

RESEARCH ARTICLE

Mixed Lagrangian formulation for modeling structures with clutched inerter devices

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Abstract

Inerters (ID) and Clutched Inerter Devices (CID) are a novel technology with demonstrated seismic control potential. However, the inherent nonlinearity and discontinuity of the clutching phenomena in CIDs can pose significant challenges for their accurate numerical modeling. In general, conventional existing methods either oversimplify the physics involved or are sensitive to the step size and thus are inherently unstable, demanding excessive numerical resources. Most relevant studies to date have focused on small-scale systems with a limited number of inerters and have used simplified models due to the lack of analysis tools. At the same time, the Mixed Lagrangian Formulation (MLF), has proven to be a powerful tool for simulating non-smooth dynamics phenomena. This paper presents an alternative way of modeling the behavior of CIDs in both MLF and conventional finite element method. We put forward an original formulation of the inerter element, clutching behavior, and the inerter-related dissipation model, as well as their associated computational scheme in MLF and the equivalent construction in FEM. The newly proposed CID element in MLF is then implemented and validated through three examples, including a single degree of freedom system, a multi 10-storey moment resisting frame (MRF), and a 10-storey self-centering concentrically braced frame (SC-CBF) with multiple rocking sections. The results are compared to those from existing models used for clutching inerter and to the proposed FE model. Finally, the advantages of using the MLF framework and salient characteristics of the structures equipped with clutched inerters are discussed. The modeling strategy proposed in this work empowers researchers to simulate structures with a larger number of degrees of freedom, equipped with a considerable amount of inerter-based devices, with reduced effort and improved computational performance.

KEYWORDS

clutched inerter damper, mixed Lagrangian formulation, non-smooth dynamics, numerical modeling, rocking structures

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1 | INTRODUCTION

As environmental challenges become more pronounced and frequent, there is a growing demand for sustainable and resilient structures. Researchers are seeking for advanced techniques to enhance the dynamic performance of structures. Inerter-based energy absorption and tuned damping devices have attracted significant interest in civil engineering over the recent years.^{1–5} Initially known in the western automotive industry as ‘J-damper’, the inerter is a device that delivers forces proportional to the relative acceleration between its two terminals,⁶ therefore, they have an inertial amplification or equivalent negative stiffness effect in a mechanical system. This amplifying effect of an ideal inerter can be quantified by the parameter $\sigma = \frac{b}{m}$ (where b is the equivalent mass of the inerter and m is the mass of the host system) which is known as ‘suppression coefficient’ and affects the inertial term in the equation of motion of a single degree of freedom system, that is, Equation (1):

$$(1 + \sigma)\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2x = f(t) \quad (1)$$

where x is the displacement coordinate of the SDOF, ω_n its natural circular frequency, ξ the damping ratio, and $f(t)$ the externally applied force.

The formalization of the inerter concept can be attributed to Smith (2002), who formulated it by leveraging the force-current analogy between electrical and mechanical networks. In Smith’s analogy, an inerter acts like a grounded capacitance in an electrical circuit, while viscous damping and stiffness are analogous to the resistance and inductance. However, the first inerter prototype was constructed by Arakaki et al.⁷ who designed and tested a ball-screw type inerter in the late 90s (before Smith’s work). In practice, inerters can use different mechanisms including mechanical (ball-screw and rack-and-pinion), hydraulic, electromagnetic, and so forth, in which the inertial demand is provided by: a rotating fly-wheel, a flowing fluid or an electromagnetic field, respectively. Ruisheng et al.⁸ provides a detailed illustration of each type of inerter stated above. Since its invention, the inerter has shown great potential in vibration suppression and control. The variety of inerter-based devices can be classified into three categories: inerter-based energy dissipators,^{9–15} inerter-based energy absorbers,^{5,16–20} inerter-based vibration isolators^{21–24} and inerter-equipped vibrating barriers^{25,26} where inerters act synergistically with dissipators, absorbers, isolators, and soil media, respectively.

It is important to note that the inertial amplifying effect of inerters may be detrimental in certain situations. Sometimes, the inerters may transfer the harvested energy back to their host structures, acting as power outputs, thus enhancing the systems’ after-peak responses and extending their vibration time.²⁷ To address this defect, Makris and Kampas²⁸ proposed a clutched inerter damper (CID) consisting of an inerter damper and a clutch system connecting the inerters to the host structure. By means of this clutch, the host structure can disengage from the inerters and cut off the undesired energy transmission from them, effectively turning CIDs into energy harvesting devices. CIDs have been proven effective in suppressing the dynamic response of SDOF systems,^{29–31} multistorey structures,³² bridge piers,³³ pedestrian bridges^{34,35} as well as rocking structures.^{27,36} However, the clutch introduces a non-smooth engagement and disengagement response, leading to difficulties in its numerical modeling. These instantaneous changes of boundary conditions can be generally defined as transient contact problems.³⁷ For clutched inerters, contact takes place between the host structure and the rotating clutch gears (ratchet). Solving transient contact problems often involves explicit or implicit time-stepping algorithms, advancing the simulation in small time steps to capture the dynamic evolution of contact conditions. Transient contact problems can be computationally demanding, especially when dealing with large-scale simulations or complex geometries.³⁸ Managing numerical instabilities, convergence issues, and contact discontinuities is a common challenge when solving these problems.

An intuitive approach to modeling the clutch engagement is non-smooth dynamics, in which the clutch discontinuous response is solved within time intervals.³⁹ In a typical non-smooth dynamic analysis, the system response is classified and partitioned into discontinuous time pieces, so-called ‘events’, accounting for every possible system state. Correspondingly, the behavior of the system is modeled by a series of discontinuous piece-wise governing equations representing each possible event. Transition criteria should also be derived to allow the jump from one event to another. During each event, the system is solved with a different set of governing equations and boundary conditions. For example, when introducing the clutched inerter, Makris and Kampas²⁸ proposed a sequential model in which the clutch shifts between engagement and disengagement states according to the acceleration or deceleration of the host structure as

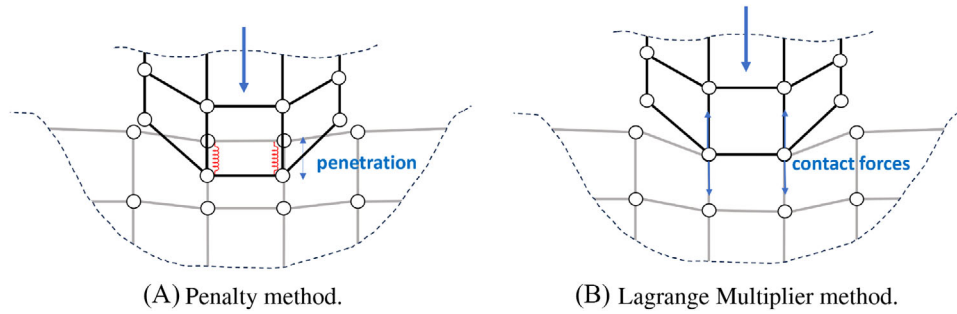


FIGURE 1 Illustration of two conventional approaches for modeling contact problems.

shown in Equation (2).

$$\begin{cases} \mathbf{F}_{CID} = 0, & \text{if } \dot{\mathbf{u}} \cdot \ddot{\mathbf{u}} < 0 \\ \mathbf{F}_{CID} = \mathbf{b}\ddot{\mathbf{u}}, & \text{if } \dot{\mathbf{u}} \cdot \ddot{\mathbf{u}} \geq 0 \end{cases} \quad (2)$$

where $\mathbf{F}_{CID}$ is the force in the CID and $\mathbf{b}$ is the equivalent mass introduced before.

The sequential model of CIDs has been widely used in subsequent research,^{30,33,36,40} although it arbitrarily assumes the sudden deceleration of the flywheels to ensure re-engagement events in each cycle. This effective damping would produce non-conservative system responses, overestimating the effectiveness of the clutched inerter. Many researches have noted flaws in the sequential model and proposed more accurate models using the principles of non-smooth dynamics.^{29,31,41,42} However, except for systems with relatively few states and simple transition criteria such as the SDOF system discussed above, non-smooth dynamics can be prone to instability issues upon event bifurcations. This can be partially compensated by converting the event transition criteria to a linear complementary problem.⁴³ When dealing with larger scale models where the number of events is large; however, the event-driven algorithm would be prohibitively time consuming. The absence of effective numerical tools appears to be one reason behind the adoption of simplified models with a relatively limited number of degrees of freedom in most studies related to CIDs to date.

To avoid discontinuous mode shifting, a considerable part of the scientific literature has resorted to solving the discontinuous contact interactions by continuous time stepping methods based on two basic techniques: the penalty method and the Lagrange multiplier method (Figure 1). The penalty contact model employs an imaginary spring which acts as a stiff compression only spring and allows object intersections, developing a penalty force proportional to their interpenetration. The instant event shifting is therefore transformed into a milder continuous process. This model is more popular since it is relatively easier to implement and computationally more efficient compared with methods relying on strict contact constraints. Málaga-Chuquitaype et al.³² used a nonlinear spring with continuous stepwise stiffness to model the interactions between twin inerter-clutch devices and a steel frame. Liang and Li²⁹ proposed a velocity-based penalty contact model in the form of a viscous damper rather than a spring. This avoids the calculation of relative displacements which increase monotonically in the flywheel, whereas that of the structure is reciprocating. However, the hysteresis effect of the penalty method may be amplified if the velocity-related contact force coefficient is not considerably large. On the other hand, the magnitude of contact stiffness must be carefully decided based on empirical laws to balance excessive penetration and numerical instability. In the Lagrange multiplier method, no-penetration contact constraints are introduced through additional terms of multipliers in the governing equations that need to be strictly satisfied. Those Lagrange multipliers are contact forces (or impulses) that prevent penetration by ensuring kinematic compatibility. However, the introduction of Lagrange multipliers may lead to a non-positive definite tangent stiffness matrix and make the formulations computationally expensive, particularly for large-scale simulations.⁴⁴ Additionally, both displacement based penalty and Lagrange multiplier methods are prone to chattering behavior when contact is impulsive and transient.³⁷

Alternatively, the Mixed Lagrangian Formulation (MLF), originally proposed by Sivaselvan and Reinhorn,⁴⁵ has shown to be a powerful tool for modeling non-smooth dynamics processes. As illustrated in the next section, the governing equations in a MLF consist of both equilibrium and velocity compatibility conditions. These are derived by Hamilton's principle or more generally by the principle of least action, where generalized coordinates contain both global displacements and internal forces as well as their derivatives. After discretization, the time step solution is reduced to a minimization problem as a weak form of the governing equations. MLF was initially proposed for transient analysis

of frames with lumped elasto-plastic hinges⁴⁵ and was later extended to the progressive collapse analysis of large-scale structures⁴⁶ through strength degradation⁴⁷ as well as elemental fracture and recontact.⁴⁸ MLF is also capable of dealing with multi-physic problems such as the thermoplastic response of structures.^{49–51} Recently, MLF was adopted in the analysis and optimization of self-centering frames due to its computational robustness and efficiency.^{52–54} Sivaselvan et al.⁴⁶ provide a summary of one-dimensional examples of generalized standard materials within the MLF framework.

Given the advantages mentioned above, MLF is ideal for modeling non-smooth mechanics problems. However, there is currently no appropriate element in the MLF framework to model the behavior of inerters and clutches explicitly. In this context, a superior numerical method for modeling inerters and their clutching mechanism is derived herein based on the MLF framework. This involves the creation of a new inerter element, the modeling of clutching behavior and the consideration of any supplemental energy dissipation within the device. Subsequently, the new MLF CID element is implemented and validated through three examples including a SDOF system, a multistorey frame and a self-centering concentrically-braced frame (SC-CBF) with multiple rocking sections. These examples demonstrate the numerical advantages and the potential for enhanced efficiency inherent in the proposed modeling method. They also reveal some interesting features of the response of structures equipped with CIDs under base motion. Besides, a more standard modeling strategy, suitable for Finite Element Analysis based on the yielding analogy is introduced. By contributing a new CID element in MLF, this paper opens the door to the numerical modeling of realistic structures equipped with a substantial array of inerter-based devices, or mechanisms in research and practice.

2 | MODELING OF CID IN MLF

2.1 | Overview of MLF

MLF is applicable to a variety of problems in structural mechanics such as geometric nonlinearities, skeletal, and continuum systems⁴⁵ as well as multi-physics problems.⁴⁹ In this section, we introduce the main aspects of the formulation through the simple case of 1-D contact gap opening and plasticity. Naturally, the new element could be combined with all other elements to form a platform capable of efficiently analyzing highly nonlinear systems with clutched inerters in a robust manner.

MLF starts with the formulation of the Lagrangian (Equation 3) and dissipation (Equation 4) functions of the system, in which both displacement and internal forces are considered as generalized coordinates.

$$L(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{J}, \dot{\mathbf{J}}) = \frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M} \dot{\mathbf{u}} + \frac{1}{2} \dot{\mathbf{J}}^T \mathbf{A} \dot{\mathbf{J}} + \mathbf{J}^T \mathbf{B} \dot{\mathbf{u}} + \mathbf{1}^T \mathbf{g}(\dot{\mathbf{J}}) + \mathbf{P}^T \mathbf{u} \quad (3)$$

$$\Phi(\dot{\mathbf{u}}, \dot{\mathbf{J}}) = \frac{1}{2} \dot{\mathbf{u}}^T \mathbf{C} \dot{\mathbf{u}} + \mathbf{1}^T \phi(\dot{\mathbf{J}}) \quad (4)$$

where L is the system's Lagrangian, Φ is the dissipation function, $\mathbf{u}$ the displacement vector in global coordinates, $\mathbf{J}$ a vector of internal impulses (i.e., time integral of the internal forces) in local coordinates, $\mathbf{M}$ and $\mathbf{C}$ are the mass and damping matrices in global coordinates, respectively, $\mathbf{A}$ is the flexibility matrix in local coordinates, $\mathbf{B}$ is the equilibrium matrix that transforms entities from local to global coordinates, and $\mathbf{P}$ is the external load vector in global coordinates. Besides, $\mathbf{g}$ and ϕ are vectors of potential energy functions for 1-D gap elements proposed by Lavan⁴⁸ and the elasto-plastic elements proposed by Sivaselvan and Reinhorn⁴⁵ as defined in Equation (5) below.

$$\mathbf{g}_i(\dot{\mathbf{J}}) = \begin{cases} 0, & \dot{\mathbf{J}}_i - \mathbf{F}_{c,i} \leq 0 \\ \infty, & \dot{\mathbf{J}}_i - \mathbf{F}_{c,i} > 0 \end{cases}, \quad \phi_i(\dot{\mathbf{J}}) = \begin{cases} 0, & |\dot{\mathbf{J}}_i| \leq \mathbf{F}_{y,i} \\ \infty, & |\dot{\mathbf{J}}_i| > \mathbf{F}_{y,i} \end{cases} \quad (5)$$

Here, subscript i denotes the potential energy function of the i th elemental force, $\mathbf{F}_{y,i}$ is the yield force of element force i and $\mathbf{F}_c$ is a vector of allowed elemental maximum tensile forces as limit of detachment. Applying Euler-Lagrange equations (Equations 6 and 7) with regards to both coordinates of displacement $\mathbf{u}$ and internal impulse $\mathbf{J}$:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{u}}} \right) - \frac{\partial L}{\partial \mathbf{u}} + \frac{\partial \Phi}{\partial \dot{\mathbf{u}}} = \mathbf{0} \quad (6)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{J}}} \right) - \frac{\partial L}{\partial \mathbf{J}} + \frac{\partial \Phi}{\partial \dot{\mathbf{J}}} = \mathbf{0} \quad (7)$$

and the governing equations of force equilibrium and velocity compatibility are obtained as (Equations 8 and 9):

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{B}\dot{\mathbf{J}} = \mathbf{P} \quad (8)$$

$$\mathbf{A}\ddot{\mathbf{J}} + \frac{\partial \phi(\dot{\mathbf{J}})}{\partial \dot{\mathbf{J}}} + \frac{d}{dt} \left\{ \frac{\partial \mathbf{g}(\dot{\mathbf{J}})}{\partial \dot{\mathbf{J}}} \right\} = \mathbf{B}^T \dot{\mathbf{u}} \quad (9)$$

Equation (8) establishes equilibrium in global coordinates and Equation (9) enforces compatibility of deformation rates in local coordinates. The first term on the left side of Equation (9) is the deformation rate of the linear elastic part, while the second term is the ideal plastic part, the third term is the gap openings speed. The right side term in this equation is the rate of deformation of all elements. Matrix $\mathbf{B}$ can be a function of $\mathbf{u}$ if geometric nonlinearity is considered.⁴⁵

By applying central differences at $t_{n+\frac{1}{2}}$ and substituting $\dot{\mathbf{u}} = \mathbf{v}$ and $\dot{\mathbf{J}} = \mathbf{F}$, the following time stepping scheme is obtained (Equations 10 and 11):

$$\mathbf{v}_{n+1} = \left(\frac{\mathbf{M}}{h} + \frac{\mathbf{C}}{2} \right)^{-1} \left[\left(\frac{\mathbf{M}}{h} - \frac{\mathbf{C}}{2} \right) \mathbf{v}_n - \mathbf{B} \left(\frac{\mathbf{F}_{n+1} + \mathbf{F}_n}{2} \right) + \left(\frac{\mathbf{P}_{n+1} + \mathbf{P}_n}{2} \right) \right] \quad (10)$$

$$\bar{\mathbf{A}}\mathbf{F}_{n+1} + \frac{1}{2} \frac{\partial \phi}{\partial \mathbf{F}} \bigg|_{n+1} + \frac{1}{h} \frac{\partial \mathbf{g}}{\partial \mathbf{F}} \bigg|_{n+1} + \bar{\mathbf{b}}_n = \mathbf{0} \quad (11)$$

where, h is the time step size, $\bar{\mathbf{A}}$ is the equivalent dynamic flexibility matrix which is dual to that of the equivalent dynamic stiffness matrix of Newmark's method with $\gamma = 1/2$ and $\beta = 1/4$, and $\bar{\mathbf{b}}_n$ is a known vector reflecting the state of the system at the current time step ($t = t_n$).

Thus, at each time step, the elemental force vector $\mathbf{F}_{n+1}$ is obtained by solving Equation (11) and subsequently updating the velocity vector $\mathbf{v}_{n+1}$ through Equation (10). Although it is troublesome to solve the two equations containing implicit functions, $\phi(\dot{\mathbf{J}})$ and $\mathbf{g}(\dot{\mathbf{J}})$, these are converted into an optimization problem with a quadratic objective function and non-linear constraints, as shown in Equation (12) below, which can be solved by various numerical optimization algorithms.⁵⁵ For linear stepwise $\phi(\dot{\mathbf{J}})$ and $\mathbf{g}(\dot{\mathbf{J}})$ in Equation (5), the nonlinear constraints can be transformed to linear constraints and form a standard quadratic programming (QP) problem.⁵⁶

$$\begin{aligned} \min \quad & \frac{1}{2} \mathbf{F}_{n+1}^T \bar{\mathbf{A}} \mathbf{F}_{n+1} + \bar{\mathbf{b}}^T \mathbf{F}_{n+1} \\ \text{s.t.} \quad & \\ & \frac{1}{2} \phi(\mathbf{F}_{n+1}) \leq 0 \\ & \frac{1}{h} \mathbf{g}(\mathbf{F}_{n+1}) \leq 0 \end{aligned} \quad (12)$$

The physical explanation of the minimization problem above can be interpreted as the principle of minimum incremental complementary potential energy: *'Of all the forces at step $n + 1$, satisfying equilibrium with prescribed external forces at the undamped quasistatic degrees of freedom and satisfying the yield conditions, the one that minimizes the incremental complementary potential energy is the one that satisfies equilibrium and compatibility in the other degrees of freedom.'*⁴⁵

2.2 | Proposed inerter element

MLF was initially proposed for the transient analysis of elastic-plastic frames.⁴⁵ A gap element was later introduced for modeling element detachment and it has been recently proved to be functional for rocking behavior.⁵² However, there is no MLF element that is capable of modeling the behavior of inerters or clutches. To fill this gap, an inerter element and a one-way clutch formulation are introduced into MLF. Note that the derivation presented here assumes all DOFs are effective and associated with dynamic masses. In practice, The mass and damping matrices can be singular and partition and condensation of DOFs should be done (see derivations by Sivaselvan and Reinhorn⁴⁵).

Consider an inerter with intertance of b_i located at the i th link of a system as shown in Figure 2, where $\ddot{u}_i, \ddot{u}_g, \ddot{u}_l, \ddot{u}_p, \ddot{u}_K$ denote the link's acceleration, gap opening, as well as inerter, yielding, and elastic deformation related accelerations, respectively. And $\dot{J}_i$ denotes the force in the link.

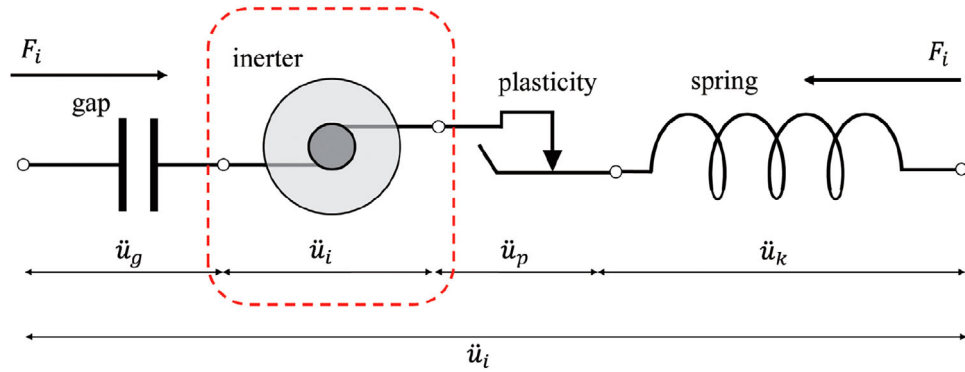


FIGURE 2 Proposed inerter element in the i th link.

According to the definition of ideal inerters, $\dot{J}_i = b_i \ddot{u}_i$, the energy stored in the inerter E_i can be derived as:

$$E_i = -W_i = - \int \dot{J}_i du_i = \int b_i \ddot{u}_i \dot{u}_i dt = b_i \dot{u}_i^2 - \int b_i \ddot{u}_i \dot{u}_i dt = \frac{1}{2} b_i \dot{u}_i^2 = \frac{(J_i + J_i^0)^2}{2b_i} \quad (13)$$

where J_i^0 is the initial momentum of the inerter which is usually assumed to be zero. Thus, the total stored energy can be expressed as:

$$E_{inerters} = \sum \frac{J_i^2}{2b_i} \quad (14)$$

Following the same procedure of MLF derivation, the governing equations involving inerters are obtained by inserting Equation (14) into the Lagrangian function and applying Euler-Lagrange equations:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{B}\dot{\mathbf{J}} = \mathbf{P} \quad (15)$$

$$\mathbf{A}\ddot{\mathbf{J}} + \frac{\partial \phi(\dot{\mathbf{J}})}{\partial \dot{\mathbf{J}}} + \frac{d}{dt} \left(\frac{\partial g(\dot{\mathbf{J}})}{\partial \dot{\mathbf{J}}} \right) + \mathbf{m}_r \dot{\mathbf{J}} = \mathbf{B}^T \dot{\mathbf{u}} \quad (16)$$

where, $\mathbf{m}_r = \begin{bmatrix} \frac{1}{b_1} & & \\ & \ddots & \\ & & \frac{1}{b_i} \end{bmatrix}$, b_i is the inertance in the i th link, while $b_j = \infty$ for the j th link without inerter.

Comparing Equations (15) and (16) with the original MLF governing equations (Equations 8 and 9), the equilibrium equation remains the same while an additional term $\mathbf{m}_r \dot{\mathbf{J}}$ appears in the compatibility equation representing the deformation rate of the inerter. Following the same discretization scheme, the following expression is obtained:

$$\bar{\mathbf{A}} \mathbf{F}_{n+1} + \frac{1}{2} \frac{\partial \phi}{\partial \mathbf{F}} \bigg|_{n+1} + \frac{1}{h} \frac{\partial g}{\partial \mathbf{F}} \bigg|_{n+1} + \bar{\mathbf{b}}_n = \mathbf{0} \quad (17)$$

where:

$$\begin{aligned} \bar{\mathbf{A}} &= \left[\frac{\mathbf{A}}{h} + \frac{\mathbf{B}^T}{2} \left(\frac{\mathbf{M}}{h} + \frac{\mathbf{C}}{2} \right)^{-1} \frac{\mathbf{B}}{2} + \frac{\mathbf{m}_r h}{4} \right] \\ \bar{\mathbf{b}}_n &= \left[-\frac{\mathbf{A}}{h} + \frac{\mathbf{B}^T}{2} \left(\frac{\mathbf{M}}{h} + \frac{\mathbf{C}}{2} \right)^{-1} \frac{\mathbf{B}}{2} + \frac{\mathbf{m}_r h}{4} \right] \mathbf{F}_n + \frac{1}{2} \frac{\partial \phi}{\partial \mathbf{F}} \bigg|_n - \frac{1}{h} \frac{\partial g}{\partial \mathbf{F}} \bigg|_n - \frac{\mathbf{B}^T}{2} \left[\left(\frac{\mathbf{M}}{h} + \frac{\mathbf{C}}{2} \right)^{-1} \left(\frac{\mathbf{M}}{h} - \frac{\mathbf{C}}{2} \right) + \mathbf{I} \right] \mathbf{v}_n \\ &\quad - \frac{\mathbf{B}^T}{2} \left(\frac{\mathbf{M}}{h} + \frac{\mathbf{C}}{2} \right)^{-1} \left(\frac{\mathbf{P}_{n+1} + \mathbf{P}_n}{2} \right) + \mathbf{m}_r \int \mathbf{F} dt \bigg|_n \end{aligned} \quad (18)$$

It is obvious that involving an inerter element does not change the structure of the time stepping scheme of MLF. The inerter-related term $\mathbf{m}_r \dot{\mathbf{J}}$ in the governing equation is resolved into coefficients $\bar{\mathbf{A}}$ and $\bar{\mathbf{b}}$ after discretization. To be

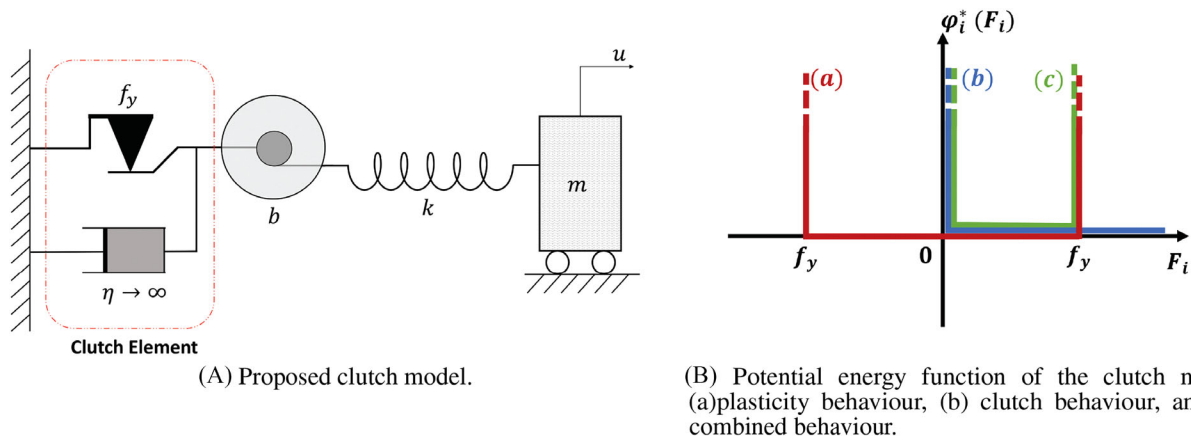


FIGURE 3 The proposed clutch element and its potential energy function.

consistent with the central difference scheme, the integration term $\int \mathbf{F} dt|_n$ can be expanded into a finite summation using the trapezoidal law. The additional term $\mathbf{m}_r h/4$ in $\bar{\mathbf{A}}$ is a non-negative diagonal matrix, thus $\bar{\mathbf{A}}$ is still positive definite, therefore, the equivalent QP problem is still valid and solvable.

2.3 | Modeling the clutching effect

The clutching effect can be physically achieved using a ratchet where force transmission occurs only when the active component's velocity exceeds that of the driven component. To facilitate the numerical modeling of the clutching effect, a plastic element is used as illustrated in Figure 3. The element comprises of a viscous plastic element with a viscous coefficient η approaching to zero.⁴⁵ The clutching effect is achieved by defining an infinite yield strength $f_{engage} = \infty$ in the direction of engagement and zero yield strength $f_{dis-engage} = 0$ in the opposite direction for disengagement. When the element force is aligned with the direction of engagement, this 'clutch element' acts as a rigid connection while the element will yield immediately and transmit no force upon force reversal. The fundamental difference between this clutch element and a contact (gap) element is that the kinematics of the clutch element are velocity-based while the kinematics of contact element are displacement-based. The clutch element is able to reconnect (re-engage) regardless of displacement which reflects the physics of a clutching mechanism. Furthermore, if a finite yield strength $f_{engage} = f_y$ is adopted, the proposed element can exhibit a combined clutching and elastic-perfect plastic behavior as depicted in the potential function shown in Figure 3.

Note-worthily, the proposed modification to the viscous plastic element model only changes the shape of $\phi(\dot{\mathbf{F}})$ to $\phi^*(\dot{\mathbf{F}})$ at the links where an inerter exists, while the structure of the QP in each time step remains unchanged. Introducing the new potential function ϕ^* into the time step discretization (Equation 17) leads to:

$$\bar{\mathbf{A}}\mathbf{F}_{n+1} + \frac{1}{2} \frac{\partial \phi}{\partial \mathbf{F}} \bigg|_{n+1} + \frac{1}{2} \frac{\partial \phi^*}{\partial \mathbf{F}} \bigg|_{n+1} + \frac{1}{h} \frac{\partial \mathbf{g}}{\partial \mathbf{F}} \bigg|_{n+1} + \bar{\mathbf{b}}_n = \mathbf{0} \quad (19)$$

The additional term $\frac{1}{2} \frac{\partial \phi^*}{\partial \mathbf{F}} \bigg|_{n+1}$ that represents clutch idling corresponds to an additional constraint in the equivalent QP problem, such that:

$$\begin{aligned} \min & \frac{1}{2} \mathbf{F}_{n+1}^\top \bar{\mathbf{A}} \mathbf{F}_{n+1} + \bar{\mathbf{b}}^\top \mathbf{F}_{n+1} \\ \text{s.t.} & \\ & |\mathbf{D}_\phi \mathbf{F}_{n+1}| \leq f_y \\ & \mathbf{D}_\phi^* \mathbf{F}_{n+1} \leq 0 \\ & \mathbf{D}_g \mathbf{F}_{n+1} \leq 0 \end{aligned} \quad (20)$$

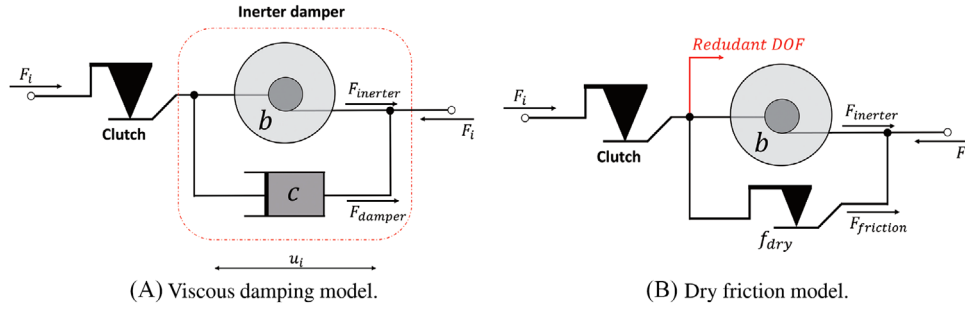


FIGURE 4 Supplemental energy dissipation models of the proposed inerter element.

where $\mathbf{D}_\phi$, $\mathbf{D}_\phi^*$ and $\mathbf{D}_g$ are sign matrices containing values of 0, 1, or -1 in their diagonal, denoting the presence and direction of the functional element: (i) 0 and 1 in $\mathbf{D}_\phi$ indicate the presence of plasticity elements, (ii) 0, 1, and -1 in $\mathbf{D}_\phi^*$ convey both the presence and the direction of clutch elements, and (iii) 0, 1, and -1 in $\mathbf{D}_g$ denote the presence of tension or compression gap elements. With this formulation, the clutching effect can be removed by removing the second constraint for modeling non-clutched inerters.

During the time stepping QP optimization process, if any of the constraints is activated in a link, it will be automatically disengaged, open (fracture) or yield since the transmitted force is restrained to be zero or f_y by the constrained optimization algorithm. The cumulative distance during disengagement or residual strain as well as gap opening can be retrieved from the Lagrange multipliers:

$$v_{plastic} = \frac{\partial \phi}{\partial \mathbf{F}} \Big|_{n+1} = 2\mathbf{D}_\phi \lambda_\phi \quad v_{clutch} = \frac{\partial \phi^*}{\partial \mathbf{F}} \Big|_{n+1} = 2\mathbf{D}_\phi^* \lambda_\phi^* \quad u_{gap} = \frac{\partial g}{\partial \mathbf{F}} \Big|_{n+1} = h\mathbf{D}_g \lambda_g \quad (21)$$

where, $v_{plastic}$ is the plastic deformation rate, v_{clutch} is the clutch idling speed and u_{gap} is the gap opening distance. λ_ϕ , λ_ϕ^* , and λ_g are the Lagrange multipliers corresponding to the three inequality constraints in Equation (20).

2.4 | Inerter energy dissipation

The inerter and clutch elements formulated above exclude any energy dissipation except for material plasticity. In reality, the energy dissipation of inerters consists of the frictional loss of mechanical transmission plus any supplemental means. The former mainly refers to friction between the ratchet and pawl or rollers and races (roller clutch) when the clutch is idling, and the latter refers to additional dissipators such as viscous dampers and frictional braking pads. Supplementary energy dissipation has proven to be vital to improve the performance of inerter devices.^{27,28} For simplicity, a constant idling friction of relative motion can be easily modeled by substituting the zero yielding force of the clutch element by a certain magnitude of dry friction force (f_{dry}). In fact, modeling dry friction through plasticity analogy is widely adopted in the mechanical engineering field.⁵⁷ Thus, the second constraint in Equation (20) becomes:

$$\mathbf{D}_\phi^* \mathbf{F}_{n+1} \leq f_r \quad (22)$$

As for supplemental energy dissipation, various options have been developed including immersing the inerter flywheel into a fluid chamber¹ or connecting the inerter with a magnetic rotor.⁵⁸ In these cases, the damper will exert viscous or dry friction forces only to decelerate the inerter without affecting other elements in the link as illustrated in Figure 4.

However, MLF is a force-based method, thus including parallel elements will significantly increase the number of unknowns (redundant DOFs as shown in Figure 4B) and potentially hamper the computational efficiency of the procedure. Therefore, an integrated inerter damper model marked by the red dashed line in Figure 4A is preferred. Formulating an integrated inerter damper model is mathematically feasible because the force compatibility and constitutive law of the proposed CID element compose a solvable first order ordinary differential equation:

$$b\ddot{u}_i + c\dot{u}_i = F_i(t) \quad (23)$$

whose solution is:

$$v_i(F_i(t), t) = e^{-ct/b} \left[v_0 + \frac{1}{b} \int e^{ct/b} F_i(t) dt \right] \quad (24)$$

Which can be expressed in matrix form for complex systems as:

$$\mathbf{V}_i(\mathbf{F}, t) = e^{-ct\mathbf{m}_r} \left[\mathbf{V}_0 + \mathbf{m}_r \int e^{ct\mathbf{m}_r} \mathbf{F} dt \right] \quad (25)$$

$\mathbf{V}_i(\mathbf{F}, t)$ represents the deformation rate of the proposed inerter damper element which is equivalent to the term $\mathbf{m}_r \mathbf{J}$ in the ideal energy conservative inerter element formulation (Equation 17). It can be shown that if a zero damping coefficient is applied ($c = 0$), the right side of Equation (25) will reduce to $\mathbf{m}_r \mathbf{J} + V_0$, where V_0 is the initial velocity of inerter. In the MLF framework, this integrated inerter damper model can be implemented by replacing the term $\mathbf{m}_r \mathbf{J}$ in Equation (25).

To model frictional energy dissipating devices (e.g., braking pads), a plasticity element with a yield strength equal to the sliding friction force can be used and connected in parallel to the inerter element as shown in Figure 4B. Due to the non-smooth hysteresis of friction, it is not feasible to derive an explicit expression for this frictional braking model. However, this type of energy dissipation can be solved by including redundant unknowns of velocities as stated previously.

2.5 | Numerical scheme

It follows from the theory and derivations presented above, that the newly proposed element integrates features of plasticity, contact, inertance, clutching effects, and energy dissipation within the MLF framework. This, in turn, will require modifications in the time stepping scheme (QP problem) of systems equipped with CIDs in the MLF framework, as follows:

$$\begin{aligned} \min \quad & \frac{1}{2} \mathbf{F}_{n+1}^\top \bar{\mathbf{A}} \mathbf{F}_{n+1} + \bar{\mathbf{b}}^\top \mathbf{F}_{n+1} \\ \text{s.t.} \quad & |\mathbf{D}_\phi \mathbf{F}_{n+1}| \leq \mathbf{f}_y \\ & \mathbf{D}_\phi^* \mathbf{F}_{n+1} \leq \mathbf{f}_r \\ & \mathbf{D}_g \mathbf{F}_{n+1} \leq \mathbf{f}_c \end{aligned} \quad (26)$$

where:

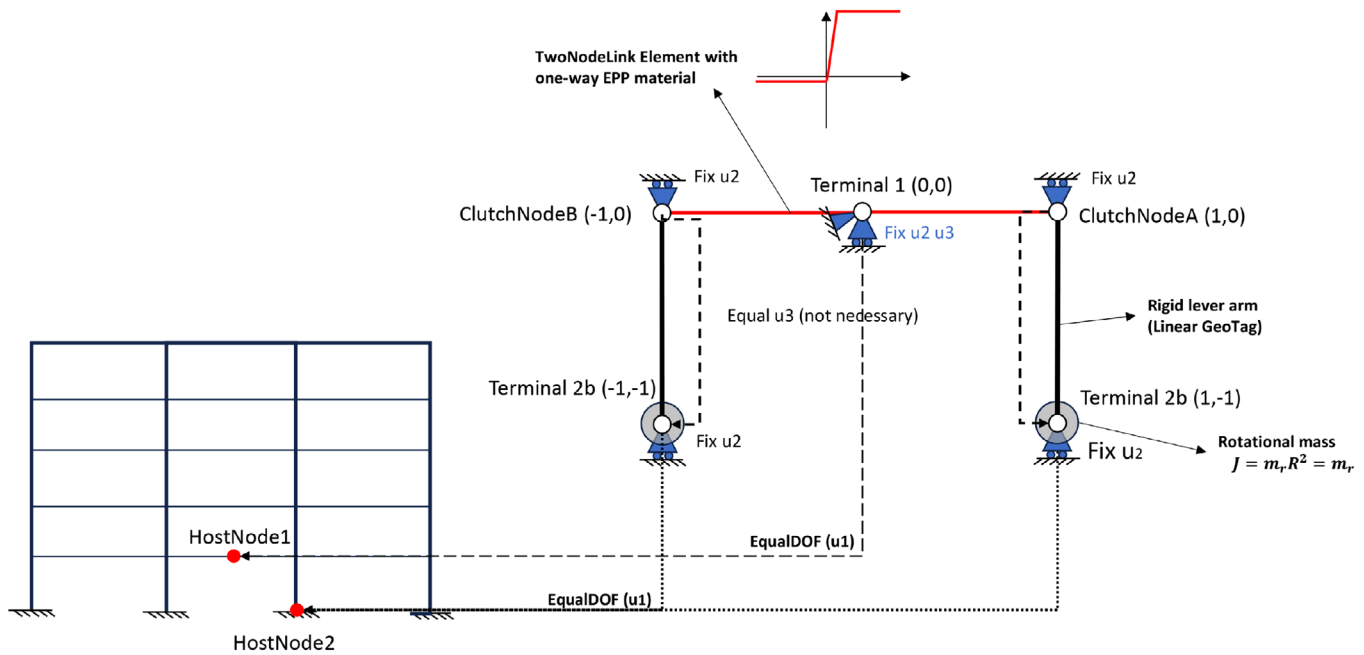
$$\begin{aligned} \bar{\mathbf{A}} &= \left[\frac{\mathbf{A}}{h} + \frac{\mathbf{B}^\top}{2} \left(\frac{\mathbf{M}}{h} + \frac{\mathbf{C}}{2} \right)^{-1} \frac{\mathbf{B}}{2} + \frac{\mathbf{m}_r h}{2} \right] \\ \bar{\mathbf{b}} &= \left[-\frac{\mathbf{A}}{h} + \frac{\mathbf{B}^\top}{2} \left(\frac{\mathbf{M}}{h} + \frac{\mathbf{C}}{2} \right)^{-1} \frac{\mathbf{B}}{2} + \frac{\mathbf{m}_r h}{2} e^{-\mathbf{c}_r \mathbf{m}_r h} \right] \mathbf{F}_n + \frac{1}{2} \frac{\partial \phi}{\partial \mathbf{F}} \Big|_n - \frac{1}{h} \frac{\partial g}{\partial \mathbf{F}} \Big|_n - \frac{\mathbf{B}^\top}{2} \left[\left(\frac{\mathbf{M}}{h} + \frac{\mathbf{C}}{2} \right)^{-1} \left(\frac{\mathbf{M}}{h} - \frac{\mathbf{C}}{2} \right) + \mathbf{I} \right] \mathbf{v}_n \\ &\quad - \frac{\mathbf{B}^\top}{2} \left(\frac{\mathbf{M}}{h} + \frac{\mathbf{C}}{2} \right)^{-1} \left(\frac{\mathbf{P}_{n+1} + \mathbf{P}_n}{2} \right) + \frac{\mathbf{m}_r}{2} [e^{-\mathbf{c}_r \mathbf{m}_r n h} + e^{-\mathbf{c}_r \mathbf{m}_r (n+1) h}] \mathbf{J}_n + \frac{1}{2} \frac{\partial \phi^*}{\partial \mathbf{F}} \Big|_n \\ \mathbf{J}_n &= \mathbf{J}_{n-1} + \frac{1}{2} [\mathbf{F}_{n-1} \cdot e^{\mathbf{c}_r \mathbf{m}_r (n-1) h} + \mathbf{F}_n \cdot e^{\mathbf{c}_r \mathbf{m}_r n h}] \\ \mathbf{v}_n &= \left(\frac{\mathbf{M}}{h} + \frac{\mathbf{C}}{2} \right)^{-1} \left[\left(\frac{\mathbf{M}}{h} - \frac{\mathbf{C}}{2} \right) \mathbf{v}_{n-1} - \mathbf{B} \left(\frac{\mathbf{F}_n + \mathbf{F}_{n-1}}{2} \right) + \left(\frac{\mathbf{P}_n + \mathbf{P}_{n-1}}{2} \right) \right] \end{aligned} \quad (27)$$

Here, $\bar{\mathbf{A}}$ is still a definite constant matrix (for constant $\mathbf{B}$), $\mathbf{f}_y$ is the yield strength vector of elemental internal forces, $\mathbf{f}_r$ is the operational inerters dry friction vector, $\mathbf{f}_c$ is the gap elements detach threshold vector, $\mathbf{c}_r$ is the inerters damping coefficient vector, $\mathbf{J}_n$ is the inerters accumulative momentum. The updating of terms $\frac{\partial \phi}{\partial \mathbf{F}}$, $\frac{\partial \phi^*}{\partial \mathbf{F}}$ and $\frac{\partial g}{\partial \mathbf{F}}$ relies on the multipliers, which have been defined in Equation (21). A standard process for the improved MLF framework involving

TABLE 1 Dynamic analysis procedure of the proposed MLF framework.

Steps	Operations
Step1	Generate characteristic matrices A , B , C , M , m_r , and indicator matrices D_φ , D_φ[*] and D_g
Step2	Initialization of F ₀ , V ₀ , $\left. \frac{\partial \phi}{\partial \mathbf{F}} \right _0$, $\left. \frac{\partial \phi^*}{\partial \mathbf{F}} \right _0$, $\left. \frac{\partial \mathbf{g}}{\partial \mathbf{F}} \right _0$
Step3	Compute $\bar{\mathbf{A}}$ and $\bar{\mathbf{b}}_n$
Step4	Compute $\mathbf{F}_{n+1}$ and multipliers λ_{ϕ} , λ_{ϕ}^* , and λ_g by solving the QP problem
Step5	March the time step ($t_n \rightarrow t_{n+1}$)
Step6	Update $\mathbf{V}_{n+1}$, $\left. \frac{\partial \phi}{\partial \mathbf{F}} \right _{n+1}$, $\left. \frac{\partial \phi^*}{\partial \mathbf{F}} \right _{n+1}$, $\left. \frac{\partial \mathbf{g}}{\partial \mathbf{F}} \right _{n+1}$
Step7	Return to step 3 until $t_n \geq t_{end}$

Abbreviations: MLF, mixed Lagrangian formulation; QP, quadratic programming.

**FIGURE 5** Equivalent OpenSees CID model. CID, clutched inerter damper.

CIDs is presented in Table 1. This has been implemented in a Matlab. To this end, the ‘QuadProg’ function with the ‘interior-point-convex’ algorithm from Matlab’s Optimization Toolbox is used (see Step4 in Table 1).

2.6 | Equivalent FEM model

The modeling strategy proposed in the previous sections is not exclusive to the MLF framework. In fact, the clatching behavior can be simulated in classic finite element software such as OpenSees through an elastic perfect plastic material response. To illustrate this, an integrated OpenSees model of a twin CID system is shown in Figure 5. This configuration was inspired by the displacement-based contact clutch model proposed by Málaga-Chuquitaype et al.³² The system consists of three terminal nodes representing the terminals of the inerter and two clutch nodes modeling the clatching behavior. An angular mass is assigned to the rotational degrees of freedom in Terminal 2a and Terminal 2b to model the inertial effect. The inerter damping can be implemented by individually assigning Rayleigh damping to these degrees of freedom as well. These terminals are connected to ClutchNodeA and ClutchNodeB through rigid lever arms modeled by elasticBeamColumn elements, which convert translational movement into rotations. A linear geometric transformation must be assigned to the lever arms to maintain their kinematics on a circular path, even when the clutch nodes have displaced far from their original positions. The clutch nodes are then connected to Terminal 1 by clutch links modeled by TwoNodeLink elements with a rigid ($E \rightarrow \infty$) Elastic-Perfectly Plastic (EPP) material. This material has effectively zero

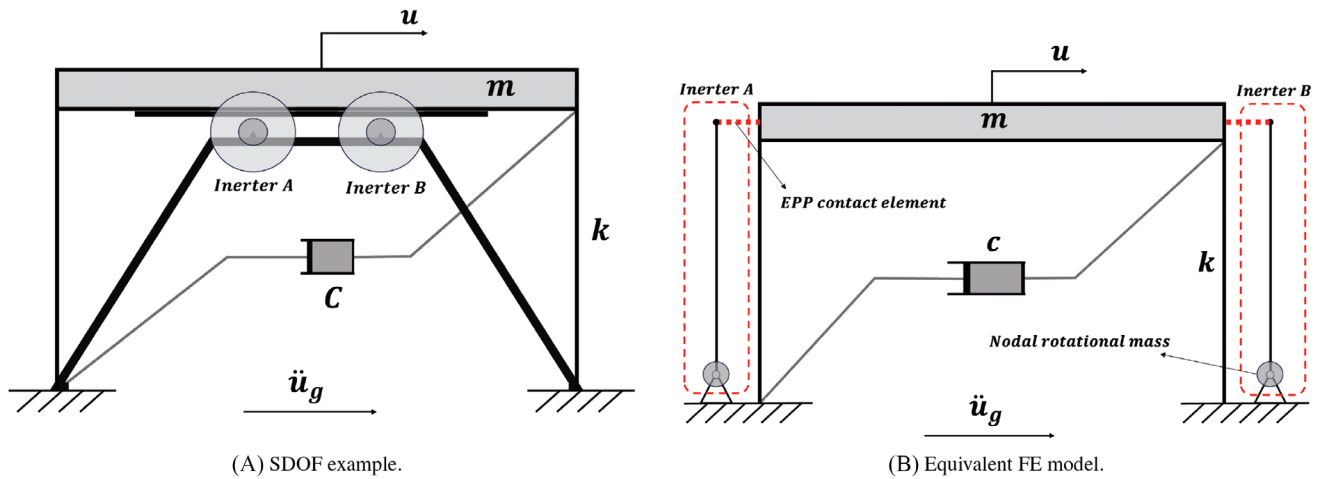


FIGURE 6 SDOF shear frame and its equivalent finite element model in OpenSees. Lateral stiffness $k = 100 \text{ N} \cdot \text{m}^{-1}$, mass $m = 1 \text{ kg}$, mass related viscous damping $C = 0.4 \text{ N} \cdot (\text{m/s})^{-1}$, inertance $b_a = b_b = 0.5 \text{ kg}$, inertance related viscous damping $c_r = 0.4 \text{ N} \cdot \text{s}^{-1}$, clutch friction $f_r = 0.02 \text{ N}$, $\ddot{u}_g = 0.5 \sin(10t)$, $t \in [0, \pi/10]$.

yield strain in one direction and effectively infinite yield strain in the opposite direction. It is important to note that the zero and infinite yield strains lead to numerical issues in OpenSees; therefore, a relatively small value should be assigned, representing the idling friction of the inerter; or conversely a relatively large value.

All vertical degrees of freedom are restrained. The horizontal displacements of Terminal 1 and Terminal 2 are set equal to the story displacement in desired installation direction. Compatibility is also enforced between the rotation of the clutch nodes and the corresponding Terminal 2 nodes to eliminate redundant degrees of freedom and maximize the rigidity of the lever arm. For convenience, all dimensions can be set to 1, making the rotational mass mathematically equal to the translational mass.

In this construction, the reactive force developed is proportional to the horizontal relative acceleration between host node 1 and 2 during engagement and includes a small value representing the idling friction during disengagement. Since this is the defining characteristic of an clutched inerter. The one-way-yield links control the engagement state. Additional mass related Rayleigh damping can be assigned to simulate viscous dissipation of the inerter.

3 | CASE STUDIES

3.1 | SDOF shear frame

To illustrate the use of the newly developed MLF CID element, we first examine a SDOF shear frame equipped with twin-clutched-inerters as shown in Figure 6A. In addition to the MLF method proposed herein, several other popular methods are implemented for comparison purposes including a sequential model,²⁸ a penalty contact model²⁷ as well as an equivalent finite element model (implemented in OpenSees) as shown in Figure 6B. Other nonlinear behavior such as gap contact and material plasticity are excluded from this example since they have been validated before.^{45,48} The purpose of this case study is to demonstrate the effectiveness of the newly proposed CID element in the MLF framework and evaluate the clutching behavior of CIDs.

The internal force vector of the structure under consideration can be written as $F = [F_k \ F_{ia} \ F_{ib}]^T$ and its local flexibility matrix as $A = \text{diag} \left[\frac{1}{k} \ 0 \ 0 \right]$, where diag = diagonal matrix of the vector, and the zeros indicate an ideal infinite contact stiffness. Assuming the clutch direction is the same as that of the displacement, the sign convention matrix is $D_\phi^* = \text{diag} [0 \ 0 \ 0]$ for the clutching inerter constraint. The inverse inertance matrix is $m_r = \text{diag} \left[0 \ \frac{1}{b} \ -\frac{1}{b} \right]$. The frame is excited by a sinusoidal impulsive load $\ddot{u}_g = 0.5 \sin(10t)$ with $t \in [0, \pi/10]$. A time step size of $h = 0.01 \text{ s}$ is adopted in the MLF solution. On the other hand, an equivalent model is built in OpenSees and solved with the Newmark-beta algorithm ($\gamma = 0.5, \beta = 0.25$) with a step size of $h = 0.001 \text{ s}$. Note that the step size used in sequential models and penalty contact models are adaptively controlled by Matlab's ODE45 function.

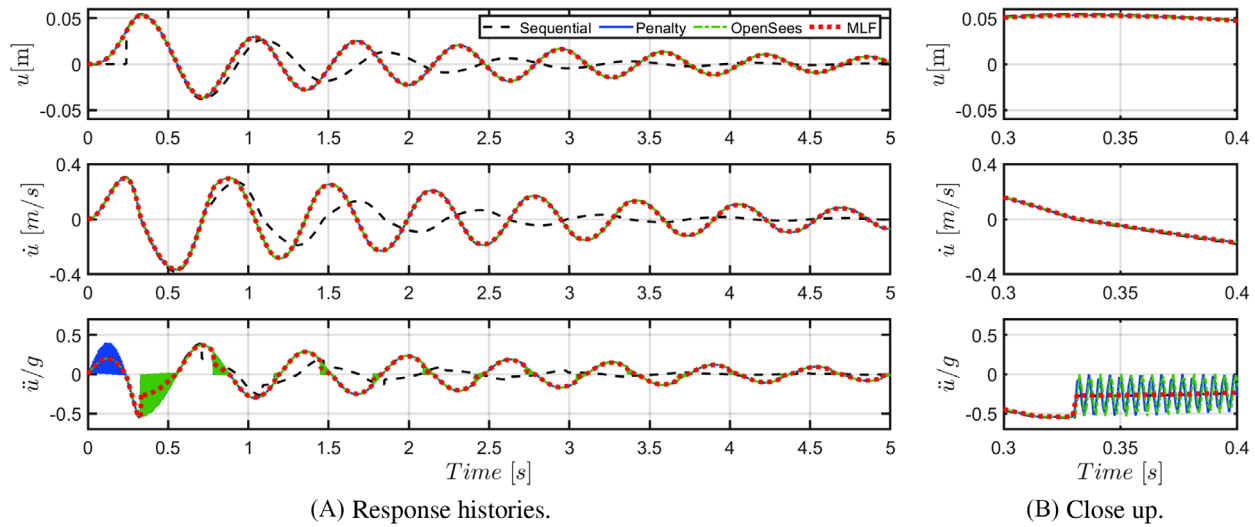


FIGURE 7 Displacement, velocity, and acceleration response histories of the single degree of freedom example predicted by sequential, penalty contact, equivalent OpenSees and MLF models. MLF, mixed Lagrangian formulation.

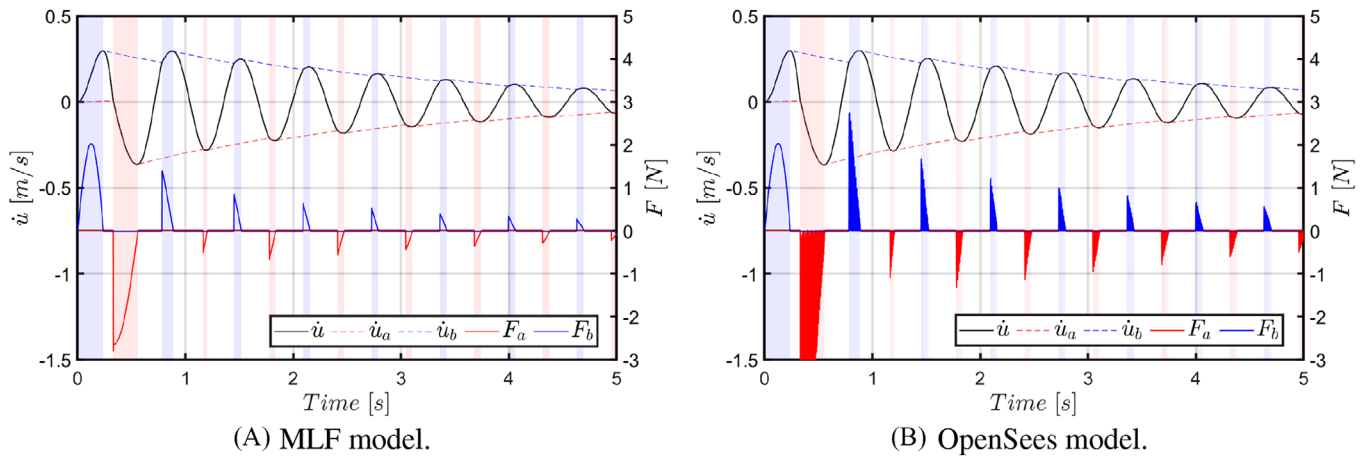


FIGURE 8 CIDs behavior predicted by the equivalent OpenSees model and the MLF model, where $\dot{u}$ is the velocity of the mass, and $\dot{u}_a, \dot{u}_b, F_a, F_b$ are the velocities and forces of left and right inerter respectively. The light shaded areas indicate the clutch engagement events. CIDs, clutched inerter devices; MLF, mixed Lagrangian formulation.

The displacement, velocity, and acceleration response histories predicted by the four methods mentioned above are presented in Figure 7. It is evident that the system responses predicted by the MLF model, the penalty contact model, and the equivalent OpenSees model are consistent with each other, showing only minor deviations between them. Notably, the acceleration response generated by the MLF exhibits a smoother profile, while chattering is observed in both the penalty contact model and the equivalent OpenSees model, leading to an over-estimation of peak acceleration and clutch force demands. The chattering phenomena and overall stability is greatly affected by the combination of contact stiffness of the EPP element and step size. Generally, a higher contact stiffness requires a smaller step size to capture the contact without delay. On the contrary, the MLF model is robust even with an infinite contact stiffness.

Additionally, the responses predicted by the MLF, penalty contact model, and OpenSees models show less attenuation when compared to the results predicted by the simplified sequential model. This difference can be explained with reference to Figure 8, where the inerter velocity experiences only limited decreases between each engagement event under the assumed damping condition. As a result, the total duration of engagement, indicated by the shaded areas in this figure, constitutes only a small proportion of the entire duration. Figure 9 shows the hysteresis of the inerter elements predicted by MLF model and Sequential model (OpenSees and Penalty contact model share the same pattern with the MLF model). It is obvious that MLF lead to less energy dissipation (enclosed area within the laces) in subsequent engagement events

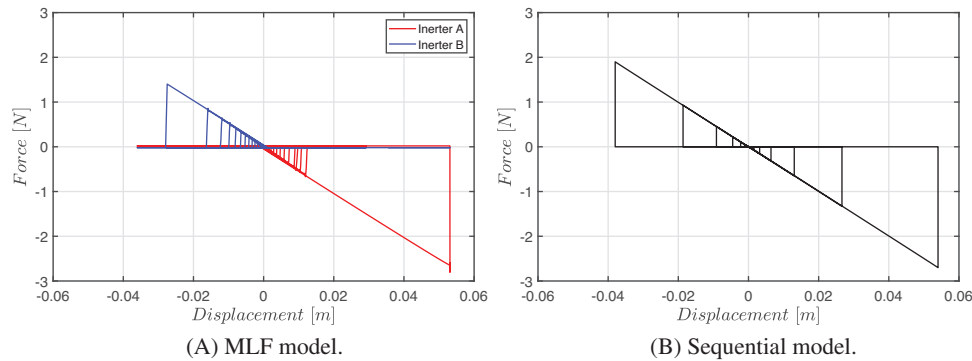


FIGURE 9 The hysteresis of CID elements in MLF model and ideal clutched inerter in sequential model. Note that OpenSees and penalty contact model share the same pattern with MLF. CID, clutched inerter damper; MLF, mixed Lagrangian formulation.

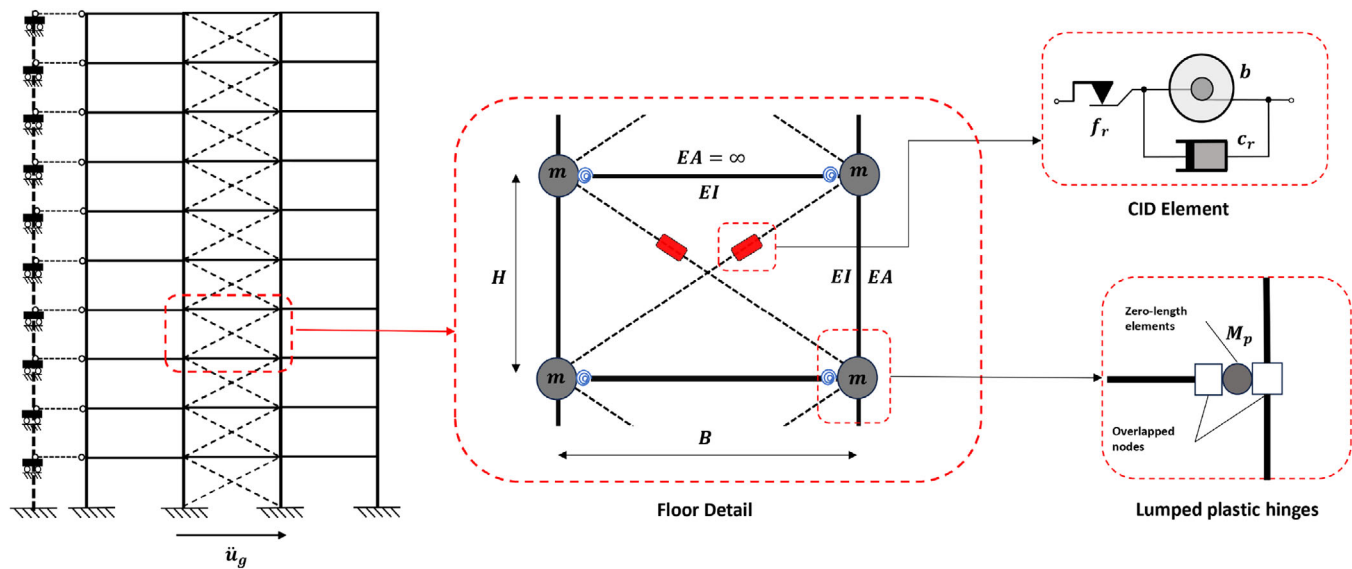


FIGURE 10 Configuration of the multistorey MRF with diagonal inerters. Bay width $B = 4$ m, storey height $H = 3$ m, columns and beams are W310 $\times$ 129 and W250 $\times$ 33, respectively, and are made of S235 steel, joint plastic moment capacity $M_p = 100$ kN $\cdot$ m, nodal mass $m = 15$ ton, suppression coefficient $\sigma = 1$, inerter operational friction $f_r = 0.01$ kN, inerter damping coefficient $c_r = 0.4$ kNm/s⁻¹. MRF, moment resisting frame.

after the first engagement. This observation aligns with the physics intuition that the assumption in the sequential model that ‘inerters dissipate all their energy in each cycle,’ is not practically easily achieved.

3.2 | Multistorey frame

The MLF method is not restricted to simple linear systems.^{45,46} In this section, a multistorey moment resisting frame (MRF), as shown in Figure 10, is analyzed to demonstrate the superiority of the MLF for larger systems. The frame’s beams and columns are modeled by means of elastic Euler-Bernoulli beam elements with lumped plasticity at beam-to-column joints with elastic perfect plastic relationships. The structure is equipped with twin CID at each floor acting in the lateral direction. Inherent structural damping is neglected so that all energy dissipation is concentrated in the CID elements and plastic hinges to facilitate comparisons. P-Delta effects are considered through the leaning column approach.⁵⁹ Note that it is feasible to implement full geometric nonlinearities by updating [B] matrix based on the current geometry at each time step, just like what we do with transformation matrix [T] in classic FE analysis. The MRF is subjected to gravity loads and ground motion of 1940 El Centro earthquake.

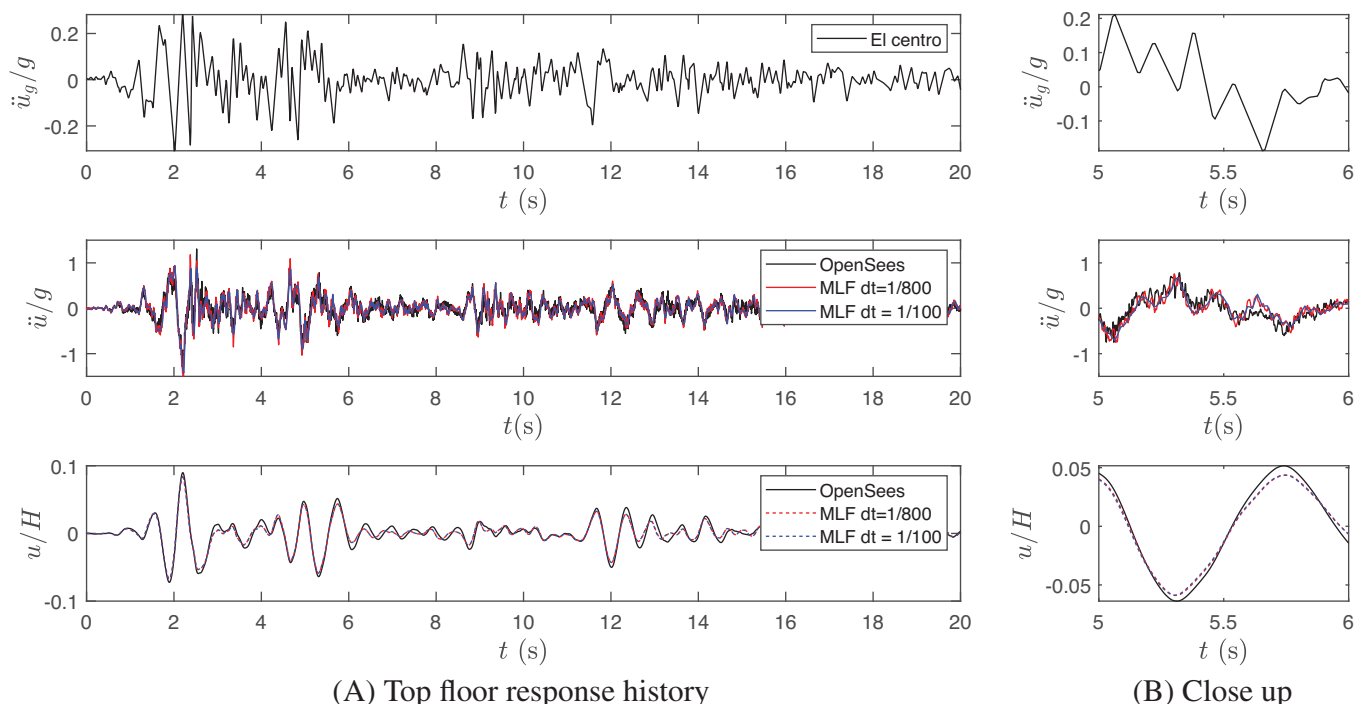


FIGURE 11 Top floor displacement, inter-story drift, acceleration response history of the 10-storey frame bracing example and close up of $T = [0.7 \ 1.1]$.

For comparison, an equivalent OpenSees model is also constructed in which the beams and columns are modeled by means of 'elasticBeamColumn' elements and lumped plastic hinges are introduced via *zero length* elements with an elastic perfect plastic material as shown in Figure 10. The modeling of inerters follows the same strategy specified in Section 2.6. In summary, the MLF model consists of 23 independent dynamic DOFs of internal forces per floor (seven beam elements and two inerter elements, $7 \times 3 + 2 = 23$), while the equivalent OpenSees model consists of 22 independent dynamic DOFs per floor (four storey nodes, six plastic hinges and four inerter nodes $4 \times 3 + 6 + 4 = 22$). It is also worth noting that lumped plasticity at column-to-column joints is inherently considered in the MLF framework even if the plastic limit is not reached, while the OpenSees model can only model the plastic hinges at beam-to-column joints. In this case, the two models have comparable matrix size.

The 10-storey MRF was subjected to the 1940 El Centro earthquake record at its base. Figure 11 shows agreements of the top floor response histories predicted by the two methods with time increments of $\Delta t = 1/100$ s and $\Delta t = 1/800$ s for MLF and $\Delta t = 1/800$ s for OpenSees, respectively. Figure 12 presents the error convergence analysis of both models in terms of Relative Root Mean Square Error (RRMSE)⁶⁰ and Normalized Prediction Error (NPE), for displacements and accelerations at all storeys. These analyses are performed with time increments ranging from 1/25 s to 1/25600 s.

According to Figure 12, both methods reach a good accuracy of RRMSE and NPE of displacement and acceleration prediction with step size decreasing. But the curves for the OpenSees model are truncated at $\Delta t = 1/800$ s because larger steps lead to numerical instability. The maximum feasible time increment is dependent on the stiffness of the EPP element (10^3 times that of the structural elements herein) and could be further reduced if a stiffer contact stiffness was used. In contrast, the MLF method, where the clutch stiffness can be ideally infinity, obtained gradual increase of error with larger step size. The displacement RRMSE and NPE predicted by MLF with a step of $\Delta t = 1/100$ s are only 0.8% and 1.0%, respectively.

Presented in Figure 13 are the computational efforts of the two models measured in terms of the average number of iterations per step and CPU time with respect to increasing matrix size. The matrix size is increased by varying the number of stories of the proposed structure from 1 to 40. Conventional mesh refinement is not adopted here because it cannot effectively scale up the quantities of nonlinear elements, including plastic hinges and clutched inerters. All results were obtained using a 4.5 GHz Intel Core i5-12500H processor and 16 GB RAM. The interior-point-convex (default option) algorithm embodied in the 'quadprog' function in Matlab Optimization Toolbox and implicit Newmark-Beta ($\gamma = 0.5$, $\beta = 0.25$) integrator in OpenSees transient solver are adopted. Note that the problem size is defined by the size of Hessian

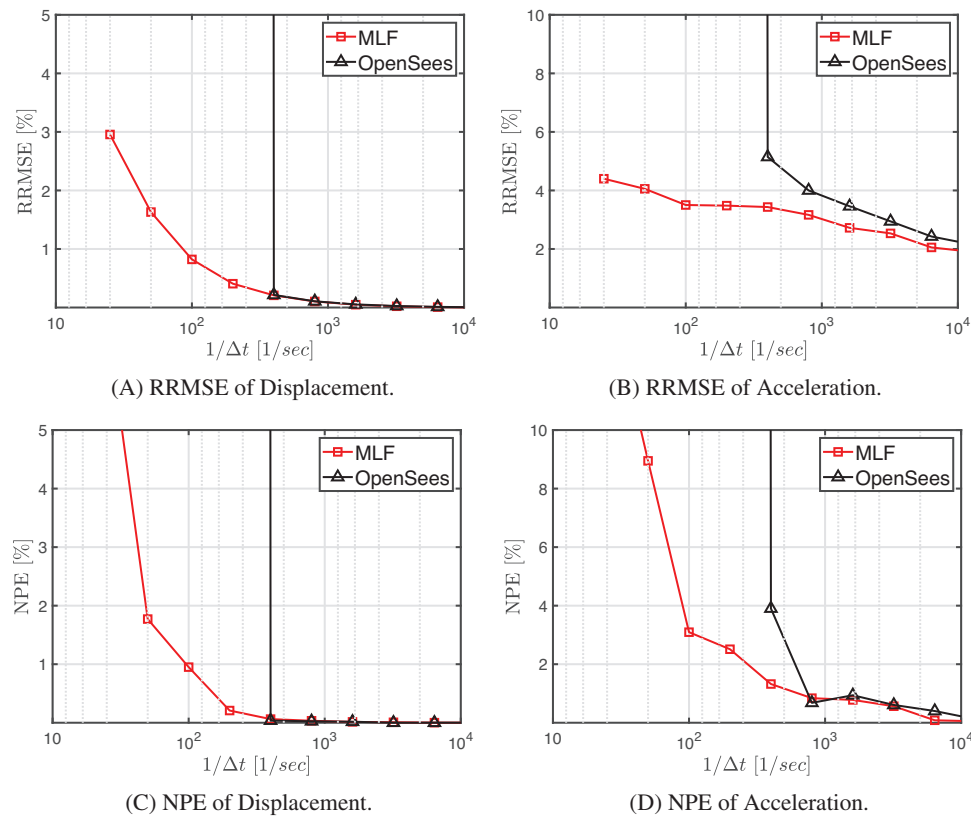


FIGURE 12 Error convergence analysis of the MLF and OpenSees models. MLF, mixed Lagrangian formulation.

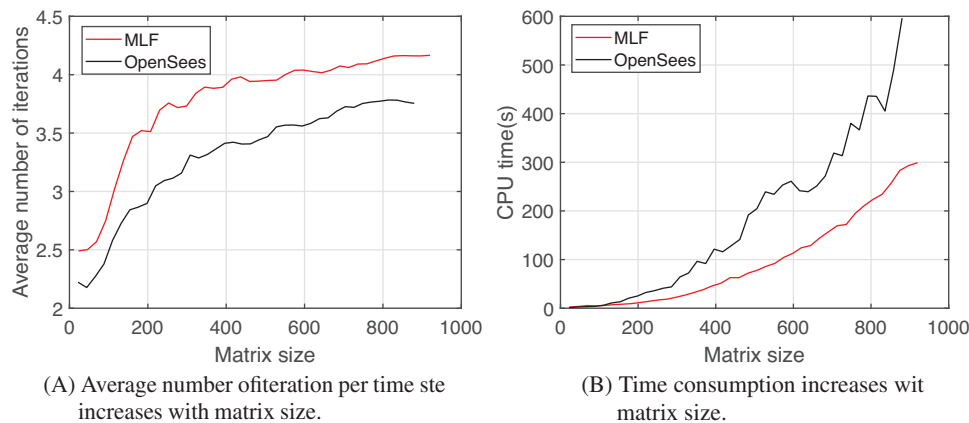


FIGURE 13 Computational effort with respect to increasing matrix size of the MLF and OpenSees models. MLF, mixed Lagrangian formulation.

matrix ($\bar{\mathbf{A}}$) of the QP problem in MLF and transient stiffness matrix in OpenSees. As noted before, the two models have comparable problem size (23 per storey for MLF and 22 per storey for OpenSees).

The computational effort of the OpenSees model using a time increment of $\Delta t = 1/800$ is compared with that of the MLF model using time increments of $\Delta t = 1/800$ s and $\Delta t = 1/100$ s for a 10-story structure, as presented in Table 2. While the MLF model with the same step size ($\Delta t = 1/800$ s) is more computationally demanding than the OpenSees model, it allows for the use of a larger time step size ($\Delta t = 1/100$ s) while maintaining sufficient accuracy and avoiding numerical instabilities. While the computational cost per time step of the MLF model exceeds that of the implicit Newmark-Beta algorithm in the OpenSees model. It should be noted that the efficiency of MLF is more pronounced when the number of global DOFs is small and the number of local state variables, for example associated with material behavior, is large.

TABLE 2 Computational efficiency comparison of two models.

Modeling method	Step size	Number of steps	Problem size ^a	Number of iterations	Total computer time ^b
OpenSees	1/800	25600	220	78036	32.3
MLF	1/800	25600	230	94616	96.8
MLF	1/100	3200	230	10762	14.4

Abbreviation: MLF, mixed Lagrangian formulation.

^aIn the MLF model, problem size $n \times m$ is defined by the size of Hessian matrix $\bar{A}$ ($n \times n$). In the FE model in OpenSees, the problem size is defined by the number of independent DOFs.

^bTotal Computer time is defined as the time consumed on iterative time marching algorithms excluded time of model assembly.

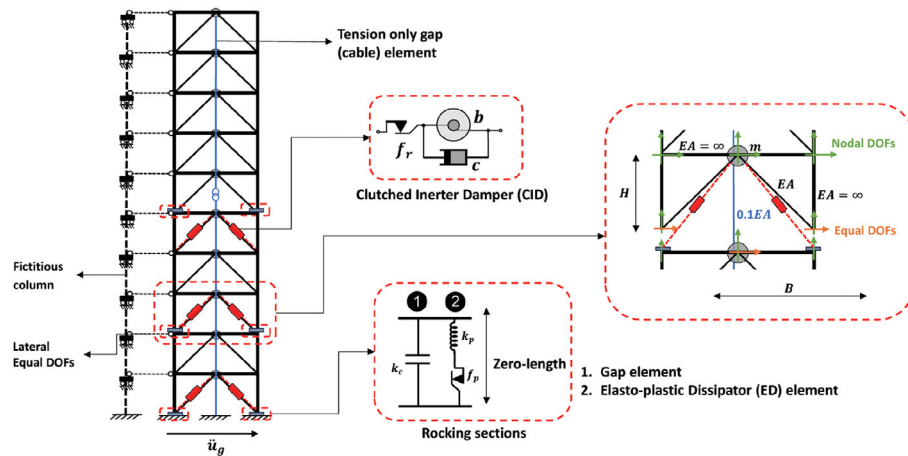


FIGURE 14 Ten-storey multi-rocking braced frame example. Bay width $B = 2$ m, storey height $H = 1$ m, axial rigidity $EA = 2 \times 10^4$ kN, nodal mass $m = 2 \times 10^3$ kg, inertance $b = 2 \times 10^3$ kg, inerter operational friction $f_r = 0.01$ kN, inerter damping coefficient $c_r = 1$ kN $\cdot$ (m/s)⁻¹, contact stiffness $k_c = 10^7$ kN/m, stiffness $k_p = 10^4$ kN/m and yield force $f_p = 5$ kN of ED element. ED, energy dissipator.

This is because the size of the optimization problem depends on the number of local forces in the system. Nonetheless, more sophisticated constitutive laws can be applied through a complex nonlinear constrain function together with more generalized optimization algorithms as described in Silvassevan et al.⁴⁶ Furthermore, according to Sivaselvan et al.,⁴⁶ the optimizer used in solving the QP problem in MLF can be tailored due to the sparse structure of Hessian matrix $\bar{A}$. In addition, MLF is implemented in Matlab, whereas OpenSees is programmed in C++ which is renowned for its computational efficiency. Therefore, it is foreseeable that programming MLF in C++ with appropriate code optimization will lead to better efficiency. Moreover, MLF also provides the option of trading off accuracy to ensure faster convergence through large step size in the early stage of the structural optimization process.^{53,54}

3.3 | Multistorey self-centering rocking frame

Rocking mechanisms can significantly reduce the excessive internal deformations and potential damage at the expense of large displacements concentrated in specific regions. Those rocking surfaces provide favorable places for energy dissipation and harvest devices including inerters.^{61,62} To demonstrate the compatibility between the proposed CID element and other nonlinear MLF elements such as gap and elasto-plastic elements, a 10 storey SC-CBF is modeled as shown in Figure 14. The structural elements including beams, columns, and bracing are modeled by truss elements. The SC-CBF includes rocking sections located at the ground, 2nd, and 5th levels. Although the dynamics of multi-rocking structures are highly nonlinear and extensively studied, choosing the optimum topology of rocking sections is out of the scope of this paper. Nonetheless, the current selection of the location of rocking interfaces is consistent with the optimized structure presented by Marzok and Lavan.⁵² At each rocking section, a twin CID device and a mild steel energy dissipator (ED), modeled by a elasto-plastic element, are deployed as illustrated in Figure 14. The CIDs are only effective in the lateral direction. A post-tensioning cable with a prestress level of 80 kN connects the foundation and top of the SC-CBF. The practicality of implementing a continued post-tensioning system for tall structures has been verified experimentally by Pei et al.⁶³ The axial deformations of beams are neglected. The horizontal DOFs at the base of columns at rocking sections are

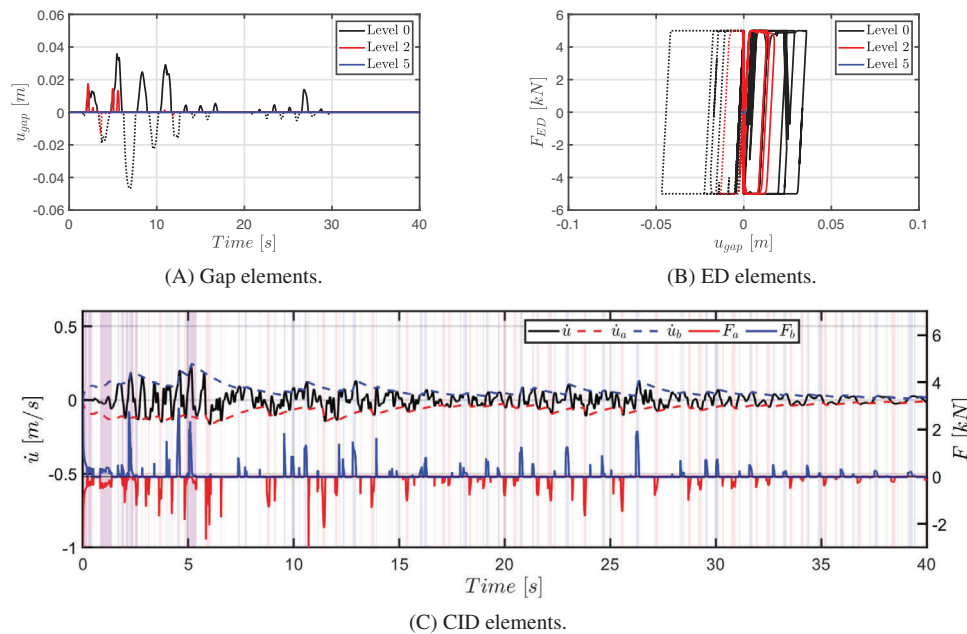


FIGURE 15 Behavior of the nonlinear elements: (A) Gap opening of gap elements, (B) hysteresis curve of ED elements, and (C) clutching behavior of ground floor CID element. Solid and dash lines indicates the elements located at left corner and right corner of the floor. CID, clutched inerter damper; ED, energy dissipator.

set to be equal to the lower floor for creating the rigid shear links. The leaning column method is used to account for P-Delta effects. The SC-CBF system is subjected to the same ground excitation as the previous example. The results were obtained for a time increment of 0.005 s and were verified through the progressive refinement of the step size afterwards observing no significant variation in the results.

Figure 15 shows the behavior of three types of nonlinear elements. The solid and dash lines indicate the elements located at left and right respectively. It is observed that, the overall rocking behavior is mainly controlled by gap openings at the ground floor. The gap elements at level 2 only opens a few times and the gap elements at level 5 remains closed throughout the action of the earthquake. This observation is also consistent with behavior of ED elements who share the same displacement with the gaps. The ED elements at the ground floor have gone through most hysteresis cycles and the ED elements at level 2 have gone through less cycles, while the ED elements at level 5 have not been activated. According to Figure 15, CIDs have effectively shaped the velocity curve during engagement and the clutch forces evolve smoothly without sharp numerically-induced oscillations.

Figure 16 shows the lateral displacement, floor rotation, and acceleration response at top floor of the SC-CBF system equipped with and without CIDs. All configuration remains the same except for the existence of inerters. Note that, since the axial deformations of the columns are ignored, the sources of displacement are constrained to the elastic deformation (of the bracing) and the uplifting (rocking) behavior, while floor rotations refer mainly to rocking. The floor rotation (in red) agree with the gap behavior in Figure 15. Rocking in the system equipped with CIDs ends at about $t = 15$ s and transitions to a full contact mode. This indicates that the equipment of CIDs effectively suppressed the rocking behavior with less influence on elastic deformation mode. In addition, the equipment of CIDs significantly mitigates the floor acceleration demands. These results are generally consistent with relevant studies on similar structures.^{27,64}

4 | CONCLUSIONS

A new modeling strategy of CIDs has been proposed in this paper. The proposed idealization can be implemented in conventional finite element software, or, for a better performance, within the MLF. To this end, the MLF formulation of the inerter element, of the clutch behavior and the dissipation within the device, as well as their corresponding computational scheme have been developed. The proposed CID model has been implemented and demonstrated through three case studies: a SDOF shear frame, a multistorey MRF and a SC-CBF.

The response histories and inerter behavior reveal that the energy harvesting effect of CIDs is significantly influenced by the dissipation. Even damped CIDs have only small periods of engagement throughout their whole response history.

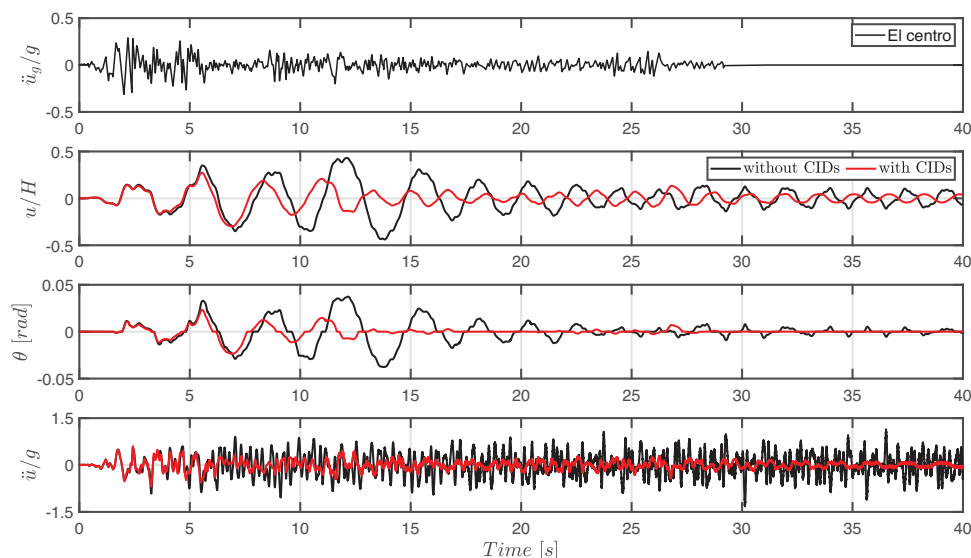


FIGURE 16 Top floor displacement, floor rotation, and acceleration response history of the 10-storey SC-CBF equipped with CIDs and without CIDs. CIDs, clutched inerter devices; SC-CBF, self-centering rocking concentrically braced frame.

Therefore, previously proposed and widely used, sequential models run the risk of grossly overestimating the energy dissipation of inerters leading to responses with much faster attenuation. Further experimental verification is required to confirm the conclusions derived from the simplified sequential models.

As a result of solving a weak form optimization problem rather than the equation of motion, the MLF method championed in this paper exhibits excellent robustness for simulating transient contact behavior, whereas other approaches, such as the penalty method and their corresponding equivalent OpenSees models, suffer from serious numerical chattering and stability problems, resulting in unstable responses and higher peak values of acceleration and clutch forces.

The error convergence analysis of Section 3.3 indicates that MLF is less sensitive to large step sizes and can produce stable results with acceptable error consistently. Although in its current implementation its numerical cost is higher per time step, this is offset by using larger step sizes and a tailored Hessian matrix. The efficiency of MLF is especially notable when few global degrees of freedom but many local state variables are involved, such as in problems involving material behavior. This is because the optimization problem's size depends on the number of local forces in the system. Moreover, a significant body of research has been dedicated to quadratic optimization problems. Many of these algorithms are readily accessible and can be easily used, which means the computational efficiency of MLF method can be increased even further.

The SC-CBF case study demonstrates the compatibility between the newly proposed CID element and other nonlinear elements like gap and elasto-plastic elements in MLF framework. This is because the proposed CID element maintains the same quadratic structure and linear constraints of the optimization problem in each time step. Our results have also proven the effectiveness of CIDs for suppressing rocking behavior and acceleration demands in multi-rocking self-centering structures.

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The authors have nothing to report.

CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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