Imperial College London

# Normal Subgroups of the Automorphism Groups of Some Homogeneous Structures

*A thesis submitted for the degree of*
*Doctor of Philosophy in Mathematics*
*by*

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September 2020

## Statement of Originality

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To my mum, who has shown me the greatest love in the world

# Acknowledgements

I could not have achieved this without the support of many people, whom I would like to thank sincerely.

First and foremost, my supervisor Prof. David Evans, who has provided me with great guidance and insight throughout my PhD. This work would not have been possible without his advice. David has always been very supportive and patient; I could not imagine a better supervisor.

My deepest gratitude goes to my mum. I would not have achieved what I have today without her love and unconditional support.

I am grateful for the President's PhD scholarship offered by Imperial College and EPSRC. I would also like to thank Imperial College for the amazing support it provides to its graduate students.

To my collaborators, Dr. Jan Hubička and Matěj Konečný, thank you for all the inspiring discussions we had and for the great effort you contributed to our work.

I would like to thank my examiners, Prof. Dugald Macpherson and Prof. Martin Liebeck, for the time they spent both reading my thesis and providing helpful comments.

I also really appreciate everyone in the Model Theory group and all the staff members at the Maths Department for all the support they gave me.

I would also like to thank the Abbott family, who have provided me with a second home in the UK.

I am fortunate in that I have so many lovely memories of my PhD; thank you to all my friends who made them with me. My time at Imperial would not have been so enjoyable without my amazing friends.

To all my lecturers at Durham, in particular Dr. Alexander Stasinski, Prof. John Parker and Prof. Herbert Gangl; your passion, patience and kindness inspired me to embark on this journey. Thank you.

# Abstract

We will study the normal subgroup structures of the automorphism groups of some homogeneous structures in this thesis. In particular, we prove the simplicity of the automorphism groups of the following homogeneous structures:

(i) all of the symmetric 'semi-free' homogeneous structures described by Cherlin in the appendix of [Che98],

(ii) some of the asymmetric 'semi-free' homogeneous structures in the appendix of [Che98],

(iii) the linear order expansions of non-trivial free homogeneous structures, and

(iv) the universal $n$-linear order for $n \geq 2$.

The methods used generalise techniques developed in a sequence of papers by Macpherson and Tent [MT11] and Tent and Ziegler [TZ13a], [TZ13b]. A key notion in these methods is that of a stationary independence relation on finite substructures of a homogeneous structure $\mathcal{M}$ defined in [TZ13a]. This generalised the idea of substructures of $\mathcal{M}$ being freely amalgamated over their intersection if $\mathcal{M}$ is free, used in [MT11].

A stationary independence relation is symmetric. We can apply Tent and Ziegler's method in [TZ13a] directly to the symmetric semi-free homogeneous structures in (i). But we cannot do so to the structures in (ii)-(iv) as they are asymmetric. One of the main contributions of our work is to drop the symmetry requirement, giving us the notion of a stationary weak independence relation. We then show how this can be used to prove the simplicity of the automorphism groups of the structures in (ii)-(iv) following a similar method of [MT11] and [TZ13a].

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# Chapter 1

# Introduction

Given a relational first-order language $\mathcal{L}$, a countable $\mathcal{L}$-structure $\mathcal{M}$ is *homogeneous* if every partial isomorphism between finite substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$. According to [Che19], this notion first appeared in the work of Urysohn [Ury25]. Fraïssé [Fra53] provided a one-to-one correspondence between such structures and *amalgamation classes* (see Definition 2.2.2), which is referred to as Fraïssé's theorem (stated in Theorem 2.2.5). We call the homogeneous structure the *Fraïssé limit* of the corresponding amalgamation class. Most of the time, we employ Fraïssé's theorem to construct a homogeneous structure. For example, the *random graph* is the Fraïssé limit of the class of all finite graphs and the *dense linear order* $(\mathbb{Q}, \leq)$ is the Fraïssé limit of the class of all finite linear orders. The *Urysohn space* is the completion of the Fraïssé limit of the class of all finite $\mathbb{Q}$-valued metric spaces. The class of all finite graphs forms a special type of amalgamation classes, called a *free* amalgamation class (see Definition 2.2.7). Its Fraïssé limit is called a *free* homogeneous structure.

The study of homogeneous structures is an area of great importance due to its connection to different areas in Mathematics. According to Macpherson [Mac11], "homogeneous structures provide a meeting-point of ideas from combinatorics, model theory, permutation group theory, and descriptive set theory, and connections to theoretical computer science and to universal algebra are beginning to emerge".

One major area of research is the study of automorphism groups of homoge-

neous structures. Homogeneous structures provide a rich source of examples of infinite permutation groups, which have nice group-theoretic properties. Many studies were conducted to study the normal subgroup structure of the automorphism groups of homogeneous structures. Truss [Tru85] proved that the automorphism group of the random graph is simple. The same statement was proved for the random poset in [GMR93] and the generic $k$-uniform hypergraphs in [Lov06]. Macpherson and Tent proved the same statement for free homogeneous structures in [MT11], stated in the following.

**Theorem 1.0.1.** *([MT11]) Let $\mathcal{M}$ be a countable free homogeneous relational structure. Suppose Aut($\mathcal{M}$) $\neq$Sym($\mathcal{M}$) and Aut($\mathcal{M}$) is transitive on $\mathcal{M}$. Then Aut($\mathcal{M}$) is simple.*

The theorem was proved by defining a ternary relation on finite substructures of a free homogeneous structure, which is motivated by non-forking in stability theory, and then using ideas from Lascar [Las92].

In [TZ13a], Tent and Ziegler axiomatised this ternary relation as a *stationary independence relation* (**SIR**, see Definition 3.1.1). This notion can be applied not only to free homogeneous structures but also to other structures including the Urysohn space and the bounded Urysohn space. Tent and Ziegler also generalised the arguments in [MT11] so that they work for homogeneous structures admitting a **SIR**. Their main result is stated in the following. The theorem uses the notion of a *moving almost maximally* automorphism (see Definition 4.0.1).

**Theorem 4.0.2.** *([TZ13a], 5.4) Suppose that $\mathcal{M}$ is a countable homogeneous structure with a* **SIR**. *If $g \in$Aut($\mathcal{M}$) moves almost maximally, then any element of Aut($\mathcal{M}$) is the product of sixteen conjugates of $g$.*

It was also shown in [TZ13a] that for a non-trivial transitive free homogeneous structure $\mathcal{M}$ and any non-trivial automorphism $g \in$Aut($\mathcal{M}$), we can find a product of conjugates of $g$ and $g^{-1}$ that moves almost maximally and thus, we can derive the simplicity of Aut($\mathcal{M}$). Tent and Ziegler also applied their theorem to the Urysohn space in [TZ13a] as well as the bounded Urysohn space in [TZ13b] using a similar approach.

Another major area of research is on classification of homogeneous structures. Henson [Hen71], and Peretyat'kin [Per73] independently, showed the

existence of $2^{\aleph_0}$ countable homogeneous directed graphs. In [LW80], Lachlan and Woodrow classified all countable homogeneous undirected graphs. Lachlan [Lac84] continued the work and classified all countable homogeneous tournaments. He also generalised his result to characterise countable homogeneous 2-tournaments. In [Che98], Cherlin provided the classification of all countable homogeneous directed graphs based on Lachlan's technique. He also produced a new proof of Lachlan and Woodrow's classification of homogeneous undirected graphs. In the appendix of [Che98], Cherlin proposed a generalisation of free homogeneous binary structures, called a *semi-free homogeneous structure* (see Definition 2.3.2). Cherlin provided 1 example for $\mathcal{L}$ consisting of three symmetric binary relations, 27 examples for $\mathcal{L}$ consisting of four symmetric binary relations and 12 examples for $\mathcal{L}$ consisting of two asymmetric binary relations. These structures were later referred to as *3-multi-graphs*, *4-multi-graphs* and *2-multi-tournaments* respectively in [Che19], which we will also use to refer to these structures in this thesis. In [Che19], Cherlin also considered the classification of *metrically homogeneous graphs*, which are connected graphs such that the associated metric space is homogeneous. It was noted in [Che11] that some 4-multi-graphs listed in [Che98] can be interpreted as metrically homogeneous graphs.

Since Tent and Ziegler's theorem can be applied to all free homogeneous structures, we ask the question whether it can be applied to Cherlin's semi-free homogeneous structures. In order to apply the theorem, we need to first find a **SIR** on these structures. We can immediately see that we cannot find a **SIR** on 2-multi-tournaments because the asymmetric relations in their language do not satisfy the Symmetry axiom of a **SIR**. This problem motivated the main result of this thesis: we will generalise the notion of **SIR** to a *stationary weak independence relation* (**SWIR**, see Definition 3.1.1), which omits the Symmetry axiom, in Section 3.1 and we then generalise Tent and Ziegler's theorem to homogeneous structures admitting a **SWIR**, as stated in the following, in Chapter 4. Note that in order to generalise the theorem, we need to generalise the notion of a moving almost maximally automorphism to an automorphism that *moves almost R-maximally* and *moves almost L-maximally* (see Definition 4.0.3).

**Theorem 4.0.4.** *Suppose $\mathcal{M}$ is a countable homogeneous structure with a* **SWIR** *and $g \in Aut(\mathcal{M})$ is such that $g$ and $g^{-1}$ move both almost R-maximally*

*and almost L-maximally. Then any element of Aut($\mathcal{M}$) can be written as a product of eight conjugates of $g$.*

We can then attempt to answer the question whether we can find a **SWIR** on Cherlin's semi-free homogeneous structures. We will show in Section 3.2 that in order to find a **SWIR** on semi-free homogeneous structures, it is helpful to define an 'order' on their language (see Definition 3.2.1). Then we can find a **SWIR** on some semi-free homogeneous structures if they satisfy some conditions (namely Condition 1 and Condition 2). We then apply the above theorem to these semi-free homogeneous structures and show the simplicity of their automorphism groups in Section 5.1. Specifically, these semi-free homogeneous structures include most structures provided by Cherlin in the appendix of [Che98] as well as some other semi-free structures and thus, we have the following theorem in Section 5.1.

**Theorem 5.1.3.** *Let $\mathcal{M}$ be one of the following structures:*

*(i) all 3-multi-graphs and 4-multi-graphs listed in the appendix of [Che98]*

*(ii) #8 to #10 of the 2-multi-tournaments listed in the appendix of [Che98].*

*Then, $\mathcal{M}$ admits a **SWIR** and Aut($\mathcal{M}$) is simple.*

We will also show, in Section 3.3, that we can find a **SWIR** on some linearly ordered homogeneous structures, including linearly ordered free homogeneous structures and the universal $n$-linear order. We then apply the Theorem 4.0.4 to prove the following in Section 5.2.

**Theorem 5.2.2.** *Let $\mathcal{M}$ be one of the following structures:*

*(i) the linear order expansion of a non-trivial free homogeneous structure,*

*(ii) the universal $n$-linear order, $n \geq 2$.*

*Then, $\mathcal{M}$ admits a **SWIR** and Aut($\mathcal{M}$) is simple.*

Note that part (i) of the above theorem was proved independently by Calderoni, Kwiatkowska and Tent in [CKT19] also by generalising Tent and Ziegler's method [TZ13a] but in a slightly different way.

Alongside the work presented in this thesis, the author together with Evans, Hubička and Konečný applied Tent and Ziegler's method from [TZ13b] to some generalised metric spaces in [EHKL19]. This result has some overlapping

part with the result in Section 5.1 as some of Cherlin's 4-multi-graphs can be interpreted as metrically homogeneous graphs.

# Chapter 2

# Background

## 2.1 Model Theory

Throughout this thesis, we fix $\mathcal{L}$ to be a first-order binary relational language. It is specified by a set of relation symbols $\{R_i : i \in I\}$, where each $R_i$ is binary. For example, the language of graphs consists of a binary relation, which identifies the pairs of vertices that are connected by an edge.

An *$\mathcal{L}$-structure* $A$ is a set $A$ together with a subset $R_i^A \subseteq A^2$ for each $i \in I$ representing the structure on $A$. For example, if $A$ is a graph, then $R_i^A$ consists of all pairs of vertices that are connected by an edge. Note that we use the same notation $A$ to represent the $\mathcal{L}$-structure and its underlying set. An $\mathcal{L}$-structure $B$ is a *substructure* of $A$ if $B \subseteq A$ and $R_i^A \cap B^2 = R_i^B$ for each $i \in I$. If $A$ is a graph, then a substructure of $A$ is an induced subgraph.

For notions introduced in this section and more definitions such as $\mathcal{L}$-formulas, $\mathcal{L}$-structures, $\mathcal{L}$-theories, we refer the readers to [Mar02].

**Definition 2.1.1.** Let $A$ be an $\mathcal{L}$-structure and $\bar{a} \in A^n$ be a $n$-tuple.

(i) The *$n$-type* of $\bar{a}$, denoted by $\mathrm{tp}(\bar{a})$, is the set of all $\mathcal{L}$-formulas $\phi(\bar{x})$ that $\bar{a}$ satisfies in $A$.

(ii) Let $B$ be a substructure of $A$. Then the *(complete) $n$-type of $\bar{a}$ over $B$*, denoted by $\mathrm{tp}(\bar{a}/B)$, is the set of all $\mathcal{L}$-formulas $\phi(\bar{x}, \bar{b})$ that $\bar{a}$ satisfies with some parameters $\bar{b} \in B^m$.

(iii) We can also define a *$n$-type* $p$ over $B$ to be a set of consistent $\mathcal{L}$-formulas

$\phi(\bar{x}, \bar{b})$ where $\bar{x}$ is an $n$-tuple and $\bar{b} \in B^m$ for some $m$.

(iv) Let $p$ be an $n$-type over some finite set $B$. If $\bar{a}$ satisfies all $\mathcal{L}$-formulas in $p$ then we say $\bar{a}$ is a *realisation* of type $p$ or $\bar{a}$ realises $p$.

**Definition 2.1.2.** For a homogeneous structure $\mathcal{M}$, we say $\mathcal{M}$ has *trivial algebraic closure* if for any 1-type over some finite set $B \subset \mathcal{M}$, it either has infinitely many realisations or one realisation that belongs to $B$.

Note that we will only consider homogeneous structures that have trivial algebraic closures in this thesis.

## 2.2 Homogeneous structures and Fraïssé's Theorem

Notions and theorems mentioned in this section and more about homogeneous structures can be found in [Mac11].

**Definition 2.2.1.** Let $\mathcal{L}$ be a relational language $\mathcal{L}$ and $\mathcal{M}$ be a countable $\mathcal{L}$-structure. Let $A, B$ be finite substructures of $\mathcal{M}$.

(i) We say $f : A \to B$ is a *partial isomorphism* of $\mathcal{M}$ if it is an isomomorphism between $A$ and $B$.

(ii) We say $\mathcal{M}$ is *homogeneous* if every partial isomorphism of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$.

Throughout this thesis, $\mathcal{M}$ always denotes a countable homogeneous relational structure. Fraïssé's Theorem [Fra53] provides one way of constructing homogeneous structures by establishing a one-to-one correspondence between such structures and *amalgamation classes*, defined as follows.

**Definition 2.2.2.** A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is an *amalgamation class* if it satisfies the following conditions:

(i) (Isomorphism property, IP) it is closed under isomorphism and there are countably many isomorphism types.

(ii) (Hereditary property, HP) if $A \in \mathcal{C}$ and $B$ is a substructure of $A$, then $B \in \mathcal{C}$.

(iii) (Joint embedding property, JEP) if $A, B \in \mathcal{C}$, then there exist $C \in \mathcal{C}$ and embeddings $f_1 : A \to C, f_2 : B \to C$.

(iv) (Amalgamation property, AP) if $A, B, C \in \mathcal{C}$ and $f_1 : B \to A, f_2 : B \to C$ are embeddings, then there exist $D \in \mathcal{C}$ and embeddings $g_1 : A \to D, g_2 : C \to D$ such that $g_1 f_1 = g_2 f_2$.

In addition, we say $\mathcal{C}$ is a *strong* amalgamation class if in AP, we also require $g_1 f_1(B) = g_2 f_2(B) = g_1(A) \cap g_2(C)$.

**Remark 2.2.3.** (i) In JEP and AP, we can take the embeddings to be inclusion maps. Hence, when we have an amalgamation class $\mathcal{C}$ and say that $B$ is a substructure of $A, C$ for $A, B, C \in \mathcal{C}$, the precise meaning is that there are embeddings from $B$ to $A, C$.

(ii) Since $\mathcal{L}$ is relational, it is enough to check all $A, B, C \in \mathcal{C}$ such that $|A \setminus B| = |C \setminus B| = 1$ in verifying AP. In such case, we can write $A, C$ as $aB, cB$ respectively where $a \in A \setminus B, c \in C \setminus B$.

(iii) Note that all amalgamation classes we consider in this thesis are strong. If an amalgamation class is strong, than its Fraïssé limit has trivial algebraic closure. Hence, all homogeneous structures we study in this thesis have trivial algebraic closure.

**Definition 2.2.4.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The *age* of $\mathcal{M}$, denoted by $\mathrm{Age}(\mathcal{M})$, is the class of all $\mathcal{L}$-structures isomorphic to some finite substructure of $\mathcal{M}$.

**Theorem 2.2.5.** *(Fraïssé's Theorem) A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is an amalgamation class if and only if $\mathcal{C} = \mathrm{Age}(\mathcal{M})$ for some countable homogeneous structure $\mathcal{M}$. In this case, $\mathcal{M}$ is determined by $\mathcal{C}$ up to isomorphism. We call $\mathcal{M}$ the* Fraïssé limit *of $\mathcal{C}$. Moreover, in this case, $\mathcal{M}$ has the* Extension Property*, i.e. for any finite substructure $A \subseteq \mathcal{M}$ and $B \in \mathcal{C}$ such that there is an embedding $f : A \to B$, there exists an embedding $g : B \to \mathcal{M}$ such that $g(f(a)) = a$ for all $a \in A$.*

A sketch proof of the theorem can be found in [Eva13].

**Example 2.2.6.** (i) Let $\mathcal{L}$ be the language of graphs, i.e. $\mathcal{L}$ consists of a single binary relation, which identifies edges. Let $\mathcal{C}$ be the class of all finite graphs.

We can check that $\mathcal{C}$ forms an amalgamation class. To check AP, let $A, B, C \in \mathcal{C}$ be such that $B$ is an induced subgraph of $A, C$. We can define $D := A \cup C$, where $A \cap C = B$ and for any $a \in A \setminus B, c \in C \setminus B$, $(a, c)$ is not connected by an edge. Then, we can see that $D$ is also a graph, i.e. $D \in \mathcal{C}$ and it contains $A, C$ as subgraphs with $A \cap C = B$. Then, $\mathcal{C}$ has AP and also JEP if we let $B = \emptyset$ in the above argument. IP and HP are straightforward to see. Hence, $\mathcal{C}$ is a strong amalgamation class. Its Fraïssé limit is the Random Graph.

In the above, $\mathcal{C}$ also forms a special amalgamation class, called a *free amalgamation class*, defined as follows.

**Definition 2.2.7.** Given finite $\mathcal{L}$-structures $A, B, C$ where $B \subseteq A, C$, the *free amalgam* of $A, C$ over $B$ is the $\mathcal{L}$-structure $D$ on the disjoint union of $A, C$ over $B$ such that for each relation $R \in \mathcal{L}$, $R^D = R^A \cup R^C$. An amalgamation class $\mathcal{C}$ is *free* if it is closed under taking free amalgams, i.e. for $A, B, C \in \mathcal{C}$ where $B \subseteq A, C$, the free amalgam of $A, C$ over $B$ is also in $\mathcal{C}$. Note that by the *disjoint union* of $A, C$ over $B$, we mean that $D = A \cup C, B = A \cap C$. This guarantees that a free amalgamation class is strong. We say that a homogeneous structure is *free* if it is the Fraïssé limit of a free amalgamation class.

Other examples of free homogeneous structures include the generic $K_n$-free graph, which is the Fraïssé limit of the class of all finite $K_n$-free graphs. We will look at a generalisation of a free amalgamation class, namely a semi-free amalgamation class, in section 2.3.

**Example 2.2.6.** *(Continued)*

*(ii) Let $\mathcal{L}$ be the language of linear orders, i.e. $\mathcal{L}$ consists of a single binary relation, which specifies the order of two elements. Let $\mathcal{C}$ be the class of all finite linear orders.*

*We can check that $\mathcal{C}$ forms an amalgamation class. To verify AP, let $A, B, C \in \mathcal{C}$ be such that $B \subseteq A, C$. We can find define $D := A \cup C$, where $A \cap C = B$. We want to define a linear order on $D$ extending the orders on $A, C$: for any $a \in A \setminus B, c \in C \setminus B$, let $a < c$ if there exists $b \in B$ such that $a < b$ in $A$ and $b < c$ in $C$ and let $a > c$ otherwise. Then it is fairly easy to see that $D$ is a linear order, i.e. $D \in \mathcal{C}$, and thus, $\mathcal{C}$*

*satisfies AP. Similarly, we can verify JEP if we let $B = \emptyset$ in the above argument. IP and HP are straightforward to see. Hence, $\mathcal{C}$ is a strong amalgamation class. Its Fraïssé limit is the dense linear order $(\mathbb{Q}, \leq)$.*

*(iii) Let $\mathcal{L} = \{R_q | q \in \mathbb{Q}\}$ be the language of rational-valued metric spaces, where each $R_q$ is a binary relation that specifies which two elements have distance $q \in \mathbb{Q}$. Note that this is an infinite language. Let $\mathcal{C}$ be the class of all finite $\mathbb{Q}$-valued metric spaces.*

*We can check that $\mathcal{C}$ forms an amalgamation class. To verify AP, for $(A, d_A), (B, d_B), (C, d_C) \in \mathcal{C}$ such that $(B, d_B)$ is a subspace of $(A, d_A)$, $(C, d_C)$, we want to find a finite $\mathbb{Q}$-valued metric space $(D, d_D)$ such that $(A, d_A), (C, d_C)$ can be embedded into $(D, d_D)$ over $(B, d_B)$. Let $D := A \cup C$, where $A \cap C = B$. Let $d_D(a, a') = d_A(a, a')$ for any $a, a' \in A$ and $d_D(c, c') = d_C(c, c')$ for any $c, c' \in C$. For $a \in A \setminus B, c \in C \setminus B$, let $d_D(a, c) = \min_{b \in B}\{d_A(a, b) + d_C(b, c)\}$. We can check that $(D, d_D)$ is a $\mathbb{Q}$-valued metric space, i.e. $(D, d_D) \in \mathcal{C}$. Hence, $\mathcal{C}$ satisfies AP. Note that in the above argument, $B$ has to be non-empty as otherwise we would not be able to define the distance between $a, c$. Hence, we cannot let $B$ be the empty set here to verify JEP as we did in the previous examples. Instead, we let $B$ consist of one element and then we can be embed it into $A, C$ and verify JEP.*

*IP and HP are straightforward to see. Hence, $\mathcal{C}$ is a strong amalgamation class. Then its Fraïssé limit is called the* rational Urysohn space *and the completion of the rational Urysohn space is called the* Urysohn space.

Alongside Fraïssé's Theorem, we can sometimes prove a structure is homogeneous by using a method called the *back-and-forth* method. This method will be used throughout the later chapters, so we demonstrate the method for $(\mathbb{Q}, \leq)$ here as an example. The same method can also be used to prove the homogeneity of the Random Graph.

**Proposition 2.2.8.** *The dense linear order $(\mathbb{Q}, \leq)$ is homogeneous.*

*Proof.* Let $\tilde{g} : A \to B$ be a partial isomorphism of $(\mathbb{Q}, \leq)$. Let $z \in \mathbb{Q} \setminus A$. We show that we can extend $\tilde{g}$ to $\tilde{g}'$ such that $z$ is in the domain and image of $\tilde{g}'$. List all elements in $A$ as $a_1, a_2, ..., a_n$, where $a_1 < a_2 < ... < a_n$. Then we have $\tilde{g}(a_1) < \tilde{g}(a_2) < ... < \tilde{g}(a_n)$.

If $a_i < z < a_{i+1}$ for some $i$, since $\mathbb{Q}$ is dense, we can find $y \in \mathbb{Q}$ such that $\tilde{g}(a_i) < y < \tilde{g}(a_{i+1})$. Similarly, if $z > a_n$, we can find $y > \tilde{g}(a_n)$. If $z < a_1$, we can find $y < \tilde{g}(a_1)$. Extend $\tilde{g}$ by sending $z$ to $y$. This is the forth direction.

For the back direction, we may assume $z \notin B$. By the same argument, if $\tilde{g}(a_i) < z < \tilde{g}(a_{i+1})$ for some $i$, we can find $x \in \mathbb{Q}$ such that $a_i < x < a_{i+1}$. Similarly, if $z > \tilde{g}(a_n)$, we can find $x > a_n$. If $z < \tilde{g}(a_1)$, we can find $x < a_1$. Extend $\tilde{g}$ by sending $x$ to $z$. Let $A' := A \cup \{x, z\}$ and $B' := B \cup \{z, y\}$. Let $\tilde{g}'$ be an isomorphism from $A'$ to $B'$ such that $\tilde{g}'(a) = \tilde{g}(a)$ for all $a \in A$ and $\tilde{g}'(z) = y, \tilde{g}'(x) = z$. Then $\tilde{g}'$ is a partial isomorphism of $\mathbb{Q}$.

Hence, we can enumerate all elements of $\mathbb{Q} \setminus A$ as $z_1, z_2, ...$ and follow the same process for each $z_i$. Let $g$ be the resulting homomorphism over all $i$. Then $g$ is an automorphism of $(\mathbb{Q}, \leq)$ as it is bijective by construction. Hence, $(\mathbb{Q}, \leq)$ is homogeneous. □

## 2.3 Semi-free amalgamation classes

In the appendix of [Che98], Cherlin introduced the idea of amalgamation classes of binary structures defined exclusively by constraints on triangles, which we can define as the following.

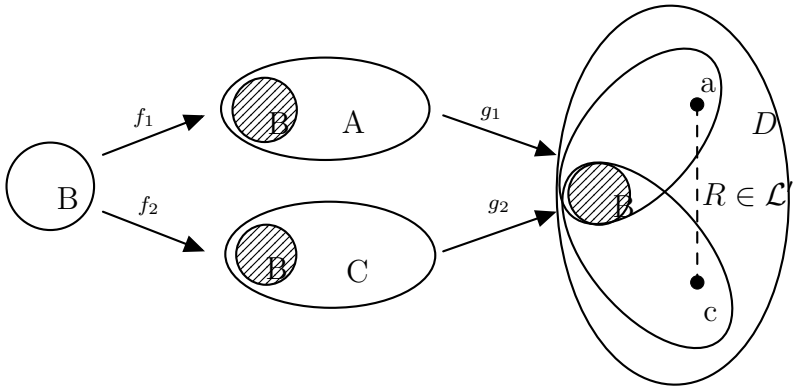
**Definition 2.3.1.** Let $\mathcal{L}$ be a binary relational language. We say an $\mathcal{L}$-structure $A$ is *complete* if every two distinct elements $a, b \in A$ are related by exactly one relation of $\mathcal{L}$. We denote this relation by $\mathbf{r}(a, b)$. An $\mathcal{L}$-structure is a *triangle* if it is a complete $\mathcal{L}$-structure on three distinct elements. Let $S$ be a set of triangles. We define $Forb_c(S)$ to be the set of all complete structures that do not embed any triangle from $S$. We call $S$ *the set of forbidden triangles* of $Forb_c(S)$.

We can think of the structures in $Forb_c(S)$ as complete edge-coloured graphs or digraphs that do not embed some coloured triangles by taking the elements of the structures as vertices and the relations as colours.

According to Cherlin, if $S$ is one of the sets of forbidden triangles listed in the appendix of [Che98], then any amalgamation of $A, C$ over $B$ where $A, B, C \in Forb_c(S)$ and $B \subseteq A, C$ can be completed by a proper subset of its language. We will give some examples of such amalgamation in Example

2.3.4. Cherlin stated that this generalises the notion of free amalgamation classes since for a free amalgamation class, we can add a relation $\bar{R}$, which represents no-relation between elements, to its language and then, any amalgamation can be completed by $\{\bar{R}\}$ in a free amalgamation class. Cherlin called this generalisation *semi-free amalgamation classes*, which we formally define as the following:

**Definition 2.3.2.** Let $\mathcal{L}$ be a relational language consisting only of binary relations. Let $\mathcal{C}$ be a strong amalgamation class of finite complete $\mathcal{L}$-structures. We say $\mathcal{C}$ is a *semi-free amalgamation class* if there exists $\mathcal{L}' \subsetneq \mathcal{L}$ such that for any finite structures $A, B, C \in \mathcal{C}$ and embeddings $f_1 : B \to A, f_2 : B \to C$, there exist $D \in \mathcal{C}$ and embeddings $g_1 : A \to D, g_2 : C \to D$ such that $g_1 f_1(B) = g_2 f_2(B) = g_1(A) \cap g_2(C)$ and for any $a \in g_1(A) \setminus g_1 f_1(B), c \in g_2(C) \setminus g_2 f_2(B)$, if $a, c$ are related by some $R \in \mathcal{L}$, then $R \in \mathcal{L}'$. We call $\mathcal{L}'$ the *set of solutions* of $\mathcal{C}$.




In the form of forbidden triangles, Cherlin provided 1 example of semi-free amalgamation classes for $\mathcal{L}$ consisting of three symmetric relations, 27 examples for $\mathcal{L}$ consisting of four symmetric relations and 12 examples for $\mathcal{L}$ consisting of two asymmetric relations. It can be checked that if $S$ is one of the sets of forbidden triangles listed by Cherlin, then $Forb_c(S)$ forms a semi-free amalgamation class. Hence, we can take its Fraïssé limit and denote it by $\mathcal{M}_S$. We will refer to them as 3-multi-graphs, 4-multi-graphs and 2-multi-tournaments as Cherlin later did in [Che19]. We provide these examples below in Figure 2.1. Each entry in the list is a set of forbidden triangles. Cherlin claimed that the list is complete if we exclude free amalgamation classes and those with a non-trivial equivalence relation on the vertices, where the latter means that there exists a proper subset $\hat{\mathcal{L}} \subsetneq \mathcal{L}$ such that for any $a, b, c \in \mathcal{M}_S$,

if $\mathbf{r}(a,b), \mathbf{r}(b,c) \in \hat{\mathcal{L}}$, then $\mathbf{r}(a,c) \in \hat{\mathcal{L}}$ and if $\mathbf{r}(a,b) \in \hat{\mathcal{L}}$, then $\mathbf{r}(b,a) \in \hat{\mathcal{L}}$. Note that for the 12 asymmetric examples given by Cherlin, one of them is finite and six of them have a definable partial order, which are not of interest here, so we omit them in the list.

To avoid any confusion in the notation, we change some of the letters used in Cherlin's original list:

3-multi-graph: $\mathcal{L} = \{R, G, X\}$

#1 RXX GGX XXX

4-multi-graph: $\mathcal{L} = \{R, G, X, Y\}$

#1 RXX GYX YXX

#2 RXX GYX YXX XXX

#3 RXX GYX YXX YYX

#4 RXX GYX YXX YYY

#5 RXX GYX YXX YYX XXX

#6 RXX GYX YXX XXX YYY

#7 RXX GYX YXX YYX YYY

#8 RXX GYX YXX YYX YYY XXX

#9 RXX GYX YYX XXX

#10 RXX GYX YYX XXX YYY

#11 RXX GGX YXX XXX

#12 RXX GGX YXX XXX YYX

#13 RXX GGX YXX XXX YYY

#14 RXX GGX YXX YYX XXX YYY

#15 RXX GYX GGX YXX XXX

#16 RXX GYX GGX YXX XXX YYX

#17 RXX GYX GGX YXX XXX YYY

#18 RXX GYX GGX YXX YYX XXX YYY

#19 RXX GYX GGX YYX XXX

#20 RXX GYX GGX YYX XXX YYY

#21 RXX RYY GYX YYX XXX

#22 RXX RYY GYX YYX YXX

#23 RXX RYY GYX YYX YXX XXX

#24 RXX RYY GYX YXX XXX YYY

#25 RXX RYY GYX YXX YYX XXX YYY

#26 RRX RXX RYY GYX GXX YYX XXX

#27 RRY RRX GYX GXX GYY YYX YXX XXX YYY

2-multi-tournament: $\mathcal{L} = \{R^{\pm}, G^{\pm}\}$

# 8 $G^+G^-G^+$ $R^+G^-G^+$

# 9 $G^+G^+G^+$ $R^+G^-G^+$

#10 $G^+G^+G^+$ $G^+G^-G^+$ $R^+G^-G^+$

#11 $R^+R^-R^+$ $G^+G^+G^+$ $G^+G^-G^+$ $G^+R^+R^+$ $G^+R^-R^-$

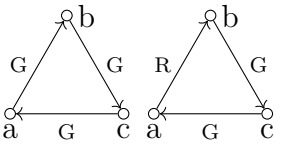
#12 $R^+R^-R^+$ $G^+G^+G^+$ $R^+G^-G^+$ $G^+R^+R^+$ $G^+R^-R^-$ $G^+R^+R^-$

Figure 2.1: Cherlin's list

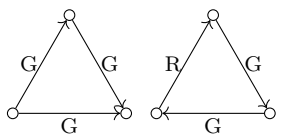
For the asymmetric examples, note that each pair of vertices has exactly one direction, i.e. for all $a, b \in M_S$, $R^+(a, b)$ if and only if $R^-(b, a)$. We set $R^+(a, b)$ if there is a directed $R$-edge from $a$ to $b$. By a triangle $abc$ with relation $R_1^+R_2^+R_3^+$, we mean $R_1^+(a, b)R_2^+(a, c)R_3^+(b, c)$. So we can interpret the lists as the following graphs:

$\mathcal{L} = \{R^{\pm}, G^{\pm}\}$

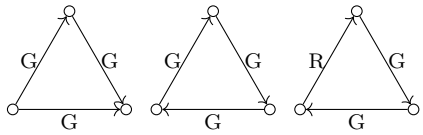
# 8 $G^+G^-G^+$ $R^+G^-G^+$



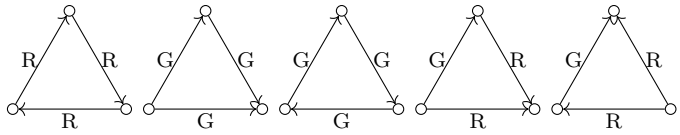

# 9 $G^+G^+G^+$ $R^+G^-G^+$



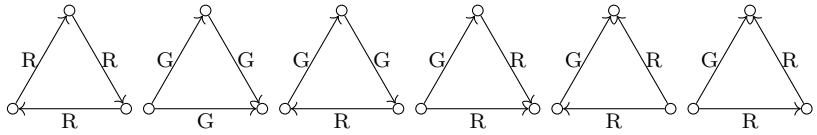

#10 $G^+G^+G^+$ $G^+G^-G^+$ $R^+G^-G^+$




#11 $R^+R^-R^+$ $G^+G^+G^+$ $G^+G^-G^+$ $G^+R^+R^+$ $G^+R^-R^-$




#12 $R^+R^-R^+$ $G^+G^+G^+$ $R^+G^-G^+$ $G^+R^+R^+$ $G^+R^-R^-$ $G^+R^+R^-$

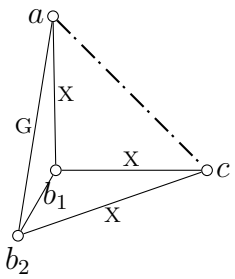



We first look at the symmetric structures listed in Figure 2.1. The following proposition can be checked by quick computation. We give an example of such check in the next example. Note that the notation $aB$ denotes a structure of $Forb_c(S)$ that $B$ can be embedded into and $|aB \setminus B| = 1$.

**Proposition 2.3.3.** *Let $S$ be one of the sets of forbidden triangles for the 3-multi-graph and 4-multi-graph listed in Figure 2.1 except # 26. Then any amalgamation problem in $Forb_c(S)$ can be completed by $\mathcal{L}' = \{R, G\}$.*

**Example 2.3.4.** Let $S$ be # 11 of the 4-multi-graphs listed in Figure 2.1. To check that we may take $\mathcal{L}' = \{R, G\}$, we want to show that any amalgamation problem can be completed by either $R$ or $G$, i.e. there does not exist an amalgamation $aB, cB \in Forb_c(S)$ over some finite substructure $B$ such that $(a, c)$ cannot be coloured by either $R$ or $G$. Suppose we have such an amalgamation. Then there exist $b_1, b_2 \in B$ such that $ab_1c$ forbids $(a, c)$ to be completed by $R$ and $ab_2c$ forbids $(a, c)$ to be completed by $G$. Since the only forbidden triangles containing $R, G$ are $RXX, GGX$, we may assume without loss of generality that $\mathbf{r}(a, b_1) = \mathbf{r}(b_1, c) = X$ and $\mathbf{r}(a, b_2) = G, \mathbf{r}(b_2, c) = X$, as shown in the graph below. However, we cannot find a colour for $b_1b_2$ without creating a forbidden triangle: if $\mathbf{r}(b_1, b_2) = G$, then $b_1ab_2$ forms a forbidden triangle $GGX$ and if $\mathbf{r}(b_1, b_2) = R, Y$, or $X$, then $b_1cb_2$ forms a forbidden triangles $RXX, YXX, XXX$ respectively. This contradicts that $aB, cB \in Forb_c(S)$ Therefore, such an amalgamation does not exist. Thus, we may take $\mathcal{L}' = \{R, G\}$. Similarly, we can prove the same statement for the

other cases.

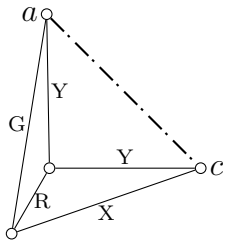



In [Che98], Cherlin stated that we can find a set of solutions $\mathcal{L}'$ consisting of two relations for all symmetric structures in the list. However, we will show in the lemma below that, for #26 the only choice we have for $\mathcal{L}'$ is $\{R, G, Y\}$.

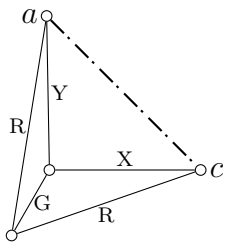
**Lemma 2.3.5.** *Let $S$ be #26 of the 4-multi-graphs listed in Figure 2.1 and $Forb_c(S)$ be the set of all complete $\mathcal{L}$-structures that do not embed any triangle in $S$. Then $\{R, G, Y\}$ is the only possible set of solutions for $Forb_c(S)$ to be a semi-free amalgamation class.*

*Proof.* We show that $\mathcal{L}'$ has to contain $G, R, Y$ by finding amalgamations where each of them is the only choice. It can be checked using an argument similar to the previous example that there does not exist an amalgamation problem $aB$, $cB$ over some finite substructure $B$ such that $(a, c)$ cannot be coloured by $R$, $G$ or $Y$.

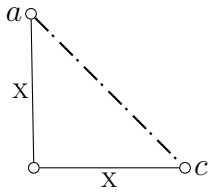
In the following amalgamation, $(a, c)$ has to be coloured by $G$ as $YY$ forbids it to be coloured $R$ or $X$ because of the forbidden triangles $RYY, YYX$ and $GX$ forbids it to be coloured $Y$ because of the forbidden triangle $GYX$. Hence $G \in \mathcal{L}'$.




Similarly, in the following amalgamation, $(a, c) \in R$ as $XY$ forbids it to be coloured $G$ or $Y$ because of the forbidden triangles $GYX, YYX$ and $RR$ forbids it to be coloured $X$ because of the forbidden triangle $RRX$. Hence $R \in \mathcal{L}'$




The following amalgamation implies $Y \in \mathcal{L}'$ since $XX$ forbids $(a, c)$ to be coloured $R$, $G$ or $X$ because of the forbidden triangles $RXX, GXX, XXX$. Hence $Y \in \mathcal{L}'$.




Therefore, we have shown that $\{R, G, Y\} \subseteq \mathcal{L}'$. □

We now look at the asymmetric examples in Figure 2.1. Using a similar argument as in Example 2.3.4, we can show the following:

**Proposition 2.3.6.** *Let $S$ be one of the 2-multi tournaments listed in Figure 2.1. Then any amalgamation problem in $Forb_c(S)$ can be completed by $\mathcal{L}' = \{R^{\pm}\}$.*

## 2.4 Automorphism groups of homogeneous structures

Let $\mathcal{M}$ be a countable homogeneous structure and $G :=\mathrm{Aut}(\mathcal{M})$. We first fix some notation and definitions regarding $G$ and then introduce some topological properties of $G$ that we will use in later chapters.

Let $B \subseteq \mathcal{M}$ be finite. We denote the pointwise stabiliser of $B$ by $G_{(B)}$ and the setwise stabiliser of $B$ by $G_{\{B\}}$. Then, since $\mathcal{M}$ is homogeneous, the set of all realisations of an $n$-type $p$ over $B$ is in fact a $G_{(B)}$-orbit on some $n$-tuple $\bar{a} \in \mathcal{M}^n$. And $\bar{a}, \bar{a}'$ have the same type over $B$ if and only if they lie in the same $G_{(B)}$-orbit. Readers can find more information about this in [Mac11].

We also say a type is *algebraic* if its set of realisations is finite, and *non-algebraic* otherwise. In all of our examples, $\mathrm{tp}(\bar{a}/B)$ is algebraic if and only if $\bar{a} \in B$.

We now look at some topological properties of $\mathrm{Aut}(\mathcal{M})$. For more information about the notions and theorems presented below, we refer the reader to [Kec12]. Note that results stated below work for all countable structures.

**Definition 2.4.1.** Let $X$ be a topological space.

(i) $X$ is *separable* if it has a countable dense subset.

(ii) $X$ is *completely metrisable* if we can define a complete metric on $X$, which generates the topology.

(iii) $X$ is *Polish* if it is a separable and completely metrisable. A topological group is *Polish* if it is also a Polish space.

For any non-empty set $X$, we can define a topology on $\mathrm{Sym}(X)$, the symmetric group of $X$, by defining the basic open sets to be cosets of pointwise stabilisers of finite sets. Suppose $X$ is countable. We can identify $X$ with $\mathbb{N}$ and define a metric space on $\mathrm{Sym}(X)$ by defining $d(g_1, g_2) = d'(g_1, g_2) + d'(g_1^{-1}, g_2^{-1})$ for distinct $g_1, g_2 \in \mathrm{Sym}(X)$, where $d'(g_1, g_2) = 1/n$ and $n$ is the smallest possible such that $g_1(n) \neq g_2(n)$. We can check that this metric space is separable and complete. Moreover, it gives the required topology. Therefore, $\mathrm{Sym}(X)$ is a Polish group. If $\mathcal{M}$ is a structure on $X$, then $\mathrm{Aut}(\mathcal{M})$ is a closed subgroup of $\mathrm{Sym}(X)$ and also a Polish group.

In the proof of our main theorem in Section 4.1, we will use some properties of Polish groups and Baire spaces, which we introduce in the following.

**Definition 2.4.2.** Given a topological space $X$ and $A$ a subset of $X$, we say

(i) $A$ is *nowhere dense* if the closure of $A$ has empty interior.

(ii) $A$ is *meagre* if it is a union of countably many nowhere dense subsets of $X$.

(iii) $A$ is *comeagre* if its complement is meagre.

(iv) $A$ is *analytic* if $X$ is Polish and $A$ is a continuous image of a Polish space.

(v) $X$ is *Baire* if it satisfies one of the following equivalent conditions:

(a) every non-empty open set in $X$ is not meagre.

(b) every comeagre set in $X$ is dense.

(c) the intersection of countably many dense open sets in $X$ is dense.

**Proposition 2.4.3.** *(i) (Baire Category Theorem) Every completely metrisable space is Baire. In particular, every Polish space is Baire.*

*(ii) An analytic subset $A$ of a Polish space has the Baire property, i.e. there is an open set $U$ such that the symmetric difference $(U \setminus A) \cap (A \setminus U)$ is meagre.*

**Theorem 2.4.4.** *(Theorem 8.46, [Kec12]) Let $X$ be a Baire space. Let $G$ be a group of homeomorphisms of $X$ such that for any non-empty open sets $U, V$ of $X$, there exists $g \in G$ such that $g(U) \cap V \neq \emptyset$. Let $A \subseteq X$ be such that $gA = A$ for all $g \in G$. If $A$ has the Baire property, then $A$ is either meagre or comeagre.*

# Chapter 3

# Stationary weak independence relations

Macpherson and Tent proved, in [MT11], the simplicity of the automorphism group of a non-trivial transitive free homogeneous structure $\mathcal{M}$. They used the relation $A \mathop{\downarrow}_B C$ for finite substructures $B \subseteq A, C \subseteq \mathcal{M}$ if the structure on $ABC \subseteq \mathcal{M}$ agrees with the free amalgamation of $A, C$ over $B$ in the free amalgamation class $\mathrm{Age}(\mathcal{M})$. Tent and Ziegler then axiomatise the notion $\mathop{\downarrow}$ to a *stationary independence relation* (**SIR**) in [TZ13a] and thus make it applicable to more structures including the Urysohn space [TZ13a] and the bounded Urysohn space [TZ13b].

In this chapter, we will first see, in Section 3.1, that we can generalise a **SIR** to a *stationary weak independence relation* (**SWIR**) by omitting the Symmetry axiom. We will also give some examples that are already known to have a **SIR** in this section. In the next two sections, we will look at some structures admitting a **SWIR**.

We will look at whether semi-free structures have a **SWIR** in Section 3.2. In order to find a **SWIR** on them, we will define an 'order' on the language $\mathcal{L}$ and thus define a *prioritised semi-free* structure in Definition 3.2.1. We will show in Theorem 3.2.4 that we can find a **SWIR** on all prioritised semi-free homogeneous structures. Note that a semi-free structure is not necessarily a prioritised semi-free structure. We will give some sufficient conditions (see Condition 1 and Condition 2) for a semi-free structure to be a prioritised semi-

free structure. We will show that all but two of Cherlin's semi-free structures listed in Figure 2.1 satisfy the conditions and thus admit a **SWIR**.

In Section 3.3, we will look at some linearly ordered homogeneous structures. We can put a linear order on any homogeneous structure $\mathcal{M}$ whose age is a strong amalgamation class so that the resulting structure is still homogeneous. We call the resulting structure the *linear order expansion* of $\mathcal{M}$ (see Definition 3.3.1). We will then show in Theorem 3.3.4 that if $\mathcal{M}$ admits a **SWIR**, then its linear order expansion also admits a **SWIR**. Some examples this theorem applies to include the linear order expansions of free homogeneous structures and the universal $n$-linear orders.

## 3.1 Definition

We define below a *stationary weak independence relation*. Readers can find some examples in Example 3.1.3. Let $\mathcal{M}$ be a homogeneous structure and $A, B \subseteq \mathcal{M}$. Note that we use the notation $AB$ to denote the substructure of $\mathcal{M}$ on the underlying set $A \cup B$ and $\bar{a}B$ denotes the substructure on $\{a_i | a_i \in \bar{a}\} \cup B$ for $\bar{a} \in \mathcal{M}^n$.

**Definition 3.1.1.** Let $\mathcal{M}$ be a countable homogeneous structure and suppose $A \mathop{\smile\!\!\!\!\!\!|}_B C$ is a ternary relation on finite substructures $A, B, C \subseteq \mathcal{M}$. We say that $\mathop{\smile\!\!\!\!\!\!|}$ is a *stationary weak independence relation* (**SWIR**) if $\mathop{\smile\!\!\!\!\!\!|}$ satisfies the following axioms:

(i) Invariance: for any $g \in \mathrm{Aut}(\mathcal{M})$, if $A \mathop{\smile\!\!\!\!\!\!|}_B C$, then $gA \mathop{\smile\!\!\!\!\!\!|}_{gB} gC$.

(ii) Monotonicity: $A \mathop{\smile\!\!\!\!\!\!|}_B CD \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}_B C,\ A \mathop{\smile\!\!\!\!\!\!|}_{BC} D.$

$$AD \mathop{\smile\!\!\!\!\!\!|}_B C \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}_B C,\ D \mathop{\smile\!\!\!\!\!\!|}_{AB} C.$$

(iii) Transitivity: $A \mathop{\smile\!\!\!\!\!\!|}_B C,\ A \mathop{\smile\!\!\!\!\!\!|}_{BC} D \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}_B D.$

$$A \mathop{\smile\!\!\!\!\!\!|}_B C,\ D \mathop{\smile\!\!\!\!\!\!|}_{AB} C \Rightarrow D \mathop{\smile\!\!\!\!\!\!|}_B C.$$

(iv) Existence: If $p$ is an $n$-type over $B$ and $C$ is a finite set, then $p$ has a realisation $\bar{a}$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_B C$.

If $p$ is an $n$-type over $B$ and $C$ is a finite set, then $p$ has a realisation $\bar{a}'$ such that $C \mathop{\smile\!\!\!\!\!\!|}_B \bar{a}'$.

(v) Stationarity: If $\bar{a}$ and $\bar{a}'$ are $n$-tuples such that $\mathrm{tp}(\bar{a}/B) = \mathrm{tp}(\bar{a}'/B)$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_B C$, $\bar{a}' \mathop{\smile\!\!\!\!\!\!|}_B C$, then $\mathrm{tp}(\bar{a}/BC) = \mathrm{tp}(\bar{a}'/BC)$.

If $\bar{a}$ and $\bar{a}'$ are $n$-tuples such that $\mathrm{tp}(\bar{a}/B) = \mathrm{tp}(\bar{a}'/B)$ and $C \mathop{\smile\!\!\!\!\!\!|}_B \bar{a}$, $C \mathop{\smile\!\!\!\!\!\!|}_B \bar{a}'$, then $\mathrm{tp}(\bar{a}/BC) = \mathrm{tp}(\bar{a}'/BC)$.

If, in addition, $\mathop{\smile\!\!\!\!\!\!|}$ satisfies Symmetry, i.e. $A \mathop{\smile\!\!\!\!\!\!|}_B C \Rightarrow C \mathop{\smile\!\!\!\!\!\!|}_B A$, we say $\mathop{\smile\!\!\!\!\!\!|}$ is a stationary independence relation (**SIR**).

**Remark 3.1.2.** (i) It can be shown from the above axioms that

$$A \underset{B}{\smile\!\!\!\!\!\!|} C \Leftrightarrow AB \underset{B}{\smile\!\!\!\!\!\!|} C \Leftrightarrow A \underset{B}{\smile\!\!\!\!\!\!|} BC.$$

$A \mathop{\smile\!\!\!\!\!\!|}_B BC \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}_B C$ follows from Monotonicity. To show that $A \mathop{\smile\!\!\!\!\!\!|}_B C \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}_B BC$, we can find $A'$ realising $\mathrm{tp}(A/BC)$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_{BC} BC$ by Existence. Then there exists $k \in G_{(BC)}$ such that $kA' = A$. Hence by Invariance, we have $kA' \mathop{\smile\!\!\!\!\!\!|}_{k(BC)} k(BC)$, which is the same as $A \mathop{\smile\!\!\!\!\!\!|}_{BC} BC$. Combining this with $A \mathop{\smile\!\!\!\!\!\!|}_B C$, we have $A \mathop{\smile\!\!\!\!\!\!|}_B BC$ by Transitivity.

Hence, we can assume without loss of generality that $B \subseteq A, C$ when we have $A \mathop{\smile\!\!\!\!\!\!|}_B C$ for arbitrary $A, B, C$.

(ii) In [Mül16], it was noted that Transitivity follows from the other axioms. Let $A, B, C, D \subset \mathcal{M}$ be finite. Suppose $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and $A \mathop{\smile\!\!\!\!\!\!|}_{BC} D$. By Existence, there exists $A'$ realising $\mathrm{tp}(A/B)$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_B CD$. By Monotonicity, we have $A' \mathop{\smile\!\!\!\!\!\!|}_B C$ and $A' \mathop{\smile\!\!\!\!\!\!|}_{BC} D$. Since $\mathrm{tp}(A/B) = \mathrm{tp}(A'/B)$ and $A \mathop{\smile\!\!\!\!\!\!|}_B C$, $A' \mathop{\smile\!\!\!\!\!\!|}_B C$, by Stationarity, we have $\mathrm{tp}(A/BC) = \mathrm{tp}(A'/BC)$. Applying Stationarity again, we have $\mathrm{tp}(A/BCD) = \mathrm{tp}(A'/BCD)$ since $A \mathop{\smile\!\!\!\!\!\!|}_{BC} D$ and $A' \mathop{\smile\!\!\!\!\!\!|}_{BC} D$. Therefore, $A \mathop{\smile\!\!\!\!\!\!|}_B CD$ by Invariance.

(iii) We can obtain Existence on the right from Existence on the left and Invariance. Let $p$ be an $n$-type over some finite set $B \subset \mathcal{M}$ and $C \subset \mathcal{M}$ is a finite set. Let $\bar{a}$ be a realisation of $p$. By Existence on the left, there exists $C'$ realising $\mathrm{tp}(C/B)$ such that $C \mathop{\smile\!\!\!\!\!\!|}_B \bar{a}$. There also exists $g \in Aut(\mathcal{M})$ such that $g$ fixes $B$ pointwise and $gC' = C$. Then by Invariance, we have $C \mathop{\smile\!\!\!\!\!\!|}_B g\bar{a}$. Since $g$ fixes $B$ pointwise, $g\bar{a}$ also realise $p$. Therefore, we have found $\bar{a}' := g\bar{a}$ realising $p$ such that $C \mathop{\smile\!\!\!\!\!\!|}_B \bar{a}'$.

(iv) In many cases, we can find a **SWIR** by defining $A \mathop{\smile\!\!\!\!\!\!|}_B C$ for some finite $B \subseteq A, C \subseteq \mathcal{M}$ if the structure on $ABC \subseteq \mathcal{M}$ agrees with some 'spe-

cial' amalgamation of $A, C$ over $B$ in the amalgamation class $\mathrm{Age}(\mathcal{M})$. Denote the amalgamation by $A \otimes_B C$. We say an amalgamation is *canonical* if for $A', B', C'$ that have the same type as $A, B, C$ respectively, then $A' \otimes_{B'} C'$ has the same type as $A \otimes_B C$.

It was noted in [TZ13a] that if for any finite structures $A, B, C$ such that $B \subseteq A, C \subseteq \mathcal{M}$, the structure on $ABC$ agrees with the canonical amalgamation of $A, C$ over $B$ in $\mathrm{Age}(\mathcal{M})$, then $\downarrow$ satisfies Existence, Invariance and Stationarity:

Existence: Let $p$ be an $n$-type over $B$ and $C$ any finite set. Let $\bar{a}$ be a realisation of $p$ and $\bar{a}B \otimes_B BC \in \mathcal{C}$ be the 'canonical' amalgamation of $\bar{a}B$ and $BC$ over $B$. Then, since we can embed $BC$ into $\bar{a}B \otimes_B BC$, by the Extension Property, we can embed $\bar{a}B \otimes_B BC$ into $\mathcal{M}$. Hence there exists $\bar{a}$ realising $p$ such that $\bar{a} \downarrow_B C$.

Invariance: Given $A, B, C \subseteq \mathcal{M}$, the type of $AB \otimes_B BC$ is dependent only on the types of $A, B, C$ because the amalgamation is 'canonical'.

Stationarity: Suppose $\bar{a}, \bar{a}'$ have the same type over $B$ and are both independent from $C$ over $B$. Then $\bar{a}B \otimes_B BC$ and $\bar{a}'B \otimes_B BC$ have the same type since the amalgamation is 'canonical'. By definition of $\downarrow$, we know that $\bar{a}BC$ is isomorphic to $\bar{a}B \otimes_B BC$ and $\bar{a}'BC$ is isomorphic to $\bar{a}'B \otimes_B BC$. Hence, $\bar{a}BC$ and $\bar{a}'BC$ have the same type.

**Example 3.1.3.** (i) Let $\mathcal{M}$ be a free homogeneous structure as defined in Definition 2.2.7. For finite substructures $A, B, C$ of $M$, we can define $A \downarrow_B C$ if the structure on $ABC \subseteq \mathcal{M}$ agrees with the free amalgamation of $AB$ and $BC$ over $B$, i.e. for any $a \in A \setminus B, c \in C \setminus B$, $(a, c)$ is not related by any relation. We can check that $\downarrow$ is a **SIR**. By part (iii) of the previous remark, we have that $\downarrow$ satisfies Existence, Invariance and Stationarity since free amalgamation is canonical. And by part (ii) of the previous remark, we automatically have Transitivity if we can prove the other axioms. So, we only prove Monotonicity and Symmetry:

Monotonicity: Let $A, B, C, D$ be finite substructures of $M$ such that $A \downarrow_B CD$. Then for any $a \in A \setminus B, c \in C \setminus B$, $(a, c)$ is not in any relation. Hence, we have $A \downarrow_B C$. For any $a \in A \setminus BC, d \in D \setminus BC$, we have $a \in A \setminus B, d \in D \setminus B$, so $(a, d)$ is not in any relation since $A \downarrow_B CD$.

Hence, we have $A \mathop{\downarrow}_{BC} D$.

Symmetry: By the definition of the free amalgamation, we have $A \mathop{\downarrow}_{B} C \Rightarrow C \mathop{\downarrow}_{B} A$.

(ii) Let $\mathcal{M}$ be the rational Urysohn space as defined in Example 2.2.6. For finite substructures $A, B, C$ of $M$, we can define $A \mathop{\downarrow}_{B} C$ if the structure on $ABC \subseteq \mathcal{M}$ agrees with the amalgamation of $AB$ and $BC$ over $B$ as defined in Example 2.2.6, i.e. for any $a \in A \setminus B, c \in C \setminus B$, $d(a,c) = \min_{x \in B}\{d(a,x) + d(x,c)\}$. We can check that $\mathop{\downarrow}$ is a **SIR**. Similarly, we check that $\mathop{\downarrow}$ satisfies Monotonicity and Symmetry:

Monotonicity: Let $A, B, C, D$ be finite substructures of $M$ such that $A \mathop{\downarrow}_{B} CD$. Then for any $a \in A \setminus B, c \in C \setminus B$, $d(a,c) = \min_{x \in B}\{d(a,x) + d(x,c)\}$. Hence, we have $A \mathop{\downarrow}_{B} C$. For any $a \in A, d \in D$, we also have $d(a,d) = \min_{x \in B}\{d(a,x) + d(x,d)\}$. Suppose $A$ is not independent from $D$ over $BC$, i.e. $d(a,d) \neq \min_{x \in BC}\{d(a,x) + d(x,d)\}$. This means that there exists $c \in C \setminus B$ such that $d(a,c) + d(c,d) < d(a,b) + d(b,d)$ for all $b \in B$. Since $d(a,c) = \min_{x \in B}\{d(a,x) + d(x,c)\}$, this implies there exists $b' \in B$ such that $d(a,c) = d(a,b') + d(b',c)$. So, we obtain $d(a,b') + d(b',c) + d(c,d) < d(a,b) + d(b,d)$ for all $b \in B$, contradicting the triangle inequality. Thus, $A \mathop{\downarrow}_{BC} D$.

Symmetry: follows from the definition of $\mathop{\downarrow}$.

## 3.2 Prioritised semi-free amalgamation classes

We now look at whether semi-free structures have a **SWIR**. Let $\mathcal{L}$ be a binary relational language. As defined in Definition 2.3.1, we let $S$ be a set of forbidden triangles and $Forb_c(S)$ be the set of all complete $\mathcal{L}$-structures that do not embed any triangle in $S$. In this section, we will only consider $S$ such that $Forb_c(S)$ forms a semi-free amalgamation class. Let $\mathcal{M}_S$ be its Fraïssé limit and $\mathcal{L}'$ be its set of solutions.

As noted in Remark 3.1.2, if we can find a canonical amalgamation of $A, C$ over $B$ for any $A, B, C \in Forb_c(S)$ such that $B \subseteq A, C$, then we can define $A \mathop{\downarrow}_{B} C$ if the structure on $ABC \subseteq \mathcal{M}_S$ agrees with this amalgamation. Then $\mathop{\downarrow}$ satisfies Existence, Invariance and Stationarity. In order to find a canonical

amalgamation, we can put a linear order on $\mathcal{L}'$ and define the following amalgamation process. We will give some examples of the amalgamation process in Example 3.2.3. Note that for any asymmetric relation $R^{\pm} \in \mathcal{L}'$, we say that $\mathcal{L}'$ consists of two relations, $R^+$ and $R^-$, instead of just $R^{\pm}$.

**Definition 3.2.1.** Let $\mathcal{L}$ be a language consisting of $n$ binary relations. Let $\mathcal{L}' \subsetneq \mathcal{L}$ and suppose $\mathcal{L}' = \{R_1, ..., R_m\}$. Suppose $\mathcal{L}'$ is ordered as $R_1 > \cdots > R_m$. Let $S$ be a set of forbidden triangles. For any $A, B, C \in Forb_c(S)$, where $B \subseteq A, C$, we define the following way to amalgamate $A$ and $C$ over $B$. For each $a \in A \setminus B, c \in C \setminus B$, first check whether $abc$ forms a forbidden triangle for any $b \in B$ if $(a, c) \in R_1$. If $B = \emptyset$ or colouring $(a, c)$ by $R_1$ does not form any forbidden triangle, we complete $(a, c)$ by $R_1$. Otherwise, we check the same thing for $(a, c) \in R_2$ and so on. In other words, we complete $(a, c)$ by $R_i$ where $i$ is the smallest possible integer such that $\mathbf{r}(a, b)\mathbf{r}(b, c)R_i \notin S$ for any $b \in B$.

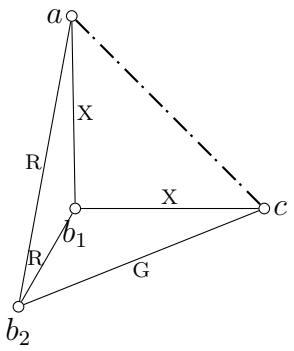
Denote the resulting amalgamation by $A \otimes_B C$. If $A \otimes_B C$ does not embed any forbidden triangle in $S$, i.e. $A \otimes_B C \in Forb_c(S)$, we call it the *prioritised semi-free amalgamation* of $A, C$ over $B$. If for any $A, B, C \in Forb_c(S)$ where $B \subseteq A, C$, we have $A \otimes_B C \in Forb_c(S)$, then we call $Forb_c(S)$ a *prioritised semi-free* amalgamation class with respect to the given ordering on $\mathcal{L}'$ and its Fraïssé limit a *prioritised semi-free* structure.

**Remark 3.2.2.** (i) Not every semi-free amalgamation class forms a prioritised semi-free amalgamation class as we will see in Example 3.2.3.

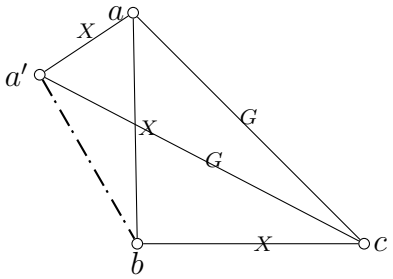
(ii) Note that in $A \otimes_B C$, a forbidden triangle can only appear in $AC \setminus B$ as we ensure by construction that there is no forbidden triangle with a vertex in $B$. Hence, it is either of the form $acc'$ for some $a \in A \setminus B, c, c' \in C \setminus B$ or $aa'c$ for some $a, a' \in A \setminus B, c \in C \setminus B$.

**Example 3.2.3.** (i) Let $\mathcal{L} = \{R, G, Y, X\}$ be the language of the 4-multi-graph and $S$ be #11 of the 4-multi-graphs listed in Figure 2.1, i.e. $S = \{RXX, GGX, YXX, XXX\}$. By Proposition 2.3.3, we may take $\mathcal{L}' = \{R, G\}$.

Suppose the order on $\mathcal{L}'$ is $R > G$. Taking the graph below as an example, if we want to amalgamate $a, c$ over $b_1b_2$, then $\mathbf{r}(a, c) \neq R$ since otherwise we would have a triangle of $RXX$. So, we have $\mathbf{r}(a, c) = G$.

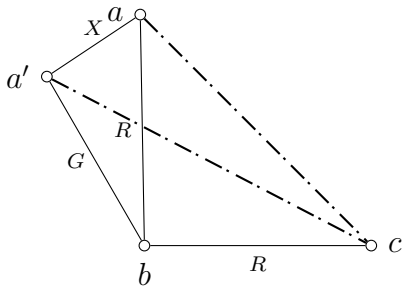



We now check that $Forb_c(S)$ forms a prioritised semi-free amalgamation class with respect to the ordering $R > G$ on $\mathcal{L}'$. Suppose for some $A, B, C$, there exists a forbidden triangle in $A \otimes_B C$. Then by the previous remark, it is either of the form $acc'$ for some $a \in A \setminus B, c, c' \in C \setminus B$ or $aa'c$ for some $a, a' \in A \setminus B, c \in C \setminus B$. Without loss of generality, we may assume it is $aa'c$ for some $a, a' \in A \setminus B, c \in C \setminus B$. Since $\mathbf{r}(a, c), \mathbf{r}(a', c) \in \mathcal{L}'$ and the only forbidden triangle containing two relations in $\mathcal{L}'$ is $GGX$, we have $\mathbf{r}(a, c) = \mathbf{r}(a', c) = G$. As the ordering we have is $R > G$, there exists $b \in B$ such that $b$ forbids $(a, c)$ to be coloured by $R$. Since the only forbidden triangle containing $R$ is $RXX$, we have $\mathbf{r}(b, a) = \mathbf{r}(b, c) = X$. However, we cannot find a colour for $a'b$ without creating a forbidden triangle: $a'ab$ forbids $(a', b)$ to be coloured by $R, Y, X$ because of the forbidden triangles $RXX, YXX, XXX$ and $a'cb$ forbids $(a', b)$ to be coloured by $G$ because of the forbidden triangle $GGX$. Therefore, there does not exist a forbidden triangle in $A \otimes_B C$. We will use a similar method to prove that $Forb_c(S)$ forms a prioritised semi-free amalgamation class if $S$ satisfies a particular condition (Condition 2) in Lemma 3.2.6.



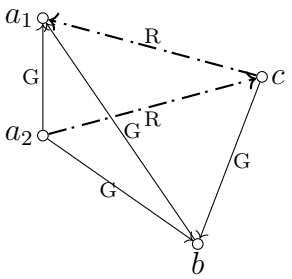

Note that $Forb_c(S)$ does not form a prioritised semi-free amalgamation class with respect to the order $G > R$ on $\mathcal{L}'$. For example, to complete the following amalgamation, we let both $(a, c), (a', c) \in G$, but we would

get a forbidden triangle $GGX$ at $aa'c$.




(ii) Let $\mathcal{L} = \{R^{\pm}, G^{\pm}\}$ be the language of 2-multi-tournament and $S$ be #8 of the 2-multi-tournaments listed in Figure 2.1, i.e. $S = \{G^+G^-G^+, R^+G^-G^+\}$. By Proposition 2.3.6, we may take $\mathcal{L}' = \{R^+, R^-\}$.

Let the ordering on $\mathcal{L}'$ be $R^+ > R^-$. Suppose we want to amalgamate $a_1a_2b$ and $bc$ over $b$ in the graph below. For $(a_1, c)$, since completing it with $R^+$ creates a forbidden triangle $R^+G^-G^+$, we complete it with $R^-$. For $(a_2, c)$, completing it with $R^+$ does not create a forbidden triangle, so we let $\mathbf{r}(a_2, c) = R^+$. Since there is no forbidden triangle with two $R$-relations, $Forb_c(S)$ is a prioritised semi-free amalgamation class with respect to the order $R^+ > R^-$ on $\mathcal{L}' = \{R^+, R^-\}$. We will use a similar method to prove that $Forb_c(S)$ forms a prioritised semi-free amalgamation class if $S$ satisfies a particular condition (Condition 1) in Lemma 3.2.5.



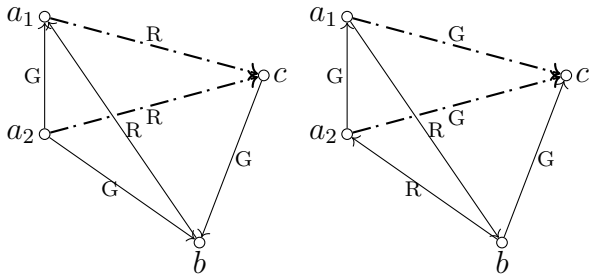

(iii) Let $\mathcal{L} = \{R^{\pm}, G^{\pm}\}$ be the language of the 2-multi-tournament and $S$ be #11 or #12 of the 2-multi-tournaments listed in Figure 2.1, i.e. $S$ is one of the following:

#11 $R^+R^-R^+$, $G^+G^+G^+$, $G^+G^-G^+$, $G^+R^+R^+$, $G^+R^-R^-$

#12 $R^+R^-R^+$, $G^+G^+G^+$, $R^+G^-G^+$, $G^+R^+R^+$, $G^+R^-R^-$, $G^+R^+R^-$

The following graphs show that $Forb_c(S)$ does not form a prioritised

semi-free amalgamation class: if we prioritise $R^+$, we would have a forbidden triangle $G^+R^+R^+$ at $a_1a_2c$ in the amalgamation illustrated in the left graph. Similarly, we would have a forbidden triangle $G^+G^+G^+$ at $a_1a_2c$ in the amalgamation illustrated in the right graph if we prioritise $G^+$. For $R^-, G^-$, we can flip the sides of these amalgamations and the same conclusion follows. Therefore, for any $\mathcal{L}' \subsetneq \mathcal{L}$, we cannot prioritise any relation in $\mathcal{L}'$.




We now prove that if $Forb_c(S)$ forms a prioritised semi-free amalgamation class with respect to some ordering on $\mathcal{L}'$, then its Fraïssé limit $\mathcal{M}_S$ admits a **SWIR**. We will then look at some sufficient conditions on $S$ for $Forb_c(S)$ to form a prioritised semi-free amalgamation class.

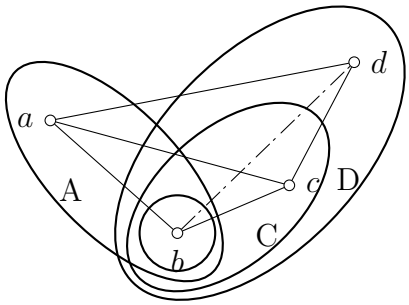
**Theorem 3.2.4.** *Let $S$ be a set of forbidden triangles such that $Forb_c(S)$ forms a prioritised semi-free amalgamation class with respect to some $\mathcal{L}' \subsetneq \mathcal{L}$ and some ordering on $\mathcal{L}'$. Let $\mathcal{M}_S$ be the Fraïssé limit of $Forb_c(S)$. For any finite substructures $A, B, C$ of $\mathcal{M}_S$, let $A \mathop{\downarrow}_B C$ if $ABC = AB \otimes_B BC$ where $AB \otimes_B BC$ is the prioritised semi-free amalgamation as defined in Definition 3.2.1. Then $\mathop{\downarrow}$ is a* **SWIR** *on $\mathcal{M}_S$. Moreover, if $\mathcal{L}$ consists only of symmetric relations, then $\mathop{\downarrow}$ is a* **SIR** *on $\mathcal{M}_S$.*

*Proof.* Invariance: Given $B \subseteq A, C \subseteq \mathcal{M}_S$, the type of prioritised semi-free amalgamation of $A, C$ over $B$ is dependent only on the types of $A, B, C$.

Monotonicity: Let $B \subseteq A \subseteq \mathcal{M}_S$ and $B \subseteq C \subseteq D \subseteq \mathcal{M}_S$. Suppose $A \mathop{\downarrow}_B CD$, i.e. $AD = A \otimes_B D$, we want to show $A \mathop{\downarrow}_B C$ and $AC \mathop{\downarrow}_C D$. By the definition of the prioritised semi-free amalgamation, the choice of relation for $(a, c)$ for any $a \in A \setminus B, c \in C \setminus B$ in $A \otimes_B C$ depends only on $B$. Hence, we have $A \otimes_B C \subseteq A \otimes_B D = AD$, so $A \otimes_B C = ABC$, i.e. $A \mathop{\downarrow}_B C$.

To show that $A \mathop{\downarrow}_{BC} D$, we want to show that $(a, d)$ for any $a \in A \backslash C, d \in D \backslash C$ is completed by the same relation in both $A \otimes_B D$ and $AC \otimes_C D$. Suppose

$(a, d)$ is completed by some $R_i \in \mathcal{L}'$ in $A \otimes_B D$ and $(a, d) \notin R_i$ in $AC \otimes_C D$. This means that there exists $c \in C \setminus B$ that forbids $(a, d)$ to be coloured by $R_i$, i.e. $R_i\mathbf{r}(a, c)\mathbf{r}(c, d)$ forms a forbidden triangle. However, $(a, c), (c, d)$ have the same colours in $A \otimes_B D$ as in $A \otimes_C D$, i.e. this would also be a forbidden triangle in $A \otimes_B D$, a contradiction.




Existence: Let $p$ be an $n$-type over $B$ and $C$ any finite set. Let $\bar{a}'$ be a realisation of $p$. Then $\bar{a}'B \in Forb_c(S)$. Let $\bar{a}'B \otimes_B BC$ be the prioritised semi-free amalgamation of $\bar{a}'B$ and $BC$ over $B$. Since $Forb_c(S)$ forms a prioritised semi-free amalgamation class, we have $\bar{a}'B \otimes_B BC \in Forb_c(S)$. By the Extension Property, since we can embed $BC$ into $\bar{a}'B \otimes_B BC$, we can find embedding $f : \bar{a}'B \otimes_B BC \to \mathcal{M}_S$ such that $f(x) = x$ for all $x \in BC$. Hence, there exists $\bar{a} := f(\bar{a}')$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_B C$ and $\bar{a}$ realises $p = \text{tp}(\bar{a}'/B)$ since $f$ fixes $B$ pointwise.

Stationarity: Suppose $\bar{a}, \bar{a}'$ have the same type over $B$ and are both independent from $C$ over $B$. Then $\bar{a}B \otimes_B BC$ and $\bar{a}'B \otimes_B BC$ have the same type by the definition of the prioritised semi-free amalgamation. Hence, $\bar{a}, \bar{a}'$ have the same type over $BC$.

By Remark 3.1.2, Transitivity follows. All the arguments work exactly the same if we swap the sides of $\mathop{\smile\!\!\!\!\!\!|}$. Therefore, we have shown that if $Forb_c(S)$ forms a prioritised semi-free amalgamation class, then there is a **SWIR** on its Fraïssé limit $\mathcal{M}_S$. Moreover, if $\mathcal{L}$ consists only of symmetric relations, then Symmetry follows immediately and hence $\mathop{\smile\!\!\!\!\!\!|}$ is a **SIR**. □

We now look at some sufficient conditions on $S$ for $Forb_c(S)$ to form a prioritised semi-free amalgamation class. In the next two lemmas, we will show that if a set of forbidden triangles $S$ satisfies one of the following two conditions,

then $Forb_c(S)$ forms a prioritised semi-free amalgamation class. We will then look at whether Cherlin's semi-free structures listed in Figure 2.1 satisfy these conditions.

**Condition 1.** Assume $\mathcal{L}' \subsetneq \mathcal{L}$ and $S$ does not contain any triangle of the form $R_iR_jR'$ where $R_i, R_j \in \mathcal{L}'$ and $R' \in \mathcal{L}$.

**Condition 2.** Assume $\mathcal{L}' = \{R_1, R_2\}$ with order $R_1 > R_2$. Assume that there exist subsets $\hat{\mathcal{L}}, \bar{\mathcal{L}} \subsetneq \mathcal{L}$ satisfying the following:

(i) $\mathcal{L}', \hat{\mathcal{L}}, \bar{\mathcal{L}}$ are pairwise disjoint,

(ii) $S$ contains all triangles of the form $R'\bar{R}^1\bar{R}^2, R_1\bar{R}^1\bar{R}^2, R_2R_2\bar{R}^1, R_2\hat{R}\bar{R}^1$, where $R' \in \mathcal{L} \setminus (\hat{\mathcal{L}} \cup \mathcal{L}'), \bar{R}^1, \bar{R}^2 \in \bar{\mathcal{L}}, \hat{R} \in \hat{\mathcal{L}}$, and

(iii) $S$ contains no other triangle involving $R_1$ or $R_2R_2$.

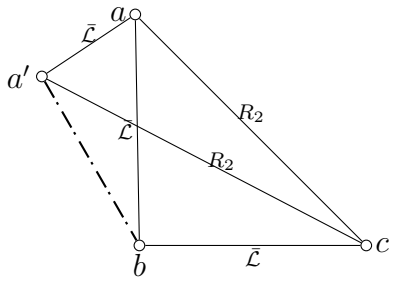
**Lemma 3.2.5.** *Let $\mathcal{L}' \subseteq \mathcal{L}$ and $S$ be as described in Condition 1. Then $Forb_c(S)$ forms a prioritised semi-free amalgamation class with respect to $\mathcal{L}'$ and any ordering on $\mathcal{L}'$.*

*Proof.* For any $A, B, C \in Forb_c(S)$ such that $B \subseteq A, C$, let $A \otimes_B C$ be the prioritised semi-free amalgamation of $A, C$ over $B$. We want to show that $A \otimes_B C \in Forb_c(S)$. Suppose there is a forbidden triangle in $A \otimes_B C$. Then by Remark 3.2.2, it is either of the form $acc'$ for some $a \in A \setminus B, c, c' \in C \setminus B$ or $aa'c$ for some $a, a' \in A \setminus B, c \in C \setminus B$. Without loss of generality, we may assume it is $aa'c$. Since $(a, c)$, $(a', c)$ are completed by relations from $\mathcal{L}'$, $\mathbf{r}(a, c)\mathbf{r}(a', c)\mathbf{r}(a, a')$ is of the form $R_iR_jR' \in S$ where $R_i, R_j \in \mathcal{L}'$ and $R' \in \mathcal{L}$, which contradicts Condition 1. Therefore, we have $A \otimes_B C \in Forb_c(S)$ and thus, $Forb_c(S)$ forms a prioritised semi-free amalgamation class. □

**Lemma 3.2.6.** *Let $\mathcal{L}', \hat{\mathcal{L}}, \bar{\mathcal{L}} \subseteq \mathcal{L}$ and $S$ be as described in Condition 2. Then $Forb_c(S)$ forms a prioritised semi-free amalgamation class with respect to $\mathcal{L}'$ and the ordering $R_1 > R_2$.*

*Proof.* For any $A, B, C \in Forb_c(S)$ such that $B \subseteq A, C$, let $A \otimes_B C$ be the prioritised semi-free amalgamation of $A, C$ over $B$. Suppose there exists a forbidden triangle in $A \otimes_B C$. As in the previous lemma, we may assume it is $aa'c$ for some $a, a' \in A \setminus B, c \in C \setminus B$ and we also have $\mathbf{r}(a, c), \mathbf{r}(a', c) \in \mathcal{L}'$. Since the only forbidden triangle involving $R_iR_j$ where $R_i, R_j \in \mathcal{L}'$ is $R_2R_2\bar{R}^1$

where $\bar{R}^1 \in \bar{\mathcal{L}}$, we have $\mathbf{r}(a,c) = \mathbf{r}(a',c) = R_2$ and $\mathbf{r}(a,a') \in \bar{\mathcal{L}}$. This implies that there exists $b \in B$ that forbids $(a,c)$ to be completed by $R_1$.




Since the only forbidden triangle containing $R_1$ is of the form $R_1\bar{R}^1\bar{R}^2$ where $\bar{R}^1, \bar{R}^2 \in \bar{\mathcal{L}}$, we have $\mathbf{r}(a,b), \mathbf{r}(b,c) \in \bar{\mathcal{L}}$. If we had $\mathbf{r}(a',b) \in \mathcal{L} \setminus (\hat{\mathcal{L}} \cup \mathcal{L}')$, then $a'ab$ would form a forbidden triangle of the form $R'\bar{R}^1\bar{R}^2$ where $\bar{R}^1, \bar{R}^2 \in \bar{\mathcal{L}}, R' \in \mathcal{L} \setminus (\hat{\mathcal{L}} \cup \mathcal{L}')$. If we had $\mathbf{r}(a',b) \in \hat{\mathcal{L}}$, then $a'bc$ would form a forbidden triangle of the form $R_2\hat{R}\bar{R}$, where $\bar{R} \in \bar{\mathcal{L}}, \hat{R} \in \hat{\mathcal{L}}$. If we had $\mathbf{r}(a',b) = R_1$, then $aa'b$ would form a forbidden triangle of the form $R_1\bar{R}^1\bar{R}^2$, where $\bar{R}^1, \bar{R}^2 \in \bar{\mathcal{L}}$. If $\mathbf{r}(a',b) = R_2$, $a'bc$ forms a forbidden triangle of the form $R_2R_2\bar{R}$, where $\bar{R} \in \bar{\mathcal{L}}$. Hence, we cannot find a colour for $a'b$ without creating a forbidden triangle, a contradiction. Thus, $A \otimes_B C \in Forb_c(S)$.

□

**Remark 3.2.7.** Note that when $S$ satisfies Condition 1, the order on $\mathcal{L}'$ does not affect whether $A \otimes_B C \in Forb_c(S)$. So $Forb_c(S)$ is a prioritised semi-free amalgamation class with respect to $\mathcal{L}'$ and any order on $\mathcal{L}'$. However for $S$ satisfying Condition 2, $Forb_c(S)$ forms a prioritised semi-free amalgamation class only with respect to $\mathcal{L}'$ and the order $R_1 > R_2$.

We now look at Cherlin's semi-free structures listed in Figure 2.1 and check whether they satisfy the conditions. By observation, we have the following lemma:

**Lemma 3.2.8.** *Let $S$ be a set of forbidden triangles listed in Figure 2.1.*

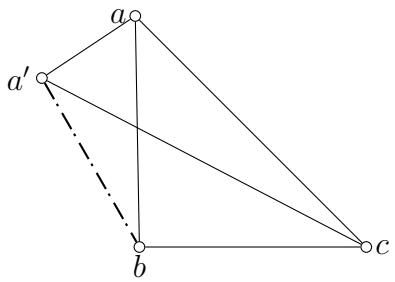
- *If $S$ is the set of forbidden triangles for the 3-multi-graph in the list, then $S$ satisfies Condition 2 with $\mathcal{L}' = \{R, G\}$, $\bar{\mathcal{L}} = \{X\}$, $\hat{\mathcal{L}} = \emptyset$ and order $R > G$.*

- *If $S$ is one of # 1 to # 10 or #21 to # 25 of the 4-multi-graphs, then $S$ satisfies Condition 1 with $\mathcal{L}' = \{R, G\}$.*
- *If $S$ is one of #11 to # 18 of the 4-multi-graphs, then $S$ satisfies Condition 2 with $\mathcal{L}' = \{R, G\}$, $\bar{\mathcal{L}} = \{X\}, \hat{\mathcal{L}} = \emptyset$ and the order $R > G$.*
- *If $S$ is # 19 or # 20 of the 4-multi-graphs, then $S$ satisfies Condition 2 with $\mathcal{L}' = \{R, G\}$, $\bar{\mathcal{L}} = \{X\}, \hat{\mathcal{L}} = \{Y\}$ and the order $R > G$.*
- *If $S$ is #27 of the 4-multi-graphs, then $S$ satisfies Condition 2 with $\mathcal{L}' = \{R, G\}$, $\bar{\mathcal{L}} = \{X, Y\}, \hat{\mathcal{L}} = \emptyset$ and the order $G > R$.*
- *If $S$ is one of #8 to # 10 of the 2-multi-tournaments, then $S$ satisfies Condition 1 with $\mathcal{L}' = \{R^+, R^-\}$ and the order $R^+ > R^-$.*

The lemma covers all structures listed in Figure 2.1 except for # 26 of the 4-multi-graphs and #11 and #12 of 2-multi-tournaments. For $S$ as # 11 or # 12 of the 2-multi-tournaments, we showed in Example 3.2.3 that $Forb_c(S)$ cannot form a prioritised semi-free amalgamation class. For $S$ as #26 of 4-multi-graphs, we showed in Lemma 2.3.5 that $\mathcal{L}' = \{R, G, Y\}$. In the following lemma, we prove that $Forb_c(S)$ forms a prioritised semi-free amalgamation class with the order $G > R > Y$.

**Lemma 3.2.9.** *Let $\mathcal{L} = \{R, G, Y, X\}$ and $S$ be #26 of the 4-multi-graphs in Figure 2.1, i.e. $S = \{RRX, RXX, RYY, GYX, GXX, YYX, XXX\}$. Then $Forb_c(S)$ forms a prioritised semi-free amalgamation class with the order $G > R > Y$ on $\mathcal{L}' = \{R, G, Y\}$.*

*Proof.* Let $A, B, C \in Forb_c(S)$ be such that $B \subseteq A, C$ and let $A \otimes_B C$ be the prioritised semi-free amalgamation of $A, C$ over $B$ as defined in Definition 3.2.1. Suppose there exists a forbidden triangle in $A \otimes_B C$. Same as before, we may assume it is $aa'c$ for some $a, a' \in A \setminus B, c \in C \setminus B$ and we have $\mathbf{r}(a, c), \mathbf{r}(a', c) \in \mathcal{L}'$. Since $\mathcal{L}' = \{R, G, Y\}$, there are five possible cases for $\mathbf{r}(a, c)\mathbf{r}(a', c)\mathbf{r}(a, a')$, as listed below. Since there is no forbidden triangle involving $GG$, we may assume $\mathbf{r}(a, c) \neq G$. Since the order on $\mathcal{L}'$ is $G > R > Y$, there exists $b \in B$ that forbids $(a, c)$ to be coloured by $G$. In each case, $(a', b)$ is the edge for which we cannot find a relation without creating a forbidden triangle and hence we have a contradiction.




I. $\mathbf{r}(a,c)\mathbf{r}(a',c)\mathbf{r}(a,a') = RRX$.

Since $\mathbf{r}(a,c) = R$, either $\mathbf{r}(b,a)\mathbf{r}(b,c) = YX$ or $\mathbf{r}(b,a)\mathbf{r}(b,c) = XY$. However, in the first case, $\mathbf{r}(a,a')\mathbf{r}(a,b) = XY$ forbids $\mathbf{r}(a',b)$ to be $Y, G$ because of the forbidden triangles $YYX, GYX$ and $\mathbf{r}(c,a')\mathbf{r}(c,b) = RX$ forbids $\mathbf{r}(a',b)$ to be $R, X$ because of the forbidden triangles $RRX, RXX$. In the second case, $\mathbf{r}(a,a')\mathbf{r}(a,b) = XX$ forbids $\mathbf{r}(a',b)$ to be $R, G, X$ because of the forbidden triangles $RXX, GXX, XXX$ and $\mathbf{r}(c,a')\mathbf{r}(c,b) = RY$ forbids $\mathbf{r}(a',b)$ to be $Y$ because of the forbidden triangle $RYY$. Hence, in both cases, we cannot find a colour for $(a',b)$ without creating a forbidden triangle.

II. $\mathbf{r}(a,c)\mathbf{r}(a',c)\mathbf{r}(a,a') = YYR$

Since $\mathbf{r}(a,c) = Y$, we have $\mathbf{r}(b,a)\mathbf{r}(b,c) = XX$. However, $\mathbf{r}(a,a')\mathbf{r}(a,b) = RX$ forbids $\mathbf{r}(a',b)$ to be $R, X$ because of the forbidden triangles $RRX$, $RXX$ and $\mathbf{r}(c,a')\mathbf{r}(c,b) = YX$ forbids $\mathbf{r}(a',b)$ to be $Y, G$ because of the forbidden triangles $YYX, GYX$. Hence, we cannot find a colour for $(a',b)$ without creating a forbidden triangle.

III. $\mathbf{r}(a,c)\mathbf{r}(a',c)\mathbf{r}(a,a') = YYX$

Similarly as in the previous cases, $\mathbf{r}(b,a)\mathbf{r}(b,c) = XX$. However, $\mathbf{r}(a,a')$ $\mathbf{r}(a,b) = XX$ forbids $\mathbf{r}(a',b)$ to be $G, R, X$ because of the forbidden triangles $GXX, RXX, XXX$ and $\mathbf{r}(c,a')\mathbf{r}(c,b) = YX$ forbids $\mathbf{r}(a',b)$ to be $Y, G$ because of the forbidden triangles $YYX, GYX$. Hence, we cannot find a colour for $(a',b)$ without creating a forbidden triangle.

IV. $\mathbf{r}(a,c)\mathbf{r}(a',c)\mathbf{r}(a,a') = YRY$

We have $\mathbf{r}(b,a)\mathbf{r}(b,c) = XX$. However, $\mathbf{r}(a,a')\mathbf{r}(a,b) = YX$ forbids $\mathbf{r}(a',b)$ to be $G, Y$ because of the forbidden triangles $GYX, YYX$ and $\mathbf{r}(c,a')\mathbf{r}(c,b) = RX$ forbids $\mathbf{r}(a',b)$ to be $R, X$ because of the forbidden

triangles $RRX, RXX$. Hence, in both case, we cannot find a colour for $(a', b)$ without creating a forbidden triangle.

V. $\mathbf{r}(a, c)\mathbf{r}(a', c)\mathbf{r}(a, a') = YGX$

In this case, we have $\mathbf{r}(a, b)\mathbf{r}(b, c) = XX$. However, $\mathbf{r}(a, a')\mathbf{r}(a, b) = XX$ forbids $\mathbf{r}(a', b)$ to be $R, G, X$ because of the forbidden triangles $RXX, GXX, XXX$ and $\mathbf{r}(a', c)\mathbf{r}(c, b) = GX$ forbids $\mathbf{r}(a', b)$ to be $Y$ because of the forbidden triangle $GYX$. Hence, we cannot find a colour for $(a', b)$ without creating a forbidden triangle.

□

Therefore, combining the results in this section, we have the following corollary:

**Corollary 3.2.10.** *Let $S$ be a set of forbidden triangles such that $Forb_c(S)$ forms a semi-free amalgamation class and let $\mathcal{M}_S$ be its Fraïssé limit. Suppose*

*(i) $S$ satisfies either Condition 1 or Condition 2, or*

*(ii) $S$ is one of the sets of forbidden triangles listed in Figure 2.1 except # 11 and #12 in the asymmetric cases.*

*Then there is a* **SWIR** *on $\mathcal{M}_S$.*

## 3.3 Linearly ordered structures

In this section, we will look at some linearly ordered homogeneous structures. We will define the *linear order expansion* of a homogeneous structure $\mathcal{M}$ below. We will then show in Theorem 3.3.4 that if $\mathcal{M}$ admits a **SWIR**, then its linear order expansion also admits a **SWIR**. We will look at some examples in Example 3.3.5.

**Definition 3.3.1.** Let $\mathcal{C}$ be a strong amalgamation class and $\mathcal{M}$ be its Fraïssé limit. We call $\mathcal{M}$ a *strong* homogeneous structure. Let $\mathcal{L}^< = \mathcal{L} \cup \{<\}$, where the new relational symbol $<$ is interpreted as a (strict) linear order. For any $A \in \mathcal{C}$, we can define a linear order on $A$. Then $A$ is an $\mathcal{L}^<$-structure. We call it a *linear order expansion* of $A$. Let $\mathcal{C}^<$ be the class of all isomorphism types of such expansions. Then $\mathcal{C}^<$ is an amalgamation class, which we will show in

the next remark. We call its Fraïssé limit, denoted by $\mathcal{M}^<$, the *linear order expansion* of $\mathcal{M}$.

**Definition 3.3.2.** Let $\mathcal{L}$ be a relational language and $\mathcal{L}^< = \mathcal{L} \cup \{<\}$. Let $\mathcal{M}$ be a strong homogeneous $\mathcal{L}$-structure and $\mathcal{M}^<$ its linear order expansion. For some finite substructure $X \subset \mathcal{M}^<$ and $n$-tuple $\bar{x}$ in $\mathcal{M}^<$, we define $\mathrm{tp}^{\mathcal{L}}(\bar{x}/X)$ to be the set of all $\mathcal{L}$-formulas satisfied by $\bar{x}$ with parameters in $X$ and $\mathrm{tp}^<(\bar{x}/X)$ to be the set of all $\{<\}$-formulas satisfied by $\bar{x}$ with parameters in $X$. We call them $\mathcal{L}$-*type* and $\{<\}$-*type* respectively.

**Remark 3.3.3.** (i) We can check that $\mathcal{C}^<$ is an amalgamation class. We can check it satisfies the amalgamation property. For any finite $B \subseteq A, C \subseteq \mathcal{M}^<$, let $A^{\mathcal{L}}, B^{\mathcal{L}}, C^{\mathcal{L}}$ be the $\mathcal{L}$-reducts of $A, B, C$. Since $\mathcal{C}$ is a strong amalgamation class, we can find $D^{\mathcal{L}} \in \mathcal{C}$ such that $A^{\mathcal{L}}, C^{\mathcal{L}}$ can be embedded into $D^{\mathcal{L}}$ where $A^{\mathcal{L}} \cap C^{\mathcal{L}} = B^{\mathcal{L}}$. Since $\mathcal{C}^<$ contains all isomorphism types of linear order expansions of $D^{\mathcal{L}}$, we can find $D \in \mathcal{C}^<$ satisfying that for $a \in A \setminus B, c \in C \setminus B$, $a < c$ if and only if there exists $b \in B$ such that $a < b < c$. Then $D$ is the amalgamation of $A, C$ over $B$. IP,HP and JEP are straightforward to see.

(ii) Also note that $\mathcal{M}^<$ satisfies the theory of the dense linear order. Hence, we can identify elements of $\mathcal{M}^<$ as elements of $\mathbb{Q}$ and have intervals of $\mathcal{M}^<$.

(iii) It is straightforward to see $\mathrm{tp}(\bar{a}/X) = \mathrm{tp}(\bar{a}'/X)$ if and only if $\mathrm{tp}^<(\bar{a}/X) = \mathrm{tp}^<(\bar{a}'/X)$ and $\mathrm{tp}^{\mathcal{L}}(\bar{a}/X) = \mathrm{tp}^{\mathcal{L}}(\bar{a}'/X)$.

(iv) Let $X$ be some finite substructure of $\mathcal{M}^<$. For any $n$-$\mathcal{L}$-type $p(\bar{x}/X^{\mathcal{L}})$ and any $n$-$\{<\}$-type $q(\bar{x}/X^<)$, $p(\bar{x}/X^{\mathcal{L}}) \cup q(\bar{x}/X^<)$ is consistent and can be realised in $\mathcal{M}^<$. This is because $\mathcal{M}^<$ embeds all isomorphism types of linear order expansions of $\bar{a}X^{\mathcal{L}} \subseteq \mathcal{M}$, where $\bar{a}$ realises $p(\bar{x}/X^{\mathcal{L}})$. Since $q(\bar{x}/X^<)$ is consistent, we can find one within these linear order expansions that satisfies $q(\bar{x}/X^<)$, i.e. we can find a realisation of $p(\bar{x}/X^{\mathcal{L}}) \cup q(\bar{x}/X^<)$ in $\mathcal{M}^<$.

(v) [CKT19] defined a more general notion. Given disjoint relational languages $\mathcal{L}_1, \mathcal{L}_2$, suppose $\mathcal{M}_i$ is a homogeneous $\mathcal{L}_i$-structure for $i = 1, 2$ and $\mathcal{M}_1, \mathcal{M}_2$ have the same underlying set $\mathcal{M}$. Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$. Then an $\mathcal{L}$-structure $\mathcal{M}^\star$ on $\mathcal{M}$ is called a *free fusion* of $\mathcal{M}_1$ and $\mathcal{M}_2$ if

(a) the $\mathcal{L}_i$-reduct of $\mathcal{M}^\star$ is $\mathcal{M}_i$ for $i = 1, 2$ and

(b) for every non-algebraic $\mathcal{L}_i$-type $p_i$ for $i = 1, 2$, their union $p_1 \cup p_2$ is realised in $\mathcal{M}^\star$.

Then, by definition and part (iv) of this remark, the linear order expansion of $\mathcal{M}$ is the free fusion of $\mathcal{M}$ and the dense linear order.

**Theorem 3.3.4.** *Let $\mathcal{L}$ be a relational language and $\mathcal{L}^< = \mathcal{L} \cup \{<\}$. Let $\mathcal{M}$ be a countable strong homogeneous structure with a* **SWIR** $\mathop{\smile\!\!\!\!\!\!|}^\star$. *For any finite substructures $A, B, C$ of its linear order expansion $\mathcal{M}^<$ with $B \subseteq A, C$, we define $A \mathop{\smile\!\!\!\!\!\!|}_B C$ if $A^\mathcal{L} \mathop{\smile\!\!\!\!\!\!|}^\star_{B^\mathcal{L}} C^\mathcal{L}$ and for any $a \in A \setminus B, c \in C \setminus B$ such that $a < c$, there exists $b \in B$ such that $a < b < c$. Then $\mathop{\smile\!\!\!\!\!\!|}$ is a* **SWIR** *on $\mathcal{M}^<$.*

*Proof.* We simplify the notation $A^\mathcal{L} \mathop{\smile\!\!\!\!\!\!|}^\star_{B^\mathcal{L}} C^\mathcal{L}$ to $A \mathop{\smile\!\!\!\!\!\!|}^\star_B C$ as it is clear when we work with $\mathop{\smile\!\!\!\!\!\!|}^\star$, we only consider the $\mathcal{L}$-reducts of substructures of $\mathcal{M}^<$.

Monotonicity: Suppose $A \mathop{\smile\!\!\!\!\!\!|}_B CD$. Then $A \mathop{\smile\!\!\!\!\!\!|}^\star_B CD$. By Monotonicity of $\mathop{\smile\!\!\!\!\!\!|}^\star$, we have $A \mathop{\smile\!\!\!\!\!\!|}^\star_B C$ and $A \mathop{\smile\!\!\!\!\!\!|}^\star_{BC} D$. We also have that for $a \in A \setminus B, c \in C \setminus B$, if $a < c$, then there exists $b \in B$ such that $a < b < c$. For any $a \in A \setminus BC, d \in D \setminus BC$, if $a < d$, then there exists $b \in B$ such that $a < b < d$. Therefore, we have $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and $A \mathop{\smile\!\!\!\!\!\!|}_{BC} D$.

Suppose $AD \mathop{\smile\!\!\!\!\!\!|}_B C$. Then $AD \mathop{\smile\!\!\!\!\!\!|}^\star_B C$. By Monotonicity of $\mathop{\smile\!\!\!\!\!\!|}^\star$, we have $A \mathop{\smile\!\!\!\!\!\!|}^\star_B C$ and $D \mathop{\smile\!\!\!\!\!\!|}^\star_{AB} C$. We also have that for $a \in A \setminus B, c \in C \setminus B$, if $a < c$, then there exists $b \in B$ such that $a < b < c$. For any $d \in D \setminus AB, c \in C \setminus AB$, if $d < c$, then there exists $b \in B$ such that $d < b < c$. Therefore, we have $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and $D \mathop{\smile\!\!\!\!\!\!|}_{AB} C$.

Transitivity: Suppose $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and $A \mathop{\smile\!\!\!\!\!\!|}_{BC} D$. By Transitivity of $\mathop{\smile\!\!\!\!\!\!|}^\star$, we have $A \mathop{\smile\!\!\!\!\!\!|}^\star_B D$. For any $a \in A \setminus B$, $d \in D \setminus B$ such that $a < d$, there exists $b \in B$ such that $a < b < d$ or there exists $c \in C \setminus B$ such that $a < c < d$. In the latter case, there exists $b \in B$ such that $a < b < c < d$ since $A \mathop{\smile\!\!\!\!\!\!|}_B C$. Hence, in both cases, there exists $b \in B$ such that $a < b < d$. Therefore, we have $A \mathop{\smile\!\!\!\!\!\!|}_B D$.

Suppose $A \mathop{\smile\!\!\!\!\!\!|}_B C$ and $D \mathop{\smile\!\!\!\!\!\!|}_{AB} C$. By Transitivity of $\mathop{\smile\!\!\!\!\!\!|}^\star$, we have $D \mathop{\smile\!\!\!\!\!\!|}^\star_B C$. For any $d \in D \setminus B$, $c \in C \setminus B$ such that $d < c$, there exists $b \in B$ such that $d < b < c$ or there exists $a \in A \setminus B$ such that $d < a < c$. In the latter case, there exists $b \in B$ such that $d < a < b < c$ since $A \mathop{\smile\!\!\!\!\!\!|}_B C$. Hence, in both

cases, there exists $b \in B$ such that $d < b < c$. Therefore, we have $D \mathop{\downarrow}_B C$.

Stationarity: Suppose $\mathrm{tp}(\bar{a}/B) = \mathrm{tp}(\bar{a}'/B)$ and $\bar{a} \mathop{\downarrow}_B C$, $\bar{a}' \mathop{\downarrow}_B C$. Then we have $\mathrm{tp}^{\mathcal{L}}(\bar{a}/B) = \mathrm{tp}^{\mathcal{L}}(\bar{a}'/B)$ and $\bar{a} \mathop{\downarrow}^{\star}_B C$, $\bar{a}' \mathop{\downarrow}^{\star}_B C$. By Stationarity of $\mathop{\downarrow}^{\star}$, we have $\mathrm{tp}^{\mathcal{L}}(\bar{a}/BC) = \mathrm{tp}^{\mathcal{L}}(\bar{a}'/BC)$. To see that $\mathrm{tp}^{<}(\bar{a}/BC) = \mathrm{tp}^{<}(\bar{a}'/BC)$, suppose otherwise. Then, there exists $a_i \in \bar{a}, a_i' \in \bar{a}'$ and $c \in C \setminus B$ such that $a_i < c < a_i'$. Since $\bar{a} \mathop{\downarrow}_B C$, there exists $b \in B$ such that $a_i < b < c < a_i'$, which contradicts $\mathrm{tp}(\bar{a}/B) = \mathrm{tp}(\bar{a}'/B)$. Therefore, we have $\mathrm{tp}^{<}(\bar{a}/BC) = \mathrm{tp}^{<}(\bar{a}'/BC)$ and thus, $\mathrm{tp}(\bar{a}/BC) = \mathrm{tp}(\bar{a}'/BC)$.

Existence: Let $p = \mathrm{tp}(x/B)$ be an $n$-type. Since by part (i) of the previous remark, we can find an amalgamation of $\bar{a}'B$ and $BC$ over $B$ where $\bar{a}'$ realises $p$ and for $a_i \in \bar{a}' \setminus B, c \in C \setminus B$, we have $a < c$ if and only if there exists $b \in B$ such that $a < b < c$. We can embed $BC$ into this amalgamation. Then by the Extension Property, we can embed this amalgamation back to $\mathcal{M}^{<}$. Hence, we can find $\bar{a} \in \mathcal{M}^{<}$ realising $p$ such that $\bar{a} \mathop{\downarrow}_B C$.

Invariance is straightforward to see. We can prove the rest by swapping the sides of $\mathop{\downarrow}$. □

**Example 3.3.5.** (i) If $\mathcal{M}$ is the Fraïssé limit of the class of all finite sets. Then $\mathcal{M}^{<}$ is the dense linear order $(\mathbb{Q}, \leq)$. We can define $A \mathop{\downarrow}^{\star}_B C$ for finite substructures $A, B, C \subseteq \mathcal{M}$ if $A \cap C \subseteq B$. We can check that $\mathop{\downarrow}^{\star}$ is a **SIR**. Hence, by the previous theorem, there is a **SWIR** $\mathop{\downarrow}$ on $(\mathbb{Q}, \leq)$: for finite $A, B, C \subseteq \mathbb{Q}$, $A \mathop{\downarrow}_B C$ if for any $a \in A \setminus B, c \in C \setminus B$ such that $a < c$, there exists $b \in B$ such that $a < b < c$. Note that this agrees with the amalgamation of finite linear orders discussed in Example 2.2.6.

(ii) Let $\mathcal{M}$ be a free homogeneous structure. Then there is a **SIR** on $\mathcal{M}$ as shown in Example 3.1.3. Hence by the previous theorem, we can find a **SWIR** $\mathop{\downarrow}$ on $\mathcal{M}^{<}$ by defining $A \mathop{\downarrow}_B C$ if for any $a \in A \setminus B, c \in C \setminus B$, $(a, c)$ is not related by any relation in $\mathcal{L}$ and if $a < c$, then there exists $b \in B$ such that $a < b < c$.

(iii) Let $\mathcal{L} = \{<_1, ..., <_n\}$, where $<_i$ is interpreted as a linear order for each $i = 1, ..., n$. Let $\mathcal{C}$ be the class of all finite $\mathcal{L}$-structures. We call the Fraïssé limit of $\mathcal{C}$, denoted by $\mathcal{M}_n$, the *universal $n$-linear order*. It is straightforward to see that $\mathcal{M}_n$ is the linear order expansion of $\mathcal{M}_{n-1}$ and $\mathcal{M}_1$ is the dense linear order. Therefore, by repeated application of

Theorem 3.3.4, starting with $\mathcal{M}_1$, we can find a **SWIR** on the universal $n$-linear order: for finite substructures $A, B, C \subseteq \mathcal{M}_n$, define $A \mathop{\smile\!\!\!\!\!\!|}_B C$ if for any $a \in A \setminus B, c \in C \setminus B$ such that $a <_i c$ for some $i$, there exists $b \in B$ such that $a <_i b <_i c$.

# Chapter 4

# Generalisation of Tent and Ziegler's Theorem

In [MT11], Macpherson and Tent proved that the automorphism of a non-trivial free homogeneous structure $\mathcal{M}$ is simple if it is transitive. They proved the result by first defining $A \mathop{\smile\!\!\!\!\!\!|}_B C$ for finite substructures $B \subseteq A, C \subseteq \mathcal{M}$ if the structure on $ABC \subseteq \mathcal{M}$ agrees with the free amalgamation of $A, C$ over $B$. Then they employed results regarding the Polish group structure of $\mathrm{Aut}(\mathcal{M})$ as well as Lascar's method from [Las92].

Tent and Ziegler then axiomatise the notion $\mathop{\smile\!\!\!\!\!\!|}$ mentioned above to a **SIR** and generalised the argument in [MT11]. They also applied the generalised argument to the Urysohn space in [TZ13a] and the bounded Urysohn space in [TZ13b]. They did this by defining the notion of a *moving almost maximally* automorphism, stated in the following. They showed that for a countable homogeneous structure $\mathcal{M}$ with a **SIR**, if an automorphism $g \in \mathrm{Aut}(\mathcal{M})$ moves almost maximally, then any element of $\mathrm{Aut}(\mathcal{M})$ can be written as a product of sixteen conjugates of $g$ and $g^{-1}$.

**Definition 4.0.1.** For a countable homogeneous structure $\mathcal{M}$ with a **SIR** $\mathop{\smile\!\!\!\!\!\!|}$, we say an automorphism $g \in \mathrm{Aut}(\mathcal{M})$ *moves almost maximally* if for any type $p$ over some finite set $X \subseteq \mathcal{M}$, there exists some realisation $\bar{a}$ of $p$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_X g\bar{a}$.

**Theorem 4.0.2.** *([TZ13a]) Suppose that $\mathcal{M}$ is a countable homogeneous structure with a* **SIR**. *If $g \in Aut(\mathcal{M})$ moves almost maximally, then any non-*

*trivial element of Aut($\mathcal{M}$) is the product of sixteen conjugates of $g$ and $g^{-1}$.*

During research, motivated by trying to apply the above theorem to Cherlin's asymmetric semi-free structures and prove the simplicity of their automorphism groups, the author generalised the above theorem to hold for a countable homogeneous structure with a **SWIR**. To do so, we need to first generalise the notion of moving almost maximally to the following:

**Definition 4.0.3.** Let $\mathcal{M}$ be a countable homogeneous structure with a **SWIR** $\downarrow$.

(i) Given a type $p$ over some finite set $X \subset \mathcal{M}$, we say that $g \in \mathrm{Aut}(\mathcal{M})$ *moves $p$ almost $R$-maximally* if there is a realisation $\bar{a}$ of $p$ such that

$$\bar{a} \underset{X}{\downarrow} g\bar{a}.$$

We say that $g$ *moves almost $R$-maximally* if for any finite set $X \subset \mathcal{M}$ and any type $p$ over $X$, $g$ moves $p$ almost $R$-maximally.

(ii) Given a type $p$ over some finite set $X \subset \mathcal{M}$, we say that $g \in \mathrm{Aut}(\mathcal{M})$ *moves $p$ almost $L$-maximally* if there is a realisation $\bar{a}$ of $p$ such that

$$g\bar{a} \underset{X}{\downarrow} \bar{a}.$$

We say that $g$ *moves almost $L$-maximally* if for any finite set $X \subset \mathcal{M}$ and any type $p$ over $X$, $g$ moves $p$ almost $L$-maximally.

Hence, we can then generalise Tent and Ziegler's theorem to the following theorem, which we will prove in Section 4.1. The proof follows a similar approach as in [TZ13a], but with more details presented here. The main modification appears in Lemma 4.1.6.

**Theorem 4.0.4.** *Suppose $\mathcal{M}$ is a countable homogeneous structure with a* **SWIR** *and $g \in$Aut($\mathcal{M}$) is such that $g$ and $g^{-1}$ move both almost $R$-maximally and almost $L$-maximally. Then any non-trivial element of Aut($\mathcal{M}$) can be written as a product of eight conjugates of $g$.*

In [TZ13a], it was then shown that for any non-trivial automorphism $g$ of a non-trivial transitive free homogeneous structure $\mathcal{M}$, we can construct, by a back-and-forth construction, a product of conjugates of $g$ and $g^{-1}$ that moves

almost maximally. Hence, we can derive the simplicity of Aut($\mathcal{M}$).

Similarly, in order to apply the above theorem to homogeneous structures $\mathcal{M}$ admitting a **SWIR**, we will go through the same process in the next Chapter, i.e. construct a product of conjugates of $g$ and $g^{-1}$ that moves both almost $R$-maximally and $L$-maximally for any non-trivial automorphism $g \in$Aut($\mathcal{M}$). We will show in Section 4.2 that we can make this step easier by weakening the condition on $g \in$Aut($\mathcal{M}$) in the above theorem in two ways (see Corollary 4.2.2 and Corollary 4.2.4).

Throughout this chapter, we let $\mathcal{M}$ be a countable homogeneous structure with a **SWIR** and let $G =$Aut($\mathcal{M}$). Let $A$ be some finite substructure of $\mathcal{M}$. We denote the pointwise stabiliser of $A$ by $G_{(A)}$. For $h, g \in G$, we write $g^h := h^{-1}gh$ and $[h, g] := h^{-1}g^{-1}hg$.

## 4.1 The proof of Theorem 4.0.4

The following two lemmas generalise Lemma 3.1 in [TZ13a]. Since the original proof does not depend on symmetry, we simply generalise by working on both sides.

**Lemma 4.1.1.** *Let $A, B, C, D$ be finite substructures of $\mathcal{M}$, if $A \downarrow_B C$ and $D$ is arbitrary, then*

*(i) there exists $D' \subseteq \mathcal{M}$ such that tp$(D'/BC)$ =tp$(D/BC)$ and $A \downarrow_B CD'$, and*

*(ii) there exists $D'' \subseteq \mathcal{M}$ such that tp$(D''/AB)$ =tp$(D/AB)$ and $AD'' \downarrow_B C$.*

*Proof.* (i) By Existence, there exists $D'$ realising tp$(D/BC)$ such that $A \downarrow_{BC} D'$. Then by Transitivity on $A \downarrow_B C$ and $A \downarrow_{BC} D'$, we have $A \downarrow_B CD'$.

(ii) Similarly, there exists $D'$ realising tp$(D/AB)$ such that $D' \downarrow_{AB} C$. Then by Transitivity, $AD' \downarrow_B C$.

□

**Lemma 4.1.2.** *Let $A, B, C$ be finite substructures of $\mathcal{M}$ such that $A \downarrow_B C$ and $g_1, ..., g_n \in G$. Then*

*(i) there exists* $e \in G_{(BC)}$ *such that* $A \downarrow_B Cg_1^e(C) \cdots g_n^e(C)$, *and*

*(ii) there exists* $f \in G_{(AB)}$ *such that* $Ag_1^f(A) \cdots g_n^f(A) \downarrow_B C$.

*Proof.* (i) By the previous lemma, there exist $C_1, ..., C_n$ such that $\text{tp}(C_1...C_n/BC)$ $=\text{tp}(g_1(C)...g_n(C)/BC)$ and $A \downarrow_B CC_1...C_n$. Then there exists $e \in G_{(BC)}$ such that $C_i = e^{-1}g_i(C)$ and thus, $g_i^e(C) = C_i$.

(ii) Similarly, there exist $A_1, ..., A_n$ such that $\text{tp}(A_1...A_n/AB) =\text{tp}(g_1(A)...$ $g_n(A)/AB)$ and $AA_1...A_n \downarrow_B C$. Then there exists $f \in G_{(AB)}$ such that $A_i = f^{-1}g_i(A) = g_i^f(A)$.

□

The following proposition uses a similar approach as Proposition 3.2 in [TZ13a]. The difference is that we require $Y_4 \downarrow_{Y_3} Y_2$ here instead of $Y_2 \downarrow_{Y_3} Y_4$ as in [TZ13a]. This is achieved in Step 4 of the proof.

**Proposition 4.1.3.** *Let* $g_1, ..., g_4 \in Aut(\mathcal{M})$ *and* $X_0, ..., X_4$ *be finite substructures of* $\mathcal{M}$ *be such that* $g_i(X_{i-1}) = X_i$. *Then for* $i = 1, ..., 4$, *there are* $a_i \in G_{(X_{i-1}X_i)}$ *and extensions* $X_i \subset Y_i$ *such that* $g_i^{a_i}(Y_{i-1}) = Y_i$ *and* $Y_0 \downarrow_{Y_1} Y_2$, $Y_4 \downarrow_{Y_3} Y_2$.

*Proof.* Step 1. Choose a finite extension $X_1'$ of $X_1$ such that $X_0 \downarrow_{X_1'} X_2X_3X_4$. Such extension exists as we can always choose $X_1' = X_0 \cup ... \cup X_4$.

Step 2. Applying part (i) of the previous lemma to $A = X_0, B = X_1', C = X_1'X_2X_3X_4$ and automorphisms $g_2, g_3g_2, g_4g_3g_2$, we obtain $e \in G_{(X_1'X_2X_3X_4)}$ such that

$$X_0 \underset{X_1'}{\downarrow} X_1'X_2X_3X_4g_2^e(X_1'X_2X_3X_4)(g_3g_2)^e(X_1'X_2X_3X_4)(g_4g_3g_2)^e(X_1'X_2X_3X_4).$$

By Monotonicity, we have

$$X_0 \underset{X_1'}{\downarrow} g_2^e(X_1')(g_3g_2)^e(X_1')(g_4g_3g_2)^e(X_1').$$

Let $X_2' := g_2^e(X_1')$, $X_3' := (g_3g_2)^e(X_1')$, $X_4' := (g_4g_3g_2)^e(X_1')$. Since $X_1 \subseteq X_1'$

and $e \in G_{(X_1' X_2 X_3 X_4)}$, we have

$$X_2' = g_2^e(X_1') \supseteq g_2^e(X_1) = e^{-1} g_2(X_1) = e^{-1}(X_2) = X_2,$$

$$X_3' = (g_3 g_2)^e(X_1') = g_3^e(X_2') \supseteq g_3^e(X_2) = X_3,$$

$$X_4' = (g_4 g_3 g_2)^e(X_1') = g_4^e(X_3') \supseteq g_4^e(X_3) = X_4.$$

Hence, we have

$$X_0 \underset{X_1'}{\downarrow} X_2' X_3' X_4'.$$

Step 3. Applying part (ii) of the previous lemma to $A = X_0 X_1', B = X_1', C = X_2' X_3' X_4'$ and automorphism $g_1^{-1}$, we have $f \in G_{(X_0 X_1')}$ such that

$$X_0 X_1' (g^{-1})^f (X_0 X_1') \underset{X_1'}{\downarrow} X_2' X_3' X_4'.$$

By Monotonicity,

$$(g_1^{-1})^f (X_1') \underset{X_1'}{\downarrow} X_2' X_3' X_4'.$$

Let $X_0' := (g_1^{-1})^f (X_1')$. Since $X_1 \subseteq X_1'$ and $f \in G_{(X_0 X_1')}$, we have

$$X_0' = (g_1^{-1})^f (X_1') \supseteq (g_1^{-1})^f (X_1) = f^{-1} g_1^{-1}(X_1) = f^{-1}(X_0) = X_0.$$

Hence, we have

$$X_0' \underset{X_1'}{\downarrow} X_2' X_3' X_4'.$$

Set $h_1 = g_1^f, h_2 = g_2^e, h_3 = g_3^e, h_4 = g_4^e$. Then we have $h_i(X_{i-1}') = X_i' \supseteq X_i$ for $i = 1, 2, 3, 4$.

Step 4. Extend $X_3'$ to $Y_3$ such that $X_4' \underset{Y_3}{\downarrow} X_2' X_1'$.

Applying part (i) of the previous lemma to $A = X_4', B = Y_3, C = Y_3 X_2' X_1'$ and automorphisms $h_3^{-1}, h_2^{-1} h_3^{-1}$, we get $a \in G_{(Y_3 X_2' X_1')}$ such that

$$X_4' \underset{Y_3}{\downarrow} Y_3 X_2' X_1' (h_3^{-1})^a (Y_3 X_2' X_1') (h_2^{-1} h_3^{-1})^a (Y_3 X_2' X_1').$$

By Monotonicity, we have

$$X_4' \underset{Y_3}{\downarrow} (h_3^{-1})^a(Y_3)(h_2^{-1}h_3^{-1})^a(Y_3).$$

Let $Y_2 := h_3^{-a}(Y_3)$ and $Y_1 := (h_2^{-1}h_3^{-1})^a(Y_3)$. Since $Y_3 \supseteq X_3'$ and $a \in G_{(Y_3X_2'X_1')}$, we have

$$Y_2 = h_3^{-a}(Y_3) \supseteq h_3^{-a}(X_3') = a^{-1}h_3^{-1}(X_3') = X_2',$$

$$Y_1 = (h_2^{-1}h_3^{-1})^a(Y_3) = h_2^{-a}(Y_2) \supseteq h_2^{-a}(X_2') = X_1'.$$

Hence, we have

$$X_4'Y_3 \underset{Y_3}{\downarrow} Y_2Y_1.$$

Step 5. Applying part (ii) of the previous lemma to $A = X_4'Y_3, B = Y_3, C = Y_1Y_2$ and $h_4$, we have $b \in G_{(Y_3X_4')}$ such that

$$Y_3X_4'h_4^b(Y_3X_4') \underset{Y_3}{\downarrow} Y_1Y_2.$$

By Monotonicity,

$$h_4^b(Y_3) \underset{Y_3}{\downarrow} Y_1Y_2.$$

Let $Y_4 := h_4^b(Y_3)$. Then we have $Y_4 = h_4^b(Y_3) \supseteq h_4^b(X_3') = X_4'$ and

$$Y_4 \underset{Y_3}{\downarrow} Y_1Y_2.$$

Step 6. By Lemma 4.1.1, since $X_0' \underset{X_1'}{\downarrow} X_2'X_3'X_4'$, we can find $Y_1'Y_2'Y_3'Y_4'$ realising $\mathrm{tp}(Y_1Y_2Y_3Y_4/X_1'X_2'X_3'X_4')$ such that $X_0' \underset{X_1'}{\downarrow} Y_1'Y_2'Y_3'Y_4'$. Then by Invariance, we also have $Y_4' \underset{Y_3'}{\downarrow} Y_1'Y_2'$. Therefore, we may assume that

$$X_0' \underset{X_1'}{\downarrow} Y_1Y_2Y_3Y_4.$$

Then, by Monotonicity,

$$X_0' \underset{Y_1}{\downarrow} Y_2Y_3Y_4.$$

Applying part (ii) of the previous lemma to $A = X_0'Y_1, B = Y_1, C = Y_2Y_3Y_4$ and automorphism $h_1^{-1}$, we find some $c \in G_{(X_0'Y_1)}$ such that

$$X_0'Y_1h_1^{-c}(X_0'Y_1) \underset{Y_1}{\downarrow} Y_2Y_3Y_4.$$

Let $Y_0 = h_1^{-c}(Y_1)$. Then $Y_0 = h_1^{-c}(Y_1) \supseteq h_1^{-c}(X_1') = X_0'$. We also have, by Monotonicity, that

$$Y_0 \underset{Y_1}{\downarrow} Y_2Y_3Y_4.$$

Therefore, we have $g_1^{fc}(Y_0) = Y_1, g_2^{ea}(Y_1) = Y_2, g_3^{ea}(Y_2) = Y_3, g_4^{eb}(Y_3) = Y_4$ and $Y_0 \underset{Y_1}{\downarrow} Y_2$, $Y_4 \underset{Y_3}{\downarrow} Y_2$.

□

The next lemma follows a similar approach as Lemma 3.5 in [TZ13a]. Lemma 4.1.6 requires some modifications from Lemma 3.6 in [TZ13a]. Combining these two lemmas, we obtain Proposition 4.1.7 the same way as obtaining Proposition 3.4 from Lemma 3.5 and 3.6 in [TZ13a].

**Lemma 4.1.4.** *Let $g \in G$ move almost $L$-maximally and $X, Y, C$ be finite sets such that $g(X) = Y$ and $X \underset{Y}{\downarrow} C$. Let $\bar{x}$ be a tuple. Then there is some $a \in G_{(XY)}$ such that*

$$g^a(\bar{x}) \underset{Y}{\downarrow} C.$$

*Proof.* Since $g$ moves almost $L$-maximally, there exists $\bar{x}'$ realising $\mathrm{tp}(\bar{x}/XY)$ such that $g\bar{x}' \underset{XY}{\downarrow} \bar{x}'$. Since $\mathrm{tp}(\bar{x}/XY) = \mathrm{tp}(\bar{x}'/XY)$, we have $a_1 \in G_{(XY)}$ such that $a_1(\bar{x}) = \bar{x}'$. Then, acting by $a_1^{-1}$ on $g\bar{x}' \underset{XY}{\downarrow} \bar{x}'$, which is the same as $ga_1\bar{x} \underset{XY}{\downarrow} a_1\bar{x}$, we have

$$g^{a_1}(\bar{x}) \underset{XY}{\downarrow} \bar{x}.$$

By Existence, there exists $\bar{y}$ realising $\mathrm{tp}(g^{a_1}(\bar{x})/\bar{x}XY)$ such that

$$\bar{y} \underset{\bar{x}XY}{\downarrow} C.$$

By Invariance, we also have $\bar{y} \underset{XY}{\downarrow} \bar{x}$. Then by Transitivity on $\bar{y} \underset{XY}{\downarrow} \bar{x}$ and

$\bar{y} \mathop{\downarrow}_{\bar{x}XY} C$, we have

$$\bar{y} \mathop{\downarrow}_{XY} C.$$

Together with $X \mathop{\downarrow}_{Y} C$, we have, by Transitivity,

$$\bar{y} \mathop{\downarrow}_{Y} C.$$

Since $\text{tp}(g^{a_1}(\bar{x})/\bar{x}XY) = \text{tp}(\bar{y}/\bar{x}XY)$, there exists $a_2 \in G_{(\bar{x}XY)}$ with $\bar{y} = a_2^{-1} g^{a_1}(\bar{x}) = g^{a_1 a_2}(\bar{x})$. □

**Remark 4.1.5.** We can also prove the following by using a symmetric argument: Let $g \in G$ move almost $R$-maximally and $X, Y, C$ be finite sets such that $g(X) = Y$ and $C \mathop{\downarrow}_{Y} X$. Let $\bar{x}$ be a tuple. Then there is some $a \in G_{(XY)}$ such that $C \mathop{\downarrow}_{Y} g^a(\bar{x})$.

**Lemma 4.1.6.** *Let $g \in G$ move almost $R$-maximally and let $X, Y$ be finite sets such that $g(X) = Y$. Let $\bar{x}$ and $\bar{y}$ be tuples such that $g\cdot tp(\bar{x}/X) = tp(\bar{y}/Y)$ and $\bar{x} \mathop{\downarrow}_{X} \bar{y}Y$, $\bar{y} \mathop{\downarrow}_{Y} X$. Then, there exists $a \in G_{(XY)}$ such that $\bar{y} = g^a(\bar{x})$.*

*Proof.* By Existence, there exists $\bar{y}'$ realising $\text{tp}(\bar{y}/XY)$ such that $\bar{y}' \mathop{\downarrow}_{XY} g(Y)$. Since $g$ moves almost $R$-maximally, there exists $\bar{y}''$ realising $\text{tp}(\bar{y}'/XYg(Y))$ such that

$$\bar{y}'' \mathop{\downarrow}_{XYg(Y)} g\bar{y}''.$$

By Invariance, we have $\bar{y}'' \mathop{\downarrow}_{XY} g(Y)$. Then, by Transitivity, we obtain

$$\bar{y}'' \mathop{\downarrow}_{XY} g(\bar{y}''Y).$$

Since $\text{tp}(\bar{y}/XY) = \text{tp}(\bar{y}'/XY) = \text{tp}(\bar{y}''/XY)$ and $\bar{y} \mathop{\downarrow}_{Y} X$, by Invariance, we have $\bar{y}'' \mathop{\downarrow}_{Y} X$. Hence, by Transitivity, we have

$$\bar{y}'' \mathop{\downarrow}_{Y} g(\bar{y}''Y),$$

which by Invariance, is the same as

$$g^{-1}(\bar{y}'') \mathop{\downarrow}_{X} \bar{y}''Y.$$

Since $\mathrm{tp}(\bar{y}''/XY) = \mathrm{tp}(\bar{y}/XY)$, there exists $b \in G_{(XY)}$ such that $b\bar{y} = \bar{y}''$, so we have

$$g^{-b}(\bar{y}) \underset{X}{\downarrow} \bar{y}Y.$$

We also have that $\mathrm{tp}(g^{-b}(\bar{y})/X) = g^{-1}\cdot\mathrm{tp}(\bar{y}/Y) = \mathrm{tp}(\bar{x}/X)$. Since $\bar{x} \underset{X}{\downarrow} yY$ and $g^{-b}(\bar{y}) \underset{X}{\downarrow} \bar{y}Y$, by Stationarity, we obtain $\mathrm{tp}(g^{-b}(\bar{y})/\bar{y}XY) = \mathrm{tp}(\bar{x}/\bar{y}XY)$. Therefore, there exists $c \in G_{(\bar{y}XY)}$ such that $\bar{x} = c^{-1}g^{-b}(\bar{y}) = g^{-bc}(\bar{y})$. □

**Proposition 4.1.7.** *Let $g_1, ..., g_4 \in G$ be such that $g_1, g_4^{-1}$ move almost $L$-maximally and $g_2^{-1}, g_3$ move almost $R$-maximally. Let $Y_0, ..., Y_4$ be finite sets such that $g_i(Y_{i-1}) = Y_i$ for $i = 1, 2, 3, 4$ and $Y_0 \underset{Y_1}{\downarrow} Y_2$, $Y_4 \underset{Y_3}{\downarrow} Y_2$. Let $\bar{x}_0$ and $\bar{x}_4$ be two tuples such that $g_4g_3g_2g_1\cdot tp(\bar{x}_0/Y_0) = tp(\bar{x}_4/Y_4)$. Then for $i = 1, 2, 3, 4$, there exist $a_i \in G_{(Y_{i-1}Y_i)}$ such that $g_4^{a_4} \dots g_1^{a_1}(\bar{x}_0) = \bar{x}_4$.*

*Proof.* Since $g_1$ and $g_4^{-1}$ move almost $L$-maximally, $g_1(Y_0) = Y_1$, $g_4^{-1}(Y_4) = Y_3$ and $Y_0 \underset{Y_1}{\downarrow} Y_2$, $Y_4 \underset{Y_3}{\downarrow} Y_2$, by Lemma 4.1.4, there exists $a_1 \in G_{(Y_0Y_1)}$ and $a_4 \in G_{(Y_3Y_4)}$ such that for $\bar{x}_1 = g_1^{a_1}(\bar{x}_0), \bar{x}_3 = g_4^{-a_4}(\bar{x}_4)$, we have

$$\bar{x}_1 \underset{Y_1}{\downarrow} Y_2, \bar{x}_3 \underset{Y_3}{\downarrow} Y_2.$$

By Existence, there exists $\bar{x}_2$ realising $g_2\cdot\mathrm{tp}(x_1/Y_1) = g_3^{-1}\cdot\mathrm{tp}(x_3/Y_3)$ such that

$$\bar{x}_2 \underset{Y_2}{\downarrow} \bar{x}_1Y_1\bar{x}_3Y_3.$$

By Monotonicity, we have $\bar{x}_2 \underset{Y_2}{\downarrow} \bar{x}_1Y_1$ and $\bar{x}_2 \underset{Y_2}{\downarrow} \bar{x}_3Y_3$.

Since we have $\bar{x}_1 \underset{Y_1}{\downarrow} Y_2$, $\bar{x}_2 \underset{Y_2}{\downarrow} \bar{x}_1Y_1$, $g_2^{-1}(Y_2) = Y_1$ and $g_2^{-1}$ moves almost $R$-maximally, we can find $a_2 \in G_{(Y_1Y_2)}$ such that $g_2^{-a_2}(\bar{x}_2) = \bar{x}_1$ by Lemma 4.1.6. Similarly, since we have $\bar{x}_3 \underset{Y_3}{\downarrow} Y_2$, $\bar{x}_2 \underset{Y_2}{\downarrow} \bar{x}_3Y_3$, $g_3(Y_2) = Y_3$ and $g_3$ moves almost $R$-maximally, we can find $a_3 \in G_{(Y_2Y_3)}$ such that $g_3a_2(\bar{x}_2) = \bar{x}_3$ by Lemma 4.1.6.

Therefore, we have found $a_i \in G_{(Y_{i-1}Y_i)}$ such that $g_4^{a_4} \dots g_1^{a_1}(\bar{x}_0) = \bar{x}_4$. □

For a partial isomorphism $u$ of $\mathcal{M}$, we denote the domain of $u$ to be $\mathrm{dom}(u)$ and the image of $u$ to be $\mathrm{Im}(u)$.

**Proposition 4.1.8.** *Let $\mathcal{M}$ be a countable structure with a* **SWIR** *and $g$ be an automorphism of $\mathcal{M}$ such that $g$ and $g^{-1}$ move both almost R-maximally and almost L-maximally. Define*

$$\phi : G^4 \to G : (h_1, ..., h_4) \mapsto g^{h_4} \dots g^{h_1}.$$

*Then for any open set $U \subseteq G^4$, there is some open set $W \subseteq G$ such that $\phi(U)$ is dense in $W$.*

*Proof.* Let $U \subseteq G^4$ be open. Then $U$ can be written as $U_1 \times ... \times U_4$, where $U_i = \mathcal{U}(u_i) = \{h \in G | u_i \subset h\}$ is the set of automorphisms that extend some finite partial isomorphism $u_i$. We can extend $u_i$ to some $a_i \in G$ such that $g^{a_i}(X_{i-1}) = X_i$, where $\mathrm{Im}(u_i) \subset X_i$ and $X_i$ is finite for $i = 1, 2, 3, 4$.

By Proposition 4.1.3, there exist $b_i \in G_{(X_{i-1}X_i)}$ and finite extensions $X_i \subset Y_i$ such that $g^{a_i b_i}(Y_{i-1}) = Y_i$ and $Y_0 \mathop{\smile\!\!\!\!\!\!|}_{Y_1} Y_2$ and $Y_4 \mathop{\smile\!\!\!\!\!\!|}_{Y_3} Y_2$.

Let $w$ be the partial isomorphism $g^{a_4 b_4} ... g^{a_1 b_1}$ restricted to $Y_0$ and $W = \mathcal{U}(w)$. We want to show that $\phi(U)$ is dense in $W$. Let $w'$ be an extension of $w$. Then $W' := \mathcal{U}(w')$ is a basic open subset of $W$. Let $\bar{x}$ be an enumeration of $\mathrm{dom}(w') \setminus Y_0$ and $\bar{y} = w'(\bar{x})$. By Proposition 4.1.7, we obtain $c_i \in G_{(Y_{i-1}Y_i)}$ such that $g^{a_4 b_4 c_4} ... g^{a_1 b_1 c_1}(\bar{x}) = \bar{y}$. Since $b_i$ and $c_i$ both fix $\mathrm{Im}(u_i)$ pointwise, we have $a_i b_i c_i \in U_i$. So $(a_1 b_1 c_1, ..., a_4 b_4 c_4) \in U$ and $\phi$ maps it to $g^{a_4 b_4 c_4} ... g^{a_1 b_1 c_1}$, which belongs to $W'$. Hence, we have shown that every basic open subset of $W$ intersects $\phi(U)$ non-trivially, i.e. $\phi(U)$ is dense in $W$. □

The remainder of the proof of Theorem 4.0.4 follows exactly from [TZ13a]. Note that it was proved in Lemma 2.8 in [TZ13a] without using Symmetry that for a countable structure $\mathcal{M}$ with a **SIR**, its automorphism group has a dense conjugacy class. Hence, the same statement can also be proved for a countable structure with a **SWIR**. We will use this result in the following proof. Moreover, it was proved in [KS19] that if $\mathcal{M}$ is a countable structure with a canonical independence relation defined in the paper, which is weaker than a **SWIR**, then $\mathrm{Aut}(\mathcal{M})$ has a cyclic dense conjugacy class. We say $G$ has *cyclic dense conjugacy class* if there exist $f, g \in G$ such that $\{f^n g f^n | n \in \mathbb{Z}\}$ is dense.

*Proof of Theorem 4.0.2.* The previous proposition implies that if $X \subseteq G$ is

nowhere dense, then its preimage $\phi^{-1}(X)$ is nowhere dense in $G^4$. Let $U$ be the interior of $\phi^{-1}(X)$ and suppose $\phi^{-1}(X)$ is not nowhere dense. Then $U$ is a non-empty open set. Then by the conclusion of the proposition, there exists a non-empty open set $W \subset G$ such that $\phi(U)$ is dense in $W$. Then we have $W \subseteq \overline{\phi(U)} \subseteq \overline{\phi(\phi^{-1}(X)} \subseteq \bar{X}$, which contradicts $X$ being nowhere dense. Therefore, for any nowhere dense set $X \subseteq G$, $\phi^{-1}(X)$ is nowhere dense.

Suppose now $\phi(G^4)$ is meagre. Then $\phi(G^4) = \bigcup_{i<\omega} X_i$ for some nowhere dense sets $X_i$. Then $G^4 = \bigcup_{i<\omega} \phi^{-1}(X_i)$, where $\phi^{-1}(X_i)$ is nowhere dense by the above argument, i.e. $G^4$ is meagre. However, by the Baire Category Theorem, $G^4$ is not meagre since $G^4$ is a non-empty complete metric space, a contradiction. Hence, we know that $\phi(G^4)$ is not meagre.

Since $\phi$ is continuous and $G^4$ is Polish, $\phi(G^4)$ is analytic and hence has the Baire property by Proposition 2.4.3. Since $G$ has a dense conjugacy class, any open sets $U, V \subseteq G$ intersect with the dense conjugacy class untrivially. Hence, if we let $G$ act on itself by conjugation, then there exists $g \in G$ such that $g(U) \cap V \neq \emptyset$. $\phi(G^4)$ is invariant under conjugation since $h^{-1}\phi(h_1, ..., h_4)h = \phi(h_1h, ..., h_4h) \in \phi(G^4)$. We also know that $G$ is Baire since it is Polish. Therefore, by Theorem 2.4.4, $\phi(G^4)$ is either meagre or comeagre. Thus, we derive that $\phi(G^4)$ is comeagre since it is not meagre.

Since $g^{-1}$ moves both almost $R$-maximally and almost $L$-maximally as well, the image of $\phi' : (h_1, \ldots, h_4) \mapsto (g^{-h_4} \ldots g^{-h_1})$ is also comeagre. So for any $f \in G$, the translation $\phi'(G^4)f$ intersects $\phi(G^4)$. Hence, every element of $G$ can be written as a product of eight conjugates of $g$. □

## 4.2 Constructing moving-maximally automorphisms

In the last section, in order for a non-trivial automorphism $g \in G$ to generate $G$ by conjugation, we require that both $g$ and $g^{-1}$ move both almost $R$-maximally and almost $L$-maximally. In this section, we show that we can weaken this condition on $g$ in two different ways.

**Lemma 4.2.1.** *Let $\mathcal{M}$ be a countable homogeneous structure with a* **SWIR** $\downarrow$ *and let $g \in G$ move both almost $R$-maximally and almost $L$-maximally.*

*Then there exists $k \in G$ such that $[g,k]$ and its inverse move both almost $R$-maximally and almost $L$-maximally.*

*Proof.* We use the back-and-forth construction. List all types of finite tuples over finite sets as $p_1, p_2, ....$ We start with the empty map. Suppose at the $i$-th stage, we have a partial isomorphism $\tilde{k} : A \to B$ such that $[\tilde{k}, g]$ and $[\tilde{k}, g]^{-1}$ move $p_1, ..., p_{i-1}$ almost $R$-maximally and almost $L$-maximally. Let $p_i$ be a type over $X$. We can extend $\tilde{k}$ so that its image and domain contain $X$. So we may assume $X \cup g(X) \subseteq A$, $X \cup g\tilde{k}(X) \subseteq B$. We may also assume $p_i$ is non-algebraic. We want to extend $\tilde{k}$ so that $[\tilde{k}, g]$ and $[\tilde{k}, g]^{-1}$ move $p_i$ almost $R$-maximally and almost $L$-maximally.

Step I. We extend $\tilde{k}$ so that the partial map $[\tilde{k}, g]$ is defined and moves $p_i$ almost $R$-maximally. Note that the arguments in this step only depend on $g$ moving almost $R$-maximally.

By Existence, there exists $\bar{a}'$ realising $p_i$ such that $\bar{a}' \mathop{\downarrow}_X A$. Since $g$ moves almost $R$-maximally, there exists $\bar{a}$ realising $\mathrm{tp}(\bar{a}'/A)$ such that $\bar{a} \mathop{\downarrow}_A g\bar{a}$. Since $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{a}'/A)$, we also have $\bar{a} \mathop{\downarrow}_X A$ by Invariance.

Again by Existence, there exists $\bar{b}$ realising $\tilde{k}\cdot\mathrm{tp}(\bar{a}/A)$ such that $\bar{b} \mathop{\downarrow}_B g^{-1}(B)$. Extend $\tilde{k}$ by sending $\bar{a}$ to $\bar{b}$. Acting by $\tilde{k}$ on $\bar{a} \mathop{\downarrow}_X A$, we obtain $\bar{b} \mathop{\downarrow}_{\tilde{k}(X)} B$ by Invariance. Then by Transitivity on $\bar{b} \mathop{\downarrow}_{\tilde{k}(X)} B$ and $\bar{b} \mathop{\downarrow}_B g^{-1}(B)$, we have $\bar{b} \mathop{\downarrow}_{\tilde{k}(X)} g^{-1}B$ since we have $\tilde{k}(X) \subseteq B$ from $X \subseteq A$.

By Existence, there exists $\bar{c}$ realising $\tilde{k}^{-1}\cdot\mathrm{tp}(g\bar{b}/\bar{b}B)$ such that $\bar{c} \mathop{\downarrow}_{\bar{a}A} g\bar{a}$. We also have $\bar{a} \mathop{\downarrow}_A g\bar{a}$, so by Transitivity, we have $\bar{c} \mathop{\downarrow}_A g\bar{a}$. Extend $\tilde{k}$ by sending $\bar{c}$ to $g\bar{b}$.

Acting by $\tilde{k}^{-1}g$ on $\bar{b} \mathop{\downarrow}_{\tilde{k}(X)} g^{-1}(B)$, we obtain $\tilde{k}^{-1}g\bar{b} \mathop{\downarrow}_{\tilde{k}^{-1}g\tilde{k}(X)} \tilde{k}^{-1}(B)$, which can be simplified to $\bar{c} \mathop{\downarrow}_{\tilde{k}^{-1}g\tilde{k}(X)} A$. Since $\tilde{k}^{-1}g\tilde{k}(X) \subseteq A$ and $c \mathop{\downarrow}_{\tilde{k}^{-1}g\tilde{k}(X)} A$, $\bar{c} \mathop{\downarrow}_A g\bar{a}$, we obtain $\bar{c} \mathop{\downarrow}_{\tilde{k}^{-1}g\tilde{k}(X)} g\bar{a}$ by Transitivity. Acting by $\tilde{k}^{-1}g^{-1}\tilde{k}$ on it, we obtain $\bar{a} \mathop{\downarrow}_X [\tilde{k}, g]\bar{a}$.

Step II. Under the assumption that $g$ moves $L$-maximally and by the same argument, but swapping the sides of $\mathop{\downarrow}$ as in Step I, we can extend $\tilde{k}$ so that $[\tilde{k}, g]$ moves $p_i$ almost $L$-maximally.

Step III. We want to extend $\tilde{k}$ so that $[g, \tilde{k}]$ moves $p_i$ almost $R$-maximally. Note that the arguments in this step only depend on $g$ moving almost $R$-maximally.

By Existence, there exists $\bar{a}$ realising $p_i$ such that $\bar{a} \mathop{\downarrow}_X g^{-1}(A)$. Since $g$ moves almost $R$-maximally, there exists $\bar{b}$ realising $\tilde{k}\cdot\mathrm{tp}(\bar{a}/A)$ such that $\bar{b} \mathop{\downarrow}_B g\bar{b}$. Again by Existence, there exists $\bar{c}$ realising $\tilde{k}\cdot\mathrm{tp}(g\bar{a}/\bar{a}A)$ such that $\bar{c} \mathop{\downarrow}_{\bar{b}B} g\bar{b}$. Extend $\tilde{k}$ by sending $\bar{a}$ to $\bar{b}$ and $g\bar{a}$ to $\bar{c}$. Acting by $\tilde{k}g$ on $\bar{a} \mathop{\downarrow}_X g^{-1}(A)$, we obtain $\bar{c} \mathop{\downarrow}_{\tilde{k}g(X)} B$ by Invariance.

By Transitivity on $\bar{b} \mathop{\downarrow}_B g\bar{b}$ and $\bar{c} \mathop{\downarrow}_{\bar{b}B} g\bar{b}$, we have $\bar{c} \mathop{\downarrow}_B g\bar{b}$. Then by Transitivity on $\bar{c} \mathop{\downarrow}_{\tilde{k}g(X)} B$, $\bar{c} \mathop{\downarrow}_B g\bar{b}$ and that $\tilde{k}g(X) \subseteq B$, we have $\bar{c} \mathop{\downarrow}_{\tilde{k}g(X)} g\bar{b}$. Acting by $g^{-1}\tilde{k}^{-1}$ on this, we obtain $\bar{a} \mathop{\downarrow}_X [g, \tilde{k}]\bar{a}$ by Invariance.

Step IV. Under the assumption that $g$ moves $L$-maximally and by the same argument as in Step III, but swapping the sides of $\mathop{\downarrow}$, we can extend $\tilde{k}$ so that $[g, \tilde{k}]$ moves $p$ almost $L$-maximally.

Let $k$ be the union of $\tilde{k}$ over all $i$. Then, $k$ is an automorphism of $\mathcal{M}$ since we ensured its domain and image contain all finite subsets of $\mathcal{M}$. Thus, we have constructed $k \in G$ such that $[\tilde{k}, g]$ and $[\tilde{k}, g]^{-1}$ move almost $L$-maximally and $R$-maximally. □

Therefore, by Theorem 4.0.4, we have the following corollary:

**Corollary 4.2.2.** *Suppose $\mathcal{M}$ is a countable homogeneous structure with a* **SWIR** *and $g \in G$ moves both almost $R$-maximally and almost $L$-maximally. Then any element of $G$ is a product of 16 conjugates of $g$ and $g^{-1}$.*

**Lemma 4.2.3.** *Let $\mathcal{M}$ be a countable homogeneous structure with a* **SWIR** $\mathop{\downarrow}$ *and $g \in G$ be such that $g$ moves almost $R$-maximally and $g^{-1}$ moves almost $L$-maximally. Then there exists $k \in G$ such that $[g, k]$ and its inverse move both almost $R$-maximally and almost $L$-maximally.*

*Proof.* We use the back-and-forth construction as before. List all types of finite tuples over finite sets as $p_1, p_2, ....$ We start with the empty map. Suppose at the $i$-th stage, we have a partial isomorphism $\tilde{k} : A \to B$ such that $[\tilde{k}, g]$ and $[\tilde{k}, g]^{-1}$ move $p_1, ..., p_{i-1}$ almost $R$-maximally and almost $L$-maximally. Let $p_i$ be a type over $X$. We can extend $\tilde{k}$ so that its image and domain contain $X$. So we may assume $X \cup g(X) \subseteq A$, $X \cup g\tilde{k}(X) \subseteq B$. We may also assume $p_i$ is non-algebraic. We want to extend $\tilde{k}$ so that the partial maps $[\tilde{k}, g]$ and $[\tilde{k}, g]^{-1}$ are defined and move $p_i$ almost $R$-maximally and almost $L$-maximally.

By Steps I and III in the proof of the previous lemma, we know that if $g$ moves almost $R$-maximally, we can extend $\tilde{k}$ so that $[\tilde{k}, g]$ and $[\tilde{k}, g]^{-1}$ move $p_i$ almost $R$-maximally. Therefore, we only have to extend $\tilde{k}$ so that $[\tilde{k}, g]$ and $[\tilde{k}, g]^{-1}$ move $p$ almost $L$-maximally.

Now we extend $\tilde{k}$ so that $[\tilde{k}, g]$ moves $p_i$ almost $L$-maximally.

By Existence, there exists $\bar{a}$ realising $p_i$ such that $A \downarrow_X \bar{a}$ and $\bar{b}'$ realising $\tilde{k}\cdot\mathrm{tp}(\bar{a}/A)$ such that $g^{-1}(B) \downarrow_B \bar{b}'$. Since $g^{-1}$ moves almost $L$-maximally, there exists $\bar{b}$ realising $\mathrm{tp}(\bar{b}'/Bg^{-1}(B))$ such that $g^{-1}\bar{b} \downarrow_{Bg^{-1}(B)} \bar{b}$. By Invariance, we have $g^{-1}(B) \downarrow_B \bar{b}$ since $\mathrm{tp}(\bar{b}/Bg^{-1}(B)) = \mathrm{tp}(\bar{b}'/Bg^{-1}(B))$. Then by Transitivity on $g^{-1}(B) \downarrow_B \bar{b}$ and $g^{-1}\bar{b} \downarrow_{Bg^{-1}(B)} \bar{b}$, we have $g^{-1}(\bar{b}B) \downarrow_B \bar{b}$. Extend $\tilde{k}$ by sending $\bar{a}$ to $\bar{b}$. By Existence, there exists $\bar{c}$ realising $\tilde{k}^{-1}\cdot\mathrm{tp}(g\bar{b}/\bar{b}B)$ such that $g\bar{a} \downarrow_{\bar{a}A} \bar{c}$. Extend $\tilde{k}$ by sending $\bar{c}$ to $g\bar{b}$.

Acting by $\tilde{k}$ on $A \downarrow_X \bar{a}$, we obtain $B \downarrow_{\tilde{k}(X)} \bar{b}$. Since $\tilde{k}(X) \subseteq B$, by Transitivity on $B \downarrow_{\tilde{k}(X)} \bar{b}$ and $g^{-1}(\bar{b}B) \downarrow_B \bar{b}$, we have $g^{-1}(\bar{b}B) \downarrow_{\tilde{k}(X)} \bar{b}$. Acting by $\tilde{k}^{-1}g$ on it, we obtain $\bar{a}A \downarrow_{\tilde{k}^{-1}g\tilde{k}(X)} \bar{c}$. Then by Transitivity on it and $ga \downarrow_{\bar{a}A} \bar{c}$, we have $g\bar{a} \downarrow_{\tilde{k}^{-1}g\tilde{k}(X)} \bar{c}$. Acting by $\tilde{k}^{-1}g^{-1}\tilde{k}$ on it, we get $[\tilde{k}, g]\bar{a} \downarrow_X \bar{a}$.

Now we extend $\tilde{k}$ so that $[g, \tilde{k}]$ moves $p_i$ almost $L$-maximally.

By Existence, there exists $\bar{a}'$ realising $p_i$ such that $g^{-1}(A) \downarrow_X \bar{a}'$. Since $g^{-1}$ moves $L$-maximally, there exists $\bar{a}$ realising $\mathrm{tp}(\bar{a}'/g^{-1}(A)X)$ such that $g^{-1}\bar{a} \downarrow_{g^{-1}(A)X} \bar{a}$. Then by Invariance, we have $g^{-1}(A) \downarrow_X \bar{a}$. By Transitivity on it and $g^{-1}\bar{a} \downarrow_{g^{-1}(A)X} \bar{a}$, we have $g^{-1}\bar{a} \downarrow_X \bar{a}$. Then acting by $g$ on it, we obtain $\bar{a} \downarrow_X g\bar{a}$. By Existence, we can find $\bar{b}$ realising $\tilde{k}\cdot\mathrm{tp}(\bar{a}/A)$ and $\bar{c}$ realising $\tilde{k}^{-1}\mathrm{tp}(g\bar{b}/\bar{b}B)$ such that $\bar{c} \downarrow_{\bar{a}A} g\bar{a}$. Since $g(X) \subseteq A$, by Transitivity on it and $\bar{a} \downarrow_X g\bar{a}$, we have $\bar{c} \downarrow_{g(X)} g\bar{a}$. Therefore, acting by $g^{-1}$ on it, we obtain $[g, \tilde{k}]\bar{a} \downarrow_X \bar{a}$.

Let $k$ be the union of $\tilde{k}$ over all $i$. Then, $k$ is an automorphism of $\mathcal{M}$ since we ensured its domain and image contain all finite subset of $\mathcal{M}$. Thus, we have constructed $k \in G$ such that $[\tilde{k}, g]$ and $[\tilde{k}, g]^{-1}$ move almost $L$-maximally and $R$-maximally. □

Therefore, we have the following corollary:

**Corollary 4.2.4.** *Suppose $\mathcal{M}$ is a countable homogeneous structure with a* **SWIR** *and $g \in G$ is such that $g$ moves almost $R$-maximally and $g^{-1}$ moves*

*almost L-maximally. Then any element of $G$ is a product of 16 conjugates of $g$ and $g^{-1}$.*

# Chapter 5

# Applications of Theorem 4.0.4

In this chapter, we will apply results proved in the last chapter to the examples studied in Chapter 3. We do this by a similar approach to [TZ13a]. In other words, we want to show that for any non-trivial automorphism $g$ of a homogeneous structure $\mathcal{M}$ admitting a **SWIR**, we can construct $h \in \mathrm{Aut}(\mathcal{M})$, by a back-and-forth construction, as a product of conjugates of $g$ and $g^{-1}$ such that $h$ and $h^{-1}$ moves both almost $R$-maximally and almost $L$-maximally.

We will apply Corollary 4.2.2 to the prioritised semi-free structures in Section 5.1. We showed in Section 3.2 that for a set of forbidden triangles $S$ satisfying either Condition 1 or Condition 2, the corresponding $\mathcal{M}_S$ is a prioritised semi-free structure. We will introduce a stronger condition (see Condition 3) and show that we can prove the simplicity of $\mathrm{Aut}(\mathcal{M}_S)$ using Corollary 4.2.2 and a back-and-forth construction if $S$ satisfies this condition. This covers all prioritised semi-free structures listed in Figure 2.1 except # 26. For # 26, we change the argument slightly to achieve the same result. Thus, we prove Theorem 5.1.3.

In Section 5.2, we will use the same method and apply Corollary 4.2.4 to some linearly ordered structures including the linear order expansions of non-trivial free homogeneous structures and the universal $n$-linear order for $n \geq 2$ and prove the simplicity of their automorphism groups.

Note that since we only consider homogeneous structure $\mathcal{M}$ with trivial algebraic closure, any 1-type $\mathrm{tp}(x/X)$ over some finite set $X \subset \mathcal{M}$ has either infinitely many realisations or exactly one realisation $a \in X$ where the type

includes the formula $x = a$. Hence, in the second case, we may simplify such notations and say that $x \in X$. Similarly, for an $n$-type $\mathrm{tp}(\bar{x}/X)$, we can say that $\bar{x} \cap X = \emptyset$ if for any $x_i \in \bar{x}$, $\mathrm{tp}(x_i/X)$ is non-algebraic. In fact it was proved in Lemma 5.1 in [TZ13a] that for a homogeneous structure admitting a **SIR**, any 1-type over some finite set has either exactly one realisation or infinitely many and the same proof applies for a homogeneous structure admitting a **SWIR**.

In the following lemma, we show that to prove that $g$ moves almost $R$-maximally (or $L$-maximally), it is enough to prove that $g$ moves all types of the form

$$\mathrm{tp}(\bar{x}/X) \text{ , where } X \text{ is finite and } \bar{x} \cap X = \emptyset, \tag{5.1}$$

almost $R$-maximally (or $L$-maximally). Thus, we can work only with types of this form later on.

**Lemma 5.0.1.** *Let $\mathcal{M}$ be a countable homogeneous structure with a* **SWIR** $\downarrow$ *and $g \in Aut(\mathcal{M})$. Suppose $g$ moves all types of the form (5.1) almost $R$-maximally. Then $g$ moves almost $R$-maximally. Similarly, if $g$ moves all types of the form (5.1) almost $L$-maximally, then $g$ moves almost $L$-maximally.*

*Proof.* Let $p$ be an algebraic 1-type over some finite set $X$. Then by the previous lemma, if $a$ is the realisation of $p$, then $a \in X$ and thus, we have $a \downarrow_X ga$ and $ga \downarrow_X a$.

Now assume $g$ moves all types of the form $\mathrm{tp}(\bar{x}/X)$, where $\bar{x} \cap X = \emptyset$, almost $R$-maximally. Let $p$ be an $n$-type. We may assume $p = \mathrm{tp}(\bar{x}\bar{y}/X)$, where $\bar{x} \in X$ and $\bar{y} \cap X = \emptyset$. We can further write $\bar{x} = \bar{x}'\bar{x}''$ where $g\bar{x}' \cap X = \emptyset$ and $g\bar{x}'' \in X$.

By Existence of $\downarrow$, we can find $\bar{a}'$ realising $\mathrm{tp}(\bar{y}/X)$ such that $\bar{a}' \downarrow_X g\bar{x}'$. We also have $\bar{a}' \cap Xg\bar{x}' = \emptyset$. Since $g$ moves all types of the form $\mathrm{tp}(\bar{x}/X)$, where $X$ is finite and $\bar{x} \cap X = \emptyset$, almost $R$-maximally, we can find $\bar{a}$ realising $\mathrm{tp}(\bar{a}'/Xg\bar{x})$ such that $\bar{a} \downarrow_{Xg\bar{x}'} g\bar{a}$. By Invariance, we have $\bar{a} \downarrow_X g\bar{x}'$. Hence by Transitivity, we have $\bar{a} \downarrow_X g(\bar{a}\bar{x}')$. Since $\bar{x}, g\bar{x}'' \in X$, we have $\bar{a}\bar{x} \downarrow_X g(\bar{a}\bar{x}'\bar{x}'')$ by Remark 3.1.2. Therefore, $\bar{x}\bar{a}$ realises $p$ and is moved almost $R$-maximally by $g$.

Hence, $g$ moves almost $R$-maximally. Similarly, if $g$ moves all types of the form $\mathrm{tp}(\bar{x}/X)$, where $X$ is finite and $\bar{x} \cap X = \emptyset$, almost $L$-maximally, by the same argument but swapping the sides of $\downarrow$, $g$ moves almost $L$-maximally. □

## 5.1 Prioritised semi-free homogeneous structures

We now look at the prioritised semi-free homogeneous structures defined in Section 3.2. In this section, we let $\mathcal{L}$ be a binary relational language. Let $S$ be a set of forbidden triangles and $Forb_c(S)$ be the set of all finite $\mathcal{L}$-structures that do not embed any triangle from $S$ as defined in Definition 2.3.1. We will only consider $S$ such that $Forb_c(S)$ forms a prioritised semi-free amalgamation class. So we let $\mathcal{L}' = \{R_1, R_2, ..., R_m\}$ be the set of solutions for $Forb_c(S)$ with order $R_1 > \cdots > R_m$. Let $\mathcal{M}_S$ be the Fraïssé limit of $Forb_c(S)$ and $G = \mathrm{Aut}(\mathcal{M}_S)$.

We will show in this section that if $S$ satisfies the following condition, then for any non-trivial automorphism $g$ of $\mathcal{M}_S$, we can construct, by a back-and-forth construction, a product of conjugates of $g$ and $g^{-1}$ that move both almost $R$-maximally and almost $L$-maximally. Then by Corollary 4.2.2, we can show the simplicity of $\mathrm{Aut}(\mathcal{M}_S)$ as stated in the next theorem.

**Condition 3.** Let $S$ be a set of forbidden triangles such that $Forb_c(S)$ forms a prioritised semi-free amalgamation class. Let $\mathcal{L}' = \{R_1, R_2, ..., R_m\}$ be the set of solutions for $Forb_c(S)$ with order $R_1 > \cdots > R_m$. Let $\mathcal{M}_S$ be the Fraïssé limit of $Forb_c(S)$. Assume that

(i) $S$ does not contain any triangle involving $R_1R_1$ or $R_1R_2$.

(ii) For $a, b, c \in \mathcal{M}_S$ and $B \subseteq \mathcal{M}_S$ such that $a, b, c$ are distinct and $a \downarrow_{bB} c$, if $\mathbf{r}(a, b) \in \mathcal{L}'$ or $\mathbf{r}(b, c) \in \mathcal{L}'$, then we have $a \downarrow_B c$.

**Remark 5.1.1.** It follows from part (ii) of the above condition that for $\bar{a}, \bar{b}, \bar{c} \in \mathcal{M}_S$ and $B \subseteq \mathcal{M}_S$ such that $\bar{a}, \bar{b}, \bar{c}, B$ are pairwise disjoint and $\bar{a} \downarrow_{\bar{b}B} \bar{c}$, if $\mathbf{r}(a_i, b_j) \in \mathcal{L}'$ for all $a_i \in \bar{a}, b_j \in \bar{b}$, then $\bar{a} \downarrow_B \bar{c}$. Similarly if we have $\mathbf{r}(b_i, c_j) \in \mathcal{L}'$ for all $b_i \in \bar{b}, c_j \in \bar{c}$, then $\bar{a} \downarrow_B \bar{c}$.

We will first prove some preparatory results about $\mathcal{M}_S$ in Subsection 5.1.1. We will then prove the following theorem in Subsection 5.1.2. Note that part(ii)

follows from part(i) by Corollary 4.2.2.

**Theorem 5.1.2.** *Let $S$ be a set of forbidden triangles such that $S$ satisfies Condition 3 and $Forb_c(S)$ forms a prioritised semi-free amalgamation class. Let $\mathcal{M}_S$ be the Fraïssé limit of $Forb_c(S)$. Then,*

*(i) For any non-trivial automorphism $g \in Aut(\mathcal{M}_S)$, there exist $h_1, h_2 \in Aut(\mathcal{M}_S)$ such that $[[h_1, g], h_2]$ moves both almost R-maximally and almost L-maximally.*

*(ii) For any non-trivial automorphism $g \in Aut(\mathcal{M}_S)$, ány automorphism of $Aut(\mathcal{M}_S)$ can be written as a product of at most 64 conjugates of $g$ and $g^{-1}$, i.e. $Aut(\mathcal{M}_S)$ is simple.*

In Subsection 5.1.3, we will see that all prioritised semi-free structures listed in Figure 2.1 except #26 satisfy Condition 3 and thus, the above theorem holds for them. For #26, we will modify the argument from Subsection 5.1.2 and obtain the same result. Altogether, we prove the following theorem.

**Theorem 5.1.3.** *Let $\mathcal{M}$ be one of the following structures:*

*(i) all 3-multi-graphs and 4-multi-graphs listed in the appendix of [Che98]*

*(ii) #8 to #10 of the 2-multi-tournaments listed in the appendix of [Che98].*

*Then, $\mathcal{M}$ admits a **SWIR** and $Aut(\mathcal{M})$ is simple.*

### 5.1.1 Preparatory results

In [TZ13a], Theorem 4.0.2 was proved using the result from [MT11] that a non-trivial automorphism of a free homogeneous structure does not fix the set of realisations of any non-algebraic type pointwise. In order to prove Theorem 5.1.2, we need to prove the same statement for $\mathcal{M}_S$, where $S$ satisfies Condition 3. In other words, we prove the following theorem:

**Theorem 5.1.4.** *Let $S$ be a set of forbidden triangles satisfying Condition 3 and let $\mathcal{M}_S$ be the Fraïssé limit of $Forb_c(S)$. Let $g \in Aut(\mathcal{M}_S)$. For any non-algebraic type $p$ over some finite set $X \subseteq \mathcal{M}_S$, $g$ moves infinitely many realisations of $p$.*

For readers familiar with model theory, the idea in this section comes from *imaginary elements*. Since the proof here does not directly involve them, I

refer interested readers to Section 16.4 and 16.5 in [Poi00]. We will show in Theorem 5.1.9 that our structure $\mathcal{M}_S$ has the *intersection property*, defined in the following. Note that for our structure $\mathcal{M}_S$, the intersection property is equivalent to having *weak elimination of imaginaries* by Theorem 16.17 in [Poi00], so this result may be of independent interest. We will use this property of $\mathcal{M}_S$ to prove the above theorem. For $S \subseteq G$, by $\langle S \rangle$, we mean the subgroup of $G$ generated by elements from $S$.

**Definition 5.1.5.** Let $\mathcal{M}$ be a homogeneous structure and $G = \mathrm{Aut}(\mathcal{M})$. We say that $\mathcal{M}$ has the *intersection property*, if for all finite subsets $A, B \subseteq \mathcal{M}$,

$$\langle G_{(A)}, G_{(B)} \rangle = G_{(A \cap B)}. \tag{5.2}$$

**Remark 5.1.6.** Note that if $g \in \langle G_{(A)}, G_{(B)} \rangle$, then $g$ fixes $A \cap B$ pointwise. So, we always have $\langle G_{(A)}, G_{(B)} \rangle \leq G_{(A \cap B)}$ and we only need to show the reverse inclusion to prove the intersection property.

We show in the following proposition that to prove the intersection property, it is enough to prove (5.2) for $A, B \subseteq \mathcal{M}$ such that $|A \setminus B| = |B \setminus A| = 1$.

**Proposition 5.1.7.** *Let $\mathcal{M}$ be a homogeneous relational structure and $G = Aut(\mathcal{M})$. Suppose (5.2) holds for all finite $A, B \subseteq \mathcal{M}$ such that $|A \setminus B| = |B \setminus A| = 1$. Then (5.2) holds for all finite subsets $A, B \subseteq \mathcal{M}$.*

*Proof.* We prove this by induction on $|A \setminus B| + |B \setminus A|$. The induction base is given by the assumption.

Suppose (5.2) holds for all $A, B$ such that $|A \setminus B| + |B \setminus A| < n$. Let $A, B$ be finite subsets of $\mathcal{M}$ such that $|A \setminus B| + |B \setminus A| = n$. We may assume $|A \setminus B| \geq 2$. Then we can pick $a \in A \setminus B$ and let $A' = (A \cap B) \cup \{a\}$ and $B' = B \cup \{a\}$. Then, $G_{(B')} \leq G_{(B)}$ and hence,

$$\langle G_{(A)}, G_{(B)} \rangle \geq \langle G_{(A)}, G_{(B')}, G_{(B)} \rangle.$$

We also have that $|A \setminus B'| = |A \setminus B| - 1$, $|B' \setminus A| = |B \setminus A|$, $|A' \setminus B| = 1$, $|B \setminus A'| = |B \setminus A|$, Then by the inductive hypothesis, we have

$$\langle G_{(A)}, G_{(B')} \rangle = G_{(A')} \text{ and } \langle G_{(A')}, G_{(B)} \rangle = G_{(A \cap B)}$$

since $A \cap B' = A', A' \cap B = A \cap B$ and $|A \setminus B'| + |B' \setminus A|, |A' \setminus B| + |B \setminus A'| < n$. Hence,

$$\langle G_{(A)}, G_{(B)} \rangle \geq \langle G_{(A)}, G_{(B')}, G_{(B)} \rangle = \langle G_{(A')}, G_{(B)} \rangle = G_{(A \cap B)}.$$

Since $\langle G_{(A)}, G_{(B)} \rangle \leq G_{(A \cap B)}$, we have $\langle G_{(A)}, G_{(B)} \rangle = G_{(A \cap B)}$. □

**Lemma 5.1.8.** *Let $\mathcal{M}$ be a homogeneous structure with a* **SWIR** $\bigcirc\!\!\!\!\!\!\downarrow$ *and $G = Aut(\mathcal{M})$. If $A \bigcirc\!\!\!\!\!\!\downarrow_{A \cap B} B$, then $\langle G_{(A)}, G_{(B)} \rangle = G_{(A \cap B)}$.*

*Proof.* Let $g \in G_{(A \cap B)}$. By Existence of $\bigcirc\!\!\!\!\!\!\downarrow$, there exists $A'$ realising $\mathrm{tp}(A/B)$ such that $A' \bigcirc\!\!\!\!\!\!\downarrow_B B \cup gB$. Since $\mathcal{M}$ is homogeneous and $\mathrm{tp}(A/B) = \mathrm{tp}(A'/B)$, there is $k \in G_{(B)}$ such that $kA = A'$ and thus, $G_{(A')} = kG_{(A)}k^{-1}$. So we have

$$\langle G_{(A)}, G_{(B)} \rangle \geq \langle G_{(A')}, G_{(B)} \rangle.$$

By Invariance from $A \bigcirc\!\!\!\!\!\!\downarrow_{A \cap B} B$, we have $kA \bigcirc\!\!\!\!\!\!\downarrow_{k(A \cap B)} kB$, which is the same as $A' \bigcirc\!\!\!\!\!\!\downarrow_{A \cap B} B$ since $k \in G_{(B)}$. By Transitivity on $A' \bigcirc\!\!\!\!\!\!\downarrow_{A \cap B} B$ and $A' \bigcirc\!\!\!\!\!\!\downarrow_B B \cup gB$, we have

$$A' \underset{A \cap B}{\bigcirc\!\!\!\!\!\!\downarrow} B \cup gB.$$

By Monotonicity, we have $A' \bigcirc\!\!\!\!\!\!\downarrow_{A \cap B} B$ and $A' \bigcirc\!\!\!\!\!\!\downarrow_{A \cap B} gB$. Since $g \in G_{(A \cap B)}$, we also have $\mathrm{tp}(B/A \cap B) = \mathrm{tp}(gB/A \cap B)$. So, by Stationarity, $\mathrm{tp}(B/A') = \mathrm{tp}(gB/A')$, i.e. there is $h \in G_{(A')}$ such that $hB = gB$. Then $g \in hG_{(B)}$ and thus, $g \in \langle G_{(A')}, G_{(B)} \rangle \leq \langle G_{(A)}, G_{(B)} \rangle$. Therefore, $G_{(A \cap B)} \leq \langle G_{(A)}, G_{(B)} \rangle$.

Since $\langle G_{(A)}, G_{(B)} \rangle \leq G_{(A \cap B)}$, we have $\langle G_{(A)}, G_{(B)} \rangle = G_{(A \cap B)}$. □

We now apply the above result to $\mathcal{M}_S$ where $S$ satisfies Condition 3:

**Theorem 5.1.9.** *Let $S$ be a set of forbidden triangles satisfying Condition 3 and $Forb_c(S)$ be the class of all finite $\mathcal{L}$-structures that do not embed any triangle from $S$. Let $\mathcal{M}_S$ be the Fraïssé limit of $Forb_c(S)$ and $G = Aut(\mathcal{M}_S)$. Then for any $a \neq c \in \mathcal{M}_S$ and finite subset $B \subseteq \mathcal{M}_S$, we have*

$$\langle G_{(aB)}, G_{(cB)} \rangle = G_{(B)}. \tag{5.3}$$

*Proof.* If $c \in B$, then we have $\langle G_{(aB)}, G_{(cB)} \rangle \geq G_{(B)}$ and hence $\langle G_{(aB)}, G_{(cB)} \rangle = G_{(B)}$.

So we may assume $c \notin aB$. Let $\downarrow$ denote the **SWIR** as defined in Section 3.2, i.e. for any $A, B, C \subseteq \mathcal{M}_S$, $A \downarrow_B C$ if $ABC = AB \otimes_B BC$ where $AB \otimes_B BC$ is the prioritised semi-free amalgamation defined in Definition 3.2.1. Then by Existence, we can find $c'$ realising $\mathrm{tp}(c/aB)$ such that $c' \downarrow_{aB} c$. Then $c' \notin acB$ and there exists $k \in G_{(aB)}$ such that $kc = c'$ and hence, $G_{(c'B)} = kG_{(cB)}k^{-1}$. So we have

$$\langle G_{(aB)}, G_{(cB)} \rangle \geq \langle G_{(c'B)}, G_{(cB)} \rangle.$$

Again by Existence, we can find $c''$ realising $\mathrm{tp}(c/c'B)$ such that $c'' \downarrow_{c'B} c$. Then $c'' \notin c'B$ and there exists $h \in G_{(c'B)}$ such that $hc = c''$ and hence, $G_{(c''B)} = hG_{(cB)}h^{-1}$. So we have

$$\langle G_{(c'B)}, G_{(cB)} \rangle \geq \langle G_{(c''B)}, G_{(cB)} \rangle.$$

Since $c' \downarrow_{aB} c$ and $c, c' \notin aB$, we have $\mathbf{r}(c', c) \in \mathcal{L}'$. Since $\mathrm{tp}(c/c'B)$ =$\mathrm{tp}(c''/c'B)$, we have $\mathbf{r}(c, c') = \mathbf{r}(c'', c') \in \mathcal{L}'$. Then by part (ii) of Condition 3, we can obtain $c'' \downarrow_B c$ from $c'' \downarrow_{c'B} c$.

Then by Lemma 5.1.8, we have $\langle G_{(c''B)}, G_{(cB)} \rangle = G_{(B)}$ since $c''B \cap cB = B$. Thus,

$$\langle G_{(aB)}, G_{(cB)} \rangle \geq \langle G_{(c'B)}, G_{(cB)} \rangle \geq \langle G_{(c''B)}, G_{(cB)} \rangle = G_{(B)}.$$

Since $\langle G_{(aB)}, G_{(cB)} \rangle \leq G_{(B)}$, we have $\langle G_{(aB)}, G_{(cB)} \rangle = G_{(B)}$.

□

Let $p$ be some type over some finite set $X$ and $P$ be its set of realisations. Then we have $G_{(X)} \leq G_{\{P\}}$ and $G_{(X)}$ is transitive on $P$. However, $X$ is not unique as $P$ can also be the set of realisations of some type over some finite set $X'$, i.e. $G_{(X')}$ is also transitive on $P$. If we are working with a countable homogeneous structure satisfying the intersection property, then we have

$$G_{(X \cap X')} = \langle G_{(X)}, G_{(X')} \rangle \leq G_{\{P\}}.$$

So, $G_{(X \cap X')}$ is also transitive on $P$ and hence $P$ is a the set of realisations of some type over $X \cap X'$. Therefore, it makes sense to define a *minimal defining set* for a type $p$ to be the smallest set $X$ such that $G_{(X)} \leq G_{\{P\}}$, where $P$ is the set of realisation of $p$.

**Corollary 5.1.10.** *Let $p$ be a type over some finite set $X \subset \mathcal{M}_S$ and $g \in Aut(\mathcal{M}_S)$. If $g$ fixes its set of realisations $P$ pointwise, then $g$ fixes $X$ setwise.*

*Proof.* Without loss of generality, we may assume $X$ is a minimal defining set for $p$. Suppose $g$ fixes $P$ pointwise. Then $G_{(gX)}$ is transitive on $P$. This is because for any $a, b \in P$, there exists $h \in G_{(X)}$ such that $ha = b$ since $G_{(X)}$ is transitive on $P$. Then $ghg^{-1}a = b$ since $g$ fixes $P$ pointwise. We also have that $ghg^{-1} \in G_{(gX)}$. Thus, $G_{(gX)}$ is transitive on $P$.

By the previous theorem, we have $G_{(X \cap gX)} = \langle G_{(X)}, G_{(gX)} \rangle$. So, $G_{(X \cap gX)}$ is transitive on $P$ and thus, $P$ is also the set of realisations of some type over $X \cap gX$. By the minimality of $X$, we have $gX = X$, i.e. $g$ fixes $X$ setwise. □

**Proposition 5.1.11.** *Let $S$ be a set of forbidden triangles satisfying Condition 3 and let $\mathcal{M}_S$ be the Fraïssé limit of $Forb_c(S)$. Let $G = Aut(\mathcal{M}_S)$ and $g \in G$. For any non-algebraic type $p$ over some finite set $X$, if $g$ fixes its set of realisations pointwise, then $g$ is trivial.*

*Proof.* Let $c \in \mathcal{M}_S$ be such that $c \bigcup X$. Then $\mathbf{r}(c, b) = R_1$ for every $b \in X$ by definition of $\bigcup$. We first show that $g$ fixes the set of realisations of $\mathrm{tp}(c/X)$ pointwise. Since $g$ fixes $X$ setwise by the previous corollary, for every $b \in X$, we have $\mathbf{r}(gc, b) = \mathbf{r}(c, b') = R_1$ for some $b' \in X$ and thus, $\mathrm{tp}(gc/X) = g{\cdot}\mathrm{tp}(c/X)$. Suppose $gc \neq c$. Then we can find $a$ realising $p$ such that $\mathbf{r}(c, a) = R_1, \mathbf{r}(gc, a) = R_2$. This is possible by part (i) of Condition 3. Since $ga = a$ as $a \in P$, we have $\mathbf{r}(a, c) = \mathbf{r}(ga, gc) = \mathbf{r}(a, gc)$, contradicting $\mathbf{r}(c, a) = R_1, \mathbf{r}(gc, a) = R_2$. Hence $gc_1 = c_1$, i.e. $g$ fixes the set of realisations of $\mathrm{tp}(c/X)$ pointwise.

Then as shown in Figure 5.1, for $x, y \notin X$ distinct, there exists $c'$ realising $\mathrm{tp}(c/X)$ such that $\mathbf{r}(x, c') = R_1, \mathbf{r}(y, c') = R_2$ since $\mathbf{r}(b, c') = R_1$ for every $b \in X$ and there does not exist a forbidden triangle containing $R_1R_1$ or $R_1R_2$. As $gc' = c'$, we have $gx \neq y$. Since $g$ fixes $X$ setwise, $gx \notin X$. Therefore, $gx = x$ for every $x \notin X$. Then for every $b, b' \in X$ distinct, there exists $d \notin X$ such that $\mathbf{r}(d, b) = R_1, \mathbf{r}(d, b') = R_2$. Hence $gb = b$ for every $b \in X$. Therefore, $ga = a$ for all $a \in \mathcal{M}_S$, i.e. $g$ is trivial.

□

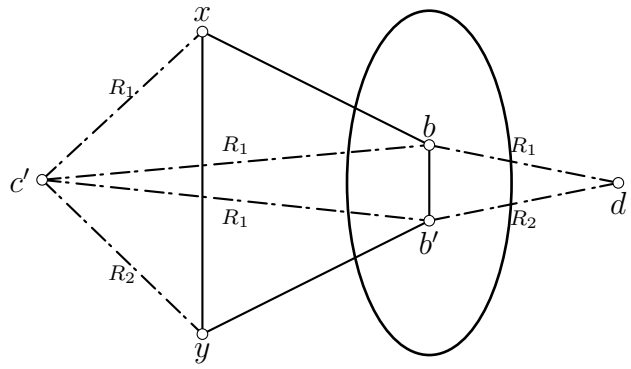



Figure 5.1:

We have shown that for any non-trivial $g \in \mathrm{Aut}(\mathcal{M}_S)$ and any non-algebraic type $p$ over some finite set $B$, $g$ moves some realisation $a$ of $p$. We can then define a new type $q$ over $aB$ which is defined the same over $B$ as $p$. Since we can repeat this process countably many times, $g$ moves countably many realisations of $p$. Hence, we have proved Theorem 5.1.4.

### 5.1.2 General cases

In this subsection, we will apply Corollary 4.2.2 to $\mathcal{M}_S$, where $S$ satisfies Condition 3 and show the simplicity of $\mathrm{Aut}(\mathcal{M}_S)$. In other words, we prove Theorem 5.1.2. We achieve this by a back-and-forth construction. Note that we will only consider $\mathcal{M}_S$ that has trivial algebraic closure.

**Lemma 5.1.12.** *Let $S$ be a set of forbidden triangles such that $Forb_c(S)$ is a semi-free amalgamation class. Let $\mathcal{M}_S$ be its Fraïssé limit. Then given any non-trivial $g \in Aut(\mathcal{M}_S)$, we can construct $h \in Aut(\mathcal{M}_S)$ such that for any non-algebraic type $p$ of the form (5.1), there exist infinitely many realisations $\bar{a}$ of $p$ such that $\mathbf{r}(a_i, [h, g]a_j) \in \mathcal{L}'$ for $a_i, a_j \in \bar{a}$.*

*Proof.* List all non-algebraic types of the form (5.1) as $p_1, p_2, ...$ with each type repeated infinitely often. We start with the empty map and use the back-and-forth method to build $h$. Suppose at the some stage, we have a partial isomorphism $\tilde{h} : A \to B$ such that for any $p \in \{p_1, ..., p_{n-1}\}$, there exists $\bar{a}$ realising $p$ such that $\mathbf{r}(a_i, [\tilde{h}, g]a_j) \in \mathcal{L}'$ for all $a_i, a_j \in \bar{a}$. We want to extend $\tilde{h}$ so that $p_n$ has a realisation $\bar{a}$ such that $\mathbf{r}(a_i, [\tilde{h}, g]a_j) \in \mathcal{L}'$ for $a_i, a_j \in \bar{a}$. Let $p_n$ be a type over some finite set $X \subseteq \mathcal{M}_S$. We may assume

$X \subseteq A$ by extending $\tilde{h}$.

Since $p_n$ is non-algebraic, by Theorem 5.1.4, $g$ does not fix the set of realisations of $p_n$ pointwise. We also have that $\mathcal{M}_S$ has trivial algebraic closure and $p_n$ is of the form (5.1). Hence $p_n$ has a realisation $\bar{a}$ that does not intersect with $A \cup g^{-1}(\bar{a}A)$ and $\tilde{h}\cdot\mathrm{tp}(\bar{a}/A)$ has a realisation $\bar{b}$ that does not intersect with $B \cup g^{-1}(\bar{b}B)$). Extend $\tilde{h}$ by sending $\bar{a}$ to $\bar{b}$.

By the Extension Property, there exists $\bar{c}$ realising $\tilde{h}^{-1}\cdot\mathrm{tp}(g\bar{b}/\bar{b}B)$ such that $\bar{c}$ and $g\bar{a}$ are semi-freely amalgamated over $\bar{a}A$. Since $\bar{b} \cap g^{-1}(\bar{b}B) = \emptyset$, which implies $g\bar{b} \cap \bar{b}B = \emptyset$, we have $\bar{c} \cap \bar{a}A = \emptyset$. We also have that $g\bar{a} \cap \bar{a}A = \emptyset$. Thus, $(c_i, ga_j)$ is coloured using relations from $\mathcal{L}'$, i.e. $\mathbf{r}(c_i, ga_j) \in \mathcal{L}'$, for all $c_i \in \bar{c}, a_j \in \bar{a}$.

Extend $\tilde{h}$ by sending $\bar{c}$ to $g\bar{b}$. Then $\bar{c} = \tilde{h}^{-1}g\tilde{h}\bar{a}$ and we have, for any $a_i, a_j \in \bar{a}$,

$$\mathbf{r}(a_i, [\tilde{h}, g]a_j) = \mathbf{r}(a_i, \tilde{h}^{-1}g^{-1}\tilde{h}ga_j) = \mathbf{r}(\tilde{h}^{-1}g\tilde{h}a_i, ga_j) = \mathbf{r}(c_i, ga_j) \in \mathcal{L}'.$$

At every other step, we can make sure $X \subset B$ by extending $\tilde{h}$. Let $h$ be the union of $\tilde{h}$ over all $n$. Then it is an automorphism since it is well defined and bijective by construction. □

**Theorem 5.1.13.** *Let $S$ be a set of forbidden triangles satisfying Condition 3 and $Forb_c(S)$ the class of all finite complete $\mathcal{L}$-structures that does not embed any triangle from $S$. Let $\mathcal{M}_S$ be its Fraïssé limit. Let $g \in Aut(\mathcal{M}_S)$ be an automorphism of $\mathcal{M}_S$ such that for any non-algebraic type $p$ of the form (5.1), there exist countably many realisations $\bar{a}$ of $p$ with $\mathbf{r}(a_i, ga_j) \in \mathcal{L}'$ for all $a_i, a_j \in \bar{a}$. Then there exists $k \in Aut(\mathcal{M}_S)$ such that $[g, k]$ moves both almost R-maximally and almost L-maximally.*

*Proof.* List all types as $p_1, p_2, ....$ We start with the empty map and use the back-and-forth method to build $k$. Suppose at some stage, we have a partial isomorphism $\tilde{k} : A \to B$ such that for $[g, \tilde{k}]$ moves $p_1, ..., p_{i-1}$ almost $R$-maximally and almost $L$-maximally. We want to extend $\tilde{k}$ so that $\tilde{k}$ moves $p_i$ almost $R$-maximally and almost $L$-maximally. Let $p_i$ be a type over some finite set $X$. By Lemma 5.0.1, we may assume $p_i$ is non-algebraic and of the form (5.1). We may also assume $X \cup gX \subseteq A$ and $g\tilde{k}X \subseteq B$ by extending $\tilde{k}$.

Since $\mathcal{M}_S$ has trivial algebraic closure and $p_i$ is non-algebraic and of the form (5.1), we can find $\bar{a}$ realising $p$ such that $\bar{a} \cap A = \emptyset$ and $\bar{a} \mathop{\downarrow}_X g^{-1}A$ by Existence. By the assumption on $g$, there exists a realisation $\bar{b}$ of $\tilde{k}\cdot\mathrm{tp}(\bar{a}/A)$ such that $\mathbf{r}(b_i, gb_j) \in \mathcal{L}'$ for all $b_i, b_j \in \bar{b}$.

Again by Existence, there exists $\bar{c}$ of $\tilde{k}\cdot\mathrm{tp}(g\bar{a}/\bar{a}A)$ such that $\bar{c} \mathop{\downarrow}_{\bar{b}B} g\bar{b}$. Since $\mathbf{r}(b_i, gb_j) \in \mathcal{L}'$ for all $b_i, b_j \in \bar{b}$, by Remark 5.1.1, we obtain $\bar{c} \mathop{\downarrow}_B g\bar{b}$.

Extend $\tilde{k}$ by sending $\bar{a}$ to $\bar{b}$ and $g\bar{a}$ to $\bar{c}$. Then acting by $\tilde{k}g$ on $\bar{a} \mathop{\downarrow}_X g^{-1}A$, we have $\tilde{k}g\bar{a} \mathop{\downarrow}_{\tilde{k}g(X)} \tilde{k}g(g^{-1}A)$, which is the same as $\bar{c} \mathop{\downarrow}_{\tilde{k}g(X)} B$. By Transitivity on it and $\bar{c} \mathop{\downarrow}_B g\bar{b}$, we have $\bar{c} \mathop{\downarrow}_{\tilde{k}g(X)} g\bar{b}$. Then by Invariance, we have

$$\bar{a} \mathop{\downarrow}_X [g, \tilde{k}]\bar{a}$$

By the same argument, but swapping the sides of $\mathop{\downarrow}$, we can also extend $\tilde{k}$ such that there is a realisation $\bar{a}'$ of $p$ such that $[g, \tilde{k}]\bar{a}' \mathop{\downarrow}_X \bar{a}'$.

At every other step, we can make sure $X \subset B$ by extending $\tilde{k}$. Let $k$ be the union of $\tilde{k}$ over all $i$. Then $[g, k]$ moves all non-algebraic types of the form (5.1) almost $R$-maximally and almost $L$-maximally. Then by Lemma 5.0.1, $[g, k]$ moves both almost $R$-maximally and almost $L$-maximally. □

Therefore, we have proved Theorem 5.1.2.

### 5.1.3 Cherlin's structures

We now look at Cherlin's semi-free structures listed in Figure 2.1. We will show in the following lemma that if $S$ satisfies Condition 1 or Condition 2, then $\mathcal{M}_S$ satisfies 3. Since we showed in Section 3.2 that all examples listed Figure 2.1 except for #26 of 4-multi-graphs and #11 and #12 of 2-multi-tournaments satisfy either Condition 1 or Condition 2, by the following lemma, we can apply Theorem 5.1.2 to these structures directly.

For #11 or #12 of the 2-multi-tournaments, we showed in Section 3.2 that they do not form a prioritised semi-free structure. For #26 of the 4-multi-graphs, we will modify the argument from the last subsection to obtain the same result.

**Lemma 5.1.14.** *Let $S$ be a set of forbidden triangles satisfying Condition 1*

*or Condition 2. Let $\mathcal{M}_S$ be the Fraïssé limit of $Forb_c(S)$. Then $S$ and $\mathcal{M}_S$ satisfy Condition 3.*

*Proof.* Part (i) of Condition 3 can be checked by observation. We only prove part (ii).

Let $a, b, c \in \mathcal{M}_S$ and $B \subseteq \mathcal{M}_S$ be such that $a, b, c$ are distinct and $a \bigcirc\!\!\!\!\!\!\perp_{bB} c$. If $\mathbf{r}(a, b) \in \mathcal{L}'$, we want to show $a \bigcirc\!\!\!\!\!\!\perp_{B} c$. If $b \in B$, the statement is trivial. Suppose $b \notin B$. Since $a \bigcirc\!\!\!\!\!\!\perp_{bB} c$, we have $abcB = abB \otimes_{bB} bcB$, where $abB \otimes_{bB} bcB$ is the prioritised semi-free amalgamation of $abB$ and $bcB$ over $bB$ as defined in Definition 3.2.1. To show that $a \bigcirc\!\!\!\!\!\!\perp_{B} c$, we want to show that $(a, c)$ is coloured by the same relation in both $a \otimes_B c$ and $a \otimes_{bB} c$. Suppose on the contrary, $(a, c) \in R_i$ in $a \otimes_B c$ for some $R_i \in \mathcal{L}'$ and $(a, c) \notin R_i$ in $a \otimes_{bB} c$. This implies that $R_i\mathbf{r}(a, b)\mathbf{r}(b, c)$ forms a forbidden triangle.

For $S$ satisfying Condition 1, since $\mathbf{r}(a, b), R_i \in \mathcal{L}'$, $R_i\mathbf{r}(a, b)\mathbf{r}(b, c)$ is of the form $R_iR_jR'$ where $R_iR_j \in \mathcal{L}', R' \in \mathcal{L}$, contradicting Condition 1.

For $S$ satisfying Condition 2, since $\mathcal{L}' = \{R_1, R_2\}$ with $R_1 > R_2$, the only possibility of $(a, c) \notin R_i$ in $a \otimes_{bB} c$ is that $\mathbf{r}(a, c) \in R_1$ in $a \otimes_B c$ and $\mathbf{r}(a, c) \in R_2$ in $a \otimes_{bB} c$. Since $\mathbf{r}(a, b) \in \mathcal{L}'$, we have $\mathbf{r}(a, b) = R_1$ or $R_2$. Hence, $R_i\mathbf{r}(a, b)\mathbf{r}(b, c)$ is a forbidden triangle involving $R_1R_1$ or $R_1R_2$, contradicting part (i) of Condition 3.

Therefore, given the assumption, it can be deduced from $a \bigcirc\!\!\!\!\!\!\perp_{bB} c$ that $a \bigcirc\!\!\!\!\!\!\perp_{B} c$. By the same argument, we can also prove that if $\mathbf{r}(b, c) \in \mathcal{L}'$, then $a \bigcirc\!\!\!\!\!\!\perp_{B} c$. □

Therefore, by Theorem 5.1.2, we obtain the following corollary:

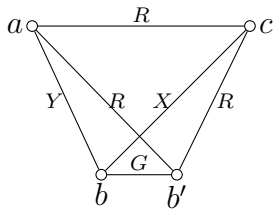
**Corollary 5.1.15.** *Let $S$ be a set of forbidden triangles and $Forb_c(S)$ the set of all finite $\mathcal{L}$-structures that do not embed any triangle from $S$. Let $\mathcal{M}_S$ be the Fraïssé limit of $Forb_c(S)$.*

*(i) If $S$ satisfies Condition 1 or Condition 2, then $Aut(\mathcal{M}_S)$ is simple.*

*(ii) In particular, if $S$ is any set listed in Figure 2.1 except for #26 of the 4-multi-graphs and #11 and #12 of the 2-multi-tournaments, then $Aut(\mathcal{M}_S)$ is simple.*

We now look at #26 in Figure 2.1. Though it does not satisfy either Condition 1 or Condition 2, we can still use a similar approach to prove the same result.

Let $S = \{RRX, RXX, RYY, GYX, GXX, YYX, XXX\}$. Note that with order $G > R > Y$ on $\mathcal{L}' = \{G, R, Y\}$, it satisfies part (i) of Condition 3. But it does not satisfy part (ii) as shown in the following example, we have $a \mathop{\downarrow}_{bb'} c$ and $\mathbf{r}(a, b) \in \mathcal{L}'$, but $a$ is not independent from $c$ over $b'$.




However, we can obtain $a \mathop{\downarrow}_{B} c$ from $a \mathop{\downarrow}_{bB} c$ if we have both $\mathbf{r}(a, b), \mathbf{r}(b, c) \in \mathcal{L}'$, which we prove in the following lemma.

**Lemma 5.1.16.** *Let $S$ be #26 of the 4-multi-graphs in Figure 2.1. Let $a, b, c \in \mathcal{M}_S$ and $B \subseteq \mathcal{M}_S$ be such that $\mathbf{r}(a, b), \mathbf{r}(b, c) \in \mathcal{L}'$ and $a \mathop{\downarrow}_{bB} c$. Then $a \mathop{\downarrow}_{B} c$.*

*Proof.* If $b \in B$, the statement is trivial. Suppose $b \notin B$. As in Lemma 5.1.14, to show that $a \mathop{\downarrow}_{B} c$, we want to show that $(a, c)$ is coloured by the same relation in both $aB \otimes_B cB$ and $abB \otimes_{bB} cbB$.

If $(a, c) \in G$ in $aB \otimes_B cB$ and $(a, c) \notin G$ in $aB \otimes_{bB} cB$, then $G\mathbf{r}(a, b)\mathbf{r}(b, c)$ forms a forbidden triangle where $\mathbf{r}(a, b), \mathbf{r}(b, c) \in \mathcal{L}'$, i.e. $\mathbf{r}(a, b), \mathbf{r}(b, c) \neq X$. However, all forbidden triangles in $S$ involving $G$ also contain $X$, a contradiction.

If $\mathbf{r}(a, c) = R$ in $aB \otimes_B cB$ and $(a, c) \notin R$ in $aB \otimes_{bB} cB$, this implies $(a, c) \in Y$ in $aB \otimes_{bB} cB$ because of the order $G > R > Y$. Then there exists $b' \in B$ such that $b'$ forbids $(a, c)$ to be coloured by $G$. Since $XX$ forbids both $G$ and $R$, we know that $\mathbf{r}(a, b')\mathbf{r}(b', c) = XY$ or $YX$. However, $XY$ forbids $(a, c)$ to be coloured by $G$ or $Y$. This implies that $R$ is the only choice for $\mathbf{r}(a, c)$ in $aB \otimes_{bB} cB$ since any amalgamation can be completed by some relation in $\mathcal{L}'$.

Therefore, given the assumption, it can be deduced from $a \mathop{\downarrow}_{bB} c$ that $a \mathop{\downarrow}_{B} c$. □

Similarly as in Remark 5.1.1, we also have that for $\bar{a}, \bar{b}, \bar{c} \in \mathcal{M}_S$ and $B \subseteq \mathcal{M}_S$ such that $\bar{a} \mathop{\downarrow}_{\bar{b}B} \bar{c}$, if $\mathbf{r}(a_i, b_j), \mathbf{r}(b_j, c_k) \in \mathcal{L}'$ for all $a_i \in \bar{a}, b_j \in \bar{b}, c_k \in \bar{c}$, then $\bar{a} \mathop{\downarrow}_{B} \bar{c}$.

Note that only Theorem 5.1.9 and 5.1.13 directly use part (ii) of Condition 3. In the proof of Theorem 5.1.9, note that we had $\mathbf{r}(c, c') = \mathbf{r}(c'', c') \in \mathcal{L}'$, so we again have $a \mathop{\downarrow}_B c$ from $a \mathop{\downarrow}_{bB} c$ by the previous lemma and the rest of the proof follows. We prove Theorem 5.1.13 for # 26 in the following:

**Theorem 5.1.17.** *Let $S$ be #26 of the 4-multi-graphs listed in Figure 2.1. Let $\mathcal{M}_S$ be its Fraïssé limit. Let $g \in Aut(\mathcal{M}_S)$ be an automorphism of $\mathcal{M}_S$ such that for any non-algebraic type $p$ of the form (5.1), there exist infinitely many realisations $\bar{a}$ of $p$ with $\mathbf{r}(a_i, ga_j) \in \mathcal{L}'$ for all $a_i, a_j \in \bar{a}$. Then there exists $k \in Aut(\mathcal{M}_S)$ such that $[g, k]$ moves both almost $R$-maximally and almost $L$-maximally.*

*Proof.* List all types as $p_1, p_2, ....$ We start with the empty map and use the back-and-forth method to build $k$. Suppose at some stage, we have a partial isomorphism $\tilde{k} : A \to B$ such that for $[g, \tilde{k}]$ moves $p_1, ..., p_{i-1}$ almost $R$-maximally and almost $L$-maximally. We want to extend $\tilde{k}$ so that $\tilde{k}$ moves $p_i$ almost $R$-maximally and almost $L$-maximally. Let $p_i$ be a type over some finite set $X$. By Lemma 5.0.1, we may assume $p_i$ is non-algebraic and of the form of the form (5.1). We may also assume $X \cup gX \subseteq A$ and $g\tilde{k}X \subseteq B$ by extending $\tilde{k}$.

By Existence, we can find $\bar{a}'$ realising $p$ such that $\bar{a}' \mathop{\downarrow}_X g^{-1}(A)$. Since $p_i$ is non-algebraic and of the form (5.1), we may assume $\bar{a}'$ does not intersect with $Ag^{-1}(A)$. Then type $\mathrm{tp}(\bar{a}'/Xg^{-1}(A))$ is of the form (5.1) since $\mathcal{M}_S$ has trivial algebraic closure. By the assumption on $g$, we can find $\bar{a}$ realising $\mathrm{tp}(\bar{a}'/Xg^{-1}(A))$ such that $\mathbf{r}(a_i, ga_j) \in \mathcal{L}'$ for all $a_i, a_j \in \bar{a}$. Then by Invariance, we also have $\bar{a} \mathop{\downarrow}_X g^{-1}(A)$. Since the type $\mathrm{tp}(\bar{a}/A)$ is of the form (5.1), there exists a realisation $\bar{b}$ of $\tilde{k}{\cdot}\mathrm{tp}(\bar{a}/A)$ such that $\mathbf{r}(b_i, gb_j) \in \mathcal{L}'$ for all $b_i, b_j \in \bar{b}$.

Again by Existence, there exists $\bar{c}$ of $\tilde{k}{\cdot}\mathrm{tp}(g\bar{a}/\bar{a}A)$ such that $\bar{c} \mathop{\downarrow}_{\bar{b}B} g\bar{b}$. Since $\mathrm{tp}(\bar{c}/\bar{b}) = \tilde{k}{\cdot}\mathrm{tp}(g\bar{a}/\bar{a})$, we have that $\mathbf{r}(c_i, b_j) \in \mathcal{L}'$ for all $c_i \in \bar{c}, b_j \in \bar{b}$. We also have that $\mathbf{r}(b_i, gb_j) \in \mathcal{L}'$ for all $b_i, b_j \in \bar{b}$. Hence, by the previous lemma, we obtain $\bar{c} \mathop{\downarrow}_B g\bar{b}$.

Extend $\tilde{k}$ by sending $\bar{a}$ to $\bar{b}$ and $g\bar{a}$ to $\bar{c}$. Then by Invariance on $\bar{a} \mathop{\downarrow}_X g^{-1}A$, we have $\tilde{k}g\bar{a} \mathop{\downarrow}_{\tilde{k}g(X)} \tilde{k}g(g^{-1}A)$, which is the same as $\bar{c} \mathop{\downarrow}_{\tilde{k}g(X)} B$. By Transitivity

on it and $\bar{c} \mathop{\downarrow}_B g\bar{b}$, we have $\bar{c} \mathop{\downarrow}_{\tilde{k}g(X)} g\bar{b}$. Then by Invariance, we have

$$\bar{a} \mathop{\downarrow}_X [g, \tilde{k}]\bar{a}$$

By the same argument, but swapping sides of $\downarrow$, we can also extend $\tilde{k}$ such that there is a realisation $\bar{a}'$ of $p$ such that $[g, \tilde{k}]\bar{a}' \mathop{\downarrow}_X \bar{a}'$.

At every other step, we can make sure $X \subset B$ by extending $\tilde{k}$. Let $k$ be the union of $\tilde{k}$ over all $i$. Then $[g, k]$ moves all non-algebraic types of the form (5.1) almost $R$-maximally and almost $L$-maximally. Then by Lemma 5.0.1, $[g, k]$ moves both almost $R$-maximally and almost $L$-maximally. □

Therefore, we obtain the same result for # 26.

**Corollary 5.1.18.** *Let $S$ be #26 of the 4-multi-graphs in Figure 2.1 and $\mathcal{M}_S$ be the Fraïssé limit of $Forb_c(S)$. Then for any non-trivial automorphism $g \in Aut(\mathcal{M}_S)$, there exist $k, h \in Aut(\mathcal{M}_S)$ such that $[[h, g], k]$ moves almost R-maximally and L-maximally. Hence, $Aut(\mathcal{M}_S)$ is simple.*

Together with Corollary 5.1.15, we proved Theorem 5.1.3.

## 5.2 Linearly ordered structures

We now look at some linearly ordered structures in this section. We let $\mathcal{M}$ be a countable strong homogeneous $\mathcal{L}$-structure admitting a **SWIR** $\mathop{\downarrow}^\star$ and let $\mathcal{L}^< = \mathcal{L} \cup \{<\}$. Let $\mathcal{M}^<$ be its linear order expansion as defined in Definition 3.3.1. It was proved in Theorem 3.3.4 that $\mathcal{M}^<$ admits a **SWIR** $\downarrow$, i.e. for any finite $A, B, C \subseteq \mathcal{M}^<$, $A \mathop{\downarrow}_B C$ if $A^{\mathcal{L}} \mathop{\downarrow}^\star_{B^{\mathcal{L}}} C^{\mathcal{L}}$ and for any $a \in A \setminus B, c \in C \setminus B$ such that $a < c$, there exists $b \in B$ such that $a < b < c$.

We will apply Corollary 4.2.4 to the linear order expansions of some homogeneous structures in this section. We will first, in Subsection 5.2.1, define some notions and prove some preparatory results applicable to the linear order expansion of all homogeneous structures. We will prove the following theorem in this subsection using a back-and-forth construction.

**Theorem 5.2.1.** *Let $\mathcal{M}$ be a countable non-trivial strong homogeneous $\mathcal{L}$-structure and $\mathcal{M}^<$ be its linear order expansion. Let $g$ be an non-trivial auto-*

*morphism of $\mathcal{M}^<$. Then there exists $h \in Aut(\mathcal{M}^<)$, a product of conjugates of $g$ and $g^{-1}$, such that $ha > a$ for all $a \in \mathcal{M}^<$.*

We will then in Subsection 5.2.2 show that if $\mathcal{M}$ is a non-trivial free homogeneous structure and $h \in \mathrm{Aut}(\mathcal{M}^<)$ is such that $ha > a$ for all $a \in \mathcal{M}^<$, then we can find a product of conjugates of $h$ and $h^{-1}$ that moves almost $R$-maximally and its inverse move almost $L$-maximally. Hence, we can apply Corollary 4.2.4 to the linear order expansion of a non-trivial free homogeneous structure and prove the simplicity of its automorphism group. The same result was proved in [CKT19] using a method also generalised from Tent and Ziegler's theorem, but in a different way from the method presented in this thesis.

In Subsection 5.2.3, we will look at the universal $n$-linear order $\mathcal{M}_n$. We will generalise the proof of the above theorem and prove that for any non-trivial automorphism $g \in \mathrm{Aut}(\mathcal{M}_n)$, where $n \geq 2$, we can find $h \in \mathrm{Aut}(\mathcal{M}_n)$, as a product of conjugates of $g$ and $g^{-1}$, such that $ha <_j a$ for all $a \in \mathcal{M}_n$ and $j = 1, ..., n$ (see Theorem 5.2.27). We will then show in Theorem 5.2.29 that such an automorphism moves almost $R$-maximally and its inverse moves almost $L$-maximally. Hence, we can apply Corollary 4.2.4 to $\mathcal{M}_n$ and prove the simplicity of $\mathrm{Aut}(\mathcal{M}_n)$ for $n \geq 2$.

Altogether,we prove the following theorem.

**Theorem 5.2.2.** *Let $\mathcal{M}$ be one of the following structures:*

*(i) the linear order expansion of a non-trivial free homogeneous structure,*

*(ii) the universal $n$-linear order, $n \geq 2$.*

*Then, $\mathcal{M}$ admits a **SWIR** and $Aut(\mathcal{M})$ is simple.*

Note that in this section, we will use the following notion as defined in Section 3.3. Let $X \subset \mathcal{M}^<$ be finite, for an $n$-tuple $\bar{x}$ in $\mathcal{M}^<$, we denote $\mathrm{tp}^{\mathcal{L}}(\bar{x}/X)$ to be the set of all $\mathcal{L}$-formulas satisfied by $\bar{x}$ with parameters in $X$, $\mathrm{tp}^<(\bar{x}/X)$ to be the set of all $\{<\}$-formulas satisfied by $\bar{x}$ with parameters in $X$ and $\mathrm{tp}(\bar{x}/X)$ to be the set of all $\mathcal{L}^<$-formulas satisfied by $\bar{x}$ with parameters.

### 5.2.1 Preparatory results

**Definition 5.2.3.** Let $S$ be a subset of $\mathcal{M}^<$. We define the *convexification* of $S$ to be the set $\{a \in \mathcal{M}^< | \exists s_1, s_2 \in S : s_1 \leq a \leq s_2\}$.

**Definition 5.2.4.** Let $g$ be an automorphism of $\mathcal{M}^<$. An *orbital* of $g$ containing some $a \in \mathcal{M}^<$ is the convexification of $\{g^n a | n \in \mathbb{Z}\}$. We say an orbital $I$ is *unbounded above* if for any $a \in \mathcal{M}^<$, there exists $b > a$ such that $b \in I$ and we say it is *unbounded below* if for any $a \in \mathcal{M}^<$, there exists $b < a$ such that $b \in I$.

Note that if $ga > a$, then $gb > b$ for all $b$ in the orbital of $g$ containing $a$. Similarly, if $ga < a$, then $gb < b$ for all $b$ in the orbital of $g$ containing $a$. So we can define the following:

**Definition 5.2.5.** (i) We say an orbital $I$ is a *+-orbital* of $g$ if $ga > a$ for all $a \in I$ and is a *−-orbital* of $g$ if $ga < a$ for all $a \in I$.

(ii) For $g \in$Aut$(\mathcal{M}^<)$, we say $g$ *has a single orbital* if for any $a \in \mathcal{M}^<$, the orbital containing $a$ is unbounded above and below.

**Remark 5.2.6.** (i) If $g \in$Aut$(\mathcal{M}^<)$ has +-orbital unbounded above (or below), then for any $b$ in the orbital, $(g^i b)_{i \in \mathbb{Z}}$ is a sequence unbounded above (or below). The same statement holds for a −-orbital.

**Lemma 5.2.7.** *Let $\mathcal{M}$ be a countable strong homogeneous structure and $\mathcal{M}^<$ be its linear order expansion. Let $p(x)$ be a 1-type over some finite set $X \subseteq \mathcal{M}^<$ and $b, c \in \mathcal{M}^<$. Suppose $p^<(x) \cup \{b < x < c\}$ is consistent. Then there exists $a \in (b, c)$ realising $p(x)$.*

*Proof.* The lemma follows from the Extension Property. Since $p^<(x) \cup \{b < x < c\}$ is consistent, we can embed $bcX$ into $abcX$ such that $a$ satisfies $p(x)$ and $b < a < c$. Then by the Extension Property, we can embed $abcX$ into $\mathcal{M}^<$. Hence, there exists $a \in (b, c)$ realising $p(x)$. □

**Corollary 5.2.8.** *Let $\mathcal{M}$ be a countable strong homogeneous structure and $\mathcal{M}^<$ be its linear order expansion. Let $p(x)$ be a 1-type over some finite set $X \subseteq \mathcal{M}^<$. Let $g \in$Aut$(\mathcal{M}^<)$.*

*(i) Suppose for all $y \in \mathcal{M}^<$, there exists $z > y$ such that $gz > z$ and $p(x)$ contains the formula $\{x > x' : x' \in X\}$. Then for any $y \in \mathcal{M}^<$, there exists $a > y$ such that $a$ realises $p(x)$ and $ga > a$.*

*(ii) Suppose for all $y \in \mathcal{M}^<$, there exists $z > y$ such that $gz < z$ and $p(x)$ contains $\{x > x' : x' \in X\}$. Then for any $y \in \mathcal{M}^<$, there exists $a > y$*

*such that $a$ realises $p(x)$ and $ga < a$.*

*(iii) Suppose for all $y \in \mathcal{M}^<$, there exists $z < y$ such that $gz > z$ and $p(x)$ contains $\{x < x' : x' \in X\}$, then for any $y \in \mathcal{M}^<$, there exists $a < y$ such that $a$ realises $p(x)$ and $ga > a$.*

*(iv) Suppose for all $y \in \mathcal{M}^<$, there exists $z < y$ such that $gz < z$ and $p(x)$ contains $\{x < x' : x' \in X\}$, then for any $y \in \mathcal{M}^<$, there exists $a < y$ such that $a$ realises $p(x)$ and $ga < a$.*

*Proof.* To prove (i), by the assumption on $g$, we can find $b > max\{y, x' : x' \in X\}$ such that $gb > b$. Since $p(x)$ contains the formula $\{x > x' : x' \in X\}$, we have that $p^<(x) \cup \{b < x < gb\}$ is consistent. So, by the previous lemma, we have $a \in (b, gb)$ realising $p(x)$. We also have that $ga > gb > a$. We can prove (ii)-(iv) by the same argument. □

**Proposition 5.2.9.** *Let $\mathcal{M}$ be a countable strong homogeneous $\mathcal{L}$-structure and let $\mathcal{L}^< = \mathcal{L} \cup \{<\}$. Let $\mathcal{M}^<$ be its linear order expansion as defined in Definition 3.3.1. Let $g \in Aut(\mathcal{M}^<)$ be such that for any $x \in \mathcal{M}^<$, there exist $y < x$ and $z > x$ such that $gy > y$ and $gz > z$. Then there exists $h \in Aut(\mathcal{M}^<)$ such that $gh^{-1}gh$ has a single $+$-orbital.*

*Proof.* List all elements of $\mathcal{M}^<$ as $x_0, x_1, ....$ We construct $h$ using the back-and-forth method and find a sequence $(a_n)_{n\in\mathbb{Z}}$ such that $gh^{-1}ghha_i = a_{i+1} > a_i$ for all $i \in \mathbb{Z}$ and for any $x \in \mathcal{M}^<$, there exist $m, n \in \mathbb{Z}$ such that $a_m < x < a_n$, i.e. the $+$-orbital of $gh^{-1}gh$ containing $a_0$ is unbounded below and above. Hence, $gh^{-1}gh$ has a single $+$-orbital.

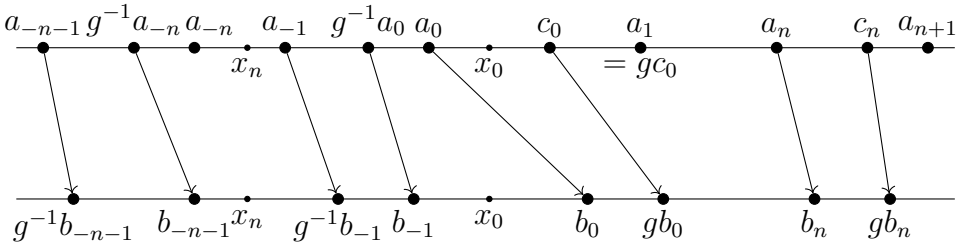



Figure 5.2: Construction of $h$

We start with $\tilde{h}$ as the empty map. Choose $a_0 < x_0 < b_0$ such that $ga_0 > a_0, gb_0 > b_0$. Extend $\tilde{h}$ by sending $a_0$ to $b_0$. By the assumption on $g$ and the

previous corollary, we can find $c_0 > x_0$ such that $c_0$ realises $\tilde{h}^{-1}\cdot\mathrm{tp}(gb_0/b_0)$ and $gc_0 > c_0$ since $\tilde{h}^{-1}\cdot\mathrm{tp}(gb_0/b_0)$ contains $\{x > a_0\}$. Extend $\tilde{h}$ by sending $c_0$ to $gb_0$. Let $a_1 := gc_0$. Then $g\tilde{h}^{-1}g\tilde{h}a_0 = a_1 = gc_0 > c_0 > x_0 > a_0$.

Now since $\tilde{h}\cdot\mathrm{tp}(g^{-1}a_0/a_0c_0)$ contains $\{x < b_0, gb_0\}$, by Corollary 5.2.8 and the assumption on $g$, we can find $b_{-1} < x_0$ such that $b_{-1}$ realises $\tilde{h}\cdot\mathrm{tp}(g^{-1}a_0/a_0c_0)$ and $gb_{-1} > b_{-1}$. Extend $\tilde{h}$ by sending $g^{-1}a_0$ to $b_{-1}$. Similarly, we can choose $a_{-1}$ realising $\tilde{h}^{-1}\cdot\mathrm{tp}(g^{-1}b_{-1}/b_0b_{-1}gb_0)$ such that $ga_{-1} > a_{-1}$ and extend $\tilde{h}$ by sending $a_{-1}$ to $g^{-1}b_{-1}$. Since $\mathrm{tp}(g^{-1}b_{-1}/b_{-1}) = \tilde{h}\cdot\mathrm{tp}(a_{-1}/g^{-1}a_0)$ and $g^{-1}b_{-1} < b_{-1}$, we have $a_{-1} < g^{-1}a_0$. Then $g\tilde{h}^{-1}g\tilde{h}a_{-1} = a_0 > g^{-1}a_0 > a_{-1}$.

Let $A_0 = \{a_{-1}, g^{-1}a_0, a_0, c_0\}$ and $B_0 = \{g^{-1}b_{-1}, b_{-1}, b_0, gb_0\}$ be the domain and image of $\tilde{h}$. Then, $x_0 \in (\min A_0, \max A_0) \cap (\min B_0, \max B_0)$. Choose $y_0$ realising $\tilde{h}\cdot\mathrm{tp}(x_0/A_0)$ and extend $\tilde{h}$ by sending $x_0$ to $y_0$. Choose $z_0$ realising $\tilde{h}^{-1}\cdot\mathrm{tp}(x_0/y_0B_0)$ and extend $\tilde{h}$ by sending $z_0$ to $x_0$. Extend $A_0$,$B_0$ to include $x_0, z_0$ and $y_0, x_0$ respectively.

Suppose at the $n$-th step, we have a partial isomorphism, $\tilde{h} : A_{n-1} \to B_{n-1}$ satisfying

(i) $A_{n-1} = \{a_{-n}, g^{-1}a_{-n+1}, ..., a_0, c_0, ..., a_{n-1}, c_{n-1}, x_0, ..., x_{n-1}, z_0, ..., z_{n-1}\}$,
$B_{n-1} = \{g^{-1}b_{-n}, b_{-n}, ...., b_{-1}, b_0, ..., b_{n-1}, gb_{n-1}, x_0, ..., x_{n-1}, y_0, ..., y_{n-1}\}$,
where $a_i := gc_{i-1} > c_{i-1}$ for all $i = 1, ..., n$.

(ii) $\min A_{n-1} = a_{-n} < g^{-1}a_{-n+1} < a_{-n+1} < \cdots < g^{-1}a_0 < a_0 < c_0 < a_1 < \cdots < a_{n-1} < c_{n-1} = \max A_{n-1}$, $\min B_{n-1} = g^{-1}b_{-n} < b_{-n} < \cdots < g^{-1}b_{-1} < b_{-1} < b_0 < gb_0 < \cdots < b_{n-1} < gb_{n-1} = \max B_{n-1}$.

(iii) $\tilde{h}$ maps $a_i$ to $b_i$, $c_i$ to $gb_i$, $x_i$ to $y_i$ and $z_i$ to $x_i$ for all $i = 0, ..., n-1$ and $\tilde{h}$ maps $a_j$ to $g^{-1}b_j$ and $g^{-1}a_{j+1}$ to $b_j$ for all $j = -1, ..., -n$.

(iv) $g\tilde{h}^{-1}g\ \tilde{h}a_i = a_{i+1} > a_i$ for all $i = -n, ..., n-1$.

Since $\tilde{h}\cdot\mathrm{tp}(a_n/A_{n-1})$ contains $\{x > b | b \in B_{n-1}\}$, by Corollary 5.2.8 and the assumption on $g$, we can choose $b_n > x_n$ realising $\tilde{h}\cdot\mathrm{tp}(a_n/A_{n-1})$ such that $gb_n > b_n$ and extend $\tilde{h}$ by sending $a_n$ to $b_n$. Similarly, we can find $c_n > x_n$ realising $\tilde{h}^{-1}\cdot\mathrm{tp}(gb_n/b_nB_{n-1})$ such that $gc_n > c_n$ and extend $\tilde{h}$ by sending $c_n$ to $gb_n$. Since $\mathrm{tp}(gb_n/b_n) = \tilde{h}\cdot\mathrm{tp}(c_n/a_n)$ and $gb_n > b_n$, we have $c_n > a_n$. Let $a_{n+1} := gc_n$. Then $g\tilde{h}^{-1}g\tilde{h}a_n = a_{n+1} = gc_n > c_n > a_n$.

Similarly, since $\tilde{h}\cdot\mathrm{tp}(g^{-1}a_{-n}/a_nc_nA_{n-1})$ contains $\{x < b | b \in b_ngb_nB_{n-1}\}$, we

can choose $b_{-n-1} < x_n$ realising $\tilde{h}\cdot\mathrm{tp}(g^{-1}a_{-n}/a_nc_nA_{n-1})$ such that $gb_{-n-1} > b_{-n-1}$. Extend $\tilde{h}$ by sending $g^{-1}a_{-n}$ to $b_{-n-1}$. Choose $a_{-n-1} < x_n$ such that $a_{-n-1}$ realises $\tilde{h}^{-1}\cdot\mathrm{tp}(g^{-1}b_{-n-1}/b_nb_{-n-1}gb_nB_{n-1})$ and $ga_{-n-1} > a_{-n-1}$. Extend $\tilde{h}$ by sending $a_{-n-1}$ to $g^{-1}b_{-n-1}$. Since we have $\mathrm{tp}(g^{-1}b_{-n-1}/b_{-n-1}) = \tilde{h}\cdot\mathrm{tp}(a_{-n-1}/g^{-1}a_{-n})$ and $g^{-1}b_{-n-1} < b_{-n-1}$, we obtain $a_{-n-1} < g^{-1}a_{-n}$.Then $g\tilde{h}^{-1}g\tilde{h}a_{-n-1} = a_{-n} > g^{-1}a_{-n} > a_{-n-1}$.

Let $A_n = A_{n-1} \cup \{a_{-n-1}, g^{-1}a_{-n}, a_n, c_n\}$ and $B_n = B_{n-1} \cup \{g^{-1}b_{-n-1}, b_{-n-1}, b_n, gb_n\}$. Then $x_n$ is in both $(\min A_n, \max A_n)$ and $(\min B_n, \max B_n)$. Find $y_n$ realising $\tilde{h}\cdot\mathrm{tp}(x_n/A_n)$ and extend $\tilde{h}$ by sending $x_n$ to $y_n$. Find $z_n$ realising $\tilde{h}^{-1}\cdot\mathrm{tp}(x_n/y_nB_n)$. Extend $A_n$,$B_n$ to include $x_n, z_n$ and $y_n, x_n$ respectively. Then, $\min A_n = a_{-n-1}$, $\max A_n = c_n$, $\min B_n = g^{-1}b_{-n-1}$, $\max B_n = gb_n$. Hence $\tilde{h} : A_n \to B_n$ satisfies the hypothesis (i)-(iv).

Let $h$ be the union of $\tilde{h}$ over all steps. Then $gh^{-1}gh$ maps $a_i$ to $a_{i+1} > a_i$ for all $i \in \mathbb{Z}$ and for any $x \in \mathcal{M}^<$, there exist $n, m \in \mathbb{Z}$ such that $x < x_n < c_n < a_{n+1}$ and $x > x_m > a_{-m}$. Therefore, the $+$-orbital containing $a_0$ is unbounded below and above, i.e. $gh^{-1}gh$ has a single $+$-orbital. □

We prove the following two propositions using a similar approach to the previous one.

**Proposition 5.2.10.** *Let $\mathcal{M}$ be a countable strong homogeneous $\mathcal{L}$-structure and let $\mathcal{L}^< = \mathcal{L} \cup \{<\}$. Let $\mathcal{M}^<$ be its linear order expansion as defined in Definition 3.3.1. Let $g \in Aut(\mathcal{M}^<)$ be such that for all $x \in \mathcal{M}^<$, there exists $y > x$ and $z < x$ such that $gy > y$ and $gz < z$. Then there exists $h \in Aut(\mathcal{M}^<)$ such that $gh^{-1}gh$ has a $+$-orbital unbounded above and a $-$-orbital unbounded below.*

*Proof.* List all elements of $\mathcal{M}^<$ as $x_0, x_1, ....$ We construct $h$ using the back-and-forth method and find sequences $(a_n)_{n\in\mathbb{Z}}, (a'_n)_{n\in\mathbb{Z}}$ such that $gh^{-1}gha_i = a_{i+1} > a_i$, $gh^{-1}gha'_i = a'_{i+1} < a'_i$ for all $i \in \mathbb{Z}$ and for any $x \in \mathcal{M}^<$, there exists $m, n \in \mathbb{Z}$ such that $a'_m < x < a_n$. Then the $+$-orbital containing $a_0$ is unbounded above and the $-$-orbital containing $a'_0$ is unbounded below.

We start with $\tilde{h}$ as the empty map. Choose $a_0, b_0 > x_0$ such that $gb_0 > b_0$. Extend $\tilde{h}$ by sending $a_0$ to $b_0$. By the assumption on $g$ and Corollary 5.2.8, we can find $c_0 > a_0$ such that $c_0$ realises $\tilde{h}^{-1}\cdot\mathrm{tp}(gb_0/b_0)$ and $gc_0 > c_0$ since $\tilde{h}^{-1}\cdot\mathrm{tp}(gb_0/b_0)$ contains $\{x > a_0\}$. Extend $\tilde{h}$ by sending $c_0$ to $gb_0$. Then we

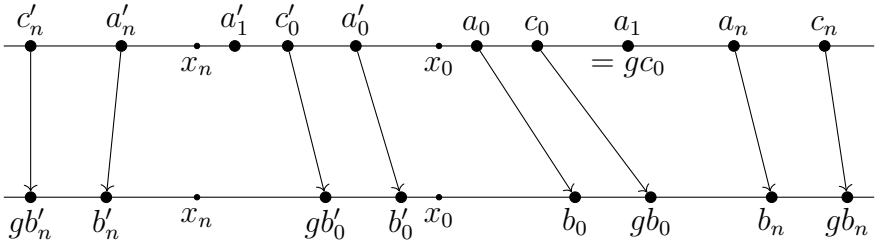



Figure 5.3: Construction of $h$

have $c_0 > a_0$ since $\mathrm{tp}(gb_0/b_0) = \tilde{h}{\cdot}\mathrm{tp}(c_0/a_0)$ and $gb_0 > b_0$. Let $a_1 := gc_0$. Then $g\tilde{h}^{-1}g\tilde{h}a_0 = a_1 = gc_0 > c_0 > a_0$.

Similarly, by Corollary 5.2.8 and the assumption on $g$, we can find $a'_0, b'_0 < x_0$ such that $b'_0$ realises $\tilde{h}{\cdot}\mathrm{tp}(a'_0/a_0c_0)$ and $gb'_0 < b'_0$. Extend $\tilde{h}$ by sending $a'_0$ to $b'_0$. We can choose $c'_0$ realising $\tilde{h}^{-1}{\cdot}\mathrm{tp}(gb'_0/b_0b'_0gb_0)$ such that $gc'_0 < c'_0$ since $\tilde{h}^{-1}{\cdot}\mathrm{tp}(gb'_0/b_0b'_0gb_0)$ contains $\{x < a'_0, a_0, c_0\}$. Extend $\tilde{h}$ by sending $c'_0$ to $gb'_0$. Then we have $c'_0 < a'_0$ since $\mathrm{tp}(gb'_0/b'_0) = \tilde{h}{\cdot}\mathrm{tp}(c'_0/a'_0)$ and $gb'_0 < b'_0$. Let $a'_1 := gc'_0$. Then $g\tilde{h}^{-1}g\tilde{h}a'_0 = a'_1 = gc'_0 < c'_0 < a'_0$.

Let $A_0 = \{c'_0, a'_0, a_0, c_0\}$ and $B_0 = \{gb'_0, b'_0, b_0, gb_0\}$ be the domain and image of $\tilde{h}$. Then, $x_0$ is in both $(\min A_0, \max A_0)$ and $(\min B_0, \max B_0)$. Choose $y_0$ realising $\tilde{h}{\cdot}\mathrm{tp}(x_0/A_0)$ and extend $\tilde{h}$ by sending $x_0$ to $y_0$. Choose $z_0$ realising $\tilde{h}^{-1}{\cdot}\mathrm{tp}(x_0/y_0B_0)$ and extend $\tilde{h}$ by sending $z_0$ to $x_0$. Extend $A_0$,$B_0$ to include $x_0, z_0$ and $y_0, x_0$ respectively.

Now suppose at the $n$-th step, we have a partial isomorphism of $\mathcal{M}$, $\tilde{h} : A_{n-1} \to B_{n-1}$ satisfying

(i) $A_{n-1} = \{c'_{n-1}, a'_{n-1}, ..., a'_0, a_0, ..., a_{n-1}, c_{n-1}, x_0, ..., x_{n-1}, z_0, ..., z_{n-1}\}$,
$B_{n-1} = \{gb'_{n-1}, b'_{n-1}, ...., b'_0, b_0, ..., b_{n-1}, gb_{n-1}, x_0, ..., x_{n-1}, y_0, ..., y_{n-1}\}$,
where $a_{i+1} := gc_i, a'_{i+1} := gc'_i$ for all $i = 0, ..., n-1$.

(ii) $\min A_{n-1} = c'_{n-1} < a'_{n-1} < \cdots < c'_0 < a'_0 < a_0 < c_0 < \cdots < a_{n-1} < c_{n-1} = \max A_{n-1}$, $\min B_{n-1} = gb'_{n-1} < b'_{n-1} < \cdots < gb'_0 < b'_0 < b_0 < gb_0 < \cdots < b_{n-1} < gb_{n-1} = \max B_{n-1}$.

(iii) $\tilde{h}$ maps $a_i$ to $b_i$, $c_i$ to $gb_i$, $a'_i$ to $b'_i$, $c'_i$ to $gb'_i$, $x_i$ to $y_i$ and $z_i$ to $x_i$ for all $i = 0, ..., n-1$.

(iv) $g\tilde{h}^{-1}g\tilde{h}a_i = a_{i+1} > a_i$, $g\tilde{h}^{-1}g\tilde{h}a'_i = a'_{i+1} < a'_i$.

Since $\tilde{h}\cdot\mathrm{tp}(a_n/A_{n-1})$ contains $\{x > b | b \in B_{n-1}\}$, we can choose $b_n > x_n$ realising $\tilde{h}\cdot\mathrm{tp}(a_n/A_{n-1})$ such that $gb_n > b_n$ and extend $\tilde{h}$ by sending $a_n$ to $b_n$. We can find $c_n > x_n$ realising $\tilde{h}^{-1}\cdot\mathrm{tp}(gb_n/b_nB_{n-1})$ such that $gc_n > c_n$ and extend $\tilde{h}$ by sending $c_n$ to $gb_n$. Then we have $c_n > a_n$ since $\mathrm{tp}(gb_n/b_n) = \tilde{h}\cdot\mathrm{tp}(c_n/a_n)$ and $gb_n > b_n$. Let $a_{n+1} := gc_n$. Then $g\tilde{h}^{-1}g\tilde{h}a_n = a_{n+1} = gc_n > c_n > a_n$.

Similarly, since $\tilde{h}\cdot\mathrm{tp}(a'_n/a_nc_nA_{n-1})$ contains $\{x < b | b \in b_ngb_nB_{n-1}\}$, we can choose $b'_n < x_n$ realising $\tilde{h}\cdot\mathrm{tp}(a'_n/a_nc_nA_{n-1})$ such that $gb'_n < b'_n$. Extend $\tilde{h}$ by sending $a'_n$ to $b'_n$. Choose $c'_n < x_n$ such that $a'_n$ realises $\tilde{h}^{-1}\cdot\mathrm{tp}(gb'_n/b_nb'_ngb_nB_{n-1})$ and $gc'_n < c'_n$. Extend $\tilde{h}$ by sending $c'_n$ to $gb'_n$. Then we have $c'_n < a'_n$ since $\mathrm{tp}(gb'_n/b'_n) = \tilde{h}\cdot\mathrm{tp}(c'_n/a'_n)$ and $gb'_n < b'_n$. Let $a'_{n+1} = gc'_n$. Then $g\tilde{h}^{-1}g\tilde{h}a'_{n-1} = a'_n = gc'_n < c'_n < a'_{n-1}$.

Let $A_n = A_{n-1} \cup \{a'_n, c'_n, a_n, c_n\}$ and $B_n = B_{n-1} \cup \{b_{-n-1}, g^{-1}b_{-n-1}, b_n, gb_n\}$. Then $x_n$ is in both $(\min A_n, \max A_n)$ and $(\min B_n, \max B_n)$. Find $y_n$ realising $\tilde{h}\cdot\mathrm{tp}(x_n/A_n)$ and extend $\tilde{h}$ by sending $x_n$ to $y_n$. Find $z_n$ realising $\tilde{h}^{-1}\cdot\mathrm{tp}(x_n/y_nB_n)$. Extend $A_n$,$B_n$ to include $x_n, z_n$ and $y_n, x_n$ respectively. Then, $\min A_n = c'_n$, $\max A_n = c_n$, $\min B_n = gb'_n$, $\max A_n = gb_n$. Hence $\tilde{h} : A_n \to B_n$ satisfies the hypothesis (i)-(iv).

Let $h$ be the union of $\tilde{h}$ over all steps. Then $gh^{-1}ghha_i = a_{i+1} > a_i$, $gh^{-1}ghha'_i = a'_{i+1} < a'_i$ for all $i \in \mathbb{Z}$ and for any $x \in \mathcal{M}^<$, there exists $m, n \in \mathbb{Z}$ such that $a'_{m+1} < c'_m < x_m < x < x_n < c_n < a_{n+1}$. Therefore, the $+$-orbital containing $a_0$ is unbounded above and the $-$-orbital containing $a'_0$ is unbounded below. □

**Proposition 5.2.11.** *Let $g \in Aut(\mathcal{M}^<)$ be such that $g$ has a $+$-orbital unbounded above and a $-$-orbital unbounded below. Then we can find $h \in Aut(\mathcal{M}^<)$ such that $[g, h]$ has a single $+$-orbital.*

*Proof.* Let $y \in \mathcal{M}^<$ be an element of the $+$-orbital unbounded above and let $z \in \mathcal{M}^<$ be an element of the $-$-orbital unbounded below, where $z < y$. Then, we have that $gx > x$ for all $x > y$ and $gx < x$ for all $x < z$. List all elements of $\mathcal{M}$ as $w_0, w_1, ....$ We construct $h$ using the back-and-forth method and find a sequence $(a_n)_{n\in\mathbb{Z}}$ such that $[g, h]a_i = a_{i+1} > a_i$ for all $i \in \mathbb{Z}$ and for any $x \in \mathcal{M}^<$, there exist $m, n \in \mathbb{Z}$ such that $a_m < x < a_n$. Then $[g, h]$ has a single $+$-orbital. We start with $\tilde{h}$ as the empty map.

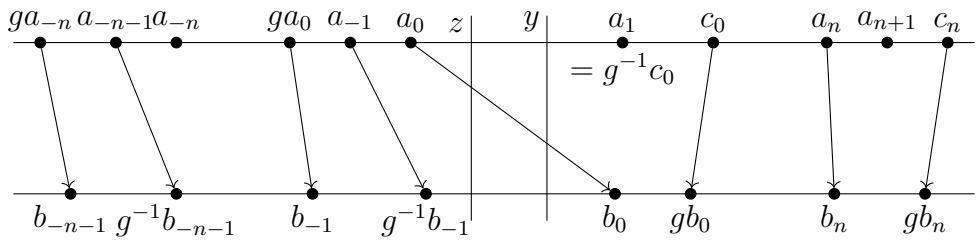



Figure 5.4: Construction of $h$

We can choose $a_0 < z, w_0$ and $b_0 > w_0, y$. Then $ga_0 < a_0$ and $gb_0 > b_0$. Extend $\tilde{h}$ by sending $a_0$ to $b_0$. By the Extension Property, we can find $c_0 > w_0, gy$ realising $\tilde{h}^{-1}\cdot\mathrm{tp}(gb_0/b_0)$ since $\tilde{h}^{-1}\cdot\mathrm{tp}(gb_0/b_0)\cup\{x > w_0, gy\}$ is consistent. Extend $\tilde{h}$ by sending $c_0$ to $gb_0$. Let $a_1 := g^{-1}c_0$. Then $g^{-1}c_0 = [g,\tilde{h}]a_0 = a_1 > y > z > a_0$.

Similarly, we can choose $b_{-1} < gz, w_0$ such that $b_{-1}$ realises $\tilde{h}\cdot\mathrm{tp}(ga_0/c_0a_0)$. Then $g^{-1}b_{-1} < z$. Extend $\tilde{h}$ by sending $ga_0$ to $b_{-1}$. Choose $a_{-1}$ realising $\tilde{h}^{-1}\cdot\mathrm{tp}(g^{-1}b_{-1}/b_{-1}b_0gb_0)$ and extend $\tilde{h}$ by sending $a_{-1}$ to $g^{-1}b_{-1}$. Then $a_{-1} < a_0$ since $g^{-1}b_{-1} < z < y < b_0$ and $\mathrm{tp}(g^{-1}b_{-1}/b_0) = \tilde{h}\cdot\mathrm{tp}(a_{-1}/a_0)$. Thus, we have $[g,\tilde{h}]a_{-1} = a_0 > a_{-1}$.

Let $A_0 = \{ga_0, a_{-1}, a_0, c_0\}$ and $B_0 = \{b_{-1}, g^{-1}b_{-1}, b_0, gb_0\}$ be the domain and image of $\tilde{h}$. Then $x_0 := w_0$ is in $(\min A_0, \max A_0) \cap (\min B_0, \max B_0)$. Choose $y_0$ realising $\tilde{h}\cdot\mathrm{tp}(x_0/A_0)$ and extend $\tilde{h}$ by sending $x_0$ to $y_0$. Choose $z_0$ realising $\tilde{h}^{-1}\cdot\mathrm{tp}(x_0/y_0B_0)$ and extend $\tilde{h}$ by sending $z_0$ to $x_0$. Extend $A_0$,$B_0$ to include $x_0, z_0$ and $y_0, x_0$ respectively.

Now suppose at the $n$-th step, we have a partial isomorphism of $\mathcal{M}$, $\tilde{h} : A_{n-1} \to B_{n-1}$ satisfying

(i) $A_{n-1} = \{ga_{-n+1}, a_{-n}, ...a_{-1}, a_0, ..., a_{n-1}, c_{n-1}, x_0, ..., x_{n-1}, z_0, ..., z_{n-1}\}$ with $\min A_{n-1} = ga_{-n+1} < a_{-n} < \cdots < ga_0 < a_{-1} < a_0 < a_1 < c_0 < \cdots < c_{n-2} < a_n < c_{n-1} = \max A_{n-1}$, where $a_i := g^{-1}c_{i-1}$ for $i = 1, ..., n$.

(ii) $B_{n-1} = \{b_{-n}, g^{-1}b_{-n}, ...., b_0, gb_0, ..., b_{n-1}, gb_{n-1}, x_0, ..., x_{n-1}, y_0, ..., y_{n-1}\}$ with $\min B_{n-1} = b_{-n} < g^{-1}b_{-n} < \cdots < b_{-1} < g^{-1}b_{-1} < b_0 < gb_0 < \cdots < b_{n-1} < gb_{n-1} = \max B_{n-1}$.

(iii) $\tilde{h}$ maps $a_i$ to $b_i$, $c_i$ to $gb_i$, $x_i$ to $y_i$ and $z_i$ to $x_i$ for all $i = 0, ..., n-1$ and $\tilde{h}$ maps $a_{-j}$ to $g^{-1}b_{-j}$ and $ga_{-j+1}$ to $b_{-j}$ for all $j = 1, ..., n$.

(iv) $[g, \tilde{h}]a_i = a_{i+1} > a_i$ for all $i = -n, ..., n-1$.

(v) $c_i > g^i a_1$, $b_i > g^{\lfloor \frac{i}{2} \rfloor} b_0$ for all $i = 1, ..., n-1$ and $a_{-j} < g^{\lfloor \frac{j}{2} \rfloor} a_0$, $b_{-j} < g^{j-1} b_{-1}$ for all $j = 2, ..., n$.

We can choose $b_n$ realising $\tilde{h}\cdot\mathrm{tp}(a_n/A_{n-1})$ and extend $\tilde{h}$ by sending $a_n$ to $b_n$. Then $gb_n > b_n$. Since $a_n > c_{n-2}$ by the inductive hypothesis and $\mathrm{tp}(b_n/gb_{n-2}) = \tilde{h}\cdot\mathrm{tp}(a_n/c_{n-2})$, we have $b_n > gb_{n-2}$. By the inductive hypothesis (v), $b_{n-2} > g^{\lfloor \frac{n-2}{2} \rfloor} b_0$. Hence, $b_n > gb_{n-2} > g^{\lfloor \frac{n}{2} \rfloor} b_0$.

We can find $c_n > x_n, g^n a_1, gc_{n-1}$ realising $\tilde{h}^{-1}\cdot\mathrm{tp}(gb_n/b_n B_{n-1})$ and extend $\tilde{h}$ by sending $c_n$ to $gb_n$. Let $a_{n+1} := g^{-1}c_n$. Then $a_{n+1} = [g, \tilde{h}]a_n > [g, \tilde{h}]a_{n-1} = a_n$ since $a_n > a_{n-1}$. We also have $a_{n+1} = g^{-1}c_n > c_{n-1}$ since $c_n > gc_{n-1}$.

Choose $b_{-n-1} < x_n, g^n b_{-1}, gb_{-n}$ realising $\tilde{h}\cdot\mathrm{tp}(ga_{-n}/a_n c_n A_{n-1})$. Extend $\tilde{h}$ by sending $ga_{-n}$ to $b_{-n-1}$. Choose $a_{-n-1}$ realising $\tilde{h}^{-1}\cdot\mathrm{tp}(g^{-1}b_{-n-1}/b_n b_{-n-1} gb_n$ $B_{n-1})$ and extend $\tilde{h}$ by sending $a_{-n-1}$ to $g^{-1}b_{-n-1}$. Then $a_{-n-1} = [\tilde{h}, g]a_{-n} < [\tilde{h}, g]a_{-n+1} = a_{-n}$ since $a_{-n} < a_{-n+1}$. Since $b_{-n-1} < gb_{-n}$ and $\mathrm{tp}(g^{-1}b_{-n-1}/b_{-n})$ $= \tilde{h}\cdot\mathrm{tp}(a_{-n-1}/ga_{-n+1})$, we have $a_{-n-1} < ga_{-n+1}$. By the inductive hypothesis (v), we have $a_{-n+1} < g^{\lfloor \frac{n-1}{2} \rfloor} a_0$. Hence, $a_{-n-1} < ga_{-n+1} < g^{\lfloor \frac{n+1}{2} \rfloor} a_0$.

Now let $A_n = A_{n-1} \cup \{a_n, c_n, ga_{-n}, a_{-n-1}\}$ and $B_n = B_{n-1} \cup \{b_{-n-1}, g^{-1}b_{-n-1}, b_n, gb_n\}$. Let $x_n$ be the first element in the list $w_0, w_1, ...$ such that $x_n \notin \{x_0, ..., x_{n-1}\}$ and $x_n$ is in $(\min A_n, \max A_n) \cap (\min B_n, \max B_n)$. Find $z_n$ realising $\tilde{h}^{-1}\cdot\mathrm{tp}(x_n/y_n B)$. Extend $A_n$,$B_n$ to include $x_n, z_n$ and $y_n, x_n$ respectively. Then, $\min B_n = b_{-n-1}$, $\max B_n = gb_n$, $\min A_n = ga_{-n}$, $\max A_n = c_n$. Hence, $\tilde{h} : A_n \to B_n$ satisfies the hypothesis (i)-(v).

Let $h$ be the union of $\tilde{h}$ over all steps. Since $g$ has a $+$-orbital unbounded above and a $-$-orbital unbounded below, by Remark 5.2.6, $(g^i a_1)_{i\in\mathbb{Z}}, (g^i b_0)_{i\in\mathbb{Z}}$ are unbounded above and $(g^i a_0)_{i\in\mathbb{Z}}, (g^i b_{-1})_{i\in\mathbb{Z}}$ are unbounded below. Hence, by the hypothesis (v), we have that $\bigcup_{n\in\mathbb{N}} (\min A_n, \max A_n) \cap (\min B_n, \max B_n) = \mathcal{M}^<$. This implies that for any $w_i$ in the list $w_0, w_1, ...$, we have $w_i = x_j$ for some $j \in \mathbb{N}$, i.e. $w_i$ is in the domain and image of $h$. Thus $h$ is bijective and $h \in \mathrm{Aut}(\mathcal{M}^<)$.

Therefore, we have that $[g, h]$ maps $a_i$ to $a_{i+1} > a_i$ for all $i \in \mathbb{Z}$ and for any $x \in \mathcal{M}^<$, there exist $n, m \in \mathbb{Z}$ such that $x < x_n < c_n < a_{n+1}$ and $x > x_m > a_{-m}$. Therefore, $[g, h]$ has a single $+$-orbital. □

For an automorphism $g \in \mathrm{Aut}(\mathcal{M}^<)$, we say $g$ is *left bounded* if there exists $a \in \mathcal{M}^<$ such that $gx = x$ for all $x < a$. We say $g$ is *right bounded* if there exists $a \in \mathcal{M}^<$ such that $gx = x$ for all $x > a$. We say $g$ is *unbounded* if it is not left bounded or right bounded.

**Theorem 5.2.12.** *Let $\mathcal{M}$ be a countable strong homogeneous $\mathcal{L}$-structure and $\mathcal{M}^<$ be its linear order expansion. Let $g$ be an unbounded automorphism of $\mathcal{M}^<$. Then there exists a product of conjugates of $g$ and $g^{-1}$ that has a single $+$-orbital. Similarly, there exists a product of conjugates of $g$ and $g^{-1}$ that has a single $-$-orbital.*

*Proof.* Suppose $g$ is an automorphism such that for any $x \in \mathcal{M}^<$, there exist $y < x$ and $z > x$ such that $gy > y$ and $gz > z$. Then, by Proposition 5.2.9, there exists $h \in \mathrm{Aut}(\mathcal{M}^<)$ such that $ghgh^{-1}$ has a single $+$-orbital. Suppose $g$ is an automorphism such that for any $x \in \mathcal{M}^<$, there exist $y < x$ and $z > x$ such that $gy < y$ and $gz < z$. Then, by applying Proposition 5.2.9 on $g^{-1}$, we can find $h \in \mathrm{Aut}(\mathcal{M}^<)$ such that $g^{-1}hg^{-1}h^{-1}$ has a single $+$-orbital.

Suppose for all $x \in \mathcal{M}^<$, there exist $y > x$ and $z < x$ such that $gy > y$ and $gz < z$. Then, by Proposition 5.2.10, we can find $h \in \mathrm{Aut}(\mathcal{M}^<)$ such that $gh^{-1}gh$ has a $+$-orbital unbounded above and a $-$-orbital unbounded below. Then by the previous proposition, we can find $k \in \mathrm{Aut}(\mathcal{M}^<)$ such that $[gh^{-1}gh, k]$ has a single $+$-orbital. Suppose for all $x \in \mathcal{M}^<$, there exist $y > x$ and $z < x$ such that $gy < y$ and $gz > z$. Then, we apply the same argument on $g^{-1}$ and find $h, k \in \mathrm{Aut}(\mathcal{M}^<)$ such that $[g^{-1}h^{-1}g^{-1}h, k]$ has a single $+$-orbital.

Since $g$ is unbounded, these are the only cases we need to consider. Hence for any $g \in \mathrm{Aut}(\mathcal{M}^<)$, there exists a product of conjugates of $g$ and $g^{-1}$ that has a single $+$-orbital. We can take the inverse of the product to obtain a product of conjugates of $g$ and $g^{-1}$ that has a single $-$-orbital. □

**Lemma 5.2.13.** *Let $\mathcal{M}$ be a non-trivial countable strong homogeneous $\mathcal{L}$-structure and $\mathcal{M}^<$ be its linear order expansion. Then for any non-trivial $g \in Aut(\mathcal{M}^<)$, $g$ does not have a non-empty open interval $I = (x, y)$ such that $ga = a$ for all $a \in I$.*

*Proof.* Since $\mathcal{M}$ is non-trivial, there is a non-trivial relation $R \in \mathcal{L}$. Suppose there exists such an open interval $I$. Then for any $b \neq c \in \mathcal{M}^< \setminus I$, we can

embed $bcxy$ into $abcxy$ where $x < a < y$ and $R(a, b)$, $\neg R(a, c)$. Then, by the Extension Property, we can embed $abcxy$ back into $\mathcal{M}^<$. So we have $ga = a$. Hence, $gb \neq c$. Therefore, $gb = b$ for any $b \in \mathcal{M}^< \setminus I$, contradicting the non-triviality of $g$. □

**Remark 5.2.14.** Hence, by the previous lemma, if $\mathcal{M}^<$ is not the dense linear order, then any non-trivial automorphism of $\mathcal{M}^<$ is unbounded.

**Corollary 5.2.15.** *Let $\mathcal{M}$ be a non-trivial countable strong homogeneous $\mathcal{L}$-structure and $\mathcal{M}^<$ be its linear order expansion. Let $g \in Aut(\mathcal{M}^<)$ be non-trivial. Then there exists a product of conjugates of $g$ and $g^{-1}$ that has a single $+$-orbital. Similarly, there exists a product of conjugates of $g$ and $g^{-1}$ that has a single $-$-orbital.*

This then immediately implies Theorem 5.2.1.

For an unbounded automorphism $g \in \mathrm{Aut}(\mathcal{M}^<)$, we have constructed $h$ as a product of conjugates of $g$ and $g^{-1}$ such that $ha > a$ for all $a \in \mathcal{M}^<$. We now prove that such $h$ has some special property, which will be useful in the later proofs.

**Lemma 5.2.16.** *Let $\mathcal{M}$ be a countable homogeneous $\mathcal{L}$-structure with a* **SWIR** $\bigcirc\!\!\!\!\!\!\downarrow^\star$ *and $\mathcal{M}^<$ be its linear order expansion. Let $p$ be an $n$-type over some finite set $X$ of the form (5.1), i.e. $p = tp(\bar{x}/X)$ where $\bar{x} \cap X = \emptyset$ and let $B$ be some finite subset of $\mathcal{M}^<$.*

*(i) Let $g \in Aut(\mathcal{M}^<)$ be such that $ga < a$ for all $a \in \mathcal{M}^<$. Then there exists $\bar{a}$ realising $p$ such that $\bar{a} \bigcirc\!\!\!\!\!\!\downarrow_X B$ and if $a_i < ga_j$ for some $a_i, a_j \in \bar{a}$, then there exists $x \in X$ such that $a_i < x < ga_j$.*

*(ii) Let $g \in Aut(\mathcal{M}^<)$ be such that $ga > a$ for all $a \in \mathcal{M}^<$. Then there exists $\bar{a}$ realising $p$ such that $B \bigcirc\!\!\!\!\!\!\downarrow_X \bar{a}$ and if $a_i > ga_j$ for some $a_i, a_j \in \bar{a}$, then there exists $x \in X$ such that $a_i > x > ga_j$.*

*Proof.* (i) Let $p$ be an $n$-type over some finite set $X$. We can write $p = \mathrm{tp}(x_1, ..., x_n/X)$ where $x_1 > \cdots > x_n$. Since $\bar{x} \cap X = \emptyset$, we can always choose $\bar{a}$ realising $p$ such that $\bar{a}X \cap g\bar{a} = \emptyset$. Let $p_k = \mathrm{tp}(x_1, ..., x_k/X)$ for $k = 1, ..., n$. We find $a_k$ inductively. For the inductive base, by Existence, we can choose $a_1$ realising $p_1$ such that $a_1 \bigcirc\!\!\!\!\!\!\downarrow_X B$. Then $ga_1 < a_1$. Now assume there exists $\bar{a}^{k-1} = (a_1, ..., a_{k-1})$ satisfying (i) $\bar{a}^{k-1}$ realises $p_{k-1}$, (ii) $\bar{a}^{k-1} \bigcirc\!\!\!\!\!\!\downarrow_X B$, and (iii)

if $a_i < ga_j$ for some $1 \leq i, j \leq k-1$, then there exists $x \in X$ such that $a_i < x < ga_j$. Note that we may assume $i, j$ are distinct as it is not possible to have $a_i < ga_i$ by the assumption on $g$.

By Existence, we can find $a_k$ realising $p_k(\bar{a}^{k-1}, x_k/X)$ and $a_k \downarrow_{\bar{a}^{k-1}X} Bg\bar{a}^{k-1}$. Then by Monotonicity, we have $a_k \downarrow_{\bar{a}^{k-1}X} B$ and by Transitivity on it and the inductive hypothesis (ii) $\bar{a}^{k-1} \downarrow_X B$, we have $\bar{a}^{k-1}a_k \downarrow_X B$. We also have $a_k \downarrow_{\bar{a}^{k-1}X} g\bar{a}^{k-1}$ by Monotonicity. Then for any $1 \leq j \leq k-1$, if $a_k < ga_j$, then either there exists $a_i \in \bar{a}^{k-1}$ such that $a_k < a_i < ga_j$ or there exists $x \in X$ such that $a_k < x < ga_j$. In the former case, by the inductive hypothesis (iii), there exists $x \in X$ such that $a_k < a_i < x < ga_j$. Hence, in both cases, we can find $x \in X$ such that $a_k < x < ga_j$. We also have that $ga_k < a_k < a_i$ for all $i = 1, ..., k-1$. Thus, we have that if $a_i < ga_j$ for some $1 \leq i, j \leq k$, then there exists $x \in X$ such that $a_i < x < ga_j$. Therefore, we have found $\bar{a}^k = (a_1, ..., a_k)$ satisfying the inductive hypothesis (i)-(iii) for $k$.

By induction, we can find a realisation of $p$ satisfying the required property. By a symmetric argument, we can prove part (ii). □

**Remark 5.2.17.** It follows from the proof that if the type $p$ is of the form $\mathrm{tp}(\bar{x}/X)$ such that $X$ is finite and for all $x_i \in \bar{X}$, $x_i \notin X$, then we can find a realisation $\bar{a}$ such that $\bar{a} \cap g\bar{a} = \emptyset$.

### 5.2.2 Linearly ordered free homogeneous structures

In this subsection, we let $\mathcal{M}$ be a non-trivial free homogeneous structure and $\mathcal{M}^<$ be its linear order expansion. As in Example 3.3.5, we can find a **SWIR** $\downarrow$ on $\mathcal{M}^<$ by defining $A \downarrow_B C$ if any tuple of $ABC$ that satisfies a relation of $\mathcal{L}$ does not contain both $a \in A \setminus B$ and $c \in C \setminus B$ and if $a < c$, then there exists $b \in B$ such that $a < b < c$.

To be able to apply Corollary 4.2.4, we will show in the next theorem that for any non-trivial $g \in \mathrm{Aut}(\mathcal{M}^<)$, we can find a product of conjugates of $g$ and $g^{-1}$ that moves $L$-maximally and its inverse moves $R$-maximally. We first prove the following lemma.

**Lemma 5.2.18.** *Let $\mathcal{M}^<$ be the linear order expansion of a non-trivial free homogeneous structure. Let $A, B, B', C$ be finite substructures of $\mathcal{M}^<$ such that*

*$A\cap B=\emptyset$ and $C\cap B=\emptyset$. Suppose $A \mathop{\smile\kern-0.7em|}_{BB'} C$ and for any $b\in B\backslash B', c\in C\backslash B'$, if $b<c$, then there exists $b'\in B$ such that $b<b'<c$. Then $A \mathop{\smile\kern-0.7em|}_{B'} C$.*

*Proof.* For $a\in A\setminus B'$ and $c\in C\setminus B'$, we have $a\in A\setminus BB'$ and $c\in C\setminus BB'$ since $A\cap B=\emptyset$ and $C\cap B=\emptyset$. Then since $A \mathop{\smile\kern-0.7em|}_{BB'} C$, $a,c$ are not related by any relation in $\mathcal{L}$. If $a<c$, then since $A \mathop{\smile\kern-0.7em|}_{BB'} C$, either there exists $b\in B\setminus B'$ such that $a<b<c$ or there exists $b'\in B'$ such that $a<b'<c$. In the former case, by the assumption, there exists $b'\in B'$ such that $a<b<b'<c$. Therefore, we have $A \mathop{\smile\kern-0.7em|}_{B'} C$. □

**Theorem 5.2.19.** *Let $g\in$Aut$(\mathcal{M}^<)$ be such that $ga<a$ for all $a\in\mathcal{M}^<$. Then there exists $k\in$Aut$(\mathcal{M}^<)$ such that $[k,g]$ moves almost $R$-maximally and $[g,k]$ moves almost $L$-maximally.*

*Proof.* We use the back-and-forth method. List all types of the form (5.1) as $p_1,p_2,\ldots$. Suppose at some stage, we have a partial isomorphism $\tilde{k}:A\to B$ such that $[\tilde{k},g]$ moves $p_1,\ldots,p_{i-1}$ almost $R$-maximally and $[g,\tilde{k}]$ moves $p_1,\ldots,p_{i-1}$ almost $L$-maximally. Suppose $p_i=$tp$(\bar{x}/X)$. By Lemma 5.0.1, we may assume $p_i$ is non-algebraic and is of the form (5.1), i.e. $\bar{x}\cap X=\emptyset$. Then we can always find realisation $\bar{a}$ such that $\bar{a}\cap g\bar{a}=\emptyset$. We may also assume that $X\cup g(X)\subseteq A$, $g\tilde{k}X\subseteq B$ by extending $\tilde{k}$.

Step 1. We extend $\tilde{k}$ so that $[\tilde{k},g]$ moves $p_i$ almost $R$-maximally. By Lemma 5.2.16, there exists $\bar{a}$ realising $p_i$ such that $\bar{a} \mathop{\smile\kern-0.7em|}_X A$ and if $a_i<ga_j$ for some $a_i,a_j\in\bar{a}$, then there exists $x\in X$ such that $a_i<x<ga_j$. By Remark 5.2.17, we may assume $\bar{a}\cap g\bar{a}=\emptyset$. Since the type $p_i$ is of the form (5.1), we may also assume $\bar{a}\cap A=\emptyset$.

By Existence, there exists $\bar{b}$ realising $\tilde{k}\cdot$tp$(\bar{a}/A)$ such that $\bar{b} \mathop{\smile\kern-0.7em|}_B g^{-1}B$. Since $\tilde{k}\cdot$tp$(\bar{a}/A)$ is non-algebraic, we may assume $\bar{b}\cap g\bar{b}=\emptyset$ by Remark 5.2.17. Extend $\tilde{k}$ by sending $\bar{a}$ to $\bar{b}$. By acting $\tilde{k}$ on $\bar{a} \mathop{\smile\kern-0.7em|}_X A$, we get $\bar{b} \mathop{\smile\kern-0.7em|}_{\tilde{k}X} B$.

Again by Existence, there exists $\bar{c}$ realising $\tilde{k}^{-1}\cdot$tp$(g\bar{b}/\bar{b}B)$ such that $\bar{c} \mathop{\smile\kern-0.7em|}_{\bar{a}A} g\bar{a}$. Extend $\tilde{k}$ by sending $\bar{c}$ to $g\bar{b}$. Then we have $\bar{c}\cap\bar{a}=\emptyset$ since $\bar{b}\cap g\bar{b}=\emptyset$. Since if $a_i<ga_j$ for some $a_i,a_j\in\bar{a}$, then there exists $x\in X$ such that $a_i<x<ga_j$, we have $\bar{c} \mathop{\smile\kern-0.7em|}_A g\bar{a}$ by Lemma 5.2.18.

By Transitivity on $\bar{b} \mathop{\smile\kern-0.7em|}_{\tilde{k}X} B$ and $\bar{b} \mathop{\smile\kern-0.7em|}_B g^{-1}B$, we have $\bar{b} \mathop{\smile\kern-0.7em|}_{\tilde{k}X} g^{-1}B$. Acting by $\tilde{k}^{-1}g$ on it, we get $\tilde{k}^{-1}g\bar{b} \mathop{\smile\kern-0.7em|}_{\tilde{k}^{-1}g\tilde{k}X} \tilde{k}^{-1}B$, which can be simplified to

$\bar{c} \mathop{\downarrow}_{\tilde{k}^{-1}g\tilde{k}X} A$. Since $\tilde{k}^{-1}g\tilde{k}X \subseteq A$, we can apply Transitivity on $\bar{c} \mathop{\downarrow}_{\tilde{k}^{-1}g\tilde{k}X} A$ and $\bar{c} \mathop{\downarrow}_A g\bar{a}$ to obtain $\bar{c} \mathop{\downarrow}_{\tilde{k}^{-1}g\tilde{k}X} g\bar{a}$. Acting by $\tilde{k}^{-1}g^{-1}\tilde{k}$ on it, we have $\bar{a} \mathop{\downarrow}_X [\tilde{k}, g]\bar{a}$.

Step 2. We extend $\tilde{k}$ so that $[g, \tilde{k}]$ moves $p_i$ almost $L$-maximally. Since $g^{-1}a > a$ for all $a \in \mathcal{M}^<$, by Lemma 5.2.16(ii), we can find $\bar{a}$ realising $p$ such that $g^{-1}(A) \mathop{\downarrow}_X \bar{a}$ and if $a_i > g^{-1}a_j$ for some $a_i, a_j \in \bar{a}$, then there exists $x \in X$ such that $a_i > x > g^{-1}a_j$. Hence, if $ga_i > a_j$ for some $a_i, a_j \in a$, then there exists $x \in X$ such that $ga_i > gx > a_j$. We may assume $\bar{a} \cap g\bar{a} = \emptyset$. By Invariance, we have $A \mathop{\downarrow}_{g(X)} g\bar{a}$.

By Existence and the fact that $\mathrm{tp}(\bar{a}/A)$ is non-algebraic, we can find $\bar{b}$ realising $\tilde{k}\cdot\mathrm{tp}(\bar{a}/A)$ such that $\bar{b} \cap g\bar{b} = \emptyset$ and $\bar{c}$ realising $\tilde{k}^{-1}\cdot\mathrm{tp}(g\bar{b}/\bar{b}B)$ such that $\bar{c} \mathop{\downarrow}_{\bar{a}A} g\bar{a}$. Extend $\tilde{k}$ by sending $\bar{a}$ to $\bar{b}$ and $\bar{c}$ to $g\bar{b}$. Then we have $\bar{c} \cap \bar{a} = \emptyset$.

Since $g(X) \subseteq A$, $\bar{c} \cap \bar{a} = \bar{a} \cap g\bar{a} = \emptyset$ and if $ga_i > a_j$ for some $a_i, a_j \in \bar{a}$, then there exists $x \in X$ such that $ga_i > gx > a_j$, we obtain $\bar{c} \mathop{\downarrow}_A g\bar{a}$ from $\bar{c} \mathop{\downarrow}_{\bar{a}A} g\bar{a}$ by Lemma 5.2.18. Then by Transitivity on $A \mathop{\downarrow}_{g(X)} g\bar{a}$ and $\bar{c} \mathop{\downarrow}_A g\bar{a}$, we have $\bar{c} \mathop{\downarrow}_{g(X)} g\bar{a}$. Therefore, acting by $g^{-1}$ on it, we obtain $[g, \tilde{k}]\bar{a} \mathop{\downarrow}_X \bar{a}$.

We can also make sure $X \subset B$ by extending $\tilde{k}$. Let $k$ be the union of all $\tilde{k}$ over all $i$. Then $k$ is bijective and thus, $k \in\mathrm{Aut}(\mathcal{M}^<)$. Then $[k, g]$ moves almost $R$-maximally and $[g, k]$ moves almost $L$-maximally. □

Therefore, we obtain the following corollary.

**Corollary 5.2.20.** *Let $\mathcal{M}^<$ be the linear order expansion of a non-trivial free homogeneous structure. Then Aut($\mathcal{M}^<$) is simple.*

*Proof.* Let $g$ be a non-trivial automorphism of $\mathcal{M}^<$. Then, by Theorem 5.2.12, there exists $h \in\mathrm{Aut}(\mathcal{M}^<)$ as a product of conjugates of $g$ and $g^{-1}$, such that $ha < a$ for all $a \in \mathcal{M}^<$. By the previous lemma, there exists $k \in\mathrm{Aut}(\mathcal{M}^<)$ such that $[k, h]$ moves almost $R$-maximally and $[h, k]$ moves almost $L$-maximally. Therefore, by Corollary 4.2.4, any element of $\mathrm{Aut}(\mathcal{M}^<)$ can be written as a product of conjugates of $g$ and $g^{-1}$. Thus, $\mathrm{Aut}(\mathcal{M}^<)$ is simple. □

### 5.2.3 Universal $n$-linear order

We now prove the same result for the universal $n$-linear orders in this subsection. As in Example 3.3.5, we can find a **SWIR** on the universal $n$-linear order $\mathcal{M}_n$, i.e. for finite substructures $A, B, C \subseteq \mathcal{M}_n$, define $A \downarrow_B C$ if $a <_i c$ for any $a \in A \setminus B, c \in C \setminus B$ and for some $i$, there exists $b \in B$ such that $a <_i b <_i c$.

We first look at the case where $n = 1$, which is the dense linear order $(\mathbb{Q}, \leq)$. The normal subgroup structure of $\mathrm{Aut}(\mathbb{Q}, \leq)$ is well-known. According to [BD85], Higman showed that the subgroup consisting of all bounded automorphisms is simple in [Hig51]. [Hol63] and [Llo64] studied the normal subgroup generated by unbounded automorphisms. We illustrate in the following how Corollary 4.2.4 can be applied to obtain an alternative proof of the same statement.

**Corollary 5.2.21.** *Let $g$ be an unbounded automorphism of $(\mathbb{Q}, \leq)$. Then the normal subgroup generated by $g$ is Aut$(\mathbb{Q}, \leq)$.*

*Proof.* Let $g$ be an unbounded automorphism of $(\mathbb{Q}, \leq)$. By Theorem 5.2.12, there exists $h \in \mathrm{Aut}(\mathbb{Q}, \leq)$, a product of conjugates of $g$ and $g^{-1}$, such that $ha < a$ for all $a \in \mathbb{Q}$. Let $p$ be an $n$-type over some finite set $X \subseteq \mathbb{Q}$. Then by Lemma 5.2.16, we can find $\bar{a}$ realising $p$ such that if $a_i < ha_j$ for some $a_i \in \bar{a} \setminus X, ha_j \in h\bar{a} \setminus X$, then there exists $x \in X$ such that $a_i < x < ha_j$. Hence $\bar{a} \downarrow_X h\bar{a}$ by definition. Similarly, we can find $\bar{a}'$ realising $p$ such that $h^{-1}\bar{a}' \downarrow_X \bar{a}'$. So, $h$ moves $R$-maximally and $h^{-1}$ moves $L$-maximally. Therefore, by Corollary 4.2.4, $\mathrm{Aut}(\mathbb{Q}, \leq)$ can be generated by products of conjugates of $g$ and $g^{-1}$. □

For $n \geq 2$, we will use a similar approach to find an automorphism that moves almost $R$-maximally and its inverse moves almost $L$-maximally. We will do this by induction and the back-and-forth method. We first define the following notions and prove the next two lemmas.

**Definition 5.2.22.** Let $A$ be a finite subset of $\mathcal{M}_n$ and $b \in \mathcal{M}_n \setminus A$. We define the *upper constraint* of $b$ with respect to $A$ on order $<_i$ to be $\min_i\{a >_i b | a \in A\}$ and the *lower constraint* of $b$ with respect to $A$ on $<_i$ to be $max_i\{a <_i b | a \in A\}$.

**Definition 5.2.23.** Let $\mathcal{M}_n$ be the universal $n$-linear order and $I \subseteq \{1, \ldots, n\}$. Define $\mathcal{M}_n^I$ to be the structure $\mathcal{M}_n$ but with the ordering $<_i$ reversed for every $i \in I$.

**Remark 5.2.24.** We have $\mathcal{M}_n^I \cong \mathcal{M}_n$ for any $I \subseteq \{1, ..., n\}$. Hence, Aut($\mathcal{M}_n$) =Aut($\mathcal{M}_n^I$).

**Lemma 5.2.25.** *Let $\mathcal{M}_n$ be the universal $n$-linear order and $a_1, ..., a_n, c_1, ..., c_n \in \mathcal{M}_n$ be such that $a_i <_i c_i$ for all $i = 1, ..., n$. Then there exists $b \in \mathcal{M}_n$ such that $a_i <_i b <_i c_i$ for all $i = 1, ..., n$.*

*Proof.* This follows from the Extension Property: we can embed $a_1...a_nc_1...c_n \in \mathcal{M}_n$ into $a_1...a_nc_1...c_nb \in \mathcal{C}$ where $a_i <_i b <_i c_i$ for all $i = 1, ..., n$ and $\mathcal{C}$ is the class of all finite $n$-linear orders. Then by the Extension Property, we can embed $a_1...a_nc_1...c_nb$ into $\mathcal{M}_n$. Hence, there exists $b \in \mathcal{M}_n$ such that $a_i <_i b <_i c_i$ for all $i = 1, ..., n$. □

**Lemma 5.2.26.** *Suppose $g, g' \in Aut(\mathcal{M}_n)$ have a single $+$-orbital on $<_i$ for some $i = 1, ..., n$. Then*

*(i) $g'g$ has a single $+$-orbital on $<_i$.*

*(ii) Any product of conjugates of $g$ has a single $+$-orbital on $<_i$.*

*Similarly, any product of conjugates of automorphisms that has a single $-$-orbital on $<_i$ has a single $-$-orbital on $<_i$.*

*Proof.* Fix some $i = 1, ..., n$. Since $g$ has a single $+$-orbital on $<_i$, there exists $(a_j)_{j\in\mathbb{Z}}$ such that $ga_j = a_{j+1} >_i a_j$ and $\cup_{j\in\mathbb{Z}}[a_j, a_{j+1})_i = \mathcal{M}_n$.

(i) Since $g'x >_i x, gx >_i x$ for all $x \in \mathcal{M}_n$, we have $g'ga_j = g'a_{j+1} >_i a_{j+1}$ for $j \in \mathbb{Z}$. Hence $(a_j, g'ga_j)_i$ intersects with $(a_{j+1}, g'ga_{j+1})_i$ for all $j \in \mathbb{Z}$. Since $g'ga_j >_i a_j$ for all $j \in \mathbb{Z}$, $g'g$ has a single $+$-orbital on $<_i$.

(ii) Let $h \in$Aut($M_n$). Then for all $j \in \mathbb{Z}$, $hgh^{-1}(ha_j) = hga_j = ha_{j+1}$. Since $a_{j+1} >_i a_j$, we have $ha_{j+1} >_i ha_j$. Since $\cup_{j\in\mathbb{Z}}[a_j, a_{j+1})_i = \mathcal{M}_n$, we also have $\cup_{j\in\mathbb{Z}}[ha_j, ha_{j+1})_i = \mathcal{M}_n$. Hence $hgh^{-1}$ has a single $+$-orbital on $<_i$. By part (i), any product of conjugates of $g$ has a single $+$-orbital on $<_i$.

□

To simplify some notations in the following proof, we denote $(a,b)_i$ for some $a,b \in \mathcal{M}_n$ and $i \in \{1,...,n\}$ to be the open interval $(a,b)$ on $<_i$ if $a <_i b$ and $(b,a)$ on $<_i$ if $b <_i a$.

**Theorem 5.2.27.** *Let $g \in Aut(\mathcal{M}_n)$, $n \geq 2$, be non-trivial. Then there exists $f$, a product of at most $4^n$ conjugates of $g$ and $g^{-1}$, and $I \subseteq \{1,...,n\}$ such that $f$ has a single $+$-orbital on $<_j$ for all $j = 1,...,n$ in $\mathcal{M}_n^I$.*

*Proof.* We prove the theorem by induction on $n$. For the inductive base, by Theorem 5.2.12, there exists $h' \in \mathrm{Aut}(\mathcal{M}_n)$, a product of conjugates of $g$ and $g^{-1}$, such that $h'$ has a single $+$-orbital on $<_1$ in $\mathcal{M}_n^{I_1}$, where $I_1 = \emptyset$.

Now for the inductive step, suppose $h \in \mathrm{Aut}(\mathcal{M}_n^{I_{m-1}})$ is such that $h$ can be written as a product of conjugates of $g$ and $g^{-1}$ and has a single $+$-orbital on $<_j$ for all $j = 1,...,m-1$ in $\mathcal{M}_n^{I_{m-1}}$ for some $I_{m-1} \subseteq \{1,...,m-1\}$. Since $\mathcal{M}_n^I \cong \mathcal{M}_n$, we may assume without loss of generality, that $I_{m-1} = \emptyset$. We will show that there exists $k \in \mathrm{Aut}(\mathcal{M}_n^{I_m})$, written as a product of conjugates of $h$ and $h^{-1}$, such that $k$ has a single $+$-orbital on $<_j$ for all $i = 1,...,m$ in $\mathcal{M}_n^{I_m}$ for some $I_m \subseteq \{1,...,m\}$. There are four possible cases to consider.

Case I. For any $x \in \mathcal{M}_n$, there exists $y <_m x$ and $z >_m x$ such that $hy >_m y$ and $hz >_m z$.

In this case, we can apply Proposition 5.2.9 and find $k_1 \in \mathrm{Aut}(\mathcal{M}_n)$ such that $hk_1hk_1^{-1}$ has a single $+$-orbital on $<_m$. By the previous lemma, $hk_1hk_1^{-1}$ also has a single $+$-orbital on $<_j$ for $j = 1,...,m-1$. Then $k = hk_1hk_1^{-1}$ has a single $+$-orbital on $<_j$ for all $i = 1,...,m$ in $\mathcal{M}_n$.

Case II. For any $x \in \mathcal{M}_n$, there exists $y <_m x$ and $z >_m x$ such that $hy <_m y$ and $hz <_m z$.

Let $I_m = \{m\}$. Then for any $x \in \mathcal{M}_n^{I_m}$, there exists $y <_m x$ and $z >_m x$ such that $hy >_m y$ and $hz >_m z$. We can again apply Proposition 5.2.9 to $h$ and find $k_1 \in \mathrm{Aut}(\mathcal{M}_n^{I_m})$ such that $hk_1hk_1^{-1}$ has a single $+$-orbital on $<_m$. By the same argument as in case I, $k = hk_1hk_1^{-1}$ has a single $+$-orbital on $<_j$ for all $i = 1,...,m$ in $\mathcal{M}_n^{I_m}$.

Case III. For any $x \in \mathcal{M}_n$, there exists $y >_m x$ and $z <_m x$ such that $hy <_m y$ and $hz >_m z$.

Let $I_m = I_{m-1} = \emptyset$. By Proposition 5.2.10, we can find $k_1 \in \mathrm{Aut}(\mathcal{M}_n)$

such that $h^{-1}k_1^{-1}h^{-1}k_1$ has a $+$-orbital unbounded above and a $-$-orbital unbounded below with respect to $<_m$. Let $k_2 = k_1^{-1}hk_1h$. Then there exists $y, z \in \mathcal{M}_n$ such that $k_2a <_m a$ for all $a >_m y$ and $k_2a >_m a$ for all $a <_m z$. By the previous lemma, we also have that $k_2$ has a single $+$-orbital on $<_j$ for all $j = 1, ..., m-1$. We want to construct $k_3$ using the back-and-forth method such that $[k_2, k_3]$ has a single $+$-orbital on $<_j$ for all $j = 1, ..., m$. We start with $\tilde{k}$ as the empty map and extend $\tilde{k}$ so it approximates to $k_3$.

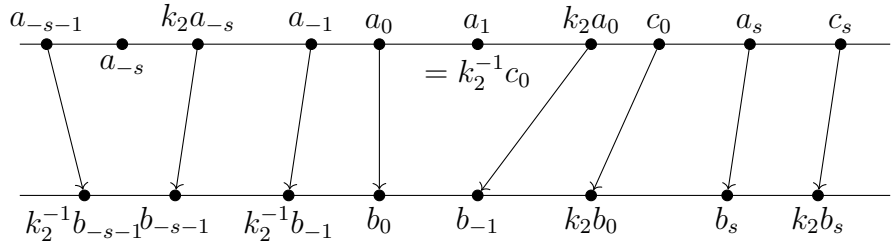



Figure 5.5: Illustration of case III and $<_j$, where $j = 1, ..., m-1$

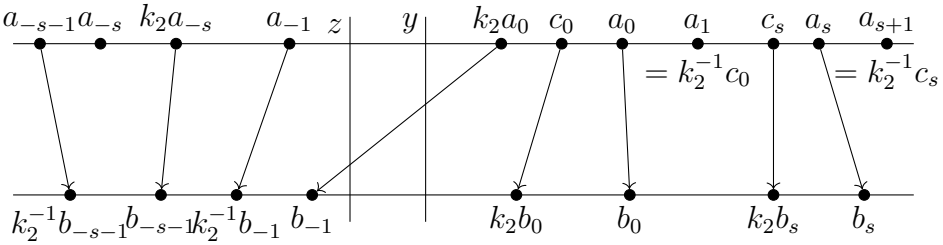



Figure 5.6: Construction of $k_3$

List all elements of $\mathcal{M}_n$ as $w_0, w_1, ....$ Choose $a_0 >_m y, w_0$. By Lemma 5.2.25, we can find $b_0 \in \mathcal{M}_n$ such that $a_0 <_j b_0 <_j k_2a_0$ for $j = 1, ..., m-1$ and $b_0 >_m y, w_0$. Then $k_2b_0 <_m b_0$ and $(a_0, k_2a_0)_j \cap (b_0, k_2b_0)_j \neq \emptyset$ for all $j = 1, ..., m-1$. Extend $\tilde{k}$ by sending $a_0$ to $b_0$. Since $\tilde{k}^{-1}\cdot\text{tp}(k_2b_0/b_0) \cup \{x >_j k_2a_0 | j = 1, ..., m\}$ is consistent, we can find $c_0 >_j k_2a_0$ for all $j = 1, ..., m$ realising $\tilde{k}^{-1}\cdot\text{tp}(k_2b_0/b_0)$. Extend $\tilde{k}$ by sending $c_0$ to $k_2b_0$. Let $a_1 := k_2^{-1}c_0$. Then for all $j = 1, ..., m$, we have $a_1 = k_2^{-1}c_0 = [k_2, \tilde{k}]a_0 >_j a_0$ since $c_0 >_j k_2a_0$. Similarly, since $\tilde{k}\cdot\text{tp}(k_2a_0/c_0a_0) \cup \{x <_m z, x_0\}$ is consistent, we can choose $b_{-1} <_m z, x_0$ realising $\tilde{k}\cdot\text{tp}(k_2a_0/c_0a_0)$. Choose $a_{-1} <_m z, x_0$ realising $\tilde{k}^{-1}\cdot\text{tp}(k_2^{-1}b_{-1}/b_{-1}b_0k_2b_0)$. Extend $\tilde{k}$ by sending $k_2a_0$ to $b_{-1}$ and $a_{-1}$ to $k_2^{-1}b_{-1}$. Then for all $j = 1, ..., m$, we have $a_{-1} = [k_2, \tilde{k}]^{-1}a_0 <_j [k_2, \tilde{k}]^{-1}a_1 = a_0$ since $a_0 <_j a_1$.

Let $A_0 = \{a_{-1}, k_2a_0, c_0, a_0\}$ and $B_0 = \{k_2^{-1}b_{-1}, b_{-1}, k_2b_0, b_0\}$. Then $S :=$

$\bigcap_{j=1}^{m}(\min_j A_0, \max_j A_0)_j \cap (\min_j B_0, \max_j B_0)_j$ is non-empty. So, we can let $x_0$ be the first element in the list $w_0, w_1, ...$ such that $x_0$ is in $S$. Choose $y_0$ realising $\tilde{k}\cdot\mathrm{tp}(x_0/A_0)$ and extend $\tilde{k}$ by sending $x_0$ to $y_0$. Choose $z_0$ realising $\tilde{k}^{-1}\cdot\mathrm{tp}(x_0/y_0B_0)$ and extend $\tilde{k}$ by sending $z_0$ to $x_0$. Extend $A_0$,$B_0$ to include $x_0, z_0$ and $y_0, x_0$ respectively.

Now suppose at the $s$-th step, we have a partial isomorphism of $\mathcal{M}_n$, $\tilde{k} : A_{s-1} \to B_{s-1}$ satisfying

(i) $A_{s-1} = \{a_{-s}, k_2a_{-s+1}, ..., c_{s-1}, a_{s-1}, x_0, ..., x_{s-1}, z_0, ..., z_{s-1}\}$,
$B_{s-1} = \{k_2^{-1}b_{-s}, b_{-s}, ...., k_2b_{s-1}, b_{s-1}, x_0, ..., x_{s-1}, y_0, ..., y_{s-1}\}$

(ii) $\tilde{k}$ maps $a_i$ to $b_i$, $c_i$ to $k_2b_i$, $x_i$ to $y_i$ and $z_i$ to $x_i$ for all $i = 0, ..., s-1$ and $\tilde{k}$ maps $a_{-i}$ to $k_2^{-1}b_{-i}$ and $k_2a_{-i+1}$ to $b_{-i}$ for all $i = 1, ..., s$,

(iii) $a_i := k_2^{-1}c_{i-1}$ for $i = 1, ..., s$ and $[k_2, \tilde{k}]a_i = a_{i+1} >_j a_i$ for all $i = -s, ..., s-1$ and $j = 1, ..., m$,

(iv) on order $<_j$, where $j = 1, ..., m-1$, we have

(a) $a_i >_j k_2^{i-1}a_0$, $a_{-i} <_j k_2^{-i+1}a_0$, $b_i >_j k_2^{\lfloor \frac{i}{2} \rfloor - 1}b_0$, $b_{-i} <_j k_2^{-\lfloor \frac{i}{2} \rfloor + 1}b_0$ for $i = 0, ..., s-1$,

(b) $\min_j A_{s-1} = a_{-s} <_j k_2a_{-s} <_j a_{-s+1} <_j k_2a_{-s+1} <_j \cdots <_j a_{s-1} <_j c_{s-2} <_j a_s <_j c_{s-1} = \max_j A_{s-1}$,

(c) $\min_j B_{s-1} = k_2^{-1}b_{-s} <_j k_2^{-1}b_{-s+1} <_j b_{-s} \cdots <_j b_{s-1} <_j k_2b_{s-2} <_j k_2b_{s-1} = \max_j B_{s-1} = k_2b_{s-1}$.

(v) on $<_m$, we have

(a) $a_i >_m k_2^{-\lfloor \frac{i}{2} \rfloor + 1}a_0$, $a_{-i-1} <_m k_2^{-i+1}a_{-1}$, $b_i >_m k_2^{-i+1}b_0$, $b_{-i-1} <_m k_2^{-\lfloor \frac{i}{2} \rfloor + 1}b_{-1}$ for $i = 0, ..., s-1$,

(b) $\min_j A_{s-1} = a_{-s} <_m k_2a_{-s} <_m a_{-s+1} <_m k_2a_{-s+1} <_m \cdots <_m a_{s-2} <_m c_{s-1} <_m a_{s-1} = \max_m A_{s-1}$, and

(c) $\min_j B_{s-1} = k_2^{-1}b_{-s} <_m k_2^{-1}b_{-s+1} <_m b_{-s} <_m \cdots <_m k_2b_{s-1} <_m b_{s-1} = \max_j B_{s-1}$,

We can choose $b_s$ such that $b_s >_m k_2^{-s+1}b_0, k_2^{-1}b_{s-1}$ and $b_s$ realises $\tilde{k}\cdot\mathrm{tp}(a_s/A_{s-1})$ since the type $\tilde{k}\cdot\mathrm{tp}(a_s/A_{s-1}) \cup \{x >_m k_2^{-s+1}b_0, k_2^{-1}b_{s-1}\}$ is consistent. Extend $\tilde{k}$ by sending $a_s$ to $b_s$. Similarly, we can find $c_s$ such that $c_s >_j k_2c_{s-1}, k_2^{s+1}a_0$ for

all $j = 1, ..., m-1$ and $c_s$ realises $\tilde{k}^{-1}\cdot\text{tp}(k_2b_s/b_sB_{s-1})$. Extend $\tilde{k}$ by sending $c_s$ to $k_2b_s$. Let $a_{s+1} := k_2^{-1}c_s$. Then we have the following:

(i) for all $j = 1, ..., m$, we have $a_{s+1} = [k_2, \tilde{k}]a_s >_j [k_2, \tilde{k}]a_{s-1} = a_s$ since $a_s >_j a_{s-1}$.

(ii) Since for any $j = 1, ..., m-1$, $c_s >_j k_2^{s+1}a_0, k_2c_{s-1}$, we have $a_{s+1} = k_2^{-1}c_s >_j k_2^s a_0, c_{s-1}$.

(iii) For $j = 1, ..., m-1$, since by the inductive hypothesis that $a_s >_j c_{s-2}$ and $\tilde{k}$ maps $a_s$ to $b_s$ and $c_{s-2}$ to $k_2b_{s-2}$, we have $b_s >_j k_2b_{s-2}$. By the inductive hypothesis that $b_{s-2} >_m k_2^{\lfloor\frac{s-2}{2}\rfloor-1}b_0$, we have $b_s >_j k_2^{\lfloor\frac{s}{2}\rfloor-1}b_0$.

(iv) Since $b_s >_m k_2^{-1}b_{s-1}$, we have $k_2b_s >_m b_{s-1}$. Since $\tilde{k}$ maps $c_s$ to $k_2b_s$ and $a_{s-1}$ to $b_{s-1}$, we have $c_s >_m a_{s-1}$. By the inductive hypothesis that $a_{s-1} >_m k_2^{-\lfloor\frac{s-1}{2}\rfloor+1}a_0$, we have $a_{s+1} = k_2^{-1}c_s >_m k_2^{-1}a_{s-1} >_m k_2^{-\lfloor\frac{s-1}{2}\rfloor}a_0$.

Similarly, we can choose $b_{-s-1}$ such that $b_{-s-1} <_j k_2b_{-s}, k_2^s b_0$ for all $j = 1, ..., m-1$ and $b_{-s-1}$ realises $\tilde{k}\cdot\text{tp}(k_2a_{-s}/a_sc_sA_{s-1})$. Extend $\tilde{k}$ by sending $k_2a_{-s}$ to $b_{-s-1}$. Choose $a_{-s-1} <_m k_2^{-1}a_{-s}, k_2^{-s}a_0$ realising $\tilde{k}^{-1}\cdot\text{tp}(k_2^{-1}b_{-s-1}/b_sb_{-s-1}$ $k_2b_sB_{s-1})$.

Choose $b_{-s-1} <_m k_2b_{-s}$ realising $\tilde{k}\cdot\text{tp}(k_2a_{-s}/a_sc_sA_{s-1})$. Extend $\tilde{k}$ by sending $k_2a_{-s}$ to $b_{-s-1}$. Choose $a_{-s-1}$ such that $a_{-s-1} <_j k_2^{-1}a_{-s}, k_2^{-s}a_0$ for all $j = 1, ..., m$ and $a_{-s-1}$ realises $\tilde{k}^{-1}\cdot\text{tp}(k_2^{-1}b_{-s-1}/b_sb_{-s-1}k_2b_sB_{s-1})$. Extend $\tilde{k}$ by sending $a_{-s-1}$ to $k_2^{-1}b_{-s-1}$. Then we have the following for all $j = 1, ..., m$:

(i) $a_{-s-1} = [k_2, \tilde{k}]^{-1}a_{-s} <_j [k_2, \tilde{k}]^{-1}a_{-s+1} = a_{-s}$ since $a_{-s} <_j a_{-s+1}$.

(ii) By the inductive hypothesis that $k_2a_{-s} <_j a_{-s+1}$ and since $\tilde{k}$ maps $k_2a_{-s}$ to $b_{-s-1}$ and $a_{-s+1}$ to $k_2^{-1}b_{-s+1}$, we have $b_{-s-1} <_j k_2^{-1}b_{-s+1}$. By the inductive hypothesis that $b_{-s+1} <_j k_2^{-\lfloor\frac{s-1}{2}\rfloor+1}b_0$, we have $b_{-s-1} <_j k_2^{-\lfloor\frac{s+1}{2}\rfloor+1}b_0$.

(iii) $k_2a_{-s-1} <_j a_{-s}$ since $a_{-s-1} <_j k_2^{-1}a_{-s}$.

Let $A_s = A_{s-1} \cup \{a_s, c_s, k_2a_{-s}, a_{-s-1}\}$ and $B_s = B_{s-1} \cup \{k_2^{-1}b_{-s-1}, b_{-s-1}, k_2b_s,$ $b_s\}$. Let $x_s$ be the first element in the list $w_0, w_1, ....$ such that $x_s \notin \{x_0, ..., x_{s-1}\}$ and $x_s$ is in $\bigcap_{i=1}^m (\min_i A_s, \max_i A_s)_i \cap (\min_i B_s, \max_i B_s)_i$. Find $y_s$ realising $\tilde{k}\cdot\text{tp}(x_s/A_s)$ and extend $\tilde{k}$ by sending $x_s$ to $y_s$. Find $z_s$ realising $\tilde{k}^{-1}\cdot\text{tp}(x_s/y_sB_s)$. Extend $A_s$,$B_s$ to include $x_s, z_s$ and $y_s, x_s$ respectively. Then, $\tilde{k} : A_s \to B_s$ satisfies the inductive hypothesis (i)-(v).

Let $k_3$ be the union of $\tilde{k}$ over all steps. Since $k_2$ has a singe $+$-orbital on $<_j$ for all $j = 1, ..., m-1$ and a $-$-orbital unbounded above and a $+$-orbital unbounded below on $<_m$. By Remark 5.2.6, $(k_2^i a_0)_{i\in\mathbb{Z}}, (k_2^i b_0)_{i\in\mathbb{Z}}$ are unbounded above and below on $<_j$ for all $j = 1, ..., m-1$. $(k_2^i a_0)_{i\in\mathbb{Z}}, (k_2^i b_0)_{i\in\mathbb{Z}}$ are unbounded above on $<_m$ and $(k_2^i a_{-1})_{i\in\mathbb{Z}}, (k_2^i b_{-1})_{i\in\mathbb{Z}}$ are unbounded below on $<_m$ Then by the inductive hypothesis (iv.a) and (v.a), we know that $(a_i)_{i\in\mathbb{Z}}$ is unbounded above and below on $<_j$ for all $j = 1, ..., m$ and

$$\bigcup_{n\in\mathbb{N}} \big( \cap_{j=1}^{m} (\min_{<_j} A_n, \max_{<_j} A_n)_j \cap (\min_{<_j} B_n, \max_{<_j} B_n)_j \big) = \mathcal{M}^{<}.$$

Thus, for any $w_i$ in the list $w_0, w_1, ...$, $w_i = x_j$ for some $j \in \mathbb{N}$, i.e. $w_i$ is in the domain and image of $k_3$. Hence, $k_3$ is bijective and $k_3 \in \mathrm{Aut}(\mathcal{M}_n^{I_m})$. Since $(a_i)_{i\in\mathbb{Z}}$, which is also $([k_2, k_3]^i a_0)_{i\in\mathbb{Z}}$, is unbounded above and below on $<_j$ for all $j = 1, ..., m$, $[k_2, k_3]$ has a single $+$-orbital on $<_j$ for all $j = 1, ..., m$ on $\mathcal{M}_n^{I_m}$.

Case IV. For any $x \in \mathcal{M}_n^{I_{m-1}}$, there exists $y >_m x$ and $z <_m x$ such that $hy >_m y$ and $hz <_m z$.

Let $I_m = I_{m-1} \cup \{m\}$. Then for any $x \in \mathcal{M}_n^{I_m}$, there exists $y >_m x$ and $z <_m x$ such that $hy <_m y$ and $hz >_m z$. By Proposition 5.2.10, we can find $k_1 \in \mathrm{Aut}(\mathcal{M}_n^{I_m})$ such that $h^{-1}k_1^{-1}h^{-1}k_1$ has a $+$-orbital unbounded above and a $-$-orbital unbounded below with respect to $<_m$. Let $k_2 = k_1^{-1}hk_1h$. Then by the same argument as in case III, there exists $k_3 \in \mathrm{Aut}(\mathcal{M}_n^{I_m})$ such that $[k_2, k_3]$ has a single $+$-orbital on $<_j$ for all $j = 1, ..., m$ on $\mathcal{M}_n^{I_m}$.

Therefore, by induction, we can find $f \in \mathrm{Aut}(\mathcal{M}_n^I)$, as a product of conjugate of $g$ and $g^{-1}$, such that $f$ has a single $+$-orbital on $<_j$ for all $j = 1, ..., n$ in $\mathcal{M}_n^I$ for some $I \subseteq \{1, ..., n\}$. □

We now show that for an automorphism $g \in \mathcal{M}_n$, if $ga <_i a$ for all $a \in \mathcal{M}_n$ and $i = 1, ..., n$, then $g$ moves almost $R$-maximally and $g^{-1}$ moves almost $L$-maximally. In order to make the notations simpler in the proof of the next theorem, we can define $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_X^{<_k} \bar{b}$ if for any $a_i \in \bar{a} \setminus X$, $b_j \in \bar{b} \setminus X$ where $a_i <_k b_j$, there exists $x \in X$ such that $a_i <_k x <_k b_j$. Also define $\mathrm{tp}^{<_i}$ Then by definition, we can prove the following lemma.

**Lemma 5.2.28.** *Suppose $\bar{a}, \bar{b} \in \mathcal{M}_n$ and $X \subseteq \mathcal{M}_n$. Then,*

1. $\bar{a} \bigcup_X \bar{b}$ *if and only if* $\bar{a} \bigcup_X^{<_k} \bar{b}$ *for all* $k = 1, ..., n$.
2. *if* $\bar{a} \bigcup_X^{<_k} \bar{b}$ *and* $\bar{a}' \bigcup_X^{<_k} \bar{b}$, *then* $\bar{a}\bar{a}' \bigcup_X^{<_k} \bar{b}$.
3. *if* $\bar{a} \bigcup_X^{<_k} \bar{b}$ *and* $\bar{a} \bigcup_X^{<_k} \bar{b}'$, *then* $\bar{a} \bigcup_X^{<_k} \bar{b}\bar{b}'$.

**Theorem 5.2.29.** *Let* $g \in Aut(\mathcal{M}_n)$. *If* $ga <_i a$ *for any* $a \in \mathcal{M}_n$ *and* $i = 1, ..., n$, *then* $g$ *moves almost R-maximally. Similarly if* $ga >_i a$ *for any* $a \in \mathcal{M}_n$ *and* $i = 1, ..., n$, *then* $g$ *moves almost L-maximally.*

*Proof.* Let $p = \text{tp}(\bar{x}/X)$ be an $l$-type over $X$. By Lemma 5.0.1, we may assume $p_i$ is of the form (5.1), i.e. $\bar{x} \cap X = \emptyset$. Firstly, by Lemma 5.2.16, we can find $\bar{a}$ realising $p$ such that $\bar{a} \bigcup_X^{<_1} g\bar{a}$. Suppose we can find $\bar{b}$ realising $p$ such that $\bar{b} \bigcup_X^{<_i} g\bar{b}$ for all $i = 1, ..., m-1$. We will show that we can find $\bar{c}$ realising $p$ such that $\bar{c} \bigcup_X^{<_i} g\bar{c}$ for $i = 1, ..., m$. Therefore, by induction on $m$, we can find $\bar{d}$ realising $p$ such that $\bar{d} \bigcup_X^{<_i} g\bar{d}$ for all $i = 1, ..., n$. Hence, by the previous lemma, we have $\bar{d} \bigcup_X g\bar{d}$.

Now fix $m$ and assume $\bar{b}$ realises $p$ and $\bar{b} \bigcup_X^{<_i} g\bar{b}$ for all $i = 1, ..., m-1$. List $\bar{b}$ as $b_1 >_m b_2 >_m ... >_m b_l$. Inductively on $k$, we change $b_k$ wherever necessary. For the inductive base, we let $c_1 = b_1$. Let $\bar{b}^k = (b_1, ..., b_k)$ and $\hat{b}^k = (b_{k+1}, ..., b_l)$. For the inductive hypothesis, suppose we now have $\bar{c}^k = (c_1, ..., c_k)$ satisfying the followings:

(i) $\text{tp}(\bar{c}^k\hat{b}^k/X) = \text{tp}(\bar{b}/X) = p$,

(ii) $\bar{c}^k \bigcup_X^{<_m} g\bar{c}^k$, and

(iii) $\text{tp}^{<_i}(\bar{c}^k\hat{b}^k g(\bar{c}^k\hat{b}^k)/X) = \text{tp}^{<_i}(\bar{b}g\bar{b}/X)$ for all $i = 1, ..., m-1$

We want to choose $c_{k+1}$ so that $\bar{c}^{k+1} = (\bar{c}^k, c_{k+1})$ satisfies (i)-(iii) for $k+1$. We divide into the following two cases.

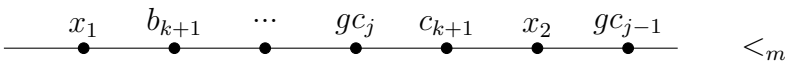
Case I. If $b_{k+1} \bigcup_X^{<_m} g\bar{c}^k$, let $c_{k+1} = b_{k+1}$. Then we automatically have (i) and (iii) for $\bar{c}^{k+1} = (\bar{c}^k, c_{k+1})$. For (ii), since $b_{k+1} \bigcup_X^{<_m} g\bar{c}^k$, $c_{k+1} = b_{k+1}$ and $\bar{c}^k \bigcup_X^{<_m} g\bar{c}^k$, by the previous lemma, we have $\bar{c}^k c_{k+1} \bigcup_X^{<_m} g\bar{c}^k$. Since $gc_{k+1} <_m c_{k+1} <_m c_k <_m .... <_m c_1$, we also have $\bar{c}^k c_{k+1} \bigcup_X^{<_m} gc_{k+1}$. Therefore, by the previous lemma, we have $\bar{c}^k c_{k+1} \bigcup_X^{<_m} g(\bar{c}^k c_{k+1})$.

Case II. $b_{k+1} \bigcup_X^{<_m} g\bar{c}^k$ does not hold, i.e. there exists $c_i \in \bar{c}^k$ such that $b_{k+1} <_m gc_i$ and there does not exist $x \in X$ such that $b_{k+1} <_m x <_m gc_i$. Let $x_1, x_2$ be the lower and upper bound of $b_{k+1}$ with respect to $X$ on $<_m$. This means that

there exist $c_{j-1}, c_j \in \bar{c}$ such that $x_1 <_m b_{k+1} <_m gc_j <_m x_2 <_m gc_{j-1}$. We also have $gc_j <_m c_k$ as otherwise, since $\bar{c}^k \bigcup_X^{<_m} g\bar{c}^k$, there would exist $x' \in X$ such that $c_k <_m x <_m gc_j$. Since $b_{k+1} <_m c_k$ from $\mathrm{tp}(\bar{c}^k\hat{b}^k/X) = \mathrm{tp}(\bar{b}/X)$, we have $x_1 <_m b_{k+1} <_m c_k <_m x' <_m gc_j <_m x_2$. This contradicts that $x_1, x_2$ are the lower and upper bound of $b_{k+1}$ with respect to $X$ on $<_m$. So, the type

$$\begin{aligned} q := & \cup_{i=1}^{m-1} \mathrm{tp}^{<_i}(b_{k+1}/\bar{c}^k\hat{b}^{k+1}Xg(\bar{c}^k\hat{b}^{k+1})g^{-1}(\bar{c}^k\hat{b}^{k+1}X)) \\ & \cup \{gc_j <_m x <_m \min_{<_m}\{x_2, c_k\}\} \\ & \cup_{i=m+1}^{n} \mathrm{tp}(b_{k+1}/\bar{c}^k\hat{b}^{k+1}X). \end{aligned}$$

is consistent since it is consistent on each $<_i$. Hence, we can choose $c_{k+1}$ realising $q$.




We show that $\bar{c}^{k+1} = (\bar{c}^k, c_{k+1})$ satisfies (i)-(iii) in the inductive hypothesis:

(i) By construction, we have $b_l <_m ... <_m b_{k+1} <_m c_{k+1} <_m \min_m\{x_2, c_k\} \leq_m c_k <_m ... <_m c_1$ and $x_1, x_2$ are the lower and upper constraint of $b_{k+1}, c_{k+1}$ with respect to $X$ on $<_m$. So, we have

$$\mathrm{tp}^{<_m}(\bar{c}^k c_{k+1}\hat{b}^{k+1}/X) = \mathrm{tp}^{<_m}(\bar{c}^k b_{k+1}\hat{b}^{k+1}/X).$$

By the inductive hypothesis (i), we have

$$\mathrm{tp}^{<_m}(\bar{c}^{k+1}\hat{b}^{k+1}/X) = \mathrm{tp}^{<_m}(\bar{c}^k b_{k+1}\hat{b}^{k+1}/X) = \mathrm{tp}^{<_m}(\bar{b}/X).$$

By the definition of type $q$, we also have

$$\mathrm{tp}^{<_i}(\bar{c}^k c_{k+1}\hat{b}^{k+1}/X) = \mathrm{tp}^{<_i}(\bar{c}^k b_{k+1}\hat{b}^{k+1}/X) = \mathrm{tp}^{<_m}(\bar{b}/X)$$

for $i \neq m$. Therefore, we have $\mathrm{tp}(\bar{c}^{k+1}\hat{b}^{k+1}/X) = \mathrm{tp}(\bar{b}/X) = p$ by the previous lemma.

(ii) Since $gc_j <_m c_{k+1} <_m \min\{x_2, c_k\} \leq_m x_2 <_m gc_{j-1}$, we have that for all $gc_i >_m c_{k+1}$, there exists $x_2 \in X$ such that $gc_i >_m x_2 >_m c_{k+1}$.

Hence, $c_{k+1} \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_X^{<_m} g\bar{c}^k$ and thus, by the previous lemma and the inductive hypothesis that $\bar{c}^k \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_X^{<_m} g\bar{c}^k$, we have $\bar{c}^k c_{k+1} \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_X^{<_m} g\bar{c}^k$. Since $gc_{k+1} <_m c_{k+1} <_m c_k <_m .... <_m c_1$, we also have $\bar{c}^k c_{k+1} \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_X^{<_m} gc_{k+1}$. Therefore, by the previous lemma, we have $\bar{c}^k c_{k+1} \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_X^{<_m} g(\bar{c}^k c_{k+1})$, which is the same as $\bar{c}^{k+1} \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_X^{<_m} g\bar{c}^{k+1}$

(iii) For any $i = 1, ..., m-1$, since $c_{k+1}$ realises $q$, we have

$$\mathrm{tp}^{<_i}(c_{k+1}/\bar{c}^k\hat{b}^{k+1}g(\bar{c}^k\hat{b}^{k+1})X) = \mathrm{tp}^{<_i}(b_{k+1}/\bar{c}^k\hat{b}^{k+1}g(\bar{c}^k\hat{b}^{k+1})X).$$

By the inductive hypothesis, we also have that

$$\mathrm{tp}^{<_i}(b_{k+1}/\bar{c}^k\hat{b}^{k+1}g(\bar{c}^k\hat{b}^{k+1})X) = \mathrm{tp}^{<_i}(b_{k+1}/\bar{b}^k\hat{b}^{k+1}g(\bar{b}^k\hat{b}^{k+1})X).$$

Hence, we have

$$\mathrm{tp}^{<_i}(c_{k+1}/\bar{c}^k\hat{b}^{k+1}g(\bar{c}^k\hat{b}^{k+1})X) = \mathrm{tp}^{<_i}(b_{k+1}/\bar{b}^k\hat{b}^{k+1}g(\bar{b}^k\hat{b}^{k+1})X).$$

We also have that

$$\begin{aligned}
&\mathrm{tp}^{<_i}(gc_{k+1}/\bar{c}^k\hat{b}^{k+1}g(\bar{c}^k\hat{b}^{k+1})X) \\
=&g \cdot \mathrm{tp}^{<_i}(c_{k+1}/g^{-1}(\bar{c}^k\hat{b}^{k+1}X)\bar{c}^k\hat{b}^{k+1})) \\
=&g \cdot \mathrm{tp}^{<_i}(b_{k+1}/g^{-1}(\bar{c}^k\hat{b}^{k+1}X)\bar{c}^k\hat{b}^{k+1})) && \text{since } c_{k+1} \text{ realises } q \\
=&\mathrm{tp}^{<_i}(gb_{k+1}/\bar{c}^k\hat{b}^{k+1}g(\bar{c}^k\hat{b}^{k+1})X) \\
=&\mathrm{tp}^{<_i}(gb_{k+1}/\bar{b}^k\hat{b}^{k+1}g(\bar{b}^k\hat{b}^{k+1})X) && \text{by the inductive hypothesis (iii)}
\end{aligned}$$

Since $gc_{k+1} <_i c_{k+1}$ and $gb_{k+1} <_i b_{k+1}$, we have

$$\mathrm{tp}^{<_i}(c_{k+1}gc_{k+1}/\bar{c}^k\hat{b}^{k+1}g(\bar{c}^k\hat{b}^{k+1})X) = \mathrm{tp}^{<_i}(b_{k+1}gb_{k+1}/\bar{b}^k\hat{b}^{k+1}g(\bar{b}^k\hat{b}^{k+1})X).$$

By rearranging the type, we have shown

$$\mathrm{tp}^{<_i}(\bar{c}^{k+1}\hat{b}^{k+1}g(\bar{c}^{k+1}\hat{b}^{k+1})/X) = \mathrm{tp}^{<_i}(\bar{b}g\bar{b}/X).$$

Hence by induction, we can find $\bar{c}$ realising $\mathrm{tp}(\bar{b}/X) = p$ such that $\mathrm{tp}^{<_i}(\bar{c}g\bar{c}/X)$ $=\mathrm{tp}^{<_i}(\bar{b}g\bar{b}/X)$ for $i = 1, ..., m-1$ and $\bar{c} \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_X^{<_n} g\bar{c}$. For all $i = 1, ..., m-1$, since $\bar{b} \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_X^{<_i} g\bar{b}$, we also have $\bar{c} \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_X^{<_i} g\bar{c}$ by Invariance. Hence, inductively, we can find

$\bar{d}$ realising $p$ such that $\bar{d} \bigcup_X^{<_i} g\bar{d}$ for all $i = 1, ..., n$. Therefore, by the previous lemma, we have $\bar{d} \bigcup_X g\bar{d}$. Hence, $g$ moves almost $R$-maximally.

By a symmetric argument, we can show that if $ga >_i a$ for any $a \in \mathcal{M}_n$ and $i = 1, ..., n$, then $g$ moves almost $L$-maximally.

□

Note that the results we have shown for $\mathcal{M}_n$ also holds for $\mathcal{M}_n^I$ since $\mathcal{M}_n \cong \mathcal{M}_n^I$. Hence, we can prove the following corollary:

**Corollary 5.2.30.** *Let $\mathcal{M}_n$ be the universal $n$-linear order for $n \geq 2$. Then Aut($\mathcal{M}_n$) is simple.*

*Proof.* Let $g$ be a non-trivial automorphism of $\mathcal{M}_n$. Then, by Theorem 5.2.27, there exists $I \subseteq \{1, ..., n\}$ and $h \in$Aut$(\mathcal{M}_n^I)$ such that $h$ can be written as a product of conjugates of $g$ and $g^{-1}$ and $h$ has a single $+$-orbital on $<_j$ for any $j = 1, ..., n$ in $\mathcal{M}_n^I$. Then $h^{-1}$ has a single $-$-orbital on $<_j$ for any $j = 1, ..., n$ in $\mathcal{M}_n^I$. By the previous theorem, $h$ moves almost $R$-maximally and $h^{-1}$ moves almost $L$-maximally. Therefore, by Corollary 4.2.4, any element of Aut$(\mathcal{M}_n^I)$ can be written as a product of conjugates of $g$ and $g^{-1}$. Since Aut$(\mathcal{M}_n)$ =Aut$(\mathcal{M}_n^I)$, any element of Aut$(\mathcal{M}_n)$ can be written as a product of conjugates of $g$ and $g^{-1}$. Thus, Aut$(\mathcal{M}_n)$ is simple. □

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