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Pinned states in Josephson arrays: A general stability theorem

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Using the lumped circuit equations, we derive a stability criterion for superconducting pinned states in two-dimensional arrays of Josephson junctions. The analysis neglects quantum, thermal, and inductive effects, but allows disordered junctions, arbitrary network connectivity, and arbitrary spatial patterns of applied magnetic flux and DC current injection. We prove that a pinned state is linearly stable if and only if its corresponding stiffness matrix is positive definite. This algebraic condition can be used to predict the critical current and frustration at which depinning occurs.

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Collective pinning occurs in a wide variety of coupled physical systems. Examples include vortices in Type-II superconductors, cracks and dislocations in solids, and charge-density waves in quasi-one-dimensional metals.[1] In each case, when the system is subjected to an external constant drive, it remains motionless until the drive exceeds a critical value (the depinning threshold) after which the system begins to move. The pinning is collective in the sense that it involves interactions among many coupled subsystems, typically in the presence of disorder. Hence it is often difficult to predict the depinning threshold theoretically.

Here we study collective pinning for a relatively tractable class of model systems: two-dimensional (2D) arrays of Josephson junctions. Besides their technological applications,[2] Josephson arrays can be used to explore fundamental questions in statistical mechanics (such as phase transitions), and in nonlinear dynamics (such as synchronization and spatiotemporal pattern formation).[3] In addition, they have been proposed as clean models for layered and granular high-T_c superconductors.[4,5] As such, their depinning could be relevant to the understanding of the onset of resistance in the current-voltage characteristics of high-T_c samples.[6]

Several advances have occurred recently in the numerical[5,7–9] and analytical[10–12] investigation of 2D Josephson arrays, thanks in part to an influx of ideas from nonlinear dynamics. In this paper, we analyze depinning in 2D arrays from this perspective. Using a compact matrix notation, we show that the linear stability problem for pinned states can be mapped onto the classical mechanical problem of small oscillations in a network of coupled, damped linear oscillators. The results apply to 2D arrays of any given topology. There are also no restrictions on the capacitances, resistances, or critical currents of the junctions, nor on the spatial patterns of DC current injection and applied magnetic flux. Our main result is that a pinned state is stable if and only if its corresponding stiffness matrix K is positive definite. This matrix K changes with the pinned configuration and depends on the connectivity and disorder of the array. A corollary is that any pinned state with all phases $|\phi_i| < \pi/2$ is guaranteed to be stable. We also prove that depinning can never occur via a Hopf bifurcation; only zero-eigenvalue bifurcations are possible.

Our analysis is based on several simplifying assumptions. First, we neglect thermal fluctuations; that is, we assume zero temperature. Second, we assume that the superconducting islands in the array are large enough that quantum (charging) effects are negligible. Thus, the phase θ_i of the complex macroscopic wavefunction at each island is a well-defined classical variable. Third, we assume that the junctions between islands are small enough that they can be approximated as lumped elements. Therefore, the junction between two islands ℓ and m can be described by a point gauge-invariant phase difference

$$\phi_i = \theta_\ell - \theta_m - \frac{2\pi}{\Phi_0}\int_\ell^m \mathbf{A}\cdot d\mathbf{l} \qquad (1)$$

where $\mathbf{A}$ is the total magnetic vector potential and $\Phi_0 = h/(2e)$ is the quantum of magnetic flux. Fourth, we model each junction by the standard RCSJ equivalent circuit[2,13] with superconducting, resistive, and capacitive channels in parallel. Then the junction dynamics obeys a damped driven pendulum equation

$$\mu_i\ddot{\phi}_i + \gamma_i\dot{\phi}_i + \eta_i \sin\phi_i = i_i^b \qquad (2)$$

with effective mass $\mu_i = \Phi_0 C_i/(2\pi I_{c0})$, damping $\gamma_i = \Phi_0/(2\pi R_i I_{c0})$, and restoring strength $\eta_i = I_{ci}/I_{c0}$. The capacitance C_i, resistance R_i, and critical current I_{ci} are fabrication- and material-dependent parameters that characterize junction i. The drive is given by the normalized current i_i^b, measured in units of $I_{c0} = \langle I_{ci}\rangle$, the average critical current of the junctions in the array.

In dealing with arrays, it is useful to introduce a vector-matrix notation,[8,9,11,14] where the variables are now vectors of three types: node vectors of dimension n (e.g. θ); edge vectors of dimension e (e.g. ϕ and i^b); and cell vectors of dimension c defined at each plaquette. (More precisely, n is the number of independent nodes, after one node is grounded and taken as reference.[14]) The edge and node variables are related through an $e \times n$ edge-node

connectivity matrix A that encodes the topology of the array, including its boundary conditions such as the presence (or absence) of edges. Similarly, an $e \times c$ edge-cell matrix B transforms between edge and cell variables, in what amounts to taking a discrete curl.

Within this framework, the nonlinear constitutive law (2) can be compactly written as

$$\mu\ddot{\phi} + \gamma\dot{\phi} + \eta \sin\phi = i^b \tag{3}$$

where $\mu = \mathrm{diag}(\mu_i)$, and γ and η are similarly defined diagonal matrices. Each junction is allowed to have a different capacitance, resistance, and critical current, as recorded in the matrices μ, γ, and η.

When junctions are interconnected to form a network, there exist topological constraints which can be expressed in terms of the connectivity matrices A and B. First, the currents must satisfy Kirchhoff's current law[14]

$$A^T i^b = i^{\mathrm{ext}}, \tag{4}$$

where the vector i^{ext} gives the balance of normalized current at each node, and reflects the particular scheme of current injection/extraction for each experimental device. For instance, in the usual experimental setup, where a uniform DC current I_{dc} is injected (extracted) at the bottom (top) nodes, all the components of i^{ext} will be zero except those at the bottom (top) boundary, which will be equal to I_{dc} ($-I_{\mathrm{dc}}$). Our analysis, however, is valid for an arbitrary injection scheme, as long as the bias currents are time-independent.

The second topological constraint is the flux quantization in each cell of the array. We assume the simplest case where all self-fields due to inductance effects are neglected. Then the flux quantization is given by

$$B^T\phi + 2\pi F = 2\pi\zeta \equiv 0, \tag{5}$$

where ζ is a cell vector of integers (topological vorticities) that have no dynamical relevance, and can be redefined as zero with no loss of generality.[11] The cell vector F records the external flux through each plaquette, measured in units of the flux quantum. In experiments, the external magnetic field is often spatially uniform across the array. Then F is a constant vector with value $f = \Phi_{\mathrm{ext}}/\Phi_0$. Our analysis holds more generally for any time-independent spatial pattern of applied flux.

For the no-inductance case assumed here, the transformation (1) between junction and island phases is given in vector form by

$$\phi = A\theta - \varphi \tag{6}$$

where φ is a time-independent edge flux vector, fixed by our choice of gauge but subject to $B^T\varphi = 2\pi F$, which follows directly from (5) noting that $B^T A \equiv 0$, from the definition of the topological matrices.[14] From (3), (4) and (6), we obtain the governing vector equation of the system:

$$A^T\mu A\,\ddot{\theta} + A^T\gamma A\,\dot{\theta} + A^T\eta\,\sin(A\theta - \varphi) = i^{\mathrm{ext}}. \tag{7}$$

From now on, we focus on the pinned states of the array. These correspond to static configurations θ^* of (7), given implicitly by

$$A^T\eta\,\sin(A\theta^* - \varphi) = i^{\mathrm{ext}}. \tag{8}$$

Typically this nonlinear algebraic system (8) has multiple solutions. Each solution depends parametrically on the external current vector i^{ext} and the applied flux vector F. (In the usual experimental setup, these are determined by the scalars I_{dc} and f, respectively.) As i^{ext} or F are varied, the linear stability of a given static configuration θ^* can change. This signals the transition to another state of the system. If the new state is still pinned, the transition corresponds to a static rearrangement of phases and currents; on the other hand, if the new state is time-dependent, it corresponds to depinning and the onset of resistance. (Because our analysis is local, it cannot distinguish between these two types of transitions.)

To study the stability of the pinned states, let $\theta = \theta^* + \alpha$ where α is a small perturbation. Linearizing (7) about θ^* yields

$$M\ddot{\alpha} + G\dot{\alpha} + K\alpha = 0, \tag{9}$$

where

$$M = A^T\mu A, \quad G = A^T\gamma A, \quad K = A^T\eta C^* A \tag{10}$$

are the mass, damping, and stiffness matrices, respectively, and

$$C^* = \mathrm{diag}(\cos\phi_i^*) \tag{11}$$

is a diagonal matrix of the cosines of the phases of the given static configuration. Both M and G are symmetric, positive definite matrices, since A is a topology matrix and μ and γ are diagonal matrices with positive masses and damping coefficients on the diagonal.[14] However, K is not necessarily positive definite since the cosines on the diagonal of C^* are not necessarily positive. We stress that the stiffness matrix K is different for each pinned state, and it also changes parametrically with the externally tunable parameters.

Equation (9) is familiar from the classical mechanical problem of small oscillations in a network of coupled, damped harmonic oscillators.[15] But the present stability problem is not as trivial as it might seem. Ordinarily one assumes that K is positive definite, but that need not be true here. Also, recall that when damping is present, normal modes cannot be used to decouple the system; in mathematical terms, one cannot simultaneously diagonalize the three symmetric matrices M, G, and K. Therefore we analyze (9) from first principles.

A given pinned state is linearly stable if and only if the perturbation $\alpha(t)$ decays to zero for all initial conditions. Equivalently, all the eigenvalues of (9) must have strictly negative real parts. The characteristic equation

$$\det(\lambda^2 M + \lambda G + K) = 0 \tag{12}$$

cannot be solved explicitly, but one can still extract useful information about the eigenvalues, as follows. Suppose that (12) holds for some λ. Then there exists a (possibly complex) eigenvector $x \neq 0$ such that $\lambda^2 Mx + \lambda Gx + Kx = 0$. Multiplying on the left by the complex conjugate transpose $x^\dagger$ yields

$$\lambda^2 m + \lambda g + k = 0, \tag{13}$$

where $m = x^\dagger Mx$, $g = x^\dagger Gx$, and $k = x^\dagger Kx$ are scalars that depend on x. Thus,

$$\lambda = \frac{-g \pm \sqrt{g^2 - 4km}}{2m}. \tag{14}$$

The key point is that $m > 0$ and $g > 0$ *for all* x, since M and G are real and symmetric (hence Hermitian) positive definite matrices. On the other hand, K is not necessarily positive definite, so k can have either sign. If $k > 0$, there are two subcases: if $g^2 - 4km < 0$, the eigenvalues are complex conjugates with $\mathrm{Re}(\lambda) = -g/(2m) < 0$; otherwise the eigenvalues are both real and negative. In either case, the eigenvalues for $k > 0$ lie in the left half plane and therefore correspond to stable modes. On the other hand, if $k < 0$, then $\lambda_- < 0$, $\lambda_+ > 0$ so the λ_+ mode is unstable. Finally, if $k = 0$, then $\lambda_- < 0$, $\lambda_+ = 0$, and the λ_+ mode is neutral.

An important qualitative conclusion from these formulas is that any eigenvalue of (12) must be either pure real, or complex with strictly negative real part. In particular, pure imaginary eigenvalues are forbidden. An immediate consequence is that pinned states can never undergo Hopf bifurcations; depinning can occur only through zero-eigenvalue bifurcations[13] such as saddle-node, transcritical, and pitchfork bifurcations.

We now prove the main result: a pinned state is linearly stable if and only if K is positive definite. To prove the "if" direction, suppose that K is positive definite. Then $k > 0$ for *all* eigenvectors x. From Eq. (14) above, $\mathrm{Re}(\lambda) < 0$ for all λ and, hence, the pinned state is linearly stable.

To prove the "only if" direction, it is equivalent to prove its contrapositive, i.e., we assume that K is not positive definite and show that the pinned state is not linearly stable. There are two cases. If $\det(K) = 0$, then $\lambda = 0$ is a solution of (12), by inspection. But $\lambda = 0$ corresponds to a neutral mode, not a decaying mode as required for linear stability. Next suppose $\det(K) \neq 0$. We outline a homotopy argument which proves that (12) has a root $\lambda > 0$. The strategy is to start with the undamped problem, where it is easy to show that there is an unstable mode if K is not positive definite. Then we continuously deform the undamped problem into Eq. (12), and show that the unstable eigenvalue *remains* unstable throughout the deformation. More precisely, consider the one-parameter family of equations

$$\det(\lambda^2 M + p\lambda G + K) = 0 \tag{15}$$

where $0 \leq p \leq 1$ is a homotopy parameter. At $p = 0$, Eq. (15) corresponds to an undamped system, and normal modes can be used to show explicitly that (15) has an eigenvalue $\lambda(0) > 0$. As p varies continuously from 0 to 1, this eigenvalue traces out a continuous curve $\lambda(p)$ in the complex plane. The curve starts on the positive real axis since $\lambda(0) > 0$, and it must stay there for all p because any eigenvalue in the right half plane must be pure real, as shown by (14). Moreover, the curve cannot cross through the origin; from (15), $\lambda(p) = 0$ for some p would imply $\det(K) = 0$, contrary to assumption. Thus $\lambda(p) > 0$ for all p. Setting $p = 1$ yields the desired result that (12) has a root $\lambda > 0$.

One consequence of this theorem is an implicit formula for the stability threshold of a pinned state θ^*. As we vary the applied current or magnetic field, θ^* and its associated matrix K will change. The theorem implies that θ^* loses stability precisely when $K = A^T \eta C^* A$ ceases to be positive definite. This threshold is reached when the following algebraic condition is satisfied for the first time:

$$\det(K) \equiv \det(A^T \eta C^* A) = 0. \tag{16}$$

Hence the stability threshold for θ^* is determined *exclusively* by the array topology, by the injection scheme and bias current (through i^{ext}), by the applied magnetic field F, and by the disorder in the junctions' critical currents (via the matrix η). On the other hand, it does not depend on the mass (capacitance) and damping matrices M and G. This means that overdamped and underdamped systems have identical depinning thresholds.

Another corollary is that if

$$\cos\phi_i^* > 0, \quad \forall i \tag{17}$$

then that configuration is stable. This follows from the fact that the diagonal matrix ηC^* of such a configuration is positive definite; therefore K is also positive definite.[14] On the other hand, since K can be positive definite even if C^* is not, (17) is only a *sufficient* (but not necessary) condition for the stability of a pinned state.

The constraint (17) has a clear physical meaning for a single, isolated Josephson junction. Recall that as the bias current is increased from zero, a single junction remains pinned until $\phi = \pi/2$, at which point it depins to a running mode.[2,13] Extrapolating naively from a single junction to an array, it is tempting to conjecture that an array should depin when its "most unstable" junction first reaches $\phi = \pi/2$. Note, however, that this heuristic depinning criterion is equivalent to $\det(C^*) = 0$, rather than the rigorous condition $\det(K) = 0$; therefore, it is not exact. Nevertheless, for the specific case of a ladder array with square plaquettes and perpendicular current injection, we have shown elsewhere[12] that it can provide a good approximation to the true depinning threshold.

The algebraic condition (16) can be used to ease the numerical determination of the depinning threshold for 2D arrays. For instance, the depinning current is usually obtained[7] through dynamical simulations that resemble

the actual experiment: the current is ramped up adiabatically and the circuit differential equations are numerically integrated until a running solution appears. In contrast, we solve (8) and (16) simultaneously to determine the critical current and the bifurcating phase configuration as functions of all the other parameters. This purely algebraic calculation can be done by Newton's method or some other rootfinding scheme. The results coincide with those found dynamically.[16]

Another theoretical approach to depinning uses thermodynamic and quasistatic calculations of pinned states.[17–19] One can show that the condition (16) is strictly equivalent to finding the point at which a given static configuration ceases to be a minimum of the potential energy

$$V = -\theta^{\dagger} i^{\mathrm{ext}} - \mathrm{Tr}(\eta C). \tag{18}$$

Thus a soft-mode condition[19] rigorously predicts depinning, while the criterion based on maximizing the quasistatic current induced by twisted boundary conditions[18] is only approximate.[7] Note also that, although stable static configurations correspond to local minima of V, we do not attempt here to obtain the *absolute* minimum of the potential energy. This problem would require global optimization methods, such as simulated annealing.

Our results open several promising lines of research. First, our analytical framework facilitates exploration of the effects of network connectivity on the depinning of Josephson arrays. The implicit condition (16) can be turned into explicit, testable predictions of the applied current and frustration at which depinning should occur. It may be possible to obtain analytical results for square and triangular arrays of identical junctions, perhaps along the lines of recent work on ladder arrays.[12] Second, one should also try to take self-fields into account. Preliminary results suggest that the formulation given here can be generalized to include inductance effects.[16] Finally, it is important to study more quantitatively how disorder affects the stability of pinned states, both as the inevitable result of fabrication irregularities and as a design tool to manipulate the response of the network in a controlled fashion.

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