Ruling out Positive Lyapunov Exponents by using the Jacobian's Second Additive Compound Matrix

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***Abstract*—Second additive compound matrices of the system's Jacobian are used to formulate sufficient conditions to rule out existence of attractors with positive Lyapunov exponents. The criteria are expressed in terms of Lyapunov dissipation inequalities or Linear Matrix Inequalities amenable to analytic verification. The results extend applicability of previous existing conditions formulated to discard periodic and almost periodic oscillations. An example of the technique to rule out chaos in certain parameters region of the Lorenz system is discussed.**

***Index Terms*—Chaotic attractors, Compound matrices, Lorenz system, Lyapunov exponents**

I. Introduction

CHARACTERISING and analysing the long-term behaviour of solutions of a dynamical system is a classical topic in the study of differential equations. The question is closely related with assessment of (global) stability and instability properties of solutions, and many complementary approaches exist to these days, including Lyapunov analysis techniques [1], contraction analysis [2], monotonicity-based considerations [3], density functions and dual Lyapunov analysis [4], to name a few. For a finite dimensional time-invariant nonlinear system defined by sufficiently smooth equations $\dot{x} = f(x)$, contraction analysis guarantees convergence of solutions towards one another by imposing a Linear Matrix Inequality constraint fulfilled by the Jacobian of the vector field $f(x)$, defining the system's equations. A related Jacobian-based approach was introduced by James Muldowney in the seminal paper [5], where conditions are provided based on the *second additive compound* of the Jacobian $Df(x)$ to rule out existence of periodic solutions in nonlinear systems of differential equations. Roughly speaking Muldowney's conditions rule out the possibility of periodic solutions on the ground of contraction assumptions prescribed on arbitrary infinitesimal bidimensional surfaces. A self-contained introduction to the topic of *additive* and *multiplicative* compound matrices was recently proposed in [6]. Indeed such matrices have been widely adopted in the qualitative study of attractors, with special focus on their geometry and dimensionality, as documented in the monograph [7]. In the same spirit, compound matrices of fractional order were recently introduced in [8]. Closer to this paper's contribution, Muldowney's results were taken up and extended in [9] to rule out periodic and almost periodic solutions (more precisely existence of invariant toruses of arbitrary dimension) and applied to the qualitative study of biochemical networks. Conditions were specifically formulated using piecewise linear Lyapunov functions, defined with respect to an auxiliary variational equation involving the second-additive compound matrix of the Jacobian:

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{\delta}^{[2]} &= \frac{\partial f}{\partial x}^{[2]}(x)\delta^{[2]}, \end{aligned} \tag{1}$$

whose precise definition will be later detailed. This auxiliary equation also serves as the starting point of our paper. Our main contribution is to show how the same Lyapunov conditions proposed in [9], and in fact in a slightly relaxed form, are also suitable to rule out existence of attractors with positive Lyapunov exponents. In this respect, they complement the results of [9] by ruling out another significant class of non-convergent dynamical behaviours and have a complementary scope of application to numerical algorithms for the computation of maximal Lyapunov exponents, (see [10] and references therein).

II. Background on Compound Matrices

Let $A \in \mathbb{R}^{n\times m}$, and take any positive integer $k \leq \min\{n,m\}$. Following [6], [5] we denote by $Q(k,n)$ the set of increasing sequences of k integers with values in $\{1,2,\dots,n\}$ listed in lexicographical order (notice that this set has cardinality $\binom{n}{k}$). For any pair $(\alpha,\beta) \in Q(k,n)\times Q(k,m)$ we select the $k\times k$ submatrix of A corresponding to the rows and columns in α and β, respectively, and denote it by $A(\alpha|\beta)$. The corresponding minor is the determinant $\det(A(\alpha|\beta))$.

Definition 1. *For a matrix $A \in \mathbb{R}^{n\times m}$ and any positive integer $k \leq \min\{n,m\}$, the k-th multiplicative compound matrix, denoted by $A^{[k]}$ is the $\binom{n}{k} \times \binom{m}{k}$ matrix whose entries are all the k dimensional minors of A listed in lexicographical order.*

Notice that, $A^{[1]} = A$, by definition, while, for a square matrix $A \in \mathbb{R}^{n\times n}$, $A^{[n]} = \det(A)$. For square matrices multiplicative compound matrices have important spectral

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properties, in particular if $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of A, then:

$$\{\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k}, 1 \le i_1 < i_2 < \ldots < i_k \le n\}$$

is the spectrum of $A^{[k]}$. Multiplicative compound matrices also allow to synthetically introduce *wedge* product operations between vectors. In particular, for any 2 vectors in $\mathbb{C}^n$ we will later use the notation $v_1 \wedge v_2 := [v_1, v_2]^{[2]}$.

Definition 2. *For a matrix $A \in \mathbb{R}^{n\times n}$ and any positive integer $k \le n$, the k-th additive compound matrix, denoted by $A^{(k)}$ is the $\binom{n}{k} \times \binom{n}{k}$ matrix defined as*

$$A^{(k)} = \frac{d}{d\varepsilon}(I + \varepsilon A)^{[k]}\bigg|_{\varepsilon=0}$$

It can be verified that $(A_1+A_2)^{(k)} = A_1^{(k)}+A_2^{(k)}$, moreover the following also holds true:

$$\frac{d}{d\varepsilon}(e^{A\varepsilon})^{[k]} = A^{(k)}.$$

For square matrices additive compound matrices have important spectral properties, in particular if $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of A, then:

$$\{\lambda_{i_1} + \lambda_{i_2} + \ldots + \lambda_{i_k}, 1 \le i_1 < i_2 < \ldots < i_k \le n\}$$

is the spectrum of $A^{(k)}$.
Besides interesting algebraic properties, compound matrices are noteworthy for their links to differential equations. For any matrix $X(t) \in \mathbb{R}^{n\times m}$, solution of a differential equation:

$$\dot{X}(t) = A(t)X(t), \tag{2}$$

with $A(t)$ a square matrix of compatible dimension, the following holds for any integer $k \le \min\{n, m\}$:

$$\dot{X}^{[k]} = A^{(k)}(t)X^{[k]}(t). \tag{3}$$

This shows that the minors of the matrix $X(t)$ also evolve according to linear dynamics, which are computed by the additive compound matrix of $A(t)$. This applies to time-varying systems, and can therefore be applied to the study of nonlinear dynamics through their variational equation.

III. Set-up and problem formulation

Consider the dynamical system described by the following set of first-order differential equations:

$$\dot{x}(t) = f(x(t)), \tag{4}$$

with state $x(t) \in X \subset \mathbb{R}^n$ (for some open set X and initial condition $x(0) = x_0$) and $f : X \to \mathbb{R}^n$ of class $\mathcal{C}^1$. As customary, one may associate to this a *variational equation*:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) \\ \dot{\delta}(t) &= \frac{\partial f}{\partial x}(x(t))\delta(t), \end{aligned} \tag{5}$$

whose $\delta(t)$ component ($\in \mathbb{R}^n$) propagates, along solutions of (4), the effect of infinitesimal perturbations of initial conditions in the direction of $\delta(0)$. A matrix version of (5) is also handy, and defined as follows:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) \\ \dot{X}(t) &= \frac{\partial f}{\partial x}(x(t))X(t), \\ X(0) &= I_n, \end{aligned} \tag{6}$$

where $X(t) \in \mathbb{R}^{n\times n}$ is the transition matrix of the 'time-varying' linear system describing the evolution of δ in (5) and I_n denotes the identity matrix of order n. In particular, $\delta(t)$, solution of (5) for a given $x(0)$ and $\delta(0)$, fulfills $\delta(t) = X(t)\delta(0)$, where $X(t)$ is a solution of (6) for the same initial condition $x(0)$ (in fact, $\frac{d}{dt}X(t)\delta(0) = \dot{X}(t)\delta(0) = \frac{\partial f}{\partial x}(x(t))X(t)\delta(0)$).

Definition 3. *Lyapunov Exponents: for a given initial condition $x(0)$ (and assuming $x(t)$ is uniformly bounded over $[0, +\infty)$), n non-zero (and orthogonal) vectors $v_1, v_2, \ldots, v_n$, and n real numbers, $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$, are defined (the Lyapunov exponents), such that:*

$$\lambda_i = \lim_{t\to+\infty} \frac{\log(|X(t)v_i|)}{t}, \tag{7}$$

where $|\cdot|$ denotes the Euclidean norm of a vector.

Conditions for existence of the Lyapunov exponents are relatively mild and have been first established in [11] and later generalised in [12]. Lyapunov exponents are invariant under fairly general coordinate transformations and along solutions, thus providing an intrinsic characterization of how an attractor responds to perturbations of initial conditions in different directions. Finding a-priori bounds on their location yields insight into the dynamics of a system and is normally approached through the study of the variational equation (5), see for instance [13], where a direct estimation of the growth rate of the variational equation is proposed.

We propose to study the Lyapunov exponents by using an alternative, possibly higher-order, linearized dynamics, involving the second-additive compound matrix $\frac{\partial f}{\partial x}^{(2)}$ of the Jacobian. In particular,

$$\begin{aligned} \dot{x}(t) &= f(x(t)) \\ \dot{\eta}(t) &= \frac{\partial f}{\partial x}^{(2)}(x(t))\eta(t), \end{aligned} \tag{8}$$

where $\eta(t) \in \mathbb{R}^{\binom{n}{2}}$, can be interpreted, when ascribable as $\delta_1(t)\wedge\delta_2(t) := [\delta_1(t), \delta_2(t)]^{[2]}$ for some solutions δ_1, δ_2 of (5), as the propagation in time of an infinitesimal area perturbation in the direction of $\delta_1(0)$, $\delta_2(0)$. In particular, if $\eta(0) = \delta_1(0)\wedge\delta_2(0)$, then $\eta(t) = \delta_1(t) \wedge \delta_2(t)$. A matrix version of (8) can also be introduced as follows:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) \\ \dot{Y}(t) &= \frac{\partial f}{\partial x}^{(2)}(x(t))Y(t), \\ Y(0) &= I_{\binom{n}{2}} \end{aligned} \tag{9}$$

where $Y(t) \in \mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$. Notice that $Y(t) = X^{[2]}(t)$, the second multiplicative compound matrix of $X(t)$. Our objective is to use (9) and its 'Lyapunov exponents' in order to gain insight into Lyapunov exponents of (6).

Definition 4. *Lyapunov Exponents of (9): for a given initial condition $x(0)$ (and assuming $x(t)$ is uniformly bounded over $[0, +\infty)$), there exists $\binom{n}{2}$ non-zero (and orthogonal) vectors $w_{ij} \in \mathbb{R}^{\binom{n}{2}}$, $1 \le i < j \le n$, and $\binom{n}{2}$ real numbers, μ_{ij}, (the Lyapunov exponents of (9)), such that:*

$$\mu_{ij} = \lim_{t \to +\infty} \frac{\log(|Y(t) w_{ij}|)}{t}, \tag{10}$$

where $|\cdot|$ denotes the Euclidean norm of a vector.

More specifically, we aim to find a bound on the maximal Lyapunov exponent of (5) by expressing suitable conditions to bound the Lyapunov exponents of (8). While the proposed approach is tight, dimension of Y grows quadratically in n, potentially making its application challenging in high-dimensional examples.

IV. Main Results

In order to accomplish our aims deriving a link between the Lyapunov spectra of (5) and (8) is essential.

Proposition 1. *For any given initial condition $x(0)$, such that $x(t)$ is uniformly bounded, the following relationship holds, (for a suitable ordering of the μ_{ij} and rescaling of vectors) :*

$$\begin{array}{rcll} \mu_{ij} & = & \lambda_i + \lambda_j, & 1 \le i < j \le n \\ w_{ij} & = & v_i \wedge v_j & \end{array} \tag{11}$$

In order to make full use of the powerful algebraic machinery of compound matrices, we adopt the following equivalent characterization of Lyapunov exponents. This will be suitable for addressing the proof of Proposition 1.

Proposition 2. *For any given initial condition $x(0)$ the Lyapunov exponents of a matrix variational equation (such as (6) or (9)) and corresponding directions are given by the spectrum of the symmetric real matrix:*

$$\bar{H} := \lim_{t \to +\infty} \frac{\log\left(X(t)^T X(t)\right)}{2t} \tag{12}$$

and by its associated orthonormal basis of eigenvectors.

Accordingly, the Lyapunov exponents of the second-additive compound variational equation (9) can be computed as given in the following proposition.

Proposition 3. *For any given initial condition $x(0) = x_0$ the Lyapunov exponents of (9) and corresponding orthogonal directions are given by the spectrum of the symmetric real matrix*

$$\bar{M} := \lim_{t \to +\infty} \frac{\log\left(Y(t)^T Y(t)\right)}{2t} \tag{13}$$

and by its associated orthonormal basis of eigenvectors.

The equivalence between the definition of the Lyapunov exponents in Definition 4 and the one in Proposition 2 can be found in [14] and [12], where the Lyapunov exponents are defined as the logarithm of the eigenvalues of the following limit matrix $\Lambda(x_0)$

$$\Lambda(x_0) = \lim_{t \to +\infty} \left(X(t, x_0)^T X(t, x_0)\right)^{\frac{1}{2t}} \tag{14}$$

Proof: (Proposition 1) By observing that $Y(t) = X^{[2]}(t)$ and exploiting the definition of the Lyapunov exponents in Proposition 2 and 3 we have

$$\begin{aligned} \bar{M} &= \lim_{t \to +\infty} \frac{\log\left(Y^T(t) Y(t)\right)}{2t} \\ &= \lim_{t \to +\infty} \frac{\log\left((X^{[2]}(t))^T X^{[2]}(t)\right)}{2t} \\ &= \lim_{t \to +\infty} \frac{\log\left((X^T(t))^{[2]} X^{[2]}(t)\right)}{2t} \\ &= \lim_{t \to +\infty} \frac{\log\left((X^T(t) X(t))^{[2]}\right)}{2t} \\ &= \lim_{t \to +\infty} \frac{\log\left((\exp^{\log(X^T(t)X(t))})^{[2]}\right)}{2t} \\ &= \lim_{t \to +\infty} \frac{\log\left(\exp^{[\log(X^T(t)X(t))]^{(2)}}\right)}{2t} \\ &= \lim_{t \to +\infty} \frac{\left[\log(X^T(t)X(t))\right]^{(2)}}{2t} = \bar{H}^{(2)} \end{aligned}$$

where the following properties of compound matrices have been exploited:

- Cauchy-Binet formula: $(AB)^{[k]} = A^{[k]} B^{[k]}$
- Transpose: $(A^{[k]})^T = (A^T)^{[k]}$
- Exponential: $e^{(A^{(k)})} = (e^A)^{[k]}$.

Recalling the spectral property of (second) additive compound matrices completes the first part of the claim. Let v_i and v_j be eigenvectors of $\bar{H}$ associated to eigenvalues λ_i and λ_j respectively. Then:

$$\bar{H}[v_i, v_j] = [v_i, v_j] \mathrm{diag}(\lambda_i, \lambda_j).$$

We exploit the latter equation in the following derivation:

$$\begin{aligned} \bar{M}(v_i \wedge v_j) = \bar{H}^{(2)}[v_i, v_j]^{[2]} &= \frac{d}{d\varepsilon}(I + \varepsilon \bar{H})^{[2]}\Big|_{\varepsilon=0} [v_i, v_j]^{[2]} \\ &= \frac{d}{d\varepsilon}\left((I + \varepsilon \bar{H})[v_i, v_j]\right)^{[2]}\Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}\left([v_i, v_j](I + \varepsilon\, \mathrm{diag}(\lambda_i, \lambda_j))\right)^{[2]}\Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}[v_i, v_j]^{[2]}(I + \varepsilon\, \mathrm{diag}(\lambda_i, \lambda_j))^{[2]}\Big|_{\varepsilon=0} \end{aligned}$$

$$= \frac{d}{d\varepsilon}[v_i, v_j]^{[2]}(1 + \varepsilon \lambda_i)(1 + \varepsilon \lambda_j)\Big|_{\varepsilon=0} = (\lambda_i + \lambda_j)(v_i \wedge v_j).$$

This concludes the proof of the Proposition. □

Proposition 1 characterizes the Lyapunov exponents of the variational equation (8). Our goal, however, is to find a bound to the maximal Lyapunov exponent of (5). To this end we exploit that whenever the ω-limit set is non trivial (see e.g. [1] for a definition) there always exists a 0 Lyapunov exponent.

This well known fact is recalled in the Lemma that follows for the sake of completeness.

Lemma 1. *Consider the non-linear dynamical system with its associate variational equation (6) with $f:\mathbb{R}^n\to\mathbb{R}^n$, $f\in\mathcal{C}^1$. Let $\omega(x_0)$ be the ω-limit set and let $f(x)\neq 0$ for all $x\in\omega(x_0)$ (i.e. the ω-limit set does not contain equilibria). Then there exists $i\in\{1,\dots,n\}$ such that $\lambda_i=0$, where $\lambda_1,\dots,\lambda_n$ are the Lyapunov exponents of the system (4).*

Proof: First, we observe that the function $f(x(t))$ is a solution of the variational equation, since

$$\dot{f}(x(t))=\frac{\partial f}{\partial x}(x(t))\dot{x}(t)=\frac{\partial f}{\partial x}(x(t))f(x(t))\ .$$

Hence, it follows that

$$f(x(t))=X(t)f(x(0))$$

For systems with bounded solutions, the ω-limit set is compact. Therefore $f(x)$ is uniformly bounded for $x\in\omega(x_0)$ and, since by hypothesis $f(x)\neq 0$ in $\omega(x_0)$, we have for sufficiently small ε and sufficiently large M:

$$0<\epsilon\leq|f(x(t))|\leq M,\qquad\forall\, t\geq 0.$$

Considering the direction $v=f(x(0))$, we have that

$$\begin{aligned}\lambda=\lim_{t\to+\infty}\frac{\log\left(|X(t)v|\right)}{t}&=\lim_{t\to+\infty}\frac{\log\left(|f(x(t))|\right)}{t}\\&\leq\lim_{t\to+\infty}\frac{\log\left(M\right)}{t}=0\end{aligned}$$

and

$$\lim_{t\to+\infty}\frac{\log\left(|f(x(t))|\right)}{t}\geq\lim_{t\to+\infty}\frac{\log\left(\epsilon\right)}{t}=0\ .$$

Hence, it follows that

$$0=\lim_{t\to+\infty}\frac{\log\left(\epsilon\right)}{t}\leq\lambda\leq\lim_{t\to+\infty}\frac{\log\left(M\right)}{t}=0$$

which implies $\lambda=0$. Therefore, there exists at least one Lyapunov exponent equal to zero.

We are now ready to state our main result.

Theorem 1. *Consider the system*

$$\begin{array}{rcl}\dot{x}(t)&=&f(x(t))\\\dot{\delta^{[2]}}(t)&=&\frac{\partial f}{\partial x}^{(2)}(x(t))\delta^{[2]}(t),\end{array}\tag{15}$$

with state $x(t)\in\mathbb{R}^n$ and initial condition $x(0)=x_0$. Assume further that $f:\mathbb{R}^n\to\mathbb{R}^n$ of class $\mathcal{C}^1$ and $f(x)\neq 0$ for all $x\in\omega(x_0)$. Let $V:\mathbb{R}^n\times\mathbb{R}^{\binom{n}{2}}\to\mathbb{R}$ of class $\mathcal{C}^1$ and let α_1, $\alpha_2:[0,+\infty)\to[0,+\infty)$ functions of class $\mathcal{K}_\infty$ such that

- $\alpha_1\left(\left|\delta^{[2]}\right|\right)\leq V\left(x,\delta^{[2]}\right)\leq\alpha_2\left(\left|\delta^{[2]}\right|\right)$
- $\dot{V}\left(x,\delta^{[2]}\right)\leq 0$

for all $x\in\mathbb{R}^n$ and $\delta^{[2]}\in\mathbb{R}^{\binom{n}{2}}$, where

$$\dot{V}(x,\delta^{[2]}):=\frac{\partial V}{\partial x}(x,\delta^{[2]})f(x)+\frac{\partial V}{\partial\delta^{[2]}}(x,\delta^{[2]})\frac{\partial f}{\partial x}^{[2]}(x)\delta^{[2]}\ .$$

Then, all Lyapunov exponents relative to the considered solution are less or equal than 0.

Proof: By Lemma 1 it follows that 0 is a Lyapunov exponent of (5) and an eigenvalue of $\left(\lim_{t\to+\infty}\frac{\log\left(X(t)^TX(t)\right)}{2t}\right)$. From the assumptions on the candidate Lyapunov function V, it follows that along any solution of (15).

$$\begin{aligned}\alpha_1(|\delta^{[2]}(t)|)&\leq V(x(t),\delta^{[2]}(t))\\&\leq V(x(0),\delta^{[2]}(0))\leq\alpha_2(|\delta^{[2]}(0)|).\end{aligned}$$

Hence $|\delta^{[2]}(t)|\leq\alpha_1^{-1}(\alpha_2(|\delta^{[2]}(0)|))$, proving uniform boundedness of $|\delta^{[2]}(t)|$. By Proposition 1 follows that

$$\mu_{ij}=\lim_{t\to+\infty}\frac{\log\left(|Y(t)w_{ij}|\right)}{t}=\lambda_i+\lambda_j$$

where $w_{ij}=v_i\wedge v_j$ and λ_i, λ_j with $1\leq i<j\leq n$ are the Lyapunov exponents of the system (4). Hence, denoting by $M:=\alpha_1^{-1}(\alpha_2(|Y(0)w_{ij}|))$ the uniform bound on $|Y(t)w_{ij}|$, we see that:

$$\lim_{t\to+\infty}\frac{\log\left(|Y(t)w_{ij}|\right)}{t}\leq\lim_{t\to+\infty}\frac{\log\left(M\right)}{t}=0$$

for all $1\leq i<j\leq n$. From the last inequality it follows that

$$\lambda_i+\lambda_j\leq 0\quad\forall\, i\neq j\ .$$

In particular, by Lemma 1, there exists at least one $\lambda_i=0$, and therefore we have

$$0+\lambda_j\leq 0\quad\forall\, j\ .$$

As a consequence, all Lyapunov exponents are less than or equal to zero, which concludes the proof of the claim. ☐

When the Lyapunov function V in Theorem 1 is quadratic with respect to $\delta^{[2]}$, one can formulate the condition on $\dot{V}$ through suitable Linear Matrix Inequalities involving the second additive compound matrix of the Jacobian. We formulate this in the next Theroem.

Theorem 2. *Consider system (4) with state $x\in\mathbb{R}^n$ and initial condition $x(0)=x_0$. Assume that $f(x)\neq 0$ for all $x\in\omega(x_0)$. Consider a state-dependent symmetric matrix $P(x)\in\mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$ such that:*

$$P(x)\geq\varepsilon I\tag{16}$$

for all x in a neighborhood of $\omega(x_0)$, and assume:

$$\left(\frac{\partial f}{\partial x}^{(2)}(x)\right)^TP(x)+P(x)\left(\frac{\partial f}{\partial x}^{(2)}(x)\right)+\dot{P}(x)\leq 0\tag{17}$$

then all Lyapunov exponents relative to the considered solution are less or equal than 0.

Proof: The result simply follows by defining function $V(x,\delta^{[2]}):=(\delta^{[2]})^TP(x)\delta^{[2]}$ and applying Theorem 1. Indeed taking derivatives of V along solutions of (15) we see that:

$$\dot{V}={\delta^{[2]}}^T\left[\frac{\partial f}{\partial x}^{(2)^T}P(x)+P(x)\frac{\partial f}{\partial x}^{(2)}+\dot{P}(x)\right]\delta^{[2]}\leq 0,$$

where the last inequality follows by (17).

V. Application Example

As an example of system which may exhibit positive Lyapunov exponents within some region of parameter space we consider the Lorenz system of equations. This is described by [15]:

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 &= x_1 x_2 - b x_3 \end{aligned}$$

where the variables $[x_1, x_2, x_3]$ have been used instead of the classical $[x, y, z]$ for consistency of notation with the previous part of the paper. The parameters $b = 8/3$ and $\sigma = 10$ are assigned, while ρ is the bifurcation parameter which we are able to vary. The system is known to be globally asymptotically stable at the origin for $0 < \rho < 1$ and exhibit a supercritical pitchfork bifurcation at $\rho = 1$. Three equilibria coexist for $\rho > 1$ and a sub-critical Hopf bifurcation occurs at $\rho = 24.74$, after which the system becomes chaotic. We seek to apply Theorem 2 in order to rule out existence of an attractor with positive Lyapunov exponents within some ranges of the parameter ρ. Clearly, such attractors exist also for $\rho < 24.74$ due to the sub-critical Hopf bifurcation which entails existence of at least one unstable periodic solution. To the best of our knowledge, it is not known for which value of ρ attractors with positive Lyapunov exponents first appear in the Lorenz equation. In order to rule out their existence we wish to apply our Theorem 2 within the globally attractive and forward invariant family of ellipsoids proposed [16]. This fulfills the equation:

$$\mathcal{E} := \{x : \lambda x_1^2 + x_2^2 + (x_3 - \lambda\sigma - \rho)^2 \leq R^2\}, \tag{18}$$

with $\lambda = \rho/5\sigma$ and $R^2 = (\lambda\sigma + \rho)^2 b^2/[4(b-1)]$, where the value selected for λ guarantees that it has minimum volume within the considered family. To this end we consider the matrix $P(x) \in \mathbb{R}^{3\times3}$ of the following form $P(x) = \mathrm{diag}(1, 1, g(x))$, where $g(x) = p_1 + p_2 x_3 + p_3 x_3^2$. Moreover, direct calculation shows that:

$$\frac{\partial f}{\partial x}^{(2)}(x) = \begin{bmatrix} -\sigma - 1 & -x_1 & 0 \\ x_1 & -b - \sigma & \sigma \\ -x_2 & \rho - x_3 & -b - 1 \end{bmatrix}.$$

Let consider the function $g(x) = 2.1462 - 0.035497 x_3 + 0.021531 x_3^2$, where the coefficient p_1, p_2 and p_3 were found by solving the minimization problem

$$\min_{p_1,p_2,p_3} \max_{x_1,x_2,x_3 \in \mathcal{E}} \max\{D_1, -D_2, D_3\}$$

where the D_i are the leading principal minors of the matrix D:

$$D = \dot{P}(x) + \left(\frac{\partial f}{\partial t}^{(2)}(x)\right)^T P(x) + P(x)\frac{\partial f}{\partial t}^{(2)}(x)$$

and fixing $\rho = 4.474$. Figure 1 shows the function

$$\max_{x_3 : (x_1,x_2,x_3) \in \mathcal{E}} \det(D)$$

inside the projection of the ellipsoid $\mathcal{E}$ with respect to the x_3 axis. This confirms that the determinant is negative and, by virtue of Theorem 2 no attractor with positive Lyapunov exponent exists for this value of ρ.

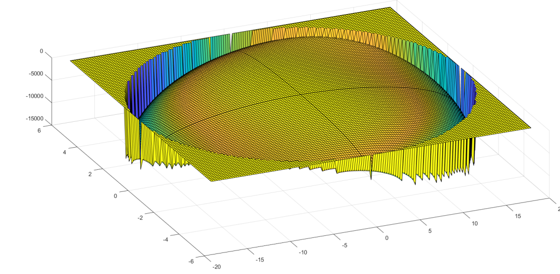

Fig. 1. Determinant of $\boldsymbol{D}$ within the ellipsoid $\boldsymbol{\mathcal{E}}$

VI. Conclusion

We studied criteria for ruling out existence of attractors with positive maximal Lyapunov exponent in finite dimensional differentiable nonlinear systems. A Lyapunov-based condition and an LMI condition are presented both based on the second additive compound matrix of the vector field's Jacobian. Our main contributions extend previous results that merely ruled out existence of invariant tori. Future research will address conditions for systems with first integrals.

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