

RESEARCH ARTICLE

Impact of truncation error and numerical scheme on the simulation of the early time growth of viscous fingering

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Summary

The truncation error associated with different numerical schemes (first order finite volume, second order finite difference, control volume finite element) and meshes (fixed Cartesian, fixed structured triangular, fixed unstructured triangular and dynamically adapting unstructured triangular) is quantified in terms of apparent longitudinal and transverse diffusivity in tracer displacements and in terms of the early time growth rate of immiscible viscous fingers. The change in apparent numerical longitudinal diffusivity with element size agrees well with the predictions of Taylor series analysis of truncation error but the apparent, numerical transverse diffusivity is much lower than the longitudinal diffusivity in all cases. Truncation error reduces the growth rate of immiscible viscous fingers for wavenumbers greater than 1 in all cases but does not affect the growth rate of higher wavenumber fingers as much as would be seen if capillary pressure were present. The dynamically adapting mesh in the control volume finite element model gave similar levels of truncation error to much more computationally intensive fine resolution fixed meshes, confirming that these approaches have the potential to significantly reduce the computational effort required to model viscous fingering.

KEYWORDS

advection-diffusion, error estimation, numerical dispersion, porous media, truncation error, viscous fingering

1 | INTRODUCTION

It has become commonplace to use numerical methods to solve the convection-diffusion equations for the many practical fluid flow problems that cannot be solved analytically. Porous media applications include the modelling of fluid flow in the subsurface for estimating the effectiveness of proposed carbon dioxide sequestration methods (Kopp et al¹), predicting the movement of ground water (Diersch and Kolditz²) and determining the performance of oil recovery schemes (Gerritsen and Durlofsky³). The most widely used methods are finite difference, finite volume, finite element and combinations thereof, in which the spatial domain of interest is discretised using a mesh of elements or grid-blocks and the partial derivative in time is expressed as the change in the property or properties of interest over a finite time interval. This reduces the partial differential equations into a set of algebraic equations that are amenable to solution on a computer.

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Of course, these numerical methods only give an approximate solution to the system of partial differential equations of interest as the discretisation results in a truncation error (Lantz,⁴ Fanchi⁵). In first order methods this truncation error appears as a numerical diffusion (eg Lantz⁴) or dispersion whose magnitude depends upon the resolution of the mesh and the size of the time step. It can therefore be reduced by refining the grid or mesh and/or reducing the time step but this significantly increases the computational expense in terms of both storage of data and operations performed to solve the equations. This has driven the development of higher order methods that aim to reduce the truncation error associated with a given discretisation (Bell and Shubin⁶).

The size of the truncation error can be estimated analytically for uniform meshes and miscible flows in finite difference methods but for more complex problems, with non-uniform meshes, varying time steps and higher order schemes, scientists and engineers will usually assess the impact of the truncation error on their solutions by performing a grid refinement study for the specific case they are studying. Papers describing new algorithms for various higher order methods will compare the accuracy of their proposed method with a standard first order scheme (eg Blunt and Rubin⁷) but, to date, there have been very few studies that compare the truncation error resulting from different numerical schemes and discretisations.

Viscous fingering or the Saffman-Taylor instability is a feature of porous media flows in which a less viscous fluid displaces a more viscous fluid (Homsy⁸), such as occurs in enhanced oil recovery processes and carbon dioxide sequestration. The number of fingers that form initially and their growth depends upon the mobility contrast between the fluids and the amount of physical diffusion or, in the case of immiscible displacements, capillary pressure. Higher levels of diffusion (capillary pressure) reduce the number of fingers that form and the growth rate of those fingers (eg Yortsos and Huang,⁹ Christie and Bond,¹⁰ Riaz and Tchelepi¹¹).

The dependence of viscous fingering on diffusion means that particular care has to be taken to ensure levels of numerical diffusion are much less than the physical diffusion in numerical simulations of these instabilities. This means that such studies need to use very fine meshes and/or higher order schemes. Some numerical studies have resorted to using spectral methods to minimise the impact of numerical errors (Tan and Homsy,¹² De Wit and Homsy,¹³ Zimmerman and Homsy¹⁴) whilst others have applied dynamic adaptive gridding to allow computational effort to be focused on resolving the fingering pattern (Edwards and Christie,¹⁵ Mulder and Gmelig Myeling,¹⁶ Adam et al¹⁷). Despite these restrictions, various authors (eg Christie and Jones,¹⁰ Christie et al,¹⁸ Davies et al¹⁹) have shown that finite difference schemes can predict the fingering patterns seen in physical experiments. Conversely this means that simulations of viscous fingering provide a convenient vehicle for investigating the impact of truncation error in different numerical schemes. Mostaghimi et al²⁰ have already shown that different numerical schemes and meshes can result in different viscous fingering patterns even when using the same fluid properties, but there appear to be few attempts to quantify and compare the truncation error between different schemes and approaches to meshing.

In this study, we perform various numerical experiments to quantify the level of numerical dispersion in different numerical schemes and meshes and its impact on the growth of viscous fingering. The numerical schemes considered are

1. First order finite volume (FOFV)
2. Higher order finite difference (HOFD)
3. Control volume finite element (CVFE)

The first two methods use structured meshes with rectangular grid blocks and we examine the impact of aligning flow with the grid as well as using a grid that is diagonal to flow. The CVFE approach can be applied using either a structured or an unstructured mesh, but in both cases the meshes are formed of triangular elements.

2 | THEORY

This section presents the underlying flow equations solved by the numerical schemes as well as the results from a perturbation analysis using those equations that can be used to estimate the impact of diffusion (numerical or physical) on the initial growth rate of viscous fingers.

2.1 | Flow equations

We consider two fluids flowing through a two-dimensional porous medium with the geometry illustrated in Figure 1. We assume flow is incompressible so the conservation of mass is given by

$$\nabla \cdot \mathbf{v} = 0 \quad (1)$$

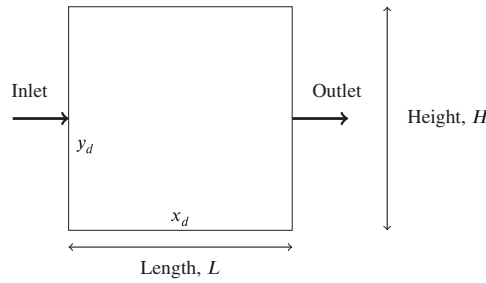


FIGURE 1 2D model used throughout this study. The inlet is along the left-hand boundary and the outlet along the right-hand side of the model respectively. No flow boundaries were imposed along the top and bottom edges

where $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ is the total interstitial velocity of the two fluids (subscript 1 denotes the displacing fluid and subscript 2 the displaced fluid).

The conservation equation for the displacing fluid is given by

$$\frac{1}{\phi} \frac{\partial S_1}{\partial t} + \nabla \cdot \mathbf{v} = 0 \quad (2)$$

where S_1 is the saturation of the displacing fluid, ϕ is the porosity and the interstitial velocities of the two fluids are given by the multi-fluid extension to Darcy's Law

$$\begin{aligned} \mathbf{v}_1 &= -K \frac{k_{r1}}{\mu_1} (\nabla p_2 - \nabla P_c) \\ \mathbf{v}_2 &= -K \frac{k_{r2}}{\mu_2} \nabla p_2 \end{aligned} \quad (3)$$

where we have ignored the effects of gravity. Here K is the absolute permeability k_{r1} is the relative permeability of fluid 1, μ_1 is the viscosity of fluid 1, p_2 is the pressure (measured in fluid 2, the original fluid saturating the porous medium) and $P_c = p_2 - p_1$, is the capillary pressure. If the two fluids are first contact miscible then Equations (2) and (3) become

$$\frac{1}{\phi} \frac{\partial c}{\partial t} + \mathbf{v} \cdot \nabla c = \mathbf{D} \nabla^2 c \quad (4)$$

and

$$\mathbf{v} = -\frac{k}{\mu} \nabla p \quad (5)$$

where c is the concentration (mass fraction) of the displacing fluid, μ is the viscosity, which depends upon concentration and $\mathbf{D}$ is the dispersion tensor, given by

$$\mathbf{D} = \begin{pmatrix} D_o + \alpha_L |\mathbf{v}| & 0 \\ 0 & D_o + \alpha_T |\mathbf{v}| \end{pmatrix}. \quad (6)$$

Here D_o is the molecular diffusivity in the porous medium, α_L is the longitudinal dispersion coefficient and α_T is the transverse dispersion coefficient. To simplify our analyses, we shall present our results in terms of dimensionless distance

$$x_d = \frac{x}{L}, \quad y_d = \frac{y}{L} \quad (7)$$

where L is the total length in x direction, and dimensionless time (pore volume injected, PVI), given by

$$t_d = \frac{Qt}{\phi AL} \quad (8)$$

where Q is the constant injection rate, t is the time elapsed since injection started and A is the cross-sectional area of the model, perpendicular to flow.

2.2 | Linear stability analysis

Miscible viscous fingering is a function of the viscosity ratio ($M = \mu_2/\mu_1$) between the fluids and the diffusion and dispersion whereas immiscible fingering is a function of the shock front mobility ratio ($M_{sf} = \lambda_{t,f}/\lambda_{t,0}$) and capillary

pressure. $\lambda_{t,f}$ is the total mobility at shock front saturation and $\lambda_{t,0}$ is the total mobility at the displacing fluid saturation ahead of the shock front where

$$\lambda_t = \frac{k_{r1}/\mu_1}{k_{r1}/\mu_1 + k_{r2}/\mu_2}$$

and the shock front saturation can be determined using the Welge construction²¹ and Buckley-Leverett analysis.²²

The initial behaviour of fingers in terms of the number that form and their growth rates can be estimated using linear stability analysis (Chuoque et al.,²³ Taylor and Saffman,²⁴ Gardner and Ypma,²⁵ Yortsos and Huang⁹). This involves writing all the parameters (S_w or c , $\mathbf{v}$, p) in the form

$$a = a_0 + a' \quad (9)$$

where $a_0(x)$ is the mean value of the quantity (averaged transverse to the flow. Here we assume the flow is in the x direction so transverse to flow is in the y direction) and $a'(x, y)$ is a fluctuation to the mean such that

$$\int_0^H a' dy = 0 \quad (10)$$

and is also so small that all non-linear terms can be neglected. Most authors then assume that all these perturbations take the form of a periodic fluctuation of a wavenumber k and search for solutions to the system of Equations (1)-(3) (immiscible flow) or (4)-(6) (miscible flow) of the form

$$a' = \Phi(x, t)e^{iky} \quad (11)$$

where $\Phi(x, t) = \phi(x, 0)e^{\omega t}$ the amplitude of the fluctuation that grows at a rate of ω .

In the absence of diffusion (miscible) or capillary pressure (immiscible) the growth rate of a finger ω increases with wavenumber k for miscible displacements according to (Christie and Bond¹⁰)

$$\omega = kv_0 \frac{M-1}{M+1} \quad (12)$$

and for immiscible displacements according to (Riaz and Tchelepi,²⁶ Yortsos and Huang⁹)

$$\omega = kv_0 \frac{f_{1,f}}{S_{1,f} - S_{1,c}} \frac{M_{sf} - 1}{M_{sf} + 1} \quad (13)$$

where v_0 is the mean interstitial velocity, $S_{1,f}$ is the displacing fluid saturation at the shock front that forms between displacing and displaced fluids in an immiscible displacement (Buckley and Leverett²²), $S_{1,c}$ is the displacing fluid saturation ahead of the shock front, $f_{1,f}$ is the fractional flow at shock front saturation given by

$$f_{1,f} = \frac{k_{r1}(S_{1,f})/\mu_1}{k_{r1}(S_{1,f})/\mu_1 + k_{r2}(S_{1,f})/\mu_2}.$$

The effect of physical diffusion (or capillary pressure) is to reduce the growth rate of higher wavenumber fingers to below the values given by Equations (12) or (13). This leads to the phenomenon of the 'most dangerous' wavenumber (the wavenumber with the highest growth rate) and the critical wavenumber (the wavenumber above which growth rate is negative) as shown schematically in Figure 2. In real physical systems no fingers can form with a wavenumber higher

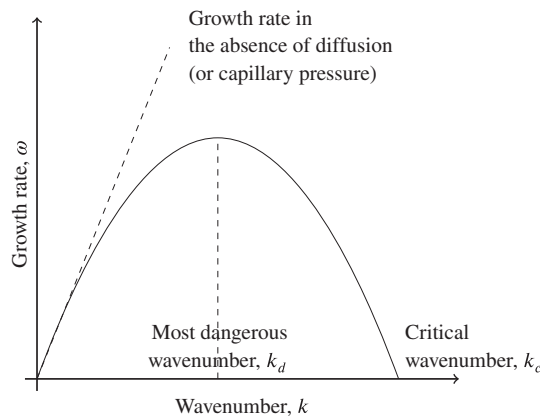


FIGURE 2 Schematic plot of initial growth rate of viscous fingers as a function of wavenumber, showing the growth rate in the absence of diffusion (or capillary pressure) and how diffusion (or capillary pressure) reduces the growth rate of higher wavenumber fingers

than this critical wavenumber. Numerical diffusion may mean that the growth rate of higher wavenumber fingers in numerical simulations is lower than expected physically or, in the worst case, may prevent any fingers growing at all.

3 | NUMERICAL SCHEMES

We aim to compare the truncation errors associated with three different numerical schemes by first quantifying the amount of transverse and longitudinal numerical diffusion found in those simulations and then comparing the early time growth rate of immiscible viscous fingers with the predictions of linear stability analysis (Equation 13).

The three numerical schemes are a first order finite volume scheme, a second order finite difference scheme and a first order control volume finite element scheme. We shall also compare the impact of using a 5 point and a 9 point stencil (Yanosik and McCracken²⁷) in the first order finite volume simulator. We will use a structured mesh with rectangular grid blocks in both the finite difference and finite volume schemes and compare the outcomes when the grid is aligned with the principal flow direction and when it is diagonal (ie at 45°) to the principal flow direction. The control volume finite element simulation uses a mesh with triangular elements and we shall compare the outcomes when using a fixed structured mesh, a fixed unstructured mesh and dynamically adapting unstructured mesh (that is more refined where saturation changes most quickly, Adam et al¹⁷).

The commercial black-oil simulator ECLIPSE 100 (Schlumberger²⁸) was used for our first order finite volume (FOFV) simulations. It employs a standard black-oil formulation using either a 5 point or 9 point stencil for calculating the pressure gradient with an option to use either an IMPES (implicit pressure, explicit saturation) or a fully implicit solver. In this study we used the fully implicit solver option. The 9 point scheme is based on that first described by Yanosik and McCracken²⁷ and reduces the grid orientation error in regular, orthogonal, Cartesian grids such as used in this work. The numerical approximation to the Laplacian operator in two dimensions is written as

$$\nabla^2 u_{i,j} = \frac{4(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}) + (u_{i+1,j-1} + u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1}) - 20u_{i,j}}{6\Delta x^2} \quad (14)$$

The diagonal transmissibilities are chosen, using the method described by Shiralkar and Stephenson,²⁹ to ensure that a) a pressure gradient in one direction does not produce flow perpendicular to that direction and b) the total x and y direction transmissibilities are preserved.

The second simulator we used was the second order, finite difference (HOFD), IMPES simulator developed by Christie and Bond¹⁰ and Christie.³⁰ This uses two-point, upstream weighting to give it second order accuracy in combination with flux corrected transport (Zalesak³¹) to prevent unphysical oscillations around the shock front. Its ability to predict viscous fingering in miscible displacements has previously been validated by comparison with experimental results by Christie and Bond,¹⁰ Christie et al,¹⁸ and Davies et al,¹⁹ amongst others. Only the 5 point stencil is available for this simulator.

The third simulator used in this study is the control volume finite element (CVFE) simulator described by Jackson et al³² and Gomes et al.³³ In this formulation, pressure, permeability, porosity and viscosity are represented using finite elements but saturation and relative permeability are described on control volumes. The use of control volumes for saturation helps ensure mass conservation. We used the $P_0(DG) - P_2$ (linear discontinuous velocity, quadratic pressure) element in this study. All equations were solved implicitly. We compared the results from this simulator when using a fixed, structured mesh, a fixed, unstructured mesh and a dynamic, adaptive, unstructured mesh. The adaptive mesh was optimised using the L_2 norm of the local interpolation error in saturation, estimated from the Hessian matrix. The mesh adaptivity at each time step proceeds by minimising a functional I , that measures the mesh quality using this interpolation error

$$I = \sum_{i \in \text{edges}} (v_i^T M v_i - 1)^2 \quad (15)$$

where

$$M_{ij} = (\det H)^{\frac{-1}{2\gamma+\delta}} \frac{|H_{ij}|}{\epsilon} \quad (16)$$

H is the Hessian, v_i are vectors describing the element edge lengths on the finite element mesh, ϵ is a normalisation constant that specifies the desired tolerance, γ is the polynomial degree of interpolation and δ is the number of dimensions in the problem. This minimisation is achieved by sequentially splitting, joining and/or shifting element nodes using the method described by Pain et al³⁴ and Mostaghimi et al³⁵ until the interpolation error is a minimum. The higher order and conservative, finite element and control-volume Galerkin projection method described by Adam et al¹⁷ was used to interpolate data from the old and the new meshes.

TABLE 1 Fluid properties for the immiscible displacement simulations

Parameter	Value
Fluid viscosity	
Fluid 2	$\mu_2 = 303$
Fluid 1	$\mu_1 = 1$
Initial condition, $t_d = 0$	
$x_d < 0.1$	$S_1 = 1 - S_2$
$x_d > 0.1$	$S_1 = S_{wc} + \frac{A_2}{2}(1 - S_{wc} - S_{or})(1 + \cos(2\pi k y_d))$
$t_d > 0$	$S_1 = 1$

TABLE 2 Summary of data used for the immiscible displacement simulations

Parameter	Maximum number of grid blocks/no. of elements	Range of element lengths (in flow direction)	Courant number
FOFV	400×400		1.0
HOFD	400×400	0.0025 - 0.005	0.4
CVFE	50000		~ 0.05

4 | METHOD

We evaluated the effect of truncation error in the different simulators first by performing a series of unit mobility ratio, first contact miscible numerical experiments (without modelling physical diffusion) and then calculating the level of longitudinal and transverse diffusion in each case for different mesh resolutions. We then illustrate how these levels of numerical error influence the pattern of viscous fingering that occurs for an adverse mobility ratio, first contact miscible displacement. Finally, we consider an unstable immiscible displacement (the fluid properties are given in Table 1) and compare the early time growth rate predicted as a function of grid resolution, mesh type and numerical scheme.

4.1 | Test cases

All simulations were performed in the square, 2D model system illustrated in Figure 1. The system was initially filled with fluid 2. Fluid 1 was then injected at constant rate Q along the left-hand face of the model. Fluid was produced at the same constant rate along the right-hand face. The top and bottom edges of the model were no-flow boundaries. Gravity was eliminated by giving the injected fluid and the displaced fluid equal densities. The permeability was constant and homogeneous in all cases. No physical diffusion (or capillary pressure in immiscible displacements) was input into any simulation so all apparent diffusion was due to truncation errors.

Numerical diffusion (longitudinal and transverse) was evaluated as a function of grid resolution by performing a series of runs using different mesh resolutions, mesh types and orientations. The finite volume and finite difference mesh simulations used Cartesian meshes with square grid block and the grid resolutions listed in Table 2. We considered both diagonal and parallel meshes as well as comparing the impact of using a 5 point versus a 9 point stencil to calculate the pressure differences in the FOFV simulations. The simulations using the CVFE simulator used triangular mesh elements. We compared the outcomes from using a fixed structured mesh, a fixed unstructured mesh and a dynamic adaptive mesh that used the method described by Adam et al.¹⁷ Figure 3 shows examples of these meshes. The total number of elements used are also listed in Table 2.

We used a CFL number of 1.0 for the first order finite volume simulations and a CFL number of 0.4 for the second order finite difference simulations. The CVFE simulations used a CFL number of 0.05. The CFL or Courant number is defined as

$$\text{CFL} = \frac{\Delta t_d}{\Delta x_d} \quad (17)$$

Truncation error is a function of time step size as well as mesh resolution: in a fully implicit scheme the larger the time step the larger the numerical diffusion (Lantz⁴) whereas numerical diffusion increases the further the time step is from the CFL limit in a scheme that is explicit in saturation.

The FOFV and CVFE simulators used here were originally developed for immiscible applications. To enable these to simulate miscible displacements, we modified the input relative permeabilities as described by Lantz.³⁶ In this case the

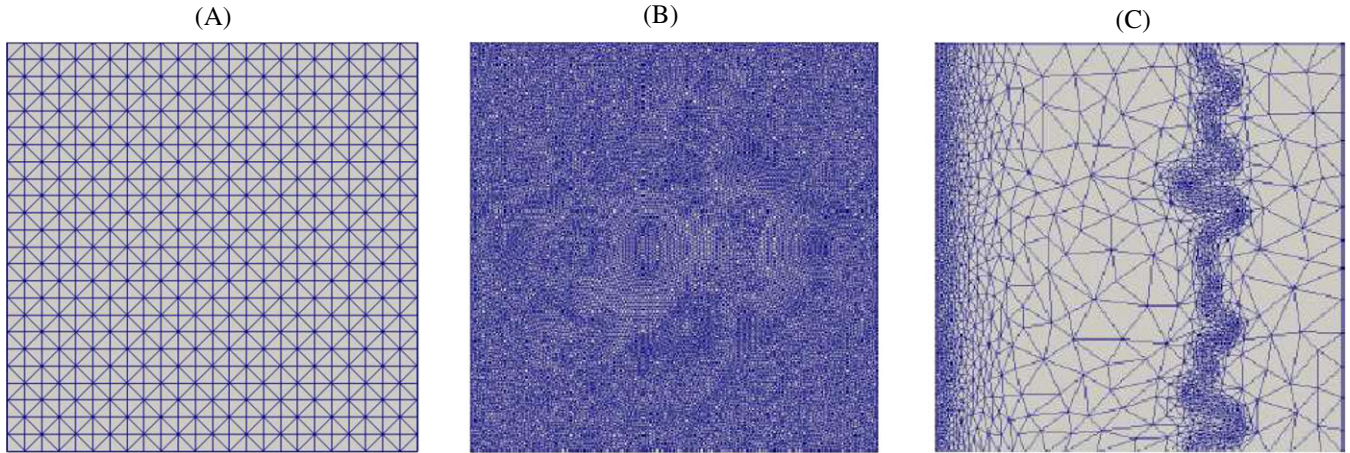


FIGURE 3 Illustrations of the different meshes used in the CVFE simulations. A, fixed structured mesh; B, fixed unstructured mesh; C, unstructured; dynamically adapting mesh (adapting to changes in displacing fluid saturation) [Colour figure can be viewed at wileyonlinelibrary.com]

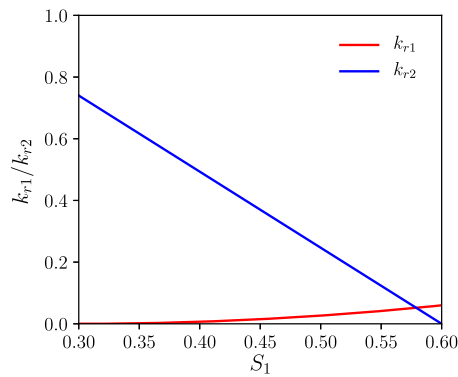


FIGURE 4 Relative permeability curves used in the immiscible viscous fingering simulations [Colour figure can be viewed at wileyonlinelibrary.com]

relative permeability curves become

$$k_{r1} = \frac{\mu_1 S_1}{\mu(S_1)} \quad (18)$$

$$k_{r2} = \frac{\mu_2 S_2}{\mu(S_2)} \quad (19)$$

where

$$\mu(S_1) = \left(\frac{S_1}{\mu^{\frac{1}{4}}} + \frac{1 - S_1}{\mu^{\frac{1}{4}}} \right)^{-4} \quad (20)$$

k_{r1} and k_{r2} are straight lines if $\mu_1 = \mu_2$. For the immiscible displacement simulations, we used the relative permeability curves shown in Figure 4. These represent a water-wet system, and are described using the Corey correlation³⁷

$$k_{r1} = 0.06 \left(\frac{S_1 - S_{or}}{1 - S_{wc} - S_{or}} \right)^2$$

$$k_{r2} = 0.74 \left(\frac{1 - S_1 - S_{or}}{1 - S_{wc} - S_{or}} \right)$$

S_{wc} was set to be 0.3 and S_{or} was 0.4. Capillary pressure was not input as we were not modelling physical diffusion and dispersion.

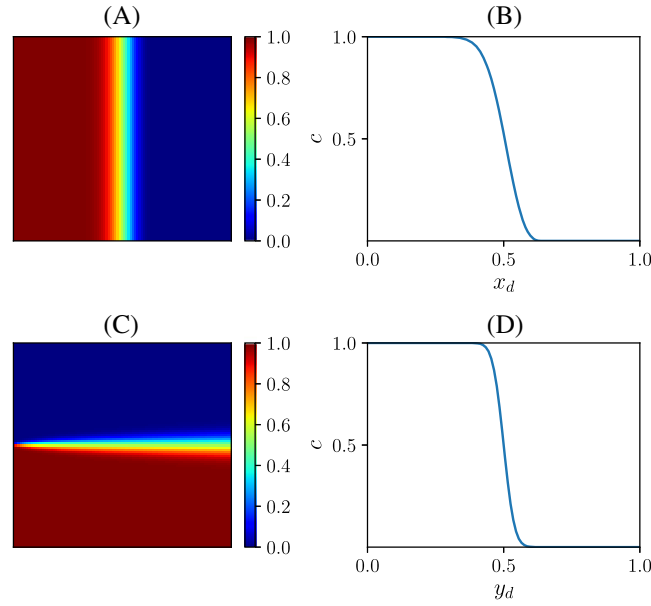


FIGURE 5 Numerical experiments used to calculate the values of longitudinal and transverse numerical dispersions. A, An example of concentration map used to determine longitudinal numerical dispersion; B, Transversely averaged concentration obtained from A; C, An example of concentration map used to determine transverse numerical dispersion; D, Concentration c at $x_d = 0.5$, as a function of vertical distance y_d obtained from C [Colour figure can be viewed at wileyonlinelibrary.com]

4.2 | Calculation of numerical diffusivity

We first performed a series of numerical experiments to quantify the amount of longitudinal and transverse numerical diffusion as a function of numerical scheme, mesh resolution and mesh orientation quantify the longitudinal and transverse dispersion values. These experiments are identical to laboratory and numerical experiments performed by many researchers including Taylor,²⁴ Ogata and Banks,³⁸ and Perkins and Johnston³⁹ for longitudinal dispersion; and Hiby⁴⁰ and Blackwell⁴¹ for transverse dispersion.

4.2.1 | Longitudinal dispersion

The model was initially fully saturated with oil, and at $t_d > 0$, we injected miscible solvent of equal viscosity into the left-hand boundary. This results in a stable but dispersed front as shown in Figure 5(b).

The analytical solution to Equation (4) for a 1D displacement in the presence of diffusion is given by

$$c = 0.5 \left(\operatorname{erfc} \left(\frac{x_d - t_d}{2\sqrt{D_{L,\text{num}}x_d}} \right) \right) \quad (21)$$

where we have replaced D with $D_{L,\text{num}}$ to emphasise that the diffusivity in our simulations is solely due to the longitudinal numerical dispersion.

Inspection of Equation (21) indicates that length of the dispersed front grows proportionally to $\sqrt{t_d}$, which is the main characteristic of dispersive-dominant flow. The numerical dispersivity $D_{L,\text{num}}$ can be found by fitting the solvent concentration c from simulation with (21) or alternatively, as demonstrated by Taylor,²⁴ by calculating the dispersivity directly from the displacement concentration contours

$$D_{L,\text{num}} = \frac{1}{t_d} \left(\frac{x_{c=0.9} - x_{c=0.1}}{3.625} \right)^2 \quad (22)$$

where $x_{c=0.1}$ and $x_{c=0.9}$ are the locations where the concentration values are $c = 0.1$ and $c = 0.9$ respectively. In this study we fitted the simulated concentration profile with (21) to find $D_{L,\text{num}}$ such that the sum of squared error was minimised.

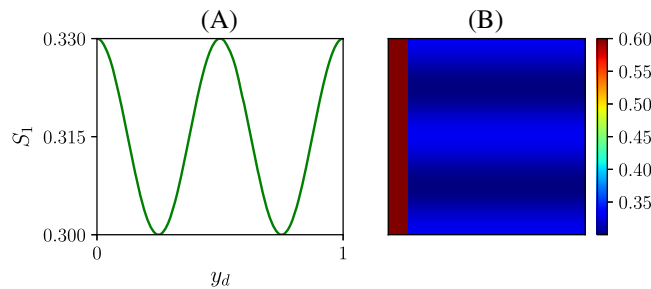


FIGURE 6 The initial conditions used to investigate the growth rate of immiscible viscous fingering. A, Initial saturation perturbation, shown here for wavenumber $N_y = 2$; B, Initial saturation map for wavenumber $N_y = 2$ [Colour figure can be viewed at wileyonlinelibrary.com]

4.2.2 | Transverse dispersion

To calculate the numerical transverse diffusivity, we continuously injected oil across the upper half of the left-hand face of the model whilst injecting solvent of equal viscosity at the same rate across the bottom half of the left-hand face, as shown in Figure 5(c). The solvent and oil were injected until all the initial reservoir fluid was displaced and a steady state distribution of oil and solvent was reached. Numerical transverse diffusion, $D_{T,num}$, mixes oil and solvent across the midline of the model and the width of this mixing zone increases from left to right.

The vertical concentration profile taken at distance $x_d = 0.5$ from the inlet is plotted in Figure 5(d) which shows a smooth transition from $c = 1$ to $c = 0$ rather than a step change because of the transverse mixing between the two fluids. This curve can be used to find $D_{T,num}$ by fitting it to the following analytical solution (Hiby⁴⁰)

$$c = 0.5 \left(\operatorname{erf} \left(\frac{y_d + 0.5}{2\sqrt{D_{T,num}x_d}} \right) + \operatorname{erf} \left(\frac{y_d - 0.5}{2\sqrt{D_{T,num}x_d}} \right) \right) \quad (23)$$

4.3 | Calculation of growth rates

To calculate the initial growth rates of immiscible viscous fingers the initial saturation field was perturbed as follows. The first 10% of the model to the right of the inlet was fully saturated with water, i.e.

$$S_1 = 1 - S_{or}; \quad x_d < 0.1. \quad (24)$$

The water saturation in the rest of the model was initialised according to the chosen wavenumber k using

$$S_1 = S_{wc} + \frac{A_s}{2}(1 - S_{wc} - S_{or})(1 + \cos(2\pi k y_d)); \quad x_d \geq 0.1 \quad (25)$$

where A_s is a scaling factor that changes the amplitude of the cosine wave (chosen to be 0.1 in this study). The saturation curve for a wavenumber k of 2 is shown in Figure 6(a) as an example, while Figure 6(b) illustrates the corresponding water saturation map. We chose to initiate the fingering part way into the model so that the early time growth of the fingers was not influenced by the inlet boundary.

We were only interested in the early time growth rate, so the simulations were only run for 0.05 PVI. To calculate the growth rate, we extracted the contour line of what would be the shock front saturation ($S_{1,f} = 0.36$) if there were no fingers. We then found the finger amplitude L_{mix} by determining the amplitude of the contour line as illustrated in Figure 7(b) at different times. The growth rate σ was then calculated using

$$\sigma = \frac{L_{mix}}{dt_d} ((1 - S_{or}) - S_{w,p}) \quad (26)$$

where L_{mix}/dt_d is the gradient of the L_{mix} plotted against t_d , and $S_{w,p}$ is the average value of the perturbation which is 0.315 (see Figure 6(a)).

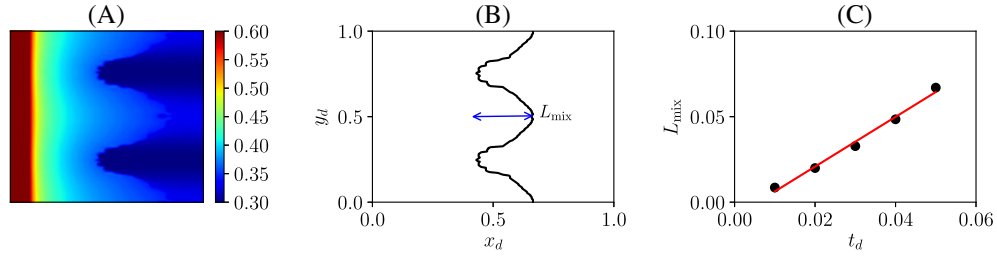


FIGURE 7 Illustration of growth rate calculation in miscible displacement for wavenumber of 2. A, Saturation map at $t_d = 0.05$; B, The mixing length L_{mix} is found from the contour line of S_w of 0.36 at various times; C, The growth rate is taken as the gradient of L_{mix} versus t_d plot [Colour figure can be viewed at wileyonlinelibrary.com]

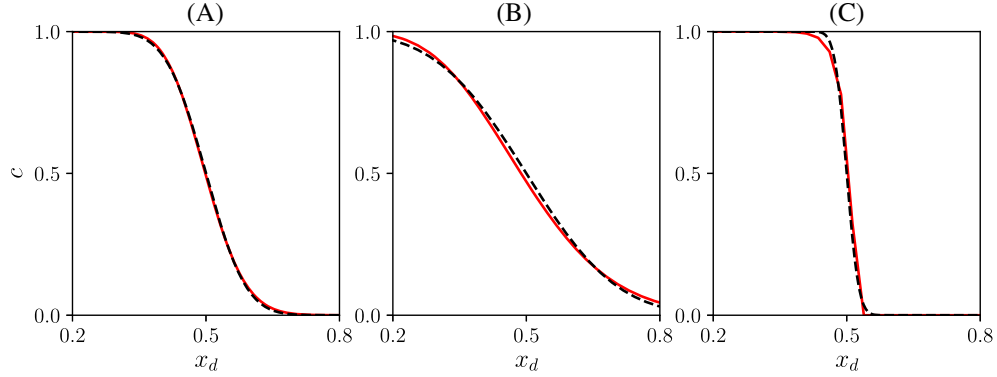


FIGURE 8 Concentration profiles from numerical simulations fitted with the analytical solution by minimising the sum of squared error. A, first-order finite (FOFV) with relatively high resolution ($\Delta x_d = 0.005$) is virtually identical to (21), indicating higher-order terms are negligible; B, Both FOFV with relatively large grid ($\Delta x_d = 0.025$); C, Higher-order finite difference ($\Delta x_d = 0.01$) deviate from the error function as the contribution from high-order terms is more significant [Colour figure can be viewed at wileyonlinelibrary.com]

5 | RESULTS AND DISCUSSION

5.1 | Numerical diffusion

Figure 8 compares the concentration profiles obtained from the FOFV and the HOFD simulations with their best fit profiles obtained from Equation (21). There is a good match between the profile obtained from the fine grid FOFV simulation and the analytical curve, (Figure 8(a), 200×200 mesh with $\Delta x_d = 1/200 = 0.005$), because the truncation error is dominated by the second order terms, as expected from the analysis of Lantz.⁴ We notice, however, that the agreement between simulation and fitted analytical is less good for the FOFV simulation with a coarse mesh (Figure 8(b), $\Delta x_d = \Delta t_d = 0.025$) and the HOFD simulation with $\Delta x_d = 0.01$. This is because the contribution from the higher order truncation errors are more significant in these cases.

Figure 9 shows the longitudinal and transverse dispersivities calculated from FOFV, HOFD and CVFE simulators as a function of grid block width and height, Δx_d and Δy_d . For the CVFE simulator, Δx_d and Δy_d are the minimum element edge lengths in the horizontal and vertical direction respectively.

Both the FOFV and CVFE methods give longitudinal numerical dispersivities that vary linearly with mesh element size, regardless of the grid orientation and stencil (5 or 9). The gradient is 1 for the FOFV method and 0.525 for the CVFE method. These values are consistent with the analysis based on a Taylor series expansion given by Lantz⁴ that predicts that in a first-order, implicit scheme

$$D_{L,\text{num}} = \frac{\Delta x_d + \Delta t_d}{2} = \frac{\Delta x_d}{2}(1 + \text{CFL}). \quad (27)$$

This is true for all mesh orientations and discretisation schemes except for simulations using both the diagonal grid with a 9 point scheme, which seem to be slightly more dispersive than expected. The numerical longitudinal dispersivities are lower in the HOFD simulations, as would be expected for a second order scheme. A best fit to the data gives a relationship $D_{L,\text{num}} = 1.16 \Delta x_d^2$.

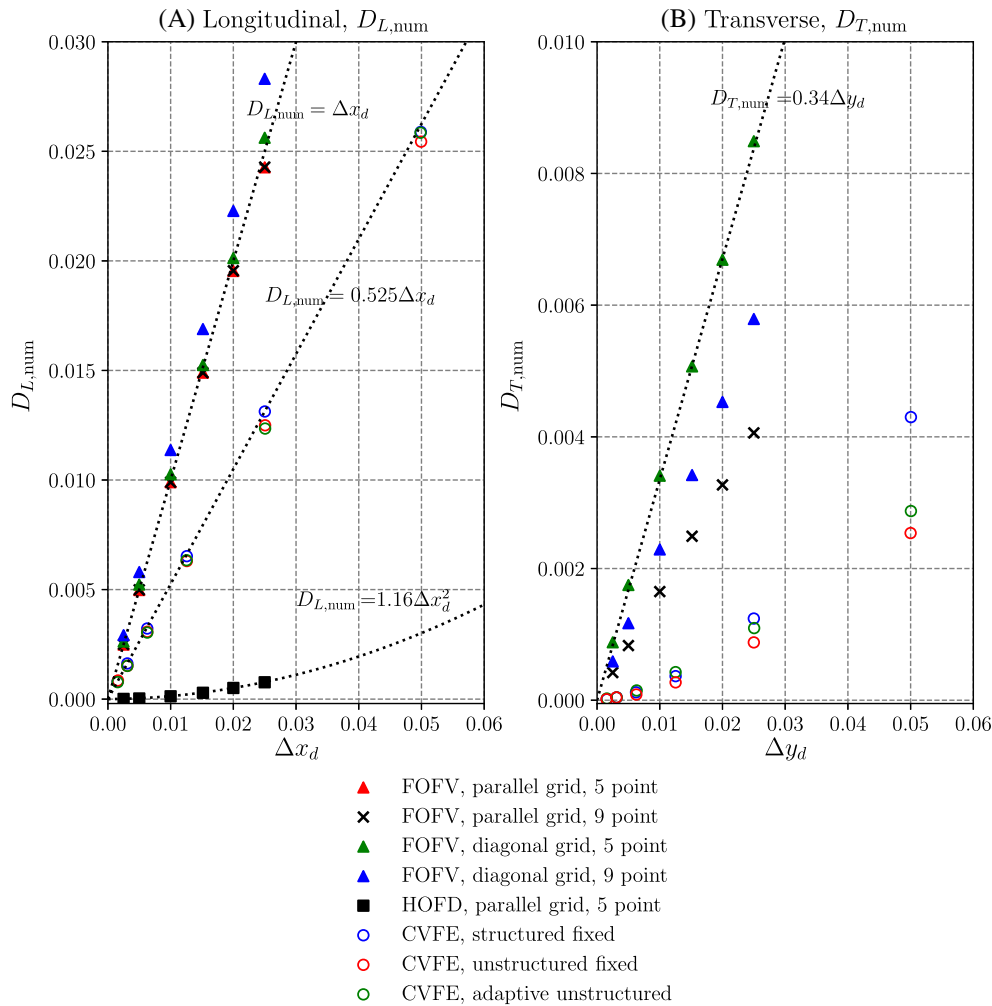


FIGURE 9 Longitudinal and transverse numerical dispersion values, $D_{L,num}$ and $D_{T,num}$, as a function of spatial resolution measured from various numerical schemes. A, $D_{L,num}$ for FOFV are in agreement with the analytical solution, although the dispersion in the diagonal grid simulations with a 9 point stencil are slightly higher. The results for the CVFE simulations with different mesh configurations are almost identical at $D_{L,num} = 0.525\Delta y_d$; B, For FOFV, $D_{T,num}$ also varies linearly with element length and is around 1/6 to 1/3 of the corresponding $D_{L,num}$ values. The CVFE simulations produce a non-linear relationship between $D_{T,num}$ and Δy_d [Colour figure can be viewed at wileyonlinelibrary.com]

The numerical transverse dispersivities, $D_{T,num}$, are systematically smaller than the longitudinal dispersivities in all cases. In the HOFD and FOFV simulations with a parallel grid and a 5 point stencil $D_{T,num} = 0$ for all grid sizes as the vertical velocity component is zero everywhere. In the CVFE simulations and the other FOFV simulations the numerical transverse diffusivity varies linearly with mesh size and are between 1/6 and 1/3 of the corresponding $D_{L,num}$. As for the numerical longitudinal dispersivity, the numerical transverse dispersivity is highest in the FOFV scheme when a diagonal grid with a 5 point stencil is used. The CVFE simulations using the dynamic adaptive unstructured mesh gave transverse dispersivities that were only slightly higher than those obtained from the fixed mesh although these simulations were significantly less expensive to run because the dynamic adaptivity reduced the total number of cells used.

These results suggest that miscible viscous fingering in a line drive may be best modelled using a finite volume or finite difference method using a grid parallel to flow as this minimises the level of numerical transverse diffusivity. Tan and Homsy⁴² have shown that lower levels of physical transverse diffusivity (compared to longitudinal diffusivity) results in more fingers being formed initially and higher growth rates. Scientists and engineers simulating viscous fingering will typically aim to choose a mesh resolution such that the input physical diffusion dominates over the numerical diffusion. They should also choose the smallest possible CFL number if their simulator is fully implicit. Reducing the amount of numerical transverse diffusivity in a simulation by choosing a parallel grid may mean that a coarser mesh resolution may be needed with a consequent saving in computational effort. We note, however, that in simulations of non-linear viscous

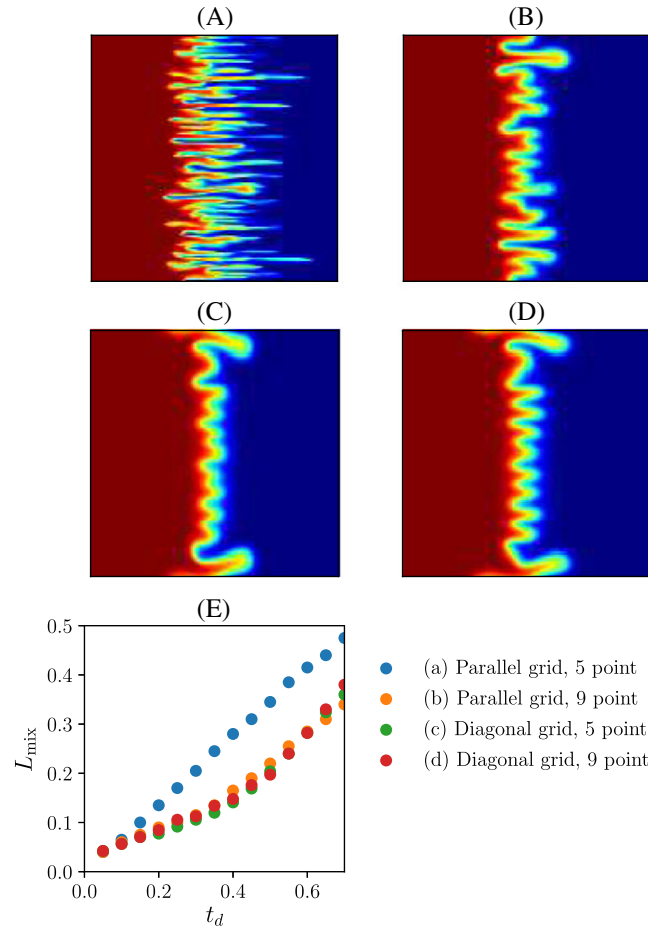


FIGURE 10 Miscible fingering for various schemes using a FOFV scheme with mobility ratio of $M = 10$ and $\Delta x_d \approx \Delta y_d \approx 0.005$ at $t_d = 0.5$. A, Parallel grid with 5 point stencil; B, Parallel grid with 9 point stencil; C, Diagonal grid with 5 point stencil; D, Diagonal grid with 9 point stencil; E, Mixing lengths L_{mix} ($x_{c=0.1} - x_{c=0.9}$) [Colour figure can be viewed at wileyonlinelibrary.com]

fingering the flow field will not always be parallel to the mesh and thus there will be non-zero transverse numerical diffusion even when using a parallel grid. It may thus be sensible to assume that the level of transverse numerical diffusion is more similar to that seen when using a diagonal grid.

Figure 10 shows how the different levels of numerical dispersivity associated with different mesh orientations and stencil in the FOFV scheme affects the non-linear fingering pattern that is predicted. These simulations used a mesh resolution such that $\Delta x_d \approx \Delta y_d \approx 0.005$. The fluids had a mobility ratio of $M = 10$. Fingers were triggered using random permeability field at the inlet. The mixing lengths L_{mix} here were taken as the distance $x_{c=0.1} - x_{c=0.9}$.

The simulations confirm the analyses of Tan and Homsy⁴² that transverse dispersivity controls the growth of viscous fingering as we see different wavelengths of fingering corresponding to the levels of transverse numerical dispersivity measured (Figure 9). The simulation using the parallel grid with the 5 point stencil predicts the largest number of fingers with the highest growth rate. This is consistent with this simulation having the lowest level of transverse numerical dispersion. The simulations with the diagonal grid have the fewest fingers with the lowest growth rate as these have the highest levels of numerical transverse dispersion. It is interesting to note the growth of an extra, large finger adjacent to the top and bottom boundaries in these simulations. These grow due to the no-flow boundaries (see Yang and Yortsos⁴³). The simulation using the parallel grid and the 9 point scheme predicts slightly more fingers than when a diagonal grid is used, consistent with the results shown in Figure 9.

5.2 | Immiscible viscous fingering

Having shown how transverse numerical dispersion influences the miscible fingering pattern predicted by numerical simulations we now investigate its influence on the initial growth rate of immiscible viscous fingers.

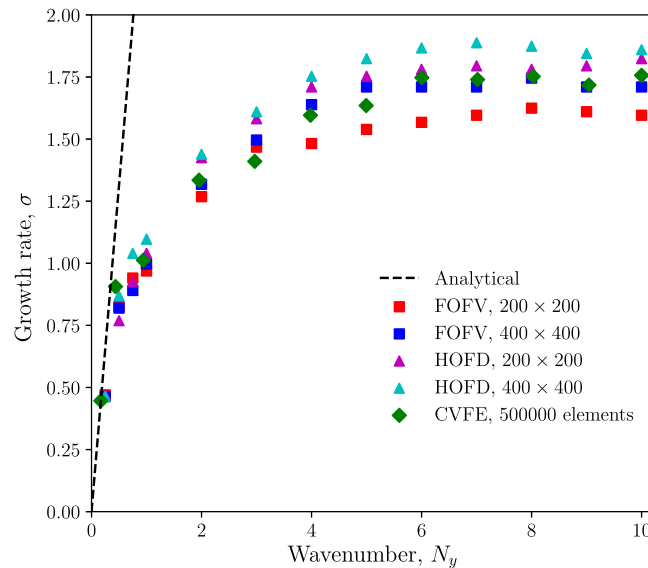


FIGURE 11 The growth rates in FOFV, HOFD and CVFE for various wavenumbers N_y . The dotted line shows the analytical solution calculated using Equation (13) taken from Yortsos and Huang⁹ [Colour figure can be viewed at wileyonlinelibrary.com]

Figure 11 shows the growth rate observed in FOFV, HOFD and CVFE simulations as a function of finger wavenumber initiated in the simulations. We compare the growth rates against the analytical solution for the case without capillarity (Equation (13)). In our simulations $M_{sf} = 1.8$, $v_0 = 1$ and $f_{1,f}/(S_{1,f} - S_{1,c}) = 9.2$. As expected there is good agreement between the analytical growth rates and those predicted by all the simulations for low wavenumbers, however the early time growth rate seen in the simulations is lower than that predicted for wavenumbers greater than 1. The wavenumber at which the simulated growth rates diverge from the analytical solution is similar for all numerical methods, meshes and stencils. It is interesting to see that the initial growth rate is then approximately constant for wavenumbers greater than 4. Based on linear stability analysis we would expect the growth rate to reduce at higher wavenumbers if numerical dispersivity behaves in the same way as capillary pressure. This apparent stabilisation may be because we did not consider wavenumbers higher than 10 in our simulations as these could not be properly resolved by the meshes we used for this study or alternatively this may be because, unlike capillary pressure, the truncation error does not vary with saturation. The FOFV simulations, using a 200×200 mesh, predict the lowest growth rate at larger wavenumbers, as we would expect as these have the highest levels of transverse numerical dispersivity. Doubling the mesh resolution results in higher predicted growth rates similar to those predicted by the HOFD simulations using a 200×200 mesh. This is consistent with the second order accuracy of this simulator.

These results confirm that very fine mesh resolutions are needed to avoid results being adversely affected by truncation error. Alternatively the use of dynamic adaptive meshing produce similar results at a fraction of the computational cost. These results further suggest that numerical diffusion cannot be used to approximate the effects of capillary pressure on viscous fingering. The initial growth rate predicted by all the numerical schemes tested deviated from the analytical solution at wavenumber of ~ 1 , despite the different levels of truncation error quantified in the previous subsection. Even the growth rate predicted by the coarser grid, first order, finite volume method was only $\sim 20\%$ lower than the best performing higher order finite difference solution on a fine grid. In addition, the growth rate of higher wavenumber fingers seemed constant for wavenumbers greater than 4 whereas the growth rate would decrease with increasing wavenumber if there was capillary pressure.

6 | CONCLUSIONS

We have compared the numerical error associated with three different numerical schemes (first-order finite volume, second-order finite difference, and CVFE) and different meshes (structured Cartesian, structured triangular, unstructured triangular, and dynamically adapting, unstructured triangular). This has been quantified as a function of mesh resolution in terms of the apparent longitudinal and transverse diffusivity seen in unit mobility ratio miscible displacements as well as in terms of the early time growth rate of immiscible viscous fingers.

Overall these results indicate that, for the numerical schemes and meshes tested, truncation error can, to first order, be characterised as a numerical diffusivity. Furthermore, these suggest that these schemes can be used to predict viscous fingering behavior provided that the input physical diffusivity or capillary pressure is larger than the underlying numerical diffusivity. For miscible displacements, a Taylor series analysis of truncation error can provide a good estimate of the longitudinal numerical diffusivity in finite difference and finite volume schemes. It seems that the transverse numerical diffusivity is typically much smaller than the longitudinal numerical diffusivity. It is also interesting to see that the level of numerical error associated with dynamic adaptive meshing is very similar to that seen in a fixed unstructured mesh with the same average element size but involving significantly more computational effort.

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