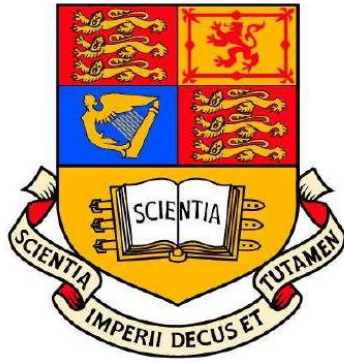


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Implementations of Fault-tolerant Quantum Devices

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Declaration of Originality

The work presented in the chapters of the second part of this thesis is the result of research carried out during the last three years as a graduate student, under supervision of Terry Rudolph and Sean Barrett.

I hope that fruitful interactions with quite a few postdocs and other researchers have added value to the thesis. People I am indebted with are mentioned later.

The rest of the contents are brief reviews I considered necessary to make this thesis minimally self-contained, and are properly referenced in the bibliography.

Abstract

Accurate control and addressability of quantum devices will come with the promise of improvement in a wide variety of theoretical and applied fields, such as chemistry, condensed matter physics, theoretical computer science, foundational physics, communications, metrology and others.

Decoherence of quantum states and the loss of quantum systems have adverse effects and deter a satisfactory usage of quantum devices. This is the main problem to be overcome, which is the goal of quantum fault tolerance. In this thesis we present a series of works that contribute to some of the fields mentioned above, in the direction of fighting decoherence and loss.

These works fall in two categories: on one hand, we looked at computer architectures which can be used to combat errors, using techniques of quantum error correcting codes. In a first project we found decoherence and loss probability thresholds below which quantum computing is provably possible. We assumed a very particular error model tailored specially to quantum dots as single photon sources and linear optics. Subsequently we looked at the problem of loss, both of heralded and unheralded, and devised some ways to fight it. The framework under which this work was done was used to develop theory which is currently being tested in a quantum optics experimental group and will be reported in an article later this year.

On the other hand, we studied how the error probability can be reduced at the physical level, thanks exclusively to the properties of the system in which information is stored, as opposed to making use of quantum codes. We looked at a particular superconducting circuit, which is potentially very well protected against some types of decoherence. In particular, we observed that the interaction with the environment become weaker for certain values of the circuit external parameters.

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The potential advantage in using the laws of quantum mechanics to simulate quantum systems was first pointed out by Feynman [1]. The states of quantum systems are described as normalized vectors in a Hilbert space of dimension D^n , where D is the dimension of an individual system and we consider n of them. One can readily see that the size of Hilbert space grows exponentially with the size of the systems we want to describe, which in turn implies an exponential increase in the parameters needed in order to define their state. Intuitively, what makes quantum systems difficult to harness is their ability to live in a superposition of states. This entails an exponential slowdown in simulation of quantum systems in classical computers. Conversely, efficient simulation of quantum systems turns out to be one of the various exciting consequences of the ability to create, maintain and control quantum superpositions, together with quantum computation and quantum cryptography.

Models of quantum computation exploit this ability to form superpositions and its generalization to the non-classical correlations which can be found in systems governed by quantum mechanics [2, 3]. This has led to new algorithms, such as Deutsch's algorithm [4], Shor's algorithm [5, 6] and Grover's algorithm [7, 8], to mention the most famous ones. They all outperform known classical

algorithms for the same problem.

One of the most striking conclusions to be drawn from the performance of these algorithms is that the complexity classes of quantum computers are fundamentally different from classical ones. In fact, the class of problems which can be solved by quantum computers efficiently (that is, in a number of time steps that is upper-bounded by some polynomial of the problem size) contains the class of those problems solved efficiently by probabilistic classical computers [9, 10]. However, the widespread belief is that quantum computers cannot solve NP-complete problems [11], which are the hardest problems for which a solution can be checked efficiently [12].

1.1 Description of Quantum States

Quantum states are represented by normalized vectors in Hilbert space. A quantum bit (qubit) in an arbitrary superposition is represented as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta|1\rangle), \quad (1.1)$$

where $\{|0\rangle, |1\rangle\}$ are the computational basis vectors and α, β are complex phases. A more general description of quantum states relies on the *density matrix*. Because of superposition, these matrices need not be diagonal in general. The density matrix can be thought of as being the generalization of quantum probability distributions to the quantum case [13]. The density matrix for the state above would be:

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}. \quad (1.2)$$

In general $\text{tr}\rho^2 \leq 1$, with equality holding only for pure states. An alternative statement is that pure states correspond to rank one density matrices.

The Pauli matrices, together with the identity, form a basis for Hermitian matrices, so the state of a qubit can thus be written as:

$$\rho = \frac{1}{2}(I + r_1X + r_2Y + r_3Z), \quad (1.3)$$

where I is the identity matrix and $\vec{r} = (r_1, r_2, r_3)$ is called the *Bloch vector*, and the Pauli matrices are¹:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.4)$$

Measurements on quantum systems are represented by a set of positive operators fulfilling the condition:

$$\Sigma_i E_i = I, \quad (1.5)$$

such that each E_i can be decomposed as the product $E_i = M_i^\dagger M_i$, where M_i are operators such that after measuring outcome i , the system, originally in state $|\psi\rangle$, is left in a state proportional to $M_i |\psi\rangle$. The probability of obtaining outcome i upon measurement is given by $p_i = \text{tr}(E_i \rho) = \text{tr}(M_i \rho M_i^\dagger)$. In general the E_i 's need not be orthogonal. If they are, $E_i = M_i^\dagger M_i = M_i$, then equation (1.5) is a orthogonal resolution of the identity and the corresponding measurement is projective, which essentially means that it can be repeated, as can be seen from the update rule of density operator after measurement outcome i :

$$\rho' = \frac{M_i \rho M_i^\dagger}{\text{tr}(M_i \rho M_i^\dagger)}. \quad (1.6)$$

1.1.1 Entanglement

It was first remarked by Einstein [2] that local measurements of certain bipartite quantum states are correlated, even if the measurement events on different parts are causally disconnected. This was taken as a paradox showing that quantum mechanics is an incomplete theory. Bell [3] showed that correlations allowed by quantum mechanics can be stronger than those allowed classically. Experiments

¹In Chapter 6, a different notation for the Pauli matrices will be used: $\sigma_X \equiv X$, $\sigma_Y \equiv Y$, $\sigma_Z \equiv Z$

confirmed that the predictions of quantum mechanics were indeed right [14]. The conclusion to be drawn is that quantum mechanics allows for stronger correlations than classical mechanics, which can be exploited in several ways [15, 16]².

This exclusive feature of quantum correlations, which is related to their ability to live in a superposition of states, is known as *quantum entanglement*. A bipartite quantum state ρ_{12} is correlated if it cannot be written as a tensor product of local quantum states, $\rho_{12} \neq \rho_1 \otimes \rho_2$. A state is classically correlated if it can be written as $\rho_{12} = \sum_i p_i \rho_{1,i} \otimes \rho_{2,i}$. It is entangled if it cannot be written as neither an uncorrelated state or a classically correlated state.

Almost all two qubit gates have some entangling power [17]. Computationally, the unitary two qubit gates controlled- X (C_X), and controlled- Z (C_Z), also known as controlled-not and controlled-phase gates, respectively, are coventionally used to entangle qubits, since their entangling power is maximal. To see how this is done consider the expressions for gates:

$$C_X = |0\rangle\langle 0|_1 I_2 + |1\rangle\langle 1|_1 X_2, \quad (1.7)$$

$$C_Z = |0\rangle\langle 0|_1 I_2 + |1\rangle\langle 1|_1 Z_2. \quad (1.8)$$

It follows that $C_X X_1 I_2 C_X^\dagger = X_1 X_2$, $C_X I_1 Z_2 C_X^\dagger = Z_1 Z_2$ and that $C_Z X_1 I_2 C_Z^\dagger = X_1 Z_2$, $C_Z I_1 X_2 C_Z^\dagger = Z_1 X_2$. For example, the transformation $C_X |+\rangle\langle +|_1 |0\rangle\langle 0|_2 C_X^\dagger \rightarrow \frac{1}{2}(|0\rangle_1 |0\rangle_2 + |1\rangle_1 |1\rangle_2)(\langle 0|_1 \langle 0|_2 + \langle 1|_1 \langle 1|_2)$ maps an initially separable state into an entangled state. Here, $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is the +1 eigenstate in the Hadamard basis $\{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$

1.1.2 Quantum Errors

Quantum systems are in general not perfectly isolated from their environment. This means that there is an interaction which correlates the dynamics of quantum systems with that of their surroundings, resulting in unwanted deviations from the

²Despite allowing for stronger correlations, quantum mechanics does not allow for information transfer faster than the speed of light, and it is reconciled with special relativity

evolution of the systems if they were completely isolated. In fact what happens in general is that the off-diagonal terms of the density matrix, which account for the ‘quantum coherence’ of the system, are damped at a rate which depends on the coupling strength between the system and its environment. Other sources of error that potentially jeopardize the quantum coherence of a system are imperfect gates and measurements, and loss errors. Imperfect operations can be modeled as perfect operations followed by some kind of error channel. In next subsection we give a physical interpretation of how errors channels arise.

Energy exchange between the system and its environment leads to fluctuations of the quantum state. Losing energy to the environment will cause the system to relax and gaining energy from it will cause the system to become excited. The channel describing this process is the *generalized amplitude damping* channel [10]. The mathematical theory giving rise to this process is extensively studied [18, 19].

Another error model which is *quantum bit-flip* channel, which will flip the label of the computational basis states with probability p :

$$\rho \rightarrow \mathcal{C}(\rho) = pX\rho X + (1 - p)\rho. \quad (1.9)$$

This is interesting whenever the computational states are degenerate. There exists a purely quantum type of error, known as *dephasing*, which arises from the ability of quantum states to form superpositions. Dephasing is the process whereby the relative phase between the energy eigenstates is destroyed. Some situations where this effect arises when the energy difference between the computational states fluctuates in time, or when the system evolves for an amount of time which is not exactly known, for example. The channel that models dephasing errors is the *quantum phase-flip* channel:

$$\rho \rightarrow \mathcal{C}(\rho) = pZ\rho Z + (1 - p)\rho, \quad (1.10)$$

also known as the *phase damping* channel. Interestingly, this channel can be seen as a bit-flip channel in the Hadamard basis.

Errors, if uncorrected, will destroy the coherence of quantum bits and spoil

their correlations, preventing quantum speed-ups.

1.1.3 Physical Interpretation of Quantum Channels

We now explain how to characterize the mechanisms giving rise to noise in quantum systems, in terms of generalized evolution of quantum states and motivate this description using a particular scenario. A detailed introduction can be found in [20, 10].

Mathematically, quantum errors arise from evolution of the system of interest into an unknown state, due to interaction with an ancilliary system to which it is coupled. This ancilliary system is sometimes a surrounding environment but in other cases can be a fictitious system introduced for mathematical convenience. Consider the time evolution operator acting on the joint system:

$$\mathcal{U} : |\psi\rangle_S |0\rangle_E \longrightarrow \sum_i M_i |\psi\rangle_S |i\rangle_E, \quad (1.11)$$

where S and E stand for “system of interest” and the “environment”, respectively. $\{|i\rangle_E\}$ is an orthonormal basis for the environment and the operators $M_i = \langle i|_E \mathcal{U} |0\rangle_E$ act on the system of interest are known as *Kraus operators*.

Noise is modeled using errors channels. An error channel acting on the state ρ_S is obtained by tracing out the environment degrees of freedom after joint evolution:

$$\begin{aligned} \mathcal{C}(\rho_S) &= \text{tr}_E [\mathcal{U}(\rho_S \otimes |0\rangle\langle 0|_E) \mathcal{U}^\dagger] \\ &= \sum_i \langle i|_E (\mathcal{U}(\rho_S \otimes |0\rangle\langle 0|_E) \mathcal{U}^\dagger) |i\rangle_E \\ &= \sum_i M_i \rho_S M_i^\dagger \end{aligned} \quad (1.12)$$

The normalization condition $\sum_i M_i^\dagger M_i = I$ arises from the fact that the channel is trace preserving, *i.e.* there is no leakage. As an example, we focus here on the phase-flip channel, introduced in previous section. Consider the channel given by

the joint evolution operator $\mathcal{U}$:

$$|0\rangle_S |0\rangle_E \longrightarrow \sqrt{1-p} |0\rangle_S |0\rangle_E + \sqrt{p} |0\rangle_S |1\rangle_E \quad (1.13)$$

$$|1\rangle_S |0\rangle_E \longrightarrow \sqrt{1-p} |1\rangle_S |0\rangle_E + \sqrt{p} |1\rangle_S |2\rangle_E, \quad (1.14)$$

where p is the *scattering* (or transition) probability. This channel is an archetypical example of a process where no energy is exchanged. The Kraus operators for this channel are:

$$M_0 = \sqrt{1-p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_1 = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \sqrt{p} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (1.15)$$

Note that the decomposition of a channel in terms of Kraus operators is not unique. Making the change $M'_1 = (M_1 + M_2)/\sqrt{2}$ and $M'_2 = (M_1 - M_2)/\sqrt{2}$ it is easy to see that the new operators leads are the same as those of the dephasing channel introduced in previous section, with a phase-flip probability $p' = \frac{p}{2}$.

As an interesting aside, and since it will be relevant later in the thesis, we explain how the loss of one photon in a state can cause loss of coherence. Imagine we start with a GHZ state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, and the ancillary system represents our knowledge about the state of the lost photon (its polarization for example). The unitary evolution is:

$$|000\rangle_S |0\rangle_A \longrightarrow \frac{1}{\sqrt{2}} |00L\rangle_S |0\rangle_A + \frac{1}{\sqrt{2}} |00L\rangle_S |1\rangle_A \quad (1.16)$$

$$|111\rangle_S |0\rangle_A \longrightarrow \frac{1}{\sqrt{2}} |11L\rangle_S |0\rangle_A - \frac{1}{\sqrt{2}} |11L\rangle_S |1\rangle_A, \quad (1.17)$$

where L stands for loss. If the photon is lost no knowledge about its state can be gained, and the operators acting on the remaining two photons are:

$$M_0 = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_1 = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1.18)$$

which is equivalent to a dephasing channel with probability $p = \frac{1}{2}$.

1.2 Quantum Error Correction

As we have seen, quantum information is exposed to both dephasing and bit-flip errors. This renders it far more delicate than classical information. Despite this fact, there exists a theory of quantum error correcting codes, which shows how quantum information can be protected from decoherence [21, 22, 23, 24, 25, 26].

Before we introduce the underlying theory of quantum codes, it will be useful to recall some basic ideas of classical error correcting codes.

1.2.1 Classical Linear Codes

In the classical setting, noise is counteracted by adding redundancy to the original information, in such a way that a receiver can measure certain subsets of the extended codeword to see if an error occurred, a procedure known as error correction [10]. A classical linear error correcting code on n physical bits, encoding k logical bits, and distance d (denoted by $[n, k, d]$) can be defined in terms of an $n \times k$ matrix G over $\mathbf{Z}_2$, called the *generator matrix*. The k logical bits are encoded in the set of codewords v of the code C , corresponding to the vector subspace spanned by the columns of G .

To correct errors, one uses a matrix called parity check matrix H , which can be obtained from the generator matrix by choosing $n - k$ linearly independent vectors orthogonal to the code subspace, and setting them as the rows of a $(n - k) \times n$ matrix. This is equivalent to saying that $H \times v = 0, \forall v \in C$, meaning that the codespace is defined as the kernel of H . Now, a set of errors e corrupt a codeword v and map it to $v' = v + e$, where the sum is modulo 2. It turns out this can be detected and corrected by measuring $s = H \times v' = H \times e$, which is known as the

error syndrome. A very straightforward example is a repetition code on 3 qubits, which takes one bit and copies. The generator matrix is:

$$G = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad (1.19)$$

so the parity check matrix is:

$$H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \quad (1.20)$$

If there is an error on the first bit, the error syndrome will be $s = H \times (v + (1, 0, 0)^T) = (0, 1)^T$, whereas an error in the second qubit would give a syndrome $s = (1, 1)^T$.

This way of obtaining information about the errors is analogous to the way quantum errors are detected and corrected in quantum stabilizer codes.

1.2.2 Stabilizer Codes

The *No-Cloning Theorem* [27] states that it is impossible to create identical copies of an unknown quantum state. The underlying idea is that unitarity of quantum theory prevents us from copying a quantum state while preserving the ‘coherences’, that is, the off-diagonal terms in the density operator. Imagine that a unitary operation does the following copying operation $U |\phi_i\rangle |e\rangle = |\phi_i\rangle |\phi_i\rangle$, for all basis vectors $|\phi_i\rangle$, then:

$$\langle \phi_i | \langle e | |\phi_j\rangle |e\rangle = \langle \phi_i | \langle e | U^\dagger U |\phi_j\rangle |e\rangle = \langle \phi_i | \langle \phi_i | |\phi_j\rangle |\phi_j\rangle \quad (1.21)$$

and since quantum states are normalized $\langle \phi_i | \phi_j \rangle = |\langle \phi_i | \phi_j \rangle|^2$, which entails that the copied vectors are either the same or they are orthogonal. This prevents techniques from classical error correction from being applied directly to quantum

states ³.

Note that violating the No-Cloning theorem would entail many unphysical consequences, such as unlimited storage capability with finite resources, since copying a state many times would allow us, by repeated measurements, to determine its parameters to arbitrary accuracy. Similarly, this would entail a violation of the uncertainty principle.

Thus, a new way of protecting quantum information against quantum errors is necessary. In a single oversimplified statement, quantum codes are clever ways to partition a large enough Hilbert space into copies of a logical subspace, in such a way that one can embed logical information in one of these subspaces and be able to recover it in the presence of errors. In what follows we explain how this is done:

The Pauli group on one qubit, $\mathcal{P}_1$ is defined as the group generated by the Pauli matrices together with the identity and the complex variable i :

$$\mathcal{P}_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}. \quad (1.22)$$

The Pauli group of n qubits, $\mathcal{P}_n$, is defined as the set of operators written as tensor product of elements of $\mathcal{P}_1$. The normalizer of the Pauli group, that is, the set of operators that leave the Pauli group invariant under conjugation, is called the *Clifford group*, and is of interest since the generators of the Clifford group plus one extra unitary transformation suffice to simulate any unitary transformation on n qubits [10].

The Stabilizer group $\mathcal{S}$ is an abelian subgroup of the Pauli group such that it does not contain $-I$ or $\pm iI$, and the subspace stabilized by these operators is called the *stabilizer code* [28]:

$$\mathcal{C} = \{|\psi\rangle \text{ s.t. } S_i |\psi\rangle = |\psi\rangle \ \forall S_i \in \mathcal{S}\}. \quad (1.23)$$

³The No-cloning Theorem can be generalized to arbitrary quantum operations by introducing an ancilla and performing a suitable unitary evolution on the extended system. This would lead to imperfect cloning, *i.e.* with finite probability.

Stabilizer codes are a special type of quantum error correcting code and represent the quantum counterpart of linear block codes. A stabilizer code on n qubits has at most $k = n - 1$ independent generators, since each operator imposes a constraint on the way the Hilbert space is partitioned.⁴

The set of stabilizer states for n qubits is exponentially large in n , and despite it being just a small subset of infinite number of general quantum states [29] their structure makes them specially well suited for error correction. Imagine the code suffers an error E such that

$$E|\psi\rangle = ES_j|\psi\rangle = -S_jE|\psi\rangle \quad (1.24)$$

for some S_j . Thus measuring the stabilizer operators will give us information about the nature of the errors that potentially occurred. This is sufficient to reverse their effect. This is the same principle as in the parity check matrix and the syndrome vector in the classical case.

Let $\mathcal{C}$ denote a code on n qubits and k stabilizers, then the set of operators which commutes with its stabilizer is denoted as:

$$\mathcal{C}^\perp = \{E \in \mathcal{P}_n \text{ s.t. } [E, S_j] = 0 \ \forall S_j \in \mathcal{S}\}. \quad (1.25)$$

The set of operators in $\mathcal{C}^\perp \setminus \mathcal{C}$ cannot be detected by measuring the stabilizer generators, yet they don't belong to the stabilizer of the code. In fact, the distance of the code is given by the minimum weight of any operator in $\mathcal{C}^\perp \setminus \mathcal{C}$. *Logical operators*, that is, operators acting on the 2^k logical subspace, are defined as the cosets in $\mathcal{C}^\perp / \mathcal{C}$. This is the same as saying that logical operators are defined modulo multiplication times a stabilizer generator.

Quantum error correcting codes are the basis on which *Quantum Fault Tolerance* is developed. Given a quantum code and some assumptions about the physical properties of a hypothetical quantum computer, it is possible to show

⁴Using the stabilizer group to describe quantum states has the advantage of being a very compact way to describe quantum states. Whereas to describe a general n qubits state one needs of order of 2^n terms, only n stabilizers are needed in order to define the state, up to a global phase

that it can be resilient against noise.

Part I

**Fault-Tolerant Quantum
Computation**

2.1 Physical Components are Noisy

For unencoded information, noise limits the length of any computation (number of logical gates applied before a failure happens at the logical level) and it will be upper-bounded by the inverse of the error rate. This is not enough to build a quantum computer, since the length of a given computation can be arbitrarily long. In order to reduce the effects of noise on quantum information at a given physical error rate, the size of the computer has to be scaled accordingly to the length of the computation being carried out.

In the context of quantum computation, the main focus of fault-tolerance is how to use noisy components, which is unavoidably the case, to simulate an error-free computation at the logical level. Whether this is at all possible is a non-trivial question, and a series of results culminating in the *threshold theorem* for quantum computation answers it positively [30, 31, 32, 33]

Fault-tolerance can also be regarded as describing a quantum to classical phase transition [34]. Below the threshold, arbitrarily long computations can be performed reliably. Above it, errors accumulate too rapidly and the quantum computation will fail with very high probability. The value of the probability threshold

depends on the particular error model assumed.

2.1.1 Threshold for Fault-Tolerant Quantum Computation

A quantum algorithm is composed of a series of time-steps at the physical level. The nature of each step can be one of the following:

- *preparation*: initialization of a physical qubits in a given state ρ_i , which can be pure or mixed.
- *gate*: steering of the initialized physical qubits into a different state. Includes single qubit and multi-qubit gates.
- *idle time*: time spent by a qubit between preparation and gate, any two gates or gate and measurement. This should be minimized by parallelization.
- *measurement*: readout of a physical qubit, which can be demolishing (the qubit is not there anymore after the measurement) or non-demolishing (it is still there).

Each physical qubit undergoes one of these processes at every step of the computation. Of course each of these processes can fail, and we assume that they all will with some probability p . Encoding the bare physical qubits in a clever way can reduce the effective probability $P^{eff}(p)$ affecting the encoded qubits so that the computation can last a little bit longer. Adding more redundancy we can reduce the effective probability even more and make it as long as we desire, at the expense of increasing the number of physical qubits. Failure will be modeled differently depending on the architecture for which a threshold is derived. Different error models will yield different thresholds, the most stringent ones giving lower thresholds.

Error correction has to be applied periodically throughout the computation to deter the propagation of errors and reverse them. The standard method is to

apply a logical gate followed by a round of error-correction. This is repeated for every logical gate, to ensure that the physical errors introduced by the logical gates are accounted for and do not accumulate.

Some general requirements for a threshold to exist are described next, although a thorough survey is available elsewhere [35, 36].

Given a code, the redundancy should be added in such a way that a physical error occurring at a given time-step cascades *at most* into a constant number of errors at a later time step. This is motivated by the fact that it should take t separate errors to cause a distance $d = 2t + 1$ code to fail. Also, preparation, gate application and measurements should be *highly parallelizable*, since otherwise errors would accumulate too fast. A heuristic argument motivating this is the following: the rate at which uncertainty is injected into the system scales proportionally to the computer's size, for each component is error-prone. Accordingly, the rate at which this uncertainty is *pumped out* of the system should also increase in a proportional way, to ensure that errors the information remains unharmed. A more subtle condition is that the classical error-correction can be done *almost immediately*, since if it were not the case, physical errors might accumulate faster than the error-correcting algorithm runtime, which would result in logical errors. This is related to the requirement of having *fast measurements*, which will give us syndrome information. Errors should be weakly correlated, to avoid either too many errors in one block, if using concatenated codes, or too long chains, for the case of topological codes.

There are further requirements that are desirable but not necessary. One of them is that any two qubits interact with an error rate independent of their separation, which need not hold in the case of topological codes. Another one is that qubits *should not leak* from the computer, although this has been shown [37, 38] to be an unnecessary requirement for fault tolerance. In fact we also address this problem in this thesis.

The existence of a threshold for a particular architecture is a constructive proof, so it depends on assumptions such as error model and the particular way error correction and measurement and post-processing is carried out. Over the

next sections we will give a simplified overview of how a threshold arises in two different schemes, namely the traditional concatenated and the topological approach.

2.2 Standard Code Concatenation

One strategy to attain a threshold is known as *code concatenation*. We start by using a $[n, 1, 3]$ code to encode a logical qubit using n physical qubits. The logical qubit is said to be at level 0, and the qubits of the code are in level 1. Now each in level 1 is encoded using the same code, which will create the concatenation level 2. Iterating this procedure amounts to concatenating the code with itself N times, which is reminiscent of classical error correction [39]. Several quantum codes exist that perform well under this scheme [40, 41].

The main idea behind code concatenation for a code correcting up to one error, is that the error effective probability at level N depends on the effective probability at level $N - 1$ as:

$$P_N^{eff} = \gamma(P_{N-1}^{eff})^2. \quad (2.1)$$

This can be understood as using quantum code to reduce the effective error probability of a qubit in the immediately lower level of concatenation. Since the probability of logical error will be the probability of two errors happening times some combinatorial constant γ which is particular to each code and error model. This gives:

$$P_N^{eff} = \frac{(\gamma p)^{2^N}}{\gamma}. \quad (2.2)$$

As can be seen in Fig. 2.1, if the physical error probability is above $1/\gamma$, the effective error probability will increase as more redundancy is added. On the contrary, if $p \leq 1/\gamma$, there is a doubly exponential decrease of the effective error probability in the number of concatenations. The number of physical qubits $Q_N = n^N$ grows exponentially in the number of concatenations, so we still have

a exponential decrease of P_N^{eff} in the number of physical qubits.

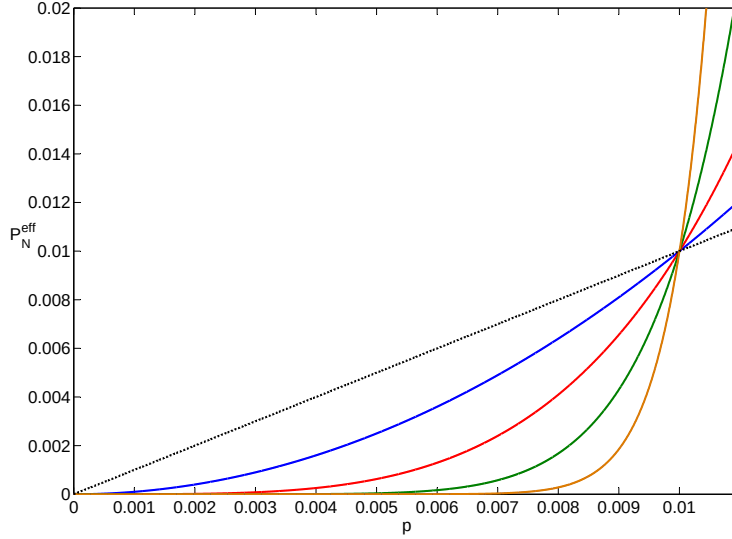


Figure 2.1: Effective error probability versus physical error probability for different concatenations, for $\gamma = 100$. The colored lines represent different levels, from 1 concatenation (blue) to 4 concatenations (yellow). The dotted line corresponds to no encoding at all. As redundancy is added, that is, as more physical qubits are used to encode a logical qubit, the dependence of effective error probability on the physical error probability approaches a Heaviside function centered at the threshold. Below the threshold, the effective error probability decreases exponentially on the number of physical qubits. On the other hand, above the threshold, increasing redundancy will add more errors than the code is actually able to correct, so errors will occur with probability 1.

2.2.1 Transversality and Measurement

In order to achieve fault-tolerance in the concatenated approach, one crucial requirement is that a single physical error at some time t_1 will create at most one error in the encoded block to which it belongs at any later time. This ensures that errors will not cascade in an uncontrolled way, since each block undergoes a periodic error-correcting stage.

In order to make encoded blocks to interact with one another while satisfying the requirement that one error cascades into at most one error on each block, we

need to use *transversal* gates. Transversal gates are those that can be done in a bit-wise fashion, in such a way that an error on the qubit i will only ever propagate to the qubit i on the other block, leaving all other qubits invariant. In order to see this, note that a block gate $U = \otimes_i U_i$ will pass through an error in qubit i E_i without propagating the error to other qubits in the block (see Fig. 2.2(b)). As an example take the controlled-not gate: remember that $C_X X_1 I_2 C_X^\dagger = X_1 X_2$, $C_X I_1 Z_2 C_X^\dagger = Z_1 Z_2$, so C_X gates *do* propagate errors, but this is fine since the errors are propagated from one block to another, which are subject to different error correction stages.

Besides protecting the quantum information from random errors, we still need to manipulate it. This demands that we still be able to perform universal gates on the encoded qubits, as well as preparation and measurement. However it is known that no code exists having a wholly transversal universal gate-set [42]. It would thus seem that it is impossible to obtain a universal gate-set. This is not true, however, since although transversality implies fault-tolerance, this is only a one-way implication. There is a way around this problem that uses the ‘gate teleportation’ protocol (explained in next chapter), which can itself be done fault tolerantly, provided we can do fault-tolerant measurements.

Fault-tolerant measurements are necessary for ancilla preparation, syndrome measurement and logical qubit readout. One way to measure an operator acting some qubits of the block is depicted in Fig. 2.2. In order to see how this is done consider the operator $U_1 U_2 U_3$ acting on a block state $|\psi\rangle$ to be measured in Fig. 2.2(a). After applying the operation $C_{U_1 U_2 U_3} = |0\rangle\langle 0| + |1\rangle\langle 1| U_1 U_2 U_3$ between the ancilla and the block, and measuring the ancilla in the Hadamard basis, the resulting state is:

$$\frac{I \pm U_1 U_2 U_3}{2} |\psi\rangle, \quad (2.3)$$

depending on which state $|\pm\rangle$ was measured. Note that if the $|\psi\rangle = \pm U_1 U_2 U_3 |\psi\rangle$, then the ancilla was then necessarily measured in $|\pm\rangle$, respectively. This allows us to learn the expected value of the operator without actually measuring the

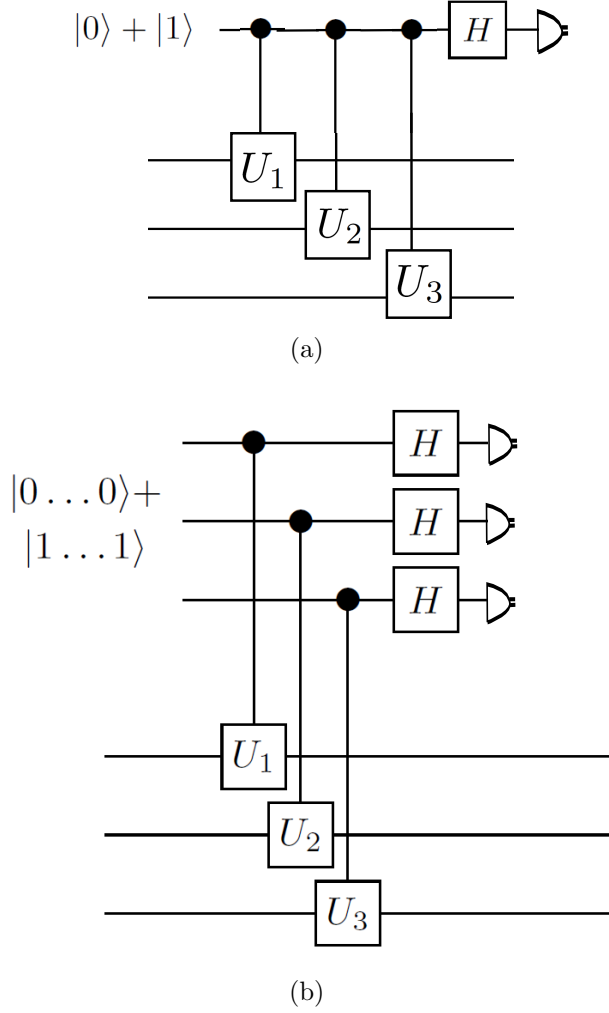


Figure 2.2: Measurement of operator $U = U_1 U_2 U_3$. The gates in the circuit are $C_{U_i} = |0\rangle\langle 0| + |1\rangle\langle 1|U_i$. In (a) this measurement is not fault-tolerant since an error in the ancilla can propagate to more than one qubit. In (b) this is no longer the case. It is assumed that the cat state is free from X errors, since the parity is measured in a pair-wise fashion before using it in the circuit. Z errors in the ancilla will not propagate to the qubits and but can change the outcome of the measurement so the measurement has to be done several times and take a majority vote.

block qubits.

This is obviously not fault-tolerant since one error in the ancilla just after preparation will cascade into many errors within the encoded block.

One way to make this procedure fault-tolerant is to substitute the single qubit ancilla by a *cat state* $\frac{1}{\sqrt{2}}(|00\dots 0\rangle + |11\dots 1\rangle)$ of the size of the encoded operator to be measured. Preparation of the cat state is not intrinsically fault-tolerant, and a way around this is to measure its pair-wise parity just before using it to perform the measurement of the operator. These parity measurements will not disturb the state since a cat state is stabilized by operators of the sort $Z_i Z_j$. Upon discovering an incorrect parity, the state is discarded and a new one is created. Z errors in the cat state will not propagate into the encoded block, but can potentially change the measurement outcome, so a majority vote is necessary in order to bring down the error probability. On the other hand a single X or Y error only propagates to one error in the encoded block.

A qubit-wise measurement of the operator is done which will change the parity of the cat state accordingly. If an error flipped the parity of the a given stabilizer operator, then the majority vote measurement would reveal this. Other methods introduced in the aforementioned references provide alternative ways around this problem [43, 44].

2.3 Topological Approach to Fault Tolerance

An alternative route to build a “clean” quantum computer robust against from errors is to have the system get rid of unwanted errors by itself. The way to accomplish this is by engineering a Hamiltonian whose ground state is the logical degenerate subspace on which the encoded gates act. In two dimensions, excitations to higher energy bands form can be defined by localized quasi-particles of fractional statistics, called anyons. This highlights its connection to topologically ordered systems [45].

Roughly speaking, a quantum system is topologically ordered [46] if its ground state degeneracy depends on the topology of the manifold on which it sits, and

more specifically the ground states are locally indistinguishable. This means that the operators which provide us with information about the ground state have support in loops belonging to a non-trivial homology class ¹ of the manifold.

To see how this non-local character of logical operators entails fault tolerance, note that local errors cannot bring us from one ground state to another. In other words, it takes a non-local chain of errors to reach and modify the logical information stored in the ground state, which is equivalent to an anyon traveling along a non-trivial path. However, local excitations are in principle harmless and can be dealt with at sufficiently low temperatures.

Now, the way quantum information stored in the ground subspace is processed depends on the anyon-model that the system supports. An anyon model [20] will consist of a list of excitations and a set of rules for braiding and fusing the anyons, which determine the computational power of the model. Essentially, how the global quantum state of system changes as two anyons are braided, whether it induces a unitary matrix acting on the state (non-abelian models) or just a global phase (abelian models), will be a signature of the power of the anyonic system. For example, the *Ising model* is non-abelian yet it is not universal, *i.e.* its braiding operations do not span the whole of $SU(2^N)$, and there are tricks to make this universal [47]. Other models, such as the *Fibonacci model*, are universal.

2.3.1 Toric Code

The first and simplest model for topological quantum computation is the toric code [48]. Consider a square lattice, one each of whose edges sits a qubit. Then the toric code is defined to be the ground state of the Hamiltonian:

$$H = -\frac{\Delta}{2}(\sum_P S_P + \sum_S S_V), \quad (2.4)$$

where $S_P = \prod_{i \in P} Z_i$ and $S_V = \prod_{i \in V} X_i$ are plaquette and vertex stabilizers, respectively. They are shown pictorially in Fig. 2.3. These stabilizers obviously

¹Loops belonging to the trivial homology class can be smoothly contracted to a single point. Loops winding around holes are not contractible.

commute, $[S_P, S_V] = 0 \forall P, V$, since they share either 0 or 2 qubits.

It is possible to define a dual lattice, by mapping each face to a vertex and viceversa. Then, each plaquette (vertex) operator is associated with a vertex (plaquette) operator in the primal lattice via a duality transformation $\sharp$, such that $S_{P^\sharp} = \prod_{i \in P^\sharp} X_i$ and $S_{V^\sharp} = \prod_{i \in V^\sharp} Z_i$.

Counting will convince us that we have the same number of independent stabilizers (plaquette and vertex) as we have qubits. This means that the state is fixed and thus no information can be encoded in it. By introducing periodic boundary conditions, we remove two constraints and a 4-fold degeneracy arises in the ground state. Logical operators would thus correspond to chains of single qubit errors winding around one of the dimensions of the torus, as depicted in Fig. 2.3(a).

Note that this degeneracy only depends on the topology of the manifold on which the lattice is defined. Surfaces other than a torus can be used as long as they are topologically non-trivial. In general these variations are called *surface codes*. Surface codes can have two types of boundary: primal boundaries are associated to primal plaquettes, whereas dual boundaries correspond to dual plaquettes (vertices). To attain non-trivial topology some of the inner stabilizer operators will not be enforced, which can be seen as a ‘hole’ in the code. Holes can also be primal or dual, depending on which type of stabilizers are not enforced. Again, not enforcing one stabilizer operator will correspond to releasing a two-dimensional subspace from the constraint imposed by the operator, which is consistent with the idea that logical codewords live in subspaces of the total Hilbert space. As shown in Fig. 2.3(b), removing four plaquette operators (and the vertex operator within them) from the Hamiltonian, while only removing four qubits, means that we have one degree of freedom associated to that hole. A dual hole is created in an analogous manner.

A Z (X) error on any one qubit anticommutes with two vertex (plaquette) operators. One such error will raise the energy of the system, as can be easily seen from:

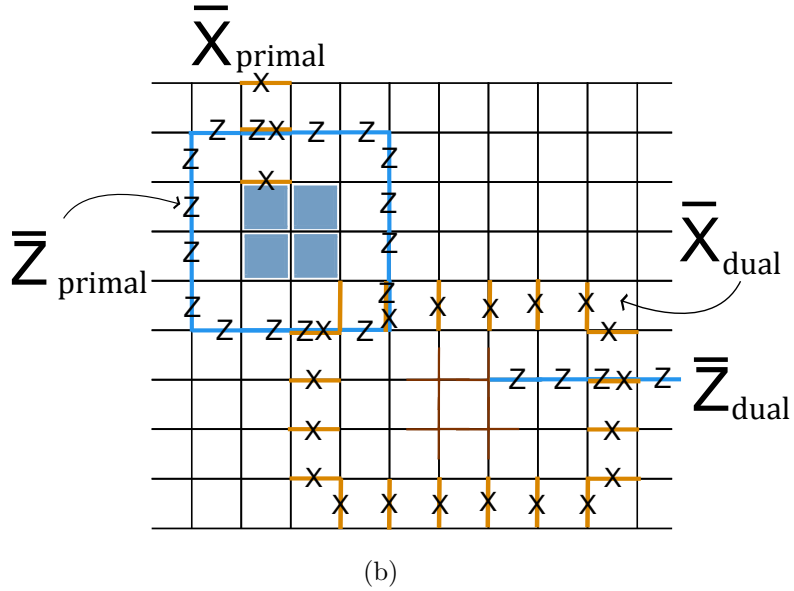
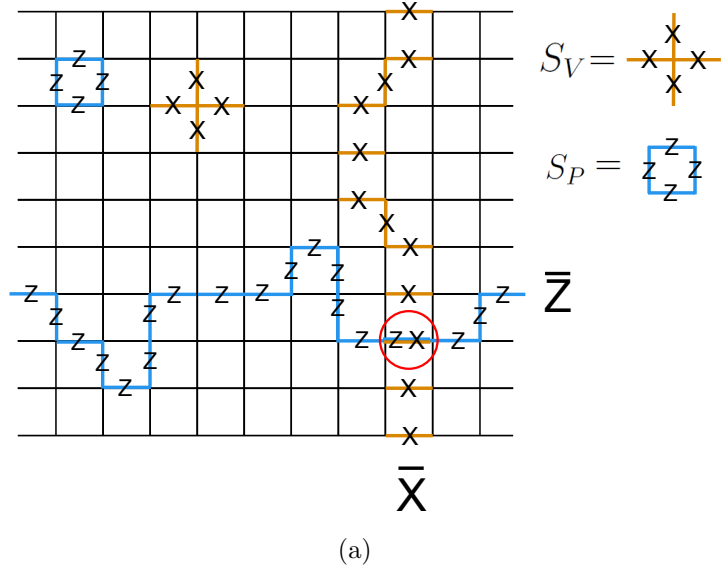


Figure 2.3: Physical qubits sit on the edges of the lattice. In (a) the boundaries of the lattice are identified, so the qubits sit in a torus. Strings of errors winding around the torus commute with the plaquette and vertex operators yet cannot be written in terms of them, acting thus on the logical information. Here only two out of the four logical operators are shown. The other two strings of operators, corresponding to the other qubit, would look similar but with the X's and Z's interchanged. As required, the anticommutation relations between the logical operators hold (see red circle). In (b) the boundaries are no longer identified. The plaquettes in shaded blue and the vertices shaded in dark yellow correspond to operators which are removed from the Hamiltonian in equation (2.4). A primal hole is created by not enforcing plaquette operators, whereas a dual hole arises from not enforcing vertex operators. They encode primal and dual qubits, respectively.

$$\langle \psi' | S_V | \psi' \rangle = \langle \psi_0 | Z S_V Z | \psi_0 \rangle = -\langle \psi_0 | S_V | \psi_0 \rangle, \quad (2.5)$$

or alternatively with X errors and plaquette operators. From the toric code Hamiltonian definition, it follows that it takes an amount of energy Δ to create an error in the system. Chains of Z (X) errors will only anticommute with the endpoint vertex (plaquette) operators. These flipped operators correspond to excitations from the ground states, which can be seen as anyonic quasi-particles, as they are created in pairs and have braiding statistics which don't correspond to either fermionic or bosonic particles. The anyon model of the toric code is abelian and has very limited computational power, although a method known as *code deformation* can be used to upgrade its computational capabilities [49].

Logical errors on the encoded information occur when an anyon follows a non-trivial loop before they recombine. A rather discouraging feature of the toric code is that it doesn't take any energy for an anyon to move around the code, since as, we've seen above, any chain of errors will only anticommute with at most two stabilizer operators. Another way to phrase this is that the energy it takes to create a pair of anyons is independent of the initial distance between them. This means that once an anyon pair has been created, they can move around the surface randomly and will potentially follow a non-trivial path before recombination, so that information will be disturbed [50, 51]. For that reason the toric code is not a *self-correcting* memory in two dimensions. By self-correcting it is meant that the likelihood of a logical error decreases as the size of the system increases. The toric code is however self-correcting in four spatial dimensions [52]. On the contrary, in two dimensions, active error correction, that is, syndrome measurements, classical processing and recovery mechanism, are necessary in order to prevent diffusion of errors [53].

Recent developments point to the possibility of there being a self-correcting memories in less than four dimensions [54, 55]. Despite this fact, some amount of error correction is always necessary at readout stages to account for errors in the protecting Hamiltonian [56].

The Hamiltonian in equation (2.4) involves four-body interactions, which are far from trivial to realize in a laboratory. For this reason, it proves useful to think of the toric code as the ground state of a fictitious Hamiltonian. One way to create this state is to operate in a regime in which a two-body hamiltonian interaction gives rise to an effective four-body interaction.

Instead of engineering a Hamiltonian whose ground state encodes the information, another possibility is to create the code itself using a quantum circuit. In the absence of Hamiltonian protection, active error correction is the only way one can preserve the information. We now focus on this approach.

In the absence of a Hamiltonian, the concept of anyonic excitation is not useful to understand how chains of errors can harm the logical information. In the context of surface codes, another way to define chains of errors is to specify on which kind of boundary they can end, *i.e.* whether it is primal or dual. For instance, a chain of Z (X) errors conforming a logical operator can only end in a dual (primal) boundary. The definition of logical operators is slightly changed, since it will depend on which boundaries we define as ‘containing’ the logical qubit. It is shown in Fig. 2.3(b).

The idea behind code deformation is that dynamically changing the topology of the lattice, that is, creating holes and braiding them, one can achieve a larger set of gates [49]. This will be vital to attain universal quantum computation, as is explained in next chapter.

For completeness, it is interesting to mention that there is an early approach to quantum computation [57] with surface codes that, instead of relying on brading of defects to achieve logical gates, assumes that several layers, each containing a logical qubit, can interact via transversal two qubits gates. Topological protection is inherited from surface codes, yet at the logical level, information is processed with circuit-like operations.

Chapter 3

Measurement-based Quantum Computation

There exist several models of quantum computation. Some of them are largely based on their classical analogs, such as quantum Turing machines [58], quantum cellular automata [59] or quantum circuits [10]. This latter has enjoyed more popularity since there exist explicit constructs to approximate arbitrary unitary evolution [60, 61].

Measurement-based quantum computation is yet another quantum computation model [62], in which evolution is carried out by performing measurements. The main primitive used in measurement-based quantum computation was introduced in [63] and is known as *gate teleportation*. It relies on there being quantum correlations before the computation starts. These correlations are consumed as measurements disentangle the qubits. Depending on how entanglement is prepared there exist several proposals of measurement-based quantum computation. One of them needs only entangled pairs [64], provided that one is able to perform projective two qubit measurements. Another one only needs single qubit measurements, but the state on which is a multi-qubit entangled state, called the *cluster state* [65, 66], or *graph state*.

Measurement-based quantum computation doesn't have a classical counter-

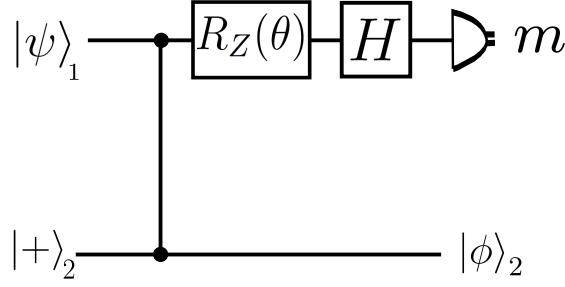


Figure 3.1: Teleportation circuit depicting how the information contained in qubit 1 is teleported into qubit 2, $|\phi\rangle_2 = X^m H R_Z(\theta) |\psi\rangle_2$, up to known corrections.

part, since the gate teleportation protocol is a genuinely quantum effect. A version of the gate teleportation circuit is depicted in Fig. 3.1.

Define a measurement in the $X - Y$ equator of the Bloch sphere:

$$M(\theta) = \{R_z(\theta)|+\rangle\langle+|R_z^\dagger(\theta), R_z(\theta)|-\rangle\langle-|R_z^\dagger(\theta)\}, \quad (3.1)$$

where $R_Z(\theta) = e^{-i\theta Z/2}$ stands for a rotation around axis Z of θ degrees. Just before the measurement, the total state is:

$$\frac{1}{\sqrt{2}}[|0\rangle_1 H R_Z(\theta) |\psi\rangle_2 + |1\rangle_1 X H R_Z(\theta) |\psi\rangle_2], \quad (3.2)$$

where H is the Hadamard gate $H = |0\rangle\langle+| + |1\rangle\langle-|$. After measuring qubit 1 in the computational basis, the resulting output state $|\phi\rangle_2$ is:

$$|\phi\rangle_2 = X^m H R_Z(\theta) |\psi\rangle_2, \quad (3.3)$$

and $m \in \{0, 1\}$ is the outcome of the measurement. This illustrates how a rotation can be teleported from one qubit to another, up to known operators, provided they share quantum correlations.

3.1 Computation on Graph States

To define graph states [67] it is useful to recall the mathematical definition of a graph $G = \{V, E\}$, where $V \subset \mathbb{N}$ are the vertices where the qubits sit. A

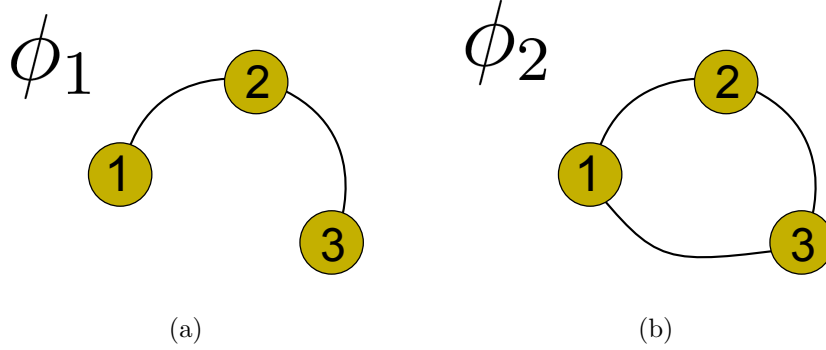


Figure 3.2: Two different graph states, defined on (a) a linear graph and (b) a circular graph.

cluster state is uniquely defined as the common eigenstate of the operators $S_i = X_i \bigotimes_{\{i,j\} \in E} Z_j, \forall i \in V$ where X_i and Z_i are the Pauli matrices applied to qubit i .

A constructive definition would be to initially set in each vertex in the state $|+\rangle$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. The symbol $E \subset \mathbf{N} \times \mathbf{N}$ corresponds to edges connecting the qubits, representing entanglement in the form of a two-qubit controlled- Z operation being applied, given by $C_Z^{ij} = \text{diag}(1, 1, 1, -1)$ for qubits i and j . C_Z gates are diagonal in the computational basis so they all commute and can be done in any order.

Whereas there is no established distinction in the literature between graph state and cluster state, we adopt the following convention: a graph state is defined as we have done above, and a cluster state is a graph state defined in a regular lattice.

To clarify the definition we look at a two different graph states composed of three qubits, depicted in Fig. 3.2. Their stabilizer description is:

$$\begin{array}{c|ccc} \phi_1 & 1 & 2 & 3 \\ \hline S_1 & X & Z & I \\ S_2 & Z & X & Z \\ S_3 & I & Z & X \end{array} \quad \begin{array}{c|ccc} \phi_2 & 1 & 2 & 3 \\ \hline S_1 & X & Z & Z \\ S_2 & Z & X & Z \\ S_3 & Z & Z & X \end{array} . \quad (3.4)$$

If two operators stabilize one state, then its product also does, $|\psi\rangle = S_i |\psi\rangle = S_i S_j |\psi\rangle$. This gives us an update rule for the set of stabilizer operators defining

the quantum state as measurements are performed, since all stabilizer operators have to commute. Performing a Z measurement in the centre qubit will change the state:

$$\begin{array}{c|ccc} \phi'_1 & 1 & 2 & 3 \\ \hline S_1 & X & Z & I \\ S_2 & I & Z & X \\ S_3 & (-1)^{m_z} I & Z & I \end{array} \quad \begin{array}{c|ccc} \phi'_2 & 1 & 2 & 3 \\ \hline S_1 & X & Z & Z \\ S_2 & Z & Z & X \\ S_3 & (-1)^{m_z} I & Z & I \end{array}, \quad (3.5)$$

where m_z is the outcome of a measurement in the Z basis. From the definition of graph states, it follows that the state ϕ_1 is unentangled and only the qubits 1 and 3 of ϕ_2 remain in a graph state. Measurements in the Z basis effectively remove the measured qubit from the graph, which otherwise remains invariant. An X measurement in the centre qubit would project the cluster into:

$$\begin{array}{c|ccc} \phi''_1 & 1 & 2 & 3 \\ \hline S_1 & X & I & X \\ S_2 & Z & X & Z \\ S_3 & (-1)^{m_x} I & X & I \end{array} \quad \begin{array}{c|ccc} \phi''_2 & 1 & 2 & 3 \\ \hline S_1 & Y & I & Y \\ S_2 & Z & X & Z \\ S_3 & (-1)^{m_x} I & X & I \end{array}, \quad (3.6)$$

and m_x is the outcome of a measurement in the X basis. So an X measurement of a given qubit does remove the measured qubit, but instead of removing the edges of the graph, it rather creates another link between the neighbors of the measured qubit, up a local correction. In fact, this is a special case of measurements in the $X - Y$ equator of the Bloch sphere which can be used to teleport rotations to neighboring qubits. It will prove very useful to describe graph states in terms of stabilizers since any operation of interest entails several qubits, so its description in terms of states becomes rapidly intractable.

This model offers some desirable properties. The most interesting one is that universal quantum computation can be achieved, as we will see, just by means of single-qubit measurements. Also, cluster states can be built using local interactions, i.e. nearest-neighbor interactions, which is also quite appealing. Some experimental realizations of graph states have already been demonstrated in a

variety of physical systems [68, 69].

The caveat is that one needs to create the graph state *prior* to computation, since it is consumed as the quantum algorithm progresses. For this reason the entanglement contained in the graph state is sometimes referred to as a resource, which can be prepared off-line.

It is not difficult to see that measurement-based quantum computation is equivalent to the quantum circuit model of computation [70]. A quantum circuit can simulate a cluster state just by creating the correlations between the qubits and then applying measurements as dictated. The conversion from circuit model to graph-state model only needs one two qubit gate per bond, so it is efficient. Conversely, a graph state can simulate efficiently a quantum circuit by carving out the circuit layout by means of Z measurements. This conversion is also efficient, as the required size of the graph scales as the circuit size and the number of measurements simulating time flow scales with the number of circuit time-steps. The way single qubit unitary operators and two-qubit gates are done in the circuit model is explained below.

3.1.1 Universality

It is a well known fact that single qubit gates and two qubit gates can generate any unitary gate in $SU(2^N)$ for arbitrary N [71, 72], albeit taking exponential many gates to do so. Since the measurement-based and circuit models are equivalent, in order to see how universality is achieved it will suffice to show how single unitary evolution is simulated and how a two qubit gate can be performed using single qubit measurements.

To see how an arbitrary single qubit gate is constructed, remember that any special unitary operator can be parameterized by three angles coming from its *Euler decomposition*, specifically, any $U \in SU(2)$ can be written as

$$U = R_z(\gamma)R_x(\beta)R_z(\alpha). \quad (3.7)$$

Using the fact that $ZH = HX$ and $ZX = -XZ$, and taking into account the

symmetry of trigonometric functions one finds that

$$R_z(\theta)X^mH = X^mR_z((-1)^m\theta)H = X^mHR_x((-1)^m\theta). \quad (3.8)$$

Iterating the result of equation (3.3) we have that, starting in a five qubit linear graph state and after a sequence of measurements:

$$\begin{aligned} |\phi\rangle_5 &= X^{m_3}HX^{m_3}HR_Z(\gamma')X^{m_2}HR_Z(\beta')X^{m_1}HR_Z(\alpha)|+\rangle \\ &= (-1)^{m_3m_2}X^{m_4+m_2}Z^{m_3+m_1}R_Z(\gamma)R_X(\beta)R_Z(\alpha)|+\rangle. \end{aligned} \quad (3.9)$$

Note that some angles have to be chosen *adaptively*, $\beta' = (-1)^{m_1}\beta$ and $\gamma' = (-1)^{m_2}\gamma$ to take into account the randomness of the prior measurement outcomes. As a consequence of the randomness of the measurement outcomes, there will always be a byproduct consisting of Pauli operators acting on the final state, but it is not a problem since it can be dealt with simply reinterpreting the results of our final measurements.

In some sense, two qubit gates are intrinsically embedded in measurement-based models, since the entanglement shared by two qubits can be seen as being created by a C_Z gate. In [66] a thorough analysis of how gates are simulated is presented. We now show that there is a way to simulate C_X gates using only single qubit measurements in the X basis. In order to do that, remember that a gate is defined by specifying how it transforms the Pauli matrices. In case of the C_X gate it is $C_X X_1 I_2 C_X^\dagger = X_1 X_2$, $C_X I_1 Z_2 C_X^\dagger = Z_1 Z_2$, where qubit 1 is the control qubit and qubit 2 is the target qubit.

It is not possible to find a universal set of transversal gates for any code [42]. As we saw in previous chapter, this doesn't mean that we cannot achieve universal fault tolerant quantum computation. In fact, it is possible to attain universality provided we can do fault tolerant preparations and measurements in the Z and X basis, and C_X gates. Measurements in the X basis is all we need to propagate information along a one-dimensional wire, up to known corrections,

and it also suffices to simulate C_X gates as we have seen. There exist certain states which cannot be described within the stabilizer formalism called *Magic States*, introduced in [73] and further studied in [74, 75]. These allow one to perform gates outside the Clifford group and yet can be distilled from a reservoir of noisy states using only C_X gates and measurements in the Z and X basis, provided they are not mixtures of the Pauli matrices eigenstates. We consider two states, $|Y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ and $|A\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$, which can be used together with the gate teleportation circuit to probabilistically implement the phase gate (within the Clifford group) and the $\frac{\pi}{8}$ gate (outside the Clifford group).

3.1.2 Clifford Operations

Despite the fact that the quantum circuit model and the measurement-based model can simulate each other with a modest overhead, the latter offers further possibilities to parallelize the gates which belong to the Clifford group.

Remember we defined Clifford operations as the set of operations that map the Pauli group to itself, $P_i O_C = O_C P_j$, where P_i, P_j belong to the Pauli group, i.e. the Clifford group is the normalizer of the Pauli group. Any gate in the Clifford group can be written in terms of the Hadamard, Phase and C_X gates.

Now, looking back at equation (3.9), it is easy to see that if measurements were not adaptive, nothing prevents us from measuring the qubits in any order. In general, all qubits in a measurement pattern giving rise to an operation in the Clifford group can be measured *at the same time*. In fact this is a reformulation of the *Knill-Gottesman Theorem* [10], which states that circuits consisting of preparations and measurements in the Pauli basis and gates in the Clifford group can be efficiently simulated classically. The byproduct operators of each single measurement can be pulled through the sequence of Clifford gates in a controlled manner, so we will still be able to reinterpret correctly the outcomes of the final measurements.

This property of measurement-based quantum computation allows us to compress the operations corresponding to non-adaptive measurements in one set of

simultaneous measurements and some classical post-processing.

3.2 Fault Tolerant Measurement Based Quantum Computer

Drawing from topological protection ideas [76, 57] Raussendorf *et al.* devised a measurement based model [77, 78, 79] which features advantages from topological codes, and attains very high thresholds [77, 80, 81]. In their scheme a cluster state is used to build a three dimensional version of the surface code [76]. This is attained by creating a three-dimensional cluster state whose qubits sit on the edges and faces of a cubic lattice $\mathcal{L}$. The basic face stabilizer is shown in Fig. 3.3, where face qubits are depicted in red and edge qubits are black. As explained next, this convention is reversed when we consider the dual lattice.

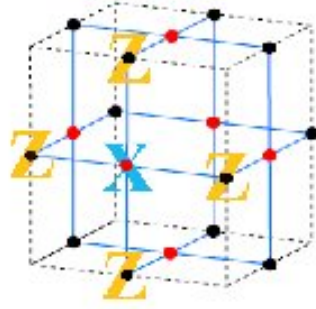


Figure 3.3: Stabilizer operator corresponding to a face qubit. The three-dimensional cluster state is composed of many adjacent unit cells, as the one depicted here, extending in all three dimensions.

3.2.1 Microscopic View

If we define a duality transformation $\sharp : \mathcal{L} \rightarrow \mathcal{L}^\sharp$, faces (edges, vertices) in the primal lattice $\mathcal{L}$ are mapped to edges (faces, cubes) in the dual lattice $\mathcal{L}^\sharp$:

$$\begin{aligned}
\#C_0 &\rightarrow \hat{C}_3 \\
\#C_1 &\rightarrow \hat{C}_2 \\
\#C_2 &\rightarrow \hat{C}_1 \\
\#C_3 &\rightarrow \hat{C}_0,
\end{aligned} \tag{3.10}$$

where C_0, C_1, C_2, C_3 and their dual counterparts stand for the set of vertices, edges, faces and cubes, respectively. Both the $\mathcal{L}$ and $\mathcal{L}^\#$ lattices are interlocked and provide a means to understand primal and dual qubits encoded in this cluster, in a similar way to surface codes introduced in last chapter.

This specific three-dimensional cluster state incorporates the surface code within its structure, as can be seen from measuring in the X basis all the face qubits of a two-dimensional section of the cluster. For this two-dimensional slice the stabilizer operators consistent with the measurement pattern in the X are the same as those of the surface code. Note that if we were considering the dual lattice, the edge qubits would be measured in the X basis, leaving the face qubits in a dual surface code. From now we will be looking only at the primal lattice, since the same applies to the dual after suitable relabeling.

Since a surface code has trivial topology it cannot encode a qubit unless some degree of freedom is released. This can be done by relaxing (not enforcing) one or more stabilizer constraints, or, in a more pictorial way, by creating holes, of *defects*, in the surface to attain a topologically nontrivial shape. Holes can be created by measuring a face qubit in the Z basis, which effectively will cause the corresponding surface code stabilizer operator not to be enforced. Remember that the effect of measuring in the Z basis amounts to removing the qubit from the cluster while leaving the rest of the correlations invariant.

To recover the encoding that we introduced in the toric code, a suitable way of encoding qubits is to measure out two separate groups of stabilizers effectively creating a two dimensional subspace in the code. Remember that this way logical

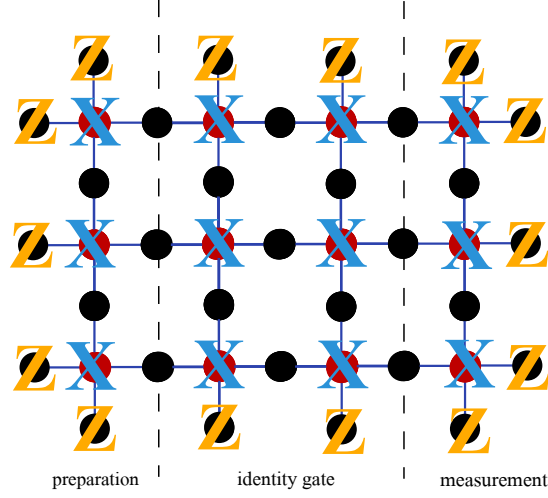


Figure 3.4: A correlation surface consists of the product of stabilizer operators defined within the surface $\mathcal{C} = \Pi Z_{\text{boundary}} X_{\text{bulk}}$. If we take time to flow from left to right, the measurement pattern giving rise to this correlation surface can also be seen as preparation and measurement of a logical qubit.

qubits are still delocalized (so only long chains will potentially between defects or around them will result in a logical error). A similar reasoning applies in one higher dimension, where one dimension can be singled out as time and the remaining two are seen as a surface in which the defects move.

The measurement pattern in the Z basis determines which of the original stabilizer operators survive as the computation proceeds. A crucial concept to understand how logic gates are carried out in this cluster state is that of *correlation surface* $\mathcal{C}$, illustrated in Fig. 3.4. A correlation surface is a product of face stabilizer elements, and as such, if it has a boundary, it must necessarily correspond to a ring of Z pauli operators (remember the pictorial definition of a face stabilizer in Fig. 3.3). Since it is a product of stabilizers the expected value of the correlation surface is 1 in the absence of unwanted errors, $\langle \mathcal{C} \rangle = 1$. Note that it is also possible to multiply by cube stabilizers and change the shape of the surface from planar to any shape which is consistent with the boundary. In fact, a pair of defects encoding a qubit (measured in the Z basis) will define a boundary in which a correlation surface must end, which can be used to propagate information

along the cluster.

The whole goal of creating this three-dimensional cluster state is to map a noisy measurement pattern to a noiseless computation. In order to do so, errors have to be detected, located and accounted for. The stabilizer structure of the cluster state will ensure that an X error occurring on the primal lattice is equivalent to a number of Z errors occurring in the dual lattice, and viceversa. This allows us to consider only Z errors, which will flip the stabilizers of the two adjacent cubes that share the faulty qubit, these stabilizers are of the form $S_{cube} = \otimes_{i=1}^6 X_i$, where we have an X for each face of the cube. Chains of Z errors will flip only the endpoints of the chain. One of the features of this scheme is that measuring the cluster qubits in the X direction will at the same time give us the syndrome, and propagate the code forwards in time through the correlation surfaces [78].

As the cluster is measured and the syndromes are read, classical post-processing of the syndrome information is needed to estimate whether an error occurred or not. In this setting, as with the surface code, a logical error occurs when a chain of Z errors ends up in two boundaries (two defects in our case) or winds around one boundary (one defect) so that the chain still commutes with the stabilizer, yet is not part of it, thus changing the value of the logical qubit. This “guessing” is done using a ‘Minimum Weight Perfect Matching’ algorithm [82, 83], which matches pairs of endpoints according to a cost function which penalizes long chains of Z errors.

The algorithms run during the classical postprocessing stage are assumed to be instantaneous, which is obviously a simplification. Although their runtime scales polynomially in the size of the cluster state, the assumption of instantaneous post-processing rapidly becomes no longer valid. In fact this can be a crucial issue operating at error rates close to the threshold, where physical errors are common. Reference [84] explains how to vastly reduce the time complexity of the decoding procedure. Also, there exists an alternative decoding algorithm which offers a trade-off between reliability and time efficiency [85].

The important point to realize is that if the errors are sufficiently sparse, then

the endpoints will be matched correctly and the errors reversed. On the other hand, if errors are too common, the chances of an incorrect matching abound and the possibility of a matching that would actually give us a logical (Z errors chain of non-trivial homology class) increases to a maximum $1/2$. There is a threshold for the error probability under which the correct decoding is asymptotically assured, around 1% [77, 80, 81]. The next chapter will dwell a bit more on thresholds for this scheme.

3.2.2 Macroscopic View

Having explained the origin in terms of stabilizer operators, it will prove useful to switch to a macroscopic viewpoint in order to understand qubit initialization and measurement, as well as how the logical C_X is done at the physical level.

The definition of logical operators is somewhat arbitrary. Since they are measured in the Z basis, we can think of defects as the boundaries of correlation surfaces. Then, logical qubit initialization and measurement in the computational basis would be achieved just by creating a simple loop of Z measurements. This is shown in Fig. 3.4. The same type of correlation surface would entail initialization and measurement of a dual defect in the rotated basis, i.e. $\{|+\rangle, |-\rangle\}$. Defining the primal and the dual lattices is a matter of convention, which accounts for this freedom to define the computational basis in the lattice. However, the anticommutation relations between logical operators and correlation surfaces will hold irrespective of this choice. A pair of defects encoding a qubit are associated to two types of logical operator, as shown in Fig. 3.5. A chain of Z errors winding around a defect will cut one correlation surface once associated with the pair of defects, thus changing its expectation value without leaving a syndrome. Likewise, a chain of Z errors joining two defects will cut another correlation surface. The logical nature of the error will depend on the choice of primal and dual lattice.

It is important to keep in mind that there exist two types of defect, *primal* and *dual*, depending upon the lattice in which they evolve. Primal and dual

defects cannot touch as they each live in a separate lattice. Thus chains of errors starting in one defect *cannot* end in a defect of a different type, which means that we can put defects of different type as close as we like without fear of logical error happening.

One of three dimensions of the cluster state is taken to be consumed in time-steps, slice by slice. The braiding of primal and dual defects along the computational direction will result in a set of correlation surfaces which are compatible with the topology of the resulting measurement pattern. Remember correlation surfaces are sheets of stabilizers operators which carry the correlations from one time-slice of the computation to another one. The correlation surfaces compatible with the braiding in Fig. 3.6 can be shown to enact a C_X gate between defects of different kind. C_X gates between primal and dual defects are not really abelian gates. The case is very similar to the discussion of the toric code in previous chapter. To make it non-abelian it has to entangle defects of the same kind. This is done by dynamically changing the topology of the code, *i.e.* by introducing defects and measuring them in the right way, which is equivalent to code deformation. A method for doing such a deformation is used in the circuit of fig. 3.7.

Fault-tolerance arises by noting that the probability of a logical error during state preparation or measurement (in the X and Z basis), and during a C_X gate, decreases as we make the defects thicker and thicker (so that a chain of errors winding around the defect is less and less likely to occur) and pull them far apart (so that a chain of errors ending in two defects is exponentially unlikely). This will be enough to map the noisy cluster state to a noiseless quantum computation pattern.

3.2.3 Universality

Using the bulk lattice allows us to perform fault-tolerant versions of preparation and measurements in the computational and rotated basis, and a C_X gate. In order to achieve fault-tolerant universal quantum computation with only a subset of

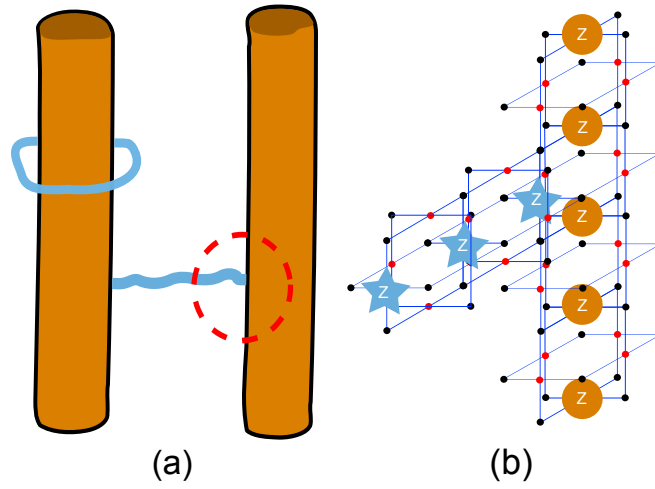


Figure 3.5: We consider only the primal lattice, focusing only on one error channel, namely Z errors. X errors will be dealt with looking at the dual lattice. (a) Macroscopic view. Defects are measured in the Z basis, whereas the rest of the qubits in the cluster will be measured in the X basis. (b) Microscopic view. The stars represent a chain of phase flip operators that goes from one defect to another. This chain doesn't belong to the stabilizer group but still commutes with all its elements, so it will act non-trivially on the qubit encoded in the defects. The same happens with the chain that winds around one of the defects, *i.e.* it will change the logical state of the encoded qubit.

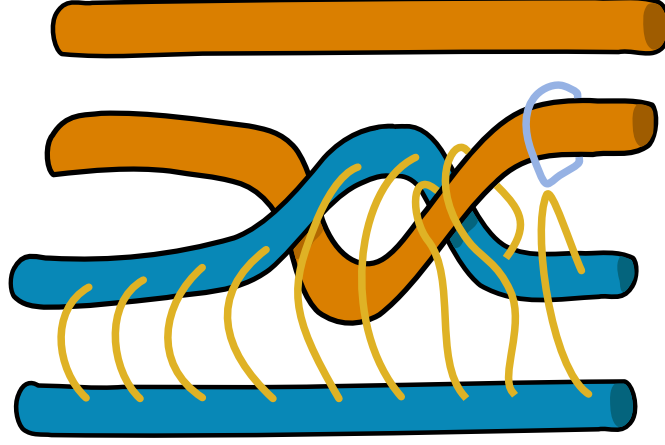


Figure 3.6: C_X gate between a primal and a dual logical qubit, as a result of braiding the respective defects. The lighter lines represent a correlation surface compatible with the Z measurements pattern. Whatever information was encoded in the dual defects will be passed on to the primal defects as a result of the X measurements.

protected gates one usually has to supplement them with magic state distillation protocols [73].

Magic states can be used to attempt phase and $\pi/8$ gates via topologically protected circuits (see fig. 3.7). In order to simulate Hadamard gates, we need three such gate teleportation circuits, thus completing the set of universal gates [79].

3.2.4 Decoding Algorithm

The authors of [77] found a threshold for resilient quantum computation in the model described above. They assumed that depolarizing noise occurs after preparation, single qubit and two-qubit gates used to build up the cluster, and during measurement.

The cluster state is measured in the X basis, which serves the dual purpose of propagating the information forward and to measure stabilizer operators which give us information about errors. Multiplying the stabilizers of the form 3.3 in a cube is another stabilizer of the code. Its parity should thus be positive unless

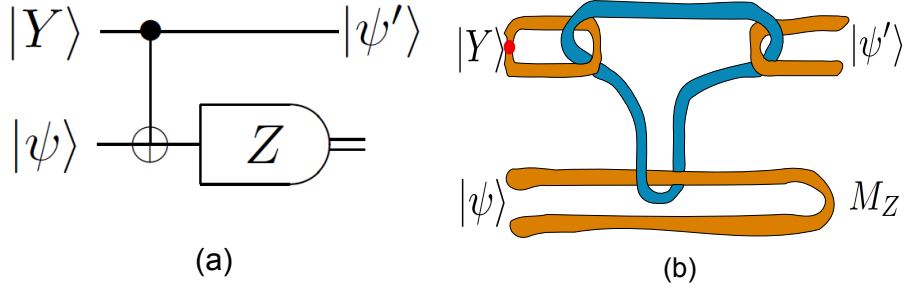


Figure 3.7: (a) A probabilistic phase gate K which uses the teleportation circuit. Given the state $|Y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, the gate teleportation circuit will return $|\psi'\rangle = X^{M_Z}K|\psi\rangle$, where $M_Z \in \{0, 1\}$. Similar results hold for $\frac{\pi}{8}$ gates. (b) Topological gate teleportation circuit.

an error has occurred. Since an error will flip the parity of two cube stabilizers, flipped-parity cubes will denote endpoints of a chain of errors and their location can be determined by processing the measurement outcomes of the cluster.

The distribution of negative parity syndromes in the reconstruction of the measured cluster state can be mapped to a statistical model called the *Random Plaquette Gauge Model*. Guessing the most likely error pattern is equivalent to finding the spin configuration which minimizes the energy in a lattice with random bond values [86]. For low error probabilities, matching the endpoints of the error chains is unambiguous and can be done very fast, using the minimum weight perfect matching algorithm, which performs this task in polynomial time. As the error probability increases the matching becomes more ambiguous and it will fail with increasing probability.

Part II

Physical Implementations

Chapter 4

Implementation of a Fault Tolerant One Way Quantum Computer

4.1 Introduction

A new implementation of the topological cluster state quantum computer is suggested, in which the basic elements are linear optics, measurements, and a two-dimensional array of quantum dots. This overcomes the need for non-linear devices to create a lattice of entangled photons. Whereas the thresholds found for computational errors are quite satisfactory (above 10^{-3}), the estimates of the minimum efficiencies needed for the detectors and quantum dots are beyond current technology's reach. This is because we heavily rely on probabilistic entangling gates, which introduce loss into the scheme irrespective of detector and quantum dots efficiencies.

4.2 Optically Addressed Quantum Dots

Architectures based on self-assembled quantum dots are a strong candidate to build scalable quantum processors and other nanoscale technologies. Self-assembled quantum dots are created by epitaxial growth of a layer of *InAs* onto a *GaAs*

substrate. The lattice mismatch between these two layers creates a strain on the atoms, causing them to accumulate in “droplets” as it were.

Self-assembly is a statistical process. Sizes of the quantum dots will oscillate around a mean value, which leads to a broadening in transition frequencies, and the locations of the dots within the wafer will also be random. This constitutes the main obstacle to scalability, although some ideas on how to counter this are referenced in [87]

It is possible to isolate a single electron within a quantum dot and use its spin as a qubit. Spin relaxation times have been shown to be of order of milliseconds, since it interacts weakly with the lattice vibrations, which lead to fluctuations in through spin-orbit coupling. Dephasing arises from precession of the spin around a fluctuating mean field caused by atomic impurities and nuclei spins, with a timescale of around a few microseconds, dominating thus the decoherence process.

Qubits in *GaAs* can be addressed optically since the typical Zeeman splitting energy is of order of a few *meV*, corresponding to wavelengths of a few hundred nanometers [87]. This is a fundamental feature since in this range, the dipole matrix element is about twenty times larger than for atoms, leading to fast control with weaker driving fields [88]. Another very important point is that the allowed transitions will be spin and polarization sensitive through the usual angular momentum selection rules, which is crucial for the scheme proposed in this chapter. Quantum dots embedded in pillar microcavities, which in principle allow for high collection efficiencies, have also been demonstrated.

4.3 The Photonic Cluster State Machine Gun

As its name suggests, the photonic cluster state machine gun fires a stream of polarized photons in a linear cluster state [89]. Although it is in principle possible to build such a device using different quantum systems[90, 91, 92], we will focus on quantum dots as their spontaneous emission rate is of order of picoseconds, the allowed transitions are better separated than in atoms, and they can be tailored

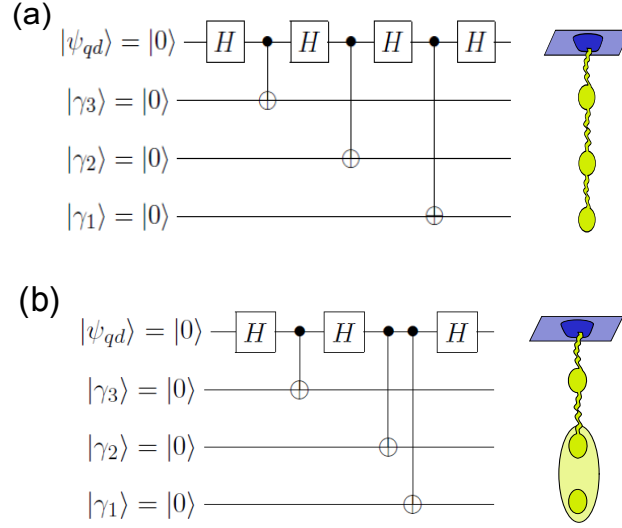


Figure 4.1: Equivalent circuit to the precession-emission process. Although this circuit obviously doesn't encompass the physics inside the dot, it provides a way to visualize how the photons fired by the dot are computationally useful. With a suitable choice of logical states, i.e. $|0\rangle \equiv |R\rangle$ and $|1\rangle \equiv -|L\rangle$, one can identify the rotation $R_Y(\frac{\pi}{2}) = e^{-iY\frac{\pi}{4}}$ with a Hadamard gate. Emission of a photon will be represented by a C_X gate, since the photon's polarization will depend on the spin of the quantum dot. (a) Three qubit linear cluster state. (b) Two qubits linear cluster state. One of them is a redundantly encoded qubit. If the quantum dots are left to precess enough time to describe a rotation $R_Y(4\pi) \equiv I$, no gate will operate on the dot, which can emit qubits in a GHZ state.

in the lab to fine tune certain parameters such as the frequency of the optical transitions [93, 94, 95].

Using quantum dots entails some disadvantages. Self-assembled quantum dots are grown in random locations within the wafer, they usually have different sizes, leading to different energy spacing, and due to stronger coupling to the environment decay faster than other quantum systems.

At the risk of oversimplifying, a quantum dot can be described as a two degenerate levels system consisting of the ground states $\{|\uparrow\rangle, |\downarrow\rangle\}$ and the excited states $\{|\uparrow\rangle, |\downarrow\rangle\}$. These represent the quantum dot with an electron in the conduction band, and the quantum dot with two electrons in the conduction band and one hole in the valence band (called *trion*), respectively. The selection rules ensure that only the transitions $|\uparrow\rangle \leftrightarrow |\uparrow\rangle$ and $|\downarrow\rangle \leftrightarrow |\downarrow\rangle$ will occur,

where the decays will be followed by emission of circularly polarized light. If we introduce a magnetic field transversal to the spin direction, say in the Y direction, the electron will precess with frequency ω_P . With a suitable choice of logical states, we can identify the rotation $R_Y(\frac{\pi}{2}) = e^{-iY\frac{\pi}{4}}$ with a Hadamard gate. The precession will implement this gate every $T_H = \pi/2\omega_P$ seconds. If at these intervals linear polarized light is shone onto the dot, a coherent superposition of trion states will be created, which will decay almost immediately. This is enough to create a linear cluster state:

$$\begin{aligned}
|\uparrow\rangle &\xrightarrow{H} |\uparrow\rangle + |\downarrow\rangle \xrightarrow{E} |\uparrow\rangle|R\rangle + |\downarrow\rangle|L\rangle \\
&\xrightarrow{H} (|\uparrow\rangle + |\downarrow\rangle)|R\rangle + (-|\uparrow\rangle + |\downarrow\rangle)|L\rangle \\
&\xrightarrow{E} (|\uparrow\rangle|R\rangle + |\downarrow\rangle|L\rangle)|R\rangle + (-|\uparrow\rangle|R\rangle + |\downarrow\rangle|L\rangle)|L\rangle \\
&\xrightarrow{H} |\uparrow RR\rangle + |\downarrow RR\rangle - |\uparrow LR\rangle + |\downarrow LR\rangle \\
&\quad - |\uparrow RL\rangle - |\downarrow RL\rangle - |\uparrow LL\rangle + |\downarrow LL\rangle.
\end{aligned} \tag{4.1}$$

Where the “H” and “E” on the arrows stand for Hadamard and emission, respectively. If we identify $|0\rangle \equiv |R\rangle$ and $|1\rangle \equiv -|L\rangle$, then expression 4.1 can be seen to represent a three qubit cluster state.

Figure 4.1 depicts the generation of a cluster state consisting of 3 photons. The cluster creation in fig. 4.1(a) is best seen within the stabilizer description:

$$\begin{aligned}
&\begin{vmatrix} \mathbf{Z} & I & I & I \\ \mathbf{I} & Z & I & I \\ \mathbf{I} & I & Z & I \\ \mathbf{I} & I & I & Z \end{vmatrix} \xrightarrow{1} \begin{vmatrix} \mathbf{X} & X & I & I \\ \mathbf{Z} & Z & I & I \\ \mathbf{I} & I & Z & I \\ \mathbf{I} & I & I & Z \end{vmatrix} \xrightarrow{2} \begin{vmatrix} \mathbf{Z} & X & I & I \\ \mathbf{X} & Z & X & I \\ \mathbf{Z} & I & Z & I \\ \mathbf{I} & I & I & Z \end{vmatrix} \xrightarrow{3} \\
&\begin{vmatrix} \mathbf{X} & X & I & X \\ \mathbf{Z} & Z & X & I \\ \mathbf{X} & I & Z & X \\ \mathbf{Z} & I & I & Z \end{vmatrix} \xrightarrow{4} \begin{vmatrix} \mathbf{Z} & X & I & X \\ \mathbf{X} & Z & X & I \\ \mathbf{Z} & I & Z & X \\ \mathbf{X} & I & I & Z \end{vmatrix} \xrightarrow{5} \begin{vmatrix} X & Z & I \\ Z & X & Z \\ I & Z & X \end{vmatrix}.
\end{aligned} \tag{4.2}$$

The quantum dot subsystem is denoted in boldface. Here, each step corresponds to a Hadamard gate applied to the quantum dot followed by an emission process, except the steps 4 and 5, which represent a single Hadamard and measuring out the quantum dot, respectively. One of the most appealing features of the machine gun is that Pauli errors in the quantum dot (due to dephasing etc.) amount to a local error in the stream of photons, which can be shown manipulating stabilizers in the equivalent circuit. How this is done is explained in [89].

In this scheme we will need to encode a qubit into several *redundantly encoded* (RE) photons. This is very easy to attain with the machine gun: one only has to let the quantum dot precess around the Y axis for a time corresponding to a 4π radian rotation before exciting it¹. Then it is easy to see that no Hadamard will act on the dot, so the new photon will be in a RE state with the previous photon(s) — see figure 4.1(b).

4.4 The Fusion Gate

The gate we present here is a slight variation of the ones proposed in [96], since here we need to create links between RE qubits rather than simply fuse them into larger RE sets. The gate is depicted in fig. 4.2.

The gate will first apply a Hadamard gate to the photons, pushing them out from the RE qubit (making them stick out like a leafy branch as it were). The next part of the gate is equivalent to a Type-I fusion gate [96].

The Type-I fusion gate consists of a polarizing beam-splitter, which reflects only vertically polarized light, followed by a polarization rotation of 45 degrees and a polarization discriminating photon detector in one of the spatial modes. Consider two two-qubit graph states, two unentangled photons of which are sent into the fusion gate. A detector with perfect fidelity would measure a photon half of the time, upon which it will project the other three photons into a linear graph

¹We can equally consider a rotation of 2π , since $R_Y(2\pi) \equiv -I$, since this can also be used to obtain RE qubits up to phases.

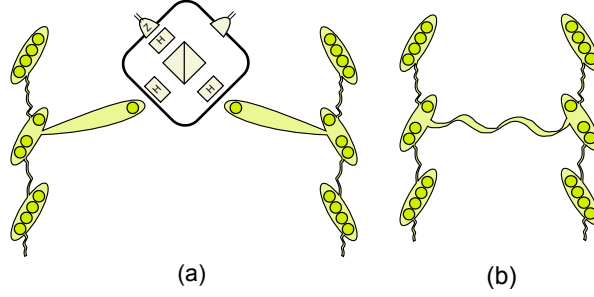


Figure 4.2: (a) One photon from each stream is used to attempt a fusion gate. (b) Upon success, a link between the streams will be created. If this is done in all directions it will give rise to an arbitrarily large, albeit incomplete, three dimensional cluster state.

state. If it fails to detect it, the resulting photons are left in a product state.

If the fusion fails it will effectively collapse the qubits in the Z basis, and that is why we need to differentiate the input photons from the rest of the RE qubit, since such failure would then destroy coherence in the RE qubit. If it is successful, there will be two RE qubits linked to a third photon. Our variation consists of a Type-I fusion gate supplemented with polarization rotators before the polarizing beam-splitter and an extra gate measuring the photon left after the fusion. In order to create a link between the two extremal RE qubits, one has to measure out the middle photon in the Y basis. This will give a cluster state, up to a phase gate K, which satisfies $KYK^\dagger = X$, $KZK^\dagger = Z$ and can be dealt with in the measurement stage.

A simplified version of how our fusion gate operates can be described considering two Bell pairs. In terms of stabilizers, the evolution exposed above takes the form:

$$\begin{vmatrix} X_1 & X_2 \\ Z_1 & Z_2 \end{vmatrix} \begin{vmatrix} X_3 & X_4 \\ Z_3 & Z_4 \end{vmatrix} \xrightarrow{T-I_3} \begin{vmatrix} X_1 & Z_2 & I_4 \\ Z_1 & X_2 & Z_4 \\ I_1 & Z_2 & X_4 \end{vmatrix} \xrightarrow{M_2^Y} \begin{vmatrix} \mathbf{Y}_1 & \mathbf{Z}_4 \\ \mathbf{Z}_1 & \mathbf{Y}_4 \end{vmatrix}. \quad (4.3)$$

The fusion gate succeeds half of the time. One possible concern is what happens when the detectors inside the fusion gate are not perfect. It can be shown that even if one or both photons are lost, the resulting state still has a

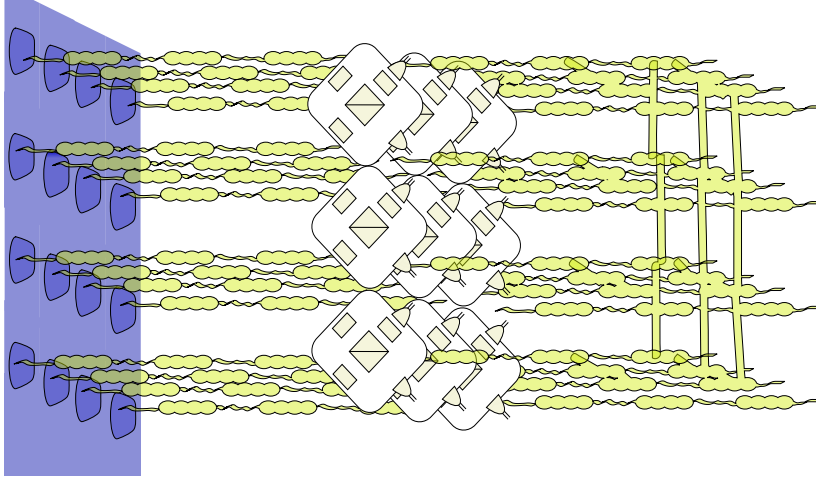


Figure 4.3: All elements considered so far assembled into a global view of our proposal. Fusion and detection stages should be connected to classical computers which can communicate in order to deal with the loss due to probabilistic gates.

(smaller) probability of being in a cluster state. However this is not very useful since we have no means of knowing whether the entanglement is there or not, and so we condition only on a successful detection of both photons.

4.5 Building up the Cluster State

In this section we will explain how to use the concepts explained in previous section to build an optical quantum computer.

In fig. 4.3 we depict the basic idea of our proposal. We consider a two-dimensional array of quantum dots that behave as photonic machine guns firing parallel streams of photons. There will be a stream of photons along each time-like edge and along each timelike face. We get the entanglement in the time-like direction for free, and for the transverse directions we need to place fusion gates between adjacent beams, so that they probabilistically will create links between them. After the photons leave this stage they should be with high probability in the cluster state described by Raussendorf (see fig. 4.3). Finally, another array of detectors will measure the incoming photons in the bases dictated by the desired quantum algorithm, creating a non-trivial topology in the cluster. The last two

stages must be controlled by classical computers, so that heralded loss at fusion can be preprocessed and accounted for as well as to undo the possible Pauli errors that the syndrome may have unveiled.

It is important to notice that we need to use RE photons to maximize the probability of link creation p_l , which goes as $p_l = 1 - 2^{-R}$, where R is the number of times we attempt a fusion, and to perform fusions in different directions. Depending on where in the cluster a qubit sits, *i.e.* in how many directions we have to try to fuse it with its neighbours, we will use a different number of RE photons. There are four types of qubit: qubits which have to be fused in the left-right direction, or in top-down direction will be composed of $2R + 1$ RE photons, qubits which have to be fused in both directions will consist of $4R + 1$ RE photons, and qubits which have to be measured out will only consist of one photon. It is easy to see that this doesn't pose any problem for synchronizing fusions. On average, a qubit will consist of $2R + 1$ RE photons, which will be either used to create links or measured out in the process, leaving a final qubit consisting uniquely of one photon.

This is in many ways similar to the proposal by [97, 98], however it overcomes the need of building a highly efficient non-linear device known as a *photonic module*. These modules are replaced with fusion gates, greatly reducing the difficulty of coupling photons at the cost of increasing the number of photons and decreasing the loss tolerance.

4.5.1 Error Model

We consider two basic types of error: computational errors and photon loss. Computational errors are modeled by one and two qubit depolarizing noise. Due to the finite lifetime of the trions, the quantum dots might precess longer than expected before emitting the photon. This results in faulty Hadamard gates with probability of depolarizing error p_1 . Imperfections in the exciting pulse will result in further dephasing of the electron in the quantum dot just before emission. This is modeled by faulty C_X gates in the equivalent circuit of fig. 4.1, which

introduce correlated depolarizing noise between two qubits with probability p_2 . A more detailed discussion can be found in [89]. A successful fusion will introduce correlated depolarizing noise with probability p'_2 . Measurements are also allowed to be noisy, again with probability p_1 .

Loss errors are themselves of two types. Loss at emission will happen with probability p_{dot} , whereas loss at detection will happen with p_{det} — this includes faulty detection in the fusion stage. For simulation purposes we identify $p_1 = p_2 = p'_2 \equiv p_C$ and $p_{dot} = p_{det} \equiv p_L$. This will enable us to find a threshold described by a curve varying only two parameters, in a way similar to a model proposed by Stace and Barrett [99, 100]. They obtain a threshold for loss errors of about 25%, assuming deterministic two qubit gates. This is, as we will see in next section, much higher than what we obtained. It is however important to keep in mind that our error model is specific to the optical setting that we have proposed, where two-qubit gates are *perforce* non-deterministic.

4.6 Decoding Algorithm Numerical Simulation

Not only does the cluster state constitute the substrate for the computation: it also provides a code for error correction. An typical error will be a chain of phase flip operators. It will be detected upon measuring wrong sign stabilizer elements at its endpoints. The error correction procedure will then be to apply a series of phase flip operators in such a way that the resulting chain belongs to the trivial homology class, which is equivalent to saying that it belongs to the stabilizer group. Alternatively, error chains that end at the boundaries of the cluster or in a defect, or wind around a defect, will not be detected since they won't flip any stabilizer. For simulation purposes it is convenient to work with a cluster with no boundary, in which case a chain of errors winding around any of the dimensions of the 3-torus will leave no syndrome.

We used the standard method for estimating the threshold, namely carrying out a large number of montecarlo simulations for the error correction procedure and sampling them at different values of the error rates. This is tractable within

reasonable time only for small cluster states, *i.e.* clusters with code distance $d \leq 15$. The simulations we present here, based on over 1500 lines of C code, were run in Imperial College High Performance Cluster during 3 months. Preliminary estimates (100 simulations per point) were taken running our algorithm on an Intel Q8200 processor at 2.33 GHz and 3.5 Gb of RAM, for up to $d \leq 13$.

The basic idea is that, for error rates below the threshold, increasing d will reduce the failure probability, since the error chain resulting from error correction will most likely belong to the trivial homology class, whereas for error rates above the threshold, going to higher d will actually cause more errors than it can correct, in the sense that non-trivial chains will be present with high probability. In the limit of $d \rightarrow \infty$, the failure probability P_F should look like:

$$\lim_{d \rightarrow \infty} P_F(p_C) = \frac{7}{8} H_S(p_C - p_{Th}) \quad (4.4)$$

where H_S is the step function and p_{Th} is the threshold probability. The factor $\frac{7}{8}$ comes from the fact that one of the eight homology classes of the 3-torus is trivial.

A comment is in order about the parameter R in our algorithm. One might be tempted to increase R in order to get a complete cluster state with high probability. This turns out to be a bad idea for a series of reasons. First, as we said before, we need qubits with $2R + 1$ RE photons on average. This means that, even if fusions were successful 100% of the times and $R = 1$, we would still have on average 3 photons per qubit, which increases the effective error probability per qubit — we confirmed this by getting a computational error threshold approximately a factor of six lower than 0.7% [78], for no loss. Also, since RE photons form a relative GHZ state, the loss of one of them will cause the rest to completely lose coherence and collapse to a maximally mixed state, which renders the qubit useless.

The simulations were carried for $R = 7$. We found that there is no benefit in setting $R \geq 8$. This probably finds an explanation in the facts exposed above, since for $R \geq 8$, $2R + 1 \geq 17$ and a loss probability of 6% will spoil the whole

cluster.

4.6.1 Thresholds

We calculated threshold estimates for computational error probability as well as for loss probability. Here we present them and provide a numerically obtained tradeoff curve for the loss and computational error thresholds.

For no loss, we found that the threshold is 0.114%, only about six times smaller than in [78] (see Fig. 4.4(a)). This is not surprising, since we are using several RE photons to encode a qubit, so the effective error rate per qubit will be necessarily higher than the error rate per photon. As explained in Section 2.1, note that for error rates below the threshold, adding more redundancy, *i.e.* increasing the distance, will reduce the effective error probability. Conversely, above the threshold, adding redundancy is counterproductive.

The threshold for loss with no computational error is 0.053% (see fig. 4.4(b)). This is, however, unrealistically small. We strongly believe that the reason for this low threshold is that we are assuming non-deterministic gates. For deterministic gates, much higher thresholds have been obtained [101]. The main problem, as far as we can see, is that our redundantly encoded qubits are very sensitive to photon loss. An encoding less naive than mere redundant encoding, as has been proposed in [102, 103] would perhaps help to improve the error.

Fig. 4.5 shows a compromise curve between loss probability and computational error probability. The thresholds showed in Fig. 4.4 correspond to the two extremal points in Fig. 4.5. The three points in the middle were each one calculated by choosing a value for p_L and then obtaining a computational threshold as usual. The area underlying the curve is where fault-tolerant quantum computation is possible with our proposed scheme.

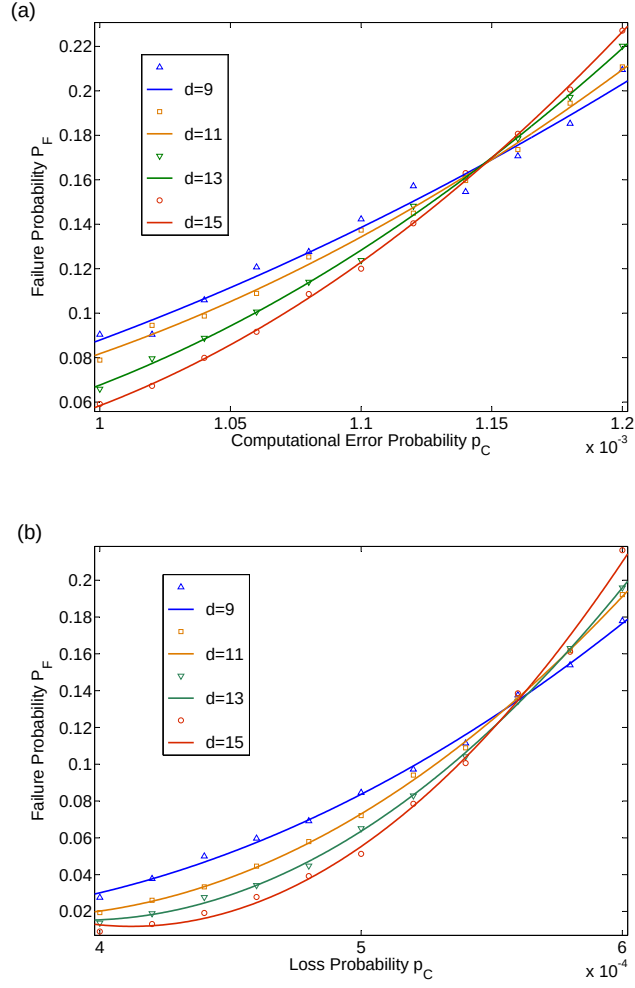


Figure 4.4: Data for $d \leq 7$ has not been included as finite size effects become too large at such low code distances. (a) Fit for computational error only. (b) Fit for loss error only. Each point corresponds to the average of 10^4 simulations (error bars not present). Crossing points of the curves for different d 's, denoting the existence of a threshold, are observed at $p_C \approx 1.14 \times 10^{-3}$ and at $p_L \approx 5.3 \times 10^{-4}$.

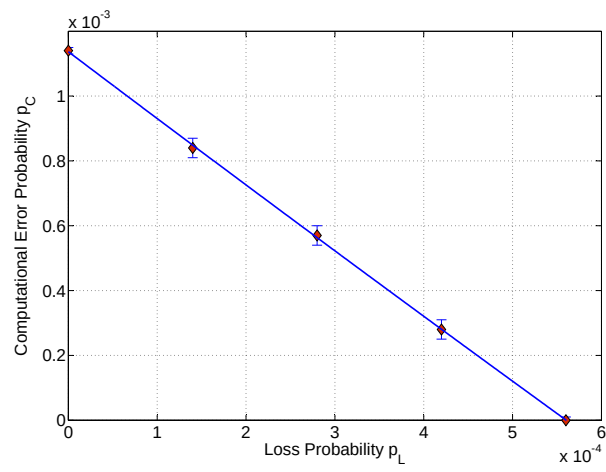


Figure 4.5: Tradeoff curve resulting from a quadratic fit of the calculated thresholds, below which fault-tolerance is achieved.

Chapter 5

Concatenated Graph Codes Against Loss Errors

5.1 Introduction

Qubit loss is a common type of error that has to be dealt with effectively if we are to build a quantum processor. Loss is arguably easier to handle than unknown computational errors, since sometimes it can be seen merely as an error which can be located. Loss can happen as a result of fluctuations of the qubit parameters that take the state of the system out of the computational subspace, or as a result of using inefficient detectors. There exist already several works [104, 105, 99, 102, 106], each making different assumptions and taking different error models, that combine loss protection with tolerance to unknown errors and are tailored for different architectures. However the approach to loss which is closer to the spirit of the present work was introduced by Varnava et al. [107], where they proved that universal quantum computation is possible even with a 50% loss rate provided there are no computational errors.

We combine ideas from measurement-based quantum computation [108, 109] and from the traditional approach to fault tolerance [36]. In particular, we use the five qubit code [110], which is the smallest quantum error correcting code

which can correct one general Pauli error [10]. In contrast to general Pauli errors, loss errors by construction can be located, a property which is crucial in our construct.

In this chapter we will show that it is possible to achieve high levels of tolerance to qubit leakage without compromising the universality of the cluster state model of computation. The whole point is to simulate a noiseless measurement pattern in a noiseless cluster state, and to this aim we encode the logical qubits using the five qubit code $[5, 1, 3]_2$ concatenated with itself, in such a way that the logical operators to be measured will be spread across many physical qubits. We will see that the logical operators only have support, *i.e.* are different from the identity, on roughly $3/5$ of the total number of qubits, which allows for loss tolerance.

We report a proof-of-principle experiment currently underway on the four qubit code, which is the smallest code which can be used to deal with loss errors. There is a general theory for graph code concatenation [111, 112] that generalizes the present approach in the code-theoretical framework.

5.2 Graph States as Error Correcting Codes

It has been shown that all stabilizer codes are locally unitary equivalent to some graph state [113]. There is a general theory for graph code concatenation [111, 112] that generalizes the present approach in the code-theoretical framework. In fact graph states can be combined with non-additive classical codes to create a larger set of quantum error-correcting codes [114].

5.2.1 A Version of the Five Qubit Code

The five qubit code $[5, 1, 3]_2$ is defined by the following operators:

	1	2	3	4	5
K_1	X	Z	Z	X	I
K_2	I	X	Z	Z	X
K_3	X	I	X	Z	Z
K_4	Z	X	I	X	Z
$\bar{X}$	X	X	X	X	X
$\bar{Z}$	Z	Z	Z	Z	Z

(5.1)

Alternatively, the logical $|0_L\rangle$ ($|1_L\rangle$) correspond to the superposition of states with odd number of zeros (ones), although using states instead of operators is rather cumbersome and abstruse, so we will describe states using their stabilizer operators. This code saturates the singleton bound [10], *i.e.* it is the smallest quantum code that protects against one Pauli error. It is not difficult to see that this is locally unitarily equivalent to a circular five qubit cluster state, defined by both of the following set of operators:

	1	2	3	4	5
K_1	X	Z	I	I	Z
K_2	Z	X	Z	I	I
K_3	I	Z	X	Z	I
K_4	I	I	Z	X	Z
K_5	Z	I	I	Z	X

 $\iff$

	1	2	3	4	5
K'_1	Y	Y	Z	I	Z
K'_2	Z	Y	Y	Z	I
K'_3	I	Z	Y	Y	Z
K'_4	Z	I	Z	Y	Y
$\bar{X}$	Z	I	I	Z	X

(5.2)

where we have renamed the stabilizers $K'_i = K_i K_{i+1}$ and chosen $\bar{X}$ to be any one K_i in the original description of the cluster, so that $[K'_i, \bar{X}] = 0, \forall i$. Note that $\bar{X}' \equiv \bar{X} K'_1 K'_3 = X_1 X_2 X_3 X_4 X_5$. We choose $\bar{Z} = Z_1 Z_2 Z_3 Z_4 Z_5$, such that $[K'_i, \bar{Z}] = 0, \forall i$, and the anticommutation relation $\{\bar{X}, \bar{Z}\} = 0$ can be readily checked. It follows that the five qubit cluster state in a pentagon is locally unitary equivalent to the usual five qubit code initialized in the logical $|+\rangle$ state. This construction can be seen as a special case of *Codeword Stabilized Codes* [114], in which the $\bar{Z}$ is the word operator.

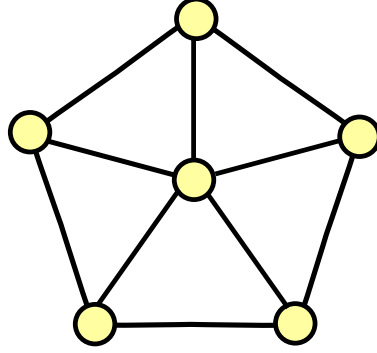


Figure 5.1: Circles are qubits and lines between them represent entanglement. Five qubits in a ring (a pentagon) entangled to a centre qubit is the basic building of our scheme.

5.2.2 Encoding and Concatenation

The way one encodes a qubit in the five qubit cluster state can be seen as a simple teleportation circuit where we have substituted one of the qubits by the pentagon cluster. Consider the general state:

$$|\psi\rangle_l = \frac{1}{2}(I + e_x X_c + e_z Z_c) |\psi\rangle_c, \quad (5.3)$$

where $e_x^2 + e_z^2 = 1$, and without loss of generality we only look at the X-Z equator of the Bloch sphere. Imagine we want to encode this state in the pentagon, as depicted in Fig. 5.2.

To encode a single qubit $|\psi\rangle_c = \alpha|0\rangle + \beta|1\rangle$ into the graph state, the logical

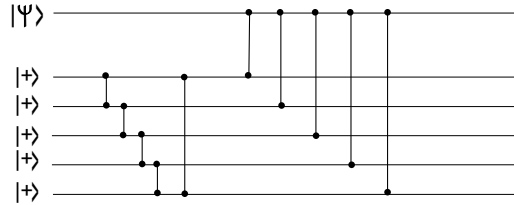


Figure 5.2: The information initially contained in the centre qubit will be spread over the whole cluster state after operation of the C_Z gates. Note that errors on concatenation level $N - 1$ will propagate at most to level N since $C_Z(X \otimes I)C_Z^\dagger = X \otimes Z$, $C_Z(Z \otimes I)C_Z^\dagger = Z \otimes I$

controlled-Z $\bar{C}_Z = C_Z^{l1} C_Z^{l2} C_Z^{l3} C_Z^{l4} C_Z^{l5}$ is applied between each of the qubits in the graph and $|\psi\rangle_l$, after which we have:

$$\begin{aligned} \bar{C}_Z |G_5\rangle |\psi\rangle_l = \\ \frac{\alpha}{\sqrt{2}}(|0_L\rangle + |1_L\rangle) |0\rangle_c + \frac{\beta}{\sqrt{2}}(|0_L\rangle - |1_L\rangle) |1\rangle_c. \end{aligned} \quad (5.4)$$

Where $|G_5\rangle = \frac{1}{\sqrt{2}}(|0_L\rangle + |1_L\rangle)$ denotes the five-qubit graph state. We then measure qubit l in the $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ basis and obtain the encoded state $\alpha |+_L\rangle + (-1)^{s_l} \beta |-_L\rangle$ within the five qubit graph state, *i.e.* we have encoded the state $|\psi\rangle$ in the Hadamard basis. Here s_l corresponds to the outcome of the measurement in the $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ basis and can be accounted for. In terms of operators, the action of the logical gate $\bar{C}_Z$ can be described as follows:

$$\bar{C}_Z \begin{bmatrix} & I_1 \\ I_5 & X_c & I_2 \\ I_4 & & I_3 \end{bmatrix} \bar{C}_Z^\dagger = \begin{bmatrix} & Z_1 \\ Z_5 & X_c & Z_2 \\ Z_4 & & Z_3 \end{bmatrix}, \quad (5.5)$$

$$\bar{C}_Z \begin{bmatrix} & Z_1 \\ X_5 & I_c & I_2 \\ Z_4 & & I_3 \end{bmatrix} \bar{C}_Z^\dagger = \begin{bmatrix} & Z_1 \\ X_5 & Z_c & I_2 \\ Z_4 & & I_3 \end{bmatrix}. \quad (5.6)$$

It is not difficult to convince oneself that the weights e_x and e_z of X_c and Z_c will, upon measurement in the $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ basis, will be effectively stored in the amplitudes of the eigenstates of $\bar{Z}$ and $\bar{X}$, respectively.

Now, for each qubit in the pentagon, we carry out the encoding procedure explained above in this section. To each pentagon qubit going out in the circuit of Fig. 5.2, we attach a copy of that same circuit and repeat iteratively. We repeat N times, so that the last level of concatenation has $Q = 5^N$ physical qubits encoding one logical qubit (see Fig. 5.3). It is important to realize that all previous $N - 1$ levels are measured out in the $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ basis, without being exposed to loss, since they are just part of the construction procedure of

the code.

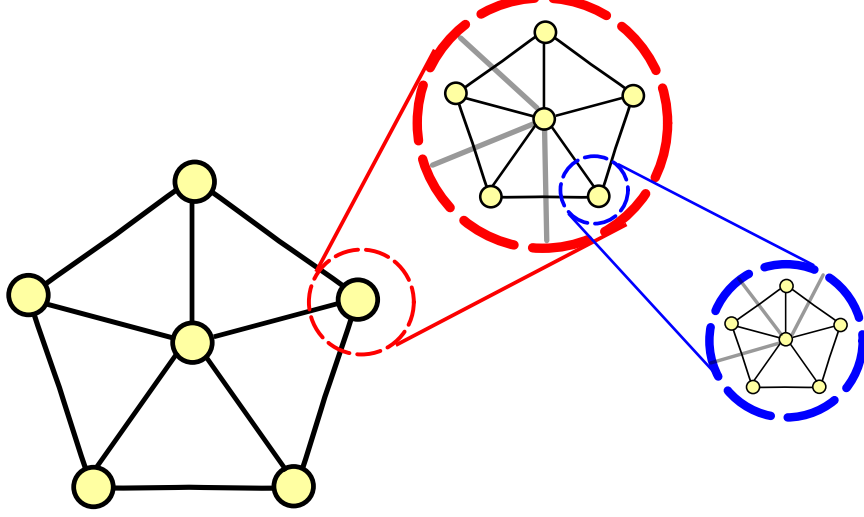


Figure 5.3: We will stop at some level of concatenation that meets our loss tolerance requirements.

5.3 Tolerance to Loss

A general result in classical and quantum information theory [10] is that a code that protects against t errors, can protect against $2t$ losses (loss can be regarded a localized error). It is important to stress that in our scheme we only allow for destructive measurements, that is, the qubit being measured will no longer be available to extract more information. Ultimately this is the reason why tolerance to Pauli errors cannot be integrated within this approach.

We classify loss into two broad categories: preannounced and non-preannounced. Preannounced loss happens when one knows prior to performing a measurement, whether the system is there or not. Non-preannounced loss means that one discovers the loss upon measuring and not getting any “click”. This categorization is not exactly the same as the “heralded-unheralded” division, since heralded loss means that we detect a loss at measurement and tag the location, whereas unheralded means that there is a loss that is not detected. To our understanding, this

concatenations. The recurrence formula

$$\begin{aligned}
P_L(N-1) &= \binom{5}{5} P_L^5(N) + \binom{5}{4} P_L^4(N) (1 - P_L(N)) \\
&+ \binom{5}{3} P_L^3(N) (1 - P_L(N))^2
\end{aligned} \tag{5.10}$$

gives us the effective loss probability at concatenation level $N-1$, $P_L(N-1)$, given that the loss probability at level N was $P_L(N)$, with $P_L(N_{top}) = p_L$ at the top level. By plotting $P_L(N)$ for different N it is possible come up with a recurrence fixed point corresponding to a threshold of 50%, as given in Fig. 5.4.

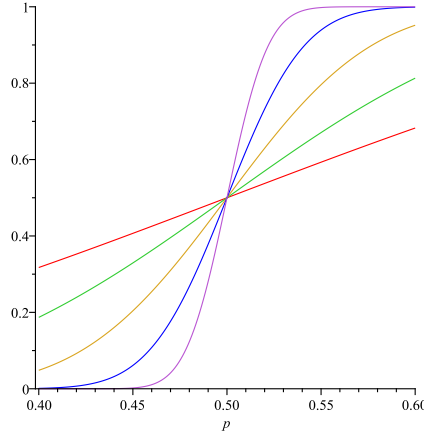


Figure 5.4: Effective loss probability versus physical loss probability for preannounced loss. Different levels of concatenation, from $N=1$ (*cf.* $Q=5$), red, to $N=5$ (*cf.* $Q=3125$), purple.

The recurrence formula giving the effective loss probability assumes that only the top level of concatenation is actually exposed to loss. This means that we should regard all the previous concatenation levels as levels of virtual qubits that help us visualize how to construct the code. These virtual levels are also useful in order to visualize the decoding procedure. Each virtual qubit in level $N-1$ is encoded in five qubits in level N , and N_{top} corresponds to the actual physical level. Being unable to recover the information stored in any pentagon belonging to level N will result in declaration of loss of the corresponding underlying qubit.

In order to gain some insight on the amount of protection given by this way of encoding for preannounced loss, the following table illustrates the number of qubits needed in to make the effective loss probability $P_L(N) \approx 10^{-8}$ or under:

	$p_L = 0.2$	$p_L = 0.3$	$p_L = 0.4$
Q^V	22188	2.3×10^5	7.6×10^6
Q	125	625	3125

(5.11)

where we have compared our results with the amount of qubits needed in [107]. We stress that in our case, these values are valid only when loss is preannounced, as opposed to [107] where loss need not be known beforehand. We nevertheless include this table to show that resources would be dramatically reduced if one could tag lost qubits prior to measurement, such as may be relevant for atoms in optical lattices.

5.3.2 Results for Non-preannounced Loss

The bound of 50% is achieved when one knows whether the qubit is there or not. If one discovers a loss while performing a computational measurement, there is a chance that the measuring pattern chosen prior to discovering the loss will be unavoidably broken and the information lost. If this happens, then one declares a loss, which can be handled in the same way in the immediately lower concatenation level.

Since each logical operator can be written in two different (commuting) ways, one can try to measure both ways at the same time and declare a loss whenever either two losses break both of them, or when a loss breaks one of them but it is impossible go back and to measure the other one. A decision tree will give us the probability of this happening . The threshold is considerably lowered to about 23% percent, which is, however, comparable to thresholds obtained for other architectures [99, 100]. It seems that, in the absence of Pauli errors, the concatenated approach would necessitate less

Measurement strategy for non-preannounced loss

The pseudocode for a $\bar{Z}$ measurement is given in the decision Tree 1. The notation “ jA ” means “qubit j is measured in the A basis”. There are similar decision trees for $\bar{X}$ and $\bar{Y}$ measurements. Using these pseudocodes recursively, *i.e.* at each level of concatenation, will give us a measurement strategy in the limit of many concatenations.

5.4 Loss-tolerant Universal Quantum Computation

To see how universality is achieved, keep in mind that a logical cluster state underlies all encodings. We assume this cluster state is two-dimensional, so that two qubit logical gates (C_Z) are naturally embedded in the encoding. As shown in Fig. 5.2, encoding can be seen as entangling the operators of the virtual qubit with the logical operators living in the pentagons. Imagine we have two such virtual qubits entangled with a C_Z gate. It is trivial to see how this translates into entanglement between the logical operators of their respective encodings.

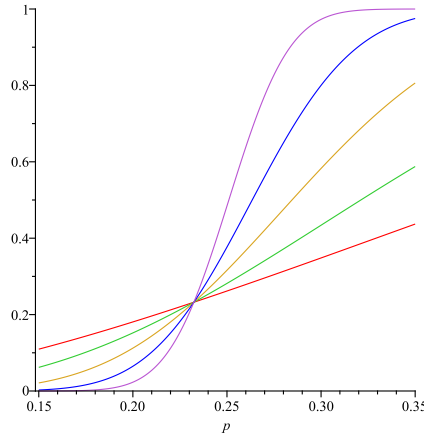


Figure 5.5: Effective loss probability versus physical loss probability for non-preannounced loss. Different levels of concatenation, from $N = 1$ (*cf.* $Q = 5$), red, to $N = 5$ (*cf.* $Q = 3125$), purple.

decision Tree 1 $\bar{Z}$ measurement

```
1: if 1X then
2:     if 2Z then
3:         if 5Z then
4:             SUCCESS
5:         else if 3Y then
6:             if 5Y then
7:                 SUCCESS
8:             else
9:                 FAILURE
10:            end if
11:        else
12:            FAILURE
13:        end if
14:    else if 3Y then
15:        if 5Y then
16:            SUCCESS
17:        else
18:            FAILURE
19:        end if
20:    else
21:        FAILURE
22:    end if
23: else if 2X then
24:     if 4Y then
25:         if 5Y then
26:             SUCCESS
27:         else
28:             FAILURE
29:         end if
30:     else
31:        FAILURE
32:    end if
33: else if 4X then
34:     if 3Z then
35:         if 5Z then
36:             SUCCESS
37:         else
38:             FAILURE
39:         end if
40:     else
41:        FAILURE
42:    end if
43: else
44:     FAILURE
45: end if
```

$$\begin{array}{cccc}
\bar{I} & \bar{I} & \bar{I} & \bar{X} \\
I & Z & X & I \\
I & X & Z & I \\
\bar{X} & \bar{I} & \bar{I} & \bar{I}
\end{array}
\Rightarrow
\begin{array}{cccc}
\bar{I} & \bar{I} & \bar{Z} & \bar{X} \\
I & Z & X & Z \\
Z & X & Z & I \\
\bar{X} & \bar{Z} & \bar{I} & \bar{I}
\end{array}
\Rightarrow
\begin{array}{cccc}
\bar{X} & \bar{I} & \bar{I} & \bar{Z} \\
I & I & X & X \\
X & X & I & I \\
\bar{Z} & \bar{I} & \bar{I} & \bar{X}
\end{array} . \quad (5.12)$$

where we start off with two virtual qubits (centre) in an entangled state. The first arrow represents the encoding of the virtual qubits (*i.e.* entangle them with logical operators), and the second arrow represents measurement of the virtual qubits in the X basis. This shows that the logical operators are entangled via a logical C_Z gate.

Unfortunately, only measurements in the X and Z basis can be done in a loss-tolerant fashion. This prevents us from doing single qubit gates with the usual prescription (*i.e.* steer a unitary by measuring in angles given by it's Euler decomposition). This can be overcome by introducing the additional set of gates $\exp(i\frac{1}{8})Z$, $\exp(i\frac{1}{4})Z$ and $\exp(i\frac{1}{4})X$. As noted in the introduction, these gates can be realized fault-tolerantly with help of magic states [73], which can be obtained via measurements in the X basis and C_X gates on a reservoir of noisy qubits, provided they are not mixtures of the Pauli matrices eigenstates.

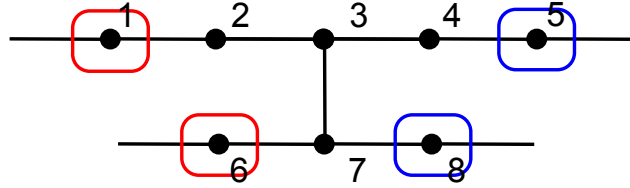


Figure 5.6: This is how a C_X gate would look like in the virtual cluster state. Red squares denote input states and blue squares denote output states. All qubits except the blue ones are measured in the X basis.

The correlations defining the C_X gate of Fig. 5.6 are:

$$X_1 I_2 X_3 I_4 X_5 I_7 X_8, \quad (5.13)$$

$$Z_1 X_2 I_3 X_4 Z_5, \quad (5.14)$$

$$I_3 X_4 Z_5 Z_6 X_7 Z_8, \quad (5.15)$$

$$X_6 I_7 X_8. \quad (5.16)$$

Measuring all qubits of Fig. 5.6 in the X basis will enact a C_X gate. Hadamard gates are also straightforward to achieve. This is clear if we again have a look at the virtual cluster state, since measuring in the X basis a qubit in state $|\psi\rangle$ will steer the next qubit in the cluster into the state $X^m H |\psi\rangle$, where m is the outcome of the measurement.

5.4.1 Overhead

The basic principle of fault tolerance using concatenated codes is that, whenever the physical qubit error probability is below the threshold, the effective error probability decreases exponentially with the number of physical qubits. However, as one gets closer to the threshold, the resources needed to maintain a given effective loss probability P_L increase very fast. The effective loss probability P_L as a function of the overhead $Q = 5^N$ and the physical loss probability p_L is summarized in the following tables. For preannounced loss we have:

Q_P	$p_L = 0.4$	$p_L = 0.3$	$p_L = 0.2$
5	0.317	0.163	0.058
25	0.187	0.033	0.002
125	0.048	3.6×10^{-4}	5.6×10^{-8}
625	0.001	4.5×10^{-10}	1.8×10^{-21}
3125	1.5×10^{-8}	9.1×10^{-28}	5.5×10^{-62}

(5.17)

whereas for non-preannounced loss we have:

Q_{NP}	$p_L = 0.15$	$p_L = 0.1$	$p_L = 0.05$
5	0.110	0.052	0.014
25	0.062	0.015	0.001
125	0.021	0.001	8.0×10^{-6}
625	0.002	1.1×10^{-5}	3.8×10^{-10}
3125	4.1×10^{-5}	7.7×10^{-10}	8.9×10^{-19}

(5.18)

5.4.2 Comparison with Tree Codes

We show now that, for low enough p_L , this approach necessitates less resources than a previous scheme which attains also the highest threshold achievable. It was introduced by M. Varnava et al. in [107] and offers protection to loss for p_L up to 50%. It is remarkable that this way of encoding a qubit protects against non-preannounced loss. The cluster they introduce consists of “trees” of qubits where at each level of the tree structure, the branching parameter is potentially different from the others. Results were obtained by exhaustive search over the branching parameter.

We investigated whether there is a regime in which the approach of concatenated pentagons performs better, in terms of overhead needed, than the tree approach. We found that this is the case for a non-preannounced loss probability below $\sim 10\%$, as can be inferred from Fig. 5.7.

To summarize, this scheme requires to our knowledge the least overhead for preannounced loss. For non-preannounced loss, it performs better than the trees below $\sim 10\%$, which is to date the scheme with the best performance for non-preannounced loss.

5.5 A Proof-Of-Principle Experiment

The ability to create certain graph states is already within reach of current technology [115]. We now suggest a way to experimentally create a graph code that could be used in a concatenated fashion to fight loss, according to the picture

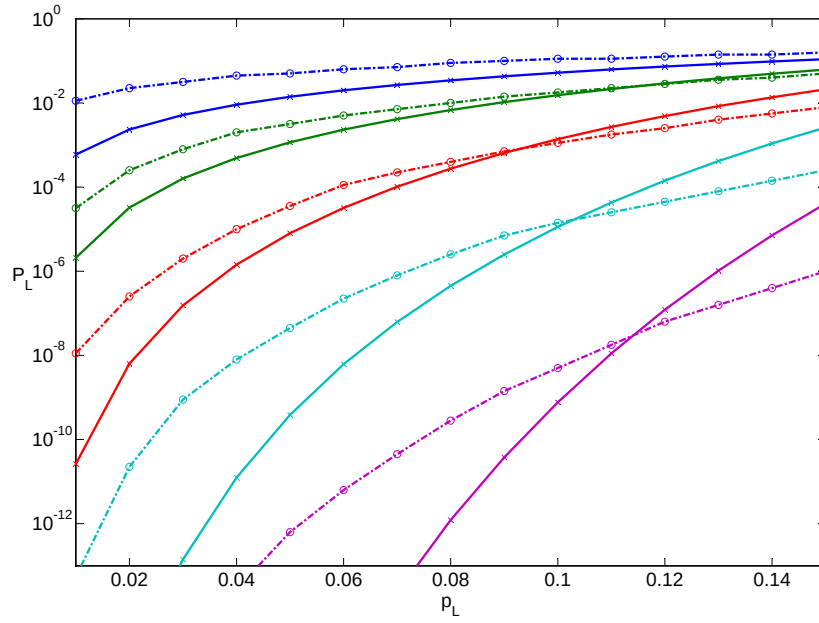


Figure 5.7: Comparison of the effective loss probability P_L as a function of the physical loss probability p_L , for both the current approach (solid lines) and the tree codes (dashed lines). The colors correspond to different numbers of redundant physical qubits Q . Blue lines correspond to $Q \approx 5$ qubits, green to $Q \approx 25$, red to $Q \approx 125$, cyan to $Q \approx 625$ and purple to $Q \approx 3125$. Clearly, the current approach yields better protection for low loss probabilities.

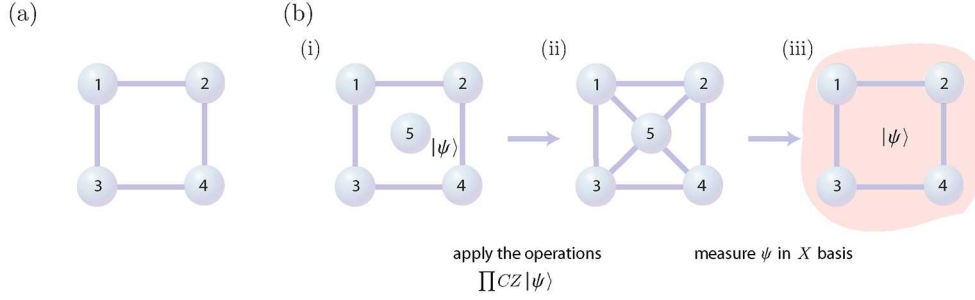


Figure 5.8: [Courtesy of Mark Tame] (a) The four qubit graph state (box cluster) that can be used as an error correction code to detect a bit-flip, phase-flip or tolerate and correct the loss of one qubit. (b) Steps taken to encode an arbitrary qubit into the graph code.

developed in previous sections.

It is a simplified version of the five qubit graph code, consisting only of four qubits, which still allows for error detection and loss correction of up to one erasure [116, 117]. This graph code can readily be implemented in the lab and what follows is an outline of an experiment currently underway in the University of Bristol, which proves the possibility of creating this type of graph codes. In order to create more levels of concatenation we need to wait for the technology to be more mature.

5.5.1 The Four Qubit Graph Code

The smallest known graph state that constitutes a code for error detection and recovering/correcting errors in the form of qubit loss is composed of four qubits and shown in Fig. 5.8 (a). This graph is also known as a box cluster state due to its shape and it forms the elementary building block for an arbitrary sized $2-d$ square lattice cluster state. The circles correspond to the qubits (vertices V) and the lines connecting them correspond to the entangling bonds (edges E). In the computational basis the graph state can be written as

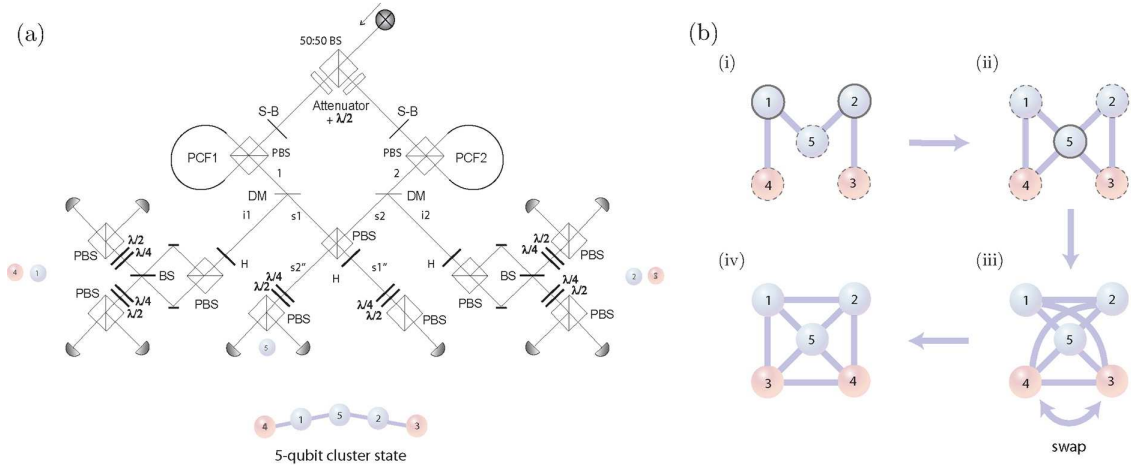


Figure 5.9: [Courtesy of Mark Tame](a) Experimental setup for generating a five-qubit graph state that can be locally converted into the desired four-qubit graph code with central qubit (for encoding) already entangled - see panel (b) step (iv). In the setup four photons are generated (requiring a four-fold coincidence), but only three photons are used for encoding qubits (the photon in mode s_1'' is measured out). The polarization degrees of freedom (blue) of each of the remaining three photons represent three of the qubits and the path degrees of freedom (red) of two of the photons represent two more qubits, making a total of five qubits. (b) Steps taken to convert the graph state generated by the setup into the desired graph state via local complementation operations applied by half- and quarter-wave plates on the photon modes.

$$|G_4\rangle = \frac{1}{2}(|+\rangle|0\rangle|+\rangle|0\rangle + |+\rangle|0\rangle|-\rangle|1\rangle + |-\rangle|1\rangle|-\rangle|0\rangle + |-\rangle|1\rangle|+\rangle|1\rangle)_{1234}. \quad (5.19)$$

In order to see the structure of the code we can rewrite this state as

$$\begin{aligned} |G_4\rangle &= [|\phi^-\rangle|\phi^-\rangle - |\psi^-\rangle|\psi^-\rangle] \\ &\quad + [|\psi^+\rangle|\phi^+\rangle + |\phi^+\rangle|\psi^+\rangle] \\ &= \frac{1}{\sqrt{2}}(|0_L\rangle + |1_L\rangle), \end{aligned} \quad (5.20)$$

where the first line represents a logical zero $|0_L\rangle$ and the second a logical one

$|1_L\rangle$ (up to normalization $1/\sqrt{2}$). It follows that the code comes naturally in the logical state $|+_L\rangle$. $|G_4\rangle$ is the common eigenstate of the stabilizer operators $S_1 = Y_1 Y_2 Z_3 Z_4 = K_1 K_2$, $S_2 = Y_1 Z_2 Y_3 Z_4 = K_1 K_3$ and $S_3 = Z_1 Z_2 Y_3 Y_4 = K_3 K_4$. We have chosen $\tilde{X} = Z_1 I_2 X_3 Z_4$ and $\tilde{Z} = Z_1 Z_2 Z_3 Z_4$ to be the logical X and Z Pauli matrices acting on the cluster, respectively, where I is the identity operation. If there is a phase flip Z on any one qubit we can *locate* the error by measuring the expectation value of all three stabilizer operators without disturbing the graph state and fix the error - as $\langle S_i \rangle = 1$ and $\langle S_i Z_j \rangle = -1$ for any given j . Similar arguments hold for a Y error. However, if there is a bit flip X on any one qubit, it can be *detected* by measuring any of the stabilizer operators, but it cannot be *located*, since a X error anticommutes with all stabilizer operators: $\langle S_i X_j \rangle = -1$ for a given j and all of the i stabilizers. This is the reason (along with a degeneracy in locating Z and Y errors) why the present code can only detect *general* errors (X , Y , Z and superpositions thereof) acting on a single qubit, but not correct them. If an error is detected by measuring the expectation value of all three stabilizer operators and comparing the values, then we discard the state and start over again by re-encoding. This is different to other codes which can locate a general error and then correct it without having to discard the state.

Before going into the details of how the graph state can also tolerate qubit loss (where the location of the lost qubit is known, *i.e.* preannounced loss), we first show how to encode a given quantum state into the code.

5.5.2 Encoding and Tolerance to Loss

To encode a single qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ into the graph state depicted in Fig. 5.8 (a) consider the following sequence, depicted in Fig. 5.8 (b), of C_Z operations applied between each of the qubits in the graph and $|\psi\rangle$, which we label qubit 5. We have

$$C_Z^{15}C_Z^{25}C_Z^{35}C_Z^{45} |G_4\rangle |\psi\rangle_5 = \frac{\alpha}{\sqrt{2}}(|0_L\rangle + |1_L\rangle) |0\rangle_5 + \frac{\beta}{\sqrt{2}}(|0_L\rangle - |1_L\rangle) |1\rangle_5. \quad (5.21)$$

We then measure qubit 5 in the $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ basis and obtain the encoded state $\alpha |+_L\rangle + (-1)^{s_5} \beta |-_L\rangle$ within the four-qubit graph state. Here s_5 corresponds to the outcome of the measurement in the $\{|+\rangle, |-\rangle\}$ basis and can be accounted for by applying the $\tilde{X}$ operation, which can be done locally. Note that if we teleport the state $|\psi\rangle$ into the code via a two-qubit Bell state measurement on $|\psi\rangle$ and qubit 5 in equation (5.21), with $\alpha = \beta = 1/\sqrt{2}$, then the state $|\psi\rangle$ will be encoded in the standard logical basis $\{|0_L\rangle, |1_L\rangle\}$.

In order to show how the graph code tolerates loss, we consider the example case of losing qubit 2. Then, due to the symmetry of the state, any other qubit can be considered to be lost, with the same recovery procedure applied upon an appropriate rotation of the labeling of the qubits corresponding to Fig. 5.8 (a).

Suppose we lose qubit 2, we can find the state of the remaining three qubits by tracing out qubit 2. One finds from $\alpha |+_L\rangle + \beta |-_L\rangle$ the state $\rho_{134} = 1/2|\phi\rangle\langle\phi| + 1/2|\phi^\perp\rangle\langle\phi^\perp|$, where

$$\begin{aligned} |\phi\rangle &= \alpha' [|0\rangle |\phi^-\rangle + |1\rangle |\psi^-\rangle] + \beta' [|0\rangle |\psi^+\rangle + |1\rangle |\phi^+\rangle], \\ |\phi^\perp\rangle &= -\alpha' [|1\rangle |\phi^-\rangle + |0\rangle |\psi^-\rangle] + \beta' [|1\rangle |\psi^+\rangle + |0\rangle |\phi^+\rangle], \end{aligned}$$

where $\alpha' = (\alpha + \beta)/\sqrt{2}$ and $\beta' = (\alpha - \beta)/\sqrt{2}$. By measuring qubit 1 in the computational basis $\{|0\rangle, |1\rangle\}$, we obtain the state $\rho_{34} = 1/2|\varphi\rangle\langle\varphi| + 1/2|\varphi^\perp\rangle\langle\varphi^\perp|$, with

$$\begin{aligned} |\varphi\rangle &= X_4^{s_1}(\alpha' |\phi^-\rangle + \beta' |\psi^+\rangle), \\ |\varphi^\perp\rangle &= X_4^{s_1}(-\alpha' |\psi^-\rangle + \beta' |\phi^+\rangle), \end{aligned}$$

and $s_1 \in \{0, 1\}$ corresponds to the computational basis measurement outcome of qubit 1. Then by measuring qubit 3 in the basis $\{|+\rangle, |-\rangle\}$ one finds the state $\rho_4 = 1/2|\tilde{\varphi}\rangle\langle\tilde{\varphi}| + 1/2|\tilde{\varphi}^\perp\rangle\langle\tilde{\varphi}^\perp|$, where $|\tilde{\varphi}\rangle = |\tilde{\varphi}^\perp\rangle = X_4^{s_1}(ZX)_4^{s_3\oplus 1}(\alpha'|+\rangle - \beta'|- \rangle) = X_4^{s_1}(ZX)_4^{s_3}Z_4(\alpha|0\rangle + \beta|1\rangle)$. Thus the final state is a pure state $\rho = |\tilde{\varphi}\rangle\langle\tilde{\varphi}|$, which if we remove the Pauli operators at the front we recover the encoded qubit using only local rotations. Note that even in the case there is no loss we can use this method to decode the qubit.

5.5.3 Stabilizer picture

The stabilizer formalism allows us to define codes in a more compact way and will also be useful to see how errors jeopardize the information encoded in the code.

A different way of writing the original state of qubit 5 is:

$$|\psi\rangle_5 = \frac{1}{2}(I + e_x X_5 + e_y Y_5 + e_z Z_5)|\psi\rangle_5 \quad (5.22)$$

where $e_x^2 + e_y^2 + e_z^2 = 1$. In order to see how this alternate description is equivalent to the one introduced in section II.B, remember that $|\psi\rangle_5 = \alpha|0\rangle_5 + \beta|1\rangle_5 = \mathcal{U}|0\rangle_5$ for some unitary $\mathcal{U}$. The projector will transform accordingly: $|\psi\rangle_5\langle\psi| = \mathcal{U}|0\rangle_5\langle 0|\mathcal{U}^\dagger = \frac{1}{2}(I + \mathcal{U}Z_5\mathcal{U}^\dagger) = \frac{1}{2}(I + e_x X_5 + e_y Y_5 + e_z Z_5)$, since the Pauli matrices form a basis for traceless matrices. Specifically we have for $\alpha = \cos\theta/2$ and $\beta = e^{i\phi}\sin\theta/2$ the correspondence $e_x = \sin\theta\cos\phi$, $e_y = \sin\theta\sin\phi$ and $e_z = \cos\theta$.

Encoding information can be seen as “expanding” the operators acting on the original qubit onto the cluster. For simplicity, we fix $e_y = 0$ and restrict the logical state to be in the $X - Z$ equator of the Bloch sphere. After tracking how the X and Z operators are expanded. Recall that the controlled- Z gate acts like $C_Z(I \otimes X)C_Z^\dagger = Z \otimes X$ and $C_Z(X \otimes I)C_Z^\dagger = X \otimes Z$. Applying the operation $\bar{C}_Z = C_Z^{15}C_Z^{25}C_Z^{35}C_Z^{45}$ gate will change the shape of the logical operators as:

$$\bar{X}_e = \bar{C}_Z X_5 \bar{C}_Z^\dagger = \begin{array}{cc} Z_1 & Z_2 \\ & X_5 \\ Z_3 & Z_4 \end{array}, \quad (5.23)$$

$$\bar{Z}_e = \bar{C}_Z Z_5 \bar{C}_Z^\dagger = \begin{array}{cc} I_1 & I_2 \\ & Z_5 \\ I_3 & I_4 \end{array}. \quad (5.24)$$

We can reshape the extended logical operators by multiplying them by the operators of which the code is an eigenstate, i.e. $\bar{X}'_e \equiv \bar{X}_e \bar{K}_i$, where $\bar{S}_i = \bar{C}_Z S_i \bar{C}_Z^\dagger$ stabilizes the new state

$$\bar{X}'_e \equiv \bar{X}_e \bar{S}_3 = \begin{array}{cc} Z_1 & Z_2 \\ & X_5 \\ Z_3 & Z_4 \end{array} \cdot \begin{array}{cc} Z_1 & Z_2 \\ & I_5 \\ Y_3 & Y_4 \end{array} = \begin{array}{cc} I_1 & I_2 \\ & X_5 \\ X_3 & X_4 \end{array}, \quad (5.25)$$

$$\bar{Z}'_e \equiv \bar{Z}_e \bar{X} = \begin{array}{cc} I_1 & I_2 \\ & Z_5 \\ I_3 & I_4 \end{array} \cdot \begin{array}{cc} Z_1 & I_2 \\ & Z_5 \\ X_3 & Z_4 \end{array} = \begin{array}{cc} Z_1 & I_2 \\ & I_5 \\ X_3 & Z_4 \end{array}. \quad (5.26)$$

where $\bar{X} = \bar{C}_Z \tilde{X} \bar{C}_Z^\dagger$ stabilizes the state by construction, since the cluster comes in the $|+_L\rangle$ state. In this way it is easy to see that the extended logical operators don't have support in qubit 2, which means that we don't need it to decode the information that was encoded in the cluster and can thus be lost. This also illustrates how encoding and decoding can be done. Qubit 5 will be measured in the X basis, which leaves the four remaining qubits in a state

$$|\psi\rangle_{1234} = \frac{1}{2}(I + e_x \tilde{X}_e + e_z \tilde{Z}_e)|\psi\rangle_{1234}. \quad (5.27)$$

The expanded logical operators $\tilde{X}_e = -I_1 I_2 X_3 X_4$ and $\tilde{Z}_e = Z_1 I_2 X_3 Z_4$ act on all four qubits. These operators are related to the original logical operators as $\tilde{X}_e = \tilde{Z} K_3 \equiv \tilde{Z}$ and $\tilde{Z}_e = \tilde{X}$. One should be able to now see the correspondence with the previous discussion where the qubit was shown to be encoded into the graph in the Hadamard basis: applying a Hadamard operation to the arbitrary qubit in equation (5.22) (with $e_y = 0$) swaps the X and Z Pauli operators, which can be seen in the encoded state in equation (5.27) using the original logical operators.

We can now find the expanded logical Y operator by using the relation $\tilde{Y}_e = i\tilde{X}_e \tilde{Z}_e$ to generalize the result. Of the qubits in the graph code, one can see from the form of the logical operators that we need to measure qubits 1 and 3 in the Z and X basis, respectively. That will leave qubit 4 in the $|\psi\rangle_4$ state, modulo some known Pauli corrections.

5.5.4 Generating a Four-Qubit Graph Code with Photons

We now describe how one could generate the graph state resource of four qubits plus the additional qubit to encode using photons. Therefore we show how to make the five-qubit graph state resource in Fig. 5.8 (b) - step (ii), *i.e.* before the qubit $|\psi\rangle$ is measured and thereby encoded into the graph. In this sense we consider the five-qubit graph state as the basic unit for encoding purposes, even though the graph code on its own only involves four qubits. In general, given an unknown qubit state, one could encode it into the four-qubit graph by making a two-qubit Bell measurement on the unknown qubit and the central qubit of the five-qubit graph unit, effectively teleporting the unknown qubit into the four-qubit code. For the moment, we consider the basic five-qubit graph state unit and that the state to encode already has the controlled- Z operations applied between it and the graph code qubits as shown in Fig. 5.8 (b) - step (ii).

The generation procedure involves three photons, with three qubits given by

the polarization degrees of freedom of the photons and two qubits given by the path degrees of freedom of two of the photons. The setup is shown in Fig. 5.9 (a) and the steps needed to convert the graph state generated by the setup into the desired five-qubit graph state are shown in Fig. 5.9 (b), where local complementation rules are used. In each step, a solid circle around a qubit denotes the central qubit i and a dotted circle denotes a neighbouring qubit j in the complementation operation given by

$$\sqrt{-iX_i} \bigotimes_{\{i,j\} \in E} \sqrt{iZ_j}. \quad (5.28)$$

These complementation operations are local and thus can be applied by inserting appropriate half- and quarter-wave plates on the photon modes before the analysis stage.

5.5.5 Loss simulation

As the experiment will be based on post-selection, we need to ensure that the three photons (plus the additional fourth heralding one) are all detected. Thus while bit- and phase-flip errors acting on the qubits of the code are easily implemented by local operations, in the case of loss we can only lose the qubits, not the actual photons. In order to go beyond this and make the code of practical use, by being able to actually lose one of the photons, all four qubits in the code would need to be represented by four separate photons. This is obviously well beyond the scope of the present work, as it involves experimental techniques that have not yet been developed. However, the proposal here is for an experiment to demonstrate the first proof-of-principle test of the code's operation and its performance. Thus it is an important first step.

In the experiment, the loss can be implemented by discarding the measurement outcomes for one of the qubits at the analysis stage, *i.e.* we consider we have lost all information about the qubit. The recovery of the encoded qubit then requires only local operations and these can be carried out with appropriate local

polarization/path rotations on the photons. For detecting bit-flip and phase-flip errors, the expectation values of the joint operators S_1 , S_2 and S_3 can be obtained locally at the analysis stage (waveplates and detectors). Although in this case we collapse the state, so that if there are no flip errors present we can't keep the code. However, this destructive check of the detection of errors is useful in testing the code's operation, regardless of whether we can or cannot use the state anymore. In principle joint detection without disturbance through quantum non-demolition measurements [118] of the code is possible, but beyond the capabilities of current technology.

Chapter 6

Decoherence of the Fluxonium Quantum Circuit

6.1 Sources of Decoherence in Superconducting Circuits

Superconducting circuits provide a way of creating ‘artificial atoms’ which can be made to strongly interact with one another in a controlled manner [119]. This is in contrast to the situation in other systems such as atoms, nuclear spins or photons, where a controlled interaction is much more difficult to achieve. However this means that superconducting circuits couple strongly to the environment, which causes them to have relatively short coherence times of about 1 microsecond [120], although this is enough to enact quantum gates on the stored information, since they can be done in the nanosecond timescale [121, 122].

The mechanisms that cause superconducting circuits to decohere are subject of intense research [123] (and references therein). It is widely accepted that impurities or material defects of different nature account for the low frequency noise¹ observed in electronic devices. Operating at frequencies far detuned from

¹Low frequency noise is relevant at timescales longer than data acquisition time of a single

the dominant resonant frequency of the defects minimizes relaxation. Dephasing is due to fluctuations of the Hamiltonian parameters, such as nominal value of the critical current, gate offset charge and external flux.

Fluctuations of the critical current (cf. Josephson energy) are understood to arise from charged impurities in the tunnel barrier that modify the transmissivity [124]. Low frequency charge noise is believed to be caused by trapping and detrapping of impurities in the insulators conforming the circuit [125]. There is not a clear understanding of how low-frequency flux noise arises from impurities but there is some evidence [126] pointing towards the idea that unpaired spins trapped in the superconductor-insulator interface could be the cause.

It remains to design circuits that are inherently insensitive to at least some sorts of noise. For instance, the *Cooper Pair Box* [127] operates at the value of the offset charge for which it is equally favorable for N and $N + 1$ Cooper pairs to exist in the island. This point is a result of an avoided crossing and receives the name of ‘sweet spot’, as it is to first order insensitive to charge fluctuations. For circuits inspired by the Cooper Pair Box, such as the *Quantronium* circuit [128], generalizations of the sweet spot to both charge and flux noise increases its coherence times. The *Transmon* also achieves a very good protection against charge noise by operating in a regime where the energy eigenstates are very little sensitive to charge variations.[129].

The quantum system we study here is the *Fluxonium* circuit, introduced in [130, 131]. It is protected from charge noise very much the same way flux qubits are, that is, by using an inductive shunt which screens charge fluctuations. We are interested in its behavior as the inductive and Josephson energies are changed, in the hope that there will be a set of optimal operating points. More precisely, we expect to see an exponential suppression of the dephasing rates between the lowest lying eigenstates, as the inductive energy is decreased. This can be understood as a consequence of the delocalization of the eigenstates, and the mechanism is formally analogous of that of the Transmon qubit.

Fluxonium also constitutes a simplified version of a more elaborated quantum experiment. This kind of noise limits the accuracy of sampling over several experiments.

tum circuit proposed by Kitaev, featuring enhanced reduction of dephasing rates, without compromising the protection against bit-flip errors [55]. For that reason, studying the Fluxonium circuit seems like a natural step towards understanding and studying how decoherence affects this device.

6.2 The Fluxonium Circuit

The shunted representation of the Fluxonium is given in Fig. 6.1(a). Following the prescriptions in [132], a Hamiltonian can be derived. The *Kirchhoff's Current Law* applied to the circuit only active node gives us:

$$C_J \ddot{\Phi} + C_g(\ddot{\Phi} + \dot{V}_g) = -I_C \sin\left(\frac{2\pi(\Phi - \Theta)}{\Phi_0}\right) - \frac{\Phi}{L}, \quad (6.1)$$

where we have used the constitutive relations $Q = C\Phi$, $I = \Phi/L$ and $I = I_C \sin(\frac{2\pi}{\Phi_0}(\Phi - \Theta))$ of the capacitor, inductance and Josephson junction, respectively. $V_g C_g$ is offset charge of the circuit. Φ is the flux variable, Θ is the external flux threading through the circuit and I_C is the critical current associated with the junction. Taking this as the equation of motion of a quantum particle described by the Φ coordinate, one can obtain a Lagrangian, using the *Euler-Lagrange* equation:

$$\frac{d\mathcal{L}}{d\Phi} - \frac{d}{dt} \frac{d\mathcal{L}}{d\dot{\Phi}} = 0. \quad (6.2)$$

The kinetic and potential energies of a hypothetical particle of mass $(C_J + C_g)$ moving in the coordinate Φ thus look like:

$$T = \frac{(C_J + C_g)\dot{\Phi}^2}{2} + C_g V_g \dot{\Phi}, \quad (6.3)$$

$$V = -\frac{I_C \Phi_0}{2\pi} \cos\left(\frac{2\pi(\Phi - \Theta)}{\Phi_0}\right) - \frac{\Phi^2}{2L}. \quad (6.4)$$

with $\mathcal{L} = T - V$. Through the canonical relations $Q = d\mathcal{L}/d\dot{\Phi}$ and $\mathcal{H} =$

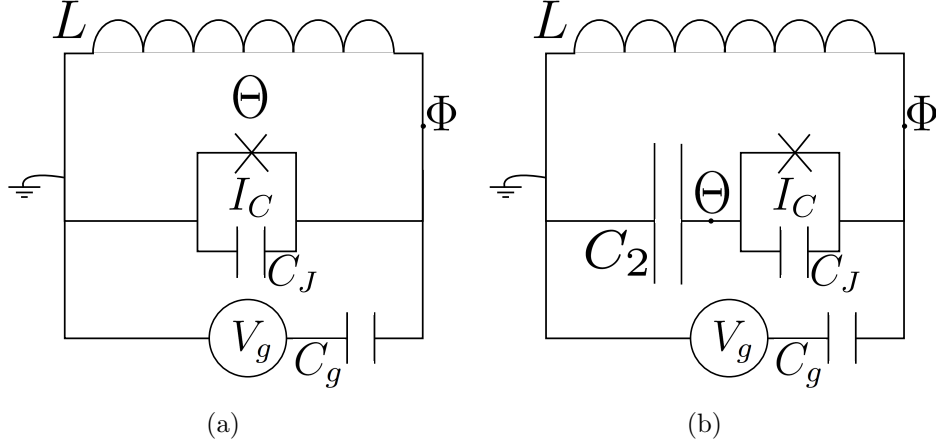


Figure 6.1: (a) The Fluxonium circuit. The inductance L is taken to be much larger than for circuits used for flux qubits, which means that the charging energy E_C will not be negligible when compared to E_L . (b) An extra capacitor $C_2 \gg C_1$ is attached to the circuit, in order to measure the current flowing across the Josephson junction.

$\dot{\Phi}Q - \mathcal{L}(\dot{\Phi}(Q), \Phi)$, it is straightforward to derive the circuit's Hamiltonian. It will prove useful to use the dimensionless variables $N = Q/2e$ and $\phi = 2\pi\Phi/\Phi_0$, denoting the number of Cooper pairs in the island and the superconducting phase, respectively. $\Phi_0 = h/(2e)$ is the *flux quantum*. In terms of these new variables the Hamiltonian reads:

$$\mathcal{H} = E_C(N - N_g)^2 - E_J \cos(\phi - \theta) + E_L \phi^2 \quad (6.5)$$

where $E_C = (2e)^2/(2(C_J + C_g))$, $E_L = (\Phi_0/2\pi)^2/2L$, $E_J = I_C \Phi_0/2\pi$ and $\theta = 2\pi\Theta/\Phi_0$ is the dimensionless flux threading through the circuit. Quantization arises as we set $[\phi, N] = i$. We will be using the conventions $2e = 1$ and $\Phi_0/2\pi = 1$, so that $\hbar = 1$. $N_g = C_g V_g/2e$ is an offset Cooper pair number.

Closing the superconducting island with an inductor means that charge carriers are no longer confined, and as a consequence the conjugate charge in the island is not necessarily an integer number of Cooper pairs that tunneled in or out of the island, as it happens in the Cooper Pair Box and derived architectures. Equivalently, since periodicity of the superconducting phase implies that the conjugate

charge must be discrete², in this case ϕ is not periodic. As explained in [130], the transformation $U|N\rangle = e^{iN_g\phi}|N\rangle = |N - N_g\rangle$ can be applied, obtaining:

$$\tilde{\mathcal{H}} = U\mathcal{H}U^\dagger = E_C N^2 + E_L \dot{\phi}^2 - E_J \cos(\phi - \theta) - \dot{N}_g \phi \quad (6.6)$$

so the spectrum of the Hamiltonian doesn't depend on the gate voltage but on its temporal variation, which renders Fluxonium specially well protected against offset charge low frequency noise. It is important to stress that this transformation wouldn't work in other devices such as the Cooper Pair Box or Transmon, since there one would need that N_g is an integer, in order to preserve the periodicity of the conjugate phase.

Equation (6.5) is formally the same as that of flux qubits. Whereas flux qubits work in the regime $E_J \gg E_C$, the operating point that we study here is $E_J \approx E_C \gg E_L$. In order to achieve high inductance, one possible way is to create chains of Josephson junctions, so that their effective inductance is very large [133].

6.3 Ground State and Persistent Current

In Ginzburg-Landau theory of superconductivity [134], the wavefunction $\Psi(\mathbf{r}) = |\Psi(\mathbf{r})|e^{i\phi(\mathbf{r})}$ describes the macroscopic Cooper pair condensate, where the parameter $\phi(\mathbf{r})$ is the superconducting phase defined above. The condition that the wavefunction be single-valued results in the quantization of flux threading a superconducting ring through the relation:

$$\oint \nabla \phi(\mathbf{r}) d\mathbf{r} = 2\pi n, \quad (6.7)$$

which means that superconducting phase can only change in multiples of 2π , corresponding to a flux quantum in $\hbar = 1$ units. The condensate wavefunction will be such that it minimizes the free energy of the system [134], so in a superconducting ring threaded by a flux, the energy is periodic in the flux number, with

²This relation is analogous to angular momentum quantization in Bohr's model

minima at integer multiples of 2π . Whenever the dimensionless flux is different from an even multiple of π , a persistent current will arise, counteracting this extra flux, whose intensity will be proportional to the bias.

In our case, one of the consequences of the ϕ not being a periodic variable is that the potential landscape in the ϕ coordinate has a parabolic shape with some wiggles whose size is proportional to the Josephson energy E_J (see Fig. 6.2).

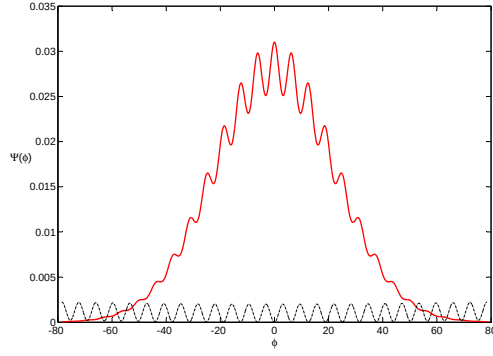
As hinted above, there is a nice mechanical analogy for quantum superconducting circuits. If one considers the superconducting phase to be the analogous of position of a particle in real space, then the capacitive energy would correspond to the kinetic energy, C and N playing the role of mass and momentum, respectively. The part of the hamiltonian dependent on ϕ can be regarded as the potential in which a quantum particle of mass C and momentum N evolves.

As shown in Fig. 6.3, the ground state exhibits a periodic dependence on the external flux θ threading through the loop. The persistent current is defined as the gradient of the ground state energy:

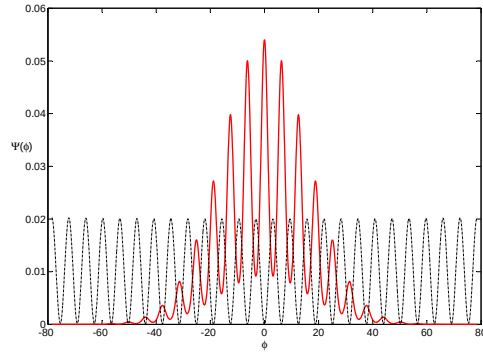
$$I_p = \frac{d}{d\theta} E_0(\theta), \quad (6.8)$$

which can intuitively be understood as a current that counteracts the energy increase due to dimensionless flux deviation from even integers of π . We are interested in studying the maximum value of I_p through the Fluxonium circuit as the ratio E_J/E_C increases. This can be engineered by varying the flux threading a SQUID-like junction [135]. The persistent current I_p is plotted in Fig. 6.4.

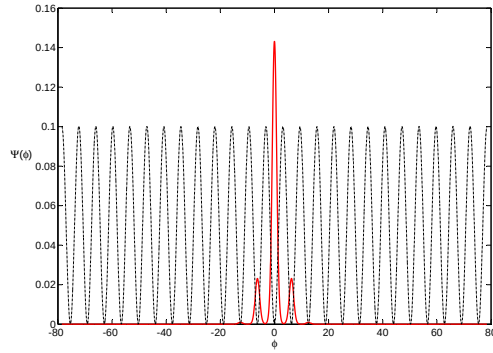
As argued above, from the single-valuedness of the wavefunction it follows that in a superconducting loop, the values of the external flux which minimize the energy of the system are multiples of 2π [134]. This means that there are several overlapping parabolas centered at values of the external flux $\theta = 2\pi n$. In the large E_J/E_C limit, Fig. 6.3 shows the overlap between two of such parabola. For small values of E_J/E_C it hints an avoided crossing. For large E_J/E_C , the value of ϕ is locked to the value of θ and varying the external flux results in dragging the localized wavefunction along a parabola, which remains locked to



(a) Weakly perturbed harmonic oscillator



(b) Intermediate regime



(c) Superconducting regime

Figure 6.2: Ground state wavefunctions are plotted in solid red line. The potential (changed by an offset value and constant scaling factor) is plotted in black dashed line. This is done for three different regimes. For $E_L/E_C = 2 \times 10^{-6}$, (a) $E_J/E_C = 0.1$ weakly perturbed harmonic oscillator, (b) $E_J/E_C = 1$, intermediate regime and (c) $E_J/E_C = 10$, superconducting regime

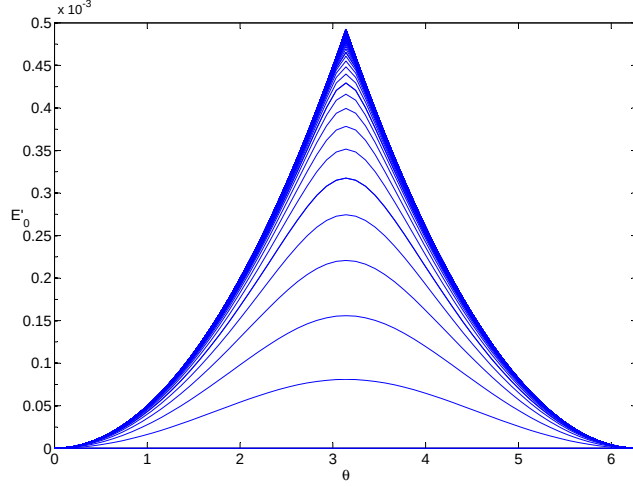


Figure 6.3: Ground state energy (with adequate offsets) as a function of the external dimensionless flux $\theta = \frac{2\pi\Phi}{\Phi_0}$, for $E_L/E_C = 10^{-4}$. The different curves correspond to values of E_J/E_C ranging from 0 to 6 for which it attains the maximum. In the large E_J/E_C limit, one can see that the ground state corresponds to an overlap of two parabolas centered at 0 and 2π , corresponding to the external flux of 0 and Φ_0 . These parabolas intersect at $\theta = \pi$.

the value of θ . At the intersection of two parabolas centered at adjacent flux numbers there is a sudden jump which creates a cusp in the ground state energy. The maximum value of the persistent current in the limit $E_J/E_C \gg 1$ will thus be:

$$I_p^{max} = \frac{d}{d\theta} E_L \theta^2|_{\theta_{max}} = \frac{\theta|_{\theta_{max}}}{L}, \quad (6.9)$$

where $\theta_{max} = (2n + 1)\pi$ is the value for which the persistent current attains its maximum, happening at the intersection between two adjacent parabolas. At small E_J/E_C values the wavefunction “feels” several minima and the dependence on the θ is not as sharp.

Another way to define the superconducting current flowing through the circuit is to attach a large capacitor C_2 to the Fluxonium circuit, as shown in Fig. 6.1(b), which acts as a measuring apparatus for the current. The Heisenberg equations of motion of the joint system are:

$$\dot{N}_\theta = -E_J \sin(\theta - \phi), \quad (6.10)$$

$$\dot{N}_\phi = -2E_L\phi + E_J \sin(\theta - \phi), \quad (6.11)$$

$$\dot{\theta} = 2E_{C_2}(N_\theta + N_\phi), \quad (6.12)$$

$$\dot{\phi} = 2E_{C_1}N_\phi, \quad (6.13)$$

where $E_{C_2} = 1/(2C_2)$. In taking the limit $E_{C_2} \rightarrow 0$ the large capacitor C_2 is made arbitrarily large, so it couples very weakly to the rest of the circuit. Equivalently, $\dot{\theta}$ can be made arbitrarily small. This is the *classical limit* on the the large capacitor, and the variable θ can thus be treated as a classical parameter of the Hamiltonian. We obtain an expression for the current flowing through the Fluxonium circuit:

$$I'_p = \langle \dot{N}_\theta \rangle = -E_J \langle \sin(\theta - \phi) \rangle \quad (6.14)$$

in the classical limit θ can be treated as an external parameter. If we restrict ourselves to the ground state, these two approaches are consistent with each other, as can be seen from straightforward application, in the classical limit, of the *Hellmann-Feynman Theorem*. The Hellmann-Feynman theorem relates the derivative of the energy with respect to some parameter to the expectation value of the derivative of the Hamiltonian with respect to that same parameter, which in our case would be θ :

$$\frac{dE_0}{d\theta} = \langle \psi_0 | \frac{d\mathcal{H}}{d\theta} | \psi_0 \rangle = \langle \psi_0 | \frac{dN_\theta}{dt} | \psi_0 \rangle. \quad (6.15)$$

In order to study numerically the properties of this circuit, we discretized the superconducting phase line and found the eigenstates using a sparse matrix diagonalization method. Either increasing the fine-graining or extending the domain of the discretization results in larger matrices to be considered, which means that longer computation times are needed.

To find a compromise between an acceptable reconstruction of the 2π periodic

part of the potential, and large enough interval such that the wavefunctions are well contained within it, while maintaining tractability of the problem, we chose to discretize the interval $\phi \in \{-50\pi, 50\pi\}$ into 10^4 points. Doubling the number of points, when tractable, does not lead to any improvement in the accuracy of our calculations. We argue this is a reasonable choice since, compared to the variance of the ground state of the quantum harmonic oscillator $\langle \phi^2 \rangle = \frac{\sqrt{E_C/E_L}}{2}$, the discretized interval is large enough so that most of the probability mass falls within it, and yet the shape of the potential is reproduced with high enough resolution. However, due to the large range explored of the parameters E_L and E_J , one particular choice of the discretization is not optimal for all cases. What follows in next sections is to our knowledge genuinely new and constitutes the main result of the chapter.

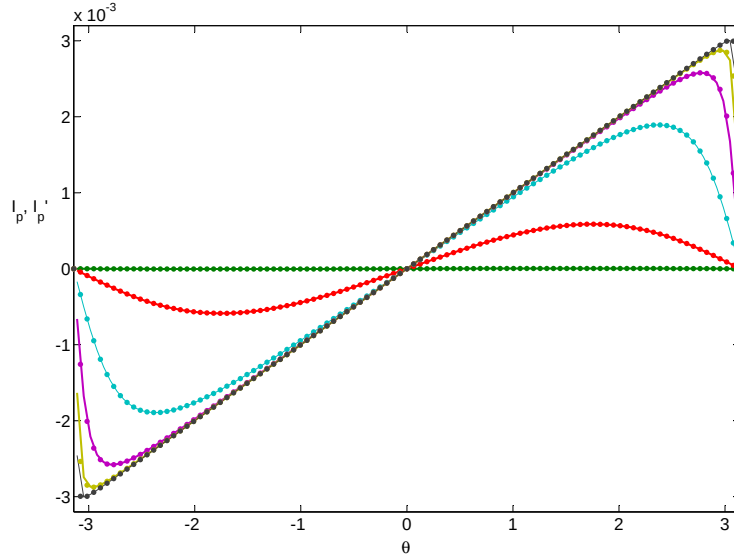


Figure 6.4: Persistent current through the circuit as function of the external dimensionless flux $\theta = \frac{2\pi\Theta}{\Phi_0}$, for $E_L/E_C = 10^{-4}$. Solid lines correspond to the calculation using equation (6.8), whereas the dots correspond to equation (6.14). Different values of the Josephson energy are plotted, starting at $E_J/E_C = 1$, in green, to $E_J/E_C = 6$, in black. Increasing E_J/E_C , due to the cusp in the ground state energy, a sawtooth dependence arises for the current which attains a maximum at $\theta = \pi$.

6.4 Superconductor-Insulator Transition

The superconducting behaviour depends on the superconducting phase ϕ of the wavefunction $\Psi(\mathbf{r})$ [134]. If the wavefunction is delocalized in ϕ -space, meaning that it does not have a definite value, then a current cannot flow through the junction. The tunneling of the quantum particle from one minimum to another of the potential in ϕ -space is called a *quantum phase slip* event [134]. Equivalently, a quantum phase-slip is the event whereby a flux quantum tunnels through a superconducting ring. This tunneling causes the superconducting phase in a Josephson junction to change as $\phi \rightarrow \phi \pm 2\pi$, depending on whether the flux quantum tunneled inwards or outwards.

As the ratio E_J/E_C is reduced, the wiggles separating adjacent minima become smaller and smaller and the phase-slip amplitude increases. Beyond a certain point, as this amplitude becomes so large that the wavefunction spreads over several minima and the junction enters an insulating phase.

We identify three different regimes and characterize them looking at the properties of the ground state wavefunction:

Insulating Regime, $E_J/E_C \leq 1$ The ground state wavefunction spreads over many local minima of the potential. Due to the large inductance, a very wide Gaussian function modulates the peaks of the wavefunction. This causes the state to be insensitive to variations of the external flux, just as the ground state of a harmonic oscillator would be. In fact, the state can be interpreted as a weakly perturbed harmonic oscillator.

Intermediate Regime, $E_J/E_C \approx 1$ Above $E_J/E_C = 1$ the quantum particle can only hop to an adjacent minimum through quantum phase-slips. As a result, the wavefunction still spreads over a few local minima so it is not possible to treat the superconducting phase as a classical variable having one definite value. Its behaviour can still be understood in terms of the *effective capacitance* approximation introduced later in this section.

Superconducting Regime, $E_J/E_C \gg 1$ The phase-slip amplitude is negligible

and the wavefunction is constrained within one well of the potential, $\langle \phi^2 \rangle \leq \pi^2$. Phase-slips are suppressed and the phase is locked to the classical value of the external flux, which can be used to control the supercurrent.

The fluctuations of the phase of the ground state are expected to be very large in the insulating regime, rendering the device insensitive to variations of the external flux. Alternatively, the fluctuations of the charge should be very small, and the number of Cooper pairs well defined. On the contrary, in the superconducting regime, phase fluctuations rapidly decrease and the fluctuations of the charge increase correspondingly, such that the uncertainty relation holds:

$$\Delta \phi^2 \Delta N^2 \geq \frac{\langle i[\phi, N] \rangle^2}{4}. \quad (6.16)$$

6.4.1 Derivation of Phase-slip Amplitude

The phase-slip amplitude can be derived analytically using instanton methods [136]. However we obtained it numerically by partially diagonalizing the Fluxonium Hamiltonian. Notice that the first two terms of equation (6.5) correspond to the Cooper Pair Box. Its solutions can be given in terms of Bloch wavefunctions [137]:

$$\psi_{b,\tilde{N}}(\phi) = e^{i\tilde{N}\phi} u_{b,\tilde{N}}(\phi), \quad (6.17)$$

where $u_{b,\tilde{N}}$ is 2π -periodic and b is the band index. The $\tilde{N}$ is the quasimomentum in a one-dimensional crystal. The Hamiltonian looks thus like:

$$\mathcal{H} = E_L(\phi + \theta)^2 + E(\tilde{N}), \quad (6.18)$$

where $E(\tilde{N})$ is the charge dispersion relation obtained from partial diagonalization, and we have provisionally made a change of variable $\phi \rightarrow \phi + \theta$. For $E_J/E_C \ll 1$ then $E(\tilde{N}) \simeq E_C(N - \tilde{N})^2$, as can be seen from the theory underlying the Cooper Pair Box [127] (see Fig. 6.5).

For $E_J/E_C \geq 1$ the approximation

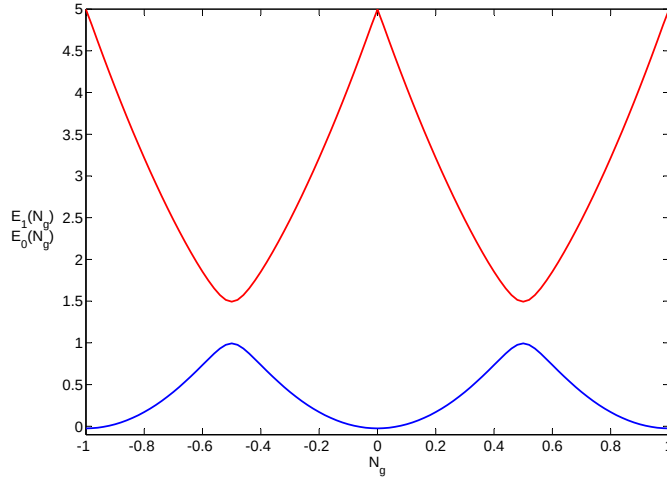


Figure 6.5: First two energies as a function of quasimomentum $\tilde{N}$, analogous in this case to the crystal momentum, for the value $E_J/E_C = 0.1$.

$$E(\tilde{N}) = -E_{QPS} \cos(2\pi\tilde{N}) \quad (6.19)$$

works outstandingly well and justifies a tight-binding treatment of the problem [137]. We define $E_{QPS} = 2t$, where t is the quantum phase-slip amplitude.

Diagonalizing the Cooper Pair Hamiltonian for different values of N_g allowed us to obtain the charge dispersion and, using equation (6.19), we then numerically derived the dependence of the quantum phase-slip amplitude t on E_J/E_C . As shown in Fig. 6.6, our numeric derivation agrees with analytic derivations using instanton methods in Ref. [136, 129], which with our energy definitions, is given by the expression:

$$t = \frac{4(2E_J^3 E_C)^{1/4}}{\sqrt{\pi}} \exp\left(-8\sqrt{\frac{E_J}{2E_C}}\right), \quad (6.20)$$

and gives us a means to define the analog of the effective mass of a flux particle moving in a one-dimensional crystal, i.e. the *effective capacitance*.

6.4.2 Effective Capacitance Approximation

In the limit $E_L/E_C \ll 1$, or alternatively, in a very wide parabolic potential, the quantum particle “feels” the periodic contribution of the Josephson junction, so an effective capacitance C^* on the circuit will be induced, defined as:

$$C^* = \left(\frac{d^2 E(\tilde{N})}{d^2 \tilde{N}} \right)^{-1} \Big|_{\tilde{N}=0} \quad (6.21)$$

where $E(N)$ is the charge dispersion relation obtained from the tight-binding approximation. This is in direct analogy with the behaviour of a particle in a one-dimensional lattice. As depicted in Fig. 6.7 the numerical and analytical expression for the effective capacitance become closer as the ratio E_J/E_C increases. The effective capacitance description will break down, however, when the wavefunction in superconducting phase basis does not expand over several minima, i.e. the quantum particle does not “see” a crystal anymore. In the charge basis, this happens when the wavefunction is so wide that the curvature of the dispersion relation evaluated at the origin $\tilde{N} = 0$, which arises in the expression for

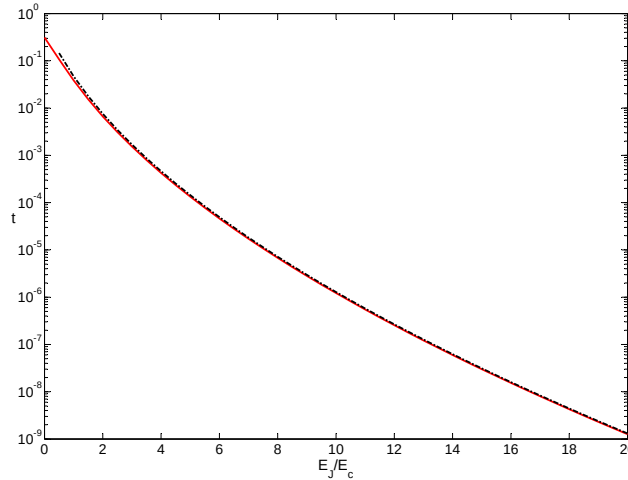


Figure 6.6: The quantum phase-slip amplitude as a function of the ratio E_J/E_C . The numerical derivation is plotted in solid line, whereas the derivation of Ref. [129] is plotted in dashed line.

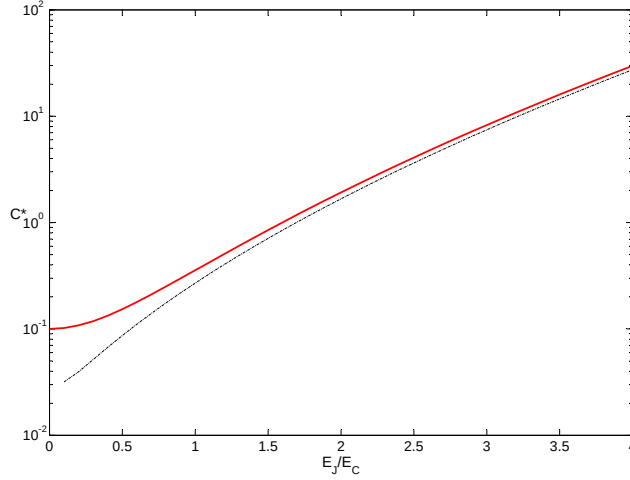


Figure 6.7: The effective capacitance C^* as a function of the ratio E_J/E_C . Numerical derivation is plotted in solid line, whereas analytic derivation in dashed line. Neglecting the effect of a large inductance, we can deduce the effective capacitance of a particle moving in ϕ space, which will to some accuracy approximate the behaviour of the circuit as a harmonic oscillator with a slightly different frequency. This will break down when the wavefunction doesn't spread over several minima anymore, invalidating the assumption that the particle feels a periodic potential

the effective capacity, cannot be evaluated. We expect this to happen when the wavefunction starts to become localized in the phase basis, so a persistent current can be measured.

The analytic expression for the effective capacitance is obtained using equation (6.20) and the tight-binding approximation in equation (6.19), allowing us to derive an expression for C^* :

$$C^* = \frac{1}{2t(2\pi)^2}. \quad (6.22)$$

For small values of the ratio E_J/E_C , the device should be independent of the external flux, and the effective capacitance gives us a quantitative way of seeing this: the corrugations in the parabola will shorten the width of the wavefunction but will not change its harmonic oscillator character. When this approximation

breaks down we expect a dependence on the external flux to arise. This is shown in Fig. 6.8.

Finally, the third regime is realized at $E_J/E_C \gg 1$. Here the phase is localized at the global minimum. Changing θ amounts to shifting the minimum along the parabola and creation of a persistent current whose absolute magnitude depends only on E_J/E_C .

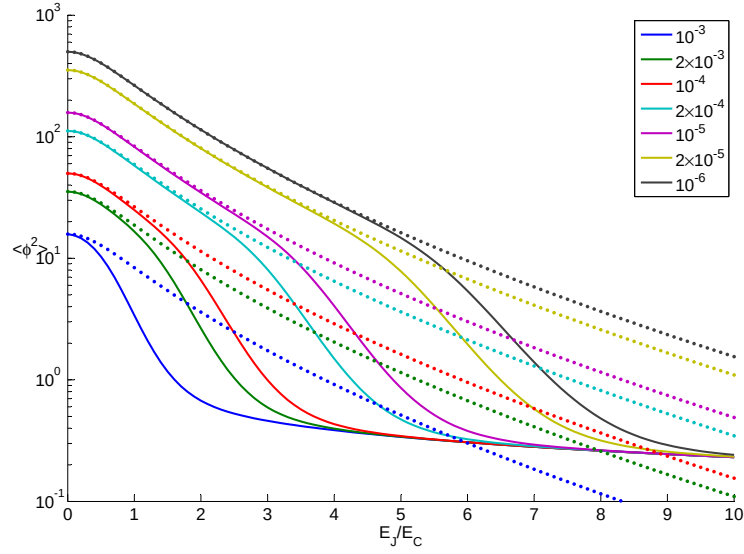


Figure 6.8: The numerically obtained variance $\langle \psi_0 | \phi^2 | \psi_0 \rangle$ of the superconducting phase in the ground state is plotted in solid lines. The different colours denote different values of E_L/E_C , which in turn correspond to different widths of the parabolic potential. The wider the parabolic potential, the more a quantum particle would “feel” the periodic potential, gaining an effective capacitance. The dotted lines correspond to the variance of an LC circuit with capacitive energy $E_{C^*} = 1/(2C^*)$, $\langle \phi^2 \rangle = \frac{\sqrt{E_{C^*}/E_L}}{2}$. This approximation describes well the circuit in the insulating regime. In the superconducting regime the effective capacitance approximation breaks down. Note that all solid lines converge to a value of $\langle \phi^2 \rangle$, corresponding to a wavefunction localized in one well of the periodic potential.

The fact that Fluxonium is very well approximated by an LC circuit with effective capacitance C^* in the insulating phase is the cause its exponential protection against low frequency noise in the external flux, *i.e.* $dE_0(\theta)/d\theta \propto \exp\left(-\sqrt{E_{C^*}/E_L}\right)$:

$$\begin{aligned}
\frac{d}{d\theta} E_0(\theta) &= -E_J \frac{d}{d\theta} \langle \cos(\phi - \theta) \rangle \\
&= -E_J \frac{d}{d\theta} \cos \theta \exp -\frac{\langle \phi^2 \rangle}{2} \\
&= E_J \sin \theta \exp \left(-\frac{\sqrt{E_{C^*}/E_L}}{4} \right), \tag{6.23}
\end{aligned}$$

which entails that the fluctuations of energy difference between the lowest-lying eigenstates will decrease as the inductance is made larger.

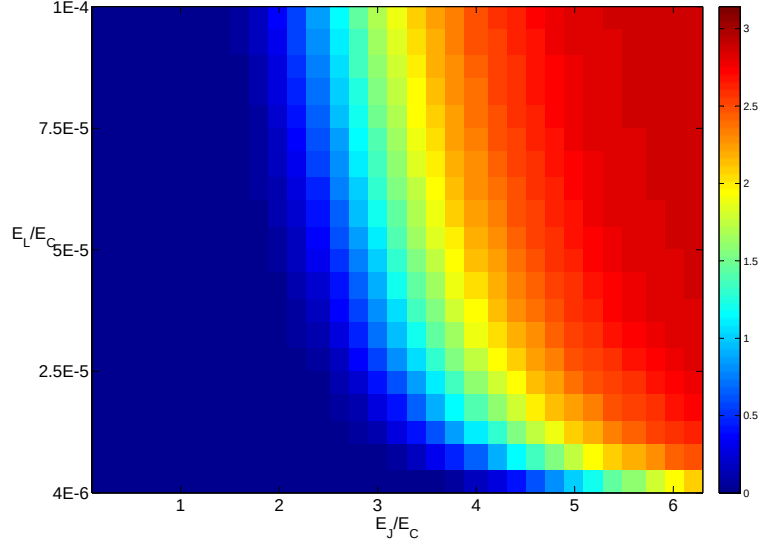
6.4.3 Transition Diagram

The normalized maximum value of the persistent current as a function of the ratios E_L/E_C and E_J/E_C is shown in Fig. 6.9(a). In Fig. 6.9(b) the superconducting regime, in which phase-slips are suppressed, is shown in green.

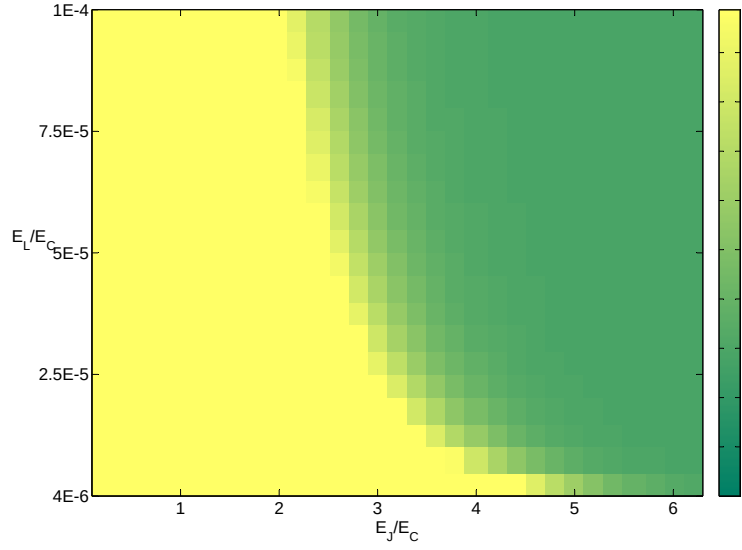
Going back to the dispersion relation, the tunneling amplitude from one minimum of $E(\tilde{N})$ to another is proportional to the maximum current flowing through the device. Once again it will prove useful to use a mechanical analogy to understand the behaviour of device. In the tight-binding approximation, one can imagine that the term $E_L \phi^2$ in equation (6.18) corresponds to the kinetic energy of a ‘charge particle’ moving in a periodic potential in the N basis, given by equation (6.19). The larger the inductance, *i.e.* the charge particle’s mass, the more difficult will it be for it to tunnel to the next minimum. This is intuitively speaking the reason why the critical current is an increasing function of E_L .

6.5 Decoherence in Fluxonium

To account for decoherence in the Fluxonium circuit, we model the coupling to an environment using a *Caldeira-Leggett* type bath of harmonic oscillators [132]. The Caldeira-Leggett model is a method to deal with the dynamics of a quantum system coupled to a dissipative environment, which can be used to recover dissi-



(a) Transition diagram



(b) Ground state variance

Figure 6.9: (a) Colormap illustrating the normalized maximum value of the persistent current $\tilde{I}_P^{max} = I_P^{max}/(2E_L)$, for different values of E_J/E_C and E_L/E_C . (b) The light yellow area corresponds to the values of the the ground state variance above π^2 . Quantum phase-slips are suppressed in the green zone

pative evolution of a particle with friction, in the classical limit. In the context of superconducting circuits, the idea is to describe a linear dissipative element (a resistor) as an infinite set of non-interacting harmonic oscillators coupled to circuit, as shown in Fig. 6.10.

We describe the coupling in the phase basis since we expect phase fluctuations to play a more important role than charge fluctuations. The complete Hamiltonian looks like:

$$\mathcal{H}_T = \mathcal{H}_F + \mathcal{H}_B + \mathcal{H}_I, \quad (6.24)$$

where $\mathcal{H}_F$ is the same as equation (6.5), except for a renormalized inductive energy $E'_L = E_L + \sum_k E_{l,k}$. The bath Hamiltonian is an infinite sum of harmonic oscillators:

$$\mathcal{H}_B = \sum_k \{E_{c,k} N_k^2 + E_{l,k} \phi_k^2\}, \quad (6.25)$$

where $E_{c,k} = 1/2c_k$ and $E_{l,k} = 1/2l_k$. The interaction Hamiltonian is:

$$\mathcal{H}_I = -2\phi \sum_k E_{l,k} \phi_k. \quad (6.26)$$

Now we define the creation and annihilation operators for the bath and Fluxonium degrees of freedom:

$$\phi = \sqrt{\frac{Z}{2}}(a^\dagger + a), \quad (6.27)$$

$$N = i\sqrt{\frac{1}{2Z}}(a^\dagger - a), \quad (6.28)$$

$$\phi_k = \sqrt{\frac{z_k}{2}}(b_k^\dagger + b_k), \quad (6.29)$$

$$N_k = i\sqrt{\frac{1}{2z_k}}(b_k^\dagger - b_k), \quad (6.30)$$

where $Z = \sqrt{E_C/E'_L}$ and $z_k = \sqrt{E_{c,k}/E_{l,k}}$. In the Fluxonium eigenbasis, $\mathcal{H}_F |i\rangle =$

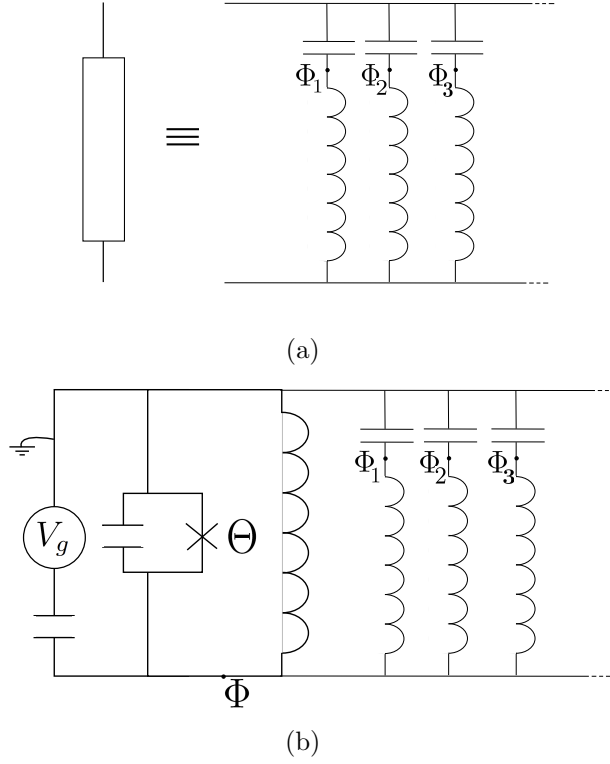


Figure 6.10: Quantum mechanics is formulated in the Hamiltonian formalism, where the total energy of the system is conserved. In order to describe dissipation, which is an irreversible process, we used the Caldeira-Leggett model, in which an infinite set of oscillators coupled to the system is used to model energy flow into the environment. (a) A linear dissipative element, a resistor, is modeled as an infinite sum of harmonic oscillators. The infinite number of degrees of freedom reconciles the irreversible behaviour of the resistor with formal reversibility of quantum mechanics. (b) Shunted representation of the circuit coupled to an environment causing flux fluctuations.

$\epsilon_i |i\rangle$, the total Hamiltonian reads:

$$\mathcal{H}_T = \sum_i \epsilon_i |i\rangle \langle i| + \sum_k \omega_k b_k^\dagger b_k + \sum_{i,i'} M_{ii'} |i\rangle \langle i'| \sum_k g_k (b_k^\dagger + b_k), \quad (6.31)$$

where $M_{ii'} = \langle i | (a^\dagger + a) | i' \rangle = \sqrt{\frac{2}{Z}} \langle i | \phi | i' \rangle$, $\omega_k = 2\sqrt{E_{c,k} E_{l,k}}$ is the k -th oscillator frequency, and the system-bath couplings are defined as $g_k = -E_{l,k} \sqrt{z_k Z}$. Having described the interaction Hamiltonian in terms of the eigenstates of the device and

the bath, we can obtain a master equation following the usual methods [138, 18].

6.5.1 Master Equation Derivation

In order to derive a Master Equation for Fluxonium we follow the approach exposed in [138]. After taking the *Born-Markov approximation*, the density operator in the interaction picture evolves according to:

$$\frac{d}{dt}\tilde{\rho}(t) = - \int_0^\infty d\tau \text{tr}_B \left\{ [\tilde{\mathcal{H}}_I(t), [\tilde{\mathcal{H}}_I(t-\tau), \tilde{\rho}(t)\rho_B]] \right\}. \quad (6.32)$$

The interaction Hamiltonian is:

$$\begin{aligned} \tilde{\mathcal{H}}_I(t) &= e^{i(\mathcal{H}_F + \mathcal{H}_B)t} \mathcal{H}_I e^{-i(\mathcal{H}_F + \mathcal{H}_B)t} = \sum_{ii'} e^{i(\epsilon_i - \epsilon'_i)t} M_{ii'} |i\rangle \langle i'| \sum_k g_k (b_k^\dagger e^{i\omega_k t} + b_k e^{-i\omega_k t}) \\ &= \sum_{ii'} \tilde{S}_{ii'}(t) \tilde{B}(t). \end{aligned} \quad (6.33)$$

$\tilde{S}_{ii'}(t) = e^{i(\epsilon_i - \epsilon'_i)t} M_{ii'} |i\rangle \langle i'|$ and $\tilde{B}(t) = \sum_k g_k (b_k^\dagger e^{i\omega_k t} + b_k e^{-i\omega_k t})$. Note that we have made the assumption of a Markovian bath. Despite this being standard in derivations of master equations for superconducting circuits, in some cases it may limit the applicability of the theory and give rise to rather qualitative statements. With this caveat in mind, equation (6.32) can then be rewritten as:

$$\frac{d}{dt}\tilde{\rho}(t) = - \sum_{ii', jj'} \int_0^t d\tau \left\{ [\tilde{S}_{ii'}(t), \tilde{S}_{jj'}(t-\tau) \tilde{\rho}(t)] C(\tau) + [\tilde{\rho}(t) \tilde{S}_{jj'}(t-\tau), \tilde{S}_{ii'}(t)] C^*(\tau) \right\}, \quad (6.34)$$

where $C(\tau)$ is the bath correlation function. Assuming that the bath is in equilibrium state, that is, $[\mathcal{H}_B, \rho_B] = 0$, the correlation function depends on the difference in times τ only, and is defined as:

$$\begin{aligned}
C(\tau) &= \text{tr}_B\{\tilde{B}(t)\tilde{B}(t-\tau)\rho_B(t)\} \\
&= \text{tr}_B\{\tilde{B}(0)\tilde{B}(-\tau)\rho_B\} \\
&= \sum_k g_k^2 [\cos \omega_k \tau \coth \beta \omega_k / 2 - i \sin \omega_k \tau] \\
&= \nu(\tau) - i\eta(\tau).
\end{aligned} \tag{6.35}$$

$\beta = 1/k_B T$, and k_B is the Boltzmann constant. In the continuum limit it reads:

$$\nu(\tau) = \int_0^\infty d\omega J(\omega) \cos \omega \tau \coth \beta \omega / 2, \tag{6.36}$$

$$\eta(\tau) = \int_0^\infty d\omega J(\omega) \sin \omega \tau, \tag{6.37}$$

where $J(\omega)$ is the bath spectral density:

$$J(\omega) = \sum_k g_k^2 \delta(\omega - \omega_k). \tag{6.38}$$

We have, in the Schrödinger picture:

$$\begin{aligned}
\frac{d}{dt}\rho(t) = & - i[\mathcal{H}_F, \rho(t)] \\
& - \sum_{ii',jj'} \int_0^\infty d\tau \{ [S_{ii'}, S_{jj'}(-\tau)\rho(t)]C(\tau) + [\rho(t)S_{jj'}(-\tau), S_{ii'}]C^*(\tau) \},
\end{aligned} \tag{6.39}$$

where $S_{jj'}(-\tau) = S_{jj'} e^{-i(\epsilon_j - \epsilon_{j'})\tau}$. Now, separating the real and imaginary parts of the correlation function:

$$\begin{aligned}
\frac{d}{dt}\rho(t) = & - i[\mathcal{H}_F, \rho(t)] \\
& - \sum_{ii',jj'} \int_0^\infty d\tau \{ [S_{ii'}, [S_{jj'}(-\tau), \rho(t)]]\nu(\tau) - i[S_{ii'}, \{S_{jj'}(-\tau), \rho(t)\}]\eta(\tau) \}.
\end{aligned} \tag{6.40}$$

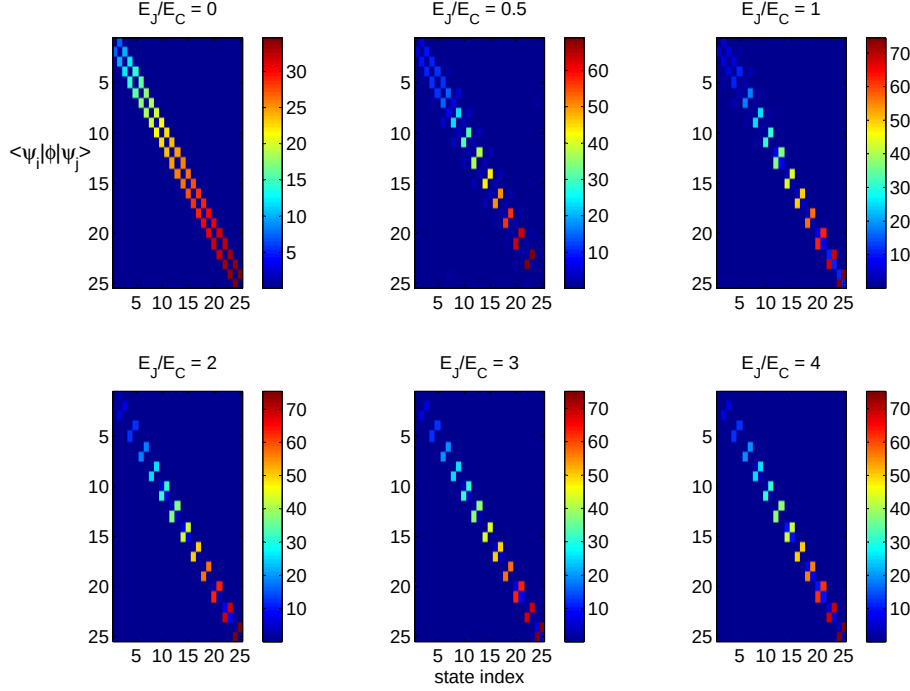


Figure 6.11: Matrix elements $\langle i | \phi | j \rangle$, which help us visualize the coupling between states $|i\rangle$ and $|j\rangle$ due to flux fluctuations. For $E_J/E_C = 0$, $\langle i | \phi | j \rangle \propto \sqrt{j} \delta_{i,j-1} + \sqrt{j+1} \delta_{i,j+1}$ as expected from a harmonic behaviour. As E_J/E_C increases, the states become localized and they are pairwise coupled.

The matrix M will be very important in determining the coupling of the circuit to its environment. For a small matrix element $M_{ii'}$, the environment will not couple strongly to the operator $|i\rangle\langle i'|$.

The first matrix elements of M for different values of E_J/E_C , $E_L/E_C = 10^{-4}$ are depicted in Fig. 6.11. One observes that as the ratio E_J/E_C increases, the states only couple significantly to one other eigenstate. This arises as a consequence of contiguous eigenstates progressively turning into degenerate symmetric and antisymmetric superpositions of opposite persistent currents, as explained in [130]. For small E_L/E_C , the harmonic character is preserved for some of the lowest-lying eigenstates, which is consistent with the energy plots in Fig. 6.12, where a linear harmonic spacing of the energies is observed.

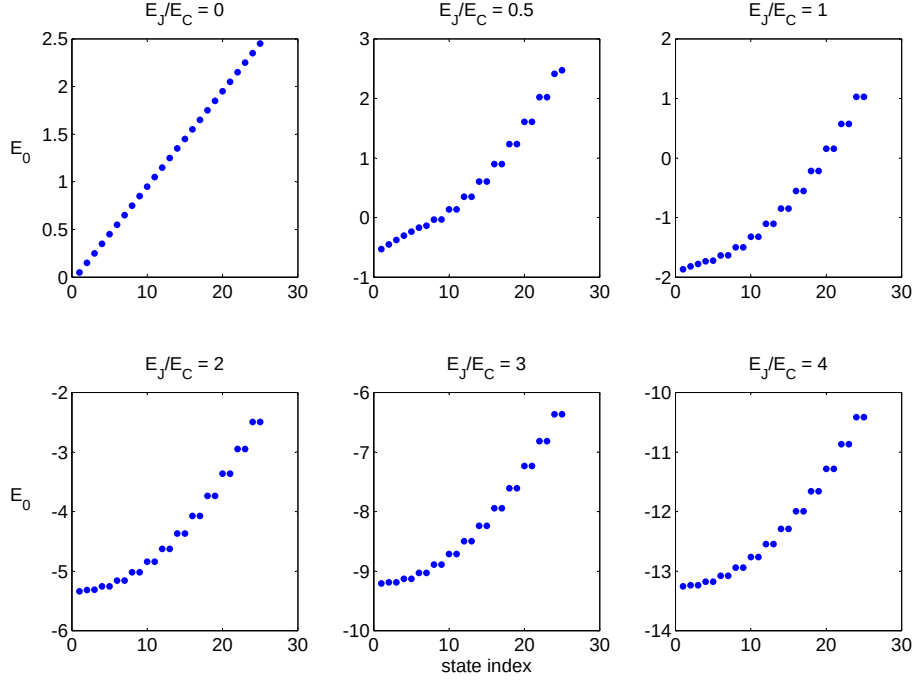


Figure 6.12: As E_J/E_C increases, the harmonic character is gradually lost, and the states become almost pairwise degenerate, corresponding to symmetric and antisymmetric superpositions of localized wavefunctions in opposite minima of the parabolic potential.

6.5.2 Two-level System Truncation

We are interested in looking at the two lowest-lying energy eigenstates of the system, which can be used to implement a qubit. To see how the environment will affect the qubit, we truncate the expression in equation (6.40) to the two eigenstates with the lowest energy. Then the total Hamiltonian is:

$$\begin{aligned} \mathcal{H}_T = & \frac{\epsilon_0 + \epsilon_1}{2} + \frac{\epsilon_0 - \epsilon_1}{2} \sigma_Z + \sum_k \omega_k b_k^\dagger b_k \\ & + \left(\frac{M_{00} + M_{11}}{2} + \frac{M_{00} - M_{11}}{2} \sigma_Z + M_{10} \sigma_+ + M_{01} \sigma_- \right) \sum_k g_k (b_k^\dagger + b_k), \end{aligned} \quad (6.41)$$

where $\sigma_+ = |1\rangle\langle 0|$ and $\sigma_- = |0\rangle\langle 1|$. The properties of these two lowest-lying eigenstates (whether their wavefunction resembles that of a harmonic oscillator, or rather a superposition of persistent currents) will determine their behaviour under flux noise. We expect the qubit to be significantly less sensitive to flux noise in the regimes $E_J/E_C \leq 1$ and $E_L/E_C \ll 1$. However, since the energy difference between the eigenstates is $\approx 2\sqrt{E_LE_C}$, we also expect the relaxation rate to increase. Conversely, one can say that in the regime $E_J/E_C \gg 1$ relaxation rates will decrease, although the persistent current states will be highly sensitive to flux noise, so dephasing rates will increase.

Fluctuations of the external flux will cause the wiggles to randomly displace along the parabola. Energy differences between the eigenstates will then fluctuate, which leads to dephasing of the qubit stored in the lowest lying eigenstates. The question is whether the increase of pure dephasing times is faster than the drop in relaxation times as we move in the transition diagram in Fig. 6.9(a). We will be concerned with studying this dephasing rate for a range of parameter values.

This truncated interaction Hamiltonian can be used to obtain a master equation in a manner analogous in previous section. In the interaction picture it reads:

$$\frac{d}{dt}\tilde{\rho}(t) = - \sum_{\substack{\alpha=1,2,3,4 \\ \beta=1,2,3,4}} \int \int_0^\infty d\tau \{ [S_\alpha, S_\beta(-\tau)\rho(t)]C(\tau) + [\rho(t)S_\beta(-\tau), S_\alpha]C^*(\tau) \}, \quad (6.42)$$

where the terms are $S_1(t) = \frac{M_{00}+M_{11}}{2}$, $S_2(t) = \frac{M_{00}-M_{11}}{2}\sigma_Z$, $S_3(t) = M_{10}\sigma_+e^{i\epsilon t}$ and $S_4 = M_{01}\sigma_-e^{-i\epsilon t}$, and $\epsilon = \epsilon_1 - \epsilon_0$. Assuming that the typical timescale $\tau_S = \epsilon^{-1}$ of the circuit is much shorter than the relaxation time τ_R of the bath will allow us to drop the terms which oscillate very fast over the time τ_R , *i.e.* with frequencies different from $\epsilon = \tau_S^{-1}$. This is the *secular approximation* [18]. The simplified master equation looks thus like:

$$\begin{aligned}
\frac{d}{dt}\tilde{\rho}(t) = & - \int_0^\infty d\tau \left\{ i \frac{M_{00}^2 - M_{11}^2}{2} \eta(\tau) [\sigma_Z, \rho] \right. \\
& + i |M_{01}|^2 (\eta(\tau) \cos \epsilon\tau - \nu(\tau) \sin \epsilon\tau) [\sigma_- \sigma_+, \rho] \\
& + i |M_{01}|^2 (\eta(\tau) \cos \epsilon\tau + \nu(\tau) \sin \epsilon\tau) [\sigma_+ \sigma_-, \rho] \\
& - \frac{(M_{00} - M_{11})^2}{4} \nu(\tau) ([\sigma_Z, [\sigma_Z, \rho]]) \\
& - \underbrace{|M_{01}|^2 (\nu(\tau) \cos \epsilon\tau - \eta(\tau) \sin \epsilon\tau)}_{\alpha} (2\sigma_- \rho \sigma_+ - \{\sigma_+ \sigma_-, \rho\}) \\
& \left. - \underbrace{|M_{01}|^2 (\nu(\tau) \cos \epsilon\tau + \eta(\tau) \sin \epsilon\tau)}_{\beta} (2\sigma_+ \rho \sigma_- - \{\sigma_- \sigma_+, \rho\}) \right\}. \tag{6.43}
\end{aligned}$$

The first three terms correspond to a *Lamb shift*, which amounts to a correction of the unitary dynamics. The last three terms describe pure dephasing, relaxation and excitation, respectively. Pure dephasing arises when the Lindblad operators are diagonal in the eigenstate basis, and as a consequence there is no energy exchange between the system and the environment. The rate for this process is:

$$\Gamma_\varphi = \frac{(M_{00} - M_{11})^2}{4} \int_0^\infty d\tau \nu(\tau) = \frac{\pi(M_{00} - M_{11})^2}{4} \lim_{\omega \rightarrow 0} J(\omega) \coth \frac{\beta\omega}{2}. \tag{6.44}$$

The pure dephasing time is defined as $T_\varphi = 1/\Gamma_\varphi$. The relaxation rate is obtained by solving the evolution equations of the matrix elements:

$$\dot{\rho}_{11} = -2\alpha\rho_{11} + 2\beta\rho_{00}, \tag{6.45}$$

$$\dot{\rho}_{00} = 2\alpha\rho_{11} - 2\beta\rho_{00}, \tag{6.46}$$

which gives us a relaxation rate $\Gamma_1 = 2(\alpha + \beta)$. For an ohmic spectral density $J(\omega) = A\omega$, which is needed in order to derive the classical dissipative behaviour, the rates Γ_φ and Γ_1 are:

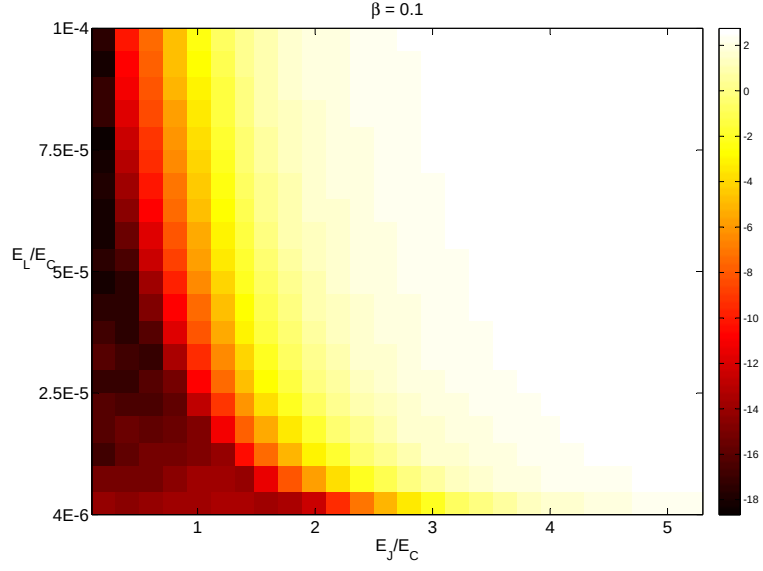
$$\Gamma_\varphi = \frac{A\pi(M_{00} - M_{11})^2}{2\beta}, \quad (6.47)$$

$$\Gamma_1 = 2A\pi|M_{01}|^2 \coth \frac{\beta|\epsilon|}{2} |\epsilon|, \quad (6.48)$$

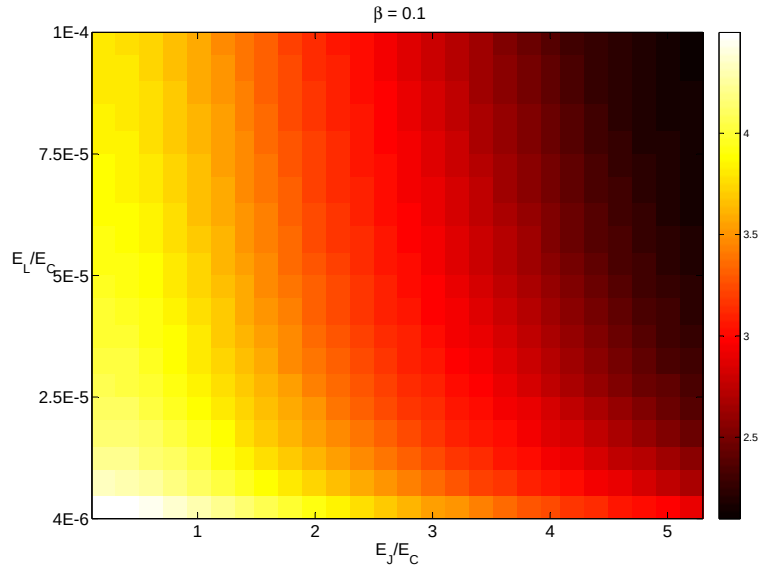
and $T_1 = 1/\Gamma_1$. Decay of the off-diagonal elements of the density operator can also be caused by relaxation, through the well known phenomenological formula $\Gamma_2 = \Gamma_1/2 + \Gamma_\varphi$ [135]. Γ_1 is not exponentially suppressed, and in fact decoherence is governed mainly by relaxation processes, which renders the exponential suppression of pure dephasing irrelevant. Coupling Fluxonium to a resonator [133] offers a regime in which decoherence is not dominated by T_1 times. Also, one could expect that by means of engineering the electromagnetic environment of the circuit, relaxation processes can be mitigated, as has been done for the Transmon [139, 140].

Their dependence on the inverse temperature is quite different. In Figs. 6.13, 6.14, 6.15 and 6.16, this difference is made patent. We averaged for different values of the external flux θ . Figs. 6.17 show the dephasing and relaxation dependence on θ .

Intuitively, one can see why relaxation decreases as the ratio E_J/E_C increases, since the wavefunctions become increasingly localized and their overlap decreases [133]. However, fluctuations of the external flux will cause the periodic part of the potential to shift randomly and relative position of the localized wavefunctions will change randomly, leading to dephasing. On the other hand, in the insulating regime, wavefunctions spread over many minima and their overlap is significantly larger, so relaxation rates increase. Fluctuations of the external flux will not change appreciably the relative position of the lowest lying eigenstates, so dephasing is suppressed.

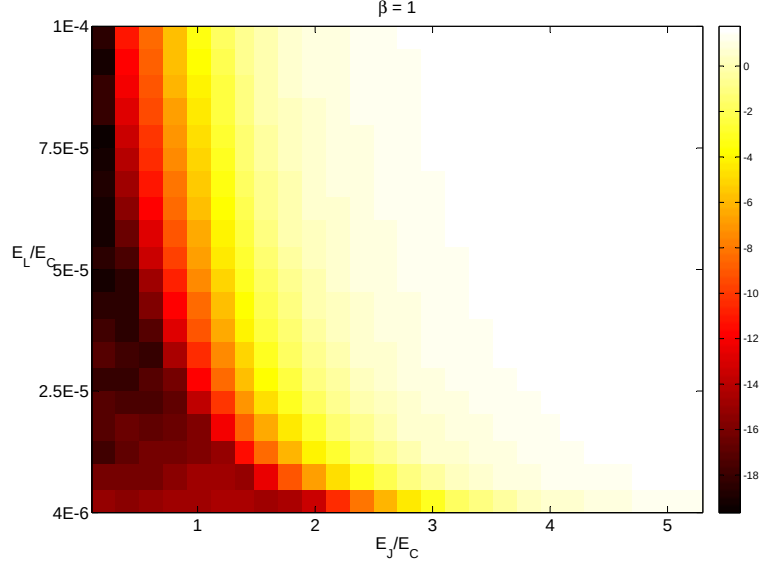


(a) $\log \Gamma_\phi$

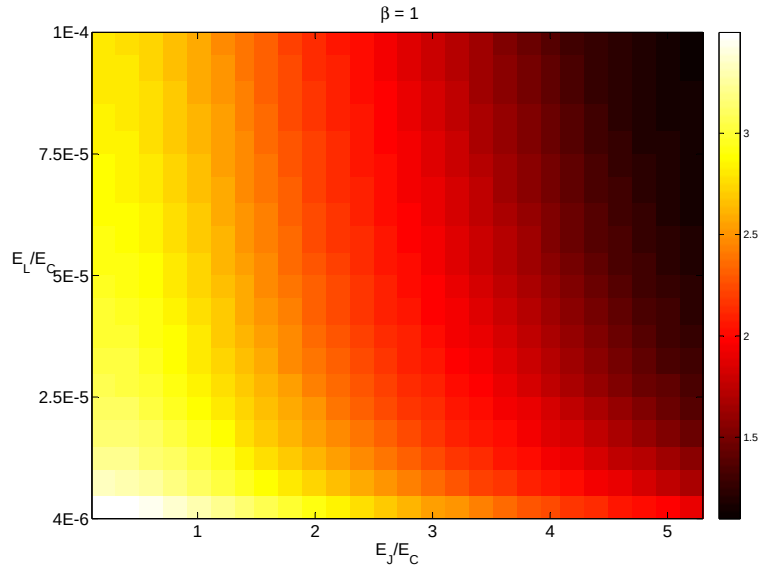


(b) $\log \Gamma_1$

Figure 6.13: Colormap of the logarithms of both dephasing and relaxation rates for different values of E_L/E_C and E_J/E_C , and averaged for different values of the external flux θ . The colorbar indicates the value of the exponents.

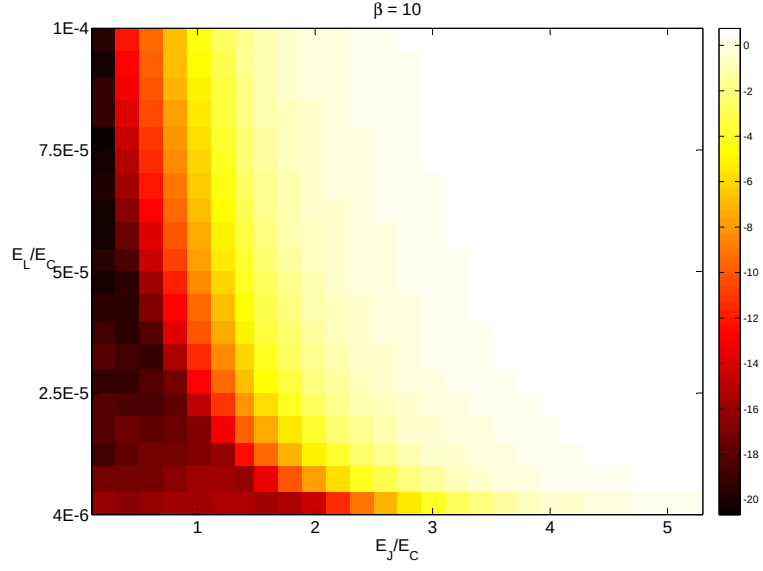


(a) $\log \Gamma_\phi$

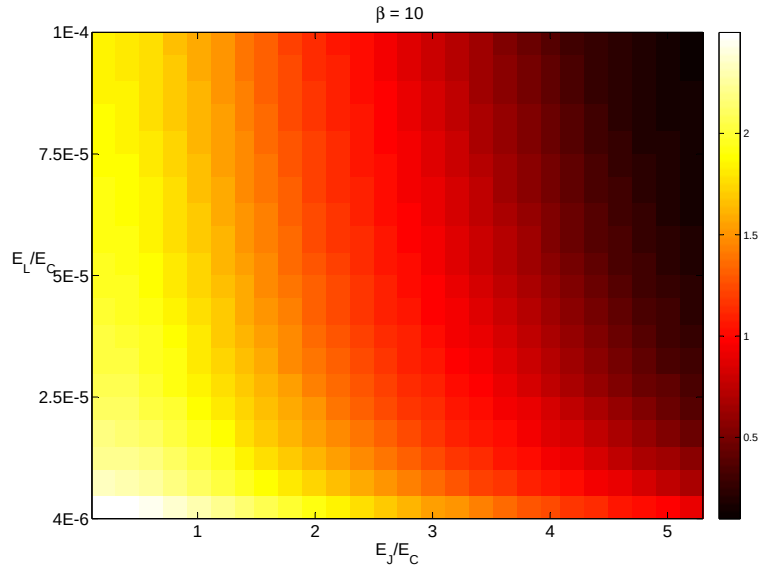


(b) $\log \Gamma_1$

Figure 6.14: Colormap of the logarithms of both dephasing and relaxation rates for different values of E_L/E_C and E_J/E_C , and averaged for different values of the external flux θ . The colorbar indicates the value of the exponents.

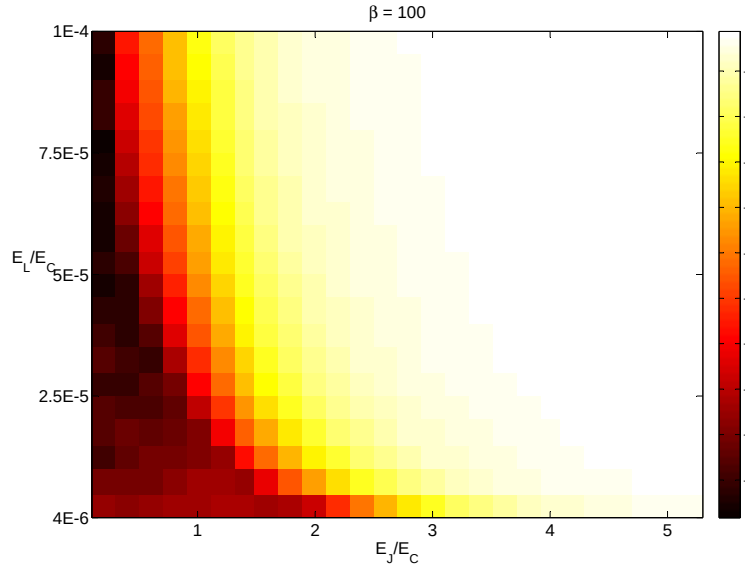


(a) $\log \Gamma_\phi$

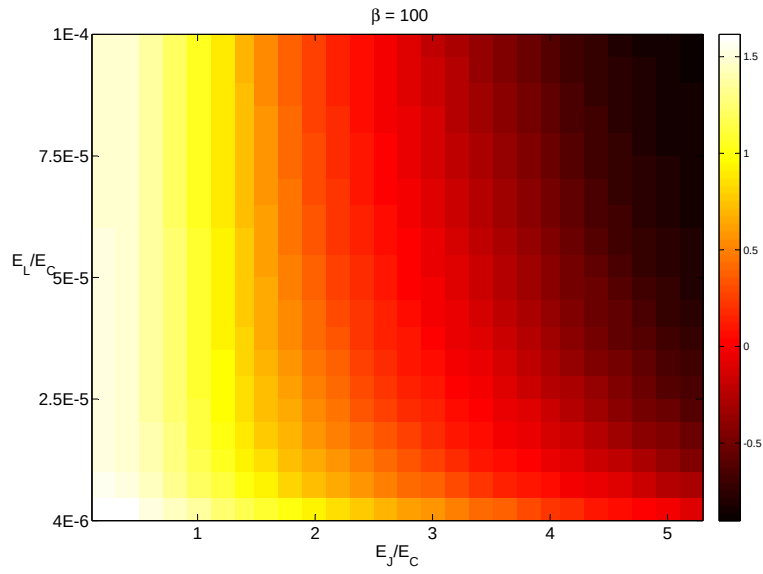


(b) $\log \Gamma_1$

Figure 6.15: Colormap of the logarithms of both dephasing and relaxation rates for different values of E_L/E_C and E_J/E_C , and averaged for different values of the external flux θ . The colorbar indicates the value of the exponents.

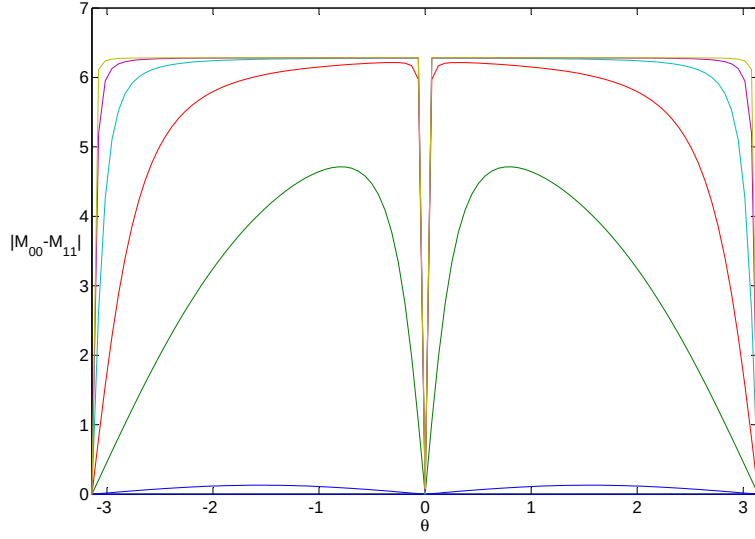


(a) $\log \Gamma_\phi$

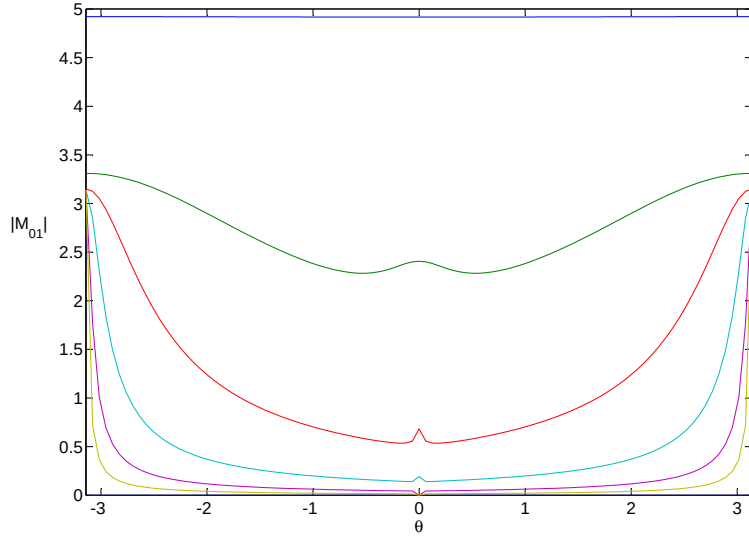


(b) $\log \Gamma_1$

Figure 6.16: Colormap of the logarithms of both dephasing and relaxation rates for different values of E_L/E_C and E_J/E_C , and averaged for different values of the external flux θ . The colorbar indicates the value of the exponents.



(a) Dephasing term



(b) Relaxation term

Figure 6.17: These two figures show the dependence of Γ_ϕ and Γ_1 on the external flux θ , for $E_L/E_C = 10^{-4}$. Different values of the Josephson energy are plotted in colors ranging from blue ($E_J/E_C = 1$) to yellow ($E_J/E_C = 6$).

6.6 Conclusion

We have studied the Fluxonium circuit from a slightly different perspective as that offered by the authors of Ref. [131, 130]. This shift is inspired by the observation by Kitaev that a circuit analogous to Fluxonium can be used to model the a topologically protected qubit [55].

We were interested in the transition from insulating to superconducting regime as the ratio E_J/E_C is increased. A description of Fluxonium in the insulating regime is given in terms of an effective capacitance picture, where the capacitance plays the analogous role of the mass of a quantum particle on a lattice in ϕ space. The breakdown of this approximation corresponds to the transition to the superconducting regime.

The anharmonicity introduced by the Josephson junctions allows to use the lowest-lying eigenstates to encode a qubit. An interesting feature of Fluxonium is that it is robust against dephasing due to low frequency noise in the insulating regime. The effective capacitance picture tells us that the circuit behaves as an LC circuit with effective capacitance C^* , whose eigenstates are exponentially insensitive to variations of the external flux.

We also showed that the pure dephasing rate due to coupling to the environment is reduced by several orders of magnitude when the circuit is in the insulating phase, as the ratio E_L/E_C is reduced, which is contrasted by the slight increase in relaxation rates. Thus, damping of the off-diagonal terms of the qubit is mainly caused by the relaxation process.

Although our results are somewhat inconclusive at the moment, they point to at least two interesting features of Fluxonium as a qubit, which little further calculations can help outline. First of them is that in a regime where decoherence is not fully dominated by relaxation processes, the dephasing times can be incremented by reducing the inductive energy of the device. This scenario can arise by reducing the coupling of the device to its electromagnetic environment so that pure dephasing becomes more important.

Another direction is to find a tradeoff between leakage errors and Pauli errors

as a function of E_L/E_C . Leakage errors are related to the anharmonicity of the system and Pauli errors to the speed at which gates can be implemented. The importance of performing very fast gates is crucial in order to reduce the error probability per gate. This is ultimately limited by the anharmonicity δ offered by the system, *i.e.* the energy difference between the first and the second transition energies. A driving field resonant with the first transtion will populate state $|2\rangle$ with a probability inversely proportional to the detuning between the field and the transition $|1\rangle \leftrightarrow |2\rangle$. This leakage probability has to be well below the threshold for loss errors [107, 99] to guarantee an error-free computation at the logical level.

Conclusions

In this thesis we surveyed the concept of fault tolerance in the context of quantum computation, and motivated the importance of design of fault-tolerant quantum computers. We identified two possible ways towards this goal.

One method is to fight errors at the *logical level* using error correcting techniques, that is, use a redundant number of physical systems in a clever way so that the logical information can be recovered even when faults in a small subset of them occur. Finding more efficient quantum error correcting codes and better measurement strategies is thus necessary in order to devise new architectures that need minimum overhead of physical qubits and encoding complexity, while maintaining the quantum character of the processed information.

On the other hand, one can address errors at the *physical level*, by refining the production process of physical systems reducing the error in one or more parameters of interest, or by designing physical systems in such a way that they are *inherently robust* against fluctuations of their physical parameters. It is highly desirable that fault tolerance is achieved at the hardware level, which will dramatically reduce the encoding complexity at the logical level.

The technical part of this thesis is roughly divided in two parts, corresponding to the two ways to fault tolerance explained above. This original work is contained

in three separate chapters in the second part of the thesis.

First, a derivation of thresholds for fault tolerance for two different architectures were obtained. Two thresholds were obtained for a version of the topological cluster state quantum computer using single photons and quantum dots as its components. Other thresholds, this time only for loss errors, were derived with more generality, without specifying the actual physical implementation.

Second, numerical analysis of a superconducting circuit was carried out. More precisely, we looked at its energy spectrum and at its eigenstates as functions of the superconducting phase. We characterized the circuit to a reasonable extent and our results agree with work done previously by other researchers. We then modeled its coupling to an environment and derived the decoherence rates. Our results point to the fact that reducing the inductive energy will result in an exponential decay of the dephasing times of a qubit stored in the two lowest-lying eigenstates of the circuit, while the relaxation rates will increase much more slowly.

Some further work we intend to consider is the design of new architectures that reduce certain requirements for fault tolerance, such as the classical post-processing times needed for error correction, and possibly a more efficient ratio between the logical qubits and the physical qubits at any given failure rate. Also, we are interested in studying physical mechanisms that provide protection in at least one error channel at the physical level. These are however future directions and, unfortunately, they cannot be fully addressed in this thesis, but we hope to work on them in the months to come.

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