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On Boundaries of ε-neighbourhoods of Planar Sets

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September 2021

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Submitted in part fulfilment of the requirements for the degree of
Doctor of Philosophy of Imperial College London and
the Diploma of Imperial College

Declaration

I declare that this thesis is entirely my own work and that where any material could be construed as the work of others, it is fully cited and referenced, and/or with appropriate acknowledgement given.

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September 2021

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Publications

The material in Chapters 2–5 of this thesis constitutes the contents of the article [LRTa], which has been submitted for publication to the Journal of the London Mathematical Society. Chapters 6–7 of the thesis have been published in the sequel [LRTb] to this article. Both Part I and Part II are available at the arXiv repository. The PhD project leading to these publications was funded by the European Union's Horizon 2020 research and innovation Programme under the Marie Skłodowska-Curie grant agreement no. 643073.

Acknowledgements

I owe my deepest gratitude to my supervisors Prof. Jeroen Lamb and Dr. Martin Rasmussen. At the most critical moment, after a tumultuous and somewhat discouraging start of my doctoral studies, they expressed their firm belief that I nevertheless had what it takes to complete my doctoral training under their guidance. Their strong and unwavering conviction that I can, and will, succeed made all the difference for me, and I do not know where I would be without it. For this I thank you from the bottom of my heart. I am also grateful for our countless discussions on mathematics, politics, psychology and a myriad of other topics that, one way or another, touched upon academic life, and life in general. I also thank you for initiating me in the veritable art form of scientific writing.

Along with my supervisors, I want to thank Professors Grigorios Pavliotis, Dmitry Turaev, Sebastian van Strien, Ale Jan Homburg, Fritz Colonius, Peter Ditlevsen, Peter Ashwin, Gabriel Sandu and Kari Astala for many helpful and enlightening discussions and suggestions regarding my research. Thank you Panu Erästö for sharing with me some of your life wisdom, which turned out to be extremely valuable.

The encouraging community at Imperial gave me their time and support when I most needed it; thank you Anderson Santos for your joyful, magical touch in helping with the everyday struggles, big and small (and for the passcode to the poster printing room). I want to thank my colleagues, Sajjad Bakrani, Michael Hartl and Christian Pangerl, with whom I shared our beloved office Huxley 635. The fantastic group of the CRITICS students will occupy a special place in my memories. Among them I now count some of my best friends. Thank you Iacopo, Courtney, Karl, Usman, Damian, Johannes, Moussa.

I am grateful to my friends Adam, William, Nik, Maxi, Toby, Wei Hao, Joel, Alexis, Sara, Paula and many others at Imperial. During the last years before COVID I also enjoyed the many conversations, both academic and otherwise, with my new office mate Gabriel Fuhrmann, who is blessed with the most unusually subtle and politically correct sense of humour. The numerous years since the start of my project were made bearable also by my close friends in London outside of academia; thank you Robert, Midhun, Pepe. Without you I would have gone insane.

A long stay abroad always has its toll on relationships back home, and I am grateful for the many friendships that have endured. Thank you Mikko M., Mikko I., Lassi, Tuomas P., Tuomas O., Vadim, Juha-Matti, Matias, Jarno, Deni, Sam, Sampo, Tuomas S., Jukka, Aapo, Juho, Kalle.

Last but not least I want to thank my family. Little Filoteea flew with us to London on her 1-year birthday in 2016, and Alexander got to start his early school career the same autumn at the age of four (five years minus nine days, luckily). Thank you Ioana, for managing through the tough times. Finally, kiitos äiti.

Abstract

We study the geometric and topological structure of boundaries of ε-neighbourhoods E_ε of compact planar sets $E \subset \mathbb{R}^2$. The main contribution of the thesis is the development of a novel geometric framework for analysing the local properties of the boundaries of ε-neighbourhoods. We utilise the established concept of a tangent cone from set-valued analysis in our definition of *outward directions*, which allow us to investigate the tangential properties of the boundary by linking them to the characteristic geometry of ε-neighbourhoods.

The technical foundation of the thesis is what we call the *local contribution property*, which intuitively asserts that the local geometry near a boundary point $x \in \partial E_\varepsilon$ is completely determined by points $y \in \partial E$ that lie at most in two different directions relative to x. This result allows us to define *local boundary representations*, which provide a local description of the boundary. These are finite collections of graphs of Lipschitz-continuous functions, defined in local coordinate systems that are adapted to the orientations of so-called extremal outward directions. This tool turns out to provide a powerful technical framework for analysing also global properties of the boundary.

Using these technical tools, we prove that the boundary points $x \in \partial E_\varepsilon$ can be classified into smooth points and eight distinct categories of singularities. This classification permits a detailed analysis of the cardinalities and topological properties of the different types of singularities. We provide examples that show that so-called wedge singularities can lie densely on boundary segments that have positive one-dimensional Hausdorff measure. One of our main results is that the set of so-called chain singularities is closed and totally disconnected, which implies that it is nowhere dense on the boundary.

Leveraging this analysis, we show that for a compact set $E \subset \mathbb{R}^2$ and $\varepsilon > 0$, the boundary ∂E_ε is the disjoint union of a possibly uncountable set $\mathcal{I}$ of inaccessible singularities and an at most countably infinite collection J of Jordan curves.

Finally, we prove the existence of curvature almost everywhere on the Jordan curve subsets of the boundary by showing that the derivatives of the functions constituting the local boundary representations are of bounded variation, and thus have a well-defined derivative at almost every point.

Our work extends the existing literature by providing a complete classification of possible boundary geometries that applies generally for any compact set $E \subset \mathbb{R}^2$ and $\varepsilon > 0$. As opposed to the existing work on the topic, we approach the properties of the boundary from a geometric viewpoint, motivated by the need to understand the geometric and dynamical properties of ε-neighbourhoods appearing in set-valued dynamical systems as models for random dynamical systems with bounded noise.

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Chapter 1

Introduction

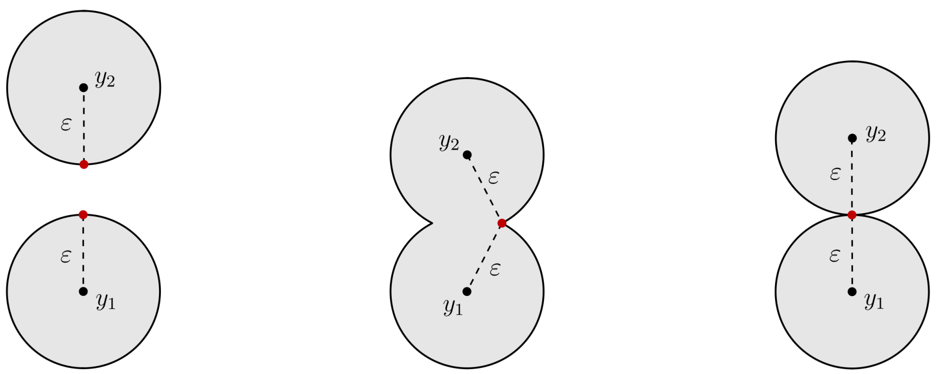
In this introductory chapter we present the background of the thesis with an overview of our main results. We begin with an elementary example of ε-neighbourhoods, followed by a review of the state of the art in this research area. We then provide an overview of the main results of the thesis and describe the methodology used for obtaining them. Finally, we consider the outlook regarding potential future research directions.

1.1 Elementary Example

For each $\varepsilon > 0$, the ε-*neighbourhood* of a subset $E \subset \mathbb{R}^d$ of the d-dimensional Euclidean space is the set

$$E_\varepsilon := \Big\{ x \in \mathbb{R}^d \,:\, d(x, E) \leq \varepsilon \Big\}, \tag{1.1.1}$$

where $d(\cdot,\cdot)$ denotes the usual Euclidean distance in $\mathbb{R}^d$. These sets, which are sometimes also called tubular neighbourhoods, ε-collars or parallel sets, are a basic object of study in



(a) $d(y_1, y_2) > 2\varepsilon$. **(b)** $d(y_1, y_2) < 2\varepsilon$. **(c)** $d(y_1, y_2) = 2\varepsilon$.

Figure 1.1. Structure of the ε-neighbourhood E_ε for $E := \{y_1, y_2\} \subset \mathbb{R}^2$. The geometric and topological properties of the boundary ∂E_ε depend on the distance of the points y_1, y_2 relative to the radius ε.

Euclidean geometry. In addition, they feature prominently in a number of applications, ranging from control theory and set-valued dynamical systems to fractal geometry and data-driven geometric inference. The motivation for the studies presented in this thesis stems from the field of random dynamical systems with bounded noise, where the boundaries of attractors appear naturally as dynamically generated ε-neighbourhoods. In this context, the geometric properties of the boundary ∂E_ε play a central role as indicators of dynamical properties of the corresponding attractor. Understanding and classifying the geometric properties of the boundary ∂E_ε is the main focus of this thesis.

The simplest ε-neighbourhood in Euclidean space consists of a closed ε-radius ball $\overline{B_\varepsilon(x)}$, centred around a single point $x \in \mathbb{R}^d$. This is a highly regular object. The situation immediately becomes more interesting if one instead considers the ε-neighbourhood of the union $E := \{y_1, y_2\} \subset \mathbb{R}^d$ of two distinct points $y_1 \neq y_2$. In this case, the structure of the boundary ∂E_ε of the corresponding ε-neighbourhood E_ε depends on whether the distance $d(y_1, y_2)$ is less than, equals, or exceeds 2ε. See Figure 1.1 for illustration of these possibilities in the planar case $d = 2$. For more general sets $E \subset \mathbb{R}^d$, understanding how the geometric, topological and analytical properties of the boundary ∂E_ε depend on the underlying set E and the radius ε becomes a highly nuanced problem.

1.2 State of the Art

The literature on ε-neighbourhoods spans a wide range of applications, and their fundamental properties have been investigated at least since the 1940s. In this section we provide a chronological review of the existing theoretical results and highlight the use of the concept in some prominent applications.

To our best knowledge, the earliest reference of ε-neighbourhoods in modern academic literature is Paul Erdős's brief note [Erd45]. In his paper, Erdős showed that for any $E \subset \mathbb{R}^d$, the d-dimensional Hausdorff measure of the boundary $\mathcal{H}^d(\partial E_\varepsilon)$ vanishes and that the $(d-1)$-dimensional measure of the boundary is finite when E is compact, i.e. the set ∂E_ε satisfies $\mathcal{H}^{d-1}(\partial E_\varepsilon) < \infty$. This result was also later documented in [OP85].

The analysis of the topological properties of the boundaries of ε-neighbourhoods was initiated in the 1970s by Morton Brown [Bro72], who studied the structure of the connected components of the *ε-boundary*

$$\partial_\varepsilon(E) := \Big\{x \in \mathbb{R}^2 \, : \, d(x, E) = \varepsilon\Big\}. \tag{1.2.1}$$

Brown's main result is the following.

Theorem 1.2.1 (**[Bro72, Theorems 2 and 3]**). *Let $E \subset \mathbb{R}^2$ be a compact planar set. Then the connected components of the ε-boundary $\partial_\varepsilon(E)$ are locally connected and for all* *but countably many $\varepsilon > 0$, each component of $\partial_\varepsilon(E)$ is either a point, a simple curve, or a simple closed curve (a Jordan curve).*

Brown's early results highlight two features that are significant for the present thesis. Firstly, by definition, the boundary of the ε-neighbourhood (1.1.1) satisfies $\partial E_\varepsilon \subset \partial_\varepsilon(E)$, but the inclusion in the other direction does not generally hold true, see Figure 1.2. In this thesis we focus on the properties of the boundary ∂E_ε. Secondly, Theorem 1.2.1 can be viewed as a kind of robustness property of the topological structure of the set $\partial_\varepsilon(E)$: it identifies a feature that applies outside a small (in this case countable) exceptional set of radii $\varepsilon > 0$. These types of result are natural and very useful for practical applications. For instance, Brown's result can readily be used to argue that if one uses an ε-neighbourhood with decreasing radii $\varepsilon > 0$ to approximate an underlying set $E \subset \mathbb{R}^2$, this approximation satisfies the mentioned topological properties for almost all ε. Up to very recently, several authors have provided increasingly refined results in a similar spirit. Our findings complement these results in the sense that they explore the geometric and topological properties of ∂E_ε for arbitrary $\varepsilon > 0$.

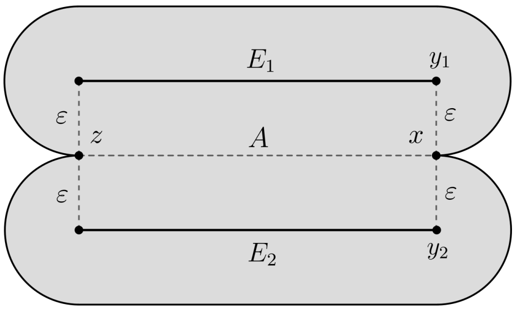


Figure 1.2. Schematic illustration of the difference between the ε-boundary $\partial_\varepsilon(E)$ and the boundary of the closed ε-neighbourhood ∂E_ε. For $E = E_1 \cup E_2$, the set $A := \partial_\varepsilon(E) \setminus \partial E_\varepsilon = \{x \in \mathbb{R}^2 : d(x, E) = \varepsilon\} \setminus \partial E_\varepsilon$ is non-empty. See also the discussion following Lemma 2.2.4 and [RW10, Example 2.1].

The manifold structure of the ε-boundary (1.2.1) was further investigated by Ferry in the 1970s as part of his doctoral thesis. This work was published separately in [Fer76]. By analysing for a fixed $E \in \mathbb{R}^d$ the properties of the distance function $d_E : \mathbb{R}^d \to [0, \infty)$,

$$d_E(x) := d(x, E), \tag{1.2.2}$$

Ferry established the following theorem, which in retrospect provides a kind of prototypical basis for subsequent refinements by other authors.

Theorem 1.2.2 (**[Fer76, Corollary 2.3]**). *Let $d \in \{2, 3\}$ and $E \subset \mathbb{R}^d$. Then $\partial_\varepsilon(E)$ is a $(d-1)$-manifold for almost all $\varepsilon > 0$.*

Gariepy and Pepe [GP72] had obtained this result a few years earlier in the case $d = 2$. While covering a further dimension ($d = 3$), Theorem 1.2.2 also provides a more specific statement compared to Theorem 1.2.1. However, this comes at the expense of replacing 'for all but countably many $\varepsilon > 0$' in Theorem 1.2.1 with 'for almost all $\varepsilon > 0$' in Theorem 1.2.2. In addition to his main result, Ferry also provides counterexamples showing that his Theorem 1.2.2 cannot be improved to hold true apart from a countable set of radii [Fer76, Example 1] or in dimensions $d > 3$ [Fer76, Example 3], and that Brown's Theorem 1.2.1 cannot be generalised to $d = 3$ [Fer76, Example 2].

Although Ferry's result provides a neat general picture of the manifold structure of the ε-boundary $\partial_\varepsilon(E)$ for arbitrary $E \subset \mathbb{R}^d$ with $d \in \{2, 3\}$, it leaves many questions unanswered. One may wonder for instance whether more could be said about the geometric properties of the boundary, what exactly happens in higher dimensions $d \geq 4$, and if the size of the exceptional set of radii (for which the statement in Theorem 1.2.2 fails) could be quantified further. Roughly a decade later, Joseph H. G. Fu provided answers to some of these questions in his landmark article [Fu85], which constitutes a part of his dissertation. As part of his novel approach, Fu leveraged the semiconcavity of the distance function (1.2.2) to establish a connection between the geometry of the boundary ∂E_ε and the set of *critical values* of the distance function (1.2.2). Fu defines this set through the notion of a *generalized (Clarke) gradient* ∂d_E, a set-valued generalisation of the classical gradient ∇d_E.

Definition 1.2.3 (**generalised (Clarke) gradient**, [**Cla75, Definition 1.1**])**.** *The* generalised gradient *of f at x, denoted $\partial f(x)$, is the convex hull of the set of limits of the form* $\lim_{i\to\infty} \nabla f(x+h_i)$*, where $h_i \to 0$ as $i \to \infty$ (here ∇f denotes the classical gradient).*

The set of critical values of the distance function is $\mathrm{cv}(d_E) := d_E(\mathrm{crit}(d_E)) \subset \mathbb{R}_+$, where

$$\mathrm{crit}(d_E) := \Big\{x \in \mathbb{R}^d \,:\, 0 \in \partial d_E(x)\Big\} \tag{1.2.3}$$

is the set of *critical points* of d_E. All other points $x \in \mathrm{reg}(d_E) := \mathbb{R}^d \setminus \mathrm{crit}(d_E)$ are called *regular points* of d_E. The main result of [Fu85] is the following.

Theorem 1.2.4 ([**Fu85, Theorem 4.1**])**.** *Let $E \subset \mathbb{R}^d$ be compact. Then for each $\varepsilon \notin \mathrm{cv}(d_E)$, the boundary ∂E_ε is a Lipschitz manifold and the complement $\overline{\mathbb{R}^d \setminus E_\varepsilon}$ has positive reach. The set of critical values $\mathrm{cv}(d_E)$ is compact and is contained in the interval $\Big[0, \sqrt{d/2(d+1)}\mathrm{diam}E\Big]$. Moreover $\mathcal{H}^{(d-1)/2}(\mathrm{cv}(d_E)) = 0$, and if $d = 2$ then for every $\delta > 0$ the entropy dimension of $\mathrm{cv}(d_E) \setminus [0, \delta)$ is $1/2$.*

Theorem 1.2.4 improves on Theorem 1.2.2 in two ways. Firstly, in dimension $d = 2$ it asserts that not only is the set C of pathological radii of zero Lebesgue measure, but that furthermore its $1/2$-dimensional Hausdorff measure on $\mathbb{R}$ vanishes as well, i.e. $\mathcal{H}^{1/2}(C) = 0$. Secondly, Fu provides new insight into the geometric structure of the boundary ∂E_ε by noting that for $\varepsilon \notin C$, the complement $\overline{\mathbb{R}^d \setminus E_\varepsilon}$ is a set of *positive reach*. This concept was developed by Federer in his famous paper [Fed59] as a natural generalisation of both convex sets and smooth manifolds. The reach of a set $A \subset \mathbb{R}^d$ is defined in terms of those points $z \in \mathbb{R}^d$ whose metric projection

$$\Pi_A(z) := \{x \in A \,:\, d(z, A) = |z - x|\}$$

onto A is a singleton. We denote this set by $\mathrm{Unp}(A)$.

Definition 1.2.5 (**[Fed59, Definition 4.1]**). *For $A \subset \mathbb{R}^d$ the (local) reach at $a \in A$ is*

$$\mathrm{reach}(A, a) := \sup\{r \geq 0 \,:\, B(a, r) \subset \mathrm{Unp}(A)\}.$$

The reach *of the set $A \subset \mathbb{R}^d$ is*

$$\mathrm{reach}\, A := \inf_{a \in A} \mathrm{reach}(A, a). \tag{1.2.4}$$

The significance of sets with positive reach stems from the fact that it is possible to formulate and prove in this context generalised analogues for the famous Steiner and kinematic formulas from convex analysis via the notion of *curvature measures*. Theorem 1.2.4 thus sheds light on the geometric structure of the boundary ∂E_ε through this concept as well.

Recent Advances

In 2009, Blokh, Misiurewicz and Oversteegen [BMO09] continued Brown's initial topological investigations of ε-boundaries by generalising his results to topological 2-manifolds.

Theorem 1.2.6 (**[BMO09, Theorem 1.1 and Corollary 1.2]**). *Let a 2-manifold M with metric d be a proper geodesic metric space. Let $E \subset M$ be compact. Then for all but countably many $\varepsilon > 0$ each component of $\partial_\varepsilon E$ is either a point, a simple closed curve disjoint from the boundary ∂M of M or an arc A such that $A \cap \partial M$ consists of both endpoints of A, and the union of arcs and simple closed curves is dense in $\partial_\varepsilon E$. If M has no boundary (e.g., if $M = \mathbb{R}^2$), then there are no arcs, and the union of simple closed curves is dense in $\partial_\varepsilon E$.*

A notable feature of Theorem 1.2.6 is how closely it resembles the formulation of Brown's original result, including the validity of the result apart from a countable set of radii ε.

Very recently, J. Rataj and L. Zajíček [RZ20] further improved on the classical results of Ferry [Fer76] and Fu [Fu85]. Using the properties of critical points of DC functions (i.e., differences of convex functions) and building on some of Ferry's original work in [Fer76], they developed a new elementary approach to quantifying the size of the set $\mathrm{cv}(d_E)$ of critical values of the distance function d_E for a compact set $E \subset \mathbb{R}^2$. To this end, they introduced the concept of BT_α-sets for subsets of the real line as follows.

Definition 1.2.7 (**[RZ20, Definition 2.2]**). *Given a compact set $K \subset \mathbb{R}$ and $\alpha > 0$, the* degree-α-gap sum *of K is*

$$G_\alpha(K) := \sum_{I \subset \mathcal{G}_K} |I|^\alpha,$$

*where $\mathcal{G}_K$ is the set of all bounded components of $\mathbb{R} \setminus K$. The set K is a BT_α-*set *if it is compact, has zero Lebesgue measure and $G_\alpha(K) < \infty$.*

Let $L_E \subset \mathbb{R}$ be the set of radii $\varepsilon > 0$ for which the ε-boundary $\partial_\varepsilon(E)$ fails to be a Lipschitz manifold. The first main result of [RZ20] provides, for a compact planar set $E \subset \mathbb{R}^2$ and $\delta > 0$, the following characterisation of the smallness of the set $\mathrm{cv}(d_E) \cap [\delta, \infty)$, i.e. the set of critical values of the distance function bounded away from 0.

Theorem 1.2.8 (**[RZ20, Theorem 1.1]**)**.** *Let $\delta > 0$ and $A \subset [\delta, \infty)$. Then the following properties are equivalent.*

(i) $A \subset L_E$ for some compact set $E \subset \mathbb{R}^2$;

(ii) $A \subset \mathrm{cv}(d_E)$ for some compact set $E \subset \mathbb{R}^2$;

(iii) $\overline{A}$ is a $BT_{1/2}$-set.

The authors also provide the following result concerning the entire (non-truncated) set of critical values $\mathrm{cv}(d_E)$ for a compact planar set.

Theorem 1.2.9 (**[RZ20, Theorem 1.2]**)**.** *Let $A \subset (0, \infty)$. Then the following conditions are equivalent.*

(i) $A \subset L_E$ for some compact set $E \subset \mathbb{R}^2$;

(ii) $A \subset \mathrm{cv}(d_E)$ for some compact set $E \subset \mathbb{R}^2$;

(iii) $\overline{A}$ is bounded, has zero Lebesgue measure and

$$\int_0^\infty G_{1/2}\Big(\overline{A} \cap [r, \infty)\Big)\sqrt{r}dr < \infty.$$

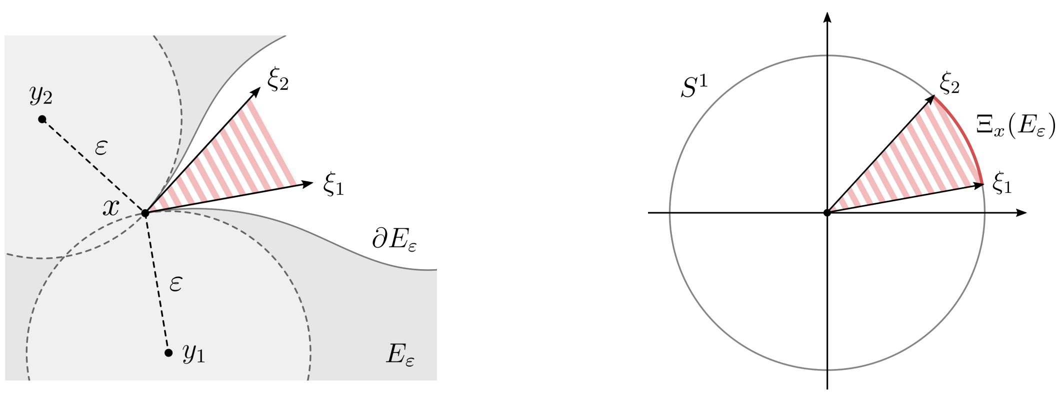
Theorem 1.2.9 clearly has relevance for situations where one wishes to obtain information about the regularity of the ε-boundary $\partial_\varepsilon(E)$ as an approximation of the underlying set E, letting the radius ε converge to zero. The characterisations of $\mathrm{cv}(d_E)$ provided by Theorems 1.2.8 and 1.2.9 are optimal, and these results furthermore provide a new proof for Fu's results in the planar case.

Apart from the topological and manifold structure of the boundary, also the volume $\mathcal{H}^d(E_\varepsilon)$ of the ε-neighbourhood E_ε and its relationship with the surface area $\mathcal{H}^{(d-1)}\partial E_\varepsilon$ is of special importance in many applications [Sta76]. This was investigated by J. Rataj and collaborators in the case of the asymptotic limit $\varepsilon \to 0$ with particular focus on self-similar fractal sets [RW10]. This relates to the general topic of analysing properties of ε-neighbourhoods of Brownian motion trajectories, known in the literature as *Wiener sausages* [RSM09, RVS09].

1.3 Main Results

Our main achievement is the development of a novel technique for analysing the geometric properties of the boundary ∂E_ε in the case of a compact planar E set and arbitrary $\varepsilon > 0$.

This technique builds on the well-established concept of a *tangent cone*, which features prominently in the field of set-valued analysis. An elementary introduction to set-valued analysis, including the theory of tangent cones, can be found for instance in [AF90]. The notion of tangent cone does not refer to a single construction, but rather to a family of definitions, all of which share the aim of describing the infinitesimal geometry of some set A at a point $x \in A$ via studying the properties of point sequences $(x_n)_{n=1}^{\infty}$ that converge to x. The choice of the 'correct' definition depends on the nature of the problem at hand, as well as the particular features of the objects under consideration. In this thesis, we define a variation of the well-known *contingent cone*, introduced by Bouligand [Bou30, Bou32]. More specifically, we define for each boundary point $x \in \partial E_\varepsilon$ the set $\Xi_x(E_\varepsilon)$ of *outward directions*, which intuitively indicate the directions one can take in order to exit the set E_ε, starting from x. Formally, outward directions are represented as unit vectors $\xi \in S^1$. See Definition 2.1.2 and Remark 2.1.3 for the precise definitions and a discussion of the connection between outward directions and contingent cones. In the case of ε-neighbourhoods, one can utilise the geometric feature that each boundary point $x \in \partial E_\varepsilon$ lies exactly at ε-distance from some $y \in \partial E$. For each $x \in \partial E_\varepsilon$, we call such points y the *contributors* of x, reflecting the contribution of the corresponding ball $\overline{B_\varepsilon(y)}$ to the geometry of the boundary ∂E_ε, see Definition 2.0.1.



(a) **(b)**

Figure 1.3. Illustration of the relationship between outward directions and extremal contributors. In **(a)** the vectors ξ_1 and ξ_2 indicate the extremal outward directions and the points y_1, y_2 the corresponding extremal contributors at the boundary point x. For $i \in \{1, 2\}$, the vectors $y_i - x$ and ξ_i are perpendicular to each other. **(b)** Topologically, the vectors ξ_1, ξ_2 form the boundary on S^1 of the set $\Xi_x(E_\varepsilon) \subset S^1$ of outward directions.

Of fundamental importance for the whole thesis are the notions of *extremal outward direction* and *extremal contributor*. At each boundary point $x \in \partial E_\varepsilon$, these form outward direction - contributor pairs (ξ, y) for which the vectors $y - x$ and ξ are perpendicular to each other, see Definition 2.1.4. The connection between outward directions and contributors allows one to tap into the particular geometric features of ε-neighbourhoods, as opposed to general closed planar sets. In particular, we leverage it in order to obtain results regarding the tangential directions at boundary points $x \in \partial E_\varepsilon$.

Since classical tangents do not necessarily exist everywhere on the boundary ∂E_ε, we adopt a set-valued definition of tangency which allows for several tangential directions to exist at each point, see Definition 2.1.1. A key ingredient in justifying our approach is Proposition 2.2.8, which asserts that the sets of one-sided tangents and extremal outward directions coincide at each boundary point. In order to study the tangential directions on the boundary ∂E_ε, one can thus restrict one's attention to the extremal outward directions.

The machinery of contributors and outward directions allows us to investigate the local geometric properties of the boundary. From a technical standpoint, the *local contribution property*, formulated in Proposition 3.1.1, is the single most important result of the thesis. Intuitively, this result guarantees that the local geometry around each boundary point $x \in \partial E_\varepsilon$ is completely determined by contributors y that lie at most in two different directions relative to x. Section 3.1 contains a more detailed discussion of this result. Combining Proposition 3.1.1 with a finite approximation scheme for the boundary enables us to define what we call *local boundary representations*, see Definition 3.2.2 and Proposition 3.2.4. These allow representing the boundary ∂E_ε around each boundary point $x \in \partial E_\varepsilon$ as a finite collection of graphs of continuous functions, defined in local coordinate systems given by extremal pairs (ξ, y). Proposition 3.2.5 asserts that these functions are Lipschitz-continuous, which turns out to be a significant partial result towards further analysis of the smoothness of the boundary.

Together, Propositions 3.1.1 and 3.2.4 form the technical foundation of the thesis, on which everything else is built. As a first application, we employ the local boundary representation to obtain results on the local structure of the complement $\mathbb{R}^2 \setminus E_\varepsilon$. Lemma 3.3.2 provides sufficient conditions under which a boundary point $x \in E_\varepsilon$ lies on the boundary of a unique connected component of the complement, and Proposition 3.3.3 provides a more detailed description of the local geometry under the same conditions.

Our first main result establishes that for any compact set $E \in \mathbb{R}^2$ and $\varepsilon > 0$, each boundary point $x \in \partial E_\varepsilon$ is either a smooth point (in the sense that, in a neighbourhood of x, ∂E_ε is a C^1-curve) or falls into exactly one of eight distinct categories of *singularities.*

Theorem 1. *Let $E \subset \mathbb{R}^2$ be compact, $\varepsilon > 0$, and let $x \in \partial E_\varepsilon$ be a boundary point of E_ε that is not smooth. Then x belongs to precisely one of the following eight categories:*

(S1)	*wedge,*	*(S5)*	*shallow-shallow singularity,*
(S2)	*sharp singularity,*	*(S6)*	*chain singularity,*
(S3)	*sharp-sharp singularity,*	*(S7)*	*chain-chain singularity,*
(S4)	*shallow singularity,*	*(S8)*	*sharp-chain singularity.*

Rigorous definitions of these categories are given in Definition 4.1.1 and Figure 1.4

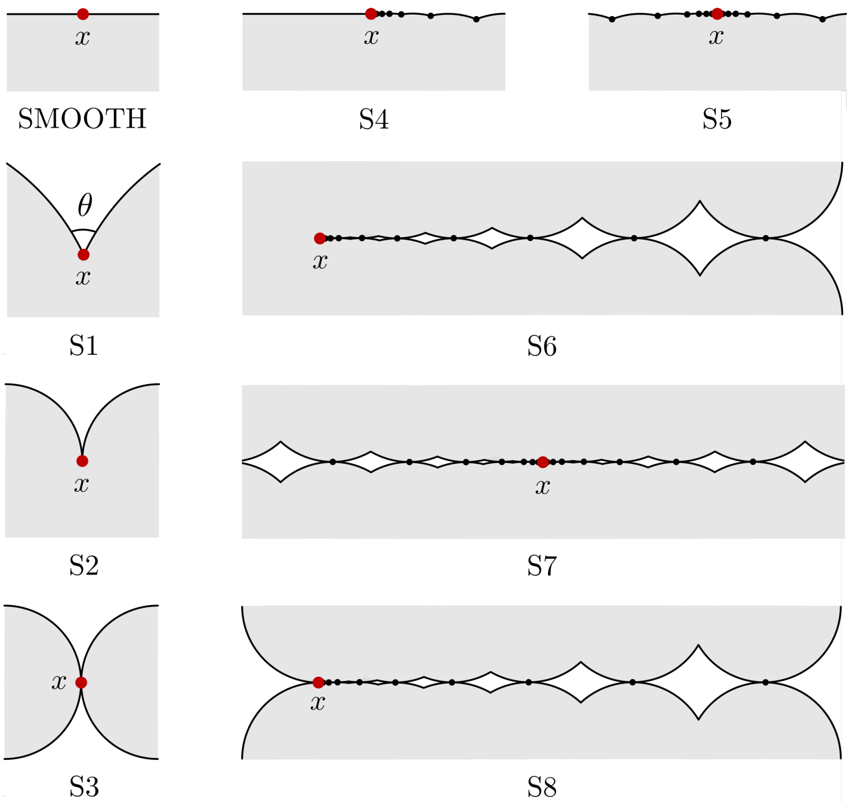


Figure 1.4. Schematic illustration of Theorem 1. The grey area represents the ε-neighbourhood E_ε, the white area the complement $\mathbb{R}^2 \setminus E_\varepsilon$. Every boundary point $x \in \partial E_\varepsilon$ either is a smooth point or belongs to exactly one of eight categories of singularities. At a *wedge* (S1) the one-sided tangents form an angle $0 < \theta < \pi$. A *sharp singularity* (S2) and a *sharp-sharp singularity* (S3) can be thought of as extremal cases of a wedge, with $\theta = 0$. A *shallow singularity* (S4) and a *shallow-shallow singularity* (S5) have a well-defined tangent, but they are accumulation points (from one or two directions, respectively) of sequences of increasingly obtuse wedges (black dots). A *chain singularity* (S6), a *chain-chain singularity* (S7) and a *sharp-chain singularity* (S8) share the geometric property of being accumulation points of sequences of increasingly acute wedges (black dots). This turns out to be equivalent (see Proposition 4.2.4) to the topological property of being the limit with respect to Hausdorff distance (see Definition 3.2.1) of a sequence of disjoint connected components of the complement E_ε^c. See also Figure 4.1.

provides indicative sketches of the different types. The proof of Theorem 1 relies heavily on the aforementioned local contribution property and local boundary representation, presented in Propositions 3.1.1 and 3.2.4. Using these same ingredients, we establish our second main result regarding the cardinality of the different types of singularities.

Theorem 2. *For any compact set $E \subset \mathbb{R}^2$ and $\varepsilon > 0$, the number of wedges (S1), sharp singularities (S2, S3 and S8), one-sided shallow singularities (S4) and chain singularities (S6) on ∂E_ε is at most countably infinite.*

Examples 5.1.5 and 5.2.1 demonstrate that the sets of shallow-shallow singularities (S5) and chain-chain singularities (S7) may be uncountable on ∂E_ε. We refer to the union of categories S6, S7 and S8 as *chain singularities* and denote the set of chain singularities on ∂E_ε by $\mathcal{C}(\partial E_\varepsilon)$. Even though the set of chain-chain singularities (S7) may in general be uncountable and can have a positive Hausdorff measure on the boundary ∂E_ε, our third main result establishes that $\mathcal{C}(\partial E_\varepsilon)$ is closed and totally disconnected.

Theorem 3. *For any compact set $E \subset \mathbb{R}^2$ and $\varepsilon > 0$, the set $\mathcal{C}(\partial E_\varepsilon)$ of chain singularities is closed and totally disconnected.*

It follows as a corollary of Theorem 3 that $\mathcal{C}(\partial E_\varepsilon)$ is nowhere dense on ∂E_ε, and is hence small in the topological sense.

Building on these results allows us to make the topology of the boundary ∂E_ε more precise in two ways. First, it is possible to give a global description of the boundary that applies for all $\varepsilon > 0$, without the need to ignore a countable set of ε-values, as was done in previous studies of *open*[1] ε-neighbourhoods [BMO09, Bro72]. Second, the local geometry of boundaries of closed ε-neighbourhoods reveals that no simple arcs exist on ∂E_ε (for any $\varepsilon > 0$), and that point components appear only in a very specific way.

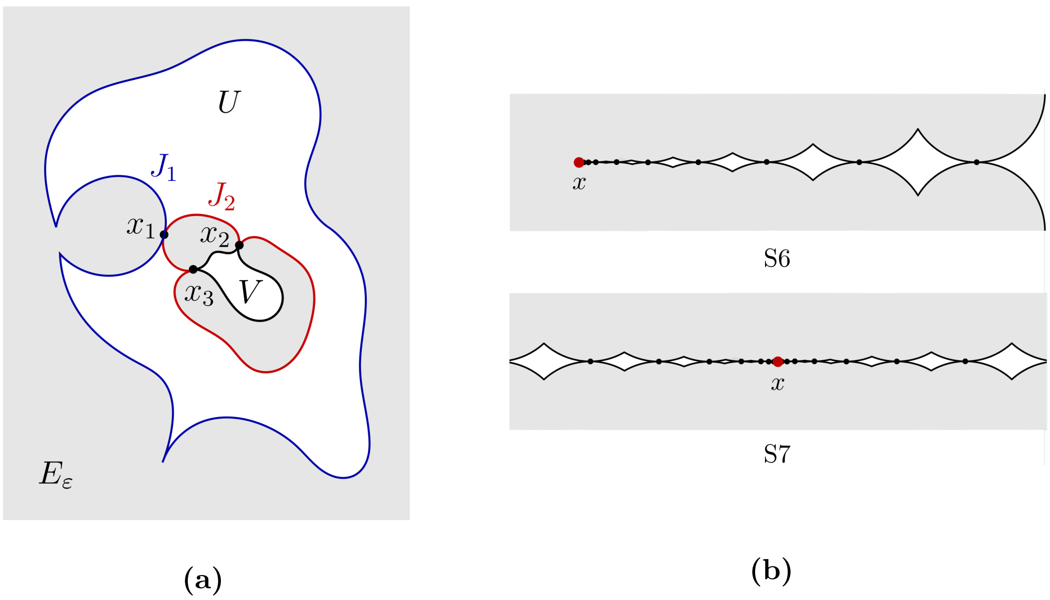


Figure 1.5. Schematic illustration of Theorem 4. **(a)**: Jordan curve subsets of the boundary of a connected component U of the complement $\mathbb{R}^2 \setminus E_\varepsilon$. Here $\partial U = J_1 \cup J_2$ and $J_1 \cap J_2 = \{x_1\}$. At the centre of the figure there is another connected component $V \subset \mathbb{R}^2 \setminus E_\varepsilon$, with $\partial V \cap \partial U = \{x_2, x_3\}$. **(b)**: The two types of inaccessible singularities. A chain singularity (S6) is the limit (in terms of Hausdorff distance) of a sequence of mutually disjoint connected components of the complement. Such a sequence approaches a chain-chain singularity (S7) on both sides. These are the only types of boundary points that do not lie on Jordan curve subsets on the boundary.

[1]The boundaries $\partial E_{<\varepsilon}$ of *open* ε-neighbourhoods $E_{<\varepsilon} := \bigcup_{x\in E} B_\varepsilon(x)$ have been referred to as the *ε-boundaries* [Fer76] or *ε-level sets* [OP85] of E. We note that $\partial E_{<\varepsilon} = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, E) = \varepsilon\}$ and that in general ∂E_ε is a closed subset of $\partial E_{<\varepsilon}$.

According to Theorem 1, each boundary point $x \in \partial E_\varepsilon$ is either smooth or belongs to exactly one of eight distinct types of singularities. From the topological point of view, these can further be divided into three groups:

(i) Point components of the boundary: the *inaccessible singularities*.

(ii) Boundary points, around which the boundary ∂E_ε can be uniquely represented as a simple curve: these are the smooth points as well as the so-called *wedges*, *sharp* and *sharp-chain singularities*[2] and *shallow singularities*.

(iii) Boundary points, around which there is more than one way to locally represent the boundary as a union of simple curves: the so-called *sharp-sharp singularities*.

Whenever sharp-sharp singularities exist on the boundary ∂E_ε, there are several ways to represent the boundary as a union of curves. It turns out, however, that the boundary of each individual connected component of the complement $\mathbb{R}^2 \setminus E_\varepsilon$ has a unique representation as a finite union of Jordan curves. On the other hand, point components correspond to boundary points $x \in \partial E_\varepsilon$ that do not lie on the boundary of any connected component of the complement $\mathbb{R}^2 \setminus E_\varepsilon$. These *inaccessible singularities* are characterised in Corollary 4.2.2. Our fourth main results is that, apart from the set of inaccessible singularities, the boundary ∂E_ε consists of a countable (possibly finite) union of Jordan curves.[3]

Theorem 4 (**Global structure of the boundary**). *For a compact set $E \subset \mathbb{R}^2$ and $\varepsilon > 0$, the boundary ∂E_ε is a disjoint union*

$$\partial E_\varepsilon = \mathcal{I} \cup J,$$

where $\mathcal{I}$ is the set of inaccessible singularities and $J = \bigcup_{i \in I} J_i$ is a countable (possibly finite) union of Jordan curves J_i. Furthermore there is a unique representation with the property that each Jordan curve J_i satisfies $J_i \subset \partial U$ for some connected component U of the complement E_ε^c.

In general, the Jordan curves J_i in Theorem 4 need not be disjoint. However, the bounded plane components defined by these Jordan curves are mutually disjoint, and the intersection of any two Jordan curves J_i is either empty or contains finitely many points.

Another fundamental property of interest is the smoothness of the boundary. It is known that for a fixed compact set $E \in \mathbb{R}^2$, the components of the boundary ∂E_ε are Lipschitz manifolds except for a zero measure set of radii $\varepsilon > 0$ [Fer76, Fu85, RZ20]. In light of Theorem 4 this property can be interpreted as the existence of well-defined tangents almost everywhere on the Jordan curve components of the boundary. It follows

[2] More precisely, the boundary of the unique connected component of the complement that touches the sharp-chain singularity can be uniquely represented by a simple curve in any sufficiently small neighbourhood, cf. Lemma 6.2.1.

[3] We call the image $\Gamma := \gamma([a, b])$ of a closed interval $[a, b]$ under a continuous map $\gamma : [a, b] \to \mathbb{R}^2$ a *curve*. If γ is injective, we call Γ a *simple curve*, and if injectivity is violated only at the endpoints, i.e. $\gamma(a) = \gamma(b)$, we say that Γ is a *simple closed curve* or a *Jordan curve*.

from Theorem 2 that tangents exist almost everywhere on the boundary outside the set of chain singularities, despite the fact that singularities of wedge type may be dense on subsets of ∂E_ε that have positive one-dimensional Hausdorff measure.

It turns out that it is possible to analyse the curvature of ∂E_ε by studying the properties of the Lipschitz functions that make up the local boundary representations at each boundary point $x \in \partial E_\varepsilon$. The existence of second derivatives (defined in a suitable way) of these functions then implies the existence of curvature for the corresponding boundary segments.

Theorem 5 (**Existence of curvature**). *Let $E \subset \mathbb{R}^2$ be compact, $\varepsilon > 0$ and let $\mathcal{J} = \partial E_\varepsilon \setminus \mathcal{I}$ where $\mathcal{I}$ is the set of inaccessible singularities. Then (signed) curvature $\kappa(x)$ exists for $\mathcal{H}^1$-almost all $x \in \mathcal{J}$, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure.*

Since the first derivative of a local boundary representation in general only exists almost everywhere, one cannot simply define the second derivative naïvely as a limit in the classical sense. We obtain Theorem 5 by showing that the derivative function is of bounded variation and consequently has a well-defined derivative almost everywhere. This second derivative allows one to define curvature locally by applying the classical formula for the curvature of a function graph.

Future Directions

The work in this thesis can be extended in a number of directions. Since our geometric line of reasoning complements the more analytic approach of studying the level sets and critical values of the distance function (1.2.2), it would be natural to explore whether results in one framework could provide insights in the other. Intuitively, a point x is a critical point of the distance function $d(\cdot, E)$, corresponding to a critical value ε, if and only if either x is a sharp (S2), sharp-sharp (S3) or a chain (S6–S8) singularity on ∂E_ε, or $x \in \partial_\varepsilon(E) \setminus \partial E_\varepsilon$, see Definitions 1.2.1 and 4.1.1.

A further classification of possible boundary geometries and topological properties of ε-neighbourhoods in higher dimensions ($d \geq 3$) is an obvious, yet challenging goal for further work. Existing results in the literature indicate that in general, the structure of ε-boundaries in dimensions $d \geq 4$ can be very complicated [Fer76]. However, understanding the properties of ε-neighbourhoods in higher dimensions, even for restricted categories, could be valuable from the point of view of specific applications.

A pressing challenge in the fields of bounded control systems and random dynamical systems with bounded noise concerns the characterisation and prediction of bifurcations of local attractors [LRR15, KMR18, GCS08, HY06]. Interpreting a random dynamical system with bounded noise as a set-valued dynamical system allows one to study these attractors as dynamically defined ε-neighbourhoods E_ε. Recent advances indicate that it might be possible to gain information about approaching bifurcations in terms of the dynamics on the boundary ∂E_ε [KMR18]. This approach holds promise of significant breakthroughs that could provide a solid theoretical foundation for stability analysis and uncertainty quantification in many applied contexts. We have included a more elaborate introduction to bifurcation theory of random dynamical systems in the next section, as this constitutes the primary motivation for our work, and is expected to benefit from the theoretical advances presented in this thesis.

1.4 Motivation

The principal motivation for this thesis stems from the central role of ε-neighbourhoods in set-valued dynamical systems as simplified frameworks for the analysis of random dynamical systems with bounded noise. In this section we motivate the use of set-valued dynamical systems in this context, and illustrate the significance of ε-neighbourhoods in the theory. We also mention some other fields of research where ε-neighbourhoods feature in a prominent role.

Random Dynamical Systems with Bounded Noise

The development of the classical dynamical systems theory over the last 150 years has been one of the success stories of modern science. This paradigm has improved the understanding of complex systems ranging from the level of molecular physics [Cea19] and neurological models for epileptic seizures [FNJ$^+$13] all the way up to entire societies [Wea19] and climate systems [Lea19]. Of particular interest is to understand when and how stable behaviours in such systems deteriorate, allowing the system to tip into other, possibly chaotic regimes [SBB$^+$09, Sch09, Sor04]. *Bifurcation theory* studies how such changes take place as system parameters are varied.

In the study of highly complex systems, it becomes necessary to distinguish between the conceptually salient components (components of intrinsic interest) from those that play the more instrumental role of grounding the model to its surroundings. A common way of accomplishing this is to find ways of systematically 'blurring' the lower level (local, micro, fast) dynamics by describing them as uncertainty or noise, while retaining the ability to model their effects on the relevant higher level (global, macro, slow) behaviours [Kue11]. In order to implement this approach in practice, sophisticated modelling methods based on stochastic differential equations (SDEs) have been developed since the first half of the 20th century. Although this approach has been a tremendously successful overall, the development of a random counterpart of the deterministic bifurcation theory for changing random systems has turned out to be a formidable challenge. In the meanwhile, studies of climate dynamics and other applications have clearly demonstrated the need for a theory that could provide a rigorous framework for understanding critical changes in random systems [TS11, Kue11, GCS08].

Quantifying system stability in the presence of random fluctuations is a topical challenge in a range of applications, from the study of the global climate system [Lea19] all the way down to neuronal dynamics of the human brain [FNJ+13]. The works of Scheffer [Sch09], Sornette [Sor04] and Gladwell [Gla00] have recently introduced the topic of system stability also to wider public discussion.

A novel approach to stability analysis was suggested in the influential paper [MHMK13], where the authors introduced the concept of *basin-stability*. This approach has been successful in illuminating stability properties particularly in networked systems, including power-grid networks [AKS+16]. However, the approach is inherently deterministic and does not allow modelling the effects of persistent noise acting on the system over periods of time. On the other hand, while unbounded (for instance Gaussian) noise is a standard model for random fluctuations, it presents a conceptual challenge for stability analysis. Unbounded noise clashes directly with the locality of macroscopic processes, giving rise to surprising artefacts [Bau07].

A profoundly different approach to modelling uncertainty—*bounded noise*—has been suggested as a promising alternative. In [KC98] the authors demonstrate the benefits of applying a bounded noise model, compared to the analogous stochastic (unbounded) model, in the study of extinction rates of endangered species. One advantage of bounded noise is that it allows the so-called *unknown-but-bounded* (UBB) approach to modelling uncertainty [BGP09, KC98]. This means that instead of having to estimate moments of an empirical noise distribution in order to calibrate a noise model, knowing or estimating the maximal noise amplitude is sufficient for performing worst-case scenario analyses. That said, a number of methods have been developed for generating bounded stochastic processes with desired probability distributions and spectral profiles [d'O13, Ch. 1].

Over the last two decades, bounded noise has steadily gained in popularity in the engineering, physics and bioscience communities [d'O13]. According to Clarivate Analytics Web of Science[4], a research database, in total 1,311 scientific articles published from 1970 to 2019 were associated with either "bounded noise" or "bounded uncertainty", with more than 80% published after 2005, and 88% in applied fields outside mathematics. The existing theoretical literature on bounded noise systems includes studies on bifurcations of stationary measures [HY10, HY06, HYG13, ZH07], early-warning signals [KMR18], connections to bounded control systems [MM, CK00, CKMS08] and characterisation of topological bifurcations [LRR15]. The last one of these has served as a central motivation for this thesis.

[4] https://clarivate.com/products/web-of-science/

Bifurcation Theory in Random Dynamical Systems

Bifurcation theory is the cornerstone of rigorous stability analysis in the classical study of deterministic dynamical systems. However, extending it to the setting of random dynamical systems has proved challenging. Formal definitions of bifurcations in the context of unbounded (for instance Gaussian) noise have been proposed [Arn98], based on phenomenology of density functions or Lyapunov exponents, but none have so far enjoyed wide-spread adoption.

The key idea underpinning the usefulness of the bounded noise approach to uncertainty quantification is to study the time evolution of system trajectories collectively under all possible noise realisations as a *set-valued dynamical system*. Asymptotically stable sets in state space then appear as *minimal invariant sets*. These coincide with the supports of the stationary measures of the corresponding random dynamical system [HY06, HY10] and (if the perturbations are interpreted as bounded controls) with the *invariant control sets* [CK00]. The boundaries of minimal invariant sets—which have a well-defined topological structure by virtue of the boundedness of the perturbations—have recently been identified as central objects for *set-valued bifurcations* [LRR15] and for developing *early warning signals* for an approaching bifurcation [KMR18]. Crucially, it has been shown that a bifurcation in a system perturbed with bounded noise is always associated with the breaking of topological conjugacy [LRR15, Thm 5.1]. This naturally links stability in the presence of bounded noise to the deterministic case via the concept of *structural stability* [Rob95, Ch. 2.6].

Discrete-time Set-valued Dynamical Systems

For a given complex system, the macroscopic dynamics of interest may be represented by a deterministic map $f : X \to X$, where the state space $X \subset \mathbb{R}^d$ is assumed to be compact. Randomness (uncertainty or noise) in the system is modelled by introducing at each time step a random kick ψ_n whose amplitude is bounded by some $\varepsilon > 0$. For a system state x_n at time $n \in \mathbb{N}$, the next state is given by

$$x_{n+1} := f(x_n) + \psi_n, \tag{1.4.1}$$

where the random variables ψ_n in the sequence $(\psi_n)_{n=1}^{\infty}$ are assumed to be independent and taking values in $\overline{B_\varepsilon(0)} := \{x \in \mathbb{R}^d : \|x\| \leq \varepsilon\}$. The function $f = f_\lambda$ is assumed to depend on a (low-dimensional) *bifurcation parameter* $\lambda \in \mathbb{R}^n$, which affects the properties of the deterministic system.

For stability analysis, one may consider the *set-valued system* (X, f, ε) associated with the time propagation of supports of the probability distributions of system states at each time step. Starting from any state $x_0 \in X$, the support of the (random) state after n

steps is given by the n-fold composition $f_\varepsilon \circ f_\varepsilon \circ \ldots \circ f_\varepsilon(x_0)$, where the *set-valued map* f_ε is defined by

$$f_\varepsilon(K) := \bigcup_{x \in K} \overline{B_\varepsilon(f(x))} \tag{1.4.2}$$

for all non-empty compact subsets $K \subset X$. A compact set M that satisfies $f_\varepsilon(M) = M$ is called an *invariant set*, and it is *minimal* if it has no proper invariant subsets, see Figure 1.6. Minimal invariant sets represent the stable regions of the set-valued system (1.4.2), and they have been shown [ZH07] to coincide with the supports of the stationary measures of the corresponding random system (1.4.1). The properties of minimal invariant sets were investigated in [Ath18, Dup10] and in a more general setting in [LRR15].

The boundaries of minimal invariant sets are central for the stability analysis of set-valued systems for two reasons. First, a bifurcation of a set-valued system (X, f, ε) occurs when the boundary of a minimal invariant set collides with the domain of attraction of the so-called *dual repeller* [LRR15]. This corresponds to the dual 'backwards' map

$$f_\varepsilon^{-1}(K) := \left\{ f^{-1}(x) \,:\, x \in \overline{B_\varepsilon(K)} \right\}.$$

In addition, recent work in the one-dimensional setting on early warning signals for approaching bifurcation suggests that analogous methods could be developed also for higher dimensions [KMR18].

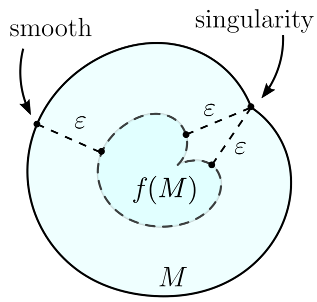


Figure 1.6. Schematic illustration of a minimal invariant set with a singularity.

Inspired by the one-dimensional results in [KMR18], the author has collaborated with a fellow PhD student Wei Hao Tey at the Dynamical Systems group at Imperial College London in order to develop theoretical foundations for a novel numerical scheme for tracking the boundaries of minimal invariant sets and computing dynamical indicators for approaching bifurcation. The dynamics on the boundary ∂M of a minimal invariant set are represented by the so-called *boundary map*. This is the (a priori set-valued) map $F : \partial M \to 2^{\partial M}$, defined by

$$F(x) := \{ z \in \partial M \,:\, \|z - f(x)\| = \varepsilon \}, \tag{1.4.3}$$

with the possibility $F(x) = \varnothing$. Whenever $F(x) \neq \varnothing$, the boundary map in fact defines a single-valued map $F : \partial M \to \partial M$, and if the underlying map f is C^1-smooth, F admits the explicit expression (written here for $d = 2$)

$$F(x) = f(x) + \frac{\varepsilon P D_f(x) P w_x}{|D_f(x) P w_x|}, \tag{1.4.4}$$

where $D_f(x)$ is the Jacobian, $w_x := \varepsilon^{-1}(y-x)$ is the inward unit normal at $x \in \partial M$ and P a suitable orthogonal rotation. A promising idea for bifurcation analysis of minimal invariant sets is to extend the map 1.4.4 into a global map defined on the unit tangent bundle over the state space $\mathbb{R}^2$. This allows interpreting the boundaries ∂M as invariant sets of this finite-dimensional map, thus reducing the complexity of the difficult set-valued problem. This representation has potential to serve as a foundation for the construction of novel numerical methods for stability analysis in noisy settings.

Geometry of Boundaries of Minimal Invariant Sets

The dynamics on the boundaries ∂M of minimal invariant sets are intimately linked to their geometry. In particular, the existence or not of singularities on the boundary ∂M has profound implications to the dynamics of the boundary map F. A key observations is that, by definition, each minimal invariant set satisfies

$$M = \{x \in \mathbb{R}^d \,:\, \mathrm{dist}(x, f(M)) \leq \varepsilon\}$$

In other words, M is an *ε-neighbourhood* of its deterministic image $f(M)$. In the context of random dynamical systems with bounded noise, it is natural to assume that the support of each noise vector is the closed ball $B_\varepsilon(0)$, which implies that the minimal invariant sets are compact. From the technical point of view, this implies in particular that the boundaries of minimal invariant sets are boundaries of the *closed* ε-neighbourhoods $\partial\overline{B_\varepsilon(f(M))}$, as opposed to the corresponding ε-boundaries $\partial_\varepsilon(f(M))$, see (1.2.1). For this reason, we focus in this thesis on the analysis of the boundaries ∂E_ε of closed ε-neighbourhoods of compact sets $E \subset \mathbb{R}^2$.

For the purposes of bifurcation analysis of the minimal invariant sets

$$M = f_\varepsilon(M) := \bigcup_{x \in M} \overline{B_\varepsilon(f_\lambda(x))} = \overline{B_\varepsilon(f_\lambda(M))},$$

one needs to relate the boundary dynamics of the minimal invariant set M to the changing geometry of the boundary ∂M, which depends in a delicate way on the bifurcation parameter $\lambda \in \Lambda$. Since the underlying set $f_\lambda(M)$ changes dynamically according to the parameter λ, one cannot directly rely on the existing literature on ε-neighbourhoods, which mostly concentrate on the manifold properties of the ε-boundary $\partial_\varepsilon(E)$ of a fixed compact set $E \subset \mathbb{R}^2$. Instead, one needs to understand the interplay between the local geometry of the boundary and the dynamics of the boundary map (1.4.3). These considerations motivate the geometric focus of this thesis as opposed to the more analytic flavour of much of the existing literature on ε-neighbourhoods.

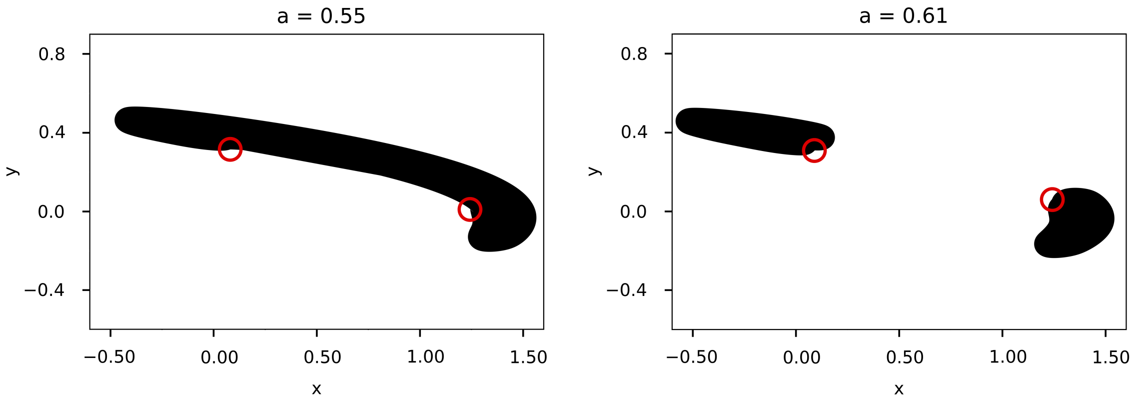


Figure 1.7. Examples of attractors of the Hénon map $f(x, y) = (1 + y - ax^2, 0.3x)$ with bounded noise with range $r = 0.0625$. At $a = 0.55$ (left), the attractor has one connected component, and at $a = 0.61$ (right), the attractor has two connected components, which are cyclically permuted. The splitting of the attractor (minimal invariant set of the set-valued system) occurs discontinuously close to $a = 0.595$ [LRR15]. The red circles highlight the presence of wedge singularities on the boundary of the attractor, see Figure 1.4. Illustration credit: Wei Hao Tey (Imperial College London).

Other Applications of ε-neighbourhoods

Ɛ-neighbourhoods arise in many branches of mathematics, ranging from convex analysis and manifold theory [Prz13] to fractal geometry [KLV13] and stochastic processes [Kla18]. In the latter, so-called Wiener sausages represent smoothed-out counterparts of Brownian motion trajectories [DV75, Ham10, RSM09, RVS09], with applications in theoretical physics [KL73, NP16, Spi64]. The interplay between the surface area and volume of ε-neighbourhoods [RW10, Sta76] and different notions of dimension [KLV13], as well as the dependence of the manifold structure of the boundary on the radius ε [Fed59, Fer76, Fu85, RZ11, RZ20] have all received considerable attention during the last decades. In particular, Rataj and collaborators [RSM09, RVS09, RW10, RZ11, RZ20] have advanced the understanding in recent years. In the context of differential inclusions, Pogodaev proved regularity properties for the boundary of the integral funnel by investigating conditions that guarantee it to be an ε-neighbourhood [Pog16]. More generally, reachable sets in control systems with bounded controls appear as ε-neighbourhoods [CK00]. In topological data-analysis, the topological (homology) properties of ε-neighbourhoods of point clouds are used to infer structural features of underlying objects via noisy datasets [CM17].

1.5 Structure of the Thesis

The thesis consists of this introductory chapter and seven subsequent chapters. In Chapters 2–5 we gradually build up the technical framework, and the reader is encouraged to proceed through these chapters more or less linearly. Chapters 6 and 7 build on the preceding results, and these can be read independently of each other.

The rest of the thesis is structured as follows. In Chapter 2 we present the basic conceptual framework and terminology that will be used throughout the thesis. Section 2.2 contains a concise introduction to the notions and basic properties of *contributors* (Definition 2.0.1) and *outward directions* (Definition 2.1.2) and their relationship with tangential properties of the boundary ∂E_ε. The main result of this section is Proposition 2.2.8, which states that the set of one-sided tangent vectors on the boundary coincides with the set of extremal outward directions (Definition 2.1.4).

In Chapter 3 we study local properties of boundary points $x \in \partial E_\varepsilon$ in small neighbourhoods $\overline{B_r(x)}$. The key result is Proposition 3.1.1, which asserts that for all $E \subset \mathbb{R}^2$ and $\varepsilon > 0$, the boundary geometry near each $x \in \partial E_\varepsilon$ is defined solely by those $y \in \partial E$ that lie near the extremal contributors of x. Building on this insight and a related approximation scheme (Definition 3.2.2) we prove Proposition 3.2.4, which asserts that the boundary can be represented locally as a finite union of graphs of continuous functions. This *local boundary representation* plays a fundamental role in the proofs of all the subsequent results. Section 3.3 contains an analysis of the topological and geometric structure of the complement $\mathbb{R}^2 \setminus E_\varepsilon$ near smooth points, wedges (S1) and shallow singularities (S4–S5), see Figure 1.4.

Chapter 4 contains the first main result of the thesis, a classification of boundary points $x \in \partial E_\varepsilon$ into smooth points and eight distinct types of singularities. We lay out the different types of singularities in Definition 4.1.1. We then show how these can be characterised in terms of the local topological structure of the complement $\mathbb{R}^2 \setminus E_\varepsilon$ and the geometric properties of the boundary ∂E_ε (Propositions 4.2.1 and 4.2.4). We also consider *inaccessible singularities* which are characterised by the property that they do not lie on the boundary of any connected component of the complement $\mathbb{R}^2 \setminus E_\varepsilon$. Corollary 4.2.2 asserts that the set of inaccessible singularities consists of one-sided chain singularities (S6) and chain-chain singularities (S7). The chapter culminates with the proof of our first main result, Theorem 1, which establishes that the classification of singularities provided in Definition 4.1.1, defines a partition on ∂E_ε.

In Chapter 5 we analyse the topological structure of the sets of different types of singularities. The second main result of the thesis, Theorem 2, asserts that for any compact $E \in \mathbb{R}^2$ and $\varepsilon > 0$, the sets of wedges (S1), sharp singularities (S2, S3 and S8) and one-sided chain singularities (S6) on ∂E_ε are at most countably infinite. The chapter concludes with the proof of Theorem 3, which states that the set $\mathcal{C}(\partial E_\varepsilon)$ of chain singularities is closed and totally disconnected. This furthermore implies that it is nowhere dense, and hence small in the topological sense.

In Chapter 6 we turn our attention to the global topological structure of the boundary ∂E_ε. We utilise the local boundary representations (Proposition 3.2.4) in order to construct Jordan curve covers for the boundaries of connected components of the complement $\mathbb{R}^2 \setminus E_\varepsilon$ (Lemma 6.2.1) and demonstrate that these covers always have a subcover of order two (Lemma 6.2.2). We use these partial results to prove Proposition 6.2.3, which states that the boundary of each connected component of the complement $\mathbb{R}^2 \setminus E_\varepsilon$ can be expressed as a finite union of Jordan curves. We close the chapter with the proof of Theorem 4, which provides a complete topological description of the boundary ∂E_ε in terms of Jordan curve subsets and inaccessible singularities.

In the final Chapter 7 we investigate the existence of curvature on the Jordan curve subsets of the boundary. We again make use of the local boundary representations given by Proposition 3.2.4, this time as a way of analysing the local smoothness of the boundary. In Section 7.1.2 we prove an elementary sufficient condition for bounded variation, which we apply to the functions in local boundary representations. The proof of the main result, Theorem 5, is based on a series of technical Lemmas 7.2.1–7.2.3, which we use to prove that the functions constituting the local boundary representations satisfy a specific lower bound for tangential directions.

Chapter 2

ε-neighbourhoods

The object of our study is the *ε-neighbourhood* $E_\varepsilon = \overline{B_\varepsilon(E)}$ of a closed subset $E \subset \mathbb{R}^d$. The main results of the thesis concern ε-neighbourhoods of planar sets $E \subset \mathbb{R}^2$, but we provide the basic definitions in a more general d-dimensional setting. Throughout the thesis we make the assumption that the underlying set $E \subset \mathbb{R}^d$ is closed[1] and $\varepsilon > 0$. Many of the results require the stronger assumption of compactness; where necessary, this will be explicitly stated in the formulation of each result.

The most immediate observation regarding the structure of the set E_ε is that each $x \in \partial E_\varepsilon$ necessarily lies on the boundary of a closed ball $\overline{B_\varepsilon(y)}$ of radius ε, centered at some $y \in \partial E$. On the other hand, for each $x \in \partial E_\varepsilon$ there may exist more than one $y \in \partial E$ with $\|y - x\| = \varepsilon$. These considerations motivate the following definition.

Definition 2.0.1 (**Contributor**). *Let $E \subset \mathbb{R}^d$ be closed. For each $x \in \partial E_\varepsilon$ we define the set of* contributors *as the collection*

$$\Pi_E(x) := \Big\{ y \in \partial E \ : \ \|y - x\| = \varepsilon \Big\}.$$

Boundary points $x \in \partial E_\varepsilon$ with only one contributor constitute the set

$$\mathrm{Unp}_\varepsilon(E) := \Big\{ x \in \partial E_\varepsilon \ : \ \Pi_E(x) = \{y\} \textit{ for some } y \in \partial E \Big\},$$

where Unp *stands for* 'unique nearest point'*, see [Fed59, Definition 4.1].*

The set of contributors $\Pi_E(x)$ consists of those points on ∂E that minimise the distance from ∂E to x. Hence Π_E can be interpreted as a restriction onto ∂E_ε of the (set-valued) projection $\mathrm{proj}_E : \mathbb{R}^2 \to E$ given by $\mathrm{proj}_E(x) := \{y \in E \ : \ \|y - x\| = \mathrm{dist}(x, E)\}$. In terms of our classification of boundary points (see Definition 4.1.1 and Figures 1.4 and 4.1) the set $\mathrm{Unp}_\varepsilon(E)$ consists of smooth points and shallow singularities (S4–S5).

[1] Note that this is not an actual restriction, since $\mathrm{dist}(x, E) = \mathrm{dist}(x, \overline{E})$ for all $x \in \mathbb{R}^d$ and $E \subset \mathbb{R}^d$.

Definition 2.0.2 (**Smooth point, singularity**). *We call a boundary point $x \in \mathrm{Unp}_\varepsilon(E)$* smooth*, if there exists a neighbourhood $B_r(x)$ for which $\partial E_\varepsilon \cap B_r(x) \subset \mathrm{Unp}_\varepsilon(E)$. If x is not smooth, we call it a* singularity *and write $x \in S(E_\varepsilon)$.*

The rationale for Definition 2.0.2 stems from the fact that any smooth $x \in \partial E_\varepsilon$ in terms of Definition 2.0.2 turns out to be equivalent to x having a neighbourhood $B_r(x)$ in which the boundary is a C^1-smooth curve, see Proposition 4.2.5.

2.1 Tangents via Outward Directions

Our first objective is to shed light on the tangential properties of individual boundary points $x \in \partial E_\varepsilon$. Acknowledging that classical tangents do not necessarily exist everywhere on the boundary, we adopt a set-valued definition of tangency which allows for several tangential directions to exist at each point. Our definition is a restriction of [Fed59, Definition 4.3] to the boundary ∂E_ε.

Definition 2.1.1 (**Tangent set**). *Let $E \subset \mathbb{R}^d$ be closed and $x \in \partial E_\varepsilon$. We define the set $T_x(E_\varepsilon)$ of unit* tangent vectors *of E_ε* at *x as all those points $v \in S^{d-1}$ for which there exists a sequence $(x_n)_{n=1}^\infty \subset \partial E_\varepsilon$ of boundary points satisfying $x_n \to x$ and*

$$\frac{x_n - x}{\|x_n - x\|} \to v, \quad \text{as } n \to \infty.$$

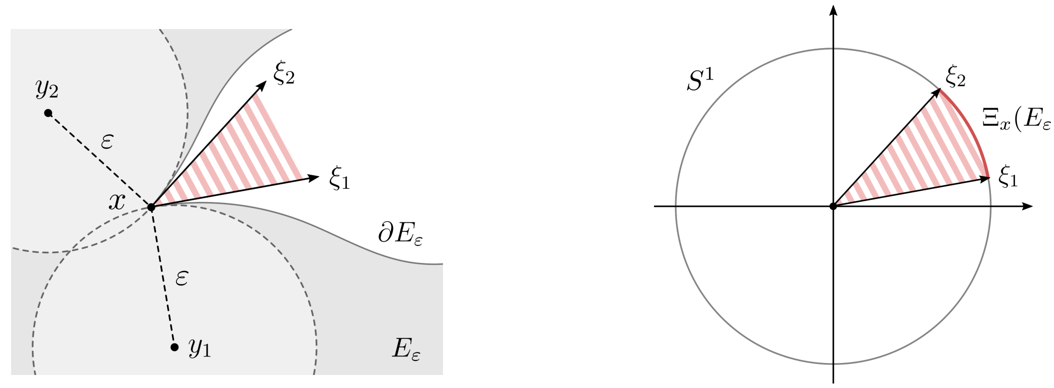
In order to study the existence of tangential directions at boundary points $x \in \partial E_\varepsilon$, we relate the set $T_x(E_\varepsilon)$ to what we call outward directions. Intuitively, the set of outward directions at each x contains the angles at which x can be approached from the complement $E_\varepsilon^c := \mathbb{R}^d \setminus E_\varepsilon$. It turns out that for an ε-neighbourhood E_ε, the extremal values of these angles coincide with the tangential directions as defined in Definition 2.1.1. Hence the existence of tangents at each $x \in \partial E_\varepsilon$ hinges on the existence and properties of corresponding outward directions, which turn out to be easier to study due to their geometric relationship with the contributors $y \in \Pi_E(x)$.

We define outward directions as points on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ but think of them rather as directional vectors in the ambient space $\mathbb{R}^d$, since we want to operate with them using the Euclidean scalar product $\langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$.

Definition 2.1.2 (**Outward direction**). *Let $E \subset \mathbb{R}^d$ be closed. We say that a point $\xi \in S^{d-1}$ is an* outward direction *from E_ε at a boundary point $x \in \partial E_\varepsilon$, if there exists a sequence $(x_n)_{n=1}^\infty \subset E_\varepsilon^c$, for which $x_n \to x$ and*

$$\xi_n := \frac{x_n - x}{\|x_n - x\|} \longrightarrow \xi \in S^{d-1} \subset \mathbb{R}^d,$$

as $n \to \infty$. We denote by $\Xi_x(E_\varepsilon)$ the set of outward directions from E_ε at x.



(a) A singularity $x \in \partial E_\varepsilon$ with $\Pi_E^{\text{ext}}(x) = \{y_1, y_2\}$ and $\Xi_x^{\text{ext}}(E_\varepsilon) = \{\xi_1, \xi_2\}$.

(b) The set $\Xi_x(E_\varepsilon) \subset S^1$ of outward directions is geodesically convex with boundary $\partial_{S^1}\Xi_x(E_\varepsilon) = \{\xi_1, \xi_2\}$.

Figure 2.1. Illustration of the relationship between outward directions and extremal contributors.

Remark 2.1.3. The concept of outward directions is a variation of the well-known contingent cone, introduced by Bouligand (see for instance [AF90, BG12, RW98] and Bouligand's original work [Bou30, Bou32]). For a boundary point $x \in \partial E_\varepsilon$ the *contingent cone (Bouligand cone)* $C_x(E_\varepsilon)$ consists of those vectors $v \in \mathbb{R}^d$, for which there exist sequences $(h_n)_{n=1}^\infty \subset \mathbb{R}_+$ and $(v_n)_{n=1}^\infty \subset \mathbb{R}^d$ for which $x + h_n v_n \in E_\varepsilon$ for all $n \in \mathbb{N}$ and

$$h_n \to 0, \quad \text{and} \quad v_n \to v$$

as $n \to \infty$. If instead of the outward directions $\Xi_x(E_\varepsilon)$ one considers at each $x \in \partial E_\varepsilon$ the contingent cone $C_x(E_\varepsilon^c)$ for the complement E_ε^c, it follows that $\mathcal{W}_x(E_\varepsilon) = C_x(E_\varepsilon^c)$, where

$$\mathcal{W}_x(E_\varepsilon) := \{s\xi \,:\, \xi \in \Xi_x(E_\varepsilon),\, s \geq 0\}$$

denotes the *outward cone* at x. We do not make use of this correspondence, but for further information on tangent cones, see for instance [Cla75, Roc79].

For each $x \in \partial E_\varepsilon$ we single out those outward directions $\xi \in \Xi_x(E_\varepsilon)$ that are perpendicular to some contributor $y \in \Pi_E(x)$—we call these the *extremal outward directions* and *extremal contributors* (see Figure 2.1). Definition 2.1.4 below emphasises this geometric relationship, while Proposition 2.2.6 in the next section confirms that extremal outward directions can equivalently be defined via the topological property of constituting the boundary of the set of outward directions $\Xi_x(E_\varepsilon)$.

Definition 2.1.4 (Extremal contributor, extremal outward direction). *Let $E \subset \mathbb{R}^d$ be closed and $x \in \partial E_\varepsilon$. If an outward direction $\xi \in \Xi_x(E_\varepsilon)$ and a contributor $y \in \Pi_E(x)$*

satisfy

$$\langle y - x, \xi \rangle = 0,$$

we call ξ *an* extremal outward direction *and* y *an* extremal contributor *at* x*. For each* $x \in \partial E_\varepsilon$*, we write* $\Xi_x^{\text{ext}}(E_\varepsilon)$ *and* $\Pi_E^{\text{ext}}(x)$ *for the sets of extremal outward directions and extremal contributors, respectively.*

The precise correspondence between extremal contributors, extremal outward directions, and tangential directions at each $x \in \partial E_\varepsilon$ is presented in Section 2.2, where we collect in one place all the basic results that we need in the remainder of the thesis. The existence of outward directions at each $x \in \partial E_\varepsilon$ is established in Proposition 2.2.1 and their geometric relationship with the contributors is explored in Lemma 2.2.4 and Proposition 2.2.6. The coincidence of the set of tangential directions $T_x(E)$ with the set of extremal outward directions $\Xi_x^{\text{ext}}(E_\varepsilon)$ is established in Proposition 2.2.8.

2.2 Properties of Contributors and Outward Directions

We collect here the basic properties of contributors and outward directions that we need in our analysis of the boundary ∂E_ε. The proofs make repeated use of convergent subsequences and scalar products and are rather elementary, although at places somewhat tedious. As before, the set $E \subset \mathbb{R}^d$ is assumed to be closed, and $\varepsilon > 0$.

Proposition 2.2.1 (**The set of outward directions is non-empty and closed**)**.** *Let* $E \subset \mathbb{R}^d$ *be closed and* $x \in \partial E_\varepsilon$*. Then the set* $\Xi_x(E_\varepsilon)$ *of outward directions is non-empty and closed.*

Proof. Let $x \in \partial E_\varepsilon$ and choose some sequence $(x_n)_{n=1}^\infty$ in E_ε^c with $x_n \to x$. This implies $x_n \neq x$ for all $n \in \mathbb{N}$ since E_ε is closed. One may hence define a sequence $(\xi_n)_{n=1}^\infty$ in S^{d-1} by setting $\xi_n := (x_n - x)/\left\|x_n - x\right\|$ for all $n \in \mathbb{N}$. The compactness of S^{d-1} implies that $(\xi_n)_{n=1}^\infty$ has a convergent subsequence, the limit of which is an element in $\Xi_x(E_\varepsilon)$.

To show that $\Xi_x(E_\varepsilon)$ is closed, let $\xi \in S^{d-1}$ and assume there exists a sequence $\left(\xi^{(n)}\right)_{n=1}^\infty \subset \Xi_x(E_\varepsilon)$ with $\xi^{(n)} \to \xi$ as $n \to \infty$. One needs to show that this implies $\xi \in \Xi_x(E_\varepsilon)$. We first use Definition 2.1.2 to identify each of the directions $\xi^{(n)}$ with a convergent sequence in E_ε^c, and then apply a kind of diagonalisation argument in order to construct a new sequence $(z_n)_{n=1}^\infty$ in E_ε^c with $(z_n - x)/\left\|z_n - x\right\| \to \xi$.

Without loss of generality, let $\left\|\xi^{(n)} - \xi\right\| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Now, for each $n \in \mathbb{N}$ one can choose a sequence $\left(x_k^{(n)}\right)_{k=1}^{\infty} \subset E_\varepsilon^c$ with $x_k^{(n)} \to x$ and

$$\xi_k^{(n)} := \frac{x_k^{(n)} - x}{\left\|x_k^{(n)} - x\right\|} \longrightarrow \xi^{(n)}, \quad \text{as } k \to \infty.$$

Consequently there exists for each $n \in \mathbb{N}$ some $K(n) \in \mathbb{N}$, for which $\left\|x_k^{(n)} - x\right\| \leq 1/n$ and $\left\|\xi_k^{(n)} - \xi^{(n)}\right\| \leq 1/n$ for all $k \geq K(n)$. Using these indices one can define a new sequence by setting $z_n := x_{K(n)}^{(n)}$ for each $n \in \mathbb{N}$. Accordingly

$$\|z_n - x\| = \left\|x_{K(n)}^{(n)} - x\right\| \leq \frac{1}{n}$$

so that $z_n \to x$. Furthermore, writing $\xi_n^z := (z_n - x)/\left\|z_n - x\right\| = \xi_{K(n)}^{(n)}$, we have

$$\|\xi_n^z - \xi\| \leq \left\|\xi_{K(n)}^{(n)} - \xi^{(n)}\right\| + \left\|\xi^{(n)} - \xi\right\| \leq \frac{2}{n} \longrightarrow 0$$

as $n \to \infty$. Thus ξ is the outward direction corresponding to the sequence $(z_n)_{n=1}^{\infty}$, which implies $\xi \in \Xi_x(E_\varepsilon)$, as required. □

Despite their simplicity, the following Lemmas 2.2.2 and 2.2.3 regarding contributors are a key ingredient in many of the subsequent proofs.

Lemma 2.2.2 (Convergence of contributors). *Let $E \in \mathbb{R}^d$ be compact, let $(x_n)_{n=1}^{\infty} \subset E_\varepsilon$ with $x_n \to x \in \partial E_\varepsilon$, and $(y_n)_{n=1}^{\infty} \subset E$ with $x_n \in \overline{B_\varepsilon(y_n)}$ for all $n \in \mathbb{N}$. Then there exists some $y \in \Pi_E(x)$ and a convergent subsequence $(y_{n_k})_{k=1}^{\infty}$, for which $y_{n_k} \to y$ as $k \to \infty$.*

Proof. Due to compactness of E, there exists a convergent subsequence $(y_{n_k})_{k=1}^{\infty}$ with $y_{n_k} \to y \in E$ as $k \to \infty$. For each $k \in \mathbb{N}$,

$$\|y - x\| \leq \|y - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| + \|x_{n_k} - x\|,$$

which implies $\|y - x\| \leq \varepsilon$, since $x_{n_k} \to x$, $y_{n_k} \to y$, and $\lim_{k\to\infty} \|y_{n_k} - x_{n_k}\| \leq \varepsilon$. On the other hand $\|y - x\| \geq \operatorname{dist}(x, E) = \varepsilon$. Hence $\|y - x\| = \varepsilon$, so that $y \in \Pi_E(x)$. □

Lemma 2.2.3 (Tails of directed sequences). *Let $E \subset \mathbb{R}^d$ be compact, $(x_n)_{n=1}^{\infty} \subset \mathbb{R}^d$ with $x_n \to x \in \partial E_\varepsilon$ and $x_n \neq x$ for all $n \in \mathbb{N}$, and assume*

$$v_n := \frac{x_n - x}{\|x_n - x\|} \longrightarrow v, \quad \textit{as } n \to \infty.$$

(i) If $\langle y - x, v\rangle > 0$ for some $y \in \Pi_E(x)$, then there exists some $N \in \mathbb{N}$, for which $x_n \in B_\varepsilon(y)$ for all $n \geq N$.

(ii) If $\langle y-x, v\rangle < 0$ for all $y \in \Pi_E(x)$, then there exists some $N \in \mathbb{N}$, for which $x_n \in E_\varepsilon^c$ for all $n \geq N$.

Proof. (i) Assume $\langle y-x, v\rangle = p > 0$ for some $y \in \Pi_E(x)$. Then $\langle y-x, v_n\rangle \to p$ due to the continuity of the scalar product. Hence there exists some $N \in \mathbb{N}$ for which $\langle y-x, v_n\rangle > \frac{1}{2}p$ and $\|x_n - x\| < p$, whenever $n \geq N$. This implies

$$\|x_n - x\| < p < 2\langle y-x, v_n\rangle$$

for all $n \geq N$ so that

$$\begin{aligned}\|y - x_n\|^2 &= \|(y-x) - (x_n - x)\|^2 \\ &= \varepsilon^2 + \|x_n - x\| \left(\|x_n - x\| - 2\langle y-x, v_n\rangle\right) < \varepsilon^2.\end{aligned}$$

(ii) Assume to the contrary that there exists a subsequence $(x_{n_k})_{k=1}^\infty \subset E_\varepsilon$. Then for each k there exists some $y_k \in E$ for which $\|x_{n_k} - y_k\| \leq \varepsilon$. Since E is compact, Lemma 2.2.2 implies the existence of some $y^* \in \Pi_E(x)$ for which $y_k \to y^*$ (if necessary, one can switch to a further convergent subsequence). By assumption $y^* \in \Pi_E(x)$ implies $\langle y^* - x, v\rangle < 0$. Due to the continuity of the scalar product there exists some $q > 0$ and $K \in \mathbb{N}$, for which $\langle y_k - x_{n_k}, v_{n_k}\rangle \leq -\frac{1}{2}q$ and $\|x_{n_k} - x\| < q$, whenever $k \geq K$. Then

$$\|x_{n_k} - x\| + 2\langle y_k - x_{n_k}, v_{n_k}\rangle < 0$$

for all $k \geq K$, and applying the triangle-inequality with respect to the points x_{n_k} yields

$$\|y_k - x\|^2 \leq \varepsilon^2 + \|x_{n_k} - x\| \left(\|x_{n_k} - x\| + 2\langle y_k - x_{n_k}, v_{n_k}\rangle\right) < \varepsilon^2.$$

This implies the contradiction $x \in \operatorname{int}(E_\varepsilon)$. □

Lemma 2.2.4 below provides a partial characterisation of outward directions $\Xi_x(E_\varepsilon)$ in terms of the contributors $\Pi_E(x)$. Geometrically it implies that outward directions point away from the vectors $y - x$ for all $y \in \Pi_E(x)$, see Figure 2.1.

Lemma 2.2.4 (Orientation of outward directions relative to contributors). *Let $E \subset \mathbb{R}^d$ be compact, $x \in \partial E_\varepsilon$ and $\xi \in S^{d-1}$. Then*

(i) if $\xi \in \Xi_x(E_\varepsilon)$, then $\langle y-x, \xi\rangle \leq 0$ for all $y \in \Pi_E(x)$,

(ii) if $\langle y-x, \xi\rangle < 0$ for all $y \in \Pi_E(x)$, then $\xi \in \Xi_x(E_\varepsilon)$.

Proof. (i) If $\xi \in \Xi_x(E_\varepsilon)$, there exists a sequence $(x_n)_{n=1}^\infty \subset E_\varepsilon^c$, for which $x_n \to x$ and $\xi_n := (x_n - x)/\|x_n - x\| \to \xi$ as $n \to \infty$. Assume contrary to the claim tha there exists some $y \in \Pi_E(x)$ with $\langle y-x, \xi\rangle > 0$. Substituting $v_n = \xi_n$ and $v = \xi$ in Lemma 2.2.3

(i) implies the existence of some $N \in \mathbb{N}$ for which $x_n \in \operatorname{int}(E_\varepsilon)$ for all $n \geq N$. This contradicts the claim.

(ii) Write $\xi_n = \frac{1}{n}\xi$, and define $x_n := x + \xi_n$, so that $(x_n - x)/\|x_n - x\| = \xi$ for all $n \in \mathbb{N}$. Substituting $v := \xi$ and $v_n := \xi_n$ in Lemma 2.2.3 (ii) implies the existence of some $N \in \mathbb{N}$ for which $x_n \in E_\varepsilon^c$ for all $n \geq N$. The sequence $(x_n)_{n=N}^\infty$ now defines the outward direction $\xi \in \Xi_x(E_\varepsilon)$. □

Note that assuming the weaker condition $\langle y - x, \xi \rangle \leq 0$ for all contributors $y \in \Pi_E(x)$ is not sufficient in Lemma 2.2.4 (ii). For example, let $E := [2,3] \times \{0,1\}$ and consider the set E_ε with $\varepsilon = 1/2$. Then $x := (3, 1/2) \in \partial E_\varepsilon$ with $\Pi(x) = \{(3,0),(3,1)\}$ and has only one outward direction $\xi = (1,0)$. Here also $\eta := (-1,0)$ satisfies $\langle y - x, \eta \rangle = 0$ for $y \in \{(3,0),(3,1)\}$, and yet $\eta \notin \Xi_x(E_\varepsilon)$. This example illustrates the difference between ∂E_ε and the ε-boundary $\partial E_{<\varepsilon}$ (see Definition (1.2.1)), since here $\partial E_{<\varepsilon} \setminus \partial E_\varepsilon = (2,3) \times \{1/2\} \subset \operatorname{int} E_\varepsilon$. See also [RW10, Example 2.1].

In order to describe the geometry of the sets of outward directions $\Xi_x(E_\varepsilon)$ on the circle S^1, we introduce the concept of a geodesic arc-segment. Intuitively, a geodesic arc-segment is the shortest curve on S^1 that connects two points $v, w \in S^1$.

Definition 2.2.5 (**Geodesic arc-segment**). *Let $v, w \in S^1 \subset \mathbb{R}^2$ and let*

$$[v,w]_{S^1} := \left\{ u \in S^1 \,:\, u = av + bw \ \text{ for some } a, b \geq 0 \right\}. \tag{2.2.1}$$

For $w \neq -v$, the set $[v,w]_{S^1}$ defines a geodesic arc-segment *between v and w. We also define the corresponding* open *geodesic arc-segment $(v,w)_{S^1} \subset S^1$ as*

$$(v,w)_{S^1} := [v,w]_{S^1} \setminus \{v,w\}. \tag{2.2.2}$$

We use the notations $[v,w]_{S^1}$ and $(v,w)_{S^1}$ in accordance with (2.2.1) and (2.2.2) also for the cases $v = w$ and $v = -w$, even though the corresponding sets in these cases are not arc-segments.

Unlike the previous results in this section, we formulate and prove the statements in Proposition 2.2.6 and Lemma 2.2.7 below only for the two-dimensional case. Note also that Proposition 2.2.6 is formulated for a compact set $E \subset \mathbb{R}^2$, but essentially the same proof works for any closed set E due to the local nature of the result.

Proposition 2.2.6 (**Structure of sets of outward directions**). *Let $E \subset \mathbb{R}^2$ be compact and $x \in \partial E_\varepsilon$. Then the set of outward directions $\Xi_x(E_\varepsilon)$ satisfies the following.*

(i) If $x \in \operatorname{Unp}_\varepsilon(E)$, then $\Xi_x(E_\varepsilon) = \{\xi \in S^1 \,:\, \langle y - x, \xi \rangle \leq 0\}$;

(ii) If $x \notin \operatorname{Unp}_\varepsilon(E)$, then $\Xi_x(E_\varepsilon) = [\xi_1, \xi_2]_{S^1}$, where ξ_1, ξ_2 are the only *extremal outward directions at x, possibly satisfying $\xi_1 = \xi_2$.*

Proof. (i) Since $\Pi_E(x) = \{y\}$, Lemma 2.2.4 (ii) implies

$$X := \{\xi \in S^1 \, : \, \langle y - x, \xi \rangle < 0\} \subset \Xi_x(E_\varepsilon).$$

Then $\partial_{S^1} X = \{\xi \in S^1 \, : \, \langle y - x, \xi \rangle = 0\}$ due to continuity of the scalar product, and Lemmas 2.2.1 and 2.2.4 (i) imply $\overline{X} = \Xi_x(E_\varepsilon)$, as claimed.

(ii) Assume then that $\Pi_E(x)$ contains at least two points. We assert that

(a) if $\xi, \eta \subset \Xi_x(E_\varepsilon)$ and $\gamma \in (\xi, \eta)_{S^1}$, then $\langle y - x, \gamma \rangle < 0$ for all $y \in \Pi_E(x)$,

(b) $\xi \in \mathrm{int}_{S^1} \Xi_x(E_\varepsilon)$ if and only if $\langle y - x, \xi \rangle < 0$ for all $y \in \Pi_E(x)$,

(c) $\Xi_x(E_\varepsilon) = [\xi_1, \xi_2]_{S^1}$.

(a) If $\Xi_x(E_\varepsilon) = \{\xi\}$ or $\Xi_x(E_\varepsilon) = \{\xi, -\xi\}$ for some $\xi \in S^1$, the claim is true since $(\xi, \xi)_{S^1} = (\xi, -\xi)_{S^1} = \varnothing$.

Let $\xi, \eta \in \Xi_x(E_\varepsilon)$ with $\eta \notin \{\xi, -\xi\}$ and define a parametrised curve $\gamma : [0, 1] \to S^1$ by

$$\gamma(t) := \frac{t\eta + (1 - t)\xi}{\|t\eta + (1 - t)\xi\|}.$$

Clearly $\gamma(0) = \xi, \gamma(1) = \eta$, and $\gamma((0, 1)) = (\xi, \eta)_{S^1}$. For each $y \in \Pi_E(x)$, consider the scalar product

$$P_y(t) := \langle y - x, t\eta + (1 - t)\xi \rangle = t\langle y - x, \eta \rangle + (1 - t)\langle y - x, \xi \rangle \tag{2.2.3}$$
$$= \langle y - x, \xi \rangle + t\langle y - x, \eta - \xi \rangle. \tag{2.2.4}$$

We show that $P_y(t) < 0$ for every $y \in \Pi_E(x)$ and all $t \in (0, 1)$. We have $P_y(t) \le 0$ for all $t \in [0, 1]$ and all $y \in \Pi_E(x)$, since Lemma 2.2.4 (i) guarantees

$$\langle y - x, \eta \rangle \le 0 \quad \text{and} \quad \langle y - x, \xi \rangle \le 0$$

for all $y \in \Pi_E(x)$. For $t \in (0, 1)$, equation (2.2.3) implies $P_y(t) < 0$ when $\langle y - x, \xi \rangle < 0$. On the other hand, if $\langle y - x, \xi \rangle = 0$, equation (2.2.4) implies

$$t\langle y - x, \eta - \xi \rangle = P_y(t) \le 0.$$

Hence the inequality $P_y(t) < 0$ holds if and only if $\eta \notin \{\xi, -\xi\}$. This shows that $P_y(t) < 0$ for arbitrary $y \in \Pi_E(x)$, whenever $t \in (0, 1)$.

(b) If $\xi \in S^1$ satisfies $\langle y - x, \xi \rangle < 0$ for all $y \in \Pi_E(x)$, then there exists some $n \in \mathbb{N}$ for which $\langle y - x, \xi \rangle < -1/n$ for all $y \in \Pi_E(x)$. To show this, assume to the contrary that for each $n \in \mathbb{N}$ there exists some $y_n \in \Pi_E(x)$ for which

$$\langle y_n - x, \xi \rangle \ge -1/n.$$

Since S^1 is compact and E is closed, there exists a convergent subsequence $(y_{n_k})_{k=1}^{\infty} \subset (y_n)_{n=1}^{\infty}$ with $y_{n_k} \to y^* \in \Pi_E(x)$. On the other hand, due to the continuity of the scalar product we have

$$\langle y^* - x, \xi \rangle = \lim_{k\to\infty} \langle y_{n_k} - x, \xi \rangle \geq 0,$$

which contradicts the assumption that $\langle y - x, \xi \rangle < 0$ for all $y \in \Pi_E(x)$.

Assume now that $\langle y - x, \xi \rangle < 0$ for all $y \in \Pi_E(x)$ and that $n \in \mathbb{N}$ has been chosen so that $\langle y - x, \xi \rangle < -1/n$ for all $y \in \Pi_E(x)$. The continuity of the scalar product implies that for some δ, depending on n, one has $\left\langle y - x, \widehat{\xi} \right\rangle < 0$ for all $\widehat{\xi}$ that satisfy $\left\| \widehat{\xi} - \xi \right\| < \delta$. Hence ξ has an open neighbourhood $B_\delta(\xi)$ satisfying $B_\delta(\xi) \cap S^1 \subset \Xi_x(E_\varepsilon)$, which implies $\xi \in \operatorname{int}_{S^1} \Xi_x(E_\varepsilon)$.

For the other direction, assume $\xi \in \operatorname{int}_{S^1} \Xi_x(E_\varepsilon)$. Then there exist $\eta_1, \eta_2 \in \Xi_x(E_\varepsilon)$, for which $\xi \in (\eta_1, \eta_2)_{S^1} \subset \operatorname{int}_{S^1} \Xi_x(E_\varepsilon)$. Step (a) consequently implies $\langle y - x, \xi \rangle < 0$ for all $y \in \Pi_E(x)$.

(c) It follows from steps (a) and (b) that $\operatorname{int}_{S^1} \Xi_x(E_\varepsilon) = (\xi_1, \xi_2)_{S^1}$. This in turn implies $\Xi_x^{\text{ext}}(E_\varepsilon) = \{\xi_1, \xi_2\}$, when $\operatorname{int}_{S^1} \Xi_x(E_\varepsilon) \neq \varnothing$, and $\Xi_x(E_\varepsilon) = \{\xi\}$ (singleton), when $\xi_1 = \xi_2$. □

Proposition 2.2.6 thus gives the following geometric picture of the set of extremal outward directions. In the case of a sharp singularity (S2) or a chain singularity (S6) (see Figure 1.4 and Definition 4.1.1), the set of extremal outward directions is a singleton $\Xi_x^{\text{ext}}(E_\varepsilon) = \{\xi\}$ for some $\xi \in S^1$. Otherwise $\Xi_x^{\text{ext}}(E_\varepsilon)$ contains two points, which may point directly away from each other or form an acute or obtuse angle.

Lemma 2.2.7 below summarises the limiting behaviour of outward directions ξ_n and contributors y_n of points x_n that appear in convergent sequences on the ε-neighbourhood boundary. In particular, Lemma 2.2.7 (ii)(a) establishes that for each $x \in \partial E_\varepsilon$ the set of tangent vectors $T_x(E_\varepsilon)$ is a subset of the set $\Xi_x^{\text{ext}}(E_\varepsilon)$ of extremal outward directions. According to Proposition 2.2.8 these sets in fact coincide for all $x \in \partial E_\varepsilon$.

Lemma 2.2.7 (Orientation in converging sequences of boundary points). *Let $E \subset \mathbb{R}^2$ be compact and let $x \in \partial E_\varepsilon$. Furthermore, let $(x_n)_{n=1}^{\infty}$ be a sequence on ∂E_ε with $x_n \to x$ and define $\xi_n := (x_n - x)/\left\|x_n - x\right\|$ for all $n \in \mathbb{N}$. Then the following statements hold true:*

(i) The sequence $(\xi_n)_{n=1}^{\infty}$ can be split into two disjoint, convergent subsequences $(\xi_{i,k})_{k=1}^{\infty}$, where $i \in \{1, 2\}$ and $\xi^{(i)} := \lim_{k\to\infty} \xi_{i,k} \in \Xi_x^{\text{ext}}(E_\varepsilon)$.

(ii) If the limit $\xi := \lim_{n\to\infty} \xi_n \in S^1$ exists, then

(a) every sequence $(y_n)_{n=1}^{\infty}$ in E with $y_n \in \Pi(x_n)$ for all $n \in \mathbb{N}$ has a convergent subsequence $(y_{n_k})_{k=1}^{\infty}$ for which $y := \lim_{k\to\infty} y_{n_k} \in \Pi_E^{\text{ext}}(x)$. Furthermore $\langle y - x, \xi \rangle = 0$ and consequently $\xi \in \Xi_x^{\text{ext}}(E_\varepsilon)$.

(b) every sequence $(\eta_n)_{n=1}^{\infty}$ *in* S^1 *with* $\eta_n \in \Xi^{\text{ext}}_{x_n}(E_\varepsilon)$ *for all* $n \in \mathbb{N}$ *satisfies*

$$\lim_{n\to\infty} \|\langle \eta_n, \xi\rangle\| = 1.$$

Proof. (ii)(a) Due to Lemma 2.2.2, there exists some $y \in \Pi_E(x)$ and a convergent subsequence $(y_{n_k})_{k=1}^{\infty} \subset (y_n)_{n=1}^{\infty}$, for which $y_{n_k} \to y$. We break the proof into three steps.

Step 1. $\langle y - x, \xi\rangle \leq 0$: Assume contrary to the claim that $\langle y - x, \xi\rangle > 0$. Then substituting $x_k := x_{n_k}$ and $v := \xi$ in Lemma 2.2.3 (1) implies that for some $K \in \mathbb{N}$ we have $x_{n_k} \in \operatorname{int}(E_\varepsilon)$ for all $k \geq K$. This contradicts the assumption $x_n \in \partial E_\varepsilon$ for all $n \in \mathbb{N}$.

Step 2. $\langle y - x, \xi\rangle \geq 0$: Assume contrary to the claim that $\langle y - x, \xi\rangle < 0$. The continuity of the scalar product then implies that there exists some $K \in \mathbb{N}$ for which $\|x_{n_k} - x\| + 2\langle y_{n_k} - x_{n_k}, \xi_{n_k}\rangle < 0$ for all $k \geq K$. Applying the triangle-inequality with respect to the points x_{n_k} yields

$$\|y_{n_k} - x\|^2 = \varepsilon^2 + \|x_{n_k} - x\| \left(\|x_{n_k} - x\| + 2\langle y_{n_k} - x_{n_k}, \xi_{n_k}\rangle\right) < \varepsilon^2$$

for all $k \geq K$. This implies the contradiction $x \in \operatorname{int}(E_\varepsilon)$.

Step 3. $\xi \in \Xi^{\text{ext}}_x(E_\varepsilon)$: For each $n \in \mathbb{N}$, write $r_n := \|x_n - x\|$. Since $(x_n)_{n=1}^{\infty} \subset \partial E_\varepsilon$ there exists for each $n \in \mathbb{N}$ some $z_n \in B_{r_n^2}(x_n) \cap E_\varepsilon^c$. Then $\xi_n^z := (z_n - x)/\|z_n - x\| \to \xi$ so that $\xi \in \Xi_x(E_\varepsilon)$. Steps *1.* and *2.* together imply $\langle y - x, \xi\rangle = 0$ so that $\xi \in \Xi^{\text{ext}}_x(E_\varepsilon)$ (see Definition 2.1.4). This concludes the proof of (ii)(a).

(ii)(b) Assume contrary to the claim that there exists some $\delta > 0$ and a subsequence $(\eta_{n_k})_{k=1}^{\infty}$, for which $\|\langle \eta_{n_k}, \xi\rangle\| < 1 - \delta$ for all $k \in \mathbb{N}$. This implies that if $y_k \in \Pi^{\text{ext}}_E(x_{n_k})$ with $\langle y_k - x_{n_k}, \eta_{n_k}\rangle = 0$, there exists some $r > 0$, depending on δ, for which

$$\|\langle y_k - x_{n_k}, \xi\rangle\| \geq r \tag{2.2.5}$$

for infinitely many $k \in \mathbb{N}$. On the other hand property (ii)(a) implies the existence of a subsequence $(y_{k_j})_{j=1}^{\infty}$ for which the limit $y := \lim_{j\to\infty} y_{k_j} \in \Pi^{\text{ext}}_E(x)$ exists and satisfies $\langle y - x, \xi\rangle = 0$. Inequality (2.2.5) now leads to the contradiction

$$\|\langle y - x, \xi\rangle\| = \lim_{j\to\infty} \left\|\left\langle y_{k_j} - x_{n_{k_j}}, \xi\right\rangle\right\| \geq r > 0.$$

(i) According to (ii)(a) every convergent subsequence $(\xi_{n_k})_{k=1}^{\infty}$ satisfies $\xi_{n_k} \to \xi \in \Xi^{\text{ext}}_x(E_\varepsilon)$ as $k \to \infty$. For $\Xi^{\text{ext}}_x(E_\varepsilon) = \{\xi\}$ (a singleton) this implies $\xi_n \to \xi$ and the claim follows. In case $\Xi^{\text{ext}}_x(E_\varepsilon) = \left\{\xi^{(1)}, \xi^{(2)}\right\}$ for some $\xi^{(1)} \neq \xi^{(2)}$, write $r = \left\|\xi^{(1)}, \xi^{(2)}\right\|$. The compactness of ∂E_ε implies that there exists some $N \in \mathbb{N}$, for which

$$\xi_n \in B_{r/3}\left(\xi^{(1)}\right) \cup B_{r/3}\left(\xi^{(2)}\right)$$

for all $n \geq N$. For each $i \in \{1,2\}$, define $N_i := \left\{n \in \mathbb{N} : \xi_n \in B_{r/3}\left(\xi^{(i)}\right)\right\}$. If N_i is finite for some $i \in \{1,2\}$, we have $\lim_{n\to\infty} \xi_n = \xi^{(j)}$ for $j \in \{1,2\} \setminus \{i\}$ and the claim follows. Otherwise the sequences $(\xi_{1,k})_{k\in N_1}$ and $(\xi_{2,k})_{k\in\mathbb{N}\setminus N_1}$ are disjoint and satisfy $\lim_{k\to\infty} \xi_{i,k} = \xi^{(i)}$ for $i \in \{1,2\}$. □

Proposition 2.2.8 (Extremal outward directions coincide with tangents). *Let $E \subset \mathbb{R}^2$ be compact and let $x \in \partial E_\varepsilon$. Then $T_x(E_\varepsilon) = \Xi_x^{\text{ext}}(E_\varepsilon)$.*

Proof. Lemma 2.2.7 (ii)(a) implies $T_x(E_\varepsilon) \subset \Xi_x^{\text{ext}}(E_\varepsilon)$, so we are left with proving the other direction. Assume $\xi \in \Xi_x^{\text{ext}}(E_\varepsilon)$. Then there exists a sequence $(x_n)_{n=1}^\infty \subset E_\varepsilon^c$, for which

$$\frac{\varphi_n}{\|\varphi_n\|} := \frac{x_n - x}{\|x_n - x\|} \longrightarrow \xi,$$

as $n \to \infty$. Since ξ is an extremal outward direction, there exists some extremal contributor $y \in \Pi_E^{\text{ext}}(x)$ for which $\langle y - x, \xi\rangle = 0$. Let $\widehat{y} := (y-x)/\|y-x\| = (y-x)/\varepsilon$ and define $H_n := h_n\xi + r_n\widehat{y}$, where

$$h_n := \|\varphi_n\| \sqrt{1 - \frac{\|\varphi_n\|^2}{4\varepsilon^2}}, \quad \text{and} \quad r_n := \frac{\|\varphi_n\|^2}{2\varepsilon}.$$

It follows from the orthogonality of ξ and $\widehat{y}$ that $\|H_n\| = \|\varphi_n\|$. Furthermore $x + H_n \in E_\varepsilon$ for all $n \in \mathbb{N}$, since $\|(x + H_n) - y\| = \varepsilon$. See figure 2.2.

Consider now the $\|\varphi_n\|$-radius circle $\partial B_{\|\varphi_n\|}(x)$ centered at x. The geodesic arc-segment (shortest path) on this circle that connects the points $x + H_n \in E_\varepsilon$ and $x + \varphi_n = x_n \in E_\varepsilon^c$ must necessarily contain a boundary point $z_n \in \partial B_{\|\varphi_n\|}(x) \cap \partial E_\varepsilon$.

Let $\delta > 0$. Since $x_n \to x$ and $\varphi_n/\|\varphi_n\| \to \xi$, there exists some $N \in \mathbb{N}$ for which

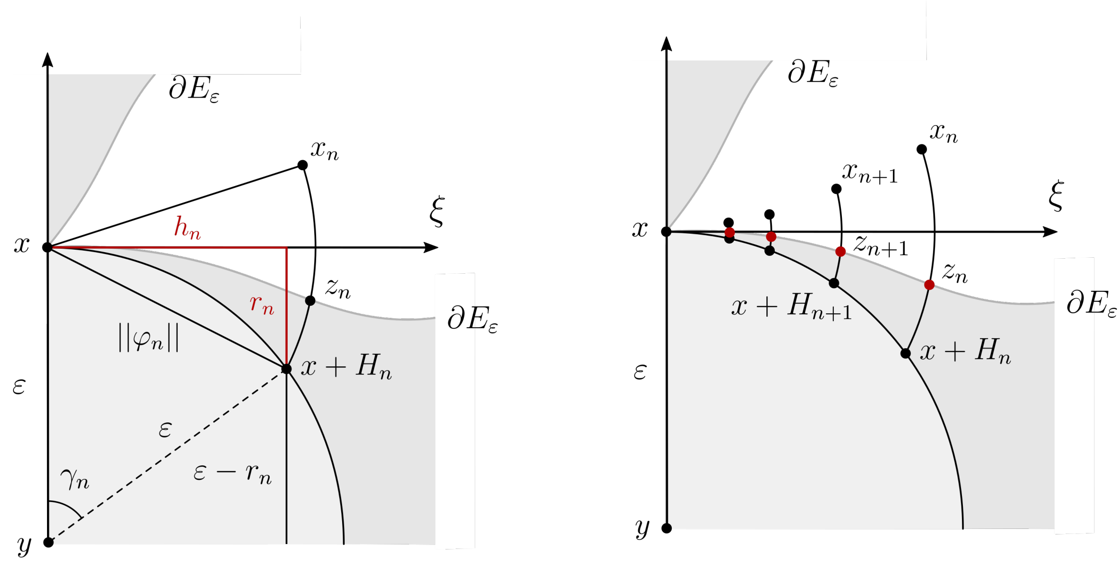
$$\|\varphi_n\| \leq \frac{\delta}{3} \quad \text{and} \quad \left\|\frac{\varphi_n}{\|\varphi_n\|} - \xi\right\| \leq \frac{\delta}{3} \tag{2.2.6}$$

whenever $n \geq N$. Since

$$\left\|\frac{H_n}{\|H_n\|} - \xi\right\|^2 = 2 - \sqrt{4 - \frac{\|\varphi_n\|^2}{\varepsilon^2}} \longrightarrow 0$$

as $\varphi_n \to 0$, we can choose some $N^* \geq N$ for which the inequality $\|H_n/\|H_n\| - \xi\| \leq \delta/3$ as well as the estimates (2.2.6) hold for all $n \geq N^*$. It follows from the definition of the points z_n that $\|z_n - (x + \varphi_n)\| \leq \|\varphi_n - H_n\|$ for all $n \in \mathbb{N}$. This allows us to obtain the estimate

$$\left\|\frac{z_n - x}{\|z_n - x\|} - \xi\right\| \leq \left\|\frac{H_n}{\|H_n\|} - \xi\right\| + 2\left\|\frac{\varphi_n}{\|\varphi_n\|} - \xi\right\| \leq \delta,$$



(a) The geometric picture. Here $r_n = \varepsilon - \varepsilon\cos(\arcsin(\frac{h_n}{\varepsilon})) = \varepsilon - \sqrt{\varepsilon^2 - h_n^2}$, and one can solve for these values so that $\|H_n\| = \|\varphi_n\|$ is satisfied.

(b) For each $n \in \mathbb{N}$, the point $z_n \in \partial E_\varepsilon$ lies on a geodesic arc-segment on $\partial B_{\|\varphi_n\|}(x)$, which connects the points x_n and $x + H_n$.

Figure 2.2. Construction of the sequence $(z_n)_{n=1}^{\infty}$. Here the point x is depicted as a wedge, but the procedure is the same for other types of boundary points.

which is valid for all $n \geq N^*$. Since also $0 < \|z_n - x\| = \|\varphi_n\| < \delta$ for all $n \geq N^*$, we see that ξ fulfils the requirements of Definition 2.1.1, so that $\xi \in T_x(E_\varepsilon)$. $\square$

Chapter 3

Local Structure of the Boundary

In this chapter we utilise the results obtained in Chapter 2 regarding outward directions and contributors in order to analyse the local properties of the boundary ∂E_ε. We begin by proving a local contribution property, Proposition 3.1.1, which intuitively states that in order to describe the local geometry of ∂E_ε near a boundary point $x \in \partial E_\varepsilon$ it suffices to consider the geometry of ∂E around the extremal contributors $y \in \Pi_E^{\text{ext}}(x)$.

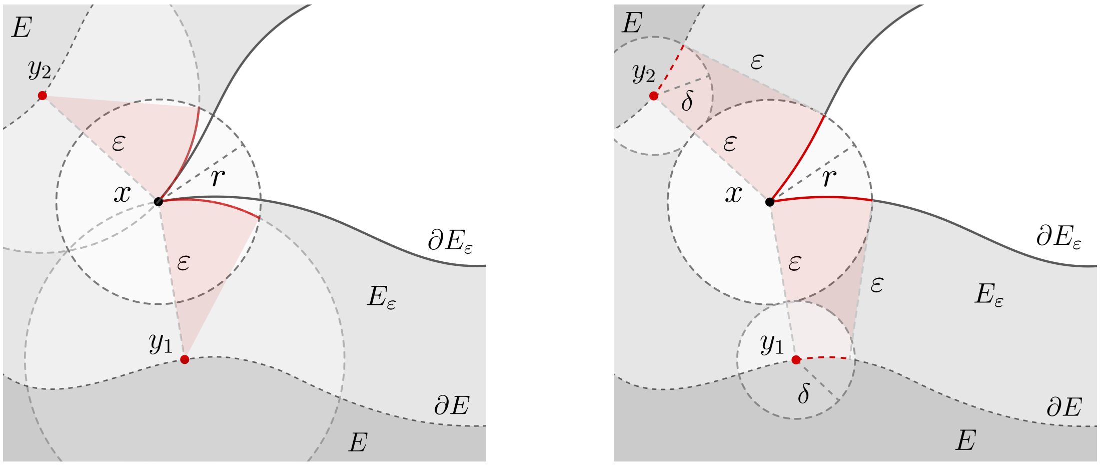
In Section 3.2 we develop a method for approximating the set E_ε with finite collections of balls $\{B_\varepsilon(d_n) : d_n \in D^n\}$ that correspond to certain finite subsets $D^n \subset E$. Combining this idea with Proposition 3.1.1 we proceed to show in Proposition 3.2.4 that local representations for the boundary ∂E_ε may be obtained using finite collections of curves that can be represented as graphs of continuous functions on a compact interval. As the first application of Proposition 3.2.4 we show in Lemma 3.3.2 that for every $x \in \mathrm{Unp}_\varepsilon(E)$ and every wedge (see Definition 4.1.1 and Figure 1.4) there exists a unique connected component V of the complement E_ε^c for which $x \in \partial V$.

3.1 Local Contribution

Intuitively, one can give a crude approximation for the boundary around each $x \in \partial E_\varepsilon$ by considering the boundaries $\partial B_\varepsilon(y)$ centered at the contributors $y \in \Pi_E(x)$, and zooming in on a suitably small neighbourhood $B_r(x)$, in which

$$\partial E_\varepsilon \cap B_r(x) \approx \partial \left(\bigcup_{y \in \Pi_E(x)} B_\varepsilon(y) \right) \cap B_r(x). \tag{3.1.1}$$

For now, the approximation in (3.1.1) can be interpreted to mean that the sets are close to each other in terms of Hausdorff distance, see Definition 3.2.1. However, this measure of distance ignores any geometric or regularity properties of the two sets. We will see shortly that (3.1.1) can be given a much more precise geometric interpretation in terms of graphs



(a) The boundaries $\partial B_\varepsilon(y_1)$ and $\partial B_\varepsilon(y_2)$ give an approximation for the local geometry of the boundary ∂E_ε inside the ball $B_r(x)$.

(b) For an exact representation, one needs to consider all the contributors within some radius $\delta > 0$ from the extremal contributors y_1 and y_2.

Figure 3.1. Idea of local contribution. The points y_1, y_2 are the extremal contributors of the wedge $x \in \partial E_\varepsilon$.

of Lipschitz-continuous functions, see Proposition 3.2.4. Inside any neighbourhood $B_r(x)$, the geometry of the ε-neighbourhood E_ε is clearly not defined solely by the positions of the contributors $y \in \Pi_E(x)$ (see Figures 3.1 and 3.2). Hence one needs to consider at least all the contributors in some neighbourhood of the set of extremal contributors $\Pi_E^{\text{ext}}(x)$. Proposition 3.1.1 below confirms that this is indeed sufficient.

We introduce here the following notation for open x-centered half-balls oriented in the direction of some $v \in S^1$:

$$U_r(x, v) := \{z \in B_r(x) \ : \ \langle z - x, v\rangle > 0\}. \tag{3.1.2}$$

Proposition 3.1.1 (**Local contribution**)**.** *Let $E \subset \mathbb{R}^2$ and $x \in \partial E_\varepsilon$ with $\Xi_x^{\text{ext}}(E_\varepsilon) = \{\xi_1, \xi_2\}$, where we allow $\xi_1 = \xi_2$. Then for all $\delta > 0$ there exists some $r > 0$ such that given $z \in B_r(x)$, we have $z \in E_\varepsilon$ if and only if either*

$$z \notin \overline{U_r(x, \xi_1)} \cup \overline{U_r(x, \xi_2)}, \text{ or} \tag{3.1.3}$$
$$z \in \overline{B_\varepsilon\Big(E \cap B_\delta(\Pi_E^{\text{ext}}(x))\Big)}. \tag{3.1.4}$$

Proof. Assume to the contrary that there exists some $\delta > 0$, for which the claim fails. This means that for all $r > 0$ there exists some $z \in B_r(x)$, for which either

1. $z \in E_\varepsilon$ and both (3.1.3) and (3.1.4) fail, or
2. $z \notin E_\varepsilon$ and one of the conditions (3.1.3) or (3.1.4) holds true.

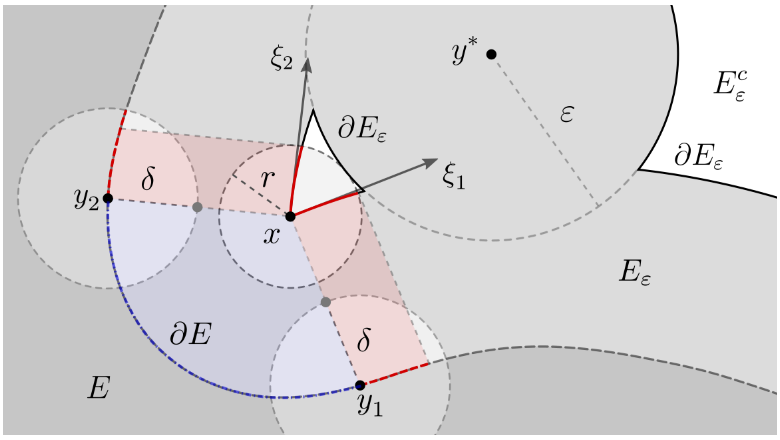


Figure 3.2. Schematic illustration of Proposition 3.1.1. For a sufficiently small $r > 0$, the balls $B_\delta(y_1)$ and $B_\delta(y_2)$ contain all the contributors $y \in \Pi_E(z)$ (red dotted lines) of those boundary points $z \in \partial E_\varepsilon$ that lie inside the ball $B_r(x)$ (red solid lines). For $\rho > r$ the boundary segment generated by the point $y^* \in E$ would lie inside the larger ball $B_\rho(x)$ and would need to be accounted for separately. Note that the geometry of the boundary ∂E_ε inside $B_r(x)$ is not affected by whether or not the points y on the blue dotted line satisfy $y \in E$.

This implies that there exists a sequence $(z_k)_{k=1}^\infty$ with $z_k \in B_{1/k}(x)$, for which either condition (1) or (2) holds true for $z = z_k$ for infinitely many indices $k \in \mathbb{N}$. A corresponding subsequence $(z_n)_{n=1}^\infty$ then satisfies $z_n \to x$ as $n \to \infty$ and either

(a) for all $n \in \mathbb{N}$ $z_n \in E_\varepsilon$ while conditions (3.1.3) and (3.1.4) both fail for $z = z_n$, or

(b) for all $n \in \mathbb{N}$ $z_n \notin E_\varepsilon$ while either condition (3.1.3) or (3.1.4) holds true for $z = z_n$.

We proceed by showing that both of these statements lead to a contradiction. In both cases one can assume, without loss of generality, the existence of the limit

$$v_z := \lim_{n\to\infty} (z_n - x)/\left\|z_n - x\right\|.$$

Assume first that (a) holds true. This means that

$$z_n \in E_\varepsilon \cap \left(\overline{U_{1/n}(x,\xi_1)} \cup \overline{U_{1/n}(x,\xi_2)}\right) \quad \text{and} \quad z_n \notin \overline{B_\varepsilon\left(E \cap B_\delta(\Pi_E^{\mathrm{ext}}(x))\right)} \tag{3.1.5}$$

for all $n \in \mathbb{N}$. Since $z_n \in E_\varepsilon$ for all $n \in \mathbb{N}$, one can choose a sequence $(y_n)_{n=1}^\infty \subset E$ with $z_n \in \overline{B_\varepsilon(y_n)}$. In addition, since $z_n \to x$, Lemma 2.2.2 guarantees the existence of a convergent subsequence $(y_{n_k})_{k=1}^\infty$ with the limit $\widehat{y} := \lim_{k\to\infty} y_{n_k} \in \Pi_E(x)$. Note that it follows from (3.1.5) that $y_n \notin B_\delta(\Pi_E^{\mathrm{ext}}(x))$ for all $n \in \mathbb{N}$, which implies $\widehat{y} \notin \Pi_E^{\mathrm{ext}}(x)$, and consequently $\Pi_E(x) \setminus \Pi_E^{\mathrm{ext}}(x) \neq \varnothing$. This rules out the possibility $\xi_1 = -\xi_2$ for the extremal outward directions $\xi_1, \xi_2 \in \Xi_x^{\mathrm{ext}}(E_\varepsilon)$, and therefore

$$\langle \xi_1, \xi_2 \rangle > -1. \tag{3.1.6}$$

The relations (3.1.5) guarantee that $z_{n_k} \notin B_\varepsilon(\Pi_E^{\mathrm{ext}}(x))$, which together with Lemma 2.2.3 (i) implies $\langle y - x, v_z\rangle \leq 0$ for all extremal contributors $y \in \Pi_E^{\mathrm{ext}}(x)$. Combined with the assumption that

$$z_{n_k} \in \overline{U_{1/k}(x,\xi_1)} \cup \overline{U_{1/k}(x,\xi_2)}$$

for all $k \in \mathbb{N}$, this implies that $v_z \in [\xi_1,\xi_2]_{S^1} = \Xi_x(E_\varepsilon)$ (see Proposition 2.2.6), where $[\xi_1,\xi_2]_{S^1}$ is a geodesic arc-segment due to (3.1.6). It follows that $\langle \widehat{y} - x, v_z\rangle < 0$, since $\widehat{y} \notin \Pi_E^{\mathrm{ext}}(x)$. But now a computation analogous to that presented in the proof of Lemma 2.2.3 (ii) leads to the contradiction $\|\widehat{y} - x\| < \varepsilon$.

Assume then that (b) holds true. Now $z_n \notin E_\varepsilon$ for all $n \in \mathbb{N}$ so that $v_z \in \Xi_x(E_\varepsilon)$. In addition, given that either (3.1.3) or (3.1.4) is satisfied for each z_n and (3.1.4) implies $z_n \in E_\varepsilon$, condition (3.1.3) necessarily holds true for all z_n. Then $\xi_1 \neq -\xi_2$, since $\xi_1 = -\xi_2$ leads to the contradiction

$$z_n \in B_{1/n}(x) \setminus \bigcup_{i\in\{1,2\}} \overline{U_{1/n}(x,\xi_i)} = B_{1/n}(x) \setminus B_{1/n}(x) = \varnothing.$$

On the other hand, if $\xi_1 \neq -\xi_2$, Proposition 2.2.6 states that $v_z \in \Xi_x(E_\varepsilon)$ can be written as a convex combination $v_z = a\xi_1 + b\xi_2$. Note that at least one of the coefficients a, b must be strictly positive, since $v_z \in S^1$. However, (3.1.3) implies $\langle z_n - x, \xi_i\rangle < 0$ for $i \in \{1,2\}$ and all $n \in \mathbb{N}$, which leads to the contradiction $\langle v_z, \xi_i\rangle \leq 0$ for $i \in \{1,2\}$. $\square$

3.2 Approximating the Boundary with Continuous Graphs

In order to study the properties of the boundary ∂E_ε, we develop a finite approximation scheme as a technical aid. The idea is to generate an expanding sequence $(D^n)_{n=1}^\infty$ of finite subsets $D^n \subset E$ and consider their ε-neighbourhoods $B_\varepsilon(D^n)$, whose boundaries approximate the actual boundary ∂E_ε uniformly with respect to Hausdorff distance.

Definition 3.2.1 (Hausdorff distance). *The* Hausdorff distance *between* $X, Y \subset \mathbb{R}^d$ *is*

$$\mathrm{dist}_H(X,Y) := \inf\Big\{\delta > 0 \,:\, X \subset Y_\delta \text{ and } Y \subset X_\delta\Big\},$$

where $X_\delta := \overline{\bigcup_{x\in X} B_\delta(x)}$.

Let $E \subset \mathbb{R}^2$ be compact, $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ with $n_0 > \frac{\ln(4/\varepsilon)}{\ln 2}$. Consider for natural numbers $n > n_0$ the partitions of $\mathbb{R}^2$ into squares

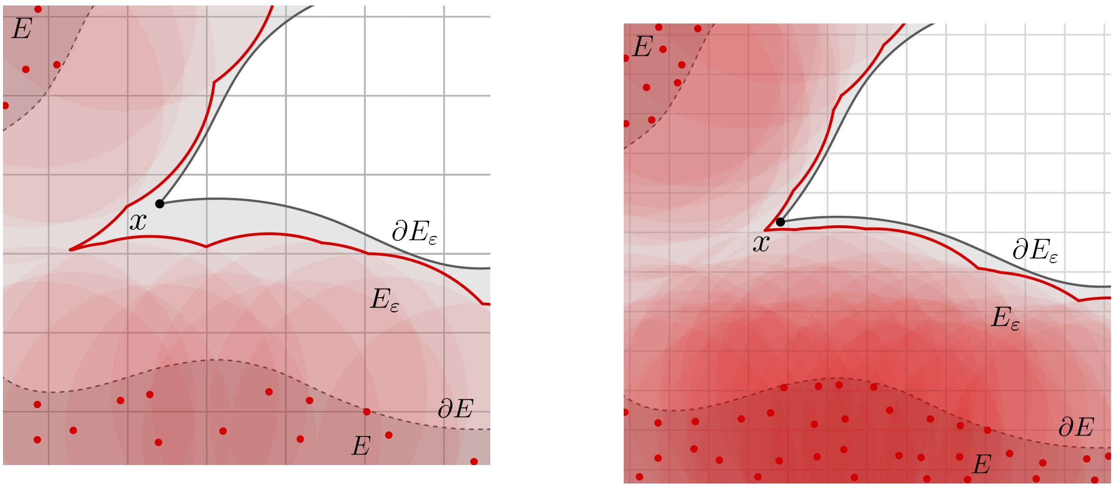
$$\mathcal{C}^n := \left\{ C_{k,\ell}^n := \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \times \left[\frac{\ell}{2^n}, \frac{\ell+1}{2^n}\right] : k, \ell \in \mathbb{Z} \right\}.$$

Due to the Axiom of Choice, there exists a non-decreasing sequence $D^{n_0} \subset D^{n_0+1} \subset \dots$ of subsets of D^n of $\mathbb{R}^2$ such that

$$D^n \cap C^n_{k,\ell} = \begin{cases} \left\{d^n_{k,\ell}\right\} & \text{if } E \cap C^n_{k,\ell} \neq \emptyset, \\ \emptyset & \text{if } E \cap C^n_{k,\ell} = \emptyset, \end{cases}$$

where the points $d^n_{k,\ell}$ are chosen arbitrarily from $E \cap C^n_{k,\ell}$. It is easy to verify that $E_\varepsilon = \bigcup_{n \geq n_0} B_\varepsilon(D^n)$ and that the approximations converge to E_ε in Hausdorff distance,

$$\lim_{n\to\infty} \operatorname{dist}_H \Big(E_\varepsilon, B_\varepsilon(D^n) \Big) = 0 \,. \tag{3.2.1}$$



(a) The boundaries $\partial B_\varepsilon(d^n_{k,\ell})$ for $d^n_{k,\ell} \in D^n$ (red dots) provide an approximation (red curve) for ∂E_ε.

(b) For D^{n+1} the approximation improves, and as $n \to \infty$, it converges uniformly to ∂E_ε.

Figure 3.3. Schematic illustration of two consecutive finite approximating sets D^n and D^{n+1} near a wedge x.

Definition 3.2.2 (Finite approximating sets). *Let $E \subset \mathbb{R}^2$ be compact and let $(D^n)_{n=1}^\infty$ be a sequence of subsets $D^n \subset E$ as described above. Then the sets D^n are called* finite approximating sets *for the set E.*

We now have the necessary ingredients in place for defining what we call *local boundary representations* near each boundary point $x \in \partial E_\varepsilon$. The local contribution property, Proposition 3.1.1, implies that in order to describe the boundary ∂E_ε near x, one only needs to consider contributors $y \in E$ around the extremal contributors $y \in \Pi_E^{\mathrm{ext}}(x)$. The extremal outward directions $\xi \in \Xi_x^{\mathrm{ext}}(E_\varepsilon)$ together with their corresponding extremal contributors form *extremal pairs* (ξ, y) that represent coordinate axes adapted to the orientation of the boundary near x.

Definition 3.2.3 (Extremal pairs). *Let $E \subset \mathbb{R}^d$ be closed, let $x \in \partial E_\varepsilon$ and denote by*

$\Xi_x^{\text{ext}}(E_\varepsilon)$ *and* $\Pi_E^{\text{ext}}(x)$ *the sets of extremal outward directions and extremal contributors, respectively. We define the set of* extremal pairs *at* x *as the collection*

$$\mathcal{P}_x^{\text{ext}}(E_\varepsilon) := \Big\{(\xi, y) \in \Xi_x^{\text{ext}}(E_\varepsilon) \times \Pi_E^{\text{ext}}(x) \,:\, \langle y - x, \xi\rangle = 0\Big\}.$$

One can thus choose a finite approximating sequence $D^n \subset E$, interpret the boundaries $\partial B_\varepsilon(D^n)$ near x as graphs of continuous functions f_n in the coordinate system $(\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$ and obtain, as a uniform limit, a continuous function $f = \lim_{n\to\infty} f_n$ whose graph serves as a representation of a part of the boundary ∂E_ε near x. Using this construction for each extremal pair $(\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$ one obtains a finite set of continuous graphs that give a complete representation of the boundary near x.

We note that the proof of Proposition 3.2.4 below makes use of Lemma 3.3.1 which we have placed in the subsequent Section 3.3, together with other results on the geometry of the complement E_ε^c.

Proposition 3.2.4 (Local boundary representation). *Let* $E \subset \mathbb{R}^2$ *be closed and let* $x \in \partial E_\varepsilon$. *For each extremal pair* $(\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$ *there exists a continuous function* $f^{\xi,y} : [0, \varepsilon/2] \to \mathbb{R}$ *and a corresponding function* $g_{\xi,y} : [0, \varepsilon/2] \to \mathbb{R}^2$, *given by*

$$g_{\xi,y}(s) := x + s\xi + f^{\xi,y}(s)(x - y), \tag{3.2.2}$$

so that the collection $\mathcal{G}(x) := \{g_{\xi,y} \,:\, (\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)\}$ *satisfies*

$$\partial E_\varepsilon \cap \overline{B_r(x)} = \bigcup_{(\xi,y)\in\mathcal{P}_x^{\text{ext}}(E_\varepsilon)} g_{\xi,y}\left(A_{\xi,y}\right) \tag{3.2.3}$$

for some $r > 0$ *and some closed* $A_{\xi,y} \subset [0, \varepsilon/2]$. *We call the collection* $\mathcal{G}(x)$ *a* local boundary representation (of radius r) *at* x. *For each extremal pair* $(\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$ *the corresponding subset* $A_{\xi,y} \subset [0, \varepsilon/2]$ *is either*

(a) an interval $[0, s_{\xi,y}]$ *for some* $0 < s_{\xi,y} \leq \varepsilon/2$, *or*

(b) a closed set whose complement in $[0, \varepsilon/2]$ *contains a sequence of disjoint open intervals with* 0 *as an accumulation point.*

For wedges (type S1) and $x \in \mathrm{Unp}_\varepsilon(E)$, *case (a) above holds true for all* $(\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$.

Proof. Define for each extremal pair $(\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$ the sets

$$\begin{aligned} C_{\xi,y} &:= \{x + s\xi + t(x - y) \,:\, (s, t) \in [-\varepsilon, \varepsilon] \times (-\infty, -\varepsilon/2]\}, \\ E_{\xi,y} &:= E \cap C_{\xi,y}. \end{aligned}$$

Consider a sequence $(D_n)_{n=1}^\infty$ of finite approximating sets for the set E (see Definition 3.2.2), and define $D_n^{\xi,y} := E_{\xi,y} \cap D_n$ for each $n \in \mathbb{N}$. Using the sets $D_n^{\xi,y}$ we define a sequence of functions $f_n^{\xi,y} : [-\varepsilon/2, \varepsilon/2] \to \mathbb{R}$ by setting

$$f_n^{\xi,y}(s) := \max\left\{t \in \mathbb{R} \,:\, \operatorname{dist}\left(x + s\xi + t(x-y), D_n^{\xi,y}\right) \le \varepsilon\right\}, \tag{3.2.4}$$

where dist $\left(\cdot, D_n^{\xi,y}\right)$ denotes the Euclidean distance from the set $D_n^{\xi,y}$. The maximum in (3.2.4) exists for all $s \in [-\varepsilon/2, \varepsilon/2]$ due to the compactness of $\overline{B_\varepsilon\left(D_n^{\xi,y}\right)}$, which also implies that the functions $f_n^{\xi,y}$ are bounded for all $n \in \mathbb{N}$. In addition, it follows from the definition of the values $f_n^{\xi,y}(s)$ that

$$x + s\xi + f_n^{\xi,y}(s)(x-y) \in \partial\overline{B_\varepsilon\left(D_n^{\xi,y}\right)}$$

for all $s \in [-\varepsilon/2, \varepsilon/2]$. Since for each $n \in \mathbb{N}$ the boundary $\partial\overline{B_\varepsilon\left(D_n^{\xi,y}\right)}$ is composed of a finite collection of arc-segments, the corresponding functions $f_n^{\xi,y}$ are continuous.

It is easy to verify that the boundaries $\partial\overline{B_\varepsilon\left(D_n\right)}$ converge to the boundary ∂E_ε uniformly in Hausdorff distance. Similarly the boundaries $\partial\overline{B_\varepsilon\left(D_n^{\xi,y}\right)}$ converge uniformly to the boundaries $\partial\overline{B_\varepsilon(E_{\xi,y})}$. Hence $\left(f_n^{\xi,y}\right)_{n=1}^{\infty}$ is a uniformly convergent, monotonically increasing sequence of continuous functions on the compact interval $[-\varepsilon/2, \varepsilon/2]$, which implies that the limiting function

$$f^{\xi,y}(s) := \lim_{n\to\infty} f_n^{\xi,y}(s) \tag{3.2.5}$$

is continuous. One can therefore define for each $(\xi, y) \in \mathcal{P}_x^{\mathrm{ext}}(E_\varepsilon)$ a function $g_{\xi,y} : [0, \varepsilon/2] \to \mathbb{R}^2$ by

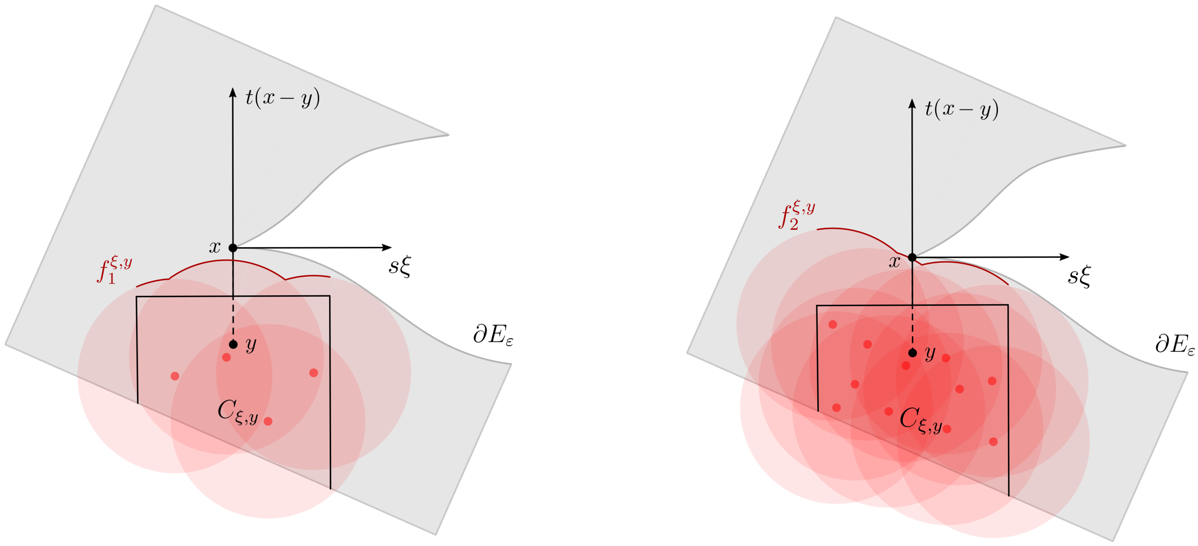
$$g_{\xi,y}(s) := x + s\xi + f^{\xi,y}(s)(x-y). \tag{3.2.6}$$

The remainder of the proof concerns the analysis of the relationship between the graphs $g_{\xi,y}([0, \varepsilon/2])$ and the boundary ∂E_ε near the point x, with the aim of identifying subsets $A_{\xi,y} \subset [0, \varepsilon/2]$ that satisfy (3.2.3).

Substituting $\delta = \varepsilon/2$ in Proposition 3.1.1 guarantees the existence of some $r > 0$, for which $z \in E_\varepsilon \cap B_r(x)$ if and only if either

$$\text{(i) } z \notin \overline{U_r(x,\xi)} \text{ for each } \xi \in \Xi_x^{\mathrm{ext}}(E_\varepsilon), \quad \text{or} \quad \text{(ii) } z \in \overline{B_\varepsilon\left(E \cap B_{\varepsilon/2}(\Pi_E^{\mathrm{ext}}(x))\right)}.$$

According to Proposition 2.2.8, the set of tangential directions $T_x(E_\varepsilon)$ coincides with the set of extremal outward directions $\Xi_x^{\mathrm{ext}}(E_\varepsilon)$. This implies that condition (i) necessarily fails for all boundary points $z \in \partial E_\varepsilon \cap B_r(x)$. Therefore condition (ii) holds true whenever $z \in \partial E_\varepsilon \cap B_r(x)$, in which case $z \in \partial\overline{B_\varepsilon(E_{\xi,y})}$ for some $(\xi, y) \in \mathcal{P}_x^{\mathrm{ext}}(E_\varepsilon)$. Hence there exists some $s_z \in [0, \varepsilon/2]$, for which $z = g_{\xi,y}(s_z)$. In other words, every boundary point $z \in \partial E_\varepsilon$ inside the neighbourhood $B_r(x)$ lies on one of the graphs $g_{\xi,y}([0, \varepsilon/2])$ with $(\xi, y) \in \mathcal{P}_x^{\mathrm{ext}}(E_\varepsilon)$.



(a) $n = 1$ **(b)** $n = 2$

Figure 3.4. First functions in the sequence $\left(f_n^{\xi,y}\right)_{n=1}^{\infty}$. The red dots represent the finite approximating sets $D_1^{\xi,y}$ and $D_2^{\xi,y}$.

It remains to be verified, for each $(\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$, which arguments $s \in [0, \varepsilon/2]$ satisfy $g_{\xi,y}(s) \in \partial E_\varepsilon \cap B_r(x)$. We divide the rest of the proof into two cases, depending on whether or not there exist extremal contributors $y_1, y_2 \in \Pi_E^{\text{ext}}(x)$ for which

$$y_1 - x = -(y_2 - x). \tag{3.2.7}$$

Condition (3.2.7) is schematically illustrated in Figure 4.1 (c) below.

(i) In case (3.2.7) is not satisfied by any $y_1, y_2 \in \Pi_E^{\text{ext}}(x)$, there is either only one contributor so that $x \in \text{Unp}_\varepsilon(E)$, or else x is a wedge (see Definition 4.1.1). In both cases we have $\mathcal{P}_x^{\text{ext}}(E_\varepsilon) = \{(\xi_1, y_1), (\xi_2, y_2)\}$ for some $\xi_1, \xi_2 \in \Xi_{\text{ext}}(x)$ and $y_1, y_2 \in \Pi_E^{\text{ext}}(x)$, where we allow for $y_1 = y_2$ in case $x \in \text{Unp}_\varepsilon(E)$. It follows from Lemma 3.3.1 that for any outward directions $\eta_1, \eta_2 \in (\xi_1, \xi_2)_{S^1}$ there exists a neighbourhood $B_\rho(x) \subset B_r(x)$ in which the graphs of g_{ξ_1,y_1} and g_{ξ_2,y_2} are separated by the cone $V_\rho(x, \eta_1, \eta_2)$. Hence

$$g_{\xi_1,y_1}([0, \varepsilon/2]) \cap g_{\xi_2,y_2}([0, \varepsilon/2]) \cap B_\rho(x) = \varnothing. \tag{3.2.8}$$

For $i \in \{1, 2\}$ we define the upper bounds

$$s_{\xi_i,y_i} := \max\left\{s \in [0, \varepsilon/2] \;:\; g_{\xi_i,y_i}\left([0, s]\right) \in \overline{B_\rho(x)}\right\}$$

and the corresponding sets $A_{\xi_i,y_i} := [0, s_{\xi_i,y_i}]$. It follows then from (3.2.8) and Proposition 3.1.1 that $g_{\xi_i,y_i}(A_{\xi_i,y_i}) \subset \partial E_\varepsilon$ for $i \in \{1,2\}$, which implies

$$\partial E_\varepsilon \cap \overline{B_\rho(x)} = \bigcup_{i\in\{1,2\}} g_{\xi_i,y_i}\left(A_{\xi_i,y_i}\right).$$

(ii) Assume then that (3.2.7) holds true for $y_1, y_2 \in \Pi_E^{\mathrm{ext}}(x)$ and consider some $\xi \in \Pi_E^{\mathrm{ext}}(x)$. In this case the graphs $g_{\xi,y_i}([0,s])$, $i \in \{1,2\}$ may generally intersect for arbitrarily small $s \in (0, \varepsilon/2]$. Due to (3.2.7) one can write for $\{i,j\} = \{1,2\}$

$$\begin{aligned} g_{\xi,y_i}(s) &= x + s\xi + \left(f^{\xi,y_j}(s) - \alpha_\xi(s)\right)(x - y_j) \\ &= g_{\xi,y_j}(s) - \alpha_\xi(s)(x - y_j), \end{aligned}$$

where $\alpha_\xi(s) := f^{\xi,y_1}(s) + f^{\xi,y_2}(s)$. Hence it follows for $i \in \{1,2\}$ from Proposition 3.1.1 and the definitions of the functions $f_n^{\xi,y}$ and $f^{\xi,y}$ (see (3.2.4) and (3.2.5)) that

$$\begin{aligned} &g_{\xi,y_i}(s) \notin \partial E_\varepsilon \text{ whenever } \alpha_\xi(s) > 0, \text{ and} \\ &g_{\xi,y_i}(s) \in \partial E_\varepsilon \text{ whenever } \alpha_\xi(s) < 0. \end{aligned}$$

For $\alpha_\xi(s) = 0$ one has $g_{\xi,y_1}(s) = g_{\xi,y_2}(s) \in \partial E_\varepsilon$ if and only if there exists a sequence $(s_n)_{n=1}^\infty$ with $s_n \to s$, for which $\alpha_\xi(s_n) < 0$ for all $n \in \mathbb{N}$. This follows from the fact that

$$\tau g_{\xi,y_1}(s) + (1-\tau) g_{\xi,y_2}(s) \in E_\varepsilon^c$$

whenever $s \in (0,r)$, $\alpha_\xi(s) < 0$ and $\tau \in (0,1)$. Equation (3.2.3) is therefore satisfied for the neighbourhood $B_r(x)$ and the sets

$$A_{\xi,y_i} := A_\xi := \overline{\{s \in [0,r] \,:\, \alpha_\xi(s) < 0\}} = \{s \in [0,r] \,:\, g_{\xi,y_1}(s), g_{\xi,y_2}(s) \in \partial E_\varepsilon\},$$

where $i \in \{1,2\}$ and $\xi \in \Xi_x^{\mathrm{ext}}(E_\varepsilon)$. Now either $[0, s_\xi] \subset A_\xi$ for some $s_\xi \in (0,r]$ or otherwise $[0,T] \setminus A_\xi \neq \varnothing$ for all $T > 0$. In any case 0 is an accumulation point of A_ξ, since $x \in \partial E_\varepsilon$ and $\xi \in T_x(E_\varepsilon)$ (see Proposition 2.2.8). □

Consider a boundary point $x \in \partial E_\varepsilon$ and the corresponding local boundary representation $\mathcal{G}(x)$ with radius $r > 0$, and let $z \in \partial E_\varepsilon \cap B_r(x)$ with

$$z = g_{\xi,y}(s_z) = x + s_z\xi + f^{\xi,y}(s_z)(x - y)$$

for some $(\xi, y) \in \mathcal{P}_x^{\mathrm{ext}}(E_\varepsilon)$ and $s_z \in [0,r]$. The construction given in Proposition 3.2.4 guarantees that, when written in the (ξ,y)-coordinates, the ξ-coordinate s_w of each contributor $w \in \Pi_E(z)$ satisfies $s_w \in [0, \varepsilon/2]$. This implies a lower and upper bound for the local growth-rate of the function $f^{\xi,y}$. Combining this observation with Proposition 2.2.8

allows us to deduce that the functions $f^{\xi,y}$ in (3.2.2) are in fact Lipschitz continuous on $[0,r]$ for all $(\xi,y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$, with a Lipschitz constant $K = 1/\sqrt{3}\varepsilon$.

Proposition 3.2.5 (Local boundary representation is Lipschitz). *Let $E \subset \mathbb{R}^2$, let $x \in \partial E_\varepsilon$ and let $\mathcal{G}(x)$ be a local boundary representation at x with radius $r > 0$. For each extremal pair $(\xi,y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$, the function $f^{\xi,y}$ in (3.2.2) is $1/\sqrt{3}\varepsilon$-Lipschitz, and the function $g_{\xi,y} \in \mathcal{G}(x)$ is $2/\sqrt{3}$-Lipschitz on the interval $[0,r]$.*

Proof. In order to work in an orthonormal coordinate system, we define for all $(\xi,y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$ the functions $h^{\xi,y} : [0,r] \to \mathbb{R}$ by setting $h^{\xi,y}(s) := \varepsilon f^{\xi,y}(s)$ so that

$$g_{\xi,y}(s) = x + s\xi + h^{\xi,y}(s)\varepsilon^{-1}(x-y)$$

for all $g_{\xi,y} \in \mathcal{G}(x)$. We will show that the functions $h^{\xi,y}$ are $1/\sqrt{3}$-Lipschitz, from which the claim follows.

Assume contrary to the claim that for some $(\xi,y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$ there exist some $p > 1/\sqrt{3}$ and $s,w \in [0,r]$ for which $|h^{\xi,y}(s) - h^{\xi,y}(w)| \geq p|s-w|$. We present the argument for the case $w < s$ and $h^{\xi,y}(s) - h^{\xi,y}(w) \geq p(s-w)$. Assuming $h^{\xi,y}(s) - h^{\xi,y}(w) \leq -p(s-w)$ leads to a contradiction through similar reasoning. Write $m := (s+w)/2$ and define

$$(w_1,s_1) := \begin{cases} (m,s) & \text{if } h^{\xi,y}(s) - h^{\xi,y}(m) > p(s-m), \\ (w,m) & \text{if } h^{\xi,y}(s) - h^{\xi,y}(m) \leq p(s-m) \end{cases}$$

so that $h^{\xi,y}(s_1) - h^{\xi,y}(w_1) \geq p(s_1 - w_1)$. For $n \in \mathbb{N}$ we define inductively $m_n := (s_n + w_n)/2$ and

$$(w_{n+1},s_{n+1}) := \begin{cases} (m_n,s_n) & \text{if } h^{\xi,y}(s_n) - h^{\xi,y}(m_n) > p(s_n-m_n), \\ (w_n,m_n) & \text{if } h^{\xi,y}(s_n) - h^{\xi,y}(m_n) \leq p(s_n-m_n) \end{cases}$$

so that for all $n \in \mathbb{N}$

$$h^{\xi,y}(s_n) - h^{\xi,y}(w_n) \geq p(s_n - w_n). \tag{3.2.9}$$

By construction, the sequence $(s_n)_{n=1}^\infty$ is non-increasing, and the sequence $(w_n)_{n=1}^\infty$ non-decreasing. Since $|s_n - w_n| = 2^{-n}|s-w| \to 0$ as $n \to \infty$ and $s_n > w_n$ for all $n \in \mathbb{N}$, there exists a unique limit $a = \lim_{n\to\infty} s_n = \lim_{n\to\infty} w_n$. From (3.2.9) it follows that

$$p \leq D_n := k_s(n)\frac{h^{\xi,y}(s_n) - h^{\xi,y}(a)}{s_n - a} + k_r(n)\frac{h^{\xi,y}(a) - h^{\xi,y}(w_n)}{a - w_n}, \tag{3.2.10}$$

where the coefficients

$$k_s(n) := \frac{s_n - a}{s_n - w_n}, \quad k_w(n) := \frac{a - w_n}{s_n - w_n}$$

satisfy $k_s(n) + k_w(n) = 1$ for all $n \in \mathbb{N}$. In case $s_N = a$ (resp. $w_N = a$) for some $N \in \mathbb{N}$,

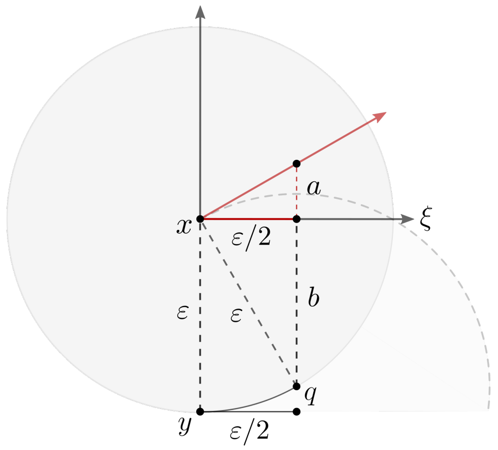


Figure 3.5. The slopes $D^{\pm}h^{\xi,y}(s)$ are bounded from above for all $s \in (0,r)$ by the ratio $D_{\max} = a/(\varepsilon/2)$, which is equal to $(\varepsilon/2)/b = (\varepsilon/2)/\sqrt{\varepsilon^2 - (\varepsilon/2)^2} = 1/\sqrt{3}$. A lower bound is given similarly by $D^{\pm}h^{\xi,y}(s) \geq D_{\min} = -1/\sqrt{3}$.

the coefficient $k_s(n)$ (resp. $k_w(n)$) vanishes for all $n \geq N$. Hence at least one and possibly both of the right and left derivatives of $h^{\xi,y}$ at a are given by

$$D^+h^{\xi,y}(a) := \lim_{n\to\infty} \frac{h^{\xi,y}(s_n) - h^{\xi,y}(a)}{s_n - a}, \quad D^-h^{\xi,y}(a) := \lim_{n\to\infty} \frac{h^{\xi,y}(a) - h^{\xi,y}(w_n)}{a - w_n}$$

and correspond to the tangential directions on ∂E_ε at $x(a) := x + a\xi + h^{\xi,y}(a)\varepsilon^{-1}(x-y)$. According to Proposition 2.2.8 these in turn coincide with the extremal outward directions ξ_a^+ and ξ_a^- at $x(a)$. Since the ξ-coordinates of the corresponding extremal contributors y_a^+, y_a^- lie on the interval $[0, \varepsilon/2]$, the directional derivatives satisfy the upper bound (see Figure 3.5)

$$D^{\pm}h^{\xi,y}(a) \leq D_{\max} := \frac{\frac{\varepsilon}{2}}{\sqrt{\varepsilon^2 - \left(\frac{\varepsilon}{2}\right)^2}} = \frac{1}{\sqrt{3}}.$$

The contradiction $p \leq \lim_{n\to\infty} D_n = 1/\sqrt{3} < p$ follows by taking the limit $n \to \infty$ in (3.2.10).

It follows from the above reasoning that for all $x \in \partial E_\varepsilon$ and each $(\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$ the functions $g_{\xi,y} \in \mathcal{G}(x)$ are $2/\sqrt{3}$-Lipschitz, since for any $w, s \in [0,r]$

$$\|g_{\xi,y}(s) - g_{\xi,y}(w)\| = \sqrt{|s-w|^2 + \varepsilon^2 |f^{\xi,y}(s) - f^{\xi,y}(w)|^2} \leq \frac{2}{\sqrt{3}}|s-w|. \qquad \square$$

3.3 Local Structure of the Complement

In this section we analyse the connectedness of the complement E_ε^c near points $x \in \mathrm{Unp}_\varepsilon(E)$ and wedges. For these points Lemma 3.3.2 guarantees the existence of a unique connected component $V \subset E_\varepsilon^c$ for which $x \in \partial V$, while Proposition 3.3.3 makes the stronger statement that in fact $E_\varepsilon^c \cap B_r(x) = V \cap B_r(x)$ for some connected $V \subset E_\varepsilon^c$ and neighbourhood $B_r(x)$.

Lemma 3.3.1 (Approximations of the outward cone). *Let $x \in \partial E_\varepsilon$, let $\xi_1, \xi_2 \in$*

$\mathrm{int}_{S^1}\Xi_x(E_\varepsilon)$ *and define for each* $r > 0$ *the truncated cone*

$$V_r(x, \xi_1, \xi_2) := \{x + sv \,:\, v \in (\xi_1, \xi_2)_{S^1},\, 0 < s < r\}\,.$$

Then there exists $r > 0$ *for which* $V_r(x, \xi_1, \xi_2) \subset E_\varepsilon^c$.

Proof. Assume to the contrary that for each $n \in \mathbb{N}$ there exists $z_n \in E_\varepsilon \cap V_{1/n}(x, \xi_1, \xi_2)$. Then $z_n \to x$ as $n \to \infty$ and we may assume without loss of generality that

$$\frac{z_n - x}{\|z_n - x\|} \to \xi \in [\xi_1, \xi_2]_{S^1}.$$

Since $\xi_1, \xi_2 \in \mathrm{int}_{S^1}\Xi_x(E_\varepsilon)$, Proposition 2.2.6 implies $\xi \in \mathrm{int}_{S^1}\Xi_x(E_\varepsilon)$, which together with Lemma 2.2.4 (i) gives $\langle y - x, \xi\rangle < 0$ for all $y \in \Pi_E(x)$. It follows then from Lemma 2.2.3 (ii) that there exists $N \in \mathbb{N}$, for which $z_n \in E_\varepsilon^c$ for all $n \geq N$, which contradicts the assumption. □

Lemma 3.3.2 (Unique connected component). *Let* $E \subset \mathbb{R}^2$ *and let* $x \in \partial E_\varepsilon$ *either be a wedge (type S1) or* $x \in \mathrm{Unp}_\varepsilon(E)$*. Then there exists a unique connected component* $V \subset E_\varepsilon^c$*, for which* $x \in \partial V$.

Proof. Since $\mathrm{int}_{S^1}\Xi(x) \neq \varnothing$ whenever x is a wedge or $x \in \mathrm{Unp}_\varepsilon(E)$, there exist outward directions $\xi_1, \xi_2 \in \mathrm{int}_{S^1}\Xi_x(E_\varepsilon)$. Lemma 3.3.1 then guarantees the existence of some $r > 0$ for which the truncated open cone

$$V_r(x, \xi_1, \xi_2) := \{x + sv \,:\, v \in (\xi_1, \xi_2)_{S^1},\, 0 < s < r\} \tag{3.3.1}$$

satisfies $V_r(x, \xi_1, \xi_2) \subset E_\varepsilon^c$. Consequently there exists a connected component $V \subset E_\varepsilon^c$ with $V_r(x, \xi_1, \xi_2) \subset V$ and $x \in \partial V$.

For each $(\xi, y) \in \mathcal{P}_x^{\mathrm{ext}}(E_\varepsilon)$, let $f^{\xi,y} : [0, \varepsilon/2] \to \mathbb{R}$ be the continuous function corresponding to the local boundary representation $\mathcal{G}(x)$ (see equation 3.2.2 in Proposition 3.2.4). To prove uniqueness, assume contrary to the claim that there exists another connected component $W \subset E_\varepsilon^c$ with $W \neq V$ and $x \in \partial W$. Then for at least one $\xi \in \Xi_x^{\mathrm{ext}}(E_\varepsilon)$ there exist arbitrarily small coordinates $s > 0$ for which

$$w(s) := x + s\xi + t_W(s)(x - y) \in W$$

for some $t_W(s) > f^{\xi,y}(s)$. Since the outward directions ξ_1, ξ_2 in the definition of the cone $V_r(x, \xi_1, \xi_2)$ above may be chosen arbitrarily close to the extremal outward directions, it follows that for all sufficiently small s there also exist coordinates $t_V(s) > t_W(s)$ for which

$$v(s) := x + s\xi + t_V(s)(x - y) \in V_r(x, \xi_1, \xi_2) \subset V.$$

It follows then from $t_W(s) > f^{\xi,y}(s)$ and the definition of the local boundary representation that in fact

$$z(s) := x + s\xi + t(x-y) \in E_\varepsilon^c$$

for all $t \in [t_W(s), t_V(s)]$. But this contradicts the assumption that V and W are both connected and $V \cap W = \varnothing$. □

Proposition 3.3.3 (Geometry of the complement). *Let $E \subset \mathbb{R}^2$ and let $x \in \partial E_\varepsilon$ either be a wedge or $x \in \mathrm{Unp}_\varepsilon(E)$. Then there exists some $r > 0$, for which*

$$E_\varepsilon^c \cap B_r(x) = V \cap B_r(x) = \bigcup_{0<\rho<r} x + A(\rho), \tag{3.3.2}$$

where $V \subset E_\varepsilon^c$ is connected and for each $\rho \in (0,r)$ either $A(\rho) = \rho(\alpha_\rho, \beta_\rho)_{S^1}$ or $A(\rho) = \rho\big(S^1 \setminus [\alpha_\rho, \beta_\rho]_{S^1}\big)$ for $\alpha_\rho, \beta_\rho \in S^1$ and $\alpha_\rho \to \xi_\alpha$, $\beta_\rho \to \xi_\beta$, where $\Xi_{ext}(x) = \{\xi_\alpha, \xi_\beta\}$.

Proof. According to Proposition 3.2.4 we may assume that x has a local boundary representation $\mathcal{G}(x)$ with radius $r > 0$ for which the functions $g_{\xi,y} \in \mathcal{G}(x)$ are of the form

$$g_{\xi,y}(s) = x + s\xi + f^{\xi,y}(s)(x-y)$$

with the functions $f^{\xi,y} : [0, \varepsilon/2] \to \mathbb{R}$ continuous. We divide the proof into two parts depending on whether x is a wedge or $x \in \mathrm{Unp}_\varepsilon(E)$.

(i) Assume first that $x \in \mathrm{Unp}_\varepsilon(E)$ with $\Pi_E(x) = \{y\}$. Then $\mathcal{G}(x) = \{g_{\xi_1,y}, g_{\xi_2,y}\}$ where $g_{\xi_i,y} : [0,r] \to \mathbb{R}^2$ and

$$g_{\xi_i,y}(s) = x + s\xi_i + f^{\xi_i,y}(s)(x-y)$$

for $i \in \{1,2\}$. Consider for each $s \in (0,r)$ the distances

$$\mathcal{D}_1(s) := \left\| g_{\xi_1,y}(s) - x \right\|, \quad \mathcal{D}_2(s) := \left\| g_{\xi_2,y}(s) - x \right\|.$$

Due to Proposition 2.2.8 and Lemma 2.2.7 (i) we can assume r to be small enough so that both $\mathcal{D}_1$ and $\mathcal{D}_2$ are strictly increasing on $(0,r)$. One can thus define for each $\rho \in (0,r)$ the points $\alpha_\rho, \beta_\rho \in S^1$ by

$$\alpha_\rho := \rho^{-1}\left(g_{\xi_1,y}\left(\mathcal{D}_1^{-1}(\rho)\right) - x\right), \qquad \beta_\rho := \rho^{-1}\left(g_{\xi_2,y}\left(\mathcal{D}_2^{-1}(\rho)\right) - x\right).$$

According to Lemma 3.3.2 there exists a unique connected component V of E_ε^c for which $x \in \partial V$, and due to Proposition 3.1.1 we can assume r to be sufficiently small so that

$$B_r(x) \cap E \subset \overline{B_\varepsilon\left(E \cap B_{\varepsilon/2}(y)\right)}.$$

This implies that for each $\rho \in (0, r)$ the geodesic curve segment

$$A(\rho) := \rho(\alpha_\rho, (x-y)/\varepsilon)_{S^1} \cup \{\rho(x-y)/\varepsilon\} \cup \rho((x-y)/\varepsilon, \beta_\rho)_{S^1} \subset \rho S^1$$

satisfies

$$x + A(\rho) \subset V \cap B_r(x) \quad \text{and} \quad x + \rho S^1 \setminus A(\rho) \subset \overline{B_\varepsilon\left(E \cap B_{\varepsilon/2}(y)\right)} \subset E_\varepsilon.$$

Hence

$$E_\varepsilon^c \cap B_r(x) = V \cap B_r(x) = \bigcup_{0<\rho<r} x + A(\rho).$$

(ii) Let then x be a wedge. As above, one can assume that the distances

$$\mathcal{D}_1(s) := \|g_{\xi_1,y_1}(s) - x\|, \quad \mathcal{D}_2(s) := \|g_{\xi_2,y_2}(s) - x\|$$

are strictly increasing in $(0, r)$. Furthermore, since $\langle \xi_1, \xi_2 \rangle > -1$, one may define the average $\xi_{\text{av}} := (\xi_1 + \xi_2)/\|\xi_1 + \xi_2\|$. Due to Lemma 2.2.7 (i) and that fact that $\xi_{\text{av}} \notin \Xi_x^{\text{ext}}(E_\varepsilon)$, we may assume r to be small enough so that the boundary segments represented by the functions g_{ξ_i,y_i} are separated by the line segment $\{x + \rho\xi_{\text{av}} : \rho \in (0, r)\}$ in the neighbourhood $B_r(x)$.

For each $\rho \in (0, r)$, we once again define the points $\alpha_\rho, \beta_\rho \in S^1$ by

$$\alpha_\rho := \rho^{-1}\left(g_{\xi_1,y}\left(\mathcal{D}_1^{-1}(\rho)\right) - x\right) \qquad \text{and} \qquad \beta_\rho := \rho^{-1}\left(g_{\xi_2,y}\left(\mathcal{D}_2^{-1}(\rho)\right) - x\right).$$

Analogously to part (i), Proposition 3.1.1 and Lemma 3.3.2 guarantee that for all $\rho \in (0, r)$ the geodesic curve segment

$$\widehat{A}(\rho) := \rho(\alpha_\rho, \xi_{\text{av}})_{S^1} \cup \{\rho\xi_{\text{av}}\} \cup \rho(\xi_{\text{av}}, \beta_\rho)_{S^1} \subset \rho S^1$$

satisfies

$$x + \widehat{A}(\rho) \subset V \cap B_r(x) \quad \text{and} \quad x + \rho S^1 \setminus \widehat{A}(\rho) \subset \overline{B_\varepsilon\left(E \cap B_{\varepsilon/2}(\{y_1, y_2\})\right)} \subset E_\epsilon,$$

where V is the unique connected component of the complement E_ε^c for which $B_r(x) \cap E_\varepsilon^c = B_r(x) \cap V$. Hence

$$E_\varepsilon^c \cap B_r(x) = V \cap B_r(x) = \bigcup_{0<\rho<r} x + \widehat{A}(\rho). \qquad \square$$

Chapter 4

Classification of Boundary Points

In this chapter we present a classification of the boundary points $x \in \partial E_\varepsilon$ based on their local geometric and topological properties. Using the results obtained in Chapters 2 and 3, we prove our first main result, Theorem 1, which states that the classification given in Definition 4.1.1 defines a partition of the boundary ∂E_ε into disjoint subsets.

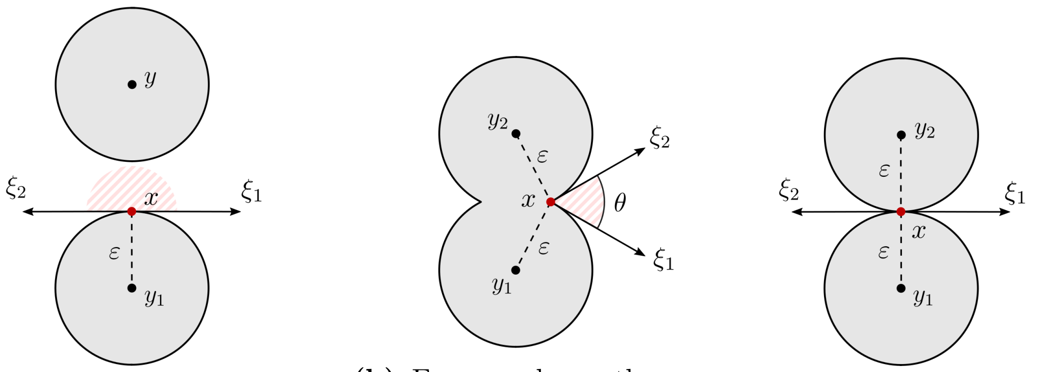
The geometric aspect of the classification scheme relies on the orientation of the extremal contributors $y \in \Pi_E^{\text{ext}}(x)$ at each boundary point $x \in \partial E_\varepsilon$. In the planar case, there are essentially three different ways this orientation can be realised, depicted schematically in Figure 4.1 below. The defining property $y_1 - x = -(y_2 - x)$ for the extremal contributors $y_1, y_2 \in \Pi_E^{\text{ext}}(x)$ in case (c) can be equivalently expressed by $\big\langle (y_1 - x)/\varepsilon, (y_2 - x)/\varepsilon \big\rangle = -1$, and we will make use of both formulations in what follows.

4.1 Types of Singularities

The classification of singularities is given in Definition 4.1.1 below. Schematic illustrations of the different types of singularities are given in Figure 1.4 in the Introduction. Recall that $U_r(x, v)$ denotes an open x-centered half-ball of radius r oriented in the direction of $v \in S^1$ (see (3.1.2)). We denote by $S(E_\varepsilon)$ the set of singularities on the boundary ∂E_ε.

Definition 4.1.1 (**Types of singularities**). *Let $E \subset \mathbb{R}^2$ be closed, let $x \in S(E_\varepsilon)$ and let $\Xi_x^{\text{ext}}(E_\varepsilon) = \{\xi_1, \xi_2\}$ be the set of extremal outward directions, where we allow for the possibility $\xi_1 = \xi_2$. We define the following eight types of singularities.*

- ***S1:*** *x is a* wedge, *if $\xi_1 \notin \{\xi_2, -\xi_2\}$, i.e. the angle θ between the vectors ξ_1, ξ_2 satisfies $0 < \theta < \pi$;*
- ***S2:*** *x is a* (one-sided) sharp singularity, *if $\xi_1 = \xi_2$, and there exists some $\delta > 0$ for which the intersection $B_\delta(x) \cap E_\varepsilon^c$ is a connected set;*
- ***S3:*** *x is a* sharp-sharp singularity, *if $\xi_1 = -\xi_2$ and for each $i \in \{1, 2\}$ there exists some $\delta_i > 0$ for which the intersection $U_{\delta_i}(x, \xi_i) \cap E_\varepsilon^c$ is a connected set;*



(a) At each $x \in \mathrm{Unp}_\varepsilon(E)$ the extremal outward directions satisfy $\xi_1 = -\xi_2$, while the set $\Xi_x(E_\varepsilon)$ spans a half-circle.

(b) For a wedge x there are two extremal contributors y_1, y_2 and two extremal outward directions ξ_1, ξ_2, forming an angle $\theta = \sphericalangle(\xi_1, \xi_2)$.

(c) For $\Pi_E^{\mathrm{ext}}(x) = \{y_1, y_2\}$ with $y_1 - x = -(y_2 - x)$, the set of extremal outward directions satisfies $\Xi_x^{\mathrm{ext}}(E_\varepsilon) \subset \{\xi_1, \xi_2\}$.

Figure 4.1. The local geometry at each boundary point $x \in \partial E_\varepsilon$ reflects the three basic scenarios (a)–(c) regarding the number and positions of contributors $y \in \Pi_E(x)$. In our classification of boundary points (see Definitions 2.0.1, 2.0.2 and 4.1.1), case (a) corresponds to smooth points and singularities of types S4 and S5, case (b) to type S1, and case (c) to types S2, S3 and S6–S8. See also Figure 1.4 and Proposition 2.2.6 regarding the structure of the set of outward directions $\Xi_x(E_\varepsilon)$.

S4: *x is a* (one-sided) shallow singularity *if $x \in \mathrm{Unp}_\varepsilon(E)$ and*

(i) $U_{\delta_1}(x, \xi_1) \cap \partial E_\varepsilon \subset \mathrm{Unp}_\varepsilon(E)$ for some $\delta_1 > 0$, and

(ii) $U_{\delta_2}(x, \xi_2) \cap \partial E_\varepsilon \not\subset \mathrm{Unp}_\varepsilon(E)$ for all $\delta_2 > 0$.

S5: *x is a* shallow-shallow singularity *if $x \in \mathrm{Unp}_\varepsilon(E)$ and $U_\delta(x, \xi_i) \cap \partial E_\varepsilon \not\subset \mathrm{Unp}_\varepsilon(E)$ for all $\delta > 0$ and $i \in \{1, 2\}$.*

S6: *x is a* (one-sided) chain singularity, *if $\xi_1 = \xi_2$ and there exists a sequence of singularities $(x_n)_{n=1}^\infty \subset S(E_\varepsilon)$, for which $x_n \to x$ and*

$$\left\langle \frac{y_n^{(1)} - x_n}{\varepsilon}, \frac{y_n^{(2)} - x_n}{\varepsilon} \right\rangle \to -1,$$

where $\left\{y_n^{(1)}, y_n^{(2)}\right\} = \Pi_E^{\mathrm{ext}}(x_n)$ is the set of extremal contributors at each x_n;

S7: *x is a* chain-chain singularity, *if $\xi_1 = -\xi_2$ and for each $i \in \{1, 2\}$ there exists some $\delta_i > 0$ and a sequence $(x_{i,n})_{n=1}^\infty \subset U_{\delta_i}(x, \xi_i) \cap S(E_\varepsilon)$, for which $x_{i,n} \to x$ and*

$$\left\langle \frac{y_{i,n}^{(1)} - x_{i,n}}{\varepsilon}, \frac{y_{i,n}^{(2)} - x_{i,n}}{\varepsilon} \right\rangle \to -1,$$

where $\left\{y_{i,n}^{(1)}, y_{i,n}^{(2)}\right\} = \Pi_E^{\mathrm{ext}}(x_{i,n})$ is the set of extremal contributors at each $x_{i,n}$;

S8: *x is a* sharp-chain singularity, *if $\xi_1 = -\xi_2$ and* *(i) there exists a $\delta_1 > 0$ for which the intersection $U_{\delta_1}(x, \xi_1) \cap E_\varepsilon^c$ is a connected set, and*

(ii) there exists some $\delta_2 > 0$ and a sequence $(x_n)_{n=1}^\infty \subset U_{\delta_2}(x, \xi_2) \cap S(E_\varepsilon)$, for which $x_n \to x$ and

$$\left\langle \frac{y_n^{(1)} - x_n}{\varepsilon}, \frac{y_n^{(2)} - x_n}{\varepsilon} \right\rangle \to -1,$$

where $\left\{y_n^{(1)}, y_n^{(2)}\right\} = \Pi_E^{\mathrm{ext}}(x_n)$ is the set of extremal contributors at each x_n.

Note that S8 may be interpreted both as a sharp singularity and as a chain singularity. Theorem 3 states that the set $\mathcal{C}(\partial E_\varepsilon) := \{x \in \partial E_\varepsilon : x \text{ is of type S6–S8}\}$ is closed, while on the other hand all the singularities of type S8 share an important property with those of type S1–S5: they all lie on the boundary ∂V of some connected component V of the complement E_ε^c. We show in Corollary 4.2.2 that this is exactly the property that is lacking from singularities of type S6 and S7 (see also Remark 4.2.3 below).

Motivated by these considerations we define a boundary point to be

(i) a *sharp singularity*, if it is of type S2, S3 or S8,

(ii) a *chain singularity*, if it is of type S6, S7 or S8, and

(iii) an *inaccessible singularity*, if it is of type S6 or S7.

The typology presented above is neither strictly topological nor strictly geometric. If one wanted to accomplish a strictly topological classification for neighbourhoods $\partial E_\varepsilon \cap B_\delta(x)$ for some $\delta := \delta(x) > 0$, types S6–S8 would necessitate an infinite tree-like classification scheme, in order to account for the potentially accumulating chain and shallow singularities in arbitrarily small neighbourhoods $B_r(x)$ with $0 < r < \delta$ (see Section 5.2 and Theorem 3).

4.2 Classification of Singularities

Proposition 4.2.1 below provides a characterisation of the topological and geometric structure of the complement E_ε^c near those singularities $x \in S(E_\varepsilon)$, whose extremal contributors y_1, y_2 satisfy $y_1 - x = -(y_2 - x)$. Geometrically these correspond to case (c) in Figure 4.1.

Proposition 4.2.1 (**Difference between sharp-type and chain-type geometry**). *Let $E \subset \mathbb{R}^2$, $x \in \partial E_\varepsilon$ and $\Pi_E^{\mathrm{ext}}(x) = \{y_1, y_2\}$ with $y_1 - x = -(y_2 - x)$. Furthermore, let $\mathcal{G}(x)$ be a local boundary representation with radius $r > 0$ at x, let $\xi \in \Xi_x^{\mathrm{ext}}(E_\varepsilon)$ and let $g_{\xi,y_1}, g_{\xi,y_2} \in \mathcal{G}(x)$ be as in (3.2.2). Then exactly one of the cases (i) and (ii) below holds true:*

(i) (sharp-type) *There exists some $r > 0$, for which $g_{\xi,y_1}(s) \neq g_{\xi,y_2}(s)$ for all $s \in (0, r)$, and*

$$E_\varepsilon^c \cap U_r(x, \xi) = V_\xi \cap U_r(x, \xi) = \bigcup_{0<s<r} x + s\Big(\alpha(s), \beta(s)\Big)_{S^1}, \tag{4.2.1}$$

where V_ξ is the unique connected component of E_ε^c intersecting $U_r(x,\xi)$, $\alpha(s), \beta(s) \in S^1$ for all $s \in (0,r)$ and $\alpha(s), \beta(s) \to \xi$ as $s \to 0$.

(ii) (chain-type) *There exists a sequence $(s_n)_{n=1}^\infty \subset \mathbb{R}_+$ with the following properties:*

(a) $s_n \to 0$ and $g_{\xi,y_1}(s_n) = g_{\xi,y_2}(s_n)$ for all $n \in \mathbb{N}$. We denote this common value by x_n.

(b) There exists some $r > 0$ and a sequence $(V_n)_{n=1}^\infty \subset U_r(x,\xi)$ of disjoint connected components of E_ε^c with $\operatorname{dist}_H(x, V_n) \to 0$ as $n \to \infty$ and $x_n \in \partial V_n$ for all $n \in \mathbb{N}$.

(c) $x_n \in S(E_\varepsilon)$ for each $n \in \mathbb{N}$, with

$$\lim_{n\to\infty} \left\langle \frac{y_n^{(1)} - x_n}{\varepsilon}, \frac{y_n^{(2)} - x_n}{\varepsilon} \right\rangle = -1, \tag{4.2.2}$$

where $\Pi_E^{\text{ext}}(x_n) = \left\{ y_n^{(1)}, y_n^{(2)} \right\}$ for all $n \in \mathbb{N}$.

Proof. Clearly, either there exists some $r > 0$, for which $g_{\xi,y_1}(s) \neq g_{\xi,y_2}(s)$ for all $s \in (0,r)$, or else there exists a sequence $(q_n)_{n=1}^\infty \subset \mathbb{R}_+$ with $q_n \to 0$, for which $g_{\xi,y_1}(q_n) = g_{\xi,y_2}(q_n)$ for all $n \in \mathbb{N}$, and these cases are mutually exclusive. The proof amounts to showing that in the former case representation (4.2.1) is valid for some connected component $V_\xi \subset E_\varepsilon^c$ and arc-segments $(\alpha(s), \beta(s))_{S^1}$, and in the latter, to identifying the prescribed sequences $(s_n)_{n=1}^\infty \subset \mathbb{R}_+$ and $(V_n)_{n=1}^\infty \subset U_r(x,\xi)$ as well as confirming the limit (4.2.2) and that $x_n \in \partial V_n$ for all $n \in \mathbb{N}$.

Consider for $i \in \{1,2\}$ the continuous functions $f^{\xi,y_i} : [0, \varepsilon/2] \to \mathbb{R}$ for which

$$g_{\xi,y_1}(s) = x + s\xi + f^{\xi,y_1}(s)(x - y_1) \neq x + s\xi + f^{\xi,y_2}(s)(x - y_2) = g_{\xi,y_2}(s)$$

(see Proposition 3.2.4). The assumption $x - y_1 = -(x - y_2)$ implies that the vector representing the difference at $s \in (0,r)$ between the graphs $g_{\xi,y_1}([0,r])$ and $g_{\xi,y_2}([0,r])$, is given by

$$g_{\xi,y_2}(s) - g_{\xi,y_1}(s) = -\left(f^{\xi,y_1}(s) + f^{\xi,y_2}(s) \right)(x - y_1). \tag{4.2.3}$$

(i) We start by assuming that there exists some $r > 0$, for which $g_{\xi,y_1}(s) \neq g_{\xi,y_2}(s)$ for all $s \in (0,r)$ which implies $\alpha_\xi(s) := f^{\xi,y_1}(s) + f^{\xi,y_2}(s) \neq 0$ for all $s \in (0,r)$. Due to continuity, this implies either

$$(1) \quad \alpha_\xi(s) > 0 \text{ for all } s \in (0,r), \qquad \text{or} \qquad (2) \quad \alpha_\xi(s) < 0 \text{ for all } s \in (0,r).$$

Note that (1) would contradict the assumption $\xi \in \Xi_x^{\text{ext}}(E_\varepsilon)$, so that (2) necessarily holds true. Hence the average $h_\xi : [0,r] \to \mathbb{R}^2$, given by

$$h_\xi(s) := \frac{g_{\xi,y_1}(s) + g_{\xi,y_2}(s)}{2} = x + s\xi + \left(f^{\xi,y_1}(s) - \frac{\alpha_\xi(s)}{2} \right)(x - y_1), \tag{4.2.4}$$

satisfies $h_\xi(s) \in E_\varepsilon^c$ for all $s \in (0,r)$. Note that although we have defined the function h_ξ in (4.2.4) in terms of the contributor y_1, we could have equally well chosen y_2 due to symmetry. Combining the facts that h_ξ is continuous, $h_\xi((0,r)) \subset E_\varepsilon^c$ and $x = h_\xi(0)$, we may deduce that there exists a connected component $V_\xi \subset E_\varepsilon^c$ for which $h_\xi((0,r)) \subset V_\xi$ and $x \in \partial V_\xi$. Emulating the reasoning presented in the proof of Lemma 3.3.2 allows one to confirm that V_ξ is the only connected component of E_ε^c that intersects $U_r(x,\xi)$.

To obtain the desired representation (4.2.1), consider for each $s \in (0,r)$ the unit vectors $h(s), \alpha(s), \beta(s) \in S^1$, given by

$$h(s) := \frac{h_\xi(s) - x}{\|h_\xi(s) - x\|}, \quad \alpha(s) := \frac{g_{\xi,y_1}(s) - x}{\|g_{\xi,y_1}(s) - x\|}, \quad \beta(s) := \frac{g_{\xi,y_2}(s) - x}{\|g_{\xi,y_2}(s) - x\|}. \tag{4.2.5}$$

Due to Proposition 2.2.8 and Lemma 2.2.7 (i) we know that the boundary ∂E_ε aligns itself with the extremal outward directions $\xi \in \Xi_x^{\mathrm{ext}}(E_\varepsilon)$ near each boundary point $x \in \partial E_\varepsilon$. We can hence assume r to be small enough such that the distances

$$\mathcal{D}_h(s) := \|h_\xi(s) - x\|\,, \quad \mathcal{D}_\alpha(s) := \|g_{\xi,y_1}(s) - x\|\,, \quad \mathcal{D}_\beta(s) := \|g_{\xi,y_2}(s) - x\|$$

appearing in the divisors in (4.2.5) are all strictly increasing in s on the interval $(0,r)$. By definition we also have $\max\left\{\mathcal{D}_h^{-1}(s), \mathcal{D}_\alpha^{-1}(s), \mathcal{D}_\beta^{-1}(s)\right\} \le s$ for all $s \in (0,r)$.

Hence, for each $s \in (0,r)$

$$h_\xi\Big(\mathcal{D}_h^{-1}(s)\Big) = x + sh\Big(\mathcal{D}_h^{-1}(s)\Big) \in x + s\Big(\alpha(\mathcal{D}_\alpha^{-1}(s)), \beta(\mathcal{D}_\beta^{-1}(s))\Big)_{S^1} \subset V_\xi \cap U_r(x,\xi).$$

In addition, we can assume Proposition 3.1.1 to apply at x with the choices $r := r$ and $\delta := \varepsilon/2$. From this it follows for $C_s := s\left(S^1 \setminus \Big(\alpha(s), \beta(s)\Big)_{S^1}\right) \cap U_r(x,\xi)$ that

$$x + C_s \subset \overline{B_\varepsilon\left(E \cap B_{\varepsilon/2}(\Pi_E^{\mathrm{ext}}(x))\right)} \subset E_\varepsilon$$

for all $s \in (0,r)$. Hence

$$E_\varepsilon^c \cap U_r(x,\xi) = V_\xi \cap U_r(x,\xi) = \bigcup_{0<s<r} x + s\Big(\alpha(s), \beta(s)\Big)_{S^1}.$$

(ii) Assume then that there exists a sequence $(q_n)_{n=1}^\infty \subset \mathbb{R}_+$ for which $g_{\xi,y_1}(q_n) = g_{\xi,y_2}(q_n)$ for all $n \in \mathbb{N}$ and $q_n \to 0$. This situation corresponds to the chain-type geometry characteristic of chain singularities (types S6–S8; see Definition 4.1.1 and Figure 1.4). Note that since $x \in \partial E_\varepsilon$ and $\xi \in \Xi_x^{\mathrm{ext}}(E_\varepsilon)$, there exists for all $s > 0$ some $0 < \lambda < s$, for which $\alpha_\xi(\lambda) = f^{\xi,y_1}(\lambda) + f^{\xi,y_2}(\lambda) < 0$. One can thus define two new sequences $(s_n)_{n=1}^\infty \subset \mathbb{R}_+$ and $(p_n)_{n=1}^\infty \subset \mathbb{R}_+$ inductively as follows. First, choose some $\lambda_1 \in (0,q_1)$

with $f^{\xi,y_1}(\lambda_1) + f^{\xi,y_2}(\lambda_1) < 0$ and define

$$s_1 := \sup\Big\{s \,:\, s > \lambda_1 \text{ and } f^{\xi,y_1}(\lambda) + f^{\xi,y_2}(\lambda) < 0 \text{ for all } \lambda \in [\lambda_1, s]\Big\},$$
$$p_1 := \inf\Big\{s \,:\, s < \lambda_1 \text{ and } f^{\xi,y_1}(\lambda) + f^{\xi,y_2}(\lambda) < 0 \text{ for all } \lambda \in [s, \lambda_1]\Big\}.$$

For the induction step, assume we have defined the points $s_1, \ldots, s_{n-1}$ and $p_1, \ldots, p_{n-1}$ for some $n \in \mathbb{N}$. One can then choose some $\lambda_n \in (0, \min\{p_{n-1}, q_n\})$ with $f^{\xi,y_1}(\lambda_n)+f^{\xi,y_2}(\lambda_n) < 0$, and define

$$s_n := \sup\Big\{s \,:\, s > \lambda_n \text{ and } f^{\xi,y_1}(\lambda) + f^{\xi,y_2}(\lambda) < 0 \text{ for all } \lambda \in [\lambda_n, s]\Big\},$$
$$p_n := \inf\Big\{s \,:\, s < \lambda_n \text{ and } f^{\xi,y_1}(\lambda) + f^{\xi,y_2}(\lambda) < 0 \text{ for all } \lambda \in [s, \lambda_n]\Big\}.$$

Then $p_n < s_n \leq p_{n-1} < s_{n-1}$ for all $n \in \mathbb{N}$ and $f^{\xi,y_1}(s) + f^{\xi,y_2}(s) = 0$ for all $s \in (s_n)_{n=1}^{\infty} \cup (p_n)_{n=1}^{\infty}$. Also, by definition, $f^{\xi,y_1}(s)+f^{\xi,y_2}(s) < 0$ for all $s \in (p_n, s_n)$ and $n \in \mathbb{N}$. This implies that for each $n \in \mathbb{N}$ the open set

$$V_n := \{\tau g_{\xi,y_1}(s) + (1-\tau) g_{\xi,y_2}(s) \,:\, \tau \in (0,1),\ s \in (p_n, s_n)\}$$

is connected and satisfies $x_n := g_{\xi,y_1}(s_n) = g_{\xi,y_2}(s_n) \in \partial V_n$. In addition $V_n \cap V_m = \varnothing$ whenever $n \neq m$, and for every $r > 0$ there exists some $N \in \mathbb{N}$, for which $0 < p_n < s_n < r$ for all $n \geq N$. It thus follows from Propositions 2.2.8 and Lemma 2.2.7 (i) that $\operatorname{dist}_H(x, V_n) \to 0$ as $n \to \infty$.

Since $x_n := g_{\xi,y_1}(s_n) = g_{\xi,y_2}(s_n)$ for each $n \in \mathbb{N}$, there exist for $i \in \{1,2\}$ extremal contributors $y_n^{(i)} \in \Pi_E^{\text{ext}}(x_n)$, for which $y_n^{(i)} \in B_{\varepsilon/2}(y_i)$. This, together with Lemma 2.2.7 (ii)(a), implies $y_n^{(i)} \to y_i$ for $i \in \{1,2\}$ as $n \to \infty$, and consequently

$$\lim_{n\to\infty} \left\langle \frac{y_n^{(1)} - x_n}{\varepsilon}, \frac{y_n^{(2)} - x_n}{\varepsilon} \right\rangle = -1. \qquad \square$$

As a consequence of Proposition 4.2.1 we obtain the following characterisation for inaccessible boundary points x, which are defined by the property that $x \notin \partial V$ for all connected components V of the complement $E_\varepsilon^c := \mathbb{R}^2 \setminus E_\varepsilon$.

Corollary 4.2.2 (**Inaccessible singularities**). *Let $E \subset \mathbb{R}^2$ and $x \in \partial E_\varepsilon$. Then $x \notin \partial V$ for all connected components V of the complement E_ε^c if and only if x is a one-sided chain singularity (S6) or a chain-chain singularity (S7).*

Proof. Assume first that $x \notin \partial V$ for all connected components V of the complement E_ε^c. Proposition 3.3.3 then implies that x is not a wedge and $x \notin \operatorname{Unp}_\varepsilon(E)$, so that the extremal contributors $y_1, y_2 \in \Pi_E^{\text{ext}}(x)$ satisfy $y_1 - x = -(y_2 - x)$.

In case x has only one extremal outward direction $\xi \in \Xi_x^{\text{ext}}(E_\varepsilon)$, Proposition 3.1.1

implies the existence of some $r < \varepsilon/2$ for which $E_\varepsilon^c \cap B_r(x) \subset U_r(x,\xi)$. By assumption there cannot exist any connected component V_ξ described in case (i) of Proposition 4.2.1, which implies that case (ii) holds. Hence x is a (one-sided) chain singularity.

If, on the other hand, there exist extremal outward directions ξ_1, ξ_2 with $\xi_1 = -\xi_2$, Proposition 4.2.1 again rules out case (i) for each one of them, and consequently x fulfils the definition of a chain-chain singularity.

Assume then that x is either a one-sided chain singularity (S6) or a chain-chain singularity (S7). In the former case $\Xi_x^{\text{ext}}(E_\varepsilon) = \{\xi\}$ for some $\xi \in S^1$, and Proposition 3.1.1 again implies the existence of some $r < \varepsilon/2$ for which $E_\varepsilon^c \cap B_r(x) \subset U_r(x,\xi)$. We aim to deduce a contradiction by assuming there exists a connected $V \subset U_r(x,\xi) \cap E_\varepsilon^c$ for which $x \in \partial V$. Note that every $z \in V$ has a representation

$$z = x + s\xi + t(s)(x - y_1) \tag{4.2.6}$$

for some $s = s(z) > 0$ and $t(s) \in \mathbb{R}$. Now choose some $s_0 > 0$ and $t(s_0)$ so that $z_0 = x + s_0\xi + t(s_0)(x - y_1) \in V$. Given that V is connected and $x \in \partial V$, it follows from $z_0 \in V$ that there exists a path $\gamma : [0,1] \to V$ for which $\gamma(0) = x$ and $\gamma(1) = z_0$. Following (4.2.6), we may write

$$\gamma(u) = x + s(u)\xi + t(s(u))(x - y_1),$$

where the coordinate $s(u)$ depends continuously on $u \in [0,1]$. Since $\gamma(u) \notin E_\varepsilon$ for all $u \in [0,1]$, Proposition 3.2.4 implies that the coordinate $t(s(u))$ satisfies $t(s(u)) \in \big(f^{\xi,y_1}(s(u)), -f^{\xi,y_2}(s(u))\big)$ for all $u \in [0,1]$, where the functions f^{ξ,y_i} are as in Proposition 3.2.4. However, Proposition 4.2.1 implies the existence of some $0 < q < s_0$ for which $f^{\xi,y_1}(q) = -f^{\xi,y_2}(q)$, and since $s(\cdot)$ is continuous as a function of u, there exists some u_q for which $s(u_q) = q$. For this coordinate we thus obtain the contradiction $t(s(u_q)) = t(q) \in \big(f^{\xi,y_1}(q), -f^{\xi,y_2}(q)\big) = \varnothing$.

In case x is a chain-chain singularity (S7), one may again follow the above reasoning to deduce that the existence of a connected component V of E_ε^c with $x \in \partial V$ would contradict the existence of a sequence $s_n \to 0$ with $f^{\xi_i,y_1}(s_n) = -f^{\xi_i,y_2}(s_n)$, which is on the other hand guaranteed for both $\xi_1, \xi_2 \in \Xi_x^{\text{ext}}(E_\varepsilon)$ by Proposition 4.2.1. $\square$

Remark 4.2.3. Corollary 4.2.2 states that it is impossible for a chain singularity (S6) or a chain-chain singularity (S7) x to lie on the boundary of any connected component $V \subset E_\varepsilon^c$, even though a sequence $(V_n)_{n=1}^\infty$ of connected components $V_n \subset E_\varepsilon^c$ converges to x in Hausdorff distance. This is the motivation for the terminology of *inaccessible* singularities and can be seen as an analogue of the distinction between accessible and inaccessible points in a Cantor set $C \subset [0,1]$, where inaccessible points do not lie on the boundary of any of the countably many removed open intervals $[a_n, b_n] \subset [0,1]$. See also

Example 5.2.1 and the discussion after Definition 4.1.1 above.

Proposition 4.2.4 (Characterisation of chain singularities). *Let $E \subset \mathbb{R}^2$, let $x \in \partial E_\varepsilon$ and let $\mathcal{G}(x)$ be a local boundary representation at x with the functions $g_{\xi,y} \in \mathcal{G}(x)$ as in (3.2.2). Then the following are equivalent:*

(i) x is a chain singularity (type S6, S7 or S8).

(ii) There exists a sequence $(V_n)_{n=1}^{\infty}$ of mutually disjoint connected components $V_n \subset E_\varepsilon^c$ for which $\operatorname{dist}_H(x, V_n) \to 0$ as $n \to \infty$.

(iii) There exists a sequence $(x_n)_{n=1}^{\infty}$ of singularities on ∂E_ε for which $x_n \to x$ and

$$\lim_{n\to\infty} \left\langle \frac{y_n^{(1)} - x_n}{\varepsilon}, \frac{y_n^{(2)} - x_n}{\varepsilon} \right\rangle = -1, \tag{4.2.7}$$

where $\Pi_E^{\text{ext}}(x_n) = \left\{y_n^{(1)}, y_n^{(2)}\right\}$ for each $n \in \mathbb{N}$.

(iv) The extremal contributors $\Pi_E^{\text{ext}}(x) = \{y_1, y_2\}$ satisfy $y_1 - x = -(y_2 - x)$ and there exist some $\xi \in \Xi_x^{\text{ext}}(E_\varepsilon)$ and corresponding functions $g_{\xi,y_1}, g_{\xi,y_2} \in \mathcal{G}(x)$ for which $g_{\xi,y_1}(s_n) = g_{\xi,y_2}(s_n)$ for all $n \in \mathbb{N}$ for some sequence $(s_n)_{n=1}^{\infty} \subset \mathbb{R}_+$ with $s_n \to 0$ as $n \to \infty$.

Proof. We begin by showing that (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii). Since clearly (i) implies (iii), the result then follows by showing that (iii) $\wedge$ (iv) $\Rightarrow$ (i).

(ii) $\Rightarrow$ (iv). Assume there exists a sequence $(V_n)_{n=1}^{\infty} \subset E_\varepsilon^c$ of mutually disjoint connected components of the complement E_ε^c with $\operatorname{dist}_H(x, V_n) \to 0$ as $n \to \infty$. It follows then from Proposition 3.3.3 that $x \notin \operatorname{Unp}_\varepsilon(E)$ and it cannot be a wedge, which implies $y_1 - x = -(y_2 - x)$ for the extremal contributors $\Pi_E^{\text{ext}}(x) = \{y_1, y_2\}$.

For each $n \in \mathbb{N}$, choose a point $v_n \in V_n$, and define $\xi_n := (v_n - x)/\left\|v_n - x\right\|$. Due to compactness, one can choose a subsequence $(v_{n_k})_{k=1}^{\infty}$ for which $\xi_{n_k} \to \xi \in S^1$. Since $v_n \in E_\varepsilon^c$ for all $n \in \mathbb{N}$, it follows from Lemma 2.2.3 that $\langle y_i - x, \xi\rangle = 0$ for $i \in \{1, 2\}$, so that $\xi \in \Xi_x^{\text{ext}}(E_\varepsilon)$.

Assume contrary to the claim that there exists $\delta > 0$ for which $g_{\xi,y_1}(s) \neq g_{\xi,y_2}(s)$ for all $s \in (0, \delta)$. According to Proposition 4.2.1 this implies

$$E_\varepsilon^c \cap U_\delta(x, \xi) = V \cap U_\delta(x, \xi) = \bigcup_{0<\rho<\delta} \rho(\alpha_\rho, \beta_\rho)_{S^1}, \tag{4.2.8}$$

where V is the unique connected component of the complement E_ε^c that intersects $U_\delta(x, \xi)$. Since $\xi_{n_k} \to \xi$ as $k \to \infty$, equation (4.2.8) now implies $v_{n_k} \in V$ for large $k \in \mathbb{N}$, which in turn contradicts the assumption that the sets V_n are connected and mutually disjoint. Hence, no such δ can exist, and (iv) follows.

(iv) $\Rightarrow$ (iii). Case (ii) in Proposition 4.2.1 now holds true, and directly implies (iii) here.

(iii) $\Rightarrow$ (ii). Write $\Pi_E^{\text{ext}}(x) = \{y_1, y_2\}$ and define $\xi_n := (x_n - x)/\|x_n - x\|$. Due to Lemma 2.2.7 (i) there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$, for which $\xi_{n_k} \to \xi \in \Xi_x^{\text{ext}}(E_\varepsilon)$ as $k \to \infty$. Furthermore, since $\Pi_E^{\text{ext}}(x_n) = \left\{y_n^{(1)}, y_n^{(2)}\right\}$ for all $n \in \mathbb{N}$, Lemma 2.2.7 (ii)(a) together with (4.2.7) implies the existence of a further subsequence $(x_m)_{m=1}^{\infty} \subset (x_{n_k})_{k=1}^{\infty}$ for which $y_m^{(i)} \to y_i \in \Pi_E^{\text{ext}}(x)$ for $i \in \{1, 2\}$. Hence $y_1 - x = -(y_2 - x)$.

To complete the argument we show that for any $\delta > 0$ there exists some $0 < s_m < \delta$ for which $x_m = g_{\xi,y_1}(s_m) = g_{\xi,y_2}(s_m)$. Once this is established, the statement follows from Proposition 4.2.1. Since $x_m \to x$ as $m \to \infty$, there exists for all $\delta > 0$ some $M \in \mathbb{N}$ for which $\|x_m - x\| \leq \delta$ for all $m > M$. Hence there exists for all $m > M$ some $s_m \leq \delta$ for which $x_m = g_{\xi,y_i}(s_m)$ for some $i \in \{1, 2\}$. But since $y_m^{(i)} \to y_i \in \Pi_E^{\text{ext}}(x)$ for $i \in \{1, 2\}$ as $m \to \infty$, where $y_1 - x = -(y_2 - x)$, we have in fact $x_m = g_{\xi,y_1}(s_m) = g_{\xi,y_2}(s_m)$ for all sufficiently large $m > M$.

(iii) $\wedge$ (iv) $\Rightarrow$ (i). Since $y_1 - x = -(y_2 - x)$ for the extremal contributors $y_1, y_2 \in \Pi_E^{\text{ext}}(x)$, it follows from Lemma 2.2.4 that the extremal outward directions $\xi_1, \xi_2 \in \Xi_x^{\text{ext}}(E_\varepsilon)$ satisfy either $\xi_1 = \xi_2$ or $\xi_1 = -\xi_2$. In the former case x is a one-sided chain singularity (S6). In the latter case we may assume without loss of generality that $\xi_n := (x_n - x)/\|x_n - x\| \to \xi_1$. Proposition 4.2.1 then states that for some $\delta > 0$, the boundary subset $\partial E_\varepsilon \cap U_\delta(x, \xi_2)$ exhibits either 'sharp'-type or 'chain'-type geometry and that these cases are mutually exclusive. In the former case x is a sharp-chain singularity (S8), in the latter a chain-chain singularity (S7). $\square$

We employ Proposition 4.2.4 to show that our definition of smooth points (see Definition 2.0.2) coincides with the property of lying on a C^1-smooth curve. By a *curve* we mean the image $\Gamma = \gamma([0, 1])$ of a continuous, injective map $\gamma : [0, 1] \to \mathbb{R}^2$.

Proposition 4.2.5 (Characterisation of smooth points). *Let $E \subset \mathbb{R}^2$ and $x \in \partial E_\varepsilon$. Then x is smooth in the sense of Definition 2.0.2 if and only if there exists a C^1-curve Γ for which $\Gamma = \partial E_\varepsilon \cap \overline{B_r(x)}$ for some $\delta > 0$.*

Proof. According to Proposition 3.2.4 there exists a local boundary representation $\mathcal{G}(x)$ with radius $r > 0$ and continuous functions $f^{\xi,y} : [0, \varepsilon/2] \to \mathbb{R}$ for which

$$g_{\xi,y}(s) = x + s\xi + f^{\xi,y}(s)(x - y)$$

for all $g_{\xi,y} \in \mathcal{G}(x)$ and $s \in [0, s_{\xi,y}]$.

(i) Assume first that x is smooth in the sense of Definition 2.0.2. Then there exists $0 < \delta < r$ for which $\overline{B_\delta(x)} \cap \partial E_\varepsilon \subset \text{Unp}_\varepsilon(E)$. Proposition 2.2.6 then implies that for all $z \in \overline{B_\delta(x)} \cap \partial E_\varepsilon$ the set of extremal outward directions satisfies $\Xi_z^{\text{ext}}(E_\varepsilon) = \{\xi_z, -\xi_z\}$ for some $\xi_z \in S^1$. According to Proposition 2.2.8, the extremal outward directions coincide with tangential directions on the boundary, and hence $\Xi_z^{\text{ext}}(E_\varepsilon) = \{\xi_z, -\xi_z\}$ implies that $g_{\xi,y}$ is differentiable at $s = s_z$, where $z = g_{\xi,y}(s_z)$. Since this is true for all $z \in \overline{B_\delta(x)} \cap \partial E_\varepsilon$, the boundary ∂E_ε inside $\overline{B_\delta(x)}$ is contained in the union of images $g_{\xi_1,y}([0, s_{\xi_1,y}]) \cup g_{\xi_2,y}([0, s_{\xi_2,y}])$ under the differentiable maps $g_{\xi_1,y}$ and $g_{\xi_2,y}$. Since $g_{\xi_1,y}([0, s_{\xi_1,y}]) \cap g_{\xi_2,y}([0, s_{\xi_2,y}]) = \{x\}$ and both images can be represented as graphs of the corresponding functions $f^{\xi_i,y}$ for $i \in \{1, 2\}$, the claim follows.

(ii) Let then Γ be a C^1-curve for which $\Gamma = \partial E_\varepsilon \cap \overline{B_\delta(x)}$ for some $\delta > 0$. Then for each $z \in \partial E_\varepsilon \cap \overline{B_\delta(x)}$, Proposition 2.2.8 implies $\Xi_z^{\text{ext}}(E_\varepsilon) = \{\xi_z, -\xi_z\}$ for some $\xi_z \in S^1$. Consider now some $z \in \partial E_\varepsilon \cap \overline{B_\delta(x)}$. Since Γ is C^1-smooth, the correspondence between tangents and extremal outward directions given by Proposition 2.2.8 implies that z is not a wedge (S1) or a sharp singularity (S2–S3). On the other hand, as a curve Γ is connected, which together with Proposition 4.2.4 implies that z cannot be a chain singularity (S6–S8). Hence it follows from Theorem 1 below that z is necessarily either a smooth point or a shallow (S4–S5) singularity, which implies $z \in \text{Unp}_\varepsilon(E)$. The same argument applies to all $z \in \partial E_\varepsilon \cap \overline{B_r(x)}$, which means that x is smooth in the sense of Definition 2.0.2. $\square$

Proof of Theorem 1

We conclude this section with the proof of our first main result, a classification of boundary points on ∂E_ε. We restate the result here for the convenience of the reader.

Theorem 1 (Classification of boundary points). *Let $E \subset \mathbb{R}^2$ be compact, $\varepsilon > 0$ and let $x \in \partial E_\varepsilon$ be a boundary point of E_ε that is not smooth. Then x belongs to precisely one of the eight categories of singularities given in Definition 4.1.1.*

Proof. Note first that for any $u, v \in S^1$ either $u = v$, $u = -v$, or $u \notin \{v, -v\}$. Hence we obtain the following categorisation of boundary point types according to the orientation of the extremal outward directions $\Xi_x^{\text{ext}}(E_\varepsilon) = \{\xi_1, \xi_2\}$:

$$\begin{aligned}
&\xi_1 \notin \{\xi_2, -\xi_2\}: && \text{type S1} \\
&\xi_1 = \xi_2: && \text{types S2 and S6} \\
&\xi_1 = -\xi_2: && \text{smooth points and types S3–S5, S7–S8}
\end{aligned}$$

These are due to Proposition 2.2.6 for $x \in \text{Unp}_\varepsilon(E)$ and Definition 4.1.1 for $x \notin \text{Unp}_\varepsilon(E)$, and they correspond to the cases (a)–(c) illustrated in Figure 4.1. It follows immediately that if x is a wedge (S1), it cannot be of any other type, and vice versa. In addition, of all the defined types of boundary points, only the shallow singularities (S4 and S5) and smooth points satisfy $\Pi_E^{\text{ext}}(x) = \{y\}$ for some $y \in \partial E$, and these types are by definition mutually exclusive.

Hence it suffices to show that the remaining types S2–S3 and S6–S8 (corresponding to case (c) in Figure 4.1) are mutually exclusive. For all these types, the set of extremal contributors $\Pi_E^{\mathrm{ext}}(x) = \{y_1, y_2\}$ satisfies $y_1 - x = -(y_2 - x)$. Proposition 4.2.1 then states that for each $\xi \in \Xi_x^{\mathrm{ext}}(E_\varepsilon)$ either

(i) there exists a connected component $V \subset E_\varepsilon^c$ and $r > 0$ for which $E_\varepsilon^c \cap U_r(x, \xi) = V \cap U_r(x, \xi)$ and $x \in \partial V$, or

(ii) there exists a sequence of singularities $(x_n)_{n=1}^\infty$ in $S(E_\varepsilon)$ with $x_n \to x$ as $n \to \infty$ and $(x_n - x)/\|x_n - x\| \to \xi$, and

$$\lim_{n\to\infty} \left\langle \frac{y_n^{(1)} - x_n}{\varepsilon}, \frac{y_n^{(2)} - x_n}{\varepsilon} \right\rangle = -1,$$

and that these situations are mutually exclusive. In other words, for each extremal outward direction $\xi \in \Xi_x^{\mathrm{ext}}(E_\varepsilon)$, the intersection $\partial E_\varepsilon \cap U_r(x, \xi)$ exhibits either 'sharp'-type or 'chain'-type geometry (see Definition 4.1.1). In case $\xi_1 = \xi_2$, the point x is hence either a sharp (S2) or a chain (S6) singularity, and in case $\xi_2 = -\xi_1$, it is either a sharp-sharp (S3), a chain-chain (S7), or a sharp-chain (S8) singularity, and all these cases are mutually exclusive. $\square$

Chapter 5

Topological Structure of the Set of Singularities

Since the categories of boundary points given in Definition 4.1.1 define a partition of the boundary, it makes sense to inquire on their cardinalities and topological structure. Our second main result, Theorem 2, states that for any compact $E \in \mathbb{R}^2$ and $\varepsilon > 0$, the sets of wedges (S1), sharp singularities (S2, S3 and S8) and one-sided chain singularities (S6) on ∂E_ε are at most countably infinite. This does not hold in general for the sets of shallow-shallow singularities (S5) or chain-chain singularities (S7), which may even have a positive one-dimensional Hausdorff measure on the boundary. In Section 5.2 we show that the set $\mathcal{C}(\partial E_\varepsilon)$ of chain singularities is nevertheless nowhere dense, and hence small in the topological sense.

5.1 Cardinalities of Sets of Singularities

In order to prove the above-mentioned results on the cardinalities of the sets of singularities, we proceed by treating one by one the cases of wedges, sharp singularities, and one-sided shallow and chain singularities. We begin with the following general result on the geometry of accumulating singularities, which is essentially a corollary of Lemma 2.2.7 on the asymptotic behaviour of sequences of boundary points.

Lemma 5.1.1 (Geometry of accumulating singularities). *Let $E \subset \mathbb{R}^2$, let $(x_n)_{n=1}^{\infty} \subset S(E_\varepsilon)$ be a sequence of pair-wise disjoint singularities with $x_n \to x \in \partial E_\varepsilon$ and let $\Xi_{x_n}^{\text{ext}}(E_\varepsilon) = \left\{\xi_n^{(1)}, \xi_n^{(2)}\right\}$ for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \left|\left\langle \xi_n^{(1)}, \xi_n^{(2)} \right\rangle\right| = 1$.*

Proof. Due to Lemma 2.2.7 (i) we can assume that the limit $\xi := \lim_{n\to\infty}(x_n - x)/\left\|x_n - x\right\|$ exists. Note that depending on the geometry at the singularities x_n, each of the extremal outward directions $\xi_n^{(i)}$ for $i \in \{1, 2\}$ may be more aligned with ξ than $-\xi$, or vice versa.

However, for each $i \in \{1,2\}$, Lemma 2.2.7 (ii)(b) implies that there exist sequences of coefficients $\left(a_n^{(i)}\right)_{n=1}^{\infty} \in \{1,-1\}^{\mathbb{N}}$ for which $\left|\left\langle \xi_n^{(i)}, \xi \right\rangle\right| = \left\langle a_n^{(i)} \xi_n^{(i)}, \xi \right\rangle$ and

$$\lim_{n\to\infty} \left\langle a_n^{(1)} \xi_n^{(1)}, \xi \right\rangle = \lim_{n\to\infty} \left\langle a_n^{(2)} \xi_n^{(2)}, \xi \right\rangle = 1. \tag{5.1.1}$$

Let $\theta_n^{(i)} \geq 0$ be the angle for which $\left\langle a_n^{(i)} \xi_n^{(i)}, \xi \right\rangle = \cos\left(\theta_n^{(i)}\right)$. Equation (5.1.1) implies $\theta_n^{(i)} \to 0$ for $i \in \{1,2\}$, and in particular there exists some $N \in \mathbb{N}$ for which $\left|\left\langle \xi_n^{(1)}, \xi_n^{(2)} \right\rangle\right| = \left\langle a_n^{(1)} \xi_n^{(1)}, a_n^{(2)} \xi_n^{(2)} \right\rangle$ for all $n \geq N$. Writing $\theta_n^{\max} := \max\left\{\theta_n^{(1)}, \theta_n^{(2)}\right\}$ for $n \in \mathbb{N}$, we have

$$\left\langle a_n^{(1)} \xi_n^{(1)}, a_n^{(2)} \xi_n^{(2)} \right\rangle = \cos\left(\theta_n^{(1)} + \theta_n^{(2)}\right) \geq \cos\left(2\theta_n^{\max}\right) \longrightarrow 1, \tag{5.1.2}$$

as $n \to \infty$, and the result follows. □

A particular consequence of Lemma 5.1.1 is that accumulating wedges (S1) become increasingly acute/obtuse as they approach a limit point, with the angles $\theta_n := \sphericalangle\left(\xi_n^{(1)}, \xi_n^{(2)}\right)$ between the extremal outward directions approaching an asymptotic value $\theta = \lim_{n\to\infty} \theta_n \in \{0, \pi\}$. It follows that for any fixed $p > 0$, there can only exist finitely many wedges whose sharpness deviates from these asymptotic values by more than p, which in turn implies that the total number of wedges can at most be countably infinite. Lemma 5.1.2 below makes this argument precise.

Lemma 5.1.2 (Number of wedges). *For any compact $E \subset \mathbb{R}^2$ and $\varepsilon > 0$, the number of wedges (S1) on ∂E_ε is at most countably infinite.*

Proof. We begin by showing that for each $p \in (0,1)$, the subset $A(p) \subset \partial E_\varepsilon$ defined by

$$A(p) := \left\{x \in \partial E_\varepsilon \;:\; \left|\left\langle \xi^{(1)}, \xi^{(2)} \right\rangle\right| \leq p \;\text{ for }\; \xi^{(1)}, \xi^{(2)} \in \Xi_x^{\text{ext}}(E_\varepsilon)\right\}$$

contains only finitely many points. Assume contrary to this that for some $p \in (0,1)$ the set $A(p)$ contains infinitely many points. This implies that there exists a pair-wise disjoint sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in A(p)$ for all $n \in \mathbb{N}$, and due to compactness of ∂E_ε we may assume that $x_n \to x \in \partial E_\varepsilon$ as $n \to \infty$. Writing $\Xi_{x_n}^{\text{ext}}(E_\varepsilon) = \left\{\xi_n^{(1)}, \xi_n^{(2)}\right\}$ for each $n \in \mathbb{N}$, Lemma 5.1.1 then implies $\lim_{n\to\infty} \left|\left\langle \xi_n^{(1)}, \xi_n^{(2)} \right\rangle\right| = 1$, contradicting the assumption that $\left|\left\langle \xi_n^{(1)}, \xi_n^{(2)} \right\rangle\right| \leq p < 1$ for all $n \in \mathbb{N}$.

By definition, each wedge $x \in \partial E_\varepsilon$ belongs to the set $A(p)$ for some $p \in (0,1)$. Hence the union

$$A := \bigcup_{n=1}^{\infty} A\left(1 - n^{-1}\right)$$

contains all the wedges on ∂E_ε. According to the reasoning above, each of the sets $A(1 - n^{-1})$ contains only finitely many points, from which the result follows. □

We next show that for a given connected component U of the complement E_ε^c, there can only exist finitely many sharp singularities on ∂U. This essentially follows from Propositions 3.2.4 and 4.2.4 which imply that any convergent sequence of pairwise disjoint sharp singularities $x_n \in \partial U$ is associated with a sequence $(U_n)_{n=1}^{\infty}$ of pairwise disjoint connected components of E_ε^c, aligned with one of the extremal outward directions $\xi \in \Xi_x^{\text{ext}}(E_\varepsilon)$ and satisfying $U_n \neq U$ and $x_n \in \partial U_n$ for all $n \in \mathbb{N}$ (or, to be precise, at least from some $N \in \mathbb{N}$ onwards). Negotiating the topological and geometric constraints of the situation would then require that $U = U_n$ for all $n \in \mathbb{N}$, which is clearly impossible.

Lemma 5.1.3 (Number of sharp singularities). *Let $E \subset \mathbb{R}^2$ be compact. For any connected component U of the complement E_ε^c, the number of sharp singularities (S2, S3 and S8) on the boundary ∂U is finite.*

Proof. Assume that the claim fails for some connected $U \subset E_\varepsilon^c$. Since ∂U is compact, this implies the existence of a pair-wise disjoint sequence $(x_n)_{n=1}^{\infty} \subset \partial U$ of sharp singularities with $x_n \to x \in \partial U$. We may furthermore assume that the sequence is ordered so that

$$\|x_{n+1} - x\| < \|x_n - x\| \tag{5.1.3}$$

for all $n \in \mathbb{N}$, and that the limit $\xi := \lim_{n\to\infty}(x_n - x)/\|x_n - x\|$ exists. Write $\Pi_E^{\text{ext}}(x_n) = \left\{y_n^{(1)}, y_n^{(2)}\right\}$. It follows from the definition of a sharp singularity that

$$y_n^{(1)} - x_n = -\left(y_n^{(2)} - x_n\right) \tag{5.1.4}$$

for all $n \in \mathbb{N}$. According to Proposition 4.2.4, also the extremal contributors $y_1, y_2 \in \Pi_E^{\text{ext}}(x)$ satisfy $y_1 - x = -(y_2 - x)$, and $y_n^{(i)} \to y_i$ for $i \in \{1, 2\}$ as $n \to \infty$. Let $\mathcal{G}(x)$ be the local boundary representation (of radius $r > 0$) at x, given by Proposition 3.2.4, and consider for $i \in \{1, 2\}$ the functions $g_{\xi,y_i} \in \mathcal{G}(x)$,

$$g_{\xi,y_i}(s) := x + s\xi + f^{\xi,y_i}(s)(x - y_i)$$

where the functions $f^{\xi,y_i} : [0, \varepsilon/2] \to \mathbb{R}$ are continuous. Due to equation (5.1.4), and since $x_n \to x$, there exists a sequence $(s_n)_{n=1}^{\infty} \subset \mathbb{R}_+$ with $s_n \to 0$ and some $N \in \mathbb{N}$, for which $n > N$ implies $x_n = g_{\xi,y_1}(s_n) = g_{\xi,y_2}(s_n)$ and consequently $f^{\xi,y_1}(s_n) + f^{\xi,y_2}(s_n) = 0$. Inequality (5.1.3) implies $s_{n+1} < s_n < s_{n-1}$ for all $n > N$, and one can define the open sets

$$\begin{aligned} S_n &:= \left\{ s \in (s_{n+1}, s_{n-1}) \;:\; f^{\xi,y_1}(s) + f^{\xi,y_2}(s) < 0 \right\}, \\ U_n &:= \{\tau g_{\xi,y_1}(s) + (1-\tau) g_{\xi,y_2}(s) \;:\; \tau \in (0,1),\, s \in S_n\}. \end{aligned}$$

For each $n > N$, the set U_n is contained in the interior $\operatorname{int} R_n$ of the closed rectangle

$$R_n := \left\{ x + s\xi + t(x - y_1) \,:\, s_{n+1} \le s \le s_{n-1},\, \inf_{s \in S_n} \left\{ f^{\xi,y_1}(s) \right\} \le t \le -\inf_{s \in S_n} \left\{ f^{\xi,y_2}(s) \right\} \right\}.$$

By definition $s_n \in S_n$ for all $n > N$, from which it follows that $x_n \notin \partial V$ for any open $V \subset E_\varepsilon^c \cap R_n^c$. On the other hand $x_n \in \partial U$ for all $n \in \mathbb{N}$ which implies $U \subset \operatorname{int} R_n$ for all $n \in \mathbb{N}$, since U is connected and $\partial R_n \cap E_\varepsilon^c = \varnothing$ by definition. However, given that $x \in \partial U \cap R_n^c$, this leads to the contradiction $U \subset R_n \cap R_n^c$ for all $n > N$. □

Lemmas 5.1.4 and 5.1.6 below state that the sets of one-sided shallow singularities (S4) and chain singularities (S6) are both at most countably infinite. The argument in both cases rests on the fact that for any finite sum $M := \sum_{x \in A} m_x < \infty$ of non-negative real numbers m_x indexed by a (potentially uncountable) set A, the index subset $A_0 := \{x \in A \,:\, m_x > 0\}$ corresponding to the positive elements in the sum is at most countably infinite.[1] In the case of shallow singularities, the numbers being summed will represent lengths (one-dimensional Hausdorff measures) $m_x := \mathcal{H}^1(I_x)$ of boundary segments $I_x \subset \partial E_\varepsilon$, and in the case of chain singularities they will stand for surface areas $m_x := \mathcal{H}^2(A_x)$ of open subsets $A_x \subset \operatorname{int} E_\varepsilon$. In each case, these numbers will be strictly positive by definition for every x, implying that the underlying index sets—corresponding to the sets of singularities in question—are themselves at most countably infinite.

Lemma 5.1.4 (Number of one-sided shallow singularities). *For a compact set $E \subset \mathbb{R}^2$, the number of one-sided shallow singularities (S4) on ∂E_ε is at most countably infinite.*

Proof. Write W for the set of one-sided shallow singularities on ∂E_ε, and consider some $x \in W$. By definition of a one-sided shallow singularity, there exists a $\xi \in \Xi_x^{\text{ext}}(E_\varepsilon)$ and $\delta > 0$ for which $J_x := U_\delta(x, \xi) \cap \partial E_\varepsilon \subset \operatorname{Unp}_\varepsilon(E)$. Due to Proposition 3.2.4 there exists a local boundary representation $\mathcal{G}(x)$ and some some $r(x) > 0$ for which $J_x = g_{\xi,y}([0, r(x)])$, where $\Pi_E^{\text{ext}}(x) = \{y\}$ and $g_{\xi,y} \in \mathcal{G}(x)$, satisfies

$$g_{\xi,y}(s) = x + s\xi + f^{\xi,y}(s)(x - y)$$

for some continuous $f : [0, \varepsilon/2] \to \mathbb{R}$. The open subsegment $I_x := g_{\xi,y}((0, r(x)/2)) \subset J_x$ contains only smooth points and has a positive, finite length $r(x)/2 < \mathcal{H}^1(I_x) < \infty$. This follows for instance from the fact that $g_{\xi,y}$ satisfies $|s - w| \le \|g_{\xi,y}(s) - g_{\xi,y}(w)\| \le 2|s - w|/\sqrt{3}$ for all $s, w \in (0, r(x)/2)$ (see Proposition 3.2.5) and since the increase of the Hausdorff measure under a Lipschitz map is bounded by the Lipschitz constant (see for instance [AFP00, Proposition 2.49]). If $z \neq x$ is another one-sided shallow singularity,

[1] This follows from the observation that the set $A_n := \{x \in A \,:\, m_x > 1/n\}$ is finite for each $n \in \mathbb{N}$ and hence $A_0 := \{x \in A \,:\, m_x > 0\} = \bigcup_{n \in \mathbb{N}} A_n$ is countable as a countable union of finite sets.

the corresponding segment I_z satisfies $I_z \cap I_x = \varnothing$ by definition. Hence, the collection $I := \bigcup_{x\in W} I_x \subset \partial E_\varepsilon$ has finite length due to the estimate

$$\mathcal{H}^1(I) = \sum_{x\in W} \mathcal{H}^1(I_x) < \mathcal{H}^1(E_\varepsilon) < \infty, \tag{5.1.5}$$

where the last inequality was established already by Erdős in [Erd45, Section 6]. According to the counting argument preceding the statement of the result, inequality (5.1.5) implies that the set W can be at most countably infinite. □

The following example demonstrates that the set of two-sided shallow singularities (S5) can be dense and have positive Hausdorff measure on the boundary ∂E_ε. The idea is to construct a suitably jagged function on the interval $[0,1]$ (say) and interpret its graph as a subset of the boundary ∂E_ε of a corresponding set $E \subset \mathbb{R}^2$.

Example 5.1.5 (Dense, positive measure set of shallow singularities). Consider a bounded, increasing function $\alpha : [0,1] \to \mathbb{R}$ that is discontinuous at every rational number $q \in \mathbb{Q} \cap [0,1]$ but continuous at every irrational number $p \in [0,1] \setminus \mathbb{Q}$.[2] As an almost everywhere continuous bounded function, every such α is Riemann-integrable, and its monotonicity implies that the integral function $I_\alpha(x) := \int_0^x \alpha(s)ds$ is convex. Most significantly for our example, I_α has a well-defined derivative at every irrational $p \in [0,1] \setminus \mathbb{Q}$, but not at any rational $q \in \mathbb{Q} \cap [0,1]$.

For any $\varepsilon > 0$, one may thus interpret the graph $G := \big\{(s, I_\alpha(s)) \,:\, s \in [0,1]\big\}$ as a subset of the ε-neighbourhood boundary ∂E_ε of a compact set E as follows. Since the one-sided derivatives

$$D^\pm I_\alpha(s) := \lim_{h\to\pm 0} \frac{I_\alpha(s+h) - I_\alpha(s)}{h}$$

exist at every $s \in [0,1]$, one can define for each $x(s) := (s, I_\alpha(s))$ the corresponding contributors $y^-(s), y^+(s) \in \Pi_E(x(s))$ by setting $y^\pm(s) := \big(s + a^\pm(s),\, I_\alpha(s) - b^\pm(s)\big)$, where for each $s \in [0,1]$

$$a^\pm(s) := \frac{\varepsilon D^\pm I_\alpha(s)}{\sqrt{1 + \big[D^\pm I_\alpha(s)\big]^2}} \qquad \text{and} \qquad b^\pm(s) := \frac{\varepsilon}{\sqrt{1 + \big[D^\pm I_\alpha(s)\big]^2}}.$$

A direct computation shows that $D^\pm I_\alpha(s) = a^\pm(s)/b^\pm(s)$ and $\|x(s) - y^\pm(s)\| = \varepsilon$ for all $s \in [0,1]$. Furthermore, the convexity of I_α implies that for each $y^\pm(s)$ the ball $\overline{B_\varepsilon(y^\pm(s))}$ intersects G only at the corresponding $x(s)$, which implies that $G \subset \partial E_\varepsilon$ for the set $E := \{y^\pm(s) \,:\, s \in [0,1]\}$.

[2] One way to construct such a function is to write $\mathbb{Q} \cap [0,1] = \big\{q_n \,:\, n \in \mathbb{N}\big\}$, define $N(x) := \{n \,:\, q_n \leq x\}$, take any positive summable sequence $(a_n)_{n=1}^\infty$, and set $\alpha(x) := \sum_{n\in N(x)} a_n$ for all $x \in [0,1]$. This way α is increasing on $[0,1]$, has a jump of amplitude a_n at each rational $x = q_n$, is continuous at every $x \in [0,1] \setminus \mathbb{Q}$, and satisfies $\alpha(1) = \sum_{n=1}^\infty a_n < \infty$.

By construction, $y^+(p) = y^-(p)$ for the irrational $p \in [0,1] \setminus \mathbb{Q}$, whereas $y^+(q) \neq y^-(q)$ with $a^-(q) < a^+(q)$ for all rational $q \in \mathbb{Q} \cap [0,1]$. Hence $x(q)$ is a wedge for each rational q while $x(p) \in \mathrm{Unp}_\varepsilon(E)$ for every irrational p, which implies that the points $x(p)$ are in fact two-sided shallow singularities (S5). Due to the continuity of I_α, these points form a dense set on G. Given that I_α is convex and absolutely continuous as an integral function, and the derivative α is bounded, I_α is in fact Lipschitz continuous. It then follows from the basic properties of Hausdorff measure (see for instance [AFP00, Proposition 2.49]) and the rectifiability of ∂E_ε (see [RW10, Proposition 2.3] and [Fal85, Corollary 3.3]) that $\mathcal{H}^1([0,1]) \leq \mathcal{H}^1(G) < \infty$ and the set $P := \Big\{(p, I_\alpha(p)) \;:\; p \in [0,1] \setminus \mathbb{Q}\Big\}$ of two-sided singularities has full measure on G.

Lemma 5.1.6 (Number of one-sided chain singularities). *For a compact set $E \subset \mathbb{R}^2$, the number of one-sided chain singularities (S6) on ∂E_ε is at most countably infinite.*

Proof. Write C for the set of one-sided chain singularities on ∂E_ε. We argue that there exists a collection $\{A_x\}_{x \in C}$ of pair-wise disjoint open sets $A_x \subset \operatorname{int} E_\varepsilon$, indexed by C. The result then follows from the counting argument discussed in the lead-up to Lemma 5.1.4 above.

For each $x \in C$, the set of outward directions is a singleton $\Xi_x(E_\varepsilon) = \{\xi\}$. Let $x \in C$ and let $\mathcal{G}(x)$ be a local boundary representation at x so that for $i \in \{1,2\}$ and $g_{\xi,y_i} \in \mathcal{G}(x)$

$$g_{\xi,y_i}(s) = x + s\xi + f^{\xi,y_i}(s)(x - y_i)$$

for some continuous functions $f^{\xi,y_i} : [0, \varepsilon/2] \to \mathbb{R}$. It follows from the definition of one-sided chain singularities that the extremal contributors $y_1, y_2 \in \Pi_E^{\mathrm{ext}}(x)$ satisfy $y_1 - x = -(y_2 - x)$ for all $x \in C$. Since $-\xi \notin \Xi_x(E_\varepsilon)$, there are two possibilities:

(i) there exists some non-extremal contributor $y \in \Pi_E(x)$ for which $\langle y - x, \xi\rangle < 0$, or else

(ii) there exists some $\delta_0 \in \mathbb{R}_+$ such that for all $\delta < \delta_0$,

$$\overline{B_\varepsilon(E \cap U_\delta(y_1, -\xi))} \cap \overline{B_\varepsilon(E \cap U_\delta(y_2, -\xi))} \setminus \{x\} \neq \varnothing$$

(see Proposition 3.1.1 for clarification). In both cases there exists some $p(x) < 0$ for which

$$Q(x) := \Big\{x + s\xi + t(x - y_1) \;:\; p(x) < s < 0,\ -\varepsilon/2 < t < \varepsilon/2\Big\} \subset \operatorname{int} E_\varepsilon. \tag{5.1.6}$$

One can then define $A_x := U_{p(x)/3}(x, -\xi)$. Note that for all $x \in C$ the set A_x is open and has a positive surface area $\mathcal{H}^2(A_x) > 0$. Furthermore, $A_x \subset Q(x) \subset \operatorname{int} E_\varepsilon$ and it follows from the construction that if $z \neq x$ is any other one-sided chain singularity, its distance from x satisfies $|z - x| \geq p(x)$, implying $A_x \cap A_z = \varnothing$. The sets A_x are thus pair-wise disjoint, open and contained in some bounded ball $B_R(0)$ due to the compactness of E_ε.

Hence the sum $\sum_{x \in C} \mathcal{H}^2(A_x)$ of their surface areas is finite, from which the result follows by the counting argument discussed in the lead-up to Lemma 5.1.4. □

Proof of Theorem 2

We conclude this section with the proof of Theorem 2, which combines Lemmas 5.1.2–5.1.4 and 5.1.6 into one statement.

Theorem 2 (Countable sets of singularities). *For any compact set $E \subset \mathbb{R}^2$ and $\varepsilon > 0$, the number of wedges (S1), sharp singularities (S2, S3 and S8) and one-sided shallow singularities (S4) and chain singularities (S6) on ∂E_ε is at most countably infinite.*

Proof. Consider the collection $\{U_i\}_{i \in I}$ of the connected components of E_ε^c. Since E is assumed to be compact, $E \subset B_R(0)$ for some $R > 0$. It follows that all but one (denote this by U_j) of the connected components U_i are bounded, so that

$$\bigcup_{i \in I \setminus \{j\}} U_i \subset B_R(0).$$

Following the counting argument presented immediately before Lemma 5.1.4 above, this implies that the index set I is at most countably infinite. By definition every sharp singularity (S2, S3, S8) $x \in \partial E_\varepsilon$ satisfies $x \in \bigcup_{i \in I} \partial U_i$. It follows then from Lemma 5.1.3 that the set of sharp singularities on ∂E_ε is countable as a countable union of finite sets. Finally, Lemmas 5.1.2, 5.1.4 and 5.1.6 guarantee that the number of wedges (S1) and one-sided shallow (S4) and chain (S6) singularities (respectively) are at most countably infinite, and the proof is complete. □

5.2 Chain Singularities Form a Totally Disconnected Set

We conclude the chapter by showing that the set $\mathcal{C}(\partial E_\varepsilon)$ of chain singularities (types S6–S8) is closed and totally disconnected. This implies that $\mathcal{C}(\partial E_\varepsilon)$ is nowhere dense, meaning that it is small in the topological sense, even though it may have a positive one-dimensional Hausdorff measure on the boundary.

Before presenting the proof of the above result, we provide a concrete example of a set $E \subset \mathbb{R}^2$ and $\varepsilon > 0$ for which the one-dimensional Hausdorff measure of the set of chain-chain singularities on ∂E_ε is positive. Essentially, we analyse [RW10, Example 2.2] from the geometric point of view.

Example 5.2.1 (A set of chain singularities with positive measure). Let $C \subset [0, 1]$ be a 'fat' Cantor set (a Cantor set with positive one-dimensional Hausdorff measure) and consider the set $E := \Big\{(s,t) \in \mathbb{R}^2 \,:\, s \in C,\, t \in \{0,1\}\Big\}$. The Cantor set is obtained by removing from the interval $[0,1]$ a certain countable collection $\mathcal{I} := \{I_n \,:\, n \in \mathbb{N}\}$ of open subintervals. By construction C is totally disconnected and thus contains no intervals itself, and since it is uncountable, most of the points $s \in C$ do not lie on the boundary of any of the removed intervals. Denote the collection of these points by $C^* := C \setminus \bigcup_{n\in\mathbb{N}} \partial I_n$. By construction, every $s \in C^*$ is however an accumulation point of $\bigcup_{n\in\mathbb{N}} I_n$. Since the sets I_n are open, it is clear that for $\varepsilon = 1/2$ the sets $V_n := \{(s,1/2) \,:\, s \in I_n\}$ satisfy $V_n \subset E_\varepsilon^c$ for all $n \in \mathbb{N}$. Thus, for every point $x \in A := \{(s,1/2) \,:\, s \in C^*\} \subset \partial E_\varepsilon$ there exists a sequence $(w_m)_{m=1}^\infty \subset E_\varepsilon^c$, where $w_m = (s_m, 1/2) \in V_m$ for some $s_m \in I_{n(m)}$, and $s_m \to s$ as $m \to \infty$. We can also assume that $I_{n(m)} \neq I_{n(m')}$ whenever $m \neq m'$. For the connected components $W_m \supset V_m$ of E_ε^c this implies $W_m \neq W_{m'}$ whenever $m \neq m'$. It is easy to see that condition (ii) in Proposition 4.2.4 thus holds true for all $x \in A$ so that $A \subset \mathcal{C}(\partial E_\varepsilon)$. Hence $\mathcal{H}^1(\mathcal{C}(\partial E_\varepsilon)) \geq \mathcal{H}^1(A) = \mathcal{H}^1(C^*) > 0$ due to the translation invariance of Hausdorff measure.

Proof of Theorem 3

The proof of our third main result builds on many of the results presented in the previous sections. To show that the set of chain singularities is closed, we combine Proposition 3.3.3 regarding the connectedness of the complement E_ε^c near wedges and $x \in \mathrm{Unp}_\varepsilon(E)$ with the characterisation of chain singularities provided by Proposition 4.2.4. The second part of the proof also makes use of the basic results established in Section 2.2.

Theorem 3 (**The set of chain singularities is closed and totally disconnected**). *For any compact set $E \subset \mathbb{R}^2$ and $\varepsilon > 0$, the set $\mathcal{C}(\partial E_\varepsilon)$ of chain singularities is closed and totally disconnected.*

Proof. We begin by showing that the complement $\partial E_\varepsilon \setminus \mathcal{C}(\partial E_\varepsilon)$ is open. To this end, consider some $x \in \partial E_\varepsilon \setminus \mathcal{C}(\partial E_\varepsilon)$. If x is a wedge (S1) or if $x \in \mathrm{Unp}_\varepsilon(E)$, Proposition 3.3.3 implies that there exists some neighbourhood $B_r(x)$ and a connected subset $V_x \subset E_\varepsilon^c$ for which

$$B_r(x) \cap E_\varepsilon^c = B_r(x) \cap V_x. \tag{5.2.1}$$

One the other hand, Proposition 4.2.4 states that each chain singularity $x \in \mathcal{C}(\partial E_\varepsilon)$ is associated with a sequence $(V_n)_{n=1}^\infty \subset E_\varepsilon^c$ of disjoint connected components of the complement E_ε^c, for which $\mathrm{dist}_H(x, V_n) \to 0$ as $n \to \infty$. Equation (5.2.1) hence implies that $B_r(x) \cap \mathcal{C}(\partial E_\varepsilon) = \varnothing$. Similarly, for a sharp singularity (type S2) or a sharp-sharp singularity (type S3), Proposition 4.2.1 implies the existence of a neighbourhood $B_r(x)$ for which $B_r(x) \cap \mathcal{C}(\partial E_\varepsilon) = \varnothing$. Hence $\partial E_\varepsilon \setminus \mathcal{C}(\partial E_\varepsilon)$ is open on the boundary.

To demonstrate that $\mathcal{C}(\partial E_\varepsilon)$ is totally disconnected, we show that for any two chain singularities $x, z \in \mathcal{C}(\partial E_\varepsilon)$ there exist disjoint open sets $A_x, A_z \subset \mathbb{R}^2$ for which $x \in A_x, z \in$

A_z and $\mathcal{C}(\partial E_\varepsilon) \subset (A_x \cup A_z) \cap \partial E_\varepsilon$. More specifically, we will consider for each $x \in \mathcal{C}(\partial E_\varepsilon)$ and $s_1 \leq 0 \leq s_2$ the sets

$$A_{x,\xi}(s_1, s_2) := \Big\{x + s\xi + t(x - y_1) \,:\, s_1 < s < s_2,\ -\varepsilon/2 < t < \varepsilon/2\Big\} \tag{5.2.2}$$

and show that for each $z \in \mathcal{C}(\partial E_\varepsilon) \setminus \{x\}$ there exist $\xi \in \Xi_x^{\mathrm{ext}}(E_\varepsilon)$ and $s_1 < 0 < s_2$ for which $\partial A_{x,\xi}(s_1, s_2) \cap \mathcal{C}(\partial E_\varepsilon) = \varnothing$ and $z \notin A_{x,\xi}(s_1, s_2)$.

Given that x is a chain singularity, Proposition 4.2.1 implies that for $r > 0$ the boundary region $\partial E_\varepsilon \cap U_r(x, \xi)$ exhibits 'chain-type' geometry near x for at least one extremal outward direction $\xi \in \Xi_x^{\mathrm{ext}}(E_\varepsilon)$. We begin by assuming that ξ is such a direction, and consider the corresponding sets $A_{x,\xi}(0, s)$ for $s > 0$. Writing $R(z) := \|z - x\|$, our aim is to find some $s < R(z)/2$ for which $\partial A_{x,\xi}(0, s) \cap \mathcal{C}(\partial E_\varepsilon) = \{x\}$. To this end, let $\mathcal{G}(x)$ be a local boundary representation at x and let $f^{\xi,y_1}, f^{\xi,y_2} : [0, \varepsilon/2] \to \mathbb{R}$ be the continuous functions for which

$$g_{\xi,y_i}(s) = x + s\xi + f^{\xi,y_i}(s)(x - y_i)$$

for every $g_{\xi,y_i} \in \mathcal{G}(x)$, $i \in \{1, 2\}$ and all $s \in [0, \varepsilon/2]$. As argued in the proof of Proposition 4.2.1 (ii), there exist sequences $(s_n)_{n=1}^\infty \subset \mathbb{R}_+$ and $(p_n)_{n=1}^\infty \subset \mathbb{R}_+$ with $s_n \to 0$ and $p_n \to 0$, and for which

- $p_n < s_n \leq p_{n-1} < s_{n-1}$ for all $n \in \mathbb{N}$,
- $f^{\xi,y_1}(s) + f^{\xi,y_2}(s) = 0$ for all $s \in (s_n)_{n=1}^\infty \cup (p_n)_{n=1}^\infty$, and
- $f^{\xi,y_1}(s) + f^{\xi,y_2}(s) < 0$ for all $s \in (p_n, s_n)$ and $n \in \mathbb{N}$.

It follows that the open set

$$V_n := \Big\{\tau g_{\xi,y_1}(s) + (1 - \tau) g_{\xi,y_2}(s) \,:\, \tau \in (0, 1),\ s \in (p_n, s_n)\Big\}$$

is a connected component of the complement E_ε^c for all $n \in \mathbb{N}$. Consequently there exists some $N \in \mathbb{N}$ for which $\mathrm{dist}_H(V_n, x) < R(z)/2$ whenever $n \geq N$. The definition of the sets V_n implies that for each $n \geq N$, the boundary point $x_n^{(1)} := g_{\xi,y_1}\left(\frac{p_n + s_n}{2}\right) \in \partial V_n \subset \partial E_\varepsilon$ has an outward direction aligned with the vector

$$\eta_n^{(1)} := g_{\xi,y_2}\left(\frac{p_n + s_n}{2}\right) - g_{\xi,y_1}\left(\frac{p_n + s_n}{2}\right) = -\left(f^{\xi,y_1}\left(\frac{p_n + s_n}{2}\right) + f^{\xi,y_2}\left(\frac{p_n + s_n}{2}\right)\right)(x - y_1)$$

(the reader is encouraged to compare this with equation (4.2.3) in the proof of Proposition 4.2.1), and the analogous expression (obtained by replacing the roles of y_1 and y_2) holds true for $x_n^{(2)}$ and $\eta_n^{(2)}$ defined similarly. We claim that there exists some $n \geq N$, for which $x_n^{(i)} \notin \mathcal{C}(\partial E_\varepsilon)$ for $i \in \{1, 2\}$. Assume this were not the case. Then it follows from the definition of chain-singularities and Proposition 2.2.6 that for at least one $i \in \{1, 2\}$ we have $\eta_n^{(i)} \in \Xi_{x_n^{(i)}}^{\mathrm{ext}}(E_\varepsilon)$ for infinitely many $n \geq N$. By virtue of the definition of the sets V_n as regions between the graphs g_{ξ,y_i}, and since $x_n^{(i)} \in \partial V_n \subset \partial E_\varepsilon$ for all $n \in \mathbb{N}$, it follows from Proposition 2.2.8 that $\xi = \lim_{n\to\infty} \left(x_n^{(i)} - x\right)\big/\left\|x_n^{(i)} - x\right\|$ for $i \in \{1,2\}$. But then, due to Lemma 2.2.7 (ii)(b), we have

$$0 = \left\langle \frac{x - y_i}{\varepsilon}, \xi \right\rangle = \lim_{n\to\infty} \left\langle \eta_n^{(i)}, \xi \right\rangle = 1,$$

which is impossible. Thus, there exists some $n \in \mathbb{N}$ for which $x_n^{(i)} \notin \mathcal{C}(\partial E_\varepsilon)$ for $i \in \{1,2\}$, which in turn implies that $\partial A_{x,\xi}\big(0, (p_n + s_n)/2\big) \cap \mathcal{C}(\partial E_\varepsilon) = \{x\}$, since

$$\partial A_{x,\xi}\left(0, \frac{p_n + s_n}{2}\right) \cap \partial E_\varepsilon = \left\{x, x_n^{(1)}, x_n^{(2)}\right\}.$$

In the remainder of the proof we consider one by one the cases of one-sided chain (S6), chain-chain (S7) and sharp-chain (S8) singularities, and identify the sets A_x and A_z mentioned in the beginning of the proof.

(i) Assume x is a one-sided chain singularity (S6), so that $\Xi_x^{\text{ext}}(E_\varepsilon) = \{\xi\}$ for some $\xi \in S^1$. By the argument presented above, there exists some $0 < s_2 < R(z)/2$ for which $\partial A_{x,\xi}\big(0, s_2\big) \cap \mathcal{C}(\partial E_\varepsilon) = \{x\}$. On the other hand, according to the reasoning presented in the proof of Lemma 5.1.6, the set

$$Q_p(x) := \Big\{x + s\xi + t(x - y_1) \,:\, p < s < 0,\ -\varepsilon/2 < t < \varepsilon/2\Big\}$$

satisfies $Q_p(x) \subset \operatorname{int} E_\varepsilon$ for some $p < 0$, see (5.1.6). By setting $s_1 := -\min\{p, R(z)/2\}$, it follows then that $\partial A_{x,\xi}(s_1, s_2) \cap \mathcal{C}(\partial E_\varepsilon) = \varnothing$ and we may define $A_x := A_{x,\xi}(s_1, s_2)$ and $A_z := \left(\mathbb{R}^2 \setminus \overline{A_x}\right)$.

(ii) Assume then that x is a chain-chain singularity (S7) with $\Xi_x^{\text{ext}}(E_\varepsilon) = \{\xi, -\xi\}$ for some $\xi \in S^1$. Since now both of the extremal outward directions ξ and $-\xi$ are associated with 'chain'-type geometry, one can again utilise the argument above in order to choose for $\xi_1 := \xi$ and $\xi_2 := -\xi$ the corresponding $s_1, s_2 > 0$ for which $\partial A_{x,\xi_i}\big(0, s_i\big) \cap \mathcal{C}(\partial E_\varepsilon) = \{x\}$ and $s_i < R(z)/2$ for $i \in \{1,2\}$. By setting $s_3 = \min\{s_1, s_2\}$ we may define $A_x := A_{x,\xi}(-s_3, s_3)$ and $A_z := \left(\mathbb{R}^2 \setminus \overline{A_x}\right)$.

(iii) Finally, assume that x is a sharp-chain singularity (S8) with $\Xi_x^{\text{ext}}(E_\varepsilon) = \{\xi, -\xi\}$ for some $\xi \in S^1$. We may assume that ξ is associated with 'chain'-type geometry, so that once again we have $\partial A_{x,\xi}\big(0, s_2\big) \cap \mathcal{C}(\partial E_\varepsilon) = \{x\}$ for some $0 < s_2 < R(z)/2$ due to the arguments presented above. For the direction $-\xi$, Proposition 4.2.1 (i) implies that there exists some $r > 0$ for which

$$E_\varepsilon^c \cap U_r(x, -\xi) = V \cap U_r(x, -\xi) = \bigcup_{0<s<r} x + s\big(\alpha(s), \beta(s)\big)_{S^1},$$

where V is the unique connected component of E_ε^c intersecting $U_r(x, -\xi)$, $\alpha(s), \beta(s) \in S^1$ for all $s \in (0, r)$ and $\alpha(s), \beta(s) \to -\xi$ as $s \to 0$. It follows then from the definition of chain singularity that $U_r(x,-\xi) \cap \mathcal{C}(\partial E_\varepsilon) = \varnothing$. By setting $s_1 = -\min\{r/2, \varepsilon/2, R(z)/2\}$ we may thus define $A_x := A_{x,\xi}(s_1, s_2)$ and $A_z := \left(\mathbb{R}^2 \setminus \overline{A_x}\right)$ and the proof is complete. $\square$

The set $\mathcal{I}$ of inaccessible singularities (types S6 and S7) inherits the properties of being totally disconnected and nowhere dense, but it may generally fail to be closed. Since the set $\mathcal{C}(\partial E_\varepsilon)$ on the other hand is compact and separable as a subset of $\mathbb{R}^2$, it follows from the Cantor-Bendixson Theorem (see [Kec95, Thm. 6.4]) that whenever the cardinality of the chain-chain singularities (type S7) is uncountable, the set $\mathcal{C}(\partial E_\varepsilon)$ can be written as a disjoint union $\mathcal{C}(\partial E_\varepsilon) = C \cup P$, where C is homeomorphic to the Cantor set and P is countable. For further information on totally disconnected spaces, see [Kec95].

Chapter 6

Global Topology of the Boundary

In this chapter we show that for a compact set $E \subset \mathbb{R}^2$ and $\varepsilon > 0$, the boundary ∂E_ε is the disjoint union of a possibly uncountable set $\mathcal{I}$ of inaccessible singularities and an at most countably infinite collection J of Jordan curves.[1] By definition, inaccessible singularities do not lie on the boundary of any connected component of the complement, and according to Corollary 4.2.2 these correspond to *chain* (S6) and *chain-chain singularities* (S7), see Figure 6.1(b).

Theorem 4 (Global structure of the boundary). *For a compact set $E \subset \mathbb{R}^2$ and $\varepsilon > 0$, the boundary ∂E_ε is a disjoint union*

$$\partial E_\varepsilon = \mathcal{I} \cup J,$$

where $\mathcal{I}$ is the set of inaccessible singularities and $J = \bigcup_{i \in I} J_i$ is a countable (possibly finite) union of Jordan curves J_i. Furthermore there is a unique representation with the property that each Jordan curve J_i satisfies $J_i \subset \partial U$ for some connected component U of the complement E_ε^c.

We leverage all the results of the previous chapters in the proof of Theorem 4. According to Theorem 1, each boundary point $x \in \partial E_\varepsilon$ is either smooth or belongs to exactly one of eight distinct types of singularities. From the topological point of view, these can further be divided into three groups:

(i) Point components of the boundary: the *inaccessible singularities.*

(ii) Boundary points, around which the boundary ∂E_ε can be uniquely represented as a simple curve: these are the smooth points as well as *wedges*, *sharp* and *sharp-chain singularities*[2] and *shallow singularities.*

[1] We call the image $\Gamma := \gamma([a, b])$ of a closed interval $[a, b]$ under a continuous map $\gamma : [a, b] \to \mathbb{R}^2$ a *curve.* If γ is injective, we call Γ a *simple curve*, and if injectivity is violated only at the endpoints, i.e. $\gamma(a) = \gamma(b)$, we say that Γ is a *simple closed curve* or a *Jordan curve.*

[2] More precisely, the boundary of the unique connected component of the complement that touches the sharp-chain singularity can be uniquely represented by a simple curve in any sufficiently small neigh-

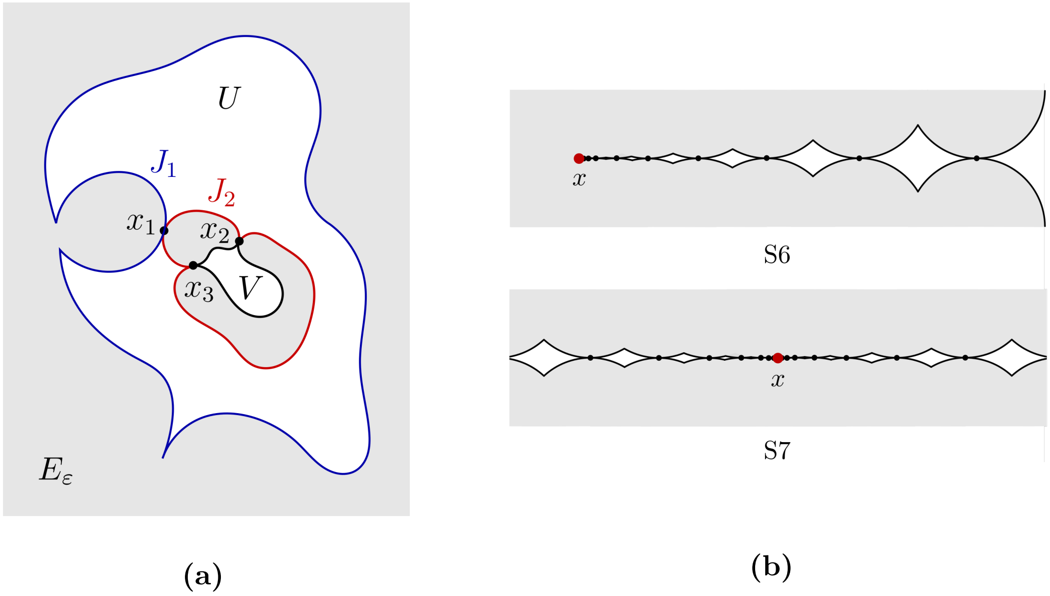


Figure 6.1. Schematic illustration of Theorem 4. **(a)**: Jordan curve subsets of the boundary of a connected component U of the complement $\mathbb{R}^2 \setminus E_\varepsilon$. Here $\partial U = J_1 \cup J_2$ and $J_1 \cap J_2 = \{x_1\}$. At the centre of the figure there is another connected component $V \subset \mathbb{R}^2 \setminus E_\varepsilon$, with $\partial V \cap \partial U = \{x_2, x_3\}$. Only the sharp-sharp singularities x_1, x_2 and x_3 allow a choice regarding how to locally continue the boundary curves in order to obtain Jordan curves. **(b)**: The two types of inaccessible singularities. A chain singularity (S6) is the limit (in terms of Hausdorff distance) of a sequence of mutually disjoint connected components of the complement. Such a sequence approaches a chain-chain singularity (S7) on both sides. These are the only types of boundary points that do not lie on Jordan curve subsets on the boundary.

(iii) Boundary points, around which there is more than one way to locally represent the boundary as a union of simple curves: the so-called *sharp-sharp singularities*.

Whenever sharp-sharp singularities exist on the boundary ∂E_ε, there are several ways to represent the boundary as a union of curves. It turns out, however, that the boundary of each individual connected component of the complement $\mathbb{R}^2 \setminus E_\varepsilon$ has a unique representation as a finite union of Jordan curves. In this chapter we construct such representations using the local boundary representations given by Proposition 3.2.4, and use them to arrive at a global representation of the boundary ∂E_ε as a union of Jordan curves and point components, cf. Figure 6.1.

Intuitively, Jordan curves can be identified on the boundary by connecting adjacent simple curves that represent local boundary segments, provided by the local boundary representations. There is an essentially unique way of connecting such curve representations around all types of boundary points, except sharp-sharp singularities, where the boundary curves 'split', see Figure 6.2. However, Proposition 6.2.3 asserts that for each connected component U of the complement E_ε^c, there exists a unique way of connecting the

bourhood, cf. Lemma 6.2.1.

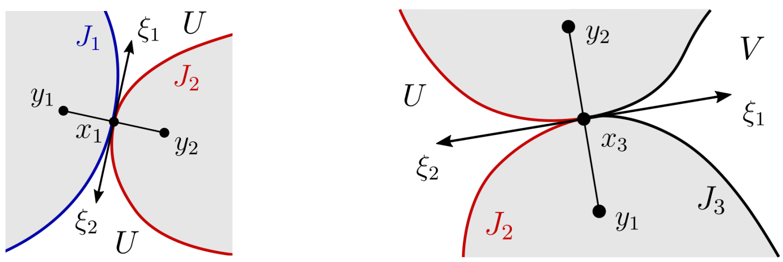
curves around sharp-sharp singularities that results in a representation of the boundary ∂U as a finite union of Jordan curves.

6.1 The Role of Sharp-sharp Singularities in Boundary Topology

Let U be a connected component of the complement E_ε^c and let Q denote the set of sharp-sharp singularities on ∂U. According to Lemma 5.1.3 the set Q is finite, and Proposition 4.2.1 implies that each $x \in Q$ lies on the boundary of at most two connected components of the complement E_ε^c. Hence,

$$Q = Q_1 \cup Q_2, \tag{6.1.1}$$

where Q_1 contains those sharp-sharp singularities that lie on the boundary of a unique connected component U of the complement E_ε^c, and Q_2 contains those $x \in Q$ for which there exist two disjoint connected components $U, V \in E_\varepsilon^c$ with $x \in \partial U \cap \partial V$. The fact that both Q_1 and Q_2 may be non-empty is the reason sharp-sharp singularities play a central role in the topology of the boundary ∂U.



(a) $x_1 \in Q_1$. **(b)** $x_3 \in Q_2$.

Figure 6.2. Topology of the complement E_ε^c around sharp-sharp singularities. The figures here are close-ups of the points x_1 and x_3 in Figure 6.1(a). **(a)** The sharp-sharp singularity $x_1 \in Q_1$ lies on the boundary of a single connected component U of the complement E_ε^c. Here the curves $\Gamma_{x_1}^1$ (blue) and $\Gamma_{x_1}^2$ (red) 'flow through' x_1 along the boundary subsets J_1 and J_2. **(b)** The point $x_3 \in Q_2$, on the other hand, lies on the boundary of two different connected components $U, V \subset E_\varepsilon^c$. The curve Γ_{x_3} (red) 'bounces back' at x_3 along the boundary subset J_2.

The boundary ∂U of a single connected component U may thus consist of more than one Jordan curve for two reasons. First, U need not be simply connected, and can thus surround several subsets $A \subset E_\varepsilon$. The boundary of each such A contains at least one Jordan curve. Second, a single curve containing the boundary subset $B_r(x) \cap \partial U$ for any $x \in Q_1$ would intersect itself at x, and would thus not be a Jordan curve.

6.2 Boundaries of Connected Components of the Complement

Let the sets U and $Q = Q_1 \cup Q_2$ be as above. Proposition 3.2.4 asserts that each $x \in \partial U$ has a neighbourhood $B_x := B_r(x)$, within which the local boundary subset $\partial U \cap \overline{B_x}$ can be represented as a union of either two or four graphs of continuous functions. We show in Lemma 6.2.1 below that it is possible to use these functions to represent the boundary subset $\partial U \cap \overline{B_x}$ with either one (for $x \in \partial U \setminus Q_1$) or two (for $x \in Q_1$) simple curves.

Lemma 6.2.1 (**Boundaries of connected components of the complement as unions of simple curves**). *Let $E \subset \mathbb{R}^2$ be compact, let $\varepsilon > 0$ and let U be a connected component of the complement E_ε^c. For each $x \in \partial U$, let $B_x := B_r(x)$ be the neighbourhood in (3.2.3), corresponding to the local boundary representation $\mathcal{G}(x)$ given by Proposition 3.2.4. In addition, denote by Q the set of sharp-sharp singularities on ∂U, and let the subsets $Q_1, Q_2 \subset Q$ be as in (6.1.1). Then*

(i) for each $x \in Q_1$ there exist continuous functions $\gamma_x^{(1)}, \gamma_x^{(2)} : [0,1] \to \partial U$ for which the images $\Gamma_x^{(i)} := \gamma_x^{(i)}([0,1])$ with $i \in \{1,2\}$ are simple curves, $\partial U \cap \overline{B_x} = \Gamma_x^{(1)} \cup \Gamma_x^{(2)}$ and $\Gamma_x^{(1)} \cap \Gamma_x^{(2)} = \{x\}$;

(ii) for each $x \in \partial U \setminus Q_1$ there exists a continuous function $\gamma_x : [0,1] \to \partial U$, for which the image $\Gamma_x := \gamma_x([0,1])$ is simple curve and $\partial U \cap \overline{B_x} = \Gamma_x$.

Proof. We begin by fixing some notations. For each $x \in \partial U$, let $\Xi_x(E_\varepsilon)$ be the set of *outward directions* and $\Pi_E(x)$ the set of *contributors* at x, see Definitions 2.0.1 and 2.1.2. We denote by $\mathcal{P}_x^{\text{ext}}(E_\varepsilon)$ the set of *extremal pairs* (see Definition 2.1.4):

$$\mathcal{P}_x^{\text{ext}}(E_\varepsilon) = \Big\{(\xi, y) \in \Xi_x(E_\varepsilon) \times \Pi_E(x) \,:\, \langle x - y, \xi\rangle = 0\Big\}.$$

Intuitively, extremal pairs define coordinate axes that are adapted to the local geometry at x, see 3.2.3. Proposition 3.2.4 asserts that for each extremal pair $(\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$ there exists a continuous function $f^{\xi,y} : [0, \varepsilon/2] \to \mathbb{R}$ and a function $g_{\xi,y} : [0, \varepsilon/2] \to \mathbb{R}^2$, given by

$$g_{\xi,y}(s) := x + s\xi + f^{\xi,y}(s)(x - y), \tag{6.2.1}$$

so that the collection $\mathcal{G}(x) := \{g_{\xi,y} \,:\, (\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)\}$ satisfies

$$\partial E_\varepsilon \cap \overline{B_r(x)} = \bigcup_{(\xi,y)\in\mathcal{P}_x^{\text{ext}}(E_\varepsilon)} g_{\xi,y}\left(A_{\xi,y}\right) \tag{6.2.2}$$

for some $r > 0$ and some closed sets $A_{\xi,y} \subset [0, r]$. We call the collection $\mathcal{G}(x)$ a *local boundary representation (of radius r)* at x.

We divide the proof into two steps.

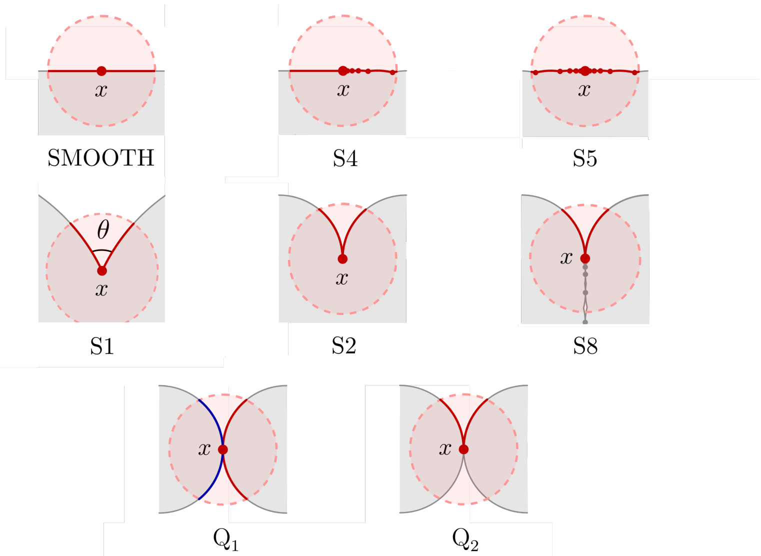


Figure 6.3. Local curve representations. In each picture, x represents the boundary point in question, the grey region represents the set E_ε, and white regions the complement E_ε^c. The red dashed balls correspond to the neighbourhoods $B_r(x)$ associated with the local boundary representations $\mathcal{G}(x)$, see (6.2.2). The dark red curves (and the dark blue curve for $x \in Q_1$) represent the simple curve representations $\Gamma_x = \gamma_x([0,1])$ of $\partial U \cap \overline{B_r(x)}$. Chain (S6) and chain-chain (S7) singularities are absent as these do not appear on boundaries of connected components of the complement, see Corollary 4.2.2. The types Q_1 and Q_2 together represent sharp-sharp (S3) singularities. Note that points of type Q_1 are the only ones for which two simple curves are needed in a local representation of the boundary of a fixed connected component U of the complement E_ε^c, see also Figure 6.2.

Step 1. In the first step, we define the local curve representations of the boundary ∂U around sharp-sharp singularities $x \in Q$. We also show that the images Γ_x and $\Gamma_x^{(i)}$ are simple (i.e. non-self-intersecting) curves, and that $\Gamma_x^{(1)} \cap \Gamma_x^{(2)} \cap B_r(x) = \{x\}$ for sufficiently small radii $r > 0$.

Consider a boundary point $x \in \partial U$, and let $\mathcal{G}(x) = \{g_{\xi,y} : (\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)\}$ be a local boundary representation at x, with the functions $g_{\xi,y}$ defined as in (6.2.1). By the definition of sharp-sharp singularities (see Definition 4.1.1), the set of extremal outward directions for each $x \in Q$ satisfies $\Xi_x^{\text{ext}}(E_\varepsilon) = \{\xi, -\xi\}$ for some $\xi \in S^1 \subset \mathbb{R}^2$. Proposition 4.2.1 furthermore states that there exist connected components $V_\xi, V_{-\xi}$ of the complement E_ε^c, for which

$$\partial U \cap \overline{B_x} \subset \Big(\partial V_\xi \cup \partial V_{-\xi}\Big) \cap \overline{B_x}. \tag{6.2.3}$$

Since $Q \subset \partial U$, we have $U \in \{V_\xi, V_{-\xi}\}$ and we may define ξ so that $U = V_\xi$. The correct way of defining the local curve representation near x now depends on whether $x \in Q_1$ or $x \in Q_2$, where $Q = Q_1 \cup Q_2$ as in (6.1.1). We deal with these cases separately as follows:

(i) Let $x \in Q_1$. This implies that U touches the point x from both directions, so that

$U = V_\xi = V_{-\xi}$. This corresponds to the situation in Figure 6.2(a). Then

$$\partial U \cap \overline{B_x} = \bigcup_{i \in \{1,2\}} \Big(g_{\xi,y_i}\big([0, s_{\xi,y_i}]\big) \cup g_{-\xi,y_i}\big([0, s_{-\xi_i,y_i}]\big)\Big). \tag{6.2.4}$$

In this case, we associate a curve segment with each extremal contributor $y_1, y_2 \in \Pi_E^{\text{ext}}(x)$. Thus, we define for each $i \in \{1,2\}$ a continuous map $\gamma_x^{(i)} : [0,1] \to \partial U \cap \overline{B_x}$ by

$$\gamma_x^{(i)}(s) := \begin{cases} g_{-\xi,y_i}\big(s_{-\xi,y_i} - sL^{(i)}\big), & \text{if } s \in \Big[0, \frac{s_{-\xi,y_i}}{L^{(i)}}\Big), \\ g_{\xi,y_i}\big(sL^{(i)} - s_{-\xi,y_i}\big), & \text{if } s \in \Big[\frac{s_{-\xi,y_i}}{L^{(i)}}, 1\Big], \end{cases} \tag{6.2.5}$$

where $g_{-\xi,y_i}, g_{\xi,y_i} \in \mathcal{G}(x)$, and $L^{(i)} := s_{\xi,y_i} + s_{-\xi,y_i}$ with s_{ξ,y_i} and $s_{-\xi,y_i}$ as in (6.2.4). We thus obtain $\partial U \cap \overline{B_x} = \Gamma_x^{(1)} \cup \Gamma_x^{(2)}$, where $\Gamma_x^{(i)} := \gamma_x^{(i)}([0,1])$ for $i \in \{1,2\}$.

(ii) Let $x \in Q_2$. Since in this case $U = V_\xi \neq V_{-\xi}$, the graphs $g([0, s_{-\xi,y_1}])$ and $g([0, s_{-\xi,y_2}])$, which correspond to the direction $-\xi$, intersect the local boundary subset $\partial U \cap B_x$ only at x, and are thus not relevant for representing ∂U. This corresponds to the situation in Figure 6.2(b). Consequently we have

$$\partial U \cap \overline{B_x} = g_{\xi,y_1}\big([0, s_{\xi,y_1}]\big) \cup g_{\xi,y_2}\big([0, s_{\xi,y_2}]\big) \tag{6.2.6}$$

for $g_{\xi,y_1}, g_{\xi,y_2} \in \mathcal{G}(x)$. We thus define

$$\gamma_x(s) := \begin{cases} g_{\xi,y_1}\big(s_{\xi,y_1} - sL\big), & \text{if } s \in \Big[0, \frac{s_{\xi,y_1}}{L}\Big), \\ g_{\xi,y_2}\big(sL - s_{\xi,y_1}\big), & \text{if } s \in \Big[\frac{s_{\xi,y_1}}{L}, 1\Big], \end{cases} \tag{6.2.7}$$

where $g_{\xi,y_1}, g_{\xi,y_2} \in \mathcal{G}(x)$, and $L := s_{\xi,y_1} + s_{\xi,y_2}$ with s_{ξ,y_1} and s_{ξ,y_2} as in (6.2.6). In this case, we obtain a representation using only one curve, since $\partial U \cap \overline{B_x} = \Gamma_x := \gamma_x([0,1])$.

For $x \in Q_1$, we define $\Gamma_x^{(i)} := \gamma_x^{(i)}([0,1])$ for $i \in \{1,2\}$, where the functions $\gamma_x^{(i)}$ are as in (6.2.5). Then $\partial U \cap \overline{B_x} = \Gamma_x^{(1)} \cup \Gamma_x^{(2)}$, and it follows directly from the definition of the functions $g_{\xi,y} \in \mathcal{G}(x)$ that the image $\Gamma_x^{(i)}$ is a simple curve for each $i \in \{1,2\}$. Note furthermore that sharp-sharp singularities by definition exhibit sharp-type geometry (as defined in Proposition 4.2.1 (i)) in the direction of both extremal outward directions $\xi_1, \xi_2 \in \Xi_x^{\text{ext}}(E_\varepsilon)$. Proposition 4.2.1 thus implies the existence of some $r_1, r_2 > 0$ for which $g_{\xi_i,y_1}(s) \neq g_{\xi_i,y_2}(s)$ for all $s \in (0, r_i)$ and $i \in \{1,2\}$. Hence, $\Gamma_x^{(1)} \cap \Gamma_x^{(2)} \cap B_r(x) = \{x\}$ for all $r < \min\{r_1, r_2\}$. By choosing $r < \min\{r_1, r_2\}$ for the local boundary representation in the first place, we obtain the equivalent property $\Gamma_x^{(1)} \cap \Gamma_x^{(2)} = \{x\}$.

For $x \in Q_2$, we define $\Gamma_x := \gamma_x([0,1])$, where the function γ_x is as in (6.2.7). The fact that $\partial U \cap \overline{B_x} = \Gamma_x$ follows from the definition of the curve γ_x. Similarly to above, Proposition 4.2.1 implies that $g_{\xi,y_1}(s) \neq g_{\xi,y_2}(s)$ for sufficiently small $s \in (0,r)$, which implies that the curve Γ_x is non-self-intersecting in every sufficiently small neighbourhood B_x.

Step 2. In the second step, we define curve representations, analogous to those defined in Step 1, for the other types of boundary points $x \in \partial U \setminus Q$.

We divide $\partial U \setminus Q$ into the following two subsets:

$$R := \Big\{x \in \partial U \;:\; x \text{ is a wedge or } x \in \mathrm{Unp}_\varepsilon(x)\Big\}, \tag{6.2.8}$$

$$P := \Big\{x \in \partial U \;:\; x \text{ is a sharp or a sharp-chain singularity}\Big\}. \tag{6.2.9}$$

According to Corollary 4.2.2, chain and chain-chain singularities (types S6 and S7) do not appear on the boundaries of connected components of the complement E_ε^c, which implies $\partial U = R \cup P \cup Q$. Furthermore, it follows from Theorem 1 that the sets R, P and Q are disjoint.

Assume first that $x \in R$, so that either $x \in \mathrm{Unp}_\varepsilon(x)$ or x is a wedge. Then $\mathcal{P}_x^{\mathrm{ext}}(E_\varepsilon) = \{(\xi_1, y_1), (\xi_2, y_2)\}$, where for $y_1 = y_2$ in case $x \in \mathrm{Unp}_\varepsilon(x)$. It follows immediately from Lemma 3.3.2 and Proposition 3.3.3 that $\overline{B_x} \cap E_\varepsilon^c = \overline{B_x} \cap U$, so that

$$\partial U \cap \overline{B_x} = g_{\xi_1,y_1}\Big([0, s_{\xi_1,y_1}]\Big) \cup g_{\xi_2,y_2}\Big([0, s_{\xi_2,y_2}]\Big), \tag{6.2.10}$$

where $g_{\xi_1,y_1}, g_{\xi_2,y_2} \in \mathcal{G}(x)$. By normalising the arguments with $L := s_{\xi_1,y_1} + s_{\xi_2,y_2}$, one can thus define a function $\gamma_x : [0,1] \to \partial U$ analogously to (6.2.7), so that $\partial U \cap \overline{B_x} = \Gamma_x := \gamma_x([0,1])$. The fact that Γ_x defines a simple curve in some neighbourhood $B_r(x)$ follows directly from Proposition 3.3.3.

Let then $x \in P$, so that $\Xi_x^{\mathrm{ext}}(E_\varepsilon) = \{\xi, -\xi\}$ for some $\xi \in S^1$. In the following, we denote by $U_r(x,v)$ an open x-centered half-ball of radius r, oriented in the direction of $v \in S^1$, see (3.1.2). For both sharp and sharp-chain singularities, there exists a unique extremal outward direction $v \in \{\xi, -\xi\}$, towards which the singularity exhibits sharp-type geometry, as defined in Proposition 4.2.1 (i). This means that there exists a unique connected component V_ξ of the complement E_ε^c, and $s_\xi > 0$, for which $U_{s_\xi}(x,\xi) \cap E_\varepsilon^c = U_{s_\xi}(x,\xi) \cap V_\xi$.

Moreover, in the case of sharp-chain singularities, one can use the local boundary representation (6.2.1) and Proposition 4.2.1 to show that there exist no connected components V of the complement E_ε^c, for which $x \in \partial(V \cap U_r(x,-\xi))$. The detailed argument can be found in the proof of Corollary 4.2.2. For one-sided sharp singularities this follows immediately from their definition, which states that for some $\delta > 0$, the set $B_\delta(x) \cap E_\varepsilon^c$ is a connected set.

Hence, for both sharp and sharp-chain singularities there exists some $r > 0$ for which

$$\partial U \cap \overline{B_r(x)} = g_{\xi,y_1}\Big([0, s_{\xi,y_1}]\Big) \cup g_{\xi,y_2}\Big([0, s_{\xi,y_2}]\Big), \tag{6.2.11}$$

where $g_{\xi,y_1}, g_{\xi,y_2} \in \mathcal{G}(x)$ are as in Proposition 3.2.4. One may thus once more define the function $\gamma_x : [0,1] \to \partial U$ analogously to (6.2.7) by concatenating the images $g_{\xi,y_i}\Big([0, s_{\xi,y_i}]\Big)$ in (6.2.11) and normalising the argument with $L := s_{\xi_1,y_1} + s_{\xi_2,y_2}$, so that $\partial U \cap \overline{B_x} = \Gamma_x := \gamma_x([0,1])$. $\square$

It follows from Lemma 6.2.1 that the boundary ∂U of every connected component U of the complement E_ε^c can be covered by the (infinite) collection

$$\mathcal{M} := \Big\{\Gamma_x \,:\, x \in \partial U \setminus Q_1\Big\} \cup \Big\{\Gamma_x^{(i)} \,:\, x \in Q_1,\, i \in \{1,2\}\Big\}$$

of simple curves. The compactness of E implies that ∂U is compact as well, and $\mathcal{M}$ thus always has a finite subcover. We show next that $\mathcal{M}$ moreover has a finite, order two[3] subcover $\mathcal{M}^*$ that contains all the curves $\Gamma_x^{(i)}$ with $i \in \{1,2\}$ which correspond to sharp-sharp singularities $x \in Q_1$.

Lemma 6.2.2 (Finite subcover of order two). *Let $E \subset \mathbb{R}^2$ be compact, let $\varepsilon > 0$ and let U be a connected component of the complement E_ε^c. In addition, let the functions γ_x, $\gamma_x^{(1)}, \gamma_x^{(2)} : [0,1] \to \partial U$ and the corresponding simple curves $\Gamma_x := \gamma_x([0,1])$ and $\Gamma_x^{(i)} := \gamma_x^{(i)}([0,1])$ with $i \in \{1,2\}$, be as in Lemma 6.2.1. Then there exists a finite subset $X^* \subset \partial U \setminus Q_1$ for which the collection*

$$\mathcal{M}^* := \Big\{\Gamma_x \,:\, x \in X^*\Big\} \cup \Big\{\Gamma_x^{(i)} \,:\, x \in Q_1,\, i \in \{1,2\}\Big\} \tag{6.2.12}$$

of simple curves is a finite, minimal[3], order two cover of ∂U.

Proof. Let the sets $R, P \subset \partial U$ be as in (6.2.8) and (6.2.9). We now use the curve representations Γ_x for $x \in R \cup P \cup Q_2$ and $\Gamma_x^{(i)}$ for $x \in Q_1$, defined above, in order to construct an order two cover for the boundary ∂U. To begin with, consider the open cover $\mathcal{U}_0 := \{B_x \,:\, x \in \partial U\}$. Since ∂U is compact, there exists a finite subcover $\mathcal{U}_1 = \{B_x : x \in X_0\} \subset \mathcal{U}_0$, corresponding to some finite subset $X_0 \subset \partial U$. We argue that in fact there exists a further subcover $\mathcal{U} \subset \mathcal{U}_1$ of order two. To see this, consider some boundary point

$$x \in \partial U \cap B_{x_1} \cap B_{x_2} \cap B_{x_3}, \tag{6.2.13}$$

where $B_{x_k} \in \mathcal{U}_1$ for $k \in \{1,2,3\}$. We claim that in this situation one of the sets B_{x_k} can always be removed from the collection so that $\mathcal{U}_1 \setminus B_{x_k}$ is still a cover for ∂U. As

[3] A cover $\mathcal{U} = \{U_\alpha \,:\, \alpha \in A\}$ of a set $X \subset \bigcup_{\alpha \in A} U_\alpha$, indexed by the set A, is said to be of *order* n, if the set $A(x) := \{\alpha \in A \,:\, x \in U_\alpha\}$ contains at most n elements for all $x \in X$, and it is said to be *minimal*, if no $U \in \mathcal{U}$ may be removed so that $\mathcal{U} \setminus \{U\}$ is still a cover for X.

presented above, for each $k \in \{1,2,3\}$ the set $X \cap \overline{B_{x_k}}$ has a parametrisation as the curve $\Gamma_{x_k} = \gamma_{x_k}([0,1])$. Hence, there exists a continuous bijection $h : [0,1] \to \partial U \cap \bigcup_{k\in\{1,2,3\}} \overline{B_{x_k}}$, for which $\partial U \cap B_{x_k} = h\left((a_k, b_k)\right)$ for some open intervals $(a_k, b_k) \subset [0,1]$ and $k \in \{1,2,3\}$. Equation (6.2.13) is then equivalent to

$$h^{-1}(x) \in \bigcap_{k\in\{1,2,3\}} (a_k, b_k).$$

On the other hand, regardless of the order in which the points a_k, b_k lie on $[0,1]$—as long as $a_k < b_k$ for all $k \in \{1,2,3\}$—one of the intervals (a_k, b_k) is always contained in the union of the other two, and can therefore be removed without affecting the cover. One can thus remove any redundant balls B_x from the cover $\mathcal{U}_1$ and obtain a minimal, order two subcover $\mathcal{U}_2$.

To complete the proof, we still need to ensure that none of the balls B_x corresponding to sharp-sharp singularities $x \in Q_1$ were removed from the initial cover $\mathcal{U}_0$ along the pruning process. To this end, note that according to Lemma 5.1.3 the set Q_1 is finite. This implies that we can assume the radii of the balls B_x to have been initially chosen sufficiently small such that $B_x \cap Q_1 = \varnothing$ for all $x \in \partial U \setminus Q_1$ and $B_x \cap Q_1 = \{x\}$ for $x \in Q_1$. This way, the inclusion of every ball B_x with $x \in Q_1$ is necessary in any subcover of $\mathcal{U}_0$.

Hence, we have arrived at the desired finite, minimal subcover $\mathcal{U}_2$ of order two, in the form

$$\mathcal{M}^* := \left\{\Gamma_x \,:\, x \in X^*\right\} \cup \left\{\Gamma_x^{(i)} \,:\, x \in Q_1,\, i \in \{1,2\}\right\}, \tag{6.2.14}$$

where the set X^* is defined by $X^* = \{x \in \partial U \setminus Q_1 \,:\, B_x \in \mathcal{U}_2\}$. □

The significance of the order two property of the cover obtained in Lemma 6.2.2, as well as the requirement that all the curves corresponding to $x \in Q_1$ are included in the cover, both stem from the need to ensure that every boundary segment in the cover has a unique successor and predecessor segment. This makes it is possible to construct Jordan curves on the boundary by following the boundary along adjacent curve segments.

Proposition 6.2.3 (**Boundaries of connected components of the complement as finite unions of Jordan curves**). *Let $E \subset \mathbb{R}^2$ be compact, let $\varepsilon > 0$ and let U be a connected component of the complement E_ε^c. Then $\partial U = \bigcup_{i=1}^M J_i$, where $M \in \mathbb{N}$ and each J_i is a Jordan curve. The representation is unique up to parametrisation, and for any $i \neq j$, the intersection $J_i \cap J_j$ contains at most one point.*

Proof. We divide the proof into three steps. First, we use the finite, order two cover given by Lemma 6.2.2 to construct the Jordan curves on the boundary. We then argue that this representation is unique up to parametrisation, and finish by showing that the intersection of any two Jordan curves in the representation is either a singleton or the

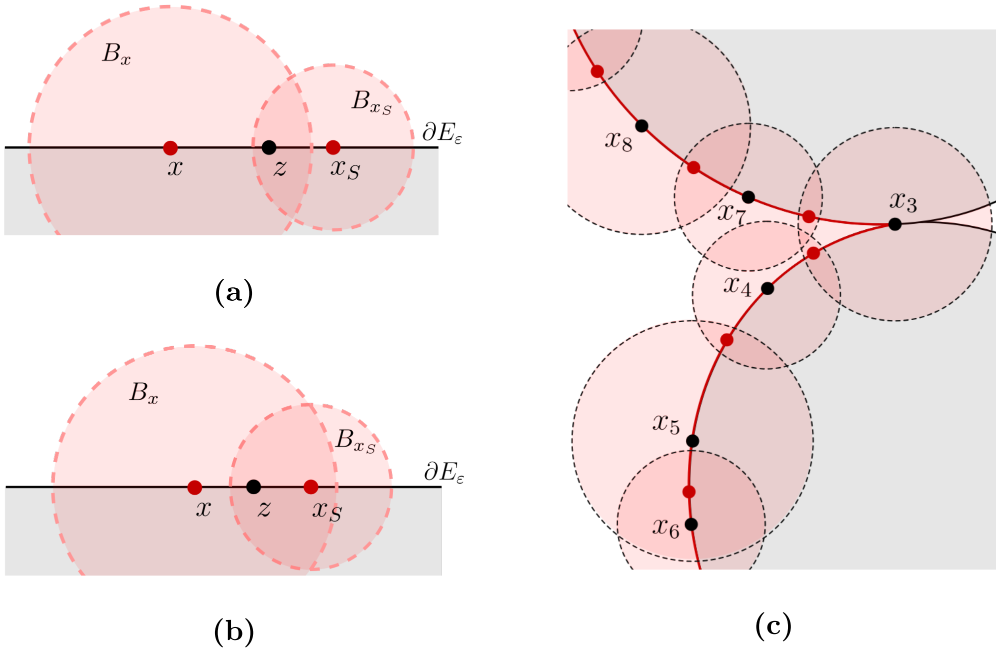


Figure 6.4. Tracking the boundary ∂U along a finite collection of adjacent balls and corresponding simple curves. In **(a)** and **(b)**, the points x and x_S correspond to simple curves $\Gamma_x, \Gamma_{x_S} \in \mathcal{M}^*$. Here Γ_{x_S} is the successor of Γ_x, and due to minimality of $\mathcal{M}^*$, no point $z \in \Gamma_x \cap \Gamma_{x_S}$ belongs to any other curve $\Gamma \in \mathcal{M}^*$. It is not ruled out that $x_S \in \Gamma_x$, as in **(b)**. In **(c)**, which is a close-up of Figure 6.1 (a), a longer curve is formed by concatenating such overlapping local representations.

empty set. The last two facts are essentially implied by the Jordan Curve Theorem and the connectedness of the set U.[4]

Step 1: Construction of Jordan Curves. Let

$$\mathcal{M}^* := \left\{\Gamma_x \,:\, x \in X^*\right\} \cup \left\{\Gamma_x^{(i)} \,:\, x \in Q_1,\, i \in \{1,2\}\right\} \tag{6.2.15}$$

be the finite, minimal, order two cover of ∂U given by Lemma 6.2.2. The sets X^* and Q_1 are as in (6.2.14), and Γ_x and $\Gamma_x^{(i)}$ are simple curves. Since $\mathcal{M}^*$ is minimal and order two, each $\Gamma \in \mathcal{A} := \left\{\Gamma_x \,:\, x \in X^*\right\}$ intersects exactly two other simple curves $\Gamma_P, \Gamma_S \in \mathcal{M}^* \setminus \{\Gamma\}$, which we call the *predecessor* and *successor*, respectively. It is possible that $\Gamma_P = \Gamma_S$. The parametrisations of individual curves $\Gamma \in \mathcal{A}$ can thus be combined to form longer curves, see Figure 6.4. Some of these eventually loop back onto themselves, forming Jordan curves, while others connect to curves $\Gamma \in \mathcal{B} := \left\{\Gamma_x^{(i)} \,:\, x \in Q_1,\, i \in \{1,2\}\right\}$. By construction, $\Gamma_x^{(1)} \cap \Gamma_x^{(2)} = \{x\}$ for each $x \in Q_1$, and in addition, each $\Gamma_x^{(i)} \in \mathcal{B}$ intersects precisely two other curves $\Gamma_P, \Gamma_S \in \mathcal{M}^* \setminus \left\{\Gamma_x^{(1)}, \Gamma_x^{(2)}\right\}$. One can thus start from any $\Gamma \in \mathcal{M}^*$ and follow the boundary along uniquely defined successor

[4]The Jordan Curve Theorem states that for any set $J \subset \mathbb{R}^2$ that is homeomorphic to the unit circle $S^1 \subset \mathbb{R}^2$, the complement $\mathbb{R}^2 \setminus J$ has precisely two connected components A_1 and A_2, one of which is bounded and the other unbounded, with the common boundary $\partial A_1 = \partial A_2 = J$.

(or predecessor) curves. The order two property and the finiteness of the collection $\mathcal{M}^*$ ensure that every such extended curve ultimately returns to Γ without self-intersections, thus forming a Jordan curve. Furthermore, there are only finitely many such curves.

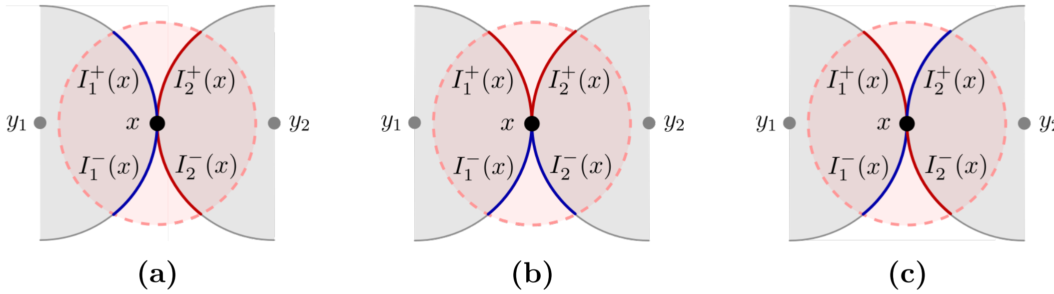


Figure 6.5. Schematic illustration of the three ways of representing the boundary ∂U near a sharp-sharp singularity $x \in Q_1$ as the union of two simple curves. Since $I_i^{\pm}(x) \subset \Gamma_x^{(i)} \subset \partial U$ for $i \in \{1,2\}$, the point x must lie at the intersection of two different Jordan curves. In **(a)** the red and blue curves are subsets of the curves $\Gamma_x^{(1)}$ and $\Gamma_x^{(2)}$ defined in Lemma 6.2.1 for $x \in Q_1$. In **(b)** the coloured curves bounce back at x and in **(c)** they cross at x. It turns out that only case **(a)** admits the continuation of such local curves into a representation of ∂U as Jordan curves.

Step 2: Uniqueness of Representation. Assume there exists another representation of ∂U as a union of Jordan curves. Since there is only one way to concatenate the curves $\Gamma \in \mathcal{A}$, the only way to obtain a representation that differs from the one constructed in Step 1 is to define the Jordan curves differently at the junction points $x \in Q_1$, where four simple curves meet at the same point. For each $x \in Q_1$ and $i \in \{1,2\}$ we write $\Gamma_x^{(i)} = I_i^+(x) \cup I_i^-(x)$ where the $I_i^{\pm}(x)$ are simple curves that intersect each other pairwise only at x. The Jordan curves constructed in Step 1 are obtained by arriving to each $x \in Q_1$ via either $I_1^-(x)$ or $I_2^-(x)$ and leaving along $I_1^+(x)$ or $I_2^+(x)$, respectively. This corresponds to following either the blue or the red curve in Figure 6.5 (a). Assume then that for some $x \in Q_1$, a Jordan curve J arrives at x via $I_1^-(x)$ (say) and leaves via either $I_2^-(x)$ or $I_2^+(x)$. This corresponds to following the blue curve either in Figure 6.5 (b) or in Figure 6.5 (c), respectively. The resulting curve would then not coincide with any of the Jordan curves constructed in Step 1. We show that both possibilities lead to a contradiction.

Assume first that J bounces back at x along $I_2^-(x)$, as in Figure 6.5 (b). Let A_1, A_2 be the connected components of $\mathbb{R}^2 \setminus J$, for which $\partial A_1 = \partial A_2 = J$. By definition, J has no self-intersections, which implies that it intersects the curves $I_1^+(x)$ and $I_2^+(x)$ only at x. It follows that $J \cap B_r(x) = I_1^-(x) \cup I_2^-(x)$, where $r > 0$ is the radius of the local boundary representation at x, see Proposition 3.2.4. Consequently, J divides the ball neighbourhood $B_r(x)$ into two connected components $C_1, C_2 \subset B_r(x)$, which satisfy $C_i \subset A_i$ for $i \in \{1,2\}$. Let $\xi \in \Xi_x^{\mathrm{ext}}(E_\varepsilon)$ be the extremal outward direction for which $\partial U \cap U_r(x,\xi) = I_1^-(x) \cup I_2^-(x)$ for some $r > 0$. Choose some $x(\xi) \in U \cap U_r(x,\xi)$ and $x(-\xi) \in U \cap U_r(x,-\xi)$. Now, since

U is connected and open, there exists a path $\gamma : [0,1] \to U$ connecting $x(\xi)$ and $x(-\xi)$. But this is impossible, because it would imply that $A_1 \cup A_2$ is connected.

Assume then that J crosses x along $I_2^+(x)$, as in Figure 6.5 (c). Since J cannot cross itself, it follows that another Jordan curve $J_1 \subset \partial U$ contains the curves $I_2^-(x)$ and $I_1^+(x)$. As before, let A_1, A_2 be the connected components of $\mathbb{R}^2 \setminus J$, for which $\partial A_1 = \partial A_2 = J$. Now $J_1 \cap A_i \neq \varnothing$ for both $i \in \{1,2\}$. But this contradicts the fact that the connected set U must be a subset of either A_1 or A_2.

Step 3: Pairwise Intersections. Assume, contrary to the claim, that there exist two Jordan curves $J_1, J_2 \subset \partial U$ and some $x_1, x_2 \in J_1 \cap J_2$ with $x_1 \neq x_2$. This implies $x_1, x_2 \in Q_1$. One may then construct a new Jordan curve $J^* \subset \partial U$ by following J_1 from x_1 to x_2 and returning back to x_1 via J_2. Let A_1, A_2 be the connected components of $\mathbb{R}^2 \setminus J^*$, for which $\partial A_1 = \partial A_2 = J^*$. Since U is connected, either $U \subset A_1$ or $U \subset A_2$. But this contradicts the fact that J^* divides the neighbourhood $B_r(x_1)$ (say) into two disjoint connected components $C_1 \subset A_1$ and $C_2 \subset A_2$, for which $C_i \cap U \neq \varnothing$ for both $i \in \{1,2\}$. □

Proof of Theorem 4

The main result of this chapter, Theorem 4, follows by combining Proposition 6.2.3 and Corollary 4.2.2.

Proof of Theorem 4. Denote by $\mathcal{U}$ the collection of connected components of the complement E_ε^c. Corollary 4.2.2 asserts that every boundary point $x \in \partial E_\varepsilon$ lies on the boundary ∂U of some $U \in \mathcal{U}$ if and only if it is not a chain (S6) or chain-chain (S7) singularity. Proposition 6.2.3 then implies that

$$\partial E_\varepsilon = \mathcal{I} \cup \bigcup_{U \in \mathcal{U}} \left(\bigcup_{n=1}^{N_U} J_n(U) \right), \tag{6.2.16}$$

where $N_U \in \mathbb{N}$ for each $U \in \mathcal{U}$ and $J_n(U) \subset \partial U$ is a Jordan curve for all $1 \leq n \leq N_U$. For each $U \in \mathcal{U}$, the collection $\{J_n(U) \,:\, n \in N_U\}$ is furthermore unique up to parametrisation.

Due to the compactness of E_ε, the complement E_ε^c contains only one unbounded connected component. All other connected components of the complement E_ε^c are contained in some bounded ball $B_M(0)$ and since they are open, each of them contains an open ball. Hence, the Lebesgue measure of each such component U satisfies $m(U) > 0$, although for each $n \in \mathbb{N}$ there are only finitely many U with $m(U) > 1/n$. It follows from this that there are at most countably many connected components of the complement, which also implies that the number of Jordan curves $J_n(U)$ in (6.2.16) is countable. □

Chapter 7

Curvature

Having established the topological structure of the boundary, we turn to investigate the curvature of the Jordan curve subsets $J \subset \partial U \subset \partial E_\varepsilon$ for connected components U of the complement E_ε^c. It is not clear from the outset that curvature can be defined on these sets even almost everywhere: Example 5.1.5 demonstrates that *shallow singularities* (S4, S5), which are defined as accumulation points of increasingly shallow wedge-type singularities, may lie densely on boundary segments that have positive one-dimensional Hausdorff measure. Even though Propositions 3.2.4 and 3.2.5 imply that it is possible to represent the boundary locally as graphs of Lipschitz-continuous functions, defining curvature at shallow singularities via limits of first derivatives is not possible, since the derivatives do not generally exist everywhere even in small neighbourhoods.

However, since there are no cusp singularities on the boundary, the tangent function on the boundary ∂U may only have jumps in one direction. In the other direction, the rate of change of the tangential direction is bounded from below by the curvature of an ε-radius ball, since every boundary point $x \in \partial U$ lies on the surface of one. It turns out that it is possible to use these properties to show that the one-sided tangent functions are locally of bounded variation, and therefore have a derivative at almost every point. One can then use these to obtain the (signed) curvature of the boundary.

7.1 Existence of Curvature via Bounded Variation

Let U be a connected component of the complement E_ε^c and let $J \subset \partial U$ be one of its Jordan curve subsets, see Theorem 4. In general, J has a well-defined tangent only at almost every point. However, Proposition 2.2.8 asserts that the directional (left and right) tangents coincide with extremal outward directions, which exist at every $x \in J$ according to Proposition 2.2.6. Lemma 6.2.2 asserts that J can be covered by a finite collection of simple curves that are constructed from the graphs of local boundary representations, given by Proposition 3.2.4. Hence, in order to establish the existence of curvature on the Jordan curve J, it suffices to work with these local representations.

In this section, we prove our last main result, Theorem 5, by applying a criterion for bounded variation to the local boundary representations of J. We present here the proof of the main theorem, and postpone the more technical proof of the criterion to Section 7.2.

7.1.1 Boundary Points in a Local Coordinate System

A local boundary representation at $x \in \partial E_\varepsilon$ is a finite collection $\mathcal{G}(x)$ of continuous functions $g_{\xi,y} : [0, \varepsilon/2] \to \mathbb{R}^2$, one for each extremal pair[1] $(\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$, with the property that

$$\partial E_\varepsilon \cap \overline{B_r(x)} = \bigcup_{(\xi,y)\in\mathcal{P}_x^{\text{ext}}(E_\varepsilon)} g_{\xi,y}\left(A_{\xi,y}\right) \tag{7.1.1}$$

for some $r > 0$ and some closed sets $A_{\xi,y} \subset [0, r]$, see Proposition 3.2.4. More precisely, for each extremal pair $(\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$ there exists a continuous function $f^{\xi,y} : [0, \varepsilon/2] \to \mathbb{R}$ for which

$$g_{\xi,y}(s) = x + s\xi + f^{\xi,y}(s)\frac{x-y}{\varepsilon}. \tag{7.1.2}$$

According to Corollary 4.2.2, chain singularities do not appear on the boundary ∂U for connected components U of the complement E_ε^c. It follows then from Proposition 3.2.4 that for each $x \in \partial U$, the sets $A_{\xi,y}$ in (7.1.1) are intervals $A_{\xi,y} = [0, s_{\xi,y}]$ for some $s_{\xi,y} \in [0, \varepsilon/2]$.

Consider a fixed $x \in J$ and an extremal pair $(\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$. To facilitate the subsequent analysis, we choose local coordinates given by the orthogonal unit vectors ξ and $(x - y)/\varepsilon$, and place the origin of this coordinate system at x. However, to simplify notation, we assume (without loss of generality) that this coordinate system coincides with the standard Euclidean coordinate system, so that $x = (0, 0)$, $\xi = (1, 0)$ and $(x - y)/\varepsilon = (0, 1)$. For each $s \in [0, s_{\xi,y}]$, we define $x_s := g_{\xi,y}(s) = \big(s, f^{\xi,y}(s)\big) \in J$, where $f^{\xi,y} : [0, s_{\xi,y}] \to \mathbb{R}$ is as in (7.1.2). The boundary subset $J \cap \overline{B_r(x)}$ can thus be expressed as the union of graphs

$$J \cap \overline{B_r(x)} = \bigcup_{(\xi,y)\in\mathcal{P}_x^{\text{ext}}(E_\varepsilon)} \Big\{\Big(s, f^{\xi,y}(s)\Big) \,:\, s \in [0, s_{\xi,y}]\Big\},$$

where each extremal pair $(\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$ defines its corresponding local coordinates.

According to Proposition 3.1.1, the extremal contributors $y(s) \in \Pi_E^{\text{ext}}(x_s)$ satisfy $y(s) \in B_{\varepsilon/2}(y)$, where $y \in \Pi_E^{\text{ext}}(x)$ is the extremal contributor corresponding to the extremal pair

[1] For each boundary point $x \in \partial E_\varepsilon$, the set of *extremal pairs* $\mathcal{P}_x^{\text{ext}}(E_\varepsilon)$ consists of all the pairs (ξ, y) of extremal outward directions $\xi \in \Xi_x^{\text{ext}}(E_\varepsilon)$ and extremal contributors $y \in \Pi_E^{\text{ext}}(x)$ for which $\langle x - y, \xi\rangle = 0$, see Definition 3.2.3

$(\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$, see Proposition 3.2.4. This implies that for each $s \in [0, s_{\xi,y}]$ the extremal outward directions $\xi_s \in \Xi_{x_s}^{\text{ext}}(E_\varepsilon)$ deviate only slightly from the ξ-axis in the local coordinate system. We call them the right and left extremal outward direction at x_s, and denote them by ξ_s^+ and ξ_s^-. According to Proposition 2.2.8, these coincide with the tangential directions at $x_s \in J$. In the local coordinates, the slopes of the one-sided tangents of the graph $\left\{\left(s, f^{\xi,y}(s)\right) : s \in [0, s_{\xi,y}]\right\}$ are given by the the left and right derivatives

$$D^\pm f^{\xi,y}(s) := \lim_{h \to \pm 0} \frac{f^{\xi,y}(s+h) - f^{\xi,y}(s)}{h}. \tag{7.1.3}$$

These correspond to the extremal outward directions $\xi_s^\pm = \left(\xi_1^\pm(s), \xi_2^\pm(s)\right) \in \Xi_{x_s}^{\text{ext}}(E_\varepsilon)$ via

$$D^\pm f^{\xi,y}(s) = \frac{\xi_2^\pm(s)}{\xi_1^\pm(s)}, \tag{7.1.4}$$

see Figure 7.1.

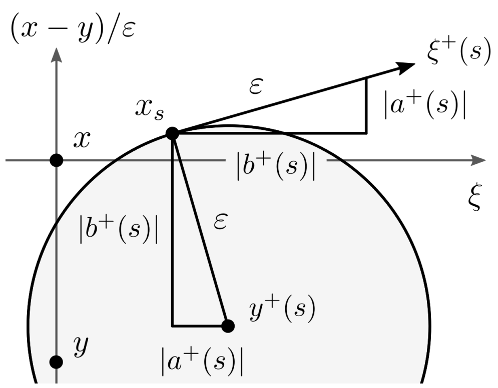


Figure 7.1. Relationship between the tangential direction ξ_s^+, slope $D^+ f^{\xi,y}(s)$ and the extremal contributor y_s^+ at $x_s = g_{\xi,y}(s) = (s, f^{\xi,y}(s))$. Here $a_s^+ < 0$, and the slope at x_s satisfies $D^+ f^{\xi,y}(s) = -a_s^+/b_s^+ = -a_s^+/\sqrt{\varepsilon^2 - [a_s^+]^2} =: p(a_s^+)$.

7.1.2 A Condition for Bounded Variation

In order to prove the existence of curvature almost everywhere on J, we consider a local boundary representation $\mathcal{G}(x)$ around an arbitrary boundary point $x \in J$, and show that for each extremal pair $(\xi, y) \in \mathcal{P}_x^{\text{ext}}(E_\varepsilon)$ the second derivative of $f^{\xi,y}$ exists almost everywhere on the interval $[0, s_{\xi,y}]$. We begin by presenting the following general criterion for bounded variation that applies to bounded functions on a closed interval.

Lemma 7.1.1 (A criterion for bounded variation). *Let $q > 0$ and let $f : [a, b] \to \mathbb{R}$ be a bounded function that satisfies*

$$f(s+h) - f(s) \geq -qh \tag{7.1.5}$$

for all $s \in [a, b)$ and $h > 0$ with $s + h \in [a, b]$. Then f is of bounded variation on $[a, b]$.

Proof. Assume to the contrary that f is not of bounded variation, and let $m > 0$ be such that $f(s) \in [-m, m]$ for all $s \in [a, b]$. Then there exists a partition $a = s_0 < s_1 < \ldots < s_K = b$ of the interval $[a, b]$ for which

$$S := \sum_{j=1}^{K} |f(s_j) - f(s_{j-1})| > 2(p + m), \tag{7.1.6}$$

where $p := \max\{q(b-a), 2m\}$. We decompose (7.1.6) into its positive and negative parts so that $S = S^+ - S^-$, where

$$S^+ := \sum_{j \in J_+} \big(f(s_j) + f(s_{j-1})\big) \qquad \text{and} \qquad S^- := \sum_{j \in J_-} \big(f(s_j) - f(s_{j-1})\big) \tag{7.1.7}$$

with index sets $J^+ := \Big\{ j \in \{1, \ldots, K\} \,:\, f(s_j) - f(s_{j-1}) > 0 \Big\}$ and $J^- = \{1, \ldots, K\} \setminus J^+$. It follows from (7.1.5) that $S^- \geq -q(b-a) \geq -p$, which together with (7.1.6) implies the lower bound $S^+ > p + 2m$. Combining these bounds leads to the contradiction

$$f(b) = f(a) + S^+ + S^- > -m + (p + 2m) - p = m. \qquad \square$$

The main step towards proving Theorem 5 is to show that the one-sided derivatives $D^\pm f^{\xi,y}$ satisfy condition (7.1.5) of Lemma 7.1.1 for a certain $q > 0$ and are thus of bounded variation on the interval $[0, s_{\xi,y}]$.

Proposition 7.1.2 (A lower bound for differences of right derivatives). *Let $E \in \mathbb{R}^2$ be a compact set and let $x \in J \subset \partial E_\varepsilon$ where J is a Jordan curve. Let $\mathcal{G}(x)$ be a local boundary representation at x, where each $g_{\xi,y} \in \mathcal{G}(x)$ satisfies*

$$g_{\xi,y}(s) = x + s\xi + f^{\xi,y}(s)\varepsilon^{-1}(x - y)$$

for some continuous function $f^{\xi,y} : [0, s_{\xi,y}] \to \mathbb{R}$. Then the left and right derivatives $D^\pm f^{\xi,y}$ satisfy the inequality

$$D^\pm f^{\xi,y}(s + h) - D^\pm f^{\xi,y}(s) \geq -\frac{8h}{3\sqrt{3}\varepsilon} \tag{7.1.8}$$

for all $s \in [0, s_{\xi,y}]$ and $s, h > 0$ with $0 \leq s + h \leq s_{\xi,y}$.

Proof of Theorem 5

We now proceed to combine Proposition 7.1.2 and Lemma 7.1.1 into a global statement about the existence of curvature on ∂E_ε. The proof of Proposition 7.1.2 is the most technical part of the proof of Theorem 5, and is presented separately in Section 7.2. We denote by $\mathcal{H}^1$ the one-dimensional Hausdorff measure, and by $\mathcal{I}$ the set of inaccessible singularities, see Corollary 4.2.2.

Theorem 5 (**Existence of curvature**). *Let $E \subset \mathbb{R}^2$ be a compact set and $\varepsilon > 0$. Then for $\mathcal{H}^1$-almost all $x \in \partial E_\varepsilon \backslash \mathcal{I}$ the (signed) curvature $\kappa(x)$ exists and is given by the formula*

$$\kappa(s_x) = \frac{\frac{d^2}{ds^2} f^{\xi,y}(s_x)}{\left(1 + \left(\frac{d}{ds} f^{\xi,y}(s_x)\right)^2\right)^{3/2}}, \tag{7.1.9}$$

where the coordinates s_x and $f^{\xi,y}(s_x)$ are associated with a local boundary representation $\mathcal{G}(z)$ at some $z \in \partial E_\varepsilon \setminus \mathcal{I}$, and $x = z + s_x\xi + f^{\xi,y}(s_x)\varepsilon^{-1}(z-y)$ for some $(\xi, y) \in \mathcal{P}_z^{\text{ext}}(E_\varepsilon)$.

Proof. According to Theorem 4, $\partial E_\varepsilon \setminus \mathcal{I}$ is a countable union $\mathcal{J}$ of Jordan curves. Since each $J \in \mathcal{J}$ is compact, there exists a finite collection Z of points $z \in J$ and corresponding boundary representations $\mathcal{G}(z)$, for which

$$J = \bigcup_{z \in Z} \left(\bigcup_{(\xi,y) \in \mathcal{P}_z^{\text{ext}}(E_\varepsilon)} g_{\xi,y}([0, s_{\xi,y}]) \right).$$

The closed intervals $[0, s_{\xi,y}]$ are as in Proposition 3.2.4. It suffices to show that for each $z \in Z$, curvature exists outside a $\mathcal{H}^1$-negligible set on each curve $g_{\xi,y}([0, s_{\xi,y}])$, where $(\xi, y) \in \mathcal{P}_z^{\text{ext}}(E_\varepsilon)$.

Let $z \in Z$ and let $g_{\xi,y} \in \mathcal{G}(z)$ for some $(\xi, y) \in \mathcal{P}_z^{\text{ext}}(E_\varepsilon)$. According to Proposition 3.2.4 there exists a continuous function $f^{\xi,y} : [0, s_{\xi,y}] \to \mathbb{R}$ for which

$$g_{\xi,y}(s) = z + s\xi + f^{\xi,y}(s)\varepsilon^{-1}(z - y).$$

Since the one-sided derivatives $D^\pm f^{\xi,y}(s)$ are related to the extremal outward directions $\xi^\pm(s)$ via (7.1.4), it follows that $D^+ f^{\xi,y}(s) = D^- f^{\xi,y}(s)$ whenever $\xi^+(s) = -\xi^-(s)$. This implies that the one-sided derivatives agree on $[0, s_{\xi,y}]$ apart from an at most countably infinite set, since it follows from Theorem 2 and Corollary 4.2.2 that the set

$$J \setminus \left\{ x \in J \,:\, \Xi_x^{\text{ext}}(E_\varepsilon) = \{\xi, -\xi\} \text{ for some } \xi \in S^1 \right\} = J \setminus \mathrm{Unp}_\varepsilon(E),$$

where $\mathrm{Unp}_\varepsilon(E)$ is as in Definition 2.0.1, is at most countably infinite.

We next confirm that the assumptions of Lemma 7.1.1 are satisfied for the left and right derivatives $D^\pm f^{\xi,y}(s)$ on the interval $[0, s_{\xi,y}]$. By the definition of a local boundary representation, the extremal contributors $y_s^\pm \in \Pi_E^{\text{ext}}(x_s)$ satisfy $y_s^\pm \in B_{\varepsilon/2}(y)$, where $y \in \Pi_E^{\text{ext}}(x)$ corresponds to the extremal pair (ξ, y). This imposes an upper bound on the angle between the ξ-axis and the corresponding extremal outward directions $\xi^\pm(s) \in \Xi_{x_s}^{\text{ext}}(E_\varepsilon)$, which in turn implies a bound on the left and right derivatives $D^{\pm}f^{\xi,y}$ on $[0, s_{\xi,y}]$ through the relationship (7.1.4). Due to inequality (7.1.8) in Proposition 7.1.2, one can thus apply Lemma 7.1.1, which implies that the one-sided derivatives $D^{\pm}f^{\xi,y}$ are of bounded variation on $[0, s_{\xi,y}]$.

It follows that each of the functions $D^{\pm}f^{\xi,y}$ has a (two-sided) derivative $\frac{d}{ds}D^{\pm}f^{\xi,y}$ almost everywhere on $[0, s_{\xi,y}]$.[2]. Since $D^{+}f^{\xi,y} = D^{-}f^{\xi,y}$ outside a countable set, this furthermore implies $\frac{d}{ds}D^{+}f^{\xi,y} = \frac{d}{ds}D^{-}f^{\xi,y}$ almost everywhere. Thus, apart from a set $W \subset [0, s_{\xi,y}]$ of zero Hausdorff measure, both the first and second (two-sided) derivatives of $f^{\xi,y}$ exist, and define curvature on the graph $\left\{\left(s, f^{\xi,y}(s)\right) : s \in [0, s_{\xi,s}]\right\}$ via (7.1.9).

According to Proposition 3.2.5, the function $s \mapsto \left(s, f^{\xi,y}(s)\right)$ is $2/\sqrt{3}$-Lipschitz on $[0, s_{\xi,y}]$. This implies $\mathcal{H}^1\left(f^{\xi,y}(W)\right) \leq 2\mathcal{H}^1(W)/\sqrt{3} = 0$, since the Hausdorff measure of a Lipschitz transformation is bounded from above by the corresponding Lipschitz constant, see [AFP00, Prop. 2.49]. Curvature thus exists outside a $\mathcal{H}^1$-negligible set on each curve $g_{\xi,y}([0, s_{\xi,y}])$. □

7.2 A Lower Bound for Differences of Tangential Directions

In this section we prove Proposition 7.1.2, which represents the key step in the proof of Theorem 5. Proposition 7.1.2 expresses the geometric observation that it is impossible for the boundary ∂E_ε to curve inwards more than a certain threshold, implied by the radius $\varepsilon > 0$.

Before proceeding with the proof, we establish some notation. We consider the local boundary representation $\mathcal{G}(x)$ around $x \in J$, where $J \subset \partial E_\varepsilon$ is a Jordan curve component of the boundary. We work in the local coordinates corresponding to an extremal pair $(\xi, y) \in \mathcal{P}_x^{\mathrm{ext}}(E_\varepsilon)$, as defined in Section 7.1.1. For each $s \in [0, s_{\xi,y}]$, define

$$x_s := g_{\xi,y}(s) = x + s\xi + f^{\xi,y}(s)\frac{x-y}{\varepsilon} = \left(s, f^{\xi,y}(s)\right) \in J,$$

where $f^{\xi,y} : [0, s_{\xi,y}] \to \mathbb{R}$ is as in (7.1.2). Each of the extremal contributors $y_s^{\pm} = \left(y_1^{\pm}(s), y_2^{\pm}(s)\right) \in \Pi_E^{\mathrm{ext}}(x_s)$ lies at the center of an ε-radius circle[3] $B_\varepsilon\left(y_s^{\pm}\right)$, whose tangent at $x_s \in \partial B_\varepsilon\left(y_s^{\pm}\right)$ coincides with the respective extremal outward direction $\xi_s^{\pm} = \left(\xi_1^{\pm}(s), \xi_2^{\pm}(s)\right) \in \Xi_{x_s}^{\mathrm{ext}}(E_\varepsilon)$, see Definition 2.1.4 and Proposition 2.2.8. Hence, for each

[2] Every function of bounded variation can be written as the difference of two non-decreasing functions, and consequently has a finite derivative at almost every point. For details, see for instance [KF75, p. 331]

[3] Due to the orientation of the extremal contributor $y \in \Pi_E^{\mathrm{ext}}(x)$ relative to x, these circles lie below the graph $g_{\xi,y}([0, s_{\xi,y}])$ in the $\left(\xi, (x-y)/\varepsilon\right)$-coordinates, and are thus uniquely defined for all $s \in [0, s_{\xi,y}]$.

$s \in [0, s_{\xi,y}]$,

$$x_s - y_s^{\pm} = \left(-\varepsilon\xi_2^{\pm}(s), \varepsilon\xi_1^{\pm}(s)\right) =: \left(a_s^{\pm}, b_s^{\pm}\right). \tag{7.2.1}$$

On the other hand, as indicated in (7.1.4), the slopes of the one-sided tangents at x_s satisfy

$$D^{\pm} f^{\xi,y}(s) = \frac{\xi_2^{\pm}(s)}{\xi_1^{\pm}(s)} = \frac{\varepsilon\xi_2^{\pm}(s)}{\sqrt{\varepsilon^2 - \left[\varepsilon\xi_2^{\pm}(s)\right]^2}} = \frac{-a_s^{\pm}}{\sqrt{\varepsilon^2 - \left[a_s^{\pm}\right]^2}} =: p\left(a_s^{\pm}\right). \tag{7.2.2}$$

In order to prove Proposition 7.1.2, we establish for all $s \in [0, s_{\xi,y}]$ and $h > 0$ with $0 \le s + h \le s_{\xi,y}$ the double inequality

$$D^{\pm} f^{\xi,y}(s+h) - D^{\pm} f^{\xi,y}(s) \ge p\left(a_s^{\pm} + h\right) - p\left(a_s^{\pm}\right) \ge -\frac{8h}{3\sqrt{3}\varepsilon}, \tag{7.2.3}$$

where the coordinates $a_s^{\pm}$ and the slope function $p : (-\varepsilon, \varepsilon) \to \mathbb{R}$ are defined in (7.2.1) and (7.2.2), respectively. We initially prove (7.2.3) for the right derivative $D^+ f^{\xi,y}$, and deduce from this the analogous inequality also for the left derivative $D^- f^{\xi,y}$. Since $D^+ f^{\xi,y}(s) = p\left(a_s^+\right)$, the key to showing inequality (7.2.3) is to demonstrate that $D^+ f^{\xi,y}(s+h) \ge p(a_s^+ + h)$ whenever $s, h \ge 0$ and $0 \le s + h \le s_{\xi,y}$. We divide the proof into the following steps, of which the first three correspond to Lemmas 7.2.1–7.2.3 and step (iv) is included in the proof of Proposition 7.1.2 below:

(i) identify a lower bound $k(T(s,h), h)$ for $D^+ f^{\xi,y}(s+h)$ in terms of

$$T(s,h) := f^{\xi,y}(s+h) - f^{\xi,y}(s);$$

(ii) show that the bound $k_h(T) := k(T,h)$ obtained in step (i) is increasing in T for a fixed h;

(iii) show that there exists $\widehat{T} := \widehat{T}(s,h) \le T(s,h)$ for which $k\left(\widehat{T}, h\right) = p(a_s^+ + h)$;

(iv) combine steps (i)–(iii) to obtain the inequality

$$D^+ f^{\xi,y}(s+h) \ge k(T,h) \ge k\left(\widehat{T}, h\right) = p(a_s^+ + h).$$

We begin by establishing a lower bound $k(T,h)$ for $D^+ f^{\xi,y}(s+h)$ in terms of $T := T(s,h)$ and the T-dependent distances

$$P(T) := \frac{\sqrt{h^2 + T^2}}{2} \quad \text{and} \quad A(T) := \sqrt{\varepsilon^2 - P^2(T)} = \sqrt{\varepsilon^2 - \frac{h^2 + T^2}{4}}. \tag{7.2.4}$$

Lemma 7.2.1 (**The functional form of the lower bound for the right derivative**). *Let $E \subset \mathbb{R}^2$ be a compact set, $\varepsilon > 0$ and let $x \in J \subset \partial E_\varepsilon$ where J is a Jordan curve. Let*

$\mathcal{G}(x)$ be a local boundary representation at x, where each $g_{\xi,y} \in \mathcal{G}(x)$ satisfies

$$g_{\xi,y}(s) = x + s\xi + f^{\xi,y}(s)\varepsilon^{-1}(x-y)$$

for some continuous function $f^{\xi,y} : [0, s_{\xi,y}] \to \mathbb{R}$. Then for all $s, h \geq 0$ with $0 \leq s+h \leq s_{\xi,y}$, the right derivative $D^+ f^{\xi,y}$ satisfies the inequality

$$D^+ f^{\xi,y}(s+h) \geq k(T,h) := \frac{TA(T) - hP(T)}{hA(T) + TP(T)}, \tag{7.2.5}$$

where $T := f^{\xi,y}(s+h) - f^{\xi,y}(s)$ and the distances $P(T), A(T)$ are as in (7.2.4).

Proof. For each $s \in [0, s_{\xi,y}]$ we write $a_s^+ := -\varepsilon\xi_2^+(s)$ and $b_s^+ := \varepsilon\xi_1^+(s)$, where $\xi_s^+ = \big(\xi_1^+(s), \xi_2^+(s)\big)$ is the right extremal outward direction at x_s. Fix then some $s, h \geq 0$ with $0 \leq s+h \leq s_{\xi,y}$ and consider the corresponding boundary points $x_s, x_{s+h} \in J$ whose local coordinates are given by

$$x_s := \Big(s, f^{\xi,y}(s)\Big), \quad x_{s+h} := \Big(s+h, f^{\xi,y}(s+h)\Big). \tag{7.2.6}$$

It follows from (7.2.1) that the right extremal contributors $y_s^+ \in \Pi_E^{\text{ext}}(x_s)$ and $y_{s+h}^+ \in \Pi_E^{\text{ext}}(x_{s+h})$ satisfy

$$y_s^+ = \Big(s - a_s^+, f^{\xi,y}(s) - b_s^+\Big) \quad \text{and} \quad y_{s+h}^+ = \Big(s+h - a_{s+h}^+, f^{\xi,y}(s+h) - b_{s+h}^+\Big). \tag{7.2.7}$$

In local coordinates, the right derivative $D^+ f^{\xi,y}(s+h)$ is the slope of the extremal outward direction ξ_{s+h}^+, which is by definition perpendicular to the vector $x_{s+h} - y_{s+h}^+$. Since y_{s+h}^+ is a contributor and thus in E, it must by definition lie outside the ε-radius ball $B_\varepsilon(x_s)$ around the boundary point $x_s \in J$. One can therefore obtain a lower bound for the right derivative $D^+ f^{\xi,y}(s+h)$ by considering how far one could rotate the contributor y_{s+h}^+ clockwise around the point x_{s+h} before it enters the ball $B_\varepsilon(x_s)$. This would happen at the point y_{s+h}^*, whose distance from both x_s and x_{s+h} equals ε, see Figure 7.2. These three points thus form an isosceles triangle whose base is the line segment

$$S := \Big\{(1-\varphi)x_s + \varphi x_{s+h} \,:\, \varphi \in [0,1]\Big\}. \tag{7.2.8}$$

Consequently, the orthogonal projection of the apex y_{s+h}^* of this triangle onto the line segment S lands on the mid-point $x_{s+h}^* := (x_s + x_{s+h})/2$ of S. Together, the points x_{s+h}^*, y_{s+h}^* and x_{s+h} in turn form a right triangle Ω, whose legs $x_{s+h} - x_{s+h}^*$ and $x_{s+h}^* - y_{s+h}^*$ satisfy

$$\Big\|x_{s+h} - x_{s+h}^*\Big\| = P(T) \quad \text{and} \quad \Big\|x_{s+h}^* - y_{s+h}^*\Big\| = A(T).$$

We use the above geometric relationships to obtain an explicit lower bound for $D^+ f^{\xi,y}(s+$

h) in terms of the lengths $P(T)$ and $A(T)$. As noted above, $y^+_{s+h} \notin B_\varepsilon(x_s)$, which implies the inequality

$$\begin{aligned}
0 &\leq \left\| y^+_{s+h} - x_s \right\|^2 - \varepsilon^2 \\
&= \left(\left(s + h - a^+_{s+h} \right) - s \right)^2 + \left(\left(f^{\xi,y}(s+h) - b^+_{s+h} \right) - f^{\xi,y}(s) \right)^2 - \varepsilon^2 \\
&= h^2 + \left(f^{\xi,y}(s+h) - f^{\xi,y}(s) \right)^2 - 2 \left(h a^+_{s+h} + \left(f^{\xi,y}(s+h) - f^{\xi,y}(s) \right) b^+_{s+h} \right).
\end{aligned}$$

Writing $T := f^{\xi,y}(s+h) - f^{\xi,y}(s)$, the above inequality can be expressed more concisely as

$$h a^+_{s+h} + T b^+_{s+h} \leq (h^2 + T^2)/2. \tag{7.2.9}$$

Since $x_{s+h} - y^+_{s+h} = \left(a^+_{s+h}, b^+_{s+h} \right)$ and $x_{s+h} - x_s = (h, T)$, (7.2.9) is furthermore equivalent to

$$\left\langle x_{s+h} - y^+_{s+h}, \frac{x_{s+h} - x_s}{\|x_{s+h} - x_s\|} \right\rangle \leq \frac{\|x_{s+h} - x_s\|}{2} = P(T). \tag{7.2.10}$$

Geometrically, inequality (7.2.10) expresses the fact that the length of the orthogonal projection of the vector $x_{s+h} - y^+_{s+h}$ onto the line $L := x_s + \operatorname{span}\{x_{s+h} - x_s\}$ is at most half the distance $\|x_{s+h} - x_s\|$. Note that when ξ^+_{s+h} is parallel to L, the scalar product on the left-hand side of (7.2.10) vanishes, and for steeper slopes it becomes negative. However, since we seek to obtain a lower bound for $D^+ f^{\xi,y}(s+h) = \tan \tau_{s+h}$, where τ_{s+h} is the angle between ξ^+_{s+h} and the ξ-axis, it is sufficient to restrict the analysis to angles τ_{s+h} that are smaller than the angle $\theta(T)$ between the line L and the ξ-axis. [4]

Consider now the vector $x^+_{s+h} - y^+_{s+h}$, where $x^+_{s+h} := \operatorname{proj}_L \left(y^+_{s+h} \right)$ is the orthogonal projection of y^+_{s+h} onto the line L. It follows from (7.2.10) that $\left\| x_{s+h} - x^+_{s+h} \right\| \leq P(T)$, and consequently

$$\left\| x^+_{s+h} - y^+_{s+h} \right\|^2 = \varepsilon^2 - \left\| x_{s+h} - x^+_{s+h} \right\|^2 \geq \varepsilon^2 - \left\| x_{s+h} - x^*_{s+h} \right\|^2 = A^2(T).$$

This implies that the angle α_{s+h} at y^+_{s+h} between the vectors $x_{s+h} - y^+_{s+h}$ and $x^+_{s+h} - y^+_{s+h}$ satisfies

$$\tan \alpha_{s+h} = \frac{\left\| x_{s+h} - x^+_{s+h} \right\|}{\left\| x^+_{s+h} - y^+_{s+h} \right\|} \leq \frac{P(T)}{A(T)} = \tan \alpha(T), \tag{7.2.11}$$

where $\alpha(T)$ is the angle at y^*_{s+h} between the vectors $x_{s+h} - y^*_{s+h}$ and $x^*_{s+h} - y^*_{s+h}$. Since the vectors $x_{s+h} - y^+_{s+h}$ and ξ^+_{s+h} are perpendicular, it furthermore follows that $\tau_{s+h} =$

[4] Additionally, as pointed out also in the proof of Theorem 5, the definition of a local boundary representation imposes an upper bound on the angle τ_{s+h} between the extremal outward direction ξ^+_{s+h} and the ξ-axis, since the extremal contributor $y^+_{s+h} \in \Pi^{\text{ext}}_E(x_{s+h})$ lies for all s, h within the ball $B_{\varepsilon/2}(y)$, where $y \in \Pi^{\text{ext}}_E(x)$.

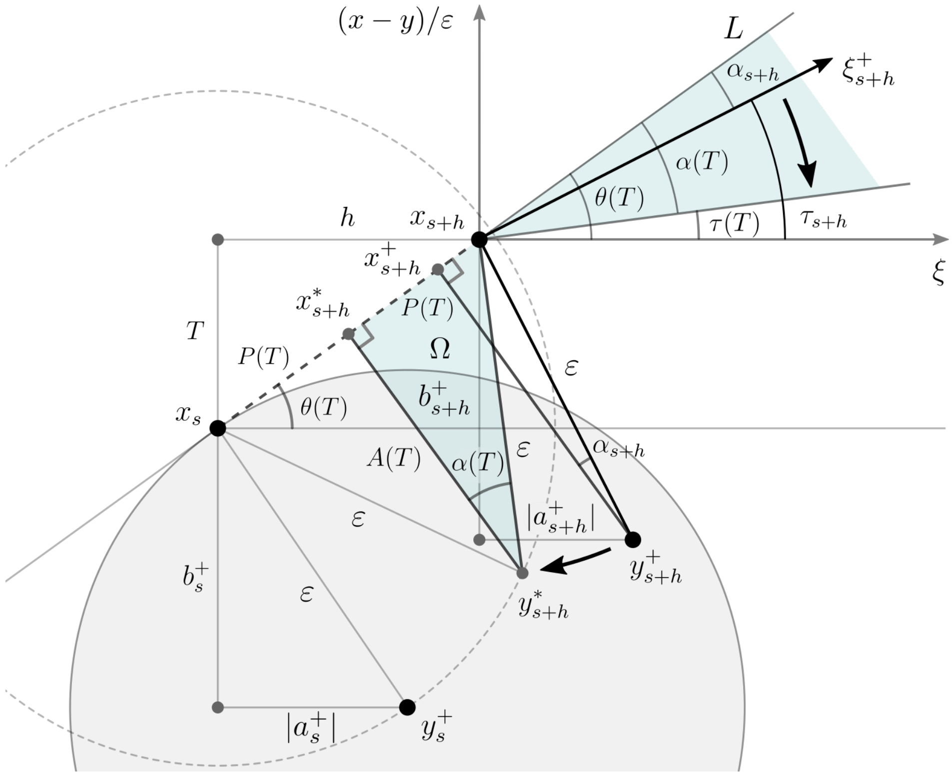


Figure 7.2. An illustration of the geometric components in the proof of Lemma 7.2.1. One can obtain a lower bound $k(T,h) := \tan\tau(T)$ for the slope $D^+f^{\xi,y}(s+h) = \tan\tau_{s+h} = \xi_2^+(s+h)/\xi_1^+(s+h)$ by considering how far one could rotate the right extremal contributor y^+_{s+h} clockwise around the point x_{s+h} before it enters the ball $B_\varepsilon(x_s)$. The line segment $S := \{(1-\varphi)x_s + \varphi x_{s+h} : \varphi \in [0,1]\}$ is marked as a dashed line. Note that the illustration here does not strictly speaking apply in the situation of Lemma 7.2.1 because the ξ-coordinate of x_{s+h} differs from that of x_s by more than $\varepsilon/2$. This artistic liberty was taken in order to improve readability through the increase of the relevant angles.

$\theta(T) - \alpha_{s+h}$. This, together with (7.2.11), leads to the lower bound

$$\begin{aligned} D^+f^{\xi,y}(s+h) &= \tan\tau_{s+h} = \tan(\theta(T)-\alpha_{s+h}) \geq \tan(\theta(T)-\alpha(T)) \\ &= \frac{\tan\theta(T)-\tan\alpha(T)}{1+\tan\theta(T)\tan\alpha(T)} = \frac{TA(T)-hP(T)}{hA(T)+TP(T)} = k(T,h), \end{aligned} \tag{7.2.12}$$

where the third last equality is due to the standard formula for the tangent of a sum of angles. □

Intuitively, Lemma 7.2.1 describes how the ε-neighbourhood geometry imposes a lower bound $k(T,h)$ on the tangential direction $D^+f^{\xi,y}(s+h)$, and how this bound depends on the increment h and the difference $T := f^{\xi,y}(s+h) - f^{\xi,y}(s)$. We show next that if the point x_s and the increment $h>0$ are fixed, $k(T,h)$ depends monotonically on T.

Lemma 7.2.2 (Monotonicity of the lower bound for the right derivative). *For* *a fixed* $h \in [0, 2\varepsilon)$ *and the corresponding functions*

$$P(t) := \frac{\sqrt{h^2 + t^2}}{2} \quad \textit{and} \quad A(t) := \sqrt{\varepsilon^2 - P^2(t)} = \sqrt{\varepsilon^2 - \frac{h^2 + t^2}{4}},$$

the lower bound function

$$t \mapsto k(t, h) := \frac{tA(t) - hP(t)}{hA(t) + tP(t)}$$

in (7.2.5) *is increasing on the interval* $\left(-\sqrt{4\varepsilon^2 - h^2}, \sqrt{\varepsilon^2 - (\varepsilon - h)^2}\right)$.

Proof. Note first that $A(t)$ is defined when $P(t) \leq \varepsilon$, which is equivalent to $|t| \leq \sqrt{4\varepsilon^2 - h^2}$. We compute the sign of the derivative $\frac{d}{dt}k(t, h)$ for a fixed $h \geq 0$. Write

$$k_1(t) := tA(t) - hP(t) \quad \text{and} \quad k_2(t) := hA(t) + tP(t),$$

so that $k(t, h) = k_1(t)/k_2(t)$. Then

$$\begin{aligned} \frac{d}{dt}k(t, h) &= \frac{k_1'(t)k_2(t) - k_1(t)k_2'(t)}{k_2^2(t)} \\ &= \frac{h\left(A^2(t) + P^2(t)\right) + (h^2 + t^2)\left(A'(t)P(t) - P'(t)A(t)\right)}{\left(hA(t) + tP(t)\right)^2}, \end{aligned}$$

where $P'(t) = t\left(2\sqrt{h^2 + t^2}\right)^{-1}$ and $A'(t) = -t\left(4\sqrt{\varepsilon^2 - \frac{h^2+t^2}{4}}\right)^{-1}$. Thus $\frac{d}{dt}k(t, h) > 0$ if and only if

$$\begin{aligned} 0 &< h\left(A^2(t) + P^2(t)\right) + \left(h^2 + t^2\right)\left(A'(t)P(t) - P'(t)A(t)\right) \\ &= h\varepsilon^2 + \left(h^2 + t^2\right)\frac{-t\left(h^2 + t^2\right) - 4t\left(\varepsilon^2 - \frac{h^2+t^2}{4}\right)}{8\sqrt{(h^2 + t^2)\left(\varepsilon^2 - \frac{h^2+t^2}{4}\right)}} \\ &= \varepsilon^2\left(h - t\frac{\sqrt{\frac{h^2+t^2}{4}}}{\sqrt{\varepsilon^2 - \frac{h^2+t^2}{4}}}\right) = \varepsilon^2\left(h - \frac{tP(t)}{\sqrt{\varepsilon^2 - P^2(t)}}\right). \end{aligned}$$

A direct computation shows that this inequality is satisfied if and only if $t < \sqrt{\varepsilon^2 - (\varepsilon - h)^2}$, where the right-hand side is defined for $h \in [0, 2\varepsilon]$. □

Lemma 7.2.3 (An explicit lower bound for the right derivative). *Let* $E \subset \mathbb{R}^2$ *be a compact set,* $\varepsilon > 0$ *and let* $x \in J \subset \partial E_\varepsilon$ *where* J *is a Jordan curve. Let* $\mathcal{G}(x)$ *be a local boundary representation at* x*, where each* $g_{\xi,y} \in \mathcal{G}(x)$ *satisfies*

$$g_{\xi,y}(s) = x + s\xi + f^{\xi,y}(s)\varepsilon^{-1}(x - y)$$

for some continuous function $f^{\xi,y} : [0, s_{\xi,y}] \to \mathbb{R}$*. In addition, let* $T := T(s,h) := f^{\xi,y}(s+h) - f^{\xi,y}(s)$ *and let*

$$k(T,h) := \frac{TA(T) - hP(T)}{hA(T) + TP(T)}$$

be as in (7.2.5)*. Then, for all* $s, h \geq 0$ *with* $0 \leq s+h \leq s_{\xi,y}$*, there exists some* $\widehat{T} \leq T$ *for which*

$$k\Big(\widehat{T}, h\Big) = p(a_s^+ + h) := \frac{-\left(a_s^+ + h\right)}{\sqrt{\varepsilon^2 - \left(a_s^+ + h\right)^2}}, \tag{7.2.13}$$

where for each $s \in [0, s_{\xi,y}]$ *the coordinate* $a_s^+ := -\varepsilon \xi_2^+(s)$ *corresponds to the right extremal outward direction* $\xi_s^+ = \Big(\xi_1^+(s), \xi_2^+(s)\Big)$ *at the boundary point* $x_s = \Big(s, f^{\xi,y}(s)\Big)$*, see* (7.2.6)*.*

Proof. Let $s, h \geq 0$ with $0 \leq s+h \leq s_{\xi,y}$, and consider the corresponding boundary points $x_s, x_{s+h} \in J$ whose local coordinates are given by

$$x_s := \Big(s, f^{\xi,y}(s)\Big), \quad x_{s+h} := \Big(s+h, f^{\xi,y}(s+h)\Big). \tag{7.2.14}$$

It follows from (7.2.1) that the right extremal contributors $y_s^+ \in \Pi_E^{\text{ext}}(x_s)$ and $y_{s+h}^+ \in \Pi_E^{\text{ext}}(x_{s+h})$ satisfy

$$y_s^+ = \Big(s - a_s^+, f^{\xi,y}(s) - b_s^+\Big) \quad \text{and} \quad y_{s+h}^+ = \Big(s+h - a_{s+h}^+, f^{\xi,y}(s+h) - b_{s+h}^+\Big). \tag{7.2.15}$$

According to Lemma 7.2.1, each $T := f^{\xi,y}(s+h) - f^{\xi,y}(s)$ is associated to a unique point $y_{s+h}^*(T)$, for which $\Big\|y_{s+h}^*(T) - x_s\Big\| = \Big\|y_{s+h}^*(T) - x_{s+h}\Big\| = \varepsilon$, and which defines geometrically the lower bound $k(T,h)$ for the right derivative $D^+ f^{\xi,y}(s+h)$. Lemma 7.2.2 on the other hand asserts that for fixed h, the function $k(T,h)$ is increasing in T. Since $y_s^+ \in \Pi_E^{\text{ext}}(x_s) \subset E$ and $h \leq \varepsilon/2$, there exists a lower bound for the value $f^{\xi,y}(s+h)$, and thus for the difference T. We aim to find an explicit value for $k(\widehat{T}, h)$ at this lower bound $\widehat{T} \leq T$.

For the boundary point $x_{s+h} \in J \subset \partial E_\varepsilon$ and the extremal contributor $y_s^+ \in \Pi_E^{\text{ext}}(x_s) \subset E$, defined in (7.2.14) and (7.2.15) respectively, the lower bound $\|x_{s+h} - y_s^+\| \geq \varepsilon$ implies

$$\widehat{T} := -b_s^+ + \sqrt{\varepsilon^2 - \left(a_s^+ + h\right)^2} \leq f^{\xi,y}(s+h) - f^{\xi,y}(s) = T, \tag{7.2.16}$$

see Figure 7.3. By construction, b_s^+ is always positive, while a_s^+ may either be positive or negative depending on the slope $D^+ f^{\xi,y}(s)$. We aim to show that $k\Big(\widehat{T}, h\Big) = p(a_s^+ + h)$.

Consider the point $\widehat{x}_{s+h} := \Big(s+h, f^{\xi,y}(s) + \widehat{T}\Big) \in \partial B_\varepsilon\left(y_s^+\right)$ and define $z_{s+h} := \Big(s+h, f^{\xi,y}(s) - b_s^+\Big)$ so that z_{s+h} shares its ξ–coordinate with $\widehat{x}_{s+h}$ and its $\varepsilon^{-1}(x-y)$–coordinate with y_s^+. Then $\widehat{x}_{s+h} - y_s^+ = \Big(\widehat{a}_{s+h}, \widehat{b}_{s+h}\Big)$, where $\widehat{a}_{s+h} := a_s^+ + h$ and $\widehat{b}_{s+h} :=$

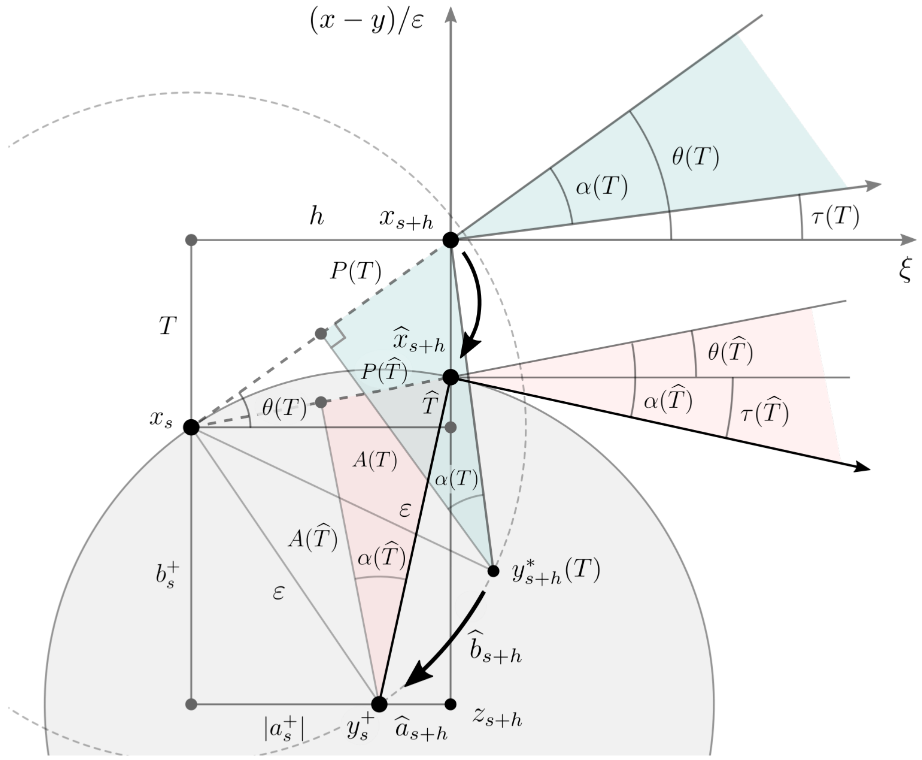


Figure 7.3. Illustration of the dependence of the function $k(T,h) = \tan\tau(T)$ on the difference $T := f^{\xi,y}(s+h) - f^{\xi,y}(s)$. As T decreases, the corresponding point $y^*_{s+h}(T)$ on the circumference $\partial B_\varepsilon(x_s)$ moves clockwise towards y_s^+. This results in a decrease in the slope $\tan\tau(T)$, where $\tau(T) := \theta(T) - \alpha(T)$. At the minimal value $\widehat{T} \leq T$, one obtains the lower bound $k(\widehat{T},h) = \tan\tau(\widehat{T}) = p(a_s^+ + h)$, which geometrically corresponds to the point $y^*_{s+h}(\widehat{T}) = y_s^+$.

$\sqrt{\varepsilon^2 - (a_s^+ + h)^2}$. This implies that

$$k(\widehat{T},h) = \tan\Big(\theta(\widehat{T}) - \alpha(\widehat{T})\Big) = \tan\tau(\widehat{T}) = \frac{-\widehat{a}_{s+h}}{\widehat{b}_{s+h}} = \frac{-\big(a_s^+ + h\big)}{\sqrt{\varepsilon^2 - \big(a_s^+ + h\big)^2}} = p\big(a_s^+ + h\big),$$

where $\tau(\widehat{T})$ is the angle at $\widehat{x}_{s+h}$ between the vectors $y_s^+ - \widehat{x}_{s+h}$ and $z_{s+h} - \widehat{x}_{s+h}$. $\square$

Combining Lemmas 7.2.1–7.2.3, we are now ready to prove Proposition 7.1.2.

Proof of Proposition 7.1.2. Let $s,h > 0$ with $0 \leq s+h \leq s_{\xi,y}$ and let $T := f^{\xi,y}(s+h) - f^{\xi,y}(s)$. In addition, define $k(T,h) := (TA(T) - hP(T))/(hA(T) + TP(T))$ as in (7.2.5) and let $\widehat{T} \leq T$ be the lower bound, given by Lemma 7.2.3, for which $k\big(\widehat{T},h\big) = p(a_s^+ + h)$. Here, the slope function $p : (-\varepsilon,\varepsilon) \to \mathbb{R}$ is given by $p(s) := -s/\sqrt{\varepsilon^2 - s^2}$ as in (7.2.2). We want to use the monotonicity of the map $T \mapsto k(T,h)$ in order to establish for each h the inequality $k(T,h) \geq k\big(\widehat{T},h\big)$. For this, we need to show that T satisfies the assumption

$$-\sqrt{4\varepsilon^2 - h^2} < T < \sqrt{\varepsilon^2 - (\varepsilon - h)^2}$$

in Lemma 7.2.2. This follows immediately by combining the Lipschitz property[5]

$$|T| = |f^{\xi,y}(s+h) - f^{\xi,y}(s)| \leq h/\sqrt{3},$$

given by Proposition 3.2.5, with the inequality $h/\sqrt{3} \leq \sqrt{\varepsilon^2 - (\varepsilon - h)^2} < \sqrt{4\varepsilon^2 - h^2}$, which is implied by $0 \leq h \leq s_{\xi,y} \leq \varepsilon/2$. The results in Lemmas 7.2.1–7.2.3 thus imply that

$$D^+ f^{\xi,y}(s+h) \geq k(T,h) \geq k\left(\widehat{T}, h\right) = p(a_s^+ + h) \tag{7.2.17}$$

whenever $s, h \geq 0$ and $0 \leq s + h \leq s_{\xi,y}$. According to (7.2.2), the slope at s satisfies $D^+ f^{\xi,y}(s) = p\left(a_s^+\right)$, which implies

$$D^+ f^{\xi,y}(s+h) - D^+ f^{\xi,y}(s) \geq p(a_s^+ + h) - p(a_s^+). \tag{7.2.18}$$

We now want to establish this inequality also for the left derivatives $D^- f^{\xi,y}(s)$ for all $s \in [0, s_{\xi,y}]$. According to Proposition 2.2.8, the left derivative $D^- f^{\xi,y}(s)$ coincides with the left extremal outward direction ξ_s^- at $x_s \in J$. Lemma 2.2.7 (ii) (b) furthermore implies that for any sequence $\left(x_{s(n)}\right)_{n=1}^{\infty} \subset J$ where $s(n) \to s$ from below, so that $x_{s(n)} \to x$ and

$$\frac{x_{s(n)} - x_s}{\left\| x_{s(n)} - x_s \right\|} \to \xi_s^-,$$

the right extremal outward directions $\xi_{s(n)}^+ \in \Xi_{x_{s(n)}}^{\mathrm{ext}}(E_\varepsilon)$ satisfy $\xi_{s(n)}^+ \to -\xi_s^-$ as $n \to \infty$. Since these in turn correspond to the right derivatives $D^+ f^{\xi,y}(s(n))$ via (7.2.2), it follows that

$$\lim_{n\to\infty} D^+ f^{\xi,y}(s(n)) = \lim_{n\to\infty} \frac{\xi_2^+(s(n))}{\xi_1^+(s(n))} = \frac{-\xi_2^-(s)}{-\xi_1^-(s)} = D^- f^{\xi,y}(s). \tag{7.2.19}$$

Since the choice of the sequence $\left(x_{s(n)}\right)_{n=1}^{\infty}$ in (7.2.19) is arbitrary and the slope function p is continuous on $[0, \varepsilon/2]$, it follows from this and (7.2.17) that for all $s, h \geq 0$ and $0 \leq s + h \leq s_{\xi,y}$,

$$D^- f^{\xi,y}(s+h) = \lim_{\varphi\to 0-} D^+ f^{\xi,y}(s+h+\varphi) \geq \lim_{\varphi\to 0-} p(a_s^+ + h + \varphi) = p(a_s^+ + h). \tag{7.2.20}$$

Due to the characterisation of the sets of extremal outward directions given in Proposition 2.2.6, there can be no cusp singularities on the boundary J. Given the relationship (7.2.2) between extremal outward directions and the one-sided derivatives, this implies that $D^- f^{\xi,y}(s) \leq D^+ f^{\xi,y}(s) = p(a_s^+)$ for all $s \in [0, s_{\xi,y}]$. Combining this inequality

[5] Note that the function $f^{\xi,y}$ here corresponds to the orthonormal coordinate system $\left(\xi, \varepsilon^{-1}(x-y)\right)$, and is thus scaled by a factor of ε compared to the corresponding function $f^{\xi,y}$ in the statement of Proposition 3.2.5, see equations (7.1.2) and (3.2.2). Hence, the Lipschitz constant here is $1/\sqrt{3}$ rather than $1/\sqrt{3}\varepsilon$.

with (7.2.17) and (7.2.20) yields the analogue of (7.2.18) for the left derivatives:

$$D^- f^{\xi,y}(s+h) - D^- f^{\xi,y}(s) \geq p(a_s^+ + h) - p(a_s^+). \tag{7.2.21}$$

A direct calculation shows that $p'(s) = (p(s)+p^3(s))/s$ and since p is an odd function, it follows that p' is even and non-positive. Then

$$p'_{\min} := \min_{|s|\leq s_{\xi,y}} p'(s) \geq \min_{|s|\leq \varepsilon/2} p'(s) = p'\left(\frac{\varepsilon}{2}\right) = -\frac{8}{3\sqrt{3}\varepsilon},$$

which implies

$$p(a_s^+ + h) - p(a_s^+) = \int_{a_s^+}^{a_s^+ + h} p'(s)ds \geq \int_{a_s^+}^{a_s^+ + h} p'_{\min} ds \geq -\frac{8h}{3\sqrt{3}\varepsilon}.$$

The result now follows from (7.2.18) and (7.2.21). □

Bibliography

[AF90] J.-P. Aubin and H. Frankowska. *Set-Valued Analysis.* Birkhäuser, Boston, 1990.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford University Press, 2000.

[AKS+16] S. Auer, K. Kleis, P. Schultz, J. Kurths, and F. Hellmann. The impact of model detail on power grid resilience measures. *The European Physical Journal Special Topics*, 225:609–625, 2016.

[Arn98] L. Arnold. *Random Dynamical Systems.* Springer, 1998.

[Ath18] A. Athorne. *Bifurcations of Set-Valued Dynamical Systems.* PhD thesis, Imperial College London, 2018. Supervisors: J. Lamb and M. Rasmussen.

[Bau07] C.A. Baumann. Harvesting in a random environment: Itô or Stratonovich calculus. *Journal of Theoretical Biology*, 244:424–432, 2007.

[BG12] F. Bigolin and G. H. Greco. Geometric characterizations of C^1-manifolds in euclidean spaces by tangent cones. *J. Math. Anal. Appl.*, 396:145–163, 2012.

[BGP09] D. Bauso, L. Giarré, and R. Pesenti. Consensus for networks with unknown but bounded disturbances. *SIAM J. Control Optim.*, 48(3):1756–1770, 2009.

[BMO09] A. Blokh, M. Misiurewicz, and L. Oversteegen. Sets of constant distance from a compact set in 2-manifolds with a geodesic metric. *Proc. Amer. Math. Soc.*, 137(2):733–743, 2009.

[Bou30] G. Bouligand. Sur les surfaces dépourvues de points hyperlimites. *Ann. Soc. Polon. Math.*, 9:32–41, 1930.

[Bou32] G. Bouligand. *Introduction á la géométrie infinitésimale directe.* Gauthier-Villars, 1932.

[Bro72] M. Brown. Sets of constant distance from a planar set. *Michigan Math. J.*, 19:321–323, 1972.

[Cea19] A. Cardellini and et al. Multi-scale approach for modeling stability, aggregation, and network formation of nanoparticles suspended in aqueous solutions. *Nanoscale*, 11:3979–3992, 2019.

[CK00] F. Colonius and W. Kliemann. *The Dynamics of Control*. Birkhäuser, Boston, 2000.

[CKMS08] F. Colonius, E. Kreuzer, A. Marquardt, and W. Sichermann. A Numerical Study of Capsizing: Comparing Control Set Analysis and Melnikov's Method. *International Journal of Bifurcation and Chaos*, 18(05):1503–1514, 2008.

[Cla75] F. H. Clarke. Generalized gradients and applications. *Trans. Am. Math. Soc.*, 205:247–262, 1975.

[CM17] F. Chazal and B. Michel. An introduction to topological data analysis: fundamental and practical aspects for data scientists. *hal-01614384*, 2017.

[d'O13] A. d'Onofrio. *Bounded Noises in Physics, Biology, and Engineering*. Birkhäuser, 2013.

[Dup10] X. Dupuis. Invariant Sets of Set-Valued Dynamical Systems. Master's thesis, Imperial College London, 2010. Supervisors: J. Lamb and M. Rasmussen.

[DV75] M. D. Donsker and S. R. S. Varadhan. Asymptotics for the Wiener Sausage. *Comm. Pure and Appl. Math.*, 28:525–565, 1975.

[Erd45] P. Erdős. Some remarks on the measurability of certain sets. *Bull. Amer. Math. Soc.*, 51(10):728–731, 1945.

[Fal85] K. J. Falconer. *The Geometry of Fractal Sets*. Cambridge University Press, 1985.

[Fed59] H. Federer. Curvature measures. *Trans. Amer. Math. Soc.*, 93:418–491, 1959.

[Fer76] S. Ferry. When ε-boundaries are manifolds. *Fund. Math.*, 90:199–210, 1976.

[FNJ+13] D. R. Freestone, D. Nešić, A. Jafarian, M. J. Cook, and D. B. Grayden. A neural mass model of spontaneous burst suppression and epileptic seizures. In *2013 35th Annual International Conference of the IEEE Engineering in Medicine and Biology Society (EMBC)*, pages 5942–5945, 2013.

[Fu85] J.H.G. Fu. Tubular neighborhoods in Euclidean spaces. *Duke Mathematical Journal*, 52(4):1025–1046, 1985.

[GCS08] M. Ghil, M. D. Chekroun, and E. Simonnet. Climate dynamics and fluid mechanics: Natural variability and related uncertainties. *Physica D: Nonlinear Phenomena*, 237(14–17):2111–2126, 2008.

[Gla00] Malcolm Gladwell. *The Tipping Point*. Little Brown, 2000.

[GP72] R. Gariepy and W. D. Pepe. On the level sets of a distance function in a minkowski space. *Proc. Amer. Math. Soc.*, 31:255–259, 1972.

[Ham10] Y. Hamana. On the expected volume of the Wiener sausage. *J. Math. Soc. Japan*, 62(1):1113–1136, 2010.

[HY06] A.J. Homburg and T.R. Young. Hard bifurcations in dynamical systems with bounded random perturbations. *Regul. Chaotic Dyn.*, 11(2):247–258, 2006.

[HY10] A.J. Homburg and T.R. Young. Bifurcations for random differential equations with bounded noise on surfaces. *Topological Methods in Nonlinear Analysis*, 35(1):77–98, 2010.

[HYG13] A.J. Homburg, T.R. Young, and M. Gharaei. Bifurcations of random differential equations with bounded noise. In A. d'Onofrio, editor, *Bounded Noises in Physics, Biology, and Engineering. Modeling and Simulation in Science, Engineering and Technology.*, pages 133–149. Birkhäuser Verlag, New York, NY, 2013.

[KC98] V. Křivan and G. Colombo. A non-stochastic approach for modelling uncertainty in population dynamics. *Bulletin of Mathematical Biology*, 60:721–751, 1998.

[Kec95] A.S. Kechris. *Classical Descriptive Set Theory*, pages 31–32, 35–36. Graduate Texts in Mathematics. Springer-Verlag, 1995.

[KF75] A. N. Kolmogorov and S. V. Fomin. *Introductory Real Analysis.* Dover Publications, 1975.

[KL73] A. Kac and J. M. Luttinger. Bose-Einstein condensation in the presence of impurities. *J. Math. Phys.*, 14:1626–1628, 1973.

[Kla18] B. Klartag. Eldan's Stochastic Localization and Tubular Neighborhoods of Complex-Analytic Sets. *The Journal of Geometric Analysis*, 28(2008–2018), 2018.

[KLV13] A. Käenmäki, J. Lehrbäck, and M. Vuorinen. Dimensions, Whitney covers, and tubular neighborhoods. *Indiana Univ. Math. J.*, 62(6):1861–1889, 2013.

[KMR18] C. Kuehn, G. Malavolta, and M. Rasmussen. Early-warning signals for bifurcations in random dynamical systems with bounded noise. *Journal of Mathematical Analysis and Applications*, 464(1), 2018.

[Kue11] C. Kuehn. A mathematical framework for critical transitions: Bifurcations, fast–slow systems and stochastic dynamics. *Physica D*, (240):1020–1035, 2011.

[Lea19] T. M. Lenton and et al. Climate tipping points — too risky to bet against. *Nature*, 575:592–595, 11 2019.

[LRR15] J.S.W. Lamb, M. Rasmussen, and C. Rodrigues. Topological bifurcations of minimal invariant sets for set-valued dynamical systems. *Proceedings of the American Mathematical Society*, 143(9):3927–3937, 2015.

[LRTa] J. S. W. Lamb, M. Rasmussen, and K. Timperi. On boundaries of ε-neighbourhoods of planar sets, part I: Singularities. *Submitted to the Journal of the London Mathematical Society. An arXiv preprint is available at https://arxiv.org/abs/2012.13515.*

[LRTb] J. S. W. Lamb, M. Rasmussen, and K. Timperi. On boundaries of ε-neighbourhoods of planar sets, part II: Global structure and curvature. *An arXiv preprint is available at https://arxiv.org/abs/2107.07544.*

[MHMK13] P. J. Menck, J. Heitzig, N. Marwan, and J. Kurths. How basin stability complements the linear-stability paradigm. *Nature Physics*, 9:89–92, 2 2013.

[MM] K. J. Meyer and R. P. McGehee. Intensity—a metric approach to quantifying attractor robustness in odes. *arXiv:2012.10786 [math.DS].*

[NP16] S. Nekovar and G. Pruessner. A Field-Theoretic Approach to the Wiener Sausage. *Journal of Statistical Physics*, 163(604–641), 2016.

[OP85] I. Ya. Oleksiv and N. I. Pesin. Finiteness of the Hausdorff measure of level sets of bounded subsets in Euclidean space. *Mat. Zametki*, 37:422–431, 1985. (in Russian).

[Pog16] N. I. Pogodaev. On the regularity of the boundary of the integral funnel of a differential inclusion. *Differential Equations*, 52(8):987–999, 2016.

[Prz13] A. Przeworski. An upper bound on density for packings of collars about hyperplanes in $\mathbb{H}^n$. *Geom. Dedicata*, 163:193–213, 2013.

[Rob95] C. Robinson. *Dynamical Systems.* CRC Press, Cambridge, 1995.

[Roc79] R. T. Rockafellar. Clarke's tangent cones and the boundaries of closed sets in $\mathbb{R}^n$. *Nonlinear Analysis: Theory, Methods and Applications*, 3(1):145–154, 1979.

[RSM09] J. Rataj, E. Spodarev, and D. Meschenmoser. Approximations of the Wiener Sausage and Its Curvature Measures. *The Annals of Applied Probability*, 19(5):1840–1859, 2009.

[RVS09] J. Rataj and E. Spodarev V. Schmidt. On the Expected Surface Area of the Wiener Sausage. *Mathematische Nachrichten*, 282(4):591–603, 2009.

[RW98] R. T. Rockafellar and R. J.-B. Wets. *Variational Analysis.* Springer-Verlag, 1998.

[RW10] J. Rataj and S. Winter. On volume and surface area of parallel sets. *Indiana University Mathematics Journal*, 59(5):1661–1685, 2010.

[RZ11] J. Rataj and L. Zajíček. Properties of distance functions on convex surfaces and applications. *Czech. Math. J.*, 61:247–269, 2011.

[RZ20] J. Rataj and L. Zajíček. Smallness of the critical values of distance functions in two-dimensional euclidean and riemannian spaces. *Mathematika*, 66:297–324, 2020.

[SBB+09] M. Scheffer, J. Bascompte, W. A. Brock, V. Brovkhin, S. R. Carpenter, V. Dakos, H. Held, E. H. van Nes, M. Rietkerk, and G. Sugihara. Early-warning signals for critical transitions. *Nature*, 461:53–59, 2009.

[Sch09] M. Scheffer. *Critical Transitions in Nature and Society*. Princeton University Press, 2009.

[Sor04] D. Sornette. *Critical Phenomena in Natural Sciences, Chaos, Fractals, Self-organization and Disorder: Concepts and Tools*. Springer Series in Synergetics, 2nd edition, 2004.

[Spi64] F. Spitzer. Electrostatic Capacity, Heat Flow, and Brownian Motion. *Z. Wahrscheinlichkeitstheorie*, 3:110–121, 1964.

[Sta76] L.L. Stachó. On the volume function of parallel sets. *Acta Sci. Math.*, 38:365–374, 1976.

[TS11] J. M. T. Thompson and J. Sieber. Predicting climate tipping as a noisy bifurcation: A review. *Int. J. of Bifurcation and Chaos*, 21(2):399–423, 2011.

[Wea19] K. Wiesner and et al. Stability of democracies: a complex systems perspective. *Eur. J. Phys.*, 40(1):014002, 2019.

[ZH07] H. Zmarrou and A.J. Homburg. Bifurcations of stationary measures of random diffeomorphisms. *Ergodic Theory and Dynamical Systems*, 27(5):1651–1692, 2007.