

Dynamic Dual Boundary Element Analyses for Cracked Mindlin Plates

Jun Li ^{a,*}, Zahra Sharif Khodaei ^a, M. H. Aliabadi ^a

^a Department of Aeronautics, Imperial College London, South Kensington Campus,
City and Guilds Building, Exhibition Road, SW7 2AZ, London, UK.

* jun.li16@imperial.ac.uk

Abstract

In this paper, a new dual boundary element formulation is presented for dynamic crack problems in Mindlin plates. The displacement and traction boundary integral equations are derived in the Laplace frequency domain to allow for a boundary-only formulation. The cracked plate is modelled with the dual boundary element method and dynamic plate bending stress intensity factors are evaluated. Four benchmark examples are presented including mode I and mixed mode deformation. Such stress intensity factors obtained are shown to be in excellent agreement with finite element results as well as published results.

Keywords: Dynamic stress intensity factor; Mindlin plates; Boundary element method (BEM); Dual boundary element method; Fracture mechanics

1. Introduction

Dynamic fracture mechanics is concerned with crack propagation in materials when inertial effects influence the behaviour of the crack such as structures subjected to impact loads. The inertial effects cannot be disregarded as stress fields in the vicinity of the crack tip change rapidly. As in the quasi-static fracture mechanics problems, the dynamic stress intensity factor (DSIF) is the key fracture parameter controlling the crack propagation behaviours, including the onset of crack extension, propagation speed, path and arrest.

Freund [1] noted the difficulty in measuring DSIFs directly in the required short time period experimentally, so numerical analysis has since played an important role in dynamic fracture mechanics [2]. However, most of the research reported so far are related to two- and three-dimensional problems. The application of dynamic fracture mechanics to plate structures is important in many fields including aeronautics and civil engineering. Different approximate theories have been constructed to deal with plates subjected to different types of loads. Tension and bending are two common loads applied to plates. When a plate is subjected to tension, the problem can be simplified into a plane stress problem. When bending is applied to a plate, the

Kirchhoff-Love plate theory [3] and the Mindlin-Reissner plate theory [4] are commonly used. According to Sih [5], the Kirchhoff-Love theory in fracture mechanics is insufficient, even for thin plates, because the theory does not satisfy all physical boundary conditions on crack surfaces. In addition, Sih noted that when the characteristic length of an area near a crack tip was similar to plate thickness, shear forces should not be disregarded in this area; and to obtain accurate DSIFs, the Mindlin-Reissner plate theory should be used.

The boundary element method (BEM) [6] has enjoyed some popularity as an alternative method to the finite element method (FEM) [7] for dynamic fracture problems due to its inherent ability to accurately model elastic waves in the vicinity of the crack tip. A method of modelling coplanar crack surfaces is known as the multi-region method [8, 9], in which artificial interfaces are introduced to divide the original domain into two regions and each region includes one crack surface. However, this method is not efficient because the extra boundaries result in the increase of the numbers of algebraic equations and boundary elements.

In order to avoid introducing the extra interfaces, the dual boundary element method (DBEM) can be used to solve crack problems in a single region [10]. In this method, one crack surface is modelled using the displacement boundary integral equation, while the opposite surface is modelled with the traction boundary integral equation. Fedelinski *et al.* [11-13] and Wen *et al.* [14-16] used the dual boundary element method (DBEM) to model cracks and solved elastodynamic equations by using the time domain, the mass matrix and the Laplace transform methods for two- and three-dimensional problems, respectively.

Application of the BEM to plate bending problems is particularly attractive because the discretization is limited to using simple line elements as in the two-dimensional BEM. Useche *et al.* [17] carried out the harmonic analysis of cracked plates using a time-domain boundary element method and the multi-region method is used to model cracks. A displacement discontinuity method, a single-region method, was adopted by Wen and Aliabadi [18] to compute the DSIFs based on the Mindlin-Reissner plate theory. However, Wen's method can only solve problems of infinite plates as it did not include the influence of plate boundaries on DSIFs. Although the DBEM is promising for finite plate problems and it has been applied to solve static bending problems [19], to the best of our knowledge, no DBEM formulation has been presented for general mixed-mode dynamic crack problems in Mindlin plates.

In this paper, a dual boundary element formulation is presented for the first time for dynamic crack problems in shear deformable plates. The problem is formulated in the Laplace frequency

domain and the fundamental solutions reported for the displacement and traction boundary integral equations by Wen and Aliabadi [20] are revisited. Numerical inversion of the Laplace transforms is conducted using Durbin's method [21]. The plate bending stress intensity factors are defined and such factors for four benchmark problems are presented where comparisons are made against solutions obtained using ANSYS implicit FEM and other solutions reported in the literature.

2. Boundary integral equations in the Laplace transform formulation

An isotropic and homogeneous plate is considered here. Figure 1 shows the sign convention for generalized displacements and tractions. Throughout this paper, Greek indices vary from 1 to 2 and Roman indices vary from 1 to 3. The x_1 - x_2 plane is located at the mid-surface of the undeformed plate, and the x_3 -axis is the normal to the mid-surface. w_3 denotes normal displacement of the plate, and w_α denotes the rotations of the normal about x_α direction. The generalized tractions are denoted as t_i , where t_α is traction from bending stress resultants and t_3 is traction from shearing stress resultants.

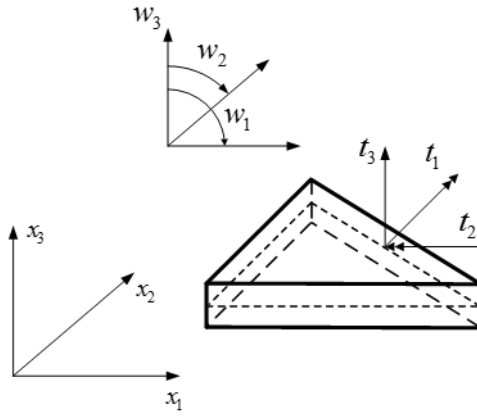


Figure 1. Sign convention for generalized displacements and tractions

Based on the plate bending theory from Mindlin [4], the equations of motion can be expressed in terms of generalized displacements w_α and w_3 as follows:

$$\frac{D}{2} \left[(1-\nu) \nabla^2 w_\alpha + (1+\nu) w_{\beta,\beta\alpha} \right] - \kappa^2 \mu h (w_{3,\alpha} + w_\alpha) = \frac{\rho h^3}{12} \frac{\partial^2 w_\alpha}{\partial t^2} \quad (1)$$

$$\kappa^2 \mu h (\nabla^2 w_3 + w_{\alpha,\alpha}) + q_3 = \rho h \frac{\partial^2 w_3}{\partial t^2} \quad (2)$$

where $D = Eh^3 / [12(1-\nu^2)]$, $\mu = E / [2(1+\nu)]$ and $\kappa^2 = \pi^2 / 12$ are the bending stiffness of the plate, the shear modulus of elasticity and the shear coefficient, respectively; E and ν are Young's modulus and Poisson's ratio; ρ and h denote the density and the thickness of the plate; q_3 is the pressure load on the plate; t denotes the time.

The Laplace transform of the above equations can be written as:

$$6D[(1-\nu)\nabla^2\tilde{w}_\alpha + (1+\nu)\tilde{w}_{\beta,\beta\alpha}] - 12\kappa^2\mu h(\tilde{w}_{3,\alpha} + \tilde{w}_\alpha) = \rho h^3 p^2 \tilde{w}_\alpha \quad (3)$$

$$\kappa^2\mu h(\nabla^2\tilde{w}_3 + \tilde{w}_{\alpha,\alpha}) + \tilde{q}_3 = \rho h p^2 \tilde{w}_3 \quad (4)$$

where p is the Laplace transform parameter. The Laplace transform of a function $f(\mathbf{x}, t)$ is defined as:

$$\mathcal{L}[f(\mathbf{x}, t)] = \tilde{f}(\mathbf{x}, p) = \int_0^\infty f(\mathbf{x}, t) e^{-pt} dt \quad (5)$$

The transformed displacements and tractions should satisfy the transformed boundary conditions:

$$\begin{aligned} \tilde{w}_i(\mathbf{x}, p) &= \tilde{w}_i^0(\mathbf{x}, p) \\ \tilde{t}_i(\mathbf{x}, p) &= \tilde{t}_i^0(\mathbf{x}, p) \end{aligned} \quad (6)$$

where $\tilde{w}_i^0(\mathbf{x}, p)$ and $\tilde{t}_i^0(\mathbf{x}, p)$ are known values of displacements and tractions on the boundary, respectively.

The boundary integral equations for rotations and normal displacement in the Laplace domain can be written as [20]:

$$\begin{aligned} c_{ij}(\mathbf{x}') \tilde{w}_j(\mathbf{x}', p) &= \int_\Gamma \tilde{U}_{ij}(\mathbf{x}', \mathbf{x}, p) \tilde{t}_j(\mathbf{x}, p) d\Gamma(\mathbf{x}) - C \int_\Gamma \tilde{T}_{ij}(\mathbf{x}', \mathbf{x}, p) \tilde{w}_j(\mathbf{x}, p) d\Gamma(\mathbf{x}) \\ &\quad + \int_\Omega \tilde{U}_{i3}(\mathbf{x}', \mathbf{X}, p) \tilde{q}_3(\mathbf{X}, p) d\Omega(\mathbf{X}) \end{aligned} \quad (7)$$

where $\mathbf{x}'$ and $\mathbf{x}$ denote a source point and a field point on the boundary Γ , respectively; $\mathbf{X}$ denotes a field point in the domain Ω ; $c_{ij}(\mathbf{x}')$ is equal to $\delta_{ij}/2$ when $\mathbf{x}'$ is located on a smooth boundary; $C \int_\Gamma$ denotes Cauchy principal value integral; $\tilde{U}_{ij}$ and $\tilde{T}_{ij}$ are Laplace transforms of fundamental solutions.

On the basis of the boundary integral equations for bending stress resultants and shearing stress resultants given in Ref. [20], when the source point $\mathbf{x}'$ is located on a smooth boundary, the traction boundary integral equations in the Laplace domain can be derived as follows:

$$\begin{aligned} \frac{1}{2} \tilde{t}_\alpha(\mathbf{x}', p) = n_\beta(\mathbf{x}') & \left[C \int_\Gamma \tilde{D}_{\alpha\beta\gamma}(\mathbf{x}', \mathbf{x}, p) \tilde{t}_\gamma(\mathbf{x}, p) d\Gamma(\mathbf{x}) + \int_\Gamma \tilde{D}_{\alpha\beta 3}(\mathbf{x}', \mathbf{x}, p) \tilde{t}_3(\mathbf{x}, p) d\Gamma(\mathbf{x}) \right. \\ & - H \int_\Gamma \tilde{S}_{\alpha\beta\gamma}(\mathbf{x}', \mathbf{x}, p) \tilde{w}_\gamma(\mathbf{x}, p) d\Gamma(\mathbf{x}) - C \int_\Gamma \tilde{S}_{\alpha\beta 3}(\mathbf{x}', \mathbf{x}, p) \tilde{w}_3(\mathbf{x}, p) d\Gamma(\mathbf{x}) \quad (8) \\ & \left. + \int_\Omega \tilde{D}_{\alpha\beta 3}(\mathbf{x}', \mathbf{X}, p) \tilde{q}_3(\mathbf{X}, p) d\Omega(\mathbf{X}) \right] \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \tilde{t}_3(\mathbf{x}', p) = n_\beta(\mathbf{x}') & \left[\int_\Gamma \tilde{D}_{3\beta\gamma}(\mathbf{x}', \mathbf{x}, p) \tilde{t}_\gamma(\mathbf{x}, p) d\Gamma(\mathbf{x}) + C \int_\Gamma \tilde{D}_{3\beta 3}(\mathbf{x}', \mathbf{x}, p) \tilde{t}_3(\mathbf{x}, p) d\Gamma(\mathbf{x}) \right. \\ & - C \int_\Gamma \tilde{S}_{3\beta\gamma}(\mathbf{x}', \mathbf{x}, p) \tilde{w}_\gamma(\mathbf{x}, p) d\Gamma(\mathbf{x}) - H \int_\Gamma \tilde{S}_{3\beta 3}(\mathbf{x}', \mathbf{x}, p) \tilde{w}_3(\mathbf{x}, p) d\Gamma(\mathbf{x}) \quad (9) \\ & \left. + \int_\Omega \tilde{D}_{3\beta 3}(\mathbf{x}', \mathbf{X}, p) \tilde{q}_3(\mathbf{X}, p) d\Omega(\mathbf{X}) \right] \end{aligned}$$

where $H \int_\Gamma$ denotes Hadamard principal value integral; $n_\beta(\mathbf{x}')$ is the unit outward normal to the boundary at the source point $\mathbf{x}'$; $\tilde{D}_{\alpha\beta\gamma}$, $\tilde{D}_{\alpha\beta 3}$, $\tilde{D}_{3\beta 3}$, $\tilde{S}_{\alpha\beta\gamma}$, $\tilde{S}_{\alpha\beta 3}$ and $\tilde{S}_{3\beta 3}$ are Laplace transforms of fundamental solutions.

The expressions of the fundamental solutions in Eqs. (7) - (9) were first given in Ref. [20]. However, these expressions include some typographical errors and some of the expressions can be further simplified. After rederiving such fundamental solutions, the modified expressions are listed in Appendix A.

The displacement integral equation (7) is used to model external boundaries of the plate, while the dual boundary element method (DBEM) is employed to model smooth crack surfaces. In the DBEM, the displacement boundary integral equation is applied when a collocation point $\mathbf{x}^+$ is located on one crack surface; the traction boundary integral equation is applied for the collocation on the opposite surface of the crack. Since there exists a point $\mathbf{x}^-$ on the opposite crack surface which coincides with the collocation point $\mathbf{x}^+$, Eq. (7) should be modified slightly to become:

$$\begin{aligned} \frac{1}{2} \tilde{w}_i(\mathbf{x}^+, p) + \frac{1}{2} \tilde{w}_i(\mathbf{x}^-, p) &= \int_\Gamma \tilde{U}_{ij}(\mathbf{x}^+, \mathbf{x}, p) \tilde{t}_j(\mathbf{x}, p) d\Gamma(\mathbf{x}) \\ &- C \int_\Gamma \tilde{T}_{ij}(\mathbf{x}^+, \mathbf{x}, p) \tilde{w}_j(\mathbf{x}, p) d\Gamma(\mathbf{x}) + \int_\Omega \tilde{U}_{i3}(\mathbf{x}^+, \mathbf{X}, p) \tilde{q}_3(\mathbf{X}, p) d\Omega(\mathbf{X}) \quad (10) \end{aligned}$$

For the same reason, when the point $\mathbf{x}^-$ becomes the collocation point, the traction boundary integral equation modified from Eqs. (8) - (9) can be written as:

$$\begin{aligned} \frac{1}{2}\tilde{t}_\alpha(\mathbf{x}^-, p) - \frac{1}{2}\tilde{t}_\alpha(\mathbf{x}^+, p) = n_\beta(\mathbf{x}^-) & \left[C \int_\Gamma \tilde{D}_{\alpha\beta\gamma}(\mathbf{x}^-, \mathbf{x}, p) \tilde{t}_\gamma(\mathbf{x}, p) d\Gamma(\mathbf{x}) \right. \\ & + \int_\Gamma \tilde{D}_{\alpha\beta 3}(\mathbf{x}^-, \mathbf{x}, p) \tilde{t}_3(\mathbf{x}, p) d\Gamma(\mathbf{x}) - H \int_\Gamma \tilde{S}_{\alpha\beta\gamma}(\mathbf{x}^-, \mathbf{x}, p) \tilde{w}_\gamma(\mathbf{x}, p) d\Gamma(\mathbf{x}) \\ & \left. - C \int_\Gamma \tilde{S}_{\alpha\beta 3}(\mathbf{x}^-, \mathbf{x}, p) \tilde{w}_3(\mathbf{x}, p) d\Gamma(\mathbf{x}) + \int_\Omega \tilde{D}_{\alpha\beta 3}(\mathbf{x}^-, \mathbf{X}, p) \tilde{q}_3(\mathbf{X}, p) d\Omega(\mathbf{X}) \right] \end{aligned} \quad (11)$$

$$\begin{aligned} \frac{1}{2}\tilde{t}_3(\mathbf{x}^-, p) - \frac{1}{2}\tilde{t}_3(\mathbf{x}^+, p) = n_\beta(\mathbf{x}^-) & \left[\int_\Gamma \tilde{D}_{3\beta\gamma}(\mathbf{x}^-, \mathbf{x}, p) \tilde{t}_\gamma(\mathbf{x}, p) d\Gamma(\mathbf{x}) \right. \\ & + C \int_\Gamma \tilde{D}_{3\beta 3}(\mathbf{x}^-, \mathbf{x}, p) \tilde{t}_3(\mathbf{x}, p) d\Gamma(\mathbf{x}) - C \int_\Gamma \tilde{S}_{3\beta\gamma}(\mathbf{x}^-, \mathbf{x}, p) \tilde{w}_\gamma(\mathbf{x}, p) d\Gamma(\mathbf{x}) \\ & \left. - H \int_\Gamma \tilde{S}_{3\beta 3}(\mathbf{x}^-, \mathbf{x}, p) \tilde{w}_3(\mathbf{x}, p) d\Gamma(\mathbf{x}) + \int_\Omega \tilde{D}_{3\beta 3}(\mathbf{x}^-, \mathbf{X}, p) \tilde{q}_3(\mathbf{X}, p) d\Omega(\mathbf{X}) \right] \end{aligned} \quad (12)$$

When the pressure load q_3 is uniform or linear distributed on the plate, the domain integrals in the above boundary integral equations can be transformed to the plate boundaries using the particular solutions given by Wen and Aliabadi [20].

The basic assumption in this paper is that there is no contact between the crack surfaces in the dynamic analyses.

3. Treatment of singularities in the transformed fundamental solutions

In order to compute the integrals in the above boundary integral equations, it is essential to understand the behaviour of these fundamental solutions in the neighbourhood of a collocation point.

It is worth mentioning that the dynamic fundamental solutions $\tilde{U}_{ij}$ and $\tilde{T}_{ij}$, as demonstrated in both isotropic problems [13, 16] and anisotropic problems [22, 23], have the same singularities as those in the corresponding static problems. Thus, when it comes to the dynamic problems in Mindlin plates, it is anticipated that these two kernels should also tend to their static counterparts. However, Ref. [20] showed that the transformed fundamental solutions $\tilde{U}_{3\alpha}$ and $\tilde{T}_{3\alpha}$ did not have this behaviour and the coefficients of their singular terms included the Laplace transform parameter p . In our study, after manipulating these coefficients, we have found that the Laplace parameter in the coefficients can be cancelled, and the kernels $\tilde{U}_{3\alpha}$ and $\tilde{T}_{3\alpha}$ still have the same singularities as those in the static kernels. Although this simplification does not change the values of the kernels and the final BEM results, it gives a correct understanding of the dynamic kernels. When it comes to the transformed fundamental solutions $\tilde{D}_{k\beta j}$ and $\tilde{S}_{k\beta j}$, the dominant singularity

is also the same as that in the static fundamental solutions. The detailed expressions of the above-mentioned singular terms can be found in Appendix B.

It should be mentioned that the transformed fundamental solutions $\tilde{S}_{\alpha\beta\gamma}$ and $\tilde{S}_{3\beta 3}$ contain both hyper-singularities (of order $1/r^2$) and weak singularities (of order $\ln(r)$). The hyper-singularities are the same as those for static problems. The weak singularities involve the Laplace transform parameter p , so their expressions are different from those for static problems.

On the basis of the above analysis, all the transformed fundamental solutions can be written in two parts:

$$\tilde{U}_{ij}(\mathbf{x}', \mathbf{x}, p) = U_{ij}^{static}(\mathbf{x}', \mathbf{x}) + \tilde{U}_{ij}^m(\mathbf{x}', \mathbf{x}, p) \quad (13)$$

$$\tilde{T}_{ij}(\mathbf{x}', \mathbf{x}, p) = T_{ij}^{static}(\mathbf{x}', \mathbf{x}) + \tilde{T}_{ij}^m(\mathbf{x}', \mathbf{x}, p) \quad (14)$$

$$\tilde{D}_{k\beta j}(\mathbf{x}', \mathbf{x}, p) = D_{k\beta j}^{static}(\mathbf{x}', \mathbf{x}) + \tilde{D}_{k\beta j}^m(\mathbf{x}', \mathbf{x}, p) \quad (15)$$

$$\tilde{S}_{k\beta j}(\mathbf{x}', \mathbf{x}, p) = S_{k\beta j}^{static}(\mathbf{x}', \mathbf{x}) + \tilde{S}_{k\beta j}^m(\mathbf{x}', \mathbf{x}, p) \quad (16)$$

where $\tilde{U}_{ij}^m$, $\tilde{T}_{ij}^m$, $\tilde{D}_{k\beta j}^m$ and $\tilde{S}_{k\beta j}^m$ are modified terms. The modified terms do not contain any singularities in terms of the distance r , except for $\tilde{S}_{\alpha\beta\gamma}^m$ and $\tilde{S}_{3\beta 3}^m$ which contain the weak singularities (of order $\ln(r)$).

The corresponding singular integrals can also be divided into two parts: static integrals including the static kernels and dynamic integrals including the modified terms. At non-crack boundaries, singularities only exist in static integrals and they can be determined using the generalized rigid body movements proposed by Rashed *et al.* [24]. At crack surfaces, singular static integrals are evaluated analytically, which has been introduced in detail in Ref. [25]. When it comes to the dynamic integrals, the weak singularities in $\tilde{S}_{\alpha\beta\gamma}^m$ and $\tilde{S}_{3\beta 3}^m$ can be tackled using the bi-cubic transformation of variable technique proposed by Telles [26].

Consider an integral with the weak singularity of order $\ln(r)$,

$$I = \int_{-1}^1 f(\eta) d\eta, \quad (17)$$

where η_s is a singular point in the interval $[-1, 1]$.

The above-mentioned variable transformation is given by:

$$\eta = \frac{\xi^3 - 3\xi_s\xi^2 + 3\xi_s^2\xi + 3\xi_s^3}{1 + 3\xi_s^2} \quad (18)$$

where $\xi_s = \sqrt[3]{(\eta_s\eta_s^* + |\eta_s^*|)} + \sqrt[3]{(\eta_s\eta_s^* - |\eta_s^*|)} + \eta_s$ and $\eta_s^* = \eta_s^2 - 1$. The Jacobian of this nonlinear transformation can cancel the weak singularity, which results in a regular integral.

All regular integrals in our study are calculated using the standard Gaussian quadrature, in which ten Gauss integration points are used.

4. Numerical implementation

The modelling strategy used in Ref. [27] is adopted in this paper. The generalized displacements and tractions on external boundaries are discretized with continuous quadratic elements, while those on crack surfaces are discretized with discontinuous quadratic elements. Geometric boundaries are represented by continuous quadratic elements.

Regardless of the pressure load $\tilde{q}_3$, the discretized displacement and traction boundary integral equations have the following forms:

$$c_{ij}^l \tilde{w}_j^l(p) = \sum_{n=1}^N \sum_{m=1}^3 \left[\tilde{t}_j^{nm}(p) \int_{-1}^1 \tilde{U}_{ij}(\xi, p) N_m(\xi) J_n(\xi) d\xi - \tilde{w}_j^{nm}(p) C \int_{-1}^1 \tilde{T}_{ij}(\xi, p) N_m(\xi) J_n(\xi) d\xi \right] \quad (19)$$

$$\begin{aligned} \frac{1}{2} \tilde{t}_\alpha^l(p) = & \sum_{n=1}^N \sum_{m=1}^3 n_\beta^l \left[\tilde{t}_\gamma^{nm}(p) C \int_{-1}^1 \tilde{D}_{\alpha\beta\gamma}(\xi, p) N_m(\xi) J_n(\xi) d\xi + \right. \\ & \tilde{t}_3^{nm}(p) \int_{-1}^1 \tilde{D}_{\alpha\beta 3}(\xi, p) N_m(\xi) J_n(\xi) d\xi - \tilde{w}_\gamma^{nm}(p) H \int_{-1}^1 \tilde{S}_{\alpha\beta\gamma}(\xi, p) N_m(\xi) J_n(\xi) d\xi \\ & \left. - \tilde{w}_3^{nm}(p) C \int_{-1}^1 \tilde{S}_{\alpha\beta 3}(\xi, p) N_m(\xi) J_n(\xi) d\xi \right] \end{aligned} \quad (20)$$

$$\begin{aligned} \frac{1}{2} \tilde{t}_3^l(p) = & \sum_{n=1}^N \sum_{m=1}^3 n_\beta^l \left[\tilde{t}_\gamma^{nm}(p) \int_{-1}^1 \tilde{D}_{3\beta\gamma}(\xi, p) N_m(\xi) J_n(\xi) d\xi + \right. \\ & \tilde{t}_3^{nm}(p) C \int_{-1}^1 \tilde{D}_{3\beta 3}(\xi, p) N_m(\xi) J_n(\xi) d\xi - \tilde{w}_\gamma^{nm}(p) C \int_{-1}^1 \tilde{S}_{3\beta\gamma}(\xi, p) N_m(\xi) J_n(\xi) d\xi \\ & \left. - \tilde{w}_3^{nm}(p) H \int_{-1}^1 \tilde{S}_{3\beta 3}(\xi, p) N_m(\xi) J_n(\xi) d\xi \right] \end{aligned} \quad (21)$$

where l is the collocation node number; N is the total number of boundary elements; $N_m(\xi)$ is the shape function and $J_n(\xi)$ is the Jacobian.

The above discretized equations can also be written in a matrix form:

$$\tilde{\mathbf{H}}(p)\tilde{\mathbf{w}}(p)=\tilde{\mathbf{G}}(p)\tilde{\mathbf{t}}(p) \quad (22)$$

where $\tilde{\mathbf{H}}$ is the boundary element influence matrix which depends on the integrals of $\tilde{T}_{ij}$ and $\tilde{S}_{k\beta j}$; the matrix $\tilde{\mathbf{G}}$ depends on the integrals of $\tilde{U}_{ij}$ and $\tilde{D}_{k\beta j}$. After imposing boundary conditions in the Laplace domain, Eq. (22) can be written as:

$$\tilde{\mathbf{A}}(p)\tilde{\mathbf{x}}(p)=\tilde{\mathbf{f}}(p) \quad (23)$$

where $\tilde{\mathbf{f}}$ is a known vector and $\tilde{\mathbf{x}}$ contains unknown displacements and tractions.

The unknown displacements and tractions in the Laplace domain are obtained after solving Eq. (23). Then, these results are converted back to the time domain using the numerical inversion of the Laplace transforms proposed by Durbin [21]. The inversion equation has the following form:

$$f(t)=\frac{e^{at}}{T}\left\{\frac{1}{2}\operatorname{Re}\left[\tilde{f}(a)\right]+\sum_{k=1}^K\operatorname{Re}\left[\tilde{f}(p)\right]\cos k\frac{\pi}{T}t-\sum_{k=1}^K\operatorname{Im}\left[\tilde{f}(p)\right]\sin k\frac{\pi}{T}t\right\} \quad (24)$$

$$, p=a+ik\frac{\pi}{T},$$

where p , a , K , and T are the Laplace transform parameter, the real part of the Laplace transform parameter, the maximum number of Laplace term, and the time period of interest, respectively. It was found that satisfactory results could be obtained when aT was equal to 3.

5. The assessment of dynamic stress intensity factors

When a plate with a through-thickness crack is loaded by tension and bending, there are three standard DSIFs: K_I , K_{II} and K_{III} with regard to three basic crack deformation modes, opening mode, sliding mode and tearing mode, respectively. According to Di Pisa and Aliabadi [28], these DSIFs can be expressed with five stress resultant intensity factors which correspond to five crack deformation modes (as shown in Figure 2).

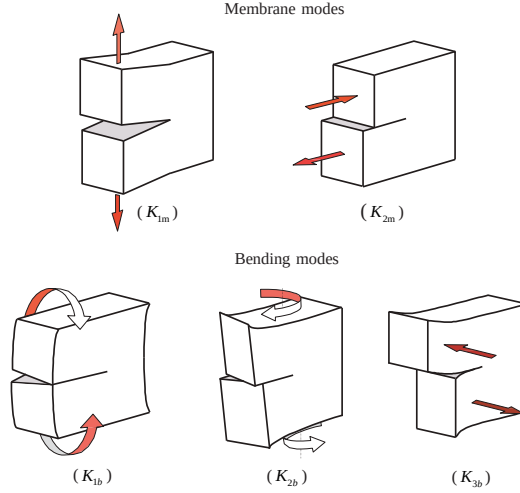


Figure 2. Crack deformation modes for stress resultant intensity factors

The relationship between the three standard DSIFs and the five stress resultant DSIFs are [28]:

$$K_I = \frac{K_{1m}}{h} + K_{1b} \frac{12}{h^3} x_3 \quad (25)$$

$$K_{II} = \frac{K_{2m}}{h} + K_{2b} \frac{12}{h^3} x_3 \quad (26)$$

$$K_{III} = \frac{3}{2h} K_{3b} \left[1 - \left(\frac{2x_3}{h} \right)^2 \right] \quad (27)$$

On the basis of the Mindlin plate bending theory, the plate bending integral equations are not coupled with the 2D plane stress equations. As a result, the membrane stress resultant DSIFs K_{1m} and K_{2m} are uncoupled with the bending stress resultant DSIFs K_{1b} , K_{2b} and K_{3b} . Fedelinski *et al.* [13] have successfully applied the Laplace transformed DBEM to the assessment of the membrane stress resultant DSIFs. Thus, in this paper, only the bending stress resultant DSIFs are considered.

According to Ref. [29], the analytical opening displacement, Δu_i , near the crack tip can be expressed as follows :

$$\begin{aligned}
\Delta u_1 &= u_1(\theta = \pi) - u_1(\theta = -\pi) = \frac{48}{Eh^3} \sqrt{\frac{2R}{\pi}} K_{2b} \\
\Delta u_2 &= u_2(\theta = \pi) - u_2(\theta = -\pi) = \frac{48}{Eh^3} \sqrt{\frac{2R}{\pi}} K_{1b} \\
\Delta u_3 &= u_3(\theta = \pi) - u_3(\theta = -\pi) = \frac{24(1+\nu)}{5Eh} \sqrt{\frac{2R}{\pi}} K_{3b}
\end{aligned} \tag{28}$$

in which the crack tip is the origin of the polar coordinate (R, θ) and u_i denotes the analytical displacement at the point (R, θ) .

It can be observed from Eq. (28) that the crack-tip displacements are of order $\sqrt{R}$. In order to represent this relationship accurately, discontinuous quarter-point elements [11], as shown in Figure 3, are used in the vicinity of the crack tip.

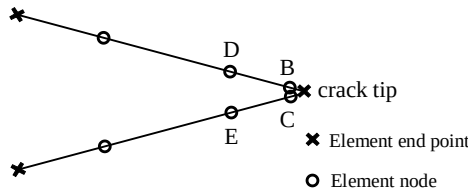


Figure 3. Discontinuous quarter-point element

The stress intensity factors are often assessed based on the crack opening displacements (CODs) at two pairs of nodes B-C and D-E. According to Ref. [30-32], the results obtained only from the node pair B-C are more accurate and less dependent on the crack mesh. Thus, this one-point formula was used in our research.

The expressions for evaluating the bending stress resultant DSIFs can be obtained from Eq. (28):

$$K_{1b} = \frac{Eh^3}{96} \sqrt{\frac{2\pi}{R^{BC}}} \Delta w_2^{BC} \tag{29}$$

$$K_{2b} = \frac{Eh^3}{96} \sqrt{\frac{2\pi}{R^{BC}}} \Delta w_1^{BC} \tag{30}$$

$$K_{3b} = \frac{5Eh}{24(1+\nu)} \sqrt{\frac{\pi}{2R^{BC}}} \Delta w_3^{BC} \tag{31}$$

where R^{BC} is the distance between the node pair B-C and the crack tip in the undeformed state.

The parameters of discontinuous quarter-point elements in Ref. [11] were adopted in our study. Thus, R^{BC} is set to $l_e/36$, where l_e denotes the length of the crack-tip element, and the bending stress resultant DSIFs become:

$$K_{1b} = \frac{Eh^3}{16} \sqrt{\frac{2\pi}{l_e}} \Delta w_2^{BC} \quad (32)$$

$$K_{2b} = \frac{Eh^3}{16} \sqrt{\frac{2\pi}{l_e}} \Delta w_1^{BC} \quad (33)$$

$$K_{3b} = \frac{5Eh}{6(1+\nu)} \sqrt{\frac{2\pi}{l_e}} \Delta w_3^{BC} \quad (34)$$

In some of the following numerical examples, the bending stress resultant DSIFs are also evaluated with the finite element method, in which continuous quarter-point crack-tip elements are used. In this case, R^{BC} is equal to $l_e/4$ and the bending stress resultant DSIFs can be expressed as:

$$K_{1b} = \frac{Eh^3}{48} \sqrt{\frac{2\pi}{l_e}} \Delta w_2^{BC} \quad (35)$$

$$K_{2b} = \frac{Eh^3}{48} \sqrt{\frac{2\pi}{l_e}} \Delta w_1^{BC} \quad (36)$$

$$K_{3b} = \frac{5Eh}{24(1+\nu)} \sqrt{\frac{2\pi}{l_e}} \Delta w_3^{BC} \quad (37)$$

6. Numerical examples

Since there are few papers regarding the bending stress resultant DSIFs for finite Mindlin plates, the results based on the present method, in the first example, are compared with those for infinite Mindlin plates obtained by Wen *et al.* [18]. In the other examples, the bending stress resultant DSIFs are calculated using the DBEM and the finite element software, ANSYS Mechanical. In the finite element analyses, 8-noded shell elements, known as Shell 281, are used because these elements can be modified easily to form the quarter-point elements near crack tips.

In the following examples, the time t is normalized with respect to l_c / c_2 , where l_c is a characteristic length and $c_2 = \sqrt{\mu / \rho}$ is the shear wave velocity. The number of Laplace terms is set to 200 for the numerical inversion of the Laplace transforms, which is sufficient to provide convergent results over the time period of interest in the following examples. The origin of the coordinate system is at the centre of a cracked plate.

6.1. An infinite plate with a central crack

Wen *et al.* [18] have applied the displacement discontinuity method to the evaluation of the DSIFs for an infinite plate with a central through crack, which is shown in Figure 4. The length of the crack and the height of the plate are denoted by $2a$ and h , respectively. The crack surfaces are subjected to a uniform bending moment $M_0 H(t)$, where M_0 is the load amplitude and $H(t)$ the Heaviside function. The material properties are the Young's modulus, $E = 69 \text{ GPa}$, the Poisson's ratio, $\nu = 0.25$, and the material density, $\rho = 290 \text{ kg/m}^3$. In order to approximate the infinite plate, an extremely large square plate is chosen as the model in the analysis with the DBEM. The edge length of the square plate is set to 100 times the crack length, which ensures that the DSIFs are not affected by the reflected elastic waves from the external boundaries during the time period of interest. In the implementation of the DBEM, each crack surface was discretized with 50 boundary elements.

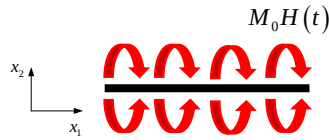


Figure 4. A crack subjected to bending load

Figure 5 presents the normalized DSIFs K_{Ib} / K_0 , where K_0 is equal to $M_0 \sqrt{\pi a}$. It is evident that the DSIFs obtained using the proposed DBEM are in very good agreement with existing results obtained by Wen *et al.* [18].

It is worth mentioning that although a finite plate model is chosen for the validation, external boundaries do not need to be considered for infinite problems using the DBEM. A more efficient strategy, in fact, is that only two crack surfaces are modelled using the DBEM. The results from such simpler model are the same as those from the finite plate model. Thus, compared with the method proposed in Ref. [18], the DBEM is not only effective in modelling cracked plates with

external boundaries, but also very suitable for solving dynamic crack problems in an infinite plate.

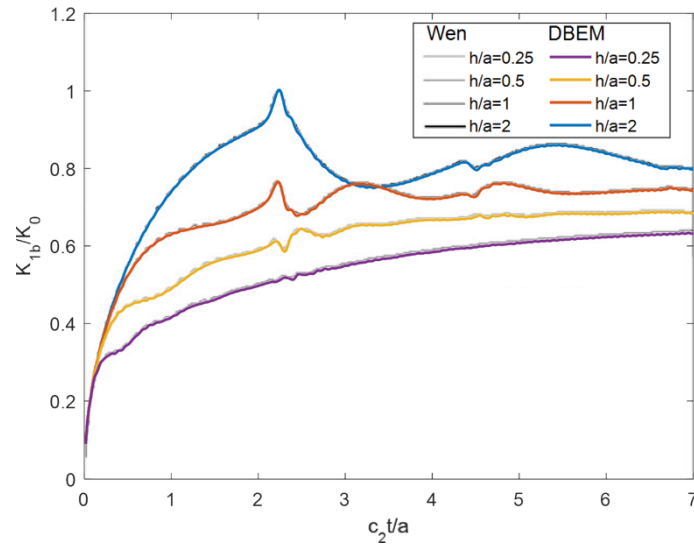


Figure 5. Normalized DSIFs for the infinite plate with a central crack

6.2. A square plate with a central crack

A square plate with a central crack (Figure 6) is loaded by edge bending $M_0 H(t)$. The material properties are: $E = 210 \text{ GPa}$, $\nu = 0.3$ and $\rho = 290 \text{ kg/m}^3$. The dimensions of the plate are: edge length $2b = 2 \text{ m}$, crack length $2a = 0.2 \text{ m}$ and plate height $h = 0.2 \text{ m}$.

The full plate was discretized with 220 boundary elements (40 elements for each edge and 30 elements for each crack surface).

In the finite element analysis with ANSYS, it was only necessary to use one quarter of the plate, modelled by 10429 shell elements, because of the biaxial symmetry.

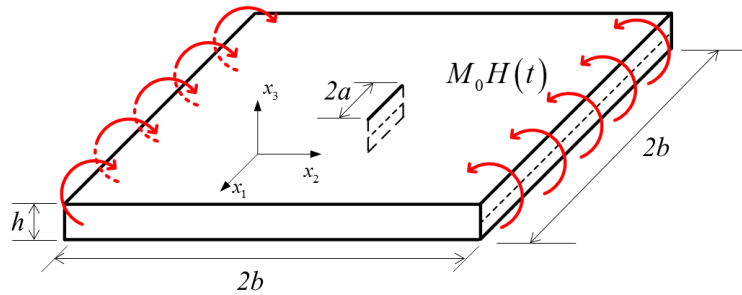


Figure 6. A square plate with a central crack

According to Thau *et al.* [33], the DSIFs depend on the propagation of elastic waves in structures. In the Mindlin plate theory, three types of flexural waves, fast flexural waves (fast waves), slow

flexural waves (slow waves) and thickness shear waves (shear waves), can be described [34]. These three waves are dispersive and the dispersion curves can be found in Ref. [20].

Figure 7 shows how the bending stress resultants $M_{22} = \int_{-h/2}^{h/2} x_3 \sigma_{22} dx_3$ at four reference points along the x_2 -axis vary in a short time period, and this variation reflects the flexural wave propagation in the plate. After the bending moments are applied on the two edges, the three types of flexural waves are generated and then propagate along the x_2 -axis. The fast wave reaches the crack surface first, and then is reflected and propagates along the opposite direction. These processes can be seen in Figure 7.

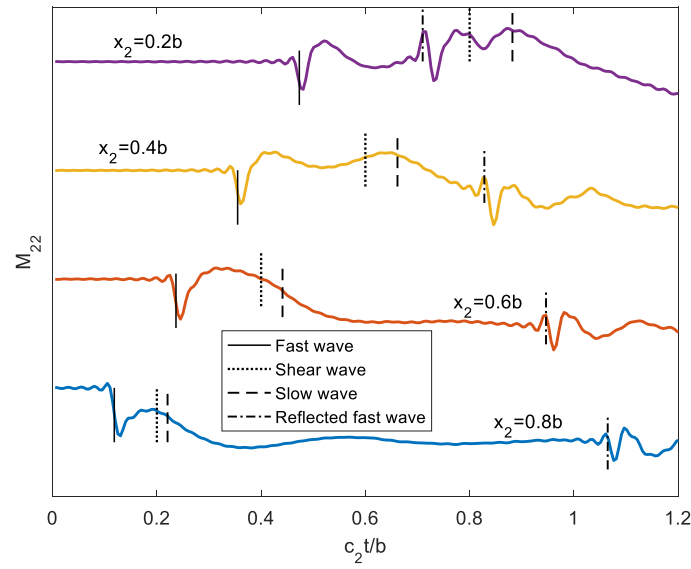


Figure 7. Moment M_{22} at different points along the x_2 -axis

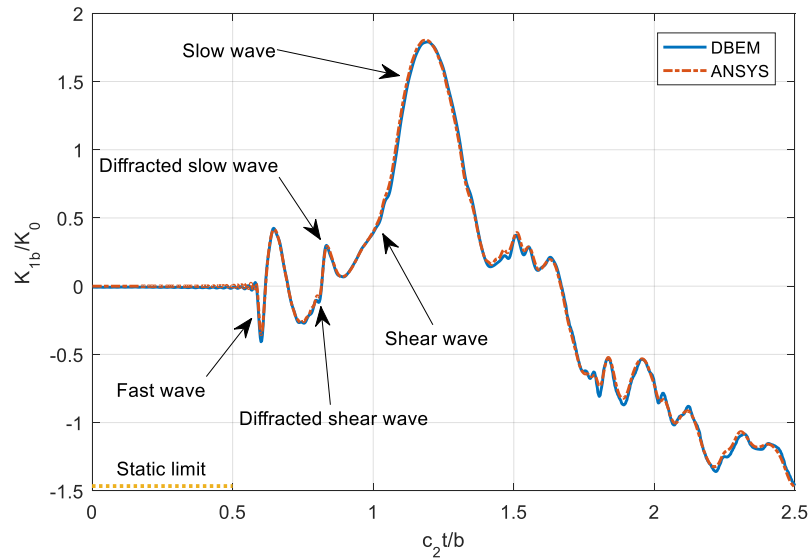


Figure 8. Normalized DSIFs for the square plate with a central crack

The normalized DSIFs K_{Ib} with respect to $K_0 = M_0 \sqrt{a}$ are given in Figure 8. It is clearly shown that the results from the boundary element analysis agree very well with the ANSYS results. The peak values obtained using the DBEM and the FEM are virtually identical.

Some kinks and wave peaks which can be seen in Figure 8 are caused by the three waves starting from the plate edges, and the diffracted waves which start from the other crack tip. The effect of the slow wave on the DSIF is greater than that of the fast wave and the shear wave. Moreover, the peak value due to the slow wave is larger than the absolute value of the static limit. The effect of the shear wave is very weak, and can be disregarded.

When the fast wave reaches one crack tip, three types of diffracted waves, diffracted fast wave, diffracted shear wave and diffracted slow wave, are generated. The DSIF at the other crack tip is affected after the diffracted waves travel to this tip. In comparison to the fast wave starting from the plate edges, the diffracted slow and shear waves have less influence over the DSIF. Moreover, the effect of the diffracted fast wave is negligible.

It is worth mentioning that time-efficiency is not the focus in this paper, so more Laplace terms and finer mesh were used in the boundary element analyses than strictly necessary, to ensure convergence. When large engineering structures are considered, in order to increase the time-efficiency, the numerical inversion of the Laplace transforms can be conducted using another formula as follows:

$$f(t) = 2 \frac{e^{at}}{T} \left\{ \frac{1}{2} \operatorname{Re}[\tilde{f}(a)] + \sum_{k=1}^K \operatorname{Re}[\tilde{f}(p)] \cos k \frac{2\pi}{T} t - \sum_{k=1}^K \operatorname{Im}[\tilde{f}(p)] \sin k \frac{2\pi}{T} t \right\} \quad (38)$$

$$, p = a + ik \frac{2\pi}{T}.$$

Both formula A from Eq. (24) and formula B from Eq. (38) were presented in Durbin's paper [21] for the numerical inversion of the Laplace transforms. If the results toward the end of the time period are not of interest, formula B would be the better choice due to higher time-efficiency.

The drawback of formula B is that the results near the end time are not accurate. The normalized DSIFs K_{Ib} / K_0 obtained using formula A and formula B are presented in Figure 9. It can be seen from Figure 9 that the results obtained using formula B are oscillatory near the end time. This problem still occurs even though the number of Laplace terms increases. Thus, in order to compare the BEM results with the FEM results over the whole period, formula B is not used in this paper.

However, the advantage of formula B is that fewer Laplace terms are required, and therefore computational time can be saved. The curves in Figure 9 show that the results based on formula B are as accurate as those from formula A, except for a short period near the end time. In this example, the number of Laplace terms used in formula B is 100, only half the number of Laplace terms used in formula A.

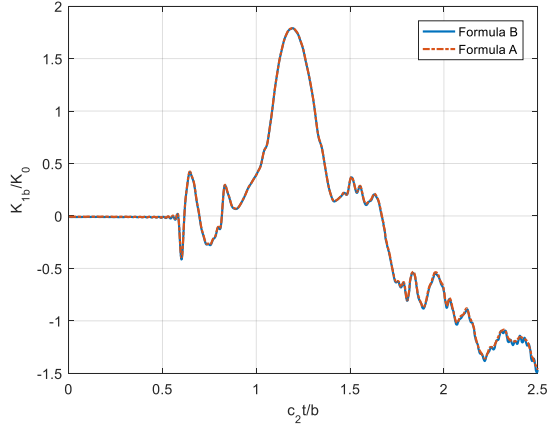


Figure 9. Normalized DSIFs obtained using different formulas

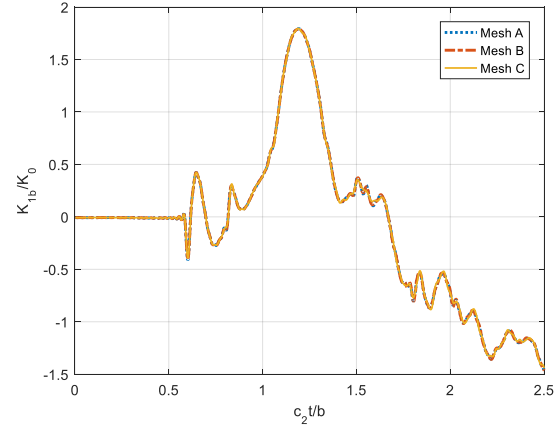


Figure 10. Normalized DSIFs obtained using different meshes

Table 1. The number of boundary elements for three different meshes

	Mesh A	Mesh B	Mesh C
Number of elements on each edge	80	40	20
Number of elements on each crack surface	50	30	20

The number of boundary elements can also be reduced if some minor details are not necessary and can be disregarded. Figure 10 shows the normalized DSIFs K_{1b} / K_0 obtained using three different meshes. The detailed mesh information is listed in Table 1. As can be seen in Figure 10, mesh B can lead to convergent results, so mesh B was used to conduct the DBEM analyses. Figure 10 also shows that the major features can be captured using mesh C. Therefore, if some weak waves are not important, a coarser mesh can be used to further reduce computational time.

6.3. A cantilever rectangular plate with a central crack

Figure 11 shows a cantilever rectangular plate of dimensions $2 \text{ m } (2b) \times 1 \text{ m } (2c) \times 0.1 \text{ m } (h)$. The length, $2a$, of the through crack is 0.1 m . A uniformly distributed shear force $Q_0 H(t)$, where Q_0 is the load amplitude, is applied on the side opposite the fixed side. The plate is made

1 of aluminium, which has a Young's modulus of 70 GPa, a Poisson's ratio (ν) of 0.33 and a
 2 density of 2700 kg/m³.

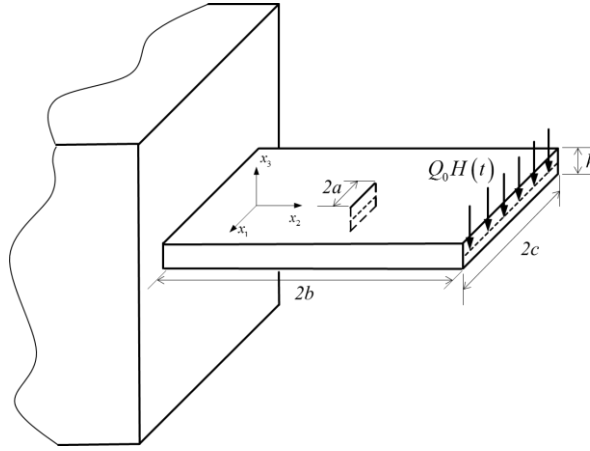


Figure 11. A cantilever rectangular plate with a central crack

5 The number of boundary elements used for each crack surface, the longer side and shorter side
 6 of the plate were 20, 60 and 30, respectively.

7 Due to geometric and loading symmetry, half plate, in which 10320 shell elements were used,
 8 was modelled to obtain FEM results.

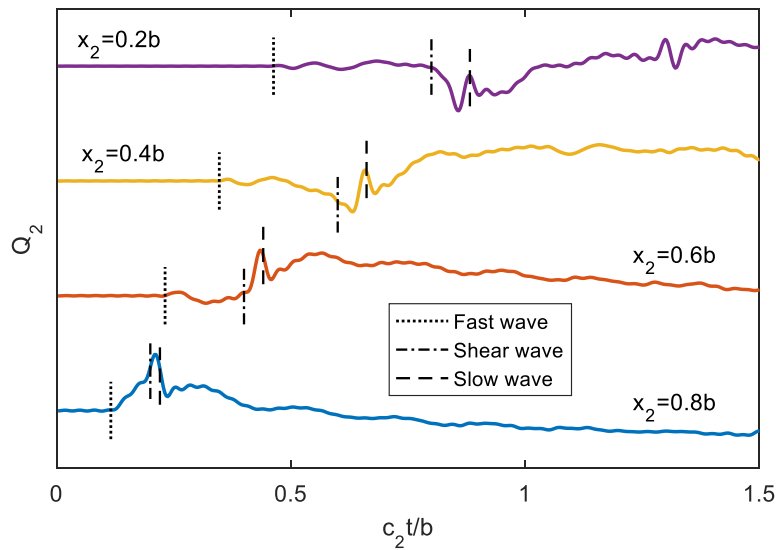


Figure 12. Shear force Q_2 at different points along the x_2 -axis

11 Figure 12 shows the time history of the shearing stress resultants $Q_2 = \int_{-h/2}^{h/2} \sigma_{23} dx_3$ at four
 12 reference points along the x_2 -axis. As can be seen in Figure 12, the three types of flexural waves
 13 propagate along the x_2 -axis.

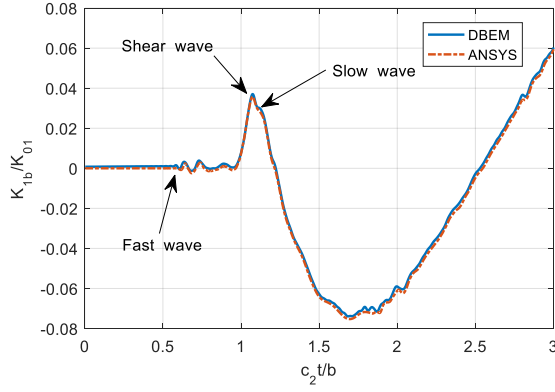


Figure 13. Normalized K_{1b} for the cantilever plate with a central crack

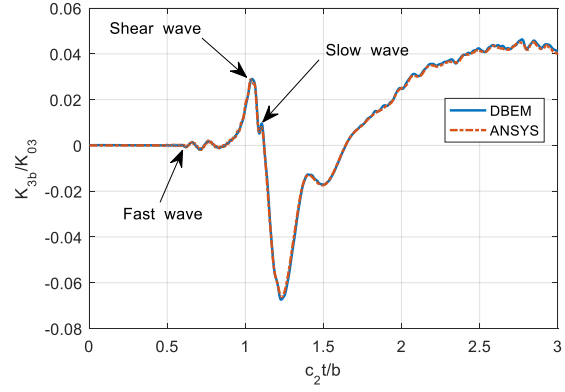


Figure 14. Normalized K_{3b} for the cantilever plate with a central crack

Figures 13 and 14 show the normalized DSIFs K_{1b} / K_{01} and K_{3b} / K_{03} respectively, where $K_{01} = 2Q_0c\sqrt{a}$ and $K_{03} = 2Q_0c\sqrt{10a}/[(1+\nu)h]$. It can be seen from these figures that the DBEM results agree very well with the FEM results. The effect of the fast wave on the DSIFs is the weakest among the three types of flexural waves.

6.4. A square plate with a slant central crack

A mixed-mode crack problem is considered in this example. A cracked square plate loaded by edge bending $M_0H(t)$, which has a length of $2b = 1$ m and a height $h = 0.1$ m, is shown in Figure 15. The crack has a length $2a = 0.1\sqrt{2}$ m and makes an angle of $\theta = \pi/4$ with the direction x_2 . The material properties are the same as those in the second example.

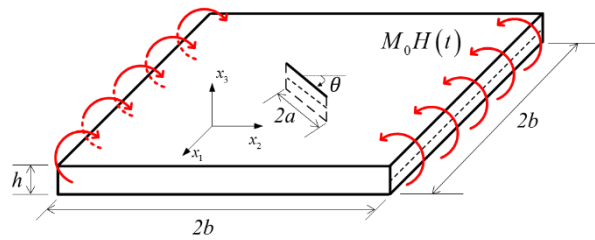


Figure 15. A square plate with a slant central crack

In the dual boundary element analysis, 220 boundary elements were used in which the crack was discretized with 30 elements on each surface, while 9977 shell elements were used for the finite element analysis.

1 The DSIFs K_{1b} and K_{2b} are normalized with respect to $K_{01} = M_0 \sqrt{a}$, while K_{3b} is normalized
2 with respect to $K_{03} = M_0 \sqrt{10a} / [(1+\nu)h]$.
3 Figures 16 to 18 present the normalized DSIFs K_{1b} / K_{01} , K_{2b} / K_{01} and K_{3b} / K_{03} , respectively.
4 As shown in these figures, the DBEM results are in excellent agreement with the FEM results.

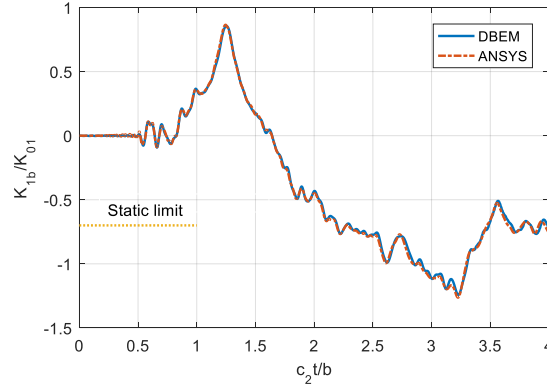


Figure 16. Normalized K_{1b} for the square plate with a slant central crack

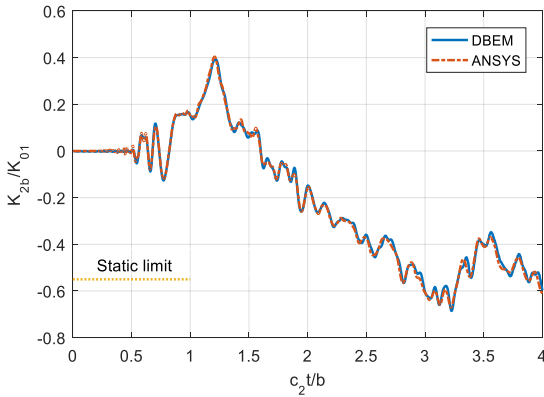


Figure 17. Normalized K_{2b} for the square plate with a
slant central crack

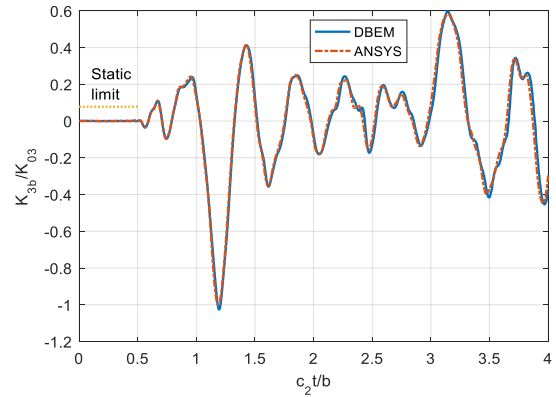


Figure 18. Normalized K_{3b} for the square plate with a
slant central crack

7 Figure 19 illustrates the effects of flexural waves on the normalized DSIF K_{1b} / K_{01} which is
8 assessed at the crack tip A. Four groups of waves, waves 1, waves 2, diffracted waves 1 and
9 diffracted waves 2, can be observed at the initial stage. Waves 1 starting from the edge A contains
10 fast wave 1, shear wave 1 and slow wave 1. Waves 2 is also composed of these three types of
11 waves which start from the edge B. When the fast wave from the edge B reaches the crack tip B,
12 diffracted waves 1 which includes diffracted shear wave 1 and diffracted slow wave 1 is generated
13 and travels to the crack tip A. Similar to the diffracted waves 1, diffracted waves 2, including
14 diffracted shear wave 2 and diffracted slow wave 2, is caused by the fast wave from the edge A.

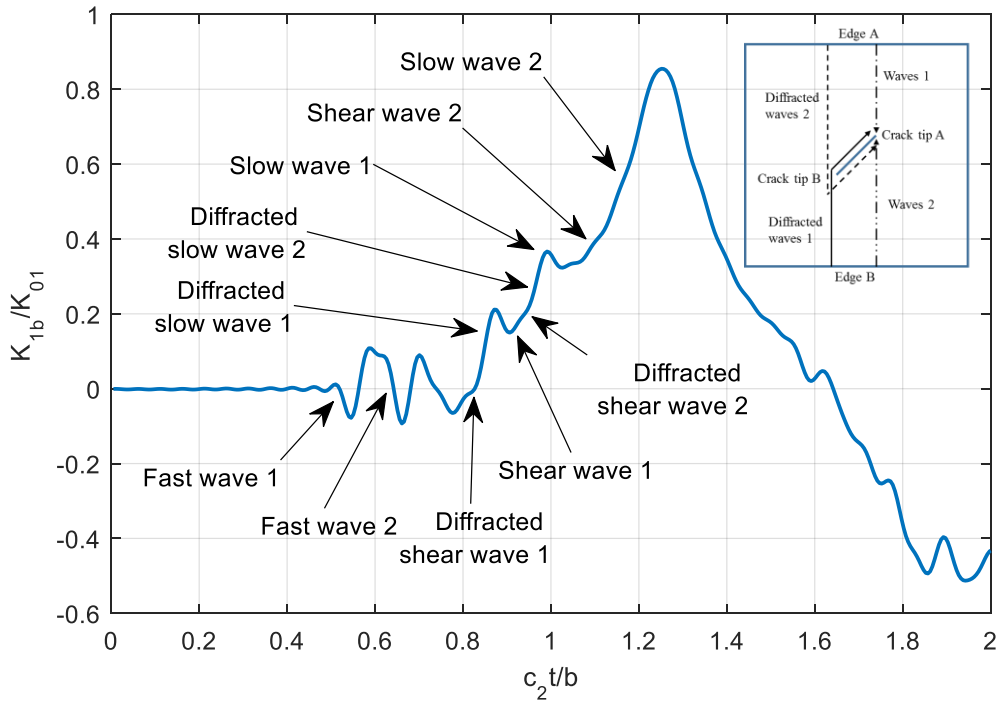


Figure 19. The effects of flexural waves on the normalized DSIF

Following the study reported in this paper, some comments can be made in relation to comparative advantageous and disadvantages of the proposed method and the FEM. A key advantage of the BEM is that only plate boundaries need to be discretized with line elements. Thus, the BEM is more efficient than the FEM especially in modelling crack problems. In the FEM, in order to deal with the stress concentration, an extremely fine mesh should be used around a crack tip region. By contrast, with the help of the DBEM, only crack surfaces, rather than a large region, are modelled and there is no need to use such fine mesh near the crack tip.

Furthermore, the DBEM is more suitable than the FEM for modelling crack problems in an infinite plate. In modelling such problems using the DBEM, the boundary conditions for infinite plates can be accurately satisfied just by removing the external boundaries. This feature makes the DBEM even more efficient because only crack surfaces are meshed. By contrast, the infinite plate can only be approximated by a large plate using the FEM.

Although domain integrals due to body forces can be transformed to boundaries when the pressure load is uniform or linear distributed on a plate, plate domain discretization is required for other types of body forces and therefore the DBEM becomes less efficient.

7. Conclusions

In this paper, a new dual boundary element formulation in the Laplace domain has been developed for dynamic bending analyses of cracked Mindlin plates. Strong and hyper-singularities in the Laplace transformed fundamental solutions are shown to have the same forms as those in static plate bending problems. Crack opening displacements were used to determine the dynamic plate bending stress intensity factors. The results obtained using the DBEM were in excellent agreement with published results and the ones obtained from finite element analysis. Therefore, the proposed DBEM was shown to be a successful method in the transient analyses of cracked plates. In the future, it is planned to apply the DBEM to harmonic analysis of cracked Mindlin plates, after which spectral elements [35] will be introduced into the DBEM to make the method more time-efficient in modelling high-frequency waves and vibrations.

Acknowledgments

This research was supported by a grant provided by the China Scholarship Council (CSC).

Appendix A

The transformed fundamental solutions in the displacement boundary integral equation can be arranged as follows:

$$\begin{aligned} \tilde{U}_{\alpha\beta} = & -\frac{1}{2\pi D(\alpha_2^2 - \alpha_1^2)} \left\{ -(a_2 - a_1) \alpha_3^2 K_0(z_3) \delta_{\alpha\beta} \right. \\ & + \left[(1 - a_2) \alpha_1 K_1(z_1) - (1 - a_1) \alpha_2 K_1(z_2) + (a_2 - a_1) \alpha_3 K_1(z_3) \right] (2r_{,\alpha} r_{,\beta} - \delta_{\alpha\beta}) \frac{1}{r} \\ & \left. + \left[(1 - a_2) \alpha_1^2 K_0(z_1) - (1 - a_1) \alpha_2^2 K_0(z_2) + (a_2 - a_1) \alpha_3^2 K_0(z_3) \right] r_{,\alpha} r_{,\beta} \right\} \end{aligned} \quad (\text{A.1})$$

$$\tilde{U}_{\alpha 3} = -\tilde{U}_{3\alpha} = -\frac{1}{2\pi D(\alpha_2^2 - \alpha_1^2)} \left[\alpha_1 K_1(z_1) - \alpha_2 K_1(z_2) \right] r_{,\alpha} \quad (\text{A.2})$$

$$\tilde{U}_{33} = \frac{1}{2\pi \kappa^2 \mu h (a_2 - a_1)} \left[(1 - a_1) K_0(z_1) - (1 - a_2) K_0(z_2) \right] \quad (\text{A.3})$$

$$\tilde{T}_{\alpha\beta} = -\frac{1-\nu}{4\pi(\alpha_2^2 - \alpha_1^2)} \left\{ \frac{2\nu}{1-\nu} \left[-(1-a_2)\alpha_1^3 K_1(z_1) + (1-a_1)\alpha_2^3 K_1(z_2) \right] r_{,\alpha} n_\beta \right. \\ \left. - \left[(1-a_2)\alpha_1 A(z_1) - (1-a_1)\alpha_2 A(z_2) + (a_2-a_1)\alpha_3 A(z_3) \right] \frac{2}{r^2} \times \right. \quad (\text{A.4})$$

$$\left[4r_{,\alpha} r_{,\beta} r_{,n} - (r_{,\alpha} n_\beta + r_{,\beta} n_\alpha + \delta_{\alpha\beta} r_{,n}) \right] + (a_2-a_1)\alpha_3^3 K_1(z_3) (r_{,\beta} n_\alpha + \delta_{\alpha\beta} r_{,n}) \\ - 2 \left[(1-a_2)\alpha_1^3 K_1(z_1) - (1-a_1)\alpha_2^3 K_1(z_2) + (a_2-a_1)\alpha_3^3 K_1(z_3) \right] \times r_{,\alpha} r_{,\beta} r_{,n} \Big\}$$

$$\tilde{T}_{\alpha 3} = \frac{\kappa^2 \mu h}{2\pi D(\alpha_2^2 - \alpha_1^2)} \left\{ \left[a_2 \alpha_1 K_1(z_1) - a_1 \alpha_2 K_1(z_2) - (a_2-a_1)\alpha_3 K_1(z_3) \right] (2r_{,\alpha} r_{,n} - n_\alpha) \frac{1}{r} \right. \quad (\text{A.5}) \\ \left. + \left[a_2 \alpha_1^2 K_0(z_1) - a_1 \alpha_2^2 K_0(z_2) - (a_2-a_1)\alpha_3^2 K_0(z_3) \right] r_{,\alpha} r_{,n} + (a_2-a_1)\alpha_3^2 K_0(z_3) n_\alpha \right\}$$

$$\tilde{T}_{3\alpha} = -\frac{(1-\nu)}{2\pi(\alpha_2^2 - \alpha_1^2)} \left\{ \left[\alpha_1 K_1(z_1) - \alpha_2 K_1(z_2) \right] (2r_{,\alpha} r_{,n} - n_\alpha) \frac{1}{r} \right. \quad (\text{A.6}) \\ \left. + \left[\alpha_1^2 K_0(z_1) - \alpha_2^2 K_0(z_2) \right] r_{,\alpha} r_{,n} + \frac{\nu}{1-\nu} \left[\alpha_1^2 K_0(z_1) - \alpha_2^2 K_0(z_2) \right] n_\alpha \right\}$$

$$\tilde{T}_{33} = -\frac{1}{2\pi(a_2 - a_1)} \left\{ (1-a_1)\alpha_1 K_1(z_1) - (1-a_2)\alpha_2 K_1(z_2) \right. \quad (\text{A.7}) \\ \left. - (1-a_1)(1-a_2) \left[\alpha_1 K_1(z_1) - \alpha_2 K_1(z_2) \right] \right\} r_{,n}$$

where $A(z_i) = 2K_1(z_i) + z_i K_0(z_i)$; $K_0(z_i)$ and $K_1(z_i)$ are modified zero- and first-order Bessel functions of the second kind, respectively. The coefficients α_i^2 and a_β in the above equations can be written as:

$$\alpha_1^2 = \frac{1}{2} \left(\frac{p}{\omega_0} \right)^2 \left[\frac{1}{S} + \frac{1}{R} + \sqrt{\left(\frac{1}{S} - \frac{1}{R} \right)^2 - \frac{4}{RS} \left(\frac{\omega_0}{p} \right)^2} \right] \quad (\text{A.8})$$

$$\alpha_2^2 = \frac{1}{2} \left(\frac{p}{\omega_0} \right)^2 \left[\frac{1}{S} + \frac{1}{R} - \sqrt{\left(\frac{1}{S} - \frac{1}{R} \right)^2 - \frac{4}{RS} \left(\frac{\omega_0}{p} \right)^2} \right] \quad (\text{A.9})$$

$$\alpha_3^2 = \left(\frac{\pi}{h} \right)^2 \left[\left(\frac{p}{\omega_0} \right)^2 + 1 \right] \quad (\text{A.10})$$

$$a_\beta = \frac{2}{1-\nu} \left(\frac{\alpha_\beta}{\alpha_3} \right)^2 \quad (\text{A.11})$$

where ω_0 is the cut-off frequency $\pi c_2/h$, and the constants S and R are defined as:

$$S = \frac{12D}{\pi^2 \mu h}, \quad R = \frac{h^2}{12} \quad (\text{A.12})$$

The fundamental solutions in the traction boundary integral equation can be arranged as follows:

$$\begin{aligned} \tilde{D}_{\alpha\beta\gamma} = & \frac{1-\nu}{4\pi(\alpha_2^2 - \alpha_1^2)} \left\{ \left[-(1-a_2)\alpha_1 A(z_1) + (1-a_1)\alpha_2 A(z_2) - (a_2-a_1)\alpha_3 A(z_3) \right] \frac{2}{r^2} \times \right. \\ & \left[4r_{,\alpha} r_{,\beta} r_{,\gamma} - (r_{,\gamma} \delta_{\alpha\beta} + r_{,\alpha} \delta_{\beta\gamma} + r_{,\beta} \delta_{\alpha\gamma}) \right] + (a_2-a_1)\alpha_3^3 K_1(z_3) (r_{,\alpha} \delta_{\beta\gamma} + r_{,\beta} \delta_{\alpha\gamma}) \\ & - 2 \left[(1-a_2)\alpha_1^3 K_1(z_1) - (1-a_1)\alpha_2^3 K_1(z_2) + (a_2-a_1)\alpha_3^3 K_1(z_3) \right] r_{,\alpha} r_{,\beta} r_{,\gamma} \\ & \left. + \frac{2\nu}{1-\nu} \left[-(1-a_2)\alpha_1^3 K_1(z_1) + (1-a_1)\alpha_2^3 K_1(z_2) \right] r_{,\gamma} \delta_{\alpha\beta} \right\} \end{aligned} \quad (\text{A.13})$$

$$\begin{aligned} \tilde{D}_{\alpha\beta 3} = & -\frac{1-\nu}{2\pi(\alpha_2^2 - \alpha_1^2)} \left\{ \left[\alpha_1 K_1(z_1) - \alpha_2 K_1(z_2) \right] \frac{1}{r} (2r_{,\beta} r_{,\alpha} - \delta_{\alpha\beta}) \right. \\ & \left. + \left[\alpha_1^2 K_0(z_1) - \alpha_2^2 K_0(z_2) \right] r_{,\alpha} r_{,\beta} + \frac{\nu}{1-\nu} \left[\alpha_1^2 K_0(z_1) - \alpha_2^2 K_0(z_2) \right] \delta_{\alpha\beta} \right\} \end{aligned} \quad (\text{A.14})$$

$$\begin{aligned} \tilde{D}_{3\beta\gamma} = & \frac{\kappa^2 \mu h}{2\pi D(\alpha_2^2 - \alpha_1^2)} \left\{ \left[a_2 \alpha_1^2 K_0(z_1) - a_1 \alpha_2^2 K_0(z_2) - (a_2-a_1)\alpha_3^2 K_0(z_3) \right] r_{,\beta} r_{,\gamma} \right. \\ & + \left[a_2 \alpha_1 K_1(z_1) - a_1 \alpha_2 K_1(z_2) - (a_2-a_1)\alpha_3 K_1(z_3) \right] (2r_{,\beta} r_{,\gamma} - \delta_{\beta\gamma}) \frac{1}{r} \\ & \left. + (a_2-a_1)\alpha_3^2 K_0(z_3) \delta_{\beta\gamma} \right\} \end{aligned} \quad (\text{A.15})$$

$$\begin{aligned} \tilde{D}_{3\beta 3} = & \frac{1}{2\pi(a_2 - a_1)} \left\{ (1-a_1)\alpha_1 K_1(z_1) - (1-a_2)\alpha_2 K_1(z_2) \right. \\ & \left. - (1-a_1)(1-a_2) \left[(\alpha_1 K_1(z_1) - \alpha_2 K_1(z_2)) \right] \right\} r_{,\beta} \end{aligned} \quad (\text{A.16})$$

$$\tilde{S}_{\alpha\beta k} = \frac{(1-\nu)D}{2} \left[\tilde{T}_{\alpha k, \beta} + \tilde{T}_{\beta k, \alpha} + \frac{2\nu}{1-\nu} \tilde{T}_{\gamma k, \gamma} \delta_{\alpha\beta} \right] \quad (\text{A.17})$$

$$\tilde{S}_{3\alpha k} = \kappa^2 \mu h (\tilde{T}_{\alpha k} + \tilde{T}_{3k, \alpha}) \quad (\text{A.18})$$

1 The derivatives of the fundamental solutions $\tilde{T}_{\alpha\beta}$ have the following forms:

$$\begin{aligned}
\tilde{T}_{\alpha\beta,\gamma} = & \frac{1-\nu}{4\pi(\alpha_2^2 - \alpha_1^2)} \left\{ \frac{2\nu}{1-\nu} \left[(1-a_2)\alpha_1^4 C(z_1) - (1-a_1)\alpha_2^4 C(z_2) \right] r_{,\alpha} r_{,\gamma} n_\beta \right. \\
& + 2 \left[(1-a_2)\alpha_1^4 C(z_1) - (1-a_1)\alpha_2^4 C(z_2) + (a_2-a_1)\alpha_3^4 C(z_3) \right] r_{,\alpha} r_{,\gamma} r_{,\beta} r_{,n} \\
& - \frac{2\nu}{1-\nu} \left[(1-a_2)\alpha_1^3 K_1(z_1) - (1-a_1)\alpha_2^3 K_1(z_2) \right] (\delta_{\alpha\gamma} - r_{,\alpha} r_{,\gamma}) \frac{n_\beta}{r} \\
& - 2 \left[(1-a_2)\alpha_1 A(z_1) - (1-a_1)\alpha_2 A(z_2) + (a_2-a_1)\alpha_3 A(z_3) \right] Q_{\alpha\beta\gamma}^a \\
& + 2 \left[(1-a_2)\alpha_1^2 B(z_1) - (1-a_1)\alpha_2^2 B(z_2) + (a_2-a_1)\alpha_3^2 B(z_3) \right] Q_{\alpha\beta\gamma}^b \\
& - 2 \left[(1-a_2)\alpha_1^3 K_1(z_1) - (1-a_1)\alpha_2^3 K_1(z_2) + (a_2-a_1)\alpha_3^3 K_1(z_3) \right] Q_{\alpha\beta\gamma}^c \\
& \left. - (a_2-a_1)\alpha_3^4 C(z_3) (r_{,\beta} n_\alpha + \delta_{\alpha\beta} r_{,n}) r_{,\gamma} + (a_2-a_1)\alpha_3^3 K_1(z_3) Q_{\alpha\beta\gamma}^d \right\}
\end{aligned} \tag{A.19}$$

3 where $B(z_i) = K_0(z_i) + z_i K_1(z_i) + 2K_1(z_i)/z_i$, $C(z_i) = K_0(z_i) + K_1(z_i)/z_i$ and

$$\begin{aligned}
Q_{\alpha\beta\gamma}^a = & \frac{1}{r^3} \left(-20r_{,\alpha} r_{,\gamma} r_{,\beta} r_{,n} + 3r_{,\alpha} r_{,\gamma} n_\beta + 3r_{,\beta} r_{,\gamma} n_\alpha + 3r_{,\gamma} r_{,n} \delta_{\alpha\beta} + 4r_{,\beta} r_{,n} \delta_{\alpha\gamma} \right. \\
& \left. + 4r_{,\alpha} r_{,n} \delta_{\beta\gamma} + 4r_{,\alpha} r_{,\beta} n_\gamma - n_\beta \delta_{\alpha\gamma} - n_\alpha \delta_{\beta\gamma} - n_\gamma \delta_{\alpha\beta} \right) \\
Q_{\alpha\beta\gamma}^b = & \frac{1}{r^2} \left(4r_{,\beta} r_{,\alpha} r_{,n} - r_{,\alpha} n_\beta - r_{,\beta} n_\alpha - r_{,n} \delta_{\alpha\beta} \right) r_{,\gamma} \\
Q_{\alpha\beta\gamma}^c = & \frac{1}{r} \left(r_{,\beta} r_{,n} \delta_{\alpha\gamma} + r_{,\alpha} r_{,n} \delta_{\beta\gamma} + r_{,\alpha} r_{,\beta} n_\gamma - 3r_{,\alpha} r_{,\gamma} r_{,\beta} r_{,n} \right) \\
Q_{\alpha\beta\gamma}^d = & \frac{1}{r} \left[(\delta_{\beta\gamma} - r_{,\beta} r_{,\gamma}) n_\alpha + (n_\gamma - r_{,\gamma} r_{,n}) \delta_{\alpha\beta} \right]
\end{aligned}$$

5 The derivatives of the fundamental solutions $\tilde{T}_{\beta 3}$, $\tilde{T}_{3\alpha}$ and $\tilde{T}_{33}$ are:

$$\begin{aligned}
\tilde{T}_{\beta 3,\gamma} = & -\frac{\kappa^2 \mu h}{2\pi D(\alpha_2^2 - \alpha_1^2)} \left\{ \left[a_2 \alpha_1 K_1(z_1) - a_1 \alpha_2 K_1(z_2) - (a_2 - a_1) \alpha_3 K_1(z_3) \right] \frac{1}{r^2} \times \right. \\
& \left(-6r_{,\beta} r_{,\gamma} r_{,n} + r_{,\gamma} n_\beta + 2r_{,n} \delta_{\beta\gamma} + 2r_{,\beta} n_\gamma \right) - (a_2 - a_1) \alpha_3^3 K_1(z_3) r_{,\gamma} n_\beta \\
& - \left[a_2 \alpha_1^2 C(z_1) - a_1 \alpha_2^2 C(z_2) - (a_2 - a_1) \alpha_3^2 C(z_3) \right] (2r_{,\beta} r_{,n} - n_\beta) \frac{r_{,\gamma}}{r} \\
& + \left[a_2 \alpha_1^2 K_0(z_1) - a_1 \alpha_2^2 K_0(z_2) - (a_2 - a_1) \alpha_3^2 K_0(z_3) \right] (r_{,n} \delta_{\beta\gamma} + r_{,\beta} n_\gamma - 2r_{,\beta} r_{,\gamma} r_{,n}) \frac{1}{r} \\
& \left. - \left[a_2 \alpha_1^3 K_1(z_1) - a_1 \alpha_2^3 K_1(z_2) - (a_2 - a_1) \alpha_3^3 K_1(z_3) \right] r_{,\beta} r_{,\gamma} r_{,n} \right\}
\end{aligned} \tag{A.20}$$

$$\begin{aligned}
\tilde{T}_{3\alpha,\beta} = & \frac{1-\nu}{2\pi(\alpha_2^2 - \alpha_1^2)} \left\{ [\alpha_1 K_1(z_1) - \alpha_2 K_1(z_2)] (-6r_{,\alpha} r_{,\beta} r_{,n} + r_{,\beta} n_{\alpha} + 2r_{,n} \delta_{\alpha\beta} + 2r_{,\alpha} n_{\beta}) \frac{1}{r^2} \right. \\
& + [\alpha_1^2 K_0(z_1) - \alpha_2^2 K_0(z_2)] (r_{,n} \delta_{\alpha\beta} + r_{,\alpha} n_{\beta} - 2r_{,\alpha} r_{,\beta} r_{,n}) \frac{1}{r} \\
& - [\alpha_1^2 C(z_1) - \alpha_2^2 C(z_2)] (2r_{,\alpha} r_{,n} - n_{\alpha}) \frac{r_{,\beta}}{r} \\
& \left. - [\alpha_1^3 K_1(z_1) - \alpha_2^3 K_1(z_2)] r_{,\alpha} r_{,\beta} r_{,n} - \frac{\nu}{1-\nu} [\alpha_1^3 K_1(z_1) - \alpha_2^3 K_1(z_2)] r_{,\beta} n_{\alpha} \right\}
\end{aligned} \tag{A.21}$$

$$\begin{aligned}
\tilde{T}_{33,\alpha} = & \frac{1}{2\pi(a_2 - a_1)} \left\{ [(1-a_1)\alpha_1 K_1(z_1) - (1-a_2)\alpha_2 K_1(z_2)] \right. \\
& - (1-a_1)(1-a_2)(\alpha_1 K_1(z_1) - \alpha_2 K_1(z_2))] (n_{\alpha} - r_{,\alpha} r_{,n}) \frac{1}{r} \\
& - [(1-a_1)\alpha_1^2 C(z_1) - (1-a_2)\alpha_2^2 C(z_2)] \\
& \left. - (1-a_1)(1-a_2)(\alpha_1^2 C(z_1) - \alpha_2^2 C(z_2)) \right] r_{,\alpha} r_{,n} \Big\}
\end{aligned} \tag{A.22}$$

Appendix B

When the distance r between the collocation point $\mathbf{x}'$ and the field point $\mathbf{x}$ tends to zero, the limits of the transformed fundamental solutions have the following forms:

$$\lim_{r \rightarrow 0} \tilde{U}_{\alpha\beta} = \frac{\nu-3}{4\pi D(1-\nu)} \ln r \delta_{\alpha\beta} = \lim_{r \rightarrow 0} U_{\alpha\beta}^{static} \tag{B. 1}$$

$$\lim_{r \rightarrow 0} \tilde{U}_{\alpha 3} = -\lim_{r \rightarrow 0} \tilde{U}_{3\alpha} = 0 = \lim_{r \rightarrow 0} U_{\alpha 3}^{static} = -\lim_{r \rightarrow 0} U_{3\alpha}^{static} \tag{B. 2}$$

$$\lim_{r \rightarrow 0} \tilde{U}_{33} = -\frac{1}{2\pi\kappa^2 \mu h} \ln r = \lim_{r \rightarrow 0} U_{33}^{static} \tag{B. 3}$$

$$\lim_{r \rightarrow 0} \tilde{T}_{\alpha\beta} = -\frac{1}{4\pi r} \left[(1-\nu)(\delta_{\alpha\beta} r_{,n} + r_{,\beta} n_{\alpha} - r_{,\alpha} n_{\beta}) + 2(1+\nu) r_{,\alpha} r_{,\beta} r_{,n} \right] = \lim_{r \rightarrow 0} T_{\alpha\beta}^{static} \tag{B. 4}$$

$$\lim_{r \rightarrow 0} \tilde{T}_{\alpha 3} = -\frac{3\kappa^2}{\pi h^2} n_{\alpha} \ln r = \lim_{r \rightarrow 0} T_{\alpha 3}^{static} \tag{B. 5}$$

$$\lim_{r \rightarrow 0} \tilde{T}_{3\alpha} = -\frac{1+\nu}{4\pi} n_{\alpha} \ln r = \lim_{r \rightarrow 0} T_{3\alpha}^{static} \tag{B. 6}$$

$$\lim_{r \rightarrow 0} \tilde{T}_{33} = -\frac{1}{2\pi r} r_{,n} = \lim_{r \rightarrow 0} T_{33}^{static} \tag{B. 7}$$

$$\lim_{r \rightarrow 0} \tilde{D}_{\alpha\beta\gamma} = \frac{1}{4\pi r} \left[(1-\nu) (\delta_{\beta\gamma} r_{,\alpha} + \delta_{\alpha\gamma} r_{,\beta} - \delta_{\alpha\beta} r_{,\gamma}) + 2(1+\nu) r_{,\alpha} r_{,\beta} r_{,\gamma} \right] = \lim_{r \rightarrow 0} D_{\alpha\beta\gamma}^{static} \quad (B. 8)$$

$$\lim_{r \rightarrow 0} \tilde{D}_{\alpha\beta 3} = -\frac{1+\nu}{4\pi} \delta_{\alpha\beta} \ln r = \lim_{r \rightarrow 0} D_{\alpha\beta 3}^{static} \quad (B. 9)$$

$$\lim_{r \rightarrow 0} \tilde{D}_{3\beta\gamma} = -\frac{3\kappa^2}{\pi h^2} \delta_{\beta\gamma} \ln r = \lim_{r \rightarrow 0} D_{3\beta\gamma}^{static} \quad (B. 10)$$

$$\lim_{r \rightarrow 0} \tilde{D}_{3\beta 3} = \frac{1}{2\pi r} r_{,\beta} = \lim_{r \rightarrow 0} D_{3\beta 3}^{static} \quad (B. 11)$$

$$\lim_{r \rightarrow 0} \tilde{S}_{3\beta\gamma} = -\frac{3\kappa^2 D(1-\nu)}{\pi h^2 r} (2r_{,\gamma} r_{,\beta} r_{,n} - n_{,\gamma} r_{,\beta}) = \lim_{r \rightarrow 0} S_{3\beta\gamma}^{static} \quad (B. 12)$$

$$\lim_{r \rightarrow 0} \tilde{S}_{\alpha\beta 3} = \frac{3\kappa^2 D(1-\nu)}{\pi h^2 r} (2r_{,\alpha} r_{,\beta} - \delta_{\alpha\beta}) r_{,n} = \lim_{r \rightarrow 0} S_{\alpha\beta 3}^{static} \quad (B. 13)$$

$$\begin{aligned} \lim_{r \rightarrow 0} \tilde{S}_{\alpha\beta\gamma} = \frac{D(1-\nu)}{4\pi r^2} \{ & (1-\nu) (\delta_{\gamma\alpha} n_{,\beta} + \delta_{\gamma\beta} n_{,\alpha} + 2n_{,\gamma} r_{,\alpha} r_{,\beta} - \delta_{\alpha\beta} n_{,\gamma}) \\ & + 2\nu [(n_{,\alpha} r_{,\beta} + n_{,\beta} r_{,\alpha}) r_{,\gamma} + \delta_{\alpha\beta} n_{,\gamma}] \} = \lim_{r \rightarrow 0} S_{\alpha\beta\gamma}^{static} \end{aligned} \quad (B. 14)$$

$$\lim_{r \rightarrow 0} \tilde{S}_{3\beta 3} = \frac{3\kappa^2 D(1-\nu)}{\pi h^2 r^2} (n_{,\beta} - 2r_{,\beta} r_{,n}) = \lim_{r \rightarrow 0} S_{3\beta 3}^{static} \quad (B. 15)$$

where U_{ij}^{static} , T_{ij}^{static} , $D_{k\beta j}^{static}$ and $S_{k\beta j}^{static}$ are fundamental solutions for static plate bending problems, which can be found in Ref. [36].

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