GAUGE THEORY AND BREAK-UP OF MAGNETIC DOMAINS

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ABSTRACT

This thesis investigates whether magnetic systems can support type II behaviour, that is whether the Gibbs free energy of their domain walls can be negative and their domain structure does break-up. In type II superconductivity the domain break-up results in the Abrikosov mixed state which is an intrinsic thermodynamic state where superconducting and normal phase co-exist in a structure which is independent of the shape and the magnetic or thermal history of the sample.

Superconductivity can be written as a U(1) gauge theory, in which the matter field is the superconductivity order parameter and the gauge field is the electromagnetic potential.

The magnetic systems which appear experimentally to exhibit some type II behaviour are spin glasses, chromium and $FeCO_3$. In spin glasses the domain walls have vanishing energy and are at best extremely ramified. In $FeCO_3$ and in chromium, below the spin flip temperature, mixed magnetic state appears over a wide range of magnetic field and is independent of the shape of the sample. In spin glasses gauge theory plays an obvious part since the fact that the spins point in different directions is an indication of the curvature of the space, which is most directly described by a gauge field.

A continuous gauge theory in magnetic models with frozen frustration lines as sources of gauge field is studied and the curvature in the spin space calculated. We shall discuss a specific model of a spin glass (the X-Y model) which is remarkably similar to the Ginzburg-Landau model of superconductivity. The field around a frustration line and the effective interaction between lines are calculated.

The problem of a random distribution of quenched frustration lines is solved by the replica method and the correlation function between replicas calculated. Finally, the stability of the mean field solution for this model is discussed.

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CHAPTER 1

SPONTANEOUS BREAKING OF SYMMETRY AND GAUGE INVARIANCE

1.I Breaking of symmetry and gauge invariance in field theory

Many of the concepts used in this thesis were first introduced in field theory and it is worth making a brief reference to them in that context before going on to the thermodynamics. As reference we list the review papers by Abers and Lee (1973) and Coleman (1975).

In field theory the system is described by the field, a function $\psi(\vec{x}, t)$, and by a Lagrangian or Hamiltonian. One is interested in the dynamics of the problem, usually small excitations above the lowest energy state. In the language of Quantum Field Theory the small excitations are the particles and the equilibrium (lowest energy) state is the vacuum. A uniform equilibrium state is given simply by the minimum of some function $V(\psi)$, the potential energy.

Let us say ψ is a scalar function and $V(\psi)$ given by

$$V = \alpha\psi^2 + \frac{\beta}{2}\psi^4 \qquad (1.1)$$

with β a positive constant.

The symmetry of the problem is evident: V is invariant under the transformation

$$\psi \longrightarrow -\psi \qquad (1.2)$$

The symmetry group is Z_2, a discrete group.

If $\alpha > 0$, $V(\psi)$ is a single well potential and $\psi = 0$ is the equilibrium. This state keeps all the symmetry of the theory.

If $\alpha < 0$, $V(\psi)$ is a double well potential and $\psi^2 = |\alpha|/\beta$ is the equilibrium. The system can choose $\psi = +\sqrt{|\alpha|/\beta}$ or $\psi = -\sqrt{|\alpha|/\beta}$, and in any case the symmetry is spontaneously broken.

With V given we can write a possible Lagrangian density

$$\mathcal{L} = \left(\frac{\partial \psi}{\partial t}\right)^2 - |\vec{\nabla}\psi|^2 - V(\psi) \tag{1.3}$$

which still keeps the Z_2 invariance.

The Euler-Lagrange equation, resulting from the minimization of the action $\int dt\, d^3x\ \mathcal{L}(\psi, \frac{\partial \psi}{\partial t}, \vec{\nabla}\psi)$, gives the dynamical equation of the classical problem.

The physics near one of the equilibrium states with broken symmetry will look very assymmetric. It is natural to take as the matter field the displacement from the equilibrium, let us say from $\psi = \psi_o = \sqrt{|\alpha|/\beta}$, by defining a new field

$$\psi' = \psi - \psi_o \tag{1.4}$$

In terms of ψ',

$$V = -\frac{|\alpha|^2}{2\beta} + 2|\alpha|\psi'^2 + 2\beta\sqrt{\frac{|\alpha|}{\beta}}\,\psi'^3 + \frac{\beta}{2}\psi'^4 \tag{1.5}$$

The cubic term would have made it difficult to imagine the symmetry by studying the small excitations about such equilibrium.

More interesting is the breaking of a continuous symmetry. As a concrete example let us take ψ as a complex field.

$$\psi = \rho\, e^{i\theta} \tag{1.6}$$

and $U(1) = SO(2)$ the symmetry group of the theory. That means the Lagrangian is invariant under transformation

$$\psi \longrightarrow \psi\, e^{i\omega} \tag{1.7}$$

where ω is a real continuous parameter labelling the operations of the group (the angle parametrizing the rotations).

For the moment we shall assume that ω is not a function of position or time. This is sometimes called gauge transformation of first kind on global gauge transformation.

The Lagrangian density can be written as

$$\mathcal{L} = \left|\frac{\partial \psi}{\partial t}\right|^2 - |\vec{\nabla}\psi|^2 - \alpha|\psi|^2 - \frac{\beta}{2}|\psi|^4 \qquad (1.8)$$

The equilibrium state for $\alpha < 0$ is

$$|\psi|^2 = \frac{|\alpha|}{\beta} \qquad (1.9)$$

with the same phase everywhere, everytime. Here again the equilibrium spontaneously breaks the symmetry by choosing one particular phase.

In terms of ρ and θ the Lagrangian density becomes

$$\mathcal{L} = \left(\frac{\partial \rho}{\partial t}\right)^2 + \rho^2\left(\frac{\partial \theta}{\partial t}\right)^2 - |\vec{\nabla}\rho|^2 - \rho^2|\vec{\nabla}\theta|^2 - \alpha\rho^2 - \beta\rho^4 \qquad (1.10)$$

The field θ appears only through its derivatives. This means the dispersion relation of the excitations associated with the field θ is gapless (or: the particle described by the field θ is massless). In this particular case, the Euler-Lagrange equation for the field $\theta(\vec{x}, t)$, when ρ is near its value at the equilibrium, is the ordinary wave equation. Those excitations are called the Goldstone modes.

Let us now enlarge the symmetry group to include transformations

$$\psi(\vec{x}, t) \longrightarrow \psi(\vec{x}, t)e^{i\omega(\vec{x}, t)} \qquad (1.11)$$

with ω a function of position and time. Those transformations are called gauge transformations of second kind or local gauge transformations.

The potential energy function

$$V = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 \tag{1.12}$$

remains an invariant in this enlarged group, but the Lagrangian density (1.8) has to be modified. The problem is with the derivatives. Under a local gauge transformation

$$\frac{\partial\psi}{\partial t} \longrightarrow \frac{\partial\psi}{\partial t}e^{i\omega} + i\frac{\partial\omega}{\partial t}\psi e^{i\omega} \tag{1.13a}$$

$$\vec{\nabla}\psi \longrightarrow (\vec{\nabla}\psi)e^{i\omega} + i(\vec{\nabla}\omega)\psi e^{i\omega} \tag{1.13b}$$

and $|\partial\psi/\partial t|^2$, $|\vec{\nabla}\psi|^2$, thus $\mathcal{L}$, are no longer invariant.

To restore the invariance we define new fields, $\vec{A}$ and φ, called gauge fields, which transform as

$$\vec{A} \longrightarrow \vec{A} + k\vec{\nabla}\omega \tag{1.14a}$$

$$\varphi \longrightarrow \varphi - \frac{k}{c}\frac{\partial\omega}{\partial t} \tag{1.14b}$$

(k and c constants)

and instead of the usual derivatives we work with the gauge co-variant derivatives

$$\vec{D}_x\psi \equiv \vec{\nabla}\psi - \frac{i}{k}\vec{A}\psi \tag{1.15a}$$

$$D_t\psi \equiv \frac{\partial\psi}{\partial t} + i\frac{c}{k}\varphi\psi \tag{1.15b}$$

These co-variant derivatives transform as

$$\vec{D}_x\psi \longrightarrow e^{i\omega}\vec{D}_x\psi \tag{1.16a}$$

$$D_t\psi \longrightarrow e^{i\omega}D_t\psi \tag{1.16b}$$

Therefore $|D_t\psi|^2$, $|\vec{D}_x\psi|^2$ are invariant, and the co-variant derivatives must replace the partial derivatives in the Lagrangian.

$\vec{A}$ and φ are new dynamical fields on their own and so invariant functions of them must, in principle, be included in the Lagrangian. The simplest ones are terms proportional to E^2 and B^2, with

$$\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \tag{1.17a}$$

$$\vec{B} = \text{curl } \vec{A} \tag{1.17b}$$

In elementary quantum mechanics we come across the problem of gauge invariance when writing the Schroedinger equation for one electron in the presence of an electromagnetic field. There the physical meaning of $\vec{A}$ and φ is that of the vector and scalar potential respectively, and the Lagrangian density is

$$\mathcal{L} = \psi^*\left(i\hbar\frac{\partial\psi}{\partial t} - e\varphi\psi\right) - \frac{1}{2m}\left|\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)\psi\right|^2 + \\ + \frac{1}{8\pi}\left[\frac{1}{c}\frac{\partial\vec{A}}{\partial t} + \vec{\nabla}\varphi\right|^2 - \frac{1}{8\pi}\left|\text{curl}\vec{A}\right|^2 \tag{1.18}$$

To see what happens with the Goldstone modes in this case we shall write $\mathcal{L}$ in terms of ρ and θ :

$$\mathcal{L} = i\hbar\rho\frac{\partial\rho}{\partial t} - \rho^2\left(\hbar\frac{\partial\theta}{\partial t} + e\varphi\right) - \frac{\hbar^2}{2m}|\vec{\nabla}\rho|^2 - \frac{\rho^2}{2m}\left|\hbar\vec{\nabla}\theta - \frac{e}{c}\vec{A}\right|^2 + \\ + \frac{1}{8\pi}\left|\frac{1}{c}\frac{\partial\vec{A}}{\partial t} + \vec{\nabla}\varphi\right|^2 - \frac{1}{8\pi}\left|\text{curl}\vec{A}\right|^2 \tag{1.19}$$

We see that $\vec{\nabla}\theta$ (and $\partial\theta/\partial t$) enters in well defined combination with $\vec{A}$ (and φ). Defining new fields

$$\vec{C}_x \equiv \frac{e}{c}\vec{A} - \hbar\vec{\nabla}\theta \tag{1.20a}$$

$$C_t \equiv e\varphi + \hbar\frac{\partial\theta}{\partial t} \tag{1.20b}$$

and noticing that

$$\frac{1}{e}\left(\frac{\partial\vec{C}_x}{\partial t} + \vec{\nabla}C_t\right) = \frac{1}{c}\frac{\partial\vec{A}}{\partial t} + \vec{\nabla}\varphi \tag{1.21a}$$

$$\frac{c}{e}\ \mathrm{curl}\ \vec{C}_x = \mathrm{curl}\ \vec{A} \tag{1.21b}$$

we can write

$$\mathcal{L} = i\hbar\rho\frac{\partial\rho}{\partial t} - \rho^2 C_t - \frac{\hbar^2}{2m}|\vec{\nabla}\rho|^2 - \frac{\rho^2}{2m}C_x^2 +$$

$$\frac{1}{8\pi e^2}\left|\frac{\partial\vec{C}_x}{\partial t} + \vec{\nabla}C_t\right|^2 - \frac{c^2}{8\pi e^2}\left|\mathrm{curl}\,\vec{C}_x\right| \tag{1.22}$$

The new feature is that the dispersion relation for C, obtained from the Euler-Lagrange equation, has a gap (Or: the particle described by C has a mass) arising from the co-variant derivative terms $\rho^2 C_t$ and $\rho^2 C_x^2/2m$. In this way, due to the requirement of local gauge invariance, the massless Goldstone mode combines with the gauge field (Eqn.(1.20)) to produce a massive particle. In quantum field theory this is known as the Higgs mechanism. The massive particle has now all three polarizations (degrees of freedom): the two tranverse polarizations of the gauge field and an additional longitudinal polarization due to the Goldstone mode.

1.II Superconductivity: An example of gauge invariance in Thermodynamics

By thermodynamics we mean, in spite of its name, the macroscopic equilibrium states of a system rather than its dynamics. Nevertheless, a few ideas developed in the previous section (like spontaneous breaking of symmetry and gauge invariance) can be found to be useful in this context. The minimization of the thermodynamic potential replaces the principle of least action and the field theoretical concepts of the previous section can, mutatis mutandis, still be applied here.

We assume the macroscopic state of the system can be described by a thermodynamical field $\psi(\vec{x})$. Because we are only interested in the equilibrium we will not be concerned with time dependence. The thermodynamical equilibrium is achieved by the minimization of the free energy functional $F(\psi)$. The value $F(\psi)_{min}$ is then the conventional thermodynamic free energy. Of course F depends on other thermodynamical variables like temperature and pressure as well. The role played by the free energy functional in thermodynamics is similar to that of the hamiltonian in conventional statistical mechanics and of the action in classical mechanics. The (total) free energy F is written as the volume integral of the free energy density f :

$$F = \int d^3x\, f \tag{1.23}$$

The symmetry of the problem will be reflected in the invariance of f under the operations of some group. So we expect it to be possible to write f in terms of the invariants of this group.

The equilibrium state may not have the full symmetry of f. As an example let us see the case of a ferromagnet. The state of a magnetic system is described by the magnetization vector $\vec{M}(\vec{x})$ and the theory is expected to have SO(3), the group of three dimensional rotations, as its symmetry group. In the ferromagnetic state $\vec{M}$ assumes a non-zero value (the same at every point) and the symmetry group is a subgroup of SO(3), the group SO(2) of rotations around the axis of $\vec{M}$. The free energy density can be expressed in terms of M^2, the invariant quantity in this case. The high temperature equilibrium, the paramagnetic state, has no magnetization $(M = 0)$ and the full symmetry is restored. The transition from para- to ferro-magnetic phase occurs at well defined temperature T_c characteristic of the material, but there is no jump in the magnetization which goes to zero as T (temperature) goes to T_c (the transition temperature) in the ferromagnetic phase.

This is a general feature of second order phase transitions (Landau and Lifshitz, 1959; Chap.XIV). The thermodynamical field (called, in this context, the order parameter) can be defined such that it is zero on average above the transition temperature and non-zero below, the transition being continuous at T_c. Near T_c the order parameter is small.

In Landau's theory of second order phase transition the free energy density f, near the critical temperature T_c, is expanded in powers of the order parameter, which in superconductivity phenomena is a complex function ψ, i.e. an amplitude <u>and</u> a phase (Ginzburg and Landau, 1950). In the absence of magnetic field, f takes the form

$$f = f_o + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 \tag{1.24}$$

where f_o is the free energy density of the normal state, and near T_c

$$\alpha = \alpha'(T - T_c) \tag{1.25}$$

with α' and β positive constants.

The minimization of f gives the equilibrium:

$$\psi = 0 \,, \quad f = f_o \qquad \text{for} \quad T > T_c \tag{1.26}$$

$$|\psi|^2 = \frac{|\alpha|}{\beta} \,, \quad f = f_o - \frac{|\alpha|^2}{2\beta} \qquad \text{for} \quad T < T_c \tag{1.27}$$

The physical meaning of ψ is that of some effective macroscopic wave function for Cooper pairs of superconductor electrons and $|\psi|^2$ the density of such pairs. So, in the presence of a magnetic field, it is necessary to take into account that associated with a non uniform order parameter, $\vec{\nabla}\psi$, there are currents interacting with the magnetic field. The coupling term has to be gauge invariant and therefore, by analogy with quantum electrodynamics, we write f in the presence of a time independent magnetic field in the form

$$f = f_o + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m^*}\left|\left(-i\hbar\vec{\nabla} - \frac{e^*}{c}\vec{A}\right)\psi\right|^2 + \frac{1}{8\pi}h^2 \tag{1.28}$$

$\vec{A}$ is the vector potential; $\vec{h} = \text{curl}\,\vec{A}$; $e^* = 2e$ the charge of a Cooper pair and m^* its effective mass.

The gauge invariant field energy $h^2/8\pi$ is also included. $\vec{h}$ is the magnetic induction (denoted usually, and above, by $\vec{B}$) and inside the sample can be different from the magnetic field, which is the same as the external magnetic field $\vec{H}$ if demagnetizing factors are neglected. Of course $\vec{h} = \vec{H}$ outside the sample.

The variational procedure to get the equilibrium is to be carried out with respect to fields ψ and $\vec{A}$. Usually the variation of $\vec{A}$ is not completely free, the constraint being given by the external source, i.e. the constant magnetic field $\vec{H}_o$. That means the free energy $F = \int d^3x f = F(T,\vec{h})$ (called the Helmholtz free energy) is not the appropriate thermodynamic potential to be minimized. Instead, we have to work with the Gibbs free energy $G = \int d^3x\, g = G(T,\vec{H})$;

$$g = f - \frac{1}{4\pi} \vec{H}.\vec{h} \tag{1.29}$$

(a Legendre transform of f).

The equilibrium equations are the Euler-Lagrange equations corresponding to the minimization of G, $\delta G = 0$:

$$\frac{c}{4\pi} \operatorname{curl} \operatorname{curl} \vec{A} - \frac{e^*\hbar}{2m^*i} (\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*) + \frac{e^{*2}}{m^*c} |\psi|^2 \vec{A} = 0 \tag{1.30b}$$

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 \psi = 0 \tag{1.30a}$$

Because $\operatorname{curl} \vec{H} = 0$ inside the sample, the difference between G and F amounts to a surface effect and the equilibrium equations are the same as if we have used F (exception: Abrikosov mixed states, to be discussed later). The importance of using the Gibbs free energy comes when discussing the exchange of stability between superconductivity and normal magnetic phase.

There are two uniform solutions inside the sample :

$$\psi = 0, \quad \operatorname{curl} \vec{A} = \vec{H} \tag{1.31}$$

corresponding to the normal state, and (below T_c) :

$$|\psi|^2 = \frac{|\alpha|}{\beta}, \quad \operatorname{curl} \vec{A} = 0 \tag{1.32}$$

corresponding to the superconductor state.

Below T_c there is a critical field H_c at which the Gibbs free energy is the same in the normal state as in the superconductor.

In the volume occupied by the sample we can write, at the equilibrium,

$$g_s = f_o - \frac{|\alpha|^2}{2\beta} \tag{1.33}$$

$$g_n = f_o - \frac{H^2}{8\pi} \tag{1.34}$$

where s and n label the superconductor and the normal state respectively.

Below a critical field H_c, given by

$$H_c^2 = 4\pi \frac{|\alpha|^2}{\beta} \tag{1.35}$$

the superconductor phase is more stable, while above H_c the normal phase is the most stable.

When there is no field present we can take $\vec{A} = 0$ and $\psi = \rho$, a pure real function. In this case the first equation gives (below T_c):

$$-|\alpha|\rho + \beta\rho^3 - \frac{\hbar^2}{2m^*}\nabla^2\rho = 0 \tag{1.36}$$

which has a characteristic length ξ, the coherence length associated with the spatial variation of the order parameter, given by

$$\xi^2 = \frac{\hbar^2}{2m^*|\alpha|} \tag{1.37}$$

The other length in the problem, the penetration depth λ, is associated with the magnetic field and a characteristic value can be got from the curl of the second equation, assuming ψ constant :

$$\text{curl curl } \vec{h} = -\frac{4\pi e^{*2}}{m^* c^2} |\psi|^2 \vec{h} \tag{1.38}$$

which gives the length

$$\frac{m^* c^2}{4\pi|\psi|^2 e^{*2}} \approx \frac{m^* c^2 \beta}{4\pi|\alpha| e^{*2}} = \lambda^2 \tag{1.39}$$

Now let us take a general state, described by $\psi(\vec{x})$ and $\vec{A}(\vec{x})$, and compare its Gibbs free energy with that of a uniform normal state:

$$\Delta G = \int d^3x (g_s - g_n)$$

$$= \int d^3x \left\{ \left[f_0 + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m^*}\left|\left(-i\hbar\vec{\nabla} - \frac{e^*}{c}\vec{A}\right)\psi\right|^2 + \frac{h^2}{8\pi} - \frac{\vec{H}\cdot\vec{h}}{4\pi}\right] - \left[f_0 - \frac{H^2}{8\pi}\right] \right\} \tag{1.40}$$

Multiplying Eqn.(1.30a) by ψ^* and integrating we have

$$\int d^3x \left[\alpha|\psi|^2 + \beta|\psi|^4 + \frac{1}{2m^*}\left|\left(-i\hbar\vec{\nabla} - \frac{e^*}{c}\vec{A}\right)\psi\right|^2 \right] = 0 \tag{1.41}$$

Using this result in Eqn.(1.40) we get

$$\Delta G = \int d^3x \left[-\frac{\beta}{2}|\psi|^4 + \frac{1}{8\pi}(\vec{H} - \vec{h})^2 \right] \tag{1.42}$$

For $H = H_c$ the free energies of uniform normal and superconductor states are the same and ΔG gives the energy of walls separating domains of different phases.

ΔG has two contributions: one positive of diamagnetic origin and another negative due to the condensation of the superconductivity. The balance between these two terms is not necessarily positive. If the system can enjoy the condensation energy in a large enough region before having to pay for the full diamagnetic energy, ΔG will be negative. Such a situation occurs when the penetration depth λ is sufficiently larger than the coherence length ξ. One defines the ratio

$$\kappa = \frac{\lambda}{\xi} \tag{1.43}$$

If $\kappa > \kappa_c = 1/\sqrt{2}$ then a plane wall separating the two phases will have ΔG negative.

Superconductor materials are classified as Type I or Type II, depending on the sign of the wall energy at fields near the critical field H_c. If this energy is positive the superconductor is said to be Type I, and if it is negative, Type II.

At $H = H_c$ there is an exchange of stability between uniform normal and superconductor phases, but H_c is not the highest field at which superconductivity can nucleate when the sample is put in a decreasing external field. To calculate that assume the superconductivity just appearing, so one can take $|\psi|$ very small and linearize the equations. For ψ we will have the equation

$$\frac{1}{2m^*}\left(-i\hbar\vec{\nabla} - \frac{e^*}{c}\vec{A}\right)^2 \psi = |\alpha|\,\psi \tag{1.44}$$

The gauge field can be

$$\vec{A} = \frac{1}{2}\vec{H} \wedge \vec{x} \tag{1.45}$$

Eqn.(1.44) is the Schroedinger equation for an electron in a magnetic field and from quantum mechanics we know its eigenvalues (Landau's levels):

$$\mathcal{E}_{nk} = (2n+1)\frac{e^*\hbar H}{2m^*c} + \frac{\hbar^2 k^2}{2m^*} \tag{1.46}$$

where n is an integer and $\vec{k}$ is a vector in the direction of H. So we write

$$|\alpha| = (2n+1)\frac{e^*\hbar H}{2m^*c} + \frac{\hbar^2 k^2}{2m^*} \tag{1.47}$$

or

$$H = \frac{2m^*c}{2n+1}\left[\,|\alpha| - \frac{\hbar^2 k^2}{2m^*}\right] \tag{1.48}$$

The highest possible value for H happens to be $n = 0,\ k = 0$:

$$H_{c_2} = \frac{2m^* c\,|\alpha|}{\hbar e^*} = \sqrt{2}\ \kappa H_c \qquad (1.49)$$

A type II superconductor will have $H_{c_2} > H_c$ and superconductivity can thus nucleate into a normal sample at fields $H > H_c$, above those for which superconductivity is a stable thermodynamic phase.

Before giving a brief account of the properties of type II superconductors we will show that the magnetic flux (or, more exactly, the fluxoid to be defined below) in a superconductor can only be a multiple of the quantum

$$\phi_o = \frac{2\pi e\hbar}{c} \qquad (1.50)$$

This is due to the single-valuedness of ψ. Equation (1.30b) can be written

$$\frac{c^2 m^*}{4\pi e^{*2} |\psi|^2} \text{ curl } \vec{h} + \vec{A} = \frac{\hbar c}{e^*} \vec{\nabla}\theta \qquad (1.51)$$

Taking the integral along a closed circuit

$$\frac{m^* c^2}{4\pi e^{*2}} \oint d\vec{\ell} \cdot \frac{\text{curl } \vec{h}}{|\psi|^2} + \oint \vec{A} \cdot d\vec{\ell} = \frac{\hbar c}{e^*} \oint (\vec{\nabla}\theta) \cdot d\vec{\ell} \qquad (1.52)$$

$\oint \vec{A} \cdot d\vec{\ell}$ is the magnetic flux ϕ and because ψ is single-valued $\oint (\vec{\nabla}\theta) \cdot d\vec{\ell} = 2n\pi$. So the quantity

$$\phi' = \phi + \frac{m^* c^2}{4\pi e^{*2}} \oint d\vec{\ell} \cdot \frac{\text{curl } \vec{h}}{|\psi|^2} \qquad (1.53)$$

called fluxoid, is quantized in units of $2\pi\hbar c/e$

If a type II superconductor is placed in a decreasing external magnetic field we expect (and observe) the superconductivity to start nucleating at fields H_{c_2} larger than H_c (Eq.1.49). Below H_{c_2} a periodic, triangular array of tube-like regions is set up, each tube carrying a unit quantum $2\pi\hbar e^*/c$ of flux (Abrikosov 1957). For small fields (or in the extreme type II situation, with $H_{c_2}/H_c \longrightarrow \infty$, $\xi \longrightarrow 0$), those normal regions can be treated as lines (vortex lines).

In this extreme type II case we can take

$$|\psi|^2 = \frac{m^* c^2}{4\pi e^{*2}\lambda^2} = \frac{|\alpha|}{\beta} \tag{1.54}$$

constant outside the core, and write for Eq.(1.51)

$$\lambda^2 \text{ curl } \vec{h} + \vec{A} = \frac{\hbar c}{e^*} \vec{\nabla}\theta \tag{1.55}$$

From which, taking the curl :

$$\lambda^2 \text{ curl curl } \vec{h} + \vec{h} = 0 \tag{1.56}$$

This equation is not valid near the vortex lines where there is a singularity. The correction term must ensure a unit of fluxoid around the line. This can be achieved by modifying Eq.(1.56) to

$$\lambda^2 \text{ curl curl } \vec{h} + \vec{h} = \hat{z} \quad \phi_o \; \delta(x) \; \delta(y) \tag{1.57}$$

with $\vec{h}$ in the z-direction.

Since $\vec{\nabla}.\vec{h} = 0$, the above equation becomes

$$-\lambda^2 \nabla^2 \vec{h} + \vec{h} = \hat{z} \quad \phi_o \; \delta(x) \; \delta(y) \tag{1.58}$$

with $\vec{h}$ independent of z.

The solution of this equation is the Green's function of the two-dimensional Helmholtz equation, that is K_o, the zero order Hankel function of imaginary argument :

$$h(r) = \frac{\phi_o}{2\pi\lambda^2} K_o(r/\lambda) \qquad (1.59).$$

CHAPTER 2

MIXED STATES IN MAGNETIC MATERIALS

In the previous chapter we discussed some properties of Type II superconductors. By Type II we mean a situation in which the non-uniform regions (walls) separating domains of different phases of the system can give a negative contribution to the total free energy. That was shown to be the case for superconductor materials with large ratio between the penetration depth λ and the coherence length ξ. In the presence of a magnetic field not larger than some critical value H_{c_2} the usual domain structure breaks-up giving rise to a mixed state (Abrikosov state) with both normal and superconductor phases coexisting in a true thermodynamic stable state.

Our purpose in this chapter is to show some evidences for the existence of the mixed state in some magnetic materials like spin glasses, chromium and $FeCO_3$. We shall take the opportunity to make a brief review of experiments and theories on spin glasses.

2.I Spin Glasses

The class of alloys known as spin glass is usually characterized experimentally by a cusp in the zero field magnetic susceptibility and the absence of long range magnetic order. The cusp occurs at temperature T_{SG} (the freezing temperature) characteristic of the material and is rounded off in the presence of a magnetic field. At high temperatures the susceptibility follows the Curie's law. Standard examples of spin glasses are CuMn in the range of 1-5 at % of concentration and AuFe with concentration below 12 at. % (Canella and Mydosh, 1972 and Canella, 1973).

Despite the absence of any observed long range magnetic order, experiments of Mossbauer effect (Borg et al. 1963) and meson absorption (Murnick et al. 1976) show the onset of local internal magnetic field at temperature roughly the same as that of the cusp in the susceptibility. The low temperature structure is pictured like an array of local magnetic moments frozen at random directions.

In the metallic alloys the magnetic impurities couple through a Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction:

$$J(r) = J_o \frac{\cos 2k_F r}{(k_F r)^3} \tag{2.1}$$

which can be positive or negative depending on the distance r between the impurities. The randomness in the position of the impurities results in a competition between ferro- and antiferro-magnetic interactions. In non metallic spin glasses, as for example cobalt and manganese aluminosilicate glasses (Rechenberg et al. 1978), interactions of shorter range than RKKY are responsible for the coupling but the picture of competing ferro- and antiferro-magnetic interactions still remains.

When a spin glass is cooled in the presence of a magnetic field a thermoremanent magnetization (TRM) is observed after removing the field. If the field is applied and removed with the sample at constant temperature the magnetization is known as isothermal remanent magnetization (IRM). TRM and IRM are different for small fields but both have the same saturation value at high fields. The remanence can last from a few seconds to about twenty minutes, with a logarithmic time dependence.

One important question in the theory and experiment on spin glasses is whether or not there is a real phase transition. Generally to a phase transition are associated critical fluctuations (divergence in the correlation function) at a well defined temperature and a manifestation of that is the non-analycity of thermodynamic quantities and the onset of long range order at that temperature. The cusp in the susceptibility strongly suggests a magnetic phase transition but the absence of long range order casts doubts on such assumption. Also the difference in the value of the critical temperature depending on the physical property measured (for example T_{SG} from the Mossbauer effect does not quite agree with that of the cusp in the susceptibility) would have to be ascribed to technical details in the experiments and sorted out.

An interpretation which does not imply the existence of phase transition can be found using Neel's theory of magnetic grains (Neel 1949). The spin glass is seen as composed of a very large number of small ferro- or antiferro-magnetic domains behaving like magnetic grains in Neel's theory. Each grain has an anisotropy energy W and the dependence of the relaxation time t with the temperature T obeys the Arrhenius law

$$t = t_o e^{W/k_B T} \tag{2.2}$$

In the interval dt after the time t the grains with anisotropy energy between W and $W + dW$ (with W given by equation 2.2) will have relaxed and the decrease dM in the magnetization will be given by

$$dM = -p(W)\, u\,(W) dW \tag{2.3}$$

where $p(W)dW$ is the probability of finding a grain with energy between W and $W + dW$ and $u(W)$ is its average contribution to the magnetization. Assuming $p(W)\, u(W)$ constant and using equation 2.2 we get

$$\frac{dM}{d(\log t)} = \text{negative constant} \tag{2.4}$$

Neel's theory explains the time dependence of the remanence and the difference between TRM and IRM, but is in difficulty with the cusp in the susceptibility. For every measurement with a certain time delay a grain with energy W has a defined blocking temperature T_b and it will be necessary to have a very narrow distribution of T_b's to explain the experimental sharpness of the transition temperature.

Apart from the approach outlined above most of the work on spin glasses assumes or tries to prove (or disprove) the existence of a phase transition.

One of the first theoretical studies on phase transition in a spin glass is due to Rivier and Adkins (Rivier and Adkins 1974). In their model the spins are assumed to be statistically independent and randomly interacting via RKKY. The probability distribution $P(\vec{H}(\vec{x}))$ for a local field $\vec{H}$ at $\vec{x}$ is then calculated from first principles (it is a random walk problem). But $P(\vec{H}(\vec{x}))$ depends on the two sites conditional probability $P(\vec{H}(\vec{x}), \vec{H}'(\vec{x}'))$, which in its turn depends on a three sites probability, and so on. The chain is broken by assuming a short range order with a local magnetization m. The result is a self-consistent equation for m with a non zero value below some temperature T_{SG}.

In 1975 Edwards and Anderson (Edwards and Anderson 1975) proposed a kind of mean field theory of spin glasses with no spatial magnetic order (even short ranged) assumed. Their definition of order parameter in spin glasses and the way of evaluating the free energy of a quenched system were bold enough and provided a lot of work on the theory.

Consider the system described by the hamiltonian

$$\mathcal{H} = - \sum_{ij} J_{ij} S_i S_j \qquad (2.5)$$

For annealed systems the equilibrium is achieved with both sets $\{S\}$ (the spins) and $\{J_{ij}\}$ (the exchange interactions) treated as thermodynamical variables. However if the system is quenched, $\{J_{ij}\}$ have a probability distribution given a priori. The ensemble free energy will be an average of partial free energies calculated for each configuration of J_{ij}'s. Let us call $Z(\{J\})$ the partition function and $P(\{J\})$ the probability for a particular configuration $\{J\}$ of J_{ij}'s in a quenched system. The free energy is the average of $-k_BT\ \ell n Z$:

$$F = -k_BT \langle \ell n Z \rangle_{av} = -k_BT \int d\{J\}\, P(\{J\})\, \ell n Z(\{J\}) \tag{2.6}$$

where $\int d\{J\} \ldots$ means an integration over all possible configurations.

Usually it is not an easy task to evaluate the average of $\ell n Z$, and the so-called "replica trick" is used. First one calculates the partition function of m replicated systems (that means with the same $\{J\}$) - it is just Z^m. Then one makes the analytical continuation from m integer to m real and use the identity:

$$\ell n Z = \lim_{m \to 0} \frac{1}{m}(Z^m - 1) \tag{2.7}$$

In this way the problem reduces to that of calculating $\langle Z^m \rangle_{av}$, usually a much easier job.

Edwards and Anderson demonstrated the existence of a second order phase transition in the mean field solution of their model, with the order parameter defined as the correlation between spins at same site and different replicas.

$$q_{\alpha\beta} = \langle S_i^{(\alpha)} S_i^{(\beta)} \rangle \quad , \quad \alpha \neq \beta \tag{2.8}$$

(α and β are labelling replicas).

From the Edwards-Anderson theory, Sherrington and Kirkpatrick proposed a "solvable" model with infinite range exchange interaction appropriately scaled with the number of spins (Sherrington and Kirkpatrick 1975). This should lead to some kind of exact mean field theory of spin glasses. There has been much discussion on the proposed solution of this model and that is why we put inverted commas on "solvable". The problem is that the solution proposed by Sherrington and Kirkpatrick gives unphysical results at low temperatures (like negative entropy for example). They assumed

$$q_{\alpha\beta} = q \quad , \; (\alpha \neq \beta) \tag{2.9}$$

at the equilibrium independent of α and β but as pointed out by de Almeida and Thouless (1978) this particular stationary point is not a minimum.

Some sort of symmetry breaking between replicas should therefore take place in the spin glass phase but the correct way of doing it is yet to be found. We are not going to pursue this point but just refer to the review by Bray and Moore (1980).

2.II Frustration

Now let us turn to the question of what is the essential feature in a spin glass that makes its thermodynamics different from that of other spin systems.

P.W. Anderson noted that "what is vital is that the interactions be of random sign, or at least that the different interactions via different paths between two given spins be competing" (Anderson 1977). This idea was developed by Toulouse with the concept of frustration (Toulouse 1977). In this section we shall be concerned with frustration and its associated gauge field in spin glasses.

Consider the hamiltonian

$$\mathcal{H} = - \sum_{ij} J_{ij} S_i S_j \tag{2.10}$$

This hamiltonian is invariant under the local transformation

$$S_i \longrightarrow - S_i \qquad \text{(for a given } i) \tag{2.11}$$

$$J_{ij} \longrightarrow - J_{ij}$$

so that, given a random distribution of positive and negative J_{ij}'s, one could attempt to recover a ferromagnetic hamiltonian (only positive J_{ij}'s) by a suitable application of transformations (2.11) wherever necessary. This turns out to be impossible in general. If it were possible the thermodynamics would be the same as that of a ferromagnet, except for the response to fields non invariant under such transformations like an external magnetic field for example.

One simple example where such transformation to an equivalent ferromagnet is possible is the Mattis model (Mattis 1976), where by definition

$$J_{ij} = \varepsilon_i \varepsilon_j \tag{2.12}$$

with $\varepsilon_i = \pm 1$ at random.

It is just enough to make the transformation

$$S_i \longrightarrow \varepsilon_i S_i \tag{2.13}$$

to see the Mattis model is thermodynamically equivalent to a ferromagnet. The disorder in this case is "irrelevant".

Unfortunately (or fortunately) it is not always possible to carry out consistently everywhere the transformation (2.11). To see that clearer we shall restrict ourselves to an Ising model ($S_i = \pm 1$)

with only nearest neighbour interaction $J_{ij} = \pm 1$, in a simple cubic lattice. Let us draw our attention to one particular face of some elementary cubic cell. If there is an even number of negative bonds there will not be any difficulty in finding a local ground state configuration satisfying all the bonds simultaneously. On the other hand if the number of negative bonds is odd there is no way of satisfying all of them at the same time. The face (usually called "plaquette") is said to be frustrated in this case (Fig.2.1).

FIGURE 2.1 - (a) Unfrustrated plaquette; (b) Frustrated plaquette.

We define a frustration function ϕ for a plaquette

$$\phi = J_{12} \; J_{23} \; J_{34} \; J_{41} \tag{2.14}$$

$\phi = +1$ for an unfrustrated plaquette and -1 for a frustrated one. ϕ is clearly an invariant under transformation (2.11), and the model cannot be transformed into an equivalent ferromagnet. Such disorder as in Fig.2.1(b) is essential. Incidentally, $\phi = 1$ in the Mattis model, as befits irrelevant disorder.

In general for a closed circuit c we define the frustration function

$$\phi = \prod_c J_{ij} \tag{2.15}$$

where the product is to be taken with the J_{ij}'s on the perimeter of the circuit c.

It is easy to see that

$$\phi = \prod_{k \in s} \phi(k) \tag{2.16}$$

with k labelling the plaquette of some surface S bounded by the circuit c.

It is always possible to thread continuous lines joining the centres of frustrated plaquettes (that means: so that the centres of successive plaquettes along the same line be no farther than one lattice constant). To show that it is enough to notice that

$$\prod_{k \in s} \phi(k) = \prod_{(ij) \in S} J_{ij}^{2} = 1 \tag{2.17}$$

for any closed surface S. Therefore the number of frustrated plaquettes is even and one can take them in pairs marking the entry and the exit of each line. Those <u>frustration lines</u> either go to infinity or close in on itself. They are the (line) "defects" into which the essential disorder is concentrated.

The above derivation is a variant of the proof given by Rivier for the existence of disclination lines (see below) in ordinary glasses (Rivier 1979).

If instead of an Ising we consider a vectorial model, the spins around a frustration line will distort themselves from the colinearity in order to minimize the energy. As an example we draw in Figure 2.2 the two topologically distinct ground states for two-dimensional spins (usually called X-Y model) in an isolated frustrated plaquette.

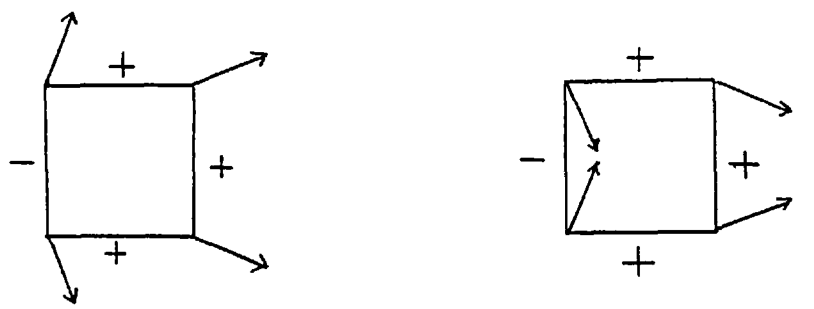

FIGURE 2.2 - Two ground state configurations for an isolated frustrated plaquette

Defining "parallel transport" of spins as that following the ground state configuration, we see that one turn around the frustration line results in a rotation of the spins by $\pm\tilde{\pi}$.

The mismatch in orientation is known in elasticity theory as disclination. (For an introduction see Harris 1977, and for the linear theory of disclinations see de Witt 1970). The identification of frustration lines with disclination lines was made by I.E. Dzyaloshinskii and G.E. Volovik to construct a continuum theory of spin glasses (Dzyaloshinskii and Volovik 1978). The information about the disorder in a particular spin glass is given by specifying some gauge field from which the density of disclination lines is derived. The problem is similar to superconductivity in the presence of a magnetic field where the induction (density of magnetic lines) is derived from a gauge field (the vector potential). In the same way the coupling between matter field (spins) and gauge field is defined by the requirement of gauge invariance (Invariance of the Hamiltonian under local transformation of the type (2.13)). Extra complications in spin glass are of two kinds: (1) The essential disorder (frustration) is quenched. It is given a priori and cannot be described simply by letting the gauge field be an ordinary dynamical variable. (2) The gauge group may be non abelian. For example, rotations in three dimensions, which certainly transform a Heisenberg spin $\vec{S}_i$ in a gauge transformation of the type (2.11), is non abelian. Superconductivity, XY spins (group U(1)) and Ising spins (group Z_2) are all abelian.

Once we have a theory with matter field and gauge field interacting one can speculate about Type II behaviour. Of course in the case of spin glasses it may not be easy to evidence it experimentally because there is no control on the sources: disclination lines are created in a random way when the sample is prepared. Nevertheless, highly ramified "domain walls" separating regions of solidarily frozen spins could be regarded as strong evidence for Type II behaviour.

In the final chapter we attack this problem for the X-Y model in detail, and show that Type II gives by far a better description of the behaviour of an actual spin glass.

2.III <u>Chromium</u> (Griessen and Fawcett 1977a and b)

Below the Neel temperature ($T_N = 312^oK$) and above the spin flip temperature ($T_{SF} = 123.5^oK$) chromium is an antiferromagnet described by a transverse spin density wave (TSDW). At T_{SF} there is a phase transition to a longitudinal spin density wave (LSDW). Below T_{SF} in a field parallel to the direction of the LSDW and above the so-called spin flop field (H_{SF}) a change in polarization is expected (since a TSDW can lower its energy more easily than LSDW in a longitudinal magnetic field). The structure near H_{SF} is usually supposed to be made of domains of different polarizations stabilised by demagnetizing energy. This multi-domain state has been successful in explaining the ultrasonic absorption peak in the antiferromagnet MnF_2 (Baryakhtar and Ivanov 1974) and has actually been observed by photographs (King and Paquette 1973).

Griessen and Fawcett measured the magnetic torque and the magnetostriction of a sample of chromium near the spin flip temperature in a magnetic field. We shall follow their papers first showing that a uniform phase model in the full range of applied field does not fit the experimental data for Cr then that the calculated range of magnetic field in which an intermediate state (with two macroscopic domains), stabilised by demagnetizing energy, can exist is too narrow compared with the observed one, and finally how a mixed state, with domain structure stabilised by negative energy walls, could explain it.

Let us consider the case of a uniaxial antiferromagnet with polarization $\vec{\eta}$ (making an angle η with the wave vector $\vec{Q}$ of the spin density wave), in a magnetic field $\vec{H}$ (making an angle θ with $\vec{Q}$). η is given by the minimization of the total energy: the anisotropy energy is $E_A = 2K\sin^2\eta$ and the magnetic energy

$$E_M = \chi^{\parallel} H^2 \cos^2(\theta + \eta) - \chi^{\perp} H^2 \sin^2(\theta + \eta) \qquad (2.18)$$

$\vec{\eta}$, $\vec{Q}$ and $\vec{H}$ are in the same plane. $\chi^{\parallel}$ and $\chi^{\perp}$ are the parallel and perpendicular components of the susceptibility tensor with $\vec{\eta}$ as the minor axis.

At the equilibrium

$$K\sin 2\eta = h^2 \sin 2(\theta + \eta) \qquad (2.19)$$

with
$$h^2 = H^2(\chi^{\perp} - \chi^{\parallel})/2 \qquad (2.20)$$

This yields the two solutions $\eta = 0$ (LSDW) and $\eta = \pi/2$ (TSDW) for $\theta = 0$. The torque per unit volume is

$$\tau = \frac{Kh^2 \sin 2\theta}{\left[(K - h^2\cos 2\theta)^2 + (h^2 \sin 2\theta)^2 \right]^{1/2}} \qquad (2.21)$$

Near T_{SF} the phase transition is modelled by assuming the anisotropy coefficient K linear in the temperature and changing the sign at T_{SF} :

$$K = K_o(T_{SF} - T) \qquad (K_o > 0) \qquad (2.22)$$

The LSDW phase $(K > 0)$ gives a positive τ irrespective of the value of the magnetic field.

The data for Cr from Griessen and Fawcett (1977) does not fit the assumption of same phase over all range of the field (see Figure 2.3). For low fields it is like LSDW, but for high fields it is like TSDW. The behaviour between H_1 and H_2 (indicated on Figure 2.3) suggests a domain structure.

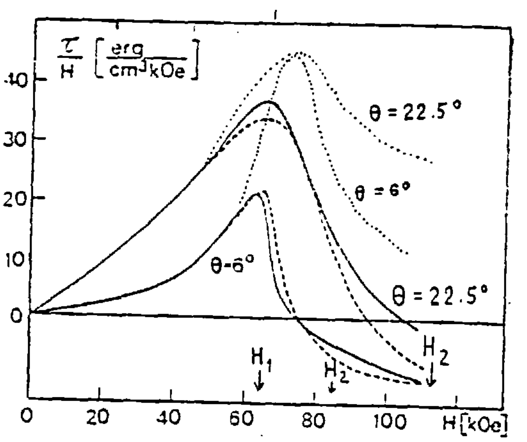

FIGURE 2.3

Normalized torque per unit volume τ/H as a function of the magnetic field at 116^oK.

Full line : experimental results.
Dotted line : theoretical results using a uniaxial antiferromagnetic model with anisotropy constant $K = 3.51 \times 10^3$ erg/cm^3 and $\chi^{\perp} - \chi^{\parallel} = 1.3 \times 10^{-6}$ esu/cm^3.
Dashed line : theoretical curve assuming negative domain wall energy; (from Griessen and Fawcett (1977a)).

If this structure were an intermediate state, the range $\Delta H = H_2 - H_1$ would be of the order of $4\pi n H_1 (\chi''_T - \chi''_L)$, with n the demagnetizing factor and the subscripts T and L referring to the transverse and longitudinal phases respectively. Using the experimental values for the susceptibility of Cr the calculated ΔH is roughly two orders of magnitude smaller than the experimentally observed.

Postulating a mixed state stabilised by the negative energy of the walls between LSDW and TSDW domains this wide range ΔH can be explained. Griessen and Fawcett assume that wall energy as

$$E_w = - \beta K x (1 - x) \tag{2.23}$$

with β a dimensionless parameter and x the fraction of TSDW domains in the sample, and got

$$\Delta H = \beta H_1 [4(1 - \beta)]^{1/2} \tag{2.24}$$

A value of $\beta \approx 0.3$ fits the experiments.

For more details on the derivation of the above expressions and comparison between theory and experiments we refer to the original papers.

2.IV $\underline{FeCO_3}$

$FeCO_3$ is antiferromagnetic below 38^oK with a very large anisotropy energy and the staggered magnetization parallel to one of the axis of symmetry. In a strong magnetic field along this axis the sample becomes paramagnetic. Dudko et al. (1975) measured the magnetization and susceptibility of $FeCO_3$ at 4.9^oK as a function of the applied field. In Fig.2.4 their results for the longitudinal magnetization are reproduced. No transverse component of the magnetization was observed.

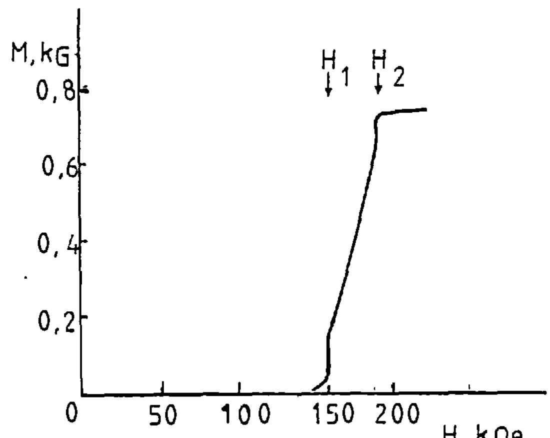

FIGURE 2.4

Dependence of the longitudinal magnetization of $FeCO_3$ on the intensity of a magnetic field directed along the axis of symmetry of the crystal (From Dudko et al. (1975)).

For a two sub-lattice model of antiferromagnetism a phenomenological thermodynamical potential F can be constructed (Bar'yakhtar et al. 1972). Neglecting transverse components one can write

$$F = -\frac{1}{2}a\,(m_1^2 + m_2^2) - bm_1m_2 - (m_1 + m_2)h \qquad (2.25)$$

m_1 and m_2 are the longitudinal components of the magnetization in the lattices 1 and 2 respectively, and h is the applied field, a and b are constants in our case. The values of m_1 and m_2 are restricted to the interval $(-1,+1)$ in units of spontaneous saturation magnetization.

Minimizing F we get the various phases depending on the value of h compared with the critical fields

$$h_1 = -(b - a) \qquad \text{and} \qquad h_2 = -(b + a) \qquad (2.26)$$

At small fields ($h < h_1$) : antiferromagnetic phase $m_1 = 1$; $m_2 = -1$.

At intermediate fields ($h_1 < h < h_2$) : $m_1 = 1$; $m_2 = -\frac{h + b}{a}$.

At large fields ($h > h_2$) : paramagnetic phase $m_1 = 1$; $m_2 = 1$.

Notice that for $a > 0$, $h_2 < h_1$ and there is a first order transition at $h_t = \frac{1}{2}(h_1 + h_2)$. For $a < 0$, $h_2 > h_1$ and a second order transition goes from h_1 to h_2. We shall call the former case Type I and the latter Type II. The constant a is related to the interaction between spins in the same lattice as can be seen from the form of the thermodynamic potential. Therefore its sign is basically determined by the second neighbour exchange interaction J_{11}. Then for Type II we can assume $J_{11} < 0$.

The absence of the transverse component of the magnetization in $FeCO_3$ makes it difficult to explain the reduction of the longitudinal component of lattice 2 in the state between h_1 and h_2 within the two sub-lattice's model. It could be thought that lattice 2 split into two new ones with magnetizations making an arbitrary angle to each other so that any intermediate value for m_2 could be achieved. This possibility is ruled out for $FeCO_3$ because there is much evidence that it behaves like an Ising model (only two states allowed to each spin). The alternative is a multilattice model described below.

The orientation of lattice 2 is determined by the inter-lattice interaction γm_1. If the external field h is large enough to offset such interaction lattice 2 can split into several sublattices. The number of sublattices with reversed spins will grow with the magnetic field allowing a smooth building-up of magnetization. Suppose there are n new sublattices, each one characterized by a field

$$h_{2i} = \sum_j b_{ij} m_j \tag{2.27a}$$

$$= a + 2 \sum_{j=1}^{k} b_{ij} \tag{2.27b}$$

$$= -a - 2 \sum_{j=k+1}^{n} b_{ij} \tag{2.27c}$$

with b_{ij} the molecular field coefficients. The summation in the second (or third) line is over the k (or $n-k$) sublattices with reversed (or not reversed) spins.

The effective field on the sublattice i is

$$h_{\mathrm{eff}\,2\,i} = h + b - h_{2i} \tag{2.28}$$

and we shall assume the orientation of the sublattice i changes at the critical field h_{c_i} such that $h_{\mathrm{eff}\,2\,i} = 0$.

$$h_{c_i} = -(b-a) + 2 \sum_{j=1}^{k} b_{ij} \tag{2.29}$$

b_{ij} is a function of the distance r_{ij} between sublattices and must decay rapidly with r_{ij}, so the summations can be restricted to spins near i. If all b_{ij} are positive, isolated spins further apart than the range of the interaction will start to change at the field $h = h_1 = -(b-a)$ and a net magnetization will begin to come out.

In the same way the other critical field h_{c_2}, the smallest field at which all sublattices can have the sign of the spins changed, is given by

$$h_{c_2} = -(b+a) - 2\sum_{j=k+1}^{n} b_{ij} \qquad (2.30)$$

and the last solitary spins will change at $h = h_2 = -(b+a)$.

Finally we will show that in the Type II case $(h_1 < h_2)$ the energy of the interface between the antiferro- and para-magnetic phases is negative.

We shall assume first (J_{12}) and second (J_{11}) neighbours interactions only. There are six of each because $FeCO_3$ has a rhombohedral structure. Let us consider a single (111) layer and the magnetic field along the [111] axis. The energy per spin can be written as

$$E_A = 6 J_{12} - 6 J_{11} \quad \text{in the antiferromagnetic phase} \qquad (2.31a)$$

$$E_p = -6 J_{12} - 6 J_{11} - \mu_0 H \quad \text{in the paramagnetic phase} \qquad (2.31b)$$

and

$$E_I = -4 J_{11} - \frac{1}{2}\mu_0 H \quad \text{in the interface} \qquad (2.31c)$$

At $H = H_t = -12 J_{12}/\mu_0$ the antiferro- and the para-magnetic phase have the same energy. At this field the excess of energy due to the interface is

$$\Delta E = 2 J_{11} \qquad (2.32)$$

As we saw before, in the Type II case J_{11} is negative and therefore the interface energy is negative also.

2.V Conclusion

In this chapter we have reviewed three magnetic systems whose experimental behaviour suggests or hints at Type II behaviour. They are spin glasses, Chromium, and $FeCO_3$. The experimental evidence for

Type II behaviour is strongest in Cr (Griessen and Fawcett 1977), for which only a mixed state hypothesis reproduces qualitatively the measurements for the torque. Such a mixed state is also independent of the shape of the sample, unlike its Type I counterpart, the intermediate state, whose domain pattern is strongly affected by shape-dependent demagnetizing fields. In $FeCO_3$, the evidence is indirect (unobservability of transverse magnetization) but Type II follows naturally from the simplest model ascribed. In spin glasses, both experimental evidence and theoretical support are extremely weak, but "domains" of solidary spins obtained by computer simulation (see, e.g., Vannimenus 1979) have a highly ramified shape, which resembles more closely the Abrikosov mixed state than the Type I intermediate state.

CHAPTER 3

CONTINUOUS GAUGE FIELDS AND SPIN GLASSES

It is almost evident that spin glasses should be described by a gauge theory. The local invariance of the Hamiltonian, first pointed out by Toulouse in the case of an Ising model and by Dzyaloshinskii and Volovik in general cases, holds under a transformation involving the spin field $\vec{S}(\vec{x})$ and a connection, the interaction $J(\vec{x},\vec{y})$ between spins. At equilibrium, the spins directions at different points define randomly different vertical axis, and a connection is required if one wishes to compare energies of configurations or excitations involving different points in space, as in standard differential geometry. From a different point of view, the ramified boundary of domains of correlated spins, exhibited by computer or model building simulations (Vannimenus et al. 1979), strongly suggests that the walls between different domains in spin glass have negative or vanishingly small positive energy, in analogy with type II superconductivity. The latter is a gauge theory, in fact it is the simplest continuous one, a U(1) gauge theory with minimal coupling to a complex matter field. The gauge field is the vector potential $\vec{A}$, and an external source of gauge field can be imposed through an external magnetic field $\vec{H}$, by the additional term in the Hamiltonian $H_{int} = -\vec{H}\cdot\vec{B}/4\pi$, where $\vec{B} = \text{curl}\,\vec{A}$.

In a spin glass we face two difficulties not encountered in superconductivity. They concern the external source of gauge fields. The sources are, clearly, the centers of frustration, the frustrated plaquettes in lattice models, which are frozen in at random when the piece of alloy has been made. Thus, the gauge field $\vec{A}$ (or the magnetic induction $\vec{B}$) are dynamical variables, but their external source ($\vec{H}$) is an external constant which happens to be random. Moreover, it can readily be verified in lattice models, and proven in general using topological arguments (Toulouse 1979, Rivier 1979, and see also Chapter 2), that frustrations are uninterrupted line defects which cannot terminate inside the material. Furthermore, they only carry one quantum of frustration and are their own antidefect. The topolocial stability of those defects is associated with the fact that, for the manifold of internal states V describing the spin glasses, the first homotopy group $\tilde{\Pi}_1(V)$ is non-trivial and equal to the two-elements group Z_2. This second fact is, of course, quite different from the electromagnetism of superconductors where $\tilde{\Pi}(U(1)) = Z$ and the external magnetic field $\vec{H}$, although still a line defect, can carry any number of flux quanta.

But the fundamental difficulty of gauge theories which have appeared previously in the literature, that of expressing the quenched nature of the disorder (Hertz 1979), will here by bypassed in an elementary fashion: the gauge fields are (annealed) dynamical variables, but their external sources (the frustration lines) are quenched once and for all when the alloy has been made by the metallurgist. Julia and Toulouse (1979) have argued for a dynamical gauge field, without explicitly constructing it, which reflects the modulo 2 character of its source, the frustration, but we believe that it is not necessary and we shall see that the bimodality will appear naturally in the dynamical variables. (it is indeed sufficient then that the homotopy group of the manifold of internal states of the field contains as a subgroup the homotopy groups describing the static defects).

We shall be concerned in this chapter with a static, thermodynamic theory of the spin glasses. The dynamics can be added on in a straightforward fashion, either by going from a time-independent to a time-dependent Ginzburg-Landau formalism, or by rewriting our free energy as a Lagrangian by including time derivative in a Lorentz-covariant form.

The time-dependent Ginzburg-Landau formalism is often favoured by solid state physicists but can be quite ambiguous (see e.g. Sarkar 1980). On the other hand, the Lorentz covariant formalism is unambiguous but fairly remote from the experimental spin waves excitations, at least in ferromagnets where they have dispersion $\omega \propto k^2$, instead of the covariant expression $\omega \propto k$.

3.I Continuous Heisenberg model

We start by showing how a continuous field theory arises from a lattice model with (local) gauge symmetry.

Consider a system of classical spins $\vec{S}(\vec{i})$ in a simple cubic lattice and described by the Heisenberg hamiltonian

$$\mathcal{H} = - \sum_{\vec{i}\vec{j}} J(\vec{i},\vec{j})\ \vec{S}(\vec{i}).\vec{S}(\vec{j}) \tag{3.1}$$

with $J(\vec{i},\vec{j}) = \pm 1$ at random for nearest neighbours and zero otherwise.

The spins can be represented by their components in any arbitrary frame of reference, but the Hamiltonian must be independent of that arbitrary choice. Thus, if one wants to write the scalar product $\vec{S}(\vec{i}).\vec{S}(\vec{j})$ as a bilinear combination of the components along the local axis, one has to introduce coefficients of connection $A^{\alpha}{}_{\beta}(\vec{i},\vec{j})$ between the basis at $\vec{i}$ and $\vec{j}$:

$$\vec{S}(\vec{i}).\vec{S}(\vec{j}) = S_{\alpha}(\vec{i})A^{\alpha}{}_{\beta}(\vec{i},\vec{j})\ S^{\beta}(\vec{j}) \tag{3.2}$$

where the Greek letters are labelling some set of local cartesian axis in which the spins are orientated ("spin space").

In Eqn.(3.2) and henceforth the convention of summation over repeated index is agreed, unless otherwise stated. Also upper and lower indices are consistently used for the contravariant and covariant components of vectors and tensors, even if the distinction is not always necessary.

The group of operations taking from one basis to another is $O(3)$ of proper and improper rotations in three dimensions.

It could be argued that as any state of $\vec{S}(\vec{i})$ can be reached by a proper rotation we should restrict ourselves to the smaller group $SO(3)$ of proper rotations. However we will see that the continuity of $\vec{S}(\vec{x})$, the spin field in the continuous model, will force us to choose the manifold of its states one for which $SO(3)$ and $O(3)$ operating on it give the same result.

Under a local transformation the components $S^\alpha(\vec{i})$ transforms accordingly :

$$S^\alpha(\vec{i}) \longrightarrow O(\vec{i})^\alpha{}_\beta \, S^\beta(\vec{i}) \tag{3.3}$$

To keep the invariance, $A^\alpha{}_\beta(\vec{i},\vec{j})$ has to transform in the following way

$$A^\alpha{}_\beta(\vec{i},\vec{j}) \longrightarrow [O(\vec{i})^{-1}]^\alpha{}_\gamma \, A^\gamma{}_\mu(\vec{i},\vec{j}) O(\vec{j})^\mu{}_\beta \tag{3.4}$$

Transformations (3.3) and (3.4) are the generalization for Heisenberg spins of the transformation (2.11) defined for the Ising case.

Here we have put the emphasis on the coefficients of connection $A^\alpha{}_\beta(\vec{i},\vec{j})$ rather than on the exchange interactions $J(\vec{i},\vec{j})$ because that will make more clear the dynamical aspect of the gauge field compared with its frozen sources.

Up to an additive constant, the scalar product can be written as the product of a difference :

$$\mathcal{H} = \frac{1}{2} \sum_{\vec{i},\vec{j}} \left[S^\alpha(\vec{i}) - A^\alpha{}_\beta(\vec{i},\vec{j}) \, J(\vec{i},\vec{j}) \, S^\beta(\vec{j})\right]\left[S_\alpha(\vec{i}) - A^\gamma{}_\alpha(\vec{i},\vec{j}) \, J(\vec{i},\vec{j}) \, S_\gamma(\vec{j})\right] \tag{3.5}$$

Near the equilibrium state and away from frustration lines, we expect $J(\vec{i},\vec{j})\vec{S}(\vec{j})$ to be almost parallel to $\vec{S}(\vec{i})$. For $J(\vec{i},\vec{j})$ negative this makes ambiguous the definition of a single-valued continuous field $\vec{S}(\vec{x})$ which takes the values $\vec{S}(\vec{i})$ and $\vec{S}(\vec{j})$ at $\vec{x}=\vec{i}$ and $\vec{x}=\vec{j}$ respectively and such that $\vec{S}(\vec{j}) \longrightarrow \vec{S}(\vec{i})$ as $\vec{j} \longrightarrow \vec{i}$ in the limit. Hence we define a double-valued field $\vec{S}(\vec{x})$, identifying $\vec{S}(\vec{x})$ with $-\vec{S}(\vec{x})$, such that $\vec{S}(\vec{j}) = J(\vec{i},\vec{j})\vec{S}(\vec{j})$. Notice that the manifold of $\vec{S}(\vec{x})$ is the projective plane P_2 and not the sphere S_2. One gets this result by taking the co-set

$$V \quad = \quad G/H \tag{3.6}$$

between the symmetry group G of the Hamiltonian and the isotropy group H which leaves $\vec{S}(\vec{x})$ invariant (Mermin 1979). In our case we can take $G = SO(3)$ and $H = D_{\infty}$ (rotations about the axis of $\vec{S}$ and rotations by π around axes perpendiculat to $\vec{S}$) since both spin orientations are identified. If we include inversion $G = O(3)$, but then $H = D_{\infty h}$ (D_{∞} plus inversion) hence V remains the same projective plane P_2.

We should wonder why starting with vectors (i.e. objects with direction <u>and</u> sense) in a discrete lattice we had finished with objects with direction but without sense in the continuous model. The reason is that to construct a continuous model one has to assume uniformity at least at local level, in order to be able to define derivatives. With negative $J(\vec{i},\vec{j})$ this is not possible if one insists on working with a single valued vector function.

Of course now in this model a line defect is topologically stable because the first homotopy group of the projective plane P_2 is the two element group Z_2.

The sum over $\vec{j}$ in (3.5) can be replaced by a sum over directions, and we shall use Latin subscripts to label directions in the real space.

In the continuous model the difference will be replaced by a covariant derivative. The j-covariant derivative of $\vec{S}(\vec{x})$ is by definition:

$$D_j S^\alpha(\vec{x}) = \frac{\partial}{\partial x^j} S^\alpha(\vec{x}) + \Gamma^\alpha_{\beta j}(\vec{x})\, S^\beta(\vec{x}) \tag{3.7}$$

where $\Gamma^\alpha_{\beta j}(\vec{i})$, the Christoffel symbols, can be related to the coefficients $A^\alpha{}_\beta\ (\vec{i},\vec{j})$ between nearest neighbours:

$$A^\alpha{}_\beta(\vec{i},\vec{j}) = \delta^\alpha_\beta - a\,\Gamma^\alpha_{\beta j}(\vec{i}) \tag{3.8}$$

where δ^α_β is the unit tensor and a is the lattice constant. Thus, in the continuous limit, we write (up to a multiplicative constant)

$$\mathcal{H} = \int d^3x\ (D_j S^\alpha)(D^j S_\alpha) \tag{3.9}$$

The spin space can be curved and a measure of its curvature is given evaluating the change in the component S^α when a "parallel transport" of $\vec{S}$ is made along a closed contour. "Parallel transport" here means parallel in the spin space. Thus the change after an infinitesimal parallel displacement is, by definition,

$$dS^\alpha = -\ \Gamma^\alpha_{\beta j}\ S^\beta\ dx^j \tag{3.10}$$

and the total change in a closed contour is

$$\Delta S^\alpha = \oint dS^\alpha = -\oint \Gamma^\alpha_{\beta j}\ S^\beta\, dx^j \tag{3.11}$$

Using Stokes theorem this becomes

$$\Delta S^{\alpha} = - \int ds^{j}\, \varepsilon_{j}^{\ \ell m} \frac{\partial}{\partial x^{\ell}} (\Gamma^{\alpha}_{\beta m} \quad S^{\beta})$$

$$= - \int ds^{j}\, \varepsilon_{j}^{\ \ell m} \quad (\frac{\partial}{\partial x^{\ell}} \Gamma^{\alpha}_{\beta m} \ ) S^{\beta} + \Gamma^{\alpha}_{\beta}{}_{m} (\frac{\partial}{\partial x^{\ell}} S^{\beta}) \tag{3.12}$$

(ds^{j} is the element of area normal to j-axis, and $\varepsilon_{j}^{\ \ell m}$ is the completely antisymmetric unit pseudotensor of rank 3).

For infinitesimal parallel displacement one can use (3.10) and write

$$\frac{\partial S^{\beta}}{\partial x^{\ell}} = - \Gamma^{\beta}_{\gamma\ell}\, S^{\gamma} \tag{3.13}$$

Taking this in Eqn.(3.12) and re-arranging the indices :

$$\Delta S^{\alpha} = - \int ds^{j}\, \varepsilon_{j}^{\ \ell m} \left[\frac{\partial}{\partial x^{\ell}} \Gamma^{\alpha}_{\beta m} \quad - \Gamma^{\alpha}_{\gamma m} \Gamma^{\gamma}_{\beta \ell} \right] S^{\beta} \tag{3.14}$$

We shall use the antisymmetric tensor $ds^{\ell m} = \quad \varepsilon_{k \ell m} dx^{\ell} dx^{m}$, and write

$$ds^{j} = \frac{1}{2} \varepsilon^{j}_{\ \ell m}\, ds^{\ell m} \tag{3.15}$$

Thus, using the identity

$$\varepsilon_{j}^{\ \ell m}\, \varepsilon^{j}_{\ ki} = \delta^{\ell}_{k} \delta^{m}_{i} - \delta^{\ell}_{i} \delta^{m}_{k} \tag{3.16}$$

Eqn.(3.14) becomes

$$\Delta S^{\alpha} = - \frac{1}{2} \int ds^{\ell m} \left[\frac{\partial}{\partial x^{\ell}} \Gamma^{\alpha}_{\beta m} - \frac{\partial}{\partial x^{m}} \Gamma^{\alpha}_{\gamma \ell} + \Gamma^{\alpha}_{\gamma \ell}\, \Gamma^{\gamma}_{\beta m} - \Gamma^{\alpha}_{\gamma m} \Gamma^{\gamma}_{\beta \ell} \right] S^{\beta}$$

(3.17)

Shrinking the circuit to a point and writing $\Delta s^{\ell m}$ for the value of the infinitesimal area enclosed by the contour Eqn.(3.17) becomes

$$\Delta s^{\alpha} = -\frac{1}{2}\left[\frac{\partial}{\partial x^{\ell}}\Gamma^{\alpha}_{\beta m} - \frac{\partial}{\partial x^{m}}\Gamma^{\alpha}_{\beta\ell} + \Gamma^{\alpha}_{\gamma\ell}\Gamma^{\gamma}_{\beta m} - \Gamma^{\alpha}_{\gamma m}\Gamma^{\gamma}_{\beta\ell}\right] s^{\beta}\Delta s^{\ell m} \tag{3.18}$$

from which we get finally

$$\Delta s^{\alpha} = -\frac{1}{2} R^{\alpha}_{\beta\ell m} s^{\beta}\Delta s^{\ell m} \tag{3.19}$$

where $R^{\alpha}_{\beta\ell m}$ is the curvature tensor:

$$R^{\alpha}_{\beta\ell m} = \frac{\partial}{\partial x^{\ell}}\Gamma^{\alpha}_{\beta m} - \frac{\partial}{\partial x^{m}}\Gamma^{\alpha}_{\beta\ell} + \Gamma^{\alpha}_{\gamma\ell}\Gamma^{\gamma}_{\beta m} - \Gamma^{\alpha}_{\gamma m}\Gamma^{\alpha}_{\beta\ell} \tag{3.20}$$

Consider now the constant vector field $\vec{S}(\vec{x})$. Constant meaning, here, they are parallel in the spin space. Then

$$s_{\beta}\, s^{\beta} = 1 \tag{3.21}$$

and

$$s_{\beta}\frac{\partial s^{\beta}}{\partial x^{\ell}} = 0 \tag{3.22}$$

For $\frac{\partial s^{\beta}}{\partial x^{\ell}}$ one can use Eqn.(3.13). Therefore

$$\Gamma^{\beta}_{\gamma\ell}\, s^{\gamma}\, s_{\beta} = 0 \tag{3.23}$$

If the local basis for the spin space is orthogonal, the covariant components of $\vec{S}$ will be the same as the contravariant ones and, in order to satisfy Eqn.(3.23), the symbols $\Gamma^{\beta}_{\gamma\ell}$ must be antisymmetric in the indices β and γ :

$$\Gamma^{\beta}_{\gamma\ell} = -\Gamma^{\gamma}_{\beta\ell} \tag{3.24}$$

We can define the new coefficients b^{γ}_{j} :

$$b^{\gamma}_{j} = \frac{1}{2}\,\varepsilon_{\alpha}{}^{\beta\gamma}\,\Gamma^{\alpha}_{\beta j} \tag{3.25}$$

or :

$$\Gamma^{\alpha}_{\beta j} = \varepsilon^{\alpha}{}_{\beta\gamma}\,b^{\gamma}_{j} \tag{3.26}$$

Instead of the curvature tensor $R^{\alpha}_{\beta\ell m}$ one can use $f^{\gamma}_{\ell m}$ (a pseudovector in spin space), defined as :

$$f^{\gamma}_{\ell m} = \frac{\partial b^{\gamma}_{m}}{\partial x^{\ell}} - \frac{\partial b^{\gamma}_{\ell}}{\partial x^{m}} - \varepsilon^{\gamma}{}_{\alpha\beta}\,b^{\alpha}_{\ell}\,b^{\beta}_{m} \tag{3.27}$$

which, as will be shown below, is related to $R^{\alpha}_{\beta\ell m}$ by :

$$f^{\gamma}_{\ell m} = \frac{1}{2}\,\varepsilon^{\gamma}{}_{\alpha}{}^{\beta}\,R^{\alpha}_{\beta\ell m} \tag{3.28}$$

To prove that we use Eqns.(3.20) and (3.26) :

$$\frac{1}{2}\,\varepsilon^{\gamma}{}_{\alpha}{}^{\beta}\,R^{\alpha}_{\beta\ell m} = \frac{1}{2}\,\varepsilon^{\gamma}{}_{\alpha}{}^{\beta}\left[\frac{\partial}{\partial x^{\ell}}\Gamma^{\alpha}_{\beta m} - \frac{\partial}{\partial x^{m}}\Gamma^{\alpha}_{\beta\ell} + \Gamma^{\alpha}_{\gamma\ell}\Gamma^{\gamma}_{\beta m} - \Gamma^{\alpha}_{\gamma m}\Gamma^{\gamma}_{\beta\ell}\right]$$

$$= \frac{1}{2}\,\varepsilon^{\gamma}{}_{\alpha}{}^{\beta}\left[\varepsilon^{\alpha}{}_{\beta}{}^{\mu}\left(\frac{\partial b^{\mu}_{m}}{\partial x^{\ell}} - \frac{\partial b^{\mu}_{\ell}}{\partial x^{m}}\right) + \varepsilon^{\alpha}{}_{\gamma\mu}\,\varepsilon^{\gamma}{}_{\beta\eta}\left(b^{\mu}_{\ell}b^{\eta}_{m} - b^{\mu}_{m}b^{\eta}_{\ell}\right)\right] \tag{3.29}$$

Using identity (3.16) and

$$\varepsilon^{\gamma}{}_{\alpha}{}^{\beta}\;\varepsilon^{\alpha}{}_{\beta}{}^{\mu} = 2\,\delta^{\gamma\mu} \tag{3.30}$$

We get :

$$\frac{1}{2}\,\varepsilon^{\gamma}{}_{\alpha}{}^{\beta}\;R^{\alpha}_{\beta\ell m} = \frac{\partial b^{\gamma}_{m}}{\partial x^{\ell}} - \frac{\partial b^{\gamma}_{\ell}}{\partial x^{m}} + \frac{1}{2}\,\varepsilon^{\gamma}{}_{\alpha}{}^{\beta}\left(-\delta^{\alpha}_{\beta}\,\delta_{\mu\eta} + \delta^{\alpha}_{\eta}\,\delta_{\mu\beta}\right)\cdot$$

$$\cdot\left(b^{\mu}_{\ell}\,b^{\eta}_{m} - b^{\mu}_{m}\,b^{\eta}_{\ell}\right) \qquad (3.31)$$

But

$$\varepsilon^{\gamma}{}_{\alpha}{}^{\beta}\;\delta^{\alpha}_{\beta} = 0 \qquad (3.32)$$

and

$$\frac{1}{2}\,\varepsilon^{\gamma}{}_{\alpha}{}^{\beta}\;\delta^{\alpha}_{\eta}\,\delta_{\mu\beta}\left(b^{\mu}_{\ell}\,b^{\eta}_{m} - b^{\mu}_{m}\,b^{\eta}_{\ell}\right) = \frac{1}{2}\,\varepsilon^{\gamma}{}_{\alpha}{}^{\beta}\left(b_{\ell\beta}\,b^{\alpha}_{m} - b_{m\beta}\,b^{\alpha}_{\ell}\right)$$

$$= -\,\varepsilon^{\gamma}{}_{\alpha\beta}\,b^{\alpha}_{\ell}\,b^{\beta}_{m} \qquad (3.33)$$

Then finally,

$$\frac{1}{2}\,\varepsilon^{\gamma}{}_{\alpha}{}^{\beta}\;R^{\alpha}_{\beta\ell m} = \frac{\partial b^{\gamma}_{m}}{\partial x^{\ell}} - \frac{\partial b^{\gamma}_{\ell}}{\partial x^{m}} - \varepsilon^{\gamma}{}_{\alpha\beta}\,b^{\alpha}_{\ell}\,b^{\beta}_{m} = f^{\gamma}_{\ell m} \qquad (3.34)$$

In electromagnetism of superconductors, both b^{γ}_{j} and f^{γ}_{lm} point in a unique direction γ in spin space. b^{γ}_{j} is the vector potential A_j and f^{γ}_{lm} is its rotational, the induction.

Now we shall put the gauge field on the same footing as the spin field taking it as a dynamical field. Hence one must include in the expression of $\mathcal{H}$ the energy of the gauge field itself, i.e. an invariant (under local gauge transformations) function of b^{γ}_{j}. The simplest invariant is $f^{\alpha}_{lm}\,f^{\alpha}_{lm}$, and the Hamiltonian reads :

$$\mathcal{H} = \int d^3x \; \left[(D_j S^\alpha)(D_j S^\alpha) \; + \; (8\pi \; k^2)^{-1} \; f^\alpha_{lm} \; f^\alpha_{lm} \right] . \tag{3.35}$$

(k is a constant).

It may look preposterous to promote b^γ_j to a dynamical field. After all, it was introduced as a coefficient of connection to take account of the arbitrariness in the choice of the frame of reference. From a field theoretical point of view it could be said that "every quantity that is not a pure numeral should be regarded as a dynamical variable, and should be varied in the Lagrangian to yield an equation of motion" (Yang and Mills, 1954, footnote). The point is that the gauge field has sources independent of the spin field. They are the frustration lines which impose curvature in spin space. But, differently to what is commonly stated in the literature, the gauge field b^γ_j (or the "induction" f^γ_{lm}) is not given by the frustration lines only, or equivalently, by a given distribution of $J(\vec{i},\vec{j})$'s. It is a dynamical variable, and labels the state of the total system as much as the matter (spin) field itself. And what is fixed by the quenching process is not b^γ_j (or f^γ_{lm}) but some "external field", now correctly described by the frustration lines. b^γ_j (or f^γ_{lm}) are response to this "external field".

3.II The X-Y model

For simplicity and to stress the analogy with superconductivity, we proceed to study the thermodynamics of a gauge invariant X-Y model. The spin field, constrained to lie on an easy plane in spin space, is described by the complex order parameter $\psi(\vec{x}) = S^X(\vec{x}) + i\,S^Y(\vec{x}) = \rho(\vec{x})\exp(i\,\theta\,(\vec{x}))$. The representative plane of the spin space (XY) is completely arbitrary in the ordinary space (x, y, z). It is chosen once and for all, and for the whole system. The symmetry group of the model is U(1) = SO(2) of rotations in the plane (XY). The gauge field to which $\psi(\vec{x})$ is coupled is here given by the "potential vector" $\vec{A}(\vec{x})$ and the invariant function of $\vec{A}$ is $|\mathrm{curl}\ \vec{A}|^2$.

The model is represented by the free energy density

$$f = \gamma \sum_j |D_j \psi(\vec{x})|^2 + \alpha |\psi(\vec{x})|^2 + \frac{\beta}{2} |\psi(\vec{x})|^4 + \frac{1}{8\pi} |\mathrm{curl}\ \vec{A}|^2 \tag{3.36}$$

where the covariant derivative is $D_j = \frac{\partial}{\partial x^j} - (i/k)A_j$, with k constant (see Eqn.1.15a), and $\alpha < 0$ if the spin have a finite length $\rho = \sqrt{|\alpha|/\beta}$. The localized XY model with well defined spin length would be given by the limit $\beta \rightarrow \infty$.

In terms of ρ and θ , f becomes

$$f = \gamma |\vec{\nabla}\rho|^2 + \gamma \rho^2 (\vec{\nabla}\theta - k^{-1}\vec{A})^2 + \alpha \rho^2 + \frac{\beta}{2}\rho^4 + \frac{1}{8\pi} |\mathrm{curl}\ \vec{A}|^2 \tag{3.37}$$

As discussed in Chapter 1, the transformation $\vec{C} = \vec{A} - k\vec{\nabla}\theta$ yields the Higgs-Kibble mechanism, i.e. a finite penetration depth $\lambda = k(1/8\pi\gamma\rho^2)^{1/2}$ and a third polarization component for the potential vector $\vec{C}$, which becomes a "massive vector particle".

External frozen sources are described by Lagrange multipliers $\vec{\Lambda}(\vec{x})$ which are localized at the sources of frustration. Since the frustrations are line defects, $\vec{\Lambda}$ is a Lagrange multiplier constraining the density of gauge field $\vec{B} = \mathrm{curl}\ \vec{A}$ to be finite along fixed lines or more properly, at the surface of fixed thin tubes which puncture the physical medium of our fields ρ and $\vec{C}$. $\vec{\Lambda}(\vec{x})$ vanishes outside the tubes.

The modified free energy density in the variational principle is, therefore, the Legendre transform of f, that is, the Gibbs free energy density g :

$$g = f - \vec{\Lambda} \cdot \frac{\delta f}{\delta(\text{curl } \vec{A})} = f - \frac{\vec{\Lambda} \cdot \text{curl } \vec{A}}{4\pi} \tag{3.38}$$

If the (Helmholtz) free energy, f, is a functional of the induction $\vec{B} = \text{curl } \vec{A}$, the Gibbs free energy, g, is a functional of the external frustration field $\vec{\Lambda}(\vec{x})$ which acts as a local chemical potential for the density of "disclination" $\vec{B}$ (Dzyaloshinskii and Volovik 1978). All this is quite familiar in superconductivity where we must work with the external field $\vec{H}$ instead of the magnetic induction $\vec{B}$ as an independent variable and with the Gibbs instead of the Helmholtz free energy.

Minimization of the total Gibbs free energy yields the Euler-Lagrange (or Ginzburg-Landau) equations:

$$-\gamma \nabla^2 \rho + (\gamma/k^2) C^2 \rho + \alpha\rho + \beta\rho^3 = 0 \tag{3.39}$$

$$8\pi(\gamma/k^2)\rho^2 \vec{C} + \text{curl curl } \vec{C} = \text{curl } \vec{\Lambda} \tag{3.40}$$

An unfrustrated (flat) model (Mattis 1976) would be represented by $\vec{\Lambda} = 0$. Then, the second Ginzburg-Landau equation has for solution $\vec{C} = 0$, thus the gauge field $\vec{A} = k\vec{\nabla}\theta$ is a pure gradient and the density of resulting disclination (the resulting magnetic induction in superconductivity) vanishes $\vec{B} = \text{curl } \vec{A} = 0$.

There are, in general, two solutions of the first Ginzburg-Landau equation:

(i) $\rho = 0$ everywhere. This is the stable, high temperature solution.

(ii) In the extreme type II limit ($\xi \longrightarrow 0$, see below) for $\alpha < 0$, $\rho = \rho_o = (-\alpha/\beta)^{1/2}$ everywhere except within a distance ξ of the core of the frustration lines. In general this solution corresponds to Abrikosov's mixed state of superconductivity.

Rescaling the fields, $p = \rho\ /(-\alpha\ /\beta)^{1/2}$, $\vec{a} = \frac{\vec{C}}{k}(\gamma\ /-\alpha)^{1/2}$, and $\vec{\ell} = \frac{\vec{\Lambda}}{k}(\gamma\ /-\alpha)^{1/2}$ emphasizes the two lengths of the problem, the matter field length ξ (the coherence length in superconductivity) and λ , the length associated with the gauge field (the penetration depth in superconductivity):

$$-\xi^{2}\ \nabla^{2} p - p + p(a^{2} + p^{2}) = 0 \tag{3.41}$$

$$\lambda^{-2}\ p^{2}\vec{a} + \text{curl curl}\ \vec{a} = \text{curl}\ \vec{\ell} \tag{3.42}$$

with

$$\xi = (\gamma\ /-\alpha)^{1/2} \quad \text{and} \quad \lambda = k(\beta\ /-8\pi\ \gamma\ \alpha\ )^{1/2} \tag{3.43}$$

The dimensionless ratio of the two lengths $\varkappa = \lambda\ /\ \xi = k\gamma\ (\beta\ /8\pi\ )^{1/2}$ determines in superconductivity whether the system is type I ($\varkappa < 1/\sqrt{2}$) or type II ($\varkappa > 1/\sqrt{2}$).

We recover the standard XY model with spin length $\rho = \rho_{o}$ = constant in the limit $\beta \longrightarrow \infty$, $|\alpha| \longrightarrow \infty$ and $\rho_{o}^{2} = |\alpha|\ /\ \beta$ finite. This because, in the limit, the probability for ρ at temperature T,

$$\exp[-(k_{B}T)^{-1}\int d^{3}x\,[\,|\alpha|\rho^{2} + \frac{1}{2}\beta\rho^{4}\,]\,] \longrightarrow$$

$$\longrightarrow [\,2\pi k_{B}T/\beta V]^{1/2} \exp[\,\frac{1}{2}\beta V\rho_{o}^{4}/(k_{B}T)\,]\ \delta(\rho^{2}-\rho_{o}^{2}) \tag{3.44}$$

(k_{B} = Boltzmann constant).

Then $\xi = (\gamma\ /\ \rho_{o}^{2}\ \beta\ )^{1/2} \longrightarrow 0$, $\lambda = k(8\ \pi\ \gamma\ \rho_{o}^{2})^{-1/2}$ remains finite, and $\varkappa \longrightarrow \infty$.

In a spin glass it is evident that we are in the extreme type II limit. Indeed, a spin glass is a system sprinkled with frustration tubes, while remaining fully magnetic, that is, ρ , the magnitude of the XY spin, is maintained approximately uniform everywhere except at the very core of each frustration tube ($\xi \longrightarrow 0$). If ρ was vanishing, the system would be paramagnetic in that region.

The microscopic inhomogeneity will be carried naturally by the gauge field $\vec{C}$ (which includes the gradient of the spin angle $\vec{\nabla}\theta$) over a finite length λ . We are therefore in the extreme type II limit where $\kappa \to \infty$.

The magnetic state of an extreme type II superconductor in a uniform external magnetic field is well known (see Chapter 1 or Tinkham 1975, Chapter 5). Superconductivity is local, and the electromagnetic properties are described by the simpler London's equation. The magnetic state is unique and depends neither on the magnetic history of the sample nor on the shape (demagnetizing effect), unlike type I superconductors, which can exhibit hysterectic and shape dependent effect.

Three critical fields are then involved: (1) The thermodynamic field Λ_c at which normal and uniform superconducting states have the same Gibbs free energy. (2) The two nucleation fields, Λ_{c_1} , for a magnetic vortex line to nucleate into a superconductor, and Λ_{c_2} , for superconductivity to nucleate into the normal state. For a magnetic field $\Lambda < \Lambda_{c_1}$, we have uniform superconductivity if $\Lambda > \Lambda_{c_2}$ the system is entirely normal, while for $\Lambda_{c_1} < \Lambda < \Lambda_{c_2}$ the system is in a mixed state of superconducting regions with normal vortex lines containing one quantum of magnetic flux. The thermodynamic critical field Λ_c lies between the two nucleation fields $\Lambda_{c_1} < \Lambda_c < \Lambda_{c_2}$. Finally the surface energy of a domain wall separating normal and superconducting regions is <u>negative</u> in type II superconductors, thus forcing normal domains to explode into quantum vortices in the mixed state.

Replace the magnetic field by frustration, and vortex lines by frustration lines, to visualize the situation in one spin glass. The difference lies in the particular boundary conditions encountered in one quenched spin glass, namely the frustration lines are given a priori, and have been put at random inside our sample. A spin glass as described by Eqns.(3.37) and (3.38) implies a concentration of frustration such that $\Lambda_{c_1} < \Lambda < \Lambda_{c_2}$. A realistic model of a spin glass requires $\varkappa = \lambda / \xi \rightarrow \infty$ to reduce the core of the frustration tubes as much as possible. A simple calculation (Tinkham 1975, Chapters 4 and 5), yields, in general,

$$\Lambda_c = (4\pi \; |\alpha|^2/\beta)^{1/2} \tag{3.45}$$

$$\Lambda_{c_2} = \sqrt{2} \; \varkappa \, \Lambda_c \tag{3.46}$$

and in the extreme type II limit $(\varkappa \rightarrow \infty)$

$$\Lambda_{c_1} = \frac{\Lambda_c}{\sqrt{2} \; \varkappa \, \ell n \, \varkappa} \tag{3.47}$$

Thus $\Lambda_{c_1} \sim 0$ and $\Lambda_{c_2} \sim \infty$ and the model given by Eqns.(3.22) and (3.23) has a comfortable region of stability.

It must be emphasized that the interface between spin glass $(\rho \neq 0)$ and paramagnetic regions (within the frustration tube, $\rho = 0$), have negative energy and thus the system naturally accommodates the frustration lines which had been put in a priori. The frustration lines can have only one single quantum of frustration, and cannot therefore split into a bundle of lines carrying a lesser amount of frustration. This leads us to look at the Lagrangian multiplier, the source of the frustration $\vec{\Lambda}(\vec{x})$ in more detail.

The frustration lines, being their own antidefect, must carry a field $\vec{\Lambda}(\vec{x})$ such that the circulation of the associated current, curl $\vec{\Lambda}$, around a path encircling the line an even number of times, vanishes. Thus, calling φ the angle of sweeping,

$$\phi(\varphi) = \int_{\varphi=0}^{\varphi} d\vec{s}\cdot \mathrm{curl}\vec{\Lambda}(\vec{x}) = 4\pi \text{ periodic function of } \varphi \tag{3.48}$$

The flux carried by a single frustration line has to be quantized for reasons analogous to that put forward in the case of superconductivity. But there are some differences. First, because the order parameter ψ is double-valued (as it was shown in Section I), the quantum is half vortex: $\phi_o = \pi k$, where k is the constant introduced in the definition of co-variant derivative in this model (Eqn.(3.36)). Second, the flux carried by a single frustration line can only be $\pm \phi_o$. This is a result of the Z_2 symmetry of the defect.

3.III <u>Gauge field around a frustration line</u>

The solution of the Ginzburg-Landau equations (3.39) and (3.40) in the case of one single frustration line can be obtained analytically in the extreme type II limit ($\xi \longrightarrow 0$) (Tinkham 1975, Chapter 5), and we simply recall it here. It is expected that, in this limit, and for the density of frustration not too close to Λ_{c_2}, the spins assume their full magnitude $\rho = \rho_o = (-\alpha/\beta)^{1/2}$, except within a distance ξ of the core of the frustration lines. The first Ginzburg-Landau equation (3.39) can indeed be rewritten as

$$-\xi^2 \nabla^2 p - p + p\left[(\xi/k)^2 C^2 + p^2\right] = 0 \tag{3.49}$$

C itself remains finite outside the frustration tube of radius ξ , since $\lambda >> \xi$ in the extreme type II limit.

In this case, it is clear from (3.49) that $p = 1$ everywhere except within the distance ξ of the core of the frustration line.

Taking the curl of the Eqn.(3.42), with $p = 1$, we obtain an equation for the density of gauge field $\vec{B} = \text{curl}\ \vec{C}$:

$$\lambda^{-2}\ \vec{B} + \text{curl}\,\text{curl}\,\vec{B} = \text{curl}\,\text{curl}\,\vec{\Lambda} \tag{3.50}$$

which corresponds to the London equation of superconductivity. Equation (3.34) is a Green's function equation for $\vec{B}$. Indeed, in the limit $\xi \longrightarrow 0$, the source in the right-hand side becomes a delta function in the plane perpendicular to the frustration line. The flux carried by the source is $\pm\,\phi_o$ *; so we write

$$\lambda^{-2}\ \vec{B} + \text{curl}\,\text{curl}\,\vec{B} = \pm\,\phi_o\ \hat{z}\ \ \delta^2(\vec{x}) \tag{3.51}$$

with the frustration line along the z-axis, and $\hat{z}$ the unit vector in that direction.

Clearly $\vec{B}$ is in the z-direction and Eqn.(3.51) is a Helmholtz equation in two-dimensions and the solution is the Hankel function $K_o(\,r\,/\,\lambda\,)$ of order zero and imaginary argument :

(* FOOTNOTE: We recall that $\vec{\Lambda}$ was introduced as a source of frustration. The term $-\ \vec{\Lambda}\cdot\text{curl}\,\vec{A}$ in the Gibbs free energy (3.38) was odd in $\vec{\Lambda}$, so that a change of sign in $\vec{\Lambda}$ implies a change of sign of $\vec{A}$ or of $\vec{B}$. This is indeed contained in Eqn.(3.50). But Eqn.(3.50) is itself a Green's function equation, and $\vec{\Lambda}$ is carried by thin tubes. The fact that $\vec{\Lambda}$ can have either sign is reflected in the orientability of the tubes and in a flux $\pm\,\phi_o$).

$$\vec{B}(\vec{x}) = \pm \hat{z} \quad \phi_o K_o(r/\lambda) \tag{3.52}$$

where r is the distance from the frustration line.

The fact that the frustration line can carry a flux $+ \phi_o$ or $- \phi_o$, means that the Ginzburg-Landau equations (3.39) and (3.40) have two possible solutions with either sign for $\vec{C}$, thus for the gauge field $\vec{A}$ and the change in circulation of the direction of the spins $\vec{\nabla}\theta$. These are the two solutions of Villain (1977) with opposite chiralities. Below, we shall show that both matter (ρ) and gauge $(\vec{C})$ fields can be integrated out to have frustration lines interacting in an Ising-like fashion, which represents a more rigorous derivation of Villain's result, generalised to a system of arbitrary dimension.

3.IV Interaction between frustration lines. Effective Hamiltonian.

The thermodynamics of the spin glass is given by the partition function

$$Z = \int \mathcal{D}\rho \; \mathcal{D}\vec{C} \; \exp\left[-(1/k_B T) \int d^d x \;\; g(\rho(\vec{x}), \vec{C}(\vec{x}), \vec{\Lambda}(\vec{x}))\right] \tag{3.53}$$

where $\int \mathcal{D}\rho \; \mathcal{D}\vec{C}$ means a functional integration over all possible configurations of ρ and $\vec{C}$, and g is the Gibbs free energy density given by Eqn.(3.38).

The functional integral in Eqn.(3.37) is dominated by the configuration of fields ρ and $\vec{C}$ which minimizes the total Gibbs free energy. We have shown that in the extreme type II limit $(\xi \longrightarrow 0)$, the magnitude of the spin field ρ could be taken as constant everywhere except in a small region very close to the core of the frustration lines. We shall take therefore $\rho = \rho_o$. Thus

$$g(\vec{x}) = g_o(\vec{x}) = \frac{\gamma \rho_o^2 C^2}{k^2} + \frac{1}{8\pi} \left| \operatorname{curl} \vec{C} \right|^2 - \frac{1}{4\pi} \vec{\Lambda} \cdot \operatorname{curl} \vec{C} \tag{3.54}$$

and the functional integration is only over configurations of $\vec{C}$

$$Z = \int \mathcal{D}\vec{C} \ \exp \left[-\frac{1}{k_B T} \int d^d x \left(\frac{\gamma \rho_o^2}{k^2} C^2 + \frac{1}{8\pi} \left| \operatorname{curl} \vec{C} \right|^2 - \frac{1}{4\pi} \vec{\Lambda} . \operatorname{curl} \vec{C} \right] \right. \tag{3.55}$$

The gauge invariance allows us to take $\vec{\nabla}\cdot\vec{C} = 0$. This and an integration by parts give

$$Z = \int \mathcal{D}\vec{C} \ \exp \left[-\frac{1}{k_B T} \int d^d x \left(\frac{\gamma \rho_o^2}{k^2} C^2 - \frac{1}{8\pi} \vec{C}.\nabla^2 \vec{C} - \frac{1}{4\pi} (\operatorname{curl} \vec{\Lambda}).\vec{C} \right) \right] \tag{3.56}$$

which can be written as

$$Z = \int \prod_i \mathcal{D}C_i \ \exp \left[-\frac{1}{2} \sum_i \int d^d x \int d^d y \ C_i(\vec{x}) \, M(\vec{x},\vec{y}) \, C_i(\vec{y}) - \sum_i \int d^d x \ X_i(\vec{x}) \, C_i(\vec{x}) \right] \tag{3.57}$$

where

$$\frac{1}{2} M(\vec{x},\vec{y}) = \frac{1}{k_B T} \delta(\vec{x}-\vec{y}) \left(\frac{\gamma \rho_o^2}{k^2} - \frac{1}{8\pi} \nabla^2 \right) \tag{3.58}$$

and

$$\vec{X}(\vec{x}) = -\frac{1}{4\pi \, k_B T} \operatorname{curl} \vec{\Lambda}(\vec{x}) \tag{3.59}$$

The Gaussian functional integral (3.57) can be evaluated without difficulty by filling the square (Edwards 1958) to yield

$$Z = N \exp \frac{1}{2} \int d^d x \int d^d y \sum_i X_i(\vec{x}) \, M^{-1}(\vec{x},\vec{y}) \, X_i(\vec{y}) \tag{3.60}$$

with the normalization

$$N = \int \Gamma_i \Gamma \, \mathcal{D} C_i \, \exp\left[-\frac{1}{2} \int d^d x \int d^d y \sum_i C_i(\vec{x}) \, M(\vec{x},\vec{y}) \, C_i(\vec{y}) \right. \tag{3.61}$$

Equation (3.60) indicates that the frustration currents, curl $\vec{\Lambda}$, of every line interact with every other through an Ising-like interaction in chirality. This was precisely the result of Villain, generalized here to the three-dimensional space, and obtained more directly. Moreover, it is clear that the specific character of the frustration lines, their bimodality, which is a fundamental feature of topological defect in glasses (Toulouse 1979, Rivier 1979), is independent of the character of the spin field and that a Heisenberg spin field and, correspondingly, a gauge field of the type introduced by Dzyaloshinskii and Volovik (1978), will not modify qualitatively the result (3,60), only the actual form of the interaction $M^{-1}(\vec{x},\vec{y})$.

The inverse operation $M^{-1}(\vec{x},\vec{y})$ is readily evaluated by using Fourier transforms :

$$M^{-1}(\vec{x},\vec{y}) = (2\pi)^{-d} \int d^d q \, d^d q' \, \exp(i\vec{q}.\vec{x}) \, M^{-1}(\vec{q},\vec{q}') \, \exp(-i\vec{q}'.\vec{y}) \tag{3.62}$$

with

$$M^{-1}(\vec{q},\vec{q}') = \frac{k_B T \quad \delta(\vec{q}-\vec{q}')}{2\left(\dfrac{\gamma \rho_o^2}{k^2} + \dfrac{q^2}{8\pi}\right)} \tag{3.63}$$

That is, with $\vec{r} = \vec{x} - \vec{y}$ and $\eta = 8\pi \gamma \rho_o^2 / k^2$

$$M^{-1}(r) = k_B T \quad J_d(r) \tag{3.64}$$

where

$$J_3(r) = e^{-\eta r}/r \quad \text{for} \quad d = 3 \tag{3.65}$$

$$J_2(r) = 2K_o(\eta r) \quad \text{for} \quad d = 2 \tag{3.66}$$

with K_o the zero order Hankel function of imaginary argument.

The effective energy of the frustration network, the exponent of Eqn.(4.59), $-E/k_BT$, becomes

$$E = -\frac{1}{2(4\pi)^2}\int d^dx \int d^dy \sum_i (\text{curl}\,\vec{\Lambda}(\vec{x}))_i \; J_d(r)(\text{curl}\,\vec{\Lambda}(\vec{y}))_i \tag{3.67}$$

That is an Ising-like interaction between the frustration currents. Eqn.(3.67) can also be written as an interaction between frustration lines:

$$E = \frac{1}{2(4\pi)^2}\int d^dx \int d^dy \sum_{km} \Lambda_k(\vec{x}) \left[\delta_{km} \sum_j \frac{\partial^2 J_d(r)}{\partial r_j^2} - \frac{\partial^2 J_d(r)}{\partial r_k \, \partial r_m} \right] \Lambda_m(\vec{y})$$

(3.68)

The range of interaction is $\eta^{-1} = (k/8\pi\gamma\rho_o^2)^{1/2} = \lambda$ (since $\rho_o \approx (-\alpha/\beta)^{1/2}$) as it should since the interaction is mediated by the gauge field.

Consider first the two dimensional problem $(d = 2)$.

The vector $\vec{\Lambda}$ is directed perpendicular to the plane and we have frustration points instead of lines. The integrations can be replaced by sums over frustration points :

$$\int dx \; \Lambda_3(x) f(x) \longrightarrow \sum_\alpha \tau_\alpha f(x_\alpha) \tag{3.69}$$

where τ_α is the frustration of the plaquette centred at α. Its sign is associated with the chirality of the plaquette. Moreover,

$$\sum_j \frac{\partial^2 K_o(\eta r)}{\partial r_j^2} - \frac{\partial^2 K_o(\eta r)}{\partial r_3 \, \partial r_3} = \eta^2 K_o(\eta r) \tag{3.70}$$

Thus

$$E = \frac{1}{2(4\pi\lambda)^2} \sum_{\alpha,\beta} \tau_\alpha K_o(r_{\alpha\beta}/\lambda)\tau_\beta \tag{3.71}$$

Equation (3.71) indicates that plaquette chirality have an antiferromagnetic, Ising-like interaction. Thus a pair of frustrated plaquettes will have opposite chirality in the ground state. This result has been obtained by Villain (1975). But here, the present of massive ($\rho \neq 0$) gauge fields gives the interaction finite range λ, even in the absence of an external field. For small distances, the interaction is logarithmic as in Villain.

$$K_o(x) \approx - \ln x \qquad (x \ll 1) \tag{3.72}$$

$$\approx \left(\frac{\pi}{2x}\right)^{1/2} \exp(-x) \qquad (x \rightarrow \infty) \tag{3.74}$$

In the extreme type II limit considered here, the matter field (the spins) playsa little role, in the absence of external magnetic field, apart from giving a mass and an additional degree of freedom to the gauge field (Higgs mechanism). It is in the presence of a magnetic field that the matter field plays its own, separate part.

The presence of an external magnetic field requires another Legendre transformation of the free energy, in addition to that imposing the correct source of frustration $\vec{\Lambda}$. Both the matter field (ρ, θ) and the gauge field $\vec{A}$ will be affected by this second Legendre transformation, in a way which is indicative of the non-orthogonality between magnetic (ρ, θ) and spin glass ($\vec{A}$) order parameters.

In three dimensions Eqn.(3.53) is more clearly rewritten as follows:

$$E = \frac{1}{2(4\pi)^2} \int d^3x \int d^3y \sum_k \Lambda_k(\vec{x}) \nabla^2 J_3(\eta r) \Lambda_k(\vec{y}) +$$

$$\frac{1}{2} \int d^3x \int d^3y \; \vec{\nabla} \cdot \vec{\Lambda}(\vec{x}) \; J_3(\eta r) \; \vec{\nabla} \cdot \vec{\Lambda}(\vec{y}) \tag{3.75}$$

Since we are taking $\vec{\Lambda}$ as lines carrying a fixed amount of frustration, $\vec{\nabla} \cdot \vec{\Lambda} = 0$. Furthermore, $\nabla^2 J_3(\ r) = \nabla^2(e^{-\eta r}/r) = \eta^2 J_3(r)$.

Thus:

$$E = \frac{1}{2(4\pi)^2} \int d^3x \int d^3y \sum_k \Lambda_k(\vec{x}) \, \eta^2 J_3(\ r) \, \Lambda_k(\vec{y}) \tag{3.76}$$

It is possible to integrate over the cross section of the frustration tube:

$$\int d^3x \;\; \Lambda(\vec{x}) \, f(\vec{x}) = \sum_\alpha \int d\vec{\rho}_\alpha \; \tau_\alpha \; f(\vec{x}_\alpha) \tag{3.78}$$

where $\vec{\rho}_\alpha$ is the line coordinate along the frustration line α and τ_α is the frustration flux carried by it. So, to an excellent approximation

$$E = \frac{1}{2(4\pi\lambda)^2} \sum_{\alpha,\beta} \int d\bar{\rho}_\alpha \cdot d\bar{\rho}'_\beta \; \tau_\alpha \; \frac{e^{-r_{\alpha\beta}/\lambda}}{r_{\alpha\beta}} \; \tau_\beta \tag{3.79}$$

3.V Conclusions

In this chapter the local gauge invariance of a system of spins was represented as nothing else other than the equivalence of all local frames of reference used to describe the spins. The gauge field is then related to the coefficients of connection between basis at different points. The advantages of this approach are twofold. First, in the absence of external magnetic field, the symmetry is exact even at small distances, therefore the use of covariant derivatives in place of differences between neighbours spins is justified. Should the local symmetry be violated at small distances, as it is argued by Dzyaloshinskii and Volovik, the use of covariant derivatives would be justified only if the matter field $\vec{S}(\vec{x})$ was defined as some average of the local spins taken over a region large enough to guarantee the fulfillment of the symmetry. Second, the dynamical character of the gauge field is not in conflict with the fixed configuration of frustration lines given a priori by the quenching process. Those lines are the external sources of the gauge field.

Another relevant point is the change in the manifold of internal states as one goes from the lattice model to the continuous one. In the former it is the sphere (S_2) while in the latter it becomes the projective plane (P_2), in which $\vec{S}(\vec{x})$ is identified with $-\vec{S}(\vec{x})$, making the order parameter double-valued. This was necessary in order to allow a proper definition of limits (and derivatives) in the continuous model.

The distinction between dynamical gauge fields and their fixed sources raises immediately the question of which thermodynamical potential to choose for the minimization procedure, which turns out to be the Gibbs free energy. At this point the similarity between the XY model of spin glass and type II superconductor in the presence of an external magnetic field is remarkable. With this model we get an effective interaction energy between frustration lines in agreement with previous work.

CHAPTER 4

MEAN FIELD SOLUTION OF THE GAUGE SPIN GLASS HAMILTONIAN

In the last chapter we introduced the gauge field in a spin glass system as a dynamical field whose sources were given by the frustration lines. Those are lines, in three dimensions, connecting the centers of neighbouring frustrated plaquettes without crossing the bonds, or traversing any unfrustrated plaquette, and in two dimensions they reduce to points at the center of the frustrated plaquettes. In particular we calculated (Section 3.III) the gauge field around a single frustration line and (Section 3.IV) the interaction energy between lines in the XY model. The trace operation in the partition function (Eqn.3.53) does not include the sources because the system is quenched: the frustration lines are frozen when the sample was prepared. Actually the observed thermodynamic quantities are the average of the equilibrium values taken over all possible configurations of frustration lines with some probability distribution given a priori. They correspond to an average over an ensemble of measurements. It is the ensemble average, over a distribution of frustration lines, which we are going to perform in this chapter. We shall use the replica method already mentioned in Chapter 2, to write the average of the free energy over configurations with a probability distribution assumed gaussian and white. The averaging introduces an effective interaction between replicas and we shall obtain the correlation functions for gauge fields and for matter fields in different replicas. In this chapter we shall also discuss the question of the stability of the symmetrical solutions in replica space. This is not a trivial question: de Almeida and Thouless (1978) have shown that the symmetrical solution of the Sherrington-Kirkpatrick model, which has been thought to be the simplest candidate for a mean field representation of the spin glass state, was unstable. In this model, we shall show that it is stable, and the mean field picture obtained in this chapter is consistent. Correlation between replicas in the absence of correlation within each replica, has been introduced by Edwards and Anderson as an index of spin glass ordering.

4.I Correlation functions for the gauge field

In the model presented in the last chapter, the partition function for a system of XY spins with a particular distribution of frustration lines is given by Eqn.(3.53):

$$Z(\vec{\Lambda}) = \int \mathcal{D}\rho \mathcal{D}\vec{C} \exp\left[-(1/k_B T)\int d^3x \, g(\rho(\vec{x}), \vec{C}(\vec{x}); \vec{\Lambda}(\vec{x}))\right] \tag{4.1}$$

with

$$g(\rho, \vec{C}; \vec{\Lambda}) = f(\rho, \vec{C}) - (1/4\pi)\vec{\Lambda}\cdot \operatorname{curl}\vec{C} \tag{4.2}$$

and

$$f(\rho, \vec{C}) = \gamma|\vec{\nabla}\rho|^2 + \alpha\rho^2 + (\beta/2)\rho^4 + (\gamma\rho^2/k^2)C^2 + (1/8\pi)|\operatorname{curl}\vec{C}|^2 \quad , \tag{4.3}$$

$$\vec{C} = \vec{A} - k\vec{\nabla}\theta \tag{4.4}$$

The quenching process introduces a random distribution of exchange interactions which can be reduced (after gauge transformations: Fradkin et al. 1978), to a random distribution of frustration lines $\vec{\Lambda}$, with some probability distribution $P(\vec{\Lambda})$.

As mentioned above, the relevant physical quantities are averages over the distribution $P(\vec{\Lambda})$. Thus, for the free energy we have

$$\langle G \rangle_{av} = -k_B T \int \mathcal{D}\vec{\Lambda}\, P(\vec{\Lambda}) \ln Z(\vec{\Lambda}) \tag{4.5}$$

The evaluation of the average of a log function is usually not an easy task and one resorts to the so-called replica method (Kac 1968; Edwards-Anderson 1975) using the identity

$$\ln Z = \lim_{n \to 0} n^{-1}(Z^n - 1) \tag{4.6}$$

Z^n can be seen as the partition function for the system replicated n times with the same $P(\vec{\Lambda})$. Eqn.(4.5) becomes

$$\langle G \rangle_{av} = -k_B T \lim_{n \to 0} n^{-1} \int \mathcal{D}\vec{\Lambda}\, P(\vec{\Lambda})(Z^n - 1) = -k_B T \lim_{n \to 0} n^{-1}(\langle Z^n \rangle_{av} - 1) \tag{4.7}$$

The limit $n \to 0$, (which may involve analytical continuation from n integer to a continuous variable) should be taken with some caution, but we will assume that it does not give aberrant results. (For a discussion of the problem, see Bray and Moore, 1979).

We will take the distribution of $\vec{\Lambda}$ gaussian and white with

$$\langle \Lambda_i(\vec{x}) \Lambda_j(\vec{x}') \rangle = e^2 \delta_{ij}\, \delta(\vec{x} - \vec{x}') \tag{4.8}$$

where e is a constant.* Thus

$$P(\vec{\Lambda}) = N \exp\left[(-1/2e^2) \int d^3x\, \Lambda^2(\vec{x}) \right] \tag{4.9}$$

with N a normalization constant.

With $P(\vec{\Lambda})$ given by Eqn.(4.8), $\langle Z^n \rangle_{av}$ becomes

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* We shall see in the discussion (Section 4.IV) that spin glass corresponds to $e^2 = -|e|^2$. The functional integrations (4.8-10) should then be regarded as a formal manipulation, with the divergence absorbed in the normalization constant N.

$$\langle Z^n\rangle_{av} = \int \mathcal{D}\vec{\Lambda}\, P(\vec{\Lambda}) \int \prod_{\alpha=1}^{n} \mathcal{D}\rho^{(\alpha)} \prod_{\alpha=1}^{n} \mathcal{D}\vec{C}^{(\alpha)} \exp\left[(-1/k_BT) \sum_{\alpha=1}^{n} g^{(\alpha)} \right]$$

$$= \int \prod_{\alpha} \mathcal{D}\rho^{(\alpha)} \int \prod_{\alpha} \mathcal{D}\vec{C}^{(\alpha)} \Big\{ \exp\left[(-1/k_BT) \sum_{\alpha} \int d^3x\, f^{(\alpha)} \right] \cdot \qquad (4.10)$$

$$\cdot N \int \mathcal{D}\vec{\Lambda} \exp\left[-\int d^3x \left((1/2e^2)\Lambda^2 - (1/4\pi k_BT)\vec{\Lambda}\cdot \sum_{\alpha} \operatorname{curl} \vec{C}^{(\alpha)} \right) \right] \Big\}$$

with the Greek letter referring to replicas and

$$f^{(\alpha)} = f(\rho^{(\alpha)}, \vec{C}^{(\alpha)}) \qquad (4.11)$$

$$g^{(\alpha)} = g(\rho^{(\alpha)}, \vec{C}^{(\alpha)}; \vec{\Lambda}) \qquad (4.12)$$

The integral in $\vec{\Lambda}$ can be done by filling the square, and Eqn.(4.10) becomes

$$\langle Z^n\rangle_{av} = \int \prod_{\alpha} \mathcal{D}\rho^{(\alpha)} \prod_{\alpha} \mathcal{D}\vec{C}^{(\alpha)} \exp(-1/k_BT) \int d^3x \left[\sum_{\alpha} f^{(\alpha)} - (e^2/32\pi^2 k_BT) \sum_{\alpha,\beta} \operatorname{curl}\vec{C}^{(\alpha)} \cdot \operatorname{curl}\vec{C}^{(\beta)} \right] \qquad (4.13)$$

It should be noticed that the averaging over $\vec{\Lambda}$ has introduced an effective interaction between replicas through their gauge fields.

$\langle Z^n\rangle_{av}$, given by Eqn.(4.13), is the partititon function for an effective free energy functional

$$G_{eff} = \int d^3x \left[\sum_{\alpha} f^{(\alpha)} - (e^2/32\pi^2 k_BT) \sum_{\alpha,\beta} \operatorname{curl}\vec{C}^{(\alpha)} \cdot \operatorname{curl}\vec{C}^{(\beta)} \right] \qquad (4.14)$$

Let us investigate the case when one neglects spatial fluctuations in ρ which will be assumed to take the constant value ρ_0 (extreme type II limit). This constant value ρ_0 is assumed further to be independent of the replica. The stability of this replica symmetric solution will be studied in the next section.

Defining the Fourier Transform

$$\vec{C}^{(\alpha)}(\vec{q}) = \int d^3x \; e^{i\vec{q}\cdot\vec{x}} \vec{C}^{(\alpha)}(\vec{x}) \tag{4.15}$$

and noticing that

$$\sum_{\alpha,\beta} \int d^3x \; \mathrm{curl}\, \vec{C}^{(\alpha)}(\vec{x}) \cdot \mathrm{curl}\, \vec{C}^{(\beta)}(\vec{x}) = \sum_{\alpha,\beta} \sum_{i,j} (2\pi)^{-3} \int d^3q \; C_i^{(\alpha)}(\vec{q}) \left(q^2 \delta_{ij} - q_i q_j \right) C_j^{(\beta)}(-\vec{q}) \tag{4.16}$$

one can write

$$\langle Z^n \rangle_{av} = Z_0 \int \prod_{\alpha} \prod_{i} \mathcal{D} C_i^{(\alpha)}(\vec{q}) \exp\left[-\frac{1}{2} \sum_{\alpha,\beta} \sum_{i,j} \int d^3q \; C_i^{(\alpha)}(\vec{q}) M_{ij}^{\alpha\beta}(\vec{q}) C_j^{(\beta)}(-\vec{q}) \right] \tag{4.17}$$

where

$$M_{ij}^{\alpha\beta}(\vec{q}) = \frac{2(2\pi)^{-3}}{k_B T} \left[\frac{\gamma \rho_0^2}{k^2} \delta_{\alpha\beta} \delta_{ij} + \left(\frac{1}{8\pi} \delta_{\alpha\beta} - \frac{e^2}{32\pi^2 k_B T} \right) \left(q^2 \delta_{ij} - q_i q_j \right) \right] \tag{4.18}$$

and $\quad Z_0 = \exp\left[-(nV/k_B T)(\alpha \rho_0^2 + \frac{1}{2} \beta \rho_0^4) \right]$

V is the volume of the sample.

The correlation function $\langle C_i^{(\alpha)}(\vec{q}) \, C_j^{(\beta)}(-\vec{q}) \rangle$ for the gauge field is the inverse of the hermitean matrix $[M_{ij}^{\alpha\beta}(\vec{q})]$ To show that we will write Eqn.(4.17) in a more compact form

$$Z_n \equiv \langle Z^n \rangle_{av} = \mathrm{Tr}\; e^{-\frac{1}{2} C^{+} M C} \tag{4.19}$$

where C here is just the column matrix formed with $\{ C_i^{(\alpha)}(\vec{q}) \}$.

Then we introduce an external field h coupling linearly with C and define $Z_n(h)$:

$$Z_n(h) = \mathrm{Tr}\, e^{-\frac{1}{2}C^{\dagger}MC + h^{\dagger}C} \tag{4.20}$$

Thus

$$\langle CC \rangle \equiv Z_n^{-1}\, \mathrm{Tr}\, CC\, e^{-\frac{1}{2}C^{\dagger}MC} = Z_n^{-1} \frac{\partial}{\partial h}\frac{\partial}{\partial h} Z_n(h)\Big|_{h=0} \tag{4.21}$$

With the transformation

$$C \rightarrow C - M^{-1}h \tag{4.22}$$

which does not affect the jacobian in the trace (4.17), one can write

$$Z_n(h) = Z_n\, e^{\frac{1}{2}h^{\dagger}M^{-1}h} \tag{4.23}$$

From Eqn.(4.23) it is easy to see that

$$\langle C_i^{(\alpha)}(\vec{q})\, C_j^{(\beta)}(-\vec{q}) \rangle = M^{-1}\,{}^{\alpha\beta}_{ij}(\vec{q}) \tag{4.24}$$

So now our problem is to invert the matrix M. That matrix can be written as

$$M_{ij}^{\alpha\beta}(\vec{q}) = \frac{2}{(2\pi)^3 k_B T}\left(A_{ij}(\vec{q})\,\delta_{\alpha\beta} + B_{ij}(\vec{q})\right) \tag{4.25}$$

with

$$A_{ij} = (\gamma\rho_0^2/k^2)\,\delta_{ij} + (1/8\pi)(q^2\delta_{ij} - q_i q_j) = \mu^2\delta_{ij} + (1/8\pi)(q^2\delta_{ij} - q_i q_j) \quad , \tag{4.26}$$

$$B_{ij} = -(e^2/32\pi^2 k_B T)(q^2\delta_{ij} - q_i q_j) = -\nu^2(q^2\delta_{ij} - q_i q_j) \tag{4.27}$$

where, for convenience, we defined

$$\mu^2 = \gamma \rho_o^2/k^2 \qquad ; \qquad \nu^2 = e^2/32\pi^2 k_B T \tag{4.28}$$

The inverse of a matrix like (4.25) is easily checked to be

$$M^{-1\,\alpha\beta}_{\;ij} = \frac{(2\pi)^3 k_B T}{2} \left([A^{-1}]_{ij}\,\delta_{\alpha\beta} - \{A^{-1}B[A+nB]^{-1}\}_{ij} \right) \tag{4.29}$$

where n is the number of replicas.

The matrix A is a 3 x 3 matrix and its inverse can be shown to be

$$A^{-1}{}_{ij} = \frac{1}{(\mu^2 + q^2/8\pi)} \left(\delta_{ij} + \frac{q_i q_i}{8\pi\,\mu^2} \right) \tag{4.30}$$

the correlation function of a massive vector boson (Landau (1972)§ 77)

Defining

$$\mu'^2 = 8\pi\mu^2/(1 - 8\pi n \nu^2) \qquad \text{and} \qquad \bar{\mu}^2 = 8\pi\,\mu^2 \tag{4.31}$$

one can write

$$[A+nB]_{ij} = \mu^2[\delta_{ij} + (1/\mu'^2)(q^2\delta_{ij} - q_i q_j)] \tag{4.32}$$

and its inverse

$$[A+nB]^{-1}{}_{ij} = \frac{1}{\mu^2(1+q^2/\mu'^2)} \left(\delta_{ij} + q_i q_j/\mu'^2\right) \tag{4.33}$$

With Eqns.(4.30) and (4.33) we finally get

$$M^{-1\,\alpha\beta}_{\;ij}(q) = \frac{(2\pi)^3 k_B T}{2} \left[\frac{\delta_{\alpha\beta}(\delta_{ij} + q_i q_j/\bar{\mu}^2)}{\mu^2(1+q^2/\bar{\mu}^2)} + \frac{\nu^2(q^2\delta_{ij} - q_i q_j)}{\mu^4(1+q^2/\bar{\mu}^2)(1+q^2/\mu'^2)} \right] \tag{4.34}$$

The first term is the correlation function for a massive vector boson. The second term represents the interaction between replicas. The correlation function in the real space can be obtained by Fourier transforming Eqn.(4.34). Let us do this term by term:

(a) $$\mathfrak{F}T\, \frac{1}{1+q^2/\bar{\mu}^2} = (2\pi)^{-3}\int d^3q\, e^{-i\vec{q}\cdot\vec{x}}\, \frac{1}{1+q^2/\bar{\mu}^2} = \frac{\bar{\mu}^2}{4\pi}\,\frac{e^{-\bar{\mu}r}}{r} \tag{4.35}$$

the Yukawa potential.

(b) $$\mathfrak{F}T\, \frac{q_i q_j}{1+q^2/\bar{\mu}^2} = \frac{1}{(-i)^2}\,\frac{\partial^2}{\partial x_i \partial x_j}\, \mathfrak{F}T\, \frac{1}{1+q^2/\bar{\mu}^2} = -\frac{\bar{\mu}^2}{4\pi}\,\frac{\partial^2}{\partial x_i\, \partial x_j}\,\frac{e^{-\bar{\mu}r}}{r}$$

$$= -\frac{\mu^4}{4\pi}\, I_{ij}(\bar{\mu}r)\,\frac{e^{-\bar{\mu}r}}{r} \tag{4.36}$$

where

$$I_{ij}(\bar{\mu}r) = \left(1 + \frac{3}{\bar{\mu}r} + \frac{3}{(\bar{\mu}r)^2}\right)\frac{x_i x_j}{r^2} - \left(\frac{1}{\bar{\mu}r} + \frac{1}{(\bar{\mu}r)^2}\right)\delta_{ij} \tag{4.37}$$

Notice that

$$\mathrm{Tr}\ I_{ij} = 1 \tag{4.38}$$

(c) $$\mathfrak{F}T\, \frac{q^2}{(1+q^2/\bar{\mu}^2)(1+q^2/\mu'^2)} = \frac{\bar{\mu}^2\mu'^2}{\mu'^2-\bar{\mu}^2}\left[\mathfrak{F}T\, \frac{1}{1+q^2/\mu'^2} - \mathfrak{F}T\, \frac{1}{1+q^2/\bar{\mu}^2}\right]$$

$$= \frac{\bar{\mu}^2\mu'^2}{\mu'^2-\bar{\mu}^2}\left[\frac{\mu'^2}{4\pi}\,\frac{e^{-\mu' r}}{r} - \frac{\bar{\mu}^2}{4\pi}\,\frac{e^{-\bar{\mu}r}}{r}\right] \tag{4.39}$$

$$\text{(d)}\quad \mathcal{F}\mathcal{T}\,\frac{q_i q_j}{(1+q^2/\bar{\mu}^2)(1+q^2/\mu'^2)} = \frac{1}{\mu'^2-\bar{\mu}^2}\left[\mu'^2\,\mathcal{F}\mathcal{T}\,\frac{q_i q_j}{1+q^2/\bar{\mu}^2} - \bar{\mu}^2\,\mathcal{F}\mathcal{T}\,\frac{q_i q_j}{1+q^2/\mu'^2}\right]$$

$$= -\frac{\mu'^2\bar{\mu}^2}{4\pi(\mu'^2-\bar{\mu}^2)}\left[\bar{\mu}^2 I_{ij}(\bar{\mu}r)\frac{e^{-\bar{\mu}r}}{r} - \mu'^2 I_{ij}(\mu' r)\frac{e^{-\mu' r}}{r}\right] \tag{4.40}$$

Collecting the terms together in Eqn.(4.34) we have

$$M^{-1\,\alpha\beta}_{\;\;ij}(\vec{x}) = \frac{(2\pi)^3 k_B T}{2}\left[\delta_{\alpha\beta}\delta_{ij}\frac{1}{\mu^2}\,\mathcal{F}\mathcal{T}\,\frac{1}{1+q^2/\bar{\mu}^2} + \delta_{\alpha\beta}\frac{1}{\mu^2\bar{\mu}^2}\,\mathcal{F}\mathcal{T}\,\frac{q_i q_j}{1+q^2/\bar{\mu}^2} + \right.$$

$$\left. + \delta_{ij}\frac{\nu^2}{\mu^4}\,\mathcal{F}\mathcal{T}\,\frac{q^2}{(1+q^2/\bar{\mu}^2)(1+q^2/\mu'^2)} - \frac{\nu^2}{\mu^4}\,\mathcal{F}\mathcal{T}\,\frac{q_i q_j}{(1+q^2/\bar{\mu}^2)(1+q^2/\mu'^2)}\right]$$

$$= \frac{(2\pi)^3 k_B T}{2}\left[\delta_{\alpha\beta}\delta_{ij}\frac{\bar{\mu}^2}{4\pi\mu^2}\frac{e^{-\bar{\mu}r}}{r} - \delta_{\alpha\beta}\frac{\bar{\mu}^2}{4\pi\mu^2}I_{ij}(\bar{\mu}r)\frac{e^{-\bar{\mu}r}}{r} + \right.$$

$$+ \delta_{ij}\frac{\nu^2\bar{\mu}^2\mu'^2}{4\pi\mu^4(\mu'^2-\bar{\mu}^2)}\left(\mu'^2\frac{e^{-\mu' r}}{r} - \bar{\mu}^2\frac{e^{-\bar{\mu}r}}{r}\right) +$$

$$\left. + \frac{\nu^2\bar{\mu}^2\mu'^2}{4\pi\mu^4(\mu'^2-\bar{\mu}^2)}\left(\bar{\mu}^2 I_{ij}(\bar{\mu}r)\frac{e^{-\bar{\mu}r}}{r} - \mu'^2 I_{ij}(\mu' r)\frac{e^{-\mu' r}}{r}\right)\right]$$

$$= (2\pi)^3 k_B T\,\frac{e^{-\bar{\mu}r}}{r}\left\{\delta_{\alpha\beta}(\delta_{ij} - I_{ij}(\bar{\mu}r)) + \right.$$

$$\left. + \frac{8\pi\nu^2}{1-\bar{\mu}^2/\mu'^2}\left[(\delta_{ij} - I_{ij}(\mu' r))\left(\frac{\mu'^2}{\bar{\mu}^2}\right)e^{-(\mu'-\bar{\mu})r} - (\delta_{ij} - I_{ij}(\bar{\mu}r))\right]\right\} \tag{4.41}$$

Now let us take the limit $n \to 0$. In this limit $\mu' \to \bar{\mu}$ and we will expand the expression in Eqn.(4.41) in powers of $\varepsilon = 8\pi n\nu^2$. To the first order in ε one can write

$$\frac{\mu'}{\bar{\mu}} = \frac{1}{\sqrt{1-\epsilon}} \cong 1 + \frac{1}{2}\epsilon \tag{4.42}$$

$$e^{-(\mu'-\bar{\mu})r} \cong 1-(\mu'-\bar{\mu})r \cong 1-\frac{1}{2}\epsilon\,\bar{\mu}r \tag{4.43}$$

and

$$(\mu'^2/\bar{\mu}^2)\,e^{-(\mu'-\bar{\mu})r} \cong (1+\epsilon)(1-\tfrac{1}{2}\epsilon\bar{\mu}r) = 1+\epsilon(1-\tfrac{1}{2}\bar{\mu}r) \quad ; \tag{4.44}$$

$$I_{ij}(\mu' r) = \left(1 + \frac{3(\bar{\mu}/\mu')}{\bar{\mu}r} + \frac{3(\bar{\mu}/\mu')^2}{(\bar{\mu}r)^2}\right)\frac{x_i x_j}{r^2} - \left(\frac{(\bar{\mu}/\mu')}{\bar{\mu}r} + \frac{(\bar{\mu}/\mu')^2}{(\bar{\mu}r)^2}\right)\delta_{ij}$$

$$= \left(1 + \frac{3(1-\epsilon/2)}{\bar{\mu}r} + \frac{3(1-\epsilon)}{(\bar{\mu}r)^2}\right)\frac{x_i x_j}{r^2} - \left(\frac{1-\epsilon/2}{\bar{\mu}r} + \frac{1-\epsilon}{(\bar{\mu}r)^2}\right)\delta_{ij}$$

$$= I(\bar{\mu}r) - \epsilon\left(\frac{3}{2\bar{\mu}r} + \frac{3}{(\bar{\mu}r)^2}\right)\frac{x_i x_j}{r^2} + \epsilon\left(\frac{1}{2\bar{\mu}r} + \frac{1}{(\bar{\mu}r)^2}\right)\delta_{ij} \tag{4.45}$$

Thus

$$(\delta_{ij} - I_{ij}(\mu' r))(\mu'^2/\bar{\mu}^2)\,e^{-(\mu'-\bar{\mu})r} =$$

$$= \delta_{ij} - I_{ij}(\bar{\mu}r) - \epsilon\left[\left(1-\frac{\bar{\mu}r}{2}\right)I_{ij}(\bar{\mu}r) + \left(\frac{3}{\bar{\mu}r} + \frac{3}{(\bar{\mu}r)^2}\right)\frac{x_i x_j}{r^2} - \left(\frac{1}{2\bar{\mu}r} \quad \frac{1}{(\bar{\mu}r)^2}\right)\delta_{ij}\right]$$

$$= \delta_{ij} - I_{ij}(\bar{\mu}r) + \epsilon\left[\delta_{ij}\left(\frac{1}{2} - \frac{\bar{\mu}r}{2}\right) + \frac{x_i x_j}{r}\left(\frac{1}{2} + \frac{\bar{\mu}r}{2}\right)\right] \tag{4.46}$$

So, with Eqn.(4.46) in Eqn.(4.41) we finally get :

$$\lim_{n\to 0} M^{-1}{}^{\alpha\beta}_{ij}(r) =$$

$$= (2\pi)^3 k_B T \, \frac{e^{-\bar{\mu} r}}{r} \left\{ \delta_{\alpha\beta}\left(\delta_{ij} - 1_{ij}(\bar{\mu} r)\right) + 4\pi\nu^2\left[\delta_{ij} + \frac{x_i x_j}{r^2} - \bar{\mu} r\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right)\right]\right\} \tag{4.47}$$

4.II Stability of the mean field solutions

So far we have assumed the matter field $\rho^{(\alpha)}$ assumes an equilibrium value ρ_0 independent of the replica. The problem we want to investigate now is the stability of such solution against the breaking of the symmetry between replicas. This study will be done within the approximation we have assumed throughout this chapter of neglecting the spatial non-uniformities in ρ_α, and investigating only non-uniformities between replicas.

Our starting point is the averaged partition function for n replicated system (Eqn.4.17).

$$\begin{aligned} \langle Z^n \rangle_{av} = & \int \prod_{\alpha=1}^{n} \mathcal{D}\rho^{(\alpha)} \exp \frac{-V}{k_B T} \sum_{\alpha} \left[\alpha \rho^{(\alpha)^2} + \frac{\beta}{2} \rho^{(\alpha)^4} \right] . \\ & . \int \prod_{\alpha, i} \mathcal{D}C_i^{(\alpha)} \exp -\frac{1}{2} \sum_{\alpha,\beta} \sum_{i,j} \int d^3q \, C_i^{(\alpha)}(\vec{q}) M_{ij}^{\alpha\beta}(\vec{q}) C_j^{(\beta)}(-\vec{q}) \end{aligned} \tag{4.48}$$

where V is the volume of the system and $M_{ij}^{\alpha\beta}(\vec{q})$ is the matrix given by Eqn.(4.18) :

$$\begin{aligned} M_{ij}^{\alpha\beta}(\vec{q}) & = \frac{2}{(2\pi)^3 k_B T} \left[\frac{\gamma \rho^{(\alpha)^2}}{k^2} \delta_{\alpha\beta} \delta_{ij} + \left(\frac{1}{8\pi} \delta_{\alpha\beta} - \frac{e^2}{32\pi^2 k_B T} \right) \left(q^2 \delta_{ij} - q_i q_j \right) \right] \\ & = \frac{1}{2(2\pi)^4 k_B T} \left[\bar{\mu}_\alpha^2 \, \delta_{\alpha\beta} \delta_{ij} + (\delta_{\alpha\beta} - 8\pi\nu^2)(q^2 \delta_{ij} - q_i q_j) \right] \end{aligned} \tag{4.49}$$

with

$$\bar{\mu}_\alpha^2 = \frac{8\pi\gamma\rho^{(\alpha)^2}}{k^2} \qquad \text{and} \qquad \nu^2 = \frac{e^2}{32\pi^2 k_B T} \tag{4.50}$$

In all integrations in $\vec{q}$, an upper cut-off Λ , of the order of the inverse of the lattice constant, is to be understood.

The integration in the gauge field is gaussian and can be evaluated by the standard method with the result

$$\int \prod_{\alpha=1}^{n} \prod_{i=1}^{3} \mathcal{D}C_i^{(\alpha)} \exp \frac{-1}{2} \sum_{\alpha,\beta} \sum_{i,j} \int d^3q \, C_i^{(\alpha)}(\vec{q}) M_{ij}^{\alpha\beta}(\vec{q}) C_j^{(\beta)}(-\vec{q}) =$$

$$= \exp V \int_{q<\Lambda} d^3q \left[\frac{3n}{2} \ln 2\pi - \frac{1}{2} \ln \det M \right] \tag{4.51}$$

To calculate $\det M$ we first notice that it is the same as $\det \widetilde{M}$, with

$$\widetilde{M}_{ij}^{\alpha\beta}(\vec{q}) = \frac{1}{2(2\pi)^4 k_B T} \left[\bar{\mu}_\alpha^2 \delta_{\alpha\beta} + (\delta_{\alpha\beta} - 8\pi\nu^2) Q_i \right] \delta_{ij} \tag{4.52}$$

where the Q_i's are the eigenvalues of $q^2\delta_{ij} - q_i q_j$. These eigenvalues are 0, q^2, q^2. We then write

$$\det M = \det \widetilde{M} = \prod_i \det_\alpha \left\{ \frac{1}{2(2\pi)^4 k_B T} \left[(\bar{\mu}_\alpha^2 - Q_i) \delta_{\alpha\beta} - 8\pi\nu^2 Q_i \right] \right\} \tag{4.53}$$

The determinant of a matrix of the type $D_\alpha \delta_{\alpha\beta} - E$ of order n, $\Delta(n)$, is related to that of order n-1 by the following recursion relation:

$$\Delta(n) = D_n \Delta(n-1) - E \prod_{\alpha=1}^{n-1} D_\alpha \tag{4.54}$$

To show that, one substracts each column from the next, and expand the determinant in the last row.

Below we illustrate for the case $n = 4$:

$$\begin{vmatrix} D_1-E & -E & -E & -E \\ -E & D_2-E & -E & -E \\ -E & -E & D_3-E & -E \\ -E & -E & -E & D_4-E \end{vmatrix} = \begin{vmatrix} D_1-E & -D_1 & 0 & 0 \\ -E & D_2 & -D_2 & 0 \\ -E & 0 & D_3 & -D_3 \\ -E & 0 & 0 & D_4 \end{vmatrix}$$

$$= D_4 \begin{vmatrix} D_1-E & -D_1 & 0 \\ -E & D_2 & -D_2 \\ -E & 0 & D_3 \end{vmatrix} - (-E) \begin{vmatrix} -D_1 & 0 & 0 \\ D_2 & -D_2 & 0 \\ 0 & D_3 & -D_3 \end{vmatrix}$$

$$= D_4\,\Delta(3) - E \prod_{\alpha=1}^{3} D_\alpha$$

One can check by induction that

$$\Delta(n) = \Big(\prod_{\alpha=1}^{n} D_\alpha\Big)\Big[1 - E\sum_{\alpha=1}^{n} \frac{1}{D_\alpha}\Big] \tag{4.55}$$

is the solution for Eqn.4. 54 , and gives the correct value for $n = 1$.
Thus

$$\det M = \left(\frac{1}{2(2\pi)^4 k_B T}\right)^{3n} \prod_\alpha (\bar{\mu}_\alpha^2 + Q_i)\left(1 - 8\pi\nu^2 \sum_\alpha \frac{1}{\bar{\mu}_\alpha^2 + Q_i}\right) \tag{4.56}$$

that is

$$\ln \det M = 3n \ln\left(\frac{1}{2(2\pi)^4 k_B T}\right) + \sum_\alpha \ln \bar{\mu}_\alpha^2 + 2\sum_\alpha \ln(\bar{\mu}_\alpha^2 + q^2) + 2\ln\left(1 - 8\pi\nu^2 \sum_\alpha \frac{q^2}{\bar{\mu}_\alpha^2 + q^2}\right) \tag{4.57}$$

The factor $\bar{\mu}_\alpha^2$ in the $\det M$, which comes from the eigenvalue $Q_i = 0$, corresponds to the longitudinal gauge modes (the modes corresponding to the eigenvalue 0 of $q^2\delta_{ij} - q_i q_j$). It is spurious in the sense that it can be taken away by a convenient choice of gauge. Indeed, all stable configurations of $\vec{C}$ (given by solutions of Eqn.3.40) satisfy, in the approximation of ρ = constant, the condition $\nabla\cdot\vec{C} = 0$, thus the longitudinal mode can be gauged away. Notice that, while such gauge can always be chosen for the field $\vec{A} = \vec{C} + k\vec{\nabla}\theta$, this is not true in general for the field $\vec{C}$, if ρ is non-uniform. Thus, in our approximation of uniform ρ, we do not have to include the term $\ln\bar{\mu}_\alpha^2$ in the exponential when evaluating the partition function. Then the partition function becomes

$$\langle Z^n\rangle_{av} = \int\prod_\alpha \mathcal{D}\bar{\mu}_\alpha \exp\frac{-V}{k_BT}\Big\{\sum_\alpha\Big[\bar{\alpha}\bar{\mu}_\alpha^2 + \frac{\bar{\beta}}{2}\bar{\mu}_\alpha^4 + k_BT\int d^3q \ln\Big(1+\frac{\bar{\mu}_\alpha^2}{q^2}\Big)\Big] + k_BT\int d^3q \ln\Big(1 - 8\pi\nu^2\sum_\alpha\frac{1}{1+\bar{\mu}_\alpha^2/q^2}\Big)\Big\} \tag{4.58}$$

where

$$\bar{\alpha} = \alpha k^2/8\pi\gamma \qquad \text{and} \qquad \bar{\beta} = \beta k^4/(8\pi\gamma)^2 \tag{4.59}$$

In the mean field approximation the equilibrium configuration is that which maximizes the integrand in the partition function, i.e., that which minimizes the free energy functional. From Eqn.4.58 , the effective free energy density is

$$g_{eff} = \sum_\alpha\Big[\bar{\alpha}\bar{\mu}_\alpha^2 + \frac{\bar{\beta}}{2}\bar{\mu}_\alpha^4 + k_BT\int d^3q \ln\Big(1+\frac{\bar{\mu}_\alpha^2}{q^2}\Big)\Big] + k_BT\int d^3q \ln\Big(1 - 8\pi\nu^2\sum_\alpha\frac{1}{1+\bar{\mu}_\alpha^2/q^2}\Big) \tag{4.60}$$

The extremization of g_{eff} is given by

$$\frac{\partial g_{eff}}{\partial\bar{\mu}_\alpha} = 2\bar{\alpha}\bar{\mu}_\alpha + 2\bar{\beta}\bar{\mu}_\alpha^3 + k_BT\int d^3q\,\frac{2\bar{\mu}_\alpha}{(q^2+\bar{\mu}_\alpha^2)} + 8\pi\nu^2k_BT\int d^3q\,\frac{2\bar{\mu}_\alpha q^2}{(q^2+\bar{\mu}_\alpha^2)^2\Big(1-8\pi\nu^2\sum_\beta\frac{1}{q^2+\bar{\mu}_\beta^2}\Big)} = 0 \tag{4.61}$$

There are two solutions symmetrical in the replicas: $\bar{\mu}_\alpha = 0$ and $\bar{\mu}_\alpha = \bar{\mu} \neq 0$. In this last case Eqn.(4.61) becomes

$$\bar{\alpha} + \beta\bar{\mu}^2 + k_BT\int d^3q\,\frac{1}{q^2+\bar{\mu}^2} + \frac{8\pi\nu^2k_BT}{1-8\pi\nu^2n}\int d^3q\,\frac{q^2}{(q^2+\bar{\mu}^2)(q^2+\mu'^2)} = 0 \tag{4.62}$$

with $\mu'^2 = \bar{\mu}^2/(1-8\pi n\nu^2)$, as in Eqn.(4.31).

After the integrations, with the cut-off Λ for q, Eqn.(4.62) becomes

$$\bar{\alpha} + \bar{\beta}\bar{\mu}^2 + k_B T \cdot 4\pi\left(\Lambda - \bar{\mu}\tan^{-1}\frac{\Lambda}{\bar{\mu}}\right) + \frac{8\pi\nu^2 k_B T}{(1-8\pi n\nu^2)} \frac{4\pi}{\mu'^2 - \bar{\mu}^2}\left[(\mu'^2-\bar{\mu}^2)\Lambda + \bar{\mu}^3\tan^{-1}\frac{\Lambda}{\bar{\mu}} - \mu'^3\tan^{-1}\frac{\Lambda}{\mu'}\right] \quad (4.63)$$

In the limit $8\pi n\nu^2 = \varepsilon \rightarrow 0$ one can write

$$\mu'^2 - \bar{\mu}^2 \cong \bar{\mu}^2\varepsilon \quad (4.64)$$

$$\mu'^3\tan^{-1}\frac{\Lambda}{\mu'} \cong \bar{\mu}^3\tan^{-1}\frac{\Lambda}{\bar{\mu}} + \bar{\mu}^3\varepsilon\left[\frac{3}{2}\tan^{-1}\frac{\Lambda}{\bar{\mu}} - \frac{\bar{\mu}/\Lambda}{2(1+\bar{\mu}^2/\Lambda^2)}\right] -$$

$$- \bar{\mu}^3\varepsilon^2\left[\frac{15}{8}\tan^{-1}\frac{\Lambda}{\bar{\mu}} - \frac{7\,\bar{\mu}/\Lambda}{8(1+\bar{\mu}^2/\Lambda^2)}\right] \quad (4.65)$$

In Eqn.(4.65) we did the expansion to second order in ε for later use, but in Eqn.(4.63) we only need the first order. Then, taking Eqn.(4.64) and Eqn.(4.65) in Eqn.(4.63) we have

$$\bar{\alpha} + \beta\bar{\mu}^2 + k_B T\, 4\pi\Lambda\left(1 - \frac{\bar{\mu}}{\Lambda}\tan^{-1}\frac{\Lambda}{\bar{\mu}}\right) + 8\pi\nu^2 k_B T\, 4\pi\Lambda\left(1 - \frac{3\bar{\mu}}{2\Lambda}\tan^{-1}\frac{\Lambda}{\bar{\mu}} + \frac{\bar{\mu}^2/\Lambda^2}{2(1+\bar{\mu}^2/\Lambda^2)}\right) = 0 \quad (4.66)$$

Λ is very large and $\bar{\mu}$, near the transition, small, so we expand the above expression in powers of $\bar{\mu}/\Lambda$. Correct to first order in $\bar{\mu}/\Lambda$, it becomes

$$\bar{\alpha} + \bar{\beta}\bar{\mu}^2 + k_B T\, 4\pi\Lambda\left(1 - \frac{\pi}{2}\frac{|\bar{\mu}|}{\Lambda}\right) + 8\pi\nu^2 k_B T\, 4\pi\Lambda\left(1 - \frac{3\pi}{4}\frac{|\bar{\mu}|}{\Lambda}\right) = 0 \quad (4.67)$$

or

$$\bar{\alpha} + \bar{\beta}\,\bar{\mu}^2 + (4\pi\Lambda k_B T + e^2\Lambda) - (2\pi^2 k_B T + 3\pi e^2/4)|\bar{\mu}| = 0 \tag{4.68}$$

We see already the two effects of the gauge field : (i) a shift in the transition temperature, and (ii) a change of the order of the phase transitions from second to first order, if $e^2 = 0$. This is due to the term linear in $|\bar{\mu}|$ in Eqn.(4.68) which corresponds, in g_{eff}, to a term in $-\gamma|\bar{\mu}|^3$ where γ is negative. The change of the transition from second order in the absence of gauge fields to (weakly) first order in their presence was discussed by Halperin, Lubensky and Ma (1974) in the case of superconductivity, and our result Eqn.(4.68) and the relevant g_{eff} are identical to theirs in the absence of frustration, $e^2 = 0 = \nu^2$ (flat model).

The frustrated case, $e^2 \neq 0$, will be discussed at the end of the section.

But we have not yet investigated the stability of the symmetrical mean field solutions given above. For that purpose we will write $\bar{\mu}_\alpha = \bar{\mu} + y_\alpha$, where y_α is a small replica dependent deviation from the symmetrical solution. Then we expand g_{eff} in powers of y_α's. The terms linear in y_α will cancel due to the extremization condition 4.61 The difference $g_{eff}(\{\bar{\mu}+y_\alpha\}) - g_{eff}(\{\bar{\mu}\})$ is written as $\sum_{\alpha,\beta} g_{\alpha\beta}\, y_\alpha y_\beta$ and the stability of the solution is investigated by finding the eigenvalues of $g_{\alpha\beta}$. If it has negative eigenvalue the solution is unstable.

Let us expand the log functions in g_{eff}. For $\bar{\mu} \neq 0$ we can write

$$\text{(a)}\quad \ln\left(1+\bar{\mu}_\alpha^2/q^2\right) = \ln\left(1+\frac{\bar{\mu}^2}{q^2} + \frac{2\bar{\mu}\,y_\alpha + y_\alpha^2}{q^2}\right)$$

$$\cong \ln\left(1+\frac{\bar{\mu}^2}{q^2}\right) + \frac{2\bar{\mu}}{\bar{\mu}^2+q^2}\,y_\alpha - \frac{2\bar{\mu}^2}{(\bar{\mu}^2+q^2)^2}\,y_\alpha^2 + \frac{y_\alpha^2}{\bar{\mu}^2+q^2} \tag{4.69}$$

$$(b)\ \ln\left(1-8\tilde{\pi}\nu^2\sum_\alpha \frac{q^2}{q^2+(\bar{\mu}+y_\alpha)^2}\right) \cong \ln\left(1-8\tilde{\pi}\nu^2\sum_\alpha \frac{q^2}{\bar{\mu}^2+q^2}\left(1-\frac{2\bar{\mu}y_\alpha+y_\alpha^2}{\bar{\mu}^2+q^2}+\frac{4\bar{\mu}^2y_\alpha^2}{(\bar{\mu}^2+q^2)^2}\right)\right)$$

$$=\ln\left(1-\frac{8\tilde{\pi}n\nu^2}{\bar{\mu}^2+q^2}+\frac{8\tilde{\pi}\nu^2q^2\,2\bar{\mu}}{(\bar{\mu}^2+q^2)}\sum_\alpha y_\alpha+\frac{8\tilde{\pi}\nu^2q^2}{(\bar{\mu}^2+q^2)}\left[\frac{1}{\bar{\mu}^2+q^2}-\frac{4\bar{\mu}^2}{(\bar{\mu}^2+q^2)^2}\right]\sum_\alpha y_\alpha^2\right.$$

$$\cong \ln\left(1-\frac{8\tilde{\pi}\nu^2nq^2}{\bar{\mu}^2+q^2}\right)+\frac{8\tilde{\pi}\nu^2q^2.\,2\bar{\mu}}{(\bar{\mu}^2+q^2)[\bar{\mu}^2+(1-8\tilde{\pi}n\nu^2)q^2]}\sum_\alpha y_\alpha\ +$$

$$+\frac{8\tilde{\pi}\nu^2q^2}{[\bar{\mu}^2+(1-8\tilde{\pi}n\nu^2)q^2]}\left[\frac{1}{\bar{\mu}^2+q^2}-\frac{4\bar{\mu}^2}{(\bar{\mu}^2+q^2)}\right]\sum_\alpha y_\alpha^2-\frac{(8\tilde{\pi}\nu^2q^2.\,2\bar{\mu})^2\left(\sum_\alpha y_\alpha\right)^2}{2[\bar{\mu}^2+(1-8\tilde{\pi}n\nu^2)q^2]^2(\bar{\mu}^2+q^2)} \qquad (4.70)$$

Taking these expansions to Eqn.(4.60), the linear terms cancel out due to Eqn.(4.61) as expected, and one writes :

$$\frac{1}{n}\left[g_{eff}(\{\bar{\mu}+y_\alpha\})-g_{eff}(\{\bar{\mu}\})\right]=\left\{\bar{\alpha}+3\bar{\beta}\bar{\mu}^2+k_BT\int d^3q\ \frac{1-\bar{\mu}^2/q^2}{q^2(1+\bar{\mu}^2/q^2)^2}\right.\ +$$

$$\left.+\,k_BT\ \frac{8\tilde{\pi}\nu^2}{1-8\tilde{\pi}n\nu^2}\int d^3q\ \frac{1}{q^2(1+\mu'^2/q^2)}\left[\frac{1}{1+\bar{\mu}^2/q^2}-\frac{4\bar{\mu}^2/q^2}{(1+\bar{\mu}^2/q^2)^2}\right]\right\}\sum_\alpha y_\alpha^2\ -$$

$$-\,2k_BT\left(\frac{8\tilde{\pi}\nu^2}{1-8\tilde{\pi}n\nu^2}\right)^2\int d^3q\ \frac{\bar{\mu}^2/q^2}{q^2(1+\mu'^2/q^2)^2(1+\bar{\mu}^2/q^2)^2}\left(\sum_\alpha y_\alpha\right)^2$$

$$=\sum_{\alpha,\beta} g_{\alpha\beta}\,y_\alpha\,y_\beta \qquad (4.71)$$

can be written as

$$g_{\alpha\beta} = g_1 \delta_{\alpha\beta} + g_2 \tag{4.72}$$

where

$$g_1 = \bar{\alpha} + 3\bar{\beta}\bar{\mu}^2 + k_B T \int d^3q \; \frac{1 - \bar{\mu}^2/q^2}{q^2(1+\bar{\mu}^2/q^2)^2} + \frac{8\pi\nu^2 k_B T}{1 - 8\pi n\nu^2} \int d^3q \; \frac{1}{q^2(1+\mu'^2/q^2)} \left[\frac{1}{1+\bar{\mu}^2/q^2} - \frac{4\bar{\mu}^2/q^2}{(1+\bar{\mu}^2/q^2)^2} \right] \tag{4.73}$$

$$g_2 = -2\left(\frac{8\pi\nu^2}{1-8\pi n\nu^2}\right)^2 k_B T \int d^3q \; \frac{\bar{\mu}^2/q^2}{q^2(1+\mu'^2/q^2)^2(1+\bar{\mu}^2/q^2)^2} \tag{4.74}$$

To get the eigenvalues of $g_{\alpha\beta}$ one has to solve the equation

$$\det | g_{\alpha\beta} - \lambda \delta_{\alpha\beta} | = \det | (g_1 - \lambda) \delta_{\alpha\beta} + g_2 | = 0 \tag{4.75}$$

The determinant of a matrix like that was obtained at the beginning of this section. We can see, by the same method, that

$$\det (g_{\alpha\beta} - \lambda \delta_{\alpha\beta}) = (g_1 - \lambda)^{n-1} (g_1 - \lambda + n g_2) \tag{4.76}$$

Therefore $g_{\alpha\beta}$ has two different eigenvalues: $\lambda_1 = g_1$, which is n-1 times degenerate and $\lambda_2 = g_1 + n g_2$. In the limit $n \to 0$, $\lambda_2 \to \lambda_1$, hence we have to analyse only $\lambda_1 = g_1$ and see if it can become negative.

Let us do the integrals in g_1, imposing the cut-off Λ for large q :

(a) $$\int d^3q \; \frac{1-\bar{\mu}^2/q^2}{q^2(1+\bar{\mu}^2/q^2)^2} = 4\pi \int_0^\Lambda dq \left[\frac{q^4}{(\bar{\mu}^2+q^2)^2} - \frac{\bar{\mu}^2 q^2}{(\bar{\mu}^2+q^2)^2} \right]$$

$$= 4\pi \left[\Lambda + \frac{\bar{\mu}^2\Lambda}{(\Lambda^2+\bar{\mu}^2)} - \frac{3\bar{\mu}}{2} \tan^{-1}\frac{\Lambda}{\bar{\mu}} - \bar{\mu}^2\left(- \frac{\Lambda}{2(\Lambda^2+\bar{\mu}^2)} + \frac{1}{2\bar{\mu}} \tan^{-1}\frac{\Lambda}{\bar{\mu}} \right) \right]$$

$$= 4\pi\Lambda \left[1 + \frac{3\bar{\mu}^2/\Lambda^2}{2(1+\bar{\mu}^2/\Lambda^2)} - 2(\bar{\mu}/\Lambda)\tan^{-1}\frac{\Lambda}{\bar{\mu}} \right] \tag{4.77}$$

(b)

$$\int d^3q \; \frac{1}{q^2(1+\mu'^2/q^2)(1+\bar{\mu}^2/q^2)} = 4\pi \int_0^\Lambda dq\, q^4 \left[\frac{1}{q^2+\bar{\mu}^2} - \frac{1}{q^2+\mu'^2} \right] \frac{1}{(\mu'^2-\bar{\mu}^2)}$$

$$= \frac{4\pi}{\mu'^2-\bar{\mu}^2} \left[(\mu'^2-\bar{\mu}^2)\Lambda + \mu^3 \tan^{-1}\frac{\Lambda}{\bar{\mu}} - \mu'^3 \tan^{-1}\frac{\Lambda}{\mu'} \right] \tag{4.78}$$

In the limit $n \to 0$ one can use the expansion 4.65 and get

$$\lim_{n\to 0} \int d^3q \; \frac{1}{q^2(1+\mu'^2/q^2)(1+\bar{\mu}^2/q^2)} = 4\pi\Lambda \left[1 - \frac{3\bar{\mu}}{2\Lambda} \tan^{-1}\frac{\Lambda}{\bar{\mu}} + \frac{\bar{\mu}^2/\Lambda}{2(1+\bar{\mu}^2/\Lambda^2)} \right] \tag{4.79}$$

The next integral is :

(c)

$$\int d^3q\, \frac{4\bar{\mu}^2/q^2}{q^2(1+\mu'^2/q^2)(1+\bar{\mu}^2/q^2)^2} = \frac{16\pi\bar{\mu}^2}{\mu'^2-\bar{\mu}^2}\int_0^\Lambda dq\, \frac{q^4}{q^2+\bar{\mu}^2}\left[\frac{1}{q^2+\bar{\mu}^2} - \frac{1}{q^2+\mu'^2}\right]$$

$$= \frac{16\pi\bar{\mu}^2}{\mu'^2-\bar{\mu}^2}\left\{\Lambda + \frac{\bar{\mu}^2\Lambda}{2(\Lambda^2+\bar{\mu}^2)} - \frac{3\bar{\mu}}{2}\tan^{-1}\frac{\Lambda}{\bar{\mu}} - \frac{1}{\mu'^2-\bar{\mu}^2}\int dq\, q^4\left[\frac{1}{q^2+\bar{\mu}^2} - \frac{1}{q^2+\mu'^2}\right]\right\}$$

$$= \frac{16\pi\bar{\mu}^2}{\mu'^2-\bar{\mu}^2}\left\{\frac{\bar{\mu}^2\Lambda}{2(\Lambda^2+\bar{\mu}^2)} - \frac{3\bar{\mu}}{2}\tan^{-1}\frac{\Lambda}{\bar{\mu}} - \frac{1}{\mu'^2-\bar{\mu}^2}\left[\bar{\mu}^3\tan^{-1}\frac{\Lambda}{\bar{\mu}} - \mu'^3\tan^{-1}\frac{\Lambda}{\mu'}\right]\right\} \tag{4.80}$$

In the limit of $n \to 0$ one has to expand the bracket up to n^2, because the factor multiplying it goes like n^{-2}. Then using (4.65) we get

$$\lim_{n\to 0}\int d^3q\, \frac{4\bar{\mu}^2 q^2}{q^2(1+\mu'^2/q^2)(1+\bar{\mu}^2/q^2)^2} = 16\pi\Lambda\left[\frac{\bar{\mu}}{\Lambda}\frac{15}{8}\tan^{-1}\frac{\Lambda}{\bar{\mu}} - \frac{7}{8}\frac{\bar{\mu}^2/\Lambda^2}{(1+\bar{\mu}^2/\Lambda^2)}\right] \tag{4.81}$$

Now we can collect the results and write :

$$g_1 = \bar{\alpha} + 3\bar{\beta}\bar{\mu}^2 + k_BT\, 4\pi\Lambda\left\{1 + \frac{3\bar{\mu}^2/\Lambda^2}{2(1+\bar{\mu}^2/\Lambda^2)} - 2(\bar{\mu}/\Lambda)\tan^{-1}(\Lambda/\bar{\mu}) + 8\pi\nu^2\left[1 - \frac{9\bar{\mu}}{\Lambda}\tan^{-1}\frac{\Lambda}{\bar{\mu}} + \frac{4\bar{\mu}^2/\Lambda^2}{(1+\bar{\mu}^2/\Lambda^2)}\right]\right. \tag{4.82}$$

Here again making an expansion to first order in $\bar{\mu}/\Lambda$ one gets

$$g_1 = \bar{\alpha} + 3\bar{\beta}\bar{\mu}^2 + 4\pi\Lambda k_BT\left[1 - (2+72\pi\nu^2)\frac{\pi}{2}\frac{|\mu'|}{\Lambda}\right] \tag{4.83}$$

Therefore, near the transition, where $\bar{\mu}/\Lambda \ll 1$, the replica symmetrical mean field solution of this model remains stable.

For $\bar{\mu} = 0$ we have to be careful with the expansions in y_α in the region of small q. The first integral can be done exactly

$$\int d^3q \ln\left(1+\frac{y_\alpha^2}{q^2}\right) = 4\pi\int^\Lambda dq\, q^2 \left[\ln(q^2+y_\alpha^2) - \ln q^2\right]$$

$$= \frac{4\pi}{3}\left[3\Lambda y_\alpha^2 - \pi|y_\alpha|^3\right] \tag{4.84}$$

which, to second order in y_α , gives the same result as Eqn.(4.69) with $\bar{\mu} = 0$.

For the other integral, $\int d^3q \ln\left(1-8\pi\nu^2\sum_\alpha \frac{q^2}{q^2+y_\alpha^2}\right)$, the expansion (4.70) for the $\ln$ will not be valid when $\bar{\mu} = 0$ for q smaller than some q of the order of the largest y_α , (let us call it Y). But the $\ln$ in the integrand is bounded (its maximum value is $\ln(1-8\pi\nu^2 n)$) so the contribution from q smaller than Y is bounded as well and of the order of $\int_{q<Y} dq\, q^2$, that is the order of Y^3. Therefore, in that integral, the dominant (to second order) contribution comes from large q, where the expansion (4.70) is valid. Thus, one can use Eqn.(4.70) to calculate g_1 in the high temperature region, where $\bar{\mu} = 0$:

$$g_1(\bar{\mu}=0) = \bar{\alpha} + 4\pi\Lambda k_B T \tag{4.85}$$

which is positive for $T > T_c$, and so there is no problem with the stability in that region.

4.III Comparison with the Hertz's model

It is worth noticing that the model presented here is not equivalent to the quenched model studied by Hertz (1978). To compare both let us calculate the g_{eff} in the Hertz model, using the replica method.

In Hertz's model the gauge field is quenched with a gaussian probability distribution

$$P(\vec{C}) = \exp -\frac{1}{2f}\int d^3x \, |\mathrm{curl}\, \vec{C}|^2 \tag{4.86}$$

and the free energy density for a given configuration of C is

$$g^H = \alpha\rho^2 + \frac{\beta}{2}\rho^4 + \frac{\gamma\rho^2 C^2}{k^2} \tag{4.87}$$

if we neglect spatial non-uniformities in ρ .

Then:

$$\langle Z^n\rangle_{av} = \int \prod_i \mathcal{D}C_i \, e^{-\frac{1}{2f}\int d^3x \, |\mathrm{curl}\,\vec{C}|^2} \prod_\alpha \mathcal{D}\rho^{(\alpha)} \, e^{-\frac{1}{k_B T}\sum_\alpha \left[\alpha\rho^{(\alpha)2} + \frac{\beta}{2}\rho^{(\alpha)4} + \frac{\gamma}{k}\rho^{(\alpha)2}C^2\right]} \tag{4.88}$$

$$= \int \prod_\alpha \mathcal{D}\rho^{(\alpha)} \, e^{\frac{-V}{k_B T}\sum_\alpha\left[\alpha\rho^{(\alpha)2} + \frac{\beta}{2}\rho^{(\alpha)4}\right]} \int \prod_i \mathcal{D}C_i \, e^{-\frac{1}{2}\sum_{i,j}\int d^3q \, C_i(\vec{q}) M^H_{ij}(\vec{q}) C_j(-\vec{q})}$$

where

$$M^H_{ij}(\vec{q}) = \frac{2}{(2\pi)^3 k_B T}\left[\frac{\gamma}{k^2}\sum_\alpha \rho^{(\alpha)2}\delta_{ij} + \frac{1}{2f}\left(q^2\delta_{ij} - q_i q_j\right)\right] \tag{4.89}$$

Notice that in this model there is only one gauge field $\vec{C}$, while in our model we have n $\vec{C}^{(\alpha)}$'s after the replication.

$\det M^H$ is easily done and gives

$$\det M^H = \left(\frac{1}{(2\pi)^3 f k_B T}\right)^3 \left(\frac{2\gamma f}{k^2}\sum_\alpha \rho^{(\alpha)2}\right)\left(q^2 + \frac{2f\gamma}{k^2}\sum_\alpha \rho^{(\alpha)2}\right)^2 \tag{4.90}$$

Here again one can exclude the longitudinal modes (represented by the second factor in 4. 90) by a convenient choice of gauge. With that taken into account we get, up to constant, the effective free energy density g^H_{eff} :

$$g^{H}_{eff} = \sum_{\alpha} \left[\alpha \rho^{(\alpha)^2} + \frac{\beta}{2} \rho^{(\alpha)^4} \right] + k_B T \int d^3q \, \ln\left(1 + \frac{2\gamma f}{k^2 q^2} \sum_{\alpha} \rho^{(\alpha)^2}\right) \tag{4.91}$$

which is higher than for the flat model (f = 0).

We can already see that g_{eff} of Eqn.(4.60) is not reducible to g^{H}_{eff}. Another point is that, in Hertz's model, the transition in the mean field approximation remains second order because the terms in ρ^3 in g^{H}_{eff}/n , arising from the presence of the gauge field and its quenching, even though negative, is of order $n^{1/2}$ and hence disappears in the limit of $n \to 0$. To see that let us take the symmetric solution $\rho_\alpha = \rho_0$ and perform the integration in Eqn.(4. 91):

$$\int d^3q \, \ln\left(1 + \frac{2\gamma f n \rho_0^2}{k^2}\right) = \frac{4\pi\Lambda^3}{3} \left\{ \ln\left(1 + \frac{2\gamma f n \rho_0^2}{k^2 \Lambda^2}\right) - \frac{1}{3} + \frac{4\gamma f}{k^2} n \frac{\rho_0^2}{\Lambda^2} \right.$$

$$\left. - 2 n^{3/2} \left(\frac{2\gamma f}{k^2}\right)^{3/2} \frac{\rho_0^3}{\Lambda^3} \tan^{-1} \frac{\Lambda}{(2\gamma f n)^{1/2} \rho_0} \right\}$$

$$\cong \frac{4\pi\Lambda^3}{3} \left\{ n \, \frac{6\gamma f}{k^2} \, \frac{\rho_0^2}{\Lambda^2} - n^{3/2} \pi \left(\frac{2\gamma f}{k^2}\right)^{3/2} \frac{\rho_0^3}{\Lambda^3} - \frac{1}{3} \right\} \tag{4.92}$$

Hence

$$\frac{1}{n} g^{H}_{eff} = \alpha \rho_0^2 + \frac{\beta}{2} \rho_0^4 + \frac{4\pi\Lambda^3}{3} k_B T \left[\frac{6\gamma f}{k^2 \Lambda^2} \rho_0^2 - n^{1/2} \pi \left(\frac{2\gamma f}{k^2}\right)^{3/2} \frac{\rho_0^3}{\Lambda^3} \right] + \text{const.} \tag{4.93}$$

Hertz remains, therefore, apparently with a second order phase transition as $n \to 0$, but was not able to find a stable fixed point in his model.

4.IV Discussion

In this chapter we gave a mean field solution for the model introduced in Chapter 3. This was done using the replica method and in the approximation of neglecting fluctuations in ρ, which is compatible with the extreme type II behaviour of the model.

The replicas are correlated in their gauge fields, even in the absence of matter field ρ, (see Eqn.4.34), but this is not a critical effect. The phase transition is driven only by ρ.

The solution symmetric in replicas is stable. Here lies an essential difference between the present gauge model and the infinite range interaction model of Sherrington-Kirkpatrick, which is considered as the most obvious candidate for a proper mean field description of the spin glass state. The replica symmetrical solution of the Sherrington-Kirkpatrick model has been shown to be unstable by de Almeida and Thouless (1978). Using the same method as theirs, we have shown that the replica symmetrical mean field solution in this model is stable.

The flat model ($e = 0$), i.e. $\vec{\Lambda}$ constrained to be vanishing, c.f. Eqn.(4.8), yields identical results to those of Halperin et al. (1974) for superconducivity, that is a weak first order phase transition, accompanied with a small shift of T_c.

If the correlation between quenched, oriented, $\vec{\Lambda}$ lines is ferromagnetic, i.e. $e^2 > 0$ in Eqn.(4.8), the first order character of the phase transition is reinforced by the quenching (see Eqn.4.83).

It has, however, been shown in Chapter 3, that the frustration lines $\vec{\Lambda}$ have antiferromagnetic interaction (mediated by matter and gauge fields). It is the configuration of the line which is quenched, not its orientation. In Eqn.(4.8), this correlation was taken to be local for simplicity. But the sign of e^2 is then the only measure of the correlation between orientations of frustration lines. It should have been treated as a dynamical variable, like an Ising spin, $e^2 = \pm |e|^2$. However, because the important difference of energy between ferro and antiferromagnetic orientation of frustration lines (Eqn.3.76), which is obtained by taking into account all the configurations of other thermodynamical variables (matter and gauge field), we can let $e^2 = -|e|^2$, and consider e as an imaginary "charge" for the frustration lines in spin glasses. (The manipulation in Eqn.(4.8 - 10) are then purely formal).

With $e^2 = -|e|^2$, then the frustration reduces the first order character of the phase transition. The term in ρ^3 remains in g_{eff}, but its coefficient becomes positive as soon as

$$|e|^2 \geqslant |e_{crit}|^2 = \frac{8}{9} k_B T_c$$

For a sufficient amount of frustration, therefore, the phase transition (in the mean field approximation) becomes of second order, as it should. Note that $e^2 < 0$ makes the free energy g_{eff} of the quenched spin glass phase higher than that of the flat model, $e^2 = 0$, (see Eqn.4. 60). This is a feature of the spin glass phase that has been emphasized by Thouless, Anderson and Palmer (1977).

Comparing our model with that of Hertz (1978) we see that, in the mean field approximation, both have a second order phase transition and the free energy increases because of the quenched disorder. Hertz could not find a stable fixed point for his model. We have not, for lack of time, analysed our model in the framework of renormalization group and look for its fixed point. This must clearly be the first test to the mean field solution presented here. But here, we have taken full account of the cubic term in ρ^3, and shown that for sufficient amount of disorder it becomes positive and the phase transition becomes of second order (Eqn.4.68).

GENERAL CONCLUSION

Invariance under local gauge transformations have not yet been fully explored in the theory of condensed matter. Of course, there is the work following Ginzburg and Landau's approach on superconductivity in which that invariance is fundamental to explain the properties of non-uniform superconductors, but only recently has the concept of local symmetry taken up the attention again among solid-state physicists, after Toulouse's work on frustration in spin glasses.

The transition from disordered (more symmetric) to ordered (less symmetric) phase is seen as the breaking of some symmetry. However if the broken symmetry is a local one, the "ordered" phase may show a very random aspect. Therefore gauge theories, dealing with local symmetry, are natural frameworks for describing random quenched phases. This remark should be valid not only for magnetic systems (our sole concern in this thesis) but also to other systems like, for example, ordinary glasses.

On the other hand a spin glass, with the magnitude of its spins fixed and their directions randomly frozen, is like a magnetic system with the usual domain structure broken-up completely. Such structure is stable if the energy of the wall between different phases is negative. The situation is similar to type II superconductivity, where, as we said, gauge invariance is essential to explain it. In this thesis we have investigated the possibility of such behaviour in magnetic systems.

Spontaneous breaking of symmetry and gauge invariance are well established topics in Field Theory, so we starte Chapter 1 by there in order to introduce the concepts to be used throughout this work. Still in Chapter 1, we reviewed Ginzburg-Landau theory of superconductivity with attention to type II superconductors, as the best example of the use of gauge invariance in condensed matter theory.

Among magnetic systems, those showing experimental evidences of a type II behaviour are spin glasses, chromium and $FeCO_3$, and they are reviewed in Chapter 2. Spin glasses deserved more attention there because their physics is apparently simpler (the Hamiltonian is possibly the simplest representative of its universality class), and they will be the systems discussed in the subsequent chapter.

The local invariance of a system of spins is explained in Chapter 3. There we work out the consequences of a very simple postulate, namely, that the Hamiltonian is invariant under any choice of local frame of reference for the spins. Then we define the gauge field in terms of the coefficients of connection between basis at different points in the space. The quenched set of exchange interactions is primarily related to the sources of gauge field. In this way the dynamical aspect of the gauge field is not in contradiction with the existence of a frozen distribution of spins. The formal similarity with type II superconductor is stressed in the case of a gauge invariant X-Y model. The difference is in the nature of the external field (sources). In the latter they are the external magnetic field, which can be controlled, while in the former they are the frustration lines, fixed at random. We also calculated the interaction between frustration lines and the result is an Ising-like interaction between the orientation of the frustration lines, or the chirality of the spin patterns around these frustration lines.

In the last chapter we used the replica method to deal with the problem of a random distribution of frustration lines. A study of the stability of the replica symmetrical mean field solution of our model indicates that it is stable, but for a more definitive answer we should have to use renormalization group and look for its fixed point.

Immediate generalization of the work done in this thesis should include the effect of an external magnetic field, which would break the gauge invariance (in this case, the relationship between gauge field $\vec{A}$ and direction of the spins θ). It would also be desirable to have a model system where the spins can point in all three dimensions, instead of restricting ourselves to a X-Y model. It is likely that a field theory in P_2 (the two-dimensional projective hyperplane) instead of P_1 would have the desirable properties.

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