Bifurcations of Set-Valued Dynamical Systems

a thesis presented for the degree of

Doctor of Philosophy of Imperial College London

and the

Diploma of Imperial College

by

Alexander Athorne

Department of

Imperial College

180 Queen's Gate, London SW7 2BZ

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Signed: _____

Copyright

The copyright of this thesis rests with the author and is made available under a Creative Commons Attribution Non-Commercial No Derivatives licence. Researchers are free to copy, distribute or transmit the thesis on the condition that they attribute it, that they do not use it for commercial purposes and that they do not alter, transform or build upon it. For any reuse or redistribution, researchers must make clear to others the licence terms of this work.

Thesis advisors: Dr Martin Rasmussen and Professor Jeroen Lamb

Bifurcations of Set-Valued Dynamical Systems

Abstract

We study families of set-valued dynamical systems and show how minimal invariant sets depend on parameters. We give a variant on the definition of attractor repeller pairs and obtain a different version of the Conley decomposition theorem. Under mild conditions on these systems we show that minimal invariant sets are related to a variation on the definition of chain components we call orbitally connected sets. We show for such systems that bifurcations can occur as a result of two orbitally connected sets colliding.

To my family and friends for all their encouragement and support, especially Hattie, Will and Hasan without whom I would probabaly have gone homeless.

Acknowledgments

The completion of this study would not have been even remotely possible without my supervisors Dr. Martin Rasmussen and Prof. Jeroen Lamb. Their persistent belief in me, their patience and their guidance was the foundation that made this endeavor feasible.

Glossary Of Terms

X	Compact subset of $\mathbb{R}^2$
$\mathcal{K}(X)$	Space of compact subsets of X
h	Hausdorff distance
$A^o, \overline{A}$	Interior and closure of a set $A \in \mathcal{K}(X)$
f	Relation, set-valued dynamical system or mapping on $X \times X$
ϵ	Size of ball each set $f(x)$ for all $x \in X$ at least contains
$\Subset, \Supset$	$A \Subset B$ if $\overline{A} \subset B^o$
$P_n(f)$	n^{th} order periodic set for a relation f
$P(f)$	$\cup_{n=1}^{\infty} P_n(f)$
$R(F)$	Recurrent set for a relation f
$\sim$	Orbitally connected equivalence relation
$\approx$	Chain connected equivalence relation
$[x]$	Equivalence class of x under $\sim$
$[[x]]$	Equivalence class of x under $\approx$

Contents

1 Introduction **1**

1.1 Set-valued Dynamical Systems, Minimal invariant sets and Bifurcations 1

1.2 Results 11

2 Setup **17**

2.1 Set-valued Dynamical Systems and Minimal Invariant Sets 17

2.1.1 The Phase Space 17

2.1.2 Relations 18

2.1.3 The Interior of f 22

2.1.4 Minimal Invariant Sets 25

2.1.5 Composition of Relations 26

2.2 The Inverse and Dual Map 27

3 The Conley Decomposition Theorems **31**

3.1 Limit Relations 32

3.1.1 Properties 34

3.1.2 Periodic and Recurrent Sets 38

3.2 Inner Attractor Repellers Pairs and Minimal Invariant Sets 40

3.3 The Open Conley Theorem 42

3.4 Open Conley Theorem for Relations Satisfying Additional Hypotheses 43

3.5 Examples 44

4 Orbital Connectivity and Minimal Invariance **46**

4.1 Periodicity 47

4.2 Orbital Connectivity and Chain Connectivity 49
4.2.1 Invariance Properties of Orbitally Connected Sets 54
4.3 Relation of Periodic Sets to Minimal Invariant Sets 56
4.3.1 Numerical Examples of Minimal Invariant Sets and Saddle Type Sets 59

5 Topological Bifurcations **62**
5.1 Bifurcations of Minimal Invariant Sets 63
5.1.1 Hausdorff Continuity 63
5.1.2 Types of Bifurcation 67
5.1.3 Continuity Properties of the Orbitally Connected and Chain Connected sets 73
5.2 Conditions for Bifurcation 78
5.2.1 Hausdorff Stability 80
5.2.2 Topological Stability 82
5.3 Examples 84
5.4 Bifurcation set 89

Bibliography **96**

A Numerical Method **97**

1

Introduction

1.1 Set-valued Dynamical Systems, Minimal invariant sets and Bifurcations

Conventionally, a dynamical system is described in two parts. The phase space of a system tells us all the possible states it may adopt and the dynamic tells us how those states transition between each other under the progression of time. Originally such systems evolved to study changes in position of physical objects such as planets or projectiles. In thus a case the phase space would usually be given as three dimensional euclidean space and the dynamical rule would tell us which point y in the phase space an object would move to in one time step if it starts at a point x in the phase space.

As a simple example we may choose the phase space to be the set of numbers between -1 and 1. Lets assume the dynamic is a rule that says given any number in the phase space we divide it by 2. This tells us for instance that the point $1/2$ goes to the point $1/4$ and the point $1/4$ goes to the point $1/8$ and so on. We call an orbit a collection of points in the phase space that is generated by repeatedly applying this dynamical rule to an initial point. Suppose we start with $\frac{1}{2}$ then we generate the orbit $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...\}$. We can write such a dynamical system more rigorously

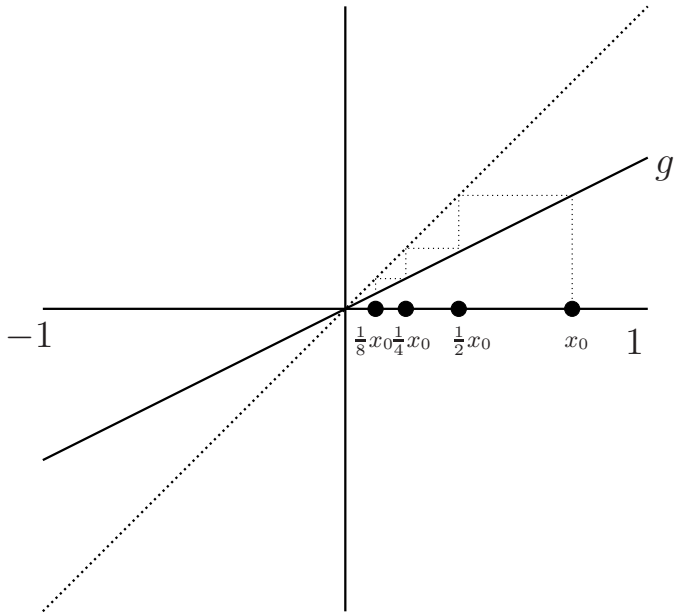


Figure 1.1: A simple example of a dynamical system g and the first couple of terms in an orbit for g starting at x_0

by saying the phase space is the set $X = [-1, 1]$ and the dynamic is given by a function $g : X \mapsto X$ such that $g(x) = \frac{1}{2}x$. Given any initial point x_0 its orbit in the space is the sequence $\{g^n(x_0)\}_{n=1}^{\infty}$ where $g^n(x_0)$ is the result of applying the rule g to the point x_0 n times. Due to the simplicity of the map we see that $g(x_0) = \frac{1}{2}x_0$, $g^2(x_0) = \frac{1}{4}x_0$ and in general $g^n(x_0) = \frac{1}{2}^n x_0$.

Assume that $x_0 = 0$ then we see that $g^n(x_0) = x_0$ for all $n \in \mathbb{N}$ we call such a point an invariant or fixed point. If x_0 is not equal to zero then as $\frac{1}{2} < 1$ the image of the point x_0 is mapped closer and closer to 0 under repeated application of g. We call 0 an attracting fixed point as any initial value in the space X is mapped towards it and in the limit onto it. Such a dynamical system is know as a single-valued dynamical system as the map g maps between single points in X.

Consider now that when applying g to a point x we also choose a random number ξ within some small distance ϵ of 0 and add it to the image of x to obtain $g(x) + \xi$. Suppose given an initial point x_0 as we evaluate the orbit of x_0 we also perturb after each application of g by a new choice of a bounded random number $\xi \in [-\epsilon, \epsilon]$. Now

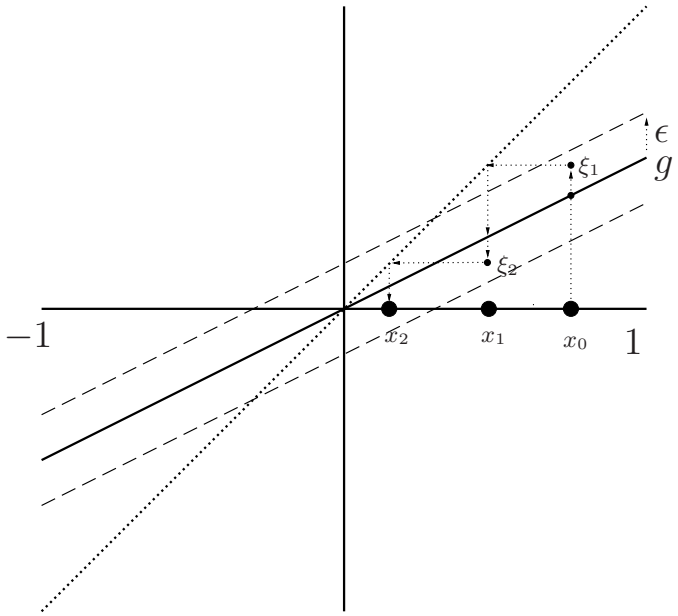


Figure 1.2: g with random perturbations bounded by ϵ

the orbit becomes

$$\{x_0, g(x_0) + \xi_1, g\left(g(x_0) + \xi_1\right) + \xi_2, ...\}$$

Where ξ_i is the random number we obtain on the i^{th} iteration of the map g. It no longer really makes sense to say that 0 is an invariant point of the mapping as now if we take zero as our initial point and draw a random value ξ_1 that not equal to 0 then we see that $g(0) + \xi_1 = \xi_1 \neq 0$ and we have jumped off the initial value 0. In fact it's clearly no longer the case that any point in X can be considered fixed as we can always choose a perturbation value ξ under which that point is no longer mapped to itself.

In order to reintroduce the notion of a fixed point into our discussion of a system like this we need to consider the set of all possible perturbations of g. Thus we introduce a new mapping f which instead of acting on points $x \in X$ acts on subsets of X. Given any $A \subset X$ we consider the rule $f(A) = \bigcup_{x \in A} \bigcup_{\xi \in [-\epsilon, \epsilon]} g(x) + \xi$. The idea here is to consider all the possible orbits that g and its perturbations can generate and then say something about there behaviour. If we choose a set A and consider $f(A)$ then in terms of g and its random perturbations we know that for any $x \in A$

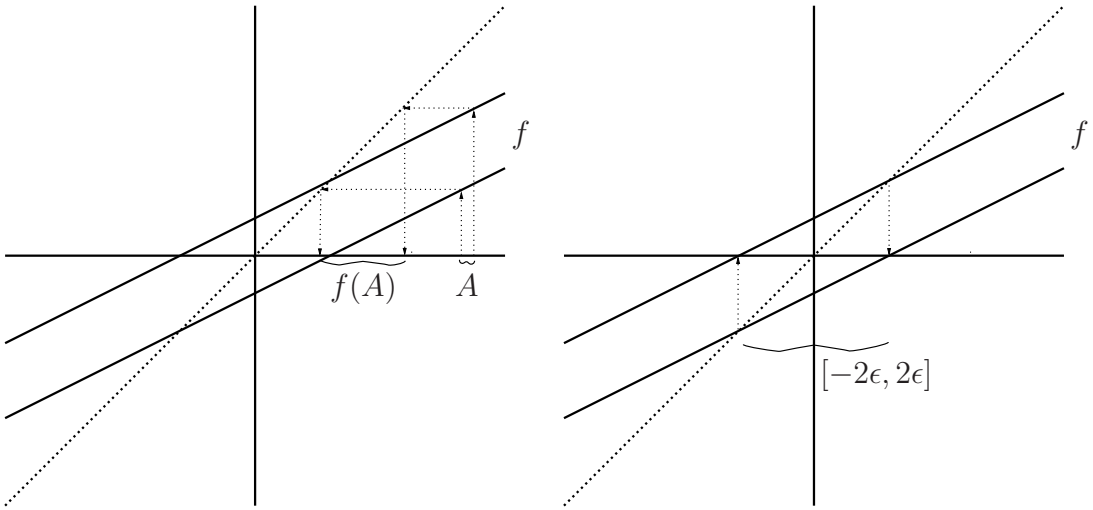


Figure 1.3: Set-valued dynamical system f bounding g and its random perturbations

and $\xi \in [-\epsilon, \epsilon]$ we have $g(x) + \xi \in f(A)$. In this way we can think of f as a map that bounds g and its random perturbations. We call the mapping f a set-valued dynamical system and it is these types of systems that interest us within this work.

The first thing to note about this new map f is that now instead of 0 being a fixed point the set $[-2\epsilon, 2\epsilon]$ is fixed under the action of f. What is more if we start with any subset A of X and map it forward under f we see that it will get closer and closer to the set $[-2\epsilon, 2\epsilon]$. This tells us that the orbits of g and their random perturbations will under repeated iteration move towards the set $[-2\epsilon, 2\epsilon]$ and in particular that any point x_0 that starts in $[-2\epsilon, 2\epsilon]$ will be bounded within $[-2\epsilon, 2\epsilon]$.

The map g is known as a deterministic dynamical system as given any point in the phase space we can extrapolate from that point all the future points it will map onto due to the action of g. When we add the random perturbations ξ at each iteration we obtain an example of a random dynamical system. These arise when modeling uncertainty in dynamical systems. For instance if we are building a model that attempts to predict the value of a stock in a financial market then we may assume some underlying deterministic relation g which represents the known aspects of the market dynamics. We may then add a random perturbation ξ to represent aspects of the system that we cannot predict. Typically one does this by assuming that ξ can perturb a state to anywhere on the whole phase space with respect to some

probability distribution where small adjustments are much more likely than large ones.

Recently interest has arisen in the study of random dynamical systems where the random element is bounded like in the example above where $\xi \in [-\epsilon, \epsilon]$. In this case one can choose to consider only the topological component of the random dynamical system by looking at the set of all random perturbations as we iterate the map g. In doing this we throw away any probabilistic considerations and look at all the possible trajectories of the dynamic rather than the likelihood of any particular one. This allows us to construct set-valued dynamical systems such as f which effectively bounds the underlying random dynamical systems orbits. For an extensive body of material on random dynamical systems see [3].

Set-valued dynamical systems initially arose during the 1930s when Zaremba and Marchaud generalized the notion of differential equations to include systems with multiple choices of derivative at each point in the phase space [27, 21]. These became known as differential inclusions and generate whole families of solutions instead of single flows which lead to the development of "generalized dynamical systems" by Barbashin and Roxin [8, 25, 24].

Set-valued dynamical systems are also naturally linked to control theory in which case we consider the set of perturbations $\xi \in [-\epsilon, \epsilon]$ to be the set of all possible alterations we can make to the dynamical system at each iteration. The fixed sets for such control systems are known as the control sets and correspond to the set of states that we can attain by choosing the correct set of perturbations to the orbit when starting at certain initial points [26, 14]. In the example above for any initial point x_0 in X each point in the set $[-2\epsilon, 2\epsilon]$ can be attained, at least in the limit, by repeatedly choosing ξ in order to steer the orbit towards that point. The minimal invariant sets have also been studied with respect to differential inclusions [5].

Set-valued dynamical systems are also referred to as relations, multi-valued mappings, general dynamical systems or families of semi-groups.

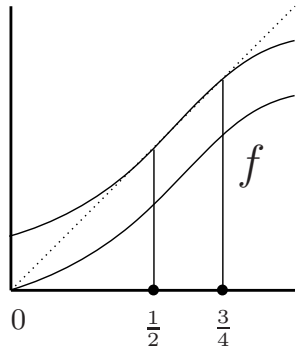


Figure 1.4: Set-valued dynamical system f with many invariant sets

In fig 1.3 above there is only one invariant set. In general things are not so mundane. For the following set-valued system f depicted in fig 1.4 we have a system which has an infinite number of invariant sets. This can be seen as for any point $b \in [\frac{1}{2}, \frac{3}{4}]$ the interval $[0, b]$ is invariant. In this particular case there are two invariant sets of more significance than the others namely the smallest one $[0, \frac{1}{2}]$ and the largest one $[0, \frac{3}{4}]$. If we take any subset of $[0, \frac{1}{2}]$ and consider the orbit generated by f on this set we see that in the limit it converges to $[0, \frac{1}{2}]$. Similarly if we take any larger set that $[0, \frac{3}{4}]$ and consider its orbit we obtain the set $[0, \frac{3}{4}]$. In the case of bounded random dynamical systems it transpires that subject to certain assumptions the smaller of these sets corresponds to points on which the equilibrium probability distribution of the system is non-zero. That means that if we take an initial distribution of points in the space and then repeatedly redistribute them by the random dynamical rule then eventually the sequence of distributions will settle on one that is contained within the smaller invariant sets and be zero outside of them [29, 16]. This means that the long term probability of an orbit staying outside of these invariant sets is zero. We call such invariant sets minimal invariant sets as they contain no smaller invariant set. Assuming certain conditions of the set-valued dynamical system and its phase space it has been show that the minimal invariant sets are compact and finite in number [17].

In this thesis we are interested in the changes that minimal invariant sets of set-valued dynamical system can undergo when we change the underlying set-valued dynamical system. Study of this nature is called bifurcation theory. Consider a

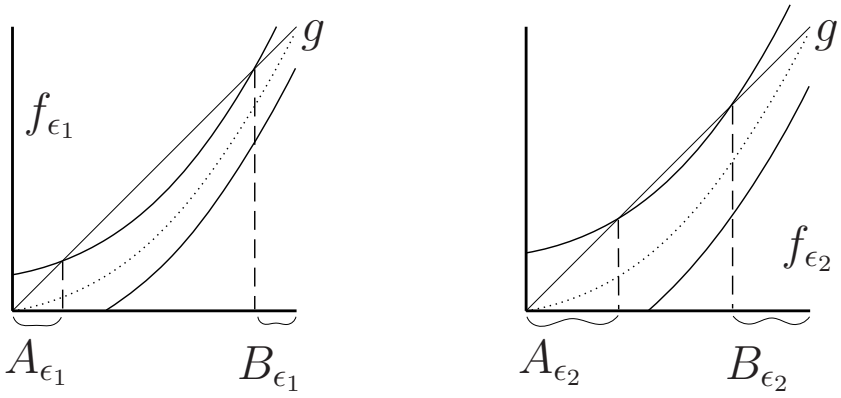


Figure 1.5: Example of behaviour we don't characterize as a bifurcation

single valued dynamical system g. Let f_ϵ be the set-valued system obtained from g by taking the set of ϵ bounded perturbations at each iteration of g given by the formula $f_\epsilon(A) = \bigcup_{x\in A}\bigcup_{\xi\in[-\epsilon,\epsilon]} g(x) + \xi$. The pair f_ϵ and g is depicted in fig 1.5 for two values of ϵ. We see that both f_{ϵ_1} and f_{ϵ_2} have a minimal invariant sets at A_{ϵ_1} and A_{ϵ_2} respectively. Similarly for each system there are sets B_{ϵ_1} and B_{ϵ_2} such that $B_{\epsilon_i} \subset f_{\epsilon_i}(B_{\epsilon_i})$ for $i = \{1, 2\}$. In fact we will show later that the B_{ϵ_i} sets are fixed under the dual systems $f^*_{\epsilon_i}$ which are the set-valued notion of inverse or the dynamical system acting backwards in time. Note that f_{ϵ_2} is obtained by slightly increasing ϵ_1 by a small amount $\Delta\epsilon = \epsilon_2 - \epsilon_1$. This change results in the set A_{ϵ_1} changing slightly in size to A_{ϵ_2} but not by much. In this case we would like to say that the system does not bifurcate as the behaviour before and after the adjustment to f is very similar.

Now take $\epsilon = \epsilon^*$as in fig 1.6 and notice that if we increase ϵ^* by even a very small amount $\Delta\epsilon$ the invariant set A_* suddenly disappears. In particular it combines with the set B_* to become the whole space X. This is an example of the type of behaviour that we want to call a bifurcation.

To develop this notion we need to be able to say when two sets in the phase space are close and when they are not. A well know metric for this purpose is the Hausdorff metric. We informally define the delta expansion $B_\delta(A)$ as the collection of all points within the euclidean distance δ of A. The set $B_\delta(A)$ looks like a slightly expanded version of A. The Hausdorff distance says two sets A and B are less than a distance

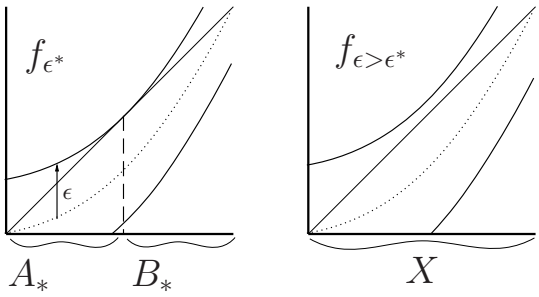


Figure 1.6: Example of a bifurcation where the minimal invariant set can become the whole space under a small change in the underlying set-valued dynamical system.

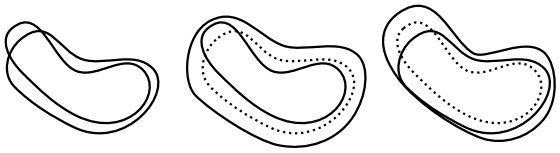


Figure 1.7: Example of two sets a distance δ apart in the Hausdorff distance

δ apart if the delta expansion of each set contains the other set. (see 1.7)

We can consider any dynamical system as a subset of the product of the phase space with itself. In the case of a single valued-dynamical system this corresponds to the graph of the dynamical rule. Namely if g is a map from $X \mapsto X$ then its graph is the subset of $X \times X$ defined by $\{(x, y)|y = g(x)\}$. Similarly in the case of a set-valued dynamical system f we can consider $\{(x, y)|y \in f(x)\}$. By doing this we can think of these systems as subsets of $X \times X$ and then use the product of the Hausdorff metric of each X to obtain a notion of closeness for two set-valued dynamical systems.

In general the notion of bifurcation is not well defined and it depends on what particular types of behaviour you want to examine. We need to specify a number of things. Firstly the type of changes that the set-valued dynamical system can undergo needs to be restricted. If we allow f to change in any way we like then typically we find that bifurcations are far too common to be of interest. To restrict the types of changes to set-valued dynamical systems that may occur requires we define a metric that tells us when two set-value dynamical systems are close. An example of a possible metric we could use is the Hausdorff metric on $X \times X$. Once we

have this we can define a collection of set-valued dynamical systems parameterized by a new variable λ. By changing λ we change the dynamic. In the above example $\lambda = \epsilon$. We will require that the parameterised collection f_λ is continuous with respect to the metric. Namely if we change λ slightly then the resultant change to f_λ is small in our metric.

In our study we are interested in how changes in the set-valued dynamical system result in changes to the minimal invariant sets. Again this requires we can talk about two sets being close and again the Hausdorff metric is a possible notion of closeness that we could use. In fact for the example above it is sufficient to describe the bifurcation that takes place for the value ϵ^*. In the first case given any small $\Delta\epsilon$ we can find δ such that $A' \subset B_\delta(A)$ and $B_\delta(A') \subset A$. In the second case A vanishes and the minimal invariant set is the whole space X in which case $A \subset B_\delta(X)$ but $X \subsetneq B_\delta(A)$ for all $\Delta\epsilon$. Hence in one instance changes to the dynamic result in small changes to the size of the minimal invariant set while in the other case small changes result in large changes to the minimal invariant set.

The notion of bifurcation we obtain is dependent on the metric that we choose. In the above case we have defined bifurcation to mean an event in which a minimal invariant set suddenly jumps to become a larger or smaller set. But if we require that the notion of bifurcation also include cases in which the minimal invariant set develops a hole then the Hausdorff metric falls short. To see this let $0 < a < b < c < 1$ and assume that we have a family of set-valued dynamical systems f_λ on $[0, 1]$ such that for $\lambda < 0$ f_λ has a minimal invariant set $[a, b]$ and for all $\lambda > 0$ f_λ has a minimal invariant set equal to $[a, c - \lambda] \cup [c + \lambda, b]$. Clearly the minimal invariant set splits in two at $\lambda = 0$. It is not hard to see that if given δ small enough we can choose a $\Delta\lambda < \delta/2$ such that $[a, c - \lambda] \cup [c + \lambda, b] \subset B_\delta([a, b])$ and $B_\delta([a, c-\lambda]\cup[c+\lambda, b]) \subset [a, b]$. Hence $\lambda = 0$ is not a point of bifurcation. If we want to define such behaviour as a bifurcation then we have to consider a different metric. Given two sets A, B a possible alternative requires that $X\backslash A$ and $X\backslash B$ be close in the Hausdorff metric as well as A and B. If we choose this to be the case then as $([a, c - \lambda] \cup [c + \lambda, b])^c = [0, a) \cup (c - \lambda, c + \lambda) \cup (b, 1]$ we see that we can no longer always choose λ small enough that $[0, a)\cup(c-\lambda, c+\lambda)\cup(b, 1] \subset B_\delta([0, a)\cup(b, 1])$. This notion of bifurcation is original to this work however it still falls short of including bifurcations that involve the development of holes as we can choose a hole to develop

on the boundary of a set and then move inside. We will detail such a possibility later in the thesis.

Our research will focus on both types of bifurcation introduced above. The aim in this thesis is to give conditions under which we know that a set-valued dynamical system is at the point of bifurcation. For instance in Figure 1.6 we observe that the bifurcation takes place as a result of the minimal invariant set colliding with a set B_*. However note that figure 1.4 depicts a system at a point of bifurcation as by changing the set-valued system to be slightly thicker results in the minimal invariant set jumping from $[0, \frac{1}{2}]$ to a new set containing $[0, \frac{3}{4}]$. Hence we want to describe the differences between these two cases.

Additional types of bifurcation for random dynamical systems have been explored in the literature usually in the context of unbounded noise. In [28] Zeeman touched upon the notion of phenomenological bifurcations for the steady state solutions of differential equations with unbounded diffusion. Such bifurcations occur when the smooth density functions for the stationary measures change qualitatively in character. Other studied bifurcations for random dynamical systems are dynamic-bifurcations which concern the change in signs of Lyaponov exponents for stationary measures. These where developed by Arnold [3, 2].

Recall that if our set-valued dynamical system bounds the orbits of a random dynamical system then subject to certain conditions the minimal invariant sets correspond to the points for which the equilibrium probability distribution is non-zero. The collection of such points are known as the support of the equilibrium measure. The types of bifurcation that involve the minimal invariant set jumping in the Hausdorff metric correspond to the support of the equilibrium measure also jumping discontinuously. Homburg and Young detail this type of bifurcation in [29, 16] (See Figure 1.8).

Finally our notion of bifurcation is very similar to that studied by Lamb, Rasmussen and Rodrigues in in [17]. However they require that the topology of the sets not change. In other words there must exists a homeomorphism between the unperturbed minimal invariant set and the perturbed minimal invariant set. This completely excludes the development of holes.

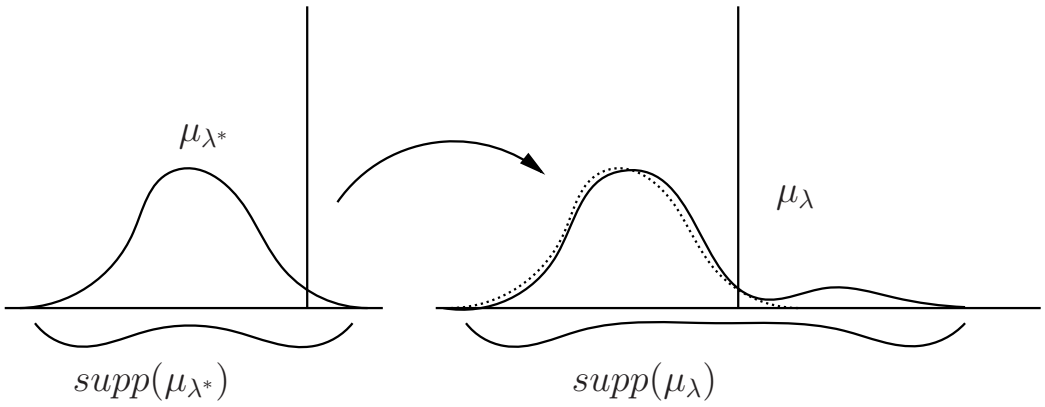


Figure 1.8: A stationary measure μ_{λ^*} for a parameterized random dynamical system bifurcates and in doing so its support jumps discontinuously.

1.2 Results

We start by adapting McGehee and Wiandts work in [23] for minimal invariant sets. In [23] they consider attractors which are invariant sets to which nearby orbits of the system move towards. These occur in natural parity with repellers which have the opposite property such that orbits move towards them under reversal of time. In figure 1.3 we have a set-valued system with an attractor equal to $[-2\epsilon, 2\epsilon]$ which coincides with the minimal invariant set. This isn't generally the case however as shown by figure 1.4 in which the attractor is the set $[0, \frac{3}{4}]$ and the minimal invariant set is equal to $[0, \frac{1}{2}]$. An example of an attractor repeller pair is given in figure 1.9 in which case one observes that orbits starting in the attractor A are bounded in A orbits starting in the repeller B can remain in B or move out of B towards A. The section of the phase space that lies in between A and B is known as the connecting orbits $\mathcal{C}$. Any point in A or B satisfies the property that if we take any slightly expanded version of f we can return to that point in finite time under the action of that expanded mapping. This is known as recurrence and the recurrent set is made up of all points in the phase space that satisfy this property. McGehee and Wiandt use these objects to prove the Conley decomposition theorem for such systems which decomposes the phase space into the connecting orbits and the recurrent set.

Recall that the attractor figure 1.4 is not a minimal invariant set which are the objects we are really interested in. Hence for our purposes the Conley decomposition

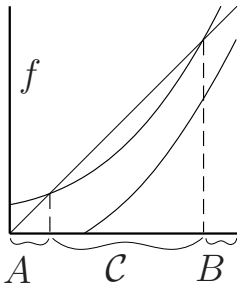


Figure 1.9

doesn't always apply. The second chapter of this thesis defines variants of each of the objects in [23] . Using these we obtain the first main original result of this work namely theorem 58. Instead of attractors we consider inner attractors which are smaller sets than attractors and allow us to give a version of the Conley decomposition theorem that relates to minimal invariant sets.

Originally Conley was motivated to develop the Conley decomposition theorem in [11, 10] in order to prove the existence of Lyaponov functions for dynamical systems [18]. These are functions that decrease along the orbits of a system. This enabled him to construct Lyaponov functions that where decreasing on the connecting orbits of the phase space in the direction of the attractors [7]. Recently work has been done to extend the notions of Lyaponov functions for random dynamical systems [4] as well as attractor repeller pairs in [12, 20, 19, 9].

An invariant set A is isolated for a set-valued dynamical system if for any neighborhood W there exists another neighborhood U of A such that $A \subset U \subset W$ and U is a neighborhood of $f(U)$. An example of an isolated set is an attractor. Conley noticed that if an invariant set is isolated then for any dynamical system g sufficiently close to f the set U will still be a neighborhood of $g(U)$. Therefore as U is mapped inside itself there will still be a invariant set inside U. Using this property one sees that changes in A due to changes to f must be bounded by U. In figure 1.10 we see a depiction of the set-valued systems f and g and sets A and U.

Using this idea the Conley decomposition tells us more than just how the connected and recurrent sets relate in the phase space it also tells us about their behaviour under perturbation of the system. Namely that if we change the underlying set-

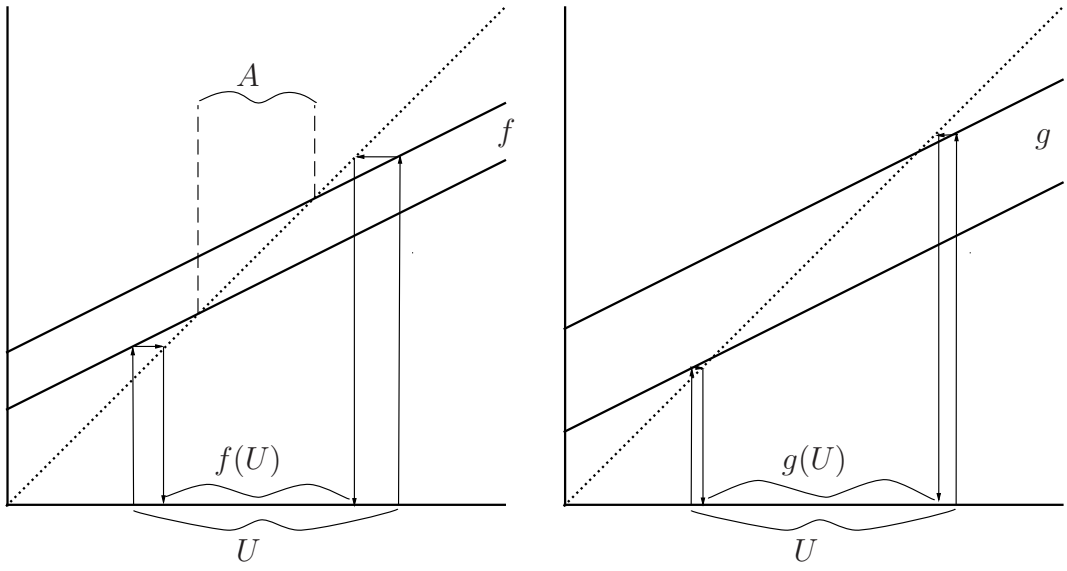


Figure 1.10: The set U is mapped inside itself under the action of f and similarly so for any set-valued system g close enough to f

valued dynamical system the resultant change to the attractor is limited by the set U. Note that in the figure 1.11 we see that f_1 has an attractor A_1 which is isolated. Hence any change to f_1 will not result in A_1 jumping to become a larger set. However notice that if we make the set-valued dynamical system smaller like f_2 then A_1 splits into an attractor repeller pair (A_1, A_2). So A_1 can still bifurcate to become a smaller set. In this example notice also that while A_1 is an attractor it is not a minimal invariant set. Theorem 58 in chapter 3 applies to the minimal invariant sets and so enables a further decomposition of A_1 into a minimal invariant set and the set that under perturbation becomes A_2.

Suppose instead of considering a neighborhood U of an attractor that maps inside itself we can find a subset V of a minimal invariant set E such that $f(V)$ is a neighborhood of V. If we can do this then assuming that the compliment of the underlying family of set-valued dynamical systems in $X \times X$ continues in the Hausdorff metric with respect to the parameter λ then we can obtain a lower bound on the change in the minimal invariant set. To see this consider figure 1.12.

If we change f in such a way that the complement of the new set-valued dynamical system is not Hausdorff close to the complement of f then we can no longer use V

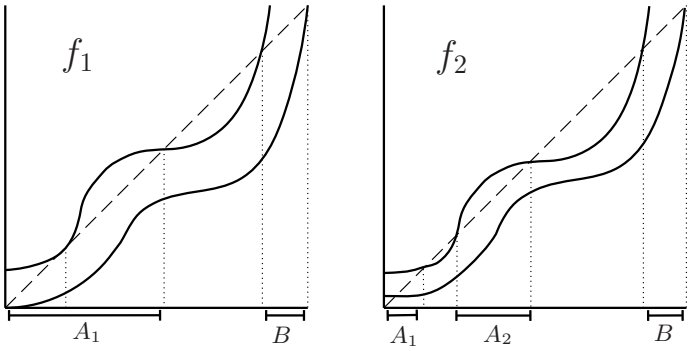


Figure 1.11

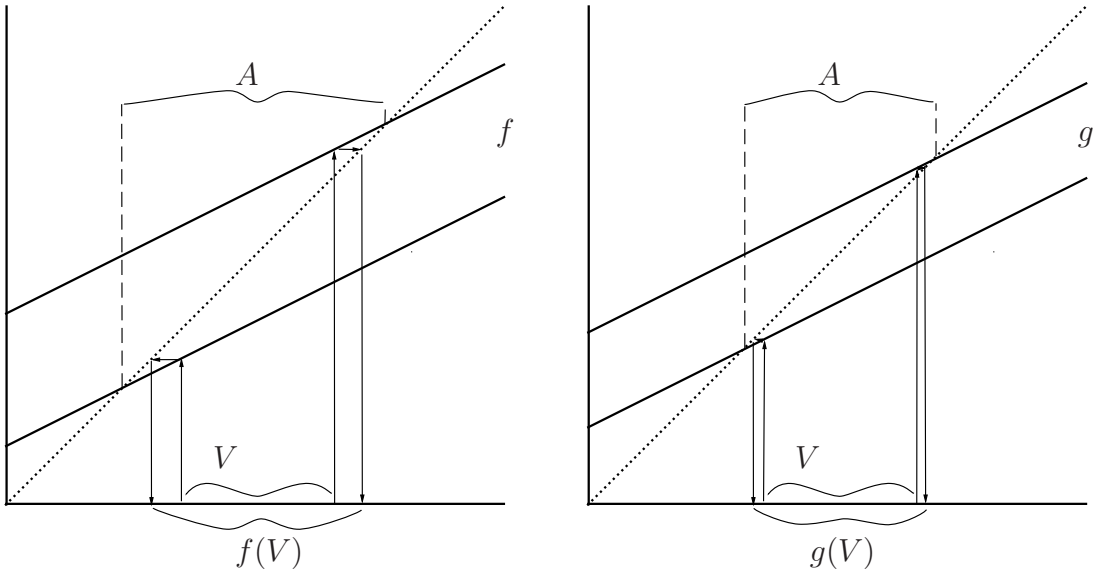


Figure 1.12: Example of a set V contained in a minimal invariant set such that $f(V)$ is a neighborhood of V

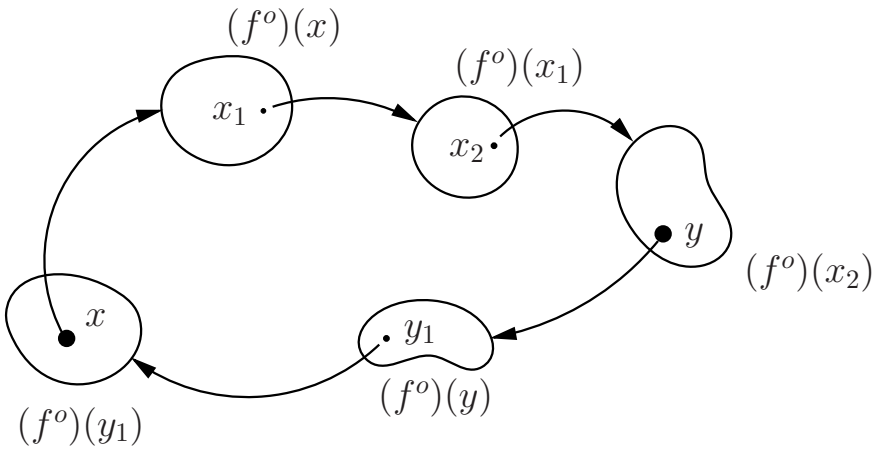


Figure 1.13: Orbitally connected points x and y

as a lower bound on the changes to the minimal invariant set. We will show later why this happens.

To construct such sets V we use a type of periodic point of the set-valued dynamical system. These are points x such that for some $n \in \mathbb{N}$ we have x contained in the interior of $f^n(x)$. We require to use the interior exactly because we need $f(V)$ to be a neighborhood of V rather than just a super-set of V. Hence any periodic point x of this type is such a set V. The second main original result in this work, theorem 88 shows that any minimal invariant set is the closure of a set of such periodic points. Thus we can use the periodic points to construct a subset of any minimal invariant set that has the required property.

In fact theorem 88 says more than this. Given a periodic point of the type described above we call its orbitally connected set the collection points y such that y is contained within the interior of $f^n(x)$ for some n and x is contained in the interior of $f^m(y)$ for some m (see fig 1.13). Theorem 88 tells us that any minimal invariant set is the closure of a set of such orbitally connected points. Note that in figure 1.9 the minimal invariant set is equal to the closure of one orbitally connected set as is the set B equal to the closure of another orbitally connected set. Thus the orbitally connected sets don't only describe the minimal invariant sets.

The orbitally connected sets are similar to the connected components of a dynamical system. Namely those points who can return to themselves in finite time if we

are allowed to make small perturbations along there forward orbits. These are another idea due to Conley and are related to the recurrent set. We obtain from the connected components and the orbitally connected components upper and lower bounds on the variation of a minimal invariant set under perturbation of the set-valued dynamical system. Not only this but we show that the orbitally connected sets cannot develop holes in there interiors.

All together these ideas are used in the main original theorem of the thesis, theorem 119 which shows under what conditions a minimal invariant set will bifurcate by discontinuously jumping in the Hausdorff metric and when it will bifurcate by developing a hole in its interior. It tells us that under certain assumptions a bifurcation in which a minimal invariant set suddenly disappears occurs as a result of two orbitally connected sets colliding. As well as this it tells us that a hole can only develop in the interior of a minimal invariant set if there exists a connected component of the system whose interior is not equal to an orbitally connected set.

These results extend ideas developed by Lamb, Rasmussen and Rodrigues in [17] where they find conditions for bifurcations of a more general class of set-valued dynamical systems.

2 Setup

Here we setup the context of our study, provide results that give insight into set-valued dynamical systems and will come in use later.

2.1 Set-valued Dynamical Systems and Minimal Invariant Sets

The main aim of this section is to define the two fundamental objects of our study. The first is the class of set-valued dynamical systems we wish to discuss, the second are the minimal invariant sets. We will also present and discuss properties that will be important. Before we can do this, we need to discuss the space and its topology on which these concepts exist.

2.1.1 The Phase Space

Instead of orbits being made up of sequences of single points as is the case for single-valued dynamical systems we wish to consider dynamics which evolve multiple collections of points in time which we represent as compact sets in the underlying space. Because of this a point in the phase space is not a singleton but a set. The natural choice for such a phase space is the collection of all non-empty compact

subsets of some compact subset of $\mathbb{R}^n$ and the natural metric to endow on this space is the Hausdorff metric.

Let X be a compact subset of $\mathbb{R}^n$ and d be the Euclidean distance, yielding a metric space (X, d). Define the space of all non-empty compact subsets of X as $\mathcal{K}(X)$. Let A be a non-empty set in $\mathcal{K}(X)$ and define $\text{dist}(x, A) := \inf\{d(x, y)|y \in A\}$. The δ-neighbourhood of the set A is $B_\delta(A) = \{x|\text{dist}(x, A) < \delta\}$. Given $A, B \in \mathcal{K}(X)$ let $\delta_{A,B} = \inf\{\delta|B \subset B_\delta(A)\}$ and finally define the *Hausdorff distance* to be $h(A, B) = \max(\delta_{A,B}, \delta_{B,A})$. We work in $(\mathcal{K}(X), h)$ which is a complete and compact metric space [15].

For a general set $A \subset X$ the complement of A is defined $A^c = X \backslash A$, the interior of a set A as $A^o = A \backslash \partial A$ and the closure as $\overline{A} = A \cup \partial A$. A neighbourhood of A is any set B such that $\overline{A} \subset B^o$. If B is a neighbourhood of A we denote this $A \Subset B$.

For any $A \in \mathcal{K}(X)$ we define the distance between $h(A, \emptyset)$ as equal to the diameter of X.

Remark 1. Note that the definition of neighbourhood differs subtly to the definition given in [23] which only requires $A \subset B^o$. If A is closed then this distinction makes no difference but for the definition in [23] the case in which A is open means that A can be a neighbourhood of itself.

2.1.2 Relations

Next we introduce the definition of relations which are generalisations of single-valued dynamical systems. These will prove to be slightly too general for the set of results we wish later to prove so we also require some additional hypothesis for these relations. First we give the definition of relation and show how it can be considered a mapping from the phase space $\mathcal{K}(X)$ to itself.

Definition 2. A relation g on a set X is a subset of $X \times X$.

Any relation g is also a mapping on sets using $g(S) = \{y|(x, y) \in g \text{ for some } x \in S\}$. We admit a slight abuse of notation by using $g(x)$ to mean $g(\{x\})$. It is easy to see that if $g(x) = \{y|(x, y) \in g\}$ then $g(S) = \bigcup_{x \in S} g(x)$. If we use g as a mapping on

sets we will write $g(S)$ for some set $S \subset X$. If we use g then we are referring to g as a set in $X \times X$.

If we are talking about relations as subsets of $X \times X$, then we can apply topological concepts such as openness, closedness and neighbourhoods in the same way as we would for any subset of $X \times X$. For instance if f and g are both relations then $f \Subset g$ if $\overline{f} \subset g^o$.

We need to show that for a closed relation f the image of any non-empty compact set is also a non-empty compact set. This will mean that we can consider closed relations as dynamical systems with phase space $\mathcal{K}(X)$.

Proposition 3. *Let f be a closed relation in $X \times X$. Given any set $A \in \mathcal{K}(X)$ then $f(A) \in \mathcal{K}(X)$.*

Proof. We require to show that for $A \in \mathcal{K}(X)$ we have that $f(A) = \bigcup_{x \in A} f(x)$ is closed. Note that clearly $f(x)$ is a closed set as otherwise f is not closed in $X \times X$. Choose $z \in f(A)^c$ then if $f(A)$ is closed thus $f(A)^c$ must be open we can find some $\delta > 0$ such that $B_\delta(z) \subset f^c(A)$. Assume for contradiction we cannot and so for every $\delta > 0$ we have $B_\delta(z) \cap f(A) \neq \emptyset$. Hence for a sequence $\{\delta_i\}_{i=1}^\infty$ such that $\lim_{i \to \infty} \delta_i = 0$ we can construct a sequence $\{x_i\}_{i=1}^\infty \subset A$ such that $f(x_i) \cap B_{\delta_i}(z) \neq \emptyset$ for all i. As $A \in \mathcal{K}(X)$ we know it is closed so there exists a sub-sequence that converges to $x' \in A$. Thus $f(x') \cap B_\delta(z) \neq \emptyset$ for all $\delta > 0$ and so as $f(x')$ is closed $z \in f(x')$. But this would mean that z is in $f(A)$ which contradicts the assumption that $z \in f(A)^c$. Thus $f(A)^c$ is open and $f(A)$ is closed. □

So as shown by the previous proposition we can consider closed relations as mappings from $\mathcal{K}(X)$ to itself. These closed relations will define the dynamics that we will consider within this thesis.

Remark 4. If f is a single-valued dynamical system from X to X then f can be written as a relation $f = \{(x, y) : y = f(x)\}$.

The definition of a relation is too general for all the results we will prove. We give a set of additional hypothesis which will be required at various points throughout but first we require to define Hausdorff continuity of set-valued mappings.

Definition 5. A set-valued mapping $f : X \mapsto \mathcal{K}(X)$ is *Hausdorff continuous* at x if given any $\epsilon > 0$ there exists δ such that if $y \in B_\delta(x)$ then $h(f(x), f(y)) < \epsilon$. f is Hausdorff continuous if f is Hausdorff continuous for all $x \in X$.

Later we will define and use upper and lower Hausdorff semi-continuity but for now Hausdorff continuity will suffice. Next we define the set of hypothesis we require for the systems of interest.

Definition 6. Let f be a relation on X

(H1) $\overline{f(x)^o} = f(x)$ for all $x \in X$.

(H2) f is Hausdorff continuous.

(H3) There exists $\epsilon > 0$ such that for all $x \in X$ there exists y such that $B_\epsilon(y) \subset f(x)$.

We will refer to systems above as either relations satisfying *(H1),(H2)* and *(H3)* or sometimes as set-valued dynamical systems.

Note that as per Remark 4 while any single-valued dynamical system is a relation they do not also satisfy *(H3)* and *(H1)*. We give a couple of examples of general relations and relations that satisfy each of *(H1),(H2)* and *(H3)*.

Example 7. (See Fig 2.1). Let $X = [0, 1]$ and $f \subset X \times X$ be the relation defined by $f = \{(x, y) : 0 \leq x < \frac{1}{2} \text{ and } \frac{1}{2}x + \frac{1}{4} \leq y \leq -\frac{1}{2}x + \frac{3}{4}\} \bigcup \{(x, y) : \frac{1}{2} \leq x \leq 1 \text{ and } -\frac{1}{2}x + \frac{3}{4} \leq y \leq \frac{1}{2}x + \frac{1}{4}\}$. It is possible to show that f is Hausdorff continuous. Note that $f(0.5)$ is equal to the single point 0.5 and so $f^o(0.5) = \emptyset$, So f does not satisfy *(H1)* or *(H3)* at 0.5.

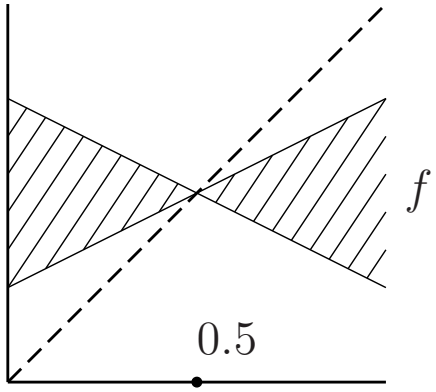


Figure 2.1: An example of a relation that does not satisfy *(H1)* or *(H3)* on all of its phase space.

Example 8. *(τ expansion of the logistic map)* Let $X = [0, 1]$ and fix $2 < \lambda < 4$. Let $g : X \to X$ be the logistic map $g(x) = \lambda x(1 - x)$, fix $\tau = (4 - \lambda)/4$ and define the relation $f = \{(x, y) \in X \times X : |y - g(x)| \leq \tau\}$. We can also define this as $f(x) = \overline{B_\tau(g(x))} \cap X$. Clearly $f(x)^o = B_\tau(g(x)) \cap X$ and so $\overline{f(x)^o} = \overline{B_\tau(g(x))} \cap X = f(x)$ and so f satisfies *(H1)*. We know that g is continuous in the Euclidean metric and therefore given some $x \in X$ and $\kappa > 0$ there exists $\delta > 0$ such that for all $y \in B_\delta(x) \cap X$ we have $|g(x) - g(y)| < \kappa$. Hence $h(\overline{B_\tau(g(x))} \cap X, \overline{B_\tau(g(y))} \cap X) < \kappa$ as any point in $\overline{B_\tau(g(y))} \cap X$ is less than κ away from a point in $\overline{B_\tau(g(x))} \cap X$ and vise versa. Thus we can use continuity of the logistic map to obtain continuity of the set-valued dynamical system f, and so f satisfies *(H2)*. If we choose $\epsilon < \tau/2$ then for all $x \in [0, 1]$ we see that $B_\epsilon(g(x)) \cap X \subset B_\tau(g(x)) \cap X \subset f$ and so f satisfies *(H3)*.

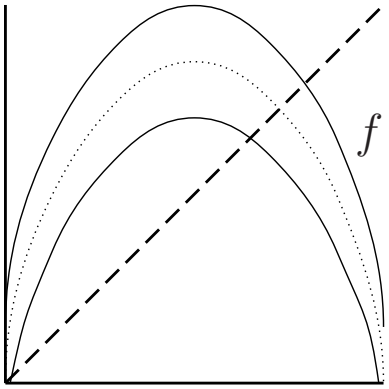


Figure 2.2: The τ expansion of the logistic map

Remark 9. Note that the above proof of Hausdorff continuity of the τ expansion of the logistic map can be generalised to all τ expansions of continuous single-valued dynamical systems.

2.1.3 The Interior of f

Given a relation satisfying *(H1),(H2)* and *(H3)* it will be important later to talk about the interior of f. We show in this section how relations satisfying the above hypothesis can also be considered as mappings on the phase space of open subsets of X instead of closed sets by using their interiors. Specifically we want the interiors of our systems to have certain properties which as shown here result from *(H1),(H2)* and *(H3).*

For clarity given a closed relation f, $f^o(x)$ is equal to $\{y|(x,y) \in f^o\}$ whereas $f(x)^o$ is the interior of the set $f(x)$. For general relations these two are not always the same as the following example shows.

Example 10. *(Discontinuous Relation)* Let $X = [0,1]$ and let f be a relation defined by $f = \{(x,y) : 0 \leq x < \frac{1}{2} \text{ and } 0 \leq y \leq \frac{1}{4}\} \cup \{(x,y) : \frac{1}{2} \leq x \leq 1 \text{ and } 0 \leq y \leq \frac{3}{4}\} \subset X \times X$. For all $x \neq 0.5$ it is possible to show that $\overline{f(x)^o} = f(x)$. If $x = 0.5$ we see that $f(x)^o = (0, \frac{3}{4})$. Now notice that $f^o = \{(x,y) : 0 \leq x \leq \frac{1}{2} \text{ and } 0 < y < \frac{1}{4}\} \cup \{(x,y) : \frac{1}{2} < x \leq 1 \text{ and } 0 < y < \frac{3}{4}\}$. So if $x = 0.5$ we see that $f^o(x) = (0, \frac{1}{4})$ and thus $f^o(x) \neq f(x)^o$. Note that the difference occurs because in one case we take the interior of f in $X \times X$ and the other the interior of $f(x)$ in X.

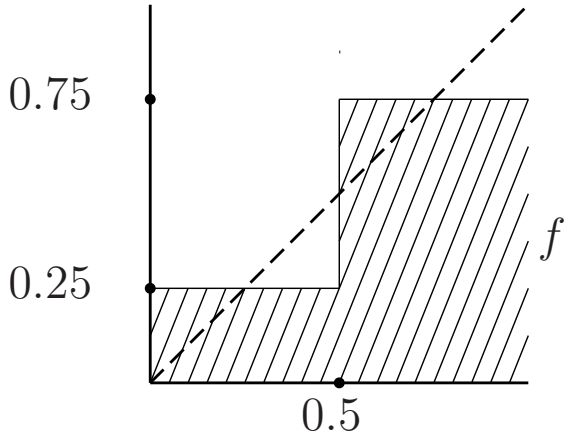


Figure 2.3: A relation discontinuous at $x = 1/2$

The following result is a consequence of *(H1).*

Proposition 11. *If a relation f satisfies (H1) then $\overline{f^o} = f$.*

Proof. Assume not and there exists f that satisfies *(H1)* but $\overline{f^o} \neq f$ then there exists $(x, y) \in f$ such that for all $\delta > 0$ we have $B_\delta((x, y)) \cap f^o = \emptyset$. This means that $y \in f(x)$ but $y \notin f^o(x) \subset f(x)^o$. Thus $B_\delta(y) \cap f(x)^o = \emptyset$ for all δ so $y \notin \overline{f(x)^o}$ which is a contradiction. □

It is not true, however, that for any relation f, the statement $\overline{f^o} = f$ implies that f satisfies *(H1).* To see this consider the following example.

Example 12. Let $X = [0, 1]$ and let $f = \{(x, y) : 0 \leq x < 1 \text{ and } 0 \leq y \leq \frac{1}{4}\} \bigcup \{(x, y) : \frac{1}{2} \leq x \leq 1 \text{ and } -\frac{1}{2}x + \frac{3}{4} \leq y \leq \frac{1}{2}x + \frac{1}{4}\}$. Clearly $\overline{f^o} = f$. Notice that $f(\frac{1}{2}) = [0, \frac{1}{4}] \cup \{\frac{1}{2}\}$ and hence $\overline{f(x)^o} = [0, \frac{1}{4}] \neq f(x)$. It is worth noting that f is discontinuous in the Hausdorff metric at $x = \frac{1}{2}$.

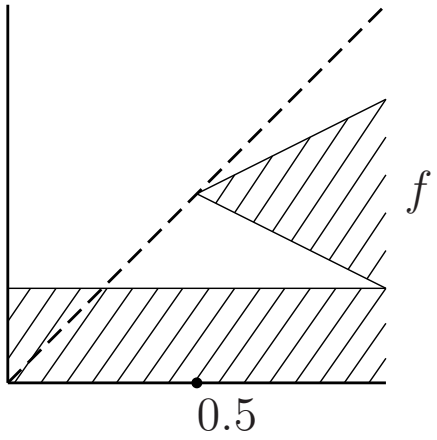


Figure 2.4: $\overline{f^o} = f$ but $\overline{f(1/2)^o} \neq f(1/2)$

In order to extend the idea of Hausdorff continuity of a relation f to its interior we note that the Hausdorff distance can be considered for any pair of sets in X. The following proposition relates subsets of X to their closures.

Proposition 13. *Let $U \subset X$ then $h(U, \overline{U}) = 0$.*

Proof. For any $x \in \overline{U}$ we have due to the definition of closure $B_\delta(x) \cap U \neq \emptyset$ for all $\delta > 0$. Hence if $x \in \overline{U}$ then $x \in B_\delta(U)$ for all $\delta > 0$ and so $\overline{U} \subset B_\delta(U)$. Similarly as $U \subset \overline{U}$ we must have $U \subset B_\delta(\overline{U})$ for all $\delta > 0$. Hence $h(\overline{U}, U) = 0$. □

Remark 14. Note that Proposition 11 implies any f satisfying *(H1)* also satisfies $f^o(x) = f(x)^o$. For any $A \subset X$ and using the extension $f^o(A) = \bigcup_{x \in A} f^o(x)$ we see that $f^o(A)$ is open in X as any union of open sets is open. Thus while f can be considered a mapping to $\mathcal{K}(X)$ we can consider f^o a mapping to $\mathcal{K}^o(X) = \{A^o | A \in \mathcal{K}(X)\}$. The definition of Hausdorff continuity can also be applied to mappings to $\mathcal{K}^o(X)$ and Proposition 13 implies that if $f : X \mapsto \mathcal{K}(X)$ is Hausdorff continuous then $f^o : X \mapsto \mathcal{K}^o(X)$ is also Hausdorff continuous. By considering f^o as acting on non-empty open subsets of X we can identify f^o with the phase space $\mathcal{K}^o(X)$. Thus f^o is a set-valued dynamical system describing a dynamic on $\mathcal{K}^o(X) \mapsto \mathcal{K}^o(X)$. So given any relation $f : \mathcal{K}(X) \mapsto \mathcal{K}(X)$ satisfying *(H1),(H2)* and *(H3)* it is interior relation $f^o : \mathcal{K}^o(X) \mapsto \mathcal{K}^o(X)$ satisfies *(H2)* and *(H3)*. Clearly f^o does not satisfy *(H1)*.

The next proposition will come in use much later.

Proposition 15. *For any relation f that satisfies (H1) and (H2) and a set $A \subset X$ we have $\overline{f^o(A)} = f(\overline{A})$*

Proof. First note that $\overline{f^o(A)} = \overline{\bigcup_{x\in A} f^o(x)}$ and $f(\overline{A}) = \bigcup_{x\in\overline{A}} f(x) \overset{(H1)}{=} \bigcup_{x\in\overline{A}} \overline{f^o(x)}$.

We show first that $f(\overline{A}) \subset \overline{f^o(A)}$. Let $z \in f(\overline{A})$ then $z \in f(x)$ for some $x \in \overline{A}$. Assume $f(x)$ is not a subset of $\overline{f^o(A)}$. If so there exists $y \in f(x)$ such that $B_\delta(y) \cap f(a) = \emptyset$ for some $\delta > 0$ and all $a \in A$ as otherwise y is in the closure of $\overline{f^o(A)}$. Hence $h(f(a), f(x)) > \delta$ for all $a \in A$ but $x \in \overline{A}$ which means contradicts Hausdorff continuity in *(H1)*. Thus $f(x) \subset \overline{f^o(A)}$ and so $z \in \overline{f^o(A)}$.

It remains to show that $\overline{\bigcup_{x\in A} f^o(x)} \subset \bigcup_{x\in\overline{A}} \overline{f^o(x)}$. Let $z \in \overline{\bigcup_{x\in A} f^o(x)}$ but $z \notin f^o(x)$ for all $x \in A$. By the definition of closure for all $\delta > 0$ we have $B_\delta(z) \cap \overline{\bigcup_{x\in A} f^o(x)} \neq \emptyset$ and hence for any $\delta > 0$ there must exist $x \in A$ such that $B_\delta(z) \cap f^o(x) \neq \emptyset$. So given a sequence $\{\delta_i\}_{i=1}^\infty$ that tends to 0 we can construct a sequence $\{x_i\}_{i=1}^\infty \subset A$ such that $B_{\delta_i}(z) \cap f^o(x_i) \neq \emptyset$. As each $x_i \in A$ we must be able to find a convergent sub-sequence who's limit x_∞ must be in $\overline{A}$. Hence we have found a point such that $B_\delta(z) \cap f^o(x_\infty) \neq \emptyset$ for all $\delta > 0$ so thus $z \in \overline{f^o(x_\infty)}$ and thus $z \in \bigcup_{x\in\overline{A}} \overline{f^o(x)}$. □

2.1.4 Minimal Invariant Sets

Usually the points of interest in single-valued dynamical systems are the fixed points. We want to find and explore properties of similar objects in set-valued systems. These are the minimal invariant sets.

Definition 16. (Minimal Invariant Set) Let f be a relation in $X \times X$ we say a set $E \in \mathcal{K}(X)$ is *invariant* if $f(E) = E$ and *minimal invariant* if for any $E' \subset E$ such that $f(E') = E'$ then $E' = E$.

While minimal invariant sets always exist they are not unique usually. They are however disjoint and if we assume *(H3)* for a relation f then $f(x)$ contains at least an ϵ-ball for all $x \in X$ and so there can only be finitely many owing to the fact that X is compact (see [17]). We define the set $\mathcal{M}$ as the union of all minimal invariant sets of f.

The ω-limit set for $A \in \mathcal{K}(X)$ is defined similarly to the single-valued ω-limit set and describes the behaviour of A under long term iteration of f.

Definition 17. Given $A \in \mathcal{K}(X)$ we denote the *ω-limit set* of A with respect to the set-valued dynamical system f as:

$$\omega(A, f) := \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} f^m(A)}$$

Remark 18. It is shown in [6] that for our set-valued dynamical systems the ω-limit set of a set A is a non-empty compact f-invariant set. From this fact we can obtain:

Proposition 19. *Given a subset A of a minimal invariant set E for a relation f satisfying (H1),(H2) and (H3) then $\omega(A, f) = E$.*

Proof. $\omega(A, f)$ must be a subset of E as $f(A) \subset f(E) = E$ and hence $\overline{\bigcup_{m \geq n} f^m(A)} \subset E$. Hence $\omega(A, f)$ is an f-invariant subset of E and hence by definition 16 of minimal invariance $\omega(A, f) = E$. □

2.1.5 Composition of Relations

The dynamics we wish to consider result from iterating set-valued dynamical systems on their phase space. The following definition tells us how to compose any two relations and is taken from [23].

Definition 20. (composition of relations) Given two relations f and g the composition $f \circ g$ is

$$f \circ g = \{(x, z) | \exists y \in X \text{ such that } (x, y) \in g \text{ and } (y, z) \in f\}$$

It is easy to show that $\overbrace{f \circ f \circ ... \circ f}^{n}(x) = f^n(x) = \bigcup_{y \in f^{n-1}(x)} f(y)$.

We state without proof Lemma 3.19 from [23] which tells us how composition acts on unions of families of relations and will be needed later as in the next section we introduce special relations which are defined as unions over the iteration of a relation.

Lemma 21. *Composition preserves inclusion and union, i.e. if* $f_1 \subset f_2 \subset X \times X$ *and* $g_1 \subset g_2 \subset X \times X$*, then* $g_1 \circ f_1 \subset g_2 \circ f_2$ *and*

$$\left(\bigcup_{g\in\mathcal{G}} g\right) \circ \left(\bigcup_{f\in\mathcal{F}} f\right) = \bigcup_{h\in\mathcal{H}} h$$

where $\mathcal{F}$ *and* $\mathcal{G}$ *are families of relations on* X *and*

$$\mathcal{H} = \{h | h = g \circ f \text{ for } g \in \mathcal{G} \text{ and } f \in \mathcal{F}\}.$$

2.2 The Inverse and Dual Map

There is no clear definition for the inverse of a relation. We define an object which we call the dual which has similar properties to the inverse and for single-valued systems is the same.

Definition 22. (Dual of a relation) Given a relation f then the *dual* of f is defined by

$$f^* = \{(y, x) \in X : (x, y) \in f\}.$$

Remark 23. Obviously $f^{**} = f$.

It easy to see that if f is a closed relation then so is f^*. Thus due to Proposition 3 we see that f^* can be considered as a set-valued dynamical system with phase space $\mathcal{K}(X)$ in the same way as we can f.

We will also require to use the interior of f^* later. The next result give an important property for the interior dual relation.

Lemma 24. *Consider a relation* f *in* $X \times X$*. Then* $f^{o*} = f^{*o}$*.*

Proof. The dual of f^* is a permutation of the x and y axis. The interior of a closed set that has been reflected across $x = y$ is the same as reflecting a closed set across $x = y$ and taking its interior. □

Note that $f^{o*}(x)$ is not always equal to $(f^*(x))^o$ instead $f^{o*}(x) \subset (f^*(x))^o$. The following example illustrates why.

Example 25. Let $X = [0,1]$ and $g : X \mapsto X$ be a single-valued dynamical system given by

$$g(x) = \begin{cases} 2x & : 0 \leq x \leq \frac{1}{4} \\ \frac{1}{2} & : \frac{1}{4} < x \leq \frac{3}{4} \\ 2x - 1 & : \frac{3}{4} < x \leq 1 \end{cases}$$

Fix $\tau < \frac{1}{10}$ and let f be the relation defined by $f = \{(x, y) : |y - g(x)| \leq \tau\} \subset X \times X$. Consider the points $z_1 = c - \tau$ and $z_2 = c + \tau$ and notice that $f^{o*}(z_i) \subset (f^*(z_i))^o$ for both $i = \{1, 2\}$ as Figure 2.5 shows.

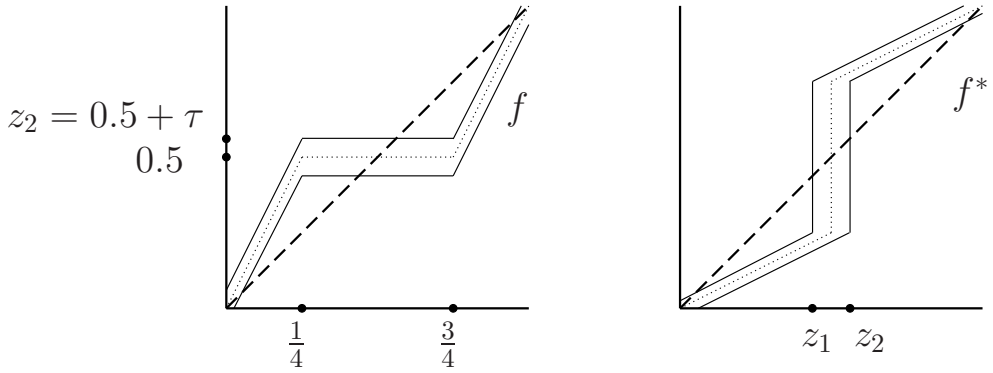


Figure 2.5: $f^{o*}(x) \neq (f^*(x))^o$ for $x = z_1$ or z_2

The above example illustrates a case where f satisfies *(H1),(H2)* and *(H3)* but f^* does not satisfy *(H1)* or *(H2)*. In fact in general if f satisfies *(H1),(H2)* and *(H3)* then f^* does not have to satisfy any of *(H1),(H2)* and *(H3)* as the following example shows.

Example 26. *(Dual logistic map)* Consider the relation f as defined in Example 8. Notice that $f^*(1) = \{0.5\}$ by the particular choice of λ and τ and thus f^*cannot satisfy *(H3)*.

Another set of objects that will be significant later for any relation f we are considering are the minimal invariant sets for its dual relation f^* which we call *dual minimal invariant set*s. Dual minimal invariant sets are not invariant under f only under f^* . This is one way in which set-valued dynamical systems and their dual

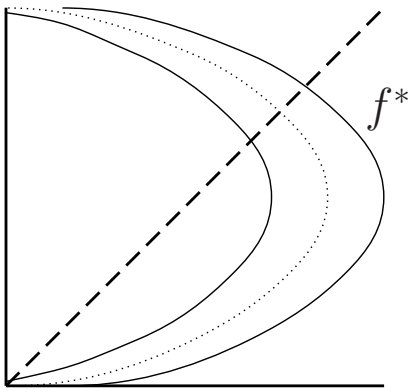


Figure 2.6: Dual system for the τ-expanded logistic map

maps differ from single-valued dynamical systems and their inverses as the set of invariant points for a single-valued map f is the same as for the inverse f^{-1}.

The next lemma is stated in [23] but not proven. It relates f^* to another object that acts like an inverse. This lemma is used in Section 3.2 .

Lemma 27. *For any relation f and for any $A \in \mathcal{K}(X)$ we have $f^*(A^c) = f^{-1}(A)^c$ where $f^{-1}(A) = \{x \in X : f(x) \subset A\}$.*

Proof. Note that $f^*(A)$ contains any point y such that $f(y) \cap A \neq \emptyset$ by definition. For a given A let

$$A_I = \{B \in \mathcal{K}(X) | B = f(x) \text{ for some } x \in X \text{ and } B \cap A \neq \emptyset\}$$

.Similarly let

$$A_S = \{B \in \mathcal{K}(X) | B = f(y) \text{ for some } x \in X \text{ and } B \subset A\}$$

.If $\mathcal{X} = \{B \in \mathcal{K}(X) | B = f(x) \text{ for some } x \in X\}$ then one has the relation $(A^c)_I = \mathcal{X} \backslash A_S$. To see this note that if $B \in (A^c)_I$ then $B \cap A^c \neq \emptyset$ and hence B cannot be a subset of A. Similarly if $B \in A_S$ then $B \subset A$ and hence $B \cap A^c = \emptyset$ which means B cannot be in $(A^c)_I$.

Each set $B \in A_I$ or A_S is the image of a point $x \in X$ by definition. To make this clear denote such B by B_x where $B_x = f(x)$. Hence $f^{-1}(B_x) = x$. Notice that

$f^*(A) = \{x|B_x \in A_I\}$ and $f^{-1}(A) = \{x|B_x \in A_S\}$. Thus:

$$f^*(A^c) = \{x|B_x \in (A^c)_I\} = \{x|B_x \in \mathcal{X}\backslash A_S\},$$

$$\{x|B_x \in \mathcal{X}\}\backslash\{x|B_x \in A_S\} = X\backslash f^{-1}(A) = f^{-1}(A)^c.$$

□

Finally we will need Lemma 3.11 from [23]:

Lemma 28. *If $\mathcal{F}$ is a family of relations on X then $(\bigcup_{f\in\mathcal{F}} f)^* = \bigcup_{f\in\mathcal{F}} f^*$ and $(\bigcap_{f\in\mathcal{F}} f)^* = \bigcap_{f\in\mathcal{F}} f^*$*

Remark 29. To see this notice that reflecting the union of sets $\bigcup_{f\in\mathcal{F}} f \subset X \times X$ across $x = y$ is equal to reflecting each set $f \in \mathcal{F}$ across $x = y$ and taking the union. The case for intersection is the same.

3

The Conley Decomposition Theorems

An important object within the theory for bifurcations of dynamical systems are attractors which are sets that trap or attract the forward orbits of the system. The Conley Decomposition theorem from [23] obtains a decomposition of the phase space that tells us about the asymptotic behavior of the system. Naturally it does this using the attractors of the system. We are interested in bifurcations of minimal invariant sets which are not always attractors. In this part of the thesis we will obtain in Theorem 58 a similar result to the Conley decomposition theorem but that applies to minimal invariant sets. We fill follow much of the analysis of McGehee and Wiandt in [23] and state some of there results including the Conley decomposition theorem in parallel to our results.

Nearly all the results in this section hold for closed relations on compact subsets X of $\mathbb{R}^n$. After the main theorem we will relate the results to the case of set-valued dynamical systems as in Definition 6. Until then we will only talk about relations.

3.1 Limit Relations

In the next section we define the limit relations for a set-valued dynamical system and give some results we will need later. The limit, $\omega-$limit and the Conley limit relation are all defined by McGehee and Wiandt in [23]. We will use these relations and introduce the Open Conley Relation in addition to them.

Definition 30. *(limit relations)*

If f is a relation on X, then the *limit relation* of f is

$$f^{\infty} \equiv \bigcap_{n\geq 0} \bigcup_{k\geq n} f^k$$

If f is a relation on X, then the *ω-limit relation* of f is

$$f^{\omega} \equiv \bigcap_{n\geq 0} \overline{\bigcup_{k\geq n} f^k}$$

If f is a relation on X, then the *Conley relation* of f is

$$f^{\Omega} \equiv \bigcap_{g \ni f} g^{\infty}$$

If f is a relation on X, then the *open Conley relation* of f is

$$f^{\Theta} \equiv \bigcup_{g \in f} g^{\infty}$$

The nature of the first three of the above relations is well documented in [23] we will look mostly at the fourth one which is original to this work.

The limit relation considers just the forward orbits while the $\omega-$limit relation considers the topology of the space X by taking the closure. The Conley relation considers the orbits with respect to larger perturbations. The open Conley relation is similar to the Conley relation but considers how the relation inhabits X with respect to smaller relations in X. Note that if f is a relation associated to a single-valued

dynamical system then $f^{\Theta} = \emptyset$ as there are no relations $g \subset X \times X$ such that $g \Subset f$.

The following example shows how the limit relations for a relation f may differ.

Example 31. let $X = [0, 1]$ and let $g : X \mapsto X$ be the single-valued dynamical system defined by

$$g(x) = \begin{cases} x^2 + \frac{1}{8} & : 0 \leq x < \frac{1}{2} \\ x - \frac{1}{8} & : \frac{1}{2} \leq x < \frac{3}{4} \\ -\left(x - \frac{5}{4}\right)^2 + \frac{7}{8} & : \frac{3}{4} \leq x \leq 1 \end{cases}$$

We let $f = \{(x, y) : |y - g(x)| \leq 1/8\}$. We can easily compute each of the limit relations for f. We focus on f^{ω},f^{Ω} and f^{Θ}. Note that f^{Θ} is open in $X \times X$.

$$f^{\omega} = \left\{(x, y) : 0 \leq x < \tfrac{1}{2} \text{ and } 0 \leq y \leq \tfrac{1}{2}\right\} \cup \left\{(x, y) : \tfrac{1}{2} \leq x < \tfrac{3}{4} \text{ and } 0 \leq y \leq x\right\}$$

$$\cup \left\{(x, y) : \frac{3}{4} \leq x < 1 \text{ and } 0 \leq y \leq \tfrac{15}{16}\right\}$$

and

$$f^{\Omega} = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq \tfrac{15}{16}\}$$

$$f^{\Theta} = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 < y < \tfrac{1}{2}\}$$

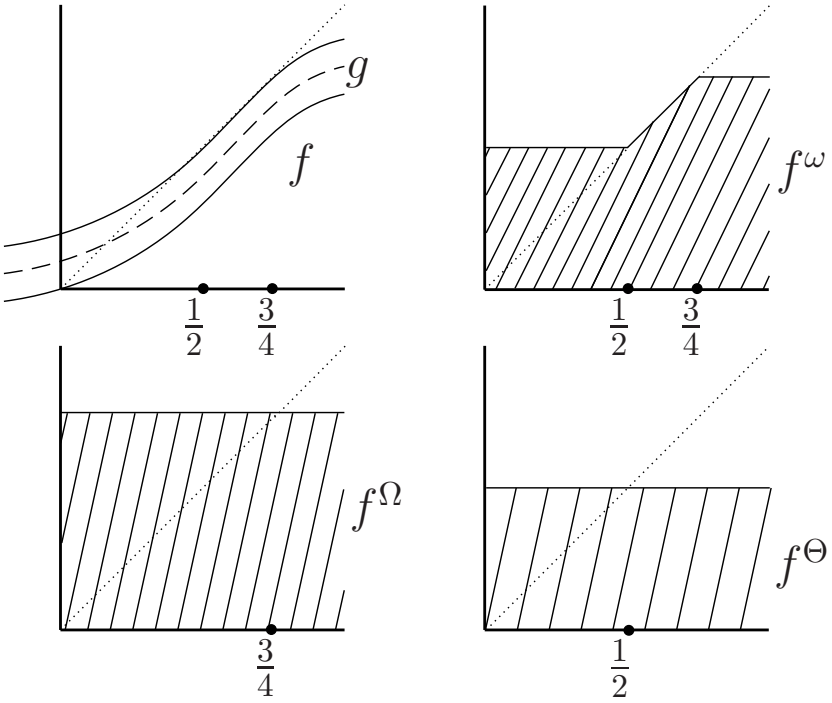


Figure 3.1: Limit relations

Remark 32. Note that $f^\omega(x)$ is not the same as $\omega(x)$ for f. This is because f^ω is a limit of a set in the product topology of $X \times X$ whereas $\omega(x)$ is a limit in X. To see this consider the example given in Remark 9.16 in [23] where we define $f = \{(x, y) \in X \times X : \ y = x^2\}$ on $X = [0, 1]$. Thus $f^\omega = \{(x, y) \in X \times X : \ y = 0 \text{ or } x = 1\}$. We see that $\omega(\{1\}) = \{1\}$ whereas $f^\omega(x) = X$. It is true in general that $\omega(x) \subset f^\omega(x)$.

3.1.1 Properties

In this section we will state and prove a collection of results for f^Θ, that are similar for $f^\infty, f^\omega, f^\Omega$ as shown by McGehee and Wiandt [23]. The aim is to set up the machinery to state and prove the Open Conley decomposition in Section 3.3.

Remark 33. Note that many of the results in [23] require f be a closed relation on a compact Hausdorff space X. We work on a compact subset of $\mathbb{R}^n$ which is a compact Hausdorff space. When presenting these results within our context we will neglect to mention this condition.

First we give a slight adaptation of [23, Lemma 7.5 on p.18] which will be used a lot throughout this section.

Lemma 34. *If f is a relation on a compact subset X of $\mathbb{R}^n$ and $f \Subset g$, then $f^\infty \subset f^\omega \Subset g^\infty$.*

Remark 35. First we point out that while the definition of neighbourhood differs slightly in [22] they coincide with out definition in this case as f is closed. (see Remark 1).

Proof. The claim mostly follows from the proof for [23, Lemma 7.5 on p.18] the statement of which identical to the above except requiring f to be closed. We are just left to show it also follows from any relation f and not just when f is closed. In our case let f be a general relation such that $f \Subset g$ thus $\overline{f} \subset g^o$ hence $\overline{f}$ and g satisfies the assumptions for [23, lemma 7.5 on p.18] and so $\overline{f}^\infty \subset \overline{f}^\omega \subset (g^\infty)^o$ and thus $f^\infty \subset f^\omega \Subset g^\infty$. □

For completeness in the next result we condense the parts of Theorems 6.2, 7.2 and Theorem 8.2 from [23] relevant to our work into a single Theorem which tells us the dual of each limit relation is the limit relation of the dual.

Theorem 36. *For any relation f on X we have*

(1) $f^{\infty *} = f^{*\infty}$

(2) $f^{\omega *} = f^{*\omega}$

(3) $f^{\Omega *} = f^{*\Omega}$

We will now prove a similar result.

Theorem 37. *For any relation f on X we have $f^{\Theta *} = f^{*\Theta}$.*

Proof. $f^{\Theta *} \equiv \left(\bigcup_{g \Subset f} g^\infty\right)^* \overset{\text{Lem } 28}{=} \bigcup_{g \Subset f} (g^\infty)^* \overset{\text{Thm } 36}{=} \bigcup_{g \Subset f} g^{*\infty}$. If $g \Subset f$ then $g^* \Subset f^*$, thus $\bigcup_{g \Subset f} g^{*\infty} = \bigcup_{g \Subset f^*} g^\infty = f^{*\Theta}$. □

The next lemma sets us up for the theorems afterward:

Lemma 38.

(1) For any open relation f on X, we have $f = \bigcup_{g \Subset f} g$.

(2) For any relation f on X, we have $f^{\Theta} = \bigcup_{g \Subset f} g^{\omega}$.

Proof. (1) ($\subset$) For any $x \in f$ there exists an open set $g \Subset f \subset X \times X$ that contains x as f is open. Hence $x \in \bigcup_{g \Subset f} g$ and so $f \subset \bigcup_{g \Subset f} g$.

($\supset$) If $x \in \bigcup_{g \Subset f} g$ then $x \in g$ for some $g \Subset f$. As f is a neighbourhood of g there exists an open set h such that $g \subset h \Subset f$. As h is open and $x \in h \subset f$, x must be in f and hence $\bigcup_{g \Subset f} g \subset f$.

(2) ($\subset$) From Definition 30 it is clear that $g^{\infty} \subset g^{\omega}$ for all g. Hence $f^{\Theta} = \bigcup_{g \Subset f} g^{\infty} \subset \bigcup_{g \Subset f} g^{\omega}$.

($\supset$) For any $g \Subset f$ there exists h_g such that $g \Subset h_g \Subset f$. Using Lemma 34 $g^{\omega} \Subset h_g^{\infty}$ and so

$$\bigcup_{g \Subset f} g^{\omega} \subset \bigcup_{g \Subset f} h_g^{\infty} \equiv \bigcup_{h \Subset f} h^{\infty} = f^{\Theta}.$$

□

In the main result theorem * we will need to use the open Conley relation to obtain invariant sets. We condense the following three theorems, 6.3,7.3 and 8.5 from [23] into one for this purpose.

Theorem 39.

If f is a relation on X, then the following inclusions hold.

(1) $f \circ f^{\infty} \subset f^{\infty}$, $f^{\infty} \circ f \subset f^{\infty}$ and $f^{\infty} \circ f^{\infty} \subset f^{\infty}$.

If f is a closed relation on X, then the following inclusions hold.

(2) $f \circ f^{\omega} \supset f^{\omega}$, $f^{\omega} \circ f \supset f^{\omega}$ and $f^{\omega} \circ f^{\omega} \supset f^{\omega}$,

(3) $f \circ f^{\Omega} = f^{\Omega}$, $f^{\Omega} \circ f = f^{\Omega}$ and $f^{\Omega} \circ f^{\Omega} = f^{\Omega}$.

We obtain the following continuation of these results for the open Conley relation:

Theorem 40. *For any open relation f on X the open Conley relation f^{Θ} satisfies:*

(1) $f \circ f^{\Theta} = f^{\Theta}$,

(2) $f^{\Theta} \circ f = f^{\Theta}$,

(3) $f^{\Theta} \circ f^{\Theta} = f^{\Theta}$.

Before proving the theorem we provide a lemma that is necessary for its proof.

Lemma 41. *For any relation f on X we have $\bigcup_{h \Subset f} \bigcup_{g \Subset f} h \circ g^{\infty} = \bigcup_{g \Subset f} g \circ g^{\infty}$ and $\bigcup_{h \Subset f} \bigcup_{g \Subset f} h \circ g^{\omega} = \bigcup_{g \Subset f} g \circ g^{\omega}$.*

Proof. The proof for each statement is identical. Firstly it is clear that $\bigcup_{g \Subset f} g \circ g^{\infty} \subset \bigcup_{h \Subset f} \bigcup_{g \Subset f} h \circ g^{\infty}$ as $g \circ g^{\infty} \subset \bigcup_{h \Subset f} h \circ g^{\infty}$ for each g.

To show $\bigcup_{h \Subset f} \bigcup_{g \Subset f} h \circ g^{\infty} \subset \bigcup_{g \Subset f} g \circ g^{\infty}$ we choose any h and note that as $h \Subset f$ there exists g such that $h \Subset g \Subset f$ and hence $h \circ g^{\infty} \subset g \circ g^{\infty}$. As this is possible for all $h \Subset f$ we see that $\bigcup_{h \Subset f} \bigcup_{g \Subset f} h \circ g^{\infty} \subset \bigcup_{g \Subset f} g \circ g^{\infty}$. □

Proof. (of Theorem 40) Using part (1) of Lemma 38 we have $f \circ f^{\Theta} = \bigcup_{h \Subset f} h \circ \bigcup_{g \Subset f} g^{\infty}$. As composition preserves union by Lemma 21 we see that

$$\bigcup_{h \Subset f} h \circ \bigcup_{g \Subset f} g^{\infty} = \bigcup_{h \Subset f} \bigcup_{g \Subset f} h \circ g^{\infty}.$$

By Lemma 41 we have $\bigcup_{h \Subset f} \bigcup_{g \Subset f} h \circ g^{\infty} = \bigcup_{g \Subset f} g \circ g^{\infty}$.

Similarly but using both parts *(1)* and *(2)* of Lemma 38 we obtain $f \circ f^{\Theta} = \bigcup_{g \Subset f} g \circ g^{\omega}$.

Firstly we have by Theorem 39 *(1)* $g \circ g^{\infty} \subset g^{\infty}$ and hence:

$$f \circ f^{\Theta} = \bigcup_{g \Subset f} g \circ g^{\infty} \subset \bigcup_{g \Subset f} g^{\infty} = f^{\Theta}.$$

Secondly note that $g \circ g^{\omega} \supset \overline{g} \circ \overline{g}^{\omega}$ and so using Theorem 39 *(2)* we have $\overline{g} \circ \overline{g}^{\omega} \supset \overline{g}^{\omega}$ and hence:

$$f \circ f^{\Theta} = \bigcup_{g \Subset f} g \circ g^{\omega} \supset \bigcup_{g \Subset f} \overline{g} \circ \overline{g}^{\omega} \supset \bigcup_{g \Subset f} \overline{g}^{\omega}.$$

by Definition 30 we see that $\overline{g}^{\omega} = g^{\omega}$ and hence $\bigcup_{g \Subset f} \overline{g}^{\omega} = \bigcup_{g \Subset f} g^{\omega} = f^{\Theta}$ and *(1)* follows.

The same argument works for *(2)* and *(3)* but using $g^{\infty} \circ g^{\infty} \subset g^{\infty}$ or $g^{\infty} \circ g \subset g^{\infty}$ rather than $g \circ g^{\infty} \subset g^{\infty}$. □

Next we show:

Lemma 42. *For any relations f, g on X such that $g \Subset f$ then $g^{\Theta} \Subset f^{\Theta}$.*

Proof. We can find h_1 and h_2 such that $g \Subset h_1 \Subset h_2 \Subset f$. Using Lemma 34 we know that $h_1^{\omega} \Subset h_2^{\infty}$ which implies there exists a relation h_3 such that $h_1^{\omega} \Subset h_3 \Subset h_2^{\infty}$. Hence we have

$$g^{\Theta} = \bigcup_{p \Subset g} p^{\infty} \subset h_1^{\infty} \subset h_1^{\omega} \Subset h_3 \Subset h_2^{\infty} \subset f^{\Theta}.$$

□

The final property we show for the open Conley relation is that it is an open relation. For this we require Lemma 34.

Lemma 43. *For any relation f on X its open Conley relation f^{Θ} is an open relation.*

Proof. We show that for any point $(x, y) \in f^{\Theta}$ we can find an open set h in f^{Θ} that contains (x, y). Choose $(x, y) \in f^{\Theta} = \bigcup_{g \Subset f} g^{\infty}$. Hence $(x, y) \in g^{\infty}$ for some $g \Subset f$. By Lemma 42 $g \Subset f$ implies $g^{\Theta} \Subset f^{\Theta}$ and hence there exists an open relation h such that $g^{\Theta} \Subset h \Subset f^{\Theta}$ and the result follows. □

3.1.2 Periodic and Recurrent Sets

We will now define periodic sets which will also be used later in section 4.1. The main result of this section is dual to [23, Theorem 9.23] so we will present them both. The following definitions and theorems show the connection between the recurrent sets and the Conley limit relations which we will use in the proof of Theorem 58.

Definition 44. (P_n set) For any relation f on X we define the n^{th} *periodic set* [23, Def 9.7 on p.24] to be

$$P_n(f) = \{x \in X | (x, x) \in f^n\}.$$

and $P(f) = \bigcup_{n=1}^{\infty} P_n(f)$.

Theorem 9.8 from [23] says that $P(f) = P_1(f^{\infty})$.

Next we state a result for f^{Ω} which first requires the definition of a recurrent set.

Definition 45. (R_n set) For any relation f on X we define the n^{th} *recurrent* set [23, Def 9.22 on p.27] by

$$R_n(f) = \bigcap_{g \ni f} P_n(g) \text{ and } R(f) = \bigcap_{g \ni f} P(g).$$

We state [23, Theorem 9.23] along with a variant we will prove. This result is needed in the open Conley theorem and the $P(\mathring{f})$ sets are required to describe minimal invariant sets.

Theorem 46.

(1) If f is a closed relation on X, then

$$R(f) = P_1(f^{\Omega}).$$

(2) If f is an open relation on X, then

$$P(f) = P_1(f^{\Theta}).$$

Proof. (Theorem 46 part (2))($\subset$): Choose $x \in P(f)$. Then there exists $n \in \mathbb{N}$ such that $(x, x) \in f^n$. As f^n is open there exists a relation $g \Subset f$ such that $(x, x) \in g^n$. Hence $(x, x) \in \bigcup_{k \geq l} g^k$ for all $l \in \mathbb{N}$ and so $(x, x) \in \bigcap_{l \geq 0} \bigcup_{k \geq l} g^k = g^{\infty}$ and $x \in P_1(f^{\Theta})$.

($\supset$): Choose $x \in P_1(f^{\Theta})$, then $(x, x) \in \bigcup_{g \Subset f} g^{\infty}$ and so there must exists $g \Subset f$ such that $(x, x) \in g^{\infty}$. Hence $(x, x) \in g^n$ for some $n \in \mathbb{N}$ and we can find an open relation h such that $g \Subset h \Subset f$ and we have $(x, x) \in h^n$ hence $x \in P(f)$. $\square$

3.2 Inner Attractor Repellers Pairs and Minimal Invariant Sets

The Conley Decomposition divides the space up using attractors [23] which we now define. In order to give the open Conley decomposition theorem we give a slightly different definition.

Definition 47. If f is a relation on X, then $A \subset X$ is called an *attractor* for f if there exists $U \ni A$ such that $f^{\Omega}(U) = A$. The set of all attractors of f will be denoted by $\mathcal{U}(f)$. The *basin of attraction* of an attractor A is the set $B(A) = \{x \in X | f^{\Omega}(x) \subset A\}$.

The variation of the above definition that we call the inner attractors is defined as follows.

Definition 48. If f is a relation on X then $A \subset X$ is called an *inner attractor* for f if $f^{\Theta}(A) = A$. The set of all *inner attractors* will be denoted by $\mathcal{V}(f)$. The *basin of attraction* of an inner attractor A is the set $B_I(A) = \{x \in X | f^{\Theta}(x) \subset A\}$.

Example 49. *(Attractors and Inner Attractors)* Recall the relation f from Example 31. We see that the set $A_1 = (0, \frac{1}{2})$ is an inner attractor as $f^{\Theta}(A_1) = A_1$. The set $A_2 = [0, \frac{3}{4}]$ is an attractor as for any open set U such that $A_2 \subset U$ we see that $f^{\Omega}(U) = A_2$. The basin of attraction for both A_1 and A_2 is the whole space X.

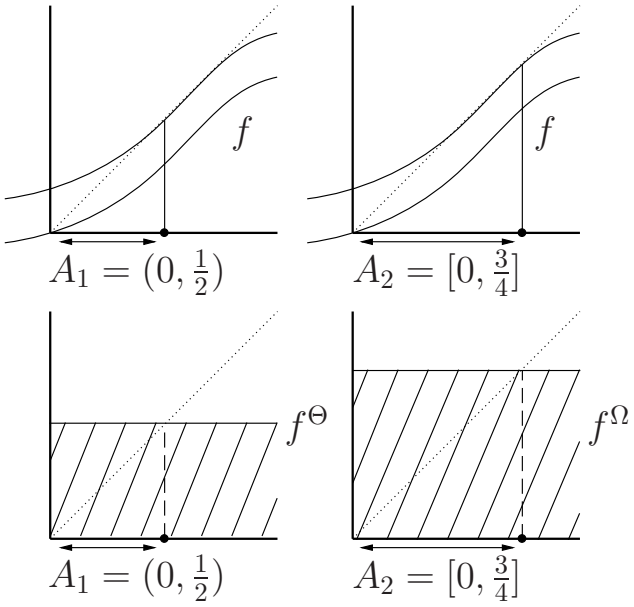


Figure 3.2: Inner Attractors and Attractors

The following proposition shows how inner attractors are invariant for open relations.

Proposition 50. *If A is an inner attractor for an open relation f then $f(A) = A$.*

Proof. As f is open, Theorem 40 states that $f \circ f^\Theta = f^\Theta$ and therefore $f(A) = f \circ f^\Theta(A) = f^\Theta(A) = A$. □

McGehee and Wiandt [23] give a collection of properties for attractors. For instance they show that attractors are closed in X. We will now focus on showing similar properties however in our case the inner attractors are open.

Lemma 51. *An inner attractor for any relation f on X is an open set in X.*

Proof. $A = f^\Theta(A)$ is open as f^Θ is open by Lemma 43. □

Next we define the dual repellers and connecting orbits for both types of attractor.

Definition 52. If A is an attractor for a closed relation f on X then the *dual repeller* of A is defined to be the set $A^* = B(A)^c$. We define the *connecting orbits* as $\mathcal{C}(A) = (A \cup A^*)^c$.

We define the same concepts for the inner attractors as so

Definition 53. If A is an inner attractor for a relation f on X then the *dual inner repeller* of A is defined to be the set $A^* = B_I(A)^c$. We define the *connecting orbits* as $\mathcal{C}_I(A) = (A \cup A^*)^c$.

Remark 54. Trivially the empty set is an inner attractor in which case $B_I(\emptyset) = \emptyset$and so $\emptyset^* = X$ and $C_I(\emptyset) = \emptyset$.

We show the inner dual repeller A^* is open in contrast to the dual repeller which is closed [23].

Lemma 55. *The inner dual repeller A^* for a relation f on X is open*

Proof. First we show that $A^* = f^{*\Theta}(A^c)$ and then note that A^* must be open as $f^{*\Theta}$ is open. Recall that for any relation we defined in Lemma 27 $f^{-1}(A) = \{x | f(x) \subset A\}$. Using Lemma 27 and part *(1)* of Theorem 36, we see that:

$$A^* = B_I(A)^c = \left((f^{\Theta})^{-1}(A)\right)^c = \left(f^{\Theta}\right)^* (A^c) = f^{*\Theta}(A^c)$$

□

Remark 56. Note that by Lemma 55 and Lemma 51 the connecting orbits of an inner attractor A is a closed set as $A \cup A^*$ is open.

3.3 The Open Conley Theorem

We can now give the adaptation of the Conley decomposition theorem for open relations. Recall that $\mathcal{U}(f)$ and $\mathcal{V}(f)$ are the collections of attractors and inner attractors for f. First we state the Conley decomposition theorem and then the open Conley Decomposition Theorem.

Theorem 57. *(Conley Decomposition Theorem) For any closed relation f on X we have*

$$R(f)^c = \bigcup_{A \in \mathcal{U}(f)} \mathcal{C}(A).$$

Theorem 58. *(Open Conley Decomposition Theorem) For any open relation f on X we have*

$$P(f)^c = \bigcup_{A \in \mathcal{V}(f)} \mathcal{C}_I(A).$$

Proof. $(\supset)$: Let $x \in \mathcal{C}_I(A)$ for some inner attractor $A \in \mathcal{V}(f)$. Since $x \in B_I(A)$, $f^\Theta(x) \subset A$. As $x \notin A$ we have $x \notin f^\Theta(x)$. Therefore, $x \notin P(f)$ by Theorem 46.

$(\subset)$: Let $x \in P(f)^c$. As $f^\Theta \circ f^\Theta(x) = f^\Theta(x)$ by Theorem 40, we see that $A := f^\Theta(x)$ an inner attractor. As $x \in P(f)^c$ then by Theorem 46 we have $(x, x) \notin f^\Theta$ and thus $x \notin A$. Conversely, $f^\Theta(x) = A$ implies $x \in B_I(A)$ which means $x \notin A^*$. Hence $x \notin A \cup A^*$, so $x \in C_I(A)$. □

3.4 Open Conley Theorem for Relations Satisfying Additional Hypotheses

Next we will put the open Conley theorem in the context of relations with the additional hypotheses *(H1),(H2)* and *(H3)*. We show how invariance and minimal invariance for such set-valued dynamical systems relates to inner attractors. Note that if a set $A \in \mathcal{K}(X)$ is invariant for such a relation f it does not follow that A^o is invariant for f^Θ. Instead the following proposition holds.

Proposition 59. *If f is a relation satisfying (H1),(H2) and (H3) and $A \in \mathcal{K}(X)$ is invariant for f then there exists an inner attractor $A' \subset A$.*

Proof. If $f(A) = A$, then for any $g \Subset f$ we have $g(A) \subset A$ and so $g^\infty(A) \subset A$, and thus $f^\Theta(A) \subset A$. Choose $x \in A$, then $A' := f^\Theta(x) \subset A$ is an inner attractor by Theorem 40. We now only require to show that this inner attractor is non-empty. Due to *(H3)* we know that $f(x)$ must contain a $\epsilon-$ball and therefore must be non-empty. Thus given $0 < \tau < \epsilon$ there must exist a closed relation g such that $g \Subset f$ and for each $x \in X$ there exists $y \in g(x)$ such that $\overline{B_\tau(y)} \subset g(x)$ so g is non-empty for each $x \in X$. We know from Remark 18 that $\omega(x, g)$ must be non-empty. As $\omega(x, g) \overset{\text{Rem 32}}{\subset} g^\omega(x) \overset{\text{Lem 38}}{\subset} f^\Theta(x)$ so $f^\Theta(x)$ is non-empty. □

We explain how inner attractors are related to minimal invariant sets.

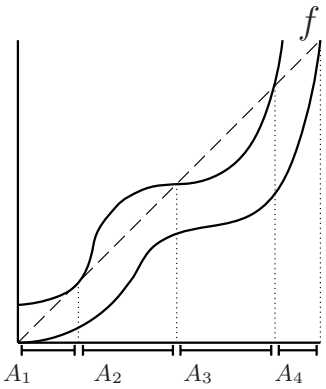
Proposition 60. *An inner attractor A for a relation f satisfying (H1),(H2) and (H3) is the interior of a minimal invariant set if it is minimal in the sense that if for any $A' \in \mathcal{V}(f)$ such that $A' \subset A$ then $A' = A$.*

Proof. Invariance of $\overline{A}$ follows from Proposition 50 and Proposition 15 . Hence we just require to show that $\overline{A}$ is a minimal invariant set. We assume that $\overline{A}$ is not minimal, i.e. there exists a closed set $A' \subsetneq \overline{A}$ such that $f(A') = A'$. Proposition 59 implies that there exists an inner attractor inside A' which contradicts the minimality of A. □

3.5 Examples

The following example shows how the inner attractors and open Conley decomposition theorem describe a set of objects related to minimal invariant sets while the Conley decomposition theorem and attractors describe a different set of objects.

Example 61. Let f be the relation defined pictorially as follows:



Each of the sets marked are closed. The collection of inner attractors are $\mathcal{V}(f) = \{\emptyset, \mathring{A}_1, (A_1 \mathring{\cup} A_2), \mathring{X}\}$ hence from Proposition 60 we see that A_1 is a minimal invariant set. Let $\{z\} = A_1 \cap A_2$. We see that $(\mathring{A}_2 \cup \mathring{A}_3 \cup \mathring{A}_4)$ is the dual repeller to $\mathring{A}_1$ and the set $\mathring{A}_4$ is dual to $(A_1 \cup A_2)$. Theorem 58 decomposes X into $\mathring{A}_1 \cup \mathring{A}_2 \cup \mathring{A}_4 = P(\mathring{f})$ and $\{z\} \cup A_3 = \bigcup_{A \in \mathcal{V}(f)} \mathcal{C}_I(A)$. In contrast we find the attrac-

tors are $\mathcal{U}(f) = \{\emptyset, A_1 \cup A_2, X\}$ where A_4 is a dual repeller to $A_1 \cup A_2$. The Conley decomposition theorem 57 gives us $A_1 \cup A_2 \cup A_4 = R(f)$ and $A_3 = \bigcup_{A \in \mathcal{U}(f)} \mathcal{C}(A)$.

4

Orbital Connectivity and Minimal Invariance

In this section we take a more detailed look at the periodic sets. In the previous section, Proposition 60 gives a connection between the inner attractors and minimal invariant sets. The inner attractors are related to $P(f^o)$ by Theorem 58. Together these two results hint at a relation between $P(f^o)$ and the minimal invariant sets.

We will first give some results for the periodic sets which will be required in later sections. The periodic sets on there own are not enough to describe the minimal invariant sets instead we find there are certain subsets of $\overline{P(f^o)}$ which are. To obtain such subsets of $\overline{P(f^o)}$ we partition it into collections of points that are connected to each other under forwards and backwards iterates of the system. That is for any two points in $P(f^o)$ we can map each onto the other under a finite number of iterations of f^o. We call this property orbital connectedness. In the final section we show that assuming f is a relation satisfying *(H1),(H2)* and *(H3)* we obtain a simple description of the minimal invariant sets of f in terms of the closures of orbitally connected sets.

4.1 Periodicity

Periodicity is an important concept within dynamical systems. Examples are fixed or periodic points for single-valued dynamical systems, i.e. x such that $f(x) = x$ or $f^n(x) = x$ for some $n \in \mathbb{N}$. The abstraction of periodic points to the context of set-valued dynamical systems will prove useful when talking about the minimal invariant sets. To introduce these ideas we first introduce orbits for such systems, a concept which arises naturally in the discussion of periodicity.

Definition 62. (Orbit) A *finite orbit* for a relation f is a sequence $\gamma = \{\gamma_i\}_{i=0}^n \subset X$ such that $\gamma_n \in f(\gamma_{n-1})$ for $i \in \{1, ..., n\}$. We say γ is an *open finite orbit* if it is an orbit for an open relation f. We say γ is an *orbit* if $\gamma = \{\gamma_n\}_{n=-\infty}^{\infty}$ and $\gamma_{n+1} \in f(\gamma_n)$ for all $n \in \mathbb{N}$. We say an orbit γ is through x if $\gamma_i = x$ for some $i \in \mathbb{N}$.

Remark 63. Note that with respect to the literature an orbit for a dynamical system is usually defined as a sequence in the dynamical systems phase space in our case $\mathcal{K}(X)$. However for our purposes the definition of orbit is a sequence of points in X.

A simple property follows from the definition.

Lemma 64. *(1) Let $\gamma = \{\gamma_i\}_{i=0}^n$ be a finite orbit for a relation f. Then $\gamma^* := \{\gamma_i^* = \gamma_{n-i}\}_{i=0}^n$ is a finite orbit for f^*.*

(2) Furthermore if $\gamma_n = \gamma_0$ then we have $\gamma_i \in f^n(\gamma_i)$ and $\gamma_i^* \in f^{*n}(\gamma_i^*)$.

Proof. If $\gamma_i \in f(\gamma_{i-1})$ then by Definition 22 we have $\gamma_{i-1} \in f^*(\gamma_i)$ and thus by reverting the order of γ we obtain an orbit for the dual system. If $\gamma_n = \gamma_0$ then γ can be extended to a full orbit γ' such that $\gamma'_{jn+i} = \gamma_i$ for all $j \in \mathbb{N}$ and $i \in \{0, ..., n\}$. Hence $\gamma_i \in f^n(\gamma_i)$ for all $i \in \{0, ..., n\}$ and by extending the reversed orbit γ^* to a full orbit $\gamma^{*\prime}$ we obtain the same result but for f^*. □

Remark 65. The exact same lemma and proof hold for finite open orbits.

The periodic sets for f are made up of the points x in X for which finite open orbits with $\gamma_n = \gamma_1 = x$ exist. We recall the notion of periodic sets which given relation f on X are the sets

$$P_n(f) = \{x \in X | (x, x) \in f^n\} \text{ and } P(f) = \bigcup_{n=1}^{\infty} P_n(f).$$

We will now explore these sets more thoroughly than in Section 3.1.2. First we show that the periodic sets are open or closed dependent on if the relation f is open or closed.

Proposition 66. *(1) Given a closed relation, f, then $P_m(f)$ is closed for all $m \in \mathbb{N}$.*

(2) Given a open relation, f, then $P_m(f)$ is open for all $m \in \mathbb{N}$.

Proof. (1):(see also [23]) Define $\iota = \{(x,x)|x \in X\} \subset X \times X$. As X is compact ι is a closed relation. f^m is also closed and thus $\iota \cap f^m$ is closed. Define the projection $\pi : X \times X \mapsto X$ given by $\pi(x,y) = x$. It is well know that projections preserve closed sets. Hence $P_m(f) = \pi(\iota \cap f^m)$ is closed.

(2): If $x \in P_m(f^o)$ then $(x,x) \in (f^o)^m$. $(f^o)^m$ is open in $X \times X$. Note that the topology on $X \times X$ is the product topology. X is a compact subset of $\mathbb{R}^n$ so is second countable . Hence there exists a countable basis $\{B_i\}_{i=1}^{\infty}$ of X and any open set in X can be written as the countable union of certain basis sets. The product topology on $X \times X$ means any open set in $X \times X$ can be written as a countable union of basis sets of the form $B_i \times B_j$ where $B_i, B_j \in \{B_i\}_{i=1}^{\infty}$. Let $\{A_i\}_{i=1}^{\infty}$ be a countable family of basis sets in $X \times X$ such that $\bigcup_{i=1}^{\infty} A_i = (f^o)^m$. As $(x,x) \in (f^o)^m$ there must exist A_j such that $(x,x) \in A_j$. A_j is of the form $B_k \times B_l$. Define $B = (B_k \cap B_l) \times (B_k \cap B_l)$ then $(x,x) \in B$, $B \subset (f^o)^m$ and B is open in $X \times X$. Notice that in particular $x \in B_k \cap B_l$ and $B_k \cap B_l \subset P_n(f^o)$. The result follows as $B_k \cap B_l$ is open. □

Remark 67. Given a relation f satisfying *(H1),(H2)* and *(H3)* then Proposition 66 implies that $P_m(f)$ is closed and $P_m(f^o)$ is open for all $m \in \mathbb{N}$.

The next proposition gives us properties for $P_n(f)$ which are important for the discussion of minimal invariance specifically 4) which we use in Theorem 88.

Proposition 68. *Given an open or closed relation f the following claims hold for all $n \in \mathbb{N}$*

1) For every $x \in P_n(f)$ there exists an orbit $\gamma = \{\gamma_i\}_{i=0}^{j} \subset P_n(f)$ such that $\gamma_0 = \gamma_j = x$. If f is open then γ is an open orbit.

2) $P_n(f) = P_n(f^*)$

3) $P_n(f) \subset f(P_n(f))$, $P_n(f) \subset f^*(P_n(f))$

4) $f(P(f)) \cap f^*(P(f)) \supset P(f)$

Proof. 1) Assume f is open. If $x \in P_n(f)$ then $x \in f^n(x)$. Clearly we can pick a open orbit $\{\gamma_i\}_{i=1}^n$ with $\gamma_1 = \gamma_n = x$ such that $\gamma_i \in f(\gamma_{i-1})$. To see that each $\gamma_i \in P_n(f)$ notice that $\gamma_i \in f^n(\gamma_i)$ and γ is open as f is open.

2) $P_n(f) = P_n(f^*)$ follows from the fact that a γ orbit for f is an γ orbit for f^* by lemma 64.

3) If $x \in P_n(f)$ then there exists an γ orbit of length at most n and each term in this orbit is in $P_n(f)$. Hence assuming w.l.o.g that the γ orbit is of length n then $x \in f(\gamma_{n-1})$ and $x \in f(P_n(f))$. As this is true for every $x \in P_n(f)$ we must have $P_n(f) \subset f(P_n(f))$. Using lemma 64 we also get $P_n(f) \subset f^*(P_n(f)))$.

4) From 3) we know that $P_n(f) \subset f(P_n(f))$ and $P_n(f) \subset f^*(P_n(f))$ and therefor $P_n(f) \subset f(P_n(f)) \cap f^*(P_n(f))$ for all $n \in \mathbb{N}$ and hence in the limit $P(f) \subset f(P(f)) \cap f^*(P(f))$. □

In order to get the opposite inclusion, $P(f) \supset f(P(f)) \cap f^*(P(f))$, of part 4 above we require to introduce orbital connectivity as in the next section.

4.2 Orbital Connectivity and Chain Connectivity

Next we define pairs of points that are connected to each other under iteration of the set-valued dynamical system which we call orbitally connected. We show that the collection of these points admit a decomposition into orbitally connected sets which are collections of points for which any pair is orbitally connected. The main result of this section is Theorem 88 which shows the relation of orbitally connected sets to minimal invariant sets.

Definition 69. (orbitally connected points and sets) Let f be an open relation on X a compact subset of $\mathbb{R}^n$. Two points $x, y \in P(f)$ are called *orbitally connected,* denoted $x \sim y$, if there exist $n_1, n_2 \in \mathbb{N}$ such that $y \in f^{n_1}(x)$ and $x \in f^{n_2}(y)$.

Remark 70. For any pair of points $x, y \in X$ if one of them is not in $P(f)$ then at least one of n_1 or n_2 cannot exist thus it is not possible to define orbital connectedness on the whole of X.

Note that orbitally connected points for f are also orbitally connected for f^* due to Lemma 64.

It is significant that we require f be open. If f is a closed relation then $x \sim y$ if x and y are orbitally connected for the interior system f^o.

We show that $\sim$ is an equivalence relation on $P(f)$ for open relations f. This will allow us to eventually describe each minimal invariant sets of f as the closure of an orbitally connected equivalence class.

Lemma 71. *If f is a relation on a compact subset $X \subset \mathbb{R}^n$ then $\sim$ is an equivalence relation.*

Proof. Clearly $x \sim x$. It follows from the definition that $x \sim y$ implies $y \sim x$, and finally if $x \sim y$ and $y \sim z$ then there exists natural numbers n_1, n_2,m_1 and m_2 such that $y \in f^{n_1}(x)$, $x \in f^{n_2}(y)$, $z \in f^{m_1}(y)$ and $y \in f^{m_2}(z)$, Hence $z \in f^{n_1+m_1}(x)$ and $x \in f^{n_2+m_2}(z)$ and thus $x \sim z$. □

Remark 72. If x, y are orbitally connected for an open relation f there exist two open finite orbits γ^1, γ^2 such that $\gamma^1_0 = x$, $\gamma^1_{n_1} = y$, $\gamma^2_0 = y$ and $\gamma^2_{n_2} = x$ for some $n_1, n_2 \in \mathbb{N}$. Due to $x \sim y$ being an equivalence relation $\gamma^1_{i_1} \sim \gamma^1_{i_2} \sim \gamma^2_{i_3} \sim \gamma^2_{i_4}$ for all $0 < i_1, i_2 < n_1$ and $0 < i_3, i_4 < n_2$ for which γ^1 and γ^2 are defined. This implies that $x, y \in P_{n_1+n_2}(f)$.

Definition 73. (Orbitally connected set) Given a relation f on a compact set $X \subset \mathbb{R}^n$ we define $[x] = \{y \in X | x \sim y\}$ as the equivalence class for any $x \in P(f^o)$. We refer to this set as the *Orbitally connected set.*

Remark 74. As $\sim$ is the same for f and f^* we see that $[x]$ is the same set for f and f^*.

Given a relation $g \in X \times X$ we may refer to the orbitally connected class of x for g as $[x]_g$.

The relation of these above sets to minimal invariant sets will be shown in the main result of the next section which states that for any minimal invariant set E, there exists an orbitally connected set $[x]$ such that $E = \overline{[x]}$.

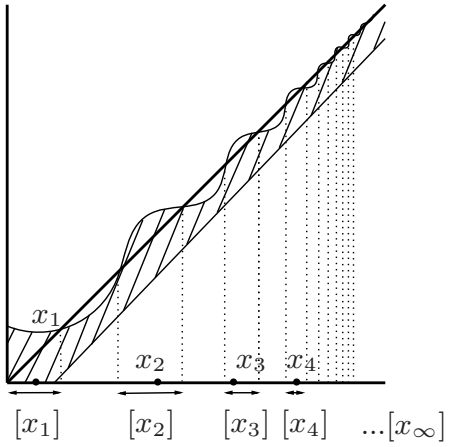


Figure 4.1: The infinite collection of sets $\{[x_i]\}_{i=1}^{\infty}$ partition $P(f^o)$

Remark 75. Given an relation f, the collection of equivalence classes for f^o is a partition of $P(f^o)$, but not necessarily a finite partition (See Figure 4.1).

Finally we give a result that carries properties we'd previously shown for the P_n sets over to the orbitally connected sets.

Lemma 76. *Given a relation f for any $[x] \subset P(f^o)$ the following hold*

1) $[x]$ is open in X

2) $[x] \subset f([x])$, $[x] \subset f^*([x])$

Proof. 1) Let $y \in [x]$ then there exists $n_1, n_2 \in \mathbb{N}$ such that $(x, y) \in (f^o)^{n_1}$ and $(y, x) \in (f^o)^{n_2}$. As f^o is open $(f^o)^n$ is open for all n. Hence there exists open sets U_1,$U_2 \subset X \times X$ such that $(x, y) \in U_1 \subset (f^o)^{n_1}$ and $(y, x) \in U_2 \subset (f^o)^{n_2}$. Thus there also exist open sets $V_1, V_2 \subset X$ such that $y \in V_1 \subset \{b|(a, b) \in U_1\}$ and $y \in V_2 \subset \{a|(a, b) \in U_2\}$. Note that $y \in V = V_1 \cap V_2$ which is an open set as both V_1 and V_2 are open. Thus we just require to show that V is contained in $[x]$. Notice that if $z \in V$ then by construction $(x, z) \in (f^o)^{n_1}$ and $(z, x) \in (f^o)^{n_2}$.

2) By definition of $[x]$ for any point $y \in [x]$ there must exist a point $z \in [x]$ such that $y \in f(z)$. Thus $f([x])$ must contain $[x]$. The same is true for f^*. □

Note that we can define orbitally connected for a closed relation f instead of f^o.

Then the orbitally connected sets become $[x]_{\text{closed}} = \{y | x \sim y \text{ for } f \text{ where } f = \overline{f}\}$. Unlike the orbitally connected sets of an open relation which are open as per Lemma 76 the orbitally connected sets for a closed relation are not necessarily closed.

We will later derive Hausdorff continuity properties for the orbitally connected sets for f^o which will be useful when proving bifurcation results. We would hope that we can also derived similar continuity properties for $[x]_{\text{closed}}$ however it turns out this is not the correct set to consider. Instead we have to define the following.

Definition 77. *(Chain connected)* Let f be a relation on X a compact subset of $\mathbb{R}^n$. Two points $x, y \in R(f)$ are called *chain connected for* f, denoted $x \approx y$, if for all $g \ni f$ we have x orbitally connected to y under g.

Chain connected points where introduced into the mathematical literature for single-valued dynamical systems by Charles Conley in [11, 10]. They are also utilized in [23] with respect to relations and our definition is equivalent to the one given there.

We start by quickly demonstrating some important features of chain connected sets. Many equivalent results to the following can be found in [23].

Lemma 78. *$\approx$ for a relation f is an equivalence relation.*

Proof. $x \approx x$ and $x \approx y \implies y \approx x$ are both obvious. If $x \approx y$ and $y \approx z$ then for all $g \ni f$ we see that $x \sim y$ and $y \sim z$ so by Lemma 71 $x \sim z$ for each g and thus $y \approx z$. □

In the same manner as we defined the orbitally connected sets so we define the chain components.

Definition 79. *(chain components)* Given a relation f on a compact set $X \subset \mathbb{R}^n$ we define $[[x]] = \{y \in X | x \approx y\}$ as the equivalence class for any $x \in R(f)$. We refer to this set as the *chain connected set.*

In the same way that the collection of orbitally connected sets partition $P(f)$ we see that the collection of chain components partition $R(f)$.

Unlike $[x]_{\text{closed}}$ it is possible to show that the chain components are actually closed. Before doing so we will need the following lemmas.

Lemma 80. *Let f be a relation on X. We have $[[x]] = \bigcap_{g \ni f} [x]_g$ where $[x]_g$ is the orbitally connected set for x.*

Proof. ($\supset$) If $y \in \bigcap_{g \ni f} [x]_g$ then $y \sim x$ for all $g \ni f$ so $y \in [[x]]$.

($\subset$) If $y \in [[x]]$ then $y \backsimeq x$ and so for all $g \ni f$ we see that $y \sim x$ for g and hence $y \in [x]_g$ for all g so $y \in \bigcap_{g \ni f} [x]_g$ □

Lemma 81. *Let g, h be open relations on X . If $h \Subset g$ then $[x]_h \Subset [x]_g$.*

Proof. Clearly $[x]_h \subset [x]_g$ as if $y \sim x$ under h then $y \sim x$ under g as $h \Subset g$. $[x]_h$ and $[x]_g$ are both open due to h and g being open and lemma 76. Thus if $\overline{[x]}_h \subset [x]_g$ then $[x]_h \Subset [x]_g$. Assume for contradiction that $\overline{[x]}_h \nsubseteq [x]_g$ and there exists $z \in \overline{[x]}_h$ such that $z \notin [x]_g$. We will show that there exists y_1 and y_2 such that $y_1, y_2 \in [x]_g$ and $z \in g(y_1)$ and $y_2 \in g(z)$ and so $z \in [x]_g$ which is a contradiction with the assumption.

First we find two points $p, q \in \overline{[x]}_h$ such that (p, z) and $(z, q) \in \overline{h}$. As $z \in \overline{[x]}_h$ there exists a sequence $\{z_i\}_{i=1}^{\infty}$ such that $z_i \in [x]_h$ for all $i \in \mathbb{N}$ and $z_i \to z$. As each $z_i \in [x]_h$ there exists $p_i, q_i \in [x]_h$ such that $(p_i, z_i) \in h$ and $(z_i, q_i) \in h$. As $p_i, q_i \in [x]_h$ there exists two sub-sequences i_k and i_l such that $p_{i_k} \to p \in \overline{[x]}_h$ and $q_{i_l} \to q \in \overline{[x]}_h$. similarly $(p_{i_k}, z_i) \to (p, z) \in \overline{h} \subset g$ and $(z_i, q_{i_l}) \to (z, q) \in \overline{h} \subset g$.

g is open so there must exist open sets U, V and W in X such that $p \in U, z \in V$ and $q \in W$ and also $U \times V \subset g$ and $V \times W \subset g$ (See Figure 4.2). We show the following property of the sets U, V and W. For any $y_1 \in U$ we have $V \subset g(y_1)$ and for any $y_2 \in V$ we have $W \subset g(y_2)$. To see this notice that if $V \subsetneq g(y_1)$ for such $y_1 \in U$ then there exists $y' \in V \backslash g(y_1)$ but this implies $(y_1, y') \in U \times V \subset g$ but $(y_1, y') \notin g$. The same method holds for any $y_2 \in V$ and W.

Next we use U, V and W to find y_1 and y_2 in $[x]_g$ that generate the contradiction.

As $p \in U$ which is open and $p \in \overline{[x]}_h$ which is closed then there must exist $y_1 \in U \cap [x]_h$. Due to the property above of U and V we must have and $z \in g(y_1)$ as $z \in V$.

Similarly as $q \in W$ which is open and $q \in \overline{[x]}_h$ which is closed then there exists $y_2 \in W \cap [x]_h$ such that $y_2 \in g(z)$ as $z \in V$.

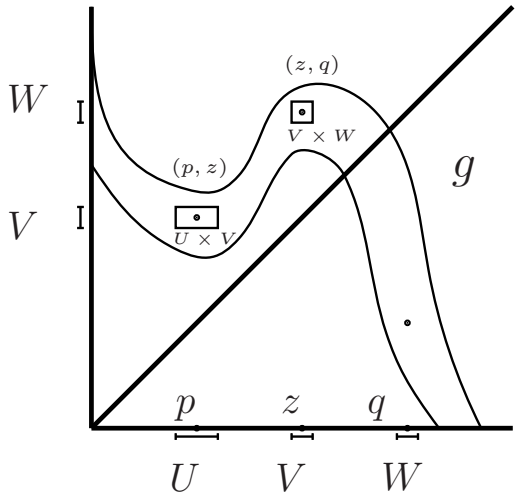


Figure 4.2: Example of choice of U, V and W for (p, z) and (z, q) in g^o

Finally as $[x]_h \subset [x]_g$ so y_1, y_2 must be in $[x]_g$ and thus $z \in [x]_g$. □

Lemma 82. *Let f be a closed relation then $[[x]]$ is a closed set.*

Proof. Recall Lemma 80 tells us $[[x]] = \bigcap_{g \supset f} [x]_g$, choose $z \in \overline{[[x]]}$. Then given any $g \ni f$ there exists open relations h_1, h_2 such that $g \ni h_1 \ni h_2 \ni f$. As $h_1 \ni h_2$ and both h_1 and h_2 are open we see from lemma 81 that $[x]_g \supset [x]_{h_1} \ni [x]_{h_2} \supset [[x]]$. Hence $[x]_g \supset [x]_{h_1} \supset \overline{[x]_{h_2}} \supset \overline{[[x]]}$ and as $[x]_{h_1}$ is open by lemma 76 we see that $z \in [x]_{h_1}$so $z \in [x]_g$. This is true for all $g \ni f$ and so $z \in \bigcap_{g \ni f} [x]_g$. □

4.2.1 Invariance Properties of Orbitally Connected Sets

In order to show the main theorem of this section that any minimal invariant set is the closure of an orbitally connected set we will require some kind of invariance for the orbitally connected set. However in general orbitally connected sets are not invariant under either f or f^o as the figure 4.3 shows.

We find however that orbitally connected sets satisfy the following property which subsumes invariance.

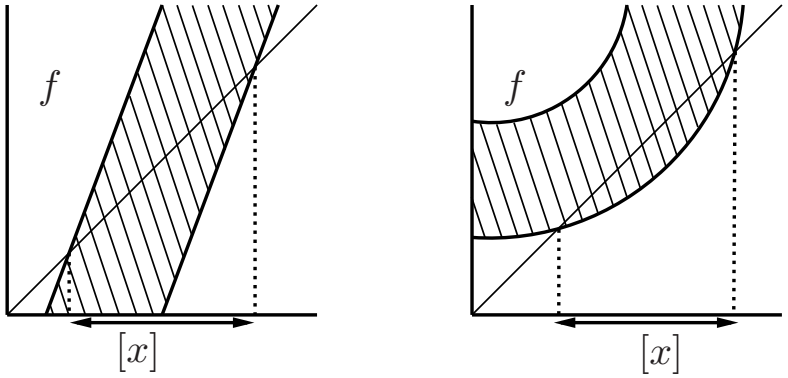


Figure 4.3: Examples of $[x]$ which are not invariant for either f or f^o.

Definition 83. For a relation f a set $E \subset X$ is called $f \cap f^*$-*invariant* if $f(E) \cap f^*(E) = E$

Remark 84. If given a relation f we assume that a subset E of $\mathcal{K}(X)$ is f-invariant then E is $f \cap f^*$ -invariant. To see how notice that if $E = f(E)$ then $E \subset f^*(E)$ and thus $f(E) \cap f^*(E) = E \cap V$ where $E \subset V$ and thus $E \cap V = E$. The same holds for f^* invariant sets.

Next we show that given any relation f any orbitally connected set for f is $f^o \cap f^{o*}$ -invariant.

Lemma 85. *Given any relation f for any $x \in P(f^o)$, the set $[x]$ is $f^o \cap f^{o*}$ -invariant.*

Proof. $f^o([x]) \cap f^{o*}([x]) \supset [x]$ follows from Lemma 76. To see that $f^o([x]) \cap f^{o*}([x]) \subset [x]$, we assume not, then $f^o([x]) \cap f^{o*}([x]) \cap [x]^c \neq \emptyset$. Hence there exists $z \in f^o([x]) \cap f^{o*}([x]) \cap [x]^c$ hence $z \in f^o([x]) \cap f^{o*}([x])$ and so $z \in f^o([x])$ and $z \in f^{o*}([x])$. Thus there exists $x_1, x_2 \in [x]$ such that $z \in f^o(x_1)$ and $z \in f^{o*}(x_2)$. Now choose any $y \in [x]$. As both x_1 and y are in $[x]$ they are orbitally connected so there exists $n_1 \in \mathbb{N}$ such that $x_1 \in (f^o)^{n_1}(y)$ and thus $z \in (f^o)^{n_1+1}(y)$ as $z \in f^o(x_1)$. Both y and x_2 are in $[x]$ as well and so there exists $n_2 \in \mathbb{N}$ such that $y \in (f^o)^{n_2}(x_2)$ and so $y \in (f^o)^{n_2+1}(z)$ as $z \in f^{o*}(x_2)$. Hence y and z are orbitally connected which contradicts $z \notin [x]$. □

4.3 Relation of Periodic Sets to Minimal Invariant Sets

We now consider orbitally connected sets with respect to relations f that satisfy *(H1),(H2)* and *(H3)*. The main result of this section, Theorem 88, says that, given a minimal invariant set E, there exists $[z]$ for some $z \in P(f^o)$ such that $E = \overline{[z]}$. We require *(H1),(H2)* and *(H3)* as

1. *(H1)* in Definition 6 to prove Proposition 86 which is used in Theorem 88 and
2. Theorem 88 requires Proposition 19 which needs Hausdorff continuity or *(H2)* of Definition 6,
3. *(H3)* of Definition 6 is required for Proposition 87

The next two propositions show that the closures of certain orbitally connected sets are minimal invariant.

Proposition 86. *Let f be a relation satisfying (H1),(H2) and (H3). If $f^o([x])\backslash\overline{[x]} = \emptyset$ then $\overline{[x]}$ is f-invariant.*

Proof. $[x] \subset f^o([x])$ by Lemma 76 and by assumption $f^o([x]) \subset \overline{[x]}$ thus $[x] \subset f^o([x]) \subset \overline{[x]}$ and so $\overline{f^o([x])} = \overline{[x]}$. Due to Proposition 15 we have $\overline{f^o([x])} = f(\overline{[x]})$ and thus $f(\overline{[x]}) = \overline{[x]}$. □

This allows us to show that the closure of f^o-invariant equivalence classes are actually minimal invariant.

Proposition 87. *Let f be a relation that satisfies (H1),(H2) and (H3). If $f(\overline{[x]}) = \overline{[x]}$, then $\overline{[x]}$ is a minimal invariant set for f.*

Proof. We only need to show minimality as $\overline{[x]}$ is invariant by assumption. Assume $\overline{[x]}$ is not minimal for contradiction, then there exists a set $A \in \mathcal{K}(X)$ such that $A \subsetneqq \overline{[x]}$ and A is invariant. We know due to *(H3)* of Definition 6 that for any $y \in A$ there exists z such that $B_\epsilon(z) \subset f(y) \subset A$. Hence $A \cap [x] \neq \emptyset$ as $B_\epsilon(z) \subset A \subset \overline{[x]}$ and $[x]$ is open. Hence there exists $a \in [x] \cap A$ and $b \in [x]\backslash A$ such that $a \nsim b$ as A is invariant. But this contradicts $[x]$ as an equivalence class. (see Fig 4.4) □

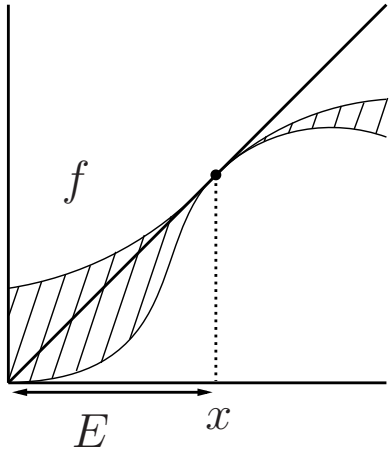


Figure 4.5: E is an invariant set and equal to $\overline{[z]}$ for some $z \in E^o$. E is not a minimal invariant set however as $x \in E$ and x is a fixed point so E is not minimal. Note that f does not satisfy *(H3)* at x.

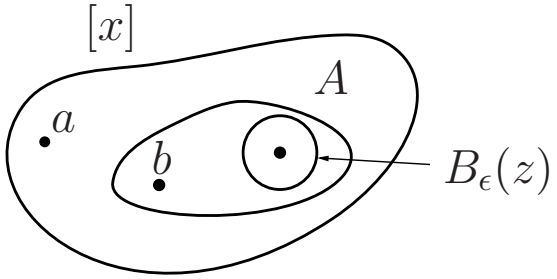


Figure 4.4: $a \nsim b$

Note that if *(H3)* of Definition 6 is not satisfied this result fails as figure 4.5 depicts.

We are now ready to show that any minimal invariant set is the closure of some equivalence class for $\sim$.

Theorem 88. *Let E be a minimal invariant set for a relation f that satisfies (H1),(H2) and (H3). Then $E = \overline{[x]}$ for some $x \in P(f^o)$.*

Proof. Firstly we prove that E must contain at least one point $x \in P(f^o)$. *(H3)* implies that there exists $z \in f(x)$ such that $B_\epsilon(z) \subset f(x)$ for every $x \in X$. E is a compact subset of $X \subset \mathbb{R}^m$ and so we can choose a cover of E by a finite number of

open balls with radius $\epsilon/2$. i.e. there exists some $n \in \mathbb{N}$ and a set of points $\{x_i\}_{i=1}^n$ such that $E \subset \bigcup_{i=1}^n B_{\epsilon/2}(x_i)$. Thus every point in E is a distance $\epsilon/2$ away from a point x_i in this collection and hence for each $i \in \{1, ..., n\}$ there exists at least one $j \in \{1, ..., n\}$ such that $f^o(x_i) \cap x_j \neq \emptyset$. We construct an open orbit of length $n+1$, call it $\gamma = \{\gamma_i\}_{i=1}^{n+1}$, such that each point in the orbit is a point in the set $\{x_i\}_{i=1}^n$. We construct the orbit with $\gamma_1 = x_1$, $\gamma_2 = x_{i_1}$ such that $x_{i_1} \in f^o(x_1)$, $\gamma_2 = x_{i_2}$ where $x_{i_2} \in f^o(x_{i_1})$ and so on. As there are only n points and the orbit γ is of length $n+1$ we must have at least one point in $\{x_i\}_{i=1}^n$ visited more than once. Hence such a point is a member of $P(f^o)$. (see Figure 4.6)

So there exists $x \in P(f^o) \subset E$. Now consider the set $[x]$ which must be a subset of E as E is invariant. Hence by Proposition 85 $f^o([x]) \cap f^{o*}([x]) = [x]$. We have three possibilities:

1) $f^o([x]) = [x]$,

2) $f^{o*}([x]) = [x]$, or

3) $f^o([x]) \cap f^{o*}([x]) = [x]$, $f^o([x]) \neq [x]$ and $f^{o*}([x]) \neq [x]$.

In case 1) as $f^o([x]) = [x]$, then $f^o([x]) \backslash \overline{[x]} = \emptyset$ and so $\overline{[x]}$ is minimal invariant by Propositions 86 and 87 and hence $E = \overline{[x]}$ due to minimality of E.

The proof for the cases 2) and 3) is the same. If $f^o([x]) \cap f^{o*}([x]) = [x]$ and $f^o([x]) \neq [x]$. If $f^o([x]) \backslash \overline{[x]} = \emptyset$ then the can consider the proof as in case 1). If $f^o([x]) \backslash \overline{[x]} \neq \emptyset$ then as $f^o([x])$ and $\overline{[x]}^c$ are both open so is $f^o([x]) \backslash \overline{[x]}$. Hence there exists $\kappa > 0$ and $y \in f^o([x]) \backslash \overline{[x]}$ such that $B_\kappa(y) \subset \left(f^o([x]) \backslash \overline{[x]} \right)$. $f^o([x]) \subset E$ by invariance of E and so $B_\kappa(y) \subset E$. For all $z \in B_\kappa(y)$ and $n \in \mathbb{N}$ we must have $(f^o)^n(z) \cap [x] = \emptyset$ as otherwise there exists an open return orbit that passes outside of $[x]$ which contradicts the definition of $[x]$ as an equivalence class of orbitally connected points. Thus as $[x]$ is an open set we see that $[x] \cap f^n(B_\kappa(y)) = \emptyset$ for all $n \in \mathbb{N}$ and hence $\omega(B_\kappa(y), f) \neq E$ a contradiction as $B_\kappa(y) \subset E$ which implies $\omega(B_\kappa(y), f) = E$ by claim 19. (see Figure 4.7) $\square$

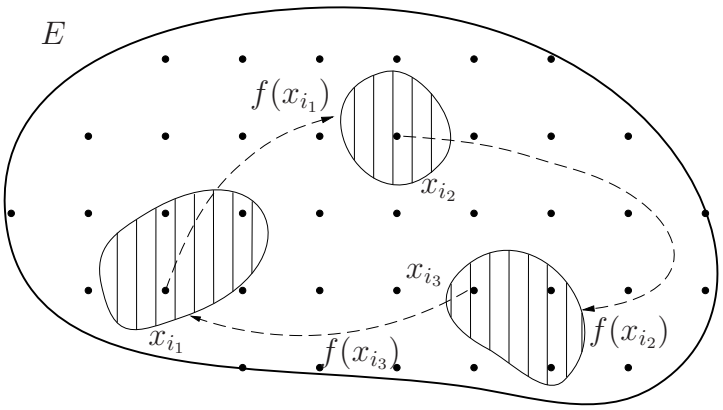


Figure 4.6: $\{x_i\}_{i=1}^n \in E$ and an open orbit constructed from the x_i

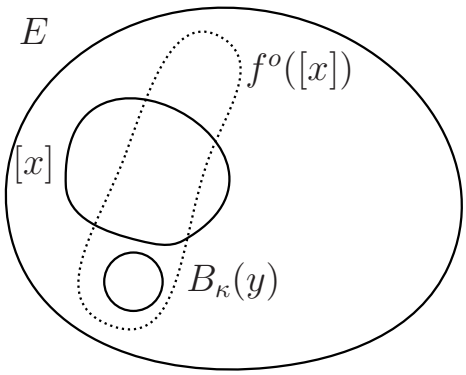


Figure 4.7: Illustration of $B_\kappa(y) \in E \backslash [x]$. Note that $(f^o)^n(B_\kappa(y)) \cap [x] = \emptyset$ for all $n \in \mathbb{N}$

We now have a description of minimal invariant sets in terms of periodicity for these set-valued dynamical systems.

4.3.1 Numerical Examples of Minimal Invariant Sets and Saddle Type Sets

Note that any point in a orbitally connected set for a relation f satisfying *(H1),(H2)* and *(H3)* is in $P_n(f^o)$ for some $n \in \mathbb{N}$. Thus we can approximate the minimal invariant sets by computing the $P_n(f^o)$ sets which we do in the following examples. (See Appendix A for details of the algorithm used)

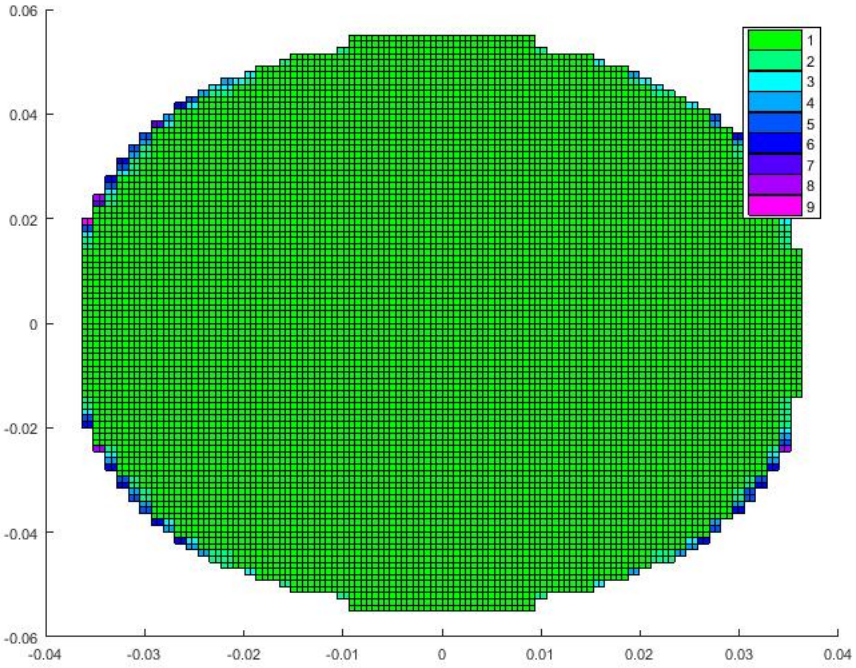


Figure 4.8: Approximation of $P_n(f)$ sets of orders 1 to 9 for f

Example 89. Let $g : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be defined by $g(x, y) = (0.4x, 0.6y)$ and let $f(x, y) = \overline{B_\epsilon(g(x, y))}$ where $\epsilon = 0.02$. We approximate the minimal invariant set for this system with $P_9(f)$. (See Figure 4.8)

In Lemma 85 we show that the $[x]$ sets satisfy the general relation $f^{o*}([x]) \cap f^o([x]) = [x]$. We give numerical evidence for the existence of such an orbitally connected set.

Example 90. Let $g : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be defined by $g(x, y) = (0.5x, 1.5y)$ and its epsilon expansion $f(x, y) = \overline{B_\epsilon(g(x, y))}$. Here we approximate such a set with $P_{11}(f)$. We also overlay its forward image under g as a collection of blue points. Each blue point must exist within ϵ of a point in the approximation of $P_{11}(f)$. (See Figure 4.9)

A more interesting example of the $P_n(f)$ sets is computed for the Henon map.

Example 91. Let $g : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be defined by $g(x, y) = (1 - x^2 + y, bx)$ and let $a = 1.4$ and $b = 0.3$. We consider the expansion f of g for $\epsilon = 0.08$ and compute the $P_n(f)$ sets for n from 1 to 9. (See Figure 4.10)

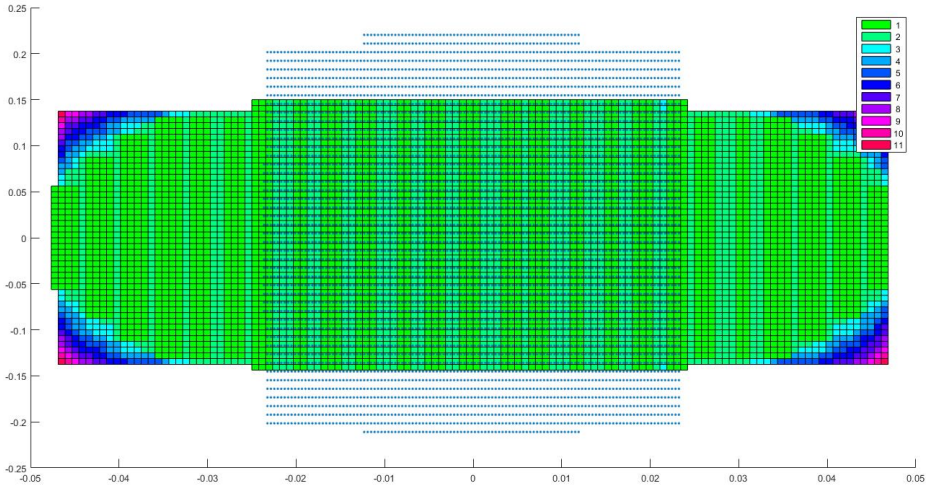


Figure 4.9: Approximation of $P_n(f)$ sets of orders 1 to 11 for f

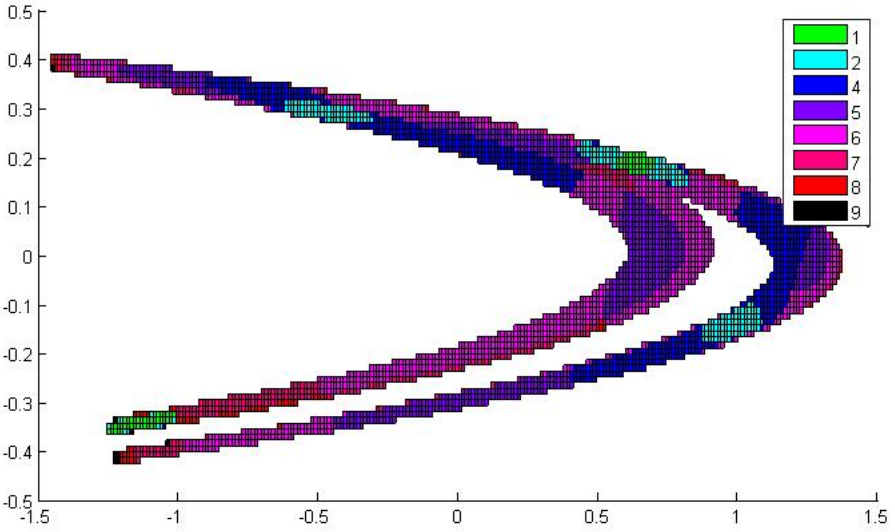


Figure 4.10: Approximation of $P_n(f)$ sets of orders 1 to 9 for f

5

Topological Bifurcations

In this chapter we define the notion of topological bifurcation for a family of set-valued dynamical systems parameterised by λ. We then explore necessary conditions on the family of systems and its minimal invariant sets such that we observe a bifurcation.

Recall that given a set-valued dynamical system f, we define $\mathcal{M}$ as the union of minimal invariant sets. For our purposes a bifurcation for f occurs if small changes to f results in large changes in $\mathcal{M}$. We use two notions of how changes in $\mathcal{M}_\lambda$ occur. The first is just continuity of $\mathcal{M}_\lambda$in the Hausdorff metric and the second is continuity of both $\mathcal{M}_\lambda$ and ${\mathcal{M}_\lambda}^c$ in the Hausdorff metric. We call this strongly Hausdorff continuous. A bifurcation occurs if $\mathcal{M}_\lambda$ or its complement is not continuous at λ^*. This looks like either a discontinuous jump in the Hausdorff metric of minimal invariant set or a minimal invariant set continuously developing a hole in its interior. (See Fig 5.1)

Once we have introduced the above concepts we then show that the orbitally connected and chain connected sets satisfy different types of Hausdorff continuity and in particular in Lemma 116 we show that if at λ^* we have $[x]_{\lambda^*} = [[x]]^o_{\lambda^*}$ then $[x]_\lambda$ will be Hausdorff continuous for families of strongly Hausdorff set-valued dynamical

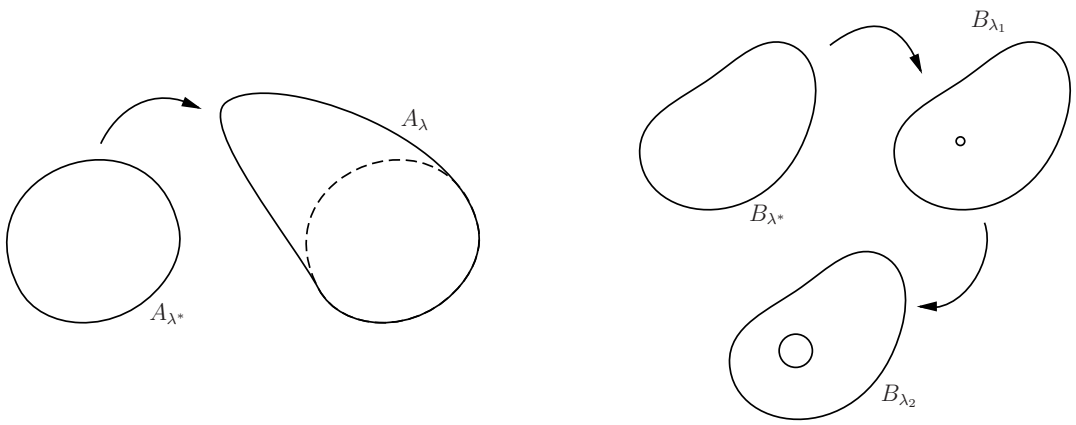


Figure 5.1: First: A minimal invariant set A_{λ^*} admitting a discontinuous bifurcation at λ^*. There exists some $\tau > 0$ such that for all $\lambda > \lambda^*$ $h(A_{\lambda^*}, A_\lambda) \geq \tau$. Second: A Minimal invariant set B_{λ^*} such that B_λ develops a hole for all $\lambda > \lambda^*$.

systems. This is enormously useful for talking about bifurcations of $\mathcal{M}_\lambda$ due to Theorem 88 which relates the minimal invariant sets to the orbitally connected sets and thus helps us give conditions for when minimal invariant sets continue in the Hausdorff metric. We do exactly this in Theorem 119.

5.1 Bifurcations of Minimal Invariant Sets

We now focus on set-valued mappings which we will use to define families of set-valued dynamical systems. We will also define the type of continuity of these mappings that we are interested in. First recall that for any $A \in \mathcal{K}(X)$ we define the distance between $h(A, \emptyset)$ as equal to the diameter of X.

5.1.1 Hausdorff Continuity

Let $Y \subset \mathbb{R}^n$ and $Z \subset \mathbb{R}^m$ be compact sets and H a set-valued mapping $H : Y \mapsto \mathcal{K}(Z)$. We define lower and upper semi-continuity of such functions.

Definition 92. (See [6, 1]) If we have a set-valued function $H : Y \mapsto \mathcal{K}(Z)$ then:

1)H is called *upper semi-continuous in the Hausdorff metric* at λ^* if for all $\sigma > 0$ there exists $\delta > 0$ such that $H(\lambda) \subset B_\sigma(H(\lambda^*))$ for all $\lambda \in B_\delta(\lambda^*)$.

2)H is called *lower semi-continuous in the Hausdorff metric* at λ^* if for all $\sigma > 0$ there exists $\delta > 0$ such that for all $H(\lambda^*) \subset B_\sigma(H(\lambda))$ for all $\lambda \in B_\delta(\lambda^*)$

The following proposition follows trivially from the Definition of Hausdorff continuity given in Definition 5

Proposition 93. *Let H be a set-valued mapping from Y to $\mathcal{K}(Z)$. Assume H is both lower semi-continuous and upper semi-continuous in the Hausdorff metric at $\lambda = 0$ then H is Hausdorff continuous at $\lambda = 0$.*

In [17] Lamb, Rasmussen and Rodrigues show that for families of set-valued dynamical systems that satisfy *(H3)* and are uniformly Hausdorff continuous in x and Hausdorff continuous in λ then any minimal invariant set for such systems changes lower semi continuously or instantaneously disappears at λ. We instead use the following form of continuity for families of set-valued dynamical systems which we define now and is required for Lemma 116.

Definition 94. A set-valued map $H : Y \mapsto \mathcal{K}(Z)$ is called *strongly Hausdorff continuous* at $y \in Y$ if H and H^c are Hausdorff continuous at $y \in Y$.

Strong Hausdorff continuity of a set-valued mapping H prevents holes appearing in the interior of $H(y)$ under change in the Y variable. This is because given an open set A and a point $x \in A$ then $h(A, A\backslash\{x\}) = 0$ whereas $h(A^c, (A\backslash\{x\})^c) = h(A^c, A^c \cup \{x\}) = d(x, A^c) > 0$. For more detail we give Example 95.

Example 95. (Hausdorff continuous set-valued mapping) Let $Y = Z = [0, 1]$ and let $A = [a, b] \subset [0, 1]$ for $0 < a < b < 1$. We define a set-valued mapping, $H : [0, 1] \mapsto \mathcal{K}(X)$ such that H is Hausdorff continuous but not strongly Hausdorff continuous (See Figure 5.2). Pick a point $c \in (a, b)$ and let $\kappa = \min\{|a - c|, |b - c|\}$. We define $H(y) = [a, c - \kappa y] \cup [c + \kappa y, b]$. Notice that $H^c(0) = [0, a) \cup (b, 1]$ but $H^c(y) = [0, a) \cup (c - \kappa y, c + \kappa y) \cup (b, 1]$ for all $y > 0$. Thus $h(H(0), H(y)) = \kappa y$ for all y and is thus Hausdorff continuous at 0 whereas the function $y \mapsto h(H^c(0), H^c(y)) = \max\{|c - \kappa y - a|, |b - c - \kappa y|\}$ for all $y > 0$ and is not continuous at $y = 0$. Thus$H$ is not strongly Hausdorff continuous at $y = 0$.

Next we give an example of a strong continuous mapping that is similar to Example 95.

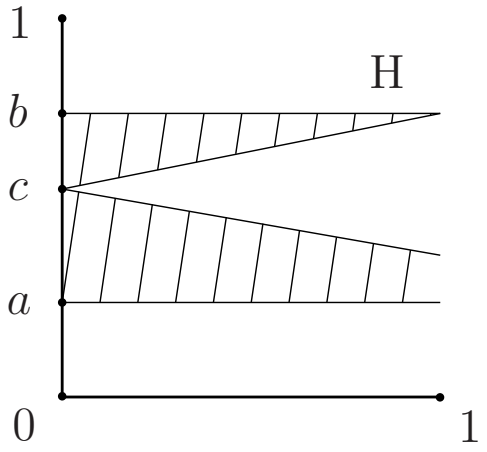


Figure 5.2: H is not strong Hausdorff continuous at $y = 0$

Example 96. (Strongly Hausdorff continuous set-valued mapping) Let $Y = Z = [0, 1]$ and let $A = [a, b] \subset [0, 1]$ for $0 < a < b < 1$. We define the following set-valued mapping H from $[0, 1] \mapsto \mathcal{K}(Z)$, such that H is strongly Hausdorff continuous. Define the single-valued map $g : Y \mapsto Z$ by $g(y) = b - \kappa y$ where $0 < \kappa < 2(b-a)/3$. Define H as $H(y) = [a, b] \backslash B_{\kappa y/2}(g(y))$ for $y > 0$ and $H(0) = [a, b]$. We see that $H(y)$ is clearly Hausdorff continuous at 0 as $B_{\kappa y/2}(g(y)) \mapsto \{b\}$ as $y \to 0$. Note that $H^c(y) = [0, a) \cup B_{\kappa y/2}(g(y)) \cup (b, 1]$ and thus $h(H^c(y), H^c(0)) = \min\{|g(y) - y\kappa/2 - a|, |b - g(y) + y\kappa/2|\}$. Thus as $y \mapsto 0$ we see that $h(H^c(y), H^c(0)) \mapsto 0$ and thus H^c is Hausdorff continuous at $y = 0$ as well.

Next we derive properties of strongly Hausdorff continuous mappings which will be useful later. The following proposition demonstrates the nature of Hausdorff continuity of the complement of set-valued mappings.

Proposition 97. *Let $H : Y \mapsto \mathcal{K}(Z)$ be a set-valued mapping such that H^c is upper semi-continuous in the Hausdorff metric at $y \in Y$. Then for any set $K \Subset H(y)$ there exists $\delta > 0$ such that $K \Subset H(y')$ for all $y' \in B_\delta(y)$.*

Proof. Let $K \Subset H(y)$. Then $H^c(y) \Subset K^c$. Which means that there exists $\tau > 0$ such that $B_\tau(H^c(y)) \Subset K^c$. Hausdorff upper semi-continuity of H^c at y implies there exists $\delta > 0$ such that for all $y' \in B_\delta(y)$ we have $H^c(y') \subset B_\tau(H^c(y))$. Hence $H^c(y') \Subset K^c$ and so $K \Subset H(y')$. □

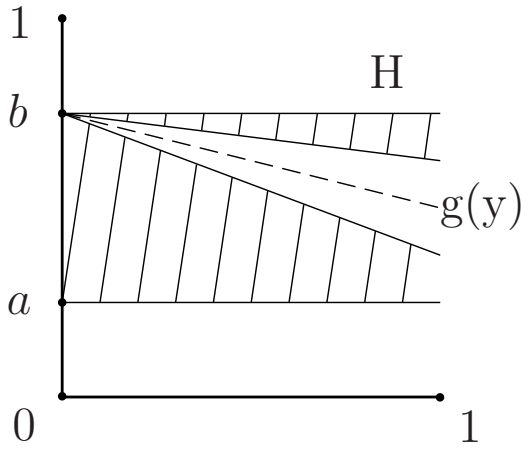


Figure 5.3: H is strong Hausdorff continuous at y

Note that Proposition 97 does not imply that H is lower semi-continuous at y. To obtain lower semi-continuity we require the following proposition which is particular to open set-valued mappings and will prove useful with respect to $[x]$.

Proposition 98. *Let $H : Y \mapsto \mathcal{K}(Z)$ be an open set-valued mapping such that H^c is upper semi-continuous in the Hausdorff metric at $y \in Y$. Then H is lower semi-continuous in the Hausdorff metric at $y \in Y$.*

Proof. Given $\sigma > 0$, if we can find a set K such that $K \Subset H(y)$ and $H(y) \Subset B_\sigma(K)$ then Proposition 97 implies there exists $\delta > 0$ such that $K \Subset H(y')$ for all $y' \in B_\delta(y)$. Hence $H(y) \subset B_\sigma(K) \subset B_\sigma(H(y'))$ and the result follows. Hence it only remains to show that we can find such a set K. Let $0 < \sigma' < \sigma$ and let $\tau = \sigma - \sigma'$. Choose a countable dense subset of $\overline{H(y)}$ denoted by $\{x_i\}_{i=0}^{\infty}$. Now let $\{B_{\sigma'}(x_i)\}_{i=1}^{\infty}$ be the collection of σ' balls with centers at each point x_i. This collection is a open cover of the set $\overline{H(y)}$ which is compact and hence there exists a finite open cover $\{B_{\sigma'}(x_{i_k})\}_{k=1}^{N}$. Each x_{i_k}must be in $\overline{H(y)}$ and hence we can choose x_j from $\{x_i\}_{i=1}^{\infty}$ such that $x_j \in H(y)$ and $d(x_{i_k}, x_j) < \tau$. Hence $B_{\sigma'}(x_{i_k}) \subset B_\sigma(x_j)$. As this is possible for each x_{i_k} we can change the finite cover of balls with centers in $\overline{H(y)}$ and radii equal to σ' into a finite cover of balls with centers in $H(y)$ and with radii σ. □

Note that by swapping H for H^c in the statement of Proposition 98 we get that if H is a closed upper semi-continuous set-valued mapping at $y \in Y$ then H^c is an open

lower semi-continuous set-valued mapping at $y \in Y$. Thus we have the following proposition.

Corollary 99. *If $H : Y \mapsto \mathcal{K}(Z)$ is closed Hausdorff continuous set-valued mapping at $y \in Y$ and H^c is upper semi-continuous at $y \in Y$ in the Hausdorff metric then H is strong Hausdorff continuous at $y \in Y$.*

5.1.2 Types of Bifurcation

In this section we study the different types of bifurcation for families of relations. Throughout we let X be a compact subset of $\mathbb{R}^n$ and $\mathcal{K}(X)$ the set of all compact subsets of X. We now define the families of relations we are interested in.

Definition 100. A *family of relations on X* is a collection of relations $\{f_\lambda\}_{\lambda\in\Lambda}$ on X where Λ is a compact metric space. We require a family of relations to also satisfy that

(H1') $\overline{f_\lambda(x)^o} = f_\lambda(x)$ for all $x \in X$ and each $\lambda \in \Lambda$,

(H2') each f_λ is Hausdorff continuous on X for all $\lambda \in \Lambda$,

(H3') there exists $\epsilon > 0$ such that for all $\lambda \in \Lambda$ and $x \in X$ there exists y such that $B_\epsilon(y) \subset f_\lambda(x)$.

We will also require that any family of relations $\{f_\lambda\}_{\lambda\in\Lambda}$ is strongly Hausdorff continuous at certain $\lambda^* \in \Lambda$.

Definition 101. Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of relations. We say that f_λ is strongly Hausdorff continuous at $\lambda^* \in \Lambda$ if the set-valued function $\lambda \mapsto f_\lambda$ is strongly Hausdorff continuous at λ^*.

Example 102. A useful example of such families is the family of set-valued dynamical system defined as follows. Let f be a relation on a compact subset X of $\mathbb{R}^2$ satisfying *(H1),(H2)* and *(H3)* and let $\delta > 0$. We define the family $\{f_\lambda\}_{\lambda\in[0,\delta]}$ given by $f_\lambda(x) = \overline{B_\lambda(f(x))}$ for $\lambda > 0$ and $f_0(x) = f(x)$.

We refer to this type of family as an λ expansion family of a set-valued dynamical system f. It is clear that f_λ satisfies *(H1'),(H2')* and *(H3')*. We show that $\lambda \mapsto f_\lambda$

must be strongly Hausdorff continuous at $\lambda = 0$. Here Corollary 99 will come in useful.

Proposition 103. *The set-valued mapping $\lambda \mapsto f_\lambda \subset X \times X$ where f_λ is the family of relations defined in Example 102 is strongly Hausdorff continuous at $\lambda = 0$.*

Proof. If $\lambda \mapsto f_\lambda$ is a closed Hausdorff continuous map at $\lambda = 0$ and $\lambda \mapsto f_\lambda^c$ is upper semi-continuous at $\lambda = 0$ then due to Corollary 99 $\lambda \mapsto f_\lambda$ is strongly Hausdorff continuous at $\lambda = 0$. Thus we show each of the conditions of Corollary 99 starting with Hausdorff continuity at $\lambda = 0$. Note that $f_\lambda = \bigcup_{x \in X} \{x\} \times \overline{B_\lambda(f(x))}$.

Given any $\epsilon > 0$ let $\delta = \epsilon$. For all $\lambda \in B_\delta(0)$ we see that for all $x \in X$ we have $h(f_0(x), f_\lambda(x)) < \epsilon$ and thus $h(f_0, f_\lambda) < \epsilon$.

Upper semi-continuity at $\lambda = 0$ follows as $f_\lambda^c \subset f_0^c$ for all $\lambda > 0$ and so given any $\epsilon > 0$ choosing $\epsilon = \delta$ means that for all $\lambda \in [0, \delta)$ we have $f_\lambda^c \subset B_\epsilon(f_0^c)$.

Finally f_λ is closed by construction. □

Remark 104. It is not true in the above that $\lambda \mapsto f_\lambda$ needs to be strongly Hausdorff continuous for all λ in some open set containing 0. To see this let $X = [0, 1]$ and f be a relation on X such that $f = \overline{\{(x, y) | y \in \cup_{n=1}^{\infty} [\frac{1}{2n}, \frac{1}{2n+1}]\}}$. f satisfies *(H1),(H2)* and *(H3)* and its expansion $f_\lambda(x) = \overline{B_\lambda(f(x))}$ is strongly Hausdorff continuous at $\lambda = 0$. However there is a sequence of points $\lambda_n = 1/n$ in $\Lambda = [0, 1]$ which tends to 0 such that f_λ is not strongly Hausdorff continuous at each λ_n and is only Hausdorff continuous.

Finally a useful property of such families of systems defined in Example 102 is that $f_{\lambda_1} \Subset f_{\lambda_2}$ for all $\lambda_1 < \lambda_2$.

Proposition 105. *Given a relation f that satisfies (H1),(H2) and (H3), and let the family $\{f_\lambda\}_{\lambda \in [0, \delta]}$ be defined as in Example 102. Then $f_{\lambda_1} \Subset f_{\lambda_2}$ for any $\lambda_1 < \lambda_2$.*

Proof. (See Figure 5.4) To see this we need to show $f_{\lambda_1} \subset f_{\lambda_2}^o$. Choose $(x, y) \in f_{\lambda_1}$ and $\sigma < \lambda_2 - \lambda_1$. As f_{λ_1} is continuous in x in the Hausdorff metric due to *(H2')* we see that there exists an open set $U \subset X$ such that x is in U and for all $y \in U$, $h(f_{\lambda_1}(x), f_{\lambda_1}(y)) < \sigma$. Hence $(x, y) \in U \times f_{\lambda_1}^o(U) \subset f_{\lambda_1 + \sigma} \subset f_{\lambda_2}$ by choice of σ. □

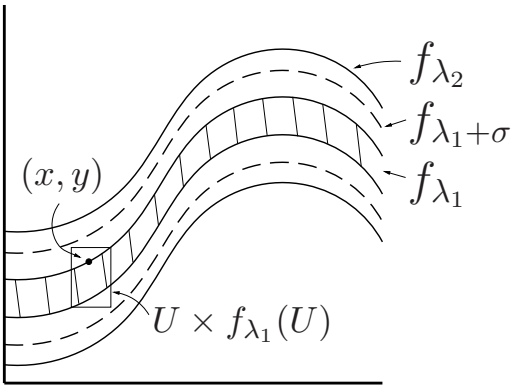


Figure 5.4: Depiction of f_{λ_1},$f_{\lambda_1+\sigma}$,f_{λ_2} and choice of U for (x, y)

Note that this property does not apply if f is not Hausdorff continuous. To see this consider the λ expansion family of the set-valued dynamical system given in Example 10.

Next we define the types of bifurcation that we consider for these families of systems. Given a family of set-valued dynamical systems $\{f_\lambda\}_{\lambda\in\Lambda}$ let $\mathcal{M}_\lambda$ denote the union of minimal invariant sets of f_λ, $\lambda \in \Lambda$.

The first main type of bifurcation involves a discontinuous jump in the Hausdorff metric of the union of minimal invariant sets.

Definition 106. (Discontinuous Topological Bifurcation) Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of relations on X satisfying *(H1'),(H2'),(H3')* that are strongly Hausdorff continuous at λ^*. We say that f_λ undergoes a discontinuous topological bifurcation at λ^* if $\lambda \mapsto \mathcal{M}_\lambda$ is discontinuous in the Hausdorff metric at λ^*.

The second type of bifurcation involves discontinuity in the complement of the set-valued mapping $\mathcal{M}_\lambda$. This involves, for instance, the development of holes on the interior of minimal invariant sets.

Definition 107. (Continuous Topological Bifurcation) Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of relations on X satisfying *(H1'),(H2'),(H3')* that are strongly Hausdorff continuous at λ^*. We say that f_λ undergoes a continuous topological bifurcation at λ^* if $\lambda \mapsto \mathcal{M}_\lambda$is continuous at λ^* and $\lambda \mapsto \mathcal{M}_\lambda^c$ is discontinuous in the Hausdorff metric at λ^*.

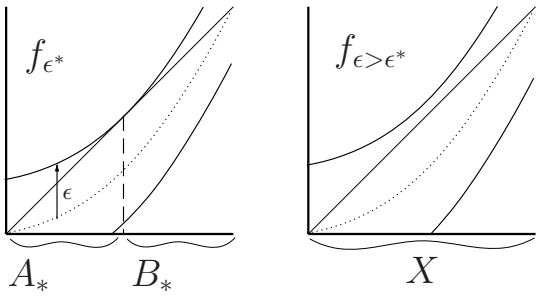


Figure 5.5: Example of a discontinuous topological bifurcation taking place for a family $\{f_\epsilon\}_{\epsilon\in\Lambda}$ when $\epsilon = \epsilon^*$. In this case at the point of bifurcation the set $\mathcal{M}_{\epsilon^*}$ is equal to A_* and becomes $\mathcal{M}_{\epsilon>\epsilon^*} = X$ after the point of bifurcation.

Next we define the notion of stability for a set-valued dynamical system. That is when any change to f does not result in a bifurcation.

Definition 108. We say that f is *Hausdorff stable (resp. inner Hausdorff stable)* if for every strongly Hausdorff continuous family of set valued dynamical systems, $\{g_\lambda\}_{\lambda\in\Lambda}$ satisfying *(H1'),(H2')* and *(H3')* such that $g_{\lambda^*} = f$ for some $\lambda^* \in \Lambda$ we have that g_λ does not undergo a discontinuous topological (resp. Continuous topological) bifurcation at λ^*.

An example of a discontinuous topological bifurcation is illustrated in fig 5.5.

The construction of an example system that undergoes a continuous bifurcation takes a little effort because one cannot construct these in one dimension.

Example 109. Given $\delta > 1$ define a box A_δ in $\mathbb{R}^2$ as $A_\delta = [-\delta, \delta] \times [-\delta, \delta]$. We will build a family of set-valued dynamical systems $\{f_\lambda\}_{\lambda\in[-1,1]}$ on the phase space $X = A_2$. We will construct this family such that it has only one minimal invariant set for all λ. However at the value $\lambda = 0$ the minimal invariant set undergoes a continuous bifurcation. Specifically a hole will emerge continuously in the Hausdorff metric at the point $(0, 0)$ which is contained within the interior of the minimal invariant set for all $\lambda < 0$. Hence $\lambda \to \mathcal{M}^c_\lambda$ will be discontinuous in the Hausdorff metric at $\lambda = 0$.

Firstly we will define g to be a single-valued function composed of two separate functions g_1 and g_2 on X to itself. The first function g_1 contracts and translates the space. Let $0 < a, b < 1$ and let $b \ll a$. Similarly we choose $c \in \mathbb{R}$ such that

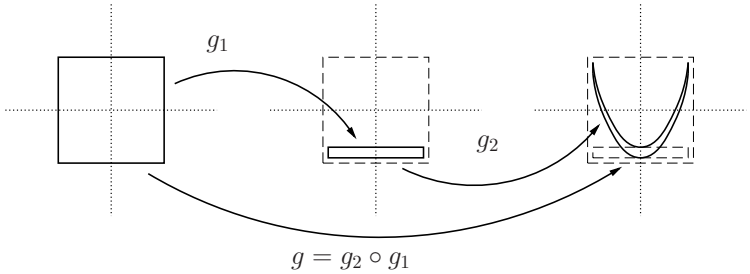


Figure 5.6: Depiction of the functions g_1,g_2 and g and there action on the unit square A_1

$\delta > c \gg b\delta$. g_1 is defined explicitly by $g_1(x, y) = (ax, by - c)$. We let $g_2(x, y) = (x, y + 2x^2)$. These two functions act to transform the unit square A_1 firstly into a narrow rectangle, much wider than it is tall and positioned slightly bellow the origin. g_2 then bends this set around into a horseshoe shape that eclipses the origin. (See Figure 5.6)

Next we will add the set-valued component which will map the set $g(A_1)$ back onto A_1. To do this we first define the set valued function $\mathcal{H}_\lambda : X \mapsto \mathcal{K}(X)$ by $\mathcal{H}_\lambda(x) = \{y | y \cdot (-\frac{x}{|x|}) \geq \lambda\}$ if $\lambda \geq 0$ and $\mathcal{H}_\lambda(x) = \{y | y \cdot (-\frac{x}{|x|}) \geq 0\}$ if $\lambda < 0$. This function takes the point x to the half plan lying in $\mathbb{R}^2$ such that the boundary tangent lies perpendicular to the vector from $(0, 0)$ to x and $d(\mathcal{H}_\lambda(x), x) = \lambda$ (See Figure 5.7). It is easy to see that $\mathcal{H}_\lambda(g(A_1)) \cap A_1 = A_1$ and yet $(0, 0) \in \partial\mathcal{H}_\lambda(x)$ for all $x \in g(A_1)$.

Finally let $s(x) = (\min(x_1, x_2) - 1)^2$ where $x = (x_1, x_2)$.

Using all of these parts we construct the family of set valued-dynamical systems.

$$f_\lambda(x) = \begin{cases} \mathcal{H}_\lambda(g(x)) \cap A_1 & x \in A_1 \\ \mathcal{H}_{\lambda(1-s(x))}(g(x)) \cap A_{s(x)+1} & x \in A_2 \backslash A_1 \end{cases}$$

It's simple to show that for each $\lambda \in [-1, 1]$ that $f_\lambda(X) = X$ and that f_λ is continuous in x. To see that $\lambda \to f_\lambda$ and $\lambda \to f_\lambda^c$ are continuous in λ notice that $\lambda \to \mathcal{H}_{\lambda(2-s(x))}(x)$ and $\lambda \to \mathcal{H}_{\lambda(2-s(x))}(x)^c$ are Hausdorff continuous in λ as are $\lambda \to A_{s(x)+1}$ and $\lambda \to A_1$ and there complements. Thus as each of them are also continuous in x we see that the intersection of these set-valued maps are Hausdorff

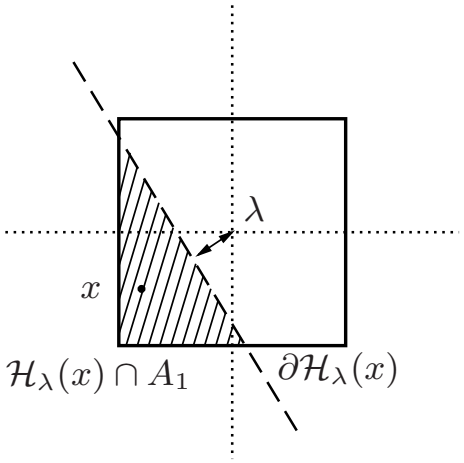


Figure 5.7: Depiction of the boundary of $\mathcal{H}_\lambda(x)$ and the set $\mathcal{H}_\lambda(x) \cap A_1$

continuous in λ as well. Hence $\lambda \to f_\lambda$is strongly Hausdorff continuous.

Notice that for $\lambda \leq 0$ we see that $f_\lambda(A_1) = A_1$ and A_1 is minimal invariant. Further more for all $\lambda \in [-1, 1]$ and for any box A_κ such that $\kappa \in (1, 2]$ then $f_\lambda(A_\kappa) = A_{(\kappa-1)^2+1}$. As $\kappa \in (1, 2]$ we see that $(\kappa - 1)^2 + 1 < \kappa$ and thus $f_\lambda(A_\kappa) \subset A_\kappa$. Hence any such box is a maps properly inside itself under the action of f_λ. Thus for all $\lambda \leq 0$ the set A_1 is an attracting minimal invariant set.

Finally notice that by the point $(0, 0)$ is in the minimal invariant set for $\lambda \leq 0$. However $(0, 0) \in \partial\mathcal{H}_0(x)$ for all $x \in A_1$ and given any $\tau > 0$ we see that $B_\tau((0, 0)) \subset {\mathcal{H}_\tau}^c(x)$ for all $x \in A_1$. As λ becomes greater than a small hole continuously opens around $(0, 0)$ in the minimal invariant set for f_λ . Hence $\lambda \to \mathcal{M}_\lambda$ is continuous in the Hausdorff metric and yet $\lambda \to \mathcal{M}_\lambda^c$ is discontinuous in the Hausdorff metric. (See Figure 5.8)

Later on in example 124 we give an example of a continuous bifurcation where we relax the condition that family of set-valued mappings $\lambda \to g_\lambda$ is strongly Hausdorff continuous at the point of bifurcation.

In [17] the following definition is given for topological bifurcation:

Remark 110. In [17] the following definition is given for topological bifurcation: Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of relations on X satisfying *(H1'),(H2'),(H3')* and such that

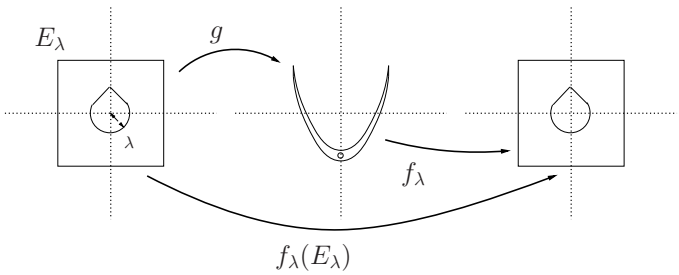


Figure 5.8: Depiction of a minimal invariant set E_λ and the action of f_λ on $E_\lambda$for $\lambda > 0$. Notice that the hole in the center of E_λ is created by the shape of $g(E_\lambda)$.

$\lambda \mapsto f_\lambda$ is Hausdorff continuous for all λ in Λ. We say f_λ admits a *topological bifurcation of minimal invariant sets at* $\lambda = \lambda^*$ if for any neighbourhood, V,of λ^* there does not exist a family of homeomorphisms $\{g_\lambda\}_{\lambda\in\Lambda}$, $g_\lambda : X \mapsto X$, depending continuously on λ, with the property that $g_\lambda(\mathcal{M}_\lambda) = \mathcal{M}_{\lambda^*}$ for all $\lambda \in V$.

The class of discontinuous and continuous topological bifurcations is not equivalent to topological bifurcation as in [17]. This is because while holes cannot form in the interior of $\mathcal{M}_\lambda$ if it is both Hausdorff and inner Hausdorff stable they can still develop on the boundary and move inside. The definition they use in [17] excludes this behaviour as in such a case there would be no family of homeomorphisms between $\mathcal{M}_{\lambda^*}$ and $\mathcal{M}_\lambda$ for $\lambda \neq \lambda^*$. To see this idea consider Figure 5.3 and assume that $H(\lambda) = \mathcal{M}_\lambda$ for some imaginary family of set-valued dynamical systems. Notice that in this case $H(\lambda)$ would be Hausdorff and inner Hausdorff stable yet admit a Topological Bifurcation at $\lambda = 0$ as per [17].

5.1.3 Continuity Properties of the Orbitally Connected and Chain Connected sets

Here we show how the orbitally connected sets and the chain connected sets for a family of set-valued dynamical systems $\{f_\lambda\}_{\lambda\in\Lambda}$ will continue under variation of λ. This will help to describe how minimal invariant sets continue in λ, which in turn will help us describe bifurcations. In this section we assume all relations are on X, a compact subset of $\mathbb{R}^m$, and Λ is a compact metric space.

It would be great if the continuation of the orbitally connected sets was clearly

defined however $\lambda \mapsto [x]_\lambda$ is ambiguous in the sense that it relies on a specific reference point x. We can extend the definition of $[x]_\lambda$ to all of$X$ if we let $[x]_\lambda = \emptyset$ if $x \notin P(f^o_\lambda)$. The next lemma shows us that if $[x]_\lambda \neq \emptyset$ then it is non-empty on some open subset of Λ containing λ.

Lemma 111. *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of relations satisfying (H1'),(H2'),(H3') and assume that $\lambda \mapsto f_\lambda$ is strongly Hausdorff continuous at λ^* then if $[x]_{\lambda^*} \neq \emptyset$ for some $x \in X$ there exists $\tau > 0$ such that for all $\lambda' \in B_\tau(\lambda^*)$ we have $[x]_{\lambda'} \neq \emptyset$.*

Proof. If $x \in P(f^o_{\lambda^*})$ then $[x]_{\lambda^*}$ must be non-empty as it at least contains x itself. Hence we show that there exists $\tau > 0$ such that $x \in P(f^o_{\lambda'})$ for all $\lambda' \in B_\tau(\lambda^*)$.

Note that x must be a point in the interior of $[x]_{\lambda^*}$ as $[x]_{\lambda^*}$ is open. As $x \in [x]_{\lambda^*}$ there exists $n \in \mathbb{N}$ and a sequence $\{x_i\}_{i=0}^n$ such that $x_n = x_0 = x$ and $(x_i, x_{i+1}) \in f^o_{\lambda^*}$ for all $i \in \{0, ..., n-1\}$. Let $K = \cup_{i=1}^{n-1}(x_i, x_{i+1})$ and notice that $K \Subset f_{\lambda^*} \subset X \times X$. As f_λ is strong Hausdorff continuous Proposition 97 implies that there exists $\tau > 0$ such that for all $\lambda' \in B_\tau(\lambda^*)$ we have that $K \Subset f^o_{\lambda'}$. So $\{x_i\}_{i=0}^n$ is still a return orbit for x and so $x \in P(f^o_{\lambda'})$ and the result follows. □

Next we show that the set-valued mapping $\lambda \mapsto [x]^c_\lambda$ is upper semi-continuous in the Hausdorff metric.

Lemma 112. *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of relations satisfying (H1'),(H2'),(H3') and assume $\lambda \mapsto f_\lambda$ is strongly Hausdorff continuous at $\lambda = \lambda^*$. Assume that $[z]_{\lambda^*} \neq \emptyset$ for some $z \in X$. Then the complement of this orbitally connected set, $[z]^c_{\lambda^*}$, is upper semi-continuous at λ^* in the Hausdorff metric.*

Proof. Given $\sigma > 0$ we require to show that there exists $\delta > 0$ such that for all $\lambda' \in B_\delta(\lambda^*)$ we have $[z]^c_{\lambda'} \subset B_\sigma([z]^c_{\lambda^*})$. First choose τ as in Lemma 111 so that $[z]_{\lambda'} \neq \emptyset$ for all $\lambda' \in B_\tau(\lambda^*)$.

Assume for contradiction that for some σ and every $\delta > 0$ there exists λ_δ such that $\lambda_\delta \in B_\delta(\lambda^*)$ and there exists $x_\delta \in [z]_{\lambda^*} \backslash B_\sigma([z]^c_{\lambda^*})$ such that $x_\delta \notin [z]_{\lambda_\delta}$. Let $\{\delta_i\}_{i=1}^\infty$ be a decreasing sequence that tends to zero. We can construct a sequence $\{x_i\}_{i=1}^\infty$ such that $x_i \in [z]_{\lambda^*} \backslash B_\sigma([z]^c_{\lambda^*})$ and $x_i \notin [z]_{\lambda_{\delta_i}}$ for each $i \in \mathbb{N}$. $B_\sigma([z]^c_{\lambda^*})$ is open so $[z]_{\lambda^*} \backslash B_\sigma([z]^c_{\lambda^*})$ is closed and a subset of $[z]_{\lambda^*}$. Compactness of $[z]_{\lambda^*} \backslash B_\sigma([z]^c_{\lambda^*})$ implies we can find a convergent sub-sequence $\{x_{i_k}\}_{k=1}^\infty$ such that $x_{i_k} \to x \in [z]_{\lambda^*} \backslash B_\sigma([z]^c_{\lambda^*})$.

We want to show that there exists some open set $K \subset X$ that contains x and which is contained in $[z]_{\lambda'}$ for all λ' close enough to λ^*. If this is true, then we obtain a contradiction by finding terms in $\{x_{i_k}\}$ that lie in $[z]_{\lambda'}$ when $\lambda_{\delta_{i_k}} < \lambda'$. To find the set K we use the fact that $f^o_{\lambda^*}$ is open and that x is in an orbitally connected set.

As x is in $[z]_{\lambda^*}$ there must exist as result of the definition for orbitally connected, n_1 and n_2 in $\mathbb{N}$ and a sequence $\{y_j\}_{j=0}^{n_1+n_2}$ such that $(y_j, y_{j+1}) \in f^o_{\lambda^*}$ and $y_0 = x$, $y_{n_1} = z$ and $y_{n_1+n_2} = x$. Let $N = n_1 + n_2$. As $f^o_{\lambda^*}$ is open in $X \times X$ we can also find a value $\kappa > 0$ such that $B_\kappa(y_i) \times B_\kappa(y_{i+1}) \Subset f^o_{\lambda^*} \subset X \times X$ for all $i \in \{0, ..., N-1\}$ and $B_\kappa(y_N) \times B_\kappa(y_0) \Subset f^o_{\lambda^*}$. Note that for any $p \in B_\kappa(y_i)$ we have $B_\kappa(y_{i+1}) \subset f^o_{\lambda^*}(p)$ as otherwise we can choose a point $q \in B_\kappa(y_{i+1}) \cap (f^o_{\lambda^*}(p))^c$ and then $(p,k) \notin f^o_{\lambda^*}$ but $(p,k) \in B_\kappa(y_i) \times B_\kappa(y_{i+1})$. Because of this property any sequence of points $\{y'_i\}_{i=1}^N$ such that $y'_i \in B_\kappa(y_i)$ satisfies $y'_{i+1} \in B_\kappa(y_{i+1}) \subset f^o_{\lambda^*}(y'_i)$ for all $i \in \{0, ..., N-1\}$ and $y'_0 \in B_\kappa(y_0) \subset f^o_{\lambda^*}(y'_N)$. Hence any point in $K = \bigcup_{i=1}^N B_\kappa(y_i)$ is also in $[z]_{\lambda^*}$. What is more the set $K_{X\times X} = B_\kappa(y_0) \times B_\kappa(y_N) \cup \left(\bigcup_{i=0}^{N-1} B_\kappa(y_i) \times B_\kappa(y_{i+1})\right)$ is an open set such that $K_{X\times X} \Subset f^o_{\lambda^*}$.

Next we use Proposition 97 to show K persists as a subset of $[z]_{\lambda'}$ for λ' close to λ^*.

As $K_{X\times X} \Subset f^o_{\lambda^*}$ and as $\lambda \mapsto f_{\lambda^*}$ is strongly Hausdorff continuous at λ^* we know $\lambda \mapsto f^c_\lambda$ is upper-semi continuous and so by Proposition 97 there exists δ such that $\tau > \delta > 0$ and for all $\lambda' \in B_\delta(\lambda^*)$ we have $K_{X\times X} \Subset f^o_{\lambda'}$. Thus by the same argument as for λ^* we see that K is a subset $[z]_{\lambda'}$. Finally note that K contains x.

Finally to obtain the contradiction we must find a point of the sub-sequence $\{x_{i_k}\}_{k=1}^\infty$ that is in K.

As $\lambda_i \to \lambda^*$ we can find $m_1 \in \mathbb{N}$ such that $d(\lambda_{\delta_i}, \lambda^*) < \delta$ for all $i \geq m_1$. Similarly as $x_{i_k} \to x$ and K is open and contains x we can choose $m_2 \in \mathbb{N}$ large enough that $x_{i_k} \in K$ for all $k > m_2$. Let $m = m_1 + m_2$ and choose $k > m$. In this case $x_{i_k} \in K$ which implies $x_{i_k} \in [z]_{\lambda_{\delta_i}}$ as λ_{δ_i} is sufficiently close to λ^*. This is a contradiction as we chose the x_i such that they where all not in $[z]_{\lambda_{\delta_i}}$. □

Lower-semi continuity of $[x]_\lambda$ follows directly from Proposition 98, Lemma 112 and Lemma 76.

Lemma 113. *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of relations satisfying (H1'),(H2'),(H3') and let $\lambda \mapsto f_\lambda$ be strongly Hausdorff continuous at λ^* then the orbitally connected sets*

$[z]_\lambda$ *are lower-semi continuous at* λ^* *in the Hausdorff metric.*

Remark 114. Note that for an family of relations $\{f_\lambda\}_{\lambda\in\Lambda}$ as $P(f^o_\lambda)$ is partitioned by the equivalence classes of $\sim$ and so $P(f^o_\lambda)$ inherits lower semi continuity from them.

So for the families of relations we are considering we now know how the orbitally connected sets for f^o_λ continue in λ. Recall that $[[x]] = \bigcap_{g\ni f}[x]_g$ due to Lemma 80. Next we show that the mapping $\lambda \mapsto [[x]]_\lambda$ is upper-semi continuous. In this case we only require Hausdorff continuity of f_λ in λ and not strong Hausdorff continuity.

Lemma 115. *Let* $\{f_\lambda\}_{\lambda\in\Lambda}$ *be a Hausdorff continuous family of relations satisfying (H1'),(H2'),(H3') and let* $\lambda \mapsto f_\lambda$ *be Hausdorff continuous at* λ^* *then the set* $[[x]]_\lambda$ *is upper-semi continuous at* λ^* *in the Hausdorff metric.*

Proof. For any $\sigma > 0$ there exists, due to Lemma 80, $g \ni f_\lambda$ such that $[x]_g \Subset B_\sigma([[x]]_\lambda)$. As $\lambda \mapsto f_\lambda$ is Hausdorff continuous at λ^* we can choose $\delta > 0$ such that for all $\lambda' \in B_\delta(\lambda^*)$ we have $f_{\lambda'} \Subset g$. Thus $[[x]]_{\lambda'} \subset [x]_g \subset B_\sigma([[x]]_\lambda)$. □

The last three lemmas set us up for the following result.

Theorem 116. *Let* $\{f_\lambda\}_{\lambda\in\Lambda}$ *be a family of relations satisfying (H1'),(H2'),(H3'), let* $\lambda \mapsto f_\lambda$ *be strongly Hausdorff continuous at* λ^* *and assume* $[x]_{\lambda^*} = [[x]]^o_{\lambda^*}$ *for some* $x \in P(f^o_{\lambda^*})$ *then* $[x]_\lambda$ *is strongly Hausdorff continuous at* λ^**.*

Proof. The proof follows from Lemma 113, 115 and 112 and Corollary 99. □

A slight difference in the above theorem yields a different result which will be useful.

Theorem 117. *Let* $\{f_\lambda\}_{\lambda\in\Lambda}$ *be a family of relations satisfying (H1'),(H2'),(H3'), let* $\lambda \mapsto f_\lambda$ *be strongly Hausdorff continuous at* λ^* *and assume* $\overline{[x]}_{\lambda^*} = [[x]]_{\lambda^*}$ *for some* $x \in P(f^o_{\lambda^*})$ *then* $[x]_\lambda$ *is Hausdorff continuous at* λ^**.*

Proof. If $\overline{[x]}_{\lambda^*} = [[x]]_{\lambda^*}$ then $[x]_{\lambda^*}$ may differ from $[[x]]^o_{\lambda^*}$. That is there might be points $z \in [[x]]^o_{\lambda^*}$ such that $z \notin [x]_{\lambda^*}$. This means that $h([x]^c_{\lambda^*}, ([[x]]^o_{\lambda^*})^c)$ might not be greater than zero in which case we don't get upper semi-continuity of $[[x]]^c_\lambda$ via $[x]_\lambda$ and Lemma 112. However Lemma 113, 115 still implies Hausdorff continuity of $\lambda \mapsto \overline{[x]}_\lambda$ at λ^*. □

Recall that Theorem 88 tells us that any minimal invariant set is the closure of an orbitally connected set. Let E_λ be a minimal invariant set for some family of set-valued dynamical systems. Assume that for some λ^* we have a orbitally connected set $[x]_{\lambda^*}$ such that $[[x]]^o_{\lambda^*} = [x]_{\lambda^*} = E^o_{\lambda^*}$. We might be temped to say that due to Theorems 116 we have $\lambda \mapsto E_\lambda$ is Hausdorff continuous at λ^*. However Hausdorff continuity of $\lambda \mapsto [x]_\lambda$does not ensure $\overline{[x]}_\lambda$ is minimal invariant as λchanges and thus we cannot use $[x]_\lambda$ to show that some minimal invariant sets E_λ exists and is close to E_{λ^*} for all λ close to λ^*. Hence we need the following proposition that provides the missing piece.

Proposition 118. *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of relations satisfying (H1'),(H2'),(H3') and let $\lambda \mapsto f_\lambda$ be strongly Hausdorff continuous at λ^*. Assume there exists $x \in P(f^o_{\lambda^*})$ such that $\overline{[x]}_{\lambda^*} = [[x]]_{\lambda^*}$and $\overline{[x]}_{\lambda^*}$ is f_{λ^*}-invariant for some $\lambda^* \in \Lambda$ then there exists $\delta > 0$ such that for all $\lambda \in B_\delta(\lambda^*)$ we have $\overline{[x]}_\lambda$ is f_λ -invariant.*

Proof. The converse of Proposition 86 implies that if $\overline{[x]}_{\lambda'}$ is not invariant for some λ' then $f^o([x]_{\lambda'}) \nsubseteq \overline{[x]}_{\lambda'}$. Assume that for all $\delta > 0$ we can find $\lambda' \in B_\delta(\lambda^*)$ such that $f^o_{\lambda'}([x]_{\lambda'}) \nsubseteq \overline{[x]}_{\lambda'}$. If this is the case we can construct a sequence $\{\lambda_i\}_{i=1}^\infty$ such that $\lambda_i \to \lambda^*$ and such that $f^o_{\lambda_i}([x]_{\lambda_i}) \nsubseteq \overline{[x]}_{\lambda_i}$. If this is true for each f_{λ_i} then there must exist z_i for each i such that $z_i \in f^o_{\lambda_i}([x]_{\lambda_i})\backslash\overline{[x]}_{\lambda_i}$. As $z_i \in f^o_{\lambda_i}([x]_{\lambda_i})$ there must exist $y \in [x]_{\lambda_i}$ such that $z_i \in f^o_{\lambda_i}(y)$. Hence $f^o_{\lambda_i}(z_i) \cap [x]_{\lambda_i} = \emptyset$ as otherwise there would exist a return orbit for y that includes z_i and thus z_i would be in $[x]_{\lambda_i}$ which would contradict the choice of z_i. Thus due to *(H3')* we see that for each $i \in \mathbb{N}$ there exists a $p_i \in f^o_{\lambda_i}(z_i)$ such that $B_\epsilon(p_i) \subset f^o_{\lambda_i}(z_i) \subset \left(f^o_{\lambda_i}\right)^2([x]_{\lambda_i})\backslash\overline{[x]}_{\lambda_i}$. Hence for all $i \in \mathbb{N}$ we have $h(f^2_{\lambda_i}(\overline{[x]}_{\lambda_i}), \overline{[x]}_{\lambda_i}) \geq \epsilon$. Now we will use continuity arguments to obtain a contradiction.

$\lambda \mapsto f_\lambda$ is strongly Hausdorff continuous at λ^* by assumption and so there exists $\delta_1 > 0$ such that for all $\lambda \in B_{\delta_1}(\lambda^*)$ and $A \in \mathcal{K}(X)$ we have $h(f^2_\lambda(A), f^2_{\lambda^*}(A)) < \frac{\epsilon}{3}$. Similarly $f^2_{\lambda^*}$ is Hausdorff continuous in x as per *(H2')* and so we can choose $\delta_2 > 0$ such that for all $A \subset B_{\delta_2}(\overline{[x]}_{\lambda^*})$ we have $h(f^2_{\lambda^*}(A), (f^2_{\lambda^*}(\overline{[x]}_{\lambda^*})) < \frac{\epsilon}{3}$. As $\overline{[x]}_{\lambda^*} = [[x]]_{\lambda^*}$ we have that Theorem 116 tells us that $\lambda \mapsto [x]_\lambda$ continues in the Hausdorff metric. Hence we can find N_2 such that for all $i \geq N_2$ we have $h(\overline{[x]}_{\lambda_i}, \overline{[x]}_{\lambda^*}) < \min\{\frac{\epsilon}{3}, \delta_2\}$ and hence $\overline{[x]}_{\lambda_i} \subset B_{\delta_2}(\overline{[x]}_{\lambda^*})$. Similarly as $\lambda_i \to \lambda^*$ there exists N_1 such that for all $i \geq N_1$ we have $\lambda_i \in B_{\delta_1}(\lambda^*)$.

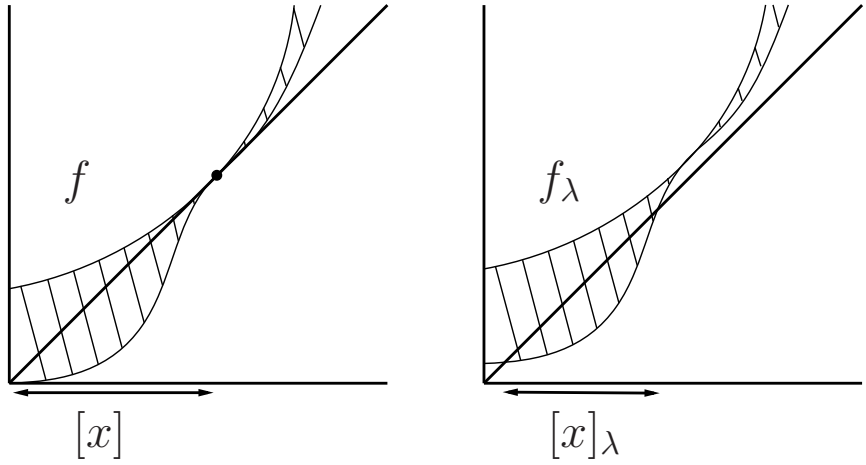


Figure 5.9: $\overline{[x]}$ is minimal invariant but $\overline{[x]}_\lambda$ isn't for all $\lambda > 0$.

Fix $i \geq \max\{N_1, N_2\}$ and let $A = \overline{[x]}_{\lambda_i}$, Thus we have $h(f^2_{\lambda^*}(\overline{[x]}_{\lambda_i}), f^2_{\lambda^*}(\overline{[x]}_{\lambda^*})) < \frac{\epsilon}{3}$ and $h(f^2_{\lambda_i}(\overline{[x]}_{\lambda_i}), f^2_{\lambda^*}(\overline{[x]}_{\lambda_i})) < \frac{\epsilon}{3}$. Using the triangle rule for h and the fact that $f^2_{\lambda^*}(\overline{[x]}_{\lambda^*}) = \overline{[x]}_{\lambda^*}$ we obtain

$$h(f^2_{\lambda_i}(\overline{[x]}_{\lambda_i}), \overline{[x]}_{\lambda_i}) \leq$$

$$h(f^2_{\lambda_i}(\overline{[x]}_{\lambda_i}), f^2_{\lambda^*}(\overline{[x]}_{\lambda_i})) + h(f^2_{\lambda^*}(\overline{[x]}_{\lambda_i}), f^2_{\lambda^*}(\overline{[x]}_{\lambda^*})) + h(\overline{[x]}_{\lambda^*}, \overline{[x]}_{\lambda_i}) < \epsilon$$

Which contradicts $h(f^2_{\lambda_i}(\overline{[x]}_{\lambda_i}), \overline{[x]}_{\lambda_i}) \geq \epsilon$ for all $i \in \mathbb{N}$. □

The above proposition isn't true if f_λ does not satisfy *(H3')*. Figure 5.9 depicts a relation $f \in X \times X$ with a orbitally connected set $[x]$ such that $f([x]) = [x]$ and $[x] = [[x]]^o$. Note that in this case the family of relations $f_\lambda = \{(x, y+\lambda) | (x, y) \in f\}$ is strongly Hausdorff continuous but while $[x]_\lambda$ continues in the Hausdorff metric it is no longer invariant under f.

5.2 Conditions for Bifurcation

We apply the results from the last couple of chapters to obtain conditions for bifurcation of the minimal invariant sets of a family of relations. The main theorem of the thesis is now stated. It tells us that bifurcations occur as a result of collisions

between orbitally connected sets and minimal invariant sets or if there exist points in the interior of minimal invariant sets that aren't periodic for f^o. Recall $\mathcal{M}$ denotes the union of minimal invariant sets of f. As there is finite minimal invariant sets for any f satisfying *(H1),(H2)* and *(H3)* and each minimal invariant set is compact we know that then $\mathcal{M}$ is also a compact set.

Theorem 119. *Let f be a set-valued dynamical systems satisfying (H1),(H2) and (H3) and assume that $R(f) = \overline{P(f^o)}$ then*

1) f is Hausdorff stable if there exists no $y \in P(f^o)\backslash\mathcal{M}$ such that $\overline{[y]} \cap \mathcal{M} \neq \emptyset$

2) f is inner Hausdorff stable if f is Hausdorff stable and there does not exist $y \in \mathcal{M}^o$ such that $y \notin P(f^o)$.

Proof. First recall that Theorem 88 tells us that any minimal invariant set for f satisfying *(H1),(H2)* and *(H3)* must be the closure of some orbitally connected set $[x]$ for $x \in P(f^o)$. Assume without loss of generality that $\mathcal{M}$ is a single minimal invariant set E. We will show *1)* and *2)* for any family of set-valued dynamical systems $\{f_\lambda\}_{\lambda\in\Lambda}$ satisfying *(H1'),(H2'),(H3')* and such that $\lambda \mapsto f_\lambda$ is strongly Hausdorff continuous at λ^* where $f_{\lambda^*} = f$ and $E_{\lambda^*} = E = \overline{[x]}$.

1) Recall that $[[x]]_\lambda$ partition $R(f_\lambda)$ and $[x]_\lambda$ partition $P(f^o_\lambda)$ and $[x]_\lambda \subset [[x]]_\lambda$for all $\lambda \in \Lambda$. Hence $\overline{[x]}_{\lambda^*} \neq [[x]]_{\lambda^*}$ if either $R(f_{\lambda^*}) \neq \overline{P(f^o_{\lambda^*})}$ or if $R(f_{\lambda^*}) = \overline{P(f^o_{\lambda^*})}$ and there exists $y \in P(f^o_{\lambda^*})\backslash[x]_{\lambda^*}$ such that $\overline{[x]}_\lambda \cap \overline{[y]}_\lambda \neq \emptyset$. In the second case notice that $[[x]]_\lambda = \overline{[x]}_\lambda \cup \overline{[y]}_\lambda$. Both of these cases are excluded by assumption. Hence $[x]_\lambda$ is Hausdorff continuous at λ^* by Theorem 117. As $\overline{[x]}_{\lambda^*}$ is f_{λ^*}-invariant then $\overline{[x]}_\lambda$ is f_λ -invariant for all λ close to λ^* by Proposition 118. Finally $\overline{[x]}_\lambda$ for λclose enough to λ^* must be minimal invariant by Proposition 87. Hence $\lambda \mapsto E_\lambda = \overline{[x]}_\lambda$ is Hausdorff stable.

2) If no such y exists then $[x]_{\lambda^*} = [[x]]^o_{\lambda^*}$ and so by Lemma 116 implies $[x]_{\lambda^*}$ is strongly Hausdorff continuous at λ^*.Hence as $E_{\lambda^*} = \overline{[x]}_{\lambda^*} = [[x]]_{\lambda^*}$ we see that E_λ is strongly Hausdorff continuous at λ^*. □

We now take a further look at each of the case in Theorem 119. For Hausdorff stability we'll first show that if a bifurcation occurs then it must do so in a certain way and then we'll derive an equivalent condition from $[[x]] = \overline{[x]}$ which is called

isolation. In the case of continuous stability we'll show how it is dependent on strong Hausdorff continuity of $\lambda \mapsto f_\lambda$.

5.2.1 Hausdorff Stability

Suppose there exists a minimal invariant set E_{λ^*} for a set-valued dynamical system f_{λ^*} belonging to a family of set-valued dynamical systems $\{f_\lambda\}_{\lambda\in\Lambda}$. Assume too that there exists $y \in P(f^o_{\lambda^*})\backslash E_{\lambda^*}$ such that $\overline{[y]}_{\lambda^*} \cap E_{\lambda^*} \neq \emptyset$ then we can say more about the nature of $[y]_{\lambda^*}$ which the following theorem does.

Theorem 120. *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of set-valued dynamical systems satisfying (H1'),(H2') and (H3'). Let $\lambda \mapsto f_\lambda$ be strongly Hausdorff continuous at λ^*. Assume there exists $x, y \in P(f^o_{\lambda^*})$ such that $\overline{[x]}_{\lambda^*} \cap \overline{[y]}_{\lambda^*} \neq \emptyset$ where $\overline{[x]}_{\lambda^*}$ is minimal invariant and $\overline{[y]}_{\lambda^*} \neq \overline{[x]}_{\lambda^*}$ then $\overline{[y]}_{\lambda^*}$ is not f_{λ^*}-invariant.*

Proof. Assume not and that $\overline{[y]}_{\lambda^*}$ is f_{λ^*} invariant. Choose $z \in \overline{[x]}_{\lambda^*} \cap \overline{[y]}_{\lambda^*}$. Due to $\overline{[y]}_{\lambda^*}$ invariance we must have $f_{\lambda^*}(z) \subset \overline{[y]}_{\lambda^*}$ the invariance of $\overline{[x]}_{\lambda^*}$ also implies $f_{\lambda^*}(z) \subset \overline{[x]}_{\lambda^*}$. Thus we must have $f_{\lambda^*}(z) \subset \overline{[x]}_{\lambda^*} \cap \overline{[y]}_{\lambda^*}$ and $f_{\lambda^*}(z) \cap ([x]_{\lambda^*} \cup [y]_{\lambda^*}) = \emptyset$ as otherwise we contradict f_{λ^*} -invariance of one or both of $[x]_{\lambda^*}$and $[y]_{\lambda^*}$. However *(H3)* implies $f_{\lambda^*}(z)$ contains a point p such that $B_\epsilon(p) \subset f_{\lambda^*}(z)$. But p must be in $\overline{[x]}_{\lambda^*} \cap \overline{[y]}_{\lambda^*}$ which means $B_\epsilon(p) \cap ([x]_{\lambda^*} \cup [y]_{\lambda^*}) \neq \emptyset$ which is a contradiction. □

Theorem 120 tells that that the above form of bifurcation results as the collision of a minimal invariant set with an orbitally connected set which is either dual invariant or satisfies $f^o([y]) \cap f^{o*}([y]) = [y]$ as per Lemma 85. This is similar to the way in which bifurcations for single-valued dynamical systems can occur as a stable attracting fixed point splitting into a attracting fixed point and a repelling fixed point.

Next we introduce the idea of isolation for a minimal invariant set which will be show to coincide with the condition in part *1)* of Theorem 119. Recall that a set A is an attractor if there exists $U \ni A$ such that $f^\Omega(U) = A$. Isolation is a weak form of attraction for a minimal invariant set. The idea of isolation originates with Conley and Easton in [11] for single-valued systems. We use the definition from [23] which is defined for relations.

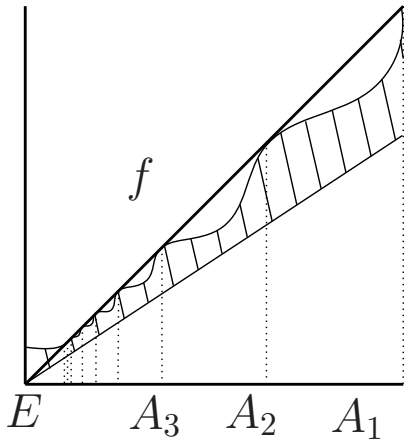


Figure 5.10: E is an isolated minimal invariant set for f. Outside of E there exists a sequence of nested attractors A_i that tend to E. Thus given any open U such that $E \Subset U$ there exists A_i such that $f^{\Omega}(U) = A_i \neq E$ hence by definition E is not an attractor. Thus E being isolated doesn't imply it is an attractor. (See Remark 122)

Definition 121. (Isolation) Let f be a relation on X. We say that a minimal invariant set E is *isolated for* f if for any proper neighbourhood U of E there is an open set $W \subset U$ such that $\bar{E} \subset W$, W contains no other minimal invariant set and $\overline{f(W)} \subset W$. We call W an *isolating set.*

Remark 122. This is not equivalent to a minimal invariant set being an attractor because one might have a sequence of nested invariant sets each attractors themselves that tend to the minimal invariant set E (See Figure 5.10). In which case E is isolated but does not satisfy the conditions to be an attractor. The converse is however clearly true namely if E is attractive then it is isolated.

Next we show that isolation of a minimal invariant set for our set-valued dynamical systems implies that the minimal invariant set is a chain component and thus isolation of minimal invariant sets for these types of set-valued dynamical systems implies Hausdorff stability by Theorem 119.

Theorem 123. *A minimal invariant set E for a set-valued dynamical system f satisfying (H1),(H2),(H3) is isolated if and only if $E = [[x]]$.*

Proof. ($\Longrightarrow$) Let W be an isolating set for $E = \overline{[x]}$ then $f(W) \Subset W$. Hence for any $g \ni f$ there exists h such that $g \ni h \ni f$ and $h(W) \Subset W$ too. Hence

$[x]_f \Subset [x]_h \Subset W$ as any $y \in W$ cannot be orbitally connected to any point outside of W. Thus as we can find such an h for any $g \ni f$ and isolation means we can choose W within any neighbourhood of E we have $\overline{[x]} = \bigcap_{g \ni f} [x]_g$.

($\Longleftarrow$) Let U be a neighbourhood of E. Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be a family of relations that satisfies *(H1'),(H2'),(H3')* and $\lambda \mapsto f_\lambda$ is strongly Hausdorff continuous at λ^* such that $f_0 = f$ and $f_{\lambda_1} \Subset f_{\lambda_2}$ for all $\lambda_1 < \lambda_2$. A simple example of such a family is given in Example 102. By Theorem 88 we know that $E = \overline{[x]}$ for some orbitally connected set. Hence by assumption we have $\overline{[x]} = E = [[x]]$ and thus by only considering f in some close neighborhood of $[x]$ we see that f and E satisfy the assumptions for the first part of Theorem 119. Hence we can choose λ close enough to λ^* such that $E_\lambda \Subset U$ and $f_\lambda(E_\lambda) = E_\lambda$. As $f \Subset f_\lambda$ we see that $f(E_\lambda) \Subset f_\lambda(E_\lambda) = E_\lambda$. Let $W = E_\lambda$ be the isolating neighborhood and we are done. □

5.2.2 Topological Stability

Suppose we have a set-valued dynamical system, f, such that f satisfies *(H1),(H2)* and *(H3).* Assume also that there exists a minimal invariant set E for f such that there exists a single point $y \in E^o$ such that $y \notin P(f^o)$. Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be any strongly continuous family of set-valued systems $\lambda \mapsto f_\lambda$ satisfying *(H1'),(H2')* and *(H3')* such that $f_{\lambda^*} = f$. Then while $\lambda \mapsto [x]_\lambda \subset E_\lambda$ will still be lower-semi continuous, $\overline{[x]}_\lambda$ does not need to be and so neither is E_λ. In this case a minimal invariant set may develop as hole at the point $y \in E^o$. An example of exactly such a case is given in Example 109. Here the unit square A_1 is a minimal invariant set for F_0 and as λ becomes greater than 0 a hole continuously appears at $(0, 0)$. In particular let $x = (0.1, 0.1)$ then $[x]_{\lambda=0} = A_1^o \backslash (0, 0)$ as the point $(0, 0)$ is not in $P(f_1^o)$ however $[[x]]_{\lambda=0} = A_1$. Hence while f_1 fulfills part one of Theorem 119 it doesn't fulfill part two and thus a hole may appear as it does for all $\lambda > 0$.

Further more if $\lambda \mapsto f_\lambda$ is only Hausdorff continuous at λ^* and not strongly Hausdorff continuous then we cannot guarantee lower-semi continuity of $E_\lambda$even under the conditions of Theorem 119 as the following example shows.

Example 124. (Continuous bifurcation for a family of Hausdorff continuous relations) Let $0 < a < 1/2$, $\epsilon > 0$ and $\lambda^* = 0$. Let $X = [-\frac{\epsilon}{1-a} - 1, \frac{\epsilon}{1-a} + 1] \subset \mathbb{R}$ and

$g : X \mapsto X$ be a single-valued map defined by $g(x) = ax$. Let $f_{\lambda^*}(x) = f_0(x) = B_\epsilon(g(x))$. The set $E_0 = [-\frac{\epsilon}{1-a}, \frac{\epsilon}{1-a}]$ is a minimal invariant set for f_{λ^*}. Let $f_\lambda = f_0$ for all $\lambda \leq 0$. For $\lambda > 0$ we will continuously (in Hausdorff metric in both x and λ) remove a section of f such that $E_\lambda = [-\frac{\epsilon}{1-a}, -h(\lambda)] \cup [h(\lambda), \frac{\epsilon}{1-a}]$ where $h(\lambda)$ is a continuous function (in euclidean metric) of λ such that $\lim_{\lambda \to 0} h(\lambda) = 0$.

Let $C = (-\epsilon/a - \delta, \epsilon/a - \delta)$ for $0 < \delta < \epsilon/a - \epsilon/(1-a)$. Note that C is wider than E_0 because $\epsilon/a > \epsilon/(1-a)$ as $0 < a < 1/2$. Define a set-valued function $H : C \mapsto \mathcal{K}(\mathbb{R})$ by

$$H_\lambda(x) = \left(-\lambda \left| \left(\frac{\epsilon}{a} - \delta \right)^2 - x^2 \right|, \lambda \left| \left(\frac{\epsilon}{a} - \delta \right)^2 - x^2 \right| \right)$$

$H_\lambda(x)$ an open set whose width gets gradually wider as x varies from $-\epsilon/a - \delta$ to 0 and then smaller until it disappears as x varies from 0 to $\epsilon/a - \delta$. At $x = 0$, $H_\lambda(0)$ is equal to $\left(-\lambda \left(\frac{\epsilon}{a} - \delta \right)^2, \lambda \left(\frac{\epsilon}{a} - \delta \right)^2 \right)$ and at it is widest. As $\lambda \to 0$ $H_\lambda \to \{0\}$ everywhere on C .

We define for $\lambda > 0$ the f_λ to be $f_\lambda(x) = f_0(x) \backslash H_\lambda(x)$. If $\lambda \to 0$ the width of $H_\lambda(C) \to 0$ and $f_\lambda \to f_0$ in the Hausdorff metric. To see the continuous bifurcation consider E_λ for $\lambda > 0$ we obtain:

$$E_\lambda = E_0 \backslash H_\lambda(\frac{\epsilon}{1-a}) \text{ or } [-\frac{\epsilon}{1-a}, -h(\lambda)] \cup [h(\lambda), \frac{\epsilon}{1-a}]$$

where $h(\lambda) = \lambda \left| \left(\frac{\epsilon}{a} - \delta \right)^2 - \left(\frac{\epsilon}{1-a} \right)^2 \right|$. This set is parameterized by λ to be continuous in the Hausdorff metric. In particular choose $x \in E_\lambda$ such that $z \neq 0$ and notice that 0 is in $[x]_\lambda$ for all $\lambda \leq 0$ but not for $\lambda > 0$. $0 \notin P(f^o)$ at $\lambda = 0$ so Theorem 119 does not apply. For the sake of clarity see Figure 5.11 .

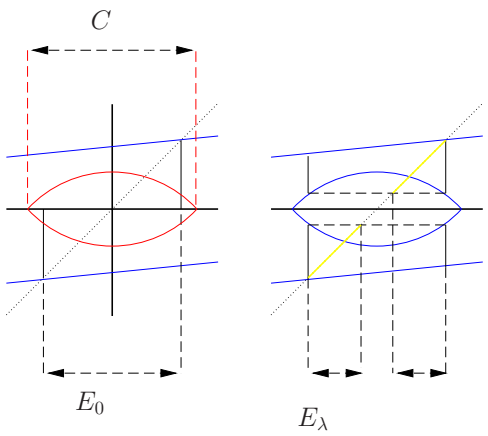


Figure 5.11: The first diagram shows in blue f_0, and in red $H_\lambda(x)$ and the minimal invariant set E_0, The second diagram shows $f_\lambda = f_0 \backslash H_\lambda$ and E_λ is marked on the $x = y$ line in yellow. As λ goes to 0 the width of H_λ goes to 0 and the two yellow lines move continuously together.

5.3 Examples

To give a better understanding of the ideas in Theorems 119, 120 and 123 we give some examples.

Example 125. For an example of the case where $R(f) \neq \overline{P(f^o)}$ recall Example 31. Here we have an Minimal invariant set given by $E = [0, \frac{1}{2}]$. There is only one orbitally connected class namely $[\frac{1}{4}] = P(f^o)$. Notice that $E = \overline{[\frac{1}{4}]}$ as expected due to Theorem 88. Similarly there is only a single chain component $[[\frac{1}{4}]] = R(f) = [0, \frac{3}{4}]$. Notably here we have $R(f) \neq \overline{P(f^o)}$ and given a small perturbation of f the minimal invariant set $[0, \frac{1}{2}]$ will jump discontinuously to become $[0, \frac{3}{4}]$. (See Figure 5.12)

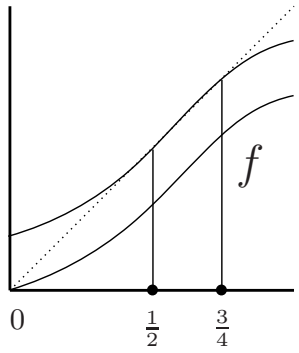


Figure 5.12: $R(f) \neq \overline{P(f^o)}$ for f

The next example of a family of set-valued dynamical systems illustrates a simple application of Theorem 119.

Example 126. (Family of set-valued dynamical systems which admit a discontinuous bifurcation for a specific parameter value) Let $X = [0,1]$ and let $\lambda \mapsto f_\lambda$ for $\lambda \in \Lambda = [\frac{1}{8}, \frac{1}{2}]$ be the family of set-valued dynamical systems defined by $f_\lambda = \{(x,y) \in X \times X : |y - x^2| \leq \lambda\}$. This family satisfies *(H1'),(H2'),(H3')* and is strongly Hausdorff continuous for all $\lambda \in \Lambda$. Notice that $P(f_\lambda^o) = P_1(f_\lambda^o)$ for all λ and we can thus obtain the set-valued mapping $P(f_\lambda^o) = \{x \in X : -\lambda < x - x^2 < \lambda\}$. For all $\lambda \in [\frac{1}{8}, \frac{1}{4}]$ we see there are two orbitally connected classes, $[0]_\lambda$ and $[1]_\lambda$ and $\overline{[0]}_\lambda$ is a minimal invariant set for f_λ . For all $\lambda \in [\frac{1}{8}, \frac{1}{4})$ there are similarly two connected components $[[0]]_\lambda$ and $[[1]]_\lambda$. For $\lambda \in [\frac{1}{8}, \frac{1}{4})$ we have $\overline{[0]}_\lambda = [[0]]_\lambda$ and $\overline{[1]}_\lambda = [[1]]_\lambda$ and by Theorem 119 f_λ is Hausdorff stable. At $\lambda = \frac{1}{4}$ we observe that $\overline{[0]}_{\frac{1}{4}} \cap \overline{[1]}_{\frac{1}{4}} = \{\frac{1}{2}\}$ and $\overline{[0]}_{\frac{1}{4}} \cup \overline{[1]}_{\frac{1}{4}} = [[0]]_{\frac{1}{4}} = [[1]]_{\frac{1}{4}}$ subsequently for all $\lambda > \frac{1}{4}$ we notice that $\overline{[0]}_\lambda$ has jumped in the Hausdorff metric and is now is the whole space meaning that $\lambda = \frac{1}{4}$ was a point of discontinuous bifurcation for f_λ. (See Figure 5.13)

Next we return to the logistic map:

Example 127. Logistic map:

Fix $\epsilon > 0$, let $X = [0, 1 + \epsilon]$ and $g_r : X \mapsto X$, where $g_r(x) = rx(1-x)$. We define the relation $f(x) = \{y : y = g_r(x) + \xi \text{ where } \xi \in [0, \epsilon]\}$. We might hope that one

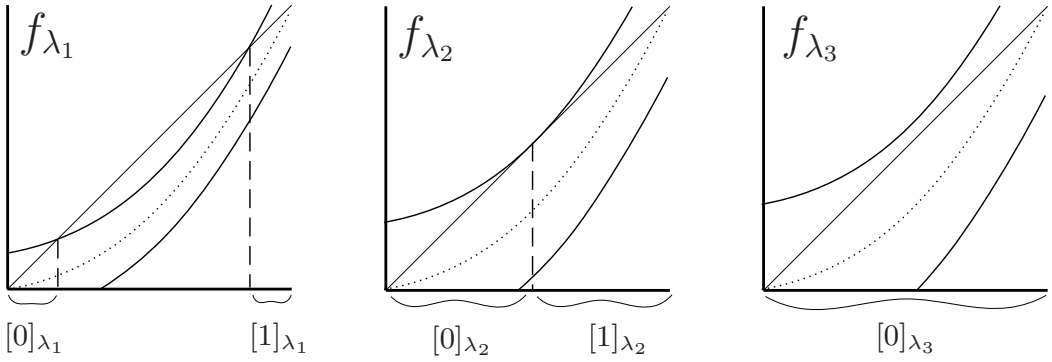


Figure 5.13: f_λ for $\lambda_1 < \lambda_2 = \frac{1}{4} < \lambda_3$. Notice the point of intersection $\{\frac{1}{2}\}$ of the closures of the orbitally connected sets for $\lambda = \frac{1}{4}$.

could simply bound the relation f by $f_\epsilon(x) = g(x) + \epsilon$ and $f_0(x) = g(x)$ and then use these bounds to compute the boundaries of $P_n(f)$. To see the problem with this let $r = 4$ and notice that at $x = \frac{1}{2}$ the maximum value of f is $1 + \epsilon$. Take $y \in f^*(1)$ we see that $1 + \epsilon \notin [\min(f_0 \circ f_0(y), f_0 \circ f_\epsilon), \max(f_\epsilon \circ f_0(y), f_\epsilon \circ f_\epsilon(y))]$ as $1 + \epsilon > \max(f_\epsilon \circ f_0(y), f_\epsilon \circ f_\epsilon(y))$. For general r any $y \in f^*(f_0(\frac{1}{2}))$ is not bound between $f_0 \circ f_0$, $f_0 \circ f_\epsilon$, $f_\epsilon \circ f_0$ and $f_\epsilon \circ f_\epsilon$. The effect of this is to truncate the peaks and troughs of higher iterates of f. Because of this and the fact that $f_\epsilon^n(x) - x$ is difficult to solve we can only reasonably describe $P_1(f)$. However if epsilon is small enough then $f^*(f_0(\frac{1}{2}))$ is not near the bifurcating minimal invariant set. We will look at a period doubling bifurcation and observe how the periodic sets and orbitally connected sets play a role. Let ϵ be very small.

We see that the equation for the $P_1(f)$ set is given by computing the fixed points of $f_0(x)$ and $f_\epsilon(x)$. Let $b_\epsilon(r) = \frac{\sqrt{4\epsilon r + r^2 - 2r + 1} + r - 1}{2r}$ and $b_0(r) = \frac{\sqrt{r^2 - 2r + 1} + r - 1}{2r}$ then we obtain

$$P_1(f) = \begin{cases} [0, b_\epsilon(r)] & : 0 < r < 1 \\ \{0\} \cup [b_0(r), b_\epsilon(r)] & : r > 1 \end{cases}$$

At $r = 1$ we see that the fixed point at 0 for $f_0(0)$ undergoes a classical bifurcation in which $\frac{df_0}{dx}(0) = 1$. Note that $P_1(f)$ is a minimal invariant set for all r between 0 and around 1 but becomes untethered from 0 at $r = 1$ (See Figure 5.14). So we don't observe a bifurcation in our sense unless we were to consider a larger X containing points below 0.

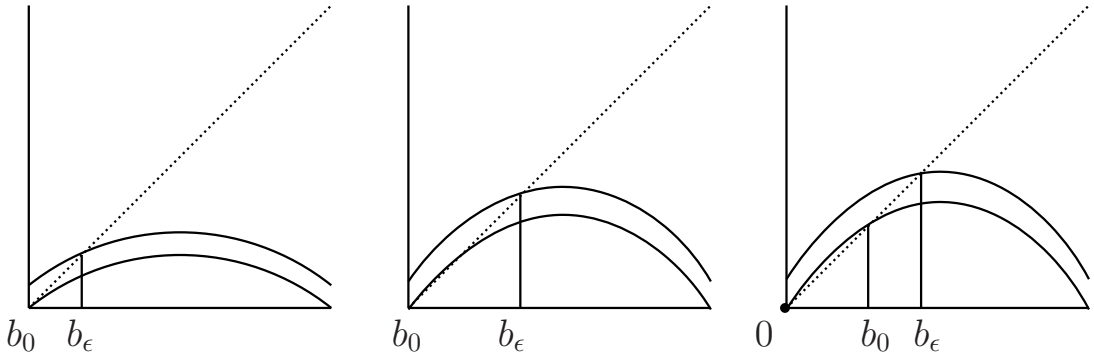


Figure 5.14: $P_1(f)$ as r passes through 1

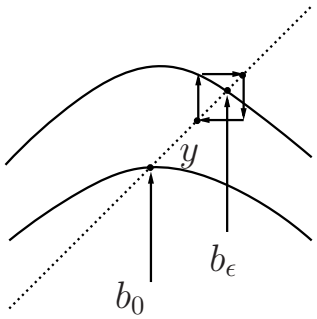


Figure 5.15: Emergence of period two points outside of $[b_0, b_\epsilon]$ as $b_\epsilon(r) > \frac{1}{2}$

While $b_\epsilon(r) < \frac{1}{2}$ we see that $f([b_0(r), b_\epsilon(r)]) = [b_0(r), b_\epsilon(r)]$. As the point $b_\epsilon(r)$ passes through $\frac{1}{2}$ for r a little less than 2 we see that $\frac{df}{dx}(b_\epsilon(r))$ moves through 0 and suddenly points $y \in [b_0(r), b_\epsilon(r)]$ that must map outside of $[b_0(r), b_\epsilon(r)]$ under f begin to appear. This is because now $b_\epsilon(r)$ has moved past the peak of f there exists $y < b_\epsilon(r)$ such that $f_\epsilon(y) > f_\epsilon(b_\epsilon(r)) = b_\epsilon(r)$ but then $f_0(f_\epsilon(y)) < b_\epsilon(r)$. This is the emergence of the $P_2(f)$ set which steadily grows on the right hand side of $[b_0(r), b_\epsilon(r)]$. Similarly as $b_0(r)$ passes $\frac{1}{2}$ at $r = 2$ the same occurs for the left hand side. Now the minimal invariant set is the portion of $P_2(f)$ orbitally connected to $[b_0(r), b_\epsilon(r)]$. (See Figure 5.15)

The set-valued period doubling bifurcation occurs after 3. At this point we see that a new orbitally connected set exists of entirely period 2 points now exists and that its boundary intersects the boundary of the orbitally connected set containing $[b_0(r), b_\epsilon(r)]$. We see that the orbitally connected set of period two points is the

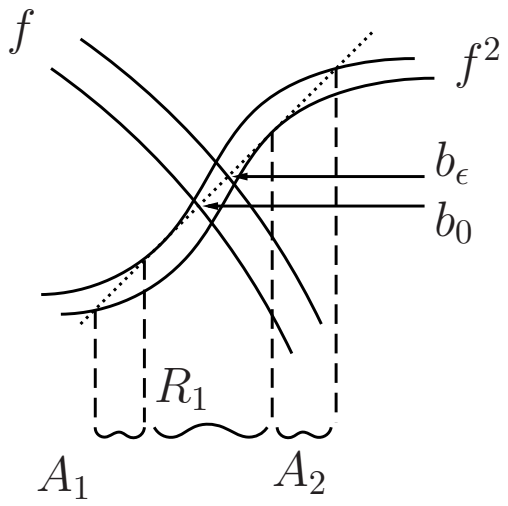


Figure 5.16: Point of bifurcation. $A_1 \cup A_2$ is a minimal invariant set and the closure of a orbitally connected set made up entirely of period two points. R_1 is a dual minimal invariant set and also the closure of another orbitally connected set. They intersect on there boundaries.

minimal invariant set and the one containing $[b_0(r), b_\epsilon(r)]$ is a dual invariant set. This is as predicted by Theorem 119. (See Figure 5.16)

The resulting bifurcation diagram for the period doubling discontinuous bifurcation is depicted in Figure 5.17.

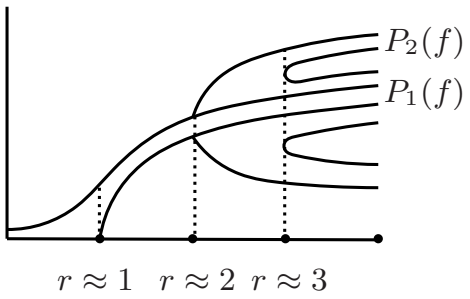


Figure 5.17: Bifurcation diagram for logistic map

5.4 Bifurcation set

We end with a brief discussion of the bifurcation set for discontinuous bifurcations. We want to know whether a relation at a point of bifurcation is isolated from other relations at points of bifurcation. In fact considering the example relation given in Figure 5.10 it is clear that by moving the relation depicted up and down slightly in the y axis one will obtain a infinite number of relations at points of bifurcations. What is more all will be close to the original relation. In fact the relation depicted in Figure 5.10 is an accumulation point of a sequence of bifurcating relations. Hence we know that relations undergoing bifurcation need not be isolated.

We will now show for a certain class of relations that the subset of them that are at points of bifurcation is dense in the class. In other words any relation is very close to bifurcation.

We will restrict ourselves to one dimension and consider first a simple class of relations defined as follows.

Definition 128. We say that set-valued dynamical system on X a compact subset of $\mathbb{R}$ is *simple* if there exists two continuous functions f_1 and f_2 from $X \mapsto X$ such that $f_1(x) < f_2(x)$ and $f(x) = [f_1(x), f_2(x)]$ for all $x \in X$. We say that a simple set-valued dynamical system f is *monotone* if f_1 and f_2 are monotonically increasing.

Example 31 is an illustration of such a set-valued dynamical system.

Next define $\mathcal{F}(X)$ to be the space all set-valued dynamical systems f on X satisfying *(H1),(H2)* and *(H3).*

Definition 129. Define the set $\mathcal{F}^s(X)$ as the subspace of $\mathcal{F}(X)$ such that each $f \in \mathcal{F}^s(X)$ is firstly simple monotonic and secondly has at least one minimal invariant set not equal to the whole space X.

Finally let the discontinuous bifurcation set be denoted as $\mathcal{B}_D(X)$.

We would like to know what the topology of $\mathcal{B}_D(X)$ is like in $\mathcal{F}^s(X)$. We will need the following result that tells us how the minimal invariant sets can be characterized in terms of f_1 and f_2.

Lemma 130. *Let f be a simple monotonic set-valued dynamical system f on X satisfying (H1),(H2) and (H3). Let E be a minimal invariant set for f then E is of the from $[a, b] \subset X$ where $a = \inf\{x \in E\}$ and $b = \sup\{x \in E\}$ and $f_1(a) = a$ and $f_2(b) = b$.*

Proof. First we show that $f_1(a) = a$ and $f_2(b) = b$. Without loss of generality we show the result for a. Assume not and $f_1(a) \neq a$ then either $f_1(a) < a$ or $f_1(a) > a$. In the first case if $f_1(a) < a$ then $f_1(a) \notin E$ as $a = \inf\{x \in E\}$ thus $f(a)$ maps points outside of E which is a contradiction. In the second case consider that as $f(E) = E$ there must exist $z \in E$ such that $a \in f(z)$. What is more $a = \min\{x \in f(z)\} = f_1(z)$ as otherwise $a < f_1(z)$ which again implies that f maps points in E outside of E. However $z \neq a$ by assumption and so $z > a$ as $z \in E$ and $a = \inf\{x \in E\}$. Thus $f_1(z) = a$ but $f_1(a) > a$ which implies $f_1(z) < f_1(a)$ but $z > a$. This contradicts simple monotonicity.

Any minimal invariant set must at least contain an interval as given $x \in E$, $f(x)$ is also in E and $f(x)$ contains an interval of size ϵ due to *(H3)*. Now assume that E isn't a interval and there exists a point $z \in [a, b]$ such that $z \notin E$. Let $a' = \inf\{x \in E | x > z\}$ and $b' = \sup\{x \in E | x < z\}$. We must have $a < b' < z < a' < b$. As $b' \in E$ we know that $f_2(b') \in E$ and thus we have either have the case that $f_2(b') \in [a, b']$ or that $f_2(b') \in [a', b]$. In the first case $f_2(b') = b'$ as monotonicity of f_2 means $f_2(b')$ is not less than b'. This would mean that $f([a, b']) = [f_1(a), f_2(b')] = [a, b']$ which contradicts the minimality of E. Thus we must have $f_2(b') \in [a', b]$. If this is the case then notice that $f(b') \subset [a', b]$ as otherwise $f_1(b')$ is either not in E or less than z which means $z \in [f_1(b'), f_2(b')] \subset E$ which we assumed it wasn't.

By a symmetric argument we can show that $f(a') = [f_1(a'), f_2(a')]$ must be a subset of $[a, b']$. But then we have $b' < a'$ and $f_1(b') > f_1(a')$ which contradicts simple monotonicity of f and hence $z \in E$. □

We now show how the set $\mathcal{B}_D(X)$ lies in $\mathcal{F}^s(X)$.

Proposition 131. *The set $\mathcal{B}_D(X)$ is dense in $\mathcal{F}^s(X)$.*

Proof. To show this result we need to show that for any relation $f \in \mathcal{F}^s(X)$ we can find $\tau > 0$ such that there exists a relation $g \in \mathcal{B}_D(X)$ such that $h(f, g) < \tau$. First

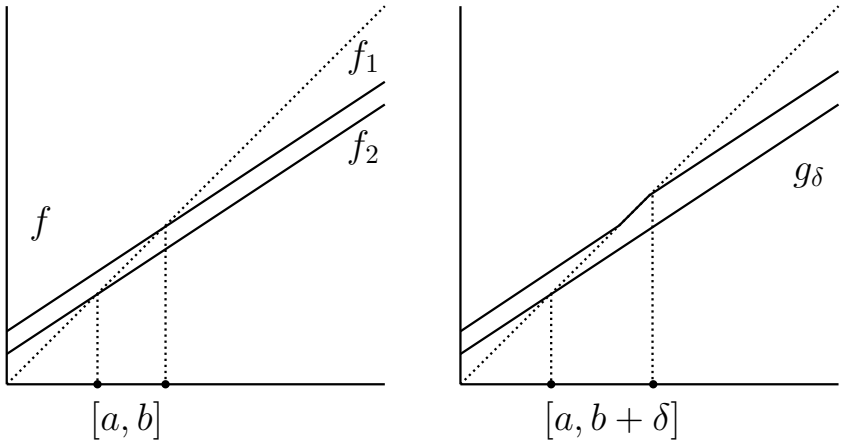


Figure 5.18: Example of the construction of g_δ from f. Notice that $[a, b]$ is a minimal invariant set for both f and g_δ but $[a, b + \delta] = [[b]]$ for g_δ.

fix some $\tau > 0$. $f \in \mathcal{F}^s(X)$ so f has at least on minimal invariant set E not equal to X. Due to Lemma 130 we know the minimal invariant set E is an interval of the form $[a, b]$. We will construct g as above by modifying the upper bound f_2 of f so that the modified relation is at a point of bifurcation and within τ of f. (See Figure 5.18).

Given $\delta > 0$ we consider the decomposition of the space into the three sets $X_{<b} = \{x \in X | x < b\}$, $[b, b + \delta]$ and $X_{>b+\delta} = \{x \in X | x > b + \delta\}$. We define g_δ on each of these sets as so:

$$g_\delta(x) = \begin{cases} f(x) & x \in X_{<b} \\ [f_1(x), x] & x \in [b, b + \delta] \\ [f_1(x), f_2(x) + (b + \delta - f_2(b + \delta))] & x \in X_{>b+\delta} \end{cases}$$

Let's first make sure that by choosing δ small enough we have $h(g_\delta, f) \leq \tau$. We will find three δ that ensure $h(g_\delta, f) \leq \tau$ on each of $X_{<b}$, $[b, b + \delta]$ and $X_{>b+\delta}$ and then pick the smallest of these.

Clearly for any $z \in X_{<b}$ we have $h(g_\delta(z), f(z)) < \delta$ thus let $\delta_1 = \delta$.

If $z \in [b, b + \delta]$ then $h(g_\delta(z), f(z)) = |f_2(z) - z|$. As $f_2(b) = b$ all we need to do is choose $\delta_2 < \frac{\tau}{2}$ such that using continuity of f_2 for any z such that $|b - z| < \delta_2$ we

have $|b - f_2(z)| < \tau/2$. Thus $|f_2(z) - z| < |b - f_2(z)| + |b - z| < \tau$.

If $z \in X_{>b+\delta}$ then $h(g_\delta(z), f(z)) = |b + \delta - f_2(b + \delta)|$. Again simply choose δ_3 to be small enough that the distance between $f_2(b + \delta)$ and $b + \delta$ is less than τ.

Now choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and hence $h(g_\delta(x), f(x)) < \tau$.

All that remains is to show that g_δ is at a point of bifurcation. To see this first pick $x \in E$. Note that by adapting the relation g_δ from f we have purposefully extended the chain connected component $[[x]]_\delta$ so that now $[[x]]_\delta = [a, b+\delta] \neq E_\delta = [a, b] = \overline{[x]}_\delta$ whereas previously it was $[[x]] = E = [a, b] = \overline{[x]}$. Hence we know due to Theorem 119 that g_δ is at a point of discontinuous bifurcation. Given any relation $f \in \mathcal{F}^s(X)$ and $\tau > 0$ we can construct such a g within τ of f we know that the set $\mathcal{B}_D(X)$ must be dense in $\mathcal{F}^s(X)$. □

Further questions arise as to whether or not the bifurcation set is dense in other more relaxed classes of systems. Certainly it seems possible that you could extend the above construction for minimal invariant sets with periodic points on the boundary such as in Example 127 for $r \in [2, 3]$. In Proposition 131 we use the fact that $f_2(a) = a$. That is the periodic orbit is completely contained within the boundary of the minimal invariant set and no internal point can map onto a. Considering this one might believe that it is possible to show that $\mathcal{B}_D(X)$ is dense in the subset $\mathcal{P}(X)$ of $\mathcal{F}(X)$ of relations whose minimal invariant sets E satisfy the property that $x \in E^o$ cannot map onto ∂E.

Note that Proposition 131 is to be expected and a similar thing happens if you allow continuous perturbations of single-valued dynamical systems. In this case any fixed point can be extended to an interval in a similar way to above. In order to really say anything interesting about the bifurcation set for such systems one has to require that the single-valued dynamical system have differentiable perturbations. In the above case one could require that given any simple monotonic set-valued dynamical system f that f_1 and f_2 be differentiable. Subsequently given any other differentiable simple monotonic set-valued dynamical system g we could say g is close to f if f_1 and g_1,f_2 and g_2, $\frac{df_1}{dx}$ and $\frac{dg_1}{dx}$ and $\frac{df_2}{dx}$ and $\frac{dg_2}{dx}$ are close. If we use this as a metric for closeness btweeen set-value dynamical systems then the construction we use in Proposition 131 no longer works. To see this notice that the distance between $\frac{dg_{\delta,2}}{dx}$

and $\frac{df}{dx}$ is bounded above zero for all $\delta > 0$.

It would certainly seem true that given a differentiable simple monotonic set-valued dynamical system f such that every minimal invariant set E for f satisfies $\frac{df_1}{dx}(a) \neq 1$ and $\frac{df_2}{dx}(b) \neq 1$ where E is of the form $[a, b] \subset X$ then f is isolated from any system at a point of discontinuous bifurcation. If however one of $\frac{df_1}{dx}(a)$ or $\frac{df_2}{dx}(b)$ is equal to 1 then we see that the construction we use in Proposition 131 does work and we can construct a sequence of systems all at points of bifurcation that accumulate at f. Such an example is given in Figure 5.10.

Bibliography

[1] E. Akin. *The general topology of dynamical systems*, volume 1. American Mathematical Soc., 2010.

[2] L. Arnold. Recent progress in stochastic bifurcation theory. In *IUTAM Symposium on Nonlinearity and Stochastic Structural Dynamics*, pages 15–27. Springer, 2001.

[3] L. Arnold. *Random dynamical systems*. Springer Science & Business Media, 2013.

[4] L. Arnold and B. Schmalfuss. Lyapunov's second method for random dynamical systems. *Journal of Differential Equations*, 177(1):235–265, 2001.

[5] J-P. Aubin and A. Cellina. *Differential inclusions: set-valued maps and viability theory*, volume 264. Springer Science & Business Media, 2012.

[6] J-P. Aubin and H. Frankowska. Set-valued analysis. systems and control: Foundations and applications, vol. 2, 1990.

[7] J. Ayala, P. Corbin, K. Mc Conville, F. Colonius, W. Kliemann, and J. Peters. Morse decomposition, attractors and chain recurrence. *Proyecciones (Antofagasta)*, 25(1):79–109, 2006.

[8] E.A. Barbashin. On the theory of generalized dynamical systems. *Uchenye Zapiski Moskovskogo Gosudarstvennogo Universiteta*, 135:110–133, 1949.

[9] X. Chen and J. Duan. Random chain recurrent sets for random dynamical systems. *Dynamical Systems*, 24(4):537–546, 2009.

[10] C. Conley. Isolated invariant sets and the morse index. cbms regional conference series in mathematics, 38. *American Mathematical Society, Providence, RI*, 16, 1978.

[11] C. Conley and R. Easton. Isolated invariant sets and isolating blocks. *Transactions of the American Mathematical Society*, 158(1):35–61, 1971.

[12] H. Crauel, L.H. Duc, and S. Siegmund. Towards a morse theory for random dynamical systems. *Stochastics and Dynamics*, 4(03):277–296, 2004.

[13] M. Dellnitz, G. Froyland, and O. Junge. The algorithms behind gaio-set oriented numerical methods for dynamical systems. *Ergodic theory, analysis, and efficient simulation of dynamical systems*, 560:145–174, 2001.

[14] T. Gayer. Control sets and their boundaries under parameter variation. *Journal of Differential Equations*, 201(1):177–200, 2004.

[15] J. Henrikson. Completeness and total boundedness of the hausdorff metric. *MIT Undergraduate Journal of Mathematics*, 1:69–80, 1999.

[16] A.J. Homburg and T. Young. Hard bifurcations in dynamical systems with bounded random perturbations. *Regular and Chaotic Dynamics*, 11(2):247–258, 2006.

[17] J. Lamb, M. Rasmussen, and C. Rodrigues. Topological bifurcations of minimal invariant sets for set-valued dynamical systems. *Proceedings of the American Mathematical Society*, 143(9):3927–3937, 2015.

[18] M. Liapounoff. *Problème général de la stabilité du mouvement.* Princeton University Press Princeton, 1949.

[19] Z. Liu. The random case of conley's theorem. *Nonlinearity*, 19(2):277, 2005.

[20] Z. Liu, S. Ji, and M. Su. Attractor–repeller pair, morse decomposition and lyapunov function for random dynamical systems. *Stochastics and Dynamics*, 8(04):625–641, 2008.

[21] A. Marchaud. Sur les champs de demi-droites et les èquations diffèrentielles du premier ordre. *Bulletin de la Sociètè Mathèmatique de France*, 62:1–38, 1934.

[22] R. McGehee. Attractors for closed relations on compact hausdorff spaces. *Indiana University Mathematics Journal*, 41(4):1165–1209, 1992.

[23] R. McGehee and T. Wiandt. Conley decomposition for closed relations. *Journal of Difference Equations and Applications*, 12(1):1–47, 2006.

[24] E. Roxin. On generalized dynamical systems defined by contingent equations. *Journal of Differential Equations*, 1(2):188–205, 1965.

[25] E. Roxin. Stability in general control systems. *Journal of Differential Equations*, 1(2):115–150, 1965.

[26] E. Roxin and V. Spinadel. Reachable zones in autonomous differential systems. *Boletín de la Sociedad Matemática Mexicana*, 5(1):125–135, 1960.

[27] S.C. Zaremba. À propos des champs de demi-cônes convexes,. *Bulletin des Sciences Mathèmatiques*, 64:5â12, 1940.

[28] E.C. Zeeman. Stability of dynamical systems. *Nonlinearity*, 1(1):115, 1988.

[29] H. Zmarrou and A.J. Homburg. Bifurcations of stationary measures of random diffeomorphisms. *Ergodic Theory and Dynamical Systems*, 27(5):1651–1692, 2007.

A

Numerical Method

Here we give an idea of the algorithm we used to compute the periodic sets in Figures 4.10, 4.9 and 4.8.

Let f be a relation on a compact subset X of $\mathbb{R}^m$ satisfying *(H1),(H2)* and *(H3)*. Points in $P_n(f)$ satisfy a finite property, namely x is in $P_n(f)$ if $x \in f^n(x)$, and therefor these objects are simpler to compute than say $P(f)$ or $[x]$. Hence the focus of our numerical methods is to compute $P_n(f)$ to a high degree in order to approximate minimal invariant sets for f. An ideal platform for doing so is the package for Matlab called GAIO or global analysis of invariant objects [13]. GAIO provides data structures and algorithms for computing invariant objects in dynamical systems. The data structure GAIO allows users to manipulate is a binary tree that represents a partition of the space X into boxes of a certain size.

Denote $\mathbb{B}(d)$ to be a partition of X into 2^d boxes where $d \in \mathbb{N}$. For each box $B_i \in \mathbb{B}(d)$ the GAIO package allows one to compute whether $f(B_i)$ intersects B_j. We compute a transition matrix T from the boxes $\mathbb{B}(d)$ by letting $T_{ij} = 1$ if and only if $f(B_i) \cap B_j \neq \emptyset$ and $T_{ij} = 0$ otherwise. T tells us how points in X can move around in the partition under the influence of f.

We can represent a some collection of boxes in the partition by using a vector v

such that a box B_i is identified with v if $v_i > 0$ and not identified with v otherwise. Thus we can approximate orbits of f using the matrix T by applying them to such vectors v. Suppose $A \in \mathcal{K}(X)$, denote v the vector defined by $v_i = 1$ if $A \cap B_i \neq \emptyset$ and $v_i = 0$ otherwise. Then by construction of T we see that the orbit $f^n(A)$ is approximated by $T^n v$ where the accuracy depends on the size of the boxes.

If for instance we have a collection of boxes given by a vector v such that $(Tv)_i = 0$ or 1 if $v_i = 1$ and $(Tv)_i = 0$ if $v_i = 0$ then T maps the set of boxes represented by v into a set contained within v thus it is reasonable to assume that there exists some invariant set for f contained in those boxes represented by v.

Of particular interest to us is the following set computed from a partition $\mathbb{B}(d)$ for f:

$$\tilde{P}_n(f) = \{i \in \mathrm{diag}(T^n)\}$$

Any box B_i such that $i \in \tilde{P}_n^d(f)$ has the property that there exists a point $x \in B_i$ such that $f^n(x) \cap B_i \neq \emptyset$. In this way $\tilde{P}_n(f)$ is an approximation of $P_n(f)$. This is how we have produced the pictures showing how the periodic sets partition the minimal invariant set.