

HYPERELLIPTIC INTEGRALS AND MIRRORS OF THE JOHNSON–KOLLÁR DEL PEZZO SURFACES

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ABSTRACT. For all integers $k > 0$, we prove that the hypergeometric function

$$\widehat{I}_k(\alpha) = \sum_{j=0}^{\infty} \frac{((8k+4)j)!j!}{(2j)!(2k+1)j!^2((4k+1)j)!} \alpha^j$$

is a period of a pencil of curves of genus $3k+1$. We prove that the function $\widehat{I}_k$ is a generating function of Gromov–Witten invariants of the family of anticanonical del Pezzo hypersurfaces $X = X_{8k+4} \subset \mathbb{P}(2, 2k+1, 2k+1, 4k+1)$. Thus, the pencil is a Landau–Ginzburg mirror of the family. The surfaces X were first constructed by Johnson and Kollár. The feature of these surfaces that makes our mirror construction especially interesting is that $|-K_X| = |\mathcal{O}_X(1)| = \emptyset$. This means that there is no way to form a Calabi–Yau pair (X, D) out of X and hence there is no known mirror construction for X other than the one given here. We also discuss the connection between our construction and work of Beukers, Cohen and Mellit on hypergeometric functions.

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1. INTRODUCTION

1.1. Main result. We begin by stating our main result. More detail will be given in Sec. 2 and 4.

For all $k > 0$ integer, consider the hypergeometric function defined by the power series:

$$(1.1) \quad \widehat{I}_k(\alpha) = \sum_{j=0}^{\infty} \frac{((8k+4)j)!j!}{(2j)!((2k+1)j)!^2((4k+1)j)!} \alpha^j$$

The function $\widehat{I}_k$ satisfies a differential equation of order $6k+2$ written out explicitly in Remark 2.9 below.

For all $k > 0$ integer, consider the family of anticanonical del Pezzo hypersurfaces

$$(1.2) \quad X_{8k+4} \subset \mathbb{P}(2, 2k+1, 2k+1, 4k+1)$$

These surfaces form the main series of the classification of anticanonical quasi-smooth and wellformed two-dimensional weighted hypersurfaces [28].

In this paper, we give positive answers to the following two questions which — as we explain below — are for us equivalent:

Question 1.1. Is $\widehat{I}_k$ a period of a pencil of curves?

Question 1.2. Does the family of anticanonical del Pezzo hypersurfaces of Equation (1.2) have a Landau–Ginzburg (LG) mirror?

The connection between the two questions is that the hypergeometric function $\widehat{I}_k$ is a generating series for Gromov–Witten invariants of X_{8k+4} . More precisely:

Proposition 1.3. *The hypergeometric function $\widehat{I}_k$ is, up to a shift in α , a specialisation of the regularised quantum period of the family of Equation (1.2).*

The exact meaning of the proposition, including the definition of the regularised quantum period and the connection with mirror symmetry, is elaborated on in Sec. 2 and App. B; its proof is an exercise in Gromov–Witten theory that we carry out in App. B.

For all $k > 0$ integer, our answer to both questions is given by the one-parameter family of hyperelliptic curves:

$$(1.3) \quad Y_k = (\alpha y^2 - h_{k,\alpha}(t_0, t_1) = 0) \subset \mathbb{P}(1, 1, 3k+2) \times \mathbb{C}^\times$$

where t_0, t_1, y are coordinates on $\mathbb{P}(1, 1, 3k+2)$, α is a coordinate on $\mathbb{C}^\times$ and

$$(1.4) \quad \begin{aligned} h_{k,\alpha}(t_0, t_1) = & t_1(4t_1^{2k+1} + \alpha t_0^{2k+1}) \times \\ & \times (-64t_1^{4k+2} + t_0 t_1^{4k+1} - 32\alpha t_0^{2k+1} t_1^{2k+1} - 4\alpha^2 t_0^{4k+2}) \end{aligned}$$

together with the projection $w_k: Y_k \rightarrow \mathbb{C}^\times$ to the second factor.

Remark 1.4. For $\alpha \neq \alpha_{k,0}$, where

$$(1.5) \quad \alpha_{k,0} = \frac{(4k+1)^{4k+1}}{4^{8k+3}(2k+1)^{2(2k+1)}},$$

the fibre $Y_{k,\alpha} = w_k^{-1}(\alpha)$ is a nonsingular hyperelliptic curve of genus $3k+1$.

Our main result is:

Theorem 1.5 (Main Theorem). *For all $k > 0$ integer, $\widehat{I}_k(\alpha) = \pi_k(\alpha)$, where*

- (A) $\widehat{I}_k(\alpha)$ is the hypergeometric function of Equation (1.1) above, and
- (B) $\pi_k(\alpha)$ is the period of the family Y_k from (1.3) given by:

$$(1.6) \quad \pi_k(\alpha) = \frac{1}{2\pi i} \oint_{\gamma_{k,\alpha}} \frac{t^{2k} dt}{y}$$

where $\gamma_{k,\alpha} \subset Y_{k,\alpha}$ is the explicit cycle described in Sec. 3 and $t = t_1/t_0$.

The solutions of the differential equation satisfied by $\widehat{I}_k$ are sections of an irreducible complex local system on $U_k = \mathbb{P}^1 \setminus \{0, \alpha_{k,0}, \infty\}$ that we denote by $\mathbb{H}_k$, see Remark 2.9. Write

$$Y_{U_k} = w_k^{-1}(U_k)$$

and denote by $w_{U_k}: Y_{U_k} \rightarrow U_k$ the restriction. By Theorem 1.5 and Remark 2.5 below, $\mathbb{H}_k$ is a subquotient of the local system $R^1 w_{U_k} \star \mathbb{C}$. Since the two local systems have the same rank, we obtain:

Corollary 1.6. *For all $k > 0$ integer, $\mathbb{H}_k$ is isomorphic to $R^1 w_{U_k} \star \mathbb{C}$.*

Remark 1.7. Theorem 1.5 answers both Question 1 and 2 in the affirmative. In particular, it states that the pencil of hyperelliptic curves of Equations 1.3 and 1.4 is a LG mirror of the corresponding family of del Pezzo hypersurfaces. We explain our terminology on LG mirrors in Sec. 2.1 below. Thus, our LG mirror of the family of del Pezzo surfaces X_{8k+4} above is a pencil of curves of genus $3k+1$. It could not be otherwise, since $\widehat{I}_k$ satisfies an irreducible differential equation of order $6k+2$. The fact that the mirror is a pencil of curves of genus > 1 is a reflection of the fact that the del Pezzo surfaces are singular, and it is consistent with the picture put forward in [3]. As discussed in detail in Sec. B.2 (Step 2) the quantum period of the family X_{8k+4} is a function of α that depends on $6k$ parameters. It interesting to

ask whether the whole quantum period, and not just the specialisation considered here, is motivic, *see* [3, Conjecture B].

Corollary 1.6 states that the natural formulation of the two questions in terms of local systems also holds.

1.2. Connections to the work of Beukers, Cohen and Mellit. For all $k > 0$ integer, consider the manifold W_k :

$$(1.7) \quad (\alpha \cdot (u_1 + u_2 + u_3 + u_4 - 1) - u_1^2 u_2^{2k+1} u_3^{2k+1} u_4^{4k+1} = 0) \subset \mathbb{T}^4 \times \mathbb{C}^\times$$

where $\mathbb{T}^4 \cong (\mathbb{C}^\times)^4$ is a 4-dimensional torus with coordinates $u_1, \dots, u_4$ and α is a coordinate on $\mathbb{C}^\times$. Denote by $v_k: W_k \rightarrow \mathbb{C}^\times$ the second projection. The 3-dimensional hypersurface $W_{k,\alpha} = v_k^{-1}(\alpha) \subset \mathbb{T}^4$ is nonsingular if and only if $\alpha \neq \alpha_{k,0}$.

A very special case of the work of Beukers, Cohen and Mellit [4] relates point counting in characteristic p on $W_{k,\alpha}$ to finite analogs of the function $\hat{I}_k$ of Equation (1.1). Morally, this says that the hypergeometric function $\hat{I}_k$ is a period of the family of 3-folds $W_{k,\alpha}$ ($\alpha \in U_k$) or, equivalently, that the variation $H^3(W_{k,\alpha}, \mathbb{Q})$ is related to the local system $\mathbb{H}_k$. In Sec. 4 we prove:

Theorem 1.8. *For all $k > 0$ integer, write $W_{U_k} = v_k^{-1}(U_k)$ and denote by $v_{U_k}: W_{U_k} \rightarrow U_k$ the restriction. Then:*

$$(1.8) \quad \mathrm{gr}_3^W R^3 v_{U_k}! \mathbb{Q}(1) = R^1 w_{U_k} \star \mathbb{Q}$$

Remark 1.9. Since an irreducible local system supports at most one rational variation of Hodge structure [15, Proposition 2.1], our main theorem and its corollary imply that $\mathbb{H}_k = \mathrm{gr}_3^W R^3 v_{U_k}! \mathbb{Q}(1) = R^1 w_{U_k} \star \mathbb{Q}$ as variations of Hodge structure.

1.3. Perspectives from mirror symmetry. Our mirrors of the surfaces $X = X_{8k+4} \subset \mathbb{P}(2, 2k+1, 2k+1, 4k+1)$ are not covered by any known mirror construction. They fall outside the range of the Givental/Hori–Vafa construction for mirrors of toric complete intersections. Furthermore, since $H^0(X, -K_X) = H^0(X, \mathcal{O}_X(1)) = (0)$, there is no Calabi–Yau pair (X, D) , and hence the intrinsic mirror symmetry program [20] does not apply.

The Fano/LG correspondence (Sec. 2.1) suggests that the mirror of the family X_{8k+4} should be a function $w_k: Y_k \rightarrow \mathbb{C}^\times$ with one-dimensional fibres together with an identification of $R^1 w_{U_k}! \mathbb{C}$ with $\mathbb{H}_k$. This motivates our Question 2.

1.4. Perspectives from Hodge theory. Katz has shown that the local systems $\mathbb{H}_k^{\mathrm{red}}$ support a canonical rational variation of Hodge structures (VHS) [29, Theorem 5.4.4]. By the criterion of [14, 18], the variation has Hodge weight one. It is natural to *wonder* if there is a morphism $w_k: Y_k \rightarrow \mathbb{C}^\times$ with one-dimensional fibres such that $\mathbb{H}_k^{\mathrm{red}} = \mathrm{gr}_3^W R^3 v_{U_k}! \mathbb{Q}(1) = R^1 w_{U_k}! \mathbb{Q}$. Our Theorem 1.8 answers this in the affirmative: the required morphism w_k is our family of hyperelliptic curves.

On the other hand, there was no reason to *expect* that $\mathrm{gr}_3^W R^3 v_{U_k}! \mathbb{Q}(1)$ is the variation of the H^1 of a pencil of curves. Looking for an explanation, one is led to wonder whether the 3-folds $W_{k,\alpha}$ are rational. This is probably not the case. In Sec. A.4 we construct a conic bundle in the Mori category, birational to $W_{k,\alpha}$. It will be interesting to study the geometry of this conic bundle and conclude that $W_{k,\alpha}$ is not rational.

1.5. Notes on the proof. Our proof of the main theorem is elementary: we expand the period in power series with the help of the residue theorem. The ‘real work’ was to come up with the equation of Y_k and the integration cycles γ_k .

For Theorem 1.8, we begin with the pencil of 3-folds $v_k: W_k \rightarrow \mathbb{C}^\times$ of Equation (1.7). Morally, the work of Beukers–Cohen–Mellit [4] identifies the local system $\mathbb{H}_k$ with $\mathrm{gr}_3^W R^3 v_{U_k}! \mathbb{C}$.

At the beginning of Sec. 5 we produce a partial compactification $W_{k,\alpha} \subset \widehat{W}_{k,\alpha}$ and a del Pezzo fibration $\phi: \widehat{W}_{k,\alpha} \rightarrow \mathbb{C}^\times$. This del Pezzo fibration becomes visible after a monomial substitution of coordinates, *see* Equation (5.6). In App. A, we explain how we discovered the monomial substitution and the del Pezzo fibration by running a minimal model program for a partial resolution of a compactification of $W_{k,\alpha}$.

Once we have the del Pezzo fibration, the proof of Theorem 1.8 is an exercise in mixed Hodge theory. The constructions in Sec. 5 make it clear how the curve $Y = Y_{k,\alpha}$ arises naturally from the geometry of the del Pezzo fibration $\phi: \widehat{W}_{k,\alpha} \rightarrow \mathbb{C}^\times$.

The key point in the proof is Lemma 5.5, which constructs an algebraic cycle $Z \subset \mathrm{CH}_2(Y \times_{\mathbb{C}^\times} \widehat{W}_{k,\alpha})$ inducing an isomorphism $p_* \mathbb{Q}_Y \rightarrow R^2 \phi_* \mathbb{Q}_{\widehat{W}_{k,\alpha}}(1)$.

It is natural to ask if our computation of the middle cohomology of $W_{k,\alpha}$ in terms of the del Pezzo fibration $\phi: \widehat{W}_{k,\alpha} \rightarrow \mathbb{C}^\times$ really is the easiest way to prove Theorem 1.8. We think that it is. The conic bundle birational to $W_{k,\alpha}$ that we construct in Sec. A.4 gives another, richer but more complicated, way to understand the geometry of $W_{k,\alpha}$.

Finally, Proposition 1.3 is an exercise in orbifold Gromov–Witten theory made possible by the “quantum Lefschetz theorem without convexity” recently proved by Wang [34].

1.6. Further questions. It may be possible to determine all hypergeometric functions that are periods of a pencil of curves, and describe the pencils explicitly.

There are several contexts where a construction of mirror symmetry is available that produces a mirror of the wrong dimension: one of these is general toric complete intersections [4]; another is the abelian/nonabelian correspondence [7]. Our ideas here can form the basis of a systematic method to extract from these constructions mirrors of the correct dimension. In particular it would be extremely attractive to eliminate the assumptions of Hori–Vafa [22] and obtain mirrors for all toric complete intersections.

It will be interesting to see that our mirrors satisfy Homological Mirror Symmetry.

1.7. Structure of the paper. All sections in this paper logically depend on the introduction; other than that, they are logically mutually independent and can be read in any order. In Sec. 2 we expand on the notion of mirror symmetry for Fano varieties with quotient singularities. The families of surfaces (1.2) are an example of such varieties, and our main theorem provides mirrors of these families. In Sec. 3 we prove the main theorem. In Sec. 4 we discuss how our Equation (1.7) arises from the work [4] as a mirror of the wrong dimension. In Sec. 5 we prove Theorem 1.8. In App. A we explain how we discovered the del Pezzo fibration $\phi: \widehat{W} \rightarrow \mathbb{C}^\times$. The appendix ends with the construction of a compactification $W \subset W'$ with a conic bundle structure $\psi: W' \rightarrow \mathbb{F}_1$. In App. B we prove Proposition 1.3.

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2. FANO VARIETIES AND MIRROR SYMMETRY

In this section we spell the notion of mirror symmetry for Fano varieties with quotient singularities that we use. The families of surfaces $X = X_{8k+4}$ of the introduction are examples of such varieties; our Theorem 1.5 provides mirrors of these families.

In this section we denote a variety with quotient singularities by a roman capital, and by the corresponding fraktur capital the unique smooth wellformed orbifold¹ that has it as its coarse moduli space.

Let X be a Fano variety with quotient singularities, that is, a variety with quotient singularities such that $-K_X$ is ample. We call a Fano surface with quotient singularities a *log del Pezzo surface*.

We denote by $\mathbb{P}(a_0, a_1, \dots, a_m)$ the weighted projective m -space with weights $a_0 \leq a_1 \leq \dots \leq a_m$. We simply write $\mathbb{P}$ when m and the weights are clear from the context. For $X = X_{d_1, \dots, d_c}$ a quasismooth [23, Definition 6.3] and wellformed [23, Definition 6.9] weighted complete intersection of degrees $d_1, \dots, d_c$, the adjunction formula for the canonical sheaf K_X [23, Section 6.14] states that

$$K_X = \mathcal{O}_X\left(\sum_{i=1}^c d_i - \sum_{i=0}^m a_i\right).$$

By definition X is Fano if and only if $-K_X$ is ample, i.e., if and only if $\sum_{i=1}^c d_i < \sum_{i=0}^m a_i$. We say that X is *anticanonical* if $-K_X = \mathcal{O}_X(1)$, that is, $\sum_{i=1}^c d_i = -1 + \sum_{i=0}^m a_i$.

2.1. Our notion of mirror symmetry. We introduce the key ingredients to state what we mean by mirror of a Fano variety X of dimension n with quotient singularities.

2.1.1. The quantum period. Let $\mathfrak{X}$ be a Fano orbifold with coarse moduli space X and let $I\mathfrak{X}$ its inertia stack. Let $\text{Box}(\mathfrak{X})$ be the set indexing the components of $I\mathfrak{X}$ and for b in $\text{Box}(\mathfrak{X})$ denote by $\text{age}(b)$ the *age* of the component $\mathfrak{X}_b$ [6, Section 3.2]. Let $H_{\text{orb}}^\bullet(\mathfrak{X})$ be the ‘even part’ of the *Chen–Ruan cohomology* of $\mathfrak{X}$, that is, the $\mathbb{Q}$ -vector space with grading:

$$(2.1) \quad H_{\text{orb}}^p(\mathfrak{X}) = \bigoplus_{\substack{b \in \text{Box}(\mathfrak{X}) \\ p - 2\text{age}(b) \in 2\mathbb{Z}}} H^{p-2\text{age}(b)}(\mathfrak{X}_b, \mathbb{Q}) \quad \forall p \in \mathbb{Q}$$

¹By definition, a wellformed orbifold is a Deligne–Mumford stack where the set of points with nontrivial stabiliser has codimension ≥ 2 .

Given $d \in H_2(X, \mathbb{Z})$, $n \geq 0$, $\alpha_1, \dots, \alpha_n \in H_{\text{orb}}^\bullet(\mathfrak{X})$, and $k_1, \dots, k_n$ non-negative integers, one can define the *Gromov–Witten invariant* [1, 2, 5]:

$$(2.2) \quad \langle \alpha_1 \psi^{k_1}, \dots, \alpha_n \psi^{k_n} \rangle_{0,n,d}$$

Write $\text{Eff}(\mathfrak{X}) \subset H_2(X, \mathbb{Z})$ for the submonoid generated by the homology classes in X that are images of representable maps from complete stacky curves to $\mathfrak{X}$.

Definition 2.1. [32, Definition 3.2] Let $\mathfrak{X}$ be a Fano orbifold and let $H_{\text{orb}}^{<2}(\mathfrak{X}) = \mathbb{Q}\mathbf{1}_{b_1} \oplus \dots \oplus \mathbb{Q}\mathbf{1}_{b_r}$ be the subspace of $H_{\text{orb}}^\bullet(\mathfrak{X})$ generated by the identity classes of the components $\mathfrak{X}_{b_i}$ of the inertia stack $I\mathfrak{X}$ such that $0 < \text{age}(b_i) < 1$. For $i = 1, \dots, r$ let $\mathbf{1}_{b_i}$ be the identity cohomology class of the component $\mathfrak{X}_{b_i}$. For $d \in \text{Eff}(\mathfrak{X})$, $n \in \mathbb{N}$, and $1 \leq i_1, \dots, i_n \leq r$ write:

$$\delta_{d, i_1, \dots, i_n} = -K_X \cdot d + \sum_{j=1}^n (1 - \text{age}(b_{i_j})) \in \mathbb{Q}$$

The *quantum period* of $\mathfrak{X}$ is:

$$(2.3) \quad G_{\mathfrak{X}}(x; \alpha) = 1 + \sum_{d \in \text{Eff}(\mathfrak{X})} \sum_{n=0}^{\infty} \sum_{1 \leq i_1, \dots, i_n \leq r} \langle \mathbf{1}_{b_{i_1}}, \dots, \mathbf{1}_{b_{i_n}}, \frac{\phi_{\text{vol}}}{1-\psi} \rangle_{0,n+1,d} \frac{x_{i_1} \dots x_{i_n}}{n!} \alpha^{\delta_{d, i_1, \dots, i_n}}$$

where $\phi_{\text{vol}} \in H^{2\dim \mathfrak{X}}(\mathfrak{X}, \mathbb{Q})$ is the cohomology class of a point, and $\alpha, x = (x_1, \dots, x_r)$ are formal variables.

Remark 2.2. [32, Proposition 3.4] shows that $G_{\mathfrak{X}}(x; \alpha) \in \mathbb{Q}[x_1, \dots, x_r][[\alpha]]$. Thus one can regard $G_{\mathfrak{X}}(x; \alpha)$ as a formal power series:

$$(2.4) \quad G_{\mathfrak{X}}(x; \alpha) = \sum_{\delta=0}^{\infty} c_{\delta}(x) \alpha^{\delta}$$

Definition 2.3. The *regularised quantum period* of $\mathfrak{X}$ is the Fourier–Laplace transform:

$$(2.5) \quad \hat{G}_{\mathfrak{X}}(x; \alpha) = \sum_{\delta=0}^{\infty} \delta! c_{\delta}(x) \alpha^{\delta}$$

2.1.2. The period of a LG model.

Definition 2.4. A *Landau–Ginzburg (LG) model* is a tuple (Y^n, w, ω, γ) where:

- (i) Y is a smooth algebraic manifold of dimension n ;
- (ii) $w: Y \rightarrow \mathbb{C}^\times$ is a projective morphism. For $\alpha \in \mathbb{C}^\times$ we denote by $Y_\alpha = w^{-1}(\alpha)$ the fibre.

Denote by $U \subset \mathbb{C}^\times$ the set of regular values of w , and by $w_U: Y_U = w^{-1}(U) \rightarrow U$ the restriction.

- (iii) $\omega \in \Gamma(U, w_U \star \Omega_{Y_U/U}^{n-1})$. For $\alpha \in U$, we write $\omega_\alpha \in H^0(Y_\alpha, \Omega_{Y_\alpha}^{n-1})$ the corresponding form;
- (iv) $\gamma \in \Gamma(D^\times, R^{n-1} w_U! \mathbb{Q}_{Y_U})$ where $0 \in D \subset \mathbb{C}$ is a small disk and $D^\times = D \setminus \{0\}$. For $\alpha \in D^\times$ we denote by $\gamma_\alpha \in H_{n-1}(Y_\alpha, \mathbb{Q})$ the corresponding cycle.

The *period* of the LG model is the function

$$(2.6) \quad \pi(\alpha) = \int_{\gamma_\alpha} \omega_\alpha \quad (\alpha \in D^\times)$$

Remark 2.5. Let $\mathcal{D}_U$ be the sheaf of polynomial differential operators on U . It follows directly from Definition 2.4 that the period π of Equation (2.6) is annihilated by differential operators in $\mathcal{D}_U$ and that the local system of solutions to the annihilator ideal of π is a subquotient of $R^{n-1}w_U!\mathbb{C}$.

Indeed, write

$$\mathcal{E} = (R^{n-1}w_U\star\mathbb{Q}) \otimes \mathcal{O}_U.$$

The sheaf $\mathcal{E}$ carries an algebraic connection $\nabla: \mathcal{E} \rightarrow \Omega_U^1 \otimes \mathcal{E}$ (the Gauss–Manin connection) by means of which we can regard it as a $\mathcal{D}_U$ -module, and

$$R^{n-1}w_U\star\mathbb{C} = \underline{\mathrm{Hom}}_{\mathcal{D}_U}(\mathcal{O}_U, \mathcal{E}), \quad \text{and} \quad R^{n-1}w_U!\mathbb{C} = \underline{\mathrm{Hom}}_{\mathcal{D}_U}(\mathcal{E}, \mathcal{O}_U)$$

are the local systems of flat sections and of solutions of $\mathcal{E}$. Now $w_U\star\Omega_{Y_U/U}^{n-1} \subset \mathcal{E}$ is the last piece of the Hodge filtration and we regard ω as a section of $\mathcal{E}$ by means of this inclusion. On the other hand we regard γ as a solution of $\mathcal{E}$ and recover the period as $\pi = \gamma(\omega)$. We have an inclusion and a surjection of $\mathcal{D}_U$ -modules:

$$\mathcal{D}_U \cdot \omega \subset \mathcal{E}, \quad \text{and} \quad \gamma: \mathcal{D}_U \cdot \omega \rightarrow \mathcal{D}_U \cdot \pi$$

So $\underline{\mathrm{Hom}}_{\mathcal{D}_U}(\mathcal{D}_U \cdot \pi, \mathcal{O}_U) \subset \underline{\mathrm{Hom}}_{\mathcal{D}_U}(\mathcal{D}_U \cdot \omega, \mathcal{O}_U)$ and $R^{n-1}w_U!\mathbb{C} = \underline{\mathrm{Hom}}_{\mathcal{D}_U}(\mathcal{E}, \mathcal{O}_U) \twoheadrightarrow \underline{\mathrm{Hom}}_{\mathcal{D}_U}(\mathcal{D}_U \cdot \omega, \mathcal{O}_U)$.

2.1.3. Mirror symmetry. We state what we mean by mirror of a Fano variety with quotient singularities.

Definition 2.6. Let X be a Fano variety of dimension n with quotient singularities. A LG model (Y^n, w, ω, γ) is *mirror* of X if there is an affine-linear isomorphism $\varphi: \mathbb{C} \rightarrow \mathbb{C}$ such that a specialisation $\hat{G}_{\mathfrak{X}}(\bar{x}; \cdot)$ of $\hat{G}_{\mathfrak{X}}$ and π satisfy for all α :

$$(2.7) \quad \hat{G}_{\mathfrak{X}}(\bar{x}; \varphi(\alpha)) = \pi(\alpha)$$

Remark 2.7. If $X = X_{d_1, \dots, d_c} \subset \mathbb{P} = \mathbb{P}(a_0, \dots, a_m)$ is a quasismooth wellformed anticanonical weighted complete intersection and $a_0 \geq 2$, then $H^0(X, -K_X) = H^0(X, \mathcal{O}_X(1)) = (0)$.

It follows from this that X does not admit a qG-degeneration to a toric variety.

We briefly sketch the argument here. If $f: \mathcal{X} \rightarrow T$ is a locally stable family [30, Definition 2.2] over a 1-dimensional regular base T , then for all $n \in \mathbb{Z}$ $\omega_{\mathcal{X}/T}^{[m]}$ has the base change property [30, Proposition 2.76]. If the fibres are Fano varieties, then by Kawamata–Viehweg vanishing for all $t \in T$ and for all $i > 0$ $H^i(\mathcal{X}_t, -K_{\mathcal{X}_t}) = (0)$. It follows from these two facts and the theorems on cohomology and base change that $h^0(\mathcal{X}_t, -K_{\mathcal{X}_t})$ is locally constant in T . Now we just observed that $h^0(X, -K_X) = 0$, whereas it is always > 0 for a toric variety — at the very least, the toric boundary is an element of $|-K|$.

Remark 2.8. The surfaces we study in this paper, *see* Equation (1.2), have one $\frac{1}{4k+1}(1, 1)$ point, and four $\frac{1}{2k+1}(1, k)$ points. These singularities are locally qG-rigid in the sense of [3], hence for these surfaces our Definition 2.6 fits inside the context of [3, Conjecture B].

2.2. Anticanonical del Pezzo weighted hypersurfaces. Johnson and Kollár classify all anticanonical quasismooth wellformed del Pezzo surfaces in weighted projective 3-spaces [28]. Their classification consists of 22 sporadic cases and the sequence (1.2), where $k \in \mathbb{N}$, $k > 0$. The 22 sporadic cases are all listed in [28, Theorem 8].

Our Proposition 1.3 states that for all $k > 0$ integer the hypergeometric function $\widehat{I}_k$ of Equation (1.1) is up to a shift in α a specialisation of the regularised quantum period of all quasismooth surfaces of the family (1.2). We prove Proposition 1.3 in Appendix B.

Remark 2.9. The hypergeometric function $\widehat{I}_k$ of Equation (1.1) is annihilated by an irreducible regular hypergeometric differential operator on $\mathbb{P}^1$ which we denote by H_k . Explicitly,

$$H_k = \alpha_{k,0} \cdot \prod_{j=0}^{2k} \left(D - \frac{j}{2k+1} \right) \prod_{j=0}^{4k} \left(D - \frac{j}{4k+1} \right) - \alpha \cdot \prod_{\substack{j=1 \\ 4 \nmid j, j \neq 4k+2}}^{8k+3} \left(D + \frac{j}{8k+4} \right)$$

where $\alpha_{k,0}$ is as in Equation (1.5) and $D = \alpha d/d\alpha$. The operator H_k is singular on $\Sigma_k = \{0, \alpha_{k,0}, \infty\}$ and has order $6k+2$. It follows from [29, Corollary 3.2.1] that the local system $\mathbb{H}_k$ of solutions to H_k is an irreducible complex local system of rank $6k+2$ on $U_k = \mathbb{P}^1 \setminus \Sigma_k$.

Remark 2.10. In light of the definitions in Sec. 2.1 and Proposition 1.3, the data $(Y_k, w_k, \omega_k, \gamma_k)$ at the beginning of Sec. 3 define a LG model and our Theorem 1.5 asserts that this LG model is a mirror to the corresponding family of Johnson and Kollár. Also, by Remarks 2.5 and 2.9, $\mathbb{H}_k \simeq R^1 w_{U_k \star} \mathbb{C}$, as stated in Corollary 1.6.

3. PROOF OF THE MAIN THEOREM

In this section, we prove the main theorem stated in the Introduction. We begin by giving data to construct the period integral.

3.1. The period integral. Fix an integer $k > 0$. We define data $(Y_k, w_k, \omega_k, \gamma_k)$ as follows:

- (i) $Y_k \subset \mathbb{P}(1, 1, 3k+2) \times \mathbb{C}^\times$ is the 2-dimensional manifold given by (1.3) and (1.4);
- (ii) $w_k: Y_k \rightarrow \mathbb{C}^\times$ is the projection on the second factor.

Let $\alpha_{k,0}$ be as in Equation (1.5) and consider $\alpha \neq \alpha_{k,0}$.

- (iii) $\omega_{k,\alpha} = \frac{1}{2\pi i} \frac{t^{2k} dt}{y}$, where $t = t_1/t_0$;
- (iv) $\gamma_{k,\alpha}$ is the cycle that we describe next.

From this data we construct the period integral:

$$\pi_k(\alpha) = \oint_{\gamma_{k,\alpha}} \omega_{k,\alpha}$$

This is the period of Y_k of Equation (1.6).

Remark 3.1. This data defines a LG model, according to Definition 2.4, and the period (1.6) is the period of the LG model (2.6). Our main theorem can be interpreted as stating that this LG model is a mirror of the family of surfaces X_k .

3.2. The cycle of integration. Let us denote by $p_{k,\alpha}: Y_{k,\alpha} \rightarrow \mathbb{P}^1$ the $2:1$ cover; $p_{k,\alpha}$ is branched at the $6k+4$ roots of the polynomial

$$h_{k,\alpha}(t) = t(4t^{2k+1} + \alpha) (-64t^{4k+2} + t^{4k+1} - 32\alpha t^{2k+1} - 4\alpha^2)$$

Lemma 3.2. For $|\alpha| \ll 1$ the polynomial $h_{k,\alpha}$ has:

- A root at $t = 0$;

- $4k + 1$ roots of norm $\sim |\alpha|^{\frac{2}{4k+1}}$;
- $2k + 1$ roots of norm $\sim |\alpha|^{\frac{1}{2k+1}}$;
- A root of norm $\sim \frac{1}{64}$.

Proof. Clearly $h_{k,\alpha}$ has a root at $t = 0$ and $2k + 1$ roots of norm $|\alpha|^{\frac{1}{2k+1}}$. Consider the polynomial $g_{k,\alpha}(t) = -64t^{4k+2} + t^{4k+1} - 32\alpha t^{2k+1} - 4\alpha^2$; then as $\alpha \rightarrow 0$ $g_{k,\alpha}(t)$ has a root $t_\alpha \sim \frac{1}{64}$ and $4k + 1$ roots t_α of norm $\sim |\alpha|^{\frac{2}{4k+1}}$. Indeed

$$\lim_{\alpha \rightarrow 0} g_{k,\alpha}(t) = t^{4k+1}(-64t + 1)$$

hence for $|\alpha| \ll 1$ $g_{k,\alpha}$ has a root that tends to $\frac{1}{64}$ and $4k + 1$ roots that tend to 0. Now t_α is a root if and only if

$$t_\alpha^{4k+1} = 4(4t_\alpha^{2k+1} + \alpha)^2$$

and if $t_\alpha \rightarrow 0$, then $|t_\alpha|^{4k+1} \sim |\alpha|^2$. $\square$

Choose a continuous function $\rho_k : (0, 1) \rightarrow \mathbb{R}$ such that for $r \ll 1$:

$$r^{\frac{2}{4k+1}} \ll \rho_k(r) \ll r^{\frac{1}{2k+1}}$$

For $|\alpha| \ll 1$ let $\bar{\gamma}_{k,\alpha}$ be the circle of radius $\rho_k(|\alpha|)$ around the origin in $\mathbb{C}$ starting at $t_0 = \rho_k(|\alpha|)$. By Lemma 3.2 this circle divides $\mathbb{P}^1$ into two regions each containing an even number of branch points of $p_{k,\alpha}$, see Figure 1. Hence $p_{k,\alpha}^{-1}(\bar{\gamma}_{k,\alpha}) \subset Y_{k,\alpha}$ consists of two disjoint circles: we take $\gamma_{k,\alpha}$ to be the lift along which the power series expansion in Equation (3.1) is valid, see Figure 2.²

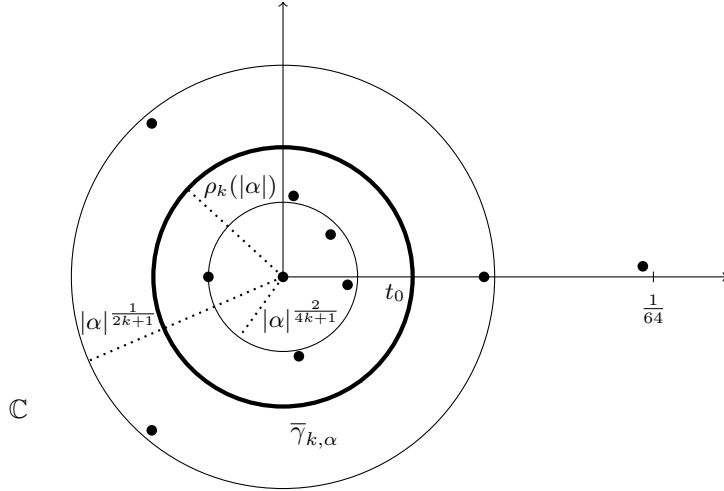


FIGURE 1. The circle $\bar{\gamma}_{k,\alpha} \subset \mathbb{C}$. The marked points represent the roots of $h_{k,\alpha}$ for $k = 1$.

²Alternatively we choose the base point for our lift such that $y > 0$.

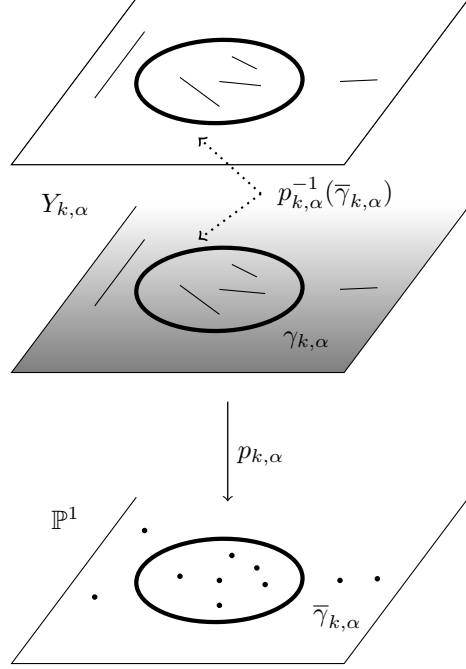


FIGURE 2. The circle $\gamma_{k,\alpha}$ on $Y_{k,\alpha}$. The gray sheet of the cover represents the one where (3.1) is valid. The line segments on each sheet indicate a choice of branch cuts for $k = 1$.

3.3. Proof of the main theorem. An immediate consequence of the next two lemmas. $\square$

Lemma 3.3. *Let $\pi_k(\alpha)$ be the period integral described above. Then, setting $m = (2k + 1)j$:*

$$(3.1) \quad \pi_k(\alpha) = \sum_{j=0}^{\infty} \binom{-1/2}{m} (-1)^m 4^{3m-j} \sum_{p=j}^{2m} \binom{-1/2}{p-j} \binom{2m}{p} \alpha^j$$

Proof. By a small manipulation we write the period as:

$$(3.2) \quad \pi_k(\alpha) = \frac{1}{2\pi i} \oint_{\bar{\gamma}_{k,\alpha}} \frac{1}{\sqrt{\left(1 + \frac{4t^{2k+1}}{\alpha}\right)}} \frac{1}{\sqrt{\left(1 - t \left(8 + \frac{2\alpha}{t^{2k+1}}\right)^2\right)}} \frac{dt}{t}.$$

By the defining inequalities of the function ρ_k , and our choice of $\gamma_{k,\alpha}$, both the following power series expansions hold:

$$\begin{aligned} \left(1 + \frac{4t^{2k+1}}{\alpha}\right)^{-\frac{1}{2}} &= \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{4^n t^{(2k+1)n}}{\alpha^n} \\ \left(1 - t \left(8 + \frac{2\alpha}{t^{2k+1}}\right)^2\right)^{-\frac{1}{2}} &= \sum_{m=0}^{\infty} \binom{-1/2}{m} (-1)^m 4^{3m} t^m \left(1 + \frac{\alpha}{4t^{2k+1}}\right)^{2m} \end{aligned}$$

Plugging the two power series in (3.2), switching the series and the integral signs and using the binomial theorem we obtain:

$$\begin{aligned} \pi_k(\alpha) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{2m} \binom{-1/2}{n} \binom{-1/2}{m} \binom{2m}{p} (-1)^m 4^{3m+n-p} \alpha^{p-n} \times \\ &\quad \times \frac{1}{2\pi i} \oint_{\gamma_{k,\alpha}} \frac{dt}{t} t^{m-(2k+1)(p-n)}. \end{aligned}$$

Finally, applying the residue theorem around $t = 0$ and setting $j = p - n$, we obtain the result. $\square$

Lemma 3.4. *For all $k > 0$ and $j \geq 0$ integers, setting $m = (2k + 1)j$:*

$$(3.3) \quad \binom{-1/2}{m} (-1)^m 4^{3m-j} \sum_{p=j}^{2m} \binom{-1/2}{p-j} \binom{2m}{p} = \frac{((8k+4)j)! j!}{(2j)! ((2k+1)j)!^2 ((4k+1)j)!}$$

Proof. Clearly (3.3) holds when $j = 0$, thus we assume $j \geq 1$. In what follows we repeatedly use the identity:

$$(3.4) \quad (2l-1)!! = \frac{(2l)!}{2^l \cdot l!} \quad (l > 0 \text{ integer})$$

We have:

$$(3.5) \quad \binom{-1/2}{m} = \frac{(-1)^m (2m-1)!!}{2^m m!} = \frac{(-1)^m}{4^m} \binom{2m}{m}$$

We set $i = p - j$ and we write the finite sum on the left hand side of (3.3) as:

$$\begin{aligned} (3.6) \quad \sum_{p=j}^{2m} \binom{-1/2}{p-j} \binom{2m}{p} &= \sum_{i=0}^{2m-j} \binom{-1/2}{i} \binom{2m}{j+i} = \\ &= \sum_{i=0}^{2m-j} \binom{-1/2}{i} \binom{2m}{2m-j-i} = \binom{2m-1/2}{2m-j} \end{aligned}$$

where the last equality in (3.6) follows from the Chu–Vandermonde formula for generalised binomial coefficients:

$$\sum_{i=0}^n \binom{\beta}{i} \binom{\alpha}{n-i} = \binom{\beta+\alpha}{n} \quad (\alpha, \beta \in \mathbb{C}, n \in \mathbb{N})$$

Plugging (3.5) and (3.6) in (3.3) and using that $m = (2k+1)j$ we can rewrite (3.3) as:

$$(3.7) \quad 4^{(4k+1)j} ((4k+2)j)! \binom{(4k+2)j-1/2}{(4k+1)j} = \frac{((8k+4)j)! j!}{(2j)! ((4k+1)j)!}$$

Now we note that

$$\binom{(4k+2)j-1/2}{(4k+1)j} = \frac{((8k+4)j-1)!!}{(2j-1)!! 2^{(4k+1)j} ((4k+1)j)!}$$

Using this equality combined with (3.4) for $l = j$, we simplify (3.7) as:

$$(3.8) \quad 2^{(4k+2)j} ((4k+2)j)! ((8k+4)j-1)!! = ((8k+4)j)!$$

Since (3.8) manifestly holds (it is (3.4) for $l = (4k+2)j$), the result is proved. $\square$

4. RELATIONS TO THE WORK OF BEUKERS, COHEN, MELLIT

In this section we show how our Equation (1.7) arises from [4] as a mirror of the wrong dimension.

4.1. Finite hypergeometric functions and point counting. We summarise the main result in [4].

Let $v, w \in \mathbb{Q}^d$ be such that for all $i, j \in \{1, \dots, d\}$ $v_i \neq w_j \bmod \mathbb{Z}$ and the polynomials $\prod_{j=1}^d (x - e^{2\pi i v_j})$ and $\prod_{j=1}^d (x - e^{2\pi i w_j})$ are products of cyclotomic polynomials. Then there exist natural numbers $p_1, \dots, p_r$ and $q_1, \dots, q_s$, with $p_1 + \dots + p_r = q_1 + \dots + q_s$, such that

$$\prod_{j=1}^d \frac{x - e^{2\pi i v_j}}{x - e^{2\pi i w_j}} = \frac{\prod_{j=1}^r x^{p_j} - 1}{\prod_{j=1}^s x^{q_j} - 1}$$

and the analytic hypergeometric function

$${}_dF_{d-1}(v, w|\lambda) = \sum_{n=0}^{\infty} \frac{(v_1)_n \cdots (v_d)_n}{(w_1)_n \cdots (w_d)_n} \lambda^n$$

where

$$(x)_n = \begin{cases} x(x+1) \cdots (x+n-1) & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}$$

can be rewritten in the form:

$$(4.1) \quad {}_dF_{d-1}(v, w|\lambda) = \sum_{j=0}^{\infty} \frac{(p_1 j)! \cdots (p_r j)!}{(q_1 j)! \cdots (q_s j)!} M^{-j} \lambda^j, \quad M = \frac{p_1^{p_1} \cdots p_r^{p_r}}{q_1^{q_1} \cdots q_s^{q_s}}$$

Similarly, the finite hypergeometric function $H_q(v, w|\lambda)$ — where q is a prime power coprime with $\text{hcf}(v, w)$ — can be written in terms of p_i, q_j only [4, Definition 1.1, Theorem 1.3].

For all $\alpha \in \mathbb{F}_q^\times$, Beukers, Cohen and Mellit consider the quasiprojective (in fact affine) variety $W_\alpha \subset \mathbb{G}_m^{r+s}$ given by the homogeneous equations:

$$\begin{cases} y_1 + y_2 + \cdots + y_r - x_1 - \cdots - x_s = 0 \\ \alpha \cdot y_1^{p_1} \cdots y_r^{p_r} = x_1^{q_1} \cdots x_s^{q_s} \end{cases}$$

and prove the following, *see* [4, Theorem 1.5] for the precise statement and [4, Section 5] for its proof:

Theorem 4.1. *If $\gcd(p_1, \dots, p_r, q_1, \dots, q_s) = 1$ and $M \cdot \alpha \neq 1$, there exists a nonsingular completion $\overline{W}_\alpha$ of W_α such that $|\overline{W}_\alpha(\mathbb{F}_q)| = H_q(v, w, M \cdot \alpha)$ up to factors depending only on p_i, q_j .*

4.2. The family of 3-folds of Theorem 1.8. We provide some additional context for our statement of Theorem 1.8. Specifically, we build a precise connection with Theorem 4.1.

Fix $k > 0$ integer. We specialise the above discussion to the pair of vectors v_k, w_k in $\mathbb{Q}^{d_k}$, $d_k = 6k + 2$:

$$v_k = \left(\frac{j}{8k+4} \right)_{j \in \{1, \dots, 8k+3\} \setminus (\{4i\}_{i \in \{1, \dots, 2k\}} \cup \{4k+2\})}$$

$$w_k = \left(\frac{l}{2k+1}, \frac{m}{4k+1} \right)_{l \in \{1, \dots, 2k+1\}, m \in \{1, \dots, 4k+1\}}$$

This leads to

$$p_k = (8k+4, 1), \quad q_k = (2, 2k+1, 2k+1, 4k+1)$$

For $\lambda = M_k \cdot \alpha$, by (4.1)

$$d_k F_{d_k-1}(v_k, w_k | \lambda) = \sum_j \frac{((8k+4)j)! j!}{(2j)! ((2k+1)j)!^2 ((4k+1)j)!} M_k^{-j} \lambda^j = \widehat{I}_k(\alpha)$$

that is, the function $\widehat{I}_k$ of Equation (1.1).

Still following the above discussion, for all α in $\mathbb{C}^\times$ consider the quasiprojective (in fact, affine) 3-fold $W_{k,\alpha} \subset \mathbb{C}^{\times 6}$ given by the homogeneous equations:

$$(4.2) \quad \begin{cases} y_1 + y_2 - (x_1 + x_2 + x_3 + x_4) = 0 \\ \alpha \cdot y_1^{8k+4} y_2 = x_1^2 x_2^{2k+1} x_3^{2k+1} x_4^{4k+1} \end{cases}$$

Next, we manipulate the Equations (4.2) to rewrite them as in the introduction. By solving the first equation for y_2 and writing $x_0 = y_1$, for all α in $\mathbb{C}^\times$ the system of Equations (4.2) leads to the equation of a 3-fold in $\mathbb{P}^4$:

$$\alpha \cdot x_0^{8k+4} (x_1 + x_2 + x_3 + x_4 - x_0) = x_1^2 x_2^{2k+1} x_3^{2k+1} x_4^{4k+1}$$

The 3-fold $W_{k,\alpha}$ is the intersection of the projective 3-fold defined by the equation above with the torus $\mathbb{T}^4 \subset \mathbb{P}^4$ and, in the affine chart $\{x_0 \neq 0\}$ with coordinates $u_i = x_i/x_0$, it is described by the equation:

$$\alpha \cdot (u_1 + u_2 + u_3 + u_4 - 1) - u_1^2 u_2^{2k+1} u_3^{2k+1} u_4^{4k+1} = 0 \quad (\forall i \ u_i \neq 0)$$

This is the same as Equation (1.7); hence the meaning of the symbol $W_{k,\alpha}$ is the same as in the introduction.

Theorem 4.1 computes $|\overline{W}_{k,\alpha}(\mathbb{F}_q)|$ in terms of the finite analog of $\widehat{I}_k$. The statement can be interpreted as asserting that the manifold W_k of Equation (1.7) is a LG mirror of the family of del Pezzo surfaces $X_{8k+4} \subset \mathbb{P}(2, 2k+1, 2k+1, 4k+1)$ of the wrong dimension. Indeed, by [16] the only interesting cohomology group of $\overline{W}_{k,\alpha}$ occurs in degree 3, hence by the Weil conjectures the main contribution to $|\overline{W}_{k,\alpha}(\mathbb{F}_q)|$ comes from $H^3(\overline{W}_{k,\alpha}, \mathbb{Q})$. Presumably for this completion we also have $H^3(\overline{W}_{k,\alpha}, \mathbb{Q}) = \text{gr}_3^W H^3(W_{k,\alpha}, \mathbb{Q})$ — as we do in Lemma 5.2.

5. PROOF OF THEOREM 1.8

In this section we prove Theorem 1.8 of the introduction.

Fix $k > 0$ integer and let $U_k = \mathbb{C}^\times \setminus \{\alpha_{k,0}\}$, as in the introduction. We prove that for all $\alpha \in U_k$ there is an isomorphism of pure Hodge structures:

$$(5.1) \quad \text{gr}_3^W H_c^3(W_{k,\alpha}, \mathbb{Q})(1) = H^1(Y_{k,\alpha}, \mathbb{Q})$$

5.1. Notation. For convenience in what follows we set $a = 1/\alpha$. We write Equation (1.7) as

$$(5.2) \quad u_1 + u_2 + u_3 + u_4 - 1 - au_1^2 u_2^{2k+1} u_3^{2k+1} u_4^{4k+1} = 0$$

and Equation (1.3) as

$$(5.3) \quad a^2 y^2 - t_1(4at_1^{2k+1} + t_0^{2k+1})(a^2 t_0 t_1^{4k+1} - 4(4at_1^{2k+1} + t_0^{2k+1})^2) = 0$$

We denote by $W_{k,a}$ and $Y_{k,a}$ the fibres of (5.2) and (5.3) over a , so $W_{k,a} = W_{k,\alpha}$ and $Y_{k,a} = Y_{k,\alpha}$ where $\alpha = 1/a$.

For the rest of this section we fix $k > 0$ and $a \neq a_{k,0}$, where $a_{k,0} = 1/\alpha_{k,0}$, and we omit all reference to k and a . Also, we write $p_{\mathbb{C}^\times} : Y_{\mathbb{C}^\times} = p^{-1}(\mathbb{C}^\times) \rightarrow \mathbb{C}^\times$ for the restriction of the $2 : 1$ cover $p : Y \rightarrow \mathbb{P}^1$.

5.2. Proof of Theorem 1.8. Let $W \subset \widehat{W}$ be the partial compactification of Equation (5.8) and $\phi : \widehat{W} \rightarrow \mathbb{C}^\times$ the projective degree 2 del Pezzo fibration of Lemma 5.3. Applying in sequence Lemmas 5.2, 5.4, 5.5 we see:

$$\begin{aligned} \mathrm{gr}_3^W H_c^3(W, \mathbb{Q}) &= \mathrm{gr}_3^W H_c^3(\widehat{W}, \mathbb{Q}) = \\ &= \mathrm{gr}_3^W H_c^1(\mathbb{C}^\times, R^2 \phi_* \mathbb{Q}) = (\mathrm{gr}_1^W H_c^1(\mathbb{C}^\times, p_{\mathbb{C}^\times *} \mathbb{Q}))(-1) \end{aligned}$$

and the latter group is $H^1(Y, \mathbb{Q})(-1)$. $\square$

5.3. The partial compactification. We construct a partial compactification $\widehat{W} \subset \mathbb{P}(1, 1, 2, 2)/\mu_2 \times \mathbb{C}^\times$ of W (where μ_2 acts on $\mathbb{P}(1, 1, 2, 2)$ as described below). Lemma 5.2 states that $\mathrm{gr}_3^W H_c^3(W, \mathbb{Q}) = \mathrm{gr}_3^W H_c^3(\widehat{W}, \mathbb{Q})$ hence for our purpose we might as well work with $\widehat{W}$ in place of W . The advantage of working with $\widehat{W}$ is that, as stated in Lemma 5.3, the second projection $\phi : \widehat{W} \rightarrow \mathbb{C}^\times$ is a fibration with fibres del Pezzo surfaces, and the point of the substitution (5.6) is precisely to make this structure manifest. (In Appendix A we explain how we discovered the del Pezzo fibration structure, and the substitution, by the methods of the minimal model program.)

Consider the weighted projective space $\mathbb{P}(1, 1, 2, 2)$ with weighted homogeneous coordinates x_1, x_2, y_1, y_2 , and the quotient $\mathbb{P}(1, 1, 2, 2)/\mu_2$ where the group μ_2 acts on the affine coordinates $x = x_2/x_1, y = y_1/x_1^2, z = y_2/x_1^2$ by

$$(5.4) \quad (x, y, z) \mapsto (-x, -y, -z)$$

Here and in what follows we write

$$(5.5) \quad G = \mathbb{P}(1, 1, 2, 2)/\mu_2 \times \mathbb{C}^\times$$

and we denote by t the coordinate on $\mathbb{C}^\times$. Note that G is a (noncompact) toric variety and the 4-dimensional torus $\mathbb{T}_G \subset G$ is the locus $(x, y, z, t \neq 0)$ in the affine piece $x_1 \neq 0$. The substitution

$$(5.6) \quad u_1 = x^{-1}y \quad u_2 = x^3 z^{-1}t \quad u_3 = x^{-1}z^{-1} \quad u_4 = x^{-1}z$$

identifies W with

$$(5.7) \quad (-at^{2k+1}y^2 + yz + z^2 + 1 - xz + tx^4 = 0) / \mu_2 \subset \mathbb{T}_G$$

We denote by $\widehat{W}$ the closure of W in G , given by the weighted homogeneous equation:

$$(5.8) \quad (-at^{2k+1}y_1^2 + y_1 y_2 + y_2^2 + x_1^4 - x_1 x_2 y_2 + t x_2^4 = 0) / \mu_2 \subset G$$

The 3-fold $\widehat{W}$ is a partial compactification of W .

Remark 5.1. The 3-fold $\widehat{W}$ is quasismooth and we think at it as a smooth orbifold. More precisely $\widehat{W}$ has nonisolated quotient singularities: although it is singular, it is a rational homology manifold. Because of this, for the purpose of cohomological computations, we can pretend that $\widehat{W}$ is smooth. Below we take the convention that the set of regular values of the map $\phi: \widehat{W} \rightarrow \mathbb{C}^\times$ is the set of values $t \in \mathbb{C}^\times$ such that the corresponding fibre $\widehat{W}_t$ is quasismooth.

Lemma 5.2. *There is an identity of pure Hodge structures:*

$$(5.9) \quad \mathrm{gr}_3^W H_c^3(W, \mathbb{Q}) = \mathrm{gr}_3^W H_c^3(\widehat{W}, \mathbb{Q})$$

Proof. Note first that the 3-fold W is nonsingular but noncompact, thus $H_c^3(W, \mathbb{Q})$ is a mixed Hodge structure with weights ≤ 3 , and so is $H_c^3(\widehat{W}, \mathbb{Q})$. Consider the divisor $D = \widehat{W} \setminus W$, and denote by $i: D \hookrightarrow \widehat{W}$ and $j: W \hookrightarrow \widehat{W}$ the inclusions. We have a long exact sequence of mixed Hodge structures

$$\cdots \rightarrow H_c^2(D, \mathbb{Q}) \rightarrow H_c^3(W, \mathbb{Q}) \rightarrow H_c^3(\widehat{W}, \mathbb{Q}) \rightarrow H_c^3(D, \mathbb{Q}) \rightarrow \cdots$$

To prove (5.9), we check that $\mathrm{gr}_3^W H_c^2(D, \mathbb{Q}) = \mathrm{gr}_3^W H_c^3(D, \mathbb{Q}) = (0)$. To this end, we study the geometry of the surface D ; D is the union $D = \bigcup_{i=1}^4 D_i$, where:

$$D_1 = \widehat{W} \cap (x_1 = 0) \quad D_2 = \widehat{W} \cap (x_2 = 0) \quad D_3 = \widehat{W} \cap (y_1 = 0) \quad D_4 = \widehat{W} \cap (y_2 = 0)$$

One can check that $D \subset \widehat{W}$ is (locally the quotient of) a simple normal crossing divisor with no 0-dimensional stata. By setting

$$D^{[1]} = \bigsqcup_i D_i \quad \text{and} \quad D^{[2]} = \bigsqcup_{i < j} D_{ij}, \quad \text{with } D_{ij} = D_i \cap D_j$$

we get a strict simplicial resolution $D^{[2]} \rightrightarrows D^{[1]} \rightarrow D$ and the long exact sequence:

$$(5.10) \quad \begin{aligned} \cdots \rightarrow \bigoplus_i H_c^{m-1}(D_i, \mathbb{Q}) &\rightarrow \bigoplus_{i < j} H_c^{m-1}(D_{ij}, \mathbb{Q}) \rightarrow \\ &\rightarrow H_c^m(D, \mathbb{Q}) \rightarrow \bigoplus_i H_c^m(D_i, \mathbb{Q}) \rightarrow \cdots \end{aligned}$$

It follows that $H_c^2(D, \mathbb{Q})$ has weights ≤ 2 . Now choose $m = 3$ in (5.10) and examine $H_c^3(D, \mathbb{Q})$. On the left hand side, $\bigoplus_{i < j} H_c^2(D_{ij}, \mathbb{Q})$ has weights ≤ 2 . On the right hand side, $\bigoplus_i H_c^3(D_i, \mathbb{Q})$ a priori has weights ≤ 3 . To conclude, we next show that, in fact, for $i = 1, 2, 3, 4$, $H_c^3(D_i, \mathbb{Q})$ has weights < 3 . Consider first the surface D_1 , given by

$$(-at^{2k+1}y_1^2 + y_2^2 + y_1y_2 + tx_2^4 = 0) / \mu_2 \subset \mathbb{P}(1, 2, 2) / \mu_2 \times \mathbb{C}^\times$$

This is the same as the surface

$$(5.11) \quad (-at^{2k+1}y_1^2 + y_2^2 + y_1y_2 + tz_2^2 = 0) / \mu_2 \subset \mathbb{P}^2 / \mu_2 \times \mathbb{C}^\times$$

where z_2, y_1, y_2 are homogeneous coordinates of $\mathbb{P}^2$ and μ_2 acts as $(y_1, y_2) \mapsto (-y_1, -y_2)$ on the affine piece ($z_2 = 1$). Note that the quotient of $\mathbb{P}^2$ by the μ_2 -action is the weighted projective space $\mathbb{P}(1, 1, 2)$ with homogeneous coordinates $y_1, y_2, w_2 = z_2^2$. In $\mathbb{P}(1, 1, 2)$ (5.11) becomes

$$(at^{2k+1}y_1^2 - y_2^2 - y_1y_2 = tw_2) \subset \mathbb{P}(1, 1, 2)$$

thus, since $t \neq 0$, we conclude that $D_1 \simeq \mathbb{P}^1 \times \mathbb{C}^\times$ and then $H_c^3(D_1, \mathbb{Q})$ is a pure Hodge structure of weight 2. An almost identical argument holds for D_2 . The surface D_4 is given by

$$(-at^{2k+1}y_1^2 + x_1^4 + tx_2^4 = 0) / \mu_2 \subset \mathbb{P}(1, 1, 2) / \mu_2 \times \mathbb{C}^\times$$

which is the same as the surface

$$(-at^{2k+1}w_1 + z_1 + tz_2 = 0) \subset \mathbb{P}^2 \times \mathbb{C}^\times$$

where z_1, z_2, w_1 are homogeneous coordinates on $\mathbb{P}^2$, thus also $D_4 \simeq \mathbb{P}^1 \times \mathbb{C}^\times$ and $H_c^3(D_4, \mathbb{Q})$ is a pure Hodge structure of weight 2. To conclude, consider the surface D_3 , given by

$$(y_2^2 + x_1^4 - x_1x_2y_2 + tx_2^4 = 0) / \mu_2 \subset \mathbb{P}(1, 1, 2) / \mu_2 \times \mathbb{C}^\times$$

By means of the substitution $y_2 \mapsto y_2 - x_1x_2/2$ write this as

$$\left(y_2^2 + x_1^4 - \frac{x_1^2x_2^2}{4} + tx_2^4 = 0\right) / \mu_2 \subset \mathbb{P}(1, 1, 2) / \mu_2 \times \mathbb{C}^\times$$

and note that this is the same as the surface

$$(5.12) \quad \left(w_2^2 = z_1z_2(-z_1^2 + \frac{z_1z_2}{4} - tz_2^2)\right) \subset \mathbb{P}(1, 1, 2) \times \mathbb{C}^\times$$

where $z_1 = x_1^2, z_2 = x_2^2, w_2 = y_2^2$ are homogeneous coordinates on $\mathbb{P}(1, 1, 2)$. A natural compactification of (5.12) is the surface $\overline{D}_3$ given by

$$\left(w_2^2 = z_1z_2(-t_0^2z_1^2 + t_0^2\frac{z_1z_2}{4} - t_0t_1z_2^2)\right) \subset \mathbb{P}(1, 1, 2) \times \mathbb{P}^1$$

Note that $\overline{D}_3$ is a 2 : 1 cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along a divisor in $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 2)|$, thus $\overline{D}_3$ is a rational surface and hence $H^3(\overline{D}_3, \mathbb{Q}) = (0)$. Then $H^3(D_3, \mathbb{Q})$ has weights ≤ 2 , since, by setting $\Gamma = \overline{D}_3 \setminus D_3$, we have the long exact sequence of mixed Hodge structures

$$\cdots \rightarrow H_c^2(\Gamma, \mathbb{Q}) \rightarrow H_c^3(D_3, \mathbb{Q}) \rightarrow H^3(\overline{D}_3, \mathbb{Q}) \rightarrow \cdots$$

and $\text{gr}_3^W H^2(\Gamma, \mathbb{Q}) = (0)$. □

Lemma 5.3. *Let $\phi: \widehat{W} \rightarrow \mathbb{C}^\times$ be the projection onto the second factor. Denote by Δ the set of critical values of ϕ and write $\Omega = \mathbb{C}^\times \setminus \Delta$ for the set of regular values. Let $j: \Omega \hookrightarrow \mathbb{C}^\times$ be the natural inclusion and denote by $\phi_\Omega: \widehat{W}_\Omega = \phi^{-1}(\Omega) \rightarrow \Omega$ the induced morphism. Let $\delta_1, \delta_2, \delta \in \mathbb{C}[t]$ be the polynomials:*

$$(5.13) \quad \delta_1(t) = 4at^{2k+1} + 1 \quad \delta_2(t) = a^2t^{4k+2} - 4t(4at^{2k+1} + 1)^2 \quad \delta = \delta_1 \cdot \delta_2$$

Write $\Omega_1 = \mathbb{C}^\times \setminus \{\delta_1 = 0\}$, $\widehat{W}_{\Omega_1} = \phi^{-1}(\Omega_1)$. Denote by K the function field $k(\mathbb{C}^\times) = \mathbb{C}(t)$. Then:

(1) $\Delta = \{\delta = 0\}$. If t is a root of δ_1 , $\widehat{W}_t$ has a unique non quasismooth point $p_t = (0 : 0 : -2 : 1)$; if t is a root of δ_2 , $\widehat{W}_t$ has a unique non quasismooth point

$$(5.14) \quad q_t = \left(1 : \sqrt{\frac{2\delta_1(t)}{at^{2k+1}}} : \frac{1}{\delta_1(t)} : \frac{2at^{2k+1}}{\delta_1(t)}\right)$$

In both cases the non quasismooth point is an ordinary double point.

(2) After the change of coordinates:

$$(5.15) \quad y_1 \mapsto \frac{y_1}{2} - \frac{x_1 x_2}{2(4at^{2k+1} + 1)} \quad \text{and} \quad y_2 \mapsto \frac{y_1}{2} + y_2 - \frac{x_1 x_2}{2}$$

the equation of the fibre $\widehat{W}_t \subset \mathbb{P}(1, 1, 2, 2)/\mu_2$ over $t \in \Omega_1$ is:

$$(5.16) \quad \left(-(4at^{2k+1} + 1)y_1^2 + y_2^2 + x_1^4 + tx_2^4 - \frac{at^{2k+1}}{4at^{2k+1} + 1}(x_1 x_2)^2 = 0 \right) / \mu_2$$

(3) For all $t \in \Omega$ the fibre $\widehat{W}_t$ is a quasismooth del Pezzo surface of degree 2 with two $1/4 \cdot (1, -1)$ points p_t^+, p_t^- , on $(x_1 = x_2 = 0)$, and intersecting $(y_1 = y_2 = 0)$ in two points q_t^+ and q_t^- . In the coordinates of Equation (5.16),

$$(5.17) \quad p_t^\pm = (0 : 0 : 1 : \pm \sqrt{\delta_1(t)}) \quad \text{and} \quad q_t^\pm = \left(\sqrt{\frac{at^{2k+1} \pm \sqrt{\delta_2(t)}}{2 \cdot \delta_1(t)}} : 1 : 0 : 0 \right)$$

(4) For all $t \in \Omega$ the fibre $\widehat{W}_t$ has Picard rank $r = h^2(\widehat{W}_t, \mathbb{Q}) = 2$. More specifically, $\widehat{W}_t$ contains a configuration of lines as pictured in Figure 3, and a basis of $\text{Pic}(\widehat{W}_t)_{\mathbb{Q}}$ is given the classes of the curves $C_{t,1}$ and $C_{t,2}$.

(5) In the variables of Equation (5.16), the map $\phi_1: Y_1 = \widehat{W} \cap (x_1 = x_2 = 0) \rightarrow \mathbb{C}^\times$ is a $2 : 1$ branched cover with branch locus $\{\delta_1 = 0\}$, and the map $\phi_2: Y_2 = \widehat{W}_{\Omega_1} \cap (y_1 = y_2 = 0) \rightarrow \Omega_1$ is a $2 : 1$ branched cover with branch locus $\{\delta_2 = 0\}$. In particular,

$$(5.18) \quad \phi_1^{-1}(t) = \begin{cases} p_t^\pm & \text{if } \delta_1(t) \neq 0 \\ p_t & \text{if } \delta_1(t) = 0 \end{cases} \quad \text{and} \quad \phi_2^{-1}(t) = \begin{cases} q_t^\pm & \text{if } \delta_2(t) \neq 0 \\ q_t & \text{if } \delta_2(t) = 0 \end{cases}$$

(6) The Picard rank of $\widehat{W}_K$ is $\rho = 1$.

(7) $R^3 \phi_* \mathbb{Q} = R^1 \phi_* \mathbb{Q} = (0)$.

(8) The natural homomorphism $R^2 \phi_* \mathbb{Q} \rightarrow j_* j^* R^2 \phi_* \mathbb{Q} = j_* R^2 \phi_{\Omega*} \mathbb{Q}$ is an isomorphism.

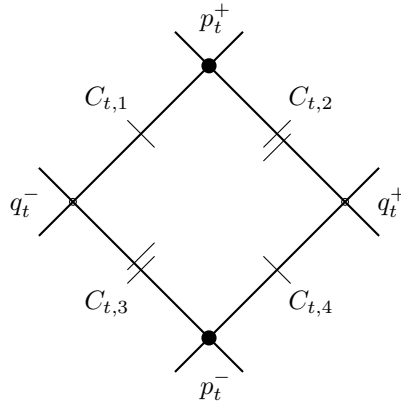


FIGURE 3. A quasismooth fibre $\widehat{W}_t$ and the four points $p_t^\pm, q_t^\pm$ in $\widehat{W}_t$. Numerically equivalent curves on $\widehat{W}_t$ are marked with the same symbol.

Proof. To prove (1), fix $t \in \mathbb{C}^\times$ and compute the Jacobian of the polynomial in Equation (5.8). To prove that for all $t \in \Delta$ the non quasismooth point of $\widehat{W}_t$ is an ordinary double point, one can check that the Hessian of (5.8) at that point is invertible.

Assertion (2) is a simple substitution.

To prove (3), fix a point $t \in \Omega$ and consider the quotient map $\sigma: \mathbb{P}(1, 1, 2, 2) \rightarrow \mathbb{P}(1, 1, 2, 2)/\mu_2$. Let $\widehat{V}_t = \sigma^{-1}(\widehat{W}_t)$ be the preimage. Note that:

$$-K_{\widehat{V}_t} = \sigma^* \left(-K_{\widehat{W}_t} \right) \quad \text{and} \quad K_{\widehat{V}_t}^2 = 2 \cdot K_{\widehat{W}_t}^2$$

since σ is $2:1$ and étale in codimension 1. We have $-K_{\widehat{V}_t} = \mathcal{O}(2)$ and $-K_{\widehat{V}_t}^2 = 4$, thus $\widehat{V}_t \subset \mathbb{P}(1, 1, 2, 2)$ is a del Pezzo surface of degree 4. The orbifold points of $\widehat{V}_t$, of type $1/2 \cdot (1, 1)$, are the two points $P_t^\pm$ of $\widehat{V}_t$ on the line $(x_1 = x_2 = 0)$; in the coordinates of Equation (5.16),

$$P_t^\pm = \left(0 : 0 : 1 : \pm \frac{\sqrt{\delta_1(t)}}{2} \right)$$

Note that, by (5.4), on the affine piece $(y_1 = 1)$ the group μ_2 acts by

$$(x_1, x_2, y_2) \rightarrow (ix_1, -ix_2, y_2)$$

Hence $\widehat{W}_t \subset \mathbb{P}(1, 1, 2, 2)/\mu_2$ is a del Pezzo surface of degree 2 with two $1/4 \cdot (1, -1)$ points $p_t^\pm = \sigma(P_t^\pm)$, as in (5.17); these are the only orbifold points of $\widehat{W}_t$ since $(1 : 0 : 0 : 0)$ and $(0 : 1 : 0 : 0)$ do not satisfy (5.16). Setting $y_1 = y_2 = 0$ in Equation (5.16), one finds that $\widehat{W}_t \cap (y_1 = y_2 = 0)$ is given by the two points $q_t^\pm$ in (5.17).

To prove (4), note that the crepant resolution $\widetilde{W}_t$ of $\widehat{W}_t$ is a smooth weak del Pezzo surface of degree 2. By Demazure's Theorem this is the blow-up of $\mathbb{P}^2$ in 7 nongeneral points, thus it has Picard rank $r = 8$. Then the surface $\widehat{W}_t$, obtained by blowing down 6 exceptional curves on $\widetilde{W}_t$, has Picard rank $r = 2$. We explain how the geometry of $\widehat{W}_t$ singles out a distinguished basis of generators of $\text{Pic}(W_t)_\mathbb{Q}$. Setting

$$\nu_\pm(t) = \sqrt{\frac{at^{2k+1} \pm \sqrt{\delta_2(t)}}{2 \cdot \delta_1(t)}}$$

on $\mathbb{P}(1, 1, 2, 2)$ Equation (5.16) factors as:

$$\begin{aligned} & (y_2 - \sqrt{\delta_1(t)}y_1)(y_2 + \sqrt{\delta_1(t)}y_1) + \\ & + (x_1 - \nu_+(t)x_2)(x_1 + \nu_+(t)x_2)(x_1 - \nu_-(t)x_2)(x_1 + \nu_-(t)x_2) = 0 \end{aligned}$$

This exhibits eight lines on $\widehat{V}_t$, four of which passing through P_t^+ , the other four passing through P_t^- , as pictured on the left of Figure 4. The four lines on $\widehat{V}_t$ through P_t^+ correspond to two orbits under the μ_2 action, and the same holds for the four lines through P_t^- . Namely, in $\mathbb{P}(1, 1, 2, 2)/\mu_2$ Equation (5.16) can only be factored as:

$$(y_2 - \sqrt{\delta_1(t)}y_1)(y_2 + \sqrt{\delta_1(t)}y_1) + (x_1^2 - \nu_+^2(t)x_2^2)(x_1^2 - \nu_-^2(t)x_2^2) = 0$$

This exhibits four lines on the surface $\widehat{W}_t$, as pictured in the middle of Figure 4, two of which passing through p_t^+ :

$$(5.19) \quad \begin{aligned} C_{t,1}: & \left(y_2 - \sqrt{\delta_1(t)}y_1 = x_1^2 - \nu_+^2(t)x_2^2 = 0 \right) \\ C_{t,2}: & \left(y_2 - \sqrt{\delta_1(t)}y_1 = x_1^2 - \nu_-^2(t)x_2^2 = 0 \right) \end{aligned}$$

the other two passing through p_t^- :

$$(5.20) \quad \begin{aligned} C_{t,3}: & \left(y_2 + \sqrt{\delta_1(t)}y_1 = x_1^2 - \nu_+^2(t)x_2^2 = 0 \right) \\ C_{t,4}: & \left(y_2 + \sqrt{\delta_1(t)}y_1 = x_1^2 - \nu_-^2(t)x_2^2 = 0 \right) \end{aligned}$$

To see that $C_{t,1} \sim C_{t,4}$ note that, on $\widetilde{W}_t$, the union of the strict transform $\widetilde{C}_{t,1}$ and the three exceptional curves above p_t^+ , as pictured on the right of Figure 4, supports a fibre of a conic bundle where another fibre is supported on the union of the strict transform $\widetilde{C}_{t,4}$ and the three exceptional curves above p_t^- . Taking direct image this implies indeed that $C_{t,1} \sim C_{t,4}$. Similarly, $C_{t,2} \sim C_{t,3}$. Then the two curves $C_{t,1}$ and $C_{t,2}$ form a basis of $\text{Pic}(\widehat{W}_t)_{\mathbb{Q}}$.

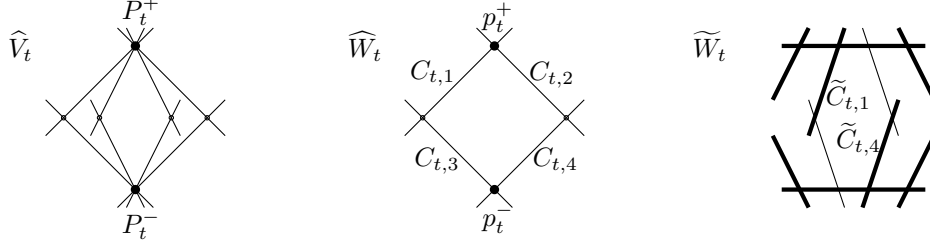


FIGURE 4. The surfaces $\widehat{V}_t$, $\widehat{W}_t$ and $\widetilde{W}_t$. The eight lines on $\widehat{V}_t$ descend to four lines on $\widehat{W}_t$. The two sets of bold lines on $\widetilde{W}_t$ correspond to two distinct fibers of a conic bundle over $\mathbb{P}^1$.

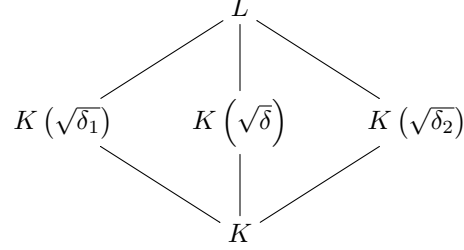
Statement (5) follows immediately from Equation (5.16), and by (1) and (3). In particular, to prove (5.18) one can check by (5.15) that, in the variables of Equation (5.8), if t is a root of δ_1 then $\phi_1^{-1}(t) = (0 : 0 : 2 : -1) = p_t$, and if t is a root of δ_2 then $\phi_2^{-1}(t)$ is given by

$$\left(\sqrt{\frac{at^{2k+1}}{2\delta_1(t)}} : 1 : \sqrt{\frac{at^{2k+1}}{2\delta_1^3(t)}} : \sqrt{\frac{2(at^{2k+1})^3}{\delta_1^3(t)}} \right)$$

which is the point q_t in (5.14).

We prove (6) as follows. Note that the function fields of Y_1 and Y_2 are $k(Y_1) = K(\sqrt{\delta_1})$ and $k(Y_2) = K(\sqrt{\delta_2})$. Set $L = K(\sqrt{\delta_1}, \sqrt{\delta_2})$ and consider the lattice of

Galois field extensions:



The Galois group $\text{Gal}(L/K) \simeq C_2 \times C_2$ is generated by the two automorphisms σ_1 and σ_2 , where:

$$\sigma_{1|K(\sqrt{\delta_1})} = \text{id} \quad \sigma_1(\sqrt{\delta_2}) = -\sqrt{\delta_2} \quad \text{and} \quad \sigma_{2|K(\sqrt{\delta_2})} = \text{id} \quad \sigma_2(\sqrt{\delta_1}) = -\sqrt{\delta_1}$$

By (4), $\text{Pic}(\widehat{W}_L)$ is generated by the classes of the curves C_1 and C_2 , of Equation (5.19). Then by Galois descent $\text{Pic}(\widehat{W}_K)$ has only one generator, given by the class of the curve $C_1 + C_2 + C_3 + C_4 \subset \widehat{W}_K$, since $C_1 + C_2 + C_3 + C_4$ is the only curve in $\widehat{W}_L$ which is invariant with respect to the induced action of $\text{Gal}(L/K)$.

Assertion (7) follows from the fact that, by (1) and (4), $\forall t \in \mathbb{C}^\times$ $H^1(\widehat{W}_t, \mathbb{Q}) = H^3(\widehat{W}_t, \mathbb{Q}) = (0)$.

To prove (8), consider a singular value $s \in \Delta$ and the fibre $\widehat{W}_s$. To check that $R^2\phi_*\mathbb{Q} \rightarrow j_*R^2\phi_{\Omega*}\mathbb{Q}$ is an isomorphism in a neighbourhood of s , it is enough to show that the natural homomorphism

$$H^2(\widehat{W}_s, \mathbb{Q}) \rightarrow H^2(\widehat{W}_t, \mathbb{Q})^{T_s}$$

is an isomorphism, where $t \in \mathbb{C}^\times$ is near s , $T_s: H^2(\widehat{W}_t, \mathbb{Q}) \rightarrow H^2(\widehat{W}_t, \mathbb{Q})$ is the local monodromy operator at s , and $H^2(\widehat{W}_t, \mathbb{Q})^{T_s}$ denotes the group of monodromy invariants. Indeed on the one hand, by the proper base change theorem, $H^2(\widehat{W}_s, \mathbb{Q})$ is the fibre at s of $R^2\phi_*\mathbb{Q}$, and on the other hand $H^2(\widehat{W}_t, \mathbb{Q})^{T_s}$ is the fibre at s of $j_*R^2\phi_{\Omega*}\mathbb{Q}$.

We have an exact triangle of constructible complexes on $\widehat{W}_s$:

$$\mathbb{Q} \rightarrow \psi\mathbb{Q} \rightarrow \varphi\mathbb{Q} \xrightarrow{+1}$$

where ψ and φ are the nearby and vanishing cycle functors [21, Exposé I]. Since $\widehat{W}_s$ has isolated hyperquotient singularities — in fact by (4) it only has one non quasismooth point — $\varphi\mathbb{Q}$ is supported at the non quasismooth point of $\widehat{W}_s$ and is concentrated in degree 2 [31]. Thus from the exact sequence:

$$(0) = H^1(\widehat{W}_s, \varphi\mathbb{Q}) \rightarrow H^2(\widehat{W}_s, \mathbb{Q}) \rightarrow H^2(\widehat{W}_s, \psi\mathbb{Q}) = H^2(\widehat{W}_t, \mathbb{Q}) \rightarrow \dots$$

we conclude that the natural homomorphism $H^2(\widehat{W}_s, \mathbb{Q}) \rightarrow H^2(\widehat{W}_t, \mathbb{Q})$ is injective. By the local invariant cycle Theorem [17, Theorem 1.4.1] then $H^2(\widehat{W}_s, \mathbb{Q})$ is the group of monodromy invariant cycles in $H^2(\widehat{W}_t, \mathbb{Q})$. $\square$

Lemma 5.4. *There is an identity of mixed Hodge structures:*

$$(5.21) \quad H_c^3(\widehat{W}, \mathbb{Q}) = H_c^1(\mathbb{C}^\times, R^2\phi_*\mathbb{Q})$$

Proof. Note that the two functors ϕ_* and $\phi_!$ coincide, since the map ϕ is proper. Consider the Leray spectral sequence of the morphism ϕ with second page $E_2^{p,q} = H_c^p(R^q\phi_*\mathbb{Q}) \implies H_c^{p+q}(\widehat{W}, \mathbb{Q})$, which is known to degenerate at the second page. The groups contributing to $H_c^3(\widehat{W}, \mathbb{Q})$ are:

$$H_c^0(\mathbb{C}^\times, R^3\phi_*\mathbb{Q}), H_c^1(\mathbb{C}^\times, R^2\phi_*\mathbb{Q}), H_c^2(\mathbb{C}^\times, R^1\phi_*\mathbb{Q})$$

Identity (5.21) holds if and only if $H_c^0(\mathbb{C}^\times, R^3\phi_*\mathbb{Q}) = H_c^2(\mathbb{C}^\times, R^1\phi_*\mathbb{Q}) = (0)$, and indeed this is so, by Lemma 5.3(7). $\square$

Lemma 5.5. *There is an isomorphism of mixed sheaves on $\mathbb{C}^\times$:*

$$(5.22) \quad p_{\mathbb{C}^\times \star} \mathbb{Q}_{Y_{\mathbb{C}^\times}} \rightarrow R^2\phi_* \mathbb{Q}_{\widehat{W}}(1)$$

Proof. The sketch of the proof is as follow: let $\Omega \subset \mathbb{C}^\times$ and $j: \Omega \hookrightarrow \mathbb{C}^\times$ be as in Lemma 5.3 and let $p_\Omega: Y_\Omega = p^{-1}(\Omega) \rightarrow \Omega$ be the induced morphism; we first construct a homomorphism

$$(5.23) \quad z: p_{\Omega \star} \mathbb{Q}_{Y_\Omega} \rightarrow R^2\phi_{\Omega \star} \mathbb{Q}_{\widehat{W}_\Omega}(1)$$

and then we show that it is an isomorphism. This concludes the proof since on the one hand it is obvious that $j_* p_{\Omega \star} \mathbb{Q}_{Y_\Omega} = p_{\mathbb{C}^\times \star} \mathbb{Q}_{Y_{\mathbb{C}^\times}}$, on the other hand by Lemma 5.3(8) $j_* R^2\phi_{\Omega \star} \mathbb{Q}_{\widehat{W}_\Omega} = R^2\phi_* \mathbb{Q}_{\widehat{W}}$.

Below we construct a cycle

$$Z \in \text{CH}_2(Y_\Omega \times_\Omega \widehat{W}_\Omega)$$

inducing the homomorphism z via the natural maps:

$$\begin{aligned} \text{CH}_2(Y_\Omega \times_\Omega \widehat{W}_\Omega) &\rightarrow H_4^{BM}(Y_\Omega \times_\Omega \widehat{W}_\Omega) = \\ &= \text{Hom}_{D_{cc}^b}(Rp_{\Omega \star} \mathbb{Q}_{Y_\Omega}, R\phi_{\Omega \star} \mathbb{Q}_{\widehat{W}_\Omega}[2](1)) \rightarrow \text{Hom}(p_{\Omega \star} \mathbb{Q}_{Y_\Omega}, R^2\phi_{\Omega \star} \mathbb{Q}_{\widehat{W}_\Omega}(1)) \end{aligned}$$

Consider the diagram

$$\begin{array}{ccc} & Y_\Omega \times_\Omega \widehat{W}_\Omega & \\ \swarrow & & \searrow p'_\Omega \\ Y_\Omega & & \widehat{W}_\Omega \\ \searrow p_\Omega & & \swarrow \phi_\Omega \\ & \Omega & \end{array}$$

Let $\overline{Z} \subset \widehat{W}_\Omega$ be the union over all $t \in \Omega$ of the four curves $C_{t,1}, \dots, C_{t,4}$ in the fibre $\widehat{W}_t$ described in Lemma 5.3, that is:

$$\overline{Z} = \left(-(4at^{2k+1} + 1)y_1^2 + y_2^2 = x_1^4 - \frac{at^{2k+1}}{4at^{2k+1} + 1}(x_1x_2)^2 + tx_2^4 = 0 \right) / \mu_2 \subset \widehat{W}_\Omega$$

We have that:

$$Y_\Omega \times_\Omega \overline{Z} = Z_1 + Z_2 \subset Y_\Omega \times_\Omega \widehat{W}_\Omega$$

is the sum of two irreducible components and we take Z to be one of these components. In order to see this, let $K = \mathbb{C}(t)$ and $L = K(\sqrt{\delta_1}, \sqrt{\delta_2})$, as in the proof of Lemma 5.3, and consider the field extensions $K \subset K(\sqrt{\delta}) \subset L$. The key point is

to notice that Y_Ω is a $2 : 1$ cover of Ω with function field $k(Y_\Omega) = K(\sqrt{\delta})$. Indeed, by (5.3) Y is a $2 : 1$ cover of $\mathbb{P}^1$ branched at the $6k + 4$ roots of the polynomial

$$t \left(4at^{2k+1} + 1 \right) \left(a^2t^{4k+1} - 4(4at^{2k+1} + 1)^2 \right)$$

By (5.13) this is the polynomial $\delta(t) = \delta_1(t) \cdot \delta_2(t)$. Then, since the generator of the Galois group $\text{Gal}(L/K(\sqrt{\delta}))$ exchanges C_1 with C_4 and C_2 with C_3 on $\widehat{W}_L$, by Galois descent the cycle $C_1 + C_2 + C_3 + C_4$ on $\widehat{W}_{k(Y_\Omega)}$ splits into two components $C_1 + C_4$ and $C_2 + C_3$, each defined over $k(Y_\Omega)$ and corresponding to the two irreducible components Z_1 and Z_2 . Note also that $p'_{\Omega*}Z = \overline{Z}$.

It remains to show that the induced homomorphism z is an isomorphism. This can be checked at the generic point η , or indeed at any point $t \in \Omega$. This is precisely the statement in Lemma 5.3(4) that the set $\{C_1 + C_4, C_2 + C_3\}$ is a basis of $\text{Pic}(\widehat{W}_t)_\mathbb{Q} = H^2(\widehat{W}_t, \mathbb{Q})$. $\square$

APPENDIX A. TORIC MMP

This appendix is not logically necessary for making sense of the statements of our main results — Theorem 1.5 and 1.8 — or their proofs.

Let W be the fibre of (5.2) over a . The goal of the appendix is to explain how we came up with the substitution (5.6), which led us to the partial compactification $\widehat{W} \subset G = \mathbb{P}(1, 1, 2, 2)/_{\mu_2} \times \mathbb{C}^\times$ and the del Pezzo fibration $\phi: \widehat{W} \rightarrow \mathbb{C}^\times$ (Sec. 5).

The study of the geometry of the del Pezzo fibration led us to the statement of Theorem 1.8 and the discovery of the ‘true’ mirror of X_{8k+4} , namely the one-parameter family of hyperelliptic curves of Equation 1.3. Thus, the constructions described in this appendix opened the way for us to state and prove the main results of this paper.

We explicitly construct a partial desingularisation of W and run a minimal model program. The minimal model program terminates with the del Pezzo fibration. The method is of independent interest and, we expect, will be useful in constructing mirrors in situations similar to the one that is our main concern here.

The space $\widehat{W}$ has canonical but not terminal singularities. In Sec. A.4 we construct a birational model W' with terminal singularities and a Mori fibration $\psi: W' \rightarrow \mathbb{F}_1$. Because it has terminal singularities, this $\psi: W' \rightarrow \mathbb{F}_1$ is ‘officially’ an object of interest to classification theory. It would be nice, for example, to prove that W is not rational by studying the geometry of $\psi: W' \rightarrow \mathbb{F}_1$. However, even though the space $\widehat{W}$ is slightly more singular than W' , it makes it easier to prove Theorem 1.8.

As in Sec. 5, we fix once and for all an integer $k > 0$ and a value $a \neq a_{k,0}$, and we omit all reference to k and a in what follows.

A.1. Notation. We set up our notation for toric varieties. For a lattice L , we denote by $L_\mathbb{R} = L \otimes_\mathbb{Z} \mathbb{R}$ the associated real vector space. For a torus $\mathbb{T}$, we denote by $M = \text{Hom}(\mathbb{T}, \mathbb{C}^\times)$ the character lattice and let $N = \text{Hom}(M, \mathbb{Z})$.

For a fan $\Sigma \subset N_\mathbb{R}$, we denote by F_Σ the associated toric variety. We denote by ρ_i the primitive generators of the 1-dimensional cones of Σ , and by $D_i \subset F_\Sigma$ the corresponding divisors.

If F is a proper toric variety and $D = \sum a_i D_i$ is a Weil divisor on F , then we denote by

$$P_D = \bigcap_i \{m : \langle \rho_i, m \rangle \geq -a_i\} \subset M_{\mathbb{R}}$$

the polytope of D ; it is well known that $P_D \cap M$ is a basis of $H^0(F, D)$.

Given a rational polytope $P \subset M_{\mathbb{R}}$, we denote by Σ_P the normal fan of P . If P is not full dimensional Σ_P is a generalised fan, that is, all the cones of Σ_P contain a fixed vector subspace σ_0 with associated lattice $N_0 = N \cap \sigma_0$. We denote by F_P the toric variety for the fan $\Sigma_P/\sigma_0 \subset (N/N_0)_{\mathbb{R}}$.

We recall two well-known facts about toric varieties which we use repeatedly:

Fact 1. There is a 1-to-1 correspondence between:

- (I) the set of polarised toric varieties (F, D) , that is, pairs of a proper toric variety F and (torus invariant) ample $\mathbb{Q}$ -divisor D , and
- (II) the set of full-dimensional rational polytopes $P \subset M_{\mathbb{R}}$.

Given a pair (F, D) , the corresponding polytope is $P = P_D$. Conversely, given P , we let $F = F_P$ and $D = \sum b_i D_i$ where $P = \bigcap_i \{m : \langle \rho_i, m \rangle \geq -b_i\}$ is the unique facet presentation of P .

Fact 2. Suppose that D is a nef divisor on a proper toric variety F . Let $P = P_D$ be the polytope of D and (F_P, D_P) the corresponding polarised pair. Then (F_P, D_P) is an *ample model* of (F, D) , in other words:

- (i) There is a proper toric morphism $f : F \rightarrow F_P$, and
- (ii) $D = f^* D_P$.

A.2. The toric variety associated to W . Let $M = \text{Hom}(\mathbb{T}^4, \mathbb{C}^\times)$ be the group of characters of the torus $\mathbb{T}^4$ in Equation (1.7) and let N be its dual lattice. Denote by $\{e_1, e_2, e_3, e_4\}$ the basis of N dual to the basis of M consisting of the coordinate functions u_1, u_2, u_3, u_4 on $\mathbb{T}^4$.

In this section we study the Newton polytope $P \subset M_{\mathbb{R}}$ of the polynomial in Equation (5.2), defining the variety $W \subset \mathbb{T}^4$, and the compactification of W in the toric variety $F = F_P$.

The polytope P is a 4-dimensional lattice polytope with six vertices $u_0 = \underline{0}$, u_i for $i \in \{1, 2, 3, 4\}$ and $u_5 = (2, 2k+1, 2k+1, 4k+1)$, and facet presentation:

$$P = \bigcap_{i=1}^8 \{m \in M_{\mathbb{R}} : \langle \rho_i, m \rangle \geq -a_i\} \subset M_{\mathbb{R}}$$

where $(\rho_i, a_i) \in N \times \mathbb{Z}$ are:

$$(A.1) \quad \left\{ \begin{array}{ll} (\rho_i, a_i) = & (e_i, 0) \quad \text{for } i = 1, 2, 3, 4 \\ (\rho_5, a_5) = & ((4k+1, -1, -1, -1), 1) \\ (\rho_6, a_6) = & ((-1, 3, -1, -1), 1) \\ (\rho_7, a_7) = & ((-1, -1, 3, -1), 1) \\ (\rho_8, a_8) = & ((-(4k+1), -(4k+1), -(4k+1), 4k+3), 4k+1) \end{array} \right.$$

Hence the fan $\Sigma = \Sigma_P$ is a complete fan with six 4-dimensional cones and eight rays generated by the vectors ρ_i , and F is a compact 4-dimensional toric variety. Note that P does not depend on a and so neither do Σ and F .

The compactification of W in F corresponds to an element of the linear system $|O_F(D)|$, where $D = \sum_{i=1}^8 a_i D_i$ is the ample divisor of F associated to P (as in Fact 1 above).

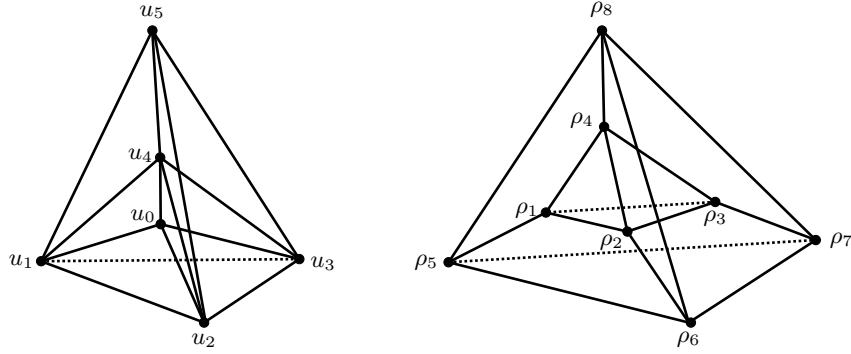


FIGURE 5. On the left, a picture of the polytope $P \subset M_{\mathbb{R}}$; on the right, a picture of the normal fan $\Sigma \subset N_{\mathbb{R}}$ with rays ρ_i in $N_{\mathbb{R}}$.

Remark A.1. The variety F is not $\mathbb{Q}$ -Gorenstein. Let $F' \rightarrow F$ be any small birational morphism such that $K_{F'}$ is $\mathbb{Q}$ -Cartier. Then F' has noncanonical singularities. Indeed, consider for instance the affine open subsets F'_{τ_1} and F'_{τ_2} corresponding to the cones

$$\tau_1 = \langle \rho_5, \rho_8 \rangle_+ \quad \text{and} \quad \tau_2 = \langle \rho_4, \rho_8 \rangle_+$$

of the fan of F' . Then

$$F'_{\tau_1} \cong \mathbb{C}^{\times 2} \times \frac{1}{4k+2}(1, 1) \quad \text{and} \quad F'_{\tau_2} \cong \mathbb{C}^{\times 2} \times \frac{1}{4k+1}(1, 2k)$$

The key point here is that the surface quotient singularities $\frac{1}{4k+2}(1, 1)$ and $\frac{1}{4k+1}(1, 2k)$ are not canonical: the vectors $\frac{1}{4k+2}(1, 1)$ and $\frac{1}{4k+1}(1, 2k)$ correspond to valuations with discrepancies $-\frac{4k}{4k+2}$ and $-\frac{2k}{4k+1}$. The weighted blow ups with weights $\frac{1}{4k+2}(1, 1)$ and $\frac{1}{4k+1}(1, 2k)$ are the minimal canonical partial resolutions of these singularities.

Our aim is to construct a birational model for F with canonical singularities, as final product of a variant of the minimal model program.

A.3. A Mori fibre space structure. In this section our plan is to:

- (i) choose and construct a $\mathbb{Q}$ -Gorenstein partial resolution $f: \tilde{F} \rightarrow F$ of F ;
- (ii) run the minimal model program with scaling for $(\tilde{F}, \tilde{D})$, where $\tilde{D} = f^*(D)$;
- (iii) study the singularities of the final product $(\bar{F}, \bar{D})$ and the equation of W in $\bar{F}$.

A.3.1. *The $\mathbb{Q}$ -Gorenstein partial resolution.* Set $\rho_9 = \frac{1}{4k+2}(\rho_5 + \rho_8) = (0, -1, -1, 1)$ and $a_9 = 1$. (For $i \in \{1, \dots, 8\}$ let a_i be as in (A.1).) For $\varepsilon \geq 0$ consider the polytope:

$$(A.2) \quad P(\varepsilon) = \bigcap_{i=1}^9 \{m \in M_{\mathbb{R}} : \langle \rho_i, m_i \rangle \geq -a_i + \varepsilon\} \subset M_{\mathbb{R}}$$

For $0 < \varepsilon \ll 1$, $P(\varepsilon)$ is 4-dimensional and the presentation in Equation (A.2) is the facet presentation of $P(\varepsilon)$, and the normal fan of $P(\varepsilon)$ is independent of ε ; denote it by $\tilde{\Sigma}$ and let $\tilde{F} = F_{P(\varepsilon)}$ be the corresponding toric variety. Let $\tilde{D} = \sum_{i=1}^9 a_i D_i$; by Fact 1 above the divisor

$$(A.3) \quad \tilde{D}(\varepsilon) = \sum_{i=1}^9 (-\varepsilon + a_i) D_i = \varepsilon K_{\tilde{F}} + \tilde{D}$$

is ample on $\tilde{F}$.

Lemma A.2. (a) $\tilde{F}$ is $\mathbb{Q}$ -Gorenstein and the obvious map $f: \tilde{F} \dashrightarrow F$ is a morphism;
(b) $\tilde{D} = f^*(D)$.

Sketch of proof. Note that $\tilde{D}$ is nef and $\tilde{D} = \tilde{D}(0)$. For $\varepsilon = 0$ we have that

$$P(0) = \bigcap_{i=1}^9 \{m \in M_{\mathbb{R}} : \langle \rho_i, m_i \rangle \geq -a_i\} = \bigcap_{i=1}^8 \{m \in M_{\mathbb{R}} : \langle \rho_i, m_i \rangle \geq -a_i\} = P$$

where the second equality above follows from the choice $a_9 = 1$; thus by Fact 2 above (F, D) is the ample model of $(\tilde{F}, \tilde{D})$: in other words there is a morphism $f: \tilde{F} \rightarrow F$ and $\tilde{D} = f^*(D_P)$. Since for $0 < \varepsilon \ll 1$ the divisor $\tilde{D}(\varepsilon) = \varepsilon K_{\tilde{F}} + \tilde{D}$ is ample on $\tilde{F}$, $K_{\tilde{F}} \sim_f 1/\varepsilon \tilde{D}(\varepsilon)$ is f -ample. Finally, one can check explicitly that $\tilde{F}$ is $\mathbb{Q}$ -Gorenstein. $\square$

A.3.2. *The minimal model program.* For $0 < \varepsilon \ll 1$ the divisors $\tilde{D}(\varepsilon)$ are ample on $\tilde{F}$, since $K_{\tilde{F}}$ is f -ample and $D \subset F$ is ample. We recover $\tilde{F}$ as the toric variety whose spanning fan is the normal fan of the polytopes $P(\varepsilon)$ for small values of $0 < \varepsilon$.

The minimal model program with scaling consists of an inductively defined finite sequence of toric varieties F_j , divisors D_j , and a strictly increasing sequence of rational numbers ε_j , and birational maps:

$$F_0 \dashrightarrow \dots \dashrightarrow F_j \xrightarrow{t_j} F_{j+1} \dashrightarrow \dots \dashrightarrow F_r = \overline{F}$$

where:

- (1) We start with $F_0 = \tilde{F}$, $D_0 = \tilde{D}$, and $\varepsilon_0 = \max\{\varepsilon > 0 \mid \varepsilon K_{\tilde{F}} + \tilde{D} \text{ is nef on } \tilde{F}\}$;
- (2) For $j \geq 0$, $t_j: F_j \dashrightarrow F_{j+1}$ is the divisorial contraction or flip of the face $R_j \subset \text{NE}(F_j)$ with $(\varepsilon_j K_j + D_j)|_{R_j} = 0$ and $(K_j + D_j)|_{R_j} < 0$;
- (3) For $j > 0$ D_j is the proper transform of D_{j-1} on F_j , $K_j = K_{F_j}$, and:

$$\varepsilon_j = \max\{\varepsilon > \varepsilon_{j-1} \mid \varepsilon K_j + D_j \text{ is nef on } F_j\}$$

- (4) The program ends at $F_r = \overline{F}$ where **either**:

The pair $(\overline{F}, \overline{D})$ is a minimal model: that is $\bar{\varepsilon} = 1$ and $K_{\overline{F}} + \overline{D}$ is nef; **or**

The pair $(\overline{F}, \overline{D})$ is a Mori fibre space (Mfs): that is $\overline{\varepsilon} < 1$, the contraction $\psi: \overline{F} \rightarrow S$ of R_r has relative dimension > 0 and $K_{\overline{F}} + \overline{D}$ is ψ -ample. In this case $\overline{\varepsilon}$ is the quasi-effective threshold of the pair $(\tilde{F}, \tilde{D})$:

$$\overline{\varepsilon} = \sup\{t \mid tK_{\tilde{F}} + \tilde{D} \in \overline{\text{Eff}}(\tilde{F})\}$$

By Fact 1, we recover F_j as the toric variety whose spanning fan is the normal fan of the polytopes $P(\varepsilon)$ for $\varepsilon \in (\varepsilon_{j-1}, \varepsilon_j)$, since the divisors $\varepsilon K_j + D_j$ are ample on F_j ; the threshold values $\cdots < \varepsilon_j < \varepsilon_{j+1} < \cdots$ are those where the polytope $P(\varepsilon)$ changes shape.

A.3.3. The final product. Returning to the polytopes of Equation (A.2), $P(1) = \emptyset$, thus the minimal model program just described will end with a Mfs. The sequence of threshold values is³:

$$\frac{1}{2}, \frac{4k+1}{6k+2}, \frac{2}{3}$$

Thus $\overline{\varepsilon} = 2/3$,

$$P\left(\frac{2}{3}\right) = \left[\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 1 \right), \left(\frac{2}{3}, \frac{2k+1}{3}, \frac{2k+1}{3}, \frac{4k+1}{3} \right) \right]$$

and $S = \mathbb{P}^1$. More precisely, the normal fan of $P(2/3)$ has two maximal cones:

$$C_1 = \{n \in N_{\mathbb{R}} : n_2 + n_3 + 2n_4 \geq 0\} \quad \text{and} \quad C_2 = \{n \in N_{\mathbb{R}} : n_2 + n_3 + 2n_4 \leq 0\}$$

intersecting along the hyperplane $C_0 = \{n \in N_{\mathbb{R}} : n_2 + n_3 + 2n_4 = 0\} \subset N_{\mathbb{R}}$.

For all $\varepsilon \in ((2k+1)/(3k+2), 2/3)$, $P(\varepsilon)$ is a 4-dimensional polytope with 9 vertices and 7 facets, whose normals are generated by the vectors $\rho_1, \rho_2, \rho_3, \rho_5, \rho_6, \rho_7, \rho_9$; thus its normal fan $\overline{\Sigma}$ has 7 rays and 9 maximal cones, which are listed in Figure 6. One can check that the 4-dimensional toric variety $\overline{F}$ is $\mathbb{Q}$ -Gorenstein but not $\mathbb{Q}$ -factorial, and has canonical singularities.

The toric morphism $\psi: \overline{F} \rightarrow \mathbb{P}^1$ is induced by the quotient map $N \rightarrow N/N_0$, where $N_0 = N \cap C_0$, sending

$$(n_1, n_2, n_3, n_4) \mapsto n_2 + n_3 + 2n_4$$

The preimage via $\psi: \overline{F} \rightarrow \mathbb{P}^1$ of the torus $\mathbb{C}^\times \subset \mathbb{P}^1$ is the non-compact toric variety G associated to the subfan $\Delta \subset \overline{\Sigma}$ given by the cones of $\overline{\Sigma}$ contained in C_0 . The fan Δ has four 3-dimensional cones (see Figure 6):

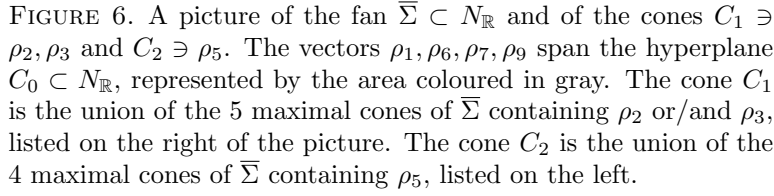
$$\langle \rho_6, \rho_1, \rho_9 \rangle_+ \quad \langle \rho_7, \rho_1, \rho_9 \rangle_+ \quad \langle \rho_6, \rho_7, \rho_1 \rangle_+ \quad \langle \rho_6, \rho_7, \rho_9 \rangle_+$$

Since the quotient N/N_0 is torsion-free, we may choose a splitting $N = N_0 \oplus N_1$ and regard Δ as a fan Δ_0 in N_0 ; we have $\Delta = \Delta_0 \oplus \{0\}$, thus $G = G_0 \times \mathbb{C}^\times$, where G_0 is the 3-dimensional toric variety with spanning fan $\Delta_0 \subset N_0$. To determine G_0 , note that the vectors $\rho_1, \rho_6, \rho_7, \rho_9$ spanning the hyperplane C_0 satisfy the linear

³We computed the threshold values as follows. Consider the 5-dimensional lattice $N' = N \oplus \mathbb{Z}$ and the dual lattice M' , whose elements are pairs (m, r) with $m \in M$ and $r \in \mathbb{Z}$; also denote by $q: M' \rightarrow \mathbb{Z}$ the projection to the second factor. Let $\rho'_i = (\rho_i, -1) \in N'$, $i = 1, \dots, 9$ and let P' be the 5-dimensional polyhedron:

$$P' = \bigcap_{i=1}^9 \{m' \in (M')_{\mathbb{R}} : \langle m', \rho'_i \rangle \geq -a_i\} \subset (M')_{\mathbb{R}}$$

Then $\forall \varepsilon$ $P(\varepsilon) = q_{\mathbb{R}}^{-1}(\varepsilon) \cap P'$ and the threshold values occur at the vertices of P' .



This is the substitution (5.6), thus W identifies with the 3-fold of Equation (5.7) in $\mathbb{T}_G$. The del Pezzo fibration $\phi: \widehat{W} \rightarrow \mathbb{C}^\times$ of Sec. 5 is the restriction of the toric morphism $\varphi: \overline{F} \rightarrow \mathbb{P}^1$.

A.4. A conic bundle structure. The del Pezzo fibration $\phi: \widehat{W} \rightarrow \mathbb{C}^\times$ is not a Mori fibre space in the Mori category:⁴ $\widehat{W}$ has strictly canonical singularities⁵ along two sections of ϕ corresponding to the orbifold points $p_t^\pm \in \widehat{W}_t$ (all t) of Lemma 5.3(3).

In this section we construct a birational model W' with terminal singularities and a Mori fibration $\psi: W' \rightarrow \mathbb{F}_1$ in the Mori category.

A.4.1. Construction. Consider the toric variety G' with weight matrix⁶:

$$\begin{array}{ccccccc} t_1 & t_2 & v_1 & v_2 & z_1 & z_2 & z \\ \hline 1 & 1 & 0 & -1 & 0 & -k-1 & -k \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}$$

and irrelevant ideal $(t_1, t_2)(v_1, v_2)(z, z_1, z_2)$. Note the morphism $G' \rightarrow \mathbb{F}_1$. Set

$$\tilde{\delta}_1(t_1, t_2) = 4at_1^{2k+1} + t_2^{2k+1}$$

and consider the hypersurface $W' \subset G'$ given by

$$(A.4) \quad -z_1^2 + t_2 \tilde{\delta}_1 z_2^2 + v_1 v_2 z^2 \left(\tilde{\delta}_1 v_1^2 - at_1^{2k+1} t_2 v_1 v_2 + t_1 t_2 \tilde{\delta}_1 v_2^2 \right) = 0$$

Theorem A.3. (A) W' has terminal singularities and the morphism $\psi: W' \rightarrow \mathbb{F}_1$ is a Mori fibre space in the Mori category;
(B) There is a commutative diagram:

$$\begin{array}{ccc} \widehat{W} & \dashrightarrow & W' \\ \phi \downarrow & & \downarrow \psi \\ \mathbb{C}^\times & \dashleftarrow & \mathbb{F}_1 \end{array}$$

Proof. (A) It is possible, and not too hard, to verify explicitly that W' has isolated cA singularities.

(B) We start by exhibiting an explicit birational map from a chart of G to a chart of G' .

Consider the chart $(t_2 = v_2 = z = 1) \subset G'$; this is isomorphic to $\mathbb{C}^4$ with coordinate functions t_1, v_1, z_1, z_2 . In this chart, W' is given by the equation:

$$-z_1^2 + \delta_1 z_2^2 + v_1 (\delta_1 v_1^2 - at_1^{2k+1} v_1 + t_1 \delta_1) = 0$$

Next consider the chart $(x_2 = 1) \subset G$; this is isomorphic to $\mathbb{G}_m \times \frac{1}{2}(1, 1, 1)$ with coordinate ring

$$\mathbb{C}[t] \otimes \mathbb{C}[x_1, y_1, y_2]^{\mu_2}$$

⁴A Mori fibre space in the Mori category is a morphism $f: X \rightarrow S$ where X is a projective variety with $\mathbb{Q}$ -factorial terminal singularities and

(i) $f_* \mathcal{O}_X = \mathcal{O}_S$;
(ii) $-K_X$ is f -ample and generates $\text{Pic}(X/S) \otimes \mathbb{Q}$;
(iii) $\dim S < \dim X$.

Mori fibre spaces in the Mori category are ‘official’ objects of interest to classification theory.

⁵That is, $\widehat{W}$ has slightly worse than terminal singularities.

⁶This matrix defines an action of $\mathbb{G}_m^3$ on $\mathbb{A}^7$ and G' is a GIT quotient.

generated by the functions $t, x_1^2, x_1 y_1$, &c. In this chart, $\widehat{W}$ is given by the equation:

$$-\delta_1 y_1^2 + y_2^2 + x_1^4 - \frac{at^{2k+1}}{\delta_1} x_1^2 + t = 0$$

We define a rational map from the chart in G to the chart in G' by:

$$v_1, z_1, z_2 \mapsto x_1^2, \delta_1 x_1 y_1, x_1 y_2$$

This map is in fact birational with inverse given by:

$$x_1^2, x_1 y_1, y_1^2, x_1 y_2, y_1 y_2, y_2^2 \mapsto v_1, \frac{z_1}{\delta_1}, \frac{z_1^2}{v_1 \delta_1^2}, z_2, \frac{z_1 z_2}{v_1 \delta_1}, \frac{z_2^2}{v_1}$$

It is easy to see that the equation for $\widehat{W}$ is transformed into the equation for W' . $\square$

Remark A.4. The variety W' has many singular points (more precisely, above $\widetilde{\delta}_1 = v_j = 0, t_1 = v_1 = 0, t_2 = v_j = 0, j = 1, 2$); thus the conic bundle $\psi: W' \rightarrow \mathbb{F}_1$, although it is a Mori fibre space in the Mori category, it is not ‘standard’ (in the birational geometry literature, a 3-fold Mori fibre space $f: X \rightarrow S$ is a *standard conic bundle* if $\dim S = 2$ and X is nonsingular, which implies that S is also nonsingular, all fibres are conics, and the discriminant is normal crossing). Hence the conjectural rationality criteria [27, 26, 24, 25] do not directly apply, though they suggest that W' is nonrational (and possibly even birationally rigid). It would be interesting to study the geometry of $\psi: W' \rightarrow \mathbb{F}_1$ further and prove that W' is not rational.

Remark A.5. The conic bundle $\psi: W' \rightarrow \mathbb{F}_1$ is an analog of a construction that in the singularity theory and mirror symmetry literature is called a *double suspension*.

The model $\widehat{W}$, although slightly more singular than W' , makes it easier to prove Theorem 1.8.

APPENDIX B. PROOF OF PROPOSITION 1.3

In this appendix we prove Proposition 1.3 by computing part of the quantum orbifold cohomology of $X_{8k+4} \subset \mathbb{P}(2, 2k+1, 2k+1, 4k+1)$.

The stack X_{8k+4} is a hypersurface in the stack $\mathbb{P}(2, 2k+1, 2k+1, 4k+1)$. Our computation is an exercise in the quantum orbifold cohomology of toric stacks [13, 8, 9, 10] and quantum Lefschetz [12, 11, 33] — made possible by the recent work [34] removing the convexity assumption from quantum Lefschetz for stacks — that follows closely calculations in [13, 32].

In fact, the main result of [34] implies that [13, Lemma 6.1] holds (in the proof, invoke [34] at the right place), meaning that the hypergeometric function I_k is a specialisation of Givental’s J -function of X . Here we show that I_k is a specialisation of the quantum period G , itself a specialisation of J .

Below a fraktur capital denotes a projective smooth Deligne–Mumford stack, and the corresponding roman capital denotes its coarse moduli space.

As in Sec. 2.1 we denote by $H_{\text{orb}}^\bullet(\mathfrak{X})$ the ‘even part’ of the Chen–Ruan cohomology of $\mathfrak{X}$ (with rational coefficients):

$$(B.1) \quad H_{\text{orb}}^p(\mathfrak{X}) = \bigoplus_{\substack{b \in \text{Box}(\mathfrak{X}) \\ p - 2\text{age}(b) \in 2\mathbb{Z}}} H^{p-2\text{age}(b)}(\mathfrak{X}_b, \mathbb{Q}) \quad \forall p \in \mathbb{Q}$$

For $b \in \text{Box}(\mathfrak{X})$ we denote by $\mathfrak{X}_b \subset I\mathfrak{X}$ the corresponding component of the inertia stack, by $\text{age}(b)$ its *age*, and by $\mathbf{1}_b \in H_{\text{orb}}(\mathfrak{X})$ the corresponding unit class. Note that $\deg \mathbf{1}_b = 2 \text{age}(b)$. The Poincaré pairing on $H_{\text{orb}}^\bullet$ is denoted by $(\cdot, \cdot)_{\text{orb}}$. We also write $\text{Eff}(\mathfrak{X}) \subset H_2(X, \mathbb{Z})$ for the submonoid generated by the classes of representable maps from stack curves.

B.1. Summary of the theory. We summarise the theory required to carry out our computation.

Givental's J -function. Let $\mathfrak{X}$ be a proper smooth Deligne–Mumford stack. Let z be a formal variable, and let $\mathbb{C}((z^{-1}))$ be the fraction field of the power series ring $\mathbb{C}[[z^{-1}]]$. The *Novikov ring* of $\mathfrak{X}$, which we denote by $\Lambda(\mathbb{C}((z^{-1})))$, is the completion of the algebra $\mathbb{C}((z^{-1}))[\text{Eff}(\mathfrak{X})]$ with respect to the valuation v defined by a fixed polarisation on X [33, Definition 2.5.4].

Following [19, 33], consider the symplectic vector space $(\mathcal{H}_{\mathfrak{X}}, \Omega_{\mathfrak{X}})$ given by the infinite-dimensional $\mathbb{C}$ -vector space:

$$\mathcal{H}_{\mathfrak{X}} = H_{\text{orb}}^\bullet(\mathfrak{X}) \otimes \Lambda(\mathbb{C}((z^{-1})))$$

together with the symplectic form $\Omega_{\mathfrak{X}}(f, g) = -\text{Res}_{z=\infty} (f(-z), g(z))_{\text{orb}} dz$ for $f, g \in \mathcal{H}_{\mathfrak{X}}$.

The genus-zero Gromov–Witten invariants of $\mathfrak{X}$ are encoded by a Lagrangian submanifold $\mathcal{L}_{\mathfrak{X}}$ of $(\mathcal{H}_{\mathfrak{X}}, \Omega_{\mathfrak{X}})$, called the *Givental cone* of $\mathfrak{X}$ [33, Section 3.1], [9, Appendix B]. The cone $\mathcal{L}_{\mathfrak{X}}$ determines and is determined by the *Givental J -function*:

$$(B.2) \quad J_{\mathfrak{X}}(\gamma; z) = z + \gamma + \sum_{d \in \text{Eff}(\mathfrak{X})} \sum_{n=0}^{\infty} \sum_{\epsilon=1}^N \frac{Q^d}{n!} \left\langle \gamma, \dots, \gamma, \frac{\phi^\epsilon}{z - \varphi} \right\rangle_{0, n+1, d} \phi_\epsilon$$

where $\gamma \in H_{\text{orb}}^\bullet(\mathfrak{X})$, Q^d denotes the element of $\Lambda(\mathbb{C}((z^{-1})))$ corresponding to $d \in \text{Eff}(\mathfrak{X})$, and $\{\phi^1, \dots, \phi^N\}$ and $\{\phi_1, \dots, \phi_N\}$ are dual homogeneous bases of $H_{\text{orb}}^\bullet(\mathfrak{X})$.

The cone $\mathcal{L}_{\mathfrak{X}}$ determines $J_{\mathfrak{X}}$ since $J_{\mathfrak{X}}(\gamma; -z)$ is the unique point on $\mathcal{L}_{\mathfrak{X}}$ of the form:

$$(B.3) \quad J_{\mathfrak{X}}(\gamma; -z) = -z + \gamma + O\left(\frac{1}{z}\right)$$

The quantum period $G_{\mathfrak{X}}$ — introduced in Sec. 2.1, Equation (2.3) — is the specialisation of the component of $J_{\mathfrak{X}}$ along $\mathbf{1}_0 \in H^0(X, \mathbb{Q}) \subset H_{\text{orb}}^0(\mathfrak{X})$ obtained setting $Q^d = \alpha^{-K_X \cdot d}$, $z = 1$, and $\gamma = \sum_{i=1}^r \alpha^{1-\text{age}(b_i)} x_i \mathbf{1}_{b_i}$.

The I -function of weighted projective hypersurfaces. Let $Y = \mathbb{P}(a_0, \dots, a_m)$ and let $X \in |\mathcal{O}_Y(d)|$ be a quasismooth and wellformed hypersurface of degree d [23]. We denote by $\mathfrak{X}$ and $\mathfrak{Y}$ the natural projective smooth Deligne–Mumford stacks with moduli spaces X, Y .

Denoting for all $x \in \mathbb{R}$, $\langle x \rangle$, $\lceil x \rceil$ and $\lfloor x \rfloor$ the fractional part, the round up and the round down of x , we have, following [32, Section 5] and [13]:

(1) There is a canonical identification $\text{Box}(\mathfrak{Y}) = B$ where

$$B = \{b \in \mathbb{Q} \mid 0 \leq b < 1 \text{ and } \exists i: ba_i \in \mathbb{Z}\} \subset \mathbb{Q}$$

For $b \in B$, $\text{age}(b) = \sum_{i=0}^m \langle -a_i b \rangle$.⁷ We denote by $\mathfrak{Y}_b = \mathbb{P}(a_i \mid a_i b \in \mathbb{Z})$ and $\mathbf{1}_b \in H_{\text{orb}}^\bullet(\mathfrak{Y})$ the corresponding component of inertia and class.

⁷We use the sign convention of [13]

(2) Writing

$$\Lambda E = \{l \in \mathbb{Q} \mid l \geq 0 \text{ and } \langle l \rangle \in B\}, \text{ and } \mathbf{P} = c_1(\mathcal{O}_{\mathfrak{Y}}(1)) \in H^2(\mathfrak{Y})$$

define the I -function by:

$$(B.4) \quad I_{\mathfrak{X}, \mathfrak{Y}}(\tau; z) = ze^{\mathbf{P}\tau/z} \sum_{l \in \Lambda E} Q^l e^{l\tau} \square_l \mathbf{1}_{\langle l \rangle} \in H_{\text{orb}}^{\bullet}(\mathfrak{Y}; \Lambda(\mathbb{C}((z^{-1}))))$$

where

$$\square_l = \frac{\prod_{a: \langle a \rangle = \langle dl \rangle} (d\mathbf{P} + az)}{\prod_{i=0}^m \prod_{a: \langle a \rangle = \langle a_i l \rangle} (a_i \mathbf{P} + az)}$$

The following special case of [34, Theorem 1.1] is the key result allowing one to compute bona fide Gromov–Witten invariants of $\mathfrak{X}$ from the I -function:

Theorem B.1. *Denote by $\iota: \mathfrak{X} \rightarrow \mathfrak{Y}$ the natural inclusion. For all τ , $\iota^* I_{\mathfrak{X}, \mathfrak{Y}}(\tau; z) \in \mathcal{L}_{\mathfrak{X}}$. $\square$*

Remark B.2. Theorem B.1 can be used to compute at least part of the J -function of $\mathfrak{X}$. For example, if $\iota^* I_{\mathfrak{X}, \mathfrak{Y}}(\tau; z) = z\mathbf{1}_0 + F(\tau) + \mathcal{O}(z^{-1})$, then by Equation (B.3) $I_{\mathfrak{X}, \mathfrak{Y}}$ determines a specialisation of the J -function of $\mathfrak{X}$. If, in addition, the cohomology classes $\mathbf{1}_b \in H_{\text{orb}}^0(\mathfrak{X})$ that appear in this specialisation have age < 1 , then $I_{\mathfrak{X}, \mathfrak{Y}}$ also determines a specialisation of the quantum period $G_{\mathfrak{X}}$.

Proposition B.3. *Let $X = X_d \subset Y = \mathbb{P}(a_0, \dots, a_3)$ be a 2-dimensional quasi-smooth wellformed anticanonical hypersurface. The function $I_{\mathfrak{X}, \mathfrak{Y}}$ is of the form:*

$$(B.5) \quad I_{\mathfrak{X}, \mathfrak{Y}}(\tau; z) = z\mathbf{1}_0 + \tau \mathbf{P} + \left(\frac{d!}{a_0! a_1! a_2! a_3!} Q e^{\tau} \right) \mathbf{1}_0 + \sum_{b \in B} c_b \mathbf{1}_b + \mathcal{O}\left(\frac{1}{z}\right)$$

where for all b $c_b \in \mathbb{Q}$ and if $c_b \neq 0$ then $\text{age}(\iota^* \mathbf{1}_b) < 1$.

Proof. For $l \in \Lambda E$ we are interested in studying the possibility that

$$\deg_z \square_l = \lceil dl \rceil - \lceil a_0 l \rceil - \dots - \lceil a_3 l \rceil$$

is ≥ -1 .

Case 1. The class $\mathbf{1}_{\langle l \rangle}$ is supported on a closed point $P \in \mathfrak{Y}$. There is no loss of generality assuming that $P = P_3 = (0 : 0 : 0 : 1) \in \mathfrak{Y}$, that is, $a_3 l \in \mathbb{Z}$ but $a_0 l, a_1 l, a_2 l \notin \mathbb{Z}$. Because X is quasismooth at P then **either** $P \notin X$, **or** $P \in X$.

Now $P \notin X$ if and only if $dl \in \mathbb{Z}$ and then

$$\begin{aligned} \deg_z \square_l &= dl - \lceil a_0 l \rceil - \lceil a_1 l \rceil - \lceil a_2 l \rceil - a_3 l = \\ &= (d - a_0 - a_1 - a_2 - a_3)l - \langle -a_0 l \rangle - \langle -a_1 l \rangle - \langle -a_2 l \rangle = -l - \text{age}(\mathbf{1}_{\langle l \rangle}) \leq -1 \end{aligned}$$

and if $= -1$ then $l = \langle l \rangle \in B$ and $\iota^* \mathbf{1}_{\langle l \rangle} = 0$ so the statement holds in this case.

If, on the other hand, $P \in X$, then, because X is quasismooth at P , for some $i = 0, 1, 2$ a monomial $x_i x_3^M$ appears in the equation of X , and there is no loss of generality in assuming that $i = 0$. This means that $d = a_0 + M a_3$ so $dl \equiv a_0 l \pmod{\mathbb{Z}}$, and then we have

$$\begin{aligned} \deg_z \square_l &= dl - a_0 l - \lceil a_1 l \rceil - \lceil a_2 l \rceil - a_3 l = \\ &= (d - a_0 - a_1 - a_2 - a_3)l - \langle -a_1 l \rangle - \langle -a_2 l \rangle = -l - \text{age}(\iota^* \mathbf{1}_{\langle l \rangle}) \leq -1 \end{aligned}$$

and if $= -1$ then $l = \langle l \rangle \in B$ and $\text{age}(\iota^* \mathbf{1}_{\langle l \rangle}) < 1$. This concludes the proof in Case 1.

Case 2. The class $\mathbf{1}_{\langle l \rangle}$ is supported on a curve. No generality is lost in assuming that the curve in question is $\Gamma = \{x_0 = x_1 = 0\}$, that is, $a_2l, a_3l \in \mathbb{Z}$ but $a_0l, a_1l \notin \mathbb{Z}$. Because X is wellformed, $dl \in \mathbb{Z}$ [23, 6.10], and then we have

$$\begin{aligned} \deg_z \square_l &= dl - \lceil a_0l \rceil - \lceil a_1l \rceil - a_2l - a_3l = \\ &= (d - a_0 - a_1 - a_2 - a_3)l - \langle -a_0l \rangle - \langle -a_1l \rangle = -l - \text{age}(\mathbf{1}_{\langle l \rangle}) \leq -1 \end{aligned}$$

and if $= -1$ then $l = \langle l \rangle \in B$ and $\text{age}(\iota^* \mathbf{1}_{\langle l \rangle}) = \text{age}(\mathbf{1}_{\langle l \rangle}) < 1$. This concludes the proof in Case 2.

Case 3. $\langle l \rangle = 0$. This case is easy and the proof omitted. $\square$

B.2. Proof of Proposition 1.3. The statement really is an immediate consequence of Proposition B.3. We go through the steps rather explicitly:

B.2.1. Step 1: Box($\mathfrak{Y}$). In our case $X = X_{8k+4} \subset Y = \mathbb{P}(2, 2k+1, 2k+1, 4k+1)$. Clearly

$$\text{Box}(\mathfrak{Y}) = \left\{ 0, \frac{1}{2}, \frac{n}{2k+1}, \frac{m}{4k+1} \mid 1 \leq n \leq 2k, 1 \leq m \leq 4k \right\}$$

and:

- (i) $\mathfrak{Y}_{\frac{1}{2}}$ is supported on $\{P_1\}$ where $P_1 = (1 : 0 : 0 : 0)$ is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$;
- (ii) for all $1 \leq n \leq 2k$, $\mathfrak{Y}_{\frac{n}{2k+1}}$ is supported on $\overline{P_2 P_3} = \{(0 : y_1 : y_2 : 0)\}$, a line all of whose points have a neighbourhood analytically isomorphic to $\mathbb{A}^1 \times \frac{1}{2k+1}(-2, 1)$;
- (iii) for all $1 \leq m \leq 4k$, $\mathfrak{Y}_{\frac{m}{4k+1}}$ is supported on $\{P_4\}$ where $P_4 = (0 : 0 : 0 : 1)$ is a singularity of type $\frac{1}{4k+1}(-2, 2k, 2k)$.

B.2.2. Step 2: Box($\mathfrak{X}$). Denoting by x, y_1, y_2, z the coordinates on Y , the equation of X is a polynomial of the form:

$$(B.6) \quad f(x, y_1, y_2, z) = x^{4k+2} + y_1^4 + y_2^4 + xz^2$$

It follows that $P_1 \notin X$ and $P_4 \in X$ is a singularity of type $\frac{1}{4k+1}(2k, 2k)$. Moreover, since $f(0, y_1, y_2, 0) = y_1^4 + y_2^4$, X intersects $P_2 P_3$ in four points Q_i ($i = 1, 2, 3, 4$), and each $Q_i \in X$ is a singularity of type $\frac{1}{2k+1}(-2, 1)$.

Thus, $\text{Box}(\mathfrak{X})$ has $12k + 1$ elements that we denote as follows:

- (1) the element 0 corresponding to the generic component;
- (2) $4 \times 2k$ elements $(i, \frac{n}{2k+1})$ ($i = 1, 2, 3, 4$ and $n = 1, \dots, 2k$) supported on $\{Q_i\}$ and of age $\frac{n}{2k+1} + \left\langle \frac{-2n}{2k+1} \right\rangle$;
- (3) $4k$ elements $\frac{m}{4k+1}$ ($m = 1, \dots, 4k$) supported on $\{P_4\}$ and of age $2 \left\langle \frac{2km}{4k+1} \right\rangle$.

It is easy to see from the above discussion that the elements of $\text{Box}(\mathfrak{X})$ of age < 1 are: the $4k$ elements $(i, \frac{n}{2k+1})$ with $1 \leq n \leq k$, and the $2k$ elements $\frac{m}{4k+1}$ with m odd. Thus the quantum period is a function of α and $6k$ parameters that we write as $G(x_1, \dots, x_{6k}, \alpha)$.

B.2.3. *Step 3: conclusion.* Since by Equation (B.3) the J -function of $\mathfrak{X}$ is the unique point on $\mathcal{L}_{\mathfrak{X}}$ with asymptotic $z + \gamma + \mathcal{O}(z^{-1})$, it follows from Theorem B.1, Proposition B.3 and the above considerations that:

$$(B.7) \quad J_{\mathfrak{X}}(\iota^* \bar{\zeta}; z) = \exp\left(-\frac{1}{z} \frac{(8k+4)!}{2!(2k+1)!^2(4k+1)!} Q\right) \iota^* I_{\mathfrak{X}, \mathfrak{Y}}(0; z)$$

where

$$\bar{\zeta} = \sum_{n=1}^k c_{\frac{n}{2k+1}} Q^{\frac{n}{2k+1}} \mathbf{1}_{\frac{n}{2k+1}}^{\mathfrak{Y}} + \sum_{j=0}^{2k-1} c_{\frac{2j+1}{4k+1}} Q^{\frac{2j+1}{4k+1}} \mathbf{1}_{\frac{2j+1}{4k+1}}^{\mathfrak{Y}}$$

Clearly

$$(B.8) \quad \iota^* \bar{\zeta} = \sum_{n=1}^k \sum_{i=1}^4 c_{\frac{n}{2k+1}} Q^{\frac{n}{2k+1}} \mathbf{1}_{(i, \frac{n}{2k+1})}^{\mathfrak{X}} + \sum_{j=0}^{2k-1} c_{\frac{2j+1}{4k+1}} Q^{\frac{2j+1}{4k+1}} \mathbf{1}_{\frac{2j+1}{4k+1}}^{\mathfrak{X}}$$

where every cohomology class has age < 1 , and then

$$(B.9) \quad G_{\mathfrak{X}}(\bar{x}; \alpha) = \exp\left(-\frac{(8k+4)!}{2!(2k+1)!^2(4k+1)!} \alpha\right) \sum_{l=0}^{\infty} \frac{((8k+4)l)!}{(2l)!((2k+1)l)!((4k+1)l)!} \alpha^l$$

where

$$\bar{x} = \left(4 \times c_{\frac{1}{2k+1}}, 4 \times c_{\frac{2}{2k+1}}, \dots, 4 \times c_{\frac{k}{2k+1}}, c_{\frac{1}{4k+1}}, c_{\frac{3}{4k+1}}, \dots, c_{\frac{4k-1}{4k+1}}\right)$$

and the statement easily follows from this.

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