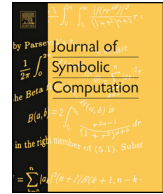




Contents lists available at ScienceDirect

Journal of Symbolic Computation

journal homepage: www.elsevier.com/locate/jsc

Tropical combinatorics of max-linear Bayesian networks

Carlos Améndola, Kamillo Ferry

Technical University Berlin, Germany



ARTICLE INFO

Article history:

Received 15 November 2024

Received in revised form 9 August 2025

Accepted 3 October 2025

Available online 8 October 2025

Dataset link: <https://zenodo.org/records/14166530>

MSC:

05C12

14T90

52B11

62R01

Keywords:

Hyperplane arrangement

Max-linear Bayesian network

Polytrope

Tropical convexity

ABSTRACT

A polytrope is a tropical polyhedron that is also classically convex. We study the tropical combinatorial types of polytropes associated to weighted directed acyclic graphs (DAGs). This family of polytropes arises in algebraic statistics when describing the model class of max-linear Bayesian networks. We show how the edge weights of a network directly relate to the facet structure of the corresponding polytrope. We also give a classification of polytropes from weighted DAGs at different levels of equivalence. These results give insight on the statistical problem of identifiability for a max-linear Bayesian network.

© 2025 The Author(s). Published by Elsevier Ltd. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

1. Introduction

A Bayesian network is a type of directed graphical model (in the sense of Lauritzen, 1996), modeling causal relations between random variables through a directed acyclic graph (DAG). Max-linear Bayesian networks (MLBNs) were introduced by Gissibl and Klüppelberg, to describe causal relations between variables that take large observed values. A common example is modeling a river network where the extreme values represent possible flooding events (Asadi et al., 2015; Tran et al., 2024; Gissibl and Klüppelberg, 2018), to describe causal relations between variables that take large observed

E-mail addresses: amendola@math.tu-berlin.de (C. Améndola), ferry@math.tu-berlin.de (K. Ferry).

<https://doi.org/10.1016/j.jsc.2025.102518>

0747-7171/© 2025 The Author(s). Published by Elsevier Ltd. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

values. A common example is modeling a river network where the extreme values represent possible flooding events (Asadi et al., 2015; Tran et al., 2024). The starting point of our work is that the structural equations that govern these relations are formulated in the language of tropical geometry (Joswig, 2021; Maclagan and Sturmfels, 2015) and can be described by weighted DAGs (Améndola et al., 2021, 2022; Tran, 2022).

Specifically, let $[n] = \{1, 2, \dots, n\}$ and consider a weighted DAG $G = ([n], E)$ with edge weights $w \in \mathbb{R}_{\geq 0}^E$. Choosing a labeling of G gives rise to a weighted adjacency matrix $C \in \mathbb{R}_{\geq 0}^{n \times n}$, such that $c_{ij} = w(j \rightarrow i) > 0$ if and only if $j \rightarrow i$ is in the edge set E of G . Denote $a \vee b = \max\{a, b\}$. Then G describes a *max-linear Bayesian network* $X = (X_1, \dots, X_n)$ where the random variables X_i satisfy the recursive equations

$$X_i = \bigvee_{j=1}^n c_{ij} X_j \vee Z_i \quad (1)$$

for $c_{ij} \geq 0$, $1 \leq i, j \leq n$ and independent random variables Z_i .

Solving the structural equations (1) purely in terms of the error terms Z_i gives equations

$$X_i = \bigvee_{j=1}^n c_{ij}^{\otimes} Z_j \quad (2)$$

where $c_{ij}^{\otimes}$ denotes the (i, j) entry of the *max-times Kleene star* $C^{\otimes} := I_n \vee C \vee C^{\otimes 2} \vee \dots \vee C^{\otimes n}$.

Interpreting (2) as a matrix-vector product and applying a coordinate-wise negative logarithmic transformation allows us to make use of tropical geometry over $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot, \infty, 0)$, which is called the *min-plus semiring* and is the standard setting in Joswig (2021); Maclagan and Sturmfels (2015).

Similarly to a classical polytope, a tropical polyhedron is a set P which is the tropical convex hull of finitely many points. But the tropical convex hull of a point set V amounts to taking all possible tropical linear combinations of points in V . For the tropical polytopes we are interested in, the corresponding matrix is a Kleene star, which means that P is not only tropically convex, but also classically convex. This kind of polyhedron is called a *polytrope*.

Example 1.1. Let $G = \kappa_n$ be the complete directed acyclic graph on n nodes, that is, κ_n has an edge $j \rightarrow i$ whenever $j < i$. For $n = 3$, κ_3 is the *directed triangle*

$$\begin{array}{ccc} 1 & & \\ \downarrow c & \nearrow a & 2 \\ 3 & \nwarrow b & \end{array} \quad \text{with lower-triangular weight matrix } C = \begin{pmatrix} 0 & \infty & \infty \\ a & 0 & \infty \\ c & b & 0 \end{pmatrix}.$$

The recursive equations $X = C \odot X \oplus Z$ associated to this network are

$$\begin{aligned} X_1 &= Z_1, \\ X_2 &= aX_1 \oplus Z_2, \\ X_3 &= cX_1 \oplus bX_2 \oplus Z_3. \end{aligned}$$

The explicit solution to this recursive system of equations is given by

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} Z_1 \\ aZ_1 \oplus Z_2 \\ (c \oplus ab)Z_1 \oplus bZ_2 \oplus Z_3 \end{pmatrix} = Z_1 \odot \begin{pmatrix} 0 \\ a \\ c \oplus ab \end{pmatrix} \oplus Z_2 \odot \begin{pmatrix} \infty \\ 0 \\ b \end{pmatrix} \oplus Z_3 \odot \begin{pmatrix} \infty \\ \infty \\ 0 \end{pmatrix},$$

which shows that X lies in the tropical polytope spanned by the columns of the (min-plus) Kleene star matrix

$$C^* = \begin{pmatrix} 0 & \infty & \infty \\ a & 0 & \infty \\ c \oplus ab & b & 0 \end{pmatrix}. \quad \square$$

Initially, much work on polytropes has been done in the bounded case. Develin and Sturmfels (2004) defined tropical polytopes and identified polytropes as their basic building blocks, without using the term ‘polytrope’ yet. They have been studied under the name *alcoved polytopes* by Lam and Postnikov (2007) as polytopes arising from deformations of affine Coxeter arrangements of type A_n . Joswig and Kulas (2010) connected these two points of view and coined the name *polytrope* (Joswig and Kulas, 2010) connected these two points of view and coined the name *polytrope*.

However, the polytropes from max-linear Bayesian networks are necessarily unbounded, because the corresponding matrix of the weighted DAG is (tropically) lower-triangular. More general work covering the unbounded case includes Fink and Rincón (2015) who studied Stiefel tropical linear spaces with arbitrary support. While polytropes were not their main focus, tropical polyhedra generated by n vertices and n -dimensional Stiefel tropical linear spaces in dimension d are related to each other as both can be specified in terms of the column space of a matrix, and the combinatorics of both objects can be described in terms of regular subdivisions of subpolytopes of products of simplices $\Delta_{n-1} \times \Delta_{d-1}$. This generalizes the previous results to unbounded polytropes in tropical affine space.

Joswig and Loho (2016) took a different approach and characterized the same regular subdivisions as projections from a generalization of weighted digraph polyhedra called the *envelope*. One of our main results concerns the tropical polytrope generated by a matrix C , and the tropical polytrope generated by the Kleene star C^* . We are able to characterize how the combinatorial data of the former already determines the combinatorial data of the latter. The necessary definitions for this are laid out in Section 2.

Theorem 3.5. *Let $C \in \mathbb{T}^{n \times n}$ be supported on a DAG G . The polytrope $\text{tconv}(C^*) \subseteq \text{tconv}(C)$ coincides with the closed cell $S_L(C)$ such that L is the perfect matching $(1, 2, \dots, n)$ in the support graph $B(C)$.*

Regarding enumeration of alcoved polytopes, Joswig and Kulas classified the combinatorial types of alcoved polytopes in dimensions 2 and 3. Then, Tran (2017) described the combinatorial types of alcoved polytopes in terms of cones of a certain Gröbner fan in $\mathbb{R}^{n^2-n}$.

On the other hand, Joswig and Schröter (2022, 2019) studied polytropes in the context of *shortest-path problems*. In particular, they studied the combinatorics of polytropes with polynomial parametrization and gave effective methods to compute moduli spaces for polytropes.

In a previous version Joswig and Schröter (2019), Joswig and Schröter characterized combinatorial types of polytropes in terms of central subdivisions of the fundamental polytope P_G defined in Section 3. Their version of the result focused mainly on the case where G is the complete directed graph, or in other words, where C has no entry equal to ∞ . We adapt their proof to allow for infinite entries, giving the following result.

Theorem 3.9. *The tropical combinatorial types of full-dimensional polytropes in $\mathbb{T} \mathbb{A}^{n-1}$ are in bijection with the regular central subdivisions of the fundamental polytope P_G , where G ranges over transitive directed graphs G on n nodes.*

In Section 4 we introduce machinery in the form of the *weighted transitive reduction* of a weighted DAG, to characterize a minimal facet description for polytropes associated to DAGs.

Theorem 4.10. *The weighted transitive reduction G_w^b is the unique minimal weighted DAG such that the weighted digraph polyhedra $Q(C)$ and $Q(C^b)$ coincide.*

Theorem 4.10 has consequences for statistical inference of the parameters of an MLBN. Namely, we can only uniquely recover the parameters of an MLBN for edges in the weighted transitive reduction, see Remark 4.11.

Our main goal is to deeply understand polytropes coming from max-linear Bayesian networks. This is done through the existing theory of tropical polyhedra and tropical hyperplane arrangements. From this we obtain *open strata* $\mathcal{M}_G^*$ for a moduli space of polytropes coming from a fixed DAG G as a polyhedral fan. There is similar work by Orlarte et al. (2019) on valuated matroids with a prescribed basis and the *local Dressians* parametrizing those.

Using the results above, we give a classification of polytropes coming from max-linear Bayesian networks on n nodes in terms of their tropical combinatorial types. We also give a procedure to enumerate generic tropical combinatorial types to summarize this classification. Another consequence is a description of the moduli space of such polytropes in terms of unions of polyhedral fans. There are two natural ways of grouping the corresponding cones, which we also describe.

Outline The paper is structured as follows. In Section 2, we give the necessary background on tropical polytopes and their dual subdivisions. In Section 3, we recall the specifics related to polytropes and give characterizations of tropical combinatorial equivalence for polytropes. In Section 4, we turn these characterizations into descriptions of a moduli space $\mathcal{M}_{\nabla,n}^*$. In Section 5 we investigate $\mathcal{M}_{\nabla,n}^*$ for small n by enumerating generic combinatorial types. We conclude with an outlook including questions guiding future research directions.

Acknowledgments

The authors thank Melody Chan, Alex Fink, Anne Frühbis-Krüger, Cécile Gachet, Sami Halaseh, Michael Joswig, Francesco Nowell, Yue Ren, Benjamin Schröter and Lena Weis for many helpful discussions and suggestions along the way. We would also like to thank the two anonymous reviewers for their many helpful comments.

The authors are funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689), Project AA3-16.

2. Tropical preliminaries

We work mostly over the *min-plus tropical semiring* $\mathbb{T} = \mathbb{T}_{\min} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ where $\oplus = \min$ and $\odot = +$. Where necessary, we distinguish between min-plus and max-plus semiring by decorating with min resp. max and using $\boxplus$ for max-tropical addition.

We denote by $\mathbb{T}\mathbb{A}^{d-1}$ the *tropical affine space* which is the set of points $x \in \mathbb{R}^d$ identified via

$$(x_1, \dots, x_n) \sim \lambda \odot (x_1, \dots, x_d) = (x_1 + \lambda, \dots, x_d + \lambda)$$

for all $\lambda \in \mathbb{R}$. This is also called the *tropical projective torus* in the literature. We can also interpret $\mathbb{T}\mathbb{A}^{d-1}$ as the quotient $\mathbb{R}^d / \mathbb{R}\mathbf{1}$ of Euclidean space by the subspace spanned by the all-ones vector $\mathbf{1} = (1, \dots, 1)$. Then, the *tropical projective space* $\mathbb{T}\mathbb{P}^{d-1}$ is defined as

$$\mathbb{T}\mathbb{P}^{d-1} := (\mathbb{T}^n \setminus \{(\infty, \dots, \infty)\}) / \mathbb{R}\mathbf{1}.$$

The space $\mathbb{T}\mathbb{P}^{d-1}$ has a stratification by Euclidean spaces in the following way. For an arbitrary subset $Z \subsetneq [d]$ the *stratum* of $\mathbb{T}\mathbb{P}^{d-1}$ given by Z is defined as

$$\mathbb{T}\mathbb{P}^{d-1}(Z) := \{x \in \mathbb{T}\mathbb{P}^{d-1} \mid x_i = \infty \text{ if and only if } i \in Z\}.$$

We can recover $\mathbb{T}\mathbb{A}^{d-1}$ as the stratum $\mathbb{T}\mathbb{P}^{d-1}(\emptyset)$ and call this the *interior stratum* of $\mathbb{T}\mathbb{P}^{d-1}$. On the other hand if $Z \neq \emptyset$, then $\mathbb{T}\mathbb{P}^{d-1}(Z) \cong \mathbb{T}\mathbb{A}^{d-1-\#Z}$ is a *boundary stratum* of $\mathbb{T}\mathbb{P}^{d-1}$.

Since our graphs are finite, we write $\mathbb{R}^E$ for the set of possible weights on a specific digraph $G = (V, E)$ and consider a stratification of the set $\mathbb{T}^E$ of weights on G possibly allowing ∞ . We define the strata indexed by subsets $E' \subseteq E$ of the edge sets, or equivalently, by subgraphs $H \subseteq G$ as

$$\mathbb{T}^E(H) := \{x \in \mathbb{T}^E \mid x_{j \rightarrow i} < \infty \text{ if and only if } j \rightarrow i \in H\}.$$

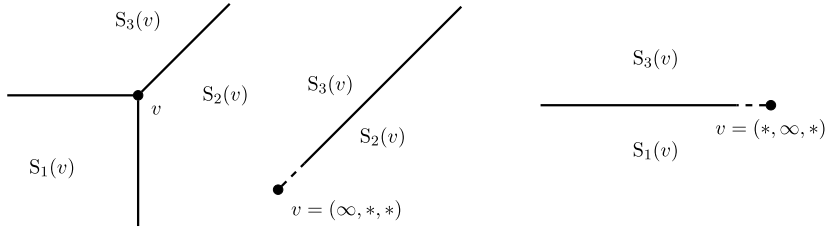


Fig. 1. Max-tropical hyperplanes $\mathcal{T}(v)$ in $\mathbb{T}\mathbb{A}^2$ and their cells. In the last two examples, the apices are at infinity.

We call a set V of n points in $\mathbb{T}\mathbb{P}^{d-1}$ a *tropical point configuration*. This can also be interpreted as a matrix $V \in \mathbb{T}^{d \times n}$.

2.1. Tropical polyhedra in affine space

We now define the basic objects of consideration. Following Butkovič (2010), we say that a tropical matrix $V \in \mathbb{T}^{d \times n}$ is *doubly $\mathbb{R}$ -astic* if every tropical row and column sum is finite.

Definition 2.1. The *tropical polyhedron* defined by a doubly $\mathbb{R}$ -astic matrix $V \in \mathbb{T}^{d \times n}$ in $\mathbb{T}\mathbb{A}^{d-1}$ is the image under left multiplication by V , that is, the (min-plus)-linear span of the columns of V :

$$\text{tconv}(V) := \left\{ V \odot \lambda \mid \lambda \in \mathbb{T}\mathbb{A}^{n-1} \right\} \subseteq \mathbb{T}\mathbb{A}^{d-1}.$$

If $\text{tconv}(V)$ is bounded, it is called a *tropical polytope*.

The condition that V is doubly $\mathbb{R}$ -astic ensures that $\text{tconv}(V)$ is a subset of $\mathbb{T}\mathbb{A}^{d-1}$ instead of living in a boundary stratum of $\mathbb{T}\mathbb{P}^{d-1}$. A set A is called *tropically convex* if it contains for any two points $p, q \in A$ the *tropical line segment* $\text{tconv}(p, q)$.

Every min-tropical polyhedron can be expressed in terms of max-tropical hyperplanes (Fink and Rincón, 2015, Section 6).

Definition 2.2. For any point $v = (v_1, \dots, v_d) \in \mathbb{T}\mathbb{P}^{d-1}$, the *tropical hyperplane* $\mathcal{T}(v)$ defined by v is the set of points $x \in \mathbb{T}\mathbb{A}^{d-1}$ such that the maximum in the (max-tropical) linear form

$$\alpha_v(x) := -v_1 \odot x_1 \boxplus -v_2 \odot x_2 \boxplus \dots \boxplus -v_d \odot x_d,$$

is attained at least twice.

The choice of negative signs in α_v makes v the *apex* of $\mathcal{T}(v)$.

For any $i \in [d]$ with $v_i \neq \infty$, the *i-th open (max-)cell* of $\mathcal{T}(v)$ is the set

$$S_i^\circ(v) := \left\{ x \in \mathbb{T}\mathbb{A}^{d-1} \mid -v_i \odot x_i > -v_\ell \odot x_\ell \text{ for all } 1 \leq \ell \leq d, i \neq \ell \right\},$$

that is, the set of points $x \in \mathbb{T}\mathbb{A}^{d-1}$ for which $-v_i \odot x_i$ realizes the maximum in $\alpha_v(x)$ uniquely. The *i-th closed (max-)cell* $S_i(v)$ of $\mathcal{T}(v)$ is the set of points $x \in \mathbb{T}\mathbb{A}^{d-1}$ for which $-v_i \odot x_i$ realizes the maximum in $\alpha_v(x)$. Any hyperplane $\mathcal{T}(v)$ divides $\mathbb{T}\mathbb{A}^{d-1}$ into up to d sectors each with apex at v , one for each monomial of α_v , as seen in Fig. 1.

For any point configuration $V = \{v^{(1)}, \dots, v^{(n)}\} \subset \mathbb{T}\mathbb{P}^{d-1}$, the union of the hyperplanes $\mathcal{T}_j := \mathcal{T}(v^{(j)})$ gives a *tropical hyperplane arrangement* $\mathcal{T}(V)$ in $\mathbb{T}\mathbb{A}^{d-1}$. Its *defining polynomial* $\mathcal{Q}_V$ is obtained by multiplying the linear forms $\alpha_{v^{(j)}}(x)$ for the tropical hyperplanes $\mathcal{T}_j$ where $1 \leq j \leq n$. That is,

$$\mathcal{Q}_V := \bigodot_{j=1}^n \alpha_{v^{(j)}}(x).$$

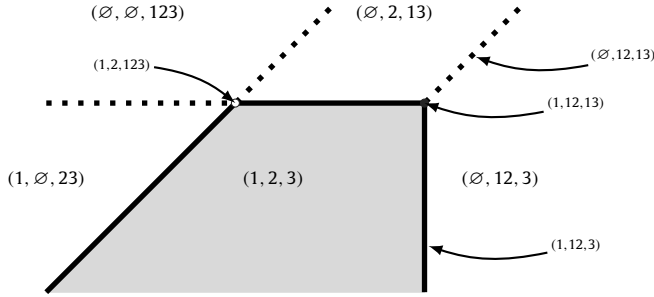


Fig. 2. Max-tropical hyperplane arrangements with some affine covectors. The shaded region is $\text{tconv}(V)$ and the solid dot is the tropical vertex $v^{(1)}$.

Example 2.3. The directed triangle κ_3 from Example 1.1 describes a point configuration V given by the columns of the weighted adjacency matrix

$$C = \begin{pmatrix} 0 & \infty & \infty \\ a & 0 & \infty \\ c & b & 0 \end{pmatrix}.$$

This point configuration gives rise to the tropical hyperplane arrangement $\mathcal{T}(V)$ with defining tropical polynomial

$$\begin{aligned} \mathcal{Q}_V &= \alpha_{v^{(1)}} \odot \alpha_{v^{(2)}} \odot \alpha_{v^{(3)}} \\ &= (x_1 \boxplus -a \odot x_2 \boxplus -c \odot x_3) \odot (x_2 \boxplus -b \odot x_3) \odot x_3 \\ &= x_1 x_2 x_3 \boxplus -a \odot x_2^2 x_3 \boxplus -(c \oplus ab) \odot x_2 x_3^2 \boxplus -b \odot x_1 x_3^2 \boxplus -(bc) \odot x_3^3. \end{aligned}$$

The tropical hyperplane arrangement for $c < a + b$ is sketched in Fig. 2 with $\text{tconv}(V)$ shown as the shaded region.

The polyhedral subdivision of $\mathbb{T}\mathbb{A}^{d-1}$ induced by $\mathcal{T}(V)$ is called the *covector decomposition* $\mathcal{H}(V)$ induced by V and it is the common refinement of the subdivisions for each $\mathcal{T}_j$ (Fink and Rincón, 2015). The *covector cells* of $\mathcal{H}(V)$ may be described combinatorially in the following way.

Define the *support graph* $B(V)$ of V as the bipartite graph on $[n] \sqcup [d]$ with edges

$$\left\{ (j, i) \in [n] \times [d] \mid v_i^{(j)} \neq \infty \right\}.$$

A (d, n) -covector L is a subgraph of the support graph $B(V)$. If clear from the context, we might refer to L as just a *covector*. For a covector L , write L_i for the predecessors of the node $i \in [d]$ in L . This also allows for a compact notation of a covector as $L = (L_1, \dots, L_d)$, compare with Joswig (2021, Section 6.3.). For example, in compact notation, $(1, \emptyset, 23)$ refers to the covector

$$\begin{array}{ccc} 1 & 2 & 3 \\ \downarrow & \searrow & \downarrow \\ 1 & 2 & 3 \end{array}.$$

Any polyhedral cell of $\mathcal{H}(V)$ can be labeled in the following way.

Definition 2.4. The *affine covector* of a point $p \in \mathbb{T}\mathbb{A}^{d-1}$ relative to V is the covector $\text{tc}(p, V)$ containing edges (j, i) if and only if p is contained in the sector $S_i(v^{(j)})$ of the vertex $v^{(j)}$.

The *open (covector) cell* $S_L^\circ(V)$ is the intersection of the open cells $S_i^\circ(v^{(j)})$ for every edge $(j, i) \in L$. Similarly, the *closed (covector) cell* $S_L(V)$ is the intersection of the closed cells $S_i(v^{(j)})$ for all $(j, i) \in L$.

Both inclusion $L \subseteq L'$ and union $L \cup L'$ of (d, n) -covectors are to be understood as the corresponding operations on the edge sets. From the definitions it follows that

$$S_{L \cup L'}(V) = S_L(V) \cap S_{L'}(V). \quad (3)$$

In particular, the full-dimensional open cells correspond to covectors where every vertex $j \in [n]$ only has one outgoing edge.

Remark 2.5. Each full-dimensional open cell corresponds to a monomial $x^a = x_1^{a_1} \dots x_d^{a_d}$ in the following way. Let S be a full-dimensional open cell of a covector decomposition $\mathcal{H}$. A generic point $x \in S$ has a covector $\text{tc}(x, V) = (L_1, \dots, L_d)$ which encodes the fact that

$$\alpha_j(x) = -v_{ij} \odot x_i$$

whenever $j \in L_i$. Since $Q_V = \odot_{j=1}^n \alpha_j$, this implies that for a generic point $x \in S$ that

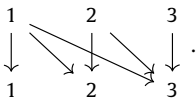
$$Q_V(x) = \bigodot_{j=1}^n (-v_{ij} \odot x_i)$$

which is exactly the monomial of Q_V with $x_i^{a_i}$ for $a_i = \#L_i$. In the context of tropical hyperplane arrangements, Dochtermann et al. referred to the vector of counts $a = (\#L_1, \dots, \#L_d)$ as *coarse type* of the point x and made the connection between coarse types and exponent vectors of monomials in Q_V (Dochtermann et al., 2012).

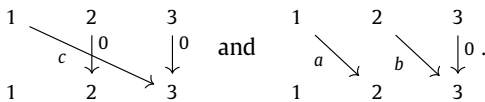
Then, the relationship between open sectors and covectors in (3) implies that the corresponding coarse type of a lower-dimensional open sector S does not correspond to a monomial of Q_V . Instead, if $S = \cap_{i \in I} S_i$ where S_i are full-dimensional closed sectors, then the coarse type of S is the least common multiple of the coarse types of each S_i .

Note that while the correspondence between full-dimensional open cells and monomials is one-to-one, there are possibly multiple covectors leading to the same monomial of Q_V . This will be discussed in more detail in the next section.

Example 2.6. Continuing from Example 2.3, Fig. 2 contains a non-exhaustive set of examples for affine covectors relative to the point configuration V . The support graph $B(V)$ of this point configuration is given by



With respect to Remark 2.5, there are two covectors giving rise to the monomial $x_2 x_3^2$, namely



The edges $j \rightarrow i$ of both covectors are annotated with the values for c_{ij} . Multiplying together the terms of each α_j corresponding in both covectors gives

$$(-(c \odot 0 \odot 0) \boxplus -(a \odot b \odot 0)) x_2 x_3^2 = -(c \oplus ab) x_2 x_3^2.$$

This suggests that the coefficient of $x_2x_3^2$ arises as minus the minimum over the sums of weights for covectors (L_1, L_2, L_3) with $a_i = \#L_i$. Similarly, all five maximal-dimensional cells correspond to the five monomials of the defining polynomial Q_V . $\square$

It follows from (3) that the poset of affine covectors of a point configuration $V \subset \mathbb{TP}^{d-1}$ is dual to the poset of cells in $\mathcal{H}(V)$. This justifies the following definition, the terminology following Fink and Rincón (2015).

Definition 2.7. The *tropical combinatorial type* $\text{TC}(V)$ of a point configuration V is defined as the poset of affine covectors of V .

A min-tropical polytope $\text{tconv}(V)$ inherits its tropical combinatorial type from the max-tropical hyperplane arrangement $\mathcal{T}(V)$ in the following way.

Proposition 2.8 (Joswig, 2021, Thm. 6. 14.). For $V \subset \mathbb{TP}^{d-1}$ finite, the tropical polyhedron $\text{tconv}(V)$ in $\mathbb{TA}^{d-1}$ is the union of closed cells with covector L such that every $L_i \neq \emptyset$ and $\bigcup_{i=1}^d L_i = [n]$. In other words, every node in L has an incoming and outgoing edge.

Example 2.9. The covectors for the arrangement $\mathcal{T}(V)$ from Example 2.6 such that no node in the (3, 3)-covector is isolated are

$$(1, 2, 3), (1, 12, 3), (1, 2, 23), (1, 2, 13), (1, 12, 13), \text{ and } (1, 2, 123).$$

These are precisely the covectors associated to the closed shaded region in Fig. 2.

If the tropical vertex set $V \subset \mathbb{TA}^{d-1}$, that is, every vertex has only finite coordinates, the above statement would admit the slogan version ' $\text{tconv}(V)$ is the union of the bounded cells in $\mathcal{H}(V)$ ' which is an important result by Develin and Sturmfels (2004, Theorem 15).

Definition 2.10. Two point configurations $V = (v_1, \dots, v_n)$ and $W = (w_1, \dots, w_n)$ of points in $\mathbb{TP}^{d-1}$ are *tropically equivalent* if there is a pair of permutations $(\tau, \sigma) \in \mathbb{S}_n \times \mathbb{S}_d$ acting on the vertices and coordinates such that the induced map on covectors is a poset isomorphism between $\text{TC}(V)$ and $\text{TC}(W)$.

Similarly, two tropical polyhedra $\text{tconv}(V)$ and $\text{tconv}(W)$ are *tropically equivalent* if the point configurations of V and W are tropically equivalent.

2.2. Regular dual subdivisions

A polyhedral subdivision of a finite point set A in $\mathbb{R}^d$ is called *regular* if it is induced by a height function $h \in \mathbb{R}^A$. We denote this subdivision by $\mathcal{S}(A, h)$. For further details, see Joswig (2021, Section A. 4.).

Definition 2.11. Let $\mathcal{S}$ be a subdivision of a point configuration $A \subset \mathbb{R}^d$ and consider all regular subdivisions $\mathcal{S}(A, h)$ that are refined by $\mathcal{S}$.

The associated height functions form the *secondary cone*

$$\Sigma(A, \mathcal{S}) = \left\{ h \in \mathbb{R}^A \mid \mathcal{S} \text{ refines } \mathcal{S}(A, h) \right\}.$$

The collection of all secondary cones forms the *secondary fan* $\Sigma(A)$.

Example 2.12. Consider the point configuration $A = \{0, 1, 2, 3\} \subset \mathbb{R}$. This has four possible regular subdivisions shown in Fig. 3. Its secondary fan $\Sigma(A)$ in $\mathbb{R}^4$ is a complete (i.e. covering the entire

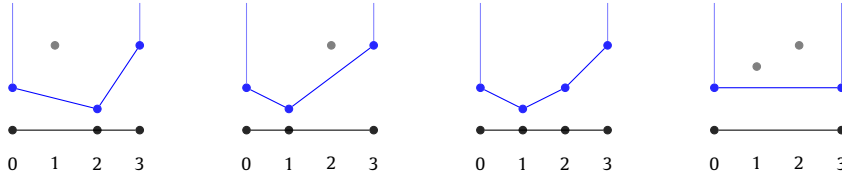


Fig. 3. The four possible regular subdivisions of the point configuration $A = \{0, 1, 2, 3\} \subset \mathbb{R}$ in Example 2.12.

space) full-dimensional polyhedral fan. It has four cones corresponding to the four possible regular subdivisions.

Denote by $\text{supp}(F)$ the support of a tropical polynomial $F \in \mathbb{T}[x_1, \dots, x_d]$, i.e. the set of exponent vectors $s \in \mathbb{Z}^d$ such that the coefficient $c_s \neq \infty$. The *Newton polytope* $\mathcal{N}(F)$ of a tropical polynomial F is the convex hull of $\text{supp}(F)$ in $\mathbb{R}^d$.

Note, that if $V \subset \mathbb{T}\mathbb{A}^{d-1}$, then $\text{supp}(\mathcal{Q}_V) = n \cdot \Delta_{d-1}$, as $\mathcal{Q}_V$ is a product of linear forms with Newton polytope Δ_{d-1} each. In the general case, Fink and Rincón (2015) gave the following description of $\mathcal{N}(\mathcal{Q}_V)$ in terms of *mixed subdivisions*. For details we refer to De Loera et al. (2010, Section 9. 2. 2.).

Proposition 2.13 (Fink and Rincón, 2015, Proposition 4. 1). *The covector decomposition $\mathcal{H}(V)$ is dual to the mixed regular subdivision of $\mathcal{N}(\mathcal{Q}_V)$ with height function given by the coefficients of $\mathcal{Q}_V$. A face of $\mathcal{H}(V)$ labeled by an affine covector L is dual to the cell of $\mathcal{N}(\mathcal{Q}_V)$ obtained as the Minkowski sum*

$$\mathcal{N}(\mathcal{Q}_V) = \sum_{j=1}^n \Delta_{L_j} \quad (4)$$

where Δ_{L_i} denotes the simplex with vertices e_j for $j \in L_i$.

Instead of enumerating every regular subdivision of $\mathcal{N}(\mathcal{Q}_V)$ and checking whether a given subdivision is actually mixed, we can use the *Cayley trick* (see Santos, 2005) to check regular subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$ and its subpolytopes instead.

Definition 2.14. Any subpolytope of $\Delta_{n-1} \times \Delta_{d-1}$ is called a *root polytope*. The *root polytope* Γ_V associated to point configuration $V \in \mathbb{T}^{d \times n}$ is given by

$$\Gamma_V := \text{conv} \left\{ (e_i, e_j) \mid v_i^{(j)} < \infty \right\}.$$

Through the Cayley trick we get a bijection between subdivisions of Γ_V and mixed subdivisions of $\mathcal{N}(\mathcal{Q}_V)$. The following result has been termed the *structure theorem of tropical convexity* by Joswig and Loho (2016).

Theorem 2.15 (Joswig and Loho, 2016, Cor. 34). *The covector decomposition $\mathcal{H}(V)$ of $\mathbb{T}\mathbb{A}^{d-1}$ is dual to the regular subdivision $\mathcal{S}$ of the root polytope Γ_V with height function given by V .*

Moreover, the covector decomposition of the tropical polytope $\text{tconv}(V)$ is dual to $\mathcal{S}$ intersected with the interior of $\Delta_{n-1} \times \Delta_{d-1}$.

Corollary 2.16. *There is a bijection between combinatorial types of tropical polytopes with n vertices in $\mathbb{T}\mathbb{A}^{d-1}$ and regular subdivisions of root polytopes in $\Delta_{n-1} \times \Delta_{d-1}$.*

In what follows, we specialize this result to the case of polytopes to later enumerate combinatorial types of polytopes in Section 5.

2.3. Compactifications of polyhedral fans

We recall the definitions of polyhedral spaces and compactifications of polyhedra in tropical toric varieties. The definitions follow Kastner et al. (2023).

Definition 2.17. For a rational polyhedral fan $\Delta \subset N_{\mathbb{R}}$ the *tropical toric variety* $N_{\mathbb{R}}(\Delta)$ of $N_{\mathbb{R}}$ with respect to Δ is

$$N_{\mathbb{R}}(\Delta) := \coprod_{\sigma \in \Delta} N_{\mathbb{R}}/\text{span}(\sigma),$$

where $N_{\mathbb{R}}(\sigma) := N_{\mathbb{R}}/\text{span}(\sigma)$ is the *stratum* associated to σ . We equip $N_{\mathbb{R}}(\Delta)$ with the unique topology such that

- the inclusions $N_{\mathbb{R}}/\text{span}(\sigma) \hookrightarrow N_{\mathbb{R}}(\Delta)$ are continuous for any cone $\sigma \in \Delta$, and
- for any $x, v \in N_{\mathbb{R}}$, the sequence $(x + nv)_{n \in \mathbb{N}}$ converges in $N_{\mathbb{R}}(\Delta)$ if and only if v is contained in the support of the fan Δ .

For any cone $\sigma \in \Delta$ we denote by π_{σ} the natural projection maps $N_{\mathbb{R}} \twoheadrightarrow N_{\mathbb{R}}(\sigma)$ onto each stratum.

Remark 2.18. The tropical toric variety we are mostly interested in is given by the cone Δ over the $(d-1)$ -simplex Δ_{d-1} . In this case, $N_{\mathbb{R}}(\Delta) = \mathbb{T}^d$. Similarly, in the case where we take the cone Δ inside $\mathbb{R}^E$ spanned by all positive coordinate directions, we get $N_{\mathbb{R}}(\Delta) = \mathbb{T}^E$. Note that in this case, we index the strata of $\mathbb{T}^E$ by the coordinates that are not equal to ∞ , as opposed to the charts of $N_{\mathbb{R}}(\Delta)$ where a cone σ denotes the coordinates that are set to infinity.

Definition 2.19 (Osserman and Rabinoff, 2013, Def 3.1). Let $\mathcal{P}$ be a finite collection of polyhedra in $N_{\mathbb{R}}$, and Δ be a pointed fan. The fan Δ is said to be *compatible* with $\mathcal{P}$ if for all $P \in \mathcal{P}$ and all cones $\sigma \in \Delta$, either $\sigma \subset \text{relint}(P)$ or $\text{relint}(\sigma) \cap \text{rec}(P) = \emptyset$, where $\text{rec}(P)$ denotes the recession cone of P . If for all $P \in \mathcal{P}$, $\text{rec}(P)$ is a union of cones in Δ , we call Δ a *compactifying fan* for $\mathcal{P}$.

A special case for compactifying fans is the compactification of a single polyhedron P . In this case, reasonable choices of compactifying fans are given by faces of the recession cone. This leaves us with the following description of the compactification.

Lemma 2.20 (Osserman and Rabinoff, 2013, Lemma 3.9). Let Δ be a compactifying fan for the polyhedron P . Then, its compactification $\overline{P}$ in $N_{\mathbb{R}}(\Delta)$ is

$$\overline{P} := \coprod_{\sigma \in \Delta, \text{relint}(\sigma) \cap \text{rec}(P) \neq \emptyset} \pi_{\sigma}(P).$$

Recall from Jell et al. that a polyhedral subspace is the underlying set of a polyhedral complex in $\mathbb{T}^n$. The definition of a polyhedral complex in $\mathbb{T}^n$ is analogous to the definition of a polyhedral complex in Euclidean space, where the comprising polyhedra in $\mathbb{T}^n$ are taken as closures of polyhedra in the strata of $\mathbb{T}^n$ (with respect to the topology as a tropical toric variety). Thus we can make the following definition, which we include for completeness.

Definition 2.21. A *polyhedral space* X is a paracompact, second-countable Hausdorff space with an atlas of charts $(\varphi_{\alpha} : U_{\alpha} \rightarrow \Omega_{\alpha} \subset X_{\alpha})_{\alpha \in A}$ such that:

- (1) The U_{α} are open subsets of X , the Ω_{α} are open subsets of polyhedral subspaces $X_{\alpha} \subset \mathbb{T}^{r_{\alpha}}$, and the maps $\varphi_{\alpha} : U_{\alpha} \rightarrow \Omega_{\alpha}$ are homeomorphisms for all $\alpha \in A$, and

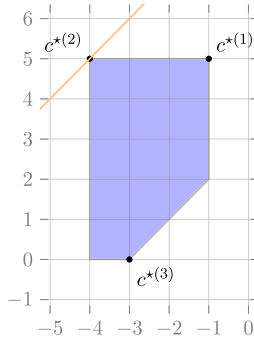


Fig. 4. The polytrope from Example 3.2 together with the non-facet defining line defined by $x_3 - x_2 = 9$.

(2) for all $\alpha, \beta \in A$, the transition maps

$$\varphi_\alpha \circ \varphi_\beta^{-1}: \varphi_\beta(U_\alpha \cap U_\beta) \rightarrow \varphi(U_\alpha \cap U_\beta)$$

are extended affine linear maps in the sense of Jell et al. (2019).

3. Polytopes from DAGs

We now shift our focus to point configurations $C \in \mathbb{T}^{n \times n}$ arising from weighted DAGs. By this we mean that for a DAG $G = ([n], E)$ with weights $w \in \mathbb{R}^E$ we fix a labeling compatible with the topological ordering induced by E . This choice results in an embedding $\mathbb{R}^E \rightarrow \mathbb{T}^{n \times n}$ for which we have $c_{ij} = w(j \rightarrow i)$ and $c_{ii} = 0$.

Following Joswig (2021, Section 6.5.) we can generalize the original definition from Joswig and Kulas (2010) and call a subset $P \subseteq \mathbb{T}\mathbb{A}^{d-1}$ a *polytrope* if $P = \text{tconv}(C)$ for some finite $C \subset \mathbb{T}\mathbb{P}^{d-1}$ and P is classically convex under the identification $\mathbb{T}\mathbb{A}^{d-1} \cong \mathbb{R}^{d-1}$.

A polytrope is also an ordinary polyhedron, which means it has a description as an intersection of classical halfspaces in the following way. For a matrix $C \in \mathbb{T}^{n \times n}$ its *weighted digraph polyhedron* $Q(C)$ is given by the (ordinarily convex) set

$$Q(C) = \{x \in \mathbb{T}\mathbb{A}^{n-1} \mid x_i - x_j \leq c_{ij} \text{ for all } 1 \leq i, j \leq n, i \neq j\}. \quad (5)$$

If a matrix $C \in \mathbb{T}^{n \times n}$ is the weighted adjacency matrix of a digraph G , calculating the shortest paths between all pairs of nodes corresponds to the evaluation of the polynomial

$$C^* = I \oplus C \oplus C^{\odot 2} \oplus \dots \oplus C^{\odot(n-1)} \oplus \dots$$

This infinite sum converges to the expression $C^* = (I \oplus C)^{\odot(n-1)}$ if the corresponding weighted digraph has no negative cycles (Joswig, 2021, Lemma 3.23). For our purposes, we focus on DAGs which means there are no cycles at all and thus C^* always converges.

The matrix C^* is called the (min-plus) *Kleene star* of C ; we denote by $G^* = ([n], E^*)$ the graph on which this matrix is supported, and by $w^* \in \mathbb{R}^{E^*}$ the corresponding vector of weights on G^* . In the literature, G^* is sometimes called the *transitive closure* of G .

If the Kleene star C^* converges, we have $c_{ii} = 0$ and its entries satisfy the triangle inequality $c_{ij} \leq c_{ik} + c_{kj}$. This also implies that taking the Kleene star is idempotent, i.e. $(C^*)^* = C^*$, and multiplication by a Kleene star is also idempotent, so $C^* \odot C^* = C^*$.

Proposition 3.1 (de la Puente, 2013, Lemma 6). For $C \in \mathbb{T}^{n \times n}$, we have $Q(C) = Q(C^*)$.

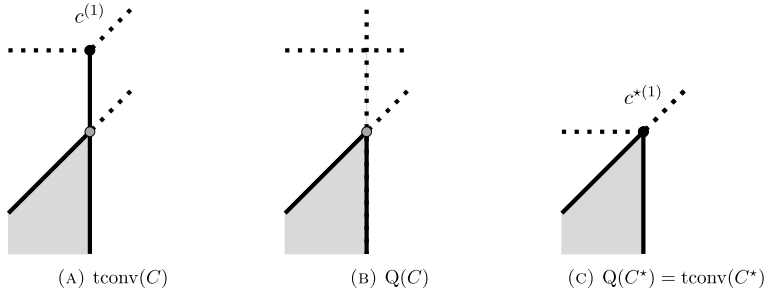


Fig. 5. Comparison of the hyperplane arrangements in Example 3.4.

Example 3.2 (Joswig, 2021, Example 3.28.). Consider the matrix

$$C = \begin{pmatrix} 1 & 4 & 0 \\ -1 & 0 & -3 \\ 5 & \infty & \infty \end{pmatrix}$$

with Kleene star

$$C^* = \begin{pmatrix} 0 & 4 & 0 \\ -1 & 0 & -3 \\ 5 & 9 & 0 \end{pmatrix}.$$

Evaluating Equation (5) for this particular matrix C^* yields the six defining inequalities

$$\begin{array}{lll} x_2 - x_1 \leq -1, & x_1 - x_2 \leq 4, & x_1 - x_3 \leq 0, \\ x_3 - x_1 \leq 5, & x_3 - x_2 \leq 9, & x_2 - x_3 \leq -3. \end{array}$$

Note how the (3, 2)-th entry is the only off-diagonal entry not agreeing between C and C^* and how the inequality $x_3 - x_2 \leq 9$ is redundant. This is shown as the line in Fig. 4.

Theorem 3.3. A tropical polytope $P \subset \mathbb{T} \mathbb{A}^{n-1}$ is a polytrope if and only if

$$P = \text{tconv}(C^*) = Q(C)$$

for some $C \in \mathbb{T}^{n \times n}$.

Proof. See Section 3.4. from Joswig and Loho (2016). $\square$

As a consequence of this result, we may assume that a polytrope is generated by a square matrix C such that $c_{ii} = 0$. This also means that the construction of the weighted digraph polyhedron does not distinguish between a matrix C and its Kleene star C^* , as seen in Proposition 3.1 and Example 3.2. On the other hand, the tropical convex hulls of a matrix and its Kleene star do differ in general. This can also be seen with the tropical (non-)equivalence of C and C^* as in the following example.

Example 3.4. Continuing from Example 2.6, we consider the case $c > a + b$ which yields the Kleene star

$$C^* = \begin{pmatrix} 0 & \infty & \infty \\ a & 0 & \infty \\ a+b & b & 0 \end{pmatrix}.$$

The corresponding hyperplane arrangements for $\mathcal{T}(C)$ and $\mathcal{T}(C^*)$ are shown in Fig. 5. The point configurations given by C and C^* are not tropically equivalent since their tropical combinatorial types $\text{TC}(C)$ and $\text{TC}(C^*)$ are not isomorphic as posets.

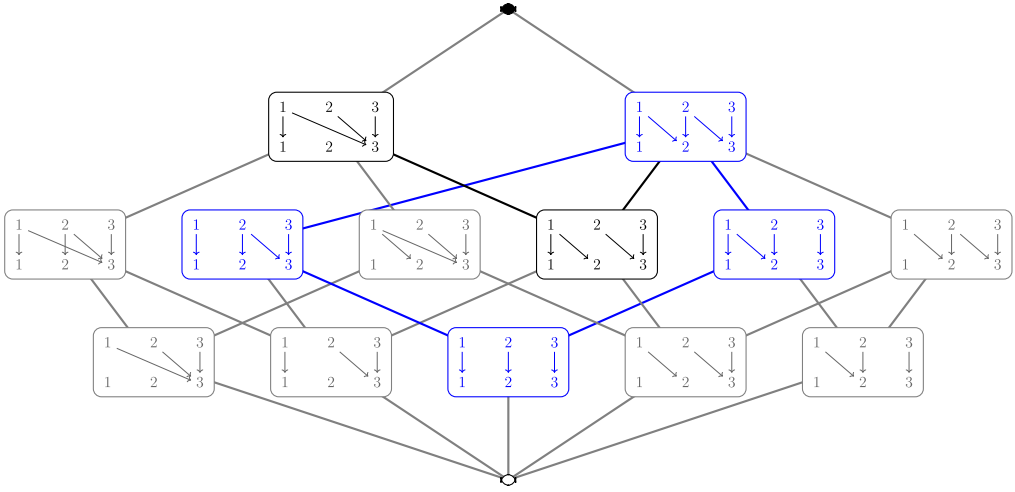


Fig. 6. Hasse diagram of $TC(C)$ from Example 3.4. The subposet corresponding to $tconv(C)$ is marked in black and blue, while the subposet of $Q(C)$ is marked in blue. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

We can see in Fig. 6 that $TC(C)$ has five atoms (i.e., elements covering the minimum) corresponding to the five chambers. On the other hand, $TC(C^*)$ has four atoms corresponding to the four chambers in Fig. 5.(c). Compared to $TC(C)$, the covectors $(\emptyset, 1, 23)$ and $(1, 1, 23)$ are missing and $(\emptyset, 1, 123)$ gets identified with $(\emptyset, 12, 23)$.

Yet, $Q(C)$ does occur in the hyperplane arrangement of $\mathcal{T}(C)$ and as a subpolytope of $tconv(C)$, namely as the two-dimensional unbounded cell in Fig. 5.

Note that a weighted digraph polyhedron can be expressed as the set of points $x \in \mathbb{T}\mathbb{P}^{d-1}$ satisfying the tropical linear inequalities $x \leq C \odot x$ since

$$x_i - x_j \leq c_{ij} \iff x_i \leq x_j + c_{ij}$$

for all $i \in [n]$ and since the right inequality holds for all $j \in [n]$,

$$x_i \leq \min_j \{x_j + c_{ij}\} = c_i \odot x \quad (6)$$

also holds.

Everything until this point has set the stage for our first main result, which now follows without much effort. This allows us to recover the polytope for a given point configuration from the combinatorial data of the tropical hyperplane arrangement. Recall that a *perfect matching* in a graph is a subset of the edges that covers every node exactly once.

Theorem 3.5. *Let $C \in \mathbb{T}^{n \times n}$ be supported on a DAG G . The polytope $tconv(C^*) \subseteq tconv(C)$ coincides with the closed cell $S_L(C)$ such that L is the perfect matching $(1, 2, \dots, n)$ in the support graph $B(C)$.*

Proof. By assumption, $B(C)$ contains the perfect matching $(1, 2, \dots, n)$ as C is assumed to be supported on a DAG. Also, $S_L(C) \subseteq tconv(C)$ by Proposition 2.8 which means we are left to prove that $S_L(C) = tconv(C^*)$.

Since C is supported on a DAG, the polytope $P = tconv(C^*)$ is non-empty and can be described as the set of points satisfying the inequalities in (6) for all $i, j \in [n]$. Rearranging those inequalities gives

$$x_j \geq \max_{1 \leq i \leq n} \{-c_{ij} + x_i\} = \alpha_j(x)$$

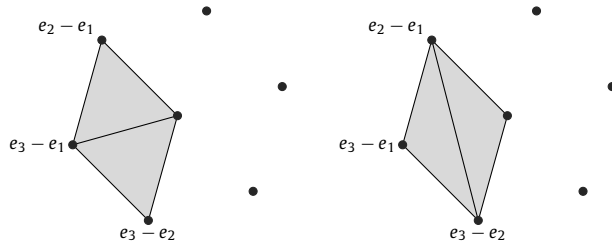


Fig. 7. Two triangulations of the fundamental polytope P_G for $G = \kappa_3$. The left triangulation is central, while the right triangulation is not.

for every $j \in [n]$. Thus, any point $x \in P$ is inside every closed sector $S_j(v^{(j)})$ for each $j \in [n]$. By (3), the intersection of all these sectors is the covector with edges (j, j) for every $j \in [n]$. $\square$

Another approach was taken by Joswig and Schröter (2019). Instead, they characterized combinatorial types of polytropes in terms of *central triangulations* of the following lower-dimensional point configuration.

Definition 3.6. For a digraph G , the associated *fundamental polytope* is defined as

$$P_G = \text{conv}(\{e_i - e_j \mid j \rightarrow i \in G\} \cup \{0\}).$$

The fundamental polytope P_G provides an appropriate restriction of the regular subdivisions of $\mathcal{N}(\mathcal{Q}_C)$ when $C \in \mathbb{T}^{n \times n}$ is supported on a DAG G . The relationship between P_G and $\mathcal{N}(\mathcal{Q}_C)$ is of the following nature. Recall from Remark 2.5 the correspondence between the open covector cells of $\mathcal{H}(C)$ and the monomials of $\mathcal{Q}_C$. For a covector L with coarse type a such that $a_j = 0$, $a_i = 2$ and otherwise $a_k = 1$, L must be (since G contains no cycles) of the form

$$\begin{array}{ccccccc} 1 & 2 & \dots & j & i & \dots & n \\ \downarrow & \downarrow & & \searrow & \downarrow & & \downarrow \\ 1 & 2 & \dots & j & i & \dots & n \end{array}.$$

The coefficient of x^a in $\mathcal{Q}_C$ is given by the weight of the above covector L and is precisely c_{ij} , which means that the vertex a is also lifted to height c_{ij} in the regular subdivision of $\mathcal{N}(\mathcal{Q}_C)$. But then, $a = e_i - e_j + \mathbf{1}$, which gives P_G as a translated subpolytope of $\mathcal{N}(\mathcal{Q}_C)$ with agreeing height function.

Motivated by this correspondence, we say that a subdivision of the fundamental polytope is *central* if the origin $\mathbf{0}$ is a vertex of every maximal cell. This terminology comes from the fact that for the complete digraph K_n , the origin is the unique interior lattice point of both P_G and $\mathcal{N}(\mathcal{Q}_V)$.

Example 3.7. Fig. 7 shows two subdivisions for P_G where $G = \kappa_3$. The first triangulation is central because the origin is a vertex of every maximal cell. This triangulation appears for all weights $w \in \mathbb{R}^E$ such that $w(1 \rightarrow 3) < w(1 \rightarrow 2) + w(2 \rightarrow 3)$.

On the other hand, if $w(1 \rightarrow 3) > w(1 \rightarrow 2) + w(2 \rightarrow 3)$ we obtain the second triangulation which is not central.

Remark 3.8. The fundamental polytope P_G corresponds to a subset of the A_n root system together with the origin. In that case, a matrix $C \in \mathbb{T}^{n \times n}$ supported on a DAG G induces a regular subdivision on the A_n root system which involves precisely the vertices of P_G while omitting the other vertices by lifting them to infinite height.

Recall that a directed graph G is *transitive* if for any pair of edges $j \rightarrow k$ and $k \rightarrow i$, G also contains the edge $j \rightarrow i$. That is, viewing the edges of G as a relation on the nodes gives a transitive relation.

Sometimes, such a graph is also called *transitively closed*. We now adapt the proof of Joswig and Schröter (2019, Theorem 22) to the case of arbitrary support.

Theorem 3.9. *The tropical combinatorial types of full-dimensional polytropes in $\mathbb{T}\mathbb{A}^{n-1}$ are in bijection with the regular central subdivisions of P_G where G ranges over transitive directed graphs G on n nodes.*

Proof. By Theorem 3.3, we know that for any given non-empty polytrope $P \subseteq \mathbb{T}\mathbb{A}^{d-1}$ there is a matrix $C \in \mathbb{T}^{n \times n}$ with $P = \text{tconv}(C)$ and $C = C^*$. Denote by G the underlying directed graph of C . By Proposition 2.13, P is dual to the regular subdivision $\mathcal{S}$ of $\mathcal{N}(\mathcal{Q}_C) \subseteq n \cdot \Delta_{n-1}$ induced by the coefficients of $\mathcal{Q}_C$. Thus, the height $h(a)$ of a vertex $a = \sum_{i=1}^n a_i e_i \in \mathcal{N}(\mathcal{Q}_C)$ is given by

$$h(a) = \min_L \sum_{j=1}^n c_{ij} \quad (7)$$

where L is a covector corresponding to a full-dimensional cell in $\mathcal{H}(C)$ such that $\#L_i = a_i$. In particular, any such covector L has a unique edge $j \rightarrow i$ for each $j \in [n]$.

Any Kleene star matrix satisfies the triangle inequality $c_{ij} \leq c_{ik} + c_{kj}$ and has zero diagonal. This means the central vertex $\mathbf{1}$ is lifted to height zero since the weight of the matching $(1, 2, \dots, n)$ evaluates to the sum of the diagonal entries of C in (7).

If there was another matching L' with lower weight, then C would have a negative cycle, which contradicts P being non-empty. Using the triangle inequality $0 = c_{ii} \leq c_{ij} + c_{ji}$, we can see that the central vertex lies below each line spanned by any two vertices $a, a' \in \mathcal{N}(\mathcal{Q}_C)$. Since $\mathbf{1}$ is the unique interior lattice point of $\mathcal{N}(\mathcal{Q}_C)$ with respect to $n \cdot \Delta_{n-1}$, this means $\mathbf{1}$ is a vertex of every maximal cell. Thus, $\mathcal{S}$ is a central mixed regular subdivision of $\mathcal{N}(\mathcal{Q}_C)$.

Now, the matrix C is the weighted adjacency matrix of a graph G . Since P_G can be seen as a translated subpolytope of $\mathcal{N}(\mathcal{Q}_C)$, we can restrict $\mathcal{S}$ to P_G . This is precisely the regular subdivision of P_G induced by C since the height function on P_G agrees with the height function on $\mathcal{N}(\mathcal{Q}_C)$.

This means the maximal-dimensional cells of $\mathcal{N}(\mathcal{Q}_C)$ are in bijection with the maximal dimensional cells of P_G . As a consequence, the central mixed subdivisions of $\mathcal{N}(\mathcal{Q}_C)$ are mapped bijectively to the central subdivisions of P_G . $\square$

Connecting back to Remark 3.8, we can see that the combinatorial types of polytropes correspond to regular central subdivisions of those subsets of the A_n root system corresponding to transitive DAGs on n nodes.

4. Moduli space of polytropes from DAGs

We have established two characterizations for tropical equivalence of polytropes, via special subcomplexes in regular subdivisions of root polytopes and duals of central regular subdivisions of fundamental polytopes. This suggests that certain subfans of the secondary fans $\Sigma(\Gamma_G)$ resp. $\Sigma(P_G)$ classify combinatorial types of polytropes.

If G is transitive, this subfan is cut out by the region of Kleene star matrices on G , which in the case of the complete graph has been noted by Tran (2017). If G is not transitive, this region is not useful, as taking the Kleene star of a matrix inserts edges in the supporting graph which enlarges both Γ_{G^*} and P_{G^*} as compared to Γ_G and P_G . In other words, there are no Kleene star matrices supported precisely on G .

At the same time, specific choices of edge weights in C might give a polytrope $Q(C)$ exhibiting the same face structure as a polytrope supported on a subgraph $H \subset G$. We can illustrate this phenomenon already with our running example.

Example 4.1. There are two possible cases for the directed triangle κ_3 with weight matrix

$$C = \begin{pmatrix} 0 & \infty & \infty \\ a & 0 & \infty \\ c & b & 0 \end{pmatrix}.$$

If $c \geq a + b$, then one can see in Fig. 5 that the hyperplane $x_3 - x_2 = c$ does not define a facet and is redundant. Thus, one can replace the value of c by ∞ and delete the edge $1 \rightarrow 3$ while keeping the resulting polytrope intact.

On the other hand, in Fig. 2, the case $c < a + b$ is shown. There, $x_3 - x_2 = c$ is a facet-defining hyperplane.

Example 4.1 suggests that an edge not contributing to a shortest path between any pair of nodes does not contribute to the polytrope. Such a path is not *critical* in the sense of Améndola et al. (2022). Thus, the solution to finding a non-redundant weight matrix C involves identifying the graph giving the minimal non-redundant facet description.

4.1. Transitive reduction

The characterization of non-redundant facets of $Q(C)$ and edges of G relies on the interplay between *transitive closure* and *covering relations* of the edge set E seen as a binary relation.

Definition 4.2. For a given DAG G , the *transitive reduction* G^{tr} contains an edge $j \rightarrow i \in G$ if there is no other path between j and i .

For a DAG G with weights $w \in \mathbb{R}^E$, the *weighted transitive reduction* G_w^b is the digraph containing an edge $j \rightarrow i$ with weight $w(j \rightarrow i)$ whenever $j \rightarrow i$ is the unique shortest path between j and i in G .

We denote by $w^b = w|_{G_w^b}$ the vector of weights on G_w^b . If C is the weighted adjacency matrix of a DAG G , denote by C^b the weighted adjacency matrix of G_w^b .

The weighted transitive reduction G_w^b has appeared in Gissibl (2018) as the *minimum max-linear* DAG. The sets of edges E^{tr} of G^{tr} resp. E_w^b of G_w^b are called the *covering relations* resp. *weighted covering relations* of G .

Remark 4.3. The transitive reduction G^{tr} is the smallest directed graph with the same reachability relation as G . The weighted transitive reduction imposes the additional constraint that the appearing edges already realize the shortest paths between their endpoints uniquely, thus we have $G_w^b \supseteq G^{\text{tr}}$.

Example 4.4. Continuing from Example 4.1, the only strict subgraph of the directed triangle $G = \kappa_3$ with the same reachability between all nodes is the path graph

$$H: 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

since for both G and H , any node i is accessible from any other node j if $i > j$. Thus, for the transitive reduction we have $G^{\text{tr}} = H$. If $c \geq a + b$, then G^{tr} and $G_{(a,b,c)}^b$ agree since $1 \rightarrow 2 \rightarrow 3$ is a shortest path. On the other hand if $c < a + b$, then $1 \rightarrow 3$ is the unique shortest path connecting its endpoints, which means $G = G_{(a,b,c)}^b \supsetneq G^{\text{tr}}$.

It has been shown by Aho et al. (1972) that for finite unweighted DAGs the transitive reduction G^{tr} always exists and is unique. We can extend this to finite weighted DAGs. For this, we recall the following definition from Joswig and Schröter (2022).

Definition 4.5. If $T^{(s)}$ is a spanning tree for the set of vertices reachable from s (including s itself), we say that $T^{(s)}$ is a *shortest-path tree at (source) s* in a DAG G with weights $w \in \mathbb{R}^E$ if s has no incoming edges and for every edge $j \rightarrow i$,

$$w(j \rightarrow i) + p_j \not\leq p_i,$$

where p_i is the length of the path $s \rightsquigarrow i$ in $T^{(s)}$.

Proposition 4.6. The weighted transitive reduction G_w^b of an weighted DAG G with weights $w \in \mathbb{R}^E$ always exists and is unique. It is given by

$$G^b = \bigcup_{s \in [n]} \bigcap \{ T^{(s)} \mid T^{(s)} \text{ is a shortest-path tree at } s \text{ for } w \}.$$

Proof. The right hand side H contains only edges in G that are contained in a shortest path connecting two nodes, thus $H \supseteq G^b$. On the other hand, consider an edge $j \rightarrow i \notin H$ with i reachable from j . This means that $j \rightarrow i$ is not contained in every shortest-path tree at j , so there exists another shortest path connecting j to i . But this means $j \rightarrow i \notin G^b$, which proves the entire statement. $\square$

The following lemma gives the condition under which an edge does not contribute a facet to $Q(C)$ and thus can be deleted from G .

Lemma 4.7. Let G be a DAG with weights $w \in \mathbb{R}^E$ and fix a labeling of G . Then, the following are equivalent:

- (1) $Q(C) = Q(C')$ for all $w' \in w + \mathbb{R}_{>0} e_{j \rightarrow i}$,
- (2) there exists $w' \in w + \mathbb{R}_{>0} e_{j \rightarrow i}$ such that $Q(C) = Q(C')$,
- (3) $j \rightarrow i \notin G^b$,

where C and C' denote the weighted adjacency matrices arising from w resp. w' .

Proof. We may assume without loss of generality that $w(j \rightarrow i) = w^*(j \rightarrow i)$. The implication (1) $\implies$ (2) is immediate. For the converse direction, if $w' \in w + \mathbb{R}_{>0} e_{j \rightarrow i}$ as required exists, then the hyperplane $\{x_i - x_j = c_{ij}\}$ does not define a facet of $Q(C)$, which means that $Q(C) = Q(C')$ for any value $w'(j \rightarrow i) \geq w(j \rightarrow i)$.

If $w'(j \rightarrow i) \geq w(j \rightarrow i)$ but $Q(C) = Q(C')$ for some $w' \in \mathbb{R}^E$, we know that there exists another path with weight $w(j \rightarrow i)$ from j to i in G with weights w' since $(w')^*(j \rightarrow i) = w(j \rightarrow i)$ has to be satisfied. Thus, $j \rightarrow i$ was not the unique shortest path. Since G is acyclic, any other path does not use the edge $j \rightarrow i$. The same reasoning works in reverse, proving the equivalence (2) $\iff$ (3). $\square$

Corollary 4.8. Let G be a DAG with weights $w \in \mathbb{R}^E$, $G' = G \setminus j \rightarrow i$ be the DAG obtained from G by deleting the edge $j \rightarrow i$ and denote by C the weight matrix obtained from w . Then, $Q(C) = Q(C|_{G'})$ if and only if $j \rightarrow i \notin G_w^b$.

Proof. The reasoning is the same as in the proof for Lemma 4.7, if $j \rightarrow i \notin G_w^b$ then the $x_i - x_j$ -hyperplane does not define a facet of $Q(C)$. Thus removing this hyperplane does not change $Q(C)$ and thus $Q(C) = Q(C|_{G'})$. If $Q(C) = Q(C|_{G'})$ we may assume without loss of generality that $w(j \rightarrow i) = w^*(j \rightarrow i)$ which means that there is another shortest path from j to i . But this means $j \rightarrow i \notin G_w^b$. $\square$

In other words, Lemma 4.7 describes that under the stated conditions the polytrope $Q(C)$ stays the same along rays in certain coordinate directions $e_{j \rightarrow i}$. After compactifying this ray by adding a point at infinity, which corresponds to the edge $j \rightarrow i$ being deleted from G , Corollary 4.8 asserts that the polytrope does not change.

On the other hand, starting with a fixed G with weights w , we may apply Lemma 4.7 successively to every edge $j \rightarrow i \in G \setminus G^b$ to get a pointed polyhedral cone. Taking into account the relationship $Q(C) = Q(C^*)$ from Proposition 3.1, leads to another of our main results.

Theorem 4.9. Fix a DAG G with weights $w \in \mathbb{R}^E$.

- (1) The set of weights $w' \in \mathbb{R}^{E^*}$ such that $Q(C) = Q(C')$ is the pointed polyhedral cone

$$\mathcal{C}(G, w) := w^* + \text{pos}(e_{j \rightarrow i} \mid j \rightarrow i \in G^* \setminus G^b).$$

- (2) The compactification of $\mathcal{C}(G, w)$ inside $\mathbb{T}^{E^*}$ with respect to its recession cone is the set of all weights w' for any DAG $H = (V, E')$ satisfying $G_w^b \subseteq H \subseteq G^*$ such that $Q(C) = Q(C')$.
 (3) The set of $w' \in \mathbb{R}^E$ such that $Q(C) = Q(C')$ is a face of the compactification $\overline{\mathcal{C}(G, w)}$.

Proof. (1) In the case that w' is chosen from the polyhedral cone $\mathcal{C}(G, w)$ we can apply Lemma 4.7 for every $j \rightarrow i \notin G^b$ to conclude $Q(C') = Q(C^*)$ where C' is the weight matrix associated to w' . If on the other hand $w' \in \mathbb{R}^E$ such that $Q(C') = Q(C)$, we know from Proposition 3.1 that additionally $Q(C') = Q(C^*) = Q(C)$ must be satisfied. If $\{x_i - x_j = c'_{ij}\}$ is a facet-defining hyperplane of $Q(C)$, we must have $c'_{ij} = c^*_{ij} = c_{ij}$. On the other hand, if $\{x_i - x_j = c'_{ij}\}$ does not define a facet, then $c'_{ij} \geq c^*_{ij}$. In other words, $w'(j \rightarrow i) = w^*(j \rightarrow i) + \varepsilon$ for some $\varepsilon \in [0, \infty]$ which by Lemma 4.7 is only possible for positive ε if $j \rightarrow i \notin G^b$.

- (2) First, note that since $\mathcal{C}(G, w)$ is a pointed polyhedral cone, it is equal to its own recession cone. In addition, the ray generators of $\mathcal{C}(G, w)$ are given by those coordinate directions $e_{j \rightarrow i}$ for which $j \rightarrow i \notin G^b$. By Lemma 2.20 this means that the compactification $\overline{\mathcal{C}(G, w)}$ inside $\mathbb{T}^E$ only intersects those strata indexed by subgraphs $G_w^b \subseteq H \subseteq G^*$. Now let $w' \in \overline{\mathcal{C}(G, w)}$ be weights for such a subgraph. We know that

$$w'(j \rightarrow i) = \begin{cases} w^*(j \rightarrow i), & j \rightarrow i \in G^b, \\ w^*(j \rightarrow i) + \lambda_{j \rightarrow i} e_{j \rightarrow i}, & j \rightarrow i \in H \setminus G_w^b. \end{cases}$$

Let $w'' \in \mathbb{R}^{E^*}$ be weights on G^* such that $w''|_H = w'$ and $w''(j \rightarrow i) = w^*(j \rightarrow i)$ otherwise. We may apply Corollary 4.8 to obtain that $Q(C') = Q(C'') = Q(C)$.

If $Q(C') = Q(C)$ for some graph $G_w^b \subseteq H \subseteq G^*$, then also $Q(C') = Q(C^*)$. Applying Lemma 4.7 when $w'(j \rightarrow i) > w^*(j \rightarrow i)$ and Corollary 4.8 when $w'(j \rightarrow i) = w^*(j \rightarrow i)$ gives $w' \in \overline{\mathcal{C}(G, w)}$.

- (3) This follows from the second point since $G_w^b \subseteq G \subseteq G^*$. $\square$

Theorem 4.10. The weighted transitive reduction G_w^b is the unique minimal weighted DAG and w^b the unique weights such that $Q(C)$ and $Q(C^b)$ coincide.

Proof. First, we get $Q(C) = Q(C^b)$ by applying Corollary 4.8 for every edge $j \rightarrow i \notin G_w^b$. Theorem 4.9.(2) implies that G_w^b is the minimal graph for which this is possible. Assume that w' is another vector of weights for G_w^b and C' the corresponding weight matrix. If $Q(C') = Q(C^b)$, we know by Lemma 4.7 that for every $j \rightarrow i \in G_w^b$ $c'_{ij} = c^b_{ij}$. But since w' and w^b are weights for G_w^b , we already have $w' = w^b$. $\square$

Remark 4.11. With a view towards max-linear Bayesian networks, Theorem 4.10 describes exactly which weights of a network are identifiable given sufficiently good data. Since $Q(C) = Q(C^*)$, sampling from a polytrope only allows us to recover C^* .

If $C^* = C^b$, this means that all original weights of C are identifiable. On the other hand, if there are edges $j \rightarrow i \notin G^b$, it is not possible to estimate the corresponding weight c_{ij} .

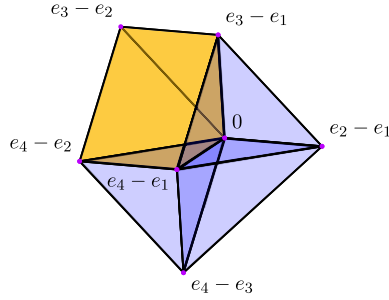


Fig. 8. The central regular subdivision from Example 4.13 with height function $(0, w) = (0, 1, 2, 2, 3, 3, 6)$. The weights w satisfy $G_w^b = G$ but the subdivision is not a triangulation due to the orange cell.

As a consequence, the weighted transitive reduction C^b gives a finer correspondence between polytropes $Q(C)$ and the corresponding DAGs G than using the Kleene star, which would only associate a transitive DAG to each C . This also refines Theorem 3.9 in the following way.

Corollary 4.12. *If the regular subdivision $\mathcal{S}$ of P_G induced by $(0, w) \in \mathbb{R} \times \mathbb{R}^E$ is a central triangulation, then $G_w^b = G$.*

Proof. If $C \in \mathbb{T}^{n \times n}$ induces a central triangulation $\mathcal{S}$ on P_G , for any $j \rightarrow i \in G$, the edge between $e_i - e_j$ and the origin occurs in $\mathcal{S}$. Any such edge is dual to a hyperplane $\{x_i - x_j = c_{ij}\}$ and describes a facet of $Q(C)$. But no hyperplane being redundant implies that $C = C^b$, and therefore $G_w^b = G$. $\square$

Example 4.13. The following example shows that the converse of Corollary 4.12 fails for $n \geq 4$. Let $G = \kappa_4$ be the complete DAG on 4 nodes and choose as weights $w = (1, 2, 2, 3, 3, 6)$. That is, the associated polytrope $Q(C)$ is given by the matrix

$$C = \begin{pmatrix} 0 & \infty & \infty & \infty \\ 1 & 0 & \infty & \infty \\ 2 & 3 & 0 & \infty \\ 2 & 3 & 6 & 0 \end{pmatrix}.$$

A straightforward calculation shows that C is idempotent, so $C^* = C$ and in particular $G_w^b = G$. The corresponding subdivision of P_G is shown in Fig. 8. This subdivision is central, but it contains the maximal cell with vertices

$$\{0, e_3 - e_1, e_3 - e_2, e_4 - e_1, e_4 - e_2\}$$

which is colored orange in Fig. 8, and clearly not a simplex. This means that the subdivision induced by $(0, w)$ is central, but not a triangulation.

Remark 4.14. In practice, for $C \in \mathbb{T}^{n \times n}$ supported on a DAG, one can calculate the weighted transitive reduction by calculating C^* and deleting every edge $j \rightarrow i$ for which $c_{ij} \neq c_{ij}^*$. This amounts to the usual $\mathcal{O}(n^3)$ running time for calculating the Kleene star.

Using Algorithm 2 of Joswig and Schröter from Joswig and Schröter (2022), one can calculate from a given DAG G the possible weighted transitive reductions with the corresponding inequalities. This may be computed in $\mathcal{O}(|E|^3 \cdot \sigma + n^2)$ time where σ is the maximum number of possible shortest-path trees at any node.

4.2. Moduli space and its stratifications

We discuss in the following the constructions of a moduli space $\mathcal{M}_{\nabla,n}^*$ for $(n-1)$ -dimensional polytropes associated to DAGs. For our construction, we use the secondary fan $\Sigma(P_G)$ in $\mathbb{R} \times \mathbb{R}^E$ of the fundamental polytope P_G restricted to height functions $(0, w)$ where $w \in \mathbb{R}^E$ are the original weights on G . We have two choices for how to write $\mathcal{M}_{\nabla,n}^*$ as a collection of polyhedral fans.

- (1) Classifying weights up to the Kleene stars of matrices $C \in \mathbb{T}^{n \times n}$ gives a covering by polyhedral fans indexed by transitive DAGs G .
- (2) Using the weighted transitive reduction C^b , which results in a finer characterization in terms of minimal non-redundant facet descriptions and the associated DAGs. This gives an atlas of charts for $\mathcal{M}_{\nabla,n}^*$ as a polyhedral space

Definition 4.15. For any DAG G , define the *open polytrope region* $\mathcal{K}(G)$ as the set of weights $w \in \mathbb{R}^E$ such that G with weights w agrees with its weighted transitive reduction G_w^b .

From the definition of the weighted transitive reduction it follows that the *open polytrope region* $\mathcal{K}(G)$ is an open polyhedral cone:

$$\mathcal{K}(G) = \left\{ w \in \mathbb{R}^E \mid w(j \rightarrow i) < \sum_{i=0}^{\ell} w(v_i \rightarrow v_{i+1}) \text{ for every path } (j = v_0, v_1, \dots, v_{\ell} = i) \right. \\ \left. \text{in } G \right\}.$$

Restricting $\Sigma(P_G)$ to $0 \times \mathcal{K}(G)$ gives a polyhedral fan $\mathcal{M}_G^*$ which we call the *open stratum* for G and which by definition parametrizes polytropes $Q(C)$ where C^b is supported precisely on G .

Remark 4.16. One can also see the open polytrope region as a region in the tropical hypersurface arrangement given by the entries of the Kleene star for a generic matrix supported on G .

Theorem 4.17. The polyhedral subspace of $\mathbb{T}^{E(\kappa_n)}$ given by the union of polyhedral fans

$$\mathcal{M}_{\nabla,n}^* := \coprod_{[G]} \mathcal{M}_G^* \subset \mathbb{T}^{E(\kappa_n)}$$

where $[G]$ is taken over isomorphism classes of directed acyclic graphs on n nodes is a moduli space $\mathcal{M}_{\nabla,n}^*$ for $(n-1)$ -dimensional polytropes associated to labeled DAGs. This means, every cone of $\mathcal{M}_{\nabla,n}^*$ forms an equivalence class of polytropes up to tropical equivalence.

Proof. If P is a polytrope associated to a DAG, then there is a DAG G with weights $w \in \mathbb{R}^E$ and weight matrix $C \in \mathbb{T}^{n \times n}$ such that $P = Q(C)$. By Theorem 4.10, G_w^b is the minimal DAG with unique weights w^b with this property. As such, P corresponds to a point in the open stratum $\mathcal{M}_G^*$.

Set-theoretically each open stratum $\mathcal{M}_G^*$ is equal to the open polytrope region $\mathcal{K}(G)$, which is an open subset of $\mathbb{R}^E$. Thus, the $\mathcal{M}_G^*$ already provide an atlas of charts for $\mathcal{M}_{\nabla,n}^*$. Each chart is disjoint meaning there is nothing left to check for the transition maps.

If $Q(C)$ and $Q(C')$ are two tropically equivalent polytropes such that C and C' are supported on DAGs G resp. G' , we know there is a pair of permutations $(\tau, \sigma) \in \mathbb{S}_n \times \mathbb{S}_n$ inducing an isomorphism on the poset of covectors. We may assume without loss of generality that $Q(C) = Q(C')$ up to a permutation of the coordinates. Then, $Q(C^b) = Q((C')^b)$ up to a permutation of the coordinates and in particular $C^b = (C')^b$ up to permutation of rows and columns. This means that $w = w'$ up to permutation, and the tropical equivalence of those two polytropes induces an isomorphism between G and G' preserving the weights.

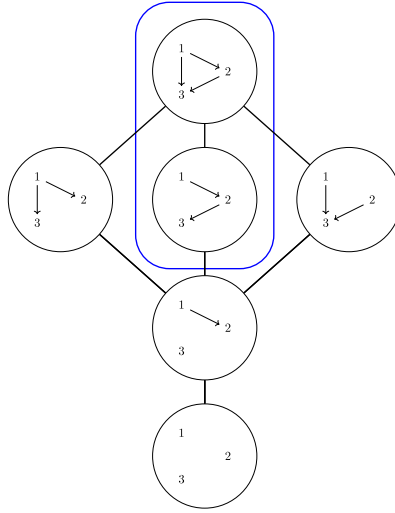


Fig. 9. The lattice of DAGs on $n=3$ nodes up to isomorphism. The single non-trivial closed stratum is circled.

On the other hand, if G and G' are isomorphic DAGs and again assume without loss of generality that $w = w'$ up to permutation, then the associated weight matrices C resp. C' only differ by a permutation of the rows and columns. But this gives a pair of permutations (τ, σ) to certify tropical equivalence. $\square$

If G is transitive, we can compare the weighted transitive reduction to the Kleene star. In this case, the defining inequalities of $\mathcal{K}(G)$ are just the strict triangle inequalities on G . These are weights $w \in \mathbb{R}^E$ for which the associated weight matrix C satisfies both $C^* = C$ and $C^\flat = C$.

The set of weights such that the associated weight matrix satisfies $C = C^*$ but not necessarily $C^\flat = C$ corresponds to weights satisfying the non-strict triangle inequalities and thus this set is the closure of $\mathcal{K}(G)$ in $\mathbb{R}^E$. Restricting a secondary fan $\Sigma(P_G)$ for a transitive DAG G to the closed polytrope region $0 \times \overline{\mathcal{K}(G)}$ results in a polyhedral fan $\overline{\mathcal{M}}_G^*$ which we call the *closed stratum* for G .

The closed strata give a covering of $\mathcal{M}_{\nabla, n}^*$ using less charts than in Theorem 4.17 in the following way.

Proposition 4.18. *Let G be a transitive DAG and $G^\flat \subseteq H \subseteq G^*$. Then, $\mathcal{M}_H^*$ can be identified with a subfan of $\overline{\mathcal{M}}_G^*$ supported on the boundary of $\mathcal{K}(G)$.*

Proof. For $w \in \mathcal{M}_H^*$, define the inclusion of $\varphi: \mathcal{M}_H^* \hookrightarrow \overline{\mathcal{M}}_G^*$ by $w \mapsto w^*$. Since $G^\flat \subseteq H$, w^* will be a point in $\mathbb{R}^{E^*}$ and by construction w^* satisfies the non-strict triangle inequalities, so $w^* \in \overline{\mathcal{M}}_G^*$.

Since $H \subsetneq G$, w^* will satisfy the triangle equality $w^*(j \rightarrow i) = w^*(j \rightarrow k) + w^*(k \rightarrow i)$, for some edges $j \rightarrow k, k \rightarrow i \in H$ and $j \rightarrow i \in G$, meaning that w^* lies on the boundary of $\mathcal{K}(G)$.

We need to show that cones of $\mathcal{M}_H^*$ get mapped to cones in $\overline{\mathcal{M}}_G^*$. Note that the mapping $w \mapsto w^*$ defines an extension of the regular subdivision $\mathcal{S}(P_G, (0, w))$ to a regular subdivision $\mathcal{S}(P_{G^*}, (0, w^*))$.

Any new vertex $e_i - e_j$ from P_{G^*} is inserted at a height $w^*(j \rightarrow i)$ which is equal to the sum of heights of other vertices already present in P_G . This means that any new vertex $e_i - e_j$ only extends maximal cells of $\mathcal{S}(P_G, (0, w))$ without subdividing those any further. This means there is a one-to-one mapping of maximal cells of $\mathcal{S}(P_G, (0, w))$ and $\mathcal{S}(P_{G^*}, (0, w^*))$. But this means that two points in the image of φ give equivalent regular subdivisions on P_{G^*} if and only if points in their preimage give equivalent regular subdivisions of P_G . $\square$

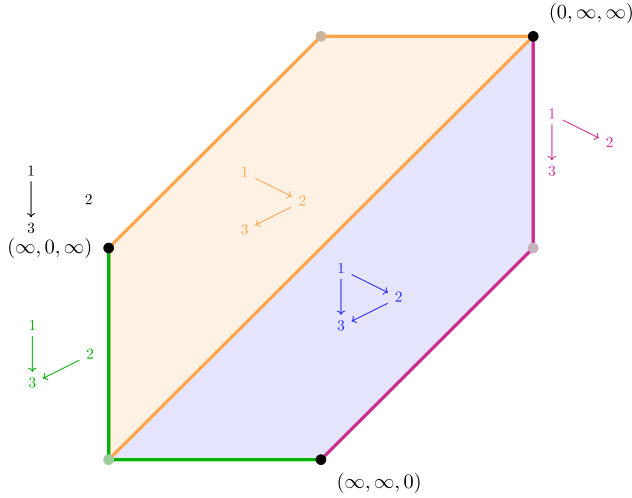


Fig. 10. The tropical projective plane $\mathbb{TP}^2$ as the space of weights w subdivided by combinatorial types. For each colored region, the corresponding weighted transitive reduction G_w^b is shown. For the vertices of $\mathbb{TP}^2$, only one tropical vertex (in black) is marked since the combinatorial type is the same for all tropical vertices. The interior of $\mathbb{TP}^2$ corresponds to weights on the complete DAG κ_3 , while the blue region is the open polytrope region $\mathcal{K}(\kappa_3)$ and the blue region together with the orange line segment is the closure of $\mathcal{K}(\kappa_3)$.

Corollary 4.19. *The union of closed strata $\overline{\mathcal{M}_G^*}$ where G ranges over isomorphism classes of transitive DAGs is a covering of the moduli space $\mathcal{M}_{\nabla, n}^*$.*

Proof. Note that any DAG is (non-uniquely) a subgraph of a transitive DAG. Proposition 4.18 then gives the result. $\square$

Example 4.20. The graphs indexing the open strata are shown in Fig. 9. There is only one non-trivial closed stratum which is circled. This is precisely the stratum that we discussed in Example 4.4.

For comparison, Fig. 10 shows the sketch of a subdivision of points $w \in \mathbb{TP}^2$ according to the combinatorial type of polytrope $P = Q(C)$. In the case $n = 3$, combinatorial types are in one-to-one correspondence with types of weighted transitive reductions G_w^b . The blue circled interval of graphs from Fig. 9 corresponds to the interior of $\mathbb{TP}^2$ in Fig. 10.

The secondary fan of P_{κ_3} for the directed triangle has two maximal cones given by the two half-spaces of the hyperplane

$$c = a + b.$$

The maximal cones correspond to the inequalities from Example 4.4. The restriction of $\Sigma(P_{\kappa_3})$ to the open polytrope region is precisely the halfspace given by $c < a + b$, which corresponds to the blue region in Fig. 10.

Furthermore, the restriction to the closed polytrope region includes the cone where $c = a + b$, which corresponds to $\mathcal{M}_H^*$ for the path graph H . This is the orange line in the interior of $\mathbb{TP}^2$ of Fig. 10.

The orange sector extends from the interior of $\mathbb{TP}^2$ to a boundary component with $c = \infty$. This is due to Theorem 4.9 which states that for every point in the orange region, there is a ray extending in the c -coordinate direction along which $Q(C)$ is constant. Then, the compactification of this ray adds a point on the orange boundary component.

Any other graph G on three nodes does not correspond to the closed stratum $\overline{\mathcal{M}_{\kappa_3}^*}$, so there is no copy of $\mathcal{M}_G^*$ in $\overline{\mathcal{M}_{\kappa_3}^*}$.

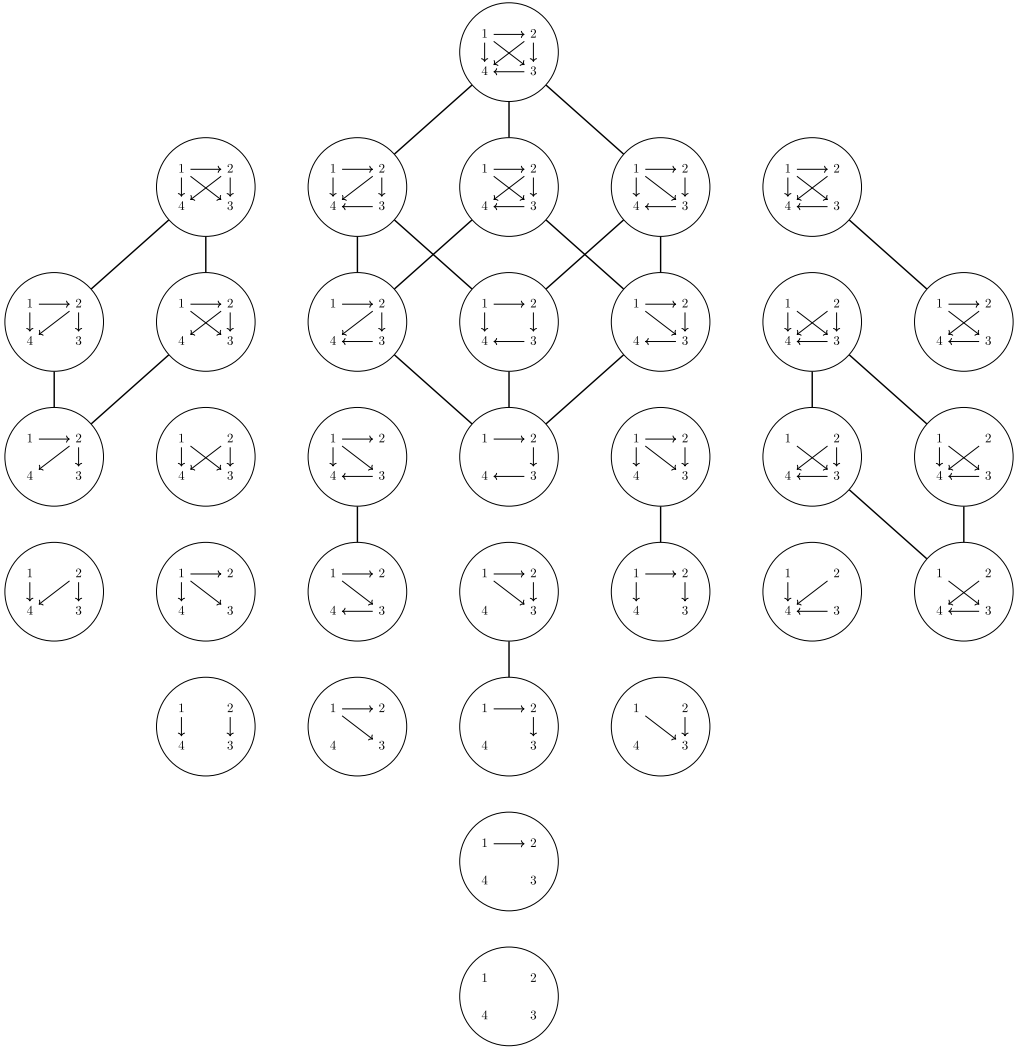


Fig. 11. Representation of the lattice of DAGs on $n = 4$ nodes up to isomorphism. Only the covering relations between DAGs associated with each closed stratum are shown. Equivalently, each connected component represents a closed local moduli space $\overline{\mathcal{M}}_G^*$ according to Corollary 4.19.

For $n = 4$, only a partial representation is shown in Fig. 11 for the sake of readability. Only the covering relations for the graphs associated to each closed stratum are shown, making each stratum visible as a connected component. Also, each connected component shows the structure of each closed stratum $\overline{\mathcal{M}}_{G^*}^*$ in terms of the open strata $\mathcal{M}_H^*$.

5. Enumerating combinatorial types

Enumerating the tropical combinatorial types of polytopes coming from DAGs on n nodes amounts to checking secondary fans ranging over a suitable set of DAGs. Due to the two different stratifications, we have the choice between two different procedures to enumerate.

In light of Corollary 4.19, we can check the subdivisions of Γ_G for each transitive DAG G on n nodes. This gives the possible tropical combinatorial types for tropical polytopes. Then, using Theo-

Table 1

Number of generic combinatorial types for polytropes associated to max-linear Bayesian networks in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ and the number of checked DAGs

n	1	2	3	4	5	6	7
#triangulations	1	2	6	32	512	65240	276733462
#DAGs	1	2	6	31	302	5984	243668
#transitive DAGs	1	2	5	16	63	318	2045

rem 3.5, we need to take the face figure of the subdivision contributing to the polytrope and eliminate duplicates. It can be seen in Table 1 that this reduces the number of DAGs to be checked.

One disadvantage of this approach is that we need to enumerate all regular subdivisions and can only filter duplicates afterwards. Table 1 gives a lower bound for the amount of subdivisions that need to be stored. It has been noted by Joswig and Schröter (2019) that the root polytope Γ_G is at most $2n$ -dimensional with $\mathcal{O}(n^2)$ vertices and lattice volume bounded by $\binom{2n-2}{n-1}$. Passing to the fundamental polytope P_G reduces the dimensionality and the number of vertices.

Using Theorem 4.17 we can instead enumerate the regular central subdivisions of P_G for every graph G without filtering, which alleviates the storage issue. When further restricting to regular central triangulations, we can take advantage of the software `mptopcom` by Jordan et al. (2018) that has been optimized for parallel enumeration of central triangulations.

Passing from subdivisions to triangulations restricts our focus to *generic* combinatorial types in the sense that the corresponding polytrope has the maximal number of faces for each dimension among all polytropes associated to a fixed graph G . In other words, the generic combinatorial types correspond to the maximal cones of each $\mathcal{M}_G^*$. Tran (2017) also enumerated the maximal cones of a specific Gröbner fan to count the number combinatorial types that are generic for the complete graph K_n for $n \leq 5$, which she termed as maximal polytropes.

To enumerate central regular triangulations, we generated the input for `mptopcom` using Julia 1.10.9 and OSCAR 1.4.1 (Decker et. al, 2025) and used `mptopcom` 1.4 for the enumeration. This way, we can take advantage of the techniques implemented in `mptopcom` to avoid checking unnecessary triangulations and to enumerate orbits of triangulations.

The result of the enumerations are shown in Table 1, with the number of DAGs that had to be enumerated listed for each method. The code for performing the enumeration can be found under <https://zenodo.org/records/14166530>.

Comparing with the results from Tran (2017), there are 1013 combinatorial types of bounded polytropes for $n = 4$ and 27248 combinatorial types of generic bounded polytropes for $n = 5$. In both cases we count significantly fewer generic combinatorial types of polytropes associated to DAGs.

6. Outlook

In this work we have given different notions of equivalence for polytropes coming from max-linear Bayesian networks. For a polytrope $Q(C)$ its minimal non-redundant facet description corresponds to a weight matrix $C^\flat$ which is associated to the weighted transitive reduction. If G is the underlying DAG of C , the weighted transitive reduction has a natural interpretation in terms of those edges which are essential for realizing the shortest paths in G . This provides an equivalence relation by labeling a polytrope with a DAG which is more specific than the underlying graph of the Kleene star. This graph is always a transitive DAG and does not see the redundancy of hyperplanes for defining facets.

We focused on DAGs because they serve as a description for max-linear Bayesian networks, and our construction of the weighted transitive reduction makes use of the absence of cycles. With respect to previous work on polytropes which focused on the complete graph, a characterization for all other graphs which contain cycles is missing.

Question 1. How does one characterize a minimal non-redundant facet description for arbitrary digraphs, thus generalizing the notion of the weighted transitive reduction to graphs with cycles?

Seeing a polytrope $Q(C)$ as a subcomplex of the covector decomposition $\mathcal{H}(C)$ allowed us to characterize tropical equivalence in terms of regular subdivisions, meaning the secondary fan is the object of choice for constructing a parameter space. While this idea has been known in tropical geometry for some time, the construction with the secondary fan requires a fixed support graph $B(C)$. Hence, the main challenge when classifying all polytropes of max-linear Bayesian networks consists in fitting together the information from all necessary secondary fans for the possible support graphs.

Related work by Olarte et al. studied in depth *local Dressians* (Olarte et al., 2019) studied in depth *local Dressians* for a given matroid. In particular, they investigated the structure given by the secondary fan of the matroid polytope and by the tropical variety cut out by three-term Plücker relations, showing these structures agree. From this, the next step for max-linear Bayesian networks is to investigate the structure coming from the Plücker relations.

Another point of view is that since matrices associated to networks are lower-triangular, we can consider $Q(C)$ as a flag of tropical polytopes. Tropical flag varieties have been studied by Brandt et al. (2021) who found incidence-Plücker relations in addition to the usual Plücker relations.

Question 2. *In what way do our local moduli spaces $\mathcal{M}_G^*$, the local Dressians for the cycle matroid of G and the tropical varieties of complete flags relate to each other?*

Finally, in terms of statistical applications, knowing that the support of a max-linear Bayesian network is a polytrope gives us additional information useful for modeling extreme data. As stated in Remark 4.11, only the weights associated to edges in the weighted transitive reduction can theoretically be recovered from samples. We also would like to know what further statistical consequences can be obtained from understanding the combinatorial structure, ideally shedding light on questions considered by Améndola et al. (2021, 2022).

Question 3. *To what extent does the combinatorics of the polytrope associated to a max-linear Bayesian network reveal information about the conditional independence statements that hold among the variables (and viceversa)?*

CRedit authorship contribution statement

Carlos Améndola: Writing – review & editing. **Kamillo Ferry:** Writing – original draft, Software.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The accompanying code is made available on Zenodo under <https://zenodo.org/records/14166530> and GitHub.

References

- Aho, Alfred V., Garey, Michael R., Ullman, Jeffrey D., 1972. The transitive reduction of a directed graph. *SIAM J. Comput.* (ISSN 0097-5397) 1 (2), 131–137. <https://doi.org/10.1137/0201008>. Publisher: Society for Industrial and Applied Mathematics.
- Améndola, Carlos, Hollering, Benjamin, Sullivant, Seth, Tran, Ngoc Mai, 2021. Markov equivalence of max-linear Bayesian networks. In: *Uncertainty in Artificial Intelligence*. PMLR, pp. 1746–1755.
- Améndola, Carlos, Klüppelberg, Claudia, Lauritzen, Steffen, Tran, Ngoc Mai, 2022. Conditional independence in max-linear Bayesian networks. *Ann. Appl. Probab.* 32 (1), 1–45. <https://doi.org/10.1214/21-AAP1670>.
- Asadi, Peiman, Davison, Anthony C., Engelke, Sebastian, 2015. Extremes on river networks. *Ann. Appl. Stat.*, 2023–2050. <https://doi.org/10.1214/15-AOAS863>.
- Brandt, Madeline, Eur, Christopher, Zhang, Leon, 2021. Tropical flag varieties. *Adv. Math.* (ISSN 0001-8708) 384, 107695. <https://doi.org/10.1016/j.aim.2021.107695>.

- Butkovič, Peter, 2010. *Max-Linear Systems: Theory and Algorithms*. Springer Monographs in Mathematics. London. Springer. ISBN: 978-1-84996-298-8 978-1-84996-299-5.
- De Loera, Jesús A., Rambau, Jörg, Santos, Francisco, 2010. *Triangulations*, vol. 25. *Algorithms and Computation in Mathematics*. Springer Berlin Heidelberg, Berlin, Heidelberg.
- Decker, Wolfram, Eder, Christian, Fieker, Claus, Horn, Max, Joswig, Michael (Eds.), 2025. *The Computer Algebra System OSCAR. Algorithms and Examples*. *Algorithms and Computation in Mathematics*, vol. 32. Springer, Cham.
- Develin, Mike, Sturmfels, Bernd, 2004. Tropical convexity. *Doc. Math.* 9, 1–27. <https://doi.org/10.4171/dm/154>. ISSN: 1431-0635, 1431-0643.
- Dochtermann, Anton, Joswig, Michael, Sanyal, Raman, 2012. Tropical types and associated cellular resolutions. *J. Algebra* (ISSN 0021-8693) 356 (1), 304–324. <https://doi.org/10.1016/j.jalgebra.2011.12.028>.
- Fink, Alex, Rincón, Felipe, 2015. Stiefel tropical linear spaces. *J. Comb. Theory, Ser. A* (ISSN 0097-3165) 135, 291–331. <https://doi.org/10.1016/j.jcta.2015.06.001>.
- Gissibl, Nadine, 2018. *Graphical modeling of extremes*. PhD thesis, Technische Universität München, p. 131.
- Gissibl, Nadine, Klüppelberg, Claudia, 2018. Max-linear models on directed acyclic graphs. *Bernoulli* (ISSN 1350-7265) 24 (4), 2693–2720. <https://doi.org/10.3150/17-BEJ941>.
- Jell, Philipp, Shaw, Kristin, Smacka, Jascha, 2019. Superforms, tropical cohomology, and Poincaré duality. *Adv. Geom.* 19 (1), 101–130. <https://doi.org/10.1515/advgeom-2018-0006>. ISSN: 1615-7168, 1615-715X.
- Jordan, Charles, Joswig, Michael, Kastner, Lars, 2018. Parallel enumeration of triangulations. *Electron. J. Comb.* (ISSN 1077-8926) 25 (3), 3.6. <https://doi.org/10.37236/7318>.
- Joswig, Michael, 2021. *Essentials of Tropical Combinatorics*. *Graduate Studies in Mathematics*, vol. 219. American Mathematical Society, Providence, RI.
- Joswig, Michael, Kulas, Katja, 2010. Tropical and ordinary convexity combined. *Adv. Geom.* (ISSN 1615-7168) 10 (2), 333–352. <https://doi.org/10.1515/advgeom.2010>. Publisher: de Gruyter.
- Joswig, Michael, Loho, Georg, 2016. Weighted digraphs and tropical cones. *Linear Algebra Appl.* (ISSN 0024-3795) 501, 304–343. <https://doi.org/10.1016/j.laa.2016.02.027>.
- Joswig, Michael, Schröter, Benjamin, 2022. Parametric shortest-path algorithms via tropical geometry. *Math. Oper. Res.* 47 (3), 2065–2081. <https://doi.org/10.1287/moor.2021.1199>.
- Joswig, Michael, Schröter, Benjamin, 2019. *The tropical geometry of shortest paths*. [arXiv:1904.01082v2](https://arxiv.org/abs/1904.01082v2).
- Kastner, Lars, Shaw, Kris, Winz, Anna-Lena, 2023. Tropical compactification via Ganter's algorithm. *Discrete Comput. Geom.* (ISSN 1432-0444). <https://doi.org/10.1007/s00454-023-00581-2>.
- Lam, Thomas, Postnikov, Alexander, 2007. Alcoved polytopes, I. *Discrete Comput. Geom.* (ISSN 1432-0444) 38 (3), 453–478. <https://doi.org/10.1007/s00454-006-1294-3>.
- Lauritzen, Steffen L., 1996. *Graphical Models*, vol. 17. Clarendon Press.
- MacLagan, Diane, Sturmfels, Bernd, 2015. *Introduction to Tropical Geometry*. *Graduate Studies in Mathematics*, vol. 161. American Mathematical Society, Providence, Rhode Island.
- Olarte, Jorge Alberto, Panizzut, Marta, Schröter, Benjamin, 2019. On local Dressians of matroids. In: *Algebraic and Geometric Combinatorics on Lattice Polytopes. Summer Workshop on Lattice Polytopes*. World Scientific, Osaka, Japan, pp. 309–329.
- Osserman, Brian, Rabinoff, Joseph, 2013. Lifting nonproper tropical intersections. In: Amini, Omid, Baker, Matthew, Faber, Xander (Eds.), *Contemporary Mathematics*, vol. 605. American Mathematical Society, Providence, Rhode Island, pp. 15–44. ISBN: 978-1-4704-1021-6 978-1-4704-1410-8.
- de la Puente, María Jesús, 2013. On tropical Kleene star matrices and alcoved polytopes. *Kybernetika* (ISSN 0023-5954) 49 (6), 897–910. <https://dml.cz/handle/10338.dmlcz/143578>.
- Santos, Francisco, 2005. The Cayley trick and triangulations of products of simplices. *Contemp. Math.* 374, 151–178.
- Tran, Ngoc Mai, 2017. Enumerating polytropes. *J. Comb. Theory, Ser. A* (ISSN 0097-3165) 151, 1–22. <https://doi.org/10.1016/j.jcta.2017.03.011>.
- Tran, Ngoc Mai, 2022. *The tropical geometry of causal inference for extremes*. [arXiv:2207.10227](https://arxiv.org/abs/2207.10227).
- Tran, Ngoc Mai, Buck, Johannes, Klüppelberg, Claudia, 2024. Estimating a directed tree for extremes. *J. R. Stat. Soc., Ser. B Stat. Methodol.* (ISSN 1369-7412) 86 (3), 771–792. <https://doi.org/10.1093/jrssi/bkqad165>.