

Some aspects of hypersonic nozzle flows.

by

David George Petty, B.Sc(Eng.)., A.C.G.I.

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ABSTRACT

This thesis describes work carried out on some aspects of continuum non-equilibrium nozzle flows. In particular vibrational "de-excitation" and "condensation" shocks are considered.

Part I describes the structure of de-excitation shocks (Blythe (1967)) and shows, by representing these shocks on a modified Rayleigh line diagram, that discontinuous solutions are possible. Thus for a certain range of finite back pressures adiabatic shocks are embedded within the de-excitation shocks. It is also shown that these de-excitation shocks will not choke the flow. The shock solutions are then compared with some numerical solutions of the full equations and it is shown that most of the discrepancies are due to the effects of finite area ratio.

The flow of a condensing vapour is considered in part II. The various types of flow are discussed and the various limiting solutions identified. The near equilibrium and near frozen limits for both the impure and pure vapours are considered, and it is shown for both that the near equilibrium limit represents a singular perturbation problem, the solution of which is then obtained for both cases. In the near frozen limit it is found that the solutions are similar to that for de-excitation shocks, although differences in the rate equation make the analysis more complex. These solutions show the possible existence of condensation shocks in typical hypersonic facilities. The structure of these shocks is found to be similar to those considered in Part I.

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CONTENTS.

General Introduction.	4.
Part I. De-excitation shocks.	8.
Part II The condensation of pure and impure vapours.	60.
References.	138.

General Introduction

In recent years much interest has been shown in the continuum non equilibrium aspects of hypersonic nozzle flows, especially in those flows which exhibit vibrational relaxation, dissociation and ionization (see Hall and Treanor (1968) for a general survey).

In many cases these flows can be characterised by a freezing point (Bray 1959) where the flow departs rapidly from the equilibrium state and approaches an asymptotic frozen state.

Many attempts to predict the variation of these frozen asymptotic levels with the rate parameter Λ have been made (Λ is a characteristic ratio of the flow time to the time scale of the rate process). A few of the simpler problems have been solved analytically, mostly with the use of matched asymptotic expansions (Van Dyke(1964)). For example the vibrational relaxation case has been discussed by Blythe (1963) (1964a) (1964b) and (1967) and dissociation by Freeman (1958), Blythe (1964a) and Cheng and Lee(1966).

When investigating the near frozen limit for a vibrationally relaxing gas Blythe (1967) discovered that instead of remaining frozen or eventually decaying to equilibrium downstream (Eschenroeder(1962)), the flow could return rapidly to equilibrium through a "de-excitation shock". It was found that these shocks, which are continuous supersonic-supersonic transitions, and fall in the general class of aerothermodynamic shocks discussed by Polachek and Seeger (1958) occurred when

$$\int \frac{dq}{T},$$

became unbounded (q is the heat input due to the rate process and T is the local temperature of the active state of the gas).

This integral was termed the "pseudo entropy" integral and in the case of a vibrationally relaxing gas represents, when the flow is analysed using the α gas plus heat addition concept of Johannesen (1961), the entropy of the α -gas.

Integrals of this type play a dominant role in the solutions presented here and the concept of "pseudo entropy" first introduced by Broer (1951) with respect to an equilibrium flow will be used, except that here it will be defined in terms of a frozen flow. As noted above for the vibrational case when the rate process only involves energy transfer the "pseudo entropy" can be uniquely defined in terms of Johannesen's α gas and represents the entropy of the α gas. For rate processes involving mass transfer as well as energy transfer it is not in general possible to uniquely define the "pseudo entropy", however the concept is still useful in obtaining the position of ~~any~~ aerothermodynamic shocks in the nozzle.

Following the discovery by Blythe (1967) of the de-excitation shocks a further investigation has been carried out. It is shown that these shocks can be described by a modified Rayleigh line diagram. This indicated that although the shocks described by Blythe were continuous, discontinuous solutions of the equations were possible, and it was noted that for a certain range of finite back pressures adiabatic shocks would be embedded within the de-excitation shock.

The zeroth order structure of the de-excitation shocks was then obtained and compared with some numerical solutions of the full equations. Although these results compared well it was found that the shocks, in the vibrational case, occurred at very large distances downstream and any hope of finding these experimentally is slight. It should be noted, however, that for other rate processes involving larger amounts of energy there is a strong possibility that similar shocks could be found in hypersonic nozzles.

One such shock is the "condensation shock" which has been found experimentally, and so in the second part of this thesis the flow of a condensing vapour in a hypersonic nozzle is considered.

The main difference between this problem and that for de-excitation is in the form of the rate equation. For the vibrational case the rate of energy transfer is a function of the local flow variables, but in the classical condensation theory the corresponding rate is a functional of the flow conditions and the energy transfer depends on the full history of the flow.

Solutions in both the near equilibrium and near frozen limits have been found. For the near equilibrium limit it is shown that the flow, although starting along the frozen isentrope, rapidly returns to equilibrium through a very narrow region.

In the near frozen case solutions are obtained which are very similar to those obtained by Blythe (1967) and indicate the occurrence of condensation shocks for various values of the flow parameters. Some numerical calculations have been performed, and although it is shown that these shocks occur in a finite distance, it is observed that most

of those found experimentally have been of the near equilibrium type.

The results obtained here were derived for quasi-one-dimensional flow. As regards two-dimensional effects, Bartlma (1965), obtained approximate numerical solutions for two-dimensional nozzle flows in a relaxing gas. He noted the apparent existence of both weak (oblique) and strong (normal) shocks. Mohammad (1967) has also observed a non uniqueness in some characteristic calculations for a one dimensional unsteady flow. The "shocks" discussed by Blythe (1967) and in this thesis differ from those reported by Bartlma and by Mohammad in that their structure is continuous. In this sense these shocks can be termed fully dispersed (Lighthill (1956)).

The precise connection between the one dimensional results discussed here and these two dimensional calculations is not clear and the problem of shock formation in expanding two-dimensional non equilibrium gas flows is of considerable current interest.

A further and even more complex problem is the interaction between these various non equilibrium effects. For example would a "condensation shock" precipitate a de-excitation shock, and the combination because of two-dimensional effects produce a Rankine Hugoniot shock. No attempt will be made to answer these questions here.

PART I

DE-EXCITATION SHOCKS.

INDEX.

1. Introduction.	10.
2. The shock equations.	14.
3. The Rayleigh line and related diagrams.	20.
4. The shock structure.	28.
4.1. The de-excitation shock.	28.
4.2..Relaxation region behind α -shock.	32.
5. Numerical solutions and higher approximations.	37.
5.1. The numerical solutions.	37.
5.2. The shock position.	38.
5.3. The temperature maxima.	41.
5.4. Application to hypersonic facilities.	48.
6. Discussion.	49.
Appendix I.	50.
Appendix II.	52.
Appendix III.	53.
Notation.	58.

1. Introduction

In a paper by Blythe (1967), it was shown that a phenomenon closely analogous to a condensation shock may occur in expanding nozzle flows in a vibrationally relaxing gas. This phenomenon is characterized by a sudden and rapid de-excitation of the lagging internal mode; associated with the de-excitation is a weak compression. The bulk of the energy from the vibrational mode is fed into the translational and rotational degrees of freedom, which are assumed to remain in a local equilibrium state.

These de-excitation shocks, or δ -shocks, fall into the general class of aerothermodynamic shocks discussed by Polachek and Seeger (1958). The shocks occur in near frozen situations when the low temperature behaviour of the relaxation time τ adopts a certain form.

For power law nozzles, and when

$$\tau^{-1} \sim \rho T^{\delta}, \quad T \rightarrow 0, \quad 1.1.$$

where ρ is the density and T the translational temperature, then a δ -shock exists if $\delta > 1$ and

$$\mathcal{L} = 1 - \nu \{ 2 - \gamma + (\gamma - 1) \delta \} > 0. \quad 1.2.$$

Here γ is the specific heat ratio for the active modes and ν is the exponent associated with the rate of growth of the nozzle.

It has been pointed out by Johannesen (1961) that continuum non-equilibrium flows are in general equivalent to the flow of some perfect gas (the α -gas) with heat addition. Since the shock region

corresponds to a local heat release the equivalence with condensation shocks is immediately apparent. With an appropriate modification in the mechanism of the heat input, the analysis of Polachek and Seeger (1958) applies equally well to the structure of either of these shocks.

Although the δ -shock solutions described by Blythe (1967) were continuous, and the flow through the shock remained supersonic, discontinuous solutions of the equations are possible. Blythe assumed that the nozzle back pressure was zero, or at least less than the pressure at the downstream limit of the δ -shock. For finite back pressures, conventional adiabatic shocks (α -shocks) may be embedded within the de-excitation shock. Solutions applicable to this case are outlined in $\beta 3$ and $\beta 4.2$.

It is useful to consider the possible transitions through the δ -shock by means of a modified Rayleigh-line diagram. Instead of the conventional temperature - entropy co-ordinates, a more convenient transition diagram is the temperature-vibrational energy plane (see $\beta 3$). In this diagram α -shocks occur at fixed values of the vibrational energy.

It is well known that for the flow of a perfect gas with heat addition there is an upper bound to the magnitude of the heat input, above which the flow chokes. An important result derived within shows that, irrespective of the amount of energy in the lagging mode, the present class of non-equilibrium flows does not break down in this way: choking is prevented by the strong coupling between the energy and rate equations. This means that the effective heat input is not independent of the local flow conditions.

It appears that there is no experimental evidence either to confirm or to deny the existence of the δ -shocks in nozzle flows with vibrational relaxation. Moreover, numerical calculations based on the quasi-one dimensional relations were not available for the conditions under which the shock should occur.

In order to verify both, that solutions of the full equations did possess regions of rapid de-excitation (for appropriate conditions), and that the approximate solution did predict the structure and position of these regions, some numerical calculations for the full equations have been carried out. At the same time detailed calculations based on the approximate solutions which are outlined in sections 2 and 3, were made.

The full numerical solutions do indeed show the existence of the δ -shocks. Moreover, the agreement with the simple theory for the position of the shocks is good. The shock structure is also predicted reasonably well by the simple theory, though values of the local maximum in the translational temperature at the downstream end of the shock are in general noticeably different from the values given by the approximate solution. It was conjectured that for sufficiently large values of ν , this discrepancy was mainly due to the effect of a finite area change through the shock. An analysis confirming this conjecture is presented in § 5, where it is found that first order theory introduces a similarity parameter for ν large; if ν is small other factors have to be taken into account and the limit $\nu \rightarrow 0$ is also discussed in § 5.

The theoretical analysis of Blythe was concerned with the near frozen limit when the rate parameter Λ was small. With respect to this limit the shock occurs at distances $O(\Lambda^{1/2})$ i.e. far downstream. A significant feature of the numerical calculations has been the magnitude of this distance. For a large number of practical situations it turned out that the shock position was typically $10^5 - 10^6$ throat diameters downstream of the minimum section, but for some cases shocks were found to occur at around 10^3 diameters. This latter distance is roughly equivalent to the maximum length of some nozzles in current hypersonic facilities.

In general however the numerical calculations suggest that δ -shocks in a vibrationally relaxing gas will not occur in conventional nozzles. One important consequence for rocket nozzles is that thrust is unlikely to be recovered in this way.

Although these results for vibrational de-excitation would seem to exclude the possibility of experimental verification, the length scales for other rate processes are not necessarily so large, and it has been shown in Blythe et al (1968) that the shock position analysis can be extended to include a fairly general class of rate processes.

2. The shock equations

The governing equations for the flow, through a quasi one-dimensional nozzle, of a gas, in vibrational non equilibrium, can be written in non dimensional variables (Johannesen 1961, Blythe 1967) as

$$\rho u A = m, \quad 2.1$$

$$\frac{1}{\rho} \frac{d\rho}{dx} - \frac{1}{(\gamma-1)T} \frac{dT}{dx} = \frac{1}{T} \frac{d\epsilon}{dx}, \quad 2.2$$

$$\frac{\gamma}{\gamma-1} T + \epsilon + \frac{u^2}{2} = \frac{\gamma}{\gamma-1} T_\infty + \epsilon_\infty + \frac{u_\infty^2}{2}, \quad 2.3$$

$$\text{and} \quad \begin{aligned} p &= \rho T, & 2.4 \\ u \frac{d\epsilon}{dx} &= \frac{1}{\rho} \Omega(T) [\bar{\epsilon}(T) - \epsilon], & 2.5 \end{aligned}$$

where ρ is the density, u the velocity, A the cross-sectional area at any station x , ϵ the vibrational energy, $\bar{\epsilon}$ the local equilibrium value of ϵ corresponding to the translational temperature T and Ω the relaxation frequency. γ is the specific heat ratio for the active modes of translation, and m is a non-dimensional mass flow rate.

The suffix ∞ denotes initial conditions where the flow is not necessarily in equilibrium. If, however $u_\infty = 0$ then $\epsilon_\infty = \bar{\epsilon}_\infty$. It is this particular case which corresponds to flow through a convergent divergent nozzle.

The above non-dimensional variables are defined by,

$$\begin{aligned} p &= \frac{p'}{p'_\infty}, \quad \rho = \frac{\rho'}{\rho'_\infty}, \quad T = \frac{T'}{T'_\infty}, \quad \Omega = \frac{\Omega'}{\Omega'_\infty}; \quad x = \frac{x'}{h'}, \\ \epsilon &= \frac{\epsilon'}{RT'_\infty}, \quad u = \frac{u'}{\sqrt{RT'_\infty}}, \quad m = \frac{m'}{\rho'_\infty \sqrt{RT'_\infty} A'_f}, \quad A = \frac{A'}{A'_f}, \end{aligned}$$

where primes denote the dimensional quantity, R is the gas constant, h' is the throat radius and suffix t defines conditions at the geometric throat,

Λ is the rate parameter defined by

$$\Lambda = \frac{p_{\infty}' \Omega_{\infty}' h'}{\sqrt{RT_{\infty}'}} = \sqrt{8} \frac{h'/a_{\infty}'}{\tau_{\infty}'}, \quad 2.6.$$

where τ' is the relaxation time and a' is the frozen sound speed. The limit $\Lambda \rightarrow 0$ implies near frozen flow and it is results arising from this case which are of interest here.

It is assumed that the equilibrium distribution of vibrational energy is given by the usual expression

$$\bar{\delta} = \frac{\Theta_v}{\exp[\Theta_v/T] - 1}, \quad 2.7.$$

which holds for a system of harmonic oscillators, and where Θ_v is the non-dimensional characteristic temperature of vibration.

It will also be assumed that at low temperatures

$$\Omega \sim BT^{\delta}, \quad 2.8.$$

as $T \rightarrow 0$ for $\delta \geq 0$.

The nozzle area ratio has been chosen so that far downstream the asymptotic nozzle shape is

$$A \sim Cx^{\nu}, \quad 2.9.$$

The near frozen case with the above assumptions was considered by Blythe(1967) and it was found that the perturbation solution became invalid far downstream and in particular when $\delta > 1$ there occurred compression regions which were termed "de-excitation shocks" (δ -shocks)

It is the structure and behaviour of these shocks, which is of interest here.

To obtain the shock equations the following scaled variables, which are similar to Blythe's were used,

$$\left. \begin{aligned} \lambda &= \frac{l}{u_0(\infty)} \left\{ \frac{M_0}{u_0(\infty)} \right\}^{\frac{1-k}{\gamma}} \\ Y &= Y_S \left(1 + \lambda^{\frac{\gamma}{\gamma-1}} Y_S^{\frac{\gamma-1}{\gamma}} \right), \\ p &= \lambda^{\frac{\gamma}{\gamma-1}} \frac{M_0}{u_0(\infty)} \frac{\pi}{Y_S^{\frac{\gamma}{\gamma-1}}}, \\ u &= u_0(\infty) V, \end{aligned} \right\} 2.10$$

with $u_0(\infty) = \frac{2\gamma}{\gamma-1} + u_{\infty}^2,$

and where $Y = \lambda^{\frac{\gamma}{\gamma-1}} X$ is a modified length scale used to obtain the shock position Y_S , which is given by Blythe (1967) as

$$Y_S = \left\{ \frac{l}{[(\delta-1)(\gamma-1) \epsilon_{\infty}]^{\frac{\gamma-1}{\gamma}} \left[\frac{M_0}{u_0(\infty)} \right]^{\gamma-1}} \right\}^{\frac{1}{\gamma}}. \quad 2.11$$

The following parameters are as used in Blythe (1967)

$$\left. \begin{aligned} k &= 1 - \nu(1 + (\gamma - 1)\delta), \\ \ell &= 1 - \nu(2 - \gamma + (\gamma - 1)\delta), \\ b &= (\gamma - 1)\nu/\ell, \\ d &= b(\delta - 1). \end{aligned} \right\} 2.12$$

The rate parameter has been modified to include the form of Ω .

$$\Lambda = \frac{\rho_{\infty}' \bar{\Omega}_{\infty}' h'}{\sqrt{RT_{\infty}'}} , \quad \text{where } \bar{\Omega}_{\infty}' = \Omega_{\infty}'/B,$$

and M_0 is a scaled version of the frozen mass flow m

$$M_0 = \frac{m}{C} = \left\{ \gamma^{1/2} \left[\frac{2}{\gamma + 1} \right]^{\frac{\gamma + 1}{2(\gamma - 1)}} \right\} \cdot C^{-1}$$

The governing equations, for the flow through the shocks, are then ($\delta > 1$)

$$\pi V = 1, \quad 2.13$$

$$\pi \left\{ \frac{T}{u_0^2(\infty)} + V^2 \right\} = 1, \quad 2.14$$

$$\frac{\gamma}{\gamma - 1} T + \frac{u_0^2(\infty) V^2}{2} + \epsilon = \frac{u_0^2(\infty)}{2} + \epsilon_{\infty}, \quad 2.15$$

and

$$\frac{dG}{d\epsilon} = \frac{\gamma}{V} \Omega^*(\tau) [\bar{\epsilon}(\tau) - \epsilon], \quad 2.16$$

where $\Omega^*(\tau) = [U_0(\infty)/M_0]^{(\gamma-1)\delta} \tau^\delta$, and it has been assumed that $\bar{\Omega} = \tau^\delta$ throughout the shock.

The right hand sides of these equations were obtained by matching with the downstream "pseudo entropy" solutions of Blythe (1967 (Part II Eqn. 3.3)).

The present choice of dependent and independent variables is preferred to that used in Blythe (1967) since the explicit dependence of the shock solution on Y_S is eliminated. For a given $U_0(\infty)$ and γ the solution depends solely on the parameters contained in the modified relaxation frequency $\Omega^*(\tau)$ and G_∞ . In this thesis attention is confined to the case

$$U_\infty = 0, \\ \text{i.e. } U_0(\infty) = \sqrt{\frac{2\gamma}{\gamma-1}}, \quad 2.17$$

If the flow is initially in equilibrium

$$G_\infty = \bar{G}_\infty(\Theta_V) = \frac{\Theta_V}{\exp[\Theta_V] - 1}, \quad 2.18$$

where Θ_V is the characteristic temperature of vibration

Consequently, for a given γ together with the conditions (2.8), (2.17), and (2.18), the shock solution will depend on the

parameters Θ_V and δ only and can be written (see Polachek and Seeger, 1958)

$$\left. \begin{aligned} V &= \frac{1}{\pi} = \frac{\gamma \pm \sqrt{1-Q}}{\gamma+1}, \\ \frac{T}{u_0^2(\infty)} &= \frac{\gamma - (1-Q) \mp (\gamma-1)\sqrt{1-Q}}{(\gamma+1)^2}, \end{aligned} \right\} 2.19$$

where
$$\Theta(\delta, \Theta_V) = \frac{2(\gamma^2-1)(\delta_{\infty}-\delta)}{u_0^2(\infty)}, \quad 2.20$$

and the upper signs in 2.19 correspond to the continuous supersonic solution. The lower sign represents the subsonic solution and the relevance of this result will be considered in §3.

Q is given as a function of ξ from the integrated form of the rate equation

$$\xi + C = \int_{Q_I}^Q \frac{V^2(q)}{M_\infty^\delta T(q) [\bar{Q}(q, \Theta_V) - q]} dq, \quad 2.21$$

where the lower limit Q_I has been inserted for convenience, and

$$\bar{Q} = Q(\delta, \Theta_V), \text{ with } M = \left\{ \frac{u_0(\infty)}{M_0} \right\}^{\gamma-1}$$

Equation 2.21 defines the structure of the de-excitation shock, and will be examined in § 4.1.

3. The Rayleigh-line and related diagrams

Diabatic flows of the general kind considered here are usually discussed with the aid of a temperature entropy plot. For the present class of non equilibrium flows entropy, in its usual sense, is not clearly defined. However the quantity

$$\Delta_A S = S - S_A = - \int_{S_A}^S \frac{dS}{T}, \quad 3.1$$

can be referred to as the change in the "pseudo entropy" (Broer 1951): S is identifiable with the entropy of the α gas of Johannesen (1961). (For a further discussion see part II of this thesis).

A convenient datum point for the "pseudo entropy" is the end point on the subsonic branch of the $T-Q$ relation, where $Q=0$, $T>0$,

From 2.19

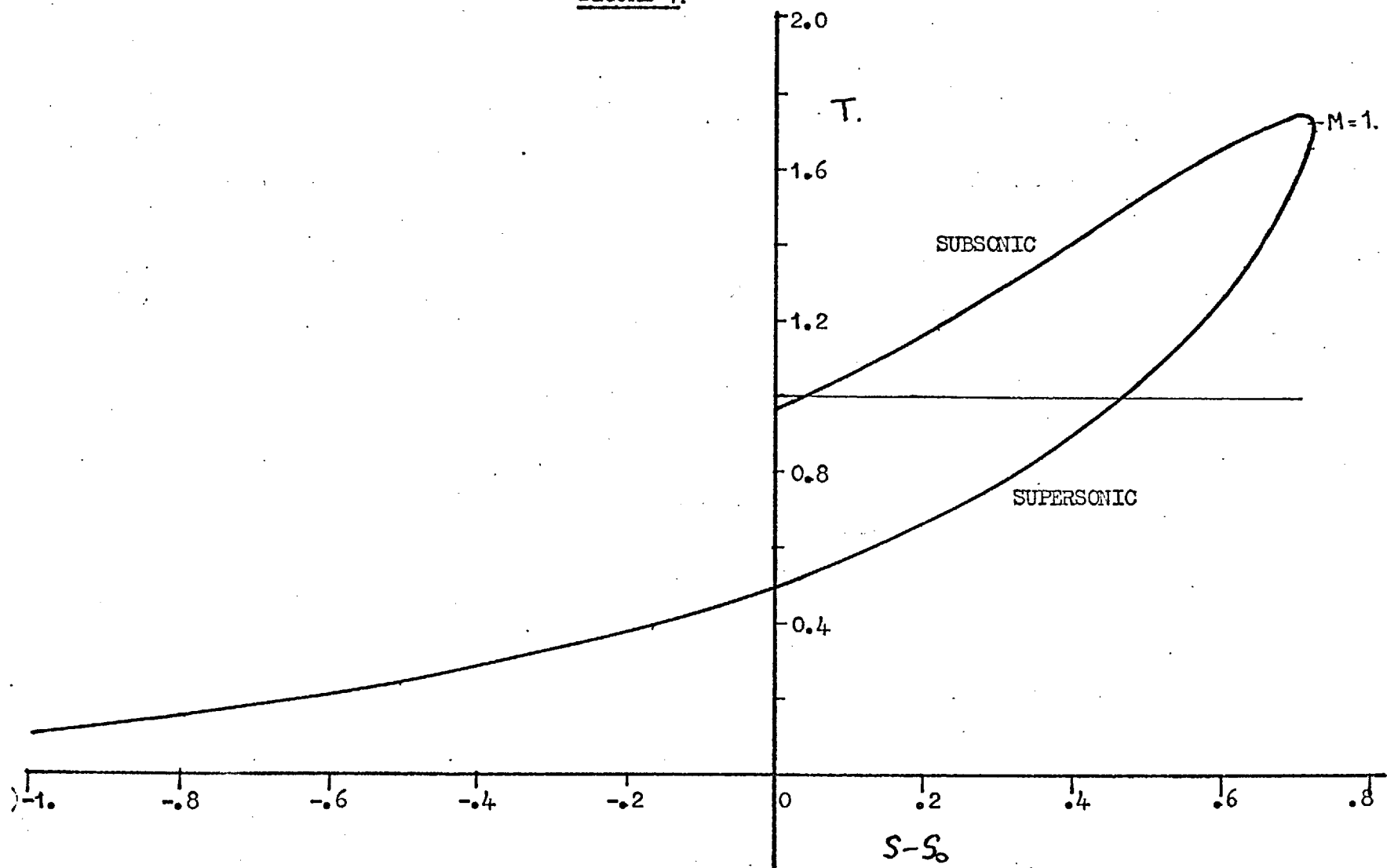
$$\Delta_0 S = \pm \frac{1}{\gamma-1} \log \left[\left\{ \frac{\gamma \pm \sqrt{1-Q}}{\gamma-1} \right\}^{\frac{\gamma}{\gamma-1}} \frac{2}{1 \mp \sqrt{1-Q}} \right]. \quad 3.2.$$

A plot of temperature against $\Delta_0 S$ is shown in figure 1. The upper branch of the curve corresponds to subsonic solutions and the lower branch to supersonic solutions

$Q=1$ defines the sonic point with respect to the frozen sound speed $\bar{c}$. Note also that on the supersonic branch S is singular as $T \rightarrow 0$ [$Q \rightarrow 0$].

* This should be understood throughout.

FIGURE 1.



RAYLEIGH LINE DIAGRAM.

For near-frozen flows in which de-excitation shocks occur the initial conditions are always supersonic and the starting point is $Q = 0$.

If the nozzle back pressure is zero the solution remains supersonic; but if the back pressure is finite the supersonic flow in the divergent part of the nozzle may be terminated by a conventional adiabatic shock (the α shock). The solution downstream of the α shock is then defined by the subsonic branch of the solution for the δ -shock. (In the limit considered here the thickness of the α shock which is strictly defined by the diffusive effects of viscosity and heat conduction is negligible in comparison with the thickness of the δ -shock).

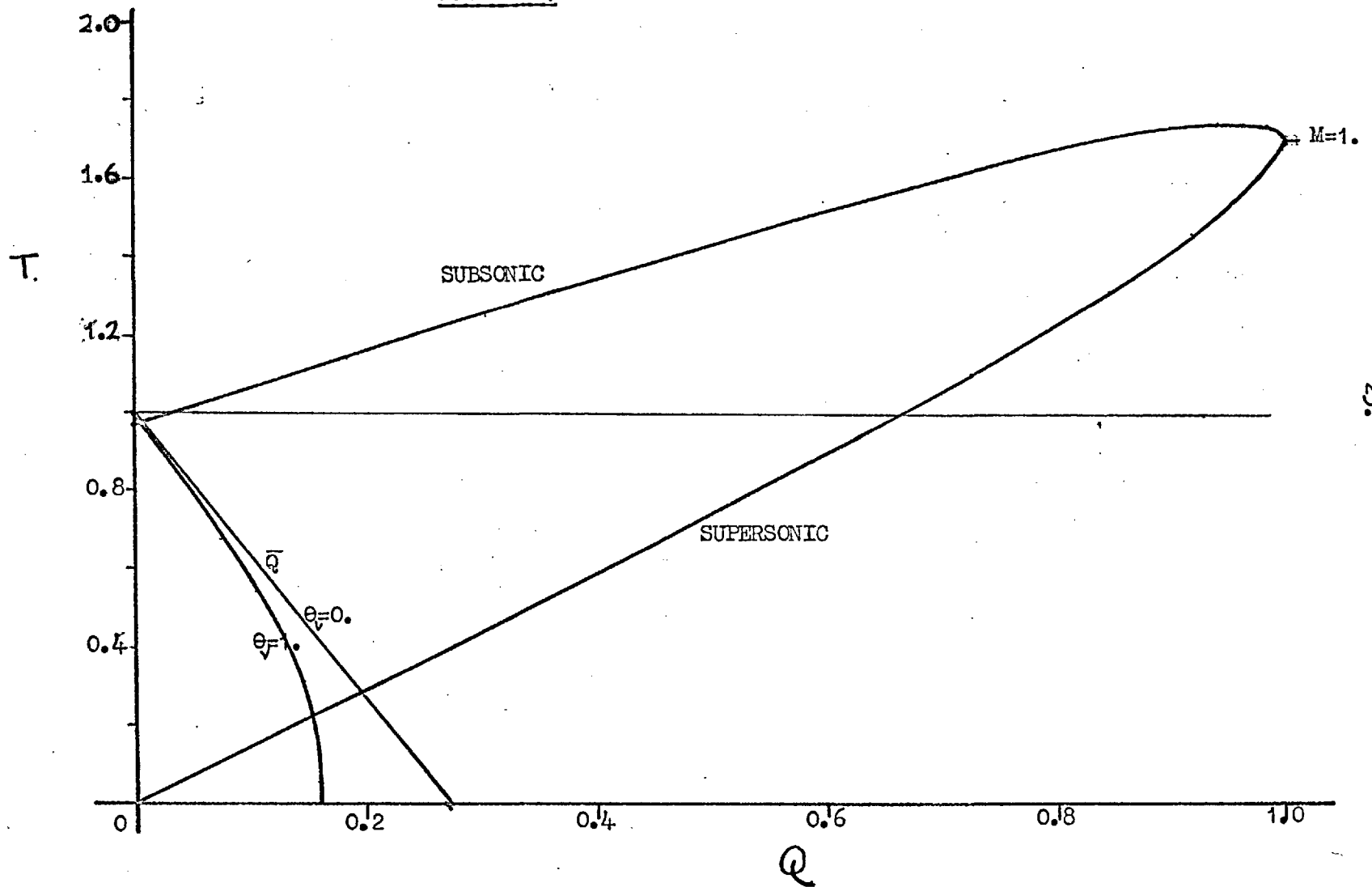
The transition between the supersonic and subsonic branches occurs at constant Q . Note that $\Delta_0 S$ increases across the α shock. Since the transition across the α shock is very simply defined in the T, Q , plane, it is more convenient to use this plane rather than the T, S plane when discussing flow fields involving α shocks.

Figure 2 shows the T - Q diagram. $Q=0$ always defines conditions upstream of the shock. Conditions downstream of the shock are given by the equilibrium value $Q = \bar{Q}$. Lines on which $Q = \bar{Q}$ are

$$\frac{T}{Q_v} = \frac{1}{\log [1 + Q_v(600 - cQ)]}, \quad 3.3$$

$$\text{where } c = \gamma(\gamma+1)^{-1}(\gamma-1)^{-2}. \quad 3.4.$$

FIGURE 2.



T-Q DIAGRAM.

In deducing (3.3) the usual equilibrium relation (2.7) for an ideal vibrationally relaxing gas has been used. These equilibrium paths are also sketched in figure 2 for various Θ_v . Note that there are always two possible final states corresponding to supersonic and subsonic flow respectively.

If the back pressure is zero the flow remains supersonic and the shock structure is as described in § 4. For finite values of the back pressure p_b there are three possibilities.

i.) $p_e > p_b$ where p_e is the equilibrium pressure at the downstream end of the supersonic

δ - shock. In this case any α -shock lies downstream of the δ -shock and the solution through the δ -shock remains supersonic.

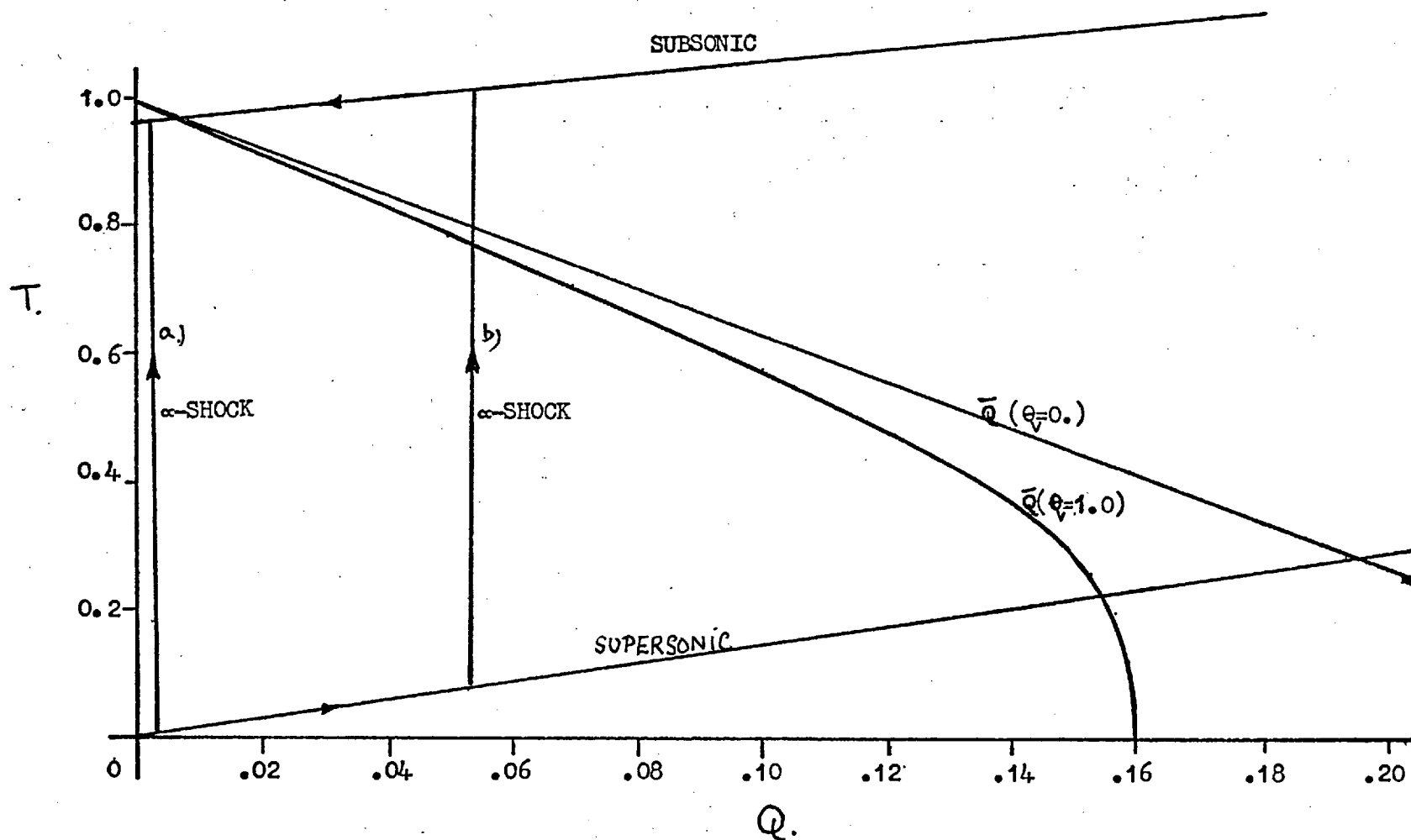
ii.) p_e and p_b are of comparable orders of magnitude with $p_e < p_b$. The α shock is now embedded within the δ -shock. Upstream of the α shock the supersonic branch of the

δ -shock is appropriate, but downstream the solution switches to the subsonic branch.

Typical paths are shown in figures 3(a) and (b).

iii.) $p_e \ll p_b$ The α shock terminates the region of supersonic frozen flow. Downstream of

FIGURES 3a & b.



POSSIBLE α -SHOCK PATHS

of the α shock there will be a return to
 equilibrium conditions.

Case (ii) is probably of most interest and will be considered more fully in § 4.2. It should be noted here, however, that for the supersonic case it was shown by Blythe (1967) that the flow, at least initially, remained effectively in equilibrium downstream of the δ - shock. It can also be shown that once the solution is on the subsonic branch equilibrium flow is the only possible limiting condition. (In the supersonic case there may again be an eventual departure from the equilibrium solution culminating in a second de-excitation shock. Since this occurs exponentially far downstream of the first δ -shock it is of no practical interest.)

In the classical treatment of flows with heat addition, solutions of this type do not exist for $Q > 1$: the sonic point, $Q = 1$, corresponds to thermal choking. For the non-equilibrium flows considered here the mechanism of heat input is not independent of the flow. Moreover for diatomic gases it was noted by Blythe (1967) that

$$Q_{MAX} < \frac{(\gamma-1)^2(\gamma+1)}{\gamma} < 1,$$

for $\gamma = 7/5$. However for an arbitrary number of vibrational modes it is not immediately apparent that $Q_{MAX} < 1$ but the proof is straight forward.

For simplicity it is assumed first that the flow starts from rest, and hence is initially in equilibrium, and that no α -shock occurs. From the initial conditions upstream of the δ -shock $G > \bar{G}$ and it follows from the rate equation that G is a monotonically decreasing function of x ; its end value in the one-dimensional de-excitation region is defined by the equilibrium value $G = G_e = \bar{G}_e$. Since $\bar{G}$ is a monotonically increasing function of T it follows that $T_e < 1$ (the initial value of $G = \bar{G}_\infty$ corresponds to $T = 1$ and $\bar{G}_e = \bar{G}_\infty$). Consequently it can be seen from figure 2 that the end point of the supersonic branch always occurs for $Q < 1$.

If the flow starts with a finite speed and an arbitrary value of G_∞ (i.e. not necessarily in equilibrium) these initial conditions are referred back to some hypothetical equilibrium reservoir conditions with $u = 0$. All variables are now normalized with respect to these new reservoir conditions. It is easily seen that the solution is formally identical with the previous solution. No difficulty arises in re-applying the above arguments and again $Q_{MAX} < 1$.

If an α shock occurs within the δ -shock then, by definition $Q_\alpha < \bar{Q}_e(u_{sup}) < 1$. On the subsonic branch $T_{sub} \geq T_{sup}$. Moreover $\bar{Q}$ is a monotonically decreasing function of T . Hence, $Q = \bar{Q}$ intersects the subsonic branch at a smaller value of Q . Thus $Q_{MAX} < 1$ on the subsonic branch.

Even though $Q_{MAX} < 1$, irrespective of the number of vibrational modes n , it can be shown that as $n \rightarrow \infty$, $1 - T_{MAX} = O(1/n)$.

4. The shock structure

4.1 The de-excitation shock

As noted in § 2 the structure of the δ -shock is given by the integrated form of the rate equation.

$$\xi + C_r = \int_{Q_I}^Q f(q) dq, \quad 4.1$$

where

$$f(Q) = \frac{V^2}{M_\delta T_\delta (\bar{Q} - Q)}, \quad 4.2$$

and the lower limit of Q_I has been inserted for convenience.

The above integral (4.1) must be computed numerically and in order to simplify this calculation in the vicinity of $Q = 0$, the integrand $f(Q)$ is expanded as follows,

$$f(Q) = Q^{-\delta} (a_0 + a_1 Q + a_2 Q^2 + \dots) \quad 4.3$$

where a_0, a_1, a_2 are given in Appendix I.

Thus, when $\delta \neq n$, where n is some integer, the integrand can be re-arranged to give

$$f = F_{r-1} + G_{r-1}, \quad 4.4$$

for $\nu > \delta > \nu - 1$ and where $g_{\nu-1} = \sum_{s=0}^{\nu} a_s q^{s-\delta}$, with

$$F_{\nu-1} = f - g_{\nu-1}, \quad 4.5$$

which is convergent as $Q \rightarrow 0$

As however, $F_{\nu-1}$ becomes unbounded as $Q \rightarrow \bar{Q}_e$ the final equilibrium value of Q , $f(Q)$ is expanded near $Q = \bar{Q}_e$ to give

$$f(Q) = \frac{b_0}{\bar{Q}_e - Q} + b_1 + b_2(\bar{Q}_e - Q), \quad 4.6$$

where b_0, b_1, b_2 are as given in Appendix II.

The integrand can now be written as

$$f(Q) = H_{\nu-1}(Q) + \frac{b_0}{\bar{Q}_e - Q} + g_{\nu-1}(Q), \quad 4.7$$

$$\text{where } H_{\nu-1} = F_{\nu-1} - \frac{b_0}{\bar{Q}_e - Q}. \quad 4.8$$

$H_{\nu-1}$, is bounded at both $Q = 0$ and $Q = \bar{Q}_e$,

and thus the integral can be written as

$$\xi + C_\nu = G_{\nu-1} - b_0 \log \left\{ \frac{\bar{Q}_e - Q}{\bar{Q}_e} \right\} + \int_0^Q H_{\nu-1}(q) dq, \quad 4.9$$

$$\text{where } G_{\nu-1} = \sum_{s=0}^{\nu-1} \frac{a_s Q^{s+1-\delta}}{s+1-\delta}. \quad 4.10$$

The lower limit has now been chosen as $Q_I = 0$, thus

$H_{\nu-1}$ and its integral are convergent as $Q \rightarrow 0$.

It can then be shown by matching with the upstream "pseudo entropy" solution of Blythe (1967) that for this case

$$C_v = 0.$$

When $\delta = n$, where n is some integer a similar analysis holds except that now

$$\xi + C_v = G_v - b_0 \log \left\{ \frac{\bar{Q}_e - Q}{\bar{Q}_e} \right\} + \int_0^Q H_v(q) dq, \quad 4.11$$

where

$$G_v = \frac{a_0 Q^{1-\delta}}{1-\delta} + \dots + a_n \log Q, \quad 4.12$$

and

$$H_v = f - g_v - \frac{b_0}{\bar{Q}_e - Q},$$

The lower limit Q_1 has again been set to zero, as H_v and its integral are convergent as $Q \rightarrow 0$.

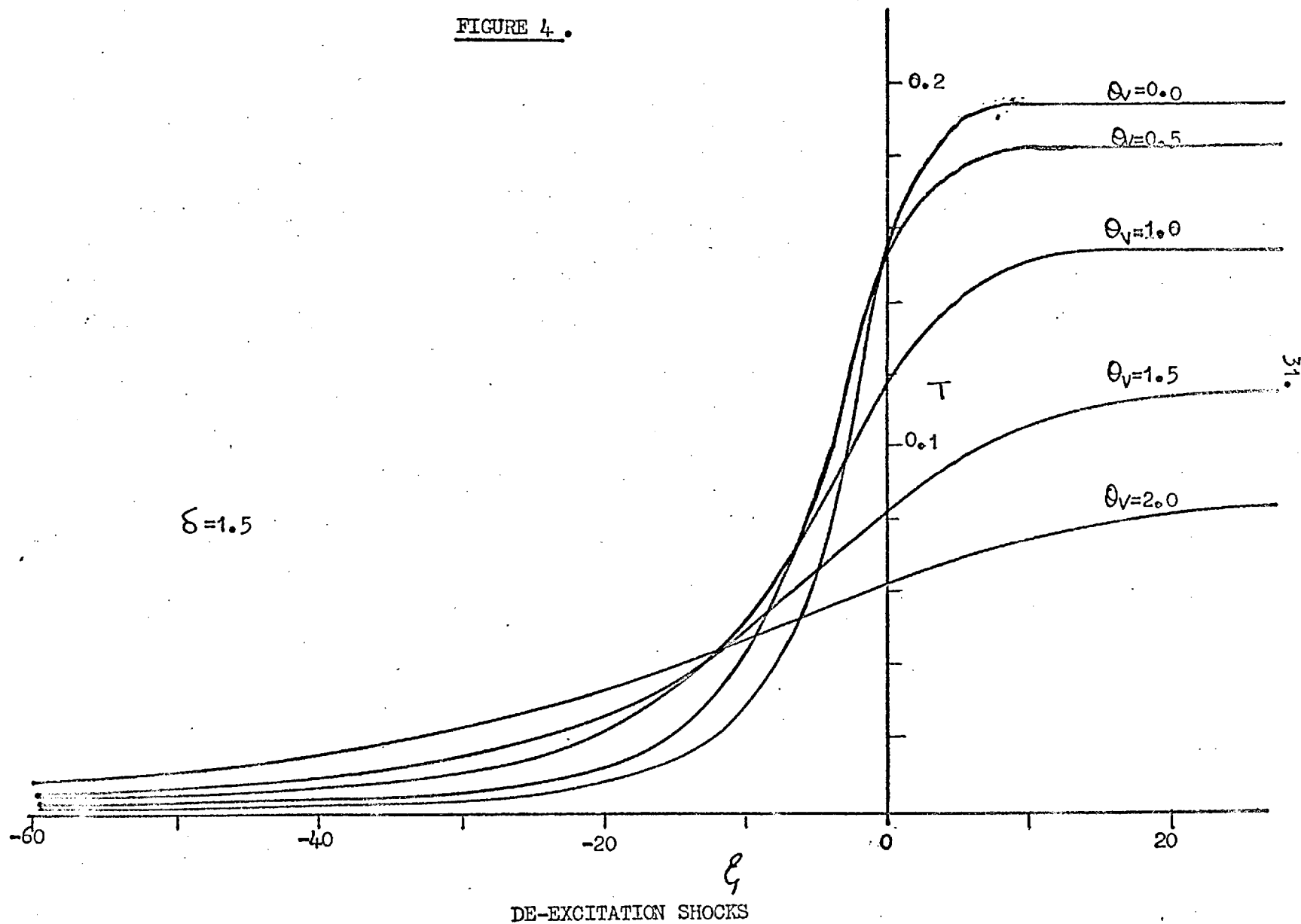
It can now be shown by matching with the upstream solution that for $\delta = n$ then

$$C_v = a_{\delta-1} \frac{(\delta-1)v}{\ell} \log \lambda,$$

where $a_{\delta-1}$ is the coefficient of $Q^{\delta-1}$ in the expansion of the integrand as $Q \rightarrow 0$ (see Appendix I).

Figure 4 shows some typical results for the shock profile when $\delta = 1.5$, for various Θ_v . Results for other values of δ show no qualitative difference. Comparison with some exact numerical calculations for the full equations is deferred until § 5.

FIGURE 4.



4.2 Relaxation region behind an α -shock

Across an α -shock Q does not change but T and V are now defined by the other roots of the equations.

Thus behind the α -shock the equations governing the relaxation zone become

$$\frac{dQ}{d\xi} = \frac{M^{\delta} T^{\delta} (\bar{Q} - Q)}{V^2}, \quad 4.13$$

$$\text{with } V = \frac{\gamma - \sqrt{1-Q}}{\gamma+1},$$

$$\text{and } \frac{T}{u_0^2(\infty)} = \frac{(\gamma-1) + Q + (\gamma-1)\sqrt{1-Q}}{(\gamma+1)^2}, \quad 4.14$$

This can be integrated to give

$$\xi - \xi_s = \int_{Q_s}^Q \frac{V^2}{M^{\delta} T^{\delta} (\bar{Q} - q)} dq, \quad 4.15$$

where Q_s is the value of Q at the α -shock, whose position is

$\xi = \xi_s$. Comparing the local values of the rate parameter Λ across the α -shock shows that

$$\frac{\Lambda_2}{\Lambda_1} = \left(\frac{T_2}{T_1} \right)^{\delta-1/2}$$

where subscript 1 is before the shock and subscript 2 is behind the shock. This ratio will in general be large and therefore the relaxation zone behind the α -shock will be narrow in comparison to the δ -shock.

The rate equation (eqn. 4.15) has been integrated numerically using a Runge-Kutta routine, and some typical results are shown in figures 5a and b. Figure 5a shows several typical zones in relation to the δ -shock, and figure 5b shows the detailed structure of one of the relaxation zones.

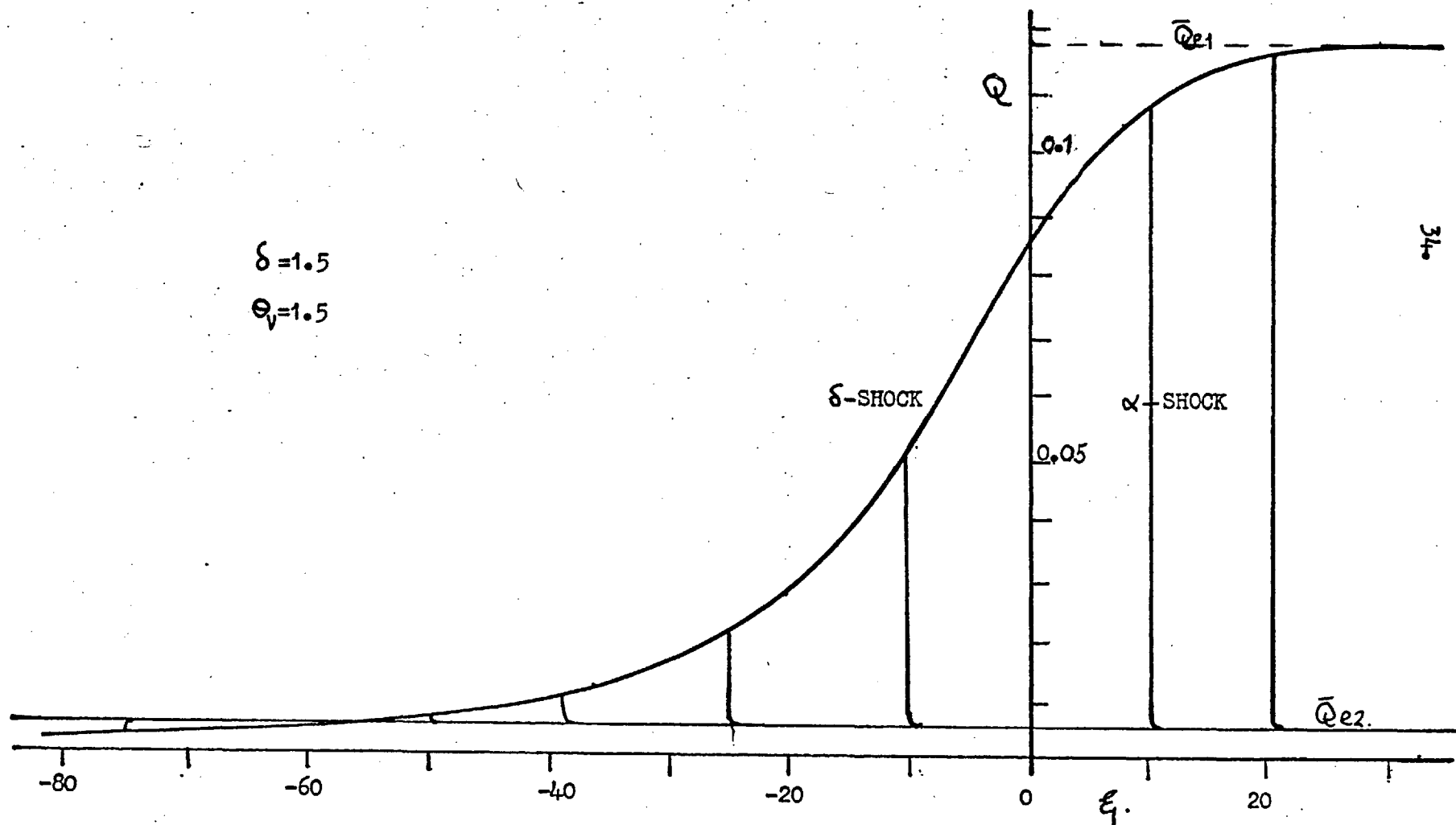
It is often necessary to obtain the shock position for a given

back pressure, and a simple inverse scheme for obtaining this is to assume that the α -shock occurs at a certain value of Q , say Q_α , on the lower (supersonic) branch of the modified Rayleigh line curve. The value of the back pressure p_b at the exit of the nozzle is then obtained for this position, and this process is repeated until the shock position is found. Possible paths in the T - Q plane were shown earlier in figure 3.

Note that for given reservoir conditions, the down stream limit of the one-dimensional α -shock Relaxation zone is always defined by the same equilibrium pressure p_e , irrespective of the value of Q . However, since the shock position, i.e., its distance from the nozzle exit plane, will depend on Q_s , each value of the back pressure p_b corresponds to a different value of Q_s , (see fig. 6).

From the transition paths sketched in figure 3, and the pressure temperature relation through the shock, it is apparent that the pressure may initially decrease behind the shock, but for nozzles of sufficient length it will ultimately rise following the subsonic equilibrium curve.

FIGURE 5a.



ADIABATIC SHOCKS AND THEIR RELAXATION ZONES.

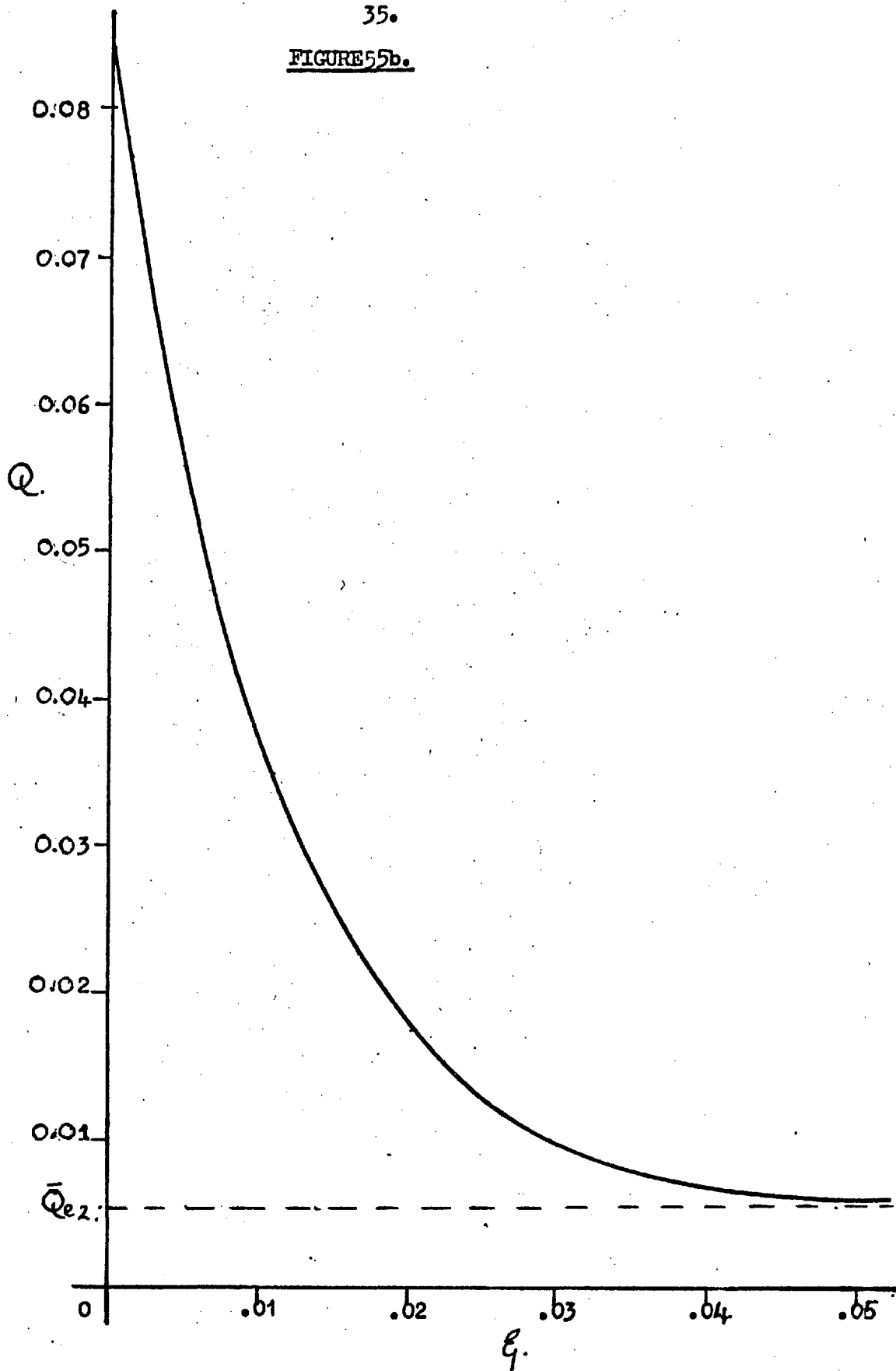
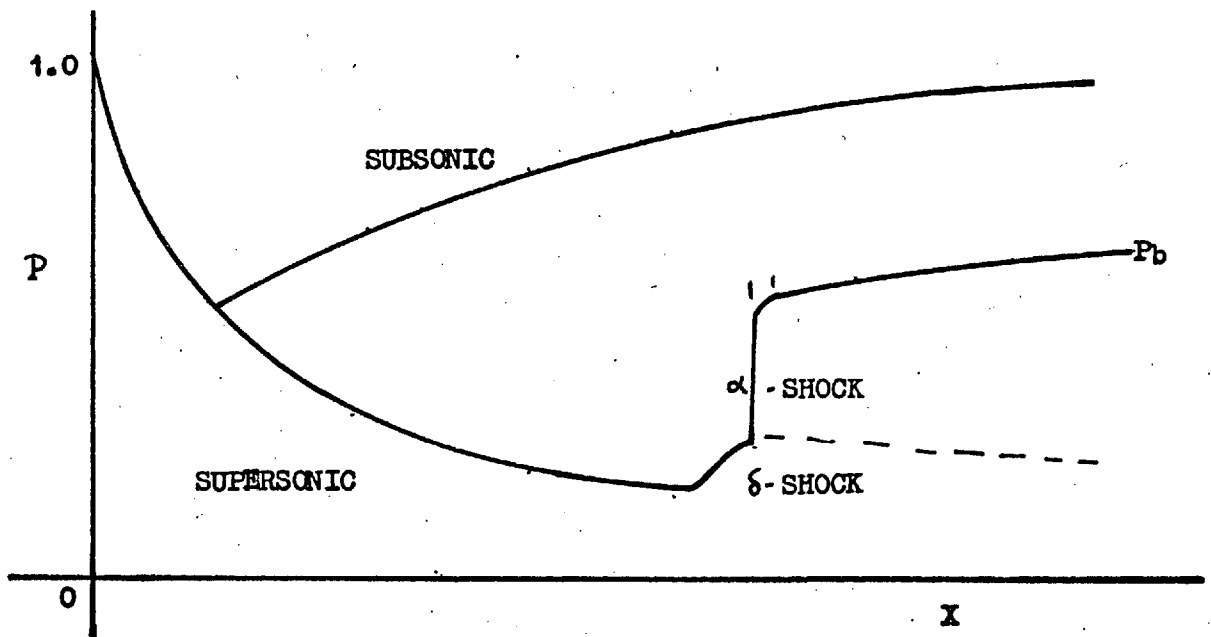
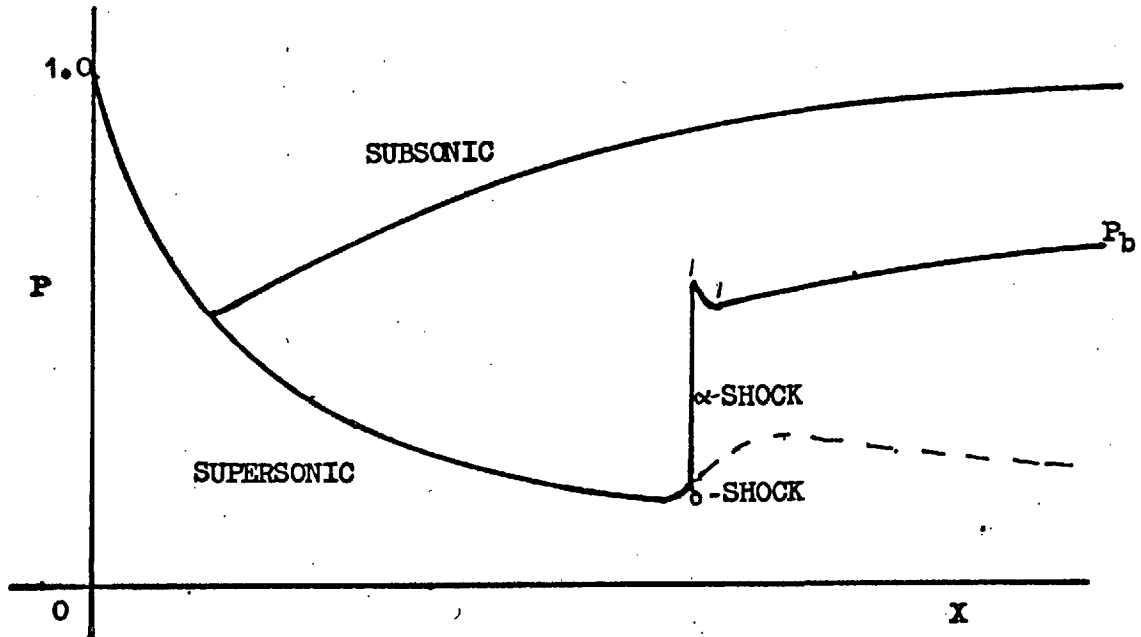
FIGURE 55b.

FIGURE 6.

SOME TYPICAL PRESSURE DISTRIBUTIONS.

5. Numerical solutions and higher approximations

5.1 The numerical solutions

To test the simple analytical solutions described by Blythe (1967) and in §§ 2, 3, and 4 of this thesis, some numerical solutions of the full equations have been obtained. The basis for the calculations was an adaptation of the programme described in Wilson, Schoofield and Lapworth (1967).

For the analytical solutions the asymptotic nozzle shape must have the form

$$A \sim Cx^v. \quad 5.1$$

Convergent-divergent nozzle shapes used in the numerical solutions were all of the form

$$A = 1 + x^2, \quad x \leq x_0, \quad 5.2$$

$$A = C(x-c)^v, \quad x \geq x_0, \quad 5.3$$

and hence (5.1) is always true for large x .

Both A , and dA/dx are continuous at x_0 , or

$$\left. \begin{aligned} C &= \left\{ \frac{2x_0}{v} \right\}^v (1 + x_0^2)^{1-v}, \\ c &= x_0 - v(x_0^{-1} + x_0)/2. \end{aligned} \right\} \quad 5.4$$

Previous numerical calculations based on the quasi-one-dimensional relations do not appear to encounter this phenomena, principally

because they did not cover the conditions for which the shocks should occur. Initial numerical experiments showed that in situations when δ -shocks should occur the region of frozen flow was extensive.

Figure 7 shows one such case where the rate of change of δ remains very small up to 10^5 diameters downstream of the throat. However continuation of the calculation up to 10^8 diameters reveals the situation shown in figure 8. With respect to the overall length, the near frozen behaviour collapses suddenly and the solution rapidly approaches an equilibrium state.

Also shown in figure 8 is the analytical solution described earlier. The agreement between this approximate solution and the exact solution is good, particularly for the shock position, although there is a significant difference in the values for the maximum temperature at the downstream end of the shock. The initial equilibrium decay downstream of the shock, apart from the shift in temperature, agrees well with theoretical predictions.

5.2 The shock position

Although the δ -shock position given by equation 2.11 is well defined, the numerical shock position is not. In order to compare the numerical and analytical solutions in a quantitative manner, the "shock position" for the numerical solution was defined by the point at which the translational temperature attains the value corresponding to $\xi = 0$ ($\alpha = \lambda^2 \gamma_s$) in the analytical solution.

FIGURE 7.

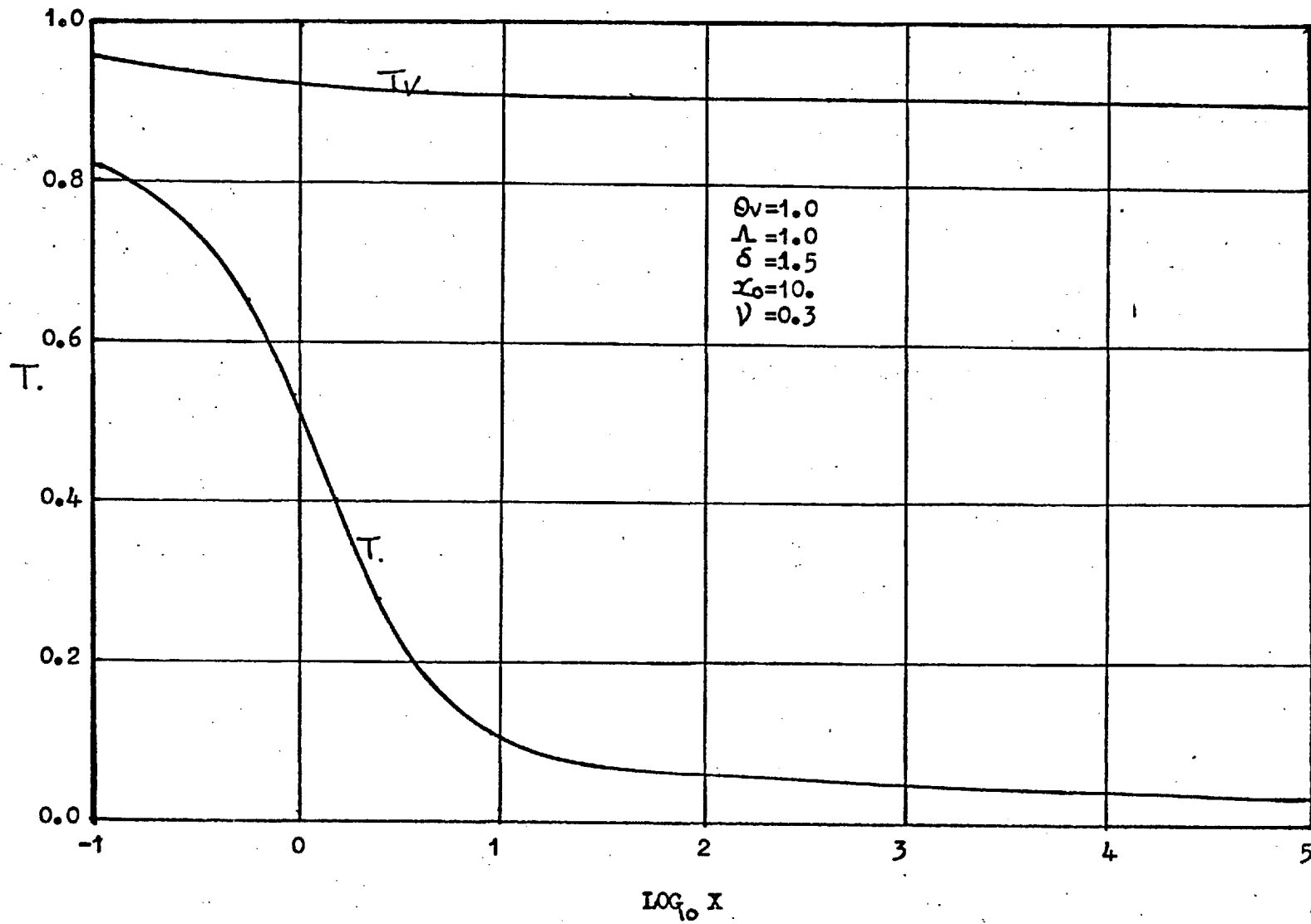
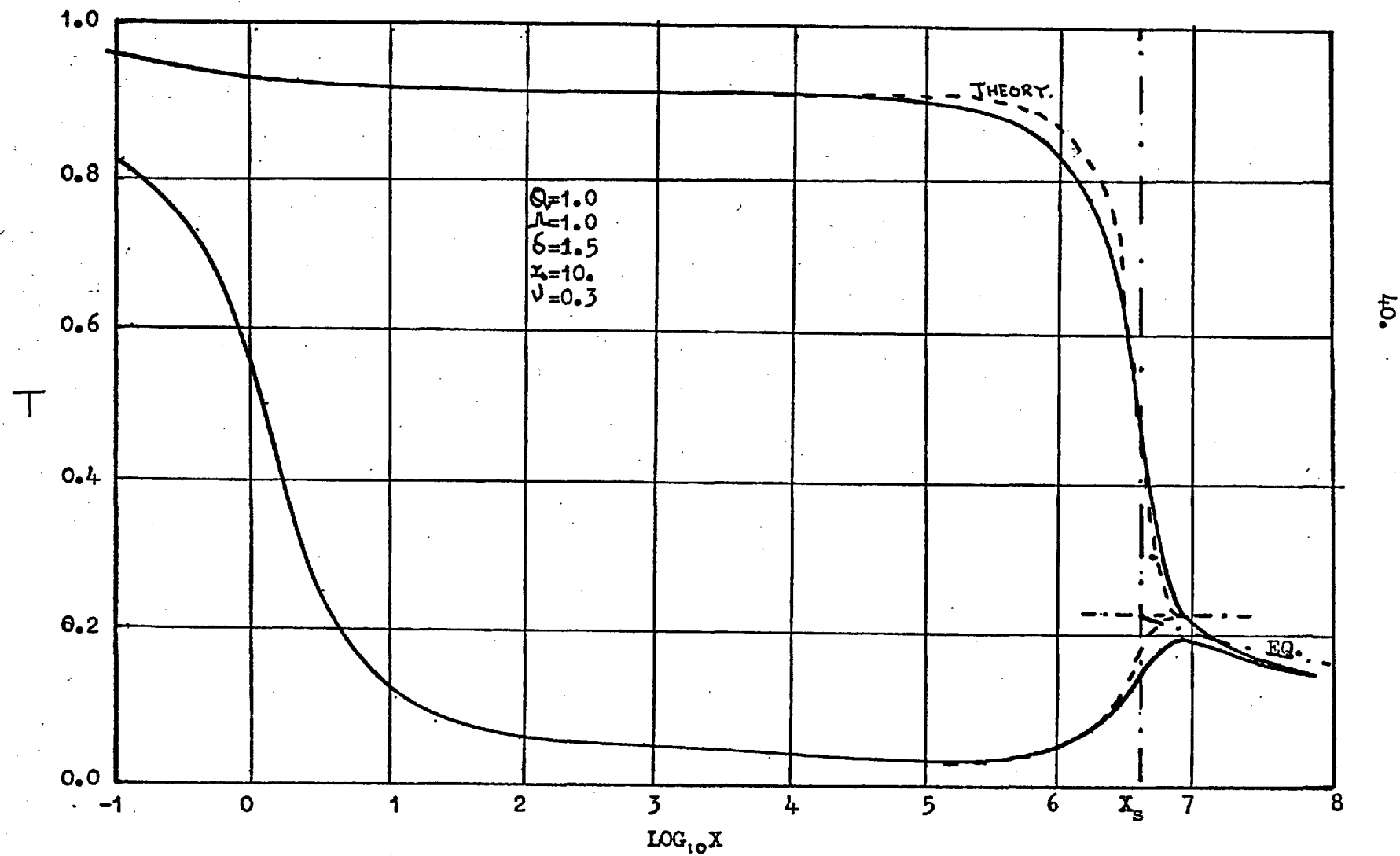


FIGURE 8.



A DE-EXCITATION SHOCK.

According to the simple theory

$$\chi_s = \lambda^{-1/2} \gamma_s (\gamma, \theta_v, \nu, \delta, C), \quad 5.5$$

where γ_s is as given by (2.11). This expression, however can be re-arranged to give

$$\chi_s = K X_s (\gamma, \nu, \delta), \quad 5.6$$

$$\text{where } X_s = \left\{ \frac{\ell u_0(\infty)}{(\delta-1)(\gamma-1)} \frac{\int m}{u_0(\infty)} \right\}^{(1-\gamma)/2}, \quad 5.7a$$

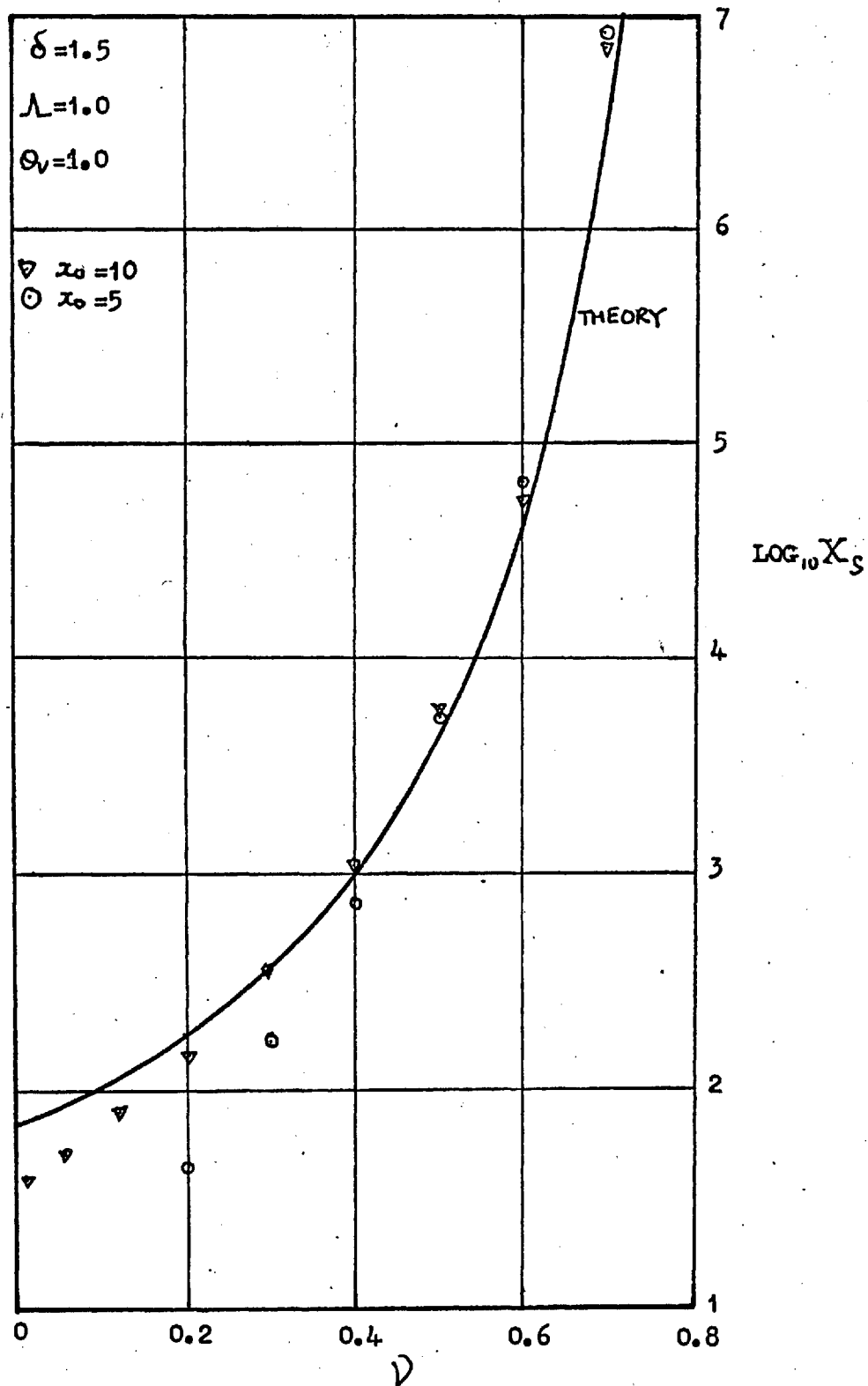
$$\text{and } K = (\Lambda \epsilon_\infty C^{(\ell-1)/2})^{-1/2}. \quad 5.7b$$

Figures 9, 10 and 11 show various tests of this similarity rule, the first for various values of the rate parameter Λ , the second for several values of the matching point χ_0 , and the third for some values of the characteristic temperature θ_v . The shock position is reasonably predicted in most cases and the various scaling laws are correct except at small values of ν . The limit $\nu \rightarrow 0$ corresponds to a straight channel and it is not surprising that the asymptotic state is dependent, not only on the asymptotic shape, but also on the initial conditions at the entrance to the channel, defined by χ_0 . For a further discussion of this point see below § 5.3.

5.3 The temperature maxima

As noted above the major discrepancy between the numerical

42
FIGURE 9.



COMPARISON OF X_s FOR VARIOUS x_0 .

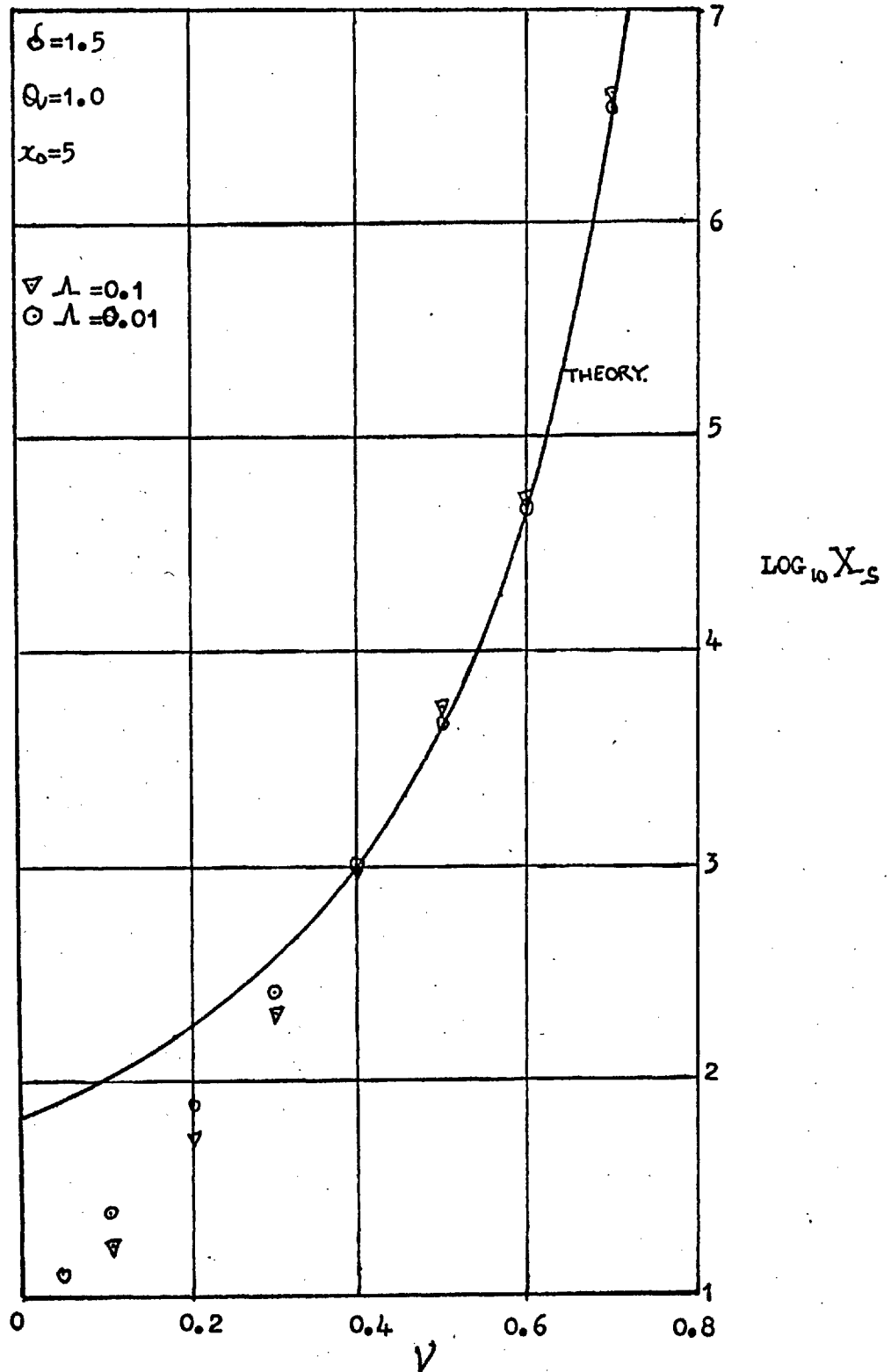
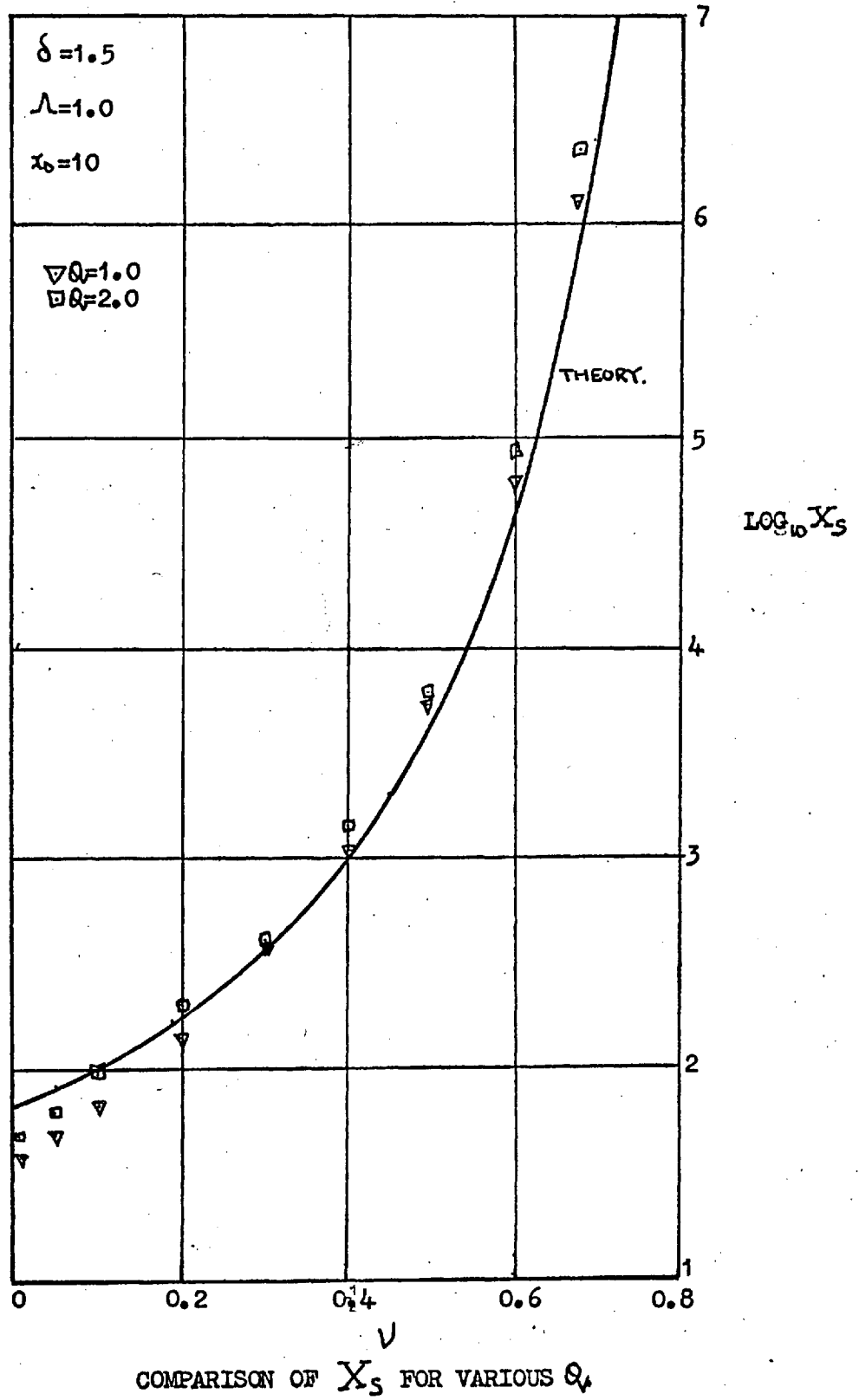
FIGURE 10.COMPARISON OF X_s FOR VARIOUS Λ .

FIGURE 11.



solutions and the theoretical predictions was for the temperature maxima at the downstream end of the shock. Figure 11 shows a plot of T_{MAX} against the shape parameter ν for a range of $\chi_0'S$. Reservoir conditions, defined by Λ and G_V are fixed and the gas parameter δ is also fixed.

It is apparent that there is a definite departure from the results of the simple theory. Similar graphs for other values of δ and the reservoir conditions show corresponding trends.

Provided ν is not small it seems likely that deviations from the simple solution are mainly due to the effects of area change through shock. If this is so then the continuity equation can be replaced by

$$\left. \begin{aligned} \pi V &= [1 + \lambda^d \gamma_s^{\nu-1} \xi]^{-\nu} \\ \text{or } \pi V &\sim 1 - \nu \lambda^d \gamma_s^{\nu-1} \xi + \dots, \\ \text{for } \nu \lambda^d \gamma_s^{\nu-1} \xi &\ll 1. \end{aligned} \right\} 5.8$$

It is the second term on the right hand side which will govern the perturbation to the shock solution. Apparently the only parameter in these perturbations will be

$$\alpha = \nu \lambda^d \gamma_s^{\nu-1}. \quad 5.9$$

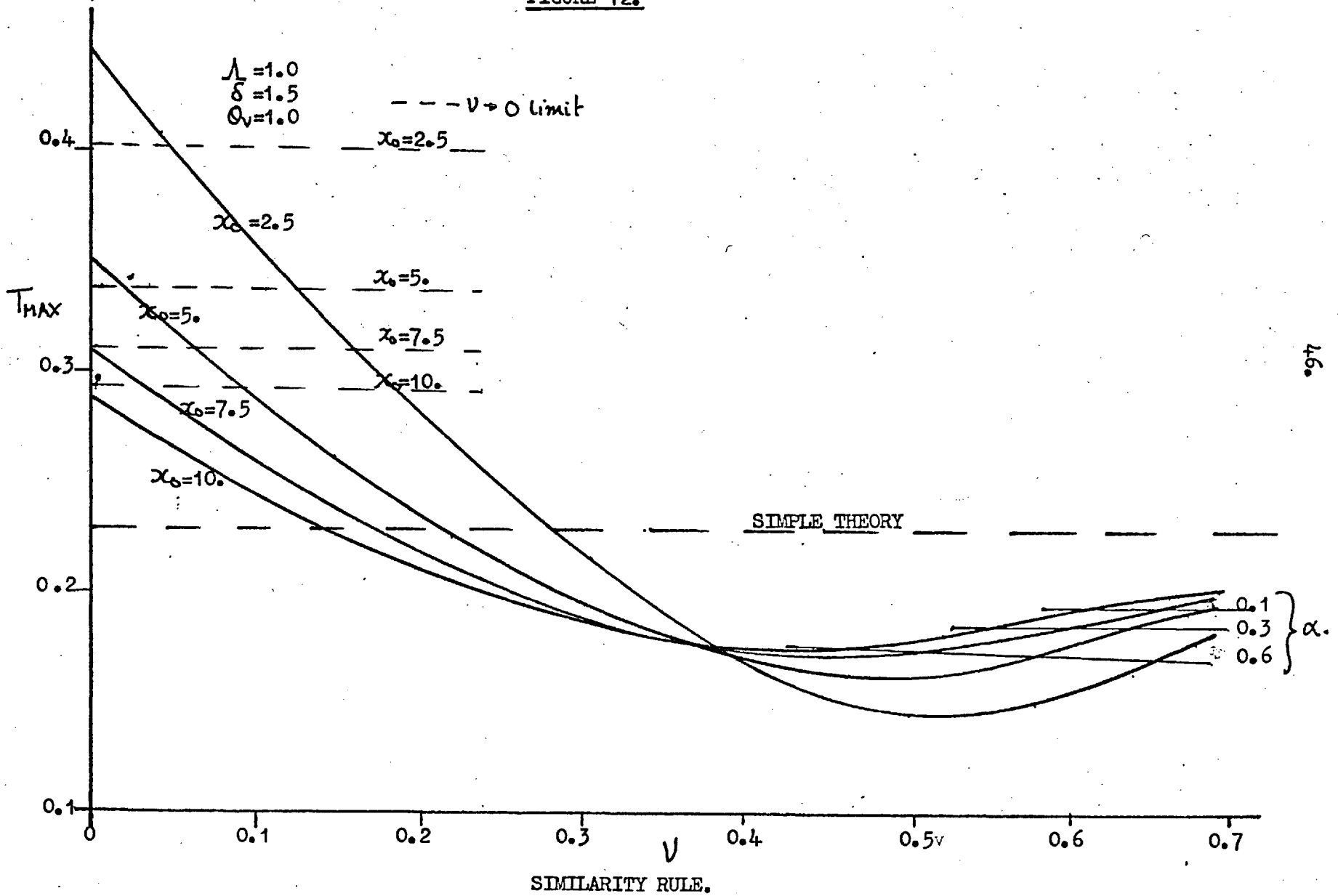
This means that to first order

$$\pi = \pi(\xi, \alpha), \quad 5.10$$

and hence for the fixed α the solution in the neighbourhood of the

δ -shock should be identical if area change is indeed the dominant perturbation. Some solutions for the perturbation

FIGURE 12.



quantities in the vicinity of the δ -shock are discussed briefly in Appendix III. Note that strictly for the expansion to be meaningful $\alpha \ll 1$.

Lines of constant α are cross plotted on figure 12. These lines show quite clearly that the data does indeed collapse for sufficiently large values of V . This similarity rule does not, however hold for small values of V , which as noted above in § 5.2, are situations more strongly dependent on χ_0 . In the parallel pipe limit $V \rightarrow 0$, perturbations are presumably dominated by "entrance" conditions defined by χ_0 .

For the one-dimensional flow limit, with conditions at χ_0 corresponding to frozen flow between the reservoir and χ_0 , it can be shown that

$$V = \frac{\gamma \epsilon + \sqrt{1 + \gamma^2(\epsilon^2 - 1)} - Q}{\gamma + 1}, \quad 5.11$$

$$\text{and } \frac{T}{u_0^2(\omega)} = \frac{(\gamma - 1)(1 + \gamma(1 - \epsilon^2)) + Q - \epsilon(\gamma - 1)\sqrt{1 + \gamma^2(\epsilon^2 - 1)} - Q}{(\gamma + 1)^2},$$

$$\text{where } \epsilon = \frac{u_{\chi_0}}{u_0(\omega)} \left\{ \frac{T_{\chi_0}}{u_{\chi_0}^2} + 1 \right\}, \quad \left. \vphantom{\frac{u_{\chi_0}}{u_0(\omega)}} \right\} 5.12$$

$$\text{or } \epsilon = \left\{ 1 + \frac{2}{(\gamma - 1)m_0} \right\}^{-1/2} \left\{ 1 + \frac{1}{\gamma m_0^2} \right\},$$

where m_0 is the frozen mach number and the subscript χ_0 refers to conditions at χ_0 .

The solution of equation 5.11 together with the equilibrium curve of T ($T = \bar{T}$ eqn 3.3) gives values of T_{MAX} which are in reasonable agreement with the numerical computations (see figure 12).

As $\chi_0 \rightarrow \infty$, $\epsilon \rightarrow 1$ and these values converge on the simple theory limit.

5.4 Application to hypersonic facilities

For the example shown (figure 8) it would appear that experimental confirmation is not feasible. Typical nozzle lengths of hypersonic facilities do not exceed 10^4 throat diameters.

Certain situations (e.g. larger values of λ) will lead to shocks at $10^3 - 10^4$ throat diameters but the shock structure becomes more diffuse as its position moves further upstream.

One of the disappointing features, from a practical view point, of the results is the shock position. In many cases it occurs at very large distances downstream.

However for other types of rate process involving a large amount of energy the results may not be quite so discouraging, and the shock position has been derived for a general rate process, where the rate equation is only a function of the local flow conditions in Blythe et al (1968).

Similar results have also been obtained for condensation, which involves a more complex rate process, in Part II of this thesis.

6. Discussion

The structure of the de-excitation shocks discussed by Blythe (1967) has been obtained and it has been shown that for a certain range of finite back pressures an adiabatic shock may be embedded within the de-excitation shock. The structure of the relaxation zone behind such an adiabatic shock has been computed.

These solutions have been compared with some numerical results for the full equations, and it has been shown that these de-excitation shocks exist. The theory predicted the position and structure of the shocks well, except for very slender nozzles, any discrepancies being mainly due to the effects of area change.

However it appears that for the vibrational rate process considered, there is very little chance of finding these shocks experimentally, although it has been noted that for a rate process involving considerably more energy such shocks might be found in a physical situation. One such process is the condensation shock, which is considered in detail in Part II of this thesis.

Appendix I

The coefficients of the expansion of the integrand $f(Q)$ in the neighbourhood of $Q=0$.

For this expansion there are two cases to consider. The first is the general case, when $Q_v > 0$ and $\bar{\epsilon}$ is exponentially small near $Q=0$, and the second is the case for $Q_v=0$ when $\bar{\epsilon}=1$.

Case I $Q_v > 0$

The integrand $f(Q)$ can be expanded by considering the various components individually and the expansion is then found to be

$$f(Q) = Q^{-\delta} (a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3 + \dots),$$

where $a_0 = q_0 t_0 / M^\delta,$

$$a_1 = a_0 (v_1 + t_1 + q_1),$$

$$a_2 = a_0 (v_2 + t_2 + q_2 + v_1 t_1 + v_1 q_1 + q_1 t_1),$$

$$a_3 = a_0 (v_3 + t_3 + q_3 + v_1 t_2 + v_1 q_2 + t_1 q_2 + t_1 v_2 + q_1 v_2 + q_1 t_2 + v_1 q_1 t_1),$$

and the various coefficients above are given by

$$v_1 = -\frac{1}{\delta+1}, \quad v_2 = \frac{-\delta}{4(\delta+1)^2},$$

$$v_3 = -\frac{\delta}{8(\delta+1)^2}, \quad v_4 = \frac{-5\delta}{64(\delta+1)^2}.$$

$$t_0 = \left\{ \frac{u_0^2(\infty)}{2(\delta+1)} \right\}^{-\delta}, \quad t_1 = -\frac{\delta}{4} \left\{ \frac{\delta-1}{\delta+1} \right\},$$

$$t_2 = -\frac{\delta}{8} \left\{ \frac{\delta-1}{\delta+1} \right\} \left\{ 1 - \left\{ \frac{\delta+1}{4} \right\} \left\{ \frac{\delta-1}{\delta+1} \right\} \right\},$$

$$t_3 = -\frac{5\delta}{64} \left\{ \frac{\delta-1}{\delta+1} \right\} \left[1 - \frac{2(\delta+1)}{5} \left\{ \frac{\delta-1}{\delta+1} \right\} + \frac{(\delta+1)(\delta+2)}{30} \left\{ \frac{\delta-1}{\delta+1} \right\}^2 \right],$$

and $q_i = \bar{Q}_0^{-(i+1)},$ where $\bar{Q}_0 = \bar{Q}(T_0).$

Case II $Q_v = 0$

The expansion is similar, but the q_i^{ave} replaced by q'_i , where

$$q'_0 = \bar{Q}_0^{-1}, \quad q'_1 = \gamma \bar{Q}_0^{-2},$$

$$q'_2 = \gamma^2 \bar{Q}_0^{-3} (1 + (\gamma - 1)(2\gamma^2 u_0^2(\infty))^{-1}),$$

$$q'_3 = \gamma^3 \bar{Q}_0^{-4} \left\{ 1 + \frac{(\gamma - 1)^3}{\gamma^2 u_0^2(\infty)} + \frac{(\gamma - 1)^4}{16 \gamma^3 u_0^4(\infty)} \right\}.$$

Appendix II

The coefficients of the expansion of $f(Q)$ near $Q=\bar{Q}_e$ are derived, in a similar manner to those in Appendix I, and the expansion is

$$f(Q) = (\bar{Q}_e - Q)^{-1} [b_0 + b_1 (\bar{Q}_e - Q) + b_2 (\bar{Q}_e - Q)^2 + \dots],$$

where

$$b_0 = \frac{V_e^2}{M_0^2 T_e^2 E},$$

$$b_1 = (V_1 + T_1 + e_1),$$

$$b_2 = (V_2 + T_2 + e_2 + V_1 T_1 + V_1 e_1 + T_1 e_1),$$

and the various coefficients are given by

$$V_e = \frac{\gamma + \sqrt{1 - \bar{Q}_e}}{\gamma + 1},$$

$$V_1 = \frac{1}{(\gamma + 1)^2 V_e^2} \left\{ 1 + \frac{\gamma}{(1 - \bar{Q}_e)^{1/2}} \right\},$$

$$V_2 = \frac{1}{(\gamma + 1)^2 V_e^2} \left\{ \frac{\gamma}{4(1 - \bar{Q}_e)^{3/2}} \right\},$$

$$\frac{T_e}{u_0^2(\infty)} = \frac{(\gamma - 1) + \bar{Q}_e - (\gamma - 1)\sqrt{1 - \bar{Q}_e}}{\gamma + 1^2},$$

$$\frac{T_e^*}{u_0^2(\infty)} = \frac{(\gamma - 1) + \bar{Q}_e - (\gamma - 1)\sqrt{1 - \bar{Q}_e}}{(\gamma + 1)^2},$$

$$T_1 = \frac{\delta}{T_e^*} \left\{ 1 + \frac{(\gamma - 1)}{2\sqrt{1 - \bar{Q}_e}} \right\},$$

$$T_2 = \frac{\delta(\delta + 1)}{2T_e^*} \left\{ 1 + \frac{(\gamma - 1)}{2\sqrt{1 - \bar{Q}_e}} - \frac{\delta}{T_e^*} + \frac{\bar{Q}_e'' t_1^2}{T_e^2} \right\},$$

$$E_0 = \frac{\bar{Q}_e' t_1}{T_e}, \quad E_1 = \frac{\bar{Q}_e' t_2}{T_e} + \frac{\bar{Q}_e'' t_1^2}{T_e^2},$$

$$E_2 = \frac{\bar{Q}_e' t_3}{T_e} + \frac{\bar{Q}_e''}{T_e^2} 2t_1 t_2 + \frac{\bar{Q}_e''' t_1^3}{T_e^3},$$

$$E = 1 + E_0, \quad e_1 = -\frac{E_1}{E}, \quad e_2 = \frac{E_1^2}{E^2} - \frac{E_2}{E}.$$

The following coefficients are also used

$$\begin{aligned} t_1 &= - \frac{1}{T_e^*} \left\{ 1 + \frac{(\gamma-1)}{2\sqrt{1-Q_e}} \right\}, \\ t_2 &= \frac{1}{T_e^*} \left\{ \frac{\gamma-1}{8(\sqrt{1-Q_e})^3} \right\}, \\ t_3 &= - \frac{1}{T_e^*} \left\{ \frac{\gamma-1}{16(\sqrt{1-Q_e})^5} \right\}. \end{aligned}$$

The subscript e refers to the equilibrium end conditions, and

$$\bar{Q}_e' = \left\{ \frac{d\bar{Q}}{dT} \right\}_e, \text{etc.}$$

Appendix III. The effects of area change.

If the effects of area change are introduced into the shock solutions, the following equations in the scaled variables, are obtained

$$\pi V = [\bar{\pi} + \lambda^d Y_s^{\nu-1} \xi]^{-\nu}, \quad A.1$$

$$\frac{d}{d\xi} (\pi T) + u_0^2(\infty) \pi V \frac{dV}{d\xi} = 0, \quad A.2$$

$$\frac{\gamma}{\gamma-1} T + \frac{u_0^2(\infty) V^2}{2} + \sigma = \frac{u_0^2(\infty)}{2} + \sigma_\infty, \quad A.3$$

$$\text{and} \quad \frac{d\sigma}{d\xi} = \frac{\pi \Omega^*(T)}{V} [\bar{\sigma}(T) - \sigma]. \quad A.4$$

Rewriting these equations in terms of the variable Q (§2 eqn.2.20)

gives

$$\pi V = 1 - \alpha \xi + \dots, \quad A.5$$

$$\frac{d}{d\xi} (\pi T) + u_0^2(\infty) \pi V \frac{dV}{d\xi} = 0, \quad A.6$$

$$\frac{2\gamma(\gamma+1)T}{u_0^2(\infty)} + (\gamma^2-1)(V^2-1) = Q, \quad A.7$$

$$\text{and} \quad \frac{dQ}{d\xi} = \frac{\pi M^d T^d}{V} (\bar{Q} - Q), \quad A.8$$

where $\alpha = \nu \lambda^d Y_s^{\nu-1}$ is the similarity parameter discussed in §5.3.

An appropriate set of variables in this region is

$$\left. \begin{aligned} \pi &= \pi_0 + \alpha \pi_1 + \dots, \\ V &= V_0 + \alpha V_1 + \dots, \\ T &= T_0 + \alpha T_1 + \dots, \\ Q &= Q_0 + \alpha Q_1 + \dots, \end{aligned} \right\} A.9$$

If these are inserted into the shock equations, (eqns.A.5-A.8)

then to zeroth order the original shock equations (2 eqns 2.13-2.15) are recovered. The solution of these zeroth order equations is discussed in §§ 2 and 4.1.

The equations for the first order solution are

$$\pi_1 V_0 = \pi_0 V_1 = -\epsilon, \quad A.10$$

$$\frac{d}{d\epsilon} \left\{ \frac{\pi_1 T_0}{u_0^2(\infty)} + \frac{\pi_0 T_1}{u_0^2(\infty)} \right\} + (\pi_1 V_0 + \pi_0 V_1) \frac{dV_0}{d\epsilon} + \pi_0 V_0 \frac{dV_1}{d\epsilon} = 0. \quad A.11$$

$$\frac{2\gamma(\gamma+1)T_1}{u_0^2(\infty)} + 2(\gamma^2-1)V_0 V_1 = Q_1, \quad A.12$$

and

$$\frac{dQ_1}{d\epsilon} = \frac{M_0^6 T_0^6}{V_0^2} \left[\left\{ \delta \frac{T_1}{T_0} - 2 \frac{V_1}{V_0} \right\} (\bar{Q}_0 - Q_0) + (\bar{Q}_1 - Q_1) \right], \quad A.13$$

$$\text{where } \bar{Q} = \bar{Q}_0 + \alpha Q_1, \text{ and } \bar{Q}_1 = -\frac{2(\gamma^2-1)}{u_0^2(\infty)} \bar{\epsilon}_0^2 e^{\alpha V_1/T_0} \frac{T_1}{T_0^2}. \quad A.14$$

Combining equation A.11 with equations A.10 and 2.13 gives

$$\frac{d}{d\epsilon} \left\{ \frac{\pi_1 T_0}{u_0^2(\infty)} + \frac{\pi_0 T_1}{u_0^2(\infty)} \right\} - \epsilon \frac{dV_0}{d\epsilon} + \frac{dV_1}{d\epsilon} = 0; \quad A.15$$

which can be integrated to give

$$\frac{\pi_1 T_0}{u_0^2(\infty)} + \frac{\pi_0 T_1}{u_0^2(\infty)} + V_1 = \int_{V_{0I}}^{V_0} \epsilon dV_0 + \Gamma, \quad A.16$$

where Γ is the constant of integration, and the lower limit V_{0I} has been inserted for convenience.

Equation A.16 can now be written, by making use of the zeroth order solution as

$$\pi_1 T_0 + \pi_0 T_1 + u_0^2(\infty) V_1 = S \quad A.17$$

where

$$S = u_0^2(\infty) \int_{Q_{0I}}^{Q_0} \frac{dV_0}{dQ_0} \left\{ \int_{Q_{0I}}^{Q_0} \frac{V_0^2}{M_0^6 T_0^6 (\bar{Q}_0 - Q)} dQ - C_1 \right\} dQ_0 + \Gamma \quad A.18$$

Thus combining equations A.10, A.12 and A.17, the following set of relations for the perturbation quantities is obtained,

$$\left. \begin{aligned} \left[1 - \frac{u_0^2(\omega) V_0^2}{\gamma T_0} \right] \frac{V_1}{V_0} &= \frac{(\gamma-1)}{\gamma} M^* - S^* + L, \\ \left[1 - \frac{u_0^2(\omega) V_0^2}{\gamma T_0} \right] \frac{\pi_1}{\pi_0} &= \frac{(\gamma-1)}{\gamma} M^* + S^* - \frac{u_0^2(\omega) V_0^2}{\gamma T_0} L, \\ \left[1 - \frac{u_0^2(\omega) V_0^2}{\gamma T_0} \right] \frac{T_1}{T_0} &= \frac{(\gamma-1)}{\gamma} M^* \left[1 - \frac{u_0^2(\omega) V_0^2}{\gamma T_0} \right] - \frac{u_0^2(\omega) V_0^2}{\gamma T_0} (L + S^*), \end{aligned} \right\} \text{A.19}$$

where

$$\left. \begin{aligned} L &= -e_1, \\ M^* &= u_0^2(\omega) Q_1 / 2(\gamma^2 - 1) T_0 = -G_1 / T_0, \\ S^* &= S / \pi_0 T_0. \end{aligned} \right\} \text{A.20}$$

The rate equation can similarly be rewritten as

$$\frac{dQ_1}{dQ_0} = \Psi(Q_0) - \Phi(Q_0) \cdot Q_1,$$

where

$$\Psi(Q_0) = \frac{2(L - S^*)}{(1 - m_0^2)} - \frac{\delta m_0^2 (L + S^*)}{(1 - m_0^2)} +$$

$$\frac{1}{Q_0 - Q_0} \left[\frac{2(\gamma^2 - 1) \bar{G}_0^2 e^{Q_1/T_0}}{u_0^2 \omega} \right] \frac{m_0^2 (L + S^*)}{(1 - m_0^2)}, \quad \text{A.22}$$

and

$$\Phi(Q_0) = \frac{(\gamma-1)}{\gamma(1-m_0^2)} \left\{ 2 - \delta(1-\gamma m_0^2) \right\} \frac{u_0^2 \omega}{2(\gamma^2-1)T_0} +$$

$$\frac{1}{Q_0 - Q_0} \left[1 + \frac{\bar{G}_0^2 e^{Q_1/T_0} \gamma - 1}{T_0} \frac{(1 - \gamma m_0^2)}{\gamma (1 - m_0^2)} \right],$$

with

$$m_0 = \sqrt{\frac{u_0^2(\omega) V_0^2}{\gamma T_0}}.$$

The rate equation, A.21, must be numerically integrated owing to the complex form of the functions $\Psi(Q_0)$ and $\Phi(Q_0)$, the constants of integration being obtained by matching with the appropriate terms in the "pseudo entropy" solution of Blythe (1967).

This process is, however, very complex, and the computational time involved is of the same order of magnitude as that used in obtaining the numerical solutions to the full equations.

The effects of area change on the shock solution were therefore tested by using the similarity^{rule,} _{λ} and the results are discussed in § 5.3.

Notation.

A	area ratio	$M = \{u(\infty)/M_0\}^{\gamma-1}$	
a_i	constants see (4.3).	p	pressure,
B	constant see (2.8).	Q	effective heat input (2.20).
$b = (\gamma-1)v/c$		$\bar{Q}$	equilibrium value of Q.
b_i	constants see (4.9).	q	dummy variable.
C_v	constant see (4.1).	R	gas constant.
C	constant see (2.9).	S_A	pseudo entropy (3.1).
c	constant see (3.4).	S	see Appendix III.
C	constant see (5.4).	T	temperature.
$d = b(\delta-1)$		T_c	constants see App. III.
E_c	constants see App. II.	T_e^*	constant see App. II.
F	} see § 4.1.	t_i	constants see App. I.
f		u	velocity.
g		V	$= u/u_0(\infty)$.
H		V_c	constants see App. I.
h'	throat height.	x	distance through nozzle.
K	see (5.7b).	x_0	constant see (5.3).
k	exponent $1 - \sqrt{1 + (\gamma-1)\delta}$.	$\bar{X}_s$	see (5.7a).
L	see Appendix III.	$\gamma = \lambda^{\frac{1}{2}} x$	
ℓ	exponent $1 - \sqrt{2 - \gamma + (\gamma-1)\delta}$.	γ_s	shock position.
M^*	see Appendix III	γ	specific heat ratio (δ^2).
M_0	scaled mass flow (δ^2).	δ	exponent see (1.1).
m	mass flow.	ϵ	see (5.12).
m_0	frozen mach number.	Q_v	characteristic vib. temperature.

Λ rate parameter (2.6).

λ modified Λ (2.10).

ν exponent (2.9).

ξ scaled length (2.10).

π scaled density (2.10).

ρ density.

ϵ vibrational energy.

$\bar{\epsilon}$ equilibrium value of ϵ .

τ relaxation time.

$\left. \begin{matrix} \Phi \\ \Psi \end{matrix} \right\}$ see Appendix III.

Ω relaxation frequency.

Subscripts.

0, 1, denote perturbation quantities in the various solutions.

Dimensional quantities.

All variables defined above are non dimensionalised according to the scheme outlined in (p 2.). Any dimensional quantities are denoted by primed variables, eg. ϵ' , τ' , etc.

PART II

THE CONDENSATION OF PURE AND IMPURE VAPOURS

INDEX.

1. Introduction.	663.
2. Equations describing flow.	68.
2.1 Equations of motion.	68.
2.2 Rate equations.	72.
2.3 Non dimensionalisation of equations.	76.
2.4 Condensation point.	77.
3. The flow regimes and their limits.	78.
3.1 The scaled equations.	78.
3.2 Regimes for the impure vapour.	80.
3.3 Regimes for the pure vapour.	81.
4. Near equilibrium solutions.	87.
4.1 Scaling in the near equilibrium regime.	87.
4.2 The impure vapour solution.	88.
4.3 The pure vapour solution.	89.
5. Near frozen solution for an impure vapour.	94.
5.1 The perturbation solution.	94.
5.2 Regime 1. The condensation shock.	98
5.3 Regime 2. Gradual condensation.	105.

6. Near frozen solutions for a pure vapour.	109.
6.1 The perturbation equations.	109.
6.2 The limit $K^2 \gg \bar{\lambda}_p^b$.	110.
6.2.1 The perturbation solution.	110.
6.2.2 Regime 1. The condensation shock.	112.
6.2.3 Regime 2. Gradual Condensation.	113.
6.3 The limit $K^2 \approx \bar{\lambda}_p^b$.	113.
6.3.1 The perturbation solution.	113.
6.3.2 Regime 1. The condensation shock.	115.
7. Numerical interpretation.	121.
7.1 The scaled variables.	121.
7.2 Numerical examples.	123.
8. Discussion.	130.
Appendix I.	131.
Appendix II.	132.
Notation.	135.

Introduction

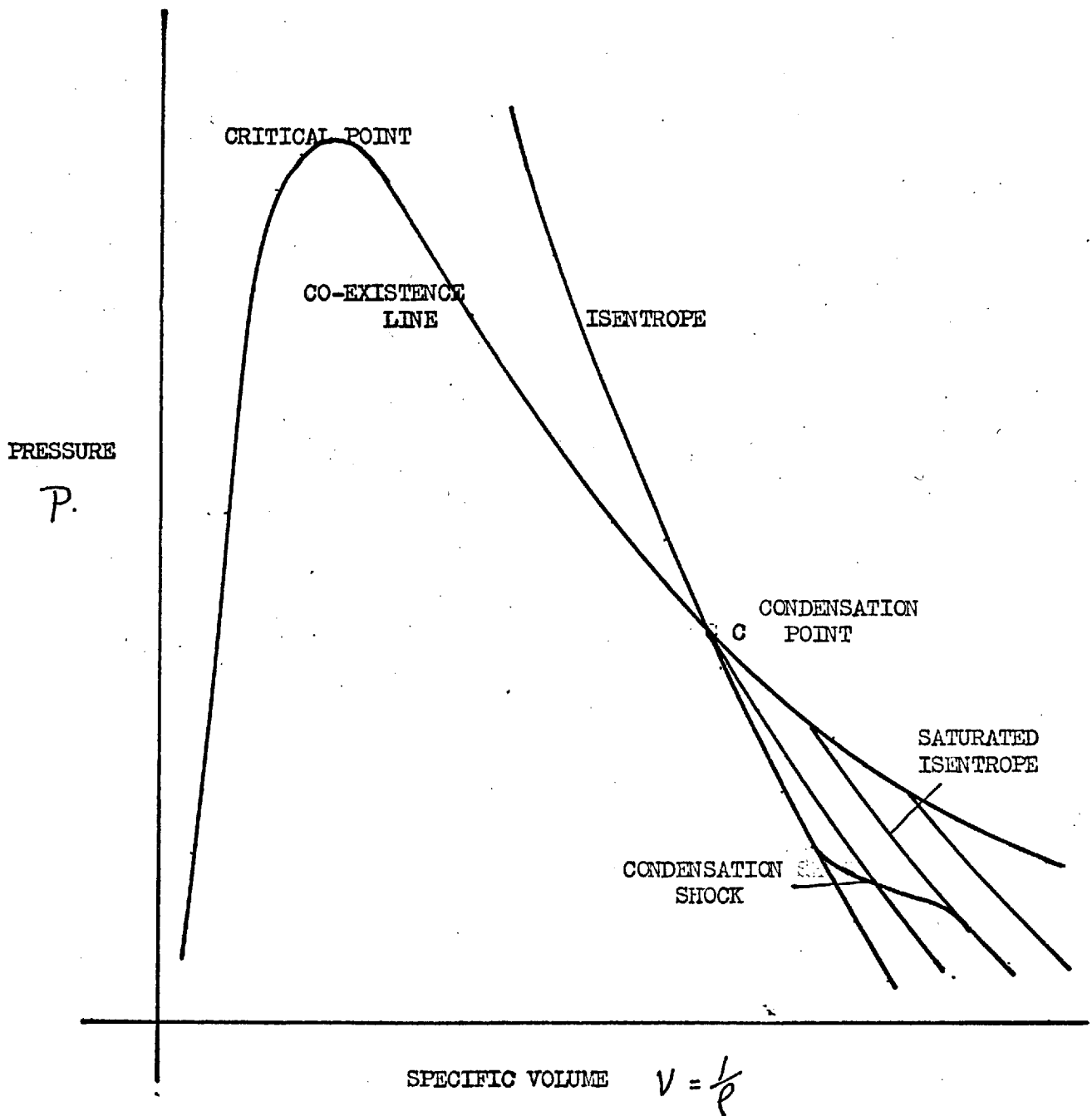
In many high speed wind tunnels condensation is a possible limiting factor in their operation at relatively low temperatures. Since this process proceeds at a finite rate, condensate does not form immediately downstream of the saturation point, and a super-saturated state may exist. It has been observed^{*} that this region is sometimes terminated by a sudden return to local equilibrium conditions. This narrow region is usually called a "condensation shock".

The condensation process can most easily be described in the pressure-specific volume plane (figure 1). The flow follows the condensate-free isentrope until it crosses the co-existence line. (The liquid-vapour equilibrium line). If it then remained in equilibrium it would follow the saturated isentrope beginning at the condensation point "C". As, however, the vapour condenses at a finite rate the flow will initially follow the original frozen isentrope and may then, depending on the flow conditions cross via a "condensation shock" to a saturated equilibrium isentrope, which it will either approach asymptotically or diverge from, depending on the limiting flow conditions.

It has been noted^{*} that there are two types of rate process involved.

1. Heterogeneous nucleation:- In this process large numbers of impurities exist in the flow and the vapour condenses onto these,

.....
^{*} For a review of most of the experimental results see Wegener and Mack(1958)

FIGURE 1.**PRESSURE SPECIFIC VOLUME DIAGRAM.**

without forming nuclei by self nucleation.

2. Homogeneous nucleation :- Here the gas forms, by random molecular processes, drops which are larger than the critical radius $\hat{r}_c'$. These form nuclei on which the vapour then condenses.

If the flow has large numbers of impurities in it, the process of heterogeneous nucleation will take precedence over that of homogeneous nucleation, and this former case will be referred to as the impure case. When however, the flow is free from impurities, heterogeneous nucleation will not occur, and condensate will be formed as a result of homogeneous nucleation. This will be referred to as the "pure" case.

The latter case has been analysed numerically (for example, Clark, 1963, and Griffin and Sherman, 1965), and it has been shown that the phenomena described above do occur as a result of the homogeneous nucleation theory.

There has, however, been no systematic theoretical attempt to analyse this phenomenon. Although mention should be made of the solutions of Daum and Gymarthy (1968) in which the simplification of a linear droplet radius growth law has been made. The position of the Wilson point (the co-ordinate at which the maximum supersaturation occurs) was then found by trial and error. This solution is in some ways analagous to the asymptotic near equilibrium solution presented here.

It is convenient to discuss the flow of a condensing vapour in terms of continuum non equilibrium gas dynamics. Here attention is principally confined to the limiting solutions for near

equilibrium and near frozen flow.

In the near equilibrium limit (when a characteristic rate parameter Λ is large) the problem is of the classical singular perturbation type, i.e. the flow initially follows the frozen isentrope but then rapidly returns to the equilibrium (for a similar problem see Bloom and Ting(1960)). It should be noted that this condensation jump is not a condensation shock in the classical sense of Wegener and Mack(1958), in that it is not completely one dimensional.

The near frozen (Λ small) analysis is similar to that given by Blythe (1967) for a de-excitation shock, although in the condensation case it is more complex as the rate equation depends on the particle history as well as the local flow conditions. For the impure vapour it can be shown from this analysis that in a nozzle whose area ratio A is given by $A = \mathcal{L}x^D$ then, when $k > 0$ where

$$k = 1 - \frac{1}{2}(\gamma + 1),$$

a condensation shock occurs. This shock is of the classical one-dimensional form described by Wegener and Mack(1958). If $k < 0, v > 1$ however vapour liquid equilibrium is established through a gradual condensation region. For $v > 1$ the flow always remains in a near frozen state.

For a pure vapour, however, there are two types of near frozen limit depending on the rate of droplet production. In the first of these nuclei are formed over a narrow region in the nozzle and droplet production ceases before the perturbation solution breaks

down, and in this limit the solution can be reduced to that for an impure vapour.

For the other case, however, nuclei are formed for most of the nozzle length and thus droplet production plays an important part in the breakdown of both the perturbation and pseudo entropy solutions.

For $J_2 > 0$ where

$$J_2 = 2k + 1 - \nu,$$

then a condensation shock occurs. For $J_2 < 0$ the solution can again be reduced to that for an impure vapour.

In the cases above, when a condensation shock exists it can be shown that this ^{is} a classical one dimensional region in which the flow returns to equilibrium, and the structure of which is similar to that described by Blythe, Petty, Schofield and Wilson (1968). It is also possible to show that the flow will not choke due to condensation.

Numerical computations have been carried out to illustrate the application of this theory to realistic nozzle flows.

It should be noted that the analysis presented here is for the condensation of a single vapour. It is however, possible to carry out an almost identical analysis for a mixture of vapours, including air and water vapour, in which one of the components condenses. The only difference occurs in the definition of the specific heats and the molecular weight. (Although these are functions of the mass fraction, they would, be constant to the appropriate order of magnitude in the present analysis.)

2. Equations describing flow

2.1 Equations of motion

The equations of motion are those which describe the flow of an ideal gas, part of which has condensed into liquid form. These are supplemented by a rate equation describing the formation of the condensate.

The derivation of these equations is described in several references and that due to Wegener and Mack (1958) will be followed. For convenience this derivation will be outlined in full.

In the continuity equation there are two terms, the mass of the condensible vapour m'_V , which passes a given location in the nozzle in unit time, and the mass per unit time of the condensed phase m'_L . The sum of these two mass rates is constant and thus

$$m'_V + m'_L = m' \quad 2.1$$

The ratio of the volume occupied by the liquid phase to that of the vapour phase is

$$\frac{V'_L}{V'_V} = \frac{\rho'_V}{\rho'_L} \cdot \frac{m'_L}{m'_V} = \frac{\rho'_V}{\rho'_L} \frac{\hat{g}}{1 - \hat{g}},$$

where $\hat{g} = \frac{m'_L}{m'}$ is the mass fraction of the condensed phase, V' the volume and ρ' the density of the respective phases. As $\rho'_V/\rho'_L \ll 1$ the

volume occupied by the liquid can be neglected. Further defining

a density ρ_{*L}' where the condensed mass is referred to the same volume as the vapour then

$$\rho' = \rho_v' + \rho_{*L}', \quad 2.2$$

and using this density ρ' the continuity equation becomes

$$\rho' u' A' = m', \quad 2.3$$

where u' is the velocity and A' the cross-sectional area.

The equation of state is

$$p' = \rho_v' \frac{R}{\mu_v'} T', \quad 2.4$$

with μ_v' the molecular weight of the vapour and where p' is the vapour pressure. Writing ρ_v' in terms of ρ' and $\hat{g} = m_L'/m'$, the mass fraction of the condensed phase, then,

$$p' = \rho' (1 - \hat{g}) \frac{R}{\mu_v'} T'. \quad 2.5$$

If it is assumed that the condensate moves with the stream velocity (see Appendix I), the momentum equation is

$$-\rho' u' du' = dp'. \quad 2.6$$

The energy equation can be derived from the first law of thermodynamics which applied to a unit mass of the mixture is

$$de' + p' d \left\{ \frac{1}{\rho'} \right\} = 0, \quad 2.7$$

where e' is the specific internal energy.

Rewriting this in terms of specific enthalpy h' and using equation 2.6.

$$d \left\{ h' + \frac{u'^2}{2} \right\} = 0. \quad 2.8$$

Applying this to the mass m' of the mixture and writing out in terms of the enthalpy of the separate components gives

$$d \left\{ \frac{u'^2}{2} + \frac{m'_v h'_v + m'_l h'_l}{m'} \right\} = 0. \quad 2.9$$

In general the liquid and vapour are at different temperatures. When the two phases are in thermodynamic equilibrium the ~~enthalpy~~ difference of the two phases is equal to the latent heat of vaporization L' . However when the temperatures are different there is an additional term for the liquid phase. thus,

$$h'_l = h'_v - L' - C'_L (T'_v - T'_l), \quad 2.10$$

where C'_L is the specific heat of the liquid.

The ratio of the last term to h'_l is given by

$$\frac{C'_L (T'_v - T'_l)}{h'_l} = \frac{T'_v - T'_l}{T'_l},$$

and if the vapour is assumed to be at the same temperature as the drop then the last term is zero (see Wegener and Mack (1958)).

The energy equation then becomes

$$C'_{pv} T' + \frac{u'^2}{2} - g L' = C'_{pv} T'_0, \quad 2.11$$

where T'_0 is the stagnation temperature.

If the vapour is assumed to be a perfect gas then

$$C'_{pv} = \frac{\gamma}{\gamma - 1} \frac{R}{M_v}, \quad 2.12$$

$$\text{and} \quad \gamma = \frac{C'_{pv}}{C'_{vv}}, \quad 2.13$$

is the specific heat ratio of the vapour without any condensate present.

Equations 2.3, 2.5, 2.6, and 2.11, are the basic equations of motion describing the flow of a condensing vapour. Similar equations can be derived (Wegener and Mack (1958)) for a mixture of an inert gas and a condensing vapour, for example water vapour in air. The only differences occur in C_p' and the molecular weight ν' and thus an identical analysis could be carried out for water vapour condensation. (In the present analysis C_p' and ν' become constant at their frozen values for both of the limits under consideration.)

It is now useful to define the concept of "pseudo entropy", which will be used in the near frozen analysis.

From equation 2.7

$$de' + p' d\left\{\frac{1}{\rho'}\right\} = 0, \quad 2.14$$

which can be rewritten as

$$de'_\alpha + p' d\left\{\frac{1}{\rho'}\right\} = -e'_g d\hat{g}, \quad 2.15$$

where

$$de'_\alpha = e'_p dp' + e'_T dT', \quad 2.16$$

$$\text{and } e' = e'(\rho', T, \hat{g}).$$

Then with respect to the frozen composition it is possible to define a "pseudo entropy"

$$T' dS'_\alpha = de'_\alpha + p' d\left\{\frac{1}{\rho'}\right\} = -e'_g d\hat{g} \quad 2.17$$

where S'_α is the "pseudo entropy".

For the ideal vibrationally relaxing gas this corresponds to entropy production in Johannesen's (1961) α -gas. Here, however,

it is one of several possible definitions depending on the thermodynamic variable used.

Re-arranging 2.17 with 2.16, gives

$$\frac{1}{\rho'} \frac{d\rho'}{dx'} - \frac{e' + T'}{\{\rho' - \rho' e'\}} \frac{1}{T'} \frac{dT'}{dx'} = \frac{\hat{g} T'}{\{\rho' - \rho' e'\}} \frac{1}{T'} \frac{d\hat{g}}{dx'}, \quad 2.18$$

For a condensing vapour

$$h' = e' + p'/\rho' = C_p' T' - \hat{g} L'$$

and thus

$$e' = C_v' T' + \hat{g} R T' - L' \hat{g},$$

where $R = R_0/\mu'v$,

and 2.18 becomes

$$\frac{1}{\rho'} \frac{d\rho'}{dx'} - \frac{C_v' + R\hat{g}}{R(1-\hat{g})} \frac{1}{T'} \frac{dT'}{dx'} = \frac{R T' - L'}{R(1-\hat{g}) T'} \frac{d\hat{g}}{dx'}, \quad 2.19$$

which is independent of the variables chosen in the definition of "pseudo entropy". This equation will hereafter be called the "pseudo entropy" equation. It should be noted that this definition is the same as that used in Blythe et al (1968) for a de-excitation shock.

2.2 The rate equations

For an impure vapour the rate equation is

$$\hat{g} = \frac{N G'(x')}{m'} \quad 2.20a$$

where N is the number of nuclei present and $G'(x)$ is the mass of each droplet at some station x .

For pure vapour the corresponding expression is

$$\hat{g} = \frac{1}{m'} \int_0^x G'(x, \xi) J'(\xi) A'(\xi) d\xi, \quad 2.20b$$

where $G'(x, \xi)$ is the mass at x of a droplet formed at ξ and $J'(\xi)$ is the rate of formation of droplets of critical radius r_x^* . $A'(\xi)$ is the area ratio at ξ .

In these rate equations there are two main terms, the rate of formation of drops $J'(\xi)$ and the subsequent growth of these drops $G'(x, \xi)$.

For the impure vapour it will be assumed that $J'(\xi)$ is zero, and that the vapour condenses solely on the nuclei already present, due to impurities.

In the case of the pure vapour, however, there are no impurities present and an expression for $J'(\xi)$ must be obtained.

Several authors have considered this problem among them Volmer and Weber (1926), Farkas (1929), Becker and Döring (1935), Volmer (1939) Frenkel (1946) and Stever (1958).

The expressions they derive from kinetic theory are all similar in form and that given by Wegener and Mack* (1958) is

$$J' = \left\{ \frac{2}{\pi} \right\}^{1/2} \frac{N_A^{3/2}}{R^2} \left(\frac{c' p'}{p_L} \right)^{1/2} \left\{ \frac{p'}{T'} \right\}^2 \exp \left[-\frac{16\pi}{3} \frac{N_A}{R^3} \left\{ \frac{c'^3 p'}{p_L^2} \right\}^{1/2} \frac{1}{T'^3 m^2 \phi} \right], \dots$$

* Equation 5.30 p 418 in Wegener and Mack 2.20b.
should read as above

where N_A is Avogadro's number, σ' is the surface tension and ϕ is the saturation ratio which is defined as

$$\phi = \frac{p'}{p'_s(T)}$$

with $\log_e p'_s = -L'\nu'/RT' + \bar{C}'$, being the equation of the co-existence line.

$$r'_x = \frac{2\sigma'\nu'}{\rho'_L R T' \log_e \phi} \quad 2.21b$$

Assuming that ρ'_L and σ' are constant the expression above reduces to

$$J' = K' \left\{ \frac{p'}{T'} \right\}^2 \exp \frac{\beta'}{T'^3 \log_e \phi} \quad 2.22$$

where

$$K' = \left\{ \frac{2}{\pi} \right\} \frac{N_A^{3/2} (\sigma'\nu')^{1/2}}{R^2 \rho'^L}$$

and

$$\beta' = \frac{16\pi}{3} \frac{N_A}{R^3} \left\{ \frac{\sigma'^3 \nu'^2}{\rho'^L{}^2} \right\}$$

Once the drops are formed their rate of growth $G'(\omega')$ must be computed.

Only drops in the free molecular regime will be considered, and noting that

$$G' = \rho'_L \cdot \frac{4}{3} \pi \hat{v}'^3,$$

where $\hat{v}'$ is the drop radius, then, considering mass impingement terms only the expression due to Wegener and Mack (1958) is obtained

$$\frac{d\hat{v}'}{d\omega'} = \alpha \frac{\rho'_L}{\rho'_L} \left\{ \frac{R}{2\pi} \right\}^{1/2} \frac{1}{\omega'} \quad 2.23a$$

where ρ'_L is the vapour density.

This expression does not contain any non-equilibrium terms. If, however, kinetic considerations are added to the problem then the following expressions may be obtained

Buhler, 1952,

$$\frac{d\hat{v}'}{dx'} = \frac{C_p}{L'} \frac{p_v'}{p_L'} \left\{ \frac{R}{2\pi} \right\}^{\frac{1}{2}} \frac{T'}{u'} \frac{(T_s' - T')}{1 - \frac{2\epsilon}{\sqrt{p_L'} L'}}, \quad 2.23b$$

or Frenkel, 1946,

$$\frac{d\hat{v}'}{dx'} = \frac{\alpha}{L'} \frac{p'}{p_L'} \left\{ \frac{2}{\pi} \right\}^{\frac{1}{2}} \left\{ \frac{R}{T'} \right\}^{\frac{1}{2}} \frac{T_s' - T'}{u'}, \quad 2.23c$$

where T_s' is the surface temperature of the drop, and is equal to the equilibrium temperature with respect to the local pressure.

If p_L' and ϵ' are both assumed constant these expressions both adopt the form

$$\frac{d\hat{v}'}{dx'} = \frac{D' p' T'^{\frac{1}{2}}}{u'} (T_s' - T'), \quad 2.24$$

(provided that $\hat{g}$ is small i.e. $p' \approx p' R T'$, $p' \approx p_v'$)

The rate equation for an impure vapour now becomes

$$\hat{g} = \frac{N \bar{D}'}{m'} \left[\int_{x_c'}^{x'} \frac{p' T'^{\frac{1}{2}}}{u'} [T_s' - T'] ds' \right], \quad 2.25a.$$

where the radius of the impurity has been assumed to be zero.

That for a pure vapour is :-

$$\hat{g} = \frac{K' \bar{D}'}{m'} \int_{x'_c}^{x'} \left\{ \frac{p'}{T'} \right\}^2 \exp \left[\frac{-p'}{T'^{3/2} \phi} \right] \left[\int_{T'}^{x'} \frac{e'^{T'/2} (T'_S - T') ds'}{u'} + \frac{A'_*}{T' \log \phi} \right]^3 A'(\xi) d\xi, \quad 2.25b$$

where $\bar{D}' = \frac{4}{3} \pi \rho_L' D'^3$, and $A'_* = \frac{2 \sigma' \rho_L'}{\rho_L' R D'}$,
and x'_c is the point at which condensation commences.

2.3 Non dimensionalization of equations.

If the equations given in the last two sections are non dimensionalised as follows

$$\left. \begin{aligned} \rho &= \frac{\rho'}{\rho_{\infty}'}, \quad T = \frac{T'}{T_{\infty}'}, \quad p' = \frac{p'}{\rho_{\infty}'}, \quad u = \frac{u'}{\sqrt{RT_{\infty}'}} \\ m' &= \frac{m'}{\rho_{\infty}' \sqrt{RT_{\infty}' A_T}}, \quad A = \frac{A'}{A_T}, \quad x = \frac{x'}{h_T}, \quad L = \frac{L'}{RT_{\infty}'} \end{aligned} \right\} \quad 2.26$$

where h_T is the throat height.

The basic equations of motion are

$$\rho u A = m \quad 2.27$$

$$- \rho u du = dp \quad 2.28a$$

$$\frac{\gamma}{\gamma-1} T + \frac{u^2}{2} - L \hat{g} = \frac{\gamma}{\gamma-1} \quad 2.29$$

$$p = \rho (1 - \hat{g}) T \quad 2.30$$

and it is sometimes convenient to replace the momentum equation by

$$\frac{1}{\rho} \frac{d\rho}{dx} - \frac{(\gamma-1) + \hat{g}}{(1-\hat{g})T} \frac{dT}{dx} = \frac{T-L}{(1-\hat{g})T} \frac{d\hat{g}}{dx}, \quad 2.28b$$

and the rate equations are

1) For an impure vapour, for $x \geq x_c$ $T \leq T_S$,

$$\hat{g} = A_i \left[\int_{x_c}^x \frac{\rho T'^{1/2} (T'_S - T') ds}{u} \right]^3 \quad 2.31a$$

where $\mathcal{L}_i = N \bar{D}' \rho_\infty'^2 T_\infty'^{5/2} h_t'^3 / A_t' R^2,$

and x_c is the condensation point.

ii) For a pure vapour

$$\hat{g} = \frac{\mathcal{L}_p}{m} \int_{x_c}^x \left\{ \frac{p}{1} \right\}^2 \exp \left\{ \frac{-B}{T^{3/2} \log_e \phi} \right\} \left[\int_{\eta}^x \frac{e^{-1/2} (T_s - T) d\zeta}{u} + \frac{\mathcal{L}_v}{T \log_e \phi} \right] A(\eta) d\eta, \quad 2.31b$$

where $\mathcal{L}_p = K' \bar{D}' \rho_\infty'^4 T_\infty'^{5/2} h_t'^4$, $\mathcal{L}_v = A_t' R^{1/2} / \rho_\infty' T_\infty'^2 h_t'$,

and $\beta = \beta' / T_\infty'^3$, $\phi = \frac{p}{p_s} = p/p_s$, with $p_s = C e^{-L/T}$
and $C = \exp \bar{C}' / p_s$

2.4 Condensation Point

Initially the flow in the nozzle is condensation free until the co-existence line is crossed, and the point at which this line is crossed will be defined as x_c

Prior to x_c , $\hat{g} \equiv 0$, this is obvious from the rate equations since even if drops were formed in the flow $T > T_g$ so that their rate of growth would be negative. Thus upstream of x_c the equations of motion are those for the isentropic flow of a perfect gas where p, ρ, T and u are given by the usual relations.

At the condensation point x_c , $p = p_s$

thus $p_c = C e^{-L/T_c}$

giving $\frac{p_c}{T_c} = C e^{-L/T_c}$

or $\gamma/\gamma-1 \log_e T_c = \log_e C - L/T_c \quad 2.32$

which can be re-written in terms of the mach number M_c at the point x_c

$$\frac{\gamma}{\gamma-1} \log_e \left[1 + \frac{\gamma-1}{2} M_c^2 \right] + \log_e L - L \left[1 + \frac{\gamma-1}{2} M_c^2 \right] = 0, \quad \dots 2.33$$

thus the three quantities M_c , A_c and x_c can be found.

It will be assumed for simplicity that $M_c > 1$ so that x_c is downstream of any geometric throat.

$$\text{Hence } M = \gamma \left\{ \frac{2}{\gamma+1} \right\}^{\frac{\gamma+1}{2(\gamma-1)}}.$$

3. The flow regimes and their limits

3.1 The scaled equations

The non dimensionalized equations are given in β 2.3 equations 2.27 - 2.3 lb. It is however more useful to scale the equations with respect to conditions at the condensation point i.e.

$$\left. \begin{aligned} \psi &= \frac{p}{p_c}, \quad \pi = \frac{\rho}{\rho_c}, \quad \theta = \frac{T}{T_c}, \quad z = \frac{x}{x_c}, \quad g = H\hat{g}, \\ V &= \frac{u}{\sqrt{T_c}}, \quad \bar{A} = \frac{A}{A_c}, \quad H = \frac{L}{T_c}, \quad \text{and } \bar{m} = \frac{\dot{m}}{A_c \rho_c T_c} \end{aligned} \right\} 3.1.$$

where the subscript c refers to the point x_c

The following set of equations are then obtained :-

$$\pi \cdot V \bar{A} = \bar{m}, \quad 3.2$$

$$\pi \cdot V \cdot dV = -d\psi, \quad 3.3a$$

$$\frac{\gamma}{\gamma-1} \Theta + \frac{V^2}{2} - g = \frac{V_0^2(\infty)}{2}, \quad 3.4$$

where $V_0^2(\infty) = \frac{2\gamma}{\gamma-1} + V_c^2$,

$$\psi = \pi \left\{ 1 - \frac{g}{H} \right\} \Theta, \quad 3.5$$

and the "pseudo entropy" equation, which will be used instead of the momentum equation in the near frozen limit, can be re-written as

$$\frac{1}{\pi} \frac{d\pi}{dz} - \frac{1}{(\gamma-1)\Theta} \frac{d\Theta}{dz} = -\frac{1}{\Theta} \frac{dg}{dz} + \frac{1}{H} \left\{ \frac{g}{\Theta} \frac{d\Theta}{dz} + \frac{g}{\pi} \frac{d\pi}{dz} - \frac{dg}{dz} \right\}. \quad 3.3b.$$

The rate equations now become

a) for an impure vapour

$$g = \lambda_i \left[\int_1^2 \frac{\pi \Theta^{1/2} (\Theta_s - \Theta) ds}{V} \right]^3, \quad 3.6a$$

where $\lambda_i = \frac{H \Lambda_i [\rho_c x_c]^3}{m}$,

b) for a pure vapour

$$g = \lambda_p \int_1^2 \left[\frac{\psi}{\Theta} \right]^2 \exp \left[\frac{-B}{\Theta^3 \log \phi} \right] \left[\int_{\xi}^2 \frac{\pi \Theta^{1/2} (\Theta_s - \Theta) (s + \lambda_v)}{V \Theta \log \phi} \right]^3 \bar{A}(\xi) d\xi, \quad 3.6b.$$

where

$$\lambda_p = \frac{H \Lambda_p [\rho_c^5 T_c^3 A_c x_c^4]}{m},$$

$$\lambda_v = \frac{\Lambda_v}{\rho_c T_c^2 x_c},$$

and

$$B = \frac{B}{T_c^3}, \quad \text{with } \Theta_s = [1 - H^{-1} \log \psi]^{-1},$$

$$\text{and } \log \phi = H \left[\frac{1}{\Theta} - 1 \right] + \log \psi.$$

In these equations the variables π, θ and ψ are $O(1)$ and $V=O(V_c)$. Thus using these equations the various flow regimes and their limits can be considered.

It will be assumed from now on that

$$H \gg |\log_e \lambda|.$$

This is not in practice a very restrictive assumption as typical values of H are, for nitrogen, between 15 and 25.

3.2 Regimes for the impure vapour

For an impure vapour, writing $V = V_R \bar{V}$ where $\bar{V}$ is $O(1)$ and V_R is a reference velocity, ($V_R = V_c$ in the near frozen limit), and $V_R = V_{eq}$ in the near equilibrium limit), then the rate equation becomes

$$g = \bar{\lambda}_i \left[\int_1^z \frac{\pi \theta^{1/2} (\theta_s - \theta) ds}{\bar{V}} \right]^3, \quad 3.7$$

where $\pi, \theta, \bar{V}$ and ψ are now $O(1)$ and

$$\bar{\lambda}_i = \frac{\lambda_i}{V_R^3}, \quad \text{is an effective rate parameter}$$

based on local flow conditions.

This effective rate parameter defines the various flow regimes and their limits.

When $\bar{\lambda}_i$ is large the flow will be in the near equilibrium regime. In this limit the flow will initially follow the frozen adiabatic isentrope, but then rapidly return to a near equilibrium

flow, in which the flow conditions are given to zeroth order by the equilibrium flow solution. This solution is considered more fully in § 4.

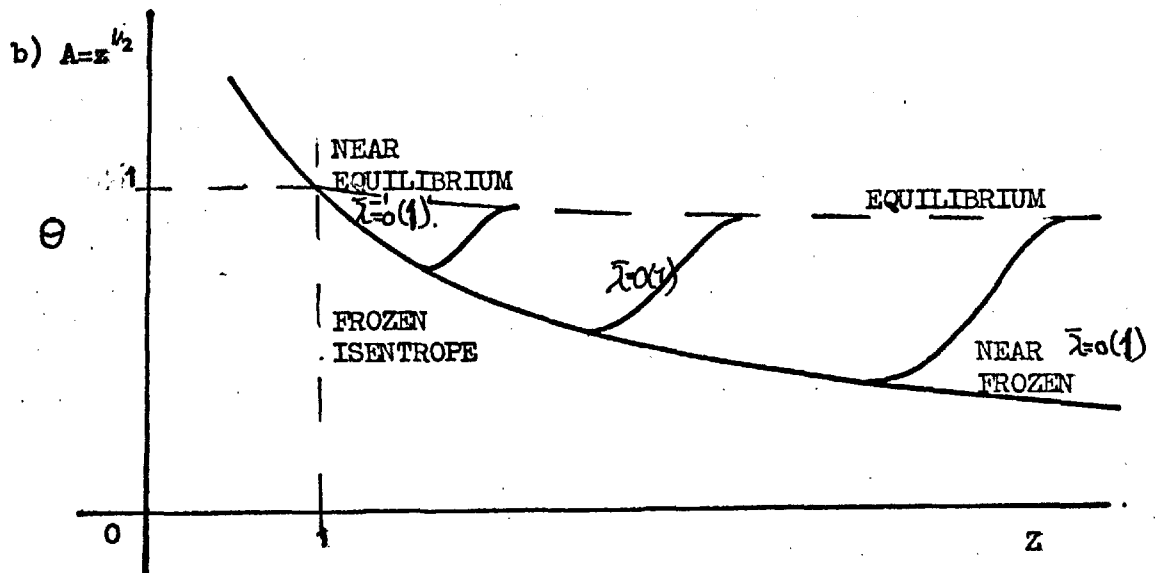
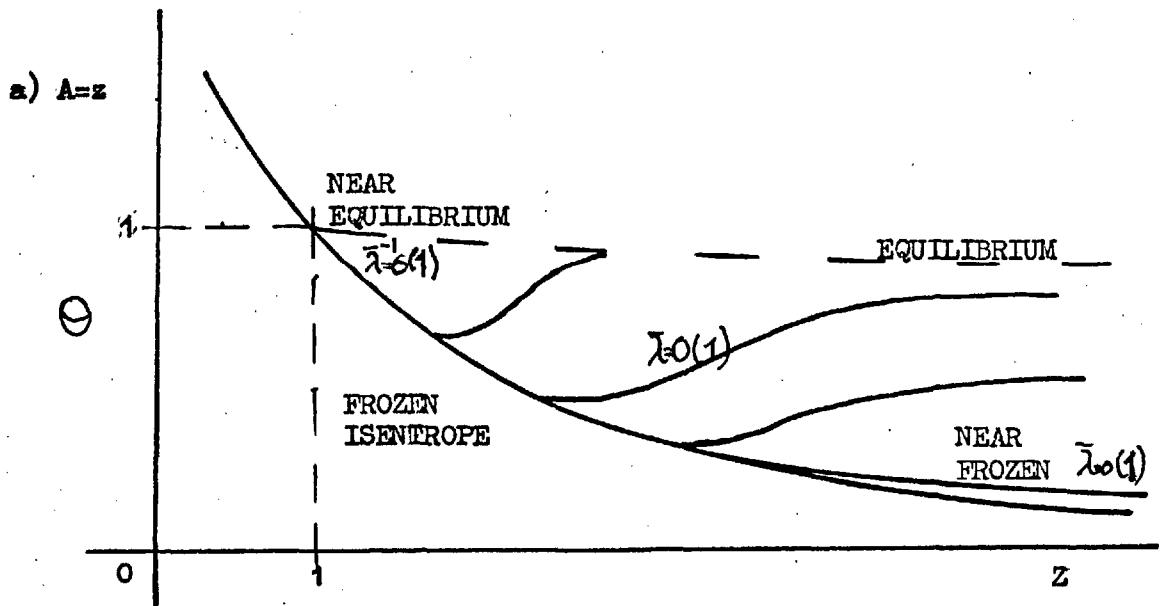
If however $\bar{\lambda}_i$ is small a near frozen solution results. For this regime the flow remains in a near frozen state until far downstream. Then depending on the convergence of a "pseudo entropy" integral, it may in a similar manner to that found by Blythe (1967) for de-excitation shocks, either return to equilibrium through a condensation shock, move towards equilibrium in a gradual condensation zone or remain in a modified near frozen state. These solutions are discussed in more detail in § 5.

For the case $\bar{\lambda}_i \rightarrow 0(1)$, no simplification in the equations is possible and they can only be solved by numerical integration of the full set. This flow regime will, however, show many of the facets of both near equilibrium and near frozen flow. These solutions are indicated qualitatively in figure 2 for a wedge shaped and a parabolic nozzle. For both nozzles the near equilibrium solution indicates a rapid return towards equilibrium, and the near frozen shows that for the wedge the temperature tends to some constant value, and for the parabolic, that the flow returns to equilibrium through a condensation shock (see § 5.).

3.3 Regimes for the pure vapour

For a pure vapour the problem of defining the regimes is more complex, as now instead of having one rate process there are two interacting rate processes, that of the formation of nuclei character-

FIGURE 2.



TYPICAL FLOW PATTERNS.

ised by a rate parameter λ_N and that of the subsequent growth of the drops, which it will be seen is associated with the rate parameter $\bar{\lambda}_p$.

Thus if large numbers of nuclei are formed and the growth rate is large the flow would be in a near equilibrium regime, but for small nucleation and growth rates near frozen flow will prevail. Similarly if both rates are $O(1)$ the flow will exhibit characteristics of both regimes.

If however, one rate is large and the other small the situation becomes more complex, and whether the flow is near equilibrium, near frozen or neither depends on the relative rates of the two processes involved. For example if the rate of drop production is small then the growth $\bar{\lambda}_p$ will have to be very large (i.e. $\bar{\lambda}_p \rightarrow \infty$) to produce a near equilibrium flow, if it is large (i.e. $\bar{\lambda}_p = O(\lambda_N^{-1})$) then the overall rate will be $O(1)$ and if it is $O(1)$ or smaller then the flow will be near frozen.

In order to define more precisely the parameters λ_N and $\bar{\lambda}_p$ the rate equation must be examined in detail. Writing $V = V_R \bar{V}$ as in § 3.2 gives

$$g = \bar{\lambda}_p \left[\int_1^z \pi^2 \exp \left\{ \frac{-K^2}{\theta(1-\theta)^2} \right\} \int_{\frac{1}{\theta}}^z \frac{\pi \theta^{1/2} (1-\theta) ds}{V} + \frac{\bar{\lambda}_v}{H(1-\theta)} \right] \bar{A}(\xi) d\xi, \quad 3.8$$

where it has been assumed that, $H \gg |\log \lambda|$ and

where $\bar{\lambda}_p = \frac{\lambda_p}{V_R^3}$, $\bar{\lambda}_v = \lambda_v V_R$, and $K^2 = \frac{B}{H^2}$

Examination of this equation shows that the parameter $\bar{\lambda}_p$ can be identified with the rate of growth of the drops, and the parameter K controls the rate of production of the nuclei. To evaluate this effect the integral defining the total number, N , of drops formed must be considered, where

$$N = \int_1^{\infty} \pi^2 \exp \left\{ \frac{-K^2}{O(1-O)^2} \right\} \bar{A}(\eta) d\eta, \quad 3.9$$

and thus it can be easily seen that $\lambda_N \propto N$. A total effective rate parameter can now be defined as

$$\lambda_T = \bar{\lambda}_p \cdot N. \quad 3.10$$

If K is large, then the integral defining N will in general be of the steepest descents type and thus

$$N = O(K^{-1}), \quad 3.11$$

$$\text{and } \lambda_T \text{ is } O[\bar{\lambda}_p K^{-1}]. \quad 3.12$$

If K is $O(1)$ then N is $O(1)$ and λ_T is $O(\bar{\lambda}_p)$

For K small, however, the evaluation of the integral becomes more complex. Considering the near frozen case (this will in general produce the maximum N for any nozzle $(\bar{A} = z^v)$).

$$\text{then } N = \int_1^{\infty} O^{\frac{3}{2}-1} \exp \left\{ \frac{-K^2}{O(1-O)^2} \right\} \eta^v d\eta. \quad 3.13$$

This integral can be split into three regions

$$\left. \begin{array}{l} 1) \quad 1-K < O < 1, \\ 2) \quad K^2 < O < 1-K, \\ 3) \quad 0 < O < K^2. \end{array} \right\} 3.14.$$

It can be shown that in region 1

$$N \propto K, \quad 3.15a$$

and that in regions 2 and 3

$$N \propto K^{-\frac{2(1-\nu)}{\nu(\gamma-1)}}, \quad 3.15b$$

thus it can easily be seen that N is $O[K^{-\frac{2(1-\nu)}{\nu(\gamma-1)}}]$

and therefore Λ_T is $O[\lambda_p K^{-\frac{2(1-\nu)}{\nu(\gamma-1)}}]$; 3.16

These results can now be summarised as follows :

For $\Lambda_T \gg 1$ the flow will behave in a near equilibrium manner (§4)
and for $\Lambda_T \propto 1$ the flow will contain characteristics of both solutions.
These solutions will adopt the same form as for the impure vapour and
are shown qualitatively by figure 2.

For $\Lambda_T \ll 1$ the flow will be a near frozen flow, there are
however, two limits in this regime.

The first of these is that $\bar{\lambda}_{p=0}(1), K^2 \gg \lambda_p^b$ and in
this case the drop production is completed prior to the breakdown of
the perturbation solution. The solution then adopts the same form
as for the impure vapour.

The second limit is when $\bar{\lambda}_{p=0}(1)$ and $K^2 \propto (\bar{\lambda}_p^b)$
and in this case the exponential term can be replaced by unity (except
in a region $O(K)$ near $\Theta=1$) for the perturbation solution. This means
that the perturbation solution breaks down when drop production is still
important. These two cases are considered in §6.

Before, however, considering the flow solutions the parameter
 $\bar{\lambda}_p$ controlling the initial size of the drops must be discussed.

This parameter will, in this analysis, be taken as $\cdot O(1)$ or less. The term contributed by the initial radius $\hat{v}_*$, will then be $O(H^{-1})$ and can be neglected to zeroth order in the following analysis.

This means that the initial radius is small compared with the final radius, and this has been demonstrated in the numerical calculations of Griffin and Sherman (1965), who showed that the radius increased by a factor of approximately 20 in the condensation region.

4. Near equilibrium solutions.

4.1 Scaling in the near equilibrium regime.

In the near equilibrium limit the flow initially follows the frozen isentrope, but rapidly returns to a near equilibrium solution. For a nozzle such that $A = Cx^v (\bar{A} = z^v)$ downstream of the condensation point, and assuming that $H \gg 1/\alpha_2^2$, an appropriate set of variables in this regime is,

$$\left. \begin{aligned} \pi &= 1 - \epsilon_1, \\ \theta &= 1 - t, \\ V &= V_c (1 + v), \\ g &= g, \\ Z &= 1 + n, \end{aligned} \right\} \quad 4.1$$

Substituting these into the equations of motion (eqns. 3.2-3.6) and dropping squares and products of small terms, gives

$$-\epsilon_1 + V = -v n, \quad 4.2$$

$$-\epsilon_1 + \frac{t}{\gamma-1} = -g, \quad 4.3$$

$$-\frac{\gamma}{\gamma-1} t + V_c^2 v - g = 0, \quad 4.4$$

which when solved give

$$V = -v n + g + \frac{t}{\gamma-1}, \quad 4.5$$

$$\text{and } t = \frac{v(\gamma-1)n}{[1 - \gamma/V_c^2]} - (\gamma-1)g \frac{[1 - \gamma/V_c^2]}{[1 - \gamma/V_c^2]}. \quad 4.6a$$

$$\text{But } V_c^2 = \frac{u_c^2}{T_c}, \quad \text{i.e. } V_c^2 = \gamma M_c^2,$$

$$\text{and then } t = v(\gamma-1)m_1 n - (\gamma-1)m_2 g, \quad 4.6b$$

$$\left. \begin{aligned} \text{where } m_1 &= \frac{M_c^2}{M_c^2 - 1}, \\ m_2 &= \frac{[8M_c^2 - 1]}{8[M_c^2 - 1]}, \end{aligned} \right\} \quad 4.6c$$

The rate equations now become

for the impure vapour

$$g = \frac{\lambda_i}{V_c^3} \left[\int_0^\eta t(\eta) d\eta \right]^3, \quad 4.7a$$

and for the pure vapour

$$g = \frac{\lambda_p}{V_c^3} \int_0^\eta \exp \left\{ \frac{-K^2}{t(\eta)^2} \right\} \left[\int_\eta^\eta t(s) ds \right]^3 d\eta, \quad 4.7b$$

where $K^2 = B/H^2$.

These two cases will now be considered separately.

4.2 The impure vapour solution.

Equation 4.7a can be rewritten by putting $g = v^3$,

$$\text{then } v = \frac{\lambda_i^{1/3}}{V_c} \int_0^\eta t(\eta) d\eta, \quad 4.8a$$

and differentiating gives

$$\frac{dv}{d\eta} = \frac{\lambda_i^{1/3}}{V_c} [\bar{\alpha}\eta - \bar{\beta}v^3], \quad 4.8b$$

$$\begin{aligned} \text{where } \bar{\alpha} &= v(\gamma-1)m_1, \\ \bar{\beta} &= (\gamma-1)m_2. \end{aligned}$$

Then changing variables to N and R

$$\begin{aligned} \text{where } \eta &= \left\{ \frac{\lambda_i}{V_c^3} \right\}^{1/5} \bar{\alpha}^{-3/5} \bar{\beta}^{-1/5} N, \\ v &= \left\{ \frac{\lambda_i}{V_c^3} \right\}^{1/5} \bar{\alpha}^{1/5} \bar{\beta}^{2/5} R, \end{aligned}$$

$$\text{gives } \frac{dR}{dN} = N - R^3, \quad 4.9$$

with b.c. $R=0, N=0$.

This first order non linear differential equation is a form of Abel's equation (Davis (1960)), and has been solved numerically, using a Runge Kutta technique.

These results are shown in figure 3 and from this it can be seen that although g is initially frozen, it rapidly returns to a near equilibrium state.

$$\text{i.e. } g \sim \sqrt{m_1/m_2} \cdot n.$$

That this matches with the near equilibrium solution is easily shown as (Buhler (1952))

$$g_e = \frac{\gamma}{\gamma-1} \Theta (\log \Theta + H(1-\Theta)), \quad 4.10$$

$$\text{i.e. } g_e \sim H t_e \text{ in these variables.}$$

Hence using equations 4.2 to 4.4

$$t_e = \frac{\sqrt{(\gamma-1)m_1}}{1 + (\gamma-1)m_2 H}, \quad 4.11$$

and thus for H large

$$\left. \begin{aligned} t_e &\sim \frac{\sqrt{m_1}}{m_2 H} n, \\ \text{and } g_e &\sim \sqrt{\frac{m_1}{m_2}} n. \end{aligned} \right\} \quad 4.12$$

4.3 The pure vapour solution

For the pure, the rate equation becomes

$$g(n) = \frac{2p}{V_c^{2/3}} \int_0^n \exp \left\{ \frac{-k^2}{t(\eta)^2} \right\} \left[\int_\eta^n t(s) ds \right]^3 d\eta, \quad 4.7b$$

and scaling the variables

$$\begin{aligned} n &= \left\{ \frac{2p}{V_c^{2/3}} \right\}^{-1/6} \bar{\alpha}^{2/3} \bar{\beta}^{-1/6} N, \\ g &= \left\{ \frac{2p}{V_c^{2/3}} \right\}^{-1/6} \bar{\alpha}^{2/3} \bar{\beta}^{-1/6} G, \\ t &= \left\{ \frac{2p}{V_c^{2/3}} \right\}^{-1/6} \bar{\alpha}^{2/3} \bar{\beta}^{-1/6} T, \end{aligned}$$

then

$$G(N) = \int_0^N \exp \left\{ \frac{-k^{*2}}{(\eta - G)^2} \right\} \left[\int_{\eta}^N (S - G(s)) ds \right]^3 d\eta, \dots 4.13$$

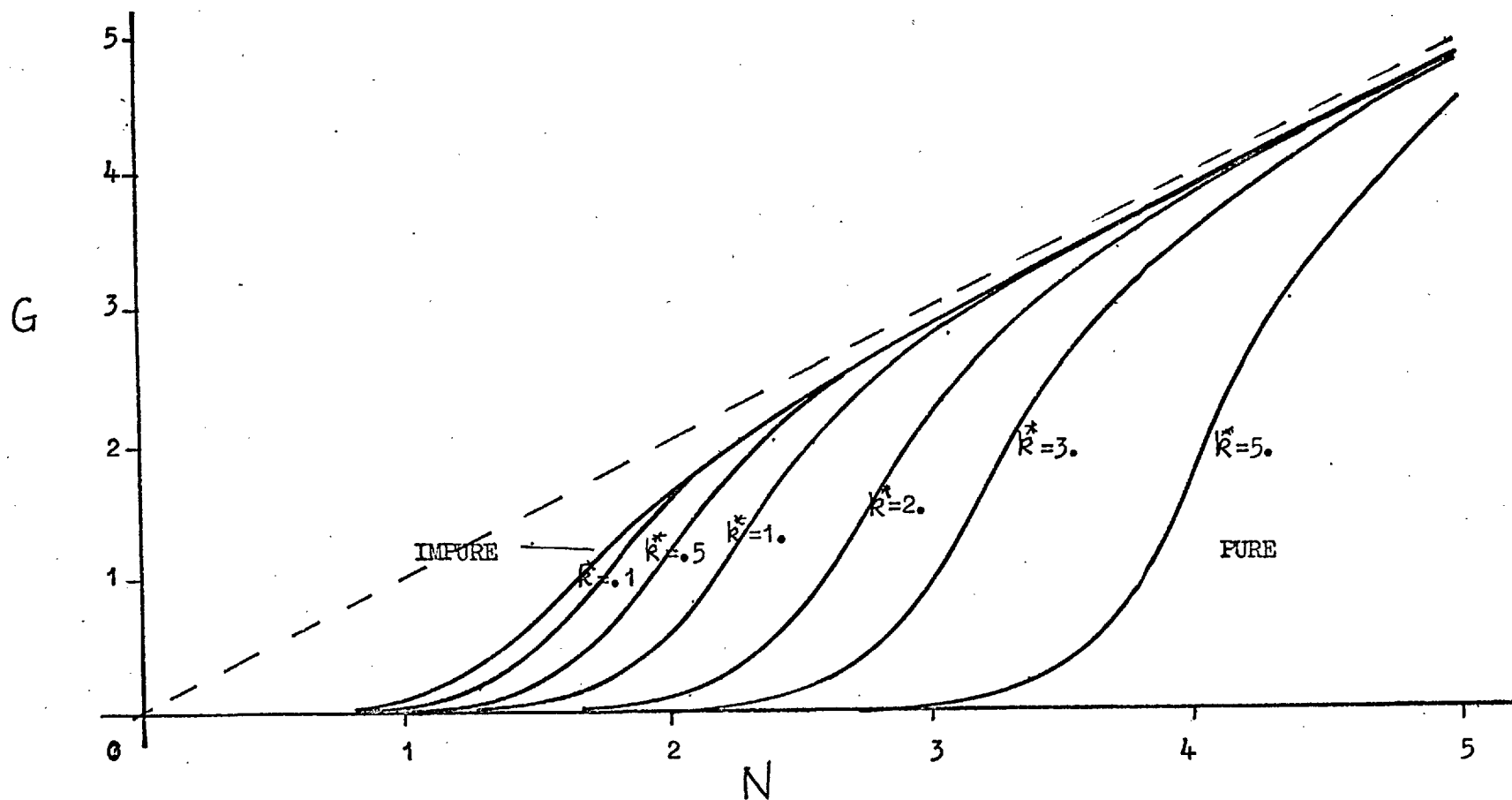
$$\text{where } k^{*2} = \frac{K^2}{\left\{ \frac{\lambda_0}{V_c^3} \right\}^{-1/6} \bar{\alpha}^{-2/3} \bar{\beta}^{-1/6}} \quad 4.14$$

This is a non linear integral equation and has been solved numerically using a predictor corrector technique.

The results for several values of k^* are shown in figures 3a and 3b. From this it can be seen that the behavior^y is similar to that found in the impure case, and that for k^* increasing the flow remains frozen longer. It should be noted that $k^*=0$ does not correspond to the impure vapour case.

The maximum temperature perturbation ΔT , which is a measure of the maximum supersaturation, also increases with k^* , and a plot of ΔT vs k is shown in figure 4. This behaviour was also noted by Griffin and Sherman in their full numerical solutions.

FIGURE 3a.



NEAR EQUILIBRIUM SOLUTION.

FIGURE 3b.

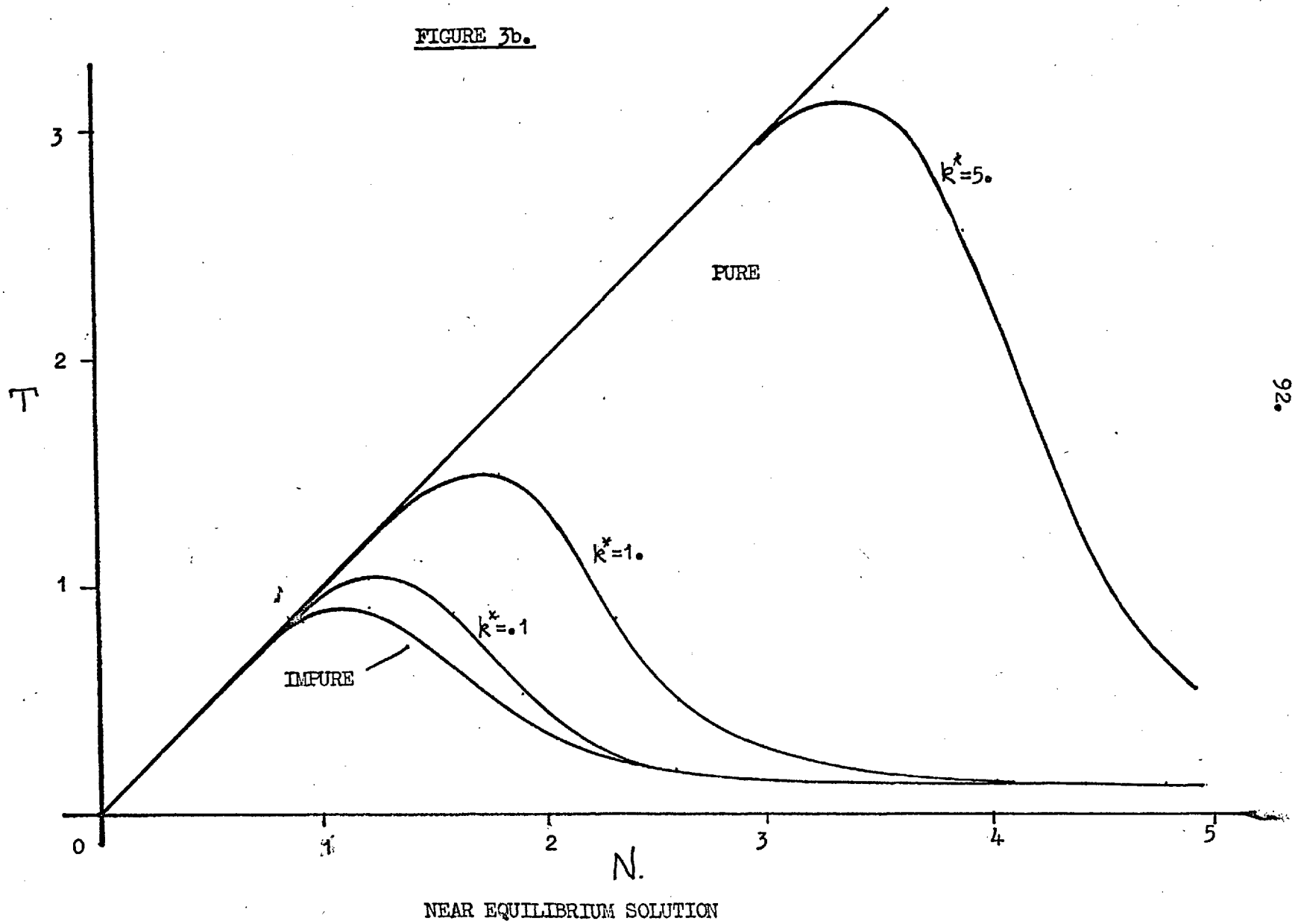
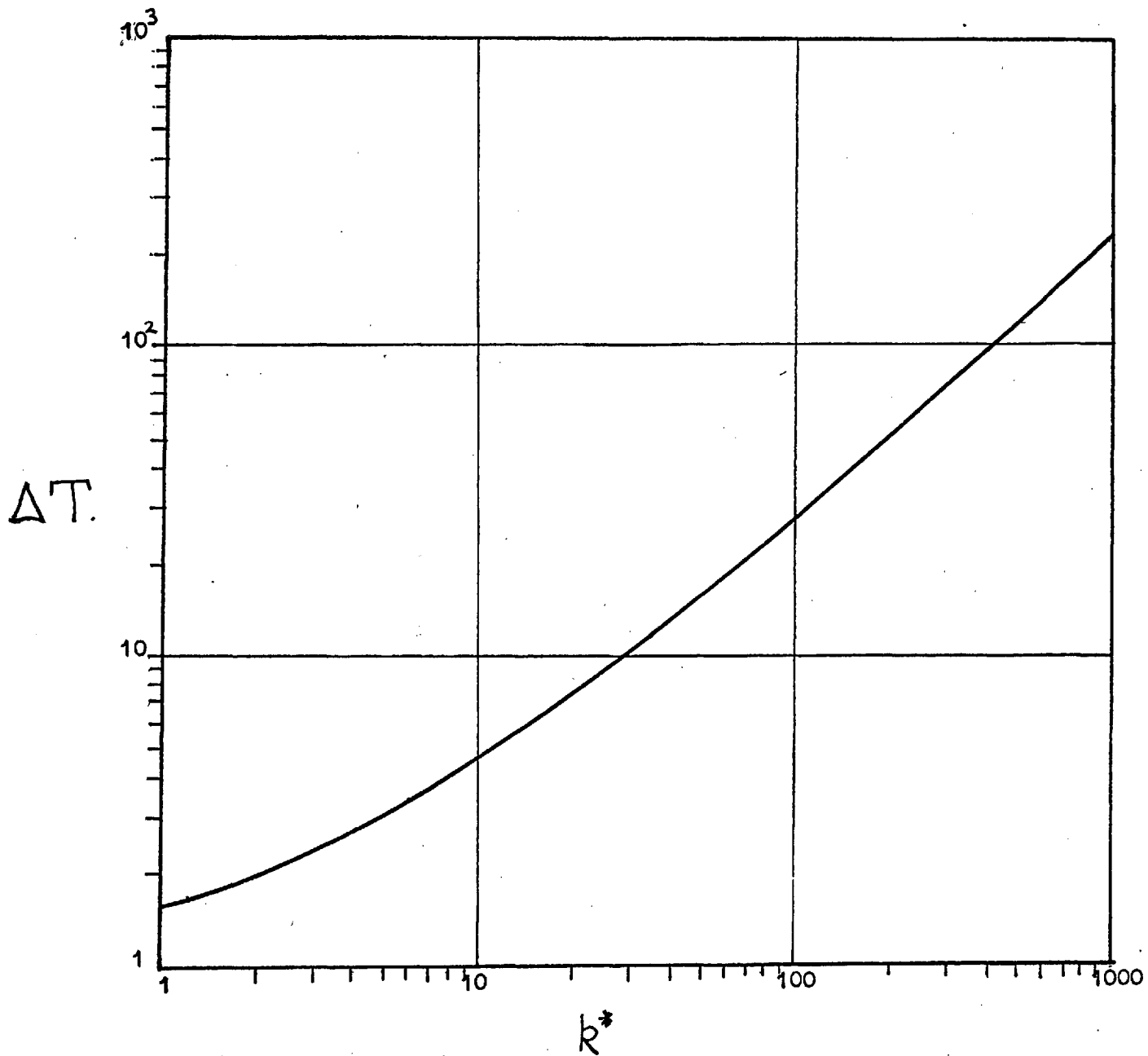


FIGURE 5.

MAXIMUM TEMPERATURE PERTURBATION.

5. Near frozen solution for an impure vapour

5.1 The perturbation solution

In the near frozen limit the flow follows the frozen isentrope initially, and it is possible to assume that a conventional near frozen expansion holds. Then considering a nozzle, such that

$$A = C_x^V [\bar{A} = z^V] \quad \text{downstream of the condensation point, and}$$

assuming that $\lambda_i \ll 1$ and $H \gg |\log \lambda|$ then the appropriate expansions are

$$\begin{aligned} \Theta &= \Theta_0(z) + \lambda_i \Theta_1(z), \\ \pi &= \pi_0(z) + \lambda_i \pi_1(z), \\ \psi &= \psi_0(z) + \lambda_i \psi_1(z), \\ V &= V_0(z) + \lambda_i V_1(z), \\ g &= \lambda_i g_1(z). \end{aligned} \quad 5.1$$

Substituting these into equations 3.2 to 3.6 gives for the zeroth order solution, the fully frozen result.

$$\pi_0 = \Theta_0^{\frac{\gamma}{\gamma-1}}, \quad 5.2$$

$$V_0 = [V_c^2 + \frac{2\gamma}{\gamma-1} (1 - \Theta_0)]^{1/2}, \quad 5.3$$

$$\text{and. } \Theta_0^{\frac{\gamma}{\gamma-1}} [V_c^2 + \frac{2\gamma}{\gamma-1} (1 - \Theta_0)]^{1/2} = \frac{\dot{m}}{z^V}. \quad 5.4$$

The equations for the first order terms now are

$$\frac{\pi_1}{\pi_0} + \frac{V_1}{V_0} = 0, \quad 5.5$$

$$\frac{\gamma}{\gamma-1} \Theta_1 + V_1 V_0 = g_1, \quad 5.6$$

$$\frac{\pi_1}{\pi_0} - \frac{1}{\gamma-1} \frac{\Theta_1}{\Theta_0} = \int_1^z \frac{dg_1}{\Theta_0}, \quad 5.7$$

which when solved give solutions of the type

$$\left\{1 - \frac{\gamma Q_0}{V_0^2}\right\} \frac{Q_1}{Q_0} = (\gamma-1) \int_1^Z \frac{dg_1}{Q_0} - (\gamma-1) \frac{g_1}{V_0^2}, \quad 5.8$$

where $g_1 = v_1^3$

and $v_1 = \int_1^Z \frac{\pi_0 Q_0^{1/2} (1 - Q_0)}{V_0} ds, \quad 5.9$

where $Q_s = 1 + \frac{\log_e Q}{H} + \dots$, ie $Q_{s0} = 1$,
as $H \gg |\log_e \lambda|$.

For large Z these solutions become $\pi_0 = Q_0^{k-1} \sim m Z^{-v}$,

where $m = \frac{\bar{m}}{V_0(\infty)} = \frac{V_c}{V_0(\infty)} = \frac{u_c}{u_0(\infty)}, \quad 5.10$

and $V_0 \sim V_0(\infty), \quad Q_0 \sim m^{\gamma-1} Z^{-v(\gamma-1)}$

and thus if $k > 0$ where $k = 1 - \frac{1}{2}(\gamma+1)$ the integral defining g_1 diverges for large Z and

$$g_1 \sim \frac{m^{3(\gamma+1)/2}}{V_0^3(\infty)} \frac{Z^{3k}}{k^3}, \quad k > 0 \quad 5.11a$$

For $k = 0$

$$g_1 \sim \frac{m^{3(\gamma+1)/2}}{V_0^3(\infty)} [\log_e Z]^3 \quad 5.11b$$

If however $k < 0$ the integral converges and v_1 is given asymptotically as $Z \rightarrow \infty$ by

$$v_1 \sim \left\{ v_{1\infty} + \frac{m^{(\gamma+1)/2}}{V_0(\infty)} \frac{Z^k}{k} \right\}, \quad k < 0,$$

where $v_{1\infty} = \int_1^\infty \frac{\pi_0 Q_0^{1/2} (1 - Q_0)}{V_0} ds, \quad 5.12$

and thus $g_1 \sim v_{1\infty}^3 + \frac{3m^{\gamma+1/2}}{V_0(\infty)} v_{1\infty}^2 \frac{Z^k}{k}, \quad k < 0 \quad 5.11c$

Similarly, the integral defining the temperature perturbation

Θ_1 , gives for $L_1 > 0$ $k > 0$

$$\Theta_1/\Theta_0 \sim \frac{3(\gamma-1)m^{(\gamma+5)/2} L_1}{V_0(\omega)^3 L_1 k^2} Z \quad 5.13a$$

where $L_1 = 3k + v(\gamma-1) = 3 - \frac{1}{2}(\gamma+5),$

if $L_1 > 0$ but $k = 0$ then

$$\Theta_1/\Theta_0 \sim \frac{3(\gamma-1)m^{(\gamma+5)/2} v(\gamma-1)}{V_0 V_0(\omega)^3} Z^{v(\gamma-1)} [\log_e Z]^2, \quad 5.13b \quad k=0$$

If however $k < 0$ $\frac{dg_1}{dZ} \sim \frac{3m^{(\gamma+1)/2}}{V_0(\omega)} v_{1\infty}^2 Z^{k-1},$

and thus $\Theta_1/\Theta_0 \sim \frac{3(\gamma-1)m^{(\gamma-3)/2}}{V_0(\omega)} v_{1\infty}^2 \frac{Z^{L_2}}{L_2}, \quad 5.13c$

where $L_2 = k + v(\gamma-1) = 1 - \frac{1}{2}(\gamma-3).$

For $L_2 < 0$ $k < 0$ then the integrals defining Θ_1 and g_1 are both convergent and a near frozen flow results.

There are thus two "pseudo entropy" regimes in which the perturbation solution breaks down for large Z .

Regime 1

This regime is characterised by both the condensation and the temperature or "pseudo entropy" integrals diverging for large Z .

The solution in this regime, § 5.2, shows that the temperature returns to equilibrium through a "condensation shock" analogous to that found by Blythe (1967) for vibrational de-excitation

Regime 2

In this regime only the "pseudo entropy" integral diverges, and the solution, β 5.3, shows that instead of a "condensation shock" there is a gradual condensation region, through which the temperature rises, though not necessarily to equilibrium. For $V \gg 1$ the flow always remains frozen. (The solution in this regime in fact reduces to that obtained by Blythe (1967) for $\delta = 1/2$.)

The two "pseudo entropy" regimes mentioned above will now be considered in turn.

5.2 Regime 1 The condensation shock

In this regime $\frac{\theta}{\theta_0} \sim 1 + \frac{\lambda_i m^{(\gamma+5)/2} 3(\gamma-1) z^{\frac{\gamma}{2}}}{V_0(\omega)^3 4k^2},$

and thus an appropriate scaling for z is

$$Y_1 = [\bar{\lambda}_i]^{\frac{1}{2}} z, \quad 5.14$$

where $\bar{\lambda}_i$ is a new rate parameter based on $V_0(\omega)$

and

$$\bar{\lambda}_i = \frac{\lambda_i m^{(\gamma+5)/2}}{V_0(\omega)^3}. \quad 5.15$$

Then

$$\left. \begin{aligned} \frac{\theta}{\theta_0} &\sim 1 + \frac{3(\gamma-1) Y_1^{\frac{\gamma}{2}}}{4k^2}, \\ \text{and } g_1 &\sim \bar{\lambda}_i^{\frac{\gamma}{2}} \frac{Y_1^{3\gamma} m^{\gamma-1}}{k^3}, \end{aligned} \right\} 5.16.$$

where $\frac{1}{2} = (\gamma-1)v/l_1.$

Thus as $\theta \sim \lambda_i^{\frac{\gamma}{2}}$ when $Y_1 \propto 1$ an appropriate set of variables in this region is

$$\left. \begin{aligned} \pi &= \bar{\lambda}_i^{\frac{v}{l_1}} m \Pi_1, \\ V &= V_0(\omega) \bar{V}_1, \\ \theta &= \bar{\lambda}_i^{\frac{\gamma}{2}} m^{\gamma-1} \Theta_1, \\ g &= \bar{\lambda}_i^{\frac{\gamma}{2}} m^{\gamma-1} G_1, \end{aligned} \right\} 5.17$$

and this gives for the zeroth order terms in the pseudo entropy region,

$$\Pi_{10} \bar{V}_{10} Y^{\nu} = 1, \quad 5.18$$

$$\frac{1}{\Pi_{10}} \frac{d\Pi_{10}}{dY_1} - \frac{1}{\gamma-1} \frac{1}{\Theta_{10}} \frac{d\Theta_{10}}{dY_1} = -\frac{1}{\Theta_{10}} \frac{dG_{10}}{dY_1}, \quad 5.19$$

and $\bar{V}_{10} = 1, \quad 5.20$

$$G_{10} = \left[\int_{Y_0}^{Y_1} \frac{\Pi_{10} \Theta_{10}^{1/2}}{\bar{V}_{10}} dY_1 \right]^3 \quad 5.21$$

where $\Pi_{10} = \Pi_{10} + o(1)$ etc.

From these equations we find that

$$\bar{V}_{10} = 1, \quad \Pi_{10} = Y_1^{-\nu} \quad 5.22$$

and thus for this regime the "pseudo entropy" equation becomes

$$\frac{\nu}{Y_1} + \frac{1}{\gamma-1} \frac{d\Theta_{10}}{dY_1} = \frac{3R_{10}^2}{\Theta_{10}} \frac{dR_{10}}{dY_1}, \quad 5.23$$

and $G_{10} = R_{10}^3, \quad 5.24$

where $R_{10} = \int_{Y_0}^{Y_1} Y_1^{-\nu} \Theta_{10}^{1/2} dY_1 + C, \quad 5.24b$

where the lower limit has been inserted for convenience and the constant C is obtained by matching with the downstream expansion.

Rewriting the equation 5.23 in terms of R_{10} gives

$$\frac{1}{Y_1^a} \frac{d}{dY_1} \left\{ Y_1^a \frac{dR_{10}}{dY_1} \right\} = \frac{3(\gamma-1)}{2} \frac{R_{10}^2}{Y_1^{2\nu}}, \quad 5.25$$

where $a = \frac{\nu}{2}(\gamma+1).$

Transforming the variables so that

$$Y_1^{\alpha+1} \frac{dR_{10}}{dY_1} = \tau, \quad Y_1^\alpha R_{10} = \xi, \quad 5.26$$

where $\alpha = 2(1-\nu)$, reduces 5.25 to

$$\frac{d\tau}{d\xi} = \frac{\mu \xi^2 + \tau}{\alpha \xi + \tau}, \quad \text{where } \mu = 3(\gamma-1)/2.$$

Examination of this equation as both τ and ξ tend to zero shows that $\tau \sim k \xi$,

which matches with the downstream solution in the perturbation region.

Considering the behaviour for large τ and ξ it can be shown that

$$\tau \sim \sqrt{\frac{2N}{3}} \xi^{3/2}, \quad 5.28$$

and thus
$$Y^{\alpha+1} \frac{dR_{10}}{dY_1} \sim \sqrt{\gamma-1} Y^{3\alpha/2} R_{10}^{3/2}$$

which when integrated gives

$$R_{10} \sim 4 \left\{ C - \frac{(\gamma-1)^{1/2}}{1-\nu} Y_1^{1-\nu} \right\}^{-2}, \quad 5.29a$$

where the undetermined constant C must be

included as otherwise R_{10} becomes negative as Y_1 increases which is not physically plausible

$$\text{As } Y_1 \rightarrow Y_S \text{ where } Y_S = \left[\frac{(1-\nu)C}{(\gamma-1)^{1/2}} \right]^{1/(1-\nu)}$$

It is seen that R_{10} becomes unbounded. Y_S is identified with the shock position (see Blythe (1967)). Rewriting 5.29a in terms of the shock position gives

$$R_{10} \sim 4 \frac{(1-\nu)^2}{(\gamma-1)} Y_S^{-2(1-\nu)} \eta^{-2}, \quad 5.29b$$

where $Y_1 = Y_S (1-\eta)$.

If equation 5.27 is numerically integrated τ can be obtained as a function of ξ . Using the relation between R_{10} and ξ the following expression for Y_1 as a function of ξ is found,

$$\log Y_1 = \int_0^S \frac{(k\xi - T) ds}{(T + \alpha\xi)(k + \alpha)\xi} + \frac{1}{\ell_1} \log \xi \quad 5.30$$

Similarly the expression for τ can now be rewritten as

$$\frac{dR_{10}}{dY_1} = \frac{\tau}{Y_1^{\alpha+1}} = Y^{-\nu} \Theta_{10}^{1/2}, \quad 5.31$$

and so $\Theta_{10}^{-1/6} = Y^{1-\nu} \tau^{-1/3}. \quad 5.32$

Y_1 and Θ_{10} can now be obtained as functions of β by numerical integration, and thus Y_S found. A plot of Y_S vs ν is shown in figure 5. It should be noted that although Y_S is small for ν large, Z_S will increase with ν for $\bar{\lambda}_i$ small as $Z_S = Y_S \bar{\lambda}_i^{1/2}$, and L_1 decreases with increasing ν .

The behaviour within the shock itself must now be considered.

Upstream of the shock the limiting behaviour as $Y_1 \rightarrow Y_S$ is

$$\Theta_{10} \sim (Y_S - Y_1)^{-6}, \quad 5.33$$

and an appropriate length scale consistent with Θ and g being $O(1)$ near $Y_1 = Y_S$ is defined by

$$Y_1 = Y_S (1 + \bar{\lambda}_i^{1/6} Y_S^{\nu-1} y). \quad 5.34$$

The downstream expansions of the pseudo entropy solutions are

$$\Theta \sim m^{\delta-1} \frac{64(1-\nu)^3}{(\delta-1)^2} (-y)^{-6}, \quad 5.35$$

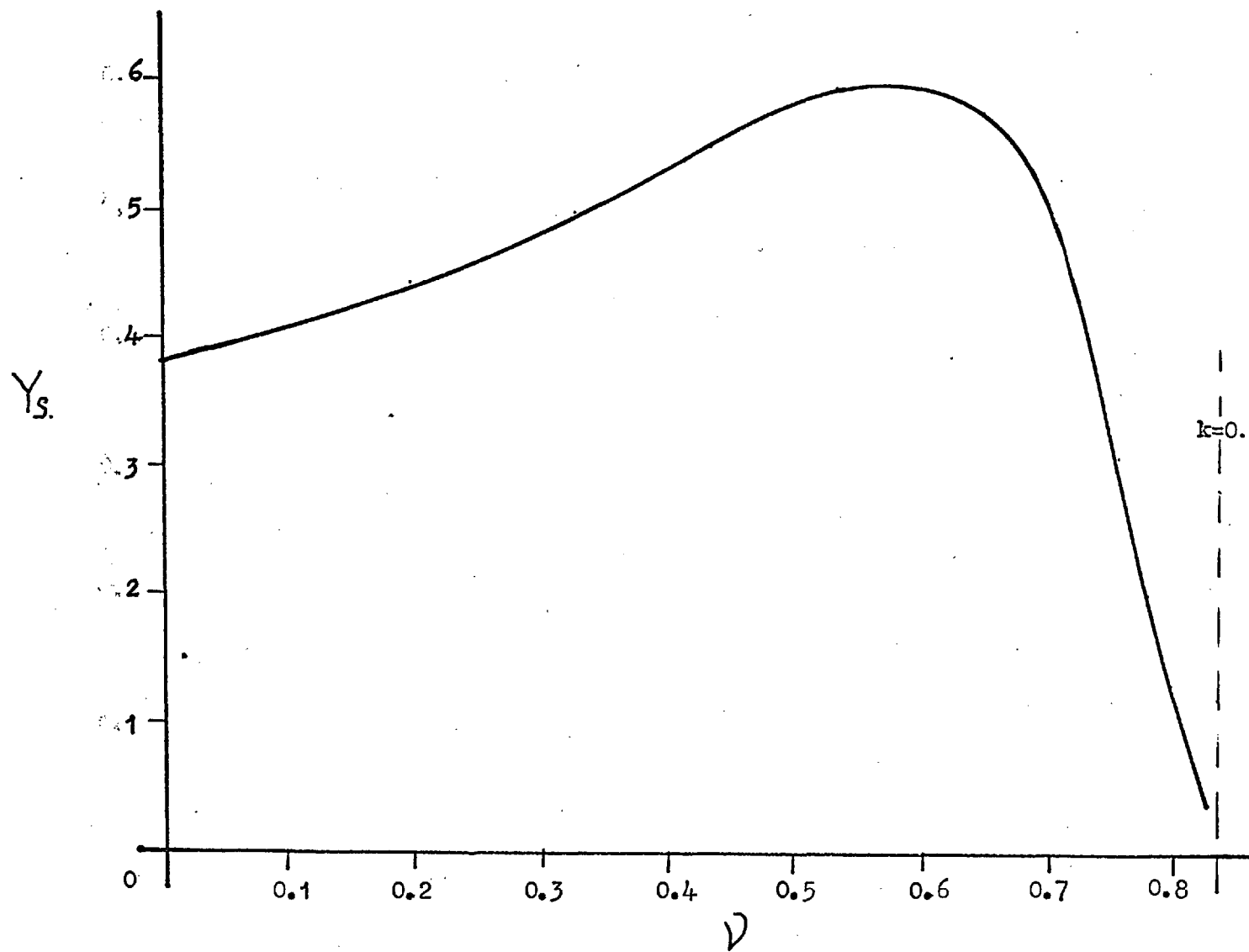
$$g \sim m^{\delta-1} \frac{64(1-\nu)}{(\delta-1)^3} (-y)^{-6}, \quad 5.36$$

where terms $O(1)$ with respect to $\bar{\lambda}_i$ have been retained. The corresponding expressions for Π_1 and $\bar{V}_1$ are particularly simple.

If Π is now scaled so that

$$\Pi_1 Y_S^\nu = \Pi. \quad 5.37$$

FIGURE 5



THE SHOCK POSITION Y_s vs. ν .

Then the following one dimensional equations are obtained for the flow through the condensation shock.

$$\pi \nabla = 1, \quad 5.38$$

$$\pi \left\{ \frac{Q}{V_0^2(\infty)} + \nabla^2 \right\} = 1, \quad 5.39$$

$$\frac{\gamma}{\gamma-1} Q + \frac{V_0^2(\infty)}{2} \nabla^2 + g = \frac{V_0^2(\infty)}{2}, \quad 5.40$$

$$g = m^{-(\gamma-1)/2} \left\{ \int_{-\infty}^y \frac{\pi Q^{1/2}}{\nabla} [Q_S - Q] dy \right\}^3 \quad 5.41$$

where $Q_S = 1 + O\left\{\frac{\log \bar{\lambda}_i}{H}\right\}, [H \gg \log \bar{\lambda}_i]$

The constants on the right hand side follow formally by matching with the downstream expansion of the pseudo entropy solution as $y \rightarrow -\infty$

Such shocks have been discussed by Wegener and Mack (1958). Here, however, as in Blythe (1967), considerable simplifications can be made as the upstream conditions correspond to infinite mach number.

The appropriate solutions of the algebraic relations above are

$$\nabla = \frac{1}{\pi} = \frac{\gamma \pm \sqrt{1-Q}}{\gamma+1}, \quad 5.42a$$

$$\frac{Q}{V_0^2(\infty)} = \frac{(\gamma-1) + Q \mp (\gamma-1)\sqrt{1-Q}}{(\gamma+1)^2}, \quad 5.42b$$

where

$$Q = \frac{2(\gamma^2-1)g}{V_0^2(\infty)}$$

and the upper sign corresponds to the continuous supersonic solution.

Conditions at the downstream end, of the shock. are defined by $Q = Q_S$ and as here Q and π are $O(1)$

$$Q_S = 1 + O\left\{\frac{\log \bar{\lambda}_i}{H}\right\},$$

and then the equilibrium end state is to zeroth order $[H \gg \log \bar{\lambda}]$

$$\theta = \theta_S = 1.$$

It can be seen that the above solution only remains valid if

$$Q \leq 1.$$

$$\text{thus } \frac{\theta}{V_0^2(\infty)} \leq \frac{\gamma}{(\gamma+1)^2}.$$

The maximum value of θ is 1 from the equilibrium end conditions,

and the minimum value of V_0^2 is

$$V_0^2 = 2\gamma/\gamma-1,$$

(i.e. $V_0 = 0$ and the condensation point is in the reservoir).

$$\text{then } \frac{\gamma-1}{2\gamma} \leq \frac{\gamma}{(\gamma+1)^2},$$

which is satisfied for all realistic values, of γ (i.e. $1 < \gamma \leq 5/3$).

Therefore it is not possible to choke the flow in a condensation shock of the present type.

The actual shock structure can be determined by a numerical integration of the above equations. However, as a similar example for the vibrational de-excitation shock was considered in Part II, this will not be attempted here.

Thus it can be seen that in Regime 1 the pseudo entropy solution predicts that a condensation shock will occur in the flow.

Downstream of the shock the flow will initially be in equilibrium. Whether it remains so or not depends (as in Blythe(1967)) on the local flow conditions.

5.3 Regime 2 Gradual Condensation

In this regime

$$\frac{Q}{Q_0} \sim 1 + \frac{\lambda_i 3(\gamma-1)m^{(3-\gamma)/2} v_{1\infty}^2 l_2}{V_0(\infty)} \quad 5.43$$

and thus the appropriate scaling now is $Y_2 = [\hat{\lambda}_i]^{1/l_2} Z$

where
$$\hat{\lambda}_i = \frac{3m^{3-\gamma/2} v_{1\infty}^2 \lambda_i}{V_0(\infty)}$$

and then
$$\frac{Q}{Q_0} \sim 1 + (\gamma-1) \frac{Y_2^{l_2}}{l_2}, \quad 5.44a$$

and
$$g \sim \hat{\lambda}_i R_{\infty}^3 + \hat{\lambda}_i^{b_2} m^{\gamma-1} Y_2^k / k,$$

where

$$R_{\infty} = v_{1\infty} \left\{ V_0(\infty) / 3m^{3-\gamma/2} v_{1\infty}^2 \right\}^{1/3}$$

also
$$r \sim \hat{\lambda}_i^{1/3} R_{\infty} + \frac{\hat{\lambda}_i^{b_2} m^{(\gamma+1)/2} R_{\infty} Y_2^k}{V_0(\infty) v_{1\infty} k}, \quad 5.44b$$

where
$$b_2 = (1/3 - k/l_2),$$

$$b_2 = v(\gamma-1)/l_2,$$

and as $Q_0 \sim \lambda_i^{1/2}$ when $Y_2 \rightarrow 0(1)$,

the appropriate set

of variables in this regime is

$$\left. \begin{aligned} \pi &= \hat{\lambda}_i^{1/l_2} m \pi_2, \\ V &= V_0(\infty) \bar{V}_2, \\ Q &= \hat{\lambda}_i^{b_2} m^{(\gamma+1)/2} \bar{Q}_2, \\ R &= \hat{\lambda}_i^{1/3} R_{\infty} + \frac{\hat{\lambda}_i^{b_2} m^{(\gamma+1)/2} R_{\infty} R_2}{V_0(\infty) v_{1\infty}} \end{aligned} \right\} \quad 5.45$$

This then gives

$$\Pi_{20} \bar{V}_{20} \gamma_2^v = 1, \quad 5.46$$

$$\frac{1}{\Pi_{20}} \frac{d\Pi_{20}}{d\gamma_2} - \frac{1}{\gamma-1} \frac{1}{\Theta_{20}} \frac{d\Theta_{20}}{d\gamma_2} = -\frac{1}{\Theta_{20}} \frac{dR_{20}}{d\gamma_2}, \quad 5.47$$

$$\bar{V}_{20} = 1, \quad 5.48$$

and
$$\frac{dR_{20}}{d\gamma_2} = \frac{\Pi_{20} \Theta_{20}^{1/2}}{\bar{V}_{20}},$$

where
$$\Pi_2 = \Pi_{20} + o(1) \text{ etc.}$$

These equations have the solutions (Blythe (1967)).

$$\left. \begin{aligned} \bar{V}_{20} &= 1, \quad \Pi_{20} = \gamma_2^{-v} \\ \Theta_{20}^{1/2} &= K \gamma_2^{-v(\gamma-1)/2} + (\gamma-1)/2 \gamma_2^{(1-v)} \\ R_{20} &= \int_{\gamma_{20}}^{\gamma_2} S^{-v} \Theta_{20}^{1/2}(S) dS + L, \end{aligned} \right\} \quad 5.49$$

where K and L are arbitrary constants, S is a dummy variable and the lower limit has been inserted for convenience.

It is easily verified by matching that $K = 1$.

The leading term in the expansion of R_{20} matches with the appropriate leading term in the near frozen solution and L is defined as the finite part of

$$\int_{\gamma_2}^{\gamma_{20}} S^{-v} \Theta_{20}^{1/2} dS, \quad 5.50$$

unless a term $O(\lambda^{1/2})$ occurs in the downstream expansion, which will not

in general happen unless $\hat{b}_2$ is a positive integer other than a unity. If this is so this expression must be modified to include any contribution from the higher order terms.

The solutions above show that there are two particular cases, and it will be convenient to divide the discussion into two parts.

$$a) \quad v > 1.$$

The temperature distribution for large γ_2 is given by

$$\Theta_{20} \sim \gamma^{(1-v)/2}$$

and thus as the exponent $(1-v)/2$ is negative this solution remains valid for all γ_2 , and R_{20} tends asymptotically to some constant value.

$$b) \quad v < 1.$$

The temperature distribution for large γ_2 is as given above, but the exponent is now positive and the temperature becomes unbounded in this limit, and an alternative solution must be sought in this region.

Thus there will be a partial return towards equilibrium, though not as great as in regime 1.

In the downstream region Θ becomes $O(1)$ when $\gamma_2 = O(\hat{\lambda}_i^{\hat{b}_2/2(1-v)})$ or $z = O(\hat{\lambda}_i^{-1/(1-v)})$. Hence a suitable length scale is defined by

$$\hat{z} = \hat{\lambda}_i^{-1/(1-v)} \hat{\gamma}_2 \quad 5.51$$

in this region $\hat{\gamma} = O(1)$, $\pi = O(\hat{\lambda}_i^{v/(1-v)})$ and it is convenient to write

$$\pi = \hat{\lambda}_i^{v/(1-v)} \hat{\pi} \quad 5.52$$

The remaining dependent variables Θ , q and $\underline{V}$ are $O(1)$ in this region, and it is implicit that a significant degree of condensation will occur. The downstream expansions are now

$$\left. \begin{aligned}
 \Theta &\sim m^{\gamma-1} \left[(\gamma-1)/2b_2 \right]^2 \hat{Y}_2^{2(1-\nu)}, \\
 \hat{\Pi} &\sim \hat{Y}_2^{-\nu}, \\
 \bar{V} &\sim 1., \\
 g &\sim \Delta \hat{Y}_2^{6(1-\nu)}, \\
 \Delta &= m^{\gamma-1} \frac{(\gamma-1)}{2b_2} \frac{1}{2(1-\nu)}.
 \end{aligned} \right\} 5.53$$

where

these relations provide initial conditions as $\hat{Y}_2 \rightarrow 0$ for the solution when $\hat{Y}_2 = O(1)$ and the equations obtained in this region are the full equations, except that as $H \gg |\log \lambda|$ the equation of state simplifies.

6. Near frozen solutions for a pure vapour

6.1 The perturbation equations

As for the near frozen flow of a pure vapour the rate parameter controlling the rate of growth of the drops λ_p is small, it is possible, as in §5.1. to use a conventional near frozen expansion. Thus the appropriate expansions are, assuming $H \gg \log \lambda_p$ and $\bar{A} = z^\nu$ as before.

$$\left. \begin{aligned} \Theta &= \Theta_0(z) + \lambda_p \Theta_1(z) + \dots \\ \pi &= \pi_0(z) + \lambda_p \pi_1(z) + \dots \\ \psi &= \psi_0(z) + \lambda_p \psi_1(z) + \dots \\ V &= V_0(z) + \lambda_p V_1(z) + \dots \\ g &= \lambda_p g_1(z) + \dots \end{aligned} \right\} 6.1$$

When these are substituted into the equations of motion 3.2 to 3.6, they again give for the zeroth order solution the fully frozen result.

$$\left. \begin{aligned} \pi_0 &= \Theta_0^{\frac{1}{\gamma}-1} \\ V_0 &= [V_c^2 + 2\gamma/(\gamma-1) \cdot (1-\Theta_0)]^{1/2} \\ \Theta_0^{\frac{\gamma}{\gamma-1}} [V_c^2 + 2\gamma/(\gamma-1) \cdot (1-\Theta_0)]^{1/2} &= \bar{m}/z^\nu \end{aligned} \right\} 6.2$$

The solutions of the first order equations are again of the form

$$\left\{ 1 - \frac{\gamma \Theta_0}{V_0^2} \right\} \frac{\Theta_1}{\Theta_0} = (\gamma-1) \int_1^z \frac{dg_1}{\Theta_0} - (\gamma-1) g_1 / V_0^2, \quad 6.3$$

where g_1 is now given by

$$g_1 = \int_1^z \Theta_0^{\frac{\gamma}{\gamma-1}} \exp \left\{ \frac{-K^2}{\Theta_0(1-\Theta_0)^2} \right\} \left[\int_{\xi}^z \frac{\Theta_0^{\frac{\gamma}{\gamma-1}} (1-\Theta_0) ds}{V_0} \right]^3 \xi^\nu d\xi. \quad 6.4$$

It is now possible to identify the two regimes for K , which were mentioned in § 3.3.

In section 5.1 it was shown that for an impure vapour the temperature perturbation became invalid when $\Theta = O(\bar{\lambda}_i^{b_1})$ and thus it can be seen from equation 6.4 that when Θ is $O(\bar{\lambda}_p^b)$, if $K^2 \gg \bar{\lambda}_p^b$ then the rate of production of drops will be exponentially small. It will be shown in § 6.2. that in this case the solution adopts the same form as that for the impure vapour.

When, however, $K^2 = O(\bar{\lambda}_p^b)$ then the exponential term governing the rate of production of drops is, to first order, unity (except in a region $O(K)$ near $\Theta = 1$.) and this shows that the production of drops is a dominant factor in the formation of the flow solution far downstream. This case will be considered in § 6.3.

6.2 The limit $K^2 \gg \bar{\lambda}_p^b$

6.2.1 The perturbation solution

The rate equation 6.4 can now be re-written as

$$g_1 = \int_1^z F(\eta) [\bar{R}(z) - R(\eta)]^3 d\eta, \quad 6.5$$

$$\text{where } F(x) = Q_0(x) x^{\frac{3}{2}-1} \exp \left\{ \frac{-K^2}{Q_0(1-Q_0)^2} \right\},$$

$$\text{and } R(x) = \int_1^x Q_0^{(x+1)/2(x-1)} V_0^{-1} (1-Q_0) ds.$$

Then expanding 6.5 gives

$$g_1 = R(z)^3 \int_1^z F(\eta) d\eta - 3R^2(z) \int_1^z R(\eta) F(\eta) d\eta + 3R(z) \int_1^z R^2(\eta) F(\eta) d\eta - \int_1^z R^3(\eta) F(\eta) d\eta. \quad 6.6$$

Now for $k > 0$ where $k = 1 - \frac{V}{2}(\gamma+1)$,

$$R(x) \sim \frac{m^{\gamma+1/2}}{V_0(\infty)} \frac{x^k}{k}, \text{ as } x \rightarrow \infty,$$

and in this limit as $K^2 \gg \bar{\lambda}_p^b$, and the solution is being sought in the region in which $\theta = O(\bar{\lambda}_p^b)$ then

$$F(x) \rightarrow 0,$$

as x becomes large.

Thus it can be seen that the four integrals above converge for z large owing to the exponential term in the rate of drop production.

Then for $k > 0$ the correct asymptotic expansion is

$$g_1 \sim \frac{Nm^{3(\gamma+1)/2}}{V_0(\infty)^3} \frac{z^{3k}}{k^3} \quad 6.7$$

where N , the total number of drops formed in the flow, is given by

$$N = \int_1^\infty \theta_0^{\frac{3}{\gamma}-1} \eta^v \exp\left\{\frac{-K^2}{\theta_0(1-\theta_0)}\right\} d\eta, \quad 6.8$$

For $k < 0$ however the integral defining $R(x)$ converges and

$$R(x) \sim v_{1,\infty} - \bar{R}(x),$$

where
$$v_{1,\infty} = \int_1^\infty \frac{\theta_0^{(\gamma+1)/2(\gamma-1)}}{V_0} (1-\theta) ds,$$

and
$$\bar{R}(x) = \int_x^\infty \frac{\theta_0^{(\gamma+1)/2(\gamma-1)}}{V_0} (1-\theta) ds,$$

$$\sim -\frac{m^{\gamma+1/2}}{V_0(\infty)} x^k, \quad x \rightarrow \infty.$$

Thus re-writing the rate equation gives

$$\begin{aligned} g_1 = & \int_1^z \bar{R}^3(\eta) F(\eta) d\eta - 3\bar{R}(z) \int_1^z \bar{R}(\eta) F(\eta) d\eta \\ & + 3\bar{R}^2(z) \int_1^z \bar{R}(\eta) F(\eta) d\eta - \bar{R}^3(z) \int_1^z F(\eta) d\eta. \end{aligned} \quad 6.9$$

Again the four integrals converge for large Z owing to the exponential term, and the asymptotic expansion now is

$$g_1 \sim B_N + \frac{3m^{x+1/2} C_N Z^k}{V_0(\omega) k}, \quad 6.10$$

where

$$B_N = \int_1^\infty \bar{R}(\eta)^3 F(\eta) d\eta, \quad 6.11a$$

and

$$C_N = \int_1^\infty \bar{R}^2(\eta) F(\eta) d\eta. \quad 6.11b$$

Similarly the integral defining the temperature perturbation gives for $l_1 > 0$ $k > 0$

$$\frac{\theta_1}{\theta_0} \sim N \frac{3(\gamma-1)m^{(x+5)/2}}{V_0(\omega)^3} \frac{Z^{l_1}}{l_1 k_1} \quad 6.12$$

and for $k < 0 < l_2$

$$\frac{\theta_1}{\theta_0} \sim C_N \frac{3(\gamma-1)m^{(3-x)/2}}{V_0(\omega)} \frac{Z^{l_2}}{l_2} \quad 6.13$$

and for $l_2 < 0$ the integral defining θ_1 , converges and a near frozen flow results.

There are therefore, as in the impure vapour case, two regimes in which the perturbation solution breaks down for large Z .

6.2.2. Regime 1 The condensation shock

This regime is, as before, characterised by the divergence of both the temperature and the condensation integrals.

If the perturbation solutions for this and the impure case $\int$ 6.2 are compared it can be seen that if an effective rate parameter λ_{ep} is defined as

$$\lambda_{ep} = N\lambda_p$$

then the two perturbation solutions become the same

$$\text{i.e. } \frac{\theta_1}{\theta_0} \sim 1 + \lambda_{ep} 3(\gamma-1) m^{(8+5)/2} \frac{z \ell_1}{4k^2} \quad . \quad 6.14$$

If the equations are now scaled in the appropriate manner (i.e. as in § 5.2), then the same "pseudo entropy" equation will result. This will be followed by a similar shock solution. (Although more drops will be formed in the shock itself they will only effect the error terms involved).

Thus in Regime 1 there will be a condensation shock at

$$z_s = \gamma_s \bar{\lambda}_{ep} \ell_1,$$

where γ_s is as given in Fig. 5.

6.2.3 Regime 2 Gradual condensation

In this regime only the temperature perturbation breaks down for large z . If this case is compared with regime 2 for the impure vapour it can be seen that if the constant B_N is considered equivalent to γ_N^3 and if an effective rate parameter is now defined as $\lambda_{ap} = \lambda_p$ then the two perturbation solutions are again the same. If the equations are now scaled as in § 5.3 then the same pseudo entropy equation will be obtained, and the remarks on this solution apply equally well here.

6.3 The limit $K_{\infty}^2 (\bar{\lambda}_p^3)$.

6.3.1 The perturbation solution

As mentioned in § 6.1 the production of nuclei plays a dominant role in the flow solution. The difference between this solution and that in § 6.2 is that here nuclei ^{are} still being produced when the perturbation solution becomes invalid, whereas in § 6.2 the production

of nuclei had been completed before this occurred.

The rate equation can again be written as

$$g_1 = \int_1^z F(\xi) [R(z) - R(\xi)]^3 d\xi, \quad 6.15$$

where

$$F(x) = Q_0^{x/2-1} x^0 \exp\left\{\frac{-K^2}{(Q_0(1-Q_0))^2}\right\} \quad \left. \vphantom{\int_1^z} \right\} 6.16$$

and

$$R(x) = \int_1^x \frac{Q_0^{(x+1)/2(x-1)}}{V_0} (1-Q_0) ds.$$

Expanding as in 6.2.1 it is easy to show that the appropriate asymptotic expansions are (see Appendix II).

$$g_1 \sim \frac{m^{x+7/2} A_N z^j}{V_0^3 k^3} + \dots \quad j > 0, \quad 6.17$$

$$g_1 \sim B_N + \frac{m^{x+7/2} A_N z^j}{V_0^3 k^3}, + \dots \quad R < j < 0, \quad 6.18$$

$$g_1 \sim B_N + \frac{3m^{x+1/2} C_N z^k}{V_0^3 k}, + \dots \quad j < k < 0, \quad 6.19$$

where

$$\left. \begin{aligned} A_N &= \left\{ \frac{1}{1-v} - \frac{3}{k+1-v} + \frac{3}{2k+1-v} - \frac{1}{j} \right\}, \\ B_N &= \int_1^\infty \tilde{R}^3(\xi) F(\xi) d\xi, \quad C_N = \int_1^\infty \tilde{R}^2(\xi) F(\xi) d\xi. \end{aligned} \right\} 6.20$$

Similarly the integral defining the temperature perturbation gives for $j > k$.

$$\frac{Q_1}{Q_0} \sim \frac{(\gamma-1) A m^{9-\gamma/2}}{V_0^3} \frac{1}{l_3} \frac{z^{l_3}}{k^3} \quad 6.21$$

$$\text{where } l_3 = j + v(\gamma-1) = 4 - \frac{v}{2}(7+\gamma).$$

If $j < k < 0$ then for $l_2 > 0$ where $l_2 = 1 + \frac{1}{2}(\gamma - 3)$,

$$\frac{\theta_1}{\theta_0} \sim \frac{3(\gamma-1)m}{V_0(\infty)} \frac{e^{\frac{l_2}{k} z}}{l_2}, \quad 6.22$$

and for $l_2 < 0$ the integrals defining both θ_1 and q_1 , converge and the flow remains near frozen. In this case the exponential term in the rate equation will eventually cause the drop production to cease.

Thus once again there are two main "pseudo entropy" regimes to be considered.

Regime 1

In this regime $j > k$ and although for some values of γ , $k < j < 0$ the condensation integral converges the temperature integral always diverges. This regime is similar in form to that described in § 6.2.2, and will itself be considered in § 6.3.2

Regime 2

If the perturbation solutions in this regime are compared with those for the other limit they are found to be identical in this regime, and thus the remarks made about regime 2 in § 6.2.2 apply equally well to this situation.

6.3.2 Regime 1 The condensation shock

in this regime

$$\frac{\theta_1}{\theta_0} \sim 1 + \frac{2p(\gamma-1)A_N m}{V_0(\infty)^3} \frac{e^{\frac{l_3}{k} z}}{l_3 k^3},$$

and thus an appropriate scaling for z is

$$Y_3 = [\lambda p]^{\frac{1}{3}} z, \quad 6.23$$

where $\bar{\lambda}_p = \frac{\lambda_p A_N m^{9/2}}{V_0 \omega_0}$.

Then $\theta/\theta_0 \sim 1 + (\gamma-1) \frac{Y^{\gamma/2}}{l_3 k^3}$, 6.24

and $g_j \sim \bar{\lambda}_p^{b_3} m^{\gamma-1} \frac{Y^j}{k^3}$, $j > 0$, 6.25a

or $g_j \sim B_N + \bar{\lambda}_p^{b_3} m^{\gamma-1} \frac{Y^j}{k^3}$, $k < j < 0$, 6.25b

where $b_3 = (\gamma-1) \nu / l_3$.

Thus as $\theta_0 \sim \bar{\lambda}_p^{b_3}$ when $Y=0(1)$, an appropriate set of variables in this region is

$$\left. \begin{aligned} \pi &= \bar{\lambda}_p^{1/3} m \pi_3, \\ v &= V_0(\omega_0) \bar{V}_3 \\ \theta &= \bar{\lambda}_p^{b_3} m^{\gamma-1} \Theta_3, \\ g &= \bar{\lambda}_p^{b_3} m^{\gamma-1} G_3, \\ \text{or } g &= B_N + \bar{\lambda}_p^{b_3} m^{\gamma-1} G_3, \quad k < j < 0. \end{aligned} \right\} 6.26$$

Inserting these into the equations of motion 3.2 to 3.6 gives for the zeroth order terms in the pseudo entropy region

$$\pi_{30} \bar{V}_{30} Y_3^\nu = 1, \quad 6.27$$

$$\frac{1}{\pi_{30}} \frac{d\pi_{30}}{dY_3} - \frac{1}{\gamma-1} \frac{1}{\Theta_{30}} \frac{d\Theta_{30}}{dY_3} = -\frac{1}{\Theta_{30}} \frac{dG_{30}}{dY_3}, \quad 6.28$$

$$\bar{V}_{30} = 1, \quad 6.29$$

and $G_{30} = \left[\int_{Y_0}^{Y_3} \frac{\pi_{30}^2}{\bar{V}_{30}} \left[\int_y^{Y_3} \frac{\pi_{30} \Theta_{30}^{1/2}}{\bar{V}_{30}} ds \right]^3 y^\nu dy \right] + L$, 6.30

where $\Theta_3 = \Theta_{30} + o[\Theta_{30}]$ etc., and where L is an arbitrary constant determined by matching. The lower limit Y_0 has been

inserted for convenience.

Inserting the values obtained from equations 6.27 and 6.29 into the "pseudo entropy" equation 6.28 gives

$$\frac{Y}{Y_3} + \frac{1}{(\gamma-1)} \frac{1}{H_{30}} \frac{dH_{30}}{dY_3} = \frac{1}{H_{30}} \frac{d}{dY_3} \left[\int_{Y_0}^{Y_3} Y^v \left[\int_Y^{Y_3} S^{-v} H_{30}^{1/2} dS \right] dY \right], \quad 6.31$$

Then writing

$$f(Y) = Y^{-v}, \quad 6.32$$

$$R_{30}(Y) = \int_{Y_0}^Y S^{-v} H_0^{1/2}(S) dS,$$

equation 6.31 can be re-arranged to give

$$R_{30} = \frac{3(\gamma-1)}{2} \int_0^{Y_3} Y^{-\frac{1}{2}(\gamma+1)} \left[\int_0^Y S^{-\frac{1}{2}(\gamma-1)} [R_{30}(S) I_1 - 2R_{30}(S) I_2 + I_3] dS \right] dY, \quad 6.33$$

where

$$I_1 = \int_0^{Y_3} f(Y) dY, \quad I_2 = \int_0^{Y_3} R_{30}(Y) f(Y) dY, \text{ and } I_3 = \int_0^{Y_3} R_{30}^2(Y) f(Y) dY. \quad 6.34.$$

I_1, I_2, I_3 are convergent as $Y_3 \rightarrow 0$ from matching with the upstream perturbation solution.

The above expression, 6.33, is in a convenient form for numerical computation.

As however R_{30} will become unbounded at some value of $Y_3 = Y_S$ the behaviour of 6.33 as $Y \rightarrow Y_S$ must be found.

Differentiating 6.33 three times

$$\frac{d}{dY_3} \left[\left\{ \frac{dR_{30}}{dY_3} \right\}^{-1} \frac{d}{dY_3} \left\{ Y^b \frac{d}{dY_3} \left\{ Y^a \frac{dR_{30}}{dY_3} \right\} \right\} \right] = \frac{3(\gamma-1)}{2} \frac{dR_{30}}{dY_3} I_1, \quad 6.35$$

where $b = 2v - a$, and $I_1 = \int_0^{Y_3} f(Y) dY = \frac{Y_3^{1-v}}{1-v}$.

$a = \frac{1}{2}(\gamma+1),$

Assuming that as $Y_3 \rightarrow Y_S$,

$$R_{30} \sim A \eta^{-\alpha},$$

$$\text{where } Y_3 = Y_S (1 - \eta),$$

then substituting into the L.H.S differentiating and retaining the dominant terms as $\eta \rightarrow 0$ it can be easily shown that

$$R_{30} \sim 4(1-v) Y_S^{-3(1-v)} \eta^{-2} + \dots, \quad 6.36$$

as $Y_3 \rightarrow Y_S$.

Equation 6.35 has been numerically integrated and using 6.39 the position of Y_S has been obtained, and is shown graphically in fig. 6.

The behaviour within the shock itself must now be considered.

Upstream of the shock the limiting behaviours of the "pseudo entropy" solutions as $Y_3 \rightarrow Y_S$ are

$$R_{30} \sim 4 \frac{(1-v)}{8-1} Y_S^{-3(1-v)} \eta^{-2} + \dots, \quad 6.36$$

$$\Theta_{30} \sim 2^6 \frac{(1-v)^2}{(8-1)^2} Y_S^{-8(1-v)} \eta^{-2} + \dots, \quad 6.37$$

$$G_{30} \sim 2^6 \frac{(1-v)^2}{(8-1)^3} Y_S^{-8(1-v)} \eta^{-6} + \dots, \quad 6.38a$$

$$\text{i.e. } G_{30} \sim \frac{Y_S^{1-v}}{1-v} R_{30}^3 + \dots, \quad 6.38b$$

This shows that as $Y_3 \rightarrow Y_S$ the number of drops N tends to a constant

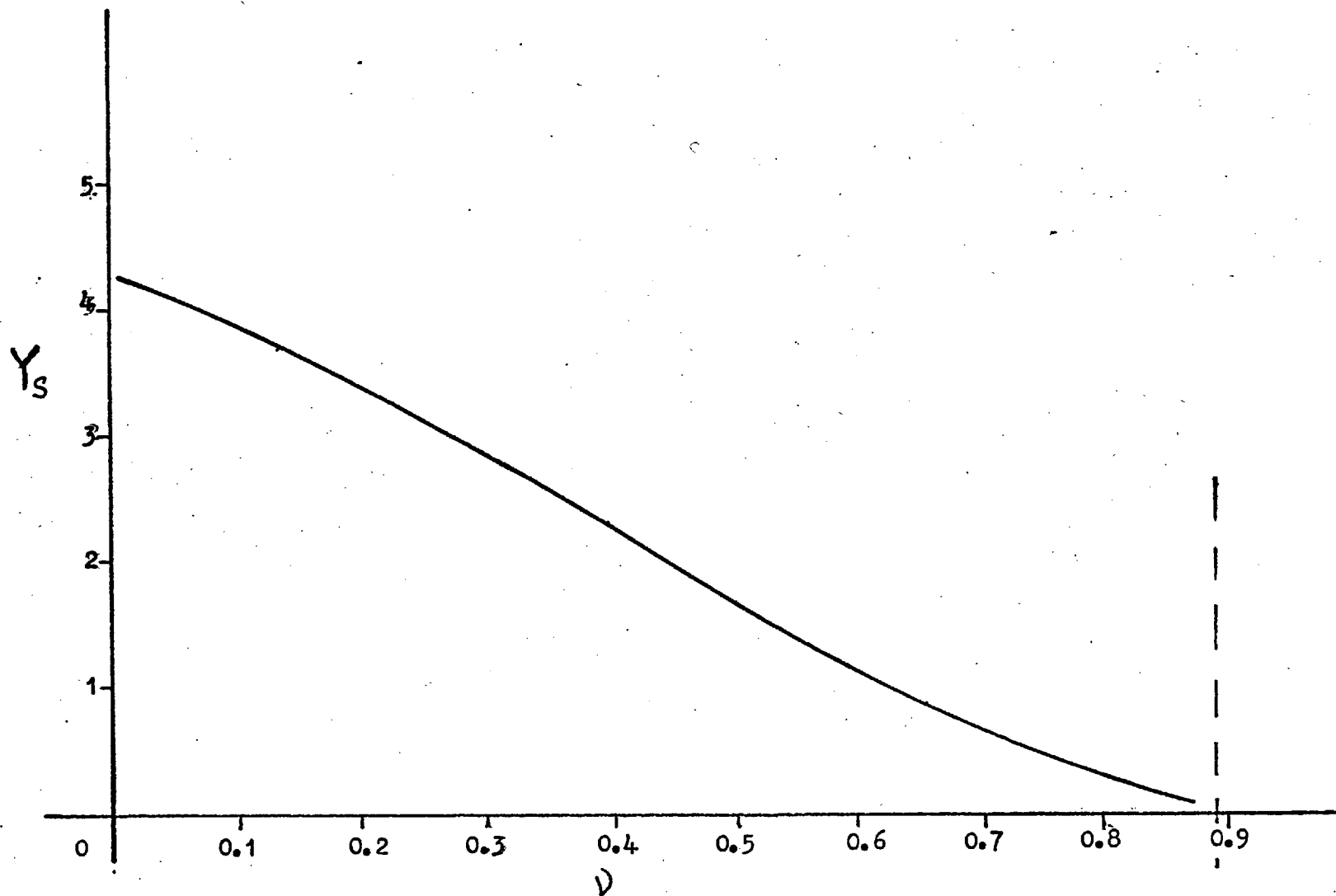
$$\text{i.e. } N \rightarrow Y_S^{1-v} / (1-v).$$

As both Θ and g are $O(1)$ near $Y_1 = Y_S$ a suitable length scale is defined by

$$Y = Y_S \left(1 + \bar{\lambda}_p^{1/6} \frac{Y_S^\alpha}{(1-v)^{2/3(1-v)}} \eta \right), \quad 6.39$$

$$\text{where } \alpha = 4/3(v-1).$$

FIGURE 6.



THE SHOCK POSITION Y_S vs. v

The downstream expansions of the pseudo entropy solutions are

$$\theta \sim m^{\gamma-1} 2^6 (1-\nu)^2 (\gamma-1)^2 (-y)^{-6},$$

$$g \sim m^{\gamma-1} 2^6 (1-\nu)^2 (\gamma-1)^3 (-y)^{-6},$$

where terms $O(1)$ with respect to $\bar{\lambda}_p$ have been retained.

If Π is now scaled as in § 5.2 so that

$$\Pi Y_S^\nu = \Pi$$

6.40

then the equations describing the flow are identical to those obtained in § 5.2 for the shock structure, and the results obtained in that section apply equally well here.

7. Numerical interpretation.

7.1 The scaled variables.

In the previous chapters asymptotic solutions have been obtained for the flow of a condensing vapour both in the near equilibrium and the near frozen limits. These solutions have involved the use of a large number of scaled variables, and in order to facilitate the use of these solutions these scalings will be set out below.

The basic factors are the rate parameters λ_l and λ_p

$$\text{where } \lambda_l = \frac{N \bar{D}'^{1/2}}{m R^2} \frac{T_\infty'^{5/2}}{p_0'} \frac{\rho_c'^3}{T_c'} \frac{x_c'^3}{A_t'}, \quad 7.1$$

$$\text{and } \lambda_p = \frac{K' \bar{D}'^{1/2}}{m R} \frac{\rho_c'^5}{p_0'} \frac{T_c'^3}{T_\infty'^{3/2}} \frac{A_c'}{A_t'} x_c'^4, \quad 7.2$$

$$\text{with } \bar{D}' = \frac{4}{3} \pi \rho_L' \left[\frac{\alpha}{L'} \frac{R}{\rho_L'} \left\{ \frac{2R}{\pi} \right\}^{1/2} \right]^3, \quad 7.3$$

$$\text{and } K' = \frac{2}{\pi} \frac{N A'^{3/2}}{R^2} \frac{(G' N')^{1/2}}{\rho_L'}, \quad 7.4$$

where primed quantities are dimensional, the subscript ∞ refers to stagnation conditions, and c to conditions at the condensation point.

In the near equilibrium limit the solutions are given

by

$$T' = T_c' (1 - \tau T'), \quad 7.5$$

$$x' = x_c' (1 + \eta N), \quad 7.6$$

$$\text{and } \hat{g} = \hat{g} G / H, \quad 7.7$$

$$\text{where } \left. \begin{aligned} \tau &= \left\{ \frac{\lambda_l}{N c^3} \right\}^{-1/5} \frac{\alpha}{2} \frac{\beta_1^{-1/5}}{\beta_1} \\ \eta &= \left\{ \frac{\lambda_p}{N c^3} \right\}^{-1/5} \frac{\alpha}{2} \frac{\beta_1^{-1/5}}{\beta_1} \\ \hat{g} &= \left\{ \frac{\lambda_p}{N c^3} \right\}^{-1/5} \frac{\alpha}{2} \frac{\beta_1^{-1/5}}{\beta_1} \end{aligned} \right\} \quad 7.8$$

for an impure vapour, and

$$\left. \begin{aligned} \tau &= \left\{ \frac{\lambda_0}{V_c^3} \right\} \bar{\alpha}^{2/3} \bar{\beta}^{-1/6} \\ \eta &= \left\{ \frac{\lambda_0}{V_c^3} \right\} \bar{\alpha}^{-1/3} \bar{\beta}^{-1/6} \\ \hat{y} &= \left\{ \frac{\lambda_0}{V_c^3} \right\} \bar{\alpha}^{2/3} \bar{\beta}^{-7/6} \end{aligned} \right\} \quad 7.9.$$

for a pure vapour, where

$$\bar{\alpha} = \frac{v(\gamma-1)Mc^2}{Mc^2-1}, \quad \bar{\beta} = \frac{(\gamma-1)[\gamma Mc^2-1]}{\gamma[Mc^2-1]}$$

$$H = \frac{L'}{RT_c}, \quad \text{and } V_c = \sqrt{\gamma} Mc. \quad 7.10$$

The factor k in the pure vapour case being given by

$$k^2 = K^2 \tau^{-1}, \quad 7.11a$$

$$\text{where } K^2 = \frac{16\pi}{3} \frac{N_A}{RL} \frac{\epsilon'^3 T_c^{1/2}}{\rho_c^{1/2}}. \quad 7.11b$$

For the near frozen case the position of the condensation shock is given by

$$x'_S = x_c' \gamma_S (\bar{\lambda}^{1/2}). \quad 7.12$$

In the impure vapour solution

$$\bar{\lambda}_c = \frac{m^{(\gamma+5)/2}}{V_{0(\infty)}^3} \lambda_{ci}, \quad 7.13$$

$$\text{where } m = V_c/V_{0(\infty)}, \quad V_c = \gamma^{1/2} Mc, \quad \text{and}$$

$$V_{0(\infty)}^2 = V_c^2 + 2\gamma/\gamma-1, \quad 7.14$$

$$\text{for this case } l_1 = 3 - \frac{V_c}{2}(\gamma+5), \quad 7.15$$

and γ_s is given in figure 5.

The pure vapour solution has two cases in the near frozen limit, the first of these is when

$$K^2 \gg \bar{\lambda}_{ep}^{b_1} \quad 7.16$$

$$\left. \begin{array}{l} \text{where } \bar{\lambda}_{ep} = N m^{(\gamma+5)/2} \lambda_p, \\ b_1 = (\gamma-1)v/l_1 \end{array} \right\} \quad 7.17$$

and N is defined in § 3.3 equation 3.13. The solution is then the same as for the impure vapour and γ_s is again given by figure 5.

The second case is when

$$K^2 \ll \bar{\lambda}_p^{b_3} \quad 7.18$$

$$\left. \begin{array}{l} \text{where } \bar{\lambda}_p = \frac{A_N m^{9-\gamma/2}}{V_0(\omega)^3} \lambda_p, \\ b_3 = (\gamma-1)v/l_3, \\ l_3 = 4 - \frac{1}{2}(\gamma+7), \end{array} \right\} \quad 7.19$$

and A_N is defined in § 6.31 equation 6.20. The shock position γ_s is now shown in figure 6.

Using these results some calculations have been performed to illustrate the use of these solutions, and they will now be discussed.

7.2 Numerical examples.

In order to illustrate the theory described in this thesis, and to show its relevance to physical nozzle flows, some

numerical results have been computed for several nozzles. The test gas used was nitrogen and its properties have been taken from the National Bureau of Standards Circular 564 and International Critical Tables. The values of the surface tension required extrapolation below 70°K and unfortunately, as will be noted later, the theory is extremely sensitive to the magnitude of the surface tension.

In the literature there are some numerical calculations which cover the near equilibrium limit but none which describe the near frozen, and direct comparison has only been possible in the former case. The results used were those of Glark (1963), and a plot of the flow conditions for various stagnation pressures is shown in figure 7. A graph of the maximum temperature perturbation is shown in figure 8, for several stagnation pressures and temperatures.

Any agreement between these solutions, is, however, probably entirely fortuitous as the solutions are very dependant on the value of the surface tension. A ten percent change resulting in a thirty percent in the solution.

For the near frozen case a pair of axi-symmetric nozzles were used, whose area ratio distributions were as follows

$$A = 1 + \tan^2 \theta^{1/2} x^2 \quad x < x_0$$

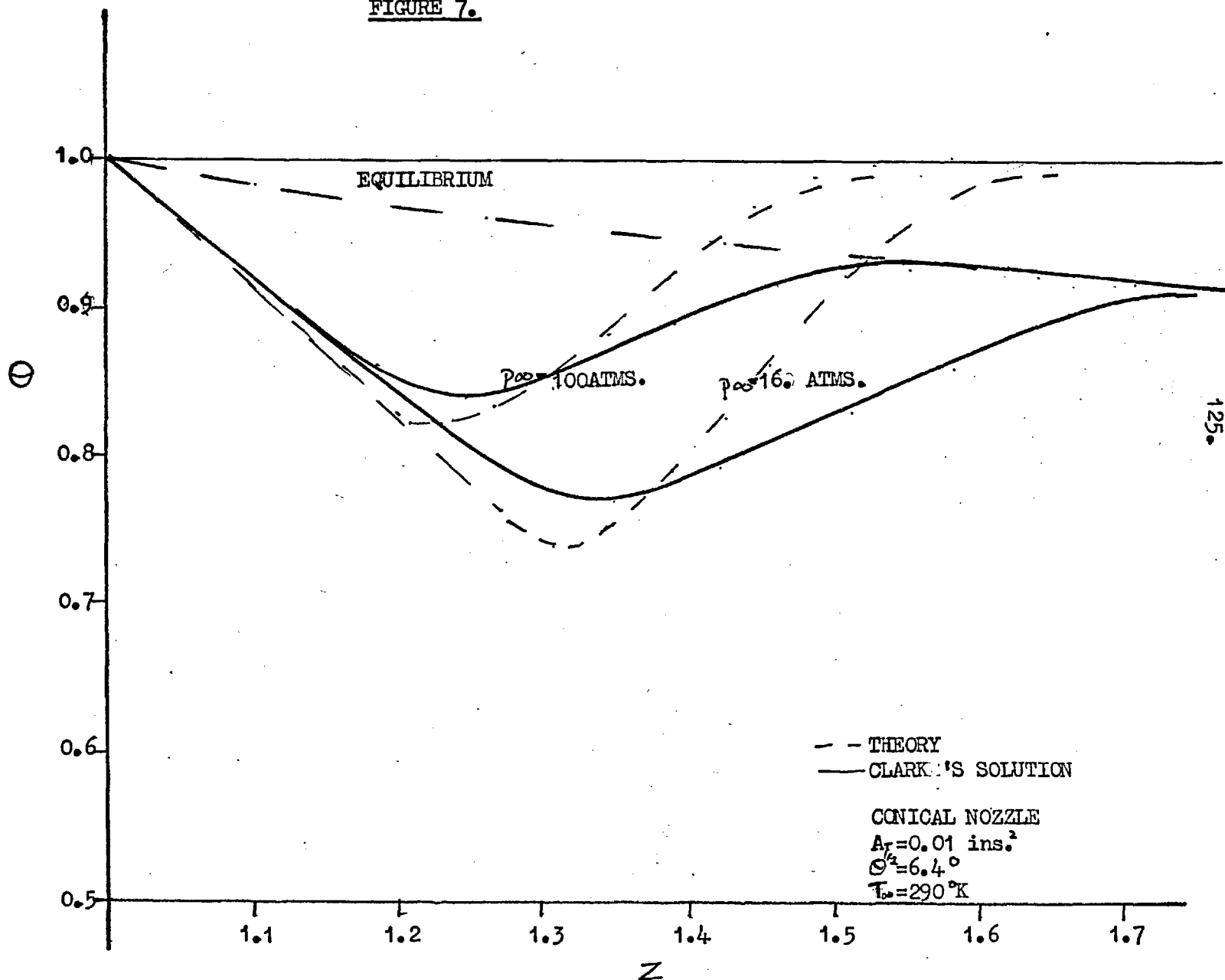
$$A = C(x - c)^v \quad x > x_0$$

where $\theta^{1/2}$ is the maximum half angle of the nozzle.

The two cases considered were $x_0=100$, $v=0.5$, and $x_0=200$, $v=0.1$, with throat diameters of 0.032 ins., and $\theta^{1/2}=10^\circ$.

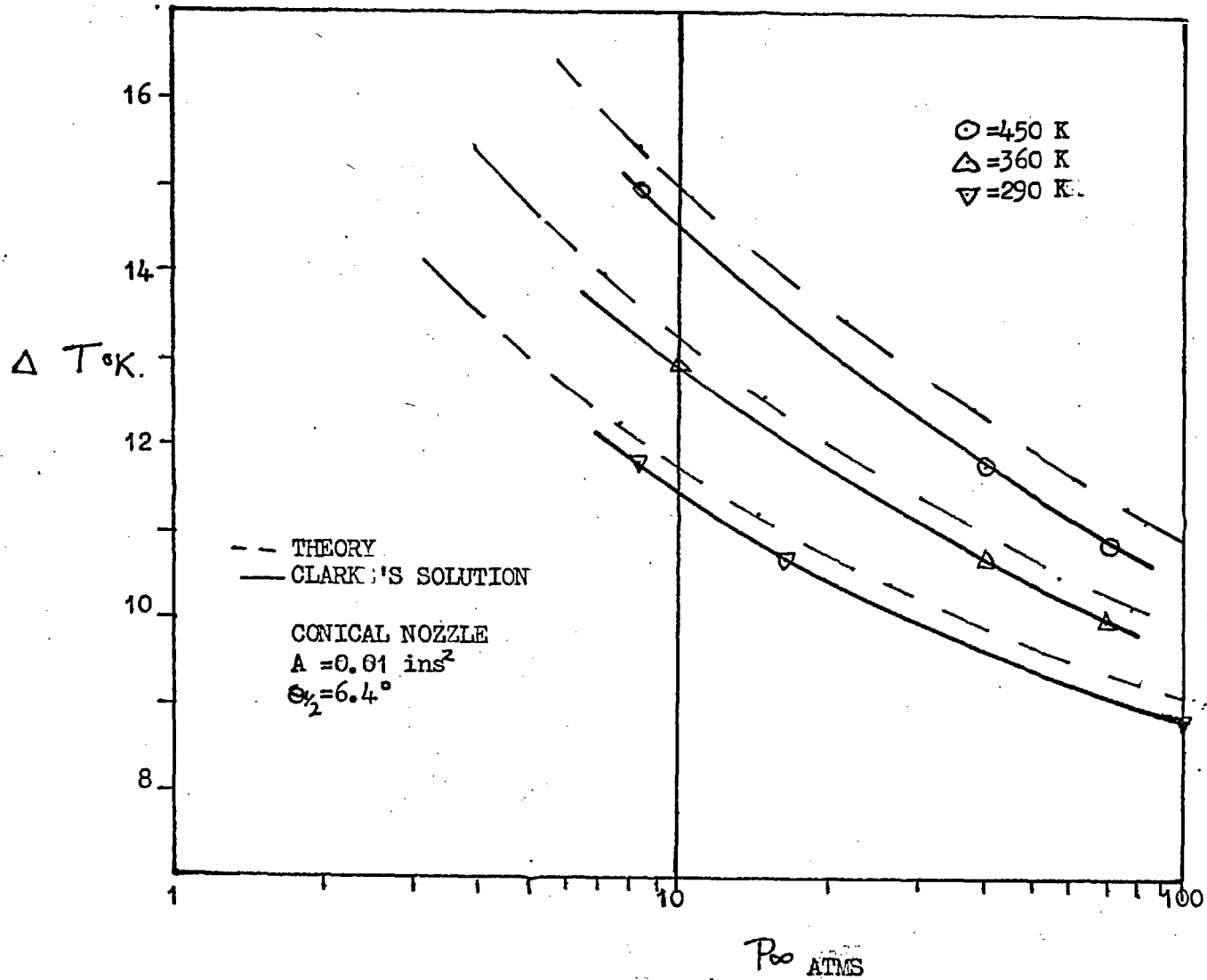
The results for the shock position in these two nozzles are

FIGURE 7.



COMPARISON OF NEAR EQUILIBRIUM SOLUTIONS.

FIGURE 8.

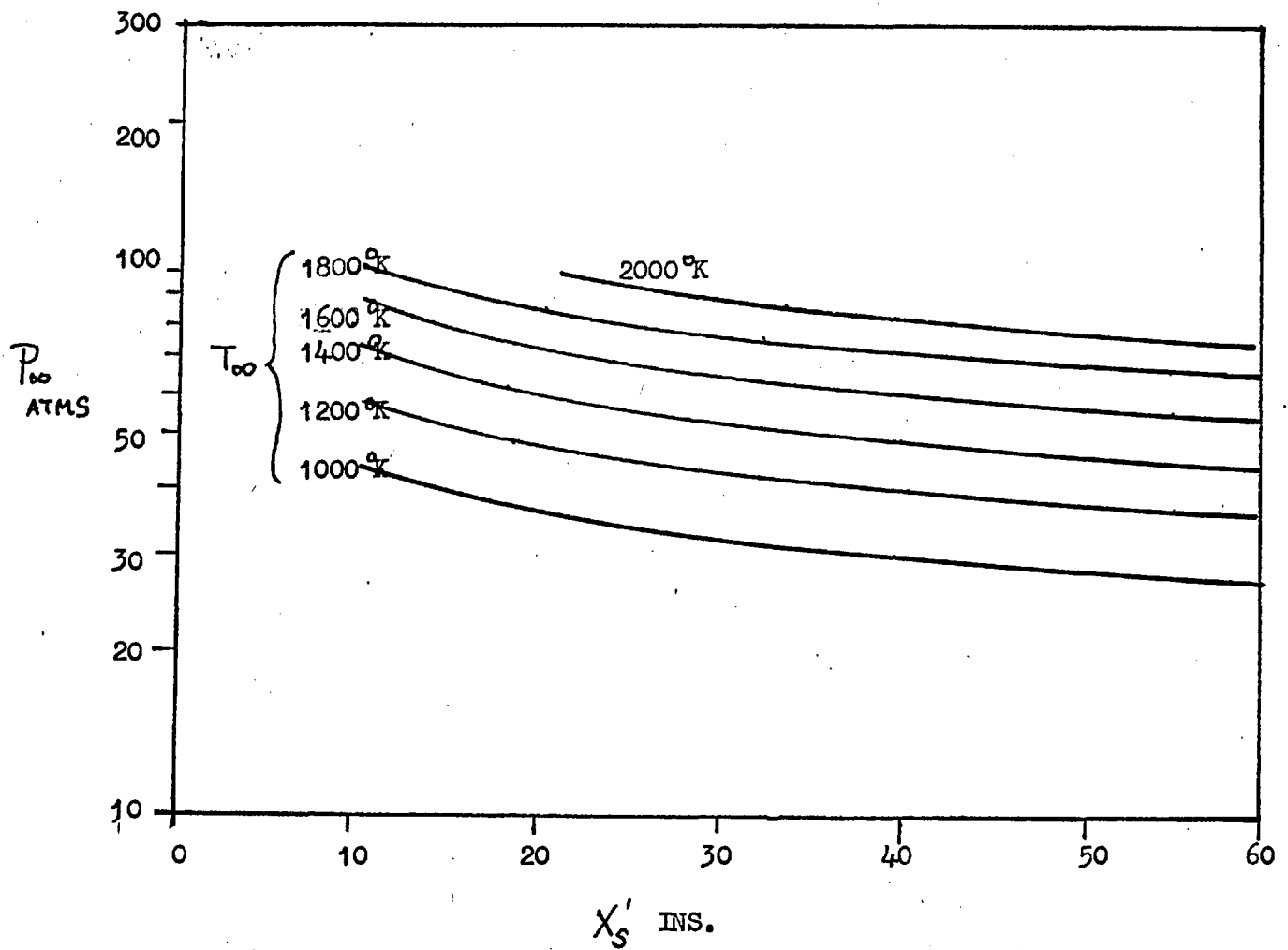
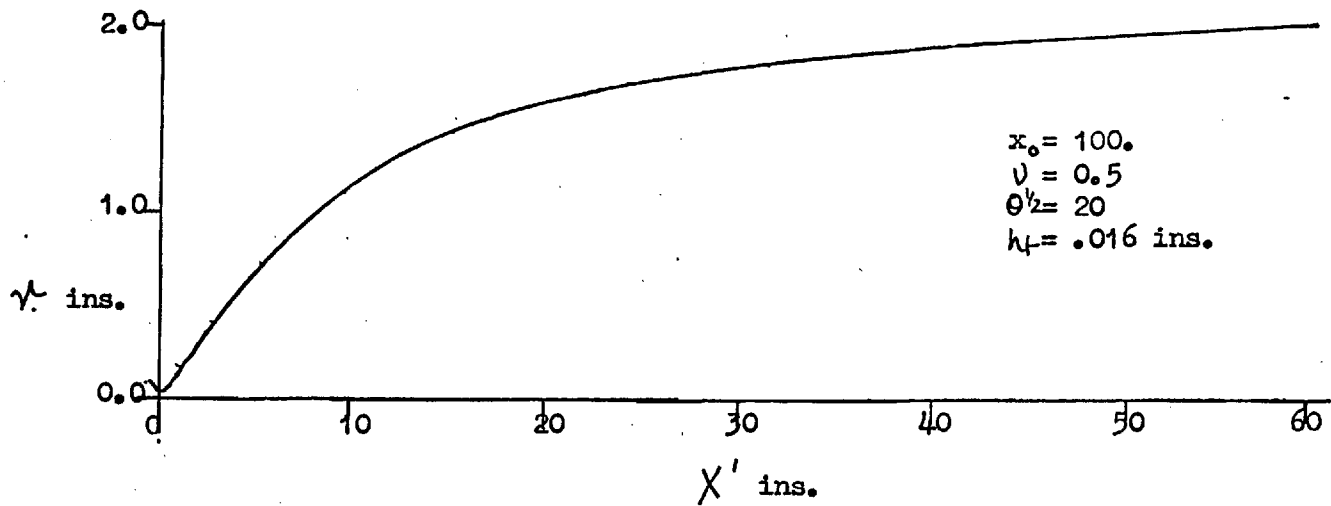


COMPARISON OF MAXIMUM TEMPERATURE PERTURBATION.

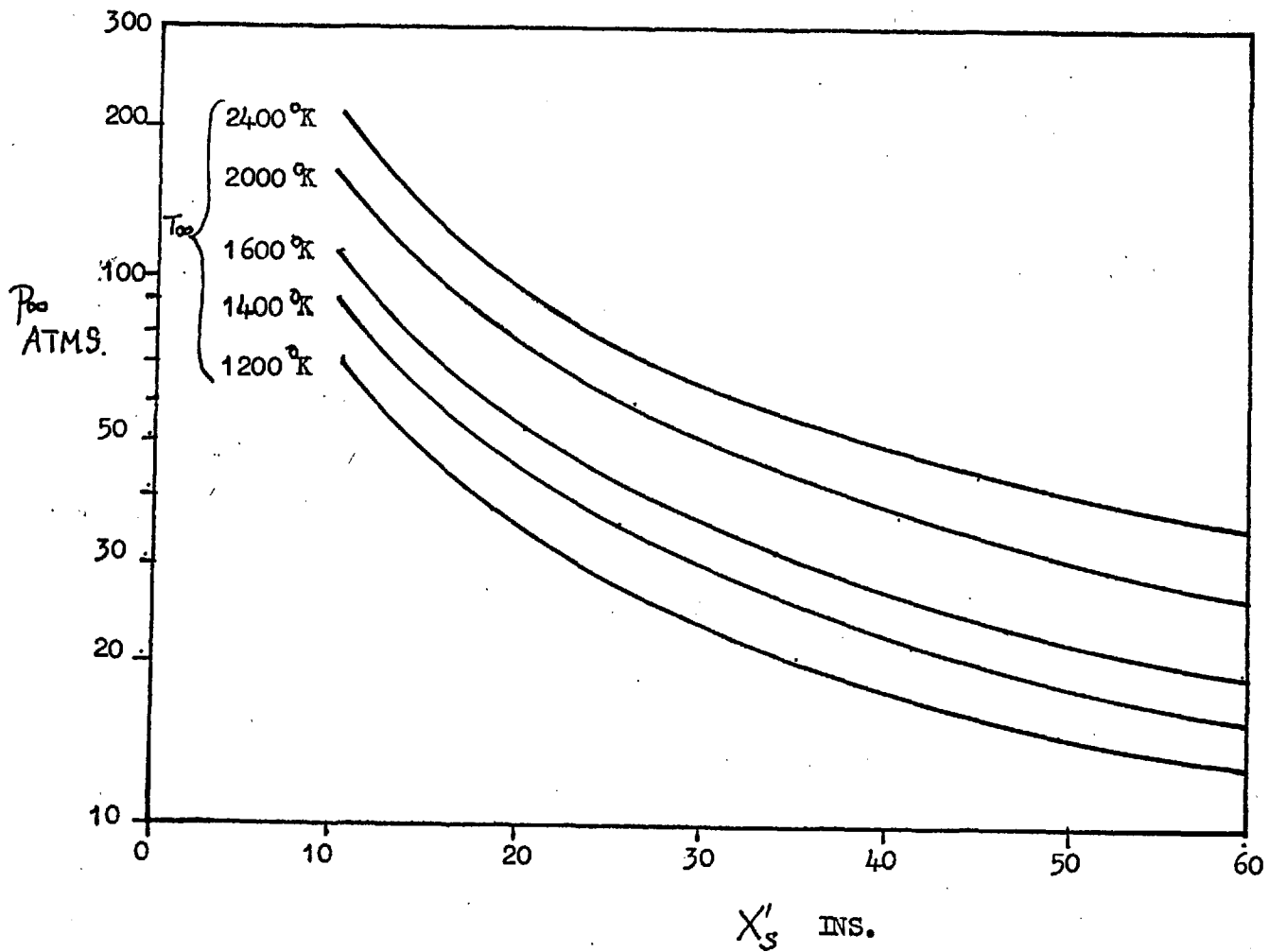
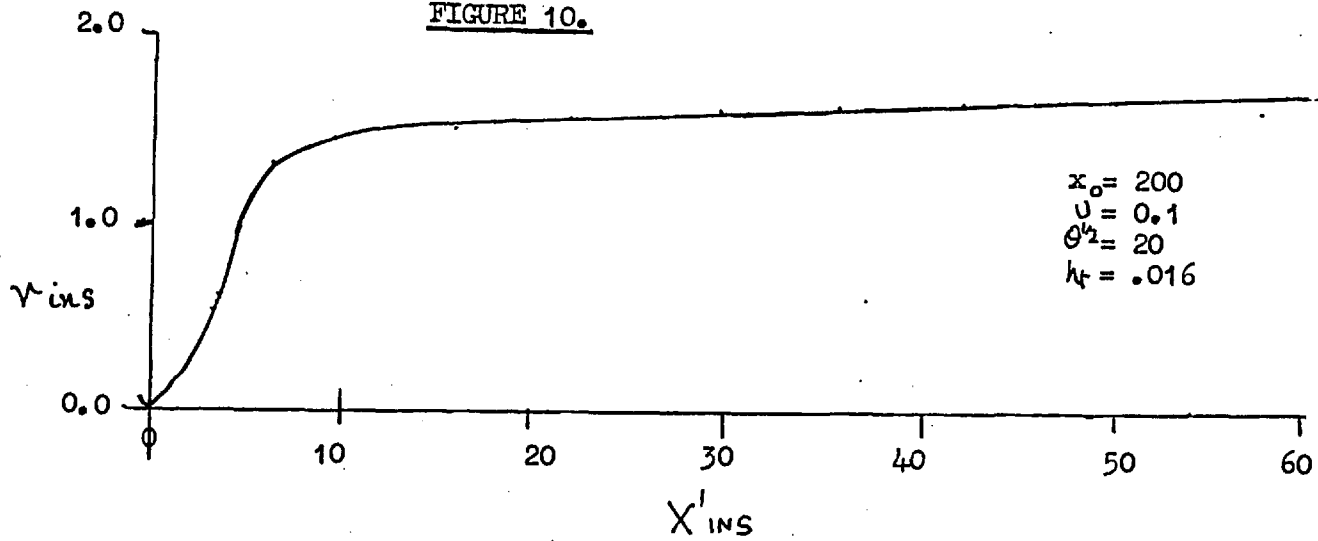
shown in figures 9 and 10. The value of the parameter K in these calculations was found to be numerically close to unity, and thus neither limit was completely applicable. It was found however, that for $V=0.5$ the first limit ($K^2 \gg \bar{\lambda}_p^b$) was the more sensible one and gave the shocks at the greatest distance downstream. For $V=0.1$, however, this limit gave the shocks too close to the throat as the number of drops was overestimated, thus the other limit ($K^2 \approx \bar{\lambda}_p^b$) was more applicable in this case.

These results do, however, show that for certain flow conditions "condensation shocks" will occur far down stream in hypersonic nozzles, and it may prove possible to study these shocks experimentally.

128.
FIGURE 9.



Typical shock positions.

FIGURE 10.

TYPICAL SHOCK POSITIONS.

8. Discussion

It has been shown that the flow of a condensing vapour can be analysed systematically using matched asymptotic expansions within the framework of continuum non equilibrium flows.

The various ^{limits}so obtained have been discussed and their solutions found. These are set out schematically in the table below.

Gas	Rate parameter	Limit	Solution
IMPURE	$\bar{\lambda}_i \gg 1$	Near equilibrium	Boundary layer solution, $\Theta = 1 - t$, etc.
	$\bar{\lambda}_i \ll 1$	Near frozen	Pseudo entropy $\gamma_0(1) \gamma = \bar{\lambda}_i^{1/2} Z$, $k > 0$ Condensation shock, $k < 0$ Gradual condensation.
PURE	$\Lambda_T \gg 1$	Near equilibrium	Boundary layer solution, $\Theta = 1 - t$, etc.
	$\Lambda_T \ll 1$ $\bar{\lambda}_p \ll 1$ $K^2 \gg \bar{\lambda}_p$	Near frozen I	Pseudo entropy (as impure), $k > 0$ Condensation shock, $k < 0$ Gradual condensation.
	$\Lambda_T \ll 1$ $\bar{\lambda}_p \ll 1$ $K^2 = O(\bar{\lambda}_p)^p$	Near frozen II	Pseudo entropy $\gamma_0(1) \gamma = \bar{\lambda}_p^{1/2} Z$, $j > k$ Condensation shock, $j < k$ Gradual condensation.

For those cases in which a shock occurs it has been shown that the shock is a one dimensional region in which the flow returns to equilibrium, without the flow being choked by the heat release involved.

Appendix I

It has been assumed that the drop moves with the stream velocity u_v . If it does not, then, since the Reynolds number based on the droplet radius is small, the drag of one drop is given by

$$D_v = 6\pi\eta(u' - u'_v) = \frac{4}{3}\pi r'^3 \rho'_L \frac{du'_v}{dt}$$

using Stokes law, where η is the gas viscosity.

Then
$$\frac{du'_v}{dx'} = \frac{9}{2} \frac{\eta}{\rho'_L} \frac{1}{r'^2} \left\{ \frac{u' - u'_v}{u'_v} \right\}$$

Writing $u_v = u'(1-\epsilon)$ and assuming that ϵ is small and constant gives

$$\frac{du'}{dx'} = \frac{9}{2} \frac{\eta}{\rho'_L} \frac{1}{r'^2} \frac{\epsilon}{(1-\epsilon)^2}$$

and if ϵ is small

$$\epsilon = \frac{2}{9} \frac{\rho'_L}{\eta'} r'^2 \frac{du'}{dx'}$$

If this is non-dimensionalised with respect to conditions at the condensation point, i.e.

$$u' = u'_c V,$$

$$x' = x'_c Z,$$

and r' is non dimensionalised with respect to an average drop size $\bar{r}'_d$,

then

$$\epsilon = \frac{2}{9} \frac{\rho'_L}{\eta'} \frac{\bar{r}'_d{}^2 u'_c}{x'_c} \frac{V^2}{Z} \frac{dV}{dZ}$$

where V , V , and Z are $O(1)$.

Thus
$$\epsilon = \frac{2}{9} \text{Red} \frac{\bar{r}'_d}{x'_c} \frac{\bar{r}'_d{}^2}{Z} \frac{dV}{dZ}$$

where Red is a Reynolds number based on the droplet radius. This means that

$$\epsilon = O \left[\text{Red} \frac{\bar{r}'_d}{x'_c} \right]$$

and so is in general small.

Appendix II

To obtain the asymptotic expansions in 6.3.1. the rate equation is written as

$$g_1 = \int_1^z F(\xi) [R(z) - R(\xi)]^3 d\xi,$$

where $F(x) = O_0^{2k-1}(\infty) x^V \exp \left\{ \frac{-K^2}{(O_0(1-O_0))^2} \right\},$

and $R(x) = \int_1^x \frac{O_0^{(x+1)/2(x-1)} (1-O_0)}{V_0} ds.$

Expanding this gives

$$g_1 = R(z) \int_1^z F(\xi) d\xi - 3R^2(z) \int_1^z R(\xi) F(\xi) d\xi \\ + 3R(z) \int_1^z R^2(\xi) F(\xi) d\xi - \int_1^z R^3(\xi) F(\xi) d\xi.$$

For $k > 0$ then

$$R(x) \sim \frac{m^{(x+1)/2}}{V_0(\infty)} \frac{x^k}{k}, \text{ as } x \rightarrow \infty,$$

However in this limit as $k \rightarrow 0(\bar{\lambda}_p^b)$ and the solution is being sought in the region in which $O = O(\bar{\lambda}_p^b)$ then

$$F(x) \sim m^2 x^{-V}, \quad O \geq O(\bar{\lambda}_p^b).$$

Treating each of these integrals in turn

- 1) $\int_1^z F(\xi) d\xi \sim m^2 z^{1-V} / (1-V),$
- 2) $\int_1^z R(\xi) F(\xi) d\xi \sim \frac{m^{2 + \frac{(x+1)}{2}} z^{k+1-V}}{V_0(\infty) (k+1-V) k},$
- 3) $\int_1^z R^2(\xi) F(\xi) d\xi \sim \frac{m^{2 + (x+1)} z^{2k+1-V}}{V_0(\infty) (2k+1-V) k},$
- 4) $\int_1^z R^3(\xi) F(\xi) d\xi \sim \frac{m^{2 + 3(x+1)/2} z^{3k+1-V}}{V_0(\infty) (3k+1-V)},$

and thus for $k > 0$ the expansion is

$$g_1 \sim \frac{A m^{\frac{x+7}{2}}}{V_0(\infty)} \frac{z^j}{k^3},$$

where $j = 3k + 1 - v$,

and
$$A_N = \left\{ \frac{1}{1-v} - \frac{3}{k+1-v} + \frac{3}{2k+1-v} - \frac{1}{j} \right\}.$$

For $k < 0$, however, the integrals defining $R(x)$ do not diverge and writing

$$R(x) \sim v_{\infty} - \bar{R}(x),$$

where v_{∞} and $\bar{R}(x)$ were defined in §6.2.1, gives

$$g_1 = \int_1^z \bar{R}^3(\eta) F(\eta) d\eta - 3\bar{R}(z) \int_1^z \bar{R}^2(\eta) F(\eta) d\eta \\ + 3\bar{R}^2(z) \int_1^z \bar{R}(\eta) F(\eta) d\eta - \bar{R}^3(z) \int_1^z F(\eta) d\eta,$$

and again treating each of these integrals in turn

$$1) \int_1^z \bar{R}^3(\eta) F(\eta) d\eta \sim \frac{-m^{2+(8+1)/2} z^{3k+1-v}}{v_0^3(\infty) (3k+1-v) k^3},$$

for $3k+1-v > 0$.

However, for $3k+1-v < 0$ the integral converges and the expansion now becomes

$$\int_1^z \bar{R}^3(\eta) F(\eta) d\eta \sim B_N - \frac{m^{(8+1)/2} z^{3k+1-v}}{v_0^3(\infty) (3k+1-v) k^3},$$

$$\text{where } B_N = \int_1^{\infty} \bar{R}^3(\eta) F(\eta) d\eta.$$

$$2) -3\bar{R}(z) \int_1^z \bar{R}^2(\eta) F(\eta) d\eta \sim \frac{3m^{(8+1)/2} z^{3k+1-v}}{v_0^3(\infty) (2k+1-v) k^3},$$

for $2k+1-v > 0$;

for $2k+1-v < 0$, i.e. $k > 3k+1-v$ the integral converges and the expansion becomes

$$-3\bar{R}(z) \int_1^z \bar{R}(\eta) F(\eta) d\eta \sim \frac{3m^{(\delta+1)/2}}{V_0(\infty)} C_N \frac{z^k}{k} + \frac{3m^{(\delta+7)/2}}{V_0(\infty)^3} \frac{z^{3k+1-\nu}}{(2k+1-\nu)k^2},$$

$$\text{where } C_N = \int_1^\infty \bar{R}(\eta) F(\eta) d\eta.$$

$$3) \quad 3\bar{R}^2(z) \int_1^z \bar{R}(\eta) F(\eta) d\eta \sim \frac{3m^{(\delta+7)/2}}{V_0(\infty)^3} \frac{z^{3k+1-\nu}}{(k+1-\nu)k^3},$$

for $k+1-\nu > 0$,

for $k+1-\nu < 0$ the integral converges and the expansion becomes

$$3\bar{R}^2(z) \int_1^z \bar{R}(\eta) F(\eta) d\eta \sim \frac{3m^{\delta+1/2}}{V_0(\infty)^2} \frac{z^{2k}}{k^2} D_N - \frac{3m^{\delta+7/2}}{V_0(\infty)^3} \frac{z^{3k+1-\nu}}{(k+1-\nu)k^3},$$

$$\text{where } D_N = \int_1^\infty \bar{R}(\eta) F(\eta) d\eta,$$

and finally

$$4) \quad -\bar{R}^3(z) \int_1^z F(\eta) d\eta \sim \frac{m^{\delta+7/2}}{V_0(\infty)^3} \frac{z^{3k+1-\nu}}{(1-\nu)k^3}, \quad \text{for } \nu < 1$$

and for $\nu > 1$.

$$-\bar{R}^3(z) \int_1^z F(\eta) d\eta \sim -\frac{Nm^{3(\delta+1)/2}}{V_0(\infty)} \frac{z^{3k}}{k^3} + \frac{m^{\delta+7/2}}{V_0(\infty)^3} \frac{z^{3k+1-\nu}}{(1-\nu)k^3},$$

$$\text{where } N = \int_1^\infty F(\eta) d\eta.$$

Thus the asymptotic expansions can easily be seen to be

$$g_1 \sim \frac{m^{\delta+7/2} A_N z^j}{V_0(\infty)^3 k^3}, \quad j > 0,$$

$$g_1 \sim B_N + \frac{m^{\delta+7/2} A_N z^j}{V_0(\infty)^3 k^3}, \quad k < j < 0,$$

and

$$g_1 \sim B_N + \frac{3m^{\delta+1/2} C_N z^k}{V_0(\infty)}, \quad j < k < 0.$$

Notation.

A area ratio.	H scaled latent heat (3.1).
A_N constant see (6.20).	h enthalpy.
$\bar{A}$ scaled area ratio.	h_t throat height.
A'_* constant see (2.25b).	J droplet production rate (p2).
B_N constant see (6.11a).	J_1 exponent $k+1-\nu$.
B constant see (4.11b).	J_2 exponent $2k+1-\nu$.
$b = (\gamma-1)\nu/L$.	J_3 exponent $3k+1-\nu$.
C constant $A=Cx^\nu$.	k exponent $1-\frac{1}{2}(\gamma+1)$.
C_N constant see (6.11b).	k^* constant see (4.14).
$C \exp(\bar{c}'/\rho_0')$ see (2.22).	K constant see (3.13).
c constant see (f7).	k' constant see (2.22).
$\bar{c}$ constant see (2.21b).	L latent heat.
C_p specific heat at const. p.	ℓ exponent $3-\frac{1}{2}(\gamma+5)$.
C_v specific heat at const. v.	ℓ_2 exponent $1+\frac{1}{2}(\gamma-3)$.
C_L specific heat of liquid.	ℓ_3 exponent $4-\frac{1}{2}(\gamma+7)$.
D' constant see (2.24).	$\bar{m}$ scaled mass see (3.1).
$\bar{D}'$ constant see (2.25b).	Mc mach no. at x_c .
D_N constant see AppII.	m frozen mach no.
e internal energy.	m_1 constants see (4.6c).
F see p6.	N number of drops.
G scaled mass fraction.	N scaled length p4.
G see p7.	N_A Avogadro's number.
G' mass of condensate p3.	n scaled length p4.
$g = H\hat{g}$.	p pressure.
$\hat{g}$ mass fraction.	p_s saturation pressure.
	Q effective heat input (5.42).

- $\cdot R$ universal gas constant,
- R gas constant.
- R scaled radius §4.
- R scaled radius §586.
- $\hat{r}$ drop radius.
- $\hat{r}_*$ critical drop radius.
- S dummy variable (4.13).
- S_x pseudo entropy.
- $\mathcal{S}$ see (5.26).
- T temperature.
- $\mathcal{T}$ scaled temperature §4.
- t temp. perturbation §4.
- u velocity.
- V scaled velocity §526.
- $\bar{V}$ scaled velocity §526.
- v velocity perturbation §4.
- x distance through nozzle.
- Y scaled length §586.
- $\hat{Y}$ scaled length §586.
- y length scale in shock. §5.
- Z scaled length see (3.4).
- α constant (2.23c).
- $\bar{\alpha}$ constant (4.8b).
- β constant (2.22).
- $\cdot \bar{\beta}$ constant see (4.8b).
- γ specific heat ratio.
- Δ constant see (5.53).
- η dummy variable.
- $\textcircled{H}$ scaled temperature §526.
- Θ scaled temperature. §3 to 6.
- Θ_S saturation temperature.
- $\mathcal{L}$ rate parameter §3.3.
- $\mathcal{L}$ rate parameter §2.
- λ scaled rate parameter
- $\bar{\lambda}$ scaled rate parameter
- $\hat{\lambda}$ scaled rate parameter } §586.
- N molecular weight.
- ν exponent in $A = Cx^\nu$.
- ξ scaled length.
- π scaled density (3.4).
- $\hat{\pi}$ scaled density (5.57).
- Π scaled density (5.17).
- $\mathcal{I}$ scaled density (5.37).
- ρ density.
- σ surface tension.
- τ see (5.26).
- ϕ saturation ratio P/P_s .
- ψ scaled pressure (3.4).

Subscripts.

0,1, denote perturbation quantities in the various solutions.

v vapour.

i impure

l liquid.

p pure.

c condensation point.

oo stagnation conditions.

r radius.

Dimensional quantities.

All variables are non-dimensionalised according to the scheme out lined in (§ 2.) any dimensional quantities are denoted by primed variables, eg. u', ρ' , etc.

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