well-balanced SQUID design, it is hard to conclude on the type of transport modes and the feasibility of temperature tunability.

Chapter 7

Conclusion and outlook

In the exploration of integrating topological insulators (TIs) with superconducting qubits, the research described and analyzed in the previous chapters laid a foundation for the development and understanding of topological insulator Josephson junctions (TI JJs) within circuit quantum electrodynamics (cQED) systems. The chapters have delineated the fabrication, characterization, and operational achievements and challenges of fixed-frequency and flux-tunable TImon devices, paving the way for potential advancements in quantum information processing and Majorana zero modes (MZMs) detection. This chapter summarizes the key points in the previous chapters and offers experimental insights on future work.

Chapter 4 confirmed the compatibility of TI JJs with cQED through effective control and coherence in a fixed-frequency TI-based transmon (TImon) device (BST-1959). The device exhibits a low lifetime, indicating lifetime-limiting noises local to the qubit. The measured anharmonicity α is approximately the same as the charging energy E_C , suggesting the predominantly diffusive nature of the TI JJs was evidenced. The integration of a SQUID loop in another flux-tunable TImon device (BST-2162) further supports the diffusive nature, with its SIS-like flux-tuning pattern. Comparative analysis of simulated and experimental flux tuning patterns suggests minimal ballistic contributions, with a ratio of ballistic to diffusive contributions (r) less than 0.3.

Chapter 5 evaluated the TImon platform's feasibility for MZM detection by testing the field compatibility of the fixed-frequency TImon device (BST-1959). Enduring an in-plane field

of 900 mT—close to the threshold for MZM emergence—the device’s qubit transition linewidth of 100 MHz sets a critical lower energy threshold for Majorana coupling, E_M . However, definitive MZM signatures remained elusive, with observed non-trivial field anisotropy likely due to a geometrically dependent interference effect, indicative of the long diffusive limit in the TI JJs.

Chapter 6 tackled the challenge posed by the diffusive nature of TI JJs by attempting to modulate the diffusive supercurrent via high temperatures in another fixed-frequency TImon device (BST-1960) and a TI flux-tunable DC device. Although the field dependences presented in the TImon and the DC device support the long junction limit of TI JJs, they did not successfully demonstrate the temperature tunability on the transport nature of the TI JJs due to the current dispersive qubit readout scheme’s incompatibility with high temperatures and the limitations of the balanced SQUID design.

Addressing the issue of diffusiveness remains a pivotal objective. Future efforts could focus on demonstrating temperature tunability within a qubit device, necessitating an optimized qubit design compatible with higher temperatures and incorporating an imbalanced SQUID to showcase ballistic supercurrent presence. Notably, the flux-tunable TImon device (BST-2162) meets these criteria, with an operational frequency above 16 GHz, aligning with the temperature stability of the readout cavity at 1.6 K, setting the stage for subsequent field dependence measurements.

Parallel to temperature-based strategies, integrating electrostatic gating with TImon qubits could further reduce the diffusive contribution. Electrostatic gates, placed over a TI nanoribbon, can finely tune the Fermi level away from the bulk states, diminishing their diffusive contribution. Additionally, such gates can mitigate potential mismatches across TI JJs, potentially enhancing surface states’ transmission efficiency and reducing backscattering.

Bibliography

- [1] P. Schüffegen, D. Rosenbach, C. Li, T. W. Schmitt, M. Schleenvoigt, A. R. Jalil, S. Schmitt, J. Kölzer, M. Wang, B. Bennemann, U. Parlak, L. Kibkalo, S. Trellenkamp, T. Grap, D. Meertens, M. Luysberg, G. Mussler, E. Berenschot, N. Tas, A. A. Golubov, A. Brinkman, T. Schäpers, and D. Grützmacher. Selective area growth and stencil lithography for in situ fabricated quantum devices. *Nature Nanotechnology*, 14(9):825–831, 2019.
- [2] G. E. Moore. Cramming more components onto integrated circuits, reprinted from electronics, volume 38, number 8, april 19, 1965, pp.114 ff. *IEEE Solid-State Circuits Society Newsletter*, 11(3):33–35, 2006.
- [3] G. E. Moore. Progress in digital integrated electronics [technical literaiture, copyright 1975 ieee. reprinted, with permission. technical digest. international electron devices meeting, ieee, 1975, pp. 11-13.]. *IEEE Solid-State Circuits Society Newsletter*, 11(3):36–37, 2006.
- [4] Y. Han, H. Xu, M. Lu, H. Wang, J. Huang, Y. Wang, Y. Wang, F. Min, Q. Liu, M. Liu, and N. Sun. The big chip: Challenge, model and architecture. *Fundamental Research*, 2023.
- [5] R. P. Feynman. Simulating physics with computers. *International Journal of Theoretical Physics*, 21(6):467–488, Jun 1982.
- [6] P. Shor. Algorithms for quantum computation: discrete logarithms and factoring. In *Proceedings 35th Annual Symposium on Foundations of Computer Science*, pages 124–134, 1994.

- [7] G. Wendin. Quantum information processing with superconducting circuits: a review. *Reports on Progress in Physics*, 80(10):106001, Sept. 2017.
- [8] P. Krantz, M. Kjaergaard, F. Yan, T. P. Orlando, S. Gustavsson, and W. D. Oliver. A quantum engineer’s guide to superconducting qubits. *Applied Physics Reviews*, 6(2):021318, 2019.
- [9] I. Acharya, Rajeev Aleiner and et al. Suppressing quantum errors by scaling a surface code logical qubit. *Nature*, 614(7949):676–681, Feb 2023.
- [10] A. Y. Kitaev. Unpaired majorana fermions in quantum wires. *Physics-Uspekhi*, 44(10S):131–136, Oct. 2001.
- [11] Anton Akhmerov, Jay Sau, Bernard van Heck, Sebastian Rubbert, Rafał Skolasinski, Bas Nijholt, Irfan Muhammad, Tómas Örn Rosdahl. topology in condensed matter: tying quantum knots, 2021.
- [12] E. Witten. Quantum field theory and the Jones polynomial. *Communications in Mathematical Physics*, 121(3):351 – 399, 1989.
- [13] G. Moore and N. Read. Nonabelions in the fractional quantum hall effect. *Nuclear Physics B*, 360(2):362–396, 1991.
- [14] M. Z. Hasan and C. L. Kane. Colloquium: Topological insulators. *Reviews of Modern Physics*, 82:3045, 2010.
- [15] P. M. Ostrovsky, I. V. Gornyi, and A. D. Mirlin. Interaction-induced criticality in z_2 topological insulators. *Phys. Rev. Lett.*, 105:036803, Jul 2010.
- [16] L. Fu and C. L. Kane. Superconducting proximity effect and majorana fermions at the surface of a topological insulator. *Physical Review Letters*, 100(9), 2008.
- [17] L. A. Jauregui, M. T. Pettes, L. P. Rokhinson, L. Shi, and Y. P. Chen. Magnetic field-induced helical mode and topological transitions in a topological insulator nanoribbon. *Nat Nanotechnol*, 11(4):345–51, 2016. Jauregui, Luis A Pettes, Michael T Rokhinson,

- Leonid P Shi, Li Chen, Yong P eng Research Support, Non-U.S. Gov't Research Support, U.S. Gov't, Non-P.H.S. England Nat Nanotechnol. 2016 Apr;11(4):345-51. doi: 10.1038/nnano.2015.293. Epub 2016 Jan 18.
- [18] A. Cook and M. Franz. Majorana fermions in a topological-insulator nanowire proximity-coupled to ans-wave superconductor. *Physical Review B*, 84(20), 2011.
 - [19] N. Read and D. Green. Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum hall effect. *Phys. Rev. B*, 61:10267–10297, Apr 2000.
 - [20] K. T. Law, P. A. Lee, and T. K. Ng. Majorana fermion induced resonant andreev reflection. *Phys. Rev. Lett.*, 103:237001, Dec 2009.
 - [21] S. D. Sarma, M. Freedman, and C. Nayak. Majorana zero modes and topological quantum computation. *npj Quantum Information*, 1(1):15001, Oct 2015.
 - [22] H. Pan and S. Das Sarma. Physical mechanisms for zero-bias conductance peaks in majorana nanowires. *Phys. Rev. Res.*, 2:013377, Mar 2020.
 - [23] C.-X. Liu, J. D. Sau, T. D. Stanescu, and S. Das Sarma. Andreev bound states versus majorana bound states in quantum dot-nanowire-superconductor hybrid structures: Trivial versus topological zero-bias conductance peaks. *Phys. Rev. B*, 96:075161, Aug 2017.
 - [24] C. Moore, C. Zeng, T. D. Stanescu, and S. Tewari. Quantized zero-bias conductance plateau in semiconductor-superconductor heterostructures without topological majorana zero modes. *Phys. Rev. B*, 98:155314, Oct 2018.
 - [25] M. Valentini, F. Peñaranda, A. Hofmann, M. Brauns, R. Hauschild, P. Krogstrup, P. San-Jose, E. Prada, R. Aguado, and G. Katsaros. Nontopological zero-bias peaks in full-shell nanowires induced by flux-tunable andreev states. *Science*, 373(6550):82–88, 2021.
 - [26] M. T. Deng, C. L. Yu, G. Y. Huang, M. Larsson, P. Caroff, and H. Q. Xu. Anomalous zero-bias conductance peak in a nb–insb nanowire–nb hybrid device. *Nano Letters*, 12(12):6414–6419, Dec 2012.

- [27] M. T. Deng, S. Vaitiekėnas, E. B. Hansen, J. Danon, M. Leijnse, K. Flensberg, J. Nygård, P. Krogstrup, and C. M. Marcus. Majorana bound state in a coupled quantum-dot hybrid-nanowire system. *Science*, 354(6319):1557–1562, 2016.
- [28] W. Yu, R. Haenel, M. A. Rodriguez, S. R. Lee, F. Zhang, M. Franz, D. I. Pikulin, and W. Pan. Zero-bias conductance peak in dirac semimetal-superconductor devices. *Phys. Rev. Res.*, 2:032002, Jul 2020.
- [29] R. M. Lutchyn, J. D. Sau, and S. Das Sarma. Majorana fermions and a topological phase transition in semiconductor-superconductor heterostructures. *Phys. Rev. Lett.*, 105:077001, Aug 2010.
- [30] L. Fu. Electron teleportation via majorana bound states in a mesoscopic superconductor. *Phys. Rev. Lett.*, 104:056402, Feb 2010.
- [31] L. P. Rokhinson, X. Liu, and J. K. Furdyna. The fractional a.c. josephson effect in a semiconductor–superconductor nanowire as a signature of majorana particles. *Nature Physics*, 8(11):795–799, Nov 2012.
- [32] I. T. Rosen, C. J. Trimble, M. P. Andersen, E. Mikheev, Y. Li, Y. Liu, L. Tai, P. Zhang, K. L. Wang, Y. Cui, M. A. Kastner, J. R. Williams, and D. Goldhaber-Gordon. Fractional ac josephson effect in a topological insulator proximitized by a self-formed superconductor, 2021.
- [33] J. Wiedenmann, E. Bocquillon, R. S. Deacon, S. Hartinger, O. Herrmann, T. M. Klapwijk, L. Maier, C. Ames, C. Brüne, C. Gould, A. Oiwa, K. Ishibashi, S. Tarucha, H. Buhmann, and L. W. Molenkamp. 4π -periodic josephson supercurrent in hgte-based topological josephson junctions. *Nature Communications*, 7(1):10303, Jan 2016.
- [34] C.-K. Chiu and S. Das Sarma. Fractional josephson effect with and without majorana zero modes. *Physical Review B*, 99(3), 2019.
- [35] D. Sticlet, J. D. Sau, and A. Akhmerov. Dissipation-enabled fractional josephson effect. *Phys. Rev. B*, 98:125124, Sep 2018.

- [36] E. Ginossar and E. Grosfeld. Microwave transitions as a signature of coherent parity mixing effects in the majorana-transmon qubit. *Nature Communications*, 5(1):4772, 2014.
- [37] F. Hassler, A. R. Akhmerov, and C. W. J. Beenakker. The top-transmon: a hybrid superconducting qubit for parity-protected quantum computation. *New Journal of Physics*, 13(9):095004, 2011.
- [38] P. Bonderson and R. M. Lutchyn. Topological quantum buses: Coherent quantum information transfer between topological and conventional qubits. *Phys. Rev. Lett.*, 106:130505, Mar 2011.
- [39] H. London and F. London. The electromagnetic equations of the supraconductor. *Proceedings of the Royal Society A*, 149:71–88, 1935.
- [40] M. Tinkham. *Introduction to Superconductivity, Chapter 4*. Dover Publications, 2 edition, June 2004.
- [41] L. N. Cooper. Bound electron pairs in a degenerate fermi gas. *Phys. Rev.*, 104:1189–1190, Nov 1956.
- [42] J. Bardeen, L. N. Cooper, and J. R. Schrieffer. Theory of superconductivity. *Phys. Rev.*, 108:1175–1204, Dec 1957.
- [43] N. N. Bogoljubov. On a new method in the theory of superconductivity. *Il Nuovo Cimento (1955-1965)*, 7(6):794–805, Mar 1958.
- [44] M. Tinkham. *Introduction to Superconductivity, Chapter 3*, chapter 3. Dover Publications, 2 edition, June 2004.
- [45] M. Tinkham. *Introduction to Superconductivity, Chapter 10*. Dover Publications, 2 edition, June 2004.
- [46] H. Zhang, C.-X. Liu, X.-L. Qi, X. Dai, Z. Fang, and S.-C. Zhang. Topological insulators in Bi_2Se_3 , Bi_2Te_3 and Sb_2Te_3 with a single dirac cone on the surface. *Nature Physics*, 5(6):438–442, 2009.
- [47] C.-X. Liu, X.-L. Qi, H. Zhang, X. Dai, Z. Fang, and S.-C. Zhang. Model hamiltonian for topological insulators. *Phys. Rev. B*, 82:045122, Jul 2010.

- [48] K. Moors, P. Schüffegen, D. Rosenbach, T. Schmitt, T. Schäpers, and T. L. Schmidt. Magnetotransport signatures of three-dimensional topological insulator nanostructures. *Physical Review B*, 97(24), 2018.
- [49] D. Heffels, D. Burke, M. R. Connolly, P. Schüffegen, D. Grützmacher, and K. Moors. Robust and fragile majorana bound states in proximitized topological insulator nanoribbons. *Nanomaterials*, 13(4), 2023.
- [50] P. Roushan, J. Seo, C. V. Parker, Y. S. Hor, D. Hsieh, D. Qian, A. Richardella, M. Z. Hasan, R. J. Cava, and A. Yazdani. Topological surface states protected from backscattering by chiral spin texture. *Nature*, 460(7259):1106–1109, Aug. 2009.
- [51] S.-Q. SHEN, W.-Y. SHAN, and H.-Z. LU. Topological insulator and the dirac equation. *SPIN*, 01(01):33–44, June 2011.
- [52] T. D. Stanescu, J. D. Sau, R. M. Lutchyn, and S. Das Sarma. Proximity effect at the superconductor–topological insulator interface. *Phys. Rev. B*, 81:241310, Jun 2010.
- [53] A. R. A. Christoph W Groth, Michael Wimmer and X. Waintal. Kwant: a software package for quantum transport. *New Journal of Physics*, 16, 2014.
- [54] M. Kotulla and U. Zülicke. Manipulating topological-insulator properties using quantum confinement. *New Journal of Physics*, 19(7):073025, 2017.
- [55] F. Münning, O. Breunig, H. F. Legg, S. Roitsch, D. Fan, M. Rößler, A. Rosch, and Y. Ando. Quantum confinement of the dirac surface states in topological-insulator nanowires. *Nature Communications*, 12(1):1038, Feb 2021.
- [56] Y. Zhang, K. He, C.-Z. Chang, C.-L. Song, L.-L. Wang, X. Chen, J.-F. Jia, Z. Fang, X. Dai, W.-Y. Shan, S.-Q. Shen, Q. Niu, X.-L. Qi, S.-C. Zhang, X.-C. Ma, and Q.-K. Xue. Crossover of the three-dimensional topological insulator Bi_2Se_3 to the two-dimensional limit. *Nature Physics*, 6(8):584–588, Aug 2010.
- [57] Y.-Y. Li, G. Wang, X.-G. Zhu, M.-H. Liu, C. Ye, X. Chen, Y.-Y. Wang, K. He, L.-L. Wang, X.-C. Ma, H.-J. Zhang, X. Dai, Z. Fang, X.-C. Xie, Y. Liu, X.-L. Qi, J.-F. Jia,

- S.-C. Zhang, and Q.-K. Xue. Intrinsic topological insulator Bi_2Te_3 thin films on Si and their thickness limit. *Advanced Materials*, 22(36):4002–4007, 2010.
- [58] B. Josephson. Possible new effects in superconductive tunnelling. *Physics Letters*, 1(7):251–253, 1962.
- [59] B. Josephson. Supercurrents through barriers. *Advances in Physics*, 14(56):419–451, 1965.
- [60] B. Pannetier and H. Courtois. Andreev reflection and proximity effect, 1999.
- [61] M. Hays. *Andreev Reflection, Andreev Levels, and the Josephson Effect*, pages 69–80. Springer International Publishing, 2021.
- [62] M. Snelder, M. Veldhorst, A. A. Golubov, and A. Brinkman. Andreev bound states and current-phase relations in three-dimensional topological insulators. *Phys. Rev. B*, 87:104507, Mar 2013.
- [63] M. Tinkham. *Introduction to Superconductivity, Chapter 6*. Dover Publications, 2 edition, June 2004.
- [64] J. Koch, T. M. Yu, J. Gambetta, A. A. Houck, D. I. Schuster, J. Majer, A. Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf. Charge-insensitive qubit design derived from the Cooper pair box. *Physical Review A*, 76(4):042319, 2007. PRA.
- [65] V. Bouchiat, D. Vion, P. Joyez, D. Esteve, and M. H. Devoret. Quantum coherence with a single Cooper pair. *Physica Scripta*, 1998(T76):165, Jan 1998.
- [66] Y. Nakamura, Y. A. Pashkin, and J. S. Tsai. Coherent control of macroscopic quantum states in a single-Cooper-pair box. *Nature*, 398(6730):786–788, Apr 1999.
- [67] A. Blais, A. L. Grimsmo, S. M. Girvin, and A. Wallraff. Circuit quantum electrodynamics. *Rev. Mod. Phys.*, 93:025005, May 2021.
- [68] A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf. Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation. *Physical Review A*, 69(6), 2004.

- [69] H. Walther, B. T. H. Varcoe, B.-G. Englert, and T. Becker. Cavity quantum electrodynamics. *Reports on Progress in Physics*, 69(5):1325, apr 2006.
- [70] K. Watanabe, K. Yoshida, T. Aoki, and S. Kohjiro. Kinetic inductance of superconducting coplanar waveguides. *Japanese Journal of Applied Physics*, 33(Part 1, No. 10):5708–5712, 1994.
- [71] J. Burnett, L. Faoro, and T. Lindström. Analysis of high quality superconducting resonators: consequences for tls properties in amorphous oxides. *Superconductor Science and Technology*, 29(4):044008, 2016.
- [72] D. P. Pappas, M. R. Vissers, D. S. Wisbey, J. S. Kline, and J. Gao. Two level system loss in superconducting microwave resonators. *IEEE Transactions on Applied Superconductivity*, 21(3):871–874, 2011.
- [73] A. Kringhøj, L. Casparis, M. Hell, T. W. Larsen, F. Kuemmeth, M. Leijnse, K. Flensberg, P. Krogstrup, J. Nygård, K. D. Petersson, and C. M. Marcus. Anharmonicity of a superconducting qubit with a few-mode josephson junction. *Physical Review B*, 97(6), 2018.
- [74] M. Devoret, B. Huard, R. Schoelkopf, and L. F. Cugliandolo. *Quantum Machines: Measurement and Control of Engineered Quantum Systems: Lecture Notes of the Les Houches Summer School: Volume 96, July 2011*. Oxford University Press, 06 2014.
- [75] J. Johansson, P. Nation, and F. Nori. Qutip 2: A python framework for the dynamics of open quantum systems. *Computer Physics Communications*, 184(4):1234–1240, 2013.
- [76] A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf. Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation. *Physical Review A*, 69(6), 2004.
- [77] J. M. Martinis, K. B. Cooper, R. McDermott, M. Steffen, M. Ansmann, K. D. Osborn, K. Cicak, S. Oh, D. P. Pappas, R. W. Simmonds, and C. C. Yu. Decoherence in josephson qubits from dielectric loss. *Phys. Rev. Lett.*, 95:210503, Nov 2005.

- [78] S. E. de Graaf, L. Faoro, L. B. Ioffe, S. Mahashabde, J. J. Burnett, T. Lindström, S. E. Kubatkin, A. V. Danilov, and A. Y. Tzalenchuk. Two-level systems in superconducting quantum devices due to trapped quasiparticles. *Science Advances*, 6(51):eabc5055, 2020.
- [79] H. London, G. R. Clarke, and E. Mendoza. Osmotic pressure of he^3 in liquid he^4 , with proposals for a refrigerator to work below 1°k. *Phys. Rev.*, 128:1992–2005, Dec 1962.
- [80] E. Jaynes and F. Cummings. Comparison of quantum and semiclassical radiation theories with application to the beam maser. *Proceedings of the IEEE*, 51(1):89–109, 1963.
- [81] M. C. Dartiailh, T. Kontos, B. Douçot, and A. Cottet. Direct cavity detection of majorana pairs. *Physical Review Letters*, 118(12), 2017.
- [82] J. I. Väyrynen, G. Rastelli, W. Belzig, and L. I. Glazman. Microwave signatures of majorana states in a topological josephson junction. *Physical Review B*, 92(13), 2015.
- [83] G. Kunakova, A. P. Surendran, D. Montemurro, M. Salvato, D. Golubev, J. Andzane, D. Erts, T. Bauch, and F. Lombardi. Topological insulator nanoribbon Josephson junctions: Evidence for size effects in transport properties. *Journal of Applied Physics*, 128(19):194304, 11 2020.
- [84] A. Q. Chen, M. J. Park, S. T. Gill, Y. Xiao, D. Reig-I-Plessis, G. J. Macdougall, M. J. Gilbert, and N. Mason. Finite momentum cooper pairing in three-dimensional topological insulator josephson junctions. *Nature Communications*, 9(1), 2018.
- [85] M. Kayyalha, M. Kargarian, A. Kazakov, I. Miotkowski, V. M. Galitski, V. M. Yakovenko, L. P. Rokhinson, and Y. P. Chen. Anomalous low-temperature enhancement of supercurrent in topological-insulator nanoribbon josephson junctions: Evidence for low-energy andreev bound states. *Physical Review Letters*, 122(4), 2019.
- [86] J. Kampmeier, C. Weyrich, M. Lanius, M. Schall, E. Neumann, G. Mussler, T. Schäpers, and D. Grützmacher. Selective area growth of bi_2te_3 and sb_2te_3 topological insulator thin films. *Journal of Crystal Growth*, 443:38–42, 2016.
- [87] P. Schüffelgen, T. Schmitt, M. Schleenvoigt, D. Rosenbach, P. Perla, A. R. Jalil, G. Musler, M. Lepsa, T. Schäpers, and D. Grützmacher. Exploiting topological matter for

- majorana physics and devices. *Solid-State Electronics*, 155:99–104, 2019. Selected Papers from the Future Trends in Microelectronics (FTM-2018) Workshop.
- [88] M. Franz. Majorana’s wires. *Nature Nanotechnology*, 8(3):149–152, 2013.
- [89] T. W. Schmitt, M. R. Connolly, M. Schleenvoigt, C. Liu, O. Kennedy, J. M. Chávez-Garcia, A. R. Jalil, B. Bennemann, S. Trellenkamp, F. Lentz, E. Neumann, T. Lindström, S. E. De Graaf, E. Berenschot, N. Tas, G. Mussler, K. D. Petersson, D. Grützmacher, and P. Schüffegen. Integration of topological insulator josephson junctions in superconducting qubit circuits. *Nano Letters*, 2022.
- [90] S. Probst, F. B. Song, P. A. Bushev, A. V. Ustinov, and M. Weides. Efficient and robust analysis of complex scattering data under noise in microwave resonators. *Review of Scientific Instruments*, 86(2):024706, 2015.
- [91] J. Burnett, A. Bengtsson, D. Niepce, and J. Bylander. Noise and loss of superconducting aluminium resonators at single photon energies. *Journal of Physics: Conference Series*, 969:012131, 2018.
- [92] L. Casparis, M. R. Connolly, M. Kjaergaard, N. J. Pearson, A. Kringhoj, T. W. Larsen, F. Kuemmeth, T. Wang, C. Thomas, S. Gronin, G. C. Gardner, M. J. Manfra, C. M. Marcus, and K. D. Petersson. Superconducting gatemon qubit based on a proximitized two-dimensional electron gas. *Nat Nanotechnol*, 13(10):915–919, 2018. Casparis, Lucas Connolly, Malcolm R Kjaergaard, Morten Pearson, Natalie J Kringhoj, Anders Larsen, Thorvald W Kuemmeth, Ferdinand Wang, Tiantian Thomas, Candice Gronin, Sergei Gardner, Geoffrey C Manfra, Michael J Marcus, Charles M Petersson, Karl D eng Research Support, Non-U.S. Gov’t England Nat Nanotechnol. 2018 Oct;13(10):915-919. doi: 10.1038/s41565-018-0207-y. Epub 2018 Jul 23.
- [93] A. Kringhøj, L. Casparis, M. Hell, T. W. Larsen, F. Kuemmeth, M. Leijnse, K. Flensberg, P. Krogstrup, J. Nygård, K. D. Petersson, and C. M. Marcus. Anharmonicity of a superconducting qubit with a few-mode josephson junction. *Physical Review B*, 97(6), 2018.

- [94] G. de Lange, B. van Heck, A. Bruno, D. J. van Woerkom, A. Geresdi, S. R. Plissard, E. P. A. M. Bakkers, A. R. Akhmerov, and L. DiCarlo. Realization of microwave quantum circuits using hybrid superconducting-semiconducting nanowire josephson elements. *Phys. Rev. Lett.*, 115:127002, Sep 2015.
- [95] D. J. Van Harlingen, T. L. Robertson, B. L. T. Plourde, P. A. Reichardt, T. A. Crane, and J. Clarke. Decoherence in josephson-junction qubits due to critical-current fluctuations. *Phys. Rev. B*, 70:064517, Aug 2004.
- [96] J. J. Burnett, A. Bengtsson, M. Scigliuzzo, D. Niepce, M. Kudra, P. Delsing, and J. Bylander. Decoherence benchmarking of superconducting qubits. *npj Quantum Information*, 5(1), 2019.
- [97] J. Burnett, L. Faoro, I. Wisby, V. L. Gurtovoi, A. V. Chernykh, G. M. Mikhailov, V. A. Tulin, R. Shaikhaidarov, V. Antonov, P. J. Meeson, A. Y. Tzalenchuk, and T. Lindström. Evidence for interacting two-level systems from the $1/f$ noise of a superconducting resonator. *Nature Communications*, 5(1):4119, Jun 2014.
- [98] R. Ilan, J. H. Bardarson, H.-S. Sim, and J. E. Moore. Detecting perfect transmission in josephson junctions on the surface of three dimensional topological insulators. *New Journal of Physics*, 16(5):053007, may 2014.
- [99] C.-Z. Li, C. Li, L.-X. Wang, S. Wang, Z.-M. Liao, A. Brinkman, and D.-P. Yu. Bulk and surface states carried supercurrent in ballistic nb-dirac semimetal cd_3as_2 nanowire-nb junctions. *Phys. Rev. B*, 97:115446, Mar 2018.
- [100] M. Veldhorst, M. Snelder, M. Hoek, T. Gang, V. K. Guduru, X. L. Wang, U. Zeitler, W. G. van der Wiel, A. A. Golubov, H. Hilgenkamp, and A. Brinkman. Josephson supercurrent through a topological insulator surface state. *Nature Materials*, 11(5):417–421, Feb. 2012.
- [101] G. Schubert, H. Fehske, L. Fritz, and M. Vojta. Fate of topological-insulator surface states under strong disorder. *Phys. Rev. B*, 85:201105, May 2012.
- [102] K. Yavilberg, E. Ginossar, and E. Grosfeld. Differentiating majorana from andreev bound states in a superconducting circuit. *Phys. Rev. B*, 100:241408, Dec 2019.

- [103] J. Lisenfeld, A. Bilmes, A. Megrant, R. Barends, J. Kelly, P. Klimov, G. Weiss, J. M. Martinis, and A. V. Ustinov. Electric field spectroscopy of material defects in transmon qubits. *npj Quantum Information*, 5(1), 2019.
- [104] D. Bothner, T. Gaber, M. Kemmler, D. Koelle, R. Kleiner, S. Wünsch, and M. Siegel. Magnetic hysteresis effects in superconducting coplanar microwave resonators. *Physical Review B*, 86(1), 2012.
- [105] H. Peng, K. Lai, D. Kong, S. Meister, Y. Chen, X.-L. Qi, S.-C. Zhang, Z.-X. Shen, and Y. Cui. Aharonov-Bohm interference in topological insulator nanoribbons. *Nature Materials*, 9(3):225–229, Mar 2010.
- [106] A. Q. Chen, M. J. Park, S. T. Gill, Y. Xiao, D. Reig-I-Plessis, G. J. Macdougall, M. J. Gilbert, and N. Mason. Finite momentum cooper pairing in three-dimensional topological insulator josephson junctions. *Nature Communications*, 9(1), 2018.
- [107] S. Hart, H. Ren, M. Kosowsky, G. Ben-Shach, P. Leubner, C. Brüne, H. Buhmann, L. W. Molenkamp, B. I. Halperin, and A. Yacoby. Controlled finite momentum pairing and spatially varying order parameter in proximitized hgte quantum wells. *Nature Physics*, 13(1):87–93, Jan 2017.
- [108] N. F. Q. Yuan and L. Fu. Zeeman-induced gapless superconductivity with a partial fermi surface. *Physical Review B*, 97(11), 2018.
- [109] L.-X. Wang, Y. Yan, L. Zhang, Z. Liao, H. Wu, and D. Yu. Zeeman effect on surface electron transport in topological insulator bi₂se₃ nanoribbons. *Nanoscale*, 7 40:16687–94, 2015.
- [110] M. Tinkham. *Introduction to Superconductivity*. Dover Publications, 2 edition, June 2004.
- [111] A. Murani, A. Kasumov, S. Sengupta, Y. A. Kasumov, V. T. Volkov, I. I. Khodos, F. Brisset, R. Delagrange, A. Chepelianskii, R. Deblock, H. Bouchiat, and S. Guéron. Ballistic edge states in bismuth nanowires revealed by squid interferometry. *Nature Communications*, 8(1):15941, Jul 2017.

- [112] E. Zimmermann, A. R. Jalil, M. Schleenvoigt, J. Karthein, B. Frohn, G. Behner, F. Lentz, S. Trelenkamp, E. Neumann, P. Schüffegen, H. Lüth, D. Grützmacher, and T. Schäpers. Topological insulator based axial superconducting quantum interferometer structures, 2024.
- [113] R. Luca. Quantum interference in josephson junctions. *Journal of Modern Physics*, 6:526–535, 2015.
- [114] G. Montambaux. Interference pattern of a long diffusive josephson junction, 2007.
- [115] J. G. Kroll, F. Borsoi, K. L. Van Der Enden, W. Uilhoorn, D. De Jong, M. Quintero-Pérez, D. J. Van Woerkom, A. Bruno, S. R. Plissard, D. Car, E. P. A. M. Bakkers, M. C. Cassidy, and L. P. Kouwenhoven. Magnetic-field-resilient superconducting coplanar-waveguide resonators for hybrid circuit quantum electrodynamics experiments. *Physical Review Applied*, 11(6), 2019.
- [116] A. P. Surendran, D. Montemurro, G. Kunakova, X. Palermo, K. Niherysh, E. Trabeldo, D. S. Golubev, J. Andzane, D. Erts, F. Lombardi, and T. Bauch. Current-phase relation of a short multi-mode bi2se3 topological insulator nanoribbon josephson junction with ballistic transport modes. *Superconductor Science and Technology*, 36(6):064003, may 2023.
- [117] J. Gao, J. Zmuidzinas, C. I. of Technology. Division of Engineering, and A. Science. *The Physics of Superconducting Microwave Resonators*. CIT theses. California Institute of Technology, 2008.
- [118] W. J. Wallace and R. H. Silsbee. Microstrip resonators for electron-spin resonance. *Review of Scientific Instruments*, 62(7):1754–1766, 1991.
- [119] K. D. Usadel. Generalized diffusion equation for superconducting alloys. *Phys. Rev. Lett.*, 25:507–509, Aug 1970.
- [120] D. Sticlet, B. Nijholt, and A. Akhmerov. Robustness of majorana bound states in the short-junction limit. *Physical Review B*, 95(11), 2017.

Appendix A

Sample Fabrication

Figure A.1 illustrates our process for the growth of nm-sized $(\text{Bi,Sb})_2\text{Te}_3$ nanoribbons and their in situ niobium (Nb) metallization forming Josephson junctions which allow for their integration into superconducting circuits. A selective area growth mask of 5 nm SiO_2 and 12 nm Si_3N_4 is prepared on a highly resistive Si(111) substrate ($> 2000 \Omega\text{cm}$). Using electron beam lithography and reactive ion etching, four separated nanotrenches are etched into the Si_3N_4 which define the position and the shape of the TI [shown for one trench in Figure A.1a]. A stencil mask of SiO_2 and Si_3N_4 is deposited and patterned locally over the nanotrenches prior to growth. By etching the sacrificial layer of SiO_2 isotropically with hydrofluoric acid, the Si_3N_4 stencil mask over the trench is almost completely suspended and rests only on four small SiO_2 pillars [Figure A.1b]. Nanoribbons of $(\text{Bi}_{0.06}\text{Sb}_{0.94})_2\text{Te}_3$ are grown via molecular beam epitaxy (MBE) selectively in the nanotrenches with substrate rotation ensuring continuous TI growth under the stencil mask [Figure A.1c]. Without breaking the vacuum, 50 nm Nb is deposited without rotation and the sample surface is covered except the small area shadowed by the stencil mask [Figure A.1d]. In this way, both ends of the nanoribbon are contacted with Nb leaving a narrow gap of bare TI that forms a Josephson junction. Finally, a stoichiometric 5 nm aluminum oxide (Al_2O_3) passivation layer is deposited under rotation. This layer protects the TI surface in the weak link area thus the sample can subsequently be taken to ambient conditions.

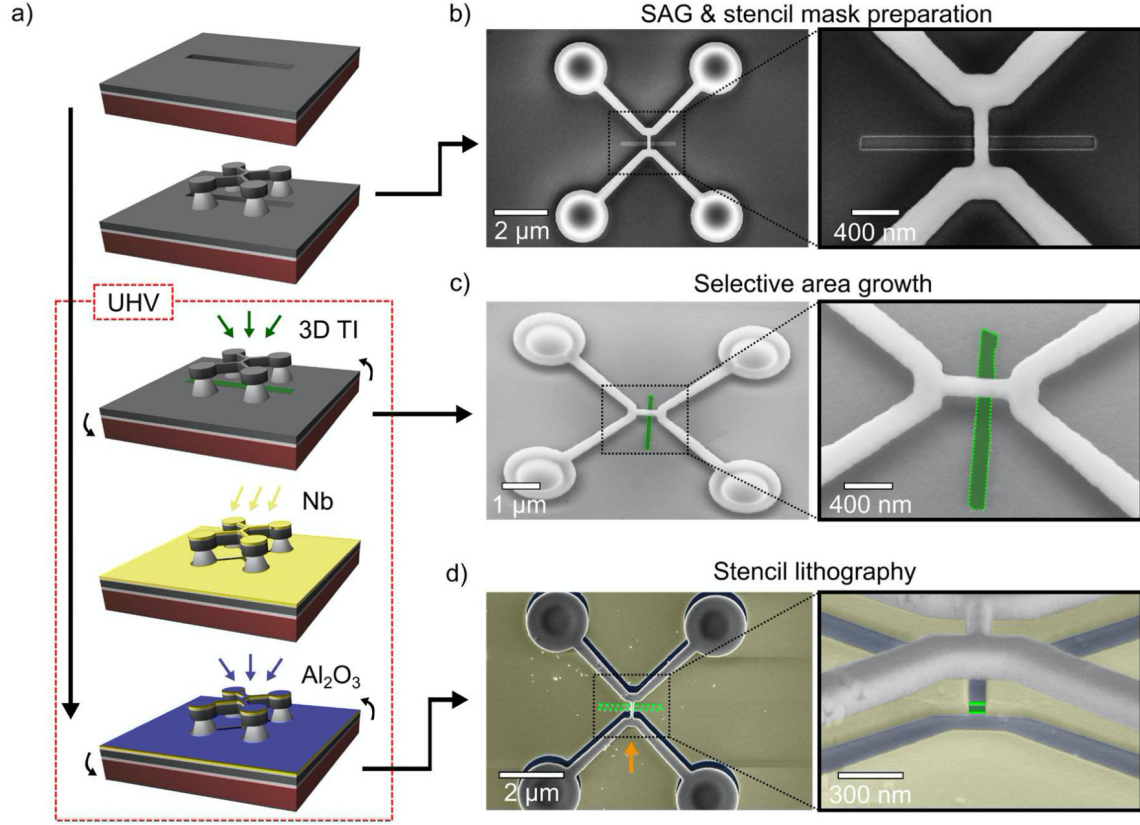


Figure A.1: Selective area epitaxy and in situ local stencil lithography fabrication process to integrate 3D TI Josephson junctions into superconducting quantum circuits. a) A mask of SiO₂ (light grey)/ Si₃N₄ (dark grey) is prepared on a Si(111) (red) substrate and trenches for selective area growth are etched. A sacrificial SiO₂ (light grey) layer and a Si₃N₄ (dark grey) stencil mask layer are prepared and structured. After dry etching of the Si₃N₄ mask, the exposed SiO₂ is etched isotopically with hydrofluoric acid such that the stencil mask is suspended and only rests on four SiO₂ pillars. Subsequently, 3D TI (green) is selectively grown under rotation via MBE followed by a deposition of Nb (yellow) without rotation. The shadow of the bridge defines the uncovered section of TI, forming the Josephson junction, while the rest of the chip is covered with a Nb film. Finally, the device surface is passivated by Al₂O₃ (blue) and can be taken to ambient conditions. b-d) False colour scanning electron micrographs of a suspended stencil mask above a trench for selective area epitaxy (b) followed by selective area epitaxy of 3D TI (c) and Nb + Al₂O₃ deposition (d). Images in (c) were taken at an angle to show selective area growth under the stencil mask. The close-up in d) was taken at a different angle (indicated by the orange arrow) to show the Josephson junction.

Appendix B

Comsol Simulation

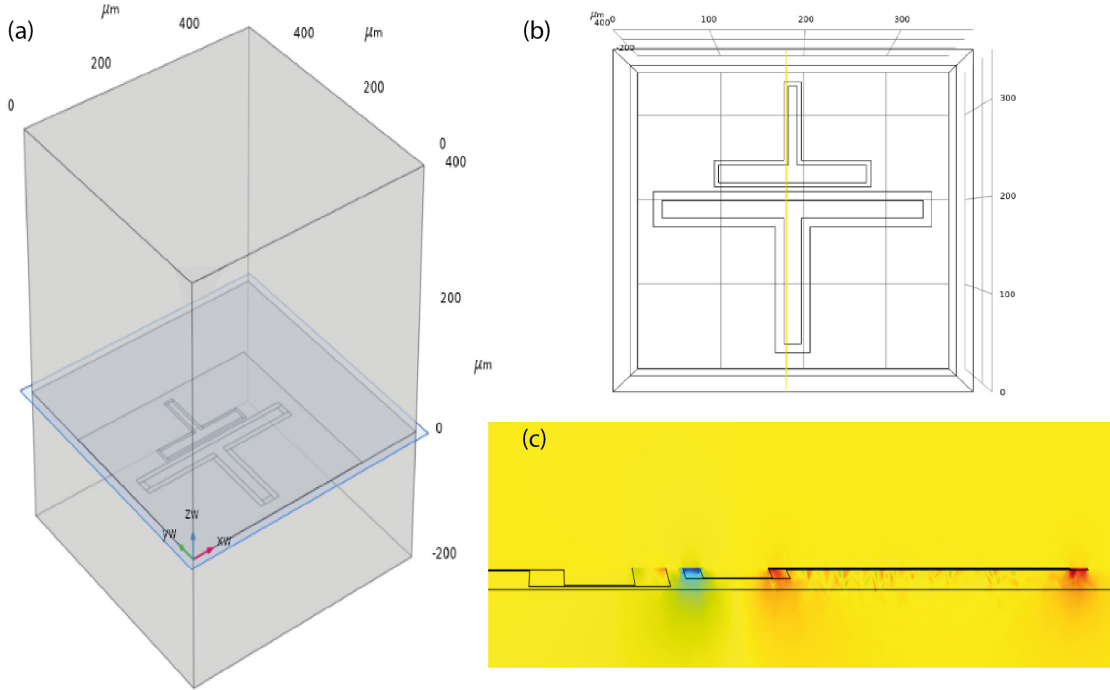


Figure B.1: Comsol simulations. (a) The 3d model used in the simulation routine. The stack is separated by a middle plane which contains the circuitry pattern shown in (b). The material of the bottom half is silicon while the top is air.(b) Two T-islands with insulating boundaries (white region) to create a capacitive coupling between the two. In real-device circuitry, the smaller T-island on the top is connected to the resonator while the bottom one is connected to the qubit. (c) The simulated potential in yz plane along the yellow line in (b).

To simulate the coupling strength of our device, I uploaded the plane geometry of the resonator and qubit T-islands [Figure C.1b], which together define C_g and C_B into a 3D COMSOL model. The model also includes two blocks of materials surrounding the working plane [Figure C.1a]. Ideally, the blocks should be multilayered to reflect the actual stack structure. However, for simplicity, we used single layer. The material below the plane is set to be Si(111), and

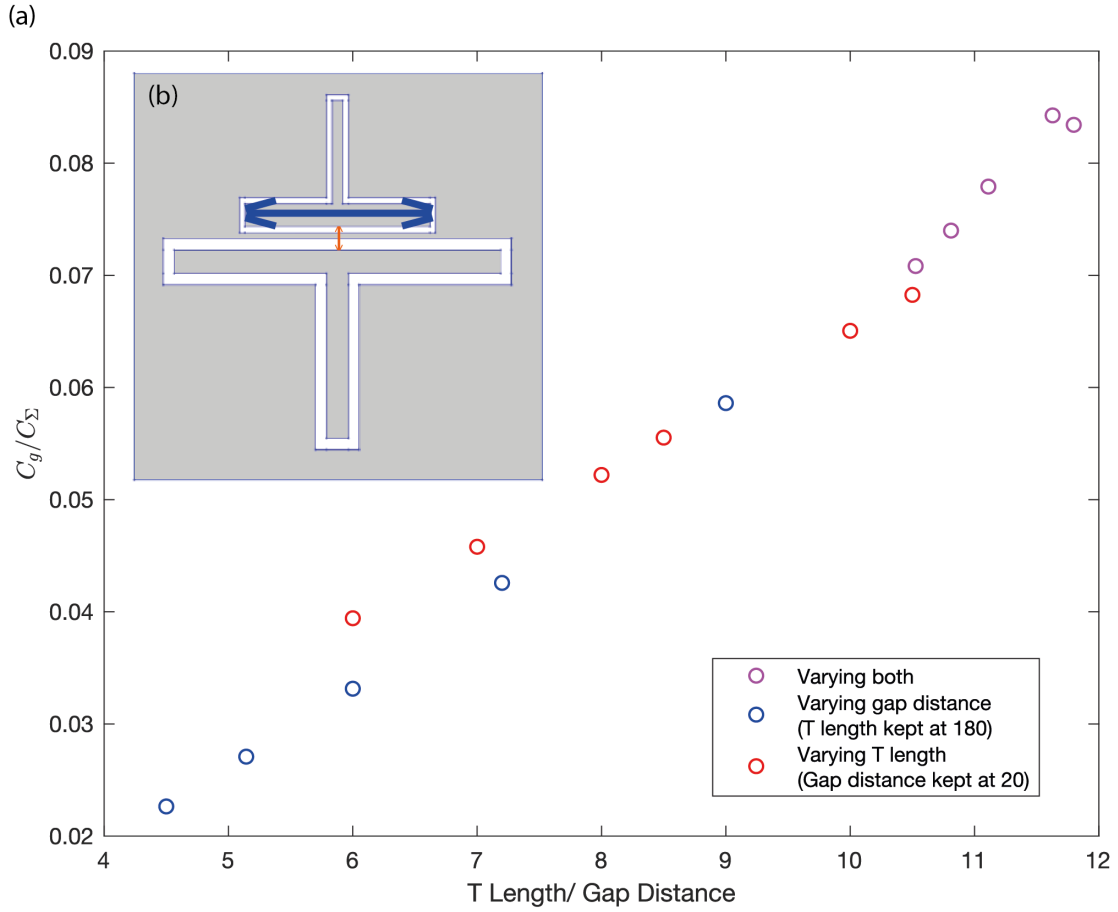


Figure B.2: The ratio of the mutual capacitance between the two T-islands over the self capacitance of the qubit T-island (C_g/C_Σ) as a function of geometric parameters (the resonator T-island length over the separation between the two T-islands). The data are separated into three sets (represented in different colors) according to which parameter is being tuned. The inset shows the two geometry parameters. The blue segment shows the T length while the red segment gives the gap distance between the T-islands.

the material above is air. This simplification might lead to inaccuracy in calculations of g , depending on how much the single-layered structure differs from a multi-layered one in dielectric constant.

By setting the two islands as voltage terminals, we could do electrostatics simulations in the 3D model space, which allows us to extract capacitances of the two terminals and hence C_g and C_B . Figure C.1c shows simulated potential in the y-z plane along the yellow line in figure C.1b, when the resonator island is set at 1V. For our design when T Length = $180 \mu\text{m}$ and Gap Distance = $20 \mu\text{m}$ [See the inset in Figure B.2], $C_g = 4.67 \times 10^{-15} \text{F}$ and $C_B = 7.96 \times 10^{-14} \text{F}$.

Appendix C

Fridge Setup

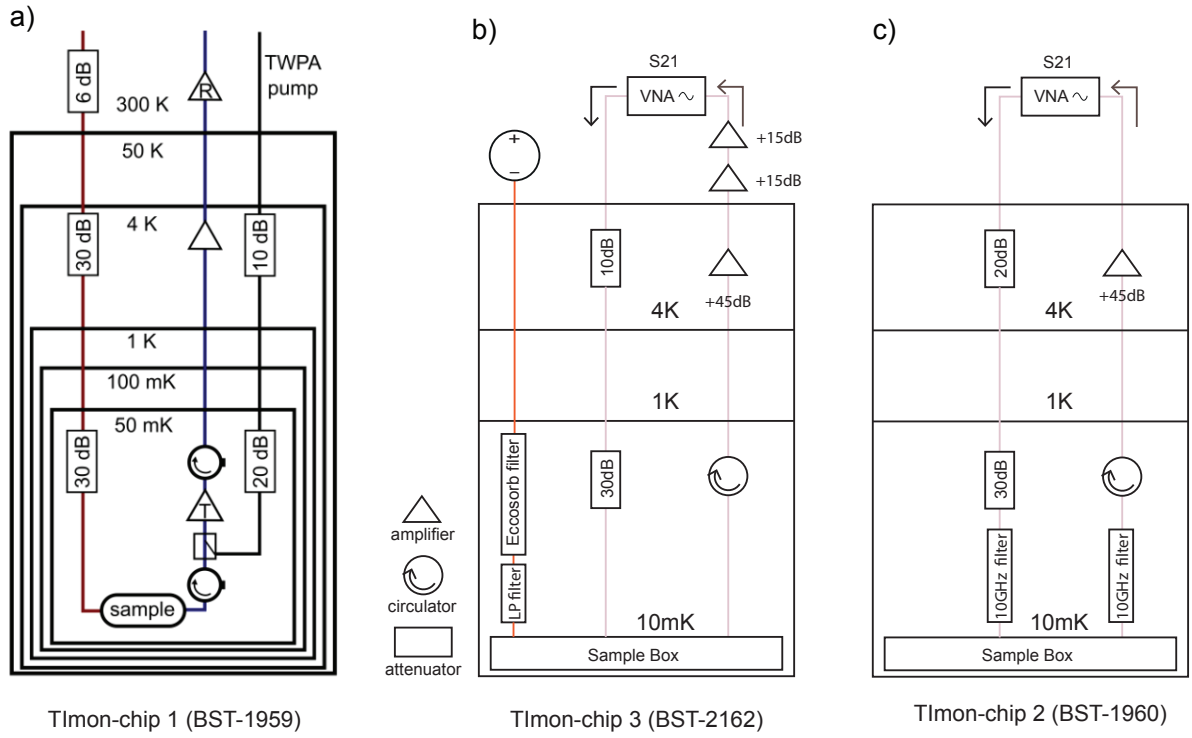


Figure C.1: Fridge setup for qubit measurements. (a) Fridge setup for fixed-frequency TImon chip BST-1959. (b) Fridge setup for flux-tunable TImon chip BST-2162. (c) Fridge setup for fixed-frequency TImon chip BST-1960.

Appendix D

Simulation

All the python scripts of the simulations below are given in the github repository (<https://github.com/chenl/Simulation.git>).

D.1 3DTI Simulation

The python scripts used for 3DTI simulations are given in the jupyter notebook (Topological_regime_no_TRS.ipynb). 3DTI simulations use Kwant [53] to build a tight-binding model that describes an infinitely long cuboid 3DTI nanoribbon. The nanoribbon model is implemented in kwant by attaching two infinite 3DTI leads with translational symmetry to a 100 nm long cubic scattering region. When discretizing the 3DTI Hamiltonian given in Eq 2.22, lattices of 1 nm cubic units are used. The choices of non-zero 3DTI parameters are given in Table D.1. Parameters in Eq 2.22 that are not mentioned in Table D.1 are all zeros. Kwant

$A_{\perp}$	3 eV
A_z	3 eV
M_0	0.3 eV
M_z	15 eV

Table D.1: 3DTI Hamiltonian Model Parameters

is also capable of peierls substitution given a magnetic field, which allows us to simulate the 3DTI system as a function of magnetic field. During the simulations, apart from the applied field as a free parameter, we can also control the chemical potentials of the two leads separately and the middle scattering region.

D.1.1 Band Structure

Kwant has a built-in function (*gauge*) to calculate the band structure of an infinite system. When simulating Figure 2.3b, we look at the band structures of the leads with the applied field to the leads varied and all the other parameters fixed. The chemical potential of the simulated lead is set at 0.

D.1.2 Energy-phase Relation

To simulate the energy-phase relation of supercurrent transport in 3DTI JJs, we use the scattering matrix approach [120]. Instead of attaching superconducting 3DTI leads to the normal 3DTI scattering region to directly build a 3DTI JJ system in kwant, we add Andreev reflection boundary conditions to the normal 3DTI scattering region to emulate 3DTI JJs by using the previous 3DTI nanoribbon system. During the simulation of Figure 2.6b, the applied field is fixed to pierce a flux quantum through the nanoribbon. The chemical potentials of the left lead and the scattering region are both at 0.15 eV. To induce backscattering, a chemical potential mismatch is introduced by setting the chemical potential of the right lead at 0.2 eV. Meanwhile, we also add a Gaussian potential fluctuation of 0.005 eV in the scattering region on top of 0.15 eV to simulate uncorrelated chemical disorders.

D.2 Qubit Simulation

QuTiP is a python software package that allows the simulation of open quantum systems given their Hamiltonians. We can use it to simulate qubit/cavity spectra, time-evolution of qubit/cavity states, and the cavity absorption/emission spectra. The general form of our qubit Hamiltonian is given by Eq 2.52. The charging energy term can be easily written in the charge space. The Josephson energy term can vary depending on the junction type. In general, we can use numerical methods to find out the exact energy-phase relation of the junction and rewrite the Josephson energy term in the charge space.

D.2.1 Numerical SQUID Simulation

When simulating the SQUID junction energy in Figure 2.7, Eq 2.40 is used. Φ is fixed at a few values with varying φ_1 . In the simulation of Figure 2.7a, the sinusoidal SIS energy-phase relation is used, $E_{J,1} = 0.6$, and $E_{J,2} = 0.4$ to achieve an asymmetry factor $d = 0.2$. In the simulation of Figure 2.7b, the SNS energy-phase relation [Eq 2.35] is used, $\Delta_1 = 0.6$, and $\Delta_2 = 0.4$ to achieve an asymmetry factor $d = 0.2$. The transmission coefficient $T = 0.95$.

In the simulation of Figure 2.8a-b, we repeat the simulations of Figure 2.7a for an array of Φ -values ranging from 0 to $2\Phi_0$. Again $E_{J,1} = 0.6$ and $E_{J,2} = 0.4$ are used. At each Φ point, we fourier transform the simulated SIS SQUID energy-phase trace to find out its effective Josephson energy E_{eff} . To find the critical current (I_c), the same energy-phase trace is differentiated to get the current-phase trace. The maximum current found in the current-phase trace is the critical current. In Figure 2.8c-d, all the settings remain the same except that $E_{J,1} = E_{J,2} = 0.5$. We also apply the same I_c simulation procedure to an SNS SQUID. In Figure 2.8b, $\Delta_1 = 0.6$ and $\Delta_2 = 0.4$. In Figure 2.8d, $\Delta_1 = \Delta_2 = 0.5$.

D.2.2 Flux-tunable SIS/SNS Qubit

QuTiP allows us to calculate qubit spectrum by inserting the simulated junction energy-phase relation at an arbitrary flux point into a qubit Hamiltonian. The parameters used in the simulation of Figure 2.12 are given in Tables D.2 and D.3. The python scripts of the simulations in this section and the previous section are given in the jupyter notebook (Transmon_simulation.ipynb).

E_C	500 MHz
$E_{J,1} + E_{J,2}$	50 GHz
d	0, 0.2

Table D.2: Flux-tunable SIS Qubit Parameters

E_C	500 MHz
T	0.95
$\Delta_1 + \Delta_2$	80 GHz
d	0, 0.2

Table D.3: Flux-tunable SNS Qubit Parameters

D.2.3 Flux-tunable Hybrid Qubit

In the simulation of Figure 4.6c-d, a hybrid junction model [Eq 4.2] is used in the qubit Hamiltonian. The parameters used in Figure 4.6c are given in Table D.4. To simulate the dressed cavity frequency in Figure 4.6d, Eqs 2.66 and 2.70 are used. Two additional parameters f_r and $g_0 = g(E_{J,0})/f_r$ need to be assumed. The choices are also included in Table D.4. Meanwhile, $\Delta_1 + \Delta_2$ are swept from 0 to 200 GHz in 31 steps. The simulation scripts are given in the jupyter notebook (Python plot APS.ipynb).

E_C	229.2 MHz
$E_{J,1} + E_{J,2}$	215.8 GHz
d	0.27
T	0.9
f_r	6.7 GHz
g_0	0.018 Hz ⁻¹
$\Delta_1 + \Delta_2$	0 GHz, 86.7 GHz

Table D.4: Hybrid Qubit Simulation Parameters

D.2.4 Qubit-cavity Interaction

When simulating the cavity dispersive shift in Figure 6.3d, we include the qubit-cavity interaction term in the QuTiP Hamiltonian. For simplicity, we treat our qubit as a two-level system and use Eq 2.67 for the total Hamiltonian. The python scripts are given in the jupyter notebook (PhD simulation.ipynb). The parameters used in the simulation are summarized in Table D.5.

f_q	8 GHz
f_c	7 GHz
g	100 MHz
$\kappa_\downarrow$	0.005
$\Gamma_\downarrow$	0.05

Table D.5: Qubit-cavity System Parameters

D.3 Field-dependence Simulation

In the simulations of Figure 5.6b-c, the isotropic trivial magnetic effect follows Eq 2.1 and contributes to the field dependence in the form of $E_{J,0}\sqrt{1 - \frac{B^2}{B_c^2}}$ as a prefactor in front of Eqs 5.1 and 5.2. The qubit frequency f_q is calculated from $8\sqrt{E_J E_C}$. The simulation scripts are included in the jupyter notebook (TI-mon in field simulation.ipynb). The parameters used in Figure 5.6b-c are given by Table D.6.

$E_{J,0}$	30 GHz
E_C	230 MHz
B_c	3 T
S_x	1800 nm ²
S_y	2200 nm ²

Table D.6: Field-dependence Simulation Parameters