Simulation-driven fixed-order controller tuning via moment matching

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***Abstract*— We propose a controller tuning method based on the data-driven model reduction by moment matching theory. By selecting a reference closed-loop transfer function, a moment matching data-driven model reduction algorithm is used to synthesize a fixed-order controller, via the identification of a model of the inverse transfer function of the controller, *i.e.* the transfer function from the controlled input of the system to the mismatch error signal. The controller is finally obtained by inverting this transfer function. The fixed-order controller is guaranteed to match the steady-state behavior of the ideal controller at certain pre-selected frequencies. The effectiveness of the resulting design method is assessed on a control problem for the 48-th order model of the Los Angeles University Hospital building.**

I. Introduction

With the increasing complexity of modern engineering systems, mathematical models to be used for control design are seldom available and their derivation might be involved and time consuming. *Data-driven modeling* techniques have thus been proposed over the years to rapidly and automatically estimate approximate models for dynamical systems using available process data. However, system identification methods are generally analysis-oriented, thus leading to models that accurately describe the system dynamics, but might still be over-complicated for control purposes. For instance, traditional model-based control techniques may be not suitable because they usually yield controllers of the same order of the employed model of the plant, which may be very high.

The complexity of models has fostered an increasing interest on techniques for *model reduction* [1]. Several strategies are available in the literature for the reduction of linear models, *e.g.,* the singular value decomposition based approach in [12] and the moment matching techniques in [3], [9], [11]. Moreover, continuous effort is put into the investigation of model-reduction policies for nonlinear systems, see *e.g.,* [2], [8], [10], [19]. In the last decade some data-driven approaches have also been proposed to directly derive reduced-order models from data, see *e.g.,* [13], [16], which rely upon frequency domain measurements, and [16], [18], that exploit time-domain data.

As an alternative research path to deal with fixed-order controller design for complex systems, *direct data-driven control* approaches have been developed, where modeling is fully skipped and data are directly mapped onto the controller parameters. Following the preliminary work [21], many data-driven techniques have been proposed over the years for both linear and nonlinear systems. Among the most popular methods, some of them, like Virtual Reference Feedback Tuning and Noniterative Correlation-based Tuning [4], [20], consider a model-reference formulation of the control problem. Although such methods have proved to work well in many real-world applications, for example when the desired closed-loop model is properly selected [6], [7], poor performance is usually obtained otherwise, *e.g.,* when the reference model is too demanding [20].

In this paper a novel method for direct data-driven control design aimed at *matching selected moments* of the desired closed-loop behavior is proposed. The data used for control design are assumed to be collected on a complex simulator of the plant. This is a realistic hypothesis as, in many critical applications, it is not possible to perform experiments on the plant, while a high-fidelity simulator is available for analysis, control design and testing. These simulators might still be too complex or might not be accessible for privacy reasons, so that data-driven strategies are needed. Given such a simulation environment, design methods could be developed within a purely deterministic setting.

The paper is organized as follows. The control problem is formulated in Section II. In Section III, the data-driven design technique by moment matching is described and discussed in detail. A simulation case study is presented in Section IV to show the performance of the method and compare the obtained results with standard model-reference techniques. The paper is concluded by some concluding remarks.

Notation: $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural, real and complex numbers, respectively. $\mathbb{C}_{<0}$ indicates the set of complex numbers with negative real part and $\mathbb{C}_0$ indicates the set of complex numbers with real part equal to zero. $\mathbb{R}^n$ denotes the set of real vectors of dimension n. Given a matrix $A \in \mathbb{R}^{n\times m}$, A' indicates its transpose, $\|A\|_2$ its ℓ_2-norm and, if $n = m$, $\sigma(A)$ denotes its spectrum. $\mathrm{vec}(A)$ and $\mathrm{rank}(A)$ indicate the vectorization and the rank of the matrix A, respectively. Given a matrix $B \in \mathbb{R}^{p\times q}$, $A \otimes B \in \mathbb{R}^{mp\times nq}$ is the Kronecker product between A and B. The symbols $I_n \in \mathbb{R}^{n\times n}$ and $0_{m\times n} \in \mathbb{R}^{m\times n}$ indicate the identity matrix and the zero matrix of dimension n and $m \times n$, respectively. Finally, $s \in \mathbb{C}$ denotes the Laplace variable.

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II. Problem formulation

Consider a *Linear Time-Invariant* (LTI) *Single-Input Single-Output* (SISO) continuous-time system $\mathcal{G}_\text{o}$ described by the state-space model

$$\mathcal{G}_\text{o}: \ \dot{x} = A_p x + B_p u, \quad y = C_p x, \tag{1}$$

where $A_p \in \mathbb{R}^{n_x \times n_x}$, $B_p \in \mathbb{R}^{n_x}$, $C_p \in \mathbb{R}^{1 \times n_x}$, and $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$ are the state, the input and the output of $\mathcal{G}_\text{o}$ at time t, respectively. Let, with a slight abuse of notation, $G_\text{o}(s) = C_p(sI_{n_x} - A_p)^{-1}B_p$ be the associated transfer function such that

$$\mathcal{G}_\text{o}: \ Y(s) = G_\text{o}(s)U(s), \tag{2}$$

where $Y(s)$ and $U(s)$ are the Laplace transforms of $y(t)$ and $u(t)$, respectively. We assume that the system $\mathcal{G}_\text{o}$ is *asymptotically stable* and its transfer function $G_\text{o}(s)$ is *strictly minimum-phase*.

In this paper we aim at designing a controller $\mathcal{C}$ for $\mathcal{G}_\text{o}$ to achieve the desired closed-loop behavior $\mathcal{M}$ given by

$$\mathcal{M}: \ Y_m(s) = M(s)R(s), \tag{3}$$

where $R(s)$ and $Y_m(s)$ are the Laplace transforms of $r(t) \in \mathbb{R}$ (the reference signal at time t) and $y_m(t) \in \mathbb{R}$ (the desired output at time t), respectively. The controller $\mathcal{C}$ attaining the desired closed-loop behavior is sought within the family of causal, finite-dimensional, and fixed-order LTI systems

$$\mathcal{C}: \ U(s) = C(s;\theta)E(s), \tag{4}$$

where $\theta \in \mathbb{R}^{n_\theta}$ is an unknown set of parameters to be determined, and $E(s)$ is the Laplace transform of the tracking error $e(t) = r(t) - y(t)$ at time t.

By introducing the model-reference criterion

$$J_{MR}(\theta) = \left\| \frac{G_\text{o}(s)C(s;\theta)}{1 + G_\text{o}(s)C(s;\theta)} - M(s) \right\|_2^2, \tag{5}$$

the problem we aim to solve can be formally stated as follows.

Problem 1 (Closed-loop model-matching): Let $\mathcal{G}_\text{o}$ be an *unknown* system to be controlled. Find a controller $\mathcal{C}$ with the structure in (4) that minimizes the model-reference objective (5) with respect to the design parameters θ. ■

If there exists an ideal controller $C^\star(s)$ such that the desired behavior $\mathcal{M}$ is realized in closed-loop, *i.e.,* if there exists $C^\star(s)$ such that

$$\frac{G_\text{o}(s)C^\star(s)}{1 + G_\text{o}(s)C^\star(s)} = M(s), \tag{6}$$

then the optimal controller is

$$C^\star(s) = \left(G_\text{o}(s) \frac{1 - M(s)}{M(s)} \right)^{-1}. \tag{7}$$

If $\mathcal{M}$ is unachievable with any controller within the selected class, *e.g.,* if the control objective is too demanding, then $C^\star(s)$ cannot be matched *exactly*. Nevertheless, by solving Problem 1 we obtain the best controller within (4).

Assume that no information is available about $\mathcal{G}_\text{o}$, other than a set $\mathcal{D}_N = \{u(t_i), y(t_i)\}_{i=1}^N$ of input/output measurements collected at time instants $\{t_i\}_{i=1}^N$ from open-loop simulation experiments on the system. Note that, even if $C^\star(s)$ exists, the controller $C^\star(s)$ in (7) depends on the system $\mathcal{G}_\text{o}$ to be controlled, which is supposed to be *unknown*, thus $C^\star(s)$ cannot be explicitly computed. For this reason, in this paper we consider the weaker model-matching problem illustrated next.

Problem 2 (Closed-loop moment-matching): Let $\mathcal{D}_N = \{u(t_i), y(t_i)\}_{i=1}^N$ be a set of open-loop simulated data generated by the *unknown* system $\mathcal{G}_\text{o}$. Find a controller $\mathcal{C}$ with the structure in (4), that *matches* the response of the ideal model-reference controller (7) at some pre-specified frequencies s_i, $i = 1, \ldots, v < n_x$. ■

The above formulation shows that the fixed-order controller design problem can be tackled as a model-reduction problem. Therefore, in the following section we derive a way to design the inverse controller, solving Problem 2, starting from the moment-matching model reduction method presented in [16].

III. Fixed-order controller design by moment-matching

Let the best controller with the required parameterization be $C(s;\theta_\text{o})$ and let its inverse be described by the state space model

$$\mathcal{C}_\text{o}^{-1}: \quad \dot{x}_c = A_c x_c + B_c u, \quad e = C_c x_c + D_c u, \tag{8}$$

with $A_c \in \mathbb{R}^{n_{x_c} \times n_{x_c}}$, $B_c \in \mathbb{R}^{n_{x_c} \times 1}$, $C_c \in \mathbb{R}^{1 \times n_{x_c}}$ and $D_c \in \mathbb{R}$. We assume that (A_c, B_c) is controllable and (A_c, C_c) is observable.

The objective of this work is to find an inverse controller of order ν, *i.e.,*

$$\mathcal{C}_\nu^{-1}: \quad \dot{\xi} = A_\nu \xi + B_\nu u, \quad \varepsilon = C_\nu \xi + D_\nu u, \tag{9}$$

with $A_\nu \in \mathbb{R}^{\nu \times \nu}$, $B_\nu \in \mathbb{R}^{\nu \times 1}$, $C_\nu \in \mathbb{R}^{1 \times \nu}$ and $D_\nu \in \mathbb{R}$ to be determined, that matches the so-called *moments* of $\mathcal{C}_\text{o}^{-1}$ in (8) at the set of complex points $\{s_i\}_{i=1}^v$. The moments of $\mathcal{C}_\text{o}^{-1}$ are defined as follows [16, Def. 3].

Definition 1: Let $s_i \in \mathbb{C}/\sigma(A_c)$, with $i = 1, \ldots, v$. The 0-moment of system (8) is the complex number $\eta_0(s_i) = C(s_i;\theta^\text{o})^{-1}$. The k-moment of system (8) at s_i, for $k \in \mathbb{N}$, is the complex number

$$\eta_k(s_i) = \frac{(-1)^k}{k!} \left[\frac{d^k}{ds^k} C(s;\theta_\text{o})^{-1} \right]_{s=s_i}. \tag{10}$$

■

Assuming that in our simulations we can arbitrarily choose the input, let u in (8) be generated by the initialized autonomous system $\mathcal{S}$, described by the state-space model

$$\mathcal{S}: \ \dot{\omega}(t) = S\omega(t), \quad u(t) = L\omega(t), \tag{11}$$

with $L \in \mathbb{R}^{1 \times \nu}$, $S \in \mathbb{R}^{\nu \times \nu}$ a non-derogatory[1] matrix and the triple $(L, S, \omega(0))$ minimal, see [15]. According to [17], the

[1] A matrix is non-derogatory if its characteristic and minimal polynomial are equal.

signal generator $\mathcal{S}$ in (11) is selected so that its characteristic polynomial $p(S)$ is given by

$$p(S) = \prod_{i=1}^{v} (s - s_i)^{k_i}, \tag{12}$$

with the eigenvalues of the matrix S in (11) satisfying $\sigma(S) \subset \mathbb{C}_0$. Therefore, the order ν of the inverse controller $\mathcal{C}_\nu^{-1}$ is given by

$$\nu = \sum_{i=1}^{v} k_i, \tag{13}$$

where k_i denotes the highest moment to be matched at s_i, for $i = 1, \dots, v$. Additionally, we impose that $0 \notin \sigma(S)$. This restriction allows reducing the order ν of the controller, while exploiting one of the parameters to be identified (namely the scalar D_ν) to match the 0-moment of $\mathcal{C}_{\mathrm{o}}^{-1}$ at zero.

Thanks to these choices there exists a one-to-one relation between the moments of the ideal inverse controller $\mathcal{C}_{\mathrm{o}}^{-1}$ and the matrix $\Xi := C_c\Pi + D_cL$, with $\Pi \in \mathbb{R}^{n_{x_c} \times \nu}$ the unique solution of the Sylvester equation

$$A_c\Pi + B_cL = \Pi S. \tag{14}$$

Similarly, there exists a one to one relation between the moments of the fixed-order inverse controller $\mathcal{C}_\nu^{-1}$ and the matrix $C_\nu P + D_\nu L$, with $P \in \mathbb{R}^{\nu \times \nu}$ the unique solution of the Sylvester equation

$$A_\nu P + B_\nu L = PS. \tag{15}$$

Then $\mathcal{C}_{\mathrm{o}}^{-1}$ and $\mathcal{C}_\nu^{-1}$ have the same moments at S if

$$C_\nu P + D_\nu L = C_c\Pi + D_cL. \tag{16}$$

A family of models satisfying (15) and (16) for $P = I_\nu$ is described by

$$\mathcal{C}_\nu^{-1} : \begin{cases} \dot{\xi}(t) = (S - B_\nu L)\xi(t) + B_\nu u(t), \\ \varepsilon(t) = (\Xi - D_\nu L)\xi(t) + D_\nu u(t), \end{cases} \tag{17}$$

with B_ν any matrix such that $\sigma(S) \cap \sigma(S - B_\nu L) = \emptyset$ and D_ν any scalar. To build the fixed-order model $\mathcal{C}_\nu^{-1}$ that matches the moments of $\mathcal{C}_{\mathrm{o}}^{-1}$ at S, it is sufficient to estimate Ξ. Nonetheless, since B_ν and D_ν are free parameters, we would like to seek the "best" model, in some sense to be defined, among the family of fixed order models (17). To this end, we estimate all the parameters $[\Xi,\ B_\nu,\ D_\nu]$ from data.

A. Data-driven estimation of Ξ

Consider the problem of identifying Ξ given the data-set $\mathcal{D}_N = \{u(t_i), y(t_i)\}_{i=1}^N$. Based on our assumptions, the interconnection between $\mathcal{S}$ and $\mathcal{C}_{\mathrm{o}}^{-1}$ has an invariant manifold described by $\{(x_c, \omega) \in \mathbb{R}^{n_{x_c}} \times \mathbb{R}^\nu : x_c = \Pi\omega\}$. The output of this interconnection is thus given by

$$e(t) = (C_c\Pi + D_cL)\omega(t) + C_c e^{A_c t}(x_c(0) - \Pi\omega(0)), \tag{18}$$

with $e_{tr}(t) = e^{A_c t}(x_c(0) - \Pi\omega(0))$ an exponentially decaying signal. Under the additional hypothesis that the number N of available samples is sufficiently large so that the transient of $\mathcal{C}_{\mathrm{o}}^{-1}$ is negligible, equation (18) can be approximated by

$$e(t) \approx \Xi\,\omega(t). \tag{19}$$

As explained in [16], this approximation can be used to estimate Ξ, by considering windows of samples of length $w \geq \nu$, with $w \in \mathbb{N}$. For each window we construct the matrices

$$R_i = \begin{bmatrix} \omega(t_{i-w+1}) & \cdots \omega(t_{i-1}) & \omega(t_i) \end{bmatrix}' \in \mathbb{R}^{w \times \nu}, \tag{20}$$

$$\Gamma_i = \begin{bmatrix} e(t_{i-w+1}) & \cdots & e(t_{i-1}) & e(t_i) \end{bmatrix}' \in \mathbb{R}^{w}, \tag{21}$$

that contain the states of $\mathcal{S}$ and the tracking errors up to the sample $t_i \in \mathbb{N}$, respectively. Then, for each window, the estimation of Ξ can be carried out by solving the least-squares problem

$$\widehat{\Xi}_i' = \arg\,\min_{\Xi'} \left\| \Gamma_i - R_i\Xi' \right\|_2^2. \tag{22}$$

If the signal generator and the length w of the window are properly chosen, *i.e.,* if R_i is full column rank, then the closed form solution of (22) is given by

$$\widehat{\Xi}_i' = (R_i'R_i)^{-1}R_i\Gamma_i. \tag{23}$$

The estimation of $\widehat{\Xi}_i$ can alternatively be carried out in a moving-horizon fashion, by shifting the window and exploiting the recursive formulas

$$\widehat{\Xi}_i' = \widehat{\Xi}_{i-1}' + \phi_i\omega(t_i)\varepsilon_{i-1}(t_i) - \phi_i\omega(t_{i-w})\varepsilon_{i-1}(t_{i-w}), \tag{24a}$$

with $\varepsilon_{i-1}(t_i) = y(t_i) - \omega(t_i)'\widehat{\Xi}_{i-1}'$. The matrix ϕ_i is also updated recursively as

$$\phi_i = (R_i'R_i)^{-1} = \psi_i + \psi_i\omega(t_{i-w})\Phi_i^{-1}\omega(t_{i-w})'\psi_i, \tag{24b}$$

$$\begin{aligned} \psi_i &= (\phi_i^{-1} + \omega(t_i)\omega(t_i)')^{-1} = \\ &= \phi_{i-1} - \phi_{i-1}\omega(t_i)\Psi_i^{-1}\omega(t_i)'\phi_{i-1}, \end{aligned} \tag{24c}$$

where

$$\Phi_i = 1 + \omega(t_{i-w})'\psi_i\omega(t_{i-w}), \tag{24d}$$

$$\Psi_i = 1 + \omega(t_i)'\phi_{i-1}\omega(t_i). \tag{24e}$$

This iterative procedure is summarized in Algorithm 1 (at the top of the next page), with Step 3.2 introduced to automatically tune the window length w. Note that both Step 3.2.1 and Step 3.2.2 require no inversion, making the iterative procedure computationally more efficient than the off-line approach. Indeed, the computation of the batch solution as in (23) requires the inversion of a square ν-dimensional matrix. The estimate $\widehat{\Xi}$ obtained through Algorithm 1 can then be used to construct the family of estimated fixed-order inverse controllers $\hat{\mathcal{C}}_\nu^{-1}$ as in (17), with $\Xi = \hat{\Xi}$. By construction, all the controllers in this family preserve the steady-state response of the ideal inverse controller $\mathcal{C}_{\mathrm{o}}^{-1}$, with the parameters $[B_\nu,\ D_\nu]$, such that $\sigma(S) \cap \sigma(S - B_\nu L) = \emptyset$, still left to be tuned.

Algorithm 1 Data-driven estimation of Ξ

Input: Initial window length $w \geq n_{x_c}\nu$, training set $\mathcal{D}_N = \{u(t_i), y(t_i)\}_{i=1}^N$, initial estimate $\widehat{\Xi}_w$.

1. **Build** the reference signal r as in [5];
2. **Construct** $e(t_i) = r(t_i) - y(t_i)$ for $i = 1, \ldots, N$;
3. **for** $i = w+1, \ldots, N$

3.1. **Build** R_i and Γ_i as in (20) and (21), respectively;

3.2. **if** $\text{rank}(R_i) = \nu$

3.2.1. **Update** Ψ_i and ψ_i as in (24e) and (24c), respectively;

3.2.2. **Update** Φ_i and ϕ_i as in (24d) and (24b), respectively;

3.2.3. **Update** $\widehat{\Xi}_i$ as in (24a) ;

3.3. **else**

3.3.1. **Increase** w, *e.g.,* $w = w + 1$;

3.3.2. **if** $w < N$ **go to** Step 3;

Output: Estimated vector $\widehat{\Xi} := \widehat{\Xi}_N$.

B. Data-driven design of the fixed-order controllers

The family of fixed order controllers (17), with $\Xi = \widehat{\Xi}$ and $\widehat{\Xi}$ estimated as described in Section III-A, does not provide any guarantee on the preservation of the transient response of $\mathcal{C}_{\text{o}}^{-1}$. Therefore, the matrices $[B_\nu,\ D_\nu]$ are chosen so that the inverse fixed-order controller $\mathcal{C}_\nu^{-1}$ preserves the transient response of $\mathcal{C}_{\text{o}}^{-1}$.

According to [17], to accomplish this goal we consider the family of fixed-order inverse controllers in (9), which satisfies equation (15) and the approximated moment matching condition

$$C_\nu P + D_\nu L = \widehat{\Xi}. \tag{25}$$

This family is similar to the one described by (17), for $\Xi = \hat{\Xi}$, with the change of coordinates given by the matrix P. In order to determine the conditions for choosing the matrices $[B_\nu,\ D_\nu]$, while preserving the transient response of $\mathcal{C}_{\text{o}}^{-1}$, we thus solve the problem of identifying the matrices $[A_\nu,\ B_\nu,\ C_\nu,\ D_\nu]$ of the family of inverse controllers in (9).

Specifically, the matrices A_ν and C_ν are selected to reproduce the exponentially decaying signal

$$e_{tr}(t) = C_c e^{A_c t}(x_c(0) - \Pi\omega(0)), \tag{26}$$

which can be seen as the output of the autonomous state-space model

$$\dot{\chi}(t) = A_c\chi(t), \quad z(t) = C_c\chi(t), \tag{27}$$

with state $\chi(t) = x_c(t) - \Pi\omega(t) \in \mathbb{R}^{n_{x_c}}$. The unknowns $[A_\nu,\ C_\nu]$ can be estimated by identifying an autonomous model of order ν, *i.e.,*

$$\dot{\xi}(t) = A_\nu\chi(t), \quad \psi(t) = C_\nu\xi(t), \tag{28}$$

that preserves the free response of the autonomous system (27). To estimate this model from data, a new data-set $\tilde{\mathcal{D}}_N = \{e_{tr}(t_i)\}_{i=1}^N$ has to be built that comprises samples of the transient response of (27). By using the definition of the error provided in (18) and the estimate $\widehat{\Xi}$ obtained through Algorithm 1, the elements of $\tilde{\mathcal{D}}_N$ can be built from $\mathcal{D}_N$ as:

$$e_{tr}(t_i) = e(t_i) - \widehat{\Xi}\omega(t_i). \tag{29}$$

Once the data-set $\tilde{\mathcal{D}}_N$ is created, standard techniques for state-space model identification, *e.g.,* the $NS4ID$ algorithm [14], can be used to determine $\hat{A}_\nu \in \mathbb{R}^{\nu\times\nu}$ and $\hat{C}_\nu \in \mathbb{R}^{1\times\nu}$ minimizing the error

$$\tilde{\varepsilon}(t_i) = e_{tr}(t_i) - \psi(t_i). \tag{30}$$

The estimates $[\hat{A}_\nu,\ \hat{B}_\nu,\ \hat{C}_\nu,\ \hat{D}_\nu]$ must also preserve the moments at the selected interpolating frequencies $\{s_i\}_{i=1}^v$. As a consequence, these matrices must satisfy the condition

$$\hat{C}_\nu\hat{P} + \hat{D}_\nu L = \widehat{\Xi}, \tag{31}$$

which requires that the estimated quantities satisfy the approximate moment matching condition for some $\hat{P}$, that solves

$$\hat{A}_\nu\hat{P} + \hat{B}_\nu L = \hat{P}S. \tag{32}$$

Additionally, we impose that the fixed-order inverse controller matches the DC gain of the ideal inverse controller. Note that this is equivalent to requiring that the moment at zero is matched, namely $\eta_0(0) = -C_c A_c^{-1}B_c + D_c$. To enforce this condition, we repeat the procedure of the previous section for $S = 0$ (scalar) and $\omega(0) \in \mathbb{R} \setminus \{0\}$ to obtain an estimate $\overline{\Xi}$ of $\eta_0(0)$. Then, to preserve the moment $\eta_0(0)$ we impose the condition

$$-\hat{C}_\nu\hat{A}_\nu^{-1}\hat{B}_\nu + \hat{D}_\nu = \overline{\Xi}, \tag{33}$$

where the left-hand side is the value of the transfer function of $\mathcal{C}_\nu^{-1}$ at $s = 0$.

The conditions (31)-(33) are then exploited to determine $\hat{B}_\nu$ and $\hat{D}_\nu$. To this end, assume that $(\hat{A}_\nu, \hat{C}_\nu)$ is observable and that $\hat{A}_\nu$ satisfies $\sigma(\hat{A}_\nu) \in \mathbb{C}_{<0}$ and $\sigma(\hat{A}_\nu) \cap \sigma(S) = \emptyset$. The matrices $\hat{B}_\nu$, $\hat{D}_\nu$ and $\hat{P}$ verifying (31), (32) and (33) are unique and they are given by

$$\begin{bmatrix} \text{vec}(\hat{P}) \\ \hat{B}_\nu \\ \hat{D}_\nu \end{bmatrix} = \Lambda^{-1}\begin{bmatrix} 0_{\nu^2\times 1} \\ \widehat{\Xi}' \\ \overline{\Xi} \end{bmatrix}, \tag{34}$$

where

$$\Lambda = \begin{bmatrix} I_\nu \otimes \hat{A}_\nu - S' \otimes I_\nu & L' \otimes I_\nu & 0_{\nu^2\times 1} \\ I_\nu \otimes \hat{C}_\nu & 0_{\nu\times\nu} & L' \\ 0_{1\times\nu^2} & -\hat{C}_\nu\hat{A}_\nu^{-1} & 1 \end{bmatrix}. \tag{35}$$

The final fixed-order inverse controller is thus given by

$$\hat{\mathcal{C}}_\nu^{-1}:\ \dot{\xi}(t) = \hat{A}_\nu\xi(t) + \hat{B}_\nu u(t), \quad \varepsilon(t) = \hat{C}_\nu\xi(t) + \hat{D}_\nu u(t). \tag{36}$$

Note that $\hat{D}_\nu \neq 0$ because the inverse controller is such that $D_c \neq 0$. Assuming that (36) is minimum phase, a controller $C(s, \hat{\theta})$ in the class $\mathcal{C}$ can finally be found by inverting $\hat{\mathcal{C}}_\nu^{-1}$. The method for simulation-driven control design via moment matching is summarized in Algorithm 2.

Algorithm 2 Simulation-driven control design via moment matching

Input: Training set $\mathcal{D}_N = \{u(t_i), y(t_i)\}_{i=1}^N$; chosen reference model $M(s)$ and interpolating points $\{s_i\}_{i=1}^{\nu}$.

1. **Design** the signal generator $\mathcal{S}$, defined as in (11);
2. **Identify** $\widehat{\Xi}$ through Algorithm 1;
3. **Construct** the new data-set $\tilde{D}_N$, with samples given by (29);
4. **Identify** the $\hat{A}_\nu, \hat{C}_\nu$ in (28), *e.g.,* by using $N4SID$;
5. **Identify** $\bar{\Xi}$ through Algorithm 1 with $S = 0$;
6. **Find** the $\hat{B}_\nu, \hat{D}_\nu$ by solving (34);
7. **Invert** $\hat{\mathcal{C}}_\nu^{-1}$ to find the controller.

Output: Simulation-driven controller $\hat{\mathcal{C}}_\nu$.

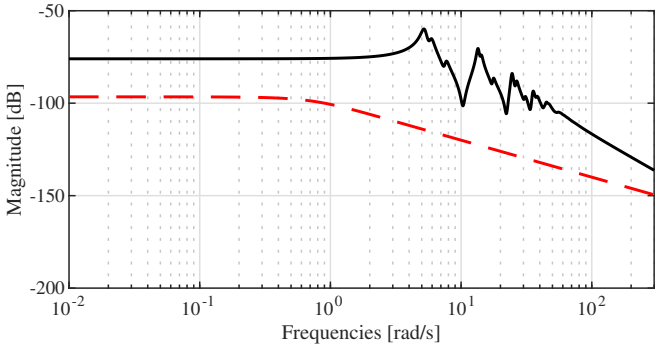


Fig. 1. Open-loop system vs closed-loop desired behavior. Bode plots of the system $\mathcal{G}_o$ (blue) and desired closed-loop response $\mathcal{M}$ (dashed red).

Remark 1: Commonly to other model-reference design strategies, the proposed simulation-driven approach requires the selection of a reference signal r to construct the tracking error e. One possible choice is to select a *virtual reference* signal from the available dataset $\mathcal{D}_N$ as the reference signal that would have fed the system during the experiment if data were generated in closed-loop. For more details see [5]. ■

IV. Simulation Results

The performance of the approach described in Section III is assessed by considering the vibration control problem for the Los Angeles University Hospital building, which is a building of 8 floors, each one having 3 degrees of freedom. The system $\mathcal{G}_o$ can be described, for small vibrations, by a state-space model (1) of order $n_x = 48$.

We consider the desired closed-loop behavior given by the transfer function

$$\mathcal{M}: \; M(s) = 10^{-5} \frac{s + 0.95}{s^2 + 1.44s + 0.64}, \tag{37}$$

that is, our aim is to force the 48-th order system $\mathcal{G}_o$ to behave as a second-order one, which is able to damp out the excitations due to undesired exogenous factors like wind or earthquakes. The Bode diagrams of $\mathcal{G}_o$ and $\mathcal{M}$ are compared in Fig. 1, providing an insight on the differences between the open-loop dynamics and the desired closed-loop. We would like to stress here that the system model is *not* used for designing the controller but only as a ground-truth to evaluate its performance. Note also that the ideal controller (6) (of order 48) is anti-causal and, consequently, it could not be used in practice even if it was computable.

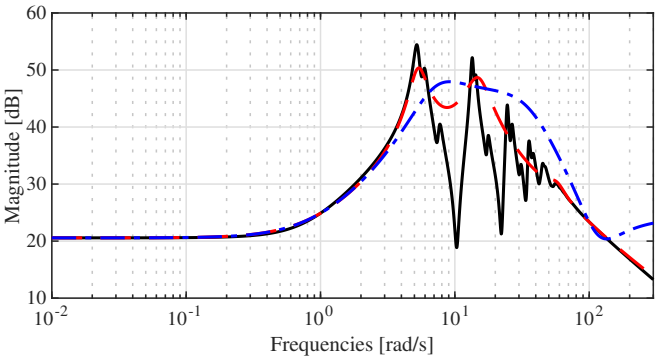


Fig. 2. Ideal vs estimated inverse controllers. Bode plots of ideal inverse controller (black) and estimated inverse controllers $C_1^{-1}(s)$ (dashed-dotted blue) and $C_2^{-1}(s)$ (dashed red).

The data used to design the controller are collected through open-loop simulations of the system $\mathcal{G}_o$. Two different sets of inputs are initially considered, by selecting two signal generators $\mathcal{S}_1$ and $\mathcal{S}_2$, both defined as in (11). Two controllers $C_1(s)$ and $C_2(s)$ are thus trained, matching the moments of the ideal controller at $\mathcal{N}_1 = \{\pm 1\imath, \pm 100\imath\}$ and $\mathcal{N}_2 = \{\pm 1\imath, \pm 50\imath, \pm 100\imath\}$, respectively. Under the assumption that no information is available except the data-set $\mathcal{D}_N$ and the chosen reference model, these interpolating points $\mathcal{N}_1$ and $\mathcal{N}_2$ are chosen by looking at the desired closed-loop behavior and by trading off the complexity of the controller with the quality of the attained closed-loop behavior in simulation. It is worth remarking that the selection of the interpolating frequencies is crucial for the performance attained by the closed-loop system and further investigation is required to properly choose these points according to the control objective at hand. The matrices S of the signal generators $\mathcal{S}_1$ and $\mathcal{S}_2$ are chosen so that their eigenvalues are equal to $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. The initial conditions $\{\omega_i(0)\}_{i=1}^2$ and the matrices $\{L_i\}_{i=1}^2$ are then selected as realizations of uniformly distributed random vectors which guarantee the minimality of $\{(S_i, L_i, \omega_i(0))\}_{i=1}^2$. Based on the chosen sets $\mathcal{N}_1$ and $\mathcal{N}_2$ of interpolating points, $C_1^{-1}(s)$ and $C_2^{-1}(s)$ are a 4-th and a 6-th order controller, respectively, while the ideal controller is of order 48. The Bode plots of these inverse controllers are compared in Fig. 2, showing that, as expected, the response of the fixed-order controllers $C_1^{-1}(s)$ and $C_2^{-1}(s)$ matches the one of the ideal controller at the selected interpolating frequencies. Fig. 2 also shows that $C_2^{-1}(s)$ is more accurate in approximating the overall behavior of the ideal controller, especially at higher frequencies. This result is somehow expected, as it is well-known that the inclusion of additional points does not influence the resulting controller only at the additional frequencies, but it reshapes the entire controller response. In light of these considerations, the Bode plots of the desired closed-loop behavior $\mathcal{M}$ and the one obtained with $C_2^{-1}(s)$ are compared in Fig. 3 and they show the effectiveness of the design strategy. It can be

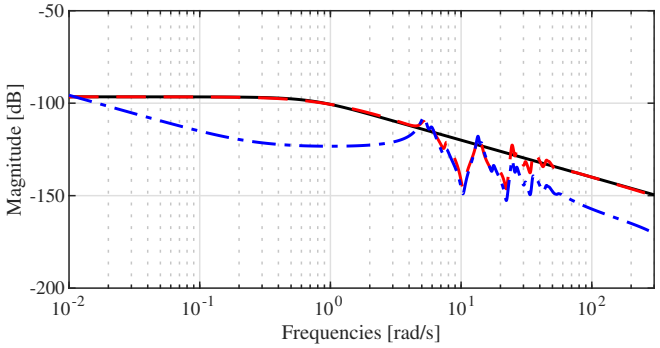


Fig. 3. Achieved vs desired closed-loop response. Magnitude of $\mathcal{M}$ (black) and the closed-loop responses achieved with $C_2(s)$ (dashed red) and $C_{vrft}(s)$ (dashed-dotted blue).

noted that the building can be forced to behave similarly to the second-order system in (37). Due to the complexity of $\mathcal{G}_o$ and the relatively low order of the designed controller, obviously $C_2(s)$ cannot remove the oscillations characterizing the open-loop response of the building. The proposed moment-matching approach is further compared with a state-of-the-art data-driven control method. Specifically, the comparison is performed with the *Virtual Reference Feedback Tuning* (VRFT) approach of [4], [5]. Note that this method cannot be employed with the input sequence $\mathcal{D}_N$, since the persistency of excitation conditions required in [4] are not satisfied. We thus perform a new simulation with a band-limited Gaussian signal, with frequency band within 1 Hz and 700 Hz. For a fair comparison, the structure of the controller $C_{vrft}(s)$ to be trained with the VFRT method is chosen to correspond to the one of $C_2(s)$. The closed-loop response obtained with the different controllers are compared in Fig. 3, showing that the response achieved with $C_2(s)$ is closer to the one of $\mathcal{M}$ than the response obtained with $C_{vrft}(s)$. This result might be explained considering that the VRFT method attempts to match $\mathcal{M}$ over all the frequencies, thus providing a trade-off solution which is on average worse. We should stress, however, that the proposed moment-matching approach strongly depends on the choice of the interpolating frequencies, that are currently not yet systematically selected, whereas the VRFT method is a fully automatic procedure. Furthermore, the approach is not yet ready to be applied to noisy data, but it is suitable only within the proposed simulation environment. Although promising, the proposed results should then be considered only as a preliminary step towards data-driven control design by moment matching.

V. Conclusions

This paper presents the first direct data-driven control design approach for fixed-order controllers based on moment matching. With the proposed method, data collected on a complex simulator of a plant can be used to directly design a low order controller, like a PID, for an arbitrarily complex plant, by concentrating on the interpolation of a few points in the frequency domain. Due to the preliminary nature of this contribution there are open problems to be addressed to make the method competitive with the state-of-the-art in applications. Among the others, the choice of the interpolation points and the analysis of the effects of noise.

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