

GAUGE SYMMETRIES IN QUANTUM GRAVITY

by

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ABSTRACT

Methods are given for the derivation of gravitational Ward identities. These are used to derive amongst others one that is verified by explicit calculation to second order in the gravitational coupling constant. Several methods for quantising general relativity are reviewed and the Feynman rules are given for what appears to be two of the most tractable formulations of the gravitational perturbation expansion. Divergent integrals are handled by a particularly convenient adaptation of t'Hooft and Veltman's 'dimensional regularisation' procedure. Tadpole diagrams and matters concerning the possible renormalisation of gravitation that obtain to this calculation are also discussed.

To a Vole, a Bear, and a Kingfisher.

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PREFACE

The work presented in this thesis was carried out in the Department of Theoretical Physics, Imperial College, between October 1970 and July 1973 under the supervision of Professor T.W.B. Kibble. The material presented is the original work of the author, except where stated in the text, and has not been submitted to this or any other University for any other degree.

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Some Definitions and Conventions

$\eta^{\alpha\beta}$ is the usual Minkowski flat space metric.

$$Q^{\alpha\beta, \mu\nu} \equiv \frac{1}{2}(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\mu\nu})$$

$\hat{Q}^{\alpha\beta, \mu\nu} \circ \underline{A}$ is a convenient shorthand for

$$\hat{Q}^{\alpha\beta, \mu\nu} \circ \underline{A} \equiv \frac{1}{2}(\eta^{\alpha\mu}A^{\beta\nu} + \eta^{\alpha\nu}A^{\beta\mu} - \eta^{\alpha\beta}A^{\mu\nu}),$$

where $\underline{A}$ is a symmetric two index object.

$\hat{Q}^{\alpha\beta, \mu\nu} \circ \underline{Q} \circ \hat{Q}^{\kappa\lambda, \sigma\tau}$ stands for the action of the left hand $\hat{Q}$ operating on the left hand pair of indices of Q followed by the action of the right hand $\hat{Q}$ operating on the right hand set of indices. The $\hat{Q}$ symbols have only direct relevance to Appendix II.

The measures of the functional integrals are defined

$$(dg) \equiv \prod_{x, \mu < \nu} dg^{\mu\nu}(x)$$

$$(d\Gamma) \equiv \prod_{x, \mu < \nu} d\Gamma_{\mu\nu}^{\chi}(x)$$

In sections two and three DeWitts 'condensed' notation is employed for relevant derivations and is explained in section two when first encountered.

The notation $\delta/\delta\zeta^{\alpha}$ is frequently used to express infinitesimal co-ordinate variations e.g.,

Under the infinitesimal co-ordinate transformation

$$x^{\mu} \rightarrow x'^{\mu} = x^{\mu} + \delta\zeta^{\mu}$$

the metric tensor transforms

$$g^{\mu\nu}(x) \rightarrow g'^{\mu\nu}(x') = g^{\mu\nu}(x) + \frac{\delta g^{\mu\nu}}{\delta \xi^\alpha} \cdot \delta \xi^\alpha$$

1. INTRODUCTION

This thesis is principally concerned with the consequences in terms of Ward identities of the necessary inclusion into the quantum theory of gravitation of the original general co-ordinate invariance of the Einstein gravitational field equations. However there are other lines of inquiry either necessitated by, consequent upon or at some distance removed from this general theme. The situation is best clarified by first discussing the various motivations that have led up to this.

Since the original work of Feynman⁽¹⁾ there has been a proliferation of literature concerned with the quantisation of the gravitational field. All this has taken place in the absence of any experimental data. There is as yet no concrete evidence that gravitational radiation exists let alone what it does. Of course this need have no bearing on the development of the theory as such but it might at first sight appear a little odd that so much effort be expended in what amounts to a vacuum. The reasons for wanting to quantise general relativity in the first place are both numerous and varied. Most are with substantial foundation. Others might be better described as justifications after the event. I shall deal only with those that pertain to this thesis. There are now many excellent review articles⁽²⁾ that deal with the remainder.

Firstly, when viewed as a massless, spin two, self-interacting field theory, there would appear no a priori reason why the gravitational field should not be amenable

to the analysis applied with great success to other, better established, quantum field theories. Certainly one would expect exaggerated ultraviolet and infrared divergence problems and its renormalisability would be uncertain. The principal stumbling block to this conception is founded in the non-abelian character of the general co-ordinate transformation group. Whereby quantum gravity is unlike most other field theories with the notable exception of the Yang-Mills field that similarly possesses a non-abelian symmetry group. Fortunately with the work of Faddeev and Poppov⁽³⁾ this difficulty appears finally to have been resolved. The Yang-Mills field has been shown to possess a unitary S-matrix and is renormalisable⁽⁴⁾. However the same cannot be said for the gravitational field.. Its renormalisability becomes increasingly in doubt and unitarity has only been established in terms of formal arguments involving path integrals⁽⁵⁾. Moreover very few actual calculations have been performed and none to my knowledge exhibit the necessary transversality of the quantised amplitudes. In other words, no justification in terms of calculations has been given for the use of Fadeev and Poppov's prescription for dealing with the gauge symmetry in terms of the now legendary fictitious particle loops. If one embarks on such a program it very soon becomes apparent as to why this might be the case. The vertices specifying the graviton self interactions are extremely complicated. The complication is only one of sheer bulk but as such it is quite sobering enough to act as a formidable deterrent.

Herein lies the motivation for the greater part of this thesis: to obtain a tractable set of Feynman rules for the gravitational field and to exhibit at least some evidence of self-consistency by actual computation of a particular Ward identity. I shall leave aside discussion of how far I am able to achieve this intention until I review the composition of the individual sections of this thesis in the latter half of this introduction.

The next motivation to be discussed has to do with the seemingly fantastic notion that quantum gravity might actually prove useful. It was Pauli who first remarked that quantum fluctuations in the light cone could serve to suppress the characteristic ultraviolet divergencies encountered in conventional quantum field theory. Recently with the work of Salam, Isham and Strathdee⁽⁶⁾ to do with the regularising capabilities of non-polynomial Lagrangians this idea has received a very positive revival of interest. By coupling the gravitational field to electrodynamics and performing a 'superpropagator' calculation they exploit the intrinsically non-polynomiality of the Einstein lagrangian to do just this - at least they certainly obtain a finite answer. However it is only with gravitation playing essentially the role of a background field that the calculation is performed. Its own dynamical degrees of freedom are for the most part suppressed. At one stage I was concerned with finding a sophistication of this approach to incorporate more of the structure of the gravitational field. A prerequisite of any such enlargement

is having at one's disposal a tractable theory of quantum gravity in the first place. One is then led back to the considerations of the previous paragraph and I have very little to say concerning the 'infinity suppression' capabilities of quantum gravity as such. There are however one or two points of interest relating to this that are dealt with in section 8.

I suppose the overriding motivation is best summarised as follows. Quantum gravity has become a reality, but it is a reality beset by a plethora of different approaches. I should perhaps mention at this point that I have nothing to say concerning that aspect of 'quantum gravity' that has to do with the rapidly expanding subject of closed universes⁽⁷⁾ and theories of superspace. It is only in the sense of general relativity being regarded as a conventional quantum field theory that this phrase should be interpreted. Even in this restricted sense one encounters considerable diversification. The resolution of the gauge problem has taken a number of different guises. What are the practical advantages of working with a manifestly covariant theory as opposed to the more usual 'asymptotically flat' formulations. What are the relative merits in the various choices of variable to quantise and do they lead to the same results. What do the infinite number of gravitational Ward identities look like, how are they derived and can they be verified.

To summarise thus far: quantum gravity, though a conjectured and formally proven reality, has comparatively

little calculational foundation. It is the intention of this thesis to attempt to provide some of this, to indicate possible extensions and to answer the above questions. In the absence of any experimental results such a discussion necessarily centres upon the realisation of the original co-ordinate invariance of general relativity in terms of Ward Identities in the quantum theory. Their validity or otherwise provides the only available check on the self consistency of the theory quite apart from their fundamental role in the as yet absent renormalisation program.

I shall now give a brief summary of the separate sections that constitute this thesis. Each section is preceded by a short introduction of its own and I include this, at the risk of being repetitive, for the purposes of obtaining a single coherent picture.

The most natural starting point is, of course, the quantisation procedures themselves. In section 2 I briefly review three of these and establish a formal equivalence of their final results in as much as they are all derivable from Fadeev and Poppov's method. One is then confronted with a choice as to which variables to quantise and which form of the Einstein Action to work with. Here there are essentially two quite different directions that one can take_ the first and second order formalisms. Section 2 deals with the more usual of these notably and for reasons about which I am unclear the second order. The first order formalism is dealt with in some detail in section 6.

Of the possible forms for the Einstein action applicable to the second order formulation by far the most algebraically simple is that due to Goldberg⁽⁸⁾. At the end of section 2 I derive the associated Feynman rules and fictitious particle structure for what seems to be the most convenient choice of gauge condition.

Section 3 is devoted to both obtaining the particular Ward identity that I later calculate to second order in the gravitational coupling constant and the derivation of more complex Ward identities involving open fictitious particle lines. The usefulness in this context of the manifestly covariant theory of DeWitt is stressed and I show that it leads to precisely the same identities obtained by more conventional means. Identities are all obtained for the same choice of gauge condition and basic variables as used in section 2 but the extension of the method to alternative choices is obvious.

Fundamental to obtaining a meaningful result for the proposed calculation to be found in sections 5 and 6 is the use of a gauge invariant regularisation procedure to deal with the acute ultraviolet divergencies that are encountered. In section 4 I develop an adaptation of the 'dimensional regularisation' technique of t'Hooft and Veltman that is particularly useful for gravitational calculations. In order to provide an illustration of this adaptation, that might at first sight appear a little unwieldy, I include a calculation of the second order Yang-Mills self energy diagrams, the results for which were

first given by t'Hooft⁽⁴⁾. The comparative simplicity of this regularisation procedure is crucial in facilitating the ensuing analagous gravitational calculation.

This calculation is performed first in section 5 and then again in section 6 using the first order formalism once the requisite Feynman rules have been obtained. Apart from one minor difference discussed in section 6 the results are identical. The finite, logarithmically and quadratically divergent, parts are shown to be individually transverse. The only reservations are that a set of ambiguous tadpole diagrams are not evaluated in both instances. Moreover, again for both calculations, there appears in the answer a non-transverse quartic divergence whose algebraic structure is that of the free graviton propogator. Also in section 5 is given a prescription for dealing with the algebraic calculations. Some such prescription is necessitated by the complexity of the vertices that are recorded in detail in Appendix I. That given is certainly most effective; whether or not it corresponds to the most simple is hard to say. Finally, in order to lend more credibility to the answer which is of a sufficiently complicated form to warrant it, I give in some detail some of the intermediate stages of the separate calculations.

The tadpole diagrams and unwanted quartic divergence are discussed at some length in section 7. In particular the tadpole diagrams are shown to be excessively ambiguous. Possible reasons for the occurrence and thereby possible

means of removal of the quartic divergence are also discussed.

Section 8, as has already been mentioned, deals with the practicability of improving non-polynomial gravitational techniques. Most of this section is of a rather general nature not out of keeping, one might say, with its subject matter. However there is given an explicit derivation of the manifestly covariant bare vertex functions that have some bearing on these matters.

2. QUANTIZATION OF THE GRAVITATIONAL FIELD.

The major obstacle to the successful quantization of gravitation is the general co-ordinate invariance of the classical Einstein action. In the quantum theory this manifests itself as a non-abelian gauge group. This has the immediate consequence that the graviton propagator is not uniquely defined. Alternatively one might say that the Euler-Lagrange equations are undetermined and do not give rise to a well defined initial-value problem that is necessary for the internal consistency of the quantization programme. In quantum electrodynamics this ambiguity does not affect the physical results of the theory. It was Feynman⁽¹⁾ who first remarked that this is not so for non-abelian gauge symmetries such as one has with gravitation: there are diagrams with closed loops, depending non trivially on the choice of propagator, which affect unitarity. In order to restore unitarity Feynman found it necessary to introduce, in addition to the graviton lines, lines corresponding to vector fermions.

The resolution of this difficulty has taken two main directions. Either to extract from the metric tensor the true (as opposed to gauge) degrees of freedom and impose canonical commutation relations upon them and their canonical conjugates and to solve the remaining constraint equations. This procedure, known conventionally as A.D.M.⁽⁹⁾ is not the one that I shall adopt in this thesis. The alternative is to impose a supplementary gauge condition from the outset using, essentially, a Lagrange multiplier

and to take account of the non-abelian nature of the gauge group in terms of fictitious, vector fermion particles. This essentially is the method employed by DeWitt⁽¹⁰⁾, Faddeev and Popov⁽³⁾, Fradkin and Tyutin⁽⁵⁾, and Honerkamp⁽¹¹⁾. However, with the possible exception of DeWitt and Honerkamp, the work of these various authors appears very different in form. It is pleasing, from the point of view of obtaining a single consistent theory, that by proceeding from Faddeev and Popov's method, which treats the problem of quantising a system with a non-abelian gauge symmetry in isolation, one can establish a 'formal' equivalence between these various approaches. Formal in the sense that it rests upon properties of functional integrals. But in as much as the various methods themselves rely upon similar properties it would not seem out of place. Indeed it would appear to be the price that one has to pay if one wants to quantise gravitation by this, probably the most direct, approach of the inclusion of a gauge condition. However it is only a price in terms of the lack of a rigorous definition of the functional integrals; their use introduces a considerable simplification, as may be seen from DeWitt's paper⁽¹⁰⁾ that concludes with the functional integral representation of the S-matrix. The derivation of this final representation is extremely complex.

Before proceeding to establish this equivalence and a discussion of the resulting Feynman rules I shall first outline very briefly the methods mentioned above.

Fradkin and Tyutin's⁽⁵⁾ Method.

The basic idea is to introduce an additional Lagrange multiplier field, $B^\mu(x)$, and to choose it in such a way so that it satisfies a free field equation. Thereby the B^μ field is not involved in the scattering and the S-matrix is unitary in the physical subspace.

In order to achieve this, to the original action, $S(g)$, is added a non-local gauge fixing term:

$$S(g) \rightarrow S(g) + S_1^\psi \equiv \int d^4x (L + L_1^\psi) \quad \text{where,}$$

$$L_1^\psi \equiv \psi_\mu(x, g) \bar{B}_\psi^\mu + \frac{1}{2\alpha} \bar{B}_\psi^\mu \eta_{\mu\nu} \bar{B}^\nu_\psi.$$

$\psi_\mu(x, g)$ is an arbitrary function describing a class of gauges, and $\bar{B}_\psi^\mu(x, g) = \int d^4y D_\psi(x, y, g) B_\nu(y)$. The function D_ψ is to be chosen so that the B field obeys a free field equation.

The field equations and the contracted Bianchi identities, $\frac{\delta S}{\delta f^\alpha(x)} = 0$ then imply:

$$Q_{\mu\nu}^\psi \cdot \bar{B}^\nu = 0 \quad \text{where,}$$

$Q_{\mu\nu}^\psi \equiv R_{\mu(k\lambda)} \frac{\delta \psi_\nu(y)}{\delta g_{k\lambda}(x)}$ and $R_{\mu(k\lambda)}$ is the differential operator corresponding to a gauge variation of the metric tensor viz:

$$\frac{\delta S}{\delta g^{\mu\nu}} \equiv R_{\mu(k\lambda)} \frac{\delta S}{\delta g^{k\lambda}} = \frac{\delta S}{\delta g^{k\lambda}} \cdot \frac{\delta g^{k\lambda}}{\delta g^{\mu\nu}}.$$

Defining

$$Q_{\psi\mu\nu}^{(0)} \equiv Q_{\psi\mu\nu} \Big|_{g_{\mu\nu} = \eta_{\mu\nu}}$$

and $\mathcal{D}_\psi^\mu \equiv [Q_\psi^{-1}]^{\mu\lambda} \overset{\leftarrow}{Q}_{\psi\lambda\nu}^{(0)}$, the field B is constrained to satisfy $Q_{\psi\mu\nu}^0 B^\nu = 0$.

The resulting generating functionals for the greens functions are then given by: -

$$(\alpha \neq 0) \quad Z_\alpha^\psi = \int (dg) \exp \left\{ i \int d^4x \left(\mathcal{L} - \frac{1}{2\alpha} \psi^\mu \eta_{\mu\nu} \psi^\nu + g_{\mu\nu} T^{\mu\nu} \right) + \text{Tr.} \ln (Q_\psi \overset{\leftarrow}{Q}_\psi^{0-1}) \right\} \quad (2.1)$$

$(\alpha = 0)$

$$Z_0^\psi = \int (dg) \delta(\psi_\mu) \exp \left\{ i \int d^4x \left(\mathcal{L} + g_{\mu\nu} T^{\mu\nu} \right) + \text{Tr.} \ln (Q_\psi Q_\psi^{0-1}) \right\} \quad (2.2)$$

The vacuum-to-vacuum amplitude $\langle 0 \text{ out} | 0 \text{ in} \rangle^\psi$ is obtained from the above by setting J equal to zero viz:

$$\langle 0 \text{ out} | 0 \text{ in} \rangle_\alpha^\psi = Z_\alpha^\psi \Big|_{J=0}, \quad \text{similarly for } \alpha = 0.$$

DeWitt's Method (10).

DeWitt achieves a manifestly covariant quantisation of the gravitational field by introducing a classical background field, $g_{\mu\nu}$, on which the quantum fluctuations propagate. The gauge symmetry of the quantum field, $g_{\mu\nu}$, is broken, as it must be, but in such a way so as to preserve manifest covariance with respect to the classical background. The transition to the more usual formulation, where one has an asymptotically flat region with the Poincaré group as a symmetry group, is achieved at each stage in the formalism by setting $g_{\mu\nu} = \eta_{\mu\nu}$, the usual Minkowski metric.

In order to avoid the characteristic plethora of indices it is useful to work with DeWitt's 'condensed notation'. By which the field indices $(\mu\nu)$, $(\kappa\lambda)$, etc. are denoted by $i, j, \dots$. The first part of the Greek alphabet is reserved for group indices; that is those indices corresponding to arbitrary co-ordinate transformations, $\delta\xi_\alpha$. Primes on indices are to distinguish different space-time points and the summation convention for repeated indices is to extend to integrations over space time variables. Thus following this convention the Einstein action, S , is expanded about the background field as follows:

$$S(g_c + g) = S(g_c) + S_{,i}(g_c)g^i + \frac{1}{2}S_{,ij'}(g_c)g^i g^{j'} + \dots$$

The condition that the action functional is a group invariant is then written:

$$\frac{\delta S}{\delta \xi^\alpha} = S_{,i} R_\alpha^i \equiv 0 \quad , \text{ where } R_\alpha^i = \frac{\delta g_c^i}{\delta \xi^\alpha} . \quad (2.3)$$

Functionally differentiating (2.3) we obtain:

$$S_{,ij} R_\alpha^j + S_{,j} R_{\alpha,i}^j = 0 .$$

Thus when the field equations are satisfied, $S_{,j} = 0$, $S_{,ij}$ is a singular operator, possessing the R_α^i as zero-eigenvalue eigenvectors. The singularity of $S_{,ij}$ is now seen to be a direct consequence of the contracted Bianchi identities, $\frac{\delta S}{\delta \xi^\alpha} \equiv 0$. At this stage DeWitt imposes a general gauge condition specifically on the retarded and advanced solutions to the field equations. For our purposes it is sufficient to consider only the effect of this whereby the operator

S_{ij} is replaced by the non-singular operator F_{ij} where:

$$F_{ij} \equiv S_{ij} + \gamma_{ik} \gamma_{jl} R_{\alpha}^k \tilde{\gamma}^{-1\alpha\beta} R_{\beta}^l .$$

γ_{ij} is an arbitrary⁽¹²⁾ matrix, used to lower field indices, having the group transformation law suggested by the position of its indices.

$\tilde{\gamma}_{\alpha\beta}$ and its unique inverse (by definition), $\tilde{\gamma}^{-1\alpha\beta}$, likewise are arbitrary matrices used to lower and raise group indices respectively.

It is the arbitrariness in these γ -matrices that enables DeWitt to compute the 'radiative corrections' or the additional fictitious particle structure, necessary to ensure unitarity of the S-matrix.

The consequent expression for the vacuum-to-vacuum amplitude is as follows: -

$$\begin{aligned} \langle 0_{out} | 0_{in} \rangle = & \frac{\int (dg) \exp i \left\{ S(g_c + g) - S(g_c) - S_{,i}(g_c) g^i + \frac{1}{2} R_{ik} R_j^{\alpha} g^i g^j \right\}}{\int (d\psi)(d\psi^*) \exp i \left\{ R_{\alpha}^i(g_c + g) R_{i\beta} \psi^{*\alpha} \psi^{\beta} \right\}} \\ & (2.4) \end{aligned}$$

The numerator of this expression is simply the original action, with the replacement $S_{ij} \rightarrow F_{ij}$ and with those parts, $S(g_c) + S_{,i}(g_c) g^i$, that do not contribute to the scattering process, removed. The denominator is the encumbent fictitious particle term.

Faddeev and Poppov's⁽³⁾ Method.

The essence of this procedure is to extract from the functional integral representing $\langle 0 \text{ out} / 0 \text{ in} \rangle$ the "volume of an orbit of the gauge group". This comes about from the following considerations:

The action $S(g)$ is invariant under the replacement $g(x) \rightarrow g^\Omega(x)$, where $g^\Omega(x)$ is the result of applying an element of the gauge group, G , (in the case of gravitation the general co-ordinate transformation group) to the field $g(x)$. viz:

$$\mathcal{N}: g \rightarrow g^\Omega.$$

The orbit of g under G is defined to be

$$\text{Orb}(g) \equiv \{ g^\Omega \mid \forall \mathcal{N} \in G, \text{fixed } g \}.$$

Thus one can say that the action is constant on the orbits of G . Hence $\langle 0 \text{ out} / 0 \text{ in} \rangle$ is proportional to the volume of an orbit, i.e.

$$\langle 0 \text{ out} | 0 \text{ in} \rangle = \int (dg) \exp i S(g) \propto \int (d\Omega) \times \int F(g) (dg). \quad (2.5)$$

It is the functional form for $F(g)$ that has to be determined. One can then factor out the redundant orbit volume and take the remaining expression as a definition for the vacuum-to-vacuum amplitude in the presence of a gauge group. To do this Faddeev and Poppov choose to integrate $S(g)$ in (2.5) over hypersurface, $\psi(g) = 0$, in the manifold of all possible 'points' g . $\psi(g)$ is chosen so that distinct orbits intersect it once only.

First then the functional $\Delta_\psi(g)$ is defined by:

$$\Delta_{\Psi}(g) \int \delta(\Psi(g^{\Omega})) (d\Omega) = \text{constant} \quad (2.6)$$

Now incorporate (2.6) under the integral (2.5) and perform a gauge transformation on the g field. This leads to the required factorisation, viz:-

$$\langle O_{out} | O_{in} \rangle = \int (d\Omega) \times \int (dg) \exp i \{ S(g) \} \Delta_{\Psi}(g) \delta(\Psi(g)) . \quad (2.7)$$

It being understood that the measure (dg) is a group invariant. Expression (2.7) can now be interpreted as the functional $\psi(g)$ fixing the gauge, $\psi(g) = 0$, and the functional $\Delta_{\Psi}(g)$, determined by (2.6), as giving rise to the fictitious particle structure.

We are now in a position to establish the equivalence of the final results of these three methods. First we show that (2.7) and (2.6) together imply (2.1) and (2.2).

Incorporating the relevant indices into (2.6), $\Delta_{\Psi}(g)$ is defined for the gravitational field by:-

$$\Delta_{\Psi}^{-1} = \int \delta(\Psi_{\mu}(g^{\kappa\lambda\Omega})) (d\Omega)$$

for some suitable choice of constant. We can make a functional Fourier transform of the δ -function in this expression to give:

$$\Delta_{\Psi}^{-1} = \int (dc) (d\Omega) \exp i \left\{ \int d^4x C^{\mu}(x) \Psi_{\mu}(x, g^{\Omega}) \right\}$$

where this integral, for the moment is evaluated on the subspace $\psi_{\mu}(g) = 0$. Expanding $\psi_{\mu}(g^{\Omega})$ in powers of Ω we obtain:-

$$\Delta_{\psi}^{-1} =$$

$$\int (dC)(d\mathcal{L}) \exp i \left\{ \int d^4x d^4y C^{\mu}(x) \left(\frac{\delta \Psi_{\mu}(x)}{\delta g^{\kappa\lambda}(y)} \cdot g^{\kappa\lambda}(y) \right)_{,\alpha} \mathcal{L}^{\alpha}(y) + O(\mathcal{L}^2) \right\} \quad (2.8)$$

If one now regards the argument of the exponential in this integral to be the Lagrangian of the C and $\mathcal{L}$ fields coupled to the gravitational field, the free part is given by:-

$$\int d^4x d^4y C^{\mu}(x) \left[\frac{\delta \Psi_{\mu}(x)}{\delta g^{\kappa\lambda}(y)} g^{\kappa\lambda}(y) \right]_{,\alpha} \mathcal{L}^{\alpha}(y) \quad g^{\kappa\lambda} = \eta^{\kappa\lambda}$$

Recalling the definition of $Q_{\psi\mu\alpha}^{(0)}$ on page (14) this may be written:

$$\int d^4x d^4y C^{\mu}(x) Q_{\psi\mu\alpha}^{(0)}(x-y) \mathcal{L}^{\alpha}(y)$$

Thus there is only a $\langle C, \mathcal{L} \rangle$ propagator, given by $Q_{\psi\mu\alpha}^{(0)}$, and the terms in $\mathcal{L}^2$ and higher powers, occurring in 2.8 can be discarded as they do not contribute to the value of the integral evaluated perturbatively. Again recalling the definition of $Q_{\mu\alpha}^{\psi}$, we may now rewrite equation (2.8) as follows:

$$\Delta_{\psi}^{-1} = \int (d\mathcal{L}) \delta(Q_{\psi\mu\alpha} \mathcal{L}^{\alpha}) \quad (2.9)$$

This is evaluated via the transformation,

$$\omega_{\mu} = Q_{\mu\alpha}^{\psi} \mathcal{L}^{\alpha}, \quad (d\omega) = \det(Q_{\mu\alpha}^{\psi}) (d\mathcal{L})$$

We then obtain $\Delta_{\psi}^{-1} = \det^{-1}(Q_{\mu\alpha}^{\psi})$ and can write

$$\det(Q_{\mu\alpha}^{\psi}) = \exp \text{tr.} \left\{ \ln Q^{\psi} \bar{Q}_{\psi}^{(0)-1} + \ln Q_{\psi}^0 \right\}$$

The $\ln Q_{\psi}^0$ term may be effectively ignored⁽¹³⁾. Substituting the resulting expression for Δ_{ψ} into equation (2.7) we reproduce equation (2.2). It should be understood that the Q_{ψ} term is evaluated on the subspace $\psi_{\mu} = 0$. This is equivalent to the condition $\alpha = 0$ in equation (2.2).

In order to obtain equation (2.1) we observe that we are free to choose the gauge condition $\psi_{\mu}(g) = \beta_{\mu}$, where $\beta_{\mu}(x)$ is some arbitrary function, add an arbitrary weight function, $\exp i \{-\frac{1}{2\alpha} B_{\mu} B^{\mu}\}$, to equation (2.7) and integrate out over B_{μ} without altering its value (save possibly for its normalisation). We then obtain instead of equation (2.7):-

$$\langle 0_{out} | 0_{in} \rangle^{\psi} = \int (dg) \exp i \left\{ S(g) - \frac{1}{2\alpha} \psi_{\mu} \eta^{\mu\nu} \psi_{\nu} \right\} \Delta_{\psi}(g) \Big|_{\psi \neq 0}. \quad (2.10)$$

This immediately leads to equation 2.1, $\Delta_{\psi}(g)$ is now evaluated from (2.9) with $\psi_{\mu} \neq 0$.

Next we shall derive DeWitt's expression for the vacuum-to-vacuum amplitude given in equation (2.4).

We shall use as our starting point equation (2.10). We choose to incorporate the gauge:

$$\psi_{\alpha}(g) \equiv \left\{ R_{\alpha}^i(g_c) (g_i - g_{ic}) - B_{\alpha} \right\} = 0$$

For the arbitrary weight function we choose

$$\exp i \left\{ -\frac{1}{2\alpha} B_{\alpha} \gamma^{-1\alpha\beta} B_{\beta} \right\}.$$

As for equation (2.8) Δ_ψ^{-1} is now given by: $\Delta_\psi^{-1} =$

$$\int (dc)(d\mathcal{L}) \exp i \left\{ \int d^4x d^4y C_\alpha^\lambda(x) R_\alpha^i(g_c) R_{i\beta}(g) \mathcal{L}^\beta(y) \right\}. \quad (2.11)$$

It is an assumed property of functional integrals that they are translation invariant i.e. that

$$\int (dg) F(g) = \int (dg) F(g+g_c).$$

Thus if we substitute the values for ψ_α and the weight function given above into equation (2.10) and make a translation in the argument over the classical field g_c we obtain: (for $\alpha = -1$) $\langle 0_{out} | 0_{in} \rangle =$

$$\int (dg) \exp i \left\{ S(g+g_c) + \frac{1}{2} R_{i\alpha} \tilde{\gamma}^{-1\alpha\beta} R_{j\beta} g^i g^j \right\} \Delta_\psi(g+g_c). \quad (2.12)$$

$\Delta_\psi(g+g_c)$ is now evaluated from the translated equation (2.11) . viz:

$$\Delta_\psi^{-1}(g+g_c) = \int (dc)(d\mathcal{L}) \exp i \left\{ \int d^4x d^4y C_\alpha^\lambda(x) R_\alpha^i(g_c) R_{i\beta}(g+g_c) \mathcal{L}^\beta \right\}$$

Proceeding now as before to evaluate $\Delta_\psi(g+g_c)$ we obtain:

$$\Delta_\psi(g+g_c) = \det (R_\alpha^i(g_c) R_{i\beta}(g+g_c)).$$

If we now make use of the representation of the inverse of the infinite determinant $\det(A_{\alpha\beta})$ in terms of fermion fields given by:

$$\det^{-1}(A_{\alpha\beta}) = \int (d\psi^*)(d\psi) \exp i \left\{ \psi^{*\alpha} A_{\alpha\beta} \psi^\beta \right\} \quad (3)$$

we obtain

$$\Delta_{\psi}^{-1}(g+g_c) =$$

$$\int (d\psi^*) (d\psi) \exp i \left\{ R_{\alpha}^i(g+g_c) R_{i\beta}(g_c) \psi^{*\alpha} \psi^{\beta} \right\}. \quad (2.13)$$

We can remove the superfluous term, $S(g_c) + S_{,i}(g_c)g^i$ from the exponential in equation (2.12) and substitute for Δ_{ψ} from (2.13) into (2.12) to obtain equation (2.4).

It is worth remarking at this stage that the structure of the fictitious particle term contained in equation (2.13) is very simple. If we expand the argument of the exponential (regarded now as the fictitious particle Lagrangian) about the classical background we obtain

$$i \left\{ \psi^{*\alpha} \psi^{\beta} \left(R_{\alpha}^i(g_c) R_{i\beta}(g_c) + R_{\alpha,j}^i(g_c) R_{i\beta}(g_c) g^j \right) \right\}.$$

All other expansion terms vanish since $R_{\alpha}^i(g)$ is linear in g .

Finally it is well to point out that by setting $g_c = n$ in equation (2.4) and choosing the corresponding gauge in equation (2.1) one does obtain identical expressions for $\langle 0 \text{ out} / 0 \text{ in} \rangle$.

I have not included a discussion of Honerkamp's () formulation as it is very similar to DeWitt's and is derivable in much the same way. Indeed the above derivation of equation (2.4) is to some extent based on his work.

The Feynman Rules.

The usual form for the Einstein Action is given by:

$$S = \frac{1}{\kappa^2} \int d^4x (-g)^{1/2} R \equiv \int d^4x L(g). \quad (2.14)$$

$g^{\mu\nu}$ is the metric tensor; $g = \det g$, R is the contracted Ricci tensor, $g^{\mu\nu} R_{\mu\nu} \equiv g^{\mu\nu} g^{\sigma\tau} R_{\sigma\mu\nu\tau}$.

The Riemann tensor $R_{\sigma\mu\nu}{}^{\tau}$ is given by:

$$R_{\sigma\mu\nu}{}^{\tau} = \Gamma_{\mu\nu,\sigma}^{\tau} - \Gamma_{\sigma\nu,\mu}^{\tau} + \Gamma_{\mu\nu}^{\rho} \Gamma_{\sigma\rho}^{\tau} - \Gamma_{\sigma\nu}^{\rho} \Gamma_{\mu\rho}^{\tau},$$

$$\Gamma_{\mu\nu}^{\sigma} = \frac{1}{2} g^{\mu\tau} (g_{\mu\tau,\nu} + g_{\nu\tau,\mu} - g_{\mu\nu,\tau}). \quad (2.15)$$

In all that has been said so far in this section it has been tacitly assumed that the Einstein action is a functional of the metric tensor alone. That is to say the Ricci tensor is evaluated in terms of g by substituting the expression for the connection $\Gamma_{\mu\nu}^{\sigma}$ given above. There is no necessity to do this. The action S can be considered as a functional of the independent fields g and Γ ; the field equations resulting from independent variations of these quantities then provide the relation (2.15)⁽¹⁴⁾.

This procedure is alternatively known as the 'Palatini formalism' or the 'first order formalism'. Fradkin and Tyutin and Faddeev and Poppov both give prescriptions for obtaining the Feynman rules corresponding to this formalism in the references already quoted. I shall not discuss this any further here but reserve it for the calculation in section (6).

It has also been assumed thus far that the separation of S into its free and interaction parts is achieved by choosing $g^{\mu\nu}$ as the variable to be quantised and setting $g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}$ or in DeWitt's case, $g^{\mu\nu} = g_c^{\mu\nu} + \kappa h^{\mu\nu}$,

where κ is the gravitational coupling constant. This parametrisation of the metric tensor, known as the "rational parametrisation" is the one that I shall adopt but there are advantages in using the 'exponential parametrisation'; $g^{\mu\nu} = \exp(\kappa h)^{\mu\nu}$ (15). The only other variation is in the choice of variable to be quantised. The most common being, instead of choosing $g^{\mu\nu}$, to choose $(-g)^n g^{\mu\nu}$. We shall examine the possible consequences of this now.

Firstly the methods for quantisation outlined in the first part of this section are only slightly altered and one obtains the same equivalence of results in much the same way as before. The second consequence is to introduce a possible simplification in the form of the action (2.14).

Let us define

$$\tilde{g}^{\alpha\beta} \equiv (-g)^n g^{\alpha\beta}, \quad \tilde{g}_{\alpha\beta} \equiv (-g)^{-n} g_{\alpha\beta}$$

then we have $\tilde{g}^{\alpha\beta} \tilde{g}_{\beta\gamma} = \delta_{\gamma}^{\alpha}$ and we can write (2.14) as follows:

$$\begin{aligned} S = \frac{1}{8\kappa^2} \int d^4x \left\{ g^{(\frac{1}{2}-n)} \tilde{g}_{\alpha_1\beta_1}{}^{,\rho} \tilde{g}_{\alpha_2\beta_2}{}^{,\sigma} \left(\frac{4n(10n-4)}{(4n-1)^2} \tilde{g}^{\rho\sigma} \tilde{g}_{\alpha_1\beta_1} \tilde{g}_{\alpha_2\beta_2} \right. \right. \\ \left. \left. - \tilde{g}^{\rho\sigma} \tilde{g}_{\alpha_1\alpha_2} \tilde{g}_{\beta_1\beta_2} + \frac{4(2n-1)}{(4n+1)} \delta_{\alpha_1}^{\rho} \delta_{\beta_1}^{\sigma} \tilde{g}_{\alpha_2\beta_2} \right. \right. \\ \left. \left. - 4 \delta_{\alpha_1}^{\sigma} \delta_{\alpha_2}^{\rho} \tilde{g}_{\beta_1\beta_2} \right) + (4\text{-divergence}) \right\} \quad (2.16) \end{aligned}$$

The most convenient choice for n is clearly $n = \frac{1}{2}$. Ex-

pression (2.16) then reduces to Goldberg's⁽⁸⁾ form of the Einstein Action.

If one separates (2.16) into its free and interaction parts by setting $\tilde{g}^{\alpha\beta} = \eta^{\alpha\beta} + \kappa \hat{h}^{\alpha\beta}$. It is amusing, but little more, that terms involving κ^{10} and above vanish for the choice $n = 5/18$ and terms involving κ^{12} and above vanish for the choice $n = 5/22$. DeWitt remarks upon this in his paper⁽¹⁰⁾.

For the remainder of this section and for the calculation in section (5) I shall choose to work with the Goldberg form of the Einstein Action. To be specific: -

Defining $\hat{g}^{\alpha\beta} = (-g)^{1/2} g^{\alpha\beta}$, $\hat{g}_{\alpha\beta} = (-g)^{-1/2} g_{\alpha\beta}$, we have

$$S(\hat{g}) = \frac{1}{8\kappa^2} \int d^4x \left\{ \hat{g}^{\alpha_1\beta_1}{}_{,\rho} \hat{g}^{\alpha_2\beta_2}{}_{,\sigma} \left(\hat{g}^{\rho\sigma} \hat{g}_{\alpha_1\beta_1} \hat{g}_{\alpha_2\beta_2} - \hat{g}^{\rho\sigma} \hat{g}_{\alpha_1\alpha_2} \hat{g}_{\beta_1\beta_2} - 4 \delta_{\alpha_1}^\sigma \delta_{\alpha_2}^\rho \hat{g}_{\beta_1\beta_2} \right) \right\} \quad (2.17)$$

To obtain the requisite perturbation series we set,

$\hat{g}^{\alpha\beta} = \eta^{\alpha\beta} + \kappa \hat{h}^{\alpha\beta}$, where $\hat{h}^{\alpha\beta} = \hat{h}^{\beta\alpha}$ and $\eta^{\alpha\beta}$ is the Minkowski flat space metric, which is now used to raise and lower the indices of the quantum $\hat{h}$ field (2.17) can now be expanded in powers of κ to any order we please. The expansion of $\hat{g}_{\alpha\beta}$ being determined by the relation $\hat{g}^{\alpha\beta} \hat{g}_{\beta\gamma} = \delta^\alpha_\gamma$. This leads to:

$$\hat{g}_{\alpha\beta} = \eta_{\alpha\beta} - \kappa \hat{h}_{\alpha\beta} + \kappa^2 \hat{h}_{\alpha\gamma} \hat{h}^\gamma_\beta - \kappa^3 \hat{h}_{\alpha\gamma} \hat{h}^{\gamma\delta} \hat{h}_{\delta\beta} + \dots$$

Thus L is expanded $L = L_0 + \kappa L_1 + \kappa^2 L_2 + O(\kappa^3)$.

If we define

$$Q_{\alpha_1 \beta_1, \alpha_2 \beta_2} \equiv \frac{1}{2} (\eta_{\alpha_1 \alpha_2} \eta_{\beta_1 \beta_2} + \eta_{\alpha_1 \beta_2} \eta_{\beta_1 \alpha_2} - \eta_{\alpha_1 \beta_1} \eta_{\alpha_2 \beta_2})$$

then L_0 is given by:

$$L_0 = \frac{1}{4} \hat{h}^{\alpha_1 \beta_1}_{, \rho} \hat{h}^{\alpha_2 \beta_2}_{, \sigma} \left(\eta^{\rho \sigma} Q_{\alpha_1 \beta_1, \alpha_2 \beta_2} - 2 \delta^{\sigma}_{\alpha_1} \delta^{\rho}_{\alpha_2} \eta_{\beta_1 \beta_2} \right)$$

As we should expect L_0 does not give rise to a well defined propagator - we have yet to fix the gauge. Before doing this and evaluating the fictitious particle term we shall first examine the structure of the bare vertex functions of the interaction terms in the expansion of (2.17). We are able to do this independantly of considerations of gauge since the gauge we shall eventually adopt will contribute only to L_0 .

In x-space we define the n-graviton bare vertex function as follows:

$$\hat{\Gamma}_{\alpha_1 \beta_1, \dots, \alpha_n \beta_n}(x_1, \dots, x_n) = \frac{\delta^n}{\delta \hat{h}^{\alpha_1 \beta_1}_{(x_1)} \dots \delta \hat{h}^{\alpha_n \beta_n}_{(x_n)}} \int d^4 x \kappa^n L_n(\hat{h}) \quad (2.18)$$

L_n in the expansion of (2.17) is of the form:

$$L_n = \int d^4 x \hat{h}^{\alpha_1 \beta_1}_{(x)} \dots \hat{h}^{\alpha_{n-2} \beta_{n-2}}_{(x)} \hat{h}^{\alpha_{n-1} \beta_{n-1}}_{(x), \rho} \hat{h}^{\alpha_n \beta_n}_{(x), \sigma} A^{\rho \sigma}_{\alpha_1 \beta_1, \dots, \alpha_n \beta_n}$$

There are only two derivatives and they are confined to $\hat{h}^{\alpha_{n-1} \beta_{n-1}}_{(x), \rho}$ and $\hat{h}^{\alpha_n \beta_n}_{(x), \sigma}$. The term A is a pure algebraic combination of η 's.

Substituting this expression into (2.18) and using the relation,

$$\frac{\delta^n}{\delta \hat{h}^{\alpha_1 \beta_1}(x_1) \dots \delta \hat{h}^{\alpha_n \beta_n}(x_n)} \left(\hat{h}^{\alpha_1 \beta_1}(x) \dots \hat{h}^{\alpha_n \beta_n}(x) \right)_{,\sigma} =$$

$$S_{\alpha_1 \beta_1} \dots S_{\alpha_n \beta_n} P_n! \left(\delta(x-x_1) \delta(x-x_2) \dots \delta(x-x_n) \right)_{,\sigma},$$

(where $S_{\alpha_i \beta_i}$ stands for a symmetrisation to be performed in the index pair $(\alpha_i \beta_i)$, and $P_n!$ stands for a summation to be carried out over the $n!$ permutations of the subscripts $(1, 2, \dots, n)$) we arrive at the following expression for ${}^n\Gamma$:

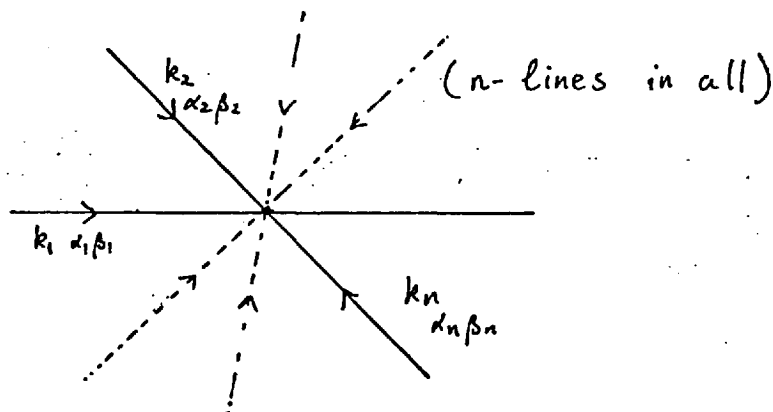
In momentum space,

$${}^n\Gamma(k_1, \dots, k_n)_{\alpha_1 \beta_1, \dots, \alpha_n \beta_n} =$$

$$S_{\alpha_1 \beta_1} \dots S_{\alpha_n \beta_n} P_n! (-k_{n-1\rho} k_{n\sigma}) \delta(k_1 + k_2 + \dots + k_n) A^{\rho\sigma}_{\alpha_1 \beta_1 \dots \alpha_n \beta_n}.$$

The explicit forms for $n = 3$ and $n = 4$ are given in appendix I.

Diagrammatically this corresponds to the vertex



all momenta being defined inwards.

As may be imagined these vertices contain a prodigious number of terms. The only point in their favour is that ${}^3\Gamma$ and ${}^4\Gamma$ contain significantly fewer than the equivalent expressions given by DeWitt⁽¹⁶⁾ for the choice of variable $g^{\mu\nu}$.

We now turn to the evaluation of the fictitious particle contribution to the action 2.17. If we choose to impose the Dedonder gauge, $g^{\mu\nu}{}_{,\nu} = 0$, directly on the action (2.17) we would have for the vacuum-to-vacuum amplitude:

$$\langle 0 \text{ out} / 0 \text{ in} \rangle_{\text{Dedonder}} = \int (dg) (dC) \exp i \left\{ S(\hat{g}) + C_{\mu} \hat{g}^{\mu\nu}{}_{,\nu} \right\} \Delta_{\Psi}(\hat{g}).$$

The lagrange multiplier field C_{μ} necessarily becomes a dynamical variable and its elimination leads to the transverse propagator:

$$\langle 0 | T \hat{h}^{\alpha_1 \beta_1} \hat{h}^{\alpha_2 \beta_2} | 0 \rangle_D = \frac{2}{k^2} Q^{\alpha_1 \beta_1}{}_{\alpha \beta} \left(\eta^{\alpha_2 \alpha} \eta^{\beta_2 \beta} - \frac{2 k^{\alpha} k^{\beta}}{k^2} Q^{\lambda \beta, \alpha_2 \beta_2} \right) \quad (2.19)$$

This has two major disadvantages. Firstly, from considerations of performing calculations, it contains rather too many terms for comfort. Secondly (if it matters)

there is no well defined rule for how to go round the pole in the term $\frac{k^{\alpha} k^{\lambda}}{k^2} Q$. Fortunately we can achieve a

simplification by incorporating the above gauge condition via equation (2.10), i.e. set $\psi^{\mu} = \hat{g}^{\mu\nu}{}_{,\nu}$ and choose the

free parameter, α , of equation (2.10) to be -1. This has

the effect of cancelling the term, $-\frac{1}{2} \hat{h}^{\alpha_1 \beta_1}{}_{,\rho} \hat{h}^{\alpha_2 \beta_2}{}_{,\sigma} \delta^{\sigma}_{\alpha_1} \delta^{\rho}_{\alpha_2} \eta_{\beta_1 \beta_2}$

from L_0 . The resulting graviton propagator is then

$$\langle 0 | T \hat{h}^{\alpha_1 \beta_1} \hat{h}^{\alpha_2 \beta_2} | 0 \rangle = \frac{2}{k^2} Q^{\alpha_1 \beta_1, \alpha_2 \beta_2}.$$

This is essentially the same as DeWitt⁽¹⁰⁾ obtains from his requirement that the matrices γ_{ij} , $\tilde{\gamma}^{\alpha\beta}$ be chosen so as to give a propagator with minimal singularities on the light-cone.

The calculation of the fictitious particle term is straightforward proceeding from equation (2.8) and remembering that $\hat{g}^{\mu\nu}_{, \nu} \neq 0$. The gauge variation in $\hat{g}^{\kappa\lambda}$ is given by

$$\frac{\delta}{\delta \mathcal{L}^\alpha} \hat{g}^{\kappa\lambda} = -\partial_\alpha \hat{g}^{\kappa\lambda} + \hat{g}^{\kappa\gamma} \partial_\gamma \delta_\alpha^\lambda + \hat{g}^{\lambda\gamma} \partial_\gamma \delta_\alpha^\kappa - \hat{g}^{\kappa\lambda} \partial_\alpha$$

Hence

$$\begin{aligned} \Delta_\psi^{-1} = & \int (dC)(d\Omega) \exp i \left\{ \int d^4x \ C_\nu \eta^{\kappa\lambda} \Omega^\nu_{, \kappa\lambda} + \right. \\ & \left. + \kappa \left(\hat{h}^{\mu\gamma} C_\nu \Omega^\nu_{, \mu\gamma} - C_\nu \Omega^\gamma \hat{h}^{\mu\nu}_{, \mu\gamma} + C_\nu \hat{h}^{\mu\gamma}_{, \mu} \Omega^\nu_{, \gamma} - C_\nu \hat{h}^{\mu\nu}_{, \mu} \Omega^\gamma_{, \gamma} \right) \right\} \end{aligned} \quad (2.20)$$

We may take the argument of the exponential in (2.20) to be the fictitious particle Lagrangian provided we take account of the possible expression of Δ_ψ^{-1} in terms of fermion fields (cf. equation 2.13) by attaching a factor of (-1) to every closed loop of C , Ω lines. The fact that there are two fields C and Ω involved in (2.19) and only a $\langle C \Omega \rangle$ propagator, gives rise to 'directed loops' of ficti-

tious particles.

The propagator and vertex function corresponding to (2.20) are given in Appendix I.

This concludes the discussion of the quantisation of the gravitational field. As was mentioned earlier we shall return to the 'first order formalism' in section (6).

3. GRAVITATIONAL WARD IDENTITIES

One would hope that the original co-ordinate invariance of general relativity would be reflected in Ward Identities relating the quantised amplitudes of the quantum theory. Indeed this is the case, but it is a considerably more complicated situation than one has in electrodynamics for example. In the quantisation procedures outlined in section (2) the interaction Lagrangian for the gravitational field possesses an infinite number of distinct vertices and the gauge group in question is non-abelian. The validity of these identities is however vital to the self consistency of the quantum theory and so far, in the absence of any experimental results, provides the only check on the predictions of the theory.

It turns out that one can formally derive as many gravitational Ward Identities as one pleases. There are several ways⁽¹⁷⁾ in which this may be accomplished and in this section I shall deal with two specific approaches. The first is an adaption of t'Hooft's⁽⁴⁾ work on the Yang Mills field. J.C. Taylor⁽⁴⁾ gives a comprehensive account of this approach and it is essentially the analogue for the gravitational field of this method that I shall employ. The second is a consequence of DeWitt's⁽¹⁰⁾ manifestly covariant theory of quantum gravity.

Firstly, let us consider t'Hooft's approach. Recalling the expression for the vacuum-to-vacuum amplitude, given by equation (2.10), written now in the modified Dedonder gauge, $\hat{g}^{\mu\nu}_{, \nu} = B^\mu$, we have:

$$\langle 0_{out} | 0_{in} \rangle = \int (d\mathcal{B})(d\hat{g}) \exp i \left\{ S(\hat{g}) + \frac{1}{2} \mathcal{B}_\mu \eta^{\mu\nu} \mathcal{B}_\nu \right\} \delta(\mathcal{B}^\mu - \hat{g}^{\mu\nu}{}_{,\nu}) \Delta_\Psi(\hat{g}).$$

We can now exploit our freedom to choose a different gauge and add an arbitrary source function $J^\mu(x)$ the δ -function occurring in this expression. If we now perform the $\mathcal{B}$ -integration we obtain

$$\langle 0_{out} | 0_{in} \rangle_J = \int (d\hat{g}) \Delta_\Psi(\hat{g}) \exp i \left\{ S(\hat{g}, J) \right\} \quad (3.1)$$

where

$$S(\hat{g}, J) = S(\hat{g}) + \frac{1}{2} \hat{g}^{\mu\kappa}{}_{,\mu} \eta_{\kappa\lambda} \hat{g}^{\nu\lambda}{}_{,\nu} - J_\nu \hat{g}^{\mu\nu}{}_{,\mu} + J_\mu \eta^{\mu\nu} J_\nu$$

For the purposes of the calculations in sections (5) and (6) expression (3.1) is sufficient to obtain the particular Ward Identity that we need. This is achieved by requiring that the expansion of (3.1) in powers of J identically vanishes after the first term that is independent of J . This is a simple consequence of the gauge invariance of (3.1). In particular the term quadratic in J must satisfy the relation

$$\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} \langle 0_{out} | 0_{in} \rangle_J \bigg|_{J=0} = 0$$

Performing the requisite differentiations we obtain

$$\partial_\mu^{(x)} \partial_\nu^{(y)} \langle 0_{out} | \hat{h}^{\mu\rho}_{(x)} \hat{h}^{\nu\tau}_{(y)} | 0_{in} \rangle - \eta^{\rho\tau} \langle 0_{out} | 0_{in} \rangle = 0 \quad (3.2a)$$

In momentum space we can write this as follows:

$$k_\mu k_\nu \left\{ \frac{\langle 0_{out} | \hat{h}^{\mu\rho} \hat{h}^{\nu\tau} | 0_{in} \rangle}{\langle 0_{out} | 0_{in} \rangle} - \frac{2}{k^2} Q^{\mu\rho, \nu\tau} \right\} = 0 \quad (3.2b)$$

The inclusion of the free propagator into (3.2b) is a consequence of the algebraic structure of the Q 's viz.

$$2 k_\mu k_\nu Q^{\mu\rho, \nu\tau} = k^2 \eta^{\rho\tau}$$

The above is essentially t'Hooft's path integral derivation of Ward Identities given in reference (4). J.C. Taylor enlarges on this approach for the Yang Mills field and I shall now give the analogue for the gravitational field.

To (3.1) is added a source term enabling one to define the Greens functions in the J -dependent gauge. The resulting generating functional is given by

$$Z(J, j) = \int (d\hat{g}) \Delta_\Psi(\hat{g}) \exp i \left\{ S(J, \hat{g}) + j_{\mu\nu} \hat{g}^{\mu\nu} \right\}. \quad (3.3)$$

One next takes J to be infinitesimal and makes an infinitesimal gauge transformation of (3.3) by

$$\begin{aligned} \hat{g}^{\mu\nu} &\rightarrow \hat{g}^{\mu\nu} - R^{\mu\nu}_\alpha \Omega^\alpha \\ R^{\mu\nu}_\alpha &\equiv \frac{\delta}{\delta \Omega^\alpha} \hat{g}^{\mu\nu} = (-\partial_\alpha \hat{g}^{\mu\nu} + \hat{g}^{\mu\gamma} \partial_\gamma \delta^\nu_\alpha + \\ &\quad + \hat{g}^{\nu\gamma} \partial_\gamma \delta^\mu_\alpha - \hat{g}^{\mu\nu} \partial_\alpha) \end{aligned} \quad (3.4)$$

Ω^α is determined by the condition

$$\partial_\mu R^{\mu\nu}_\alpha \mathcal{L}^\alpha = \bar{J}^\mu \quad (3.5)$$

Under this transformation the measure $(d\hat{g})_{\Delta_\psi}(\hat{g})$ is invariant() and (3.3) is replaced by

$$\begin{aligned} Z'(\bar{J}, j) = & \int (d\hat{g}) \Delta_\psi(\hat{g}) \exp i \left\{ S(\hat{g}) + \right. \\ & \left. + \frac{1}{2} \hat{g}^{\mu\kappa}_{,\mu} \eta_{\kappa\lambda} \hat{g}^{\nu\lambda}_{,\nu} + j_{\mu\nu} \hat{g}^{\mu\nu} - j_{\mu\nu} R^{\mu\nu}_\alpha \mathcal{L}^\alpha \right\} \end{aligned} \quad (3.6)$$

One can now derive Ward Identities by equating the Greens functions defined alternatively by (3.3) or (3.6).

To facilitate discussion of identities resulting from differentiations of the j-source it is convenient to introduce the transverse and longitudinal projection operators T and L defined as follows:-

$$T^{\alpha\beta, \kappa\lambda} = \delta^{\alpha\beta, \kappa\lambda} - \frac{2}{\partial^2} Q^{\alpha\beta, \sigma}_\rho \partial_\sigma \partial_T \delta^{TP, \kappa\lambda}$$

$$L^{\alpha\beta, \kappa\lambda} = \delta^{\alpha\beta, \kappa\lambda} - T^{\alpha\beta, \kappa\lambda}$$

To prove that these satisfy the usual properties,

$$\partial \cdot T = 0, \quad T^2 = T, \quad T \cdot L = 0, \quad L^2 = L$$

it suffices to show

$$\partial \cdot T = 0 \quad \text{and} \quad L^2 = L$$

This may be seen to be the case as follows:-

$$\partial_\alpha T^{\alpha\beta, \kappa\lambda} = \partial_\alpha \delta^{\alpha\beta, \kappa\lambda} - \frac{2}{\partial^2} \partial_\alpha \partial_\sigma \partial_T Q^{\alpha\beta, \sigma}_\rho \delta^{TP, \kappa\lambda}$$

$$\begin{aligned}
&= \partial_\alpha \delta^{\alpha\beta, \kappa\lambda} - \frac{2}{\partial^2} \left(\frac{1}{2}\right) \partial_\alpha \partial_\sigma \eta^{\alpha\sigma} \partial_T \delta_\rho^\beta \delta^{TP, \kappa\lambda} \\
&= \partial_\alpha \delta^{\alpha\beta, \kappa\lambda} - \partial_T \delta^{T\beta, \kappa\lambda} = 0.
\end{aligned}$$

$$\begin{aligned}
L^{\alpha\beta, \mu\nu} L_{\mu\nu, \kappa\lambda} &= \frac{4}{\partial^4} Q^{\alpha\beta, \sigma}_\rho \partial_\sigma \partial_T \partial_{\sigma'} \partial_{T'} Q^{TP, \sigma'}_{\rho'} \delta^{T'\rho', \kappa\lambda} \\
&= \frac{2}{\partial^2} Q^{\alpha\beta, \sigma}_\rho \partial_\sigma \partial_{T'} \delta^{T'\rho, \kappa\lambda} = L^{\alpha\beta, \kappa\lambda}.
\end{aligned}$$

We can now write $j_{\mu\nu} R^{\mu\nu}_\alpha \Omega^\alpha$ of expression (3.6) in terms of these functions viz:-

$$j_{\mu\nu} R^{\mu\nu}_\alpha \Omega^\alpha \equiv j_{\mu\nu} (T^{\mu\nu}_{, \kappa\lambda} + L^{\mu\nu}_{, \kappa\lambda}) R^{\kappa\lambda}_\alpha \Omega^\alpha \quad (3.7)$$

Evaluating the right hand side of this expression:-

$$\begin{aligned}
L^{\mu\nu}_{\kappa\lambda} R^{\kappa\lambda}_\alpha \Omega^\alpha &= \frac{2}{\partial^2} Q^{\mu\nu, \sigma}_\rho \partial_\sigma \partial_T R^{TP}_\alpha \Omega^\alpha \\
&= \frac{2}{\partial^2} \partial_\sigma Q^{\sigma\rho, \mu\nu} T_\rho.
\end{aligned}$$

This follows from equation (3.5).

For the evaluation of the transverse term, we shall expand $R^{\kappa\lambda}_\alpha(\hat{g})$ in terms of η and $\hat{h}$.

Recalling the definition of $\hat{h}$ viz, $\hat{g}^{\kappa\lambda} = \eta^{\kappa\lambda} + \kappa \hat{h}^{\kappa\lambda}$

$$R^{\kappa\lambda}_\alpha(\hat{g}) = R^{\kappa\lambda}_\alpha(\eta) + \kappa R^{\kappa\lambda}_\alpha(\hat{h})$$

$$\begin{aligned}
R^{\kappa\lambda}_\alpha(\eta) &= \eta^{\kappa\gamma} \partial_\gamma \delta^\lambda_\alpha + \eta^{\lambda\gamma} \partial_\gamma \delta^\kappa_\alpha - \eta^{\kappa\lambda} \partial_\alpha \\
&= 2 \partial_\gamma Q^{\gamma, \kappa\lambda}_\alpha.
\end{aligned}$$

It is easily established that

$$T^{\mu\nu}_{\kappa\lambda} \partial_\gamma Q^\gamma_{\alpha}{}^{\kappa\lambda} = 0$$

We can now re-write identity (3.7),

$$j_{\mu\nu} R^{\mu\nu}_{\alpha}(\hat{q}) \mathcal{N}^\alpha = j_{\mu\nu} \frac{2}{\partial^2} \partial_\sigma Q^{\sigma\rho,\mu\nu} \mathcal{T}_\rho - \kappa j_{\mu\nu} T^{\mu\nu}_{\kappa\lambda} R^{\kappa\lambda}_{\alpha}(\hat{h}) \mathcal{N}^\alpha. \quad (3.8)$$

Let us now consider the particular identity

$$\frac{\delta}{\delta \mathcal{T}} \frac{\delta}{\delta j} \left\{ Z(\mathcal{T}, j) - Z'(\mathcal{T}, j) \right\}_{\mathcal{T}=j=0} = 0$$

We evaluate this using equations (3.3), (3.6) and (3.8).

In momentum space we obtain the result

$$k_\mu \langle 0_{out} | T \hat{h}^{\mu\nu} \hat{h}^{\kappa\lambda} | 0_{in} \rangle = 2 \frac{k_\mu}{k^2} Q^{\mu\nu,\kappa\lambda} \langle 0_{out} | 0_{in} \rangle - \kappa T^{\kappa\lambda}_{\sigma\tau}(k) \langle 0_{out} | R^{\sigma\tau}_{\alpha}(\hat{h}) \frac{\delta \mathcal{N}^\alpha}{\delta \mathcal{T}^\nu} | 0_{in} \rangle \quad (3.9)$$

Equation (3.9) clearly reproduces equation (3.2). This is a consequence of the final term on the right hand side of (3.9) being transverse. I shall not discuss the implications of this term at the moment save to say that it corresponds to open fictitious particle lines that result from the inversion of equation (3.5). I shall give a more explicit representation of this term when deriving essentially the same Ward Identity using DeWitts manifestly covariant

formalism.

Before moving on to this it is perhaps relevant at this stage to comment on the derivatives standing outside the time ordered products that have occurred thus far. The functional integral method is not sufficiently discriminating or well-defined to allow for a rigorous evaluation of the possible consequences of this with terms such as $\partial_\alpha \theta(x^0 - y^0)$. In all that has been said so far and is to follow I shall ignore such contributions by adopting the consistent approach of meaning by expressions such as (3.2a) the integral

$$\int (d\hat{h}) (\partial_\mu \hat{h}^{\mu\rho} \partial_\nu \hat{h}^{\nu\tau} - \eta^{\rho\tau}) \exp i \{ S \hat{g} \} \Delta_\psi(\hat{g}).$$

That is to say I shall evaluate such expressions in the integral form in which they arise by calculating the necessary Gaussian functional Integrals. Thus in general I use the expression

$$\partial_\alpha \dots \partial_\beta \langle 0 \text{ out} | T A(g) | 0 \text{ in} \rangle$$

as a convenient shorthand for the expression,

$$\int (dg) \partial_\alpha \dots \partial_\beta A(g) \exp i \{ S'(g) \} \Delta_\psi(g).$$

Let us now proceed to discover how one obtains Ward Identities using the manifestly covariant theory. First let us rewrite equation (2.4) using the variables $\hat{g}^{\mu\nu}$.

We shall then have

$$\begin{aligned} \langle 0 \text{ out} | 0 \text{ in} \rangle = & \int (d\hat{g})(d\chi) \exp i \left\{ S(\hat{g}, \phi) + \right. \\ & \left. + \frac{1}{2} \tilde{f}_{\alpha\beta} \chi^\alpha \chi^\beta \right\} \Delta_\psi(\hat{g}, \phi). \end{aligned} \quad (3.10)$$

Where:-

$$\hat{g}^i \equiv (-g)^{1/2} g^i, \quad \hat{g}_i \equiv (-g)^{-1/2} g_i$$

$$\phi^i \equiv (-g_c)^{1/2} g_c^i, \quad \phi_i \equiv (-g_c)^{-1/2} g_{ci}$$

$$S(\hat{g}, \phi) \equiv S(\hat{g} + \phi) - S(\phi) - S_{,i}(\phi) \hat{g}^i + \frac{1}{2} R_{i\alpha} \tilde{\gamma}^{-1\alpha\beta} R_{j\beta} \hat{g}^i \hat{g}^j$$

$$\Delta_T(\hat{g}, \phi) \equiv \int (dC) (d\mathcal{L}) \exp(-i) \left\{ R_{i\alpha}^i(\hat{g} + \phi) R_{i\beta}(\phi) C^\alpha \mathcal{L}^\beta \right\}$$

$$R_{i\alpha}^i(\phi) \equiv \frac{\delta \phi^i}{\delta \xi^\alpha}, \quad R_{j\alpha} \equiv \gamma_{ji} R_{i\alpha}^i.$$

The factor

$$\int (d\chi) \exp i \left\{ \frac{1}{2} \tilde{\gamma}_{\alpha\beta} \chi^\alpha \chi^\beta \right\}$$

is included to ensure invariance of (3.10) under arbitrary transformations of the $\tilde{\gamma}$ matrix. It is simply a determinant factor (see DeWitt p. 1237, ref. (10)).

I choose to write the fictitious particle term in the numerator as this facilitates later interpretations. One can regard it (as written in 3.10) as being an addition to the gravitational action $S(\hat{g}, \phi)$ as mentioned in section 2 provided one incorporates a factor of (-1) for each closed loop of fictitious particles.

As in section 2 the matrices γ_{ij} , $\tilde{\gamma}_{\alpha\beta}$ and their inverses are used to lower and raise field and group indices respectively. They are symmetric in their indices.

Finally

$$R_{i\alpha}^i(\hat{g} + \phi) \equiv R_{i\alpha}^i(\hat{g}) + R_{i\alpha}^i(\phi) = R_{i\alpha}^i(\phi) + R_{i\alpha,j}^i(\phi) \hat{g}^j$$

The arguments of the ubiquitous R terms should from now on taken to be ϕ unless otherwise stated.

I have made the dependence on the classical background field more explicit by writing ϕ where before we had g_c .

The methods now available for producing Ward Identities are numerous. Broadly speaking I shall be content to derive the particular identities (3.2) and (3.9) and to comment on their more obvious extensions.

Let us then first derive the covariant equivalent of (3.2). We shall in any case need this for the later derivation of the equivalent to (3.9).

One possibility open to us for this derivation is to exploit the invariance of (3.10) under arbitrary changes in the matrix $\tilde{\gamma}_{\alpha\beta}$. This is particularly simple since unlike γ_{ij} , $\tilde{\gamma}_{\alpha\beta}$ does not explicitly occur in the fictitious particle term. We may therefore write

$$\frac{\delta}{\delta \tilde{\gamma}^{-1\alpha\beta}} \langle 0_{out} | 0_{in} \rangle = 0$$

This is readily evaluated yielding the result:-

$$R_{i\alpha} R_{j\beta} \langle 0_{out} | \hat{g}^i \hat{g}^j | 0_{in} \rangle - \tilde{\gamma}_{\alpha\beta} \langle 0_{out} | 0_{in} \rangle = 0 \quad (3.11)$$

We have made use of the relation

$$\tilde{\gamma}_{\alpha\gamma} \delta \tilde{\gamma}^{-1\alpha\beta} + \tilde{\gamma}^{-1\alpha\beta} \delta \tilde{\gamma}_{\alpha\gamma} = 0$$

To see that (3.11) corresponds to (3.2) we must evaluate

it in the limit of the background field being set equal to η . We shall also need to know the flat space values of the matrices γ_{ij} and $\tilde{\gamma}_{\alpha\beta}$. Any choice for these matrices corresponds to a choice of gauge. In order to implement the same gauge for expression (3.10) as we used in the derivation of equations (3.2) and (3.9) we now make the choices for $\gamma_{\alpha\beta}$ and γ_{ij} as follows:-

In the limit $\phi_{\mu\nu} = \eta_{\mu\nu}$

$$\gamma_{\kappa\lambda, \mu\nu} = \frac{1}{2} Q_{\kappa\lambda, \mu\nu}$$

$$\tilde{\gamma}_{\alpha\beta} = \eta_{\alpha\beta}$$

The corresponding covariant quantities $\gamma_{ij}(\phi)$, $\tilde{\gamma}_{\alpha\beta}(\phi)$ are essentially those given by DeWitt in the table on page (20) of reference (10). It is only the flat space values that interest us and I do not bother to record their extensions.

With the above choices for the γ -matrices and the expression for R^i_α given by equation (3.4) we obtain the following

$$R^{\kappa\lambda}_\alpha(\eta) = 2\partial_\gamma Q^\gamma_{\alpha, \kappa\lambda}$$

$$R_{\mu\nu, \alpha}(\eta) = \gamma_{\mu\nu, \kappa\lambda} R^{\kappa\lambda}_\alpha = \partial_\gamma \delta^\gamma_{\alpha, \mu\nu}$$

(3.11) thus reduces to

$$\partial_\alpha \partial_\beta \langle 0_{out} | T \hat{h}^{\alpha\kappa} \hat{h}^{\beta\lambda} | 0_{in} \rangle - \eta^{\kappa\lambda} \langle 0_{out} | \partial_\alpha \partial_\beta | 0_{in} \rangle = 0$$

It only remains to prove that in the relevant limit the action of (3.10) reduces to that used in the calculation of (3.2). This is indeed the case as may be seen as follows

$$\begin{aligned} \frac{1}{2} R_{i\alpha} R_j^\alpha \hat{g}^i \hat{g}^j \Big|_{\phi=\eta} &= \frac{1}{2} \eta^{\alpha\beta} \partial_\gamma \delta_{\alpha,\mu\nu}^\gamma \partial_\tau \delta_{\beta,\mu\nu}^\tau \hat{g}^{\mu\nu} \hat{g}^{\kappa\lambda} \\ &= \frac{1}{2} \hat{g}^{\mu\kappa}{}_{,\mu} \eta_{\kappa\lambda} \hat{g}^{\nu\lambda}{}_{,\nu} \end{aligned}$$

$$\Delta_\psi(\hat{g}, \eta) = \det \left\{ R_{i\alpha}^i(\eta) R_{i\beta}^i(\eta) + \kappa R_{i\alpha}^i R_{i\beta,j} \hat{h}^j \right\}$$

$$\begin{aligned} R_{i\alpha}^i(\eta) R_{i\beta}^i(\eta) &= 2 \partial_\gamma \delta_{\beta,\mu\nu}^\gamma \partial_\tau Q_{\alpha}^{\tau,\mu\nu} = 2 \partial_\gamma \partial_\tau Q_{\alpha}^{\tau,\gamma}{}_{\beta} \\ &= \partial^2 \eta_{\alpha\beta} \end{aligned}$$

$$\begin{aligned} R_{i\alpha}^i R_{i\beta,j} \hat{h}^j &= R_{i\alpha}^i(\eta) R_{i\beta}^i(\hat{h}) \\ &= \partial_\gamma \delta_{\alpha,\mu\nu}^\gamma \left(-\partial_\beta \hat{h}^{\mu\nu} + \hat{h}^{\mu\sigma} \partial_\sigma \delta_\beta^\nu + \hat{h}^{\nu\sigma} \partial_\sigma \delta_\beta^\mu \right. \\ &\quad \left. - \hat{h}^{\mu\nu} \partial_\beta \right) . \end{aligned}$$

This is precisely the expression for Δ_ψ given by equation (2.20).

Therefore (3.11) is exactly the same equation as (3.2).

It is a very pleasant fact that employing the ϕ dependent choices for the γ -matrices (3.11) reduces to the compact form

$$D_\alpha(\phi) D_\beta(\phi) \langle 0_{out} | T \hat{g}^{\alpha\kappa} \hat{g}^{\beta\lambda} | 0_{in} \rangle - \tilde{\gamma}^{-1 \kappa\lambda} \langle 0_{out} | 0_{in} \rangle$$

Where $D_\alpha(\phi)$ is the covariant derivative defined as a function of the background field, viz:

$$D_\alpha(\phi) = \partial_\alpha - \hat{\Gamma}_\alpha^\mu(\phi) - \hat{\Gamma}_\alpha^\nu(\phi)$$

$$\hat{\Gamma}_{\alpha\nu}^\mu = \frac{1}{2} \phi^{\mu\tau} \left(\phi_{\alpha\tau,\nu} + \phi_{\nu\tau,\alpha} - \phi_{\alpha\nu,\tau} \right) .$$

For the choice of variables $g^{\mu\nu}$, $D_\alpha(\phi)$ ($= D_\alpha(g_c^{\mu\nu})$) is replaced by the usual covariant derivative $D_\alpha(g_c)$. The only remaining point to note is that in the above derivation of (3.11) the space time arguments have been suppressed. They are easily incorporated if required; they simply correspond to δ -functions.

The derivation of the covariant form of (3.9), as one might imagine is a little more complicated. We shall require some additional notation and definitions.

We define the covariant fictitious particle propagator $\hat{G}^{\alpha\beta}$ from the free part of the fictitious particle action of (3.10) as follows

$$R^i_\alpha R_{i\beta} \hat{G}^{\beta\gamma}(\phi) = \delta^\gamma_\alpha \quad (3.12)$$

The generalisations of the transverse and longitudinal projection operators $T^{\alpha\beta,\mu\nu}$, $L^{\alpha\beta,\mu\nu}$ are not strictly necessary for what follows but they are nice to have around. We define these as follows

$$T^m_n(\phi) = \delta^m_n - \hat{G}^{\alpha\beta} R^m_\alpha R_{n\beta}$$

$$L^m_n(\phi) = \delta^m_n - T^m_n(\phi).$$

They satisfy the relations

$$T^m_n T^n_\ell = T^m_\ell, \quad T^m_n L^n_\ell = 0, \quad T^m_n R^n_\alpha = 0$$

These properties all stem from equation (3.12) as may easily be proven. In the limit $\phi = \eta$ they correspond exactly

to the functions.

$$T^{\alpha\beta,\mu\nu} \text{ and } L^{\alpha\beta}{}_{,\mu\nu}.$$

This again is easily proven using the flat space forms of the γ -matrices already given and the expression for $\hat{g}^{\alpha\beta}$ as given in Appendix I.

One last piece of notation also proves useful viz:

$$\langle A(q) \rangle \equiv \frac{\langle 0_{out} | T(A(q)) | 0_{in} \rangle}{\langle 0_{out} | 0_{in} \rangle}$$

We now exploit the invariance of equation (3.10) under arbitrary infinitesimal variations in the matrix γ_{ij} to give

$$\frac{\delta}{\delta \gamma_{ij}} \langle 0_{out} | 0_{in} \rangle = 0 \quad (3.13)$$

The matrix γ_{ij} occurs both in the gravitational and the fictitious particle action. It is this that now gives rise to open fictitious particle lines. Strictly speaking we should write

$$\gamma_{ij} = \gamma_{ij'} = \gamma_i(x, x')_j$$

thereby indicating that the indices i and j are associated with different space-time points. This is easily incorporated into the ensuing derivation if required.

(3.13) now becomes

$$\begin{aligned} & R^{i\beta} R_{m\beta} \langle \hat{g}^j \hat{g}^m \rangle + R^{j\beta} R_{m\beta} \langle \hat{g}^i \hat{g}^m \rangle \\ &= \langle C^\alpha (R^i_\alpha(\phi + \hat{g}) R^j_\beta + R^j_\alpha(\phi + \hat{g}) R^i_\beta) \mathcal{R}^\alpha \rangle \end{aligned}$$

If we now multiply this equation by $\hat{G}_\delta^\gamma R_{j\gamma}$ and employ equation (3.12) we can write the above

$$\begin{aligned} & \hat{G}_\delta^\gamma R^{i\beta} R_{j\gamma} R_{m\beta} \langle \hat{g}^j \hat{g}^m \rangle + R_{m\delta} \langle \hat{g}^m \hat{g}^i \rangle \\ &= \langle C^\alpha (R_\alpha^i (\phi + \hat{g}) \tilde{\gamma}_{\delta\beta} + \hat{G}_\delta^\gamma R_\alpha^j (\phi + \hat{g}) R_{j\gamma} R_\beta^i) \Omega^\beta \rangle \end{aligned}$$

We next make use of equation (3.11) to rewrite this

$$\begin{aligned} R_\gamma^i \hat{G}_\delta^\gamma + R_{m\delta} \langle \hat{g}^m \hat{g}^i \rangle &= \\ (R_\alpha^i \tilde{\gamma}_{\delta\beta} + R_\beta^i \tilde{\gamma}_{\delta\alpha}) \langle C^\alpha \Omega^\beta \rangle + \\ + \kappa \langle C^\alpha (R_{\alpha,j}^i \tilde{\gamma}_{\delta\beta} + \hat{G}_\delta^\gamma R_{\alpha,j}^m R_{m\gamma} R_\beta^i) \hat{g}^j \Omega^\beta \rangle \end{aligned}$$

We next separate $\langle C^\alpha \Omega^\beta \rangle$ into its free part, $\hat{G}^{\alpha\beta}$, and self energy part to give

$$\begin{aligned} R_{m\delta} \langle \hat{g}^m \hat{g}^i \rangle - R_\gamma^i \hat{G}_\delta^\gamma &= \\ 2 R_\alpha^i \tilde{\gamma}_{\delta\beta} (\langle C^\alpha \Omega^\beta \rangle - \hat{G}^{\alpha\beta}) + \\ + \langle C^\alpha (R_{\alpha,j}^i \tilde{\gamma}_{\delta\beta} + \hat{G}_\delta^\gamma R_{\alpha,j}^m R_{m\gamma} R_\beta^i) \hat{g}^j \Omega^\beta \rangle \end{aligned} \tag{3.14}$$

One could stop here but the structure of (3.14) is not particularly revealing and we have yet to write it in

a form that corresponds to (3.9). To do this we need a fictitious particle Ward Identity. This is contained within equation (3.14). If we multiply (3.14) by $\hat{G}_\eta^\epsilon R_{i\epsilon}$ the left hand side is zero as a direct consequence of equation (3.11). (3.14) thus becomes

$$2 \tilde{\gamma}_{\eta\alpha} \tilde{\gamma}_{\delta\beta} (\langle C^\alpha \mathcal{L}^\beta \rangle - \hat{G}^{\alpha\beta}) + \\ + \kappa \langle C^\alpha (\hat{G}_\eta^\epsilon R_{\alpha,j}^\mu R_{m\epsilon} \tilde{\gamma}_{\delta\beta} + \hat{G}_\delta^\gamma R_{\alpha,j}^\mu R_{m\gamma} \tilde{\gamma}_{\eta\beta}) \hat{g}^j \mathcal{L}^\alpha \rangle = 0 \quad (3.15)$$

Substituting (3.15) into (3.14) we obtain

$$R_{m\delta} \langle \hat{g}^m \hat{g}^i \rangle - R_{\gamma}^i \hat{G}_\delta^\gamma = \\ \kappa \langle C^\alpha (R_{\alpha,j}^i \tilde{\gamma}_{\delta\beta} - R_{\gamma}^i \hat{G}_\eta^\epsilon R_{m\epsilon} R_{\alpha,j}^\mu \tilde{\gamma}_{\delta\beta}) \hat{h}^j \mathcal{L}^\alpha \rangle$$

Or

$$R_{m\delta} \langle \hat{g}^m \hat{g}^i \rangle - R_{\gamma}^i \hat{G}_\delta^\gamma = \\ \kappa (\delta_m^i - R_{\gamma}^i \hat{G}_\eta^\epsilon R_{m\epsilon}) \langle C^\alpha (R_{\alpha,j}^\mu \tilde{\gamma}_{\delta\beta}) \hat{h}^j \mathcal{L}^\alpha \rangle \quad (3.16)$$

The factor $(\delta_m^i - R_{\gamma}^i \hat{G}_\eta^\epsilon R_{m\epsilon})$ on the right hand side of (3.16) is precisely T_m^i . If one now takes the limit $\phi = \eta$ for equation (3.16) one obtains in momentum space

$$k_\mu \langle \hat{h}^{\mu\delta} \hat{h}^{\kappa\lambda} \rangle - \frac{2k_\mu}{k^2} Q^{\mu\delta, \kappa\lambda} =$$

$$\kappa T^{\kappa\lambda}_{\sigma\tau}(k) \langle C^\alpha (R^{\sigma\tau}_{\alpha, \rho\nu}) \hat{h}^{\rho\nu} \mathcal{L}^\alpha \rangle$$

This is exactly equation (3.9) when one has carried out the differentiation $\frac{\delta \mathcal{L}}{\delta J}$.

Clearly (3.16) is consistent with (3.11). One multiplies (3.16) by $R_{i\gamma}$ and uses the property $R_{i\gamma} T_m^i = 0$.

The only other point that I would draw attention to is that one also obtains as a by-product of this approach the Ward Identity relating the fictitious particle Vertex to the fictitious particle propagator. This is expressed by equation (3.15).

Despite its being a little cumbersome at times the manifestly covariant theory is an extremely powerful approach. Certainly for the purposes of generating Ward Identities: we have not yet employed its major asset; notably the ability to functionally differentiate with respect to the background field. Nor yet have we used the transformation properties of the covariant Greens functions that one simply inferred from the position of their indices. This last method is that used by DeWitt⁽¹⁶⁾ himself to obtain the Ward Identity relating the three-graviton vertex to the complete propagator. It has immediate and obvious extensions to identities relating the n -graviton vertex to the $(n-1)$ vertex. I shall comment on this briefly at the end of this section. It is first necessary to see how the covariant Greens functions can be defined by differentiations with respect to the background field. This also has relevance if one wishes to extend the identities (3.11) and (3.16) to higher order vertices.

Unfortunately the necessary modification of the

original action by the inclusion of the fictitious particle term complicates matters considerably. In order to focus attention on what one might hope to achieve, consider the situation as given by DeWitt where the general co-ordinate invariance group is absent. The free 'graviton' propagator is then defined by

$$S_{,ij} G^{jk} = -\delta_i^k$$

Leaving aside the gauge terms in (3.10) for the moment we apply the operator $G^{ki} \frac{\delta}{i\delta\phi^i}$ to the vacuum-to-vacuum amplitude to give

$$G^{ki} \frac{\delta}{i\delta\phi^i} \langle 0_{out} | 0_{in} \rangle = \int (d\hat{g}) G^{ki} (S_{,i}(\phi + \hat{g}) - S_{,i}\phi - S_{,ji} \hat{g}^j) \exp i S(\hat{g}) . \quad (3.17)$$

If we employ the identity

$$\int (d\hat{g}) G^{ki} \frac{\delta}{i\delta\hat{g}^i} \exp i S(\hat{g}) = 0$$

we can rewrite (3.17)

$$G^{ki} \frac{\delta}{i\delta\phi^i} \langle 0_{out} | 0_{in} \rangle = \int (d\hat{g}) G^{ki} (S_{,ij} \hat{g}^j) \exp i S(\hat{g}) = \langle 0_{out} | \hat{g}^k | 0_{in} \rangle . \quad (3.18)$$

If we now carry through exactly this procedure when the invariance group is present, i.e. we use the full action given by (3.10) and the free graviton propagator is now defined by $(S_{,ij} + R_{i\alpha} R_j^\alpha) G^{jk} = -\delta_i^k$ we obtain instead

of (3.18)

$$\begin{aligned}
 G^{ki} \frac{\delta}{i\delta\phi^i} \langle 0_{out} | 0_{in} \rangle &= \langle \hat{g}^k \rangle + \\
 &+ G^{ki} (R_{m\alpha} R_{n,i}^{\alpha} \langle \hat{g}^m \hat{g}^n \rangle) \\
 &- G^{ki} \langle C^{\alpha} (R_{\alpha(\phi+g)}^m R_{m\beta,i}) \Omega^{\beta} \rangle
 \end{aligned}
 \tag{3.19}$$

In the derivation of (3.19) we have not differentiated either of the γ -matrices since these terms must cancel as a consequence of the γ -invariance of the theory. Again in equation (3.19) one sees the emergence of open fictitious particle lines. In general these terms must be included though they can be written in many different forms. By employing the identity (3.16) one such is given by

$$\begin{aligned}
 G^{ki} \frac{\delta}{i\delta\phi^i} \langle 0_{out} | 0_{in} \rangle &= \langle \hat{g}^k \rangle + \\
 &- G^{ki} \left\{ L_m^{\alpha} R_{n\alpha,i} R_{\alpha',j}^m \langle C^{\alpha'} \hat{g}^j \Omega^{\alpha} \rangle \right\}
 \end{aligned}
 \tag{3.20}$$

Higher order Greens functions are obtained by successive applications of the operator

$$G^{ki} \frac{\delta}{i\delta\phi^i}$$

I shall not discuss this further here save only to mention DeWitts statement⁽¹⁰⁾ that on mass shell transverse g -lines may be inserted by the replacement of G^{ij} in equation (3.19) by the propagator, G^{Tij} , of physical quanta only.

Presumably in terms of the functions T and L this corresponds

to the replacement of G^{ij} by something like:

$$G^{ij} \sim T^i_k G^{kl} T^j_l$$

One then recovers the simple form of (3.19) notably (3.18) with the g-line now transverse and on mass shell. This clearly ought to be the case but the necessary formalism is formidable and I have not worked out the details. For the purposes of obtaining off mass shell amplitudes equation (3.20) and its extensions are sufficient albeit complex. It is worth mentioning also that formal derivations of on mass shell amplitudes presupposes assumptions to do with the gravitational infrared divergencies. I have not made any attempt to deal with these in this thesis.

To summarise thus far: the identities (3.11) and (3.16) are readily extended by differentiations with respect to the background field. Apart from alterations to the factors standing outside the quantum amplitudes this essentially corresponds to the inclusion of additional g-lines with some extra fictitious particle structure. On mass shell identities should be properly obtained only by taking the limit $k^2 \rightarrow 0$ at the end of any such calculation.

I shall now briefly outline how one obtains Ward Identities from considerations of the transformation properties of the covariant Greens functions. The possibility of doing this is certainly one of the principal justifications for working with the manifestly covariant theory. In all that has been said so far we have not exploited the 'covariance' of the theory as such. We have

simply made use of the arbitrariness of choosing the gauge condition as incorporated via the γ -matrices.

The essential ingredients of this last approach are the transformation properties of the bare vertex functions of the theory. These may be derived by successively differentiating the contracted Bianchi identities,

$$S_{,i} R_{\alpha}^i \equiv 0.$$

We then obtain

$$S_{,ij} R_{\alpha}^i + S_{,i} R_{\alpha,j}^i = 0$$

$$S_{,ijk} R_{\alpha}^i + S_{,ik} R_{\alpha,j}^i + S_{,ij} R_{\alpha,k}^i = 0 \quad (3.21)$$

If we now define the complete Greens function

$$S^{ij} = \langle \hat{g}^i \hat{g}^j \rangle$$

and its inverse

$$S^{-1}_{jk} S^{ki} = \delta_k^i \quad (3.22)$$

then S^{-1}_{ij} must transform under the action of the gauge group in a contragradient fashion, i.e. as does $S_{,ij}$ in equation (3.21). Thus:-

$$\frac{\delta}{\delta \phi^{\alpha}} S^{-1}_{ij} = S^{-1}_{ij,k} R_{\alpha}^k \quad \text{satisfies}$$

$$S^{-1}_{ij,k} R_{\alpha}^k + S^{-1}_{ik} R_{\alpha,j}^k + S^{-1}_{kj} R_{\alpha,i}^k = 0$$

We can use (3.22) to write

$$S^{-1}_{ij,k} = - S^{-1}_{im} S^{-1}_{jn} \frac{\delta}{\delta \phi^k} \langle \hat{g}^m \hat{g}^n \rangle$$

In the appropriate limit one then has a relation between the 3-graviton vertex function and the graviton propagator. There will in general also be some open fictitious particle lines to take account of. Clearly this method is readily extended to higher orders.

On this formal note I shall conclude this section on Ward Identities. I should, perhaps, stress that in subsequent sections it is only equation (3.2) or equivalently equation (3.11) that I shall attempt to verify to second order in the gravitational coupling constant κ .

4. REGULARISATION OF DIVERGENT INTEGRALS

Equation (3.2) contains terms that are logarithmically, quadratically and quartically divergent. It is essential to the verification of (3.2), and other Ward Identities, to have a gauge-invariant regularisation procedure. Indeed (3.2) is meaningless without such a procedure. Thanks largely to the work of t'Hooft and Veltman⁽¹⁸⁾ there now exists a powerful method for achieving just this. I refer to the process now known as dimensional regularisation. To quote t'Hooft, the applicability of the method rests upon the statement "Ward Identities are independent of dimension". Presumably what is meant by this is the following. If one regards the action (2.14) as being n -dimensional, that is to say $g^{\mu\nu}$ is the metric tensor on an n -dimensional manifold, then it is invariant under the n -dimensional gauge group of general co-ordinate transformations on that manifold. This is in fact the case. The only time that one becomes aware of the dimensionality of space-time is in evaluating the contraction $g^\mu_\mu = n$ and in deriving Ward Identities for the action (2.14) nowhere do such contractions occur. Thus the equation,

$$k_\mu k_\nu \left\{ \frac{\langle 0_{out} | T \hat{h}^{\mu\kappa} \hat{h}^{\nu\lambda} | 0_{in} \rangle}{\langle 0_{out} | 0_{in} \rangle} - \frac{1}{k^2} Q^{\mu\kappa, \nu\lambda} \right\} = 0 \quad (4.1)$$

holds equally as well in n -dimensions as it does (if it does) in 4-dimensions.

The basic dimensional regularisation procedure may be summarised as follows. One evaluates equation (4.1) in some region of the complex n -plane, where the integrals are finite, and defines its value in the physical region $n = 4$ by analytic continuation. The customary divergencies are then revealed as the necessary subtractions of the pole terms in the Laurent expansion of (4.1) about $n = 4$. Now in the evaluation of (4.1) any contractions δ_{μ}^{μ} that occur must be ascribed the value $\delta_{\mu}^{\mu} = n$; the loop integrations become n -dimensional loop integrations and only at the final stage of the calculation when one evaluates the residues of the pole terms can one take $n = 4$. As will be seen this is not immediately ideal for the gravitational situation.

This section will be devoted to obtaining a modification to the above procedure to be used in the calculations in sections 5 and 7. First we shall see both why this is necessary and how it arises.

It may be seen by referring to the Feynman rules in Appendix I that a typical integral occurring in the second order calculation of (4.1) would be

$$\int (d^4 p) \frac{f(4) p_{\mu} p_{\nu} p_{\lambda}}{p^2 (p+k)^2} \quad (4.2)$$

Where, k , is the external momentum and $f(4)$ the numerical factor arising from contractions $\delta_{\mu}^{\mu} = 4$. If one straightforwardly adopts the dimensional regularisation procedure outlined above and uses the Feynman parameter method and

Integral formulae given in Appendix III, ignoring for the moment that the theory is massless, one would eventually be left taking the residue at $n = 4$ of the term:

$$\int_0^1 dx \frac{i \pi^{\eta/2} \Gamma(2-\eta/2) f(n)}{(k^2 x (1-x))^{2-\eta/2}} x$$

$$\times \left\{ -x^3 k_\mu k_\nu k_\lambda + \frac{x^3 k^2}{(2-n)} (\eta_{\mu\nu} k_\lambda + \eta_{\mu\lambda} k_\nu + \eta_{\nu\lambda} k_\mu) \right\}$$

This has the two-fold disadvantage both of being infrared divergent and involving a knowledge of $f(n)$; the n -dependence arising from δ_μ^μ contractions of the algebraic terms. In the case of gravitation $f(n)$ is highly non-trivial. In particular $Q^{\alpha\beta, \mu\nu}$ is no longer its own inverse. This has the consequence that the propagator acquires an additional term $\frac{(n-4)}{(n-2)} \eta^{\alpha\beta} \eta^{\mu\nu}$. The algebraic calculations that rest heavily on this property of the Q 's for their simplicity are increased enormously in complexity. The characteristic structure for $f(n)$ is that of a quartic polynomial in n with a few factors of $(n-2)$ in the denominator for good measure.

In an earlier paper⁽⁴⁾ dealing with the Yang Mills field t'Hooft exploits the freedom to extend the dimension to introduce regulator masses by adding a fixed fifth component to the momenta occurring inside closed loops. We shall consider this approach in some detail since we shall use it to provide a useful illustration of the modified dimensional regularisation techniques to be introduced later.

The Yang-Mills lagrangian is given by:-

$$L = -\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a$$

where $G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g \epsilon_{abc} W_\mu^b W_\nu^c$.

t'Hooft then regards the fields W_μ^a as fifteen component quantities. The W_5^a component he treats as an extra particle, its vertices and propagator being determined by the 4+1 decomposition of the now 5-dimensional lagrangian. For example:-

$$\begin{aligned} k_\mu W_\nu^a k_\mu W_\nu^a &= \bar{k}_\mu W_\nu^a \bar{k}_\mu W_\nu^a + M^2 W_\nu^a W_\nu^a \\ &+ \bar{k}_\mu W_5^a \bar{k}_\mu W_5^a + M^2 W_5^a W_5^a, \end{aligned}$$

where a bar under something indicates that it is explicitly in 4-dimensions. M denotes the fixed fifth component of the momentum k_μ .

Overall this has the effect of reproducing all the diagrams of the original theory, now with massive propagators, plus a set of regulator diagrams involving the propagation of the fifth component of the W -field (the fifth space-time component that is). From these regulator diagrams he extracts a minimal set and proceeds to a calculation. A 'minimal' set in the sense that he leaves out those regulator diagrams that are themselves gauge invariant. The net result for the second order Yang-Mills calculation analogous to equation (4.1) is to evaluate the set of diagrams:

$$\frac{1}{2} \text{---} \bigcirc \text{---} + \frac{1}{2} \text{---} \bigcirc \text{---} - \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} \quad (4.3)$$

Where — represents an ordinary 4-dimension Y.M. particle.

----- represents a fictitious particle.

+++++ represents the fifth component of the W-field.

There are no tadpole diagrams as these are algebraically zero, from terms such as $\epsilon_{a_1 a_2 a_3} \delta^{a_2 a_3}$.

In the set of diagrams (4.3) the 'minimal set' corresponds to one diagram only. It should also be noted that the masses, M , involved are true regulator masses - they are used to define the otherwise divergent integrals. That is to say there is no analytic continuation involved in t'Hoofts calculation of (4.3).

If one were to follow this procedure with gravitation there would be a problem in that the minimal set of regulator diagrams were it easily extracted would be very large.

So much then for what has been done. I shall now proceed to derive an adaptation of these methods suitable for the gravitational calculation of (4.1). In order to make things clearer I shall continue to work with the Yang-Mills field as an illustration.

The first observation that we might make is that, with dimensional regularisation procedures at our disposal, there is no need to carry out the separation of the Lagrangian into original and regulator diagrams at the

initial stage of the calculation: it can be done at the end. Of course there is no need to incorporate regulator masses at all but we shall continue to do so as this will prove useful. What I propose then is to perform the calculation of the diagrams in (4.3) with the exception of the last one where the following rules are to be obeyed.

Momenta occurring inside closed loops are to have $n+m$ components. The first n components being the straightforward extension from four to n -dimensions. The $(n+1)^{\text{th}}$ component is fixed equal to M ; the remaining $(m-1)$ components are fixed equal to zero.

Thus:

$$k^{(n+m)} = (k^{(n)}, M, 0^{(m-1)})$$

External momenta are to be in 4-dimensions. Thus:

$$\delta_{\mu}^{\mu} = (n+m)$$

$$k^2 \text{ (inside a closed loop)} = k^{(n)2} + M^2$$

$$k_{\alpha} \text{ (where } \alpha \text{ is an external index)} = k_{\alpha}^{(n)} = (k_{\alpha}^{(4)}, 0^{n-4})$$

In the above I have simply married together the two methods outlined at the beginning of this section. I have made an extension to n -dimensions, necessary for dimensional regularisation procedures, and have further extended the dimension to incorporate mass terms. With regard to these mass terms we also have the following rules viz:-

Momentum conservation requires that the masses must be the same for each propagator inside a closed loop.

Apart from this consideration different masses M_i are to

be incorporated as follows:-

If after the algebraic calculations one has the term

$$\frac{g(M^2)}{(p^2 + M^2)} \quad \text{it is to be replaced by} \quad \sum_i \frac{e_i g(M_i^2)}{(p^2 + M_i^2)},$$

where the signs, e_i , and masses, M_i , are chosen such that:-

$$e_0 = 1, \quad M_0 = 0$$

$$\sum_i e_i = 0, \quad \sum_i e_i M_i^2 = 0$$

$$\sum_{i \neq 0} e_i \log M_i^2 = A, \quad \sum_{i \neq 0} e_i M_i^2 \log M_i^2 = B. \quad (4.4)$$

It should be stressed that the masses, so incorporated, in this context are not used to regulate the integrals; they are, for the moment, simply additional parameters that we are free to incorporate if we please.

Let us now proceed with the calculation (4.3). Making use of the Feynman rules given by t'Hooft⁽⁴⁾ we obtain for the separate diagrams of (4.3) the following expressions:-

$$\int d^n p d^n q \delta(k+p+q) g^2 \delta_{aa'} \sum_i e_i \left\{ \frac{1}{(p^2 + M_i^2)} \cdot \frac{1}{(q^2 + M_i^2)} \right\}^x$$

$$\times \left(\eta_{\alpha\alpha'} (4k^2 + 2(p^2 + \Pi_i^2)) + (n+m) (4p_\alpha p_{\alpha'} + 2k_\alpha p_{\alpha'}) - 4k_\alpha k_{\alpha'} - 6p_\alpha p_{\alpha'} - 2k_\alpha p_{\alpha'} \right) \} \quad (4.5)$$

$$\int d^n p d^n q \delta(k+p+q) g^2 \delta_{\alpha\alpha'} \sum_i e_i \left\{ \frac{1}{(p^2 + \Pi_i^2)} \cdot \frac{1}{(q^2 + \Pi_i^2)} \times \right. \\ \left. \times (-2k_\alpha p_{\alpha'} - 2p_\alpha p_{\alpha'}) \right\} . \quad (4.6)$$

(the large arrows indicate the orientation of the fictitious particle vertices)

$$\int d^n p g^2 \delta_{\alpha\alpha'} \sum_i e_i \left\{ \frac{1}{(p^2 + \Pi_i^2)} \times \right. \\ \left. \times (-2(n+m) \eta_{\alpha\alpha'} - 2 \eta_{\alpha\alpha'}) \right\} . \quad (4.7)$$

We next perform the n -dimensional loop integration using the formulae given in Appendix III. I shall not record the details of this for the individual diagrams (4.5),

(4.6) and (4.7), but instead exhibit the important general feature that results by returning for a moment to the integral (4.2).

Applying the revised calculational rules given above, those terms that led to the integral (4.2) now give

$$\sum_i e_i \int d^n p \frac{f_{(n+m)} p_\mu p_\nu p_\lambda}{(p^2 + M_i^2)((p+k)^2 + M_i^2)} \quad (4.8)$$

Upon integration this becomes

$$\sum_i e_i \int_0^1 dx \frac{i \pi^{n/2} \Gamma(2-n/2) f_{(n+m)} G(n, k, x, M_i)}{(M_i^2 + k^2 x(1-x))^{2-n/2}} \quad (4.9)$$

where

$$G(n, k, x, M_i) \equiv \left\{ -x^3 k_\mu k_\nu k_\lambda + \frac{(M_i^2 + k^2 x(1-x))x}{(n-2)} (\gamma_{\mu\nu} k_\lambda + \dots) \right\}$$

The ultraviolet divergence (for $n=4$) of (4.8) is revealed as a simple pole in (4.9) arising from the $\Gamma(2-n/2)$ factor.

We next expand (4.9) in a Laurent series about $(n=4)$ as follows -

$$\begin{aligned} (4.9) &= \sum_i e_i \int_0^1 dx \ i \pi^2 \left(\frac{1}{n-4} \right) \left(f_{(4+m)} + (n-4) f'_{(4+m)} + O(n-4)^2 \right) \\ &\quad \times \exp \left\{ \frac{1}{2} (4-n) \log(M_i^2 + k^2 x(1-x)) \right\} \times \\ &\quad \times \left(G(4, k, x, M_i) + (n-4) G'(4, k, x, M_i) + O(n-4)^2 \right) \end{aligned}$$

The signs e_i and masses M_i are chosen in such a way that the only term in this expansion that contributes anything at all is the term

$$- \sum_i e_i \int_0^1 dx \, i\pi^2 f(4+m) \cdot \frac{1}{2} \log(M_i^2 + k^2 x(1-x)) G(4, k, x, M_i).$$

The important feature being that the function $f(4+m)$ survives but none of its derivatives appear. This is the case for all loop integrations that we shall have to deal with. It is a consequence of there being only simple poles at $(n=4)$ in the formulae given in Appendix III. It is this property that enables us to perform all the algebraic calculations in 4-dimensions where we come to deal with gravitation. The final result is obtained by taking the limit $M_{i \neq 0} \rightarrow \infty$.

Let us now return to the illustrative Yang-Mills calculation where we have performed the necessary algebra in $(n+m)$ dimensions. In particular it is the m dependence of $f(4+m)$ that I wish to discuss. The only terms that contain such a factor come from the results (4.5) and (4.7). If we separate these out from the remainder for particular attention and perform the necessary integrations etc. they may, collectively, be written:-

$$\sum_i e_i \int_0^1 dx \, (4+m) \left\{ \left[(4x^2 - 2x) k_\alpha k_{\alpha'} - 2\eta_{\alpha\alpha'} (M_i^2 + k^2 x(1-x)) \right] \right. \\ \left. \times \log(M_i^2 + k^2 x(1-x)) + 2\eta_{\alpha\alpha'} M_i^2 \log M_i^2 \right\}. \quad (4.10)$$

If we now invoke the statement, "Ward Identities are independent of dimension" we should discover that (4.10) is separately transverse. We shall prove this in a moment but let us first discuss some of the implications. We are clearly free to choose the value of m from the outset. The choice, $m=0$, corresponds in t'Hooft's approach to discarding all those regulator diagrams that were themselves gauge-invariant. This is to be understood in the sense that the sum, (4.5) + (4.6) + (4.7), with $m=0$, $n=4$, corresponds exactly to t'Hooft's evaluation of the sum (4.3). The choice $m = -4$ implies that, for the purposes of proving gauge invariance, one can ignore all those terms involving contractions $\delta_{\mu}^{\mu}=4$. Equivalently one can say that all terms involving such contractions conspire to add together to give a transverse result.

We shall now exhibit the transversality of (4.10) in the limit $M_{i \neq 0} \rightarrow \infty$. The only terms that remain are as follows:-

$$\begin{aligned}
 (4+m) \int_0^1 dx \left\{ \left[(4x^2 - 2x) k_{\alpha} k_{\alpha'} - 2\eta_{\alpha\alpha'} k^2 x(1-x) \right] \left[A + \log k^2 \right] \right. \\
 + \left[(4x^2 - 2x) k_{\alpha} k_{\alpha'} - 2\eta_{\alpha\alpha'} k^2 x(1-x) \right] \left[\log x(1-x) \right] \\
 \left. + \sum_{i \neq 0} e_i \left[-2\eta_{\alpha\alpha'} k^2 x(1-x) \right] \right\}
 \end{aligned}$$

Performing the x integrations we are left with:-

$$(4+m) \left[\frac{1}{3} (A + \log k^2) - \frac{8}{9} \right] (k_{\alpha} k_{\alpha'} - k^2 \eta_{\alpha\alpha'}) , \quad (4.11)$$

which is the required result.

Evaluation of those terms remaining in the sum (4.5) + (4.6) + (4.7) yields:-

$$\left[\frac{14}{3} (A + \log k^2) - \frac{88}{9} \right] (k^2 \eta_{\alpha\alpha'} - k_\alpha k_{\alpha'}) \quad (4.12)$$

It can now be seen that the sum (4.12) + (4.11)|_{m=0} reproduces t'Hooft's result viz:-

$$\left[\frac{10}{3} \log k^2 + \frac{10}{3} A - \frac{56}{9} \right] (k^2 \eta_{\alpha\alpha'} - k_\alpha k_{\alpha'})$$

(For some reason he chooses to define $\Sigma \epsilon_i \log M_i^2 = -A$).

We have also omitted an irrelevant overall factor. The only point that remains, is to decide what corresponds to the diagram in t'Hooft's approach given by:



(4.13)

This is an easy matter to deal with. The only occasion that we have had to employ the rule, p^2 (inside a closed loop) = $p^2 + M^2$, to a term in the numerator of the sum (4.5) + -- + (4.7), arises in the first term in (4.5).

This addition corresponds exactly to the inclusion of diagram (4.13) in t'Hooft's method. One can see this by evaluating (4.13) with the requisite Feynman rules as given by t'Hooft. I should perhaps point out that, from considerations of performing the equivalent gravitational calculation, the above is deceptively simply. It has been included and been performed with the unnecessary rule,

$\delta_{\mu}^{\mu} = n+m$, simply to serve as an illustration of the method that I propose to use for gravitation.

I shall now quickly review those rules that are to be employed in the calculations in sections 5 and 6.

- (1) All algebraic calculations performed before integrating over loop momenta are done in 4-dimensions, i.e. $\delta_{\mu}^{\mu} = 4$.
- (2) k^2 (inside a closed loop) is to be replaced by $k^2 + M^2$.
- (3) External algebraic indices are to be in 4-dimensions.
- (4) If after the algebraic calculations one has the term, $\frac{g(M^2)}{(p^2 + M^2)}$, it is to be replaced by $\sum_i e_i \frac{g(M_i^2)}{(p^2 + M_i^2)}$ where the signs e_i and masses M_i are chosen such that:

$$e_0 = 1, \quad M_0 = 0$$

$$\sum_i e_i = 0, \quad \sum_i e_i M_i^2 = 0, \quad \sum_i e_i M_i^4 = 0$$

$$\sum_{i \neq 0} e_i \log M_i^2 = A, \quad \sum_{i \neq 0} e_i M_i^2 \log M_i^2 = B, \quad \sum_{i \neq 0} e_i M_i^4 \log M_i^2 = C.$$

- (5) After performing the loop integrations and taking residues at $n=4$, one takes the limit $M_{i \neq 0} \rightarrow \infty$.

The masses M_i so introduced are not regulator masses in the conventional sense, but rather serve only to parametrise the ultraviolet divergencies.

I have not said anything about infrared divergencies. By incorporating masses and writing the integrals in the form (4.8) they are, for calculational purposes discreetly

avoided. Were one to attempt to verify Ward Identities with lines on mass shell such as (3.12) they would have to be taken account of. I do not do so here.

I do not choose, as it seems I might, to ignore all those terms that involve contractions $\delta_{\mu}^{\mu} = 4$. In other words for the function $f(4+m)$ in expression (4.9) I make the choice $m = 0$, and thereby calculate the actual second order contribution to the graviton propagator as opposed to some 'regulator diagram reduced' amplitude.

Finally it is worth noting that with the rules (4.14) one can effectively cancel terms of the form $\frac{p^2 + M_i^2}{p^2 + M_i^2}$, from the integrands, i.e.

$$\begin{aligned} & \sum_i e_i \int d^n p d^n q \frac{(p^2 + M_i^2) h(p, q, M_i) \delta(p+q+k)}{(p^2 + M_i^2) (q^2 + M_i^2)} \\ &= \sum_i e_i \int d^n p d^n q \frac{h(p, q, M_i) \delta(p+q+k)}{(q^2 + M_i^2)} \end{aligned} \quad (4.15)$$

At least this is the case for all functions, h , that arise in the ensuing calculations. I shall not bother to prove this. It is easily, if a little untidily, shown to be the case for individual terms.

There is one last point concerning equation (4.15). It might be argued that this cancellation is obvious and that there is nothing to prove. Certainly it is highly desirable for the consistency of the regularisation procedure that such a cancellation takes place. Were the

four dimensional integrals corresponding to those in (4.15) finite the cancellation clearly is obvious. However they are not. For $h=1$ they are pure quadratic divergencies. For some suitable value of n they are finite and well defined and no doubt one can establish their equivalence irrespective of how the masses are introduced. That this equivalence is preserved in the values that the separate integrals take in the physical subspace ($n=4$) does depend upon how this inclusion is effected and equation (4.15) should correctly be taken as an assertion of that fact. That is to say the rules given for the incorporation of additional masses are consistent with what one would expect; equation (4.15) holds.

As to whether or not this is obvious is to some extent a matter of taste. In conclusion I would say that equation (4.15) with no masses introduced is not at all trivial to establish, see for example reference (24). Nor is it the case that the further cancellation

$$\sum e_i \int d^n p d^n q \frac{(p^2 + m_i^2)}{(p^2 + m_i^2)} \delta(p+q+k)$$

$$= \sum e_i \int d^n p d^n q \delta(p+q+k)$$

is obvious or even meaningful. The right hand side is undefined. This last point is dealt with in some detail in section 6 where such an integral occurs.

5. SECOND ORDER GRAVITON SELF-ENERGY CALCULATION

With the Feynman rules derived from the Goldberg form of the Einstein Action, evaluated in the modified Dedonder gauge, we now attempt to verify equation (3.4). The divergencies will be handled by the dimensional regularisation procedure described in section (4). The only problem still confronting us is the algebraic complexity of the vertices as given in Appendix I. One can go a considerable way to alleviating this but the calculation is still rather lengthy.

The relevant Feynman diagrams for the second order contribution to $\langle 0 \text{ out} / T \hat{h}^i \hat{h}^j / 0 \text{ in} \rangle$ are as follows:-

$$\begin{aligned}
 & \frac{1}{2} \text{---} \text{---} \text{---} \text{---} \text{---} + \frac{1}{2} \text{---} \text{---} \text{---} \text{---} \text{---} - \text{---} \text{---} \text{---} \text{---} \text{---} \\
 & + \frac{1}{2} \text{---} \text{---} \text{---} \text{---} \text{---} - \text{---} \text{---} \text{---} \text{---} \text{---}
 \end{aligned}
 \tag{5.1}$$

We shall not deal with the tadpole terms (4) and (5) in this section but shall reserve discussion of them until section (7). For the moment we shall let it suffice to say that they are not a priori well defined.

Concerning the algebraic manipulations. As given in appendix I, ${}^4r(k_1 k_2 k_3 k_4)$ contains 2,344 terms. Diagram (1) in expression (5.1) contains 11,664 terms. Clearly something has to be done. Fortunately these numbers can be significantly reduced by employing the following procedure.

The symmetrisations, $S_{\alpha_i \beta_i}$, are either absorbed into the symmetric propagators, Q , in any given calculation or they stand outside the final answer. The $P_n!$ term, standing outside the vertices, is removed on each combinatorically distinct algebraic term (call it A) by first finding the subgroup, G , of permutations that leave A invariant and then writing $P_n!$ as the sum over distinct left cosets of G . To clarify matters we shall give one example of this last procedure.

Consider one of the terms in the expression for ${}^4r$ given in Appendix I, namely:-

$$S_{\alpha_1 \beta_1} \dots S_{\alpha_4 \beta_4} P_4! (k_3 \cdot k_4 \eta_{\alpha_4 \alpha_1} \eta_{\alpha_2 \beta_1} \eta_{\beta_2 \beta_4} \eta_{\alpha_3 \beta_3}) \quad (5.2)$$

The elements of the group of permutations on the set of suffices $\{1,2,3,4\}$ can be written as products of 'transpositions' or 'two-cycles'. The transposition (i,j) is defined:-

$$(i,j) A_{ijk\ell} = A_{jik\ell} = (j,i) A_{ijk\ell}$$

Products of transpositions (where the order is, in general, important) are similarly defined viz:-

$$(k,j)(i,j) A_{ijk\ell} = (k,j) A_{jik\ell} = A_{kij\ell}$$

Returning to our example:-

$S_{\alpha_1\beta_1} \dots S_{\alpha_4\beta_4} \eta_{\alpha_4\alpha_1} \eta_{\alpha_2\beta_1} \eta_{\beta_2\beta_4} \eta_{\alpha_3\beta_3}$, is invariant under the subgroup, $G \equiv \{(1), (12), (14), (24), (12)(14), (12)(24)\}$

We then write $P_{4,1} = G + (13)G + (23)G + (34)G$

If we now use momentum conservation to compute the action of G on k_3, k_4 we have:-

$$\begin{aligned} G(k_3, k_4) &= (k_3, k_4 + k_3, k_4 + k_3, k_1 + k_3, k_2 + k_3, k_2 + k_3, k_1) \\ &= 2 k_3 \cdot (k_1 + k_2 + k_4) = -2k_3^2 \end{aligned}$$

Thus (5.2) becomes:

$$S_{\alpha_1\beta_1} \dots S_{\alpha_4\beta_4} \left((1) + (12) + (13) + (14) \right) \left(-2k_3^2 \eta_{\alpha_4\alpha_1} \eta_{\alpha_2\beta_1} \eta_{\beta_2\beta_4} \eta_{\alpha_3\beta_3} \right)$$

The factors $S_{\alpha_i\beta_i}$ are then to be dealt with as mentioned earlier.

In this way ${}^4r$ is effectively reduced to about 28 terms. Probably even more important is that this is done in such a way that it is not easy to make mistakes. Which is just as well since the eventual verification of (3.4) is extremely sensitive to the slightest error.

Similarly the diagram, , eventually corresponds to approximately 20 separate calculations. These are however still a little lengthy - or might appear so. In actual fact if one judiciously exploits the properties of the Q 's they are mostly surprisingly simple. To

give one last example; a typical term in the calculation of the above diagram is the following:-

$$Q^{\alpha\beta, \alpha_1\beta_1} Q^{\alpha'_1\beta'_1, \alpha'_1\beta'_1} k_{2\alpha_1} k_{3\beta_1} k_{3\alpha'_2} k_{1\beta'_2} Q_{\alpha_2\beta_2, \alpha_3\beta_3} Q_{\alpha'_3\beta'_3, \alpha'_1\beta'_1} \\ \times Q^{\alpha_2\beta_2, \alpha'_2\beta'_2} Q^{\alpha_3\beta_3, \alpha'_3\beta'_3}.$$

One can immediately reduce this to:

$$k_{2\alpha_1} k_{3\beta_1} Q^{\alpha_1\beta_1, \alpha\beta} k_{3\alpha'_1} k_{1\beta'_1} Q^{\alpha'_1\beta'_1, \alpha'_1\beta'_1}.$$

It is crucial in this radical simplification that

$$Q_{\alpha_1\beta_1, \alpha_2\beta_2} Q^{\alpha_2\beta_2, \alpha'_1\beta'_1} = \delta_{\alpha_1\beta_1}^{\alpha'_1\beta'_1}$$

This very convenient property is destroyed if one performs the algebra instead with the rule $\delta_{\mu}^{\mu} = n$ and is, as has already been mentioned, one reason for seeking an adaptation of this method.

We are now in a position to calculate the set of diagrams (1) + (2) + (3) of expression (5.1). After performing the requisite algebra, but before doing the loop integrals there are 30 combinatorically distinct algebraic terms and we shall not record the details of these. Instead, to lend a little more credence to the final result, we shall record the situation once the integrals have been performed and the residues taken. One is then left with simply the Feynman parameter integration, $\int_0^1 dx$, to perform. For the individual diagrams these are as follows:

(1)

$$\frac{1}{2} \xrightarrow{\alpha\beta} \xrightarrow{k} \text{ (loop) } \xrightarrow{k} \xrightarrow{\alpha'\beta'} =$$

$$\begin{aligned}
& \left[\frac{1}{4} (2 + 38x - 51x^2 - 6x^3 + 5x^4) k^4 \eta_{\alpha\beta} \eta_{\alpha'\beta'} \right. \\
& \quad + \frac{1}{2} (1 - 4x + 19x^2 - 14x^3 + 5x^4) k^4 \eta_{\alpha\alpha'} \eta_{\beta\beta'} \\
& \quad + \frac{1}{2} (-44x + 19x^2 - 14x^3 + 5x^4) k^2 k_\alpha k_\beta \eta_{\alpha'\beta'} + \frac{1}{2} (4x - 4x^2 - 56x^3 + 40x^4) k^2 k_\alpha k_{\alpha'} \eta_{\beta\beta'} \\
& \quad + (14x - 16x^2 + 10x^4) k_\alpha k_\beta k_{\alpha'} k_{\beta'} + \frac{1}{2} (16 + 3x - 5x^2) M_i^2 k^2 \eta_{\alpha\beta} \eta_{\alpha'\beta'} \\
& \quad + (12 + 4x - 20x^2) M_i^2 k_\alpha k_{\alpha'} \eta_{\beta\beta'} + \frac{5}{4} M_i^4 \eta_{\alpha\beta} \eta_{\alpha'\beta'} \\
& \quad \left. + \frac{5}{2} M_i^4 \eta_{\alpha\alpha'} \eta_{\beta\beta'} \right] \times \log(M_i^2 + k^2 x(1-x)) \\
& - \left[M_i^2 k^2 \eta_{\alpha\beta} \eta_{\alpha'\beta'} + 7 M_i^2 k^2 \eta_{\alpha\alpha'} \eta_{\beta\beta'} - 4 M_i^2 k_\alpha k_\beta \eta_{\alpha'\beta'} \right. \\
& \quad \left. - 8 M_i^2 k_\alpha k_{\alpha'} \eta_{\beta\beta'} \right] \times \log M_i^2 .
\end{aligned}$$

(2)

$$\frac{1}{2} \xrightarrow{\alpha\beta} \xrightarrow{k} \text{ (loop) } \xrightarrow{k} \xrightarrow{\alpha'\beta'} =$$

$$\begin{aligned}
& \left[-\frac{11}{2} M_i^2 k^2 \eta_{\alpha\beta} \eta_{\alpha'\beta'} + 9 M_i^2 k^2 \eta_{\alpha\alpha'} \eta_{\beta\beta'} + 10 M_i^2 k_\alpha k_\beta \eta_{\alpha'\beta'} \right. \\
& \quad \left. - 12 M_i^2 k_\alpha k_{\alpha'} \eta_{\beta\beta'} - \frac{5}{2} M_i^4 \eta_{\alpha\beta} \eta_{\alpha'\beta'} \right] \times \log M_i^2 .
\end{aligned}$$

(3)

$$(-1) \xrightarrow{\alpha\beta} \xrightarrow{k} \begin{array}{c} \text{---} \nearrow \text{---} \\ \text{---} \searrow \text{---} \end{array} \xrightarrow{k} \xrightarrow{\alpha'\beta'} =$$

$$\begin{aligned} & \left[-\frac{1}{4} (-3x - x^2 + 12x^3 + 4x^4) k^4 \eta_{\alpha\beta} \eta_{\alpha'\beta'} - (2x^2 - 4x^3 + 2x^4) k^4 \eta_{\alpha\alpha'} \eta_{\beta\beta'} \right. \\ & - (3x - 15x^2 + 2x^3 + 8x^4) k^2 k_{\alpha} k_{\beta} \eta_{\alpha'\beta'} \\ & - (-5x + 25x^2 - 36x^3 + 16x^4) k^2 k_{\alpha'} k_{\beta'} \eta_{\alpha\beta} \\ & - (2x + 10x^2 - 20x^3 + 8x^4) k_{\alpha} k_{\beta} k_{\alpha'} k_{\beta'} \\ & - \frac{1}{4} (-5 - 12x - 8x^2) \Pi_i^2 k^2 \eta_{\alpha\beta} \eta_{\alpha'\beta'} - (-5 + 20x - 16x^2) \Pi_i^2 k_{\alpha} k_{\alpha'} \eta_{\beta\beta'} \\ & \left. - \Pi_i^4 \eta_{\alpha\beta} \eta_{\alpha'\beta'} - 2 \Pi_i^4 \eta_{\alpha\alpha'} \eta_{\beta\beta'} \right] \times \log (\Pi_i^2 + k^2 x (1-x)) \\ & + \left[-3 \Pi_i^2 k^2 \eta_{\alpha\beta} \eta_{\alpha'\beta'} + 2 \Pi_i^2 k_{\alpha} k_{\beta} \eta_{\alpha'\beta'} \right. \\ & \left. + 2 \Pi_i^4 \eta_{\alpha\beta} \eta_{\alpha'\beta'} \right] \times \log \Pi_i^2 . \end{aligned}$$

We have omitted an overall factor. There should also be included in all of the above results the factor:

$$\int_0^1 dx \ S_{\alpha\beta} S_{\alpha'\beta'} S_{(\alpha\beta),(\alpha'\beta')} \sum_i e_i$$

$S_{(\alpha\beta),(\alpha'\beta')}$ stands for a symmetrisation in the index pairs

$(\alpha\beta)$ and $(\alpha'\beta')$.

There is only one comment to make on the form of these expressions. Notably, with the exception of diagram, (2) they fall into two parts; one with a factor $\log(M_i^2 + k^2x(1-x))$, the other with a factor $\log M_i^2$. This arises from the application of the cancellation procedure (4.15).

We next take the limit $M_{i \neq 0} \rightarrow \infty$ with the restrictions on the signs, e_i , and masses given at the end of section (4). Finally the x -integration is performed. The eventual result is written as a sum of terms of the form:-

$$\frac{1}{k^4} \pi(A, B, C) S_{\alpha\beta} S_{\alpha'\beta'} S_{(\alpha\beta)(\alpha'\beta')} \times$$

$$\left(c_1 k^4 \eta_{\alpha\beta} \eta_{\alpha'\beta'} + c_2 k^4 \eta_{\alpha\alpha'} \eta_{\beta\beta'} + c_3 k^2 k_\alpha k_\beta \eta_{\alpha'\beta'} \right.$$

$$\left. + c_4 k^2 k_\alpha k_{\alpha'} \eta_{\beta\beta'} + c_5 k_\alpha k_\beta k_{\alpha'} k_{\beta'} \right)$$

Where $\pi(A, B, C)$ is some function of A, B, C and constant terms and the C_i are just numerical coefficients.

The relevant condition on the coefficients c_i that such a term is transverse, (i.e. that when multiplied by $k^\alpha k^{\alpha'}$ the answer is zero) is given by the following:-

$$\text{both } 2c_2 + c_4 = 0 \text{ and } c_1 + c_3 + c_5 - c_2 = 0 \quad (5.3)$$

Thus the final result for the sum of diagrams (1) + (2) + (3) of expression (5.1) is given by:-

$$\begin{aligned}
& \frac{1}{k^4} S_{\alpha\beta} S_{\alpha'\beta'} S_{(\alpha\beta),(\alpha'\beta')} \times \\
& \left[\frac{1}{60} (A + \log k^2) (23 k^4 \eta_{\alpha\beta} \eta_{\alpha'\beta'} + 81 k^4 \eta_{\alpha\alpha'} \eta_{\beta\beta'} \right. \\
& \quad \left. - 106 k^2 k_{\alpha} k_{\beta} \eta_{\alpha'\beta'} - 162 k^2 k_{\alpha} k_{\alpha'} \eta_{\beta\beta'} + 164 k_{\alpha} k_{\beta} k_{\alpha'} k_{\beta'} \right) \\
& + \frac{1}{900} (-1448 k^4 \eta_{\alpha\beta} \eta_{\alpha'\beta'} - 2046 k^4 \eta_{\alpha\alpha'} \eta_{\beta\beta'} \\
& \quad + 4696 k^2 k_{\alpha} k_{\beta} \eta_{\alpha'\beta'} + 4092 k^2 k_{\alpha} k_{\alpha'} \eta_{\beta\beta'} - 5294 k_{\alpha} k_{\beta} k_{\alpha'} k_{\beta'}) \\
& - \frac{113}{6} (-k^2 \eta_{\alpha\beta} \eta_{\alpha'\beta'} + k^2 \eta_{\alpha\alpha'} \eta_{\beta\beta'} + 2 k_{\alpha} k_{\beta} \eta_{\alpha'\beta'} \\
& \quad - 2 k_{\alpha} k_{\alpha'} \eta_{\beta\beta'}) \\
& \left. - \frac{C}{2} Q_{\alpha\beta, \alpha'\beta'} \right] \quad (5.4)
\end{aligned}$$

(Note: In my paper⁽¹⁹⁾, the final term appears with a factor $(-\frac{C}{4})$. The discrepancy arises from the alternative definition of Q that has been used in this thesis.)

We now have the truly remarkable result that the first three terms of the above are individually transverse, as may be seen by showing that they satisfy condition (5.3). Regrettably the term $-\frac{C}{2} Q_{\alpha\beta 1\alpha' 1\beta'}$ is not transverse. Indeed, from dimensional considerations, one must have the result that the quartic divergence does not appear in the answer. We shall discuss the possible implications of this in section 7. Before moving on to this there are one or two remarks to be made concerning expression (5.4).

On the question of the renormalisability of gravitation there is not a great deal that can be inferred from this calculation alone except possibly to confirm that one has to introduce counterterms involving four derivatives⁽²⁰⁾. These additional terms would not correspond to the original form of the Einstein action. Gravitation would not then appear to be renormalisable in the conventional sense. I shall comment a little more on this in section (9).

Secondly one might well ask whether or not (5.4) can be reduced to a more simple form. Or at least to a form where its transversality is made manifest. It would for example be very pleasant if one could write it as a sum of terms of the form of the functions $T^{\alpha\beta, \alpha'\beta'}, L^{\alpha\beta, \alpha'\beta'}$ that were introduced in section (3).

Consider for a moment the equivalent Yang-Mills calculation given in section 4. In particular consider all possible forms of terms that can occur in the answer. Clearly there are only two viz: $\eta_{\mu\nu}$ and $\frac{k_\mu k_\nu}{k^2}$ and these are linearly independent. Thus we can say that the space of all possible forms of the answer corresponds to a vector space of dimension 2. It is spanned by the particular functions given above. I shall call this vector space the 'answer space'. Note also that in this answer space there is only one transverse combination of terms. Everything is very simple and the answer is very simple. Let us now similarly describe the gravitational situation.

Even with the relevant symmetrisation factors the answer space is of dimension 5. The terms

$$\eta_{\alpha\beta} \eta_{\alpha'\beta'}, \quad \eta_{\alpha\alpha'} \eta_{\beta\beta'}, \quad \frac{k_\alpha k_\beta}{k^2} \eta_{\alpha'\beta'},$$

$$\frac{k_\alpha k_{\alpha'}}{k^2} \eta_{\beta\beta'}, \quad \frac{k_\alpha k_\beta k_{\alpha'} k_{\beta'}}{k^4}, \quad (5.5)$$

constitute a basis. Moreover, in order to establish transversality, multiplication of the terms in (5.5) by $k_\alpha k_{\alpha'}$ now leaves two linearly independent terms notably

$$\eta_{\alpha\beta} \quad \text{and} \quad \frac{k_\alpha k_\beta}{k^2}$$

Hence there are two conditions on the c_i necessary to ensure transversality of (5.4).

To ask the question "can 5.4 be written in a more simple or manifestly transverse form" is to ask "can one choose a different basis from (5.5) in which transversality is manifest". The functions T and L with appropriate symmetrisations are obvious choices. The problem is what to choose for the remaining 3. T specifies a particular transverse combination of the terms in (5.5) but unlike the Yang-Mills situation there is now more than one linearly independent transverse combination possible. Thus T alone is insufficient.

Clearly it is possible to make (5.4) appear transverse. To make it appear both transverse and simple has thus far eluded me; there are some rather curious coefficients to take account of and one can imagine the eventual revision of the answer as still being rather complicated. As such expression (5.4) does not seem to

correspond to frequently cited statement about quantum gravity "the results are so simple there must exist an easier way". If there does exist an "easier way" then (5.4) would seem to imply that it should not involve conventional perturbation theory, both from the standpoint of reaching the answer and from the form of the answer itself. It has long been known that deriving the answers can prove a little lengthy. In the light of this, higher order calculations, though straightforward, recommend the use of perturbation theory even less.

6. FIRST ORDER FORMALISM

By using the Goldberg Lagrangian instead of the action (2.14) one can undoubtedly produce a considerable simplification in the vertices and encumbent algebra of the gravitational perturbation expansion. However there is a possible source of worry in employing the procedure that we have hitherto adhered to. Notably that the Goldberg Lagrangian as given by (2.17) is explicitly dependent upon its being understood that it is in four space-time dimensions. In other words; in order to obtain the form (2.17) one has to employ the rule $\delta_{\mu}^{\mu} = 4$. This is worrying in that the regularisation procedure that is employed in the calculation of (5.1) rests upon the statement that "Ward Identities are independent of dimension". We have consequently introduced a possible inconsistency by proceeding initially from a Lagrangian that is dependent for its form on being in a certain number of dimensions. It is not the case that (2.17) is invariant under the n -dimensional gauge group of general co-ordinate transformations for arbitrary n . I shall discuss this further in section 7. In particular I shall show that the necessary additions to (2.17) to make it n -dimensionally gauge invariant cannot affect the final answer for the calculation of (5.1) nor are they inconsistent with the regularisation procedure employed.

For reasons both historical and corroborative I chose to re-calculate equation (4.1) using a form of the Einstein Action that is invariant under the n -dimensional

gauge group of general co-ordinate transformations: notably the original form (2.14) but treating the metric tensor and Riemannian connection as independent quantities. It is to this end that this section is devoted.

This first order or Palatini formalism is not without interest in its own right. Certainly by choosing the variables $\hat{g}^{\mu\nu}$ rather than $g^{\mu\nu}$ we should reproduce at the very least the finite part of expression 5.4. But more than that: at first sight things appear quite attractive. There is only one vertex and that is rather simple. It is just conceivable that this might provide a more tractable perturbation expansion than we have met hitherto. For the purposes of developing a renormalisation program for gravitation there would appear to be advantages in working with this form rather than trying to establish the interdependence of an infinite number of vertex renormalisation factors from an infinite number of Ward Identities.

I do not propose to say anymore about this last point in this thesis. But of the former we certainly recover the results of 5.4 in their entirety. Whether or not they are arrived at more simply I would not care to say. There is perhaps a little more algebra to be done. Certainly it is the case that the calculation is much more ordered. By which I mean that the individual diagrams readily reveal their own particular finite or divergent structure.

Before proceeding to the actual calculation of (4.1) which I shall give in some detail, we must first establish the requisite Feynman rules and indeed show that (3.2) is

the relevant Ward Identity for this first order formalism.

As usual the action is given by

$$S = \frac{1}{\kappa^2} \int d^4x \hat{g}^{\mu\nu} R_{\mu\nu} \quad (6.1)$$

where

$$R_{\mu\nu} = \Gamma_{\mu\rho,\nu}^{\rho} - \Gamma_{\mu\nu,\rho}^{\rho} + \Gamma_{\mu\sigma}^{\rho} \Gamma_{\nu\rho}^{\sigma} - \Gamma_{\rho\sigma}^{\sigma} \Gamma_{\mu\nu}^{\rho}$$

Independent variation of (6.1) with respect to $\hat{g}$ and Γ yields the customary Einstein field equations.

To obtain a perturbation expansion we now set

$$\hat{g}^{\alpha\beta} = \eta^{\alpha\beta} + \kappa \hat{h}^{\alpha\beta} \quad (\text{exactly as in section 2})$$

and $\Gamma_{\alpha\beta}^{\gamma} = \kappa \Gamma_{\alpha\beta}^{\gamma}.$

With this prescription (6.1) splits into the free and interaction parts given in Appendix II. The salient feature being that there is now only one vertex,

$$\kappa \hat{h}^{\mu\nu} \left(\Gamma_{\mu\sigma}^{\rho} \Gamma_{\nu\rho}^{\sigma} - \Gamma_{\rho\sigma}^{\sigma} \Gamma_{\mu\nu}^{\rho} \right).$$

In choosing the gauge condition there now exists a choice. In as much as we can restrict $\hat{g}$ or Γ or some combination of the two. If we implement precisely the same gauge condition as we did in deriving the Feynman rules in section 2 only the $\hat{g}$ field is affected. This has the advantage of reproducing exactly the same fictitious particle structure, i.e. $\Delta\psi(\hat{g})$ is once more given by equation (2.20). (7.1) is affected by the addition of the

term

$$-\frac{1}{2\alpha} \hat{h}^{\mu\sigma}_{,\mu} \eta_{\sigma\tau} \hat{h}^{\nu\tau}_{,\nu}$$

to the free part of the Lagrangian. We are still free to choose the value of α in this term and it should come as not surprise that the relevant choice is again $\alpha = -1$. Details of this are given in Appendix II along with the derivation of the three free propagators dependent upon this gauge condition.

I assume that these propagators are something like the simplest possible for this first order approach. In which case life is not quite as rosy as one might have hoped. To use DeWitt's phrase, they have only 'minimal singularities on the light cone' but the $\langle T, T \rangle$ propagator in particular is a very long winded affair, consisting of a 27 term $(\frac{1}{r^2})$ potential and a 27 term contact potential. It seems to be the case that what one gains on the vertices one loses on the propagators.

Finally we have to show that equation (3.4) still holds. This is clearly the case with only obvious modifications. One can carry through a precisely equivalent derivation following t'Hooft's method as given in section (3). Alternatively one can still work with the manifestly covariant theory provided the action and functional integral measure are suitably adapted. This would not be the case had we imposed the gauge condition on the T 's.

The requisite expression for the vacuum-to-vacuum amplitude if one is to follow the former approach is given

by:-

$$\langle 0_{out} | 0_{in} \rangle_T = \int (d\hat{g})(d\Gamma) \Delta_\psi(\hat{g}) \times \\ \times \exp i \left\{ S(\hat{g}, \Gamma) + \frac{1}{2} \hat{g}^{\mu\kappa}_{,\mu} \eta_{\kappa\lambda} \hat{g}^{\nu\lambda}_{,\nu} + \bar{J}_\nu \hat{g}^{\mu\nu}_{,\mu} + \bar{J}^2 \right\}$$

The derivation of equation (4.1) is carried through as before.

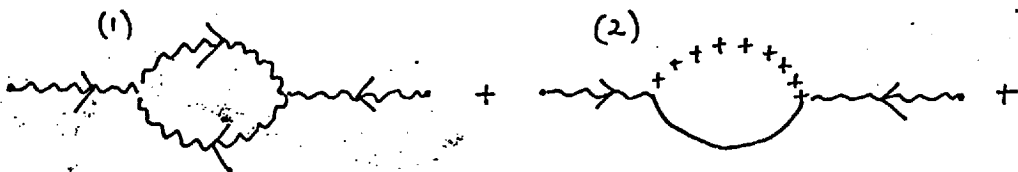
We now turn to the calculation of (4.1). For reasons that will become apparent, it is convenient to separate out from the $\langle \Gamma, \Gamma \rangle$ propagator the contact potential contribution. We shall represent this diagrammatically by attaching a cross to the $\langle \Gamma, \Gamma \rangle$ line where it is only the contact potential that is operative. Thus

$$++++++ \quad \text{corresponds to} \quad \frac{2}{k^2} k_\sigma k_\tau \hat{Q}^{\sigma\gamma_1}_{\alpha_1\beta_1} \circ \underline{Q} \circ \hat{Q}^{\tau\gamma_2}_{\alpha_2\beta_2}$$

$$+++X++++ \quad \text{corresponds to} \quad - \hat{Q}^{\sigma\gamma_1}_{\alpha_1\beta_1} \circ \left(\frac{2}{3} \delta^{\rho\gamma_2}_{\alpha_2\beta_2} \underline{Q}_{\rho\sigma} - \delta^\gamma_\sigma \underline{Q}_{\alpha_2\beta_2} \right)$$

With this convention the relevant diagrams in the calculation of (3.4) are as follows:-

(diagrammatic conventions are given in Appendix I)



(3) (4)

(5) (6)

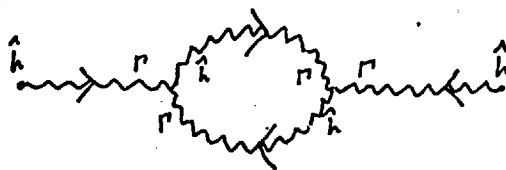
(7) (8)

(9) (10)

(11) + (tadpoles)

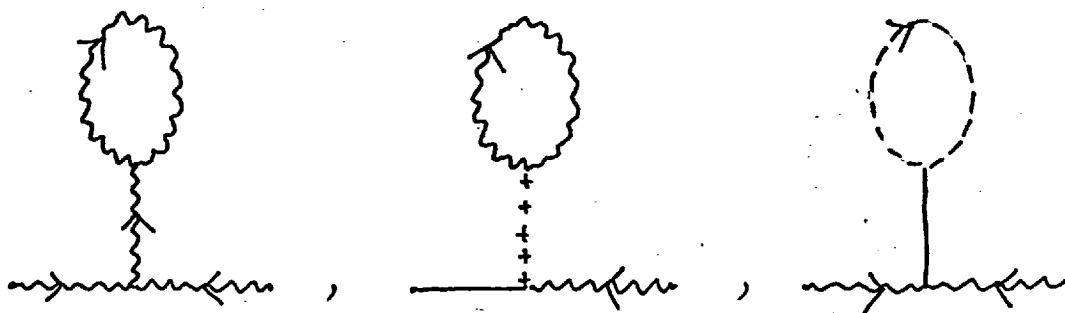
(6.2)

The correct orientation of the above diagrams is made explicit by the arrows on $\langle h \rangle$ lines. All diagrams begin and terminate in an $\hat{h}$. Diagram (1), for instance, written more explicitly is as follows



Momentum flows from left to right in the external legs of all diagrams. I do not include the numerous tadpole

diagrams since I do not propose to make any attempt to evaluate them. There are several varieties, typically:



(6) is the fictitious particle contribution exactly as for diagram (3) of (5.1).

(7),(8) and (9) are the contact potential contributions from (2),(3) and (4) respectively.

(10) and (11) are the contact potential contributions from diagram (5).

I shall now give the intermediate results for diagrams (1) --- (10) of (6.2). Diagram (11) warrants special consideration and I shall discuss this later.

Indices and momenta are included in all diagrams in exactly the same way viz:-

e.g. diagram (1)



First then the situation following the algebraic contractions.

The results are as follows:

The factor,

$$k^2 \int d^4 k_2 d^4 k_3 \delta(k+k_2+k_3) S_{\alpha\beta} S_{\alpha'\beta'} S_{(\alpha\beta),(\alpha'\beta')} \times \frac{1}{(k_2^2 + m^2)(k_3^2 + m^2)}$$

is to be included in the results for (1) --- (6) given below.

(1)

$$k_K k_\lambda k_{2\sigma} k_{3\tau} \left(-2 \eta^{\alpha\beta} \eta^{\alpha'\beta'} \eta^{\kappa\sigma} \eta^{\lambda\tau} + 20 \eta^{\alpha\alpha'} \eta^{\beta\beta'} \eta^{\kappa\sigma} \eta^{\lambda\tau} \right. \\ \left. - 40 \eta^{\alpha\alpha'} \eta^{\kappa\sigma} \eta^{\beta\lambda} \eta^{\beta'\tau} + 28 \eta^{\alpha\kappa} \eta^{\alpha'\lambda} \eta^{\beta\sigma} \eta^{\beta'\tau} \right)$$

(2)

$$k_K k_\lambda k_{2\sigma} k_{2\tau} \left(-8 \eta^{\alpha\beta} \eta^{\alpha'\beta'} \eta^{\kappa\sigma} \eta^{\lambda\tau} + 20 \eta^{\alpha\alpha'} \eta^{\beta\beta'} \eta^{\kappa\sigma} \eta^{\lambda\tau} \right. \\ + 24 \eta^{\alpha\beta} \eta^{\kappa\sigma} \eta^{\alpha'\lambda} \eta^{\beta'\tau} - 48 \eta^{\alpha\alpha'} \eta^{\kappa\sigma} \eta^{\beta\lambda} \eta^{\beta'\tau} - 20 \eta^{\alpha\beta} \eta^{\kappa\lambda} \eta^{\alpha'\sigma} \eta^{\beta'\tau} \\ + 16 \eta^{\alpha\alpha'} \eta^{\kappa\lambda} \eta^{\beta\sigma} \eta^{\beta'\tau} + 8 \eta^{\alpha\kappa} \eta^{\alpha'\lambda} \eta^{\beta\sigma} \eta^{\beta'\tau} + 8 \eta^{\alpha\kappa} \eta^{\beta\lambda} \eta^{\alpha'\sigma} \eta^{\beta'\tau} \\ \left. + 2 \eta^{\sigma\tau} \eta^{\kappa\lambda} \eta^{\alpha\alpha'} \eta^{\beta\beta'} + 5 \eta^{\sigma\tau} \eta^{\kappa\lambda} \eta^{\alpha\beta} \eta^{\alpha'\beta'} - 4 \eta^{\sigma\tau} \eta^{\alpha\beta} \eta^{\alpha'\kappa} \eta^{\lambda\beta'} \right)$$

(3) & (4)

$$k_\mu k_{2\sigma} k_{2\tau} k_{3\kappa} \left(-12 \eta^{\alpha\beta} \eta^{\alpha'\beta'} \eta^{\kappa\sigma} \eta^{\mu\tau} + 28 \eta^{\alpha\alpha'} \eta^{\beta\beta'} \eta^{\kappa\sigma} \eta^{\mu\tau} \right. \\ + 24 \eta^{\alpha\beta} \eta^{\kappa\sigma} \eta^{\alpha'\mu} \eta^{\beta'\tau} - 40 \eta^{\alpha\alpha'} \eta^{\kappa\sigma} \eta^{\beta\mu} \eta^{\beta'\tau} - 8 \eta^{\alpha\beta} \eta^{\kappa\mu} \eta^{\alpha'\sigma} \eta^{\beta'\tau} \\ + 40 \eta^{\alpha\kappa} \eta^{\alpha'\mu} \eta^{\beta\sigma} \eta^{\beta'\tau} + 6 \eta^{\alpha\beta} \eta^{\alpha'\beta'} \eta^{\kappa\mu} \eta^{\sigma\tau} - 8 \eta^{\alpha\beta} \eta^{\sigma\tau} \eta^{\alpha'\kappa} \eta^{\beta'\mu} \\ \left. + 8 \eta^{\alpha\alpha'} \eta^{\sigma\tau} \eta^{\beta\kappa} \eta^{\beta'\mu} - 8 \eta^{\alpha\beta} \eta^{\mu\tau} \eta^{\alpha'\kappa} \eta^{\beta'\sigma} \right)$$

(5)

$$k_2 \sigma k_{2T} k_{3K} k_{3\lambda} \left(-2 \eta^{\alpha\beta} \eta^{\alpha'\beta'} \eta^{\kappa\sigma} \eta^{\lambda\tau} + 4 \eta^{\alpha\alpha'} \eta^{\beta\beta'} \eta^{\kappa\sigma} \eta^{\lambda\tau} \right. \\ \left. + 2 \eta^{\alpha\alpha'} \eta^{\beta\beta'} \eta^{\kappa\lambda} \eta^{\sigma\tau} + 20 \eta^{\alpha\sigma} \eta^{\alpha'\tau} \eta^{\beta\kappa} \eta^{\beta'\lambda} \right)$$

$$(6) \left(-4 k^\alpha k^{\alpha'} k^\beta k_2^{\beta'} + 2 k^\alpha k^2 k_2^\beta \eta^{\alpha'\beta'} + 2 k^\alpha k^\beta (k \cdot k_2) \eta^{\alpha'\beta'} \right. \\ - k^2 (k \cdot k_2) \eta^{\alpha\beta} \eta^{\alpha'\beta'} - 20 k^\alpha k^{\alpha'} k_2^\beta k_2^{\beta'} + 20 k^\alpha k_2^\beta (k \cdot k_2) \eta^{\alpha'\beta'} \\ - 5 (k \cdot k_2)^2 \eta^{\alpha\beta} \eta^{\alpha'\beta'} - 40 k^\alpha k_2^\beta k_2^{\beta'} + 20 k^\alpha k_2^\beta k_2^2 \eta^{\alpha'\beta'} \\ + 20 k_2^\alpha k_2^\beta (k \cdot k_2) \eta^{\alpha'\beta'} - 10 k_2^2 (k \cdot k_2) \eta^{\alpha\beta} \eta^{\alpha'\beta'} - 16 k_2^\alpha k_2^\beta k_2^\alpha k_2^{\beta'} \\ \left. + 16 k_2^\alpha k_2^\beta k_2^2 \eta^{\alpha'\beta'} - 4 k_2^4 \eta^{\alpha\beta} \eta^{\alpha'\beta'} \right)$$

(7)

$$k^2 \int (d^n k_3) \delta_{\alpha\beta} \delta_{\alpha'\beta'} \delta_{(\alpha\beta),(\alpha'\beta')} \times \frac{1}{k_3^2 + M^2} \times$$

$$k_\kappa k_\lambda \left(11 \eta^{\kappa\lambda} \eta^{\alpha\beta} \eta^{\alpha'\beta'} - 18 \eta^{\kappa\lambda} \eta^{\alpha\alpha'} \eta^{\beta\beta'} - 20 \eta^{\kappa\alpha} \eta^{\lambda\beta} \eta^{\alpha'\beta'} \right. \\ \left. + 24 \eta^{\kappa\alpha} \eta^{\lambda\alpha'} \eta^{\beta\beta'} \right)$$

(8) & (9)

These diagrams integrate to zero from the factor

$$\int d^n k_3 \frac{k_{3\mu}}{(k_3^2 + M^2)}$$

and I have not evaluated them explicitly.

$$(10) \quad k^2 \int d^4 k_3 \quad S_{\alpha\beta} S_{\alpha'\beta'} S_{(\alpha\beta)(\alpha'\beta')} \times \frac{1}{k_3^2 + M^2} \times \\ k_{3\kappa} k_{3\lambda} \left(-20 \eta^{\alpha\beta} \eta^{\alpha'\kappa} \eta^{\beta'\lambda} + [\text{a term involving } \eta^{\kappa\lambda}] \right)$$

(The $\eta^{\kappa\lambda}$ term again integrates to zero and I have not evaluated it).

I shall next give the results once the different masses M_i and signs e_i have been inserted and the integrations performed. That is to say the equivalent stage to the results given for the evaluation of the sum (5.1) of section (5). For the purposes of comparison I shall extract from the sum (1) - (10) given above an overall factor of $(-2 i\pi^2)$. The sum (1) - (10) of (6.2) then has precisely the same overall factor as has to be included in the results of section (5). Also, as for the intermediate results given in section (5), there is now a factor

$$\int_0^1 dx \quad S_{\alpha\beta} S_{\alpha'\beta'} S_{(\alpha\beta),(\alpha'\beta')} \sum_i e_i$$

to be included in all the results given below. These are as follows:-

$$(1) + (2) + (3) + (4) + (5) =$$

$$\begin{aligned}
& \left[\frac{1}{4} (4x + 21x^2 - 26x^3 + 5x^4) k^4 \eta^{\alpha\beta} \eta^{\alpha'\beta'} + \right. \\
& \quad + \frac{1}{2} (24x - 23x^2 - 10x^3 + 5x^4) k^4 \eta^{\alpha\alpha'} \eta^{\beta\beta'} \\
& \quad + (-4x + 10x^2 - 26x^3 + 10x^4) k^\alpha k^\beta \eta^{\alpha'\beta'} \\
& \quad + (-18x + 22x^2 - 20x^3 + 20x^4) k^\alpha k^{\alpha'} \eta^{\beta\beta'} \\
& \quad + (14x - 16x^2 + 10x^4) k^\alpha k^\beta k^{\alpha'} k^{\beta'} + \frac{1}{2} (4 + 13x - 5x^2) M_i^2 k^2 \eta^{\alpha\beta} \eta^{\alpha'\beta'} \\
& \quad + (2 + 5x - 5x^2) M_i^2 k^2 \eta^{\alpha\alpha'} \eta^{\beta\beta'} + (-4 + 8x - 10x^2) M_i^2 k^\alpha k^\beta \eta^{\alpha'\beta'} \\
& \quad \left. + (2 - 20x^2) M_i^2 k^\alpha k^{\alpha'} \eta^{\beta\beta'} + \frac{5}{4} M_i^4 \eta^{\alpha\beta} \eta^{\alpha'\beta'} + \frac{5}{2} M_i^4 \eta^{\alpha\alpha'} \eta^{\beta\beta'} \right] x \\
& \quad \times \log (M_i^2 + k^2 x(1-x)) \\
& - \left[-\frac{5}{2} M_i^2 k^2 \eta^{\alpha\beta} \eta^{\alpha'\beta'} + 13 M_i^2 k^2 \eta^{\alpha\alpha'} \eta^{\beta\beta'} + 10 M_i^2 k^\alpha k^\beta \eta^{\alpha'\beta'} \right. \\
& \quad \left. - 20 M_i^2 k^\alpha k^{\alpha'} \eta^{\beta\beta'} \right] \log M_i^2 .
\end{aligned}$$

(6) This is exactly as given in section (5).

(7)

$$\begin{aligned}
& \left[-\frac{1}{2} M_i^2 k^2 \eta^{\alpha\beta} \eta^{\alpha'\beta'} + 9 M_i^2 k^2 \eta^{\alpha\alpha'} \eta^{\beta\beta'} + 10 M_i^2 k^\alpha k^\beta \eta^{\alpha'\beta'} \right. \\
& \quad \left. - 12 M_i^2 k^\alpha k^{\alpha'} \eta^{\beta\beta'} \right] x \log M_i^2 .
\end{aligned}$$

$$(10) \quad \left[-\frac{5}{2} M_i^4 \eta^{\alpha\beta} \eta^{\alpha'\beta'} \right] \times \log M_i^2.$$

We are now in a position to make a comparison with the results of section (5). There is one point to be born in mind when doing this. The expressions involving integrations over the Feynman parameter variable x are not unique. This non-uniqueness arises from the possibility of shifting the k_2 integration variables over the external momentum k . This has the effect of introducing factors of $(1-2x)$ or $(1-2x)^3$ into, for example, the above result for the diagrams (1) + (2) + (3) + (4) + (5). It does not of course affect the final result since

$$\int_0^1 dx (1-2x)^{2n+1} \log x(1-x) = 0, \text{ for } n = 0 \text{ or a positive}$$

integer. Thus a strict comparison can only be made at this stage by expanding the x -dependent terms about $x = \frac{1}{2}$ and comparing even powers of $(1-2x)$. It can then be seen that the results for (1) + (2) + (3) + (4) + (5) are not the same as the results for diagram (1) of (5.1). They do however integrate to exactly the same value. It can also be seen that the results for (7) + (10) above are precisely those for diagram (2) of (5.1).

To summarise: Diagrams (1) --- (10) of (7.2) reproduce exactly the result for diagrams (1) - (3) of (5.1). The answer is that given by expression (5.4). Thus we again have the unwanted quartically divergent term

$$- \frac{c}{2} Q_{\alpha\beta, \alpha'\beta'} .$$

Of course we have not yet evaluated diagram (11). The first point to note is that diagram (11) will never cancel the term

$$\frac{c}{2} Q_{\alpha\beta, \alpha'\beta'}$$

The algebraic factor corresponding to diagram 11 is given by

$$5 \delta_{\alpha\beta, \alpha'\beta'}$$

The momentum dependence before any integrations are performed is given by

$$\sum e_i \int (d^{n+1}k) \delta(k_{n+1} - M_i) \quad (6.3)$$

The evaluation of this integral, which in conventional regularisation procedures corresponds to a $\delta^4(0)$ term is at best ambiguous.

If one immediately performed the integration $\int dk_{n+1} \delta(k_{n+1} - M_i)$ there would be no way of parametrising the associated quartic divergence within the framework of section 4.

Alternatively one could transform to spherical coordinates, but then one would have to give a prescription for the pole term $(k^2 + M^2)^{-\alpha}$.

The only consistent approach appears to be to write (6.3) as follows

$$(6.3) = \sum e_i \int (d^{n+1}k) \delta(k_{n+1} - M_i) \times \frac{k^2}{k^2 + i\varepsilon} =$$

$$= \sum_i e_i \int d^n k \frac{k^2 + M_i^2}{k^2 + M_i^2}$$

In which case it is zero as may be seen by applying the formulae in Appendix III.

Thus if the contact potential term of Appendix I is multiplied by the factor $(\frac{k^2}{k^2 + i\epsilon})$ the results for (6.2) are exactly those for (5.1). It should be noted that this prescription for the evaluation of diagram (11) in no way alters the results for diagrams (7), (8), (9), (10). This is due to the 'cancellation' property referred to in equation (4.15). Also in the derivation of the contact potential term in Appendix II it is not clear that such a factor should not be incorporated anyway. It is a matter of taste whether or not one substitutes in terms of integral or differential equations.

To summarise the results of this section: we have shown that with a suitable prescription for the evaluation of $\int d^n k$, we reproduce exactly the results (5.4). Again we have not evaluated the tadpole diagrams. Thus by proceeding from an n-dimensional gauge invariant Lagrangian where there is no possibility of an inconsistency with the regularisation procedures employed we recover the same result as we had when working with the Goldberg Lagrangian in section 5.

7. TADPOLE DIAGRAMS AND QUARTIC DIVERGENCIES

In order that the discussion of tadpole diagrams and quartic divergencies in the latter part of this section should refer both to the calculations of section 5 and section 6 I shall first establish that there is no inconsistency with regard to the regularisation procedure if one chooses to work with the Goldberg form of the Einstein Action. There are more or less complicated ways of showing this depending upon the starting point one chooses. The most straightforward is to re-derive the Goldberg form starting from the original taking care to perform the algebra in n-dimensions. This may be accomplished as follows: We take as our starting point

$$\mathcal{L}(g) = g^{\frac{1}{2}} g^{\mu\nu} \left\{ \Gamma_{\mu\rho,\nu}^{\rho} - \Gamma_{\mu\nu,\rho}^{\rho} + \Gamma_{\mu\sigma}^{\rho} \Gamma_{\nu\rho}^{\sigma} - \Gamma_{\rho\sigma}^{\sigma} \Gamma_{\mu\nu}^{\rho} \right\} \quad (7.1)$$

where we have

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2} g^{\rho\lambda} (g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}) \quad (7.2)$$

The first two terms of the bracket in (7.1) can be written

$$\partial_{\lambda} (g^{\frac{1}{2}} g^{\mu\lambda} \Gamma_{\mu\rho}^{\rho} - g^{\frac{1}{2}} g^{\mu\nu} \Gamma_{\mu\nu}^{\lambda}) - \Gamma_{\rho\mu}^{\rho} \partial_{\nu} (g^{\frac{1}{2}} g^{\mu\nu}) + \Gamma_{\mu\nu}^{\rho} \partial_{\rho} (g^{\frac{1}{2}} g^{\mu\nu}).$$

We can discard the divergent term. The remainder can be shown to equal

$$2 (\Gamma_{\rho\sigma}^{\sigma} \Gamma_{\mu\nu}^{\rho} - \Gamma_{\mu\sigma}^{\rho} \Gamma_{\nu\rho}^{\sigma}) g^{\frac{1}{2}} g^{\mu\nu}$$

One then obtains instead of (7.1) the effective Lagrangian

$$L(g) = g^{1/2} g^{\mu\nu} (\Gamma_{\sigma\rho}^{\sigma} \Gamma_{\mu\nu}^{\rho} - \Gamma_{\mu\sigma}^{\rho} \Gamma_{\nu\rho}^{\sigma}) \quad (7.3)$$

This form is relatively well known. It is for example the usual form taken as a starting point for the A.D.M. (9) quantisation procedure. For our purposes it is important that nowhere have we made use of the relation $\delta_{\mu}^{\mu} = n$.

The next step is to rewrite (7.2) in terms of derivatives of $g^{\alpha\beta}$. Making use of the relation,

$$\delta g_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta} \delta g^{\alpha\beta}$$

we obtain,

$$\Gamma_{\mu\nu}^{\rho} = -\frac{1}{2} \left\{ \delta_{\alpha}^{\rho} g_{\mu\beta} \delta_{\nu}^{\beta} + \delta_{\alpha}^{\rho} g_{\nu\beta} \delta_{\mu}^{\beta} - g^{\rho\gamma} g_{\mu\alpha} g_{\nu\beta} \right\} g^{\alpha\beta}_{,\gamma}$$

Substituting this into (7.3) $L(g)$ becomes

$$L(g) = \frac{1}{4} g^{1/2} g^{\alpha\beta}_{,\sigma} g^{\alpha'\beta'}_{,\tau} \left\{ 2 g_{\alpha'\beta'} \delta_{\alpha}^{\sigma} \delta_{\beta}^{\tau} - 2 g_{\alpha\alpha'} \delta_{\beta'\beta}^{\sigma} \delta_{\beta}^{\tau} + g^{\sigma\tau} g_{\alpha\alpha'} g_{\beta\beta'} - g^{\sigma\tau} g_{\alpha\beta} g_{\alpha'\beta'} \right\} \quad (7.4)$$

Again this is, so far, independent of dimension.

The final stage of the derivation is to replace the variables $g^{\alpha\beta}$, $g_{\alpha\beta}$ by the variables $\hat{g}^{\alpha\beta}$, $\hat{g}_{\alpha\beta}$ where $\hat{g}^{\alpha\beta} = g^{1/2} g^{\alpha\beta}$, $\hat{g}_{\alpha\beta} = g^{-1/2} g_{\alpha\beta}$.

This is achieved by consideration of the well known relation

$$\delta g = -g g_{\mu\nu} \delta g^{\mu\nu}$$

Thus:

$$\begin{aligned}\delta g &= -g g_{\mu\nu} \delta(\bar{g}^{-1/2} \hat{g}^{\mu\nu}) \\ &= -g g_{\mu\nu} \left(-\frac{1}{2} \bar{g}^{-3/2} \hat{g}^{\mu\nu} \delta g + \bar{g}^{-1/2} \delta \hat{g}^{\mu\nu} \right) \\ &= \frac{1}{2} \delta g - \hat{g}_{\mu\nu} \delta \hat{g}^{\mu\nu}\end{aligned}$$

or

$$\delta g = \frac{2}{(n-2)} \hat{g}_{\mu\nu} \delta \hat{g}^{\mu\nu}$$

Thus we may write

$$g^{\alpha\beta}_{,\sigma} = \hat{g}^{\alpha\beta}_{,\sigma} \left\{ \delta^{\alpha\beta}_{\alpha_1\beta_1} - \frac{1}{(n-2)} \hat{g}_{\alpha_1\beta_1} \hat{g}^{\alpha\beta} \right\} g^{-1/2}.$$

Substituting this into expression (7.4) yields the final result:-

$$\begin{aligned}\mathcal{L}(g) &= \frac{1}{8} \hat{g}^{\alpha_1\beta_1}_{,\sigma} \hat{g}^{\alpha_2\beta_2}_{,\tau} \left\{ 2 \hat{g}^{\sigma\tau} \hat{g}_{\alpha_1\alpha_2} \hat{g}_{\beta_1\beta_2} \right. \\ &\quad \left. - \frac{2}{(n-2)} \hat{g}^{\sigma\tau} \hat{g}_{\alpha_1\beta_1} \hat{g}_{\alpha_2\beta_2} - 4 \hat{g}_{\alpha_1\alpha_2} \delta^{\sigma}_{\beta_2} \delta^{\tau}_{\beta_1} \right\}.\end{aligned}\tag{7.5}$$

For $n = 4$ (7.5) reduces to (2.17), the form that was used in the calculation of (5.1). We can now ask the question, 'what would be the effect of using (7.5) instead of (2.17)?'

Firstly and most crucially, the term in (7.5) that is subtracted off by the gauge condition is independent of n . That is the term

$$\frac{1}{8} \hat{g}^{\alpha_1\beta_1}_{,\sigma} \hat{g}^{\alpha_2\beta_2}_{,\tau} \left(-4 \eta_{\alpha_1\alpha_2} \delta^{\sigma}_{\beta_2} \delta^{\tau}_{\beta_1} \right).$$

Thus there is no alteration to the gauge condition nor to the derivation of the Ward Identities.

Secondly if we expand the n dependent term about $(n = 4)$ we obtain

$$\frac{2}{(n-2)} = 1 - \frac{(n-4)}{(n-2)} + O(n-4)^2$$

Consequently the vertices obtain additional factors:

$$\hat{\Gamma} \rightarrow \hat{\Gamma}_{\text{original}} + (n-4) \hat{\Gamma}_1 + O(n-4)^2$$

The terms $(n-4)\Gamma_1$, etc, cannot contribute to the answer for precisely the same reason that we were able to perform the algebraic calculations specifically in 4 dimensions (c.f. the discussion following equation 4.9).

Finally we have to decide what happens to the propagator. The algebraic structure of the propagator given in Appendix I is determined by the inverse of the term

$$(\eta_{\alpha_1 \alpha_2} \eta_{\beta_1 \beta_2} + \eta_{\alpha_1 \beta_2} \eta_{\beta_1 \alpha_2} - \eta_{\alpha_1 \beta_1} \eta_{\alpha_2 \beta_2})$$

When working with (7.5) we have to take the inverse of,

$$\left(\eta_{\alpha_1 \alpha_2} \eta_{\beta_1 \beta_2} + \eta_{\alpha_1 \beta_2} \eta_{\beta_1 \alpha_2} - \frac{2}{(n-2)} \eta_{\alpha_1 \beta_1} \eta_{\alpha_2 \beta_2} \right) \quad (7.6)$$

Working still with the rule $\delta_{\mu}^{\mu} = n$ the result now is even better viz:-

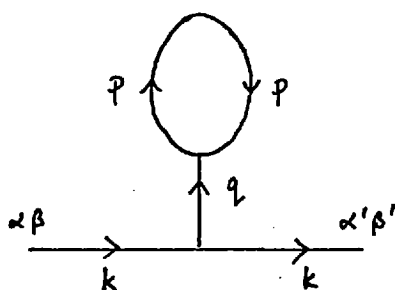
$$(7.6)^{-1} = Q^{\alpha\beta, \alpha_1\beta_1}$$

Thus overall we would expect no change in results (5.4)

were we to work with (7.5) instead of (2.17). This conclusion is reinforced (not that it needs it) by the calculation of section 6.

Let us now turn to considerations of the tadpole diagrams. Specifically we shall deal with diagrams (4) and (5) of the sum (5.1). The first observation that one can make is that these diagrams are purely quadratic or quartic divergencies. This arises from the single loop integration necessary in any evaluation of them. The only point in evaluating them would be to cancel the quartic divergence in the results (5.4). This is not at all straightforward or well defined as I mentioned earlier. The reason for this is as follows:

Consider diagram (4) of (5.1). If we proceed to crudely insert momenta we have to evaluate the diagram



Employing the Feynman Rules in Appendix I this corresponds to

$$\int dq \frac{\delta(q)}{q^2} Q^{\alpha\beta, \alpha_1\beta_1} Q^{\alpha'\beta', \alpha'_1\beta'_1} {}^3\Gamma_{\alpha_1\beta_1, \alpha_2\beta_2, \alpha'_1\beta'_1}(k, -q, -k) \cdot Q^{\alpha_2\beta_2, \alpha'_2\beta'_2} \\ \times {}^3\Gamma_{\alpha'_2\beta'_2, \alpha_3\beta_3, \alpha'_3\beta'_3}(q, p, -p) \times \int d^4p \frac{Q^{\alpha_3\beta_3, \alpha'_3\beta'_3}}{(p^2 + m^2)}$$

There are three major faults with this expression.

- (1) It contains terms of the form

$$\int dq \frac{\delta(q)}{q^2}$$

that are infrared divergent.

- (2) It contains terms of the form

$$\int dq \delta(q) \frac{q^\mu q^\nu}{q^2}$$

that are finite, but depend on the direction in which the limit $q_\mu \rightarrow 0$ is taken.

- (3) One cannot unambiguously employ momentum conservation at the Vertices.

Suppose, for instance, one had a term p^2 arising from $r(q,p,-p)$. One could equally well write for this term, $(p+q)^2$ and obtain an additional contribution

$$\delta(q) (q^2 + 2q \cdot p) / q^2$$

that is in general non zero.

Faced with such a threefold ambiguity it does not seem worthwhile performing the very considerable algebra simply to cancel an unwanted quartic divergence from just one calculation. Conceivably, with such flexibility in definition, this would be comparatively easy to arrange. It would however be a purely arbitrary procedure valid for this calculation alone. The very much more attractive alternative is to set all such diagrams equal to zero. One then has to decide what to do with the quartic divergence

occurring in expression (5.4).

Fortunately, here, there are some grounds for hope. The algebraic structure of this quartic divergence is precisely that of the propagator. Let us recall at this stage the particular Ward identity that was to be verified. This was given by

$$k_\alpha k_{\alpha'} \left\{ \langle 0_{out} | \hat{h}^{\alpha\beta} \hat{h}^{\alpha'\beta'} | 0_{in} \rangle - \frac{Q^{\alpha\beta, \alpha'\beta'}}{k^2} \langle 0_{out} | 0_{in} \rangle \right\} = 0 \quad (7.7)$$

or more explicitly

$$k_\alpha k_{\alpha'} \frac{1}{k^4} Q^{\alpha\beta, \alpha_1\beta_1} Q^{\alpha'\beta', \alpha'_1\beta'_1} \overline{\Pi}_{\alpha_1\beta_1, \alpha'_1\beta'_1} = 0$$

So far we have not considered in any detail the normalisation of the functional integrals. In particular the term

$$\langle 0_{out} | 0_{in} \rangle \quad (7.8)$$

occurring in equation (7.7).

If one were to imagine an expansion of (7.8) of the form

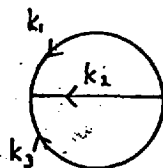
$$\langle 0_{out} | 0_{in} \rangle = 1 + a\kappa^2 M^4 \log M^2 +$$

one would only have to choose the value of 'a' so as to cancel the quartic divergence occurring in expression (5.4). I say 'choose' but it doesn't in fact correspond to a choice. Consider the diagrams necessary for the evaluation of (7.8) to second order in κ . These are as follows:-

$$\begin{aligned}
 & \frac{1}{2} \text{ (1) } + \frac{1}{2} \text{ (2) } - \text{ (3) } \\
 & + \frac{1}{2} \text{ (4) } - \text{ (5) } \quad (7.9)
 \end{aligned}$$

If we are consistent in setting tadpole diagrams equal to zero we must discard diagrams (4) and (5) and thereby remove the only source of ambiguity in (7.9).

Unfortunately the evaluation of (1) + (2) + (3) of (7.9) does not follow from the evaluation of (1) + (2) + (3) of (5.1). This is related to the overlapping divergencies encountered in the evaluation of (7.9). The situation for (5.1) where the relevant diagrams had external legs was easy in that the additional masses M_i were to be incorporated only on internal lines. Momentum conservation at the vertices required that these masses be the same. In 7.9 we must again incorporate Masses on all internal lines but there are now three internal lines at the vertex of diagram (1) for example. If in diagram (1) we write the momenta as follows



where k_1, k_2, k_3 have masses M_1, M_2, M_3 associated with them respectively, then momentum conservation requires

$M_1 + M_2 + M_3 = 0$ and we have the possibility of different masses on different lines.

t'Hooft and Veltman⁽¹⁸⁾ have shown that their dimensional regularisation method deals with overlapping divergencies without any trouble. I have yet to work out the details of whether or not the adaptation of their method employed in this thesis is similarly and easily extended to take account of this problem. I can see no reason why this should not be the case. More than that, at the moment, I cannot say. It is, of course, a considerable amount of work simply to compute a single numerical factor.

There is one other alternative approach and that is to calculate the diagrams (5.1) not with the propagator $k^{\frac{1}{2}} Q^{\alpha\beta} \alpha'\beta'$, but instead with the propagator as given by equation (2.19). The fictitious particle vertex is necessarily altered since one is now employing the Dedonder gauge directly. In fact it is simplified. Unfortunately the algebra of the graviton diagrams is increased three-fold and there are some very unpleasant sixth order momentum integrals to perform. The virtue of doing this is that the identity (3.9) is replaced by

$$k_\mu \langle \hat{h}^{\mu\nu} \hat{h}^{\kappa\lambda} \rangle = \kappa T^{\kappa\lambda}_{\sigma\tau} \langle R^{\sigma\tau}_\alpha \frac{\delta \mathcal{L}^\alpha}{\delta J^\nu} \rangle$$

The term proportional to the graviton propagator no longer appears since it is now transverse. It is still not clear if the normalisation of this identity is unimportant but there is no longer any scope for its removing possible unwanted quartic divergencies. This is evident from

dimensional considerations. Of course by so doing one is now faced with the problem mentioned earlier in section (2); that of negotiating the second pole term in the propagator (2.19). Perhaps there is a good reason for not worrying about this too much but I have not yet come across it.

8. INFINITY SUPPRESSION AND RELATED TOPICS

The major significant advance made in non-polynomial lagrangian techniques since the original work of Salam, Isham and Strathdee⁽⁶⁾ has been that of Honerkamp⁽²¹⁾. By working with a manifestly covariant chiral theory he is able to define and use to great effect a covariant chiral superpropogator. In the course of a series of papers⁽²¹⁾ dealing with manifestly covariant perturbation expansions and their applications he derives for gravitation one such that is identical to that of DeWitt, characterised by the functional integral (2.4). Moreover he states that apart from some technical difficulties his method of construction of his covariant chiral superpropogator is in theory applicable to the gravitational situation. Not least of these difficulties is the apparent lack of any method for easily obtaining the covariant n -point bare graviton vertex functions. They are easily defined by successive differentiations of the original action. However the subsequent computation by this method is extremely arduous beyond the third and the general form of the n^{th} vertex is unclear. In this section I show how these functions can be readily evaluated. This has relevance to this thesis in general; their usefulness as regards the possible construction of a covariant graviton superpropogator, as we shall see, is severely limited. We proceed as follows:

The essence of obtaining a manifestly covariant perturbation series is to expand the Einstein action about a

classical background field as a power series in the quantum field. It is convenient to work with a slightly altered notation than we have used hitherto, viz:

We shall take the classical field to be $g_{\mu\nu}$, and the quantum field $\hat{g}_{\mu\nu}$. (Note this is just the metric tensor there are no weight functions attached.) The usual Einstein Action is taken as a function of $\tilde{g}_{\mu\nu}$ where

$$S[\tilde{g}] = \int d^4x \tilde{g}^{1/2} \tilde{g}^{\mu\nu} \tilde{R}_{\mu\nu}$$

and the expansion is obtained by setting

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \hat{g}_{\mu\nu}$$

It also proves useful to define

$$\gamma_{\alpha, \mu\nu}^{\sigma} = \frac{1}{2} (g_{\alpha\mu} \delta_{\nu}^{\sigma} + g_{\alpha\nu} \delta_{\mu}^{\sigma} - g_{\mu\nu} \delta_{\alpha}^{\sigma}) \quad (8.1)$$

In what follows, $\tilde{\gamma}$ is the function obtained by replacing g by $\tilde{g}$ in (8.1); $\hat{\gamma}$ - replace g by $\hat{g}$. The same also applies to other quantities, e.g. the Ricci tensor

$$\tilde{R}_{\mu\nu} \equiv R(\tilde{g}) \quad , \quad R_{\mu\nu} \equiv R_{\mu\nu}(g).$$

The connection $\tilde{\Gamma}_{\mu\nu}^{\lambda}$ may now be written in terms of $\tilde{\gamma}$ thus:-

$$\tilde{\Gamma}_{\mu\nu}^{\lambda} = \tilde{g}^{\lambda\kappa} \partial_{\sigma} \tilde{\gamma}_{\kappa, \mu\nu}^{\sigma}$$

This is then re-arranged as follows:-

$$\tilde{\Gamma}_{\mu\nu}^{\lambda} = \tilde{g}^{\lambda\kappa} (\partial_{\sigma} \tilde{\gamma}_{\kappa, \mu\nu}^{\sigma} + \partial_{\sigma} \hat{\gamma}_{\kappa, \mu\nu}^{\sigma}) =$$

$$\begin{aligned}
&= \tilde{g}^{\lambda\kappa} (g_{\rho\kappa} \Gamma_{\mu\nu}^{\rho} + D_{\sigma} \hat{g}_{\kappa,\mu\nu}^{\sigma} + \hat{g}_{\rho\kappa} \Gamma_{\mu\nu}^{\rho}) \\
&= \Gamma_{\mu\nu}^{\lambda} + \tilde{g}^{\lambda\kappa} D_{\sigma} \hat{g}_{\kappa,\mu\nu}^{\sigma}
\end{aligned}$$

where D_{σ} is the covariant derivative defined as a function of the background field. (8.2) now makes obvious the expansion of $\tilde{r}$ as a power series in $\hat{g}$.

Let us define the tensor

$$\bar{\Gamma}_{\mu\nu}^{\lambda} \equiv \tilde{g}^{\lambda\kappa} D_{\sigma} \hat{g}_{\kappa,\mu\nu}^{\sigma}$$

and employ (8.2) to write the Ricci tensor

$$\begin{aligned}
\tilde{R}_{\mu\kappa} &= R_{\mu\kappa} + (\partial_{\mu} \bar{\Gamma}_{\lambda\kappa}^{\lambda} - \Gamma_{\mu\kappa}^{\rho} \bar{\Gamma}_{\lambda\rho}^{\lambda}) - \\
&(\partial_{\rho} \bar{\Gamma}_{\mu\kappa}^{\rho} - \Gamma_{\rho\kappa}^{\lambda} \bar{\Gamma}_{\lambda\mu}^{\rho} - \Gamma_{\rho\mu}^{\lambda} \bar{\Gamma}_{\lambda\kappa}^{\rho} + \Gamma_{\rho\lambda}^{\rho} \bar{\Gamma}_{\mu\kappa}^{\lambda}) + \bar{\Gamma}_{\mu\lambda}^{\rho} \bar{\Gamma}_{\kappa\rho}^{\lambda} - \bar{\Gamma}_{\mu\kappa}^{\rho} \bar{\Gamma}_{\lambda\rho}^{\lambda},
\end{aligned}$$

or,

$$\tilde{R}_{\mu\kappa} = R_{\mu\kappa} + D_{\mu} \bar{\Gamma}_{\lambda\kappa}^{\lambda} - D_{\rho} \bar{\Gamma}_{\mu\kappa}^{\rho} + \bar{\Gamma}_{\mu\lambda}^{\rho} \bar{\Gamma}_{\kappa\rho}^{\lambda} - \bar{\Gamma}_{\mu\kappa}^{\rho} \bar{\Gamma}_{\lambda\rho}^{\lambda}. \quad (8.3)$$

We now have

$$S[\tilde{g}] = \int d^4x \tilde{g}^{1/2} \tilde{g}^{\mu\nu} \tilde{R}_{\mu\kappa}$$

where $\tilde{R}_{\mu\kappa}$ is given by (8.3). This expression can be further simplified by writing

$$\int d^4x \tilde{g}^{1/2} \tilde{g}^{\mu\kappa} (D_{\mu} \bar{\Gamma}_{\lambda\kappa}^{\lambda} - D_{\rho} \bar{\Gamma}_{\mu\kappa}^{\rho}) =$$

$$= \int d^4x \, g^{1/2} \mathcal{D}_\mu \left[\tilde{g}^{-1/2} \tilde{g}^{1/2} (\tilde{g}^{\mu\kappa} \bar{\Gamma}_{\lambda\kappa}^\lambda - \tilde{g}^{\rho\kappa} \bar{\Gamma}_{\rho\kappa}^\mu) \right] \\ - \int d^4x \left[\mathcal{D}_\mu (\tilde{g}^{1/2} \tilde{g}^{\mu\kappa}) \bar{\Gamma}_{\lambda\kappa}^\lambda - \mathcal{D}_\rho (\tilde{g}^{1/2} \tilde{g}^{\mu\kappa}) \bar{\Gamma}_{\mu\kappa}^\rho \right]$$

The first term on the right hand side of this expression is transformed by Stoke's Theorem to give an asymptotically vanishing surface integral. The second can be shown to equal

$$- 2 \tilde{g}^{1/2} \tilde{g}^{\mu\kappa} (\bar{\Gamma}_{\mu\lambda}^\rho \bar{\Gamma}_{\kappa\rho}^\lambda - \bar{\Gamma}_{\mu\kappa}^\rho \bar{\Gamma}_{\lambda\rho}^\lambda)$$

by making use of the relations

$$\mathcal{D}_\sigma \tilde{g}^{1/2} = \frac{1}{2} \tilde{g}^{1/2} \tilde{g}^{\mu\nu} \mathcal{D}_\sigma \hat{g}^{\mu\nu} = \tilde{g}^{1/2} \tilde{g}^{\mu\nu} \mathcal{D}_\tau \hat{g}^{\tau}_{\mu,\sigma\nu}$$

$$\mathcal{D}_\sigma \tilde{g}^{\mu\nu} = -\frac{1}{2} (\tilde{g}^{\mu\kappa} \tilde{g}^{\nu\beta} + \tilde{g}^{\mu\beta} \tilde{g}^{\nu\kappa}) \mathcal{D}_\sigma \hat{g}_{\kappa\beta} \\ = -(\tilde{g}^{\mu\kappa} \tilde{g}^{\nu\beta} + \tilde{g}^{\mu\beta} \tilde{g}^{\nu\kappa}) \mathcal{D}_\tau \hat{g}^{\tau}_{\alpha,\sigma\beta}$$

Thus we arrive at a form for $S(\tilde{g})$

$$S[\tilde{g}] = \int d^4x \left[\tilde{g}^{1/2} \tilde{g}_{\mu\nu} R^{\mu\nu} + \right. \\ \left. \tilde{g}^{1/2} \tilde{g}^{\nu\beta} (\tilde{g}^{\kappa\lambda} \tilde{g}^{\mu\alpha} - \tilde{g}^{\kappa\mu} \tilde{g}^{\lambda\alpha}) \mathcal{D}_\sigma \hat{g}^{\sigma}_{\alpha,\mu\nu} \mathcal{D}_\tau \hat{g}^{\tau}_{\beta,\kappa\lambda} \right] \quad (8.4)$$

The expansion of this is both manifestly covariant (the only derivatives that explicitly occur are covariant derivatives) and straightforward via

$$\tilde{g}^{\kappa\lambda} = g^{\kappa\lambda} - (g \cdot \hat{g} \cdot g)^{\kappa\lambda} + (g \cdot \hat{g} \cdot g \cdot \hat{g} \cdot g)^{\kappa\lambda} - \dots \text{ etc.}$$

One appreciates the comparative ease by which bare vertices are obtained from (8.4) once one has tried ones hand at functionally differentiating the original action a few times.

It is perhaps interesting to observe that this form for $S(\tilde{g})$, is essentially that obtained by applying the 'principal of equivalence' to the case where one expands about a Minkowski background metric.

i.e. one expands

$$S(\tilde{g}) = \int d\lambda \tilde{g}^{1/2} \tilde{g}^{\nu\beta} (\tilde{g}^{\kappa\lambda} \tilde{g}^{\mu\alpha} - \tilde{g}^{\kappa\mu} \tilde{g}^{\lambda\alpha}) \partial_\sigma \tilde{\gamma}^\sigma_{\alpha,\mu\nu} \partial_\tau \tilde{\gamma}^\tau_{\beta,\kappa\lambda}$$

as a power series in $\hat{g}$ where now

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \hat{g}_{\mu\nu}$$

$$\hat{g}^{\mu\nu} = \eta^{\mu\nu} - (\eta \cdot \hat{g} \cdot \eta)^{\mu\nu} + \dots \text{etc.}$$

Thus the bare covariant vertices, readily obtainable from (8.4) are as bad (or as good depending on how one looks at it) as those when one has a flat background metric as for example in the derivation of the Feynman rules for the Goldberg Lagrangian of section 2. Indeed by a change of variables (8.4) can be written in an equivalently simple form. There is certainly nothing very pleasant about them from the point of view of calculating a covariant graviton superpropogator. At least we may no longer cherish false hopes.

What then goes wrong or rather why is Honerkamp able to produce such an object for the chiral Lagrangian. There are two reasons. Firstly the form of the chiral lagrangian,

secondly, the fact that covariant derivatives of the chiral Riemann tensor vanish. These enable him to find a convenient form for the vertices. Maybe a convenient form for the gravitational vertices exist, I haven't shown that it does not just that it's rather unlikely. There remains a possibility in that one can arrange for the covariant derivative of the Riemann tensor to vanish by defining it as a function of a metric that is a metric on a symmetric space. Thus one would have to choose instead of the metric $g_{\mu\nu}$ some combination of the g 's that corresponds to a metric on a symmetric space. This isn't too difficult to arrange, the combination

$$g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta}$$

or something very like it⁽²²⁾ corresponds to just that. (The 'points' of the manifold are now the functions g). The problem arises in arranging both for this to happen and to arrange the derivative structure of the Einstein Action to correspond to the form of the chiral lagrangian

viz.
$$\int dx \, \xi_{ij} \, \partial_\mu \pi^i \partial^\mu \pi^j$$

My efforts to arrange this have not been fruitful.

9. CONCLUSION

Above all I would hope that this thesis provides a clarification of the problems encountered in treating gravitation as a spin two field theory and gives considerable insight into the more or less tractable forms that these problems can take. My intentions throughout have been to portray quantum gravity as a field theory like any other. One is then able to make a direct comparison of the import of its own individual features. Particularly useful in this respect has been the regularisation procedure employed. Although the masses introduced therein are not strictly speaking 'cut off masses' the eventual answer is revealed in the more commonplace form. The necessary counter terms to be added to the original Lagrangian are thereby made rather more transparent than would be the case had one employed dimensional regularisation directly. I shall discuss this point in a little more detail when I come to talk of the status of renormalisation theory in the light of the calculations of this thesis.

The principle inference to be drawn from this work is I think that quantum gravity differs quantitatively but not qualitatively from other better established quantum field theories. In this respect it would conform to the hope expressed in the introduction. In many ways this is disappointing. There seems little new 'physics' to be gained from this still unwieldy interpretation of general relativity. For although I have shown calculations to be practically possible they remain so only for low orders of

the perturbation expansion. The linearised field equations of classical general relativity reveal little of the structure of a complete solution to the full Einstein field equations and so it is now with perturbation theory. Unless there is some method of summing sets of diagrams to all orders there would appear little hope of achieving startling results. The superpropogator method is one such but at present this corresponds to the simplest possible inclusion of the gravitational field into the framework of quantum electrodynamics and extensions of this method prove hard to come by. There is even then a principal defect of this approach of summing particular sets of diagrams in that one has no way of knowing if the particular set chosen is of any greater or lesser importance than those discarded. General relativity is by its very definition an all embracing subject. It does not seem to be doing justice to the conceptual elegance of that theory to make scrappy attempts at evaluating arbitrary parts of it. Alas it appears to be the case that for the moment it is all that one can do. These factors should be borne in mind but they should not prevent one from discussing the quantum theory as such and that after all is the purpose of the preceeding pages.

Faced with the necessity of dealing with the formal properties of functional integrations it has been my policy to show that what at first sight appear different approaches reduce to a single consistent method for obtaining the Feynman rules. Moreover that Ward Identities derived from

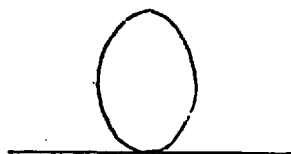
two different formulations take an identical form. Of course this is as it should be. DeWitt's manifestly covariant theory has been shown to be derivable from Fadeev and Poppov's general prescription for dealing with non-abelian gauge symmetries and one would not expect any divergence of inference once the former has been reduced by restricting the background field to be flat. However it is satisfying that this should happen with such a striking resemblance. Compare, for example, equations (3.16) and (3.9). It seems to be the case that it is very much more evident precisely what is happening when one uses the manifestly covariant theory. The role of the fictitious particle propagator in equation (3.16) is an example. There is one other remark that I would make at this stage but would be uncertain as to its significance. This has to do with the omnipresence throughout this thesis of the particular algebraic structure characterised by the function $Q^{\alpha\beta, \alpha^1\beta^1}$: It is the form of the graviton propagator. It enters into the gauge variation of the metric tensor. It is also precisely the form of the Riemannian connection. The properties of the Q 's greatly facilitate the reduction of the algebra in evaluating Feynman diagrams. I could go on for some time expounding upon the virtues of this function. One simply wonders why this should be or rather if there is some transformation to be made that would simplify matters. I do not know the answer.

I now turn to the calculations themselves exhibited in sections five and six. I have shown at least in part

that the inclusion of fictitious particle diagrams correctly produces transverse amplitudes. On balance I would say that it is easier to work with the Goldberg Lagrangian to obtain these results. As I mentioned earlier the first order formalism is of no doubt more use in discussing renormalisation. It is again pleasing that both approaches should yield the same result. I might add that it is a very testing result to reproduce. Moreover it is important in that it underlines the fact that there is no inconsistency in working with the Goldberg lagrangian and using the regularisation procedure given in section 4. This combination is a particularly happy one for ease in computations. I have not extended it to cover the case of overlapping divergencies but it is to be hoped that this will prove possible. The form of the results of expression (5.4) are very complicated and from the discussion at the end of that section it would seem that this is necessarily the case. Although it does not run to the bizarre lengths of producing numerical factors such as $2^{647}/450$; 2647 is a prime number! Nevertheless the first three terms of (5.4) are transverse. The only fly in the ointment is the quartic divergence. From dimensional considerations this term should not be there. Or rather the whole term inside square brackets is of the fourth degree composed jointly of the external momentum k and the mass parameter M^2 . No algebraic combination of the minkowski metric (there are only two permissible) can yield zero on multiplication by $k_\alpha k_{\alpha'}$. However the particular algebraic structure of the quartic

divergence appearing in (5.4) is not any combination it is of the form of the graviton propagator. Considerations that this might not be pure chance are dealt with in section 7.

Also in section 7 I discuss the regrettable tadpole diagrams. These diagrams are certainly pure renormalisation terms if they are anything. There is one point that I would add here apart from the discussion of section 7. Capper has performed a similar calculation to that of section 5⁽²³⁾ but using t'Hooft and Veltman's dimensional regularisation procedure directly. In a later paper⁽²⁴⁾ he discusses tadpole diagrams in particular. Here he describes the diagram



(9.1)

as a tadpole diagram. He has very good reason for doing this. It too is a pure renormalisation term. In his approach however, it is also ambiguous and he gives a prescription for its removal. The point I would make is that by incorporating masses as in section 4, the diagram (9.1) has a well defined value and has to be incorporated. Thus for the purposes of this thesis diagram (9.1) is not included in those diagrams that I term 'tadpole diagrams'. It is if you like the only well defined example of those that arise. Returning to my terminology: since those a priori undefined tadpoles are just renormalisation terms,

in developing any renormalisation theory I am at liberty to define them to be zero. I must then establish the consistency of this with the calculation of equation (3.2). The greater part of this has been done in proving all but the quartic divergence of (5.4) to be transverse independently of the tadpole diagrams. It remains to be shown that the quartic divergence is cancelled by the normalisation of the functional integral involved or by the alternative discussed in section 7. As yet this matter remains unresolved.

Finally there are one or two additional remarks to be made beyond those at the end of section 5 concerning the possible renormalisation of quantum gravity. Both DeWitt⁽²⁰⁾ and Honerkamp⁽¹¹⁾ argue that there should be added to the original Einstein action counter terms proportional to the square of the Riemann tensor. Moreover that this addition should be associated with the logarithmic divergence. One thereby obtains counter terms involving four derivatives that do not correspond to the form of the original. From the results (5.4) it is evident that there is nothing very remarkable in these terms being associated with the logarithmic divergence. On dimensional grounds one cannot have it otherwise. It would be remarkable were one to show that necessary addition of four derivative counter terms corresponding to those associated with the logarithmic divergence of expression (5.4) had something to do with the square of the Riemann tensor.

The other matter has to do with the evaluation of

the 'non-causal chains' (DeWitts phrase) of diagrams, (7.9), that appear necessary for the cancellation of the quartic divergence. DeWitt draws attention to just these sets of diagrams⁽¹⁰⁾. In particular he expresses the hope that they can be ignored. Maybe eventually they can but superficially at least it is a conclusion of this thesis that it is far from obvious that they are irrelevant.

This effectively concludes all that I have to say on this subject for the moment. Evidently there are many questions left unanswered. However, I think it is the case that within the limitations imposed by the vast scope of the subject my original objectives cited in the introduction have been achieved with a considerable degree of success.

APPENDIX I - Feynman Rules.

(1) Second Order Formalism

(1.1) Free Graviton Propogator.

$$\langle 0 | T \hat{h}^{\alpha_1 \beta_1} \hat{h}^{\alpha_2 \beta_2} | 0 \rangle_{(\alpha=0)} = \frac{2}{k^2} Q^{\alpha_1 \beta_1, \mu \nu} \left(\eta^{\mu \alpha_2} \eta^{\nu \beta_2} - 2 \frac{k^\mu k^\nu}{k^2} Q^{\sigma \nu, \alpha_2 \beta_2} \right).$$

$$\langle 0 | T \hat{h}^{\alpha_1 \beta_1} \hat{h}^{\alpha_2 \beta_2} | 0 \rangle_{(\alpha=-1)} = \frac{2}{k^2} Q^{\alpha_1 \beta_1, \alpha_2 \beta_2}.$$

Both of these expressions are evaluated in what is essentially the Dedonder gauge. $\alpha=0$ corresponds to no term being added to the original free Lagrangian. ($\alpha=-1$) indicates both that such a term is added and gives the coefficient of that term.

(1.2) Bare Vertex Functions.

$$\begin{aligned} {}^3\Gamma_{\alpha_1 \beta_1, \alpha_2 \beta_2, \alpha_3 \beta_3}(k_1, k_2, k_3) &= \frac{\kappa}{8} S_{\alpha_1 \beta_1} S_{\alpha_2 \beta_2} S_{\alpha_3 \beta_3} P_3! (-k_{2\rho} k_{3\sigma}) \times \\ &\times \delta(k_1 + k_2 + k_3) \left(2 \delta_{\alpha_1}^\rho \delta_{\beta_1}^\sigma Q_{\alpha_2 \beta_2, \alpha_3 \beta_3} + \right. \\ &\left. - 4 \eta^{\rho\sigma} \eta_{\alpha_1 \alpha_2} Q_{\beta_1 \beta_2, \alpha_3 \beta_3} + 4 \delta_{\alpha_2}^\sigma \delta_{\alpha_3}^\rho \eta_{\alpha_1 \beta_2} \eta_{\beta_1 \beta_3} \right) \end{aligned}$$

$$\begin{aligned} {}^4\Gamma_{\alpha_1 \beta_1, \alpha_2 \beta_2, \alpha_3 \beta_3, \alpha_4 \beta_4}(k_1, k_2, k_3, k_4) &= \frac{\kappa^2}{8} S_{\alpha_1 \beta_1} \dots S_{\alpha_4 \beta_4} P_4! \delta(k_1 + k_2 + k_3 + k_4) \\ &\times (-k_{3\rho} k_{4\sigma}) \cdot \left(4 (\eta^{\rho\sigma} \eta_{\alpha_1 \alpha_3} - \delta_{\alpha_1}^\rho \delta_{\beta_1}^\sigma \eta_{\alpha_2 \alpha_3}) Q_{\beta_2 \beta_3, \alpha_4 \beta_4} \right. \\ &\left. + \dots \right) \end{aligned}$$

$${}^4P \text{ (cont.)} \quad \dots + \eta^{\rho\sigma} \eta_{\alpha_1 \alpha_3} \eta_{\beta_2 \beta_4} (2 \eta_{\alpha_2 \beta_3} \eta_{\beta_1 \alpha_4} - \eta_{\beta_1 \beta_3} \eta_{\alpha_2 \alpha_4}) \\ - \delta_{\alpha_3}^{\sigma} \delta_{\alpha_4}^{\rho} \eta_{\alpha_1 \beta_3} \eta_{\alpha_2 \beta_4} \eta_{\beta_1 \beta_2}) .$$

(1.3) Fictitious Particle Propogator.

$$\langle 0 | T (C^{\kappa} \mathcal{J}^{\lambda}) | 0 \rangle = \frac{1}{k^2} \eta^{\kappa\lambda}$$

(1.4) Fictitious Particle Vertex.

$$V(\overset{\hat{h}}{\underset{\alpha_1 \beta_1}{k_1}}, \underset{\alpha_2}{k_2}, \underset{\alpha_3}{k_3}) = \kappa \delta_{\alpha_1 \beta_1} \delta(k_1 + k_2 + k_3) \times \\ (\eta_{\alpha_2 \alpha_3} k_{2\alpha_1} k_{3\beta_1} - \eta_{\alpha_3 \alpha_1} k_{3\alpha_2} k_{1\beta_1}) .$$

A factor of (-1) is to be included for each closed loop of fictitious particles.

(2) First Order Formalism

(2.1) Free Propogators. (the diagrammatic representations are included)

$$\langle 0 | T \overset{\hat{h}}{\underset{\alpha_1 \beta_1}{h}} \overset{\hat{h}}{\underset{\alpha_2 \beta_2}{h}} | 0 \rangle = \frac{2}{k^2} Q^{\alpha_1 \beta_1, \alpha_2 \beta_2} \\ (\text{---})$$

$$\langle 0 | T \overset{\hat{h}}{\underset{\alpha_1 \beta_1}{h}} \overset{\gamma_2}{\underset{\alpha_2 \beta_2}{r}} | 0 \rangle = -\frac{2i}{k^2} k_{\sigma} \overset{\hat{Q}}{Q}^{\sigma \gamma_2, \alpha_2 \beta_2} \delta^{\alpha_1 \beta_1} \\ (\text{---}\rightarrow\text{---})$$

(arrow indicates that the line begins in $\hat{h}$ and ends in r)

$$\begin{aligned}
 \langle 0 | T (\Gamma_{\alpha_1 \beta_1}^{\gamma_1} \Gamma_{\alpha_2 \beta_2}^{\gamma_2}) | 0 \rangle &= \frac{2}{k^2} k_\sigma k_\tau \hat{Q}_{\alpha_1 \beta_1}^{\sigma \gamma_1} \circ \underline{Q} \circ \hat{Q}_{\alpha_2 \beta_2}^{\tau \gamma_2} \\
 (++++++) &- \hat{Q}_{\alpha_1 \beta_1}^{\sigma \gamma_1} \circ \left(\frac{2}{3} \delta_{\alpha_2 \beta_2}^{\rho \gamma_2} \underline{Q}_{\rho \sigma} - \delta_{\sigma}^{\gamma_2} \underline{Q}_{\alpha_2 \beta_2} \right)
 \end{aligned}$$

(2.2) Bare Vertex Function.

$$\begin{aligned}
 W_{\alpha_1 \beta_1, \gamma_2, \gamma_3}^{\alpha_2 \beta_2, \alpha_3 \beta_3} (k_1, k_2, k_3) &= (-\kappa) S_{\alpha_1 \beta_1} S_{\alpha_2 \beta_2} S_{\alpha_3 \beta_3} \delta(k_1 + k_2 + k_3) \times \\
 &\times \left(2 \delta_{\alpha_1}^{\alpha_2} \delta_{\gamma_3}^{\beta_2} \delta_{\beta_1}^{\alpha_3} \delta_{\gamma_2}^{\beta_3} - \delta_{\gamma_2}^{\alpha_2} \delta_{\gamma_3}^{\beta_2} \delta_{\alpha_1}^{\alpha_3} \delta_{\beta_1}^{\beta_3} - \delta_{\alpha_1}^{\alpha_2} \delta_{\beta_1}^{\beta_2} \delta_{\gamma_2}^{\alpha_3} \delta_{\gamma_3}^{\beta_3} \right)
 \end{aligned}$$

(2.3) Fictitious Particle Structure.

This is exactly the same as far (1.3) and (1.4) given above.

In all of the above rules there is to be included the usual factors of $1/(2\pi)^4 i$ for each propagator and $(2\pi)^4 i$ for each Vertex.

APPENDIX II - Derivation of the Feynman Rules for the first order formalism.

This appendix is included partly in its own right and partly to serve as an illustration of the algebraic conventions involving the $\hat{Q}$'s.

From the gauge dependent Lagrangian, which we shall write for the moment as,

$$\mathcal{L} = \frac{1}{k^2} \left(\hat{g}^{\mu\nu} R_{\mu\nu} - \frac{1}{2\alpha} \hat{h}^{\mu\sigma}{}_{,\mu} \eta_{\sigma\tau} \hat{h}^{\nu\tau}{}_{,\nu} \right), \quad (\text{II.1})$$

We separate the free and interaction parts, given respectively by,

$$\mathcal{L}_0 = \eta^{\mu\nu} (\Gamma_{\mu\sigma}^{\rho} \Gamma_{\nu\rho}^{\sigma} - \Gamma_{\rho\sigma}^{\sigma} \Gamma_{\mu\nu}^{\rho}) + \hat{h}^{\mu\nu} (\Gamma_{\mu\rho,\nu}^{\rho} - \Gamma_{\mu\nu,\rho}^{\rho}) - \frac{1}{2\alpha} \hat{h}^{\mu\sigma}{}_{,\mu} \eta_{\sigma\tau} \hat{h}^{\nu\tau}{}_{,\nu}$$

$$\mathcal{L}_{\text{int}} = \kappa \hat{h}^{\mu\nu} (\Gamma_{\mu\sigma}^{\rho} \Gamma_{\nu\rho}^{\sigma} - \Gamma_{\rho\sigma}^{\sigma} \Gamma_{\mu\nu}^{\rho})$$

where we shall choose the numerical factor, $(\frac{1}{\alpha})$, so as to give the simplest possible propagators.

If we next add to (II.1) the source terms, $\bar{J}_{\mu\nu} \hat{h}^{\mu\nu}$ and $K^{\alpha\beta}_{\gamma} \Gamma^{\gamma}_{\alpha\beta}$ the free propagators are then defined as follows:-

$$\langle 0 | \hat{h}^{\alpha_1\beta_1}(x_1) \hat{h}^{\alpha_2\beta_2}(x_2) | 0 \rangle = \frac{\delta}{i \delta \bar{J}_{\alpha_1\beta_1}(x_1)} \frac{\delta}{i \delta \bar{J}_{\alpha_2\beta_2}(x_2)} \int (d\hat{h}) (d\Gamma) \exp i (\mathcal{L}_0 + \bar{J} \cdot \hat{h} + K \cdot \Gamma) \Big|_{\bar{J}=K=0}$$

The $\langle h, \Gamma \rangle$ and $\langle \Gamma, \Gamma \rangle$ propagators are similarly defined.

Thus we may write:

$$\langle 0 | \hat{h}^{\alpha_1\beta_1}(x_1) \hat{h}^{\alpha_2\beta_2}(x_2) | 0 \rangle = \frac{\delta}{i \delta \bar{J}_{\alpha_2\beta_2}(x_2)} \langle 0 | \hat{h}^{\alpha_1\beta_1}(x_1) | 0 \rangle$$

$\langle 0 | \hat{h}^{\alpha_1 \beta_1}_{(x_1)} | 0 \rangle$ is evaluated from the conditions

$$\int (d\hat{h})(d\Gamma) \frac{\delta}{\delta \hat{h}^{\mu\nu}} \exp i \left(\int dx (\mathcal{L}_0 + \bar{U} \cdot \hat{h} + K \cdot \Gamma) \right) = 0 \quad (II.2)$$

$$\int (d\hat{h})(d\Gamma) \frac{\delta}{\delta \Gamma^{\alpha\beta}_\gamma} \exp i \left(\int dx (\mathcal{L}_0 + \bar{U} \cdot \hat{h} + K \cdot \Gamma) \right) = 0 \quad (II.3)$$

To simplify matters we shall from now on write

$$\langle 0 | \hat{h}^{\alpha_1 \beta_1}_{(x_1)} | 0 \rangle = \hat{h}^{\alpha_1 \beta_1}_{(x_1)}, \quad \langle 0 | \Gamma^{\gamma_1}_{\alpha_1 \beta_1}(x_1) | 0 \rangle = \Gamma^{\gamma_1}_{\alpha_1 \beta_1}(x_1).$$

(II.2) and (II.3) then become:

$$\begin{aligned} \Gamma^{\sigma}_{\mu\sigma,\nu} + \Gamma^{\sigma}_{\nu\sigma,\mu} - 2\Gamma^{\sigma}_{\mu\nu,\sigma} + \frac{1}{\alpha} \left(\hat{h}^{\sigma}_{\nu,\mu} + \hat{h}^{\sigma}_{\mu,\nu} \right)_{,\sigma} \\ = -2\bar{U}_{\mu\nu} \end{aligned} \quad (II.4)$$

$$\begin{aligned} \eta^{\kappa\beta} \Gamma^{\alpha}_{\kappa\gamma} + \eta^{\kappa\alpha} \Gamma^{\beta}_{\kappa\gamma} - \eta^{\alpha\beta} \Gamma^{\sigma}_{\sigma\gamma} - \delta^{\alpha\beta}_{\sigma\gamma} \Gamma^{\sigma}_{\kappa\lambda} \eta^{\kappa\lambda} + \\ + \hat{h}^{\alpha\beta}_{,\gamma} - \delta^{\alpha\beta}_{\tau\gamma} \hat{h}^{\sigma\tau}_{,\sigma} = -K^{\alpha\beta}_{,\gamma}. \end{aligned} \quad (II.5)$$

(II.5) may be simplified by first lowering α & β then taking the sum; $(\gamma \leftrightarrow \alpha)II.5 + (\gamma \leftrightarrow \beta)II.5 - II.5$. This then yields the equation:-

$$\begin{aligned} 2Q_{\alpha\beta}{}^{\kappa\lambda} \Gamma_{(\kappa\lambda)\gamma} - 2Q_{\alpha\beta}{}^{\kappa\gamma} \Gamma^{\sigma}_{\sigma\kappa} = \eta_{\alpha\beta} \partial_{\sigma} \hat{h}^{\sigma}_{\gamma} \\ - (\partial_{\alpha} \hat{h}_{\beta\gamma} + \partial_{\beta} \hat{h}_{\alpha\gamma} - \partial_{\gamma} \hat{h}_{\alpha\beta}) - (K_{(\alpha\gamma)\beta} + K_{(\beta\gamma)\alpha} - K_{(\alpha\beta)\gamma}) \end{aligned}$$

Next, multiply through by $Q_{\mu\nu}{}^{\alpha\beta}$ and raise γ to get:

$$\begin{aligned} 2(\Gamma^{\gamma}_{\mu\nu} - \delta^{\kappa\gamma}_{\mu\nu} \Gamma^{\sigma}_{\sigma\kappa}) = -(\partial_{\mu} \hat{h}^{\gamma}_{\nu} + \partial_{\nu} \hat{h}^{\gamma}_{\mu} - \partial^{\gamma} \hat{h}_{\mu\nu}) - \frac{1}{2} \eta_{\mu\nu} \partial^{\gamma} \hat{h}^{\alpha}_{\alpha} \\ - Q_{\mu\nu}{}^{\alpha\beta} (K^{(\gamma)}_{\alpha}\beta + K^{(\gamma)}_{\beta}\alpha - K_{\alpha\beta}\gamma). \end{aligned}$$

We can then remove the $\Gamma^{\sigma}_{\sigma\kappa}$ term by multiplying by δ^{ν}_{γ} ,

to achieve the final expression:-

$$\Gamma_{\mu\nu}^{\gamma} = \frac{1}{2} \partial_{\sigma} Q^{\sigma\gamma}_{\mu\nu} \hat{h}_{\alpha}^{\alpha} - \frac{1}{2} (\partial_{\mu} \hat{h}_{\nu}^{\gamma} + \partial_{\nu} \hat{h}_{\mu}^{\gamma} - \partial^{\gamma} \hat{h}_{\mu\nu}) \\ + \left(\frac{1}{3} \delta_{\mu\nu}^{\rho\gamma} Q_{\rho\sigma}^{\alpha\beta} - \frac{1}{2} \delta_{\sigma}^{\gamma} Q_{\mu\nu}^{\alpha\beta} \right) (K_{\alpha}^{\sigma}{}_{\beta} + K_{\beta}^{\sigma}{}_{\alpha} - K_{\alpha\beta}{}^{\sigma}) \quad (\text{II.6})$$

At this stage we shall introduce the additional notation of the $\hat{Q}$'s.

We define the product $\hat{Q}^{\alpha\beta}_{\kappa\lambda} \circ \underline{A}$, where A is a symmetric two index object, as follows:-

$$\hat{Q}^{\alpha\beta}_{\kappa\lambda} \circ \underline{A} \equiv \frac{1}{2} (\delta_{\kappa}^{\alpha} A_{\lambda}^{\beta} + \delta_{\lambda}^{\alpha} A_{\kappa}^{\beta} - \eta^{\alpha\beta} A_{\kappa\lambda})$$

One could, of course, not work with the $\hat{Q}$'s but instead define $q^{\alpha\beta}_{\kappa\lambda, \sigma\tau}$,

$$q^{\alpha\beta}_{\kappa\lambda, \sigma\tau} \equiv \frac{1}{2} (\delta_{\kappa}^{\alpha} \delta^{\sigma\tau, \beta}_{\lambda} + \delta_{\lambda}^{\alpha} \delta^{\sigma\tau, \beta}_{\kappa} - \delta^{\alpha\beta} \delta^{\sigma\tau}_{\kappa\lambda})$$

In which case,

$$\hat{Q}^{\alpha\beta}_{\kappa\lambda} \circ \underline{A} \equiv q^{\alpha\beta}_{\kappa\lambda, \sigma\tau} A_{\sigma\tau}$$

The above simply avoids the extra indices.

One should note that $\hat{Q}^{\alpha\beta}_{\kappa\lambda}$, unlike $Q^{\alpha\beta}_{\kappa\lambda}$, is not symmetric in both sets of indices. The bar under α is simply to emphasise this fact. Note also the relations

$$\hat{Q}^{\alpha\beta}_{\kappa\lambda} \circ \underline{\eta} = Q^{\alpha\beta}_{\kappa\lambda}$$

$$\hat{Q}^{\alpha\beta}_{\kappa\lambda} \circ \underline{\delta}_{\mu\nu} = \hat{Q}^{\beta\alpha}_{\mu\nu} \circ \underline{\delta}_{\kappa\lambda}$$

If we now employ this notation, (II.6) is rewritten:-

$$\begin{aligned} \Gamma_{\mu\nu}^{\gamma} &= \frac{1}{2} \partial_{\sigma} \hat{Q}^{\sigma\gamma}_{\mu\nu} \circ \eta \cdot \eta_{\alpha\beta} \hat{h}^{\alpha\beta} - \partial_{\sigma} \hat{Q}^{\sigma\gamma}_{\mu\nu} \circ \delta_{\alpha\beta} \hat{h}^{\alpha\beta} \\ &+ 2 \left(\frac{1}{3} \delta_{\mu\nu}^{\rho\sigma} Q_{\rho\sigma}^{\alpha\beta} - \frac{1}{2} \delta_{\sigma}^{\gamma} Q_{\mu\nu}^{\alpha\beta} \right) \hat{Q}^{\sigma\omega}_{\kappa\lambda} \circ \delta_{\alpha\beta} K^{\kappa\lambda}_{,\omega} \end{aligned}$$

or equivalently

$$\begin{aligned} \Gamma_{\mu\nu}^{\gamma} &= - \partial_{\sigma} \hat{Q}^{\sigma\gamma}_{\mu\nu} \circ Q_{\alpha\beta} \hat{h}^{\alpha\beta} + \\ &+ 2 \hat{Q}^{\sigma\omega}_{,\kappa\lambda} \circ \left(\frac{1}{3} \delta_{\mu\nu}^{\rho\sigma} Q_{\rho\sigma} - \frac{1}{2} \delta_{\sigma}^{\gamma} Q_{\mu\nu} \right) K^{\kappa\lambda}_{,\omega} \quad (II.7) \end{aligned}$$

If we now substitute for Γ in equation II.4 we obtain:-

$$\begin{aligned} &\frac{1}{2} \eta_{\mu\nu} \partial^2 \hat{h}^{\alpha}_{\alpha} - \partial^2 \hat{h}_{\mu\nu} + 2 \partial_{\sigma} \hat{Q}^{\sigma\omega}_{\kappa\lambda} \circ Q_{\mu\nu} K^{\kappa\lambda}_{,\omega} \\ &= - \left(\frac{1}{\alpha+1} \right) \left(\hat{h}^{\sigma}_{\nu,\mu\sigma} + \hat{h}^{\sigma}_{\mu,\nu\sigma} \right) - 2 \bar{J}_{\mu\nu} \quad (II.8) \end{aligned}$$

We can now choose $\alpha = -1$ and thereby eliminate the first term on the right hand side of (II.8). This choice then gives precisely the same subtraction term for the Lagrangian (II.1) as was the case in the derivation of the Feynman rules for the second order formalism given in Section 2. This is of course to be expected. However, it is not so immediately apparent which terms are eliminated from the original Lagrangian when one does not solve for the Γ 's in terms of the g 's.

Finally, with this choice for α , we achieve the desired form for equation II.8 viz:-

$$\partial^2 \hat{h}_{\alpha\beta} = 2 Q_{\alpha\beta}^{\mu\nu} \bar{J}_{\mu\nu} + 2 \partial_{\sigma} \hat{Q}^{\sigma\omega}_{\kappa\lambda} \circ \delta_{\alpha\beta} K^{\kappa\lambda}_{,\omega} \quad (II.9)$$

Substituting this expression for $\hat{h}$ in equation II.8 we obtain:-

$$\begin{aligned} \Gamma_{\mu\nu}^{\gamma} = & -\frac{2\partial_{\sigma}}{\partial^2} \hat{Q}_{\mu\nu}^{\sigma\gamma} \circ \delta^{\alpha\beta} \bar{J}_{\alpha\beta} - \frac{2\partial_{\sigma}\partial_{\tau}}{\partial^2} \hat{Q}_{\mu\nu}^{\sigma\gamma} \circ \underline{Q} \circ \hat{Q}_{\kappa\lambda}^{\tau\omega} K^{\kappa\lambda}_{\omega} \\ & + 2 \hat{Q}_{\kappa\lambda}^{\sigma\omega} \circ \left(\frac{1}{3} \delta_{\mu\nu}^{\rho\gamma} Q_{\rho\sigma} - \frac{1}{2} \delta_{\sigma}^{\gamma} Q_{\mu\nu} \right) K^{\kappa\lambda}_{\omega} \end{aligned} \quad (\text{II.10})$$

The propagators, in x space, are determined from (II.9) and (II.10) as mentioned earlier viz.

$$\langle 0 | T \hat{h}^{\alpha_1\beta_1} \hat{h}^{\alpha_2\beta_2} | 0 \rangle = \frac{1}{i} \frac{\delta}{\delta \bar{J}_{\alpha_2\beta_2}} \langle \hat{h}^{\alpha_1\beta_1}(x_1) \rangle$$

$$\langle 0 | T \hat{h}^{\alpha_1\beta_1} \Gamma_{\alpha_2\beta_2}^{\gamma_2} | 0 \rangle = \frac{1}{i} \frac{\delta}{\delta K_{\alpha_2\beta_2}^{\gamma_2}(x_2)} \langle \hat{h}^{\alpha_1\beta_1}(x_1) \rangle$$

etc.

The corresponding momentum - space expressions are as given in Appendix I.

APPENDIX III - Integral Formulae⁽¹⁸⁾.

$$\int d^n p \frac{1}{(p^2 + 2k \cdot p + M^2)^2} = \frac{i \pi^{n/2} \Gamma(2 - \frac{1}{2}n)}{(M^2 - k^2)^{2 - \frac{1}{2}n}} \equiv C$$

$$\int d^n p \frac{p_\mu}{(p^2 + 2k \cdot p + M^2)^2} = C(-k_\mu)$$

$$\int d^n p \frac{p_\mu p_\nu}{(p^2 + 2k \cdot p + M^2)^2} = C \left(k_\mu k_\nu - \frac{\eta_{\mu\nu}(M^2 - k^2)}{(n-2)} \right)$$

$$\int d^n p \frac{p_\mu p_\nu p_\lambda}{(p^2 + 2k \cdot p + M^2)^2} = C \left(-k_\mu k_\nu k_\lambda + \right. \\ \left. + (\eta_{\mu\nu} k_\lambda + \eta_{\mu\lambda} k_\nu + \eta_{\nu\lambda} k_\mu) \frac{(M^2 - k^2)}{(n-2)} \right)$$

$$\int d^n p \frac{p_\mu p_\nu p_\kappa p_\lambda}{(p^2 + 2k \cdot p + M^2)^2} = C \left(k_\mu k_\nu k_\kappa k_\lambda - \right. \\ \left. - (\eta_{\mu\nu} k_\kappa k_\lambda + \eta_{\mu\lambda} k_\kappa k_\nu + \eta_{\nu\lambda} k_\kappa k_\mu + \right. \\ \left. + \eta_{\mu\kappa} k_\lambda k_\nu + \eta_{\nu\kappa} k_\lambda k_\mu + \eta_{\kappa\lambda} k_\mu k_\nu) \frac{(M^2 - k^2)}{(n-2)} \right. \\ \left. + (\eta_{\mu\nu} \eta_{\kappa\lambda} + \eta_{\mu\kappa} \eta_{\nu\lambda} + \eta_{\nu\kappa} \eta_{\mu\lambda}) \frac{(M^2 - k^2)^2}{n(n-2)} \right).$$

$$\int d^n p \frac{1}{(p^2 + M^2)} = i \pi^{n/2} \Gamma(1 - n/2) \cdot M^{2(n/2 - 1)}$$

$$\int d^n p \frac{p_\mu p_\nu}{(p^2 + M^2)} = i \pi^{n/2} \Gamma(-n/2) \cdot \left(\frac{1}{2} g_{\mu\nu} M^2\right) \cdot M^{2(n/2 - 1)}$$

Integrals of for example $(p^{(n)})^2 p_\kappa$ are obtained from the above by contracting $p_\mu p_\nu$ with $\eta^{\mu\nu}$, where $\delta_\mu^\mu = n$.

The Feynman parameter method corresponds to the replacement

$$(M^2 - k^2) \rightarrow (M^2 + k^2 x(1-x))$$

$$k_\mu \rightarrow x k_\mu$$

in the above formulae, and performing the integration $\int_0^1 dx$. For the purposes of the Feynman parameter integrations the following proves useful:

Defining
$$I_n = \int_0^1 dx x^n \log x(1-x)$$

then
$$I_n = \left(\frac{n}{n+1}\right) I_{n-1} - \frac{(n-1)}{n(n+1)^2} .$$

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