

BUCKLING AND POSTBUCKLING BEHAVIOUR OF
GENERALLY LAYERED COMPOSITE PANELS

by

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Finally, may I express my thanks to Mr. K.A. Stevens for occasional but very helpful discussions involving various aspects of this investigation.

ABSTRACT

In the present thesis an analytical investigation is presented for initial buckling and postbuckling behaviour of unstiffened flat and cylindrically curved panels of composite materials. A very general lay-up arrangement in the panels is considered in conjunction with various boundary and loading conditions. Extensive results are obtained for panels with different geometry, different lay-ups and different composite materials.

The governing equations are obtained by using the virtual displacement and virtual force principles to allow for a general lay-up situation in the panel.

The main discoveries and improvements made in the current investigation may be summarised as follows :

High precision of the beam functions is proved to be necessary for the orthogonality of the functions and, therefore, for buckling and postbuckling investigations of composite panels.

A new kind of relationship between critical in-plane compressive ratio and shear ratio is found in initial buckling problems. It is shown that the applied shear in a certain direction may greatly benefit the stability of the panel against compression.

Both double sine series and double sine squared series are taken to express the deflection when investigating the post-buckling behaviour of curved panels under compression in order to represent the tendency for inward buckling.

The investigation of panels under shear reveals that the applied shear direction has significant influence on the post-

buckling behaviour of such panels.

The bending effect during initial loading, caused by certain lay-up arrangements, is investigated for both flat and curved panels under compression and shear.

Imperfections in panel geometry and combined lateral loading are also considered.

BUCKLING AND POSTBUCKLING BEHAVIOUR OF GENERALLY LAYERED COMPOSITE PANELS

The Author wishes to include the following amendments and comments:

- px $\beta = a/b$
- pxv for Fig IV,5, and elsewhere, for 'inner force' read 'stress resultant'
- pxvi for Fig IV,11, and elsewhere, for 'inner moment' read 'stress couple'
- p2 l8 omit 'lateral'; l16 for 'strength' read 'stiffness'
- p4 l2 Donnell
- p11 l7 for 'solution' read 'problem'
- p12 l5 details of STAGS may be found in Ref 99
- p13 l3 " ... which is not to be seen ... "
- p14 l8 " ... to the shape and magnitude of the initial imperfection"
- p18 after (II,6) Q_{ij} should be $\bar{Q}_{ij}$
- p19 l9 for ' k_{th} ' read ' k_{th} '
- p24 last expression of (II,24) should commence: $\int_0^a \{ (B_{12} \dots$
- p28 the derivation of eqns (II,33) and (II,34) assumes that the initial shape with imperfections is stress free
- p29 l2 Ashton et al is Ref 42, Chia et al is Ref 76
- Eqn (II,37) is obtained by substitution of (II,35) into (II,36)
- p30 l6 the 6% error is relative
- last l 15 figures were adopted since this was the limit imposed by the computer used. No tests were made with a smaller number of figures
- p36 note that for the boundary conditions in (III,3) - (III,6) a trivial solution is being found and hence $\partial^2 F / \partial \eta^2$ (etc) = 0. In effect the uniform stress state has been subtracted
- p40 the purpose of the indices I and J in (III,17) is to convert the infinite series of terms in (III,12) and (III,13) into a finite number of terms expressed in matrix form
- p47 penultimate l) G_{lt} should read G_{lt}
- p48 l1) G_{lt} should read G_{lt}
- p49 Section f) note that Figs (III,9-13) are for clamped boundaries
- l9 for (II,9) read (III,9)
- note that in Figs (III,5,10, & 13) the curves should not be continuous. Symmetric $\pm 45^\circ$ lay-ups can only have 4,8,12 ... layers; antisymmetric $\pm 45^\circ$ lay-ups can only have 2,4,6 ... layers; one layer at 45° is, of course, symmetric
- p58 last l the second d_{26} should be d_{22}
- p59 in eqn (III,41) the second sine term in u should be $\sin l \pi \eta$, and in v should be $\sin n \pi \eta$
- p64 l1 the above comment re symmetry and numbers of layers also applies to Fig (III,36)
- l12 for 'crossply' read 'all 0° or all 90° '
- p65 l6 in Fig (III,40) the results for the 2 and 4-layer plates are not identical, as indicated, but too close to be separated with the scale used
- p76 Figs (III,9,10,11,12,13,25,26,27,28,29) should have $|P_{xy}|$ on the vertical axis. Fig (III,12) is for positive shear
- etc In Figs (III,14-20, 30-35, & 38-42), if required, the values of R_{xo} and R_{so} can be found from the earlier curves for the particular lay-up under compression-only or shear-only loading
- p112 l3 of (IV,1) upper value should read $|\xi|=1$
- p113 (IV,3) note that the conditions for $\partial^2 F / \partial \eta^2$ etc are now different to before, since the uniform stress state is included
- In (IV,4) the $\sin^2 \xi \sin^2 \eta$ term is included to represent the tendency for inward buckling [15]

pl19 l2 for 'inflexible' read 'undeflected'
 pl12 last l ".... Figs (IV,1-20) below. ... "
 pl23 l2 one 'run' will establish the complete postbuckled path,
 about 20 steps are needed to get a typical curve.
 In Figs (IV,1-4) the results become inaccurate for $\omega/h > 4$.
 pl24 l3 to obtain the cusps and loops it is usually necessary to re-
 run the analysis using smaller step lengths in these regions
 pl25 l8 because of their complexity it is difficult to illustrate
 the buckled mode shapes. Many components will be present and the
 relative proportions of these components will alter as the mode
 changes.
 pl71 l17 should be $\{M_0\}$ and $\{O_0\}$
 l22/3 in Appendix F (p264) last three terms should be T_0^i, T_1^i, T_2^i
 pl73 l3 the analysis is started with a high step length (0.5) and if
 the results indicate that regions of rapid change have been missed
 then the process is repeated using a smaller step size.
 pl77 (V,6) in equation for w the first term should be $w_{pq}^{(1)}$
 pl78 (V,11) last diagonal term should be $\{w^{(0)}\}^T$
 pl79 next-to-last eqn should read $[B^{(3)}] = [I] + [[J]$ etc ...
 pl86 (V,18) first row should end 'at $\xi = 0$ and 1'
 second row should end 'at $\eta = 0$ and 1'
 pl91 para 4 in Fig (V,17) the shear direction is positive and the term
 on the vertical axis should be $P_{xy}b^2/E_th^3$
 pl92 last l for 'non-square' read 'rectangular'
 p225 l8 for 'appliance' read 'test rig'
 p226 Ref 6 'Vorlesungen Physik und Mechanik'
 p227 Ref 10 for 'American Elsevier Publishing Co' read 'Chatto and
 Windus Ltd'
 Ref 12 for 'Frein' read 'Freien'
 Ref 13 Galerkin, B.G.
 p229 Ref 28 insert '.. London, 1968'
 Ref 29 insert '.. Stamford, Connecticut, 1969, ... '
 p231 Ref 42 insert '.. Stamford, Connecticut, 1970'
 p232 Ref 51 should be 'Iyengar'
 p233 Ref 62 for 'Orothotropic' read 'Orthotropic'
 p234 Ref 71 insert '.. Washington D.C., 1975'
 p238 Ref 99 add at end '.. Dayton, Ohio'

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NOMENCLATURE

A, B, D
A*, B*, D*

A_{ij}, B_{ij}, D_{ij}
 $A_{ij}^*, B_{ij}^*, D_{ij}^*$
 $\bar{A}_{ij}, \bar{B}_{ij}, \bar{D}_{ij}$

[A], [B] ... [Z]

$[A^{(1)}], [A^{(2)}], [A^{(3)}]$

$[B^{(1)}], [B^{(2)}], [B^{(3)}]$

$[C^{(1)}], [C^{(2)}], [C^{(3)}]$

$C_j^{mi} (j=0,1,2,3)$

E_l, E_t

F

F_{mn}

$\{F\}$

$[F], [F^{(1)}], [F^{(2)}]$

$F_j^{mi} (j=0,1,2,3)$

G_{lt}

H_o^{mi}

I, I_1

J, J_1

$J_j^{mi} (j=0,1,2,3)$

$J_j^{kri} (j=4,5,6)$

K

$[K^{(i)}]$

Stiffness matrices in constitutive relations

Reduced stiffness matrices

Elements of **A, B, D**

Elements of **A*, B*, D***

Nondimensional A_{ij}^*, B_{ij}^* and D_{ij}^*
 defined in expressions (II,26)

Coefficient matrices defined in

Appendix E

Coefficient matrices

Coefficient matrices

Coefficient matrices

Constants obtained from certain
 integrals

Elastic moduli of composite material
 along and perpendicular to fibre direction

Nondimensional stress function

Components of stress function

Column matrix of F_{mn}

Coefficient matrices

Constants obtained from certain integrals

Shear modulus of composite material

Constants obtained from certain integral

Integers (used as subscripts)

Integers (used as subscripts)

Constants obtained from certain integrals

Constants obtained from certain integrals

Loading parameter

Coefficient matrices

K_x	Nondimensional compression parameter (x direction)
K_y	Nondimensional compression parameter (y direction)
K_s	Nondimensional shear parameter
K_R	Nondimensional curvature parameter
L_O^i	Constants obtained from certain integral
L_i ($i = 1, 2, \dots, 12$)	Differential operators
M	Certain integer selected
M	Moment and torque column matrix
M_x, M_y, M_{xy}	Moments and torque per unit width
$M_\xi, M_\eta, M_{\xi\eta}$	Nondimensional moments and torque
N	Certain integer selected
N	Membrane force column matrix
N_x, N_y, N_{xy}	Membrane forces per unit width
$N_\xi, N_\eta, N_{\xi\eta}$	Nondimensional membrane forces
$\bar{N}_x, \bar{N}_y$	Nondimensional external forces in x and y directions
P_x, P_y	External compression in x and y directions
P_{xy}	External shear
P_{x0}, P_{y0}, P_{xy0}	Corresponding simple critical loads (without combination with other loads)
Q_O^{mi}	Constants obtained from certain integral
Q	Nondimensional lateral load
Q_{ij}	Stiffness coefficients of composite layers
$\bar{Q}_{ij}$	Reduced stiffness coefficients of layers
R_x, R_s	critical compressive and shear ratio
R	Radius of curved panel
R_j^{mi} ($j=0, 1, 2, 3$)	Constants obtained from certain integral
$\bar{S}$	Nondimensional external shear
S_j^{mi} ($j=0, 1$)	Constants obtained from certain integrals

T_j^i ($j=0,1,2,3$)	Constants obtained from certain integrals
$[T_w^{(1)}]$ $[T_w^{(2)}]$ $[T_w]$ $[T_w^{(0)}]$	Matrices related to deflection coefficients
Z_{mn}^{rki} ($m=0,1,2$; $n=0,1,2, \dots, 7$)	Constants obtained from certain integrals
a, b	Length and width of panels
a_i, b_i	Nondimensional coefficients related to $A_{ij}^*, B_{ij}^*, D_{ij}^*$
a_{ij}, b_{ij}	Nondimensional coefficients related to A_{ij}, B_{ij} and D_{ij}
c	$\cos \theta$
f	Stress function in general form
h	Thickness of panel
$i, j, k, l, m, n, p, q, r, s$	Integers
k	Flexural curvature column matrix
K_x, K_y, K_{xy}	Flexural curvatures of panel
$\bar{q}$	Lateral pressure per unit area
s	$\sin \theta$
t	Transposed matrices (used for superscript)
u_o, v_o, w_o	Displacements of panel in x, y and z directions
u, v, w	Nondimensional displacements
w (in figures)	deflection of panel
$\bar{w}_o$	Geometrical imperfection of panel
$\bar{w}$	Nondimensional imperfection
$\bar{w}_{ij}$	Components of nondimensional imperfection
$w_{pq}, w_{pq}^{(1)}, w_{pq}^{(2)}$	Components of deflection function
$\{w\}, \{w^{(1)}\}, \{w^{(2)}\}, \{w^{(0)}\}$	Column matrix with elements of $w_{pq}, w_{pq}^{(1)}, w_{pq}^{(2)}, \bar{w}_{ij}$

x, y, z	Cartesian coordinates
$[\Gamma], [\Lambda], [\Psi], [\Lambda^{(i)}]$	Coefficient matrices
Π	Total potential energy
α_i, γ_i	Constants in beam functions
β	Aspect ratio of panel
δ_j^{mi}	Constants in Appendix B
δ	Variation
ε^o	Midplane strain column matrix
$\varepsilon_x^o, \varepsilon_y^o, \gamma_{xy}^o$	Strains of midplane
$\varepsilon_x, \varepsilon_y, \gamma_{xy}$	Strains of panel
θ	Fibre angle
μ_x, μ_y, μ_s	Loading ratio parameters
ν_{lt}, ν_{tl}	Poisson ratios of composite materials in two directions
ξ, η	Nondimensional Cartesian coordinates
$\sigma_x, \sigma_y, \tau_{xy}$	Stresses in panel
λ	Nondimensional loading parameter ($\lambda = \frac{1}{K}$)
ϕ	Stress function
ω_1	$\omega_1 = m\pi$
ω_2	$\omega_2 = n\pi$

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CHAPTER I

INTRODUCTION

A. Objective

Composite materials are ideal for many structural applications, because of their "designable" performance and light weight with high strength. Especially in aircraft and other weight-sensitive industries they have become more and more important in recent decades. They are now used for many primary load carrying structures such as wing surfaces, fuselage sections and missile skins. Flat plates and curved panels made of laminated composite of fibre reinforced plastics are probably the most important structural members. To exploit the efficiency of these materials with high performance and high cost, scientists and engineers have made continuous efforts to develop analytical and experimental techniques to reveal and to predict the buckling and postbuckling behaviour of such plates and panels. The present investigation is a small part of these efforts.

In general, laminated plates and shells consist of a number of layers, in which unidirectional aligned fibres made of strong materials such as carbon, boron, graphite, glass and so on are embedded in resin plastics. The performance of these structures depends not only on the properties of the composite, but also the lay-up arrangements in the plates and shells. The layers are usually considered to be orthotropic.

Previous work done in this area was mostly directed towards the study of the flat plates with special lay-ups, such as crossply, antisymmetric lay-ups, and so forth. Because of the complexity of the problem, postbuckling studies, even for plates having special lay-ups, are few and far between.

In contrast, the present study is devoted to an analytic investigation for both initial buckling and postbuckling behaviour of flat plates and curved panels with totally general lay-up arrangements.

B. Literature Survey

The theory of elastic stability is now an important branch of applied mechanics. Since Euler^[1] formulated the first theoretical solution for the lateral buckling of a column compressed on the ends, in the middle of the 18th century, stability research for all kinds of elastic systems have been very well developed. A wealth of published research may be found in the literature, including hundreds of papers and reports and tens of books. A brief review is given below of this particular field related to the present study. It is, however, not intended to be complete.

It is the modern use of high strength materials which made buckling problems of slender structural members one of practical importance. Stability of columns, beam-columns with various loading and end conditions was first under consideration by scientists and engineers in the latter half of the last century^[2]. Since then the problems of buckling of columns have become widely known. The theoretical solutions and experimental verifications for such elements showed that linear theory is usually enough to give a precise prediction of buckling behaviour for engineering purposes. The results may be found in various books^{[3] [4] [5]}.

In the theory of thin plates, Kirchhoff^[6] made one of the most important contributions. He formulated the theory based on the hypothesis that the normal to the plate remains undeformed after plate deformation. This hypothesis is now known

by his name and is still accepted as the basis of most theoretical investigations of thin plates and shells.

The buckling study of isotropic plates started with Bryan^[7]. He solved the initial buckling problems of a compressed rectangular plate with simply supported edges by assuming a series function for deflection. His paper gave a foundation for consequent theory of elastic stability of thin-walled structures. Investigations after that show that linear theory on thin plates usually produces a good prediction of onset of buckling, though it is not able to describe the behaviour of thin plates after buckling. It was Von Karman who built up the non-linear theory of plates by introducing the squared terms of the first derivatives of deflection into the strain expressions of the plates. The initial buckling and postbuckling behaviour for isotropic plates have been very well investigated for different boundary, loading and geometry conditions, and they may be found in a number of books, say references [8] [9] [10] and elsewhere.

In stability study it is not always possible to obtain exact solutions. The approximate methods based on the stationary energy principle became probably the most important analytic tools in such cases. Rayleigh^[11]-Ritz^[12] method and Galerkin^[13] method both are widely used since the early years of this century. They play active roles in stability studies of all kinds.

The classical analysis of stability usually takes the form of an eigenvalue problem for the lowest load. When tests were carried out on thin shell structures the classical theory became suspect because experimental buckling loads were much lower than theory predicted and testing results were

widely scattered as well. An explanation for this was given by Donell^[14] in 1934, in which the effect of initial deviation of the test model geometry from that of the perfect cylinder was discussed, but the yielding stresses of the materials in the shell, developed in the loading process, was mistaken as the criterion of the collapse. In 1941 Von Karman and Tsien^[15] made a very important advance in our understanding of this problem. Their research revealed that the large discrepancies between test and theory for the buckling of certain types of thin shell structures were due to the highly unstable postbuckling behaviour of these structures. The sound foundations of the non-linear theory of the stability of shell structures were since set up. In 1950 Donell and Wan^[16] employed approximate techniques to investigate the effect of imperfections on postbuckling of thin cylinders and columns under axial compression by assuming initial imperfection shapes. This put the investigation on a correct basis and gave a reasonable explanation of the experimental behaviour of shell structures.

Starting with Von Karman and Tsien's work, a large number of numerical calculations in the large-deflection range have been made. The research resulted in successively more accurate estimates of the minimum postbuckling load on the basis that this minimum load would provide a conservative engineering estimate of the buckling strength of practical cylindrical structures. However, this approach proved to be fruitless since it was eventually discovered that the minimum postbuckling load approached zero for very thin shells^[17].

At roughly the same time, in wartime Holland, another very important contribution to general stability theory of elastic systems was made by Koiter^[18], although his work

attracted only little attention until the early 1960s. He developed the initial postbuckling approach by asymptotic techniques to investigate postbuckling behaviour in the neighbourhood of the bifurcation point. Koiter's research spawned numerous investigations in the past two decades. Pope^[19] ^[20] worked on the postbuckling analysis of slightly curved panels, including the effect of transverse shear deformation. He^[21] also discussed the general characteristics of initial postbuckling for isotropic structural members.

Experimental research is another significant aspect of the stability study of elastic systems. Perhaps the most important concept in this field is one due to Southwell^[22]. He proposed a method for obtaining the theoretical buckling stress of a perfect column from experiments on real columns containing small initial imperfections. It has been proved that Southwell's method may be extended to the prediction of the elastic critical load for plates, shells and other structures^[23] ^[8] ^[24].

According to reference [2], investigation for orthotropic plates was first launched by Gehring in 1860, but the book by Huber^[25] in 1929 might be the first one to consider the buckling problems of orthotropic plates. Most early work in this field was directed towards wooden plates^[26] ^[27]. Russian scientists made important contributions to the theory of isotropic plates and orthotropic plates, but because of the different language the work of Leknitskii^[28] and Ambartsumyan^[29] was not known in the western world until the latter years of the 1960s. Leknitskii systematically established the basic theory of classical analysis for

homogeneous orthotropic and anisotropic plates. Ambartsumyan's complicated analysis included the effect of transverse shear in orthotropic plates. But these classical analyses were not suitable for generally laminated non-homogeneous plates, though Lekhnitskii^[28] and Smith^[30] declared that their theory would be suitable for use with plates composed of unsymmetric crossply orthotropic layers. They concluded that such a plate would behave as a homogeneous orthotropic one.

In 1961 Reissner and Stavsky^[31] discovered the coupling between in-plane stress resultants and out-of-plane stress couples in general composite plates. This work, based upon the latter's dissertation^[32], provided the theoretical groundwork for the study of composite plates. The subsequent work of Stavsky^[33] [34] makes the theory complete. Starting with this a number of investigators have made important progress in revealing the effects of such coupling.

The year 1969 marks the beginning of a large rate of progress in composite plate buckling analysis. Whitney and Leissa^[35], based on the former's dissertation^[36], presented the closed form exact solutions for antisymmetric crossply and angle-ply plates having simply supported rectangular boundaries. Simultaneously, Ashton and Waddoups^[37] showed the results of applying the Ritz method for rectangular plates loaded in biaxial compression and shear and compared the theoretical results with experimental ones obtained by Ashton and Love^[38] [39]. Probably, however, the earliest experimental investigation for fibre reinforced composite plates was due to Mandell^[40]. Ashton^[41] suggested using the stationary energy principle in conjunction with proper beam characteristic functions to solve the buckling problems of anisotropic plates.

The outstanding book by Ashton and Whitney^[42] includes various results of buckling investigations of laminated plates.

A linear theory for generally laminated anisotropic plates and shells was also presented by Dong et al^[43].

Ashton^[44] suggested an approximate analysis for unsymmetrical laminated plates, in which in-plane and out-of-plane coupling terms (B_{ij}^*) were deleted, to analyse non-symmetric laminates as homogeneous plates with reduced bending stiffness D_{ij}^* replacing bending stiffness D_{ij} . This suggestion was originally made by Reissner and Stavsky^[31], but Whitney's research pointed out that for certain laminated plates the reduced bending stiffness approximation does not give acceptable agreement with coupled laminate solutions.^[45]

In industry, midplane symmetrically laminated plates are widely used because they exhibit no in-plane and out-of-plane coupling. But there usually exists coupling between tension and shear in such laminates when the fibre orientations of layers are not coincident with the edges of laminated plates. These plates have mathematically the same governing equations as the homogeneous anisotropic plates. The main difficulties in analysing such plates results from the tension and shear coupling terms (D_{16} , D_{26} or D_{16}^* , D_{26}^*). Wang^[46] concluded that for such plates there exists no deformation surface which may be represented mathematically in a separable form in order to satisfy simply supported boundary conditions if the Kirchhoff-Love non-deformable normal hypothesis is adopted.

In order to analyse such plates various methods are used to investigate approximately their response. Chamis^[47] [48] introduced the Galerkin method into buckling analysis for such anisotropic plates. The equilibrium equation then took a more general form, though there were a few mistakes in his formulations

as pointed out by Hsu^[49]. Ashton^[50] found the governing equation of anisotropic plates to be mathematically similar to that of skew isotropic ones in oblique coordinates. An analogy technique was suggested by him to obtain exact solutions for anisotropic plates, compared with existing results of skew plates in a few special cases^[51] ^[52]. Furthermore, Whitney et al ^[53] ^[54] ^[55] modified the Fourier series method, which was suggested by Green^[56] for homogeneous isotropic plates, to investigate anisotropic plates. This approach is a useful tool for solving such problems. The Ritz procedure used by Ashton^[57] proves to be also very useful for investigating the gross response of anisotropic plates, such as deflection, buckling, frequencies of natural vibration, and so on.

A great deal of initial buckling analyses for plates with various lay-ups, different loading, boundary and geometry conditions may be found in the literature. Jones et al^[58] and Jones himself^[59] used exact solutions to study the buckling of rectangular plates with unsymmetrical crossply and angle-ply lay-up. His results show that the effects of unsymmetrical laminating can be more severe in reducing the buckling load than exists for the antisymmetric case. Turvey^[60] ^[61] extended the buckling analysis to include the effect of a general Winkler/Pasternak foundation. Fraser et al^[62] used a generalised Ritz method, assuming a complete Fourier series in conjunction with the Lagrange multiplier technique of minimization to obtain upper and lower bounds of the buckling load of orthotropic clamped plates. Kicher and Mandell^[63] did an experimental and theoretical study for simply supported laminated plates. More buckling research may be found in the literature^[64] ^[65] ^[66].

The effect of transverse shear deformation on buckling behaviour of symmetrically laminated plates was investigated by Whitney^[67] and Vinson and Chou^[68]. They showed that transverse shear deformation can be significant, not only in altering the magnitudes of the buckling loads, but also in changing their mode shapes when the ratio of thickness to the other dimensions is relatively large. Turvey^[69] combined the reduced stiffness concept and transverse shear deformation theory to analyse the stability of unsymmetrically laminated plates. The effect of variable thickness and non-linear stress-strain relationship on the buckling of laminated plates may be found in references [41] and [70]. Additional theoretical results are contained in references [71], [72], [73] and [74].

The experimental study^{[38] [39] [63] [75]} of the buckling of laminated plates shows that linear buckling analyses usually give a reasonably good measure of critical load. In most cases the "Southwell plot" technique was used to determine these buckling loads.

As far as the postbuckling study of laminated plates is concerned, the literature in this field is considerably limited, though an excellent book by Chia^[76] appeared in 1980. In this book there are two chapters dealing with the postbuckling behaviour of laminated plates with symmetric or unsymmetric lay-ups.

In 1952 Yusuff published his paper^[77] in which the large deflection theory for homogeneous orthotropic rectangular plates in the Von Karman sense was established. Initial deviation in the plate under edge compression is also taken into account. In 1964 Stavsky^[34] presented a general non-linear theory for

laminated plates. Variable thickness, initial deviation and a variable elastic foundation are all considered in conjunction with the in-plane and out-of-plane effects of generally laminated plates. The governing system is complicated, and no general solution was suggested.

According to reference [78], the practical study of postbuckling behaviour of laminates started in the later years of the 1960s^[79]. In the same year of 1971, Turvey^[80], Chan^[81] and Islam^[82] presented their Ph.D. dissertations, in which postbuckling behaviour of laminated plates was studied. Later on Turvey and Wittrick^[83] published theoretical results of the dynamic relaxation method, based on the former's thesis. It was concluded that for antisymmetric angle-ply laminates the in-plane and out-of-plane coupling terms (B_{ij}) do not affect the postbuckling plate stiffness, though they do greatly reduce the critical load. Harris^[84] found, consequently, that the stiffness change of postbuckled laminates was mainly caused by a change in mode shape of deformation.

Prabhakara and Chia^[85] [86] introduced the characteristic beam functions into postbuckling analysis for orthotropic plates. Later on Chia and Prabhakara^[87] used proper beam function series for both the stress function and deflection function, as Coan^[88] did for the simple case of isotropic plates, to deal with the postbuckling investigation of unsymmetric laminates. Then, computation results were obtained for the postbuckling behaviour of clamped and simply supported, unsymmetrical, angle-ply plates and clamped, unsymmetrical, cross-ply plates. Subsequently, Prabhakara^[89]

presented an analysis for the postbuckling behaviour of simply supported laminated general cross-ply rectangular plates subjected to biaxial compression. Because of unbalanced layers oriented at 0° and 90° with respect to plate axes, the coupling phenomenon is found to be such that there is bending in the laminate immediately in-plane load is applied. The solution exhibited is no longer bifurcational, which is different from previous work^[90]. The behaviour of laminated plates under combined transverse and in-plane loads has also been investigated^[91] ^[92]. But only orthotropic plates are considered, therefore no coupling effect is included. The effect of membrane boundary conditions and initial deviation on the postbuckling behaviour of orthotropic plates was considered by Banks et al^[93].

Experimental research on postbuckling behaviour of laminated plates is sparse^[94] ^[95] ^[96] ^[97]. In most cases experiments were carried out for either orthotropic or quasi-isotropic plates. It is reported that the experimental results agree reasonably well with theoretical prediction. So far it seems that there exists little testing data for laminated plates with unsymmetric lay-ups^[63].

Both analytical and experimental studies for cylindrically curved panels of laminated composite materials are far fewer than for flat plates. Kudva^[98] presented a finite element analysis for the investigation of the postbuckling behaviour of symmetrically laminated curved panels. Because of the demands of large computer storage and high computing costs, even for symmetrically laminated panels, only a few examples were computed in his Ph.D. thesis of 1979. The effects of boundary conditions, loading conditions, geometry conditions

and stretching-bending coupling on the postbuckling behaviour were not considered. Becker^[99] did an analytical/experimental investigation on the instability of cylindrical panels in 1979. His M.Sc. thesis is even simpler. A ready-made computer code, STAGS (Structural Analysis of General Shells), was used to obtain critical loads for symmetric laminated curved panels, which were compared with experimental results. Experimental research for buckling of laminated curved panels under compression and shear was performed by Wilkins^[100] [101]. The results are also for symmetric lay-ups.

C. Scope of the Present Study

In the current investigation initial buckling and post-buckling behaviour is studied for unstiffened flat and cylindrically curved panels of general laminated composite materials. The study is analytical and extensive results are obtained from computations.

Chapter II gives the theoretical background of the present study. The governing system in the Von Karman sense is obtained from the virtual displacement and virtual force principles. Because of the completely general lay-up considered, all the 36 coefficients (21 independent) in the constitutive relations are taken into account, and the effect of boundary moments is included in the governing equations. In the third section of this chapter the precision of the beam functions used by previous workers is discussed, and those with improved precision are given for subsequent use.

Chapter III is devoted to initial buckling investigations. In the first section of this chapter curved panels with movable edges are considered. Extensive parameter studies are carried

out for simply supported and clamped boundaries. A new relationship between combined in-plane compression and shear is found, which is to be seen in other work. It shows that combined shear in certain directions can greatly stabilise the laminated panels against compression rather than always destabilising, as for isotropic and orthotropic plates. It is found to be true for curved panels and flat plates alike. The second section discusses panels with fixed edges. The emphasis is clearly put on the effect of alternate shear directions on the initial buckling characteristics of laminated panels. It is proved that the same effect exists even for combined biaxial compressive and shear loading.

Chapter IV deals with postbuckling behaviour. Von Karman form nonlinear terms are included in the governing system. Both double sine series and double sine squared series are assumed to express the deformed surface of the panel with simply supported edges. The double sine squared series is intended to represent the tendency for inward buckling of curved elements. This series has not been found in previous work concerning postbuckling analysis of laminated panels. In the following section of Chapter IV in-plane shear loading is considered. The results show that the applied shear directions significantly affect the postbuckling behaviour of laminated plates. This fact probably has never been reported in the literature.

Chapter V describes the bending effect of laminated panels under in-plane loading. First of all, laminated panels with unsymmetric crossply layers under compression, and those with unsymmetric angle layers under shear are discussed. In these

cases the panel starts bending immediately load is applied. It means that no bifurcation characteristics are found when load increases. To the author's knowledge, the second case has not been discussed in the literature, even for flat laminates. The effect of initial imperfections is discussed in the following section. The results show that the post-buckling behaviour of curved panels of laminated composite is very sensitive to the initial imperfection. The remaining section of this chapter discusses the behaviour of generally laminated curved panels under the combination of in-plane and lateral loading. The effect of direction of combined shear is also demonstrated. The topics in the last two sections are not found to be discussed for such general lay-ups in previous work elsewhere. At the same time the effect of shear direction on laminated panel behaviour seems not to have been studied in any existing non-linear analyses either.

In the last chapter the author presents the main conclusions which are drawn from the present study and gives some ideas for further investigations in this particular field.

A part of this work has been summarised in several papers [102] [103] [104] which have been, or will be, published in the open literature.

CHAPTER II

THEORY

A. General Remarks

A nonlinear theory for buckling and postbuckling behaviour of generally laminated, cylindrically curved panels under compression and shear is derived in this chapter.

The panel is assumed to consist of layers of composite material (fibre reinforced plastic). These layers are considered to have different thicknesses, different physical properties and different orientations of orthotropic axes with respect to the generator. It is also assumed that the material of the layers is homogeneous and linearly elastic, the total thickness of the panel is small compared with its other dimensions.

Because of the small thickness, the deformation of the panel is considered to obey the well known Kirchhoff hypothesis (mentioned in Chapter I). In the present analysis all thirty-six coefficients in constitutive relations (21 independent) are taken into account to suit the general lay-up situation.

In buckling and postbuckling analysis of isotropic shells and curved panels it has been proved that shallow shell theory [105] usually produces results having good agreement with those of more exact theory^{[106] [107]}. This is due to the large number of halfwaves in buckled shells. Here shallow shell theory is also adopted for the study of buckling and postbuckling behaviour of laminated curved panels.

The governing system is developed on the basis of the virtual displacement principle and the virtual force principle. To allow for the general lay-up, the effect of boundary moments is taken into the governing system. The resulting pair of

coupled nonlinear partial differential equations are both fourth order. The variables in the governing equations are the Airy stress function and the transverse deflection of the panel. To be general, the governing equations are expressed in non-dimensional form.

In this chapter characteristic beam functions, which are to be used for deflection and stress function in the following chapters, are discussed. The beam functions are originally obtained as the normal modes of vibration of beams. They were used for initial buckling analysis of laminated panels by Ashton et al^[37] ^[42] and for postbuckling analysis by Chia et al^[87] ^[76]. In the current analysis, they are used for both initial buckling and postbuckling analysis of laminated curved panels. These functions are required to be orthogonal. The effect of certain constants in the beam functions on their orthogonality is discussed. Using constants with high precision allows us to obtain more exact results in both initial buckling and postbuckling investigations.

B. Governing System

Consider a cylindrically curved panel with total thickness h , length a , mid-surface arc b , and radius R (see Figure (II, 1)). The mid-surface of the plate before deformation contains the reference coordinate system Oxy with the origin at one of the corners. A nonlinear analysis is presented below.

1. Constitutive Relations

An arbitrary point (x, y) on the mid-surface of the panel experiences displacements u_o , v_o and w_o after deformation in x , y and z directions respectively. The strains at this point may be expressed as :

$$\begin{aligned}
\varepsilon_x^0 &= \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \\
\varepsilon_y^0 &= \frac{\partial v_0}{\partial y} + \frac{1}{2} \left(\frac{\partial w_0}{\partial y} \right)^2 - \frac{w_0}{R} \\
\gamma_{xy}^0 &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y}
\end{aligned} \tag{II,1}$$

It is seen that the effect of the curvature has been taken into account on the basis of shallow shell theory.

According to the Kirchhoff hypothesis, the strains at any point of the panel may be written in matrix form :-

$$\{\varepsilon\} = \{\varepsilon^0\} + z \{k\} \tag{II,2}$$

where

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}, \quad \{\varepsilon^0\} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix}, \quad \{k\} = \begin{Bmatrix} -\frac{\partial^2 w_0}{\partial x^2} \\ -\frac{\partial^2 w_0}{\partial y^2} \\ -2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \tag{II,3}$$

Because of the orthotropic properties of the layers, the stresses at the point can be written in terms of the strains in a local Cartesian coordinate system, in which the coordinate l is taken along the fibre direction, t perpendicular to that, i.e.

$$\begin{Bmatrix} \sigma_l \\ \sigma_t \\ \tau_{lt} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_l \\ \varepsilon_t \\ \gamma_{lt} \end{Bmatrix} \tag{II,4}$$

where the coefficients Q_{ij} are related to the elastic properties of the composite.

$$\begin{aligned}
Q_{11} &= \frac{E_l}{1 - \nu_{lt} \nu_{tl}} \\
Q_{22} &= \frac{E_t}{1 - \nu_{lt} \nu_{tl}} \\
Q_{12} &= \frac{\nu_{lt}}{1 - \nu_{lt} \nu_{tl}} E_t = \frac{\nu_{tl}}{1 - \nu_{lt} \nu_{tl}} E_l \\
Q_{66} &= G_{lt}
\end{aligned} \tag{II,5}$$

in which E_l and E_t are elastic moduli along and perpendicular to the fibre directions, respectively.

ν_{lt} and ν_{tl} are Poisson's ratios in two directions and, G_{lt} is the shear modulus.

The stress at the point may also be written in the general coordinate system of the panel

i.e.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{II,6}$$

The coefficients Q_{ij} are obtained from the following matrix expression:

$$\begin{Bmatrix} \bar{Q}_{11} \\ \bar{Q}_{12} \\ \bar{Q}_{22} \\ \bar{Q}_{16} \\ \bar{Q}_{26} \\ \bar{Q}_{66} \end{Bmatrix} = \begin{bmatrix} c^4 & 2c^2s^2 & s^4 & 4c^2s^2 \\ c^2s^2 & c^4+s^4 & c^2s^2 & -4c^2s^2 \\ s^4 & 2c^2s^2 & c^4 & 4c^2s^2 \\ c^3s & cs^3-c^3s & -cs^3 & -2cs(c^2s^2) \\ cs^3 & c^3s-cs^3 & -c^3s & 2cs(c^2s^2) \\ c^2s^2 & -2c^2s^2 & c^2s^2 & (c^2-s^2)^2 \end{bmatrix} \begin{Bmatrix} Q_{11} \\ Q_{12} \\ Q_{22} \\ Q_{66} \end{Bmatrix} \tag{II,7}$$

where $c = \cos \theta$

$s = \sin \theta$

(II,8)

with θ being the angle of the fibres with respect to the panel

coordinates (see Figure (II,1)).

The stress resultants and couples of the panel are defined by

$$\begin{aligned} (N_x, N_y, N_{xy}) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x^{(k)}, \sigma_y^{(k)}, \tau_{xy}^{(k)}) dz \\ (M_x, M_y, M_{xy}) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x^{(k)}, \sigma_y^{(k)}, \tau_{xy}^{(k)}) z dz \end{aligned} \quad (\text{II},9)$$

in which N_x , N_y and N_{xy} are membrane forces, M_x , M_y and M_{xy} are bending and twisting moments, all per unit width (see Figure (II,2)). On the right hand side the superscript (k) indicates the values of the stresses referred to the k_{th} layer in the panel.

Substituting the stress expression (II,6) into (II,9), considering (II,2) and (II,3), the constitutive relations of the panel may be obtained in the following matrix form:

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^0 \\ \mathbf{k} \end{Bmatrix} \quad (\text{II},10)$$

where

$$\mathbf{N} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}, \quad \mathbf{M} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}, \quad \boldsymbol{\varepsilon}^0 = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix}, \quad \mathbf{k} = \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \quad (\text{II},11)$$

$$\text{and } k_x = -\frac{\partial^2 w_0}{\partial x^2}, \quad k_y = -\frac{\partial^2 w_0}{\partial y^2}, \quad k_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}$$

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{21} & A_{22} & A_{26} \\ A_{61} & A_{62} & A_{66} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{21} & B_{22} & B_{26} \\ B_{61} & B_{62} & B_{66} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{21} & D_{22} & D_{26} \\ D_{61} & D_{62} & D_{66} \end{bmatrix} \quad (\text{II},12)$$

The elements in the matrices **A**, **B** and **D** are defined by

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{Q}_{ij}^{(k)}(1, z, z^2) dz$$

where $\bar{Q}_{ij}^{(k)}$ are the reduced stiffnesses of the k_{th} layer.

A_{ij} are known as the membrane rigidities of the panel, B_{ij} the coupling rigidities and D_{ij} the flexural rigidities.

The constitutive relations (II,10) are widely used in the mechanics of composite materials, and may be found almost in every book concerning composite response [71] [108] [109].

In fact, the partial inverted version of the constitutive relations of (II,10) are more convenient to use in the present investigation. This takes the form

$$\begin{Bmatrix} \epsilon^0 \\ M \end{Bmatrix} = \begin{bmatrix} A^* & B^* \\ -(B^*)^t & D^* \end{bmatrix} \begin{Bmatrix} N \\ k \end{Bmatrix} \quad (\text{II},13)$$

$$\text{in which } A^* = A^{-1}, B^* = -A^{-1}B, D^* = D - BA^{-1}B \quad (\text{II},14)$$

where

$$A^* = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{21}^* & A_{22}^* & A_{26}^* \\ A_{61}^* & A_{62}^* & A_{66}^* \end{bmatrix} \quad B^* = \begin{bmatrix} B_{11}^* & B_{12}^* & B_{16}^* \\ B_{21}^* & B_{22}^* & B_{26}^* \\ B_{61}^* & B_{62}^* & B_{66}^* \end{bmatrix} \quad D^* = \begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ D_{21}^* & D_{22}^* & D_{26}^* \\ D_{61}^* & D_{62}^* & D_{66}^* \end{bmatrix} \quad (\text{II},15)$$

The A_{ij}^* , B_{ij}^* and D_{ij}^* are known as the reduced rigidities of the panels. The matrices **A**^{*} and **D**^{*} are symmetric, but **B**^{*} is not symmetric.

For special lay-ups the constitutive relations become simpler. For example, for specially orthotropic panels $B_{ij} = 0$ (therefore $B_{ij}^* = 0$) and $A_{16}^* = A_{26}^* = A_{61}^* = A_{62}^* = D_{16}^* = D_{26}^* = D_{61}^* = D_{62}^* = 0$ (also $A_{16}^* = A_{26}^* = A_{61}^* = A_{62}^* = D_{16}^* = D_{26}^* = D_{61}^* = D_{62}^* = 0$).

2. Governing Equations

One of the most important principles in the mechanics of an elastic system subjected to conservative loading might be the stationary potential energy principle. The virtual displacement principle and virtual force principle are its different versions.

In order to obtain the equilibrium equation, the virtual displacement principle is used for the composite panel under combined compressive and shear loading,

i.e.

$$\delta\pi = \delta U - \delta W \quad (\text{II}, 16)$$

where $\delta\pi$ variation of total potential energy of the panel

δU variation of inner potential energy of the panel

δW variation of external work.

Now

$$\delta U = \int_0^a \int_0^b (N_x \delta \epsilon_x^0 + N_y \delta \epsilon_y^0 + N_{xy} \delta \gamma_{xy}^0 + M_x \delta k_x + M_y \delta k_y + M_{xy} \delta k_{xy}) dx dy \quad (\text{II}, 17)$$

$$\delta W = - \int_0^b P_x \delta u_0 \Big|_{x=0}^{x=a} dy + \int_0^a P_{xy} \delta u_0 \Big|_{y=0}^{y=a} dx + \int_0^b P_{xy} \delta v_0 \Big|_{x=0}^{x=a} dy \quad (\text{II}, 18)$$

where P_x is the applied axial compression, P_{xy} the applied shear, all per unit width (see Figure II,1).

Taking the variation of the displacements, the strain variations can be obtained, for example,

$$\delta \epsilon_x^0 = \delta \left[\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \right] = \frac{\partial}{\partial x} (\delta u_0) + \frac{\partial w_0}{\partial x} \frac{\partial}{\partial x} (\delta w_0) \quad (\text{II}, 19)$$

The other strain variations may be found in Appendix A.

Substituting the resultant strain variations into equation (II,17), considering (II,18), then integrating by parts, finally, (II,16) becomes as follows (see Appendix A):

$$\begin{aligned}
\delta \Pi = & - \int_0^a \int_0^b \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u_0 dx dy - \int_0^a \int_0^b \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) \delta v_0 dx dy \\
& - \int_0^a \int_0^b \left\{ \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{N_y}{R} + N_x \frac{\partial^2 w_0}{\partial x^2} + N_y \frac{\partial^2 w_0}{\partial y^2} \right. \\
& + 2 N_{xy} \frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial w_0}{\partial x} \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) + \frac{\partial w_0}{\partial y} \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) \left. \right\} \delta w_0 dx dy \\
& + \int_0^b \left[(N_x + P_x) \delta u_0 \right] \Big|_{x=0}^{x=a} dy + \int_0^a \left[(N_{xy} - P_{xy}) \delta u_0 \right] \Big|_{y=0}^{y=b} dx \\
& + \int_0^b \left[(N_{xy} - P_{xy}) \delta v_0 \right] \Big|_{x=0}^{x=a} dy + \int_0^a \left[\left(N_x \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) \delta w_0 \right] \Big|_{x=0}^{x=a} dy \\
& + \int_0^a \left[\left(N_y \frac{\partial w_0}{\partial y} + N_{xy} \frac{\partial w_0}{\partial x} \right) \delta w_0 \right] \Big|_{y=0}^{y=b} dx + \int_0^a (N_y \delta v_0) \Big|_{y=0}^{y=b} dx \\
& + \int_0^b \left[\frac{\partial M_x}{\partial x} \delta w_0 - M_x \delta \left(\frac{\partial w_0}{\partial x} \right) \right] \Big|_{x=0}^{x=a} dy + \int_0^a \left[\frac{\partial M_y}{\partial y} \delta w_0 - M_y \delta \left(\frac{\partial w_0}{\partial y} \right) \right] \Big|_{y=0}^{y=b} dx \\
& + \int_0^b \left[\frac{\partial M_{xy}}{\partial y} \delta w_0 - M_{xy} \delta \left(\frac{\partial w_0}{\partial y} \right) \right] \Big|_{x=0}^{x=a} dy + \int_0^a \left[\frac{\partial M_{xy}}{\partial x} \delta w_0 - M_{xy} \delta \left(\frac{\partial w_0}{\partial x} \right) \right] \Big|_{y=0}^{y=b} dx = 0
\end{aligned} \tag{II, 20}$$

The first two double integrals in the last equation represent the equilibrium of inplane forces in x and y directions respectively. The third one is that for transverse equilibrium. The line integrals are known as natural boundary conditions.

Introducing the stress function ϕ defined by the following expressions

$$N_x = \frac{\partial^2 \phi}{\partial y^2}, \quad N_y = \frac{\partial^2 \phi}{\partial x^2}, \quad N_{xy} = - \frac{\partial^2 \phi}{\partial x \partial y} \tag{II, 21}$$

the first two double integrals become zero automatically, and the third integral is simplified at the same time.

In the present investigation panels with simply supported or clamped edges are under consideration. The kinematic boundary

conditions require no deflection on the edges. Also on the boundaries it is necessary to keep the inner force of the panel in equilibrium with the external force. In that case, equation (II,20) is simplified as follows:

$$\int_0^a \int_0^b \left[\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{N_y}{R} + N_x \frac{\partial^2 w_0}{\partial x^2} + N_y \frac{\partial^2 w_0}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w_0}{\partial x \partial y} \right] \delta w_0 dx dy$$

$$+ \int_0^b \left[M_x \delta \left(\frac{\partial w_0}{\partial x} \right) \right] \Big|_{x=0}^{x=a} dy + \int_0^a \left[M_y \delta \left(\frac{\partial w_0}{\partial y} \right) \right] \Big|_{y=0}^{y=b} dx = 0 \quad (\text{II}, 22)$$

in which the two line integrals represent the effect of the boundary moments. With these terms, the zero boundary moment requirements of the simply supported edges may be satisfied in the mean when the deflection function does not satisfy them.

From the three equations of the expressions (II,1) the displacement u_0 and v_0 may be eliminated by properly taking the derivatives of ξ_x^0 , ξ_y^0 and γ_{xy}^0 with respect to x and y . The compatibility equation may, then, be obtained from the virtual force principle,

$$\text{i.e.} \quad \int_0^a \int_0^b \left[\frac{\partial^2 \xi_x^0}{\partial y^2} + \frac{\partial^2 \xi_y^0}{\partial x^2} - \frac{\partial^2 \gamma_{xy}^0}{\partial x \partial y} - \left(\frac{\partial^2 w_0}{\partial x \partial y} \right)^2 + \frac{\partial^2 w_0}{\partial x^2} \frac{\partial^2 w_0}{\partial y^2} + \frac{1}{R} \frac{\partial^2 w_0}{\partial x^2} \right] \delta \phi dx dy = 0$$

(II,23)

The pair of equations (II,22) and (II,23) make up the governing system of the present investigation.

Using the partial inverted form of the constitutive relations (II,13), the governing system can be expressed in terms of two variables - the stress function ϕ and the deflection function w_0 of the panel,

i.e.

$$\begin{aligned}
& \int_0^a \int_0^b \left[D_{11}^* \frac{\partial^4 w_0}{\partial x^4} + 4D_{16}^* \frac{\partial^4 w_0}{\partial x^3 \partial y} + 2(D_{12}^* + 2D_{66}^*) \frac{\partial^4 w_0}{\partial x^2 \partial y^2} + 4D_{26}^* \frac{\partial^4 w_0}{\partial x \partial y^3} + D_{22}^* \frac{\partial^4 w_0}{\partial y^4} - \frac{1}{R} \frac{\partial^2 \phi}{\partial x^2} \right. \\
& + B_{21}^* \frac{\partial^4 \phi}{\partial x^4} + (2B_{26}^* - B_{61}^*) \frac{\partial^4 \phi}{\partial x^3 \partial y} + (B_{11}^* + B_{22}^* - 2B_{66}^*) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + (2B_{16}^* - B_{62}^*) \frac{\partial^4 \phi}{\partial x \partial y^3} \\
& + B_{12}^* \frac{\partial^4 \phi}{\partial y^4} - \frac{\partial^2 \phi}{\partial y^2} \frac{\partial^2 w_0}{\partial x^2} + 2 \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial^2 w_0}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 w_0}{\partial y^2} \left. \right] \delta w_0 dx dy \\
& + \int_0^b \left[(B_{11}^* \frac{\partial^2 \phi}{\partial y^2} + B_{21}^* \frac{\partial^2 \phi}{\partial x^2} - B_{61}^* \frac{\partial^2 \phi}{\partial x \partial y} + D_{11}^* \frac{\partial^2 w_0}{\partial x^2} + D_{12}^* \frac{\partial^2 w_0}{\partial y^2} + 2D_{16}^* \frac{\partial^2 w_0}{\partial x \partial y}) \cdot \delta \left(\frac{\partial w_0}{\partial x} \right) \right] \Big|_{x=0}^{x=a} dy \\
& + \int_0^a \left[(B_{12}^* \frac{\partial^2 \phi}{\partial y^2} + B_{22}^* \frac{\partial^2 \phi}{\partial x^2} - B_{62}^* \frac{\partial^2 \phi}{\partial x \partial y} + D_{21}^* \frac{\partial^2 w_0}{\partial x^2} + D_{22}^* \frac{\partial^2 w_0}{\partial y^2} + 2D_{26}^* \frac{\partial^2 w_0}{\partial x \partial y}) \cdot \delta \left(\frac{\partial w_0}{\partial y} \right) \right] \Big|_{y=0}^{y=b} dx \\
& = 0
\end{aligned} \tag{II, 24}$$

$$\begin{aligned}
& \int_0^a \int_0^b \left[A_{22}^* \frac{\partial^4 \phi}{\partial x^4} - 2A_{26}^* \frac{\partial^4 \phi}{\partial x^3 \partial y} + (2A_{12}^* + A_{66}^*) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} - 2A_{16}^* \frac{\partial^4 \phi}{\partial x \partial y^3} + A_{11}^* \frac{\partial^4 \phi}{\partial y^4} \right. \\
& - B_{21}^* \frac{\partial^4 w_0}{\partial x^4} - (2B_{26}^* - B_{61}^*) \frac{\partial^4 w_0}{\partial x^3 \partial y} - (B_{11}^* + B_{22}^* - 2B_{66}^*) \frac{\partial^4 w_0}{\partial x^2 \partial y^2} \\
& - (2B_{16}^* - B_{62}^*) \frac{\partial^4 w_0}{\partial x \partial y^3} - B_{12}^* \frac{\partial^4 w_0}{\partial y^4} + \frac{1}{R} \frac{\partial^2 w_0}{\partial x^2} - \left(\frac{\partial^2 w_0}{\partial x \partial y} \right)^2 + \frac{\partial^2 w_0}{\partial x^2} \frac{\partial^2 w_0}{\partial y^2} \left. \right] \delta \phi dx dy \\
& = 0
\end{aligned} \tag{II, 25}$$

3. Non-dimensional Form of the Governing Equations

To obtain generality, the governing equations (II,24) and (II,25) are nondimensionalised.

Introducing the following nondimensional parameters and constants

$$\left. \begin{aligned}
\bar{x} &= \frac{x}{a}, & \bar{y} &= \frac{y}{b}, & \bar{\beta} &= \frac{a}{b}, \\
\bar{w} &= \frac{w_0}{h}, & \bar{F} &= \frac{\phi}{A_{22} h^2}, & \bar{K}_R &= \frac{b^2}{R h}, \\
\bar{A}_{ij}^* &= A_{22}^* A_{ij}^*, & \bar{B}_{ij}^* &= B_{ij}^* / h, & \bar{D}_{ij}^* &= D_{ij}^* / A_{22} h^2,
\end{aligned} \right\}$$

$$\begin{aligned}
a_1 &= 2\beta \bar{A}_{26}^* / \bar{A}_{22}^* , & a_2 &= \beta^2 (2\bar{A}_{12}^* + \bar{A}_{66}^*) / \bar{A}_{22}^* , & a_3 &= 2\beta^3 \bar{A}_{16}^* / \bar{A}_{22}^* , \\
a_4 &= \beta^4 \bar{A}_{11}^* / \bar{A}_{22}^* , & a_5 &= \bar{B}_{21}^* / \bar{A}_{22}^* , & a_6 &= \beta (2\bar{B}_{26}^* - \bar{B}_{61}^*) / \bar{A}_{22}^* , \\
a_7 &= \beta^2 (\bar{B}_{11}^* + \bar{B}_{22}^* - 2\bar{B}_{66}^*) / \bar{A}_{22}^* , & a_8 &= \beta^3 (2\bar{B}_{16}^* - \bar{B}_{62}^*) / \bar{A}_{22}^* , & a_9 &= \beta^4 \bar{B}_{12}^* / \bar{A}_{22}^* , \\
b_1 &= 4\beta \bar{D}_{16}^* / \bar{D}_{11}^* , & b_2 &= 2\beta^2 (\bar{D}_{12}^* + 2\bar{D}_{66}^*) / \bar{D}_{11}^* , & b_3 &= 4\beta^3 \bar{D}_{26}^* / \bar{D}_{11}^* , \\
b_4 &= \beta^4 \bar{D}_{22}^* / \bar{D}_{11}^* , & b_5 &= \bar{B}_{21}^* / \bar{D}_{11}^* , & b_6 &= \beta (2\bar{B}_{26}^* - \bar{B}_{61}^*) / \bar{D}_{11}^* , \\
b_7 &= \beta^2 (\bar{B}_{11}^* + \bar{B}_{22}^* - 2\bar{B}_{66}^*) / \bar{D}_{11}^* , & b_8 &= \beta^3 (2\bar{B}_{16}^* - \bar{B}_{62}^*) / \bar{D}_{11}^* , & b_9 &= \beta^4 \bar{B}_{12}^* / \bar{D}_{11}^* , \\
b_{10} &= \beta^2 \bar{B}_{11}^* / \bar{D}_{11}^* , & b_{11} &= -\beta \bar{B}_{61}^* / \bar{D}_{11}^* , & b_{12} &= \beta^2 \bar{D}_{12}^* / \bar{D}_{11}^* , \\
b_{13} &= \beta^2 \bar{B}_{22}^* / \bar{D}_{11}^* , & b_{14} &= -\beta^3 \bar{B}_{62}^* / \bar{D}_{11}^* , & b_{15} &= \beta^2 \bar{D}_{21}^* / \bar{D}_{11}^* ,
\end{aligned}
\tag{II, 26}$$

The governing system of (II.22) and (II,23) can be written in the following form:

$$\begin{aligned}
& \int_0^1 \int_0^1 \left\{ \frac{\partial^4 W}{\partial \xi^4} + b_1 \frac{\partial^4 W}{\partial \xi^3 \partial \eta} + b_2 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + b_3 \frac{\partial^4 W}{\partial \xi \partial \eta^3} + b_4 \frac{\partial^4 W}{\partial \eta^4} + b_5 \frac{\partial^4 F}{\partial \xi^4} + b_6 \frac{\partial^4 F}{\partial \xi^3 \partial \eta} + b_7 \frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} \right. \\
& \quad + b_8 \frac{\partial^4 F}{\partial \xi \partial \eta^3} + b_9 \frac{\partial^4 F}{\partial \eta^4} - \frac{\beta^2 K_R}{\bar{D}_{11}^*} \frac{\partial^2 F}{\partial \xi^2} - \left(\frac{\beta^2}{\bar{D}_{11}^*} \right) \left(\frac{\partial^2 W}{\partial \xi^2} \frac{\partial^2 F}{\partial \eta^2} + \frac{\partial^2 W}{\partial \eta^2} \frac{\partial^2 F}{\partial \xi^2} - 2 \frac{\partial^2 W}{\partial \xi \partial \eta} \frac{\partial^2 F}{\partial \xi \partial \eta} \right) \Big\} \delta W d\xi d\eta \\
& \quad + \int_0^1 \left[\left(b_{10} \frac{\partial^2 F}{\partial \eta^2} + b_5 \frac{\partial^2 F}{\partial \xi^2} + b_{11} \frac{\partial^2 F}{\partial \xi \partial \eta} + \frac{\partial^2 W}{\partial \xi^2} + b_{12} \frac{\partial^2 W}{\partial \eta^2} + \frac{b_1}{2} \frac{\partial^2 W}{\partial \xi \partial \eta} \right) \cdot \delta \left(\frac{\partial W}{\partial \xi} \right) \right] \Big|_{\xi=0}^{\xi=1} d\eta \\
& \quad + \int_0^1 \left[\left(b_9 \frac{\partial^2 F}{\partial \eta^2} + b_{13} \frac{\partial^2 F}{\partial \xi^2} + b_{14} \frac{\partial^2 F}{\partial \xi \partial \eta} + b_{15} \frac{\partial^2 W}{\partial \xi^2} + b_4 \frac{\partial^2 W}{\partial \eta^2} + \frac{b_3}{2} \frac{\partial^2 W}{\partial \xi \partial \eta} \right) \cdot \delta \left(\frac{\partial W}{\partial \eta} \right) \right] \Big|_{\eta=0}^{\eta=1} d\xi \\
& = 0
\end{aligned}
\tag{II, 27}$$

$$\begin{aligned}
& \int_0^1 \int_0^1 \left\{ \frac{\partial^4 F}{\partial \xi^4} - a_1 \frac{\partial^4 F}{\partial \xi^3 \partial \eta} + a_2 \frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} - a_3 \frac{\partial^4 F}{\partial \xi \partial \eta^3} + a_4 \frac{\partial^4 F}{\partial \eta^4} - a_5 \frac{\partial^4 W}{\partial \xi^4} - a_6 \frac{\partial^4 W}{\partial \xi^3 \partial \eta} - a_7 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} \right. \\
& \quad - a_8 \frac{\partial^4 W}{\partial \xi \partial \eta^3} - a_9 \frac{\partial^4 W}{\partial \eta^4} + \frac{\beta^2 K_R}{\bar{A}_{22}^*} \frac{\partial^2 W}{\partial \xi^2} - \left(\frac{\beta^2}{\bar{A}_{22}^*} \right) \left[\left(\frac{\partial^2 W}{\partial \xi \partial \eta} \right)^2 - \frac{\partial^2 W}{\partial \xi^2} \frac{\partial^2 W}{\partial \eta^2} \right] \Big\} \delta F d\xi d\eta \\
& = 0
\end{aligned}
\tag{II, 28}$$

These two equations are both non-linear fourth order partial differential equations. They are coupled with each other and, therefore, need to be solved simultaneously.

When the panel has special lay-ups, these equations will be simplified. For example, for panels having midsurface symmetric lay-ups we have $a_5=a_6=a_7=a_8=a_9=b_5=b_6=b_7=b_8=b_9=b_{10}=b_{11}=b_{13}=b_{14}=0$; for panels having general crossply situations we have $a_1=a_3=a_6=a_8=b_1=b_3=b_6=b_8=b_{11}=b_{14}=0$. For panels having midsurface antisymmetric angle lay-ups we have $a_1=a_3=a_5=a_7=a_9=b_1=b_3=b_5=b_7=b_9=b_{10}=b_{13}=0$.

At the same time the membrane forces and moments may also be nondimensionalised. They take the following form:

$$\begin{Bmatrix} N_{\xi} \\ N_{\eta} \\ N_{\xi\eta} \end{Bmatrix} = \frac{b^2}{A_{22}h^2} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\beta^2} & 0 \\ 0 & 0 & \frac{1}{\beta} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 F}{\partial \eta^2} \\ \frac{\partial^2 F}{\partial \xi^2} \\ -\frac{\partial^2 F}{\partial \xi \partial \eta} \end{Bmatrix} \quad (\text{II}, 29)$$

$$\begin{Bmatrix} M_{\xi} \\ M_{\eta} \\ M_{\xi\eta} \end{Bmatrix} = \frac{b^2}{A_{22}h^3} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} -B_{11}^* & -B_{21}^* & -B_{61}^* \\ -B_{12}^* & -B_{22}^* & -B_{62}^* \\ -B_{16}^* & -B_{26}^* & -B_{66}^* \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\beta^2} & 0 \\ 0 & 0 & \frac{1}{\beta} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 F}{\partial \eta^2} \\ \frac{\partial^2 F}{\partial \xi^2} \\ -\frac{\partial^2 F}{\partial \xi \partial \eta} \end{Bmatrix} \\ + \begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ D_{21}^* & D_{22}^* & D_{26}^* \\ D_{61}^* & D_{62}^* & D_{66}^* \end{bmatrix} \begin{bmatrix} \frac{1}{\beta^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\beta} \end{bmatrix} \begin{Bmatrix} -\frac{\partial^2 W}{\partial \xi^2} \\ -\frac{\partial^2 W}{\partial \eta^2} \\ -2\frac{\partial^2 W}{\partial \xi \partial \eta} \end{Bmatrix} \quad (\text{II}, 30)$$

The external loads may be similarly nondimensionalised,

giving compression parameter and shear parameter as defined by

$$\begin{aligned}\bar{N}_x &= P_x b^2 / A_{22} h^2 \\ \bar{S} &= P_{xy} b^2 / A_{22} h^2\end{aligned}\tag{II,31}$$

where P_x , P_{xy} are external compressive load and shear load, respectively.

4. Effect of Imperfections

The effect of small initial imperfections on the post-buckling behaviour was studied by Yamaki^{[116] [117]} for isotropic flat plates. A similar procedure, however, may be used for the postbuckling study of curved panels of laminated composite materials.

In a thin imperfect panel the strains at any point of the middle surface may be described as follows instead of by (II,1) for a perfect panel,

$$\begin{aligned}\epsilon_x^o &= \frac{\partial u_o}{\partial x} + \frac{1}{2} \left(\frac{\partial \bar{w}_o}{\partial x} \right)^2 + \frac{\partial \bar{w}_o}{\partial x} \frac{\partial \bar{w}_o}{\partial x} \\ \epsilon_y^o &= \frac{\partial u_o}{\partial y} + \frac{1}{2} \left(\frac{\partial \bar{w}_o}{\partial y} \right)^2 - \frac{\bar{w}_o}{R} + \frac{\partial \bar{w}_o}{\partial y} \frac{\partial \bar{w}_o}{\partial y} \\ \gamma_{xy}^o &= \frac{\partial u_o}{\partial y} + \frac{\partial \bar{v}_o}{\partial x} + \frac{\partial \bar{w}_o}{\partial x} \frac{\partial \bar{w}_o}{\partial y} + \frac{\partial \bar{w}_o}{\partial x} \frac{\partial \bar{w}_o}{\partial y} + \frac{\partial \bar{w}_o}{\partial y} \frac{\partial \bar{w}_o}{\partial x}\end{aligned}\tag{II,32}$$

where $\bar{w}_o$ is used to express the geometry imperfection of the panel before deformation.

Following the same procedure as that presented for the perfect panels in the former paragraphs of this chapter, the governing equations in nondimensionalised form are expressed as

$$\begin{aligned}
& \int_0^1 \int_0^1 \left\{ \frac{\partial^4 F}{\partial \xi^4} - a_1 \frac{\partial^4 F}{\partial \xi^3 \partial \eta} + a_2 \frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} - a_3 \frac{\partial^4 F}{\partial \xi \partial \eta^3} + a_4 \frac{\partial^4 F}{\partial \eta^4} - a_5 \frac{\partial^4 W}{\partial \xi^4} - a_6 \frac{\partial^4 W}{\partial \xi^3 \partial \eta} - a_7 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} \right. \\
& - a_8 \frac{\partial^4 W}{\partial \xi \partial \eta^3} - a_9 \frac{\partial^4 W}{\partial \eta^4} + \frac{\beta^2 K_R}{A_{22}^*} \frac{\partial^2 W}{\partial \xi^2} - \frac{\beta^2}{A_{22}^*} \left[\left(\frac{\partial^2 W}{\partial \xi \partial \eta} \right)^2 - \frac{\partial^2 W}{\partial \xi^2} \frac{\partial^2 W}{\partial \eta^2} \right. \\
& \left. \left. - \frac{\partial^2 W}{\partial \xi^2} \frac{\partial^2 \bar{W}}{\partial \eta^2} - \frac{\partial^2 W}{\partial \eta^2} \frac{\partial^2 \bar{W}}{\partial \xi^2} + 2 \frac{\partial^2 W}{\partial \xi \partial \eta} \frac{\partial^2 \bar{W}}{\partial \xi \partial \eta} \right] \right\} \delta F dx dy \\
& = 0
\end{aligned}$$

(II, 33)

$$\begin{aligned}
& \int_0^1 \int_0^1 \left\{ \frac{\partial^4 W}{\partial \xi^4} + b_1 \frac{\partial^4 W}{\partial \xi^3 \partial \eta} + b_2 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + b_3 \frac{\partial^4 W}{\partial \xi \partial \eta^3} + b_4 \frac{\partial^4 W}{\partial \eta^4} + b_5 \frac{\partial^4 F}{\partial \xi^4} + b_6 \frac{\partial^4 F}{\partial \xi^3 \partial \eta} + b_7 \frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} \right. \\
& + b_8 \frac{\partial^4 F}{\partial \xi \partial \eta^3} + b_9 \frac{\partial^4 F}{\partial \eta^4} - \frac{\beta^2 K_R}{D_{11}^*} \frac{\partial^2 F}{\partial \xi^2} - \left(\frac{\beta^2}{D_{11}^*} \right) \left[\frac{\partial^2 W}{\partial \xi^2} \frac{\partial^2 F}{\partial \eta^2} + \frac{\partial^2 W}{\partial \eta^2} \frac{\partial^2 F}{\partial \xi^2} \right. \\
& \left. - 2 \frac{\partial^2 W}{\partial \xi \partial \eta} \frac{\partial^2 F}{\partial \xi \partial \eta} + \frac{\partial^2 \bar{W}}{\partial \xi^2} \frac{\partial^2 F}{\partial \eta^2} + \frac{\partial^2 \bar{W}}{\partial \eta^2} \frac{\partial^2 F}{\partial \xi^2} - 2 \frac{\partial^2 \bar{W}}{\partial \xi \partial \eta} \frac{\partial^2 F}{\partial \xi \partial \eta} \right] \right\} \delta W d\xi d\eta \\
& + \int_0^1 \left[\left(b_{10} \frac{\partial^2 F}{\partial \eta^2} + b_5 \frac{\partial^2 F}{\partial \xi^2} + b_{11} \frac{\partial^2 F}{\partial \xi \partial \eta} + \frac{\partial^2 W}{\partial \xi^2} + b_{12} \frac{\partial^2 W}{\partial \eta^2} + \frac{b_1}{2} \frac{\partial^2 W}{\partial \xi \partial \eta} \right) \delta \left(\frac{\partial W}{\partial \xi} \right) \right]_{\xi=0}^{\xi=1} d\eta \\
& + \int_0^1 \left[\left(b_9 \frac{\partial^2 F}{\partial \eta^2} + b_{13} \frac{\partial^2 F}{\partial \xi^2} + b_{14} \frac{\partial^2 F}{\partial \xi \partial \eta} + b_{15} \frac{\partial^2 W}{\partial \xi^2} + b_4 \frac{\partial^2 W}{\partial \eta^2} + \frac{b_3}{2} \frac{\partial^2 W}{\partial \xi \partial \eta} \right) \delta \left(\frac{\partial W}{\partial \eta} \right) \right]_{\eta=0}^{\eta=1} d\xi \\
& = 0
\end{aligned}$$

(II, 34)

where $\bar{w} = \frac{\bar{w}_0}{h}$ is the nondimensional imperfection.

The expressions for nondimensional inner forces and moments and external loads have the same form as those for perfect panels, that is, the equations (II,29), (II,30) and (II,31) are still valid for the imperfect panel.

C. Beam Functions

Beam eigenfunctions (or characteristic beam functions) were originally obtained as the normal modes of vibration of beams. As mentioned before, they have been used extensively

to solve problems of initial buckling and postbuckling of laminated panels by Ashton et al and Chia et al. These functions are proved to be very effective and convenient to use. One of the most important reasons is that they are orthogonal functions.

The beam functions may have different expressions corresponding to different boundary condition requirements. In the present study the selected functions are defined by

$$f_i(\xi) = \cosh \alpha_i \xi - \cos \alpha_i \xi - \gamma_i (\sinh \alpha_i \xi - \sin \alpha_i \xi) \quad (i=1, 2, \dots) \quad (\text{II}, 35)$$

where α_i and γ_i are constants. With certain values of α_i and γ_i the beam functions satisfy the following boundary conditions:

At $\xi = 0, 1$

$$f(\xi) = \frac{df(\xi)}{d\xi} = 0 \quad (\text{II}, 36)$$

The values of α_i and γ_i may be obtained by solving the following two equations:

$$\begin{aligned} \cosh \alpha_i - \cos \alpha_i - \gamma_i (\sinh \alpha_i - \sin \alpha_i) &= 0 \\ \sinh \alpha_i + \sin \alpha_i - \gamma_i (\cosh \alpha_i - \cos \alpha_i) &= 0 \end{aligned} \quad (\text{II}, 37)$$

Therefore α_i are the roots of the following equation:

$$(\cosh \alpha_i - \cos \alpha_i)^2 - (\sinh^2 \alpha_i - \sin^2 \alpha_i) = 0 \quad (\text{II}, 38)$$

γ_i , then, may be obtained as

$$\gamma_i = (\cosh \alpha_i - \cos \alpha_i) / (\sinh \alpha_i - \sin \alpha_i) \quad (\text{II}, 39)$$

With such α_i and γ_i the orthogonality of the beam functions is also obtained, that is,

$$\int_0^1 f_i(\xi) f_j(\xi) d\xi = \begin{cases} 0 & \text{when } i \neq j \\ 1 & \text{when } i = j \end{cases} \quad (\text{II}, 40)$$

The values of α_i and γ_i have been given in references [110], [76] and listed in Table(II,1). These values of α_i and γ_i have six

significant figures. But it is found, however, that the precision of these constants has significant influence on the results. The values of α_i and γ_i with 6 significant figures is sometimes not enough to give an exact solution for buckling and postbuckling studies of laminated panels. (In the next chapter it is seen that in certain examples more than 6% error can result). This is because with only six significant figures for α_i and γ_i the orthogonality requirement for the beam functions may not be satisfied adequately.

For instance $\int_0^1 f_4(\xi) f_4(\xi) d\xi$ is 0.965 instead of 1 and $\int_0^1 f_4(\xi) f_5(\xi) d\xi$ is 0.032 instead of 0. The reason can be seen by rewriting $f_i(\xi)$ as

$$f_i(\xi) = \frac{1-\gamma_i}{2} e^{\alpha_i \xi} + \frac{1+\gamma_i}{2} e^{-\alpha_i \xi} - \cos \alpha_i \xi + \gamma_i \sin \alpha_i \xi \quad (\text{II},38)$$

Taking 1.00000 for the value of γ_i implies that the first term on the right hand side of (II,38) is deleted in computation. However, since the exponent α_i may be very large, this term is not zero if $\gamma_i \neq 1$.

More exact values for α_i and γ_i , therefore, are suggested for use in the beam functions when relatively large values of i are used. The α_i and γ_i with 15 significant figures are given in Table(II,1).

Table (II,1). The values of α_i and γ_i

Values used in $[110]$ and $[76]$			Values used in present work	
i	α_i	γ_i	α_i	γ_i
1	4.73004	0.982502	4.73004074486270	0.982502214576238
2	7.85320	1.00078	7.85320462409584	1.000777311907269
3	10.9956	0.999967	10.9956078380016	0.999966450125409
4	14.1372	1.00000	14.1371654912575	1.000001449897656
5	$(11/2)\pi$	1.00000	17.2787596573995	0.999999937344383
6	$(13/2)\pi$	1.00000	20.4203522456260	1.000000002707595
7	$(15/2)\pi$	1.00000	23.5619449020404	0.999999999882994
8	$(17/2)\pi$	1.00000	26.7035375555082	1.000000000005056
9	$(19/2)\pi$	1.00000	29.8451302091033	0.999999999999781

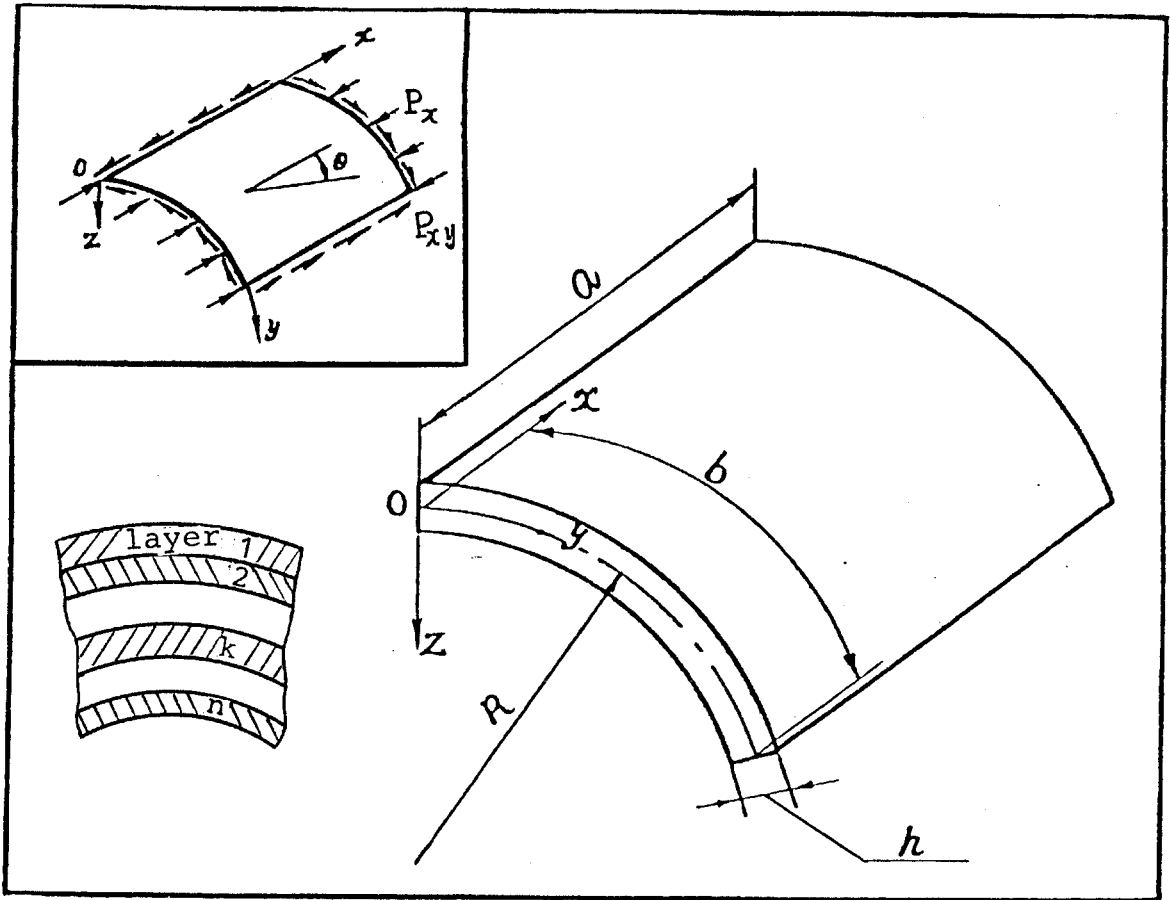


Fig. (II,1) Definition of geometry and external forces

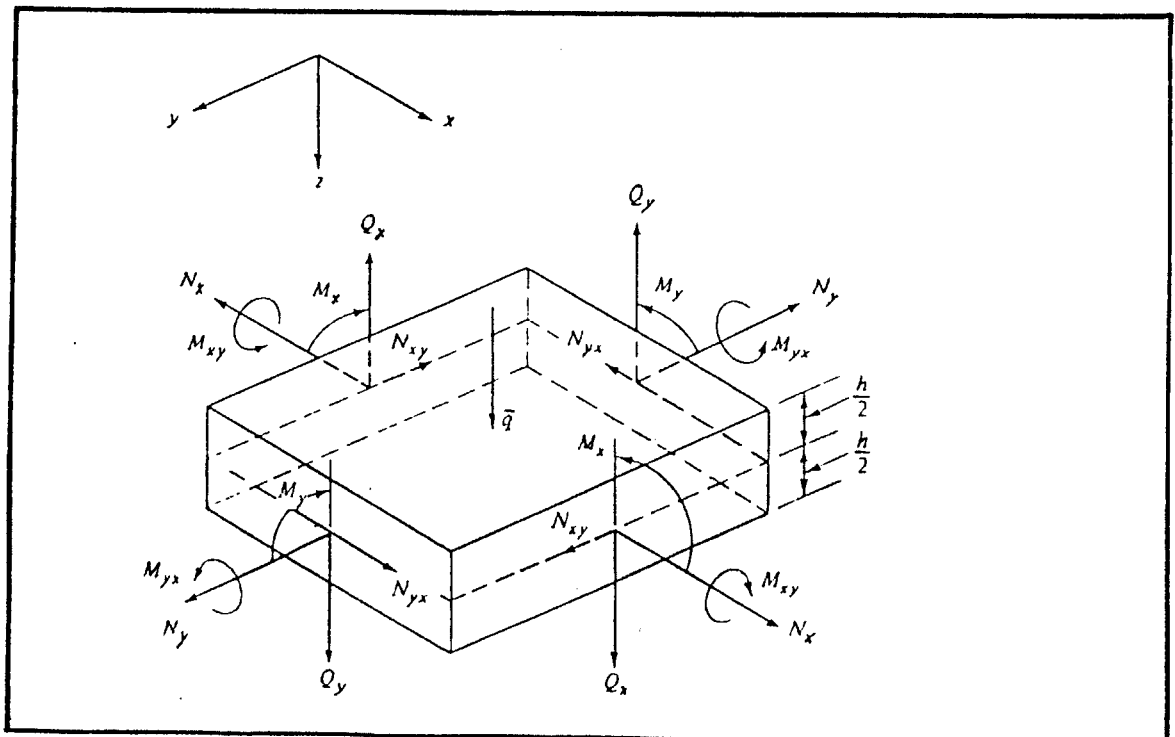


Fig. (II,2) Sense of positive stress resultants and couples in rectangular Cartesian system

CHAPTER III

INITIAL BUCKLING

A. General Remarks

This chapter deals with the initial buckling of laminated panels. Initial buckling analysis is important, not only because it creates a starting point for postbuckling study, but also because it gives an adequate measure for stability of the investigated system itself. A parameter study for initial buckling load may give engineers certain ideas of how to arrange the materials for highest resistance against buckling.

In this chapter panels with different boundary conditions are investigated. Section B of this chapter is devoted to those with movable edges (inplane stress zero on the boundaries of the panels), whilst section C is for those with immovable edges (inplane movements completely restrained on the boundaries). In section B four different flexural boundary conditions are involved, that is, all edges of the panel are simply supported, all edges are clamped, curved edges are simply supported, straight edges clamped, and vice versa.

Initial buckling problems may usually be reduced mathematically to one of finding the eigenvalues of the load parameter and the corresponding eigenvectors. The lowest load given by these eigenvalues is usually of most interest and is known as the critical load of the panel. In the current investigation a set of linear governing equations is solved simultaneously by assuming adequate beam functions for the stress function and transverse deflection function. From the resultant characteristic equations in matrix form the QZ algorithm is used to obtain the critical load and corresponding eigenvector which represents the buckling mode shape of the panel at the moment

of buckling.

In section B of this chapter extensive results are given from computations. The effects of curvature, lay-up arrangements, applied shear directions, boundary conditions and aspect ratios of the panels are studied. The compressive and shear loading is assumed to be applied to the panels, either simultaneously or separately.

A new relationship between the compressive load and combined shear load is found. It shows that the combined shear directions have significant influence on the buckling resistance of the panel against compressive loading. The applied shear in a certain direction may greatly stabilise the panels under compression rather than always being destabilising, as for isotropic or orthotropic panels. This effect is found for panels with midplane symmetric lay-ups. It is true for flat plates and curved panels alike, although panel curvature reduces the effect. This effect becomes greater when the ratio between major and minor modulus of the layers (E_l/E_t) increases and when the number of alternate layers decreases. It seems to the author that this kind of effect has not yet been found in any previous work, though it is pointed out by Ashton et al that there is a preferred shear direction for laminated flat plates under pure shear^[39].

In section C of this chapter special emphasis is put on the effect of shear direction on the buckling of panels with immovable edges. As expected, the same effect is again found in panels with midplane symmetric lay-ups. In addition to that, the effect of applied shear direction is also found in plates with asymmetric lay-ups and ones under the combination of bidirectional compression and shear.

B. Initial Buckling of Curved Panels of Generally Layered Composite Materials without In-plane Restraint

1. Governing System

For a thin, generally laminated, cylindrically curved panel (see Fig. (II,1)) under axial compression and shear, the nondimensional governing equations for initial buckling may be obtained from equations (II,27) and (II,28) by dropping all nonlinear terms and replacing the second derivative terms of the stress function F by corresponding load parameters. The governing system will then take the following form:

$$\int_0^1 \int_0^1 \left\{ \frac{\partial^4 F}{\partial \xi^4} - a_1 \frac{\partial^4 F}{\partial \xi^3 \partial \eta} + a_2 \frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} - a_3 \frac{\partial^4 F}{\partial \xi \partial \eta^3} + a_4 \frac{\partial^4 F}{\partial \eta^4} - a_5 \frac{\partial^4 W}{\partial \xi^4} - a_6 \frac{\partial^4 W}{\partial \xi^3 \partial \eta} - a_7 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} - a_8 \frac{\partial^4 W}{\partial \xi \partial \eta^3} - a_9 \frac{\partial^4 W}{\partial \eta^4} + \frac{\beta^2 K_R}{A_{22}^*} \frac{\partial^2 W}{\partial \xi^2} \right\} \delta F d\xi d\eta = 0 \quad (\text{III}, 1)$$

$$\begin{aligned} & \int_0^1 \int_0^1 \left\{ \frac{\partial^4 W}{\partial \xi^4} + b_1 \frac{\partial^4 W}{\partial \xi^3 \partial \eta} + b_2 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + b_3 \frac{\partial^4 W}{\partial \xi \partial \eta^3} + b_4 \frac{\partial^4 W}{\partial \eta^4} + b_5 \frac{\partial^4 F}{\partial \xi^4} + b_6 \frac{\partial^4 F}{\partial \xi^3 \partial \eta} + b_7 \frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} + b_8 \frac{\partial^4 F}{\partial \xi \partial \eta^3} + b_9 \frac{\partial^4 F}{\partial \eta^4} - \frac{\beta^2 K_R}{D_{11}^*} \frac{\partial^2 F}{\partial \xi^2} - \frac{\beta^2}{D_{11}^*} \left(-\bar{N}_x \frac{\partial^2 W}{\partial \xi^2} + 2\beta \bar{S} \frac{\partial^2 W}{\partial \xi \partial \eta} \right) \right\} \delta W d\xi d\eta \\ & + \int_0^1 \left\{ \left(b_{10} \frac{\partial^2 F}{\partial \eta^2} + b_5 \frac{\partial^2 F}{\partial \xi^2} + b_{11} \frac{\partial^2 F}{\partial \xi \partial \eta} + \frac{\partial^2 W}{\partial \xi^2} + b_{12} \frac{\partial^2 W}{\partial \eta^2} + \frac{b_1}{2} \frac{\partial^2 W}{\partial \xi \partial \eta} \right) \cdot \delta \left(\frac{\partial W}{\partial \xi} \right) \right\} \Big|_{\xi=0}^{\xi=1} d\eta \\ & + \int_0^1 \left\{ \left(b_9 \frac{\partial^2 F}{\partial \eta^2} + b_{13} \frac{\partial^2 F}{\partial \xi^2} + b_{14} \frac{\partial^2 F}{\partial \xi \partial \eta} + b_{15} \frac{\partial^2 W}{\partial \xi^2} + b_4 \frac{\partial^2 W}{\partial \eta^2} + \frac{b_3}{2} \frac{\partial^2 W}{\partial \xi \partial \eta} \right) \cdot \delta \left(\frac{\partial W}{\partial \eta} \right) \right\} \Big|_{\eta=0}^{\eta=1} d\xi \\ & = 0 \quad (\text{III}, 2) \end{aligned}$$

The nondimensional parameters and constants are given in (II,26) and (II,31).

It can be seen that the equations for a flat plate will be recovered when the curvature parameter K_R ($K_R = \frac{b^2}{Rh}$) vanishes as R becomes infinite. The two line integrals in equation (III,2) represent the effect of boundary moments.

The objective of initial buckling analysis is to find the lowest eigenvalues of the compressive parameter $\bar{N}_x$ and shear parameter $\bar{S}$.

In order to solve the equations (III,1) and (III,2), two series of functions are assumed for the stress function F and the flexural deflection w according to the boundary conditions of the panel. Here four kinds of boundaries are under consideration. These boundary conditions in terms of nondimensional parameters may be written as:

a) All edges clamped

$$\begin{aligned} W=0, \quad \frac{\partial W}{\partial \xi}=0, \quad \frac{\partial^2 F}{\partial \eta^2}=0, \quad \frac{\partial^2 F}{\partial \xi \partial \eta}=0 & \quad \text{at } \xi=0,1 \\ W=0, \quad \frac{\partial W}{\partial \eta}=0, \quad \frac{\partial^2 F}{\partial \xi^2}=0, \quad \frac{\partial^2 F}{\partial \xi \partial \eta}=0 & \quad \text{at } \eta=0,1 \end{aligned} \quad (III,3)$$

b) All edges simply supported (s-s)

$$\begin{aligned} W=0, \quad M_\xi=0, \quad \frac{\partial^2 F}{\partial \eta^2}=0, \quad \frac{\partial^2 F}{\partial \xi \partial \eta}=0 & \quad \text{at } \xi=0,1 \\ W=0, \quad M_\eta=0, \quad \frac{\partial^2 F}{\partial \xi^2}=0, \quad \frac{\partial^2 F}{\partial \xi \partial \eta}=0 & \quad \text{at } \eta=0,1 \end{aligned} \quad (III,4)$$

c) Curved edges clamped, straight edges s-s

$$\begin{aligned} W=0, \quad \frac{\partial W}{\partial \xi}=0, \quad \frac{\partial^2 F}{\partial \eta^2}=0, \quad \frac{\partial^2 F}{\partial \xi \partial \eta}=0 & \quad \text{at } \xi=0,1 \\ W=0, \quad M_\eta=0, \quad \frac{\partial^2 F}{\partial \xi^2}=0, \quad \frac{\partial^2 F}{\partial \xi \partial \eta}=0 & \quad \text{at } \eta=0,1 \end{aligned} \quad (III,5)$$

d) Straight edges clamped, curved edges s-s

$$\begin{aligned} W=0, \quad M_{\xi}=0, \quad \frac{\partial^2 F}{\partial \eta^2}=0, \quad \frac{\partial^2 F}{\partial \xi \partial \eta}=0 \quad \text{at } \xi=0,1 \\ W=0, \quad \frac{\partial W}{\partial \eta}=0, \quad \frac{\partial^2 F}{\partial \xi^2}=0, \quad \frac{\partial^2 F}{\partial \xi \partial \eta}=0 \quad \text{at } \eta=0,1 \end{aligned} \quad (\text{III},6)$$

The governing equations (III,1) and (III,2) should be solved simultaneously in conjunction with the appropriate boundary conditions. The solutions for the different boundary conditions are separately formulated below.

2. Solutions

a) All edges clamped

In this case the nondimensional stress function and transverse deflection may be selected as follows:

$$\begin{aligned} F &= \sum_m \sum_n F_{mn} X_m(\xi) Y_n(\eta) \\ W &= \sum_p \sum_q W_{pq} X_p(\xi) Y_q(\eta) \end{aligned} \quad (\text{III},7)$$

($m, n, p, q = 1, 2, \dots, \infty$)

in which F_{mn} and w_{pq} are undetermined coefficients, and X_i and Y_i are the beam functions defined by (II,35). These now take the following form:

$$\begin{aligned} X_i(\xi) &= \cosh \alpha_i \xi - \cos \alpha_i \xi - \gamma_i' (\sinh \alpha_i \xi - \sin \alpha_i \xi) \\ Y_i(\eta) &= \cosh \alpha_i \eta - \cos \alpha_i \eta - \gamma_i' (\sinh \alpha_i \eta - \sin \alpha_i \eta) \end{aligned} \quad (\text{III},8)$$

where α_i and γ_i' are the constants given in Table (II-1)

(see page 31). The beam functions have the following properties:

$$\begin{aligned} X_i(0) = X_i(1) = \frac{dX_i(\xi)}{d\xi} \Big|_{\xi=0} = \frac{dX_i(\xi)}{d\xi} \Big|_{\xi=1} = 0 \\ Y_i(0) = Y_i(1) = \frac{dY_i(\eta)}{d\eta} \Big|_{\eta=0} = \frac{dY_i(\eta)}{d\eta} \Big|_{\eta=1} = 0 \end{aligned} \quad (\text{III},9)$$

and

$$\frac{d^4 X_i(\xi)}{d\xi^4} = \alpha_i^4 X_i(\xi) \quad (\text{III}, 10)$$

$$\frac{d^4 Y_i(\eta)}{d\eta^4} = \alpha_i^4 Y_i(\eta)$$

$$\int_0^1 X_i(\xi) X_j(\xi) d\xi = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad (\text{III}, 11)$$

$$\int_0^1 Y_i(\eta) Y_j(\eta) d\eta = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

With these properties the double series solutions (III,7) for F and w completely satisfy the boundary conditions (III,3) for the panels with all edges clamped. In that case the line integrals in the governing equation (III,2) will vanish because

$$\left. \frac{\partial w}{\partial \xi} \right|_{\xi=0,1} = 0 \quad \text{and} \quad \left. \frac{\partial w}{\partial \eta} \right|_{\eta=0,1} = 0$$

Upon substituting the solutions (III,7) into the governing system (III,1) and (III,2), performing the integration and noticing the properties (III,10) and (III,11) of the assumed functions, the governing system may be written in the following form:

$$\sum_m \sum_n F_{mn} \left[\alpha_m^4 J_0^{mi} J_0^{nj} - a_1 \alpha_m^3 \alpha_n J_3^{mi} J_1^{nj} + a_2 \alpha_m^2 \alpha_n^2 J_2^{mi} J_2^{nj} - a_3 \alpha_m \alpha_n^3 J_1^{mi} J_3^{nj} + a_4 \alpha_n^4 J_0^{mi} J_0^{nj} \right] = \sum_p \sum_q W_{pq} \left[a_5 \alpha_p^4 J_0^{pi} J_0^{qj} + a_6 \alpha_p^3 \alpha_q J_3^{pi} J_1^{qj} + a_7 \alpha_p^2 \alpha_q^2 J_2^{pi} J_2^{qj} + a_8 \alpha_p \alpha_q^3 J_1^{pi} J_3^{qj} + a_9 \alpha_q^4 J_0^{pi} J_0^{qj} - \frac{\beta^2 K_R}{A_{22}^*} \alpha_p^2 \alpha_q J_2^{pi} J_2^{qj} \right] \quad (\text{III}, 12)$$

$$(i, j = 1, 2, \dots)$$

$$\begin{aligned}
& \sum_p \sum_q W_{pq} \left[\alpha_p^4 J_0^{pi} J_0^{qj} + b_1 \alpha_p^3 \alpha_q J_3^{pi} J_1^{qj} + b_2 \alpha_p^2 \alpha_q^2 J_2^{pi} J_2^{qj} + b_3 \alpha_p \alpha_q^3 J_1^{pi} J_3^{qj} \right. \\
& \quad \left. + b_4 \alpha_q^4 J_0^{pi} J_0^{qj} \right] + \sum_m \sum_n F_{mn} \left[b_5 \alpha_m^4 J_0^{mi} J_0^{nj} + b_6 \alpha_m^3 \alpha_n J_3^{mi} J_1^{nj} \right. \\
& \quad \left. + b_7 \alpha_m^2 \alpha_n^2 J_2^{mi} J_2^{nj} + b_8 \alpha_m \alpha_n^3 J_1^{mi} J_3^{nj} + b_9 \alpha_n^4 J_0^{mi} J_0^{nj} - \frac{\beta^2 K_R}{D_{11}^*} \alpha_m^2 J_2^{mi} J_0^{nj} \right] \quad (\text{III}, 13) \\
& = \sum_p \sum_q W_{pq} \left[-K_x \frac{\beta^2 E_t h^3}{D_{11}^*} \alpha_p^2 J_2^{pi} J_0^{qj} + K_s \frac{2\beta^3 E_t h^3}{D_{11}^*} \alpha_p \alpha_q J_1^{pi} J_1^{qj} \right] \\
& \quad (i, j = 1, 2, \dots)
\end{aligned}$$

in which the loading parameters K_x and K_s are defined as

$$\begin{aligned}
K_x &= \frac{P_x b^2}{E_t h^3} \\
K_s &= \frac{P_{xy} b^2}{E_t h^3}
\end{aligned} \quad (\text{III}, 14)$$

The constants J 's are obtained by integrating the following expressions:

$$\begin{aligned}
J_0^{mi} &= \int_0^1 X_m(\xi) X_i(\xi) d\xi \\
J_1^{mi} &= \int_0^1 \bar{X}_m(\xi) X_i(\xi) d\xi \\
J_2^{mi} &= \int_0^1 \bar{\bar{X}}_m(\xi) X_i(\xi) d\xi \\
J_3^{mi} &= \int_0^1 \bar{\bar{\bar{X}}}_m(\xi) X_i(\xi) d\xi
\end{aligned} \quad (\text{III}, 15)$$

where

$$\begin{aligned}
X_m(\xi) &= \cosh \alpha_m \xi - \cos \alpha_m \xi - \gamma_m (\sinh \alpha_m \xi - \sin \alpha_m \xi) \\
&= \frac{1-\gamma_m}{2} e^{\alpha_m \xi} + \frac{1+\gamma_m}{2} e^{-\alpha_m \xi} - \cos \alpha_m \xi + \gamma_m \sin \alpha_m \xi \\
\bar{X}_m(\xi) &= \frac{1}{\alpha_m} \frac{dX_m(\xi)}{d\xi} = \frac{1-\gamma_m}{2} e^{\alpha_m \xi} - \frac{1+\gamma_m}{2} e^{-\alpha_m \xi} + \sin \alpha_m \xi + \gamma_m \cos \alpha_m \xi \\
\bar{\bar{X}}_m(\xi) &= \frac{1}{\alpha_m^2} \frac{d^2 X_m(\xi)}{d\xi^2} = \frac{1-\gamma_m}{2} e^{\alpha_m \xi} + \frac{1+\gamma_m}{2} e^{-\alpha_m \xi} + \cos \alpha_m \xi - \gamma_m \sin \alpha_m \xi \\
\bar{\bar{\bar{X}}}_m(\xi) &= \frac{1}{\alpha_m^3} \frac{d^3 X_m(\xi)}{d\xi^3} = \frac{1-\gamma_m}{2} e^{\alpha_m \xi} - \frac{1+\gamma_m}{2} e^{-\alpha_m \xi} - \sin \alpha_m \xi - \gamma_m \cos \alpha_m \xi
\end{aligned} \quad (\text{III}, 16)$$

The resultant expressions for the J's may be found in Appendix B.

The system of equations (III,12) and (III,13) contain an infinite number of linear algebraic equations which are to be solved simultaneously. In practice only a finite number of equations may be used to obtain the critical load to a specified accuracy. The following indicial expressions are, therefore, introduced.

$$\begin{aligned} I &= (i-1)M+j \\ J &= (m-1)M+n \end{aligned} \tag{III,17}$$

where

$$\begin{aligned} i &= 1, 2, \dots, M \\ j &= 1, 2, \dots, M \\ m &= 1, 2, \dots, M \\ n &= 1, 2, \dots, M \end{aligned} \tag{III,18}$$

and

$$\begin{aligned} p &= m \\ q &= n \end{aligned} \tag{III,19}$$

In the above expressions M refers to how many terms of the functions F and w are taken into the series (III,7). For instance, $M=5$ means that the first 25 terms are used in (III,7).

Then the final set of equations (III,12) and (III,13) can be rewritten in matrix form, i.e.

$$\begin{aligned} [K^{(1)}] \{w\} + [K^{(2)}] \{F\} &= [K_x [K^{(3)}] + K_s [K^{(4)}]] \{w\} \\ [K^{(5)}] \{F\} &= [K^{(6)}] \{w\} \end{aligned} \quad (\text{III}, 20)$$

The elements of $[K^{(1)}]$, $[K^{(2)}]$... $[K^{(6)}]$ may be found in Appendix C.

This system of matrix equations (III,20) may be reduced to a very simple form by substituting $\{F\}$ solved from the second equation of (III,20) into the first one,

i.e.

$$[\Lambda] \{w\} = K [\Gamma] \{w\} \quad (\text{III}, 21)$$

where

$$\begin{aligned} [\Lambda] &= [K^{(1)}] + [K^{(2)}] [K^{(5)}]^{-1} [K^{(6)}] \\ [\Gamma] &= \mu_x [K^{(3)}] + \mu_s [K^{(4)}] \end{aligned} \quad (\text{III}, 22)$$

and

$$\begin{aligned} K_x &= \mu_x K \\ K_s &= \mu_s K \end{aligned} \quad (\text{III}, 23)$$

The loading parameters μ_x and μ_s are determined by the required ratios of compressive and shear loading. It is obvious that $\mu_x = 1$ and $\mu_s = 0$ represents the panel under compression only, while $\mu_x = 0$ and $\mu_s = 1$ corresponds to a pure shear loading.

Equation (III,21) is a typical characteristic equation in matrix form. The buckling loads are determined by finding the lowest eigenvalue of K in this equation.

b) All edges simply supported

In this case the functions F and w are taken to be

$$\begin{aligned} F &= \sum_m \sum_n F_{mn} X_m(\xi) Y_n(\eta) \\ w &= \sum_p \sum_q W_{pq} \sin p\pi\xi \sin q\pi\eta \end{aligned} \quad (\text{III}, 24)$$

($m, n, p, q = 1, 2, \dots, \infty$)

As in the case of all edges clamped, the function F satisfies the boundary conditions (III,4) for the panel with all edges simply supported.

However, the function w satisfies only the kinematic condition ($w = 0$) on the boundaries and does not satisfy the zero moment conditions on the boundaries when the panel has general lay-ups. Fortunately, however, the effect of boundary moments has been considered in the governing equation (III,2). So it is necessary only to set the solutions (III,24) into the governing system (III,1) and (III,2). Following the same procedure as in the previous section, for the panel with all edges clamped, we can also obtain the same characteristic equation in matrix form.

$$[\Lambda] \{w\} = K[\Gamma] \{w\}$$

The matrices $[\Lambda]$ and $[\Gamma]$ are defined by (III,22). The loading parameter K is given by (III,23) and (III,14). But at the same time, the elements of the matrices $[K^{(1)}]$, $[K^{(2)}]$... $[K^{(6)}]$ are different from those for the panel with all edges clamped. They are given in Appendix C, together with certain integral constants given in Appendix B.

c) Curved edges clamped, straight edges simply supported

In this case the functions F and w may be taken as

$$F = \sum_m \sum_n F_{mn} X_m(\xi) Y_n(\eta) \quad (III,25)$$

$$w = \sum_p \sum_q w_{pq} X_p(\xi) \sin q\pi\eta$$

($m, n, p, q = 1, 2, \dots, \infty$)

The assumed function F satisfies the boundary conditions (III,5) completely, but the function w satisfies only the zero deflection and rotation condition. But, as we mentioned before,

with the line integrals in the governing equation (III,2) representing the effect of boundary moments, the zero moment condition is satisfied in the mean. So we directly set the solutions (III,25) into the governing system (III,1) and (III,2). Following the same procedure as that for the panel with all edges clamped, we obtain the equations (III,21), (III,22) and (III,23). The elements of the matrices $[K^{(1)}]$, $[K^{(2)}]$... $[K^{(6)}]$, however, are different from those for the panel with all edges clamped or with all edges simply supported. The elements are included in Appendix C and the constants obtained from certain integrals which are used in the expressions of these elements are given in Appendix B.

d) Straight edges clamped, curved edges simply supported

The following series are taken as the solutions in this case.

$$F = \sum_m \sum_n F_{mn} X_m(\xi) Y_n(\eta) \quad (\text{III,26})$$

$$W = \sum_p \sum_q W_{pq} \sin p\pi\xi Y_q(\eta) \quad (m, n, p, q = 1, 2, \dots, \infty)$$

Similarly to the last case for the panel with longitudinal edges simply supported and curved edges clamped, the governing equations may also be reduced to the matrix form (III,21), (III,22) and (III,23). The elements of the matrices in these equations are given in Appendix C, and corresponding constants representing certain integrals are also given in Appendix B.

3. Computation and Results

For the four kinds of boundary conditions described before the same system of characteristic equations has been obtained, i.e.

$$\{\Lambda\} \{W\} = K \{\Gamma\} \{W\} \quad (\text{III,21})$$

It is obvious that the lowest eigenvalue of the loading parameter K and corresponding eigenvector are of most interest in initial buckling investigations. The critical load and buckling mode of the panel can be obtained as long as the lowest K and corresponding eigenvector are known.

An iterative scheme, such as the power method described in reference [48], is often used to produce the largest eigenvalue of characteristic equations. This method could also be used here if equation (III,21) were rewritten in the form

$$\lambda \{w\} = [\Psi] \{w\} \quad (\text{III},27)$$

where

$$[\Psi] = [\Lambda]^{-1} [\Gamma] \quad (\text{III},28)$$

$$\lambda = \frac{1}{K} \quad (\text{III},29)$$

The largest eigenvalue of λ then gives the lowest buckling load. But if the matrix $[\Lambda]$ is singular, or near singular, the inversion of $[\Lambda]$ may be troublesome and the power method may fail to give any correct results.

The QZ algorithm is a general method for the characteristic equation **$AX = \lambda BX$** and, since it does not involve any inversion, it is not so sensitive to the singularity of the matrices in the characteristic equation. A detailed description of the QZ algorithm may be found in references [111], [112]. In the present investigation the QZ algorithm is selected to obtain the characteristic solutions.

a) Precision of Constants

As mentioned before, the precision of the constants α_i

and γ_i influences the orthogonality of the beam functions. Therefore the buckling loads are also affected by them. Comparison between the results of the present work and those given in references [87], [76], which used six significant figures for α_i and γ_i , shows that the error in the critical load may reach 6.4% (see Table (III,1)). In this particular example both investigations should have given the same results. The calculations refer to a flat boron/epoxy plate with the four layer antisymmetric arrangement ($\pm 45^\circ, \pm 45^\circ$). It is found that the more terms that are taken in F and w, the larger is the error.

Table (III,1) Comparison of critical value of $P_x b^2 / E_t h^3$

	Terms (M^2) in (III, 7)	Present work	References [76] & [87]	
			Results	error %
Clamped angle-ply plate	4	113.10	114.71	1.8
	9	106.87	113.67	6.4
Clamped cross-ply plate	4	118.08	118.08	0.0
	9	109.90	112.65	2.5

b) Material Properties and Dimensions

A series of computations have been performed in order to investigate the influence on buckling loads of various factors, including curvature, stacking sequence, boundary conditions and aspect ratio of the panels. The initial buckling of curved panels (or flat plates as $R \rightarrow \infty$) with midplane symmetric or unsymmetric layers under the action of compression only, shear

only or combined loading is investigated.

The calculations are carried out using data appropriate to boron/epoxy composites, unless special declaration is given for glass/epoxy, carbon/epoxy for comparison with previous work. The elastic constants of these materials are listed in Table (III,2). These restrictions are imposed solely to reduce computing cost since the various parameters are affected by the relative values of the elastic moduli.

Table (III,2) Elastic constants for calculations

	Boron/Epoxy	Carbon/Epoxy	Glass/Epoxy
E_l (GN/m ²)	206.9	206.9	53.8
E_t (GN/m ²)	20.7	5.2	17.9
ν_{lt}	0.3	0.25	0.25
G_{lt} (GN/m ²)	5.2	2.6	8.9

where E_l and E_t are tensile modulus in or perpendicular to the filament direction;

G_{lt} is the shear modulus;

ν_{lt} is the Poisson's ratio.

In the calculations the total thickness of the panel is always kept constant at 2.50 mm, no matter how many layers are considered.

The calculations refer to panels with length a and arc b both equal to 250 mm and radius R equal to 500 mm unless otherwise stated. However, the length a is varied when the influence of aspect ratio is investigated.

c) Number of Terms in Solutions

Generally speaking, the value of the critical load obtained

from the calculations depends not only on the assumed functions for F and w for the buckling mode, but also on the number of terms taken in them. In the present work most computations are carried out with the first 49 terms ($M = 7$ in expression (III,18)) in each series of F and w . To obtain some idea about the convergence of the present solutions, comparison of the critical compression and shear loadings with 25, 49 and 81 terms for a boron/epoxy panel with four layers at $(\pm 45^\circ)_s$ has been made in Table (III,3). It can be seen that 49 terms give satisfactory accuracy for the results, whilst still keeping computation cost within reasonable bounds. Computations are performed on a CDC 6500 computer system. It takes about 17 sec to get the results for each panel.

Table (III,3) Critical compression loads (N/mm) with different number of terms in F and w

Boundary Conditions	Number of terms	Radius R (mm)				
		250	500	1000	2000	∞
All edges clamped	25	565	321	199	163	150
	49	549	320	198	162	149
	81	548				
All edges s-s	25	322	181	116	91	81
	49	311	177	115	90	81
	81	307				

d) Verification of Flat Plate Solution

Fig. (III,1) and Fig. (III,2) are for a flat plate with the parameters given in Reference [42]. In that reference $E_t/E_l = 0.1$, $G_{lt}/E_l = 0.03$, $\nu_{lt} = 0.3$ and $a/b = 1.13$ are used for panels with all edges simply supported, while $E_t/E_l = 0.09$,

$G_{lt}/E_t = 0.03$, $\nu_{lt} = 0.3$ and $a/b = 1.0$ are used for all the other situations. Comparison shows that the present results are in excellent agreement with those obtained by Ashton et al. The sets of curves from the two approaches are completely coincident.

e) Critical Compressive Load

The variation of critical compressive load with fibre angle is shown for unidirectional ply panels in Fig. (III,3). It is seen that the critical compressive load for any fibre angle arrangement increases with panel curvature.

Fig. (III,4) shows the variation of compressive buckling load with fibre angle for laminated panels with twenty alternate layers $(\pm\theta)_{ss}$. It is found that for all values of curvature the alternately layered panels have higher buckling loads than the unidirectional panels.

The effect of the number of $\pm 45^\circ$ layers is shown in Figs. (III,5) and (III,6). Fig. (III,5) shows that as the number of alternate layers in a panel increases, keeping constant thickness of the panel at the same time, the critical compressive load for both symmetric and antisymmetric lay-ups becomes larger. As the number of layers becomes larger both symmetric and antisymmetric lay-ups tend towards the orthotropic solution corresponding to an infinite number of layers, i.e. $D_{16} = D_{26} = 0$. Fig. (III,6) shows how critical load varies with curvature. It is seen that for all values of curvature a two layer anti-symmetric angle ply arrangement gives a lower load than a single layer.

Fig. (III,7) repeats Fig. (III,6) but for unidirectional panels; as expected, the higher critical load occurs for a panel with fibres aligned in the load direction (0°). Fig. (III,8) shows the effect of panel aspect ratios on the critical loads of unidirectional panels. It is seen that the critical

load tends to a constant value as the panel length increases, although small fluctuations may occur in the curves.

f) Critical shear load

Figs. (III,9) - (III,13) describe similar effects to those presented in the previous section. However, with shear loading the direction of the force proves to be an additional parameter [113] [39]. The assumed convention for positive shear and fibre directions is shown in Fig. (II,1) of Chapter II.

Fig. (II,9) shows that for a unidirectional panel, with all fibres at $+\theta$, a higher critical load is obtained for a negative than for a positive shear. This is because in the former case the compressive component of the shear is more nearly aligned with the fibres. The maximum occurs when the fibres are at 45° for a square flat plate, but slightly changes from this angle as panel curvature is involved. For fibres oriented at $-\theta$ the curves of Fig. (III,9) would correspond to the opposite direction of shear forces. Thus this figure can be used to determine the preferred shear direction for a given lay-up or the optimum lay-up direction for a given load.

Fig. (III,10) shows that as the number of symmetric layers alternately at $\pm 45^\circ$ increases, the influence of shear force direction is reduced. Fig. (III,10) and Fig. (III,11) both show that the less alternate layers a panel has, the more it benefits from correctly orienting the fibres. Fig. (III,10) also shows that the difference between the two shears is reduced gradually by increasing the number of layers. Fig. (III,11) demonstrates that the influence of shear force direction on the critical loads of the curved panels with different curvatures has almost the same importance as on those of the flat plate.

Fig. (III,12) shows the variations of the critical shear forces with aspect ratio for single layer panels. As expected the critical values converge to constant values when the ratio becomes large.

Fig. (III,13) is a repetition of Fig. (III,10) but for an antisymmetric lay-up. In this case the direction of the shear force is shown to be unimportant.

g) Combined Compression and Shear

Fig. (III,14) to Fig. (III,20) investigate the influence of shear direction on the relationship between critical compressive and shear loadings, when acting in combination, for different panel curvatures, different fibre angles and different lay-ups. In these figures R_x represents the ratio of the critical value of compressive load when combined with shear to that without shear, R_s that of shear load when combined with compression to that without compression. It is worth pointing out that a critical compressive ratio R_x larger than one means that the combined shear stabilises rather than destabilises the panel. In a metal structure combined shear is always destabilising. This stabilising effect is caused by the different stiffness of the laminate in different directions with respect to the structural axes. The combined shear in a certain sense may produce a tension in the direction with the lowest stiffness and the panel therefore becomes stronger than when it is subjected to compression only. This strengthening effect is demonstrated for unidirectional flat plates, Fig. (III,14), and curved panels, Fig. (III,15) and (III,16). The only exceptions are panels with unidirectional layers at 0° and 90° , Fig. (III,17) and antisymmetric layers at $\pm 45^\circ$, Fig. (III,19). The influence of elastic properties can be seen clearly from Fig. (III,20), in

which the relationship between compressive and shear loading has been obtained for three materials with different ratios of E_l to E_t . It is found that the higher the ratio of E_l to E_t , the higher the curve will be.

h) Edge Conditions

Figs. (III,21) to (III,35) are obtained for different boundary conditions. As expected, it is always found that the buckling load of the panel with all edges clamped is the highest for these four kinds of boundary conditions. All edges simply supported gives the lowest critical load. Which of the remaining two is higher depends on the panel parameters such as fibre angles, curvature, aspect ratio and so on. The information presented in these figures essentially repeats that given previously in Figs. (III,1) - (III,20), which referred to all edges clamped. Thus Figs. (III,21) - (III,24) are for compressive loading, Figs. (III,25) - (III,29) for shear loading and Figs. (III,30) - (III,35) for combined shear and compressive loading. Figs. (III,21), (III,23), (III,26), (III,32), (III,33) and (III,34) refer to situations with mixed clamped and simply supported edges.

i) Comparison with other workers

Computations are also carried out for comparison with the experimental results of other workers. The results are listed in Tables (III,4), (III,5) and (III,6). It has been found that the results of the present work are in fairly good agreement with these tests. Table (III,4) is for panels with all edges simply supported using the parameters given in reference [99], i.e. $R=12$ in, $b=12.6$ in, $a=12$ in, $E_l = 20.5448 \times 10^6$ lbf/in², $E_t = 1.33305 \times 10^6$ lbf/in², $\nu_{lt} = 0.33524$, $G_{lt} = 0.751556 \times 10^6$ lbf/in², $h = 8 \times 0.005$ in .

Table (III,4) Comparison of $P_x a^2 / E_l h^3$ for panels with all edges simply supported

Lay-up	Reference [99]		Current work	
	Theor.	Exp.	Theor.	$\frac{\text{Exp. [99]}}{\text{Current work}}$
$(90,0)_{2s}$	33.2	24	28.9	0.83
$(90,\pm 45,0)_s$	45.4	25	31.1	0.80

Table (III,5) is for panels with all edges clamped, using the parameters given in reference [100], ($R = 12$ in, $b = 8$ in, $a = 12$ in, $E_l = 20 \times 10^6$ lb/in², $E_t = 2.1 \times 10^6$ lb/in², $\nu_{lt} = 0.21$, $G_{lt} = 0.85 \times 10^6$ lb/in²).

Table (III,5) Comparison of Critical load $P_x b$ lbf for panels with all edges clamped

Lay-up	Reference [100]	Current Work	
	Exp.	Theor.	$\frac{\text{Exp [100]}}{\text{Current Theor.}}$
$(0^\circ, 90^\circ)_{2s}$	6740	6972	0.97
$(\pm 45^\circ)_{2s}$	5980	6878	0.87
$(0^\circ, 90^\circ)_s$	1175	1419	0.83
$(\pm 45^\circ)_s$	1625	2161	0.75
$(-45^\circ)_{2s}$	1480	2065	0.71
$(+30^\circ)_{4s}$	4985	5167	0.96
$(0^\circ)_{4s}$	7300	7836	0.93
$(0^\circ)_{4s}$	6600	7691	0.86
$(0^\circ)_{6s}$	20560	19778	1.04
$(0^\circ, 45^\circ, 90^\circ, -45^\circ)_s$	7100	7177	0.99
$(0^\circ, \pm 60^\circ)_s$	3730	4516	0.83
$(0^\circ, \pm 60^\circ)_s$	3530	3876	0.91

Table (III,6) is obtained for panels with straight edges simply supported and curved edges clamped, using the parameters given in reference [100], as before. Comparison is made here for only 15 of the 72 panels tested. The remaining panels gave similar results.

It is clear that the current theory produces a better estimate of initial buckling load than that used in references [100] and [99]. However, since the details of these theories are not known, it is not possible to comment on the reasons for the improvement.

Table (III,6) Comparison of critical load P_{xb} lbf. for panels with straight edges s-s, curved edges clamped

Lay-up	Reference [100]			Current Work	
	Theor.	Exp.	%	Theor.	$\frac{\text{Exp [100]}}{\text{Current Theor.}}$
$(0^\circ, 90^\circ)_{2s}$	7200	6680	93	5597	1.19
$(0^\circ, 90^\circ)_{2s}$	5900	4865	83	5503	0.88
$(\pm 45^\circ)_{2s}$	12400	8660	70	8141	1.06
$(\pm 45^\circ)_{2s}$	12700	9000	71	8371	1.08
$(\pm 45^\circ)_{2s}$	13000	8820	68	8497	1.03
$(\pm 45^\circ)_{2s}$	13200	8760	66	8624	1.02
$(\pm 45^\circ)_{2s}$	9500	5740	60	6226	0.92
$(0, 90^\circ)_s$	1530	985	64	1209	0.81
$(0, 90^\circ)_s$	1470	925	63	1151	0.80
$(\pm 45^\circ)_s$	4000	1870	47	1917	0.98
$(\pm 45^\circ)_s$	4180	1610	38	2023	0.80
$(\pm 45^\circ)_s$	3780	1590	42	1740	0.91
$(\pm 45^\circ)_s$	4130	1850	45	1983	0.93
$(\pm 45^\circ)_s$	2950	1280	43	1365	0.94
$(\pm 45^\circ)_{3s}$	27700	17760	64	17165	1.03

4. Conclusions

Summarising the previous computing results, the following conclusions may be drawn:

- (i) More precise values of the coefficients α_i and γ_i should be used in the beam function X_i and Y_i in order to obtain better values for the critical loads.
- (ii) Curving the panel is always beneficial to its initial buckling resistance - no matter to what kind of load it is subjected, axial compression, shear forces, or both. The critical load increases with decreasing radius of curvature R and the critical value for a flat plate is recovered when R becomes infinite.
- (iii) Angle panels having unidirectional fibres have preferred direction for applied shear forces. The difference in buckling loads obtained for alternate shear directions is so large that one of them may be several times that of the other. The difference is affected greatly by filament angle but only slightly by curvature. For a constant direction of shear the fibre direction must be correctly selected to obtain a high critical load.
- (iv) For midplane symmetric laminated curved panels, conclusion (iii) is also true, but for midplane antisymmetric panels alternate directions of applied shear do not make much difference to the buckling load.
- (v) Orthotropic solutions are always higher than anisotropic solutions for both midplane symmetrically and antisymmetrically laminated panels under compression and also for antisymmetrically laminated panels under shear. But they may be much higher or much lower than anisotropic solutions for symmetrically laminated panels under shear, depending on the

direction of shear forces. The critical loads of anisotropic panels converge to the orthotropic solutions as the number of alternate laminae oriented at $\pm \theta$, with respect to generator, increases. The rate of convergence is slow for symmetrically laminated panel under shear. This means that symmetrically laminated panels under shear may not be treated as orthotropic unless a large number of alternate layers are used.

(vi) A combined shear load in a certain direction may greatly stabilise symmetrically laminated panels under compression. This happens when the combined shear is in the same direction as that giving the higher critical load in a shear-only situation, and when the magnitude of the combined shear is not close to the critical value. The higher the ratio of E_l to E_t for the composite material of the panel, the higher the critical compression ratio that can be reached.

C. Initial Buckling of Laminated Plates with Immovable Edges

In the last section initial buckling of laminated panels with movable edges (i.e. unrestrained in the plane of the panel) was investigated. It is found that the direction of applied shear is very important for the buckling behaviour of such panels. This is true for curved panels and flat plates alike.

The present section deals with the initial buckling of generally laminated plates with immovable edges (i.e. completely restrained in the plane of the panel). Flat plates under the combination of in-plane compressive and shear loading are considered. The emphasis is put on the effect of alternative shear directions.

The governing system of three coupled equations, including the influence of edge moments, is solved by the use of multiple

Fourier series for the displacements of the plates. The results obtained in the present work show that the direction of combined shear again has significant influence on the critical compressive load. In addition to that, it is found that the direction of combined shear is as important for plates under bidirectional compression as for ones under unidirectional compression.

1. Governing System

A rectangular plate, having length a , width b and thickness h is considered to consist of n generally laminated unidirectional composite layers. The mid-plane of the plate is taken for the reference plane Oxy with the origin O on one corner. The plate is assumed thin and elastic so that the Kirchhoff hypothesis is valid. The constitutive relations of plate are shown in the expression (II,10) of Chapter II. Similarly to the equation (II,22), the governing equations of this problem may be written on the basis of the virtual displacement principle, in which the influence of edge moments is included,

i.e.

$$\int_0^a \int_0^b \left\{ \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right\} \delta u_0 dx dy = 0 \quad (\text{III}, 30)$$

$$\int_0^a \int_0^b \left\{ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right\} \delta v_0 dx dy = 0 \quad (\text{III}, 31)$$

$$\begin{aligned} \int_0^a \int_0^b \left\{ \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - P_x \frac{\partial^2 w_0}{\partial x^2} - P_y \frac{\partial^2 w_0}{\partial y^2} + 2 P_{xy} \frac{\partial^2 w_0}{\partial x \partial y} \right\} \delta w_0 dx dy \\ + \int_0^b \left[M_x \delta \left(\frac{\partial w_0}{\partial x} \right) \right]_{x=0}^{x=a} dy + \int_0^a \left[M_y \delta \left(\frac{\partial w_0}{\partial y} \right) \right]_{y=0}^{y=b} dx = 0 \end{aligned} \quad (\text{III}, 32)$$

where N_x, N_y, N_{xy} are inplane forces

M_x, M_y, M_{xy} are bending moments and torque

P_x, P_y are applied compression

P_{xy} is applied shear.

Introducing the following nondimensional parameters

$$\begin{aligned} \xi &= \frac{x}{a}, \quad \eta = \frac{y}{b}, \quad \beta = \frac{a}{b}, \\ u &= \frac{u_0 b}{h^2}, \quad v = \frac{v_0 b}{h^2}, \quad w = \frac{w_0}{h}, \\ a_{ij} &= \frac{A_{ij}}{A_{22}}, \quad b_{ij} = \frac{B_{ij}}{A_{22}h}, \quad d_{ij} = \frac{D_{ij}}{A_{22}h^2}, \\ K_x &= \frac{P_x b^2}{E_t h^3}, \quad K_y = \frac{P_y b^2}{E_t h^3}, \quad K_s = \frac{P_{xy} b^2}{E_t h^3} \end{aligned} \quad (\text{III}, 33)$$

Upon substitution of the constitutive relations (II,10) into the equation system (III,30), (III,31) and (III,32), the following nondimensional equations are obtained:

$$\int_0^1 \int_0^1 \{ \beta L_1 u + \beta L_2 v - L_3 w \} \delta u d\xi d\eta = 0 \quad (\text{III}, 34)$$

$$\int_0^1 \int_0^1 \{ \beta L_2 u + \beta L_4 v - L_5 w \} \delta v d\xi d\eta = 0 \quad (\text{III}, 35)$$

$$\begin{aligned} & \int_0^1 \int_0^1 \{ \beta L_3 u + \beta L_5 v - L_6 w \} \delta w d\xi d\eta \\ & + \int_0^1 \left\{ \left[\beta L_7 u + \beta L_8 v - L_9 w \right] \delta \left(\frac{\partial w}{\partial \xi} \right) \right\} \Big|_{\xi=0}^{\xi=1} d\eta \\ & + \int_0^1 \left\{ \left[\beta^3 L_{10} u + \beta^3 L_{11} v - \beta^2 L_{12} w \right] \delta \left(\frac{\partial w}{\partial \eta} \right) \right\} \Big|_{\eta=0}^{\eta=1} d\xi \\ & = \int_0^1 \int_0^1 \left\{ \frac{E_t h}{A_{22}} \left[K_x \beta^2 \frac{\partial^2 w}{\partial x^2} + K_y \beta^4 \frac{\partial^2 w}{\partial y^2} - 2K_s \beta^3 \frac{\partial^2 w}{\partial x \partial y} \right] \right\} \delta w d\xi d\eta \end{aligned} \quad (\text{III}, 36)$$

where L_i are the linear differential operators which are defined by the following expressions:

$$L_1 = a_{11} \frac{\partial^2}{\partial \xi^2} + 2\beta a_{16} \frac{\partial^2}{\partial \xi \partial \eta} + \beta^2 a_{66} \frac{\partial^2}{\partial \eta^2}$$

$$L_2 = a_{16} \frac{\partial^2}{\partial \xi^2} + \beta(a_{12} + a_{66}) \frac{\partial^2}{\partial \xi \partial \eta} + \beta^2 a_{26} \frac{\partial^2}{\partial \eta^2}$$

$$L_3 = b_{11} \frac{\partial^3}{\partial \xi^3} + 3\beta b_{16} \frac{\partial^3}{\partial \xi^2 \partial \eta} + \beta^2(b_{12} + 2b_{66}) \frac{\partial^3}{\partial \xi \partial \eta^2} + \beta^3 b_{26} \frac{\partial^3}{\partial \eta^3}$$

$$L_4 = a_{66} \frac{\partial^2}{\partial \xi^2} + 2\beta a_{26} \frac{\partial^2}{\partial \xi \partial \eta} + \beta^2 \frac{\partial^2}{\partial \eta^2}$$

$$L_5 = b_{16} \frac{\partial^3}{\partial \xi^3} + \beta(b_{12} + 2b_{66}) \frac{\partial^3}{\partial \xi^2 \partial \eta} + 3\beta^2 b_{26} \frac{\partial^3}{\partial \xi \partial \eta^2} + \beta^3 b_{22} \frac{\partial^3}{\partial \eta^3}$$

$$L_6 = d_{11} \frac{\partial^4}{\partial \xi^4} + 4\beta d_{16} \frac{\partial^4}{\partial \xi^3 \partial \eta} + 2\beta^2(d_{12} + 2d_{66}) \frac{\partial^4}{\partial \xi^2 \partial \eta^2} + 4\beta^3 d_{26} \frac{\partial^4}{\partial \xi \partial \eta^3} + \beta^4 d_{22} \frac{\partial^4}{\partial \eta^4}$$

(III, 37)

$$L_7 = b_{11} \frac{\partial}{\partial \xi} + b_{16} \beta \frac{\partial}{\partial \eta}$$

$$L_8 = b_{16} \frac{\partial}{\partial \xi} + b_{12} \beta \frac{\partial}{\partial \eta}$$

$$L_9 = d_{11} \frac{\partial^2}{\partial \xi^2} + 2d_{16} \beta \frac{\partial^2}{\partial \xi \partial \eta} + d_{12} \beta^2 \frac{\partial^2}{\partial \eta^2}$$

$$L_{10} = b_{21} \frac{\partial}{\partial \xi} + b_{26} \beta \frac{\partial}{\partial \eta}$$

$$L_{11} = b_{26} \frac{\partial}{\partial \xi} + b_{22} \beta \frac{\partial}{\partial \eta}$$

$$L_{12} = d_{21} \frac{\partial^2}{\partial \xi^2} + 2d_{26} \beta \frac{\partial^2}{\partial \xi \partial \eta} + d_{26} \beta^2 \frac{\partial^2}{\partial \eta^2}$$

The boundary conditions used for inplane restrained simply supported edges are

$$\begin{aligned} u=0, \quad v=0, \quad w=0, \quad M_{\xi}=0 \quad \text{at } \xi=0,1 \\ u=0, \quad v=0, \quad w=0, \quad M_{\eta}=0 \quad \text{at } \eta=0,1 \end{aligned} \quad (\text{III}, 38)$$

with the nondimensional forces and moments having the form

$$\begin{aligned} (N_{\xi}, N_{\eta}, N_{\xi\eta}) &= \frac{b^2}{A_{22}h^2} (N_x, N_y, N_{xy}) \\ (M_{\xi}, M_{\eta}, M_{\xi\eta}) &= \frac{b^2}{A_{22}h^3} (M_x, M_y, M_{xy}) \end{aligned} \quad (\text{III}, 39)$$

The constitutive relations are then written as follows:

$$\begin{Bmatrix} N_{\xi} \\ N_{\eta} \\ N_{\xi\eta} \\ M_{\xi} \\ M_{\eta} \\ M_{\xi\eta} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} & b_{11} & b_{12} & b_{16} \\ a_{21} & a_{22} & a_{26} & b_{21} & b_{22} & b_{26} \\ a_{61} & a_{62} & a_{66} & b_{61} & b_{62} & b_{66} \\ b_{11} & b_{12} & b_{16} & d_{11} & d_{12} & d_{16} \\ b_{21} & b_{22} & b_{26} & d_{21} & d_{22} & d_{26} \\ b_{61} & b_{62} & b_{66} & d_{61} & d_{62} & d_{66} \end{bmatrix} \begin{Bmatrix} \frac{1}{\beta} \frac{\partial u}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \\ \frac{\partial u}{\partial \eta} + \frac{1}{\beta} \frac{\partial v}{\partial \xi} \\ -\frac{1}{\beta^2} \frac{\partial^2 w}{\partial \xi^2} \\ -\frac{\partial^2 w}{\partial \eta^2} \\ -\frac{2}{\beta} \frac{\partial^2 w}{\partial \xi \partial \eta} \end{Bmatrix} \quad (\text{III}, 40)$$

2. Solutions

The assumed solutions for the governing system (III,34), (III,35) and (III,36) take the following form

$$\begin{aligned} u &= \sum_k \sum_l u_{kl} \sin k\pi\xi \sin l\pi\eta \\ v &= \sum_m \sum_n v_{mn} \sin m\pi\xi \sin n\pi\eta \\ w &= \sum_p \sum_q w_{pq} \sin p\pi\xi \sin q\pi\eta \end{aligned} \quad (\text{III}, 41)$$

(k, l, m, n, p, q = 1, 2, \dots, \infty)

It is obvious that the solutions of u and v in (III,41) satisfy the boundary conditions (III,38) but not the solution of w . As mentioned before, however, the influence of edge moments has been considered in the governing equation (III,36). The zero moment conditions on the boundaries of the plate will be satisfied in the mean when setting the assumed solutions (III,41) into the governing system. It is obvious that the more terms that are taken in the solutions (III,41), the better the boundary conditions are satisfied.

Truncating the first MXM terms for the solutions of u , v and w in equation (III,41) and setting the solutions into the equation system (III,34), (III,35) and (III,36), the resulting equations may be represented in matrix form,

i.e.

$$\begin{aligned} [A^{(1)}] \{u\} + [A^{(2)}] \{v\} + [A^{(3)}] \{w\} &= 0 \\ [B^{(1)}] \{u\} + [B^{(2)}] \{v\} + [B^{(3)}] \{w\} &= 0 \\ [C^{(1)}] \{u\} + [C^{(2)}] \{v\} + [C^{(3)}] \{w\} &= K [\Gamma] \{w\} \end{aligned} \quad (\text{III,42})$$

with the new indices I, J defined by

$$\begin{aligned} I &= (i-1)M + j \\ J &= (m-1)M + n \end{aligned} \quad (\text{III,43})$$

and

$$\begin{aligned} i &= k = m = p = 1, 2, \dots, M \\ j &= l = n = q = 1, 2, \dots, M \end{aligned} \quad (\text{III,44})$$

The elements of the coefficient matrices $[A^{(i)}]$, $[B^{(i)}]$ and $[C^{(i)}]$ may be given as follows:

$$A_{IJ}^{(1)} = -(a_{11}\beta\omega_1^2 + a_{66}\beta^3\omega_2^2)R_1^{mi}R_1^{nj} + 2a_{16}\beta^2\omega_1\omega_2R_2^{mi}R_2^{nj}$$

$$A_{IJ}^{(2)} = -(a_{16}\beta\omega_1^2 + a_{26}\beta^3\omega_2^2)R_1^{mi}R_1^{nj} + (a_{12} + a_{66})\beta^2R_2^{mi}R_2^{nj}$$

$$A_{IJ}^{(3)} = [b_{11}\omega_1^3 + (b_{12} + 2b_{66})\beta^2\omega_1\omega_2^2]R_2^{mi}R_1^{nj} + (3b_{16}\beta\omega_1^2\omega_2 + b_{26}\beta^3\omega_2^3)R_1^{mi}R_2^{nj}$$

$$B_{IJ}^{(1)} = -(a_{16}\beta\omega_1^2 + a_{26}\beta^3\omega_2^2)R_1^{mi}R_1^{nj} + (a_{12} + a_{66})\beta^2\omega_1\omega_2R_2^{mi}R_2^{nj}$$

$$B_{IJ}^{(2)} = -(a_{66}\beta\omega_1^2 + \beta^3\omega_2^2)R_1^{mi}R_1^{nj} + 2a_{26}\beta^2\omega_1\omega_2R_2^{mi}R_2^{nj}$$

$$B_{IJ}^{(3)} = (b_{16}\omega_1^3 + 3b_{26}\beta^2\omega_1\omega_2^2)R_2^{mi}R_1^{nj} + [(b_{12} + 2b_{66})\beta\omega_1^2\omega_2 + b_{22}\beta^3\omega_2^3]R_1^{mi}R_2^{nj}$$

(III, 45)

$$C_{IJ}^{(1)} = -[(b_{11}\beta\omega_1^3 + (b_{12} + 2b_{66})\beta^3\omega_1\omega_2^2)R_2^{mi}R_1^{nj} - (3b_{16}\beta^2\omega_1^2\omega_2 + b_{26}\beta^4\omega_2^3)R_1^{mi}R_2^{nj}$$

$$+ b_{11}\beta\omega_1R_3^{mi}R_1^{nj} + b_{26}\beta^4\omega_2R_1^{mi}R_3^{nj}$$

$$C_{IJ}^{(2)} = -(b_{16}\beta\omega_1^3 + 3b_{26}\beta^3\omega_1\omega_2^2)R_2^{mi}R_1^{nj} - [(b_{12} + 2b_{66})\beta^2\omega_1^2\omega_2 + b_{22}\beta^4\omega_2^3]R_1^{mi}R_2^{nj}$$

$$+ b_{16}\beta\omega_1R_3^{mi}R_1^{nj} + b_{22}\beta^4\omega_2R_1^{mi}R_3^{nj}$$

$$C_{IJ}^{(3)} = [-d_{11}\omega_1^4 - 2(d_{12} + 2d_{66})\beta^2\omega_1^2\omega_2^2 - d_{22}\beta^4\omega_2^4]R_1^{mi}R_1^{nj}$$

$$+ 4(d_{16}\beta\omega_1^3\omega_2 + d_{26}\beta^3\omega_1\omega_2^3)R_2^{mi}R_2^{nj}$$

$$- 2d_{16}\beta\omega_1\omega_2R_3^{mi}R_2^{nj} - 2d_{26}\beta^3\omega_1\omega_2R_2^{mi}R_3^{nj}$$

$$\Gamma_{IJ} = \frac{E_t h}{A_{22}} \left[-(\mu_x \beta^2\omega_1^2 + \mu_y \beta^4\omega_2^2)R_1^{mi}R_1^{nj} - 2\mu_s \beta^3\omega_1\omega_2R_2^{mi}R_2^{nj} \right]$$

where

$$\omega_1 = m\pi$$

(III,46)

$$\omega_2 = n\pi$$

$$R_1^{mi} = \int_0^1 \sin m\pi \xi \sin i\pi \xi d\xi = \int_0^1 \sin m\pi \eta \sin i\pi \eta d\eta$$

$$R_2^{mi} = \int_0^1 \cos m\pi \xi \sin i\pi \xi d\xi = \int_0^1 \cos m\pi \eta \sin i\pi \eta d\eta \quad (\text{III,47})$$

$$R_3^{mi} = [i\pi \cos m\pi \xi \cos i\pi \xi] \Big|_{\xi=0}^{\xi=1} = [i\pi \cos m\pi \eta \cos i\pi \eta] \Big|_{\eta=0}^{\eta=1}$$

and

$$K_x = \mu_x K, \quad K_y = \mu_y K, \quad K_s = \mu_s K \quad (\text{III,48})$$

As stated before, the loading ratio parameter μ_x , μ_y and μ_s is selected according to the required loading conditions. Different sets of μ_x , μ_y and μ_s are able to represent different ratios of compression in x,y directions and shear. With μ_x , μ_y and μ_s selected, all the elements of the coefficient matrices may be calculated as long as properties of the composite layers are known.

Solving for $\{u\}$ and $\{v\}$ in terms of $\{w\}$ from the first two equations of the system (III,42) and setting the obtained $\{u\}$ and $\{v\}$ into the remaining equation of (III,42), the characteristic equation in matrix form may be established as:

$$[\Lambda] \{w\} = K [\Gamma] \{w\} \quad (\text{III,49})$$

where

$$[\Lambda] = -[C^{(1)}][F^{(1)}] - [C^{(2)}][F^{(2)}] + [C^{(3)}]$$

$$[F^{(1)}] = [D^{(2)}] - [D^{(1)}][F^{(2)}]$$

$$[F^{(2)}] = [E^{(1)}]^{-1}[E^{(2)}]$$

$$\left. \begin{aligned}
 [E^{(1)}] &= [B^{(2)}] - [B^{(1)}] [D^{(1)}] \\
 [E^{(2)}] &= [B^{(3)}] - [B^{(1)}] [D^{(2)}] \\
 [D^{(1)}] &= [A^{(1)}]^{-1} [A^{(2)}] \\
 [D^{(2)}] &= [A^{(1)}]^{-1} [A^{(3)}]
 \end{aligned} \right\} \quad (\text{III}, 50)$$

Similarly to the previous section in this chapter, the characteristic equation (III,49) may be solved by the iterative scheme based on the QZ algorithm by Moler et al^[111] [112]. The lowest eigenvalue of the loading parameter K and the corresponding eigenvector are usually required in initial buckling problems. The critical loading corresponding to the particular μ_x , μ_y and μ_s may be obtained as long as these characteristic solutions are found. With μ_x , μ_y and μ_s varying, relationships between the critical compressive loads and combined shear will be known.

3. Results and Discussion

In order to investigate the effect of the directions of shear on the buckling behaviour a series of computations have been carried out for plates with different lay-ups and materials. The results are presented in a series of figures below. In the computation a forty-nine term approximation is used for every displacement in the solutions (III,41), i.e. $M = 7$ in equations (III,43) and (III,44). As before, the computation is carried out on a CDC 6500 computer system. The computing cost is not high; it takes about 15 seconds per run.

In the computations three different materials are involved, the properties of which may be found in Table (III,2) of this Chapter (see page 46).

Fig. (III,36) shows the influence of shear direction on the critical load, when only shear is applied, to mid-plane symmetrically laminated plates composed of $\pm 45^\circ$ layers. The parameters used here are the same as for Fig. (5-10) in reference [42]. The results of both works show complete coincidence with each other.

Fig. (III,37) shows the variation of critical shear, for square boron/epoxy plates having symmetric lay-ups, as fibre angles are varied. It can be seen that the alternate shear directions have significant influence on the critical value of the applied shear unless either the number of layers in the plates become very large or crossply layers are used for the plates. It is again seen that the laminated plates have a preferred shear direction. The largest difference of two critical shear forces corresponding to different shear directions is for $\pm 45^\circ$ layers of a square plate. The difference is reduced gradually by increasing the number of alternate layers.

In Figs. (III,38) to (III,42) the relationship between critical compressive ratio R_x and critical shear ratio R_s is presented. Fig. (III,38) is obtained for square boron/epoxy plates with symmetric lay-ups. It is found that there are two relationships between applied compression and shear, corresponding to alternate directions of the shear. It is seen that, as for unrestrained plates, the shear in a certain direction may greatly stabilise the plates with symmetric lay-ups when under compression. Such effects decrease as the number of layers in the plate increases. It may be expected that when the number of layers becomes very large, the plates will behave as if they were orthotropic. In that case the alternate directions of shear will not have any effect on the

buckling behaviour of the plates under the combination of compression and shear. Fig. (III,39) is obtained for a square plate with four symmetric layers at $(\pm 45)_s$, made of different composite materials. It can be seen that the higher the ratio of E_l to E_t , the greater the effect of the shear direction. Fig. (III,40) refers to square boron/epoxy plates with antisymmetric layers at $(\pm 45^\circ)$ and $(\pm 45^\circ, \pm 45^\circ)$. It is found that the alternate directions of shear make no difference to the buckling behaviour of the plates with such lay-ups. This is to be expected because when an antisymmetric plate under negative shear is turned over, that under positive shear is recovered.

Fig. (III,41) is obtained for square boron/epoxy plates having asymmetric lay-ups. It can be seen that the shear direction is important for the plates with such layer arrangements, especially when the two outer layers have the same filament direction.

As pointed out in section B of this chapter for laminated panels with movable edges, the shear direction is also very important to the buckling behaviour of the symmetrically laminated plates with full inplane restraint.

Calculations carried out for plates under a combination of bidirectional compression and shear show that the direction of shear is just as important as for plates under unidirectional compression and shear. Fig. (III,42) is obtained for square plates with four symmetric layers at $(\pm 45^\circ)_s$ under equal bidirectional compression ($P_x = P_y$) and shear. The ordinate $R_{(x+y)}$ represents the critical ratio of equal bidirectional compression, i.e. $R_{(x+y)} = \frac{P_x}{P_{x_0}} = \frac{P_y}{P_{y_0}}$, P_x and P_y being equal to each other in this case. They denote the critical value of compressive loads when combined with shear, P_{x_0} and P_{y_0} are

also equal to each other, but they are the critical compression without shear. Comparison between Fig. (III,39) and Fig. (III,42) shows that the same relationship between the critical compressive and shear ratios are obtained, although the loading conditions and the absolute values of critical loads are different in the two cases.

In section B of this chapter we have given a physical explanation for the effect of shear direction on the buckling behaviour of laminated panels. Here we may have a few words more. In fact, the bending stiffness of laminated plates with symmetric or asymmetric lay-ups may vary dramatically with direction to the axes of the plates. The bending stiffness along one of the diagonals of the plate may be much larger than in the other diagonal direction. At the same time, however, the in-plane shear produces a compressive resultant in one of the diagonals and a tensile resultant in the other. If the tensile resultant of the shear is coincident with or close to the direction of the smaller bending stiffness, the plate will be stabilised. Otherwise the plate will be destabilised.

In practice the shear direction may not alter, that is the shear in a certain direction may be prescribed for the plate considered. In that case, correctly selecting the fibre directions for the layers becomes very important. For instance, $(\pm 45^\circ)_s$ or $(\mp 45^\circ)_s$ for a four layer plate having symmetric lay-ups makes the buckling behaviour of the plates much different. In order to obtain the higher buckling loads it is suggested that a plate subjected to a combination of compression and shear should have such a lay-up that the larger bending stiffness in the two diagonal directions of

the plate is coincident with the compressive resultant of the shear. To do so the outer layers may be selected to have the fibre direction coincident with, or close to, the compressive resultant of the applied shear.

4. Conclusions

For generally laminated plates with immovable edges the following conclusions as to initial buckling behaviour may be drawn.

(i) The shear direction has significant influence on the buckling behaviour of plates with mid-plane symmetry. The combined shear in a certain direction may greatly stabilise the plate under compression, whether unidirectional or bidirectional.

(ii) The effect of the shear direction varies with the properties of the composite materials, the number of layers and fibre angles. The higher the ratio of E_l to E_t for the composite and the less the number of layers in the plate, the greater the shear effect.

(iii) The alternate shear directions make no difference for plates with antisymmetric layers, as is the case for isotropic and orthotropic plates.

(iv) For prescribed shear the lay-up of the plates under the combination of compression and shear should be so selected that the larger bending stiffness of the plate is in, or close to, the diagonal direction corresponding to the compressive resultant of the shear.

(v) The shear direction has significant influence on plates with asymmetric lay-ups, shear combined with compression may greatly stabilise such plates having symmetric outer layers.

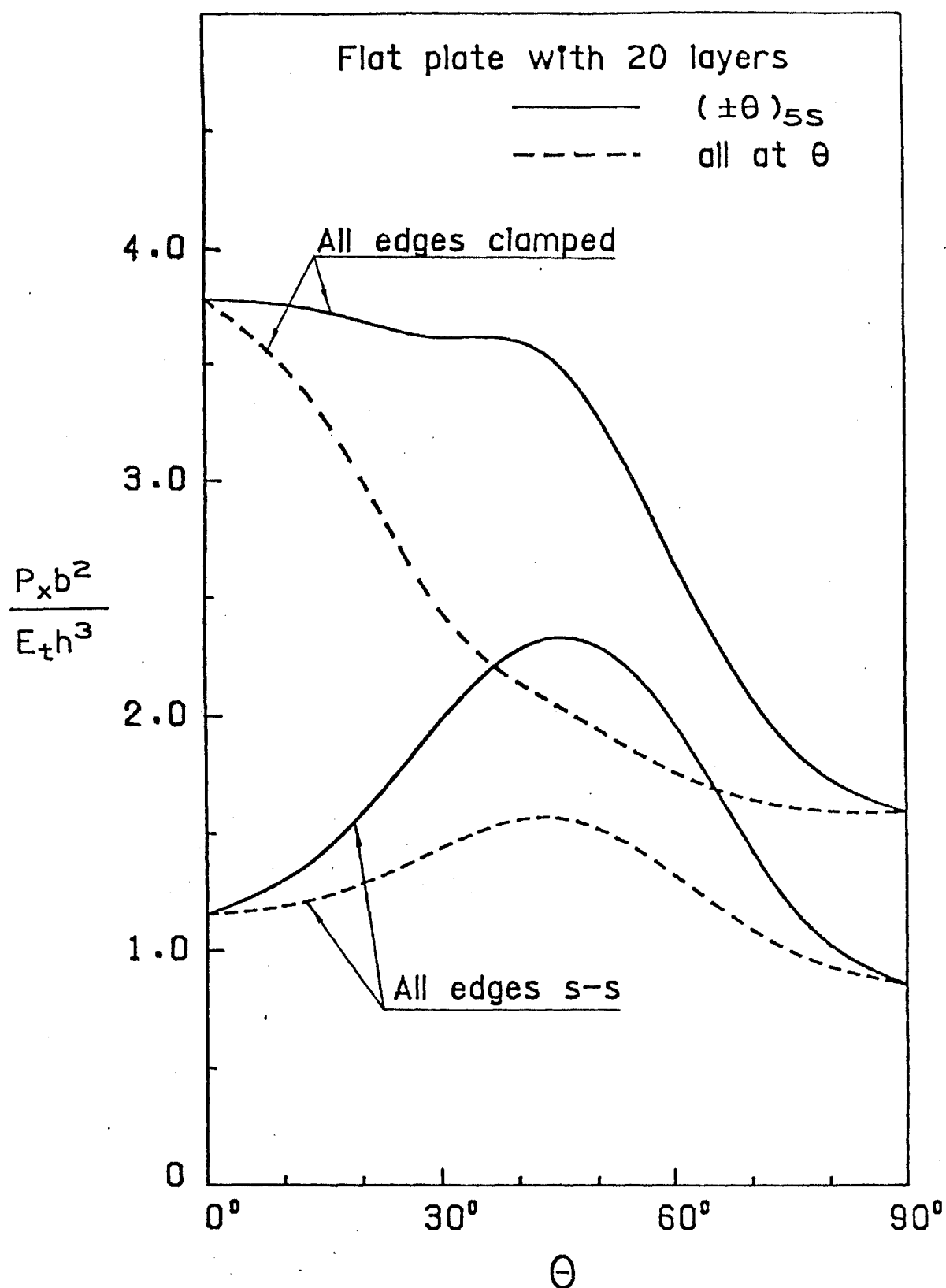


Fig. (III,1) Flat plate: variation of critical compressive load (P_x) with fibre angle; clamped or simply supported edges

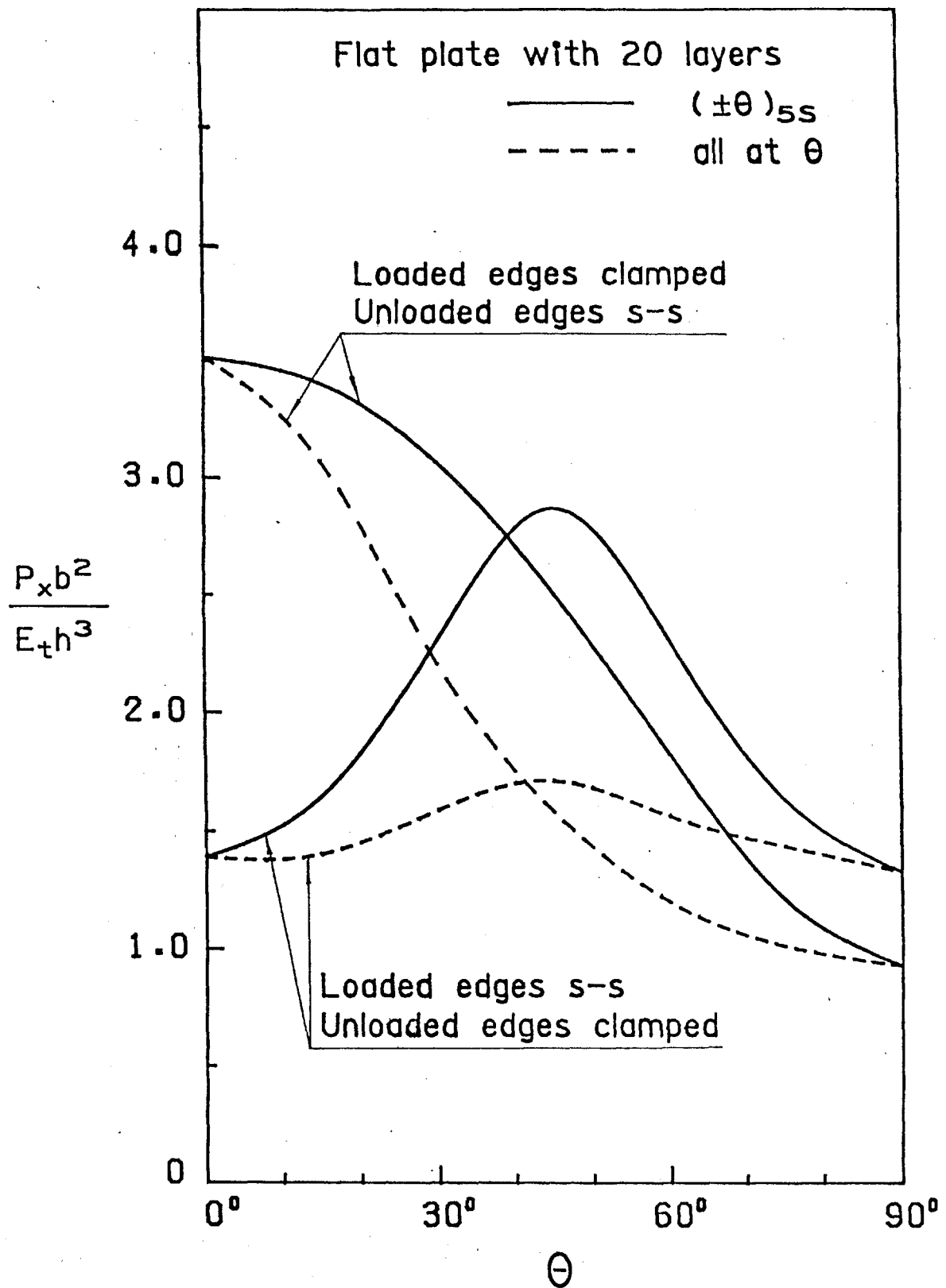


Fig. (III,2) Flat plate: variation of critical compressive load (P_x) with fibre angle; mixed edge conditions

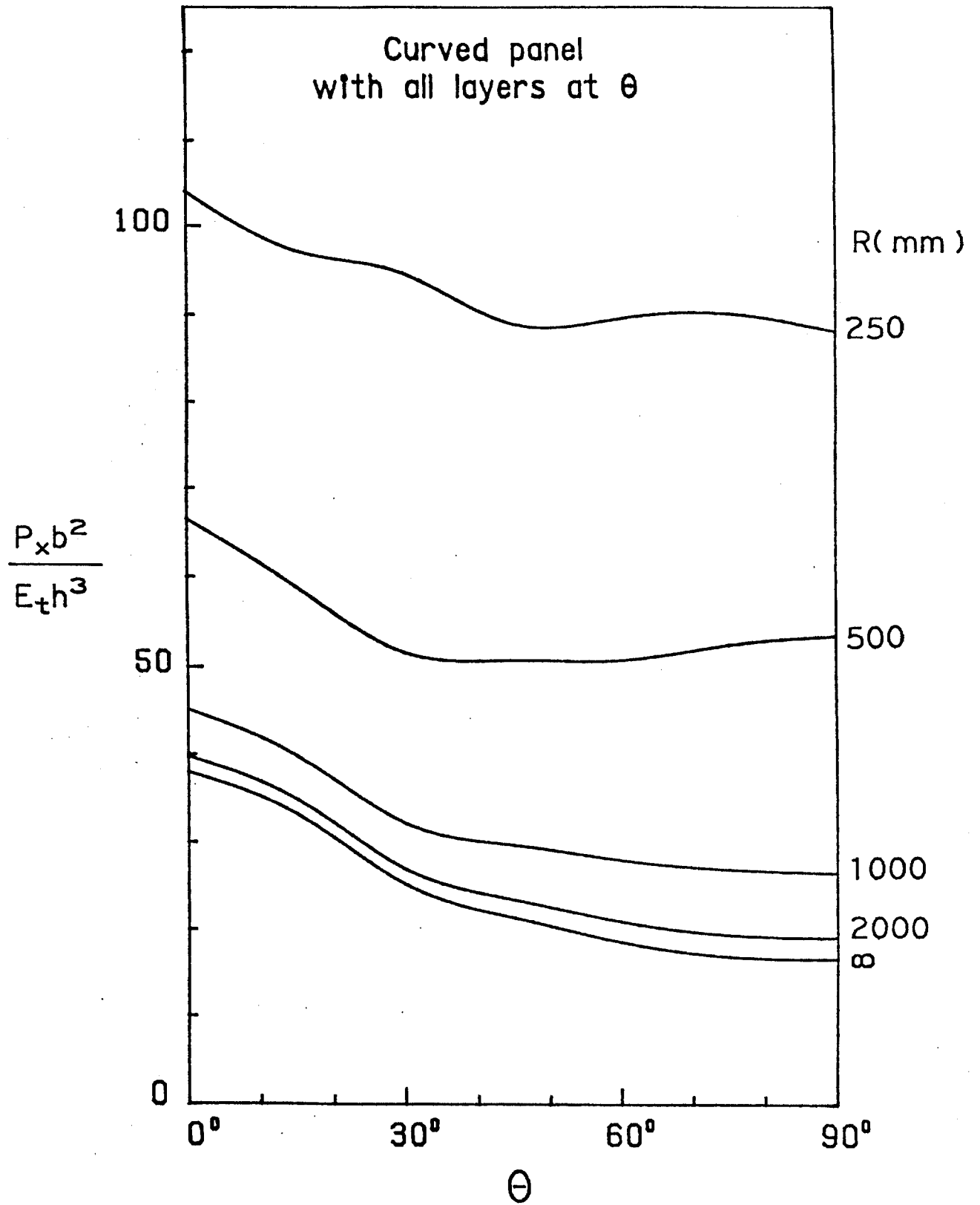


Fig. (III,3) Curved panel with unidirectional lay-up:
variation of P_x with fibre angle

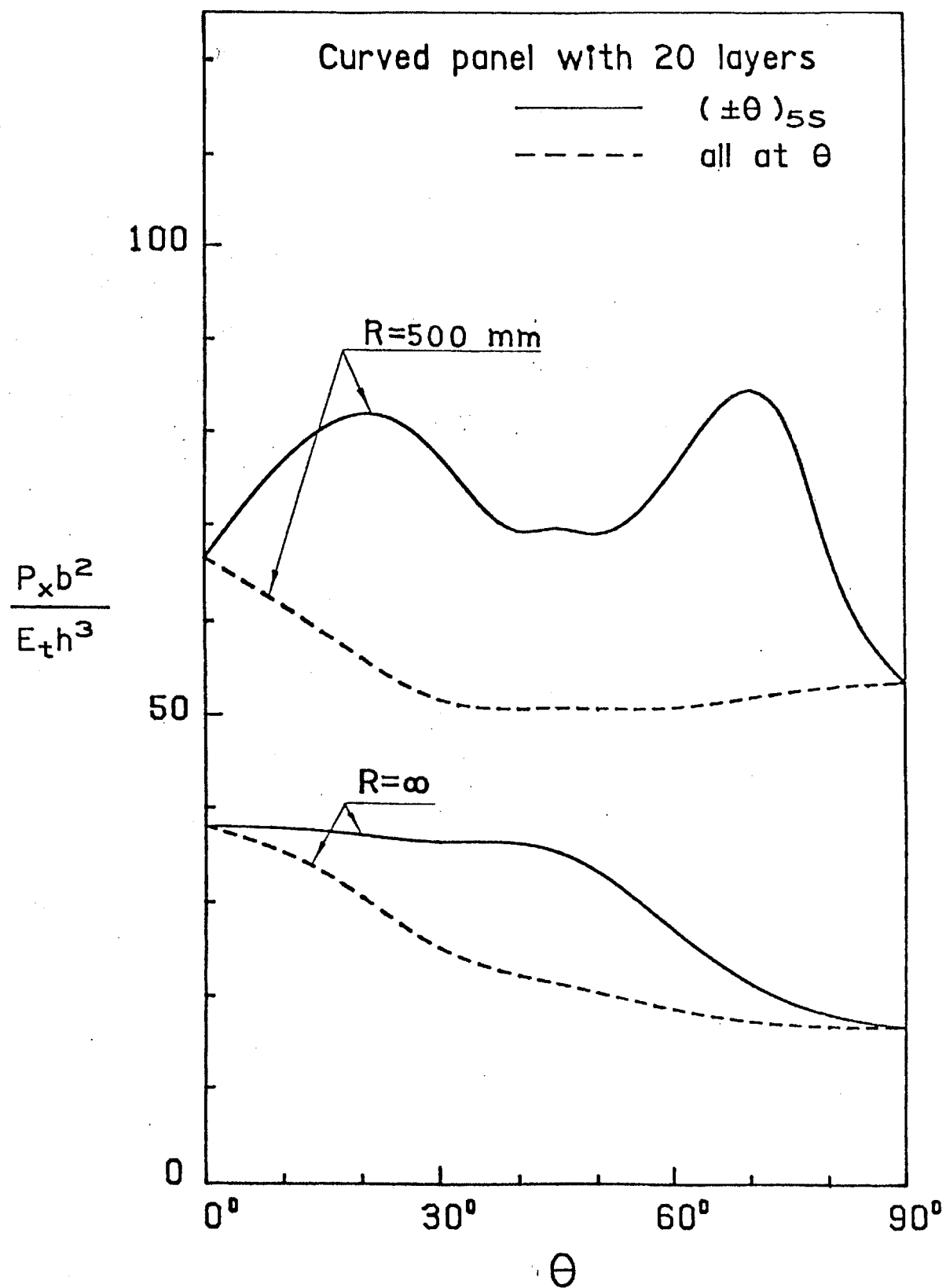


Fig. (III,4) Curved panel with alternate layers:
 variation of P_x with fibre angle

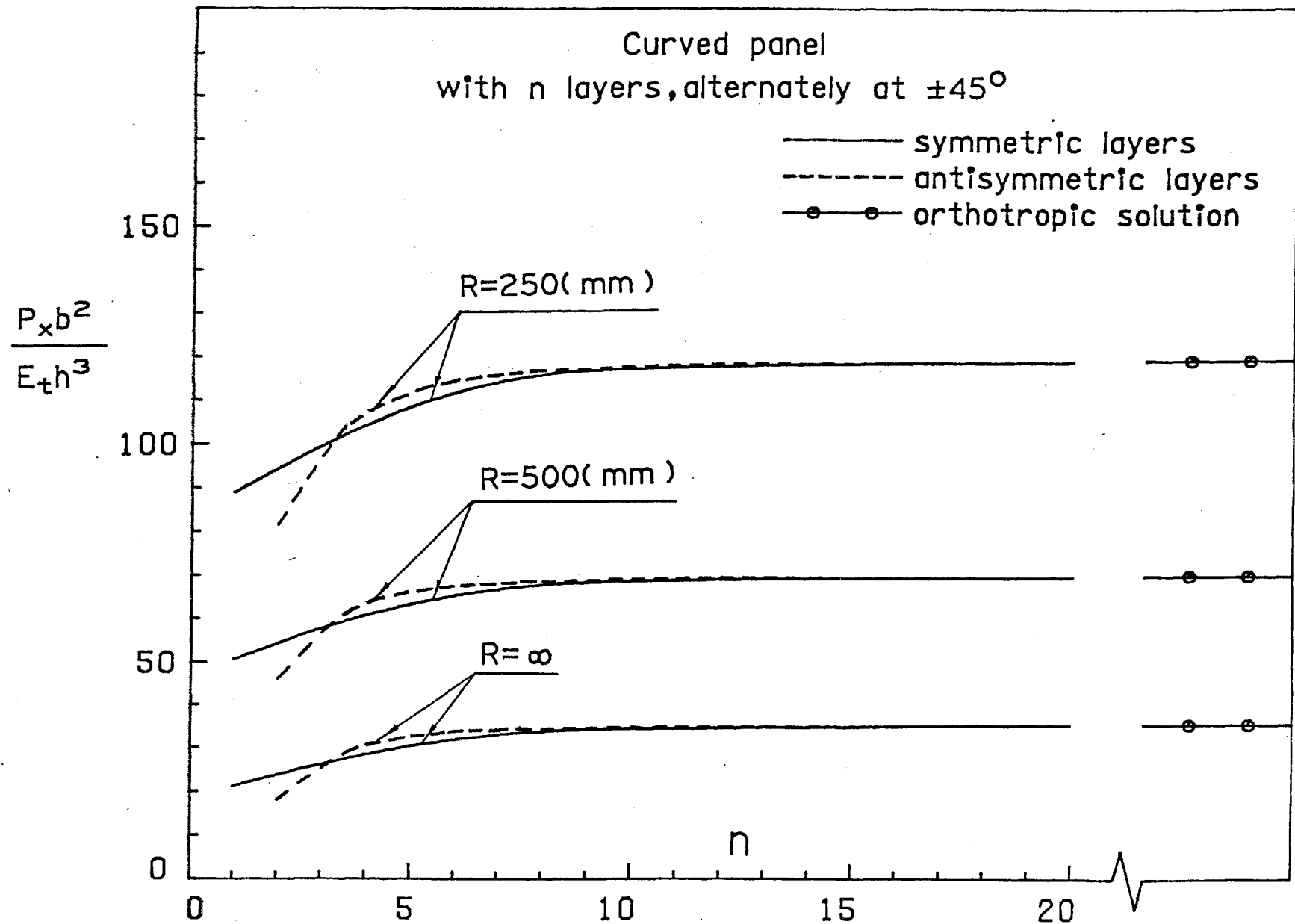


Fig. (III,5) Curved panel with symmetric and antisymmetric lay-up: variation of P_x with number of layers

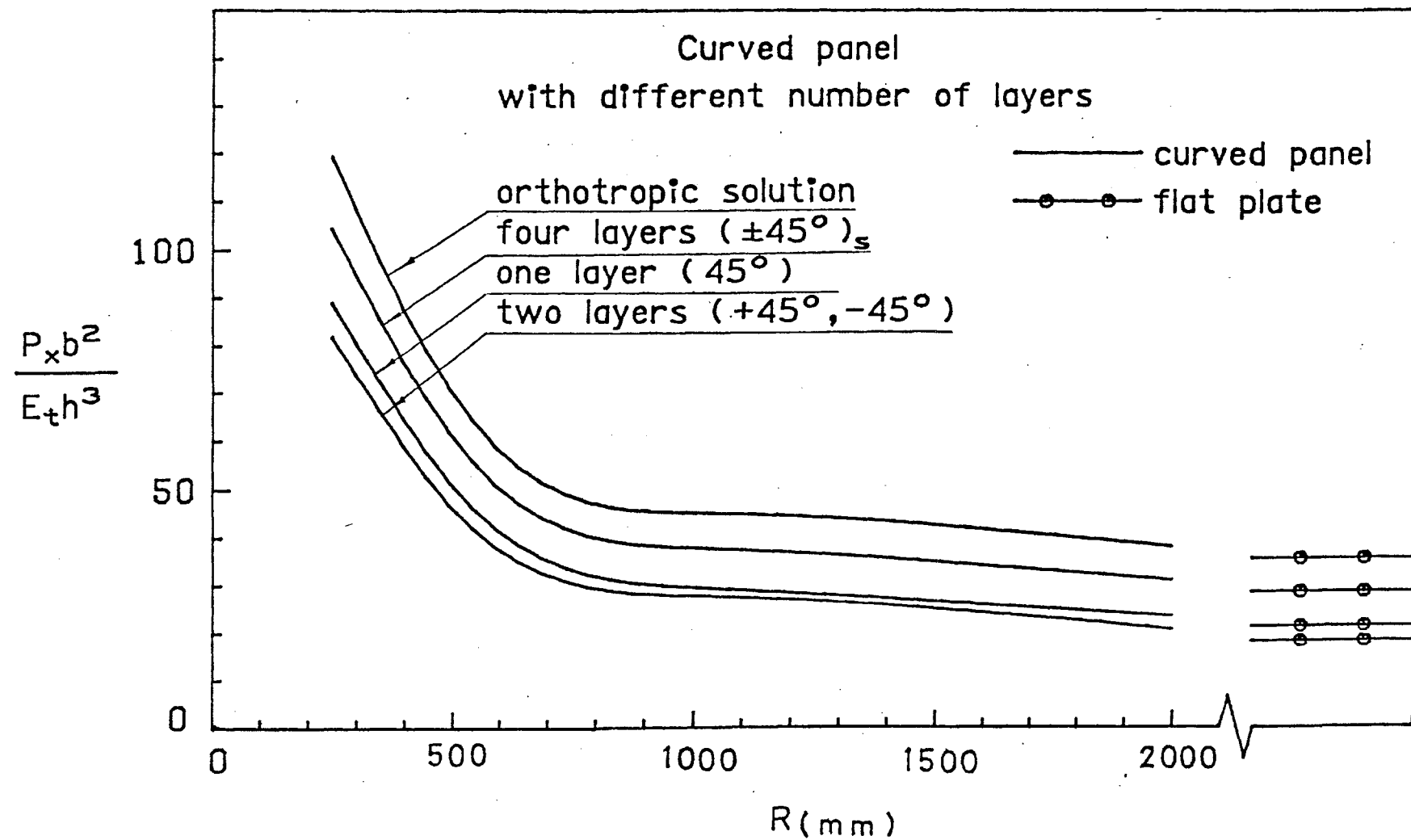


Fig. (III,6) Curved panel with alternate layers: variation of P_x with radius of curvature

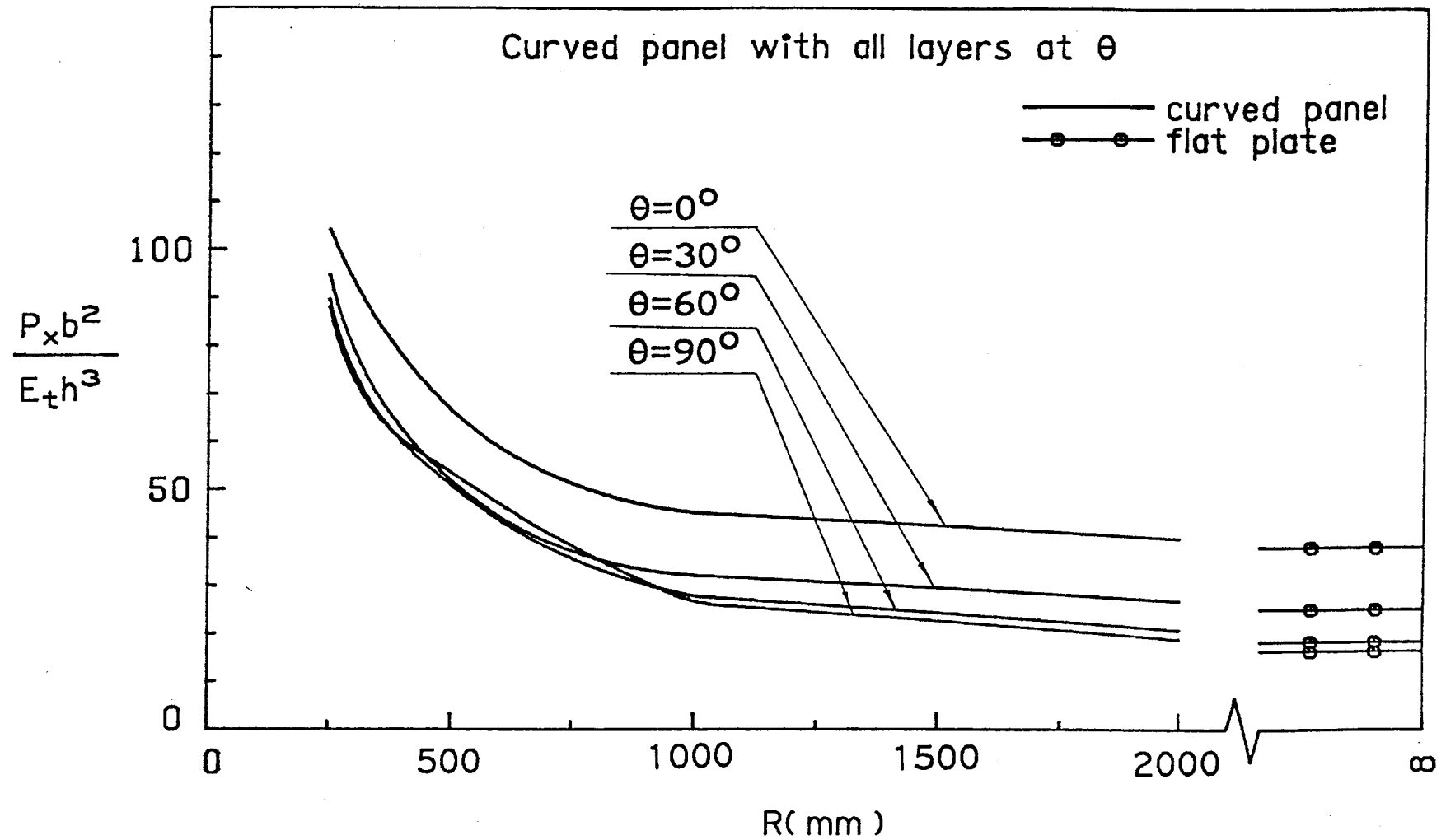


Fig. (III,7) Curved panel with unidirectional lay-up:
variation of P_x with radius of curvature

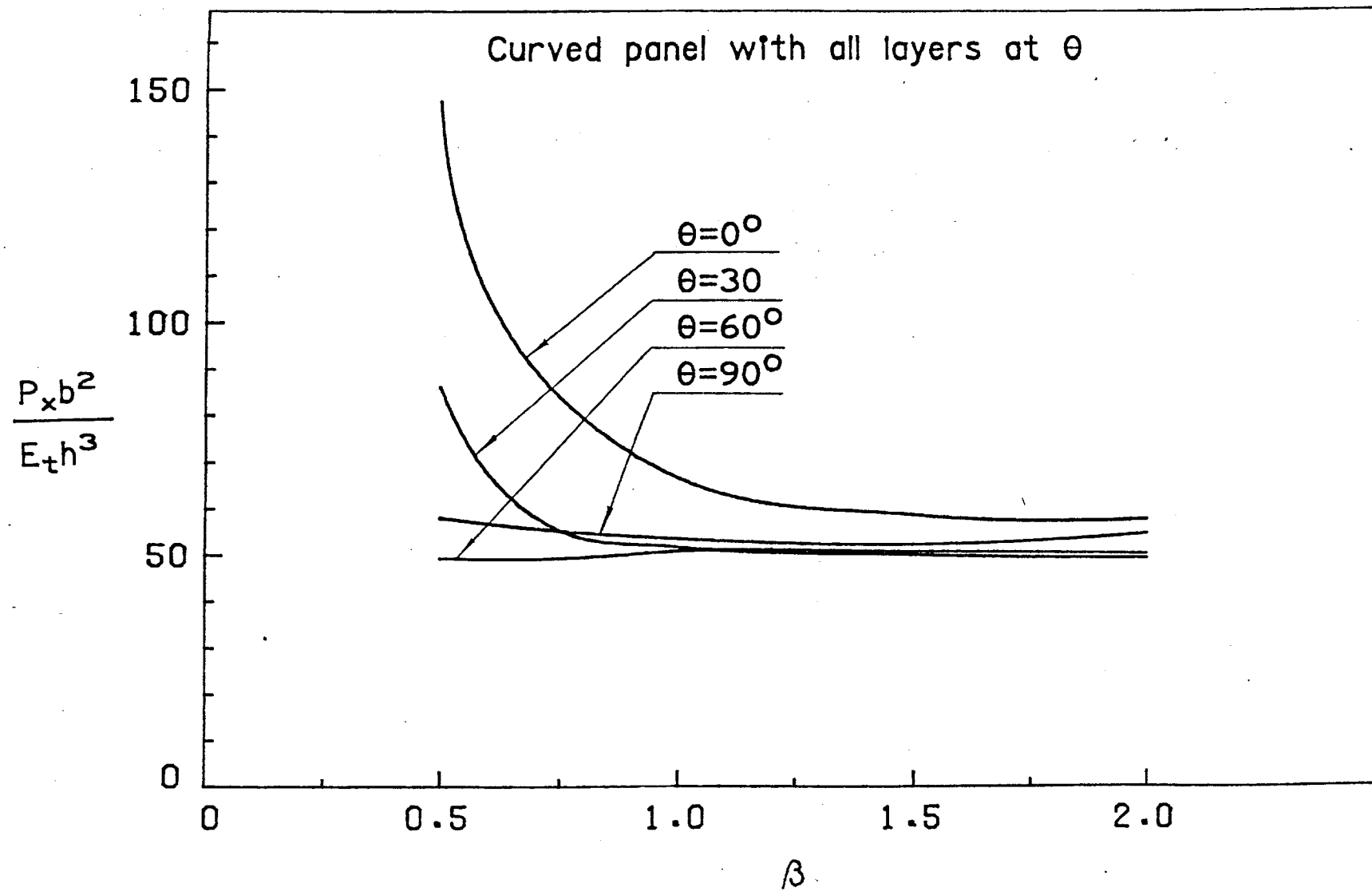


Fig. (III,8) Curved panel with unidirectional lay-up:
variation of P_x with aspect ratio

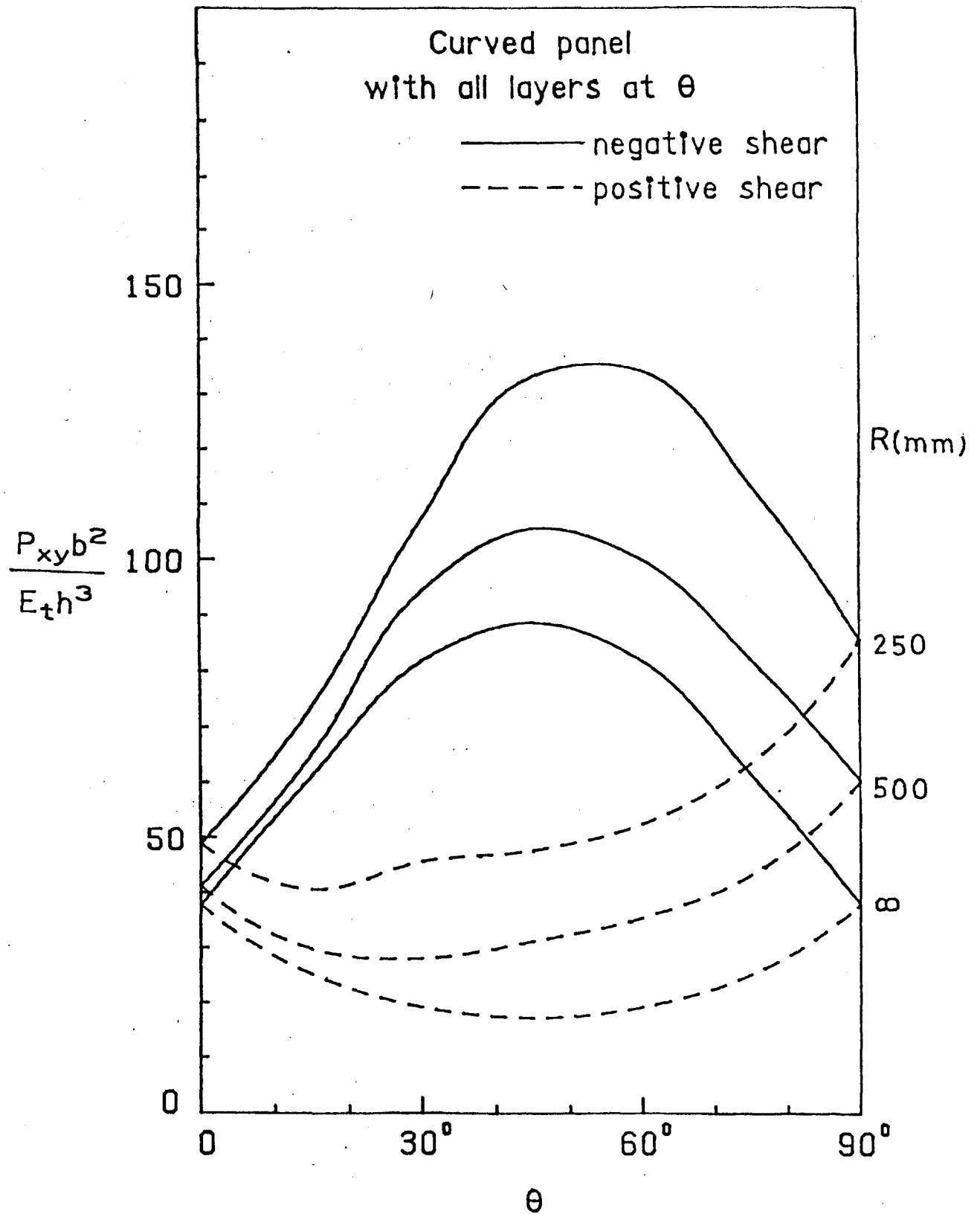


Fig. (III,9) Curved panel with unidirectional lay-up:
variation of critical shear load (P_{xy})
with fibre angle

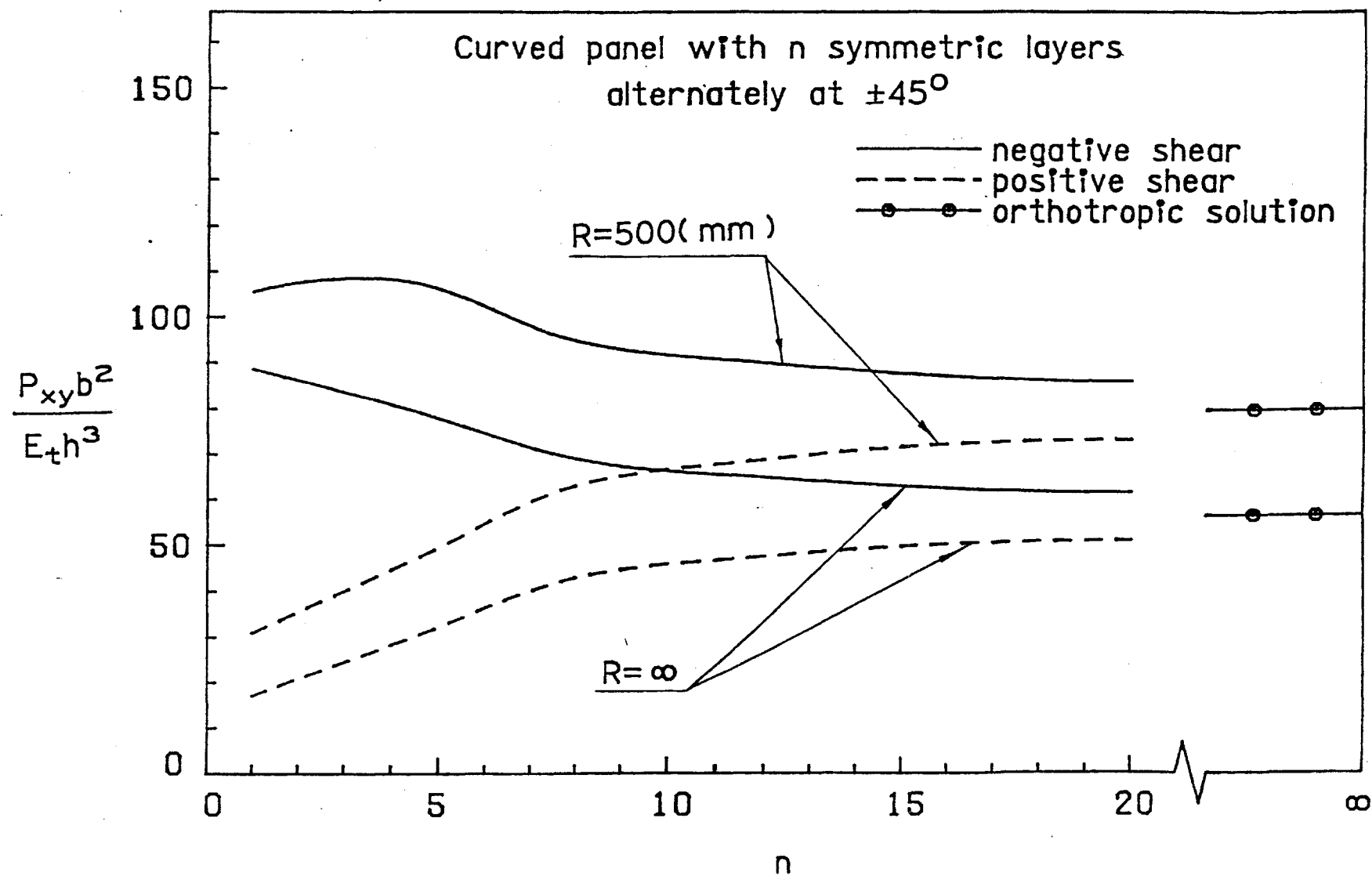


Fig. (III,10) Curved panel with symmetric lay-up:
variation of P_{xy} with number of layers

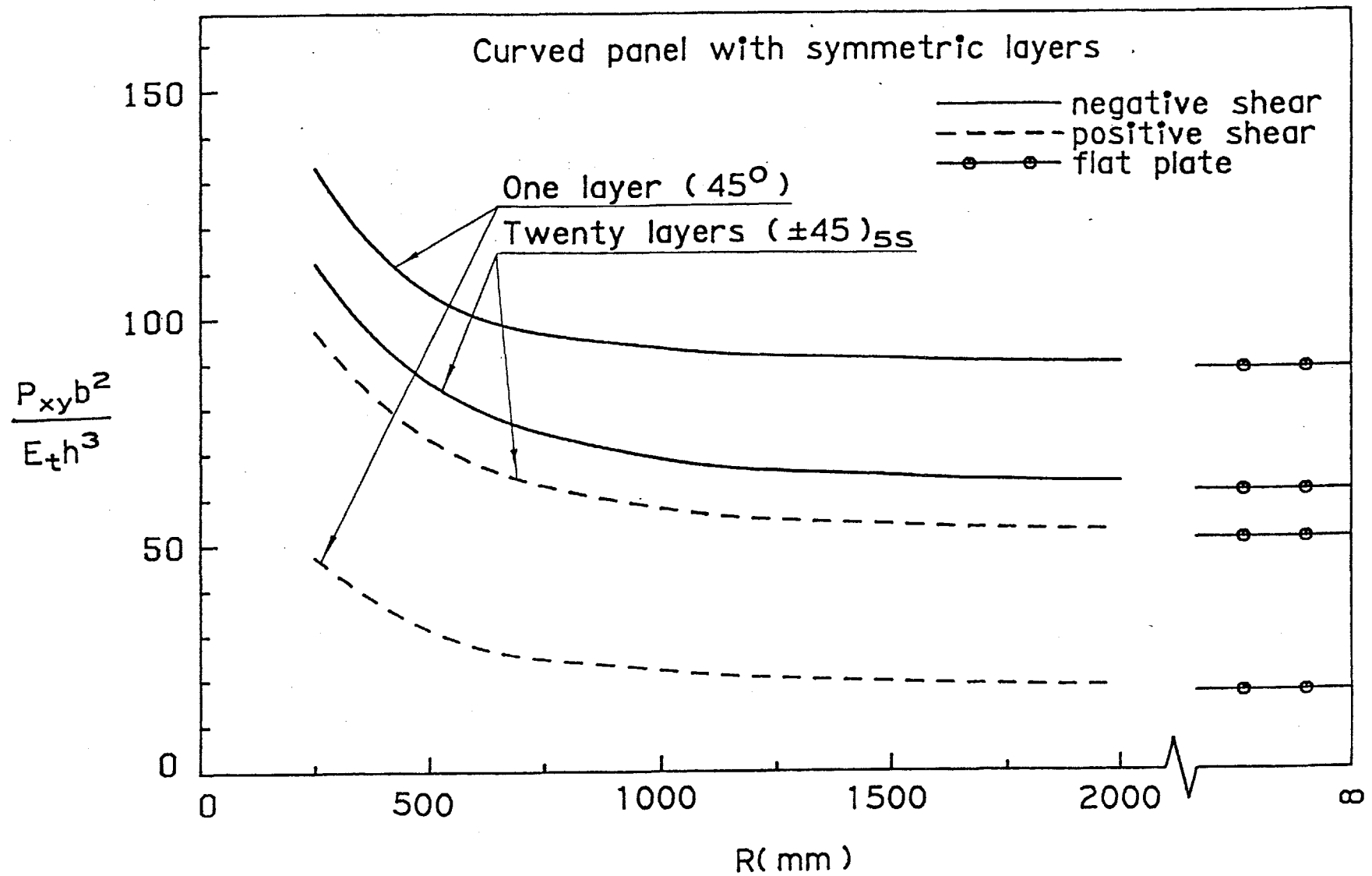


Fig. (III,11) Curved panel with symmetric lay-up:
variation of P_{xy} with radius of curvature

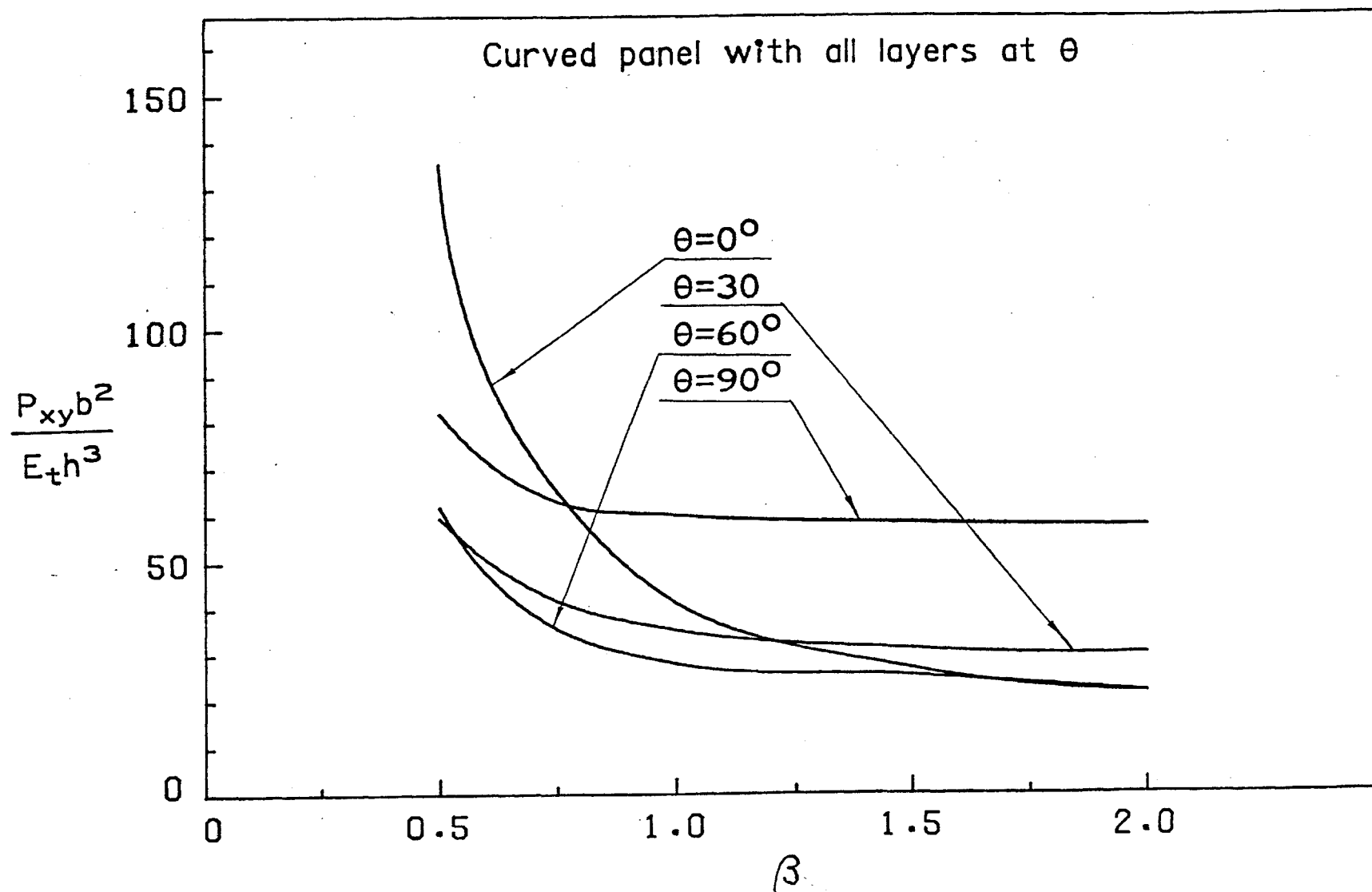


Fig. (III,12) Curved panel with unidirectional lay-up:
variation of P_{xy} with aspect ratio

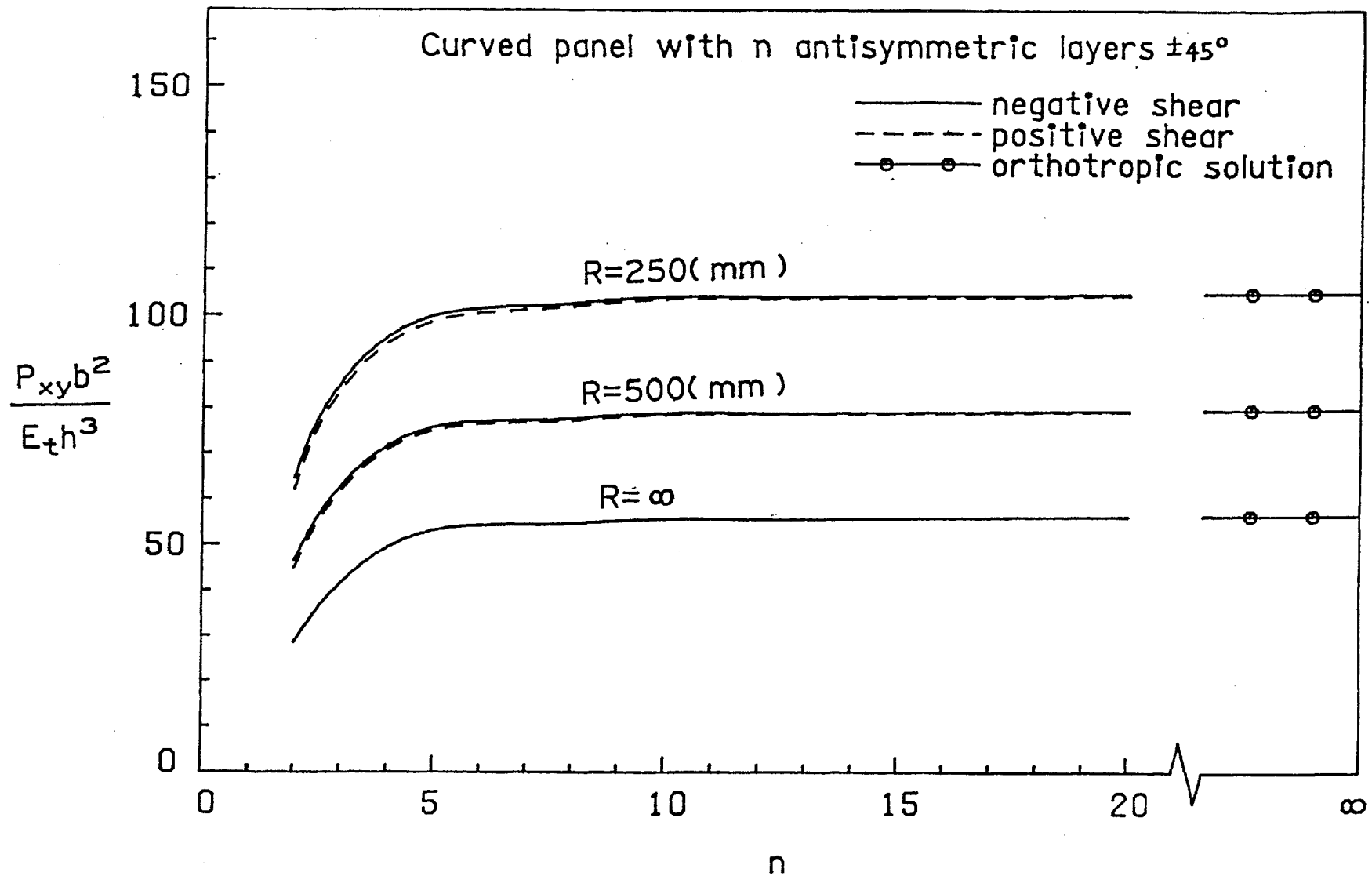


Fig. (III,13) Curved panel with antisymmetric lay-up:
variation of P_{xy} with number of layers

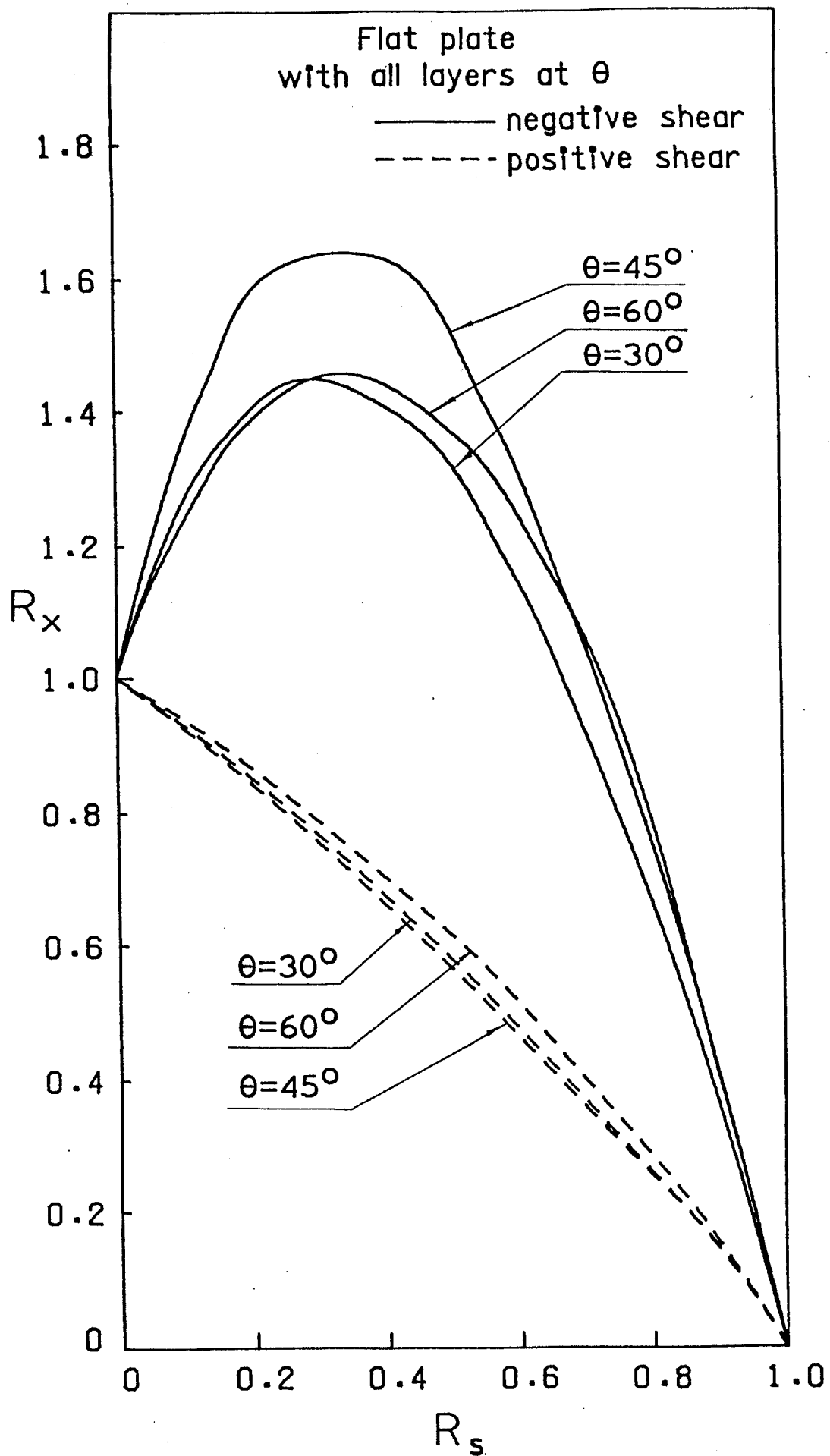


Fig. (III, 14) Flat plate with unidirectional lay-up:
combined shear and compressive loading

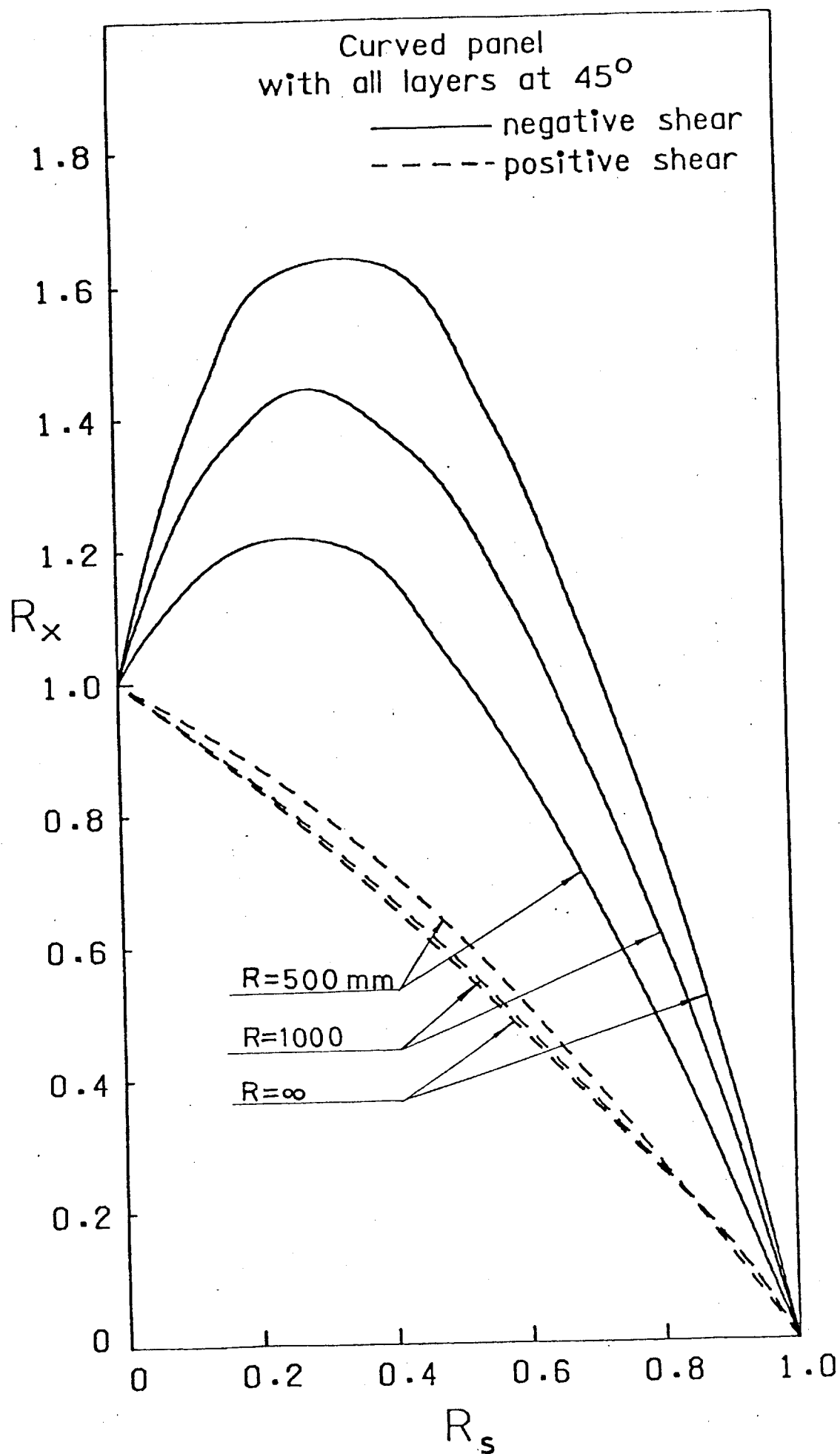


Fig. (III,15) Curved panel with 45° lay-up: combined shear and compressive loading

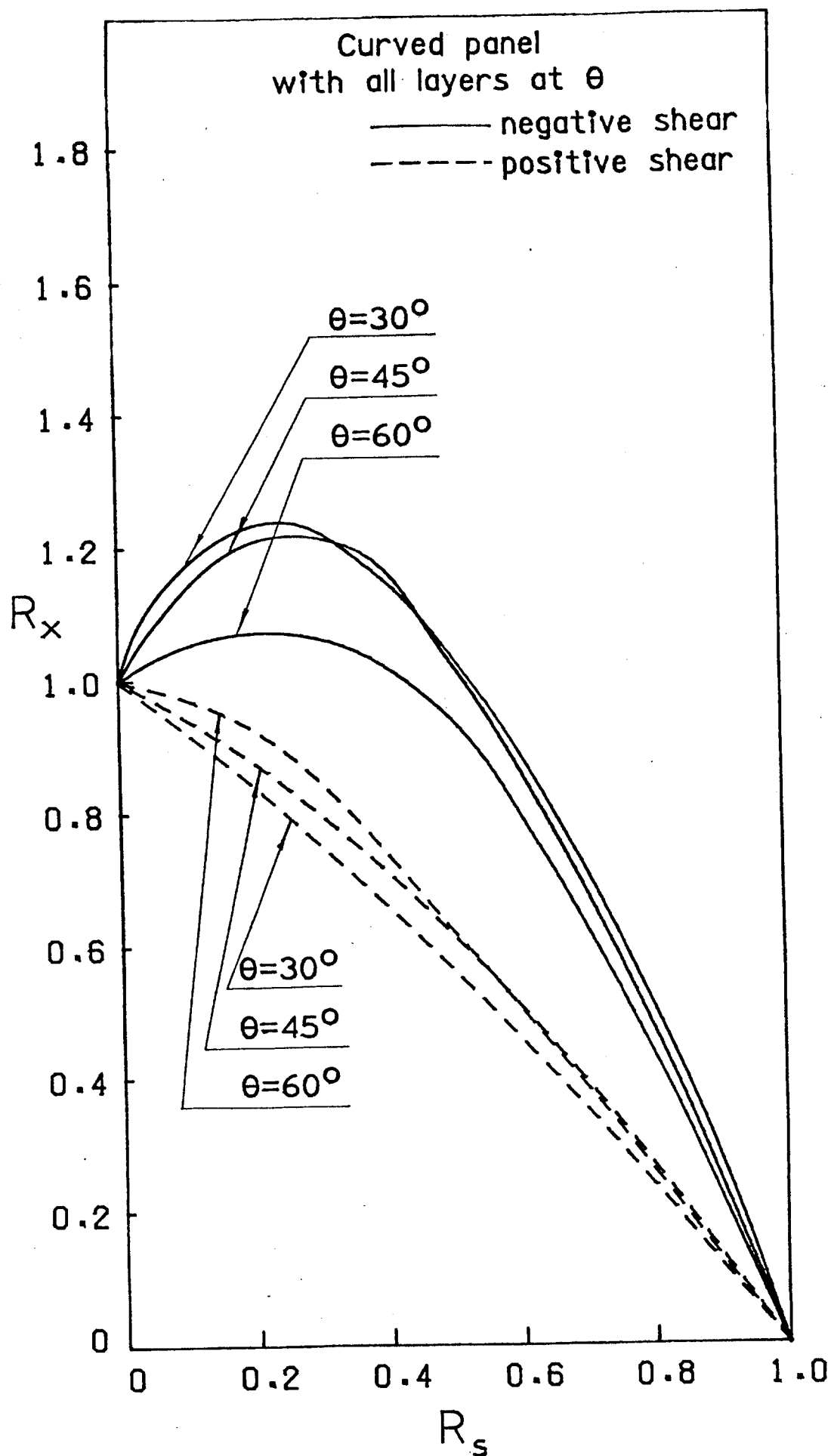


Fig. (III,16) Curved panel with unidirectional lay-up:
combined shear and compressive loading

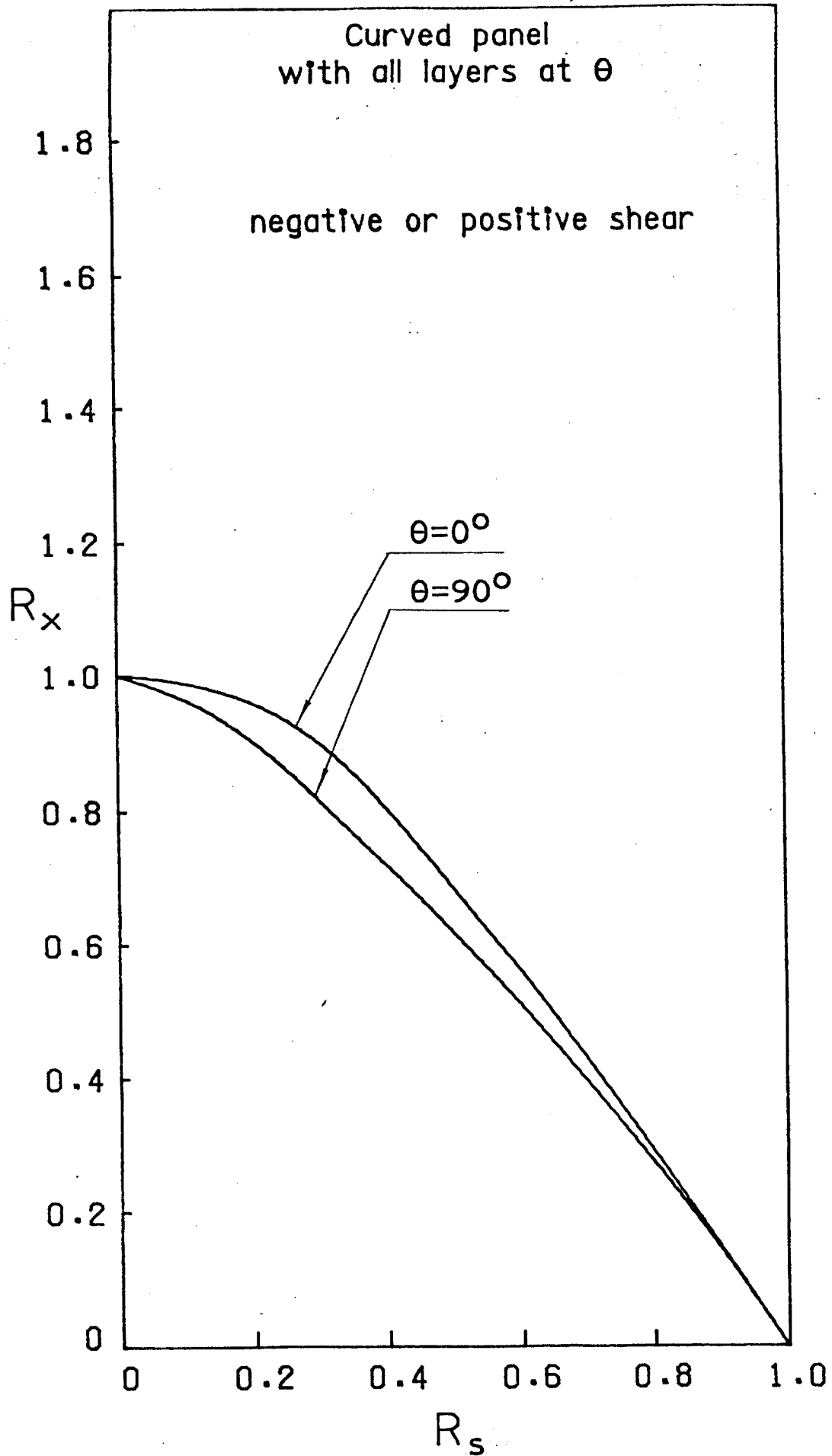


Fig. (III,17) Curved panel with 0° and 90° lay-up :
combined shear and compressive loading

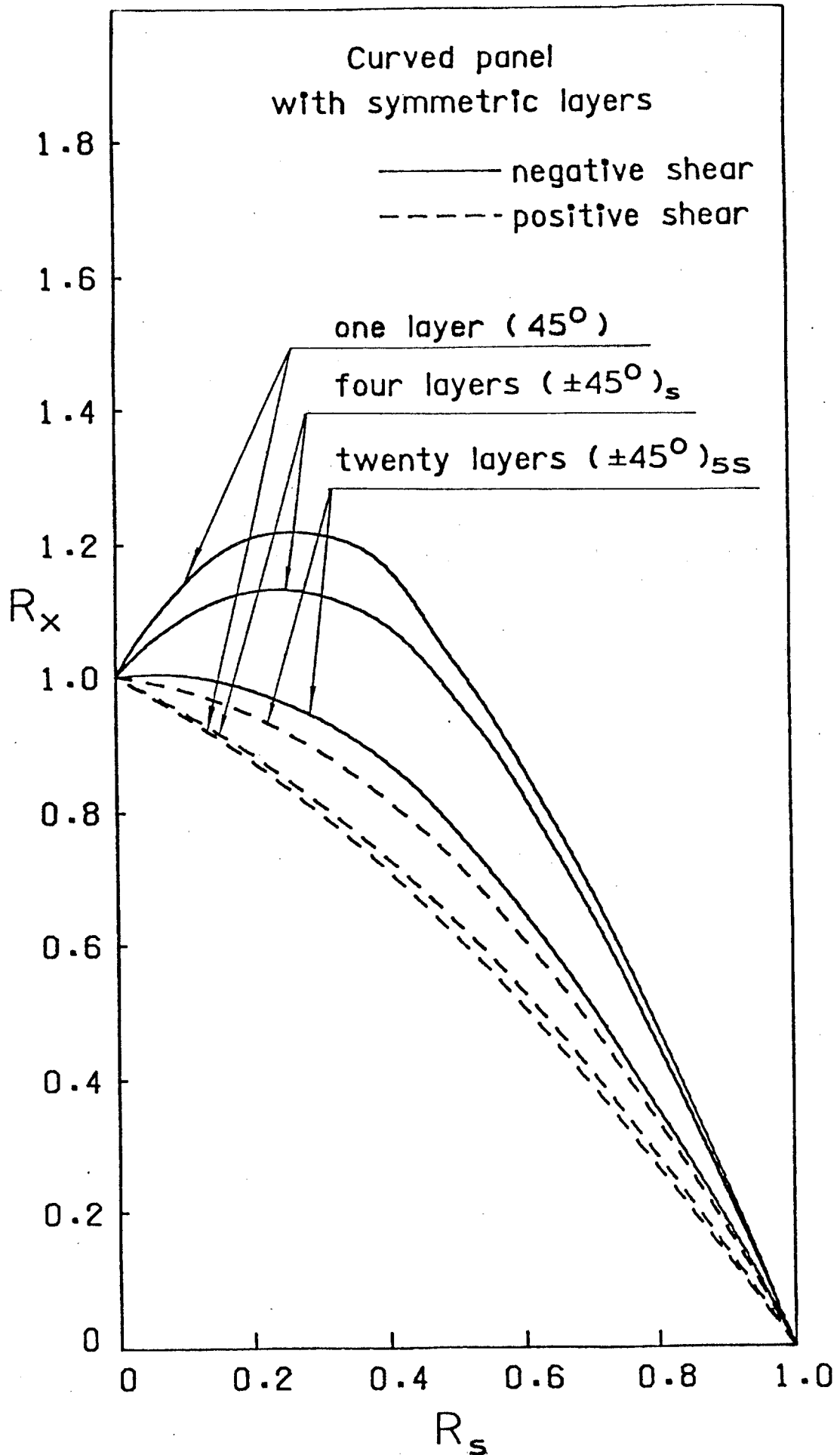


Fig. (III,18) Curved panel with symmetric lay-up:
combined shear and compressive loading

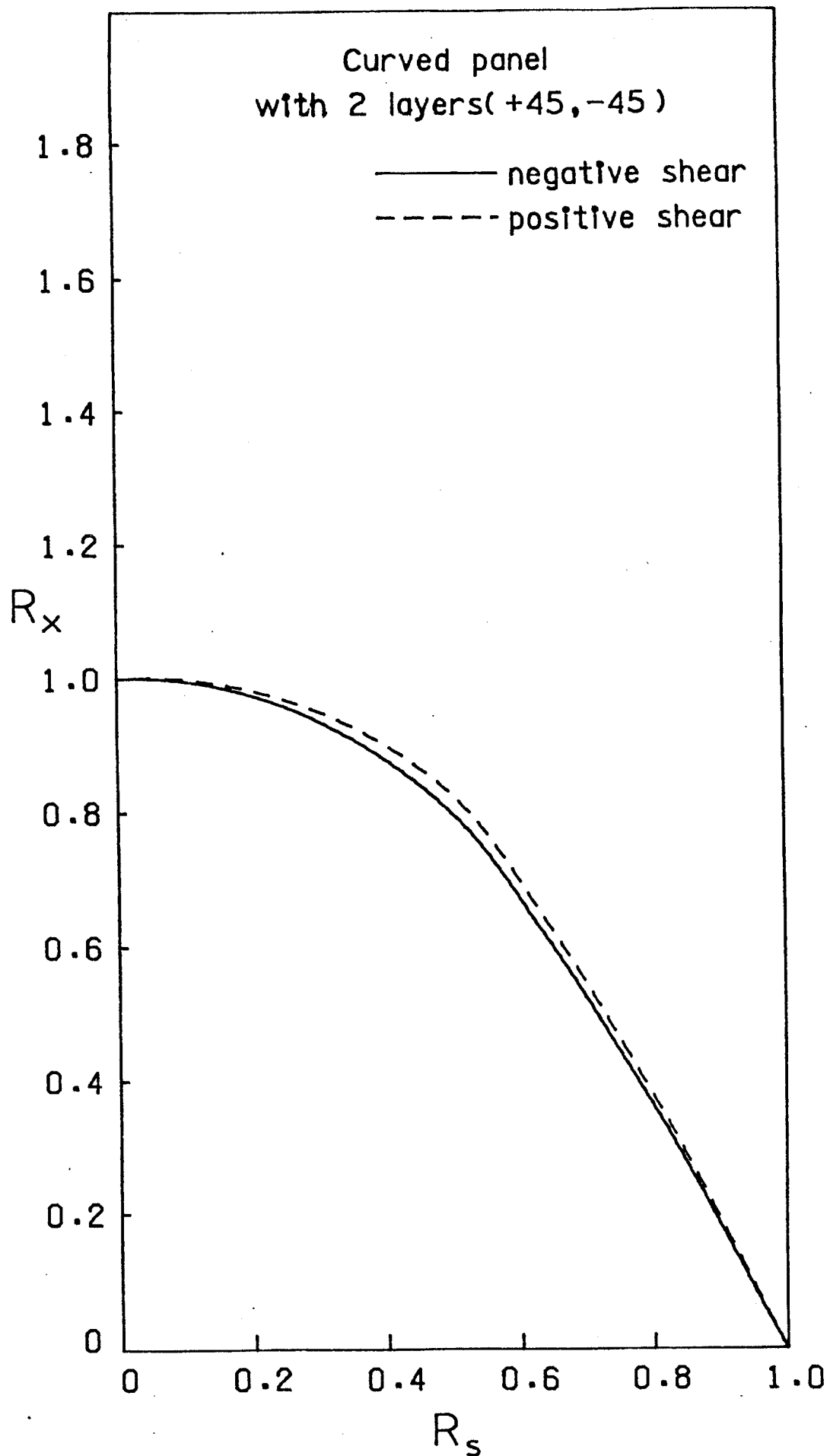


Fig. (III,19) Curved panel with $\pm 45^\circ$ lay-up: combined shear and compressive loading

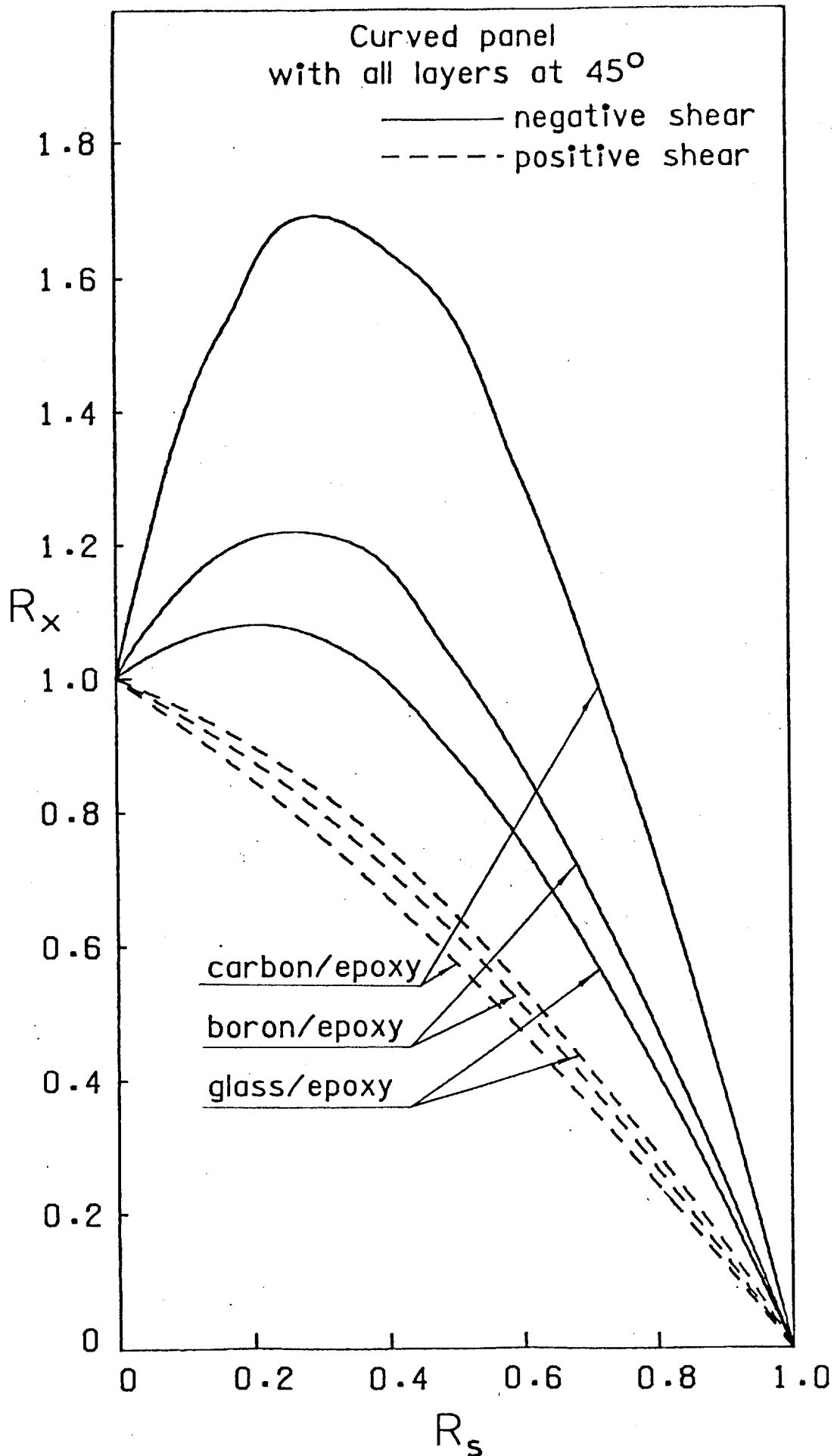


Fig. (III,20) Curved panel with 45° lay-up: combined shear and compressive loading; effect of material properties

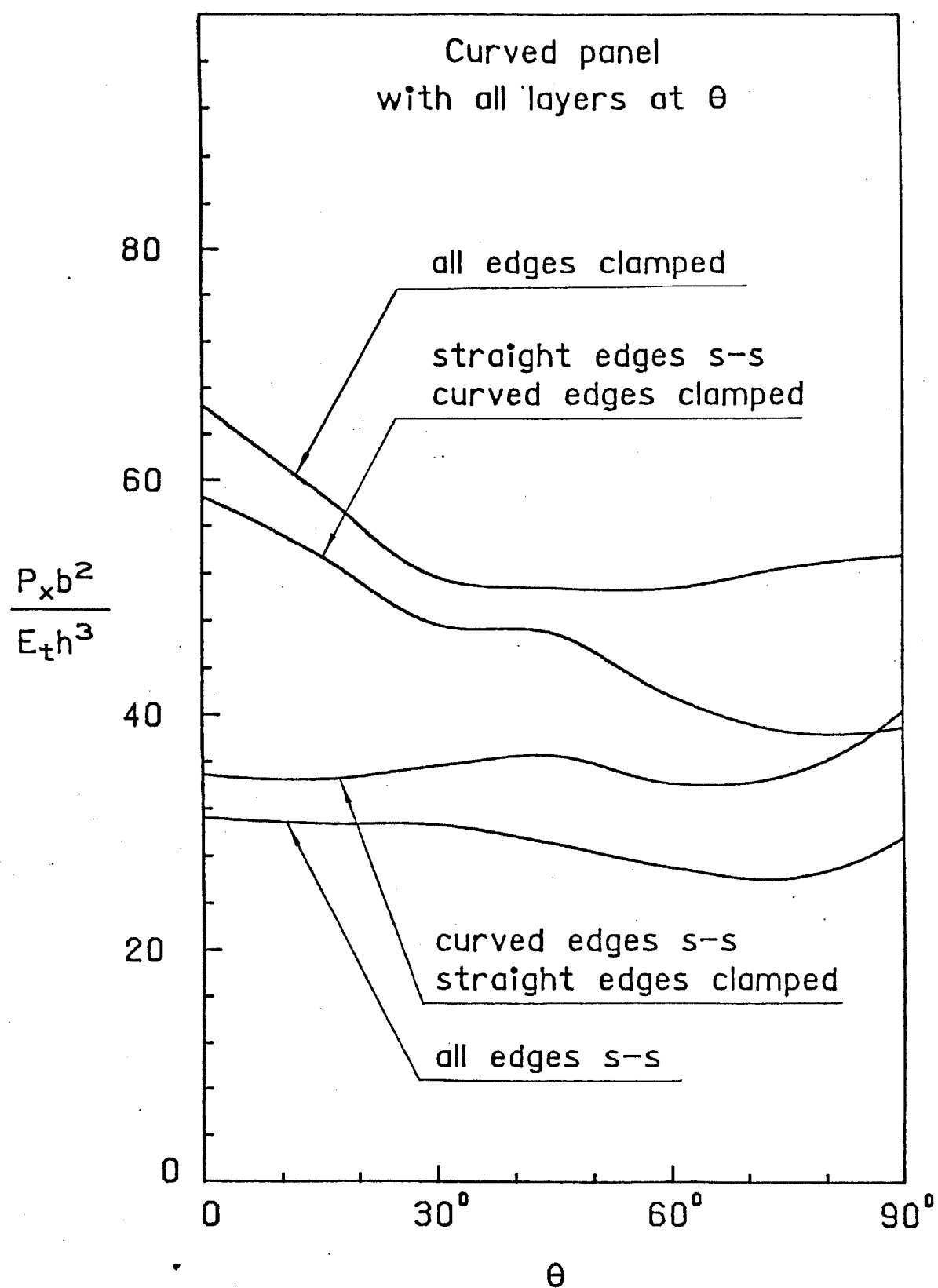


Fig. (III,21) Curved panel with unidirectional lay-up:
variation of P_x with fibre angle; effect
of boundary conditions

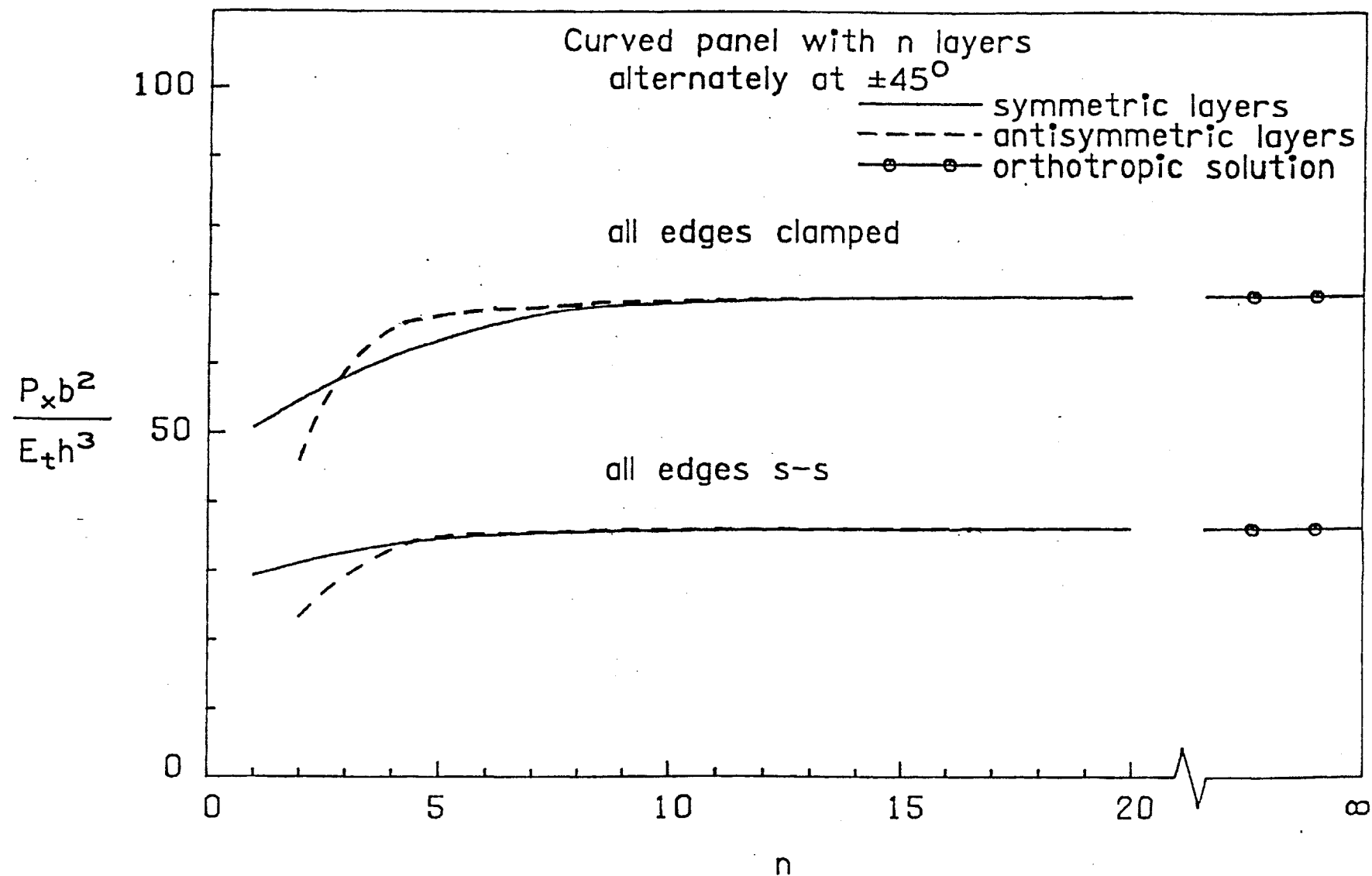


Fig. (III,22) Curved panel with alternate layers:
variation of P_x with number of layers;
clamped and simply supported edges

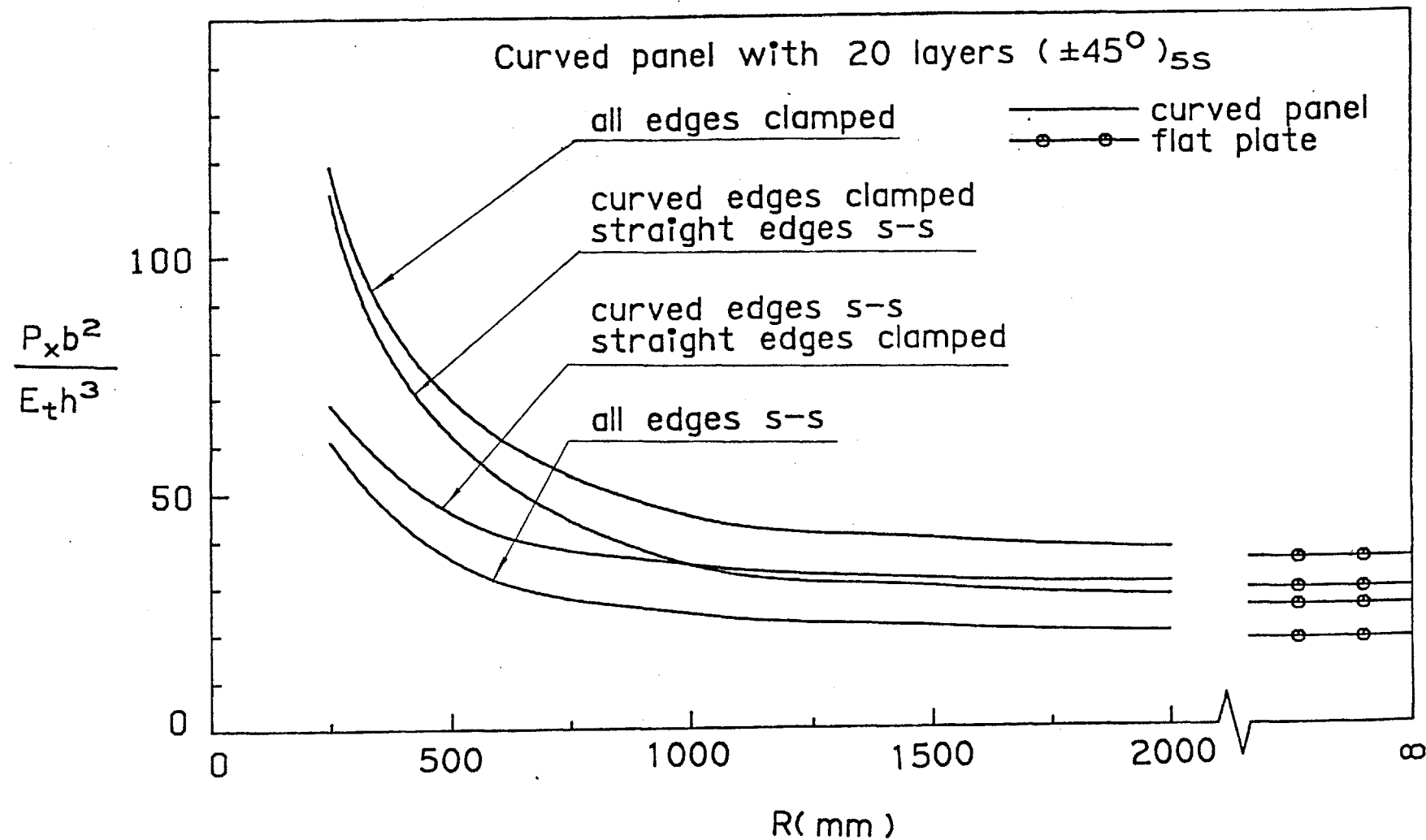


Fig. (III,23) Curved panel with $\pm 45^\circ$ symmetric lay-up:
 variation of P_x with radius of curvature;
 effect of boundary conditions

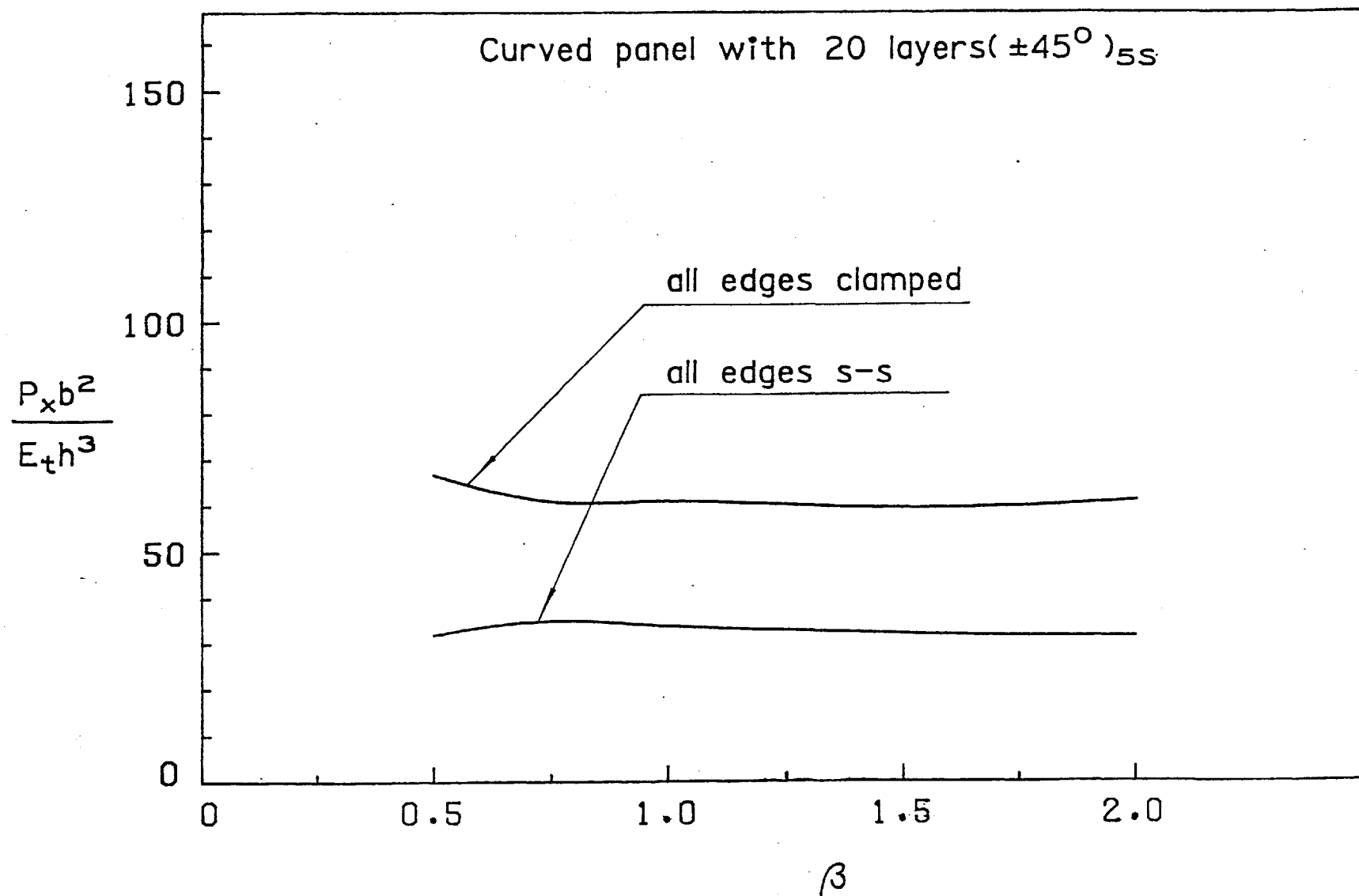


Fig. (III,24) Curved panel with $\pm 45^\circ$ symmetric lay-up:
variation of P_x with aspect ratio; clamped
and simply supported edges

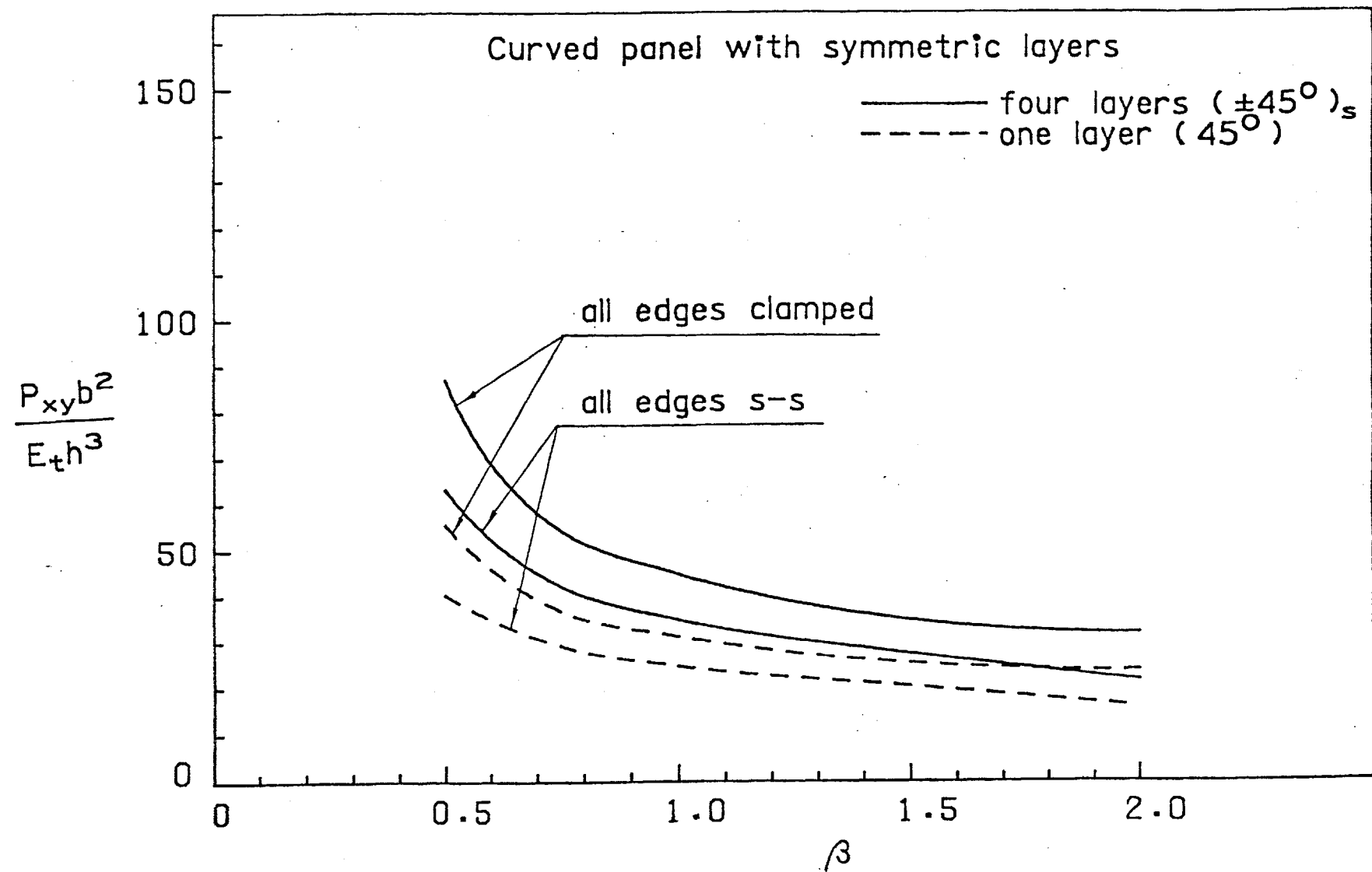


Fig. (III,25) Curved panel with symmetric lay-up:
 variation of P_{xy} with aspect ratio; clamped
 and simply supported edges

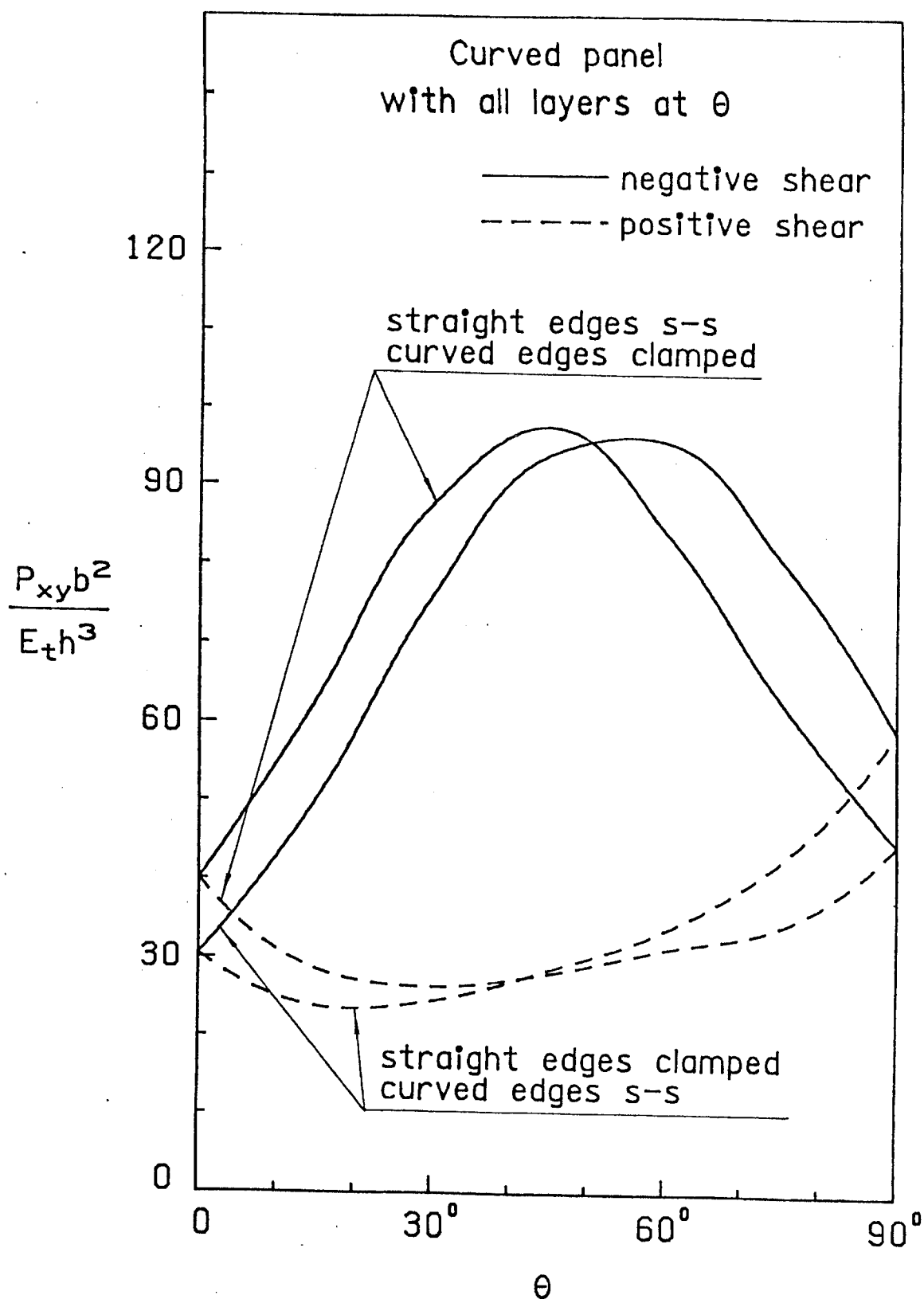


Fig. (III,26) Curved panel with unidirectional lay-up: variation of P_{xy} with fibre angle; mixed edge conditions

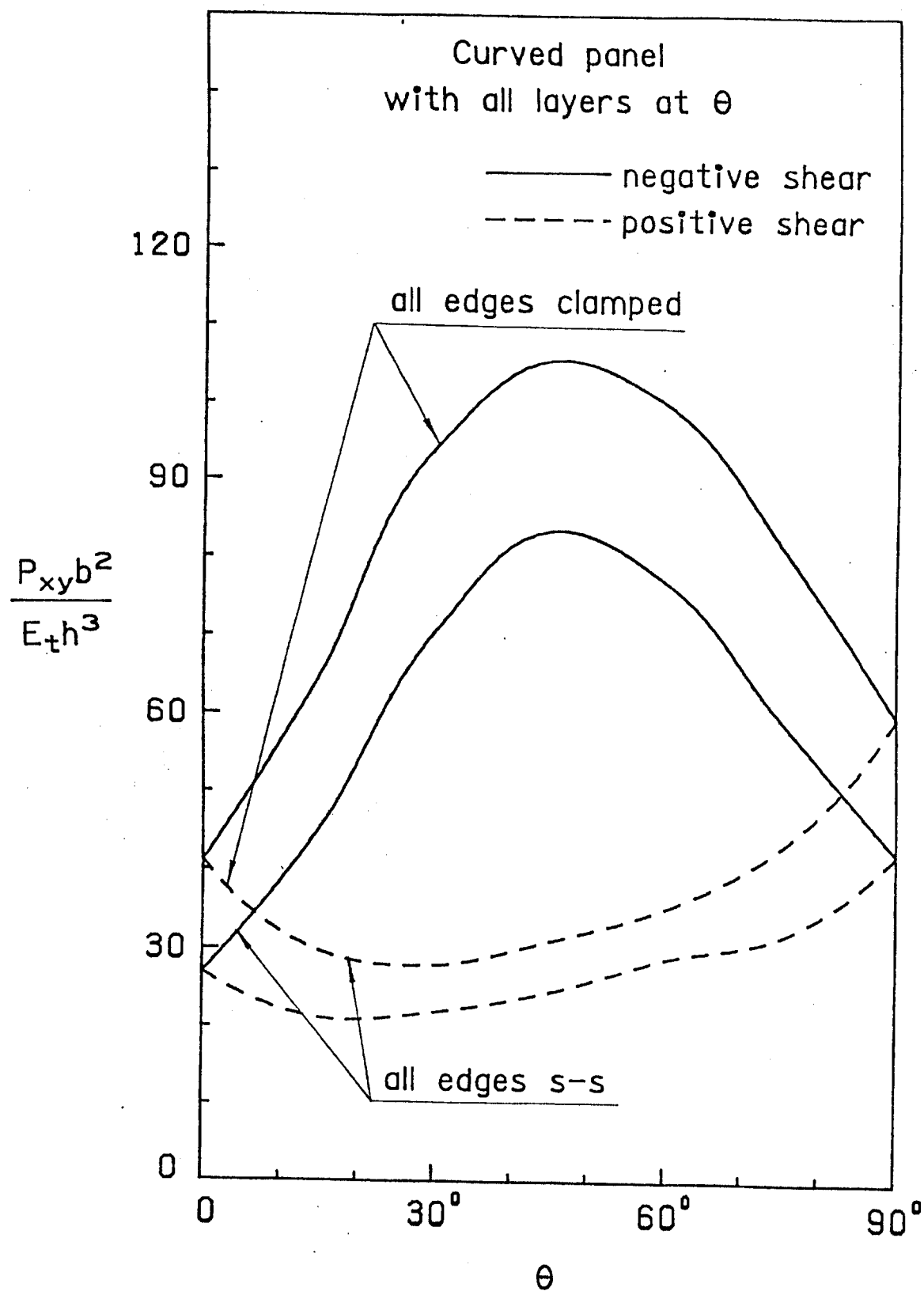


Fig. (III,27) Curved panel with unidirectional lay-up:
variation of P_{xy} with fibre angle;
clamped and simply supported edges

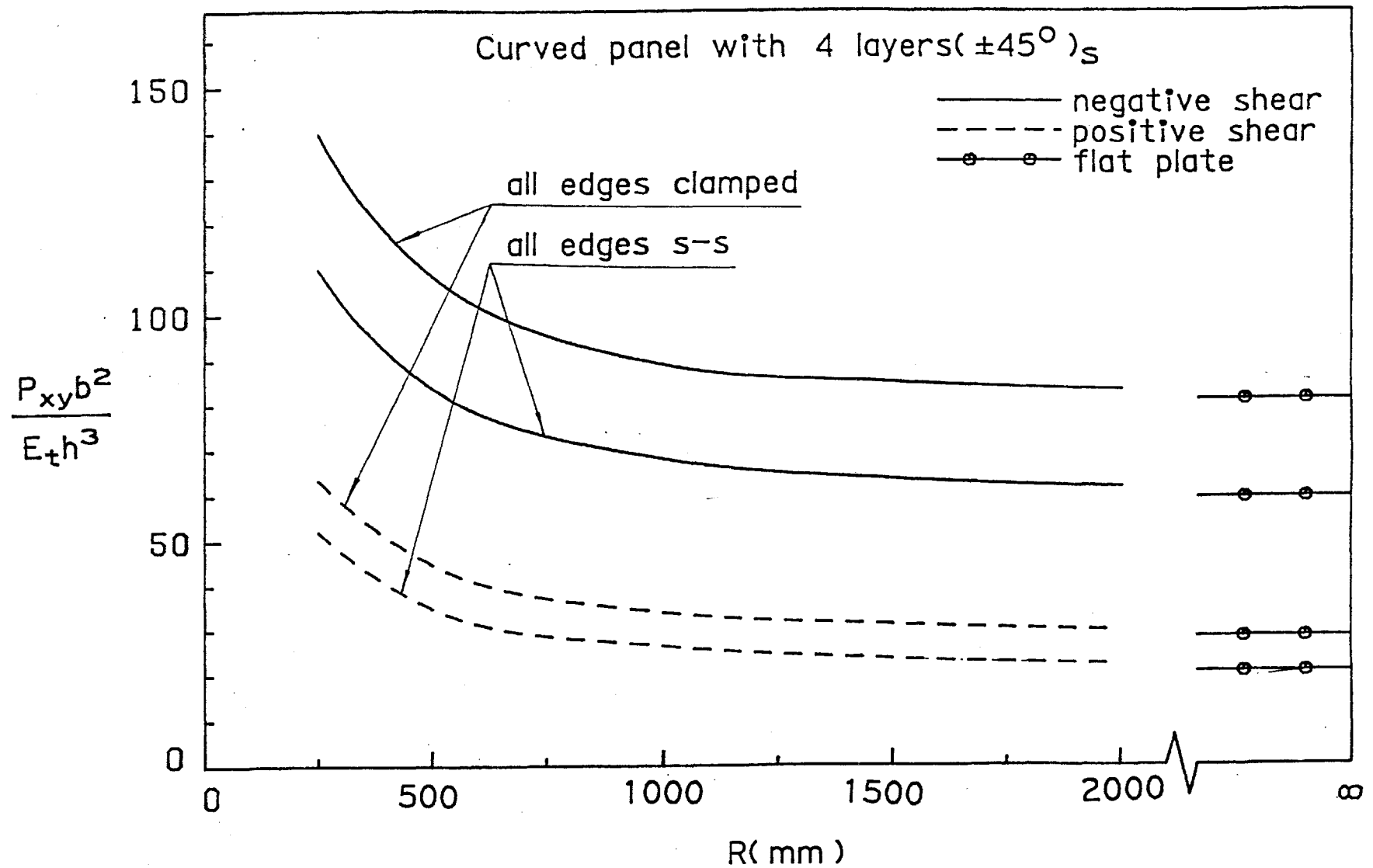


Fig. (III,28) Curved panel with $\pm 45^\circ$ symmetric lay-up:
variation of P_{xy} with radius of curvature;
clamped and simply supported edges

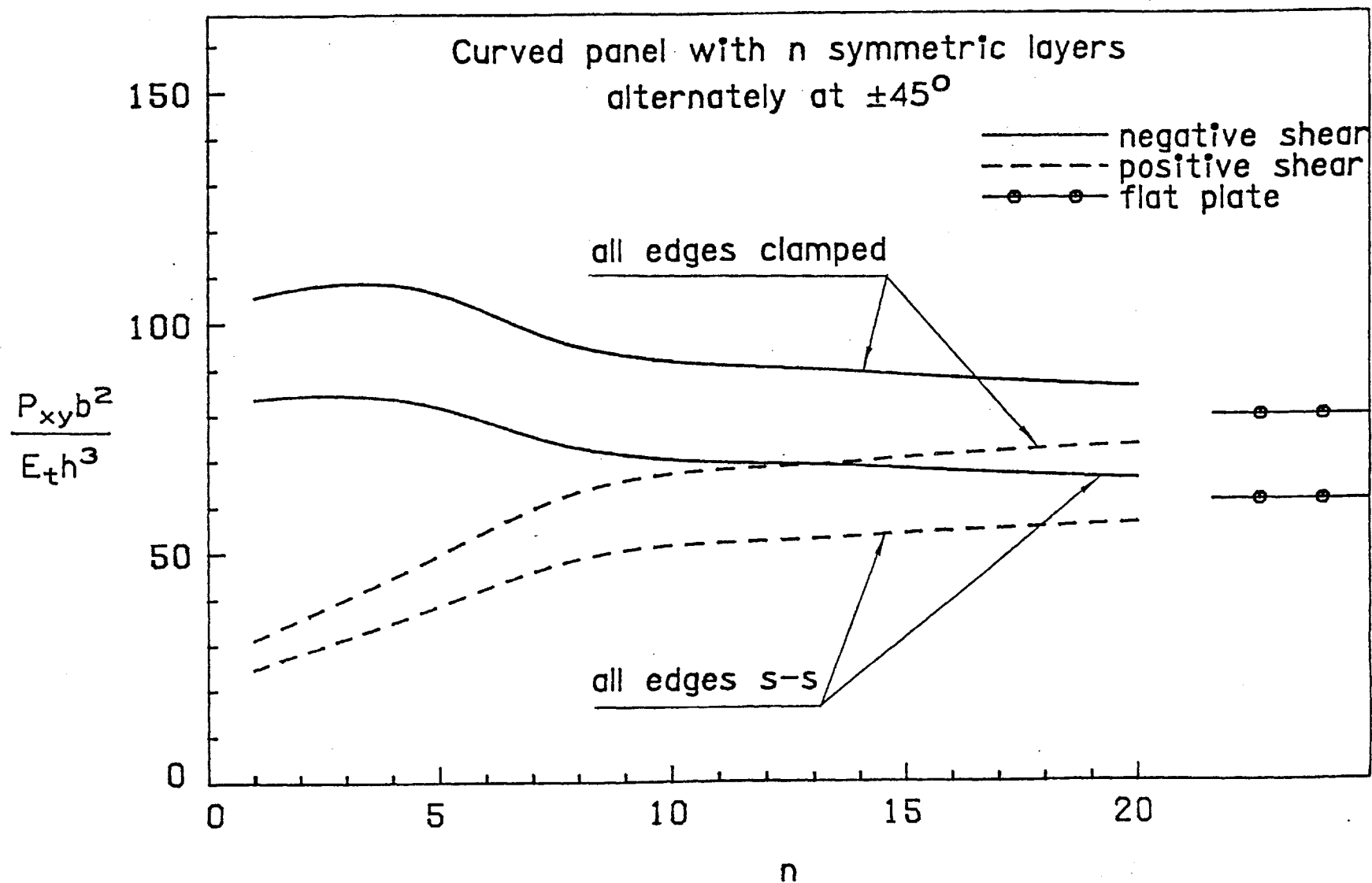


Fig. (III,29) Curved panel with $\pm 45^\circ$ symmetric lay-up:
variation of P_{xy} with number of layers;
clamped and simply supported edges

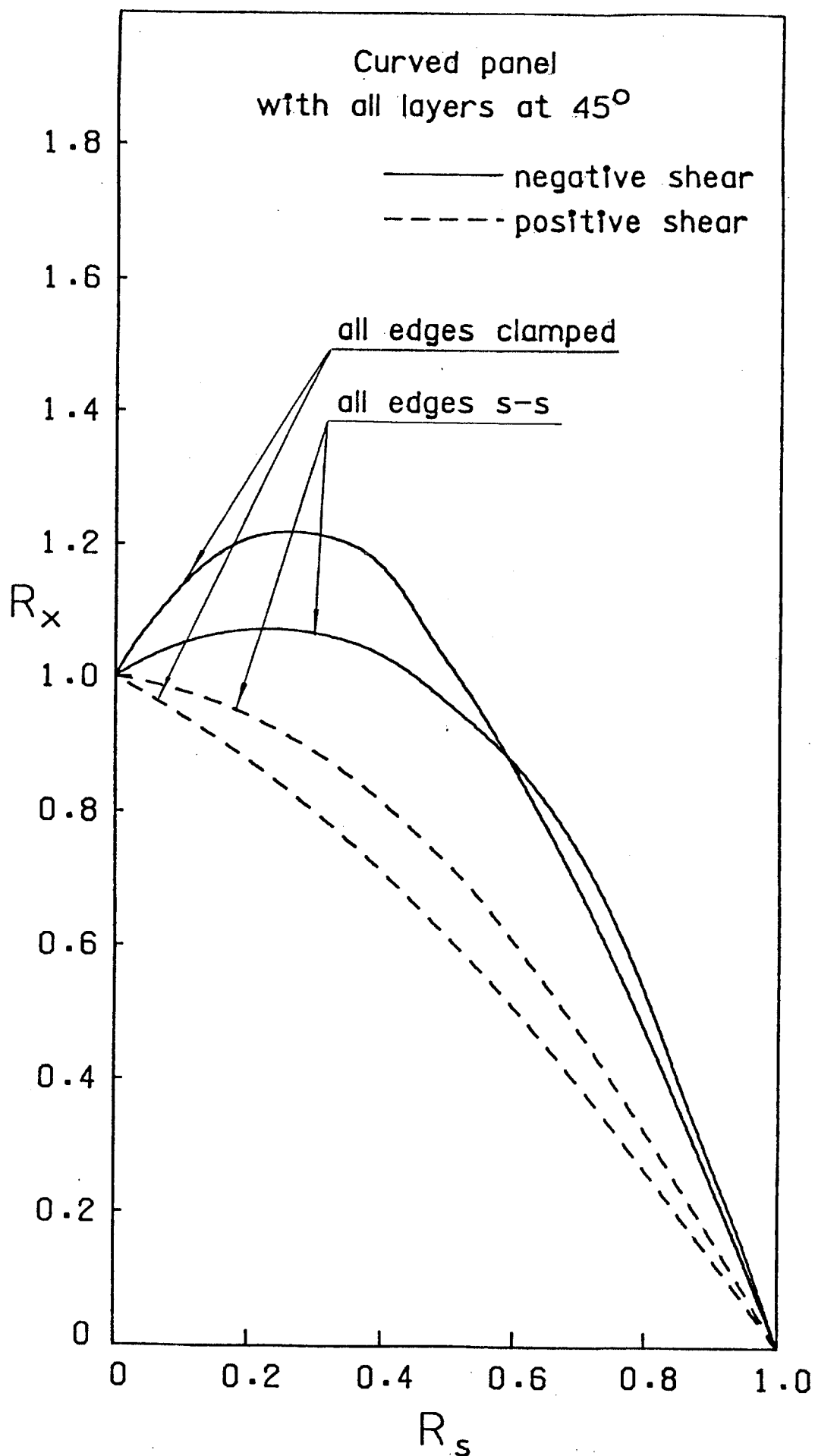


Fig. (III,30) Curved panel with 45° lay-up: combined shear and compressive loading; clamped and simply supported edges

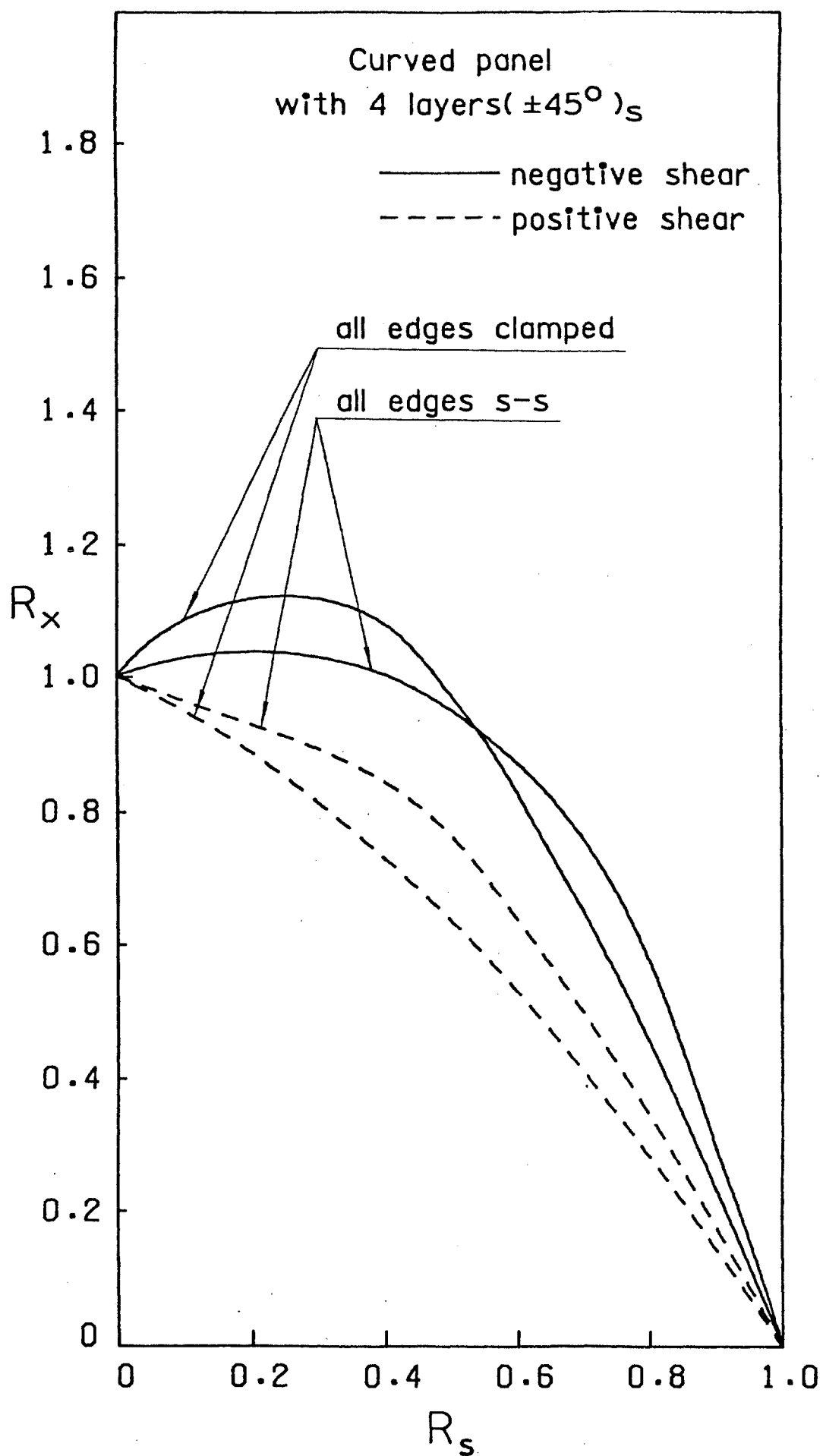


Fig. (III,31) Curved panel with $\pm 45^\circ$ symmetric lay-up:
combined shear and compressive loading;
clamped and simply supported edges

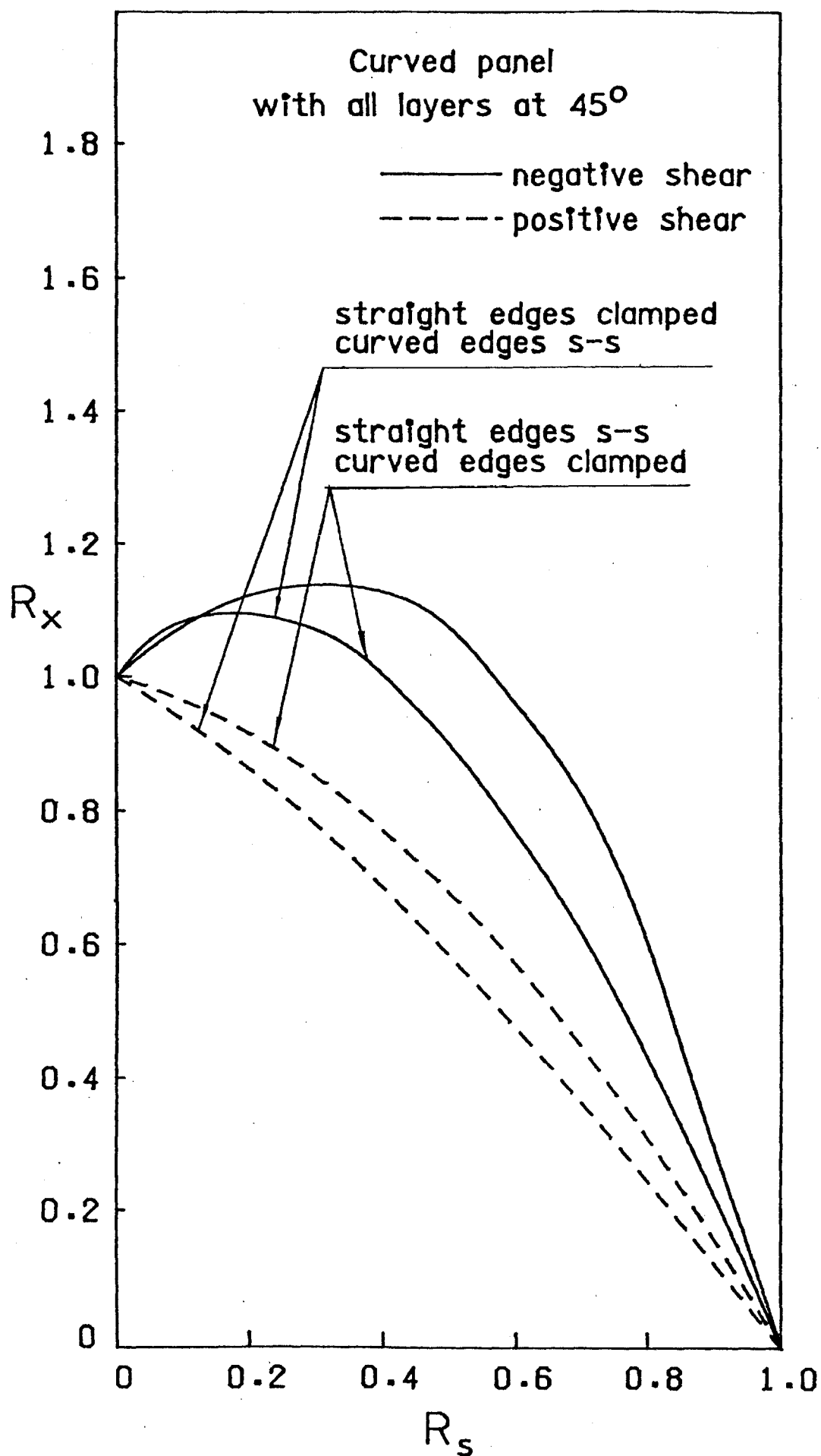


Fig. (III,32) Curved panel with 45° lay-up: combined shear and compressive loading; mixed edge conditions

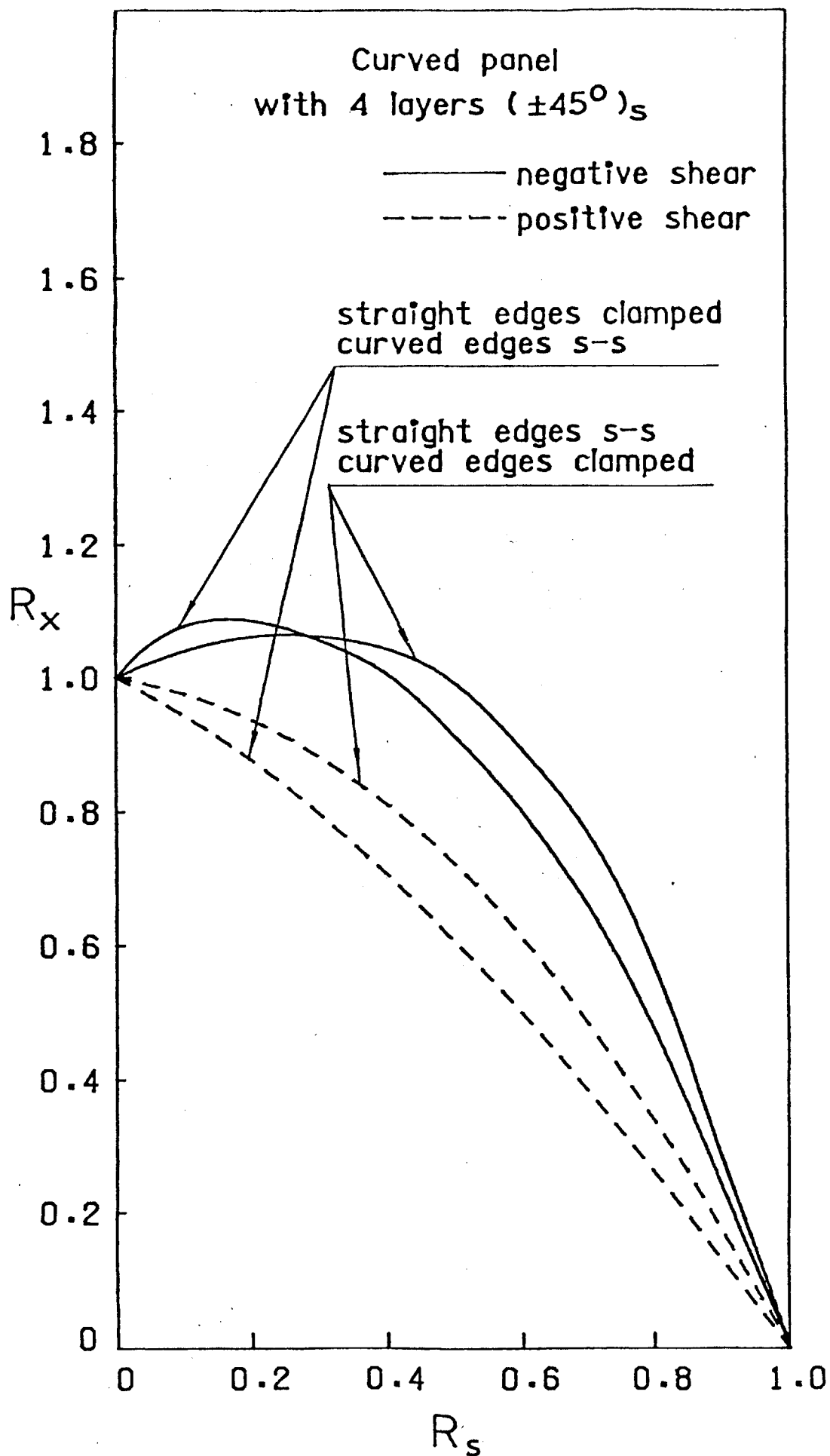


Fig. (III,33) Curved panel with $\pm 45^\circ$ symmetric lay-up:
combined shear and compressive loading;
mixed edge conditions

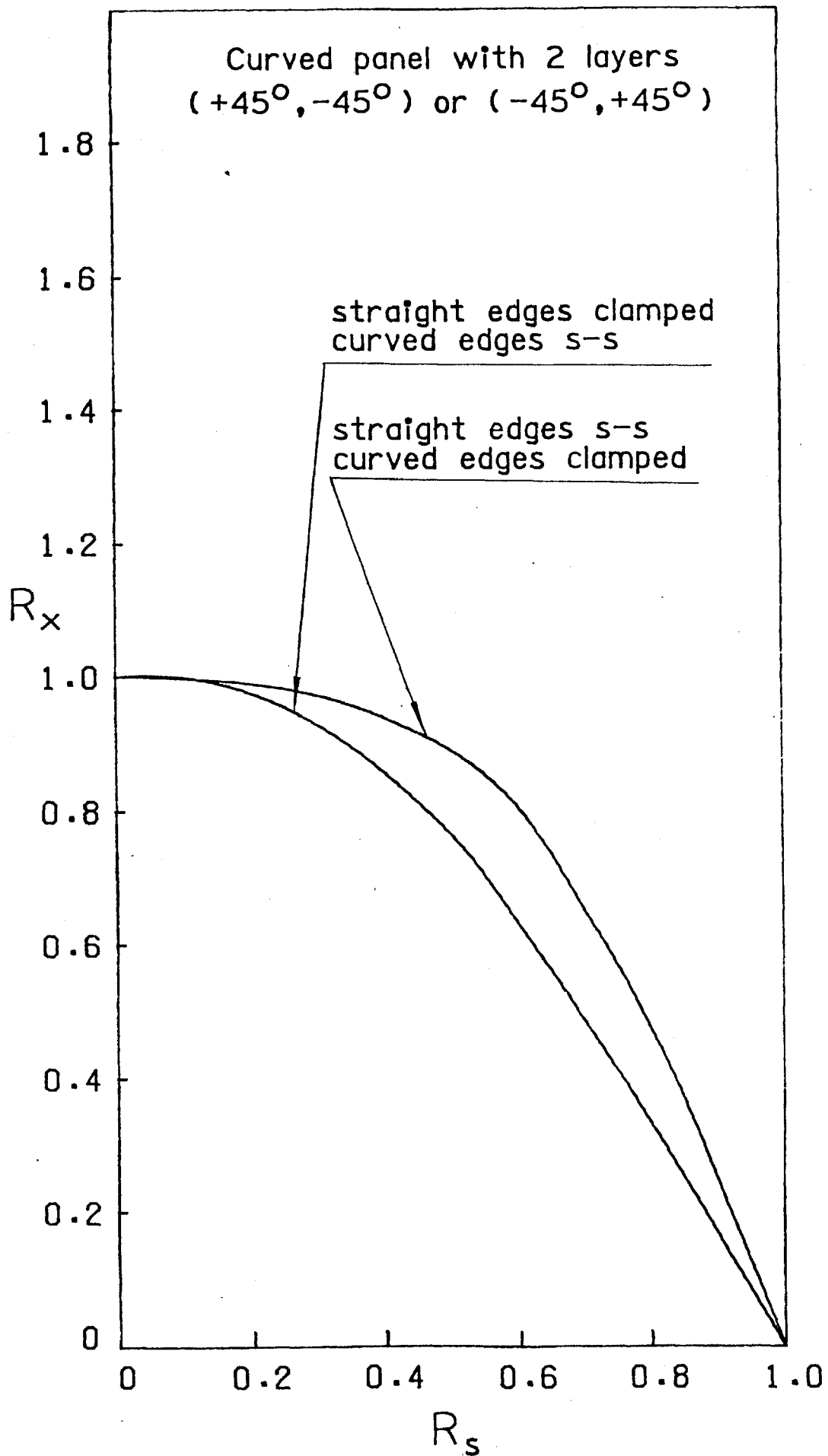


Fig. (III,34) Curved panel with two $\pm 45^\circ$ layers:
 combined shear and compressive loading;
 mixed edge conditions

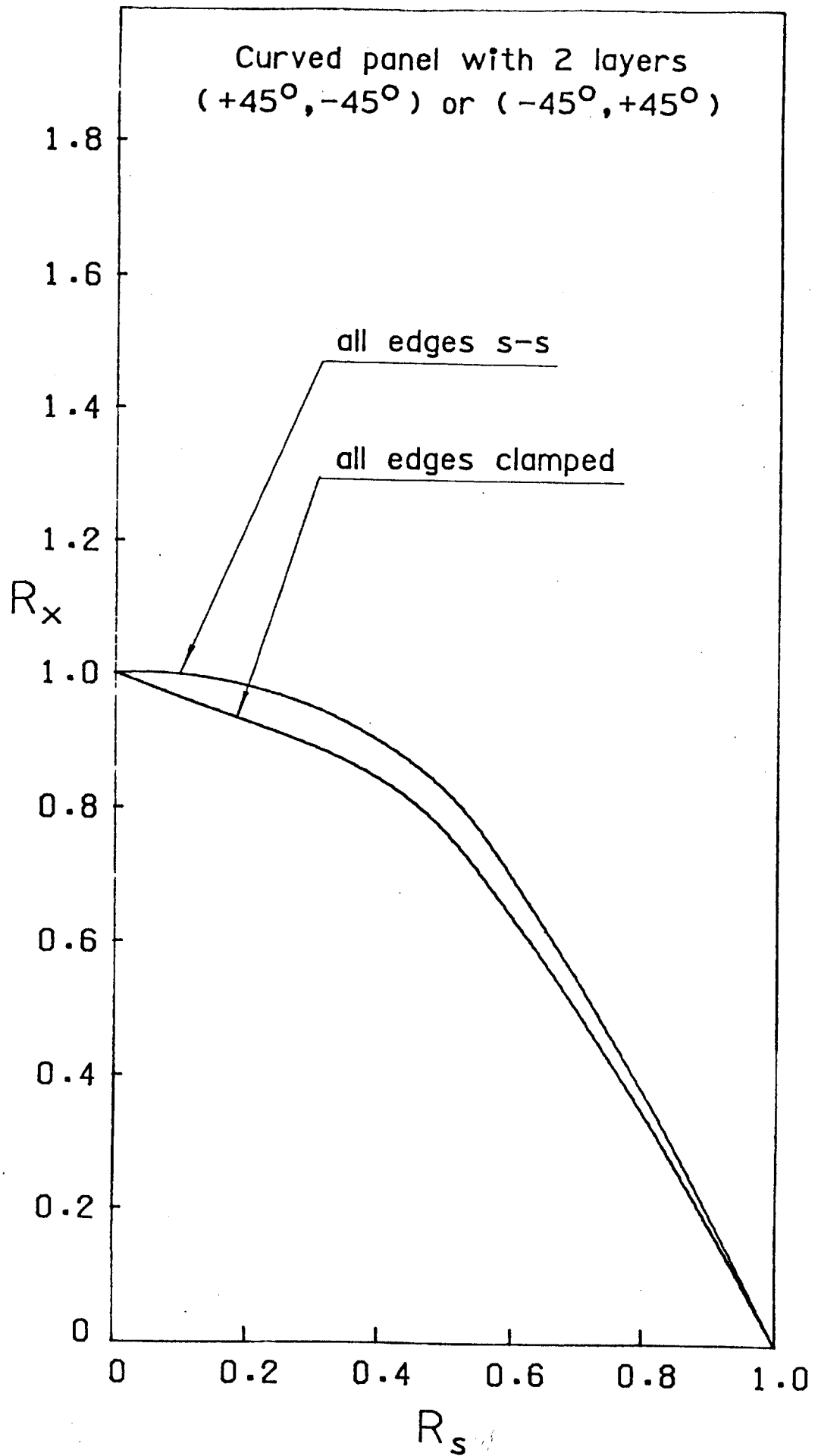


Fig. (III,35) Curved panel with two $\pm 45^\circ$ layers:
 combined shear and compressive loading;
 clamped and simply supported edges

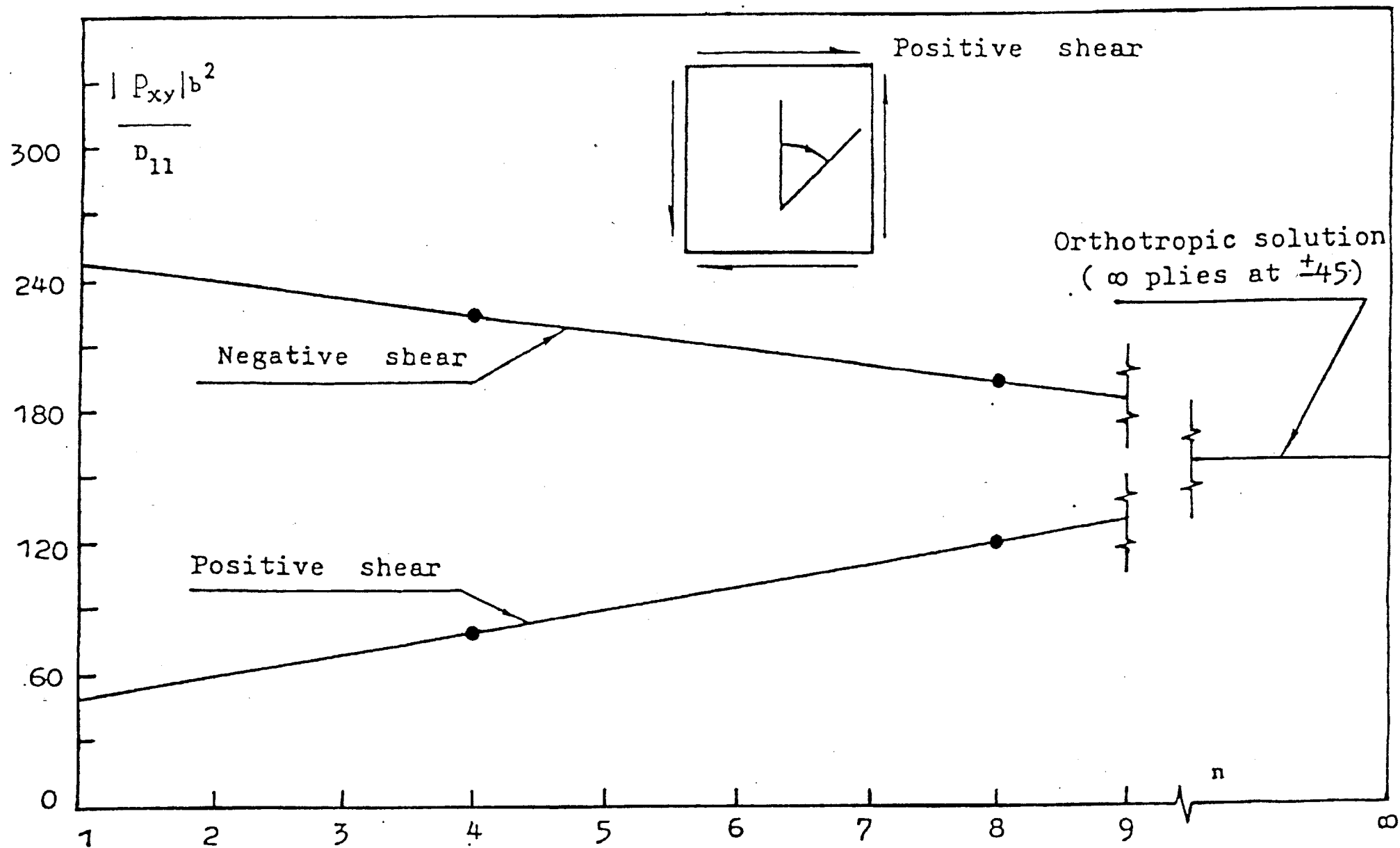


Fig. (III,36) Variation of critical shear with number of layers (n) for square plates;
mid-plane symmetric lay-up , alternate shear directions

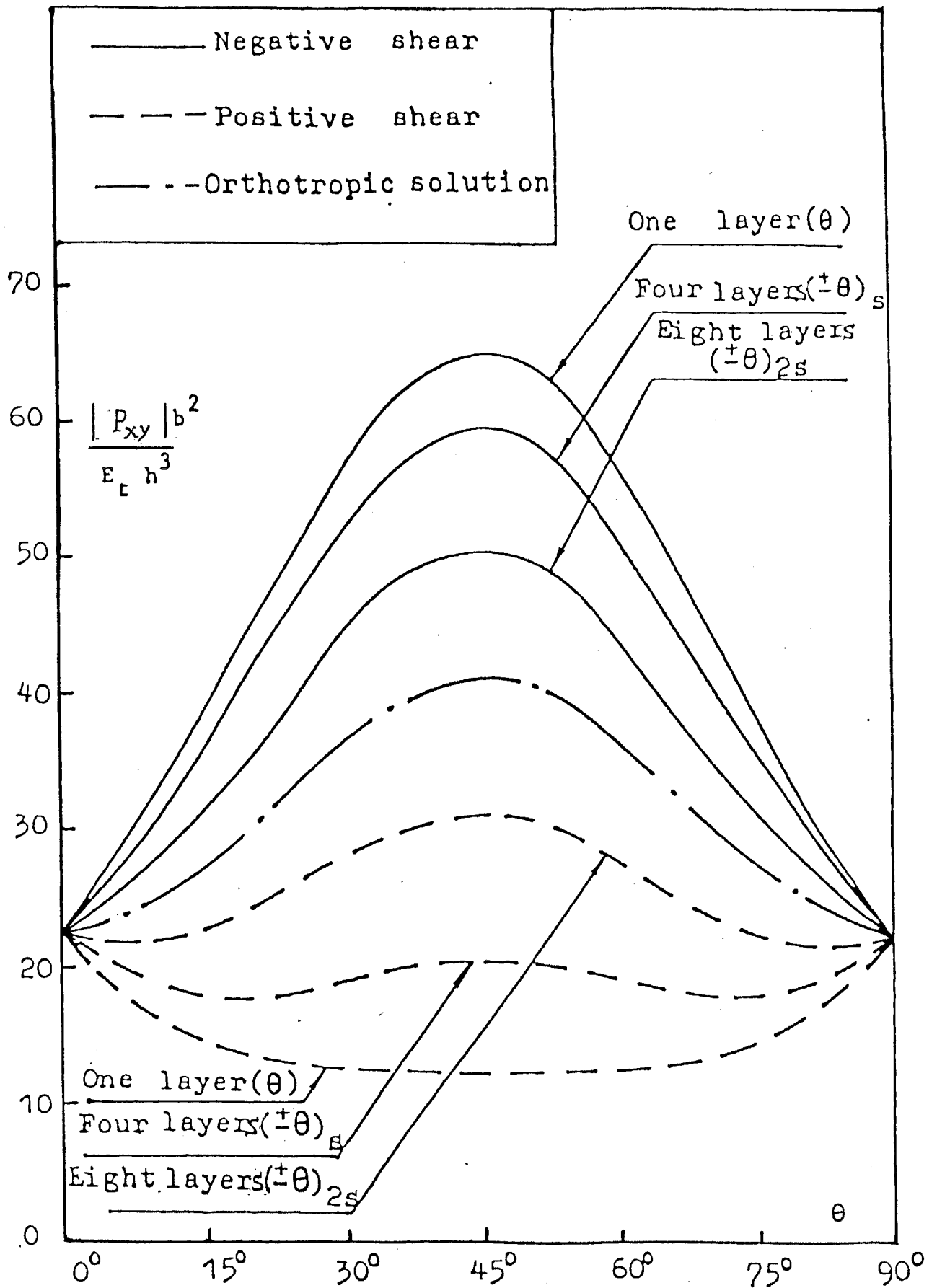


Fig. (III,37) Variation of critical shear with fibre angle; square plates, mid-plane symmetric lay-up , alternate shear directions

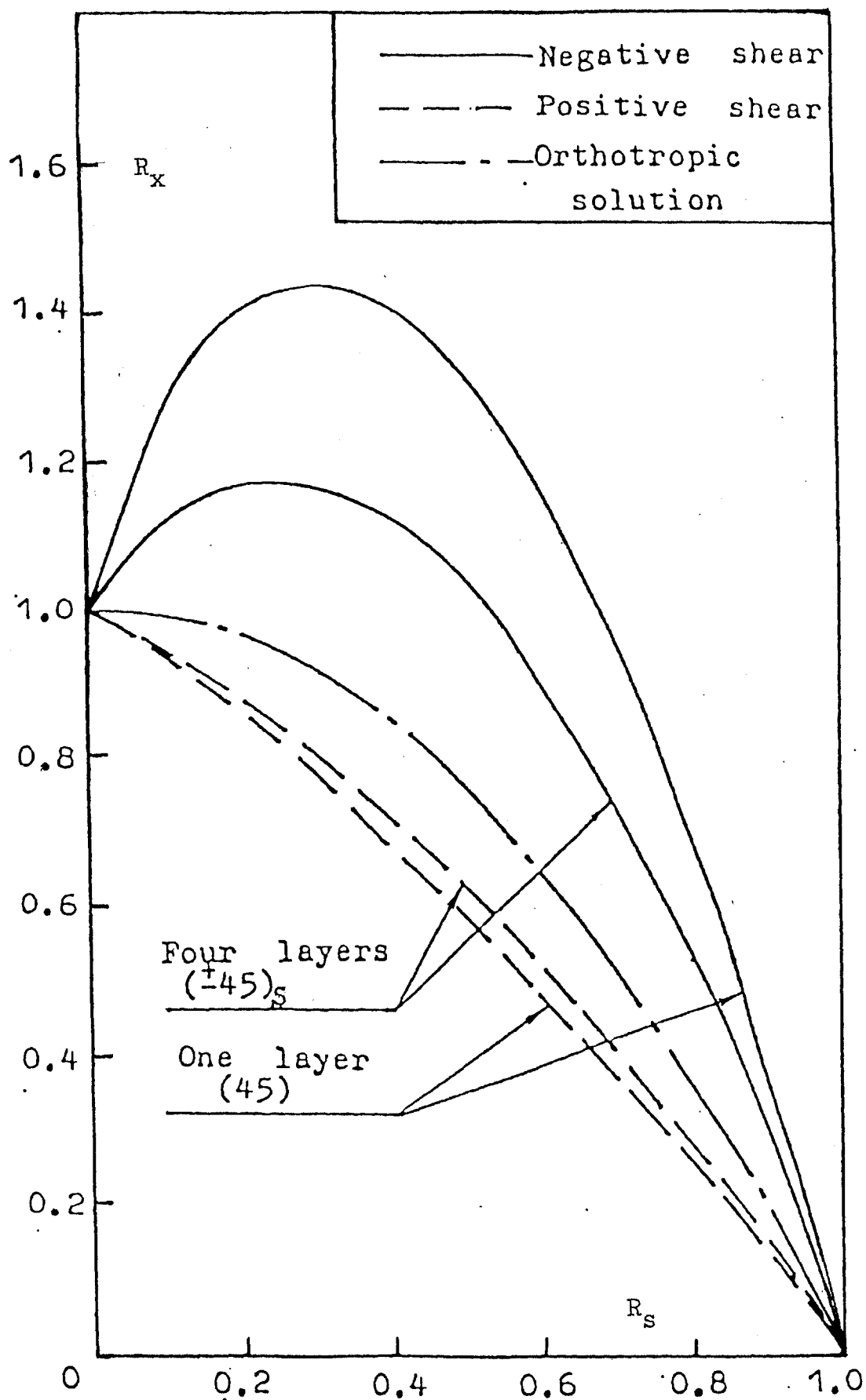


Fig. (III,38) Relationship between critical compressive ratio R_x and critical shear ratio R_s ; square BFRP plates, mid-plane symmetric lay-up, combination of unidirectional compression and shear

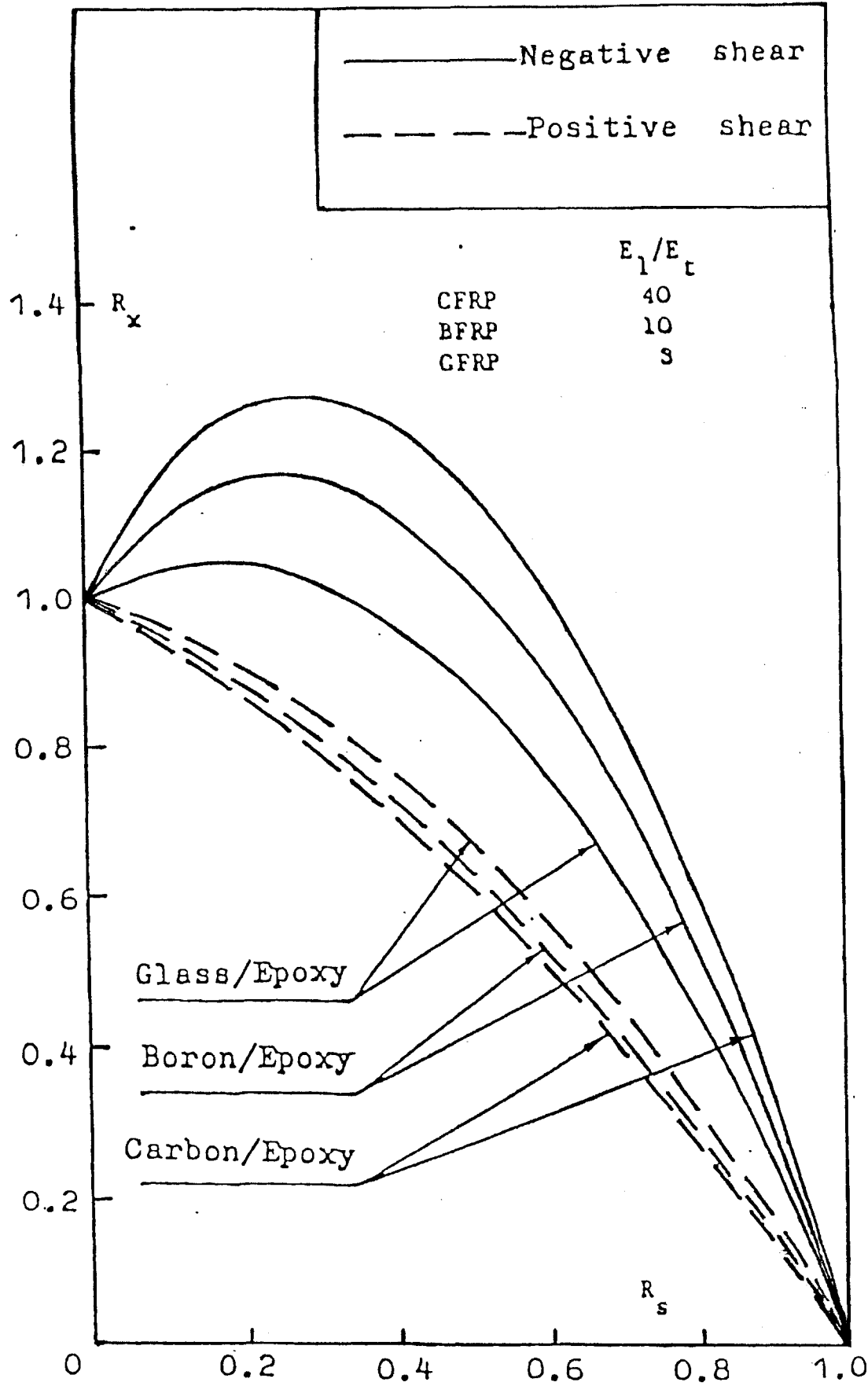


Fig. (III,39) Relationship between critical compressive ratio R_x and critical shear ratio R_s ; square four-layer plates $(\pm 45^\circ)_s$ of different composites, combination of uni-directional compression and shear

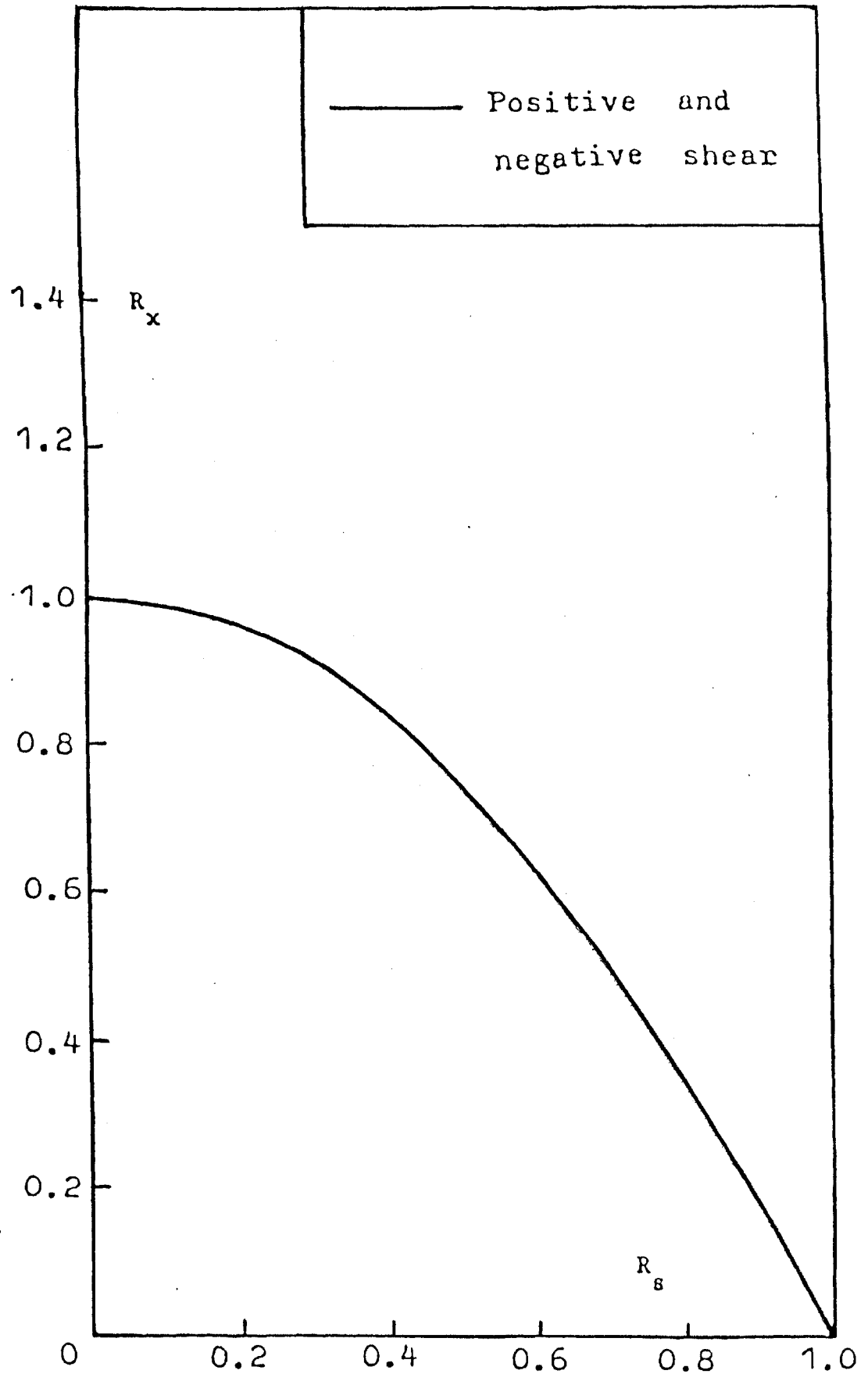


Fig. (III,40) Relationship between critical compressive ratio R_x and critical shear ratio R_s ; square BFRP plates, antisymmetric lay-up ($\pm 45^\circ$) & ($\pm 45^\circ, \pm 45^\circ$), combination of uni-directional compression and shear

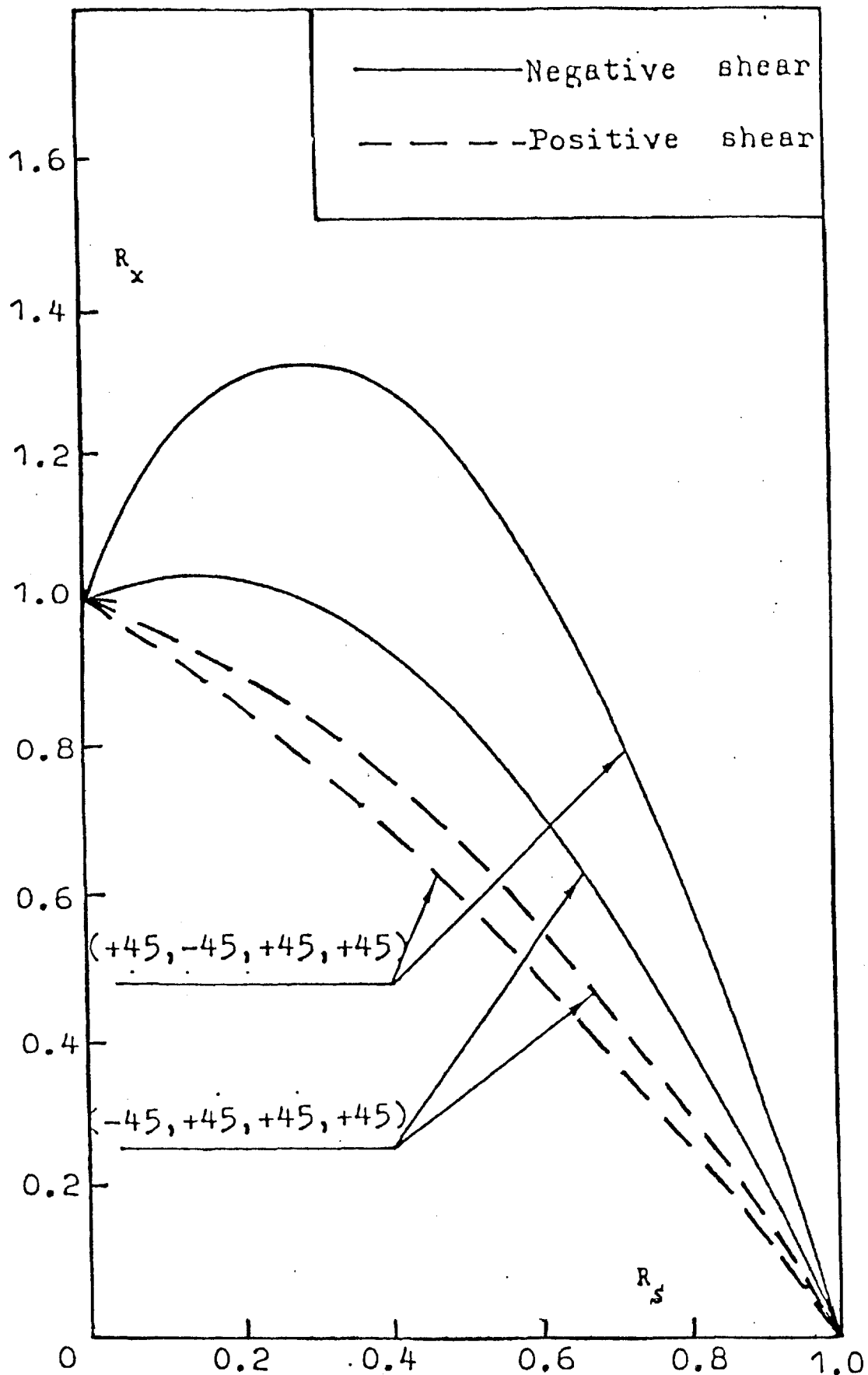


Fig. (III,41) Relationship between critical compressive ratio R_x and critical shear ratio R_s ; square BFRP plates, asymmetric lay-up, combination of unidirectional compression and shear

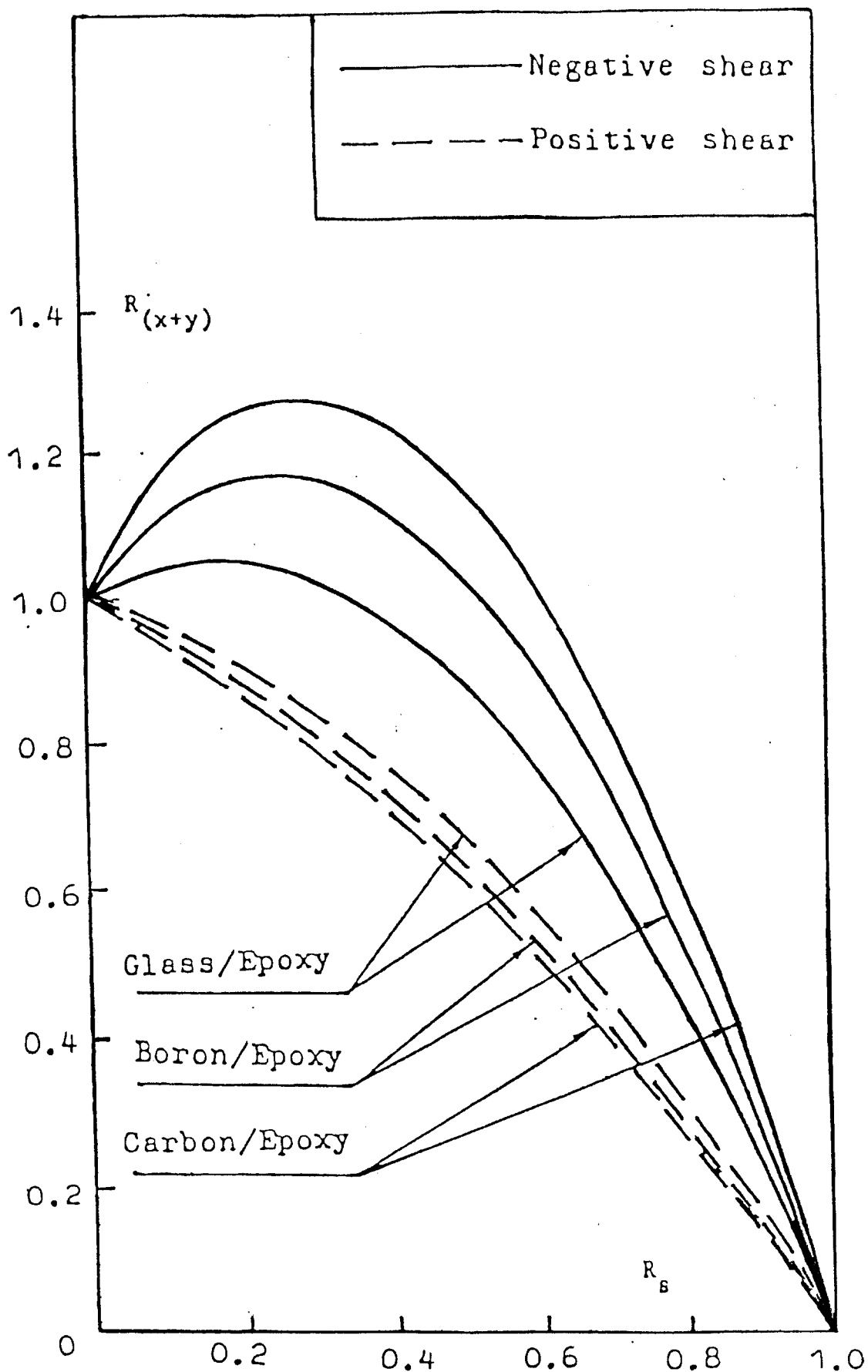


Fig. (III,42) Relationship between critical compressive ratio $R_{(x+y)}$ and critical shear ratio R_s ; square four-layer plates $(\pm 45^\circ)_s$ of different composites, combination of bi-directional compression and shear

CHAPTER IV

POSTBUCKLING BEHAVIOUR

A. General Remarks

In order to exploit the efficiency of composite structures the postbuckling behaviour of laminated anisotropic plates and curved panels needs to be considered. During the last decade constant efforts have been made in this field by a few investigators^{[83][84][87][76]}. But these workers have mostly concentrated on cross-ply or midplane antisymmetrically laminated flat plates. Because of the complexity of postbuckling problems, references to the postbuckling behaviour of curved panels made of generally laminated composites can only rarely be found in the literature.

This chapter of the current work, however, deals with the postbuckling behaviour of flat and curved panels of generally layered composite materials. All the 36 coefficients (21 independent) in the constitutive relations of the panels are taken into account.

Section B of this chapter is devoted to the postbuckling behaviour of curved panels under compressive loading while section C discusses that of flat plates under shear loading. In both cases a pair of coupled nonlinear partial differential equations including the effect of boundary moments are solved in conjunction with simply supported edges. In section B the double sine series and double sine-squared series are assumed, for the deformed surface of curved panels, to represent the tendency for inward buckling. This series has not been found in previous work concerning layered composite panels and

cylinders. In section C the emphasis is put on the effect of shear directions on postbuckling behaviour of composite plates. Shear direction proves to be very important in that case. This fact probably has never been reported in the literature.

B. Postbuckling Behaviour of Curved Panels of Generally Layered Composite Materials under Compression

A cylindrically curved panel with midsurface radius R and thickness h , having length a and arc b , is shown in Fig. (II-1). The panel is assumed to consist of composite layers having different thicknesses, elastic properties and arbitrary orientations of orthotropic axes with respect to the generators of the panel. The panel is subjected to axial compression denoted by P_x per unit width.

1. Governing System

The panel is assumed thin so that the constitutive relations (II,10) or (II,13) based on the Kirchhoff hypothesis are considered valid.

The governing system for the postbuckling behaviour of curved panels of generally layered composite materials has been developed on the basis of the virtual displacement principle and virtual force principle in Chapter II. All thirty six coefficients (21 independent) of the constitutive relations have been taken in the theory to allow for general lay-ups. A pair of nonlinear equations in the Von Karman sense are both fourth order partial differential ones with two variables - one a deflection function, the other a stress function.

The nondimensional governing system takes the following form (see equations (II,27) and (II,28)).

$$\begin{aligned}
& \int_0^1 \int_0^1 \left\{ \frac{\partial^4 W}{\partial \xi^4} + b_1 \frac{\partial^4 W}{\partial \xi^3 \partial \eta} + b_2 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + b_3 \frac{\partial^4 W}{\partial \xi \partial \eta^3} + b_4 \frac{\partial^4 W}{\partial \eta^4} + b_5 \frac{\partial^4 F}{\partial \xi^4} + b_6 \frac{\partial^4 F}{\partial \xi^3 \partial \eta} + b_7 \frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} \right. \\
& + b_8 \frac{\partial^4 F}{\partial \xi \partial \eta^3} + b_9 \frac{\partial^4 F}{\partial \eta^4} - \frac{\beta^2 K_R}{D_{11}^*} \frac{\partial^2 F}{\partial \xi^2} - \left(\frac{\beta^2}{D_{11}^*} \right) \left[\frac{\partial^2 W}{\partial \xi^2} \frac{\partial^2 F}{\partial \eta^2} + \frac{\partial^2 W}{\partial \eta^2} \frac{\partial^2 F}{\partial \xi^2} - 2 \frac{\partial^2 W}{\partial \xi \partial \eta} \frac{\partial^2 F}{\partial \xi \partial \eta} \right] \} \delta W d\xi d\eta \\
& + \int_0^1 \left\{ \left[b_{10} \frac{\partial^2 F}{\partial \eta^2} + b_5 \frac{\partial^2 F}{\partial \xi^2} + b_{11} \frac{\partial^2 F}{\partial \xi \partial \eta} + \frac{\partial^2 W}{\partial \xi^2} + b_{12} \frac{\partial^2 W}{\partial \eta^2} + \frac{b_1}{2} \frac{\partial^2 W}{\partial \xi \partial \eta} \right] \delta \left(\frac{\partial W}{\partial \xi} \right) \right\} \Big|_{\xi=0}^{\xi=1} d\eta \\
& + \int_0^1 \left\{ \left[b_9 \frac{\partial^2 F}{\partial \eta^2} + b_{13} \frac{\partial^2 F}{\partial \xi^2} + b_{14} \frac{\partial^2 F}{\partial \xi \partial \eta} + b_{15} \frac{\partial^2 W}{\partial \xi^2} + b_4 \frac{\partial^2 W}{\partial \eta^2} + \frac{b_3}{2} \frac{\partial^2 W}{\partial \xi \partial \eta} \right] \delta \left(\frac{\partial W}{\partial \eta} \right) \right\} \Big|_{\eta=0}^{\eta=1} d\xi \\
& = 0
\end{aligned}$$

(IV, 1)

$$\begin{aligned}
& \int_0^1 \int_0^1 \left\{ \frac{\partial^4 F}{\partial \xi^4} - a_1 \frac{\partial^4 F}{\partial \xi^3 \partial \eta} + a_2 \frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} - a_3 \frac{\partial^4 F}{\partial \xi \partial \eta^3} + a_4 \frac{\partial^4 F}{\partial \eta^4} - a_5 \frac{\partial^4 W}{\partial \xi^4} - a_6 \frac{\partial^4 W}{\partial \xi^3 \partial \eta} - a_7 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} \right. \\
& - a_8 \frac{\partial^4 W}{\partial \xi \partial \eta^3} - a_9 \frac{\partial^4 W}{\partial \eta^4} + \frac{\beta^2 K_R}{A_{22}^*} \frac{\partial^2 W}{\partial \xi^2} - \left(\frac{\beta^2}{A_{22}^*} \right) \left[\left(\frac{\partial^2 W}{\partial \xi \partial \eta} \right)^2 - \frac{\partial^2 W}{\partial \xi^2} \frac{\partial^2 W}{\partial \eta^2} \right] \} \delta F d\xi d\eta \\
& = 0
\end{aligned}$$

(IV, 2)

in which w, F are nondimensional deflection and stress function respectively, K_R is the nondimensional curvature parameter, β is aspect ratio, ξ, η are nondimensional coordinates in x, y directions respectively. The coefficients a_i, b_i, A_{22}^* and D_{11}^* are nondimensional constants related to the properties of the composite layers. The definition of all these parameters and constants are given in expressions (II, 26).

At the same time the membrane forces and moments have also been nondimensionalised and are given in expressions (II, 29)

and (II,30).

2. Solutions

The panel under consideration is assumed to have simply supported edges without inplane restraint. The boundary conditions may be written in terms of nondimensional parameters as follows:

$$\begin{aligned} W=0, \quad M_{\xi}=0, \quad \frac{\partial^2 F}{\partial \eta^2}=-\bar{N}_x, \quad \frac{\partial^2 F}{\partial \xi \partial \eta}=-\beta \bar{S} \quad \text{at } \xi = 0, 1 \\ W=0, \quad M_{\eta}=0, \quad \frac{\partial^2 F}{\partial \xi^2}=0, \quad \frac{\partial^2 F}{\partial \xi \partial \eta}=-\beta \bar{S} \quad \text{at } \eta = 0, 1 \end{aligned} \quad (\text{IV}, 3)$$

where $\bar{N}_x = P_x b^2 / A_{22} h^2$, $\bar{S} = P_{xy} b^2 / A_{22} h^2$ are the nondimensional edge compressive and shear loads.

The assumed functions for the nondimensionalised deflection w and stress function F take the form:

$$F = -\bar{N}_x \frac{\eta^2}{2} - \bar{S} \beta \xi \eta + \sum_m \sum_n F_{mn} X_m(\xi) Y_n(\eta) \quad (\text{IV}, 4)$$

$$W = \sum_p \sum_q \left[w_{pq}^{(1)} \sin p\pi \xi \sin q\pi \eta + w_{pq}^{(2)} \sin^2 p\pi \xi \sin^2 q\pi \eta \right] \quad (m, n, p, q = 1, 2, \dots, \infty)$$

in which F_{mn} , $w_{pq}^{(1)}$ and $w_{pq}^{(2)}$ are undetermined coefficients.

X_i and Y_i are the beam functions defined by

$$X_i(\xi) = \cosh \alpha_i \xi - \cos \alpha_i \xi - \gamma_i (\sinh \alpha_i \xi - \sin \alpha_i \xi) \quad (\text{IV}, 5)$$

$$Y_i(\eta) = \cosh \alpha_i \eta - \cos \alpha_i \eta - \gamma_i (\sinh \alpha_i \eta - \sin \alpha_i \eta)$$

where α_i and γ_i are constants given in Table (II,1). With these values of α_i and γ_i the characteristic beam functions $X_i(\xi)$ and $Y_i(\eta)$ have the following properties:

$$X_i(0)=X_i(1)=\left.\frac{dX_i(\xi)}{d\xi}\right|_{\xi=0}=\left.\frac{dX_i(\xi)}{d\xi}\right|_{\xi=1}=0$$

(IV,6)

$$Y_i(0)=Y_i(1)=\left.\frac{dY_i(\eta)}{d\eta}\right|_{\eta=0}=\left.\frac{dY_i(\eta)}{d\eta}\right|_{\eta=1}=0$$

$$\frac{d^4 X_i(\xi)}{d\xi^4} = \alpha_i^4 X_i(\xi)$$

(IV,7)

$$\frac{d^4 Y_i(\eta)}{d\eta^4} = \alpha_i^4 Y_i(\eta)$$

and

$$\int_0^1 X_i(\xi) X_j(\xi) d\xi = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

(IV,8)

$$\int_0^1 Y_i(\eta) Y_j(\eta) d\eta = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

It is seen that the stress function F satisfies the boundary conditions (IV,3) completely, but the function w satisfies only the zero deflection conditions. As mentioned previously, however, the effect of boundary moments has been considered in the governing equation (IV,1) in terms of two line integrals. The zero moment conditions may be satisfied in the mean. So what we need to do is simply to set the solution (IV,4) into the governing equations (IV,1) and (IV,2).

The derivatives and their products of the assumed functions (IV,4) appearing in equation (IV,1) and (IV,2) are listed in Appendix D.

The resulting equations may be written as follows:

$$\begin{aligned}
\sum_m \sum_n F_{mn} A_{mn}^{ij} = & \sum_p \sum_q W_{pq}^{(1)} B_{pq}^{ij} - \sum_p \sum_q W_{pq}^{(2)} C_{pq}^{ij} + \sum_k \sum_l \sum_r \sum_s W_{kl}^{(1)} W_{rs}^{(1)} D_{klrs}^{ij} \\
& + \sum_k \sum_l \sum_r \sum_s W_{kl}^{(2)} W_{rs}^{(1)} E_{klrs}^{ij} + \sum_k \sum_l \sum_r \sum_s W_{kl}^{(1)} W_{rs}^{(2)} F_{klrs}^{ij} + \sum_k \sum_l \sum_r \sum_s W_{kl}^{(2)} W_{rs}^{(2)} G_{klrs}^{ij} \\
& (i, j = 1, 2, \dots)
\end{aligned} \tag{IV, 9}$$

$$\begin{aligned}
\sum_p \sum_q W_{pq}^{(1)} H_{pq}^{ij} + \sum_p \sum_q W_{pq}^{(2)} I_{pq}^{ij} + \sum_m \sum_n F_{mn} J_{mn}^{ij} = & \sum_k \sum_l \sum_r \sum_s W_{kl}^{(1)} F_{rs} K_{klrs}^{ij} \\
& + \sum_k \sum_l \sum_r \sum_s W_{kl}^{(2)} F_{rs} L_{klrs}^{ij} + K_x M_o^{ij} + K_x \sum_k \sum_l W_{kl}^{(1)} M_{kl}^{ij} - K_x \sum_k \sum_l W_{kl}^{(2)} N_{kl}^{ij} \\
& + K_s O_o^{ij} + K_s \sum_k \sum_l W_{kl}^{(1)} O_{kl}^{ij} + K_s \sum_k \sum_l W_{kl}^{(2)} P_{kl}^{ij} \\
& (i, j = 1, 2, \dots)
\end{aligned} \tag{IV, 10}$$

$$\begin{aligned}
\sum_p \sum_q W_{pq}^{(1)} Q_{pq}^{ij} + \sum_p \sum_q W_{pq}^{(2)} R_{pq}^{ij} + \sum_m \sum_n F_{mn} S_{mn}^{ij} = & \sum_k \sum_l \sum_r \sum_s W_{kl}^{(1)} F_{rs} T_{klrs}^{ij} + \sum_k \sum_l \sum_r \sum_s W_{kl}^{(2)} F_{rs} U_{klrs}^{ij} \\
& + K_x \sum_k \sum_l W_{kl}^{(1)} V_{kl}^{ij} - K_x \sum_k \sum_l W_{kl}^{(2)} X_{kl}^{ij} + K_s \sum_k \sum_l W_{kl}^{(1)} Y_{kl}^{ij} + K_s \sum_k \sum_l W_{kl}^{(2)} Z_{kl}^{ij} \\
& (i, j = 1, 2, \dots)
\end{aligned} \tag{IV, 11}$$

in which

$$K_x = \frac{P_x b^2}{E_t h^3}, \quad K_s = \frac{P_{xy} b^2}{E_t h^3}$$

The coefficients A_{mn}^{ij} , B_{pq}^{ij} ... Z_{kl}^{ij} are given in Appendix E, while the integral constants appearing in Appendix E are listed in Appendix F.

The set of equations (IV,9), (IV,10) and (IV,11) contains an infinite number of equations. In fact, only a finite number

of equations can be solved in practical applications. The solutions (IV,4) must, therefore, be truncated and $m, p=1, 2, \dots, M$ and $n, q=1, 2, \dots, N$ are taken. The largest integers M and N (usually $M=N$ is used) are determined according to the required accuracy of results and computing cost. With certain indicial expressions properly introduced, the equation system of (IV,9), (IV,10) and (IV,11) may be written in matrix form as follows:

$$\begin{aligned} [A] \{F\} = [B] \{w^{(1)}\} - [C] \{w^{(2)}\} + [T_w^{(1)}] [D] \{w^{(1)}\} \\ + [T_w^{(2)}] [E] \{w^{(1)}\} + [T_w^{(1)}] [F] \{w^{(2)}\} \end{aligned} \quad (\text{IV}, 12)$$

$$+ [T_w^{(2)}] [G] \{w^{(2)}\}$$

$$\begin{aligned} [H] \{w^{(1)}\} + [I] \{w^{(2)}\} + [J] \{F\} = [T_w^{(1)}] [K] \{F\} \\ + [T_w^{(2)}] [L] \{F\} + K_x \{ [M_0] + [M] \{w^{(1)}\} - [N] \{w^{(2)}\} \} \end{aligned} \quad (\text{IV}, 13)$$

$$+ K_s \{ [O_0] + [O] \{w^{(1)}\} + [P] \{w^{(2)}\} \}$$

$$\begin{aligned} [Q] \{w^{(1)}\} + [R] \{w^{(2)}\} + [S] \{F\} = [T_w^{(1)}] [T] \{F\} \\ + [T_w^{(2)}] [U] \{F\} + K_x \{ [V] \{w^{(1)}\} - [X] \{w^{(2)}\} \} \end{aligned} \quad (\text{IV}, 14)$$

$$+ K_s \{ [Y] \{w^{(1)}\} + [Z] \{w^{(2)}\} \}$$

The elements of the coefficient matrices $[A], [B] \dots [Z]$ are related to the coefficients $A_{mn}^{ij}, B_{pq}^{ij} \dots Z_{kl}^{ij}$ with indicial expressions. They are all given in Appendix E.

The elements $F_I, w_I^{(1)}, w_I^{(2)}$ of the column matrices of unknowns are related to the coefficients $F_{mn}, w_{pq}^{(1)}, w_{pq}^{(2)}$ of the solutions (IV,4) with the help of the indicial expression

$$I = (m - 1)M + n = (p - 1)M + q.$$

The matrices $[T_W^{(1)}]$ and $[T_W^{(2)}]$ in equations (IV,12), (IV,13) and (IV,14) are defined by

$$[T_W^{(1)}] = \begin{bmatrix} \{w^{(1)}\}^t & & & \\ & \{w^{(1)}\}^t & & \\ & & \ddots & \\ & & & \{w^{(1)}\}^t \end{bmatrix} \quad (\text{IV,15})$$

$$[T_W^{(2)}] = \begin{bmatrix} \{w^{(2)}\}^t & & & \\ & \{w^{(2)}\}^t & & \\ & & \ddots & \\ & & & \{w^{(2)}\}^t \end{bmatrix} \quad (\text{IV,16})$$

in which the superscript t indicates transposition of the column matrices.

To simplify the set of equations (IV,12), (IV,13) and (IV,14), the column matrix $\{F\}$ may be solved for $\{w^{(1)}\}$ and $\{w^{(2)}\}$ from equation (IV,12). It takes the form

$$\{F\} = [A^{(1)}] \{w^{(1)}\} + [A^{(2)}] \{w^{(2)}\} \quad (\text{IV,17})$$

where

$$\begin{aligned} [A^{(1)}] &= [A]^{-1} \left([B] + [T_W^{(1)}][D] + [T_W^{(2)}][E] \right) \\ [A^{(2)}] &= [A]^{-1} \left(-[C] + [T_W^{(1)}][F] + [T_W^{(2)}][G] \right) \end{aligned} \quad (\text{IV,18})$$

Setting the expression (IV,17) into equations (IV,13) and (IV,14) produces the following equation system in matrix form,

i.e.

$$\begin{aligned} ([B^{(1)}] - K[B^{(2)}])\{w^{(1)}\} + ([B^{(3)}] - K[B^{(4)}])\{w^{(2)}\} &= K[\mu_x\{M_o\} + \mu_s\{O_o\}] \\ ([C^{(1)}] - K[C^{(2)}])\{w^{(1)}\} + ([C^{(3)}] - K[C^{(4)}])\{w^{(2)}\} &= 0 \end{aligned} \quad (\text{IV,19})$$

where

$$[B^{(1)}] = [H] + ([J] - [T_w^{(1)}][K] - [T_w^{(2)}][L])[A^{(1)}]$$

$$[B^{(2)}] = \mu_x [M] + \mu_s [O]$$

$$[B^{(3)}] = [I] + ([J] - [T_w^{(1)}][K] - [T_w^{(2)}][L])[A^{(2)}]$$

$$[B^{(4)}] = -\mu_x [N] + \mu_s [P]$$

(IV,20)

$$[C^{(1)}] = [Q] + ([S] - [T_w^{(1)}][T] - [T_w^{(2)}][U])[A^{(1)}]$$

$$[C^{(2)}] = \mu_x [V] + \mu_s [Y]$$

$$[C^{(3)}] = [R] + ([S] - [T_w^{(1)}][T] - [T_w^{(2)}][U])[A^{(2)}]$$

$$[C^{(4)}] = -\mu_x [X] + \mu_s [Z]$$

and

$$K_x = \mu_x K$$

$$K_s = \mu_s K$$

Noticing the expressions (IV,20), (IV,18), (IV,15) and (IV,16), it is easy to find that all the equations contained in the system (IV,19) are third order algebraic ones, of unknowns $w_{pq}^{(1)}$ and $w_{pq}^{(2)}$, which are to be solved simultaneously.

3. Computation and Results

A series of computations is carried out for panels with midplane symmetric and antisymmetric lay-ups under compressive loading only. In these cases, $\mu_x = 1.0$, $\mu_s = 0$ and the right hand side of the first equation of the system (IV,19) vanishes.

The postbuckling of the panels exhibit purely bifurcational-type behaviour. This means that the panels will remain inflexible as compressive load increases until the load reaches the level of the bifurcational or initial buckling load.

An iterative scheme is used to obtain the postbuckling behaviour of the panels. The computational procedure is briefly described here.

a) Bifurcation Solution

The bifurcation point is determined by dropping all the nonlinear terms in the system (IV,19). That is, all the terms involving $[T_w^{(1)}]$ and $[T_w^{(2)}]$ in the expressions (IV,18) and (IV,20) vanish. A set of linear algebraic equations is then obtained,

i.e.

$$\begin{aligned} [B^{(1)}]\{w^{(1)}\} + [B^{(3)}]\{w^{(2)}\} &= K \left([B^{(2)}]\{w^{(1)}\} + [B^{(4)}]\{w^{(2)}\} \right) \\ [C^{(1)}]\{w^{(1)}\} + [C^{(3)}]\{w^{(2)}\} &= K \left([C^{(2)}]\{w^{(1)}\} + [C^{(4)}]\{w^{(2)}\} \right) \end{aligned} \quad (\text{IV}, 21)$$

in which

$$\begin{aligned} [B^{(1)}] &= [H] + [J][A]^{-1}[B] \\ [B^{(2)}] &= [M] \\ [B^{(3)}] &= [I] - [J][A]^{-1}[C] \\ [B^{(4)}] &= -[N] \\ [C^{(1)}] &= [Q] + [S][A]^{-1}[B] \\ [C^{(2)}] &= [V] \\ [C^{(3)}] &= [R] - [S][A]^{-1}[C] \\ [C^{(4)}] &= -[X] \end{aligned} \quad (\text{IV}, 22)$$

The equation system (IV,21) may be written in a simpler form as follows:

$$[\Lambda]\{w\} = K[\Gamma]\{w\} \quad (\text{IV},23)$$

where

$$[\Lambda] = \left[\begin{array}{c|c} B^{(1)} & B^{(3)} \\ \hline C^{(1)} & C^{(3)} \end{array} \right] \quad (\text{IV},24)$$

$$[\Gamma] = \left[\begin{array}{c|c} B^{(2)} & B^{(4)} \\ \hline C^{(2)} & C^{(4)} \end{array} \right]$$

We are familiar with the form of equation (IV,23), which has been discussed in the last chapter. The QZ algorithm may, therefore, be used to obtain the lowest eigenvalue K_0 and the corresponding eigenvector $\{w^{(1)}\}_0$ and $\{w^{(2)}\}_0$. As long as these characteristic solutions at the bifurcation point are known the first point on the postbuckling path is set. $\{w^{(1)}\}_0$ and $\{w^{(2)}\}_0$ represent the buckling mode at the bifurcation point, although they only give the ratios among the components of the deflection function.

b) Postbuckling Path

In order to get the postbuckling path, an iterative scheme based on the modified Newton method is used. First, to give an estimate for a new point on the path, the largest component of $\{w^{(1)}\}_0$ and $\{w^{(2)}\}_0$ is prescribed equal to a certain step length while keeping the same ratios among the components of $\{w^{(1)}\}_0$ and $\{w^{(2)}\}_0$ as in the eigenvector obtained. At the same time the eigenvalue of the loading parameter K_0 is kept. Starting from this first estimate, the iterative computation produces

a true solution for a new point on the path. For the next point, the true solution of the last point is used as the starting value of iteration, with an increase of one step length in the prescribed component. After the first three points on the path have been obtained an extrapolation scheme based on the true values of $\{w^{(1)}\}$, $\{w^{(2)}\}$ and K at these points is introduced to produce a better estimate of the following points. As the prescribed components increase step by step a series of points on the path is obtained.

Accordingly, the column matrix $\{F\}$ may be calculated from the equation (IV,17), the stress function and transverse deflection of any point (ξ, η) of the panel are obtained from expressions (IV,4) and the membrane forces and moments on that point can be found from equations (II,29) and (II,30). All these equations are also expressed in matrix form for convenience of computation.

Numerical computations are performed for panels with different lay-ups, different curvatures and different materials. The elastic constants of the materials used for the computations are shown in Table (III,2) (see page 46)

c) Step Length and Finite-Term Approximation

In computation the step length of the prescribed component in $\{w^{(1)}\}$ and $\{w^{(2)}\}$ needs to be properly selected. It is obvious that using a larger step length saves computing time to get the postbuckling path, although more iterations may be needed to obtain every point of it. Too large a step length, however, may cause the iterative procedure to fail to give correct convergence. For the examples given below the step length of prescribed components varies from 0.1 - 0.6.

In most calculations it is 0.5. The convergence criterion of the iterative procedure in the current computation is that the sum of the squares of the residuals of the non-linear algebraic equations is less than 0.01 per cent.

In the computations a nine-term approximation for the stress function F and eighteen-term approximation for the deflection function w are used ($m = p = 1, 2, 3$, $n = 1, 2, 3$ in solution (IV,4)). In order to get some idea about the convergence of the current solutions, numerical results for the bifurcation solutions of boron/epoxy panels with different lay-ups and curvatures are given in Table (IV,1), using different numbers of terms. A panel with length and width both equal to 250 mm and thickness 2.50 mm is considered.

Table (IV,1) The Convergence of Bifurcation Load

P_x (N/mm)

Lay-up	K_R	2 terms (M=1)	8 terms (M=2)	18 terms (M = 3)
symmetric four layers (45°, -45°, -45°, 45°)	0	438	83	82
	25	450	121	119
	50	505	192	189
antisymmetric four layers (45°, -45°, 45°, -45°)	0	385	87	84
	25	397	117	115
	50	456	174	173

d) Numerical Results

As examples, a series of computations have been performed for various panels. In these calculations the aspect ratio of the panels is kept equal to one and the ratio of width to thickness kept equal to one hundred. The results of computations are summarised in the figures below. Computation is

carried out on the CDC 6500 computer system. It takes 30-40 minutes per run, depending on the panel parameters and lay-ups.

Figs. (IV,1) - (IV,4) represent the load-central deflection behaviour of the panels after buckling. Fig. (IV,1) corresponds to boron/epoxy panels having unidirectional layers at 45° . Fig. (IV,2) corresponds to boron/epoxy panels having four layers arranged at $(\pm 45^\circ)_s$. The effect of panel curvature on the load-deflection relations can be seen clearly from these two figures. As for isotropic panels, curvature raises the bifurcation load but makes the monotonic nature of the load-deflection relation disappear. Compared to the behaviour of a flat plate, this means that a "jump" of the deflection may occur for curved panels when the compressive load increases towards the bifurcation point. This kind of snap-through occurs in curved panels because there are more than one equilibrium states of the panels corresponding to a certain loading level.

Fig. (IV,3) shows the influence of lay-up on the postbuckling behaviour of curved panels. The curves in the figure are obtained for boron/epoxy composites with the curvature parameter $K_R = 25$. It may be seen that lay-up has a significant influence on the postbuckling behaviour when the number of layers in the panel is small. But a panel may behave in a similar fashion to an orthotropic one, corresponding to an infinite number of layers, if it contains more than four layers. At the same time, an antisymmetric two-layer arrangement always gives the lowest postbuckling path.

Fig. (IV,4) is for curved panels with $(\pm 45^\circ)_s$ lay-up

and $K_R = 25$. As expected, different materials have different postbuckling paths.

In Figs. (IV,1) - (IV,4) certain cusp points may be found on the curves. It is obvious that these points correspond to changes in buckling mode.

Figs. (IV,5) - (IV,10) represent the variation of inplane forces with external compression. Fig. (IV,5) shows the influence of curvature on the in-plane forces at the centre of boron/epoxy panels with $(\pm 45^\circ)_s$ lay-up. It is seen that a loop exists in the plots of both inner forces. A "jump" of inner force may also be expected for curved panels when the load is close to the bifurcation point.

The influence of lay-up on the in-plane force variation may be clearly seen in Fig. (IV,6) and Fig. (IV,7). These two figures also represent the values at the centre of the boron/epoxy panel with $(\pm 45^\circ)_s$ lay-up and $K_R = 25$. Fig. (IV,8) shows the influence of composite properties. The figure is obtained for panels with $(\pm 45^\circ)_s$ lay-up and $K_R = 25$.

Figs. (IV,9) and (IV,10) show the variation of in-plane forces at different points of the panel. The panel consists of unidirectional boron/epoxy layers at 45° with $K_R = 25$. In these figures the point of buckling mode change can also be clearly seen on the postbuckling paths.

In Figs. (IV,11) - (IV,12) load-bending moment relations are given. The results in the figures are obtained for the antisymmetric lay-up arrangement with layers alternately at $\pm 45^\circ$ and show the variation of M_x at different points of the boron/epoxy panel. Similar curves exist for M_y at these points but are not given in the present thesis. Fig. (IV,13) shows the effect on M_x of the number of alternate layers.

The distribution of deflection over the panels is also calculated. Fig. (IV,14) gives the results on a quarter of a boron/epoxy panel with unidirectional layers at 45° and $K_R = 25$, when the centre deflection is equal to 8.91 h. Fig. (IV,15) shows the increase of deflection on circumferential and longitudinal sections passing through the panel centre. It can be seen that on the longitudinal section, BD, the buckling mode of the panel changes twice; the letters on the various curves correspond to the similar plotted regions of Fig. (IV,1).

The distribution of inner forces on the panels is also illustrated. The results shown in Figs. (IV,16) and (IV,17) refer to a quarter of a flat boron/epoxy plate with four layers $(\pm 45^\circ)_s$. It is observed from Fig. (IV,16) that the inner force N_x is uniform on the loaded edge AB but varies dramatically with increasing load on the symmetric section CD. N_x may even become tensile in the small area surrounding the panel centre when the load is large enough. At the same time a very large compressive stress resultant N_x occurs near the longitudinal boundaries. This fact was found by Timoshenko^[3] for an isotropic plate. The current analysis shows that the fact is true also for laminated anisotropic composite panels, flat or curved alike.

Fig. (IV,17) shows the distribution of N_y on the same panel as Fig. (IV,15). It may be seen that a large N_y occurs on the loaded edges. The development of N_y is also shown in the figure.

Fig. (IV,18) shows the distribution of bending moments M_x for a boron/epoxy panel with lay-up $(\pm 45^\circ, \pm 45^\circ)$ and $K_R = 25$.

It is found that the largest M_x occurs neither on the section of symmetry nor on the boundaries. It may be expected that M_y has a similar distribution in this panel.

e) Comparison with Previous Work

Fig. (IV,19) gives the comparison between the results of the present work and previous analysis^[87] for a flat boron/epoxy plate with four antisymmetric layers alternately at $\pm 45^\circ$. As expected, some difference between the post-buckling paths obtained from the two analyses is seen. This is caused by using a different deflection function and different precision of the constants α_i and γ_i in the beam functions. As mentioned before, the double sine-squared terms are included in the present work, but not in reference [87]. Fifteen significant digits are used for α_i and γ_i in the present work, in contrast to the six significant digits in reference [87]. The reason for using more exact values of these constants has been stated in Chapter II.

In Fig. (IV,20) the results of the present analysis are compared with experimental data for a flat unidirectional glass-polyester composite plate. The experimental data are taken from Fig. 10 of reference [94]. Both in-plane stresses and bending stresses at the centre of the plate, in x or y direction respectively, are shown in the figures. It may be seen that the current analysis gives a good prediction for the postbuckling stresses on the centre of the tested plates, though the inplane boundary conditions on the loaded boundaries in reference [94] are of uniform displacement instead of the uniform compression for the current work. An explanation for this is that the tested plate has an aspect ratio $\beta = 1.36$ and therefore the stresses measured at the

panel centre are not very sensitive to slight difference in the in-plane condition at the loaded boundaries.

4. Conclusions

From the numerical computations the following conclusions may be drawn:

(i) As for isotropic plates, the curvature of a laminated panel raises the bifurcation load but makes the monotonic form of the load deflection relation, load-inner forces and moment relations disappear. This is because more than one equilibrium state can exist at a certain load level. Snap-through on the panel happens when the load reaches the bifurcation load.

(ii) On the postbuckling path the buckling mode of the panel may change distinctively, as may the inner forces and moments.

(iii) The lay-up of a panel has a significant influence on the postbuckling behaviour when the number of alternate layers at $\pm 45^\circ$ in the panel is small. The larger the number of layers, the closer to that of an orthotropic one does the postbuckling behaviour of the panel become. The two anti-symmetric layer arrangement always results in the lowest postbuckling path.

(iv) After buckling the inner forces and moments of the panels (flat and curved alike) become non-uniform in the panels. In the circumferential sections, apart from the loaded edge, N_x varies dramatically. N_x on a small area near the panel centre may become tensile, the affected area becoming larger as the load increases. At that time the increased load is withstood by only the parts near the longitudinal boundaries of the panels.

C. Postbuckling Behaviour of Anisotropic Laminated Plates under Pure Shear and Shear combined with Compressive Loading

The postbuckling behaviour of panels under compression has been discussed in the last section. In this section postbuckling behaviour of anisotropic plates under pure shear and combined compressive and shear loading is investigated.

The postbuckling behaviour of composite plates under shear loading has been studied experimentally by a few investigators^{[95][114][115]}. It was reported that the ultimate loads of plates made of boron/epoxy, carbon/epoxy or graphite/epoxy could be considerably higher than the initial buckling loads. But so far few postbuckling analyses are available for anisotropic plates under shear or combined loading.

In previous work by Ashton et al^{[39][42]} it was pointed out that the shear direction has significant influence on the initial buckling of anisotropic plates under pure shear. In Chapter II of this thesis the effect of applied shear direction on the initial buckling behaviour of laminated panels under combined shear and compressive loading is discussed. The combined shear direction proves to be very important for laminated panels under compression. The shear may greatly stabilise or destabilise the panels. So it is natural to raise the question: how does the shear direction affect the postbuckling behaviour of the anisotropic plates? The answer could not be found in the literature. This section is intended, therefore, to give such an answer.

In this section a postbuckling analysis for flat plates with mid-plane symmetric lay-up composites is presented. The governing equations in the Von Karman sense are solved in conjunction with simply supported boundaries. The effect of boundary moments is taken into account in the system to allow for general anisotropy. As examples, a series of computations are carried out for plates having different fibre directions, different stacking sequences of layers, different materials and different aspect ratios. The results reveal that the direction of applied shear with respect to the fibres of the layers is very important for postbuckling behaviour of anisotropic plates under shear loading or the combination of compression and shear. This means that the postbuckling behaviour of anisotropic plates will depend greatly upon the direction of the shear acting on the plates.

1. Governing System

Consider a rectangular plate (see Fig. (IV,21)) consisting of laminated anisotropic composite materials with a mid-plane symmetric lay-up. It is assumed that the materials are elastic and the thickness h of the plate is very small compared with the length a and width b . The mid-plane of the plates before buckling contains the reference coordinate system Oxy with the origin at one of the corners.

In Chapter II, the nondimensional governing system (II,27) and (II,28) for the general case has been obtained, based on the virtual displacement and virtual force principles. In the particular case for the current investigation the governing system may be simplified, i.e.

$$\begin{aligned}
& \int_0^1 \int_0^1 \left\{ \frac{\partial^4 F}{\partial \xi^4} - a_1 \frac{\partial^4 F}{\partial \xi^3 \partial \eta} + a_2 \frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} - a_3 \frac{\partial^4 F}{\partial \xi \partial \eta^3} + a_4 \frac{\partial^4 F}{\partial \eta^4} \right. \\
& \quad \left. - \left(\frac{\beta^2}{A_{22}^*} \right) \left[\left(\frac{\partial^2 W}{\partial \xi \partial \eta} \right)^2 - \frac{\partial^2 W}{\partial \xi^2} \frac{\partial^2 W}{\partial \eta^2} \right] \right\} \delta F d\xi d\eta \\
& = 0
\end{aligned} \tag{IV, 25}$$

$$\begin{aligned}
& \int_0^1 \int_0^1 \left\{ \frac{\partial^4 W}{\partial \xi^4} + b_1 \frac{\partial^4 W}{\partial \xi^3 \partial \eta} + b_2 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + b_3 \frac{\partial^4 W}{\partial \xi \partial \eta^3} + b_4 \frac{\partial^4 W}{\partial \eta^4} \right. \\
& \quad \left. - \left(\frac{\beta^2}{D_{11}^*} \right) \left[\frac{\partial^2 W}{\partial \xi^2} \frac{\partial^2 F}{\partial \eta^2} + \frac{\partial^2 W}{\partial \eta^2} \frac{\partial^2 F}{\partial \xi^2} - 2 \frac{\partial^2 W}{\partial \xi \partial \eta} \frac{\partial^2 F}{\partial \xi \partial \eta} \right] \right\} \delta W d\xi d\eta \\
& + \int_0^1 \left\{ \left[\frac{\partial^2 W}{\partial \xi^2} + b_{12} \frac{\partial^2 W}{\partial \eta^2} + \frac{b_1}{2} \frac{\partial^2 W}{\partial \xi \partial \eta} \right] \delta \left(\frac{\partial W}{\partial \xi} \right) \right\} \Big|_{\xi=0}^{\xi=1} d\eta \\
& + \int_0^1 \left\{ \left[b_{15} \frac{\partial^2 W}{\partial \xi^2} + b_4 \frac{\partial^2 W}{\partial \eta^2} + \frac{b_3}{2} \frac{\partial^2 W}{\partial \xi \partial \eta} \right] \delta \left(\frac{\partial W}{\partial \eta} \right) \right\} \Big|_{\eta=0}^{\eta=1} d\xi \\
& = 0
\end{aligned} \tag{IV, 26}$$

where F and w are nondimensional stress function and deflection function; ξ and η are nondimensional coordinates in x and y direction, β is the aspect ratio of the plate. The coefficients a_i , b_i , A_{22}^* and D_{11}^* are constants related to the properties of the composite layers in the plates.

All these parameters and constants are defined in the expressions (II, 26) above.

The two line integrals in expression (IV, 26) represent the effect of boundary moments.

The nondimensional forces and moments are now taken in the form

$$\begin{Bmatrix} N_{\xi} \\ N_{\eta} \\ N_{\xi\eta} \end{Bmatrix} = \frac{b^2}{A_{22}h^2} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\beta^2} & 0 \\ 0 & 0 & \frac{1}{\beta} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 F}{\partial \eta^2} \\ \frac{\partial^2 F}{\partial \xi^2} \\ -\frac{\partial^2 F}{\partial \xi \partial \eta} \end{Bmatrix} \quad (\text{IV}, 27)$$

$$\begin{Bmatrix} M_{\xi} \\ M_{\eta} \\ M_{\xi\eta} \end{Bmatrix} = \frac{b^2}{A_{22}h^2} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ D_{21}^* & D_{22}^* & D_{26}^* \\ D_{61}^* & D_{62}^* & D_{66}^* \end{bmatrix} \begin{bmatrix} \frac{1}{\beta^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\beta} \end{bmatrix} \begin{Bmatrix} -\frac{\partial^2 W}{\partial \xi^2} \\ -\frac{\partial^2 W}{\partial \eta^2} \\ -2\frac{\partial^2 W}{\partial \xi \partial \eta} \end{Bmatrix} \quad (\text{IV}, 28)$$

The boundary conditions for simply supported edges may be written in terms of nondimensional parameters and constants as follows

$$W=0, \quad M_{\xi}=0, \quad \frac{\partial^2 F}{\partial \eta^2} = -\bar{N}_x, \quad \frac{\partial^2 F}{\partial \xi \partial \eta} = -\beta \bar{S} \quad \text{at } \xi = 0, 1 \quad (\text{IV}, 29)$$

$$W=0, \quad M_{\eta}=0, \quad \frac{\partial^2 F}{\partial \xi^2} = -\beta^2 \bar{N}_y, \quad \frac{\partial^2 F}{\partial \xi \partial \eta} = -\beta \bar{S} \quad \text{at } \eta = 0, 1$$

where

$$\begin{aligned} \bar{N}_x &= \frac{P_x b^2}{A_{22} h^2} \\ \bar{N}_y &= \frac{P_y b^2}{A_{22} h^2} \\ \bar{S} &= \frac{P_{xy} b^2}{A_{22} h^2} \end{aligned} \quad (\text{IV}, 30)$$

P_x, P_y being boundary compressive loads in the x or y direction and P_{xy} the boundary shear load.

2. Solutions

The solutions to the governing system (IV,25) and (IV,26) may be assumed to have the following form:

$$F = -\bar{N}_x \frac{\eta^2}{2} - \bar{N}_y \beta^2 \frac{\xi^2}{2} - \bar{S} \beta \xi \eta + \sum_m \sum_n F_{mn} X_m(\xi) Y_n(\eta)$$

(IV,31)

$$W = \sum_p \sum_q W_{pq} \sin p\pi\xi \sin q\pi\eta$$

(m, n, p, q = 1, 2, ..., ∞)

where $X_i(\xi)$ and $Y_i(\eta)$ are the characteristic beam functions i.e.

$$X_i(\xi) = \cosh \alpha_i \xi - \cos \alpha_i \xi - \gamma_i (\sinh \alpha_i \xi - \sin \alpha_i \xi)$$

(IV,32)

$$Y_i(\eta) = \cosh \alpha_i \eta - \cos \alpha_i \eta - \gamma_i (\sinh \alpha_i \eta - \sin \alpha_i \eta)$$

in which α_i and γ_i are the constants given in Table (II,1).

The properties of these beam functions have been stated in equations (IV,6), (IV,7) and (IV,8) of section B in this chapter.

It may be seen that the assumed solutions (IV,31) satisfy the kinematic and in-plane boundary conditions of (IV,29), but not the zero moment conditions on the edges due to the coupling terms involving D_{16}^* and D_{26}^* . As mentioned before, however, the effects of edge moments are taken into account in equation (IV,26). Hence, it is necessary only to set the solution (IV,31) into the system (IV,25) and (IV,26). The resultant equations are shown below:

$$\sum_m \sum_n F_{mn} A_{mn}^{ij} = \sum_k \sum_l \sum_r \sum_s W_{kl} W_{rs} B_{klrs}^{ij}$$

(IV,33)

$$\sum_p \sum_q W_{pq} C_{pq}^{ij} = \sum_k \sum_l \sum_r \sum_s W_{kl} F_{rs} D_{klrs}^{ij} + K \sum_m \sum_n W_{mn} T_{mn}^{ij}$$

(i, j = 1, 2, ..., ∞)

where

$$A_{mn}^{ij} = (\alpha_m^4 + a_4 \alpha_n^4) J_0^{mi} J_0^{nj} - a_1 \alpha_m^3 \alpha_n J_3^{mi} J_1^{nj} + a_2 \alpha_m^2 \alpha_n^2 J_2^{mi} J_2^{nj} - a_3 \alpha_m \alpha_n^3 J_1^{mi} J_3^{nj}$$

$$B_{klrs}^{ij} = \frac{\beta^2 \pi^4}{A_{22}^*} (klrs Z_{05}^{ikr} Z_{05}^{jls} - k^2 s^2 Z_{01}^{ikr} Z_{01}^{jls})$$

$$C_{pq}^{ij} = \frac{\pi^4}{4} (p^4 + b_2 p^2 q^2 + b_4 q^4) J_0^{pi} J_0^{qj} - b_1 p q \pi^2 (p^2 \pi^2 H_0^{pi} + i \pi Q_0^{pi}) H_0^{qj} \\ - b_3 p q \pi^2 (q^2 \pi^2 H_0^{qj} + j \pi Q_0^{qj}) H_0^{pi} \quad (\text{IV}, 34)$$

$$D_{klrs}^{ij} = -\frac{\beta^2 \pi^2}{D_{11}^*} (k^2 \alpha_s^2 Z_{01}^{rki} Z_{21}^{slj} + l^2 \alpha_r^2 Z_{21}^{rki} Z_{01}^{slj} + 2kl \alpha_r \alpha_s Z_{11}^{rki} Z_{11}^{slj})$$

$$T_{mn}^{ij} = \beta^2 \pi^2 \left[\frac{\mu_x m^2 + \mu_y \beta^2 n^2}{4} J_0^{mi} J_0^{nj} + 2\beta \mu_s m n H_0^{mi} H_0^{nj} \right] \cdot \left(\frac{E_t h^3}{D_{11}^*} \right)$$

$$K = \frac{P b^2}{E_t h^3}$$

and μ_x , μ_y and μ_s are loading ratio parameters defined by

$$P_x = \mu_x P$$

$$P_y = \mu_y P$$

$$P_{xy} = \mu_s P$$

(IV, 35)

By properly choosing the ratio between μ_x , μ_y and μ_s , one may have any required combination of boundary loading. As long as the loading parameter K is obtained in computation P , P_x , P_y and P_{xy} can be known at once.

J_0 , J_1 , J_2 , J_3 , H_0 , Q_0 , Z_{01} , Z_{05} , Z_{11} , Z_{21} are obtained from integrals of the functions X_i and Y_i and their derivatives. The values of these constants are listed in Appendix F.

The system (IV, 33) contains an infinite number of equations. To solve this system for practical application,

the solutions (IV,31) need to be truncated properly.

First, let $m=p=1,2,\dots,M$ and $n=q=1,2,\dots,N$ in the solutions (IV,31) (usually $M = N$ is used), that means that the first M^2 terms in (IV,31) are taken. At the same time, the number of equations contained in the system (IV,33) are reduced to M^2 as well.

Secondly, write the system (IV,33) into matrix form, i.e.

$$\begin{aligned} [A] \{F\} &= [T_w] [B] \{W\} \\ [C] \{W\} &= [T_w] [D] \{F\} + K [T] \{W\} \end{aligned} \quad (\text{IV},36)$$

where

$$[T_w] = \begin{bmatrix} \{W\}^t & & & \\ & \{W\}^t & & \\ & & \ddots & \\ & & & \{W\}^t \end{bmatrix} \quad (\text{IV},37)$$

in which the superscript t indicates the transposition of the columns.

The elements of the matrices in the system (IV,36) are given below.

With the indices I defined by

$$I = (m-1)M + n = (p-1)M + q \quad (\text{IV},38)$$

we have

$$\begin{aligned} F_I &= F_{mn} \\ W_I &= W_{pq} = W_{mn} \end{aligned} \quad (\text{IV},39)$$

With the indices I, J defined by

$$I = (i-1)M + j \quad (\text{IV},40)$$

$$J = (m-1)M + n = (p-1)M + q$$

we have

$$\begin{aligned}
A_{IJ} &= A_{mn}^{ij} \\
C_{IJ} &= C_{pq}^{ij} \\
T_{IJ} &= T_{mn}^{ij}
\end{aligned}
\tag{IV,41}$$

With the indices I,J defined by

$$\begin{aligned}
I &= (I_1 - 1)M + J_1 \\
J &= (r - 1)M + s
\end{aligned}
\tag{IV,42}$$

where

$$\begin{aligned}
I_1 &= (i - 1)M + j \\
J_1 &= (k - 1)M + l
\end{aligned}
\tag{IV,43}$$

then

$$\begin{aligned}
B_{IJ} &= B_{klrs}^{ij} \\
D_{IJ} &= D_{klrs}^{ij}
\end{aligned}
\tag{IV,44}$$

3. Computation and Results

As in section B of this chapter, an iterative scheme based on the modified Newton method is used to obtain the postbuckling path. In our case the postbuckling behaviour of the panel is of the pure bifurcational type. The starting point of the iteration is based on the characteristic solutions of bifurcation, corresponding to the lowest critical load.

a) Bifurcation point

By dropping the nonlinear terms from the governing equations of the system (IV,36) the characteristic equation is simply established,

i.e.

$$[C] \{w\} = K [\tau] \{w\}
\tag{IV,45}$$

It can be seen that the stress function F has no influence on the characteristic solutions of the plates with symmetric lay-ups. In other words, in that case there is no coupling between in-plane forces and out-of-plane bending in the range of linear theory.

The QZ algorithm is used as before to obtain the lowest eigenvalue K_0 and the corresponding eigenvector $\{w\}_0$ which represents the buckling mode for initial buckling of the plate.

b) Postbuckling Path

The method for solving the postbuckling path, as mentioned before, consists of prescribing one of the components of $\{w\}$ and letting it increase step by step. Correspondingly, the other components of $\{w\}$ and the load parameter K , from the governing system (IV,36) are found by the use of the iterative scheme based on the modified Newton method. Thus a series of points on the postbuckling path can be obtained. At the beginning of the process the first estimate used for the iteration for each new point is the solution belonging to the last point except for the prescribed component of $\{w\}$. After three points are obtained an extrapolation scheme is introduced to make a better estimate for every next point. In the process of finding the postbuckling path the function F is solved from the first expression of the system (IV,36) and set in the second one of that set.

The component having the largest absolute value in the characteristic solutions $\{w\}_0$ is chosen as the prescribed one. The step length varies from 0.1 to 0.6 for different plates. The convergence criterion used for iteration is that the sum of the squares of the residuals of the governing

equations is less than 0.01 percent.

c) Results

A series of computations is carried out for various plates. In this section the first 25 terms approximation for the solution (IV,31) is used. Only plates with symmetric lay-ups are considered here and such plates demonstrate the pure bifurcation behaviour described above. Plates with other lay-ups may, however, bend immediately load is applied. This phenomenon will be discussed later in Chapter V.

The material properties of boron/epoxy, carbon/epoxy and glass/epoxy used for the computation are shown in Table (III,2) (see page 46).

The computation results are summarised in the figures below and all refer to plates with the ratio of width to thickness $b/h = 100$.

Figs (IV,22) and (IV,23) are obtained for square boron/epoxy plates under pure shear. The figures show how the deflection at the centre of the plates varies with increasing shear loading after buckling. It can be seen from the figures that there are two postbuckling paths for each plate when the applied shear directions are altered, unless the plate is orthotropic. In Fig. (IV,22) it is shown that for a four layer plate even fibre angles as small as 15° in the alternate layers may cause a large difference in the postbuckling behaviour of the plate. In Fig. (IV,23) the effect of the number of layers in the plate with alternate 45° layers is shown. It may be seen that the less the number of layers, the larger are the differences that exist between the postbuckling paths corresponding to alternate directions of applied shear. A plate will behave as if orthotropic only when the

number of layers in it becomes very large.

Fig. (IV,24) is obtained for four-layer $(\pm 45^\circ)_S$ boron/epoxy plates having different aspect ratios. The load-central deflection curves on the figure show that the postbuckling behaviour is also different for non-square plates when alternating the shear directions.

Fig. (IV,25) refers to square plates having a $(\pm 45^\circ)_S$ lay-up of carbon/epoxy and glass/epoxy composite materials. The curves of deflection at the centre of the plate versus the shear loading indicate that for the carbon/epoxy plate, which has the higher ratio of E_l/E_t for composite, the difference between the postbuckling paths for alternate shear direction is larger than for the glass/epoxy plate.

Fig. (IV,26) shows the deflection at the centre of a square boron/epoxy plate, having a $(\pm 45^\circ)_S$ lay-up, under combined loading. It can be seen that the postbuckling behaviour of the plate is also very sensitive to the direction of the combined shear, whether unidirectional compression or bidirectional compression is applied to it.

Fig. (IV,27) shows the internal shear force at the centre of the plate under shear loading. The composite material, lay-up and geometry of the plate is the same as for Fig.(IV,26). As expected, the nonlinear postbuckled internal forces are also greatly affected by the applied shear directions.

Fig. (IV,28) is calculated for the same plate as for Fig. (IV,26). The distribution of the postbuckled deflection and internal shear force on a quarter of the plate, under positive shear, corresponding to the point C in Fig. (IV,27) occurring on the selected sections, may be connected approximately by a straight line. The line is close to, but not

coincident with, one of the diagonals because of the anisotropy of the plate. In the figure it is also seen that the distribution of internal shear force becomes non-uniform at all the sections selected, apart from the edge.

In order to determine the accuracy of the 25 term approximation of the solutions, the results for a square boron/epoxy plate with $(\pm 45^\circ)_s$ lay-up, under positive shear, is shown in Fig. (IV,29) for different numbers of terms. It is seen that the 25-term solution is sufficient for most engineering purposes.

4. Conclusions

The following conclusions may be drawn from the current investigation:

(i) For anisotropic plates of symmetrically laminated composite materials loaded in shear, there are two postbuckling paths corresponding to alternate shear directions. The difference between these two paths is affected greatly by the lay-up of laminate, the properties of the composite and the aspect ratios of the plates. A high ratio of E_l/E_t and small number of layers will result in a big difference between the two paths.

(ii) The postbuckling behaviour of anisotropic plates under combined compression and shear loading are also significantly affected by the direction of the shear, whether unidirectional or bidirectional compression acts on the plates.

(iii) The higher performance of a composite plate corresponds to the direction of applied shear forces which produces a tension resultant in the diagonal of the plate having the lower bending stiffness. For a prescribed shear, however, the performance of the plate depends on the direction of the fibre angles of the layers, especially of the outer layers.

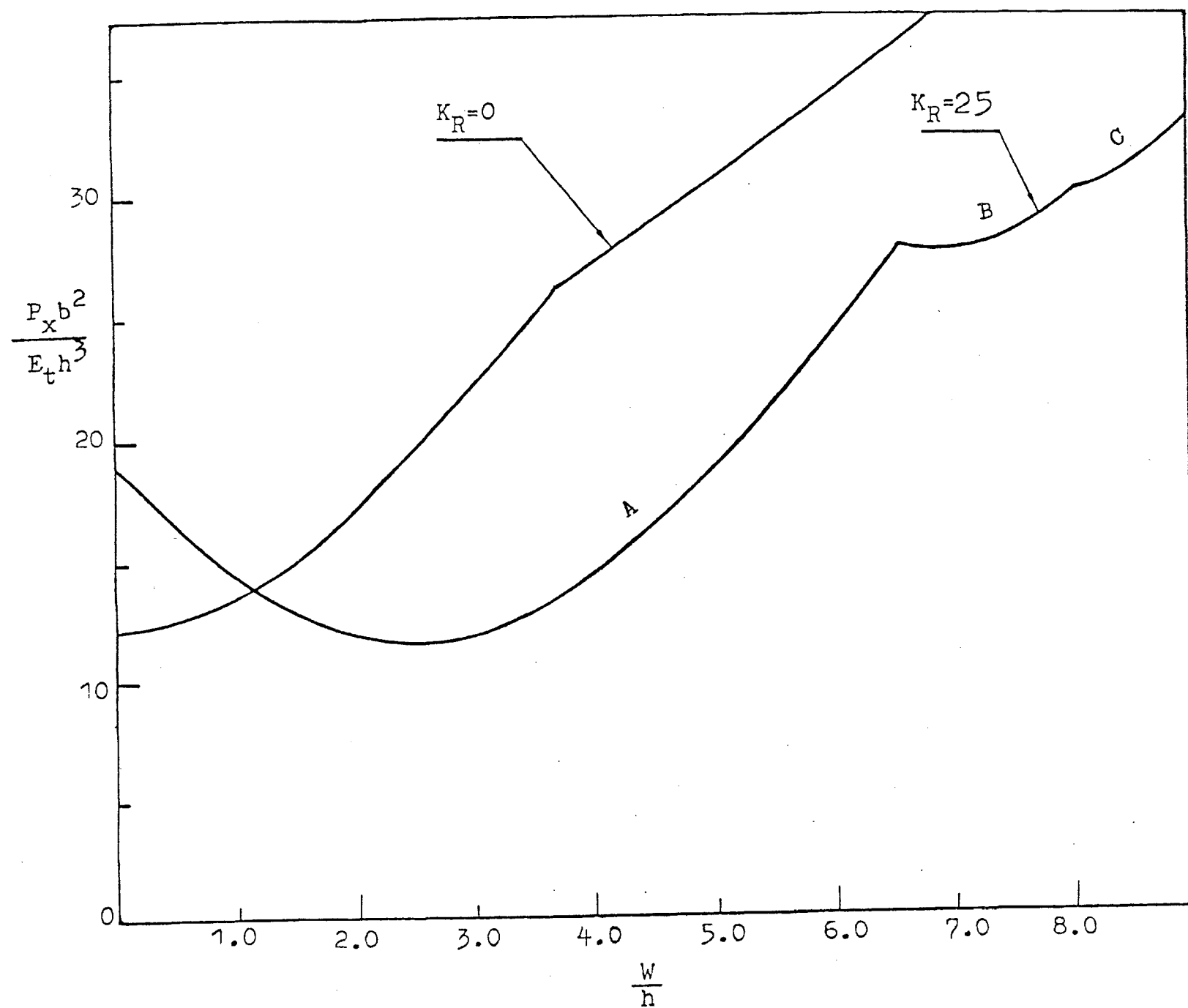


Fig. (IV,1)
 Variation of central
 deflection of boron/
 epoxy panels;
 unidirectional layers
 at 45°

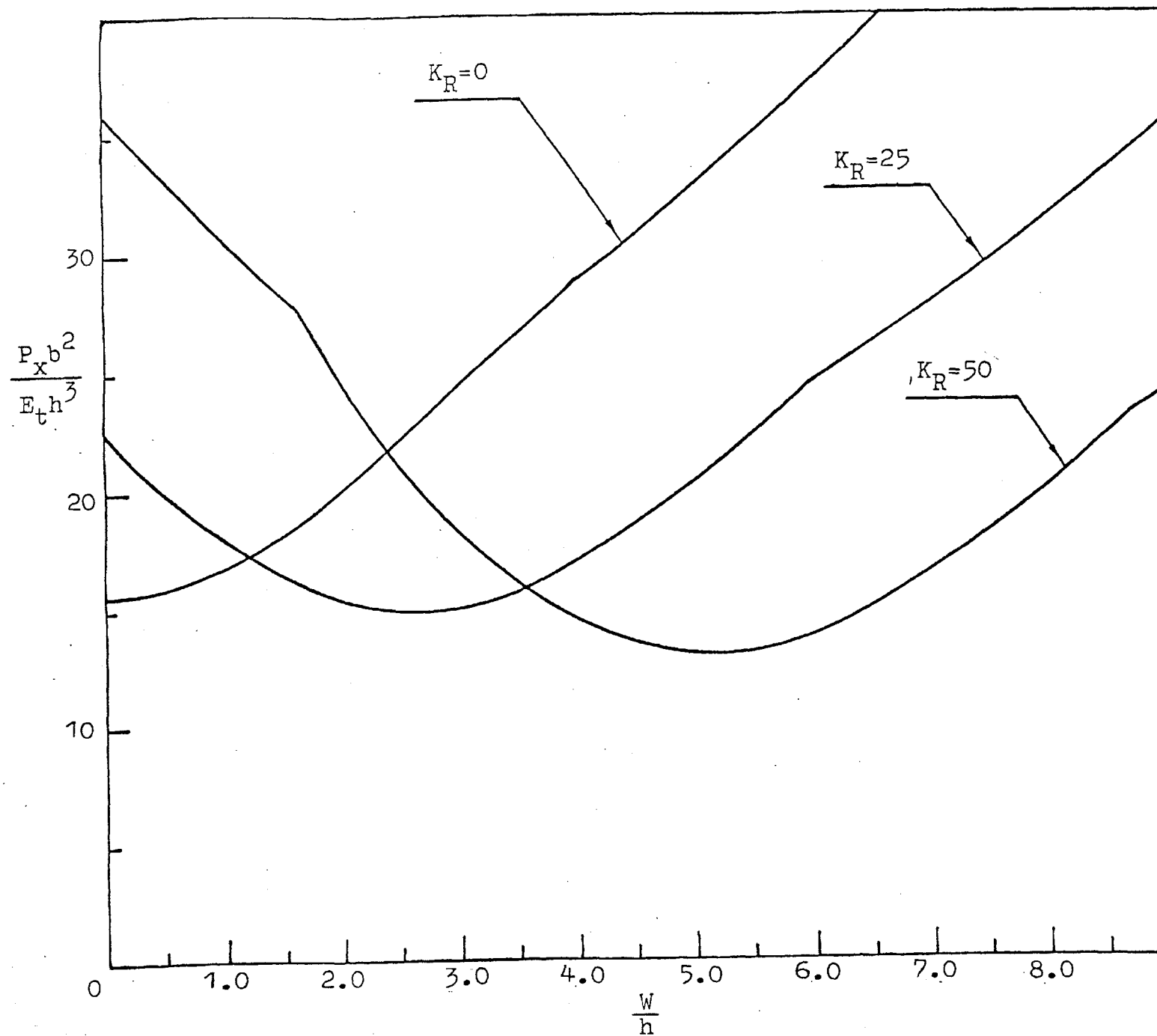


Fig. (IV,2)
Variation of central
deflection of boron/
epoxy panels;
 $(\pm 45^\circ)_s$ lay-up

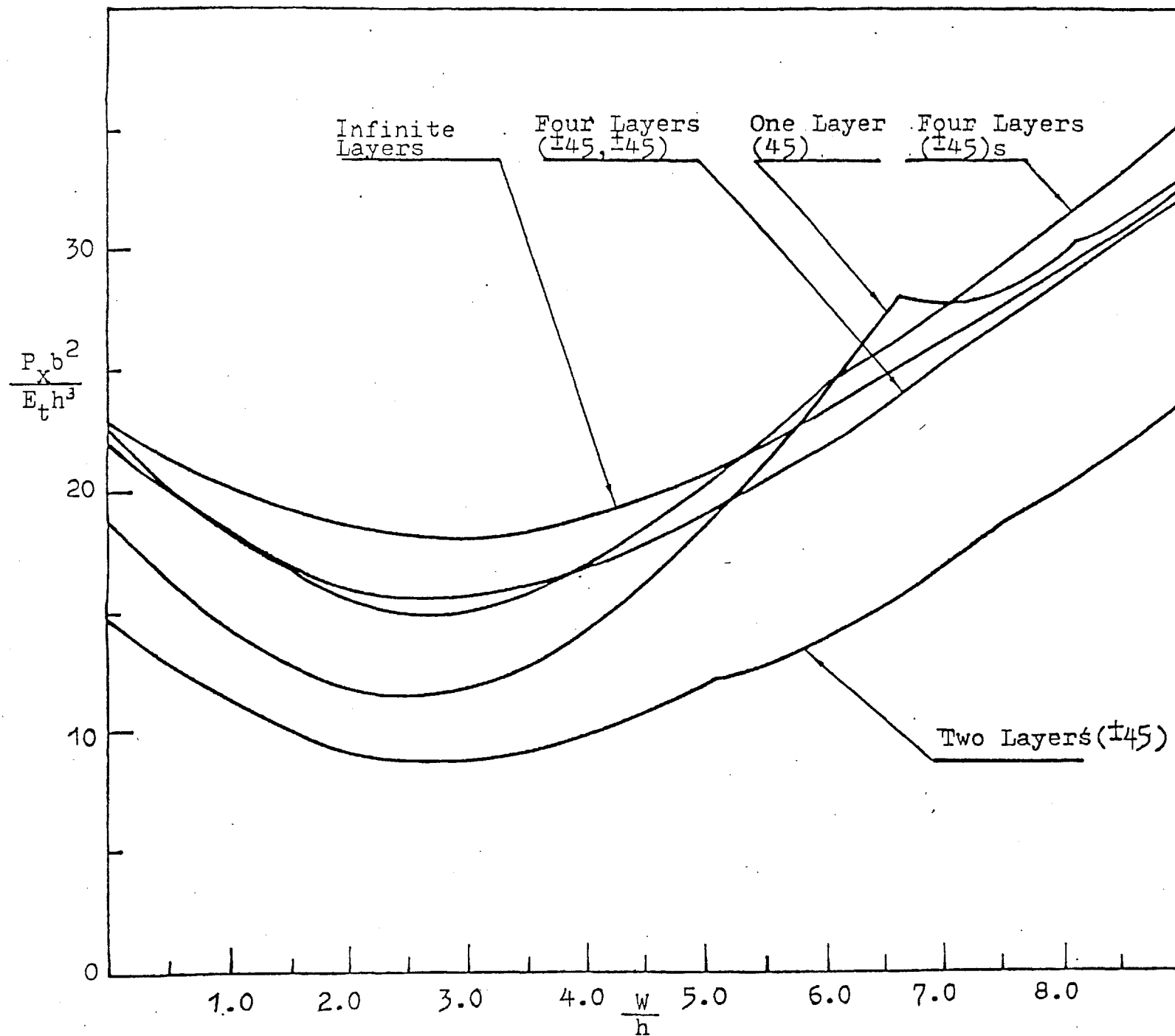


Fig. (IV,3)

Variation of central
deflection of boron/
epoxy panels; different
lay-ups and $K_R=25$

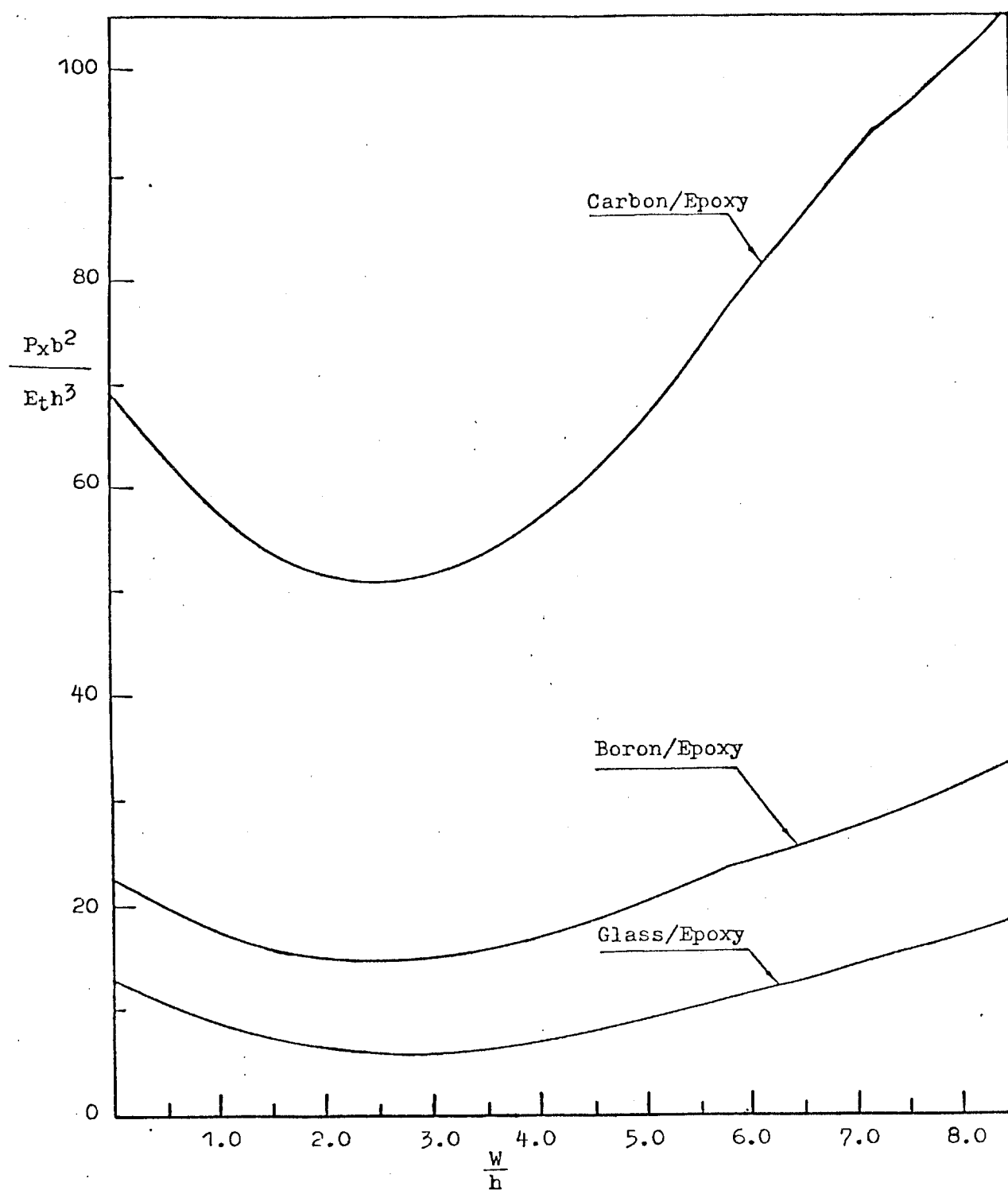


Fig. (IV,4) Variation of central deflection of panels made of different materials; $(\pm 45^\circ)_s$ lay-up and $K_R = 25$

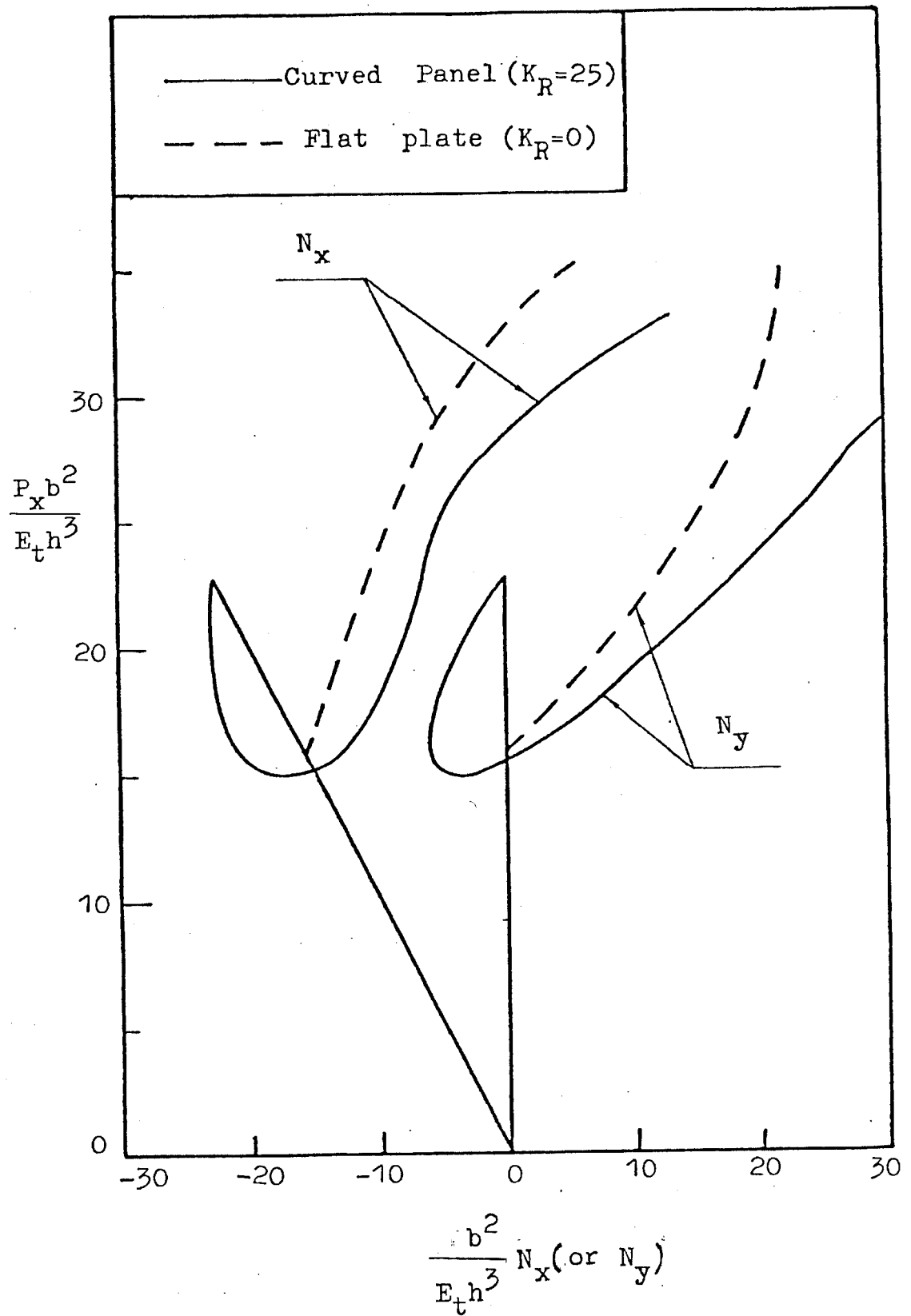


Fig. (IV,5) Variation of inner forces at panel centre;
four symmetric boron/epoxy layers ($\pm 45^\circ$)_s

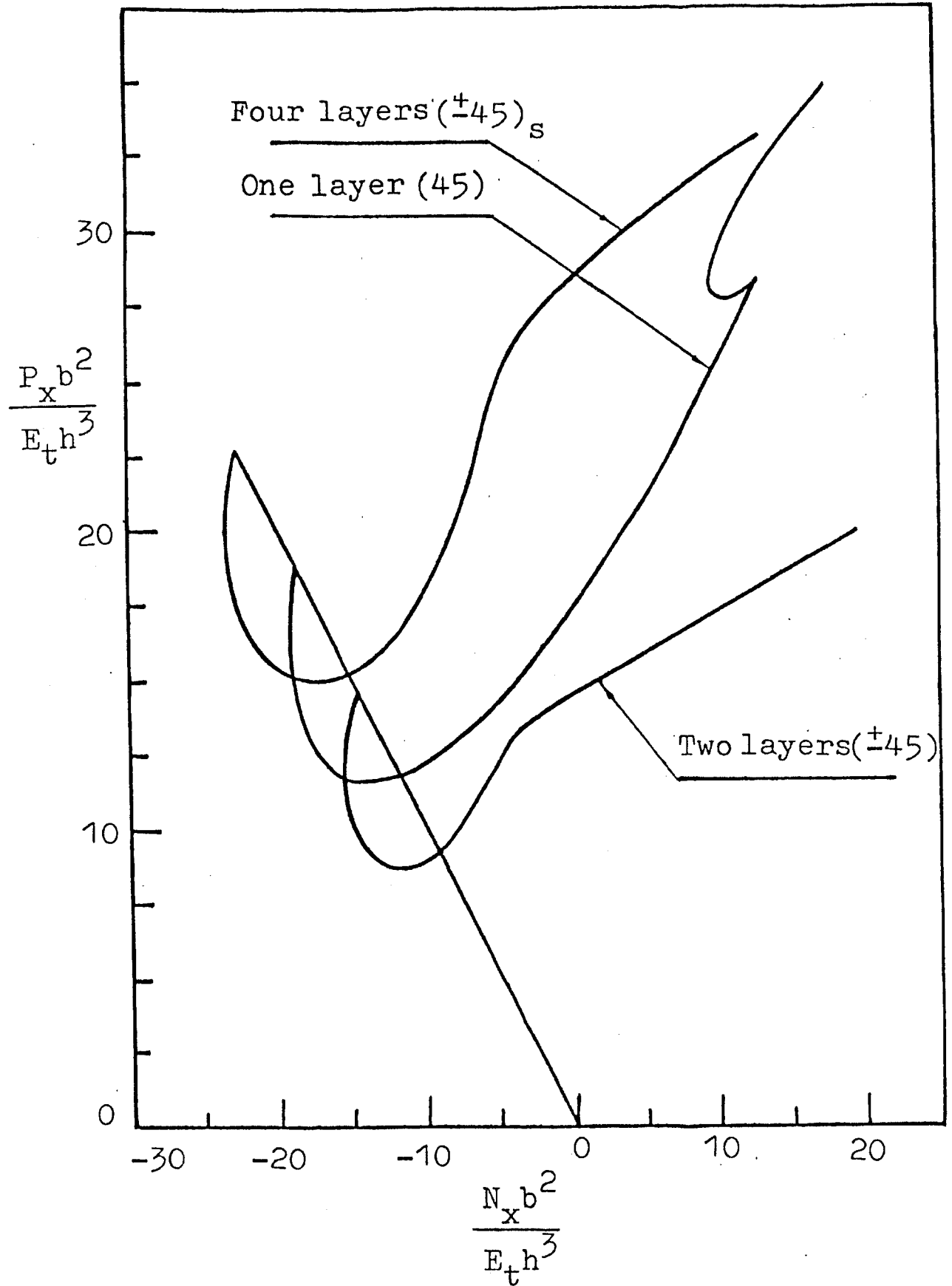


Fig. (IV,6) Variation of inner forces N_x at the centre of boron/epoxy panels; different lay-ups

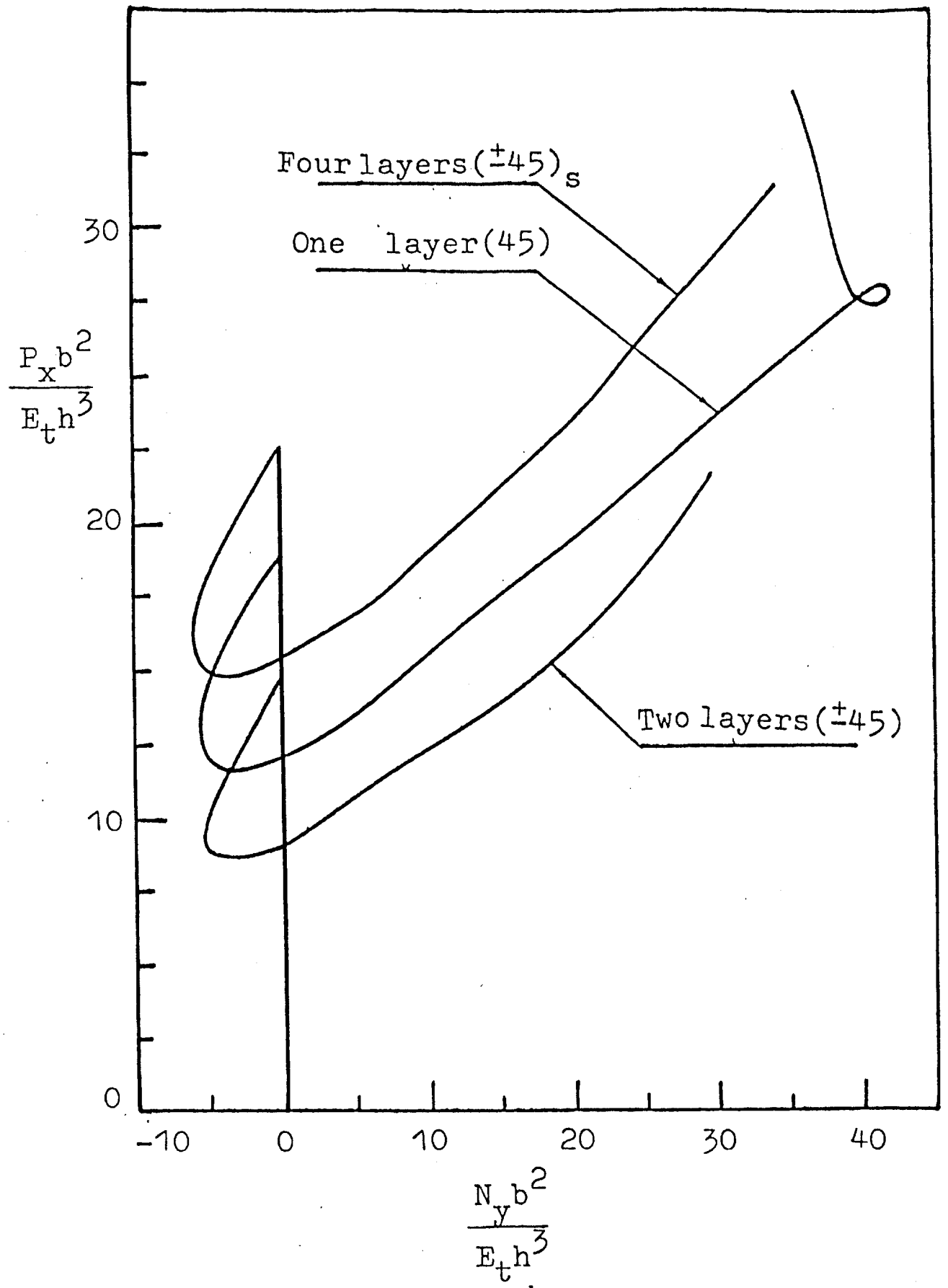


Fig. (IV,7) Variation of inner forces N_y at the centre of boron/epoxy panels; different lay-ups

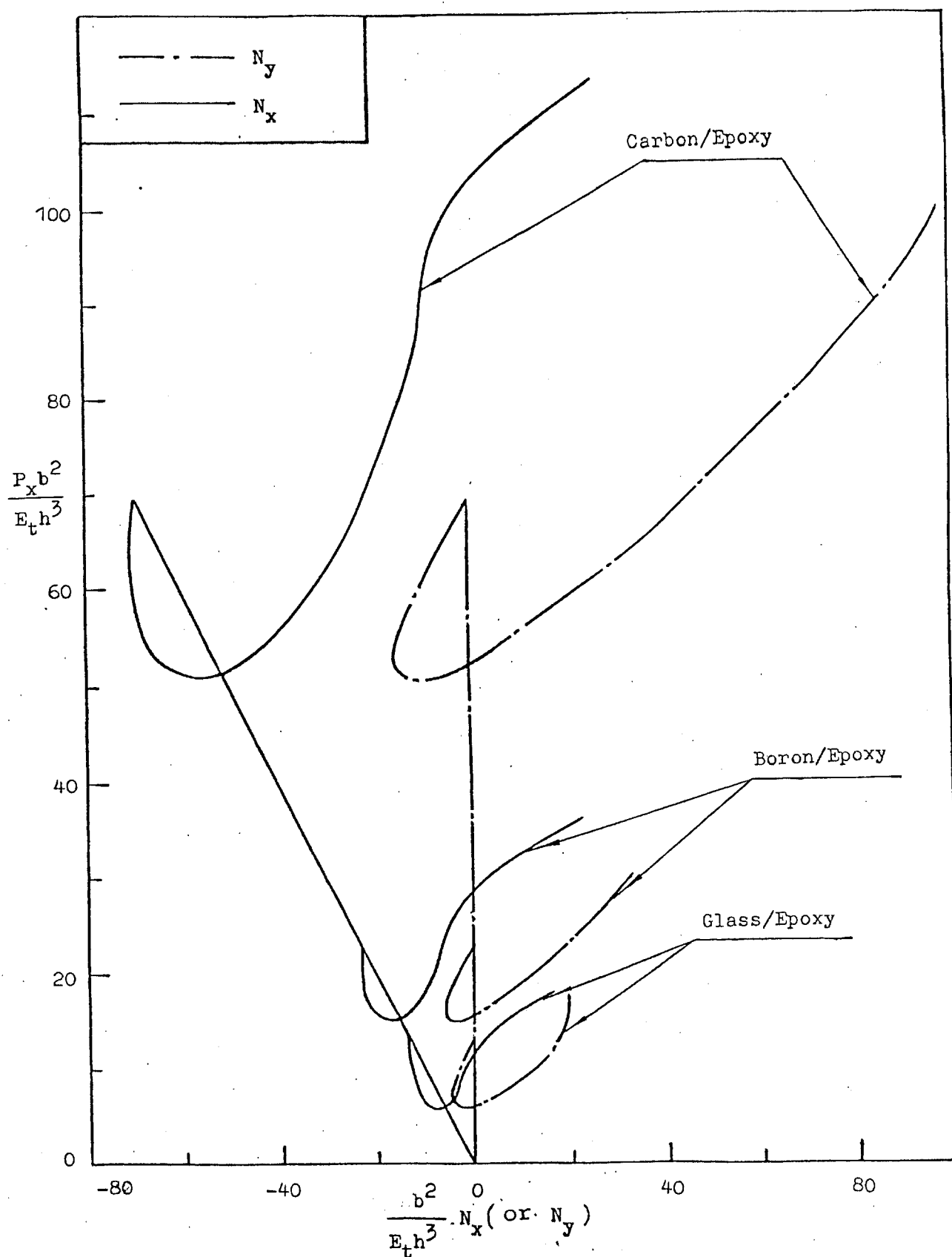


Fig. (IV,8) Variation of inner forces N_x and N_y at panel centre; different materials, four symmetric layers $(\pm 45^\circ)_s$

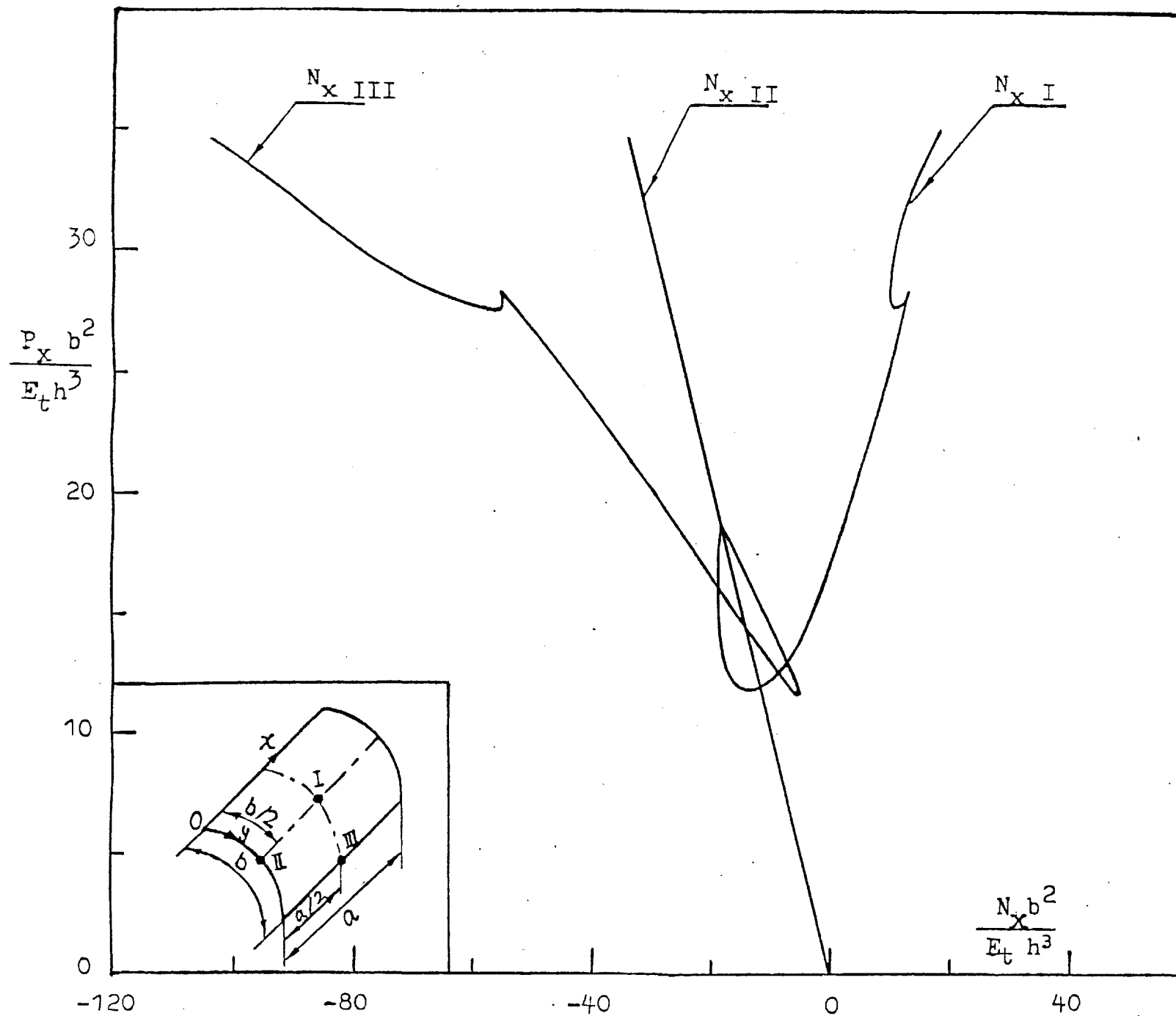


Fig. (IV,9)

Variation of inner forces
 N_x at different points
 of a panel; four symmetric
 boron/epoxy layers $(\pm 45^\circ)_s$

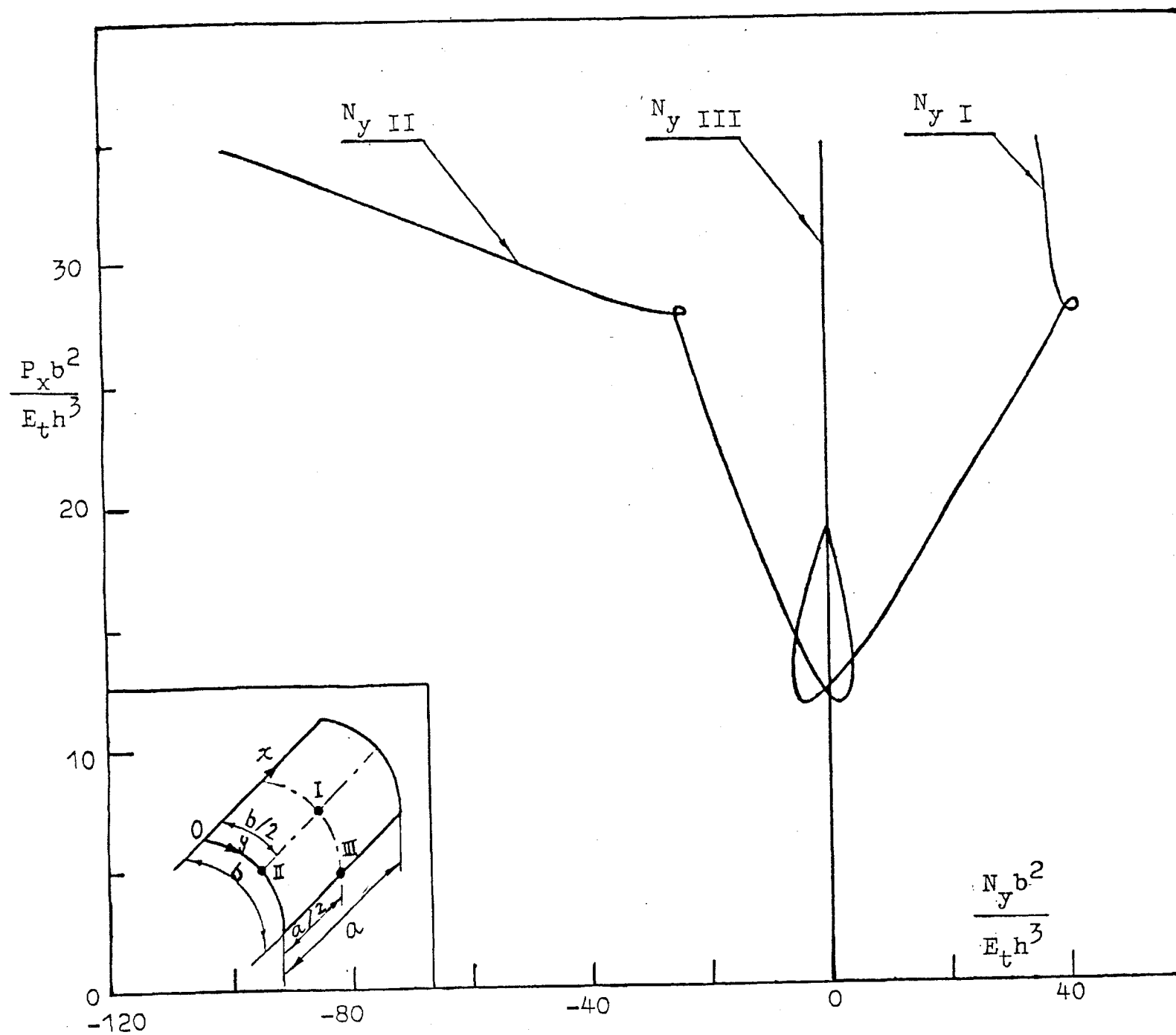


Fig. (IV,10)

Variation of the inner forces N_y at different points of a panel; four symmetric boron/epoxy layers $(\pm 45^\circ)_s$

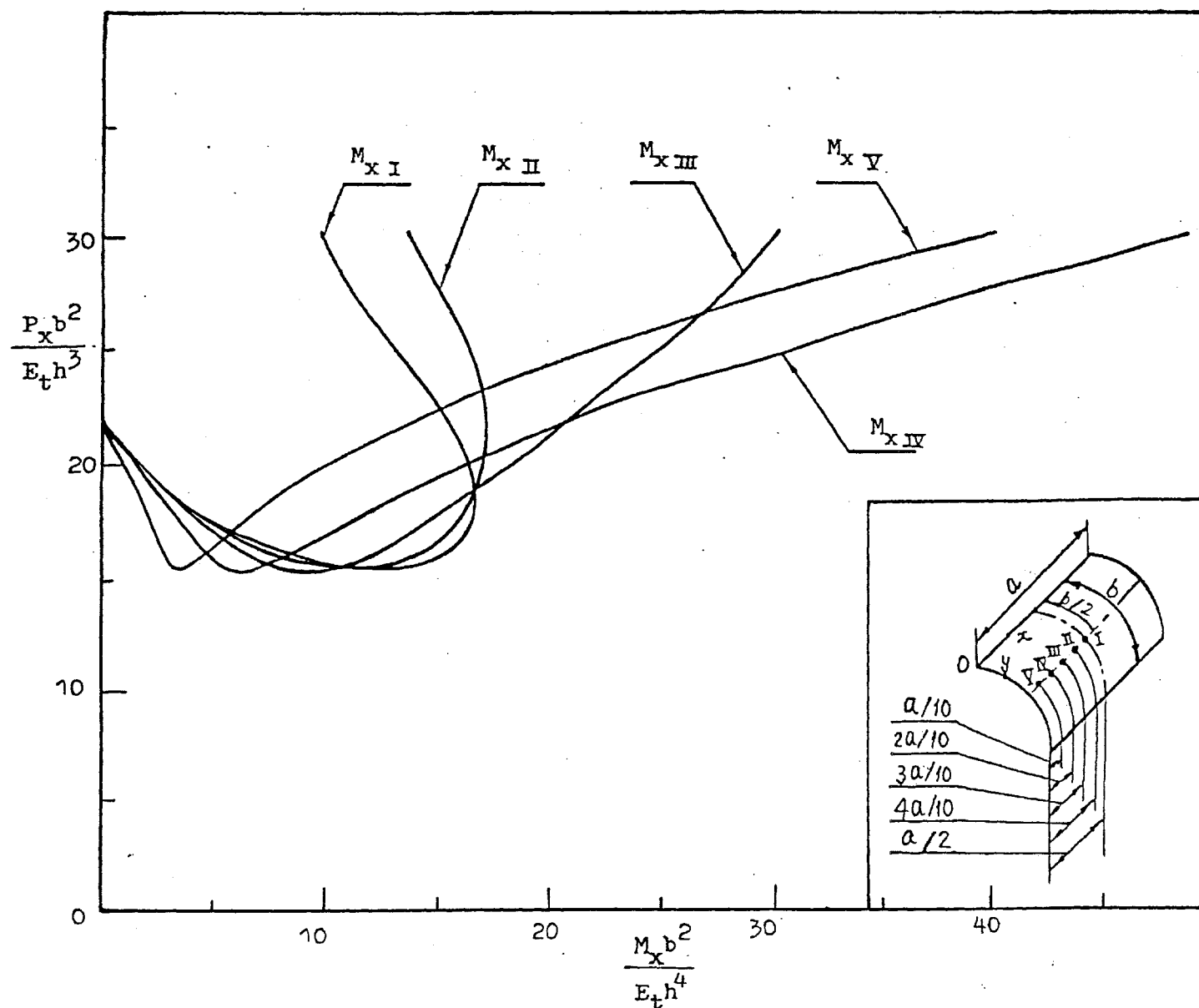


Fig. (IV,11) Variation of inner moments M_x at different points along the length of a boron/epoxy panel; four layers at $(\pm 45^\circ, \pm 45^\circ)$

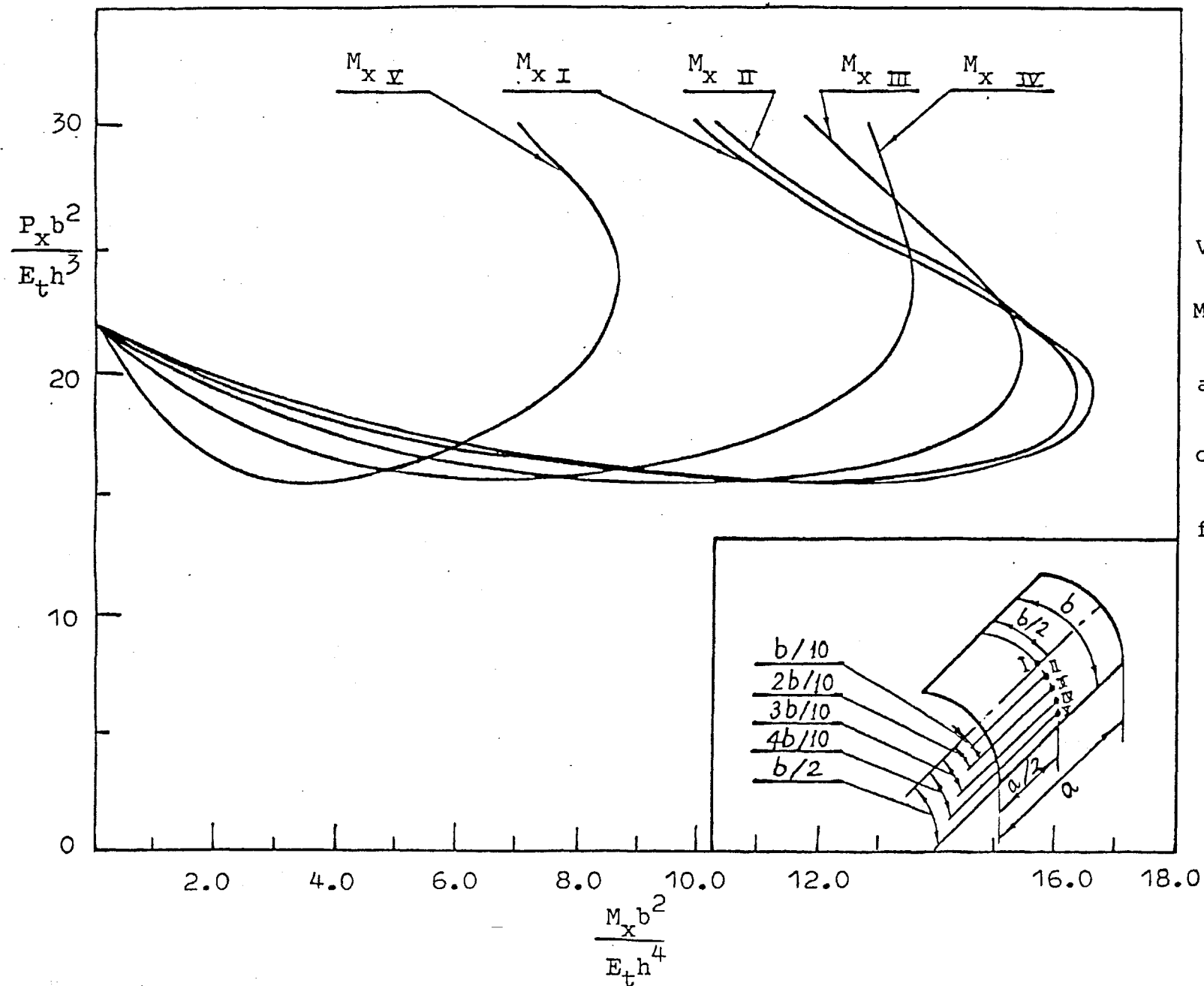


Fig. (IV,12)

Variation of inner moments
 M_x at different points
 along the circumference
 of a boron/epoxy panel;
 four layers at $(\pm 45^\circ, \pm 45^\circ)$

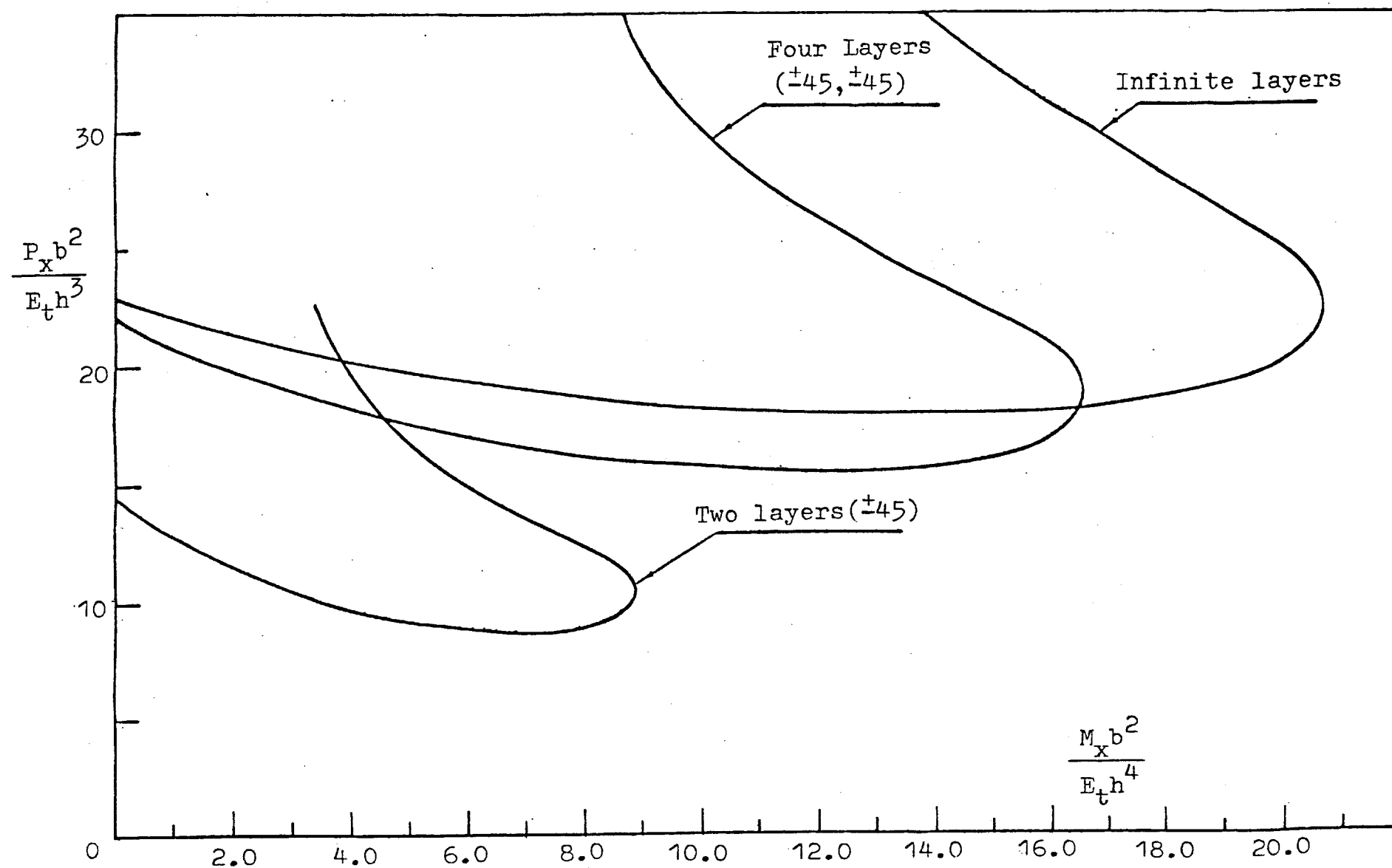


Fig. (IV,13) Variation of inner moments M_x at the centre of boron/epoxy panels; different lay-ups

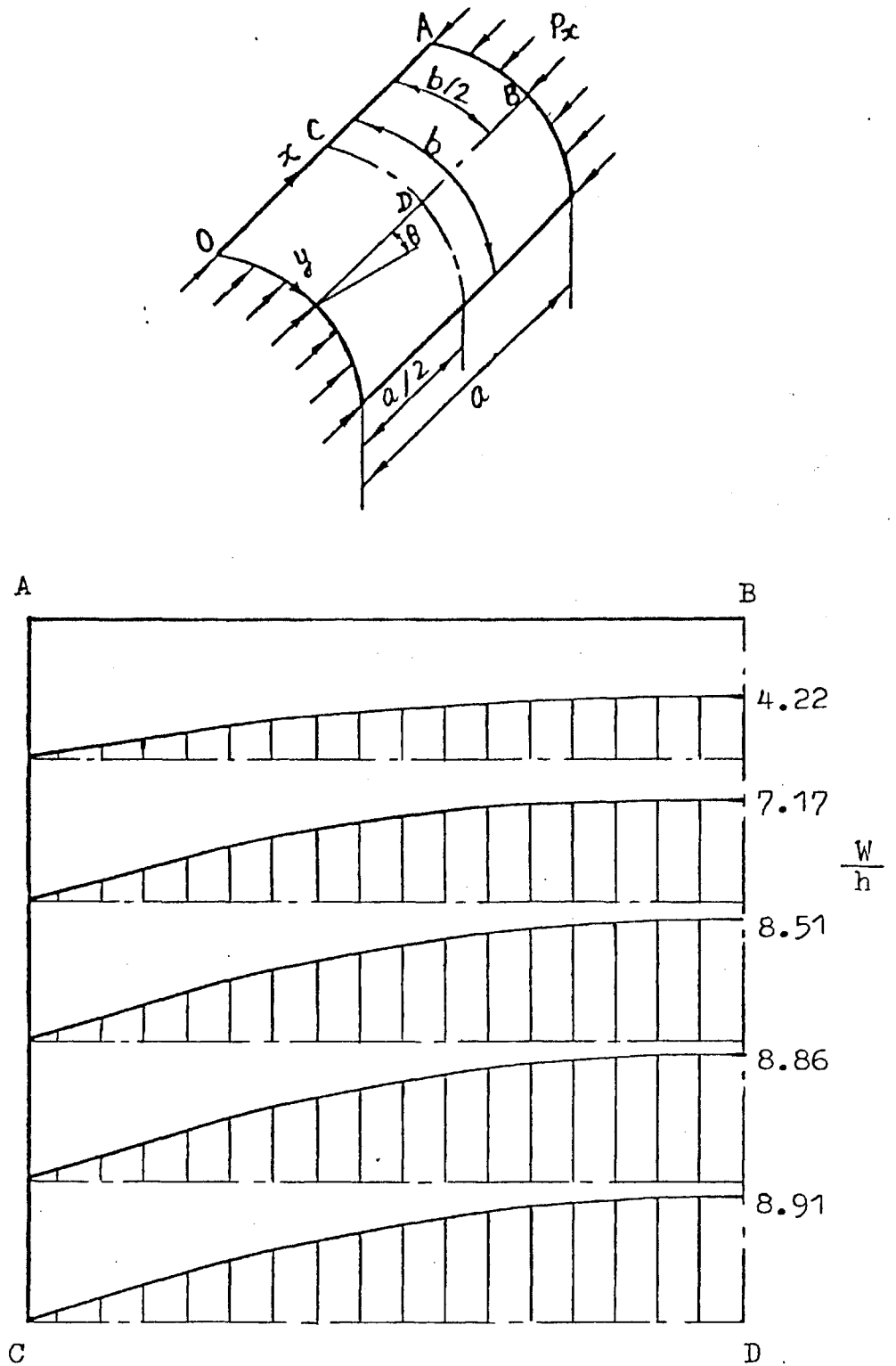


Fig. (IV,14) Distribution of deflection of a boron/epoxy panel; unidirectional layers at $\theta = 45^\circ$ and $K_R = 25$

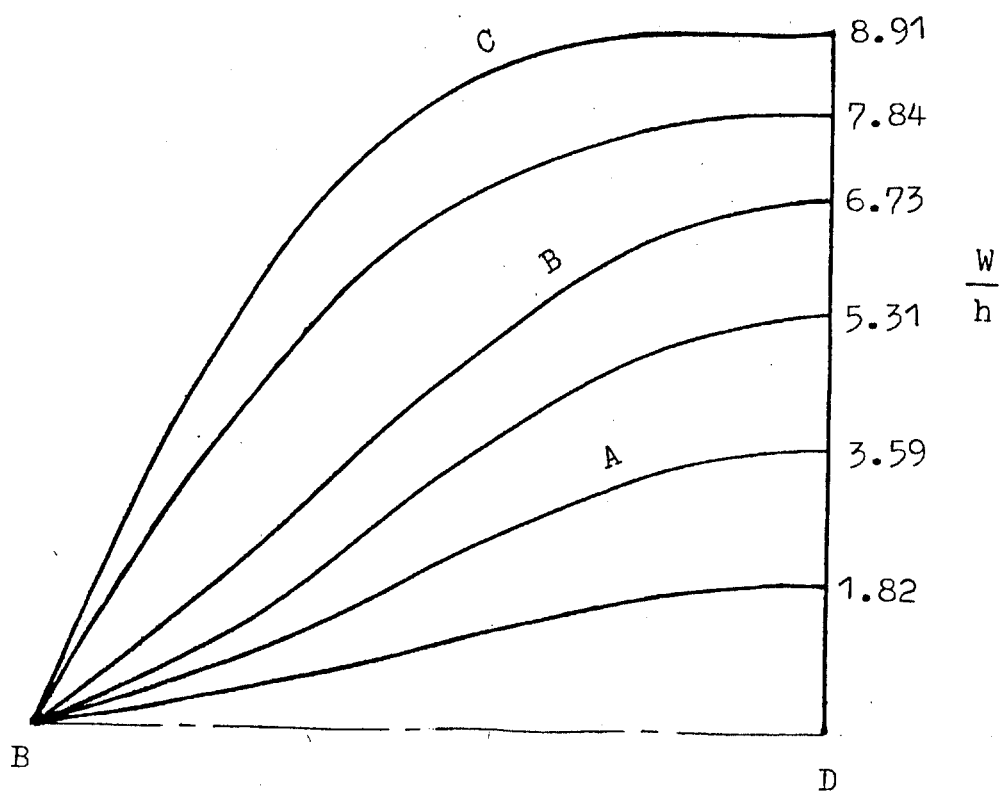
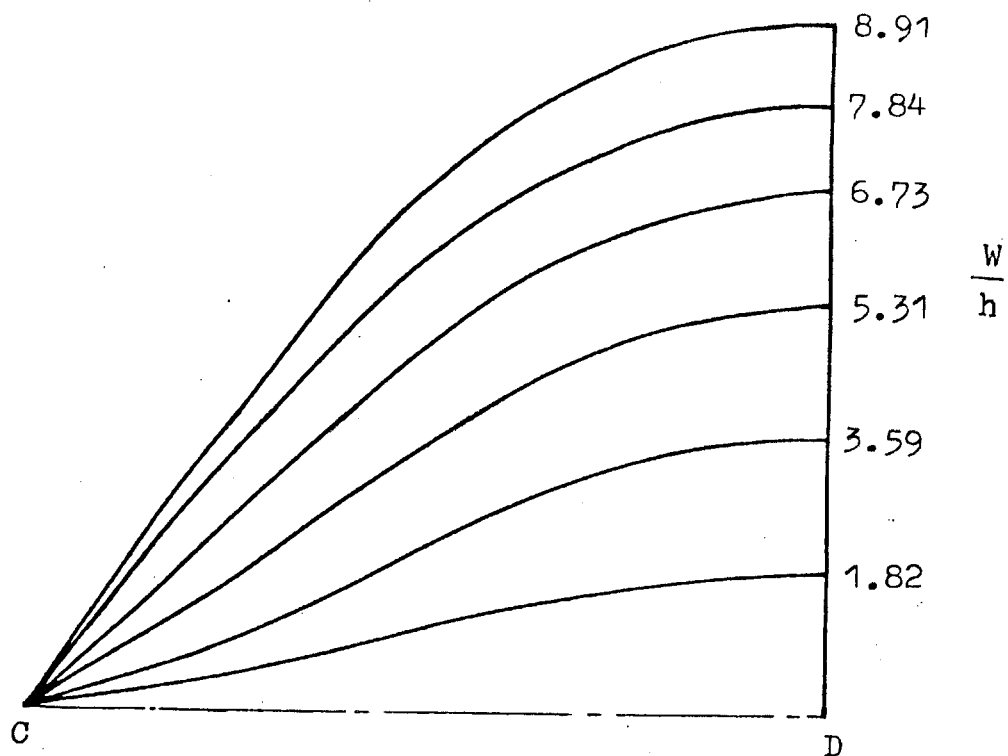


Fig. (IV,15) Development of deflection on a boron/epoxy panel; unidirectional layers at $\theta = 45^\circ$ and $K_R = 25$

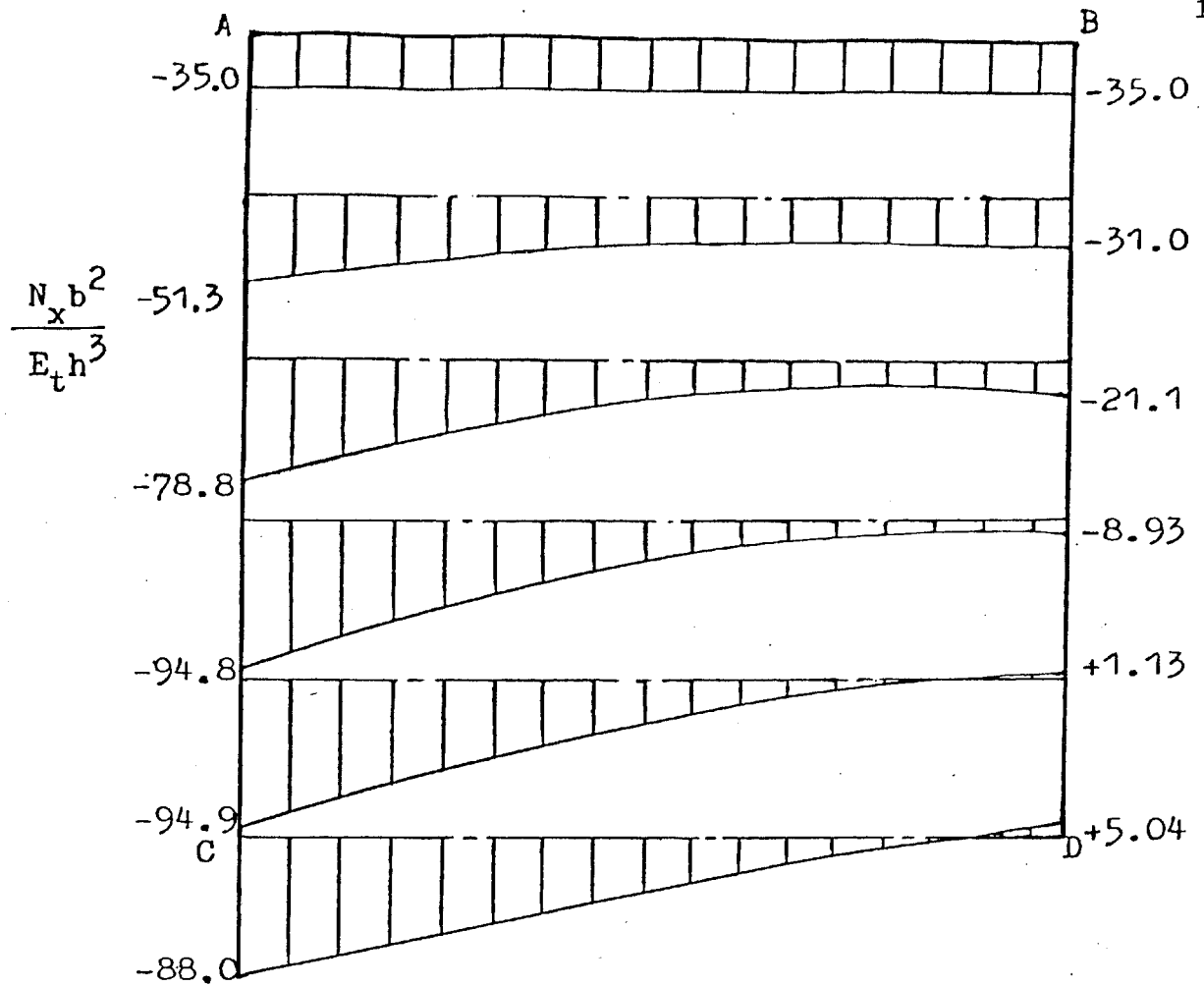


Fig. (IV, 16) Distribution and development of inner force N_x on a flat boron/epoxy panel; $(\pm 45^\circ)_s$ lay-up

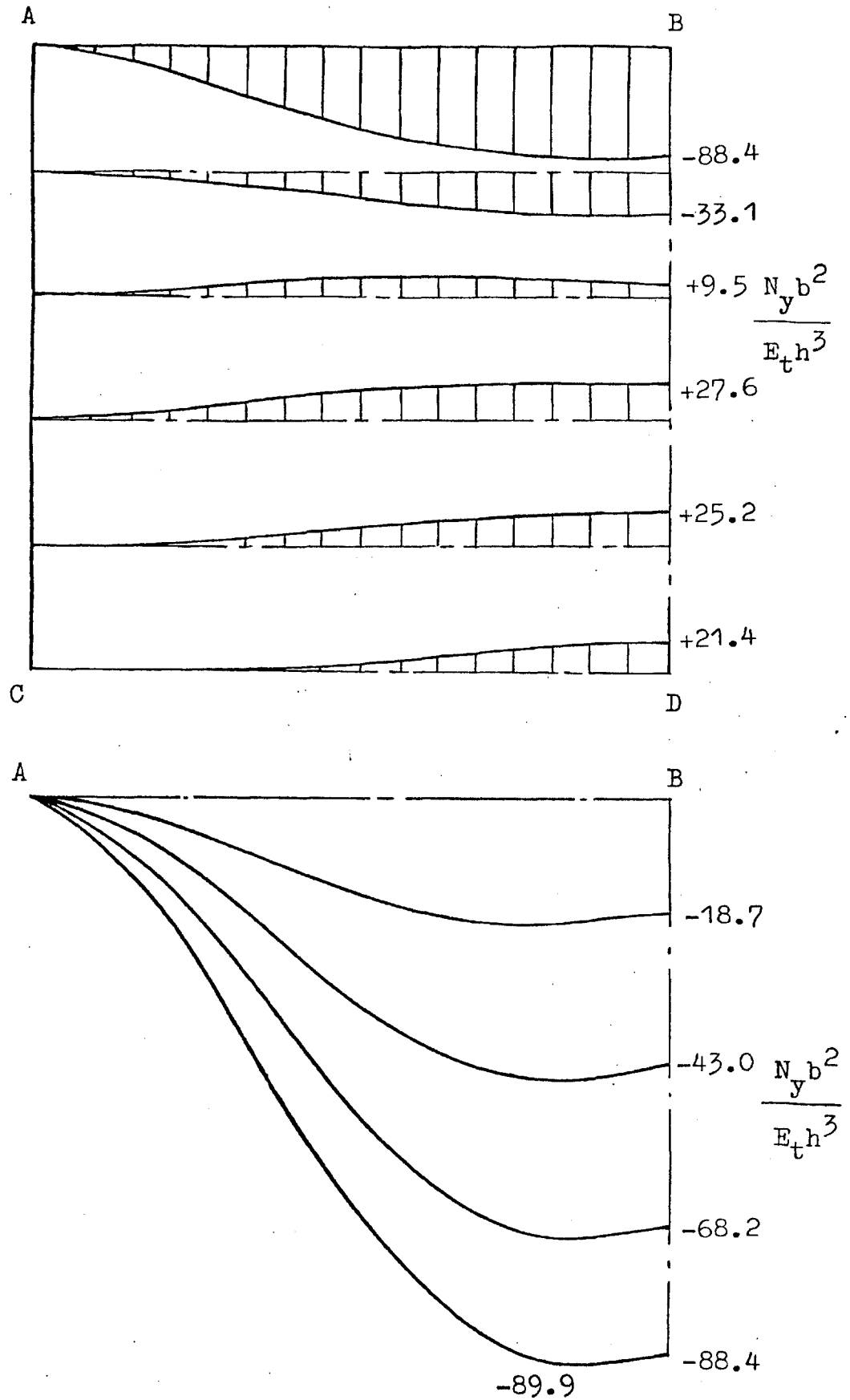


Fig. (IV,17) Distribution and development of inner force N_y on a flat boron/epoxy panel; $(\pm 45^\circ)_s$

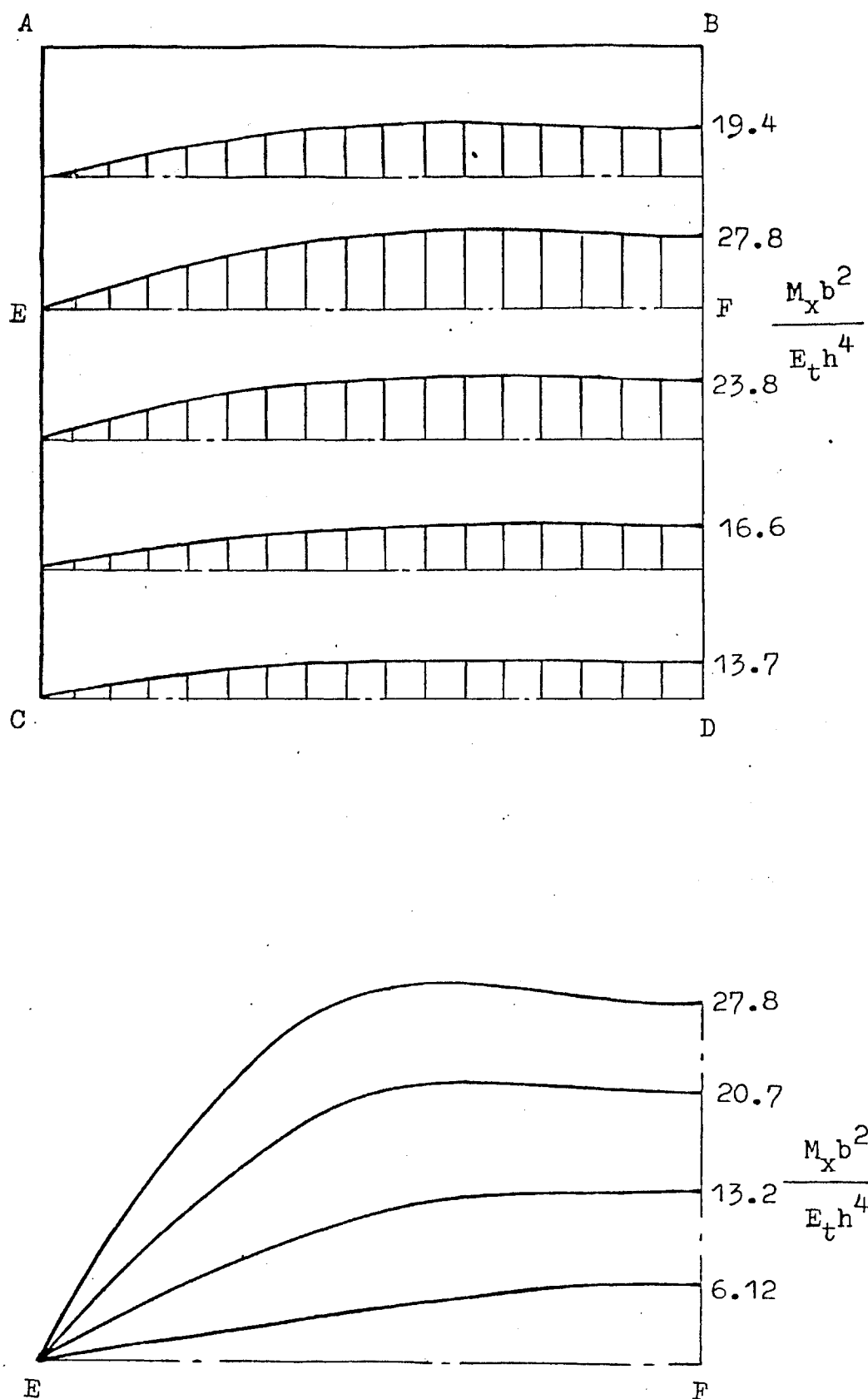


Fig. (IV,18) Distribution and development of inner moment M_x on boron/epoxy panel; ($\pm 45^\circ$, $\pm 45^\circ$) lay-up and $K_R = 25$

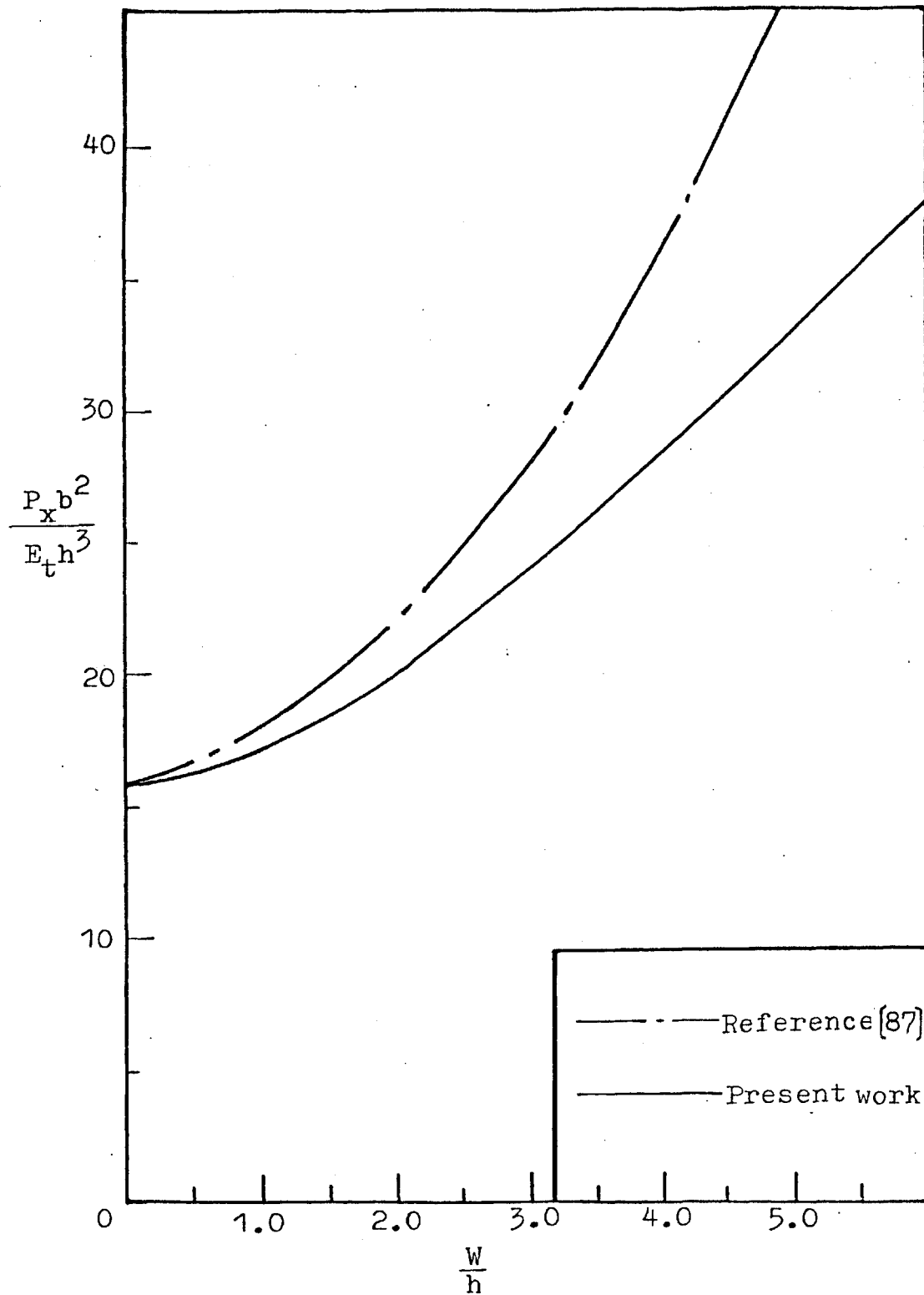


Fig. (IV,19) Variation of deflection at the centre of a panel; four antisymmetric layers ($\pm 45^\circ$, $\pm 45^\circ$), Comparison of the present solution with previous work

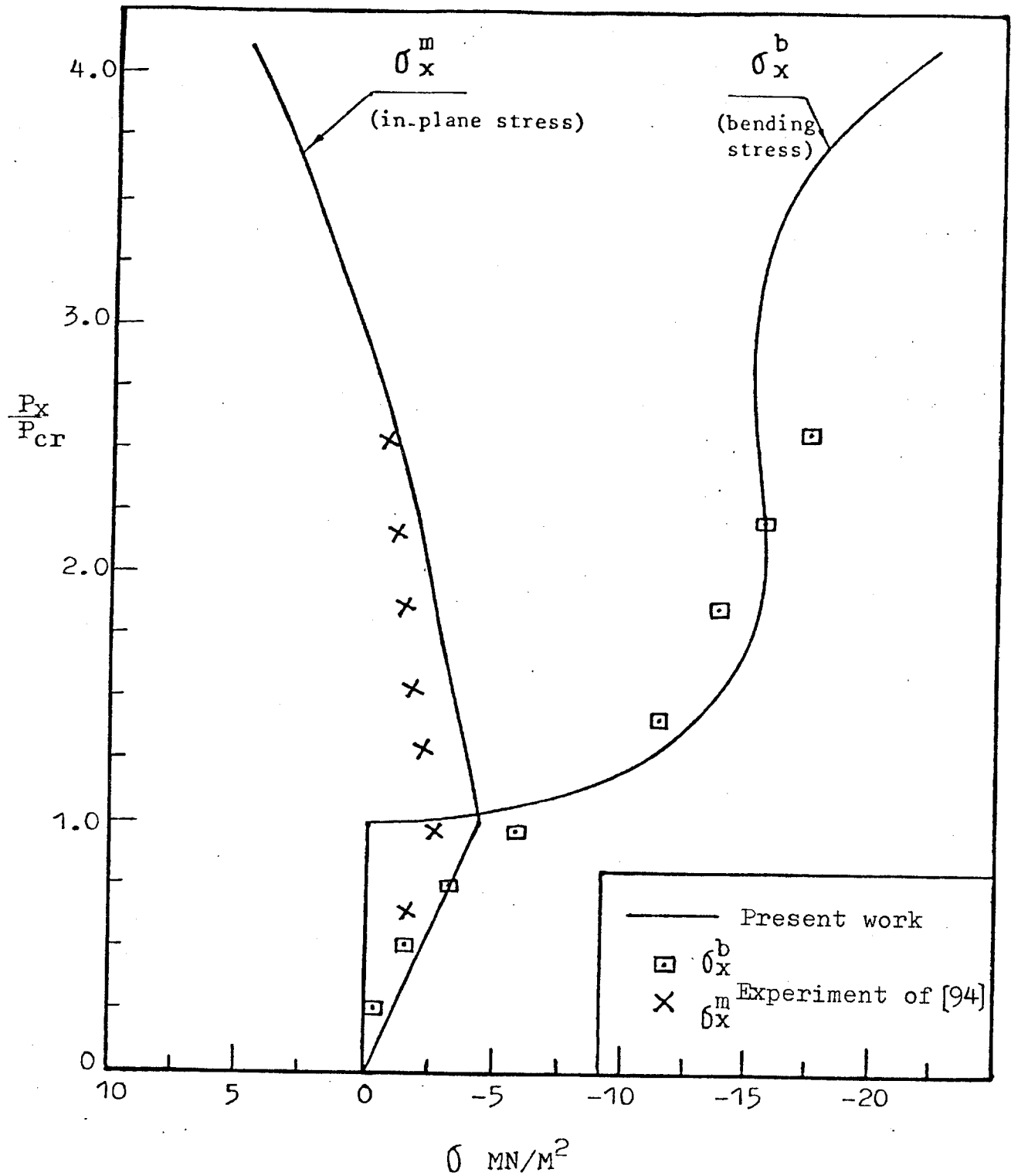


Fig. (IV,20) Variation of stress σ_x at the centre of a unidirectional glass-polyester composite plate. Comparison of the present solution with experimental data

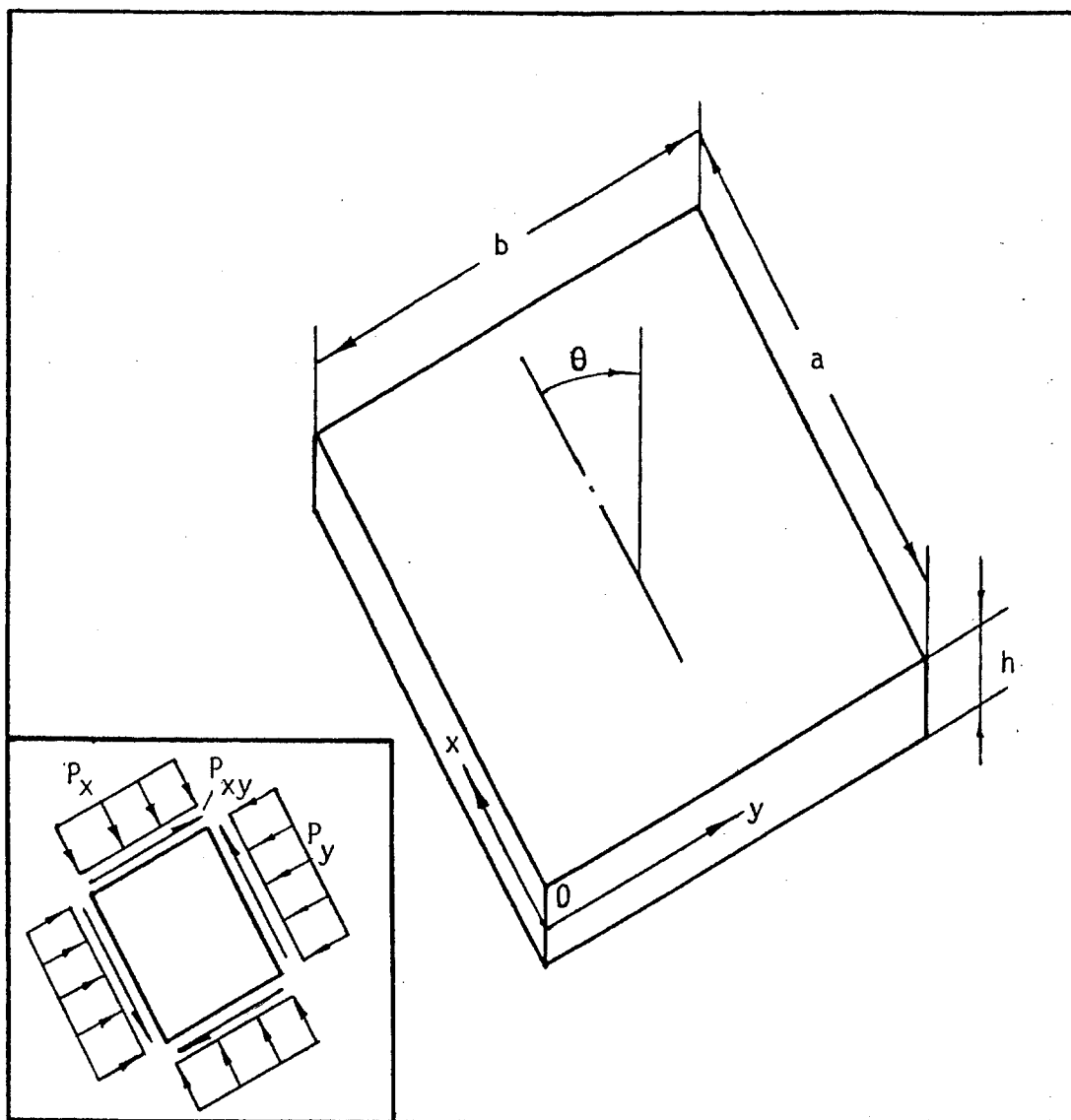


Fig. (IV,21) Positive loading and geometry of a flat laminated plate

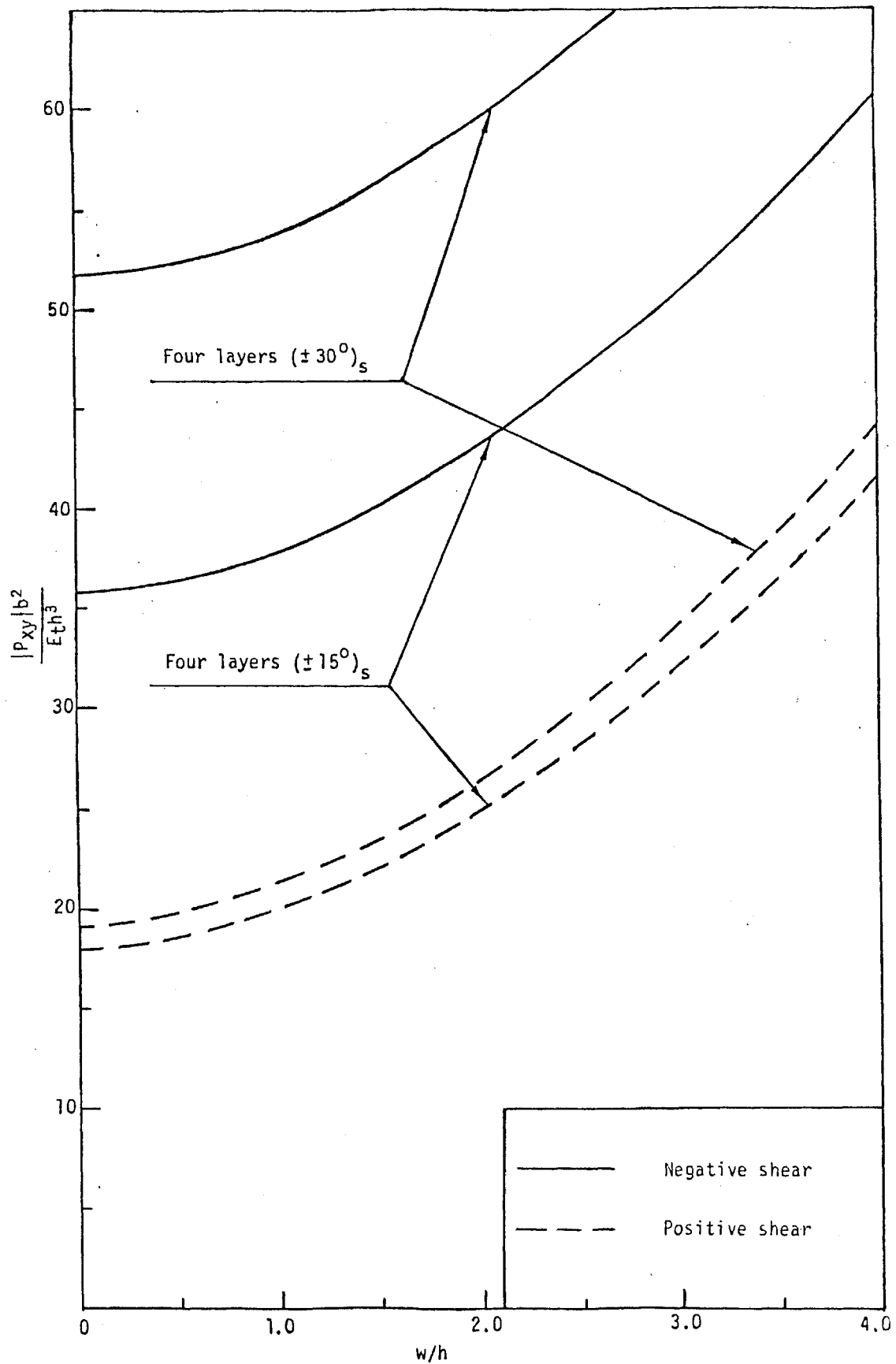


Fig. (IV,22) Shear loading - central deflection relations of square boron/epoxy plates; different fibre angles

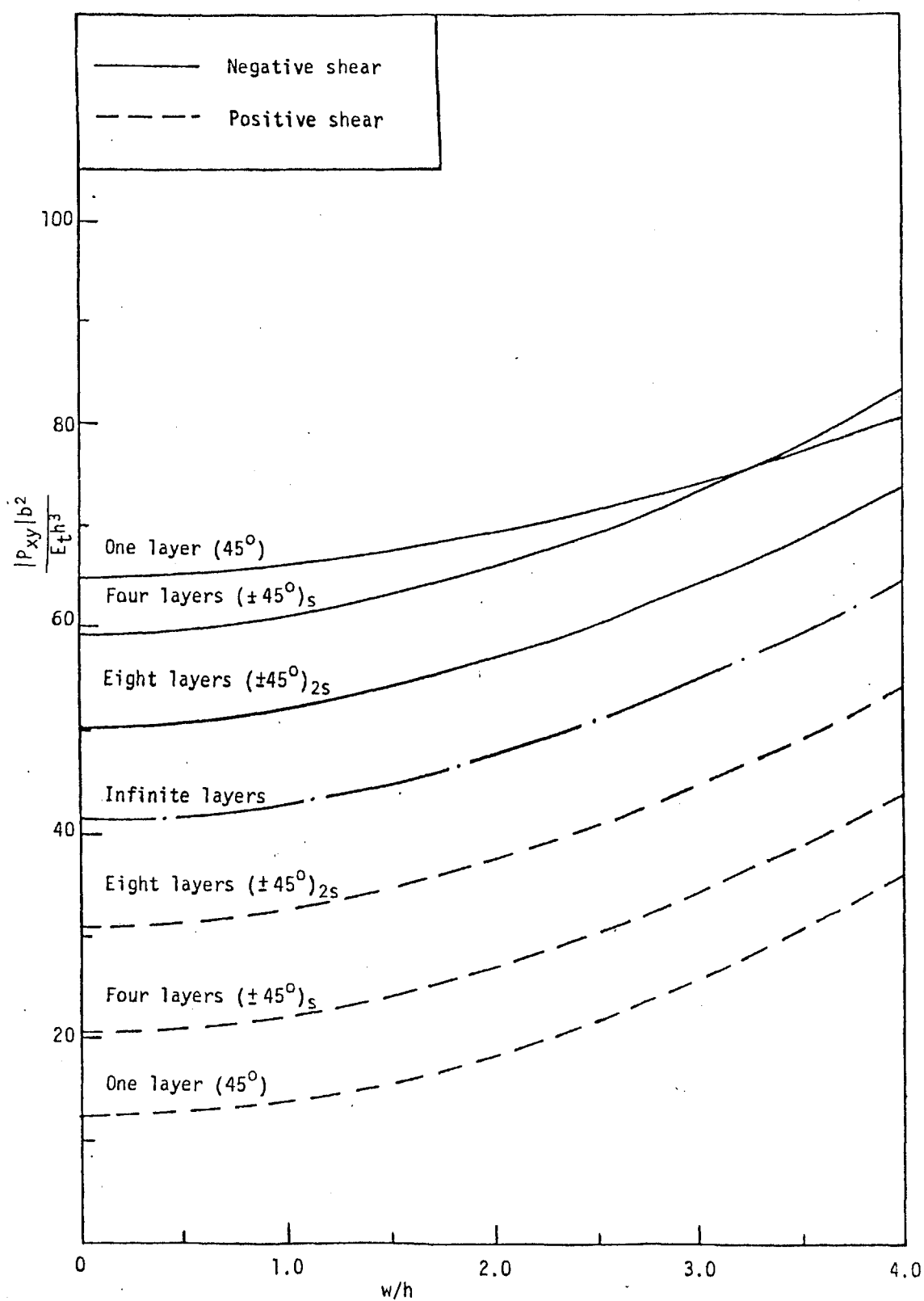


Fig. (IV,23) Shear loading - central deflection relations of square boron/epoxy plates; alternate $\pm 45^\circ$ layers

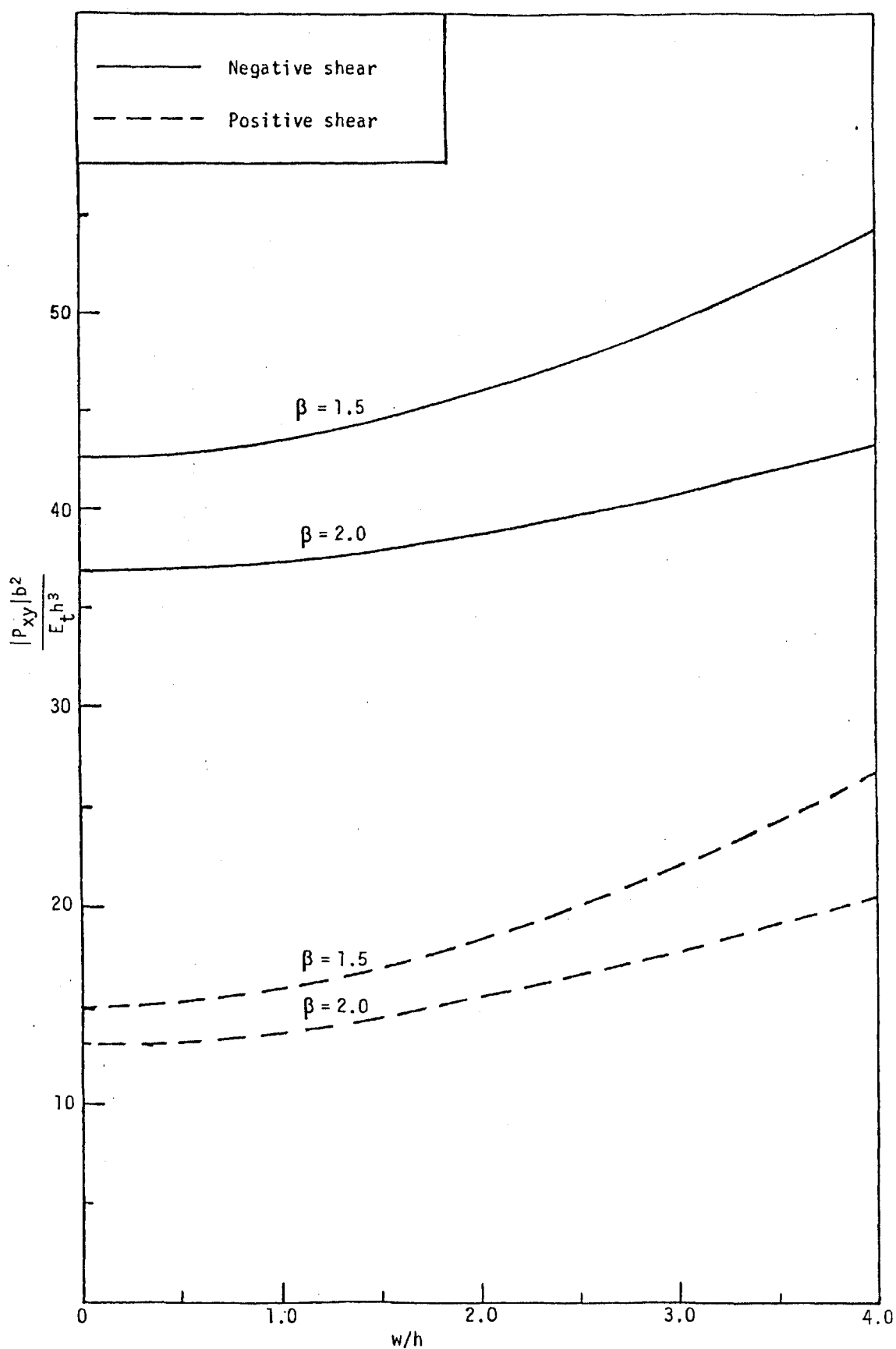


Fig. (IV,24) Shear loading - central deflection relations of boron/epoxy plates; different aspect ratios

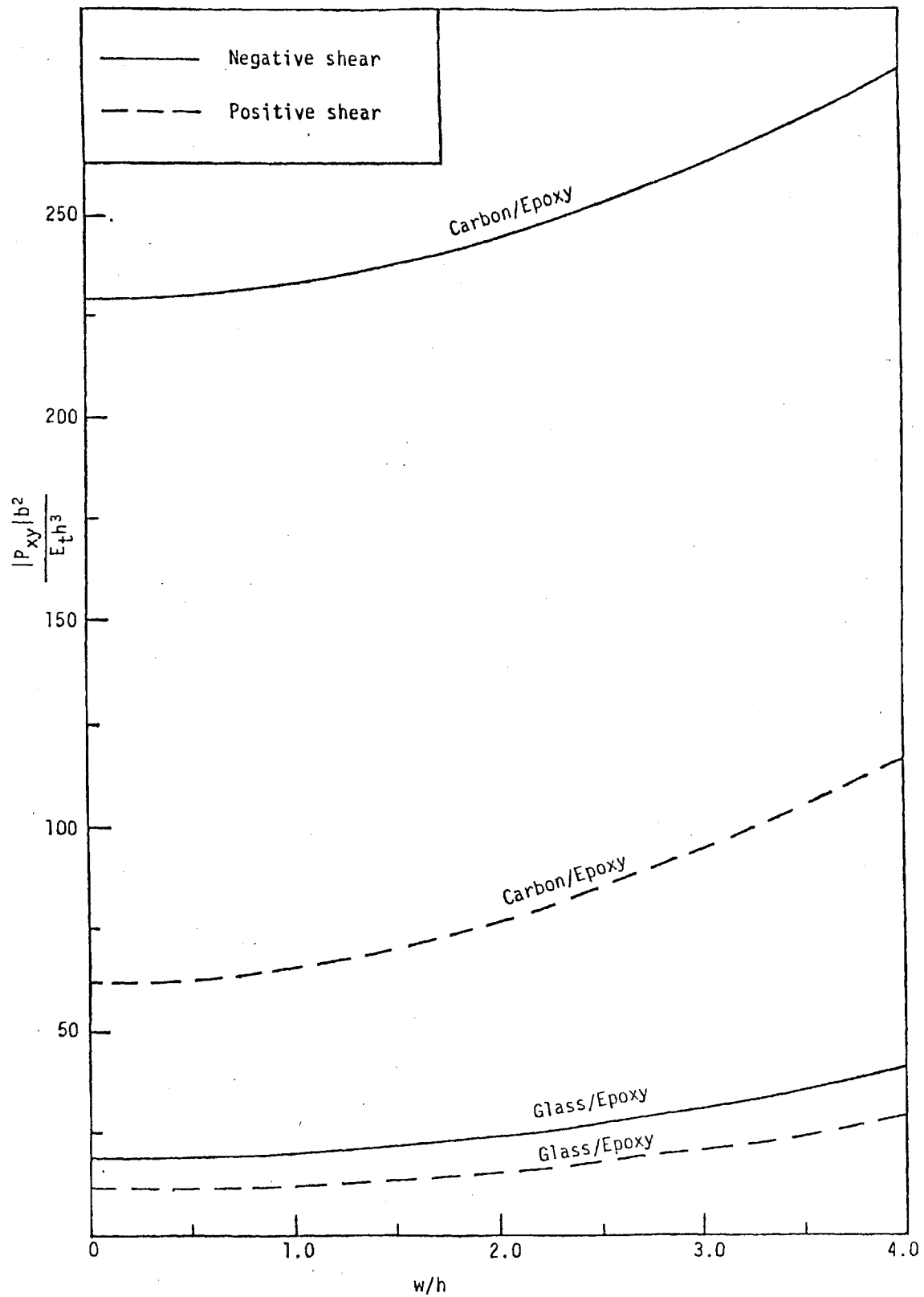


Fig. (IV,25) Shear loading - central deflection relations of square plates; different composite materials

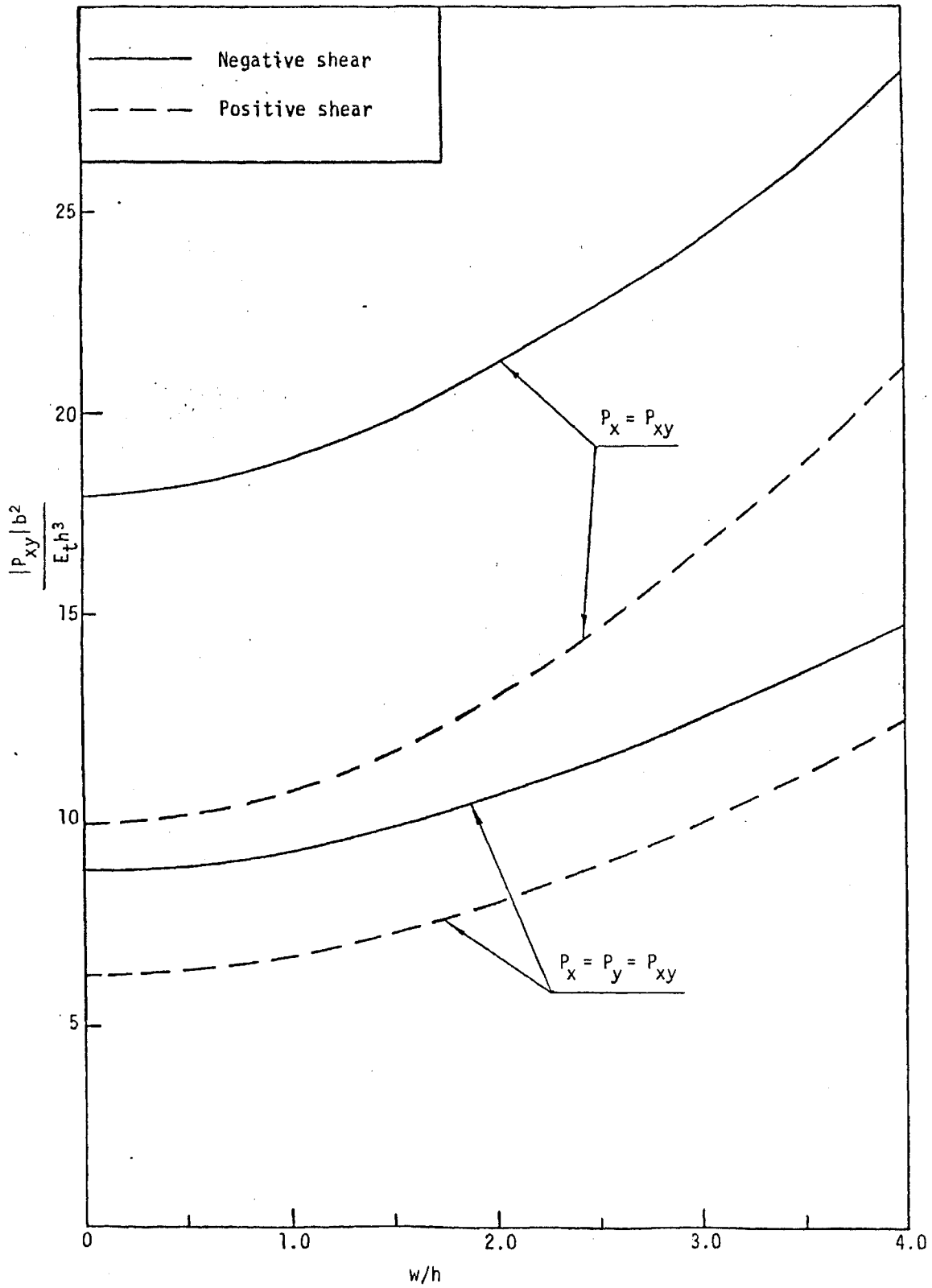


Fig. (IV,26) Loading - central deflection relations of square boron/epoxy plates under combined loading

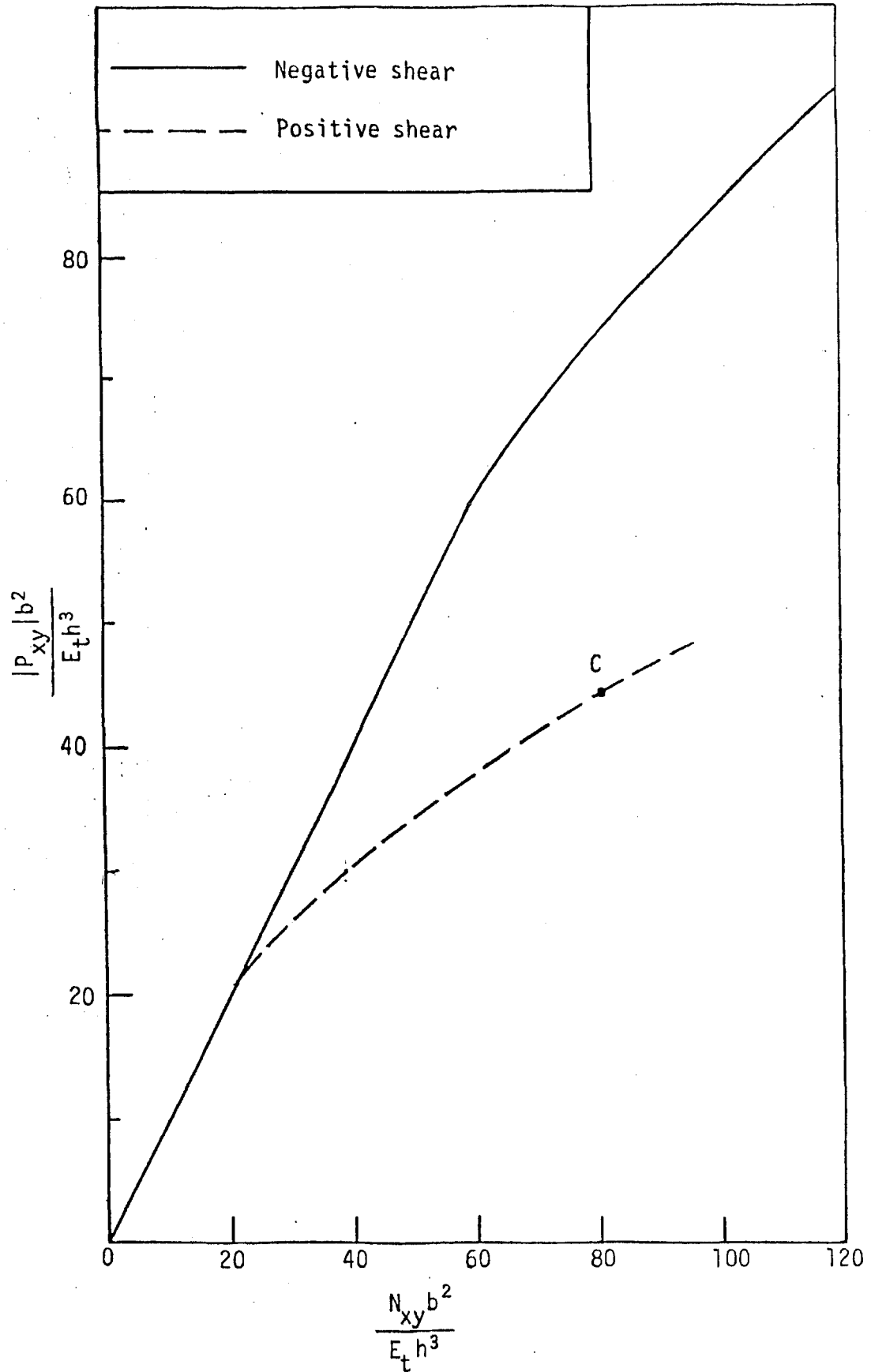


Fig. (IV,27) Variation of internal shear at the centre of square boron/epoxy plates with external shear loading

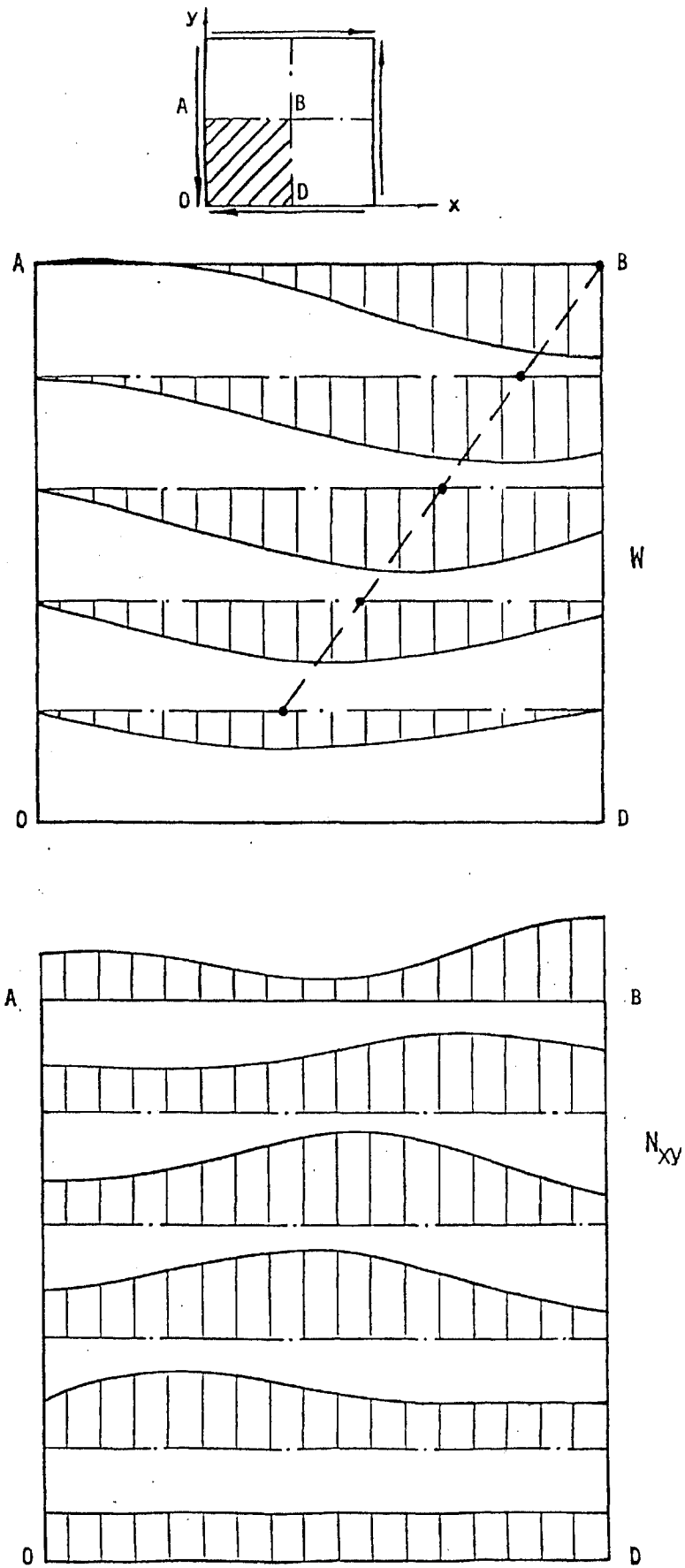


Fig. (IV,28) Distribution of lateral deflection and internal shear force on a quarter of a square boron/epoxy plate

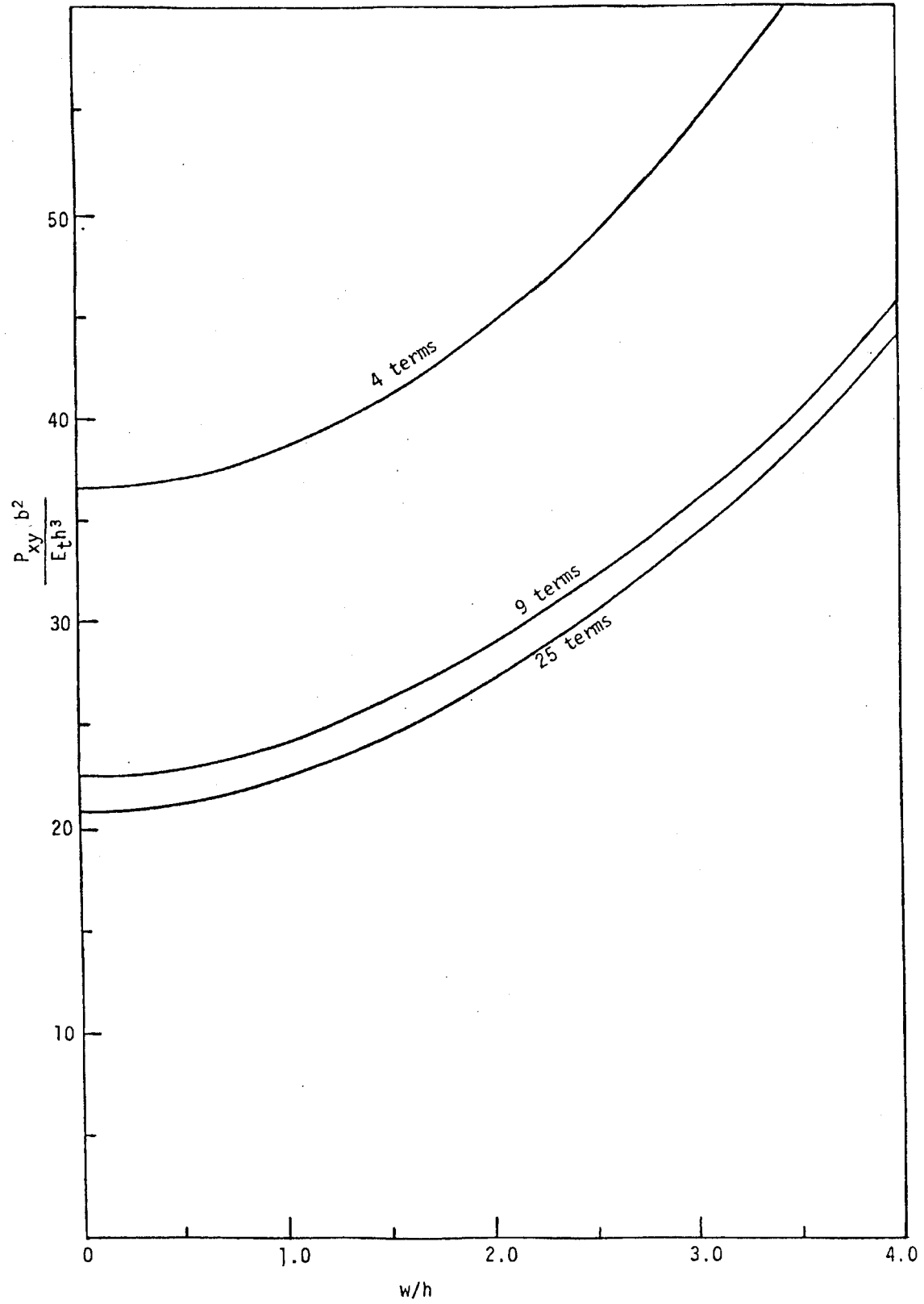


Fig. (IV,29) Comparison of the results obtained for different solution approximations

CHAPTER V

BENDING EFFECTS

A. General Remarks

In the last two chapters the bifurcational type of initial buckling and postbuckling behaviour of flat and curved composite panels under in-plane loading have been studied. But the bending effect of the panels is not always able to be ignored. In certain practical applications bending takes place immediately the in-plane loading is applied to the panels. In such cases, the buckling and postbuckling behaviour may not be studied correctly and completely without considering the bending effect during the loading process. In this chapter three different kinds of bending effects are investigated below. They are the bending effects caused by unsymmetric layers in the panel, initial geometrical imperfections of the panel and lateral loading.

[89]

In 1976 Prabhakara investigated the bending behaviour of unsymmetric crossply plates under in-plane loading. It was pointed out by him that the coupling phenomenon is such that there is bending of such laminates when subjected to only in-plane forces. But his work is only for flat plates under compressive loading. So far there seems to be no investigation existing which deals with the bending effect of panels under in-plane shear. In the current investigation the bending effects of both flat plates and curved panels under compression and shear are considered. For laminated curved panels with unsymmetric crossply arrangements under compressive load we find that a snap-through may occur, following the bending of

the panel in the loading process, and the maximum load corresponding to the snap-through is much less than the bifurcational load. For shear loading we find that bending of the laminated plates starts from the beginning of the loading process if the plates have unsymmetric angle-ply arrangements.

In this chapter bending effects caused by initial geometrical imperfection and lateral loading are also discussed in detail. It is seen that the initial irregularity of geometry has significant influence on the buckling and post-buckling behaviour of curved panels. The initial imperfection may greatly reduce the resistance of curved panels to collapse. For combined in-plane and lateral loading it is found that alternate applied shear directions result in different bending behaviour of the panels under the combination of lateral and in-plane loading. This is shown to be true for laminated panels with symmetric and asymmetric lay-ups, but not for antisymmetric lay-ups. It seems that no previous work has discussed these effects for general laminated panels.

B. Bending Effect in Unsymmetric Laminated Panels under In-Plane Loading

In Chapter IV, the postbuckling behaviour of laminated curved panels under compression and shear was investigated. As we mentioned before, the analysis presented there may be used for panels with general lay-up arrangements.

Recalling equation (IV,19) in the last chapter, we have

$$\left\{ [B^{(1)}] - K[B^{(2)}] \right\} \{ w^{(1)} \} + \left\{ [B^{(3)}] - K[B^{(4)}] \right\} \{ w^{(2)} \} = K \left[\mu_x \{ M_0 \} + \mu_s \{ O_0 \} \right]$$

$$\left\{ [C^{(1)}] - K[C^{(2)}] \right\} \{ w^{(1)} \} + \left\{ [C^{(3)}] - K[C^{(4)}] \right\} \{ w^{(2)} \} = 0 \quad (V, 1)$$

On the right hand side of the first one of the last set of equations there is a term of the form $K[\mu_x\{M_o\}+\mu_s\{O_o\}]$, which involves neither deflection function W nor stress function F . This term is clearly proportional to the in-plane loading parameter K defined by

$$\begin{aligned}\mu_x K &= K_x = \frac{P_x b^2}{E_t h^3} \\ \mu_s K &= K_s = \frac{P_{xy} b^2}{E_t h^3}\end{aligned}\tag{V,2}$$

in which μ_x and μ_s are constants determined by the specified loading ratio.

It may be expected that when in-plane load is applied to the panel, the deflection w may occur immediately if $\{M_o\}$ and $\{O_o\}$ are not all zero. The elements of $\{M_o\}$ and $\{O_o\}$ can be obtained from Appendix E; they take the form:

$$\begin{aligned}M_{o_I} &= \frac{E_t h}{A_{22}} (b_{10} L_o^i T_1^j + b_9 L_o^j T_1^i) \\ O_{o_I} &= \frac{E_t h}{A_{22}} (b_{11} L_o^i T_1^j + b_{14} L_o^j T_1^i)\end{aligned}\tag{V,3}$$

where M_{o_I} and O_{o_I} are the I^{th} elements in the column matrices M_o and O_o respectively. The relation between the indices I and i, j is

$$I = (i-1)M + j$$

with M representing the constants referring to how many terms are used in the assumed solutions (IV,4).

L_o^i and T_1^j in equations (V,3) are constants only, which are given in Appendix F.

b_i are defined in the expressions (II,26). They take

the form:

$$\begin{aligned}
 b_9 &= \beta^4 \bar{B}_{12}^* / \bar{D}_{11}^* = \beta^4 B_{12}^* A_{22} h / D_{11}^* \\
 b_{10} &= \beta^2 \bar{B}_{11}^* / \bar{D}_{11}^* = \beta^2 B_{11}^* A_{22} h / D_{11}^* \\
 b_{11} &= -\beta \bar{B}_{61}^* / \bar{D}_{11}^* = -\beta B_{61}^* A_{22} h / D_{11}^* \\
 b_{14} &= -\beta^3 \bar{B}_{62}^* / \bar{D}_{11}^* = -\beta^3 B_{62}^* A_{22} h / D_{11}^*
 \end{aligned}
 \tag{V, 4}$$

Now, it can be concluded that if the stretching-bending coupling terms B_{11}^* and B_{12}^* of the panel are not equal to zero, $\{M_o\}$ will not be zero and bending will occur when axial compression is applied to them. At the same time, if the coupling terms B_{61}^* and B_{62}^* are not equal to zero, $\{0_o\}$ will not be zero and bending of the panels will exist when in-plane shear loading is applied. In those cases the behaviour of the panels under in-plane loading is no longer of the bifurcational type, and the load-deflection relations are curved directly from the origin. In order to obtain the correct convergent results, in contrast to what is done in the last chapter, the undeformed state of the panel (zero loading and zero deflection) is taken as the starting point, instead of the bifurcational point, for iterative computation. The iterative scheme itself and subsequent extrapolation scheme described in Chapter IV can still be used here. But the step length of the prescribed component of the deflection function must be selected very carefully at the beginning of computation. This is because the load-deflection curves resulting from the bending effect, in many cases, are very steep at the beginning of the loading process. Too large a step length may cause the iterative scheme to fail, and too

small a step length results in high computing cost. The step lengths used in the examples vary from panel to panel with a range of 0.001-0.5.

As is very well known, for panels with antisymmetric cross-ply arrangements B_{11}^* is not equal to zero. Therefore, bending occurs when the panels are subjected to axial compression. In this investigation the bending effect of such panels is considered. At the same time, general crossply situations are discussed in examples as well. In that case B_{11}^* and B_{12}^* are both non-zero.

It is known that for the panels with antisymmetric angle-ply layers B_{61}^* and B_{62}^* are not zero. In examples, the bending effect of such panels under shear loading is also investigated.

Computation is carried out for boron/epoxy and carbon/epoxy. The properties of these materials used for calculations are the same as in the previous chapters, which may be found in Table (III,2). The panels are assumed to have length a and arc b both equal to 250 mm. The total thickness of the panels is kept to 2.5 mm, no matter how many layers are in the panels.

In the computation four-term approximation for the stress function F and eight-term approximation for the deflection function w are used. Because there is no term involving the coupling stiffness D_{16} and D_{26} in the examples, such approximations for F and W are enough to show the tendency of bending effects in the panels, whilst still keeping the computing cost within reasonable limits.

The results of computation are summarised in Figs. (V,1) to (V,7). In all these figures the influence of B_{ij}^* terms can be seen. The dashed curves marked $B_{ij}^* = 0$ represent the approximate results obtained by deleting B_{ij}^* terms. As mentioned before

in some previous work, it is often suggested that these terms be deleted, although they are not zero in realistic panels. In this investigation big differences have been found for the behaviour of the panels with or without such terms.

Fig. (V,1) is obtained for panels with two layers (90° , 0°) of boron/epoxy composite under compression. The load-central deflection results of flat plates and curved panels are shown. The dashed lines represent the results of the same panels with zero coupling terms B_{ij} . It can be clearly seen that because of coupling terms B_{ij} ($B_{11} \neq 0$, $B_{12} \neq 0$) the behaviour of the laminated panels under in-plane loading becomes different. Bending exists from the beginning of the loading process. For a flat plate ($K_R = 0$) with such a lay-up the deflection increases monotonically with increasing load. It means that for such a plate under axial compression, bending occurs instead of buckling. But for a curved panel ($K_R = 25$) a snap-through may occur when the applied compression reaches a certain level. It is worth pointing out, however, that because of the coupling rigidities B_{ij} , the maximum load for snap-through is much lower than the bifurcation load, i.e. these terms greatly reduce the resistance of the panel against collapse.

Figs. (V,2) - (V,6) are all obtained for carbon/epoxy panels under axial compression. The load-central deflection relations are shown on the figures. Fig. (V,2) repeats Fig. (V,1) except for a different composite material. The same phenomena are seen in Fig. (V,2), though the absolute values of load-deflection are different in both cases.

Fig. (V,3) shows the effect of the number of layers in carbon/epoxy panels with $K_R = 25$, which are arranged

alternately at 90° and 0° . It is found that the bending effect decreases with increasing number of layers. That is, the more alternate layers in the panel, the higher the performance of the panel, and the less the bending effect is involved. Fig. (V,4) shows the influence of panel curvature. The figure is obtained for two layer (90° , 0°) carbon/epoxy panels. It is seen that the maximum snap-through load decreases when the panel becomes less curved and no snap-through load is found when the panel becomes very flat (say, $K_R = 12.5$). Figs. (V,5) and (V,6) are obtained for four layer carbon/epoxy panels having general crossply arrangements. It can be seen that for such panels, the coupling terms B_{ij} also have significant influence. The bending effect of such panels resulting from these terms is sometimes even more severe than that for anti-symmetric crossply arrangements.

Fig. (V,7) is obtained for flat carbon/epoxy plates with antisymmetric angle-ply arrangements. The effect of increasing the number of alternate layers on the behaviour of the plates under shear is demonstrated. The load-central deflection curves indicate that for plates with such lay-ups, bending occurs immediately in-plane shear loading is applied. For the two layer arrangement the bending effect is most severe, but decreases as the number of alternate layers increases. For plates with different composite materials, different curvatures and different lay-ups, similar investigations to those for compressive loading can be done for shear loading and the similar conclusions may be drawn to those from Figs. (V,1) - (V,6). In order to limit the computing cost, such investigations are not included in this work.

From these examples the following conclusions may be

summarised:

(i) Bending effects exist in the composite panels with unsymmetric crossply arrangements under in-plane compression during loading.

(ii) The bending effect is also found to exist when panels under shear loading have antisymmetric angle-ply arrangements.

(iii) The bending effect may be reduced by increasing the number of alternate layers in the panel while keeping constant thickness. The two layer unsymmetric arrangement results in the lowest performance of the panel.

C. Bending Effect of Imperfect Panels of Laminated Composites

Since Donnell and Wan^[16] did an investigation on the effect of imperfection on the buckling of thin cylinders under axial compression the sensitivity of thin plates and cylinders to imperfections of geometry has been studied by a number of investigators^{[88][116][117][118][119]}. It is indicated that for isotropic and orthotropic plates and shells the bending effect is induced by geometric imperfection when the plates and shells are subjected to compression. Naturally, it may also be expected that geometric imperfection may have certain influence on the buckling and postbuckling behaviour of curved panels of generally laminated composite materials. The author has found no analysis available on this subject.

In Chapter II we developed a theory for the postbuckling behaviour of curved laminated panels with perfect and imperfect geometry. In Chapter IV, a detailed investigation is presented for postbuckling behaviour of perfect panels. Following the same procedure as in Chapter IV, here we investigate the effect of imperfections on the postbuckling behaviour of the panels.

The governing equations of this problem in terms of non-dimensional parameters are given in (II,33) and (II,34).

The geometrical imperfection is assumed to have a form described by the following double sine series in terms of non-dimensional coordinates

$$\bar{W} = \sum_m \sum_n \bar{W}_{mn} \sin m\pi\xi \sin n\pi\eta \quad (V,5)$$

$(m, n = 1, 2, \dots)$

With the coefficients $\bar{W}_{mn}$ given, the initial imperfection in the panel is prescribed. The imperfection described by the expression (V,5) is very general.

The same solutions to the governing equations (II,33) and (II,34) as those for perfect panels are taken here, i.e.

$$F = -\bar{N}_x \frac{\eta^2}{2} - S\beta\xi\eta + \sum_m \sum_n F_{mn} X_m(\xi) Y_n(\eta) \quad (V,6)$$

$$W = \sum_p \sum_q \left[W_{pq} \sin p\pi\xi \sin q\pi\eta + W_{pq}^{(2)} \sin^2 p\pi\xi \sin^2 q\pi\eta \right] \quad (m, n, p, q = 1, 2, \dots, \infty)$$

Substituting expressions (V,5) and (V,6) into the governing system, (II,33) and (II,34), following the same procedure as that presented in section B of Chapter IV, the resultant equations in matrix form, including the effect of geometric imperfection, may be obtained as follows:

$$\begin{aligned} [A] \{F\} &= [B] \{W^{(1)}\} - [C] \{W^{(2)}\} + [T_W^{(1)}] [D] \{W^{(1)}\} \\ &+ [T_W^{(2)}] [E] \{W^{(1)}\} + [T_W^{(1)}] [F] \{W^{(2)}\} \\ &+ [T_W^{(2)}] [G] \{W^{(2)}\} + \underline{[T_W^{(0)}] [\Lambda^{(1)}] \{W^{(1)}\}} \\ &\underline{+ [T_W^{(0)}] [\Lambda^{(2)}] \{W^{(2)}\}} \end{aligned} \quad (V,7)$$

$$\begin{aligned}
[H] \{w^{(1)}\} + [I] \{w^{(2)}\} + [J] \{F\} &= [T_w^{(1)}] [K] \{F\} \\
&+ [T_w^{(2)}] [L] \{F\} + K_x \{M_o\} + K_x [M] \{w^{(1)}\} - K_x [N] \{w^{(2)}\} \\
&+ K_s \{O_o\} + K_s [O] \{w^{(1)}\} + K_s [P] \{w^{(2)}\} \\
&\underline{+ K_x [\wedge^{(3)}] \{w^{(0)}\} + K_s [\wedge^{(4)}] \{w^{(0)}\} + [T_w^{(0)}] [\wedge^{(5)}] \{F\}}
\end{aligned} \tag{V, 8}$$

$$\begin{aligned}
[Q] \{w^{(1)}\} + [R] \{w^{(2)}\} + [S] \{F\} &= [T_w^{(1)}] [T] \{F\} \\
&+ [T_w^{(2)}] [U] \{F\} + K_x [V] \{w^{(1)}\} - K_x [X] \{w^{(2)}\} \\
&+ K_s [Y] \{w^{(1)}\} + K_s [Z] \{w^{(2)}\} \\
&\underline{+ K_x [\wedge^{(6)}] \{w^{(0)}\} + K_s [\wedge^{(7)}] \{w^{(0)}\} + [T_w^{(0)}] [\wedge^{(8)}] \{F\}}
\end{aligned} \tag{V, 9}$$

The definition of most of the matrices and parameters may be found in Chapter IV and Appendix E. They are the same as for perfect panels discussed before.

The effect of imperfection is represented by the terms which are underlined.

The elements of the matrices $\{w^{(0)}\}$ and $[T_w^{(0)}]$ are the coefficients in the imperfection expression (V,5).

$$W_I^{(0)} = \overline{W}_{mn} \tag{V, 10}$$

where $I = (m - 1)M + n$ with M referring to how many terms are taken in (V,5) or (V,6) and

$$[T_w^{(0)}] = \begin{bmatrix} \{w^{(0)}\}^t & & & \\ & \{w^{(0)}\}^t & & \\ & & \ddots & \\ & & & \{w^{(0)}\} \end{bmatrix} \tag{V, 11}$$

The elements of the matrices $[\Lambda^{(i)}]$ may be found in Appendix G.

Solving $\{F\}$ from equation (V,7), substituting the result into equations (V,8) and (V,9), a set of equations for computation is established as follows:

$$[E^{(1)}] \{W^{(1)}\} + [E^{(2)}] \{W^{(2)}\} = K[\bar{\Lambda}^{(1)}] \{W^{(0)}\} + K\{H^{(1)}\} \quad (V,12)$$

$$[E^{(3)}] \{W^{(1)}\} + [E^{(4)}] \{W^{(2)}\} = K[\bar{\Lambda}^{(2)}] \{W^{(0)}\}$$

where

$$[E^{(1)}] = [B^{(1)}] - K[B^{(2)}]$$

$$[E^{(2)}] = [B^{(3)}] - K[B^{(4)}]$$

$$[E^{(3)}] = [C^{(1)}] - K[C^{(2)}]$$

$$[E^{(4)}] = [C^{(3)}] - K[C^{(4)}]$$

(V,13)

$$[H^{(1)}] = \mu_x \{M_0\} + \mu_s \{O_0\}$$

$$[\bar{\Lambda}^{(1)}] = \mu_x [\Lambda^{(3)}] + \mu_s [\Lambda^{(4)}]$$

$$[\bar{\Lambda}^{(2)}] = \mu_x [\Lambda^{(6)}] + \mu_s [\Lambda^{(7)}]$$

and

$$[B^{(1)}] = [H] + ([J] - [T_w^{(1)}][K] - [T_w^{(2)}][L] - [T_w^{(0)}][\Lambda^{(5)}])[A^{(1)}]$$

$$[B^{(2)}] = \mu_x [M] + \mu_s [O]$$

$$[B^{(3)}] = [I] - ([J] - [T_w^{(1)}][K] - [T_w^{(2)}][L] - [T_w^{(0)}][\Lambda^{(5)}])[A^{(2)}]$$

$$[B^{(4)}] = -\mu_x [N] + \mu_s [P]$$

$$\begin{aligned}
 [C^{(1)}] &= [Q] + ([S] - [T_W^{(1)}][T] - [T_W^{(2)}][U] \\
 &\quad - [T_W^{(0)}][\Lambda^{(8)}])[A^{(1)}] \\
 [C^{(2)}] &= \mu_x [V] + \mu_s [Y] \\
 [C^{(3)}] &= [R] + ([S] - [T_W^{(1)}][T] - [T_W^{(2)}][U] \\
 &\quad - [T_W^{(0)}][\Lambda^{(8)}])[A^{(2)}] \\
 [C^{(4)}] &= -\mu_x [X] + \mu_s [Z] \\
 [A^{(1)}] &= [A]^{-1} ([B] + [T_W^{(1)}][D] + [T_W^{(2)}][E] + [T_W^{(0)}][\Lambda^{(1)}]) \\
 [A^{(2)}] &= [A]^{-1} (-[C] + [T_W^{(1)}][F] + [T_W^{(2)}][G] + [T_W^{(0)}][\Lambda^{(2)}])
 \end{aligned}
 \tag{V, 14}$$

In expressions (V,12) (V,13) and (V,14) μ_x , μ_s and K are defined by

$$\begin{aligned}
 \mu_x K &= K_x = \frac{P_x b^2}{E_t h^3} \\
 \mu_s K &= K_s = \frac{P_{xy} b^2}{E_t h^3}
 \end{aligned}
 \tag{V, 15}$$

The iterative scheme described in Chapter IV and section B of this chapter may be used to solve equation (V,12) to predict the response of the imperfect panels. For the imperfect panel, the starting point of the iterative procedure is the undeformed state of the panel (zero load and zero deflection), instead of the bifurcation point for the perfect panel with symmetric and antisymmetric angle layers. For the first few steps, the step length of the prescribed component of the deflection is taken relatively small because the load-deflection

relation curves are very steep at the beginning of the loading process. In the examples given below, the first component $w_{11}^{(1)}$ of the second equation in (V,6) is taken as the prescribed component. The step length of this component varies from 0.001 to 0.5 for different panels. It is found that for a panel with small imperfections the iterative procedure may fail to give correct convergent results if a large step length is used.

The computing results are summarised in figures (V,8) - (V,16). The properties of composite materials appropriate to boron/epoxy, carbon/epoxy and glass/epoxy are taken from Table (III,2). In order to limit the computing cost, the main purpose of these examples is restricted to showing the varying tendency of the parameters and their effects on each other rather than to giving exact values. So, a four term approximation for the stress function F and an eight term approximation for the deflection w are used in the computations. The results show that such approximations are adequate to serve for this purpose.

In the examples the thickness of the panel is held constant at 2.5 mm whilst the length a of the panel and the arc b of mid-surface are both taken equal to 250 mm.

Fig. (V,8) is obtained for four layer panels of $(\pm 45^\circ)_s$ boron/epoxy composite material with different curvatures. The dotted lines indicate the behaviour of perfect panels, and solid lines show the results of imperfect panels in which the first component $\bar{w}_{11}$ of the imperfection expression (V,5) is equal to 0.5 and the other components vanish. It can be seen that because of the initial imperfection the behaviour of the panels under compression is no longer bifurcational. Bending occurs as soon as the load is applied. This is true, both for flat plates

and curved panels. But for flat plates with geometrical imperfections the load-central deflection relation is still monotonic. This means that the equilibrium state is always unique corresponding to a certain level of load. For curved panels, however, there may be more than one equilibrium state which corresponds to the same load level. This means that snap-through may occur in such panels.

It may be of interest to point out that the net deflection of the imperfect panels may be smaller than that of the perfect ones but the total deflection (net deflection plus initial imperfection) is always larger than that of the perfect panels (for perfect panels net deflection equals total deflection). This phenomenon was pointed out by Yamaki^[116] for isotropic plates. Now it is found to be true also for laminated curved panels. It is seen in all the following figures in this section as well.

Figs. (V,9), (V,10) and (V,11) illustrate the effect of the magnitude of the imperfection on the postbuckling behaviour of the panels. These panels are assumed to have only an imperfection of the form $\bar{w} \sin \pi \xi \sin \pi \eta$ (nondimensional form). Fig. (V,9) is obtained for four layer $(\pm 45^\circ)_s$ boron/epoxy panels having the curvature parameter $K_R = 50$. Fig. (V,10) is for four layer $(\pm 45^\circ)_s$ glass/epoxy flat plates ($K_R = 0$), and Fig. (V,11) is for four layer $(\pm 45^\circ)_s$ carbon/epoxy panels with $K_R = 25$. All these results obtained for panels of different materials and different curvatures indicate that the bending effect becomes severe when initial imperfection is large. It is also found that laminated curved panels are imperfection sensitive. Small imperfections may considerably reduce the resistance of the panel against buckling. When initial

imperfection in a curved panel is relatively large, the load-deflection curve becomes very flat and snap-through may no longer happen in the loading process. The load-central deflection curves in Fig. (V,12) demonstrate the effect of the coupling terms D_{16} and D_{26} on the postbuckling behaviour of imperfect panels. The figure is calculated for panels with symmetric boron/epoxy layers having $K_R = 25$. It is obvious that the largest D_{16} and D_{26} terms appear in the panel having unidirectional layers at 45° , and they decrease with increasing number of alternate layers at $\pm 45^\circ$ in the panel. It is found that the coupling terms D_{16} and D_{26} degrade the panel performance both for perfect panels or imperfect ones under compression unless the deflection is very large.

Fig. (V,13) is obtained for boron/epoxy panels with anti-symmetric angle layers. The effect of the coupling terms B_{16} and B_{26} can be seen clearly in this figure. Panels having two layer antisymmetric arrangement have the lowest performance, no matter whether the panel is perfect or imperfect.

Fig. (V,14) shows the effect of different shapes of imperfection. It can be seen that the sensitivity of the panel to the different imperfection shapes is quite different. For this four layer $(\pm 45^\circ)_s$ boron/epoxy panel with the curvature parameter $K_R = 50$, the components $\bar{w}_{11}$ and $\bar{w}_{21}$ of imperfection have more influence than $\bar{w}_{21}$ and $\bar{w}_{22}$.

Figs. (V,15) and (V,16) are obtained for boron/epoxy panels with unsymmetric crossply arrangement at $(90^\circ, 0^\circ)$. As we discussed in the previous sections of this chapter, even for panels with perfect geometry under compression, bending of the panels is induced by the unsymmetric layers. The dotted lines show the behaviour of such perfect panels. For imperfect

geometry the bending effect of the panel is even more severe. It follows that the unsymmetric crossply arrangement and geometric imperfection both reduce the resistance of the curved panel against collapse.

From these examples the following conclusions may be summarised:

- (i) Initial geometrical imperfection in composite panels has an important influence on the behaviour of the panels under in-plane loading. It may greatly reduce the resistance of the curved panels against collapse.
- (ii) The coupling terms D_{16} , D_{26} or B_{16} , B_{26} degrade the panel performance both for perfect and imperfect panels.
- (iii) Laminated panels have different sensitivity to different types of imperfection shape.
- (iv) Imperfection brings more bending effect to the panels having certain unsymmetric lay-ups under compressive or shear loading.

D. Bending Behaviour of Laminated Anisotropic Plates under the Combination of Lateral and in-plane Loading

In previous work^{[91][92]} the bending behaviour of laminated plates under combined lateral and compressive loading was investigated. The in-plane shear loading, however, was not included.

In Chapter IV of the current work the in-plane shear direction is proved to be an important factor for the buckling and postbuckling behaviour of laminated panels under shear loading itself or the combination of compressive and shear loading. So it is natural to ask how the shear direction affects the bending behaviour of laminated panels under combined lateral and shear loading? This section is intended to provide

some answer to that question, which the author believes has not yet been covered in the literature.

Here, a flat plate of generally laminated composite under combined lateral and in-plane loading is considered. The edges of the plate are considered to be loosely clamped (in-plane stress zero but deflection and rotation restrained on the boundaries).

With the lateral load per unit area denoted by $\bar{q}$, the governing system of the problem may be written (similar to (II,22) and (II,23)):

$$\int_0^a \int_0^b \left\{ \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + N_x \frac{\partial^2 w_0}{\partial x^2} + N_y \frac{\partial^2 w_0}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w_0}{\partial x \partial y} + \bar{q} \right\} \delta w_0 dx dy$$

$$+ \int_0^b \left[M_x \delta \left(\frac{\partial w_0}{\partial x} \right) \right] \Big|_{x=0}^{x=a} dy + \int_0^a \left[M_y \delta \left(\frac{\partial w_0}{\partial y} \right) \right] \Big|_{y=0}^{y=b} dx = 0$$

(V,15)

$$\int_0^a \int_0^b \left\{ \frac{\partial^2 \epsilon_x^0}{\partial y^2} + \frac{\partial^2 \epsilon_y^0}{\partial x^2} - \frac{\partial^2 \gamma_{xy}^0}{\partial x \partial y} - \left(\frac{\partial^2 w_0}{\partial x \partial y} \right)^2 + \frac{\partial^2 w_0}{\partial x^2} \frac{\partial^2 w_0}{\partial y^2} \right\} \delta \phi dx dy = 0$$

Following the same procedure as in Chapter II, the non-dimensional form of the governing equations is obtained:

$$\int_0^1 \int_0^1 \left\{ \frac{\partial^4 F}{\partial \xi^4} - a_1 \frac{\partial^4 F}{\partial \xi^3 \partial \eta} + a_2 \frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} - a_3 \frac{\partial^4 F}{\partial \xi \partial \eta^3} + a_4 \frac{\partial^4 F}{\partial \eta^4} - a_5 \frac{\partial^4 W}{\partial \xi^4} \right.$$

$$- a_6 \frac{\partial^4 W}{\partial \xi^3 \partial \eta} - a_7 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} - a_8 \frac{\partial^4 W}{\partial \xi \partial \eta^3} - a_9 \frac{\partial^4 W}{\partial \eta^4}$$

$$\left. - \frac{\beta^2}{A_{22}^*} \left[\left(\frac{\partial^2 W}{\partial \xi \partial \eta} \right)^2 - \frac{\partial^2 W}{\partial \xi^2} \frac{\partial^2 W}{\partial \eta^2} \right] \right\} \delta F d\xi d\eta = 0$$

(V,16)

$$\begin{aligned}
& \int_0^1 \int_0^1 \left\{ \frac{\partial^4 W}{\partial \xi^4} + b_1 \frac{\partial^4 W}{\partial \xi^3 \partial \eta} + b_2 \frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} + b_3 \frac{\partial^4 W}{\partial \xi \partial \eta^3} + b_4 \frac{\partial^4 W}{\partial \eta^4} + b_5 \frac{\partial^4 F}{\partial \xi^4} + b_6 \frac{\partial^4 F}{\partial \xi^3 \partial \eta} \right. \\
& \quad \left. + b_7 \frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} + b_8 \frac{\partial^4 F}{\partial \xi \partial \eta^3} + b_9 \frac{\partial^4 F}{\partial \eta^4} - \frac{\bar{q} a^4}{D_{11}^* h} \right\} \delta W d\xi d\eta \\
& + \int_0^1 \left\{ \left[b_{10} \frac{\partial^2 F}{\partial \eta^2} + b_5 \frac{\partial^2 F}{\partial \xi^2} + b_{11} \frac{\partial^2 F}{\partial \xi \partial \eta} + \frac{\partial^2 W}{\partial \xi^2} + b_{12} \frac{\partial^2 W}{\partial \eta^2} + \frac{b_1}{2} \frac{\partial^2 W}{\partial \xi \partial \eta} \right] \delta \left(\frac{\partial W}{\partial \xi} \right) \right\} \Big|_{\xi=0}^{\xi=1} d\eta \\
& + \int_0^1 \left\{ \left[b_9 \frac{\partial^2 F}{\partial \eta^2} + b_{13} \frac{\partial^2 F}{\partial \xi^2} + b_{14} \frac{\partial^2 F}{\partial \xi \partial \eta} + b_{15} \frac{\partial^2 W}{\partial \xi^2} + b_4 \frac{\partial^2 W}{\partial \eta^2} + \frac{b_3}{2} \frac{\partial^2 W}{\partial \xi \partial \eta} \right] \delta \left(\frac{\partial W}{\partial \eta} \right) \right\} \Big|_{\eta=0}^{\eta=1} d\xi \\
& = \int_0^1 \int_0^1 \left\{ \left(\frac{\beta^2}{D_{11}^*} \right) \left[\frac{\partial^2 W}{\partial \xi^2} \frac{\partial^2 F}{\partial \eta^2} + \frac{\partial^2 W}{\partial \eta^2} \frac{\partial^2 F}{\partial \xi^2} - 2 \frac{\partial^2 W}{\partial \xi \partial \eta} \frac{\partial^2 F}{\partial \xi \partial \eta} \right] \right\} \delta W d\xi d\eta
\end{aligned} \tag{V, 17}$$

The boundary conditions of the plate with clamped edges may be expressed as follows:

$$\begin{aligned}
W=0 \quad \frac{\partial W}{\partial \xi}=0 \quad \frac{\partial^2 F}{\partial \eta^2}=-\bar{N}_x \quad \frac{\partial^2 F}{\partial \xi \partial \eta}=-\beta \bar{S} \\
W=0 \quad \frac{\partial W}{\partial \eta}=0 \quad \frac{\partial^2 F}{\partial \xi^2}=-\beta^2 \bar{N}_y \quad \frac{\partial^2 F}{\partial \xi \partial \eta}=-\beta \bar{S}
\end{aligned} \tag{V, 18}$$

where

$$\begin{aligned}
\bar{N}_x &= P_x b^2 / A_{22} h^2 \\
\bar{N}_y &= P_y b^2 / A_{22} h^2 \\
\bar{S} &= P_{xy} b^2 / A_{22} h^2
\end{aligned} \tag{V, 19}$$

in which P_x , P_y are compressive loads in x and y directions, per unit length and P_{xy} is the shear load per unit length.

The solutions to (V,16) and (V,17) are assumed in the following form:

$$F = -\bar{N}_x \frac{\eta^2}{2} - \bar{N}_y \beta^2 \frac{\xi^2}{2} - \bar{S} \beta \xi \eta + \sum_m \sum_n F_{mn} X_m(\xi) Y_n(\eta) \tag{V, 20}$$

$$W = \sum_p \sum_q W_{pq} X_p(\xi) Y_q(\eta)$$

where

$$\begin{aligned}
X_i(\xi) &= \cosh \alpha_i \xi - \cos \alpha_i \xi - \gamma_i (\sinh \alpha_i \xi - \sin \alpha_i \xi) \\
Y_i(\eta) &= \cosh \alpha_i \eta - \cos \alpha_i \eta - \gamma_i (\sinh \alpha_i \eta - \sin \alpha_i \eta)
\end{aligned} \tag{V, 21}$$

Considering the properties (IV,6), (IV,7) and (IV,8) of the beam function it can be seen that the solution (V,20) completely satisfies the boundary conditions (V,18). Therefore the two line integrals in the governing equation (V,17) vanish automatically.

Substituting solution (V,20) into the governing system, (V,16) and (V,17), a similar algebraic procedure to that of Chapter IV produces the following equations for computation:

$$\begin{aligned} \sum_m \sum_n A_{mn}^{ij} F_{mn} = & \sum_p \sum_q B_{pq}^{ij} w_{pq} + \sum_k \sum_l \sum_r \sum_s D_{klrs}^{ij} w_{kl} w_{rs} \\ & (i, j = 1, 2, \dots) \end{aligned} \quad (V, 22)$$

$$\begin{aligned} \sum_p \sum_q H_{pq}^{ij} w_{pq} + \sum_m \sum_n I_{mn}^{ij} F_{mn} - \sum_k \sum_l \sum_r \sum_s K_{klrs}^{ij} w_{kl} F_{rs} \\ - \bar{N} \sum_k \sum_l T_{kl}^{ij} w_{kl} - \bar{Q} G_{ij} = 0 \\ & (i, j = 1, 2, \dots) \end{aligned}$$

where F_{mn} , w_{pq} (or w_{kl}) are the undetermined coefficients in the stress function and deflection function (V,20). The non-dimensional lateral load is

$$\bar{Q} = \frac{\bar{q} b^4}{E_t h^4} \quad (V, 23)$$

This set of equations contains an infinite number of equations, but in fact only a finite number of equations need to be solved in practice.

Letting $m = p = k = 1, 2, \dots, M$, $n = q = l = 1, 2, \dots, M$, $i = 1, 2, \dots, M$ and $j = 1, 2, \dots, M$, the equations (V,22) can be expressed in matrix form:

$$[A]\{F\}=[B]\{W\}+[T_W][D]\{W\} \quad (V,24)$$

$$[H]\{W\}+[I]\{F\}-[T_W][K]\{F\}-\bar{N}[T]\{W\}-\bar{Q}\{G\}=0$$

where

$$[T_W]=\begin{bmatrix} \{W\}^t \\ \{W\}^t \\ \dots \\ \{W\}^t \end{bmatrix} \quad (V,25)$$

The elements of the coefficient matrices in equations (V,24) are given below.

With the indices I, J defined by

$$I=(i-1)M+j \quad (V,26)$$

$$J=(m-1)M+n=(p-1)M+q=(k-1)M+l$$

$$\begin{aligned} A_{IJ} &= (\alpha_m^4 + a_4 \alpha_n^4) J_0^{mi} J_0^{nj} - a_1 \alpha_m^3 \alpha_n J_3^{mi} J_1^{nj} + a_2 \alpha_m^2 \alpha_n^2 J_2^{mi} J_2^{nj} - a_3 \alpha_m \alpha_n^3 J_1^{mi} J_3^{nj} \\ B_{IJ} &= (a_5 \alpha_p^4 + a_9 \alpha_q^4) J_0^{pi} J_0^{qj} + a_6 \alpha_p^3 \alpha_q J_3^{pi} J_1^{qj} + a_7 \alpha_p^2 \alpha_q^2 J_2^{pi} J_2^{qj} + a_8 \alpha_p \alpha_q^3 J_1^{pi} J_3^{qj} \\ H_{IJ} &= (\alpha_p^4 + b_4 \alpha_q^4) J_0^{pi} J_0^{qj} + b_1 \alpha_p^3 \alpha_q J_3^{pi} J_1^{qj} + b_2 \alpha_p^2 \alpha_q^2 J_2^{pi} J_2^{qj} + b_3 \alpha_p \alpha_q^3 J_1^{pi} J_3^{qj} \\ I_{IJ} &= (b_5 \alpha_m^4 + b_9 \alpha_n^4) J_0^{mi} J_0^{nj} + b_6 \alpha_m^3 \alpha_n J_3^{mi} J_1^{nj} + b_7 \alpha_m^2 \alpha_n^2 J_2^{mi} J_2^{nj} + b_8 \alpha_m \alpha_n^3 J_1^{mi} J_3^{nj} \\ T_{IJ} &= \frac{\beta^2}{D_{11}^*} \left[-\mu_x \alpha_k^2 J_2^{ki} J_0^{lj} - \mu_y \beta^2 \alpha_l^2 J_0^{ki} J_2^{lj} + 2\mu_s \beta \alpha_k \alpha_l J_1^{kl} J_1^{ij} \right] \end{aligned} \quad (V,27)$$

With the indices I defined by

$$I=(i-1)M+j \quad (V,28)$$

$$G_I = \frac{E_t h^3}{D_{11}^*} \beta^4 J_6^i J_6^j \quad (V,29)$$

With the indices I, J, I_1, J_1 defined by

$$\begin{aligned} I_1 &= (i-1)M + j \\ J_1 &= (k-1)M + j \\ I &= (I_1-1)M + J_1 \\ J &= (r-1)M + s \end{aligned} \quad (V, 30)$$

$$\begin{aligned} D_{IJ} &= \frac{\beta^2}{A_{22}^*} (\alpha_k \alpha_l \alpha_r \alpha_s J_4^{kri} J_4^{lsj} - \alpha_k^2 \alpha_s^2 J_5^{kri} J_5^{lsj}) \\ K_{IJ} &= \frac{\beta^2}{D_{11}^*} (\alpha_k^2 \alpha_s^2 J_5^{kri} J_5^{lsj} + \alpha_l^2 \alpha_r^2 J_5^{rki} J_5^{lsj} - 2\alpha_k \alpha_l \alpha_r \alpha_s J_4^{kri} J_4^{lsj}) \end{aligned} \quad (V, 31)$$

and

$$\begin{aligned} \bar{N} &= \frac{P b^2}{A_{22} h^2} \\ \mu_x P &= P_x \\ \mu_y P &= P_y \\ \mu_s P &= P_{xy} \end{aligned} \quad (V, 32)$$

where

$$\left. \begin{aligned} J_0^{mi} &= \int_0^1 \bar{X}_m(\xi) X_i(\xi) d\xi \\ J_1^{mi} &= \int_0^1 \bar{\bar{X}}_m(\xi) X_i(\xi) d\xi \\ J_2^{mi} &= \int_0^1 \bar{\bar{\bar{X}}}_m(\xi) X_i(\xi) d\xi \\ J_3^{mi} &= \int_0^1 \bar{\bar{\bar{\bar{X}}}}_m(\xi) X_i(\xi) d\xi \end{aligned} \right\} \quad (V, 33)$$

$$\begin{aligned}
J_4^{kri} &= \int_0^1 \bar{X}_k(\xi) \bar{X}_r(\xi) X_i(\xi) d\xi \\
J_5^{kri} &= \int_0^1 \bar{\bar{X}}_k(\xi) X_r(\xi) X_i(\xi) d\xi \\
J_6^i &= \int_0^1 X_i(\xi) d\xi
\end{aligned}$$

and

$$\begin{aligned}
\bar{X}_i &= \frac{1}{\alpha_i} \frac{dX_i(\xi)}{d\xi} \\
\bar{\bar{X}}_i &= \frac{1}{\alpha_i^2} \frac{d^2X_i(\xi)}{d\xi^2} \\
\bar{\bar{\bar{X}}}_i &= \frac{1}{\alpha_i^3} \frac{d^3X_i(\xi)}{d\xi^3}
\end{aligned} \tag{V,34}$$

Solving $\{F\}$ for $\{w\}$ from the first equation of (V,24), substituting the resultant $\{F\}$ into the second equation of (V,24) we obtain

$$[E]\{w\} = \bar{N}[T]\{w\} + \bar{Q}\{G\} \tag{V,35}$$

where

$$[E] = [H] + ([I] - [T_w][K])[A^{(1)}] \tag{V,36}$$

$$[A^{(1)}] = [A]^{-1}([B] + [T_w][D]) \tag{V,37}$$

(V,35) may also be written as follows:

$$[E]\{w\} = \frac{P b^2}{E_t h^3} ([T_1]\{w\} + C_K\{G\}) \tag{V,38}$$

where

$$\begin{aligned}
[T_1] &= \frac{E_t h}{A_{22}} [T] \\
C_K &= \frac{\frac{\bar{q} b^4}{E_t h^4}}{\frac{P b^2}{E_t h^3}} = \frac{\bar{q} b^2}{P h}
\end{aligned} \tag{V,39}$$

Selecting properly the parameters C_K , μ_x , μ_y and μ_s we may create any kind of loading combinations we need.

The iterative procedure described in Chapter IV is used here to solve equation (V,38). In order to limit the computing cost, a four term approximation for both stress function F and deflection function W is used in the examples. Because the assumed solutions (V,20) completely satisfy the boundary conditions (V,18), it can be expected that the results with such an approximation should be adequate to show the tendency of the load-deflection relationships.

In the calculations the properties of the composite materials are as listed in Table (III,2). The thickness of the panels is assumed constant at 2.5 mm. A constant width b equal to 250 mm and a variable length a are also assumed. The length a depends on the required aspect ratios. In most cases $a = b$ is taken (square plate).

Fig. (V,17) is obtained for a square four layer $(\pm 45)_s$ carbon/epoxy plate. The loading condition is combined in-plane shear and lateral loading ($\mu_x = \mu_y = 0$, $\mu_s = 1.0$, $C_K \neq 0$). The shear direction is not altered. The shear-central deflection curves are shown on the figure. The different curves correspond to different ratios C_K of lateral load to shear load. It is seen that the panel deflection increases with increasing value of C_K .

Figs. (V,18) and (V,19) are obtained for square carbon/epoxy plates with different lay-ups under the combination of in-plane shear and lateral loading. $C_K = 1.0$, $\mu_x = \mu_y = 0$, $\mu_s = 1.0$ are used in these cases. Fig. (V,18) corresponds to plates with symmetric lay-up arrangements. The effect of shear directions is investigated. The load-central deflection relations of the

plates show that alternate shear directions have a very significant influence on the behaviour of symmetric laminates under such combined loading. It is also seen that the fewer layers there are, the greater is the influence. When the number of layers in the plate becomes very large, the plate behaves as an orthotropic one and the alternate shear directions make little difference. As the limit the orthotropic solution, which corresponds to infinite layers, is also given in Fig. (V,18).

Fig. (V,19) shows the behaviour of plates with antisymmetric angle-ply arrangements. The load-central deflection relations indicate that the alternate shear directions have no influence on the panel behaviour. This is not difficult to understand because alternating shear directions is only equivalent to turning over the plate.

Fig. (V,20) is for carbon/epoxy plates with asymmetric lay-ups. It can be seen that the shear direction is also important for the behaviour of these plates. For plates with symmetric outer layers the influence of the shear directions seems more severe.

Fig. (V,21) investigates the behaviour of square plates of four layer (± 45)_s carbon/epoxy composites under the general combination of in-plane load and lateral load. $C_K = \mu_x = \mu_s = 1.0$, $\mu_y = 0$ are used for the combination of unidirectional compression, shear and lateral loading (denoted by $P_x + P_{xy} + Q$ in the figure). $C_K = \mu_x = \mu_y = \mu_s = 1.0$ are used for the combination of bidirectional compression, shear and lateral loading (denoted by $P_x + P_y + P_{xy} + Q$). It is found that the alternative shear direction corresponds to different bending behaviour of such plates.

Fig. (V,22) is obtained for non-square plates of four

layer $(\pm 45)_s$ carbon/epoxy composite. The figure corresponds to the combined loading of in-plane shear and lateral pressure ($C_K = \mu_s = 1.0$, $\mu_x = \mu_y = 0$). The load-central deflection curves show that the alternate shear directions are as important to non-square plates as to square ones.

Figure (V,23) shows results for square plates with four symmetric layers $(\pm 45)_s$ of different composites. The loading condition is the same as in Figure (V,20) ($C_K = \mu_s = 1.0$, $\mu_x = \mu_y = 0$). The results confirm the importance of the in-plane shear direction.

The effect of altering the shear direction is equivalent to that of changing the sign of fibre angle in the layers with constant shear direction. Therefore it is important for designers to select correctly the fibre directions to obtain the higher performance of the panel under the combined loading with prescribed load directions.

Summarising the examples, the following conclusions may be drawn:

(i) Alternate applied shear directions produce different behaviour of laminated panels under the combination of shear and lateral loading. This is also true for panels under the general combined loading, including unidirectional and bidirectional compression.

(ii) Alternate shear directions are important for panels with symmetric and asymmetric lay-ups, but not for anti-symmetric lay-ups.

(iii) The effect of alternate shear directions may be reduced when a large number of alternate layers is used in symmetric panels.

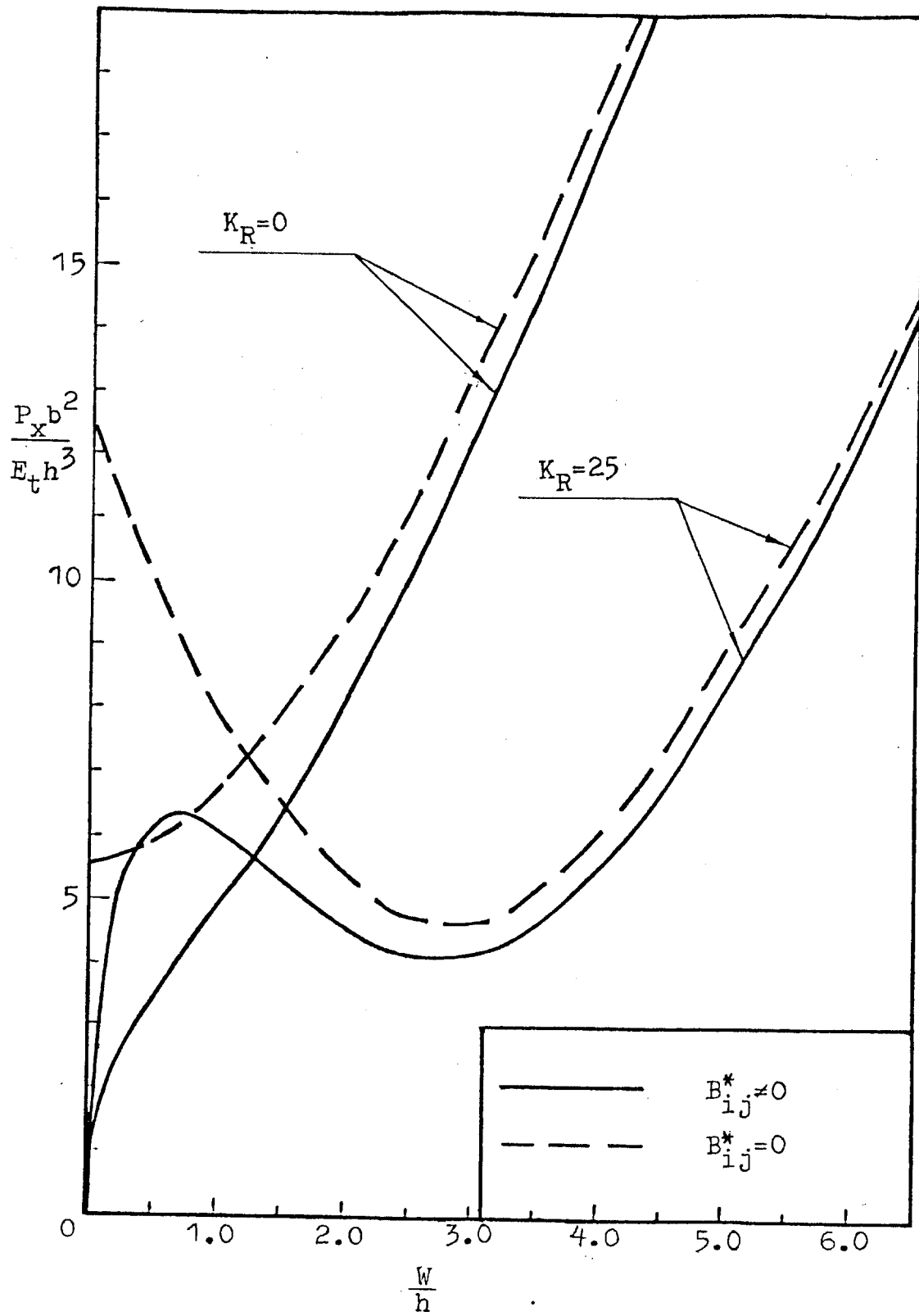


Fig. (V,1) Variation of central deflection with axial compression of boron/epoxy panels; (90°, 0°) lay-up, different curvatures

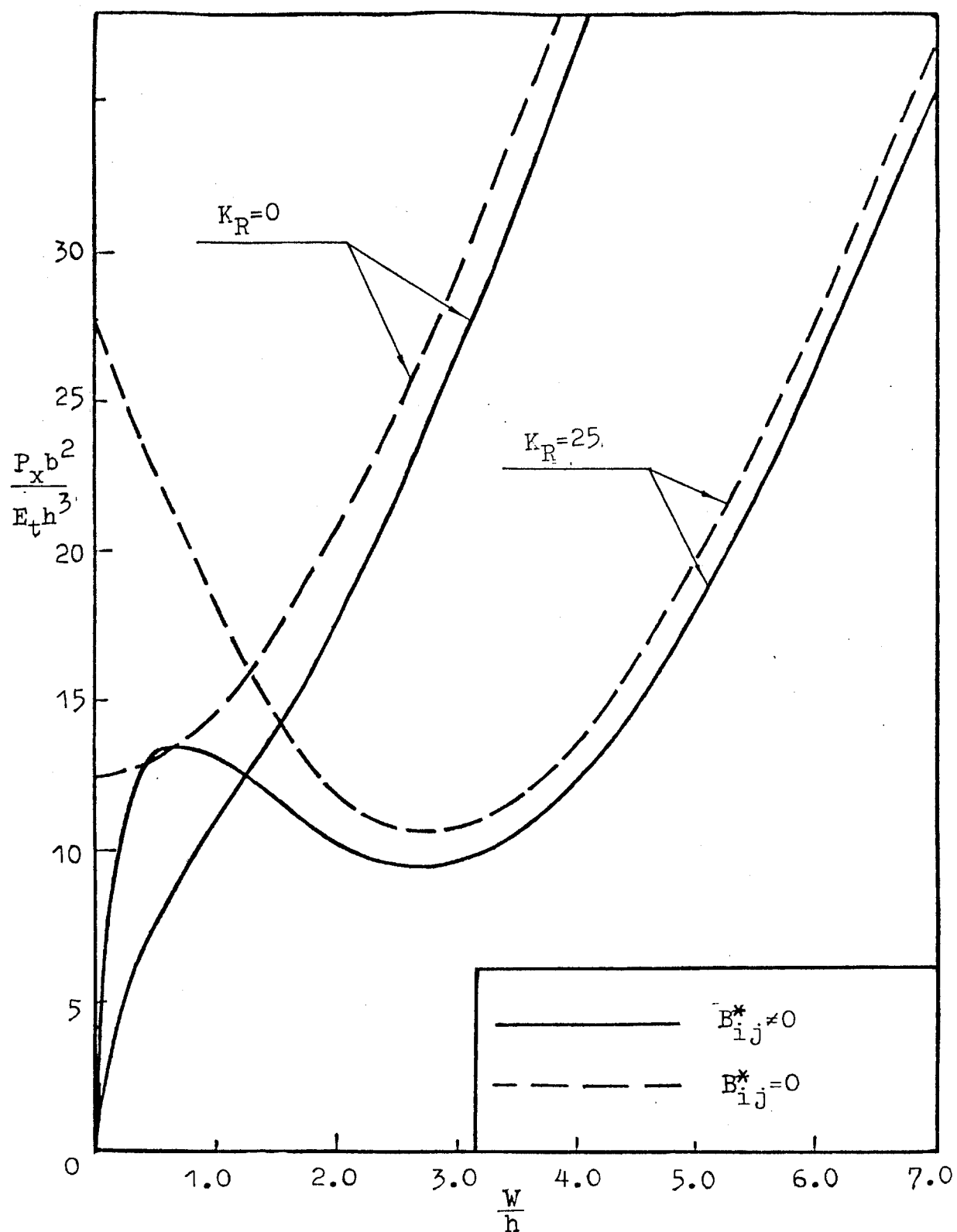


Fig. (V,2) Variation of central deflection with axial compression of carbon/epoxy panels; (90°, 0°) lay-up, different curvatures

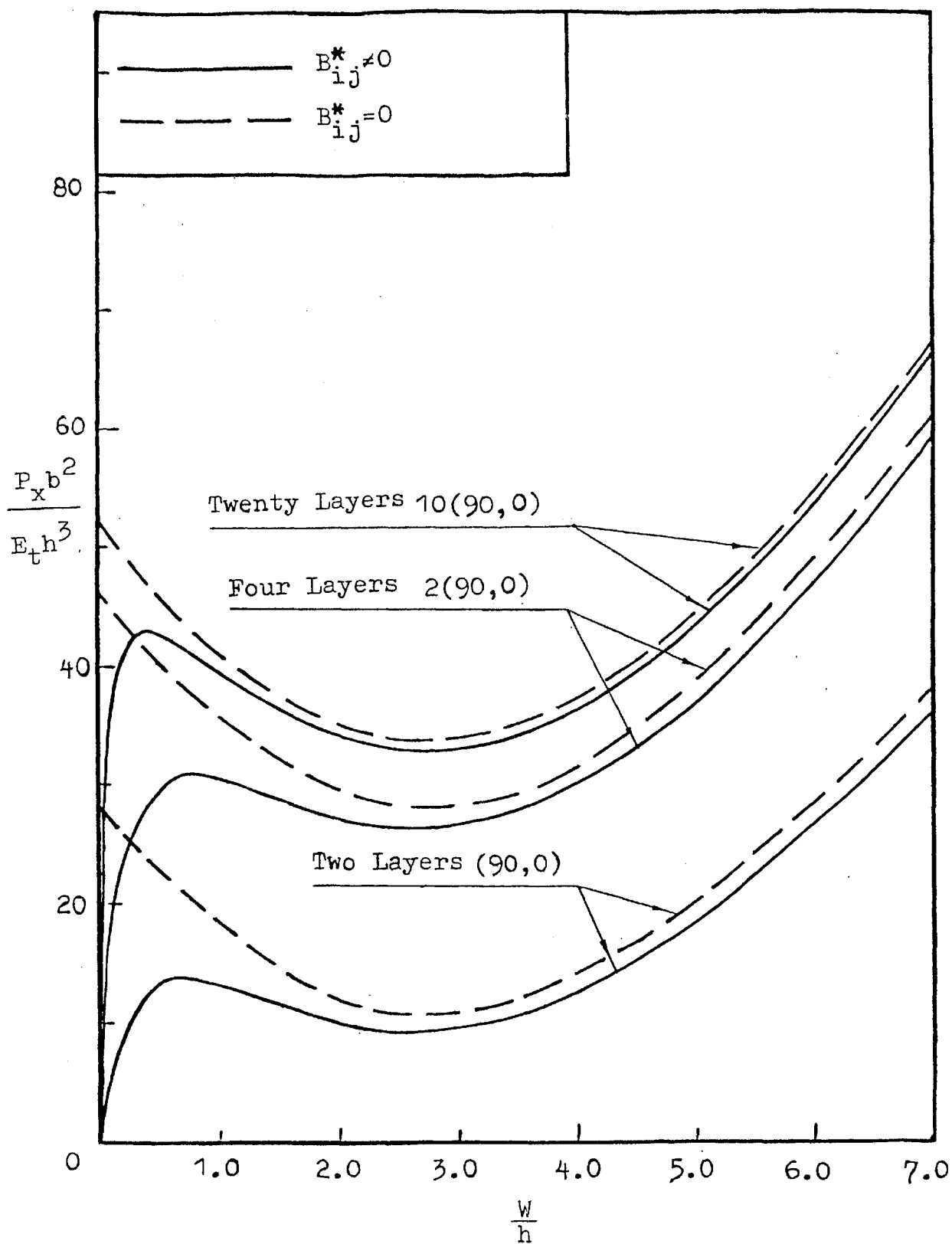


Fig. (V,3) Variation of central deflection with axial compression of carbon/epoxy panels; alternate layers at 90° and 0° and $K_R=25$

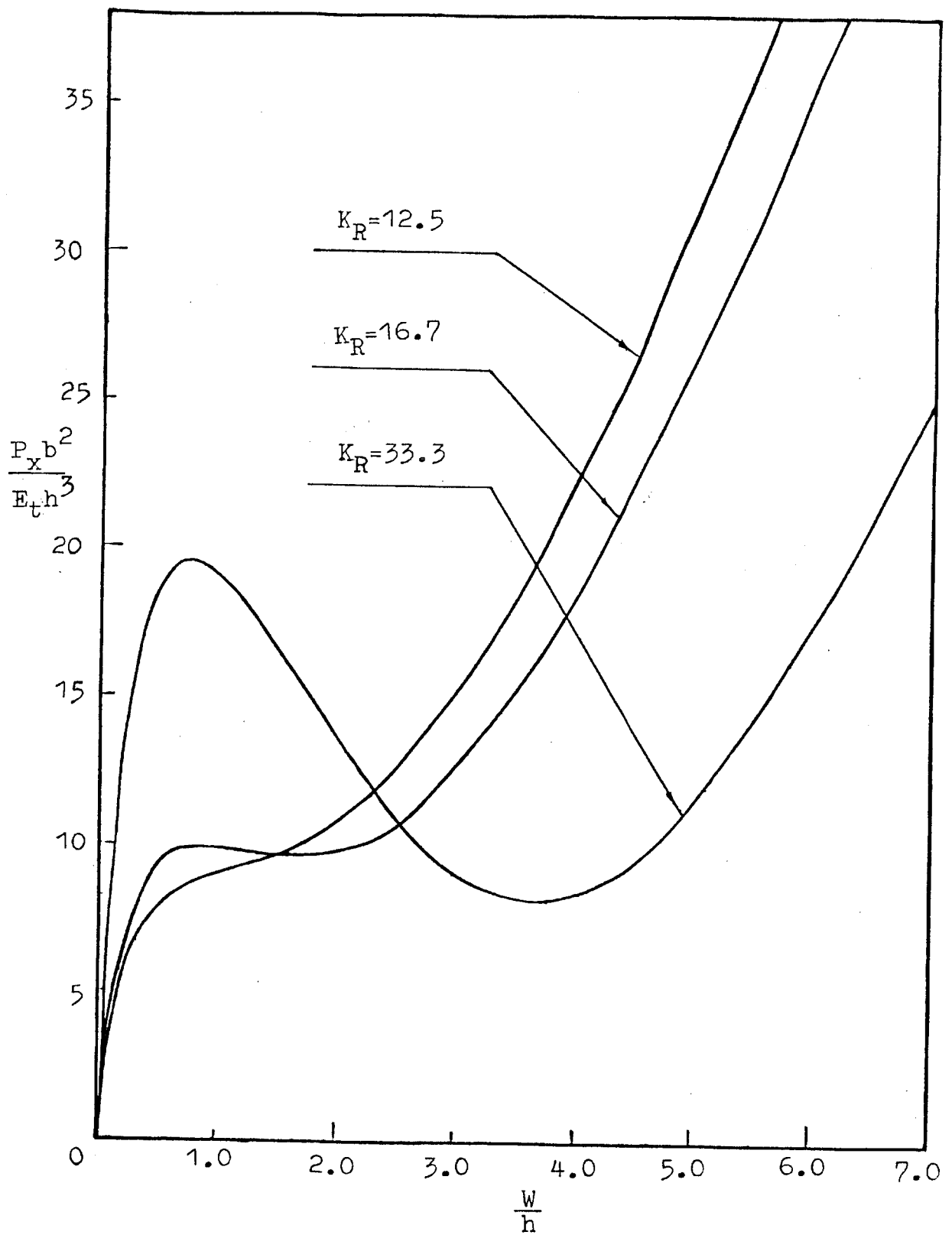


Fig. (V,4)

Variation of central deflection with
axial compression of carbon/epoxy panels;
(90°, 0°) lay-up, different curvatures

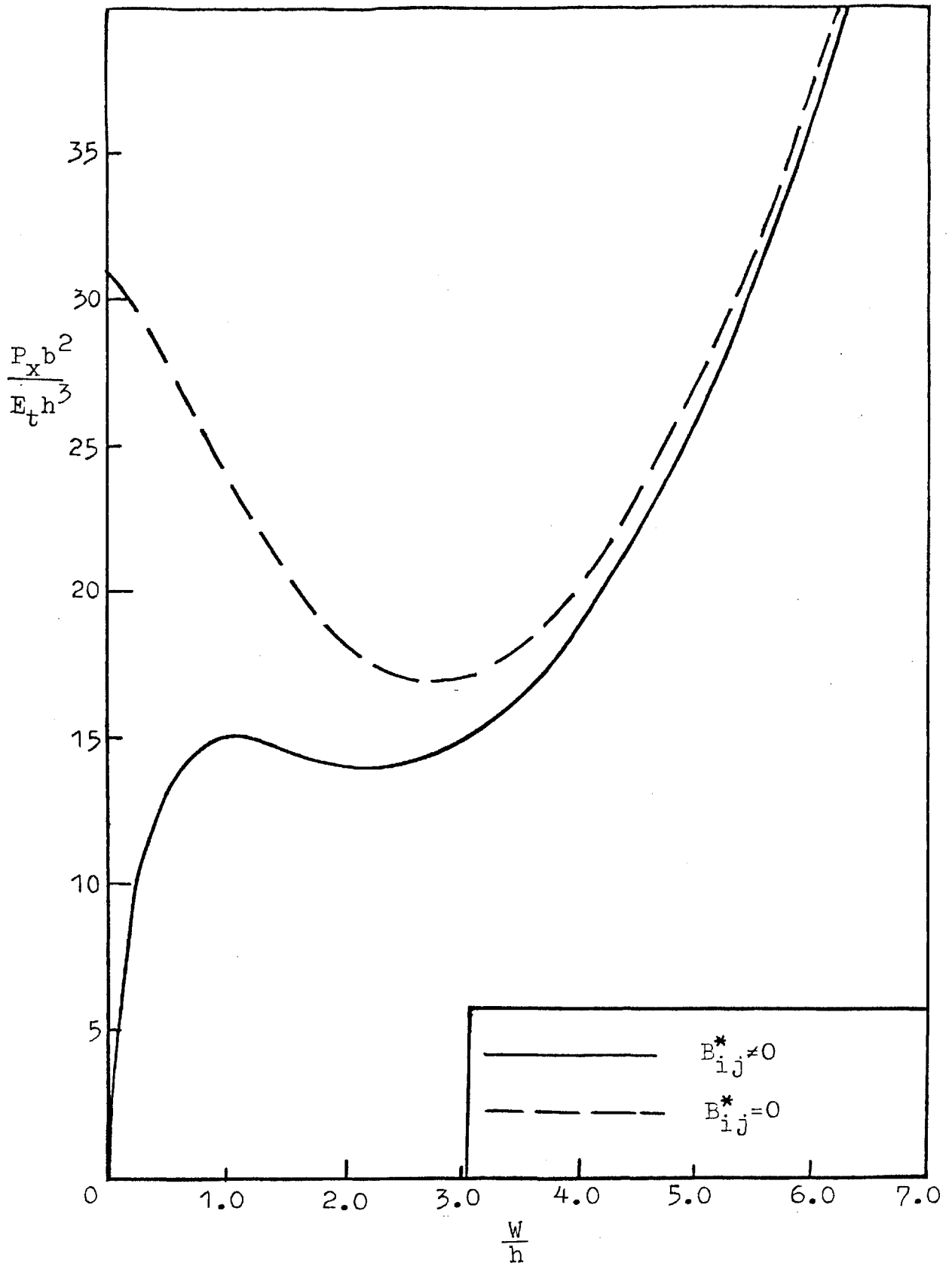


Fig. (V,5)

Variation of central deflection with
axial compression of carbon/epoxy panels;
(90°, 90°, 90°, 0°) lay-up and $K_R = 25$

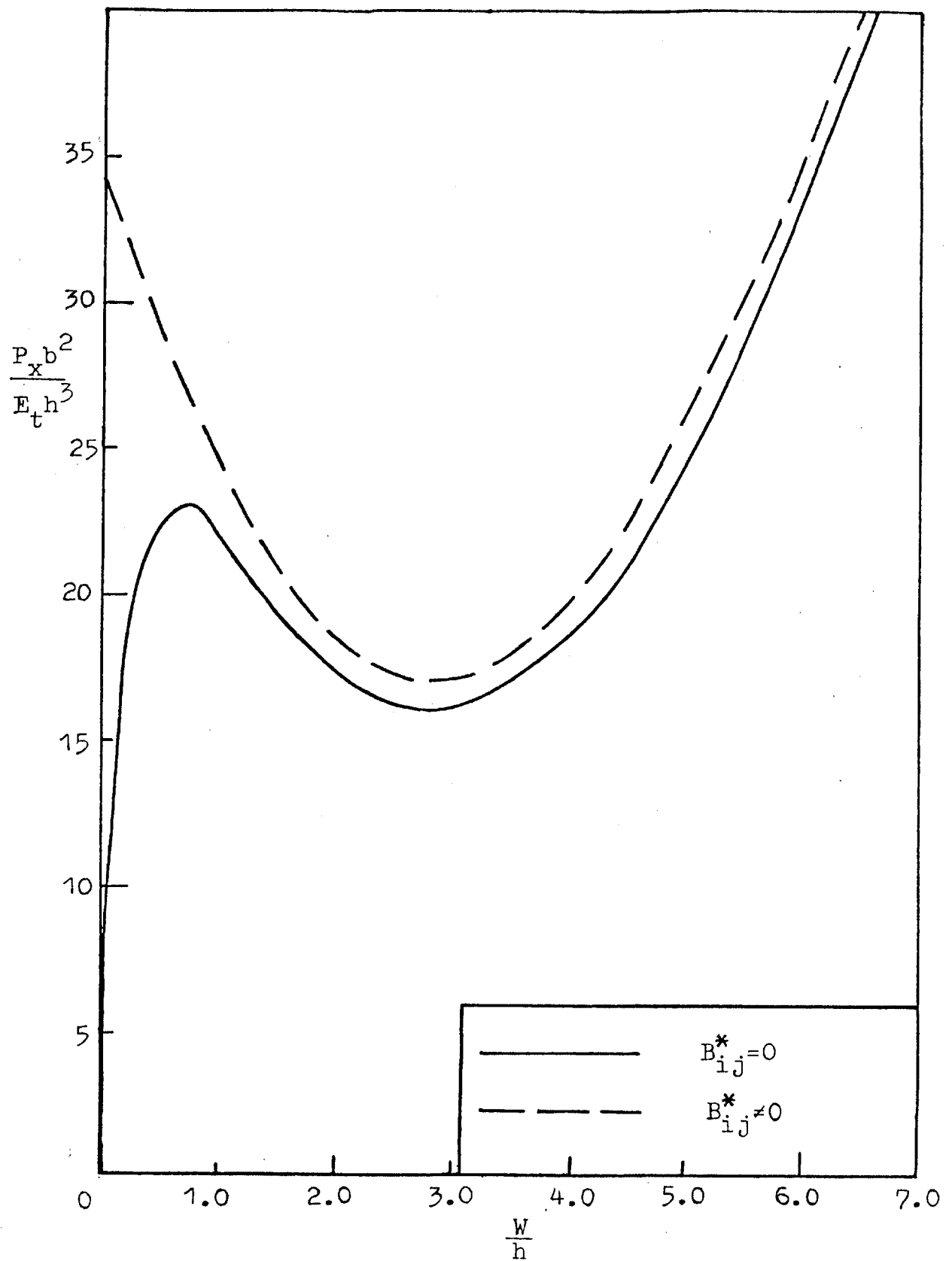


Fig. (V,6) Variation of central deflection with axial compression of carbon/epoxy panels; $(0^\circ, 0^\circ, 0^\circ, 90^\circ)$ lay-up and $K_R = 25$

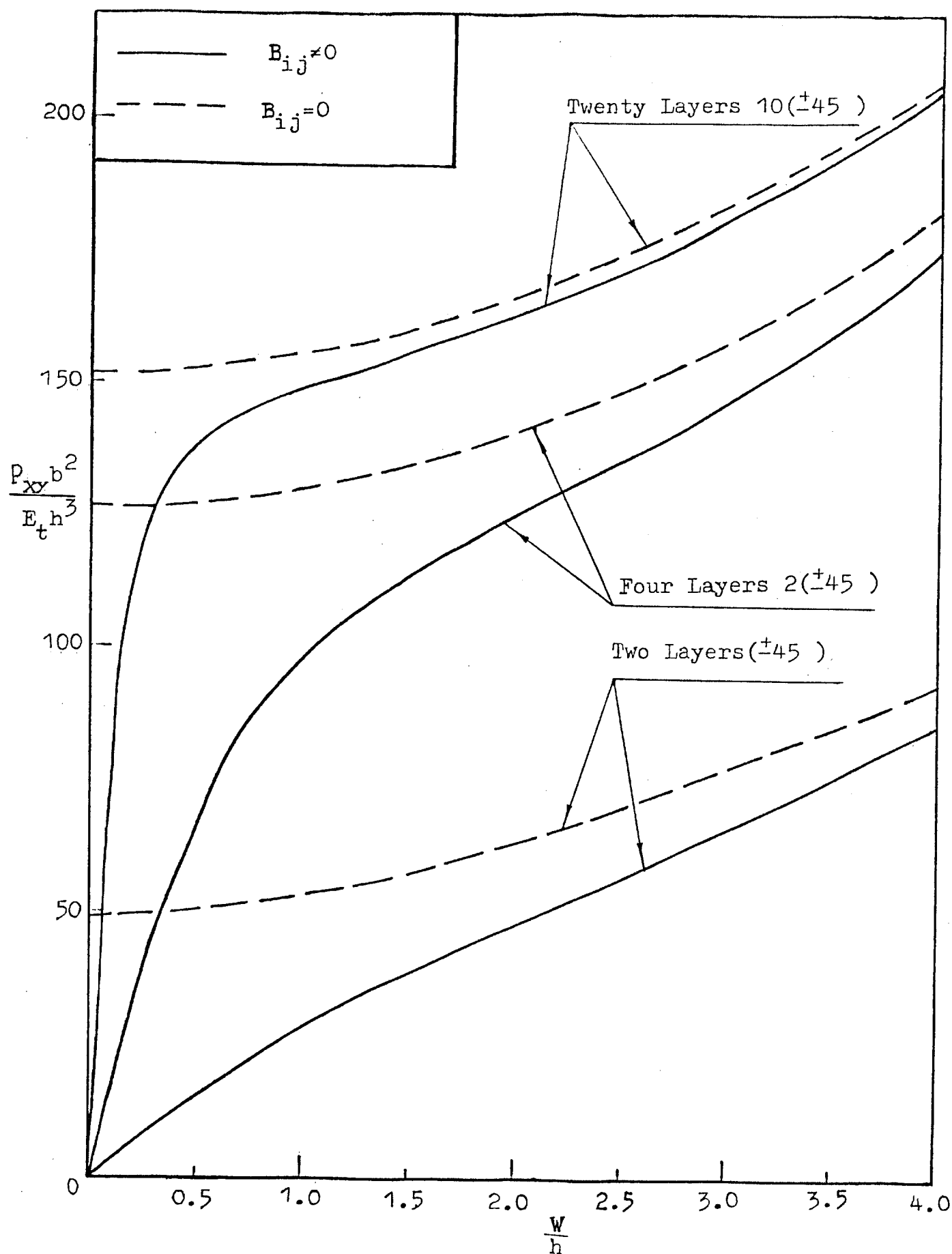


Fig. (v,7) Variation of central deflection with applied shear of carbon/epoxy plates; antisymmetric lay-up

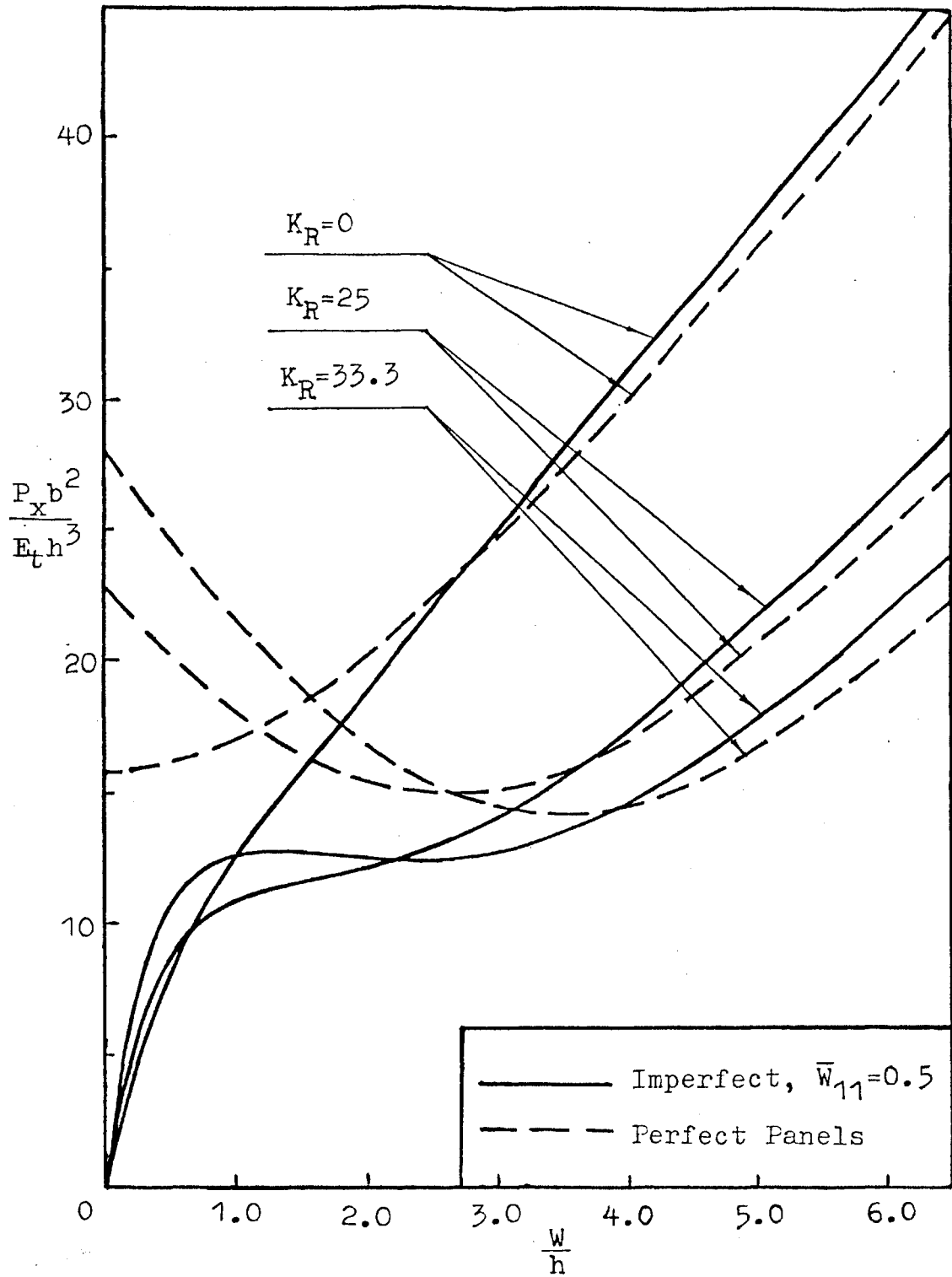


Fig. (V,8) Variation of central deflection with axial compression of imperfect boron/epoxy panels; $(\pm 45^\circ)_s$ lay-up, different curvatures

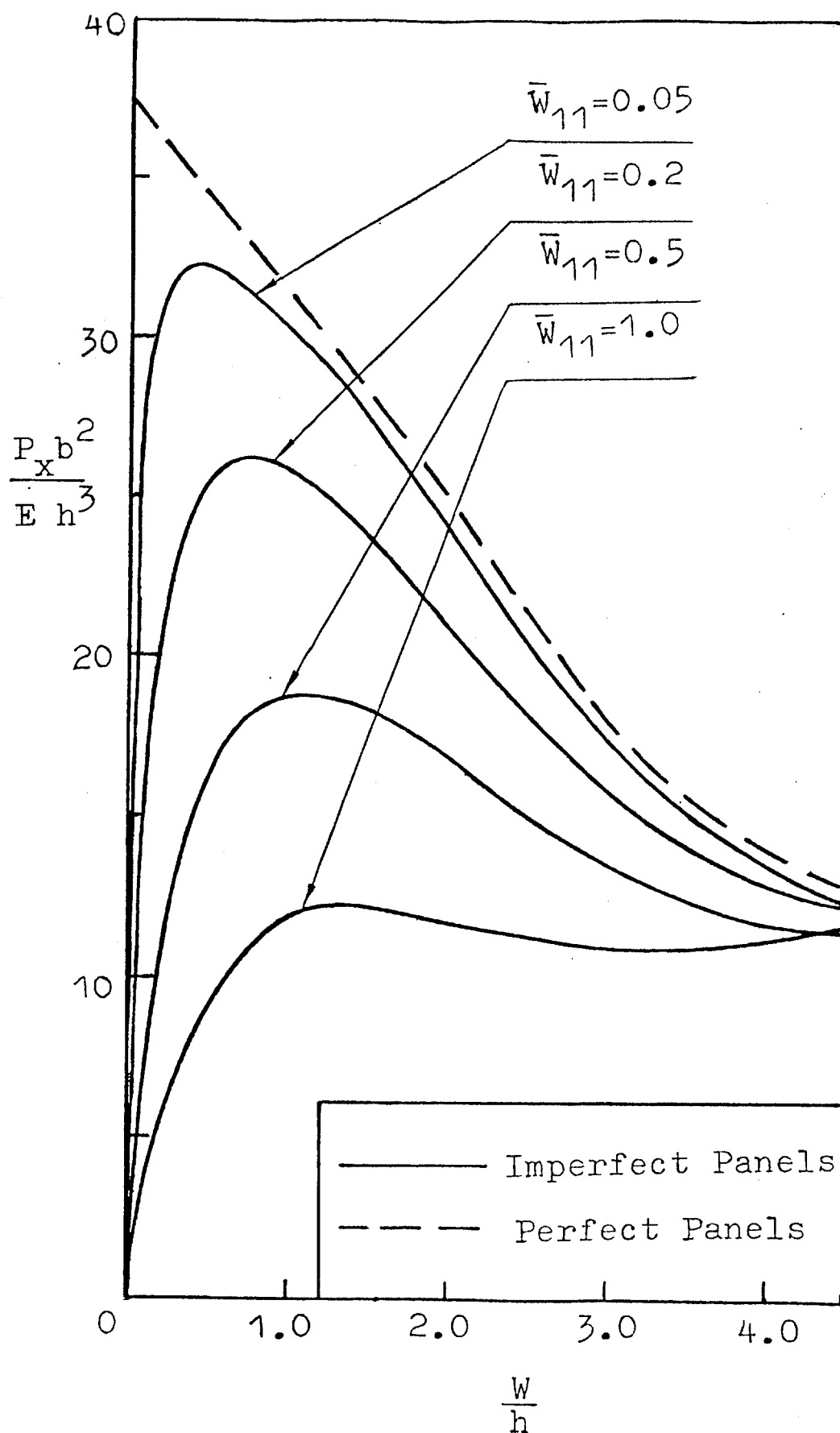


Fig. (V,9)

Variation of central deflection with axial compression of imperfect boron/epoxy panels; $(\pm 45^\circ)_s$ lay-up and

$$K_R = 50$$

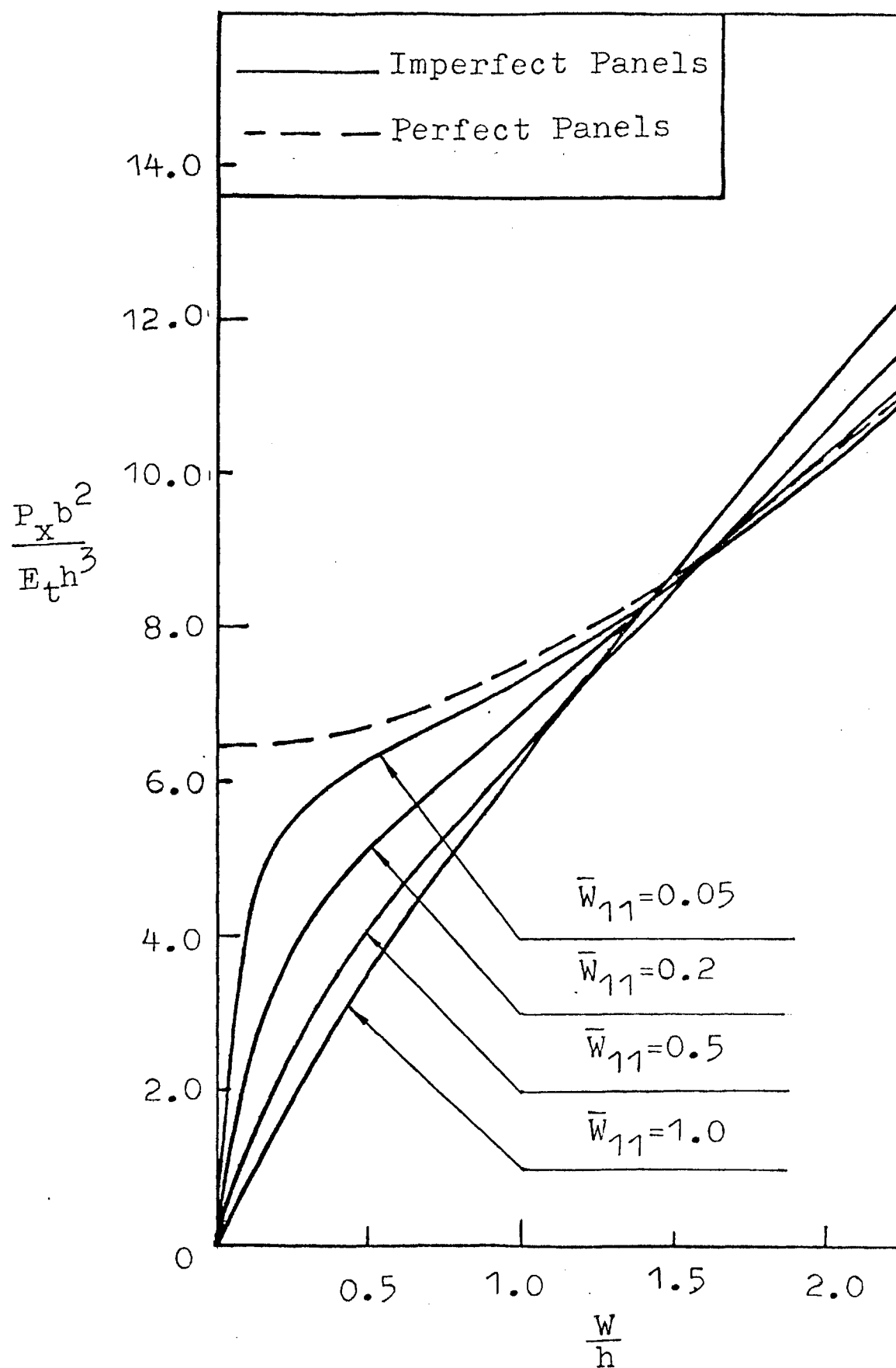


Fig. (V,10) Variation of central deflection with axial compression of imperfect glass/epoxy flat plates; $(\pm 45^\circ)_s$ lay-up

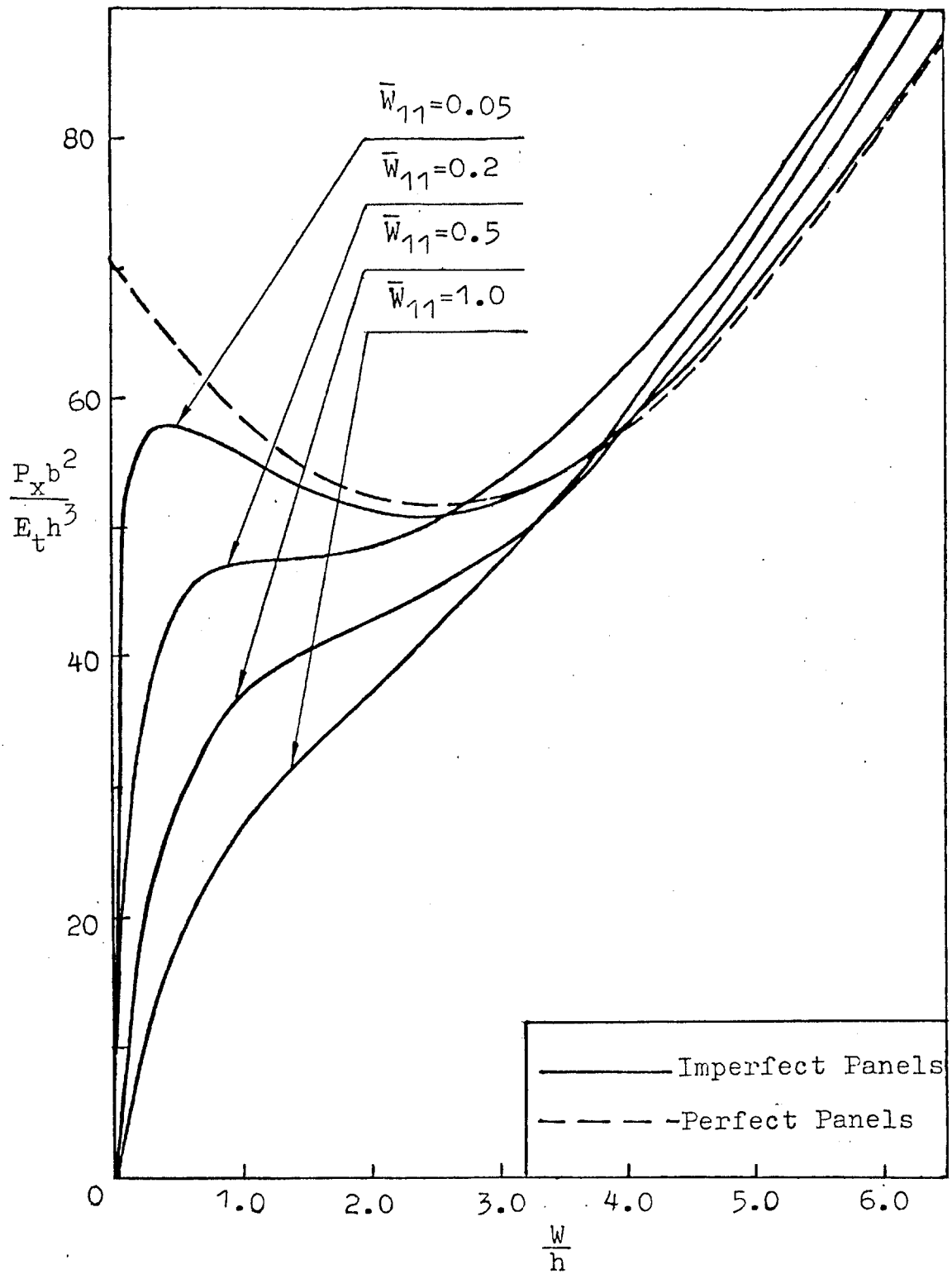


Fig. (V,11) Variation of central deflection with axial compression of imperfect carbon/epoxy panels; $(\pm 45^\circ)_s$ lay-up and $K_R = 25$

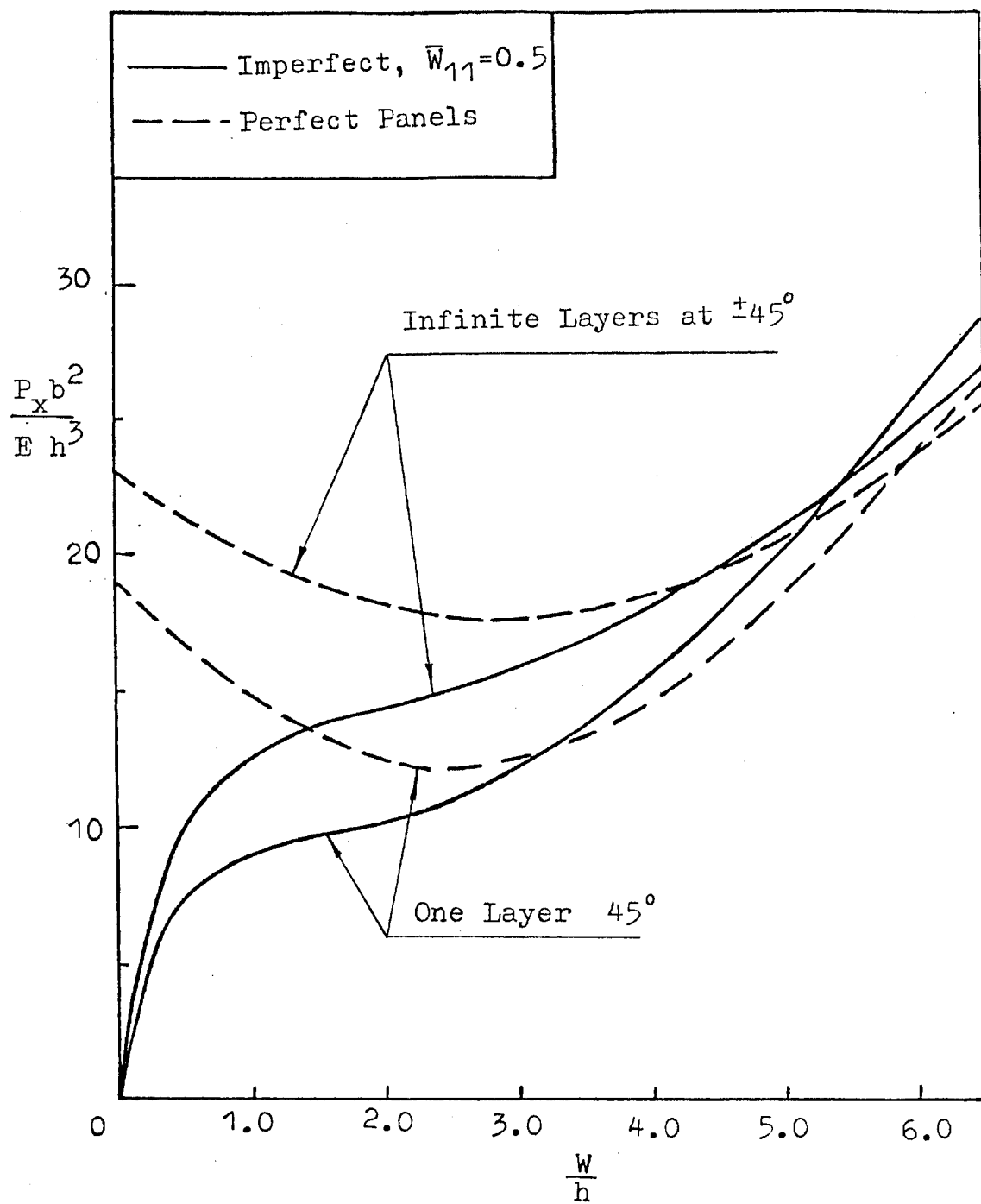


Fig. (V,12) Variation of central deflection with axial compression of imperfect boron/epoxy panels; different lay-ups and $K_R = 25$

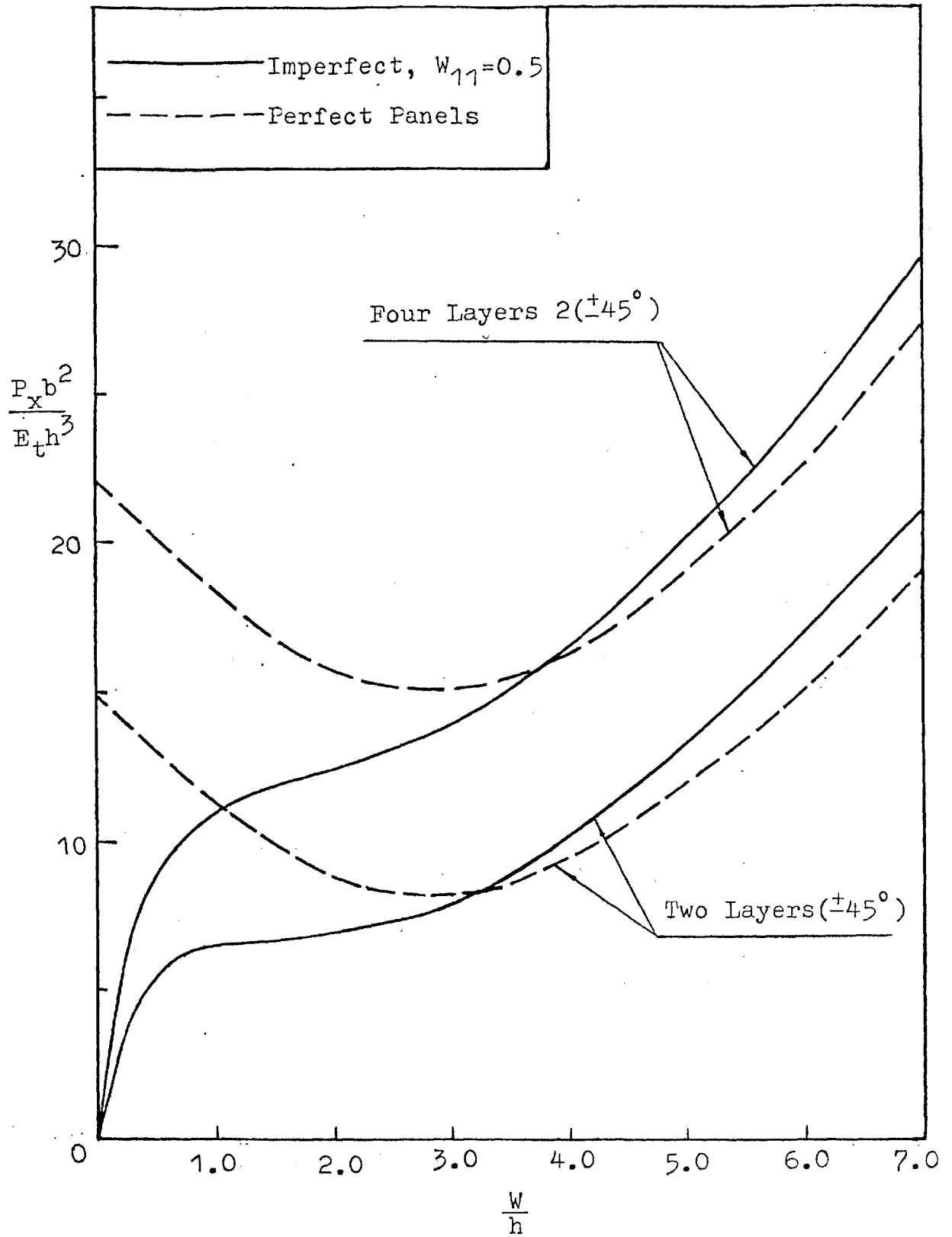


Fig. (V,13) Variation of central deflection with axial compression of imperfect boron/epoxy panels; antisymmetric lay-up and $K_R = 25$

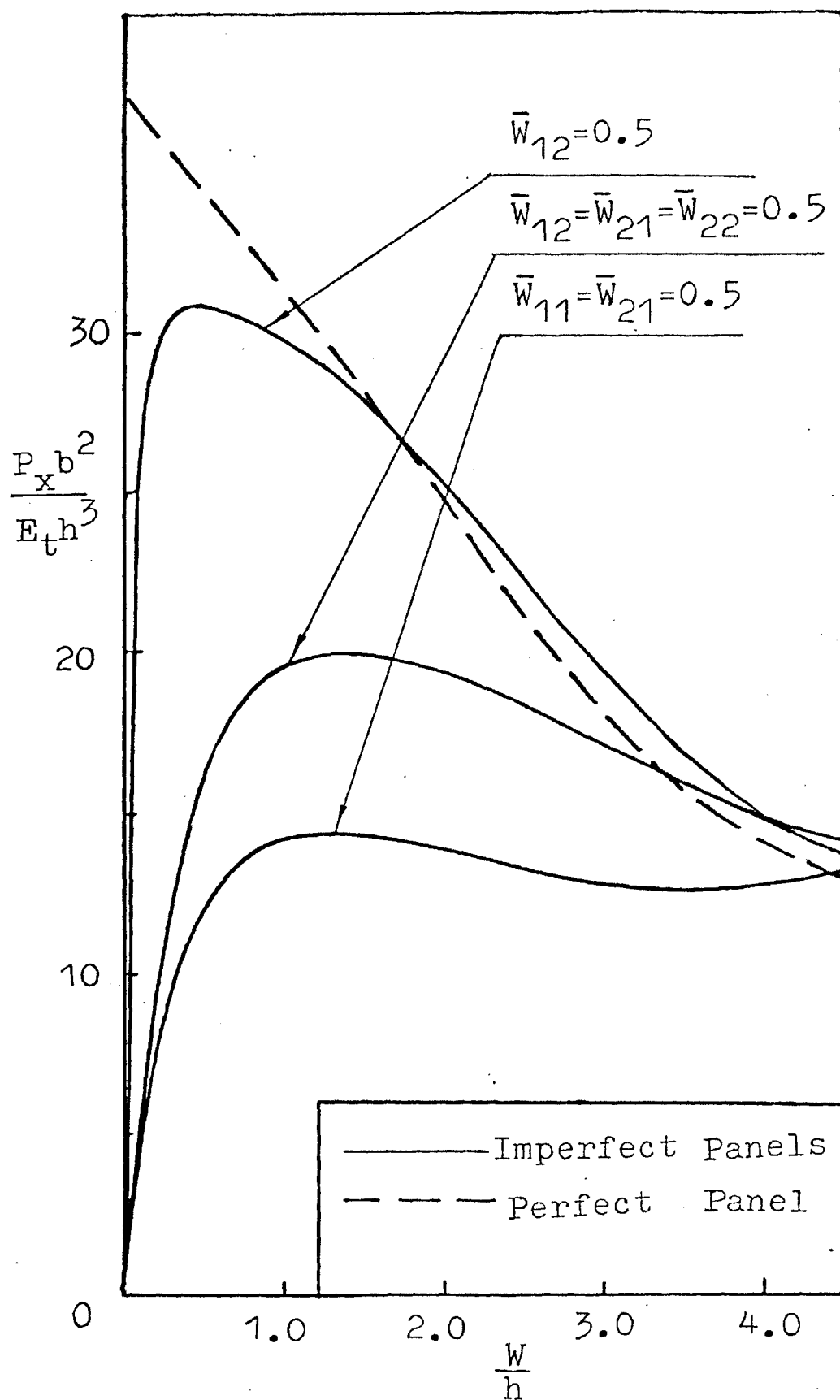


Fig. (V,14) Variation of central deflection with axial compression of imperfect boron/epoxy panels; $(\pm 45^\circ)_s$ lay-up and $K_R = 50$

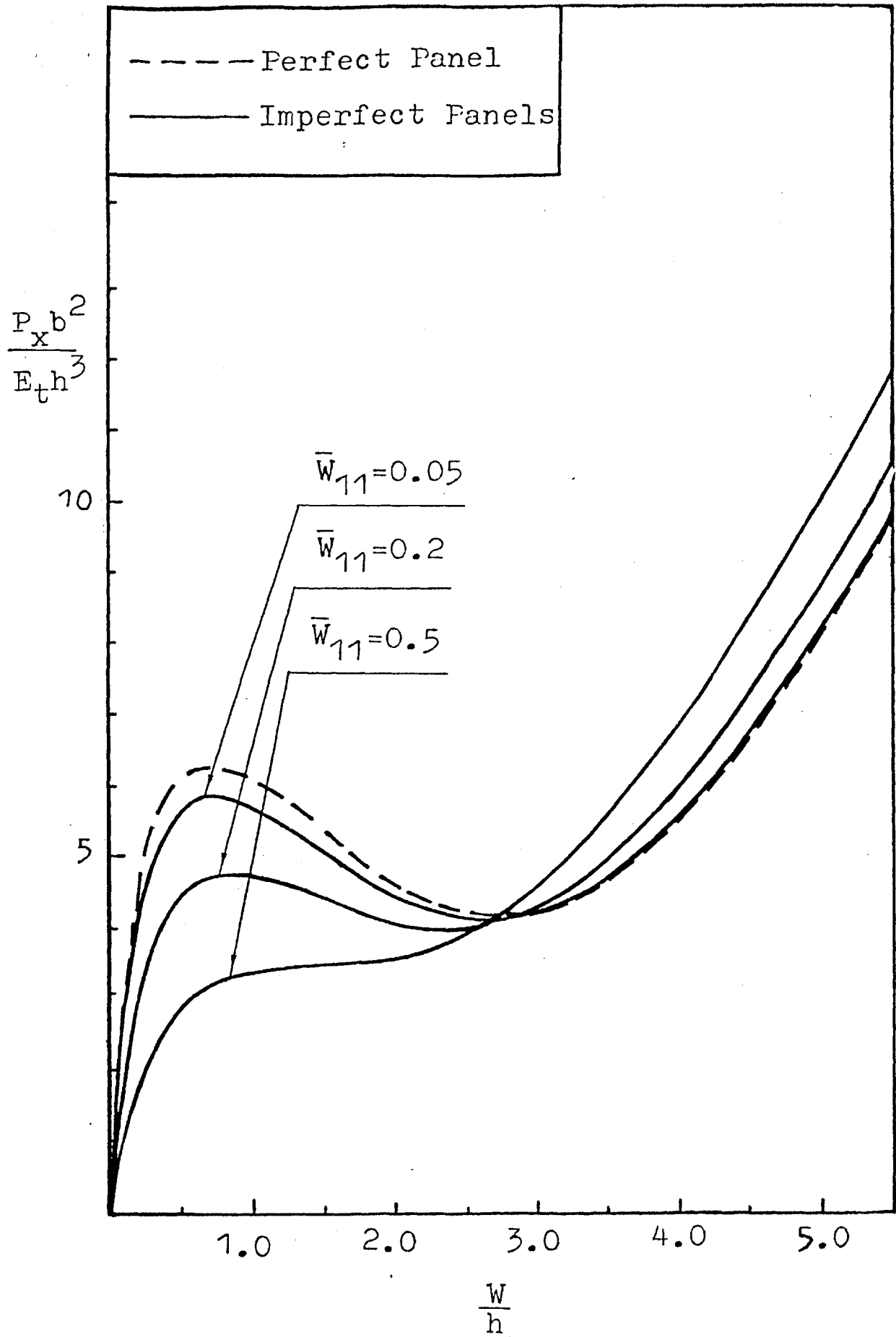


Fig. (V,15) Variation of central deflection with axial compression of boron/epoxy panels; $(90^\circ, 0^\circ)$ lay-up and $K_R = 25$

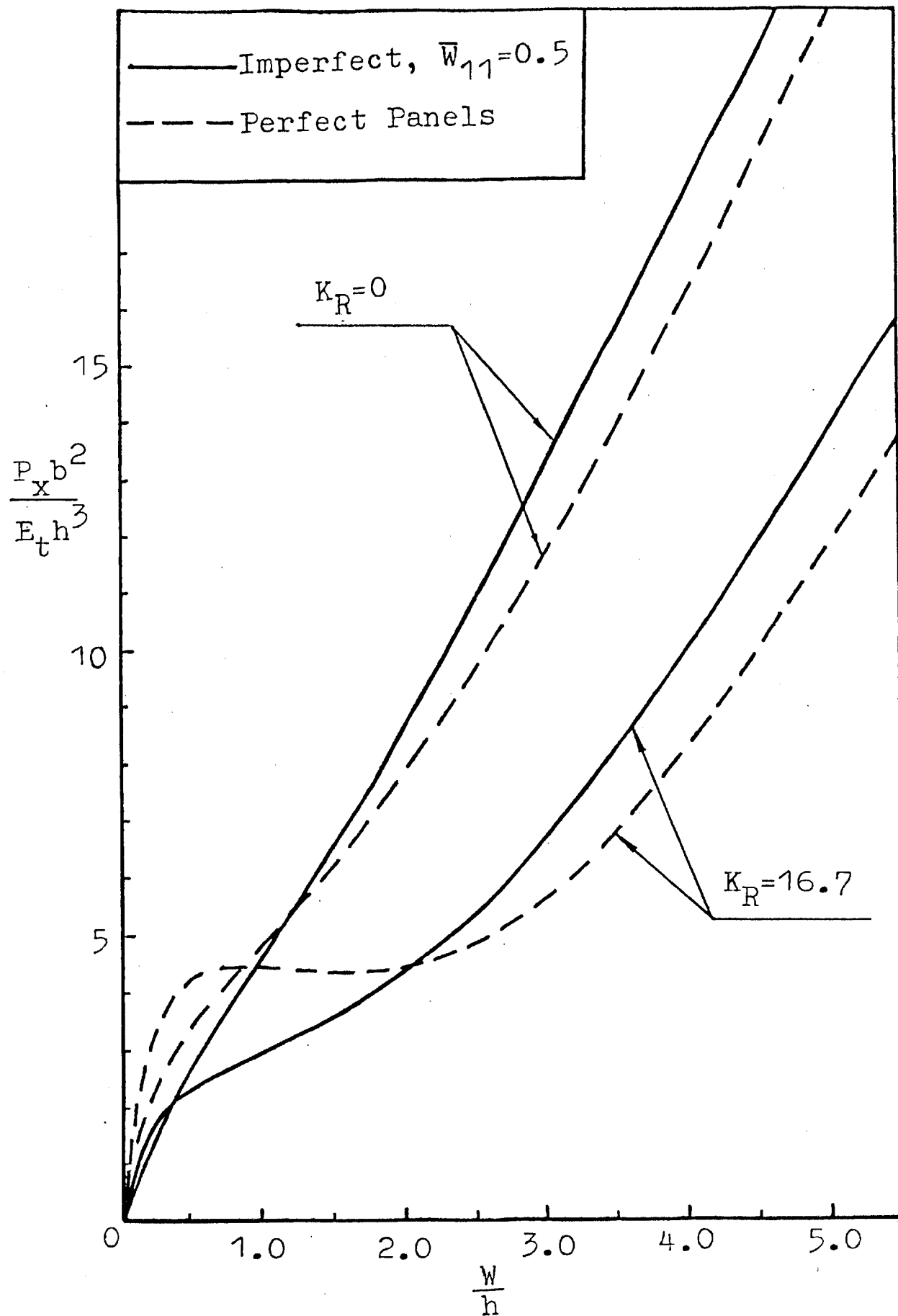


Fig. (V,16) Variation of central deflection with axial compression of boron/epoxy panels; $(90^\circ, 0^\circ)$ lay-up, different curvatures

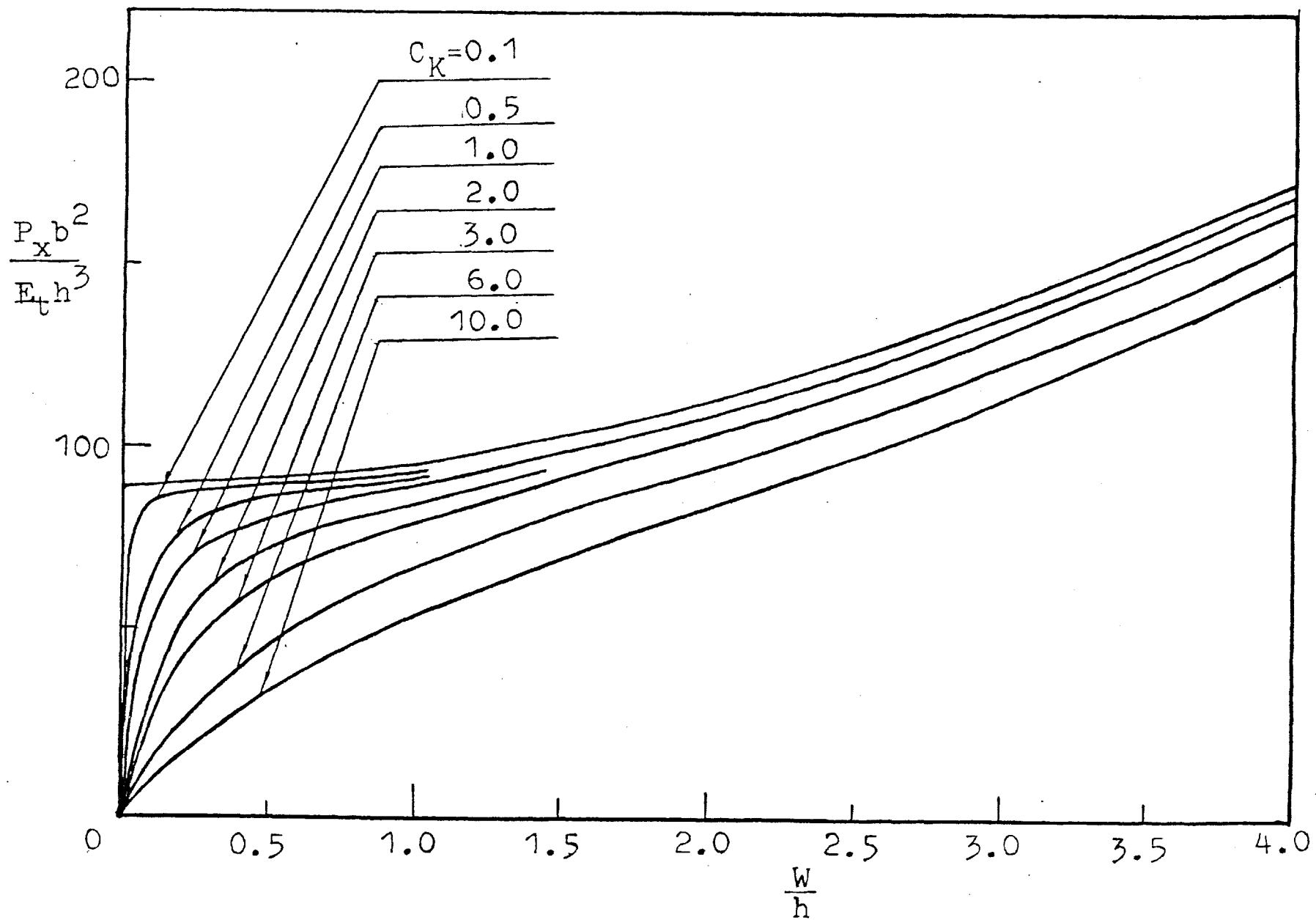


Fig. (V,17) Variation of central deflection with loading of square carbon/epoxy plates; $(\pm 45^\circ)_s$ lay-up, under the combination of in-plane shear and lateral loading

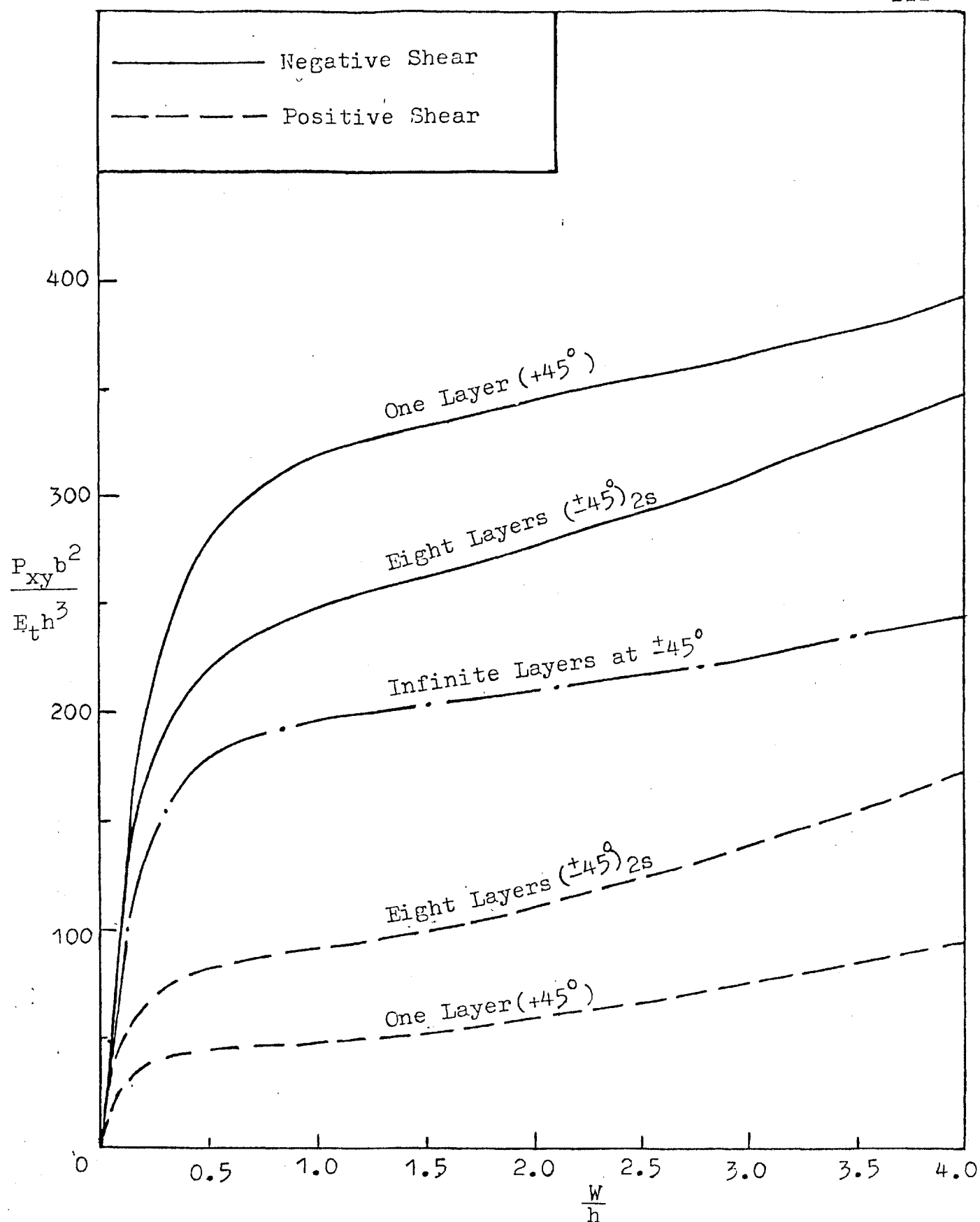


Fig. (V,18) Variation of central deflection with loading of square carbon/epoxy plates; symmetric lay-up, under the combination of in-plane shear and lateral loading

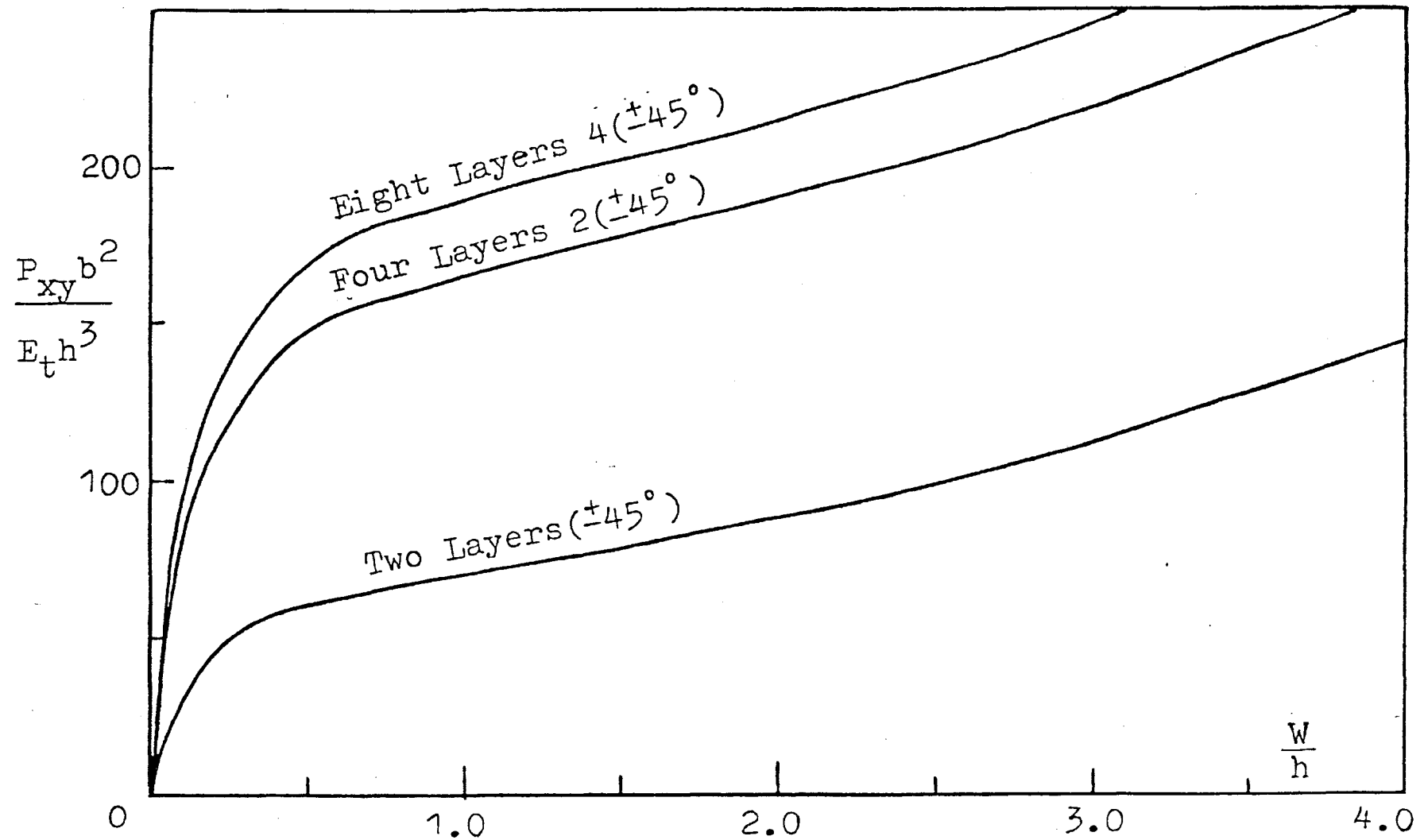


Fig. (V,19) Variation of central deflection with loading of square carbon/epoxy plates; antisymmetric lay-up, under the combination of in-plane shear and lateral loading

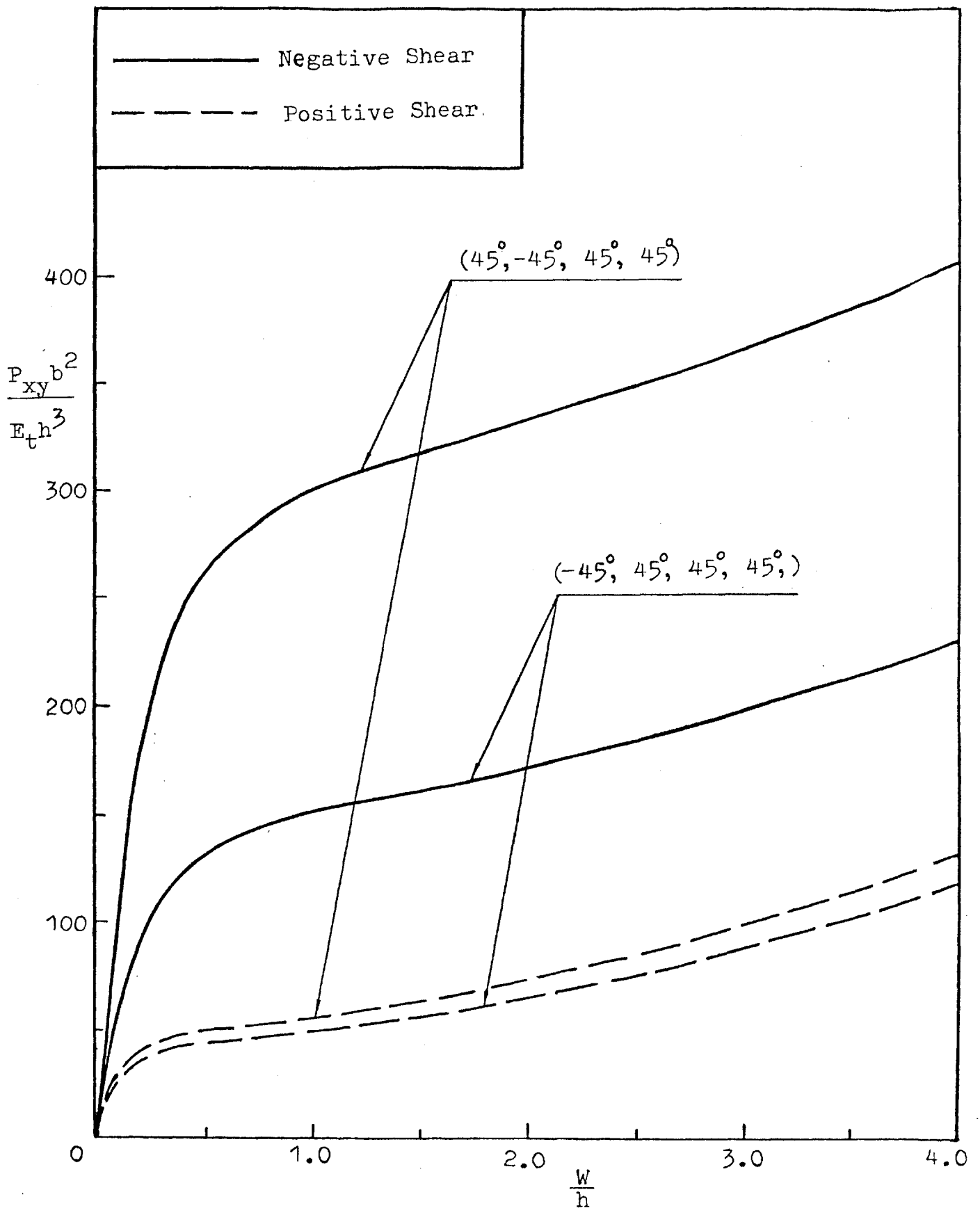


Fig. (V,20) Variation of central deflection with loading of square Carbon/epoxy plates; asymmetric lay-up, under the combination of in-plane shear and lateral loading

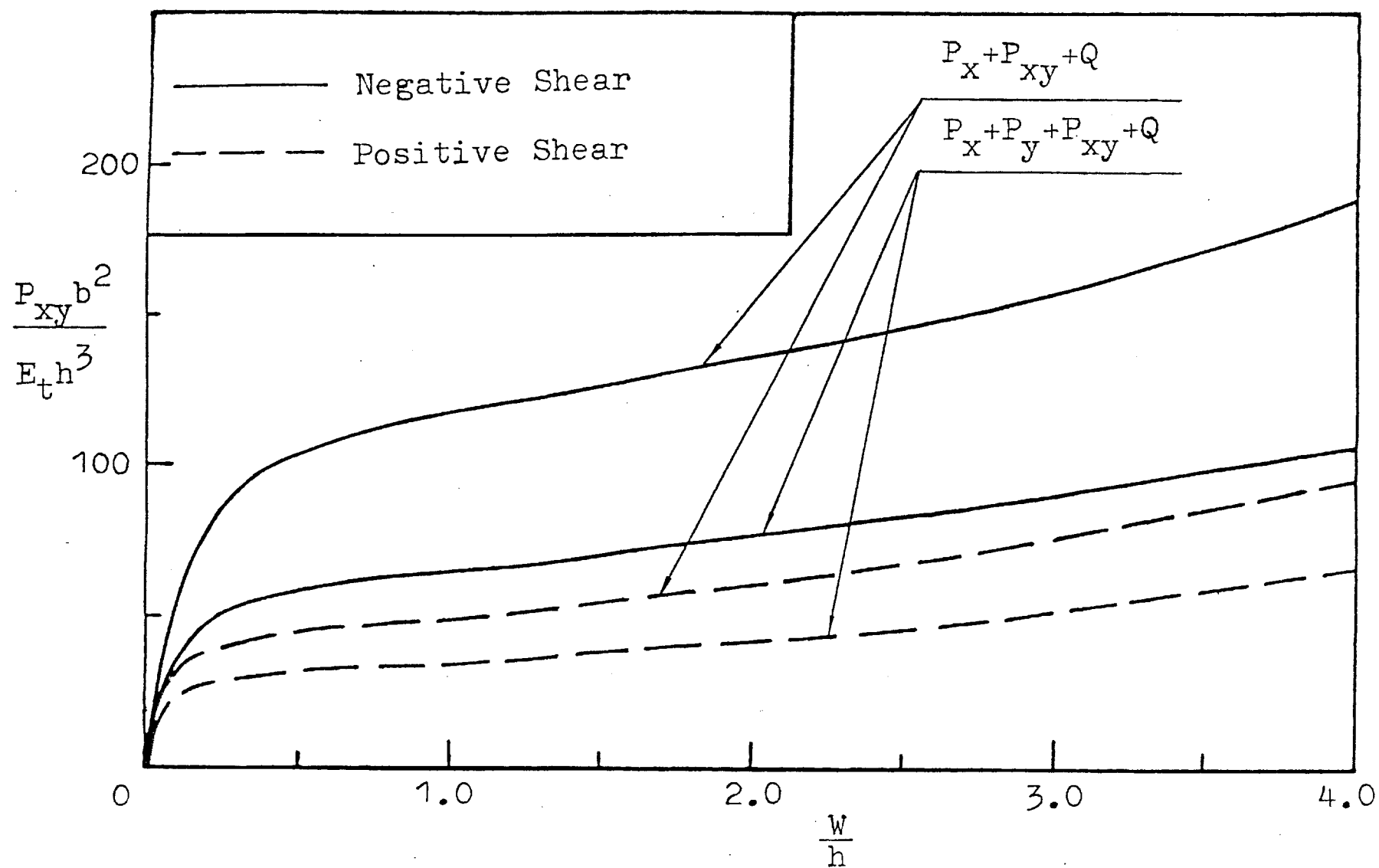


Fig. (V,21) Variation of central deflection with loading of square carbon/epoxy plates; ($\pm 45^\circ$) lay-up, under the combination of in-plane compression, shear and lateral loading

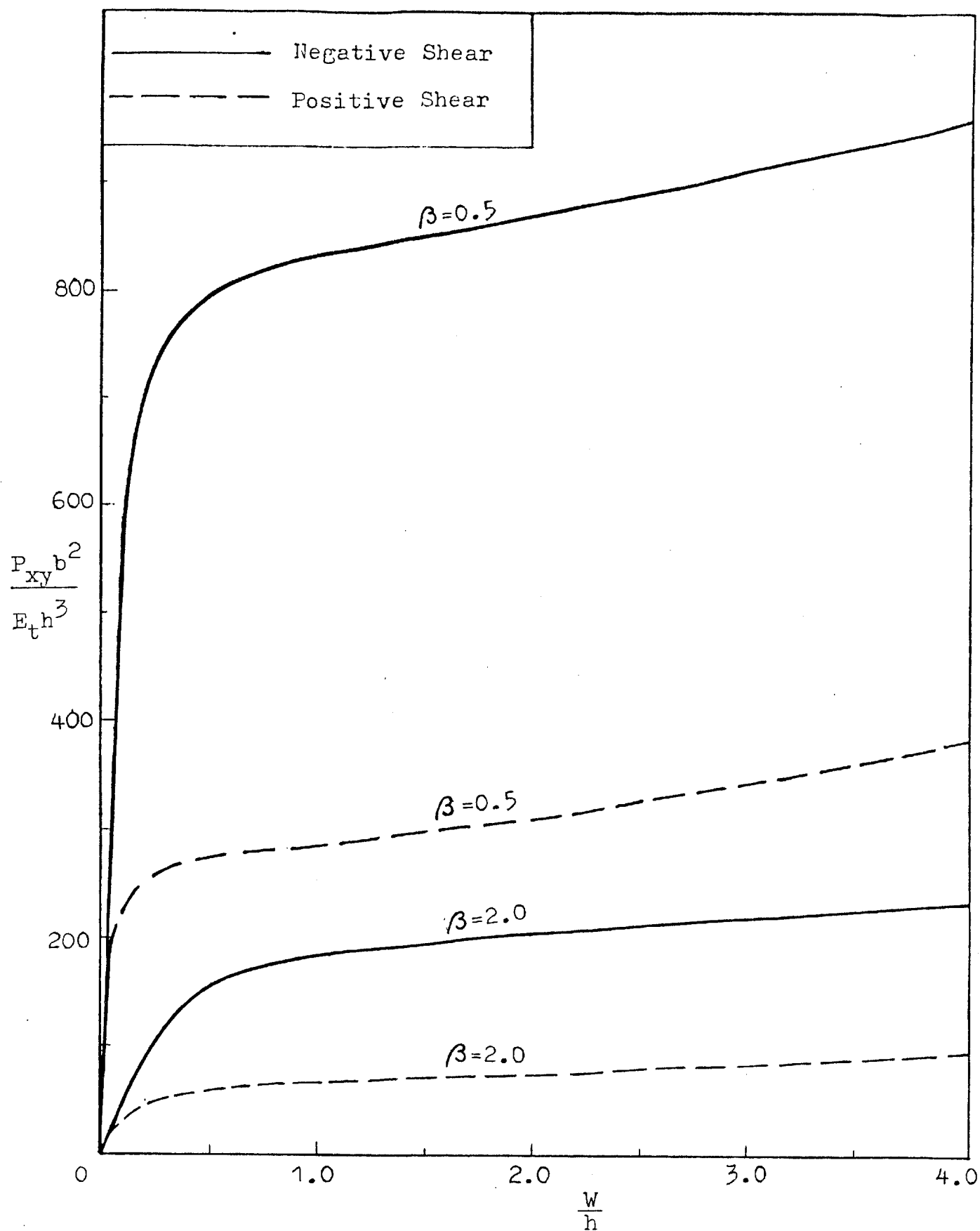


Fig. (V,22) Variation of central deflection with loading of carbon/epoxy plates; $(\pm 45^\circ)_s$ lay-up, different aspect ratios, under the combination of in-plane shear and lateral loading

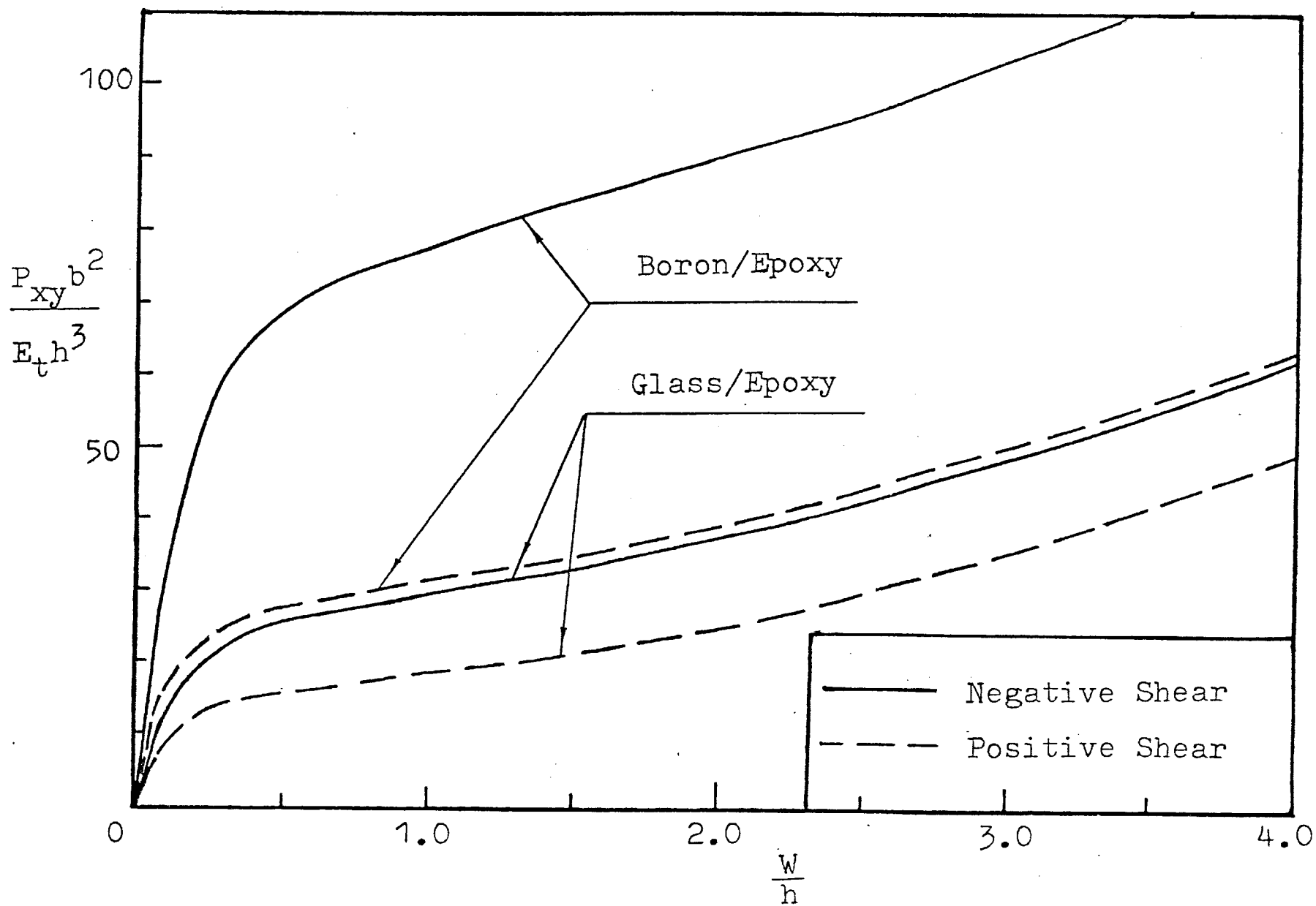


Fig. (V,23) Variation of central deflection with loading of square plates made of different materials; $(\pm 45^\circ)_s$ lay-up, under the combination of in-plane shear and lateral loading

CHAPTER VI

CONCLUDING REMARKS

A. Main Results of the Present Work

In the previous chapters of this thesis, initial buckling and postbuckling behaviour have been studied for unstiffened flat and curved panels of laminated composite materials under compressive and shear loading. Extensive results obtained from computations refer to different geometry, different loading, different boundaries and different lay-ups. A number of conclusions have been drawn in individual projects of this thesis. Now they are reviewed below.

1. Effect of Curvature

The curvature of composite panels has significant influence on the initial buckling and postbuckling behaviour, as for isotropic panels. The initial buckling load increases with increasing panel curvature (decreasing radius). But for curved panels, the load-deflection, load-inner force and moment relations become no longer monotonic, and snap-through may, therefore, occur when load becomes large. This is because there exists more than one equilibrium state of the panel corresponding to a certain loading level. "Jump" from one state to another is possible.

2. The Effect of Shear Direction

In-plane applied shear direction is proved to be an extra factor for initial buckling and postbuckling behaviour of laminated composite panels. Shear in a certain direction may greatly raise the initial buckling load of the panel against compression. Besides, composite panels demonstrate

a preferred shear direction not only for initial buckling but also for postbuckling behaviour. The difference between two postbuckling paths corresponding to different directions of applied shear is sometimes very big. The effect of shear direction is also found for panels under the combination of shear, compressive and lateral loading.

The effect of applied shear direction is important for certain composite panels because the bending stiffness varies dramatically with the direction relative to the axes of the panels. The bending stiffness in one of the diagonal directions may be much higher than that in the other diagonal. When shear forces produce a tension along the diagonal with the lower stiffness, then a strengthening is produced.

For a constant shear loading it becomes important to select the lay-up correctly to obtain the higher performance of the panel.

3. The Effect of Lay-up

The lay-up arrangement in a laminated composite panel is important for its initial buckling and postbuckling behaviour. In general, increasing the number of alternate layers in the panel, while keeping the other conditions constant, gives higher performance of the panel. When the number of layers is very large, the panel may behave as if it were orthotropic. In contrast to this, a two layer antisymmetric arrangement always results in the lowest performance of the panel.

For symmetric and asymmetric lay-ups the direction of applied shear is important for the behaviour of the panel, but not important when the panel is layered antisymmetrically. The effects of shear direction mentioned in the last paragraph are gradually reduced with increasing numbers of layers.

4. Bending Effects in Certain Unsymmetric Composite Panels

For general crossply panels under compression, bending may occur immediately the load is applied. In that case, the response of the panel under in-plane loading is no longer of the bifurcational type. The deflection of the panel varies with load from the beginning of the loading process.

As far as pure shear loading is concerned, such bending effect is also found when the panel has antisymmetrical angle lay-ups.

The bending effect may considerably reduce the resistance of curved panels against collapse, unless the number of alternate layers in the panel becomes very large.

5. The Effect of Geometric Imperfection

Imperfection of geometry will influence the postbuckling behaviour of composite panels. It may greatly reduce the resistance of the panel against collapse under in-plane loading. Bending is involved in geometrically imperfect panels immediately load is applied. "Snap-through" type of collapse may happen in curved panels with small imperfections but may not happen in curved panels with large geometrical imperfections and imperfect flat plates. In the latter case, the behaviour of the panels under in-plane loading is not of the bifurcational type.

The shape of the imperfection is proved to be important for the behaviour of imperfect panels. Different shapes of geometrical imperfections can make considerable difference in reducing the resistance of the panel against collapse.

6. Distribution of Inner Forces and Moments

After buckling, inner forces and moments in a composite panel under uniform edge loading are no longer uniform at

different points of it. On cross-sections near the centre of the panel under compression the inner force N_x varies dramatically and in the small area surrounding the panel centre N_x may become tensile on overall compression of the panel. In the latter case the increasing load is withstood only by the parts of the panel near the side boundaries.

Additionally, after buckling, the largest moment may occur at a point other than the panel centre, even for a square panel.

7. The Effect of Boundary Conditions

Initial buckling and postbuckling performance of a composite panel is affected by its boundary conditions. Clamped edges always produce a higher performance of the panel than simply supported edges. But of the different kinds of mixed edges, the arrangement which produces the highest performance depends on the lay-up and geometry of the panel.

8. The Effect of Composite Elastic Properties

The initial buckling and postbuckling behaviour of composite panels depends on the properties of the layers of which the panel is composed. But in all the cases investigated in this thesis, conclusions 1-7 mentioned in this section remain valid. This means that the properties of the materials do not change the trends of initial buckling and postbuckling behaviour of the panels, although they do make a difference to the absolute values.

Additionally, it is found that the effect of shear direction becomes more severe for a composite with a higher ratio of E_l to E_t than for one with a lower ratio.

9. The Effect of Aspect Ratio

The aspect ratio of a composite panel is found to have some influence on its initial buckling and postbuckling behaviour,

unless the panel length is large. Generally speaking, shorter plates have a higher initial buckling load than that of longer plates, if the other dimensions of the plates are kept constant. But certain small fluctuations may also be found in the initial buckling load - aspect ratio curves (see Figure (III,8)). In the cases studied in this thesis it is found that the previous conclusions 1-8 pointed out in this section are valid for different aspect ratios, although different absolute values result in the different cases.

10. The Precision of the Beam Functions

In this thesis the beam function defined by

$$f(x) = \cosh \alpha_i x - \cos \alpha_i x - \gamma_i (\sinh \alpha_i x - \sin \alpha_i x)$$

where α_i and γ_i are constants, is used.

It is found that the precision of α_i and γ_i is important for the orthogonality of the beam functions and extremely precise values of α_i and γ_i need to be used to produce acceptable results for the buckling investigations. The results with 15 significant figures are, therefore, introduced in this investigation.

B. Suggestion for future work

The initial buckling and postbuckling analysis presented in this thesis is based on the virtual displacement principle and the virtual force principle. The governing equations including the effect of boundary moments are solved by assuming certain series solutions for two unknowns - the deflection function and stress function. This procedure is proved to be feasible for most engineering purposes. But it seems that the following improvements might be taken into account in future work in this particular field:

1. Exact solution

As mentioned before, so far no solution is known which satisfies exactly the simply supported boundary conditions of a generally laminated panel. And in that case no solutions may be expressed in terms of separable functions in x and y directions when the coupling terms D_{16} and D_{26} are not zero. The solutions assumed for the deflection function in this thesis satisfy only kinematic boundary conditions, though the moment boundary conditions are satisfied in the mean with the help of the virtual displacement principle. So efforts could be made to find an exact solution, which is not in the separate form of the function in x and y directions. It may be expected that such a solution would bring a most important advance in these problems. But at the same time the exact solution, if found, would introduce extra difficulties in mathematical procedure and numerical evaluation.

2. Complicating Effects

The panels under consideration in this thesis are assumed to obey the Kirchhoff hypothesis. This means that the effect of transverse shear deformation is ignored. Generally speaking it is true that such a hypothesis produces adequate results for most structural applications in aircraft structures. But in certain special cases, such as short panels, the transverse shear (or interface shear between composite layers) of the panel might no longer be ignorable. In that case the present analysis needs to be extended to include the effect of such a shear. But it is obvious that any improvement made for considering transverse shear must bring additional complexity to these problems.

In the present analysis a linear elastic relationship between stress and strain is assumed. This is, no doubt, of

primary significance in buckling and postbuckling investigations. In practice, however, non-linear elastic relations may sometimes occur. A more sophisticated analysis to allow arbitrary composite materials would need to include the effect of non-linear elasticity.

There are other effects such as thermal strains and local delaminations, which may also have some influence on the buckling and postbuckling response of composite panels in engineering practice. An investigation into the behaviour of composite panels including such effects might be of significance both theoretically and practically. It is, however, not an easy task.

3. In-plane Boundary Conditions on Loaded Edges

In this thesis several different kinds of flexural (transverse) boundary conditions of panels are considered for the postbuckling investigation. At the same time the in-plane boundary conditions on the two loaded edges are taken as those of uniform loading. Uniform loading conditions are certainly one of the most common situations assumed in all kinds of postbuckling investigations of compressed plates, either isotropic or orthotropic. Detailed investigation of such conditions is of primary significance. But it needs to be pointed out that uniform axial displacement conditions (keeping the edges always straight when loading) may frequently be found in analytical and experimental investigations. It seems that the uniform displacement conditions are the easiest to realise in experiments.

In order to consider the uniform displacement conditions in postbuckling analysis of generally laminated composite plates, it appears more convenient to use a governing system consisting of three equations with the variables as displacements u , v and w in x , y and z directions respectively. With u , v , w

properly assumed, the governing system may be transformed to a set of algebraic equations in matrix form. Solutions may then be obtained using an iterative procedure. In this case the iteration will almost certainly take more computing time than for the uniform loading situation because there are three matrix equations to be solved simultaneously instead of two.

Finally it can be expected that almost all the conclusions drawn in the current investigation will remain true for uniform displacement conditions.

4. Prebuckling State of Panels

In buckling and postbuckling investigations of plates and shells, most work, including the current investigation, has been based on the assumption that there exists a uniform field of stress in the prebuckling state. But, as pointed out by a previous survey^[119], theoretical analysis, including non-linear prebuckling behaviour, has produced improved agreement between theoretical prediction and experimental results for buckling behaviour of isotropic cylinders. It is, therefore, expected that an analysis for curved composite panels taking such prebuckling response into account would create better results. But, of course, the complexity resulting from this improvement might be extreme.

5. Experiments

In the present investigation an extensive study has been carried out for the buckling and postbuckling behaviour of curved panels of generally layered composite materials. As mentioned before, certain theoretical improvements and discoveries have been made in this thesis. At the same time it should be pointed out that a large part of the conclusions drawn in the present work have not yet been verified by reliable

experiments so far, although in a few special cases comparison between the present analysis and previous experiments indicates encouraging agreement. One of the most important facts found in this thesis is probably the effect of applied shear direction on the buckling and postbuckling behaviour of composite panels under combined compression and shear loading. Verifying this fact with carefully designed experiments is of primary significance. For this purpose a special appliance must be developed to suit simultaneous loading in compression and shear. Additionally, the proper boundary conditions need to be provided. It appears not at all easy to realise these requirements completely in practice.

6. Ultimate Failure of Panels

In this thesis, the postbuckling investigations are carried out on the basis of the assumption that there is no damage in the layers which occurs during postbuckling of the panel. This is true when the strains in the composite layers are not too large. But it may not be the case when the strains become very large. In the latter case, damage may occur in the individual layers or between these layers and the stiffnesses of the panels may be greatly changed. Then the postbuckling behaviour of the panels may be quite different from that predicted for non-damaged ones. To consider such effects a more sophisticated theory is needed, which should include both the postbuckling behaviour analysis and a strength analysis of the panels. But, of course, this makes the problem much more complicated.

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Appendix A

Derivation of Equilibrium Equation

The equilibrium equation for postbuckling behaviour of a curved panel of laminated composites may be obtained from the virtual displacement principle

$$\delta \Pi = \delta U - \delta W = 0 \quad (A, 1)$$

where $\delta \Pi$ variation of total potential energy of the panel

δU variation of inner potential energy of the panel

δW variation of external work

$$\delta U = \int_0^a \int_0^b \left[N_x \delta \epsilon_x^0 + N_y \delta \epsilon_y^0 + N_{xy} \delta \gamma_{xy}^0 + M_x \delta k_x + M_y \delta k_y + M_{xy} \delta k_{xy} \right] dx dy \quad (A, 2)$$

$$\delta W = - \int_0^b (P_x \delta u_0) \Big|_{x=0}^{x=a} dy + \int_0^a (P_{xy} \delta u_0) \Big|_{y=0}^{y=b} dx + \int_0^b (P_{xy} \delta v_0) \Big|_{x=0}^{x=a} dy \quad (A, 3)$$

where

$$\epsilon_x^0 = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2$$

$$\epsilon_y^0 = \frac{\partial v_0}{\partial y} + \frac{1}{2} \left(\frac{\partial w_0}{\partial y} \right)^2 - \frac{w_0}{R}$$

$$\gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y} \quad (A, 4)$$

$$k_x = - \frac{\partial^2 w_0}{\partial x^2}$$

$$k_y = - \frac{\partial^2 w_0}{\partial y^2}$$

$$k_{xy} = -2 \frac{\partial^2 w_0}{\partial x \partial y}$$

Setting (A,4) into (A,2), developing (A,2) term by term, noticing the rules of integrating by parts, we have

$$\begin{aligned}
\int_0^a \int_0^b N_x \delta \varepsilon_x^o dx dy &= \int_0^a \int_0^b N_x \delta \left[\frac{\partial u_o}{\partial x} + \frac{1}{2} \left(\frac{\partial w_o}{\partial x} \right)^2 \right] dx dy \\
&= \int_0^a \int_0^b N_x \left[\frac{\partial}{\partial x} (\delta u_o) + \frac{\partial w_o}{\partial x} \frac{\partial}{\partial x} (\delta w_o) \right] dx dy \\
&= \int_0^b (N_x \delta u_o) \Big|_{x=0}^{x=a} dy - \int_0^a \int_0^b \frac{\partial N_x}{\partial x} \delta u_o dx dy \\
&\quad + \int_0^b (N_x \frac{\partial w_o}{\partial x} \delta w_o) \Big|_{x=0}^{x=a} dy - \int_0^a \int_0^b (N_x \frac{\partial^2 w_o}{\partial x^2} + \frac{\partial N_x}{\partial x} \frac{\partial w_o}{\partial x}) \delta w_o dx dy
\end{aligned}$$

$$\begin{aligned}
\int_0^a \int_0^b N_y \delta \varepsilon_y^o dx dy &= \int_0^a \int_0^b N_y \delta \left[\frac{\partial v_o}{\partial y} + \frac{1}{2} \left(\frac{\partial w_o}{\partial y} \right)^2 - \frac{w_o}{R} \right] dx dy \\
&= \int_0^a \int_0^b N_y \left[\frac{\partial}{\partial y} (\delta v_o) + \frac{\partial w_o}{\partial y} \frac{\partial}{\partial y} (\delta w_o) - \frac{1}{R} (\delta w_o) \right] dx dy \\
&= \int_0^a (N_y \delta v_o) \Big|_{y=0}^{y=b} dx - \int_0^a \int_0^b \frac{\partial N_y}{\partial y} \delta v_o dx dy + \int_0^a (N_y \frac{\partial w_o}{\partial y} \delta w_o) \Big|_{y=0}^{y=b} dx \\
&\quad - \int_0^a \int_0^b (N_y \frac{\partial^2 w_o}{\partial y^2} + \frac{\partial N_y}{\partial y} \frac{\partial w_o}{\partial y}) \delta w_o dx dy - \int_0^a \int_0^b \frac{N_y}{R} \delta w_o dx dy
\end{aligned}$$

$$\begin{aligned}
\int_0^a \int_0^b N_{xy} \delta \gamma_{xy}^o dx dy &= \int_0^a \int_0^b N_{xy} \delta \left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + \frac{\partial w_o}{\partial x} \frac{\partial w_o}{\partial y} \right) dx dy \\
&= \int_0^a \int_0^b N_{xy} \left[\frac{\partial}{\partial y} (\delta u_o) + \frac{\partial}{\partial x} (\delta v_o) + \frac{\partial w_o}{\partial x} \frac{\partial}{\partial y} (\delta w_o) + \frac{\partial w_o}{\partial y} \frac{\partial}{\partial x} (\delta w_o) \right] dx dy \\
&= \int_0^a (N_{xy} \delta u_o) \Big|_{y=0}^{y=b} dx - \int_0^a \int_0^b \frac{\partial N_{xy}}{\partial y} \delta u_o dx dy + \int_0^b N_{xy} \delta v_o \Big|_{x=0}^{x=a} dy \\
&\quad - \int_0^a \int_0^b \frac{\partial N_{xy}}{\partial x} \delta v_o dx dy + \int_0^a N_{xy} \frac{\partial w_o}{\partial x} \delta w_o \Big|_{y=0}^{y=b} dx - \int_0^a \int_0^b (N_{xy} \frac{\partial^2 w_o}{\partial x \partial y} + \frac{\partial N_{xy}}{\partial y} \frac{\partial w_o}{\partial x}) \delta w_o dx dy \\
&\quad + \int_0^b (N_{xy} \frac{\partial w_o}{\partial y} \delta w_o) \Big|_{x=0}^{x=a} dy - \int_0^a \int_0^b (N_{xy} \frac{\partial^2 w_o}{\partial x \partial y} + \frac{\partial N_{xy}}{\partial x} \frac{\partial w_o}{\partial y}) \delta w_o dx dy
\end{aligned}$$

$$\begin{aligned}
\int_0^a \int_0^b M_x \delta k_x dx dy &= \int_0^a \int_0^b M_x \delta \left(-\frac{\partial^2 W_0}{\partial x^2} \right) dx dy \\
&= - \int_0^a \int_0^b M_x \frac{\partial}{\partial x} \left[\delta \left(\frac{\partial W_0}{\partial x} \right) \right] dx dy \\
&= - \int_0^b \left[M_x \delta \left(\frac{\partial W_0}{\partial x} \right) \right] \Big|_{x=0}^{x=a} dy + \int_0^a \int_0^b \frac{\partial M_x}{\partial x} \delta \left(\frac{\partial W_0}{\partial x} \right) dx dy \\
&= - \int_0^b \left[M_x \delta \left(\frac{\partial W_0}{\partial x} \right) \right] \Big|_{x=0}^{x=a} dy + \int_0^a \left[\frac{\partial M_x}{\partial x} \delta W_0 \right] \Big|_{x=0}^{x=a} dy - \int_0^a \int_0^b \frac{\partial^2 M_x}{\partial x^2} \delta W_0 dx dy
\end{aligned}$$

$$\begin{aligned}
\int_0^a \int_0^b M_y \delta k_y dx dy &= \int_0^a \int_0^b M_y \delta \left(-\frac{\partial^2 W_0}{\partial y^2} \right) dx dy = - \int_0^a \int_0^b M_y \frac{\partial}{\partial y} \left[\delta \left(\frac{\partial W_0}{\partial y} \right) \right] dx dy \\
&= - \int_0^a \left[M_y \delta \left(\frac{\partial W_0}{\partial y} \right) \right] \Big|_{y=0}^{y=b} dx + \int_0^a \int_0^b \frac{\partial M_y}{\partial y} \delta \left(\frac{\partial W_0}{\partial y} \right) dx dy \\
&= - \int_0^a \left[M_y \delta \left(\frac{\partial W_0}{\partial y} \right) \right] \Big|_{y=0}^{y=b} dx + \int_0^a \left[\frac{\partial M_y}{\partial y} \delta W_0 \right] \Big|_{y=0}^{y=b} dx - \int_0^a \int_0^b \frac{\partial^2 M_y}{\partial y^2} \delta W_0 dx dy
\end{aligned}$$

$$\begin{aligned}
\int_0^a \int_0^b M_{xy} \delta k_{xy} dx dy &= \int_0^a \int_0^b M_{xy} \delta \left(-2 \frac{\partial^2 W_0}{\partial x \partial y} \right) dx dy = - \int_0^a \int_0^b M_{xy} \left\{ \frac{\partial}{\partial x} \left[\delta \left(\frac{\partial W_0}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[\delta \left(\frac{\partial W_0}{\partial x} \right) \right] \right\} dx dy \\
&= - \int_0^b \left[M_{xy} \delta \left(\frac{\partial W_0}{\partial y} \right) \right] \Big|_{x=0}^{x=a} dy + \int_0^a \int_0^b \frac{\partial M_{xy}}{\partial x} \delta \left(\frac{\partial W_0}{\partial y} \right) dx dy - \int_0^a \left[M_{xy} \delta \left(\frac{\partial W_0}{\partial x} \right) \right] \Big|_{y=0}^{y=b} dx \\
&\quad + \int_0^a \int_0^b \frac{\partial M_{xy}}{\partial y} \delta \left(\frac{\partial W_0}{\partial x} \right) dx dy \\
&= \int_0^b \left[\frac{\partial M_{xy}}{\partial y} \delta W_0 - M_{xy} \delta \left(\frac{\partial W_0}{\partial y} \right) \right] \Big|_{x=0}^{x=a} dy + \int_0^a \left[\frac{\partial M_{xy}}{\partial x} \delta W_0 - M_{xy} \delta \left(\frac{\partial W_0}{\partial x} \right) \right] \Big|_{y=0}^{y=b} dx \\
&\quad - \int_0^a \int_0^b 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \delta W_0 dx dy
\end{aligned}$$

(A, 5)

Finally, $\delta \Pi$ may be written in the following form :

$$\begin{aligned}
\delta \Pi = & - \int_0^a \int_0^b \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u_0 dx dy - \int_0^a \int_0^b \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) \delta v_0 dx dy \\
& - \int_0^a \int_0^b \left\{ \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{N_y}{R} + N_x \frac{\partial^2 w_0}{\partial x^2} + N_y \frac{\partial^2 w_0}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w_0}{\partial x \partial y} \right. \\
& + \frac{\partial w_0}{\partial x} \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) + \frac{\partial w_0}{\partial y} \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) \left. \right\} \delta w_0 dx dy \\
& + \int_0^b \left[(N_x + P_x) \delta u_0 \right]_{x=0}^{x=a} dy + \int_0^a \left[(N_{xy} - P_{xy}) \delta u_0 \right]_{y=0}^{y=b} dx \\
& + \int_0^b \left[(N_{xy} - P_{xy}) \delta v_0 \right]_{x=0}^{x=a} dy + \int_0^a \left[\left(N_x \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y} \right) \delta w_0 \right]_{x=0}^{x=a} dy \\
& + \int_0^a \left[\left(N_y \frac{\partial w_0}{\partial y} + N_{xy} \frac{\partial w_0}{\partial x} \right) \delta w_0 \right]_{y=0}^{y=b} dx + \int_0^a \left[N_y \delta v_0 \right]_{y=0}^{y=b} dx \\
& + \int_0^b \left[\frac{\partial M_x}{\partial x} \delta w_0 - M_x \delta \left(\frac{\partial w_0}{\partial x} \right) \right]_{x=0}^{x=a} dy + \int_0^a \left[\frac{\partial M_y}{\partial y} \delta w_0 - M_y \delta \left(\frac{\partial w_0}{\partial y} \right) \right]_{y=0}^{y=b} dx \\
& + \int_0^b \left[\frac{\partial M_{xy}}{\partial y} \delta w_0 - M_{xy} \delta \left(\frac{\partial w_0}{\partial y} \right) \right]_{x=0}^{x=a} dy + \int_0^a \left[\frac{\partial M_{xy}}{\partial x} \delta w_0 - M_{xy} \delta \left(\frac{\partial w_0}{\partial x} \right) \right]_{y=0}^{y=b} dx \\
& = 0
\end{aligned} \tag{A, 6}$$

Appendix B

Integrals for Initial Buckling

The constants used in the resultant expressions for the initial buckling of curved panels without in-plane edge restraint (section B of Chapter III) are defined by the following integral expressions

$$C_0^{mi} = \int_0^1 X_m(\xi) \sin i\pi \xi d\xi$$

$$C_1^{mi} = \int_0^1 \overline{X}_m(\xi) \sin i\pi \xi d\xi$$

$$C_2^{mi} = \int_0^1 \overline{\overline{X}}_m(\xi) \sin i\pi \xi d\xi$$

$$C_3^{mi} = \int_0^1 \overline{\overline{\overline{X}}}_m(\xi) \sin i\pi \xi d\xi$$

$$J_0^{mi} = \int_0^1 X_m(\xi) X_i(\xi) d\xi = 2 \int_0^1 \sin i\pi \xi \sin j\pi \xi d\xi$$

$$J_1^{mi} = \int_0^1 \overline{X}_m(\xi) X_i(\xi) d\xi$$

$$J_2^{mi} = \int_0^1 \overline{\overline{X}}_m(\xi) X_i(\xi) d\xi$$

$$J_3^{mi} = \int_0^1 \overline{\overline{\overline{X}}}_m(\xi) X_i(\xi) d\xi$$

$$H_0^{mi} = \int_0^1 \cos m\pi \xi \sin i\pi \xi d\xi$$

$$R_0^{mi} = \int_0^1 X_m(\xi) \cos i\pi \xi d\xi$$

where

$$\chi_i(\xi) = \cosh \alpha_i \xi - \cos \alpha_i \xi - \gamma_i (\sinh \alpha_i \xi - \sin \alpha_i \xi)$$

and

$$\bar{\chi}_i(\xi) = \frac{1}{\alpha_i} \frac{d\chi_i(\xi)}{d\xi}$$

$$\bar{\bar{\chi}}_i(\xi) = \frac{1}{\alpha_i^2} \frac{d^2\chi_i(\xi)}{d\xi^2}$$

$$\bar{\bar{\bar{\chi}}}_i(\xi) = \frac{1}{\alpha_i^3} \frac{d^3\chi_i(\xi)}{d\xi^3}$$

It is obvious that with η replacing ξ these integrals are still true.

These integrals may be approximately obtained by numerical methods. They may be exactly obtained by integrating directly. The results of direct integration are given below.

$$C_0^{mi} = \frac{1-\gamma_m}{2} \delta_1^{mi} + \frac{1+\gamma_m}{2} \delta_2^{mi} - \delta_7^{mi} + \gamma_m \delta_8^{mi}$$

$$C_1^{mi} = \frac{1-\gamma_m}{2} \delta_1^{mi} - \frac{1+\gamma_m}{2} \delta_2^{mi} + \delta_8^{mi} + \gamma_m \delta_7^{mi}$$

$$C_2^{mi} = \frac{1-\gamma_m}{2} \delta_1^{mi} + \frac{1+\gamma_m}{2} \delta_2^{mi} + \delta_7^{mi} - \gamma_m \delta_8^{mi}$$

$$C_3^{mi} = \frac{1-\gamma_m}{2} \delta_1^{mi} - \frac{1+\gamma_m}{2} \delta_2^{mi} - \delta_8^{mi} - \gamma_m \delta_7^{mi}$$

$$J_0^{mi} = \begin{cases} 0 & m \neq i \\ 1 & m = i \end{cases}$$

$$\begin{aligned}
J_1^{mi} = & \frac{(1-\gamma_m)(1-\gamma_i)}{4} \delta_9^{mi} - \frac{(1+\gamma_m)(1-\gamma_i)}{4} \delta_{10}^{mi} + \frac{(1-\gamma_i)}{2} \delta_{13}^{mi} + \frac{(1-\gamma_i)\gamma_m}{2} \delta_{14}^{im} \\
& + \frac{(1-\gamma_m)(1+\gamma_i)}{4} \delta_{10}^{im} - \frac{(1+\gamma_m)(1+\gamma_i)}{4} \delta_{11}^{mi} + \frac{(1+\gamma_i)}{2} \delta_{15}^{im} + \frac{(1+\gamma_i)\gamma_m}{2} \delta_{16}^{im} \\
& - \frac{(1-\gamma_m)}{2} \delta_{14}^{mi} + \frac{(1+\gamma_m)}{2} \delta_{16}^{mi} - \delta_{17}^{mi} - \gamma_m \delta_{19}^{im} + \frac{(1-\gamma_m)\gamma_i}{2} \delta_{13}^{mi} \\
& - \frac{(1+\gamma_m)\gamma_i}{2} \delta_{15}^{mi} + \gamma_i \delta_{18}^{im} + \gamma_i \gamma_m \delta_{17}^{mi}
\end{aligned}$$

$$\begin{aligned}
J_2^{mi} = & \frac{(1-\gamma_i)(1-\gamma_m)}{4} \delta_9^{mi} + \frac{(1+\gamma_m)(1-\gamma_i)}{4} \delta_{10}^{mi} + \frac{(1-\gamma_i)}{2} \delta_{14}^{im} - \frac{(1-\gamma_i)\gamma_m}{2} \delta_{13}^{im} \\
& + \frac{(1-\gamma_m)(1+\gamma_i)}{4} \delta_{10}^{im} + \frac{(1+\gamma_i)(1+\gamma_m)}{4} \delta_{11}^{mi} + \frac{(1+\gamma_i)}{2} \delta_{16}^{im} - \frac{(1+\gamma_i)\gamma_m}{2} \delta_{15}^{im} \\
& - \frac{(1-\gamma_m)}{2} \delta_{14}^{mi} - \frac{(1+\gamma_m)}{2} \delta_{16}^{mi} - \delta_{19}^{im} + \gamma_m \delta_{17}^{im} + \frac{(1-\gamma_m)\gamma_i}{2} \delta_{13}^{mi} \\
& + \frac{(1+\gamma_m)\gamma_i}{2} \delta_{15}^{mi} + \gamma_i \delta_{17}^{mi} - \gamma_m \gamma_i \delta_{18}^{im}
\end{aligned}$$

$$\begin{aligned}
J_3^{mi} = & \frac{(1-\gamma_m)(1-\gamma_i)}{4} \delta_9^{mi} - \frac{(1+\gamma_m)(1-\gamma_i)}{4} \delta_{10}^{mi} - \frac{(1-\gamma_i)}{2} \delta_{13}^{im} - \frac{(1-\gamma_i)\gamma_m}{2} \delta_{14}^{im} \\
& + \frac{(1-\gamma_m)(1+\gamma_i)}{4} \delta_{10}^{im} - \frac{(1+\gamma_m)(1+\gamma_i)}{4} \delta_{11}^{mi} - \frac{(1+\gamma_i)}{2} \delta_{15}^{im} - \frac{(1+\gamma_i)\gamma_m}{2} \delta_{16}^{im} \\
& - \frac{(1-\gamma_m)}{2} \delta_{14}^{mi} + \frac{(1+\gamma_m)}{2} \delta_{16}^{mi} + \delta_{17}^{im} + \gamma_m \delta_{19}^{im} + \frac{(1-\gamma_m)\gamma_i}{2} \delta_{13}^{mi} \\
& - \frac{(1+\gamma_m)\gamma_i}{2} \delta_{15}^{mi} - \gamma_i \delta_{18}^{im} - \gamma_i \gamma_m \delta_{17}^{mi}
\end{aligned}$$

$$H_0^{mi} = \delta_{12}^{mi}$$

$$R_0^{mi} = \frac{(1-\gamma_m)}{2} \delta_3^{mi} + \frac{(1+\gamma_m)}{2} \delta_4^{mi} - \delta_5^{mi} + \gamma_i \delta_6^{mi}$$

$$S_0^{mi} = (-1)^i \bar{X}_m(1) - \bar{X}_m(0)$$

$$Q_0^{mi} = -\frac{1}{2} [(-1)^{m+i} - 1]$$

where

$$\delta_1^{mi} = \begin{cases} \frac{i\pi}{\alpha_m^2 + (i\pi)^2} (1 + e^{\alpha_m}) & \text{when } i \text{ odd} \\ \frac{i\pi}{\alpha_m^2 + (i\pi)^2} (1 - e^{\alpha_m}) & \text{when } i \text{ even} \end{cases}$$

$$\delta_2^{mi} = \begin{cases} \frac{i\pi}{\alpha_m^2 + (i\pi)^2} (1 + e^{-\alpha_m}) & \text{when } i \text{ odd} \\ \frac{i\pi}{\alpha_m^2 + (i\pi)^2} (1 - e^{-\alpha_m}) & \text{when } i \text{ even} \end{cases}$$

$$\delta_3^{mi} = \begin{cases} \frac{-\alpha_m}{\alpha_m^2 + (i\pi)^2} (e^{\alpha_m} + 1) & \text{when } i \text{ odd} \\ \frac{\alpha_m}{\alpha_m^2 + (i\pi)^2} (e^{\alpha_m} - 1) & \text{when } i \text{ even} \end{cases}$$

$$\delta_4^{mi} = \begin{cases} \frac{\alpha_m}{\alpha_m^2 + (i\pi)^2} (e^{-\alpha_m} + 1) & \text{when } i \text{ odd} \\ \frac{-\alpha_m}{\alpha_m^2 + (i\pi)^2} (e^{-\alpha_m} - 1) & \text{when } i \text{ even} \end{cases}$$

$$\delta_5^{mi} = \begin{cases} \frac{-\alpha_m \sin \alpha_m}{\alpha_m^2 - (i\pi)^2} & \text{when } i \text{ odd} \\ \frac{\alpha_m \sin \alpha_m}{\alpha_m^2 - (i\pi)^2} & \text{when } i \text{ even} \end{cases}$$

$$\delta_6^{mi} = \begin{cases} \frac{\alpha_m (1 + \cos \alpha_m)}{\alpha_m^2 - (i\pi)^2} & \text{when } i \text{ odd} \\ \frac{\alpha_m (1 - \cos \alpha_m)}{\alpha_m^2 - (i\pi)^2} & \text{when } i \text{ even} \end{cases}$$

$$\delta_7^{mi} = \begin{cases} \frac{-i\pi (1 + \cos \alpha_m)}{\alpha_m^2 - (i\pi)^2} & \text{when } i \text{ odd} \\ \frac{-i\pi (1 - \cos \alpha_m)}{\alpha_m^2 - (i\pi)^2} & \text{when } i \text{ even} \end{cases}$$

$$\delta_8^{mi} = \begin{cases} \frac{-i\pi \sin \alpha_m}{\alpha_m^2 - (i\pi)^2} & \text{when } i \text{ odd} \\ \frac{i\pi \sin \alpha_m}{\alpha_m^2 - (i\pi)^2} & \text{when } i \text{ even} \end{cases}$$

$$\delta_9^{mi} = \frac{e^{(\alpha_m + \alpha_i)} - 1}{\alpha_m + \alpha_i}$$

$$\delta_{10}^{mi} = \begin{cases} \frac{e^{(\alpha_i - \alpha_m)} - 1}{\alpha_i - \alpha_m} & \text{when } i \neq m \\ 1 & \text{when } i = m \end{cases}$$

$$\delta_{11}^{mi} = \frac{1 - e^{-(\alpha_m + \alpha_i)}}{\alpha_m + \alpha_i}$$

$$\delta_{12}^{mi} = \begin{cases} \frac{2i}{\pi(i^2 - m^2)} & \text{when } (i + m) \text{ odd} \\ 0 & \text{when } (i + m) \text{ even} \end{cases}$$

$$\delta_{13}^{mi} = \frac{1}{\alpha_m^2 + \alpha_i^2} \left[e^{\alpha_m} (\alpha_m \sin \alpha_i - \alpha_i \cos \alpha_i) + \alpha_i \right]$$

$$\delta_{14}^{mi} = \frac{1}{\alpha_m^2 + \alpha_i^2} \left[e^{\alpha_m} (\alpha_m \cos \alpha_i + \alpha_i \sin \alpha_i) - \alpha_m \right]$$

$$\delta_{15}^{mi} = \frac{1}{\alpha_m^2 + \alpha_i^2} \left[e^{-\alpha_m} (-\alpha_m \sin \alpha_i - \alpha_i \cos \alpha_i) + \alpha_i \right]$$

$$\delta_{16}^{mi} = \frac{1}{\alpha_m^2 + \alpha_i^2} \left[e^{-\alpha_m} (-\alpha_m \cos \alpha_i + \alpha_i \sin \alpha_i) + \alpha_m \right]$$

$$\delta_{17}^{mi} = \begin{cases} \frac{1}{4\alpha_m} (1 - \cos 2\alpha_m) & \text{when } i = m \\ \frac{1 - \cos(\alpha_i - \alpha_m)}{2(\alpha_i - \alpha_m)} + \frac{1 - \cos(\alpha_i + \alpha_m)}{2(\alpha_i + \alpha_m)} & \text{when } i \neq m \end{cases}$$

$$\delta_{18}^{mi} = \begin{cases} \frac{1}{2} - \frac{1}{4\alpha_m} \sin 2\alpha_m & \text{when } i = m \\ \frac{\sin(\alpha_m - \alpha_i)}{2(\alpha_m - \alpha_i)} - \frac{\sin(\alpha_m + \alpha_i)}{2(\alpha_m + \alpha_i)} & \text{when } i \neq m \end{cases}$$

$$\delta_{19}^{mi} = \begin{cases} \frac{1}{2} + \frac{1}{4\alpha_m} \sin 2\alpha_m & \text{when } i = m \\ \frac{\sin(\alpha_m - \alpha_i)}{2(\alpha_m - \alpha_i)} + \frac{\sin(\alpha_m + \alpha_i)}{2(\alpha_m + \alpha_i)} & \text{when } i \neq m \end{cases}$$

Appendix C

Elements of Matrices $[K^{(i)}]$ for Initial Buckling

Introduce the following initial expressions

$$I = (i-1)M + j$$

$$J = (m-1)M + n = (p-1)M + q$$

where

$$i = 1, 2, \dots, M$$

$$j = 1, 2, \dots, M$$

$$m = p = 1, 2, \dots, M$$

$$n = q = 1, 2, \dots, M$$

The elements of the matrices in the resultant system (III, 20) for initial buckling of curved composite panels with different boundaries are written as follows :

1) All edges clamped

$$K_{IJ}^{(1)} = \alpha_p^4 J_0^{pi} J_0^{qj} + b_1 \alpha_p^3 \alpha_q J_3^{pi} J_1^{qj} + b_2 \alpha_p^2 \alpha_q^2 J_2^{pi} J_2^{qj} \\ + b_3 \alpha_p \alpha_q^3 J_1^{pi} J_3^{qj} + b_4 \alpha_q^4 J_0^{pi} J_0^{qj}$$

$$K_{IJ}^{(2)} = b_5 \alpha_m^4 J_0^{mi} J_0^{nj} + b_6 \alpha_m^3 \alpha_n J_3^{mi} J_1^{nj} + b_7 \alpha_m^2 \alpha_n^2 J_2^{mi} J_2^{nj} \\ + b_8 \alpha_m \alpha_n^3 J_1^{mi} J_3^{nj} + b_9 \alpha_n^4 J_0^{mi} J_0^{nj} - (\beta^2 K_R / \bar{D}_{11}^*) \alpha_m^2 J_2^{mi} J_0^{nj}$$

$$K_{IJ}^{(3)} = -(\beta^2 E_t h^3 / D_{11}^*) \alpha_p^2 J_2^{pi} J_0^{qj}$$

$$K_{IJ}^{(4)} = (2/\beta^3 E_t h^3 / D_{11}^*) \alpha_p \alpha_q J_1^{pi} J_1^{qj}$$

$$K_{IJ}^{(5)} = \alpha_m^4 J_0^{mi} J_0^{nj} - a_1 \alpha_m^3 \alpha_n J_3^{mi} J_1^{nj} + a_2 \alpha_m^2 \alpha_n^2 J_2^{mi} J_2^{nj} \\ - a_3 \alpha_m \alpha_n^3 J_1^{mi} J_3^{nj} + a_4 \alpha_n^4 J_0^{mi} J_0^{nj}$$

$$K_{IJ}^{(6)} = a_5 \alpha_p^4 J_0^{pi} J_0^{qj} + a_6 \alpha_p^3 \alpha_q J_3^{pi} J_1^{qj} + a_7 \alpha_p^2 \alpha_q^2 J_2^{pi} J_2^{qj} \\ + a_8 \alpha_p \alpha_q^3 J_1^{pi} J_3^{qj} + a_9 \alpha_q^4 J_0^{pi} J_0^{qj} - (\beta^2 K_R / A_{22}^*) \alpha_p^2 J_2^{pi} J_0^{qj}$$

2) All edges simply supported

$$K_{IJ}^{(1)} = \frac{1}{4} (m^4 + b_2 m^2 n^2 + b_4 n^4) J_0^{mi} J_0^{nj} - b_1 m n (m^2 H_0^{mi} + \frac{i Q_0^{pi}}{\pi}) H_0^{nj} \\ - b_3 m n (n^2 H_0^{nj} + \frac{j Q_0^{qj}}{\pi}) H_0^{mi}$$

$$K_{IJ}^{(2)} = [b_5 \alpha_m^2 C_0^{nj} (\alpha_m^2 C_0^{mi} + i \pi S_0^{mi}) + b_9 \alpha_n^2 C_0^{mi} (\alpha_n^2 C_0^{nj} + j \pi S_0^{nj}) \\ + b_6 \alpha_m^3 \alpha_n C_3^{mi} C_1^{nj} + b_7 \alpha_m^2 \alpha_n^2 C_2^{mi} C_2^{nj} + b_8 \alpha_m \alpha_n^3 C_1^{mi} C_3^{nj} \\ - \frac{\beta^2 K_R}{D_{11}^*} \alpha_m^2 C_2^{mi} C_0^{nj}] \cdot \frac{1}{\pi^4}$$

$$K_{IJ}^{(3)} = \frac{i^2 \beta^2}{4 \pi^2} \frac{E_t h^3}{D_{11}^*} J_0^{mi} J_0^{nj}$$

$$K_{IJ}^{(4)} = \frac{2 \beta^3}{\pi^2} \frac{E_t h^3}{D_{11}^*} m n H_0^{mi} H_0^{nj}$$

$$K_{IJ}^{(5)} = (\alpha_m^4 + a_4 \alpha_n^4) J_0^{mi} J_0^{nj} - a_1 \alpha_m^3 \alpha_n J_3^{mi} J_1^{nj} + a_2 \alpha_m^2 \alpha_n^2 J_2^{mi} J_2^{nj} \\ - a_3 \alpha_m \alpha_n^3 J_1^{mi} J_3^{nj}$$

$$K_{IJ}^{(6)} = \pi^4 [(a_5 m^4 + a_7 m^2 n^2 + a_9 n^4) C_0^{im} C_0^{jn} - (a_6 m^3 n + a_8 m n^3) R_0^{im} R_0^{jn} \\ + \frac{\beta^2 m^2 K_R}{A_{22}^* \pi^2} C_0^{im} C_0^{jn}]$$

- 3) Longitudinal edges simply supported; curved edges clamped

$$K_{IJ}^{(1)} = \frac{1}{2} (\alpha_m^4 + b_4 n^4 \pi^4) J_0^{mi} J_0^{nj} + b_1 \alpha_m^3 (n\pi) J_3^{mi} H_0^{nj} - b_2 \alpha_m^2 \left(\frac{n^2 \pi^2}{2} \right) J_2^{mi} J_0^{nj} - b_3 \alpha_m (n\pi^2) (n^2 \pi H_0^{nj} + j Q_0^{nj}) J_1^{mi}$$

$$K_{IJ}^{(2)} = b_5 \alpha_m^4 J_0^{mi} C_0^{nj} + b_9 \alpha_n^2 J_0^{mi} (\alpha_n^2 C_0^{nj} + j \pi S_0^{nj}) + b_6 \alpha_m^3 \alpha_n J_3^{mi} C_1^{nj} + b_7 \alpha_m^2 \alpha_n^2 J_2^{mi} C_2^{nj} + b_8 \alpha_m \alpha_n^3 J_1^{mi} C_1^{nj} - \frac{\beta^2 K_R}{D_{11}^*} \alpha_m^2 J_2^{mi} C_0^{nj}$$

$$K_{IJ}^{(3)} = - \frac{\beta^2}{2} \left(\frac{E_t h^3}{D_{11}^*} \right) \alpha_m^2 J_2^{mi} J_0^{nj}$$

$$K_{IJ}^{(4)} = 2\beta^3 (n\pi) \alpha_m J_1^{mi} H_0^{nj} \cdot \left(\frac{E_t h^3}{D_{11}^*} \right)$$

$$K_{IJ}^{(5)} = (\alpha_m^4 + a_4 \alpha_n^4) J_0^{mi} J_0^{nj} - a_1 \alpha_m^3 \alpha_n J_3^{mi} J_1^{nj} + a_2 \alpha_m^2 \alpha_n^2 J_2^{mi} J_2^{nj} - a_3 \alpha_m \alpha_n^3 J_1^{mi} J_3^{nj}$$

$$K_{IJ}^{(6)} = (a_5 \alpha_m^4 + a_9 n^4 \pi^4) J_0^{mi} C_0^{jn} + a_6 \alpha_m^3 n \pi J_3^{mi} R_0^{jn} - a_7 \alpha_m^2 n^2 \pi^2 J_2^{mi} C_0^{jn} - a_8 \alpha_m n^3 \pi^3 J_1^{mi} R_0^{jn} - \frac{\beta^2 K_R}{A_{22}^*} \alpha_m^2 J_2^{mi} C_0^{jn}$$

- 4) Longitudinal edges clamped, curved edges simply supported

$$K_{IJ}^{(1)} = \frac{1}{2} (m^4 \pi^4 + b_4 \alpha_m^4) J_0^{mi} J_0^{nj} - b_1 \alpha_n m \pi^2 (m^2 \pi H_0^{mi} + i Q_0^{mi}) J_1^{nj} - b_2 \frac{m^2 \pi^2}{2} \alpha_n^2 J_0^{mi} J_2^{nj} + b_3 m \pi \alpha_n^3 H_0^{mi} J_3^{nj}$$

$$K_{IJ}^{(2)} = b_5 \alpha_m^2 J_0^{nj} (\alpha_m^2 C_0^{mi} + i \pi S_0^{mi}) + b_9 \alpha_n^4 C_0^{mi} J_0^{nj} + b_6 \alpha_m^3 \alpha_n C_3^{mi} J_1^{nj} + b_7 \alpha_m^2 \alpha_n^2 C_2^{mi} J_2^{nj} + b_8 \alpha_m \alpha_n^3 C_1^{mi} J_3^{nj} - \frac{\beta^2 K_R}{D_{11}^*} \alpha_m^2 C_2^{mi} J_0^{nj}$$

$$K_{IJ}^{(3)} = \frac{1}{2} i^2 \pi^2 \beta^2 \left(\frac{E_t h^3}{D_{11}^*} \right) J_0^{mi} J_0^{nj}$$

$$K_{IJ}^{(4)} = 2\beta^3 m \pi \alpha_n \left(\frac{E_t h^3}{D_{11}^*} \right) H_0^{mi} J_1^{nj}$$

$$K_{IJ}^{(5)} = (\alpha_m^4 + a_4 \alpha_n^4) J_0^{mi} J_0^{nj} - a_1 \alpha_m^3 \alpha_n J_3^{mi} J_1^{nj} + a_2 \alpha_m^2 \alpha_n^2 J_2^{mi} J_2^{nj} \\ - a_3 \alpha_m \alpha_n^3 J_1^{mi} J_3^{nj}$$

$$K_{IJ}^{(6)} = (a_5 m^4 \pi^4 + a_9 \alpha_n^4) C_0^{im} J_0^{nj} - a_6 m^3 \pi^3 \alpha_n R_0^{im} J_1^{nj} \\ - a_7 m^2 \pi^2 \alpha_n^2 C_0^{im} J_2^{nj} + a_8 m \pi \alpha_n^3 R_0^{im} J_3^{nj} + \frac{\beta^2 K_R}{A_{22}^*} m^2 \pi^2 C_0^{im} J_0^{nj}$$

Appendix D

Derivatives of Deflection and Stress Functions

Assume the deflection and stress function as follows:

$$F = -\bar{N}_x \eta^2 / 2 - \beta \bar{S} \xi \eta + \sum_m \sum_n F_{mn} X_m(\xi) Y_n(\eta)$$

$$W = \sum_p \sum_q \left[W_{pq}^{(1)} \sin p\pi \xi \sin q\pi \eta + W_{pq}^{(2)} \sin^2 p\pi \xi \sin^2 q\pi \eta \right]$$

The derivatives of these functions used for this investigation may be written :-

$$\frac{\partial F}{\partial \xi} = -\beta \bar{S} \eta + \sum_m \sum_n F_{mn} \alpha_m \bar{X}_m(\xi) Y_n(\eta)$$

$$\frac{\partial F}{\partial \eta} = -\bar{N}_x \eta - \beta \bar{S} \xi + \sum_m \sum_n F_{mn} \alpha_n X_m(\xi) \bar{Y}_n(\eta)$$

$$\frac{\partial^2 F}{\partial \xi^2} = \sum_m \sum_n F_{mn} \alpha_m^2 \bar{X}_m(\xi) Y_n(\eta)$$

$$\frac{\partial^2 F}{\partial \xi \partial \eta} = -\beta \bar{S} + \sum_m \sum_n F_{mn} \alpha_m \alpha_n \bar{X}_m(\xi) \bar{Y}_n(\eta)$$

$$\frac{\partial^2 F}{\partial \eta^2} = -\bar{N}_x + \sum_m \sum_n F_{mn} \alpha_n^2 X_m(\xi) \bar{Y}_n(\eta)$$

$$\frac{\partial^4 F}{\partial \xi^4} = \sum_m \sum_n F_{mn} \alpha_m^4 X_m(\xi) Y_n(\eta)$$

$$\frac{\partial^4 F}{\partial \xi^3 \partial \eta} = \sum_m \sum_n F_{mn} \alpha_m^3 \alpha_n \bar{X}_m(\xi) \bar{Y}_n(\eta)$$

$$\frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} = \sum_m \sum_n F_{mn} \alpha_m^2 \alpha_n^2 \bar{X}_m(\xi) \bar{Y}_n(\eta)$$

$$\frac{\partial^4 F}{\partial \xi \partial \eta^3} = \sum_m \sum_n F_{mn} \alpha_m \alpha_n^3 \bar{X}_m(\xi) \bar{Y}_n(\eta)$$

$$\frac{\partial^4 F}{\partial \eta^4} = \sum_m \sum_n F_{mn} \alpha_n^4 X_m(\xi) Y_n(\eta)$$

$$\frac{\partial W}{\partial \xi} = \sum_p \sum_q \left[W_{pq}^{(1)} p\pi \cos p\pi \xi \sin q\pi \eta + W_{pq}^{(2)} \frac{p\pi}{2} \sin 2p\pi \xi (1 - \cos 2q\pi \eta) \right]$$

$$\frac{\partial W}{\partial \eta} = \sum_p \sum_q \left[W_{pq}^{(1)} q\pi \sin p\pi \xi \cos q\pi \eta + W_{pq}^{(2)} \frac{q\pi}{2} (1 - \cos 2p\pi \xi) \sin 2q\pi \eta \right]$$

$$\frac{\partial^2 W}{\partial \xi^2} = \sum_p \sum_q \left[-W_{pq}^{(1)} p^2 \pi^2 \sin p\pi \xi \sin q\pi \eta + W_{pq}^{(2)} p^2 \pi^2 \cos 2p\pi \xi (1 - \cos 2q\pi \eta) \right]$$

$$\frac{\partial^2 W}{\partial \xi \partial \eta} = \sum_p \sum_q \left[W_{pq}^{(1)} p q \pi^2 \cos p\pi \xi \cos q\pi \eta + W_{pq}^{(2)} p q \pi^2 \sin 2p\pi \xi \sin 2q\pi \eta \right]$$

$$\frac{\partial^2 W}{\partial \eta^2} = \sum_p \sum_q \left[-W_{pq}^{(1)} q^2 \pi^2 \sin p\pi \xi \sin q\pi \eta + W_{pq}^{(2)} q^2 \pi^2 (1 - \cos 2p\pi \xi) \cos 2q\pi \eta \right]$$

$$\frac{\partial^3 W}{\partial \xi^3} = \sum_p \sum_q \left[-W_{pq}^{(1)} p^3 \pi^3 \cos p\pi \xi \sin q\pi \eta - W_{pq}^{(2)} (2p^3 \pi^3) \sin 2p\pi \xi (1 - \cos 2q\pi \eta) \right]$$

$$\frac{\partial^3 W}{\partial \xi^2 \partial \eta} = \sum_p \sum_q \left[-W_{pq}^{(1)} p^2 q \pi^3 \sin p\pi \xi \cos q\pi \eta + W_{pq}^{(2)} (2p^2 q \pi^3) \cos 2p\pi \xi \sin 2q\pi \eta \right]$$

$$\frac{\partial^3 W}{\partial \xi \partial \eta^2} = \sum_p \sum_q \left[-W_{pq}^{(1)} p q^2 \pi^3 \cos p\pi \xi \sin q\pi \eta + W_{pq}^{(2)} (2p q^2 \pi^3) \sin 2p\pi \xi \cos 2q\pi \eta \right]$$

$$\frac{\partial^3 W}{\partial \eta^3} = \sum_p \sum_q \left[-W_{pq}^{(1)} q^3 \pi^3 \sin p\pi \xi \cos q\pi \eta - W_{pq}^{(2)} (2q^3 \pi^3) (1 - \cos 2p\pi \xi) \sin 2q\pi \eta \right]$$

$$\frac{\partial^4 W}{\partial \xi^4} = \sum_p \sum_q \left[W_{pq}^{(1)} p^4 \pi^4 \sin p\pi \xi \sin q\pi \eta - W_{pq}^{(2)} (4p^4 \pi^4) \cos 2p\pi \xi (1 - \cos 2q\pi \eta) \right]$$

$$\frac{\partial^4 W}{\partial \xi^3 \partial \eta} = \sum_p \sum_q \left[-W_{pq}^{(1)} p^3 q \pi^4 \cos p\pi \xi \cos q\pi \eta - W_{pq}^{(2)} (4p^3 q \pi^4) \sin 2p\pi \xi \sin 2q\pi \eta \right]$$

$$\frac{\partial^4 W}{\partial \xi^2 \partial \eta^2} = \sum_p \sum_q \left[W_{pq}^{(1)} p^2 q^2 \pi^4 \sin p\pi \xi \sin q\pi \eta + W_{pq}^{(2)} (4p^2 q^2 \pi^4) \cos 2p\pi \xi \cos 2q\pi \eta \right]$$

$$\frac{\partial^4 W}{\partial \xi \partial \eta^3} = \sum_p \sum_q \left[-W_{pq}^{(1)} p q^3 \pi^4 \cos p\pi \xi \cos q\pi \eta - W_{pq}^{(2)} (4p q^3 \pi^4) \sin 2p\pi \xi \sin 2q\pi \eta \right]$$

$$\frac{\partial^4 W}{\partial \eta^4} = \sum_p \sum_q \left[W_{pq}^{(1)} q^4 \pi^4 \sin p\pi \xi \sin q\pi \eta - W_{pq}^{(2)} (4q^4 \pi^4) (1 - \cos 2p\pi \xi) \cos 2q\pi \eta \right]$$

where

$$X_i(\xi) = \cosh \alpha_i \xi - \cos \alpha_i \xi - \gamma_i (\sinh \alpha_i \xi - \sin \alpha_i \xi)$$

$$Y_i(\eta) = \cosh \alpha_i \eta - \cos \alpha_i \eta - \gamma_i (\sinh \alpha_i \eta - \sin \alpha_i \eta)$$

$$\bar{X}_i(\xi) = \frac{1}{\alpha_i} \frac{dX_i(\xi)}{d\xi}$$

$$\bar{\bar{X}}_i(\xi) = \frac{1}{\alpha_i^2} \frac{d^2 X_i(\xi)}{d\xi^2}$$

$$\bar{\bar{\bar{X}}}_i(\xi) = \frac{1}{\alpha_i^3} \frac{d^3 X_i(\xi)}{d\xi^3}$$

$$\bar{Y}_i(\eta) = \frac{1}{\alpha_i} \frac{dY_i(\eta)}{d\eta}$$

$$\bar{\bar{Y}}_i(\eta) = \frac{1}{\alpha_i^2} \frac{d^2 Y_i(\eta)}{d\eta^2}$$

$$\bar{\bar{\bar{Y}}}_i(\eta) = \frac{1}{\alpha_i^3} \frac{d^3 Y_i(\eta)}{d\eta^3}$$

Appendix E

Elements of Matrices for Postbuckling

(1) With the indices I, J defined as follows

$$I = (i-1)M + j$$

$$J = (m-1)M + n = (p-1)M + q = (k-1)M + l$$

where

$$i = 1, 2, \dots, M$$

$$j = 1, 2, \dots, M$$

$$m = p = k = 1, 2, \dots, M$$

$$n = q = l = 1, 2, \dots, M$$

then we have

$$A_{IJ} = (\alpha_m^4 + a_4 \alpha_n^4) J_0^{mi} J_0^{nj} - a_1 \alpha_m^3 \alpha_n J_3^{mi} J_1^{nj} + a_2 \alpha_m^2 \alpha_n^2 J_2^{mi} J_2^{nj} \\ - a_3 \alpha_m \alpha_n^3 J_1^{mi} J_3^{nj}$$

$$B_{IJ} = \pi^4 \left[(a_5 p^4 + a_7 p^2 q^2 + a_9 q^4 + \frac{\beta^2 K_R}{A_{22}^*} \frac{p^2}{\pi^2}) C_0^{ip} C_0^{jq} \right. \\ \left. - (a_6 p^3 q + a_8 p q^3) R_0^{ip} R_0^{jq} \right]$$

$$C_{IJ} = 4\pi^4 \left[(a_5 p^4 + \frac{\beta^2 K_R}{A_{22}^*} \frac{p^2}{4\pi^2}) G_0^{ip} T_0^{jq} + a_9 q^4 G_0^{jq} T_0^{ip} \right. \\ \left. + (a_6 p^3 q + a_8 p q^3) G_1^{ip} G_1^{jq} - (a_5 p^4 + a_7 p^2 q^2 + a_9 q^4 + \frac{\beta^2 K_R}{A_{22}^*} \frac{p^2}{4\pi^2}) G_0^{ip} G_0^{jq} \right]$$

$$H_{IJ} = \frac{\pi^4}{4} (p^4 + b_2 p^2 q^2 + b_4 q^4) J_0^{pi} J_0^{qj} - b_1 p q \pi^3 (p^2 \pi H_0^{pi} + i Q_0^{pi}) H_0^{qj} \\ - b_3 p q \pi^3 (q^2 \pi H_0^{qj} + j Q_0^{qj}) H_0^{pi}$$

$$\begin{aligned}
I_{IJ} = & 4\pi^4 \left[-p^4 G_2^{pi} T_1^j - b_4 q^4 T_1^i G_2^{qj} - (b_1 p^3 q + b_3 p q^3) G_3^{pi} G_3^{qj} \right. \\
& + (p^4 + b_2 p^2 q^2 + b_4 q^4) G_2^{pi} G_2^{qj} + \frac{i p^2}{4\pi} S_1^{pi} G_6^{qj} \\
& \left. + b_4 \frac{j q^4}{4\pi} S_1^{qj} G_6^{ip} \right]
\end{aligned}$$

$$\begin{aligned}
J_{IJ} = & b_5 \alpha_m^2 C_0^{nj} (\alpha_m^2 C_0^{mi} + i\pi S_0^{mi}) + b_9 \alpha_n^2 C_0^{mi} (\alpha_n^2 C_0^{nj} + j\pi S_0^{nj}) \\
& + b_6 \alpha_m^3 \alpha_n C_3^{mi} C_1^{nj} + b_7 \alpha_m^2 \alpha_n^2 C_2^{mi} C_2^{nj} + b_8 \alpha_m \alpha_n^3 C_1^{mi} C_3^{nj} \\
& - \frac{\beta^2 K_R}{D_{11}^*} \alpha_m^2 C_0^{mi} C_0^{nj}
\end{aligned}$$

$$M_{IJ} = \frac{\beta^2 \pi^2}{4} k^2 J_0^{ki} J_0^{lj} \left(\frac{E_t h^3}{D_{11}^*} \right)$$

$$N_{IJ} = \beta^2 \pi^2 k^2 G_2^{ki} G_6^{lj} \left(\frac{E_t h^3}{D_{11}^*} \right)$$

$$O_{IJ} = 2\beta^3 \pi^2 k l H_0^{ki} H_0^{lj} \left(\frac{E_t h^3}{D_{11}^*} \right)$$

$$P_{IJ} = 2\beta^3 \pi^2 k l G_3^{kl} G_3^{ij} \left(\frac{E_t h^3}{D_{11}^*} \right)$$

$$Q_{IJ} = \pi^4 \left[(p^4 + b_2 p^2 q^2 + b_4 q^4) G_6^{pi} G_6^{qj} - (b_1 p^3 q + b_3 p q^3) G_4^{pi} G_4^{qj} \right]$$

$$\begin{aligned}
R_{IJ} = & 4\pi^2 \left[-p^4 G_5^{pi} T_2^j - b_4 q^4 T_2^i G_5^{qj} - (b_1 p^3 q + b_3 p q^3) G_7^{pi} G_7^{qj} \right. \\
& \left. + (p^4 + b_2 p^2 q^2 + b_4 q^4) G_5^{pi} G_5^{qj} \right]
\end{aligned}$$

$$\begin{aligned}
S_{IJ} = & (b_5 \alpha_m^4 + b_9 \alpha_n^4) F_0^{mi} F_0^{nj} + b_6 \alpha_m^3 \alpha_n F_3^{mi} F_1^{nj} + b_7 \alpha_m^2 \alpha_n^2 F_2^{mi} F_2^{nj} \\
& + b_8 \alpha_m \alpha_n^3 F_1^{mi} F_3^{nj} - \frac{K_R \beta^2}{D_{11}^*} \alpha_m^2 F_2^{mi} F_0^{nj}
\end{aligned}$$

(2) With the indices I defined as follows :

$$I = (i-1)M + j$$

$$i = 1, 2, \dots, M$$

$$j = 1, 2, \dots, M$$

we have

$$M_{0I} = (b_{10} L_0^i T_1^j + b_{19} L_0^j T_1^i) \cdot \left(\frac{E_t h}{A_{22}} \right)$$

$$O_{0I} = (b_{11} L_0^i T_1^j + b_{14} L_0^j T_1^i) \cdot \left(\frac{E_t h}{A_{22}} \right)$$

(3) With the indices I, J defined as follows

$$I = (I_1 - 1)M + J_1$$

$$J = (r - 1)M + s$$

and $I_1 = (i - 1)M + j$

where $J_1 = (k - 1)M + l$

$$i = 1, 2, \dots, M$$

$$j = 1, 2, \dots, M$$

$$k = 1, 2, \dots, M$$

$$l = 1, 2, \dots, M$$

$$r = 1, 2, \dots, M$$

$$s = 1, 2, \dots, M$$

We have

$$D_{klrs}^{ij} = \frac{\beta^2 \pi^4}{A_{22}^*} (klrs Z_{05}^{ikr} Z_{05}^{jls} - k^2 s^2 Z_{01}^{ikr} Z_{01}^{jls})$$

$$E_{klrs}^{ij} = \frac{\beta^2 \pi^4}{A_{22}^*} (klrs Z_{06}^{ikr} Z_{06}^{jls} + k^2 s^2 Z_{02}^{ikr} Z_{03}^{jls})$$

$$F_{klrs}^{ij} = \frac{\beta^2 \pi^4}{A_{22}^*} (klrs Z_{06}^{irk} Z_{06}^{jsl} + k^2 s^2 Z_{03}^{irk} Z_{02}^{jsl})$$

$$G_{klrs}^{ij} = \frac{\beta^2 \pi^4}{A_{22}^*} (klrs Z_{07}^{ikr} Z_{07}^{jls} - k^2 s^2 Z_{04}^{ikr} Z_{04}^{jls})$$

$$K_{klrs}^{ij} = \frac{\beta^2 \pi^2}{D_{11}^*} (-k^2 \alpha_s^2 Z_{01}^{rki} Z_{21}^{slj} - l^2 \alpha_r^2 Z_{21}^{rki} Z_{01}^{slj} - 2kl \alpha_r \alpha_s Z_{11}^{rki} Z_{11}^{slj})$$

$$L_{klrs}^{ij} = \frac{\beta^2 \pi^2}{D_{11}^*} (k^2 \alpha_s^2 Z_{02}^{rki} Z_{22}^{slj} + l^2 \alpha_r^2 Z_{22}^{rki} Z_{02}^{slj} - 2kl \alpha_r \alpha_s Z_{12}^{rki} Z_{12}^{slj})$$

where the constants C_i^{ij} , J_i^{ij} , G_i^{ij} , Z_i^{ij} , F_i^{ij} , T_i^i , S_i^{mi} and H_o^{mi} , R_o^{ij} , Q_o^{ij} , L_o^i are obtained from the integrals defined in Appendix F.

Appendix F

Integrals for Postbuckling

The constants appearing in the elements which are listed in Appendix E are as follows :

$$C_0^{mi} = \int_0^1 X_m(\xi) \sin i\pi\xi d\xi$$

$$C_1^{mi} = \int_0^1 \bar{X}_m(\xi) \sin i\pi\xi d\xi$$

$$C_2^{mi} = \int_0^1 \bar{\bar{X}}_m(\xi) \sin i\pi\xi d\xi$$

$$C_3^{mi} = \int_0^1 \bar{\bar{\bar{X}}}_m(\xi) \sin i\pi\xi d\xi$$

$$F_0^{mi} = \int_0^1 X_m(\xi) (1 - \cos 2i\pi\xi) d\xi$$

$$F_1^{mi} = \int_0^1 \bar{X}_m(\xi) (1 - \cos 2i\pi\xi) d\xi$$

$$F_2^{mi} = \int_0^1 \bar{\bar{X}}_m(\xi) (1 - \cos 2i\pi\xi) d\xi$$

$$F_3^{mi} = \int_0^1 \bar{\bar{\bar{X}}}_m(\xi) (1 - \cos 2i\pi\xi) d\xi$$

$$J_0^{mi} = \int_0^1 X_m(\xi) X_i(\xi) d\xi$$

$$J_1^{mi} = \int_0^1 \bar{X}_m(\xi) X_i(\xi) d\xi$$

$$J_2^{mi} = \int_0^1 \bar{\bar{X}}_m(\xi) X_i(\xi) d\xi$$

$$J_3^{mi} = \int_0^1 \bar{\bar{\bar{X}}}_m(\xi) X_i(\xi) d\xi$$

$$H_0^{mi} = \int_0^1 \cos m\pi\xi \sin i\pi\xi d\xi$$

$$R_0^{mi} = \int_0^1 X_m(\xi) \cos i\pi\xi d\xi$$

$$G_0^{mi} = \int_0^1 X_m(\xi) \cos 2i\pi\xi d\xi$$

$$G_1^{mi} = \int_0^1 X_m(\xi) \sin 2i\pi\xi d\xi$$

$$G_2^{mi} = \int_0^1 \cos 2m\pi\xi \sin i\pi\xi d\xi$$

$$G_3^{mi} = \int_0^1 \sin 2m\pi\xi \sin i\pi\xi d\xi$$

$$G_4^{mi} = \int_0^1 \cos m\pi\xi (1 - \cos 2i\pi\xi) d\xi$$

$$G_5^{mi} = \int_0^1 \cos 2m\pi\xi (1 - \cos 2i\pi\xi) d\xi$$

$$G_6^{mi} = \int_0^1 \sin m\pi\xi (1 - \cos 2i\pi\xi) d\xi$$

$$G_7^{mi} = \int_0^1 \sin 2m\pi\xi (1 - \cos 2i\pi\xi) d\xi$$

$$G_8^{mi} = \int_0^1 (1 - \cos 2m\pi\xi) (1 - \cos 2i\pi\xi) d\xi$$

$$L_0^i = i\pi [(-1)^i - 1]$$

$$Q_0^{mi} = -\frac{1}{2} [(-1)^{(m+i)} - 1]$$

$$S_0^{mi} = (-1)^i \bar{X}_m(1) - \bar{X}_m(0)$$

$$S_1^{mi} = (-1)^{(2m+i)} - 1$$

$$T_0^{mi} = \int_0^1 X_i(\xi) d\xi$$

$$T_1^{mi} = \int_0^1 \sin i\pi\xi d\xi$$

$$T_2^{mi} = \int_0^1 (1 - \cos 2i\pi\xi) d\xi$$

$$Z_{01}^{rki} = \int_0^1 X_r(\xi) \sin k\pi\xi \sin i\pi\xi d\xi$$

$$Z_{02}^{rki} = \int_0^1 X_r(\xi) \cos 2k\pi\xi \sin i\pi\xi d\xi$$

$$Z_{03}^{rki} = \int_0^1 X_r(\xi) \sin k\pi\xi (1 - \cos 2i\pi\xi) d\xi$$

$$Z_{04}^{rki} = \int_0^1 X_r(\xi) \cos 2k\pi\xi (1 - \cos 2i\pi\xi) d\xi$$

$$Z_{05}^{rki} = \int_0^1 X_r(\xi) \cos k\pi\xi \cos i\pi\xi d\xi$$

$$Z_{06}^{rki} = \int_0^1 X_r(\xi) \sin 2k\pi\xi \cos i\pi\xi d\xi$$

$$Z_{07}^{rki} = \int_0^1 X_r(\xi) \sin 2k\pi\xi \sin 2i\pi\xi d\xi$$

$$Z_{11}^{rki} = \int_0^{1-} \bar{X}_r(\xi) \cos k\pi\xi \sin i\pi\xi d\xi$$

$$Z_{12}^{rki} = \int_0^{1-} \bar{X}_r(\xi) \sin 2k\pi\xi \sin i\pi\xi d\xi$$

$$Z_{13}^{rki} = \int_0^{1-} \bar{X}_r(\xi) \cos k\pi\xi (1 - \cos 2i\pi\xi) d\xi$$

$$Z_{14}^{rki} = \int_0^{1-} \bar{X}_r(\xi) \sin 2k\pi\xi (1 - \cos 2i\pi\xi) d\xi$$

$$Z_{21}^{rki} = \int_0^{1=} \bar{X}_r(\xi) \sin k\pi\xi \sin i\pi\xi d\xi$$

$$Z_{22}^{rki} = \int_0^{1=} \bar{X}_r(\xi) (1 - \cos 2k\pi\xi) \sin i\pi\xi d\xi$$

$$Z_{23}^{rki} = \int_0^{1=} \bar{X}_r(\xi) (1 - \cos 2k\pi\xi) (1 - \cos 2i\pi\xi) d\xi$$

where

$$X_i(\xi) = \cosh \alpha_i h - \cos \alpha_i \xi - \gamma_i (\sinh \alpha_i \xi - \sin \alpha_i \xi)$$

$$\bar{X}_i(\xi) = \frac{1}{\alpha_i} \frac{dX_i(\xi)}{d\xi}$$

$$\bar{\bar{X}}_i(\xi) = \frac{1}{\alpha_i^2} \frac{d^2 X_i(\xi)}{d\xi^2}$$

$$\bar{\bar{\bar{X}}}_i(\xi) = \frac{1}{\alpha_i^3} \frac{d^3 X_i(\xi)}{d\xi^3}$$

It is obvious that with ξ replaced by η , all the expressions in this Appendix are still true.

The indices m, i, r, k are integers other than 0.

Appendix G

Elements of Matrices $[\Lambda^{(i)}]$ for Behaviour of Imperfect Panels

With the indices I, J defined by

$$I = (i-1)M + j$$

$$J = (k-1)M + l$$

where

$$i = 1, 2, \dots, M$$

$$j = 1, 2, \dots, M$$

$$k = 1, 2, \dots, M$$

$$l = 1, 2, \dots, M$$

we have

$$\Lambda_{IJ}^{(3)} = \frac{1}{4} \beta^2 k^2 \pi^2 \left(\frac{E_t h^3}{D_{11}^*} \right) J_o^{ki} J_o^{lj}$$

$$\Lambda_{IJ}^{(4)} = 2\beta^3 k l \pi^2 \left(\frac{E_t h^3}{D_{11}^*} \right) H_o^{ki} H_o^{lj}$$

$$\Lambda_{IJ}^{(6)} = \beta^2 k^2 \pi^2 \left(\frac{E_t h^3}{D_{11}^*} \right) G_6^{ki} G_6^{lj}$$

$$\Lambda_{IJ}^{(7)} = 2\beta^3 k l \pi^2 \left(\frac{E_t h^3}{D_{11}^*} \right) G_4^{ki} G_4^{lj}$$

With the indices I, J defined as follows

$$I = (I_1 - 1)M + J_1$$

$$J = (r - 1)M + s$$

where

$$I_1 = (i - 1)M + j$$

$$J_1 = (k - 1)M + l$$

and

$$i = 1, 2, \dots, M$$

$$j = 1, 2, \dots, M$$

$$k = 1, 2, \dots, M$$

$$l = 1, 2, \dots, M$$

$$r = 1, 2, \dots, M$$

$$s = 1, 2, \dots, M$$

we have

$$\Lambda_{IJ}^{(1)} = \frac{\beta^2 \pi^4}{A_{22}^*} \left\{ -l^2 r^2 Z_{01}^{ikr} Z_{01}^{jls} - k^2 s^2 Z_{01}^{ikr} Z_{01}^{jls} + 2klrs Z_{05}^{ikr} Z_{05}^{jls} \right\}$$

$$\Lambda_{IJ}^{(2)} = \frac{\beta^2 \pi^4}{A_{22}^*} \left\{ l^2 r^2 Z_{02}^{irk} Z_{03}^{jls} + k^2 s^2 Z_{03}^{ikr} Z_{02}^{jls} + 2klrs Z_{06}^{irk} Z_{06}^{jls} \right\}$$

$$\Lambda_{IJ}^{(5)} = -\frac{\beta^2 \pi^2}{D_{11}^*} \left\{ k^2 \alpha_s^2 Z_{01}^{rki} Z_{21}^{slj} + l^2 \alpha_r^2 Z_{21}^{rki} Z_{01}^{slj} + 2kl\alpha_r \alpha_s Z_{11}^{rki} Z_{11}^{slj} \right\}$$

$$\Lambda_{IJ}^{(8)} = -\frac{\beta^2 \pi^2}{D_{11}^*} \left\{ k^2 \alpha_s^2 Z_{03}^{rki} Z_{21}^{slj} + l^2 \alpha_r^2 Z_{22}^{rik} Z_{03}^{slj} + 2kl\alpha_r \alpha_s Z_{13}^{rki} Z_{13}^{slj} \right\}$$