

THESIS

Imperial College

of

Science and Technology

"SYNCHRONOUS MACHINE ANALYSIS —

Including Unbalanced and Transient Conditions"

Y. K. Ching

Thesis Submitted for the Diploma of the Imperial
College of Science and Technology

Electrical Engineering Department,
Imperial College of Science and
Technology,
London, April, 1953.

ACKNOWLEDGEMENTS

The work was carried out in the Imperial College under the guidance of Prof. Willis Jackson, D.Sc., D.Phil., M.I.E.E., F.Inst.P., F.R.S, Mr. B. Adkins, M.A., M.I.E.E. and Mr. M.W. Humphrey Davies, M.Sc., M.I.E.E. The author would like to express his utmost gratitude to them for their assistance, encouragement and constructive criticism without which the work could not have been completed. He is also indebted to Dr. G.R. Slemon, Ph.D., M.A., B.Sc., A.M.I.E.E., who gave him many useful suggestions at the early stage of his research, and to Mr. G.W. Chilton, A.M.I.E.C., who rendered him invaluable help in the laboratory. Finally, he would like to thank his friends Messrs. S.L. Chen, B.Sc., C.W. Yu, B.Sc., D.I.C. and Y.K. Pun, B.Sc., D.I.C., who all had a hand in helping to reproduce the oscillograms in the course of preparing the thesis.

CONTENT

(1) INTRODUCTION	Page 1
PART I STEADY-STATE SOLUTION	
(2) GENERAL EQUATIONS AND THEIR STEADY-STATE SOLUTION	5-16
(2.1) General Equations of a Three-phase Machine	5
(2.2) Approximate Solutions for the Fundamental, Third- and Fifth- harmonic Currents	10
(2.3) Corrections for Effects of Currents of Higher Frequencies	14
(2.4) Sources of Harmonic Currents	15
(3) SIMPLIFICATION OF THE SOLUTION	17-27
(3.1) Assumptions and Conventional Machine Constants	17
(3.2) Simplified Solution in terms of Conventional Machine Constants	18
(3.3) Balanced Three-phase Operation and Two-reaction Theory	20
(3.4) Single-phase Operations — Approximate Solution	23
(3.5) Single-phase Operations with Negligible Armature Resistance — Exact Solution	24
(3.6) Further Possible Assumptions	26
(4) METHOD OF SYMMETRICAL COMPONENTS	28-44
(4.1) General Considerations	28
(4.2) Sequence Equations	29
(4.3) Single-phase Operations	31
(4.4) Synchronous Motor on Unbalanced Mains	33
(4.5) Negative-sequence Reactance	34
(4.6) Simplified Method of Solution	36
(4.7) Experimental Results	38

(5) FIELD CURRENT, DAMPER CURRENTS & ELECTROMAGNETIC TORQUE	45-52
(5.1) Field and Damper Currents	45
(5.2) Electromagnetic Torque	48
(6) STEADY-STATE EQUIVALENT CIRCUITS & THEIR APPLICATIONS	53-89
(6.1) Types of Equivalent Circuits	53
(6.2) Unsymmetrical Short Circuits	60
(6.3) Unsymmetrical Short Circuits with Capacitive Loads	72
(6.4) Two or More Machines in Parallel	85
PART II TRANSIENT-STATE SOLUTION	
(7) GENERAL OPERATIONAL SOLUTION	90-111
(7.1) General Equations	90
(7.2) Operational Equations	92
(7.3) 1, 2, 0 Components	94
(7.4) Operational Solution in terms of 1, 2, 0 Components	96
(7.5) , , 0 Components	98
(7.6) General Operational Solution in terms of α , β , 0 Components	100
(7.7) Other Important Reference Axes	106
(7.8) Field and Damper Currents	109
(7.9) Electromagnetic Torque	110
(8) SYMMETRICAL THREE-PHASE OPERATION	112-129
(8.1) Three-phase Short Circuit of an Open-circuit Machine	112
(8.2) Solution for Armature Currents with Small Resistance in the Armature Circuit	114
(8.3) Field and Damper Currents	117
(8.4) Short-circuit Torque	120

(8.5) Short Circuit of a Loaded Machine with Initial Balanced Loading	121
(8.6) Short Circuit through External Impedances	124
(9) SINGLE-PHASE SHORT CIRCUITS	130-152
(9.1) Line-to-neutral Short Circuit	130
(9.2) Operational Solution	131
(9.3) Steady-state Component of Short-circuit Current	132
(9.4) Transient and Subtransient Components of Short- circuit Current	134
(9.5) Armature-transient Component of Short-circuit Current	136
(9.6) Field and Damper Currents	138
(9.7) Open-phase Voltages	141
(9.8) Short-circuit Torque — Pulsating Component	143
(9.9) Single-phase Line-to-line Short Circuit	145
(9.10) Experiment Results	147
(10) DOUBLE-LINE-TO-NEUTRAL SHORT CIRCUIT	153-177
(10.1) Terminal Conditions in terms of i_d , i_q , i_0 Components	153
(10.2) Independent i_d - and i_q component Equations and their Operational Solutions	154
(10.3) Steady-state Components of Short-circuit Currents	156
(10.4) Transient and Subtransient Components of Short- circuit Currents	160
(10.5) Armature-transient Components of Short-circuit Currents	162
(10.6) Field and Damper Currents	169
(10.7) Open-phase Voltage	172
(10.8) Short-circuit Torque — Pulsating Component	173
(10.9) Experimental Results	175

(11) EQUIVALENT CIRCUITS FOR TRANSIENT-STATE CURRENTS	178-202
(11.1) Transient-state Component of Current	178
(11.2) Odd-harmonic Equivalent Circuits	180
(11.3) Even-harmonic Equivalent Circuits	187
(11.4) Short-circuit Torque — Unidirectional Components	197
(12) CONCLUSION	203
(13) REFERENCES	205
(14) APPENDICES	207-214
(14.1) Steady-state Solution	207
(14.2) Single-phase Operations	211
(14.3) Two Additional Proofs Concerning the Solution of the Recurrence Equation (Art. 7.6)	212

(1) INTRODUCTION

The principal object of this thesis is to present a detailed linear analysis of a synchronous machine based on the fundamental electric circuit theory. For the sake of convenience in presentation, the material is divided into two parts, one on steady state and the other on transient state. This is done also because in finding the steady-state solution some attempt has been made to take into account the effects of space harmonics whereas there has been no such attempt in finding the transient-state solution.

At steady state, all authors have used the method of symmetrical components⁽¹⁾ for dealing with unbalanced loading conditions, though very often the applicability of the method has been taken for granted. In fact, the method of symmetrical components, if applied only to the fundamental-frequency quantities without due consideration of the harmonic quantities, is only an approximate process. Its accuracy rests on the choice of the negative-sequence reactance of the machine, which has no definite value and is dependent on the entire system to which the machine is connected. Fortunately, its value does not, in general, vary very much and the approximate way of applying the method of symmetrical components can be used without much error. The worst conditions arise when there is capacitance in the armature circuit which may make it impossible to assess an appropriate value for the negative-sequence reactance without giving full consideration to the effects of the harmonics.

Usually the transient theory of a synchronous machine is concerned with the transient-state currents and voltages following a sudden change of the terminal condition of the machine, though Concordia⁽²⁾ has drawn particular attention to the transient torque. Park's equations⁽³⁾ have been able to transform a three-phase synchronous machine into an equivalent d.c. machine with two sets of brushes^(4,5), thus enabling the transient state current of a synchronous machine operating under balanced conditions to be derived in a more logical and rigorous manner by solving these operational equations⁽⁶⁾. Earlier, the transient-state current has been derived by using the constant-flux theorem⁽⁷⁾, which Doherty and Nickle have used with advantage in their study of the single-phase short circuits⁽⁸⁾. Concordia, following the same line of reasoning, has extended the method to the double-line-to-neutral short circuit⁽²⁾.

In the thesis, the differential voltage equations of the synchronous machine are established in accordance with Kirchhoff's first law. They are linear ~~equation~~ equations with periodic coefficients. The equations are solved by applying the method of the Laplace transformation⁽⁹⁾. The fact that the coefficients of the differential equations are not all constants makes it impossible (except for the symmetrical three-phase operations of the machine) to get a closed non-series solution for the equations.

At steady state, the armature current can be expressed in the form of an odd-harmonic series so that the steady-state solution can be obtained by substituting the harmonic series

in the equations and proceeding to determine the coefficients of the harmonic series. With the neglect of the space-harmonic effects, a general operational solution is obtained by solving a recurrence equation (derived from the general differential equations) prescribing the general operation of a synchronous machine under linear conditions. The operational solution is in the form of an infinite series involving also continued fractions. To find the inverse transform of the operational solution, which gives the explicit expression for the armature currents, in a rigorous manner is by no means a simple process. An approximate process is illustrated by the examples of a symmetrical three-phase short circuit, a line-to-neutral short circuit and a double-line-to-neutral short circuit.

The derivation of an explicit solution from the operational solution is extremely complicated. It is done with the primary aim of showing the fact that the well-known solutions for the simpler cases of transient problems can be obtained by a more rigorous mathematical approach. For dealing with more complicated problems, it is imperative to have a simpler yet dependable method of solution. With this in mind, equivalent circuits relating the current and voltage components at different frequencies and of different sequences are developed which can be applied to the steady-state and the transient-state problems. The average torque can also be obtained readily with the aid of the equivalent circuits.

The equivalent circuits can be applied to more complicated problems by suitably connecting the component circuits together.

The final complete equivalent circuit is in general an infinite ladder network. The numerical solution for a particular problem can be found approximately by using a limited number of meshes of the infinite network. The minimum number of meshes that can give a sufficiently accurate answer varies with the type of problem as well as the constants of the machine and must be determined by one's own discretion. If a suitable type of network analyser is available (one that can represent negative resistance), this can be done by trial and error without spending too much labour.

As has been pointed out, the subject of the thesis is the linear theory of the synchronous machine, the assumptions of no magnetic saturation and no speed variation are made throughout the whole thesis. The magnetic saturation can be allowed for by choosing machine constants appropriate to the condition being considered. The speed variation of the machine following a sudden change can be approximately taken into account by dividing the period of time under consideration into a number of intervals with appropriate constant values of speed.

PART I STEADY-STATE SOLUTION

(2) GENERAL EQUATIONS AND THEIR STEADY-STATE SOLUTION

(2.1) General Equations of a Three-phase Machine

When magnetic non-linearity is neglected, a synchronous machine may be regarded as two or more magnetically coupled electric circuits with time-varying circuit parameters. Because of rotation the windings on one side of the air gap constantly change their positions with respect to the windings on the other side of the air gap with the result that all the mutual inductances between any two windings belonging to different members (the stator and the rotor) of the machine vary periodically with the rotor position. With the usual symmetries of the magnetic and electric circuits within the machine, these mutual inductances can be expressed as a Fourier series consisting of a fundamental and odd-harmonic components, the rotor position being the variable. If the effects of winding slots are neglected, the self-inductances of all windings and the mutual inductances between any two windings on the same side of the air gap of a round-rotor machine are constants because they remain unaffected by the rotation of the moving member of the machine. If the machine is one of salient-pole construction, the self- and mutual inductances of the windings on the field structure are still constants but those of the armature windings become functions of rotor position owing to the periodic change of the reluctance of the magnetic circuits linking with them. These functions can usually be expressed as a Fourier series consisting of a constant and even-harmonic components. When

the machine is running, the rotor position varies with time and the inductances which are functions of rotor position become functions of time.

Apart from the peculiarity of the time variation of some of the self- and mutual inductances, there is no basic difference between a static network and a synchronous machine. In fact, any electric circuit theorem, if not pertaining only to circuits with constant parameters, applies equally well to the synchronous machine. Based on this reasoning, it is possible to establish the general differential equations of the synchronous machine from the knowledge of the basic electric circuit theory.

Fig. 2.1 shows the schematic diagram of a three-phase salient-pole synchronous machine with one damper circuit in each of the pole and interpole axes.

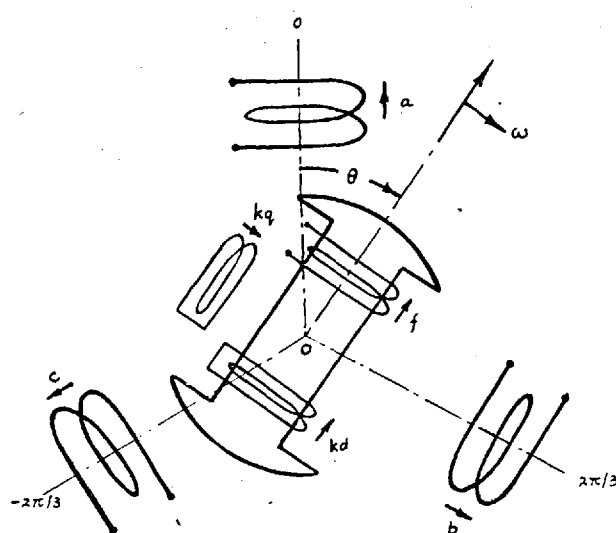


Fig. 2.1

The armature windings are in the stator and are labelled a, b and c, a being the leading phase. The field and damper windings are in the rotor

and are labelled f, kd and kq. The arrows indicate the direction of the mmf's set up by the currents in the respective windings.

With the rotor in the position shown, i.e., with the axis

of the field winding leading the axis of phase a by an angle θ , the expressions for the various self- and mutual inductances are, in the form of Fourier series, as follows.

$$\begin{aligned} L_a(\theta) &= \text{self-inductance of winding a} \\ &= L_{a0} + L_{a2}\cos 2\theta + L_{a4}\cos 4\theta + \dots \end{aligned}$$

$$\begin{aligned} L_b(\theta) &= \text{self-inductance of winding b} \\ &= L_a(\theta - 2\pi/3) \end{aligned}$$

$$\begin{aligned} L_c(\theta) &= \text{self-inductance of winding c} \\ &= L_a(\theta + 2\pi/3) \end{aligned}$$

$$\begin{aligned} M_{ab}(\theta) &= \text{mutual inductance between windings a and b} \\ &= M_{s0} + M_{s2}\cos(2\theta + \pi/3) + M_{s4}\cos(4\theta + 2\pi/3) + \dots \end{aligned}$$

$$\begin{aligned} M_{bc}(\theta) &= \text{mutual inductance between windings b and c} \\ &= M_{ab}(\theta - 2\pi/3) \end{aligned}$$

$$\begin{aligned} M_{ca}(\theta) &= \text{mutual inductance between windings c and a} \\ &= M_{ab}(\theta + 2\pi/3) \end{aligned}$$

$$\begin{aligned} M_{af}(\theta) &= \text{mutual inductance between windings a and f} \\ &= M_{af1}\cos\theta + M_{af3}\cos 3\theta + M_{af5}\cos 5\theta + \dots \end{aligned}$$

$$\begin{aligned} M_{bf}(\theta) &= \text{mutual inductance between windings b and f} \\ &= M_{af}(\theta - 2\pi/3) \end{aligned}$$

$$\begin{aligned} M_{cf}(\theta) &= \text{mutual inductance between windings c and f} \\ &= M_{af}(\theta + 2\pi/3) \end{aligned}$$

$$\begin{aligned} M_{akd}(\theta) &= \text{mutual inductance between windings a and kd} \\ &= M_{akd1}\cos\theta + M_{akd3}\cos 3\theta + M_{akd5}\cos 5\theta + \dots \end{aligned}$$

$$\begin{aligned} M_{bkd}(\theta) &= \text{mutual inductance between windings b and kd} \\ &= M_{akd}(\theta - 2\pi/3) \end{aligned}$$

$$\begin{aligned} M_{ckd}(\theta) &= \text{mutual inductance between windings c and kd} \\ &= M_{akd}(\theta + 2\pi/3) \end{aligned}$$

$$\begin{aligned} M_{akq}(\theta) &= \text{mutual inductance between windings a and kq} \\ &= M_{akq1} \cos(\theta + \pi/2) + M_{akq3} \cos(3\theta - \pi/2) \\ &\quad + M_{akq5} \cos(5\theta + \pi/2) + \dots \end{aligned}$$

$$\begin{aligned} M_{bkq}(\theta) &= \text{mutual inductance between windings b and kq} \\ &= M_{akq}(\theta - 2\pi/3) \end{aligned}$$

$$\begin{aligned} M_{ckq}(\theta) &= \text{mutual inductance between windings c and kq} \\ &= M_{akq}(\theta + 2\pi/3) \end{aligned}$$

With the machine running steadily at an angular speed of ω , $\theta = \omega t + \lambda$, where λ is the angle by which the field axis is ahead of the axis of phase a at $t = 0$. This relation can be substituted into the above expressions to change them from functions of rotor position to functions of time.

Only those parameters which are functions of rotor position have been listed above. The remaining parameters, namely, the armature resistances R_a , R_b , R_c , which are assumed equal, the field and damper resistances R_f , R_{kd} , R_{kq} , the self-inductance of the field and damper windings, L_f , L_{kd} , L_{kq} , and the mutual inductance between the field and the direct-axis damper windings M_{fkd} are independent of the rotor position.

In the above Fourier series expressions, the even-harmonic components represent saliency effects and the odd-harmonic components (excepting the fundamental component) represent space-harmonic effects. When either or both effects are negligible the corresponding harmonic components will be reduced to zero.

When the armature phases a, b and c are connected through

external impedances $R_{ae}, L_{ae}, R_{be}, L_{be}$ and R_{ce}, L_{ce} to a three-phase voltage source v_a, v_b and v_c , there will be six separate closed circuits all inductively coupled within the machine. Six voltage equations, therefore, can be established which are sufficient for the determination of the six unknown currents. When the exciting current has reached its steady value of I_{fo} , these equations may be written in matrix form as (the inductance components higher than the third harmonic have been neglected for simplicity)

$$[e] = [R] \cdot [i] + \frac{d}{dt} \{ [L] \cdot [i] \} \quad (2.1)$$

$$[e] = \begin{matrix} a \\ b \\ c \\ f \\ kd \\ kq \end{matrix} \begin{bmatrix} V_a + e_a \\ V_b + e_b \\ V_c + e_c \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.2); [i] = \begin{matrix} a \\ b \\ c \\ f \\ kd \\ kq \end{matrix} \begin{bmatrix} i_a \\ i_b \\ i_c \\ i_f \\ i_{kd} \\ i_{kq} \end{bmatrix} \quad (2.3); [R] = \begin{matrix} & a & b & c & f & kd & kq \\ \begin{matrix} a \\ b \\ c \\ f \\ kd \\ kq \end{matrix} & \begin{bmatrix} R_a + R_{ae} & & & & & \\ & R_a + R_{be} & & & & \\ & & R_a + R_{ce} & & & \\ & & & R_f & & \\ & & & & R_{kd} & \\ & & & & & R_{kq} \end{bmatrix} \end{matrix} \quad (2.4)$$

$$[L] = \begin{matrix} & a & b & c & f & kd & kq \\ \begin{matrix} a \\ b \\ c \\ f \\ kd \\ kq \end{matrix} & \begin{bmatrix} L_{ae} + L_{ao} + L_{ax} \cos 2\theta & M_{so} + M_{sx} \cos(2\theta + \pi/3) & M_{so} + M_{sx} \cos(2\theta - \pi/3) & M_{af1} \cos \theta + M_{af3} \cos 3\theta & M_{akd1} \cos \theta + M_{akd3} \cos 3\theta & -M_{akq1} \cos(\theta - \pi/2) - M_{akq3} \cos(3\theta + \pi/2) \\ M_{so} + M_{sx} \cos(2\theta + \pi/3) & L_{be} + L_{ao} + L_{ax} \cos(2\theta + 2\pi/3) & -M_{sx} \cos 2\theta & M_{af1} \cos(\theta - 2\pi/3) + M_{af3} \cos 3\theta & M_{akd1} \cos(\theta - 2\pi/3) + M_{akd3} \cos 3\theta & -M_{akq1} \cos(\theta + 5\pi/6) - M_{akq3} \cos(3\theta + \pi/2) \\ M_{so} + M_{sx} \cos(2\theta - \pi/3) & -M_{sx} \cos 2\theta & L_{ce} + L_{ao} + L_{ax} \cos(2\theta - 2\pi/3) & M_{af1} \cos(\theta + 2\pi/3) + M_{af3} \cos 3\theta & M_{akd1} \cos(\theta + 2\pi/3) + M_{akd3} \cos 3\theta & -M_{akq1} \cos(\theta + \pi/6) - M_{akq3} \cos(3\theta + \pi/2) \\ M_{af1} \cos \theta + M_{af3} \cos 3\theta & M_{af1} \cos(\theta - 2\pi/3) + M_{af3} \cos 3\theta & M_{af1} \cos(\theta + 2\pi/3) + M_{af3} \cos 3\theta & L_f & M_{fkd} & \\ M_{akd1} \cos \theta + M_{akd3} \cos 3\theta & M_{akd1} \cos(\theta - 2\pi/3) + M_{akd3} \cos 3\theta & M_{akd1} \cos(\theta + 2\pi/3) + M_{akd3} \cos 3\theta & M_{fkd} & L_{kd} & \\ -M_{akq1} \cos(\theta - \pi/2) - M_{akq3} \cos(3\theta + \pi/2) & -M_{akq1} \cos(\theta + 5\pi/6) - M_{akq3} \cos(3\theta + \pi/2) & -M_{akq1} \cos(\theta + \pi/6) - M_{akq3} \cos(3\theta + \pi/2) & & & L_{kq} \end{bmatrix} \end{matrix} \quad (2.5)$$

In the voltage matrix, $e_a = \omega M_{af1} I_{fo} \sin \theta + 3 \omega M_{af3} I_{fo} \sin 3\theta$, $e_b = e_a(\theta - 2\pi/3)$ and $e_c = e_a(\theta + 2\pi/3)$. Higher harmonic components of the self- and mutual inductances may be added to

the equations when a truer representation of the machine is desired. When there are mutuals among the external impedances, they can be included in the equations by adding extra terms to the appropriate elements of the inductance matrix $[L]$ and the resistance matrix $[R]$. The current i_f' is the harmonic (and transient) component of the total field current, the steady component I_{f0} having been deducted.

(2.2) Approximate Solutions for the Fundamental, Third- and Fifth- harmonic Currents

Before the steady-state solution of the general equations is given, it is desirable to state the assumptions made in deriving it. It is assumed

1. That the mutual inductance between the armature and the field windings is the same as that between the armature and the direct-axis-damper windings, when they are expressed on the same basis (say the per-unit values), i.e., $M_{af}(\theta) = M_{akd}(\theta)$; and

2. That the field and damper resistances are negligible compared with the field and damper reactances at twice or higher multiples of the fundamental frequency.

The first assumption is nearly true for salient-pole machine of normal construction, though the error may be larger for round-rotor machines. This assumption really means that there is a flux common to all, or any two, of the armature, the field and the direct-axis-damper windings when their axes coincide. The second assumption is very nearly true with respect to the

field winding though the error may be larger for the damper winding. This assumption, however, is not essential, though it helps to simplify the expression of the solution.

With these assumptions, the fundamental components of the armature currents, expressed in terms of complex numbers, are given by the following matrix equation whose derivation is given in detail in the Appendix, Art. 14.1.

$$\begin{bmatrix} V_{a1} + E_{a1} \\ V_{b1} + E_{b1} \\ V_{c1} + E_{c1} \end{bmatrix} = \begin{bmatrix} Z_{a1} & -Z_{m1}^* & Z_{m1} & Z_u & Z_w & h^2 Z_w \\ Z_{m1} & Z_{b1} & -Z_{m1}^* & Z_w & h Z_u & h Z_w \\ -Z_{m1}^* & Z_{m1} & Z_{c1} & h^2 Z_w & h Z_w & h^2 Z_u \end{bmatrix} \cdot \begin{bmatrix} I_{a1} \\ I_{b1} \\ I_{c1} \\ I_{a1}^* \\ I_{b1}^* \\ I_{c1}^* \end{bmatrix} \quad (2.6)$$

$$E_{a1} = -j\omega M_{af1} I_{f0} / \sqrt{2}$$

$$E_{b1} = h^2 E_{a1}$$

$$E_{c1} = h E_{a1}$$

$$h = e^{j2\pi/3}$$

$$V_{a1}, V_{b1}, V_{c1} = \text{fundamental components of the three-phase voltage source}$$

$$Z_{a1} = R_{ae} + R_a + j\omega(L_{ae} + L_l)$$

$$Z_{b1} = R_{be} + R_a + j\omega(L_{be} + L_l)$$

$$Z_{c1} = R_{ce} + R_a + j\omega(L_{ce} + L_l)$$

$$L_l = L_{a0} - \frac{1}{4} M_{fs1} - \frac{1}{2} M_{fs3}$$

$$M_{fs1} = M_{af1}^2 / L_{fe} + M_{akq1}^2 / L_{kq}$$

$$M_{fs3} = M_{af3}^2 / L_{fe} + M_{akq3}^2 / L_{kq}$$

$$L_{fe} = (L_f L_{kd} - M_{fkd}^2) / (L_f + L_{kd} - 2M_{fkd})$$

$$Z_{m1} = \frac{\sqrt{3}}{8} \omega M_{fs1} + j\omega (M_{s0} + \frac{1}{8} M_{fs1} - \frac{1}{2} M_{fs3})$$

$$Z_u = j\omega (L_{a2} - M_{md}) / 2$$

$$Z_w = -j h^2 \omega (M_{s2} - \frac{1}{2} M_{md}) / 2$$

$$M_{md} = M_{af1} M_{af3} / L_{fe} - M_{akq1} M_{akq3} / L_{kq}$$

(2.7)

The third-harmonic components of the armature currents are given by the matrix equation

$$\begin{bmatrix} V_{a3} + E_{a3} \\ V_{b3} + E_{b3} \\ V_{c3} + E_{c3} \end{bmatrix} = \begin{bmatrix} Z_{a3} & Z_{m3} & Z_{m3} \\ Z_{m3} & Z_{b3} & Z_{m3} \\ Z_{m3} & Z_{m3} & Z_{c3} \end{bmatrix} \cdot \begin{bmatrix} I_{a3} \\ I_{b3} \\ I_{c3} \end{bmatrix} \quad (2.8)$$

$$\begin{aligned} E_{a3} &= -j3\omega M_{af3} I_{f0}/\sqrt{2} + j3\omega \cdot \frac{1}{2} (-L_{a2} I_{a1} + M_{s2} h^2 I_{b1} + M_{s2} h I_{c1}) + j3\omega \cdot \frac{1}{2} M_{fd1} (I_{a1} + h^2 I_{b1} + h I_{c1}) \\ &\quad + j3\omega \cdot \frac{1}{2} M_{md} (I_{a1} + I_{b1} + I_{c1}) + j3\omega \cdot \frac{1}{2} M_{ms} (I_{a1}^* + I_{b1}^* + I_{c1}^*) \\ M_{fd1} &= M_{af1}^2 / L_{fe} - M_{akq1} / L_{kq} \\ M_{ms} &= M_{af1} M_{af3} / L_{fe} + M_{akq1} M_{akq3} / L_{kq} \\ E_{b3} &= -j3\omega M_{af3} I_{f0}/\sqrt{2} + j3\omega \cdot \frac{1}{2} (M_{s2} I_{a1} - L_{a2} h^2 I_{b1} + M_{s2} h I_{c1}) h^2 + j3\omega \cdot \frac{1}{2} M_{fd1} (I_{a1} + h^2 I_{b1} + h I_{c1}) h^2 \\ &\quad + j3\omega \cdot \frac{1}{2} M_{md} (I_{a1} + I_{b1} + I_{c1}) h + j3\omega \cdot \frac{1}{2} M_{ms} (I_{a1}^* + I_{b1}^* + I_{c1}^*) \\ E_{c3} &= -j3\omega M_{af3} I_{f0}/\sqrt{2} + j3\omega \cdot \frac{1}{2} (M_{s2} I_{a1} + M_{s2} h^2 I_{b1} - L_{a2} h I_{c1}) h + j3\omega \cdot \frac{1}{2} M_{fd1} (I_{a1} + h^2 I_{b1} + h I_{c1}) h \\ &\quad + j3\omega \cdot \frac{1}{2} M_{md} (I_{a1} + I_{b1} + I_{c1}) h^2 + j3\omega \cdot \frac{1}{2} M_{ms} (I_{a1}^* + I_{b1}^* + I_{c1}^*) \end{aligned} \quad (2.9)$$

$V_{a3}, V_{b3}, V_{c3} =$ Third-harmonic components of the three-phase voltage source

$$\begin{aligned} Z_{a3} &= R_{ae} + R_a + j3\omega (L_{ae} + L_3) \\ Z_{b3} &= R_{be} + R_a + j3\omega (L_{be} + L_3) \\ Z_{c3} &= R_{ce} + R_a + j3\omega (L_{ce} + L_3) \\ L_3 &= L_{a0} - \frac{1}{2} M_{fs1} - \frac{1}{2} M_{fs3} \\ Z_{m3} &= j3\omega (M_{s0} + \frac{1}{2} M_{fs1} - \frac{1}{2} M_{fs3}) \end{aligned}$$

Similarly the fifth-harmonic components of the armature currents are given by the matrix equation

$$\begin{bmatrix} V_{a5} + E_{a5} \\ V_{b5} + E_{b5} \\ V_{c5} + E_{c5} \end{bmatrix} = \begin{bmatrix} Z_{a5} & Z_{m5} & Z_{m5} \\ Z_{m5} & Z_{b5} & Z_{m5} \\ Z_{m5} & Z_{m5} & Z_{c5} \end{bmatrix} \cdot \begin{bmatrix} I_{a5} \\ I_{b5} \\ I_{c5} \end{bmatrix} \quad (2.10)$$

where

$$E_{a5} = j5\omega \frac{1}{2} M_{ms} (2I_{a1} - hI_{b1} - h^2 I_{c1}) + j5\omega \frac{1}{2} M_{fd3} (I_{a1}^* + I_{b1}^* + I_{c1}^*) + j5\omega \frac{1}{2} (-L_{a2} I_{a3} + M_{s2} h^2 I_{b3} + M_{s2} h I_{c3}) \\ + j5\omega \frac{1}{2} M_{fd1} (I_{a3} + h^2 I_{b3} + h I_{c3}) + j5\omega \frac{1}{2} M_{md} (2I_{a3} - h^2 I_{b3} - h I_{c3})$$

$$M_{fd3} = M_{af3}/L_{fe} - M_{akq3}/L_{kq}$$

$$E_{b5} = j5\omega \frac{1}{2} M_{ms} (-I_{a1} + 2hI_{b1} - h^2 I_{c1})h + j5\omega \frac{1}{2} M_{fd3} (I_{a1}^* + I_{b1}^* + I_{c1}^*) + j5\omega \frac{1}{2} (M_{s2} I_{a3} - L_{a2} h^2 I_{b3} + M_{s2} h I_{c3})h^2 \\ + j5\omega \frac{1}{2} M_{fd1} (I_{a3} + h^2 I_{b3} + h I_{c3})h^2 + j5\omega \frac{1}{2} M_{md} (-I_{a3} + 2h^2 I_{b3} - h I_{c3})h^2$$

$$E_{c5} = j5\omega \frac{1}{2} M_{ms} (-I_{a1} - hI_{b1} + 2h^2 I_{c1})h^2 + j5\omega \frac{1}{2} M_{fd3} (I_{a1}^* + I_{b1}^* + I_{c1}^*) + j5\omega \frac{1}{2} (M_{s2} I_{a3} + M_{s2} h^2 I_{b3} - L_{a2} h I_{c3})h \\ + j5\omega \frac{1}{2} M_{fd1} (I_{a3} + h^2 I_{b3} + h I_{c3})h + j5\omega \frac{1}{2} M_{md} (-I_{a3} - h^2 I_{b3} + 2h I_{c3})h$$

$$V_{a5}, V_{b5}, V_{c5} = \text{Fifth-harmonic components of the three-phase voltage source}$$

$$Z_{a5} = R_{ae} + R_a + j5\omega (L_{ae} + L_s)$$

$$Z_{b5} = R_{be} + R_a + j5\omega (L_{be} + L_s)$$

$$Z_{c5} = R_{ce} + R_a + j5\omega (L_{ce} + L_s)$$

$$L_s = L_{a0} - \frac{1}{2} M_{fd1} - \frac{1}{2} M_{fd3}$$

$$Z_{ms} = j5\omega (M_{s0} + \frac{1}{2} M_{fd1} - \frac{1}{2} M_{fd3})$$

Similar equations for the determination of current components of higher frequencies can be derived without much difficulty following the same procedure used in deriving Equ. 2.6. But it may be necessary to include more harmonic components of the various inductances in the voltage matrix [e] and the inductance matrix [L], as they may become significant for higher-order harmonic currents.

(2.3) Corrections for Effects of Currents of Higher Frequencies

The equations given in the preceding article for determining the armature currents are not exact expressions. One can see that the equations for the fundamental currents do not involve harmonic currents, the equations for the third-harmonic currents involve only the fundamental and the third-harmonic currents, and the equations for the fifth-harmonic currents involve only

currents of the fundamental, third- and fifth- harmonic frequencies. Not that there are no reactions from currents of higher frequencies on currents of lower frequencies but that their effects have been omitted. This has been done in order to make the equations reasonable simple. If a more accurate answer is desired, particularly when the external impedances are small, corrections can be made to the first set of results obtained by the equations in the last article. The correction is achieved by adding to the first set of results current increments due to the additional voltages representing the reactions from currents of higher frequencies.

For the correction of the fundamental armature current, Equ. 2.6 can be used by putting current increments ΔI_{a1} , ΔI_{b1} , ΔI_{c1} , ΔI_{a1}^* , ΔI_{b1}^* and ΔI_{c1}^* in place of the original currents and additional voltages ΔE_{a1} , ΔE_{b1} , ΔE_{c1} in place of the original voltages. The additional voltages are given by

$$\left. \begin{aligned} \Delta E_{a1} &= j\omega \frac{1}{2} (-L_{a2} I_{a3} + M_{s2} h I_{b3} + M_{s2} h^2 I_{c3}) + j\omega \frac{1}{4} M_{fd1} (I_{a3} + h I_{b3} + h^2 I_{c3}) \\ &\quad + j\omega \frac{1}{4} M_{ms} (2 I_{a5} - h^2 I_{b5} - h I_{c5}) + j\omega \frac{1}{4} M_{fd3} (I_{a5}^* + I_{b5}^* + I_{c5}^*) \\ \Delta E_{b1} &= j\omega \frac{1}{2} (M_{s2} I_{a3} - L_{a2} h I_{b3} + M_{s2} h^2 I_{c3}) h + j\omega \frac{1}{4} M_{fd1} (I_{a3} + h I_{b3} + h^2 I_{c3}) h \\ &\quad + j\omega \frac{1}{4} M_{ms} (-I_{a5} + 2 h^2 I_{b5} - h I_{c5}) h^2 + j\omega \frac{1}{4} M_{fd3} (I_{a5}^* + I_{b5}^* + I_{c5}^*) \\ \Delta E_{c1} &= j\omega \frac{1}{2} (M_{s2} I_{a3} + M_{s2} h I_{b3} - L_{a2} h^2 I_{c3}) h^2 + j\omega \frac{1}{4} M_{fd1} (I_{a3} + h I_{b3} + h^2 I_{c3}) h^2 \\ &\quad + j\omega \frac{1}{4} M_{ms} (-I_{a5} - h^2 I_{b5} + 2 h I_{c5}) h + j\omega \frac{1}{4} M_{fd3} (I_{a5}^* + I_{b5}^* + I_{c5}^*) \end{aligned} \right\} (2.12)$$

Similarly the correction of the third-harmonic currents due to the effect of the fifth-harmonic currents can be obtained from Equ. 2.8 by putting

$$\begin{aligned} \Delta E_{a3} &= j3\omega \frac{1}{2} (L_{a2} I_{a5} - M_{s2} h I_{b5} - M_{s2} h^2 I_{c5}) + j3\omega \frac{1}{4} M_{fd1} (I_{a5} + h I_{b5} + h^2 I_{c5}) \\ &\quad + j3\omega \frac{1}{4} M_{ms} (2 I_{a5} - h I_{b5} - h^2 I_{c5}) \end{aligned}$$

$$\left. \begin{aligned} \Delta E_{b3} &= j3\omega \frac{1}{2} (-M_{s2} I_{a5} + L_{a2} h I_{b5} - M_{s2} h^2 I_{c5}) h + j3\omega \frac{1}{4} M_{fd1} (I_{a5} + h I_{b5} + h^2 I_{c5}) h \\ &\quad + j3\omega \frac{1}{4} M_{md} (-I_{a5} + 2h I_{b5} - h^2 I_{c5}) h \\ \text{and } \Delta E_{c3} &= j3\omega \frac{1}{2} (-M_{s2} I_{a5} - M_{s2} h I_{b5} + L_{a2} h^2 I_{c5}) h^2 + j3\omega \frac{1}{4} M_{fd1} (I_{a5} + h I_{b5} + h^2 I_{c5}) h^2 \\ &\quad + j3\omega \frac{1}{4} M_{md} (-I_{a5} - h I_{b5} + 2h^2 I_{c5}) h^2 \end{aligned} \right\} (2.13)$$

The complete solution for the steady-state armature currents by is therefore given by the set of equations

$$[E_1] + [\Delta E_1] = [Z_1] \cdot [I_1]$$

$$[E_3] + [\Delta E_3] = [Z_3] \cdot [I_3]$$

$$[E_5] + [\Delta E_5] = [Z_5] \cdot [I_5]$$

.....

In these equations, $[E_1]$, $[E_3]$, $[E_5]$, etc. are the equivalent fundamental, third-harmonic, fifth-harmonic, etc. voltages due to current components at lower than the respective harmonic frequencies. $[\Delta E_1]$, $[\Delta E_3]$, $[\Delta E_5]$, etc. are the equivalent fundamental, third-harmonic, fifth-harmonic, etc. voltages due to current components at higher than the respective harmonic frequencies.

(2.4) Sources of Harmonic Currents

If magnetic non-linearity is neglected, there remain three sources responsible for the generation of harmonic currents in a synchronous machine. They are the non-sinusoidal variation of the mutual inductances between the armature and the rotor (field and damper) windings, the variation of the self- and mutual inductances of the armature windings, and the unbalanced fundamental armature currents. In other words, the harmonic currents in an unsaturated synchronous machine may be due to the space-harmonic effect, the armature saliency

effect and the unbalanced loading effect.

To make the statement clearer, consider the equivalent third-harmonic voltage E_{a3} (Equ. 2.9), which comprises the following four components:

1. $-j3\omega M_{af3} I_{f0} / \sqrt{2}$
2. $-j3\omega \frac{1}{2} (L_{a0} I_{a1} - M_{s2} h^2 I_{b1} - M_{s2} h I_{c1})$
3. $j3\omega \frac{1}{4} (M_{af1}^2 / L_{fe} - M_{akq1}^2 / L_{kq}) (I_{a1} + h^2 I_{b1} + h I_{c1})$
4. $j3\omega \frac{1}{4} (M_{af1} M_{af3} / L_{fe} - M_{akq1} M_{akq3} / L_{kq}) (I_{a1} + I_{b1} + I_{c1})$
 $+ j3\omega \frac{1}{4} (M_{af1} M_{af3} / L_{fe} + M_{akq1} M_{akq3} / L_{kq}) (I_{a1}^* + I_{b1}^* + I_{c1}^*)$

The first component is due to space harmonics; if there is no space harmonic, $M_{af3} = 0$ and this component disappears. The second component is due to armature saliency effect; if there is no saliency, $L_{a2} = M_{s2} = 0$ and this component disappears. The third component is due to unbalanced loading; if the armature circuits are balanced, there will be no negative-sequence currents and again this component vanishes. The fourth component is due to the combined action of space harmonics and unbalanced loading, and the elimination of either the space harmonics or the zero-sequence armature currents will reduce it to zero.

(3) SIMPLIFICATION OF THE SOLUTION

(3.1) Assumptions and Conventional Machine Constants

A great simplification of the solution given in Chapter 2 can be realised if the space-harmonic effects are neglected. By space harmonics are usually meant the harmonic components of the spatial distribution of the air-gap flux, but in the expressions of the general parameters (Article 2.1), it is their effects, rather than the space harmonics themselves, which are expressed explicitly. These space-harmonic effects are represented by the harmonic components of the mutual inductances between windings belonging to different members of the machine, such as M_{af3} , M_{akq3} , etc. The relative values of L_{a2} , M_{s2} , etc. in the expressions for the self-inductances of the armature windings are also affected by space harmonics.

Although space harmonics are in general not negligible in themselves, particularly for salient-pole machines, their effects are nevertheless negligible in a well-designed machine. In fact, the space-harmonic effects can be completely eliminated if the armature windings are so distributed and chorded as to make the winding factors for the harmonic voltages equal to zero.

In the event of negligible space-harmonic effects, the expressions of the general parameters are simplified considerably, as M_{af3} , M_{akq3} , etc. will be reduced to zero and, as has been shown elsewhere, $M_{s2} = -L_{a2}$. With these simplifications the conventional machine constants X_o , X_d , X_q , X_d'' and X_q'' can now be introduced. Define

$$\begin{aligned}
 X_o &= \omega(L_{ao} + 2M_{so}) \\
 &= \text{zero-sequence reactance} \\
 X_d &= \omega(L_{ao} - M_{so} + 3L_{a2}/2) \\
 &= \text{direct-axis synchronous reactance} \\
 X_q &= \omega(L_{ao} - M_{so} - 3L_{a2}/2) \\
 &= \text{quadrature-axis synchronous reactance} \\
 X_d'' &= X_d - 3\omega M_{afl}^2 / 2L_{fe} \\
 &= \text{direct-axis subtransient reactance} \\
 X_q'' &= X_q - 3\omega M_{akql}^2 / 2L_{kq} \\
 &= \text{quadrature-axis subtransient reactance}
 \end{aligned}
 \tag{3.1}$$

which can be used in place of the five constants L_{ao} , M_{so} , L_{a2} , M_{afl}^2/L_{fe} and M_{akql}^2/L_{kq} remaining in the steady-state solution (Eqs. 2.7, 2.9 and 2.11) after making the assumption of negligible space-harmonic effects.

(3.2) Simplified Solution in terms of Conventional Machine Constants

When the conventional machine constants (X_o , X_d , X_q , X_d'' and X_q'') are introduced, it can be shown that in Equ. 2.7

$$\begin{aligned}
 \omega L_1 &= \frac{1}{3} \left\{ \frac{1}{2}(x_d + x_q) + \frac{1}{2}(x_d'' + x_q'') + x_o \right\} \\
 Z_m &= \frac{\sqrt{3}}{6} \left\{ \frac{1}{2}(x_d + x_q) - \frac{1}{2}(x_d'' + x_q'') \right\} - j \frac{1}{6} \left\{ \frac{1}{2}(x_d + x_q) + \frac{1}{2}(x_d'' + x_q'') - 2x_o \right\} \\
 Z_u &= j \frac{1}{6} (x_d - x_q)
 \end{aligned}
 \tag{3.2}$$

and $Z_w = h^2 Z_u$

In Equ. 2.9,

$$\begin{aligned}
 \omega L_3 &= \frac{1}{3} (x_d'' + x_q'' + x_o) \\
 Z_m &= -j \frac{1}{2} (x_d'' + x_q'' - 2x_o) \\
 E_{a3} &= -j \frac{3}{2} (x_d'' - x_q'') I_m
 \end{aligned}
 \tag{3.3}$$

$$E_{b3} = h^2 E_{a3}$$

and $E_{c3} = h E_{a3}$

Similarly for the $(2m+1)$ th harmonic ($m = 1, 2, 3, \dots$)

$$\left. \begin{aligned} \omega L_{(2m+1)} &= \frac{1}{3} (x_d'' + x_q'' + x_o) \\ Z_{m(2m+1)} &= -j \frac{1}{6} (2m+1) (x_d'' + x_q'' - 2x_o) \\ E_{a(2m+1)} &= -j \frac{1}{2} (2m+1) (x_d'' - x_q'') I_{n(2m-1)} \\ E_{b(2m+1)} &= h^2 E_{a(2m+1)} \\ \text{and } E_{c(2m+1)} &= h E_{a(2m+1)} \\ \text{where } I_{n(2m+1)} &= \frac{1}{3} (I_{a(2m-1)} + h^2 I_{b(2m-1)} + h I_{c(2m-1)}) \end{aligned} \right\} (3.4)$$

Also the additional voltages (Equ. 2.12) representing the corrections due to currents of higher frequencies are reduced to

$$\left. \begin{aligned} \Delta E_{a1} &= -j \frac{1}{2} (x_d'' - x_q'') I_{p3} \\ \Delta E_{b1} &= h \Delta E_{a1} \\ \text{and } \Delta E_{c1} &= h^2 \Delta E_{a1} \end{aligned} \right\} (3.5)$$

and in general

$$\left. \begin{aligned} \Delta E_{a(2m-1)} &= -j \frac{1}{2} (2m-1) (x_d'' - x_q'') I_{p(2m+1)} \\ \Delta E_{b(2m-1)} &= h \Delta E_{a(2m-1)} \\ \text{and } \Delta E_{c(2m-1)} &= h^2 \Delta E_{a(2m-1)} \\ \text{where } I_{p(2m+1)} &= \frac{1}{3} (I_{a(2m+1)} + h I_{b(2m+1)} + h^2 I_{c(2m+1)}) \end{aligned} \right\} (3.5 a)$$

Thus, when the effects of space harmonics are neglected, the interaction between currents of different frequencies can be expressed in the simple general form given by Eqs. 4 and 5a. If the source voltages v_a , v_b and v_c are sinusoidal, the $(2m+1)$ th-harmonic current will depend on the product of $(x_d'' - x_q'')$ and the negative-sequence, $(2m-1)$ th-harmonic current. The sub-transient saliency ($x_d'' \neq x_q''$) is, therefore, the most important factor in determining the relative magnitudes of the harmonic currents when a machine is operating under unbalanced conditions, in the absence of space harmonic effects in the machine

itself and time-harmonic effects in the source voltages.

(3.3) Balanced Three-phase Operation and Two-reaction Theory

When the external impedances connected to the armature phases are equal, i.e., when $R_{ae} = R_{be} = R_{ce}$ and $X_{ae} = X_{be} = X_{ce}$ and the source voltages are balanced, i.e., $V_{a1} = hV_{b1} = h^2V_{c1}$, the fundamental armature currents are also balanced, and the set of equations represented by Equ. 2.6 can be reduced to a single equation representing one phase only. Omitting the subscript 1 and using the conventional constants, the voltage equation of phase a is

$$V_a + E_{a1} = \left\{ R_a + R_e + j(x_e + \frac{x_d + x_q}{2}) \right\} I_a + j \frac{x_d - x_q}{2} I_a^* \\ \text{or} \quad V_{at} + E_{a1} = \left\{ R_a + j \frac{1}{2}(x_d + x_q) \right\} I_a + j \frac{1}{2}(x_d - x_q) I_a^* \quad (3.6)$$

where $V_{at} = V_a - Z_e I_a = \text{terminal voltage of phase a}$

When the source voltages are sinusoidal, there will be no harmonic components in the armature currents and Equ. 3.6 is an exact solution.

Because of the presence of I_a^* , I_a cannot be solved for as a single mathematical entity from Equ. 3.6. The most natural way of solving the equation is to resolve I_a into its real and imaginary components, thereby yielding two equations of real numbers. This process coincides with the two-reaction theory because, as will be shown, the real and imaginary components of the armature current are in fact the direct- and quadrature- axis currents introduced by the two-reaction theory. Thus using the conventional subscripts d and q to denote the real and imaginary components of the armature current and

voltage, the two equations of real numbers derived from Equ. 3.6 are ($E_a = -jE_{ao}$)

$$V_d - E_{ao} = X_d I_d + R_a I_q \quad (3.7)$$

and $V_d = R_a I_d - X_q I_q$

which are the same as the two axis-equations given by the two-reaction theory.

The difference between Eqs. 3.6 and 3.7 can also be compared graphically from the vector diagrams drawn accordingly. Fig. 3.1 is the composite diagram comprising the vector diagrams representing the equations based on both theories.

In the two-reaction theory, the armature current I_a is resolved into two components, the direct-axis component I_d and the quadrature-axis component I_q . The voltage pentagon OABDEO is completed by adding to the terminal voltage - V_{at} the direct-axis synchronous impedance drop $(R_a + jX_d)I_d$, represented in the diagram by the vectors AB and CD, and the quadrature-axis synchronous impedance drop $(R_a + jX_q)I_q$, represented in the diagram by the vectors BC and DE. The open-circuit voltage of the machine is given by the vector OE.

If Equ. 3.6 is used, instead of the axis currents, a conjugate current, I_a^* , is introduced. The skew voltage pentagon OACFEO is completed by adding to the terminal voltage - V_{at} the resistance drop AC and reactance drop CF due to the armature current I_a and another reactance drop FE due to the conjugate current I_a^* . The open-circuit voltage of the machine is again given by the vector OE. That the point E determined by both voltage pentagons coincides can be checked by consider-

ing the magnitudes and directions of the various component voltage vectors, as marked in the vector diagram.

There is an advantage of the diagram based on Equ. 3.6 over the two-axis diagram based on the two-reaction theory for the

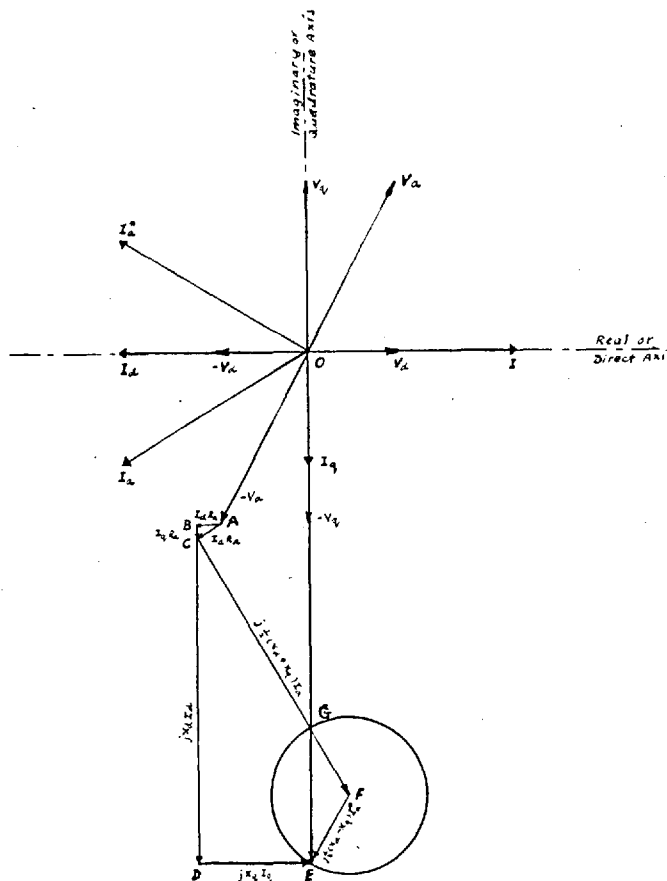


Fig. 3.1

graphical determination of the voltage regulation of a synchronous machine.

When the terminal voltage and the armature current are known, the voltage vectors OA, AC and CF are fixed. The magnitude of the voltage vector FE is also known. A circle can be drawn with the point F as centre and the length of FE as radius, intercepting CF at G. The line joining the points

O and G is in the direction of the open-circuit voltage vector OE. The point E, or the end of the vector OE, is the other point of intersection of the circle and the line OG or its protraction. The skew voltage pentagon OACFEO is thus completed. On

the other hand, if the two-axis diagram is followed, I_a must be resolved into its axis components I_d and I_q , which cannot be done since the direction of OE is unknown. Consequently, it is impossible to complete the voltage pentagon OABDEO by purely graphical means.

(3.4) Single-phase Operations — Approximate Solution

The single-phase operations (line-to-neutral and line-to-line) are the simplest cases of unbalanced operations of a three-phase synchronous machine, as there is in effect one closed armature circuit. When the machine is loaded from line to neutral, the fundamental armature current is readily found by putting $I_{b1} = I_{c1} = 0$ in Equ. 2.6. Thus, using the conventional machine constants,

$$V_{a1} + E_{a1} = \left\{ R_a + R_e + j \left[x_e + \frac{1}{3} \left(x_e + \frac{x_d + x_q}{2} + \frac{x_d'' + x_q''}{2} \right) \right] \right\} I_{a1} + j \frac{1}{6} (x_d - x_q) I_{a1}^* \quad (3.8)$$

Similarly, putting $I_{b3} = I_{c3} = 0$ in Equ. 2.8 gives the approximate third-harmonic current:

$$V_{a3} - j \frac{1}{2} (x_d'' - x_q'') I_{a3} = \left\{ R_a + R_e + j 3 \left(x_e + \frac{x_d'' + x_q'' + x_e}{3} \right) \right\} I_{a3} \quad (3.9)$$

In general, the $(2m+1)$ th-harmonic current is approximately given by

$$V_{a(2m+1)} - j \frac{1}{6} (2m+1) (x_d'' - x_q'') I_{a(2m+1)} = \left\{ R_a + R_e + j (2m+1) \left(x_e + \frac{x_d'' + x_q'' + x_e}{3} \right) \right\} I_{a(2m+1)} \quad (3.10)$$

The general expression of the equivalent voltage representing the corrections due to currents of higher frequencies is given by

$$\Delta E_{a(2m+1)} = -j \frac{1}{6} (2m+1) (x_d'' - x_q'') I_{a(2m+1)} \quad (3.11)$$

When the machine is loaded from line to line, expressions similar to Eqs. 3.10 and 3.11 can be obtained by subtracting

the voltage equation of phase c from that of phase b and using the relations $I_a = 0$ and $I_c = -I_b$ (valid for all frequencies). It can be shown that the fundamental current is approximately given by

$$V_{bc1} + E_{bc1} = \left\{ 2R_a + R_c + j \left(x_c + \frac{x_d + x_q}{2} + \frac{x_d'' + x_q''}{2} \right) \right\} I_{b1} - j \frac{1}{2} (x_d - x_q) I_{b1}^* \quad (3.12)$$

the $(2m+1)$ th-harmonic current is approximately given by

$$V_{bc(2m+1)} + j \frac{1}{2} (2m+1) (x_d'' - x_q'') I_{b(2m+1)} = \left\{ 2R_a + R_c + j (2m+1) \left(x_c + \frac{x_d + x_q}{2} + \frac{x_d'' + x_q''}{2} \right) \right\} I_{b(2m+1)} \quad (3.13)$$

and the equivalent voltage for current correction is given by

$$\Delta E_{bc(2m+1)} = j \frac{1}{2} (2m+1) (x_d'' - x_q'') I_{b(2m+1)} \quad (3.14)$$

When the external impedance R_e, X_e is large, or the subtransient saliency of the machine is small, the harmonic currents are small and Eqs. 3.8 and 3.12 will give nearly the correct values of the fundamental currents. Otherwise corrections must be made by using Eqs. 3.11 and 3.14.

(3.5) Single-phase Operation with Negligible Armature Resistance—Exact Solution

When the resistance in the armature circuit is negligible and the source voltage is sinusoidal, an 'exact' solution for the armature current can be found as shown in the Appendix (Article 14.2). It is shown that for the line-to-neutral case, the fundamental and harmonic components of the armature current are given by

$$3(V_{a1} + E_{a1}) = j \left\{ \frac{1}{2} (x_d + x_q + x_0 + 3x_e) + \sqrt{[x_d^2 + \frac{1}{4}(x_0 + 3x_e)] [x_q^2 + \frac{1}{4}(x_0 + 3x_e)]} \right\} I_{a1} + j \frac{1}{2} (x_d - x_q) I_{a1}^* \quad (3.15)$$

where

$$I_{a(2m+1)} = b_m I_{a1}$$

$$m = 1, 2, 3, \dots$$

$$\text{and } b_m = \frac{\sqrt{x_q^2 + \frac{1}{4}(x_0 + 3x_e)} - \sqrt{x_d^2 + \frac{1}{4}(x_0 + 3x_e)}}{\sqrt{x_d^2 + \frac{1}{4}(x_0 + 3x_e)} + \sqrt{x_q^2 + \frac{1}{4}(x_0 + 3x_e)}} \quad (3.16)$$

Similarly, for the line-to-line case, the fundamental and

the harmonic components of the armature current are given by

$$V_{bc1} + E_{bc1} = j \left\{ \frac{1}{2} (x_d + x_q + x_e) + \sqrt{(x_d'' + \frac{1}{2} x_e)(x_q'' + \frac{1}{2} x_e)} \right\} I_{b1} - j \frac{1}{2} (x_d - x_q) I_{b1}^* \quad (3.17)$$

where $I_{b(\lambda m+1)} = (-b_1)^m I_{b1}$

$$\text{and } b_1 = \frac{\sqrt{x_q'' + \frac{1}{2} x_e} - \sqrt{x_d'' + \frac{1}{2} x_e}}{\sqrt{x_d'' + \frac{1}{2} x_e} + \sqrt{x_q'' + \frac{1}{2} x_e}} \quad (3.18)$$

Comparing Equ. 3.8, the approximate solution for the fundamental component of current for the line-to-neutral operation, and Equ. 3.15, the corresponding 'exact' solution for negligible resistance in the armature circuit, it is clear that the only difference between them, after neglecting the R's in Equ. 3.8, is in the coefficient of I_{a1} . The two equations would be identical if the quantity

$$\frac{1}{2} (X_d'' + X_q'') \quad (3.19)$$

which is part of the coefficient of I_{a1} in Equ. 3.8, were equal to

$$\sqrt{[x_d'' + \frac{1}{2}(x_o + 3x_e)][x_q'' + \frac{1}{2}(x_o + 3x_e)]} - \frac{1}{2}(x_o + 3x_e) \quad (3.20)$$

the corresponding quantity in Equ. 3.15. Evidently Expressions 3.19 and 3.20 are equal only when $X_d'' = X_q''$, that is, when the machine has zero subtransient saliency and consequently zero harmonic currents. The actual difference between them depends on the relative magnitudes of X_d'' , X_q'' , X_o and X_e , and will be largest when X_e is equal to zero, that is, when the machine is directly short-circuited from line to neutral.

Similarly, Eqs. 3.12 and 3.17 can be compared for the line-to-line case. It can be shown that the expression corresponding to Expression 3.20 is

$$\sqrt{(x_d'' + \frac{1}{2} x_e)(x_q'' + \frac{1}{2} x_e)} - \frac{1}{2} x_e \quad (3.21)$$

which differs from Expression 3.19 by the largest amount when

X_e is equal to zero. The difference will then be the difference between the arithmetical mean and the geometrical mean of the direct-axis and the quadrature-axis subtransient reactances.

(3.6) Further Possible Assumptions

When the effects of space harmonics are neglected, the solution for the steady-state currents of a synchronous machine is greatly simplified (Art. 3.2) and can be expressed in terms of five derived machine constants (Equ. 3.1). There are other possible assumptions which will further reduce the number of machine constants and simplify the solution. The assumptions may reduce the accuracy of the result but they may be essential when it becomes necessary to have as simple a representation of the synchronous machine as possible, as in the power system studies.

If necessary, the synchronous saliency ($X_d \neq X_q$) may be neglected. This is equivalent to neglecting the terms involving the conjugate currents in the solution for the fundamental currents and taking the average value of X_d and X_q as the single synchronous reactance. The appropriate value of this synchronous reactance is, in fact, some sort of a mean value of X_d and X_q , depending on the phase of the armature currents, and the arithmetical mean is but one of the possible values.

The subtransient saliency may also be neglected for simplicity. This is equivalent to neglecting the harmonic currents. A single subtransient reactance (which is virtually the same thing as the negative-sequence reactance) may be used and

some sort of a mean value between X_d'' and X_q'' may be assigned to it. Its appropriate value depends on the harmonic currents and can be worked out for any particular case.

When both kinds of saliency (synchronous and subtransient) are neglected, there remain only three distinct reactances, i.e., the zero-sequence reactance X_0 , the synchronous reactance (mean value of X_d and X_q) and the subtransient or negative-sequence reactance (mean value of X_d'' and X_q''), and the solution may be considerably simplified.

(4) METHOD OF SYMMETRICAL COMPONENTS

(4.1) General Considerations

In the simplified solution (Article 3.2, negligible space-harmonic effects) the positive- and negative- sequence currents ($I_{p(2m-1)}$ and $I_{n(2m-1)}$) automatically come in although no attempt has been made to introduce the method of symmetrical components at that stage. This suggests that there must be some gain if the method is applied.

The form of the machine impedance matrices (Eqs. 2.6, 2.8 and 2.10 with external impedances omitted and space-harmonic terms neglected) also shows that it can be diagonalised when pre-multiplied by $[S]^{-1}$ and post-multiplied by $[S]$, where $[S]$ is the well-known symmetrical-component transformation matrix defined by

$$[S] = \begin{matrix} & \begin{matrix} p & n & o \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ h^2 & h & 1 \\ h & h^2 & 1 \end{bmatrix} \end{matrix}$$

When the external impedance matrix is of the general form

$$\begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} Z & Z_I & Z_{II} \\ Z_{II} & Z & Z_I \\ Z_I & Z_{II} & Z \end{bmatrix} \end{matrix}$$

its inclusion does not affect the process of diagonalisation, and may be treated as part of the machine impedance matrices. When the external impedance matrix does not fall into this category, no material gain will be realised by the use of the

method of symmetrical components.

Without the knowledge of the actual form of the external impedance matrix, the external impedances may be omitted if the machine terminal voltages v_{at} , v_{bt} and v_{ct} are used in place of the source voltages v_a , v_b and v_c . This is a rigorous process though the terminal voltages of the machine may not be known without first finding the armature currents.

Since the method of symmetrical components is in itself a rigorous mathematical transformation process, its application will in all cases give correct results, provided it is applied in a rigorous way. In other words, the method is always valid if in addition to the fundamental-frequency quantities the harmonic-frequency quantities are also included in the course of the transformation.

(4.2) Sequence Equations

When the symmetrical component transformation is made to the voltage, current and impedance matrices (Equs. 2.6, 2.8 and 2.10 with the elements of the voltage and impedance matrices simplified to Equs. 3.2 and 3.4 by neglecting space-harmonic effects) according to the formulae

$$[I]' = [s]^{-1} \cdot [I]$$

$$[E]' = [s]^{-1} \cdot [E]$$

$$\text{and } [Z]' = [s]^{-1} \cdot [Z] \cdot [s]$$

where the dashed ' quantities are those after the transformation, and the interaction of currents of higher frequencies (Equs. 3.5 and 3.5.a) are included (i.e., the exact solution),

it can be shown that the fundamental voltage equation is

$$\begin{bmatrix} V_{p1t} - jE_{20} \\ V_{n1t} \\ V_{o1t} \end{bmatrix} = \begin{bmatrix} R_a + j \frac{x_d + x_q}{2} & & & j \frac{x_d - x_q}{2} & \\ & R_a + j \frac{x_d'' + x_q''}{2} & & & j \frac{x_d'' - x_q''}{2} \\ & & R_o + j x_o & & \end{bmatrix} \cdot \begin{bmatrix} I_{p1} \\ I_{n1} \\ I_{o1} \\ I_{p1}^* \\ I_{p3} \end{bmatrix} \quad (4.1)$$

and the $(2m+1)$ th-harmonic voltage equation is

$$\begin{bmatrix} V_{p(2m+1)t} \\ V_{n(2m+1)t} \\ V_{o(2m+1)t} \end{bmatrix} = \begin{bmatrix} R_a + j(2m+1) \frac{x_d + x_q}{2} & & & j(2m+1) \frac{x_d - x_q}{2} & \\ & R_a + j(2m+1) \frac{x_d'' + x_q''}{2} & & & j(2m+1) \frac{x_d'' - x_q''}{2} \\ & & R_o + j(2m+1) x_o & & \end{bmatrix} \cdot \begin{bmatrix} I_{p(2m+1)} \\ I_{n(2m+1)} \\ I_{o(2m+1)} \\ I_{n(2m-1)} \\ I_{p(2m+3)} \end{bmatrix} \quad (4.2)$$

where the subscripts p, n and o denote the positive-, negative- and zero- sequence quantities.

It is clear from Eqs. 4.1 and 4.2 that there are independent circuits for the positive-sequence, fundamental-frequency component of current and the zero-sequence component of current at all frequencies. The negative-sequence, fundamental-frequency component and the positive- and negative- sequence harmonic components of current are all mutually related. This fact has been shown elsewhere based on the concept of rotating fields. The equations (4.1 and 4.2) may, therefore, serve as a mathematical proof of the rotating-field theory.

Since the positive-sequence, fundamental-frequency current is independent of the armature current components of other sequences or frequencies, the two synchronous reactances X_d

and X_q are not affected by the armature currents (remembering the assumption of no magnetic saturation). Similarly the zero-sequence reactance X_0 is also independent of the armature currents. The negative-sequence reactance, however, has no definite values, being dependent on the magnitude and phase of the harmonic currents. When the harmonic currents become negligible, it is equal to the arithmetical mean of the direct- and quadrature- axis subtransient reactances X_d'' and X_q'' , as can be seen from Equ. 4.1.

It is now clear that the quantity given by Expression 3.20, which is originally picked out from the armature reactance as the part which is affected by the harmonic currents, is in fact the negative-sequence reactance appropriate to the case (single-phase line-to-line) considered. Similarly the quantity given by Expression 3.21 is the appropriate negative-sequence reactance for the single-phase line-to-line operation with negligible resistance and capacitance in the armature circuit and sinusoidal source voltage.

(4.3) Single-phase Operations

The solution for single-phase operations can also be found from the sequence equations given in the preceding article in addition to the more direct process shown in Article 3.4. The use of the sequence equations for these simpler cases may be a roundabout process, but it does serve to show, as an example, the application of the method of symmetrical components.

It can be shown that for the line-to-neutral case, the

positive-, negative- and zero- sequence currents are equal and each has one third the value of the phase current. This condition is valid for currents of all frequencies, so that

$$I_p(2m-1) = I_n(2m-1) = I_o(2m-1) = I_a(2m-1) \quad (4.3)$$

Substituting the relation given by Equ. 4.3 into each of the sequence equations represented by Equ. 4.1 and adding, gives

$$V_{a1t} + E_{a1} = \left\{ 3R_a + j \left[\frac{1}{2}(X_d + X_q) + \frac{1}{2}(X_d'' + X_q'') + X_o \right] \right\} I_{a1}/3 \\ + j(X_d - X_q) I_{a1}^*/6 + j(X_d'' - X_q'') I_{a3}/6 \quad (4.4)$$

which is equal to Equ. 3.8 with Z_e omitted and ΔE_{a1} added.

Similarly, from Equ. 4.2,

$$V_{a(2m+1)t} = j(2m+1)(X_d'' - X_q'') I_{a(2m-1)}/6 \\ + \left\{ 3R_a + j(2m+1)(X_d'' + X_q'' + X_o) \right\} I_{a(2m+1)}/3 \\ + j(2m+1)(X_d'' - X_q'') I_{a(2m+3)}/6 \quad (4.5)$$

which is equal to Equ. 3.9 with Z_e omitted and $\Delta E_{a(2m+1)}$ added.

When R_a is neglected and $V_{a(2m+1)t} = 0$, the following relation can be derived from Equ. 4.5 (Referring to Appendix 14.2):

$$I_{a(2m+1)} = b_o I_{a(2m-1)} \quad (4.6)$$

where $b_o = (\sqrt{X_d'' + \frac{1}{2}X_o} - \sqrt{X_q'' + \frac{1}{2}X_o}) / (\sqrt{X_d'' + \frac{1}{2}X_o} + \sqrt{X_q'' + \frac{1}{2}X_o}) \quad (4.7)$

Putting $m = 1$ in Equ. 4.6 and substituting into the negative-sequence equation of Equ. 4.1, I_{a3} can be eliminated. The negative-sequence equation becomes, with R_e neglected,

$$V_{n1} = j \left\{ \sqrt{(X_d'' + \frac{1}{2}X_o)(X_q'' + \frac{1}{2}X_o)} - \frac{1}{2}X_o \right\} I_{n1} \quad (4.8)$$

which shows that the negative-sequence reactance for the line-to-line loading without external impedance should be

$$X_2 = \sqrt{(X_d'' + \frac{1}{2}X_o)(X_q'' + \frac{1}{2}X_o)} - \frac{1}{2}X_o \quad (4.9)$$

which is the same as Expression 3.20 when $X_e = 0$.

Adding the positive- and zero- sequence equations from Equ. 4.1 and the negative-sequence equation (Equ. 4.8) gives Equ. 3.15 for the fundamental-frequency current. The equation for the harmonic currents (Equ. 3.16) can also be derived from Equ. 4.5 by using Equ. 4.6.

Similar considerations can be given to the line-to-line case. With the current restraints

$$i_a = 0 \text{ and } i_b = -i_c$$

the sequence currents are

$$I_{p(2m-1)} = -I_{n(2m-1)} = jI_{b(2m-1)}/\sqrt{3}$$

$$\text{and } I_{o(2m-1)} = 0$$

Using these current relations and proceeding in like manner as in the line-to-neutral case, it can be shown that Eqs. 3.12 and 3.13 (with X_e omitted and ΔE_{a1} and $\Delta E_{a(2m+1)}$ added) can be derived. It can also be shown that when R_a is neglected and $V_{bc(2m-1)t} = 0$ the appropriate negative-sequence reactance is

$$X_2 = \sqrt{X_d'' X_q''} \quad (4.10)$$

which is the same as Expression 3.21 when $X_e = 0$.

(4.4) Synchronous Motor on Unbalanced Mains

As another example for the application of the method of symmetrical components, consider the case of a synchronous motor operating with unbalanced sinusoidal applied voltages. It is clear, from Equ. 4.2, that when $V_{p(2m-1)} = V_{n(2m-1)} = V_{o(2m-1)} = 0$, no harmonic currents higher than the third will exist. Thus, instead of an infinite number of equations as for the

general case there remain only four equations, among them two independent, relating the sequence fundamental currents and the positive-sequence, third-harmonic current:

$$V_{p1} - jE_{a0} = \{R_a + j \frac{1}{2}(x_d + x_q)\} I_{p1} + j \frac{1}{2}(x_d - x_q) I_{p1}^* \quad (4.11)$$

$$V_{n1} = \{R_a + j \frac{1}{2}(x_d'' + x_q'')\} I_{n1} + j \frac{1}{2}(x_d'' - x_q'') I_{p3} \quad (4.12)$$

$$0 = \{R_a + j \frac{3}{2}(x_d'' + x_q'')\} I_{p3} + j \frac{3}{2}(x_d'' - x_q'') I_{n1} \quad (4.13)$$

and $V_{01} = (R_a + j x_0) I_{01} \quad (4.14)$

In Equ. 4.12, the third-harmonic current I_{p3} can be eliminated by using Equ. 4.13. If R_a is neglected compared with $3(x_d'' + x_q'')/2$, Equ. 4.12 is reduced to

$$V_{n1} = \{R_a + j 2 x_d'' x_q'' / (x_d'' + x_q'')\} I_{n1} \quad (4.15)$$

which shows that the appropriate negative-sequence reactance is

$$X_2 = 2X_d''X_q''/(X_d'' + X_q'') \quad (4.16)$$

The third-harmonic current is approximately given by

$$I_{p3} = I_{n1} (x_d'' - x_q'') / (x_d'' + x_q'') \quad (4.17)$$

(4.5) Negative-sequence Reactance

In connection with the solution for the particular cases, a number of possible values (Expressions 3.19, 3.20, 3.21, 4.9, 4.10 and 4.16) for the negative-sequence reactance have been worked out. It is interesting to note that all these values lie within two limits, the arithmetical mean (3.19) and the harmonic mean (4.16) of the direct- and quadrature- axis subtransient reactances. It is thus possible to conclude that the relation

$$\frac{1}{2}(x_d'' + x_q'') \geq x_2 \geq 2x_d''x_q''/(x_d'' + x_q'') \quad (4.18)$$

for the negative-sequence reactance holds in general provided

that the source voltages are sinusoidal and that there is no capacitance in the armature circuits.

Although X_2 depends on the harmonic currents and varies within the limits given by Expression 4.18, a definite value must be assigned to it before an 'independent' negative-sequence equation can be obtained. A possible value for X_2 is the geometrical mean of the two limits, which is also the geometrical mean of X_d'' and X_q'' and the appropriate value for the line-to-line short circuit. The difference among these three sorts of mathematical mean value (arithmetical, geometrical and harmonic) of X_d'' and X_q'' depends on the relative magnitudes of X_d'' and X_q'' . For a machine with damper windings in both the direct and the quadrature axes, the difference between X_d'' and X_q'' is generally small and the three sorts of mean value will not differ much from one another, and the use of a single X_2 for all possible operation (except with capacitances in the armature circuits) of the machine will not cause much error. Much greater error will be introduced when the machine has negligible quadrature-axis damping, as X_q'' will then have the same value as X_q . External inductive impedances will reduce the difference between the overall subtransient reactances in the direct and quadrature axes, and are therefore favourable to the use of a single X_2 . Table I gives a comparison of the three sorts of mean value for a certain range of the ratio X_q''/X_d'' .

TABLE I

X_q''/X_d''	A.M./ X_d''	G.M./ X_d''	H.M./ X_d''
1.0	1.000	1.000	1.000
1.5	1.250	1.225	1.200
2.0	1.500	1.414	1.333
2.5	1.750	1.581	1.429
3.0	2.000	1.732	1.500
4.0	2.500	2.000	1.600

(4.6) Simplified Method of Solution

For the sake of simplicity, both or either of the synchronous or subtransient saliency of a synchronous machine may be neglected as discussed in Article 3.6. The neglect of the subtransient saliency and the use of a single X_2 results in the possibility of an independent negative-sequence equation. If the synchronous saliency is retained, the steady-state fundamental-frequency armature currents of a synchronous machine can be approximately found by the following four equations:

$$V_q - E_{ao} = X_d I_d + R_a I_q \quad (4.19)$$

$$V_d = R_a I_d - X_q I_q \quad (4.20)$$

$$V_n = (R_a + jX_2) I_n \quad (4.21)$$

$$V_o = (R_a + jX_o) I_o \quad (4.22)$$

where V_d and V_q are the direct- and quadrature- axis components of the positive-sequence terminal voltage, I_d and I_q are real numbers and are the real and imaginary parts of the positive-sequence current I_p . I_n and I_o are complex numbers.

To simplify the equations further still, R_a may be neglected compared with X_d and X_q . The direct- and quadrature- axis equations (4.19 and 4.20) thus become independent of each other.

The machine may therefore be represented by four independent equivalent circuits, three of them are simple passive networks each containing a single impedance, as shown in Fig. 4.1.

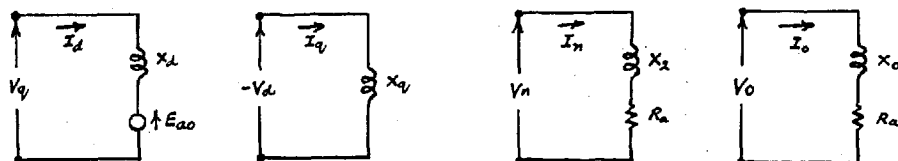


Fig. 4.1

If the synchronous saliency is also neglected, the positive-sequence equation reduces to

$$V_p + E_a = (R_a + jX_1)I_p \quad (4.23)$$

where X_1 is the positive-sequence reactance. Eqs. 4.23, 4.21 and 4.22 are sufficient for the determination of the fundamental-frequency armature currents. Parallel to the negative-sequence reactance, the value of X_1 may be taken as the geometrical mean of X_d and X_q . Since the ratio X_d/X_q is seldom larger than 1.5, there is not much difference whether the arithmetical mean, the geometrical mean or the harmonic mean of X_d and X_q is used as X_1 .

It is interesting to note the analogy regarding X_1 and X_2 . The success of using a single X_1 depends on the neglect of the synchronous saliency and that of using a single X_2 depends on the neglect of the subtransient saliency. If the machine has a symmetrical rotor (one with both synchronous and subtransient symmetries such as the rotor of an induction motor), there will be no saliency of either sort and Eqs. 4.23, 4.21 and 4.22 give exact results. A solid (unlaminated) round rotor is synchronously symmetrical and approximately subtransiently symmetrical, so the three sequence equations can give good results

for a machine equipped with such a rotor. For a salient-pole machine with damper windings in both the pole and the interpole axes, the synchronous saliency is appreciable but the subtransient saliency is small, and Eqs. 4.19, 4.20, 4.21 and 4.22 can be used without much error.

In the event of appreciable space-harmonic effects, the fundamental currents may still be determined by the method of symmetrical components as discussed above, as the effects of space harmonics on the fundamental currents are small (referring to Equ. 2.7, if $M_{af3} = 0.1M_{af1}$, $M_{af3}^2/L_{fe} = 0.01M_{af1}^2/L_{fe}$, etc.). The harmonic currents are much more affected by the presence of space harmonics. The most prominent term due to space harmonics in E_{a3} (Equ. 2.9) is the term involving the field excitation unless the zero-sequence, fundamental-frequency current is unduly large. This suggests that the harmonic currents may still be found by Equ. 4.2 provided the terminal voltage $V_{a(2m+1)t}$ includes the component $-j(2m+1)M_{af(2m+1)}I_{fo}/\sqrt{2}$. This process has the physical meaning of neglecting the armature reaction on the harmonic flux (in space due to the field excitation). This is approximately true when the degree of unbalance of the armature circuits is not too large, as the positive-sequence, fundamental-frequency current cannot modify the harmonic flux, a concept parallel to the fact that the negative-sequence, fundamental-frequency current cannot modify the fundamental flux.

(4.7) Experimental Results

(a) Round-rotor machine

The machine was a synchronous-induction machine with negligible synchronous saliency and damping effects. The space-harmonic effects were also negligible. The fundamental and derived constants of the machine shown in Table II were measured by a.c. measurement.

TABLE II

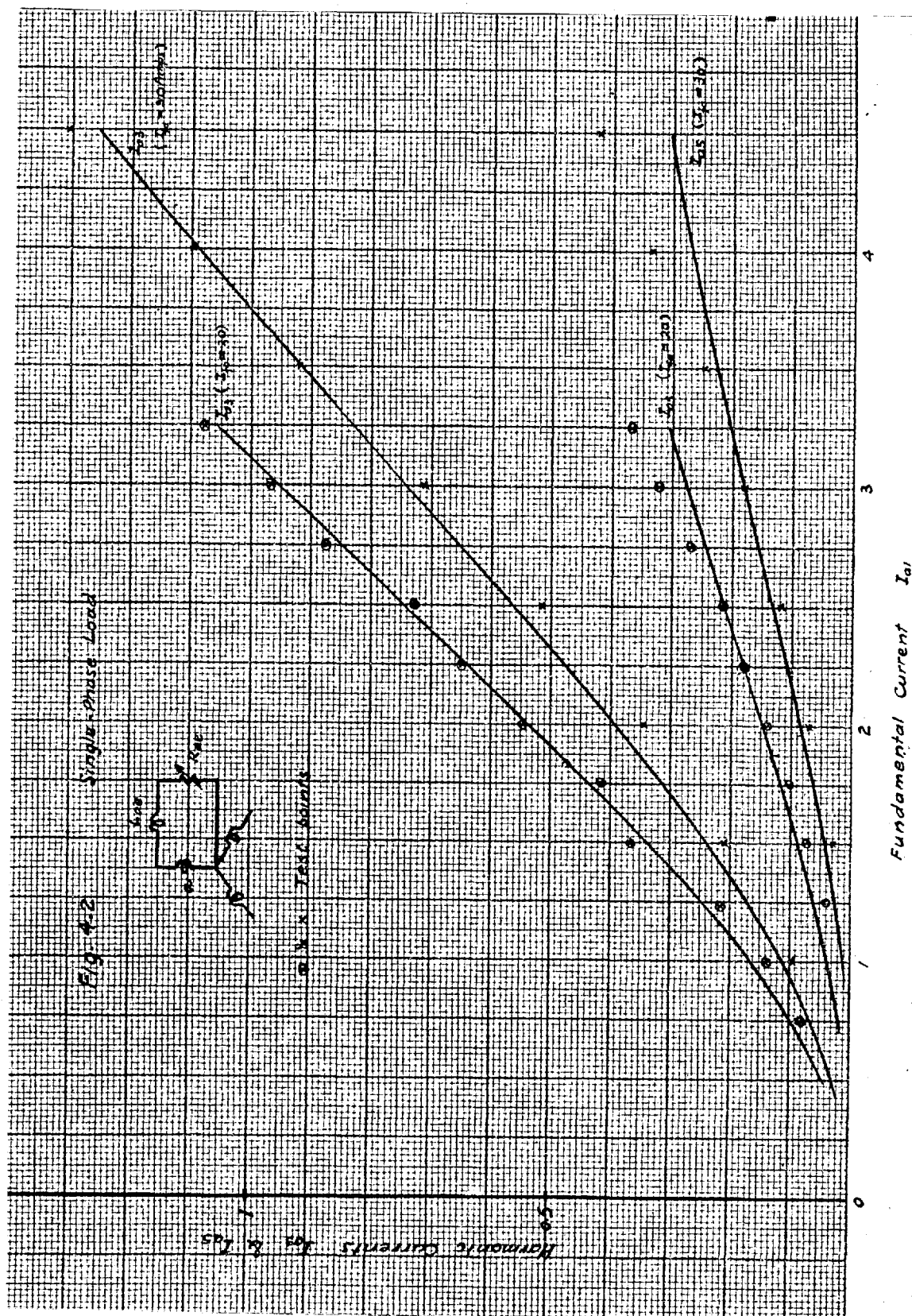
$R_a = 0.68$ ohm	$\omega L_{a0} = 87$ ohms	$\omega M_{s0} = -40$ ohms
$\omega L_f = 4.79$ ohms (including the exciter inductance)		
$\omega M_{af1} = 18.2$ ohms	$\omega/2\pi = 50$ cps	
$X_d = X_q = X_q'' = 127$ ohms	$X_d'' = 23.2$ ohms	$X_0 = 7$ ohms

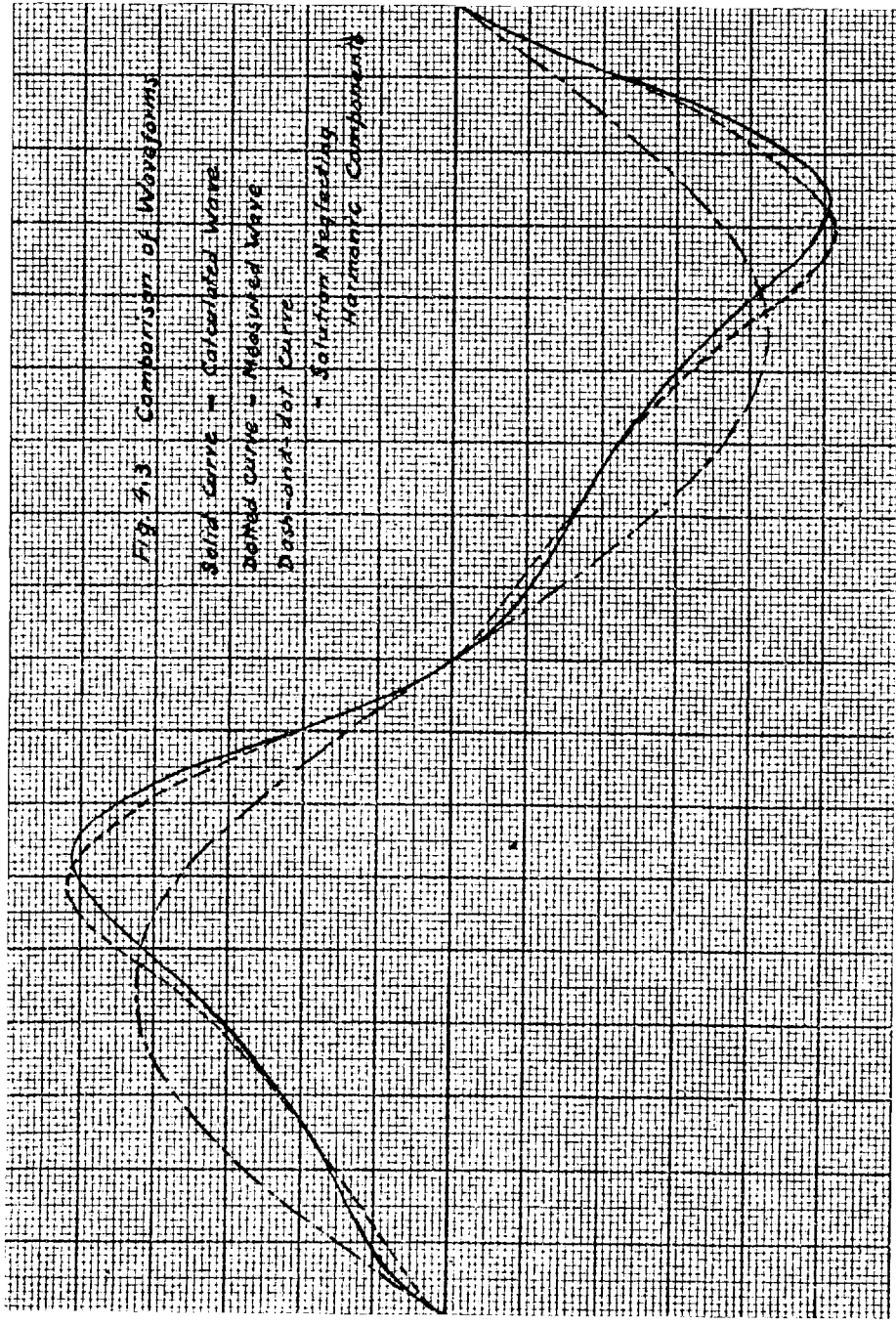
With a line-to-neutral inductive load consisting of a fixed inductor of 11.6 ohms at 50 cps in series with a variable resistor for load adjustment, and an excitation of 30 Amps., the calculated results, using the equations given in Article 3.4 and making the correction once, are shown in Table III and Fig. 4.2. The effects of harmonics higher than the fifth have been neglected in obtaining the results given in Table III.

TABLE III

R_e	0	20	40	70	100	140	200
I_{a1}	5.05 <u>$\angle -179.4^\circ$</u>	4.87 <u>$\angle -164.7^\circ$</u>	4.43 <u>$\angle -152.6^\circ$</u>	3.66 <u>$\angle -136.9^\circ$</u>	3.02 <u>$\angle -127.3^\circ$</u>	2.38 <u>$\angle -118.7^\circ$</u>	1.78 <u>$\angle -111.3^\circ$</u>
I_{a3}	1.45 <u>$\angle -173.9^\circ$</u>	1.39 <u>$\angle -158.2^\circ$</u>	1.24 <u>$\angle -138.6^\circ$</u>	0.98 <u>$\angle -115.0^\circ$</u>	0.75 <u>$\angle -92.6^\circ$</u>	0.53 <u>$\angle -79.9^\circ$</u>	0.33 <u>$\angle -62.7^\circ$</u>
I_{a5}	0.34 <u>$\angle -179.2^\circ$</u>	0.33 <u>$\angle -156.1^\circ$</u>	0.30 <u>$\angle -134.2^\circ$</u>	0.25 <u>$\angle -106.4^\circ$</u>	0.18 <u>$\angle -84.0^\circ$</u>	0.13 <u>$\angle -60.2^\circ$</u>	0.08 <u>$\angle -33.9^\circ$</u>

The waveforms of the calculated and measured currents corresponding to $R_e = 100$ ohms are compared as shown in Fig. 4.3. The measured values of the third- and fifth- harmonic currents





are given in Fig. 4.2. These values were measured by an harmonic analyser.

The values of the fundamental current found by using $X_2 = \sqrt{x_d''x_q''} = 51.4$ ohms and neglecting the harmonic currents are given in Table IV.

TABLE IV

R_e	0	20	40	70	100	140	200
I_{a1}	5.26 $\angle -179.5^\circ$	5.06 $\angle -164.3^\circ$	4.60 $\angle -151.0^\circ$	3.78 $\angle -136.1^\circ$	3.10 $\angle -126.1^\circ$	2.46 $\angle -117.9^\circ$	1.80 $\angle -110.1^\circ$

For $R_e = 0$, the 'exact' value of I_{a1} and I_{a3} found by Eqs. 3.15 and 3.16 are $5.07 \angle -180^\circ$ and $1.48 \angle -180^\circ$ respectively ($x_2 = 59.7$ and $b_0 = 0.293$).

The measured and calculated results with an excitation of 20 Amps. are also shown in Fig. 4.2.

(b) Salient-pole machine

The machine was a salient-pole machine with negligible quadrature-axis damping. The fundamental parameters and the derived machine constants (neglecting space-harmonic effects) are shown in Table V.

TABLE V

$R_a = 0.03$ ohm	$\omega L_{a0} = 0.51$ ohm	$\omega M_{s0} = -0.19$ ohm
$\omega L_{a2} = 0.07$ ohm	$\omega M_{s2} = -0.06$ ohm	$\omega M_{af1} = 34.5$ ohms
$\omega M_{af3} = 1.81$ ohms	$\omega M_{af1}^2 / L_{fe} = 0.39$ ohm	$\omega M_{af1} M_{af3} / L_{fe} = 0.02$ ohm
$X_d = 0.81$ ohm	$X_q = x_q'' = 0.60$ ohm	$X_0 = 0.13$ ohm
		$x_d'' = 0.33$ ohm

The quantity involving the square of M_{af3} is small and neglected.

The connected load impedance were

$$Z_{ae} = 0.09 + j1.0, \quad Z_{be} = 1.68, \quad Z_{ce} = 0.74 + j1.0$$

at 50 cps. The field excitation was 1 Amp.

The calculated and measured values of the fundamental currents are compared as shown in Table VI.

TABLE VI

Phase	a	b	c
Using Equ. 2.6	$\frac{34.7}{/-179.1^\circ}$	$\frac{22.9}{/133.7^\circ}$	$\frac{25.1}{/-39.5^\circ}$
Neglecting Synchronous Saliency & Space-harmonic Effects	$\frac{33.6}{/-175.7^\circ}$	$\frac{24.0}{/136.5^\circ}$	$\frac{26.4}{/-40.2^\circ}$
Test (magnitude only)	34.7	23.1	25.6

The various components (as classified in Article 2.4) of the equivalent third-harmonic voltages and the third-harmonic currents are shown in Table VII.

TABLE VII

Phase	a	b	c
Voltage component due to space harmonic & excitation	$\frac{7.56}{/-90.0^\circ}$	$\frac{7.56}{/-90.0^\circ}$	$\frac{7.56}{/-90.0^\circ}$
Voltage component due to synchronous saliency	$\frac{1.94}{/-104.2^\circ}$	$\frac{1.20}{/102.5^\circ}$	$\frac{1.26}{/-11.3^\circ}$
Voltage component due to negative-sequence current	$\frac{5.14}{/141.0^\circ}$	$\frac{5.14}{/21.0^\circ}$	$\frac{5.14}{/-99.0^\circ}$
Voltage component due to zero-sequence current	$\frac{0.47}{/-90.0^\circ}$	$\frac{0.47}{/-90.0^\circ}$	$\frac{0.47}{/-90.0^\circ}$
Total third-harmonic voltage	$\frac{7.45}{/-126.8^\circ}$	$\frac{6.89}{/-47.9^\circ}$	$\frac{13.4}{/-87.8^\circ}$
Third-harmonic current	$\frac{1.89}{/157.7^\circ}$	$\frac{4.06}{/-82.7^\circ}$	$\frac{3.43}{/-162.5^\circ}$
Test (current, magnitude only)	1.96	4.28	3.52

The results given in Table VI for the fundamental current components show that the error introduced by neglecting the synchronous saliency and space-harmonic effects is not large. This close result is partly due to the fact that the effects

of synchronous saliency and those of the space harmonic are of opposite nature, which is clear from Equ. 2.7; and partly due to the effect of the external reactances.

Comparing the various components of the third-harmonic voltages given in Table VII confirms the statement made in the last paragraph of Article 4.6 that the most prominent terms are those due to the field excitation and the negative-sequence current. The component due to synchronous saliency becomes wholly dependent on the negative-sequence current when $M_{s2} = -L_{s2}$, a condition which is very nearly true unless the effects of space harmonics are unduly large, and may be taken as part of the component due to the negative-sequence current. Thus, with these two prominent components included and the others (part of the second component and the fourth component in Table VII) neglected, the error introduced will be of the order of the fourth component (due to zero-sequence current) in Table VII.

(5) FIELD CURRENT, DAMPER CURRENTS & ELECTROMAGNETIC TORQUE

(5.1) Field and Damper Currents

In the process of deriving the steady-state solution for the armature currents, the field and damper currents have been eliminated, so that these currents are missing in the solution given in the previous chapters. They can, after the armature currents have been determined, be found by substituting the values of the armature currents in the field- and damper-circuit voltage equations.

At steady state, each of the currents i_f^1 (the field excitation I_{f0} excluded), i_{kd} and i_{kq} can be expressed by an even-harmonic series without the constant term (d.c. component), so that the field and damper resistances may be neglected compared with the reactances at twice and higher multiples of the fundamental frequency. Thus, neglecting the field and damper resistances and integrating, the field- and damper- circuit voltage equations may be reduced to

$$0 = M_{af1} \left[i_a \cos \omega t + i_b \cos (\omega t - 2\pi/3) + i_c \cos (\omega t + 2\pi/3) \right] \\ + M_{af3} (i_a + i_b + i_c) \cos 3\omega t + L_f i_f' + M_{fkd} i_{kd} \quad (5.1)$$

$$0 = M_{af1} \left[i_a \cos \omega t + i_b \cos (\omega t - 2\pi/3) + i_c \cos (\omega t + 2\pi/3) \right] \\ + M_{af3} (i_a + i_b + i_c) \cos 3\omega t + M_{fkd} i_f' + L_{kd} i_{kd} \quad (5.2)$$

and $0 = -M_{aq1} \left[i_a \sin \omega t + i_b \sin (\omega t - 2\pi/3) + i_c \sin (\omega t + 2\pi/3) \right] \\ + M_{aq3} (i_a + i_b + i_c) \sin 3\omega t + L_{kq} i_{kq} \quad (5.3)$

Solving Eqs. 5.1 and 5.2 simultaneously for i_f^1 and i_{kd} gives

$$i_f' = -\frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{1}{L_{fe}} \left\{ M_{af1} [i_a \cos \omega t + i_b \cos(\omega t - 2\pi/3) + i_c \cos(\omega t + 2\pi/3)] + M_{af3} (i_a + i_b + i_c) \cos 3\omega t \right\} \quad (5.4)$$

$$\text{and } i_{kd} = -\frac{L_f - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{1}{L_{fe}} \left\{ M_{af1} [i_a \cos \omega t + i_b \cos(\omega t - 2\pi/3) + i_c \cos(\omega t + 2\pi/3)] + M_{af3} (i_a + i_b + i_c) \cos 3\omega t \right\} \quad (5.5)$$

where $L_{fe} = (L_f L_{kd} - M_{fkd}^2) / (L_f + L_{kd} - 2M_{fkd})$

Also from Equ. 5.3,

$$i_{kq} = \frac{1}{L_{kq}} \left\{ M_{aq1} [i_a \sin \omega t + i_b \sin(\omega t - 2\pi/3) + i_c \sin(\omega t + 2\pi/3)] - M_{aq3} (i_a + i_b + i_c) \sin 3\omega t \right\} \quad (5.6)$$

Let the steady-state armature currents be expressed as

$$\left. \begin{aligned} i_a &= \sqrt{2} \sum_{m=0}^{\infty} I_{a(2m+1)} \cos[(2m+1)\omega t + \varphi_{a(2m+1)}] \\ i_b &= \sqrt{2} \sum_{m=0}^{\infty} I_{b(2m+1)} \cos[(2m+1)\omega t + \varphi_{b(2m+1)}] \\ \text{and } i_c &= \sqrt{2} \sum_{m=0}^{\infty} I_{c(2m+1)} \cos[(2m+1)\omega t + \varphi_{c(2m+1)}] \end{aligned} \right\} \quad (5.7)$$

Substituting Equ. 5.7 in Equ. 5.3 and omitting the constant terms (since the d.c. components do not exist at steady state) gives the harmonic components of the steady-state i_f' :

$$\begin{aligned} i_f' &= -\frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{M_{af1}}{L_{fe}} \cdot \sqrt{2} \sum_{m=0}^{\infty} \left\{ \frac{1}{2} I_{a(2m+1)} \cos(2\overline{m+1}\omega t + \varphi_{a(2m+1)}) + \frac{1}{2} I_{b(2m+1)} \cos(2\overline{m+1}\omega t + \varphi_{b(2m+1)} - 2\pi/3) \right. \\ &\quad \left. + \frac{1}{2} I_{c(2m+1)} \cos(2\overline{m+1}\omega t + \varphi_{c(2m+1)} + 2\pi/3) \right. \\ &\quad \left. + \frac{1}{2} I_{a(2m+3)} \cos(2\overline{m+1}\omega t + \varphi_{a(2m+3)}) + \frac{1}{2} I_{b(2m+3)} \cos(2\overline{m+1}\omega t + \varphi_{b(2m+3)} + 2\pi/3) \right. \\ &\quad \left. + \frac{1}{2} I_{c(2m+3)} \cos(2\overline{m+1}\omega t + \varphi_{c(2m+3)} - 2\pi/3) \right\} \\ &\quad - \frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{M_{af3}}{L_{fe}} \cdot \sqrt{2} \left\{ \frac{1}{2} I_{a1} \cos(2\omega t - \varphi_{a1}) + \frac{1}{2} I_{b1} \cos(2\omega t - \varphi_{b1}) + \frac{1}{2} I_{c1} \cos(2\omega t - \varphi_{c1}) \right. \\ &\quad \left. + \frac{1}{2} I_{a3} \cos(2\omega t + \varphi_{a3}) + \frac{1}{2} I_{b3} \cos(2\omega t + \varphi_{b3}) + \frac{1}{2} I_{c3} \cos(2\omega t + \varphi_{c3}) \right. \\ &\quad \left. + \sum_{m=0}^{\infty} \left[\frac{1}{2} I_{a(2m+1)} \cos(2\overline{m+2}\omega t + \varphi_{a(2m+1)}) + \frac{1}{2} I_{b(2m+1)} \cos(2\overline{m+2}\omega t + \varphi_{b(2m+1)}) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} I_{c(2m+1)} \cos(2\overline{m+2}\omega t + \varphi_{c(2m+1)}) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} I_{a(2m+3)} \cos(2\overline{m+2}\omega t + \varphi_{a(2m+3)}) + \frac{1}{2} I_{b(2m+3)} \cos(2\overline{m+2}\omega t + \varphi_{b(2m+3)}) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} I_{c(2m+3)} \cos(2\overline{m+2}\omega t + \varphi_{c(2m+3)}) \right] \right\} \quad (5.8) \end{aligned}$$

The second-harmonic component of the field current can be represented symbolically in terms of complex numbers as

$$I'_{f2} = -\frac{1}{2} \frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{1}{L_{je}} \left\{ M_{af1} (I_{a1} + h^2 I_{b1} + h I_{c1} + I_{a3} + h I_{b3} + h^2 I_{c3}) \right. \\ \left. + M_{af3} (I_{a1}^* + I_{b1}^* + I_{c1}^* + I_{a3} + I_{b3} + I_{c3}) \right\} \\ = -\frac{3}{2} \frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{1}{L_{je}} \left\{ M_{af1} (I_{n1} + I_{p1}) + M_{af3} (I_{a1}^* + I_{a3}) \right\} \quad (5.9)$$

Similarly the 2mth (m is any positive integer except 0 and 1) harmonic can be represented in symbolic form as

$$I'_{f(2m)} = -\frac{3}{2} \frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{1}{L_{je}} \left\{ M_{af1} (I_{n(2m-1)} + I_{p(2m+1)}) + M_{af3} (I_{a(2m-1)} + I_{a(2m+1)}) \right\} \\ m = \text{positive integers except 0 and 1} \quad (5.10)$$

Comparing Eqs. 5.4 and 5.5 it is clear that the steady-state direct-axis-damper current is given by

$$I_{kd} = \frac{L_f - M_{fkd}}{L_{kd} - M_{fkd}} I'_f \quad (5.11)$$

Similarly, substituting Equ. 5.7 in Equ. 5.6 and neglecting the constant terms gives the quadrature-axis-damper current:

$$i_{kq} = \frac{M_{aq1}}{L_{kq}} \sqrt{2} \sum_{m=0}^{\infty} \left\{ \frac{1}{2} I_{a(2m+1)} \sin(2 \overline{m+1} \omega t + \varphi_{a(2m+1)}) + \frac{1}{2} I_{b(2m+1)} \sin(2 \overline{m+1} \omega t + \varphi_{b(2m+1)} - 2\pi/3) \right. \\ \left. + \frac{1}{2} I_{c(2m+1)} \sin(2 \overline{m+1} \omega t + \varphi_{c(2m+1)} + 2\pi/3) \right. \\ \left. - \frac{1}{2} I_{a(2m+3)} \sin(2 \overline{m+1} \omega t + \varphi_{a(2m+3)}) - \frac{1}{2} I_{b(2m+3)} \sin(2 \overline{m+1} \omega t + \varphi_{b(2m+3)} + 2\pi/3) \right. \\ \left. - \frac{1}{2} I_{c(2m+3)} \sin(2 \overline{m+1} \omega t + \varphi_{c(2m+3)} - 2\pi/3) \right\} \\ - \frac{M_{aq3}}{L_{kq}} \sqrt{2} \left\{ \frac{1}{2} I_{a1} \sin(2\omega t - \varphi_{a1}) + \frac{1}{2} I_{b1} \sin(2\omega t - \varphi_{b1}) + \frac{1}{2} I_{c1} \sin(2\omega t - \varphi_{c1}) \right. \\ \left. - \frac{1}{2} I_{a3} \sin(2\omega t + \varphi_{a3}) - \frac{1}{2} I_{b3} \sin(2\omega t + \varphi_{b3}) - \frac{1}{2} I_{c3} \sin(2\omega t + \varphi_{c3}) \right. \\ \left. + \sum_{m=0}^{\infty} \left[\frac{1}{2} I_{a(2m+1)} \sin(2 \overline{m+2} \omega t + \varphi_{a(2m+1)}) + \frac{1}{2} I_{b(2m+1)} \sin(2 \overline{m+2} \omega t + \varphi_{b(2m+1)}) \right. \right. \\ \left. \left. + \frac{1}{2} I_{c(2m+1)} \sin(2 \overline{m+2} \omega t + \varphi_{c(2m+1)}) \right. \right. \\ \left. \left. - \frac{1}{2} I_{a(2m+3)} \sin(2 \overline{m+2} \omega t + \varphi_{a(2m+3)}) - \frac{1}{2} I_{b(2m+3)} \sin(2 \overline{m+2} \omega t + \varphi_{b(2m+3)}) \right. \right. \\ \left. \left. - \frac{1}{2} I_{c(2m+3)} \sin(2 \overline{m+2} \omega t + \varphi_{c(2m+3)}) \right] \right\} \quad (5.12)$$

In symbolic form, the second-harmonic component of the steady-state quadrature-axis-damper current is given by

$$I_{kq2} = \frac{1}{2} \frac{M_{aq1}}{L_{kq}} \left\{ -j(I_{a1} + h^2 I_{b1} + h I_{c1}) + j(I_{a3} + h I_{b3} + h^2 I_{c3}) \right\} \\ - \frac{1}{2} \frac{M_{aq3}}{L_{kq}} \left\{ -j(I_{a1}^* + I_{b1}^* + I_{c1}^*) + j(I_{a3} + I_{b3} + I_{c3}) \right\} \\ - j \frac{3}{2} \frac{M_{aq1}}{L_{kq}} (I_{n1} - I_{p1}) + j \frac{3}{2} \frac{M_{aq3}}{L_{kq}} (I_{a1}^* - I_{a3}) \quad (5.13)$$

and the 2mth harmonic is given by

$$I_{kq(2m)} = -j \frac{3}{2} \cdot \frac{M_{akq1}}{L_{ky}} (I_{n(2m-1)} - I_{p(2m+1)}) + j \frac{3}{2} \cdot \frac{M_{akq3}}{L_{ky}} (I_{o(2m-3)} - I_{o(2m+3)})$$

$m = \text{positive integers except } 0 \text{ and } 1$ (5.14)

When the space-harmonic effect is neglected, $M_{af3} = M_{akq3} = 0$ and the 2mth (m is any non-zero positive integer) harmonic components of the field and damper currents are reduced to

$$\left. \begin{aligned} I'_{f(2m)} &= -\frac{3}{2} \frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{M_{af}}{L_{fe}} (I_{n(2m-1)} + I_{p(2m+1)}) \\ I_{kd(2m)} &= -\frac{3}{2} \frac{L_f - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{M_{af}}{L_{fe}} (I_{n(2m-1)} + I_{p(2m+1)}) \end{aligned} \right\} (5.15)$$

and $I_{kq(2m)} = -j \frac{3}{2} \cdot \frac{M_{aq}}{L_{ky}} (I_{n(2m-1)} - I_{p(2m+1)})$

(5.2) Electromagnetic Torque

The electromagnetic torque developed by a synchronous machine can be obtained by the application of the law of conservation of energy as shown in the following.

The voltage equations of the machine can be expressed in terms of the terminal voltages and currents as

$$[v_t] = [R_m] \cdot [i_t] + \frac{d}{dt} \{ [L_m] \cdot [i_t] \} \quad (5.15.a)$$

where $[v_t]$ contains the terminal voltages v_{at} , v_{bt} , v_{ct} and e_f , $[i_t]$ contains the terminal currents i_a , i_b , i_c , i_f , i_{kd} and i_{kq} , and $[R_m]$ and $[L_m]$ are given by Eqs. 2.4 and 2.5 with the external resistances and inductances omitted.

Pre-multiplied through by the transpose of $[i_t]$ and also multiplied through by the scalar dt , Equ. 5.15.a becomes the energy equation

$$\begin{aligned} [i_t]_t \cdot [v_t] dt &= [i_t]_t \cdot [R_m] \cdot [i_t] dt + [i_t]_t \cdot \frac{d}{dt} \{ [L_m] \cdot [i_t] \} dt \\ &= [i_t]_t \cdot [R_m] \cdot [i_t] dt + \frac{1}{2} \cdot \frac{d}{dt} \{ [i_t]_t \cdot [L_m] \cdot [i_t] \} dt \\ &\quad + \frac{1}{2} [i_t]_t \cdot \frac{d}{dt} \{ [L_m] \cdot [i_t] \} dt - \frac{1}{2} \frac{d[i_t]_t}{dt} \cdot [L_m] \cdot [i_t] dt \end{aligned} \quad (5.16)$$

In Equ. 5.16, the term on the left-hand side is the electrical energy input to the machine, the first term on the right-hand side is the energy dissipated as heat in the machine, the second term is the energy stored in the magnetic field in the machine, so that the last two terms must be the mechanical energy output converted by the machine. Consequently the mechanical output of the machine is given by

$$P_m = \frac{1}{2} [i]_t \cdot \frac{d}{dt} \{ [L_m] [i] \} - \frac{1}{2} \frac{d[i]_t}{dt} \cdot [L_m] \cdot [i] \quad (5.17)$$

Since L_m is symmetrical, $[i]_t \cdot [L_m] \cdot \frac{d[i]}{dt} = \frac{d[i]_t}{dt} \cdot [L_m] \cdot [i]$, and Equ. 5.17 reduces to

$$P_m = \frac{1}{2} [i]_t \cdot \frac{d[L_m]}{dt} \cdot [i] \quad (5.18)$$

The mechanical power output is also equal to $\tau d\theta/dt$, where τ is the electromagnetic torque of the machine. Thus

$$\tau = P_m \frac{dt}{d\theta} = \frac{1}{2} [i]_t \cdot \frac{d[L_m]}{d\theta} \cdot [i] \quad (5.19)$$

Comparing Eqs. 5.18 and 5.19 shows that at constant speed, the torque is numerically equal to the power when it is multiplied by ω , since $d\theta/dt = \omega$. When the power is expressed in watts, the torque, after being multiplied by ω , is in synchronous watts.

When the inductance matrix $[L_m]$ given by Equ. 2.5 (external inductances omitted) is substituted in Equ. 5.19, the explicit form of the electromagnetic torque can be found. The expression will, however, be very complicated. If the effects of the space harmonics are neglected by putting $M_{af3} = M_{akq3} = 0$ and $M_{s2} = -L_{a2}$, the expression of the electromagnetic torque can be simplified to

$$\tau = \frac{3}{2} (L_d - L_q) i_d i_q + \frac{3}{2} M_{af} (i_f + i_{kd}) i_q - \frac{3}{2} M_{aq} i_{kq} i_d \quad (5.20)$$

$$\text{or } \tau = \frac{3}{2} (L_d - L_q) i_d i_q + \frac{3}{2} M_{af} I_{f0} i_q + \frac{3}{2} M_{af} (i_f' + i_{kd}) i_q - \frac{3}{2} M_{aq} i_{kq} i_d \quad (5.21)$$

$$\text{where } i_d = \frac{2}{3} \{ i_a \cos \omega t + i_b \cos(\omega t - 2\pi/3) + i_c \cos(\omega t + 2\pi/3) \} \quad (5.22)$$

$$\text{and } i_q = -\frac{2}{3} \{ i_a \sin \omega t + i_b \sin(\omega t - 2\pi/3) + i_c \sin(\omega t + 2\pi/3) \}$$

In the second expression (Equ. 5.21) for τ , it has been deliberately divided into several physically distinct components. The first term on the right-hand side is the reluctance torque due to the synchronous saliency, the second term is the synchronous torque due to the field excitation I_{f0} and the last two terms are damping torques like that developed in an induction machine.

Equ. 5.20 may be written more compactly as

$$\tau = \Psi_d i_q - \Psi_q i_d \quad (5.23)$$

$$\text{where } \Psi_d = \frac{3}{2} \{ L_d i_d + M_{af} (i_f + i_{kd}) \} \quad (5.24)$$

$$\text{and } \Psi_q = \frac{3}{2} (L_q i_q + M_{aq} i_{kq})$$

Since at steady state, each of the four quantities Ψ_d , Ψ_q , i_d and i_q in Equ. 5.23 can be represented by an even-harmonic series, the torque can also be represented by an even-harmonic series. The complete expression for the torque involves the evaluation of the product of two infinite harmonic series and is rather complicated. It is not difficult, however, to find an expression for the average torque, since it is contributed only by the equal-frequency components of Ψ_d and i_q and Ψ_q and i_d . Expressed in the unit of power, say the synchronous-watts, the average torque is given by

$$\tau_{ave} = \text{Real part of } \sum_{m=0}^{\infty} \{ \omega \Psi_{d(2m)}^* I_{q(2m)} - \omega \Psi_{q(2m)} I_{d(2m)}^* \} \quad (5.25)$$

Substituting Equ. 5.7 in Equ. 5.22, it can be shown that the d.c. component of i_d is

$$\begin{aligned} I_{do} &= \sqrt{2} \text{ Real part of } I_{p1} \\ &= (I_{p1} + I_{p1}^*)/\sqrt{2} \end{aligned} \quad (5.26)$$

and the 2mth (m is any non-zero positive integer) harmonic is

$$I_{d(2m)} = I_{n(2m-1)} + I_{p(2m+1)} \quad (5.27)$$

Also the d.c. component of i_q is

$$\begin{aligned} I_{qo} &= j\sqrt{2} \text{ Imaginary part of } I_{p1} \\ &= - (I_{p1} - I_{p1}^*)/\sqrt{2} \end{aligned} \quad (5.28)$$

and the 2mth harmonic is

$$I_{q(2m)} = j (I_{n(2m-1)} - I_{p(2m+1)}) \quad (5.29)$$

$$\left. \begin{aligned} \omega \Psi_{do} &= \frac{3}{\sqrt{2}} \left(x_d \frac{I_{p1} + I_{p1}^*}{2} + E_{ao} \right) \\ \omega \Psi_{qo} &= -j \frac{3}{\sqrt{2}} x_q \frac{I_{p1} - I_{p1}^*}{2} \\ \omega \Psi_{d(2m)} &= \frac{3}{2} x_d'' (I_{n(2m-1)} + I_{p(2m+1)}) \\ \omega \Psi_{q(2m)} &= j \frac{3}{2} x_q'' (I_{n(2m-1)} - I_{p(2m+1)}) \end{aligned} \right\} \quad (5.30)$$

and $\omega \Psi_{q(2m)} = j \frac{3}{2} x_q'' (I_{n(2m-1)} - I_{p(2m+1)})$

Consequently in Equ. 5.25,

$$\left. \begin{aligned} \text{Real part of } (\omega \Psi_{do}^* I_{qo} - \omega \Psi_{qo} I_{do}^*) &= 3 \text{ Real part of } \mathcal{E}_o^* I_{p1} \\ \text{and } \text{Real part of } \sum_{m=1}^{\infty} (\omega \Psi_{d(2m)}^* I_{q(2m)} - \omega \Psi_{q(2m)} I_{d(2m)}^*) &= 3 \text{ Real part of } \sum_{m=1}^{\infty} \mathcal{E}_{2m}^* (I_{p(2m+1)} - I_{n(2m-1)}) \end{aligned} \right\} \quad (5.31)$$

where $\mathcal{E}_o = -E_{ao} + j \frac{1}{2} (x_d - x_q) (I_{p1} + I_{p1}^*)$

and $\mathcal{E}_{2m} = j \frac{1}{2} (x_d'' - x_q'') (I_{n(2m-1)} + I_{p(2m+1)})$ (5.32)

and the average torque can be expressed in terms of the armature sequence currents as

$$T_{ave} = 3 \times \text{Real part of } \left\{ \mathcal{E}_o^* I_{p1} + \sum_{m=1}^{\infty} \mathcal{E}_{2m}^* (I_{p(2m+1)} - I_{n(2m-1)}) \right\}$$

Note that the quantity $\mathcal{E}_{2m}$ is the voltage developed in

Substituting Equ. 5.7 in Equ. 5.22, it can be shown that the d.c. component of i_d is:

$$\begin{aligned} I_{do} &= \sqrt{2} \text{ Real part of } I_{p1} \\ &= (I_{p1} + I_{p1}^*)/\sqrt{2} \end{aligned} \quad (5.26)$$

and the 2mth (m is any non-zero positive integer) harmonic is

$$I_{d(2m)} = I_{n(2m-1)} + I_{p(2m+1)} \quad (5.27)$$

Substituting Eqs. 5.15, 5.26, 5.27, 5.28 and 5.29 in Equ.

5.24 gives the constant and alternating components of Ψ_d and Ψ_q as

$$\left. \begin{aligned} \omega \Psi_{do} &= \frac{3}{\sqrt{2}} \left(x_d \frac{I_{p1} + I_{p1}^*}{2} + E_{ao} \right) \\ \omega \Psi_{qo} &= -j \frac{3}{\sqrt{2}} x_q \frac{I_{p1} - I_{p1}^*}{2} \\ \omega \Psi_{d(2m)} &= \frac{3}{2} x_d'' \left(I_{n(2m-1)} + I_{p(2m+1)} \right) \end{aligned} \right\} \quad (5.30)$$

$$\text{and } \omega \Psi_{q(2m)} = j \frac{3}{2} x_q'' \left(I_{n(2m-1)} - I_{p(2m+1)} \right)$$

Consequently in Equ. 5.25,

$$\left. \begin{aligned} \text{Real part of } (\omega \Psi_{do}^* I_{qo} - \omega \Psi_{qo} I_{do}^*) &= 3 \text{ Real part of } \mathcal{E}_o^* I_{p1} \\ \text{and } \text{Real part of } \sum_{m=1}^{\infty} (\omega \Psi_{d(2m)}^* I_{q(2m)} - \omega \Psi_{q(2m)} I_{d(2m)}^*) &= 3 \text{ Real part of } \sum_{m=1}^{\infty} \mathcal{E}_{2m}^* (I_{p(2m+1)} - I_{n(2m-1)}) \end{aligned} \right\} \quad (5.31)$$

$$\text{where } \mathcal{E}_o = -E_{ao} + j \frac{1}{2} (x_d - x_q) (I_{p1} + I_{p1}^*)$$

$$\text{and } \mathcal{E}_{2m} = j \frac{1}{2} (x_d'' - x_q'') (I_{n(2m-1)} + I_{p(2m+1)}) \quad (5.32)$$

and the average torque can be expressed in terms of the armature sequence currents as

$$T_{ave} = 3 \times \text{Real part of } \left\{ \mathcal{E}_o^* I_{p1} + \sum_{m=1}^{\infty} \mathcal{E}_{2m}^* (I_{p(2m+1)} - I_{n(2m-1)}) \right\}$$

Note that the quantity $\mathcal{E}_{2m}$ is the voltage developed in

the mutual reactance $\frac{1}{2}(X_d'' - X_q'')$ between the negative-sequence $(2m-1)$ th-harmonic current $I_n(2m-1)$ and the positive-sequence $(2m+1)$ th-harmonic current $I_p(2m+1)$. Physically it represents a field rotating at $2m$ times the speed of the fundamental rotating field with respect to the field structure. This rotating field reacts with the current $I_p(2m+1)$ to produce a positive average torque and with the current $I_n(2m-1)$ to produce a negative average torque.

The quantity $\mathcal{E}_0$ is corresponding to the forward rotating field (normal speed with respect to the armature) which reacts only with the positive-sequence, fundamental-frequency component of current to produce the main average torque of the machine.

The average torque produced by the components of the armature current other than the positive-sequence, fundamental-frequency component depends on the subtransient saliency of the machine as it is proportional to the quantity $X_d'' - X_q''$. This is in close resemblance to the reluctance torque, produced by the positive-sequence, fundamental frequency component of current, which depends on the synchronous saliency of the machine.

(6) STEADY-STATE EQUIVALENT CIRCUITS AND THEIR APPLICATIONS

(6.1) Types of Equivalent Circuit

The steady-state solution (with the space-harmonic effects neglected) given in Chapter 4 can be represented by the equivalent circuits to be described below. These equivalent circuits are derived directly from the simplified equations so that they are in fact another form of statement of the simplified solution. Nevertheless, the equivalent circuits can sometimes give a clearer picture of the physical phenomena than the equations themselves. Moreover, they can be set up in a network analyser to avoid the labour of solving the equations.

Three different types of equivalent circuit relating the armature voltage and current components of different sequences and at different frequencies are necessary. The first type of equivalent circuit gives the relation between the positive-sequence, fundamental-frequency components of the armature voltage and current. There is only one circuit belonging to this type. The second type of equivalent circuit gives the relation between the negative-sequence, $(2m-1)$ th-harmonic (m is any non-zero positive integer) components and the positive-sequence, $(2m+1)$ th-harmonic component of the armature voltage and current. There is an infinite number of such circuits relating the voltage and current components at frequencies of two consecutive harmonic orders. The third type of equivalent circuit gives the relation between the zero-sequence

components of the armature voltage and current at any frequency. There is again an infinite number of such circuits but there is no mutual between two voltage or current components at different frequencies.

(a) Equivalent circuit of the first type

The positive-sequence, fundamental-frequency voltage equation is, from Equ. 4.1,

$$V_{p1} - jE_{ao} = (R_a + j \frac{x_d + x_q}{2}) I_{p1} + j \frac{x_d - x_q}{2} I_{p1}^* \quad (6.1)$$

Its conjugate equation is

$$V_{p1}^* + jE_{ao} = (R_a - j \frac{x_d + x_q}{2}) I_{p1}^* - j \frac{x_d - x_q}{2} I_{p1}$$

which may be rewritten as

$$-V_{p1}^* - jE_{ao} = j \frac{x_d - x_q}{2} I_{p1} + (-R_a + j \frac{x_d + x_q}{2}) I_{p1}^* \quad (6.2)$$

Eqs. 6.1 and 6.2 are, of course, consistent and not independent, but they may be considered as the voltage equations of a two-mesh circuit with I_{p1} and I_{p1}^* as the mesh currents.

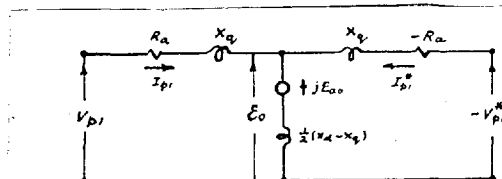


Fig. 6.1.

Evidently the circuit shown in Fig. 6.1 is equivalent to these two equations.

Fig. 6.1 is the equivalent circuit for the positive-sequence, fundamental-frequency components of the armature voltage and current. When a machine is operating under balanced condition, it is the steady-state equivalent circuit of the machine as voltage and current components of other sequences and at other

frequencies do not exist. For a machine where there is no synchronous saliency, $X_d = X_q$ and the right-hand mesh of Fig. 6.1 can be omitted.

The average torque contributed by the positive-sequence, fundamental-frequency current is obtainable from the equivalent circuit by taking the average power involved in the current I_{p1} and the voltage $\mathcal{E}_0$.

The chief disadvantage of the circuit shown in Fig. 6.1 is the presence of the negative resistance $-R_a$, a quantity not all types of network analyser can represent. An unsuccessful attempt has been made to find another equivalent circuit with the negative resistance but still suitable for use in conjunction with the second and third types of equivalent circuits to be described in the subsequent sections.

(b) Equivalent circuit of the second type

Putting $m-1$ for m , and dividing through by $2m-1$, the second equation of Equ. 4.2 becomes

$$V_{n(2m-1)}/(2m-1) = \left[R_a/(2m-1) + j(x_d'' + x_q'')/2 \right] I_{n(2m-1)} + j \frac{x_d'' - x_q''}{2} I_{p(2m+1)} \quad (6.3)$$

Also, dividing through by $2m+1$, the first equation of Equ. 4.2 becomes

$$V_{p(2m+1)}/(2m+1) = j \frac{x_d'' - x_q''}{2} I_{n(2m-1)} + \left[R_a/(2m+1) + j(x_d'' + x_q'')/2 \right] I_{p(2m+1)} \quad (6.4)$$

Eqs. 6.3 and 6.4 can be represented by the equivalent circuit shown in Fig. 6.2, which is a familiar T-network. This circuit shows the mechanism by which the harmonic voltage and current components are produced when a machine is operating under unbalanced conditions. This process of producing higher-

frequency components may go on indefinitely so long as $I_{n(2m-1)}$ exists, except for a machine where there is no subtransient saliency when $X_d'' = X_q''$.

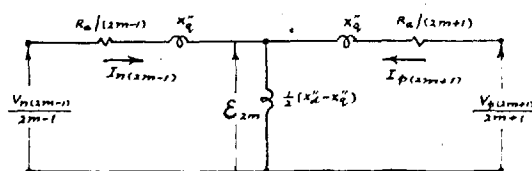


Fig. 6.2

In the actual application of this equivalent circuit, a number of circuits of this type are connected together according to the terminal conditions imposed by the particular problem, and in general an infinite ladder network will result. This infinite ladder network will not be equal to an ordinary infinite filter network, because not all the circuit elements in a particular mesh are identical with those in another mesh. However, in some problems, such as the unsymmetrical short circuits of a machine directly at its terminals or through inductances, R_o may be neglected so that the infinite ladder network may be reduced to an ordinary infinite filter network and the theory of the latter may be applied directly.

Finally the average torque contributed by the harmonic currents can be obtained from the equivalent circuit by taking the difference of the power involved in the current $I_{p(2m+1)}$ and the voltage E_{2m} and the power involved in the current $I_{n(2m-1)}$ and the voltage E_{2m} .

(c) Equivalent circuit of the third type

Dividing through by $2m+1$, the third equation of Equ. 4.2 becomes

$$V_o(2m+1)/(2m+1) = [R_o/(2m+1) + jX_o] I_o(2m+1) \quad (6.5)$$

which can be represented by the simple series circuit shown in Fig. 6.3. This type of

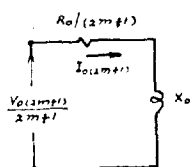


Fig. 6.3

equivalent circuit is the simplest of the three, which can be expected as the zero-sequence current does not

produce any flux crossing the air gap, thus it has no coupling with the circuits on the other side of the air gap. There is no torque involved in this circuit.

(d) Effects of iron losses and the field- and damper-circuit copper losses.

So far the iron losses of the machine have not been considered and the copper losses in the field and damper circuits have been neglected, owing to the difficulty of including them in a simple manner in the equations. However, these losses can be represented readily by putting suitable series or shunt resistance in the equivalent circuits.

The eddy-current loss in the armature iron can be included by suitably increasing the value of R_a and the eddy-current loss in the field iron can be included by using suitable values of R_{kd} and R_{kq} . The hysteresis loss, however, cannot be represented in this manner, as it is proportional to the frequency of the current. It can be represented by adding a shunt resistance R_h in the second type of equivalent circuit in parallel with the element $\frac{1}{2}(X_d'' - X_q'')$. The resistance R_h , unlike others, is not divided by the factor $2m+1$.

The copper losses in the field and damper circuits can be

represented approximately by including a resistance R_d defined by

$$R_d = \frac{M_d^2}{L_{fd}^2} \left\{ \left(\frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \right)^2 R_f + \left(\frac{L_f - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \right)^2 R_{kd} \right\} \quad (6.6)$$

in series with X_d'' and a resistance R_q defined by

$$R_q = \frac{M_q^2}{L_{kq}^2} R_{kq} \quad (6.7)$$

in series with X_q'' . In the equivalent circuit, a resistance of $(R_d + R_q)/4m$ may be put in series with the mutual branch $\frac{1}{2}(X_d'' - X_q'')$.

With these losses included, the second type of equivalent circuit is modified to the circuit shown in Fig. 6.4. It is

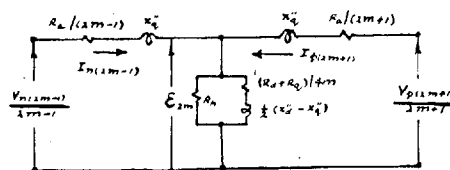


Fig. 6.4.

clear that, if the harmonic currents are neglected, the equivalent negative-sequence resistance is given by

$$R_2 = R_a + (R_d + R_q)/4 \quad (6.8)$$

(e) Change of reference axis

The reference axis for the steady-state solution and consequently for the equivalent circuits given in the previous sections has been the field excitation I_{f0} . In other words, I_{f0} is a pure real number and acts to determine the real axis to whose reference all the voltages and currents, which are generally complex numbers, are expressed. The choice of reference axis is, of course, purely optional, and that the field excitation has been chosen is because it can give simpler expressions for the solution.

Let the new reference ^{axis} lag the old one (field excitation)

by an angle δ . Multiplied through by $e^{j\delta}$, Equ. 6.1 becomes

$$V_{p1} e^{j\delta} - jE_{a0} e^{j\delta} = (R_a + j \frac{x_d + x_q}{2}) I_{p1} e^{j\delta} + j \frac{x_d - x_q}{2} I_{p1}^* e^{j\delta}$$

or $V_{p1}' - jE_{a0} e^{j\delta} = (R_a + j \frac{x_d + x_q}{2}) I_{p1}' + j \frac{x_d - x_q}{2} I_{p1}'^* e^{j2\delta}$ (6.9)

where $V_{p1}' = V_{p1} e^{j\delta}$ and $I_{p1}' = I_{p1} e^{j\delta}$ are the terminal voltage and current components expressed along the new reference axis. Similarly Equ. 6.2 can be reduced to

$$-V_{p1}'^* e^{j2\delta} - jE_{a0} e^{j\delta} = j \frac{x_d - x_q}{2} I_{p1}' + (-R_a + j \frac{x_d + x_q}{2}) I_{p1}'^* e^{j2\delta}$$
 (6.10)

If it is remembered that the voltages and currents are now expressed along the new reference axis, the dashes may be dropped for simplicity. Thus, when the reference axis is lagging the field axis by the angle δ , the first type of equivalent circuit becomes that shown in Fig. 6.5.

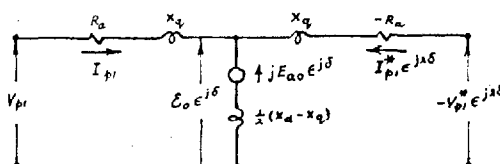


Fig. 6.5

By similar reasoning, it can be shown that the second and the third types of equivalent circuits can be reduced to Figs.

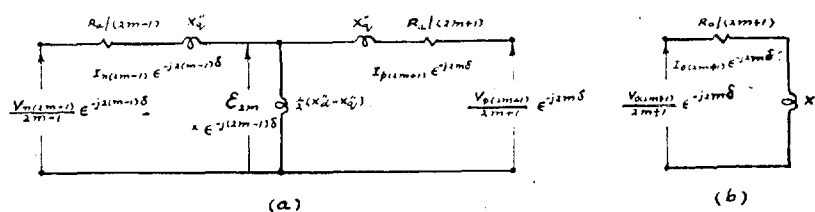


Fig. 6.6

6.6a and 6.6.b when the currents and voltages are referred to the new axis.

(6.2) Unsymmetrical Short Circuits

In order to show how the equivalent circuits are connected together, the three typical unsymmetrical short circuits (line-to-line, line-to-neutral and double-line-to-neutral), which are the simpler cases of the unsymmetrical operation of a synchronous machine, will be considered. The approximate expressions for the steady-state voltages and currents neglecting resistances will be derived.

(a) Line-to-line short circuit

If the terminals b and c are short-circuited, the terminal conditions are, for the fundamental as well as the harmonic components, as follows.

$$\begin{aligned} I_a &= 0 \\ I_b &= -I_c \end{aligned} \quad (6.11)$$

and $V_b = V_c$

In terms of symmetrical components, the terminal conditions become

$$\begin{aligned} I_o &= 0 \\ I_p - I_n &= 0 \end{aligned} \quad (6.12)$$

and $V_p - V_n = 0$

The pattern of connection, which shows the manner in which

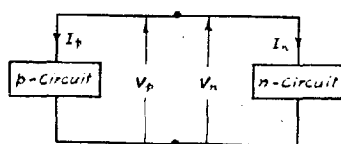


Fig. 6.7.

the circuits are connected together, can be formed in accordance with Equ. 6.12 as shown in Fig. 6.7. Using this pattern of connection the harmonic equivalent

lent circuits can be connected together as shown in Fig. 6.8. The negative-sequence impedance, Z_2 , which is equal to the

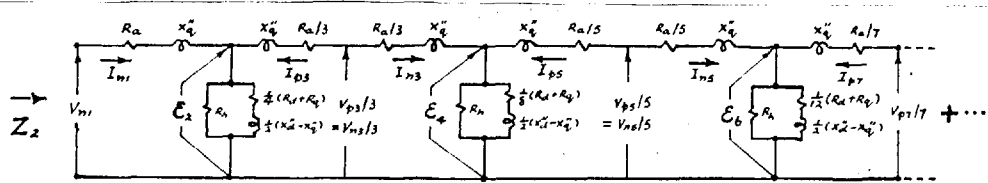


Fig. 6.8.

ratio V_{n1}/I_{n1} , is given by the equivalent impedance viewed from the V_{n1} end of the circuit shown in Fig. 6.8.

The complete equivalent circuit is shown in Fig. 6.9. The impedance $-Z_2^*$ is represented by the same form of equivalent circuit as shown in Fig. 6.8 except that all the resistances

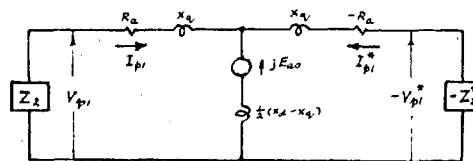


Fig. 6.9.

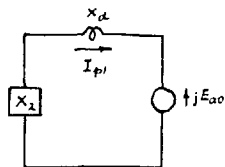


Fig. 6.10.

become negative. If the resistances are neglected, $-Z_2^*$ will be equal to Z_2 as they will be pure imaginaries. In addition, with the resistances neglected, I_{p1} will be a pure real number so that $I_{p1} = I_{p1}^*$, and the complete equivalent circuit can be reduced to Fig. 6.10.

The final equivalent circuit shown in Fig. 6.10 shows the validity of the ordinary way of applying the method of symmetrical components to the problem by considering the fundamental-frequency quantities alone. The accuracy of the method in determining the fundamental-frequency components rests entirely on the choice of the value of X_2 .

To determine the proper expression for X_2 from Fig. 6.8 (the resistances neglected), consider the general infinite filter

network shown in Fig. 6.11. It has been shown elsewhere that if the current in the infinite mesh tends to zero, the ratio

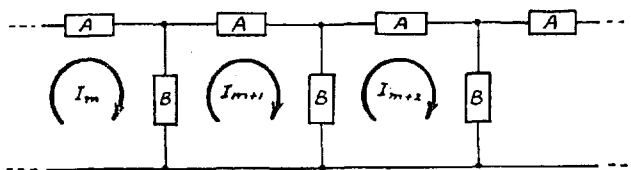


Fig. 6.11.

of the currents in the $(m+1)$ th and the m th meshes is given by

$$\left. \begin{aligned} I_{m+1}/I_m = b \end{aligned} \right\} \quad (6.13)$$

where $b = \frac{1}{2B} \left\{ A + 2B - \sqrt{(A+2B)^2 - 4B^2} \right\}$

Putting $A = j2X_q''$ and $B = j\frac{1}{2}(X_d'' - X_q'')$ in Equ. 6.13, the geometric ratio b for the line-to-line short circuit is

$$\begin{aligned} b &= \frac{1}{j(X_d'' - X_q'')} \left\{ j(X_d'' + X_q'') - \sqrt{-(X_d'' + X_q'')^2 + (X_d'' - X_q'')^2} \right\} \\ &= \frac{\sqrt{X_d''} - \sqrt{X_q''}}{\sqrt{X_d''} + \sqrt{X_q''}} = -b_1 \end{aligned} \quad (6.14)$$

The $(2m+1)$ th harmonic of current is thus given by

$$I_{n(2m+1)} = (-b_1)^m I_{n1} \quad (6.15)$$

The voltage equation of the first mesh in Fig. 6.8 (the resistances neglected) is

$$\begin{aligned} V_{n1} &= j\frac{1}{2}(X_d'' + X_q'')I_{n1} + j\frac{1}{2}(X_d'' - X_q'')I_{p1} \\ &= j\frac{1}{2} \left\{ (X_d'' + X_q'') - (X_d'' - X_q'') \frac{\sqrt{X_d''} - \sqrt{X_q''}}{\sqrt{X_d''} + \sqrt{X_q''}} \right\} I_{n1} \end{aligned}$$

so that the negative-sequence reactance is given by

$$X_2 = V_{n1}/jI_{n1} = \sqrt{X_d'' X_q''} \quad (6.16)$$

The positive-sequence, fundamental-frequency component of current is given by, from Fig. 6.10,

$$I_{p1} = -E_{ao}/(X_d + \sqrt{X_d'' X_q''}) \quad (6.17)$$

The fundamental component of the current in phase b is

$$\begin{aligned} I_{b1} &= h^2 I_{p1} + h I_{n1} + I_{o1} \\ &= j\sqrt{3}E_{ao}/(X_d + \sqrt{X_d'' X_q''}) \end{aligned} \quad (6.18)$$

and the $(2m+1)$ th-harmonic component is

$$I_{b(2m+1)} = (-b_1)^m I_{b1} \quad (6.19)$$

The Fourier expression for the steady-state short-circuit current is therefore

$$\begin{aligned} i_b &= \frac{\sqrt{3} E_{am}}{x_a + \sqrt{x_d'' x_q''}} \left\{ \cos(\omega t + \frac{\pi}{2}) - b_1 \cos(3\omega t + \frac{\pi}{2}) + b_1^2 \cos(5\omega t + \frac{\pi}{2}) - \dots \right\} \\ &= -\frac{\sqrt{3} E_{am}}{x_a + \sqrt{x_d'' x_q''}} \sum_{m=0}^{\infty} (-b_1)^m \sin(2m+1)\omega t \end{aligned} \quad (6.20)$$

where $E_{am} = \sqrt{2} E_{a0}$

The sequence voltages are, from Fig. 6.8,

$$\begin{aligned} \frac{V_{p(2m+1)}}{2m+1} &= \frac{V_{n(2m+1)}}{2m+1} \\ &= j \frac{1}{2} (x_d'' + x_q'') I_{n(2m+1)} + j \frac{1}{2} (x_d'' - x_q'') I_{p(2m+1)} \\ &= -j \sqrt{x_d'' x_q''} I_{p(2m+1)} \end{aligned}$$

so that the $(2m+1)$ th-harmonic component of the terminal voltage of phase a, the open phase, is

$$\begin{aligned} V_{a(2m+1)} &= V_{p(2m+1)} + V_{n(2m+1)} \\ &= \frac{j(-b_1)^m (2m+1) \cdot 2 \sqrt{x_d'' x_q''}}{x_a + \sqrt{x_d'' x_q''}} E_{a0} \end{aligned} \quad (6.21)$$

The Fourier expression for the open-phase voltage is therefore given by

$$v_a = -\frac{2 \sqrt{x_d'' x_q''} E_{a0}}{x_a + \sqrt{x_d'' x_q''}} \sum_{m=0}^{\infty} (-b_1)^m (2m+1) \sin(2m+1)\omega t \quad (6.22)$$

The short-circuit current and the open-phase voltage given by Eqs. 6.20 and 6.22 will also be given later as components of the complete solution after a sudden short circuit when the transients are also considered.

(b) Line-to-neutral short circuit

If phase a is short-circuited, the terminal conditions are

$$I_b = I_c = 0 \quad (6.23)$$

and $V_a = 0$

In terms of symmetrical components, the terminal conditions

become

$$I_p = I_n = I_o \quad (6.24)$$

and $V_p + V_n + V_o = 0$

The pattern of connected satisfying Equ. 6.24 is shown in

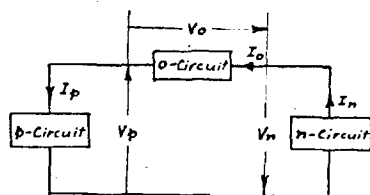


Fig. 6.12

Fig. 6.12. According to this pattern of connection the harmonic equivalent circuits, with the resistances neglected, can be connected together as shown

in Fig. 6.13. The geometric ratio can be found by putting

$A = j(2X_d'' + X_o)$ and $B = j\frac{1}{2}(X_d'' - X_q'')$ in Equ. 6.13. Its expression is

$$b = \frac{\sqrt{X_d'' + \frac{1}{2}X_o} - \sqrt{X_q'' + \frac{1}{2}X_o}}{\sqrt{X_d'' + \frac{1}{2}X_o} + \sqrt{X_q'' + \frac{1}{2}X_o}} = -b_o \quad (6.25)$$

The $(2m+1)$ th-harmonic component is given by

$$I_{p(2m+1)} = b_o^m I_{p1} \quad (6.26)$$

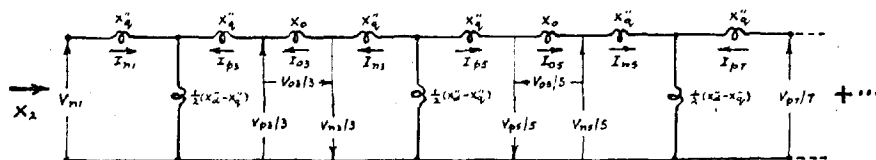


Fig. 6.13

The voltage equation of the first mesh in Fig. 6.13 is

$$\begin{aligned} V_{n1} &= j\frac{1}{2}(X_d'' + X_q'') I_{n1} + j\frac{1}{2}(X_d'' - X_q'') I_{p3} \\ &= j\frac{1}{2} \left\{ (X_d'' + X_q'') + (X_d'' - X_q'') \frac{\sqrt{X_d'' + \frac{1}{2}X_o} - \sqrt{X_q'' + \frac{1}{2}X_o}}{\sqrt{X_d'' + \frac{1}{2}X_o} + \sqrt{X_q'' + \frac{1}{2}X_o}} \right\} I_{n1} \end{aligned}$$

so that the negative-sequence reactance is

$$\begin{aligned} X_2 &= V_{n1} / j I_{n1} \\ &= \sqrt{(X_d'' + \frac{1}{2}X_o)(X_q'' + \frac{1}{2}X_o)} - \frac{1}{2}X_o \end{aligned} \quad (6.27)$$

The complete equivalent circuit is shown in Fig. 6.14. The fundamental component of current is

$$I_{p1} = - \frac{E_{ao}}{x_d + \frac{1}{2}x_o + \sqrt{(x_d'' + \frac{1}{2}x_o)(x_q'' + \frac{1}{2}x_o)}} \quad (6.28)$$

and the $(2m+1)$ th-harmonic component is

$$I_{p(2m+1)} = - \frac{b_o^m E_{ao}}{x_d + \frac{1}{2}x_o + \sqrt{(x_d'' + \frac{1}{2}x_o)(x_q'' + \frac{1}{2}x_o)}} \quad (6.29)$$

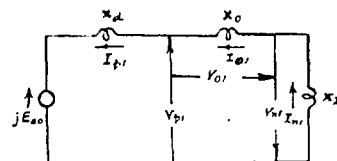


Fig. 6-14.

Remembering that the phase current is equal to three times the sequence current, the short-circuit current is found to be

$$\begin{aligned} i_a &= - \frac{3E_{ao}}{x_d + \frac{1}{2}x_o + \sqrt{(x_d'' + \frac{1}{2}x_o)(x_q'' + \frac{1}{2}x_o)}} \left\{ \cos \omega t + b_o \cos 3\omega t + b_o^4 \cos 5\omega t + \dots \right\} \\ &= - \frac{3E_{ao}}{x_d + \frac{1}{2}x_o + \sqrt{(x_d'' + \frac{1}{2}x_o)(x_q'' + \frac{1}{2}x_o)}} \sum_{m=0}^{\infty} b_o^m \cos (2m+1)\omega t \end{aligned} \quad (6.30)$$

The positive-sequence, $(2m+1)$ th-harmonic terminal voltage is given by

$$\begin{aligned} \frac{V_{p(2m+1)}}{2m+1} &= -j \left\{ \frac{1}{2} (x_d'' + x_q'') + x_o \right\} I_{p(2m+1)} - j \frac{1}{2} (x_d'' - x_q'') I_{p(2m+3)} \\ &= -j \left\{ \frac{1}{2} (x_d'' + x_q'') + x_o + \frac{1}{2} (x_d'' - x_q'') \frac{\sqrt{x_q'' + \frac{1}{2}x_o} - \sqrt{x_d'' + \frac{1}{2}x_o}}{\sqrt{x_d'' + \frac{1}{2}x_o} + \sqrt{x_q'' + \frac{1}{2}x_o}} \right\} I_{p(2m+1)} \\ &= -j \left\{ \sqrt{(x_d'' + \frac{1}{2}x_o)(x_q'' + \frac{1}{2}x_o)} + \frac{1}{2}x_o \right\} I_{p(2m+1)} \end{aligned} \quad (6.31)$$

Similarly the negative-sequence, $(2m+1)$ th-harmonic terminal voltage is given by

$$\begin{aligned} \frac{V_{n(2m+1)}}{2m+1} &= j \frac{1}{2} (x_d'' + x_q'') I_{p(2m+1)} + j \frac{1}{2} (x_d'' - x_q'') I_{p(2m+3)} \\ &= j \left\{ \sqrt{(x_d'' + \frac{1}{2}x_o)(x_q'' + \frac{1}{2}x_o)} - \frac{1}{2}x_o \right\} I_{p(2m+1)} \end{aligned} \quad (6.32)$$

The zero-sequence, $(2m+1)$ th-harmonic terminal voltage is given by

$$\frac{V_{o(2m+1)}}{2m+1} = j x_o I_{p(2m+1)} \quad (6.33)$$

The $(2m+1)$ th-harmonic components of the terminal voltages of phase b and c, the open phases, are therefore given by

$$\begin{aligned} V_{b(2m+1)} &= h^2 V_{p(2m+1)} + h V_{n(2m+1)} + V_{o(2m+1)} \\ &= (2m+1) \left\{ -\sqrt{3} \sqrt{(x_d'' + \frac{1}{2}x_o)(x_q'' + \frac{1}{2}x_o)} + j \frac{3}{2}x_o \right\} I_{p(2m+1)} \end{aligned} \quad (6.34)$$

and $V_{c(2m+1)} = h V_{p(2m+1)} + h^2 V_{n(2m+1)} + V_{o(2m+1)}$

$$= (2m+1) \left\{ \sqrt{3} \sqrt{(x_d'' + \frac{1}{2}x_o)(x_q'' + \frac{1}{2}x_o)} + j \frac{3}{2}x_o \right\} I_{p(2m+1)} \quad (6.35)$$

The Fourier expressions for the terminal voltage of phases b and c are

$$V_b = \frac{E_{am}}{x_a + \frac{1}{2}x_o + \sqrt{(x_a^2 + \frac{1}{4}x_o)(x_q^2 + \frac{1}{4}x_o)}} \left\{ \sqrt{3} \sqrt{(x_a^2 + \frac{1}{4}x_o)(x_q^2 + \frac{1}{4}x_o)} \sum_{m=0}^{\infty} b_o^m(2m+1) \cos(2m+1)\omega t \right. \\ \left. + \frac{1}{2}x_o \sum_{m=0}^{\infty} b_o^m(2m+1) \sin(2m+1)\omega t \right\} \quad (6.36)$$

$$\text{and } V_c = \frac{E_{am}}{x_a + \frac{1}{2}x_o + \sqrt{(x_a^2 + \frac{1}{4}x_o)(x_q^2 + \frac{1}{4}x_o)}} \left\{ -\sqrt{3} \sqrt{(x_a^2 + \frac{1}{4}x_o)(x_q^2 + \frac{1}{4}x_o)} \sum_{m=0}^{\infty} b_o^m(2m+1) \cos(2m+1)\omega t \right. \\ \left. + \frac{1}{2}x_o \sum_{m=0}^{\infty} b_o^m(2m+1) \sin(2m+1)\omega t \right\} \quad (6.37)$$

(c) Double-line-to-neutral short circuit

If both the phases b and c are short-circuited to the neutral, the terminal conditions are

$$I_a = 0 \quad (6.38)$$

$$\text{and } V_b = V_c = 0$$

which become, when expressed in terms of symmetrical components,

$$I_p + I_n + I_o = 0 \quad (6.39)$$

$$\text{and } V_p = V_n = V_o$$

The pattern of connection satisfying Equ. 6.39 is shown in Fig. 6.15. It is plain that when the O-branch is deleted, Fig. 6.15 reduces to Fig. 6.7, so that the double-line-to-neutral case may be reduced to the line-to-line case by putting $X_o = \infty$.

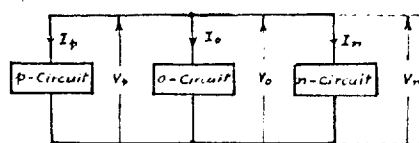


Fig. 6.15

Also when the O-branch is short-circuited, all the current components will be equal to zero except the positive-sequence, fundamental-frequency component, as

there is no internal voltage in any component circuit except the positive-sequence, fundamental-frequency circuit. Therefore the double-line-to-neutral short circuit may be reduced to the symmetrical three-phase short circuit by putting $R_o = X_o = 0$.

According to the pattern of connection given by Fig. 6.15,

the harmonic equivalent circuits can be connected together as shown in Fig. 6.16.a to form the negative-sequence reactance X_2 (resistances neglected). Since there are two different kinds of meshes in Fig. 6.16.a, it is necessary to reduce them to a

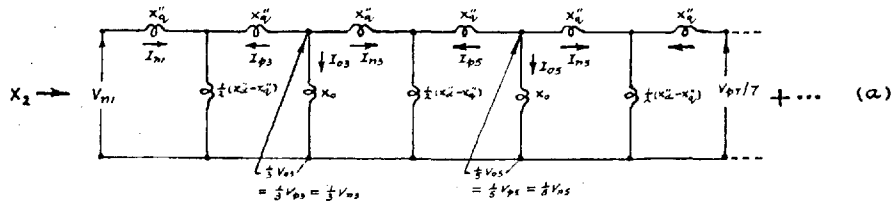
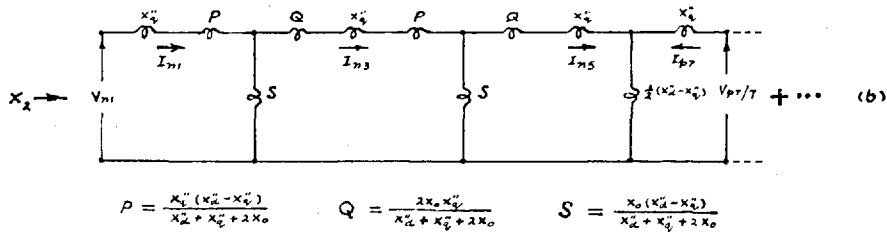


Fig. 6.16.



single kind before the expression for X_2 can be derived by the method used in the two previous sections. This can be achieved by the delta-star transformation and the result is shown in Fig. 6.16.b.

In order to derive the expression for X_2 , the geometric ratio b is first determined by putting $A = j(P + Q + X''_q)$ and $B = jS$ in Equ. 6.13. Thus,

$$b = \frac{\sqrt{x''_d(x''_q + 2x_0)} - \sqrt{x''_q(x''_d + 2x_0)}}{\sqrt{x''_d(x''_q + 2x_0)} + \sqrt{x''_q(x''_d + 2x_0)}} = b_2 \quad (6.40)$$

The negative-sequence, $(2m+1)$ th-harmonic component is given by

$$I_{n(2m-1)} = b_2^m I_{n1} \quad (6.41)$$

The voltage equation of the first mesh in Fig. 6.16.b is

$$\begin{aligned} V_{n1} &= j(P + S + X''_q)I_{n1} - jSI_{n3} \\ &= j(P + S + X''_q - b_2S)I_{n1} \end{aligned}$$

so that the equivalent negative-sequence reactance is

$$\begin{aligned} x_2 &= P + S + x_q'' - b_2 S \\ &= \frac{x_d'' x_q'' + \sqrt{x_d'' x_q'' (x_d'' + 2x_o)(x_q'' + 2x_o)}}{x_d'' + x_q'' + 2x_o} \end{aligned} \quad (6.42)$$

The complete equivalent circuit is shown in Fig. 6.17. Re-

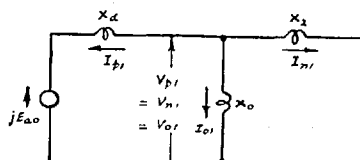


Fig. 6.17.

ferring to this circuit, the fundamental-frequency sequence currents are given by

$$\begin{aligned} I_{p1} &= - \frac{E_{ao}}{x_d + \frac{x_2 x_o}{x_2 + x_o}} \\ I_{n1} &= - \frac{x_o}{x_2 + x_o} I_{p1} \\ \text{and } I_{o1} &= - \frac{x_2}{x_2 + x_o} I_{p1} \end{aligned} \quad (6.43)$$

Equ. 6.43 also gives the $(2m+1)$ th-harmonic components of current when it is multiplied by b_2^m .

The fundamental-frequency component of current in phase b is given by

$$\begin{aligned} I_{b1} &= h^4 I_{p1} + h I_{n1} + I_{o1} \\ &= - \left\{ \frac{3}{2} \cdot \frac{x_2}{x_2 + x_o} + j \frac{\sqrt{3}}{2} \cdot \frac{x_2 + 2x_o}{x_2 + x_o} \right\} I_{p1} \\ &= \frac{\sqrt{3}}{2} \left\{ \sqrt{3} + j \left(1 + \frac{2x_o}{x_2} \right) \right\} \cdot \frac{E_{ao}}{x_d \left(1 + \frac{x_o}{x_2} \right) + x_o} \\ &= \sqrt{3} \left\{ \sqrt{3} + j \frac{\sqrt{(x_d'' + 2x_o)(x_q'' + 2x_o)}}{x_d'' x_q''} \right\} \cdot \frac{E_{ao}}{x_d \left[1 + \frac{\sqrt{(x_d'' + 2x_o)(x_q'' + 2x_o)}}{x_d'' x_q''} \right] + 2x_o} \end{aligned} \quad (6.44)$$

and the $(2m+1)$ th-harmonic component is given by

$$I_{b(2m+1)} = b_2^m I_{b1} \quad (6.45)$$

Similarly, the fundamental-frequency component of current in phase c is given by

$$\begin{aligned} I_{c1} &= h I_{p1} + h^2 I_{n1} + I_{o1} \\ &= \sqrt{3} \left\{ \sqrt{3} - j \frac{\sqrt{(x_d'' + 2x_o)(x_q'' + 2x_o)}}{x_d'' x_q''} \right\} \cdot \frac{E_{ao}}{x_d \left[1 + \frac{\sqrt{(x_d'' + 2x_o)(x_q'' + 2x_o)}}{x_d'' x_q''} \right] + 2x_o} \end{aligned} \quad (6.46)$$

and the $(2m+1)$ th-harmonic component is given by

$$I_{c(2m+1)} = b_2^m I_{c1} \quad (6.47)$$

The Fourier expressions of the phase currents are

$$i_b = \frac{\sqrt{3} E_{am}}{x_d \left[1 + \sqrt{(x_d'' + 2x_o)(x_q'' + 2x_o)/x_d'' x_q''} \right] + 2x_o} \cdot \left\{ \sqrt{3} \sum_{m=0}^{\infty} b_2^m \cos(2m+1)\omega t - \sqrt{(x_d'' + 2x_o)(x_q'' + 2x_o)/x_d'' x_q''} \sum_{m=0}^{\infty} b_2^m \sin(2m+1)\omega t \right\} \quad (6.48)$$

$$i_c = \frac{\sqrt{3} E_{am}}{x_d \left[1 + \sqrt{(x_d'' + 2x_o)(x_q'' + 2x_o)/x_d'' x_q''} \right] + 2x_o} \cdot \left\{ \sqrt{3} \sum_{m=0}^{\infty} b_2^m \cos(2m+1)\omega t + \sqrt{(x_d'' + 2x_o)(x_q'' + 2x_o)/x_d'' x_q''} \sum_{m=0}^{\infty} b_2^m \sin(2m+1)\omega t \right\} \quad (6.49)$$

Referring to Fig. 6.16.b, it is clear that the $(2m+1)$ th-harmonic components of the sequence voltages are

$$\frac{V_{p(2m+1)}}{2m+1} = \frac{V_{n(2m+1)}}{2m+1} = \frac{V_{o(2m+1)}}{2m+1} = j x_{o(2m+1)}$$

or $V_{p(2m+1)} = V_{n(2m+1)} = V_{o(2m+1)}$

$$\begin{aligned} &= j \frac{b_2^m (2m+1) x_o E_{ao}}{x_d (1 + x_o/x_d) + x_o} \\ &= j \frac{b_2^m (2m+1) 2x_o E_{ao}}{x_d \left[1 + \sqrt{(x_d'' + 2x_o)(x_q'' + 2x_o)/x_d'' x_q''} \right] + 2x_o} \end{aligned} \quad (6.50)$$

The Fourier expression for the terminal voltage of the open phase (phase a) is thus

$$V_a = - \frac{6x_o E_{am}}{x_d \left[1 + \sqrt{(x_d'' + 2x_o)(x_q'' + 2x_o)/x_d'' x_q''} \right] + 2x_o} \sum_{m=0}^{\infty} b_2^m (2m+1) \sin(2m+1)\omega t \quad (6.51)$$

(d) Numerical examples and experimental results

The calculation of the open-phase voltages and their experimental check for the three types of unsymmetrical short circuits are given in this section. But before giving these examples Eqs. 6.22, 6.36, 6.37 and 6.51 are first reduced to other forms which are more convenient for computation. This is found to be necessary because in the examples to be given the ratio x_q''/x_d'' is large so that the higher-order harmonic terms are not negligible. It can be shown that by making use

of the mathematical identities

$$\left. \begin{aligned} \sum_{n=0}^{\infty} b^n \cos(2n+1)\theta &\equiv \frac{(A+\sqrt{AB}) \cos \theta}{A+B+(A-B) \cos 2\theta} \\ \text{and } \sum_{n=0}^{\infty} (-b)^n \sin(2n+1)\theta &\equiv \frac{(A+\sqrt{AB}) \sin \theta}{A+B-(A-B) \cos 2\theta} \end{aligned} \right\} \quad (6.52)$$

$$\text{where } b = (\sqrt{B}-\sqrt{A})/(\sqrt{A}+\sqrt{B})$$

Eqs. 6.22, 6.36, 6.37 and 6.51 can be expressed alternatively as $(\theta = \omega t)$

$$v_a = 4x_q'' \cdot \frac{x_d'' + \sqrt{x_d'' x_q''}}{x_d'' + \sqrt{x_d'' x_q''}} \cdot E_{am} \cdot \frac{(x_q'' - 2x_d'') \sin \theta - (x_q'' - x_d'') \sin^3 \theta}{[x_d'' + x_q'' + (x_q'' - x_d'') \cos 2\theta]^2} \quad (6.53)$$

$$v_b = - \frac{x_d'' + \frac{1}{2}x_0 + \sqrt{(x_d'' + \frac{1}{2}x_0)(x_q'' + \frac{1}{2}x_0)}}{x_d'' + \frac{1}{2}x_0 + \sqrt{(x_d'' + \frac{1}{2}x_0)(x_q'' + \frac{1}{2}x_0)}} \cdot E_{am} \cdot \left\{ 2\sqrt{3}(x_q'' + \frac{1}{2}x_0) \frac{(x_q'' - 2x_d'' - \frac{1}{2}x_0) \cos \theta - (x_q'' - x_d'') \cos^3 \theta}{[x_d'' + x_q'' + x_0 - (x_q'' - x_d'') \cos 2\theta]^2} \right. \\ \left. + 3x_0 \frac{(x_d'' - 2x_q'' - \frac{1}{2}x_0) \sin \theta + (x_q'' - x_d'') \sin^3 \theta}{[x_d'' + x_q'' + x_0 - (x_q'' - x_d'') \cos 2\theta]^2} \right\} \quad (6.54)$$

$$v_c = \frac{x_d'' + \frac{1}{2}x_0 + \sqrt{(x_d'' + \frac{1}{2}x_0)(x_q'' + \frac{1}{2}x_0)}}{x_d'' + \frac{1}{2}x_0 + \sqrt{(x_d'' + \frac{1}{2}x_0)(x_q'' + \frac{1}{2}x_0)}} \cdot E_{am} \cdot \left\{ 2\sqrt{3}(x_q'' + \frac{1}{2}x_0) \frac{(x_q'' - 2x_d'' - \frac{1}{2}x_0) \cos \theta - (x_q'' - x_d'') \cos^3 \theta}{[x_d'' + x_q'' + x_0 - (x_q'' - x_d'') \cos 2\theta]^2} \right. \\ \left. - 3x_0 \frac{(x_d'' - 2x_q'' - \frac{1}{2}x_0) \sin \theta + (x_q'' - x_d'') \sin^3 \theta}{[x_d'' + x_q'' + x_0 - (x_q'' - x_d'') \cos 2\theta]^2} \right\} \quad (6.55)$$

$$\text{and } v_a = -3x_q'' x_0 \cdot \frac{x_d'' [1 + \sqrt{(x_d'' + 2x_0)(x_q'' + 2x_0)}/x_d'' x_q''] + 2x_0}{x_d'' [1 + \sqrt{(x_d'' + 2x_0)(x_q'' + 2x_0)}/x_d'' x_q''] + 2x_0} \cdot E_{am} \\ \times \frac{[x_d'' x_q'' - 2x_0(x_q'' - 2x_d'')] \sin \theta + 2x_0(x_q'' - x_d'') \sin^3 \theta}{[x_d'' x_q'' + x_0(x_d'' + x_q'') + x_0(x_q'' - x_d'') \cos 2\theta]^2} \quad (6.56)$$

respectively.

Tests were performed on a $7\frac{1}{2}$ H.P. induction motor (operated as a synchronous machine) with the following constants:

$$X_d = X_q = 6 \text{ ohms; } X_0 = 0.9 \text{ ohm and } X_d'' = 0.54 \text{ ohm}$$

With $X_q'' = 6$ ohms (no quadrature-axis damping) the oscillograms of the open-phase voltages of the line-to-line, line-to-neutral and double-line-to-neutral short circuits were obtained as shown by the solid curves in Figs. 6.18.a, 6.19 and 6.20. The solid curve in Fig. 6.18.b was the open-phase voltage for the

line-to-line case when the value of X_q'' was 1.08 ohms.

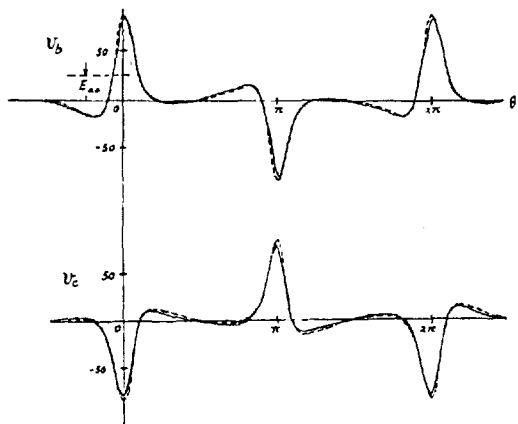
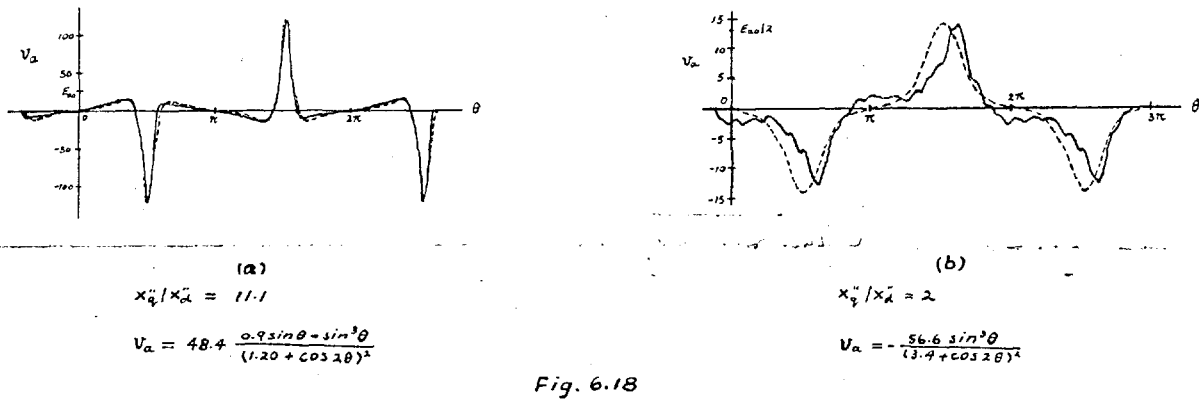


Fig. 6.19
Open-phase voltages of line-to-neutral short circuit

$$X_q''/X_d = 11.1$$

Solid curves : Test result

Dotted curves : Calculated result.

Equations for the dotted curves :

$$V_b = -63.4 \frac{0.819 \cos \theta - \cos^3 \theta}{(1.36 - \cos 2\theta)^2} + 7.11 \frac{2.18 \sin \theta - \sin^3 \theta}{(1.36 - \cos 2\theta)^2}$$

$$V_c = 63.4 \frac{0.819 \cos \theta - \cos^3 \theta}{(1.36 - \cos 2\theta)^2} + 7.11 \frac{2.18 \sin \theta - \sin^3 \theta}{(1.36 - \cos 2\theta)^2}$$

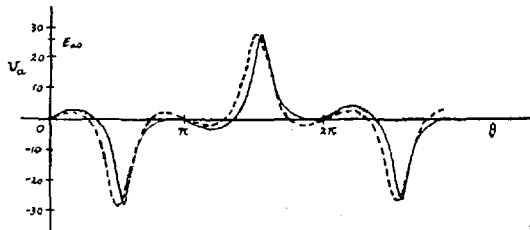


Fig. 6.20

$$X_q''/X_d = 11.1$$

$$V_a = 4.65 \frac{0.571 \sin \theta - \sin^3 \theta}{(1.857 + \cos 2\theta)^2}$$

Substituting the numerical values for X_d , X_q , X_o , X_d'' and X_q'' in Eqs. 6.53, 6.54, 6.55 and 6.56 gives the expression for the open-phase voltages for the three cases of unsymmetrical short circuits.

These voltages are shown by the dotted curves in Figs. 6.18.a, 6.18.b, 6.19 and 6.20, where their numerical expressions are also given. The discrepancies of the calculated results from the test results are mainly due to the neglect of resistances.

(6.3) Unsymmetrical Short Circuits with Capacitive Loads

When there are capacitance in the armature circuits of a synchronous machine, as when the machine is connected to a long transmission line, some of the harmonics produced by an unsymmetrical short circuit may reach disproportionately high

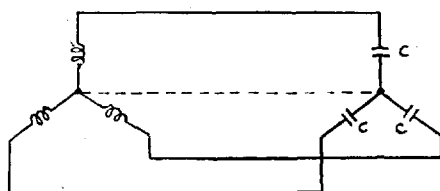


Fig. 6.21

values if the capacitances are of such values as to give a partial resonance condition to these harmonics. Fig. 6.21 shows schematically a machine connected to a bank

of capacitors used to simulate the long transmission line or other terminal capacitance of the machine. This idealised circuit will be used as the basis for studying the effects of capacitive loads on unsymmetrical short circuit.

With capacitances in the armature circuits, the equivalent negative-sequence reactance of the machine can no longer be safely assessed by taking some sort of mean value of X_d'' and X_q'' , as it may vary considerably owing to its dependence on the external capacitive reactances.

The three typical cases of unsymmetrical short circuits will be considered in the subsequent sections in the order of increasing complexity, with the simplest case first.

(a) Line-to-line short circuit.

If terminals b and c are short-circuited, Fig. 6.21 is reduced to Fig. 6.22 from which the terminal conditions of the machine can be found as

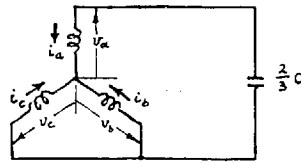


Fig. 6.22

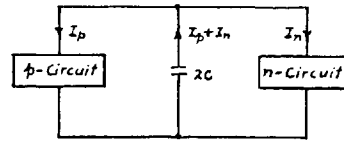


Fig. 6.23

$$I_{a(2m+1)} + I_{b(2m+1)} + I_{c(2m+1)} = 0$$

$$V_{b(2m+1)} - V_{c(2m+1)} = 0$$

and $V_{a(2m+1)} - V_{b(2m+1)} = j \frac{3}{2} \cdot \frac{X_c}{2m+1} I_{a(2m+1)}$

} (6.57)

In terms of symmetrical components the terminal conditions can be expressed as

$$I_{0(2m+1)} = 0$$

and $V_{p(2m+1)} = V_{n(2m+1)} = j \frac{1}{2} \cdot \frac{X_c}{2m+1} (I_{p(2m+1)} + I_{n(2m+1)})$

} (6.58)

The pattern of connection of the equivalent circuits satisfying Equ. 6.58 is shown in Fig. 6.23. According to this pattern of connection the harmonic equivalent circuits can be

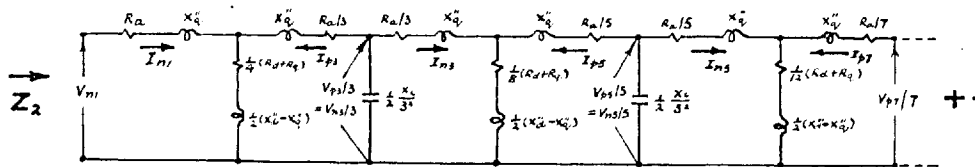


Fig. 6.24

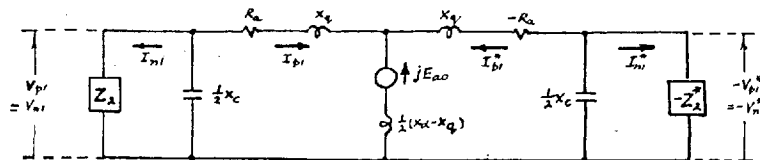


Fig. 6.25

connected together as shown in Fig. 6.24, and the complete equivalent circuit is shown in Fig. 6.25.

Referring to Fig. 6.24, it is clear that when the subtran-

sient saliency is not large (the ratio X_q''/X_d'' is not much larger than 1) so that the mutual dependence of the current components at different frequencies is relatively small, the $(2m+1)$ th-harmonic current will reach its highest value when X_c is approximately equal to $\frac{1}{2}(2m+1)^2(X_d'' + X_q'')$. When the ratio X_q''/X_d'' is large, the value of X_c may depart considerably from $\frac{1}{2}(2m+1)^2(X_d'' + X_q'')$ to give a largest $(2m+1)$ th-harmonic current. Fig. 6.24 also shows that when the sum of the currents $I_{p(2m+1)}$ and $I_{n(2m+1)}$ reaches its largest value, the voltage $V_{a(2m+1)}$ also reaches its largest value.

When the value of X_c is of such value that a certain harmonic component of v_a , say the third, has its largest value, the higher harmonic components, say from the seventh or the ninth upwards, may become smaller than they would be for a plain line-to-line short circuit, i.e., when there were no terminal capacitances, owing to the shunting effect of the capacitances to higher-harmonic voltages. With a small ratio of X_q''/X_d'' , the voltage v_a is substantially contributed by its lower harmonic components; consequently the maximum value of v_a is in general increased with capacitance load on the healthy phase. The fact that with small resistances in the armature circuits the peaks of the harmonic components of voltage are very nearly additive makes the maximum values of v_a occur at the same points where the principal harmonic components, the third and the fifth, have their largest values. Fig. 6.26 shows the oscillograms of v_a for different values of X_c with $X_q'' = 2X_d''$.

When the ratio X_q''/X_d'' is considerably larger than unity, the

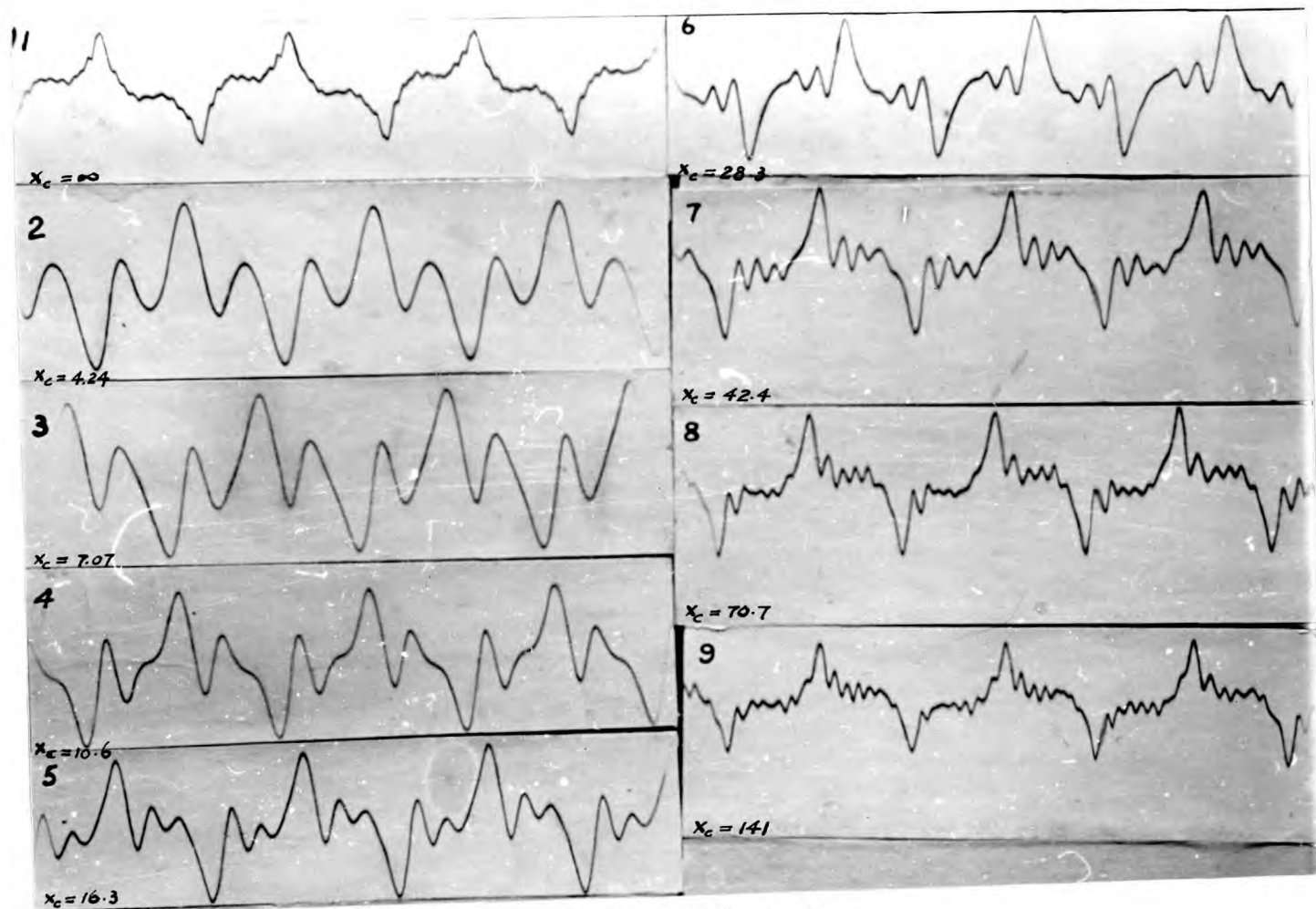


Fig. 6.26 ($x''_q = 2x''_d$)

Oscillogram No. 1 is same as that in Fig. 6.18.b.

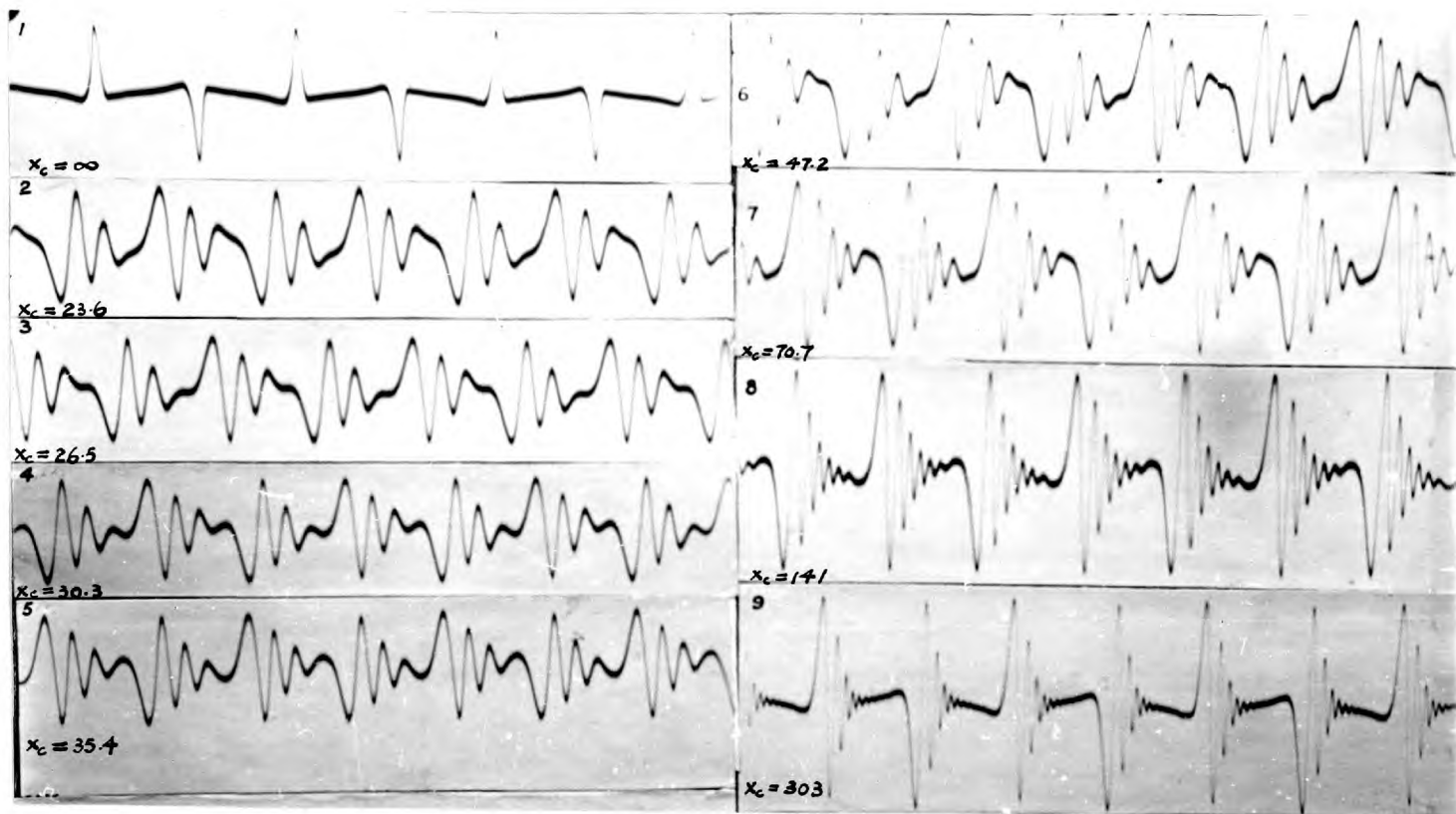


Fig. 6-27 ($x_q'' = 11.1 x_d''$)

Oscillogram No.1 same as that in Fig. 6-18.a

peak value of v_a will actually become smaller than that of a plain line-to-line short circuit with small values of X_0 , as shown in Fig. 6.27 ($X_d''/X_d'' = 11.1$). The reason is that the higher-harmonic components of v_a now contribute appreciably to its peak value, and the increase in the third or fifth harmonic component cannot make up for the decrease in the rest of the harmonic components.

As a numerical example, the voltage shown by the second oscillogram in Fig. 6.27 is calculated up to the ninth-harmonic component with the aid of Fig. 6.24 (The values of the resistances: $R_d = 0.6$ ohm; $R_q = 0$; $R_a = 0.09$ ohm). The equation of this voltage is found to be

$$v_a = 17.6\cos(\theta + 52^\circ) - 29\cos(3\theta + 4^\circ) + 52\cos(5\theta - 60^\circ) \\ + 28.3\cos(7\theta - 31^\circ) + 6.2\cos(9\theta - 16^\circ)$$

A comparison of the calculated result and the test result is shown in Fig. 6.28, where the solid curve is the test result.

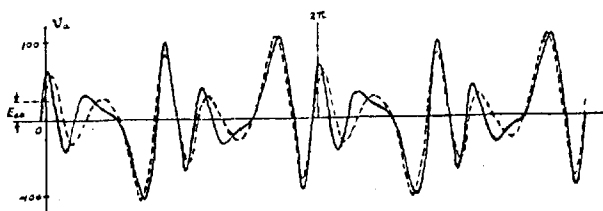


Fig. 6.28

In this example, the equivalent impedance for the negative-sequence, fundamental-frequency current (Fig. 6.24) is found to be approximately $1.58/46.5$, a value which is obviously unpredictable from the values of X_d'' (0.54) and X_q'' (6).

(b) Double-line-to-neutral short circuit

If phases b and c are both short-circuited to the neutral, Fig. 6.21 is reduced to Fig. 6.29. The terminal conditions are

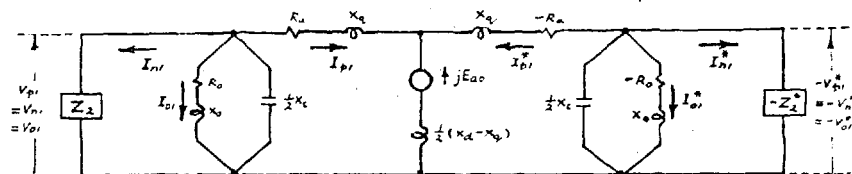
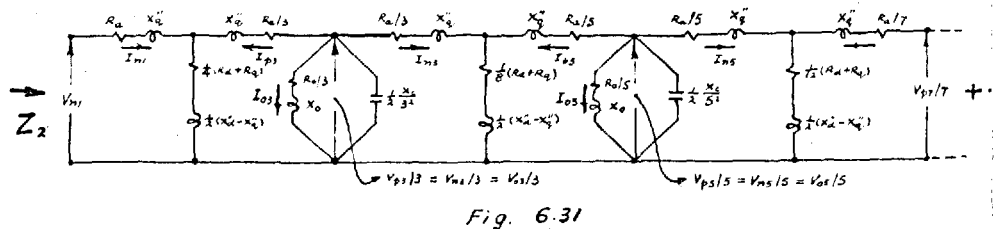
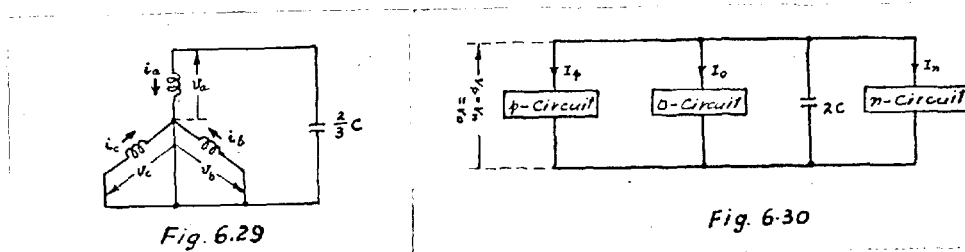
$$V_{b(2m+1)} = V_{c(2m+1)} = 0$$

$$\text{and } V_{a(2m+1)} = j \frac{3}{2} \cdot \frac{x_c}{2m+1} I_{a(2m+1)}$$

Or, in terms of symmetrical components,

$$V_{p(2m+1)} = V_{n(2m+1)} = V_{o(2m+1)} = j \frac{1}{2} \cdot \frac{x_c}{2m+1} \{ I_{p(2m+1)} + I_{n(2m+1)} + I_{o(2m+1)} \}$$

The pattern of connection is shown in Fig. 6.30. Figs. 6.31 and 6.32 show the harmonic equivalent circuit and the complete equivalent circuit. It is clear that the difference between



the equivalent circuits of the previous type and the present type of short circuit is that there is an additional branch, Z_o , in parallel with the capacitance for the present type.

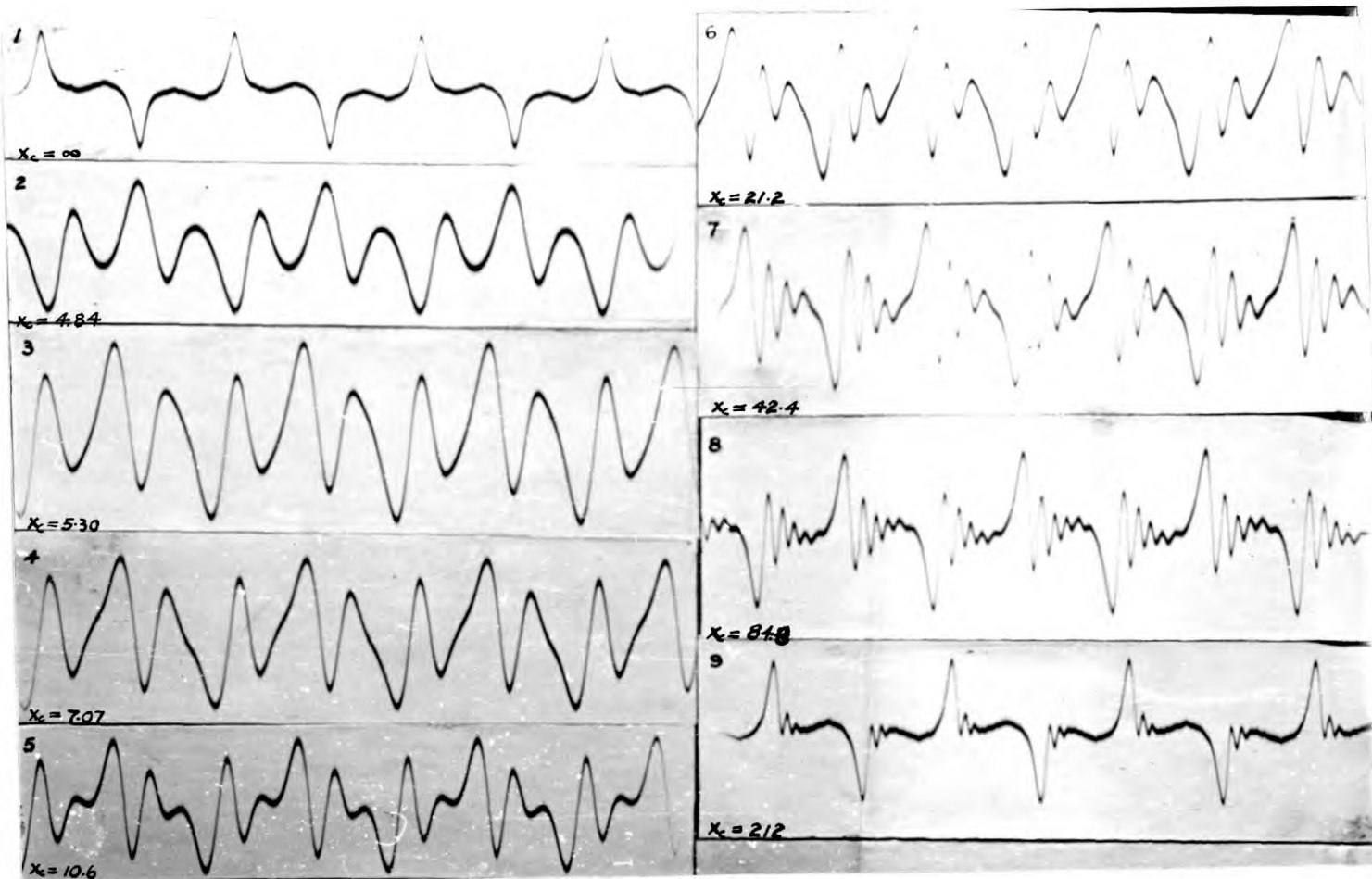


Fig. 6.33

Oscillogram No. 1 same as that in Fig. 6.20

The oscillograms of the voltage v_a for different values of X_c are shown in Fig. 6.33. They are largely similar to those of a line-to-line short circuit with a smaller ratio of X_d''/X_q'' .

(c) Line-to-neutral short circuit

There are two different conditions for this type of short circuit, with or without the connection between the neutral points of the machine and the capacitors. If phase a is short-

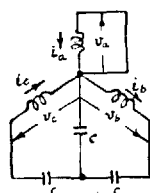


Fig. 6.34

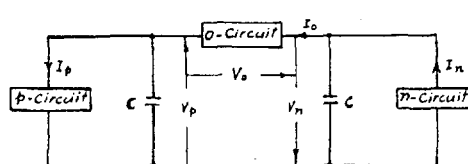


Fig. 6.35

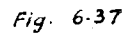
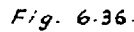
circuited and there is no neutral connection, Fig. 6.21 is reduced to Fig. 6.34. The terminal conditions are

$$\left. \begin{aligned} V_{a(2m+1)} &= 0 \\ V_{b(2m+1)} &= j \frac{X_c}{2m+1} (2 I_{b(2m+1)} + I_{c(2m+1)}) \\ \text{and } V_{c(2m+1)} &= j \frac{X_c}{2m+1} (I_{b(2m+1)} + 2 I_{c(2m+1)}) \end{aligned} \right\} (6.59)$$

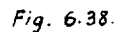
or, in terms of symmetrical components,

$$\left. \begin{aligned} V_{p(2m+1)} + V_{n(2m+1)} + V_{0(2m+1)} &= 0 \\ V_{p(2m+1)} &= j \frac{X_c}{2m+1} (I_{p(2m+1)} - I_{0(2m+1)}) \\ \text{and } V_{n(2m+1)} &= j \frac{X_c}{2m+1} (I_{n(2m+1)} - I_{0(2m+1)}) \end{aligned} \right\} (6.60)$$

The pattern of connection satisfying Equ. 4.70 is shown in Fig. 6.35. The harmonic equivalent circuit and the complete equivalent circuit shown in Figs. 6.36 and 6.37 are connected together in accordance with the pattern shown in Fig. 6.35.



conditions are now



(6.61)

$$\text{and } V_{c(2m+1)} = j \frac{x_c}{2m+1} I_{c(2m+1)}$$

or, in terms of symmetrical components,

(6.62)

and
$$V_{0(2m+1)} = j \frac{1}{3} \cdot \frac{x_0}{2m+1} (-I_{p(2m+1)} - I_{n(2m+1)} + 2 I_{o(2m+1)})$$

The pattern of connection satisfying Equ. 6.62 is shown in Fig. 6.39.a, which may be reduced to Fig. 6.39.b by means of the star-delta transformation. Comparing Figs. 6.39.b and 6.35 shows the difference between the two conditions, with and without neutral connections. With neutral connection, there is an additional capacitance branch in parallel with the zero-sequence circuit. Figs. 6.40 and 6.41 show the harmonic equivalent cir-

cuit and the complete equivalent circuit connected in accordance with the pattern shown in Fig. 6.39.b.

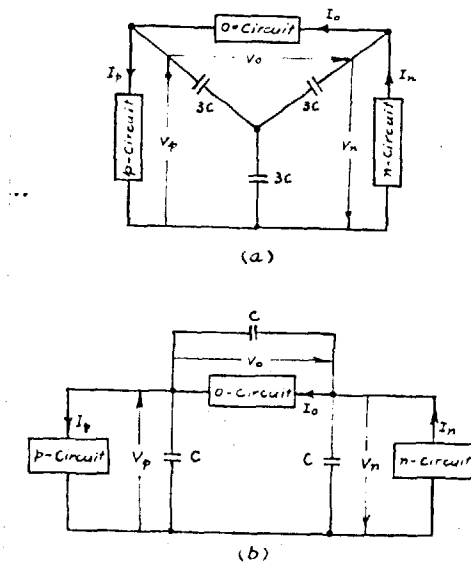


Fig. 6.39

There is an interesting point about the circuit shown in Fig. 6.40. When $X_c = (2m+1)^2 X_o$, i.e., when L_o and C are in parallel resonance at the $(2m+1)$ th-harmonic frequency, harmonics higher than the $(2m+1)$ th will be cut off.

Figs. 6.42.a and 6.42.b show the oscillograms of the voltages of the unshortcircuited phases for different values of X_c .

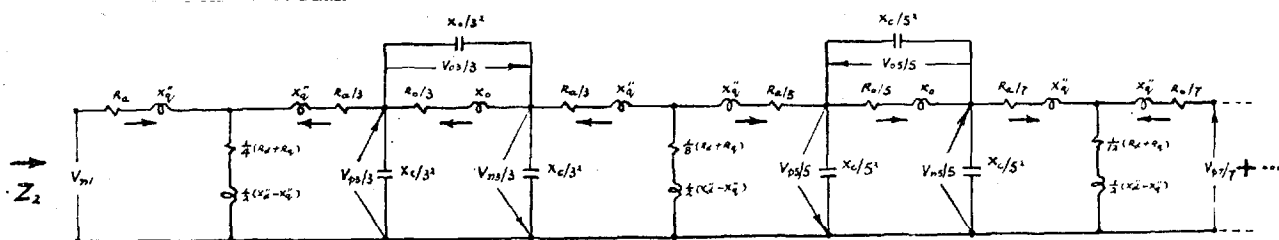


Fig. 6.40

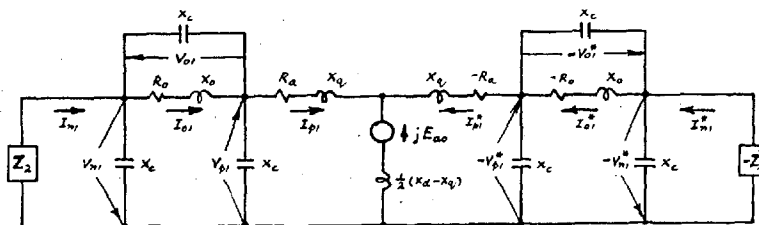


Fig. 6.41

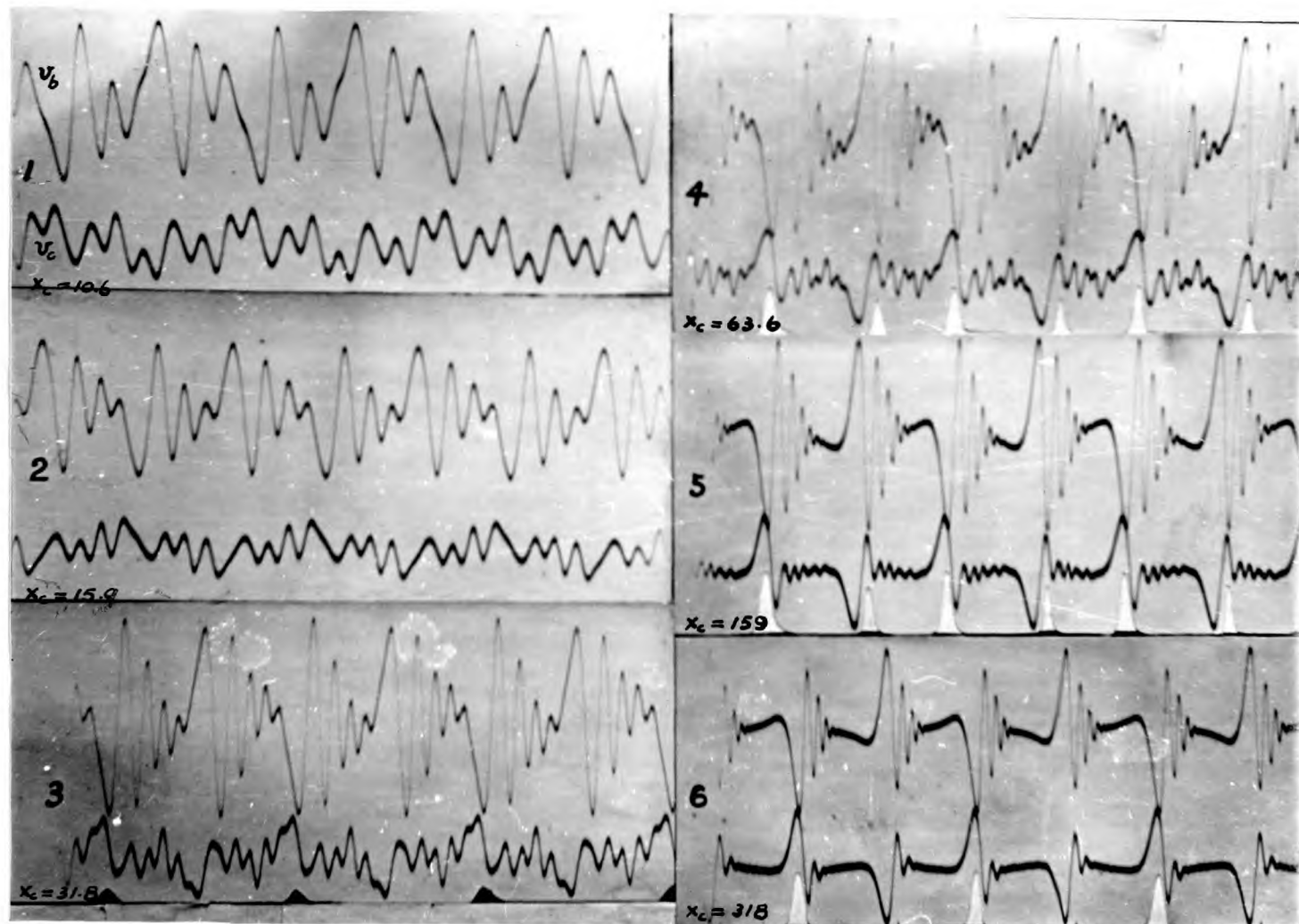


Fig. 6.42. a (No Neutral Connection)

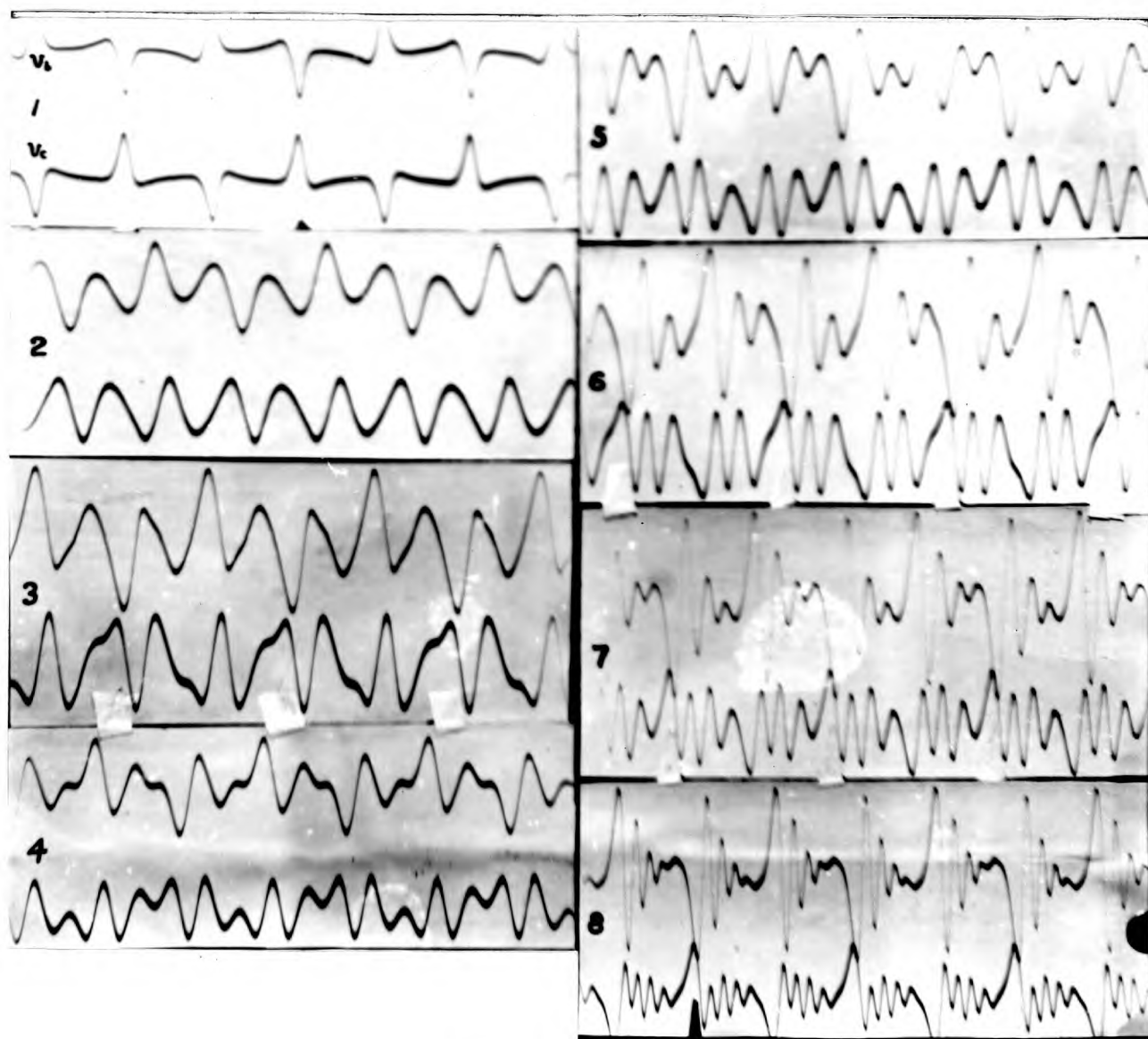


Fig. 6.42.b
(With Neutral Connection)

1. $X_c = \infty$ (Same oscillograms as in Fig. 6.19)
2. $X_c = 10.6$
3. $X_c = 13.8$
4. $X_c = 22.4$
5. $X_c = 29.2$
6. $X_c = 44.2$
7. $X_c = 72.3$
8. $X_c = 159$

$$x_d = x_q = 6, \quad x_d'' = 0.54, \quad x_q'' = 6, \quad x_o = 0.9.$$

(6.4) Two or More Machines in Parallel

When there are more than one machine in question, equivalent circuits representing the whole system may be constructed just as for the single-machine problems considered before. The equivalents are, however, more difficult to represent owing to the introduction of the ideal phase shifter for the purpose of connecting together two circuits whose terminal voltages are equal in magnitude but out of phase.

(a) A number of machines connected to an infinite bus

When several machines are connected to an infinite bus through balanced reactances as shown

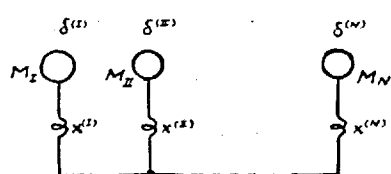


Fig. 6.43.

in Fig. 6.43, with the field axes of the machines I, II, ... N leading an arbitrarily chosen axis by the

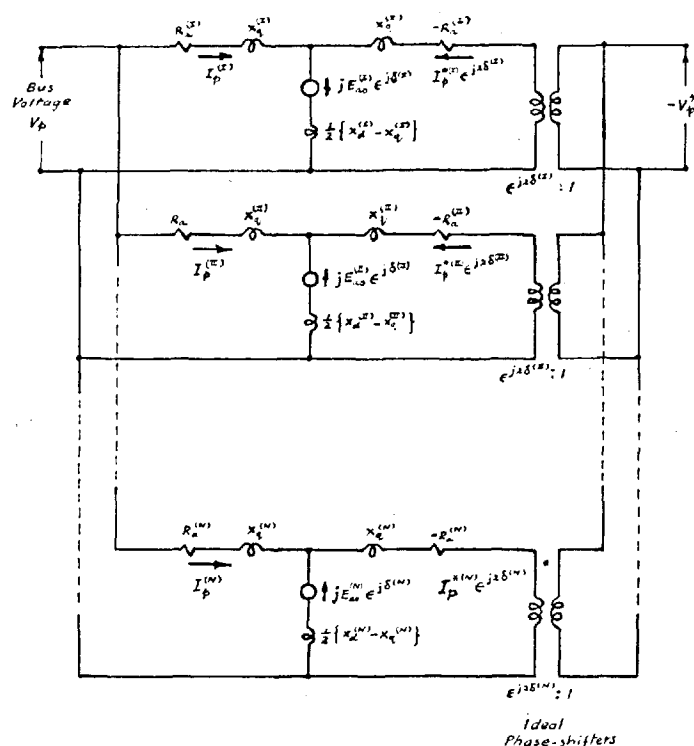


Fig. 6.44.

angles $\delta^{(I)}$, $\delta^{(II)}$, ... $\delta^{(N)}$, the equivalent circuit for the whole system is shown in Fig. 6.44.

(b) Unsymmetrical short circuits of two machines in parallel

When an unsymmetrical short circuit occurs in a system with two machines operating in parallel, both machines must be con-

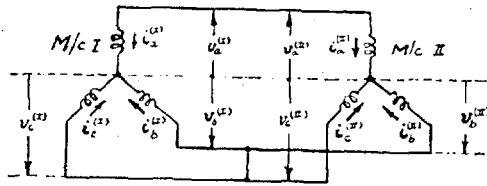


Fig. 6.45.

sidered as a whole because the unshort-circuited phases still connect one machine to the other. Let the line-to-line short circuit be

considered first. If lines b and c are short-circuited, as shown in Fig. 6.45, the terminal conditions of the machine are

$$\left. \begin{aligned} V_a^{(I)} - V_b^{(I)} &= V_a^{(II)} - V_b^{(II)} \\ V_b^{(I)} - V_c^{(I)} &= V_b^{(II)} - V_c^{(II)} = 0 \\ I_b^{(I)} + I_c^{(I)} &= I_b^{(II)} + I_c^{(II)} = 0 \end{aligned} \right\} (6.63)$$

and $I_a^{(I)} + I_a^{(II)} = 0$

In terms of symmetrical components, the terminal conditions can be expressed as

$$\left. \begin{aligned} V_p^{(I)} &= V_n^{(I)} = V_p^{(II)} = V_n^{(II)} \\ I_p^{(I)} + I_n^{(I)} + I_p^{(II)} + I_n^{(II)} &= 0 \end{aligned} \right\} (6.64)$$

and $V_o^{(I)} = V_o^{(II)} = 0$

Fig. 6.46 shows the pattern of connection. Fig. 6.47 shows the harmonic equivalent circuits of both machines connected

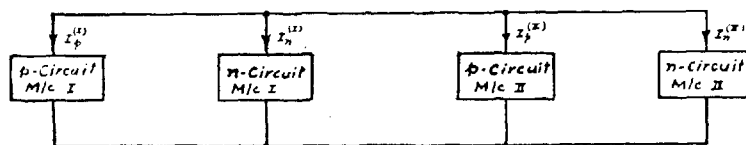


Fig. 6.46.

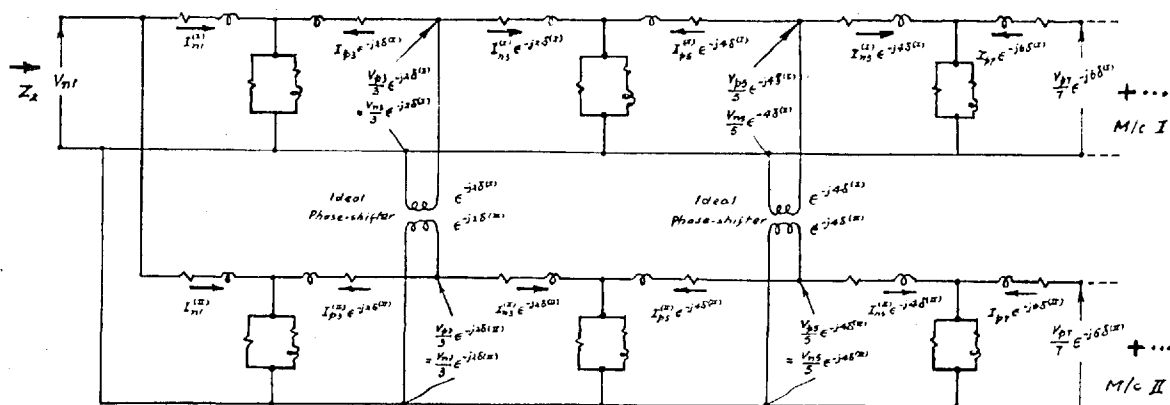


Fig. 6.47.

together, which when viewed from the V_{n1} end, represents the equivalent negative-sequence impedance of the entire system. The ideal phase-shifters introduced for the purpose of connecting together the harmonic quantities have different ratios for different harmonics. The complete equivalent circuit is shown in Fig. 6.48.

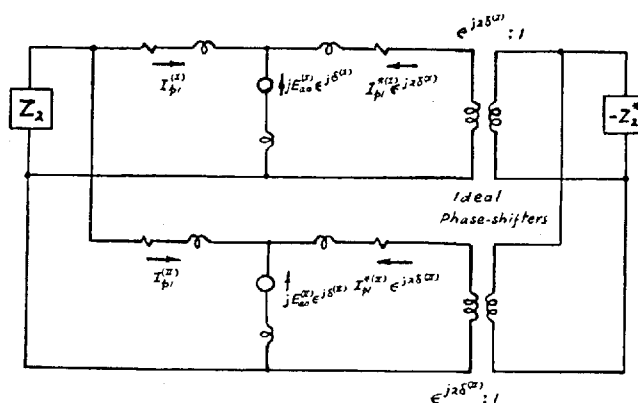


Fig. 6.48

The equivalent circuit for the double-line-to-neutral short circuit is very similar to that for the line-to-line short circuit. The pattern of connection, the harmonic equivalent circuit and the complete equivalent circuit are shown in Figs. 6.49, 6.50 and 6.51 respectively.

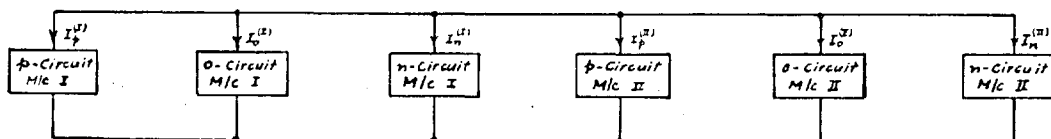


Fig. 6.49

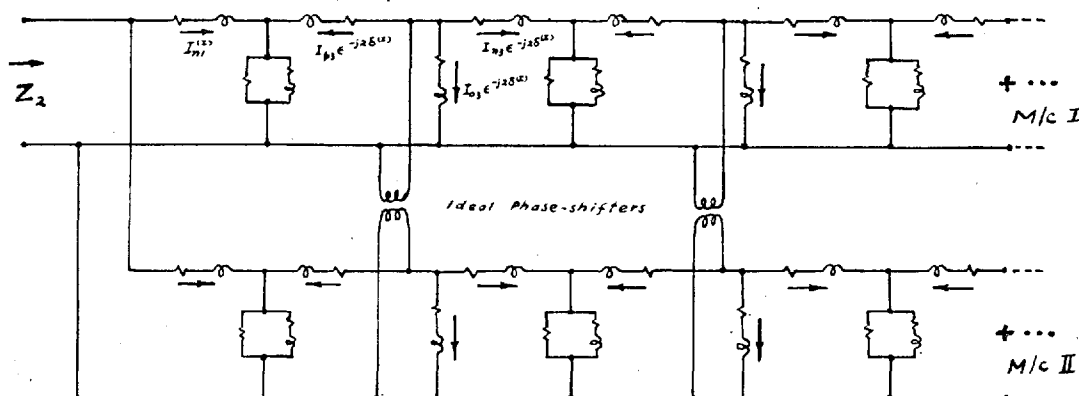


Fig. 6.50.

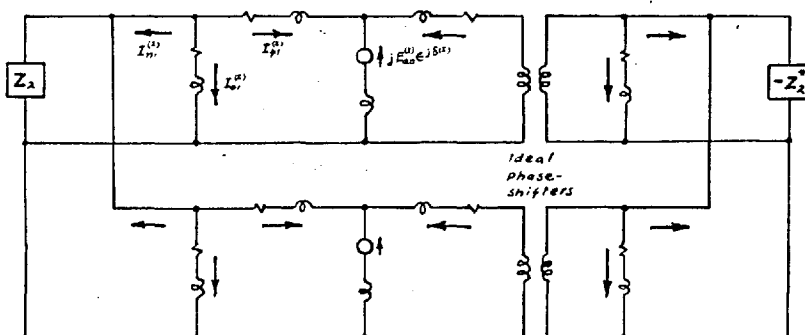


Fig. 6.51

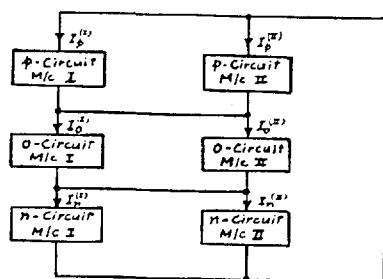


Fig. 6.52.

For the line-to-neutral short circuit, the pattern of connection, the harmonic equivalent circuit and the complete equivalent circuit are shown in Figs. 6.52, 6.53 and 6.54 respectively.

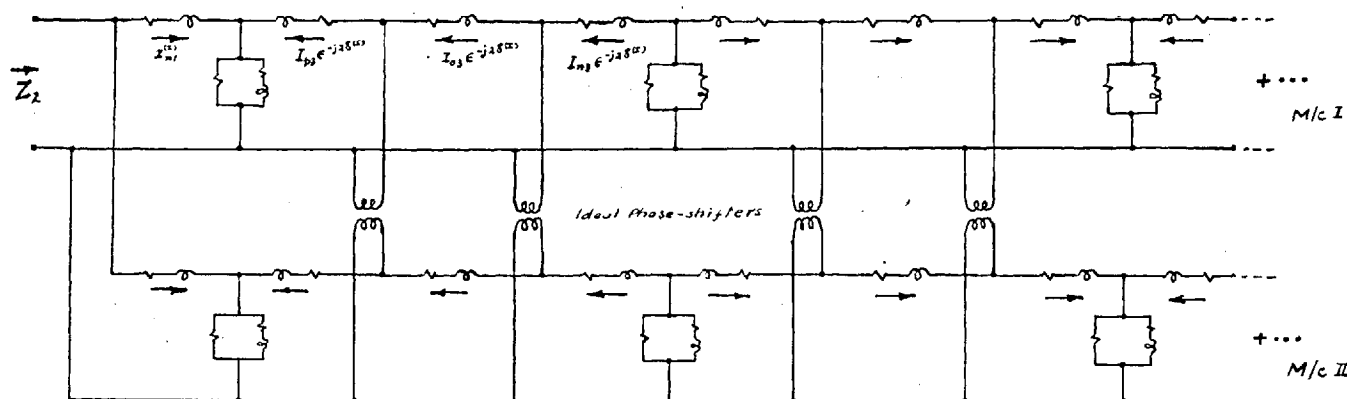


Fig. 6.53

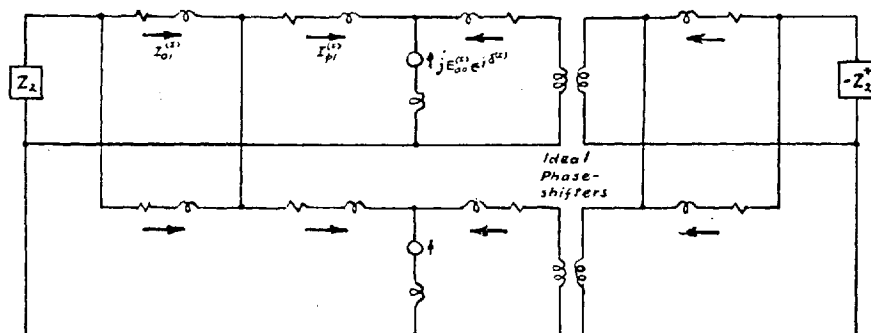


Fig. 6.54

Although the equivalent circuits given above are for two machines only, similar equivalent circuits can be derived without difficulty for more than two machines. The representation of Z_2 will, of course, become more and more complicated when the number of machines involved increases.

PART II TRANSIENT-STATE SOLUTION

(7) GENERAL OPERATIONAL SOLUTION

(7.1) General Equations with Negligible Space-harmonic Effects

In addition to the neglect of the non-linearities of magnetic saturation and momentary variation of speed, which are the basic assumptions underlying the general equations given in Article 2.4, the assumption of negligible space-harmonic effects, introduced in the later chapters in Part I, must be made from the beginning when the transient-state problems are considered, in order that the general equations may be handled with reasonable ease.

It has been shown in Part I that with the space-harmonic effects neglected, the mutuals between current components at different frequencies are limited to only the two components of consecutive harmonic orders, so that the steady-state solution can be materially simplified. This is also true for the transient-state currents and is the reason for making the additional assumption from the beginning.

When a synchronous machine is running steadily at an angular velocity of ω and the field current has reached its steady value of I_{f0} , and the armature terminals are connected through external impedances to a voltage source, the voltage equations of the system may be represented by the matrix equation

$$[v] = \{[Z_{mc}] + [Z_{ex}]\} \cdot [i] \quad (7.1)$$

The machine impedance matrix, $[Z_{mc}]$, is, assuming as before a common mutual between any two of the armature (a, b or c),

field (f) and direct-axis-damper (kd) axes, given by

$$[Z_{mc}] = \begin{matrix} & \begin{matrix} a & b & c & f & kd & kg \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ f \\ kd \\ kg \end{matrix} & \begin{bmatrix} R_a + L_{aa} \frac{d}{dt} & M_{so} \frac{d}{dt} & M_{so} \frac{d}{dt} & \frac{d}{dt} M_{af} \cos \theta & \frac{d}{dt} M_{af} \cos \theta & -\frac{d}{dt} M_{ag} \sin \theta \\ + \frac{d}{dt} L_{a2} \cos 2\theta & + \frac{d}{dt} L_{a2} \cos(2\theta - \frac{2\pi}{3}) & + \frac{d}{dt} L_{a2} \cos(2\theta + \frac{2\pi}{3}) & & & \\ M_{so} \frac{d}{dt} & R_a + L_{aa} \frac{d}{dt} & M_{so} \frac{d}{dt} & \frac{d}{dt} M_{af} \cos(\theta - \frac{2\pi}{3}) & \frac{d}{dt} M_{af} \cos(\theta - \frac{2\pi}{3}) & -\frac{d}{dt} M_{ag} \sin(\theta - \frac{2\pi}{3}) \\ + \frac{d}{dt} L_{a2} \cos(2\theta - \frac{2\pi}{3}) & + \frac{d}{dt} L_{a2} \cos(2\theta + \frac{2\pi}{3}) & + \frac{d}{dt} L_{a2} \cos 2\theta & & & \\ M_{so} \frac{d}{dt} & M_{so} \frac{d}{dt} & R_a + L_{aa} \frac{d}{dt} & \frac{d}{dt} M_{af} \cos(\theta + \frac{2\pi}{3}) & \frac{d}{dt} M_{af} \cos(\theta + \frac{2\pi}{3}) & -\frac{d}{dt} M_{ag} \sin(\theta + \frac{2\pi}{3}) \\ + \frac{d}{dt} L_{a2} \cos(2\theta + \frac{2\pi}{3}) & + \frac{d}{dt} L_{a2} \cos 2\theta & + \frac{d}{dt} L_{a2} \cos(2\theta - \frac{2\pi}{3}) & & & \\ \frac{d}{dt} M_{af} \cos \theta & \frac{d}{dt} M_{af} \cos(\theta - \frac{2\pi}{3}) & \frac{d}{dt} M_{af} \cos(\theta + \frac{2\pi}{3}) & R_f + L_f \frac{d}{dt} & M_{fkd} \frac{d}{dt} & \\ \frac{d}{dt} M_{af} \cos \theta & \frac{d}{dt} M_{af} \cos(\theta - \frac{2\pi}{3}) & \frac{d}{dt} M_{af} \cos(\theta + \frac{2\pi}{3}) & M_{fkd} \frac{d}{dt} & R_{kd} + L_{kd} \frac{d}{dt} & \\ -\frac{d}{dt} M_{ag} \sin \theta & -\frac{d}{dt} M_{ag} \sin(\theta - \frac{2\pi}{3}) & -\frac{d}{dt} M_{ag} \sin(\theta + \frac{2\pi}{3}) & & & R_{kg} + L_{kg} \frac{d}{dt} \end{bmatrix} \end{matrix} \quad (7.2)$$

where $\theta = \omega t + \lambda$, λ being the angle by which the axis of the field is ahead of the axis of phase a when time begins to count.

The external impedance matrix $[Z_{ex}]$ in Equ. 7.1 has the form

$$[Z_{ex}] = \begin{matrix} & \begin{matrix} a & b & c & f & kd & kg \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ f \\ kd \\ kg \end{matrix} & \begin{bmatrix} Z_{aa}(t) & Z_{ab}(t) & Z_{ac}(t) & & & \\ Z_{ba}(t) & Z_{bb}(t) & Z_{bc}(t) & & & \\ Z_{ca}(t) & Z_{cb}(t) & Z_{cc}(t) & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{matrix} \quad (7.3)$$

where each component may be expressed as

$$Z(t) = R + L(d/dt)$$

The voltage matrix in Equ. 7.1 is

$$[v] = \begin{matrix} \begin{matrix} a \\ b \\ c \\ f \\ kd \\ kg \end{matrix} & \begin{bmatrix} v_a + e_a \\ v_b + e_b \\ v_c + e_c \\ \\ \\ \end{bmatrix} \end{matrix} \quad (7.4)$$

where v_a , v_b and v_c are the source voltages and $e_a = \omega M_{af} I_{f0} \sin \theta$,
 $e_b = \omega M_{af} I_{f0} \sin(\theta - 2\pi/3)$, $e_c = \omega M_{af} I_{f0} \sin(\theta + 2\pi/3)$.

Finally the current matrix in Equ. 7.1 is

$$[i] = \begin{bmatrix} i_a \\ i_b \\ i_c \\ i_f' \\ i_{kd} \\ i_{kq} \end{bmatrix} \quad (7.5)$$

where the current i_f' does not include the steady component I_{f0} .

It is well known that the natural reference axes for the armature currents, i.e., the phase axes a, b and c, are not convenient for the purpose of studying the performance of a synchronous machine, and some simplifications and transformations are needed. In the course of simplifying and transforming the general equations, it is convenient to leave out the external impedances at first and consider their effects afterwards. Thus $[Z_{ex}]$ may be omitted from Equ. 7.1 but the source voltages v_a , v_b and v_c must be replaced by the machine terminal voltages v_{at} , v_{bt} and v_{ct} respectively. These two sets of voltages are related by the matrix equation

$$[v_t] = [v] - [Z_{ex}] \cdot [i] \quad (7.6)$$

(7.2) Operational Equations

It can be shown that when the Laplace transformation, defined by

$$\bar{x}(p) = \int_0^{\infty} e^{-pt} x(t) dt$$

is performed on Equ. 7.1 ($[Z_{ex}]$ omitted) and the field and damper currents are eliminated, the operational voltage equations of the armature circuits are found to be

$$[\bar{v}(p)] = [\bar{Z}_{mc}(p)] \cdot [\bar{i}(p)] \quad (7.7)$$

where

	a	b	c
a	$R_a + \frac{1}{3} p \{ L_o + \bar{L}_s(p+j\omega) + \text{Conj.} \}$	$\frac{1}{3} p \{ L_o + h^2 \bar{L}_s(p+j\omega) + \text{Conj.} \}$	$\frac{1}{3} p \{ L_o + h \bar{L}_s(p+j\omega) + \text{Conj.} \}$
b	$\frac{1}{3} p \{ L_o + h \bar{L}_s(p+j\omega) + \text{Conj.} \}$	$R_a + \frac{1}{3} p \{ L_o + \bar{L}_s(p+j\omega) + \text{Conj.} \}$	$\frac{1}{3} p \{ L_o + h^2 \bar{L}_s(p+j\omega) + \text{Conj.} \}$
c	$\frac{1}{3} p \{ L_o + h^2 \bar{L}_s(p+j\omega) + \text{Conj.} \}$	$\frac{1}{3} p \{ L_o + h \bar{L}_s(p+j\omega) + \text{Conj.} \}$	$R_a + \frac{1}{3} p \{ L_o + \bar{L}_s(p+j\omega) + \text{Conj.} \}$

(7.8)

a	$\bar{v}_{ax}(p) + \bar{e}_a(p) - \frac{1}{3} p \{ e^{-j2\lambda} \bar{L}_D(p+j\omega) [\bar{e}_a(p+j2\omega) + h \bar{e}_b(p+j2\omega) + h^2 \bar{e}_c(p+j2\omega)] + \text{Conj.} \}$
b	$\bar{v}_{bx}(p) + \bar{e}_b(p) - \frac{1}{3} p \{ h e^{-j2\lambda} \bar{L}_D(p+j\omega) [\bar{e}_a(p+j2\omega) + h \bar{e}_b(p+j2\omega) + h^2 \bar{e}_c(p+j2\omega)] + \text{Conj.} \}$
c	$\bar{v}_{cx}(p) + \bar{e}_c(p) - \frac{1}{3} p \{ h^2 e^{-j2\lambda} \bar{L}_D(p+j\omega) [\bar{e}_a(p+j2\omega) + h \bar{e}_b(p+j2\omega) + h^2 \bar{e}_c(p+j2\omega)] + \text{Conj.} \}$

(7.9)

and

a	$\bar{e}_a(p)$
b	$\bar{e}_b(p)$
c	$\bar{e}_c(p)$

(7.9)

In the impedance and voltage matrices, Eqs. 7.8 and 7.9,

$$L_o = L_{ao} + 2 M_{so}$$

$$\bar{L}_s(p) = \frac{1}{2} \{ \bar{L}_d(p) + \bar{L}_q(p) \}$$

$$\bar{L}_D(p) = \frac{1}{2} \{ \bar{L}_d(p) - \bar{L}_q(p) \}$$

$$\begin{aligned} \bar{L}_d(p) &= L_{ao} - M_{so} + \frac{3}{2} L_{d2} - \frac{3}{2} \cdot \frac{M_{d2}^2 p}{\bar{Z}_d(p)} \\ &= L_d - \frac{M_d^2 p}{\bar{Z}_d(p)} \end{aligned}$$

$$\begin{aligned} \bar{L}_q(p) &= L_{ao} - M_{so} - \frac{3}{2} L_{q2} - \frac{3}{2} \cdot \frac{M_{q2}^2 p}{\bar{Z}_q(p)} \\ &= L_q - \frac{M_q^2 p}{\bar{Z}_q(p)} \end{aligned}$$

$$\bar{Z}_d(p) = \{ (R_f + L_f p)(R_{kd} + L_{kd} p) - M_{fkd}^2 p^2 \} / \{ R_f + R_{kd} + (L_f + L_{kd} - 2 M_{fkd}) p \}$$

$$\bar{Z}_q(p) = R_{kq} + L_{kq} p$$

(7.10)

$$\begin{aligned} L_d &= L_{ao} - M_{so} + \frac{3}{2} L_{a2} \\ L_q &= L_{ao} - M_{so} - \frac{3}{2} L_{a2} \\ M_d &= \sqrt{3/2} M_{af} \\ M_q &= \sqrt{3/2} M_{aq} \end{aligned}$$

The terms involving $\bar{i}_a(p+j2\omega)$, etc., have been considered as part of $\bar{v}(p)$ and included therein.

(7.3) 1, 2, 0 Components — Diagonalisation of the Machine Operational Impedance Matrix

The form of the machine operational impedance matrix (Equ. 7.8) suggests that it can be diagonalised when pre-multiplied by $[S]^{-1}$ and post-multiplied by $[S]$, $[S]$ being the well-known

$$[S] = \begin{matrix} & \begin{matrix} 1 & 2 & 0 \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ h^2 & h & 1 \\ h & h^2 & 1 \end{bmatrix} \end{matrix} \quad (7.11)$$

$[S]^{-1}$, the inverse of $[S]$, is equal to one third the conjugate of the transpose of $[S]$.

Carrying out the process of diagonalisation, which is equivalent to changing the reference axes from the phase axes a, b and c to the symmetrical-component axes 1, 2 and 0, it can be shown that Equ. 7.8 is transformed to

$$[\bar{v}(p)]' = [Z_{mc}(p)]' \cdot [\bar{i}(p)]' \quad (7.12)$$

where the new voltage matrix is

$$[\bar{v}(p)]' = [S]^{-1} \cdot [\bar{v}(p)] = \begin{matrix} 1 & \bar{v}_{1t}(p) + \bar{e}_1(p) - p \epsilon^{j2\lambda} \bar{L}_D(p-j\omega) \bar{L}_2(p-j2\omega) \\ 2 & \bar{v}_{2t}(p) + \bar{e}_2(p) - p \epsilon^{-j2\lambda} \bar{L}_D(p+j\omega) \bar{L}_1(p+j2\omega) \\ 0 & \bar{v}_{0t}(p) \end{matrix} \quad (7.13)$$

$$\left. \begin{aligned}
 \bar{v}_{1c}(p) &= \frac{1}{3} \{ \bar{v}_{ac}(p) + h \bar{v}_{bc}(p) + h^2 \bar{v}_{cc}(p) \} \\
 \bar{v}_{2c}(p) &= \frac{1}{3} \{ \bar{v}_{ac}(p) + h^2 \bar{v}_{bc}(p) + h \bar{v}_{cc}(p) \} \\
 &\text{Conjugate of } \bar{v}_{1c}(p) \\
 \bar{v}_{0c}(p) &= \frac{1}{3} \{ \bar{v}_{ac}(p) + \bar{v}_{bc}(p) + \bar{v}_{cc}(p) \} \\
 \bar{e}_1(p) &= \frac{1}{3} \{ \bar{e}_a(p) + h \bar{e}_b(p) + h^2 \bar{e}_c(p) \} \\
 \bar{e}_2(p) &= \text{Conjugate of } \bar{e}_1(p)
 \end{aligned} \right\} \quad (7.14)$$

the new impedance matrix is

$$[\bar{z}_{mc}(p)]' = [S]^{-1} \cdot [\bar{z}_{mc}(p)] \cdot [S] = \begin{matrix} & \begin{matrix} 1 & 2 & 0 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 0 \end{matrix} & \begin{bmatrix} R_a + p \bar{L}_S(p-j\omega) & & \\ & R_a + p \bar{L}_S(p+j\omega) & \\ & & R_a + p L_o \end{bmatrix} \end{matrix} \quad (7.15)$$

and the new current matrix is

$$[\bar{i}(p)]' = \begin{matrix} & \begin{bmatrix} \bar{i}_1(p) \\ \bar{i}_2(p) \\ \bar{i}_0(p) \end{bmatrix} \\ \begin{matrix} 1 \\ 2 \\ 0 \end{matrix} & \end{matrix} \quad (7.16)$$

which is related to the old current matrix by the formula

$$[\bar{i}(p)] = [S] \cdot [\bar{i}(p)]' \quad (7.17)$$

The 1, 2 and 0 components, having been obtained by the symmetrical-component transformation, may be called the positive, the negative and the zero components respectively. They should be distinguished from the positive-, the negative- and the zero- sequence components as the sequence quantities apply only to steady-state sinusoidal quantities already expressed in symbolic form (complex number or electric vector).

It is clear from Equs. 7.13, 7.14 and 7.15 that the positive voltage and impedance components are the conjugate of the nega-

tive voltage and impedance components respectively, consequently the positive and negative current components, $\bar{i}_1(p)$ and $\bar{i}_2(p)$, are conjugate functions.

(7.4) Operational Solution in terms of 1, 2, 0 Components
— for particular form of external impedance matrix

Although Equ. 7.12 has a much simpler form than the original equation (Equ. 7.7), it is still not readily solvable owing to the presence of the functions $\bar{i}_1(p+j2\omega)$ and $\bar{i}_2(p-j2\omega)$ in the voltage matrix. Whether or not it can further be simplified depends on the form of the external impedance matrix. When the external impedance matrix is of the form

$$[\bar{Z}_{ex}(p)] = \begin{array}{c|cc} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{array}{ccc} \bar{Z}(p) & \bar{Z}_I(p) & \bar{Z}_{II}(p) \\ \bar{Z}_{II}(p) & \bar{Z}(p) & \bar{Z}_I(p) \\ \bar{Z}_I(p) & \bar{Z}_{II}(p) & \bar{Z}(p) \end{array} \end{array} \quad (7.18)$$

it can be diagonalised when expressed along the 1, 2, 0 axes as

$$[\bar{Z}_{ex}(p)]' = \begin{array}{c|cc} & \begin{matrix} 1 & 2 & 0 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 0 \end{matrix} & \begin{array}{ccc} \bar{Z}_{ex1}(p) & & \\ & \bar{Z}_{ex2}(p) & \\ & & \bar{Z}_{exo}(p) \end{array} \end{array} \quad (7.19)$$

where

$$\begin{aligned} \bar{Z}_{ex1}(p) &= \bar{Z}(p) + h^2 \bar{Z}_I(p) + h \bar{Z}_{II}(p) \\ \bar{Z}_{ex2}(p) &= \bar{Z}(p) + h \bar{Z}_I(p) + h^2 \bar{Z}_{II}(p) \\ \bar{Z}_{exo}(p) &= \bar{Z}(p) + \bar{Z}_I(p) + \bar{Z}_{II}(p) \end{aligned} \quad (7.20)$$

and the quantities $\bar{i}_1(p+j2\omega)$ and $\bar{i}_2(p-j2\omega)$ in the voltage matrix of Equ. 7.12 can be eliminated as shown below.

When the external impedances are included, $[\bar{Z}_{ex}(p)]'$ is

added to $\bar{Z}_{mc}(p)$ and the terminal voltages $\bar{v}_{1t}(p)$, $\bar{v}_{2t}(p)$ and $\bar{v}_{ot}(p)$ become the source voltages $\bar{v}_1(p)$, $\bar{v}_2(p)$ and $\bar{v}_o(p)$ respectively, $\bar{v}_1(p)$, $\bar{v}_2(p)$ and $\bar{v}_o(p)$ being given by the first three expressions of Equ. 7.14 with the subscript t in each term omitted. Equ. 7.12 thus becomes

$\bar{v}_1(p) + \bar{e}_1(p) - p\epsilon^{j2\lambda} \bar{L}_D(p-j\omega) \bar{i}_2(p-j2\omega)$
$\bar{v}_2(p) + \bar{e}_2(p) - p\epsilon^{-j2\lambda} \bar{L}_D(p+j\omega) \bar{i}_1(p+j2\omega)$
$\bar{v}_o(p)$

$$= \begin{array}{|c|c|c|} \hline R_a + \bar{Z}_{ex1}(p) + p\bar{L}_S(p-j\omega) & & \\ \hline & R_a + \bar{Z}_{ex2}(p) + p\bar{L}_S(p+j\omega) & \\ \hline & & R_a + \bar{Z}_{ex0}(p) + L_o p \\ \hline \end{array} \cdot \begin{array}{|c|} \hline \bar{i}_1(p) \\ \hline \bar{i}_2(p) \\ \hline \bar{i}_o(p) \\ \hline \end{array} \quad (7.21)$$

Putting $p-j2\omega$ in place p in the second equation and eliminating $\bar{i}_2(p-j2\omega)$ in the first equation of Equ. 7.21 and grouping terms gives the equation containing the positive component of current only:

$$\begin{aligned} \bar{v}_1(p) + \bar{e}_1(p) - \frac{p\epsilon^{j2\lambda} \bar{L}_D(p-j\omega)}{R_a + \bar{Z}_{ex2}(p-j2\omega) + (p-j2\omega)\bar{L}_S(p-j\omega)} \cdot \{ \bar{v}_2(p-j2\omega) + \bar{e}_2(p-j2\omega) \} \\ = \left\{ R_a + \bar{Z}_{ex1}(p) + p\bar{L}_S(p-j\omega) - \frac{p(p-j2\omega) \bar{L}_D(p-j\omega)}{R_a + \bar{Z}_{ex2}(p-j2\omega) + (p-j2\omega)\bar{L}_S(p-j\omega)} \right\} \bar{i}_1(p) \end{aligned} \quad (7.22)$$

Similarly, for the negative component of current,

$$\begin{aligned} \bar{v}_2(p) + \bar{e}_2(p) - \frac{p\epsilon^{-j2\lambda} \bar{L}_D(p+j\omega)}{R_a + \bar{Z}_{ex1}(p+j2\omega) + (p+j2\omega)\bar{L}_S(p+j\omega)} \cdot \{ \bar{v}_1(p+j2\omega) + \bar{e}_1(p+j2\omega) \} \\ = \left\{ R_a + \bar{Z}_{ex2}(p) + p\bar{L}_S(p+j\omega) - \frac{p(p+j\omega) \bar{L}_D(p+j\omega)}{R_a + \bar{Z}_{ex1}(p+j2\omega) + (p+j2\omega)\bar{L}_S(p+j\omega)} \right\} \bar{i}_2(p) \end{aligned} \quad (7.23)$$

The zero-component quantities are related by the third equation of Equ. 7.21 which is

$$\bar{v}_o(p) = \{ R_a + \bar{Z}_{ex0}(p) + L_o p \} \bar{i}_o(p) \quad (7.24)$$

Thus three independent equations (Eqs. 7.22, 7.23 and 7.24)

are obtained and the component currents can be found in operational form for given source voltages and external impedances. Since $\bar{i}_1(p)$ and $\bar{i}_2(p)$ are conjugate functions, only one of them needs to be evaluated. $\bar{i}_0(p)$ is generally equal to zero unless there is zero-component source voltage.

(7.5) $\alpha, \beta, 0$ Components — Equivalent two-phase machine

When the external impedance matrix is of the particular form given by Equ. 7.18, the three armature voltage equations have in effect been reduced to two equations (Eqs. 7.22 or 7.23 and 7.24) in terms of the 1, 2, 0 components. When the external impedance matrix does not fall into this category, it becomes impossible to separate the positive- and negative-component voltage equations. This is to be expected as there should result in three distinct equations (Eqs. 7.22 and 7.23 are not distinct as one of them can be derived from the other) whatever would be the transformation.

The most natural way of solving an equation with a complex variable and its conjugate function is to resolve the variable into its real and imaginary parts. This applies to the present case as there is a term involving $\bar{i}_2(p-j2\omega)$, conjugate of $\bar{i}_1(p+j2\omega)$, in the positive-component voltage equation (first equation of Equ. 7.12). Let $\bar{i}_\alpha(p)$ and $\bar{i}_\beta(p)$ be twice the real and imaginary parts of $\bar{i}_1(p)$. Thus

$$\bar{i}_1(p) = \frac{1}{2} [\bar{i}_\alpha(p) + j\bar{i}_\beta(p)] \quad (7.25)$$

$$\text{and } \bar{i}_2(p) = \frac{1}{2} [\bar{i}_\alpha(p) - j\bar{i}_\beta(p)]$$

Inverse-ly $\bar{i}_\alpha(p)$ and $\bar{i}_\beta(p)$ are given by

$$\bar{i}_\alpha(p) = \bar{i}_1(p) + \bar{i}_2(p)$$

(7.26)

$$\text{And } \bar{i}_\beta(p) = -j [\bar{i}_1(p) - \bar{i}_2(p)]$$

The direct transformation from the phase axes a, b and c to the α , β , 0 axes may be achieved by using the matrix [T] defined by

$$[T] = \begin{matrix} & \begin{matrix} 1 & 2 & 0 \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ h^2 & h & 1 \\ h & h^2 & 1 \end{bmatrix} \end{matrix} = \begin{matrix} & \begin{matrix} \alpha & \beta & 0 \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2}j & 0 \\ \frac{1}{2} & -\frac{1}{2}j & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix} = \begin{matrix} & \begin{matrix} \alpha & \beta & 0 \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 1 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 \end{bmatrix} \end{matrix} \quad (7.27)$$

The inverse of [T] is

$$[T]^{-1} = \frac{1}{3} \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} \alpha \\ \beta \\ 0 \end{matrix} & \begin{bmatrix} 2 & -1 & -1 \\ 0 & \sqrt{3} & -\sqrt{3} \\ 1 & 1 & 1 \end{bmatrix} \end{matrix} \quad (7.28)$$

Transforming Equ. 7.7 by using the matrix [T], or substituting Eqs. 7.25 in 7.12, the operational voltage equations in terms of the α , β , 0 components can be found as:

$$\begin{aligned} \bar{v}_{\alpha t}(p) + \bar{e}_\alpha(p) &= \left\{ R_a + \frac{1}{2} p [\bar{L}_s(p+j\omega) + \text{Conj}] \right\} \bar{i}_\alpha(p) - \frac{1}{2} p \left\{ j \bar{L}_s(p+j\omega) + \text{Conj} \right\} \bar{i}_\beta(p) \\ &\quad + \frac{1}{2} p \left\{ \epsilon^{-j2\lambda} \bar{L}_D(p+j\omega) \bar{i}_\alpha(p+j2\omega) + \text{Conj} \right\} \\ &\quad + \frac{1}{2} p \left\{ j \epsilon^{-j2\lambda} \bar{L}_D(p+j\omega) \bar{i}_\beta(p+j2\omega) + \text{Conj} \right\} \end{aligned} \quad (7.29)$$

$$\begin{aligned} \bar{v}_{\beta t}(p) + \bar{e}_\beta(p) &= \frac{1}{2} p \left\{ j \bar{L}_s(p+j\omega) + \text{Conj} \right\} \bar{i}_\alpha(p) + \left\{ R_a + \frac{1}{2} p [\bar{L}_s(p+j\omega) + \text{Conj}] \right\} \bar{i}_\beta(p) \\ &\quad + \frac{1}{2} p \left\{ j \epsilon^{-j2\lambda} \bar{L}_D(p+j\omega) \bar{i}_\alpha(p+j2\omega) + \text{Conj} \right\} \\ &\quad - \frac{1}{2} p \left\{ \epsilon^{-j2\lambda} \bar{L}_D(p+j\omega) \bar{i}_\beta(p+j2\omega) + \text{Conj} \right\} \end{aligned} \quad (7.30)$$

$$\text{and } \bar{v}_{0t}(p) = (R_a + L_0 p) \bar{i}_0(p) \quad (7.31)$$

The zero-component equation is unchanged. The α - and β -component equations are more complicated than the 1- and 2-component equations as there is mutual coupling between $\bar{i}_\alpha(p)$ and $\bar{i}_\beta(p)$.

One can see that Eqs. 7.29 and 7.39 are identical with the voltage equations of a symmetrical two-phase synchronous machine. Consequently replacing the a, b, c components by the α , β , 0 components is equivalent to replacing the three-phase armature by a two-phase armature and an independent zero-component network. The α and β components are known as, nothing more appropriately, the two-phase components. As a tentative suggestion, they may be called respectively the direct-phase and the quadrature-phase components in parallel to the direct-axis and the quadrature-axis components (the d and q components to be described later).

The chief advantage of using the α and β components is the possibility of eliminating all the mutuals between these two circuits whatever may be the external impedances. It is thus possible to obtain an equation with only one variable, say $\bar{i}_\alpha(p)$, and its derived functions, $\bar{i}_\alpha(p+j2\omega)$ and $\bar{i}_\alpha(p-j2\omega)$.

(7.6) General Operational Solution in terms of α , β , 0 Components

Provided that the external impedances are linear and static, the operational external impedance matrix may be expressed in terms of the α , β , 0 components in the general form

$$[\bar{z}_{ex}(p)] = \begin{array}{c} \alpha \\ \beta \\ 0 \end{array} \begin{array}{|c|c|c|} \hline \alpha & \beta & 0 \\ \hline \bar{z}_{\alpha\alpha}(p) & \bar{z}_{\alpha\beta}(p) & \bar{z}_{\alpha 0}(p) \\ \hline \bar{z}_{\beta\alpha}(p) & \bar{z}_{\beta\beta}(p) & \bar{z}_{\beta 0}(p) \\ \hline \bar{z}_{0\alpha}(p) & \bar{z}_{0\beta}(p) & \bar{z}_{00}(p) \\ \hline \end{array} \quad (7.32)$$

Using the relation $[v_t] = [v] - [Z_{ex}] \cdot [i]$ and substituting the values of $\bar{v}_t(p)$, $\bar{v}_t(p)$ and $\bar{v}_{ot}(p)$ in Eqs. 7.29, 7.30 and 7.31 and eliminating $\bar{i}_o(p)$ gives

$$\bar{f}_\alpha(p) = \bar{Z}_\alpha(p) \bar{I}_\alpha(p) + \bar{Z}_{\alpha\beta}(p) \bar{I}_\beta(p) + \frac{1}{2} \epsilon^{-j2\lambda} \bar{M}_{\alpha\alpha}(p) \{ \bar{I}_\alpha(p+j2\omega) + j \bar{I}_\beta(p+j2\omega) \} + \frac{1}{2} \epsilon^{j2\lambda} \bar{M}_{\alpha\alpha}^*(p) \{ \bar{I}_\alpha(p-j2\omega) - j \bar{I}_\beta(p-j2\omega) \} \quad (7.33)$$

$$\bar{f}_\beta(p) = \bar{Z}_{\beta\alpha}(p) \bar{I}_\alpha(p) + \bar{Z}_\beta(p) \bar{I}_\beta(p) + \frac{1}{2} j \epsilon^{-j2\lambda} \bar{M}_{\alpha\alpha}(p) \{ \bar{I}_\alpha(p+j2\omega) + j \bar{I}_\beta(p+j2\omega) \} - \frac{1}{2} j \epsilon^{j2\lambda} \bar{M}_{\alpha\alpha}^*(p) \{ \bar{I}_\alpha(p-j2\omega) - j \bar{I}_\beta(p-j2\omega) \} \quad (7.34)$$

where

$$\left. \begin{aligned} \bar{f}_\alpha(p) &= \bar{e}_\alpha(p) + \bar{v}_\alpha(p) - \frac{\bar{f}_{\alpha o}(p) \cdot \bar{v}_o(p)}{R_a + p L_o + \bar{f}_{oo}(p)} \\ \bar{f}_\beta(p) &= \bar{e}_\beta(p) + \bar{v}_\beta(p) - \frac{\bar{f}_{\beta o}(p) \cdot \bar{v}_o(p)}{R_a + p L_o + \bar{f}_{oo}(p)} \\ \bar{Z}_\alpha(p) &= R_a + \frac{1}{2} p [\bar{L}_s(p+j\omega) + \text{Conj}] - \frac{\bar{f}_{\alpha o}(p) \bar{f}_{\alpha o}(p)}{R_a + p L_o + \bar{f}_{oo}(p)} + \bar{f}_{\alpha\alpha}(p) \\ \bar{Z}_\beta(p) &= R_a + \frac{1}{2} p [\bar{L}_s(p+j\omega) + \text{Conj}] - \frac{\bar{f}_{\beta o}(p) \bar{f}_{\beta o}(p)}{R_a + p L_o + \bar{f}_{oo}(p)} + \bar{f}_{\beta\beta}(p) \\ \bar{Z}_{\alpha\beta}(p) &= -\frac{1}{2} p [j \bar{L}_s(p+j\omega) + \text{Conj}] - \frac{\bar{f}_{\alpha o}(p) \bar{f}_{\beta o}(p)}{R_a + p L_o + \bar{f}_{oo}(p)} + \bar{f}_{\alpha\beta}(p) \\ \bar{Z}_{\beta\alpha}(p) &= \frac{1}{2} p [j \bar{L}_s(p+j\omega) + \text{Conj}] - \frac{\bar{f}_{\beta o}(p) \bar{f}_{\alpha o}(p)}{R_a + p L_o + \bar{f}_{oo}(p)} + \bar{f}_{\beta\alpha}(p) \\ \bar{M}_{\alpha\alpha}(p) &= p \bar{L}_D(p+j\omega) \\ \bar{M}_{\alpha\alpha}^*(p) &= p \bar{L}_D(p-j\omega) \end{aligned} \right\} \quad (7.35)$$

Multiplying Equ. 7.34 by j and adding it to Equ. 7.33, then putting $p+j2\omega$ in place of p in the resultant equation gives

$$\bar{f}_\alpha(p+j2\omega) + j \bar{f}_\beta(p+j2\omega) = \{ \bar{Z}_\alpha(p+j2\omega) + j \bar{Z}_{\beta\alpha}(p+j2\omega) \} \bar{I}_\alpha(p+j2\omega) + j \{ \bar{Z}_\beta(p+j2\omega) - j \bar{Z}_{\alpha\beta}(p+j2\omega) \} \bar{I}_\beta(p+j2\omega) + \epsilon^{j2\lambda} \bar{M}_{\alpha\alpha}^*(p+j2\omega) \{ \bar{I}_\alpha(p) - j \bar{I}_\beta(p) \} \quad (7.36)$$

The conjugate equation of Equ. 7.36 is

$$\bar{f}_\alpha(p-j2\omega) - j \bar{f}_\beta(p-j2\omega) = \{ \bar{Z}_\alpha(p-j2\omega) - j \bar{Z}_{\beta\alpha}(p-j2\omega) \} \bar{I}_\alpha(p-j2\omega) - j \{ \bar{Z}_\beta(p-j2\omega) + j \bar{Z}_{\alpha\beta}(p-j2\omega) \} \bar{I}_\beta(p-j2\omega) + \epsilon^{-j2\lambda} \bar{M}_{\alpha\alpha}(p-j2\omega) \{ \bar{I}_\alpha(p) + j \bar{I}_\beta(p) \} \quad (7.37)$$

Eliminating $\bar{I}_\beta(p+j2\omega)$ and $\bar{I}_\beta(p-j2\omega)$ from Eqs. 7.33 and 7.34 by using Eqs. 7.36 and 7.37 gives

$$\bar{f}'_{\alpha}(p) = \bar{z}'_{\alpha}(p) \bar{t}_{\alpha}(p) + \bar{z}'_{\alpha\beta}(p) \bar{t}_{\beta}(p) + \frac{1}{2} e^{-j2\lambda} \bar{M}'_{\alpha\alpha}(p) \bar{t}_{\alpha}(p+j2\omega) + \frac{1}{2} e^{j2\lambda} \bar{M}'_{\alpha\alpha}^*(p) \bar{t}_{\alpha}(p-j2\omega) \quad (7.38)$$

$$\bar{f}'_{\beta}(p) = \bar{z}'_{\beta\alpha}(p) \bar{t}_{\alpha}(p) + \bar{z}'_{\beta}(p) \bar{t}_{\beta}(p) + \frac{1}{2} j e^{-j2\lambda} \bar{M}'_{\alpha\alpha}(p) \bar{t}_{\alpha}(p+j2\omega) - \frac{1}{2} j e^{j2\lambda} \bar{M}'_{\alpha\alpha}^*(p) \bar{t}_{\alpha}(p-j2\omega) \quad (7.39)$$

where

$$\left. \begin{aligned} \bar{f}'_{\alpha}(p) &= \bar{f}_{\alpha}(p) - \frac{1}{2} \cdot \frac{e^{-j2\lambda} [\bar{f}_{\alpha}(p+j2\omega) + j \bar{f}_{\beta}(p+j2\omega)] \cdot \bar{M}_{\alpha\alpha}(p)}{\bar{z}_{\beta}(p+j2\omega) - j \bar{z}_{\alpha\beta}(p+j2\omega)} - \text{Conj.} \\ \bar{f}'_{\beta}(p) &= \bar{f}_{\beta}(p) - \frac{1}{2} \cdot \frac{j e^{-j2\lambda} [\bar{f}_{\alpha}(p+j2\omega) + j \bar{f}_{\beta}(p+j2\omega)] \cdot \bar{M}_{\alpha\alpha}(p)}{\bar{z}_{\beta}(p+j2\omega) - j \bar{z}_{\alpha\beta}(p+j2\omega)} - \text{Conj.} \\ \bar{z}'_{\alpha}(p) &= \bar{z}_{\alpha}(p) - \frac{1}{2} \cdot \frac{\bar{M}_{\alpha\alpha}(p) \cdot \bar{M}_{\alpha\alpha}^*(p+j2\omega)}{\bar{z}_{\beta}(p+j2\omega) - j \bar{z}_{\alpha\beta}(p+j2\omega)} - \text{Conj.} \\ \bar{z}'_{\beta}(p) &= \bar{z}_{\beta}(p) - \frac{1}{2} \cdot \frac{\bar{M}_{\alpha\alpha}(p) \cdot \bar{M}_{\alpha\alpha}^*(p+j2\omega)}{\bar{z}_{\beta}(p+j2\omega) - j \bar{z}_{\alpha\beta}(p+j2\omega)} - \text{Conj.} \\ \bar{z}'_{\alpha\beta}(p) &= \bar{z}_{\alpha\beta}(p) + \frac{1}{2} \cdot \frac{j \bar{M}_{\alpha\alpha}(p) \cdot \bar{M}_{\alpha\alpha}^*(p+j2\omega)}{\bar{z}_{\beta}(p+j2\omega) - j \bar{z}_{\alpha\beta}(p+j2\omega)} + \text{Conj.} \\ \bar{z}'_{\beta\alpha}(p) &= \bar{z}_{\beta\alpha}(p) - \frac{1}{2} \cdot \frac{j \bar{M}_{\alpha\alpha}(p) \cdot \bar{M}_{\alpha\alpha}^*(p+j2\omega)}{\bar{z}_{\beta}(p+j2\omega) - j \bar{z}_{\alpha\beta}(p+j2\omega)} - \text{Conj.} \\ \bar{M}'_{\alpha\alpha}(p) &= \bar{M}_{\alpha\alpha}(p) \left\{ 1 - \frac{\bar{z}_{\alpha}(p+j2\omega) + j \bar{z}_{\beta\alpha}(p+j2\omega)}{\bar{z}_{\beta}(p+j2\omega) - j \bar{z}_{\alpha\beta}(p+j2\omega)} \right\} \\ \bar{M}'_{\alpha\alpha}^*(p) &= \bar{M}_{\alpha\alpha}^*(p) \left\{ 1 - \frac{\bar{z}_{\alpha}(p-j2\omega) - j \bar{z}_{\beta\alpha}(p-j2\omega)}{\bar{z}_{\beta}(p-j2\omega) + j \bar{z}_{\alpha\beta}(p-j2\omega)} \right\} \end{aligned} \right\} \quad (7.40)$$

Eliminating $\bar{i}_{\beta}(p)$ from Eqs. 7.38 and 7.39 gives

$$\bar{F}_{\alpha}(p) = \bar{t}_{\alpha}(p) + \frac{1}{2} e^{-j2\lambda} \bar{K}_{\alpha}(p) \bar{t}_{\alpha}(p+j2\omega) + \frac{1}{2} e^{j2\lambda} \bar{K}_{\alpha}^*(p) \bar{t}_{\alpha}(p-j2\omega) \quad (7.41)$$

where

$$\left. \begin{aligned} \bar{F}_{\alpha}(p) &= \frac{\bar{z}'_{\beta}(p) \bar{f}'_{\alpha}(p) - \bar{z}'_{\alpha\beta}(p) \bar{f}'_{\beta}(p)}{\bar{z}'_{\alpha}(p) \bar{z}'_{\beta}(p) - \bar{z}'_{\alpha\beta}(p) \bar{z}'_{\beta\alpha}(p)} \\ \bar{K}_{\alpha}(p) &= \frac{[\bar{z}'_{\beta}(p) - j \bar{z}'_{\alpha\beta}(p)] \cdot \bar{M}'_{\alpha\alpha}(p)}{\bar{z}'_{\alpha}(p) \bar{z}'_{\beta}(p) - \bar{z}'_{\alpha\beta}(p) \bar{z}'_{\beta\alpha}(p)} \\ \bar{K}_{\alpha}^*(p) &= \frac{[\bar{z}'_{\beta}(p) + j \bar{z}'_{\alpha\beta}(p)] \cdot \bar{M}'_{\alpha\alpha}^*(p)}{\bar{z}'_{\alpha}(p) \bar{z}'_{\beta}(p) - \bar{z}'_{\alpha\beta}(p) \bar{z}'_{\beta\alpha}(p)} = \text{Conjugate of } \bar{K}_{\alpha}(p) \end{aligned} \right\} \quad (7.42)$$

Equ. 7.41 involves only one variable, $\bar{i}_{\alpha}(p)$, and its derived functions $\bar{i}_{\alpha}(p+j2\omega)$ and $\bar{i}_{\alpha}(p-j2\omega)$, so that it is a kind of recurrence equations and its general solution can be found. To solved this equation, let the form of its solution be first considered. If p is replaced by $p \pm j2\omega$ in Equ. 7.41 and the resultant expressions are substituted in the original equation,

a new equation having terms involving $\bar{F}_\alpha(p)$, $\bar{F}_\alpha(p \pm j2\omega)$, $\bar{i}_\alpha(p)$ and $\bar{i}_\alpha(p \pm j4\omega)$ will be obtained. Continuing the same process an infinite number of new equations will result with the general one involving $\bar{F}_\alpha(p)$, $\bar{F}_\alpha(p \pm j2n\omega)$, $\bar{i}_\alpha(p)$ and $\bar{i}_\alpha(p \pm j2n\omega)$. It can be concluded, therefore, that the solution of Equ. 7.41 has the form

$$\bar{i}_\alpha(p) = \sum_{n=-\infty}^{\infty} \bar{A}_n(p) \bar{F}_\alpha(p + j2n\omega) \quad n = \text{integer} \quad (7.43)$$

A formal proof for the validity of Equ. 7.43 can be found in the Appendix (Article 14.3).

Substituting this expression for $\bar{i}_\alpha(p)$ in Equ. 7.42 and equating coefficients gives

$$\bar{A}_n(p) + \frac{1}{2} e^{-j2\lambda} \bar{K}_\alpha(p) \bar{A}_{n-1}(p+j2\omega) + \frac{1}{2} e^{j2\lambda} \bar{K}_\alpha^*(p) \bar{A}_{n+1}(p-j2\omega) = 0 \quad (7.44)$$

except for $n = 0$ when

$$\bar{A}_0(p) + \frac{1}{2} e^{-j2\lambda} \bar{K}_\alpha(p) \bar{A}_{-1}(p+j2\omega) + \frac{1}{2} e^{j2\lambda} \bar{K}_\alpha^*(p) \bar{A}_1(p-j2\omega) = 1 \quad (7.45)$$

From the known physical phenomenon that generally the magnitudes of the current components decrease with increasing frequencies, the condition of convergence

$$\lim_{n \rightarrow \infty} \bar{A}_n(p) = 0$$

can be imposed on Equ. 7.44. When n is positive and is sufficiently large, the last term in Equ. 7.44 may be treated as a remainder and omitted for the moment, thus leaving

$$\bar{A}_n(p) + \frac{1}{2} e^{-j2\lambda} \bar{K}_\alpha(p) \bar{A}_{n-1}(p+j2\omega) \approx 0 \quad (7.46)$$

Replacing n by $n - 1$ in Equ. 7.44 and eliminating $\bar{A}_n(p-j2\omega)$ by using Equ. 7.46 gives

$$\left\{ 1 - \frac{1}{2} \bar{K}_\alpha^*(p) \bar{K}_\alpha(p-j2\omega) \right\} \bar{A}_{n-1}(p) + \frac{1}{2} e^{-j2\lambda} \bar{K}_\alpha(p) \bar{A}_{n-2}(p+j2\omega) = 0 \quad (7.47)$$

which is similar to Equ. 7.46 except the coefficient of the

first term. Similarly, replacing n by $n - 2$ in Equ. 7.44 and eliminating $\bar{A}_{n-1}(p-j2\omega)$ by using Equ. 7.47 gives

$$\left\{ 1 - \frac{\frac{1}{2}\bar{K}_\alpha^*(p)\bar{K}_\alpha(p-j2\omega)}{1 - \frac{1}{2}\bar{K}_\alpha^*(p-j2\omega)\bar{K}_\alpha(p-j4\omega)} \right\} \bar{A}_{n-2}(p) + \frac{1}{2}e^{-j2\lambda}\bar{K}_\alpha(p)\bar{A}_{n-3}(p+j2\omega) = 0 \quad (7.48)$$

Repeating the process shows that the correct form of Equ. 7.46 should be

$$\bar{x}_\alpha(p) \bar{A}_n(p) + \frac{1}{2}e^{-j2\lambda}\bar{K}_\alpha(p)\bar{A}_{n-1}(p+j2\omega) = 0 \quad n = \text{non-zero positive integer} \quad (7.49)$$

$$\text{where } \bar{x}_\alpha(p) = 1 - \frac{\frac{1}{2}\bar{K}_\alpha^*(p)\bar{K}_\alpha(p-j2\omega)}{\bar{x}_\alpha(p-j2\omega)}$$

$$= 1 - \frac{\frac{1}{2}\bar{K}_\alpha^*(p)\bar{K}_\alpha(p-j2\omega)}{1 - \frac{\frac{1}{2}\bar{K}_\alpha^*(p-j2\omega)\bar{K}_\alpha(p-j4\omega)}{1 - \dots}} \quad (7.50)$$

Equ. 7.49 is a rigorous alternative expression of Equ. 7.44. The proof is given in Appendix 14.3.

Similarly, for negative values of n ,

$$\bar{x}_\alpha^*(p)\bar{A}_n(p) + \frac{1}{2}e^{j2\lambda}\bar{K}_\alpha^*(p)\bar{A}_{n+1}(p-j2\omega) = 0 \quad n = \text{non-zero negative integer} \quad (7.51)$$

Or, changing the sign of n ,

$$\bar{x}_\alpha^*(p)\bar{A}_{-n}(p) + \frac{1}{2}e^{j2\lambda}\bar{K}_\alpha^*(p)\bar{A}_{-n+1}(p-j2\omega) = 0 \quad n = \text{non-zero positive integer} \quad (7.52)$$

where $\bar{x}_\alpha^*(p)$ = Conjugate function of $\bar{x}_\alpha(p)$

$$= 1 - \frac{\frac{1}{2}\bar{K}_\alpha(p)\bar{K}_\alpha^*(p+j2\omega)}{\bar{x}_\alpha^*(p+j2\omega)} \quad (7.53)$$

Putting $n = 1$ in Eqs. 7.49 and 7.52 and substituting them in Equ. 7.45 to eliminate $\bar{A}_{-1}(p+j2\omega)$ and $\bar{A}_1(p-j2\omega)$ gives

$$\bar{A}_0(p) = 1 / \{ \bar{x}_\alpha(p) + \bar{x}_\alpha^*(p) - 1 \} \quad (7.54)$$

Repeated use of Eqs. 7.49 and 7.52 gives

$$\bar{A}_n(p) = \left(-\frac{1}{2}\right)^n e^{-j2n\lambda} \bar{A}_0(p+j2n\omega) \prod_{\nu=0}^{n-1} \frac{\bar{K}_\alpha(p+j2\nu\omega)}{\bar{x}_\alpha(p+j2\nu\omega)} \quad (7.55)$$

and

$$\bar{A}_{-n}(p) = \left(-\frac{1}{2}\right)^n \epsilon^{j2n\lambda} \bar{A}_0(p+j2n\omega) \prod_{\nu=0}^{n-1} \frac{\bar{K}_\alpha^*(p-j2\nu\omega)}{\bar{z}_\alpha^*(p-j2\nu\omega)} \quad (7.56)$$

Finally the solution of Equ. 7.41 is

$$\bar{i}_\alpha(p) = \bar{A}_0(p)\bar{F}_\alpha(p) + \sum_{n=1}^{\infty} \left\{ \bar{A}_n(p)\bar{F}_\alpha(p+j2n\omega) + \bar{A}_{-n}(p)\bar{F}_\alpha(p-j2n\omega) \right\} \quad (7.57)$$

Since $\bar{A}_n(p)$ and $\bar{A}_{-n}(p)$ are conjugate functions, the two infinite series in Equ. 7.57 are conjugates so long as $\bar{F}_\alpha(p)$ is real, which is often true. Consequently it is generally sufficient to evaluate only one of the series.

Like Equ. 7.41 for the α component, a similar equation for the β component can be derived. If, using Eqs. 7.36 and 7.37, $\bar{i}_\alpha(p+j2\omega)$ and its conjugate are eliminated in Eqs. 7.33 and 7.34 instead of $\bar{i}_\beta(p+j2\omega)$ and its conjugate, then corresponding to Eqs. 7.38 and 7.39 are the following equations:

$$\bar{f}_\alpha''(p) = \bar{z}_\alpha''(p)\bar{t}_\alpha(p) + \bar{z}_{\alpha\beta}''(p)\bar{t}_\beta(p) + \frac{1}{2}j\epsilon^{-j2\lambda}\bar{M}_{\alpha\alpha}''(p)\bar{t}_\beta(p+j2\omega) - \frac{1}{2}j\epsilon^{j2\lambda}\bar{M}_{\alpha\alpha}''^*(p)\bar{t}_\beta(p-j2\omega) \quad (7.58)$$

$$\bar{f}_\beta''(p) = \bar{z}_{\beta\alpha}''(p)\bar{t}_\alpha(p) + \bar{z}_\beta''(p)\bar{t}_\beta(p) - \frac{1}{2}\epsilon^{-j2\lambda}\bar{M}_{\alpha\alpha}''(p)\bar{t}_\beta(p+j2\omega) - \frac{1}{2}\epsilon^{j2\lambda}\bar{M}_{\alpha\alpha}''^*(p)\bar{t}_\beta(p-j2\omega) \quad (7.59)$$

where

$$\left. \begin{aligned} \bar{f}_\alpha''(p) &= \bar{f}_\alpha(p) - \frac{1}{2} \cdot \frac{\epsilon^{-j2\lambda} [\bar{f}_\alpha(p+j2\omega) + j\bar{f}_\beta(p+j2\omega)] \cdot \bar{M}_{\alpha\alpha}(p)}{\bar{z}_\alpha(p+j2\omega) + j\bar{z}_{\beta\alpha}(p+j2\omega)} - \text{Conj} \\ \bar{f}_\beta''(p) &= \bar{f}_\beta(p) - \frac{1}{2} \cdot \frac{j\epsilon^{j2\lambda} [\bar{f}_\alpha(p+j2\omega) + j\bar{f}_\beta(p+j2\omega)] \cdot \bar{M}_{\alpha\alpha}(p)}{\bar{z}_\alpha(p+j2\omega) + j\bar{z}_{\beta\alpha}(p+j2\omega)} - \text{Conj} \\ \bar{z}_\alpha''(p) &= \bar{z}_\alpha(p) - \frac{1}{2} \cdot \frac{\bar{M}_{\alpha\alpha}(p) \cdot \bar{M}_{\alpha\alpha}^*(p+j2\omega)}{\bar{z}_\alpha(p+j2\omega) + j\bar{z}_{\beta\alpha}(p+j2\omega)} - \text{Conj} \\ \bar{z}_\beta''(p) &= \bar{z}_\beta(p) - \frac{1}{2} \cdot \frac{\bar{M}_{\alpha\alpha}(p) \cdot \bar{M}_{\alpha\alpha}^*(p+j2\omega)}{\bar{z}_\alpha(p+j2\omega) + j\bar{z}_{\beta\alpha}(p+j2\omega)} - \text{Conj} \\ \bar{z}_{\alpha\beta}''(p) &= \bar{z}_{\alpha\beta}(p) - \frac{1}{2} \cdot \frac{j\bar{M}_{\alpha\alpha}(p) \cdot \bar{M}_{\alpha\alpha}^*(p+j2\omega)}{\bar{z}_\alpha(p+j2\omega) + j\bar{z}_{\beta\alpha}(p+j2\omega)} + \text{Conj} \\ \bar{z}_{\beta\alpha}''(p) &= \bar{z}_{\beta\alpha}(p) - \frac{1}{2} \cdot \frac{j\bar{M}_{\alpha\alpha}(p) \cdot \bar{M}_{\alpha\alpha}^*(p+j2\omega)}{\bar{z}_\alpha(p+j2\omega) + j\bar{z}_{\beta\alpha}(p+j2\omega)} - \text{Conj} \\ \bar{M}_{\alpha\alpha}''(p) &= \bar{M}_{\alpha\alpha}(p) \left\{ 1 - \frac{\bar{z}_\beta(p+j2\omega) - j\bar{z}_{\alpha\beta}(p+j2\omega)}{\bar{z}_\alpha(p+j2\omega) + j\bar{z}_{\beta\alpha}(p+j2\omega)} \right\} \\ \bar{M}_{\alpha\alpha}''^*(p) &= \bar{M}_{\alpha\alpha}^*(p) \left\{ 1 - \frac{\bar{z}_\beta(p-j2\omega) + j\bar{z}_{\alpha\beta}(p-j2\omega)}{\bar{z}_\alpha(p-j2\omega) - j\bar{z}_{\beta\alpha}(p-j2\omega)} \right\} \end{aligned} \right\} \quad (7.60)$$

Eliminating $\bar{i}_\alpha(p)$ in Eqs. 7.58 and 7.59 gives

$$\bar{F}_\beta(p) = \bar{i}_\beta(p) + \frac{1}{2} \epsilon^{-j\lambda} \bar{K}_\beta(p) \bar{i}_\beta(p+j\omega) + \frac{1}{2} \epsilon^{j\lambda} \bar{K}_\beta^*(p) \bar{i}_\beta(p-j\omega) \quad (7.61)$$

$$\begin{aligned} \text{where} \quad \bar{F}_\beta(p) &= \frac{\bar{Z}_\alpha''(p) \bar{f}_\beta''(p) - \bar{Z}_\beta''(p) \bar{f}_\alpha''(p)}{\bar{Z}_\alpha''(p) \bar{Z}_\beta''(p) - \bar{Z}_{\alpha\beta}''(p) \bar{Z}_{\beta\alpha}''(p)} \\ \bar{K}_\beta(p) &= - \frac{[\bar{Z}_\alpha''(p) + j \bar{Z}_{\beta\alpha}''(p)] \cdot \bar{M}_{\alpha\alpha}''(p)}{\bar{Z}_\alpha''(p) \bar{Z}_\beta''(p) - \bar{Z}_{\alpha\beta}''(p) \bar{Z}_{\beta\alpha}''(p)} \\ \bar{K}_\beta^*(p) &= - \frac{[\bar{Z}_\alpha''(p) - j \bar{Z}_{\beta\alpha}''(p)] \cdot \bar{M}_{\alpha\alpha}''^*(p)}{\bar{Z}_\alpha''(p) \bar{Z}_\beta''(p) - \bar{Z}_{\alpha\beta}''(p) \bar{Z}_{\beta\alpha}''(p)} = \text{Conjugate of } \bar{K}_\beta(p) \end{aligned} \quad (7.62)$$

Equ. 7.61 has exactly the same form as Equ. 7.41 and its solution is obtainable from Equ. 7.57 with $\bar{F}_\alpha(p)$, $\bar{K}_\alpha(p)$ and $\bar{K}_\alpha^*(p)$ replaced by $\bar{F}_\beta(p)$, $\bar{K}_\beta(p)$ and $\bar{K}_\beta^*(p)$ respectively.

(7.7) Other Important Reference Axes

In addition to the phase axes a, b, c, the symmetrical-component axes 1, 2, 0 and the two-phase-component axes α , β , 0 which have already been dealt with, there are two other sets of reference axes in common use. They are the F, B, 0 component and the d, q, 0 component axes. The F, B, 0 component axes can be defined readily from the 1, 2, 0 axes and the d, q, 0 component axes from the F, B, 0 axes.

The current components $\bar{i}_F(p)$ and $\bar{i}_B(p)$ can be defined as

$$\bar{i}_F(p) = \epsilon^{-j\lambda} \bar{i}_1(p+j\omega) \quad (7.63)$$

$$\text{and} \quad \bar{i}_B(p) = \epsilon^{j\lambda} \bar{i}_2(p-j\omega)$$

Since $\bar{i}_1(p)$ and $\bar{i}_2(p)$ are conjugate functions, so are $\bar{i}_F(p)$ and $\bar{i}_B(p)$. Expressed as functions of time, Eqs. 7.63 become

$$i_F(t) = \epsilon^{-j\theta} i_1(t) \quad (7.64)$$

$$\text{and} \quad i_B(t) = \epsilon^{j\theta} i_2(t)$$

where $\theta = \omega t + \lambda$.

The phase components and the F, B, O components are related by, in operational form,

$$\begin{bmatrix} \tau_a(p) \\ \tau_b(p) \\ \tau_c(p) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ h^2 & h & 1 \\ h & h^2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \epsilon^{j\lambda} \tau_F(p-j\omega) \\ \epsilon^{-j\lambda} \tau_B(p+j\omega) \\ \tau_o(p) \end{bmatrix} \quad (7.65)$$

or in explicit form,

$$\begin{bmatrix} i_a(t) \\ i_b(t) \\ i_c(t) \end{bmatrix} = \begin{bmatrix} \epsilon^{j\theta} & \epsilon^{-j\theta} & 1 \\ \epsilon^{j(\theta-\frac{2\pi}{3})} & \epsilon^{-j(\theta-\frac{2\pi}{3})} & 1 \\ \epsilon^{j(\theta+\frac{2\pi}{3})} & \epsilon^{-j(\theta+\frac{2\pi}{3})} & 1 \end{bmatrix} \cdot \begin{bmatrix} i_F(t) \\ i_B(t) \\ i_o(t) \end{bmatrix} \quad (7.66)$$

The current components $\bar{i}_d(p)$ and $\bar{i}_q(p)$ can be considered as twice the real and imaginary parts of $\bar{i}_F(p)$, just as $\bar{i}_\alpha(p)$ and $\bar{i}_\beta(p)$ have been considered as twice the real and imaginary parts of $\bar{i}_1(p)$. Thus,

$$\bar{i}_F(p) = \frac{1}{2} \{ \bar{i}_d(p) + j\bar{i}_q(p) \} \quad (7.67)$$

$$\text{and } \bar{i}_B(p) = \frac{1}{2} \{ \bar{i}_d(p) - j\bar{i}_q(p) \}$$

Consequently $\bar{i}_d(p)$ and $\bar{i}_q(p)$ are given by

$$\bar{i}_d(p) = \bar{i}_F(p) + \bar{i}_B(p) \quad (7.68)$$

$$\text{and } \bar{i}_q(p) = -j \{ \bar{i}_F(p) - \bar{i}_B(p) \}$$

The phase components and the d, q, O components are related by, using Eqs. 7.65 and 7.67,

$$\begin{bmatrix} \tau_a(p) \\ \tau_b(p) \\ \tau_c(p) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2}j & -\frac{1}{2}j & 1 \\ \frac{1}{2}h^2 & \frac{1}{2}h & \frac{1}{2}jh^2 & -\frac{1}{2}jh & 1 \\ \frac{1}{2}h & \frac{1}{2}h^2 & \frac{1}{2}jh & -\frac{1}{2}jh^2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \epsilon^{j\lambda} \tau_d(p-j\omega) \\ \epsilon^{-j\lambda} \tau_d(p+j\omega) \\ \epsilon^{j\lambda} \tau_q(p-j\omega) \\ \epsilon^{-j\lambda} \tau_q(p+j\omega) \\ \tau_o(p) \end{bmatrix} \quad (7.69)$$

The inverse transform of Equ. 7.69 gives the relation between the phase components and the d, q, 0 components in explicit form:

$$\begin{bmatrix} i_a(t) \\ i_b(t) \\ i_c(t) \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 1 \\ \cos(\theta - \frac{2\pi}{3}) & -\sin(\theta - \frac{2\pi}{3}) & 1 \\ \cos(\theta + \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) & 1 \end{bmatrix} \cdot \begin{bmatrix} i_d(t) \\ i_q(t) \\ i_o(t) \end{bmatrix} \quad (7.70)$$

The difference between the 1, 2 and α , β axes and the F, B and d, q axes is entirely the result of the manner in which they are fixed in space. The 1, 2 and α , β axes are stationary with respect to the armature whereas the F, B and d, q axes are stationary with respect to the field structure and consequently rotating with respect to the armature. Each sort of axes, stationary or rotating, has its own merit in application.

As far as the solution of the armature currents is concerned, the 1, 2 and α , β axes are more advantageous. As a matter of fact, only when the currents are expressed along the α , β , 0 axes can the general operational solution like Equ. 7.57 be obtained.

The d, q components, being fixed to the field structure, have a more direct relation with the field and damper currents. Furthermore, the electromagnetic torque can be expressed in a simpler way when the currents are expressed along the d, q axes. These two points will be illustrated in the next two articles.

Because of the readiness the different sets of reference axes can be interchanged, each of the armature voltage equation, the field and damper voltage equations and the electromagnetic

torque equation need not be expressed along the same set of reference axes but can always be expressed in its most convenient form for any particular operating condition of the machine.

(7.8) Field and Damper Currents

So far only the armature currents have been dealt with and the field and damper currents have been missing as they have been eliminated from the beginning. In order to find the field and damper currents, it is necessary to go back to Equ. 7.1 where the field and damper voltage equations still exist.

The operational form of the field voltage equation is

$$0 = \frac{1}{2} p M_{af} \{ e^{-j\lambda} \tau_a(p+j\omega) + \text{conj} + h e^{-j\lambda} \tau_b(p+j\omega) + \text{conj} + h^2 e^{-j\lambda} \tau_c(p+j\omega) + \text{conj} \} \\ + (R_f + L_f p) \bar{i}'_f(p) + p M_{fkd} \bar{i}_{kd}(p) \quad (7.71)$$

Or, introducing i_1 and i_2 , it reduces to

$$0 = \frac{3}{2} p M_{af} \{ e^{-j\lambda} \tau_1(p+j\omega) + e^{j\lambda} \tau_2(p-j\omega) \} + (R_f + L_f p) \bar{i}'_f(p) + p M_{fkd} \bar{i}_{kd}(p) \quad (7.72)$$

Or, introducing i_d (Eqs. 7.63 and 7.68), it further reduces to

$$0 = \frac{3}{2} p M_{af} \tau_d(p) + (R_f + L_f p) \bar{i}'_f(p) + p M_{fkd} \bar{i}_{kd}(p) \quad (7.73)$$

Similarly the operational damper voltage equations are

$$0 = \frac{3}{2} p M_{af} \tau_d(p) + p M_{fkd} \bar{i}'_f(p) + (R_{kd} + L p) \bar{i}_{kd}(p) \quad (7.74)$$

$$\text{and } 0 = \frac{3}{2} p M_{aq} \tau_q(p) + (R_{kq} + L_{kq} p) \bar{i}_{kq}(p) \quad (7.75)$$

When the armature currents have been obtained in terms of whichever set of components considered and proved to be the most convenient, $\bar{i}_d(p)$ and $\bar{i}_q(p)$ can be determined. The field current $\bar{i}'_f(p)$ and the two damper currents $\bar{i}_{kd}(p)$ and $\bar{i}_{kq}(p)$ can be found by solving the equations (7.73, 7.74 and 7.75) above and expressed in terms of $\bar{i}_d(p)$ and $\bar{i}_q(p)$. Thus, solving

Equs. 7.73 and 7.74 simultaneously and using the symbols defined in Equ. 7.10 gives the currents $\bar{i}_f(p)$ and $\bar{i}_{kd}(p)$ as

$$\bar{i}_f(p) = -\sqrt{\frac{3}{2}} \cdot M_d \cdot \frac{R_{kd} + (L_{kd} - M_{fkd})p}{R_f + R_{kd} + (L_f + L_{kd} - 2M_{fkd})p} \cdot \frac{p \bar{I}_d(p)}{\bar{Z}_d(p)} \quad (7.76)$$

and $\bar{i}_{kd}(p) = -\sqrt{\frac{3}{2}} \cdot M_d \cdot \frac{R_f + (L_f - M_{fkd})p}{R_f + R_{kd} + (L_f + L_{kd} - 2M_{fkd})p} \cdot \frac{p \bar{I}_d(p)}{\bar{Z}_d(p)} \quad (7.77)$

Also from Equ. 7.75,

$$\bar{i}_{kq}(p) = -\sqrt{\frac{3}{2}} \cdot M_q \cdot \frac{p \bar{I}_q(p)}{\bar{Z}_q(p)} \quad (7.78)$$

It is clear from Equ. 7.76 and 7.77 that because of the presence of two windings in the pole axis, the current in that axis, which is the image of the armature current component $\bar{i}_d(p)$, divides itself in the two windings in inverse ratio as the leakage impedances of the respective windings. The image of the armature current component $\bar{i}_q(p)$ is the whole quadrature-axis-damper current $\bar{i}_{kq}(p)$, as shown in Equ. 7.78.

(7.9) Electromagnetic Torque

The expression for the electromagnetic torque has been given in Article 5.2 (Equ. 5.23) as a function of the armature and rotor currents. It is possible to derive an alternative expression for the torque which does not involve the rotor currents except the steady d.c. excitation I_{f0} . This can be achieved by taking the Laplace transforms of Equ. 5.24, which are

$$\left. \begin{aligned} \bar{\Psi}_d(p) &= \frac{3}{2} [L_d \bar{i}_d(p) + M_{df} \bar{i}_f(p) + M_{dk} \bar{i}_{kd}(p)] + \frac{3}{2} M_{df} \frac{I_{f0}}{p} \\ \text{and } \bar{\Psi}_q(p) &= \frac{3}{2} [L_q \bar{i}_q(p) + M_{aq} \bar{i}_{kq}(p)] \end{aligned} \right\} \quad (7.79)$$

and substituting into them (Equs. 7.79) the expressions for

$\bar{i}_f(p)$, $\bar{i}_{kd}(p)$ and $\bar{i}_{kq}(p)$ given by Eqs. 7.76, 7.77 and 7.78, and referring back to Equ. 7.10. Thus, ψ_d and ψ_q may be expressed as

$$\left. \begin{aligned} \psi_d &= \frac{3}{2} \mathcal{L}^{-1} \bar{L}_d(p) \bar{\tau}_d(p) + \frac{3}{2} M_{af} I_{fo} \\ \text{and } \psi_q &= \frac{3}{2} \mathcal{L}^{-1} \bar{L}_q(p) \bar{\tau}_q(p) \end{aligned} \right\} (7.80)$$

The alternative expression for the torque is therefore given by

$$\tau = \frac{3}{2} \left[\mathcal{L}^{-1} \bar{L}_d(p) \bar{\tau}_d(p) + M_{af} I_{fo} \right] \dot{i}_q - \frac{3}{2} \mathcal{L}^{-1} \bar{L}_q(p) \bar{\tau}_q(p) \cdot \dot{i}_d \quad (7.81)$$

which is often more convenient than Equ. 5.20 when the transient-state torque is to be found.

(8) SYMMETRICAL THREE-PHASE OPERATION

(8.1) Three-phase Short Circuit of an Open-circuit Machine

When the symmetrical short circuit occurs to a machine which is initially open-circuited, at the instant when the field axis is ahead of the axis of phase a by an angle λ , the positive component of the armature short-circuit current can be found directly from Equ. 7.22 by putting $\bar{v}_1(p) = \bar{v}_2(p) = 0$ and $\bar{z}_{ex1}(p) = \bar{z}_{ex2}(p) = 0$. The voltage components $\bar{e}_1(p)$ and $\bar{e}_2(p)$ are given by

$$\begin{aligned}\bar{e}_1(p) &= -\frac{1}{2}E_{am}j\epsilon^{j\lambda}/(p-j\omega) \\ \text{and } \bar{e}_2(p) &= \frac{1}{2}E_{am}j\epsilon^{-j\lambda}/(p+j\omega)\end{aligned}\quad (8.1)$$

where $E_{am} = \omega M_{af} I_{fo}$

Using Equ. 8.1 the left-hand member of Equ. 7.22 becomes

$$\begin{aligned}\bar{e}_1(p) &= \frac{p\epsilon^{j\lambda}\bar{L}_D(p-j\omega)}{R_a + (p-j2\omega)\bar{L}_S(p-j\omega)} \cdot \bar{e}_2(p-j2\omega) \\ &= \left\{ 1 + \frac{p\bar{L}_D(p-j\omega)}{R_a + (p-j2\omega)\bar{L}_S(p-j\omega)} \right\} \frac{E_{am}}{2} \cdot \frac{-j\epsilon^{j\lambda}}{p-j\omega} \\ &= \frac{R_a + (p-j\omega)\bar{L}_D(p-j\omega) - j\omega\bar{L}_q(p-j\omega)}{R_a + \frac{1}{2}(p-j2\omega)[\bar{L}_d(p-j\omega) + \bar{L}_q(p-j\omega)]} \cdot \frac{E_{am}}{2} \cdot \frac{-j\epsilon^{j\lambda}}{p-j\omega}\end{aligned}\quad (8.2)$$

and the coefficient of $\bar{i}_1(p)$ reduces to

$$\begin{aligned}R_a + p \left\{ \bar{L}_S(p-j\omega) - \frac{(p-j2\omega)\bar{L}_D^2(p-j\omega)}{R_a + (p-j2\omega)\bar{L}_S(p-j\omega)} \right\} \\ = \frac{R_a^2 + R_a(p-j\omega)[\bar{L}_d(p-j\omega) + \bar{L}_q(p-j\omega)] + p(p-j2\omega)\bar{L}_d(p-j\omega)\bar{L}_q(p-j\omega)}{R_a + \frac{1}{2}(p-j2\omega)[\bar{L}_d(p-j\omega) + \bar{L}_q(p-j\omega)]}\end{aligned}\quad (8.3)$$

Equ. 7.22 thus becomes

$$\bar{i}_1(p) = \frac{E_{am}}{2} \cdot \frac{R_a + (p-j\omega)\bar{L}_D(p-j\omega) - j\omega\bar{L}_q(p-j\omega)}{R_a^2 + R_a(p-j\omega)[\bar{L}_d(p-j\omega) + \bar{L}_q(p-j\omega)] + p(p-j2\omega)\bar{L}_d(p-j\omega)\bar{L}_q(p-j\omega)} \cdot \frac{-j\epsilon^{j\lambda}}{p+j\omega}\quad (8.4)$$

As Equ. 8.4 involves only polynomials of p in both the numerator and denominator, its inverse transform can be found by resolving it into partial fractions whose inverse transforms

are known. This involves finding the roots of the equation

$$R_a^2 + R_a(p-j\omega) [\bar{L}_d(p-j\omega) + \bar{L}_q(p-j\omega)] + p(p-j2\omega)\bar{L}_d(p-j\omega)\bar{L}_q(p-j\omega) = 0 \quad (8.5)$$

The degree of Equ. 8.5 depends on the form of the operational inductances $\bar{L}_d(p-j\omega)$ and $\bar{L}_q(p-j\omega)$. With only one damper circuit in each of the pole and interpole axes, as has been assumed in establishing the general equations, Equ. 8.5 is a quintic whose general solution cannot be found. In view of this difficulty, it is necessary to introduce some simplifying assumptions in order to obtain an approximate expression for $i_1(t)$ which is reasonably simple.

There is no difficulty in deriving the steady-state component from Equ. 8.4, as $p - j\omega$ is already a factor. The coefficient of the partial fraction $1/(p-j\omega)$ is simply given by

$$[(p-j\omega)\bar{i}_1(p)]_{p=j\omega}.$$

It is advantageous to introduce at this stage new forms of the operational inductances $\bar{L}_d(p)$ and $\bar{L}_q(p)$, which will prove to be the most convenient. $\bar{L}_q(p)$ may be expressed as (see Equ. 7.10)

$$\bar{L}_q(p) = \frac{1 + T_q'' p}{1 + T_{qo}'' p} L_q \quad (8.6)$$

where $T_{qo}'' = L_{kq}/R_{kq}$

$$\text{and } T_q'' = (L_{kq} - \frac{M_q^2}{L_f})/R_{kq} \quad (8.7)$$

$\bar{L}_d(p)$ may be expressed as

$$\bar{L}_d(p) = \frac{(1 + T_d' p)(1 + T_d'' p)}{(1 + T_{do}' p)(1 + T_{do}'' p)} L_d \quad (8.8)$$

where $-1/T_{do}'$ and $-1/T_{do}''$ are the roots, $-1/T_{do}'$ being the smaller one (in absolute value), of the quadratic

$$(L_f L_{kd} - M_{fkd}^2) p^2 + (R_f L_{kd} + R_{kd} L_f) p + R_f R_{kd} = 0 \quad (8.9)$$

and $-1/T_d'$ and $-1/T_d''$ are the roots, $-1/T_d'$ being the smaller one (in absolute value), of another quadratic

$$\left[L_f L_{kd} - M_{fkd}^2 - \frac{M_d^2}{L_d} (L_f + L_{kd} - 2 M_{fkd}) \right] p^2 + \left[R_f \left(L_{kd} - \frac{M_d^2}{L_d} \right) + R_{kd} \left(L_f - \frac{M_d^2}{L_d} \right) \right] p + R_f R_{kd} = 0 \quad (8.10)$$

If the two roots of each quadratic are vastly different from each other, which will be true if R_{kd} is much larger than R_f when expressed in a common basis (say the per-unit value), T_{do}' , T_{do}'' , T_d' and T_d'' may be expressed approximately as

$$\left. \begin{aligned} T_{do}' &\approx L_f / R_f \\ T_{do}'' &\approx \left(L_{kd} - \frac{M_{fkd}^2}{L_f} \right) / R_{kd} \\ T_d' &\approx \left(L_f - \frac{M_d^2}{L_d} \right) / R_f \\ \text{and } T_d'' &\approx \left(L_{kd} - \frac{M_{fkd}^2 L_d + M_d^2 L_f - 2 M_d^2 M_{fkd}}{L_f L_d - M_d^2} \right) / R_{kd} \end{aligned} \right\} \quad (8.11)$$

It can be anticipated that the advantage of expressing $\bar{L}_d(p)$ and $\bar{L}_q(p)$ like Eqs. 8.8 and 8.6 lies on their product form and the form of the factors like $1 - T_d'p$, which lends itself to ready approximation for a large range of the value of p . When $p = 0$, the factor simply equals to 1. When p is large, the factor reduces to $T_d'p$, since 1 may become negligible.

(8.2) Solution for Armature Currents with Small Armature Resistance

When the armature resistance is small, R_a may be neglected and Equ. 8.4 becomes, using Eqs. 8.8 and 8.6 for $\bar{L}_d(p)$ and $\bar{L}_q(p)$,

$$\bar{I}_a(p) \approx \frac{(p-j\omega) [1 + T_d'(p-j\omega)] [1 + T_d''(p-j\omega)] [1 + T_q'(p-j\omega)] L_d - j\omega [1 + T_{do}'(p-j\omega)] [1 + T_{do}''(p-j\omega)] [1 + T_q'(p-j\omega)] L_q}{p(p-j\lambda\omega) [1 + T_d'(p-j\omega)] [1 + T_d''(p-j\omega)] [1 + T_q'(p-j\omega)] L_d L_q} \cdot \frac{E_{am}}{2} \cdot \frac{-j e^{j\lambda}}{p-j\omega}$$

which can be expanded into partial fraction as.

(8.12)

$$\begin{aligned} \tau_1(p) \approx & -\frac{E_{am} \epsilon^{j\lambda}}{2\omega} \left\{ \left(\frac{1}{L_d''} - \frac{1}{L_d'} \right) \frac{T_d''}{1+T_d''(p-j\omega)} + \left(\frac{1}{L_d'} - \frac{1}{L_d} \right) \frac{T_d'}{1+T_d'(p-j\omega)} + \frac{1}{L_d} \frac{1}{p-j\omega} \right\} \\ & + \frac{E_{am} \epsilon^{j\lambda}}{4\omega} \cdot \frac{L_d''+L_q''}{L_d''L_q''} \cdot \frac{1}{p} - \frac{E_{am} \epsilon^{j\lambda}}{4\omega} \cdot \frac{L_d'-L_q'}{L_d'L_q'} \cdot \frac{1}{p-j2\omega} \end{aligned} \quad (8.13)$$

where $L_d'' = (T_d'T_d''/T_{d0}'T_{d0}'')L_d$, $L_d' = (T_d'/T_{d0}')L_d$ and $L_q'' = (T_q''/T_{q0}'')L_q$, assuming $T_d' \gg T_d''$, $T_d' \gg T_{d0}'$, $\omega \gg 1/T_d'$ or $1/T_d''$, also neglecting $1/\omega$ compared with unity

The last two terms in Equ. 8.13 are not correct as they correspond to sustained d.c. and double-frequency components, a result of omitting the damping, R_a , in the armature circuit. The correct expressions for these two components can be found directly from Equ. 8.4. When p is approximately equal to 0 or $j2\omega$, $\bar{L}_d(p-j\omega) \approx L_d''$ and $\bar{L}_q(p-j\omega) \approx L_q''$ and Equ. 8.4 reduces to

$$\tau_1(p) \approx \frac{E_{am}}{2} \cdot \frac{R_a + (p-j\omega)L_d'' - j\omega L_q''}{R_a^2 + R_a(p-j\omega)(L_d''+L_q'') + p(p-j2\omega)L_d''L_q''} \cdot \frac{-je^{j\lambda}}{p-j\omega} \quad (8.14)$$

The roots of the quadratic in the denominator of Equ. 8.14 are given by

$$\begin{aligned} p = & -\frac{R_a(L_d''+L_q'') - j2\omega L_d''L_q''}{2L_d''L_q''} \pm \frac{\sqrt{R_a^2(L_d''-L_q'')^2 - 4\omega^2 L_d''^2 L_q''^2}}{2L_d''L_q''} \\ \approx & -\frac{R_a(L_d''+L_q'')}{2L_d''L_q''} \quad \text{and} \quad -\frac{R_a(L_d''+L_q'')}{2L_d''L_q''} + j2\omega \end{aligned} \quad (8.15)$$

since $4(\omega L_d''L_q'')^2 \gg R_a^2(L_d''-L_q'')^2$. Defining

$$T_a(\beta) = 2L_d''L_q''/(L_d'' + L_q'')R_a \quad (8.16)$$

and finding the coefficients of the partial fractions of Equ. 8.14 in the usual manner, the proper forms of the d.c. and the double-frequency components are found to be

$$\begin{aligned} & \frac{E_{am} \epsilon^{j\lambda}}{4\omega} \cdot \frac{L_d''+L_q''}{L_d''L_q''} \cdot \frac{1}{\frac{1}{T_a(\beta)} + p} \\ \text{and} \quad & -\frac{E_{am} \epsilon^{j\lambda}}{4\omega} \cdot \frac{L_d''-L_q''}{L_d''L_q''} \cdot \frac{1}{\frac{1}{T_a(\beta)} + p-j2\omega} \end{aligned}$$

if $1/T_a(\beta)$ is neglected compared with ω and R_a is neglected

compared with $\omega(L_d'' \pm L_q'')$.

Equ. 8.13 is thus modified to

$$\begin{aligned} \bar{i}_1(p) \approx & -\frac{E_{am}e^{j\lambda}}{2\omega} \left\{ \left(\frac{1}{L_d''} - \frac{1}{L_d} \right) \frac{1}{\frac{1}{T_d''} + p - j\omega} + \left(\frac{1}{L_d''} - \frac{1}{L_d} \right) \frac{1}{\frac{1}{T_d''} + p - j\omega} + \frac{1}{L_d} \cdot \frac{1}{p - j\omega} \right\} \\ & + \frac{E_{am}e^{j\lambda}}{4\omega} \cdot \frac{L_d'' + L_q''}{L_d'' L_q''} \cdot \frac{1}{\frac{1}{T_{a(0)}} + p} - \frac{E_{am}e^{j\lambda}}{4\omega} \cdot \frac{L_d'' - L_q''}{L_d'' L_q''} \cdot \frac{1}{\frac{1}{T_{a(0)}} - p - j2\omega} \end{aligned} \quad (8.18)$$

Since $\bar{v}_0(p)$ is equal to zero the armature phase current $\bar{i}_a(p)$ is equal to $\bar{i}_1(p)$ and its conjugate. $i_a(t)$ is given by

$$\begin{aligned} i_a(t) \approx & -\frac{E_{am}}{\omega} \left\{ \left(\frac{1}{L_d''} - \frac{1}{L_d} \right) e^{-t/T_d''} + \left(\frac{1}{L_d''} - \frac{1}{L_d} \right) e^{-t/T_d} + \frac{1}{L_d} \right\} \cos(\omega t + \lambda) \\ & + \frac{E_{am} \cos \lambda}{2\omega} \left(\frac{1}{L_d''} + \frac{1}{L_q''} \right) e^{-t/T_{a(0)}} + \frac{E_{am}}{2\omega} \left(\frac{1}{L_d''} - \frac{1}{L_q''} \right) e^{-t/T_{a(0)}} \cos(2\omega t + \lambda) \end{aligned} \quad (8.19)$$

The current in phase b is given by

$$\bar{i}_b(p) = h \bar{i}_1(p) - h \bar{i}_2(p)$$

which is equivalent to putting $\lambda - 2\pi/3$ in place of λ in Equ. 8.18 and adding the conjugate of the resultant expression. $i_b(t)$ is therefore obtainable from Equ. 8.19 simply by putting $\lambda - 2\pi/3$ in place of λ wherever it occurs. Similarly, replacing λ by $\lambda + 2\pi/3$ in Equ. 8.19 gives $i_c(t)$.

Equ. 8.19 shows that the symmetrical short-circuit current consists of three different transient-state components, as there are three different circuits involved (the three armature circuits are identical and the quadrature-axis-damper circuit has no d.c. component of current) in determining the damping coefficients. The components of current whose damping coefficients are proportional to R_{kd} and R_f are known as the direct-axis subtransient and the transient components respectively. When there is d.c. component of current in the quadrature-axis-damper circuit (Article 8.5), there is an additional component of current whose damping coefficient is proportional to R_{kq} .

Both the d. c. and the double-frequency components have a common damping coefficient proportional to the resistance in the armature circuit. They may be called the "armature-transient" component. It will be shown later that the general form of the armature-transient component is an even-harmonic series, the higher harmonics being equal to zero for the symmetrical three-phase short circuit.

(8.3) Field and Damper Currents

The field and damper currents have been expressed in terms of the armature currents as given by Eqs. 7.76, 7.77 and 7.78. Since $\bar{i}_1(p)$ is known, $\bar{i}_d(p)$ and $\bar{i}_q(p)$ are also known, and the derivation of the operational expressions for the field and damper currents is merely a matter of substitution. It should be remembered, however, that whereas Eqs. 7.76, 7.77 and 7.78 are exact, the final simplified expression of $\bar{i}_1(p)$ is only approximate. In order to get simple and consistent expressions for the field and damper currents, it is necessary to make the same simplifying assumptions.

The d- and q- component currents can be found by using Equ. 8.18 for $\bar{i}_1(p)$ in Equ. 7.63 and also using Equ. 7.68. Thus:

$$\begin{aligned}\bar{i}_d(p) &= \epsilon^{-j\lambda} \bar{i}_1(p+j\omega) + \text{conj} \\ &= -\frac{E_{am}}{\omega} \left\{ \left(\frac{1}{L'_d} - \frac{1}{L_d} \right) \frac{1}{\frac{1}{T'_d} + p} + \left(\frac{1}{L_d} - \frac{1}{L_d} \right) \frac{1}{\frac{1}{T_d} + p} + \frac{1}{L_d} \frac{1}{p} \right\} \\ &\quad + \frac{E_{am}}{2\omega} \cdot \frac{1}{L'_d} \cdot \frac{1}{\frac{1}{T_{d(2)}} + p + j\omega} + \text{conj} \quad (8.20)\end{aligned}$$

$$\begin{aligned}\bar{i}_q(p) &= -j \left[\epsilon^{-j\lambda} \bar{i}_1(p+j\omega) - \text{conj} \right] \\ &= \frac{E_{am}}{2\omega} \cdot \frac{1}{L'_q} \cdot \frac{-j}{\frac{1}{T_{q(2)}} + p + j\omega} + \text{conj} \quad (8.21)\end{aligned}$$

Substituting $\bar{i}_d(p)$ in Equ. 7.76, with $\bar{Z}_d(p)$ expressed in

explicit form (Equ. 7.10) and the function $(L_f L_{kd} - M_{fkd}^2)p^2 + (R_f L_{kd} + R_{kd} L_f)p + R_f R_{kd}$ factorised (Eqs. 8.9 and 8.11), gives the field current:

$$i_f'(p) = \sqrt{\frac{3}{2}} \cdot M_d \cdot \frac{E_{am}}{\omega} \cdot \frac{R_{kd} + (L_{kd} - M_{fkd})p}{R_f R_{kd} (1 + T_{do}' p)(1 + T_{do}'' p)} \times \left\{ \left(\frac{1}{L_d'} - \frac{1}{L_d} \right) \frac{T_d' p}{1 + T_d' p} + \left(\frac{1}{L_d} - \frac{1}{L_d''} \right) \frac{T_d'' p}{1 + T_d'' p} + \frac{1}{L_d} - \frac{1}{2L_d} \cdot \frac{T_{a(0)} p}{1 + T_{a(0)}(p + j\omega)} - \text{Conj.} \right\} \quad (8.22)$$

Similarly the damper currents are

$$i_{kd}(p) = \sqrt{\frac{3}{2}} \cdot M_d \cdot \frac{E_{am}}{\omega} \cdot \frac{R_f + (L_f - M_{fkd})p}{R_f R_{kd} (1 + T_{do}' p)(1 + T_{do}'' p)} \times \left\{ \left(\frac{1}{L_d'} - \frac{1}{L_d} \right) \frac{T_d' p}{1 + T_d' p} + \left(\frac{1}{L_d} - \frac{1}{L_d''} \right) \frac{T_d'' p}{1 + T_d'' p} + \frac{1}{L_d} - \frac{1}{2L_d} \cdot \frac{T_{a(0)} p}{1 + T_{a(0)}(p + j\omega)} - \text{Conj.} \right\} \quad (8.23)$$

$$\text{and } i_{kq}(p) = \sqrt{\frac{3}{2}} \cdot M_q \cdot \frac{E_{am}}{2\omega L_{qy}''} \cdot \frac{p}{R_{kq}(1 + T_{qo}' p)} \cdot \left\{ \frac{-j T_{a(0)}}{1 + T_{a(0)}(p + j\omega)} + \text{Conj.} \right\} \quad (8.24)$$

The expression for the field current (Equ. 8.22) can be resolved into partial fractions. The fractions with $1 + T_{do}' p$ and $1 + T_{do}'' p$ as denominators are very nearly equal to zero. Thus, writing $E_{am} = \omega M_{af} I_{fo} = \omega \sqrt{2/3} M_d I_{fo}$, Equ. 8.23 is reduced to

$$i_f'(p) = M_d^2 I_{fo} \cdot \left\{ \left(\frac{L_{kd} - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1}{L_d'} - \frac{1}{L_f} \cdot \frac{1}{L_d} \right) \cdot \frac{1}{\frac{1}{T_d'} + p} + \frac{1}{L_f} \cdot \frac{1}{L_d} \cdot \frac{1}{\frac{1}{T_d''} + p} - \frac{1}{2} \cdot \frac{L_{kd} - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1}{L_d''} \left(\frac{1}{\frac{1}{T_{a(0)}} + p + j\omega} + \text{Conj.} \right) \right\} \quad (8.25)$$

Adding the steady exciting component I_{fo} to the inverse transform of Equ. 8.25 gives the explicit form of the total field current:

$$i_f(t) = I_{fo} + M_d^2 I_{fo} \cdot \left\{ \left(\frac{L_{kd} - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1}{L_d'} - \frac{1}{L_f} \cdot \frac{1}{L_d} \right) e^{-t/T_d'} + \frac{1}{L_f} \cdot \frac{1}{L_d} e^{-t/T_d''} - \frac{L_{kd} - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1}{L_d''} e^{-t/T_{a(0)}} \cos \omega t \right\} \quad (8.26)$$

Thus the total field current has a form in close resemblance to the armature current. It also consists of a subtransient, a transient and a steady-state component, all of the same frequency, and an armature-transient component of another frequency. The quantity $(L_{kd} - M_{fkd}) / (L_f L_{kd} - M_{fkd}^2)$ appearing in Equ. 8.26 may be written as:

$$\frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{1}{\frac{L_f L_{kd} - M_{fkd}^2}{L_f + L_{kd} - 2M_{fkd}}} = \frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{1}{L_{fe}}$$

for the benefit of interpreting its physical significance. The quantities $L_f - M_{fkd}$ and $L_{kd} - M_{fkd}$ are the leakage inductances of the field and direct-axis-damper windings with respect to each other. The quantity $L_{fe} = (L_f L_{kd} - M_{fkd}^2) / (L_f + L_{kd} - 2M_{fkd})$ is the equivalent inductance of these two windings in parallel, neglecting resistances. Thus the term $(L_{kd} - M_{fkd}) / (L_f + L_{kd} - 2M_{fkd})$ corresponds to the term $1/L_f L_d$, the extra factor $(L_{kd} - M_{fkd}) / (L_f + L_{kd} - 2M_{fkd})$ is there to indicate the division of current in the field circuit.

Similarly Eqs. 8.23 and 8.24 may be reduced to

$$\bar{i}_{kd}(p) = M_d^2 I_{fo} \cdot \frac{L_f - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1}{L_d''} \left\{ \frac{1}{\frac{1}{T_d} + p} - \frac{1}{2} \left(\frac{1}{\frac{1}{T_{adn}} + p + j\omega} + \text{conj} \right) \right\} \quad (8.27)$$

$$\text{and } \bar{i}_{kq}(p) = \frac{M_d M_q I_{fo}}{2 L_{kq} L_q''} \left\{ \frac{j}{\frac{1}{T_{adn}} + p + j\omega} + \text{conj} \right\} \quad (8.28)$$

In explicit forms, the transient-state damper currents are

$$i_{kd}(t) = M_d^2 I_{fo} \cdot \frac{L_f - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1}{L_d''} \left\{ e^{-t/T_d'} - e^{-t/T_{adn}} \cos \omega t \right\} \quad (8.29)$$

$$\text{and } i_{kq}(t) = \frac{M_d M_q I_{fo}}{L_{kq} L_q''} e^{-t/T_{adn}} \sin \omega t \quad (8.30)$$

Immediately after the short circuit, the total direct current in the field axis is given by

$$\left[\text{D.c. component of } i_f(t) + i_{kd}(t) \right]_{t=0_+} = I_{fo} + \frac{M_d^2 I_{fo}}{L_{fe} L_d''} = \frac{L_d}{L_d''} I_{fo} \quad (8.31)$$

which may be many times the exciting current. Although this current is not entirely in the field circuit, it is nevertheless in the field axis and its reaction on the armature circuit is the same regardless of its actual distribution. Consequently the ratio L_d/L_d'' also gives the relative magnitudes of the initial and final values of the fundamental armature current.

After the subtransient component has died away the d.c.com-

ponent of the field current is given by

$$I_{f0} + \frac{M_A^2 I_{f0}}{L_f L_A} = \frac{L_A}{L_f} I_{f0} \quad (8.32)$$

(8.4) Short-circuit Torque

Substituting $\bar{i}_d(p)$ and $\bar{i}_q(p)$ in Equ. 7.80 and reducing gives

$$\begin{aligned} \psi_d &= \sqrt{\frac{3}{2}} M_A I_{f0} e^{-t/\tau_{Ad}} \cos \omega t \\ \text{and } \psi_q &= -\sqrt{\frac{3}{2}} M_A I_{f0} e^{-t/\tau_{Aq}} \sin \omega t \end{aligned} \quad (8.33)$$

The explicit forms of i_d and i_q are, from Eqs. 8.20 and 8.21,

$$\begin{aligned} i_d &= -\sqrt{\frac{3}{2}} \frac{M_A I_{f0}}{L_d} \left\{ g(t) - e^{-t/\tau_{Ad}} \cos \omega t \right\} \\ \text{and } i_q &= -\sqrt{\frac{3}{2}} \frac{M_A I_{f0}}{L_q} e^{-t/\tau_{Aq}} \sin \omega t \end{aligned} \quad (8.34)$$

$$\text{where } g(t) = \left(1 - \frac{\tau_d''}{\tau_{d0}}\right) e^{-t/\tau_d} + \frac{\tau_d'}{\tau_{d0}} \left(1 - \frac{\tau_d'}{\tau_{d0}}\right) e^{-t/\tau_d'} + \frac{\tau_d' \tau_d''}{\tau_{d0}^2} \quad (8.35)$$

Substituting Eqs. 8.33 and 8.34 for ψ_d , ψ_q , i_d and i_q in Equ. 5.23, the short-circuit torque is found to be

$$\tau = -M_A^2 I_{f0}^2 g(t) e^{-t/\tau_{Ad}} \sin \omega t + \frac{1}{2} M_A^2 I_{f0}^2 \left(\frac{1}{L_d} - \frac{1}{L_q}\right) e^{-2t/\tau_{Aq}} \sin 2\omega t \quad (8.36)$$

which is a damped pulsating torque consisting of a fundamental-frequency component and a double-frequency component. The complete expression for the short-circuit torque consists of a unidirectional component in addition to the pulsating component given by Equ. 8.36. The fact that the unidirectional component of torque is missing in using Equ. 5.23 is the result of neglecting the resistances in determining the phase of the short-circuit currents (and hence the flux-linkages), which has caused all the equal-frequency components of ψ_d and i_q , or ψ_q and i_d , to become in time quadrature with each other. With the presence of the resistances, these components will not in exact time quadrature and consequently their products will consist a constant term (or more strictly, a monotonously decreasing term)

representing the unidirectional component of torque. The exact exact expressions for the short-circuit currents with resistances taken into account are difficult to obtain, so that the unidirectional component of torque can only be found approximately. Its approximate expression are given in Chapter 11 when the equivalent circuits are considered.

(8.5) Short Circuit of A Loaded Machine with Initial Balanced Loading

If before the short circuit, the machine is operating balancedly with a load angle δ , the short-circuit armature current excluding the pre-short steady-state component can be found from Equ. 7.22 by putting $\bar{v}_1(p) = \bar{v}_2(p) = 0$, $Z_{ex1} = Z_{ex2} = 0$ and replacing $\bar{e}_1(p)$ and $\bar{e}_2(p)$ by

$$\left. \begin{aligned} \bar{e}_1'(p) &= \frac{V_{am}}{Z} \cdot \frac{-j\epsilon^{j(\lambda-\delta)}}{p-j\omega} \\ \text{and } \bar{e}_2'(p) &= \frac{V_{am}}{Z} \cdot \frac{j\epsilon^{-j(\lambda-\delta)}}{p+j\omega} \end{aligned} \right\} (8.37)$$

respectively. The left-hand member of Equ. 7.22 becomes

$$\begin{aligned} \bar{e}_1'(p) &= \frac{p\epsilon^{j\lambda} \bar{L}_D(p-j\omega)}{R_a + (p-j\omega) \bar{L}_S(p-j\omega)} \cdot \bar{e}_2'(p-j\omega) \\ &= \left\{ 1 + \frac{p \bar{L}_D(p-j\omega)}{R_a + (p-j\omega) \bar{L}_S(p-j\omega)} \right\} \cdot \frac{V_{am} \cos \delta}{Z} \cdot \frac{-j\epsilon^{j\lambda}}{p-j\omega} \\ &\quad - \left\{ 1 - \frac{p \bar{L}_D(p-j\omega)}{R_a + (p-j\omega) \bar{L}_S(p-j\omega)} \right\} \cdot \frac{V_{am} \sin \delta}{Z} \cdot \frac{\epsilon^{j\lambda}}{p-j\omega} \end{aligned} \quad (8.38)$$

The coefficient of $\bar{i}_1(p)$ remains unchanged as given by Equ. 8.3. Comparing Eqs. 8.38 and 8.2, the equivalent voltage functions, it is clear that the former equation has one more component than the latter. The first component of Equ. 8.38 corresponds to Equ. 8.2 except that $V_{am} \cos \delta$ replaces E_{am} . As the coefficient of $\bar{i}_1(p)$, the equivalent impedance function, is the same

in both cases, it is clear that the part of the armature current $i_a(t)$ corresponding to the first component of Equ. 8.38 is the same as Equ. 8.19 with E_{am} replaced by $V_{am} \cos \delta$. All remains to be found is, therefore, that part of the armature current corresponding to the second part of Equ. 8.38. Denoting this part of current by $\bar{i}_1(p)$,

$$\bar{i}_1(p) = - \frac{V_{am} \sin \delta}{2} \cdot \frac{R_a - j\omega L_a(p-j\omega) + (p-j\omega) L_q(p-j\omega)}{R_a^2 + R_a(p-j\omega) [L_a(p-j\omega) + L_q(p-j\omega)] + p(p-j\omega) L_a(p-j\omega) L_q(p-j\omega)} \cdot \frac{e^{j\lambda}}{p-j\omega} \quad (8.39)$$

Equ. 8.39 is very similar to Equ. 8.4 and following the same steps used in obtaining Equ. 8.18 it can be shown that Equ. 8.39 is reduced to

$$\begin{aligned} \bar{i}_1(p) \approx & \frac{V_{am} \sin \delta}{2\omega} \cdot \frac{j e^{j\lambda}}{L_q} \left\{ \left(\frac{1}{L_q'} - \frac{1}{L_q} \right) \cdot \frac{1}{\frac{1}{T_q} + p - j\omega} + \frac{1}{L_q} \cdot \frac{1}{p - j\omega} \right\} \\ & - \frac{V_{am} \sin \delta}{4\omega} \cdot \frac{L_a' + L_q'}{L_a' L_q'} \cdot \frac{j e^{j\lambda}}{\frac{1}{T_{a0}} + p} - \frac{V_{am} \sin \delta}{4\omega} \cdot \frac{L_a' - L_q'}{L_a' L_q'} \cdot \frac{j e^{j\lambda}}{\frac{1}{T_{a0}} + p - j\omega} \end{aligned} \quad (8.40)$$

where $L_q'' = (T_q''/T_{q0}'') L_q$. The current $i_a'(t)$, given by the inverse transform of the sum of $\bar{i}_1(p)$ and its conjugate function, is

$$\begin{aligned} i_a'(t) \approx & - \frac{V_{am} \sin \delta}{\omega} \left\{ \left(\frac{1}{L_q'} - \frac{1}{L_q} \right) e^{-t/T_q''} + \frac{1}{L_q} \right\} \sin(\omega t + \lambda) \\ & + \frac{V_{am} \sin \delta \sin \lambda}{2\omega} \left(\frac{1}{L_a'} + \frac{1}{L_q'} \right) e^{-t/T_{a0}} - \frac{V_{am} \sin \delta}{2\omega} \left(\frac{1}{L_a'} - \frac{1}{L_q'} \right) e^{-t/T_{a0}} \sin(2\omega t + \lambda) \end{aligned} \quad (8.41)$$

The original steady-state component can be found in the usual way by using the steady-state theory. Alternatively, it may be derived from Equ. 7.22 by putting $\bar{v}_1(p) = -\bar{e}_1'(p)$, $\bar{v}_2(p) = -\bar{e}_2'(p)$, $Z_{ex1} = Z_{ex2} = 0$, and evaluating the function $(p-j\omega)\bar{i}_1(p)$ at $p = j\omega$ to give the coefficient of the fraction $1/(p-j\omega)$, the quantity corresponding to the steady-state component. As an example to show how readily the steady-state component can be derived from the operational solution, Equ. 7.22, the latter course is followed.

Substituting the expressions of $\bar{e}_1(p)$, $\bar{e}_2(p)$, $\bar{e}_1'(p)$ and $\bar{e}_2'(p)$, the left-hand member of Equ. 7.22 becomes

$$\begin{aligned} \bar{e}_1(p) + \bar{v}_1(p) &= \frac{p \epsilon^{j\lambda} \bar{L}_D(p-j\omega)}{R_a + (p-j\omega) \bar{L}_S(p-j\omega)} \{ \bar{e}_2(p-j\omega) + \bar{v}_2(p-j\omega) \} \\ &= \frac{E_{am} - V_{am} \cos \delta}{2} \cdot \frac{R_a + (p-j\omega) \bar{L}_d(p-j\omega) - j\omega \bar{L}_q(p-j\omega)}{R_a + \frac{1}{2}(p-j\omega) [\bar{L}_d(p-j\omega) + \bar{L}_q(p-j\omega)]} \cdot \frac{-j\epsilon^{j\lambda}}{p-j\omega} \\ &\quad + \frac{V_{am} \sin \delta}{2} \cdot \frac{R_a - j\omega \bar{L}_d(p-j\omega) + (p-j\omega) \bar{L}_q(p-j\omega)}{R_a + \frac{1}{2}(p-j\omega) [\bar{L}_d(p-j\omega) + \bar{L}_q(p-j\omega)]} \cdot \frac{\epsilon^{j\lambda}}{p-j\omega} \end{aligned} \quad (8.42)$$

and using Equ. 8.3,

$$\begin{aligned} \bar{v}_1(p) &= \frac{E_{am} - V_{am} \cos \delta}{2} \cdot \frac{R_a + (p-j\omega) \bar{L}_d(p-j\omega) - j\omega \bar{L}_q(p-j\omega)}{R_a^2 + R_a(p-j\omega) [\bar{L}_d(p-j\omega) + \bar{L}_q(p-j\omega)] + p(p-j\omega) \bar{L}_d(p-j\omega) \bar{L}_q(p-j\omega)} \cdot \frac{-j\epsilon^{j\lambda}}{p-j\omega} \\ &\quad - \frac{V_{am} \sin \delta}{2} \cdot \frac{R_a - j\omega \bar{L}_d(p-j\omega) + (p-j\omega) \bar{L}_q(p-j\omega)}{R_a^2 + R_a(p-j\omega) [\bar{L}_d(p-j\omega) + \bar{L}_q(p-j\omega)] + p(p-j\omega) \bar{L}_d(p-j\omega) \bar{L}_q(p-j\omega)} \cdot \frac{\epsilon^{j\lambda}}{p-j\omega} \end{aligned} \quad (8.43)$$

When $p = j\omega$, $\bar{L}_d(p-j\omega) = L_d$ and $\bar{L}_q(p-j\omega) = L_q$, hence

$$[(p-j\omega) \bar{v}_1(p)]_{p=j\omega} = \frac{E_{am} - V_{am} \cos \delta}{2} \cdot \frac{-j\epsilon^{j\lambda} (R_a - j\omega L_q)}{R_a^2 + \omega^2 L_d L_q} + \frac{V_{am} \sin \delta}{2} \cdot \frac{\epsilon^{j\lambda} (R_a - j\omega L_d)}{R_a^2 + \omega^2 L_d L_q} \quad (8.44)$$

If R_a is neglected, the steady-state component of $\bar{i}_1(p)$ is

$$\bar{v}_1(p) = -\frac{E_{am} - V_{am} \cos \delta}{2} \cdot \frac{1}{\omega L_d} \cdot \frac{\epsilon^{j\lambda}}{p-j\omega} - \frac{V_{am} \sin \delta}{2} \cdot \frac{1}{\omega L_q} \cdot \frac{j\epsilon^{j\lambda}}{p-j\omega} \quad (8.45)$$

and the original steady-state armature current is

$$i_a(t) = \int^{-1} [\bar{v}_1(p) + \text{conj}] = -\frac{E_{am} - V_{am} \cos \delta}{\omega L_d} \cos(\omega t + \lambda) + \frac{V_{am} \sin \delta}{\omega L_q} \sin(\omega t + \lambda) \quad (8.46)$$

Finally the total armature current in phase a is given by summing up Eqs. 8.19 (with E_{am} replaced by $V_{am} \cos \delta$), 8.41 and 8.46. Thus

$$\begin{aligned} i_a(t) &= -\frac{V_{am} \cos \delta}{\omega} \left\{ \left(\frac{1}{L_d} - \frac{1}{L_q} \right) \epsilon^{-t/\tau_d''} + \left(\frac{1}{L_d} - \frac{1}{L_q} \right) \epsilon^{-t/\tau_d'} \right\} \cos(\omega t + \lambda) - \frac{E_{am}}{\omega L_d} \cos(\omega t + \lambda) \\ &\quad - \frac{V_{am} \sin \delta}{\omega} \left(\frac{1}{L_q} - \frac{1}{L_d} \right) \epsilon^{-t/\tau_q''} \sin(\omega t + \lambda) \\ &\quad + \frac{V_{am} \cos(\lambda - \delta)}{2\omega} \left(\frac{1}{L_d} + \frac{1}{L_q} \right) \epsilon^{-t/\tau_{a(1)}} \\ &\quad + \frac{V_{am}}{2\omega} \left(\frac{1}{L_d} - \frac{1}{L_q} \right) \epsilon^{-t/\tau_{a(2)}} \cos(2\omega t + \lambda + \delta) \end{aligned} \quad (8.47)$$

Equ. 8.47 is the expression for the short-circuit current in

phase a starting from any balanced loading condition. It is clear that the steady-state component, $(-E_{am}/\omega L_d)\cos(\omega t + \lambda)$, is the same in Equ. 8.47 as that in Equ. 8.19 when the short circuit starts from no load, as the short circuit of a machine under load is merely the short circuit of the machine from no load in more than one step, and the steady-state current is the same irrespective of the previous history, provided, of course, no change of excitation has taken place.

(8.6) Short Circuit through External Impedances

For the sake of simplicity, only the case of equal and bilateral impedances in each armature circuit is considered, so that

$$\bar{Z}_{ex1}(p) = \bar{Z}_{ex2}(p) = R_e + pL_e \quad (8.48)$$

Also, whenever R_e and R_a go together, they are denoted by R_t , the total resistance in the armature circuit.

The steady-state component can be obtained without difficulty from Equ. 7.22 irrespective of the relative magnitudes of the external resistance and inductance. The transient component and, provided the total resistance is small, the subtransient component can be derived with ease from the steady-state component. This is contrary to the usual practice when everything starts from the determination of an initial current obtained by neglecting all resistances of the machine.

The derivation of the steady-state component from the operational solution is a straightforward process as shown in the last article. When an open-circuit machine is suddenly short-

circuited through external impedance defined by Equ. 8.48, the the positive-component current is, from Equ. 7.22,

$$\bar{i}_1(p) = \bar{\varphi}(p) \frac{E_{am}}{\lambda} \frac{-je^{j\lambda}}{p-j\omega} \quad (8.49)$$

$$\bar{\varphi}(p) = \frac{R_t + (p-j\omega)L_e + (p-j\omega)\bar{L}_d(p-j\omega) - j\omega\bar{L}_q(p-j\omega)}{\left[(R_t + pL_e) \left\{ R_t + (p-j\omega)L_e + \frac{1}{\lambda}(p-j\omega)[\bar{L}_d(p-j\omega) + \bar{L}_q(p-j\omega)] \right\} + \frac{1}{\lambda}p[\bar{L}_d(p-j\omega) + \bar{L}_q(p-j\omega)][R_t + (p-j\omega)L_e] + p(p-j\omega)\bar{L}_d(p-j\omega)L_q(p-j\omega) \right]} \quad (8.50)$$

The coefficient of the steady-state component is found by evaluating the function $\bar{\varphi}(p)$ at $p = j\omega$. Thus,

$$\bar{\varphi}(j\omega) = \frac{R_t - j\omega(L_q + L_e)}{R_t^2 + \omega^2(L_d + L_e)(L_q + L_e)} \quad (8.51)$$

which is similar to the result with no external impedance (Equ. 8.44). Only the armature resistance R_a has been replaced by the total resistance R_t and the inductances L_d and L_q increased by the external inductance L_e .

To find the transient component, let it be of the form

$$\bar{i}_1[\text{Transient}] = \frac{k}{\alpha + p - j\omega} \quad (8.52)$$

where both k and α are to be determined. The quantity k is generally a complex ^{number} and has a direct bearing on the magnitude of the transient component. The quantity α is the damping coefficient and is a positive real number whose value lies between $1/T_{d0}'$ and $1/T_d'$, as they are the damping coefficients of the field circuit corresponding to the two extreme values of the armature external impedance, the infinite external impedance, or open circuit, and the zero external impedance, or short circuit. Strictly, α is a complex number thus modifying slightly the frequency of the transient component. This will be understood when attempting to factorise the denominator of

of $\bar{\varphi}(p)$. This frequency aberration is, however, very small, and may usually be neglected.

If it were possible to get the factor $\alpha + p - j\omega$ from the denominator of $\bar{\varphi}(p)$, the quantity k could be found with ease by evaluating the function $(\alpha + p - j\omega)\bar{i}_1(p)$ at $p = -\alpha + j\omega$. Since this is not possible, an approximate method of finding k as well as α has to be devised. When α is small compared with ω , the substitution of $p = -\alpha + j\omega$ in $\bar{\varphi}(p)$ will give largely the same result as the substitution of $p = j\omega$ (Equ. 8.51) except the two factors $1 + T_d^1(p - j\omega)$ and $1 + T_{d0}^1(p - j\omega)$ of the function $\bar{L}_d(p - j\omega)$, which cannot be assessed without knowing the value of α , assuming as before that $R_{kd} \gg R_f$. Thus, when p is in the vicinity of $j\omega$, the value of $\bar{\varphi}(p)$ is obtained approximately by replacing L_d in Equ. 8.51 by $[1 + T_d^1(p - j\omega)]L_d/[1 + T_{d0}^1(p - j\omega)]$, and the value of $\bar{i}_1(p)$ is

$$\bar{i}_1(p) = \frac{E_{am}}{\lambda} \cdot \frac{R_t - j\omega(L_q + L_e)}{R_t^2 + \omega^2 \left[\frac{1 + (p - j\omega)T_d^1}{1 + (p - j\omega)T_{d0}^1} L_d + L_e \right] (L_q + L_e)} \cdot \frac{-je^{j\lambda}}{p - j\omega} \quad (8.53)$$

which can be rewritten as

$$\bar{i}_1(p) = \frac{E_{am}}{\lambda} \cdot \frac{[R_t - j\omega(L_q + L_e)] [1 + T_{d0}^1(p - j\omega)]}{[R_t^2 + \omega^2(L_d + L_e)(L_q + L_e)] [1 + T_{d3}^1(p - j\omega)]} \cdot \frac{-je^{j\lambda}}{p - j\omega} \quad (8.54)$$

where $T_{d3}^1 = \frac{R_t^2 + \omega^2(L_d + L_e)(L_q + L_e)}{R_t^2 + \omega^2(L_d + L_e)(L_q + L_e)} T_{d0}^1 = \frac{1}{\alpha} \quad (8.55)$

Equ. 8.54 has the factor $1 + T_{d3}^1(p - j\omega)$, or $\alpha + p - j\omega$, in the denominator and its partial fraction with this factor as denominator is the required transient component. This is found to be

$$\bar{i}_1(p) [Transient] = \frac{E_{am}}{\lambda} \left\{ R_t - j\omega(L_q + L_e) \right\} \left\{ \frac{1}{R_t^2 + \omega^2(L_d + L_e)(L_q + L_e)} - \frac{1}{R_t^2 + \omega^2(L_d + L_e)(\alpha + p - j\omega)} \right\}$$

With exactly the same argument, it is clear that the sub-

transient components can be obtained by putting $[1 + T_d''(p-j\omega)]I_d'$ $\div [1 + T_{d0}''(p-j\omega)]$ and $[1 + T_q''(p-j\omega)]L_q/[1 + T_{q0}''(p-j\omega)]$ in place of L_d and L_q respectively in the steady-state component wherever they occur, and proceeding in like manner to determine their magnitudes and damping coefficients. Difficulty arises, however, in expressing them in reasonably simple forms, owing to the fact that a quadratic equation in $p-j\omega$ has to be solved, unless the total resistance in the armature circuit is small, or the machine has only one damper circuit. Obviously when R_t is negligible, the factor involving L_q in Equ. 8.51 will be cancelled out and no quadratic equation will result. Or, when the machine has only one damper circuit, say in the pole axis, the function $\bar{L}_q(p)$ becomes the constant L_q and the subtransient component can be found in exactly the same manner as the transient component. In fact, it can be derived directly from Equ. 8.56 by replacing L_d' and L_d by L_d'' and L_d' respectively, and also replacing T_{dz}' by T_{dz}'' , T_{dz}'' being given by Equ. 8.55 with L_d' L_d and T_{d0}' replaced by L_d'' , L_d' and T_{d0}'' respectively.

Although it is difficult to find general expressions for the subtransient components when the total resistance in the armature circuit is appreciable, their numerical values can nevertheless be found with reasonable ease. Replacing L_d and L_q by $[1 + T_d''(p-j\omega)]L_d'/[1 + T_{d0}''(p-j\omega)]$ and $[1 + T_q''(p-j\omega)]L_q/[1 + T_{q0}''(p-j\omega)]$ in the steady-state component and reducing, gives

$$i_a(p) = [1 + T_{d0}''(p-j\omega)] \frac{(R_t - j\omega L_e) [1 + T_{q0}''(p-j\omega)] - j\omega L_q [1 + T_d''(p-j\omega)]}{(p - \alpha_1 - j\omega)(p - \alpha_2 - j\omega)} \cdot \frac{E_{am}}{\lambda} \cdot \frac{-je^{j\lambda}}{p-j\omega} \quad (8.57)$$

where $-\alpha_1$ and $-\alpha_2$ are the roots of the equation

$$0 = R_t^2 + \omega^2(L_d + L_e)(L_q + L_e) + \left\{ R_t^2(T_{d0}'' + T_{q0}'') + \omega^2 L_d(T_{d0}'' + T_{q0}'') + \omega^2 L_q(T_{q0}'' + T_{d0}'') + 2\omega^2 L_e(T_{d0}'' + T_{q0}'') \right\} (p - j\omega) \\ + (R_t^2 T_{d0}'' T_{q0}'' + \omega^2 L_d T_{d0}'' T_{q0}'' + \omega^2 L_q T_{q0}'' T_{d0}'' + 2\omega^2 L_e T_{d0}'' T_{q0}'') (p - j\omega)^2 \quad (8.58)$$

The two subtransient components are therefore given by (in operational form)

$$\left. \begin{aligned} & [(p + \alpha_1 - j\omega) \bar{i}_1(p)]_{p = \alpha_1 + j\omega} \\ \text{and } & [(p + \alpha_2 - j\omega) \bar{i}_1(p)]_{p = \alpha_2 + j\omega} \end{aligned} \right\} \quad (8.59)$$

where $\bar{i}_1(p)$ is given by Equ. 8.57.

The difficulty of getting simple expressions for the subtransient components in general terms may be looked at from the physical point of view. One can see from Eqs. 8.51 and 8.56 that they both consist of the factor $j\epsilon^{j\lambda} [R_t - j\omega(L_q + L_e)]$ which may be reduced to $\epsilon^{j(\lambda + \theta_z)}$, where $\theta_z = \cot^{-1} \omega(L_q + L_e)/R_t$, so that there is no change in the relative phase position of the steady-state and the transient components. With no quadrature damping, the phase angle of the current at the instant of short circuit is also $\lambda + \theta_z$, as $L_q'' = L_q$. It is thus possible to express the armature current in several components of equal frequency and fixed phase position, differing only in magnitude and decrement factor. On the other hand, when there is quadrature damping, the phase angle at the instant of short circuit becomes $\lambda + \theta_z'$, where $\theta_z' = \cot^{-1} \omega(L_q'' + L_e)/R_t$, while the steady-state phase angle is still $\lambda + \theta_z$. It becomes, therefore, impossible to express the complete story of the current variation from its initial value to its final value by components of equal frequency and fixed phase position, or discontinuity must be introduced, which is contrary to the physical phenomenon. The current, for a short period of time immediately follow-

ing the short circuit, must be represented by a periodic function with varying magnitude and phase, or by what amounts to the same thing, two periodic functions of suitable magnitude and slightly different frequencies, as those given by the inverse transforms of Equ. 8.59.

The method of obtaining the armature-transient component has been given in Article 8.2 for the case of no external impedance. The arguments there still apply when there is external impedance, so that the armature-transient component of $\bar{i}_1(p)$ is obtainable from Equ. 8.17 by replacing R_a , L_d'' and L_q'' by R_t , $(L_d'' + L_e)$ and $(L_q'' + L_e)$ respectively.

(9) SINGLE-PHASE SHORT CIRCUITS

(9.1) Line-to-neutral Short Circuit

Single-phase short circuits are the simplest types of unsymmetrical operations of a synchronous machine. The short circuit may be from line to neutral or line to line. These two cases are virtually the same in matter of finding their solutions although the solution for the line-to-line case has a simpler expression owing to the adoption of the three-phase constants. The line-to-neutral case is more general and will be treated in detail. The solution for the line-to-line case can be obtained from that for the line-to-neutral case by simple modifications.

When a machine, initially open-circuited, is short-circuited from line to neutral at the instant when the field axis is ahead of the axis of the short-circuited phase by an angle λ , the operational voltage equation of the armature circuit can be obtained from Equ. 7.7 by putting $\bar{v}_{at}(p) = 0$, $\bar{i}_b(p) = \bar{i}_c(p) = 0$ and deleting the axes b and c. Thus,

$$\bar{e}_a(p) = \left\{ R_a + \frac{1}{3} p [L_0 + L_s(p+j\omega) + \cos\gamma] \right\} \bar{i}_a(p) + \frac{1}{3} p e^{-j\lambda} \bar{L}_D(p+j\omega) \bar{i}_a(p+j\omega) + \frac{1}{3} p e^{j\lambda} \bar{L}_D(p-j\omega) \bar{i}_a(p-j\omega) \quad (9.1)$$

Alternatively, Equ. 9.1 can be obtained from the α , β , 0 component equations (Eqs. 7.29, 7.30 and 7.31) by using the terminal conditions

$$\left. \begin{aligned} \bar{i}_\beta(p) &= 0 \\ \bar{i}_\alpha(p) &= 2 \bar{i}_0(p) \end{aligned} \right\} \quad (9.2)$$

and $\bar{v}_{\alpha t}(p) + \bar{v}_{0 t}(p) = 0$

This is, of course, a roundabout process, but it does serve to

show the validity of the transformation performed.

(9.2) Operational Solution

Equ. 9.1 may be written as

$$\bar{F}_a(p) = \bar{L}_a(p) + \frac{1}{2} e^{-j2\lambda} \bar{K}_a(p) \bar{L}_a(p+j2\omega) + \frac{1}{2} e^{j2\lambda} \bar{K}_a^*(p) \bar{L}_a(p-j2\omega) \quad (9.3)$$

$$\left. \begin{aligned} \text{where } \bar{F}_a(p) &= \frac{\bar{G}_a(p)}{R_a + \frac{1}{3} p [L_o + \bar{L}_S(p+j\omega) + \bar{L}_S(p-j\omega)]} \\ \bar{K}_a(p) &= \frac{2}{3} \cdot \frac{p \bar{L}_D(p+j\omega)}{R_a + \frac{1}{3} p [L_o + \bar{L}_S(p+j\omega) + \bar{L}_S(p-j\omega)]} \\ \text{and } \bar{K}_a^*(p) &= \frac{2}{3} \cdot \frac{p \bar{L}_D(p-j\omega)}{R_a + \frac{1}{3} p [L_o + \bar{L}_S(p+j\omega) + \bar{L}_S(p-j\omega)]} = \text{Conjugate of } \bar{K}_a(p) \end{aligned} \right\} \quad (9.4)$$

which has the same form as Equ. 7.41. The solution of Equ. 9.3 is therefore deducible from Equ. 7.57, the solution of Equ. 7.41. Thus,

$$\bar{L}_a(p) = \bar{B}_o(p) \bar{F}_a(p) + \sum_{n=1}^{\infty} \left\{ \bar{B}_n(p) \bar{F}_a(p+j2n\omega) + \text{Conj} \right\} \quad (9.5)$$

$$\left. \begin{aligned} \text{where } \bar{B}_o(p) &= \frac{1}{\bar{x}_a(p) + \bar{x}_a^*(p) - 1} \\ \bar{B}_n(p) &= \left(-\frac{1}{2}\right)^n e^{-j2n\lambda} \bar{B}_o(p+j2n\omega) \prod_{\nu=0}^{n-1} \frac{\bar{K}_a(p+j2\nu\omega)}{\bar{x}_a(p+j2\nu\omega)} \\ \bar{x}_a(p) &= 1 - \frac{\frac{1}{2} \bar{K}_a^*(p) \bar{K}_a(p-j2\omega)}{\bar{x}_a(p-j2\omega)} \\ &= 1 - \frac{\frac{1}{2} \bar{K}_a^*(p) \bar{K}_a(p-j2\omega)}{1 - \frac{\frac{1}{2} \bar{K}_a^*(p-j2\omega) \bar{K}_a(p-j4\omega)}{1 - \dots}} \\ \text{and } \bar{x}_a^*(p) &= 1 - \frac{\frac{1}{2} \bar{K}_a(p) \bar{K}_a^*(p+j2\omega)}{\bar{x}_a^*(p+j2\omega)} \end{aligned} \right\} \quad (9.6)$$

Because of the complicated nature of $\bar{i}_a(p)$, Equ. 9.5, some simplifying assumptions must be made if a reasonably simple form of $i_a(t)$, the inverse transform of $\bar{L}_a(p)$, is to be obtained. The subsequent articles will show how the various components of $i_a(t)$ are obtained and the assumptions involved.

(9.3) Steady-state Component of Short-circuit Current

Substituting $\bar{e}_a(p) = (\omega \cos \lambda + p \sin \lambda) E_{am} / (p^2 + \omega^2)$ in the expression for $\bar{F}_a(p)$, Equ. 9.4, it is clear that the steady-state component of the armature current is an odd-harmonic series since the function $\bar{F}_a(p+j2n\omega)$ will have the factors $p + j(2n+1)\omega$ and $p + j(2n-1)\omega$ in its denominator. The steady-state component, in operational form, may thus be represented by

$$\bar{i}_a(p) \left[\text{steady} \right] = \sum_{n=0}^{\infty} \left\{ \frac{C_n}{p+j(2n+1)\omega} + C_n j \right\} \quad (9.7)$$

where $C_n = \left[\{ p+j(2n+1)\omega \} \bar{i}_a(p) \right]_{p=-j(2n+1)\omega} \quad (9.8)$

It is not necessary to include all terms of $\bar{i}_a(p)$ given by Equ. 9.5 for the evaluation of C_n , since not all of them have in their denominators the factor $p + j(2n+1)\omega$. In fact, only the two terms $\bar{B}_n(p) \bar{F}_a(p+j2n\omega)$ and $\bar{B}_{n+1}(p) \bar{F}_a(p+j2n+1\omega)$ which have such a factor in their denominators. Consequently Equ. 9.8 can be reduced to

$$C_n = \left[\{ p+j(2n+1)\omega \} \left\{ \bar{B}_n(p) \bar{F}_a(p+j2n\omega) + \bar{B}_{n+1}(p) \bar{F}_a(p+j2n+1\omega) \right\} \right]_{p=-j(2n+1)\omega} \quad (9.9)$$

The quantity C_n is now expressed in a form ready for its evaluation. In order to show the evaluation of C_n in a clearer manner, let each factor of Equ. 9.9 be considered separately. Thus, provided R_a is negligible compared with ωL_d , ωL_q , $\omega L_d''$ or $\omega L_q''$ and $2\omega T_d'$, $2\omega T_{d0}'$, $2\omega T_d''$, $2\omega T_{d0}''$, $2\omega T_q''$ and $2\omega T_{q0}''$ are large compared with unity,

$$\left[\{ p+j(2n+1)\omega \} \bar{F}_a(p+j2n\omega) \right]_{p=-j(2n+1)\omega} \approx - \frac{3 E_{am} e^{-j\lambda}}{\omega (L_d + L_q + L_d'' + L_q'' + 2L_o)} \quad (9.10)$$

$$\left[\{ p+j(2n+1)\omega \} \bar{F}_a(p+j2n+1\omega) \right]_{p=-j(2n+1)\omega} \approx - \frac{3 E_{am} e^{j\lambda}}{\omega (L_d + L_q + L_d'' + L_q'' + 2L_o)} \quad (9.11)$$

$$\bar{B}_n(-j \overline{2\nu+1} \omega) = \left(-\frac{1}{2}\right)^n \frac{e^{-j2n\lambda}}{\bar{x}_a(-j\omega) + \bar{x}_a^*(-j\omega) - 1} \prod_{\nu=1}^n \frac{\bar{K}_a(-j \overline{2\nu+1} \omega)}{\bar{x}_a(-j \overline{2\nu+1} \omega)} \quad (9.12)$$

$$\bar{B}_{n+1}(-j \overline{2\nu+1} \omega) = \left(-\frac{1}{2}\right)^{n+1} \frac{e^{-j2(n+1)\lambda}}{\bar{x}_a(j\omega) + \bar{x}_a^*(j\omega) - 1} \prod_{\nu=0}^n \frac{\bar{K}_a(-j \overline{2\nu+1} \omega)}{\bar{x}_a(-j \overline{2\nu+1} \omega)} \quad (9.13)$$

In Eqs. 9.12 and 9.13 the values of $\bar{x}_a(\pm j\omega)$, $\bar{x}_a^*(\pm j\omega)$, $\bar{x}_a(-j \overline{2\nu+1} \omega)$ and $\bar{K}_a(-j \overline{2\nu+1} \omega)$ are yet to be found. Using Equ. 9.4 and making similar simplifications as in Eqs. 9.10 and 9.11, it can be shown that their values are given by

$$\left. \begin{aligned} \bar{K}_a(\pm j \overline{2\nu+1} \omega) &= \bar{K}_a^*(\mp j \overline{2\nu+1} \omega) \approx \frac{L_d^* - L_q^*}{L_d + L_q + L_d^* + L_q^* + 2L_o} & \nu = \text{non-zero positive integer} \\ \bar{K}_a(j\omega) &= \bar{K}_a^*(-j\omega) \approx \frac{2(L_d^* - L_q^*)}{L_d + L_q + L_d^* + L_q^* + 2L_o} \\ \bar{K}_a(-j\omega) &= \bar{K}_a^*(j\omega) \approx \frac{-2(L_d - L_q)}{L_d + L_q + L_d^* + L_q^* + 2L_o} \\ \bar{x}_a(-j \overline{2\nu+1} \omega) &\approx \frac{1}{2} \left\{ 1 + \frac{2\sqrt{(L_d + \frac{1}{2}L_o)(L_q^* + \frac{1}{2}L_o)}}{L_d^* + L_q^* + L_o} \right\} & \nu = \text{non-zero positive integer} \\ \bar{x}_a(-j\omega) &= \bar{x}_a^*(j\omega) \approx \frac{L_d + L_q + L_o + 2\sqrt{(L_d + \frac{1}{2}L_o)(L_q^* + \frac{1}{2}L_o)}}{L_d + L_q + L_d^* + L_q^* + 2L_o} \\ \bar{x}_a(j\omega) &= \bar{x}_a^*(-j\omega) \approx 1 - \frac{(L_d - L_q)^2}{(L_d + L_q + L_d^* + L_q^* + 2L_o) [L_d + L_q + L_o + 2\sqrt{(L_d + \frac{1}{2}L_o)(L_q^* + \frac{1}{2}L_o)}]} \end{aligned} \right\} \quad (9.14)$$

Consequently in Eqs. 9.12 and 9.13, the fractions have the values

$$\frac{1}{\bar{x}_a(-j\omega) + \bar{x}_a^*(-j\omega) - 1} = \frac{1}{\bar{x}_a(j\omega) + \bar{x}_a^*(j\omega) - 1} \approx \frac{1}{\frac{1}{4}(L_d + L_q + L_d^* + L_q^* + 2L_o)} \left\{ \frac{1}{L_d + \frac{1}{2}L_o + \sqrt{(L_d + \frac{1}{2}L_o)(L_q^* + \frac{1}{2}L_o)}} + \frac{1}{L_q + \frac{1}{2}L_o + \sqrt{(L_d + \frac{1}{2}L_o)(L_q^* + \frac{1}{2}L_o)}} \right\} \quad (9.15)$$

and the products have the values

$$\left. \begin{aligned} \left(-\frac{1}{2}\right)^n \prod_{\nu=1}^n \frac{\bar{K}_a(-j \overline{2\nu+1} \omega)}{\bar{x}_a(-j \overline{2\nu+1} \omega)} &\approx b_o^n \\ \text{and } \left(-\frac{1}{2}\right)^{n+1} \prod_{\nu=0}^n \frac{\bar{K}_a(-j \overline{2\nu+1} \omega)}{\bar{x}_a(-j \overline{2\nu+1} \omega)} &\approx -b_o^n \cdot \frac{L_d - L_q}{L_d + L_q + L_o + 2\sqrt{(L_d + \frac{1}{2}L_o)(L_q^* + \frac{1}{2}L_o)}} \end{aligned} \right\} \quad (9.16)$$

where $b_o = \frac{\sqrt{L_q^* + \frac{1}{2}L_o} - \sqrt{L_d^* + \frac{1}{2}L_o}}{\sqrt{L_d^* + \frac{1}{2}L_o} + \sqrt{L_q^* + \frac{1}{2}L_o}}$

Evaluating Equ. 9.9 by using Eqs. 9.10, 9.11, 9.12, 9.13, 9.15 and 9.16 gives

$$C_n \approx -\frac{3E_{am}}{\lambda\omega} \cdot \frac{b_o^n e^{-j(2n+1)\lambda}}{L_d + \frac{1}{2}L_o + \sqrt{(L_d + \frac{1}{2}L_o)(L_q^* + \frac{1}{2}L_o)}} \quad (9.17)$$

The steady-state component of current is thus given by

$$\bar{i}_a(p) \left[\begin{smallmatrix} \text{steady} \\ \text{state} \end{smallmatrix} \right] \approx -\frac{3E_{am}}{\lambda\omega} \frac{1}{L_d + \frac{1}{2}L_o + \sqrt{(L_d + \frac{1}{2}L_o)(L_q^2 + \frac{1}{2}L_o)}} \sum_{n=0}^{\infty} \left\{ \frac{b_o^n e^{-j(2n+1)\lambda}}{p + j(2n+1)\omega} + \text{Conj} \right\} \quad (9.18)$$

whose inverse transform is

$$i_a(t) \left[\begin{smallmatrix} \text{Steady} \\ \text{State} \end{smallmatrix} \right] \approx -\frac{3E_{am}}{\omega} \frac{1}{L_d + \frac{1}{2}L_o + \sqrt{(L_d + \frac{1}{2}L_o)(L_q^2 + \frac{1}{2}L_o)}} \sum_{n=0}^{\infty} b_o^n \cos(2n+1)(\omega t + \lambda) \quad (9.19)$$

Equ. 9.19 shows that in the odd-harmonic series the magnitudes of the harmonics are in a geometric progression with the common ratio b_o , and the relative phase relations of the harmonics are simple as they all reach their positive peaks at $t = (2n\pi - \lambda)/\omega$. The simple form of the harmonic series is possible only when the resistances are negligible. The higher the frequency of the harmonic, the nearer will be its magnitude approaching b_o times the magnitude of the harmonic of its preceding order, and the truer will they reach their positive peaks at the same instants, as the resistances become less and less significant.

(9.4) Transient and Subtransient Components of Short-circuit Current

Following the same arguments made in Article 8.6, the transient component can be derived from the steady-state component by replacing L_d by $[1 + T_d^1(p \pm j2n+1)\omega] L_d / [1 + T_{d0}^1(p \pm j2n+1)\omega]$ and proceeding in similar manner to determine the magnitude and the decrement factor, provided as before that $R_{kd} \gg R_f$ and $1/T_d^1$ and $1/T_{d0}^1$ are small compared with ω . Thus, from Equ. 9.18, when p is in the vicinity of $\pm j(2n+1)\omega$,

$$\bar{i}_a(p) \approx -\frac{3E_{am}}{\lambda\omega} \sum_{n=0}^{\infty} \left\{ \frac{1}{\frac{1+T_d^1(p \pm j2n+1)\omega}{1+T_{d0}^1(p \pm j2n+1)\omega} L_d + \frac{1}{2}L_o + \sqrt{(L_d + \frac{1}{2}L_o)(L_q^2 + \frac{1}{2}L_o)}} \cdot \frac{b_o^n e^{-j(2n+1)\lambda}}{p + j(2n+1)\omega} + \text{Conj} \right\}$$

which may be reduced to

$$\bar{I}_a(p) \approx -\frac{3E_{am}}{\lambda\omega} \sum_{n=0}^{\infty} \left\{ \frac{1 + T_{d0}'(p+j(2n+1)\omega)}{[L_d' + \frac{1}{2}L_o + \sqrt{(L_d' + \frac{1}{2}L_o)(L_q' + \frac{1}{2}L_o)] T_{d0}'} \cdot \frac{1}{p+j(2n+1)\omega} \cdot \frac{b_o^n e^{-j(2n+1)\lambda}}{\frac{1}{T_{d(1-n)}} + p+j(2n+1)\omega} + \text{Conj} \right\} \quad (9.20)$$

$$\text{where } T_{d(1-n)}' = \frac{L_d' + \frac{1}{2}L_o + \sqrt{(L_d' + \frac{1}{2}L_o)(L_q' + \frac{1}{2}L_o)}}{L_d' + \frac{1}{2}L_o + \sqrt{(L_d' + \frac{1}{2}L_o)(L_q' + \frac{1}{2}L_o)}} T_{d0}' \quad (9.21)$$

which is independent of n. The transient component is given by

$$\begin{aligned} \bar{I}_a(p) [\text{Transient}] &= \sum_{n=0}^{\infty} \left[\left(\frac{1}{T_{d(1-n)}} + p+j(2n+1)\omega \right) \bar{I}_a(p) \right]_{p=-\frac{1}{T_{d(1-n)}}-j(2n+1)\omega} \cdot \frac{1}{\frac{1}{T_{d(1-n)}} + p+j(2n+1)\omega} + \text{Conj} \\ &\approx -\frac{3E_{am}}{\lambda\omega} \left\{ \frac{1}{L_d' + \frac{1}{2}L_o + \sqrt{(L_d' + \frac{1}{2}L_o)(L_q' + \frac{1}{2}L_o)}} - \frac{1}{L_d + \frac{1}{2}L_o + \sqrt{(L_d' + \frac{1}{2}L_o)(L_q' + \frac{1}{2}L_o)}} \right\} \\ &\quad \times \sum_{n=0}^{\infty} \left\{ \frac{b_o^n e^{-j(2n+1)\lambda}}{\frac{1}{T_{d(1-n)}} + p+j(2n+1)\omega} + \text{Conj} \right\} \quad (9.22) \end{aligned}$$

whose inverse transform is

$$\begin{aligned} i_a(t) [\text{Transient}] &\approx -\frac{3E_{am}}{\omega} \left\{ \frac{1}{L_d' + \frac{1}{2}L_o + \sqrt{(L_d' + \frac{1}{2}L_o)(L_q' + \frac{1}{2}L_o)}} - \frac{1}{L_d + \frac{1}{2}L_o + \sqrt{(L_d' + \frac{1}{2}L_o)(L_q' + \frac{1}{2}L_o)}} \right\} \\ &\quad \times e^{-t/T_{d(1-n)}} \sum_{n=0}^{\infty} b_o^n \cos(2n+1)(\omega t + \lambda) \quad (9.23) \end{aligned}$$

Similarly, providing $1/T_d''$ and $1/T_{d0}''$ are small compared with ω , the subtransient component can be obtained by putting $[1 + T_d''(p+j2n+1\omega)] L_d' / [1 + T_{d0}''(p+j2n+1\omega)]$ in place of L_d in the operational form of the steady-state component and proceeding the same as in deriving the transient component. The result will be similar to Equ. 9.22 except L_d'' , L_d' and $T_{d(1-n)}''$ will take the place of L_d' , L_d and $T_{d(1-n)}'$ respectively, $T_{d(1-n)}''$ being given by

$$T_{d(1-n)}'' = \frac{L_d'' + \frac{1}{2}L_o + \sqrt{(L_d'' + \frac{1}{2}L_o)(L_q' + \frac{1}{2}L_o)}}{L_d'' + \frac{1}{2}L_o + \sqrt{(L_d'' + \frac{1}{2}L_o)(L_q' + \frac{1}{2}L_o)}} T_{d0}'' \quad (9.24)$$

which is obtainable from Equ. 9.21 by putting L_d'' , L_d' and T_{d0}'' in place of L_d' , L_d and T_{d0}' respectively. Hence the subtransient component of current is, similar to Equ. 9.23,

$$\begin{aligned} i_a(t) [\text{Subtransient}] &\approx -\frac{3E_{am}}{\omega} \left\{ \frac{1}{L_d'' + \frac{1}{2}L_o + \sqrt{(L_d'' + \frac{1}{2}L_o)(L_q' + \frac{1}{2}L_o)}} - \frac{1}{L_d + \frac{1}{2}L_o + \sqrt{(L_d' + \frac{1}{2}L_o)(L_q' + \frac{1}{2}L_o)}} \right\} \\ &\quad \times e^{-t/T_{d(1-n)}''} \sum_{n=0}^{\infty} b_o^n \cos(2n+1)(\omega t + \lambda) \quad (9.25) \end{aligned}$$

As the steady-state, transient and subtransient components

all have the same series expression, they can be combined to form the odd-harmonic series of the short-circuit current as

$$i_a(t) [_{series}^{Odd}] \approx -\frac{3E_{am}}{\omega} \left\{ \left[\frac{1}{L_d'' + \frac{1}{2}L_o + \sqrt{(L_d'' + \frac{1}{2}L_o)(L_q'' + \frac{1}{2}L_o)}} - \frac{1}{L_d' + \frac{1}{2}L_o + \sqrt{(L_d' + \frac{1}{2}L_o)(L_q' + \frac{1}{2}L_o)}} \right] e^{-t/T_d''(\omega)} \right. \\ \left. + \left[\frac{1}{L_d'' + \frac{1}{2}L_o + \sqrt{(L_d'' + \frac{1}{2}L_o)(L_q'' + \frac{1}{2}L_o)}} - \frac{1}{L_d + \frac{1}{2}L_o + \sqrt{(L_d + \frac{1}{2}L_o)(L_q + \frac{1}{2}L_o)}} \right] e^{-t/T_d'(\omega)} \right. \\ \left. + \frac{1}{L_d + \frac{1}{2}L_o + \sqrt{(L_d + \frac{1}{2}L_o)(L_q + \frac{1}{2}L_o)}} \right\} \times \sum_{n=1,3,5}^{\infty} b_o^n \cos((2n+1)(\omega t + \lambda)) \quad (9.26)$$

which, apart from the harmonics, has a close resemblance to the fundamental-frequency components (comprising the steady-state, transient and subtransient components) of the symmetrical three-phase short-circuit current through external inductances. In fact, as far as the fundamental-frequency terms in Equ. 9.26 are concerned, the single-phase line-to-neutral short circuit may be treated as a symmetrical three-phase short circuit through an external inductance of $\sqrt{(L_d'' + \frac{1}{2}L_o)(L_q'' + \frac{1}{2}L_o)} + \frac{1}{2}L_o$, and the open-circuit voltage is three times as large (see Article 8.6).

(9.5) Armature-transient Component of Short-circuit Current

It is clear that when the armature resistance is neglected, the function $\overline{F_a}(p+j2n\omega)$ has a factor $p + j2n\omega$ in its denominator in addition to the factors already considered in connection with the steady-state component, which indicates that there is an even-harmonic series in the short-circuit current. This series is the armature-transient component and, when the armature resistance is not neglected, will decay with a damping coefficient proportional to the armature resistance. As has been anticipated, the armature-transient component of the

single-phase line-to-neutral short-circuit current has a more general expression than the corresponding component of the symmetrical three-phase short-circuit current, where the series stops at the second-harmonic term.

The armature-transient component can be derived from the operational solution (Equ. 9.5) directly in a manner very similar to the derivation of the transient or the subtransient components. When p is in the vicinity of $-j2n\omega$, or when $p = \epsilon - j2n\omega$, the general term of Equ. 9.5 is

$$\left[\bar{B}_n(p) \bar{F}_a(p+j2n\omega) \right]_{p=\epsilon-j2n\omega} \approx \frac{3E_{am} \cos \lambda}{2\omega} \cdot \frac{1}{\sqrt{(L_d'' + \frac{1}{2}L_0)(L_q'' + \frac{1}{2}L_0)}} \cdot \frac{b_0^n \epsilon^{-j2n\lambda}}{\frac{1}{T_{a(t-n)}} + p + j2n\omega} \quad (9.27)$$

where

$$T_{a(t-n)} = \frac{2}{3} \cdot \frac{\sqrt{(L_d'' + \frac{1}{2}L_0)(L_q'' + \frac{1}{2}L_0)}}{R_a} \quad (9.28)$$

Equ. 9.27 is obtained by putting $p = -j2n\omega$ in the expression of $\bar{B}_n(p)\bar{F}_a(p+j2n\omega)$ except for the factor $p + j2n\omega$, which is left unchanged. This approximation is justified because ϵ is a small quantity compared with ω .

Substituting Equ. 9.27 in Equ. 9.5, the armature-transient component is

$$\bar{i}_a(p) \left[\text{Armature transient} \right] \approx \frac{3E_{am} \cos \lambda}{2\omega} \cdot \frac{1}{\sqrt{(L_d'' + \frac{1}{2}L_0)(L_q'' + \frac{1}{2}L_0)}} \left\{ \frac{1}{\frac{1}{T_{a(t-n)}} + p} + \sum_{n=1}^{\infty} \left[\frac{b_0^n \epsilon^{-j2n\lambda}}{\frac{1}{T_{a(t-n)}} + p + j2n\omega} + \text{conj} \right] \right\} \quad (9.29)$$

whose inverse transform is

$$i_a(t) \left[\text{Armature transient} \right] \approx \frac{3E_{am} \cos \lambda}{\omega \sqrt{(L_d'' + \frac{1}{2}L_0)(L_q'' + \frac{1}{2}L_0)}} \epsilon^{-t/T_{a(t-n)}} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} b_0^n \cos 2n(\omega t + \lambda) \right\} \quad (9.30)$$

Note that in the even-harmonic series (armature-transient component) the coefficients of the harmonics are also progressing with the same geometric ratio b_0 (except for the zero-frequency or d.c. term, which is usually singular in an even-harmonic series) as in the odd-harmonic series of the short-

circuit current.

Finally, the total short-circuit current is given by the sum of the two series, the odd-harmonic series (Equ. 9.26), which decreases from its initial value to its steady-state value with two time constants, and the even-harmonic series (Equ. 9.30) which decays with a single time constant and dies away completely when the steady state is reached.

(9.6) Field and Damper Currents

Since the field and damper currents have been expressed in terms of the d and q components of the armature current (Eqs. 7.76, 7.77 and 7.78), it is necessary to find the expressions for $\bar{i}_d(p)$ and $\bar{i}_q(p)$ before proceeding to determine $\bar{i}_f(p)$, $\bar{i}_{kd}(p)$ and $\bar{i}_{kq}(p)$. Since $\bar{i}_1(p) = \bar{i}_a(p)/3$, so that

$$\begin{aligned} \bar{i}_d(p) &= \epsilon^{-j\lambda} \bar{i}_1(p+j\omega) + \text{Conj.} = \frac{1}{3} \epsilon^{-j\lambda} \bar{i}_a(p+j\omega) + \text{Conj.} \\ &\approx -\frac{E_{am}}{\omega} \left\{ \frac{1}{L_d'' + \frac{1}{2}L_o + \sqrt{(L_d'' + \frac{1}{2}L_o)(L_q'' + \frac{1}{2}L_o)}} - \frac{1}{L_d + \frac{1}{2}L_o + \sqrt{(L_d + \frac{1}{2}L_o)(L_q + \frac{1}{2}L_o)}} \right\} \left\{ \frac{1}{\frac{1}{T_{d(t-n)}} + p} + \frac{1+b_o}{2} \sum_{n=1}^{\infty} \left[\frac{b_o^{n-1} \epsilon^{-j2n\lambda}}{\frac{1}{T_{d(t-n)}} + p + j2n\omega} + \text{Conj.} \right] \right\} \\ &\quad - \frac{E_{am}}{\omega} \left\{ \frac{1}{L_d'' + \frac{1}{2}L_o + \sqrt{(L_d'' + \frac{1}{2}L_o)(L_q'' + \frac{1}{2}L_o)}} - \frac{1}{L_d + \frac{1}{2}L_o + \sqrt{(L_d + \frac{1}{2}L_o)(L_q + \frac{1}{2}L_o)}} \right\} \left\{ \frac{1}{\frac{1}{T_{d(t-n)}} + p} + \frac{1+b_o}{2} \sum_{n=1}^{\infty} \left[\frac{b_o^{n-1} \epsilon^{-j2n\lambda}}{\frac{1}{T_{d(t-n)}} + p + j2n\omega} + \text{Conj.} \right] \right\} \\ &\quad - \frac{E_{am}}{\omega} \cdot \frac{1}{L_d + \frac{1}{2}L_o + \sqrt{(L_d + \frac{1}{2}L_o)(L_q + \frac{1}{2}L_o)}} \left\{ \frac{1}{p} + \frac{1+b_o}{2} \sum_{n=1}^{\infty} \left[\frac{b_o^{n-1} \epsilon^{-j2n\lambda}}{p + j2n\omega} + \text{Conj.} \right] \right\} \\ &\quad + \frac{E_{am} \cos \lambda \cdot (1+b_o)}{2\omega \sqrt{(L_d'' + \frac{1}{2}L_o)(L_q'' + \frac{1}{2}L_o)}} \sum_{n=0}^{\infty} \left\{ \frac{b_o^n \epsilon^{-j(2n+1)\lambda}}{\frac{1}{T_{d(t-n)}} + p + j(2n+1)\omega} + \text{Conj.} \right\} \end{aligned} \quad (9.31)$$

$$\begin{aligned} \bar{i}_q(p) &= -j \left[\epsilon^{-j\lambda} \bar{i}_1(p+j\omega) - \text{Conj.} \right] = -\frac{1}{3} j \left[\epsilon^{-j\lambda} \bar{i}_a(p+j\omega) - \text{Conj.} \right] \\ &\approx \frac{jE_{am}(1-b_o)}{2\omega} \left\{ \frac{1}{L_d'' + \frac{1}{2}L_o + \sqrt{(L_d'' + \frac{1}{2}L_o)(L_q'' + \frac{1}{2}L_o)}} - \frac{1}{L_d + \frac{1}{2}L_o + \sqrt{(L_d + \frac{1}{2}L_o)(L_q + \frac{1}{2}L_o)}} \right\} \sum_{n=1}^{\infty} \left\{ \frac{b_o^{n-1} \epsilon^{-j2n\lambda}}{\frac{1}{T_{d(t-n)}} + p + j2n\omega} - \text{Conj.} \right\} \\ &\quad + \frac{jE_{am}(1-b_o)}{2\omega} \left\{ \frac{1}{L_d'' + \frac{1}{2}L_o + \sqrt{(L_d'' + \frac{1}{2}L_o)(L_q'' + \frac{1}{2}L_o)}} - \frac{1}{L_d + \frac{1}{2}L_o + \sqrt{(L_d + \frac{1}{2}L_o)(L_q + \frac{1}{2}L_o)}} \right\} \sum_{n=1}^{\infty} \left\{ \frac{b_o^{n-1} \epsilon^{-j2n\lambda}}{\frac{1}{T_{d(t-n)}} + p + j2n\omega} - \text{Conj.} \right\} \\ &\quad + \frac{jE_{am}(1-b_o)}{2\omega} \cdot \frac{1}{L_d + \frac{1}{2}L_o + \sqrt{(L_d + \frac{1}{2}L_o)(L_q + \frac{1}{2}L_o)}} \sum_{n=1}^{\infty} \left\{ \frac{b_o^{n-1} \epsilon^{-j2n\lambda}}{p + j2n\omega} - \text{Conj.} \right\} \\ &\quad - \frac{jE_{am} \cos \lambda \cdot (1-b_o)}{2\omega \sqrt{(L_d'' + \frac{1}{2}L_o)(L_q'' + \frac{1}{2}L_o)}} \sum_{n=0}^{\infty} \left\{ \frac{b_o^n \epsilon^{-j(2n+1)\lambda}}{\frac{1}{T_{d(t-n)}} + p + j(2n+1)\omega} - \text{Conj.} \right\} \end{aligned} \quad (9.32)$$

Substituting Equ. 9.31 in Equ. 7.76 and again assuming $R_{kd} \gg R_f$ and neglecting the resistances compared with the reactances at normal (fundamental) or higher frequencies, the field current (excluding I_{f0}) is found to be

$$\begin{aligned} \bar{I}_f(p) \approx M_d^1 I_{f0} \left\{ \left[\frac{L_{kd} - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1}{L_d + \frac{1}{2} L_o + \sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} - \frac{1}{L_f} \cdot \frac{1}{L_d + \frac{1}{2} L_o + \sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} \right] \cdot \frac{1}{\frac{1}{T_{d(l-n)}} + p} \right. \\ + \frac{1}{L_f} \cdot \frac{1}{L_d + \frac{1}{2} L_o + \sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} \cdot \frac{1}{\frac{1}{T_{d(l-n)}} + p} \\ + \frac{L_{kd} - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1+b_o}{2} \cdot \left[\frac{1}{L_d + \frac{1}{2} L_o + \sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} - \frac{1}{L_d + \frac{1}{2} L_o + \sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} \right] \sum_{n=1}^{\infty} \left[\frac{b_o^{n-1} e^{-j2n\lambda}}{\frac{1}{T_{d(l-n)}} + p + j2n\omega} + \text{Conj} \right] \\ - \frac{L_{kd} - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1+b_o}{2} \cdot \left[\frac{1}{L_d + \frac{1}{2} L_o + \sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} - \frac{1}{L_d + \frac{1}{2} L_o + \sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} \right] \sum_{n=1}^{\infty} \left[\frac{b_o^{n-1} e^{-j2n\lambda}}{\frac{1}{T_{d(l-n)}} + p + j2n\omega} + \text{Conj} \right] \\ + \frac{L_{kd} - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1+b_o}{2} \cdot \frac{1}{L_d + \frac{1}{2} L_o + \sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} \sum_{n=1}^{\infty} \left[\frac{b_o^{n-1} e^{-j2n\lambda}}{p + j2n\omega} + \text{Conj} \right] \\ - \frac{L_{kd} - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1+b_o}{2} \cdot \frac{\cos \lambda}{\sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} \sum_{n=0}^{\infty} \left[\frac{b_o^n e^{-j(2n+1)\lambda}}{\frac{1}{T_{d(l-n)}} + p + j(2n+1)\omega} + \text{Conj} \right] \left. \right\} \quad (9.33) \end{aligned}$$

Substituting Equ. 9.31 in Equ. 7.77 and making similar simplifications gives the direct-axis-damper current:

$$\begin{aligned} \bar{I}_{kd}(p) \approx M_d^1 I_{f0} \frac{L_f - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \left\{ \frac{1}{L_d + \frac{1}{2} L_o + \sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} \cdot \frac{1}{\frac{1}{T_{d(l-n)}} + p} \right. \\ + \frac{1+b_o}{2} \cdot \left[\frac{1}{L_d + \frac{1}{2} L_o + \sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} - \frac{1}{L_d + \frac{1}{2} L_o + \sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} \right] \sum_{n=1}^{\infty} \left[\frac{b_o^{n-1} e^{-j2n\lambda}}{\frac{1}{T_{d(l-n)}} + p + j2n\omega} + \text{Conj} \right] \\ + \frac{1+b_o}{2} \cdot \left[\frac{1}{L_d + \frac{1}{2} L_o + \sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} - \frac{1}{L_d + \frac{1}{2} L_o + \sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} \right] \sum_{n=1}^{\infty} \left[\frac{b_o^{n-1} e^{-j2n\lambda}}{\frac{1}{T_{d(l-n)}} + p + j2n\omega} + \text{Conj} \right] \\ + \frac{1+b_o}{2} \cdot \frac{1}{L_d + \frac{1}{2} L_o + \sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} \sum_{n=1}^{\infty} \left[\frac{b_o^{n-1} e^{-j2n\lambda}}{p + j2n\omega} + \text{Conj} \right] \\ - \frac{1+b_o}{2} \cdot \frac{\cos \lambda}{\sqrt{(L_d^2 + \frac{1}{4} L_o)(L_f^2 + \frac{1}{4} L_o)}} \sum_{n=0}^{\infty} \left[\frac{b_o^n e^{-j(2n+1)\lambda}}{\frac{1}{T_{d(l-n)}} + p + j(2n+1)\omega} + \text{Conj} \right] \left. \right\} \quad (9.34) \end{aligned}$$

Similarly the quadrature-axis-damper current can be obtained by substituting Equ. 9.32 in Equ. 7.78 and reducing:

$$\begin{aligned} \bar{t}_{kq}(p) \approx & -\frac{M_d M_q I_{f0}(1-b_0)}{2 L_{kq}} \left\{ \left[\frac{1}{L_d' + \frac{1}{2} L_0 + \sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} - \frac{1}{L_d' + \frac{1}{2} L_0 + \sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} \right] \sum_{n=1}^{\infty} \left[\frac{j b_0^{n-1} e^{-j 2 n \lambda}}{\frac{1}{T_{d'(1-n)}} + p + j 2 n \omega} + \text{Conj} \right] \right. \\ & + \left[\frac{1}{L_d' + \frac{1}{2} L_0 + \sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} - \frac{1}{L_d' + \frac{1}{2} L_0 + \sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} \right] \sum_{n=1}^{\infty} \left[\frac{j b_0^{n-1} e^{-j 2 n \lambda}}{\frac{1}{T_{d'(1-n)}} + p + j 2 n \omega} + \text{Conj} \right] \\ & + \frac{1}{L_d' + \frac{1}{2} L_0 + \sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} \sum_{n=1}^{\infty} \left[\frac{j b_0^{n-1} e^{-j 2 n \lambda}}{p + j 2 n \omega} + \text{Conj} \right] \\ & \left. - \frac{\cos \lambda}{\sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} \sum_{n=0}^{\infty} \left[\frac{j b_0^n e^{-j (2n+1) \lambda}}{\frac{1}{T_{d'(1-n)}} + p + j (2n+1) \omega} + \text{Conj} \right] \right\} \end{aligned} \quad (9.35)$$

Expressed explicitly, the field and damper current are

$$\begin{aligned} i_f(t) = & I_{f0} + \mathcal{L}^{-1} \bar{t}_f(p) \\ \approx & I_{f0} + M_d^2 I_{f0} \left\{ \left[\frac{L_{kd} - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1}{L_d' + \frac{1}{2} L_0 + \sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} - \frac{1}{L_f} \cdot \frac{1}{L_d' + \frac{1}{2} L_0 + \sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} \right] e^{-t/T_{d'(1-n)}} \right. \\ & \left. - \frac{1}{L_f} \cdot \frac{1}{L_d' + \frac{1}{2} L_0 + \sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} e^{-t/T_{d'(1-n)}} \right\} \\ & + M_d^2 I_{f0} \frac{L_{kd} - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{(1+b_0) g_0(t)}{L_d' + \frac{1}{2} L_0 + \sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} \sum_{n=1}^{\infty} b_0^{n-1} \cos 2n\theta \\ & - M_d^2 I_{f0} \frac{L_{kd} - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{(1+b_0) \cos \lambda}{\sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} e^{-t/T_{d'(1-n)}} \sum_{n=0}^{\infty} b_0^n \cos (2n+1)\theta \end{aligned} \quad (9.36)$$

$$\text{where } g_0(t) = \left(1 - \frac{T_{d'(1-n)}}{T_{d0}}\right) e^{-t/T_{d'(1-n)}} + \left(1 - \frac{T_{d'(1-n)}}{T_{d0}}\right) \cdot \frac{T_{d'(1-n)}}{T_{d0}} e^{-t/T_{d'(1-n)}} + \frac{T_{d'(1-n)}}{T_{d0}} \cdot \frac{T_{d'(1-n)}}{T_{d0}} \quad (9.37)$$

$$\begin{aligned} i_{kd}(t) = & \mathcal{L}^{-1} \bar{t}_{kd}(p) \\ \approx & M_d^2 I_{f0} \cdot \frac{L_f - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1}{L_d' + \frac{1}{2} L_0 + \sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} e^{-t/T_{d'(1-n)}} \\ & - M_d^2 I_{f0} \cdot \frac{L_f - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{(1+b_0) g_0(t)}{L_d' + \frac{1}{2} L_0 + \sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} \sum_{n=1}^{\infty} b_0^{n-1} \cos 2n\theta \\ & - M_d^2 I_{f0} \cdot \frac{L_f - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{(1+b_0) \cos \lambda}{\sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} e^{-t/T_{d'(1-n)}} \sum_{n=0}^{\infty} b_0^n \cos (2n+1)\theta \end{aligned} \quad (9.38)$$

$$\begin{aligned} \text{and } i_{kq}(t) = & \mathcal{L}^{-1} \bar{t}_{kq}(p) \\ \approx & -\frac{M_d M_q I_{f0}}{L_{kq}} \cdot \frac{(1-b_0) g_0(t)}{L_d' + \frac{1}{2} L_0 + \sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} \sum_{n=1}^{\infty} b_0^{n-1} \sin 2n\theta \\ & + \frac{M_d M_q I_{f0} (1-b_0) \cos \lambda}{L_{kq} \sqrt{(L_d' + \frac{1}{2} L_0)(L_q' + \frac{1}{2} L_0)}} e^{-t/T_{d'(1-n)}} \sum_{n=0}^{\infty} b_0^n \sin (2n+1)\theta \end{aligned} \quad (9.39)$$

Comparing Eqs. 9.36 and 9.38 shows that the expressions for the field and direct-axis-damper currents differs chiefly in the d.c. components when the resistances R_f and R_{kd} , which have been assumed vastly different, play the major role in determining the relative magnitudes of the currents. The a.c. components merely divide themselves in the two circuits in

the inverse ratio as the leakage inductances when the resistances become of minor importance.

(9.7) Open-phase Voltages

The terminal voltages of the two open phases, v_{bt} and v_{ct} , are related to the open-circuit voltages and the phase currents by the second and third equations represented by Equ. 7.7. As $i_b = i_c = 0$, v_{bt} and v_{ct} become functions of the short-circuit current in phase a and their open-circuit voltages only. Thus,

$$\bar{v}_{be}(p) = -\bar{e}_b(p) + \frac{1}{3} p \left\{ L_0 + h^2 \bar{L}_s(p+j\omega) + \text{Conj} \right\} \bar{i}_a(p) + \frac{1}{3} p h \epsilon^{-j\lambda} \bar{L}_D(p+j\omega) \bar{i}_a(p+j\omega) + \text{Conj} \quad (9.40)$$

$$\bar{v}_{ce}(p) = -\bar{e}_c(p) + \frac{1}{3} p \left\{ L_0 + h^2 \bar{L}_s(p+j\omega) + \text{Conj} \right\} \bar{i}_a(p) + \frac{1}{3} p h \epsilon^{-j\lambda} \bar{L}_D(p+j\omega) \bar{i}_a(p+j\omega) + \text{Conj} \quad (9.41)$$

Substituting the expressions for $\bar{e}_b(p)$ and $\bar{i}_a(p)$ in Equ.

9.40 and reducing gives

$$\begin{aligned} \bar{v}_{be}(p) = & \frac{E_{om}}{4} \left\{ 3L_0 - j 2\sqrt{3} \sqrt{(L_d' + \frac{1}{2}L_0)(L_q' + \frac{1}{2}L_0)} \right\} \\ & \times \left\{ \left[\frac{1}{L_d' + \frac{1}{2}L_0 + \sqrt{(L_d' + \frac{1}{2}L_0)(L_q' + \frac{1}{2}L_0)}} - \frac{1}{L_d' + \frac{1}{2}L_0 + \sqrt{(L_d' + \frac{1}{2}L_0)(L_q' + \frac{1}{2}L_0)}} \right] \sum_{n=0}^{\infty} \frac{j(2n+1)b_0^n \epsilon^{-j(2n+1)\lambda}}{\frac{1}{T_{d'(L_n)}} + p + j(2n+1)\omega} \right. \\ & + \left[\frac{1}{L_d' + \frac{1}{2}L_0 + \sqrt{(L_d' + \frac{1}{2}L_0)(L_q' + \frac{1}{2}L_0)}} - \frac{1}{L_d' + \frac{1}{2}L_0 + \sqrt{(L_d' + \frac{1}{2}L_0)(L_q' + \frac{1}{2}L_0)}} \right] \sum_{n=0}^{\infty} \frac{j(2n+1)b_0^n \epsilon^{-j(2n+1)\lambda}}{\frac{1}{T_{d'(L_n)}} + p + j(2n+1)\omega} \\ & \left. - \frac{1}{L_d' + \frac{1}{2}L_0 + \sqrt{(L_d' + \frac{1}{2}L_0)(L_q' + \frac{1}{2}L_0)}} \sum_{n=0}^{\infty} \frac{j(2n+1)b_0^n \epsilon^{-j(2n+1)\lambda}}{p + j(2n+1)\omega} \right\} + \text{Conj} \\ & - \frac{E_{om} \cos \lambda}{2} \left\{ \frac{3L_0}{2\sqrt{(L_d' + \frac{1}{2}L_0)(L_q' + \frac{1}{2}L_0)}} - j\sqrt{3} \right\} \sum_{n=1}^{\infty} \frac{j 2n b_0^n \epsilon^{-j 2n\lambda}}{\frac{1}{T_{d'(L_n)}} + p + j 2n\omega} - \text{Conj} \end{aligned} \quad (9.42)$$

whose inverse transform gives the explicit expression for the terminal voltage of phase b:

$$\begin{aligned} v_{bt}(t) = & \frac{\sqrt{3} E_{om}}{2} \cdot \frac{g_0(t)}{L_d' + \frac{1}{2}L_0 + \sqrt{(L_d' + \frac{1}{2}L_0)(L_q' + \frac{1}{2}L_0)}} \left\{ \sqrt{3} L_0 \sum_{n=0}^{\infty} (2n+1) b_0^n \sin(2n+1)\theta \right. \\ & \left. + 2 \sqrt{(L_d' + \frac{1}{2}L_0)(L_q' + \frac{1}{2}L_0)} \sum_{n=0}^{\infty} (2n+1) b_0^n \cos(2n+1)\theta \right\} \\ & - \frac{3L_0 E_{om} \cos \lambda}{2 \sqrt{(L_d' + \frac{1}{2}L_0)(L_q' + \frac{1}{2}L_0)}} e^{-t/T_{d'(L_n)}} \sum_{n=1}^{\infty} 2n b_0^n \sin 2n\theta \\ & - \sqrt{3} E_{om} \cos \lambda e^{-t/T_{d'(L_n)}} \sum_{n=1}^{\infty} 2n b_0^n \cos 2n\theta \end{aligned} \quad (9.43)$$

The terminal voltage of phase c, the other open phase, is directly obtainable from Equ. 9.43 simply by altering the sign of the cosine series. As the magnitudes of the sine and the cosine series are not in the simple ratio $1/\sqrt{3}$, the two open-phase voltages no longer have a $2\pi/3$ radian relation between them.

Equ. 9.43 has a more complicated expression than the short-circuit current, as both the sine and the cosine series exist. Moreover, the magnitudes of the voltage harmonics are augmented by a factor corresponding to their frequencies, which is an obvious fact as the harmonic voltages of the open phases depend on the rate of change of the harmonic currents in the short-circuited phase (the field and damper currents having been eliminated).

The fact that both the sine and the cosine series exist in the expressions for the open-phase voltages makes it difficult to obtain a simple expression for their maximum values. However, it will not be difficult to find the maximum value of the sum or the difference of these two voltages as only the sine series or the cosine series will exist. The difference of these two voltages is in effect the voltage across the terminals b and c, so let it be considered. This voltage is given by

$$\begin{aligned} v_{bct}(t) &= v_{bc}(t) - v_{ct}(t) \\ &= \frac{2\sqrt{3} E_{am} \sqrt{(L_d + \frac{1}{2}L_o)(L_q + \frac{1}{2}L_o)}}{L_d + \frac{1}{2}L_o + \sqrt{(L_d + \frac{1}{2}L_o)(L_q + \frac{1}{2}L_o)}} g_o(t) \sum_{n=0}^{\infty} (2n+1) b_n'' \cos(2n+1)\theta \\ &\quad - 2\sqrt{3} E_{am} \cos \lambda \cdot e^{-t/T_{a1(n)}} \sum_{n=1}^{\infty} 2n b_n'' \cos 2n\theta \end{aligned} \quad (9.44)$$

The magnitude of v_{bct} depends on the switching angle λ as well as the time t . If the decrement factors are neglected,

i.e., if $g_0(t)$ and $e^{-t/Ta(e-n)}$ are kept at their initial value of unity, the maximum value of $v_{bct}(t)$ occurs at $\lambda = 0$ and $\theta = \pi$ (considering only the values of the angles smaller than 2π), that is, at half a cycle after the short circuit at minimum voltage (or maximum flux-linkage) of the short-circuited phase, when it has the value

$$v_{bct} = - \left(\lambda \frac{L_q'' + \frac{1}{2}L_o}{L_d'' + \frac{1}{2}L_o} - 1 \right) \sqrt{3} E_{am} \quad (9.45)$$

which depends on the ratio $(L_q'' + L_o/2)/(L_d'' + L_o/2)$. When the machine has zero subtransient saliency ($L_d'' = L_q''$), this voltage reduces to $\sqrt{3}E_{am}$, the maximum open-circuit value. When the machine has negligible quadrature damping, L_q'' is considerably larger than L_d'' , the maximum voltage across the open-circuit terminals may be several times its normal value.

As the maximum voltage given by Equ. 9.45 occurs half a cycle after the short circuit, this value will not be reached in practice because both $g_0(t)$ and $e^{-t/Ta(e-n)}$ are no longer equal to unity at $t = \pi/\omega$. Consequently with $L_d'' = L_q''$ the voltage across the open-circuit terminals is always less than its open-circuit value of $\sqrt{3}E_{am}$.

(9.8) Short-circuit Torque — Pulsating Component

As in the case of a symmetrical three-phase short circuit, the short-circuit torque in the present case also consists of two components, the pulsating component, which can be found from Equ. 5.23 by using the approximate expressions for the currents and the flux-linkages, and the unidirectional compo-

ment, which can be found by considering the losses of the machine. Only the pulsating component of torque is considered for the present. The unidirectional component will be considered in Chapter 11.

The explicit form of the d and q components of current are, from Eqs. 9.31 and 9.32,

$$i_d \approx -\frac{E_{am}}{\omega} \frac{1}{L'_d + \frac{1}{2}L_o + \sqrt{(L'_d + \frac{1}{2}L_o)(L'_q + \frac{1}{2}L_o)}} g_o(t) \left\{ 1 + (1+b_o) \sum_{n=1}^{\infty} b_o^{n-1} \cos 2n\theta \right\} \\ - \frac{E_{am}(1+b_o)\cos\lambda}{\omega \sqrt{(L'_d + \frac{1}{2}L_o)(L'_q + \frac{1}{2}L_o)}} e^{-t/T_{a(l-n)}} \sum_{n=0}^{\infty} b_o^n \cos(2n+1)\theta \quad (9.46)$$

$$i_q \approx \frac{E_{am}}{\omega} \frac{1-b_o}{L'_d + \frac{1}{2}L_o + \sqrt{(L'_d + \frac{1}{2}L_o)(L'_q + \frac{1}{2}L_o)}} g_o(t) \sum_{n=1}^{\infty} b_o^{n-1} \sin 2n\theta \\ - \frac{E_{am}(1-b_o)\cos\lambda}{\omega \sqrt{(L'_d + \frac{1}{2}L_o)(L'_q + \frac{1}{2}L_o)}} e^{-t/T_{a(l-n)}} \sum_{n=0}^{\infty} b_o^n \sin(2n+1)\theta \quad (9.47)$$

Substituting $\bar{i}_d(p)$ and $\bar{i}_q(p)$ in Equ. 7.84 and reducing gives

$$\Psi_d \approx \sqrt{\frac{3}{2}} M_d I_{f_o} \left\{ \frac{\frac{1}{2}L_o + \sqrt{(L'_d + \frac{1}{2}L_o)(L'_q + \frac{1}{2}L_o)}}{L'_d + \frac{1}{2}L_o + \sqrt{(L'_d + \frac{1}{2}L_o)(L'_q + \frac{1}{2}L_o)}} g_o(t) - \frac{L'_d(1+b_o)g_o(t)}{L'_d + \frac{1}{2}L_o + \sqrt{(L'_d + \frac{1}{2}L_o)(L'_q + \frac{1}{2}L_o)}} \sum_{n=1}^{\infty} b_o^{n-1} \cos 2n\theta \right. \\ \left. + \frac{L'_d(1+b_o)\cos\lambda}{\sqrt{(L'_d + \frac{1}{2}L_o)(L'_q + \frac{1}{2}L_o)}} e^{-t/T_{a(l-n)}} \sum_{n=0}^{\infty} b_o^n \cos(2n+1)\theta \right\} \quad (9.48)$$

$$\Psi_q \approx \sqrt{\frac{3}{2}} M_d I_{f_o} L'_q \left\{ \frac{(1-b_o)g_o(t)}{L'_d + \frac{1}{2}L_o + \sqrt{(L'_d + \frac{1}{2}L_o)(L'_q + \frac{1}{2}L_o)}} \sum_{n=1}^{\infty} b_o^{n-1} \sin 2n\theta \right. \\ \left. - \frac{(1-b_o)\cos\lambda}{\sqrt{(L'_d + \frac{1}{2}L_o)(L'_q + \frac{1}{2}L_o)}} e^{-t/T_{a(l-n)}} \sum_{n=0}^{\infty} b_o^n \sin(2n+1)\theta \right\} \quad (9.49)$$

Substituting Eqs. 9.46, 9.47, 9.48 and 9.49 for i_d , i_q , Ψ_d and Ψ_q in Equ. 5.23 and reducing, the pulsating component of torque is found to be

$$T_{(l-n)} = \Psi_d i_q - \Psi_q i_d \\ \approx -M_d^2 I_{f_o}^2 \frac{2\cos\lambda}{L'_d + \frac{1}{2}L_o + \sqrt{(L'_d + \frac{1}{2}L_o)(L'_q + \frac{1}{2}L_o)}} g_o(t) e^{-t/T_{a(l-n)}} \sum_{n=0}^{\infty} (2n+1) b_o^n \sin(2n+1)\theta \\ + M_d^2 I_{f_o}^2 \left\{ \frac{1+b_o}{L'_d + \frac{1}{2}L_o + \sqrt{(L'_d + \frac{1}{2}L_o)(L'_q + \frac{1}{2}L_o)}} g_o^2(t) + \frac{2b_o\cos^2\lambda}{(L'_d + \frac{1}{2}L_o)(L'_q + \frac{1}{2}L_o)} e^{-2t/T_{a(l-n)}} \right\} \sum_{n=1}^{\infty} n b_o^{n-1} \sin 2n\theta \quad (9.50)$$

To get the closed form for the torque as Equ. 9.50, it is necessary to evaluate the product of two infinite harmonic

series. The four types of products involved are given below for reference.

$$\left. \begin{aligned} \sum_{n=1}^{\infty} b_o^{n-1} \sin 2n\theta \times \sum_{n=1}^{\infty} b_o^{n-1} \cos 2n\theta &= \frac{1}{2} \sum_{n=1}^{\infty} n b_o^{n-1} \sin 2(n+1)\theta \\ \sum_{n=0}^{\infty} b_o^n \sin (2n+1)\theta \times \sum_{n=0}^{\infty} b_o^n \cos (2n+1)\theta &= \frac{1}{2} \sum_{n=1}^{\infty} n b_o^{n-1} \sin 2n\theta \\ \sum_{n=0}^{\infty} b_o^n \sin (2n+1)\theta \times \sum_{n=1}^{\infty} b_o^{n-1} \cos 2n\theta &= \frac{1}{2} \sum_{n=0}^{\infty} \left(n - \frac{b_o}{1+b_o} \right) b_o^{n-1} \sin (2n+1)\theta \\ \sum_{n=0}^{\infty} b_o^n \cos (2n+1)\theta \times \sum_{n=1}^{\infty} b_o^{n-1} \sin 2n\theta &= \frac{1}{2} \sum_{n=0}^{\infty} \left(n + \frac{b_o}{1+b_o} \right) b_o^{n-1} \sin (2n+1)\theta \end{aligned} \right\} (9.51)$$

Equ. 9.50 shows that the pulsating component of torque consists of an odd-harmonic and an even-harmonic series. The switching angle λ has a large bearing on the pulsating component of torque, particularly the odd-harmonic series. The even-harmonic series is of the same order of magnitude as the odd-harmonic series. Even with zero subtransient saliency so the $b_o = 0$, a large second-harmonic torque still exist owing to the inevitable second-harmonic currents in the field and damper circuits.

(9.9) Single-phase Line-to-line Short Circuit

When the machine is short-circuited from line to line from no load, at the instant when the field axis is ahead of the open phase by an angle λ , the operational voltage equation of the armature circuits can be obtained by putting $i_a = 0$ in Equ. 7.7 and deleting the axis a. Thus the operational voltage equations of the two short-circuited phases are

$$\begin{aligned} \bar{v}_{bc}(p) + \bar{e}_b(p) &= \left\{ R_a + \frac{1}{2} p [L_o + L_s(p+j\omega) + \text{Conj}] \right\} \bar{i}_b(p) + \frac{1}{2} p \left\{ L_o + L^2 L_s(p+j\omega) + \text{Conj} \right\} \bar{i}_c(p) \\ &\quad + \frac{1}{2} p e^{j\lambda} L_b(p+j\omega) \left\{ L^2 \bar{i}_b(p+j\omega) + \bar{i}_c(p+j\omega) \right\} + \text{Conj} \end{aligned} \quad (9.52)$$

$$\text{and } \bar{v}_{ce}(p) + \bar{e}_c(p) = \frac{1}{3} p \{ L_0 + h \bar{L}_s(p+j\omega) + \text{Conj} \} \bar{L}_b(p) + \{ R_a + \frac{1}{3} p [L_0 + L_s(p+j\omega) + \text{Conj}] \} \bar{L}_c(p) \\ + \frac{1}{3} p e^{j2\lambda} \bar{L}_b(p+j\omega) \{ \bar{L}_b(p+j\omega) + h \bar{L}_c(p+j2\omega) \} + \text{Conj} \quad (9.53)$$

Subtracting Equ. 9.53 from Equ. 9.52 and putting $\bar{v}_{bt}(p)$
- $\bar{v}_{ct}(p) = 0$ and $i_c = -i_b$ gives

$$\bar{e}_b(p) - \bar{e}_c(p) = \{ 2R_a + p [L_s(p+j\omega) + \text{Conj}] \} \bar{L}_b(p) + p e^{-j2(\lambda - \frac{\pi}{2})} \bar{L}_b(p+j\omega) \bar{L}_b(p+j2\omega) + \text{Conj} \quad (9.54)$$

Alternatively this equation can be directly derived from the
 β -component equation by putting $\bar{v}_{\beta t}(p) = 0$ and $\bar{i}_{\alpha}(p) = 0$.

Comparing Eqs. 9.54 and 9.1, and remembering that $e_p = e_c$
 $= \sqrt{3} E_{am} \sin(\omega t + \lambda - \frac{1}{2}\pi)$, it is clear that the solution, as a func-
of time, of Equ. 9.54 is obtainable from Eqs. 9.26 and 9.30
by omitting L_0 , doubling R_a , trebling L_d , L_d' , L_d'' and L_q'' , and
putting $\sqrt{3} E_{am}$ and $\lambda - \pi/2$ in place of E_{am} and λ . Thus

$$i_b(t) = -i_c(t) \\ \approx -\frac{\sqrt{3} E_{am}}{\omega} \frac{1}{L_d' + \sqrt{L_d' L_q'}} g_1(t) \sum_{n=0}^{\infty} (-b_1)^n \sin(2n+1)\theta - \frac{\sqrt{3} E_{am} \sin \lambda}{\omega \sqrt{L_d' L_q'}} e^{-t/T_{a1}(t)} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} (-b_1)^n \cos 2n\theta \right\} \quad (9.55)$$

$$g_1(t) = \left(1 - \frac{T_{d1}(t)}{T_{d0}} \right) e^{-t/T_{d1}(t)} + \left(1 - \frac{T_{d1}(t)}{T_{d0}} \right) \frac{T_{d1}(t)}{T_{d0}} e^{-t/T_{d1}(t)} + \frac{T_{d1}(t)}{T_{d0}} \frac{T_{d1}(t)}{T_{d0}} \\ \left. \begin{aligned} T_{d1}(t) &= \frac{L_d' + \sqrt{L_d' L_q'}}{L_d' + \sqrt{L_d' L_q'}} T_{d0} \\ T_{d1}'(t) &= \frac{L_d' + \sqrt{L_d' L_q''}}{L_d' + \sqrt{L_d' L_q''}} T_{d0}' \\ T_{d1}''(t) &= \sqrt{L_d' L_q''} / R_a \\ b_1 &= \frac{\sqrt{L_q''} - \sqrt{L_d'}}{\sqrt{L_d'} + \sqrt{L_q''}} \end{aligned} \right\} \quad (9.56)$$

The field and damper currents are obtainable from Eqs. 9.36,
9.38 and 9.39 by omitting L_0 , altering the time constants and
replacing b_0 , $g_0(t)$ and λ by b_1 , $g_1(t)$ and $\lambda - \pi/2$ respective-
ly. Thus,

$$i_j(t) \approx I_{j0} + M_d^2 I_{j0} \left\{ \left[\frac{L_{kd} - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1}{L_d + \sqrt{L_d^2 L_q^2}} - \frac{1}{L_f} \cdot \frac{1}{L_d + \sqrt{L_d^2 L_q^2}} \right] e^{-t/T_{d'(L-L)}} + \frac{1}{L_f} \cdot \frac{1}{L_d + \sqrt{L_d^2 L_q^2}} e^{-t/T_{d'(L-L)}} \right\} \\ + M_d^2 I_{j0} \frac{L_{kd} - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1+b_1}{L_d + \sqrt{L_d^2 L_q^2}} g_1(t) \sum_{n=1}^{\infty} (-1)^n b_1^{n-1} \cos 2n\theta \\ - M_d^2 I_{j0} \frac{L_{kd} - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{(1+b_1) \sin \lambda}{\sqrt{L_d^2 L_q^2}} e^{-t/T_{d'(L-L)}} \sum_{n=0}^{\infty} (-b_1)^n \sin(2n+1)\theta \quad (9.57)$$

$$i_{kd}(t) \approx M_d^2 I_{j0} \frac{L_f - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{1}{L_d + \sqrt{L_d^2 L_q^2}} \left\{ e^{-t/T_{d'(L-L)}} + (1+b_1) g_1(t) \sum_{n=1}^{\infty} (-1)^n b_1^{n-1} \cos 2n\theta \right\} \\ - M_d^2 I_{j0} \frac{L_f - M_{fkd}}{L_f L_{kd} - M_{fkd}^2} \cdot \frac{(1+b_1) \sin \lambda}{\sqrt{L_d^2 L_q^2}} e^{-t/T_{d'(L-L)}} \sum_{n=0}^{\infty} (-b_1)^n \sin(2n+1)\theta \quad (9.58)$$

$$\text{and } i_{kq}(t) \approx - \frac{M_d M_q I_{j0}}{L_{kq}} \cdot \frac{1-b_1}{L_d + \sqrt{L_d^2 L_q^2}} g_1(t) \sum_{n=1}^{\infty} (-1)^n b_1^{n-1} \sin 2n\theta \\ + \frac{M_d M_q I_{j0} (1-b_1) \sin \lambda}{L_{kq} \sqrt{L_d^2 L_q^2}} e^{-t/T_{d'(L-L)}} \sum_{n=0}^{\infty} (-1)^{n+1} b_1^n \cos(2n+1)\theta \quad (9.59)$$

The open-phase voltage is obtainable from Equ. 9.44 with similar modifications but the factor $\sqrt{3}$ is also neglected as in this case the voltage is measured from line to neutral instead of line to line. Thus,

$$v_{at}(t) \approx - \frac{2E_{am} \sqrt{L_d^2 L_q^2}}{L_d + \sqrt{L_d^2 L_q^2}} g_1(t) \sum_{n=0}^{\infty} (-b_1)^n (2n+1) \sin(2n+1)\theta + 2E_{am} \sin \lambda \cdot e^{-t/T_{d'(L-L)}} \sum_{n=1}^{\infty} (-b_1)^n 2n \cos 2n\theta \quad (9.60)$$

The maximum value of the open-phase voltage, with the decrement factors neglected, occurs also at half a cycle after the short circuit at minimum voltage across the short-circuited phases, i.e., at $t = \pi/\omega$ and $\theta = 3\pi/2$. This maximum value is

$$v_{at}(\max) = (2L_d''/L_d'' - 1)E_{am} \quad (9.61)$$

The pulsating component of the short-circuit torque is obtainable from Equ. 9.50 with similar modifications. Thus,

$$T_{(L-L)} \approx - M_d^2 I_{j0}^2 \frac{2 \sin \lambda}{L_d + \sqrt{L_d^2 L_q^2}} g_1(t) \cdot e^{-t/T_{d'(L-L)}} \sum_{n=0}^{\infty} (-1)^{n+1} b_1^n (2n+1) \cos(2n+1)\theta \\ + M_d^2 I_{j0}^2 \left\{ \frac{1+b_1}{L_d + \sqrt{L_d^2 L_q^2}} g_1^2(t) + \frac{2b_1 \sin^2 \lambda}{\sqrt{L_d^2 L_q^2}} e^{-2t/T_{d'(L-L)}} \right\} \sum_{n=1}^{\infty} (-1)^n b_1^{n-1} n \sin 2n\theta \quad (9.62)$$

(9.10) Experimental Results

Before the experimental results are given, it is desirable to derive the alternative expressions for the short-circuit currents, the open-phase voltages and the field currents which are more convenient for the purpose of computing their values at a particular instant for both cases of single-phase short circuit. It can be shown that by making use of the following mathematical identities

$$\left. \begin{aligned} \sum_{n=0}^{\infty} b^n \sin(2n+1)\theta &\equiv \frac{(B+\sqrt{AB}) \sin \theta}{A+B+(A-B) \cos 2\theta} \\ \sum_{n=0}^{\infty} b^n \cos(2n+1)\theta &\equiv \frac{(A+\sqrt{AB}) \cos \theta}{A+B+(A-B) \cos 2\theta} \\ \sum_{n=0}^{\infty} b^n \sin 2(n+1)\theta &\equiv \frac{1}{2} \cdot \frac{(\sqrt{A}+\sqrt{B})^2 \sin 2\theta}{A+B+(A-B) \cos 2\theta} \\ \sum_{n=0}^{\infty} b^n \cos 2(n+1)\theta &\equiv \frac{\sqrt{AB}(\sqrt{A}+\sqrt{B})^2}{B-A} \left(\frac{1}{A+B+(A-B) \cos 2\theta} - \frac{1}{2\sqrt{AB}} \right) \\ b &= (\sqrt{B}-\sqrt{A})/(\sqrt{A}+\sqrt{B}) \end{aligned} \right\} (9.63)$$

the quantities i_a , v_b , v_c and i_f' for the line-to-neutral short circuit can be expressed as

$$\left. \begin{aligned} i_a &= - \frac{3E_{am}}{x_d'' + x_q'' + x_0 - (x_d'' - x_q'') \cos 2\theta} \left(g_0(t) \cos \theta - e^{-t/T_d'(c-n)} \cos \lambda \right) \\ v_b &= \sqrt{3} E_{am} g_0(t) \frac{\sqrt{3} x_0 [(x_d'' - x_q'' + \frac{1}{2} x_0) \sin \theta - (x_d'' - x_q'') \sin^3 \theta] - 2(x_d'' + \frac{1}{2} x_0) [(x_d'' - 2x_q'' - \frac{1}{2} x_0) \cos \theta - (x_d'' - x_q'') \cos^3 \theta]}{[x_d'' + x_q'' + x_0 - (x_d'' - x_q'') \cos 2\theta]^2} \\ &\quad - \sqrt{3} E_{am} (x_d'' - x_q'') e^{-t/T_d'(c-n)} \cos \lambda \frac{\sqrt{3} x_0 \sin 2\theta - (x_d'' - x_q'') [x_d'' - x_q'' - (x_d'' + x_q'' + x_0) \cos 2\theta]}{[x_d'' + x_q'' + x_0 - (x_d'' - x_q'') \cos 2\theta]^2} \\ v_c &= \sqrt{3} E_{am} g_0(t) \frac{\sqrt{3} x_0 [(2x_d'' - x_q'' + \frac{1}{2} x_0) \sin \theta - (x_d'' - x_q'') \sin^3 \theta] + 2(x_d'' + \frac{1}{2} x_0) [(x_d'' - 2x_q'' - \frac{1}{2} x_0) \cos \theta - (x_d'' - x_q'') \cos^3 \theta]}{[x_d'' + x_q'' + x_0 - (x_d'' - x_q'') \cos 2\theta]^2} \\ &\quad - \sqrt{3} E_{am} (x_d'' - x_q'') e^{-t/T_d'(c-n)} \cos \lambda \frac{\sqrt{3} x_0 \sin 2\theta + (x_d'' - x_q'') [x_d'' - x_q'' - (x_d'' + x_q'' + x_0) \cos 2\theta]}{[x_d'' + x_q'' + x_0 - (x_d'' - x_q'') \cos 2\theta]^2} \\ i_f' &= (x_d - x_d'') I_{f0} \left\{ \left(\frac{L_{ad} - M_{fd}}{L_f + L_{ad} - 2M_{fd}} \frac{1}{x_d'' + \frac{1}{2} x_0 + \sqrt{(x_d'' + \frac{1}{2} x_0)(x_q'' + \frac{1}{2} x_0)}} - \frac{L_{fd}}{L_f} \frac{1}{x_d'' + \frac{1}{2} x_0 + \sqrt{(x_d'' + \frac{1}{2} x_0)(x_q'' + \frac{1}{2} x_0)}} \right) e^{-t/T_d'(c-n)} \right. \\ &\quad \left. + \frac{L_{fd}}{L_f} \frac{1}{x_d'' + \frac{1}{2} x_0 + \sqrt{(x_d'' + \frac{1}{2} x_0)(x_q'' + \frac{1}{2} x_0)}} e^{-t/T_d'(c-n)} \right\} \\ &\quad + \frac{L_{ad} - M_{fd}}{L_f + L_{ad} - 2M_{fd}} \cdot 2(x_d - x_d'') I_{f0} \left\{ \frac{x_d'' + \frac{1}{2} x_0}{x_d'' - x_q''} \left(\frac{1}{x_d'' + x_q'' + x_0 - (x_d'' - x_q'') \cos 2\theta} - \frac{1}{2\sqrt{(x_d'' + \frac{1}{2} x_0)(x_q'' + \frac{1}{2} x_0)}} \right) g_0(t) \right. \\ &\quad \left. - \frac{\cos \lambda \cos \theta}{x_d'' + x_q'' + x_0 - (x_d'' - x_q'') \cos 2\theta} e^{-t/T_d'(c-n)} \right\} \end{aligned} \right\} (9.64)$$

Also the quantities i_b , i_c , v_a and i_f' for the line-to-line short circuit can be expressed as

$$\begin{aligned}
 i_b = -i_c &= -\frac{\sqrt{3} E_{am}}{x_d + x_q' + (x_q' - x_d) \cos 2\theta} \left(g_1(t) \sin \theta - e^{-t/T_{d0}} \sin \lambda \right) \\
 V_a &= 4 x_q' E_{am} g_1(t) \frac{(x_q' - 2x_d) \sin \theta - (x_q' - x_d) \sin^3 \theta}{[x_d + x_q' + (x_q' - x_d) \cos 2\theta]^2} - 2(x_q' - x_d) E_{am} e^{-t/T_{d0}} \sin \lambda \frac{x_q' - x_d + (x_d + x_q') \cos 2\theta}{[x_d + x_q' + (x_q' - x_d) \cos 2\theta]^2} \\
 i_f &= (x_d - x_d') I_{f0} \left\{ \left(\frac{L_{fd} - M_{fd}}{L_f + L_{fd} - 2M_{fd}} \cdot \frac{1}{x_d + \sqrt{x_d x_q'}} - \frac{L_f}{L_f} \cdot \frac{1}{x_d + \sqrt{x_d x_q'}} \right) e^{-t/T_{d0}} + \frac{L_f}{L_f} \cdot \frac{1}{x_d + \sqrt{x_d x_q'}} e^{-t/T_{d0}(1-L)} \right\} \\
 &\quad + \frac{L_{fd} - M_{fd}}{L_f + L_{fd} - 2M_{fd}} \cdot 2(x_d - x_d') I_{f0} \left\{ \frac{x_q'}{x_q' - x_d} \left(\frac{1}{x_d + x_q' + (x_q' - x_d) \cos 2\theta} - \frac{1}{2(x_d x_q')} \right) g_1(t) - \frac{\sin \lambda \sin \theta}{x_d + x_q' + (x_q' - x_d) \cos 2\theta} e^{-t/T_{d0}(1-L)} \right\}
 \end{aligned}$$

The machine under test had the following constants:

$$X_d = X_q = 6, \quad X_0 = 0.9, \quad X_d' = X_d'' = 0.54, \quad X_{d1}'' = 6,$$

$$R_a = 0.15 \quad \text{and} \quad T_{d0}' = 0.0735.$$

The measured and calculated results for the line-to-neutral and the line-to-line short circuits are shown in Figs. 9.1, 9.2, 9.3 and 9.4, where the numerical expressions for the various currents and voltages are also shown.

The results for the line-to-line short circuit with $X_{d1}'' = 1.08$ are shown in Figs. 9.5 and 9.6.

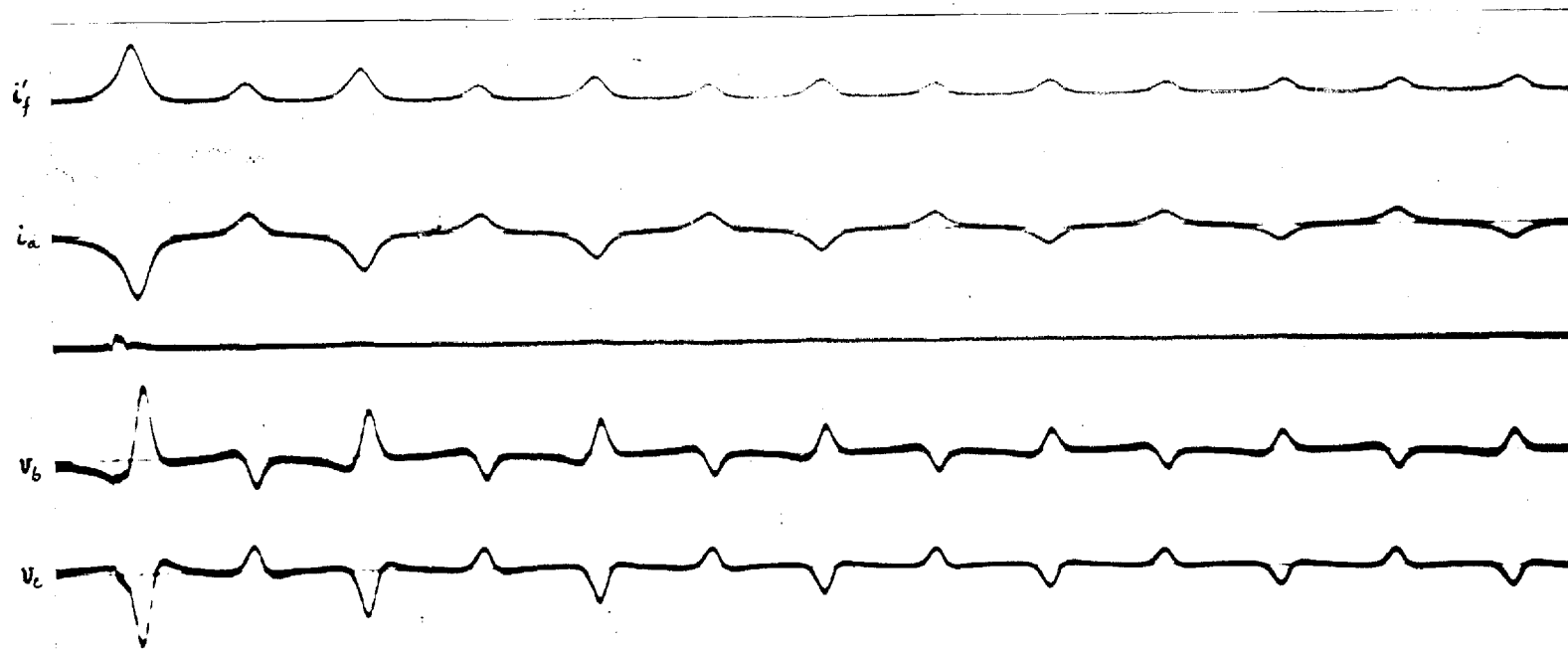


Fig. 9.1

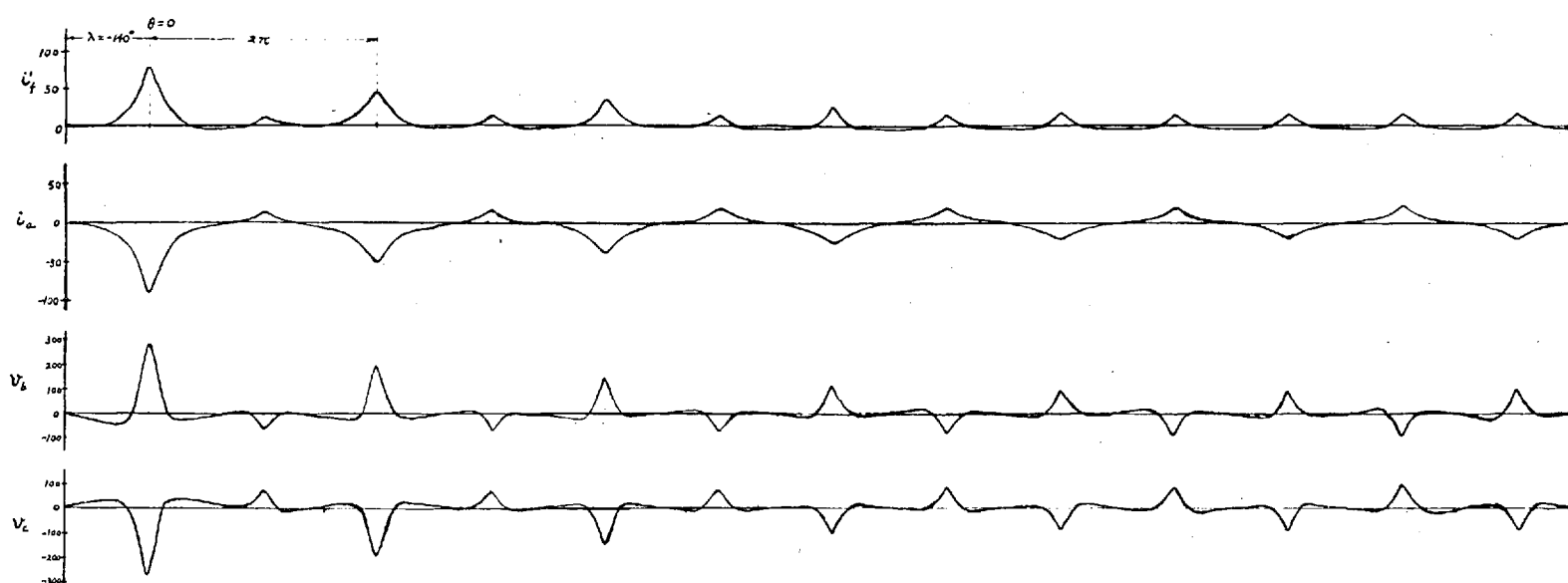


Fig. 9.2. Line-to-neutral Short Circuit

$$i_f' = 15.5 e^{-t/0.0288} + 23.6 \left(\frac{1}{1.36 - \cos 2\theta} - 1.08 \right) (0.607 e^{-t/0.288} + 0.393) + \frac{15.3 \cos \theta}{1.36 - \cos 2\theta} e^{-t/0.0358}$$

$$i_a = -\frac{20.2}{1.36 - \cos 2\theta} \left\{ (0.607 e^{-t/0.0288} + 0.393) \cos \theta + 0.766 e^{-t/0.0358} \right\}$$

$$v_b = \frac{39.6 \sin \theta - 18.1 \sin^3 \theta - 123 \cos \theta + 150 \cos^3 \theta}{(1.36 - \cos 2\theta)^2} (0.607 e^{-t/0.0288} + 0.393) \\ - \frac{48.7 - 13.4 \sin 2\theta - 66.3 \cos 2\theta}{(1.36 - \cos 2\theta)^2} e^{-t/0.0358}$$

$$v_c = \frac{39.6 \sin \theta - 18.1 \sin^3 \theta + 123 \cos \theta - 150 \cos^3 \theta}{(1.36 - \cos 2\theta)^2} (0.607 e^{-t/0.0288} + 0.393) \\ + \frac{48.7 + 13.4 \sin 2\theta - 66.3 \cos 2\theta}{(1.36 - \cos 2\theta)^2} e^{-t/0.0358}$$

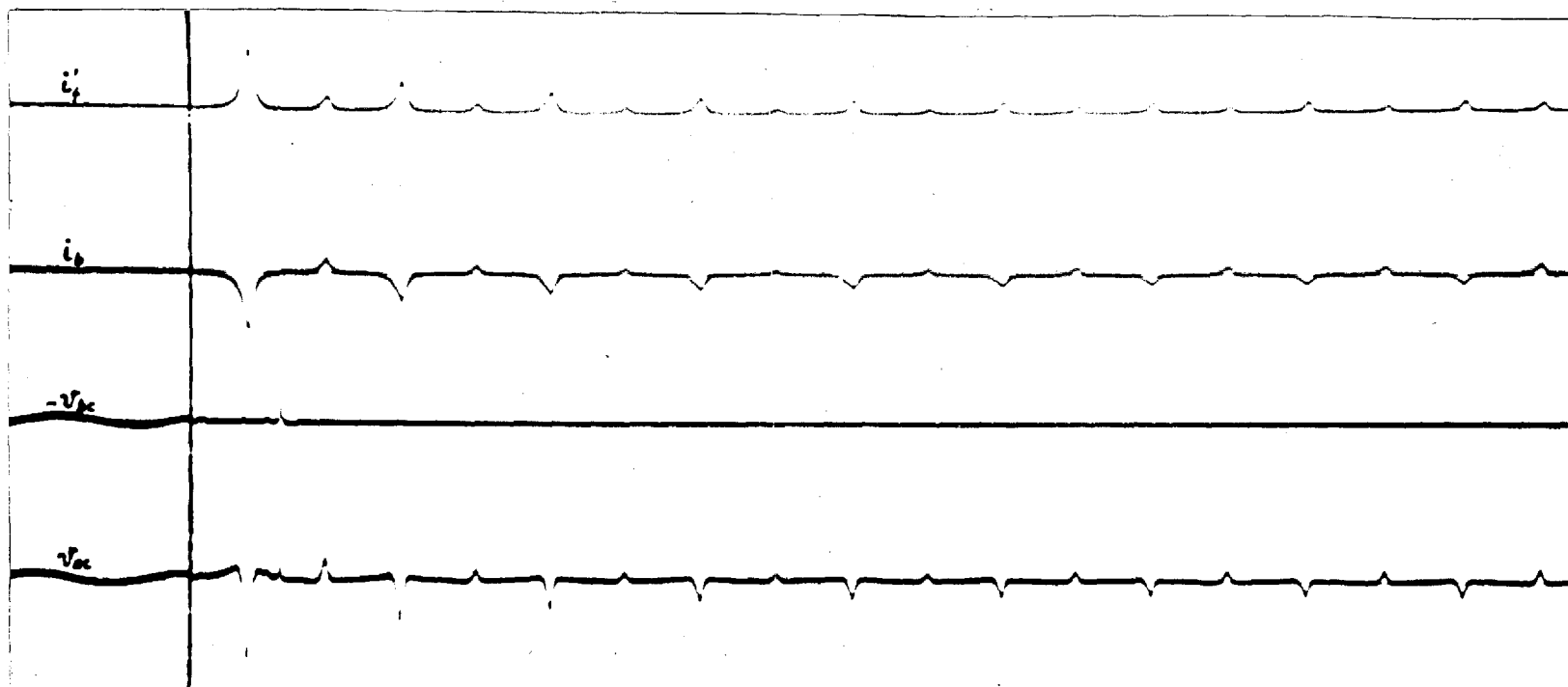


Fig. 9.3

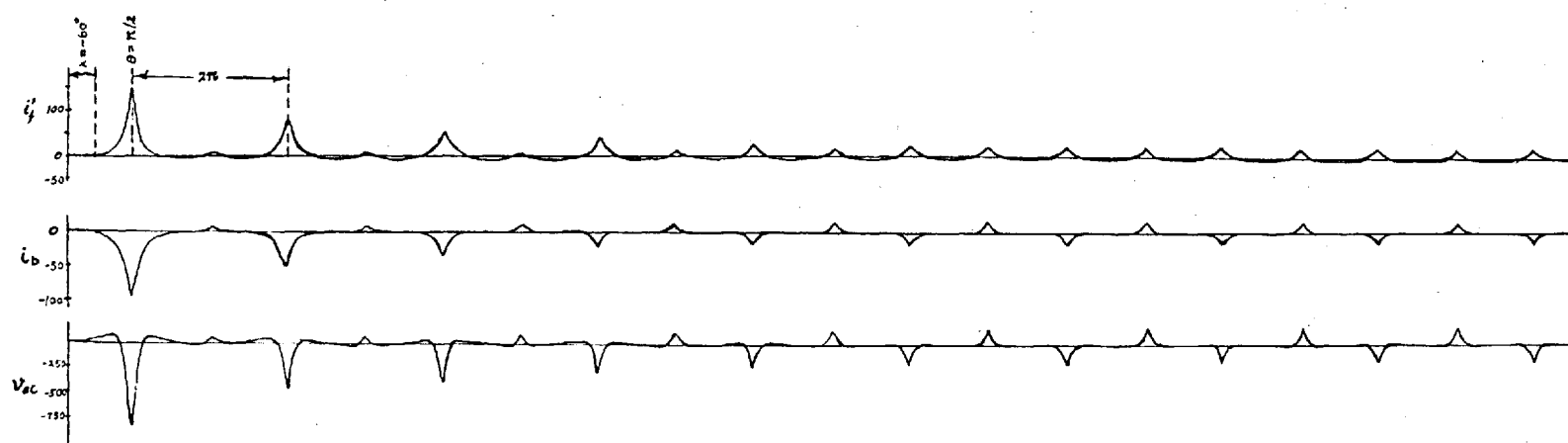


Fig. 9.4. Line-to-line Short Circuit ($x_q'' = 11.1 x_d''$)

$$i_f' = 23.6 e^{-t/0.0219} + \left(\frac{22}{1.20 + \cos 2\theta} - 33.8 \right) (0.702 e^{-t/0.0219} + 0.298) + \frac{17.3 \sin \theta}{1.20 + \cos 2\theta} e^{-t/0.0376}$$

$$i_b = -i_c = -\frac{11.7}{1.20 + \cos 2\theta} \left\{ \sin \theta (0.702 e^{-t/0.0219} + 0.298) + 0.866 e^{-t/0.0376} \right\}$$

$$V_{ac} = \frac{3}{2} V_a \quad (\text{After the short circuit})$$

$$V_a = \frac{146 \sin \theta - 161 \sin^3 \theta}{(1.20 + \cos 2\theta)^2} (0.702 e^{-t/0.0219} + 0.298) + \frac{63.8 + 76.3 \cos 2\theta}{(1.20 + \cos 2\theta)^2} e^{-t/0.0376}$$

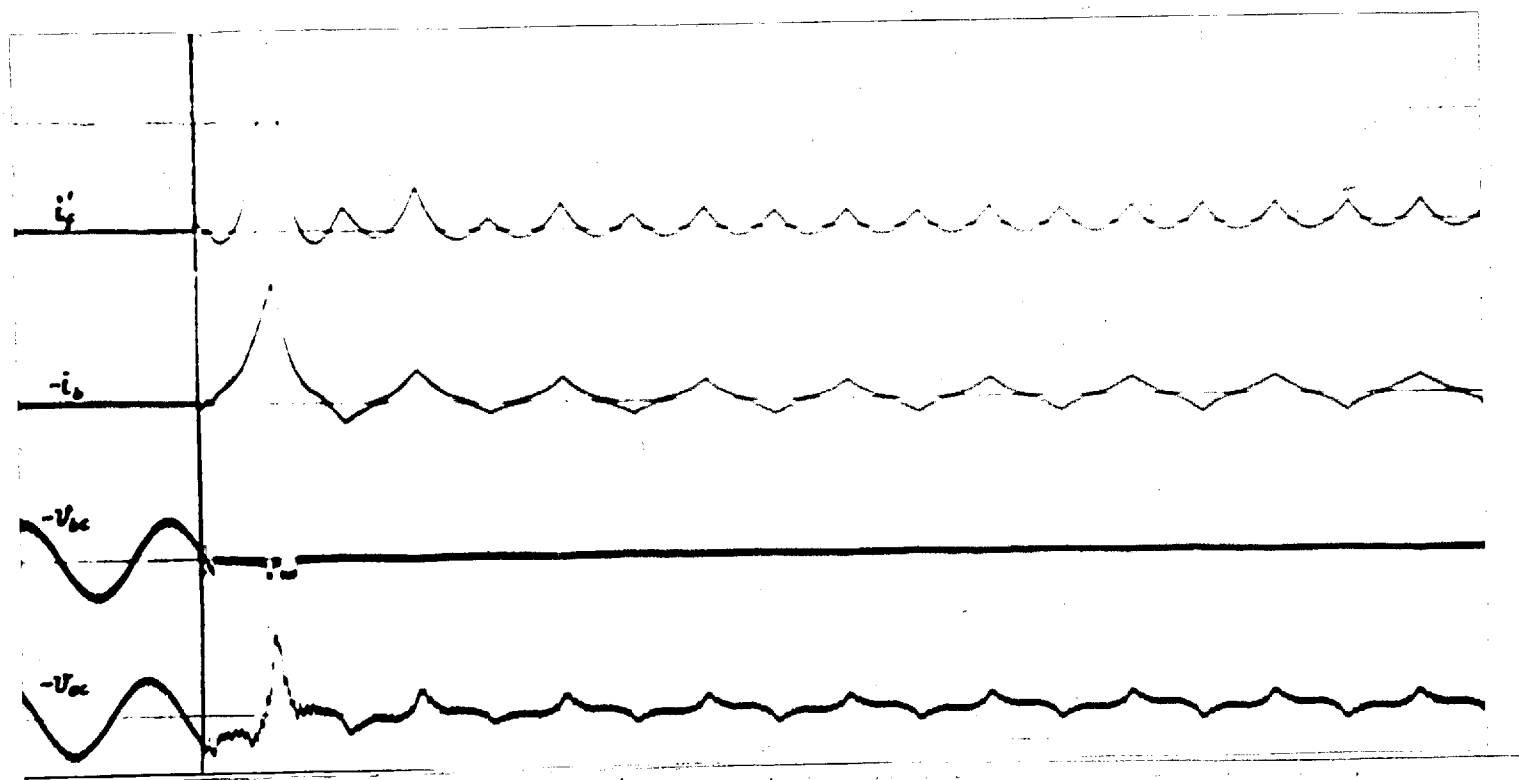


Fig. 9.5

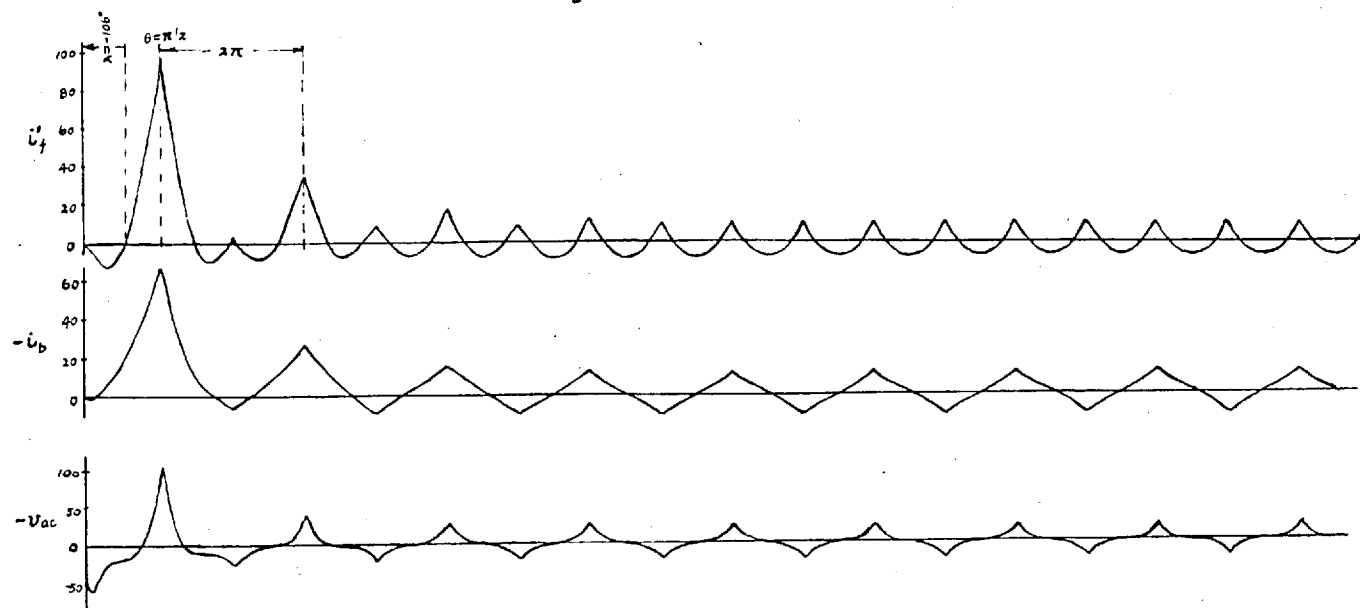


Fig. 9.6 Line-to-line Short Circuit ($x_q'' = 2x_d''$)

$$i_f' = 41.9 e^{-t/0.0142} + 219 \left(\frac{1}{1.62 + 0.54 \cos 2\theta} - 0.655 \right) (0.807 e^{-t/0.0142} + 0.193) + \frac{105 \sin \theta}{1.62 + 0.54 \cos 2\theta} e^{-t/0.0162}$$

$$i_b = -i_c = -\frac{63.7}{1.62 + 0.54 \cos 2\theta} \left\{ \sin \theta (0.807 e^{-t/0.0142} + 0.193) + 0.961 e^{-t/0.0162} \right\}$$

$$V_{ac} = \frac{3}{2} V_a \quad (\text{After the short circuit})$$

$$V_a = -\frac{89 \sin^2 \theta}{(1.62 + 0.54 \cos 2\theta)^2} (0.807 e^{-t/0.0142} + 0.193) + \frac{20.6 + 61.8 \cos 2\theta}{(1.62 + 0.54 \cos 2\theta)^2} e^{-t/0.0162}$$

(10) DOUBLE-LINE-TO-NEUTRAL SHORT CIRCUIT

(10.1) Terminal Conditions in terms of α , β , 0 Components

The single-phase short circuit, whether from line to neutral or from line to line, is a very particular case of unsymmetrical operation of a synchronous machine as there is only one closed armature circuit. As a consequence, the α , β , 0 components do not offer appreciable advantages over the phase components a, b and c. This is not so for the case of a double-line-to-neutral short circuit as there are two closed armature circuits and hence two different currents involved.

If phases b and c are the short-circuited phases, the terminal conditions are

$$i_a = i_\alpha + i_0 = 0$$

$$v_{bt} = -\frac{1}{2} v_{\alpha t} + \frac{\sqrt{3}}{2} v_{\beta t} + v_{0t} = 0$$

$$\text{and } v_{ct} = -\frac{1}{2} v_{\alpha t} - \frac{\sqrt{3}}{2} v_{\beta t} + v_{0t} = 0$$

$$\left. \begin{array}{l} \\ \\ \end{array} \right\} (10.1)$$

which can be expressed in operational form as

$$\bar{i}_\alpha(p) + \bar{i}_0(p) = 0 \quad (10.2)$$

$$\bar{v}_{\alpha t}(p) - 2 \bar{v}_{0t}(p) = 0 \quad (10.3)$$

$$\text{and } \bar{v}_{\beta t}(p) = 0 \quad (10.4)$$

Substituting the relation $\bar{v}_{0t}(p) = (R_0 + L_0 p) \bar{i}_0(p)$ (R_0 may include three times the earthing resistance and L_0 may be increased by three times the earthing inductance) in Eqs. 10.2 and 10.3, the zero-component current and voltage can be eliminated. Thus,

$$\bar{v}_{\alpha t}(p) + 2(R_0 + L_0 p) \bar{i}_\alpha(p) = 0 \quad (10.5)$$

Substituting Eqs. 10.5 and 10.4 for $\bar{v}_{\alpha t}(p)$ and $\bar{v}_{\beta t}(p)$ in Eqs. 7.29 and 7.30 gives the α - and β - component voltage equations:

$$\bar{e}_{\alpha}(p) = \left\{ R_a + 2R_o + \frac{1}{2}p[4L_o + L_s(p+j\omega) + \text{Conj}] \right\} \bar{i}_{\alpha}(p) - \frac{1}{2}p \left\{ jL_s(p+j\omega) + \text{Conj} \right\} \bar{i}_{\beta}(p) + \frac{1}{2}p e^{-j2\lambda} L_b(p+j\omega) \left\{ \bar{i}_{\alpha}(p+j2\omega) + j\bar{i}_{\beta}(p+j2\omega) \right\} + \text{Conj} \quad (10.6)$$

$$\bar{e}_{\beta}(p) = \frac{1}{2}p \left\{ jL_s(p+j\omega) + \text{Conj} \right\} \bar{i}_{\alpha}(p) + \left\{ R_a + \frac{1}{2}p[L_s(p+j\omega) + \text{Conj}] \right\} \bar{i}_{\beta}(p) + \frac{1}{2}p e^{j2\lambda} L_b(p+j\omega) \left\{ \bar{i}_{\alpha}(p+j2\omega) + j\bar{i}_{\beta}(p+j2\omega) \right\} + \text{Conj} \quad (10.7)$$

The double-line-to-neutral short circuit of a three-phase machine is therefore equivalent to an unsymmetrical short circuit of an equivalent two-phase machine, with one phase directly short-circuited and the other phase short-circuited through an external impedance of $2(R_o + L_o p)$. The absence of an external impedance in one phase of the equivalent two-phase machine and the zero external coupling between the two phases indicate the extent to which the double-line-to-neutral short circuit of a three-phase machine departs from the most general unsymmetrical operation.

(10.2) Independent α - and β - Component Equations and their Operational Solution

Eqs. 10.6 and 10.7 are similar to Eqs. 7.33 and 7.34 so that $\bar{i}_{\alpha}(p)$ and $\bar{i}_{\beta}(p)$ can be separated and two independent equations, one for the α -component quantities and the other for the β -component quantities, can be obtained. The α -component equation is, from Equ. 7.41,

$$\bar{f}_{\alpha}(p) = \bar{i}_{\alpha}(p) + \frac{1}{2} e^{-j2\lambda} \bar{K}_{\alpha}(p) \bar{i}_{\alpha}(p+j2\omega) + \frac{1}{2} e^{j2\lambda} \bar{K}_{\alpha}^*(p) \bar{i}_{\alpha}(p-j2\omega) \quad (10.8)$$

where

$$\begin{aligned}
\bar{E}_\alpha(p) &= \frac{1}{\bar{D}_\alpha(p)} \left\{ R_\alpha + \frac{1}{2} p \left[\frac{R_\alpha \bar{L}_5(p+j\omega) + (p+j2\omega) \bar{L}_d(p+j\omega) \bar{L}_q(p+j\omega)}{R_\alpha + (p+j2\omega) \bar{L}_5(p+j\omega)} + \text{Conj} \right] \right\} \\
&\quad \times \left\{ \frac{E_{am}}{2} \cdot \frac{j e^{-j\lambda}}{p+j\omega} \cdot \frac{R_\alpha + (p+j\omega) \bar{L}_d(p+j\omega) + j\omega \bar{L}_q(p+j\omega)}{R_\alpha + (p+j2\omega) \bar{L}_5(p+j\omega)} + \text{Conj} \right\} \\
&\quad - \frac{1}{\bar{D}_\alpha(p)} \left\{ j \frac{1}{2} p \frac{R_\alpha \bar{L}_5(p+j\omega) + (p+j2\omega) \bar{L}_d(p+j\omega) \bar{L}_q(p+j\omega)}{R_\alpha + (p+j2\omega) \bar{L}_5(p+j\omega)} + \text{Conj} \right\} \\
&\quad \times \left\{ \frac{E_{am}}{2} \cdot \frac{e^{-j\lambda}}{p+j\omega} \cdot \frac{R_\alpha + (p+j\omega) \bar{L}_d(p+j\omega) + j\omega \bar{L}_q(p+j\omega)}{R_\alpha + (p+j2\omega) \bar{L}_5(p+j\omega)} + \text{Conj} \right\} \\
\bar{K}_\alpha(p) &= - \frac{1}{\bar{D}_\alpha(p)} \left\{ R_\alpha + p \frac{R_\alpha \bar{L}_5(p-j\omega) + (p-j2\omega) \bar{L}_d(p-j\omega) \bar{L}_q(p-j\omega)}{R_\alpha + (p-j2\omega) \bar{L}_5(p-j\omega)} \right\} \cdot \frac{2 [R_\alpha + (p+j2\omega) L_0] p \bar{L}_5(p+j\omega)}{R_\alpha + (p+j2\omega) \bar{L}_5(p+j\omega)} \\
\bar{K}_\alpha^*(p) &= \text{Conjugate of } \bar{K}_\alpha(p) \\
\bar{D}_\alpha(p) &= \left\{ R_\alpha + 2R_0 + \frac{1}{2} p \left[4L_0 + \frac{R_\alpha \bar{L}_5(p+j\omega) + (p+j2\omega) \bar{L}_d(p+j\omega) \bar{L}_q(p+j\omega)}{R_\alpha + (p+j2\omega) \bar{L}_5(p+j\omega)} + \text{Conj} \right] \right\} \\
&\quad \times \left\{ R_\alpha + \frac{1}{2} p \left[\frac{R_\alpha \bar{L}_5(p+j\omega) + (p+j2\omega) \bar{L}_d(p+j\omega) \bar{L}_q(p+j\omega)}{R_\alpha + (p+j2\omega) \bar{L}_5(p+j\omega)} + \text{Conj} \right] \right\} \\
&\quad - \frac{1}{4} p^2 \left\{ \frac{R_\alpha \bar{L}_5(p+j\omega) + (p+j2\omega) \bar{L}_d(p+j\omega) \bar{L}_q(p+j\omega)}{R_\alpha + (p+j2\omega) \bar{L}_5(p+j\omega)} - \text{Conj} \right\}^2
\end{aligned} \tag{10.9}$$

Similarly, from Equ. 7.61, the β -component equation is found to be

$$\bar{F}_\beta(p) = \bar{L}_\beta(p) + \frac{1}{2} \epsilon^{-j2\lambda} \bar{K}_\beta(p) \bar{L}_\beta(p+j2\omega) + \frac{1}{2} \epsilon^{j2\lambda} \bar{K}_\beta^*(p) \bar{L}_\beta(p-j2\omega) \tag{10.10}$$

$$\begin{aligned}
\bar{F}_\beta(p) &= - \frac{1}{\bar{D}_\beta(p)} \left\{ R_\alpha + 2R_0 + \frac{1}{2} p \left[4L_0 + \frac{(R_\alpha + 2R_0) \bar{L}_5(p+j\omega) + (p+j2\omega) [2L_0 \bar{L}_5(p+j\omega) + \bar{L}_d(p+j\omega) \bar{L}_q(p+j\omega)]}{R_\alpha + 2R_0 + (p+j2\omega) [2L_0 + \bar{L}_5(p+j\omega)]} + \text{Conj} \right] \right\} \\
&\quad \times \left\{ \frac{E_{am}}{2} \cdot \frac{e^{-j\lambda}}{p+j\omega} \cdot \frac{R_\alpha + 2R_0 + 2(p+j2\omega) L_0 + (p+j\omega) \bar{L}_d(p+j\omega) + j\omega \bar{L}_q(p+j\omega)}{R_\alpha + 2R_0 + (p+j2\omega) [2L_0 + \bar{L}_5(p+j\omega)]} + \text{Conj} \right\} \\
&\quad - \frac{1}{\bar{D}_\beta(p)} \left\{ j \frac{1}{2} p \frac{(R_\alpha + 2R_0) \bar{L}_5(p+j\omega) + (p+j2\omega) [2L_0 \bar{L}_5(p+j\omega) + \bar{L}_d(p+j\omega) \bar{L}_q(p+j\omega)]}{R_\alpha + 2R_0 + (p+j2\omega) [2L_0 + \bar{L}_5(p+j\omega)]} + \text{Conj} \right\} \\
&\quad \times \left\{ \frac{E_{am}}{2} \cdot \frac{j e^{-j\lambda}}{p+j\omega} \cdot \frac{R_\alpha + 2R_0 + 2(p+j2\omega) L_0 + (p+j\omega) \bar{L}_d(p+j\omega) + j\omega \bar{L}_q(p+j\omega)}{R_\alpha + 2R_0 + (p+j2\omega) [2L_0 + \bar{L}_5(p+j\omega)]} + \text{Conj} \right\} \\
\bar{K}_\beta(p) &= - \frac{1}{\bar{D}_\beta(p)} \left\{ R_\alpha + 2R_0 + p \left[2L_0 + \frac{(R_\alpha + 2R_0) \bar{L}_5(p-j\omega) + (p-j2\omega) [2L_0 \bar{L}_5(p-j\omega) + \bar{L}_d(p-j\omega) \bar{L}_q(p-j\omega)]}{R_\alpha + 2R_0 + (p-j2\omega) [2L_0 + \bar{L}_5(p-j\omega)]} \right] \right\} \\
&\quad \times \frac{2 [R_\alpha + (p+j2\omega) L_0] p \bar{L}_5(p+j\omega)}{R_\alpha + 2R_0 + (p+j2\omega) [2L_0 + \bar{L}_5(p+j\omega)]} \\
\bar{K}_\beta^*(p) &= \text{Conjugate of } \bar{K}_\beta(p) \\
\bar{D}_\beta(p) &= \left\{ R_\alpha + 2R_0 + \frac{1}{2} p \left[4L_0 + \frac{(R_\alpha + 2R_0) \bar{L}_5(p+j\omega) + (p+j2\omega) [2L_0 \bar{L}_5(p+j\omega) + \bar{L}_d(p+j\omega) \bar{L}_q(p+j\omega)]}{R_\alpha + 2R_0 + (p+j2\omega) [2L_0 + \bar{L}_5(p+j\omega)]} + \text{Conj} \right] \right\} \\
&\quad \times \left\{ R_\alpha + \frac{1}{2} p \left[\frac{(R_\alpha + 2R_0) \bar{L}_5(p+j\omega) + (p+j2\omega) [2L_0 \bar{L}_5(p+j\omega) + \bar{L}_d(p+j\omega) \bar{L}_q(p+j\omega)]}{R_\alpha + 2R_0 + (p+j2\omega) [2L_0 + \bar{L}_5(p+j\omega)]} + \text{Conj} \right] \right\} \\
&\quad - \frac{1}{4} p^2 \left\{ \frac{(R_\alpha + 2R_0) \bar{L}_5(p+j\omega) + (p+j2\omega) [2L_0 \bar{L}_5(p+j\omega) + \bar{L}_d(p+j\omega) \bar{L}_q(p+j\omega)]}{R_\alpha + 2R_0 + (p+j2\omega) [2L_0 + \bar{L}_5(p+j\omega)]} - \text{Conj} \right\}^2
\end{aligned} \tag{10.11}$$

The operational solutions of the α - and β - component equations are, from Equ. 7.57,

$$i_{\alpha}(p) = \bar{A}_0(p) \bar{E}_{\alpha}(p) + \sum_{n=1}^{\infty} \left\{ \bar{A}_n(p) \bar{E}_{\alpha}(p+j2n\omega) + \text{Conj} \right\} \quad (10.12)$$

where

$$\begin{aligned} \bar{A}_0(p) &= \frac{1}{\bar{x}_{\alpha}(p) + \bar{x}_{\alpha}^*(p) - 1} \\ \bar{A}_n(p) &= \left(-\frac{1}{2}\right)^n e^{-j2n\lambda} \bar{A}_0(p+j2n\omega) \prod_{\nu=0}^{n-1} \frac{\bar{K}_{\alpha}(p+j2\nu\omega)}{\bar{x}_{\alpha}(p+j2\nu\omega)} \\ \bar{x}_{\alpha}(p) &= 1 - \frac{\frac{1}{2} \bar{K}_{\alpha}^*(p) \bar{K}_{\alpha}(p-j2\omega)}{\bar{x}_{\alpha}(p-j2\omega)} \\ &= 1 - \frac{\frac{1}{2} \bar{K}_{\alpha}^*(p) \bar{K}_{\alpha}(p-j2\omega)}{1 - \frac{\frac{1}{2} \bar{K}_{\alpha}^*(p-j2\omega) \bar{K}_{\alpha}(p-j4\omega)}{1 - \dots}} \end{aligned} \quad (10.13)$$

$$\bar{x}_{\alpha}^*(p) = \text{Conjugate of } \bar{x}_{\alpha}(p)$$

and

$$i_{\beta}(p) = \bar{B}_0(p) \bar{E}_{\beta}(p) + \sum_{n=1}^{\infty} \left\{ \bar{B}_n(p) \bar{E}_{\beta}(p+j2n\omega) + \text{Conj} \right\} \quad (10.14)$$

and

where

$$\begin{aligned} \bar{B}_0(p) &= \frac{1}{\bar{x}_{\beta}(p) + \bar{x}_{\beta}^*(p) - 1} \\ \bar{B}_n(p) &= \left(-\frac{1}{2}\right)^n e^{-j2n\lambda} \bar{B}_0(p+j2n\omega) \prod_{\nu=0}^{n-1} \frac{\bar{K}_{\beta}(p+j2\nu\omega)}{\bar{x}_{\beta}(p+j2\nu\omega)} \\ \bar{x}_{\beta}(p) &= 1 - \frac{\frac{1}{2} \bar{K}_{\beta}^*(p) \bar{K}_{\beta}(p-j2\omega)}{\bar{x}_{\beta}(p-j2\omega)} \\ &= 1 - \frac{\frac{1}{2} \bar{K}_{\beta}^*(p) \bar{K}_{\beta}(p-j2\omega)}{1 - \frac{\frac{1}{2} \bar{K}_{\beta}^*(p-j2\omega) \bar{K}_{\beta}(p-j4\omega)}{1 - \dots}} \end{aligned} \quad (10.15)$$

and

$$\bar{x}_{\beta}^*(p) = \text{Conjugate of } \bar{x}_{\beta}(p)$$

Having got the operational solutions, the α - and β - component currents can be derived from them in exactly the same manner as in the case of a single-phase line-to-neutral short circuit. As the functions $\bar{K}(p)$ and $\bar{F}(p)$ are more complicated in the present case, so are the expressions for the currents. Moreover, as there are two closed armature circuits instead of one, there are two distinct armature-transient time constants as shown later.

(10.3) Steady-state Components of Short-circuit Currents

Referring to Article 9.3 it is clear that the steady-state

component of the α component of current is given by

$$\bar{I}_\alpha(p) \left[\begin{smallmatrix} \text{Steady} \\ \text{state} \end{smallmatrix} \right] = \sum_{n=-\infty}^{\infty} \left\{ \frac{C_n(\alpha)}{p+j(2n+1)\omega} + \text{Conj} \right\} \quad (10.16)$$

$$\text{where } C_n(\alpha) = \left[\{p+j(2n+1)\omega\} \left\{ \bar{A}_n(p) \bar{F}_\alpha(p+j2n\omega) + \bar{A}_{n+1}(p) \bar{F}_\alpha(p+j2\overline{n+1}\omega) \right\} \right]_{p=-j(2n+1)\omega} \quad (10.17)$$

The determination of the value of $C_n(\alpha)$ involves the evaluation of the following functions, which have been simplified by neglecting the resistances compared with the reactances at normal or higher frequencies. Thus,

$$\left[\{p+j(2n+1)\omega\} \bar{F}_\alpha(p+j2n\omega) \right]_{p=-j(2n+1)\omega} = -\frac{E_{om}e^{-j\lambda}}{2\omega} \cdot \frac{L_d L_d' L_q'}{L_d L_q L_d' L_q' + \frac{1}{2} L_o [L_d L_q (L_d' + L_q') + L_d' L_q' (L_d + L_q)]}$$

$$\left[\{p+j(2n+1)\omega\} \bar{F}_\alpha(p+j2\overline{n+1}\omega) \right]_{p=-j(2n+1)\omega} \approx -\frac{E_{om}e^{j\lambda}}{2\omega} \cdot \frac{L_d L_d' L_q'}{L_d L_q L_d' L_q' + \frac{1}{2} L_o [L_d L_q (L_d' + L_q') + L_d' L_q' (L_d + L_q)]}$$

$$\bar{K}_\alpha(j\omega) = \bar{K}_\alpha^*(-j\omega) \approx -\frac{L_d L_q L_o (L_d' - L_q')}{L_d L_q L_d' L_q' + \frac{1}{2} L_o [L_d L_q (L_d' + L_q') + L_d' L_q' (L_d + L_q)]}$$

$$\bar{K}_\alpha(-j\omega) = \bar{K}_\alpha^*(j\omega) \approx -\frac{L_d L_q L_o (L_d - L_q)}{L_d L_q L_d' L_q' + \frac{1}{2} L_o [L_d L_q (L_d' + L_q') + L_d' L_q' (L_d + L_q)]}$$

$$\bar{K}_\alpha(\pm j \overline{2\nu+1} \omega) = \bar{K}_\alpha^*(\mp j \overline{2\nu+1} \omega) \approx -\frac{L_o (L_d' - L_q')}{L_d' L_q' + L_o (L_d' + L_q')} \quad \nu = \text{non-zero positive integer}$$

$$\bar{x}_\alpha(j\omega) = \bar{x}_\alpha^*(-j\omega) \approx 1 - \frac{1}{2} \cdot \frac{L_d' L_q' L_o^2 (L_d - L_q)^2}{L_d L_q L_d' L_q' + \frac{1}{2} L_o [L_d L_q (L_d' + L_q') + L_d' L_q' (L_d + L_q)]} \\ \times \frac{1}{L_d L_q L_d' L_q' + L_d' L_q' L_o (L_d + L_q) + L_d L_q \sqrt{L_d' L_q' (L_d' + 2L_o)} (L_q' + 2L_o)}$$

$$\bar{x}_\alpha(-j\omega) = \bar{x}_\alpha^*(j\omega) \approx \frac{1}{2} \cdot \frac{L_d L_q L_d' L_q' + L_d' L_q' L_o (L_d + L_q) + L_d L_q \sqrt{L_d' L_q' (L_d' + 2L_o)} (L_q' + 2L_o)}{L_d L_q L_d' L_q' + \frac{1}{2} L_o [L_d L_q (L_d' + L_q') + L_d' L_q' (L_d + L_q)]}$$

$$\bar{x}_\alpha(-j \overline{2\nu+1} \omega) \approx \frac{1}{2} \left\{ 1 + \frac{\sqrt{L_d' L_q' (L_d' + 2L_o)} (L_q' + 2L_o)}{L_d' L_q' + L_o (L_d' + L_q')} \right\} \quad \nu = \text{non-zero positive integer}$$

$$\frac{1}{\bar{x}_\alpha(j\omega) + \bar{x}_\alpha^*(j\omega) - 1} = \frac{1}{\bar{x}_\alpha(-j\omega) + \bar{x}_\alpha^*(-j\omega) - 1} \approx 2 \frac{L_d L_q L_d' L_q' + \frac{1}{2} L_o [L_d L_q (L_d' + L_q') + L_d' L_q' (L_d + L_q)]}{L_d L_q L_d' L_q' + L_d' L_q' L_o (L_d + L_q) + L_d L_q \sqrt{L_d' L_q' (L_d' + 2L_o)} (L_q' + 2L_o)} \\ \times \frac{L_d L_q L_d' L_q' + L_d' L_q' L_o (L_d + L_q) + L_d L_q \sqrt{L_d' L_q' (L_d' + 2L_o)} (L_q' + 2L_o)}{L_d' L_q' L_o (L_d + 2L_o) + L_d L_q \sqrt{L_d' L_q' (L_d' + 2L_o)} (L_q' + 2L_o)}$$

$$\frac{\bar{K}_\alpha(-j\omega)}{\bar{x}_\alpha(-j\omega)} \approx -\frac{2 L_d L_q L_o (L_d - L_q)}{L_d L_q L_d' L_q' + L_d' L_q' L_o (L_d + L_q) + L_d L_q \sqrt{L_d' L_q' (L_d' + 2L_o)} (L_q' + 2L_o)}$$

$$\frac{\bar{K}_\alpha(-j \overline{2\nu+1} \omega)}{\bar{x}_\alpha(-j \overline{2\nu+1} \omega)} \approx -2 \frac{\sqrt{L_d' L_q' (L_d' + 2L_o)} - \sqrt{L_q' (L_d' + 2L_o)}}{\sqrt{L_d' L_q' (L_d' + 2L_o)} + \sqrt{L_q' (L_d' + 2L_o)}} = -2 b_\lambda \quad \nu = \text{non-zero positive integer}$$

$$\left(-\frac{1}{2}\right)^n \prod_{\nu=1}^n \frac{\bar{K}_\alpha(-j \overline{2\nu+1} \omega)}{\bar{x}_\alpha(-j \overline{2\nu+1} \omega)} \approx b_\lambda^n$$

$$\left(-\frac{1}{2}\right)^{n+1} \prod_{\nu=0}^n \frac{\bar{K}_\alpha(-j \overline{2\nu+1} \omega)}{\bar{x}_\alpha(-j \overline{2\nu+1} \omega)} \approx b_\lambda^n \frac{L_d L_q L_o (L_d - L_q)}{L_d L_q L_d' L_q' + L_d' L_q' L_o (L_d + L_q) + L_d L_q \sqrt{L_d' L_q' (L_d' + 2L_o)} (L_q' + 2L_o)}$$

Substituting the values of the functions given by Equ. 10.18 in Equ. 10.17 and reducing gives

$$C_n(\alpha) = - \frac{b_2^n E_{am} e^{-j(2n+1)\lambda}}{\omega} \cdot \frac{1}{L_d(1+k) + 2L_o} \quad (10.19)$$

where $k = \sqrt{(L_d + 2L_o)(L_q^2 + 2L_o)} / L_d L_q$ (10.20)

Substituting $C_n(\alpha)$ in Equ. 10.16 gives the steady-state component of $\bar{i}_\alpha(p)$:

$$\bar{i}_\alpha(p) \left[\text{Steady} \right] = - \frac{E_{am}}{\omega} \cdot \frac{1}{L_d(1+k) + 2L_o} \sum_{n=0}^{\infty} \left\{ \frac{b_2^n e^{-j(2n+1)\lambda}}{p + j(2n+1)\omega} + \text{conj} \right\} \quad (10.21)$$

In explicit form, the steady-state component of the α -component current is

$$\begin{aligned} i_\alpha(t) \left[\text{Steady} \right] &= \mathcal{L}^{-1} \bar{i}_\alpha(p) \left[\text{Steady} \right] \\ &= - \frac{2E_{am}}{\omega} \cdot \frac{1}{L_d(1+k) + 2L_o} \sum_{n=0}^{\infty} b_2^n \cos(2n+1)\theta \end{aligned} \quad (10.22)$$

Similarly, the steady-state component of the β component of current is given by

$$\bar{i}_\beta(p) \left[\text{Steady} \right] = \sum_{n=0}^{\infty} \left\{ \frac{C_n(\beta)}{p + j(2n+1)\omega} + \text{conj} \right\} \quad (10.23)$$

where $C_n(\beta) = \left[\left\{ p + j(2n+1)\omega \right\} \left\{ \bar{B}_n(p) \bar{F}_\beta(p + j2n\omega) + \bar{B}_{n+1}(p) \bar{F}_\beta(p + j2(n+1)\omega) \right\} \right]_{p=-j(2n+1)\omega}$ (10.24)

The functions evaluated at the proper values of p necessary for the determination of the value of $C_n(\beta)$ are, after the simplification of neglecting the resistances compared with the reactances at normal or higher frequencies, as follows.

$$\left[\left\{ p + j(2n+1)\omega \right\} \bar{F}_\beta(p + j2n\omega) \right]_{p=-j(2n+1)\omega} \approx -j e^{j\lambda} \frac{E_{am}}{\omega} \cdot \frac{(L_q + 2L_o)(L_d + 2L_o)(L_q^2 + 2L_o)}{[L_d L_q + L_o(L_d + L_q)](L_d + 2L_o)(L_q^2 + 2L_o) + [L_d L_q + L_o(L_d + L_q)](L_d + 2L_o)(L_q + 2L_o)}$$

$$\left[\left\{ p + j(2n+1)\omega \right\} \bar{F}_\beta(p + j2(n+1)\omega) \right]_{p=-j(2n+1)\omega} \approx j e^{j\lambda} \frac{E_{am}}{\omega} \cdot \frac{(L_q + 2L_o)(L_d + 2L_o)(L_q^2 + 2L_o)}{[L_d L_q + L_o(L_d + L_q)](L_d + 2L_o)(L_q^2 + 2L_o) + [L_d L_q + L_o(L_d + L_q)](L_d + 2L_o)(L_q + 2L_o)}$$

$$\bar{K}_\beta(j\omega) = \bar{K}_\beta^*(-j\omega) \approx - \frac{2L_o(L_d - L_q)(L_d + 2L_o)(L_q^2 + 2L_o)}{[L_d L_q + L_o(L_d + L_q)](L_d + 2L_o)(L_q^2 + 2L_o) + [L_d L_q + L_o(L_d + L_q)](L_d + 2L_o)(L_q + 2L_o)}$$

$$\bar{K}_\beta(-j\omega) = \bar{K}_\beta^*(j\omega) \approx - \frac{2L_o(L_d - L_q)(L_d + 2L_o)(L_q^2 + 2L_o)}{[L_d L_q + L_o(L_d + L_q)](L_d + 2L_o)(L_q^2 + 2L_o) + [L_d L_q + L_o(L_d + L_q)](L_d + 2L_o)(L_q + 2L_o)}$$

$$\bar{K}_\beta(\pm j 2\nu\pi\omega) = \bar{K}_\beta^*(\mp j 2\nu\pi\omega) \approx - \frac{L_o(L_d - L_q)}{L_d L_q + L_o(L_d + L_q)} \quad \nu = \text{non-zero positive integer}$$

$$\begin{aligned}\bar{x}_\beta(j\omega) &= \bar{x}_\beta^*(-j\omega) = 1 - \frac{L_0^2(L_d - L_q)^2(L_d^2 + 2L_0^2)(L_q^2 + 2L_0^2)}{[L_d L_q + L_0(L_d + L_q)](L_d^2 + 2L_0)(L_q^2 + 2L_0) + [L_d^2 L_q + L_0(L_d^2 + L_q^2)](L_d + 2L_0)(L_q + 2L_0)} \\ &\quad \times \frac{1}{[L_d L_q + L_0(L_d + L_q)](L_d^2 + 2L_0)(L_q^2 + 2L_0) + (L_d + 2L_0)(L_q + 2L_0)\sqrt{L_d L_q}(L_d^2 + 2L_0)(L_q^2 + 2L_0)} \\ \bar{x}_\beta(-j\omega) &= \bar{x}_\beta^*(j\omega) = \frac{[L_d L_q + L_0(L_d + L_q)](L_d^2 + 2L_0)(L_q^2 + 2L_0) + (L_d + 2L_0)(L_q + 2L_0)\sqrt{L_d L_q}(L_d^2 + 2L_0)(L_q^2 + 2L_0)}{[L_d L_q + L_0(L_d + L_q)](L_d^2 + 2L_0)(L_q^2 + 2L_0) + [L_d^2 L_q + L_0(L_d^2 + L_q^2)](L_d + 2L_0)(L_q + 2L_0)} \\ \bar{x}_\beta(-j\frac{\omega}{2\nu+1}) &= \frac{1}{2} \left\{ 1 + \frac{\sqrt{L_d L_q}(L_d^2 + 2L_0)(L_q^2 + 2L_0)}{L_d L_q + L_0(L_d + L_q)} \right\} \\ \frac{\bar{x}_\beta(j\omega) + \bar{x}_\beta^*(j\omega) - 1}{\bar{x}_\beta(j\omega) - \bar{x}_\beta^*(j\omega) - 1} &= \frac{1}{\bar{x}_\beta(j\omega) + \bar{x}_\beta^*(j\omega) - 1} \\ &= \frac{[L_d L_q + L_0(L_d + L_q)](L_d^2 + 2L_0)(L_q^2 + 2L_0) + [L_d^2 L_q + L_0(L_d^2 + L_q^2)](L_d + 2L_0)(L_q + 2L_0)}{L_d(L_q + 2L_0)(L_d^2 + 2L_0)(L_q^2 + 2L_0) + (L_d + 2L_0)(L_q + 2L_0)\sqrt{L_d L_q}(L_d^2 + 2L_0)(L_q^2 + 2L_0)} \\ &\quad \times \frac{[L_d L_q + L_0(L_d + L_q)](L_d^2 + 2L_0)(L_q^2 + 2L_0) + (L_d + 2L_0)(L_q + 2L_0)\sqrt{L_d L_q}(L_d^2 + 2L_0)(L_q^2 + 2L_0)}{L_q(L_d + 2L_0)(L_d^2 + 2L_0)(L_q^2 + 2L_0) + (L_d + 2L_0)(L_q + 2L_0)\sqrt{L_d L_q}(L_d^2 + 2L_0)(L_q^2 + 2L_0)} \\ \frac{\bar{K}_\beta(-j\frac{\omega}{2\nu+1})}{\bar{x}_\beta(-j\frac{\omega}{2\nu+1})} &\approx -2b_2 \quad \text{where } b_2 \text{ is defined in Equ. 10.18 and } \nu \text{ is a non-zero positive integer} \\ \frac{\bar{K}_\beta(-j\omega)}{\bar{x}_\beta(-j\omega)} &= - \frac{2L_0(L_d - L_q)(L_d^2 + 2L_0)(L_q^2 + 2L_0)}{[L_d L_q + L_0(L_d + L_q)](L_d^2 + 2L_0)(L_q^2 + 2L_0) + (L_d + 2L_0)(L_q + 2L_0)\sqrt{L_d L_q}(L_d^2 + 2L_0)(L_q^2 + 2L_0)} \\ (-\frac{1}{2})^n \prod_{\nu=1}^n \frac{\bar{K}_\beta(-j\frac{\omega}{2\nu+1})}{\bar{x}_\beta(-j\frac{\omega}{2\nu+1})} &\approx b_2^n \\ (-\frac{1}{2})^{n+1} \prod_{\nu=0}^n \frac{\bar{K}_\beta(-j\frac{\omega}{2\nu+1})}{\bar{x}_\beta(-j\frac{\omega}{2\nu+1})} &\approx b_2^{n+1} \frac{L_0(L_d - L_q)(L_d^2 + 2L_0)(L_q^2 + 2L_0)}{[L_d L_q + L_0(L_d + L_q)](L_d^2 + 2L_0)(L_q^2 + 2L_0) + (L_d + 2L_0)(L_q + 2L_0)\sqrt{L_d L_q}(L_d^2 + 2L_0)(L_q^2 + 2L_0)}\end{aligned} \quad (10.25)$$

Substituting the values of the functions given by Equ. 10.25 in Equ. 10.24 and reducing gives

$$C_n(\beta) = - \frac{b_2^n E_{am} j e^{-j(2n+1)\lambda}}{\omega} \cdot \frac{k}{L_d(1+k) + 2L_0} \quad (10.26)$$

where k is given by Equ. 10.20. Substituting in Equ. 10.23 gives the operational form of the steady-state component of i_β :

$$\bar{i}_\beta(\beta) \left[\begin{smallmatrix} \text{Steady} \\ \text{State} \end{smallmatrix} \right] = - \frac{E_{am}}{\omega} \cdot \frac{k}{L_d(1+k) + 2L_0} \sum_{n=0}^{\infty} \left\{ \frac{b_2^n j e^{-j(2n+1)\lambda}}{\beta + j(2n+1)\omega} + \text{Conj} \right\} \quad (10.27)$$

Its explicit form is

$$\begin{aligned}i_\beta(t) \left[\begin{smallmatrix} \text{Steady} \\ \text{State} \end{smallmatrix} \right] &= \int^{-1} \bar{i}_\beta(\beta) \left[\begin{smallmatrix} \text{Steady} \\ \text{State} \end{smallmatrix} \right] \\ &= - \frac{2E_{am}}{\omega} \cdot \frac{k}{L_d(1+k) + 2L_0} \sum_{n=0}^{\infty} b_2^n \sin(2n+1)\theta\end{aligned} \quad (10.28)$$

Eqs. 10.22 and 10.28 show that the harmonic series for the α and β components have the same geometric ratio b_2 . This is obviously so because the harmonic currents produced in any armature circuit are the result of the asymmetry of the same

rotor circuits.

The double-line-to-neutral short circuit can be reduced to a symmetrical short circuit or a single-phase line-to-line short circuit by allowing L_0 to take two extreme values. If $L_0 = 0$, b_2 is reduced to zero and k to unity, so the i_α and i_β become the phase currents of an equivalent two-phase machine symmetrically short-circuited. If $L_0 = \infty$, $i_\alpha = 0$ and i_β becomes $2/\sqrt{3}$ times the single-phase line-to-line short-circuit current, since

$$\lim_{L_0 \rightarrow \infty} b_2 = \lim_{L_0 \rightarrow \infty} \frac{\sqrt{L_d(L_q + 2L_0)} - \sqrt{L_q(L_d + 2L_0)}}{\sqrt{L_d(L_q + 2L_0)} + \sqrt{L_q(L_d + 2L_0)}} = \frac{\sqrt{L_d} - \sqrt{L_q}}{\sqrt{L_d} + \sqrt{L_q}} = -b_1$$

$$\text{and } \lim_{L_0 \rightarrow \infty} \frac{k}{L_d(1+k) + 2L_0} = \lim_{L_0 \rightarrow \infty} \frac{\sqrt{(L_d + 2L_0)(L_q + 2L_0)}/L_d L_q}{L_d \{1 + \sqrt{(L_d + 2L_0)(L_q + 2L_0)}/L_d L_q\} + 2L_0} = \frac{1}{L_d + \sqrt{L_d L_q}}$$

(10.4) Transient and Subtransient Components of Short-circuit Currents

As before, the transient component can be derived from the steady-state component by replacing L_d by $[1 + T_d'(p \pm j2n+1\omega)]L_d / [1 + T_{d0}'(p \pm j2n+1\omega)]$ and proceeding to find its magnitude as well as its decrement factor, with the assumptions that $R_{kd} \gg R_f$ and $1/T_d'$ and $1/T_{d0}'$ are small compared with ω . Thus, from Equ. 10.21, when p is in the vicinity of $\pm j(2n+1)\omega$,

$$\bar{i}_\alpha(p) \approx -\frac{E_{am}}{\omega} \sum_{n=0}^{\infty} \left\{ \frac{1}{\frac{1 + T_d'(p + j2n+1\omega)}{1 + T_{d0}'(p + j2n+1\omega)} L_d(1+k) + 2L_0} \cdot \frac{b_1^n e^{-j(2n+1)\omega t}}{p + j(2n+1)\omega} + \text{Conj} \right\}$$

which may be reduced to

$$\bar{i}_\alpha(p) \approx -\frac{E_{am}}{\omega} \sum_{n=0}^{\infty} \left\{ \frac{1 + (p + j2n+1\omega) T_{d0}'}{[L_d(1+k) + 2L_0] T_{d0}'} \cdot \frac{1}{p + j(2n+1)\omega} \cdot \frac{b_1^n e^{-j(2n+1)\omega t}}{\frac{1}{T_d'(1-k)} + p + j(2n+1)\omega} + \text{Conj} \right\} \quad (10.29)$$

$$\text{where } T_{d'(1-k)} = \frac{L_d(1+k) + 2L_0}{L_d(1+k) + 2L_0} T_{d0}' \quad (10.30)$$

which is independent of n . This means that all the harmonic components have the same time constant $T_d'(11-n)$. The transient component is given by

$$\begin{aligned} \bar{i}_\alpha(p) [Transient] &= \sum_{n=0}^{\infty} \left[\left(\frac{1}{T_d'(11-n)} + p + j \frac{2n+1}{2} \omega \right) \bar{i}_\alpha(p) \right]_{p = -\frac{1}{T_d'(11-n)} - j \frac{(2n+1)}{2} \omega} \cdot \frac{1}{T_d'(11-n) + p + j \frac{(2n+1)}{2} \omega} + \text{Conj} \\ &= -\frac{E_{am}}{\omega} \left\{ \frac{1}{L_d'(1+k) + 2L_o} - \frac{1}{L_d(1+k) + 2L_o} \right\} \sum_{n=0}^{\infty} \left\{ \frac{b_1^n e^{-j(2n+1)\theta}}{\frac{1}{T_d'(11-n)} + p + j \frac{(2n+1)}{2} \omega} + \text{Conj} \right\} \quad (10.31) \end{aligned}$$

whose inverse transform is the explicit expression of transient component of i_α :

$$i_\alpha(t) [Transient] = -\frac{2E_{am}}{\omega} \left\{ \frac{1}{L_d'(1+k) + 2L_o} - \frac{1}{L_d(1+k) + 2L_o} \right\} e^{-t/T_d'(11-n)} \sum_{n=0}^{\infty} b_1^n \cos(2n+1)\theta \quad (10.32)$$

Similarly, by assuming $1/T_d''$ and $1/T_{d0}''$ small compared with ω and substituting $[1 + T_d''(p \pm j \frac{2n+1}{2} \omega)] L_d' / [1 + T_{d0}''(p \pm j \frac{2n+1}{2} \omega)]$ for L_d in the operational form of the steady-state component, it can be shown that the subtransient component of i_α is

$$i_\alpha(t) [Subtransient] = -\frac{2E_{am}}{\omega} \left\{ \frac{1}{L_d'(1+k) + 2L_o} - \frac{1}{L_d(1+k) + 2L_o} \right\} e^{-t/T_d''(11-n)} \sum_{n=0}^{\infty} b_1^n \cos(2n+1)\theta \quad (10.33)$$

$$\text{where } T_{\alpha''}(11-n) = \frac{L_d'(1+k) + 2L_o}{L_d(1+k) + 2L_o} T_{d0}'' \quad (10.34)$$

Similarly the transient and subtransient components of i_β are respectively

$$i_\beta(t) [Transient] = -\frac{2kE_{am}}{\omega} \left\{ \frac{1}{L_d'(1+k) + 2L_o} - \frac{1}{L_d(1+k) + 2L_o} \right\} e^{-t/T_d'(11-n)} \sum_{n=0}^{\infty} b_1^n \sin(2n+1)\theta \quad (10.35)$$

$$\text{and } i_\beta(t) [Subtransient] = -\frac{2kE_{am}}{\omega} \left\{ \frac{1}{L_d'(1+k) + 2L_o} - \frac{1}{L_d(1+k) + 2L_o} \right\} e^{-t/T_d''(11-n)} \sum_{n=0}^{\infty} b_1^n \sin(2n+1)\theta \quad (10.36)$$

The complete odd-harmonic series of i_α and i_β may be written more compactly as

$$i_\alpha(t) [Odd \text{ series}] = -\frac{2E_{am}}{\omega} \cdot \frac{1}{L_d'(1+k) + 2L_o} g_\alpha(t) \sum b_1^n \cos(2n+1)\theta \quad (10.37)$$

$$\text{and } i_\beta(t) [Odd \text{ series}] = -\frac{2E_{am}}{\omega} \cdot \frac{k}{L_d'(1+k) + 2L_o} g_\beta(t) \sum b_1^n \sin(2n+1)\theta \quad (10.38)$$

$$\text{where } g_\alpha(t) = \left(1 - \frac{T_d''(11-n)}{T_{d0}''}\right) e^{-t/T_d''(11-n)} + \left(1 - \frac{T_d'(11-n)}{T_{d0}'}\right) \frac{T_d''(11-n)}{T_{d0}''} e^{-t/T_d'(11-n)} + \frac{T_d'(11-n)}{T_{d0}'} \cdot \frac{T_d''(11-n)}{T_{d0}''} \quad (10.39)$$

Since $i_o = -i_\alpha$, the odd-harmonic series of the phase compo-

nents of the short-circuit armature current are

$$\begin{aligned} i_{\beta}(t) [_{series}^{Odd}] &= -\frac{3}{\lambda} i_{\alpha}(t) [_{series}^{Odd}] + \frac{\sqrt{3}}{\lambda} i_{\beta}(t) [_{series}^{Odd}] \\ &= \frac{\sqrt{3} E_{sm}}{\omega} \cdot \frac{1}{L_d(1+k) + \lambda L_o} g_1(t) \sum_{n=0}^{\infty} b_n^{\lambda} \left\{ \sqrt{3} \cos(\lambda n + 1) \theta - k \sin(\lambda n + 1) \theta \right\} \end{aligned} \quad (10.40)$$

$$\begin{aligned} \text{and } i_{\alpha}(t) [_{series}^{Odd}] &= -\frac{3}{\lambda} i_{\alpha}(t) [_{series}^{Odd}] - \frac{\sqrt{3}}{\lambda} i_{\beta}(t) [_{series}^{Odd}] \\ &= \frac{\sqrt{3} E_{sm}}{\omega} \cdot \frac{1}{L_d(1+k) + \lambda L_o} g_1(t) \sum_{n=0}^{\infty} b_n^{\lambda} \left\{ \sqrt{3} \cos(\lambda n + 1) \theta + k \sin(\lambda n + 1) \theta \right\} \end{aligned} \quad (10.41)$$

(10.5) Armature-transient Components of Short-circuit Currents

The armature-transient components of i_{α} and i_{β} can be derived from the operational solutions given by Eqs. 10.12 and 10.14 by the same method used in the previous cases. By inspection of Eq. 10.12, the operational solution for $\bar{i}_{\alpha}(p)$, it is clear that, from the expressions of $\bar{F}_{\alpha}(p)$ and $\bar{K}_{\alpha}(p)$, the 2nth-harmonic component of $\bar{i}_{\alpha}(p)$ is contributed not only by the term $\bar{A}_n(p) \bar{F}_{\alpha}(p + j2n\omega)$ but also by the terms $\bar{A}_{n\pm 1}(p) \bar{F}_{\alpha}(p + j2n\omega \pm 1)$. Same can be said for the β component of current $\bar{i}_{\beta}(p)$.

When p is in the vicinity of $-j2n\omega$, or $p = \epsilon - j2n\omega$, where ϵ is a real quantity small compared with ω , the values of these three terms contributing to the 2nth-harmonic component of $\bar{i}_{\alpha}(p)$ are

$$\begin{aligned} \left[\bar{A}_{n-1}(p) \bar{F}_{\alpha}(p + j2(n-1)\omega) \right]_{p=\epsilon-j2n\omega} &= \left(-\frac{1}{\lambda}\right)^{n-1} \epsilon^{-j2(n-1)\lambda} \prod_{\nu=2}^n \frac{\bar{K}_{\alpha}(-j2\nu\omega)}{\bar{X}_{\alpha}(-j2\nu\omega)} \cdot \left[\frac{\bar{F}_{\alpha}(p + j2(n-1)\omega)}{\bar{X}_{\alpha}(p + j2(n-1)\omega) + \bar{X}_{\alpha}^*(p + j2(n-1)\omega) - 1} \right]_{p=\epsilon-j2n\omega} \\ &= \left(-\frac{1}{\lambda}\right)^{n-1} \epsilon^{-j2(n-1)\lambda} \prod_{\nu=2}^n \frac{\bar{K}_{\alpha}(-j2\nu\omega)}{\bar{X}_{\alpha}(-j2\nu\omega)} \cdot \left[\left\{ \frac{\frac{1}{\lambda} \bar{K}_{\alpha}(p + j2n\omega) \bar{K}_{\alpha}(p + j2(n-1)\omega)}{\bar{X}_{\alpha}(p + j2(n-1)\omega) - \frac{1}{\lambda} \bar{K}_{\alpha}(p + j2n\omega) \bar{K}_{\alpha}(p + j2(n-1)\omega)} + 1 \right\} \frac{\bar{F}_{\alpha}(p + j2(n-1)\omega)}{\bar{X}_{\alpha}(p + j2(n-1)\omega)} \right]_{p=\epsilon-j2n\omega} \end{aligned} \quad (10.42)$$

$$\begin{aligned} \left[\bar{A}_n(p) \bar{F}_{\alpha}(p + j2n\omega) \right]_{p=\epsilon-j2n\omega} &= \left(-\frac{1}{\lambda}\right)^n \epsilon^{-j2n\lambda} \prod_{\nu=2}^n \frac{\bar{K}_{\alpha}(-j2\nu\omega)}{\bar{X}_{\alpha}(-j2\nu\omega)} \cdot \left[\frac{\bar{K}_{\alpha}(p + j2(n-1)\omega)}{\bar{X}_{\alpha}(p + j2(n-1)\omega)} \cdot \frac{\bar{F}_{\alpha}(p + j2n\omega)}{\bar{X}_{\alpha}(p + j2n\omega) + \bar{X}_{\alpha}^*(p + j2n\omega) - 1} \right]_{p=\epsilon-j2n\omega} \\ &= \left(-\frac{1}{\lambda}\right)^n \epsilon^{-j2n\lambda} \prod_{\nu=2}^n \frac{\bar{K}_{\alpha}(-j2\nu\omega)}{\bar{X}_{\alpha}(-j2\nu\omega)} \cdot \left[\frac{\bar{K}_{\alpha}(p + j2(n-1)\omega) \bar{F}_{\alpha}(p + j2n\omega)}{\bar{X}_{\alpha}(p + j2(n-1)\omega) - \frac{1}{\lambda} \bar{K}_{\alpha}(p + j2n\omega) \bar{K}_{\alpha}(p + j2(n-1)\omega)} \right]_{p=\epsilon-j2n\omega} \end{aligned} \quad (10.43)$$

and

$$\begin{aligned} \left[\bar{A}_{n+1}(p) \bar{E}_\alpha(p+j2n\pi\omega) \right]_{p=\epsilon-j2n\omega} &= \left(-\frac{1}{2} \right)^{n+1} e^{-j2(n+1)\lambda} \prod_{\nu=2}^n \frac{\bar{K}_\alpha(-j2\nu\omega)}{\bar{Z}_\alpha(-j2\nu\omega)} \cdot \left[\frac{\bar{K}_\alpha(p+j2n\omega)}{\bar{Z}_\alpha(p+j2n\omega)} \cdot \frac{\bar{K}_\alpha(p+j2\pi\omega)}{\bar{Z}_\alpha(p+j2\pi\omega)} \cdot \frac{\bar{E}_\alpha(p+j2\pi\omega)}{\bar{Z}_\alpha(p+j2\pi\omega) + \bar{Z}_\alpha^*(p+j2\pi\omega) - 1} \right]_{p=\epsilon-j2n\omega} \\ &= \left(-\frac{1}{2} \right)^{n+1} e^{-j2(n+1)\lambda} \prod_{\nu=2}^n \frac{\bar{K}_\alpha(-j2\nu\omega)}{\bar{Z}_\alpha(-j2\nu\omega)} \cdot \left[\frac{\bar{K}_\alpha(p+j2n\omega) \bar{K}_\alpha(p+j2\pi\omega)}{\bar{Z}_\alpha(p+j2\pi\omega) - \frac{1}{2} \bar{K}_\alpha(p+j2n\omega) \bar{K}_\alpha(p+j2\pi\omega)} \cdot \frac{\bar{E}_\alpha(p+j2\pi\omega)}{\bar{Z}_\alpha(p+j2\pi\omega)} \right]_{p=\epsilon-j2n\omega} \quad (10.44) \end{aligned}$$

Since ϵ is an unknown quantity so that the evaluation of Eqs. 10.42, 10.43 and 10.44 cannot be done by a direct substitution of $\epsilon - j2n\omega$ for p . As ϵ is small compared with ω , this can be done by simply putting $p = -j2n\omega$ except when p is in the factor $p + j2n\omega$, when it is left unchanged. Thus, neglecting the resistances compared with the reactances at normal or higher frequencies, it can be shown that the values of the functions on the right-hand side of Eqs. 10.42, 10.43 and 10.44 are as follows.

$$\begin{aligned} \left(-\frac{1}{2} \right)^{n+1} \prod_{\nu=2}^n \frac{\bar{K}_\alpha(-j2\nu\omega)}{\bar{Z}_\alpha(-j2\nu\omega)} &\approx b_2^{n+1} \\ \left[\bar{K}_\alpha(p+j2n\omega) \right]_{p=\epsilon-j2n\omega} &\approx - \frac{\frac{2L_0(L_2-L_0)}{L_0+L_2} (p+j2n\omega)}{R_0+2R_0+(p+j2n\omega) \left\{ 2L_0 + \frac{2L_0L_2}{L_2+L_0} \right\}} \\ \left[\bar{K}_\alpha(p+j2\pi\omega) \right]_{p=\epsilon-j2n\omega} &\approx - \frac{\frac{2L_0L_2(L_2-L_0)}{L_0+L_2} \left\{ R_0 + (p+j2n\omega)L_0 \right\}}{R_0 \left\{ L_0L_2 + L_0 \left(\frac{L_2+L_0}{2} + \frac{2L_0L_2}{L_2+L_0} \right) \right\} + (p+j2n\omega) L_0L_2 \left\{ 2L_0 + \frac{2L_0L_2}{L_2+L_0} \right\}} \\ \left[\bar{Z}_\alpha(p+j2\pi\omega) \right]_{p=\epsilon-j2n\omega} &\approx \frac{1}{2} \frac{R_0 \left\{ L_0L_2 + 2L_0 \frac{2L_0L_2}{L_2+L_0} + \sqrt{K_0L_2(L_2+2L_0)} \right\} + (p+j2n\omega) \frac{2L_0L_2}{L_2+L_0} \left\{ L_0L_2 + L_0(L_2+L_0) + \sqrt{K_0L_2(L_2+2L_0)} \right\}}{R_0 \left\{ L_0L_2 + L_0 \left(\frac{L_2+L_0}{2} + \frac{2L_0L_2}{L_2+L_0} \right) \right\} + (p+j2n\omega) L_0L_2 \left\{ 2L_0 + \frac{2L_0L_2}{L_2+L_0} \right\}} \\ \left[\bar{E}_\alpha(p+j2\pi\omega) \right]_{p=\epsilon-j2n\omega} &\approx - \frac{E_{0m} e^{j\lambda}}{2\omega} \cdot \frac{L_0L_2(L_2-L_0)/(L_2+L_0)}{R_0 \left\{ L_0L_2 + L_0 \left(\frac{L_2+L_0}{2} + \frac{2L_0L_2}{L_2+L_0} \right) \right\} + (p+j2n\omega) L_0L_2 \left\{ 2L_0 + \frac{2L_0L_2}{L_2+L_0} \right\}} \\ \left[\bar{E}_\alpha(p+j2n\omega) \right]_{p=\epsilon-j2n\omega} &\approx \frac{E_{0m} \cos \lambda}{\omega} \cdot \frac{1}{R_0+2R_0+(p+j2n\omega) \left\{ 2L_0 + \frac{2L_0L_2}{L_2+L_0} \right\}} \\ \left[\bar{E}_\alpha(p+j2\pi\omega) \right]_{p=\epsilon-j2n\omega} &\approx - \frac{E_{0m} e^{j\lambda}}{2\omega} \cdot \frac{L_0L_2(L_2-L_0)/(L_2+L_0)}{R_0 \left\{ L_0L_2 + L_0 \left(\frac{L_2+L_0}{2} + \frac{2L_0L_2}{L_2+L_0} \right) \right\} + (p+j2n\omega) L_0L_2 \left\{ 2L_0 + \frac{2L_0L_2}{L_2+L_0} \right\}} \end{aligned} \quad (10.45)$$

Substituting Equ. 10.45 in Eqs. 10.42, 10.43 and 10.44 and adding gives

$$\left[\bar{E}_\alpha(p) \right]_{p=\epsilon-j2n\omega} = \left[\bar{A}_{n+1}(p) \bar{E}_\alpha(p+j2\pi\omega) + \bar{A}_n(p) \bar{E}_\alpha(p+j2n\omega) + \bar{A}_{n-1}(p) \bar{E}_\alpha(p+j2\pi\omega) \right]_{p=\epsilon-j2n\omega}$$

$$= - \frac{b_1^{n-1} E_{am}}{2\omega} \cdot \frac{L_d - L_q}{L_d L_q + L_o (L_d + L_q) + \sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)}} \cdot \frac{e^{-j(2n-1)\lambda}}{\frac{1}{T_{\beta}(l-n)} + p + j2n\omega}$$

$$+ \frac{b_1^{n-1} E_{am} \cos \lambda}{2\omega} \cdot \frac{(L_d^2 - L_q^2) \left\{ L_d L_q + 2L_o \frac{2L_d L_q}{L_d + L_q} + \sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)} \right\}}{\sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)} \left\{ L_d L_q + L_o (L_d + L_q) + \sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)} \right\}} \cdot \frac{\{R_o + (p + j2n\omega)L_o\} e^{-j2n\lambda}}{\left\{ \frac{1}{T_{\alpha}(l-n)} + p + j2n\omega \right\} \left\{ \frac{1}{T_{\beta}(l-n)} + p + j2n\omega \right\}}$$

(10.46)

where $T_{\alpha}(l-n) = \frac{1}{R_o + 2R_o} \cdot \frac{2L_o \sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)}}{\sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)} - L_d L_q} = \frac{1}{R_o + 2R_o} \cdot \frac{2kL_o}{k-1}$

(10.47)

$$T_{\beta}(l-n) = \frac{1}{R_o} \cdot \frac{2L_o L_d L_q}{\sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)} - L_d L_q} = \frac{1}{R_o} \cdot \frac{2L_o}{k-1}$$

(10.48)

Resolving the second term on the right-hand side of Equ. 10.46 into partial fractions and grouping gives

$$[\bar{I}_{\alpha}(p)]_{p=\epsilon-j2n\omega} \approx - \frac{b_1^{n-1} E_{am}}{2\omega} \cdot \frac{L_d - L_q}{L_d L_q + L_o (L_d + L_q) + \sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)}} \times \left\{ \frac{L_d L_q \cos \lambda}{\sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)}} \cdot \frac{e^{-j2n\lambda}}{\frac{1}{T_{\alpha}(l-n)} + p + j2n\omega} + \frac{(\epsilon^{j\lambda} - \cos \lambda) e^{-j2n\lambda}}{\frac{1}{T_{\beta}(l-n)} + p + j2n\omega} \right\}$$

(10.49)

or $[\bar{I}_{\alpha}(p)]_{p=\epsilon-j2n\omega} \approx - \frac{b_1^n E_{am}}{2\omega L_o} \left\{ \frac{L_d L_q \cos \lambda}{\sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)}} \cdot \frac{e^{-j2n\lambda}}{\frac{1}{T_{\alpha}(l-n)} + p + jn\omega} - \frac{j \sin \lambda e^{-j2n\lambda}}{\frac{1}{T_{\beta}(l-n)} + p + j2n\omega} \right\}$

(10.50)

Equ. 10.50 does not hold for $n = 0$, as $\bar{A}_0(p) \bar{F}_{\alpha}(p)$ is a singular term in the operational solution. For $n = 0$, or for the d.c. component, the terms $\bar{A}_0(p) \bar{F}_{\alpha}(p)$, $\bar{A}_1(p) \bar{F}_{\alpha}(p + j2\omega)$ and its conjugate are considered similarly for a value of p in the vicinity of 0 (say ϵ). It can be shown that

$$[\bar{I}_{\alpha}(p)]_{p=\epsilon} = \left[\bar{A}_0(p) \bar{F}_{\alpha}(p) + \bar{A}_1(p) \bar{F}_{\alpha}(p + j2\omega) + \text{conj} \right]_{p=\epsilon}$$

$$= \text{2nd term of Equ. 10.46 (n=0)} \times (-2) \left[\frac{\bar{I}_{\alpha}(p-j2\omega)}{\bar{K}_{\alpha}(p-j2\omega)} \right]_{p=\epsilon}$$

$$\approx \frac{E_{am} \cos \lambda}{\omega} \cdot \frac{L_d + L_q}{2\sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)}} \cdot \frac{L_d L_q + 2L_o \frac{2L_d L_q}{L_d + L_q} + \sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)}}{L_d L_q + L_o (L_d + L_q) + \sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)}} \cdot \frac{1}{\frac{1}{T_{\alpha}(l-n)} + p}$$

$$= \frac{E_{am} \cos \lambda}{2\omega L_o} \cdot \frac{\sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)} - L_d L_q}{\sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)}} \cdot \frac{1}{\frac{1}{T_{\alpha}(l-n)} + p}$$

(10.51)

The complete expression for the armature-transient component of $\bar{I}_{\alpha}(p)$ is obtained by adding Equ. 10.51 to Equ. 10.50 (and its conjugate) summed from $n = 1$ to $n = \infty$. Thus, replacing $\sqrt{(L_d^2 + 2L_o)(L_q^2 + 2L_o)}/L_d L_q$ by k ,

$$i_{\alpha}(p) \left[\begin{array}{l} \text{Armature} \\ \text{transient} \end{array} \right] \approx \frac{E_{am} \cos \lambda}{2 \omega L_o} \left\{ \left(1 - \frac{1}{k}\right) \frac{1}{\frac{1}{T_{a(u-n)}} + p} - \frac{1}{k} \sum_{n=1}^{\infty} \left[\frac{b_1^n e^{-j \lambda n}}{\frac{1}{T_{a(u-n)}} + p + j \lambda n \omega} + \text{Conj} \right] \right\} \\ - \frac{E_{am} \sin \lambda}{2 \omega L_o} \sum_{n=1}^{\infty} \left\{ \frac{b_2^n j e^{-j \lambda n}}{\frac{1}{T_{\beta(u-n)}} + p + j \lambda n \omega} + \text{Conj} \right\} \quad (10.52)$$

The inverse transform of Equ. 10.52 gives the explicit expression for the armature-transient component of i_{α} :

$$i_{\alpha}(t) \left[\begin{array}{l} \text{Armature} \\ \text{transient} \end{array} \right] \approx \frac{E_{am} \cos \lambda}{\omega L_o} e^{-t/T_{a(u-n)}} \left\{ \frac{1}{k} \left(1 - \frac{1}{k}\right) - \frac{1}{k} \sum_{n=1}^{\infty} b_1^n \cos 2 \pi \theta \right\} \\ - \frac{E_{am} \sin \lambda}{\omega L_o} e^{-t/T_{\beta(u-n)}} \sum_{n=1}^{\infty} b_2^n \sin 2 \pi \theta \quad (10.53)$$

As a check, put $R_o = L_o = 0$ so that i_{α} becomes the symmetrical short-circuit current of a two-phase machine. The time constants become, from Eqs. 10.47 and 10.48,

$$\lim_{\substack{R_o \rightarrow 0 \\ L_o \rightarrow 0}} T_{a(u-n)} = \lim_{\substack{R_o \rightarrow 0 \\ L_o \rightarrow 0}} \frac{1}{R_a + 2R_o} \cdot \frac{2kL_o}{k-1} \\ = \lim_{\substack{R_o \rightarrow 0 \\ L_o \rightarrow 0}} \frac{1}{R_a + 2R_o} \cdot \frac{(L_d + 2L_o)(L_q + 2L_o) + \sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)}}{L_d + L_q + 2L_o} \\ = \frac{1}{R_a} \cdot \frac{2L_d L_q}{L_d + L_q}$$

$$\text{and } \lim_{\substack{R_o \rightarrow 0 \\ L_o \rightarrow 0}} T_{\beta(u-n)} = \lim_{\substack{R_o \rightarrow 0 \\ L_o \rightarrow 0}} \frac{1}{R_a} \cdot \frac{2L_o}{k-1} \\ = \lim_{\substack{R_o \rightarrow 0 \\ L_o \rightarrow 0}} \frac{1}{R_a} \cdot \frac{L_d L_q + \sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)}}{L_d + L_q + 2L_o} \\ = \frac{1}{R_a} \cdot \frac{2L_d L_q}{L_d + L_q}$$

which are equal to $T_a(3)$ (Equ. 8.16). The d.c. component is reduced to, from Equ. 10.53,

$$\lim_{\substack{R_o \rightarrow 0 \\ L_o \rightarrow 0}} \frac{E_{am} \cos \lambda}{2 \omega L_o} e^{-t/T_{a(u-n)}} \left(1 - \frac{1}{k}\right) \\ = \frac{E_{am} \cos \lambda}{2 \omega} e^{-t/T_{a(u-n)}} \lim_{L_o \rightarrow 0} \frac{2(L_d + L_q)}{(L_d + 2L_o)(L_q + 2L_o) + \sqrt{L_d L_q (L_d + 2L_o)(L_q + 2L_o)}} \\ = \frac{E_{am} \cos \lambda}{2 \omega} e^{-t/T_{a(u-n)}} \left(\frac{1}{L_d} + \frac{1}{L_q}\right)$$

as in Equ. 8.19. The double-frequency component is reduced to, from Equ. 10.53,

$$- \lim_{\substack{R_o \rightarrow 0 \\ L_o \rightarrow 0}} \left\{ \frac{E_{am} \cos \lambda}{\omega L_o} e^{-t/T_{a(u-n)}} \cdot \frac{b_1}{k} \cos 2 \theta + \frac{E_{am} \sin \lambda}{\omega L_o} e^{-t/T_{\beta(u-n)}} b_2 \sin 2 \theta \right\} \\ = - \frac{E_{am}}{\omega} e^{-t/T_{a(u-n)}} \lim_{L_o \rightarrow 0} \frac{2(L_d - L_q)}{[\sqrt{L_d L_q (L_d + 2L_o)} + \sqrt{L_q L_d (L_d + 2L_o)}]^2} \left\{ \frac{1}{k} \cos 2 \theta \cos \lambda + \sin 2 \theta \sin \lambda \right\} \\ = \frac{E_{am}}{2 \omega} e^{-t/T_{a(u-n)}} \left(\frac{1}{L_d} - \frac{1}{L_q}\right) \cos (2 \theta - \lambda)$$

as in Equ. 8.19. The harmonics higher than the second given by Equ. 10.53 are all equal to zero for $L_0 = 0$.

Similarly the 2nth-harmonic component of the armature-transient component of $\bar{i}_\beta(p)$ is given by

$$[\bar{i}_\beta(p)]_{p=\epsilon-j2n\omega} = \left[\bar{B}_{n-1}(p) \bar{F}_\beta(p+j2(n-1)\omega) + \bar{B}_n(p) \bar{F}_\beta(p+j2n\omega) + \bar{B}_{n+1}(p) \bar{F}_\beta(p+j2(n+1)\omega) \right]_{p=\epsilon-j2n\omega} \quad (10.54)$$

and its conjugate. The functions on the right-hand side of Equ. 10.54 can be expressed similar to Equ. 10.42, 10.43 and 10.44 as.

$$\left. \begin{aligned} [\bar{B}_{n-1}(p) \bar{F}_\beta(p+j2(n-1)\omega)]_{p=\epsilon-j2n\omega} &= (-\frac{1}{2})^{n-1} e^{-j2(n-1)\lambda} \prod_{\nu=2}^n \frac{\bar{K}_\beta(-j2\nu\omega)}{\bar{X}_\beta(-j2\nu\omega)} \left[\left\{ \frac{\frac{1}{2} \bar{K}_\beta(p+j2n\omega) \bar{K}_\beta(p+j2(n-1)\omega)}{\bar{X}_\beta(p+j2(n-1)\omega) - \frac{1}{2} \bar{K}_\beta(p+j2n\omega) \bar{K}_\beta(p+j2(n-1)\omega)} + 1 \right\} \frac{\bar{F}_\beta(p+j2(n-1)\omega)}{\bar{X}_\beta(p+j2(n-1)\omega)} \right]_{p=\epsilon-j2n\omega} \\ [\bar{B}_n(p) \bar{F}_\beta(p+j2n\omega)]_{p=\epsilon-j2n\omega} &= (-\frac{1}{2})^n e^{-j2n\lambda} \prod_{\nu=2}^n \frac{\bar{K}_\beta(-j2\nu\omega)}{\bar{X}_\beta(-j2\nu\omega)} \left[\frac{\bar{K}_\beta(p+j2(n-1)\omega) \bar{F}_\beta(p+j2n\omega)}{\bar{X}_\beta(p+j2(n-1)\omega) - \frac{1}{2} \bar{K}_\beta(p+j2n\omega) \bar{K}_\beta(p+j2(n-1)\omega)} \right]_{p=\epsilon-j2n\omega} \\ [\bar{B}_{n+1}(p) \bar{F}_\beta(p+j2(n+1)\omega)]_{p=\epsilon-j2n\omega} &= (-\frac{1}{2})^{n+1} e^{-j2(n+1)\lambda} \prod_{\nu=2}^n \frac{\bar{K}_\beta(-j2\nu\omega)}{\bar{X}_\beta(-j2\nu\omega)} \left[\frac{\frac{1}{2} \bar{K}_\beta(p+j2n\omega) \bar{K}_\beta(p+j2(n+1)\omega)}{\bar{X}_\beta(p+j2(n-1)\omega) - \frac{1}{2} \bar{K}_\beta(p+j2n\omega) \bar{K}_\beta(p+j2(n+1)\omega)} \cdot \frac{\bar{F}_\beta(p+j2(n+1)\omega)}{\bar{X}_\beta(p+j2(n+1)\omega)} \right]_{p=\epsilon-j2n\omega} \end{aligned} \right\} \quad (10.55)$$

The functions involved in Equ. 10.55 can be evaluated by putting $-j2n\omega$ for p except where p is in the factor $p+j2n\omega$ when it is retained. It can be shown that with the resistances neglected compared with the reactances at normal or higher frequencies, the values of these functions are as follows.

$$\left. \begin{aligned} (-\frac{1}{2})^{n-1} \prod_{\nu=2}^n \frac{\bar{K}_\beta(-j2\nu\omega)}{\bar{X}_\beta(-j2\nu\omega)} &\approx b_2^{n-1} \\ [\bar{K}_\beta(p+j2n\omega)]_{p=\epsilon-j2n\omega} &\approx -\frac{\frac{2L_0(L'_2-L'_q)}{L'_2+L'_q+4L_0} (p+j2n\omega)}{R_a+(p+j2n\omega) \cdot \frac{L'_2+L'_q+L_0(L'_2+L'_q)}{L'_2+L'_q+4L_0}} \\ [\bar{K}_\beta(p+j2(n-1)\omega)]_{p=\epsilon-j2n\omega} &\approx -\frac{2(L'_2+2L_0)(L'_q+2L_0) \frac{L'_2-L'_q}{L'_2+L'_q} [R_0+(p+j2n\omega)L_0]}{(R_a+2R_0)[(L'_2+2L_0)(L'_q+2L_0)+L_0(\frac{2L'_2L'_q}{L'_2+L'_q}-\frac{L'_2+L'_q}{2})] + (p+j2n\omega)(L'_2+2L_0)(L'_q+2L_0)(2L_0+\frac{2L'_2L'_q}{L'_2+L'_q})} \\ [\bar{X}_\beta(p+j2(n-1)\omega)]_{p=\epsilon-j2n\omega} &= \frac{\frac{(L'_2+2L_0)(L'_q+2L_0)}{L'_2+L'_q} [L'_2L'_q+L_0(L'_2+L'_q)+\sqrt{L'_2L'_q(L'_2+2L_0)(L'_q+2L_0)}] (\frac{1}{T_{2(11-2)}}+p+j2n\omega)}{(R_a+2R_0)[(L'_2+2L_0)(L'_q+2L_0)+L_0(\frac{2L'_2L'_q}{L'_2+L'_q}-\frac{L'_2+L'_q}{2})] + (p+j2n\omega)(L'_2+2L_0)(L'_q+2L_0)(2L_0+\frac{2L'_2L'_q}{L'_2+L'_q})} \\ [\bar{F}_\beta(p+j2(n-1)\omega)]_{p=\epsilon-j2n\omega} &\approx -\frac{E_{am}j e^{-j\lambda}}{2\omega} \cdot \frac{(L'_2+2L_0)(L'_q+2L_0)(L'_2-L'_q)/(L'_2+L'_q)}{(R_a+2R_0)[(L'_2+2L_0)(L'_q+2L_0)+L_0(\frac{2L'_2L'_q}{L'_2+L'_q}-\frac{L'_2+L'_q}{2})] + (p+j2n\omega)(L'_2+2L_0)(L'_q+2L_0)(2L_0+\frac{2L'_2L'_q}{L'_2+L'_q})} \\ [\bar{F}_\beta(p+j2n\omega)]_{p=\epsilon-j2n\omega} &\approx \frac{E_{am} \sin \lambda}{\omega} \cdot \frac{1}{R_a+(p+j2n\omega) \frac{L'_2L'_q+L_0(L'_2+L'_q)}{L'_2+L'_q+4L_0}} \\ [\bar{F}_\beta(p+j2(n+1)\omega)]_{p=\epsilon-j2n\omega} &= \text{Conjugate of } [\bar{F}_\beta(p+j2(n-1)\omega)]_{p=\epsilon-j2n\omega} \end{aligned} \right\} \quad (10.56)$$

Substituting Eqs. 10.55 and 10.56 in Equ. 10.54 and reducing gives

$$\begin{aligned} [\bar{i}_\beta(p)]_{p=j2n\omega} \approx & -\frac{b_2^{n-1} E_{am}}{\lambda \omega} \cdot \frac{L_d'' - L_q''}{L_d'' L_q'' + L_0(L_d'' + L_q'') + \sqrt{L_d'' L_q'' (L_d'' + 2L_0)(L_q'' + 2L_0)}} \cdot \frac{j e^{-j(2n-1)\lambda}}{\frac{1}{T_{a(u-n)}} + p + j2n\omega} \\ & + \frac{b_2^{n-1} E_{am} \sin \lambda}{\lambda \omega} \cdot \frac{(L_d''^2 - L_q''^2) \{ L_d'' L_q'' + 2L_0 \frac{2L_d'' L_q''}{L_d'' + L_q''} + \sqrt{L_d'' L_q'' (L_d'' + 2L_0)(L_q'' + 2L_0)} \}}{L_d'' L_q'' \{ L_d'' L_q'' + L_0(L_d'' + L_q'') + \sqrt{L_d'' L_q'' (L_d'' + 2L_0)(L_q'' + 2L_0)} \}^2} \cdot \frac{\{ R_0 + (p + j2n\omega) L_0 \} e^{-j2n\lambda}}{\{ \frac{1}{T_{a(u-n)}} + p + j2n\omega \} \{ \frac{1}{T_{\beta(u-n)}} + p + j2n\omega \}} \end{aligned} \quad (10.57)$$

Or, resolving into partial fractions,

$$\begin{aligned} [\bar{i}_\beta(p)]_{p=j2n\omega} &= -\frac{b_2^{n-1} E_{am}}{\lambda \omega} \cdot \frac{L_d'' - L_q''}{L_d'' L_q'' + L_0(L_d'' + L_q'') + \sqrt{L_d'' L_q'' (L_d'' + 2L_0)(L_q'' + 2L_0)}} \left\{ \frac{(j e^{j\lambda} + \sin \lambda) e^{-j2n\lambda}}{\frac{1}{T_{a(u-n)}} + p + j2n\omega} - \frac{\sqrt{L_d'' L_q'' (L_d'' + 2L_0)(L_q'' + 2L_0)}}{L_d'' L_q''} \cdot \frac{\sin \lambda e^{-j2n\lambda}}{\frac{1}{T_{\beta(u-n)}} + p + j2n\omega} \right\} \\ &= -\frac{b_2^{n-1} E_{am}}{\lambda \omega L_0} \left\{ \frac{j \cos \lambda e^{-j2n\lambda}}{\frac{1}{T_{a(u-n)}} + p + j2n\omega} - \frac{\sqrt{L_d'' L_q'' (L_d'' + 2L_0)(L_q'' + 2L_0)}}{L_d'' L_q''} \cdot \frac{\sin \lambda e^{-j2n\lambda}}{\frac{1}{T_{\beta(u-n)}} + p + j2n\omega} \right\} \end{aligned} \quad (10.58)$$

The sum of Equ. 10.58 and its conjugate function gives the 2nth-harmonic component of the armature-transient component of $\bar{i}_\beta(p)$. The d.c. component of $\bar{i}_\beta(p)$ is given by, similar to Equ. 10.51,

$$\begin{aligned} [\bar{i}_\beta(p)]_{p=0} &= [\bar{B}_0(p) \bar{F}_\beta(p) + \bar{B}_0(p) \bar{F}_\beta(p + j2\omega) + \text{Conj}]_{p=0} \\ &= \text{2nd term of Equ. 10.57 } (n=0) \times (-2) \left[\frac{\bar{x}_\beta(p - j2\omega)}{\bar{K}_\beta(p - j2\omega)} \right]_{p=0} \\ &\approx \frac{E_{am} \sin \lambda}{\lambda \omega L_0} \cdot \frac{\sqrt{L_d'' L_q'' (L_d'' + 2L_0)(L_q'' + 2L_0)} - L_d'' L_q''}{L_d'' L_q''} \cdot \frac{1}{\frac{1}{T_{\beta(u-n)}} + p} \end{aligned} \quad (10.59)$$

The complete expression for the armature-transient component is given by adding Equ. 10.59 to Equ. 10.55 and its conjugate, summed from $n = 1$ to $n = \infty$. Thus,

$$\begin{aligned} \bar{i}_\beta(p) [\text{Armature transient}] &= -\frac{E_{am} \cos \lambda}{\lambda \omega L_0} \sum_{n=1}^{\infty} \left\{ \frac{b_2^n j e^{-j2n\lambda}}{\frac{1}{T_{a(u-n)}} + p + j2n\omega} + \text{Conj} \right\} \\ &\quad + \frac{E_{am} \sin \lambda}{\lambda \omega L_0} \left\{ (k-1) \frac{1}{\frac{1}{T_{\beta(u-n)}} + p} + k \sum_{n=1}^{\infty} \left[\frac{b_2^n e^{-j2n\lambda}}{\frac{1}{T_{\beta(u-n)}} + p + j2n\omega} + \text{Conj} \right] \right\} \end{aligned} \quad (10.60)$$

where $k = \sqrt{(L_d'' + 2L_0)(L_q'' + 2L_0) / L_d'' L_q''}$. The inverse transform of Equ. 10.60 is

$$\begin{aligned} i_\beta(t) [\text{Armature transient}] &\approx -\frac{E_{am} \cos \lambda}{\omega L_0} e^{-t/T_{a(u-n)}} \sum_{n=1}^{\infty} b_2^n \sin 2n\theta \\ &\quad + \frac{E_{am} \sin \lambda}{\omega L_0} e^{-t/T_{\beta(u-n)}} \left\{ \frac{1}{\lambda} (k-1) + k \sum_{n=1}^{\infty} b_2^n \cos 2n\theta \right\} \end{aligned} \quad (10.61)$$

Equ. 10.61 can be reduced to the armature-transient component of the symmetrical short-circuit current of a two-phase machine when R_0 and L_0 are put to zero, as has been shown for i_α . As a further check, put $L_0 = \infty$ so that i_β corresponds to the single-phase line-to-line short-circuit current ($2/\sqrt{3}$ times as large) and see if the result agrees with that obtained previously. When $L_0 = \infty$, the sine series in Equ. 10.61 goes out, b_2 becomes $-b_1$ and

$$\lim_{L_0 \rightarrow \infty} \frac{k}{L_0} = \lim_{L_0 \rightarrow \infty} \sqrt{\frac{(L_d^2 + 2L_0)(L_q^2 + 2L_0)}{L_d L_q L_0}} = \frac{2}{\sqrt{L_d L_q}}$$

$$\lim_{L_0 \rightarrow \infty} T_{\beta(u-n)} = \lim_{L_0 \rightarrow \infty} \frac{1}{R_a} \cdot \frac{L_d L_q + \sqrt{L_d L_q (L_d + 2L_0)(L_q + 2L_0)}}{L_d + L_q + 2L_0} = \frac{\sqrt{L_d L_q}}{R_a} = T_{au(u)}$$

so that Equ. 10.61 becomes

$$\frac{2E_{am} \sin \lambda}{\omega \sqrt{L_d L_q}} e^{-t/T_{au(u)}} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} (-b_1)^n \cos 2n\theta \right\}$$

which is equal to $2/\sqrt{3}$ times the armature-transient component of Equ. 9.55.

The phase components of the armature currents are given by, since $i_0 = -i_\alpha$,

$$i_b(t) \left[\begin{smallmatrix} \text{Armature} \\ \text{transient} \end{smallmatrix} \right] = -\frac{3}{2} i_\alpha(t) \left[\begin{smallmatrix} \text{Armature} \\ \text{transient} \end{smallmatrix} \right] + \frac{\sqrt{3}}{2} i_\beta(t) \left[\begin{smallmatrix} \text{Armature} \\ \text{transient} \end{smallmatrix} \right]$$

$$\approx -\frac{\sqrt{3} E_{am} \cos \lambda}{2\omega L_0} e^{-t/T_{\alpha(u-n)}} \left\{ \frac{\sqrt{3}}{2} \left(1 - \frac{1}{k} \right) - \frac{1}{k} \sum_{n=1}^{\infty} b_1^n \left[\sqrt{3} \cos 2n\theta + k \sin 2n\theta \right] \right\}$$

$$+ \frac{\sqrt{3} E_{am} \sin \lambda}{2\omega L_0} e^{-t/T_{\beta(u-n)}} \left\{ \frac{1}{2} (k-1) + \sum_{n=1}^{\infty} b_1^n \left[\sqrt{3} \sin 2n\theta + k \cos 2n\theta \right] \right\} \quad (10.62)$$

$$i_c(t) \left[\begin{smallmatrix} \text{Armature} \\ \text{transient} \end{smallmatrix} \right] = -\frac{3}{2} i_\alpha(t) \left[\begin{smallmatrix} \text{Armature} \\ \text{transient} \end{smallmatrix} \right] - \frac{\sqrt{3}}{2} i_\beta(t) \left[\begin{smallmatrix} \text{Armature} \\ \text{transient} \end{smallmatrix} \right]$$

$$\approx -\frac{\sqrt{3} E_{am} \cos \lambda}{2\omega L_0} e^{-t/T_{\alpha(u-n)}} \left\{ \frac{\sqrt{3}}{2} \left(1 - \frac{1}{k} \right) - \frac{1}{k} \sum_{n=1}^{\infty} b_1^n \left[\sqrt{3} \cos 2n\theta - k \sin 2n\theta \right] \right\}$$

$$+ \frac{\sqrt{3} E_{am} \sin \lambda}{2\omega L_0} e^{-t/T_{\beta(u-n)}} \left\{ -\frac{1}{2} (k-1) + \sum_{n=1}^{\infty} b_1^n \left[\sqrt{3} \sin 2n\theta - k \cos 2n\theta \right] \right\} \quad (10.63)$$

which are rather complicated expressions. The fact that the two armature time constants as given by Eqs. 10.47 and 10.48 have no simple relation with each other even for the simpler case of no earthing resistance ($R_0 = R_g$) makes it impossible to reduce Eqs. 10.62 and 10.63 any further.

Finally, the total short-circuit currents in phase b and c are given by the sum of the odd-harmonic series given by Equ. 10.41 and 10.40 and the even-harmonic series given by Eqs. 10.62 and 10.63.

(10.6) Field and Damper Currents

The d and q components of the armature current are first determined before using Eqs. 7.76, 7.77 and 7.78 to find the field and damper currents. These components of current are given by, in terms of the α and β components,

$$\begin{aligned} \tau_d(p) &= e^{-j\lambda} \tau_{\alpha}(p+j\omega) + \text{Conj} \\ &= \frac{1}{2} e^{-j\lambda} [\tau_{\alpha}(p+j\omega) + j \tau_{\beta}(p+j\omega)] + \text{Conj} \end{aligned} \quad (10.64)$$

$$\begin{aligned} \text{and } \tau_q(p) &= -j [e^{-j\lambda} \tau_{\alpha}(p+j\omega) - \text{Conj}] \\ &= -\frac{1}{2} j e^{-j\lambda} [\tau_{\alpha}(p+j\omega) + j \tau_{\beta}(p+j\omega)] + \text{Conj} \end{aligned} \quad (10.65)$$

Substituting the expressions for $\bar{i}_{\alpha}(p)$ and $\bar{i}_{\beta}(p)$ in Eqs. 10.64 and 10.65 and reducing gives

$$\begin{aligned} \tau_d(p) &\approx -\frac{E_{am}}{\omega} \frac{1+k}{(1+k)\bar{L}_d(p) + 2L_o} \cdot \frac{1}{p} \\ &\quad - \frac{E_{am}}{2\omega} [1+b_2 - k(1-b_2)] \sum_{n=0}^{\infty} \left\{ \frac{1}{(1+k)\bar{L}_d(p+j2(n+1)\omega) + 2L_o} \cdot \frac{b_2^n e^{-j2(n+1)\lambda}}{p+j2(n+1)\omega} + \text{Conj} \right\} \\ &\quad - \frac{E_{am} \cos \lambda}{4\omega L_o} \cdot \frac{1+b_2 - k(1-b_2)}{k} \sum_{n=0}^{\infty} \left\{ \frac{b_2^n e^{-j(2n+1)\lambda}}{\frac{1}{\bar{L}_{\alpha}(p-j\omega)} + p+j(2n+1)\omega} + \text{Conj} \right\} \\ &\quad - \frac{E_{am} \sin \lambda}{4\omega L_o} [1+b_2 - k(1-b_2)] \sum_{n=0}^{\infty} \left\{ \frac{b_2^n e^{-j(2n+1)\lambda}}{\frac{1}{\bar{L}_{\beta}(p-j\omega)} + p+j(2n+1)\omega} + \text{Conj} \right\} \end{aligned} \quad (10.66)$$

$$\begin{aligned} \text{and } \tau_q(p) \approx & \frac{E_{am}}{2\omega} [1-b_2 - k(1+b_2)] \sum_{n=0}^{\infty} \left\{ \frac{1}{(1+k) \bar{L}_d(p+j2(n+1)\omega) + 2L_0} \frac{b_2^n j e^{-j2(n+1)\lambda}}{p+j2(n+1)\omega} + \text{conj} \right\} \\ & - \frac{E_{am} \cos \lambda}{4\omega L_0} \cdot \frac{1-b_2 - k(1+b_2)}{k} \sum_{n=0}^{\infty} \left\{ \frac{b_2^n j e^{-j(2n+1)\lambda}}{\frac{1}{T_d(u-n)} + p+j(2n+1)\omega} + \text{conj} \right\} \\ & - \frac{E_{am} \sin \lambda}{4\omega L_0} [1-b_2 - k(1+b_2)] \sum_{n=0}^{\infty} \left\{ \frac{b_2^n e^{-j(2n+1)\lambda}}{\frac{1}{T_d(u-n)} + p+j(2n+1)\omega} + \text{conj} \right\} \quad (10.67) \end{aligned}$$

In Eqs. 10.66 and 10.67 the direct-axis inductance is retained in its operational form with the view to avoid the lengthy writing of the steady-state, transient and subtransient components separately.

Referring to Eqs. 7.76, 7.77 and 7.78, the expressions for the field and damper currents in terms of $\bar{i}_d(p)$ and $\bar{i}_q(p)$, it is clear that except for the d.c. component of $\bar{i}_d(p)$ (first term on the right-hand side of Equ. 10.66) when p is in the vicinity of zero, the resistances can be neglected compared with the reactances at twice or higher multiples of the normal frequencies. Thus, the a.c. components of the field and direct-axis-damper currents are approximately given by

$$\bar{i}_f'(p) [\text{a.c.}] \approx -\sqrt{\frac{3}{2}} \cdot \frac{M_d}{L_{fe}} \cdot \frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \bar{i}_d(p) [\text{a.c.}] \quad (10.68)$$

$$\text{and } \bar{i}_{kd}(p) [\text{a.c.}] \approx -\sqrt{\frac{3}{2}} \cdot \frac{M_d}{L_{fe}} \cdot \frac{L_f - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \bar{i}_d(p) [\text{a.c.}] \quad (10.69)$$

and the quadrature-axis-damper current (no d.c. component) is approximately given by

$$\bar{i}_{kq}(p) \approx -\sqrt{\frac{3}{2}} \cdot \frac{M_q}{L_{kq}} \cdot \bar{i}_q(p) \quad (10.70)$$

Consequently the quadrature-axis-damper current and the a.c. components of the field and direct-axis-damper currents can be obtained simply by multiplying i_q and the a.c. components of i_d by appropriate constants as given by Eqs. 10.68, 10.69

and 10.70. If E_{am} is replaced by $\sqrt{2/3} M_d I_{fo}$ in Eqs. 10.66 and 10.67, the a.c. components of the field and damper currents are, in explicit form, as follows.

$$i_f'(t) [a.c.] \approx \frac{M_d^2 I_{fo}}{L_{fe}} \cdot \frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot [1 + b_2 - k(1 - b_2)] \times \left\{ \frac{1}{L_d''(1+k) + 2L_o} g_2(t) \sum_{n=0}^{\infty} b_2^n \cos 2(n+1)\theta + \frac{\cos \lambda}{2kL_o} e^{-t/T_{u(u-n)}} \sum_{n=0}^{\infty} b_2^n \cos(2n+1)\theta + \frac{\sin \lambda}{2L_o} e^{-t/T_{p(u-n)}} \sum_{n=0}^{\infty} b_2^n \sin(2n+1)\theta \right\} \quad (10.71)$$

$$i_{kd}(t) [a.c.] \approx \frac{L_f - M_{fkd}}{L_{kd} - M_{fkd}} \times \text{Equ. 10.71} \quad (10.72)$$

$$i_{kq}(t) \approx -\frac{M_d M_q I_{fo}}{L_{kq}} \cdot [1 - b_2 - k(1 + b_2)] \times \left\{ \frac{1}{L_d''(1+k) + 2L_o} g_2(t) \sum_{n=0}^{\infty} b_2^n \sin 2(n+1)\theta + \frac{\cos \lambda}{2kL_o} e^{-t/T_{u(u-n)}} \sum_{n=0}^{\infty} b_2^n \sin(2n+1)\theta - \frac{\sin \lambda}{2L_o} e^{-t/T_{p(u-n)}} \sum_{n=0}^{\infty} b_2^n \cos(2n+1)\theta \right\} \quad (10.73)$$

The d.c. components of i_f' and i_{kd} can be found as in the previous cases by using Eqs. 7.76 and 7.77. It can be shown that with the assumption of $R_{kd} = R_f$, the expressions for the d.c. components of the field and direct-axis-damper currents are given by

$$i_f(t) [d.c.] = I_{fo} + i_f'(t) [d.c.] \approx I_{fo} + M_d^2 I_{fo} \left\{ \left[\frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{1+k}{L_d''(1+k) + 2L_o} - \frac{1}{L_f} \cdot \frac{1+k}{L_d(1+k) + 2L_o} \right] e^{-t/T_d''(u-n)} + \frac{1}{L_f} \cdot \frac{1+k}{L_d(1+k) + 2L_o} e^{-t/T_d(u-n)} \right\} \quad (10.74)$$

$$i_{kd}(t) [d.c.] \approx M_d^2 I_{fo} \cdot \frac{L_f - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{1+k}{L_d''(1+k) + 2L_o} e^{-t/T_d''(u-n)} \quad (10.75)$$

The complete expressions for the field and direct-axis-damper currents are given by the sums of Eqs. 10.71 and 10.74 and Eqs. 10.72 and 10.75 respectively.

Comparing with the single-phase cases, it is clear that the

field and dampen currents for the double-line-to-neutral short circuit have an extra odd-harmonic series decaying with the time constant of the extra armature circuit, i.e., the α - or the β - component circuit.

(10.7) Open-phase Voltage

The terminal voltage of phase a, the open-circuit phase, is given by, using Equ. 7.27,

$$\bar{v}_{at}(p) = \bar{v}_{\alpha t}(p) + \bar{v}_{ot}(p)$$

which may be reduced to, by using Eqs. 10.3 and 10.5,

$$\bar{v}_{at}(p) = -3(R_0 + L_0 p) \bar{i}_{\alpha}(p) \quad (10.76)$$

If the resistance R_0 is neglected, the voltage of the open phase, $v_{at}(t)$, is equal to $-3L_0$ times the derivative of $i_{\alpha}(t)$. If the resistance is not neglected, the voltage is simply increased by $-3R_0 i_{\alpha}(t)$. Thus, neglecting R_0 and also neglecting the derivatives of $g_2(t)$, $e^{-t/\tau_{\alpha}(u-n)}$ and $e^{-t/\tau_{\beta}(u-n)}$ compared with $g_2(t)$, $e^{-t/\tau_{\alpha}(u-n)}$ and $e^{-t/\tau_{\beta}(u-n)}$, the terminal voltage of phase a is found to be

$$v_{at}(t) = -\frac{6L_0 E_{am}}{L'_k(1+k) + 2L_0} g_2(t) \sum_{n=0}^{\infty} (2n+1) b_1^n \sin(2n+1)\theta$$

$$- \frac{3E_{am} \cos \lambda}{k} e^{-t/\tau_{\alpha}(u-n)} \sum_{n=1}^{\infty} 2n b_1^n \sin 2n\theta + 3E_{am} \sin \lambda e^{-t/\tau_{\beta}(u-n)} \sum_{n=1}^{\infty} 2n b_1^n \cos 2n\theta \quad (10.77)$$

It should be noted that the voltage is measured from earth to terminal a when R_0 and L_0 include the earthing impedance, if any. Consequently when L_0 is put to infinity to reduce the present case to the case of a single-phase line-to-line short circuit, Equ. 10.77 will not be equal to Equ. 9.75, but will be $3/2$ times as large.

(10.8) Short-circuit Torque — Pulsating Component

As in the previous cases the short-circuit torque consists of two components, the pulsating component and the unidirectional component. The pulsating component of torque can be obtained by using the approximate expression for $\bar{i}_d(p)$ and $\bar{i}_q(p)$ (Equs. 10.66 and 10.67) directly in the torque equation. To find the unidirectional component of torque, however, requires the consideration of the further effects of the resistances other than their damping effect. Only the pulsating component of torque is considered for the moment.

The explicit forms of i_d and i_q are, from Equs. 10.66 and 10.67,

$$\begin{aligned} i_d \approx & -\sqrt{\frac{2}{3}} M_d I_{f0} \frac{1+k}{L_d''(1+k)+2L_o} g_2(t) \\ & -\sqrt{\frac{2}{3}} M_d I_{f0} [1+b_2-k(1-b_2)] \left\{ \frac{g_2(t)}{L_d''(1+k)+2L_o} \sum_{n=0}^{\infty} b_2^n \cos 2(n+1)\theta \right. \\ & \quad + \frac{\cos \lambda}{2kL_o} e^{-t/T_{d''(1+k)}} \sum_{n=0}^{\infty} b_2^n \cos (2n+1)\theta \\ & \quad \left. + \frac{\sin \lambda}{2L_o} e^{-t/T_{d''(1+k)}} \sum_{n=0}^{\infty} b_2^n \sin (2n+1)\theta \right\} \end{aligned} \quad (10.78)$$

$$\begin{aligned} \text{and } i_q \approx & \sqrt{\frac{2}{3}} M_d I_{f0} [1-b_2-k(1+b_2)] \left\{ \frac{g_2(t)}{L_d''(1+k)+2L_o} \sum_{n=0}^{\infty} b_2^n \sin 2(n+1)\theta \right. \\ & \quad + \frac{\cos \lambda}{2kL_o} e^{-t/T_{d''(1+k)}} \sum_{n=0}^{\infty} b_2^n \sin (2n+1)\theta \\ & \quad \left. - \frac{\sin \lambda}{2L_o} e^{-t/T_{d''(1+k)}} \sum_{n=0}^{\infty} b_2^n \cos (2n+1)\theta \right\} \end{aligned} \quad (10.79)$$

For the a.c. components of $\bar{i}_d(p)$ and $\bar{i}_q(p)$, $\bar{L}_d(p)$ and $\bar{L}_q(p)$ are approximately equal to L_d'' and L_q'' , so that the pulsating components of the flux-linkages are,

$$\psi_d [\text{pulsating}] \approx \frac{3}{2} L_d'' i_d [\text{a.c.}] \quad (10.80)$$

$$\text{and } \psi_q \approx \frac{3}{2} L_q'' i_q \quad (10.81)$$

The unidirectional component of the flux-linkage $\bar{\psi}_a(p)$ is

given by

$$\begin{aligned}\tilde{\Psi}_d(p) [\text{unidirectional}] &= \frac{3}{2} \tilde{L}_d(p) \tilde{L}_d(p) [d.c.] + \frac{3}{2} M_d I_{f0} \cdot \frac{1}{p} \\ &\approx -\sqrt{\frac{3}{2}} M_d I_{f0} \frac{(1+k) \tilde{L}_d(p)}{(1+k) \tilde{L}_d(p) + 2L_0} \cdot \frac{1}{p} + \sqrt{\frac{3}{2}} M_d I_{f0} \cdot \frac{1}{p} \\ &\approx \sqrt{\frac{3}{2}} M_d I_{f0} \cdot 2L_0 \left\{ \left[\frac{1}{L_d''(1+k) + 2L_0} - \frac{1}{L_d'(1+k) + 2L_0} \right] \cdot \frac{1}{\frac{1}{T_d(u-n)} + p} \right. \\ &\quad \left. + \left[\frac{1}{L_d'(1+k) + 2L_0} - \frac{1}{L_d''(1+k) + 2L_0} \right] \cdot \frac{1}{\frac{1}{T_d(u-n)} + p} + \frac{1}{L_d'(1+k) + 2L_0} \cdot \frac{1}{p} \right\} \quad (10.82)\end{aligned}$$

$$\text{so that } \Psi_d [\text{unidirectional}] \approx \sqrt{\frac{3}{2}} M_d I_{f0} \frac{2L_0}{L_d''(1+k) + 2L_0} g_d(t) \quad (10.83)$$

The pulsating component of torque is therefore given by

$$\begin{aligned}T_{(u-n)} &= \Psi_d i_q - \Psi_q i_d \\ &= M_d^2 I_{f0}^2 \frac{1-b_2-k(1+b_2)}{L_d''(1+k) + 2L_0} g_d(t) \left\{ L_q''(1+k) + 2L_0 \right\} \left\{ \frac{1}{L_d''(1+k) + 2L_0} g_d(t) \sum_{n=0}^{\infty} b_2^n \sin 2(n+1)\theta \right. \\ &\quad \left. - \frac{\cos \lambda}{2kL_0} e^{-k/T_d(u-n)} \sum_{n=0}^{\infty} b_2^n \sin (2n+1)\theta \right. \\ &\quad \left. - \frac{\sin \lambda}{2L_0} e^{-k/T_d(u-n)} \sum_{n=0}^{\infty} b_2^n \cos (2n+1)\theta \right\} \\ &\quad - M_d^2 I_{f0}^2 \{1+b_2-k(1-b_2)\} \{1-b_2-k(1+b_2)\} (L_d' - L_q') \\ &\quad \times \left\{ \frac{1}{L_d''(1+k) + 2L_0} g_d(t) \sum_{n=0}^{\infty} b_2^n \cos 2(n+1)\theta + \frac{\cos \lambda}{2kL_0} e^{-k/T_d(u-n)} \sum_{n=0}^{\infty} b_2^n \cos (2n+1)\theta \right. \\ &\quad \left. + \frac{\sin \lambda}{2L_0} e^{-k/T_d(u-n)} \sum_{n=0}^{\infty} b_2^n \sin (2n+1)\theta \right\} \\ &\quad \times \left\{ \frac{1}{L_d''(1+k) + 2L_0} g_d(t) \sum_{n=0}^{\infty} b_2^n \sin 2(n+1)\theta + \frac{\cos \lambda}{2kL_0} e^{-k/T_d(u-n)} \sum_{n=0}^{\infty} b_2^n \sin (2n+1)\theta \right. \\ &\quad \left. - \frac{\sin \lambda}{2L_0} e^{-k/T_d(u-n)} \sum_{n=0}^{\infty} b_2^n \cos (2n+1)\theta \right\} \quad (10.84)\end{aligned}$$

Using the identities given by Equ. 9.21 and also the following products of infinite harmonic series

$$\left. \begin{aligned}\sum_{n=0}^{\infty} b_2^n \cos 2(n+1)\theta \times \sum_{n=0}^{\infty} b_2^n \cos (2n+1)\theta &= \frac{1}{2} \sum_{n=0}^{\infty} \left(n+1 + \frac{1}{1-b_2} \right) b_2^{n+1} \cos (2n+1)\theta \\ \sum_{n=0}^{\infty} b_2^n \sin 2(n+1)\theta \times \sum_{n=0}^{\infty} b_2^n \sin (2n+1)\theta &= -\frac{1}{2} \sum_{n=0}^{\infty} \left(n+1 - \frac{1}{1-b_2} \right) b_2^{n+1} \cos (2n+1)\theta \\ \left\{ \sum_{n=0}^{\infty} b_2^n \cos (2n+1)\theta \right\}^2 &= \frac{1}{2} \left\{ \frac{1}{1-b_2} + \sum_{n=1}^{\infty} \left(n + \frac{2b_2}{1-b_2} \right) b_2^{n-1} \cos 2n\theta \right\} \\ \left\{ \sum_{n=0}^{\infty} b_2^n \sin (2n+1)\theta \right\}^2 &= \frac{1}{2} \left\{ \frac{1}{1-b_2} - \sum_{n=1}^{\infty} \left(n - \frac{2b_2}{1-b_2} \right) b_2^{n-1} \cos 2n\theta \right\}\end{aligned}\right\} \quad (10.85)$$

it can be shown that Equ. 10.84 is reduced to

$$T_{(u-n)} = M_d^2 I_{f0}^2 \left\{ L_d''(1+k) + 2L_0 \right\} \left\{ 1 - b_2 - k(1+b_2) \right\} \\ \times \left\{ \frac{\cos \lambda}{2kL_0} \cdot \frac{1}{L_d''(1+k) + 2L_0} \cdot g_2(t) e^{-t/T_{\alpha(u-n)}} \sum_{n=0}^{\infty} (2n+1) b_2^n \sin(2n+1)\theta \right. \\ - \frac{\sin \lambda}{2L_0} \cdot \frac{1}{L_d''(1+k) + 2L_0} g_2(t) e^{-t/T_{\beta(u-n)}} \sum_{n=0}^{\infty} (2n+1) b_2^n \cos(2n+1)\theta \\ - \frac{b_2 \sin 2\lambda}{4kL_0} e^{-(\frac{1}{T_{\alpha(u-n)}} + \frac{1}{T_{\beta(u-n)}})t} \sum_{n=0}^{\infty} n b_2^{n-1} \cos 2n\theta \\ \left. + \left[\frac{g_2^2(t)}{[L_d''(1+k) + 2L_0]^2} + \frac{b_2 \cos^2 \lambda}{4k^2 L_0^2} e^{-2t/T_{\alpha(u-n)}} - \frac{b_2 \sin^2 \lambda}{4L_0^2} e^{-2t/T_{\beta(u-n)}} \right] \times \sum n b_2^{n-1} \sin 2n\theta \right\} \quad (10.86)$$

It can be shown that when L_0 is put equal to ∞ , Equ. 10.86 may be reduced to Equ. 9.77, the expression for the pulsating component of torque for the single-phase line-to-line short circuit.

(10.9) Experimental Results

Using Equ. 9.63 the short-circuit currents, the open-phase voltage and the field current can be expressed alternatively as:

$$i_b = \frac{\sqrt{3}}{2} E_{om} g_2(t) \frac{\sqrt{3} x_q'' \cos \theta - (x_q'' + 2x_0) \sin \theta}{x_d x_q'' + x_0(x_d'' + x_q'') + x_0(x_q'' - x_d'') \cos 2\theta} \\ - \frac{\sqrt{3}}{4} E_{om} e^{-t/T_{\alpha(u-n)}} \cos \lambda \frac{\sqrt{3} [x_d'' + x_q'' + (x_q'' - x_d'') \cos 2\theta] + (x_q'' - x_d'') \sin 2\theta}{x_d x_q'' + x_0(x_d'' + x_q'') + x_0(x_q'' - x_d'') \cos 2\theta} \\ + \frac{\sqrt{3}}{4} E_{om} e^{-t/T_{\beta(u-n)}} \sin \lambda \frac{x_d'' + x_q'' + 4x_0 - (x_q'' - x_d'') \cos 2\theta - \sqrt{3} (x_q'' - x_d'') \sin 2\theta}{x_d x_q'' + x_0(x_d'' + x_q'') + x_0(x_q'' - x_d'') \cos 2\theta} \\ i_c = \frac{\sqrt{3}}{2} E_{om} g_2(t) \frac{\sqrt{3} x_q'' \cos \theta + (x_q'' + 2x_0) \sin \theta}{x_d x_q'' + x_0(x_d'' + x_q'') + x_0(x_q'' - x_d'') \cos 2\theta} \\ - \frac{\sqrt{3}}{4} E_{om} e^{-t/T_{\alpha(u-n)}} \cos \lambda \frac{\sqrt{3} [x_d'' + x_q'' + (x_q'' - x_d'') \cos 2\theta] - (x_q'' - x_d'') \sin 2\theta}{x_d x_q'' + x_0(x_d'' + x_q'') + x_0(x_q'' - x_d'') \cos 2\theta} \\ - \frac{\sqrt{3}}{4} E_{om} e^{-t/T_{\beta(u-n)}} \sin \lambda \frac{x_d'' + x_q'' + 4x_0 - (x_q'' - x_d'') \cos 2\theta + \sqrt{3} (x_q'' - x_d'') \sin 2\theta}{x_d x_q'' + x_0(x_d'' + x_q'') + x_0(x_q'' - x_d'') \cos 2\theta} \\ V_a = 3x_0 x_q'' E_{om} g_2(t) \frac{[2x_0(x_q'' - 2x_d'') - x_d x_q''] \sin \theta - 2x_0(x_q'' - x_d'') \sin^3 \theta}{[x_d x_q'' + x_0(x_d'' + x_q'') + x_0(x_q'' - x_d'') \cos 2\theta]^2} \\ + 3x_0 x_d x_q'' (x_q'' - x_d'') E_{om} e^{-t/T_{\alpha(u-n)}} \frac{\cos \lambda \sin 2\theta}{[x_d x_q'' + x_0(x_d'' + x_q'') + x_0(x_q'' - x_d'') \cos 2\theta]^2} \\ - 3x_0(x_q'' - x_d'') E_{om} e^{-t/T_{\beta(u-n)}} \sin \lambda \frac{x_0(x_q'' - x_d'') + [x_d x_q'' + x_0(x_d'' + x_q'')] \cos 2\theta}{[x_d x_q'' + x_0(x_d'' + x_q'') + x_0(x_q'' - x_d'') \cos 2\theta]^2}$$

$$\begin{aligned}
 \text{and } i_f' = & (x_d - x_d'') I_{f0} \left\{ \left[\frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{1+k}{x_d''(1+k) + 2x_0} - \frac{L_{fe}}{L_f} \cdot \frac{1+k}{x_d(1+k) + 2x_0} \right] e^{-L/T_{d1}(u-n)} - \frac{L_{fe}}{L_f} \cdot \frac{1+k}{x_d(1+k) + 2x_0} e^{-L/T_{d1}(u-n)} \right\} \\
 & + \frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{(x_d - x_d'') I_{f0}}{x_d''(1+k) + 2x_0} \cdot \left(1+k + \frac{1-k}{b_2} \right) g_2(t) \left(\frac{k x_d'' x_d''}{x_d'' x_d'' + x_0(x_d'' + x_d'') + x_0(x_d'' - x_d'') \cos 2\theta} - \frac{1}{2} \right) \\
 & + \frac{L_{kd} - M_{fkd}}{L_f + L_{kd} - 2M_{fkd}} \cdot \frac{(x_d - x_d'') I_{f0}}{4 x_0} [(1+k)b_2 + 1-k] \frac{x_d''(1+k) + 2 x_0}{x_d'' x_d'' + x_0(x_d'' + x_d'') + x_0(x_d'' - x_d'') \cos 2\theta} \\
 & \times \left\{ \frac{1}{k} x_d'' e^{-L/T_{d1}(u-n)} \cos \lambda \cos \theta + (x_d'' + 2x_0) e^{-L/T_{d1}(u-n)} \sin \lambda \sin \theta \right\}
 \end{aligned}$$

The test was performed on the same machine as that described in Article 9.10. The test results and the calculated results are shown in Figs. 10.1 and 10.2.

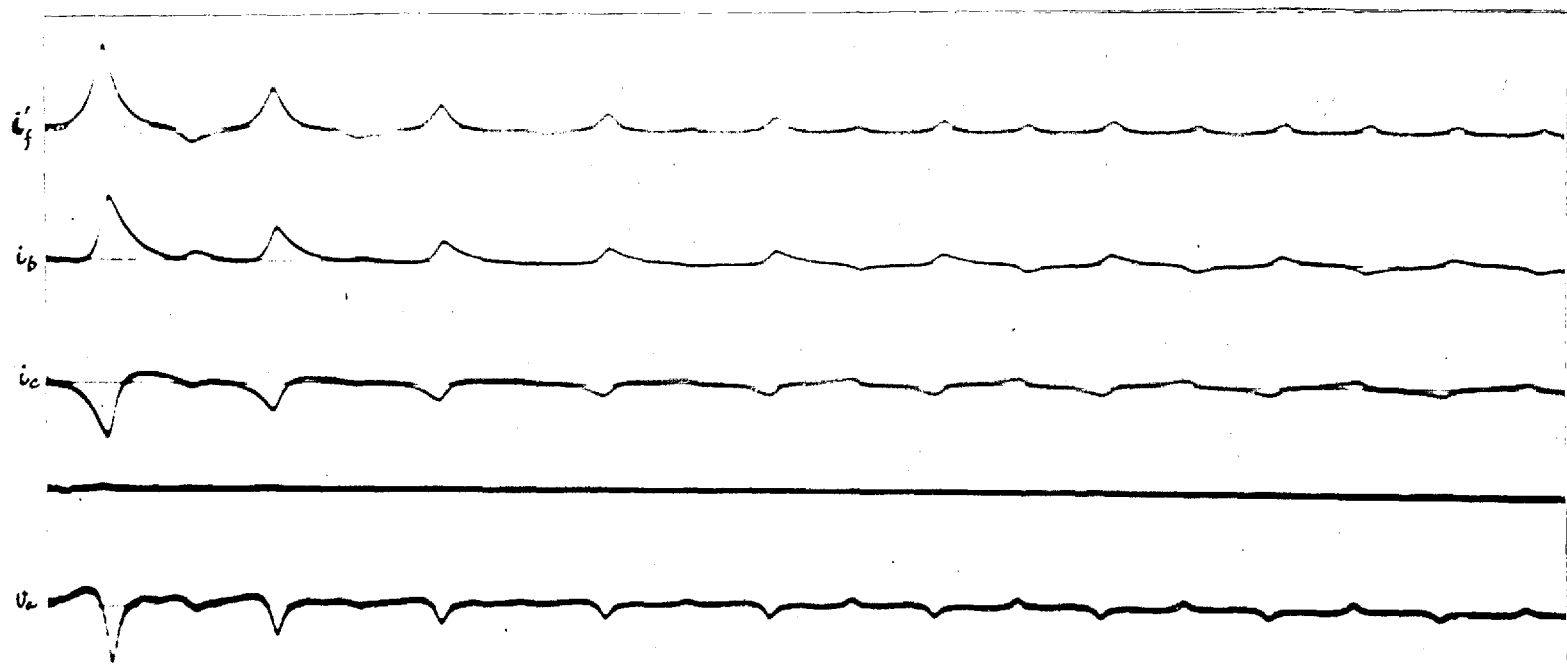


Fig. 10.1

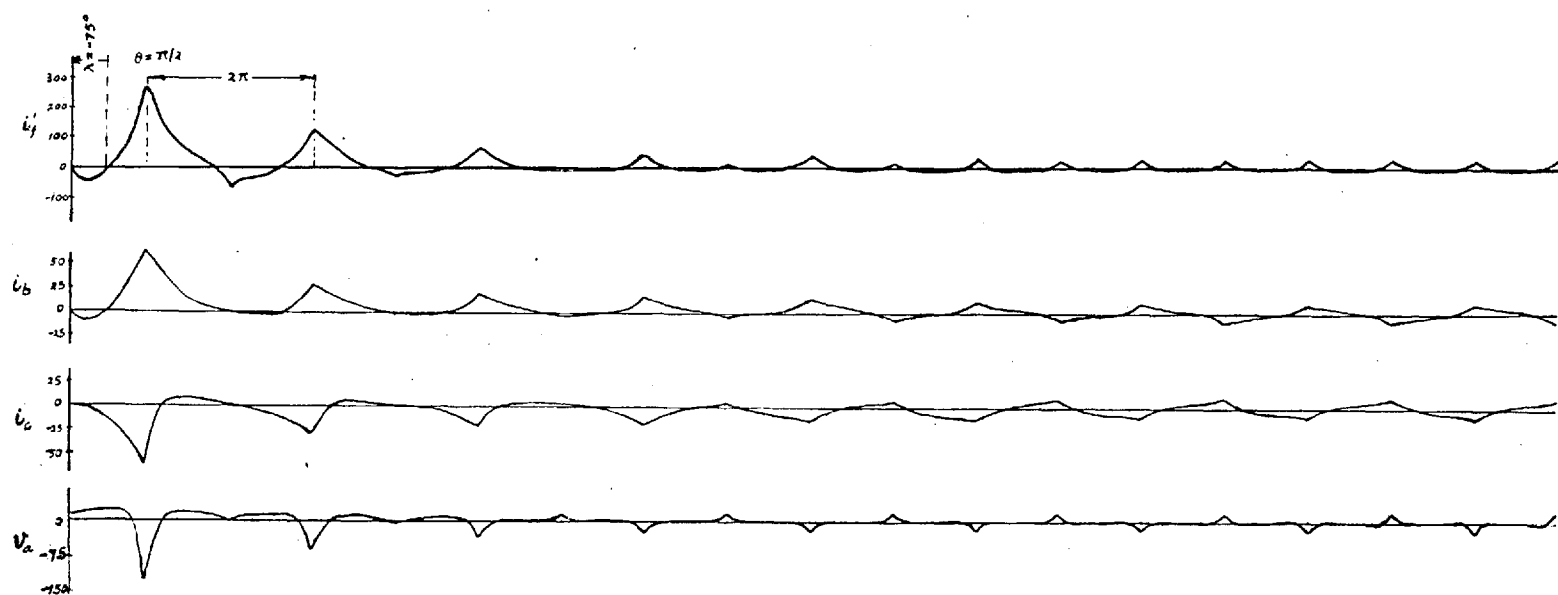


Fig. 10.2. Double-line-to-neutral Short Circuit

$$\begin{aligned}
 i_f' &= 50.1 e^{-t/0.0121} + \left(\frac{188}{1.86 + \cos 2\theta} - 60 \right) (0.835 e^{-t/0.0121} + 0.165) - \frac{175 \cos \theta \cdot e^{-t/0.022}}{1.86 + \cos 2\theta} + \frac{20.1 \sin \theta \cdot e^{-t/0.0278}}{1.86 + \cos 2\theta} \\
 i_b &= \frac{67.2 \cos \theta - 50.5 \sin \theta}{1.86 + \cos 2\theta} (0.835 e^{-t/0.0121} + 0.165) - \frac{13.3 + 10.5 \cos 2\theta + 6.4 \sin 2\theta}{1.86 + \cos 2\theta} e^{-t/0.022} \\
 &\quad - \frac{22.7 - 12.2 \cos 2\theta - 20.1 \sin 2\theta}{1.86 + \cos 2\theta} e^{-t/0.0278} \\
 i_c &= \frac{67.2 \cos \theta + 50.5 \sin \theta}{1.86 + \cos 2\theta} (0.835 e^{-t/0.0121} + 0.165) - \frac{13.3 + 10.5 \cos 2\theta - 6.4 \sin 2\theta}{1.86 + \cos 2\theta} e^{-t/0.022} \\
 &\quad + \frac{22.7 - 12.2 \cos 2\theta + 20.1 \sin 2\theta}{1.86 + \cos 2\theta} e^{-t/0.0278} \\
 v_a &= \frac{139 \sin \theta - 243 \sin^3 \theta}{(1.86 + \cos 2\theta)^2} (0.835 e^{-t/0.0121} + 0.165) + \frac{68.1 \sin 2\theta \cdot e^{-t/0.022}}{(1.86 + \cos 2\theta)^2} + \frac{(286 + 53 / \cos 2\theta) e^{-t/0.0278}}{(1.86 + \cos 2\theta)^2}
 \end{aligned}$$

(11) EQUIVALENT CIRCUITS FOR TRANSIENT-STATE CURRENTS

(11.1) Transient-state Component of Current

In the previous four chapters it has been shown that the complete solutions for the simpler types of transient problem connected with a synchronous machine can be obtained by a more rigorous mathematical approach. The fact that the same solutions as those obtained by other simpler but less rigorous method have ultimately been arrived at after making similar similar simplifying assumptions shows the validity of the simpler methods which are more readily adaptable to more complicated problems. The application of the equivalent circuits similar to those given in Chapter 6 provides one of such simpler method for finding the transient-state current of a synchronous machine following a sudden change of terminal conditions.

As it has been shown in the previous examples, the transient-state component of current of a synchronous machine consists of, in the absence of capacitance in the armature circuits, an odd-harmonic series and an even-harmonic series. The assumptions generally made in simpler methods are that the decay of the odd-harmonic series depends on the decay of the positive-sequence, fundamental-frequency component of the armature current (or the d.c. component of the field and damper currents) and that the decay of the even-harmonic series depends on the decay of the d.c. component of the armature current. These assumptions are equivalent to neglecting all resistances

with respect to the harmonic components of current. The physical significance of the first assumption is that the odd-harmonic series of the transient-state current dies away in the same manner as the forward rotating flux, which is contributed only by the positive-sequence, fundamental-frequency component of the armature current and the d.c. components of the field and damper currents (assuming that the field structure is the moving member of the machine). The physical significance of the second assumption is that the even-harmonic series dies away in the same manner as the flux initially trapped in the armature (strictly, that part of the initial trapped-flux which does die away) which determines the d.c. component of the armature current.

Apart from the peculiarities of the frequency conversion (i.e., the difference of the frequencies of the currents on the different sides of the air gap) and the production of higher currents, the transient-state short-circuit current of a synchronous machine corresponds closely to the transient current in a simple series R-L circuit with its applied voltage suddenly taken away. This suggests the possibility of finding the transient-state current of a synchronous machine following a sudden change by determining its magnitude and decrement factor separately, a process identical with the application of the constant flux theorem. The way of determining the magnitude and decrement factor of the transient-state current with the aid of equivalent circuits will be shown in the subsequent articles. The unidirectional component of torque

will also be considered.

(11.2) Odd-harmonic Equivalent Circuits

Except the positive-sequence, fundamental-frequency component, which must be found by new equivalent circuits, the other components (the harmonic components and the negative- and zero-sequence, fundamental-frequency components) of the odd-harmonic series of the transient-state armature current can be found by the steady-state equivalent circuits given in Chapter 6, provided the time constants of the transient-state components are sufficiently longer than the time of a complete cycle at fundamental frequency.

The positive-sequence, fundamental-frequency component of the armature-current, being responsible for the production of a unidirectional space mmf stationary with respect to the field structure, interacts with the field and damper currents as if they were currents in a multi-winding transformer. Consequently except at steady state when there is no induced ^{direct} currents in the field and damper circuits, the reaction of the field and damper circuits to the armature circuit must be considered.

For the sake of simplicity, there have been defined three distinct direct-axis reactances and two quadrature-axis reactances. These various reactances may be defined alternatively from the equivalent circuits' point of view. Fig. 11.1.a shows the equivalent circuit for the direct-axis currents. The branches a , f and k_d are the leakage reactances of the armature, the field and the direct-axis-damper windings respectively.

tively and the branch md is the common mutual between any pair of these windings in the direct axis. The synchronous reactance X_d is the equivalent reactance with no induced current in the field and damper circuits, so that it is defined by leaving the branches f and kd open as shown. The transient

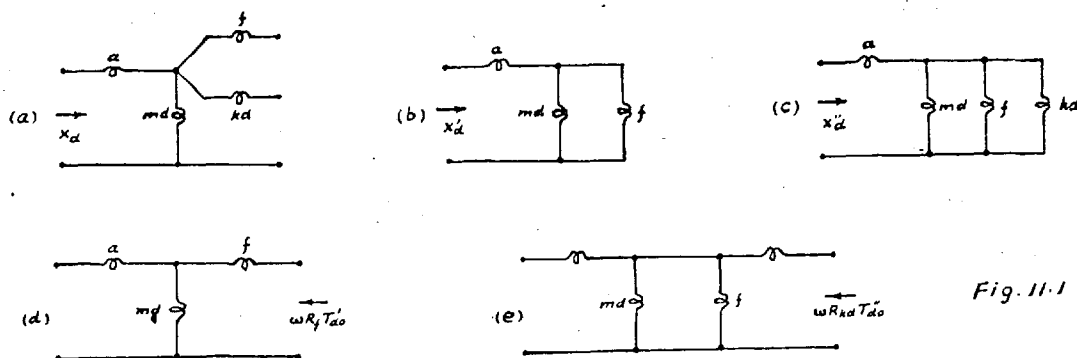


Fig. 11.1

reactance X_d' is the equivalent reactance with induced current in the field circuit but the induced current in the damper circuit already died away, so that it is defined by closing the branch f as shown in Fig. 11.1.b. The subtransient reactance X_d'' is the equivalent reactance with induced currents in both the field and the damper circuits so that it is defined by closing both of the branches f and kd as shown in Fig. 11.1.c. All these reactances are the equivalent reactances viewed from the armature side (in the direct axis) of the machine and are sufficient for the purpose of determining the initial subtransient and transient values as well as the steady-state value of the direct-axis current. To determine the decrement factors, two further reactances must be defined. Fig. 11.1.d shows the equivalent reactance viewed from the field circuit with the branches a and kd left open. It is equal to $\omega R_f T_{d0}'$, where

T_{d0}^I is the open-circuit transient time constant. Fig. 11.1.e shows the equivalent reactance viewed from the damper circuit with the branch f closed. It is equal to $\omega R_{kd} T_{d0}''$, where T_{d0}'' is the open-circuit subtransient time constant. Similarly, in

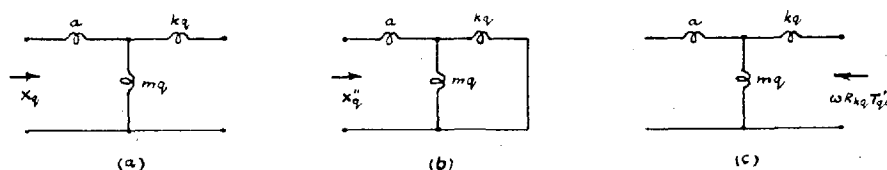


Fig. 11.2

the quadrature axis, the quantities X_q , X_q'' and T_{q0}'' are given in Fig. 11.2, where the branches a, kq and mq are respectively the leakage reactances of the armature and the quadrature-axis-damper windings and the mutual reactance between them.

It has been shown in Chapter 6 that if the armature resistance is neglected, the positive-sequence, fundamental-frequency component of the steady-state currents for the three typical

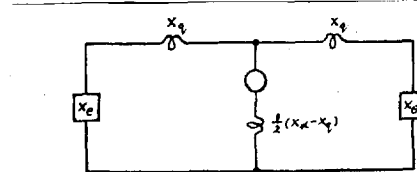


Fig. 11.3

unsymmetrical short circuits of a synchronous machine can be determined by the same sort of equivalent circuit as shown in Fig.

11.3, where X_e is an appropriate equivalent external reactance. For the line-to-line short circuit, X_e is equal to $X_2 (= \sqrt{X_d'' X_q''})$. For the line-to-neutral short, X_e is equal to X_0 and $X_2 (= \sqrt{(X_d'' + \frac{1}{2}X_0)(X_q'' + \frac{1}{2}X_0)} - \frac{1}{2}X_0)$ in series. For the double-line-to-neutral short circuit, X_e is equal to X_0 and $X_2 (= \frac{x_d'' x_q'' + \sqrt{x_d'' x_q'' (x_d'' + \frac{1}{2}X_0)(x_q'' + \frac{1}{2}X_0)}}{x_d'' + x_q'' + 2X_0})$ in parallel. For a symmetrical three-phase short circuit through an external reactance, X_e is the actual external reactance. In short, as far as

the positive-sequence, fundamental-frequency component of current is concerned, the three typical unsymmetrical short circuits are equivalent to a symmetrical short circuit through

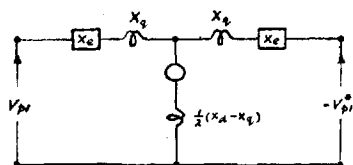


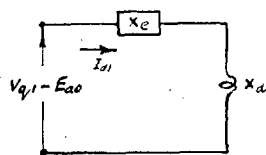
Fig. 11.4

an appropriate external reactance.

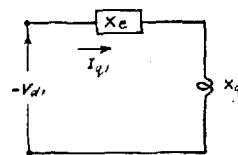
This concept may be extended to the unsymmetrical operation of a synchronous machine in general, so that the positive-sequence,

fundamental-frequency component may be determined by the circuit shown in Fig. 11.4, where X_e is the equivalent external reactance appropriate to the operating condition being considered.

As a result of the neglect of the armature resistance, the real and imaginary parts of I_{p1} , i.e., the direct-axis and the quadrature-axis components of current I_{d1} and I_{q1} , can be determined by separate equivalent circuits as shown in Fig. 11.5. These circuits are derived from the circuit shown in Fig. 11.4 by putting $I_{p1} = I_{d1} + jI_{q1}$ and $V_{p1} = V_{d1} + jV_{q1}$ and separating



(a)



(b)

Fig. 11.5

the real and imaginary quantities. These forms of equivalent circuit are more convenient for the determination of the decrement factors of the transient-state current as I_{d1} and I_{q1} are the currents in branch a of the equivalent circuits shown in Figs. 11.1 and 11.2 respectively.

To find the initial subtransient value of the total armature current, i.e., the value of the current immediately following the sudden change with the induced currents in the field and damper circuits having their highest values, the synchronous reactances X_d and X_q in Fig. 11.5 are replaced by the subtransient reactances X_d'' (Fig. 11.1.c) and X_q'' (Fig. 11.2.b) respectively. The initial currents in the two axes are thus

$$\begin{aligned} I_{d1} &= (V_{q1} - E_{a0}) / (X_d'' + X_e) \\ \text{and } I_{q1} &= - V_{d1} / (X_q'' + X_e) \end{aligned} \quad (11.1)$$

After the induced currents in the damper circuits have practically died away so that the damper circuits can be omitted, the currents in the two axes can be found from Fig. 11.5 by replacing X_d by X_d' (Fig. 11.1.b), X_q being unchanged. Thus

$$\begin{aligned} I_{d1} &= (V_{q1} - E_{a0}) / (X_d' + X_e) \\ \text{and } I_{q1} &= - V_{d1} / (X_q - X_e) \end{aligned} \quad (11.2)$$

And finally the steady-state currents are

$$\begin{aligned} I_{d1} &= (V_{q1} - E_{a0}) / (X_d + X_e) \\ \text{and } I_{q1} &= - V_{d1} / (X_q + X_e) \end{aligned} \quad (11.3)$$

If it is assumed that in the direct axis, the duration of the induced current in the damper circuit is so short compared with the duration of the induced current in the field circuit that the induced field current changes little from its initial value while the induced damper current has practically died away, the difference of the values of I_{d1} given by Eqs. 11.1 and 11.2, i.e., the subtransient component of current,

$$I_{d1}[\text{Subtransient}] = (V_{q1} - E_{a0}) \left[\frac{1}{X_d'' + X_e} - \frac{1}{X_d' + X_e} \right] \quad (11.4)$$

will be the counterpart of the induced current in the direct-

axis-damper circuit. The subtransient component of current thus decays with the same decrement factor as the induced damper current, which is determined by the damper resistance R_{kd} and the equivalent inductance viewed from the damper circuit as shown in Fig. 11.6 (T''_{dz} is the subtransient time constant).

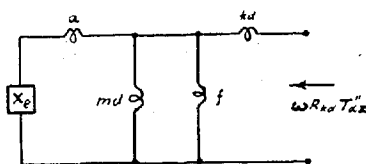


Fig. 11.6

To find the expression for T''_{dz} , let the reactance of branch a be A, that of branch kd be B and that of branches md and f in parallel be C.

Thus, referring to Fig. 11.6,

$$\begin{aligned}\omega R_{kd} T''_{dz} &= B + \frac{(A + X_e)C}{A + C + X_e} \\ &= \frac{AB + BC + CA + X_e(B + C)}{A + C + X_e}\end{aligned}\quad (11.5)$$

Also from Figs. 11.1.b, 11.1.c and 11.1.e,

$$\begin{aligned}x'_d &= A + C \\ x''_d &= A - \frac{BC}{B + C} = \frac{AB + BC + CA}{B + C} \\ \omega R_{kd} T''_{do} &= B + C\end{aligned}\quad (11.6)$$

Eliminating the quantities A, B and C in Equ. 11.5 by using Equ. 11.6 gives the subtransient time constant

$$T''_{dz} = \frac{x'_d + x_e}{x''_d + x_e} \cdot T''_{do} \quad (11.7)$$

Similarly, the transient component of current

$$I_{d1} [\text{Transient}] = (V_{q1} - E_{ao}) \left[\frac{1}{x'_d + x_e} - \frac{1}{x''_d + x_e} \right] \quad (11.8)$$

dies away in the same manner as the induced field current and its decrement factor is determined by the field resistance R_f and the equivalent inductance viewed from the field circuit

as shown in Fig. 11.7 (T_{dz}^1 is the transient time constant).

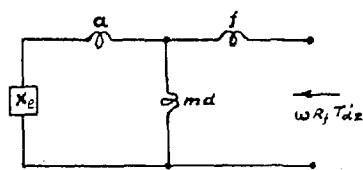


Fig. 11.7

It can be shown that by using Figs. 11.1.a, 11.1.b and 11.1.d, the transient time constant can be expressed as, from Fig. 11.7,

$$T_{dz} = \frac{x_d' + x_e}{x_d + x_e} T_{d0}' \quad (11.9)$$

The complete expression for the direct-axis component of the positive-sequence, fundamental-frequency component of current is therefore given by

$$I_{d1} = (V_{q1} - E_{a0}) \left\{ \left(\frac{1}{x_d' + x_e} - \frac{1}{x_d + x_e} \right) e^{-t/T_{dz}'} + \left(\frac{1}{x_d' + x_e} - \frac{1}{x_d + x_e} \right) e^{-t/T_{d0}'} + \frac{1}{x_d + x_e} \right\} \quad (11.10)$$

Similarly, it can be shown that the quadrature-axis component is given by

$$I_{q1} = -V_{d1} \left\{ \left(\frac{1}{x_q' + x_e} - \frac{1}{x_q + x_e} \right) e^{-t/T_{qz}''} + \frac{1}{x_q + x_e} \right\} \quad (11.11)$$

where $T_{qz}'' = \frac{x_q' + x_e}{x_q + x_e} T_{q0}'' \quad (11.12)$

The positive-sequence, fundamental-frequency current is thus given by

$$I_{p1} = I_{d1} + j I_{q1}$$

or $i_{p1} = \sqrt{2} I_{d1} \cos \theta - \sqrt{2} I_{q1} \sin \theta \quad (11.13)$

The solution given by Equ. 11.13 for the positive-sequence, fundamental-frequency current holds for all types of symmetrical or unsymmetrical operation of a synchronous machine provided the appropriate value of X_e is used and the condition of small resistance in the armature circuit is satisfied. The fundamental-frequency currents of other sequences and the harmonic currents are given by the currents distributed in the

various branches of the infinite ladder network representing X_0 , as described in Chapter 6.

As an example, let the transient time constant of a double-line-to-neutral short circuit be determined from Equ. 11.9. The equivalent external reactance is,

$$\begin{aligned} X_e &= \frac{X_d X_0}{X_d + X_0} \\ &= \frac{2X_0 \cdot X_d^2 X_q'' + \sqrt{X_d^2 X_q'' (X_d^2 + 2X_0)(X_q'' + 2X_0)}}{X_d^2 X_q'' + (X_d^2 + 2X_0)(X_q'' + 2X_0) + 2\sqrt{X_d^2 X_q'' (X_d^2 + 2X_0)(X_q'' + 2X_0)}} \\ &= \frac{2X_0 X_d^2 X_q'' (1+k)}{X_d^2 X_q'' (1+k^2 + 2k)} = \frac{2X_0}{1+k} \end{aligned}$$

where $k = \sqrt{(X_d^2 + 2X_0)(X_q'' + 2X_0)/X_d^2 X_q''}$. The transient time constant is therefore given by

$$T'_{d(u-n)} = \frac{X_d + 2X_0/(1+k)}{X_d + 2X_0/(1+k)} T'_{do} = \frac{X_d(1+k) + 2X_0}{X_d(1+k) + 2X_0} T'_{do}$$

which is the same expression as that derived in Chapter 10 (Equ. 10.30).

(11.3) Even-harmonic Equivalent Circuit

Except the d.c. component, which must be determined by another type of equivalent circuit, the other components of the even-harmonic series of the transient-state current can be determined by the steady-state odd-harmonic equivalent circuits given in Chapter 6, provided the modification regarding the frequency of the currents in the circuits is suitably made. The modified equivalent circuits shown in Fig. 11.8 are appropriate to the even-harmonic currents.

Since for a particular operating condition of the machine, the same terminal condition apply to the odd-harmonic quantities as well as the even-harmonic quantities, the even-harmonic

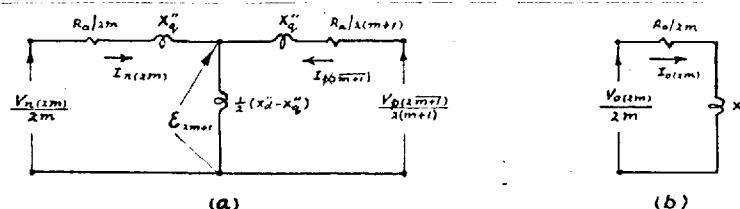


Fig. 11.8

equivalent circuits shown in Fig. 11.8 must be connected together in accordance with the same pattern of connection as the odd-harmonic equivalent circuits. Consequently, if the resistances in the circuits are neglected, the negative-sequence reactance derived from the resultant infinite ladder network is the same for both the odd-harmonic and the even-harmonic series of current.

The d.c. component, which in turn produces the even-harmonic components, of the transient-state current is caused by the flux initially trapped in the armature circuits. To understand this, let the simple series R-L circuit shown in Fig. 11.9 be considered. It is well known that if the voltage across the

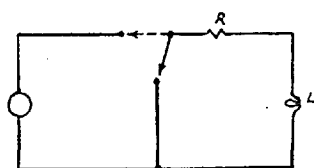


Fig. 11.9

circuit is suddenly taken away (the circuit remains closed), the transient current in the circuit is equal to $I_0 e^{-Rt/L}$, where I_0 is the initial current in the circuit.

Alternatively, this transient current may be considered as the result of an impulsive flux-linkage $\psi_0 \delta(t)$ suddenly applied to the circuit, where ψ_0 is the initial flux-linkage and $\delta(t)$ is the impulsive or Dirac function defined by

$$\begin{aligned} \delta(t) &= 0 & \text{for } 0 \geq t \geq \epsilon \\ &= \frac{1}{\epsilon} & \text{for } \epsilon > t > 0 \end{aligned}$$

With this impulsive flux-linkage suddenly applied to the

circuit, the differential equation is

$$\Psi_0 \delta(t) = (R + L \frac{d}{dt}) i$$

whose Laplace transform is

$$\Psi_0 = (R + Lp) \bar{i}(p)$$

$$\text{or } \bar{i}(p) = \frac{\Psi_0}{L} \cdot \frac{1}{p + R/L}$$

The transient current is therefore given by $\frac{\Psi_0}{L} e^{-Rt/L}$ or $I_0 e^{-Rt/L}$ since $\Psi_0 = LI_0$. If the magnitude of the current is represented symbolically as if it were a sinusoidal current, it can be determined by assuming a sinusoidal voltage of $j\omega\Psi_0$ applied to the circuit with the resistance omitted, as obviously the current will then be $j\omega\Psi_0/j\omega L = I_0$.

Following the arguments in the previous paragraph, the equivalent circuits for the d. c. component of the transient-state current of a synchronous machine are found as shown in Figs. 11.10.a and 11.10.b. These two circuits are not independent and one of them can be derived from the other. In the circuits

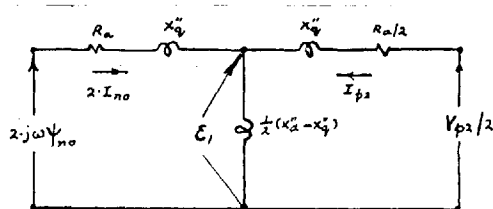


Fig. 11.10 a

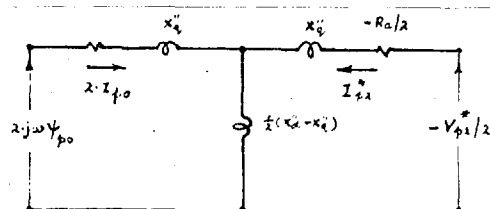


Fig. 11.10 b

Ψ_{po} and Ψ_{no} are the positive- and negative- sequence components of the initially trapped flux-linkage in the armature circuits. The resistances are left in the d.c. component meshes only for the benefit of determining the decrement factor and they should be omitted in finding the magnitude of the currents. The equivalent circuit for the zero component of the d.c. com-

ponent of current is shown in Fig. 11.11, where Ψ_{00} is the zero component of the initially trapped flux-linkage. Again

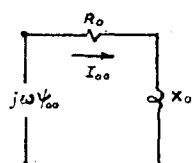


Fig. 11.11

the resistance in the circuit should be omitted when the magnitude of the current is to be determined.

The decrement factor (or factors) for the d.c. component of current, which is also the decrement factor (or factors) for the entire even-harmonic series, is given by e^{-t/T_a} , where

$$T_a = \Psi_0 / I_0 R_t \quad (11.14)$$

where Ψ_0 = applied flux-linkage

I_0 = d.c. component of current due to Ψ_0

and R_t = total series resistance in the d.c. mesh

In order to show the application of these equivalent circuits, the even-harmonic series (armature-transient component) of the short-circuit currents of the symmetrical three-phase short circuit, the line-to-line short circuit, the line-to-neutral short circuit and the double-line-to-neutral short circuit are considered.

In a symmetrical three-phase short circuit, the terminal voltage of the machine is equal to zero, so that the complete

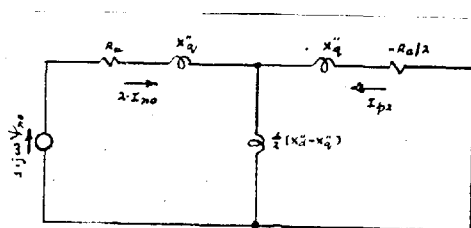


Fig. 11.12

even-harmonic circuit is given by Fig. 11.12. Harmonics higher than the second do not exist so that the even-harmonic series consists of the d.c.

and the second-harmonic components only.

The initial flux-linkages in the armature circuits are

$$\left. \begin{aligned} \psi_{ao} &= \frac{E_{am}}{\omega} \cos \lambda \\ \psi_{bo} &= \frac{E_{am}}{\omega} \cos (\lambda - 2\pi/3) \\ \text{and } \psi_{co} &= \frac{E_{am}}{\omega} \cos (\lambda + 2\pi/3) \end{aligned} \right\} (11.15)$$

which can be expressed in terms of symmetrical components as

$$\left. \begin{aligned} \psi_{so} &= 0 \\ \psi_{po} &= \frac{1}{2} \frac{E_{am}}{\omega} e^{j\lambda} \\ \text{and } \psi_{no} &= \frac{1}{2} \frac{E_{am}}{\omega} e^{-j\lambda} \end{aligned} \right\} (11.16)$$

Referring to Fig. 11.12, the following voltage equations can be written:

$$\left. \begin{aligned} jE_{am} e^{-j\lambda} &= j(x_d'' + x_q'') I_{no} + j \frac{1}{2} (x_d'' - x_q'') I_{p2} \\ \text{and } 0 &= j(x_d'' - x_q'') I_{no} + j \frac{1}{2} (x_d'' + x_q'') I_{p2} \end{aligned} \right\} (11.17)$$

Solving these equations gives

$$\left. \begin{aligned} I_{no} &= \frac{x_d'' + x_q''}{4x_d''x_q''} E_{am} e^{-j\lambda} \\ \text{and } I_{p2} &= \frac{x_q'' - x_d''}{2x_d''x_q''} E_{am} e^{-j\lambda} \end{aligned} \right\} (11.18)$$

Referring to Fig. 11.10 and Equ. 11.16, it is clear that I_{p0} is the conjugate of I_{no} . I_{o0} is equal to zero as $\psi_{o0} = 0$.

The phase currents are therefore given by

$$\left. \begin{aligned} I_{a0} &= \frac{x_d'' + x_q''}{2x_d''x_q''} E_{am} \cos \lambda \\ I_{b0} &= \frac{x_d'' + x_q''}{2x_d''x_q''} E_{am} \cos (\lambda - 2\pi/3) \\ I_{c0} &= \frac{x_d'' + x_q''}{2x_d''x_q''} E_{am} \cos (\lambda + 2\pi/3) \\ I_{a2} &= \frac{x_q'' - x_d''}{2x_d''x_q''} E_{am} e^{-j\lambda} \\ I_{b2} &= \frac{x_q'' - x_d''}{2x_d''x_q''} E_{am} e^{-j(\lambda + 2\pi/3)} \\ I_{c2} &= \frac{x_q'' - x_d''}{2x_d''x_q''} E_{am} e^{-j(\lambda - 2\pi/3)} \end{aligned} \right\} (11.19)$$

The time constant is given by, using Equ. 11.14,

$$T_{a(3)} = \frac{1}{R_a} \cdot \frac{\psi_{no}}{I_{no}} = \frac{1}{\omega R_a} \cdot \frac{2x_d''x_q''}{x_d'' + x_q''} \quad (11.20)$$

Finally the complete expressions for the armature-transient component of the phase currents are ($\theta = \omega t + \lambda$)

$$\left. \begin{aligned} i_a &= \frac{E_{am}}{2} e^{-t/\tau_a(n)} \left\{ \left(\frac{1}{x_d''} + \frac{1}{x_q''} \right) \cos \lambda - \left(\frac{1}{x_d''} - \frac{1}{x_q''} \right) \cos (2\theta - \lambda) \right\} \\ i_b &= \frac{E_{am}}{2} e^{-t/\tau_a(n)} \left\{ \left(\frac{1}{x_d''} + \frac{1}{x_q''} \right) \cos (\lambda - 2\pi/3) + \left(\frac{1}{x_d''} - \frac{1}{x_q''} \right) \cos (2\theta - \lambda - 2\pi/3) \right\} \\ \text{and } i_c &= \frac{E_{am}}{2} e^{-t/\tau_a(n)} \left\{ \left(\frac{1}{x_d''} + \frac{1}{x_q''} \right) \cos (\lambda + 2\pi/3) + \left(\frac{1}{x_d''} - \frac{1}{x_q''} \right) \cos (2\theta - \lambda + 2\pi/3) \right\} \end{aligned} \right\} \quad (11.21)$$

as derived before.

For a line-to-line short circuit, the equivalent circuit is shown in Fig. 11.13.

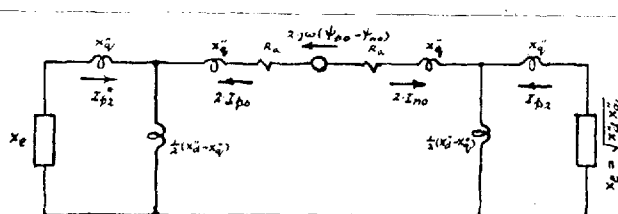


Fig. 11.13

The initial flux-linkage of the short-circuited windings (b and c) is

$$\begin{aligned} \psi_{bo} - \psi_{co} &= \frac{E_{am}}{\omega} \left\{ \cos (\lambda - 2\pi/3) - \cos (\lambda + 2\pi/3) \right\} \\ &= \frac{\sqrt{3} E_{am} \sin \lambda}{\omega} \end{aligned} \quad (11.22)$$

so that in terms of symmetrical components,

$$\psi_{bo} - \psi_{no} = j \frac{E_{am}}{\omega} \sin \lambda \quad (11.23)$$

The voltage equations of the middle and the right-hand meshes in the circuit shown in Fig. 11.13 are, since $I_{po} = -I_{no}$,

$$\begin{aligned} 2E_{am} \sin \lambda &= j2(x_d'' + x_q'') I_{no} + j \frac{1}{2} (x_d'' - x_q'') (I_{p1} - I_{p2}) \\ 0 &= j(x_d'' - x_q'') I_{no} + j \frac{1}{2} (x_d'' + x_q'' + 2\sqrt{x_d'' x_q''}) I_{p2} \end{aligned} \quad (11.24)$$

Solving these two equations simultaneously gives

$$\begin{aligned} I_{no} &= -j \frac{E_{am} \sin \lambda}{2 \sqrt{x_d'' x_q''}} \\ \text{and } I_{p2} &= j \frac{\sqrt{x_d''} - \sqrt{x_q''}}{\sqrt{x_d''} + \sqrt{x_q''}} \cdot \frac{E_{am} \sin \lambda}{\sqrt{x_d'' x_q''}} \end{aligned} \quad (11.25)$$

The d.c. and second-harmonic phase currents are

$$\left. \begin{aligned} I_{bo} &= -I_{co} = j\sqrt{3} I_{no} \\ &= \frac{\sqrt{3} E_{am} \sin \lambda}{2 \sqrt{x_d x_q}} \\ \text{and } I_{b2} &= -I_{c2} = j\sqrt{3} I_1 \\ &= \frac{\sqrt{x_d} - \sqrt{x_q}}{\sqrt{x_d} + \sqrt{x_q}} \cdot \frac{\sqrt{3} E_{am} \sin \lambda}{\sqrt{x_d x_q}} = -b_1 \frac{\sqrt{3} E_{am} \sin \lambda}{\sqrt{x_d x_q}} \end{aligned} \right\} (11.26)$$

where $b_1 = (\sqrt{x_d} - \sqrt{x_q}) / (\sqrt{x_d} + \sqrt{x_q})$

As it has been shown in Chapter 6 that the currents in two consecutive meshes of the infinite ladder network representing X_θ are related by the geometric ratio $-b_1$, so that the higher harmonic components are given by

$$\begin{aligned} I_{b,2m} &= -I_{c,2m} \\ &= (-b_1)^m \cdot \frac{\sqrt{3} E_{am} \sin \lambda}{\sqrt{x_d x_q}} \quad (m \neq 0) \end{aligned} \quad (11.27)$$

The time constant is given by, using Equ. 11.14,

$$\begin{aligned} T_{a(c)} &= \frac{1}{\lambda R_a} \cdot \frac{\psi_{po} - \psi_{no}}{I_{bo}} \\ &= \sqrt{x_d x_q} / \omega R_a \end{aligned} \quad (11.28)$$

which applies to the entire even-harmonic series. The complete expression for the armature-transient component of current is thus given by

$$\begin{aligned} i_b &= -i_c \\ &= \frac{\sqrt{3} E_{am} \sin \lambda}{\sqrt{x_d x_q}} \cdot e^{-t/T_{a(c)}} \left\{ \frac{1}{2} + \sum_{m=1}^{\infty} (-b_1)^m \cos 2\theta \right\} \end{aligned} \quad (11.29)$$

as derived before.

For a line-to-neutral short circuit, the equivalent circuit is shown in Fig. 11.14.

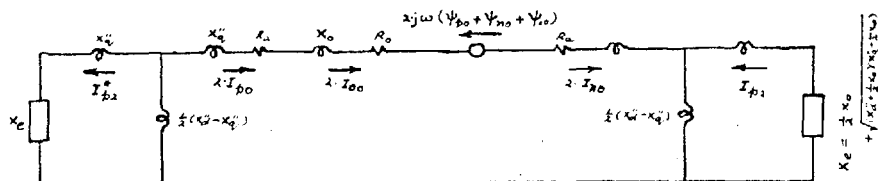


Fig. 11.14

The initial flux-linkage is

$$\psi_{a0} = \frac{E_{am}}{\omega} \cos \lambda \quad (11.30)$$

so that $\psi_{p0} + \psi_{n0} + \psi_{o0} = \frac{E_{am}}{\omega} \cos \lambda \quad (11.31)$

Since $I_{p0} = I_{n0} = I_{o0}$, the voltage equations of the middle and the right-hand meshes in the circuit shown in Fig. 11.14 are

$$\left. \begin{aligned} j 2 E_{am} \cos \lambda &= j 2 (x_d'' + x_q'' + x_o) I_{n0} + j \frac{1}{2} (x_d'' - x_q'') (I_{p2} + I_{p2}') \\ 0 &= j (x_d'' - x_q'') I_{n0} + j \frac{1}{2} (x_d'' + x_q'' + 2x_o) I_{p2} \end{aligned} \right\} (11.32)$$

and

Solving these two equations simultaneously gives

$$\left. \begin{aligned} I_{n0} &= \frac{E_{am} \cos \lambda}{2 \sqrt{(x_d'' + \frac{1}{2}x_o)(x_q'' + \frac{1}{2}x_o)}} \\ I_{p2} &= b_o \cdot \frac{E_{am} \cos \lambda}{\sqrt{(x_d'' + \frac{1}{2}x_o)(x_q'' + \frac{1}{2}x_o)}} \end{aligned} \right\} (11.33)$$

and

where $b_o = \frac{\sqrt{x_q'' + \frac{1}{2}x_o} - \sqrt{x_d'' + \frac{1}{2}x_o}}{\sqrt{x_d'' + \frac{1}{2}x_o} + \sqrt{x_q'' + \frac{1}{2}x_o}}$

The higher harmonic components are given by

$$I_{n(am)} = b_o^m \cdot \frac{E_{am} \cos \lambda}{\sqrt{(x_d'' + \frac{1}{2}x_o)(x_q'' + \frac{1}{2}x_o)}} \quad (11.34)$$

The d.c. component of the phase current is

$$I_{a0} = \frac{3 E_{am} \cos \lambda}{2 \sqrt{(x_d'' + \frac{1}{2}x_o)(x_q'' + \frac{1}{2}x_o)}} \quad (11.35)$$

and the harmonic components are

$$I_{a(am)} = b_o^m \cdot \frac{3 E_{am} \cos \lambda}{\sqrt{(x_d'' + \frac{1}{2}x_o)(x_q'' + \frac{1}{2}x_o)}} \quad (11.36)$$

The time constant is, using Equ. 11.14,

$$\begin{aligned} T_{a(L-n)} &= \frac{1}{2R_a + R_o} \cdot \frac{\psi_{p0} + \psi_{n0} + \psi_{o0}}{I_{p0}} \\ &= \frac{2 \sqrt{(x_d'' + \frac{1}{2}x_o)(x_q'' + \frac{1}{2}x_o)}}{\omega (2R_a + R_o)} \end{aligned} \quad (11.37)$$

If $R_o = R_a$, that is, if there is no earthing resistance, the time constant is reduced to $2 \sqrt{(x_d'' + \frac{1}{2}x_o)(x_q'' + \frac{1}{2}x_o)} / 3 \omega R_a$.

The complete expression for the armature-transient component

of the short-circuit current is

$$i_a = \frac{3E_{sm} \cos \lambda}{\sqrt{(x_d'' + \frac{1}{2}x_0)(x_q'' + \frac{1}{2}x_0)}} \left\{ \frac{1}{2} + \sum_{m=1}^{\infty} b_m^m \cos 2m\theta \right\} \quad (11.38)$$

as derived before.

In the case of a double-line-to-neutral short circuit, the equivalent circuit for the even-harmonic series is shown in Fig. 11.15.

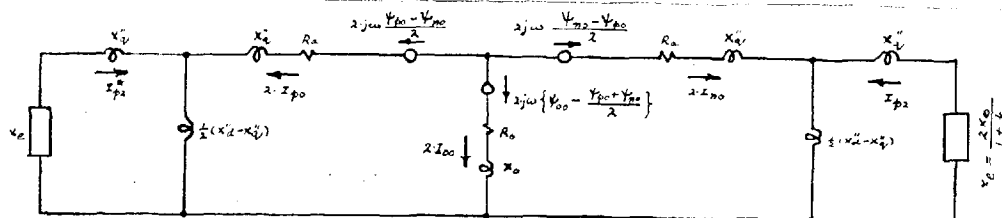


Fig. 11.15

The initial flux-linkages are

$$\Psi_{bo} = \frac{E_{sm}}{\omega} \cos(\lambda - 2\pi/3) \quad \left. \vphantom{\Psi_{bo}} \right\} (11.39)$$

and $\Psi_{co} = \frac{E_{sm}}{\omega} \cos(\lambda + 2\pi/3)$

so that $\Psi_{po} - \Psi_{no} = j \frac{E_{sm}}{\omega} \sin \lambda$

and $\Psi_{po} - \frac{\Psi_{po} + \Psi_{no}}{2} = \frac{1}{2}(\Psi_{bo} + \Psi_{co})$
 $= -\frac{1}{2} \cdot \frac{E_{sm}}{\omega} \cos \lambda \quad \left. \vphantom{\Psi_{po}} \right\} (11.40)$

The voltage equations of the two middle meshes and the right-hand mesh of the circuit shown in Fig. 11.15 are, since $I_{so} = -(I_{po} + I_{no})$,

$$\left. \begin{aligned} jE_{sm} e^{j\lambda} &= j(x_d'' + x_q'' + 2x_0)I_{po} + j2x_0I_{no} + j\frac{1}{2}(x_d'' - x_q'')I_{p2} \\ jE_{sm} e^{-j\lambda} &= j2x_0I_{po} + j(x_d'' + x_q'' + 2x_0)I_{no} + j\frac{1}{2}(x_d'' - x_q'')I_{p2} \\ 0 &= j(x_d'' - x_q'')I_{no} + j\frac{1}{2}(x_d'' + x_q'' + 2x_0)I_{p2} \end{aligned} \right\} (11.41)$$

and

Solving these equations simultaneously gives

$$\left. \begin{aligned} I_{po} &= \frac{E_{sm}(k-1)}{4x_0} \left(\frac{1}{k} \cos \lambda + j \sin \lambda \right) \\ I_{no} &= \frac{E_{sm}(k-1)}{4x_0} \left(\frac{1}{k} \cos \lambda - j \sin \lambda \right) \\ \text{and } I_{p2} &= b_2 \cdot \frac{E_{sm}}{4x_0} \left(\frac{1}{k} \cos \lambda + j \sin \lambda \right) \end{aligned} \right\} (11.42)$$

where
$$b_2 = \frac{\sqrt{x_2(x_2+2x_0)} - \sqrt{x_0^2(x_2+2x_0)}}{\sqrt{x_2(x_2+2x_0)} + \sqrt{x_0^2(x_2+2x_0)}}$$

Also
$$\left. \begin{aligned} I_{00} &= -(I_{p0} + I_{n0}) = -\frac{E_{am}(k-1) \cos \lambda}{2kx_0} \\ I_{02} &= -\frac{x_2}{x_2+x_0} I_{p2} = -\frac{x_0}{x_0} I_{p2} \\ &= -\frac{2}{1+k} I_{p2} \\ \text{and } I_{n2} &= -(I_{p2} + I_{02}) = \frac{1-k}{1+k} I_{p2} \end{aligned} \right\} (11.43)$$

Since the currents of two consecutive frequency orders are related by the geometric ratio b_2 as shown in Chapter 6, the higher even-harmonic components can be expressed in general forms as

$$\left. \begin{aligned} I_{p(2m)} &= b_2^m \cdot \frac{E_{am}}{2x_0} \left(\frac{1}{k} \cos \lambda + j \sin \lambda \right) \\ I_{n(2m)} &= \frac{1-k}{1+k} I_{p(2m)} \end{aligned} \right\} (11.44)$$

and
$$I_{0(2m)} = -\frac{2}{1+k} I_{p(2m)}$$

Special considerations are needed for the present case regarding the determination of the time constants. It is clear that the real parts of I_{p0} and I_{n0} , and I_{00} (no imaginary part) are produced by the applied flux-linkage $\Psi_{00} - \frac{1}{2}(\Psi_{p0} + \Psi_{n0})$ in a circuit whose damping is R_0 in series with two R_a 's in parallel. The time constant for I_{00} and the real parts of I_{p0} and I_{n0} , which is also the time constant for $I_{0(2m)}$ and the real parts of $I_{p(2m)}$ and $I_{n(2m)}$, is thus given by (referring to Equ. 11.14)

$$\begin{aligned} T_{a(2m)} &= \frac{1}{R_0 + R_a/2} \cdot \frac{\Psi_{00} - \frac{1}{2}(\Psi_{p0} + \Psi_{n0})}{I_{00}} \\ &= \frac{1}{R_0 + R_a/2} \cdot \frac{-\frac{1}{2} \cdot \frac{E_{am}}{\omega} \cos \lambda}{-\frac{E_{am}(k-1) \cos \lambda}{2kx_0}} \\ &= \frac{1}{\omega(R_a + 2R_0)} \cdot \frac{2kx_0}{k-1} \end{aligned} \quad (11.45)$$

as in Equ. 10.47.

Similarly the imaginary parts of I_{p0} and I_{n0} are produced

only by the applied flux-linkage $\Psi_{po} - \Psi_{no}$ in a circuit whose damping is $2R_a$. Thus, the time constant for the imaginary parts of I_{po} and I_{no} , which is also the time constant for the imaginary parts of $I_{p(2m)}$ and $I_{n(2m)}$, is given by

$$\begin{aligned} T_{\beta(u-n)} &= \frac{1}{2R_a} \cdot \frac{\Psi_{po} - \Psi_{no}}{j \cdot \text{Imag. of } I_{po}} \\ &= \frac{1}{2R_a} \cdot \frac{j \frac{E_{om}}{\omega} \sin \lambda}{j \frac{E_{om}(k-1)}{4X_o} \sin \lambda} \\ &= \frac{1}{\omega R_a} \cdot \frac{2X_o}{k-1} \end{aligned} \quad (11.46)$$

as in Equ. 10.48.

Reverting the sequence currents into the phase currents after the proper decrement factors have been attached to them, it can be shown that the armature-transient components of the short-circuit currents, given by Eqs. 10.62 and 10.63, can be obtained.

(11.4) Short-circuit Torque — Unidirectional Component

In deriving the short-circuit currents, only one of the two effects of the machine resistances, i.e., their damping effect, has been considered. Their other effect, the phase-shifting effect, has been ignored for the benefit of getting simpler approximate expressions of the currents. This is the reason why the use of the approximate expressions for the currents in the torque equation fails to give the unidirectional component of torque. Since the equivalent circuits, with the resistances of the machine added, can give the average torque directly in the form of active power, the unidirectional component of torque can be found without much difficulty by the

application of the equivalent circuits.

The unidirectional component of torque consists of two components. The first component is contributed by the odd-harmonic series of current. Its magnitude depends on the equivalent armature resistance (modified by the rotor resistances) and its decrement factors depend on the rotor resistances. The second component is contributed by the even-harmonic series of current. Its magnitude depends on the rotor resistances and its decrement factors depend on the armature resistance. The first unidirectional component of torque can be found with the aid of the odd-harmonic equivalent circuits and the second unidirectional component of torque can be found with the aid of the even-harmonic equivalent circuits. The derivation of the unidirectional components of torque for the four typical short circuits of a synchronous machine are considered in detail in the subsequent paragraphs.

(a) First unidirectional component of torque

For the sake of simplicity, the initial undamped value of the transient-state currents are used in the course of deriving the unidirectional component of torque. The appropriate decrement factors will be attached to the final expressions.

The equivalent circuits for the initial odd-harmonic cur-

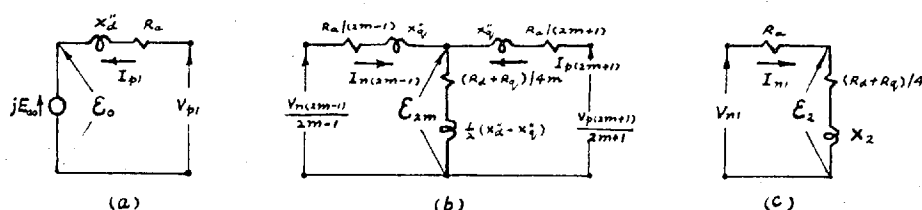


Fig. 11.16

rents are given in Figs. 11.16.a and 11.16.b. The resistance

R_d (function of R_f and R_{kd}) in series with X_d'' to represent the field and damper resistances is not included in Fig. 11.16.a for the positive-sequence, fundamental-frequency current, because the damping effect is its principal role. This is analogous to the fact that the armature resistance should be omitted in determining the magnitude of the d.c. component of the armature-transient current as described in the last article.

The first unidirectional of torque is given by

$$\Delta T = 3 \text{ Real part of } \left\{ \mathcal{E}_0^* I_{p1} + \sum_{m=1}^{\infty} \mathcal{E}_{2m}^* (I_{p(2m+1)} - I_{n(2m-1)}) \right\} \quad (11.47)$$

That is, it is given by the power involved in the voltage $\mathcal{E}_0$ and the current I_{p1} in Fig. 11.16.a and the power involved in the voltage $\mathcal{E}_{2m}$ and the difference of the currents $I_{p(2m+1)}$ and $I_{n(2m-1)}$ in Fig. 11.16.b. Obviously the real part of $\mathcal{E}_0^* I_{p1}$ is equal to $-I_{p1}^2 R_t$, where R_t is the total equivalent resistance in series in the circuit shown in Fig. 11.16.a when it is connected together with other equivalent circuits. The real part of $\mathcal{E}_{2m}^* (I_{p(2m+1)} - I_{n(2m-1)})$ has in general a more complicated expression. But if the ratio of $I_{p(2m+1)}/I_{n(2m-1)}$ is a scalar, which is approximately true for the various types of short circuits, it can be reduced to $-I_{n(2m-1)}^2 (1-b^2)(R_d + R_{d'})/4m$, where b is the scalar ratio.

If the scalar ratio b is not unduly large, the harmonic currents will be small and their effects on the unidirectional component of torque may be neglected. Doing this, the infinite number of equivalent circuits represented in Fig. 11.16.b may be replaced by a single negative-sequence equivalent circuit as shown in Fig. 11.16.c, where X_2 is the negative-sequence

reactance appropriate to the particular operating condition being considered. The unidirectional torque contributed by the negative-sequence current is $-I_{n1}^2(R_d + R_q)/4$ or $-I_{n1}^2(R_2 - R_a)$, where $R_2 = R_a + (R_d + R_q)/4$ is the equivalent negative-sequence armature resistance of the machine. The first unidirectional component of torque can therefore be found by, approximately,

$$\Delta_1 T \approx -3 I_{p1}^2 R_t - 3 I_{n1}^2 (R_2 - R_a) \quad (11.48)$$

Equ. 11.48 can be applied to all the four typical short circuits. For the symmetrical three-phase short circuit, $R_t = R_a$, $I_{n1} = 0$ and $I_{p1} = -\frac{E_{a0}}{X_d''} g(t)$, so that the first unidirectional component of torque is

$$\Delta_1 T \text{ [3-phase]} \approx -\frac{3 E_{a0}^2}{X_d''^2} R_a g^2(t) \quad (11.49)$$

For the line-to-line short circuit, $R_t = R_a + R_2$, $I_{n1} = -I_{p1}$ and $I_{p1} = \frac{E_{a0}}{X_d'' + \sqrt{X_d'' X_q''}} g_1(t)$, so that

$$\Delta_1 T \text{ [line-line]} \approx -\frac{3 E_{a0}^2}{(X_d'' + \sqrt{X_d'' X_q''})^2} \cdot 2 R_2 \cdot g_1^2(t) \quad (11.50)$$

For the line-to-neutral short circuit, $R_t = R_a + R_2 + R_0$, $I_{n1} = I_{p1}$ and $I_{p1} = -\frac{E_{a0}}{X_d'' + \frac{1}{2}X_0 + \sqrt{(X_d'' + \frac{1}{2}X_0)(X_q'' + \frac{1}{2}X_0)}} g_0(t)$, so that

$$\Delta_1 T \text{ [line-neutral]} \approx -\frac{3 E_{a0}^2}{[X_d'' + \frac{1}{2}X_0 + \sqrt{(X_d'' + \frac{1}{2}X_0)(X_q'' + \frac{1}{2}X_0)}]^2} (2 R_2 + R_0) g_0^2(t) \quad (11.51)$$

For the double-line-to-neutral short circuit, $R_t \approx R_a + (X_0/(X_2 + X_0))^2 R_2 + (X_2/(X_2 + X_0))^2 R_2$, $I_{n1} = -X_0 I_{p1}/(X_2 + X_0)$ and $I_{p1} = -\frac{E_{a0}}{X_d'' + X_2 X_0/(X_2 + X_0)} g_2(t)$, so that

$$\Delta_1 T \text{ [line-line-neutral]} \approx -\frac{E_{a0}^2}{(X_d'' + \frac{X_2 X_0}{X_2 + X_0})^2} \left\{ \left[1 - \left(\frac{X_0}{X_2 + X_0} \right)^2 \right] R_a + 2 \left(\frac{X_0}{X_2 + X_0} \right)^2 R_2 + \left(\frac{X_2}{X_2 + X_0} \right)^2 R_0 \right\} g_2^2(t) \quad (11.52)$$

It is obvious that Equ. 11.52 can be reduced to Equ. 11.49 by putting $X_0 = R_0 = 0$ and reduced to Equ. 11.50 by putting

$$X_0 = \infty.$$

(b) Second unidirectional component of torque.

The second unidirectional component of torque is contributed by the losses associated with the even-harmonic series of current. The equivalent circuits with the resistances R_d and R_q

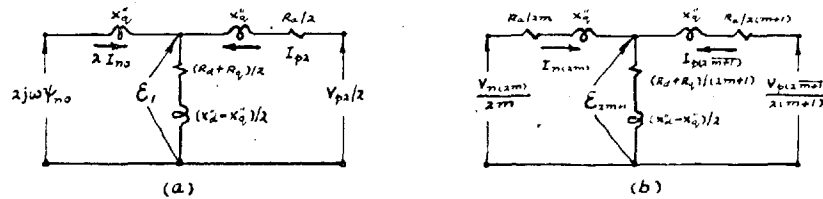


Fig. 11.17

added are shown in Fig. 11.17. The torque is given by

$$\Delta_1 T = 3 \text{ Real part of } \left\{ \mathcal{E}_1^* (I_{p1} - 2I_{n0}) + \sum_{m=1}^{\infty} \mathcal{E}_{2m+1}^* (I_{p(2m+1)} - I_{n(2m+1)}) \right\} \quad (11.53)$$

If the effect of the harmonic currents are neglected, the second unidirectional component of torque is approximately given by - 3 times the loss of the current $2I_{n0}$ in the resistance $\frac{1}{2}(R_d + R_q)$, i.e.,

$$\Delta_1 T \approx -6 I_{n0}^2 (R_d + R_q) \quad (11.54)$$

For the symmetrical three-phase short circuit,

$$I_{n0} = \frac{X_d'' + X_q''}{4X_d''X_q''} E_{a0} e^{-j\lambda} e^{-t/T_{a(1)}}$$

so that $|I_{n0}|^2 = \left(\frac{X_d'' + X_q''}{4X_d''X_q''} \right)^2 E_{a0}^2 e^{-2t/T_{a(1)}}$

$$\text{and } \Delta_1 T [3\text{-phase}] \approx -\frac{3}{8} \left(\frac{X_d'' + X_q''}{X_d''X_q''} \right)^2 E_{a0}^2 (R_d + R_q) e^{-2t/T_{a(1)}} \quad (11.55)$$

For the line-to-line short circuit,

$$I_{n0} = -j \frac{E_{a0} \sin \lambda}{2\sqrt{X_d''X_q''}} e^{-t/T_{a(1)}}$$

so that $|I_{n0}|^2 = \frac{E_{a0}^2 \sin^2 \lambda}{4X_d''X_q''} e^{-2t/T_{a(1)}}$

$$\text{and } \Delta_1 T [\text{line-line}] \approx -\frac{3E_{a0}^2 \sin^2 \lambda}{2X_d''X_q''} (R_d + R_q) e^{-2t/T_{a(1)}} \quad (11.56)$$

For the line-to-neutral short circuit,

$$I_{no} = \frac{E_{ao} \cos \lambda}{2\sqrt{(x_d + \frac{1}{2}x_o)(x_q + \frac{1}{2}x_o)}} e^{-t/T_{a(l-n)}}$$

$$\text{so that } \Delta_1 \tau [\text{line-neutral}] \approx - \frac{3E_{ao} \cos^2 \lambda}{2(x_d + \frac{1}{2}x_o)(x_q + \frac{1}{2}x_o)} (R_d + R_q) e^{-2t/T_{a(l-n)}} \quad (11.57)$$

For the double-line-to-neutral short circuit,

$$I_{no} = \frac{E_{ao}(k-1)}{4x_o} \left\{ \frac{1}{k} \cos \lambda \cdot e^{-t/T_{a(l-n)}} + j \sin \lambda \cdot e^{-t/T_{p(l-n)}} \right\}$$

$$\text{so that } |I_{no}|^2 = \frac{E_{ao}^2(k-1)^2}{16x_o^2} \left\{ \frac{1}{k^2} \cos^2 \lambda e^{-2t/T_{a(l-n)}} + \sin^2 \lambda e^{-2t/T_{p(l-n)}} \right\}$$

$$\text{and } \Delta_1 \tau [\text{line-line-neutral}] \approx - \frac{3E_{ao}^2(k-1)^2}{8x_o^2} (R_d + R_q) \left\{ \frac{1}{k^2} \cos^2 \lambda e^{-2t/T_{a(l-n)}} + \sin^2 \lambda e^{-2t/T_{p(l-n)}} \right\} \quad (11.58)$$

It can be shown that Equ. 11.58 is reduced to Equ. 11.55 by putting $X_o = R_o = 0$ and reduced to Equ. 11.56 by putting $X_o = \infty$.

(12) CONCLUSION

In the thesis, the operational method (the Laplace transformation) has been used in solving the differential equations prescribing the performance of a synchronous machine. At steady state, provided the machine in question is running at constant speed and the magnetic saturation is neglected, the currents, the terminal voltages and the electromagnetic torque can always be found to any desired degree of accuracy, irrespective of the terminal conditions of the machine, though this may mean solving a large number of simultaneous equations in complex numbers. The labour involved in finding the currents, voltages and average torque can be reduced when a suitable type of network analyser (one that can represent negative resistance) is available so that use can be made of the equivalent circuits given in Chapter 6.

The application of the operational method to the solution of the transient-state current of a machine following a sudden change of terminal conditions has also been shown in the thesis, though the general solution can only be expressed in operational form. The detailed derivation of the well-known solutions for the simpler problems, hitherto obtained by less rigorous methods (except for the symmetrical three-phase operations of the machine, see Ref. 6), has been done with the view to show the validity of these solutions and to provide an approximate method of deriving the explicit expression from the operational solution. This approximate method of finding the inverse transforms can

always be applied so long as the approximate frequency of the current component is known.

For a more complicated problem, the derivation of the inverse transforms from the operational solution may become too complicated to carry out, though a numerical solution is always possible. In this event, the equivalent circuits given in Chapter 11 may be used in practice for the determination of the transient-state quantities. As it has been shown that in general only the d.c. and the fundamental-frequency components of the armature current need special consideration at transient state, and that the negative-sequence impedance for a particular operating condition of the machine (represented by a suitable connection of the harmonic equivalent circuits) is the same for transient state as for steady state, the process of finding the complete expressions for the currents, terminal voltages and unidirectional component of torque as shown in Chapter 11 for the simpler cases can be applied to more complicated problems.

Although there has been a detailed study of the linear theory of the synchronous machine, the problems dealt with being only to a very small part of the numerous problems concerning a synchronous machine. It is hoped, however, that the material presented may be of some use and that the extension of the method employed may lead to the solution of a larger number of problems.

(13) REFERENCES.

- (1) C.F. WAGNER and R.D. EVANS: "Symmetrical Components," McGraw-Hill, 1933.
- (2) C. CONCORDIA: "Synchronous Machines — Theory and Performance," John Wiley & Sons, 1951.
- (3) R.H. PARK: "Two-reaction Theory of Synchronous Machines," Trans. A.I.E.E., 1929, Vol. 48, p.716.
- (4) G. KRON: "A Short Course in Tensor Analysis for Electrical Engineers," John Wiley & Sons, 1942.
- (5) R.E. VOWELS: "Transient Analysis of Synchronous Machines," Monograph 33, 1952.
- (6) B. ADKINS: "Transient Theory of Synchronous Generators connected to Power Systems," Proceedings I.E.E., 1951, Vol. 98, Part.II, p.510.
- (7) R.E. DOHERTY: "A Simplified Method of Analysing Short-Circuit Problems," Trans. A.I.E.E., 1923, Vol.42, p.841.
- (8) R.E. DOHERTY and C.A. NICKLE: "Synchronous Machines IV — Single-phase Short Circuits," Trans. A.I.E.E., 1930, Vol.49, p.700.
- (9) H.S. CARLAW and J.C. JAEGER: "Operational Methods in Applied Mathematics" Oxford University Press, 1941.
- (10) E. CLARKE, C. CONCORDIA, and C.N. WEYGANDT: "Overvoltages Caused by Unbalanced Short Circuits (Effect of Amortisseur Windings)," Trans. A.I.E.E., Vol. 57, 1938, p.453.

- (11) W. V. LYON: "Extention of the Method of Symmetrical Components Using Ladder Networks," A.I.E.E. Trans., Vol. 59, 1940, p.1025.
- (12) G. KRON: Equivalent Circuits of Electrical Machinery, John Wiley & Sons, 1951.
- (13) L. A. PIPES: Applied Mathematics for Engineers and Physicists, McGraw Hill, 1946.
- (14) W. J. GIBBS: "Algebra of Electric Machine Analysis," Technical Monograph, The British Thomson-Houston Co. Ltd.

(14) APPENDICES

(14.1) Steady-state solution

Assuming $M_{af}(\theta) = M_{akd}(\theta)$ and putting $\lambda = 0$ (the steady-state solution is not affected by the switching angle), then carrying out the Laplace transformation defined by

$$\bar{x}(p) = \int_0^{\infty} e^{-pt} x(t) dt$$

on the six equations represented by Equ. 2.1, they can be expressed in operational form as follows, when the initial currents are zero.

$$\begin{aligned} & \left\{ (R_a + R_{ae}) + (L_{ao} + L_{oe})p \right\} \bar{I}_a(p) + \frac{1}{2} L_{a2} p \left\{ \bar{I}_a(p+j2\omega) + \text{Conj} \right\} + M_{so} p \bar{I}_b(p) - \frac{1}{2} M_{s2} p \left\{ h \bar{I}_b(p+j2\omega) + \text{Conj} \right\} \\ & + M_{so} p \bar{I}_c(p) - \frac{1}{2} M_{s2} p \left\{ h^2 \bar{I}_c(p+j2\omega) + \text{Conj} \right\} + \frac{1}{2} M_{af1} p \left\{ \bar{I}_{fe}(p+j\omega) + \text{Conj} \right\} + \frac{1}{2} M_{af3} p \left\{ \bar{I}_{fe}(p+j3\omega) + \text{Conj} \right\} \\ & - \frac{1}{2} M_{akq1} p \left\{ j \bar{I}_{kq}(p+j\omega) + \text{Conj} \right\} + \frac{1}{2} M_{akq3} p \left\{ j \bar{I}_{kq}(p+j3\omega) + \text{Conj} \right\} = \bar{V}_a(p) + \bar{E}_a(p) \end{aligned} \quad (14.1)$$

$$\begin{aligned} & M_{so} p \bar{I}_a(p) - \frac{1}{2} M_{s2} p \left\{ h \bar{I}_a(p+j2\omega) + \text{Conj} \right\} + \left\{ (R_b + R_{be}) + (L_{bo} + L_{oe})p \right\} \bar{I}_b(p) + \frac{1}{2} L_{b2} p \left\{ h^2 \bar{I}_b(p+j2\omega) + \text{Conj} \right\} \\ & + M_{so} p \bar{I}_c(p) - \frac{1}{2} M_{s2} p \left\{ \bar{I}_c(p+j2\omega) + \text{Conj} \right\} + \frac{1}{2} M_{af1} p \left\{ h \bar{I}_{fe}(p+j\omega) + \text{Conj} \right\} + \frac{1}{2} M_{af3} p \left\{ \bar{I}_{fe}(p+j3\omega) + \text{Conj} \right\} \\ & - \frac{1}{2} M_{akq1} p \left\{ j h \bar{I}_{kq}(p+j\omega) + \text{Conj} \right\} + \frac{1}{2} M_{akq3} p \left\{ j \bar{I}_{kq}(p+j3\omega) + \text{Conj} \right\} = \bar{V}_b(p) + \bar{E}_b(p) \end{aligned} \quad (14.2)$$

$$\begin{aligned} & M_{so} p \bar{I}_a(p) - \frac{1}{2} M_{s2} p \left\{ h^2 \bar{I}_a(p+j2\omega) + \text{Conj} \right\} + M_{s2} p \bar{I}_b(p) - \frac{1}{2} M_{s2} p \left\{ \bar{I}_b(p+j2\omega) + \text{Conj} \right\} \\ & + \left\{ (R_c + R_{ce}) + (L_{co} + L_{oe})p \right\} \bar{I}_c(p) + \frac{1}{2} L_{c2} p \left\{ h \bar{I}_c(p+j2\omega) + \text{Conj} \right\} + \frac{1}{2} M_{af1} p \left\{ h^2 \bar{I}_{fe}(p+j\omega) + \text{Conj} \right\} \\ & + \frac{1}{2} M_{af3} p \left\{ \bar{I}_{fe}(p+j3\omega) + \text{Conj} \right\} - \frac{1}{2} M_{akq1} p \left\{ j h^2 \bar{I}_{kq}(p+j2\omega) + \text{Conj} \right\} - \frac{1}{2} M_{akq3} p \left\{ j \bar{I}_{kq}(p+j3\omega) + \text{Conj} \right\} \\ & = \bar{V}_c(p) + \bar{E}_c(p) \end{aligned} \quad (14.3)$$

$$\begin{aligned} & \frac{1}{2} M_{af1} p \left\{ \bar{I}_a(p+j\omega) + \text{Conj} + h \bar{I}_b(p+j\omega) + \text{Conj} + h^2 \bar{I}_c(p+j\omega) + \text{Conj} \right\} \\ & + \frac{1}{2} M_{af3} p \left\{ \bar{I}_a(p+j3\omega) + \text{Conj} + \bar{I}_b(p+j3\omega) + \text{Conj} + \bar{I}_c(p+j3\omega) + \text{Conj} \right\} \\ & + (R_f + L_f p) \bar{I}_f'(p) + M_{fkd} p \bar{I}_{kd}(p) = 0 \end{aligned} \quad (14.4)$$

$$\begin{aligned} & \frac{1}{2} M_{af1} p \left\{ \bar{I}_a(p+j\omega) + \text{Conj} + h \bar{I}_b(p+j\omega) + \text{Conj} + h^2 \bar{I}_c(p+j\omega) + \text{Conj} \right\} \\ & + \frac{1}{2} M_{af3} p \left\{ \bar{I}_a(p+j3\omega) + \text{Conj} + \bar{I}_b(p+j3\omega) + \text{Conj} + \bar{I}_c(p+j3\omega) + \text{Conj} \right\} \\ & + M_{fkd} p \bar{I}_f'(p) + (R_{kd} + L_{kd} p) \bar{I}_{kd}(p) = 0 \end{aligned} \quad (14.5)$$

$$\begin{aligned} & - \frac{1}{2} M_{akq1} p \left\{ j \bar{I}_a(p+j\omega) + \text{Conj} + j h \bar{I}_b(p+j\omega) + \text{Conj} + j h^2 \bar{I}_c(p+j\omega) + \text{Conj} \right\} \\ & + \frac{1}{2} M_{akq3} p \left\{ j \bar{I}_a(p+j3\omega) + \text{Conj} + j \bar{I}_b(p+j3\omega) + \text{Conj} + j \bar{I}_c(p+j3\omega) + \text{Conj} \right\} \\ & + (R_{kq} + L_{kq} p) \bar{I}_{kq}(p) = 0 \end{aligned} \quad (14.6)$$

where $i_{fe} = i_f^1 + i_{kd}$. Combining the field and the direct-axis-damper equations (14.4 and 14.5) gives

$$\begin{aligned} & \frac{1}{2} M_{af1} p \{ \bar{u}_a(p+j\omega) + \text{Conj} + h \bar{u}_b(p+j\omega) + \text{Conj} + h^2 \bar{u}_c(p+j\omega) + \text{Conj} \} \\ & + \frac{1}{2} M_{af3} p \{ \bar{u}_a(p+j3\omega) + \text{Conj} + \bar{u}_b(p+j3\omega) + \text{Conj} + \bar{u}_c(p+j3\omega) + \text{Conj} \} + \bar{z}_d(p) \bar{u}_{fe}(p) = 0 \end{aligned} \quad (14.7)$$

where

$$\bar{z}_d(p) = \frac{(R_f + L_f p)(R_{kd} + L_{kd}) - M_{fkd}^2 p^2}{R_f + R_{kd} + (L_f + L_{kd} - 2 M_{fkd})p} \approx R_{fe} + L_{fe} p$$

$$R_{fe} = \frac{R_f(L_{kd} - M_{fkd})^2 + R_{kd}(L_f - M_{fkd})^2}{(L_f + L_{kd} - 2 M_{fkd})^2}$$

$$L_{fe} = (L_f L_{kd} - M_{fkd}^2) / (L_f + L_{kd} - 2 M_{fkd})$$

The expressions for $\bar{i}_{fe}(p+j\omega)$, $\bar{i}_{fe}(p+j3\omega)$, $\bar{i}_{kg}(p+j\omega)$, $\bar{i}_{kg}(p+j3\omega)$ and their conjugate functions are obtained by solving $\bar{i}_{fe}(p)$ and $\bar{i}_{kg}(p)$ from Eqs. 14.7 and 14.6, then putting $p+j\omega$ and $p+j3\omega$ and their conjugates in place of p in the resultant expressions. Substituting these results in Equ. 14.1 gives

$$\begin{aligned} & \{ (R_a + R_{ae}) + (L_{ao} + L_{ae})p \} \bar{u}_a(p) + \frac{1}{2} L_{a2} p \{ \bar{u}_a(p+j2\omega) + \text{Conj} \} + M_{so} p \bar{u}_b(p) - \frac{1}{2} M_{s2} p \{ h \bar{u}_b(p+j2\omega) + \text{Conj} \} \\ & + M_{so} p \bar{u}_c(p) - \frac{1}{2} M_{s2} p \{ h^2 \bar{u}_c(p+j2\omega) + \text{Conj} \} \\ & - \frac{1}{4} M_{af1}^2 \frac{p(p+j\omega)}{R_{fe} + L_{fe}(p+j\omega)} \{ \bar{u}_a(p+j2\omega) + \bar{u}_a(p) + h \bar{u}_b(p+j2\omega) + h^2 \bar{u}_b(p) + h^2 \bar{u}_c(p+j2\omega) + h \bar{u}_c(p) \} - \text{Conj} \\ & - \frac{1}{4} M_{af1} M_{af3} \frac{p(p+j\omega)}{R_{fe} + L_{fe}(p+j\omega)} \{ \bar{u}_a(p+j4\omega) + \bar{u}_a(p-j2\omega) + \bar{u}_b(p+j4\omega) + \bar{u}_b(p-j2\omega) + \bar{u}_c(p+j4\omega) + \bar{u}_c(p-j2\omega) \} - \text{Conj} \\ & - \frac{1}{4} M_{af1} M_{af3} \frac{p(p+j3\omega)}{R_{fe} + L_{fe}(p+j3\omega)} \{ \bar{u}_a(p+j4\omega) + \bar{u}_a(p-j2\omega) + h \bar{u}_b(p+j4\omega) + h^2 \bar{u}_b(p-j2\omega) + h^2 \bar{u}_c(p+j4\omega) + h \bar{u}_c(p-j2\omega) \} - \text{Conj} \\ & - \frac{1}{4} M_{af3}^2 \frac{p(p+j3\omega)}{R_{fe} + L_{fe}(p+j3\omega)} \{ \bar{u}_a(p+j6\omega) + \bar{u}_a(p) + \bar{u}_b(p+j6\omega) + \bar{u}_b(p) + \bar{u}_c(p+j6\omega) + \bar{u}_c(p) \} - \text{Conj} \\ & - \frac{1}{4} M_{akq1}^2 \frac{p(p+j\omega)}{R_{kq} + L_{kq}(p+j\omega)} \{ -\bar{u}_a(p+j2\omega) + \bar{u}_a(p) - h \bar{u}_b(p+j2\omega) + h^2 \bar{u}_b(p) - h^2 \bar{u}_c(p+j2\omega) + h \bar{u}_c(p) \} - \text{Conj} \\ & - \frac{1}{4} M_{akq1} M_{akq3} \frac{p(p+j\omega)}{R_{kq} + L_{kq}(p+j\omega)} \{ \bar{u}_a(p+j4\omega) - \bar{u}_a(p-j2\omega) + \bar{u}_b(p+j4\omega) - \bar{u}_b(p-j2\omega) + \bar{u}_c(p+j4\omega) - \bar{u}_c(p-j2\omega) \} - \text{Conj} \\ & - \frac{1}{4} M_{akq1} M_{akq3} \frac{p(p+j3\omega)}{R_{kq} + L_{kq}(p+j3\omega)} \{ \bar{u}_a(p+j4\omega) - \bar{u}_a(p-j2\omega) + h \bar{u}_b(p+j4\omega) - h^2 \bar{u}_b(p-j2\omega) + h^2 \bar{u}_c(p+j4\omega) - h \bar{u}_c(p-j2\omega) \} - \text{Conj} \\ & - \frac{1}{4} M_{akq3}^2 \frac{p(p+j3\omega)}{R_{kq} + L_{kq}(p+j3\omega)} \{ -\bar{u}_a(p+j6\omega) + \bar{u}_a(p) - \bar{u}_b(p+j6\omega) + \bar{u}_b(p) - \bar{u}_c(p+j6\omega) + \bar{u}_c(p) \} - \text{Conj} \\ & = \bar{v}_a(p) + \bar{e}_a(p) \end{aligned} \quad (14.8)$$

Similar expressions are obtainable from Eqs. 14.2 and 14.3 when $\bar{i}_{fe}(p+j\omega)$, $\bar{i}_{fe}(p+j3\omega)$, $\bar{i}_{kg}(p+j\omega)$, $\bar{i}_{kg}(p+j3\omega)$ and their conjugate functions are eliminated. It is not necessary to write them down in details as Equ. 14.8 alone is sufficient for the purpose of

showing the process.

To find the first approximation of the steady-state solution, let it be assumed that the fundamental voltage drops are due to the fundamental currents only. Thus, let

$$i_a(t) = I_{a1x} \cos \omega t$$

then
$$\bar{i}_a(p) = I_{a1x} \frac{p}{p^2 + \omega^2}$$

Substituting in Equ. 14.8 it can be shown that the inverse transforms of the terms involving i_a are

$$\mathcal{L}^{-1} \{ (R_a + R_{ae}) + (L_{a0} + L_{ae})p \} \bar{i}_a(p) = (R_a + R_{ae}) I_{a1x} \cos \omega t - \omega (L_{a0} + L_{ae}) I_{a1x} \sin \omega t$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{2} L_{a2} p \{ \bar{i}_a(p+j2\omega) + \text{Conj} \} \right\} = -\frac{1}{2} \omega L_{a2} I_{a1x} \sin \omega t + \text{Third harmonic}$$

$$\mathcal{L}^{-1} \left\{ -\frac{1}{4} M_{a1}^2 \frac{p(p+j\omega)}{R_{fe} + L_{fe}(p+j\omega)} \bar{i}_a(p+j2\omega) - \text{Conj} \right\} = \text{Third harmonic \& transient only}$$

$$\mathcal{L}^{-1} \left\{ -\frac{1}{4} M_{a1}^2 \frac{p(p+j\omega)}{R_{fe} + L_{fe}(p+j\omega)} \bar{i}_a(p) - \text{Conj} \right\} = \frac{1}{2} \cdot \frac{\omega^2 M_{a1}^2}{R_{fe}^2 + 4\omega^2 L_{fe}^2} I_{a1x} (R_{fe} \cos \omega t + 2\omega L_{fe} \sin \omega t) + \text{Transient}$$

and so on. Also let

$$i_a(t) = -I_{a1y} \sin \omega t$$

then
$$\bar{i}_a(p) = -I_{a1y} \frac{\omega}{p^2 + \omega^2}$$

Substituting in Equ. 14.8 it can be shown that the inverse transforms of the terms involving i_a are

$$\mathcal{L}^{-1} \{ (R_a + R_{ae}) + (L_{a0} + L_{ae})p \} \bar{i}_a(p) = -(R_a + R_{ae}) I_{a1y} \sin \omega t - \omega (L_{a0} + L_{ae}) I_{a1y} \cos \omega t$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{2} L_{a2} p \{ \bar{i}_a(p+j2\omega) + \text{Conj} \} \right\} = \frac{1}{2} \omega L_{a2} I_{a1y} \cos \omega t + \text{Third harmonic}$$

$$\mathcal{L}^{-1} \left\{ -\frac{1}{4} M_{a1}^2 \frac{p(p+j\omega)}{R_{fe} + L_{fe}(p+j\omega)} \bar{i}_a(p+j2\omega) - \text{Conj} \right\} = \text{Third harmonic \& transient only}$$

$$\mathcal{L}^{-1} \left\{ -\frac{1}{4} M_{a1}^2 \frac{p(p+j\omega)}{R_{fe} + L_{fe}(p+j\omega)} \bar{i}_a(p) - \text{Conj} \right\} = \frac{1}{2} \cdot \frac{\omega^2 M_{a1}^2}{R_{fe}^2 + 4\omega^2 L_{fe}^2} I_{a1y} (-R_{fe} \sin \omega t + 2\omega L_{fe} \cos \omega t) + \text{Transient}$$

and so on. To convert the expressions into clearer forms, complex numbers can be introduced by putting 1 and -j in place of $\cos \omega t$ and $\sin \omega t$ respectively. Thus, if

$$I_{a1x} \cos \omega t \longrightarrow I_{a1x}$$

and
$$-I_{a1y} \sin \omega t \longrightarrow j I_{a1y}$$

then
$$I_{a1x} \cos \omega t - I_{a1y} \sin \omega t \longrightarrow I_{a1x} + j I_{a1y} = I_{a1}$$

Adding the above two sets of inverse transforms and introducing complex numbers, the terms involving only fundamental voltage drops are

$$\begin{aligned}\mathcal{L}^{-1}\{(R_a + R_{ae}) + (L_{ao} + L_{ae})p\} \bar{v}_a(p) &\longrightarrow \{(R_a + R_{ae}) + j\omega(L_{ao} + L_{ae})\} I_{a1} \\ \mathcal{L}^{-1}\{\frac{1}{2}L_{a2}p\} \{\bar{v}_a(p + j2\omega) + \text{conj}\} &\longrightarrow j\omega \cdot \frac{1}{2}L_{a2} I_{a1}^* \\ \mathcal{L}^{-1}\left\{-\frac{1}{4}M_{af1}^2 \frac{p(p+j\omega)}{R_{fe} + L_{fe}(p+j\omega)} - \text{conj}\right\} &\longrightarrow \frac{1}{2} \cdot \frac{\omega^2 M_{af1}^2}{R_{fe} + j2\omega L_{fe}} I_{a1} \\ &\approx -j\omega \left(\frac{1}{2} \frac{M_{af1}^2}{L_{fe}}\right) I_{a1} \quad \text{when } R_{fe} \text{ is neglected}\end{aligned}$$

and so on. Proceeding in like manner for i_b and i_c , it can be shown that corresponding to Equ. 14.1, the following approximate equation in complex numbers can be found:

$$\begin{aligned}&\{(R_a + R_{ae}) + j\omega(L_{ao} + L_{ae})\} I_{a1} + j\omega \cdot \frac{1}{2}L_{a2} I_{a1}^* + j\omega M_{so} I_{b1} - j\omega^2 \cdot \frac{1}{2}M_{s2} I_{b1}^* + j\omega M_{so} I_{c1} - j\omega^2 \cdot \frac{1}{2}M_{s2} I_{c1}^* \\ &- j\omega \frac{1}{4} \cdot \frac{M_{af1}^2}{L_{fe}} \cdot \{I_{a1} + h I_{b1} + h^2 I_{c1}\} - j\omega \frac{1}{4} \cdot \frac{M_{af1} M_{af2}}{L_{fe}} \{2 I_{a1}^* + h^2 I_{b1}^* + h I_{c1}^*\} \\ &- j\omega \frac{1}{2} \cdot \frac{M_{af2}^2}{L_{fe}} \{I_{a1} + I_{b1} + I_{c1}\} - j\omega \frac{1}{4} \cdot \frac{M_{akq1}}{L_{kq}} \{I_{a1} + h I_{b1} + h^2 I_{c1}\} \\ &+ j\omega \frac{1}{4} \cdot \frac{M_{akq1} M_{akq2}}{L_{kq}} \{2 I_{a1}^* + h^2 I_{b1}^* + h I_{c1}^*\} - j\omega \frac{1}{4} \cdot \frac{M_{akq2}^2}{L_{kq}} \{I_{a1} + I_{b1} + I_{c1}\} = -j\omega M_{af1} I_{fo} + V_{a1}\end{aligned}$$

Grouping terms and dividing I_{fo} by $\sqrt{2}$ so that the armature currents are in rms values, this equation is reduced to the first equation (axis a) represented by Equ. 2.6. The remaining two equations (axes b and c) can be found in the same way.

Similarly the equations for the harmonic currents and the additional voltages for current corrections can be derived.

(14.2) Single-phase Operation

Putting $V_a(2m+1) = 0$, $R_a = R_{ae} = 0$ and adding $\Delta E_a(2m+1)$ (Equ. 3.11) to Equ. 3.10 and reducing gives

$$B I_{a(2m-1)} + A I_{a(2m+1)} + B I_{a(2m+3)} = 0 \quad (14.9)$$

where $B = \frac{1}{2} (x_a'' - x_q'')$

and $A = x_a'' + x_q'' + x_o + 3x_e$

As m may be any non-zero positive integer, Equ. 14.9 represents an infinite number of equations of similar pattern:

$$B I_{a1} + A I_{a3} + B I_{a5} = 0 \quad (14.9.1)$$

$$B I_{a3} + A I_{a5} + B I_{a7} = 0 \quad (14.9.2)$$

....

$$B I_{a(2m-1)} + A I_{a(2m+1)} + B I_{a(2m+3)} = 0 \quad (14.9.m)$$

....

When m is sufficiently large, the quantity $B I_{a(2m+3)}$ in Equ. 14.9.m may be treated as a remainder and neglected for the moment, so that it is possible to eliminate $I_{a(2m+1)}$ from Eqs. 14.9.m-1 and 14.9.m, giving

$$B I_{a(2m-3)} + \left(A - \frac{B^2}{A} \right) I_{a(2m-1)} = 0 \quad (14.10)$$

Similarly the elimination of $I_{a(2m-1)}$ from Eqs. 14.9.m-2 and 14.10 gives

$$B I_{a(2m-5)} + \left\{ A - \frac{B^2}{A - \frac{B^2}{A}} \right\} I_{a(2m-3)} = 0$$

Proceeding the same process, and remembering that the equations represented by Equ. 14.9 are infinite in number, the general form of Equ. 14.10 should be

$$B I_{a(2m-1)} + \left\{ A - \frac{B^2}{A - \dots} \right\} I_{a(2m+1)} = 0 \quad (14.11)$$

where the coefficient of $I_{a(2m+1)}$ is a continued fraction of the second kind. To evaluate its value, let

$$x = A - \frac{B^2}{A - \dots}$$

so that $x = A - B^2/x$

and $x = \frac{1}{2} (A - \sqrt{A^2 - 4B^2})$

Equ. 14.11 thus reduces to

$$I_{a(2m+1)} = b_0 I_{a(2m-1)} \quad (14.12)$$

where $b_0 = -2B / (A - \sqrt{A^2 - 4B^2})$

$$= \frac{\sqrt{x_d^2 + \frac{1}{4}(x_0 + 3x_g)} - \sqrt{x_d^2 + \frac{1}{4}(x_0 + 3x_e)}}{\sqrt{x_d^2 + \frac{1}{4}(x_0 + 3x_e)} + \sqrt{x_d^2 + \frac{1}{4}(x_0 + 3x_e)}}$$

From Equ. 14.12, the expression relating the $(2m+1)$ th-harmonic current to the fundamental current can be derived:

$$I_{a(2m+1)} = b_0^m I_{a1} \quad (14.13)$$

Adding ΔE_{a1} (Equ. 3.11) to Equ. 3.8 and neglecting the R's gives

$$V_{a1} + E_{a1} = j \left\{ x_e + \frac{1}{3} x_0 + \frac{1}{4} (x_d + x_g) + \frac{1}{4} (x_d^2 + x_g^2) \right\} I_{a1} + j \frac{1}{6} (x_d - x_g) I_{a1}^* + j \frac{1}{6} (x_d - x_g) I_{a3} \quad (14.14)$$

which is an exact expression apart from the neglect of resistances. Eliminating I_{a3} from Equ. 14.14 by using Equ. 14.13 gives Equ. 3.15.

(14.3) Two Additional Proofs Concerning the Solution of the Recurrence Equation (Article 7.6)

(a) Prove that the recurrence equation

$$\bar{E}_\alpha(p) = \bar{L}_\alpha(p) + \frac{1}{2} \epsilon^{-j2\lambda} \bar{K}_\alpha(p) \bar{L}_\alpha(p+j2\omega) + \frac{1}{2} \epsilon^{j2\lambda} \bar{K}_\alpha^*(p) \bar{L}_\alpha(p-j2\omega) \quad (14.15)$$

has a solution of the form

$$\bar{L}_\alpha(p) = \sum_{n=-\infty}^{\infty} \bar{A}_n(p) \bar{E}_\alpha(p+j2n\omega) \quad (14.16)$$

By putting $p + j2n\omega$ for p , with n taking the value of any integer from $-\infty$ to ∞ in Equ. 14.15, an infinite number of equations of similar pattern are obtained, and they can be represented by the matrix equation

$$[y_n] = [a_{nm}] \cdot [x_m] \quad (14.17)$$

where $[y_n]$ is an infinite column matrix whose general element is

$$y_n = \bar{F}_x(p + j2n\omega) \quad (14.18)$$

$[x_m]$ is also an infinite column matrix whose general element is

$$x_m = \bar{U}_x(p + j2m\omega) \quad (14.19)$$

and $[a_{nm}]$ is an infinite two-way matrix whose general element is

$$\begin{aligned} a_{nm} &= 1 & \text{when } m-n &= 0 \\ &= \frac{1}{2} e^{-j2\lambda} \bar{K}_x(p + j2n\omega) & \text{when } m-n &= 1 \\ &= \frac{1}{2} e^{j2\lambda} \bar{K}_x^*(p + j2n\omega) & \text{when } m-n &= -1 \\ &= 0 & \text{when } |m-n| &\geq 2 \end{aligned} \quad (14.20)$$

By Cramer's rule, x_0 (i.e. $m = 0$) is given by

$$x_0 = \frac{1}{|a_{nm}|} \sum_{n=-\infty}^{\infty} (\text{Cofactor of } a_{n0}) \times y_n \quad (14.21)$$

where $|a_{nm}|$ is the determinant of the matrix $[a_{nm}]$. Substituting Eqs. 14.18 and 14.19 in Equ. 14.21 gives Equ. 14.16 and also defines the coefficient $\bar{A}_n(p)$:

$$\bar{A}_n(p) = (\text{Cofactor of } a_{n0}) \div |a_{nm}| \quad (14.22)$$

Consequently Equ. 14.16 is a solution of the recurrence equation (14.15) which exists provided a_{nm} is not singular.

(b) Prove the validity of the equation

$$\bar{x}_x(p) \bar{A}_n(p) + \frac{1}{2} e^{-j2\lambda} \bar{K}_x(p) \bar{A}_{n-1}(p + j2\omega) = 0 \quad (14.23)$$

which appears in the course of deriving the coefficient $\bar{A}_n(p)$.

Assuming that Equ. 14.23 is correct, replacing n by $n + 1$ and p by $p - j2\omega$ gives

$$\bar{x}_\alpha(p-j2\omega) \bar{A}_{n+1}(p-j2\omega) + \frac{1}{2} e^{-j2\lambda} \bar{K}_\alpha(p-j2\omega) \bar{A}_n(p) = 0$$

or $\bar{A}_n(p) + 2e^{j2\lambda} \frac{\bar{x}_\alpha(p-j2\omega)}{\bar{K}_\alpha(p-j2\omega)} \bar{A}_{n+1}(p-j2\omega) = 0$ (14.24)

Since $\bar{x}_\alpha(p) = 1 - \frac{\frac{1}{4} \bar{K}_\alpha^*(p) \bar{K}_\alpha(p-j2\omega)}{\bar{x}_\alpha(p-j2\omega)}$

so that $\frac{\bar{x}_\alpha(p-j2\omega)}{\bar{K}_\alpha(p-j2\omega)} = \frac{1}{4} \cdot \frac{\bar{K}_\alpha^*(p)}{1 - \bar{x}_\alpha(p)}$ (14.25)

Substituting Equ. 14.25 in Equ. 14.24 and multiplying through by $1 - \bar{x}_\alpha(p)$ gives

$$[1 - \bar{x}_\alpha(p)] \bar{A}_n(p) + \frac{1}{2} e^{j2\lambda} \bar{K}_\alpha^*(p) \bar{A}_{n+1}(p-j2\omega) = 0$$

(14.26)

Adding Eqs. 14.23 and 14.26 gives the original equation

$$\bar{A}_n(p) + \frac{1}{2} e^{-j2\lambda} \bar{K}_\alpha(p) \bar{A}_{n+1}(p+j2\omega) + \frac{1}{2} e^{j2\lambda} \bar{K}_\alpha^*(p) \bar{A}_{n+1}(p-j2\omega) = 0$$

from which Equ. 14.23 have been derived. The validity of Equ. 14.23 is thus proved.