

Higher genus hyperelliptic reductions of the Benney equations

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Abstract. It was shown by Gibbons and Tsarev (1996 *Phys. Lett. A* **211** 19, 1999 *Phys. Lett. A* **258** 263) that N -parameter reductions of the Benney equations correspond to N -parameter families of conformal maps. Here, we consider a specific set of these, the hyperelliptic reductions. The mapping function for this is calculated explicitly by inverting a second-kind Abelian integral on the stratum Θ_1 of the Jacobi variety of a genus g ($g \geq 3$) hyperelliptic curve. This is done using a method based on the result of Jorgenson (1992 *Israel Journal of Mathematics* **77** 273).

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1. Introduction

1.1. Reductions of the Benney Moment Equations

The Benney equations [3] are an example of an infinite system of hydrodynamic type. These can be written as a Vlasov equation [7], [15]

$$\frac{\partial f}{\partial t_2} + p \frac{\partial f}{\partial x} - \frac{\partial A^0}{\partial x} \frac{\partial f}{\partial p} = 0.$$

Here $f = f(x, p, t)$ is a distribution function and the moments are defined by

$$A^n = \int_{-\infty}^{\infty} p^n f \, dp.$$

Benney showed that this system has infinitely many conserved densities, polynomial in the moments A^n .

Following [14] and [1], we will now consider reductions of the moment equations; that is the case where only a finite number, N , of the A^n are independent. Here, the moment equations can be reduced to a diagonal system of hydrodynamic type with N Riemann invariants, $\hat{\lambda}_i$ say, dependent on N characteristic speeds, $\hat{p}_i$. We will assume that the characteristic speeds are real and distinct.

It was shown by Tsarev and one of the authors that in such a case the reductions correspond to N -parameter families of conformal mappings of slit domains. For details

of the properties of these maps and the general construction of such a domain see [8] and [9]. We will now consider a specific set of these reductions which we will call the hyperelliptic reductions.

1.2. Hyperelliptic reductions

For this set of reductions the conformal mapping $\lambda(p) : \Gamma_1 \rightarrow \Gamma_2$ is defined as follows. Let Γ_1 be the upper half p -plane with $3n$ real points marked on it, p_i ($i = 1, \dots, 2n$) and the set of characteristic speeds $\hat{p}_j$ ($j = 1, \dots, n$). These satisfy

$$p_1 < \hat{p}_2 < p_3 < p_4 < \hat{p}_3 < p_5 < \dots < p_{2n-1} < \hat{p}_n < p_{2n}.$$

The domain Γ_2 is the upper half λ -plane with n vertical slits going from the fixed real points λ_i^0 to the variable points $\hat{\lambda}_i$ ($i = 1, \dots, n$). Here, $\hat{\lambda}_i$ is the Riemann invariant associated with the characteristic speed $\hat{p}_i$ and it satisfies the relation

$$\operatorname{Re}(\hat{\lambda}_i) = \lambda_i^0.$$

We now impose the conditions

$$\lambda(p) = p + O\left(\frac{1}{p}\right) \quad \text{as } p \rightarrow \infty \quad (1)$$

and

$$\lambda(p_{2i-1}) = \lambda(p_{2i}) = \lambda_i^0 \quad (i = 1, \dots, n). \quad (2)$$

It follows that $\lambda(p)$ is a function of n independent parameters which may be taken to be $\operatorname{Im}(\hat{\lambda}_i)$ ($i = 1, \dots, n$), the varying heights of the slits $\ddagger$ and that Γ_2 is a polygonal domain. The map $p \rightarrow \lambda(p)$ is thus of Schwarz-Christoffel type:

$$\lambda(p) = p + \int_{\infty}^p [\varphi(p') - 1] dp' \quad (3)$$

where $\varphi(p)$ is given by

$$\varphi(p) = \frac{\prod_{i=1}^n (p - \hat{p}_i)}{\sqrt{\prod_{i=1}^{2n} (p - p_i)}}.$$

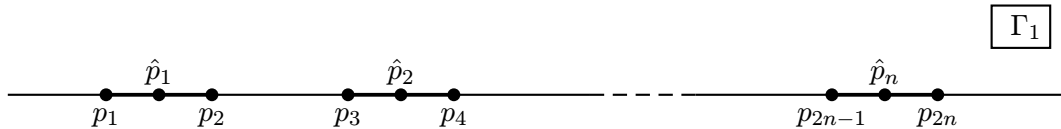


Figure 1. (The n parameter reduction) The p -plane with n branch cuts.

$\ddagger$ Note that since $\operatorname{Im}(\lambda) \geq 0 \forall p$ and the distribution function $f = -\pi \operatorname{Im}(\lambda)$, the distribution function is negative.

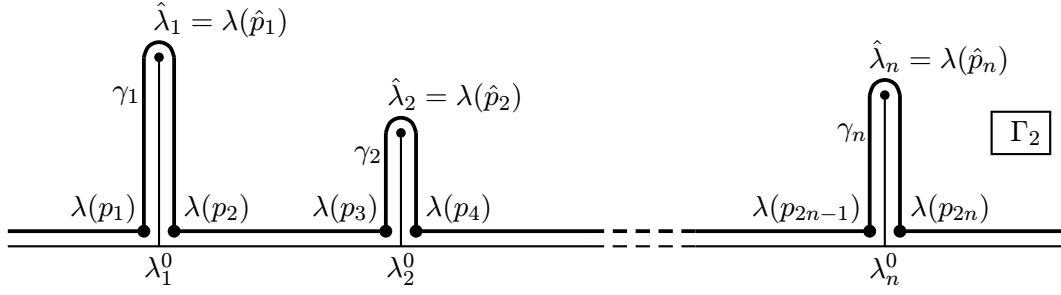


Figure 2. The λ -plane associated with figure 1.

One of the conditions in (1) and (2) may be replaced by the constraint that the residue of $\varphi(p)$, as $p \rightarrow \infty$ on either sheet, is zero. This provides a relation between the set of points p_i and the set of characteristic speeds $\hat{p}_j$. Rewriting

$$\varphi(p) = \frac{p^n - \alpha_{n-1} p^{n-1} - \alpha_{n-2} p^{n-2} - \cdots - \alpha_1 p - \alpha_0}{\sqrt{\prod_{i=1}^{2n} (p - p_i)}},$$

we find that the expansion of $\varphi(p)$ near infinity is

$$1 + \frac{\left(\frac{1}{2} \sum_{i=1}^{2n} p_i - \alpha_{n-1}\right)}{p} + O\left(\frac{1}{p^2}\right).$$

The condition on the residue is therefore satisfied when

$$\alpha_{n-1} = \frac{1}{2} \sum_{i=1}^{2n} p_i$$

that is,

$$\sum_{i=1}^n \hat{p}_i = \frac{1}{2} \sum_{i=1}^{2n} p_i. \quad (4)$$

It follows that $\varphi(p) dp$ is a *second* kind Abelian differential on the Riemann surface

$$R_g = \left\{ (p, v) : v^2 = \prod_{i=1}^{2n} (p - p_i) \right\}.$$

where $g = n - 1$. That is, the differential 1-form $\varphi(p) dp$ is meromorphic on R_g with zero residue at each singular point.

This surface may be constructed from two copies of the complex p -plane joined along the closed intervals

$$[p_{2i-1}, p_{2i}] \quad (i = 1, 2, \dots, g + 1).$$

A homology basis $(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_g; \mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_g)$ for R_g is given in figure 3.

The first three examples of these maps, $g = 0, 1, 2$, have been worked out in detail. For $g = 0$ the mapping may be calculated directly. The case of the $n = 2$ elliptic reduction was evaluated in [14] by Yu and Gibbons. The $n = 3$ genus 2 hyperelliptic reduction was studied in [1] by the authors. We now consider the case for $g \geq 3$. All such maps, once known explicitly, correspond to reductions of Benney's equations to systems

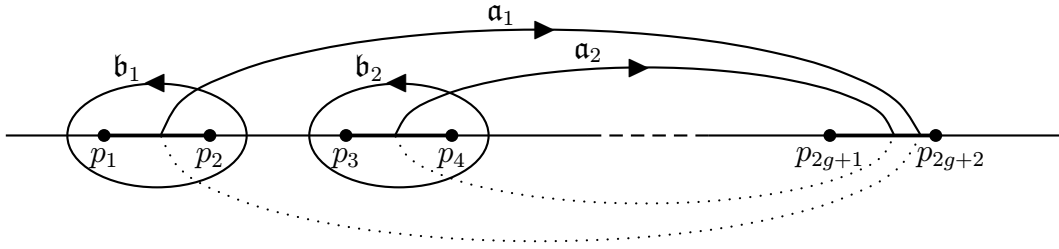


Figure 3. A homology basis on the genus g Riemann surface, R_g . The $\mathfrak{b}$ -cycles are closed loops on the first sheet and the $\mathfrak{a}$ -cycles are completed on the second sheet (broken line). These have intersection index given by $\mathfrak{a}_i \circ \mathfrak{a}_j = \mathfrak{b}_i \circ \mathfrak{b}_j = 0$, $\mathfrak{a}_i \circ \mathfrak{b}_j = -\mathfrak{a}_j \circ \mathfrak{b}_i = \delta_{ij}$.

of hydrodynamic type with finitely many Riemann invariants. Tsarev's generalised hodograph transformation [13] leads to solutions of these, in terms of the solution of an over-determined system of linear equations. The construction of n -parameter families of such maps is thus an important step towards understanding the solutions of these equations.

2. Transformation of the integral

Following [1], the integral we need to evaluate is (3):

$$\lambda(p) = p + \int_{\infty}^p \left[\frac{\prod_{i=1}^{g+1} (p' - \hat{p}_i)}{\sqrt{\prod_{i=1}^{2g+2} (p' - p_i)}} - 1 \right] dp'.$$

Setting $p = p_{2g+2} - 1/t$ in the integrand $(\varphi(p) - 1) dp$, we find

$$(\varphi(p) - 1) dp = \left(\frac{A_{g+1} t^{g+1} + A_g t^g + \dots + A_2 t^2 + A_1 t + (-1)^{g+1}}{\sqrt{\prod_{i=1}^{2g+2} [(p_{2g+2} - p_i)t - 1]}} - 1 \right) \frac{dt}{t^2} \quad (5)$$

for some constants A_i ($i = 1, 2, \dots, g+1$). We note here that

$$A_1 = (-1)^g \sum_{i=1}^{g+1} (p_{2g+2} - \hat{p}_i).$$

This may be expressed in terms of just the p_i using identity (4):

$$A_1 = \frac{(-1)^g}{2} \sum_{i=1}^{2g+1} (p_{2g+2} - p_i). \quad (6)$$

If we now remove the constant imaginary factor

$$k = \left(\frac{-4}{\prod_{i=1}^{2g+1} (p_{2g+2} - p_i)} \right)^{\frac{1}{2}}$$

from (5), then we obtain a standardized form for the irrational denominator,

$$\begin{aligned}\varphi(p) dp &= k \left(\frac{A_{g+1} t^{g+1} + A_g t^g + \cdots + A_2 t^2 + A_1 t + (-1)^{g+1}}{s} \right) \frac{dt}{t^2} \\ &= k \left(A_{g+1} t^{g-1} + A_g t^{g-2} \cdots + A_2 + \frac{A_1}{t} + \frac{(-1)^{g+1}}{t^2} \right) \frac{dt}{s}\end{aligned}\quad (7)$$

where

$$\begin{aligned}s^2 &= -k^2 + \left[k^2 \sum_{i=1}^{2g+1} (p_{2g+2} - p_i) \right] t + \cdots + \mu_{2g} t^{2g} + 4t^{2g+1} \\ &= \mu_0 + \mu_1 t + \cdots + \mu_{2g} t^{2g} + 4t^{2g+1}.\end{aligned}\quad (8)$$

The term

$$\varphi_1(p) dp = k (A_{g+1} t^{g-1} + A_g t^{g-2} + \cdots + A_2) \frac{dt}{s}$$

in (7) may be evaluated directly since the set

$$du_i = t^{i-1} \frac{dt}{s} \quad (i = 1, 2, \dots, g)$$

forms a basis of holomorphic Abelian differentials. The last two terms in $\varphi(p) dp$ can be rewritten using (6) and the definitions of μ_0 and μ_1 in (8). We have

$$\begin{aligned}\varphi_2(p) dp &= k \left[\frac{(-1)^{g+1}}{t^2} + \frac{A_1}{t} \right] \frac{dt}{s} \\ &= (-1)^{g+1} k \left[\frac{1}{t^2} - \frac{1}{2} \left(\sum_{i=1}^{2g+1} (p_{2g+2} - p_i) \right) \frac{1}{t} \right] \frac{dt}{s} \\ &= (-1)^{g+1} k \left[\frac{1}{t^2} + \frac{1}{2} \frac{\mu_1}{\mu_0} \frac{1}{t} \right] \frac{dt}{s}.\end{aligned}\quad (9)$$

This is a second kind differential on R_g . As in the genus 2 case, we can evaluate $\varphi_2(p) dp$ using a restriction of the Jacobi inversion theorem to a one complex dimensional subspace of the Jacobi variety, the one-dimensional stratum of the theta divisor, Θ_1 .

3. The Θ divisor

Following Enolski [4], [5], let $R_g(s, t)$ be the hyperelliptic curve where s and t satisfy

$$s^2 = 4 \prod_{i=1}^{2g+1} (t - t_i) = \sum_{i=0}^{2g} \mu_i t^i + 4t^{2g+1}.$$

We define a set of holomorphic and their associated set of second kind differentials on R_g to be, respectively,

$$du_i = t^{i-1} \frac{dt}{s} \quad (i = 1, 2, \dots, g) \quad (10)$$

and

$$dr_i = \sum_{k=i}^{2g+1-i} (1 + k - i) \mu_{1+i+k} \frac{t^k dt}{4s} \quad (i = 1, 2, \dots, g). \quad (11)$$

From the period integrals of these differentials we form the matrices $\omega, \omega', \eta, \eta'$:

$$\begin{aligned} 2\omega &= \left(\oint_{\mathbf{a}_i} du_j \right) & 2\omega' &= \left(\oint_{\mathbf{b}_i} du_j \right) \\ 2\eta &= \left(- \oint_{\mathbf{a}_i} dr_j \right) & 2\eta' &= \left(- \oint_{\mathbf{b}_i} dr_j \right) \end{aligned} \quad (i, j = 1, 2, \dots, g).$$

These matrices satisfy the generalized Legendre relation

$$\begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix} \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}^T = -\frac{i\pi}{2} \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix},$$

where I_g is the $g \times g$ identity matrix.

Letting $\Lambda = 2\omega \oplus 2\omega'$ be the lattice generated by the periods of the holomorphic differentials, the Jacobi variety, $\text{Jac}(R_g)$, is the g -dimensional complex torus $\mathbb{C}^g / \Lambda$. The Jacobi variety can be subdivided into k -dimensional strata, Θ_k , defined by

$$\Theta_k = \sum_{i=1}^k \int_{(t_0, s_0)}^{(t_i, s_i)} d\mathbf{u} + 2\omega \mathbf{K}_{(t_0, s_0)} \quad (k = 1, 2, \dots, g)$$

where $\mathbf{K}_{(t_0, s_0)}$ is the vector of Riemann constants with base point (t_0, s_0) . These have the structure $\text{Jac}(R_g) = \Theta_g \supset \Theta_{g-1} \supset \dots \supset \Theta_2 \supset \Theta_1$. Such stratifications have been studied by Ônishi [12] and others.

The Abel map, $\mathfrak{A} : R_g \rightarrow \text{Jac}(R_g)$, is given by $\mathbf{u}(z)$:

$$u_i(z) = \int_{z_0}^z du_i, \quad (i = 1, 2, \dots, g)$$

where the $u_i(z)$ are taken modulo Λ and the base point $z_0 = (t_0, s_0)$ is any fixed point in R_g . These create a one-dimensional image of the hyperelliptic curve in the Jacobi variety. For the inversion theorem we require an extension of this map to a set of points.

Definition 3.1 *A divisor $\mathcal{D}$ on the Riemann surface R_g is defined by the finite formal sum*

$$\mathcal{D} = \sum_{i=1}^M n_i z_i$$

where $n_i \in \mathbb{Z}$ and $z_i = (s_i, t_i) \in R_g$.

We define the Abel mapping of $\mathcal{D}$ into $\text{Jac}(R_g)$ by

$$\mathfrak{A}(\mathcal{D}) = \sum_{i=1}^M n_i \int_{z_0}^{z_i} d\mathbf{u} \quad \text{mod } \Lambda.$$

The lower limit of integration, here the point z_0 , is called the base point of the Abel map. From now we shall set this to be (∞, ∞) .

3.1. Hyperelliptic functions

Definition 3.2 *The theta function is defined by the Fourier series*

$$\theta((2\omega)^{-1}\mathbf{u}) = \sum_{\mathbf{m} \in \mathbb{Z}^g} \exp \left\{ i\pi \left[\mathbf{m}^T \tau \mathbf{m} + \mathbf{m}^T (\omega^{-1}) \mathbf{u} \right] \right\},$$

where $\tau = \omega^{-1}\omega'$ is a symmetric matrix with positive definite imaginary part.

One important property of this function is that it is zero when $\mathbf{u} = 2\omega\mathbf{K}$, the vector of Riemann constants associated with the point (∞, ∞) . For further properties see [4].

From the θ -function we define the Kleinian σ -function of the curve R_g to be

$$\sigma(\mathbf{u}) = C \exp(\mathbf{u}^T \chi \mathbf{u}) \theta((2\omega)^{-1}\mathbf{u} - \mathbf{K})$$

where

$$C = \sqrt{\frac{\pi^3}{\det 2\omega}} \left(\frac{1}{\prod_{1 \leq i < j \leq 2g+1} (t_i - t_j)} \right)^{\frac{1}{4}}$$

and $\chi = \eta(2\omega)^{-1}$ is a symmetric matrix.

In analogy to the Weierstrass $\wp$ -function, the Kleinian $\wp$ -function is defined as [4]

$$\wp_{ij} = -\frac{\partial^2}{\partial u_i \partial u_j} \ln[\sigma(\mathbf{u})] = \left(\frac{\sigma_i \sigma_j - \sigma_{ij} \sigma}{\sigma^2} \right) (\mathbf{u})$$

where

$$\sigma_i = \frac{\partial}{\partial u_i} \sigma(\mathbf{u}), \quad \sigma_{ij} = \frac{\partial^2}{\partial u_j \partial u_i} \sigma(\mathbf{u}).$$

Higher logarithmic derivatives of σ are expressed similarly. For example

$$\wp_{ijkl} = -\frac{\partial^4}{\partial u_i \partial u_j \partial u_k \partial u_l} \ln[\sigma(\mathbf{u})].$$

3.2. Jacobi Inversion formula

Theorem 1 (Jacobi inversion theorem) [4] *The Abel preimage of the point $\mathbf{u} \in \text{Jac}(R_g)$ is given by the set $S = \{(t_1, s_1), (t_2, s_2), \dots, (t_g, s_g)\} \in (R_g)^g$, where t_k are the zeros of the polynomial*

$$\mathcal{P}(t; \mathbf{u}) = t^g - t^{g-1} \wp_{g,g}(\mathbf{u}) - t^{g-2} \wp_{g,g-1}(\mathbf{u}) - \dots - \wp_{g,1}(\mathbf{u})$$

and the s_k are given by

$$s_k = - \left. \frac{\partial \mathcal{P}(t; \mathbf{u})}{\partial u_g} \right|_{t=t_k}.$$

For the integral of the differential (9), we need the preimage of $\mathbf{u}$ when the points $t_i \rightarrow \infty$ ($i = 2, \dots, g$). That is, for the case when $S = \{(t_1, s_1)\}$ and so $\mathbf{u} \in \Theta_1$:

$$\mathfrak{A}(S) = \int_{\infty}^{t_1} d\mathbf{u}.$$

This relation has been calculated from the results of Jorgenson [11] by Enolski (see Appendix A). We obtain

$$t_1 = - \frac{\sigma_1}{\sigma_2}(\mathbf{u}) \Big|_{\mathbf{u} \in \Theta_1} \quad (12)$$

where the one-dimensional stratum Θ_1 may be defined as

$$\Theta_1 = \{\mathbf{u} : \sigma(\mathbf{u}) = 0, \sigma_k(\mathbf{u}) = 0 \ (k = 3, \dots, g)\}.$$

This useful result (12) was first given by Grant in [10].

4. Evaluation of the integral

We now further transform the integrand $(\varphi_1(p) + \varphi_2(p)) dp$ using the substitution $t = (-\sigma_1/\sigma_2)(\mathbf{u})$ (12) and the definitions of the holomorphic differentials, du_i ($i = 1, 2, \dots, g$) (10).

Table 1. A list of branch points (p_i) and poles ($\infty_{\pm}$) of $\lambda(p)$ with the corresponding points in the t and $\mathbf{u}$ variables.

(p)	p_1	p_2	$\dots$	p_{2g+1}	p_{2g+2}	$\infty_{\pm}$
(t)	t_1	t_2	$\dots$	t_{2g+1}	∞	$0_{\pm}$
$(\mathbf{u})$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\dots$	$\mathbf{u}_{2g+1}$	$\mathbf{0}$	$\pm \mathbf{u}_0$

Lemma 1 Let $t = (-\sigma_1/\sigma_2)(\mathbf{u})$ where $\mathbf{u} \in \Theta_1$ and define $du_i = t^{i-1} dt/s$, a set of holomorphic differentials on R_g . Then

$$\varphi(p) dp = k (\mathbf{A}^T \cdot d\mathbf{u}) + (-1)^{g+1} k \left(\frac{\sigma_2^2}{\sigma_1^2}(\mathbf{u}) - \frac{1}{2} \frac{\mu_1}{\mu_0} \frac{\sigma_2}{\sigma_1}(\mathbf{u}) \right) \frac{dt}{s}$$

where $\mathbf{A}^T = (A_2, A_3, \dots, A_{g+1})$.

The term

$$\varphi_2(\mathbf{u}) du_1 = \left(\frac{\sigma_2^2}{\sigma_1^2}(\mathbf{u}) - \frac{1}{2} \frac{\mu_1}{\mu_0} \frac{\sigma_2}{\sigma_1}(\mathbf{u}) \right) du_1$$

is a second kind differential with a pole of order 2 at $\mathbf{u} = \pm \mathbf{u}_0$ (see Table 1). This can be verified as follows.

Since $\mathbf{u}_0$ is a regular point on the hyperelliptic curve R_g , we can evaluate the expansion of φ_2 near $\mathbf{u}_0$ in terms of the local parameter t . Setting $v_k = \mathbf{e}_k^T \cdot (\mathbf{u} - \mathbf{u}_0)$ where $(\mathbf{e}_k)_j = \delta_{kj}$, we have

$$\begin{aligned} v_k &= \int_{\infty}^t du_k - \int_{\infty}^0 du_k \\ &= \int_0^t \frac{t^{k-1}}{\sqrt{4t^{2g+1} + \mu_{2g}t^{2g} + \dots + \mu_1t + \mu_0}} dt. \end{aligned}$$

This gives

$$v_k = \left(\frac{1}{k} \frac{1}{\sqrt{\mu_0}} \right) t^k - \left(\frac{1}{2(k+1)} \frac{\mu_1}{\mu_0^{3/2}} \right) t^{k+1} + O(t^{k+2}) \quad (k = 1, 2, \dots, g)$$

and so for $k > 1$

$$v_k = \left(\frac{1}{k} \mu_0^{(k-1)/2} \right) v_1^k + O(v_1^{k+1}). \quad (13)$$

The Taylor series of φ_2 near $\mathbf{u}_0$ can thus be expressed in terms of the single parameter $v_1 = \mathbf{e}_1^T \cdot (\mathbf{u} - \mathbf{u}_0)$. We have

$$\frac{\sigma_2}{\sigma_1}(\mathbf{u}_0 - (\mathbf{u}_0 - \mathbf{u})) = \frac{(\sigma_2) + (\sigma_{12}) v_1 + \dots}{(\sigma_{11}) v_1 + \dots} = \left(\frac{\sigma_2}{\sigma_{11}} \right) v_1^{-1} + O(1)$$

and

$$\begin{aligned} \frac{\sigma_2^2}{\sigma_1^2}(\mathbf{u}_0 - (\mathbf{u}_0 - \mathbf{u})) &= \frac{\sigma_2^2 + (2\sigma_2\sigma_{12})v_1 + \dots}{\sigma_{11}^2 v_1^2 + (\sigma_{11}\sigma_{111})v_1^3 + (2\sigma_{11}\sigma_{12})v_1 v_2 + \dots} \\ &= \left(\frac{\sigma_2^2}{\sigma_{11}^2} \right) v_1^{-2} + \left(2 \frac{\sigma_2\sigma_{12}}{\sigma_{11}^2} - \frac{\sigma_2^2\sigma_{111}}{\sigma_{11}^3} - \sqrt{\mu_0} \frac{\sigma_2^2\sigma_{12}}{\sigma_{11}^3} \right) v_1^{-1} + O(1) \end{aligned}$$

(using (13)).

These expansions may be simplified by using the substitutions for $\sigma_{11}(\mathbf{u}_0)$ and $\sigma_{111}(\mathbf{u}_0)$ calculated in Appendix B. This gives

$$\left(\frac{\sigma_2^2}{\sigma_1^2} - \frac{1}{2} \frac{\mu_1\sigma_2}{\mu_0\sigma_1} \right) (\mathbf{u}_0 - (\mathbf{u}_0 - \mathbf{u})) = \left(\frac{1}{\mu_0} \right) v_1^{-2} + O(1) \quad (\forall g \geq 3). \quad (14)$$

In analogy to the genus 2 case, we now consider the function

$$\Psi(\mathbf{u}) = -\frac{1}{\mu_0} \frac{\sigma_{11}}{\sigma_1}(\mathbf{u})$$

for $\mathbf{u} \in \Theta_1$. Since $du_i = (-\sigma_1/\sigma_2)^{(i-1)} du_1$, the derivative of Ψ with respect to u_1 along $\Theta_1 = \{\mathbf{u} : \sigma = 0, \sigma_k = 0 \ (k = 3, \dots, g)\}$ is

$$\begin{aligned} \psi &= \frac{d}{du_1} \left[-\frac{1}{\mu_0} \frac{\sigma_{11}}{\sigma_1} \right] \\ &= -\frac{1}{\mu_0} \left[\sum_{i=1}^g (-1)^{i-1} \left(\frac{\sigma_1}{\sigma_2} \right)^{i-1} \left(\frac{\sigma_{11i}}{\sigma_1} - \frac{\sigma_{11}\sigma_{1i}}{\sigma_1^2} \right) \right]. \end{aligned} \quad (15)$$

This function is only singular when $\sigma_1(\mathbf{u}) = 0$, that is when $\mathbf{u} = \pm \mathbf{u}_0$.

We calculate the Taylor series of ψ near the singular point $\mathbf{u}_0$ as follows. Since just the first three terms in the sum contain negative powers of σ_1 we will rewrite $\psi(\mathbf{u})$ as

$$\psi = -\frac{1}{\mu_0} \left[(-\sigma_{11}^2) \frac{1}{\sigma_1^2} + \left(\sigma_{111} + \frac{\sigma_{11}\sigma_{12}}{\sigma_2} \right) \frac{1}{\sigma_1} + O(1) \right] \quad (\forall g \geq 3)$$

for $\mathbf{u}$ near $\mathbf{u}_0$. If we now take the limit $\mathbf{u} \rightarrow \mathbf{u}_0 \Leftrightarrow p \rightarrow \infty$, we obtain

$$\begin{aligned} \lim_{\mathbf{u} \rightarrow \mathbf{u}_0} \left[\frac{1}{\mu_0} \frac{\sigma_{11}^2}{\sigma_1^2} \right] &= \lim_{v_1 \rightarrow 0} \left[\frac{(\sigma_{11}^2) + (2\sigma_{11}\sigma_{111})v_1 + \dots}{(\mu_0\sigma_{11}^2)v_1^2 + (\mu_0\sigma_{11}\sigma_{111})v_1^3 + (2\mu_0\sigma_{11}\sigma_{12})v_1 v_2 + \dots} \right] \\ &= \lim_{v_1 \rightarrow 0} \left[\left(\frac{1}{\mu_0} \right) v_1^{-2} + \left(\frac{1}{\mu_0} \frac{\sigma_{111}}{\sigma_{11}} - \frac{1}{\sqrt{\mu_0}} \frac{\sigma_{12}}{\sigma_{11}} \right) v_1^{-1} + O(1) \right] \end{aligned}$$

and

$$\begin{aligned} \lim_{\mathbf{u} \rightarrow \mathbf{u}_0} \left[-\frac{1}{\mu_0} \left(\frac{\sigma_{111}}{\sigma_1} + \frac{\sigma_{11} \sigma_{12}}{\sigma_2 \sigma_1} \right) \right] &= \lim_{v_1 \rightarrow 0} \left[\frac{-(\sigma_{111} \sigma_2 + \sigma_{11} \sigma_{12}) + \dots}{(\mu_0 \sigma_2 \sigma_{11}) v_1 + \dots} \right] \\ &= \lim_{v_1 \rightarrow 0} \left[\left(-\frac{1}{\mu_0} \frac{\sigma_{111}}{\sigma_{11}} - \frac{1}{\mu_0} \frac{\sigma_{12}}{\sigma_2} \right) v_1^{-1} + O(1) \right]. \end{aligned}$$

Combining these gives

$$\begin{aligned} \lim_{\mathbf{u} \rightarrow \mathbf{u}_0} \psi(\mathbf{u}) &= \lim_{v_1 \rightarrow 0} \left[\left(\frac{1}{\mu_0} \right) v_1^{-2} + \left(-\frac{1}{\sqrt{\mu_0}} \frac{\sigma_{12}}{\sigma_{11}}(\mathbf{u}_0) - \frac{1}{\mu_0} \frac{\sigma_{12}}{\sigma_2}(\mathbf{u}_0) \right) v_1^{-1} + O(1) \right] \\ &= \left(\frac{1}{\mu_0} \right) v_1^{-2} + O(1) \quad (\forall g \geq 3) \end{aligned} \quad (16)$$

(using substitution (B.1)).

From the expansion of φ_2 (14) and ψ (16) near their singular points, it follows that $(\varphi_2(\mathbf{u}) - \psi(\mathbf{u}))$ is a holomorphic function on R_g . We thus have that

$$(-1)^{g+1} \varphi_2(\mathbf{u}) du_1 + \mathbf{A}^T \cdot d\mathbf{u} = (-1)^{g+1} \psi(\mathbf{u}) du_1 + \mathbf{B}^T \cdot d\mathbf{u} \quad (17)$$

for some g -vector of constants $\mathbf{B} = (B_2, B_3, \dots, B_{g+1})^T$.

5. Evaluation of the vector $\mathbf{B}$.

Following [2], let f be a function on the Riemann surface R_g . The divisor of f , (f) , is defined as

$$(f) = \sum n_i Z_i - \sum m_i P_i \quad n_i, m_i \in \mathbb{Z}^+$$

where Z_i is a zero of f of degree n_i and P_i is a pole of f of order m_i . The degree of the divisor of f is

$$\deg(f) = \sum n_i - \sum m_i.$$

For any function f and Abelian differential dv the following hold:

$$\begin{aligned} \deg(f) &= 0; \\ \deg(dv) &= 2g - 2. \end{aligned} \quad (18)$$

We will now consider the Abelian differential

$$(-1)^{g+1} [\varphi_2(\mathbf{u}) - \psi(\mathbf{u})] du_1.$$

By construction, du_1 is a first kind Abelian differential. It therefore has no poles on R_g and zeros of total degree $(2g - 2)$. From section 4, we know that the hyperelliptic function $(\varphi_2 - \psi)$ has no poles and so, by (18), it cannot have any zeros. Hence, for some constant C_0 , we have

$$C_0 du_1 = (-1)^{g+1} [\varphi_2(\mathbf{u}) - \psi(\mathbf{u})] du_1.$$

Rewriting this using identity (17) gives

$$\begin{aligned} C_0 du_1 &= (\mathbf{B} - \mathbf{A})^T \cdot d\mathbf{u} \\ \Rightarrow C_0 \frac{dt}{s} &= [(B_2 - A_2) + (B_3 - A_3)t + \dots + (B_{g+1} - A_{g+1})t^{g-1}] \frac{dt}{s}. \end{aligned}$$

Matching coefficients of t , we see

$$C_0 = B_2 - A_2$$

and so

$$B_i = A_i \quad (i = 3, \dots, g+1).$$

The value of B_2 may be found by evaluating $(\varphi_2(\mathbf{u}) - \psi(\mathbf{u}))$ at a specific point. If, for example, we take $\mathbf{u} = \mathbf{u}_0$, then we obtain

$$C_0 = \lim_{\mathbf{u} \rightarrow \mathbf{u}_0} [\varphi_2(\mathbf{u}) - \psi(\mathbf{u})] = \left(\frac{1}{\sqrt{\mu_0}} \frac{\sigma_{22}}{\sigma_2}(\mathbf{u}_0) + \frac{2}{\mu_0} \frac{\sigma_{112}}{\sigma_2}(\mathbf{u}_0) - \frac{2}{\mu_0} \frac{\sigma_{12}^2}{\sigma_2^2}(\mathbf{u}_0) \right) + O(v_1)$$

(using substitutions (B.1), (B.2) and (B.3) from Appendix B). From this we have

$$B_2 = A_2 + (-1)^{g+1} \left(\frac{1}{\sqrt{\mu_0}} \frac{\sigma_{22}}{\sigma_2}(\mathbf{u}_0) + \frac{2}{\mu_0} \frac{\sigma_{112}}{\sigma_2}(\mathbf{u}_0) - \frac{2}{\mu_0} \frac{\sigma_{12}^2}{\sigma_2^2}(\mathbf{u}_0) \right).$$

It would be possible to rewrite $\sigma_{112}(\mathbf{u}_0)$ in terms of lower order σ -derivatives using the following procedure. For each $g \geq 1$ there exists a set of PDE of the form

$$\wp_{ijkl} - f(\mu_0, \dots, \mu_{2g+1}; \wp_{mn}) = 0, \quad (19)$$

where $1 \leq i \leq j \leq k \leq l \leq g$ and $1 \leq m \leq n \leq g$ (see [4]). If we expand (19) for $\mathbf{u}$ near $\mathbf{u}_0$, then we get Taylor series equal to zero. The relations between the σ -derivatives at the point $\mathbf{u}_0 \in \Theta_1$ are then found by setting $\sigma(\mathbf{u}_0) = \sigma_1(\mathbf{u}_0) = \sigma_k(\mathbf{u}_0) = 0$ ($k = 3, \dots, g$) and equating each coefficient with zero. This process, however, cannot easily be generalized for all $g \geq 3$.

6. Result

Setting

$$k = \pm \sqrt{\mu_0} = \pm \left(\frac{-4}{\prod_{i=1}^{2g+1} (p_{2g+2} - p_i)} \right)^{\frac{1}{2}},$$

$$\tilde{B}_2 = (-1)^{g+1} \left(\frac{1}{\sqrt{\mu_0}} \frac{\sigma_{22}}{\sigma_2}(\mathbf{u}_0) + \frac{2}{\mu_0} \frac{\sigma_{112}}{\sigma_2}(\mathbf{u}_0) - \frac{2}{\mu_0} \frac{\sigma_{12}^2}{\sigma_2^2}(\mathbf{u}_0) \right)$$

and substituting

$$p = p_{2g+2} - \frac{1}{t} = p_{2g+2} + \frac{\sigma_2}{\sigma_1}(\mathbf{u})$$

into (3) we have

$$\begin{aligned} \lambda(p) &= p + \int_{\infty}^p [\varphi(p') - 1] dp' \\ &= \left(p_{2g+2} + \frac{\sigma_2}{\sigma_1}(\mathbf{u}) \right) + \int_0^{\frac{1}{(p_{2g+2}-p)}} \left[k \mathbf{A}^T \cdot d\mathbf{u} + k \tilde{B}_2 du_1 + (-1)^{g+1} k \left(\frac{d}{du_1} \Psi(\mathbf{u}) \right) du_1 - \frac{dt}{t^2} \right] \\ &= \left(p_{2g+2} + \frac{\sigma_2}{\sigma_1}(\mathbf{u}) \right) + \left[k \left(\mathbf{A} + \tilde{B}_2 \mathbf{e}_1 \right)^T \cdot \mathbf{u} + (-1)^g \frac{k}{\mu_0} \frac{\sigma_{11}}{\sigma_1} - \frac{\sigma_2}{\sigma_1}(\mathbf{u}) \right] + \tilde{C}. \end{aligned}$$

The value of the constant $\tilde{C}$ can be found by considering the limit of $(\lambda(p) - p)$ as $p \rightarrow \infty_+ \Leftrightarrow \mathbf{u} \rightarrow +\mathbf{u}_0$. Since

$$\lim_{p \rightarrow \infty} [\lambda(p) - p] = 0,$$

we have that

$$\tilde{C} = -k \left(\mathbf{A} + \tilde{B}_2 \mathbf{e}_1 \right)^T \cdot \mathbf{u}_0 + \lim_{\mathbf{u} \rightarrow \mathbf{u}_0} \left[(-1)^{g+1} \frac{k}{\mu_0} \frac{\sigma_{11}}{\sigma_1}(\mathbf{u}) + \frac{\sigma_2}{\sigma_1}(\mathbf{u}) \right].$$

Expanding the terms in this limit we obtain

$$\begin{aligned} \lim_{\mathbf{u} \rightarrow \mathbf{u}_0} \left[(-1)^{g+1} \frac{k}{\mu_0} \frac{\sigma_{11}}{\sigma_1} \right] &= (-1)^{g+1} \left(\frac{k}{\mu_0} \right) \lim_{v_i \rightarrow 0} \left[\frac{(\sigma_{11}) + (\sigma_{111}) v_1 + \dots}{(\sigma_{11}) v_1 + (\frac{1}{2} \sigma_{111}) v_1^2 + (\sigma_{12}) v_2 + \dots} \right] \\ &= (-1)^{g+1} \left(\frac{k}{\mu_0} \right) \lim_{v_1 \rightarrow 0} \left[v_1^{-1} + \left(\frac{1}{2} \frac{\sigma_{111}}{\sigma_{11}} - \frac{\sqrt{\mu_0}}{2} \frac{\sigma_{12}}{\sigma_{11}} \right) + O(v_1) \right] \\ &= (-1)^{g+1} \left(\frac{k}{\mu_0} \right) \lim_{v_1 \rightarrow 0} \left[v_1^{-1} + \left(2 \frac{\sigma_{12}}{\sigma_2} + \frac{1}{4} \frac{\mu_1}{\sqrt{\mu_0}} \right) + O(v_1) \right] \end{aligned}$$

and

$$\begin{aligned} \lim_{\mathbf{u} \rightarrow \mathbf{u}_0} \left[\frac{\sigma_2}{\sigma_1} \right] &= \lim_{v_i \rightarrow 0} \left[\frac{(\sigma_2) + (\sigma_{12}) v_1 + \dots}{(\sigma_{11}) v_1 + (\frac{1}{2} \sigma_{111}) v_1^2 + (\sigma_{12}) v_2 + \dots} \right] \\ &= \lim_{v_1 \rightarrow 0} \left[\left(\frac{\sigma_2}{\sigma_{11}} \right) v_1^{-1} + \left(\frac{\sigma_{12}}{\sigma_{11}} - \frac{1}{2} \frac{\sigma_2 \sigma_{111}}{\sigma_{11}^2} - \frac{\sqrt{\mu_0}}{2} \frac{\sigma_2 \sigma_{12}}{\sigma_{11}^2} \right) + O(v_1) \right] \\ &= \lim_{v_1 \rightarrow 0} \left[\left(-\frac{1}{\sqrt{\mu_0}} \right) v_1^{-1} + \left(\frac{1}{4} \frac{\mu_1}{\mu_0} \right) + O(v_1) \right]. \end{aligned}$$

Since $\tilde{C}$ is constant we set $k = (-1)^{g+1} \sqrt{\mu_0}$ and hence

$$\tilde{C} = (-1)^g \sqrt{\mu_0} \left(\mathbf{A} + \tilde{B}_2 \mathbf{e}_1 \right)^T \cdot \mathbf{u}_0 + \frac{2}{\sqrt{\mu_0}} \frac{\sigma_{12}}{\sigma_2}(\mathbf{u}_0) + \frac{1}{2} \frac{\mu_1}{\mu_0}.$$

This gives the following result.

Theorem 2 *Let*

$$\begin{aligned} \lambda(p) &= p + \int_{\infty}^p \frac{\prod_{i=1}^{g+1} (p' - \hat{p}_i)}{\sqrt{\prod_{i=1}^{2g+2} (p' - p_i)}} dp', \\ k &= (-1)^{g+1} \left(\frac{-4}{\prod_{i=1}^{2g+1} (p_{2g+2} - p_i)} \right)^{\frac{1}{2}}, \\ \tilde{B}_2 &= (-1)^{g+1} \left(\frac{1}{\sqrt{\mu_0}} \frac{\sigma_{22}}{\sigma_2}(\mathbf{u}_0) + \frac{2}{\mu_0} \frac{\sigma_{112}}{\sigma_2}(\mathbf{u}_0) - \frac{2}{\mu_0} \frac{\sigma_{12}^2}{\sigma_2^2}(\mathbf{u}_0) \right) \end{aligned}$$

and $\mathbf{A}^T = (A_2, A_3, \dots, A_{g+1})$ where the A_i are defined as

$$\sum_{i=0}^{g+1} A_i t^i = \prod_{i=1}^{g+1} [(p_{2g+2} - \hat{p}_i) t - 1].$$

Then, if we set

$$p = p_{2g+2} + \frac{\sigma_2}{\sigma_1}(\mathbf{u})$$

with $\mathbf{u}, \mathbf{u}_0 \in \Theta_1$ and $\sigma_1(\mathbf{u}_0) = 0$, we have

$$\begin{aligned} \lambda(p) = & (-1)^{g+1} \sqrt{\mu_0} \left(\mathbf{A} + \tilde{B}_2 \mathbf{e}_1 \right)^T \cdot (\mathbf{u} - \mathbf{u}_0) - \frac{1}{\sqrt{\mu_0}} \frac{\sigma_{11}}{\sigma_1}(\mathbf{u}) \\ & + p_{2g+2} + \frac{2}{\sqrt{\mu_0}} \frac{\sigma_{12}}{\sigma_2}(\mathbf{u}_0) + \frac{1}{2} \frac{\mu_1}{\mu_0} \end{aligned} \quad (20)$$

on sheet R_g^+ of the Riemann surface

$$R_g = \left\{ (v, p) \in \mathbb{C}^g : v^2 = \prod_{i=1}^{2g+2} (p - p_i) \right\}$$

associated with the relation $p \rightarrow \infty_+ \Leftrightarrow \mathbf{u} \rightarrow +\mathbf{u}_0$.

We note that in the $g = 2$ case the analogous solution to (20) could be rewritten using the relation

$$\frac{\sigma_{11}}{\sigma_1}(\mathbf{u}) = \frac{\sigma_1}{\sigma}(\mathbf{u} + \mathbf{u}_0) + \frac{\sigma_1}{\sigma}(\mathbf{u} - \mathbf{u}_0) = \zeta_1(\mathbf{u} + \mathbf{u}_0) + \zeta_1(\mathbf{u} - \mathbf{u}_0)$$

for $\mathbf{u} \in \Theta_1$. In the case of higher genus reductions this is not possible since $(\mathbf{u} \pm \mathbf{u}_0) \in \Theta_2$ and ζ_1 is singular everywhere on Θ_2 .

The formula (20) seems a little more complicated than the analogous results in genus 1 and 2; the reason for this is the difficulty of expanding the terms involving $\mathbf{u}_0$ in the general case. However, we consider it remarkable that essentially the same formula is valid for any genus.

Acknowledgments

We would like to thank V Z Enolski for bringing [11] to our attention and for the result given in Appendix A.

Appendix A. Reduction of the Inversion theorem to Θ_1 .

Following Enolski and Previato [6], we begin by rewriting the main result of [11] in terms of first derivatives of the σ -function.

Theorem 3 *Let $\mathbf{K}_P$ be the vector of Riemann constants associated with the point P , $\{P_1, P_2, \dots, P_{g-1}\}$ be a set of points on R_g and let $\mathbf{a} = (a_1, a_2, \dots, a_g)^T$, $\mathbf{b} = (b_1, b_2, \dots, b_g)^T \in \mathbb{C}^g$ be any nonzero vectors. Then the following identity holds*

$$\frac{\sum_{j=1}^g \sigma_j(\mathbf{u}) a_j}{\sum_{j=1}^g \sigma_j(\mathbf{u}) b_j} = \frac{\det[\mathbf{a} | d\mathbf{u}(P_1) | \dots | d\mathbf{u}(P_{g-1})]}{\det[\mathbf{b} | d\mathbf{u}(P_1) | \dots | d\mathbf{u}(P_{g-1})]}$$

where the point $\mathbf{u}$ is given by

$$\mathbf{u} = \sum_{k=1}^{g-1} \int_P^{P_k} d\mathbf{u} + 2\omega \mathbf{K}_P.$$

Here, we take the du_i to be the holomorphic differentials defined above:

$$du_i = \frac{t^{i-1}}{s} dt \quad (i = 1, \dots, g).$$

Corollary 3.1 *Let the points $P_1, P_2, \dots, P_{g-1}$ coalesce to a point P . Then we obtain by L'Hôpital's rule*

$$\frac{\sum_{j=1}^g \sigma_j (2\omega \mathbf{K}_P) a_j}{\sum_{j=1}^g \sigma_j (2\omega \mathbf{K}_P) b_j} = \frac{\det [\mathbf{a} | d\mathbf{u}(P) | d\mathbf{u}(P)^{(1)} | \dots | d\mathbf{u}(P)^{(g-2)}]}{\det [\mathbf{b} | d\mathbf{u}(P) | d\mathbf{u}(P)^{(1)} | \dots | d\mathbf{u}(P)^{(g-2)}]} \quad (\text{A.1})$$

where $d\mathbf{u}(P)^{(k)}$ denotes the column of k^{th} derivatives of the holomorphic differentials $d\mathbf{u}(P)$.

Expanding the RHS of (A.1) we find that the numerator is the determinant of the matrix

$$C \begin{bmatrix} a_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ a_2 & t & 0 & 0 & \dots & 0 & 0 \\ a_3 & t^2 & 0 & 0 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{g-1} & t^{g-2} & 0 & 1 & \dots & 0 & 0 \\ a_g & t^{g-1} & 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

for some constant C . The matrix in the denominator of the RHS is of the same form, but with b_i instead of a_i ($i = 1, \dots, g$). It follows that (A.1) can be written as

$$\frac{\sum_{j=1}^g \sigma_j (2\omega \mathbf{K}_P) a_j}{\sum_{j=1}^g \sigma_j (2\omega \mathbf{K}_P) b_j} = \frac{a_1 t - a_2}{b_1 t - b_2}. \quad (\text{A.2})$$

To evaluate t in terms of the σ_j we can therefore set $\mathbf{a} = (1, 0, \dots, 0)^T$ and $\mathbf{b} = (0, 1, 0, \dots, 0)^T$. This gives

$$\frac{\sigma_1}{\sigma_2}(\mathbf{u}) = -t$$

for $\mathbf{u} \in \Theta_1$. Further, since only a_1, a_2 and b_1, b_2 appear in the RHS of (A.2), we obtain the following definition for Θ_1 :

$$\Theta_1 = \{\mathbf{u} : \sigma(\mathbf{u}) = 0, \sigma_k(\mathbf{u}) = 0 \quad (k = 3, \dots, g)\}.$$

Appendix B. Differential relations holding at $\mathbf{u} = \mathbf{u}_0$.

For any $\mathbf{u}$ in Θ_1 we have $\sigma(\mathbf{u}) = 0$. Expanding this identity near $\mathbf{u}_0$ we obtain a Taylor series in $v_k = \mathbf{e}_k^T \cdot (\mathbf{u} - \mathbf{u}_0)$ equal to zero:

$$\begin{aligned} 0 &= \sigma(\mathbf{u}_0 - (\mathbf{u}_0 - \mathbf{u})) \\ &= \left[\frac{1}{2} \sigma_{11}(\mathbf{u}_0) \right] v_1^2 + [\sigma_2(\mathbf{u}_0)] v_2 + [\sigma_{12}(\mathbf{u}_0)] v_1 v_2 + \left[\frac{1}{6} \sigma_{111}(\mathbf{u}_0) \right] v_1^3 + \dots \end{aligned}$$

(since $\sigma(\mathbf{u}_0) = \sigma_1(\mathbf{u}_0) = \sigma_3(\mathbf{u}_0) = 0$). If we now substitute relations (13)

$$v_k = \left(\frac{1}{k} \mu_0^{(k-1)/2} \right) v_1^k + O(v_1^{k+1}) \quad (k = 2, 3, \dots, g)$$

into this expansion, then for $g \geq 3$ we have

$$0 = \left[\frac{1}{2} \sigma_{11}(\mathbf{u}_0) + \frac{1}{2} \sqrt{\mu_0} \sigma_2(\mathbf{u}_0) \right] v_1^2 + \left[\frac{1}{6} \sigma_{111}(\mathbf{u}_0) + \frac{1}{12} \mu_1 \sigma_2(\mathbf{u}_0) + \frac{1}{2} \sqrt{\mu_0} \sigma_{12}(\mathbf{u}_0) \right] v_1^3 + O(v_1^4).$$

Setting each coefficient to zero, we find

$$\sigma_{11}(\mathbf{u}_0) = -\sqrt{\mu_0} \sigma_2(\mathbf{u}_0) \quad (\text{B.1})$$

and

$$\sigma_{111}(\mathbf{u}_0) = -\frac{1}{2} \mu_1 \sigma_2(\mathbf{u}_0) - 3 \sqrt{\mu_0} \sigma_{12}(\mathbf{u}_0) \quad (\text{B.2})$$

for $\mathbf{u}_0 \in \Theta_1$ with $\sigma_1(\mathbf{u}_0) = 0$ and for $\forall g \geq 3$.

If we repeat the above procedure for the identity $\sigma_3(\mathbf{u}) = 0$ ($\forall \mathbf{u} \in \Theta_1$), then we obtain the following expansion

$$\begin{aligned} 0 &= \sigma_3(\mathbf{u}_0 - (\mathbf{u}_0 - \mathbf{u})) \\ &= [\sigma_{13}(\mathbf{u}_0)] v_1 + [\sigma_{23}(\mathbf{u}_0)] v_2 + \left[\frac{1}{2} \sigma_{113}(\mathbf{u}_0) \right] v_1^2 + \cdots \\ &= [\sigma_{13}(\mathbf{u}_0)] v_1 + O(v_1^2). \end{aligned}$$

This gives the identity

$$\sigma_{13}(\mathbf{u}_0) = 0 \quad \text{for } g \geq 3. \quad (\text{B.3})$$

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