

Integral Controlled Lagrangians for underactuated mechanical systems subject to matched and unmatched disturbances

Enrico Franco 

Department of Mechanical Engineering, Imperial College London, London, SW7 2AZ, UK

ARTICLE INFO

Keywords:

Energy shaping
Disturbances
Nonlinear systems
Underactuated mechanical systems

ABSTRACT

This work investigates the extension of the Controlled Lagrangians methodology for underactuated mechanical systems subject to matched and unmatched disturbances. A new passivity-preserving controller design that includes two dynamic extensions for disturbance rejection purposes is presented. The new controller is compared with the corresponding port-controlled Hamiltonian implementation, and an interpretation of the dynamic extensions as low-pass filters is proposed. The effectiveness of the proposed controller is demonstrated with numerical simulations.

1. Introduction

Among various methodologies for the control of underactuated mechanical systems, energy-shaping controllers are noteworthy because of their interpretability in terms of mechanical energy and their ability to accommodate nonlinearities [1]. The most notable energy-shaping methodologies include the *Interconnection and damping assignment Passivity based control* (IDA-PBC) [2], which employs the port-controlled Hamiltonian (PCH) formalism, and the *Controlled Lagrangians* (CL) [3–5], which employs the Lagrangian formalism. In both approaches, the control action is designed so that the closed-loop dynamics preserves the structure of a mechanical system and is characterized by a desired total energy. Both methodologies have been applied to a variety of systems, including robotic manipulators [6,7], soft robotic systems [8], electromechanical systems [9], surface vessels [10], and satellites [11]. One of the main challenges that characterize both the CL and the IDA-PBC methodology is the need to solve a set of partial differential equations (PDEs) to ensure matching between the open-loop dynamics and the prescribed closed-loop dynamics. Despite this being a bottleneck, several results exist in this direction, including the study of kinetic energy shaping for mechanical systems, see [12], the transformation of PDEs into Pfaffian differential equations, see [13], the algebraic solutions of the PDEs, see [9], and the numerical solution of the PDEs with reinforcement learning, see [14]. Due to their different formulation, the CL methodology and the IDA-PBC methodology typically result in slightly different PDEs, which become identical under certain conditions, see [15]. For simplicity, the original formulations of IDA-PBC and CL did not account for disturbances or model uncertainties. The effect of velocity-dependent forces was later investigated as part of the CL methodology, see [5,16]. More recently, an adaptive CL design that accounts for parametric uncertainties in the model was proposed in [17]. In parallel, the robustness of IDA-PBC to parametric uncertainties was investigated in [18], while the robustification to constant matched disturbances resulted in the so-called *integral IDA-PBC* methodology (iIDA-PBC) [19–21]. This approach was later extended to systems with state-dependent matched disturbances [22], with constant unmatched disturbances [23], and with position-dependent unmatched disturbances [24]. In spite of the close similarity between IDA-PBC and CL, see [15], few works have attempted to produce an exact equivalent of the iIDA-PBC for the CL methodology. In particular, [25] has proposed a passivity-preserving dynamic extension for the CL, termed *integral CL* (iCL), which can compensate the effect of

E-mail address: e.franco11@imperial.ac.uk

<https://doi.org/10.1016/j.jfranklin.2025.108205>

Received 7 June 2025; Received in revised form 5 September 2025; Accepted 30 October 2025

Available online 3 November 2025

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matched disturbances. However, unmatched disturbances were not considered and, to the best of the author's knowledge, no other work has investigated this aspect with the CL approach. This is a relevant problem in engineering practice where disturbances are ubiquitous. In addition, comparing iCL and iIDA-PBC can shed light on their differences and can provide conditions under which they are equivalent.

This paper investigates the extension of the CL methodology with the aim to compensate the effect of matched and unmatched disturbances. The main contributions of this work include the following points.

- A new passivity-preserving extension of the CL methodology for underactuated mechanical systems subject to matched and unmatched linearly parameterized additive disturbances. Two dynamic extensions are introduced to compensate the effects of matched and unmatched disturbances respectively.
- A detailed comparison with the corresponding iIDA-PBC implementation and a set of conditions under which the two controllers are identical.
- The interpretation of the dynamic extensions as low-pass filters.

The effectiveness of the new controller is demonstrated with numerical simulations on three examples, that is an inertia-wheel pendulum system, a pendulum-on-cart system moving on a slope, and a vertical-takeoff-and-landing aircraft.

Compared to the use of traditional observers, the proposed approach preserves the interpretation of the control action in terms of system energy. Compared to robust controllers, the proposed approach does not require prior knowledge of the disturbance bounds. Compared to prior implementations of iCL, see [25], the new controller also accounts for unmatched disturbances. While the proposed design bears similarities with the iIDA-PBC methodology, see [24], it is specific to the CL formulation hence it results in a different parameterization.

The rest of the paper is organized as follows. Section 2 gives an overview of the CL methodology. Section 3 details the new controller design. Section 4 presents a comparison with iIDA-PBC, while Section 5 provides an interpretation of the dynamic extensions as low-pass filters. Section 6 contains the simulation results, and Section 7 concludes the paper.

Notation. Function arguments are specified on first use and subsequently omitted in equations for conciseness.

2. Overview of Controlled Lagrangians

Consider an underactuated mechanical system with n degrees-of-freedom (DOFs) and the control input $u \in \mathbb{R}^m$ applied through the input matrix $G(q) \in \mathbb{R}^{n \times m}$, where $\text{rank}(G) = m < n$ for all $q \in \mathbb{R}^n$, and subject to the disturbances $\delta \in \mathbb{R}^n$. The system states are the position $q \in \mathbb{R}^n$ and the velocity $\dot{q} \in \mathbb{R}^n$. The Lagrangian function is defined as

$$L(q, \dot{q}) = \frac{1}{2} \dot{q}^\top M \dot{q} - V(q), \quad (1)$$

where the inertia matrix is $M(q) = M(q)^\top > 0$, and the potential energy is $V(q)$. The system dynamics is defined as

$$\frac{d}{dt} \partial_{\dot{q}} L(q, \dot{q}) - \partial_q L(q, \dot{q}) + \partial_{\dot{q}} \mathcal{R} = G(q)u - \delta. \quad (2)$$

Physical damping is accounted for by the dissipation function $\mathcal{R}(\dot{q})$, where $\dot{q}^\top \partial_{\dot{q}} \mathcal{R} > 0$, see [5]. In the absence of disturbances and damping (i.e., $\delta = 0$, $\mathcal{R}(\dot{q}) = 0$), the CL methodology [3] stabilizes the desired equilibrium $(q, \dot{q}) = (q^*, 0)$ by enforcing the prescribed Lagrangian $L_c(q, \dot{q}) = \frac{1}{2} \dot{q}^\top M_c \dot{q} - V_c(q)$ and the closed-loop dynamics

$$\frac{d}{dt} \partial_{\dot{q}} L_c(q, \dot{q}) - \partial_q L_c(q, \dot{q}) = 0, \quad (3)$$

with the inertia matrix $M_c(q) = M_c^\top(q) > 0$, and the potential energy $V_c(q) \geq 0$, where $V_c(q^*) = 0$ is a global minimum, that is $\partial_q V_c(q^*) = 0$ and $\partial_q^2 V_c(q^*) > 0$. The dynamics (3) is achieved by the control law

$$u_{CL} = G^\dagger (\partial_q (M \dot{q}) \dot{q} - \partial_q L) - G^\dagger M M_c^{-1} (\partial_q (M_c \dot{q}) \dot{q} - \partial_q L_c), \quad (4)$$

see [4], where $G^\dagger = (G^\top G)^{-1} G^\top$, provided that M_c and V_c verify for all $(q, \dot{q}) \in \mathbb{R}^{2n}$ the PDEs

$$G^\perp \left(-\partial_q \left(\frac{1}{2} \dot{q}^\top M \dot{q} \right) + M M_c^{-1} \partial_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right) \right) + G^\perp (\partial_q (M \dot{q}) \dot{q} - M M_c^{-1} \partial_q (M_c \dot{q})) = 0, \quad (5)$$

$$G^\perp (\partial_q V - M M_c^{-1} \partial_q V_c) = 0, \quad (6)$$

where $G^\perp$ is the full-rank left annihilator of G , defined such that $G^\perp G = 0$ and $\text{rank}(G^\perp) = n - m$, see [15]. Despite their close similarity, the IDA-PBC methodology is considered more general than the CL methodology, see [15]. Conversely, the CL methodology can result in PDEs that have a different parameterization, which might aid their solution for some classes of systems, see [3]. Nevertheless, under some conditions, the CL control law (4) is equivalent to the IDA-PBC control law, that is

$$u_{IDA} = G^\dagger (\partial_q H - M_d M^{-1} \partial_q H_d + J_2 \partial_p H_d) - k_v G^\top \partial_p H_d, \quad (7)$$

where $p = M \dot{q}$ are the momenta, $H(q, p) = \frac{1}{2} p^\top M^{-1} p + V$ and $H_d(q, p) = \frac{1}{2} p^\top M_d^{-1} p + V_d$ are the open-loop Hamiltonian and the prescribed Hamiltonian respectively, and $J_2 = -J_2^\top$ is a design parameter, see [2]. The conditions for equivalence between CL and IDA-PBC are defined in [15] as

$$M_c = M M_d^{-1} M, \quad V_c = V_d, \quad J_2 = M_d M^{-1} (\partial_q (M M_d^{-1} p)^\top - \partial_q (M M_d^{-1} p)) M^{-1} M_d. \quad (8)$$

Note that (7) contains a damping injection term for the purpose of stabilizing the equilibrium $(q, p) = (q^*, 0)$. This can also be achieved with CL by including a further term in the control law (4), that is

$$u = u_{CL} + u_{DI}, \quad u_{DI} = -k_v G^T M^{-1} M_c \dot{q}. \quad (9)$$

This yields the closed-loop dynamics

$$\frac{d}{dt} \partial_{\dot{q}} L_c(q, \dot{q}) - \partial_q L_c(q, \dot{q}) + \partial_{\dot{q}} \mathcal{R}_c = 0, \quad (10)$$

where $\partial_{\dot{q}} \mathcal{R}_c = M_c M^{-1} (\partial_{\dot{q}} \mathcal{R} + G k_v G^T M^{-1} M_c \dot{q})$, and $k_v > 0$ is a tuning parameter. Computing the time-derivative of the storage function $W = V_c + \frac{1}{2} \dot{q}^T M_c \dot{q} > 0$ along the trajectories of the closed-loop system (10) yields then

$$\dot{W} = -\dot{q}^T \partial_{\dot{q}} \mathcal{R}_c \leq 0, \quad (11)$$

provided that $\dot{q}^T \partial_{\dot{q}} \mathcal{R}_c > 0$. Invoking LaSalle's theorem it follows that the equilibrium $(q, \dot{q}) = (q^*, 0)$ is asymptotically stable. If dissipation is due to viscous friction defined by the matrix $D = D^T \geq 0$, we have $\mathcal{R}(\dot{q}) = \frac{1}{2} \dot{q}^T D \dot{q} \geq 0$, and consequently $\dot{q}^T \partial_{\dot{q}} \mathcal{R}_c > 0, \forall \dot{q} \neq 0$ provided that

$$\frac{1}{2} (M_c M^{-1} D + D M^{-1} M_c) + M_c M^{-1} G k_v G^T M^{-1} M_c > 0,$$

where the product $G k_v G^T$ is rank deficient. If $D > 0$, this condition is verified when $M_c M^{-1} D + D M^{-1} M_c > 0$ and $k_v > 0$ (i.e., substituting $M_c = M M_d^{-1} M$ yields the corresponding condition for IDA-PBC, see [26]). In the absence of physical damping (i.e., $D = 0$), the time-derivative of the storage function (11) yields $\dot{W} = y_c^T u_{DI} \leq 0$ and asymptotic stability is ensured for all $k_v > 0$ if the passive output $y_c = G^T M^{-1} M_c \dot{q}$ of the closed-loop system (10) is detectable, that is $y_c \rightarrow 0 \implies (q, \dot{q}) \rightarrow (q^*, 0)$, see [5,15].

If the disturbances are only matched, that is, $\delta = \delta_1 G G^T h_1(q)$ where $\delta_1 \in \mathbb{R}$ is an unknown scalar constant, while $h_1(q) \in \mathbb{R}^n$ is a known globally bounded and continuously differentiable function of q , they can be compensated with the iCL control law

$$u = u_{CL} + u_{DI} + G^T h_1(q) K_I (\zeta_1 - \Psi_1), \quad (12)$$

$$\Psi_1(q, \dot{q}) = h_1(q)^T G G^T M^{-1} M_c \dot{q},$$

see [25], where the time derivative of the extended state $\zeta_1 \in \mathbb{R}$ is

$$\dot{\zeta}_1 = -(\partial_{\dot{q}} \Psi_1)^T M_c^{-1} \left(\partial_q V_c + \partial_{\dot{q}} \mathcal{R}_c + \partial_q (M_c \dot{q}) \dot{q} - \partial_q \left(\frac{1}{2} \dot{q}^T M_c \dot{q} \right) \right) - (\partial_{\dot{q}} \Psi_1)^T \dot{q} + (\partial_q \Psi_1)^T \dot{q}, \quad (13)$$

and the prescribed closed-loop dynamics is

$$\frac{d}{dt} \partial_{\dot{q}} L_c(q, \dot{q}) - \partial_q L_c(q, \dot{q}) + \partial_{\dot{q}} \mathcal{R}_c(q, \dot{q}) + \partial_{\dot{q}} \Theta_1(q, \dot{q}, \zeta_1) = 0, \quad \Theta_1 = \frac{K_I}{2} \left(\zeta_1 - \Psi_1(q, \dot{q}) - \frac{\delta_1}{K_I} \right)^2. \quad (14)$$

Defining the storage function

$$W(q, \dot{q}, \zeta_1) = V_c + \frac{1}{2} \dot{q}^T M_c \dot{q} + \Theta_1 > 0, \quad (15)$$

and computing its time-derivative along the trajectories of the closed-loop system (13) and (14) yields

$$\dot{W} = -\dot{q}^T \partial_{\dot{q}} \mathcal{R}_c - (\partial_{\dot{q}} \Theta_1)^T M_c^{-1} (\partial_{\dot{q}} \Theta_1) \leq 0. \quad (16)$$

Invoking LaSalle's theorem it follows that the equilibrium point $(q, \dot{q}) = (q^*, 0)$ is locally asymptotically stable for all $K_I > 0, k_v > 0$ provided that $\dot{q}^T \partial_{\dot{q}} \mathcal{R}_c \geq 0$ and that the passive output y_c of (10) is detectable, see [25].

3. Integral Controlled Lagrangians

This section presents the new iCL design for systems with matched and unmatched disturbances.

3.1. System class

The class of underactuated mechanical systems with dynamics (2) considered in this work is qualified by the following assumptions.

Assumption 1. The PDEs (5) and (6) are solvable analytically with M_c and V_c , where $q^* = \operatorname{argmin}(V_c)$. The equilibrium is assignable, that is $G^\perp(\partial_q V + \delta) = 0$ at $q = q^*$. Physical damping is defined by the dissipation function $\mathcal{R}(\dot{q})$, such that $\dot{q}^T \partial_{\dot{q}} \mathcal{R}_c \geq 0$. In addition, the passive output $y_c = G^T M^{-1} M_c \dot{q}$ of the closed-loop system (10) is detectable. All model parameters are exactly known, and all states are measured.

The solvability of PDEs remains a major challenge for both the CL and the IDA-PBC methodologies. However, this condition is verified by several canonical examples, see [17,27]. The study of the solvability of the PDEs represents a separate topic, see [12], which is beyond the scope of this work. The remaining conditions are common to several works, see [23,24].

Assumption 2. The disturbance is defined as $\delta = \delta_1 G G^\top h_1(q) + \delta_2 G^\perp G^\perp h_2(q)$ with $\delta_1, \delta_2 \in \mathbb{R}$ unknown scalar constants, while $h_1(q), h_2(q) \in \mathbb{R}^n$ are known continuously differentiable functions of q .

Without loss of generality, disturbances can be decomposed into matched and unmatched components. The dependence on the generalized position is accounted for by the functions h_1 and h_2 (i.e., constant in case of constant disturbances). Position-dependent disturbances are representative of uncertain stiffness (i.e., in suspension systems or soft robotic systems [8]). In addition, the input matrix G might also depend on the generalized position. The disturbances are parameterized with the scalars δ_1 and δ_2 as in [24] to facilitate the comparison with the corresponding iIDA-PBC formulation, see Section 4. A different parameterization is discussed in Remark 2.

Assumption 3. Given the assignable equilibrium q^* of (2), there exists a scalar function $\Phi(q, \dot{q}, \zeta_2)$, where $\zeta_2 \in \mathbb{R}$ is an extended state employed to estimate the scalar constant δ_2 , that verifies the PDEs

$$G^\perp M M_c^{-1} \left(\partial_{\dot{q}} \Psi_2 K_I (\zeta_2 - \Psi_2) - \partial_q \Phi - \partial_{\dot{q}} \Phi - \partial_{\zeta_2} \Phi \partial_q \Psi_2 \right) = 0, \quad (17a)$$

$$\partial_{\zeta_2} \Phi \partial_q \Psi_2 + \partial_{\dot{q}} \Phi = 0, \quad (17b)$$

$$\partial_q V_c + \partial_q \Phi = 0 \quad q = q^*, \quad (17c)$$

$$\partial_q^2 V_c + \partial_q^2 \Phi > 0 \quad q = q^*, \quad (17d)$$

where $\Psi_2(q, \dot{q}) = h_2(q)^\top G^\perp G^\perp M^{-1} M_c \dot{q}$ and $K_I > 0$. In addition, $\partial_{\zeta_2} \Phi = 0$ at $q = q^*$.

The function $\Phi(q, \dot{q}, \zeta_2)$ represents the mechanical work of the closed-loop non-conservative forces resulting from the unmatched disturbance. In this respect, Φ bears some similarity to the potential energy V_c , and it can be considered its extension pertaining to the non-conservative forces. This assumption is a bottleneck of the proposed approach since, similarly to the PDEs (5) and (6), solving the PDEs 17(a) to 17(d) can be challenging. A procedure for solving these PDEs locally at the equilibrium, $q = q^*$, is outlined in Remark 1. It should be noted that the PDEs 17(a) to 17(d) are similar but, in general, not identical to those in iIDA-PBC, see [24]. Conditions for which the two sets of PDEs are identical are provided in Section 4.

3.2. Controller design

A new control law, which extends the iCL (12) to account for unmatched disturbances, is defined such that the open-loop dynamics (2) corresponds to a newly defined prescribed closed-loop dynamics. To this end, the scalar function $\Phi(q, \dot{q}, \zeta_2)$ is employed to account for the effect of the unmatched disturbances in closed loop. Two extended states, $\zeta_1 \in \mathbb{R}$ and $\zeta_2 \in \mathbb{R}$, are introduced to estimate the unknown scalar constants δ_1 and δ_2 . The new control law and the time-derivatives of ζ_1 and ζ_2 are defined to ensure matching between the open-loop and the prescribed closed-loop dynamics and to preserve the passivity property. This result is formalized by the following proposition.

Proposition 1. Consider the underactuated mechanical system (2) satisfying Assumptions 1 to 3 in closed-loop with the control law

$$u = u_{CL} + u_{DI} + G^\top h_1(q) K_I (\zeta_1 - \Psi_1) - G^\top M M_c^{-1} (\partial_q \Phi + \partial_{\dot{q}} \Phi + \partial_{\zeta_2} \Phi \partial_q \Psi_2),$$

$$\Psi_1(q, \dot{q}) = h_1(q)^\top G G^\top M^{-1} M_c \dot{q}, \quad (18)$$

$$\Psi_2(q, \dot{q}) = h_2(q)^\top G^\perp G^\perp M^{-1} M_c \dot{q},$$

where the time-derivatives of ζ_1 and ζ_2 are given by

$$\begin{aligned} \dot{\zeta}_1 = & -(\partial_{\dot{q}} \Psi_1)^\top M_c^{-1} (\partial_q V_c + \partial_q \Phi + \partial_{\dot{q}} \mathcal{R}_c) - (\partial_{\dot{q}} \Psi_1)^\top M_c^{-1} \left(\partial_q (M_c \dot{q}) \dot{q} - \partial_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right) \right) \\ & - (\partial_{\dot{q}} \Psi_1)^\top M_c^{-1} \partial_{\zeta_2} \Phi \partial_q \Psi_2 - (\partial_{\dot{q}} \Psi_1)^\top \dot{q} + (\partial_{\dot{q}} \Psi_1)^\top \dot{q}, \end{aligned} \quad (19a)$$

$$\begin{aligned} \dot{\zeta}_2 = & -(\partial_{\dot{q}} \Psi_2)^\top M_c^{-1} (\partial_q V_c + \partial_q \Phi + \partial_{\dot{q}} \mathcal{R}_c) - (\partial_{\dot{q}} \Psi_2)^\top M_c^{-1} \left(\partial_q (M_c \dot{q}) \dot{q} - \partial_q \left(\frac{1}{2} \dot{q}^\top M_c \dot{q} \right) \right) \\ & - (\partial_{\dot{q}} \Psi_2)^\top M_c^{-1} \partial_{\zeta_2} \Phi \partial_q \Psi_2 - (\partial_{\dot{q}} \Psi_2)^\top \dot{q} + (\partial_{\dot{q}} \Psi_2)^\top \dot{q}. \end{aligned} \quad (19b)$$

The following claims are in order.

- C1. The closed-loop dynamics is given by 19(a), 19(b) and by 20.
- C2. The controller (18) with the dynamic extensions 19(a) and 19(b) is passivity-preserving.
- C3. The equilibrium $(q, \dot{q}) = (q^*, 0)$ is asymptotically stable for all $K_I > 0, k_v > 0$.
- C4. ζ_1 and ζ_2 converge to δ_1/K_I and δ_2/K_I respectively provided that $h_1(q^*) \neq 0, h_2(q^*) \neq 0$.

$$\begin{aligned} \frac{d}{dt} \partial_{\dot{q}} L_c - \partial_q L_c + \partial_q \mathcal{R}_c + \partial_{\dot{q}} \Theta_1 + \partial_{\dot{q}} \Theta_2 + \partial_{\dot{q}} \Phi + \partial_{\dot{q}} \Phi + \partial_{\zeta_2} \Phi \partial_q \Psi_2 &= 0, \\ \partial_{\dot{q}} \mathcal{R}_c &= M_c M^{-1} (\partial_{\dot{q}} \mathcal{R} + G k_v G^\top M^{-1} M_c \dot{q}), \quad L_c(q, \dot{q}) = \frac{1}{2} \dot{q}^\top M_c \dot{q} - V_c, \\ \Theta_1 &= \frac{K_I}{2} \left(\zeta_1 - \Psi_1 - \frac{\delta_1}{K_I} \right)^2, \quad \Theta_2 = \frac{K_I}{2} \left(\zeta_2 - \Psi_2 - \frac{\delta_2}{K_I} \right)^2. \end{aligned} \quad (20)$$

The proof is given in Appendix A.

Remark 1. The new control law (18) extends (12) by accounting for the unmatched disturbances through the scalar function Φ and the extended state ζ_2 . Thus, the prescribed closed-loop dynamics also includes the time-derivative of ζ_2 in (19b). In addition, the time-derivative of ζ_1 in (19a) differs from (13) in that it also accounts for the scalar function Φ . Compared to (12), the new controller (18) requires satisfying *Assumption 3*, which represents a bottleneck. If the unmatched disturbances are constant, that is $h_2(q) = I^n \in \mathbb{R}^n$, and the matrices M, M_c and G are constant, it follows that $\nabla_q \Psi_2 = 0$ and $\nabla_q \Psi_2 = G^{\perp T} G^{\perp} M^{-1} M_c = \kappa_2 \in \mathbb{R}^n$. Substituting (17b) into (17a) yields then

$$\kappa_1 \kappa_2 \underbrace{K_I (\zeta_2 - \Psi_2)}_{\hat{\delta}_2} + \kappa_1 \kappa_2 \partial_{\zeta_2} \Phi - \kappa_1 \partial_q \Phi = 0, \quad (21)$$

where $\kappa_1 = G^{\perp} M M_c^{-1}$, $\kappa_1 \in \mathbb{R}^{(n-m) \times n}$, $\kappa_2 = G^{\perp T} \kappa_1$, and only Φ depends on q , while $\hat{\delta}_2$ represents the estimate of δ_2 . Note that (21) is a simpler PDE than (17a) and it admits a solution of the form

$$\Phi = K_I (\zeta_2 - \Psi_2) \kappa_3^T (q - q^*) + \kappa_4^T (q - q^*) + \frac{K_I}{2} \kappa_5^T (q - q^*) (q - q^*)^T \kappa_3, \quad (22)$$

where $n - m$ elements of $\kappa_3, \kappa_4, \kappa_5 \in \mathbb{R}^n$ are defined by (21), that is $\kappa_1 \kappa_2 = \kappa_1 \kappa_3 = \kappa_1 \kappa_5$ and $\kappa_1 \kappa_4 = 0$. In summary, substituting (22) into (21) yields a set of $n - m$ algebraic equations in $\kappa_3, \kappa_4, \kappa_5$. The remaining m elements of $\kappa_3, \kappa_4, \kappa_5$ should be defined such that 17(c) and 17(d) are verified. Note that (22) also verifies 17(b) by construction and ensures $\partial_{\zeta_2} \Phi = 0$ at $q = q^*$. In the general case, a local solution of 17(a) to 17(d), valid at the assignable equilibrium position, can be found by: (i) replacing M, M_c, G and $h_2(q)$ in 17(a) with their values at $q = q^*$, which yields (21); (ii) constructing $\Phi(q, \dot{q}, \zeta_2)$ as in (22); (iii) substituting (22) into (21) and computing $\kappa_3, \kappa_4, \kappa_5$ such that (21), 17(c) and 17(d) are verified (i.e., solving a set of algebraic equations). The solution is global if M, M_c, G and h_2 are constant. This procedure has been applied to all three examples presented in Section 6. See details in Appendix B, C, D.

Remark 2. For illustrative purpose, a different parameterization of the disturbances is considered, that is $\delta = h_1(q) G G^T \delta_1 + h_2(q) G^{\perp T} G^{\perp} \delta_2$ with $\delta_1, \delta_2 \in \mathbb{R}^n$ unknown constant vectors, while $h_1(q), h_2(q) \in \mathbb{R}$ are known continuously differentiable scalar functions of q . This yields the same control law (18) with the time-derivatives 19(a) and 19(b), where the extended states are now vectors of n elements, that is $\zeta_1, \zeta_2 \in \mathbb{R}^n$ and accordingly $\Psi_1, \Psi_2 \in \mathbb{R}^n$. The closed-loop dynamics (20) is unchanged, where Θ_1, Θ_2 are now defined as

$$\Theta_1 = \frac{K_I}{2} \left(\zeta_1 - \Psi_1 - \frac{\delta_1}{K_I} \right)^T \left(\zeta_1 - \Psi_1 - \frac{\delta_1}{K_I} \right), \quad \Theta_2 = \frac{K_I}{2} \left(\zeta_2 - \Psi_2 - \frac{\delta_2}{K_I} \right)^T \left(\zeta_2 - \Psi_2 - \frac{\delta_2}{K_I} \right).$$

The proof of *Proposition 1* is unchanged (i.e., after substituting the new expression of the disturbances).

4. Comparison with iIDA-PBC

The iIDA-PBC design for matched and unmatched disturbances, see [24], yields the control law

$$\begin{aligned} u &= u_{IDA} + G^T h_1(q) \Gamma_1 k_1 (\zeta_1 - \bar{\Psi}_1) - G^T M_d M^{-1} \left(\partial_q \Phi + \partial_q \bar{\Psi}_2 + \partial_{\zeta_2} \Phi \right), \\ \bar{\Psi}_1(q, p) &= h_1(q)^T G G^T p, \\ \bar{\Psi}_2(q, p) &= h_2(q)^T G^{\perp T} G^{\perp} p, \end{aligned} \quad (23)$$

where the time-derivatives of ζ_1 and ζ_2 are

$$\dot{\zeta}_1 = -(\partial_p \bar{\Psi}_1)^T M_d M^{-1} \left(\partial_q H_d + \partial_q \Phi + \partial_q \bar{\Psi}_2 \partial_{\zeta_2} \Phi \right) - (\partial_p \bar{\Psi}_1)^T (\Gamma_1 I - S_{22}^T) M_d^{-1} p + (\partial_q \bar{\Psi}_1)^T M^{-1} p, \quad (24a)$$

$$\dot{\zeta}_2 = -(\partial_p \bar{\Psi}_2)^T M_d M^{-1} \left(\partial_q H_d + \partial_q \Phi + \partial_q \bar{\Psi}_2 \partial_{\zeta_2} \Phi \right) - (\partial_p \bar{\Psi}_2)^T (\Gamma_1 I - S_{22}^T) M_d^{-1} p + (\partial_q \bar{\Psi}_2)^T M^{-1} p, \quad (24b)$$

with $S_{22} = G K_v G^T - J_2 + D M^{-1} M_d$ and $H_d(q, p) = \frac{1}{2} p^T M_d^{-1} p + V_d$. The scalar function $\Phi(q, p, \zeta_2)$ should verify the PDEs (see *Assumption 4* in [24])

$$G^{\perp} \left(\partial_p \bar{\Psi}_2 \Gamma_1 k_1 (\zeta_2 - \bar{\Psi}_2) - M_d M^{-1} \partial_q \Phi + \left(\partial_p \bar{\Psi}_2 \Gamma_1 - M_d M^{-1} \partial_q \bar{\Psi}_2 \right) \partial_{\zeta_2} \Phi \right) = 0, \quad (25a)$$

$$\partial_{\zeta_2} \Phi \partial_p \bar{\Psi}_2 + \partial_p \Phi = 0, \quad (25b)$$

$$\partial_q V_d + \partial_q \Phi = 0 \quad q = q^*, \quad (25c)$$

$$\partial_q^2 V_d + \partial_q^2 \Phi > 0 \quad q = q^*. \quad (25d)$$

The new controller (18) and the iIDA-PBC (23) are compared by considering the following points.

1. *PDEs.* The PDEs (5) and (6) in the CL methodology are equivalent to the corresponding PDEs in IDA-PBC provided that the conditions in (8) hold, see [15]. In addition, there is a close similarity between the PDEs 17(a) – 17(d) and 25(a) – 25(d). Substituting 17(b) into 17(a) yields

$$G^{\perp} M M_c^{-1} \left(\partial_q \Psi_2 K_I (\zeta_2 - \Psi_2) - \partial_q \Phi + \partial_{\zeta_2} \Phi \partial_q \Psi_2 - \partial_{\zeta_2} \Phi \partial_q \Psi_2 \right) = 0,$$

while substituting $\partial_q \Psi_2(q, \dot{q}) = M_c M^{-1} G^{\perp} G^{\perp} h_2(q)$ and $M_c = M M_d^{-1} M$ yields

$$G^{\perp} \left(G^{\perp} G^{\perp} h_2(q) K_I (\zeta_2 - \Psi_2) - M_d M^{-1} \partial_q \Phi + \left(G^{\perp} G^{\perp} h_2(q) - M_d M^{-1} \partial_q \Psi_2 \right) \partial_{\zeta_2} \Phi \right) = 0,$$

which is identical to 25(a) if $k_2 = K_I, \Gamma_1 = 1$, and $\partial_q \Psi_2 = \partial_q \bar{\Psi}_2$. In addition, 17(b) is identical to (25b) if $\partial_q \Psi_2 = \partial_q \bar{\Psi}_2$. Finally, 17(c) is identical to 25(c) and 17(d) is identical to 25(d). In summary, the PDEs 17(a) – 17(d) are identical to 25(a) – 25(d) if $k_2 = K_I, \Gamma_1 = 1$, and if $M_c = M_d = M$. This condition refers to the CL controller (4) and the corresponding IDA-PBC (7) only enforcing potential energy shaping. In such case, the PDE (5) is trivially verified, and the CL design becomes coincident to the corresponding IDA-PBC. In general, whether finding an analytical solution of the PDEs 17(a) – 17(d) proves easier than for the PDEs 25(a) – 25(d) depends on the specific case.

2. Design procedure. A key difference between the new iCL controller (18) and the iIDA-PBC (23) is the design procedure employed for the time-derivatives 19(a) and 19(b). The extended states ζ_1 and ζ_2 are included in the system dynamics (20) through the scalar functions Θ_1 and Θ_2 . Drawing a parallel to the dissipation function $\mathcal{R}_c$, the functions Θ_1 and Θ_2 can also be interpreted in terms of dissipation due to the residuals $z_1 = (\zeta_1 - \Psi_1 - \delta_1/K_I)$ and $z_2 = (\zeta_2 - \Psi_2 - \delta_2/K_I)$. Accordingly, $\partial_q \Theta_1$ and $\partial_q \Theta_2$ are included in (20) while $\partial_q \Theta_1$ and $\partial_q \Theta_2$ are not, which is similar to how $\partial_q \mathcal{R}_c$ is treated. Conversely, the scalar function Φ is treated in a similar way to the potential energy V_c , see 17(c) and 17(d). In addition, the closed-loop dynamics (20) also contains $\partial_q \Phi$ and $\partial_{\zeta_2} \Phi \partial_q \Psi_2$. These two terms do not lend themselves to an intuitive interpretation, but are required to enforce passivity conditions. Accordingly, the time-derivatives of ζ_1 and ζ_2 in (19a) and (19b) are computed from (A5) to enforce $\dot{W} \leq 0$, see Appendix A. This is different from the iIDA-PBC design procedure, where 24(a) and 24(b) result from enforcing a PCH structure in closed loop. In summary, the iIDA-PBC design [24] provides a more intuitive interpretation in terms of system energy compared to the proposed iCL design (18).

3. Controller parameters. Comparing (23), 24(a) and 24(b) to (18), 19(a) and 19(b) respectively highlights a similar structure and some notable differences: (i) $\partial_q \Phi$ is present in (18) but $\partial_p \Phi$ is missing in (23) thus the two control laws are not identical; (ii) the two controllers use different parameterizations, with Ψ_1, Ψ_2 in (18) containing the closed-loop inertia matrix M_c , while $\bar{\Psi}_1, \bar{\Psi}_2$ in (23) do not depend on M_d ; (iii) physical damping is accounted for with a negative sign in 19(a) and 19(b), that is $-(\partial_q \Psi_2)^{\top} M_c^{-1} \partial_q \mathcal{R}_c$, but with a positive sign in 24(a) and 24(b), that is $(\partial_p \bar{\Psi}_2)^{\top} S_{22}^{\top} M_d^{-1} p$. As a result, the new controller (18) preserves passivity for all $K_I > 0$, while the controller (23) requires a sufficiently large Γ_1 such that, given $\partial_q \mathcal{R} = D\dot{q}$ with $D \geq 0$,

$$\Gamma_1 I - \left(G k_v G^{\top} + \frac{1}{2} D M^{-1} M_d + \frac{1}{2} M_d M^{-1} D \right) > 0.$$

Increasing Γ_1 in (23) results in a slower response and a reduced control action, see [24]. Both controllers require $k_v > 0$, while decreasing k_v results in faster response.

5. Interpretation as low-pass filter

The interpretation of the dynamic extensions as low-pass filters, initially introduced in [25], is expanded here to account for unmatched disturbances, and the key differences are highlighted. Recall that a first-order low-pass filter with input x , output z , and bias Δ , is defined as

$$\dot{z} = \frac{1}{\tau} (x - z) + \Delta(t),$$

where $\tau > 0$ is the time constant. Defining $\eta_1 = (\zeta_1 - \Psi_1)$ and $\eta_2 = (\zeta_2 - \Psi_2)$ and computing their time-derivatives along the trajectories of the closed-loop system (20) yields

$$\dot{\eta}_1 = \dot{\zeta}_1 - (\partial_q \Psi_1)^{\top} \dot{q} - (\partial_q \Psi_1)^{\top} \ddot{q},$$

$$\dot{\eta}_2 = \dot{\zeta}_2 - (\partial_q \Psi_2)^{\top} \dot{q} - (\partial_q \Psi_2)^{\top} \ddot{q}.$$

Substituting $\dot{\zeta}_1$ and $\dot{\zeta}_2$ from 19(a) and 19(b), and $\ddot{q}$ from (A2) in the above expressions yields

$$\underbrace{\dot{\eta}_1}_{z_1} = \underbrace{(\partial_q \Psi_1)^{\top} M_c^{-1} (\partial_q \Psi_1) K_I}_{1/\tau_{C1}} \underbrace{\left(\frac{\delta_1}{K_I} - \eta_1 \right)}_{x_1 - z_1} - \underbrace{(\partial_q \Psi_1)^{\top} \dot{q} + (\partial_q \Psi_1)^{\top} M_c^{-1} (\partial_q \Psi_2) K_I \left(\frac{\delta_2 - \partial_{\zeta_2} \Phi}{K_I} - \eta_2 \right)}_{\Delta_{C1}}, \quad (26a)$$

$$\underbrace{\dot{\eta}_2}_{z_2} = \underbrace{(\partial_q \Psi_2)^{\top} M_c^{-1} (\partial_q \Psi_2) K_I}_{1/\tau_{C2}} \underbrace{\left(\frac{\delta_2 - \partial_{\zeta_2} \Phi}{K_I} - \eta_2 \right)}_{x_2 - z_2} - \underbrace{(\partial_q \Psi_2)^{\top} \dot{q} + (\partial_q \Psi_2)^{\top} M_c^{-1} (\partial_q \Psi_1) K_I \left(\frac{\delta_1}{K_I} - \eta_1 \right)}_{\Delta_{C2}}, \quad (26b)$$

which are representative of first-order low-pass filters with bias Δ_{C1} and Δ_{C2} respectively. In particular, the outputs η_1 and η_2 follow the inputs δ_1/K_I and $(\delta_2 - \partial_{\zeta_2} \Phi)/K_I$ respectively. Differently from [25], the bias terms, Δ_{C1} and Δ_{C2} , contain a velocity-dependent part and a coupling term. It follows from (18) that the coupling terms vanish if M and M_c are diagonal matrices, which yields $(\partial_q \Psi_2)^{\top} M_c^{-1} (\partial_q \Psi_1) = 0$. Differently from the interpretation provided for ζ_1 in [25], the expressions 26(a) and 26(b) include the term $\partial_{\zeta_2} \Phi$, which is due to the unmatched disturbances. It follows from Proposition 1 that the trajectories of the closed-loop system (19a), (19b) and (20) converge asymptotically to the set $\dot{q} = 0 \cap \partial_q \Theta_1 + \partial_q \Theta_2 + \partial_q \Phi = 0$. Evaluating 26(a) and 26(b) within this set yields $\dot{\eta}_1 = \dot{\eta}_2 = 0$. Since $\partial_{\zeta_2} \Phi = 0$ at $q = q^*$ by design, η_1 and η_2 converge to δ_1/K_I and δ_2/K_I provided that $h_1(q^*) \neq 0, h_2(q^*) \neq 0$.

For comparison purposes, the analysis is repeated for the iIDA-PBC design [24] yielding

$$\dot{\eta}_1 = \dot{\zeta}_1 - (\partial_q \bar{\Psi}_1)^\top M^{-1} p - (\partial_p \bar{\Psi}_1)^\top \dot{p},$$

$$\dot{\eta}_2 = \dot{\zeta}_2 - (\partial_q \bar{\Psi}_2)^\top M^{-1} p - (\partial_p \bar{\Psi}_2)^\top \dot{p},$$

where

$$\begin{aligned} \dot{p} = & -M_d M^{-1} \left(\partial_q H_d + \partial_q \Phi + \partial_q \bar{\Psi}_2 \partial_{\zeta_2} \Phi \right) - S_{22} M_d^{-1} p - S_{22} (\partial_p \Phi + \partial_p \bar{\Psi}_2 \partial_{\zeta_2} \Phi) + \Gamma_1 \partial_p \bar{\Psi}_2 \partial_{\zeta_2} \Phi \\ & + \partial_p \bar{\Psi}_1 \Gamma_1 k_1 \left(\zeta_1 - \bar{\Psi}_1 - \frac{\delta_1}{\Gamma_1 k_1} \right) + \partial_p \bar{\Psi}_2 \Gamma_1 k_2 \left(\zeta_2 - \bar{\Psi}_2 - \frac{\delta_2}{\Gamma_1 k_2} \right). \end{aligned}$$

Substituting $\dot{\zeta}_1$ and $\dot{\zeta}_2$ from 24(a) and 24(b) while refactoring terms yields

$$\underbrace{\dot{\eta}_1}_{\dot{z}_1} = \underbrace{(\partial_p \bar{\Psi}_1)^\top (\partial_p \bar{\Psi}_1) \Gamma_1 k_1 \left(\frac{\delta_1}{\Gamma_1 k_1} - \eta_1 \right)}_{1/\tau_{I1}} - \underbrace{(\partial_p \bar{\Psi}_1)^\top (\Gamma_1 I - (S_{22} + S_{22}^\top)) M_d^{-1} p}_{\Delta_{I1}} \quad (27a)$$

$$\underbrace{\dot{\eta}_2}_{\dot{z}_2} = \underbrace{(\partial_p \bar{\Psi}_2)^\top (\partial_p \bar{\Psi}_2) \Gamma_1 k_2 \left(\frac{\delta_2 - \partial_{\zeta_2} \Phi}{\Gamma_1 k_2} - \eta_2 \right)}_{1/\tau_{I2}} - \underbrace{(\partial_p \bar{\Psi}_2)^\top (\Gamma_1 I - (S_{22} + S_{22}^\top)) M_d^{-1} p}_{\Delta_{I2}} \quad (27b)$$

which are similar to Eqn. 26(a) and Eqn. 26(b) (i.e., the input x_2 includes the term $\partial_{\zeta_2} \Phi$), but have a different parameterization and different bias terms. In particular, Δ_{I1} and Δ_{I2} do not contain coupling terms, since it follows from (23) that $(\partial_p \bar{\Psi}_2)^\top \partial_p \bar{\Psi}_1 = 0$. In summary, the PCH formulation employed for (27(a) and 27(b) provides a more intuitive interpretation of the dynamic extensions as low-pass filters (i.e., the two filters are decoupled) compared to 26(a) and 26(b).

If $h_1(q) \neq 0, h_2(q) \neq 0$ globally, it follows that

$$\begin{aligned} 1/\tau_{C1} &= (\partial_q \bar{\Psi}_1)^\top M_c^{-1} (\partial_q \bar{\Psi}_1) K_I > 0, & 1/\tau_{C2} &= (\partial_q \bar{\Psi}_2)^\top M_c^{-1} (\partial_q \bar{\Psi}_2) K_I > 0, \quad \forall q, \\ 1/\tau_{I1} &= (\partial_p \bar{\Psi}_1)^\top (\partial_p \bar{\Psi}_1) k_1 \Gamma_1 > 0, & 1/\tau_{I2} &= (\partial_p \bar{\Psi}_2)^\top (\partial_p \bar{\Psi}_2) k_2 \Gamma_1 > 0, \quad \forall q. \end{aligned}$$

Increasing K_I or k_1, k_2 and Γ_1 scales down the input and increases the corner frequency of the low-pass filters yielding a faster convergence of the outputs η_1 and η_2 . In practice, the outputs continue updating until the bias terms vanish.

6. Simulation results

6.1. Inertia wheel pendulum

The inertia-wheel pendulum (IWP) consists of an unactuated planar inverted pendulum with an actuated wheel at the tip, see Fig. 1. The system has two DOF: the angular position of the pendulum q_1 , and the angular position of the wheel q_2 . The system dynamics is given by (2) where

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad M = \begin{bmatrix} a_1 + a_2 & a_2 \\ a_2 & a_2 \end{bmatrix}, \quad V = a_3 (\cos(q_1) + 1), \quad \mathcal{R}(\dot{q}) = 0,$$

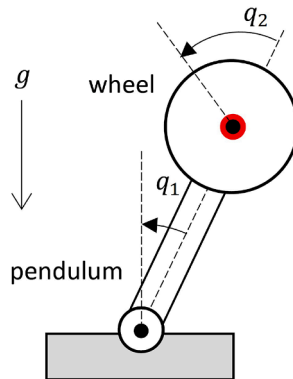


Fig. 1. Schematic of the IWP system.

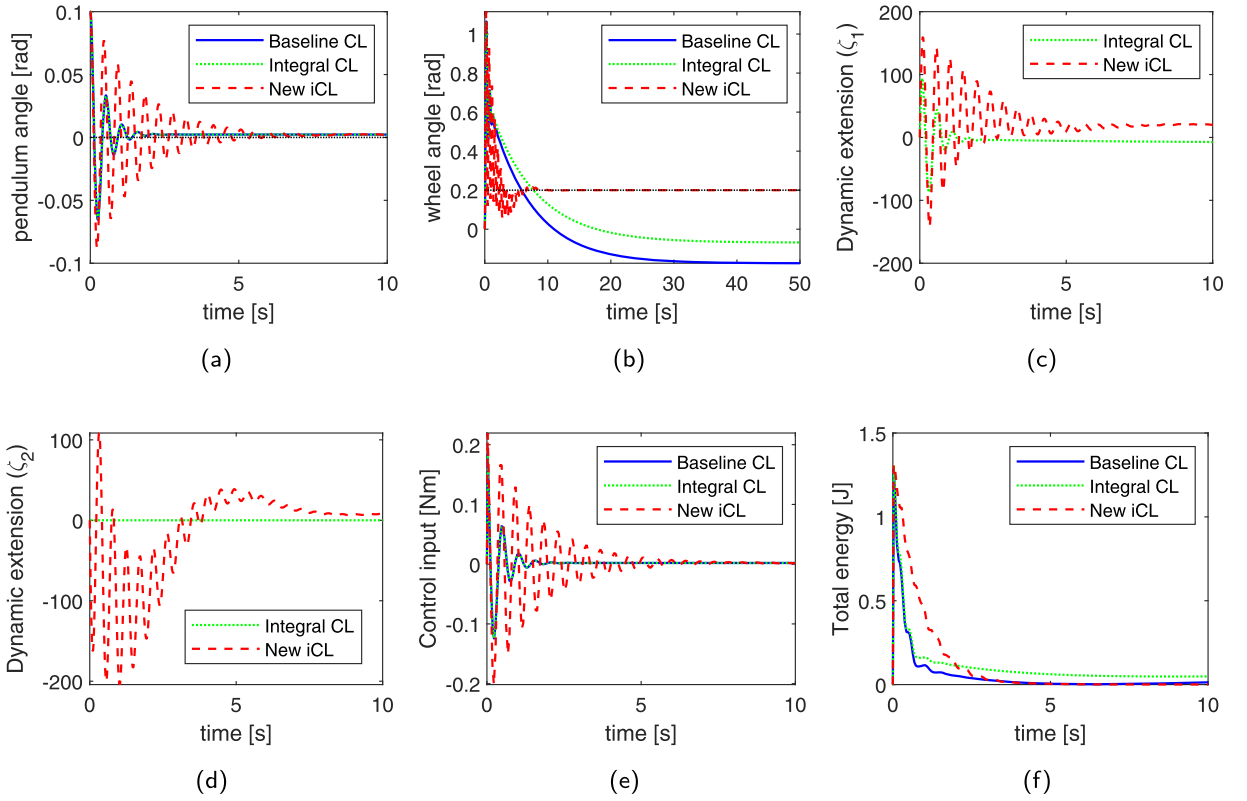


Fig. 2. Simulation results for IWP with matched and unmatched disturbances: (a) pendulum angle q_1 ; (b) wheel angle q_2 ; (c) extended state ζ_1 ; (d) extended state ζ_2 ; (e) control input u ; (f) storage function W .

and a_1, a_2, a_3 are constant parameters that depend on the size of the pendulum and of the wheel [2]. In the absence of disturbances, the control goal consists in reaching the prescribed position $(q_1, q_2) = (0, q_2^*)$, corresponding to the upright pendulum. The matrix M_c and the potential energy V_c , where k_p, ϵ, m_1, m_3 are tuning parameters, are given by

$$V_c = -a_3 \gamma_1 \cos(q_1) + \frac{1}{2} k_p (q_2 + \gamma_1 \epsilon (a_1 + a_2) q_1)^2, \quad \gamma_1 = \frac{1}{a_2(m_2 - m_1)}, \quad m_2 = m_1 \frac{a_2}{a_1} + \epsilon,$$

$$M_c = \frac{1}{(a_1 - a_2)(m_1 m_3 - m_2^2)} \begin{bmatrix} m_3 a_1^2 - 2m_2 a_1 a_2 + m_1 a_2^2 & a_2(m_1 - m_2) + a_1(m_3 - m_2) \\ a_2(m_1 - m_2) + a_1(m_3 - m_2) & a_2(m_1 - 2m_2 + m_3) \end{bmatrix},$$

and solve the PDEs (5) and (6), see [25]. A different parameterization of the system was obtained in [28] by employing structure-preserving transformations, which could result in different expressions of the PDEs (5) and (6) and corresponding solutions. Differently from [25], which only considered matched disturbances, the system is subjected to matched and unmatched disturbance, that is $\delta = \delta_1 G + \delta_2 G^\perp$, where G and $G^\perp$ are constant. The assignable equilibrium is $(q_1, q_2) = (q_1^*, q_2^*)$, where $a_3 \sin(q_1^*) = \delta_2$, and it exists provided that $|\delta_2| \leq a_3$. To implement the controller (18), q_1^* is computed from the equation

$$\sin(q_1^*) = \frac{1}{a_3} K_I (\zeta_2 - \Psi_2),$$

where $\Psi_1 = G^T M^{-1} M_c \dot{q}$ and $\Psi_2 = G^\perp M^{-1} M_c \dot{q}$. The scalar function $\Phi(q, \dot{q}, \zeta_2)$ has been computed according to the procedure outlined in Remark 1 and is given in Appendix B. This yields a global solution of (17a) to (17d) since M, M_c, G and h_2 are constant.

Simulations have been performed in MATLAB using an ODE23 solver with the model parameters $a_1 = 0.0124, a_2 = 0.0025, a_3 = 0.4446$ for illustrative purposes. The tuning parameters have been set as $k_p = 1, k_v = 0.0005, m_1 = 0.4, m_3 = 5, \epsilon = 1.08$ and $K_I = 0.0001$. The initial conditions are $(q_1, q_2, \dot{q}_1, \dot{q}_2, \zeta_1, \zeta_2) = (0.1, 0, 0, 0, 0, 0)$, and the prescribed equilibrium position is $q_2^* = 0.2$. Fig. 2 shows the system response with $\delta_1 = 0.002$ and $\delta_2 = 0.001$. Using the new iCL controller (18) with the dynamic extensions (19a) and (19b), the position reaches the assignable equilibrium $(q_1^*, q_2^*) = (0.002, 0.2)$. The extended states ζ_1 and ζ_2 converge to $\delta_1/K_I = 20$ and $\delta_2/K_I = 10$ respectively. Instead, either employing the integral CL [25] or the baseline CL (4) yields a large steady-state error on the wheel angle. The same control problem can be approached with the iIDA-PBC design [24], which yields similar performance.

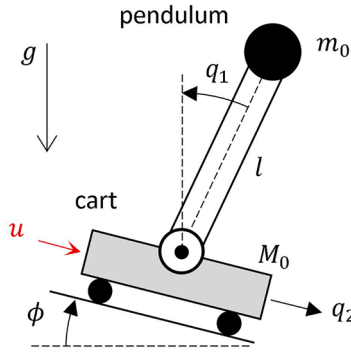


Fig. 3. Schematic of the POC system on a slope.

6.2. Pendulum-on-cart

The pendulum-on-cart (POC) consists of a pendulum of length l with a point mass m_0 at the tip, mounted on an actuated cart of mass M_0 , see Fig. 3. The cart moves on a slope of inclination ϕ , see [4]. The system has 2 DOF and the equations of motion are given by (2) with

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad M = \begin{bmatrix} m_0 l^2 & m_0 l \cos(q_1 - \phi) \\ m_0 l \cos(q_1 - \phi) & M_0 + m_0 \end{bmatrix},$$

$$V = m_0 g l \cos(q_1) - (m_0 + M_0) g q_2 \sin(\phi), \quad \mathcal{R}(\dot{q}) = 0.$$

The angle q_1 of the unactuated pendulum is measured from the vertical, while the position q_2 of the actuated cart is measured from an arbitrary origin. The matrix M_c and the potential energy V_c , where ϵ, k, ρ are tuning parameters, are

$$M_c = \begin{bmatrix} -\rho(M_0 + m_0) & -\gamma_1 \\ -\gamma_1 & -m_0 l^2 - \frac{m_0^2 l^2}{m_0 + M_0} \gamma_2 \end{bmatrix},$$

$$\gamma_1 = m_0 l \rho (k + 1) \cos(\phi - q_1), \quad \gamma_2 = (k + k^2 + (\rho - 1)(k + 1)^2) \cos^2(\phi - q_1),$$

$$V_c = g(1 - \cos(q_1)) + \frac{\epsilon m_0 g l}{2 m_0 l} (M_0 + m_0)^2 \left(q_2 + \left(k + \frac{\rho - 1}{\rho} \right) \frac{m_0 l}{M_0 + m_0} (\sin(q_1 - \phi) + \sin(\phi)) \right)^2,$$

which are a specific solution of the PDEs (5) and (6), see [4]. While other solutions are possible, this particular one is used here to highlight that the new iCL controller (18) is modular with respect to the baseline CL (4). For illustrative purposes, the system is subjected to matched and unmatched disturbance, that is $\delta = \delta_1 q_2 G + \delta_2 G^\perp$, where G and $G^\perp$ are constant. The assignable equilibrium is $(q_1^*, q_2^*) = (q_1^*, q_2^*)$, where $m_0 g l \sin(q_1^*) = \delta_2$, and it exists provided that $|\delta_2| \leq m_0 g l$. To implement the controller (18), q_1^* is computed from the equation

$$\sin(q_1^*) = \frac{1}{m_0 g l} K_I (\zeta_2 - \Psi_2),$$

where $\Psi_1 = q_2 G^\top M^{-1} M_c \dot{q}$ and $\Psi_2 = G^\perp M^{-1} M_c \dot{q}$. The scalar function $\Phi(q, \dot{q}, \zeta_2)$ has been computed according to the procedure outlined in Remark 1 and is given in Appendix C.

Simulations have been performed in MATLAB with an ODE23 solver using the parameters $M_0 = 0.44, m_0 = 0.144, l = 0.215, \phi = \pi/9$ and the initial conditions $(q_1, q_2, \dot{q}_1, \dot{q}_2, \zeta_1, \zeta_2) = (\pi/6, 1, 0, 0, 0, 0)$. The tuning parameters have been set as $k = 20, \rho = -0.02, \epsilon = 0.00001, k_v = 0.5$, and $K_I = 0.1$ for illustrative purposes. The prescribed equilibrium position is $q_2^* = 0$. Fig. 4 shows the system response with $\delta_1 = 0.002$ and $\delta_2 = 0.001$. Using the new iCL controller (18) with the dynamic extensions (19a) and (19b), the position reaches the assignable equilibrium $(q_1^*, q_2^*) = (0.003, 0)$. Note that the control input converges to a negative value because of the slope ϕ . Instead, either employing the integral CL [25] or the baseline CL (4) yields a large steady-state error on the cart position. The same control problem can be approached with the iIDA-PBC design [24]. However, for the POC system, the iIDA-PBC requires a prior partial feedback-linearization, see [29]. Instead, the proposed controller (18) is directly implementable since it builds upon the baseline CL design [4] in a modular fashion.

6.3. Vertical-take-off-and-landing aircraft

The vertical-take-off-and-landing aircraft (VTOL) is characterized by two actuators, that is (u_1, u_2) , and three DOF, that is the horizontal and vertical coordinates of the center of mass (x, y) and the roll angle θ , see Fig. 5. The mass and the moment of inertia of the VTOL are denoted with m_0 and J respectively. The parameter $0 \leq \epsilon \leq 1$ captures the effect of the "slopped" wings, inducing a

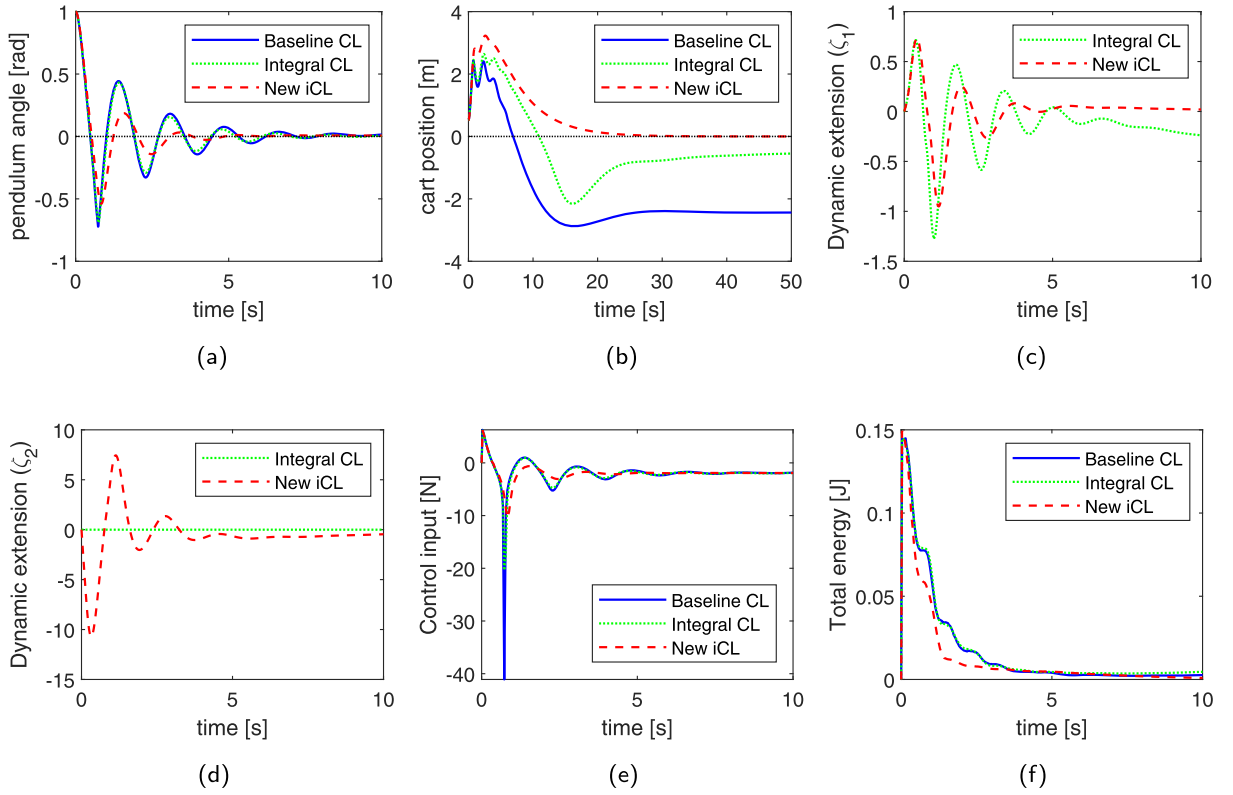


Fig. 4. Simulation results for POC with matched and unmatched disturbances: (a) pendulum angle q_1 ; (b) cart position q_2 ; (c) extended state ζ_1 ; (d) extended state ζ_2 ; (e) control input u ; (f) storage function W .

coupling between the vertical and the roll dynamics. The equations of motion of the simplified system, as detailed in [30], are given by (2) with

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \quad G = \begin{bmatrix} -\sin(q_3) & \epsilon \cos(q_3) \\ \cos(q_3) & \epsilon \sin(q_3) \\ 0 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} m_0 & 0 & 0 \\ 0 & m_0 & 0 \\ 0 & 0 & J \end{bmatrix}, \quad V = m_0 g q_2, \quad \mathcal{R}(\dot{q}) = 0,$$

where $(q_1, q_2, q_3) = (x, y, \theta)$. This is a first approximation of the complex dynamics of an aircraft, and it has been used extensively for control purposes, see [17,29]. The matrix M_c and the potential energy V_c that solve the PDEs (5) and (6) are given by

$$M_c = \begin{bmatrix} m_{11} & 0 & \sigma_2 \\ 0 & m_{22} & \sigma_1 \\ \sigma_2 & \sigma_1 & \frac{J\epsilon^2\gamma_{33} + \gamma_{11}(\gamma_{11}m_{11} - m_0)}{\epsilon^2\gamma_{33}^2} \end{bmatrix}, \quad \sigma_1 = \frac{(\gamma_{11}m_{22} - m_0)}{\epsilon\gamma_{33}} \sin(q_3), \quad \sigma_2 = \frac{(\gamma_{11}m_{11} - m_0)}{\epsilon\gamma_{33}} \cos(q_3),$$

$$V_c = \frac{k_p}{2} \left(q_1 - q_1^* + \frac{\gamma_{11} \sin(q_3)}{\epsilon\gamma_{33}} \right)^2 + \frac{k_p}{2} \left(q_2^* - q_2 + \frac{\gamma_{22}(\cos(q_3) - 1)}{\epsilon\gamma_{33}} \right)^2 + \frac{gm_0(\cos(q_3) - 1)}{\epsilon\gamma_{33}},$$

where $k_p, m_{11}, m_{22}, \gamma_{11}, \gamma_{22}, \gamma_{33}$ are tuning parameters, see [17].

For illustrative purposes, the system is subjected to matched and unmatched disturbances, that is $\delta = \delta_1 G + \delta_2 G^{\perp}$, where G and $G^{\perp}$ are state-dependent. The assignable equilibrium is (q_1^*, q_2^*, q_3^*) , where $-m_0 g \sin(q_3^*) = (\epsilon^2 + 1)\delta_2$, and it exists provided that $|\delta_2| \leq m_0 g / (\epsilon^2 + 1)$. To implement the controller (18), q_3^* is computed from the equation

$$\sin(q_3^*) = -\frac{1}{m_0 g} (\epsilon^2 + 1) K_I (\zeta_2 - \Psi_2),$$

where $\Psi_1 = G^T M^{-1} M_c \dot{q}$ and $\Psi_2 = G^{\perp} M^{-1} M_c \dot{q}$. The scalar function $\Phi(q, \dot{q}, \zeta_2)$ has been computed according to the procedure outlined in Remark 1 and is given in Appendix D.

Simulations have been performed in MATLAB with an ODE23 solver using the same model parameters and controller parameters as [17], that is $\epsilon = 1, m_0 = 1, J = 1$, and $k_p = 0.9, k_v = 3.8I, \gamma_{11} = \gamma_{22} = 3, \gamma_{33} = -1, m_{11} = m_{22} = 1.5$ and $K_I = 0.1$. The initial conditions are $(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, \zeta_1, \zeta_2) = (-5, 5, 0.1, -0.1, -0.1, 0.1, 0, 0)$. Fig. 6 shows the system response with $\delta_1 = 0.5$ and $\delta_2 = -0.2$. Employing

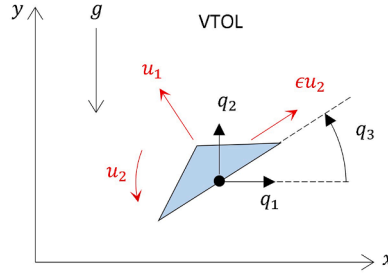


Fig. 5. Schematic of the VTOL system with $(q_1, q_2, q_3) = (x, y, \theta)$.

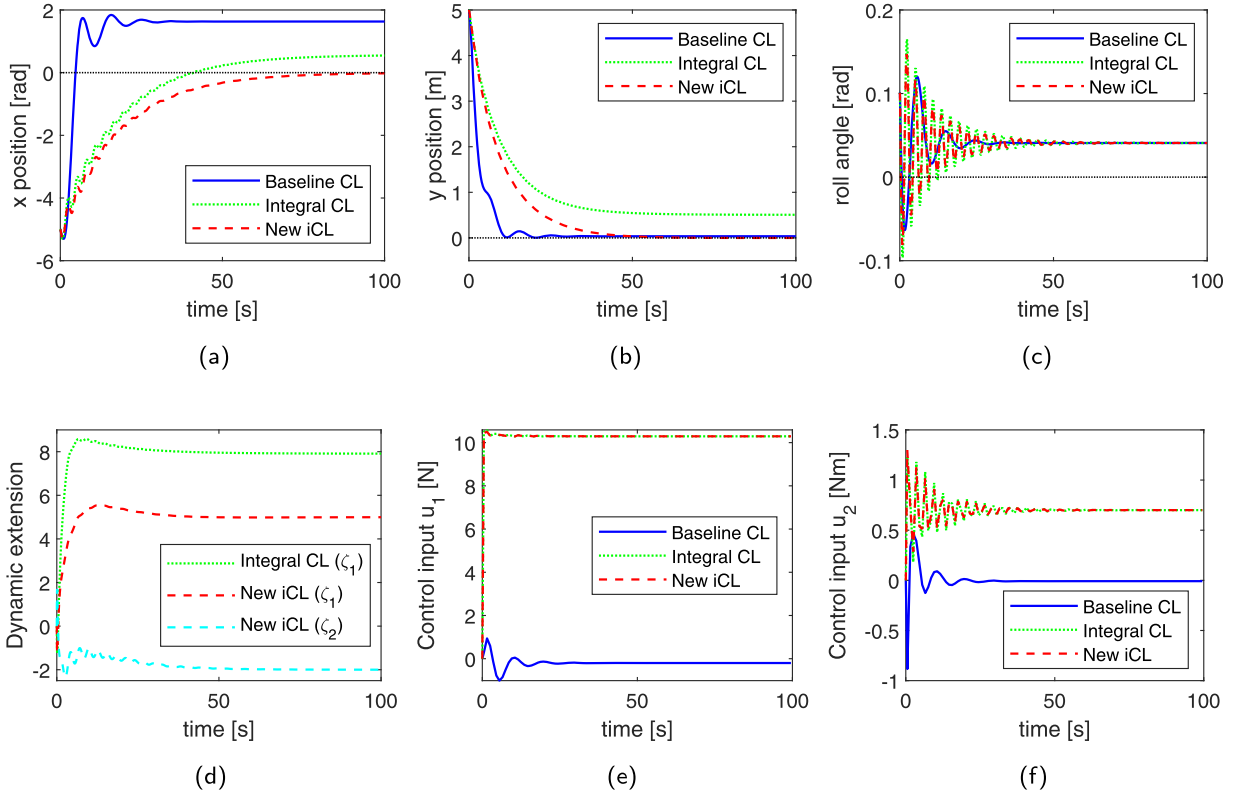


Fig. 6. Simulation results for VTOL with matched and unmatched disturbances: (a) x position; (b) y position; (c) roll angle θ ; (d) extended states ζ_1 and ζ_2 ; (e) control input u_1 ; (f) control input u_2 .

the new iCL controller (18), the position reaches the assignable equilibrium $(q_1^*, q_2^*, q_3^*) = (0, 0, 0.04)$. In addition, the extended states ζ_1 and ζ_2 converge to $\delta_1/K_I = 5$ and $\delta_2/K_I = -2$ respectively. In comparison, the baseline CL controller (4) and the integral CL [25] yield steady-state errors on the position (x, y) .

Fig. 7 shows the system response with the same disturbances while employing different values of the parameter k_v . Note that decreasing k_v yields a faster convergence of the position to the prescribed equilibrium. Conversely, the response of the extended states shows less noticeable variations. Fig. 8 shows the system response with the same disturbances while employing the iIDA-PBC design [24] with $\Gamma_1 = 10k_v$ and the same values of the remaining tuning parameters. Also in this case, decreasing k_v yields a faster convergence of the position to the prescribed equilibrium. However, the responses of q_1 and q_2 follow different transients, which is a consequence of the different parameterization employed in (18) and in [24]. A faster response can be obtained with the iIDA-PBC design by decreasing Γ_1 , provided that the condition $\Gamma_1 > k_v$ is verified, see [24]. Note finally that implementing the iIDA-PBC design [24] for the VTOL requires an input transformation, see [29]. Instead, the new controller (18) builds upon the baseline CL [17] which does not require input transformations.

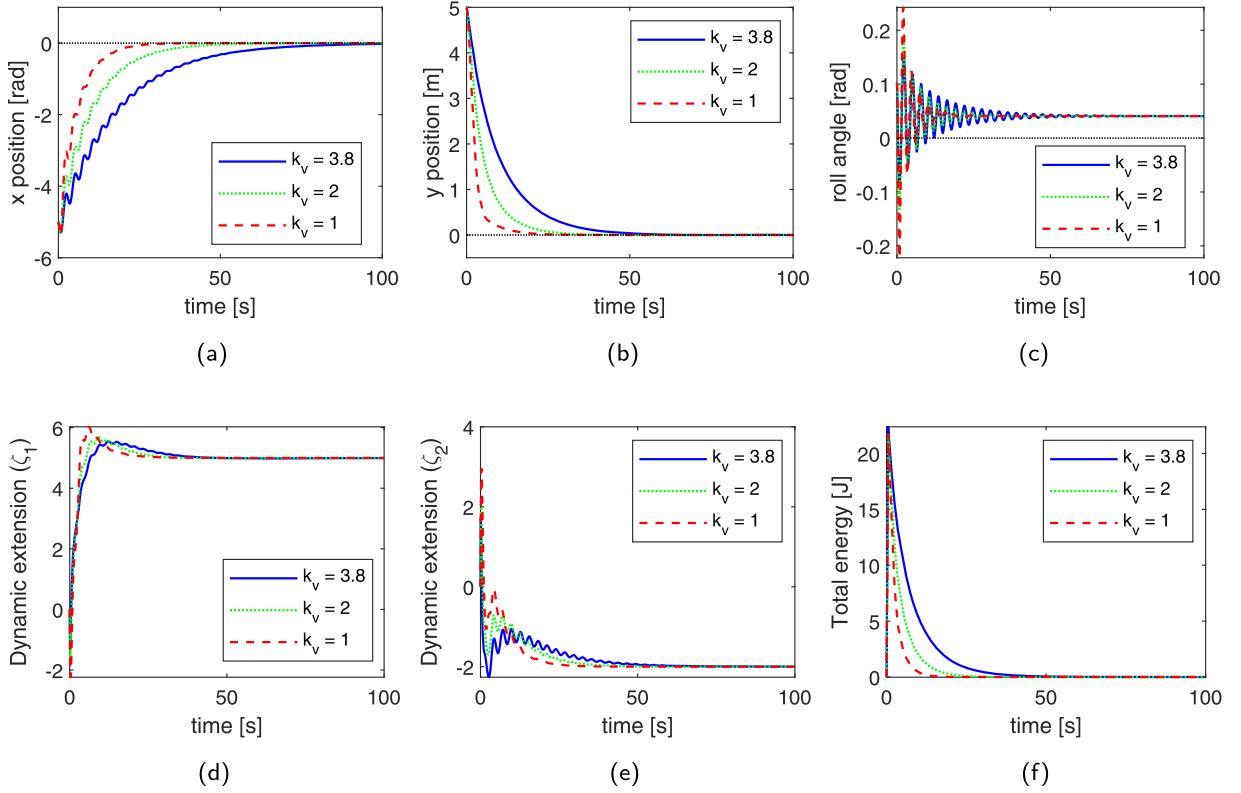


Fig. 7. Simulation results for VTOL with the new iCL controller (18) and different values of k_v : (a) x position; (b) y position; (c) roll angle θ ; (d) extended state ζ_1 ; (e) extended state ζ_2 ; (f) storage function W .

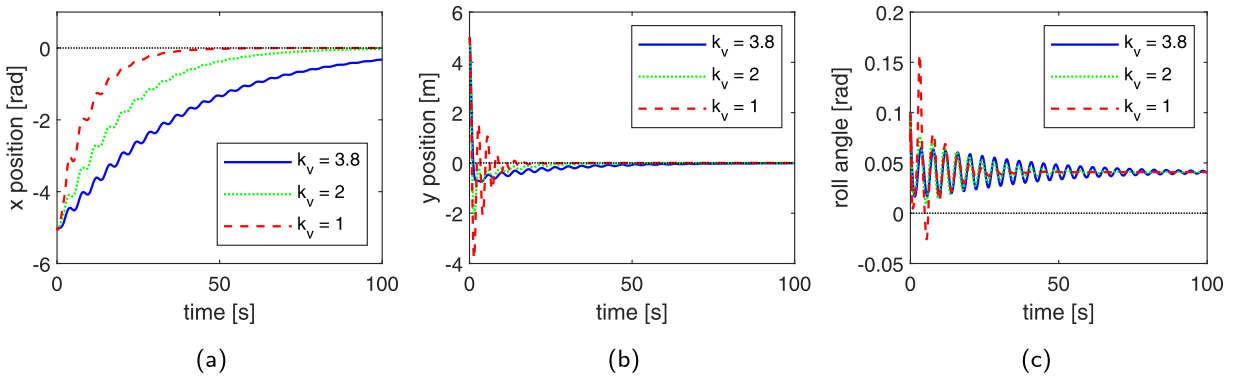


Fig. 8. Simulation results for VTOL with the iIDA-PBC controller [24]: (a) x position; (b) y position; (c) roll angle θ .

7. Conclusion

In this work we have presented a new passivity-preserving extension of the CL methodology for underactuated mechanical systems subject to a class of matched and unmatched disturbances. The new iCL controller requires solving an additional set of PDEs, which are similar but in general not identical to those resulting from the iIDA-PBC methodology. In particular, the two sets of PDEs become identical if the controllers only enforce potential energy shaping, in which case the CL design coincides with the corresponding IDA-PBC. While solving these PDEs represents a bottleneck of the proposed approach, a strategy for finding a local solution at the prescribed equilibrium has been outlined. The new iCL controller design bears similarity with the corresponding iIDA-PBC procedure, where the latter provides a more intuitive interpretation in terms of system energy. Both the new iCL controller and the corresponding iIDA-PBC offer an interpretation of the dynamic extensions as low-pass filter. Again, the iIDA-PBC design provides a clearer interpretation, without coupling terms. Simulation results on three examples demonstrate the effectiveness of the new iCL controller in accounting for unmatched disturbances. For the POC and the VTOL systems, the new iCL design can be readily applied without a prior change

of input or a partial feedback-linearization, which are instead required for the iIDA-PBC design. In this respect, the new iCL design represents an alternative to the corresponding iIDA-PBC which might be more convenient in some specific cases. Future work will investigate strategies for solving the PDEs in both formulations.

CRedit authorship contribution statement

Enrico Franco: Writing - review & editing, Writing - original draft, Visualization, Software, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

This research was supported by the [Engineering and Physical Sciences Research Council](#) (grant agreement No [EP/X033546/1](#)).

Appendix A

Proof of Proposition 1. To prove the first claim, start by substituting (1) into (2) and compute the acceleration $\ddot{q}$, which yields

$$\ddot{q} = M^{-1} \left(Gu - \delta_1 G G^T h_1(q) - \delta_2 G^{\perp T} G^{\perp} h_2(q) - \partial_q V - \partial_q \mathcal{R} - \partial_q (M \dot{q}) \dot{q} + \partial_q \left(\frac{1}{2} \dot{q}^T M \dot{q} \right) \right). \quad (A1)$$

Computing the acceleration $\ddot{q}$ from (20) yields instead

$$\ddot{q} = -M_c^{-1} \left(\partial_q V_c + \partial_q \Phi + \partial_q \Phi + \partial_q \mathcal{R}_c + \partial_q \Theta_1 + \partial_q \Theta_2 + \partial_{\zeta_2} \Phi \partial_q \Psi_2 + \partial_q (M_c \dot{q}) \dot{q} - \partial_q \left(\frac{1}{2} \dot{q}^T M_c \dot{q} \right) \right). \quad (A2)$$

Equating (A1) and (A2) while substituting Θ_1, Θ_2 and $\mathcal{R}_c$ yields

$$\begin{aligned} M^{-1} \left(Gu - \delta_1 G G^T h_1(q) - \delta_2 G^{\perp T} G^{\perp} h_2(q) - \partial_q V - \partial_q \mathcal{R} - \partial_q (M \dot{q}) \dot{q} + \partial_q \left(\frac{1}{2} \dot{q}^T M \dot{q} \right) \right) = \\ -M_c^{-1} \left(\partial_q V_c + \partial_q \Phi + \partial_q \Phi + M_c M^{-1} (\partial_q \mathcal{R} + G k_v G^T M^{-1} M_c \dot{q}) + \partial_{\zeta_2} \Phi \partial_q \Psi_2 + \partial_q (M_c \dot{q}) \dot{q} - \partial_q \left(\frac{1}{2} \dot{q}^T M_c \dot{q} \right) \right) \\ + M^{-1} G G^T h_1(q) K_I \left(\zeta_1 - \Psi_1 - \frac{\delta_1}{K_I} \right) + M^{-1} G^{\perp T} G^{\perp} h_2(q) K_I \left(\zeta_2 - \Psi_2 - \frac{\delta_2}{K_I} \right). \end{aligned} \quad (A3)$$

C1. Refactoring terms in (A3) cancels the parameters δ_1, δ_2 and the physical damping $\partial_q \mathcal{R}$. Premultiplying (A3) with $G^T M$ yields (18). Premultiplying (A3) with $G^{\perp} M$ yields the sum of the PDEs (5), (6), and (17a), which are verified by design.

To prove the remaining claims, define the storage function

$$W(q, \dot{q}, \zeta_1, \zeta_2) = \frac{1}{2} \dot{q}^T M_c \dot{q} + V_c + \Phi + \Theta_1 + \Theta_2 > 0, \quad (A4)$$

and compute its time-derivative along the trajectories of the closed-loop system 19(a), 19(b) and (20), which yields

$$\begin{aligned} \dot{W} = \partial_q \left(\frac{1}{2} \dot{q}^T M_c \dot{q} \right)^T \dot{q} + \partial_q \left(\frac{1}{2} \dot{q}^T M_c \dot{q} \right)^T \ddot{q} + (\partial_q V_c)^T \dot{q} + (\partial_q \Phi)^T \dot{q} + (\partial_q \Phi)^T \ddot{q} \\ + (\partial_q \Theta_1 + \partial_q \Theta_2)^T \dot{q} + (\partial_q \Theta_1 + \partial_q \Theta_2)^T \ddot{q} + (\partial_{\zeta_1} \Theta_1)^T \zeta_1 + (\partial_{\zeta_2} \Theta_2 + \partial_{\zeta_2} \Phi)^T \zeta_2. \end{aligned} \quad (A5)$$

Substituting $\ddot{q}$ from (A2), ζ_1, ζ_2 from (19a), (19b), and (17b) while noting that $\partial_q \Theta_1 = -\partial_{\zeta_1} \Theta_1 \partial_q \Psi_1$, $\partial_q \Theta_2 = -\partial_{\zeta_2} \Theta_2 \partial_q \Psi_2$ and $\partial_q \Theta_1 = -\partial_{\zeta_1} \Theta_1 \partial_q \Psi_1$, $\partial_q \Theta_2 = -\partial_{\zeta_2} \Theta_2 \partial_q \Psi_2$ yields finally

$$\dot{W} = -\dot{q}^T \partial_q \mathcal{R}_c - (\partial_q \Theta_1 + \partial_q \Theta_2 + \partial_q \Phi)^T M_c^{-1} (\partial_q \Theta_1 + \partial_q \Theta_2 + \partial_q \Phi). \quad (A6)$$

C2. It follows from (A6) that $\dot{W} \leq -\dot{q}^T \partial_q \mathcal{R}_c \leq y_c^T u_{DI} \leq 0$ and the controller is passivity-preserving for all $K_I > 0$.

C3. If $\dot{q}^T \partial_q \mathcal{R}_c > 0$, it follows from (A6) that $\dot{W} \leq 0$ thus the equilibrium is stable and all states are bounded. Invoking LaSalle's theorem (see Theorem 3.4 in [31]) proves that the trajectories of the closed-loop system 19(a), 19(b) and (20) converge asymptotically to the set $\dot{q} = 0 \cap \partial_q \Theta_1 + \partial_q \Theta_2 + \partial_q \Phi = 0$. Evaluating (20) within this set and noting that $\partial_q \Psi_1 = \partial_q \Psi_2 = 0$ if $\dot{q} = 0$ yields $\partial_q V_c + \partial_q \Phi = 0$ thus it follows from Assumption 3 that the equilibrium $(q, \dot{q}) = (q^*, 0)$ is asymptotically stable.

If $\dot{q}^T \partial_q \mathcal{R} = 0$, the trajectories of the closed-loop system (19a), (19b) and (20) converge asymptotically to the set $G^T M^{-1} M_c \dot{q} = 0 \cap \partial_q \Theta_1 + \partial_q \Theta_2 + \partial_q \Phi = 0$. Since the passive output $y_c = G^T M^{-1} M_c \dot{q}$ of (10) is detectable by hypothesis, evaluating (20) within this set yields again $\partial_q V_c + \partial_q \Phi = 0$, hence the equilibrium $(q, \dot{q}) = (q^*, 0)$ is asymptotically stable.

C4. Pre-multiplying $\partial_q \Theta_1 + \partial_q \Theta_2 + \partial_q \Phi = 0$ by M , substituting $\dot{q} = 0$, and refactoring terms yields the set

$$G G^T h_1(q) K_I \left(\zeta_1 - \frac{\delta_1}{K_I} \right) + G^{\perp T} G^{\perp} h_2(q) K_I \left(\zeta_2 - \frac{\delta_2 - \partial_{\zeta_2} \Phi}{K_I} \right) = 0,$$

where the two addends are linearly independent. Since $\partial_{\zeta_2} \Phi = 0$ at $q = q^*$ by design, it follows from LaSalle's theorem that ζ_1 and ζ_2 converge to δ_1/K_I and δ_2/K_I asymptotically, provided that $h_1(q^*) \neq 0, h_2(q^*) \neq 0$ $\square$

Appendix B

Scalar function Φ for the IWP.

The function Φ is given by (22) where

$$\kappa_3 = \kappa_5 = -G^{\perp \top} \gamma_1, \quad \kappa_4 = -k_p \gamma_1 (a_1 m_2 - a_2 m_1) q_1^* \begin{bmatrix} -\gamma_1 (a_1 m_2 - a_2 m_1) \\ 1 \end{bmatrix},$$

which, given that M, M_c, G and h_2 are constant, yields a global solution of (17a) to (17d), that is

$$\begin{aligned} \Phi = & \frac{K_I (q_1 - q_1^*)}{2a_2^2 (m_1 - m_2)^2} \left((q_1 - q_1^*) + 2a_2 (m_1 - m_2) \zeta_2 - \frac{2(\dot{q}_1 (a_1 m_3 - a_2 m_2) + \dot{q}_2 a_2 (m_3 - m_2))}{(a_1 - a_2)(m_1 m_3 - m_2^2)} \right) \\ & - \frac{k_p q_1^*}{a_2^2 (m_1 - m_2)} ((a_1 m_2^2 - a_2 m_1 (m_1 - m_2))((q_1 - q_1^*) - (q_2 - q_2^*)) - a_2 m_1 m_2 (q_1 - q_1^*)). \end{aligned}$$

Appendix C

Scalar function Φ for the POC.

The function Φ is given by (22) where

$$\kappa_3 = -G^{\perp \top}, \quad \kappa_5 = m_0 \kappa_3, \quad \kappa_4 = \frac{\epsilon g (l m_0 \sigma_1 (\sin(\phi - q_1^*) - \sin(\phi)) - \rho q_1^* M_T)}{\rho^2 l m_0} \begin{bmatrix} \rho M_T \\ l m_0 K \cos(\phi - q_1^*) \end{bmatrix},$$

which yields a local solution of (17a) to (17d), that is

$$\begin{aligned} \Phi = & \frac{(m_0 \rho - (q_1 - q_1^*)) \sigma_1}{\rho M_T} + \sigma_2 \sigma_3 \left(\frac{\rho(1+k)-1}{\rho M_T} - \frac{q_2 - q_2^*}{\cos(\phi - q_1^*) l m_0} \right), \quad \sigma_2 = \epsilon g \cos(\phi - q_1^*) M_T^2, \\ \sigma_1 = & K_I (\zeta_2 + \sigma_4) + \frac{K_I m_0}{2} (q_1 - q_1^*), \quad \sigma_3 = q_2^* - \frac{l m_0 (\sin(\phi - q_1^*) - \sin(\phi))(k+1-1/\rho)}{M_T}, \\ \sigma_4 = & \frac{l \dot{q}_1 (M_0 + m_0 (1 - \cos^2(\phi - q_1) (1+k-k(k+1)\rho))) + k \rho \dot{q}_2 \cos(\phi - q_1) M_T}{l M_0 + l m_0 \sin^2(\phi - q_1)}, \quad M_T = M_0 + m_0. \end{aligned}$$

Appendix D

Scalar function Φ for the VTOL.

The function Φ is given by (22) where $\gamma_{11} = \gamma_{22}$, $m_{11} = m_{22}$ and

$$\kappa_3 = \kappa_5 = \begin{bmatrix} 0 \\ 0 \\ -\frac{\epsilon^2+1}{\epsilon \gamma_{33}} \end{bmatrix}, \quad \kappa_4 = -k_p \begin{bmatrix} q_2^* - \sigma_1 \sin(q_3^*) \\ q_1^* + \sigma_1 (\cos(q_3^*) - 1) \\ \sigma_1 (q_2^* - \sigma_1 \sin(q_3^*)) \cos(q_3^*) + \sigma_1 (q_1^* + \sigma_1 (\cos(q_3^*) - 1)) \sin(q_3^*) \end{bmatrix}, \quad \sigma_1 = \frac{\gamma_{22}}{\epsilon \gamma_{33}},$$

which yields a local solution of (17a) to (17d), that is

$$\begin{aligned} \Phi = & k_p (q_1^* - \sigma_1 \sin(q_3^*)) ((q_1 - q_1^*) + \sigma_1 \cos(q_3^*)) - \frac{K_I \sigma_2 (1 + \epsilon^2 (q_3 - q_3^*))}{\epsilon \gamma_{33}} \\ & + \frac{K_I (\epsilon^2 + 1) (q_3 - q_3^*) (1 - (q_3 - q_3^*))}{2(\epsilon \gamma_{33})^2} + k_p (q_2^* + \sigma_1 (\cos(q_3^*) - 1)) ((q_2 - q_2^*) + (q_3 - q_3^*) \sigma_1 \sin(q_3^*)), \\ \sigma_2 = & \zeta_2 - \frac{1}{m_0} \sin(q_3) \left(m_{22} \dot{q}_2 - \frac{\dot{q}_3 \sin(q_3) (m_0 - \gamma_{22} m_{22})}{\epsilon \gamma_{33}} \right) - \frac{1}{m_0} \cos(q_3) \left(m_{22} \dot{q}_1 - \frac{\dot{q}_3 \cos(q_3) (m_0 - \gamma_{22} m_{22})}{\epsilon \gamma_{33}} \right) \\ & - \frac{1}{J} \left(\frac{\dot{q}_3 (\gamma_{22} (m_0 - \gamma_{22} m_{22}) - J \epsilon^2 \gamma_{33})}{\epsilon \gamma_{33}^2} + \frac{m_0 - \gamma_{22} m_{22}}{\gamma_{33}} (\dot{q}_1 \cos(q_3) + \dot{q}_2 \sin(q_3)) \right). \end{aligned}$$

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