IMPERIAL

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Theoretical Physics

Causality, Extremality, and All That

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Abstract

It is well known that general relativity, when viewed from the perspective of quantum field theory, is a non-renormalisable theory — this is central to the problem of quantum gravity. A less problematic perspective is to view general relativity from the perspective of effective field theory. As such, the Einstein-Hilbert action is considered just the lowest-order term in a derivative expansion, where higher-order terms are suppressed and agnostically parameterise the effects of an unknown UV completion. It was thought that all effective actions of this form are able to accurately capture the effect of UV physics at low energies, but certain observables within the low-energy theories can tip us off about possibly undesirable properties of UV completions. In this thesis, we are concerned with such qualitative features of low-energy gravitational effective field theories. In the first part, we will consider the restrictions that causality puts on the Wilson coefficients of an effective field theory. To that end, we first establish the use of an appropriate notion of causality and find consistency with gravitational positivity bounds. We then apply this so-called infrared causality to put constraints on the leading-order EFT of gravity on black hole and pp-wave backgrounds in five or higher dimensions. In the second part, we are concerned with qualitative features of extremal black holes, the kinematics of which are known to have interesting implications on quantum gravity. We first study deformations to the near-horizon region of known extremal black holes and use intuition from these to state a conjecture on Wilson coefficients in terms of the behaviour of the deformations near the horizon. Finally, we consider a prototypical scalar field toy example for restrictions on EFTs in the context of AdS_2 holography.

Declaration of Originality

I confirm that this dissertation is my own work and that contributions from others have been appropriately referenced. Portions of this work have appeared in published works and were products of collaborative work, with ideas originating from group discussions and calculations performed and checked by multiple individuals.

Part I of this work is based on the following publications:

[1] Calvin Y.-R. Chen et al. "A cautionary case of casual causality". *JHEP* 03 (2022), p. 025. DOI: 10.1007/JHEP03(2022)025. arXiv: 2112.05031 [hep-th].

[2] Calvin Y.-R. Chen et al. "Surfin' pp-waves with Good Vibrations: Causality in the presence of stacked shockwaves" (Sept. 2023). arXiv: 2309.04534 [hep-th].

Part II is based on the soon to be published works:

[3] Calvin Y. -R. Chen, Claudia de Rham, and Andrew J. Tolley. "Deformations of Extremal Black Holes and the UV" (Aug. 2024). arXiv: 2408.05549 [hep-th].

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Chapter 1

Introduction

Faust is a tragedy by the German writer (and polymath) Goethe, widely consider one of the greatest works in German literature. Although it was published in 1808, nearly a century before the eventual advent of QFT and GR, Goethe remarkably captures the awkward yet exciting position modern theoretical physics finds itself in presently.

In the second scene, titled *Nacht* ("Night"), we meet the protagonist, Heinrich Faust, who is sat restlessly at his desk in his cramped Gothic study. Faust is introduced as an accomplished academic, who reveals his deepest desire in a poignant monologue:

> "*Daß ich erkenne, was die Welt*
> *Im Innersten zusammenhält.*
> That I may understand what holds
> The innermost world together."

Theoretical high-energy physics not having been established as a discipline yet, Faust was driven to master philosophy, jurisprudence, medicine, and even theology, earning him many degrees, titles, and positions along the way. This desperate and misguided attempt at understanding the universe is at the core of the pursuit of knowledge as a scientific endeavour — we aim to understand nature at its most fundamental level, yet often begin with smaller questions. However, even these more manageable questions are answered over timescales spanning lifetimes.

Faust's lamentation over this realisation resonates with the challenges modern physicists are faced with due to the double-edged swords that decoupling and separation of scales represent — phenomena at different scales (be it time, distance, energy, or momenta) separate into different hierarchies which are decoupled and insensitive to each other. This is *e.g.* why Faust is able to focus on his monologue without worrying about the intricacies of quantum gravity (yet): His desk chair and the fire in his lamp are largely insensitive to the workings of the molecules they are comprised of, which are in turn agnostic to the precise details of quantum gravity. This is extremely useful, because the problem of understanding nature is naturally organised into much more manageable chunks. However, for someone like Faust, who epitomises the high-energy physics community to some degree, this is also unfortunate. Because observations

and measurements are not sensitive to the precise details of the underlying theory, they cannot serve as conclusive evidence distinguishing between one fundamental theory and another. This creates an awkward situation — by shielding us from the messy inner workings of the world, nature inevitably hides its beauty.

As tragedies go, further despair leads Faust to seek out magic, suffer from a mental breakdown, and go on a walk — the latter ultimately leads him to make a pact with the devil, Mephistopheles. Fortunately, recent years have seen a surge of interest and subsequent understanding of non-trivial connections between high- and low-energy physics that will allow us to take a more pedestrian route towards understanding the universe. This thesis summarises aspects of constraints on quantum gravity at low energies in the spirit of EFT, based on [1–3].

This chapters serves to set the scene for the forthcoming discussion. We begin reviewing EFT and consistency conditions on them in section 1.1, before discussing gravity from the perspective of field theory and EFT in sections 1.2 and 1.3 respectively. An outline of the discussions in the bulk of this work are given in section 1.4.

1.1 Effective Field Theory

All theories are effective field theories (EFTs). This rather strong statement is exaggerated but — as catchy slogans so often do — contains some truth. Separation of scales, decoupling, and universality put a name to the phenomenon that the dynamics of a large number of degrees of freedom with complicated, possibly even unknown, interactions often reduce to simple relations between a small number of relevant degrees for specific ranges of length or time scales [4, 5]. It is the reason why our approach to the natural sciences works. EFT is the framework that formalises these ideas for quantum field theory (QFT), the framework we use to understand nature at the fundamental level. This is not just a mathematical peculiarity, but is also leveraged for practical uses in the bottom-up construction of EFTs. The fact that physics at different scales decouples can be used to write down an action for the relevant degrees of freedom with all possible terms that are consistent with known symmetries, supplied with free coefficients which are successively suppressed by the cut-off up to which the effective theory ought to be valid.

The discussion in this section is based roughly on [6, 7].

1.1.1 Field Theory

For completeness, we begin by reviewing some background on classical and quantum field theory — see *e.g.* [8, 9] for more details.

For convenience, we consider a field theory with a scalar field ϕ, although ϕ could denote any collection of possibly spinning fields[1]. In field theory, it is natural to compute correlation functions — these enable us to extract physical observables. In asymptotically flat spaces, it is

[1]The discussion below generalises to fermionic fields in a straightforward manner.

particularly convenient to consider correlation functions

$$\langle \phi(x_1) \dots \phi(x_n) \rangle := \langle \Omega | \, \mathcal{T}\{\phi(x_1) \dots \phi(x_n)\} \, | \Omega \rangle \,, \tag{1.1.1}$$

which are time-ordered expectation values of Heisenberg-picture fields ϕ sandwiched between the ground states of the system, *i.e.* the true ground state of the system. The generating function for these

$$Z[J] := \sum_{n=1}^{\infty} \frac{i^n}{n!} \left(\prod_{i=1}^{n} \int \mathrm{d}^D x_i \, J(x_i) \right) \langle \phi(x_1) \dots \phi(x_n) \rangle \tag{1.1.2}$$

is known as the partition function. It computes expectation values in the true interacting vacuum by the Gell-Mann and Low theorem

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \frac{(-i)^n}{Z[0]} \frac{\delta^n Z[J]}{\delta\phi(x_i) \dots \delta\phi(x_n)} \bigg|_{J=0} . \tag{1.1.3}$$

Fortuitously, this can be computed by a path integral — generally, these define transition amplitudes under time evolution with appropriate boundary conditions. For a theory defined by a classical action $S[\phi]$, the partition function can be evaluated by a path integral

$$Z[J] = \int \mathcal{D}\phi \, e^{iS[\phi] + i \int \mathrm{d}^D x \, J(x)\phi(x)} . \tag{1.1.4}$$

This object contains virtually all information about correlation functions in a QFT. In fact, it is possible to view Lorentzian correlation functions with a fixed ordering of coordinates (known as Wightman functions) and Euclidean correlation functions (known as Schwinger functions) as different limits of a function which is analytic in the so-called forward tube [2]. Although no information is lost when analytically continuing between them, properties such as unitarity and causality are ultimately clearer in Lorentzian signature [10–13].

In perturbative QFT, it is often useful to work with other generating functions that are related to the partition function. Disconnected graphs in the Feynman graph representation of terms in the perturbative expansion can be subtracted by taking the logarithm — this defines the connected generating function

$$W[J] = -i \log \frac{Z[J]}{Z[0]}, \tag{1.1.5}$$

which generates connected correlation functions:

$$\langle \phi(x_1) \dots \phi(x_n) \rangle_{\text{conn.}} = (-i)^n \frac{\delta^n W[J]}{\delta\phi(x_i) \dots \delta\phi(x_n)} \bigg|_{J=0} . \tag{1.1.6}$$

The response of the expectation value of the field itself to the external source is

$$\varphi := \langle \phi \rangle_{J \neq 0} = \frac{\delta W}{\delta J}, \tag{1.1.7}$$

[2] For more details on the forward tube and analyticity, see section 3.1

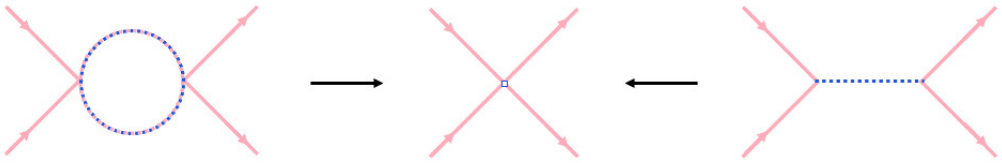


Figure 1.1: The effect of heavy modes (in blue) at tree- or loop-level is equivalent to replacement with a local vertex at low energies.

the Legendre transform with respect to which is

$$\Gamma[\varphi] := W[J] - \int \mathrm{d}^D x\, \phi(x) J(x). \tag{1.1.8}$$

This is known as the quantum effective action, which generates amputated one-particle-irreducible diagrams. Tree-level computations involving the quantum action are equivalent to computations with the original action to all orders in loops.

The quantum effective action admits a semiclassical expansion, with

$$\Gamma[\varphi] = S[\varphi] + \frac{i}{2} \operatorname{Tr} \log \left(\frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi=\varphi} \right) + \dots. \tag{1.1.9}$$

up to 1-loop.

1.1.2 Top-Down

The ethos of EFT is to exploit separation of scales and focus on the relevant degrees of freedom at hand. The starting point for doing this is the following observation: Suppose we have two fields ϕ and χ with external sources J and K — once again, these could denote any collection of spinning fields — the partition function then takes the form

$$Z[J,K] = \int \mathcal{D}\phi \mathcal{D}\chi\, e^{iS[\phi,\chi] + i\int \mathrm{d}^D x (J\phi + K\chi)}. \tag{1.1.10}$$

An almost trivial observation is that we can evaluate correlation functions involving *e.g.* just ϕ using the effective action we obtain by performing the path integral over χ

$$e^{iS_{\mathrm{EFT}}} = \int \mathcal{D}\chi\, e^{iS[\phi,\chi]}, \tag{1.1.11}$$

since

$$Z[J;0] = \int \mathcal{D}\phi\, e^{iS_{\mathrm{eff}}[\phi] + i\int \mathrm{d}^D x\, J\phi} \tag{1.1.12}$$

by definition.

It might seem miraculous that an effective action exists that is able to capture the dynamics of low-energy fields, but is even more surprising that it should take the form of a local action. The physical intuition behind this are essentially energy conservation and the uncertainty principle.

The former ensures that time evolution under the full Hamiltonian remains linear and unitary when projected onto the low-energy states, as high-energy states cannot be produced. The latter guarantees that the effect of the high-energy modes in virtual exchanges is localised to small time scales and short distances.

Example: Euler-Heisenberg Theory

A real-life example of a known EFT was derived by Euler and Heisenberg for QED in [14] — see also [15] for a review. Consider the theory of a Dirac fermion ψ with mass m (electron) coupled to a massless spin-1 or $U(1)$ gauge field A_μ (photon) in $D = 4$, described by the action

$$S_{\text{QED}} = \int \mathrm{d}^4x\sqrt{-g}\,\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D}_\mu - m)\psi\right], \tag{1.1.13}$$

where $D_\mu = \partial_\mu + iA_\mu$ and $\not{D} = \gamma^\mu D_\mu$. If we are interested in purely photon scattering in a constant background electric field at energies below the electron mass, we can instead consider the effective action after integrating out the electron

$$e^{iS_{\text{EFT}}[A]} = \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,e^{iS_{\text{QED}}}. \tag{1.1.14}$$

This is a function of the invariants

$$\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad \mathcal{G} = \frac{1}{4}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{1.1.15}$$

and takes the explicit form

$$\begin{aligned} S_{\text{EFT}}[A] &= -i\log\det\left(i\gamma^\mu D_\mu - m\right) \\ &= \int \mathrm{d}^4x\frac{\mathrm{d}s}{s^3}e^{-m^2s}\left[-e^2s^2\mathcal{G}\frac{\operatorname{Re}\cosh\left(se\sqrt{2(\mathcal{F}+i\mathcal{G})}\right)}{\operatorname{Im}\cosh\left(se\sqrt{2(\mathcal{F}+i\mathcal{G})}\right)} + 1 + \frac{2}{3}e^2s^2\mathcal{F}\right] \\ &\sim \int \mathrm{d}^4x\,e^4\frac{4\mathcal{F}^2+7\mathcal{G}^2}{45m^4} + \dots\,. \end{aligned} \tag{1.1.16}$$

Therefore, in order to compute photon or so-called light-by-light scattering in QED, we can equivalently do computations in non-linear electrodynamics. This is very convenient — instead of repeatedly doing 1-loop computations, this is a matter of using the classical equations of motion with a particular action.

1.1.3 Bottom-Up

We have seen the top-down construction of EFT, which formalises the intuition behind separation of scales, allowing us to concentrate on the physics of the degrees of freedom relevant to the problem at hand. However, EFT is not magic — it just exploits the fact that sources for decoupled fields also decouple in the partition function. This means that even when the ultraviolet (UV) completion is not known, the effective description of the high-energy dynamics should mimic its symmetries. We can use this! Given known explicit realisations of EFTs from UV theo-

ries, we can anticipate the form of and accurately write down an effective action: These should include all possible operators composed of the relevant fields of interest and their derivatives, satisfying the known symmetries of the degrees of freedom, organised in a derivative expansion, and with unknown coefficients fixed by dimensional analysis. This is known as the bottom-up construction of EFT. We can use this effective action to do Kindergarten perturbative QFT, taking care that we stay within the regime of validity of perturbation theory (by staying at weak coupling) and EFT.

In the previous example of QED, for instance, we found an effective action by explicitly evaluating the 1-loop determinant, that manages to accurately capture photon scattering at 1-loop with its classical equations of motion. However, the individual terms one could have written down in the effective action are extremely restricted. Lorentz and gauge invariance alone dictate that the action can only contain terms involving F, $\tilde{F}$ and derivatives thereof, so schematically the corrections come in the form

$$S_{\text{EFT}} \supset \int \mathrm{d}^D x\, \Lambda^D \sum_{a,b,c>2} c_{abc} \left(\frac{\nabla}{\Lambda}\right)^{2a} \left(\frac{F}{\Lambda^2}\right)^{b} \left(\frac{\tilde{F}}{\Lambda^2}\right)^{c}. \tag{1.1.17}$$

Here, Λ is the cut-off that indicates the scale at which new physics enters, while c_{abc} are Wilson coefficients — these are $\mathcal{O}(1)$-coefficients unless there are reasons to believe otherwise. As a special case, this includes the Euler-Heisenberg action from before with specifically tuned coefficients c_{abc} and $\Lambda = m$, but also encapsulates many other possible UV completions to Maxwell theory.

Spontaneous Symmetry Breaking

EFTs are fully determined by the linearly and non-linearly realised symmetries of the relevant fields — in other words, they are determined by symmetries and their symmetry-breaking pattern. This is made particularly explicit within the coset construction [16].

A theory is said to break a symmetry spontaneously, when it admits a ground state which fails to be invariant under the symmetries of the theory. What does this mean? First recall that Noether's theorem ensures the existence of a conserved current j_a^μ satisfying $\partial_\mu j_a^\mu = 0$ on-shell, for every global continuous symmetry of the action. These define charges

$$Q_a = \int_{\Sigma_t} \mathrm{d}^{D-1}\mathbf{x}\, j_a^0(t, \mathbf{x}) \tag{1.1.18}$$

which generate the symmetry transformations via the action of the commutator. Now, suppose the theory has a local order parameter. This could be some field $\phi(x)$ which transforms non-trivially under the symmetry with a non-zero vacuum expectation value $\langle\phi\rangle \neq 0$. This implies the existence of a Goldstone mode $|G\rangle$ for which

$$\langle G|\, j_a^0\, |\Omega\rangle \neq 0, \tag{1.1.19}$$

i.e. it is created from the ground state by performing a symmetry transformation. Goldstone's

theorem states that these are gapless and decouple in the low-energy limit [17].

The symmetry transformation acts inhomogeneously on the field via shifts $\phi \to \phi + c$ for constant c. To respect this symmetry, terms in the effective action describing the dynamics of the Goldstone modes need to be composed of $\partial_\mu \phi$ and its derivatives, so they take the form

$$S_{\text{EFT}} \supset \int \mathrm{d}^D x \, \Lambda^D \sum_{a,b>2} \left(\frac{\partial}{\Lambda}\right)^a \left(\frac{\partial \phi}{\Lambda^2}\right)^b \tag{1.1.20}$$

schematically.

A simple example for this comes from the Dirac-Born-Infeld (DBI) action

$$S_{\text{DBI}} = \int \mathrm{d}^4 x \, \Lambda^4 \left(1 - \sqrt{1 + \frac{(\partial\phi)^2}{\Lambda^4}}\right), \tag{1.1.21}$$

which is the lowest-order part of the EFT describing the dynamics of the transverse fluctuations of a 3-brane with vanishing field strength embedded into $\mathbb{R}^{1,4}$[18–21]. The presence of the brane spontaneously breaks the isometry group from $\mathrm{ISO}(1,4)$ down to $\mathrm{ISO}(1,3)$, and the corresponding Goldstone boson ϕ is the brane position modulus. At leading order in small fluctuations, we find

$$S_{\text{DBI}} \sim \int \mathrm{d}^4 x \left[-\frac{1}{2}(\partial\phi)^2 + \frac{1}{8\Lambda^4}(\partial\phi)^4 + \dots\right]. \tag{1.1.22}$$

We will revisit the $(\partial\phi)^4$ interaction as the leading-order operator in the Goldstone EFT operator several times.

1.1.4 Consistency Conditions

The symmetries of a theory are crucial in the bottom-up construction of EFTs — in fact, it was thought that these provided the only restrictions. In more recent years, it was realised that there are non-trivial constraints that imply that some EFTs cannot be UV completed.

Cherished physical principles such as covariance, causality, locality, and unitarity are deeply embedded into QFT as a framework historically — we will refer to theories that satisfy these properties as being "consistent". However, the modern "bootstrap" perspective on QFT is to only care about amplitudes or correlation functions as an output of QFT — this is somewhat motivated by the fact that QFT calculations often involve complicated expressions that cancel and give simple final results. Precisely what functions or distributions one can write down for the amplitudes and correlation functions is then informed or constrained solely by symmetry and the physical principles above. In fact, the various (conformal, S-matrix, cosmological *etc.*) bootstrap programs have been extremely fruitful in pinpointing and extracting physical observables using this approach [22–27].

These principles also place non-trivial constraints on EFTs. When we construct an EFT from the top-down by projecting the S-matrix of a consistent UV theory onto the low-energy states, the resulting effective theory is automatically consistent within the Hilbert space containing only

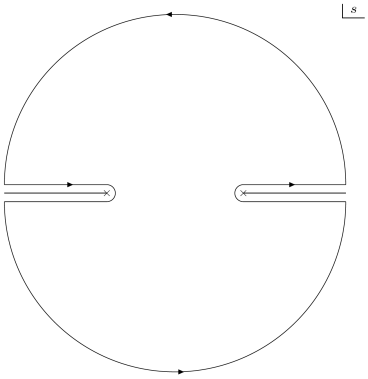


Figure 1.2: The contour $\mathcal{C}'$ used to derive dispersion relations. Figure taken from [62].

the low-energy states. In particular, the resulting theory satisfies causality, unitarity, and locality essentially due to energy conservation and the uncertainty principle. When we construct an EFT from the bottom-up, that theory is also automatically consistent within the low-energy theory itself. However, it is not obvious that these will also descend from a consistent UV theory. It is now well-understood that properties such as unitarity, locality, Lorentz invariance, and causality are non-trivially retained along renormalisation group (RG) flow and therefore put constraints on possible low-energy physics — for a review, see [28].

Positivity Bounds

One example of such constraints are positivity bounds, which are non-trivial constraints on the derivatives of scattering amplitudes through analyticity of the S-matrix [29–60].

Let us briefly review arguments for this. Consider a 4-point s-channel amplitude $\mathcal{A}(s,t)$ with external fields of masses m_i with $i \in \{1,\dots,4\}$ and exchanges involving either massive states of arbitrary spin or massless states with spin $J < 2$ in some $D = 4$-dimensional field theory, which is function of the kinematic Mandelstam variables s and t. We want to derive a dispersion relation for this [61]. As we discuss later, causality guarantees that the scattering amplitude is analytic in s at fixed t, up to singularities required by unitarity. In particular, suppose there is a branch point corresponding to a particle production threshold at $s = s_0$ with a cut going in the positive real s-direction, and a related branch cut in the u-channel due to crossing symmetry. There are also poles, which will be of no concern. By Cauchy's integral formula, we can then write

$$\frac{1}{n!}\frac{\partial^n}{\partial s^n}\mathcal{A}_s(s,t) = \oint_{\mathcal{C}} \frac{\mathrm{d}\mu}{2\pi i}\frac{\mathcal{A}_s(\mu,t)}{(\mu-s)^{n+1}} \tag{1.1.23}$$

where $\mathcal{C}$ is a circular contour around the pole at $\mu = s$. Due to analyticity away from the physical singularities, we can deform the $\mathcal{C}$ to a different contour $\mathcal{C}'$ depicted in 1.2, that goes out to $|\mu| \to \infty$ while avoiding the branch cuts due to unitarity. We can then simply divide this integral into contributions from discontinuities across the branch cuts and one from the arch $\mathcal{C}_\infty$ at infinity to give

$$\frac{1}{n!}\frac{\partial^n}{\partial s^n}\mathcal{A}_s(s,t) = \int_{\mathcal{C}_\infty}\frac{\mathrm{d}\mu}{2\pi i}\frac{\mathcal{A}_s(\mu,t)}{(\mu-s)^{n+1}} + \int_{s_0}^{\infty}\frac{\mathrm{d}\mu}{\pi}\left[\frac{\mathrm{Disc}_\mu\mathcal{A}_s(\mu,t)}{(\mu-s)^{n+1}} + \frac{\mathrm{Disc}_\mu\mathcal{A}_u(\mu,t)}{(\mu-u)^{n+1}}\right], \tag{1.1.24}$$

where $\text{Disc}_z f(z,w) = \frac{1}{2i}\left[f(z+i\epsilon,w) - f(z-i\epsilon,w)\right]$.

The dispersion relation connects a quantity computed at low energies within the EFT, $\mathcal{A}_s$, to (in principle arbitrarily) high-energy physics, the integral over $\mathcal{C}_\infty$. Non-trivial constraints on the EFT are derived precisely from this connection using input on the behaviour of the theory at high energies. Let us assume that the amplitude is bounded by μ^k at large $|k|$, then the integral over $\mathcal{C}_\infty$ vanishes for any $n \geq k$. As a further simplification, let us consider the forward scattering limit $t \to 0^-$, in which case

$$\frac{1}{n!}\frac{\partial^n}{\partial s^n}\mathcal{A}_s(s,0) = \int_{s_0}^{\infty}\frac{\mathrm{d}\mu}{\pi}\left[\frac{\text{Disc}_\mu \mathcal{A}_s(\mu,0)}{(\mu-s)^{n+1}} + \frac{\text{Disc}_\mu \mathcal{A}_u(\mu,0)}{(\mu+s-\sum_i m_i^2)^{n+1}}\right]. \tag{1.1.25}$$

Unitarity of the S-matrix guarantees that the discontinuities are positive on the positive real axis, so the left-hand side is manifestly positive for $0 < s < s_0$. This is known as an n-subtracted positivity bound!

In particular, for a theory with a mass gap as above, the Froissart-Martin bound guarantees that $k = 2$ is sufficient essentially due to unitarity [63, 64]. More subtractions are needed when the theory is gapless or contains higher-spin states, either because the Froissart-Martin bound no longer applies or because of higher-order poles due to exchanges of the latter. An illustrative example is coupling to a massless spin-2 field — the exchange has a $1/t$-pole known as the t-channel pole, and more sophisticated methods and assumptions are needed to derive bounds [41, 45, 50, 58, 60, 65–71].

Example: Goldstone EFT

The prototypical example for constraint on EFTs from positivity bounds is the Goldstone EFT, described at leading-order by

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + \frac{g}{\Lambda^4}(\partial\phi)^4. \tag{1.1.26}$$

We recognise the DBI action (1.1.21) as a special case of this with $m = 0$ and $g = 1/8$. We have given the Goldstone mode a small mass m that we will take to vanish at the end, in order for the Froissart-Martin bound to hold. We should hence think of this as describing Pseudo-Goldstone mode of a spontaneously broken approximate symmetry with small mass m. The 4-point EFT interaction first shows up in the 4-point amplitude at tree-level

$$\mathcal{A}(s,t) = \frac{2g}{\Lambda^4}\left[(s-2m^2)^2 + (t-2m^2)^2 + (u-2m^2)^2\right]. \tag{1.1.27}$$

Since

$$\lim_{t\to 0^-}\mathcal{A}(s,t) = \frac{8g}{\Lambda^4} > 0, \tag{1.1.28}$$

the twice-subtracted positivity bound above imposes $g > 0$, up to higher-order corrections. See also [72] for a holographic perspective on this.

1.2 Gravity

The other ingredient in the EFT of gravity is, of course, gravity itself. Our current best understanding of this is given by GR, which can be seen as a geometric theory describing gravitation as the fictitious force due to the curvature of spacetime induced by matter. Einstein's equation, which describe this relation, can be obtained as the classical equation of motion from the Einstein-Hilbert action. In D dimensions and with cosmological constant Λ, this is

$$S_{\rm EH} = \int \mathrm{d}^D x \sqrt{-g} \left[\frac{M_{\rm Pl}^{D-2}}{2}(R - 2\Lambda) + \mathcal{L}_{\rm matter} \right], \tag{1.2.1}$$

where the dynamics of the minimally-coupled matter fields are packaged into $\mathcal{L}_{\rm matter}$, and $M_{\rm Pl}$ is the Planck mass in D dimensions.

Particularly striking is the prediction of its own breakdown through the generic formation of curvature singularities [73]. In practical terms this means that, in order to understand dynamics at very high energies or small length scales, a UV completion to GR, *i.e.* a theory of quantum gravity, is necessary.

1.2.1 Spin-2 Field

It is useful to view GR from a different, although equivalent, perspective of a field theory describing a force mediated by massless spin-2 particles [74, 75]. From this perspective, the fully non-linear theory breaks down at sufficiently high energies because the interactions are non-renormalisable.

What we interpret as particles intuitively correspond to irreducible unitary representations of the Poincaré group, which are uniquely classified by their mass m and spin S through the Wigner classification. The field responsible for gravitational interactions is the massless spin-2 field, which is represented by a symmetric and traceless tensor field $h_{\mu\nu}$. We will see that demanding Lorentz invariance and the absence of ghosts uniquely fixes its kinetic term.

To see this, let us write down the most general local kinetic, *i.e.* two-derivative, terms that are quadratic in the fields

$$\mathcal{L}_{(2)} = h_{\mu\nu} \left[a_1 \Box h^{\mu\nu} + a_2 \partial_\alpha \partial^{(\mu} h^{\nu)\alpha} + a_3 \partial^\mu \partial^\nu h + a_4 \eta^{\mu\nu} \Box h \right]. \tag{1.2.2}$$

This is essentially fixed by the index structure and hence covariance. We can decompose a symmetric tensor field into transverse tensor and a vector part (which can further be decomposed into a transverse and longitudinal vector part) as

$$h_{\mu\nu} = h^T_{\mu\nu} + 2\partial_{(\mu}\pi_{\nu)}, \quad \partial^\mu h^T_{\mu\nu} = 0. \tag{1.2.3}$$

This introduces the following higher-derivative terms

$$\begin{aligned}\mathcal{L}_{(2)} \supset &(2a_1+a_2)\partial^\nu\pi^\mu\Box\partial_\nu\pi_\mu + 2(a_1+\frac{3}{2}a_2+2a_3+2a_4)\partial_\mu\pi^\mu\Box\partial_\nu\pi^\nu \\ &+2(2a_4+a_3)h^T\Box\partial_\lambda\pi^\lambda + 2(2a_1+a_2)h^T_{\mu\nu}\Box\partial_{(\mu}\pi_{\nu)} + 2(a_2+a_3)h^T_{\mu\nu}\partial_\lambda\partial^\mu\partial^\nu\pi^\lambda,\end{aligned} \tag{1.2.4}$$

which are typically associated to ghost instabilities. We can ensure the absence thereof by demanding that

$$a_1 = -\frac{1}{2}a_2 = \frac{1}{2}a_3 = -a_4, \tag{1.2.5}$$

so after fixing $a_1 = 1/8$, this leaves us with

$$\mathcal{L}_{(2)} = -\frac{1}{4}h^{T,\mu\nu}\,\mathcal{E}^{\alpha\beta}_{\mu\nu}h^T_{\alpha\beta}, \tag{1.2.6}$$

where

$$\mathcal{E}^{\alpha\beta}_{\mu\nu}h^T_{\alpha\beta} = -\frac{1}{2}\left[\Box h_{\mu\nu} - 2\partial^\lambda\partial_{(\mu}h_{\nu)\lambda} + \partial_\mu\partial_\nu h - \eta^{\mu\nu}\left(\Box h - \partial_\alpha\partial_\beta h^{\alpha\beta}\right)\right] \tag{1.2.7}$$

defines the Lichnerowicz operator. This is known as the Fierz-Pauli action, which is the unique kinetic term for a free massless spin-2 field. It enjoys gauge invariance under

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}, \tag{1.2.8}$$

which we recognise as linearised diffeomorphisms. In order to couple this field to matter via a stress tensor, we are forced to introduce non-linearities in the gauge transformation and the kinetic term becomes the fully non-linear Einstein-Hilbert term with

$$g_{\mu\nu} = \eta_{\mu\nu} + M_{\rm Pl}^{-\frac{D-2}{2}}h_{\mu\nu}. \tag{1.2.9}$$

Assuming that the full equations of motion contain a finite (but arbitrary) number of derivatives and satisfy an integrability condition on its divergence, it can be shown from the consistency of non-linear self-coupling that this is the unique non-linear completion to the Fierz-Pauli action [76].

Non-linear corrections to the Fierz-Pauli action from the Einstein-Hilbert action come with coupling constant $M_{\rm Pl}^{-(D-2)}$. This means that the Einstein-Hilbert action is non-renormalisable in $D > 2$, and the theory becomes strongly coupled once energies hit the Planck scale. This is how the theory predicts its own breakdown.

The Canonical Formalism

It is best to identify the degrees of freedom of a gauge theory in the canonical formalism. For gravity, this is known as the ADM formalism [77]. For a globally hyperbolic spacetime, the global time function can be used to decompose the metric as

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \gamma_{ij}\left(N^i\mathrm{d}t + \mathrm{d}x^i\right)\left(N^j\mathrm{d}t + \mathrm{d}x^j\right), \tag{1.2.10}$$

where N and N^i are known as the lapse and shift functions, and γ is the induced metric on the spatial constant-time hypersurfaces Σ_t.

In this form, the Einstein-Hilbert action becomes

$$S = \int \mathrm{d}^D x \sqrt{\gamma} \frac{M_{\mathrm{Pl}}^2}{2} \left[NR[\gamma] + N \left(K^i{}_j K^j{}_i - K^i{}_i K^j{}_j \right) \right], \tag{1.2.11}$$

up to boundary terms, and where K_{ij} is the extrinsic curvature. The lapse and shift enter the action without time derivatives and act as Lagrange multipliers, so γ_{ij} and its conjugate momentum π_{ij} are coordinates of the unconstrained phase space, with which we find the Hamiltonian

$$H = \int \mathrm{d}^{D-1} x \sqrt{\gamma} \left(N\mathcal{H} + N^i \mathcal{H}_i \right), \tag{1.2.12}$$

where $\mathcal{H}$ and $\mathcal{H}_i$ are the Hamiltonian and momentum constraints of GR respectively.

It is now clear what the degrees of freedom are. The unconstrained phase space is $D(D-1)$-dimensional. The Hamiltonian and momentum constraints are first-class constraints, so they act twice — once as a constraint, and another time as a gauge symmetry. This means that the constrained phase space has dimension $D(D-3)$, which corresponds to $D(D-3)/2$ degrees of freedom with second-order equations of motion.

1.2.2 Quantisation

From both the field theoretic and the geometric perspectives, GR breaks down at high energies and short distances. This means that a theory which is better behaved at these scales is needed. This UV completion of GR is what is typically referred to as quantum gravity. We will soon describe a modern EFT perspective on quantum gravity, but let us lose a few words on other qualitative aspects of quantum gravity first.

Local Observables

In QFT, the typical observables we are interested in are correlation functions. A natural requirement for these is that they are gauge-invariant. This is however subtle in a theory of dynamical gravity that is not perturbative around a background. The bulk ADM Hamiltonian of GR generates local diffeomorphisms and vanishes, so the theory is gauge-invariant under local diffeomorphisms, which act non-trivially on correlation functions. Local observables such as correlation functions, which are function of the operator insertion positions, therefore cannot be good observables.

There are two ways out of this. The boundary term of the ADM Hamiltonian is non-vanishing, so diffeomorphisms that reach infinity physically act on the states of the theory and are therefore not gauge transformations — these form the asymptotic symmetry group of gravity. We can therefore either define non-local observables of operators smeared along paths anchored at asymptotic infinity, or we take the operator insertions themselves and push them to asymptotic infinity. The former is analogous to Wilson lines in Yang-Mills, and the latter constructs the

S-matrix. It is sometimes said that "the only observable in quantum gravity is the S-matrix" [78].

The Gravitational Path Integral

In any theory of quantum gravity, path integrals such as the one used to compute the partition function should in principle also integrate over the dynamical metric

$$Z = \int \mathcal{D}g\, \mathcal{D}\phi\, e^{iS[g,\phi]}, \tag{1.2.13}$$

with ϕ collectively standing for the matter fields in the theory. The boundary conditions are crucial and hide the physical meaning as usual. For instance, evaluating the path integral over Riemannian manifolds with periodic Euclidean time results in the thermal partition function, which determines thermodynamic properties of the spacetime [79, 80].

As an example, let us consider the Euclidean Schwarzschild black hole (BH)

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}t_E^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} + r^2\mathrm{d}\Omega_2^2, \tag{1.2.14}$$

obtained by Wick rotating $t_L = it_E$, which agrees with its Lorentzian cousin on the $t = 0$ slice. To be a well-defined angular coordinate, Euclidean time needs to be periodic by the identification $t_E \sim t_E + \beta$, with $\beta = 8\pi M$. The Euclidean Schwarzschild black is a saddlepoint of the Euclidean action, so we can approximate the thermal partition function semiclassically as

$$Z \approx e^{-I[g_{\text{Saddle}}]}, \tag{1.2.15}$$

where I denotes the Euclidean gravitational action. The bulk contribution to this vanishes for Euclidean Schwarzschild, so contributions to the boundary are crucial — we take this to be the surface $r = r_0$, which we eventually take to infinity. The boundary term itself also diverges in this limit, so we need to include an appropriate counter-term (informed by the flat space limit)

$$I[g] = -M_{\text{Pl}}^2\left[\frac{1}{2}\int_M \sqrt{g}R + \int_{\partial M}\sqrt{h}\,(K - K_0)\right], \tag{1.2.16}$$

where K is the trace of the extrinsic curvature on ∂M and K_0 is the trace of the extrinsic curvature of a same surface with the same intrinsic geometry embedded in flat spacetime. This only cancels the divergence in the $r_0 \to \infty$ limit, but also affects the finite part. One can show that for the Euclidean Schwarzschild geometry, in units where $G_N = 1$

$$I[g_{\text{Saddle}}] = -\frac{1}{8\pi}\beta\left[(8\pi r_0 - 12\pi M) - (8\pi r_0 - 8\pi M)\right] = \frac{M\beta}{2}, \tag{1.2.17}$$

so that the thermal partition function is

$$Z(\beta) = e^{-\beta^2/16\pi}. \tag{1.2.18}$$

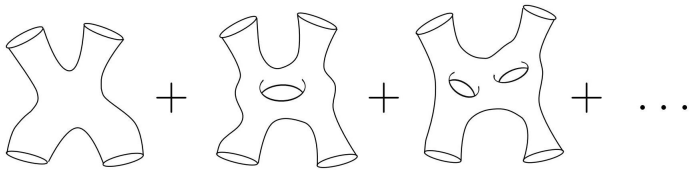


Figure 1.3: The string loop expansion. Figure taken from [86].

String Theory

We are actually in possession of explicit examples of consistent theories of quantum gravity. Arguably the most promising candidate for this is string theory.

This is a framework studying strings as fundamental particles. At length scales above the string length ℓ_s, different vibrational modes in the spectrum of the string look like fundamental particles in different representations. For historical reasons, one often uses the so-called Regge slope $\alpha' = (\ell_s/2\pi)^2$ or the string mass scale $M_s = 1/\sqrt{\alpha'}$. We can study this spectrum at weak-coupling — amongst massive higher-spins states, this crucially also includes a massless spin-2 state. With a bit more work, this can be identified as the graviton. In particular, the massless sectors are described by supergravity.

String amplitudes crucially differ qualitatively from field theory amplitudes. For example, the tree-level amplitude for scattering of four (massless) gravitons in type-II superstring theory is given by the Virasoro-Shapiro-type amplitude

$$M_{\text{Pl}}^{D-2}\mathcal{A} \sim \frac{1}{16}\alpha'^3 \frac{\Gamma\left(-\frac{1}{4}\alpha' s\right)\Gamma\left(-\frac{1}{4}\alpha' t\right)\Gamma\left(-\frac{1}{4}\alpha' u\right)}{\Gamma\left(1+\frac{\alpha' s}{4}\right)\Gamma\left(1+\frac{\alpha' t}{4}\right)\Gamma\left(1+\frac{\alpha' u}{4}\right)} K_1 K_2, \tag{1.2.19}$$

where K_i are kinematic factors that are quartic in the momenta [81, 82]. At fixed t there are poles at $s = 4n/\alpha'$ for $n \in \mathbb{N}^+$, which correspond to the masses of the states with spin up to $2(n+1)$ in the closed string spectrum, so the scattering describes a tree-level exchange of infinitely many massive higher-spin states. At very high energies, the amplitude takes the form

$$M_{\text{Pl}}^{D-2}\mathcal{A} \sim -\frac{1}{2}\exp\left[-\frac{\alpha'}{2}\left(s\log s + t\log t + u\log u\right)\right], \tag{1.2.20}$$

i.e. falls off exponentially. Any field theory amplitude exchanging only finitely many massive higher-spin states will lead to worse UV behaviour — it only has this soft UV behaviour because the individual contributions are resummed in string theory. The softened UV behaviour is often attributed to the minimal length of the string, and is in fact not just a property at tree-level. Scattering amplitudes in string theory are organised in a loop expansion as depicted in 1.3. The individual terms have been rigorously shown to be UV-finite up to two loops, and it is thought that this holds to all order in loops [83–85].

There is an intuitive picture for this [87]. In field theory, an amplitude at g loops with external momenta $\{p_k\}$ and internal momenta $\{\ell_i\}$ can be written in terms of integrals over Schwinger

parameters $\{s_i\}$ as

$$\left(\int \prod_{i=1}^{g} \mathrm{d}^D \ell_i\right) \prod_{j=1}^{r} \left(k_j^2 + m_j^2\right)^{-1} \mathcal{N}\left(\{\ell_i\}, \{p_k\}\right) = \left(\prod_{i=1}^{r} \int_0^\infty \mathrm{d}s_i\right) f(\{s_j\}), \tag{1.2.21}$$

where $\{k_j\}$ are linear combinations of the external and internal momenta. The UV divergences of the original diagrams are encoded near $s = 0$, while infrared (IR) divergences are associated to $s \to \infty$. Now, an n-point amplitude at g loops in superstring theory takes the form of an integral

$$\int_{\mathcal{M}_{g,n}} \prod_{i=1}^{\dim \mathcal{M}_{g,n}} \mathrm{d}m_i \, g(\{m_i\}) \tag{1.2.22}$$

over a $\dim \mathcal{M}_{g,n} = 6(g-1) + 2n$-dimensional moduli space parameterising a genus-g Riemann surface with n punctures of some worldsheet correlation function $g(\{m_i\})$. These integrals are divergent typically when the Riemann surface itself is singular — this turns out to be the case for when $m_i \to \infty$. Interpreted as Schwinger parameters, we therefore see that these are interpreted as IR divergences, which we can deal with.

In this sense, string theory constitutes a full UV completion to Einstein gravity.

Holography

Another fruitful avenue from which we have learned a lot about quantum gravity is from ideas that loosely fall under the term holography.

The first hint of the holographic nature of gravity comes from BH entropy. From the thermal partition function for the Euclidean Schwarzschild BH in (1.2.18), we find that the BH entropy is

$$S_{\text{BH}} = (1 - \beta\partial_\beta) \log Z(\beta) = \frac{A_{\text{BH}}}{4} \tag{1.2.23}$$

in terms of the surface area $A_{\text{BH}} = 16\pi M^2$ of the BH horizon — the relation between the entropy and surface area of a BH in fact hold more generally. Therefore, given some codimension-2 surface in a spacetime, the second law of BH thermodynamics would require that the maximum entropy we can package into the volume is bounded above by the surface area through the possibility of BH formation [88]. This motivates the holographic principle — a theory quantum gravity is secretly determined by dynamics within the boundary [89, 90]. One of the great successes of string theory as a UV completion to gravity is to reproduce the BH entropy for extremal BHs microscopically [91].

This is made much more precise by the correspondence between Anti-de Sitter (AdS) space and conformal field theories (CFTs) — a conjectured duality between dynamical theories of gravity on asymptotically AdS spacetimes and CFTs defined on the (conformal) boundary thereof via the following dictionary between the on-shell gravitational bulk and boundary field theory partition functions [92–94]

$$Z_{\text{Bulk}}[\phi, \partial M] = Z_{\text{Boundary}}[\phi], \tag{1.2.24}$$

or more concretely

$$\int_{\Phi|_{\partial M}\sim\phi} \mathcal{D}\Phi\, e^{-I[\Phi]} = \left\langle \exp\left(-\int_{\partial M} \mathcal{O}\phi\right)\right\rangle_{\text{QFT}}. \tag{1.2.25}$$

Once again, Φ collectively denotes the (light) bulk fields, while ϕ are the sources for the operators $\mathcal{O}$ in the CFT and determined as boundary conditions from the bulk fields. The correspondence itself is believed to hold more widely, but our understanding of it is in the context of certain limits of specific examples — most prominently between type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ Super-Yang-Mills theory in $D = 4$ [95, 96].

Notably, CFTs are UV complete by construction, so in principle (1.2.25) is the only non-perturbative definition for a UV-complete theory of gravity in terms of a QFT on a fixed background.

Lower-Dimensional Gravity

The Einstein-Hilbert action is renormalisable in $D = 2$. However, the counting of degrees of freedom suggests that this has -1 dynamical degrees of freedom. In fact, this is easy to see. In two dimensions, the Einstein-Hilbert action itself is a topological invariant, and relatedly

$$R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu} \tag{1.2.26}$$

is an identity.

We can however find non-trivial dynamics if we allow for scalar-tensor theories with some dilaton degree of freedom Φ[3] [97]. The most general dilaton gravity model we can write down with two-derivative action is of the form

$$S_{\text{DG}}[g, \Phi] = \int_M \mathrm{d}^2x \sqrt{|g|} \left[U_1(\tilde{\Phi})\frac{R}{2} + U_2(\tilde{\Phi}) g^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\Phi} + U_3(\tilde{\Phi})\right] \tag{1.2.27}$$

for some arbitrary functions U_1, U_2, and U_3. However, we can

- Redefine $\Phi = U_1(\tilde{\Phi})$ provided that $U_1'(\tilde{\Phi}) \neq 0$,
- Weyl transform $g \to e^{2\omega} g$ with $\omega(x) = \int^{\Phi(x)} U_2(\Phi')d\Phi'$ to cancel the kinetic term,

to bring this into the form

$$S_{\text{DG}}[g, \Phi] = \int_M \mathrm{d}^2x \sqrt{|g|} \left[\Phi\frac{R}{2} - U(\Phi)\right]. \tag{1.2.28}$$

These types of theories are well-motivated. For instance, they can arise from in spherical/dimensional reductions from higher-dimensional gravity — demanding the metric is a warped product of a $2D$ Lorentzian manifold and either S^d or T^d reduces the action to a dilaton-gravity system as above, with the warping factor playing the role of the dilaton [98–100]. This perspective is taken of in [101] to systematically find background solutions to higher-dimensional gravity coupled to p-form fields and a dilaton. Dilaton-gravity models also arise in critical string theory,

[3] Also known to be generically equivalent to $f(R)$ theories.

since the conditions for worldsheet conformal invariance can be rephrased as the equations of motion of a dilaton-gravity theory [102–107].

Jackiw-Teitelboim (JT) gravity is a specific example of a dilaton gravity theory with $U(\Phi) = \Lambda\Phi$ [108–115]. It is the simplest non-trivial such theory, and describes is obtained by spherical reduction of small perturbations for near-extremal BHs. Including the Gibbons-Hawking-York (GHY) boundary term and a holographic counterterm, the action is

$$S_{\rm JT}[g,\Phi] = \int_M \mathrm{d}^2x\sqrt{-g}\Phi\left(\frac{R}{2}-\Lambda\right) + \int_{\partial M}\mathrm{d}x\sqrt{h}\Phi(K-1) \tag{1.2.29}$$

where h is the induced metric and K is the extrinsic curvature on the boundary ∂M. This arises as the universal near-horizon limit of near-extremal BHs in higher dimensions, and is hence (more or less) of phenomenological relevance [4].

JT gravity is arguably not very closely related to Einstein gravity or its UV completion in higher dimensions — it possesses no dynamical bulk gravitational degrees of freedom. It is nevertheless a renormalisable theory of gravity, which can be consistently quantised in field theory and coupled to matter with non-trivial boundary dynamics. A crucial development in this was the improved understanding of near-AdS_2 holography in terms of JT gravity and the Sachdev-Ye-Kitaev (SYK) model [116–119]. This has earned it a lot of interest as a toy model for quantum gravity in recent years, and has been particularly useful for studying quantum aspects of BHs in the context of the information loss paradox [120–126].

1.3 The Effective Field Theory of Gravity

We have established that gravity is, in many ways, just a regular field theory. Even though it is of the non-renormalisable kind, it suggests we should be able to understand it from the perspective of EFT [7, 127–130].

1.3.1 Construction of EFT Action

To do this we follow the familiar steps. Focusing on purely gravitational degrees of freedom, the only relevant symmetry are diffeomorphisms. The effective action therefore needs to be built out of curvature tensors and their covariant derivatives. Similar to (1.1.17) and (1.1.20) for the spin-1 field and the Goldstone boson respectively, the action for the EFT of gravity is

$$S_{\rm EFT} = \int \mathrm{d}^Dx\sqrt{-g}\,M_{\rm Pl}^{D-2}\left[\frac{1}{2}R + \Lambda^2\sum_{m\geq 0,n\geq 2} c_{mn}\left(\frac{\nabla}{\Lambda}\right)^m\left(\frac{\text{Riemann}}{\Lambda^2}\right)^n\right] \tag{1.3.1}$$

schematically. More precisely, the term at fixed m and n collectively stands for all possible contractions of n different Riemann tensors with m covariant derivatives acting on a subset

[4] From the higher-dimensional perspective, the dilaton plays the role of an area, in which case the action should come with an appropriate factor of $M_{\rm Pl}$, which can however always be absorbed in Φ. We would then expect also a contribution from the background value of the dilaton Φ_0 multiplying the topological Einstein-Hilbert term. This usually contributes to the entropy.

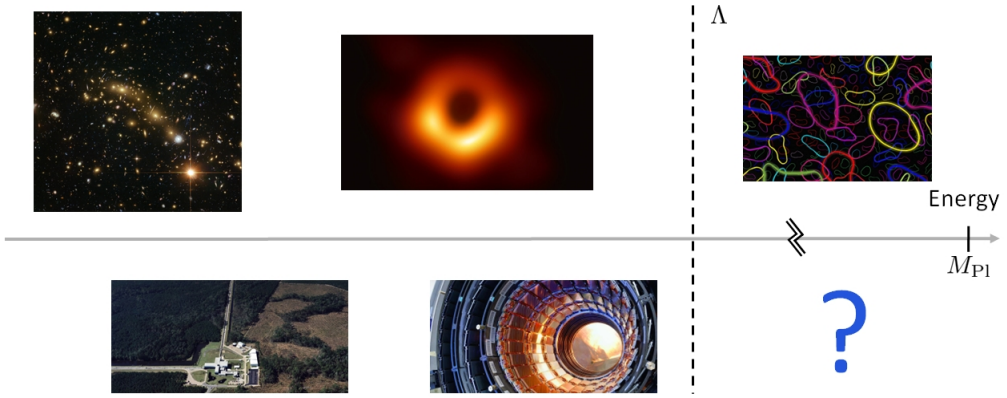


Figure 1.4: This diagram illustrates the hierarchy of different scales relevant to the EFT of gravity.

of the Riemann tensors. Stated like this, the parameterisation is abundantly redundant, as many terms will be related by field redefinitions involving the metric, integration by parts, or commutation of covariant derivatives. Further, Λ is the cut-off scale, which characterises the scale at which new physics becomes important. As before, this is a derivative expansion which parameterise the effects of a UV completion at low energies. The coefficients of the higher-dimension operators are fixed by dimensional analysis, so that the Wilson coefficients c_{mn} are $\mathcal{O}(1)$-factors unless there is a good reason they should not be — in fact, this is one way to state the cosmological constant problem. As is typical for *e.g.* weakly-coupled string theories, these corrections can arise from tree-level exchanges of higher-spin particles or loop-level exchanges of particles of all spins with mass Λ. The former will lead to $c_{mn} \sim \mathcal{O}(1)$, while in the latter case we would find $c_{mn} \sim (\Lambda/M_{\rm Pl})^{D-2}$ due to loop suppression. From the low-energy point of view, it is in general impossible to know whether a particular higher-curvature term comes from tree-level higher-spin effects or loop contributions and so we must allow for both generic countings.

The "quantum gravity" regime corresponds to when energies reach the Planck scale, so this is the scale at which we will violate perturbative unitarity if we just took the Einstein-Hilbert action at face value. Consequentially, the effect of higher-derivative corrections ought to come in when scales reach $\Lambda \leq M_{\rm Pl}$. For example in string theory, this is given by $\Lambda \sim 1/\sqrt{\alpha'}$, which is the scale of the new massive states that are exchanged in (1.2.19). At this point, we need to appeal to a UV completion, *i.e.* a theory of "quantum gravity", to accurately describe the physics.

For gravity, it is particularly useful to distinguish between weakly and strongly coupled UV completions. Heuristically, we say that UV completion is weakly coupled when EFT corrections are generated by integrating out additional heavy degrees of freedom, whereas we would expect a more drastic change in the degrees of freedom of the underlying theory for a strongly coupled UV completion — known EFTs in nature come in both varieties. For instance, quantum chromodynamics is the strongly coupled UV completion to pion theory — the underlying degrees of freedom are quarks which, in the confining phase, give rise to pions as the pseudo-Goldstone

bosons for the breaking of the approximate chiral symmetry at low energies [131–133]. We also have electroweak theory as a weakly coupled UV completion to Fermi's theory, where the effect of the underlying W^--boson is effectively described by the four-point Fermi contact interaction [134, 135]. A proxy for this is that we expect a UV completion to GR to be weakly coupled when $\Lambda \ll M_{\rm Pl}$ and strongly coupled when $\Lambda \sim M_{\rm Pl}$. It is *a priori* unclear what type of UV completion gravity needs.

Explicit Constructions

Above, we have presented the bottom-up construction of the EFT action of gravity. We would like to argue that any (partial) UV completion will give rise to a low-energy EFT action that takes this form. For non-gravitational theories, we have already seen explicit top-down constructions of EFT actions that take the form we would expect. This is also possible from UV completions to gravity such as string theory [82, 136–139].

We saw that it is possible to compute scattering amplitudes of string states represented by vertex operators. By construction, a low-energy EFT capturing the massless string states should be able to reproduce the scattering amplitudes involving the massless string states. In general, this will involve infinitely many terms suppressed in inverse powers of the string mass scale M_s. There are different ways to do this — either, one demands conformal invariance on the world-sheet (by vanishing of the β-function of the string σ-model) or one exploits spacetime symmetries with matching to amplitudes. In practice, it is more straightforward to construct the effective action perturbatively using the latter method. *E.g.* consider the four-point graviton scattering amplitude (1.2.19) in type-II superstring theory. At low energies, this takes the form

$$M_{\rm Pl}^{D-2}\mathcal{A} = \frac{1}{16}\alpha'^3\left[-\frac{64}{\alpha'^3}\frac{1}{stu} + \psi''(1) + \dots\right]K_1K_2, \tag{1.3.2}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the Digamma function. The leading-order contribution is at $\mathcal{O}\left(\alpha'^0\right)$ and quadratic in momenta, so is reproduced by the two-derivative in the type-II supergravity action (this is fixed by gauge invariance) as long as we identify the string coupling by

$$g_s = \frac{2\sqrt{\pi}\kappa_{10}}{(4\pi^2\alpha')^2}. \tag{1.3.3}$$

The first correction is octic in momenta and at $\mathcal{O}\left(\alpha'^3\right)$, and arises from a quartic curvature term with cut-off M_s. Higher-order corrections schematically enter the action as $\alpha'^n R^{n+1}$. By virtue of being derived as the low-energy approximation of a UV-complete theory, this automatically satisfies known consistency conditions.

1.3.2 Predictions from the EFT of Gravity

Once we have constructed the EFT action, we should truncate it to finite order and establish its regime of validity. Then, we can do regular perturbative QFT with that truncated EFT action to

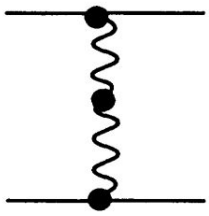


Figure 1.5: The Feynman diagram for graviton exchange between two heavy matter fields that corresponds to the Newtonian potential. The dark "blobs" are 1PI vertices. Figure taken from [140].

make predictions. In particular, we do this perturbatively by expanding around a background

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + M_{\rm Pl}^{-\frac{D-2}{2}} h_{\mu\nu}. \tag{1.3.4}$$

Since we are dealing with a non-renormalisable theory, we expect this perturbative expansion to break down eventually. This typically manifests itself by scattering amplitudes violating perturbative unitarity, as cross sections grow at high energies. These problems will however be outside of the regime of validity of the EFT.

In order to use (1.3.1) in a perturbative QFT framework, we should use power-counting arguments to establish the scale $E_{\rm breakdown}$ at which the loop expansion breaks down. For the Einstein-Hilbert term alone, this is given by $M_{\rm Pl}$. This is more complicated in the EFT of gravity, because there are non-linear interaction terms coming from both the Einstein-Hilbert action (which comes with inverse of powers of $M_{\rm Pl}$) and manifestly higher-derivative EFT corrections (which come with inverse powers of Λ). Tree-level interactions from curvature-squared corrections therefore compete with loop-level contributions from the Einstein-Hilbert term in the expansion *etc.*, so the actual scale $E_{\rm breakdown}$ controlling the loop expansion is a combination of these. Roughly,

$$\Lambda \leq E_{\rm breakdown} := M_{\rm Pl}^{x} \Lambda^{1-x} \leq M_{\rm Pl}, \tag{1.3.5}$$

for some number $x \in [0, 1]$ that can be found by examining all Feynman diagrams. The expansion is therefore still under control at the scale at which new physics enters.

There are many observational signatures of EFT corrections to gravitational phenomena. These present promising possibilities to use observations to pin down information on quantum gravity.

Correction to the Newtonian Potential

The Newtonian gravitational potential is an illustrative example for the different expansions at play in the EFT of gravity [129, 140]. To see this, consider the Einstein-Hilbert action supplied with a generic 4-derivative EFT correction.

Expanding this around flat space $\bar{g} = \eta$, it can be shown that the non-relativistic limit of the

graviton exchange diagram 1.5 up to 1-loop takes the schematic form

$$\tilde{V}(k) = \frac{1}{|\mathbf{k}|^2} - \frac{\alpha}{\Lambda^2} + \frac{\beta_1}{M_{\rm Pl}^2}\frac{1}{\sqrt{|\mathbf{k}|^2}} + \frac{\beta_2}{M_{\rm Pl}^2}\log\left(\frac{|\mathbf{k}|^2}{\mu^2}\right) + \dots, \tag{1.3.6}$$

where α is a Wilson coefficient encoding the information about the underlying high-energy degrees of freedom, and β_i are fully calculable from 1-loop diagrams in pure Einstein gravity and are particularly intriguing. As we will discuss in the next section, the four-derivative terms in the EFT action can be traded in for the Gauss-Bonnet term in the absence of matter, so effectively $\alpha = 0$ in $D = 4$ — it is however nevertheless illustrative to see the contribution to the potential it induces. If we restored $\hbar \neq 1$, we would see that β_1 is $\mathcal{O}\left(\hbar^0\right)$ while β_2 is $\mathcal{O}\left(\hbar\right)$. It is also notable that the β_i terms are both non-analytic contributions, which are associated with long-range interactions of massless particles.

The Fourier transform of (1.3.6) is

$$V(r) \sim \frac{1}{r} + \frac{\alpha}{\Lambda^2}\delta(r) + \frac{\beta_1}{M_{\rm Pl}^2 r^2} + \frac{\beta_2}{M_{\rm Pl}^2 r^3} + \dots. \tag{1.3.7}$$

As expected, the leading-term which comes from the free propagator is just the classical Newtonian potential (up to factors). The tree-level EFT correction did not give rise to a non-analytic term (although it will at loop-order), and contributes a contact term correction to the Newtonian potential. It parameterises the effects of heavy degrees of freedom, so it makes sense that the effect of these is completely local. The non-analytic terms indeed lead to predictions of long-range modifications of the Newtonian potential. The β_1-term is an example of a classical relativistic (post-Newtonian) correction, the type of which has become increasingly important in the systematic high-precision computation for gravitational wave signals from binary systems [141, 142]. On the other hand, the β_2-term is an honest quantum gravity prediction.

1.3.3 Leading EFT in Vacuum

For simplicity, we will typically focus on the EFT of gravity without matter. In $D \geq 5$ dimensions, this is described by the action

$$S_{\rm GB} = \int \mathrm{d}^D x\sqrt{-g}\, M_{\rm Pl}^{D-2}\left(\frac{1}{2}R + \frac{c_{\rm GB}}{\Lambda^2}R^2_{\rm GB} + \dots\right), \tag{1.3.8}$$

where $R^2_{\rm GB} = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$ is the GB term, and $c_{\rm GB}$ is the $\mathcal{O}(1)$ dimensionless Wilson coefficient. The vacuum Einstein-Gauss-Bonnet (EGB) equations are

$$G_{\alpha\beta} + \frac{2c_{\rm GB}}{\Lambda^2}B_{\alpha\beta} = 0, \tag{1.3.9}$$

where

$$B_{\alpha\beta} = 4R_{\sigma\alpha\beta\rho}R^{\sigma\rho} + 2R_{\alpha}{}^{\sigma\rho\kappa}R_{\beta\sigma\rho\kappa} - 4R_{\alpha\sigma}R_{\beta}{}^{\sigma} + 2RR_{\alpha\beta} - \frac{1}{2}R^2_{\rm GB}\, g_{\alpha\beta}. \tag{1.3.10}$$

Positivity bounds for this truncated theory have been considered in [36, 37] and properties adjacent to causality in [143, 144].

While other dimension-4 curvature operators (e.g. $R_{\mu\nu}^2$ or R^2) may generically enter at the same order in the EFT expansion, we are specifically interested in vacuum solutions with $R_{\mu\nu} = 0$ to leading order, so the effect of these operators on the propagation of gravitational waves (GWs) will be suppressed relative to the effect of the GB term via field redefinitions. Of course, in specific UV completions $c_{\rm GB}$ may vanish, but $c_{\rm GB}$ is generically non-zero and our discussion is predicated on $c_{\rm GB} \neq 0$. If it did vanish, then the cubic curvature terms will be leading corrections whenever supersymmetry is broken. These are considered in [145–147].

We will typically work only up to linear order in perturbations (*i.e.* in powers of the inverse cutoff scale, Λ^{-2}). Truncation at this order is a reflection of the fact that (1.3.8) represents only the first terms in an infinite series of effective operators built from scalar contractions of Riemann tensors and their covariant derivatives. For now, we are implicitly assuming it is safe to neglect these higher-dimension curvature operators because they would come suppressed by more powers of Λ^{-2} compared to the GB term. In fact, it is exactly this assumption which will define for us the "regime of validity" later on. In the meantime, we just note that corrections higher-order in $c_{\rm GB}$ would be meaningless since we would generically expect corrections from other operators at the same higher orders.

1.3.4 Consistency Conditions

A central part of our discussion on EFTs was that, beyond the symmetries of the relevant degrees of freedom, other physical properties of QFT are not guaranteed to hold *a priori* and place non-trivial constraints on the EFT interactions we can write down. We also motivated the EFT perspective on gravity and corrections to it from the UV. This suggests that there should be similar non-trivial constraints on the higher-derivative corrections in the EFT of gravity.

However, while gravity is in many ways just a regular field theory, it also holds a special position. While we are typically free to choose coupling between different matter fields (unless *e.g.* restricted by supersymmetry), the equivalence principle dictates that the gravitational field universally couples to all fields with the same strength. Since it also defines the background geometry, it also sets the causal structure for the other matter fields. The generic formation of curvature singularities and (assuming the weak cosmic censorship conjecture) horizons also leads to interesting physical implications. For these reasons, it will turn out that consistency conditions on quantum gravity are more subtle.

The "Swampland" Program

The so-called swampland program also aims to find constraints on low-energy EFTs of quantum gravity. More specifically, it aims to delineate the boundary between the string landscape and eponymous swampland or, in other words, find qualitative restrictions directly on quantum gravity informed by explicit constructions in string theory [148–151]. It consists of a web of conjectures, which are intimately connected, with varying degrees of evidence. Of particular

interest to us is the weak gravity conjecture (WGC), for which BHs play an important role.

Two of the strongest and most well-known of these conjectures are the "no global symmetries" and "weak gravity" conjectures — these were first discussed in and respectively, but have regained significant interest over more recent years .

The WGC is related to the older "no global symmetries" conjecture [152]. This states that any consistent theory of quantum gravity should not have global symmetries. Intuitively, this is because Hawking radiation — the process by which BHs decay and shed mass and charge — is relatively insensitive to their global charges. Unlike gauge symmetries, there is no chemical potential which favours the discharging of black hole. Roughly this means that BHs must destroy global charge. More precisely, we could take states charged under the global symmetry to form BHs with arbitrary charge, which would decay to BHs also with arbitrary charge and of similar size. This is a continuum of microstates which is in tension with the Bekenstein entropy bound.

The WGC is on less firm footing — it was originally formulated in [153], but has regained significant interest in recent years [154–159]. It comes in several different (related) forms, but in its mildest form it states that a consistent theory of quantum gravity should contain a state with charge-to-mass ratio

$$\frac{|q|}{m} \geq \frac{|Q|}{M}\bigg|_{\text{Extremal}} \tag{1.3.11}$$

above the extremality bound for a charged black hole. An intuitive explanation for this condition is that it provides a mechanism for extremal BHs to decay without becoming superextremal themselves — that would be in tension with weak cosmic censorship.

A particularly interesting case of the WGC is when the states satisfying the WGC are large BHs themselves. We define the extremality bound with respect to asymptotically large BHs, computed with just the two-derivative action, while finite-size BHs will be affected by the presence of EFT corrections. The WGC then translates into a statement about the Wilson coefficients of the EFT [161].

1.4 Overview of Thesis

Consistency conditions on EFTs of gravity are precisely what we shall concern ourselves with in this thesis. We will start in I by discussing restrictions that causality places on gravitational EFTs, which are complementary to gravitational positivity bounds. In part II, we will take a closer look at the properties of extremal BHs obtained by deforming the near-horizon regions of analytically known solutions, and discuss a speculative conjecture related to the WGC.

1.4.1 Part I: Causality

In part I, we discuss the causality bounds on gravitational EFTs. This based on [1, 2].

In section 1.1.4, we discussed the constraints that positivity bounds are able to put on non-gravitational EFTs. These have been useful in putting strong bounds on, for instance, the allowed

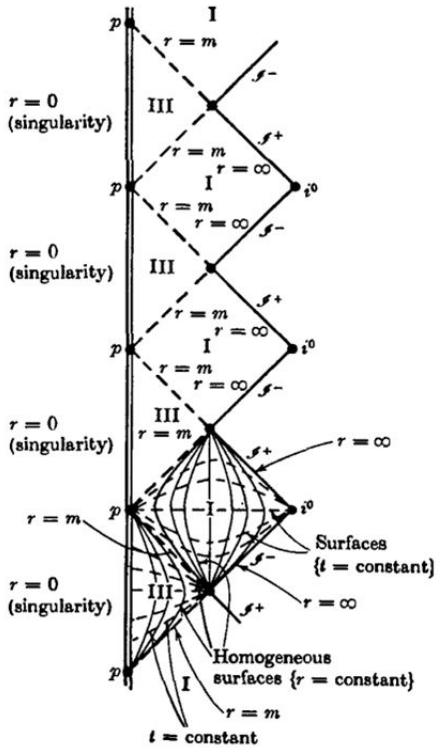


Figure 1.6: The Penrose diagram of an extremal Reissner-Nordström black hole. Figure taken from [160].

parameter subspace of the standard model EFT [162]. However, coupling to gravity complicates this because more subtractions are needed when massless spin-2 states are present in the theory and the exchange itself suffers from a t-channel pole. Another drawback of positivity bounds is that the existence of an S-matrix is necessary — this restricts us to set-ups that are asymptotically flat. One of the key realisations for these is that known flat-space bounds are therefore at the very least weakened in the presence of dynamical gravity [57, 58, 66, 67, 69–71, 163–169]. We will be interested in a complementary perspective on these gravitational positivity bounds, using the fact that they are closely related to a much more manifest form of causality: Demanding the absence of superluminal propagation [31]. This is perhaps not surprising, since a crucial ingredient in deriving these bounds was analyticity of the scattering amplitude, which is associated with causality.

In the first part of the thesis, we will make the notion of causality appropriate for putting constraints on EFT corrections in curved spacetime more precise. The starting point for this is to consider propagation speeds of fields around backgrounds that spontaneously break Lorentz invariance. Phrased in terms of propagation speeds, it is well-known that causality is equivalent to the statement that the front velocity, *i.e.* the phase velocity in the high-frequency limit, is subluminal [170]. Both the phase and group velocities themselves are allowed to be superluminal without being in tension with causality, although precisely how this happens depends on the dispersive properties of the effective medium.

The drawback of this is that propagation speeds themselves are not invariant under field redefinitions involving the metric — only their ratios are. This means that propagation speeds

are individually not physically meaningful quantities to study. Out of convenience, we shall nonetheless restrict ourselves to asymptotically flat spacetimes on which we can define an S-matrix, from which we can define a quantity known as the scattering time delay ΔT. We will shortly go into a lot more detail on what precisely we mean by this, but from a heuristic point of view, it quantifies the time difference (delay) a particle experiences due to a scattering event. We can think of it as a proxy for the propagation speed averaged over the entire trajectory. The key idea is that small amounts of apparent superluminal propagation over short distances are not in tension with causality globally. Causality can then roughly be related to a positivity condition on ΔT [61, 72, 146, 171–183]. There is a further caveat to this — waves of frequency ω are insensitive to timescales less than ω^{-1}, so negative time delays are in agreement with causality so long as their magnitude is sufficiently small. We refer to this as the resolvability criterion for the time delay. Put together, this is known as asymptotic causality.

Now, in order to leverage this to study constraints on EFTs, we need to emphasise that it is fatal to think of EFTs as full theories. For example, if the higher-derivative terms are not thought of perturbatively, they will inevitably lead to ghosts in the spectrum of the Hamiltonian. We should therefore adopt this philosophy for properties such as causality too — instead of asking whether the full theory as a whole satisfies various constraints, we should ask what corrections spoil the causality of the deep IR. Splitting the full time delay into a GR and an EFT part, we argue that any resolvable and negative EFT correction to GR will constitute a causality violation. We refer to this as IR causality. We do this on general grounds and by showing that this is consistent with known gravitational positivity bounds for a toy model.

We then study the implications of this for the EFT of gravity. One of the best tools we have at hand at physicists is to smash things into other things. In [1], we consider gravitational waves scattering off a static and spherically symmetric BH spacetime in the leading-order EFT of gravity in vacuum — as mentioned before, this is essentially EGB gravity. By deriving the time delays in a semiclassical approximation, we find that the Wilson coefficient of the Gauss-Bonnet term comes with factors of different signs in the EFT correction to the time delay. Since any non-zero value will therefore cause at least one polarisation to lead to a negative EFT correction to the time delay this was thought to mean that EGB is inconsistent with causality. We revisit this conclusion, emphasising that we need to take particular care to establish its regime of validity, when EGB is thought of perturbatively. When this is done, the would-be causality violations are outside the regime of validity of EGB gravity, and we see that EGB gravity is consistent with IR causality.

We also take a closer look at the resolvability condition in [2]. In principle, one might hope to find stronger bounds on EFT coefficients if we could get rid of this. One option is by amplifying the time delay and hence amount of causality violation by repeatedly scattering off potentials. Gravity acts highly non-linearly, so background solutions describing multiple BHs are difficult to construct analytically — luckily, this is possible in the pp-wave limit. Pp-waves are often described as exact solutions in quantum gravity, since they are solutions to the equations of motion derived from the EFT of gravity for all corrections. Shockwaves are particular pp-waves with a highly localised (in fact, singular) metric function, which can be obtained by the Aichelburg-Sexl

limit from a Schwarzschild black hole. Einstein's equations are linear in the metric functions of these pp-waves, and we can therefore stack up multiple shockwaves in a pp-wave metric.

Once again, we study gravitational waves scattering off these background geometries. These introduce non-trivial constraints to the regime of validity of the EFT, which the background itself would not have been able to probe. This requires us to regularise shockwaves, and introduce a minimum width for each shock. To determine whether IR causality is satisfied or not, we compute the null scattering time delays in this set-up within the semiclassical, eikonal, and Born approximations. For shockwaves stacked along the trajectory of the wave, we find that a large number of scattering shocks is in tension with the regime of validity of the EFT and the semiclassical approximation itself. Heuristically, scattering causes a deviation in the trajectory of the scattering wave, and we can only trust the approximations used to derive the expression for the time delay in the first place over a finite time.

To go one step further, we can consider more generic configurations in the directions transverse to the direction of propagation of the wave. By travelling along the classical equilibrium, the eikonal approximation then remains exact and one expects to be able to accumulate a sufficient amount of time delay to make do with the resolvability criterion. This would be true in a purely classical set-up with a point-particle travelling along geodesics. However, in both the semi-classical and Born approximations, which account for spreading of wavepackets, we see that the instability timescale associated to the unstable equilibrium also takes us out of the regime of validity of the respective approximations. We conclude that no special configurations can allow enough amplification to sum unresolvable amounts of causality violation into a resolvable amount.

1.4.2 Part II: Extremal Black Holes

Following [3], we discuss particular deformations to the horizons of extremal BHs and their implications on UV physics in part II.

Even though familiar examples possess smooth horizons, it is well-known that BHs generically do not. Most famously, the Majumdar-Papapetrou solution in $D \geq 5$, which describes static configurations of extremal charged BHs even have singular horizons which lead to divergent tidal forces. To see whether this is generic for all extremal BHs and not just analytically known solutions, a natural approach is to study stationary deformations to extremal BHs. The near-horizon geometry of extremal BHs universally takes the form of $\mathrm{AdS}_2 \times S^{D-2}$ and hence comes with an enhanced isometry group, so this problem simplifies drastically within the near-horizon limit — this analysis was initiated in [184, 185]. The isometries of AdS force the deformations to scale with a critical exponent determined by the effective mass of the field compactified onto AdS_2, which describes the behaviour close to the horizon. Indeed, these exhibit a variety of interesting behaviours — they can be generally non-analytic, or even lead to tidal force and curvature singularities.

We take a closer look at deformations of extremal Reissner-Nordström (RN) BHs with a particular eye towards the effect of EFT corrections. Singular solutions are particularly interesting since

it turns out they arise from the extremal limits of non-singular sub-extremal solutions and are hence physical. Interestingly, there are certain deformations — which we refer to as marginal — which are constant near the horizon and hence at the cusp of being singular. We shall specifically focus on the case for which curvature invariants appear to diverge on the horizon of the deformed black holes. For electrically charged BHs, these exist linearly in $D \geq 5$ dimensions, but are accompanied by other singular modes for $D > 5$. This makes the $D = 5$ case non-linearly marginal. When we consider EFT corrections, this means that the nature of the horizon is completely determined by the sign of the EFT contribution. This signals an extreme form of UV sensitivity, since EFT breaks down when curvature invariants become large. We argue this not to be the case. First of all, singular deformations are already present in GR (without any EFT corrections) and signal a breakdown of metric perturbation theory — the linear deformations become large near the horizon, and the non-linearities of GR become important. It suggests the possibility that this is also the case for the EFT corrections that push the marginal deformations to being singular. We confirm that this is the case by closely examining the regimes of validity for the EFT and metric perturbation theory, and seeing that metric perturbation theory clearly breaks down before.

This is supported by an example of a partial UV completion to a particular EFT correction, for which the marginal correction is pushed towards being singular when the EFT correction satisfies otherwise known positivity constraints. The problem of singularities is already there in the UV completion, where we understand it to be caused by the breakdown of metric perturbation theory (there is also no EFT to break down). If it were the EFT itself that broke down, we would have expected details from the UV completion to resolve this apparent breakdown — this is not the case.

Motivated by the EFT corrections to deformations of BHs and a scalar toy model, we then speculatively make the following conjecture. Physically consistent EFTs, *i.e.* ones which have a healthy UV completion, always push marginal deformations towards being singular. We offer a possible explanation from a holographic perspective — our speculative so-called near-horizon negativity conjecture ensures that the corresponding bulk modes remain normalisable. We then derive the bounds that this conjecture implies for the leading-order four-derivative EFT corrections to the Einstein-Maxwell action in AdS, which imply the WGC.

In the final chapter, we turn to the dynamics of the perturbations to the near-horizon region of near-extremal BHs. The dynamics of perturbations around extremal BHs, reduced to the AdS_2 -factor in the undeformed geometry, are described by JT gravity. We will study the boundary correlation functions of matter fields coupled to this JT gravity theory, with special interest in the four-point function, which is an observable related to chaos. Remarkably, we will be able to compute bulk loop contributions since the boundary is one-dimensional and loop integrals can be solved using Cauchy's theorem.

We will take the bulk matter theory to be the leading-order Goldstone EFT (1.1.20). This is the prototypical example of an EFT operator subject to non-trivial constraints from the UV in the form of positivity (analyticity) and causality bounds. We would like to understand this from the

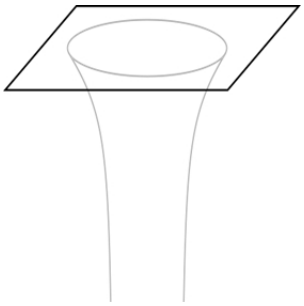


Figure 1.7: The near-horizon geometry of the near-extremal Reissner-Nordström BH is $\mathrm{AdS}_2 \times S^d$. It interpolates between the horizon and the asymptotically flat end, and diverges in proper distance as extremality is reached.

perspective of the boundary observables, such as a measure of chaos, and present preliminary results for the boundary their four-point function up to 1-loop in the bulk matter fields and Schwarzian modes.

This set-up is particularly interesting because of the different ways we can think of the Goldstone EFT operator. It turns out that the $(\partial\phi)^4$-operator can be obtained, to leading-order, as the effect of a $T\overline{T}$-deformation of a massless free scalar field. This is a rare example of an exactly solvable but irrelevant deformation due to the operator defined in [186], which has had seen a lot of interest in the context of integrability and holography. Relatedly, the $T\overline{T}$-deformation can be understood as coupling of the undeformed theory to the reference metric of two-dimensional massive gravity [187]. In that case, the Wilson coefficient of the EFT is related to the mass of the graviton.

1.A Conventions

We will work in units where $\hbar = c = 1$. Actions are denoted by S in Lorentzian signature and I in Euclidean signature. Lorentzian metrics are written in mostly-plus signature $(-,+,\dots,+)$.

On flat space, we often denote the spatial parts of Lorentzian D-vectors with boldface. We will often denote the split for position and momentum vectors as $x = (t, \mathbf{x})$ and $k = (\omega, \mathbf{k})$ respectively.

In the context of BH spacetimes, it is convenient to split D-dimensional full spacetimes into two-dimensional orbit spaces and $d = D - 2$-dimensional base/transverse space. Tensors on the full spacetime are indexed with letters from the Greek alphabet $\{\alpha, \beta, \dots\}$, while tensors on the orbit or base space are indexed with letter from the beginning and middle of the Roman alphabet, *i.e.* $\{a, b, \dots\}$ and $\{i, j, \dots\}$, respectively. On the full spacetime, covariant derivatives are denoted by ∇, and the Laplace-Beltrami operator is written $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$. On the base space, covariant derivatives (defined with respect to the pull-back of the metric to the base space) are denoted by $\hat{D}$, and the associated Laplacian is $\hat{\Delta}_d$.

Most prominently in chapter 5, we will be interested in geometries with a flat Euclidean d-dimensional submanifold. Tensors on this transverse space will be indexed with letters from the

middle of the Roman alphabet, *i.e.* $\{i, j, \dots\}$, and the Laplacian will be $\nabla_d^2 = \eta^{ij}\partial_i\partial_j$. Further, it will be convenient to isolate the $D-3 = d-1$-sphere. Tensors here will be indexed by letter from the middle of the Roman alphabet with an additional bar,*i.e.* $\{\bar{i}, \bar{j}, \dots\}$, and we denote the covariant derivative by $\bar{D}_{\bar{i}}$ and the Laplace-Beltrami operator by $\bar{\Delta}_{d-1} = \gamma^{\bar{i}\bar{j}}\bar{D}_{\bar{i}}\bar{D}_{\bar{j}}$.

Part I

Causality Constraints

Chapter 2

Introduction

The first part of the thesis concerns causality constraints on EFT corrections to GR.

In chapter 3, we give a broad overview of differing notions of causality in QFT and GR. As our weapon of choice for diagnosing causality of a field theory, we settle on the so-called time delay ΔT. At an intuitive level, this encodes the extra time it takes a wavepacket to traverse a potential (possibly sourced by spacetime curvature) relative to empty (flat) space. We say that causality is then satisfied when ΔT is not negative by an amount resolvable by the scattering wavepacket. For one peaked at a frequency ω, this means that

$$\Delta T \geq -\omega^{-1}. \tag{2.0.1}$$

We refer to this notion of causality as *asymptotic causality*.

In order to put constraints on EFT corrections, we will need to slightly adapt this notion of causality. Consider a weakly coupled UV completion to our EFT, so that the EFT corrections capture the effect of additional degrees of freedom in addition to the standard metric. This ensures that high-frequency modes propagate on a background geometry set by that metric. Due to the equivalence principle, all species propagating in a curved background will be subject to a universal GR time delay in addition to a time delay due to interactions with high-energy species. This motivates a split of the time delay

$$\Delta T = \Delta T^{\mathrm{GR}} + \Delta T^{\mathrm{EFT}}. \tag{2.0.2}$$

One approach to think about causality constraints on the EFT is to consider the decoupling limit — as $M_{\mathrm{Pl}} \to \infty$, gravity becomes non-dynamical and we are left with a field theory on a fixed (curved) background geometry, on which causality (in the sense above) is determined by the GR background geometry. This implies that

$$\Delta T^{\mathrm{EFT}} \geq -\omega^{-1}, \tag{2.0.3}$$

and is what we shall refer to as IR causality. This will prove to be the notion of causality most

closely related to gravitational positivity bounds.

Higher-derivative corrections in an EFT expansion are constructed to mimic the effect of high-energy interactions. IR causality is violated if the EFT itself induces superluminal propagation, but this means that the heavy states we integrated out must themselves be superluminal. The high-energy states within the UV completion only see the lightcone structure dictated by GR, while low-energy states are subject to additional EFT corrections. Crucially, there is an asymmetry in that a high-energy state can decay into low-energy states but not *vice versa*. By construction, we should take the background itself to be absent of superluminal propagation, so that the high-energy particles propagate causally. However, high-energy species can violate causality indirectly through low-energy particles as soon as an EFT correction leads to a resolvable support of the retarded Green's function of the low-energy species outside the lightcone.

Furthermore, $\Delta T^{\rm GR}$ is positive resolvably for causal backgrounds (in the sense above), so asymptotic causality is a weaker condition than IR causality. Requiring only asymptotic causality would allow for resolvably negative $\Delta T^{\rm EFT}$, which would be a manifest violation of causality in the decoupling limit. We will shortly also see explicitly in the Goldstone EFT toy model that asymptotic causality does not reproduce the scaling of gravitational positivity bounds, while IR causality does.

As a proof of principle, we then apply this to higher-derivative corrections of Einstein gravity. The leading-order EFT of gravity in $D \geq 5$ vacuum is EGB gravity, and will serve as the example that we shall be interested in. In order to derive causality constraints on higher-derivative corrections to Einstein gravity, we want to consider scattering of gravitational perturbations on sufficiently simple but non-trivial background geometries. For us to be able to derive any bounds, we need a non-trivial potential — at a pragmatic level, this is because scales need to be introduced to the system in order for non-trivial bounds to be possible. The trivial background, flat space, is not able to provide that. In chapter 4 we consider the next best thing, a static and spherically symmetric background Ansatz. In pure GR this would have led us to the Schwarzschild-Tangherlini solution, but we will see that the EFT modifies these [155, 188].

The equations of motion for perturbations around this background take the form of distinct Schrödinger-like master equations

$$\frac{\mathrm{d}^2\psi_M}{\mathrm{d}^2\rho^2} + W_M(\rho)\psi_M = 0, \tag{2.0.4}$$

for some appropriate radial coordinate ρ and master variables ψ_M, distinguished by their transformation properties under the isometries of the spherically symmetric background. We can solve these for the phase shift and hence time delay within the Jeffreys-Wentzel–Kramers–Brillouin (JWKB) approximation, and find that

$$\Delta T^{\rm net} = \Delta T^{\rm GR} + c_{\rm GB}\Delta T_M^{\rm GB}, \tag{2.0.5}$$

where $\Delta T_M^{\rm GB}$ is not sign-definite. This means that there is no non-zero value for the Wilson coefficient which does not lead to a negative EFT correction to the time delay. Naïvely, we

would conclude that the GB operator is in tension with IR causality.

However, given any truncated EFT, we should always make sure that we are within the regime of validity of said truncation. Properly establishing the regime of validity of the EFT, it turns out that

$$|\Delta T_M^{GB}| \lesssim \omega^{-1}, \tag{2.0.6}$$

so the EFT correction to the time delay is not resolvable when $|c_{\rm GB}| \lesssim 1$ as expected from positivity bounds — *cf.* particularly fig. 1 in [168]. The apparent violation of causality is therefore something the theory can put up with.

Stronger bounds on $c_{\rm GB}$ could be derived if we managed to amplify ΔT_M. We rule this possibility out in chapter 5. The set-up for amplifying causality violation is the following. Once again, we start by considering gravitational waves scattering off black hole spacetimes. However, we now take the Aichelburg-Sexl limit, which are sourced by stress tensors localised in null directions. The solutions to Einstein's equations (and its higher-derivative corrections) with this source is a shockwave, which is a special case of a pp-wave. Linear superpositions of generic solutions to Einstein's equations are themselves not solutions to Einstein's equations. However, remarkably, pp-waves are exact linearisations around flat space, and we can therefore superpose individual shocks. Assuming that the resulting total time delay is amplified by the number of shocks, IR causality then requires

$$N|\Delta T_M| \lesssim \omega^{-1}. \tag{2.0.7}$$

As $N \to \infty$, this effectively sets the right-hand side to zero. This set-up is studied in [189–192]. This was used to argue that the leading-order EFT correction, given by the Gauss-Bonnet operator, is inconsistent with causality unless higher-spin degrees of freedom (as one would expect from string theory) enter the theory and restore it.

We will argue that this is not possible within the approximations used to compute the time delay itself. One way to see it is the following. For N isolated scattering events, we can separate the interaction picture time evolution operators into a time-ordered product of individual time evolution operators

$$\hat{U}(t_N, t_0) = \mathcal{T} \prod_{n=1}^{N} \hat{U}(t_n, t_{n-1}). \tag{2.0.8}$$

For sufficiently long time intervals between the individual scattering events, the intermediate states approach asymptotic states, so that the total S-matrix factorises into individual S-matrices

$$\hat{S}_{\rm total} \approx \mathcal{T} \prod_{n=1}^{N} \hat{S}_n. \tag{2.0.9}$$

If (and this is crucial) we assume that the individual scatterings events are identical, then $\hat{S}_{\rm total} = (\hat{S}_1)^N$. The time delay is related to the logarithm of the S-matrix, so this becomes

$$\Delta T_{\rm total} = N \Delta T_1. \tag{2.0.10}$$

However, this is too quick — the S-matrices $\hat{S}_n$ for the individual scattering events are operators

in interaction picture, and experience time evolution under the free Hamiltonian $\hat{H}_0$ — this does not generally commute with the interaction Hamiltonian. Even when the interaction is composed of N identical impulses $\hat{K}$ separated

$$\hat{H}_{\text{int}} = \sum_{n=1}^{N} \delta[t - (t_{n-1} + a_n)]\hat{K} \tag{2.0.11}$$

for $0 < a_n < t_n - t_{n-1}$, we would find

$$\hat{S}_n = e^{i\hat{H}_0(t_{n-1}+a_n)} e^{-i\hat{K}} e^{-i\hat{H}_0(t_n+a_n)}. \tag{2.0.12}$$

This is not identical for all n. The effect of evolution under H_0 is diffusion or scattering of the original wavepacket, so this is the physical effect we can attribute to the breakdown of the approximation that naïvely lead us to the enhancement of ΔT. We will make this more precise within the semiclassical, eikonal, and Born approximations.

Details on physics in spherical symmetry, including discussions on the background solutions, decompositions, and scattering are presented in appendix A.

Chapter 3

Infrared Causality

Causality has established itself as a powerful tool to constrain physical theories. In this part of the thesis, we are concerned with the restrictions that causality imposes on gravitational EFTs.

In section 3.1, we give a brief overview over different (and ultimately related) notions of causality in GR and QFT, eventually motivating the definition of IR causality as the appropriate notion to constrain gravitational EFTs. To demonstrate this, we apply it the Galileon and Goldstone EFT as toy models in sections 3.2 and 3.3 respectively. IR causality is a condition on the time delay, and we start out only giving an intuitive description and useful expressions. A more complete discussion needed for the arguments in later sections is postponed to section 3.4.

3.1 Notions of Causality

What do we mean by causality? It is certainly not a technical term exclusive to the theoretical high-energy and gravitational physics community — causality as an idea is also of interest to statistics, philosophy, and linguistics. For the avoidance of any doubt, we briefly review different consequences, aspects, and definitions of causality in physics, before establishing which notion of causality is most useful for putting constraints on EFTs.

Apart from in this section, we will take causality to heuristically mean the absence of superluminal propagation in a theory and we will use the terms subluminality and causality interchangably.

3.1.1 Causality in GR and QFT

Causality describes the physical intuition that causes precede their effects.

One perspective is to phrase this in terms of transfer functions. Consider an impulse to a system sourced by a function $f_{\text{source}}(t)$ followed by a response $f_{\text{response}}(t)$. Newtonian, non-relativistic causality is the statement that

$$\frac{\delta f_{\text{response}}(t)}{\delta f_{\text{source}}(t')} = 0, \quad \text{if } t < t'. \tag{3.1.1}$$

Relativistic causality, on the other hand, is the statement that

$$\frac{\delta f_{\text{response}}(x)}{\delta f_{\text{source}}(y)} = 0, \quad \text{if } x^0 < y^0 \text{ or } (x-y)^2 > 0 \tag{3.1.2}$$

i.e. unless x sits in the future lightcone of y.

When we refer to causality, we will typically mean relativistic causality [1].

GR and QFT are both relativistic theories, and causality is manifested in different ways.

Retarded Green's Function

The most direct consequence of causality lies in analyticity properties of the retarded Green's function.

At a linear level, the response to an impulse f_{source} is described by

$$f_{\text{response}}(x) = \int \mathrm{d}^D y \, G_{\text{ret}}(x-y) f_{\text{source}}(y), \tag{3.1.3}$$

where G_{ret} is the retarded Green's function, which can be computed from the equations of motion governing the system. Causality can therefore be stated as

$$G_{\text{ret}}(x) = 0, \quad \text{if } x^0 < 0 \text{ or } x^2 > 0. \tag{3.1.4}$$

The Fourier transform of the retarded Green's function is then defined as

$$\tilde{G}_{\text{ret}}(k) = \int \mathrm{d}^D x \, G_{\text{ret}}(x) e^{-ik\cdot x}. \tag{3.1.5}$$

For polynomially bounded G_{ret}, the integrand is exponentially suppressed when $k \in \mathbb{C}^D$ if $\operatorname{Im} k^0 > |\operatorname{Im} \mathbf{k}| > 0$, since

$$\operatorname{Im}(k\cdot x) = -\operatorname{Im}\{k^0\}\, x^0 + \operatorname{Im}\{\mathbf{k}\}\cdot\mathbf{x} \leq x^0 \left(-\operatorname{Im} k^0 + |\operatorname{Im}\mathbf{k}|\right) \tag{3.1.6}$$

within the support of a causal G_{ret}. By construction via the Fourier transform (within its radius of convergence), $\tilde{G}_{\text{ret}}$ is therefore a holomorphic/analytic function of its argument when k is in the forward tube, *i.e.* when $\operatorname{Im} k$ is a future-pointing causal vector[194].

The non-relativistic version of this statement is obtained by restricting the Fourier integral to an integral over just k^0. This leads to the familiar statement that causality is related to analyticity of the Fourier transform of the retarded Green's function in the upper-half plane.

Commutation Relations

The following discussions are based on [193].

[1] This is not a universal convention. In [193], causality refers to Newtonian causality while relativistic causality is subluminality.

In quantum mechanics, two operators that do not influence each other commute. Indeed, microcausality is the statement that the commutator of any pair of spacelike-separated operators vanishes

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0, \quad (x-y)^2 > 0 \tag{3.1.7}$$

at the operator level. It is a non-perturbative and exact statement. In practice, given some choice of states which form a Hilbert space, we evaluate the expectation value of the operator equation between arbitrary states in said Hilbert space. In particular, it turns out that the causal response of the vacuum to an external source is directly related to the vacuum expectation value of this operator statement — this plays the role of a interacting retarded Green's function.

Unsurprisingly, the signature of the spacetime is crucial for microcausality to make sense. In Euclidean signature, the only well-defined correlation functions are Schwinger functions (time-ordered correlation functions), and we cannot access information about the expectation values of commutators. These are intrinsically Lorentzian, and the non-analyticities introduced by crossing light cones are crucial.

There are several ways to see that microcausality is intricately linked to Lorentz invariance. First of all, microcausality itself is a sufficient condition to show that the S-matrix itself is Lorentz invariant. Furthermore, the expectation value $\langle[\mathcal{O}_1(x), \mathcal{O}_2(y)]\rangle$ would not even Lorentz invariant when x and y are spacelike separated unless it vanished identically. It it also easy to see that with Lorentz invariance, microcausality for field operators has to hold even for interacting theories — this is essentially inherited from the free theory [193]. In a Lorentz invariant theory, time evolution and Lorentz boosts can be used to bring spacelike-separated operators to the initial time slice, where canonical commutation relations hold in canonical quantisation. Alternatively, we can see this from Källen-Lehmann representation, which relates the exact 2-point function of an interacting theory to that of a free theory. To be explicit, (once again) consider ϕ collectively denoting the fundamental fields of some theory. Then

$$\langle 0|\,[\phi(x), \phi(y)]\,|0\rangle_{\text{int.}} = \int_0^\infty \mathrm{d}\mu\rho(\mu^2)\,\langle 0|\,[\phi(x), \phi(y)]\,|0\rangle_{\text{free}}\Big|_{\text{mass}=\mu}, \tag{3.1.8}$$

which necessarily vanishes for spacelike separation from canonical quantisation. It is worth noting that in the arguments above, Lorentz invariance is a useful crutch but not a necessary condition to show microcausality. In fact, microcausality is also expected to hold on curved spacetimes, where Lorentz invariance holds only locally as opposed to globally.

Closed Timelike Curves

In GR, causality is encoded in the geometric properties of the spacetime manifold. An obvious way a spacetime can violate causality is by allowing closed timelike curves. These lead to plethora of possible paradoxes, by allowing observers to travel into their own past. An example of a spacetime that possesses closed timelike curves is the Gödel universe.

Spacetimes without any closed timelike curves are said to satisfy the chronology condition. More generally, a spacetime without any closed causal curves is said to satisfy the causality

condition. Despite the distinction, these two notions are equivalent generically under certain assumptions. This can be made slightly more precise by various related definitions, namely strong causality and stable causality. In all cases, the structure of the local light cones is crucial — these are determined by the (local) metric. It is one aspect of gravity and the metric holding a distinguished position amongst the dynamical fields in a theory.

An idea closely related to causality is determinism. In physically reasonable spacetimes, the dynamics of all fields should be uniquely determined by their values and finitely many derivatives on a spacelike initial value surface. This is the case for globally hyperbolic spacetimes, which is a spacetime which possesses a Cauchy hypersurface. The remark at the end of chapter 8 in [195] shows that global hyperbolicity in fact implies strong causality.

It is evident that causality and global hyperbolicity as described above are properties of a given spacetime, and are hence not useful for distinguishing between different theories of gravity. In fact, GR itself admits solutions which are contain closed timelike curves (such as the Gödel universe), and we are content with GR as a physical (and causal) theory.

Hyperbolicity

From the perspective of hyperbolic partial differential equations (PDEs), causality is determined by the characteristic hypersurfaces — the discussion below closely follows [143, 144].

Heuristically, characteristic hypersurfaces are surfaces on which the highest-order derivatives cannot be determined in terms of the lower derivatives using the equations of motion — this implies that discontinuities propagate along them. They are also the surfaces of constant phase for high-frequency waves propagating on the background. This is closely linked to the statement that causality is set by the front velocity of propagating waves. In the absence of higher-derivative corrections, hypersurfaces are characteristic if and only if they are null with respect to the metric. In the EFT of gravity (or other modifications of GR), these are however generically non-null — this is equivalent to the statement that gravitational perturbations are multi-refringent, i.e. they propagate at different speeds. A number of works have considered the implications of this apparently enlarged lightcone and is crucial to the following discussion [143, 144, 196–202].

Hyperbolicity (more precisely, strong hyperbolicity) of the equations of motion is important because it ensures well-posedness of the associated initial value problem. Einstein's equations have been shown to satisfy this, and it holds *e.g.* for Lovelock theories when curvatures remain small. It is however not clear that the same should hold for gravity with higher-derivative corrections generally — for one, the notion of hyperbolicity does not apply to higher-derivative equations of motion. For a well-posed initial value problem, we also need initial data to be spacelike with respect to the causal structure determined by the hyperbolic PDEs.

Asymptotic Causality

The causal properties of dynamical gravity are subtle, because the causal structure is not invariant under field redefinitions involving the metric itself. This motivates a notion of causality that

does not depend on the local geometry. This will turn out to be the notion of causality most useful for our purposes.

For this, we use the so-called "Eisenbud-Wigner time delay" ΔT [203–207]. Let us consider scattering off a spherical potential — this may be sourced externally or, motivated by the decoupling limit when it exists, by the background geometry itself. The effect of interaction with a potential is necessarily encoded in a phase shift δ_ℓ, so that the outgoing wavepacket takes the form

$$\psi_{\text{out}} = \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \frac{A(\mathbf{k})}{r} e^{-i(\omega t - kr)} e^{2i\delta}. \tag{3.1.9}$$

The center of the wavepacket is located at the point of stationary phase, so for an observer at fixed r the outgoing wavepacket arrives at a time $\partial_k(-\omega t + kr + 2\delta)$. It therefore experiences a time delay, a literal delay in time

$$\Delta T = \frac{2\partial\delta}{\delta\omega} \tag{3.1.10}$$

compared to a freely propagating wavepacket. In other words, it describes the amount by which the scattered wavepacket is delayed relative to the same wavepacket propagating on Minkowski spacetime. More precisely, for a partial wave labelled by the harmonic ℓ, the Eisenbud-Wigner time delay is defined as

$$\Delta T_b = \left.\frac{2\partial\delta}{\partial\omega}\right|_b. \tag{3.1.11}$$

A discussion of the time delay for partial waves is given in appendix A.3.

There is a long history of using positivity or boundedness of this time delay as a means of imposing causality going back to Eisenbud and Wigner [203, 204], and improved by Smith [205] — see for example [61, 207]. More recently in the context of relativistic field theories, in [174] the eikonal approximation was used to determine this scattering time delay in various low-energy effective theories. Subsequently, for example in [146, 175, 177, 178, 208, 209], it was generally argued on the same lines outlined by Eisenbud and Wigner that an overall negative time delay is a signal of causality violation and inconsistency of the low-energy effective theory [2]. This criterion is sometimes dubbed absence of "asymptotic superluminality" [171]. The perspective of some literature is that any time delay for which there is net negative sign $\Delta T < 0$, *i.e.* a net time advance, leads to a causality violation, since it implies propagation faster than the asymptotic spacetime, regardless of magnitude. However the precise statement of asymptotic causality should still include a notion of resolvability.

More specifically, asymptotic causality is the requirement that there is no resolvable net time advance $\Delta T \gtrsim -\omega^{-1}$. This accounts for the fact that propagating waves with frequencies peaked at ω will not be sensitive to timescales smaller than ω^{-1} — higher-order corrections to the JWKB approximation kick in at that scale. In field theory, the eikonal approximation is the exponential resummation of the t-channel exchange, which is sensible when the ladder diagrams are enhanced, *i.e.* when $|\delta| \gg 1$.

[2] In the context of non-relativistic quantum mechanics, the time delay can be negative. A well-known example for this is hard sphere scattering, and the original requirement of Wigner is only that the magnitude of the associated time advance is bounded.

It is sometimes said that violation of causality allows the construction of closed timelike curves and hence time machines [31, 174]. This would mean that a violation of asymptotic causality implies a violation of strong causality. However, this is misleading — as mentioned previously, background solutions which violate causality are also a feature of GR and we do not discard it as a theory on those grounds. The resolution to this is that solutions allowing closed timelike curves do not arise as the Cauchy development of generic initial data [144, 210]. Note that we will nevertheless use the term causality to refer to the absence (resolvable) time advances.

3.1.2 Causality in EFTs

We are interested in establishing restrictions on gravitational EFTs from causality. If we write down an effective action for gravity and think of that as defining the full theory, then we would hope that any of the aforementioned notions of causality would hold in these. This is however in tension with how we typically view EFTs — within the regime of validity of the EFT, the higher-derivative corrections should be subdominant to the main contribution and should not change the theory qualitatively. In the classical equations of motion, the highest-derivative term is critical in determining the behaviour of the solutions. In QFT, this is manifest in the scaling of the propagator in inverse powers of the momentum. It is only when we treat the higher-derivative corrections perturbatively that we can make sense of these theories. This is, for instance, done by either eliminating higher-derivative corrections in the classical equations of motion by using the lower-order equations of motion or by expanding the propagator in the powers of the momentum.

This should also be the case for causality. EFTs should not be subject to constraints on the theories viewed as a whole, but rather on the perturbative corrections only. If we were to consider gravity with EFT corrections on spacetime that already admits superluminal propagation in GR, it would be hard to distinguish the source of causality violation — is the EFT itself problematic, or have we just chosen the background poorly? We should certainly pick a causal background to study causality violations in gravitational EFT corrections.

This is related to the fact that we should be able to diagnose whether or not a theory is causal directly at low energies. To understand this, it is illustrative to consider Euler-Heisenberg theory from section 1.1 on a curved background. It was shown in [211] that this naïvely leads to a violation of causality, which is cured when the effect of the electron is revived [212–214]. This is of course obvious, as there is nothing inconsistent about placing QED on curved spacetime and integrating out the electron, so we always expected causality to be satisfied.

This is analogous to the statement that the Gauss-Bonnet operator requires higher-spin fields (as expected from string theory) in order to be consistent with causality [174]. However, whether we like it or not, physicists can only make low-energy observations, and we should be able to diagnose whether a theory is causal or not without having to refer to degrees of freedom that we are ignorant of. We will use gravitational positivity bounds, which are rigorous bounds on the resulting low-energy amplitudes, as a guiding principle for what we mean by causality in EFTs — they should, at least qualitatively, reproduce them. It was found in [180, 181] that IR

causality not to only reproduces but also complements gravitational positivity bounds.

Subluminality

We briefly mentioned that violations of strong causality are typically associated with the superluminal propagation of modes. It turns out that on flat space, these are clearly directly linked to positivity bounds. To see this, we consider the leading-order Goldstone EFT from (1.1.20), described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + g(\partial\phi)^4. \tag{3.1.12}$$

For the purposes of causality bounds, we can set $m = 0$ from the out-set. The classical equation of motion for the scalar field ϕ takes the form

$$G^{\mu\nu}\partial_\mu\partial_\nu = 0, \quad G^{\mu\nu} = \eta^{\mu\nu} - 4g\left[(\partial\phi)^2\eta^{\mu\nu} + 2\partial^\mu\phi\partial^\nu\phi\right], \tag{3.1.13}$$

so $G^{\mu\nu}$ determines the characteristic hypersurfaces. The effective metric is defined as the inverse of $G^{\mu\nu}$ — at weakling coupling,

$$G_{\mu\nu} = \eta^{\mu\nu} + 4g\left[(\partial\phi)^2\eta^{\mu\nu} + 2\partial^\mu\phi\partial^\nu\phi\right], \tag{3.1.14}$$

to leading order in g. The causal structure of the system is now determined by the cones of this effective metric G. For an arbitrary vector X^μ,

$$G_{\mu\nu}X^\mu X^\nu = \left[1 + 4g\,(\partial\phi)^2\right]X^2 + 8g\,(X_\mu\partial^\mu\phi)^2 \tag{3.1.15}$$

so at weak coupling, the null cones of the effective metric G are inside/outside the ones of the background spacetime metric η for $g > 0$ and $g < 0$ respectively. For example, for perturbations around a background solution which spontaneously breaks Lorentz invariance $\bar{\phi} = c_\alpha x^\alpha$ for some constant c, the linearised equations of motion for the perturbations $\delta\phi = \phi - \bar{\phi}$ are

$$(1 - 4c_\mu c^\mu)\,\Box\delta\phi - 8gc^\mu c^\nu\partial_\mu\partial_\nu\delta\phi = 0, \tag{3.1.16}$$

which is a wave equation with effective speed

$$v^2 = 1 - \frac{8g(c_\alpha p^\alpha)/|\mathbf{p}|^2}{1 - 4gc_\alpha c^\alpha}, \tag{3.1.17}$$

where $\delta\phi \sim e^{ip\cdot x}$. Demanding that the perturbations are subluminal therefore imposes $g > 0$ — this is precisely what the twice-subtracted positivity bound demanded previously. It is notable that this argument concerns local propagation speeds, and we are led to the same bound in a background produced by some source that looks like $\bar{\phi} \sim c_\alpha x^\alpha$ only locally.

Demanding that perturbations around a background that spontaneously breaks Lorentz invariance propagate subluminally lead us to the same bound. This however came with significantly less effort, and the subtlety of having to worry about assumptions in the spectrum. This is

promising! There are however certain drawbacks — propagation speeds themselves are not invariant under field redefinitions, only their ratios are.

Infrared Causality

To circumvent the ambiguity of propagation speeds, it is useful to adapt asymptotic causality, which is manifestly field redefinition invariant.

Since every field couples to gravity, every species will receive a contribution $\Delta T^{\rm GR}$ to the time delay from the gravitational background, what is known in $D = 4$ as the Shapiro time delay. In $D \geq 5$ the equivalent Shapiro time delay is IR finite and positive for a positive mass source. In addition, scattering particles will receive an additional time delay from interactions with heavy states. The latter effect is what is captured at low energies by the higher-dimension operators in the EFT. Thus it is natural to separate the low-energy time delay in the manner

$$\Delta T^{\rm net} = \Delta T^{\rm GR} + \Delta T^{\rm EFT}. \tag{3.1.18}$$

Infrared causality is the notion that only the sign of the EFT correction $\Delta T^{\rm EFT}$ to the time delay is relevant. This is the definition that is implicit in considering field theories in curved space, as in the investigations of causality in the EFT description of QED [211, 213, 215]. More precisely we require that $\Delta T^{\rm EFT} \gtrsim -\omega^{-1}$.

The notion of infrared causality is motivated by the fact that $\Delta T^{\rm GR}$ is universal for all particles coupled to gravity, including gravitons, and should be considered as the "reference" time delay. For instance, in cases where the EFT corrections come from integrating out loops of massive fields, which are minimally coupled to gravity in the UV, it is always the case that the causal structure is determined by the high-energy modes which are themselves minimally coupled and so lead to $\Delta T^{\rm GR}$. Overall a resolvable negative $\Delta T^{\rm EFT}$ (*i.e.* a violation of infrared causality) would indicate the fact that some species (in this case gravitons) can enjoy a spacelike geodesic which departs from the null geodesic by a resolvable amount. This would also mean that gravitons would experience a resolvable time advance as compared to other massless particles (say photons) that minimally couple to gravity.

Ultimately, the resolvability criterion can be traced back to the perturbative EFT expansion [216]. Treated as such, the support of the retarded Green's function inherits the causal structure of the uncorrected background geometry. The only possibility for causality violation to occur is when this perturbative expansion breaks down — this is the case when there are secular effects. Only their resummation can lead to significant changes to the support of the retarded Green's function, and hence measurable changes in the causal structure of the theory.

One may worry that the definition of the split $\Delta T^{\rm net} = \Delta T^{\rm GR} + \Delta T^{\rm EFT}$ is sensitive to field redefinitions of the metric that change the GR part. First, note that performing a field redefinition in the low-energy EFT also redefines the metric to which the high-energy modes couple, and in this analysis we consider a frame where at least one field couples minimally to gravity in the UV (at scales well above $\Lambda \lesssim M_{\rm Pl}$). This is further mitigated when a decoupling limit $M_{\rm Pl} \to \infty$ exists.

In general, this is a non-trivial multi-scaling limit in which all of the gravitational degrees of freedom fully decouple while keeping other non-trivial interactions intact — it strongly depends on the situation under consideration. The ambiguity of the split can then be resolved by defining

$$\Delta T^{\rm GR} = \lim_{\Lambda\to\infty} \Delta T \tag{3.1.19}$$

while keeping $\Lambda \lesssim M_{\rm Pl}$, within the low-energy EFT — this can be done at the level of the S-matrix. The definition above ensures that $\Delta T^{\rm EFT}$ only contains those terms which come from an inverse expansion in heavy masses. For an example of a non-trivial decoupling limit, consider type-II string theory from section 1.3, for which we identified $M_{\rm Pl}^8 \sim \frac{M_s^2}{g_s^2}$. A black hole solution with mass M will have horizon size

$$r_H^7 = \frac{M}{M_{\rm Pl}^8} \sim \frac{g_s M}{M_s^2}. \tag{3.1.20}$$

The naïve decoupling limit $M_s \to \infty$ leads to a flat background with vanishing $\Delta T^{\rm GR}$. Instead, we are interested in the limit where $M_s \to \infty$ at fixed g_s such that r_H is finite. This is achieved by a double scaling limit in which $M_s \to \infty$ and $M \to \infty$ simultaneously.

3.2 Toy Model: Galileon

Our proposed "infrared causality" condition would be useless were it not able to identify known field theories which violate causality. In this section we shall consider a scalar field theory in Minkowski space which is known to violate positivity bounds [44], and thus is expected to violate standard causality conditions. We will first establish the regime of validity and compute the scattering time delay to demonstrate that this example does indeed lead to resolvable time advances within the regime of validity of the EFT.

3.2.1 Action

We will consider a scalar field theory which closely resembles gravity. This is the Galileon, which is invariant under Galilean shifts $\phi \to \phi + c + v_\mu x^\mu$ for some constant vector v. The first, *i.e.* lowest derivative, invariant tensor is therefore $K_{\mu\nu} = \partial_\mu\partial_\nu\phi$. This acts like a Riemann curvature, and other similarities with gravity follow from this. We do not canonically normalise the action, but instead choose $[\phi] = 1$, so that

$$\begin{aligned} S = \int d^D x\, \Lambda^{D-4} \Bigg[&-\frac{1}{2}(\partial\phi)^2 + \frac{a_1}{\Lambda^3}(\partial\phi)^2[K] + \frac{b_1}{\Lambda^5}[K^3] + \frac{b_2}{\Lambda^5}[K^2][K] + \frac{b_3}{\Lambda^5}[K]^3 \\ &+ \frac{c_1}{\Lambda^6}(\partial\phi)^2\left([K^2]-[K]^2\right) + \frac{d_1}{\Lambda^8}[K^4] + \frac{d_2}{\Lambda^8}[K^3][K] + \frac{d_3}{\Lambda^8}[K^2][K]^2 + \frac{d_4}{\Lambda^8}[K^2]^2 + \frac{d_5}{\Lambda^8}[K]^4 \Bigg], \end{aligned} \tag{3.2.1}$$

where square brackets stand for the trace, *i.e.* $[K] = K^\mu{}_\mu$, and Λ is the cut-off of the theory. We expect the operator with more than one box in the action not to contribute, as it will have boxes at the level of the equations of motion — these are the terms with coefficients b_3, d_3, and d_5. Note that we skipped the contraction $[K^2]$, as it is actually equivalent to $[K]^2$ through integration

by parts. We will be interested in the subset of these terms invariant under Z_2: $\phi \to -\phi$, which have coefficients c_1 and d_i for $i \in \{1, \ldots, 5\}$.

It is well-known that Galileons exhibit superluminalities [31, 217, 218], although their relations with causality and consistent high-energy completions have a long history [31, 219–222].

3.2.2 Background

We consider a spherically symmetric background $\bar{\phi} = \bar{\phi}(r)$.

Solving the background equations of motion up to order Λ^{-8}, we find that the background field takes the form

$$\begin{aligned}\bar{\phi}(r) =& \frac{\alpha}{r^{D-3}} - c_1 \frac{2(D-2)(D-3)^4}{(3D-5)r^{3D-5}} \frac{\alpha^3}{\Lambda^6} \\ &+ \frac{(D-3)^3(D-1)(D-2)\big[(20-8D)d_1 - 3(D-3)d_2 - 8(D-2)d_4\big]}{3r^{3(D-1)}} \frac{\alpha^3}{\Lambda^8},\end{aligned} \tag{3.2.2}$$

where α is a constant measuring the scalar charge, analogous to the Schwarzschild mass parameter. In particular, including gravity, the canonically normalised Lagrangian of a scalar field ϕ_c coupled to the trace of the stress-tensor should be

$$\mathcal{L} \supset \frac{M_{\rm Pl}^{D-2}}{2} R - \frac{1}{2}(\partial\phi_c)^2 + \frac{c}{M_{\rm Pl}^{(D-2)/2}} \phi_c T. \tag{3.2.3}$$

Since the Newtonian potential goes like $\sim (r_g/r)^{D-3}$, then the dimensionless ratio should be

$$\frac{\phi_c}{M_{Pl}^{(D-2)/2}} \sim \left(\frac{r_g}{r}\right)^{D-3}. \tag{3.2.4}$$

Since $\Lambda^{(D-4)/2}\phi \sim \phi_c$, we then see that

$$\phi = \frac{M_{\rm Pl}^{(D-2)/2}}{\Lambda^{(D-4)/2}} \left(\frac{r_g}{r}\right)^{D-3} \longrightarrow \alpha = \frac{M_{\rm Pl}^{(D-2)/2}}{\Lambda^{(D-4)/2}} r_g^{D-3}. \tag{3.2.5}$$

3.2.3 Perturbations

Unlike the background, the perturbations can be function of all the coordinates. We shall simplify the full perturbation equations of motion by using the lower-order equations to get rid of higher derivatives in the higher-order equations.

We rescale the perturbation variable/field to get rid of the first radial derivative. This brings us into the canonical form of a master equation, that governs the dynamics of the Galileon perturbations

$$\frac{{\rm d}^2\Phi}{{\rm d}r^2} + W_\ell(r)\Phi = 0. \tag{3.2.6}$$

To improve the behaviour of the JWKB approximation at low ℓ, we can perform the so-called Langer shift effectively by changing coordinates to ρ with $r = e^\rho$. After performing a field

redefinition, we can see that this essentially just shifts the effective potential by $W_\ell \to W_\ell - \frac{1}{4r^2}$. In particular, with the Langer shift

$$\begin{aligned}
W_\ell &= e^{2\rho}\omega^2 - b^2\omega^2 \\
&\quad - \frac{3c_1\alpha^2}{\Lambda^6}(D-3)^2\big[4e^{2\rho}\omega^2(D-2)^2 + ((D-1)(3D-5) - 4b^2\omega^4)(D-1)(D-3)\big]e^{-2(D-1)\rho} \\
&\quad - \frac{d_1\alpha^2}{4\Lambda^8}(D-3)^2\big[-16e^{4\rho}\omega^4(D-2)^2 + 8e^{2\rho}\omega^2(4b^2\omega^2 + 15D^2 - 40D + 13)(D-2)(D-1) \\
&\qquad - 16b^4\omega^4(D-1)^2 - 8b^2\omega^2(D-1)(15D^3 - 67D^2 + 71D + 5) \\
&\qquad + (D-1)(127D^5 - 567D^4 + 610D^3 + 46D^2 - 33D + 9)\big]e^{-2D\rho} \\
&\quad - \frac{6d_2\alpha^2}{\Lambda^8}(D-3)^3(D-2)(D-1)\big[e^{2\rho}\omega^2(2D-1) + 2D(-b^2\omega^2 + D^2)\big]e^{-2D\rho} \\
&\quad - \frac{d_4\alpha^2}{2\Lambda^8}(D-3)^2\big[-16e^{4\rho}\omega^4(D-2)^2 + 8e^{2\rho}\omega^2(4b^2\omega^2 + 7D^2 - 12D + 1)(D-1)(D-2) \\
&\qquad - 16b^4\omega^4(D-1)^2 - 8b^2\omega^2(D-1)(7D^3 - 27D^2 + 23D + 5) \\
&\qquad + (D-1)(63D^5 - 247D^4 + 226D^3 + 46D^2 - 33D + 9)\big]e^{-2D\rho}\,,
\end{aligned} \tag{3.2.7}$$

where we have defined the impact parameter b via

$$b\omega = \ell + \frac{D-3}{2}. \tag{3.2.8}$$

3.2.4 Regime of validity

We will now establish the EFT regime of validity for the background $\bar{\phi}$. Because of the Galileon symmetry, we need only consider operators built out of $K_{\mu\nu} = \partial_\mu\partial_\nu\phi$. The generic requirement for the traditional regime of validity of the effective theory is

$$\left(\frac{\partial}{\Lambda}\right)^p \left(\frac{K}{\Lambda^3}\right)^q \ll 1\,. \tag{3.2.9}$$

This ensures that the EFT action remains under control as an asymptotic expansion. To leading order, $\bar{\phi} \sim \alpha/r^{D-3}$, and so demanding that this condition is valid at the point of closest approach $r \sim b$, we have

$$\frac{\alpha^q}{b^{p+q(D-1)}} \ll \Lambda^{p+3q}. \tag{3.2.10}$$

The first, strongest bound come from $q \to \infty$

$$b \gg r_V = \left(\frac{\alpha}{\Lambda^3}\right)^{\frac{1}{D-1}}, \tag{3.2.11}$$

where r_V is known as the Vainshtein radius. In addition, focusing on the bound $p \to \infty$ gives

$$b\Lambda \gg 1\,. \tag{3.2.12}$$

We will encounter a similar bound for gravitational EFTs later on.

In order to establish the EFT regime of validity for the perturbations $\delta\phi$ around the background,

we follow the same procedure and work in the limit of high momenta for fluctuations so that we may approximate $\partial_\alpha \delta\phi \sim i k_\alpha \delta\phi$. We then look for general scalar operators built out of the combinations

$$\left(\frac{\partial}{\Lambda}\right)^p \left(\frac{K}{\Lambda^3}\right)^q \left(\frac{k}{\Lambda}\right)^r \ll 1\,. \tag{3.2.13}$$

As $k_\alpha k^\alpha = 0$ to leading order, it is sufficient to focus on terms with the most k's contracted with $K_{\mu\nu}$, specifically $k^\mu k^\nu K_{\mu\nu}$. For instance, we may focus on the set of operators of the form

$$\Box^p [k^\alpha \partial_\alpha]^q \left(k^\mu k^\nu K_{\mu\nu}\right)^r \ll \Lambda^{2p+2q+5r}\,. \tag{3.2.14}$$

Once again, since at leading order $\bar{\phi} \sim \alpha / r^{D-3}$, in the limit $q \to \infty$, (3.2.14) gives

$$\omega \ll \Lambda^2 b. \tag{3.2.15}$$

The limit $p \to \infty$ reproduces (3.2.12). The remaining non-trivial bound ($r \to \infty$) is

$$|k^\mu k^\nu K_{\mu\nu}| \ll \Lambda^5\,, \tag{3.2.16}$$

which gives

$$\omega^2 \ll \frac{\Lambda^5 b^{D-1}}{\alpha}\,. \tag{3.2.17}$$

To summarise, equations (3.2.11) and (3.2.12) define the regime of validity for the backgrounds, and equations (3.2.17) and (3.2.15) define the regime of validity for the perturbations in our EFT.

3.2.5 Scattering Time Delay

The JWKB solution of the Langer-shifted version of (3.2.6) decays exponentially as $\rho \to -\infty$

$$\chi_\ell \approx \frac{\bar{\chi}_\ell}{(-W_\ell)^{1/4}} \exp\left(-\int_\rho^{\rho_t} \sqrt{-W_\ell} d\rho\right). \tag{3.2.18}$$

Using the connection formula, we find that for $\rho > \rho_t$

$$\chi_\ell \approx \frac{2\bar{\chi}_l}{W_l^{1/4}} \sin\left(\int_{\rho_t}^{\rho} \sqrt{W_\ell} d\rho + \frac{\pi}{4}\right). \tag{3.2.19}$$

By definition of the scattering phase shifts, we require these solutions to have following asymptotic form

$$\chi_\ell \sim \left(e^{2i\delta_\ell} e^{i\omega\rho} + e^{i\pi d/2} e^{i\pi\ell} e^{-i\omega\rho}\right). \tag{3.2.20}$$

By comparison

$$\delta_\ell = \int_{\rho_t}^{\infty} \left(\sqrt{W_\ell} - \omega e^\rho\right) d\rho - \omega r_t + \left(\ell + \frac{D-3}{2}\right)\frac{\pi}{2}. \tag{3.2.21}$$

From this, we can compute the scattering time delay using (3.2.7) and (3.1.11) which takes the form

$$\Delta T = \Bigg[- c_1 \frac{1}{b^{-3+2D}} \frac{6\sqrt{\pi}(D-1)^2(D-3)^2\Gamma\left(D-\frac{3}{2}\right)}{\Gamma\left(D-1\right)} \frac{\alpha^2}{\Lambda^6} + (d_1+2d_4)\frac{\omega^2}{b^{-3+2D}} \frac{3\sqrt{\pi}D(D-1)^2(D-3)^2\Gamma\left(D-\frac{3}{2}\right)}{\Gamma\left(D+1\right)} \frac{\alpha^2}{\Lambda^8} \Bigg] , \tag{3.2.22}$$

up to terms which are subleading in $1/\ell$ and $1/\Lambda$. We see that the first term leads to a time advance if $c_1 > 0$. Before concluding any violation of causality we must establish whether this is resolvable within the regime defined by the all four EFT bounds (3.2.11), (3.2.12), (3.2.17) and (3.2.15). Indeed we have from (3.2.11)

$$|\omega \Delta T^{(c_1)}| \ll |c_1| \omega b \, . \tag{3.2.23}$$

Imposing the condition (3.2.15) then gives

$$|\omega \Delta T^{(c_1)}| \ll |c_1| (b\Lambda)^2 \, . \tag{3.2.24}$$

We can make the combination $b\Lambda$ arbitrarily large within the regime of validity of the EFT (none of the conditions impose an upper cutoff on b since a large b pushes us towards the IR), we see that there is no difficulty in making the r.h.s. as large as desired. Thus we find that, as expected, this scalar model violates causality for $c_1 > 0$ — more precisely, we can generate a resolvably large time advance within the regime of validity of the EFT. We can cleanly reach this conclusion without the aforementioned ambiguities due to gravity being dynamical because we are working with a fixed background geometry — it is why the decoupling limit was crucial to our discussion of superluminality in dynamical gravity earlier.

Interestingly, on considering a combination of higher-dimension operators (those captured by the coefficient $d_1 + 2d_4$), we find that these make a comparable contribution to the time delay already when $\omega \sim \Lambda$, and so to infer any conclusion from the first term we need to impose

$$\omega \ll \Lambda. \tag{3.2.25}$$

Note that this does not imply a breakdown of the EFT at these scales, it rather reflects an accident of the leading Galileon terms being smaller than expected. In other words, the EFT as a whole can still be under control when (3.2.25) is violated, but the contribution from the quartic Galileon operator is simply suppressed as compared to that of other operators. Regardless, even if we impose (3.2.25) on (3.2.23), we still have

$$|\omega \Delta T^{(c_1)}| \ll |c_1| (b\Lambda) \, , \tag{3.2.26}$$

for which the r.h.s. remains arbitrarily large. This example illustrates that the resolvability criterion does not indiscriminately sweep all apparent acausalities under the carpet, but correctly identifies genuine ones.

3.3 Asymptotic vs. Infrared Causality

To clearly distinguish between the notions of asymptotic causality and infrared causality, we shall now argue that it is the latter condition which correctly reproduces the gravitational positivity bounds diagnosed in [45, 67, 69, 164, 168, 223].

3.3.1 Toy Model: Goldstone EFT

To this end, let us consider the leading-order Goldstone scalar model from (1.1.20), now on a curved spacetime

$$S = \int \mathrm{d}^D x\sqrt{-g}\left[-\frac{1}{2}(\nabla\phi)^2 + \frac{c}{\Lambda^D}(\nabla\phi)^4 + \dots\right]. \tag{3.3.1}$$

To begin with, let us assume that the stress-energy of the scalar is sufficiently subdominant to other contributions that we can treat its backreaction on the metric perturbatively. We take the metric to be a general spherically symmetric form

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{B(r)} + r^2\mathrm{d}\Omega_d^2. \tag{3.3.2}$$

We can further assume that whatever sources the spherically symmetric solution, also sources a spherically symmetric background configuration for the scalar. At leading order, this background is determined by

$$\bar{\phi}'(r) = \frac{\alpha}{r^{D-2}C(r)} + \mathcal{O}(\Lambda^{-D}), \tag{3.3.3}$$

with $C(r) = A(r)/B(r)$.

Considering fluctuations $\phi = \bar{\phi} + \delta\phi$, defining $\chi = r^{(D-2)/2}\sqrt{C}\left(1 - \frac{6c\alpha^2}{\Lambda^D r^{2D-4}A^2C^2}\right)\delta\phi$, and performing the Langer transformation $r = e^\rho$, we have for each partial wave

$$\chi_\ell''(\rho) + W_\ell(\rho)\chi_\ell(\rho) = 0, \tag{3.3.4}$$

where

$$W_\ell(\rho) = \frac{e^{2\rho}}{C^2}\omega^2\left(1 + \frac{8c\alpha^2}{\Lambda^D r^{2D-4}B^2C^2}\right) - B^2\omega^2 b^2\left(1 + \frac{8c\alpha^2}{\Lambda^D r^{2D-4}B^2C^2}\right) + \dots, \tag{3.3.5}$$

up to subleading terms negligible at large ℓ, and terms of higher order in $1/\Lambda$.

3.3.2 Scattering Time Delay

We can compute the time delay in the Goldstone toy model by following the procedure in section 3.2.5, in particular using equations (3.2.21) and (3.1.11).

Given the turning point defined by $W_\ell(\rho_t) = 0$, it is helpful to split W_ℓ as $W_\ell = \omega^2\left(U_\ell^{(1)} + \frac{c\alpha^2}{\Lambda^{D-2}}U_\ell^{(2)}\right)$ where

$$U_\ell^{(1)}(\rho,\rho_t) = \left(\frac{e^{2\rho}}{C^2(\rho)} - \frac{e^{2\rho_t}}{C^2(\rho_t)}\right) - b^2\left[B^2(\rho) - B^2(\rho_t)\right], \tag{3.3.6}$$

and

$$U_\ell^{(2)}(\rho,\rho_t)=4\left(\frac{e^{-(2D-3)\rho}}{B^2(\rho)C^3(\rho)}-\frac{e^{-(2D-3)\rho_t}}{B^2(\rho_t)C^3(\rho_t)}\right)-4b^2\left(\frac{e^{-(2D-4)\rho}}{C^2(\rho)}-\frac{e^{-(2D-4)\rho_t}}{C^2(\rho_t)}\right). \tag{3.3.7}$$

The net time delay at fixed impact parameter is given up to order $1/\Lambda^D$ by

$$\Delta T_b^{\rm net}=2\int_{\rho_t}^{\infty}\left(\sqrt{U_\ell^{(1)}(\rho,\rho_t)}-e^{\rho}\right)\mathrm{d}\rho-r_t+\frac{c\alpha^2}{\Lambda^{D-2}}\int_{\rho_t}^{\infty}\frac{U_\ell^{(2)}(\rho,\rho_t)}{\sqrt{U_\ell^{(1)}(\rho,\rho_t)}}\mathrm{d}\rho+\dots\,. \tag{3.3.8}$$

We now define the GR time delay as the time delay we would obtain in the limit $\Lambda\to\infty$, in which all corrections from heavy modes are removed. Since the turning point ρ_t in general depends on Λ, this will be written in terms of the turning point ρ_t^0, for which $A(\rho_t^0)b=e^{\rho_t^0}$. Thus

$$\Delta T_b^{\rm GR}=\lim_{\Lambda\to\infty}\Delta T_b^{\rm net}=2\int_{\rho_t^0}^{\infty}\left(\sqrt{U_\ell^{(1)}(\rho,\rho_t^0)}-e^{\rho}\right)\mathrm{d}\rho-r_t^0\,. \tag{3.3.9}$$

Putting this together, the EFT time delay at fixed impact parameter is

$$\begin{aligned}\Delta T_b^{\rm EFT}&=\frac{c\alpha^2}{\Lambda^D}\int_{\rho_t^0}^{\infty}\frac{U_\ell^{(2)}(\rho,\rho_t^0)}{\sqrt{U_\ell^{(1)}(\rho,\rho_t^0)}}\mathrm{d}\rho+2\int_{\rho_t}^{\infty}\sqrt{U_\ell^{(1)}(\rho,\rho_t)}\mathrm{d}\rho-2\int_{\rho_t^0}^{\infty}\sqrt{U_\ell^{(1)}(\rho,\rho_t^0)}\mathrm{d}\rho+\dots\\&=\frac{c\alpha^2}{\Lambda^D}\int_{\rho_t^0}^{\infty}\frac{U_\ell^{(2)}(\rho,\rho_t^0)}{\sqrt{U_\ell^{(1)}(\rho,\rho_t^0)}}\mathrm{d}\rho+(\rho_t-\rho_t^0)\int_{\rho_t^0}^{\infty}\frac{\partial_{\rho_t^0}U_\ell^{(1)}(\rho,\rho_t^0)}{\sqrt{U_\ell^{(1)}(\rho,\rho_t^0)}}\mathrm{d}\rho+\dots\,.\end{aligned} \tag{3.3.10}$$

Backreaction

Crucially, we now include the gravitational backreaction of the field $\bar{\phi}$. We remain at impact parameters for which the weak gravitational field approximation is valid, so when including the scalar field backreaction we can take $A\approx B\approx C\approx 1$. The metric in harmonic gauge then satisfies

$$\Box\left(h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h\right)=-\frac{2}{M_{\rm Pl}^{D-2}}T_{\mu\nu}\,. \tag{3.3.11}$$

This is easily solved for a spherically symmetric source (3.3.2), and schematically we have [3]

$$h\sim\frac{\alpha^2}{M_{\rm Pl}^{D-2}r^{2(D-3)}}\,. \tag{3.3.12}$$

It is straightforward to estimate the GR time delay (3.3.9) from the backreaction of the scalar as

$$\Delta T^{\phi,\rm GR}\sim\frac{\alpha^2}{M_{\rm Pl}^{D-2}b^{2(D-3)}}b\,. \tag{3.3.13}$$

[3] For the remainder of this section we omit order unity factors as our goal is to estimate orders of magnitudes. With this in mind, the arguments in this and the following section apply equally well to the time delay at fixed b or ℓ.

This adds to the contribution from any other gravitational source such as the Schwarzschild background due to a localised source of mass M with Schwarzschild radius $r_g^{D-3} \sim M/M_{\rm Pl}^{D-2}$

$$\Delta T^{M,\rm GR} \sim \frac{r_g^{D-3}}{b^{D-4}}\,, \tag{3.3.14}$$

while in the weak field limit, the EFT contribution is essentially dominated by its value in Minkowski spacetime

$$\Delta T^{\rm EFT} \sim \frac{c\alpha^2}{\Lambda^D b^{2D-5}}\,. \tag{3.3.15}$$

3.3.3 Causality

As already illustrated in section 3.2, causality can be used to put bounds on the Wilson coefficients in an effective theory. So far, preserving causality seems to imply $\Delta T_b^{\rm EFT} > 0$, which would demand $c > 0$, consistent with known positivity bounds [29–31]. However we shall see that this bound can actually be slightly violated in the presence of gravity.

Asymptotic Causality

Putting everything together, the condition for asymptotic causality, *i.e.* the absence of any resolvable time advance is

$$\Delta T^{\rm net} \sim \frac{r_g^{D-3}}{b^{D-4}} + \frac{\alpha^2}{M_{\rm Pl}^{D-2} b^{2D-6}} b\left(1 + \frac{cM_{\rm Pl}^{D-2}}{\Lambda^D b^2}\right) \gtrsim -\omega^{-1}\,. \tag{3.3.16}$$

If $c > 0$, the net time delay is always positive, but for $c < 0$ the net delay can in principle, *i.e.* ignoring the regime of validity, always be made negative by choosing b small enough. More precisely we can read this equation as a lower bound on the coefficient c, similar in spirit to the positivity bounds

$$c \gtrsim -\frac{\Lambda^D b^2}{M_{\rm Pl}^{D-2}} - \frac{\Lambda^D b^{D-1} r_g^{D-3}}{\alpha^2} - \frac{\Lambda^D b^{2D-5}}{\alpha^2 \omega}\,. \tag{3.3.17}$$

Our goal is to find the tightest version of this bound for which the r.h.s. is as large as possible. We may always choose $r_g = 0$, or α large enough so that the Schwarzschild contribution is negligible. As in the Galileon example, the largest ω we are allowed is $\Lambda^2 b$ (cf. (3.2.15)), and the largest α can be fixed from requiring $(\partial\phi)^2 \lesssim \Lambda^D$, so that $\alpha \lesssim \Lambda^{D/2} b^{D-2}$. With these choices we have

$$c \gtrsim -\frac{\Lambda^D b^2}{M_{\rm Pl}^{D-2}} - \frac{1}{\Lambda^2 b^2}\,. \tag{3.3.18}$$

Extremising the r.h.s. gives $b = \Lambda^{-1}\left(\frac{M_{\rm Pl}}{\Lambda}\right)^{(D-2)/4} \gg \Lambda^{-1}$ which in turn gives the bound

$$c \gtrsim -\left(\frac{\Lambda}{M_{\rm Pl}}\right)^{(D-2)/2}\,. \tag{3.3.19}$$

This is the most optimistic bound we can get from this analysis, and occurs at $h \sim (\Lambda/M_{\rm Pl})^{(D-2)/2} \ll 1$, well within the weak gravitational field approximation [4] .

Crucially, the bound (3.3.19) is weaker than the gravitational positivity bounds [223] observed in [45, 67, 69]. Indeed we would never expect positivity bounds to give a condition of the form (3.3.19) due to the non-analytic square root of the gravitational coupling constant.

Infrared Causality

If we now consider the infrared causality condition, a much clearer picture emerges. Again working in the weak gravitational field regime, we focus now only on the EFT time delay (3.3.15)

$$\Delta T^{\rm EFT} \sim \frac{c\alpha^2}{\Lambda^D b^{2D-5}} \,. \tag{3.3.20}$$

In order to ask whether this is resolvable within in the regime of the EFT we again use the maximum allowed frequency $\omega \ll \Lambda^2 b$ so that

$$\omega \Delta T^{\rm EFT} \ll \frac{c\alpha^2}{\Lambda^{D-2} b^{2D-6}} \,. \tag{3.3.21}$$

Now, in the absence of gravity, the only clear upper bound we can impose on α comes from demanding $(\partial\phi)^2 \ll \Lambda^D$ which would tell us

$$\omega \Delta T^{\rm EFT} \ll c(b\Lambda)^2 \,. \tag{3.3.22}$$

Since the r.h.s. can be arbitrarily large we find that if $c < 0$ we can establish an arbitrarily large resolvable time advance, regardless of how small the magnitude of c is. This is of course consistent with positivity bounds in Minkowski spacetime that demand the strict requirement [29–31]

$$c > 0 \,. \tag{3.3.23}$$

The central difference when we include gravity is that, as we increase the scale α in the field profile, there comes a point at which we can no longer neglect the gravitational backreaction of the scalar field ϕ itself. Demanding that the backreaction is under control, which was implicit in our calculation so far, amounts to

$$\frac{\alpha^2}{M_{\rm Pl}^{D-2} b^{2D-6}} \ll 1 \,. \tag{3.3.24}$$

Putting this condition into the (3.3.21) we infer that

$$\omega \Delta T_b^{\rm EFT} \ll \frac{c M_{\rm Pl}^{D-2}}{\Lambda^{D-2}} \,. \tag{3.3.25}$$

[4] If we had fixed the maximum α by requiring the weak gravitational field approximation is still valid, *i.e.* $\alpha^2 \sim M_{\rm Pl}^{D-2} b^{2D-6}$ then substituting in (3.3.17) with $r_g = 0$ and the maximum ω gives $c \gtrsim -\Lambda^D b^2 M_{\rm Pl}^{2-D} - \Lambda^{D-2} M_{\rm Pl}^{2-D}$. Then demanding that $(\partial\phi)^2 \ll \Lambda^D$ gives $b^2 \gg \Lambda^{-1} (M_{\rm Pl}/\Lambda)^{D-2}$, leading to the even weaker statement $c \gtrsim -1$.

Now the condition for infrared causality, $\Delta T_b^{\rm EFT} \gtrsim -\omega^{-1}$, becomes the statement

$$c \gtrsim -\left(\frac{\Lambda}{M_{\rm Pl}}\right)^{D-2} , \tag{3.3.26}$$

which is (up to order unity factors) exactly the gravitational positivity bound derived in [223] and conjectured in [67] (for the case of $D = 4$).

We learn something very important from the above analysis. The fact that the coefficient c is allowed to be slightly negative when we include gravity has nothing whatsoever to do with the positive contribution from $\Delta T^{\rm GR}$. It is entirely down to the negative sign in the resolvability criterion $\Delta T^{\rm EFT} \gtrsim -\omega^{-1}$. This is the most compelling evidence that causality in a gravitational theory is determined by the infrared causality condition and not the asymptotic causality condition. A similar observation was made in the EFT of gravity in $D = 4$ in [145, 147]. There, it was noted that if we only impose the criterion of asymptotic causality, we actually allow for values for the Wilson coefficients which are in conflict with positivity bounds.

3.4 Scattering time delay

In this section, we finally present the formal definition of scattering time delays and derive expressions for it in various representations that will be useful for future calculations. In particular we give a generic semi-classical expression to complement the more familiar eikonal expression.

We will initially consider stationary spacetimes, which admit a causal Killing vectors to which we can associate a conserved energy. For stationary spacetimes, the Killing vector is timelike and we will denote the conserved energy as E. For pp-waves, it is convenient to use null coordinates (u, v, x^i), where ∂_v is the Killing vector for the null time v and dynamical evolution is with respect to u.

3.4.1 Eisenbud-Wigner-Smith time delay

Consider the single-particle S-matrix $\hat{S}$. In the elastic scattering region, in which $\hat{S}^\dagger \hat{S} = 1$, the eigenvalues of $\hat{S}$ define the asymptotic phase shifts $e^{2i\delta}$. For example, in the case of spherically symmetric non-relativistic scattering, the S-matrix diagonalises into partial waves and each partial wave phase shift $\delta_\ell(E)$, which is a function of the incoming energy E, determines an eigenvalue of the S-matrix. Eisenbud and Wigner [203, 204] proposed that for non-relativistic scattering, associated with each partial wave, we can define the Eisenbud-Wigner time delay

$$\Delta T_\ell = \frac{2\partial\delta_\ell}{\partial E} , \tag{3.4.1}$$

by differentiating the partial wave phase shifts with respect to the energy — this is (3.1.11). Qualitatively, this describes the delay of a scattering wavepacket which is peaked around an energy E.

In situations with less symmetry, it is in general difficult to construct the eigenvalues and eigen-

states, so it is preferable to have a more general definition of what we mean by the time delay. This is provided by the Hermitian time delay operator, known as the Wigner-Smith operator [205, 206], which is defined in terms of the S-matrix as

$$\widehat{\Delta T} = -\frac{i}{2}\hat{S}^\dagger \frac{\partial \hat{S}}{\partial E} + \frac{i}{2}\frac{\partial \hat{S}^\dagger}{\partial E}\hat{S}. \tag{3.4.2}$$

Connection can be made with the Eisenbud-Wigner definition of the time delay by considering the elastic scattering region once again and considering the expectation value of $\widehat{\Delta T}$ on an eigenstate of $\hat{S}$ [5]. Denoting these by $|\delta\rangle$, then differentiating $\hat{S}|\delta\rangle = e^{2i\delta}|\delta\rangle$ we have

$$\frac{\partial \hat{S}}{\partial E}|\delta\rangle + (\hat{S} - e^{2i\delta})\frac{\partial |\delta\rangle}{\partial E} = e^{-2i\delta}\frac{\partial (e^{2i\delta})}{\partial E}|\delta\rangle\,, \tag{3.4.3}$$

from which we infer

$$\langle\delta|\,\widehat{\Delta T}\,|\delta\rangle = 2\frac{\partial \delta}{\partial E}\,, \tag{3.4.4}$$

which is exactly the Eisenbud-Wigner time delay.

The operator (3.4.2) can be formally extended to the full Fock space/QFT Hilbert space as follows: Since the full S-matrix $\hat{\mathbb{S}}$ commutes with the asymptotic Hamiltonian $\hat{H}_0$ for time-translation invariant systems, we may write it in terms of its spectral decomposition

$$\hat{\mathbb{S}} = \int \mathrm{d}E \sum_{I,J} |E, I\rangle \mathbb{W}_{IJ}(E) \langle E, J|\,, \tag{3.4.5}$$

where $|E, I\rangle$ denote a complete set of multi-particle energy eigenstates of $\hat{H}_0$

$$\int \mathrm{d}E \sum_I |E, I\rangle\langle I, E| = \hat{1}\,, \quad \langle E, I|E', J\rangle = \delta(E - E')\delta_{IJ}\,. \tag{3.4.6}$$

The time delay operator defined on the full Fock space is then

$$\begin{aligned}\widehat{\Delta\mathbb{T}} &= -\frac{i}{2}\int \mathrm{d}E \sum_I \hat{\mathbb{S}}^\dagger |E, I\rangle \frac{\partial}{\partial E}\left(\langle I, E|\hat{\mathbb{S}}\right) + \frac{i}{2}\int \mathrm{d}E \sum_I \frac{\partial}{\partial E}\left(\hat{\mathbb{S}}^\dagger|E, I\rangle\right)\langle I, E|\hat{\mathbb{S}} \\ &= -i\int \mathrm{d}E \sum_I \frac{\partial}{\partial \epsilon}\left[\hat{\mathbb{S}}^\dagger|E - \frac{\epsilon}{2}, I\rangle\langle E + \frac{\epsilon}{2}, I|\hat{\mathbb{S}}\right]\Big|_{\epsilon=0}\,.\end{aligned} \tag{3.4.7}$$

Given that $\hat{H}_0$ commutes with $\hat{\mathbb{S}}$ and $\hat{H}_0|E \pm \epsilon/2, I\rangle = (E \pm \epsilon/2)|E \pm \epsilon/2, I\rangle$ then formally we have

$$[\hat{H}_0, \widehat{\Delta\mathbb{T}}] = i\int \mathrm{d}E\, \frac{\partial}{\partial \epsilon}\left[\epsilon\hat{\mathbb{S}}^\dagger|E - \frac{\epsilon}{2}, I\rangle\langle E + \frac{\epsilon}{2}, I|\hat{\mathbb{S}}\right]\Big|_{\epsilon=0} = i\,, \tag{3.4.8}$$

which shows that the time delay is the operator appropriately conjugate to $\hat{H}_0$. Note that this expression only holds at a formal level since, according to Pauli's theorem [224], there can be no self-adjoint operator conjugate to a Hamiltonian with an energy spectrum bounded below. In short, if there were then it would be possible to construct a unitary operator $e^{-i\Delta_E\widehat{\Delta T}}$ which could translate energy eigenstates to arbitrarily negative energies $e^{-i\Delta_E\widehat{\Delta T}}|E\rangle = |E - \Delta_E\rangle$.

[5] Unitary operators are normal, and the spectral theorem holds for these.

Pauli's theorem is avoided, because the time delay operator is a non self-adjoint operator [225–238].

In the literature, the time delay is often associated only with the phase factor that occurs in the impact parameter representation of scattering amplitudes $\Delta T \sim 2\frac{\partial\delta(b,E)}{\partial E}$ or equivalently $\Delta v \sim 2\frac{\partial\delta(b,k_v)}{\partial k_v}$. We stress that this is only an approximation since the S-matrix does not diagonalise in impact parameter basis. Further, this definition fails to connect with the well-understood situation for small multipoles, which was the original interest of Eisenbud and Wigner [203, 204]. By contrast, the Wigner-Smith operator is meaningful and matches precisely with the Eisenbud-Wigner definition for partial waves and the impact parameter definition when the latter is a good approximation.

Null time delay

Given the symmetries of pp-wave spacetimes, it is natural to generalise the usual concept of a time delay to that of a null time delay conjugate to the conserved null momentum $\hat{P}_v$. We aim to define an operator $\widehat{\Delta\mathbb{V}}$ which acts on the asymptotic states and is conjugate to the asymptotic null momentum $\hat{P}^0_v$. It is natural to foliate the spacetime by null hypersurfaces, which corresponds to setting up our field theory scattering problem in front form, *i.e.* in lightcone quantisation [6]. This is generally subtle — for instance, first class constraints no longer necessarily generate gauge transformations [240]. These will however be inconsequential to our discussion below.

In Heisenberg picture, the quantised free-field metric fluctuations satisfy a Schrödinger-like equation with u as dynamical time. Given the null Killing vector it is natural to split the Hermitian fields into creation and annihilation operators

$$\hat{\Phi}(u,v,\mathbf{x}) = \int_0^\infty \frac{\mathrm{d}k_v}{2\pi} \sum_I \left[\hat{a}_{k_v,I} e^{ik_v v}\psi_I(u,\mathbf{x}) + \hat{a}^\dagger_{k_v,I} e^{-ik_v v}\psi^*_I(u,\mathbf{x})\right] , \tag{3.4.9}$$

with $k_v > 0$ by definition. The complex single-particle wavefunctions $\psi_I(u,\mathbf{x})$ satisfy a master equation obtained from performing a Fourier transform in the v-coordinate $\partial_v \to ik_v$. This takes the form of a non-relativistic Schrödinger equation

$$i\frac{\partial}{\partial u}\psi(u,\mathbf{x}) = -\frac{1}{2k_v}\nabla^2_d\,\psi(u,\mathbf{x}) + V(u,\mathbf{x})\psi(u,\mathbf{x})\,, \tag{3.4.10}$$

where the u-coordinate plays the role of time, $\nabla^2_d = \partial_i\partial^i$ is the Laplacian on the d-dimensional transverse Euclidean space, and the potential term arises from the curvature of the spacetime. In this language, we can treat the metric perturbations as a particle of mass k_v being scattered off a potential V sourced by the pp-wave metric. For instance, assuming the potential vanishes asymptotically, we can label the incoming states by the incident transverse momenta $\mathbf{k}$, and so

[6] Pp-waves are not generally globally hyperbolic. Causality properties are in fact determined by the fall-off of the metric components at spatial infinity [239]. Fortunately, for the cases of interest to us, global hyperbolicity is guaranteed.

the incident field may be written as

$$\lim_{u\to-\infty}\hat{\Phi}(u,v,\mathbf{x})=\int_0^\infty\frac{\mathrm{d}k_v}{2\pi}\frac{1}{\sqrt{2k_v}}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\left[\hat{a}^{\mathrm{in}}_{k_v,\mathbf{k}}e^{ik_vv}e^{i\mathbf{k}\cdot\mathbf{x}-i\frac{\mathbf{k}^2}{2k_v}u}+\hat{a}^{\mathrm{in}}_{k_v,\mathbf{k}}{}^\dagger e^{-ik_vv}e^{-i\mathbf{k}\cdot\mathbf{x}+i\frac{\mathbf{k}^2}{2k_v}u}\right], \tag{3.4.11}$$

with a similar expression for the outgoing field

$$\lim_{u\to\infty}\hat{\Phi}(u,v,\mathbf{x})=\int_0^\infty\frac{\mathrm{d}k_v}{2\pi}\frac{1}{\sqrt{2k_v}}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\left[\hat{a}^{\mathrm{out}}_{k_v,\mathbf{k}}e^{ik_vv}e^{i\mathbf{k}\cdot\mathbf{x}-i\frac{\mathbf{k}^2}{2k_v}u}+\hat{a}^{\mathrm{out}}_{k_v,\mathbf{k}}{}^\dagger e^{-ik_vv}e^{-i\mathbf{k}\cdot\mathbf{x}+i\frac{\mathbf{k}^2}{2k_v}u}\right]. \tag{3.4.12}$$

The relation between the in- and out-fields is given by the full Fock space S-matrix $\hat{\mathbb{S}}$

$$\hat{a}_{\mathrm{out}}=\hat{\mathbb{S}}^\dagger\hat{a}_{\mathrm{in}}\hat{\mathbb{S}}\,, \tag{3.4.13}$$

and the normalisation is

$$[\hat{a}^{\mathrm{in}}_{k_v,\mathbf{k}},\hat{a}^{\mathrm{in}}_{k'_v,\mathbf{k}'}{}^\dagger]=(2\pi)^d\delta^{(d)}(\mathbf{k}-\mathbf{k}')(2\pi)\delta(k_v-k'_v)\,. \tag{3.4.14}$$

In writing (3.4.13) we understand that (3.4.11) and (3.4.12) apply also to the fully interacting Heisenberg fields in the Lehmann–Symanzik–Zimmermann (LSZ) sense. One of the central virtues of the pp-wave spacetimes is that, despite being "time-dependent", the equations for fluctuations are first-order in (null) time. This means we do not need to worry about the pair-creation characteristic of time-dependent spacetimes[7] and since k_v is conserved, the S-matrix diagonalises on states of definite k_v, so the vacuum satisfies $\hat{\mathbb{S}}|\mathrm{vac}\rangle=|\mathrm{vac}\rangle$. In the elastic limit, in which we neglect particle creation, the S-matrix projected onto single-particle states is then simply the naive S-matrix $\hat{S}$ inferred from the non-relativistic Schrödinger equation,

$$\langle\mathrm{vac}|\hat{a}^{\mathrm{out}}_{k_v,\mathbf{k}'}\hat{a}^{\mathrm{in}}_{k_v,\mathbf{k}}{}^\dagger|\mathrm{vac}\rangle=\langle\mathrm{vac}|\hat{a}^{\mathrm{in}}_{k_v,\mathbf{k}'}\hat{\mathbb{S}}\hat{a}^{\mathrm{in}}_{k_v,\mathbf{k}}{}^\dagger|\mathrm{vac}\rangle=\langle\mathbf{k}'|\hat{S}|\mathbf{k}\rangle(2\pi)\delta(k'_v-k_v)\,, \tag{3.4.15}$$

with normalisation $\langle\mathbf{k}'|\mathbf{k}\rangle=(2\pi)^d\delta^{(d)}(\mathbf{k}-\mathbf{k}')$. In other words, in the absence of particle creation, we may view the entire dynamics from the perspective of the single-particle Hilbert space. More generally, single-particle unitarity holds in the sense

$$\hat{\mathbb{S}}^\dagger\hat{\mathbb{S}}=\hat{1}\Rightarrow\hat{S}^\dagger\hat{S}\leq\hat{1}\,. \tag{3.4.16}$$

To define the null time delay we repeat the derivation of the Wigner-Smith time delay almost verbatim. Formally, given the full S-matrix

$$\hat{\mathbb{S}}=\int_0^\infty\mathrm{d}k_v\sum_{I,J}|k_v,I\rangle\mathbb{W}_{IJ}(k_v)\langle k_v,J|\,, \tag{3.4.17}$$

with $|k_v,I\rangle$ a complete set of $\hat{P}^0_v$ eigenstates with eigenvalue k_v, we define the null time delay

[7] There is no Bogolyubov transformation between in- and out- creation/annihilation operators since the in- and out- vacuum states are identical $|\mathrm{in}\rangle=|\mathrm{out}\rangle=|\mathrm{vac}\rangle$,

operator as

$$\widehat{\Delta\mathbb{V}} = -i\int_0^\infty \mathrm{d}k_v \sum_I \frac{\partial}{\partial\epsilon}\left[\hat{\mathbb{S}}^\dagger|k_v - \frac{\epsilon}{2}, I\rangle\langle k_v + \frac{\epsilon}{2}, I|\hat{\mathbb{S}}\right]\Big|_{\epsilon=0}. \tag{3.4.18}$$

This operator is automatically conjugate to the asymptotic null momenta

$$\left[\hat{P}_v^0, \widehat{\Delta\mathbb{V}}\right] = i\,. \tag{3.4.19}$$

Projected onto the single-particle Hilbert space this is then

$$\widehat{\Delta v} = -\frac{i}{2}\hat{S}^\dagger\frac{\partial\hat{S}}{\partial k_v} + \frac{i}{2}\frac{\partial\hat{S}^\dagger}{\partial k_v}\hat{S}\,, \tag{3.4.20}$$

for which

$$\langle\delta|\,\widehat{\Delta v}\,|\delta\rangle = 2\frac{\partial\delta}{\partial k_v}\,. \tag{3.4.21}$$

Like the S-matrix, the time delay operator should be understood as acting on the asymptotic states. Generic asymptotic states are not eigenstates of the S-matrix, but the expectation value of the Wigner-Smith operator can always be used as a measure of the time delay.

Delay of wavepackets

To understand why the Eisenbud-Wigner-Smith operator (3.4.7) and its null extension (3.4.18) physically corresponds to a time delay, consider a generic incoming one-particle state, which can be written as a superposition of eigenstates of the single-particle S-matrix with a given k_v[8]

$$|g,\mathrm{in}\rangle = \int_0^\infty \frac{\mathrm{d}k_v}{2\pi}\sum_I g_I(k_v)\hat{a}^{\mathrm{in}}_{k_v,I}{}^\dagger|\mathrm{vac}\rangle = \int_0^\infty \frac{\mathrm{d}k_v}{2\pi}\sum_I g_I(k_v)|\delta_I(k_v)\rangle\,. \tag{3.4.22}$$

Here, $\hat{a}^{\mathrm{in}}_{k_v,I}{}^\dagger$ is the creation operator for a single-particle S-matrix eigenstate $|\delta_I(k_v)\rangle$ and the wavepacket is encoded in the profile function $g_I(k_v)$. Consider the same state with an additional null time delay Δv, which can be inferred from translating (3.4.9) by $v \to v - \Delta v$ [241]

$$|g,\mathrm{out}\rangle = e^{i\hat{P}_v\Delta v}|g,\mathrm{in}\rangle = \int_0^\infty \frac{\mathrm{d}k_v}{2\pi}\sum_I g_I(k_v)e^{ik_v\Delta v}\hat{a}^{\mathrm{in}}_{k_v,I}{}^\dagger|\mathrm{vac}\rangle\,. \tag{3.4.23}$$

Given

$$\langle\mathrm{vac}|\hat{a}^{\mathrm{in}}_{k_v',I'}\hat{\mathbb{S}}\hat{a}^{\mathrm{in}}_{k_v,I}{}^\dagger|\mathrm{vac}\rangle = 2\pi\delta_{II'}e^{2i\delta_I(k_v)}\delta(k_v' - k_v)\,, \tag{3.4.24}$$

then

$$\langle g,\mathrm{out}|\hat{\mathbb{S}}|g,\mathrm{in}\rangle = \int_0^\infty \frac{\mathrm{d}k_v}{2\pi}\sum_I |g_I(k_v)|^2 e^{2i\delta_I(k_v)-ik_v\Delta v}\,. \tag{3.4.25}$$

If we assume the incoming state is peaked at some value $\bar{k}_v$ and that further $g_I(k_v)$ is dominated by a single S-matrix eigenstate I, then the amplitude will be maximised when the phase is stationary — this is precisely when

$$\Delta v = \frac{2\partial\delta_I(\bar{k}_v)}{\partial\bar{k}_v}\,, \tag{3.4.26}$$

[8] We shall make this argument for the null time delay, the extension to the usual case being trivial.

and we recover the pp-wave generalisation of the Eisenbud-Wigner time delay. In this sense, the outgoing state $\hat{\mathbb{S}}|g,\text{in}\rangle$ is well-approximated by an asymptotic state close to indistinguishable from the incoming state were it not for the presence of the time delay. The expectation value of the Fock space Wigner-Smith operator is

$$\langle g,\text{in}|\widehat{\Delta\mathbb{V}}|g,\text{in}\rangle = \int_0^\infty \frac{\mathrm{d}k_v}{2\pi} \sum_I |g_I(k_v)|^2 \frac{2\partial\delta_I(k_v)}{\partial k_v}\,, \tag{3.4.27}$$

which is the appropriately weighted average of the Eisenbud-Wigner time delay for each S-matrix $\hat{P}_v$ eigenstate.

More precisely, given a distribution which, near its peak, is well-approximated as a Gaussian of width $\Delta k_v \ll \bar{k}_v$ for a single S-matrix eigenstate

$$g_I(k_v) \approx \frac{1}{\pi^{1/4}\Delta k^{1/2}} e^{-\frac{1}{2\Delta k_v^2}(k_v-\bar{k}_v)^2}\,, \tag{3.4.28}$$

expanding the phase around $\bar{k}_v$, the magnitude of the amplitude may be approximated as

$$|\langle g,\text{out}|\hat{\mathbb{S}}|g,\text{in}\rangle| \sim \frac{1}{\left(1+[\delta_I''(\bar{k}_v)\Delta k_v^2]^2\right)^{1/4}} e^{-\frac{1}{4}\frac{\Delta k_v^2(2\delta_I'(\bar{k}_v)-\Delta v)^2}{\left(1+[\delta_I''(\bar{k}_v)\Delta k_v^2]^2\right)}} \sim e^{-\frac{1}{4}\Delta k_v^2(2\delta_I'(\bar{k}_v)-\Delta v)^2}\,, \tag{3.4.29}$$

assuming $\delta_I''(\bar{k}_v)\Delta k_v^2 \ll 1$. Although, mathematically, the amplitude is peaked at $\Delta v = 2\delta_I'(\bar{k}_v)$, in practice, the amplitude is order unity over a range of width $1/\Delta k$ — this is the resolvability criterion coming from the uncertainty principle. Thus a more precise statement is that the time delay is

$$\Delta v_I = \frac{2\partial\delta_I(\bar{k}_v)}{\partial\bar{k}_v} \pm \mathcal{O}(\Delta k_v^{-1})\,. \tag{3.4.30}$$

Given that $\Delta k_v \ll \bar{k}_v$ in order to trust the saddle point approximation, the strongest statement of causality we can infer from this argument is

$$\Delta v_I \gtrsim -\bar{k}_v^{-1}\,. \tag{3.4.31}$$

3.4.2 Schrödinger picture

Consider the Schrödinger-like equation of the form

$$i\frac{\partial}{\partial u}\psi(u,\mathbf{x}) = -\frac{1}{2k_v}\nabla^2\psi(u,\mathbf{x}) + V(u,\mathbf{x})\psi(u,\mathbf{x})\,, \tag{3.4.32}$$

or more abstractly, in first-quantised language,

$$i\frac{\partial}{\partial u}|\psi\rangle = \hat{H}|\psi\rangle\,, \tag{3.4.33}$$

where $\hat{H} = \hat{H}_0 + \hat{H}_1$ with $\hat{H}_0 = \hat{p}^2/(2k_v)$ and $\hat{H}_1 = V(u,\hat{x})$. Let us assume without loss of generality that interactions vanish for $u < u_i$ and $u > u_f$, with the usual case recovered in the

limit $|u_i|, |u_f| \to \infty$. Then, the S-matrix is the time-evolution operator in interacting picture

$$\hat{S} = \hat{\mathcal{U}}(u_f, u_i) = \mathcal{T} e^{-i\int_{u_i}^{u_f} \mathrm{d}u \hat{H}_1^{\mathrm{int}}(u)} = e^{i\hat{H}_0 u_f} \hat{U}(u_f, u_i) e^{-i\hat{H}_0 u_i} \,, \tag{3.4.34}$$

where $\hat{H}_1^{\mathrm{int}}(u) = e^{i\hat{H}_0 u} \hat{H}_1 e^{-i\hat{H}_0 u}$ is the perturbation in interacting picture and $\hat{U}(u_f, u_i) = \mathcal{T} e^{-i\int_{u_i}^{u_f} \mathrm{d}u \hat{H}(u)}$ is the Schrödinger-picture evolution operator. To determine the time delay we need

$$\begin{aligned} \frac{\partial}{\partial k_v} \hat{S} &= \frac{\partial}{\partial k_v} \mathcal{T} e^{-i\int_{u_i}^{u_f} \mathrm{d}u \hat{H}_1^{\mathrm{int}}(u)} = -i \int_{u_i}^{u_f} \mathrm{d}u' \mathcal{T} \left[\frac{\partial \hat{H}_1^{\mathrm{int}}(u')}{\partial k_v} e^{-i\int_{u_i}^{u_f} \mathrm{d}u \hat{H}_1^{\mathrm{int}}(u)} \right] \\ &= -i \int_{u_i}^{u_f} \mathrm{d}u' \hat{\mathcal{U}}(u_f, u') \frac{\partial \hat{H}_1^{\mathrm{int}}(u')}{\partial k_v} \hat{\mathcal{U}}(u', u_i) \,, \end{aligned} \tag{3.4.35}$$

from which we can infer

$$\begin{aligned} \widehat{\Delta v} &= -i\hat{S}^\dagger \frac{\partial \hat{S}}{\partial k_v} = -\int_{u_i}^{u_f} \mathrm{d}u' \, \hat{\mathcal{U}}(u_f, u_i)^\dagger \hat{\mathcal{U}}(u_f, u') \frac{\partial \hat{H}_1^{\mathrm{int}}(u')}{\partial k_v} \hat{\mathcal{U}}(u', u_i) \\ &= -\int_{u_i}^{u_f} \mathrm{d}u' \, \hat{\mathcal{U}}(u', u_i)^\dagger \frac{\partial \hat{H}_1^{\mathrm{int}}(u')}{\partial k_v} \hat{\mathcal{U}}(u', u_i) \,. \end{aligned} \tag{3.4.36}$$

Now let us convert this into Schrödinger picture. First note since $\hat{H}_0$ commutes with $\frac{\partial \hat{H}_0}{\partial k_v}$, from the definition of the interacting picture Hamiltonian, we have

$$\begin{aligned} \frac{\partial \hat{H}_1^{\mathrm{int}}(u)}{\partial k_v} &= e^{i\hat{H}_0 u} \frac{\partial \hat{H}_1(u)}{\partial k_v} e^{-i\hat{H}_0 u} + iu \left[\frac{\partial \hat{H}_0}{\partial k_v}, \hat{H}_1^{\mathrm{int}}(u) \right] \\ &= e^{i\hat{H}_0 u} \left(\frac{\partial \hat{H}_1(u)}{\partial k_v} + iu \left[\frac{\partial \hat{H}_0}{\partial k_v}, \hat{H}_1(u) \right] \right) e^{-i\hat{H}_0 u} \,. \end{aligned} \tag{3.4.37}$$

Denoting the time delay operator with finite initial and final times as

$$\widehat{\Delta v} = -\int_{u_i}^{u_f} \mathrm{d}u' \, \hat{\mathcal{U}}(u', u_i)^\dagger \frac{\partial \hat{H}_1^{\mathrm{int}}(u')}{\partial k_v} \hat{\mathcal{U}}(u', u_u) \,, \tag{3.4.38}$$

this becomes, in terms of Schrödinger picture quantities,

$$\widehat{\Delta v} = -\int_{u_i}^{u_f} \mathrm{d}u \, e^{i\hat{H}_0 u_i} \hat{U}(u, u_i)^\dagger \left(\frac{\partial \hat{H}_1(u)}{\partial k_v} + iu \left[\frac{\partial \hat{H}_0}{\partial k_v}, \hat{H}_1(u) \right] \right) \hat{U}(u, u_i) e^{-i\hat{H}_0 u_i} \,. \tag{3.4.39}$$

At the level of the expectation value this is

$$\Delta v = \langle \mathrm{in} | \widehat{\Delta v} | \mathrm{in} \rangle = -\int_{u_i}^{u_f} \mathrm{d}u \langle \psi(u) | \hat{Q}(u) | \psi(u) \rangle \,, \tag{3.4.40}$$

where

$$\hat{Q}(u) = \frac{\partial \hat{H}_1(u)}{\partial k_v} + iu \left[\frac{\partial \hat{H}_0}{\partial k_v}, \hat{H}_1(u) \right] \,, \tag{3.4.41}$$

and

$$|\psi(u)\rangle = \hat{U}(u, u_i) e^{-i\hat{H}_0 u_i} |\text{in}\rangle \,, \tag{3.4.42}$$

which we recognise as the solution to the Schrödinger equation with initial condition

$$|\psi(u_i)\rangle = e^{-i\hat{H}_0 u_i} |\text{in}\rangle \,. \tag{3.4.43}$$

The u in the second term in (3.4.41) is specific to the convention that states in the Heisenberg picture are defined as Schrödinger states evaluated at $u = 0$, and interacting picture states are referred to similarly. A different choice $u \to u - u_0$ can be absorbed by a unitary transformation in $|\psi\rangle$, so (3.4.40) is universal when considering the set of all possible states $|\psi\rangle$.

In the situation in which all interactions vanish for $u < 0$, *i.e.* $u_i = 0$, the time delay can be written simply in terms of Heisenberg operators and the Heisenberg state $|\psi(0)\rangle = |\text{in}\rangle$,

$$\Delta v = -\int_0^{u_f} \mathrm{d}u \langle \psi(0) | \hat{Q}_H(u) | \psi(0) \rangle \,, \tag{3.4.44}$$

with

$$\hat{Q}_H(u) = \hat{U}(u, 0)^\dagger \hat{Q}(u) \hat{U}(u, 0) \,. \tag{3.4.45}$$

We may view $\hat{Q}_H(u)$ as the operator version of the classical quantity that determines the rate of increase in the v-time delay per unit u-time

$$Q(u) = -\frac{\mathrm{d}\Delta v(u)}{\mathrm{d}u} \,. \tag{3.4.46}$$

EFT time delay in Schrödinger picture

In the previous section, we defined the total time delay. This is typically a combination of the GR time delay and corrections that come from interactions which, in the EFT, are captured by contributions suppressed by inverse powers of the cutoff. More precisely, the form of the (generally u time-dependent) Hamiltonian for the non-gravitational fields can be split as

$$\hat{H} = \hat{H}_{\text{GR}} + \hat{H}_{\text{EFT}} \,, \tag{3.4.47}$$

for which

$$\lim_{\Lambda \to \infty} \hat{H}_{\text{EFT}} = 0 \,. \tag{3.4.48}$$

In many situations, the cutoff Λ must be necessarily below the Planck scale, so one should also take the limit $M_{\text{Pl}} \to \infty$. Note that, since shockwaves are exact solutions, we can always appropriately scale the metric function in that limit so as to maintain a non-trivial background.

Alternatively, a more pragmatic (and generally applicable) definition of the split is that $\hat{H}_{\text{GR}}$ denotes the terms in the Hamiltonian that come from the two- (and lower) derivative part of the action, while $\hat{H}_{\text{EFT}}$ are those terms that arise from higher-derivative (irrelevant) operators. Given this split, we may give a more local definition of the time delay which encodes the effect of the EFT interactions relative to the GR background. To do this, we choose the map from

Schrödinger to interacting picture by

$$|\psi(u)\rangle = \mathcal{T} e^{-i\int_0^u \mathrm{d}u'\,\hat{H}_{\mathrm{GR}}(u')}|\psi_I(u)\rangle\,. \tag{3.4.49}$$

Then we may define the EFT S-matrix via

$$\hat{S} = \hat{\mathcal{W}}(u_f,u_i) = \mathcal{T} e^{-i\int_{u_i}^{u_f}\mathrm{d}u\hat{H}^{\mathrm{int}}_{\mathrm{EFT}}(u)}\,, \tag{3.4.50}$$

with now

$$\hat{H}^{\mathrm{int}}_{\mathrm{EFT}}(u) = \hat{\mathcal{K}}(u,0)^\dagger \hat{H}_{\mathrm{EFT}}(u)\hat{\mathcal{K}}(u,0)\,, \tag{3.4.51}$$

and

$$\hat{\mathcal{K}}(u,u_i) = \mathcal{T} e^{-i\int_{u_i}^{u}\mathrm{d}u'\,\hat{H}_{\mathrm{GR}}(u')}\,. \tag{3.4.52}$$

The EFT time delay is then

$$\widehat{\Delta v}_{\mathrm{EFT}} = -\int_{u_i}^{u_f}\mathrm{d}u'\,\hat{\mathcal{K}}(u',u_i)^\dagger \frac{\partial \hat{H}^{\mathrm{int}}_{\mathrm{EFT}}(u')}{\partial k_v}\hat{\mathcal{K}}(u',u_u)\,. \tag{3.4.53}$$

Using

$$\begin{aligned}\frac{\partial}{\partial k_v}\hat{\mathcal{K}}(u,0) &= -i\int_0^u \mathrm{d}u'\,\hat{\mathcal{K}}(u,u')\frac{\partial \hat{H}_{\mathrm{GR}}(u')}{\partial k_v}\hat{\mathcal{K}}(u',0)\\ &= -i\int_0^u \mathrm{d}u'\hat{\mathcal{K}}(u,u')\frac{\partial \hat{H}_{\mathrm{GR}}(u')}{\partial k_v}\hat{\mathcal{K}}(u,u')^\dagger\hat{\mathcal{K}}(u,0)\,,\end{aligned} \tag{3.4.54}$$

this then becomes

$$\Delta v_{\mathrm{EFT}} = -\int_{u_i}^{u_f}\mathrm{d}u\langle\psi(u)|\hat{Q}_{\mathrm{EFT}}(u)|\psi(u)\rangle\,, \tag{3.4.55}$$

where now

$$\hat{Q}_{\mathrm{EFT}}(u) = \frac{\partial \hat{H}_{\mathrm{EFT}}(u)}{\partial k_v} + i\int_0^u \mathrm{d}u'\left[\hat{\mathcal{K}}(u,u')\frac{\partial \hat{H}_{\mathrm{GR}}(u')}{\partial k_v}\hat{\mathcal{K}}(u,u')^\dagger, \hat{H}_{\mathrm{EFT}}(u)\right]\,, \tag{3.4.56}$$

and

$$|\psi(u)\rangle = \hat{\mathcal{K}}(u,0)\hat{\mathcal{W}}(u,u_i)|\mathrm{in}\rangle\,. \tag{3.4.57}$$

For most purposes, it is sufficient to determine the EFT time delay to first order in $\hat{H}_{\mathrm{EFT}}$ and so we just directly compute

$$\Delta v_{\mathrm{EFT}} \approx -\frac{\partial}{\partial k_v}\int_{u_i}^{u_f}\mathrm{d}u\,\langle\mathrm{in}|\hat{H}^{\mathrm{int}}_{\mathrm{EFT}}(u)|\mathrm{in}\rangle\,. \tag{3.4.58}$$

3.4.3 Uncertainty of time delay

A generic incoming state is not an eigenstate of the S-matrix or time delay operator. Indeed, normalisable scattering states are always wavepackets. Since naïvely $[\Delta\hat{\mathbb{V}}, \hat{P}^0_v] = i$, there is an inevitable uncertainty relation between the time delay and the asymptotic null energy. We may

define this uncertainty in the usual way

$$\Delta_{\rm un} v^2 = \langle g, {\rm in}|\widehat{\Delta\mathbb{V}}^{\,2}|g, {\rm in}\rangle - \Big(\langle g, {\rm in}|\widehat{\Delta\mathbb{V}}|g, {\rm in}\rangle\Big)^2 \,. \tag{3.4.59}$$

Evaluating the expectation value on the single-particle wavepackets (3.4.22) gives

$$\langle g, {\rm in}|\widehat{\Delta\mathbb{V}}^{\,2}|g, {\rm in}\rangle = \int_0^\infty \frac{{\rm d}k_v}{2\pi} \sum_I \left(|g_I(k_v)|^2 \left[\frac{2\partial\delta_I(k_v)}{\partial k_v}\right]^2 + \left|\frac{\partial g_I(k_v)}{\partial k_v}\right|^2 \right) , \tag{3.4.60}$$

and so

$$\Delta_{\rm un} v^2 = \left\langle\!\!\left\langle \left(\frac{2\partial\delta_I}{\partial k_v}\right)^2 \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \frac{2\partial\delta_I}{\partial k_v} \right\rangle\!\!\right\rangle^2 + \int_0^\infty \frac{{\rm d}k_v}{2\pi} \sum_I \left|\frac{\partial g_I(k_v)}{\partial k_v}\right|^2 , \tag{3.4.61}$$

where $\langle\!\langle X_I \rangle\!\rangle = \int_0^\infty \frac{{\rm d}k_v}{2\pi} \sum_I |g_I(k_v)|^2 X_I(k_v)$. The last term is the usual uncertainty due to the fact that we are considering wavepackets in k_v, while the first two terms are contributions to the uncertainty which come directly from the scattering and vanish in the limit of no scattering. We thus have the bound

$$\Delta_{\rm un} v^2 \geq \int_0^\infty \frac{{\rm d}k_v}{2\pi} \sum_I \left|\frac{\partial g_I(k_v)}{\partial k_v}\right|^2 . \tag{3.4.62}$$

Although $k_v \geq 0$, we may view $f(k_v)$ as a wavefunction that happens to vanish for $k_v < 0$, and so by the usual reasoning we have

$$\Delta_{\rm un} v \, \Delta k_v \geq 1/2 \,, \tag{3.4.63}$$

with

$$(\Delta k_v)^2 = \langle\!\langle k_v^2 \rangle\!\rangle - \langle\!\langle k_v \rangle\!\rangle^2 \,. \tag{3.4.64}$$

This was explicit in the wavepackets considered above. The contribution to the uncertainty from scattering at fixed k_v can be expressed in terms of the single-particle Hilbert space time delay as

$$\delta v = \sqrt{\langle {\rm in}|\, \widehat{\Delta v}^{\,2} \,|{\rm in}\rangle - \langle {\rm in}|\, \widehat{\Delta v} \,|{\rm in}\rangle^2} \,, \tag{3.4.65}$$

with

$$\langle {\rm in}|\, \widehat{\Delta v}^{\,2} \,|{\rm in}\rangle = \langle {\rm in}|\, \frac{\partial \hat{S}^\dagger}{\partial k_v} \frac{\partial \hat{S}}{\partial k_v} \,|{\rm in}\rangle \ . \tag{3.4.66}$$

We shall make crucial use of this fact later.

3.4.4 Phase space formulation

In practice, the exact formula for the scattering time delay (3.4.40) is difficult to determine other than by means of an approximation or by numerical evolution. It is also relatively straightforward to infer the time delay in perturbation theory (Born series), and we shall do so in Section 5.5. Our main tool will however be the eikonal and semi-classical approximations. It proves useful to rewrite (3.4.40) in terms of Wigner functions since they are most closely tied to a (nearly) classical interpretation of the scattering. Denoting the density operator as

$$\hat{\rho}(u) = |\psi(u)\rangle\langle\psi(u)| \,, \tag{3.4.67}$$

then the total time delay is

$$\Delta v = -\int_{u_i}^{u_f} \mathrm{d}u \, \mathrm{Tr}[\hat{\rho}(u)\hat{Q}(u)] \, . \tag{3.4.68}$$

Denoting the Wigner phase-space function by

$$W(\mathbf{x}, \mathbf{p}, u) = \int \mathrm{d}^d\mathbf{a} \, e^{-i\mathbf{a}\cdot\mathbf{p}} \langle \mathbf{x} + \mathbf{a}/2 | \hat{\rho}(u) | \mathbf{x} - \mathbf{a}/2 \rangle \, , \tag{3.4.69}$$

and the associated phase space representation of $\hat{Q}(u)$ by

$$Q(\mathbf{x}, \mathbf{p}, u) = \int \mathrm{d}^d\mathbf{a} \, e^{-i\mathbf{a}\cdot\mathbf{p}} \langle \mathbf{x} + \mathbf{a}/2 | \hat{Q}(u) | \mathbf{x} - \mathbf{a}/2 \rangle \, , \tag{3.4.70}$$

then the asymptotic/total time delay can be rewritten as

$$\Delta v = -\int_{u_I}^{u_f} \mathrm{d}u \int \mathrm{d}^d\mathbf{x} \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d} W(\mathbf{x}, \mathbf{p}, u) Q(\mathbf{x}, \mathbf{p}, u) \, . \tag{3.4.71}$$

A similar expression holds for the EFT time delay

$$\Delta v_{\mathrm{EFT}} = -\int_{u_i}^{u_f} \mathrm{d}u \int \mathrm{d}^d\mathbf{x} \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d} W(\mathbf{x}, \mathbf{p}, u) Q_{\mathrm{EFT}}(\mathbf{x}, \mathbf{p}, u) \, . \tag{3.4.72}$$

In both cases, the operator $\hat{Q}$ as a function of its constituent operators can be replaced with the phase-space function as a function of the phase-space versions of the same, with operator products replaced by the Moyal product, denoted by $\star$. Similarly, the equation for the density operator in Schrödinger picture

$$i\frac{\partial \hat{\rho}(u)}{\partial u} = -\left[\hat{\rho}(u), \hat{H}\right] \, , \tag{3.4.73}$$

is replaced by the quantum Liouville equation

$$i\frac{\partial W(\mathbf{x}, \mathbf{p}, u)}{\partial u} = -\left[W(\mathbf{x}, \mathbf{p}, u) \star H(\mathbf{x}, \mathbf{p}, u) - H(\mathbf{x}, \mathbf{p}, u) \star W(\mathbf{x}, \mathbf{p}, u)\right] \, . \tag{3.4.74}$$

These expressions may appear unwieldy, but they prove to be the most useful for deriving the semi-classical approximation.

Semi-classical approximation

In the semi-classical approximation, we intuitively expect the dynamics to be dominated by that of the classical particle trajectories. This is most straightforwardly derived by reintroducing $\hbar$ in the conventional places within the Schrödinger equation, and taking the limit $\hbar \to 0$. In this limit, the Moyal product reduces to the ordinary product

$$\lim_{\hbar \to 0} A(\mathbf{x}, \mathbf{p}, u) \star B(\mathbf{x}, \mathbf{p}, u) \to A(\mathbf{x}, \mathbf{p}, u) B(\mathbf{x}, \mathbf{p}, u) \, , \tag{3.4.75}$$

and the expression for $Q = -\mathrm{d}\Delta v/\mathrm{d}u$ is given by its classical version with commutators replaced by Poisson brackets

$$\begin{aligned} Q(\mathbf{x},\mathbf{p},u) &= \frac{\partial H_1(\mathbf{x},\mathbf{p},u)}{\partial k_v} - u\left\{\frac{\partial H_0(\mathbf{x},\mathbf{p},u)}{\partial k_v}, H_1(\mathbf{x},\mathbf{p},u)\right\} \\ &= \frac{\partial}{\partial k_v}V(u,\mathbf{x}) - u\frac{\mathbf{p}}{k_v^2}\cdot\boldsymbol{\nabla}V(u,\mathbf{x})\,. \end{aligned} \tag{3.4.76}$$

Finally, at leading order in the semi-classical approximation, the dynamical equation for the Wigner function reduces to the Liouville equation

$$\frac{\partial}{\partial u}W(\mathbf{x},\mathbf{p},u) = -\{W(\mathbf{x},\mathbf{p},u), H(\mathbf{x},\mathbf{p},u)\}\,. \tag{3.4.77}$$

The solution of this linear equation is straightforward

$$W(\mathbf{x},\mathbf{p},u) = \int \mathrm{d}^d\mathbf{x_0}\int\frac{\mathrm{d}^d\mathbf{p_0}}{(2\pi)^d}W_0(\mathbf{x}_0,\mathbf{p}_0)\delta^{(d)}\big(\mathbf{x}-\mathbf{x}_{\rm cl}(\mathbf{x}_0,\mathbf{p}_0,u)\big)(2\pi)^d\delta^{(d)}\big(\mathbf{p}-\mathbf{p}_{\rm cl}(\mathbf{x}_0,\mathbf{p}_0,u)\big)\,, \tag{3.4.78}$$

where $W_0(\mathbf{x}_0,\mathbf{p}_0)$ is the Wigner function at time $u=0$, and $\big(\mathbf{x}_{\rm cl}(\mathbf{x_0},\mathbf{p_0},\mathbf{u})\,,\mathbf{p}_{\rm cl}(\mathbf{x_0},\mathbf{p_0},\mathbf{u})\big)$ denotes the solution of the classical equations of motion in phase space, given the initial data $(\mathbf{x}_{\rm cl},\mathbf{p}_{\rm cl}) = (\mathbf{x}_0,\mathbf{p}_0)$ at $u=0$.

Substituting into the expression for the time delay, we finally obtain at leading order in the semi-classical approximation,

$$(\Delta v)_{\rm semi-classical} = -\int \mathrm{d}^d\mathbf{x}_0\int\frac{\mathrm{d}^d\mathbf{p}_0}{(2\pi)^d}W_0(\mathbf{x}_0,\mathbf{p}_0)\int_{u_i}^{u_f}\mathrm{d}u\,Q(\mathbf{x}_{\rm cl}(\mathbf{x}_0,\mathbf{p}_0,u),\mathbf{p}_{\rm cl}(\mathbf{x}_0,\mathbf{p}_0,u),u)\,. \tag{3.4.79}$$

The classical approximation would further neglect the effect of uncertainty and quantum diffusion:

$$(\Delta v)_{\rm classical} = -\int_{u_i}^{u_f}\mathrm{d}u\,Q(\mathbf{x}_{\rm cl}(\mathbf{x}_0,\mathbf{p}_0,u),\mathbf{p}_{\rm cl}(\mathbf{x}_0,\mathbf{p}_0,u),u)\,. \tag{3.4.80}$$

Stated in words: In order to determine the semi-classical scattering time delay, we first infer the classical time delay (3.4.80) associated with an arbitrary classical initial condition $(\mathbf{x}_0,\mathbf{p}_0)$ in phase space. We then perform the average of those classical initial conditions according to whatever the initial wavefunction and hence initial Wigner distribution is. The latter step automatically incorporates the effects of the uncertainty principle and quantum diffusion into the calculation of the time delay. It is clear that the classical approximation can only be trusted when most of the time delay is built up well before the diffusion time, for which we can no longer neglect the quantum induced spread.

Quantum diffusion

A crucial fact which is embedded in the above discussion so far, is that a generic initial wavefunction ψ_0 can never be perfectly localised both in position and momenta by virtue of the uncertainty principle $\Delta\mathbf{x}\cdot\Delta\mathbf{k}\geq 1/2$. Thus, generic initial wavepackets have some spread in transverse space. Viewed as a field in a low-energy EFT, its spatial extent $\sigma\sim|\Delta\mathbf{x}|$ is addition-

ally bounded below by the EFT cut-off as $\sigma \gg \Lambda^{-1}$ (see appendix 5.C). Even in the absence of a potential, this initial uncertainty grows with time due to quantum diffusion and limits the amount of time delay that can be built up. Consider a spherically symmetric Gaussian initial profile

$$\psi_0(\mathbf{x}) = \frac{1}{(2\pi\sigma^2)^{d/4}} \exp\left[-\frac{|\mathbf{x}-\mathbf{b}|^2}{4\sigma^2}\right], \tag{3.4.81}$$

centred at $\mathbf{b}$ with width σ much smaller than the distance to the source(s) $\sigma \ll b$. The associated Wigner function at the initial time is

$$W_0(\mathbf{x_0}, \mathbf{p_0}) = 2^d \exp\left[-\frac{|\mathbf{x_0}-\mathbf{b}|^2}{2\sigma^2}\right] \exp\left[-\frac{\mathbf{p_0}^2}{2\Delta p^2}\right], \tag{3.4.82}$$

with

$$\sigma\Delta p = \frac{1}{2}. \tag{3.4.83}$$

After a time u under free evolution, the Wigner function will be peaked around the straight-line trajectory

$$W(\mathbf{x}, \mathbf{p}, u) = 2^d \exp\left[-\frac{|\mathbf{x}-\mathbf{p}u/k_v-\mathbf{b}|^2}{2\sigma^2}\right] \exp\left[-\frac{\mathbf{p}^2}{2\Delta p^2}\right], \tag{3.4.84}$$

and the wavepacket will diffuse to one with an effective width of $\sigma_{\text{eff}}^2(u) = \sigma^2 + (u/2k_v\sigma)^2$.

This diffusion inevitably provides a cutoff on the scattering process and the build-up of the time delay/advance. For example, if we consider the scattering of two localised objects of impact parameter b, we no longer expect a significant contribution to the time delay once the diffusion of the wavepacket is comparable to the impact parameter. This occurs at the "diffusion time" for which

$$\sigma_{\text{eff}}(u_{\text{diff}}) \sim b \quad \Longrightarrow \quad u_{\text{diff}} \sim 2k_v\sigma\sqrt{b^2-\sigma^2}. \tag{3.4.85}$$

The maximum diffusion time is then seen to be

$$u_{\text{diff}} \sim k_v b^2. \tag{3.4.86}$$

However, if the initial wavepacket is well-localised, it may be even smaller $u_{\text{diff}} \sim k_v\sigma b$. Either way, the wavefunction $\psi(u, \mathbf{x})$ can no longer be considered localised after a finite time u_{diff}. We shall confirm this explicitly by more detailed calculations below.

3.4.5 Eikonal approximation

Having written everything in terms of the Schrödinger picture makes it easier to compare with the standard eikonal approximation for the Schrödinger equation. Let us first do this for the S-matrix describing scattering from states of momenta $\mathbf{k}$ to $\mathbf{k}'$. In order to maintain symmetry between the incoming and outgoing state, we define the average momenta $\mathbf{n} = (\mathbf{k}'+\mathbf{k})/2$. The eikonal approximation is derived by first factorising out the free evolution at the average momenta

$$\psi = e^{i\mathbf{n}\cdot\mathbf{x}} e^{-i|\mathbf{n}|^2\mathbf{u}/\mathbf{2k_v}} \chi, \tag{3.4.87}$$

so that the Schrödinger equation becomes

$$i\frac{\partial}{\partial u}\chi(\mathbf{x},u) = -i\frac{\mathbf{n}}{k_v}\cdot\boldsymbol{\nabla}\chi(\mathbf{x},u) - \frac{1}{2k_v}\nabla_d^2\chi(\mathbf{x},u) + V(u,\mathbf{x})\chi(\mathbf{x},u)\,. \tag{3.4.88}$$

At leading order in the approximation, we neglect the Laplacian term, meaning that one of the conditions for the approximation to be valid is that the momentum transfer is small, which is achieved for small scattering angles. Then, changing variables to $\mathbf{x} = \mathbf{y} + \mathbf{n}u/k_v$, the resulting equation

$$i\frac{\partial}{\partial u}\chi(\mathbf{y},u) \approx V(u,\mathbf{y}+\mathbf{n}u/k_v)\chi(\mathbf{y},u)\,, \tag{3.4.89}$$

can be exactly integrated

$$\chi(\mathbf{y},u_f) \approx e^{-i\int_{u_i}^{u_f}\mathrm{d}u\,V(u,\mathbf{y}+\mathbf{n}u/k_v)}\chi(\mathbf{y},u_i)\,. \tag{3.4.90}$$

The S-matrix between initial state $\chi_i = e^{i(\mathbf{k}-\mathbf{n})\cdot\mathbf{y}}$ and final state $\chi_f = e^{i(\mathbf{k}'-\mathbf{n})\cdot\mathbf{y}}$ is then

$$\langle\mathbf{k}'|\hat{S}|\mathbf{k}\rangle = \int \mathrm{d}^d\mathbf{y}e^{-i\mathbf{q}\cdot\mathbf{y}}e^{-i\int_{u_i}^{u_f}\mathrm{d}u\,V(u,\mathbf{y}+\mathbf{n}u/k_v)}\,, \tag{3.4.91}$$

with momentum transfer $\mathbf{q} = \mathbf{k}' - \mathbf{k}$. In the familiar case, for which V is independent of u with $|u_{i,f}| \to \infty$, we may further define $\mathbf{y} = \mathbf{b} + \hat{\mathbf{n}}z$, where $\mathbf{n}\cdot\mathbf{b} = 0$. Then, by shifting $u \to u - |\mathbf{n}|z$, the S-matrix becomes

$$\langle\mathbf{k}'|\hat{S}|\mathbf{k}\rangle = 2\pi\delta(\mathbf{q}\cdot\hat{\mathbf{n}})\int \mathrm{d}^{d-1}\mathbf{b}\,e^{-i\mathbf{q}\cdot\mathbf{b}}e^{-i\int_{-\infty}^{\infty}\mathrm{d}u\,V(\mathbf{b}+\mathbf{n}u/k_v)}\,, \tag{3.4.92}$$

which we recognise to be the familiar eikonal approximation for scattering off a time-independent potential.

To determine the scattering time delay for an incoming state which is peaked at incoming momentum $\mathbf{p}_0$, it is convenient to further approximate (3.4.91) by replacing $\mathbf{n}$ by $\mathbf{p_0}$

$$\langle\mathbf{k}'|\hat{S}|\mathbf{k}\rangle \approx \int \mathrm{d}^d\mathbf{y}e^{-i\mathbf{q}\cdot\mathbf{y}}e^{-i\int_{u_i}^{u_f}\mathrm{d}u\,V(u,\mathbf{y}+\mathbf{p_0}u/k_v)}\,, \tag{3.4.93}$$

so that

$$\langle\mathbf{k}'|\hat{S}|\psi\rangle \approx \int \mathrm{d}^d\mathbf{y}e^{-i\mathbf{k}'\cdot\mathbf{y}}\psi(y)e^{-i\int_{u_i}^{u_f}\mathrm{d}u\,V(u,\mathbf{y}+\mathbf{p_0}u/k_v)}\,. \tag{3.4.94}$$

This allows the integrals to be performed, giving

$$\begin{aligned}\Delta v &= -i\langle\psi|\hat{S}^\dagger\frac{\partial}{\partial k_v}\hat{S}|\psi\rangle \\ &= -i\int \mathrm{d}^d\mathbf{k}'\langle\psi|\hat{S}^\dagger|\mathbf{k}'\rangle\frac{\partial}{\partial k_v}\langle\mathbf{k}'|\hat{S}|\psi\rangle \\ &\approx -\int_{u_i}^{u_f}\mathrm{d}u\int \mathrm{d}^d\mathbf{y}|\psi(y)|^2\left[\frac{\partial}{\partial k_v}V\left(u,\mathbf{y}+\frac{\mathbf{p_0}u}{k_v}\right) - u\frac{\mathbf{p_0}}{k_v^2}\cdot\boldsymbol{\nabla}V\left(u,\mathbf{y}+\frac{\mathbf{p_0}u}{k_v}\right)\right]\,.\end{aligned} \tag{3.4.95}$$

Given the Wigner function will be similarly peaked at p_0, we can rewrite the eikonal scattering

time delay as

$$(\Delta v)_{\text{eikonal}} \approx -\int_{u_i}^{u_f} \mathrm{d}u \int \mathrm{d}^d\mathbf{y} \int \mathrm{d}^d\mathbf{p_0}\, W_0(\mathbf{y},\mathbf{p_0}) \left[\frac{\partial}{\partial k_v} V\left(u, \mathbf{y} + \frac{\mathbf{p_0}u}{k_v}\right) - u\frac{\mathbf{p_0}}{k_v^2}\cdot\boldsymbol{\nabla} V\left(u, \mathbf{y} + \frac{\mathbf{p_0}u}{k_v}\right)\right]$$
$$\approx -\frac{\partial}{\partial k_v}\int_{u_i}^{u_f} \mathrm{d}u \int \mathrm{d}^d\mathbf{y} \int \mathrm{d}^d\mathbf{p_0}\, W_0(\mathbf{y},\mathbf{p_0}) V\left(u, \mathbf{y} + \frac{\mathbf{p_0}u}{k_v}\right). \tag{3.4.96}$$

Identifying $\mathbf{y} = \mathbf{x_0}$ as the classical position at $u = 0$, we recognise the leading-order eikonal approximation to simply be the semi-classical approximation evaluated such that the trajectories of the particles are taken to be straight lines, which would be expected to be valid for small angle scattering. As such, it neglects certain higher-order effects of scattering, which cause the mean trajectory to depart from a straight line.

If we further neglect the effect of quantum uncertainty/diffusion then this reduces to the "classical" eikonal form

$$(\Delta v)_{\text{eikonal}} \approx -\frac{\partial}{\partial k_v}\int_{u_i}^{u_f} \mathrm{d}u\, V\left(u, \mathbf{x_0} + \frac{\mathbf{p_0}u}{k_v}\right). \tag{3.4.97}$$

In realistic situations, the potential vanishes at large distances, so this integral will be maximised for $\mathbf{p_0}$ as small as possible. At the impact parameter $\mathbf{x_0} = \mathbf{b}$, the eikonal time delay reduces to

$$(\Delta v)_{\text{eikonal}} \approx -\frac{\partial}{\partial k_v}\int_{u_i}^{u_f} \mathrm{d}u\, V(u,\mathbf{b}), \tag{3.4.98}$$

which is the expression considered in [174] — we derive this expression directly in the eikonal approximation for scattering off shockwaves in section 5.A. This result may be obtained more directly by neglecting scattering in the transverse directions from the outset. This amounts to ignoring the Laplacian-term in (3.4.10) so that the particle deviates from its initial configuration only by a phase. Setting up our initial conditions at time $u = 0$ with initial particle displacement $\mathbf{x} = \mathbf{b} = b\hat{\mathbf{b}}$, then the approximate solution at a later time is

$$\psi(u,\mathbf{b}) = \psi_0 \exp\left[2i\delta(u,\mathbf{b})\right], \quad 2\delta(u,\mathbf{x}) = -\int_0^u \mathrm{d}u' V(u',\mathbf{x}), \tag{3.4.99}$$

from which the time delay is easily obtained

$$(\Delta v)_{\text{eikonal}} = 2\frac{\partial\delta(u,\mathbf{b})}{\partial k_v} = -\frac{\partial}{\partial k_v}\int_0^u \mathrm{d}u' V(u',\mathbf{b}). \tag{3.4.100}$$

It is clear from our previous derivations that (3.4.100) can only be trusted when the integral is taken over a range of u for which the effects of diffusion and scattering are neglected.

Chapter 4

Causality on Black Holes

We have established that causality ought to put non-trivial constraints on the EFT of gravity. In particular, we argued that IR causality is the appropriate notion of causality for constraining perturbative EFT corrections to Einstein gravity, as it is amenable to concrete calculations and seems in agreement with positivity bounds in a scalar toy model.

We want to put this to use for the leading-order EFT in $D \geq 5$, essentially EGB gravity. A tried and tested method of how we study objects in physics is by smashing them into each other. In section 4.1, we will therefore study the scattering of gravitational perturbations (waves) on Schwarzschild black holes, arguably some of the simplest objects in gravity. IR causality is determined by the EFT correction to the GR time delay, which we compute semiclassically in section 4.2. We take some care to establish the regime of validity of this theory on general grounds and using specific UV completions in section 4.3, and find that IR causality is satisfied as long as the EFT is weakly coupled.

Details on the spherical decomposition and the decoupling into master equations are given in 4.A. In appendix 4.B we present a derivation for the eikonal approximation of the time delay from the semiclassical approximation. We elaborate on reasons for the particular choice of additional higher-dimension operators in appendix 4.C.

4.1 Black Hole Scattering in the EFT

Let us briefly review spherically symmetric and static solutions in EGB gravity, given by the action (1.3.8).

4.1.1 Black holes in D-dimensional EFT

The vacuum Einstein-Gauss-Bonnet equations are given in (1.3.9).

Since the GB tensor $B_{\mu\nu}$ is already suppressed by Λ^{-2} compared to the Einstein tensor and we consider a Ricci-flat vacuum solution at leading order, $R_{\mu\nu} = \mathcal{O}(\Lambda^{-2})$, we can ignore any terms

in the equations of motion containing a Ricci tensor/scalar without compromising the first-order result. The static and spherically symmetric Ansatz (A.1.1) is a solution to the background equation of motion for

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{D-2}^2, \tag{4.1.1}$$

and the metric function is

$$f(r) = 1 - \left(\frac{r_g}{r}\right)^{D-3} + 2(D-3)(D-4)\frac{c_{\mathrm{GB}}}{\Lambda^2 r_g^2}\left(\frac{r_g}{r}\right)^{2D-4} + \mathcal{O}(\Lambda^{-2})\,. \tag{4.1.2}$$

The resulting solution is a static Schwarzschild-like BH, with horizon r_H set by $f(r_H) = 0$, which differs ever so slightly from the GR Schwarzschild radius r_g.

4.1.2 Metric perturbations

We are interested in the dynamics of linearised metric perturbations, denoted by $h_{\mu\nu}$, to this curved background. As we will see, due to the EFT corrections, these modes can follow geodesics that deviate slightly from being null, becoming either time-like or space-like, and thus lead to concerns about causality.

Given the spherical symmetry of the background spacetime, it is useful to parameterise the components of $h_{\mu\nu}$ according to their transformation properties under $SO(D-1)$. In appendix 4.A, we follow the procedure outlined in [242–244] to reduce the $D(D+1)/2$ components of the symmetric $h_{\mu\nu}$ tensor down to $D(D-3)/2$ scalars called "master variables". These master variables entirely encode all the propagating degrees of freedom of the massless spin-2 field associated with metric perturbations. They fall into three categories — scalar, vector or tensor — depending on the $SO(D-1)$-transformation properties of the $h_{\mu\nu}$-components from which they derive. The vector modes are the higher-dimensional analogue of the Regge-Wheeler axial mode in 4-dimensional GR, and the scalar modes correspond to the Zerilli polar mode in 4-dimensional GR [245, 246].

The evolution equations for the master variables, known as the "master equations", are derived from the first-order perturbation to the Einstein-Gauss-Bonnet equations (1.3.9). Since the GB operator famously produces second-derivative equations of motion, they can be cast in the form of a Schrödinger-like wave equation with a potential V sourced by the background curvature:

$$-\frac{\partial^2}{\partial t^2}\Phi_M + f\frac{\partial}{\partial r}\left(f\frac{\partial}{\partial r}\Phi_M\right) - V_M\Phi_M = 0, \tag{4.1.3}$$

where $M \in \{S, V, T\}$ labels the modes. As implied, the potential experienced by the master variables depends only on whether they are classed as scalars (S), vectors (V) or tensors (T). Each partial wave evolves independently because of the spherical symmetry of the background. Mode indices are suppressed, but the potentials V_M carry the dependence on the partial wave number ℓ, as in (4.1.4) below. The exact potentials for metric perturbations to static BHs are calculated for full (non-perturbative) Einstein-Gauss-Bonnet gravity in [247, 248] and for general Lovelock theories in [249, 250]. We are only interested in their leading-order behaviour in the

EFT expansion, for reasons already discussed above subsection 4.1.1. All three potentials are provided explicitly to leading order in appendix 4.A. The dynamics of the tensor modes specifically will become of particular interest in subsection 4.3.3 so we reproduce its potential here, for reference:

$$\begin{aligned}\frac{V_T}{f} =& \frac{1}{r^2}\Bigg[k_T^2\left(1+8\frac{c_{\rm GB}}{\Lambda^2 r_g^2}(D-1)\left(\frac{r_g}{r}\right)^{D-1}\right)\\ &+\frac{D(D-6)+16}{4}\left(1-32\frac{c_{\rm GB}}{\Lambda^2 r_g^2}\frac{(D-1)(D-6)}{D(D-6)+16}\left(\frac{r_g}{r}\right)^{D-1}\right)\\ &+\frac{(D-2)^2}{4}\left(\frac{r_g}{r}\right)^{D-3}\left(1-2\frac{c_{\rm GB}}{\Lambda^2 r_g^2}\frac{(D-4)\left[3D(D-3)(D-6)-32\right]}{(D-2)^2}\left(\frac{r_g}{r}\right)^{D-1}\right)\Bigg].\end{aligned} \tag{4.1.4}$$

Here, $-k_T^2$ is the eigenvalue of the Laplace-Beltrami operator on the $(D-2)$-sphere acting on a tensor-type spherical harmonic. It is related to the more familiar integer mode numbers ℓ by $k_T^2 = \ell(\ell+D-3)-2$, where $\ell = 1, 2, \ldots$. The expression for V_T is by far the simplest of the three potentials.

4.1.3 Apparent local superluminality

Before proceeding to the calculation of the scattering time delay, we can already get a local indication of the supposed acausality by inspecting the expression for the tensor potential (4.1.4). Specifically, it is apparent that the angular speed v_Ω of the GWs is modified by the GB contribution — notably, the radial speed is unaffected to leading order in the EFT. By identifying v_Ω^2 with the coefficient of k_T^2/r^2, we immediately see that

$$v_\Omega^2 = 1+8\frac{c_{\rm GB}}{\Lambda^2 r_g^2}(D-1)\left(\frac{r_g}{r}\right)^{D-1}. \tag{4.1.5}$$

If $c_{\rm GB}$ were a positive number, the angular speed of the GWs would be locally superluminal ($v_\Omega > 1$). This will contribute negatively to the total time delay.

At this point, one might be tempted to forbid $c_{\rm GB} > 0$ and content oneself that any causal UV complete theory should have $c_{\rm GB} \leq 0$ as part of its low-energy description. Unfortunately, examination of the vector potential (4.A.5) suggests that those modes would propagate superluminally if $c_{\rm GB} < 0$. Thus, it seems we should be forced to conclude that $c_{\rm GB} = 0$ is the only viable option and that the GB term is not a good causal operator.

This argument is altogether too quick for two reasons. First, local superluminalities do not by themselves necessarily indicate causality violation. What is in tension with causality is the possibility of creating closed time-like curves which requires building up superluminalities over some trajectories, which is why we will compute the scattering time delay below. Within the time delay, local superluminalities at one point could be compensated by subluminalities at other points. The second reason why local superluminalities cannot by themselves indicate whether causality is necessarily violated is because we have yet to impose the full requirements for the validity of the EFT which, as we shall see, will mitigate the level of superluminality one can hope for.

4.2 Time Delay

We can now use the equations of motion governing the perturbations on top of the black hole backgrounds to compute scattering time delays.

4.2.1 Scattering phase shifts

To calculate the scattering time delay (3.1.11), it is convenient to first calculate the phase shift δ_ℓ of a partial wave scattering to/from asymptotic infinity in the BH spacetime. Since the master equations (4.1.3) are reminiscent of the Schrödinger equation, we may utilise the semiclassical JWKB approximation to determine the phase shift. This will automatically reproduce the classical time delay and the eikonal (shock wave) time delay [146] in appropriate limits, as discussed in the appendices of [215].

It is usual to consider the BH perturbation equations in the tortoise coordinate $\hat{r}$ defined via

$$\mathrm{d}r = f(r)\mathrm{d}\hat{r}\,, \tag{4.2.1}$$

however, in order to compare with standard scattering problems (without a horizon) we rather work with Langer coordinates $r = e^\rho$ [251]. In the case of a regular source such as a star, this has the effect of mapping the origin of coordinates, $r = 0$, to $\rho = -\infty$, so that a JWKB approximation does well near $\rho = -\infty$ as well as $\rho = \infty$. This is well known to improve the form of the JWKB approximation at small ℓ. Despite the presence of a horizon, this proves to be useful here. We will, however, only be interested in the solution outside the horizon, $r > r_H$. Making a wave Ansatz for the time-dependence of the master variable $\Phi = e^{-i\omega t}\phi(\rho)$, and defining $\phi = (fe^{-\rho})^{-1/2}\chi$, the master equation (4.1.3) then takes the form

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}\rho^2} + W\left(\rho\right)\chi = 0\,, \tag{4.2.2}$$

where

$$W(\rho) = \left(\omega^2 - V(\rho)\right)\frac{e^{2\rho}}{f^2} - \frac{1}{2}\Omega''(\rho) - \frac{1}{4}(1-\Omega'(\rho))^2\,, \tag{4.2.3}$$

and $\Omega = \ln(f)$. The field χ should carry both the index M to label its potential and, additionally, a mode number ℓ. Each partial wave ℓ for each perturbation type M evolves independently according to (4.2.2).

A property of the potentials V_M (as given in appendix 4.A) is that all three agree in the limit $r_g \to 0$. This allows us to define a universal impact parameter b via

$$b^2\omega^2 = \lim_{r_g\to 0}\omega^2 e^{2\rho} - W(\rho)\,, \tag{4.2.4}$$

which results in

$$b = \omega^{-1}\left(\ell + \frac{D-3}{2}\right)\,. \tag{4.2.5}$$

The point of closest approach, or turning point, ρ_t, is determined by $W(\rho_t) = 0$. The turning

point coincides with the impact parameter b when $r_g = 0$, but the curvature-corrections will depend on whether we are considering scalar-, vector- or tensor-type perturbations.

In terms of the original radial coordinate, the phase shift is given in (3.2.21). In the limit $r_g \to 0$ this formula correctly gives $\delta_\ell(\omega) = 0$.

4.2.2 Time delay for D-dimensional black hole

Although the JWKB expression for the time delay is capable of dealing with small ℓ, it will be sufficient to focus on large $\ell \sim b\omega \gg 1$, since the high-energy regime will be important in our subsequent analysis. This is essentially the eikonal limit — see appendix 4.B to see a derivation directly from the phase shift within the JWKB approximation. We will also assume for simplicity that the scattering impact parameter is large compared to the Schwarzschild radius, $b \gg r_g$. With these assumptions, and using the effective potentials provided in Appendix 4.A, we find the following expressions for the time delays[1]

$$\Delta T_T = \Delta T^{\rm GR}\left[1 - \frac{8(D-1)}{D-3}\frac{c_{\rm GB}}{\Lambda^2 r_g^2}\left(\frac{r_g}{b}\right)^2\right], \tag{4.2.6a}$$

$$\Delta T_V = \Delta T^{\rm GR}\left[1 + \frac{4(D-1)(D-4)^2}{D-3}\frac{c_{\rm GB}}{\Lambda^2 r_g^2}\left(\frac{r_g}{b}\right)^2\right], \tag{4.2.6b}$$

$$\Delta T_S = \Delta T^{\rm GR}\left[1 + \frac{8(D-1)(D-4)^2}{D-3}\frac{c_{\rm GB}}{\Lambda^2 r_g^2}\left(\frac{r_g}{b}\right)^2\right], \tag{4.2.6c}$$

where

$$\Delta T^{\rm GR} = \frac{(D-2)\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{D-4}{2}\right)}{\Gamma\left(\frac{D-3}{2}\right)}\left(\frac{r_g}{b}\right)^{D-3} b\,, \tag{4.2.7}$$

is the time delay from GR in the same limits. Note that with particular choices for Λ and $c_{\rm GB}$, this reproduces the results in [143, 144, 174], both in terms of time delays or propagation speeds.

We are now in a position to see how GB theory could appear to violate both asymptotic causality and infrared causality if the resolvability condition was not properly accounted for in the regime of validity of the EFT. Looking at the expressions in equations (4.2.6), we see that the EFT corrections to the scalar and vector mode time delays have opposite sign to the tensor mode time delay. There are no non-zero values of $c_{\rm GB}$ which can make $\Delta T^{\rm EFT} > 0$ for all three. This immediately suggests a violation of infrared causality. To generate an asymptotic acausality, on the other hand, we need to further push $b^2 \to \Lambda^{-2}$ and then we would find ourselves in a situation where there would be at least one negative $\Delta T_M^{\rm EFT}$ that could overwhelm $\Delta T^{\rm GR}$ for any non-zero $c_{\rm GB}$. This corresponds to the situation in [174] where, in their notation, $b^2 \sim \lambda_{\rm GB}$. The central claim of what follows is that once we have properly identified the regime of validity of the EFT, neither causality conditions are violated in a resolvable way. To see why, we now turn to the consideration of the regime of validity.

[1] These are the time delays at fixed ℓ. Very similar expressions follow for the time delay at fixed b.

4.3 Validity of the gravitational EFT

In this section, we explore the constraints on the applicability of the EFT description and thereby determine its regime of validity. The discussion in this section closely mirrors the analysis in [215].

4.3.1 Constraints on background

As repeatedly emphasised, our action (1.3.8) describes just the lowest-order terms in an infinite series of effective operators generically present in the EFT of gravity. No real example of low-energy expansions of UV-complete theories [82, 136] would ever truncate after R^2–operators. Rather, they generically contain all possible dimension-6, -8, -10,... operators built out of the curvature and its derivatives, and we have parameterised the typical form in (1.2.1). Ultimately any EFT expansion is an expansion in derivatives, and so control over this expansion is lost whenever that curvature or its derivatives become too large. In practice, this means that the EFT is only a useful description of gravity in regions of spacetime where, schematically,

$$\nabla^{2m}\text{Riemann}^n \ll \Lambda^{2m+2n}, \tag{4.3.1}$$

where Λ is the cutoff of the effective theory. What is meant by this schematic expression is: All possible scalar contractions built out of $2m$ covariant derivatives and n powers of Riemann tensors.

We may first apply these constraints to the background geometry for which "Riemann" may be replaced with the Riemann tensor as calculated in the GR background spacetime, since any corrections to those components will be automatically suppressed by additional powers of $1/\Lambda$ in a region where the expansion is controlled. Since our background is Ricci-flat at leading order, it is sufficient to place these bounds on the Weyl or Riemann tensor. Thus an example requirement for the validity of the EFT expansion is

$$\Box^m \left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)^n \ll \Lambda^{2m+4n}. \tag{4.3.2}$$

For the background geometry, Riemann scales as r_g^{D-3}/r^{D-1}, and the strongest bound will be at the point of closest approach, which is related to the impact parameter b. The implication is

$$\nabla^{2m}\text{Riemann}^n \sim \frac{r_g^{(D-3)n}}{b^{(D-1)n+2m}} \ll \Lambda^{2m+2n}. \tag{4.3.3}$$

In the limit $n \to \infty$ for fixed m, this is the requirement that

$$\frac{r_g^{D-3}}{b^{D-1}} \ll \Lambda^2\,, \tag{4.3.4}$$

which is just our usual statement that the curvature is small. At the other extreme, we can consider operators with a large number of covariant derivatives and take $m \to \infty$ to obtain:

$$b \gg \Lambda^{-1}. \tag{4.3.5}$$

From this statement alone, it is already clear that there is no violation of asymptotic causality (2.0.1) in this EFT if the background is under control, no matter the scale Λ.

Taken together, (4.3.4) and (4.3.5) define the range of distance scales b for which the EFT expansion is well-defined on a Schwarzschild background. They both express the fact that our understanding of the low-energy breaks down at short scales or high energies, where the microscopic degrees of freedom that have been integrated out to lead to this EFT ought to be described in their own right. Note that *a priori* we do not require $r_g > \Lambda^{-1}$ since it is sufficient that the geometry is asymptotically Schwarzschild at distances $r, b \gg \Lambda^{-1}$, and we do not need to resolve the horizon.

4.3.2 Constraints on perturbations

In addition to the largely familiar constraints on the background, we should also ensure that higher-order operators do not spoil the equations of motion for the perturbations we have considered so far. Naively one may expect that, so long as the background is under control, the higher-order EFT corrections to the linear perturbation equations will also be under control. In practice, for backgrounds that enjoy a high level of symmetry (as is the case for the static and spherically symmetric BH solution), the perturbations are able to probe much more of the information contained in the Riemann tensor and hence can provide a richer insight on the validity of the EFT.

From the point of view of the perturbations, since we are dealing with a low-energy EFT, it is clearly not possible to consider scattered particles of arbitrarily high energy. Not only will the Riemann curvature itself be perturbed, the covariant derivatives in generic higher-derivative operators can now act both on the background and on the perturbations. Focusing on a GW of momentum k^μ, and replacing all derivatives acting on the perturbations by their form at high energies, *i.e.* $\nabla_\mu h_{\alpha\beta} = i k_\mu h_{\alpha\beta}$, the schematic form of the corrections to the equation of motion for GWs $h_{\mu\nu}$ are

$$k^2 h + \sum_{n+m+p>1} \frac{c_{nmp}}{\Lambda^{2(n+m+p-1)}} \left(\nabla^p \text{Riemann}^m\right) k^{2n+p} h = 0\,, \tag{4.3.6}$$

where we understand ∇ and Riemann to be background quantities. This is a tensor equation, *i.e.* of the form

$$\mathcal{E}^{\alpha\beta,\text{GR}}_{\mu\nu} h_{\alpha\beta} + \mathcal{E}^{\alpha\beta,\text{EFT}}_{\mu\nu} h_{\alpha\beta} = 0\,. \tag{4.3.7}$$

Nevertheless, in order estimate the size of the EFT corrections, it is sufficient to focus on the eigenvalues of $\mathcal{E}^{\alpha\beta,\text{EFT}}_{\mu\nu}$, which are all scalar quantities, relative to k^2. Thus to establish the regime of validity for perturbations it is sufficient (but not necessary) to consider scalar operators built in the manner

$$\left(\frac{\nabla}{\Lambda}\right)^p \left(\frac{\text{Riemann}}{\Lambda^2}\right)^m \left(\frac{k}{\Lambda}\right)^{2n+p} \ll 1\,, \tag{4.3.8}$$

and again we consider all possible scalar contractions. Due to the symmetries of the Riemann tensor, and the fact that $k_\mu k^\mu \approx 0$ on-shell, the tensor with the highest power of k we can build

from a single Riemann tensor is

$$A^{\mu}{}_{\nu} = R^{\mu}{}_{\alpha\nu\beta}k^{\alpha}k^{\beta}. \tag{4.3.9}$$

The trace of this object vanishes because of the symmetry of the background spacetime, but scalars can be built out of higher powers of $A^{\mu}{}_{\nu}$. The requirement that they are not too large:

$$\mathrm{Tr}\left[A^{n}\right] \ll \Lambda^{4n}\,, \tag{4.3.10}$$

translates into a bound on the frequency ω of the wavevector k_{μ}. Denoting the metric on the $(D-2)$-sphere as $\mathrm{d}\Omega_{D-2}^2 = \mathrm{d}\theta^2 + \sin(\theta)^2\mathrm{d}\Omega_{D-3}^2$, and solving the geodesic equation for a massless particle, we can parameterise the scattering momenta as

$$k^{\mu}\partial_{\mu} = \frac{\omega}{f}\partial_t + \omega\sqrt{1-\frac{fb^2}{r^2}}\partial_r + \frac{b\omega}{r^2}\partial_{\theta}\,, \tag{4.3.11}$$

so that

$$\begin{aligned} A_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} =& \frac{(D-3)r_g^{D-3}\omega^2}{r^{D-1}}\Bigg[-\frac{1}{2}\left((D-2)-(D-1)\frac{b^2}{r^2}f\right)\mathrm{d}t^2 - \frac{1}{2f^2}\left((D-2)-\frac{b^2}{r^2}f\right)\mathrm{d}r^2 \\ &+ \frac{(D-2)}{f}\sqrt{1-\frac{fb^2}{r^2}}\mathrm{d}t\,\mathrm{d}r + \left(-\frac{b}{2}\mathrm{d}t + \frac{b}{2f}\sqrt{1-\frac{fb^2}{r^2}}\mathrm{d}r + \frac{b^2}{2}\mathrm{d}\theta\right)\mathrm{d}\theta \\ &+ \frac{b^2}{D-3}\mathrm{d}\Omega_{D-3}^2\Bigg]. \end{aligned} \tag{4.3.12}$$

Evaluating equation (4.3.10) explicitly with $n=2$, we find the following bound:

$$C_D\frac{\omega^2 b^2 r_g^{D-3}}{r^{D+1}} \ll \Lambda^4, \tag{4.3.13}$$

where the coefficient C_D is a dimensionless order-1 dimension-dependent factor, $C_D^2 = D^4 - 8D^3 + 23D^2 - 26D + 6$. Again, considering this at the point of closest approach $r \sim b$, this condition becomes essentially

$$\omega^2 \ll \frac{\Lambda^4 b^{D-1}}{r_g^{D-3}}. \tag{4.3.14}$$

No further constraint is obtained from considering the limit $n \to \infty$. Equation (4.3.14) is our first upper bound on the energy of GWs which can safely propagate within the regime of validity of the EFT.

We find another type of bound on the frequency from operators with covariant derivatives. Contractions of covariant derivatives with Riemann tensor indices vanish, as do contractions of k_{μ} with $A_{\mu\nu}$, so amongst the remaining possibilities are

$$\left((k^{\mu}\nabla_{\mu})^m A^{\alpha\beta}\right)\left((k^{\nu}\nabla_{\nu})^n A_{\alpha\beta}\right) \ll \Lambda^{8+4(m+n)}, \tag{4.3.15a}$$

$$(k^{\mu}\nabla_{\mu})^m\left(W^{\alpha\beta\gamma\delta}W_{\alpha\beta\gamma\delta}\right) \ll \Lambda^{4+2m}. \tag{4.3.15b}$$

In the limit $m, n \to \infty$, the above bounds amount to $k^\mu \nabla_\mu \ll \Lambda^2$, or what is tantamount to

$$\omega \ll \Lambda^2 b \,, \tag{4.3.16}$$

which is our second upper bound on the energy of GWs in the EFT. As long as we consider scattered waves for which $b \gtrsim r_g$, the bound (4.3.16) is always stronger than (4.3.14), and so for all intents and purposes we may regard the maximum frequency for which the effective theory can be trusted to be $\omega \sim \Lambda^2 b$. This will prove important in what follows.

It is important to stress that we are allowed to consider frequencies ω larger than Λ, and still remain within the validity of the EFT. The reason for this is that ω is not a covariant/Lorentz invariant quantity, and in a Lorentz invariant theory bounds should always be placed on Lorentz scalars. The bound (4.3.16) can be intuitively understood as the bound associated with the Mandelstam invariant $k_\mu \kappa^\mu \ll \Lambda^2$, where κ^μ is the typical momentum of the coherent gravitons that make up the background. The bound (4.3.5) is similarly the requirement that $\kappa_\mu \kappa^\mu \ll \Lambda^2$.

4.3.3 Regime of validity from time delays

The arguments of the previous section on the regime of validity were largely schematic, and so it is useful to consider explicit examples of higher-dimension operators from which the bounds may be observed. We will now show how we can arrive at the same frequency bound (4.3.14) through concrete calculation with a specific higher-dimension operator. We will concentrate only on tensor-type perturbations to illustrate our point. Among all the dimension-8 curvature operators expected to enter the EFT (1.2.1), the following one is quite generic

$$S_{\rm D8} = \int \mathrm{d}^D x \sqrt{-g} M_{\rm Pl}^{D-2} \left(\frac{c_{\rm R4}}{\Lambda^6} R_{\alpha\beta\gamma\delta} R^{\gamma\delta\zeta\iota} R_{\zeta\iota\kappa\lambda} R^{\kappa\lambda\alpha\beta} \right) , \tag{4.3.17}$$

where $c_{\rm R4} \sim \mathcal{O}(1)$ is our new Wilsonian coefficient. The presence of this operator is very generic to any gravitational EFT and makes no assumption on its precise UV completion. The truncated EFT action we now consider is $S_{\rm eff} + S_{D8}$, where $S_{\rm eff}$ is given in (1.3.8). There are of course dimension-6 operators as well as other dimension-8 operators that enter the EFT, however for the purpose of this argument they have no impact as will be discussed later. We will show that by demanding the time delay induced by this particular dimension-8 is subdominant to the GB-induced time delay, a necessary condition for the latter to be taken seriously, we can exactly reproduce (4.3.14). For clarity of notation, we split the EFT time delay into two components:

$$\Delta T^{\rm EFT} = c_{\rm GB} \Delta T^{\rm GB} + c_{\rm R4} \Delta T^{\rm R4} + \dots \,, \tag{4.3.18}$$

where $\Delta T^{\rm GB} \propto \Lambda^{-2}$ is the GB-correction we have already calculated and $\Delta T^{\rm R4} \propto \Lambda^{-6}$ will arise from the newcomer.

This new operator will modify the propagation of GWs by introducing higher-derivative, $\mathcal{O}(\Lambda^{-6})$-corrections to the master equations (4.1.3). As usual in an EFT, the appearance of higher derivatives does not signal new ghostly degrees of freedom, and following [216] we can trade higher radial derivatives for higher powers of ω and ℓ using the lower-order equations of motion. The

net effect is a correction to the effective potential, which we will denote by V_T^{R4}. Since we are working only to leading order in large-ℓ with the JWKB approximation, we only need to know the large-ℓ behaviour of V_T^{R4} to calculate the time delay. This can be deduced by following only the highest-, four-derivative terms, which is done explicitly in appendix 4.C. The result is

$$\frac{V_T^{\mathrm{R4}}}{f} = -16(D-1)^2 \frac{c_{\mathrm{R4}}}{\Lambda^6 r_g^6} \frac{1}{r^2} \left(\frac{r_g}{r}\right)^{2D} k_T^4 \left(1 + \mathcal{O}\left(k_T^{-2}\right)\right). \tag{4.3.19}$$

Including these contributions to the tensor modes time delay, we find that the $\mathcal{O}(\Lambda^{-6})$-correction is

$$\Delta T_T^{\mathrm{R4}} = 16 \frac{c_{\mathrm{R4}}}{\Lambda^6 r_g^6} \frac{\sqrt{\pi}(D-1)\Gamma\left(D+\frac{1}{2}\right)}{\Gamma\left(D-1\right)} \left(\frac{r_g}{b}\right)^{2D} b^3\omega^2 + \ldots, \tag{4.3.20}$$

up to terms which are subleading at large ℓ. Now, if our original truncated EFT action (1.3.8) was to be trusted, it should be the case that ΔT_T^{D8} is negligible compared to the leading-order EFT term ΔT_T^{GB} in (4.2.6a). Assuming that the Wilson coefficients are order unity, this is the statement that

$$\frac{1}{(\Lambda r_g)^6} \left(\frac{r_g}{b}\right)^{2D} b^3\omega^2 \ll \frac{1}{(\Lambda r_g)^2} \left(\frac{r_g}{b}\right)^{D-1} b\,, \tag{4.3.21}$$

which, after a little rearrangement gives exactly the upper bound on ω given in (4.3.14).

Our reason for focusing our attention on (4.3.17) is that it is the lowest-order term in the generic EFT expansion which introduces higher powers of ω^2 in the effective potential and hence in (4.3.20) as compared to (4.2.6a) (via their conversion to a potential term featuring k_T^4). It is the contrast in frequency dependence of (4.3.20) and (4.2.6a) that leads to the non-trivial bound given in (4.3.14). The fact that no lower-dimension operator gives the same result is a quirk of the symmetries of the background spacetime, combined with the transverse-traceless structure of the tensor spherical harmonics. This eliminates all other dimension-4 and dimension-6 operators from contention, as explained in appendix 4.C. In the analogous argument of subsection 4.3.2, the dimension-4 scalar $\mathrm{Tr}[A]$ vanished and we had to look to its dimension-8 counterpart $\mathrm{Tr}[A^2]$ to place a non-trivial bound on the wave's momentum.

Of course one could imagine that the EFT happened to have $c_{\mathrm{R4}} = 0$ by virtue of symmetry or some sum-rule. However, the special properties of this dimension-8 operator (4.3.17) that provide the right bound are not unique, as previously mentioned. Any operator that gives terms in the potential depending on k_T^{2n} for $n > 1$ (after perturbatively replacing higher derivatives with the lower-order master equation) will do. This means that, to evade the upper bound ω (which, as we shall see soon is necessary for the resolution to the apparent violation of causality), one would have to set to zero essentially all higher-derivative operators, giving an EFT that could never have come from a consistent UV completion. Indeed it is now well understood from positivity bound arguments that there are upper and lower bounds on generic Wilson coefficients at every order in the derivative expansion [44, 48, 57]. This means that, in general, setting coefficients of higher-curvature operators to zero is not necessarily even an option. This conforms with general expectations from explicit loops from massive fields, where essentially every curvature invariant is generated at some order.

4.3.4 Unresolvability of time delay

We are now in a position to demonstrate our central result. To take any apparent acausality seriously, it must be resolvable within the confines of the EFT as defined by the four bounds: (4.3.4), (4.3.5), (4.3.14) and (4.3.16).

Focusing on the order of magnitude of the GB contribution, we first note that

$$\frac{|\Delta T^{\rm GB}|}{\Delta T^{\rm GR}} \sim \frac{c_{\rm GB}}{\Lambda^2 b^2}. \tag{4.3.22}$$

Within the regime of validity of the EFT (4.3.5), the GR contribution to the time delay always dominates over the EFT correction. It is therefore not possible to violate asymptotic causality in this set-up.

What about infrared causality? Within the regime of validity of the EFT, we can write

$$\omega\left|\Delta T^{\rm GB}\right| \sim \frac{1}{(\Lambda r_g)^2}\omega b\left(\frac{r_g}{b}\right)^{D-1} \sim \omega b\frac{1}{b^2\Lambda^2}\left(\frac{r_g}{b}\right)^{D-3} \ll \left(\frac{r_g}{b}\right)^{\frac{D-3}{2}} \ll 1\,, \tag{4.3.23}$$

where we have used (4.3.14) to bound the frequency ω and the fact that $b \gg r_g$. In fact, using (4.3.16), we get the stronger statement

$$\omega\left|\Delta T^{\rm GB}\right| \sim \frac{1}{(\Lambda r_g)^2}\omega b\left(\frac{r_g}{b}\right)^{D-1} \sim \frac{\omega}{b\Lambda^2}\left(\frac{r_g}{b}\right)^{D-3} \ll \left(\frac{r_g}{b}\right)^{D-3} \ll 1\,. \tag{4.3.24}$$

It is now clear that the possible time advance induced by the GB term in the effective action is unresolvable for $|c_{\rm GB}| \lesssim 1$, *i.e.* $\left|\Delta T^{\rm EFT}\right| \ll \omega^{-1}|c_{\rm GB}|$, within the regime of validity of the EFT, regardless of how small the scale Λ, or the magnitude of the BH/spherical source. In other words, Einstein-Gauss-Bonnet gravity does not violate infrared causality unless we choose to take it seriously in a regime in which it could not be viewed as having come from a consistent high-energy theory, irrespective of what its precise completion is. This is consistent with the cosmological solutions considered in [215].

We showed that the apparent violation of (infrared or asymptotic) causality is not actually observable within the regime of validity of the EFT itself. As alluded to in section 3.1.2, the (apparent) violation of causality is resolved entirely within the low-energy theory without the need to appeal to new degrees of freedom such as an infinite tower of massive higher-spin fields as suggested by [174]. In the next chapter, we will show the same result holds when considering multiple shock waves.

4.A Master Equation

In appendix A.2, we will describe the decomposition of the metric perturbations in terms of their transformation properties under spherical symmetry into tensor, vector, and scalar modes. In this appendix, we make use of this to put the perturbation equations of EGB gravity into the form of (4.1.3).

4.A.1 Master Variables

Let us consider the tensor, vector, and scalar modes separately.

Tensor Modes

To obtain the master equation in the form (4.1.3), it is simply a matter of rescaling H_T accordingly:

$$\Phi_T = r^{(D-2)/2}\left(1-4(D-4)\frac{c_{\rm GB}}{\Lambda^2 r_g^2}\left(\frac{r_g}{r}\right)^{D-1}\right)H_T. \tag{4.A.1}$$

For the tensor potential, we find

$$\begin{aligned}\frac{V_T}{f} = \frac{1}{r^2}\Bigg[&k_T^2\left(1+8\frac{c_{\rm GB}}{\Lambda^2 r_g^2}(D-1)\left(\frac{r_g}{r}\right)^{D-1}\right)\\ &+\frac{D(D-6)+16}{4}\left(1-32\frac{c_{\rm GB}}{\Lambda^2 r_g^2}\frac{(D-1)(D-6)}{D(D-6)+16}\left(\frac{r_g}{r}\right)^{D-1}\right)\\ &+\frac{(D-2)^2}{4}\left(\frac{r_g}{r}\right)^{D-3}\left(1-2\frac{c_{\rm GB}}{\Lambda^2 r_g^2}\frac{(D-4)\left[3D(D-3)(D-6)-32\right]}{(D-2)^2}\left(\frac{r_g}{r}\right)^{D-1}\right)\Bigg].\end{aligned} \tag{4.A.2}$$

We can identify the tensor angular speed from the coefficient of the k_T^2/r^2-term in (4.A.2):

$$v_{\Omega,T}^2 = 1+8\frac{c_{\rm GB}}{\Lambda^2 r_g^2}(D-1)\left(\frac{r_g}{r}\right)^{D-1}. \tag{4.A.3}$$

Vector Modes

Identifying the master variable for vectors is a more involved process than it was for tensors because of the extra variables f_a in the initial parameterisation of the metric perturbations (A.2.14).

Using the gauge-invariant observables, we find that the master variable Φ_T is directly related to F_r as

$$\Phi_V = r^{-(D-6)/2}\left(1-4(D-4)\frac{c_{\rm GB}}{\Lambda^2 r_g^2}\left(\frac{r_g}{r}\right)^{D-1}\right)f(r)F_r. \tag{4.A.4}$$

For the vector potential, we get

$$\begin{aligned}\frac{V_V}{f} = \frac{1}{r^2}\Bigg[&k_V^2\left(1-4(D-1)(D-4)\frac{c_{\rm GB}}{\Lambda^2 r_g^2}\left(\frac{r_g}{r}\right)^{D-1}\right)\\ &+\frac{D(D-6)+12}{4}\left(1+16\frac{(D-1)(D-4)(3D-5)}{D(D-6)+12}\frac{c_{\rm GB}}{(\Lambda r_g)^2}\left(\frac{r_g}{r}\right)^{D-1}\right)\\ &-\frac{3(D-2)^2}{4}\left(\frac{r_g}{r}\right)^{D-3}\left(1-\frac{2(D-4)\left[D^2(5D-57)+134D-88\right]}{3(D-2)^2}\frac{c_{\rm GB}}{\Lambda^2 r_g^2}\left(\frac{r_g}{r}\right)^{D-1}\right)\Bigg].\end{aligned} \tag{4.A.5}$$

We can identify the vector angular speed from the coefficient of the k_V^2/r^2-term in (4.A.5):

$$v_{\Omega,V}^2 = 1-4(D-1)(D-4)\frac{c_{\rm GB}}{\Lambda^2 r_g^2}\left(\frac{r_g}{r}\right)^{D-1}. \tag{4.A.6}$$

Scalar Modes

As for the vectors, identifying the master variable in terms of those in (A.2.19) becomes easier after constructing the following gauge-invariant variables.

There is a change of variables of the form

$$F = \alpha(r)\Phi_S + \beta(r)\partial_r\Phi_S, \tag{4.A.7}$$

such that all components of the perturbed Einstein-Gauss-Bonnet equations are automatically satisfied when Φ_S obeys the master equation (4.1.3). To leading order in Λ^{-1}, we need

$$\begin{aligned}
\alpha(r) = \frac{1}{4(D-2)r^{(D-2)/2}H}\Bigg\{ & 4k_S^4 + 2(D-2)(D-6)k_S^2 - 2(D-2)^2(D-4) \\
& + \left[6(D-2)k_S^2 - (D-2)^2\left(D(D-5)+10\right)\right]\left(\frac{r_g}{r}\right)^{D-3} + (D-1)(D-3)^3\left(\frac{r_g}{r}\right)^{2(D-3)}\Bigg\} \\
& + \frac{c_{\rm GB}}{(\Lambda r_g)^2}\frac{D-4}{4(D-2)r^{\frac{D}{2}-1}H^2}\Bigg\{16k_S^6 - 8(D-2)\left(D(2D-7)+12\right)k_S^4 \\
& + 16(D-2)^4\left(D(2D-7)+9\right)k_S^2 - 8(D-2)^3\left(D(2D-7)+8\right) \\
& + \Big[4(D-2)\left(D(5D-17)+24\right)k_S^4 - 24(D-2)^2\left(D(3D-10)+11\right)k_S^2 \\
& \quad + 4(D-2)\left(D(5D-17)+24\right)k_S^4 + 4(D-2)^3\left(D(13D-43)+42\right)\Big]\left(\frac{r_g}{r}\right)^{D-3} \\
& + \left[16(D-1)(D-2)^2(2D-3)k_S^2 - 2(D-1)(D-2)^3\left(D(3D+7)-18\right)\right]\left(\frac{r_g}{r}\right)^{2(D-3)} \\
& - \left[(D-1)^2(D-2)^3\left(D(D-11)+14\right)\right]\left(\frac{r_g}{r}\right)^{3(D-3)}\Bigg\},
\end{aligned} \tag{4.A.8}$$

where

$$H = k_S^2 - D + 2 + \frac{(D-1)(D-2)}{2}\left(\frac{r_g}{r}\right)^{D-3}, \tag{4.A.9}$$

and

$$\beta(r) = \frac{1}{r^{(D-4)/2}}\left[1 - \left(\frac{r_g}{r}\right)^{D-3} + 4(D-4)\frac{c_{\rm GB}}{(\Lambda r_g)^2}\left(\frac{r_g}{r}\right)^{D-1}\left(1 + \frac{D-5}{2}\left(\frac{r_g}{r}\right)^{D-3}\right)\right]. \tag{4.A.10}$$

For the scalar potential we have

$$\frac{V_S}{f} = \frac{1}{16H^2r^2}\mathcal{V}^{\rm GR} + \frac{c_{\rm GB}}{\Lambda^2 r_g^2}\frac{D-4}{16H^3r^2}\left(\frac{r_g}{r}\right)^{D-1}\mathcal{V}^{\rm GB}, \tag{4.A.11}$$

where the contribution from GR is given by

$$\begin{aligned}\mathcal{V}^{\rm GR} &= 4\left(k_S^2-D+2\right)^2\left(4k_S^2+D^2-6D+8\right)\\ &-12(D-2)\left(k_S^2-D+2\right)\left[(D-6)k_S^2+D^2(D-8)+22D-20\right]\left(\frac{r_g}{r}\right)^{D-3}\\ &+4k_S^2(D-1)(D-2)(18+D(2D-11))\left(\frac{r_g}{r}\right)^{2(D-3)}\\ &+(D-1)(D-2)^2(D(D-6)(D-13)-96)\left(\frac{r_g}{r}\right)^{2(D-3)}\\ &+(D-1)^2(D-2)^4\left(\frac{r_g}{r}\right)^{3(D-3)}\,,\end{aligned} \tag{4.A.12}$$

and that from the GB term by

$$\begin{aligned}\mathcal{V}^{\rm GB} &= -128(D-1)(k_S^2-D+2)^3\left[(D-2)^2+k_S^2\right]\\ &-8k_S^2(D-2)(k_S^2-D+2)^2(D(21D-131)+140)\left(\frac{r_g}{r}\right)^{D-3}\\ &+8(D-2)^2(k_S^2-D+2)^2(8D^3-125D^2+347D-260)\left(\frac{r_g}{r}\right)^{D-3}\\ &-36k_S^2(D-1)(D-2)^2(k_S^2-D+2)(D-3)(D-12)\left(\frac{r_g}{r}\right)^{2(D-3)}\\ &+12(D-1)(D-2)^3(k_S^2-D+2)\left(3D^2-77D+140\right)\left(\frac{r_g}{r}\right)^{2(D-3)}\\ &+2k_S^2(D-1)^2(D-2)^2\left(112-9D^3+97D^2-226D\right)\left(\frac{r_g}{r}\right)^{3(D-3)}\\ &+2(D-1)^2(D-2)^3\left(9D^3-113D^2+258D-128\right)\left(\frac{r_g}{r}\right)^{3(D-3)}\\ &-(D-1)^3(D-2)^3\left[3D(D-3)(D-6)-32\right]\left(\frac{r_g}{r}\right)^{4(D-3)}\,.\end{aligned} \tag{4.A.13}$$

Due to the complicated dependence of the scalar potential on the eigenvalue k_S^2, the scalar angular speed cannot be readily extracted from (4.A.11) unless one takes the large ℓ (k_S) limit.

4.A.2 Time Delay Expressions

We can use the master equations for higher dimensions to find the time delays.

We concentrate on the fixed-ℓ time delays to $\mathcal{O}(\Lambda^{-2})$, outside the eikonal limit, to leading order in the curvature expansion:

$$\begin{aligned}\Delta T_T &= \frac{(D-2)\sqrt{\pi}}{8r_g\omega^2}\left(\frac{r_g}{b}\right)^{D-2}\frac{\Gamma\left(\frac{D-4}{2}\right)}{\Gamma\left(\frac{D-3}{2}\right)}\Bigg[4b^2\omega^2-(D-4)(D-2)\\ &\qquad-8\frac{c_{\rm GB}}{\Lambda^2r_g^2}\left(\frac{r_g}{b}\right)^2\frac{(D-1)(4b^2\omega^2-D^2+2D+8)}{D-3}\Bigg]\end{aligned} \tag{4.A.14a}$$

$$\begin{aligned}\Delta T_V &= \frac{(D-2)\sqrt{\pi}}{8r_g\omega^2}\left(\frac{r_g}{b}\right)^{D-2}\frac{\Gamma\left(\frac{D-4}{2}\right)}{\Gamma\left(\frac{D-3}{2}\right)}\Bigg[4b^2\omega^2+3(D-4)(D-2)\\ &\qquad+4\frac{c_{\rm GB}}{\Lambda^2r_g^2}\epsilon\left(\frac{r_g}{b}\right)^2\frac{(D-1)(D-4)^2(4b^2\omega^2-D^2-6D+8)}{D-3}\Bigg]\end{aligned} \tag{4.A.14b}$$

$$\Delta T_S = \frac{(D-2)\sqrt{\pi}}{8 r_g \omega^2}\left(\frac{r_g}{b}\right)^{D-2}\frac{\Gamma\left(\frac{D-4}{2}\right)}{\Gamma\left(\frac{D-3}{2}\right)}\left[\frac{16b^4\omega^4+8(44+3(D-8)D)b^2\omega^2+3(3D-8)(D-2)^2(D-4)}{4b^2\omega^2-D(D-2)}\right.$$
$$\left.+\,8\frac{c_{\text{GB}}}{\Lambda^2 r_g^2}\epsilon\left(\frac{r_g}{b}\right)^2\frac{(D-1)(D-4)^2(4b^2\omega^2+8+3D^2-10D)}{D-3}\right]. \tag{4.A.14c}$$

4.B The Eikonal Limit

We will now see that the eikonal approximation can be obtained as a limit of the semiclassical approximation. It will allow us to find simple expressions for the time delay.

In the tortoise coordinates, we split the potential in the form

$$V_{\text{eff}}(\hat{r}) = \frac{b^2\omega^2}{\hat{r}^2} + U_{\text{eff}}(\hat{r}), \tag{4.B.1}$$

with an impact parameter that $b = (l+(d-1)/2)\omega^{-1}$ chosen to simplify the potential. The eikonal approximation corresponds to the high energy limit with $\omega^2 \gg U_{\text{eff}}(\hat{r})$, in which we can treat U_{eff} perturbatively. However since $\hat{r}_t$ itself depends on the potential, we need to be careful. To do that, we use the fact that

$$0 = W_\ell(\hat{r}_t) = \omega^2 - V_{\text{eff}}(\hat{r}_t), \tag{4.B.2}$$

so that

$$\frac{b^2\omega^2}{\hat{r}_t^2} + U_{\text{eff}}(\hat{r}_t) = \omega^2. \tag{4.B.3}$$

Solving this perturbatively in the curvature, *i.e.* r_g, we find that to leading order $r_t = b$. With this, we can expand the phase shift in equation 3.2.21 as follows

$$\begin{aligned}
\delta_l &= \int_{\hat{r}_t}^{\infty} d\hat{r}\left(\sqrt{\omega^2 - V_{\text{eff}}(\hat{r})} - \omega\right) - \omega\hat{r}_t + \frac{\pi}{2}(l+(d-1)/2) \\
&= \int_{\hat{r}_t}^{\infty} d\hat{r}\left(\sqrt{\left(1-\frac{\hat{r}_t^2}{\hat{r}^2}\right)\omega^2 - \left(U_{\text{eff}}(\hat{r}) - U_{\text{eff}}(\hat{r}_t)\frac{\hat{r}_t^2}{\hat{r}^2}\right)} - \omega\right) - \omega\hat{r}_t + \frac{\pi}{2}(l+(d-1)/2) \\
&= \int_{\hat{r}_t}^{\infty} d\hat{r}\left[\sqrt{\left(1-\frac{\hat{r}_t^2}{\hat{r}^2}\right)\omega^2} - \frac{1}{2}\frac{1}{\sqrt{\left(1-\frac{\hat{r}_t^2}{\hat{r}^2}\right)\omega^2}}\left(U_{\text{eff}}(\hat{r}) - U_{\text{eff}}(\hat{r}_t)\frac{r_{t*}^2}{\hat{r}^2}\right) + \mathcal{O}\left(\left(\frac{U_{\text{eff}}}{\omega^2}\right)^2\right) - \omega\right] \\
&\quad - \omega\hat{r}_t + \frac{\pi}{2}(l+(d-1)/2) \\
&= \frac{\pi}{2}\omega\,(b-\hat{r}_t) - \frac{1}{2}\int_{\hat{r}_t}^{\infty} d\hat{r}\frac{1}{\sqrt{\left(1-\frac{\hat{r}_t^2}{\hat{r}^2}\right)\omega^2}}\left(U_{\text{eff}}(\hat{r}) - U_{\text{eff}}(\hat{r}_t)\frac{\hat{r}_t^2}{\hat{r}^2}\right) + \mathcal{O}\left(\left(\frac{U_{\text{eff}}}{\omega^2}\right)^2\right).
\end{aligned} \tag{4.B.4}$$

Using 4.B.3, we can write

$$b - \hat{r}_t = -\frac{\hat{r}_t^2}{(b+\hat{r}_t)\omega^2}U_{\text{eff}}(\hat{r}_t) \approx -\frac{\hat{r}_t}{2\omega^2}U_{\text{eff}}(\hat{r}_t), \tag{4.B.5}$$

so that

$$\delta_l = -\frac{1}{2\omega}\int_{\hat{r}_t}^{\infty} d\hat{r}\frac{1}{\sqrt{\left(1-\frac{\hat{r}_t^2}{\hat{r}^2}\right)}}U_{\text{eff}}(\hat{r}) + \dots \quad . \tag{4.B.6}$$

To see that this is the eikonal limit, take $\hat{r}^2 = \hat{r}_t^2 + z^2$, which gives

$$\begin{aligned}\delta_l =& -\frac{1}{2\omega}\int_0^{\infty} dz\, U_{\text{eff}}\left(\sqrt{\hat{r}_t^2+z^2}\right) + \dots = -\frac{1}{4\omega}\int_{-\infty}^{\infty} dz\, U_{\text{eff}}\left(\sqrt{\hat{r}_t^2+z^2}\right) + \dots \\ \approx& -\frac{1}{4\omega}\int_{-\infty}^{\infty} dz\, U_{\text{eff}}\left(\sqrt{b^2+z^2}\right) + \dots = \delta_l^{\text{eik}},\end{aligned} \tag{4.B.7}$$

as expected.

4.C Easy as (dimension-) 4, 6, 8

We will now outline why we considered the operator in (4.3.17) as the next term in our EFT expansion. The EFT contains operators with arbitrary number of derivatives. In what follows, we shall focus on the higher-derivative terms in the equations of motion, as those are the ones that will lead to the most stringent bound on ω, and may therefore neglect non-derivative terms. In practice, this means we can freely replace covariant with partial derivatives ($\nabla \to \partial$). Moreover, since we are implementing a perturbative approach, we may freely use the GR equations of motion for the tensor fluctuations, namely $\Box h \approx 0$, in the EFT corrections.

For this section, we are specifically interested in the dynamics of tensor modes. For the tensor perturbations, h_{ij} are the only non-vanishing components. They therefore inherit transverse-tracelessness from the tensor spherical harmonics, *i.e.* $\nabla_\alpha h^\alpha{}_\mu = 0$ and $h^\alpha{}_\alpha = 0$.

4.C.1 Dimension-4 operators

First, consider the dimension-4 operators. Schematically, *i.e.* suppressing the indices,

$$\mathcal{L}_{\text{D4}} \sim \frac{R^2}{\Lambda^2}. \tag{4.C.1}$$

As previously discussed in section 4.1, R^2/Λ^2 and $R_{\mu\nu}^2/\Lambda^2$ contribute to the perturbed Einstein equations at $\mathcal{O}(\Lambda^{-6})$, which are higher-order than the leading EFT contributions.

Even though we can compute the equations of motion, it will be instructive to just construct the possible terms in them by other means. In particular, we know from dimensional analysis that $\mathcal{L}_{\text{D4}}$ will introduce terms of the schematic form

$$\nabla^4 h, \quad R\nabla^2 h, \quad R^2 h. \tag{4.C.2}$$

Of these, only the first term can possibly provide us with four derivatives. Leaving two indices free, the only possible contractions (up to commutation of covariant derivatives, which we ignore) are

$$\nabla^4 h \quad \to \quad \Box^2 h_{\mu\nu}, \quad \Box\nabla_\mu\nabla_\alpha h^\alpha{}_\nu, \quad \Box\nabla_\mu\nabla_\nu h^\alpha{}_\alpha. \tag{4.C.3}$$

Notably, we are always forced to contract at least two covariant derivatives with each other. We can see that none of the above terms are truly higher derivative in nature upon substituting $\Box \approx 0$ up to non-derivative terms at leading order.

4.C.2 Dimension-6 operators

Knowing that dimension-4 operators do not result in higher-derivative equations, we now turn our attention to dimension-6 operators.

Dimensional analysis means that $\mathcal{L}_{\text{D6}}$ will introduce terms in the perturbation equations of motion of the schematic form

$$\nabla^6 h, \quad R\nabla^4 h, \quad R^2\nabla^2 h, \quad R^3 h. \tag{4.C.4}$$

The first term inevitably has two covariant derivatives contracted with each other, giving a $\Box \approx 0$, so we only need to consider the second term. Next, since the tensor modes are transverse and traceless, the only seemingly non-trivial contraction of indices (up to symmetries of the Riemann tensor) is

$$R\nabla^4 h \to R^{\alpha\rho\beta\sigma}\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta h_{\rho\sigma}. \tag{4.C.5}$$

To find the highest-derivative term, we trade in covariant with partial derivatives and using the form of the Riemann tensor on the Schwarzschild background (A.1.3), we find that

$$\begin{aligned} R^{\alpha m\beta n}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta h_{mn} &= R^{ambn}\partial_\mu\partial_\nu\partial_a\partial_b h_{mn} + R^{imjn}\partial_\mu\partial_\nu\partial_i\partial_j h_{mn} \\ &= -\frac{f'}{2r}g^{ab}g^{mn}\partial_\mu\partial_\nu\partial_a\partial_b h_{mn} + \frac{1-f}{r^2}\left(g^{mn}g^{ij} - g^{mj}g^{ni}\right)\partial_\mu\partial_\nu\partial_i\partial_j h_{mn} \\ &= 0\,, \end{aligned}$$

where we used the fact that the tensor perturbations are transverse and traceless. This implies that we have to go to dimension-8 operators to find our first explicit example of higher-derivative term.

4.C.3 Dimension-8 operators

We now show how that next-order EFT contribution in (4.3.17) does indeed lead to genuine higher-derivative terms in the equations of motion. We denote the background equations of motion derived from this operator as $C^{\mu\nu}$, and the associated perturbation equations of motion as $\delta C^{\mu\nu}$.

Once again, we want to find terms that lead to genuine higher-derivative terms in the perturbation equations of motion. The only term with four derivatives acting on the metric perturbation for tensor modes is:

$$\delta C^{\mu\nu} = -16\frac{1}{(\Lambda r_g)^6}\frac{c_{\text{R4}}}{\Lambda^6}R^{\alpha\rho\sigma\mu}R^{\beta\kappa\lambda\nu}\nabla_\rho\nabla_\sigma\nabla_\kappa\nabla_\lambda h_{\alpha\beta} + \ldots, \tag{4.C.6}$$

where the ellipses stand for terms that are trivially lower-derivative terms for reasons similar to those in the last two subsections.

Since the tensor perturbations only have non-vanishing components in the angular directions, we pick $(\mu,\nu) = (i,j)$. Once again, we are interested in terms with the highest numbers of derivatives acting on the tensor mode, so we trade covariant for partial derivatives. Then, making use of the lower-order equations of motion $\Box \approx 0$ as well as (A.1.3), we find:

$$\begin{aligned}
&R^{m\rho\sigma i}R^{n\kappa\lambda j}\partial_\rho\partial_\sigma\partial_\kappa\partial_\lambda h_{mn} \\
&= \left[\left(\frac{f'}{2r}\right)^2 \partial_a\partial^a\partial_b\partial^b h^{ij} - \frac{(1-f)f'}{r^3}\partial_a\partial^a\partial_m\partial^m h^{ij} + \left(\frac{1-f}{r^2}\right)^2 \partial_m\partial^m\partial_n\partial^n h^{ij}\right] \\
&= \frac{r_g^{2D-6}}{r^{2D-2}}\left[\frac{(D-3)^2}{4}\partial_m\partial^m\partial_n\partial^n h^{ij} + (D-3)\partial_n\partial^n\partial_m\partial^m h^{ij} + \partial_m\partial^m\partial_n\partial^n h^{ij}\right] + \mathcal{O}\left(\Lambda^{-2}\right) \\
&= \frac{r_g^{2D-6}}{r^{2D+2}}\frac{(D-1)^2}{4}\hat{\Delta}_{D-2}^2 h^{ij} + \mathcal{O}\left(\Lambda^{-2}\right).
\end{aligned} \tag{4.C.7}$$

Including the lower-order terms presents in the action, the full perturbation equation of motion is therefore of the form

$$\begin{aligned}
0 =& \frac{1}{2}\delta G_{ij} + \delta B_{ij} + \delta C_{ij} \\
=& -\frac{1}{4}\Box h_{ij} + \frac{c_{\text{GB}}}{\Lambda^2 r_g^2}(\ldots) + \frac{c_{\text{GB}}^2}{\Lambda^4 r_g^4}(\ldots) - 4(D-1)^2\frac{c_{\text{R4}}}{\Lambda^6 r_g^6}\frac{r_g^{2D}}{r^{2D+2}}k_T^4\left[1+\mathcal{O}\left(k_T^{-2}\right)\right]h_{ij},
\end{aligned} \tag{4.C.8}$$

where the $\mathcal{O}\left(\Lambda^{-2}\right)$ terms are known from before, and we have not computed the lower-derivative corrections at $\mathcal{O}\left(\Lambda^{-4}\right)$ or $\mathcal{O}\left(\Lambda^{-6}, k_T^2\right)$.

Now, recall that the master equation is of the form

$$\Box_2\Phi - \frac{V}{f}\Phi = 0. \tag{4.C.9}$$

The metric perturbations are related to the master variable simply by a rescaling. We can therefore just read off the $\mathcal{O}\left(\Lambda^{-6}\right)$ correction to the effective potential:

$$\frac{V_T^{\text{D8}}}{f} = -16(D-1)^2 c_{\text{R4}}\frac{1}{(\Lambda r_g)^6}\frac{1}{r^2}\left(\frac{r_g}{r}\right)^{2D} k_T^4\left[1+\mathcal{O}\left(k_T^{-2}\right)\right]. \tag{4.C.10}$$

Chapter 5

Causality on pp-waves

In this chapter, we explain why the resolvability criterion cannot be circumvented by more complicated scattering configurations. These are possible within the family of pp-wave spacetimes, which is therefore the background we will consider.

We start by briefly reviewing details on pp-waves in section 5.1 and explaining their physical significance for the problem at hand. We then proceed to discuss the background solutions and perturbations in the EFT of gravity in section 5.2. Sections 5.3, 5.4, and 5.5 are dedicated to computing the time delay in the eikonal, semiclassical, and Born approximations respectively, in order to argue why scattering off localised sources and more general configurations cannot lead to sufficient amplification of unresolvable causality violation.

We discuss the Penrose limit and the related Aichelburg-Sexl limit in appendix 5.A, and study gravitational perturbations in pure GR in 5.B to argue for the use of lightcone gauge. We defer discussions about the EFT bound on the spread of a wavefunction to appendix 5.C. The decomposition into master variables for the higher-dimension EFT operators, used to check the regime of validity, and in the case of the balancing sources are presented in appendices 5.D and 5.E respectively. Additional expressions are given in appendix 5.F.

5.1 pp-Waves

Let us start with a general discussion of pp-waves. A pp-wave spacetime is a Lorentzian manifold which admits a nowhere vanishing, covariantly constant, null vector field l, and is an exact solution to Einstein's equation describing plane-fronted waves with parallel propagation/rays [252].

5.1.1 Background

If l is nowhere vanishing, we can write, without loss of generality

$$l = \frac{\partial}{\partial v}, \tag{5.1.1}$$

where v is the parameter along integral curves of l. In components, this means that

$$l^\mu = \delta^\mu_v, \quad l_\mu = g_{\mu v}. \tag{5.1.2}$$

Now, l being null means that $0 = l^2 = g_{vv}$. By l being covariantly constant, we mean that $\nabla_\mu l^\nu = 0$. Firstly, this implies that l is a Killing vector field, so the components of the metric are v-independent

$$\partial_v g_{\mu\nu} = 0. \tag{5.1.3}$$

It also means that $dl = 0$, so that that (locally) $l = \mathrm{d}f$ for some function $f = f(X^\mu)$. Choosing

$$l_\mu = g_{v\mu} = -\frac{1}{2}\partial_\mu u \tag{5.1.4}$$

with $u = u(X^\mu)$, the general form of a metric in the (u, v, x^i)-coordinates is

$$\mathrm{d}s^2 = 2\mathrm{d}u\,\mathrm{d}v + g_{uu}(u, x^i)\mathrm{d}u^2 + 2g_{ui}(u, x^i)\mathrm{d}u\,\mathrm{d}x^i + g_{ij}(u, x^i)\mathrm{d}x^i\,\mathrm{d}x^j. \tag{5.1.5}$$

There are residual coordinate transformations which leave this form of the metric invariant, with which we can set $g_{ij} = \delta_{ij}$ and eliminate g_{ui}. This is the pp-wave metric in Brinkmann coordinates [1]

$$\mathrm{d}s^2 = 2\mathrm{d}u\,\mathrm{d}v + H(u, x^i)\mathrm{d}u^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{5.1.7}$$

which has the following non-zero Christoffel symbol components

$$\Gamma^v{}_{uu} = \frac{1}{2}\partial_u H, \quad \Gamma^v{}_{ui} = \frac{1}{2}\partial_i H \quad \Gamma^i{}_{uu} = -\frac{1}{2}\partial^i H. \tag{5.1.8}$$

Curvature

Returning to Minkowski-like coordinates (t, z, x^i) with $u = (z - t)/\sqrt{2}$, $v = (z + t)/\sqrt{2}$, we can see that the metric can be written (covariantly) as [2]

$$g_{\mu\nu} = \eta_{\mu\nu} + H(u, x^i)l_\mu l_\nu. \tag{5.1.9}$$

This form is convenient for computations, since we know that η is flat. In particular, we find that the Christoffel symbol is

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}\left(l^\mu l_\alpha \partial_\beta H + l^\mu l_\beta \partial_\alpha H - l_\alpha l_\beta \partial^\mu H\right) \tag{5.1.10}$$

so that the Riemann tensor can be found to be

$$R_{\rho\sigma\mu\nu} = 2l_{[\mu}(\partial_{\nu]}\partial_{[\rho}H)l_{\sigma]}. \tag{5.1.11}$$

[1] We will mostly use Brinkmann coordinates, but an alternative set of coordinates are Rosen coordinates, in which the metric takes the form

$$\mathrm{d}s^2 = -\mathrm{d}u\,\mathrm{d}v + H_{ij}(u, x^i)\mathrm{d}x^i\mathrm{d}x^j \tag{5.1.6}$$

for an arbitrary H_{ij}.

[2] Note that [253, 254] are using the metric with $-dudv$ and are missing a factor of 4.

In particular, we note that $R_{v\mu\nu\rho} = 0$, and the only non-zero components are of the form

$$R_{uiuj} = -\frac{1}{2}\partial_i\partial_j H. \tag{5.1.12}$$

The Ricci tensor is

$$R_{\mu\nu} = -\frac{1}{2}l_\mu l_\nu \partial_\rho\partial^\rho H = -\frac{1}{2}l_\mu l_\nu \partial_i\partial^i H \tag{5.1.13}$$

since $H = H(u, x^i)$.

A special property of pp-wave backgrounds is that all curvature invariants (scalars) vanish. This makes them very useful for finding exact solutions in string theory [254]. Consider the following two types of curvature tensors:

$$(R^n)^{\alpha\beta}{}_{\mu\nu} = R^{\alpha\beta}{}_{\gamma\delta}R^{\gamma\delta}{}_{\cdot\cdot}\ldots R^{\cdot\cdot}{}_{\rho\sigma}R^{\rho\sigma}{}_{\mu\nu} \tag{5.1.14}$$

$$(\tilde{R}^n)^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} = R^{\alpha}{}_{\gamma}{}^{\beta}{}_{\delta}R^{\gamma}{}_{\cdot}{}^{\delta}{}_{\cdot}\ldots R^{\cdot}{}_{\rho}{}^{\cdot}{}_{\sigma}R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}, \tag{5.1.15}$$

where we have suppressed some of the n Riemann tensors. Note that $(R^n)_{\mu\nu\alpha\beta}$ has the same symmetries as the Riemann tensor and $(\tilde{R}^n)_{\mu\nu\alpha\beta} = (\tilde{R}^n)_{\alpha\beta\mu\nu} = (\tilde{R}^n)_{\nu\mu\beta\alpha}$. This would imply that all higher-power scalar contractions vanish.

We claim that the R^n vanishes for $n \geq 2$ and $\tilde{R}^n$ vanishes for $n \geq 3$. On a pp-wave background

$$(R^2)^{\mu\nu}{}_{\alpha\beta} = R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta} = 2R^{\mu\nu}{}_{ui}R^{ui}{}_{\alpha\beta} = 0 \tag{5.1.16}$$

so and for any $n > 2$, also

$$(R^n)^{\mu\nu}{}_{\alpha\beta} = (R^2)^{\mu\nu}{}_{\rho\sigma}(R^{n-2})^{\rho\sigma}{}_{\alpha\beta} = 0. \tag{5.1.17}$$

Whenever a pair of indices on a Riemann tensor are contracted with other Riemann tensor indices, they are both forced to be in the transverse direction, forcing the expression to vanish. Similarly, we find

$$(\tilde{R}^2)^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} = R^{\alpha}{}_{\rho}{}^{\beta}{}_{\sigma}R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu} = R^{\alpha}{}_{i}{}^{\beta}{}_{j}R^{i}{}_{\mu}{}^{j}{}_{\nu} \tag{5.1.18}$$

which is non-zero only when $(\alpha, \beta) = (v, v)$ and $(\mu, \nu) = (u, u)$, and so

$$(\tilde{R}^3)^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} = (R^2)^{\alpha}{}_{\rho}{}^{\beta}{}_{\sigma}R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu} = (R^2)^{\alpha}{}_{u}{}^{\beta}{}_{u}R^{u}{}_{\mu}{}^{u}{}_{\nu} = 0 \tag{5.1.19}$$

and for $n > 3$

$$(\tilde{R}^n)^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} = (R^3)^{\alpha}{}_{\rho}{}^{\beta}{}_{\sigma}(R^{n-3})^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu} = 0. \tag{5.1.20}$$

5.1.2 Shockwaves

Shockwave geometries are pp-wave spacetimes describing the gravitational field of an ultra-relativistic (high-energy) particle.

We can model such a particle moving in the v-direction [3] by the stress-tensor with only non-zero

[3] Since $u = 0$, $z = -t$ so the particle travels luminally in the $-z$ direction. On a Penrose diagram, this is localised

component being

$$T_{uu} = -P_u \delta(u) \delta^{(D-2)}(x^i), \tag{5.1.21}$$

where $P_u < 0$ is the momentum of the particle. The only non-trivial component of Einstein's equation is the (u,u)-component, which implies that

$$\partial_i \partial^i H = 16\pi G P_u \delta(u) \delta^{(D-2)}(x^i). \tag{5.1.22}$$

This is just the equation for a Green's function in the x^i coordinates. Repackaging the constants into $c = c(u) = 16\pi G P_u \delta(u)$, and integrating over B_r^{D-2}, the $(D-2)$-dimensional ball or radius r,

$$16\pi G P_u \delta(u) = \int_{B_r^{D-2}} \partial_i \partial^i H \, dV = \int_{\partial B_r^{D-2}} n^i \partial_i H \, dS, \tag{5.1.23}$$

where we have used Stokes' theorem. Here, n^i is the unit normal to ∂B_r^{D-2}, and $\mathrm{d}V$ and $\mathrm{d}S$ are the measures on B_r^{D-2} and ∂B_r^{D-2} respectively. Since $\partial B_r^{D-2} = S^{D-3}$, n^i just points in the radial direction, so assuming that h is only function of $r = \sqrt{x_i x^i}$, then

$$16\pi G P_u \delta(u) = \int_{S^{D-3}} \frac{\partial H}{\partial r} r^{D-3} d\Omega_{D-3} = \frac{\partial H}{\partial r} r^{D-3} \Omega_{D-3}, \tag{5.1.24}$$

where $\Omega_n = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$ is the n-volume of the n-sphere. Solving this gives

$$H(u,r) = -\frac{16\pi G P_u \delta(u)}{(D-4)\Omega_{D-3} r^{D-4}} + k \tag{5.1.25}$$

for some constant k. In $D \geq 5$, we can fix $k = 0$ by demanding $\lim_{r\to\infty} h = 0$. Putting this together, we find that in $D \geq 5$

$$H(u,r) = \frac{4\Gamma\left(\frac{D-4}{2}\right)}{\pi^{(D-4)/2}} \delta(u) \frac{G|P_u|}{r^{D-4}} \tag{5.1.26}$$

which agrees with [175].

The shockwave solution can also be derived as a boost of a Schwarzschild black hole. We discuss this in appendix 5.A.

Shapiro Time Delay

The shock wave solution metric 5.1.26 is quite peculiar because it has the delta function $\delta(u)$. We can remove it by switching to a new coordinate system $(u, v, x^i) \to (u, v', x^i)$, with

$$v' = v - \frac{2\Gamma\left(\frac{D-4}{2}\right)}{\pi^{(D-4)/2}} \theta(u) \frac{G|P_u|}{b^{D-4}}. \tag{5.1.27}$$

Under this change of coordinates, the metric function also transforms $H \to H'$, where

$$H' = g'_{uu} = \frac{\partial x^\mu}{\partial u}\frac{\partial x^\nu}{\partial u} g_{\mu\nu} = g_{uu} + 2\frac{\partial v}{\partial u} g_{uv} = \frac{4\Gamma\left(\frac{D-4}{2}\right)}{\pi^{(D-4)/2}} \delta(u) G|P_u| \left(\frac{1}{r^{D-4}} - \frac{1}{b^{D-4}}\right) \tag{5.1.28}$$

on the v-axis.

which vanishes at $r = b$. Geodesics that go through this point are continuous in this coordinate system, which means they suffer a shift

$$\Delta v = \frac{2\Gamma\left(\frac{D-4}{2}\right)}{\pi^{(D-4)/2}} \frac{G|P_u|}{b^{D-4}} \tag{5.1.29}$$

in the original coordinates. This is a manifestation of the Shapiro time delay.

Another way to derive the Shapiro time delay is to think of a quantum field ϕ probing the shock wave background. The wave equation for it will be

$$\Box\phi = -H\partial_v^2\phi + 2\partial_u\partial_v\phi + \partial_i\partial^i\phi = 0, \tag{5.1.30}$$

and as we shall later, this is indeed the equation of motion for gravitational waves. We want consider the change in $\phi(u, v, x^i)$ from $u = 0^-$ to $u = 0^+$. For this, we can ignore the transverse direction derivatives, so that

$$\partial_u\partial_v\phi = \frac{1}{2}H\partial_v^2\phi. \tag{5.1.31}$$

Using separation of variables in u, then

$$\phi(u, v, x^i) = c' e^{\frac{1}{2}\int H du\, \partial_v} \tag{5.1.32}$$

for some constant c. Therefore,

$$\phi(u = 0^+, v, x^i) = e^{\frac{1}{2}\int_{0^-}^{0^+} H du\, \partial_v}\phi(u = 0^-, v, x^i) = e^{i\Delta v p_v}\phi(u = 0^-, v, x^i) \tag{5.1.33}$$

where $\Delta v = \frac{1}{2}\int_{0^-}^{0^+} H\mathrm{d}u$ is once again the Shapiro time delay and we have Fourier transformed to write $\partial_v = ip_v$. This is trivial to integrate, and gives a result that agrees with the previous calculation.

5.2 pp-waves in the EFT of gravity

As we discuss in appendix 5.A, every metric, with a choice of null geodesic, can be associated to a pp-wave metric via the Penrose limit process [255]. This means that they can be used as analogues to study the physics of systems where the exact metric is unknown. For example, a pp-wave metric (5.1.7) with appropriately localised singularities in $H(u, \mathbf{x})$ is analogous to a multi-black hole spacetime via an Aichelburg-Sexl boost [256]. By this association, the (a)causality of an EFT operator on a pp-wave metric is actually reflective of its (a)causality in a much broader class of situations.

The pp-wave metric is an exact solution not only in GR but also to all orders in the EFT of gravity due to (5.1.17) [254]. Remarkably this is true even for the Aichelburg-Sexl shockwaves despite their delta function singularity. Since all irrelevant operators vanish, there is naïvely no constraint on the amplitude or shape of a pp-wave in order to be consistently treated within a given EFT.

However, this is no longer true as soon as the pp-wave metric is perturbed. A wave of arbi-

trarily small amplitude travelling in the opposite direction can potentially lead to an arbitrarily large curvature once it hits the wavefront of the shockwave, at which point it can no longer be described within the EFT. This fact is enough to constrain not only the energy of gravitational perturbations, but also the energy of the background pp-wave metric on which they propagate, in a similar fashion to [1]. In this section, we will describe some properties of the background pp-wave metric before introducing perturbations and their dynamics. These can then be used to ascertain the constraints from requiring that a given EFT is under control, which in turn will impose restrictions on the form of the pp-wave metric. These constraints will play a crucial role in our later discussion of scattering time delays.

5.2.1 Properties and examples of the pp-wave metrics

In what follows we shall exclusively be concerned with pp-waves of the form (5.1.7), with Riemann tensor given by (5.1.12).

Up to symmetry, the only non-zero component of the Riemann tensor associated to the pp-wave metric are

$$R_{uiuj} = -\frac{1}{2}\partial_i\partial_j H. \tag{5.2.1}$$

It follows that the only non-zero component of the Ricci tensor is $R_{uu} = -\frac{1}{2}\nabla_d^2 H$ and the Ricci scalar automatically vanishes because the inverse metric component g^{uu} is zero. The metric (5.1.7) is thus a solution to the vacuum Einstein equations if H is harmonic in transverse space $\nabla_d^2 H(u,\mathbf{x}) = 0$. More generally, we will be interested in spacetimes sourced by point particles (analogous to black holes) for which the Laplacian of H is non-zero only at a discrete set of points $\{\mathbf{b}_i\}$:

$$-\nabla_d^2 H(u,\mathbf{x}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\sum_i f_i(u)\delta^{(d)}(\mathbf{x}-\mathbf{b}_i)\,, \tag{5.2.2}$$

the solution to which is

$$H(u,\mathbf{x}) = \sum_i \frac{f_i(u)}{|\mathbf{x}-\mathbf{b}_i|^{d-2}}\,. \tag{5.2.3}$$

The functions of the u-coordinate $f_i(u)$ must be positive for such a source to satisfy the null energy condition, but are otherwise unspecified at this time. Example functions which will be useful in what follows are given in Fig. 5.1. In addition to the different configurations of shockwaves we will consider — situations which are captured through different choices of functions $f(u)$ — we also have the freedom to choose different sets of singular points $\{\mathbf{b}_i\}$. Throughout this paper, we will consider two simple cases:

1. **Point Sources:** A single or set of point sources located at the origin with spherical symmetry in the transverse directions,

$$H_{\text{pt}}(u,r) = \frac{f(u)}{r^{d-2}}\,, \tag{5.2.4}$$

   which will be discussed in detail in Section 5.3, and

2. **Balancing Case:** Two equal (sets of) sources located at $\pm\mathbf{b}$, which we shall refer to as the

"balancing" case

$$H_{\text{bal}}(u,\mathbf{x}) = f(u)\left(\frac{1}{|\mathbf{x}-\mathbf{b}|^{d-2}} + \frac{1}{|\mathbf{x}+\mathbf{b}|^{d-2}}\right) \tag{5.2.5}$$

to be considered in Section 5.4.

For the most part, the arguments in this paper apply for arbitrary functions $f(u)$. There are however some specific examples we will refer to for illustrative purposes:

a) The single "**Shockwave**",

$$f_{\text{sw}}(u) = \frac{4\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}} G P_u \delta(u). \tag{5.2.6}$$

As we discussed in 5.1.2, this corresponds to a point particle moving ultra-relativistically in the v-direction with momentum P_u, and features prominently in the discussion on causality in [174].

b) The "**Sequence of shockwaves**",

$$f_{\text{ssw}}(u) = \sum_{n=0}^{N-1} f_{\text{sw}}(u-u_n), \tag{5.2.7}$$

corresponds to series of shockwaves which occur one after another in the u-direction. This is the situation one would typically refer to as "stacking" a multitude of shockwaves, carefully placed one after the other in the "hope" of accumulating time delay/advance.

c) A **constant** $f(u) = f$, which can be seen as the limit of f_{ssw} as the number of shocks grows very large $N \to \infty$ while the u-distance between them goes to zero $u_n - u_{n+1} \to 0$. In terms of optimising the amount of time delay/advance, this situation is typically more optimal than the previous one, but can easily be considered within the same formalism.

5.2.2 Metric perturbations and their dynamics

We will now introduce metric perturbations $h_{\alpha\beta}$ on top of background pp-waves. It is their dynamics which will determine the causal properties of the GB-operator in sections 5.3 – 5.5. Perturbations spoil the exactness of the pp-wave metric to any order in the EFT. We will soon see how this gives rise to constraints on both the energy of the perturbations and the profile of the background function $H(u,\mathbf{x})$, captured by the function $f(u)$. This will play an important role for causality.

The field equations for the metric perturbations $h_{\alpha\beta}$ in this EFT are given by

$$\frac{1}{2}\delta\mathcal{G}_{\alpha\beta} + \frac{c_{\text{GB}}}{\Lambda^2}\delta B_{\alpha\beta} = 0. \tag{5.2.8}$$

The full set of equations is given in appendix 5.D. In lightcone gauge,

$$h_{v\alpha} = 0 \quad \forall\, \alpha, \tag{5.2.9}$$

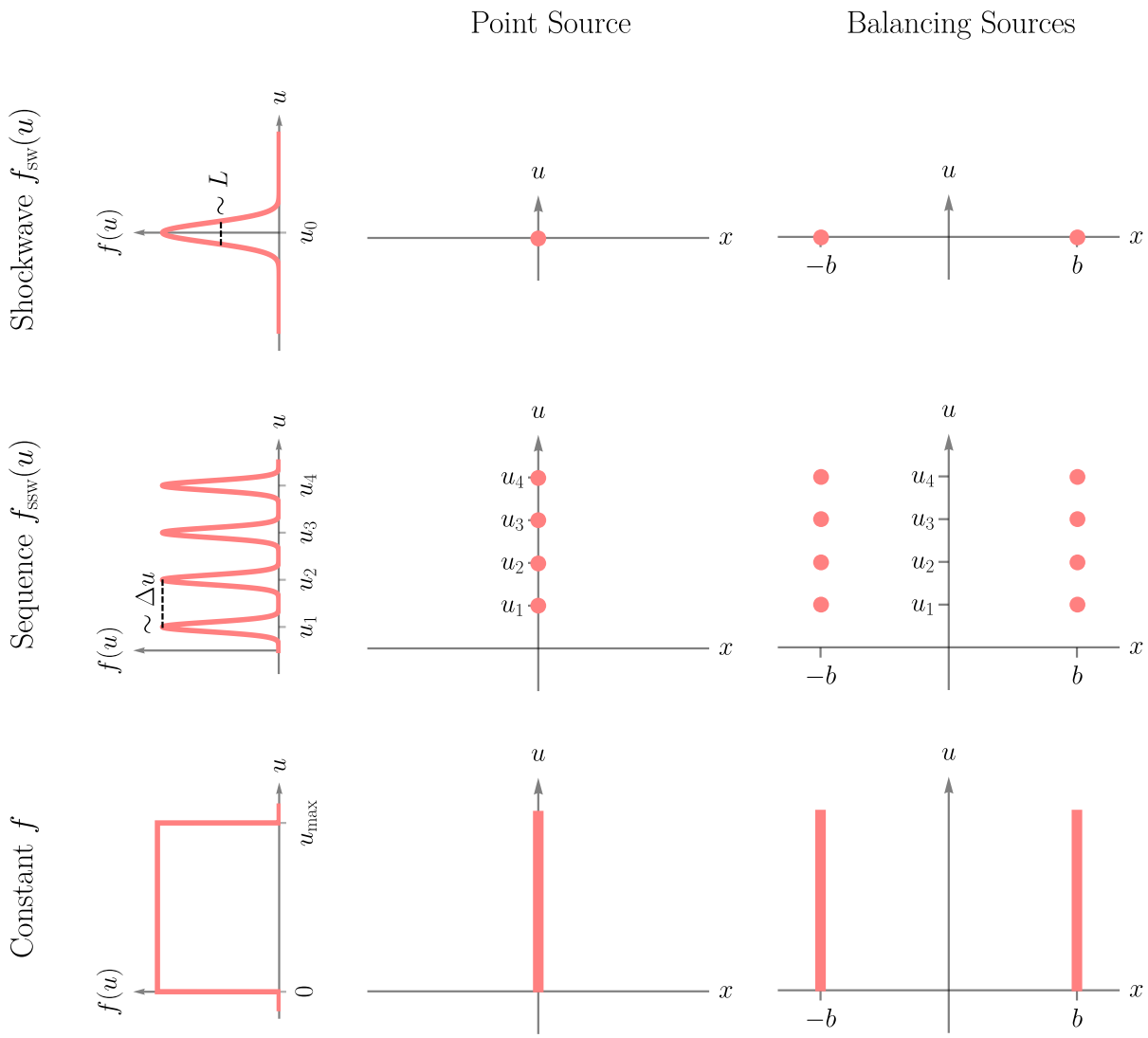


Figure 5.1: An illustration of the pp-wave sources referred to in the text. Each dot represents a Dirac delta function in the u-direction.

the (v,α)-equations reduce to two constraint equations

$$h^i{}_i = 0, \tag{5.2.10}$$
$$\partial_v h_{iu} + \partial^j h_{ij} = 0. \tag{5.2.11}$$

Taking the trace of the (i,j)-equation and applying the above two constraints produces a third constraint,

$$\partial_v h_{uu} + \partial^i h_{iu} = -4\frac{d-2}{d}\frac{c_{\rm GB}}{\Lambda^2}\partial^i\partial^j H \partial_v h_{ij}, \tag{5.2.12}$$

so that the h_{uu} and h_{ui} components are all specified in terms of h_{ij} as long as $\partial_v h_{\alpha\beta} \neq 0$. This leaves only the $D(D-3)/2$ components of the traceless h_{ij} as the dynamical degrees of freedom. Their equation of motion is given by

$$\begin{aligned} &\tilde{\Box} h_{ij} - 8\frac{c_{\rm GB}}{\Lambda^2}\partial_v^2 X_{ij} = 0, \\ &X_{ij} = \frac{1}{2}\left(h_{ik}\partial_j\partial^k H + h_{jk}\partial_i\partial^k H\right) - \frac{1}{d}\delta_{ij}h^{kl}\partial_k\partial_l H\,. \end{aligned} \tag{5.2.13}$$

Unlike the background solution, the equation for the perturbations is modified by the GB-operator in the action. In fact, it will generically receive corrections from any effective gravitational operator. Demanding that these corrections are under control will define the regime of validity of the EFT in the next section.

The decoupled "master variables" are linear combinations of the h_{ij} depending on the form of $H(u,\mathbf{x})$. In the case of the point source (5.2.4), spherical symmetry means the equations (5.2.13) are all immediately decoupled when expressed in spherical coordinates. The single caveat is that not all of the diagonal components of h_{ij} are independent because of the tracelessness condition (5.2.10). Choosing h_{DD} as the dependent component, the master variables are

$$\Phi \in \{h_{rr}, h_{r\bar{i}}, h_{\bar{i}\bar{j}} \text{ for } \bar{i}\neq\bar{j}, h_{\bar{i}\bar{i}} - h_{DD} \text{ (no sum)}\}, \tag{5.2.14}$$

where $\{\bar{i},\bar{j},\dots\}$ label the angular directions. Their master equations are

$$\tilde{\Box}\Phi + A\frac{c_{\rm GB}}{\Lambda^2}\frac{\partial_r H}{r}\partial_v^2\Phi = 0, \tag{5.2.15}$$

where A is an integer depending on the master variable under consideration

$$A \in \{8(d-2), 4(d-2), -8, -8\}\,.$$

In the balancing source case (5.2.5), it is less straightforward to identify the master variables. Some details are given in appendix 5.E but are not crucial for what follows. Suffice to say, the master equations will take a similar form to (5.2.15).

5.2.3 The EFT regime of validity

Once again, the GB-term in the EGB effective theory (1.3.8) under consideration is just the lowest-order term in an infinite series of effective operators (1.2.1). Generically, any number

of these terms could be present when UV degrees of freedom are integrated out to obtain a low-energy action. The full EFT action is an expansion in powers of the spacetime curvature ("Riemann") and its derivatives, compared to the EFT cut-off energy Λ. The expansion is only under control in regions where that ratio is small [1, 215]. In other words, the EFT is only valid in the regime where

$$\left(\frac{\nabla}{\Lambda}\right)^m \left(\frac{\text{Riemann}}{\Lambda^2}\right)^n \ll 1. \tag{5.2.16}$$

What is meant by this schematic expression on the left-hand side is all possible scalar contractions built out of m covariant derivatives ∇ and n powers of the Riemann tensor. The background pp-wave metric seemed to evade validity constraints by exactly solving the EFT to any order, so that the left-hand side of (5.2.16) simply vanishes. However, in the perturbed spacetime, the "Riemann" tensor in (5.2.16) refers to the sum of the background Riemann tensor and its perturbation. The perturbed Riemann tensor has a much richer structure than the background Riemann tensor, with none of its non-trivial components vanishing. Hence there is a non-trivial EFT regime of validity for the pp-wave spacetime as soon as metric perturbations are allowed to propagate.

The perturbations to the Riemann tensor are given by second derivatives of the metric perturbations $h_{\alpha\beta}$. Considering perturbations of momentum k^α, we may represent the contribution of the derivatives acting on the perturbations by powers of k^α. To establish the regime of validity, it is then sufficient to consider all possible scalar invariants of the form

$$\left(\frac{\nabla}{\Lambda}\right)^m \left(\frac{\bar{R}}{\Lambda^2}\right)^n \left(\frac{k}{\Lambda}\right)^p \ll 1\,, \tag{5.2.17}$$

where $\bar{R}$ represents the background Riemann tensor (5.1.12) and ∇ are covariant derivatives acting on this. There are a number of contractions which would render the left-hand side of (5.2.17) trivially zero, or lead to a much weaker constraint.

Firstly, contracting two momentum vectors together will lead to a weaker bound than if they had been contracted with other components. The reason for this is that the gravitational wave momentum is a null vector in GR, so its norm vanishes $k_\alpha k^\alpha \approx 0$ up to EFT corrections. Secondly, any contractions of indices on the same Riemann tensor produces a Ricci tensor (or scalar) which is zero in vacuum. Therefore, each of the four indices on a Riemann tensor need to be contracted with either a momentum vector k_α, a covariant derivative ∇_α or a different Riemann tensor. Also, it can be checked that the divergence of the Riemann tensor vanishes in vacuum

$$\nabla^\alpha \bar{R}_{\alpha\beta\gamma\delta} = 0 \quad \text{when} \quad \nabla_d^2 H = 0. \tag{5.2.18}$$

Finally, by symmetry, the Riemann tensor may only enter via contractions with at most two k's, at most two ∇'s and at most two other Riemann tensors.

The expressions in (5.2.21) represent the only cases where no covariant derivatives have been contracted with an index on a Riemann tensor. The remaining options are to contract one

covariant derivative with a Riemann tensor index,

$$(\bar{R}\cdot\nabla)^{2p}k^{4p}\ll\Lambda^{10p}, \tag{5.2.19}$$

or two covariant derivatives with Riemann tensor indices,

$$(\bar{R}\cdot\nabla\nabla)^{2q}k^{4q}\ll\Lambda^{12q}. \tag{5.2.20}$$

Both cases produce weaker bounds, which follow from bounds coming from contractions of the form

$$(k\cdot\nabla)^m(k\cdot\bar{R}\cdot k)^{2n}\ll\Lambda^{2m+8n}, \tag{5.2.21a}$$
$$(\nabla\cdot\nabla)^m(k\cdot\bar{R}\cdot k)^{2n}\ll\Lambda^{2m+8n}. \tag{5.2.21b}$$

A non-trivial example of (5.2.21a) with $m=n=1$, written in terms of components, is

$$k_\alpha\nabla^\alpha(k^\beta R_{\beta\gamma\delta\sigma}k^\delta k_\kappa R^{\kappa\gamma\rho\sigma}k_\rho)\ll\Lambda^{10}\,.$$

Similarly, k_α may be replaced with ∇_α to obtain an example of (5.2.21b). Taking the $n\to\infty$ limit of (5.2.21) gives us our first EFT regime of validity bound:

$$(k_v\partial_i\partial_j H k_v)^2\ll\Lambda^8\,,\qquad\text{or}\qquad\partial_i\partial_j H\ll\frac{\Lambda^4}{k_v^2}\,. \tag{5.2.22}$$

In particular, for the spherically symmetric point source, it can be written as

$$\partial_r^2 H\sim\frac{\partial_r H}{r}\ll\frac{\Lambda^4}{k_v^2}. \tag{5.2.23}$$

This represents a bound on the strength of the source at any given distance from its singularity in transverse space. Taking the $m\to\infty$ limit of (5.2.21a) gives

$$k^\alpha\nabla_\alpha\ll\Lambda^2, \tag{5.2.24}$$

where ∇ is understood to be acting on a scalar. In particular, for non-zero momentum in the v-direction k_v, we obtain a bound on the u-derivative of H:

$$\frac{\partial_u H}{H}\ll\frac{\Lambda^2}{k_v}. \tag{5.2.25}$$

Lastly, taking the $m\to\infty$ limit of (5.2.21b) gives

$$\tilde{\Box}\ll\Lambda^2, \tag{5.2.26}$$

where $\tilde{\Box}$ is understood to be acting on a scalar. Isolating the transverse Laplacian results in a distance-resolution scale (say, for the point source),

$$r\gg\Lambda^{-1}. \tag{5.2.27}$$

Below we gather the three EFT regime of validity bounds. On the left, they are written in their generic form as they would apply without specifying a background pp-wave metric. In the middle, they are specialised to the point-source metric (5.2.4). On the right-hand side, they are further specified for a particular impact parameter $\mathbf{x} = \mathbf{b} = b\,\hat{\mathbf{b}}$.

General	Point Source(s)	$\mathbf{x} = \mathbf{b} = b\hat{\mathbf{b}}$	
$\Box \ll \Lambda^2$	$r \gg \Lambda^{-1}$	$b \gg \Lambda^{-1}$	(5.2.28a)
$\partial_i \partial_j H \ll \dfrac{\Lambda^4}{k_v^2}$	$\dfrac{\partial_r H}{r} \ll \dfrac{\Lambda^4}{k_v^2}$	$\dfrac{f(u)}{b^d} \ll \dfrac{\Lambda^4}{k_v^2}$	(5.2.28b)
$k^\alpha \nabla_\alpha \ll \Lambda^2$	$\dfrac{\partial_u H}{H} \ll \dfrac{\Lambda^2}{k_v}$	$\dfrac{f'(u)}{f(u)} \ll \dfrac{\Lambda^2}{k_v}$	(5.2.28c)

The top line (5.2.28a) is relatively intuitive and represents the lower limit on the distance scales we may probe before exiting the regime of the validity of the EFT. As mentioned previously, the middle line (5.2.28b) represents a bound on the strength of the pp-wave source at a given distance from the source. Here the bound on $f(u)$ should be understood not in a point-wise sense but rather in an averaged sense,

$$\int_{u-\lambda_u}^{u+\lambda_u} |f(\tilde{u})|^p \mathrm{d}\tilde{u} \ll \left(\frac{\Lambda^4 b^d}{k_v^2}\right)^p \lambda_u \,, \tag{5.2.29}$$

for any positive power p and where $\lambda_u = P_u^{-1}$ is the wavelength integrated over in the u-direction. The last line (5.2.28c) bounds how quickly $f(u)$ may vary in u-time and should also be understood in its averaged sense,

$$\int_{u-\lambda_u}^{u+\lambda_u} |f'(\tilde{u})|^p \mathrm{d}\tilde{u} \ll \frac{\Lambda^{2p}}{k_v^p} \int_{u-\lambda_u}^{u+\lambda_u} |f(\tilde{u})|^p \mathrm{d}\tilde{u} \,. \tag{5.2.30}$$

The latter two bounds depend not only on the EFT cut-off Λ, but also on the energy k_v of the gravitational wave probing the spacetime. This underlines the role that metric perturbations play in establishing an EFT regime of validity on a pp-wave spacetime.

Notably, shockwaves (or sequences therefore) lead to divergences in the profile $f(u)$ and its derivative, and are hence object that cannot be probed within the regime of validity of the EFT without smoothing out as we discuss will shortly.

Regime of validity by explicit example

The above arguments to establish a regime of validity rely only on generic assumptions about the EFT series expansion, but were largely schematic. In this section, we will consider an explicit example of a truncated EFT expansion, reminiscent of the dimension-8 operator in chapter 4. Demanding that the higher-dimension operators in the expansion are subdominant to the GB-operator will concretely reproduce the pp-wave regime of validity given in (5.2.28).

Consider the following effective action which contains (1.3.8) as its leading-order part supple-

mented by generic dimension-6 and -8 curvature operators:

$$S_{\rm EFT} = \int \mathrm{d}^D x\sqrt{-g}\, M_{\rm Pl}^{D-2}\Bigg(\frac{1}{2}R + \frac{c_{\rm GB}}{\Lambda^2}R^2_{\rm GB} + \frac{c_{\rm R3}}{\Lambda^4}R_{\alpha\beta\gamma\delta}R^{\gamma\delta\sigma\rho}R_{\sigma\rho}{}^{\alpha\beta} + \frac{c_{\rm R4}}{\Lambda^6}R_{\alpha\beta\gamma\delta}R^{\gamma\delta\sigma\rho}R_{\sigma\rho\kappa\lambda}R^{\kappa\lambda\alpha\beta} + \dots\Bigg)\,. \tag{5.2.31}$$

The full field equations for the background and perturbations are provided in appendix 5.D. The pp-wave metric (5.1.7) of course remains an exact (background) solution to (5.2.31). Choosing lightcone gauge as before, the traceless components of the metric perturbations in the transverse directions h_{ij} remain the dynamical degrees of freedom. With the addition of the new operators, it is less straightforward to decouple the equations of motion and identify the master variables. In the case of the spherically symmetric point source, one can perform a scalar-vector-tensor (SVT) decomposition on the transverse space with the result that the tensor modes Φ_T are immediately decoupled. For our purposes, it is enough just to consider their master equation up to corrections of $\mathcal{O}(\Lambda^{-8})$:

$$0 = \tilde{\Box}\Phi_T - \frac{\kappa_T^2+2}{r^2}\Phi_T - 8\frac{c_{\rm GB}}{\Lambda^2}\frac{\partial_r H}{r}\partial_v^2\Phi_T + 24\frac{c_{\rm R3}}{\Lambda^4}\partial_v^2\Bigg[d\frac{\partial_r H}{r^2}\left(\partial_r\Phi_T + \frac{\Phi_T}{r}\right) - \frac{\partial_u\partial_r H}{r}\partial_v\Phi_T\Bigg] + 16\frac{c_{\rm R4}-12c_{\rm GB}c_{\rm R3}}{\Lambda^6}\left(\frac{\partial_r H}{r}\right)^2\partial_v^4\Phi_T\,, \tag{5.2.32}$$

where κ_T^2 is the eigenvalue of the tensor spherical harmonic on the $(d-1)$-sphere. This equation can be expressed in v-momentum space by replacing $\partial_v \to ik_v$. For the GB-operator to truly be the leading-order term in the EFT expansion, the contributions of the higher-order operators to this equation of motion ($\propto c_{\rm R3}, c_{\rm R4}$), specifically to the potential, should be subdominant to the contribution from the GB-operator itself ($\propto c_{\rm GB}$). Assuming that the Wilson coefficients $c_{\rm R3}$ and $c_{\rm R4}$ are $\mathcal{O}(c_{\rm GB})$, then control of the EFT amounts to:

$$24d\frac{c_{\rm R3}}{\Lambda^4}\frac{\partial_r H}{r^3}\partial_v^2\Phi_T \ll 8\frac{c_{\rm GB}}{\Lambda^2}\frac{\partial_r H}{r}\partial_v^2\Phi_T \quad\Longrightarrow\quad r \ll \Lambda^{-1}\,, \tag{5.2.33}$$

$$16\frac{c_{\rm R4}}{\Lambda^6}\left(\frac{\partial_r H}{r}\right)^2\partial_v^4\Phi_T \ll 8\frac{c_{\rm GB}}{\Lambda^2}\frac{\partial_r H}{r}\partial_v^2\Phi_T \quad\Longrightarrow\quad \frac{\partial_r H}{r} \ll \frac{\Lambda^4}{k_v^2}\,, \tag{5.2.34}$$

$$24\frac{c_{\rm R3}}{\Lambda^4}\frac{\partial_u\partial_r H}{r}\partial_v^3\Phi_T \ll 8\frac{c_{\rm GB}}{\Lambda^2}\frac{\partial_r H}{r}\partial_v^2\Phi_T \quad\Longrightarrow\quad \frac{\partial_u H}{H} \ll \frac{\Lambda^2}{k_v}\,, \tag{5.2.35}$$

which is exactly the regime of validity of the EFT (5.2.28) obtained in the previous section. Note, it was not strictly necessary to include the dimension-8 operator $\sim$ Riemann^4 since the subleading correction from the dimension-6 operator produced the same term. Its inclusion illustrates that there is nothing particular about our choice of higher-dimension operators in reproducing the regime of validity. There are many possible higher-dimension operators whose presence in the EFT of gravity would constrain the parameters of the spacetime in the same way.

Regulating the shockwave in the EFT of gravity

The main result from the previous two sections is that any calculation performed within the effective theory of gravity should only be trusted within the regime in which the set of bounds (5.2.28) are satisfied. The pp-wave solutions provided in section 5.2.1 need to be revisited in light of this. In particular, the singular shockwave solution $f_{\text{sw}}(u)$ (5.2.6) is clearly against the spirit of the bound (5.2.28c), or rather its average (5.2.30), on $f'(u)/f(u)$. To bring it within the remit of the EFT, the shockwave may be regulated by expressing it as a Gaussian,

$$f_{\text{sw}}^{\text{reg}}(u) = \frac{\alpha}{\sqrt{2\pi L^2}} \exp\left[-\frac{u^2}{2L^2}\right] , \tag{5.2.36}$$

where $\alpha = 4\Gamma((d-2)/2)/\pi^{(d-2)/2}\, GP_u$. Interpreting (5.2.28c) in an averaged sense, as indicated in (5.2.30), we then infer a bound on the width of the pp-wave that can be considered

$$L \gg \frac{k_v}{\Lambda^2} . \tag{5.2.37}$$

This means that the more peaked the source (smaller L), the "slower" the GW needs to be (smaller k_v) in order to probe the spacetime within the regime of the validity of the EFT. To be clear, this does not mean that k_v/Λ^2 is the width of the wave in the UV completion. We should interpret this in the following sense: When we construct a low-energy EFT in the Euclidean we naturally coarse-grain over modes with momenta larger than $\sqrt{k^2} \sim \Lambda$, so that no correlation function can resolve distances smaller than Λ^{-1}. In the Wilsonian picture of renormalisation, this scale is our choosing and need not correspond to the actual scale of new physics. The arguments of the previous section similarly imply that we cannot resolve distances in u smaller than k_v/Λ^2. It could well be that in the UV completion, the shockwave is actually more localised/peaked than this (*i.e.* has a width shorter than k_v/Λ^2). However, it cannot be resolved better than $k_v\Lambda^2$ and will always be perceived as having a width of at least k_v/Λ^2 to a low-energy observer.

The sequence of shockwaves may be regulated in an analogous way as $f_{\text{ssw}}^{\text{reg}} = \sum_i f_{\text{sw}}^{\text{reg}}(u - u_i)$. Since each regulated shock now has a finite spread $\sim L$ and height $\sim \alpha L^{-1}$, they become indistinguishable as the distance between them shrinks and approach a constant source $f(u) = f \sim \alpha L^{-1}$. Conversely, if they are to remain distinguishable, there must be a minimum u-separation between them $\Delta u \sim u_{i+1} - u_i \gg L$.

5.3 The eikonal approximation

As outlined before, there are two physical effects which limit the maximum time delay/advance that can be built up in a given experiment: (i) Quantum diffusion and (ii) Scattering. In this section we shall consider the maximum time delay/advance that can be obtained by a sequence (or continuum) of shockwaves, each of which may be viewed as a point source in the transverse direction. In this case it will turn out that the main limitation on an accumulation of time delay is from (ii) the scattering, and so we shall neglect the quantum diffusion. The eikonal approximation is well-equipped to deal with that.

5.3.1 Warm-up: Scattering due to a single point source

Before proceeding to the case of multiple (or a continuum of) shockwaves, let us consider the simplest scenario of a particle scattering off a single point shockwave. An idealised shockwave describing a boosted black hole/particle of null momentum $P_u^{(2)}$ takes the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\,\mathrm{d}v + \alpha\delta(u)\mathrm{d}u^2 + \mathrm{d}\mathbf{x}^2\,, \tag{5.3.1}$$

with $\alpha = 4\Gamma((d-2)/2)/\pi^{(d-2)/2}\,GP_u^{(2)}$. As discussed in Section 5.2.3, this solution is singular from the point of view of the EFT, and to be treated within the EFT it must be smoothed out over a null distance $L \sim k_v/\Lambda^2$ to, for example,

$$\mathrm{d}s^2 = 2\mathrm{d}u\,\mathrm{d}v + \alpha\frac{1}{\sqrt{2\pi L^2}}\exp\left[-\frac{u^2}{2L^2}\right]\mathrm{d}u^2 + \mathrm{d}\mathbf{x}^2\,. \tag{5.3.2}$$

If L is taken to be too large then we need to account for the evolution of the trajectory through the Gaussian which we do in the next section. For now we choose it to be the smallest possible consistent with the EFT. Since the potential lasts only for a short u-time, the eikonal approximation is good, and including the EFT correction, this is

$$\Delta v = \frac{\alpha}{2b^{d-2}} - A(d-2)\frac{c_{\mathrm{GB}}}{\Lambda^2}\frac{\alpha}{2b^d} \tag{5.3.3}$$

to a good approximation. It is easy to see that the GR/Shapiro contribution can be made arbitrarily large and resolvable without contradicting the validity of the EFT. This is of course as it should be, since the GR effects are experimentally observable. By contrast, in this situation, the EFT contribution to the time delay is bounded. The EFT bound on the Riemann curvature applied at the impact parameter amounts (5.2.28b) to

$$\frac{k_v^2\alpha}{Lb^d} = \frac{\Lambda^2 k_v^2\alpha}{k_v b^d} \ll \Lambda^4\,, \tag{5.3.4}$$

which simplifies to

$$\frac{k_v\alpha}{b^d} \ll \Lambda^2\,, \tag{5.3.5}$$

from which we see

$$k_v|\Delta v_{\mathrm{EFT}}| \lesssim |c_{\mathrm{GB}}|\,. \tag{5.3.6}$$

Thus, the EFT contribution to the time delay, meaning the contribution from the GB-term, is only resolvable if $|c_{\mathrm{GB}}|$ is taken to be larger than unity. This is outside of the expectations of the EFT, and more precisely the expectations of positivity/bootstrap bounds. The same argument does not apply to the GR/Shapiro contribution because it is larger by a factor of $(\Lambda b)^2$ and there is no constraint on how large b can be.

5.3.2 Scattering due to a set of point sources

We now want to extend the argument in the previous section to multiple shockwaves or extended (not localised) pp-wave configurations, including a continuum. For now, we continue to

consider just the point source (5.2.4) with arbitrary u-profile for the function $f(u)$ and hence H. This captures the main physics at play except for the possibility of an equilibrium point in the potential, which will be addressed in section 5.4. The point source potential is

$$V(u,r) = -\frac{k_v}{2}H(u,r) + Ak_v\frac{c_{\text{GB}}}{\Lambda^2}\frac{\partial_r H(u,r)}{r}. \tag{5.3.7}$$

The eikonal time delay, with diffusion and scattering neglected, accumulated up until time u is

$$\begin{aligned}(\Delta v)_{\text{eikonal}}(u) &= -\frac{\partial}{\partial k_v}\int_0^u \mathrm{d}u' V(u',\mathbf{b}) \\ &= \left(\frac{1}{2}\int_0^u \mathrm{d}u' H(u',r) - A\frac{c_{\text{GB}}}{\Lambda^2}\int_0^u \mathrm{d}u'\frac{\partial_r H(u',r)}{r}\right)\Big|_{r=b}.\end{aligned} \tag{5.3.8}$$

If we consider N copies of the previous shockwave solution, stacked together and sufficiently spaced in null time to be distinguishable, it would at first sight appear that we can generate a contribution arbitrarily large in magnitude from the GB-term and still remain consistent with the EFT validity constraints (5.2.28). This is however not the case, because in such a stacked shockwave configuration, the approximations used to obtain (5.3.8) do not hold for all time. There are two effects that limit this: One is diffusion and the second is scattering in the transverse directions. In the present case scattering in the transverse directions is the main constraint. As such, we need only concern ourselves with the magnitude of the time delay accumulated by some time u_{max} when the eikonal approximation breaks down.

Scattering time estimate

A simple estimate for this time can be obtained by considering the classical trajectory a particle of mass k_v experiencing a force due to the potential $V(u,\mathbf{x})$ would follow:

$$k_v\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}u^2} = -\boldsymbol{\nabla}V(u,\mathbf{x}). \tag{5.3.9}$$

Since the main contribution to the potential is from the GR term, we will neglect the GB-term to get an estimate on u_{max}. The amount scattered in the transverse direction is thus approximately

$$\Delta r(u) \sim -\frac{1}{k_v}\int_0^u \mathrm{d}u'\int_0^{u'}\mathrm{d}u''\partial_r V(u'',r)\Big|_{r=b} \sim -\int_0^u \mathrm{d}u'\int_0^{u'}\mathrm{d}u''\partial_r H(u'',r)\Big|_{r=b}. \tag{5.3.10}$$

When the amount scattered becomes comparable to the impact parameter b, we can no longer trust the eikonal approximation, and so u_{max} is defined by the condition

$$\Delta r(u_{\text{max}}) \sim b \quad \Longrightarrow \quad \int_0^{u_{\text{max}}}\mathrm{d}u\int_0^u \mathrm{d}u'\frac{f(u')}{b^d} \sim \mathcal{O}(1). \tag{5.3.11}$$

The above form of the time delay (5.3.8) can only be trusted for null times less than this scattering time. Imposing this upper bound, the maximum EFT time delay generated by time u_{max} is

$$|\Delta v_{\text{EFT}}(u_{\text{max}})| \sim \frac{|c_{\text{GB}}|}{\Lambda^2}\int_0^{u_{\text{max}}}\mathrm{d}u\frac{f(u)}{b^d}, \tag{5.3.12}$$

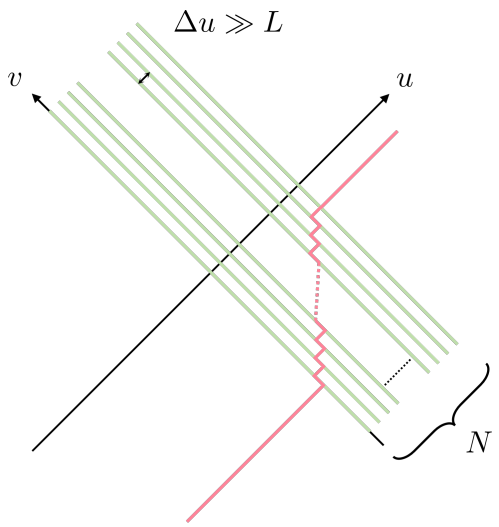


Figure 5.2: Illustration of the time delay accumulated by scattering off a pp-wave with N sources.

and making the judicious choice to express $f(u)$ as the integral of its derivative,

$$|\Delta v_{\text{EFT}}(u_{\text{max}})| \sim \frac{|c_{\text{GB}}|}{\Lambda^2} \int_0^{u_{\text{max}}} \mathrm{d}u \int_0^u \mathrm{d}u' \frac{f'(u')}{b^d} \,. \tag{5.3.13}$$

If we now apply the EFT validity bound (5.2.28c), or rather its averaged version (5.2.30), we find that

$$k_v |\Delta v_{\text{EFT}}(u_{\text{max}})| \ll |c_{\text{GB}}| \int_0^{u_{\text{max}}} \mathrm{d}u \int_0^u \mathrm{d}u' \frac{f(u')}{b^d} \ll |c_{\text{GB}}|. \tag{5.3.14}$$

That is, the EFT time delay is unresolvable for any $|c_{\text{GB}}| \lesssim \mathcal{O}(1)$. We stress that this argument not only applies to a single extended shockwave, but any number of shocks stacked together.

5.3.3 Special situation: Sequence of shockwaves

Let us run this argument in more detail for the sequence of N shockwaves, illustrated in figure 5.2 and described by the metric function

$$f_{ssw}(u) = \sum_{n=0}^{N-1} \alpha \delta(u - n\Delta u) \,. \tag{5.3.15}$$

The time delay is now N times larger than (5.3.3) and so, returning to the argument of the introduction, we may think that by making N large enough, we can make the EFT contribution resolvable. The total EFT time delay is

$$\Delta v_{\text{EFT}} \sim -NA(d-2) \frac{c_{\text{GB}}}{\Lambda^2} \frac{\alpha}{2b^d} \,. \tag{5.3.16}$$

The key observation is that the total time $N\Delta u$, over which we can trust the eikonal approximation, is bounded by the scattering time. Identifying $u_{\text{max}} = N\Delta u$, then (5.3.11) amounts to

$$N^2 \Delta u \frac{\alpha}{b^d} = u_{max}^2 \frac{\alpha}{\Delta u b^d} \sim 1 \,. \tag{5.3.17}$$

The scattering time is

$$u_{\max} = u_s = \sqrt{\frac{b^d}{f_0}}\,, \tag{5.3.18}$$

with $f_0 = \alpha/\Delta u$ being the mean value of $f(u)$ during the period of the shocks. Substituting in (5.3.16) we obtain

$$k_v|\Delta v_{\rm EFT}| \approx |c_{\rm GB}|\frac{k_v}{\Lambda^2 u_s}\,. \tag{5.3.19}$$

However, to trust the EFT, the spacing between the shocks must satisfy (5.2.28c) and (5.2.28b) both of which amount to

$$u_s \gg \frac{k_v}{\Lambda^2}\,, \tag{5.3.20}$$

which is just the statement that we cannot meaningfully consider null times smaller than the EFT cutoff k_v/Λ^2. Putting this together we obtain

$$k_v|\Delta v_{\rm EFT}| \ll |c_{\rm GB}|\,, \tag{5.3.21}$$

regardless of N. Thus, the superficial gain from multiple shocks of the same amplitude is more than compensated by the increased scattering they induce. We may, of course, think to compensate this by repositioning the location of each source to track the scattered state. A configuration which effectively does this will be considered in Section 5.4.

Breakdown of eikonal

For a more sophisticated argument, we can account for the leading corrections to the eikonal approximation. This may be done by directly computing the corrections in (3.4.80) which depart from (3.4.97). A more immediate way is to follow [174] and write the wavefunction in JWKB form

$$\psi(u,\mathbf{x}) = \psi_0 \exp\left[2i\delta(u,\mathbf{x})\right]\,, \tag{5.3.22}$$

for which the JWKB phase must obey

$$-2\frac{\partial\delta\left(u,\mathbf{x}\right)}{\partial u} = \frac{2}{k_v}\left|\boldsymbol{\nabla}\delta\left(u,\mathbf{x}\right)\right|^2 - \frac{i}{k_v}\nabla_d^2\delta\left(u,\mathbf{x}\right) + V\left(u,\mathbf{x}\right). \tag{5.3.23}$$

The leading-order solution $\delta^{(0)}\left(u,\mathbf{x}\right)$, obtained by neglecting gradient terms, is the eikonal expression already given by (3.4.99). The first correction $\delta^{(1)}\left(u,\mathbf{x}\right)$ is obtained by substituting the leading term back into (5.3.23),

$$-2\frac{\partial\delta^{(1)}\left(u,\mathbf{x}\right)}{\partial u} = \frac{2}{k_v}\left|\boldsymbol{\nabla}\delta^{(0)}\left(u,\mathbf{x}\right)\right|^2 - \frac{i}{k_v}\nabla_d^2\delta^{(0)}\left(u,\mathbf{x}\right). \tag{5.3.24}$$

Given that V is harmonic, the solution is simply

$$\delta^{(1)}\left(u,\mathbf{x}\right) = -\frac{1}{4k_v}\int_0^u \mathrm{d}u'\left|\int_0^{u'}\mathrm{d}u''\boldsymbol{\nabla}V\left(u'',\mathbf{x}\right)\right|^2\,, \tag{5.3.25}$$

and so the leading correction to the eikonal time delay is

$$(\Delta v)^{(1)}_{\text{eikonal}}(u) = -\frac{\partial}{\partial k_v}\left(\frac{1}{2k_v}\int_0^u \mathrm{d}u' \left|\int_0^{u'} \mathrm{d}u'' \boldsymbol{\nabla} V\left(u'', \mathbf{b}\right)\right|^2\right) . \tag{5.3.26}$$

This result can be obtained directly from (3.4.97), or more precisely (3.4.80), by accounting for the leading correction to the straight line trajectory. When $(\Delta v)^{(1)}_{\text{eikonal}}(u)$ becomes comparable to $(\Delta v)^{(0)}_{\text{eikonal}}(u)$, we can clearly no longer trust the leading-order eikonal result.

In a situation with repeated shocks, the integral in (5.3.25) continues to grow, meaning that there is inevitably some time u_{max} at which the eikonal approximation breaks down. At the level of the phase equation (5.3.23), we may ask when their partial-u derivatives become comparable, *i.e.*

$$\frac{1}{2k_v}\left|\int_0^{u_{\text{max}}} \mathrm{d}u \boldsymbol{\nabla} V\left(u, \mathbf{b}\right)\right|^2 \sim V\left(u_{\text{max}}, \mathbf{b}\right) . \tag{5.3.27}$$

For our spherically symmetric pp-wave this condition for u_{max} is

$$\int_0^{u_{\text{max}}} \mathrm{d}u \frac{f(u)}{b^d} \sim \sqrt{\frac{f(u_{\text{max}})}{b^d}}, \tag{5.3.28}$$

see eq. (5.3.11). The left-hand side of (5.3.28) can be directly replaced by the right-hand side in the expression for the EFT time delay (5.3.12) to give

$$|\Delta v_{\text{EFT}}(u_{\text{max}})| \sim \frac{|c_{\text{GB}}|}{\Lambda^2}\sqrt{\frac{f(u_{\text{max}})}{b^d}}. \tag{5.3.29}$$

We can now deploy a second EFT validity bound (5.2.28b), in the form

$$\sqrt{\frac{f(u)}{b^d}} \ll \frac{\Lambda^2}{k_v} , \tag{5.3.30}$$

to arrive again at the key result:

$$k_v |\Delta v_{\text{EFT}}(u_{\text{max}})| \ll |c_{\text{GB}}| . \tag{5.3.31}$$

We thus see that, provided $|c_{\text{GB}}| \lesssim \mathcal{O}(1)$, the EFT contribution to the time delay computed in the eikonal regime is necessarily unresolvable.

Neither of the above arguments required a specific choice for the time-dependence of the background spacetime, or in particular $f(u)$.

5.4 Scattering bypass by balancing shockwaves

It should now be clear that the main obstruction to generating an observably large EFT time advance within the regime of validity of the EFT is scattering in the transverse directions. This scattering is the result of an attractive gravitational potential pulling waves propagating in the v-direction towards the source.

Naïvely, it may seem like this fate could be avoided by engineering an equilibrium point in the potential at which the v-moving-waves could sit without deviation. However, point-particles should be understood as limits of highly, but crucially not perfectly, localised wavepackets — these do feel the gravitational pull towards the sources.

5.4.1 Surfing on an extremum

In order to see why it is not possible to indefinitely accumulate time delay even when the potential possesses an equilibrium point, we need to understand this set-up within the semi-classical approximation, as discussed in section 3.4.4.

The prototypical example of such a configuration with an (unstable) equilibrium is associated to the balanced metric function (5.2.5), which sets up a symmetric potential sourced by two pp-waves located at $\mathbf{x} = \pm\mathbf{b}$. The GR-term in the potential is

$$V(u,\mathbf{x}) = -\frac{k_v f(u)}{2}\left(\frac{1}{|\mathbf{x}-\mathbf{b}|^{d-2}} + \frac{1}{|\mathbf{x}+\mathbf{b}|^{d-2}}\right). \tag{5.4.1}$$

We refer to this scenario as "balancing" pp-waves, because there is an unstable equilibrium at $\mathbf{x} = \mathbf{0}$ at which the forces are perfectly balanced $\boldsymbol{\nabla} V = 0$ (see also [176]). An ideal point-particle, perfectly localised at the origin in transverse space, could in principle move in the v-direction without fear of scattering into a gravitational well.

At first glance, since a classical trajectory initially at the equilibrium can remain there indefinitely, it would seem reasonable to suppose that the only cutoff on the time delay is the quantum diffusion which occurs by $u_{\text{diff}} = k_v b^2$. Indeed, if this were the case, it is easy to see that it would be possible to generate a resolvable EFT time delay within the regime of validity of the EFT

$$k_v|\Delta v_{\text{EFT}}| \sim \frac{k_v}{\Lambda^2 b^d} f \times u_{\text{diff}} \sim (\Lambda b)^2 \frac{k_v^2 f}{\Lambda^4 b^d} \ll (\Lambda b)^2 . \tag{5.4.2}$$

However, with the potential being unstable along the $\mathbf{b}$-direction, any small perturbation will lead to an instability on a timescale associated to the second derivative of the potential in that direction. More precisely, if we consider a wavepacket initially localised near the equilibrium point $\mathbf{x} = \mathbf{0}$, then the potential may be approximated as a time-dependent unstable harmonic oscillator

$$\begin{aligned} V(u,\mathbf{x}) &\approx -\frac{k_v f(u)}{b^{d-2}} + \frac{1}{2}\frac{k_v f(u)}{b^d}(d-2)\left(\mathbf{x}^2 - d\frac{(\mathbf{x}\cdot\mathbf{b})^{\mathbf{2}}}{b^2}\right) \\ &\approx -\frac{k_v f(u)}{b^{d-2}} + \frac{1}{2}\frac{k_v f(u)}{b^d}(d-2)\left(\mathbf{x}_\perp^2 - (d-1)\mathbf{x}_\parallel^2\right) , \end{aligned} \tag{5.4.3}$$

where we split the position vector in the orthogonal and parallel directions $\mathbf{x} = \mathbf{x}_\perp + \mathbf{x}_\parallel$, with $\mathbf{x}_\perp \cdot \mathbf{b} = \mathbf{x}_\perp \cdot \mathbf{x}_\parallel = 0$. In order to determine the semi-classical time delay, we need to solve the classical equations of motion for arbitrary initial data. These are unstable in the direction parallel to $\mathbf{b}$ and stable in the direction perpendicular, and take the form

$$\frac{\mathrm{d}^2\mathbf{x}_\parallel}{\mathrm{d}u^2} = \Omega_\parallel^2(u)\mathbf{x}_\parallel , \quad \frac{\mathrm{d}^2\mathbf{x}_\perp}{\mathrm{d}u^2} = -\omega_\perp^2(u)\mathbf{x}_\perp , \tag{5.4.4}$$

with

$$\Omega_{\parallel}^2(u) = \frac{f(u)}{b^d}(d-2)(d-1)\,, \quad \omega_{\perp}^2(u) = \frac{f(u)}{b^d}(d-2)\,. \tag{5.4.5}$$

Since $\mathbf{x}_{\parallel} = x_{\parallel}\mathbf{b}/b$ is the unstable direction, it sets the maximum null time over which the time delay can be built up. In particular, once the expectation value of $x_{\parallel}^2$ becomes comparable to b^2, we can no longer trust the harmonic approximation to the potential and hence rely on the symmetry of the balancing sources to neglect scattering.

Assuming, for simplicity, that the source varies slowly so that we may use the JWKB approximation (or more precisely that, averaged over several shockwaves, the source may be taken to vary slowly), then for the solution in the parallel direction we have

$$x_{\parallel}(u) \approx x_{\parallel}(0)\sqrt{\frac{\Omega_{\parallel}(0)}{\Omega_{\parallel}(u)}}\cosh\left(\int_0^u \mathrm{d}u'\Omega_{\parallel}(u')\right) + \frac{p_{\parallel}(0)}{k_v}\frac{1}{\sqrt{\Omega_{\parallel}(u)\Omega_{\parallel}(0)}}\sinh\left(\int_0^u \mathrm{d}u'\Omega_{\parallel}(u')\right)\,. \tag{5.4.6}$$

Classically, it is possible to tune $p_{\parallel}(0) = 0$ and $x_{\parallel}(0) = 0$, so that the particle remains at $x_{\parallel}(u) = 0$ — quantum mechanically, this is not possible due to the inevitable spread in phase space. This can be captured by the parallel part of the Wigner distribution, which, assuming the initial Gaussian state considered previously, is

$$W_{\parallel,0}(x_{\parallel}(0), p_{\parallel}(0)) = 2\exp\left(-\frac{x_{\parallel}^2(0)}{2\sigma^2}\right)\exp\left(-2\sigma^2 p_{\parallel}^2(0)\right)\,. \tag{5.4.7}$$

Using this Wigner distribution to compute the expectation value of $x_{\parallel}(u)^2$, we find

$$\langle\psi|\hat{x}_{\parallel}^2(u)|\psi\rangle \approx \frac{1}{4\Omega_{\parallel}(u)}\left(\sigma^2\Omega_{\parallel}(0) + \frac{1}{4k_v^2\sigma^2\Omega_{\parallel}(0)}\right)\exp\left(2\int_0^u \mathrm{d}u'\Omega_{\parallel}(u')\right)\,. \tag{5.4.8}$$

This expectation value is minimised for

$$\sigma^2 = \frac{1}{2k_v\Omega_{\parallel}(0)}\,, \tag{5.4.9}$$

so that

$$\langle\psi|\hat{x}_{\parallel}^2(u)|\psi\rangle \gtrsim \frac{1}{4k_v\Omega_{\parallel}(u)}\exp\left(2\int_0^u \mathrm{d}u'\Omega_{\parallel}(u')\right)\,. \tag{5.4.10}$$

Clearly, we can no longer trust the harmonic treatment of the potential when this is comparable to b^2, and so the maximum null time over which the balancing time delay can be accumulated is estimated by

$$\frac{1}{k_v\Omega_{\parallel}(u_{\max})}\exp\left(2\int_0^{u_{\max}} \mathrm{d}u'\Omega_{\parallel}(u')\right) \leq b^2\,, \tag{5.4.11}$$

so that

$$\int_0^{u_{\max}} \mathrm{d}u'\Omega_{\parallel}(u') \leq \ln\left(b^2 k_v\Omega_{\parallel}(u_{\max})\right)\,. \tag{5.4.12}$$

We can again use this to bound the maximum achievable EFT time delay. The master equations in the balancing potential are very similar to the point source potential and can be found in appendix 5.E, so the parameter dependence of the EFT-induced time delay is still given by

(5.3.12). Therefore,

$$\begin{aligned} k_v|\Delta v_{\text{EFT}}(u_{\text{max}})| &\sim k_v \frac{|c_{\text{GB}}|}{\Lambda^2}\int_0^{u_{\text{max}}} \mathrm{d}u \frac{f(u)}{b^d} \\ &\leq \frac{k_v}{\Lambda^2}|c_{\text{GB}}|\text{max}\left\{\sqrt{\frac{f(u)}{b^d}}\right\}\int_0^{u_{\text{max}}} \mathrm{d}u\sqrt{\frac{f(u)}{b^d}}\,. \end{aligned} \tag{5.4.13}$$

Then, using the expression for the instability timescale (5.4.12), we arrive at:

$$k_v|\Delta v_{\text{EFT}}(u_{\text{max}})| \leq |c_{\text{GB}}|\text{max}\left\{\sqrt{\frac{k_v^2 f(u)}{\Lambda^4 b^d}}\right\}\ln\left(b^2 k_v \Omega_{||}^{\text{max}}\right)\,. \tag{5.4.14}$$

The combination $b^2 k_v \Omega_{||}^{\text{max}}$ controls quantum corrections. Using the EFT validity bound (5.2.28b) $f(u)/b^d \ll \Lambda^4/k_v^2$, we infer

$$k_v|\Delta v_{\text{EFT}}(u_{\text{max}})| \ll |c_{\text{GB}}|\ln\left(b^2 k_v \Omega_{||}^{\text{max}}\right)\,. \tag{5.4.15}$$

Provided $(b^2 k_v \Omega_{||}^{\text{max}})$ is tuned to be exponentially small to minimise the quantum diffusion as much as possible, it appears to be possible in the balancing case to generate a resolvable EFT time delay. In fact this is not the case, since we have tacitly assumed the time delay has a small intrinsic uncertainty, which is not always the case.

5.4.2 Time delay uncertainty

The Gaussian localised on the equilibrium point chosen to generate a large time delay/advance is not an eigenstate of the S-matrix, and there is an intrinsic uncertainty in the time delay associated with this, as discussed in section 3.4.3. To determine this, we consider

$$\widehat{\Delta v}^{\,2} = \frac{\partial \hat{S}^\dagger}{\partial k_v}\frac{\partial \hat{S}}{\partial k_v}\,. \tag{5.4.16}$$

Following the discussion of section 3.4.2, we can write its expectation value in Schrödinger picture as

$$\langle \text{in}|\widehat{\Delta v}^{\,2}|\text{in}\rangle = \int_{u_i}^{u_f}\mathrm{d}u\int_{u_i}^{u_f}\mathrm{d}u'\langle\psi(u)|\hat{Q}(u)\hat{U}(u,u')\hat{Q}(u')|\psi(u')\rangle\,, \tag{5.4.17}$$

or equivalently in Heisenberg picture as

$$\langle \text{in}|\widehat{\Delta v}^{\,2}|\text{in}\rangle = \int_{u_i}^{u_f}\mathrm{d}u\int_{u_i}^{u_f}\mathrm{d}u'\langle\psi(0)|\hat{Q}_H(u)\hat{Q}_H(u')|\psi(0)\rangle\,, \tag{5.4.18}$$

so that the uncertainty is

$$\begin{aligned}\delta v^2 &= \langle \mathrm{in}|\widehat{\Delta v}^{\,2}|\mathrm{in}\rangle - \langle \mathrm{in}|\widehat{\Delta v}|\mathrm{in}\rangle^2 \\ &= \int_{u_i}^{u_f} \mathrm{d}u \int_{u_i}^{u_f} \mathrm{d}u' \Big[\langle\psi(0)|\hat{Q}_H(u)\hat{Q}_H(u')|\psi(0)\rangle \\ &\qquad - \langle\psi(0)|\hat{Q}_H(u)|\psi(0)\rangle\langle\psi(0)|\hat{Q}_H(u')|\psi(0)\rangle \Big] .\end{aligned} \tag{5.4.19}$$

The main contribution to the uncertainty will come from the unstable direction along $x_\parallel$, so we neglect motion in the perpendicular direction $\mathbf{x}_\perp$. In the near-equilibrium approximation (5.4.3), the expression for $\hat{Q}$ is then

$$\hat{Q} \approx -\frac{f(u)}{b^{d-2}} - \frac{1}{2}\frac{f(u)}{b^d}(d-2)(d-1)\left(\hat{x}_\parallel^2 - \frac{u}{k_v}(\hat{p}_\parallel\hat{x}_\parallel + \hat{x}_\parallel\hat{p}_\parallel)\right) , \tag{5.4.20}$$

which may easily be translated into Heisenberg picture. The first term, which is the main contribution to the Shapiro delay, will drop out of the uncertainty. By the time the instability has kicked in, we may use the approximation

$$\hat{p}_\parallel^H \approx k_v \Omega_\parallel(u)\hat{x}_\parallel^H , \tag{5.4.21}$$

so that

$$\hat{Q}_H \approx -\frac{f(u)}{b^{d-2}} - \frac{1}{2}\frac{f(u)}{b^d}(d-2)(d-1)\left(1-2u\Omega_\parallel(u)\right)(\hat{x}_\parallel^H(u))^2 , \tag{5.4.22}$$

where

$$\hat{x}_\parallel^H(u) \approx \frac{1}{2}\left(x_\parallel(0)\sqrt{\frac{\Omega_\parallel(0)}{\Omega_\parallel(u)}} + \frac{p_\parallel(0)}{k_v}\frac{1}{\sqrt{\Omega_\parallel(u)\Omega_\parallel(0)}}\right)\exp\left(\int_0^u \mathrm{d}u'\Omega_\parallel(u')\right) . \tag{5.4.23}$$

Assuming $u_i = 0$ without loss of generality, a straightforward calculation gives

$$\begin{aligned}\delta v \approx &\left|\int_0^{u_f} \mathrm{d}u \left(1-2u\Omega_\parallel(u)\right)\Omega_\parallel(u)\exp\left(2\int_0^u \mathrm{d}u'\Omega_\parallel(u')\right)\right| \\ &\times 2^{-3/2}\Omega_\parallel(0)\left(\sigma^2 + \frac{1}{4k_v^2\Omega_\parallel^2(0)\sigma^2}\right) ,\end{aligned} \tag{5.4.24}$$

and from extremising over σ we infer the bound

$$k_v\delta v \gtrsim 2^{-3/2}\left|\int_0^{u_f} \mathrm{d}u \left(1-2u\Omega_\parallel(u)\right)\Omega_\parallel(u)\exp\left(2\int_0^u \mathrm{d}u'\Omega_\parallel(u')\right)\right| . \tag{5.4.25}$$

If u_f is taken to be such that $\exp\left(2\int_0^{u_f}\mathrm{d}u'\Omega_\parallel(u')\right) \lesssim 1$, then the right-hand side is of order unity or less, and in this situation $k_v|\Delta v_{\mathrm{EFT}}(u_{\mathrm{max}})| \lesssim |c_{\mathrm{GB}}|$. In order to make the EFT contribution to the time delay large, we must take u_f such that $\exp\left(2\int_0^{u_f}\mathrm{d}u'\Omega_\parallel(u')\right) \gg 1$. But then the integral on the r.h.s. of (5.4.25) is exponentially large, and grows faster than $k_v|\Delta v_{\mathrm{EFT}}|$. Thus, precisely at the point where it appears we can generate a resolvable EFT time delay, the uncertainty in the GR contribution to the time delay swamps it.

5.4.3 Generic Transverse Profile

Instead of the specific configuration before, let us try to generalise to a generic non-vacuum transverse profile that breaks spherical symmetry in the transverse space, *i.e.* with a potential of the form

$$V(u,x^i) = -\frac{k_v}{2}F_{\text{gen}}(u,x^i). \tag{5.4.26}$$

Suppose we have already chosen our configuration so that $x^i = 0$ is an equilibrium point, *i.e.*

$$\frac{\partial V}{\partial x^i} = -\frac{k_v}{2}\left.\frac{\partial F_{\text{gen}}}{\partial x^i}\right|_{x^i=0} = 0. \tag{5.4.27}$$

Then the equations of motion around the equilibrium are

$$k_v\frac{\mathrm{d}^2x^i}{\mathrm{d}u^2} = -\left.\frac{\partial^2 V}{\partial x^i\partial x^j}\right|_{x^i=0}x^j. \tag{5.4.28}$$

To decouple these equations, we want to diagonalise the operator

$$\Omega_{ij}x^j = \frac{1}{k_v}\left.\frac{\partial^2 V}{\partial x^i\partial x^j}\right|_{x^i=0}x^j = \omega^2 x^i, \tag{5.4.29}$$

although the coordinate transformation, *i.e.* the matrix of eigenvectors, that does this will generally be u-dependent — we will assume that this u-dependence is negligible. This is equivalent to saying that the u-dependence of the eigenvalues is negligible, so that we can take $x \sim e^{i\omega u}$ and ignore u-derivatives of ω. Suppose we have done this, and found the diagonalised matrix O_{ij} with real eigenvalues $\omega^2_{m'} \in \mathbb{R}$ and a basis of orthonormal eigenvectors $(q_{m'})^i$ (which is suppose to denote the i-component of the m'-th eigenvector), *i.e.* such that $\vec{q}_{m'}\cdot\vec{q}_{n'} = (q_{m'})^i\,(q_{n'})_i = \delta_{m'n'}$. Finally, taking $x^i = y^{m'}\,(q_{m'})^i$, we find that

$$x^i\Omega_{ij}x^j = y^{m'}\,(q_{m'})^i\,\Omega_{ij}y^{n'}\,(q_{n'})^j = y^{m'}\,(q_{m'})^i\,y^{n'}\omega^2_{n'}\,(q_{n'})_i = y^{n'}y_{n'}\omega^2_{n'}. \tag{5.4.30}$$

Recall that $(q_{m'})^i$ is the matrix of normalised eigenvectors, so that $y^{m'}$ are essentially the eigenbasis coordinates in terms of the old coordinates. Therefore, dropping the primes from now on,

$$V(u,x^i) = V(u,0) + \frac{1}{2}k_v\omega_n^2(u)y_n^2 + \dots. \tag{5.4.31}$$

The eigenvalues ω_m^2 correspond to the eigenfrequencies squared for small oscillations of the corresponding eigenmodes around this equilibrium. In particular, instabilities will exhibit themselves as negative eigenvalues of this operator. The trace of the operator is

$$\sum_n \omega_n^2 = \Omega^i{}_i = \frac{1}{k_v}\,\partial_i\partial^i V\big|_{x^i=0} = -\frac{1}{2}\,\partial_i\partial^i F_{\text{gen}}\big|_{x^i=0} \le 0, \tag{5.4.32}$$

where the equality holds in vacuum (away from sources). In this case, there will be at least one mode with negative eigenvalue ω_i^2, corresponding to unstable oscillations (unless all modes are zero modes, in which case we would need to look at the next order in the Taylor expansion). However if we allow presence of matter, the right-hand side will be negative, and we know for

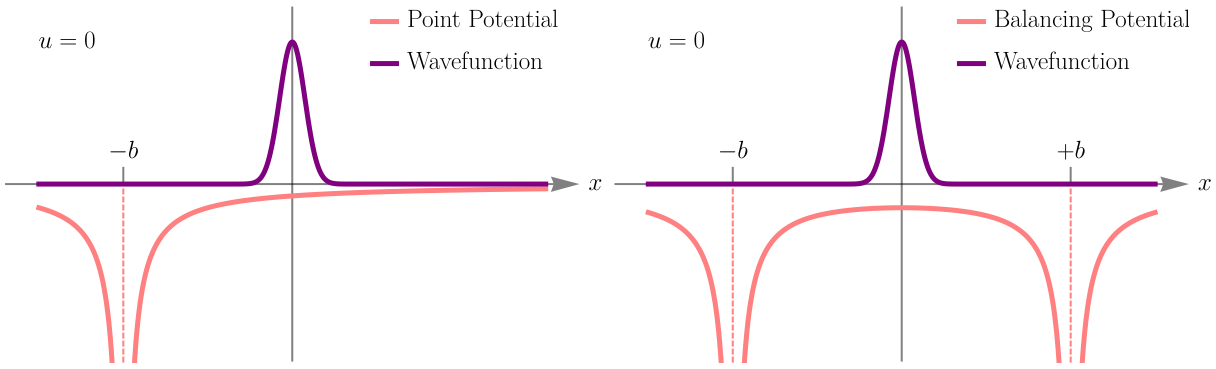


Figure 5.3: Snapshot at time $u = 0$ of the point-source and balancing potentials and the wavefunction which probes the spacetime.

certain that there will be directions with negative eigenvalue. We can then re-run the argument from before.

5.5 The Born approximation

In this section we compute the scattering time delay in perturbation theory, identifying the first few terms in the Born series. In general, the Born series has a smaller regime of validity than either the eikonal or semi-classical approximations, and so we do not draw any major conclusions from its results. However, we include it because the calculation may be performed more precisely and it is informative in terms of demonstrating the distinction between the single source and balancing shockwaves. It is also worth emphasising that almost all treatments of the eikonal approximation in the literature rely on resumming the Born amplitude, and the leading Born approximation to the time delay is identical to the eikonal result. The exponential eikonal resummation is implicit in the definition of the time delay operator.

The physical scenario for the two types of potential we have in mind is illustrated in Fig. 5.3. Then the potential is generated by either a single point source located at $\mathbf{x} = -\mathbf{b}$ or balancing sources at $\mathbf{x} = \pm\mathbf{b}$:

$$V(u, \mathbf{x}) = -\frac{k_v f(u)}{2}\left(\frac{\Theta_{\text{bal}}}{|\mathbf{x}-\mathbf{b}|^{d-2}} + \frac{1}{|\mathbf{x}+\mathbf{b}|^{d-2}}\right), \tag{5.5.1}$$

where we have introduced the notation

$$\Theta_{\text{bal}} = \begin{cases} 0, & \text{point source} \\ 1, & \text{balancing sources}\,. \end{cases} \tag{5.5.2}$$

Note that, in order to accommodate the point source case and balancing source case simultaneously, the location of the point source is shifted relative to the discussion of Section 5.2. The

interaction picture potential is

$$\hat{H}_1^{\text{int}}(u) = V\left(\hat{x} + \frac{u}{k_v}\hat{p}\right) = -\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} k_v f(u) \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{e^{i\mathbf{q}\cdot\mathbf{b}} + \Theta_{\text{bal}} e^{-i\mathbf{q}\cdot\mathbf{b}}}{\mathbf{q}^2} e^{i\mathbf{q}\cdot\left(\hat{\mathbf{x}} + \frac{u}{k_v}\hat{\mathbf{k}}\right)}. \tag{5.5.3}$$

which is conveniently rewritten as

$$\begin{aligned}\hat{H}_1^{\text{int}}(u) = -\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} k_v f(u) \int \frac{\mathrm{d}^d q}{(2\pi)^d} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int \mathrm{d}^d x \frac{e^{i\mathbf{q}\cdot\mathbf{b}} + \Theta_{\text{bal}} e^{-i\mathbf{q}\cdot\mathbf{b}}}{\mathbf{q}^2} \\ \exp\left[i\mathbf{x}\cdot(\mathbf{q}+\mathbf{k}) + \frac{iu}{k_v}\mathbf{q}\cdot\left(\frac{\mathbf{q}}{2}+\mathbf{k}\right)\right] |\mathbf{x}\rangle \langle\mathbf{k}|\,.\end{aligned} \tag{5.5.4}$$

Denoting the terms in the Dyson series as

$$\hat{S} = 1 + \underbrace{(-i)\int_{-\infty}^{+\infty} \mathrm{d}u\, \hat{H}_1^{\text{int}}(u)}_{\hat{S}^{(1)}} + \underbrace{(-i)^2 \int_{-\infty}^{+\infty} \mathrm{d}u \int_{-\infty}^{u} \mathrm{d}u'\, \hat{H}_1^{\text{int}}(u)\hat{H}_1^{\text{int}}(u')}_{\hat{S}^{(2)}} + \ldots, \tag{5.5.5}$$

the first three terms in the series expansion for the time delay are

$$\widehat{\Delta v}^{(1)} = -i\frac{\partial \hat{S}^{(1)}}{\partial k_v}, \tag{5.5.6a}$$

$$\widehat{\Delta v}^{(2)} = -i\left(\frac{\partial \hat{S}^{(2)}}{\partial k_v} + \hat{S}^{(1)\dagger}\frac{\partial \hat{S}^{(1)}}{\partial k_v}\right), \tag{5.5.6b}$$

$$\widehat{\Delta v}^{(3)} = -i\left(\frac{\partial \hat{S}^{(3)}}{\partial k_v} + \hat{S}^{(1)\dagger}\frac{\partial \hat{S}^{(2)}}{\partial k_v} + \hat{S}^{(2)\dagger}\frac{\partial \hat{S}^{(1)}}{\partial k_v}\right). \tag{5.5.6c}$$

Without ambiguity we denote the corresponding n-th order term in the expectation value for the in-state (3.4.81) simply by

$$\Delta v^{(n)} = \langle \text{in}|\, \widehat{\Delta v}^{(n)}\, |\text{in}\rangle\ . \tag{5.5.7}$$

We consider interactions over a finite time $0 < u < u_{\text{max}}$. To first order, the GR time delay is

$$\Delta v_{\text{GR}}^{(1)}(u) = \int_0^u \mathrm{d}u_1\, f(u_1)\left(1 - u_1\frac{\partial}{\partial u_1}\right) K^{(1)}(u_1)\,. \tag{5.5.8}$$

The second order GR time delay is given by

$$\begin{aligned}\Delta v_{\text{GR}}^{(2)}(u) = k_v \left(\prod_{i=1}^{2}\int_0^u \mathrm{d}u_i\, f(u_i)\right) \\ \times \left[\theta(u_1-u_2)\left(1 - u_1\frac{\partial}{\partial u_1}\right) - \theta(u_2-u_1)\left(1 - u_2\frac{\partial}{\partial u_2}\right)\right] K^{(2)}(u_1,u_2)\,,\end{aligned} \tag{5.5.9}$$

and the third order one by

$$\Delta v^{(3)}_{\text{GR}}(u) = k_v^2 \left(\prod_{i=1}^{3} \int_0^u \mathrm{d}u_i \, f(u_i) \right) \times \\ \left[\theta(u_1 - u_2)\theta(u_2 - u_3) \left(3 - u_1 \frac{\partial}{\partial u_1} - u_2 \frac{\partial}{\partial u_2} - u_3 \frac{\partial}{\partial u_3} \right) \right. \\ - \theta(u_2 - u_3) \left(2 - u_2 \frac{\partial}{\partial u_2} - u_3 \frac{\partial}{\partial u_3} \right) + \theta(u_2 - u_1) \left(1 - u_3 \frac{\partial}{\partial u_3} \right) \Big] K^{(3)}(u_1, u_2, u_3) \,, \tag{5.5.10}$$

where $\theta(u)$ is the Heaviside step function and

$$K^{(n)}(u_1, \ldots, u_n) = (-i)^{n+1} \left(-\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} \right)^n \left(\prod_{i=1}^{n} \int \frac{\mathrm{d}^d q_i}{(2\pi)^d} \frac{e^{i\mathbf{q}_i \cdot \mathbf{b}} + \Theta_{\text{bal}} e^{-i\mathbf{q}_i \cdot \mathbf{b}}}{\mathbf{q}_i^2} \right) \\ \times \exp\left[-\frac{\sigma^2}{2} \left\{ \left| \sum_{i=1}^{n} \mathbf{q}_i \right|^2 + \left| \frac{1}{2k_v \sigma^2} \sum_{i=1}^{n} u_i \mathbf{q}_i \right|^2 \right\} + \frac{i}{2k_v} \sum_{i<j=1}^{n} (u_i - u_j) \mathbf{q}_i \cdot \mathbf{q}_j \right] . \tag{5.5.11}$$

5.5.1 Shockwave Limit

As a check, let us reproduce the shockwave limit. If we take the potential to be just a delta-function in the u-direction, with a constant profile in the transverse directions, *i.e.*

$$V(u, x) = \alpha k_v \delta(u) \tag{5.5.12}$$

then we can actually find an exact solution to the S-matrix. This will be helpful as a check, so that we understand what we expect the shockwave limit to look like later.

In particular, if we neglect the diffusion term, the Schrödinger equation just reduces to

$$i \frac{\partial}{\partial u} \psi = \alpha k_v \delta(u) \psi \tag{5.5.13}$$

so that

$$\psi(u) = e^{-i \int_{-\infty}^{u} du' \, \alpha k_v \delta(u')} \psi(-\infty) = e^{-i \alpha k_v \theta(u)} \psi(-\infty) \tag{5.5.14}$$

and we can read off that

$$S = e^{-i\alpha k_v} = 1 - i\alpha k_v - \frac{1}{2}\alpha^2 k_v^2 + \frac{1}{3!} i \alpha^3 k_v^3 + \ldots . \tag{5.5.15}$$

In this toy model, we find that the perturbative expansion of the Wigner-Smith time delay gives

$$Q^{(1)} = -i \frac{\partial \hat{S}_1}{\partial k_v} = -\alpha \tag{5.5.16}$$

$$Q^{(2)} = -i \left(\frac{\partial \hat{S}_{(2)}}{\partial k_v} + \hat{S}^{\dagger}_{(1)} \frac{\partial \hat{S}_{(1)}}{\partial k_v} \right) = 0 \tag{5.5.17}$$

$$Q^{(3)} = -i \left(\frac{\partial \hat{S}_{(3)}}{\partial k_v} + S^{\dagger}_{(1)} \frac{\partial \hat{S}_{(2)}}{\partial k_v} + \hat{S}^{\dagger}_{(2)} \frac{\partial \hat{S}_{(1)}}{\partial k_v} \right) = 0 \tag{5.5.18}$$

etc., as expected. The exact time delay is just $2\partial\delta/\partial k_v = -2\alpha$, as we found in section 5.1.1!

5.5.2 The classical limit: Identifying diffusion and scattering effects

The effects of uncertainty and diffusion are captured, at every order in Δv, by the term in curly brackets in the exponential of each $K^{(n)}$ in (5.5.11). In the classical limit, where this term is negligible, the first-order term becomes

$$\lim_{\text{classical}} \Delta v^{(1)}_{\text{GR}}(u) = -\frac{1}{k_v}\int_0^u \mathrm{d}u_1 V(u_1, \mathbf{b})(1+\Theta_{\text{bal}}), \tag{5.5.19}$$

which is exactly the eikonal expression of (5.3.8). This is to be expected since the leading-order eikonal phase can be identified with the leading-order Born approximation.

After some manipulation, the classical limit of the second-order term in the time delay expansion can be written as

$$\begin{aligned}\lim_{\text{classical}} \Delta v^{(2)}_{\text{GR}}(u) = &-\left(-\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}\right)^2 k_v \int_0^u \mathrm{d}u_1 f(u_1)\int_0^{u_1}\mathrm{d}u_2 f(u_2)\int \frac{\mathrm{d}^d q_1}{(2\pi)^2}\frac{\mathrm{d}^d q_2}{(2\pi)^2}\\ &\frac{e^{i\mathbf{q}_1\cdot\mathbf{b}}+\Theta_{\text{bal}}e^{-i\mathbf{q}_1\cdot\mathbf{b}}}{\mathbf{q}_1^2}\frac{e^{i\mathbf{q}_2\cdot\mathbf{b}}+\Theta_{\text{bal}}e^{-i\mathbf{q}_2\cdot\mathbf{b}}}{\mathbf{q}_2^2}\\ &\left(\sin\left[\frac{1}{2k_v}(u_1-u_2)\mathbf{q}_1\cdot\mathbf{q}_2\right]-\frac{u_1}{2k_v}\mathbf{q}_1\cdot\mathbf{q}_2\cos\left[\frac{1}{2k_v}(u_1-u_2)\mathbf{q}_1\cdot\mathbf{q}_2\right]\right).\end{aligned} \tag{5.5.20}$$

In the balancing case, $\Theta_{\text{bal}} = 1$, it is easy to see that the integrand of (5.5.20) is odd under $\mathbf{q}_1 \to -\mathbf{q}_1$ (or $\mathbf{q}_2 \to -\mathbf{q}_2$) and thus the integral vanishes. In the point source case, there is no such anti-symmetry so the integral does not vanish.

The classical limit of the third-order term in the time delay expansion can be written as

$$\begin{aligned}\lim_{\text{classical}} \Delta v^{(3)}_{\text{GR}}(u) = &\left(-\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}\right)^3 k_v^2 \int_0^u \mathrm{d}u_1 f(u_1)\int_0^u\mathrm{d}u_2 f(u_2)\int_0^u \mathrm{d}u_3 f(u_3)\int \frac{\mathrm{d}^d q_1}{(2\pi)^2}\frac{\mathrm{d}^d q_2}{(2\pi)^2}\frac{\mathrm{d}^d q_3}{(2\pi)^2}\\ &\frac{e^{i\mathbf{q}_1\cdot\mathbf{b}}+\Theta_{\text{bal}}e^{-i\mathbf{q}_1\cdot\mathbf{b}}}{\mathbf{q}_1^2}\frac{e^{i\mathbf{q}_2\cdot\mathbf{b}}+\Theta_{\text{bal}}e^{-i\mathbf{q}_2\cdot\mathbf{b}}}{\mathbf{q}_2^2}\frac{e^{i\mathbf{q}_3\cdot\mathbf{b}}+\Theta_{\text{bal}}e^{-i\mathbf{q}_3\cdot\mathbf{b}}}{\mathbf{q}_3^2}\\ &\Bigg[\theta(u_1-u_2)\theta(u_2-u_3)\left(1-u_1\frac{\partial}{\partial u_1}\right)\\ &-\theta(u_2-u_1)\theta(u_2-u_3)\left(1-u_2\frac{\partial}{\partial u_2}\right)\\ &+\theta(u_2-u_1)\theta(u_3-u_2)\left(1-u_3\frac{\partial}{\partial u_3}\right)\Bigg]\\ &\exp\left[\frac{i}{2k_v}\left((u_1-u_2)\mathbf{q}_1\cdot\mathbf{q}_2+(u_1-u_3)\mathbf{q}_1\cdot\mathbf{q}_3+(u_2-u_3)\mathbf{q}_2\cdot\mathbf{q}_3\right)\right].\end{aligned} \tag{5.5.21}$$

The vanishing of this in the balancing case is less straightforward. First note that, as the eigenvalue of an Hermitian operator, the expression (5.5.21) should be real. We can verify that it is

real by simultaneously exchanging

$$u_1 \longleftrightarrow u_3 \quad \text{and} \quad \mathbf{q}_1 \longleftrightarrow \mathbf{q}_3, \tag{5.5.22}$$

in the integrand to retrieve its complex conjugate. Thus, the exponential factor at the bottom of (5.5.21) may be replaced by its real part, a cosine. Then, in the balancing case only, we have the following situation

$$\begin{aligned}\lim_{\text{classical}} \Delta v^{(3)}_{\text{GR}}(u) = \int_0^u \mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3 f(u_1)f(u_2)f(u_3) \Big[\theta(u_1-u_2)\theta(u_2-u_3)\left(1-u_1\frac{\partial}{\partial u_1}\right) \\ -\theta(u_2-u_1)\theta(u_2-u_3)\left(1-u_2\frac{\partial}{\partial u_2}\right) \\ +\theta(u_2-u_1)\theta(u_3-u_2)\left(1-u_3\frac{\partial}{\partial u_3}\right)\Big] G(u_1,u_2,u_3)\,,\end{aligned} \tag{5.5.23}$$

where $G(u_1,u_2,u_3)$ is a function that is invariant under any permutation of the u_i's. For example, the exchange $u_1 \leftrightarrow u_2$ in G, may be offset by the exchange of the integration variables $\mathbf{q}_1 \leftrightarrow \mathbf{q}_2$ and the change of variables $\mathbf{q}_3 \to -\mathbf{q}_3$ to leave it invariant. With this in mind, the first and third terms in the square brackets give the same contribution overall (by exchange of $u_1 \leftrightarrow u_3$). Merging these terms and additionally exchanging $u_1 \leftrightarrow u_2$ in the second term, we obtain:

$$\begin{aligned}\lim_{\text{classical}} \Delta v^{(3)}_{\text{GR}}(u) = \int_0^u \mathrm{d}u_1 f(u_1) \int_0^{u_1} \mathrm{d}u_2 f(u_2) \left(2\int_0^{u_2}\mathrm{d}u_3 - \int_0^{u_1}\mathrm{d}u_3\right) f(u_3) \\ \left(1-u_1\frac{\partial}{\partial u_1}\right) G(u_1,u_2,u_3).\end{aligned} \tag{5.5.24}$$

Now, note that

$$\begin{aligned}\int_0^{u_1}\mathrm{d}u_2\left(2\int_0^{u_2}\mathrm{d}u_3 - \int_0^{u_1}\mathrm{d}u_3\right) &= \int_0^{u_1}\mathrm{d}u_2\left(\int_0^{u_2}\mathrm{d}u_3 - \int_{u_2}^{u_1}\mathrm{d}u_3\right) \\ &= \int_0^{u_1}\mathrm{d}u_2\int_0^{u_2}\mathrm{d}u_3 - \int_0^{u_1}\mathrm{d}u_3\int_0^{u_3}\mathrm{d}u_2\,,\end{aligned} \tag{5.5.25}$$

and since the integrand is invariant under the exchange of $u_2 \leftrightarrow u_3$, the whole integral in (5.5.24) is zero.

It is expected that all higher-order corrections to the leading term (5.5.19) in the balancing-source time delay vanish in the classical limit. Again, let it be stressed that this is a reflection of the fact that there is, classically, no scattering from the unstable equilibrium point in the balancing potential. Thus, outside the classical limit, the higher-order terms $\Delta v^{(n>1)}$ may be regarded as the result of diffusion-induced scattering in the quantum problem.

5.5.3 Perturbative regime of validity

The requirement that the higher-order scattering-induced terms $\Delta v^{(n>1)}_{\text{GR}}(u)$ are under control compared to the leading-order GR term $\Delta v^{(1)}_{\text{GR}}(u)$ imposes a time cut-off u_{max} beyond which we may no longer trust *perturbation theory*. Below, we will determine the parameter "ϵ" which

controls the series expansion of $\Delta v_{\text{GR}}(u)$. As we will see, this parameter will grow with time u itself — at early times, the series expansion is under better control and $\Delta v_{\text{GR}}^{(1)}(u)$ represents a good approximation for $\Delta v_{\text{GR}}(u)$, but at late times that approximation begins to fail. The scattering timescale u_{pert} will be defined as the time at which it becomes large, $\epsilon(u_{\text{pert}}) \sim 1$.

As a concession to simplicity, consider the limit of a large number of closely spaced shock-waves for which we may approximate $f(u) = f_0$ as a constant. With this simplification, the u-derivative-terms in expressions (5.5.8)-(5.5.10) may be integrated by parts. For $u < u_{\text{max}}$, we therefore get, explicitly, at first order

$$\Delta v_{\text{GR}}^{(1)}(u) = -2\left(-\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}\right) f_0 \int_0^u \mathrm{d}u_1 \int \frac{\mathrm{d}^d q_1}{(2\pi)^d} \frac{e^{i\mathbf{q}_1\cdot\mathbf{b}} + \Theta_{\text{bal}} e^{-i\mathbf{q}_1\cdot\mathbf{b}}}{\mathbf{q}_1^2} \exp\left[-\frac{\sigma^2\mathbf{q}_1^2}{2} - \frac{u_1^2\mathbf{q}_1^2}{8k_v^2\sigma^2}\right], \tag{5.5.26}$$

at second order,

$$\begin{aligned}\Delta v_{\text{GR}}^{(2)}(u) = -2\left(-\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}\right)^2 k_v f_0^2 \int_0^u \mathrm{d}u_1 \mathrm{d}u_2 \int \frac{\mathrm{d}^d q_1}{(2\pi)^d}\frac{\mathrm{d}^d q_2}{(2\pi)^d} \\ \frac{e^{i\mathbf{q}_1\cdot\mathbf{b}} + \Theta_{\text{bal}} e^{-i\mathbf{q}_1\cdot\mathbf{b}}}{\mathbf{q}_1^2}\frac{e^{i\mathbf{q}_2\cdot\mathbf{b}} + \Theta_{\text{bal}} e^{-i\mathbf{q}_2\cdot\mathbf{b}}}{\mathbf{q}_2^2} \\ \exp\left[-\frac{\sigma^2}{2}\left|\mathbf{q}_1+\mathbf{q}_2\right|^2 - \frac{1}{8k_v^2\sigma^2}\left|u_1\mathbf{q}_1 + u_2\mathbf{q}_2\right|^2\right] \\ \sin\left[\frac{1}{2k_v}(u_1-u_2)\mathbf{q}_1\cdot\mathbf{q}_2\right],\end{aligned} \tag{5.5.27}$$

and at third order

$$\begin{aligned}\Delta v_{\text{GR}}^{(3)}(u) = 2\left(-\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}\right)^3 k_v^2 f_0^3 \int_0^u \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3 \int \frac{\mathrm{d}^d q_1}{(2\pi)^d}\frac{\mathrm{d}^d q_2}{(2\pi)^d}\frac{\mathrm{d}^d q_3}{(2\pi)^d} \\ \frac{e^{i\mathbf{q}_1\cdot\mathbf{b}} + \Theta_{\text{bal}} e^{-i\mathbf{q}_1\cdot\mathbf{b}}}{\mathbf{q}_1^2}\frac{e^{i\mathbf{q}_2\cdot\mathbf{b}} + \Theta_{\text{bal}} e^{-i\mathbf{q}_2\cdot\mathbf{b}}}{\mathbf{q}_2^2}\frac{e^{i\mathbf{q}_3\cdot\mathbf{b}} + \Theta_{\text{bal}} e^{-i\mathbf{q}_3\cdot\mathbf{b}}}{\mathbf{q}_3^2} \\ \exp\left(-\frac{\sigma^2}{2}\left|\mathbf{q}_1+\mathbf{q}_2+\mathbf{q_3}\right|^2 - \frac{1}{8k_v^2\sigma^2}\left|u_1\mathbf{q}_1 + u_2\mathbf{q}_2 + u_3\mathbf{q}_3\right|^2\right) \\ \cos\left(\frac{1}{2k_v}\left[(u_1-u_2)\mathbf{q}_1\cdot\mathbf{q}_2 + (u_1-u_3)\mathbf{q}_1\cdot\mathbf{q}_3 + (u_2-u_3)\mathbf{q}_2\cdot\mathbf{q}_3\right]\right) \\ \left[2\Theta(u_1-u_2)\Theta(u_2-u_3) - \Theta(u_2-u_1)\Theta(u_2-u_3)\right].\end{aligned} \tag{5.5.28}$$

While the cross terms mean these expressions are not readily integrated, we can estimate the parameter dependence of each $\Delta v_{\text{GR}}^{(n)}(u)$ as long as we make the following assumptions:

$$\sigma \ll b, \tag{5.5.29a}$$

$$u \ll k_v \sigma b. \tag{5.5.29b}$$

Physically, (5.5.29a) corresponds to the requirement that the width of the initial Gaussian wavepacket σ is much smaller than the distance to the source b. The second assumption (5.5.29b) is that the timescale under consideration u is smaller than the diffusion time $u_{\text{diff}} \sim$

$k_v\sigma b$ (see (3.4.85)). We also need to distinguish between the two possible cases where scattering is before diffusion and *vice versa*.

Scattering before diffusion

By switching to dimensionless integration variables $w_i = u_i/u$ and $\mathbf{p}_i = b\mathbf{q}_i$, the parameter dependence of $\Delta v_{\text{GR}}^{(n)}(u)$ can be extracted to leading order in σ/b and $u/k_v\sigma b$:

$$\Delta v_{\text{GR}}^{(1)}(u) \sim \frac{fu}{b^{d-2}}, \tag{5.5.30a}$$

$$\Delta v_{\text{GR}}^{(2)}(u) \sim \frac{f^2u^3}{b^{2d-2}}\left[1+\Theta_{\text{bal}}\left(-1+a^{(2)}\frac{\sigma^2}{b^2}+\tilde{a}^{(2)}\frac{u^2}{k_v^2b^2\sigma^2}\right)\right], \tag{5.5.30b}$$

$$\Delta v_{\text{GR}}^{(3)}(u) \sim \frac{k_v^2f^3u^3}{b^{3d-6}}\left[1+\Theta_{\text{bal}}\left(-1+a^{(3)}\frac{\sigma^2}{b^2}+\tilde{a}^{(3)}\frac{u^2}{k_v^2b^2\sigma^2}\right)\right], \tag{5.5.30c}$$

where $a^{(n)}$ and $\tilde{a}^{(n)}$ are numbers expressed in terms of $n(d+1)$ dimensionless integrals (given in appendix 5.F).

The ratio of one term $\Delta v_{\text{GR}}^{(n)}(u)$ to the next $\Delta v_{\text{GR}}^{(n+1)}(u)$ is not the same for all n. The even terms $\Delta v_{\text{GR}}^{(2n)}(u)$ are artificially suppressed by the presence of an odd sine in their integrands, compared to the even cosine in odd terms $\Delta v_{\text{GR}}^{(2n+1)}(u)$. In other words, the third-order term $\Delta v_{\text{GR}}^{(3)}(u)$ is not necessarily subdominant to the second-order term $\Delta v_{\text{GR}}^{(2)}(u)$, and so on. This means that the ratio which truly controls the expansion is not the ratio of one term to the next $\Delta v_{\text{GR}}^{(n+1)}(u)/\Delta v_{\text{GR}}^{(n)}(u)$ but instead it is of even (or equally, odd) terms $\Delta v_{\text{GR}}^{(n+2)}(u)/\Delta v_{\text{GR}}^{(n)}(u)$.

Although the explicit parameter dependence of the generic n-th term is not given here, it is apparent from the general expression (5.5.11) and the examples (5.5.30) that the series expansion of the time delay is controlled by

$$\epsilon(u) \sim \frac{\Delta v_{\text{GR}}^{(n)}(u)}{\Delta v_{\text{GR}}^{(n+2)}(u)} \sim \left(\frac{k_v fu}{b^{d-2}}\right)^2. \tag{5.5.31}$$

The S-matrix expansion is under control when $\epsilon(u) \ll 1$ and so we may identify the time at which perturbation theory breaks down as

$$\epsilon(u_{\text{pert}}) \sim 1 \quad \Longrightarrow \quad u_{\text{pert}} \sim \frac{1}{k_v}\frac{b^{d-2}}{f}. \tag{5.5.32}$$

Unlike the estimates (5.3.11), (5.3.28) and (5.4.12), this estimate for the cut-off timescale depends on the v-momentum k_v, which is a property of the test wave itself and not of the background. In fact

$$\frac{u_{\text{pert}}}{u_s} \sim \frac{u_s}{u_{\text{diff}}}, \tag{5.5.33}$$

and so for $u_s < u_{\text{diff}}$ the Born series breaks down before the eikonal approximation, which is why we regard it as less informative.

Using the expressions for the two timescales, (3.4.85) and (5.5.32), and the approximation for

the leading-order time delay (5.5.30), we find

$$u_{\rm pert} \ll u_{\rm diff} \quad \Longleftrightarrow \quad k_v \left|\Delta v^{(1)}_{\rm GR}(u_{\rm diff})\right| \gg 1. \tag{5.5.34}$$

In other words, the assumption that perturbation theory breaks down before the diffusion time is consistent, as long as the GR time delay generated by $u_{\rm diff}$ is resolvable.

In this formalism, the EFT contribution to the time delay can generally be estimated as

$$\Delta v^{(1)}_{\rm EFT}(u) \sim \frac{c_{\rm GB}}{b^2\Lambda^2}\Delta v^{(1)}_{\rm GR}(u) \sim c_{\rm GB}\frac{fu}{b^d\Lambda^2}\,. \tag{5.5.35}$$

The total EFT time delay by the time at which perturbation theory breaks down is

$$\Delta v^{(1)}_{\rm EFT}(u_{\rm pert}) \sim \frac{c_{\rm GB}}{b^2\Lambda^2}\frac{1}{k_v}\,, \tag{5.5.36}$$

and, given the EFT-validity bound $b\Lambda \gg 1$, satisfies

$$k_v|\Delta v^{(1)}_{\rm EFT}(u_{\rm pert})| \ll |c_{\rm GB}|\,. \tag{5.5.37}$$

Diffusion before scattering

We have already seen that the assumption $u_{\rm scatt} < u_{\rm diff}$ is self-consistent, and we will now show that it is actually the only regime of interest. Assuming that we can trust the expressions for $\Delta v^{(n)}_{\rm GR}(u)$ given in (5.5.30) all the way up until $u \sim u_{\rm diff}$, we can write

$$\Delta v^{(3)}_{\rm GR}(u_{\rm diff}) \sim k_v^2\left(\Delta v^{(1)}_{\rm GR}(u_{\rm diff})\right)^3\left[1+\Theta_{\rm bal}\left(-1+a^{(3)}\frac{\sigma^2}{b^2}+\tilde{a}^{(3)}\right)\right]. \tag{5.5.38}$$

Of the parts that are exclusive to the balancing case, $\tilde{a}^{(3)}$ will dominate over $a^{(3)}\,(\sigma/b)^2$, so we can write-off the totality of the square brackets as an $\mathcal{O}(1)$ number plus suppressed corrections. If the scattering time comes after the diffusion time then, by definition, the first-order term $\Delta v^{(1)}_{\rm GR}(u)$ should still dominate the series expansion by $u = u_{\rm diff}$. However, asking for the third-order term to be sub-dominant, $\left|\Delta v^{(3)}_{\rm GR}(u_{\rm diff})\right| < \left|\Delta v^{(1)}_{\rm GR}(u_{\rm diff})\right|$ leads to the conclusion

$$\left|k_v^2\left(\Delta v^{(1)}_{\rm GR}(u_{\rm diff})\right)^3\right| < \left|\Delta v^{(1)}_{\rm GR}(u_{\rm diff})\right| \quad \Longrightarrow \quad k_v\left|\Delta v^{(1)}_{\rm GR}(u_{\rm diff})\right| < 1, \tag{5.5.39}$$

i.e. the first-order GR time delay itself is unresolvable. As hinted at previously, we see that if we try to slow down the effects of scattering for long enough such that diffusion takes hold first, then the price to pay is a potential so weak (*i.e.* f so small) that even the leading-order GR time delay cannot be resolved. Since the EFT contribution is further suppressed by $1/b^2\Lambda^2 \ll 1$ compared to the GR contribution, it will certainly not be resolvable.

5.A Penrose Limit

Pp-waves are not just solutions, but also have a physical interpretation. This is due to a result by Penrose, who showed that for every spacetime with metric g and a choice of null geodesic γ, we can associate a pp-wave spacetime [255]. To see this, we take the so-called Penrose limit.

In any conjugate point free portion γ' of γ, we can use adapted (Penrose) coordinates to write the metric as

$$\mathrm{d}s^2 = 2\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v} + a(\tilde{u},\tilde{v},y^i)\mathrm{d}\tilde{v}^2 + 2b_i(\tilde{u},\tilde{v},y^i)\mathrm{d}\tilde{v}\,\mathrm{d}y^i + g_{ij}(\tilde{u},\tilde{v},y^i)\mathrm{d}y^i\mathrm{d}y^j. \tag{5.A.1}$$

Then, we perform the change of coordinates to $(\tilde{u},\tilde{v},y^i) = (u,\Omega^2 v,\Omega x^i)$ for $\Omega \in \mathbb{R}$. Then the Penrose limit metric $\bar{g}$ is defined to be

$$\bar{g}_{\mu\nu} = \lim_{\Omega\to 0}\Omega^{-2}g_{\mu\nu}(\Omega) = 2\mathrm{d}u\,\mathrm{d}v + \bar{g}_{ij}(u)\mathrm{d}x^i\mathrm{d}x^j, \tag{5.A.2}$$

where $g_{\mu\nu}(\Omega)$ is the metric g in (u,v,x^i) coordinates and $\bar{g}_{ij}(u) = g_{ij}(\tilde{u},0,0)$. This is the pp-wave metric in Rosen coordinates, which is a change of coordinates away from the pp-wave metric in Brinkmann coordinates.

Aichelburg-Sexl Boost

An alternative way to view the shockwave solution is by considering the Penrose limit of a Schwarzschild spacetime. Physically, we think of boosting a black hole to the speed of light, while simultaneously decreasing its mass to make sure energy stays finite. We will use the same notation for coordinates as before, but now restrict ourselves to $D = 4$.

It is well known that the Schwarzschild solution describes the spacetime outside a mass. The Schwarzschild metric for a mass M in isotropic coordinates is

$$\mathrm{d}s^2 = -\left(\frac{1-A}{1+A}\right)^2\mathrm{d}t^2 + (1+A)^4\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right), \tag{5.A.3}$$

where $A = 2M/\rho$ and $\rho^2 = x^2+y^2+z^2$. Now we apply a Lorentz transformation $(t,x,y,z) \to (\bar{t},\bar{x},\bar{y},\bar{z})$, specifically a boost with speed v in the x-direction so that

$$t = (1-v^2)^{-1/2}(\bar{t} - v\bar{x}) \tag{5.A.4}$$
$$x = (1-v^2)^{-1/2}(\bar{x} - v\bar{t}) \tag{5.A.5}$$
$$y = \bar{y} \tag{5.A.6}$$
$$z = \bar{z}. \tag{5.A.7}$$

In these coordinates, the metric becomes

$$\mathrm{d}s^2 = (1+A)^4\,\eta_{\mu\nu}\mathrm{d}\bar{x}^\mu\mathrm{d}\bar{x}^\nu + \left[(1+A)^4 - \left(\frac{1-A}{1+A}\right)^2\right]\frac{(\mathrm{d}\bar{t} - v\mathrm{d}\bar{x})^2}{1-v^2} \tag{5.A.8}$$

To keep the energy p finite, we define $p = m(1-v^2)^{-1/2}$. In the new coordinates,

$$A = \frac{M}{2\rho} = \frac{p(1-v^2)}{2\left[(\bar{x}-v\bar{t})^2 + (1-v^2)R^2\right]^{1/2}}, \tag{5.A.9}$$

where $R^2 = \bar{y}^2 + \bar{z}^2$. We are interested in boosting the black hole to the speed of light, *i.e.* $v \to 1$. It is therefore helpful to split $A = (1-v^2)B$, where

$$\lim_{v\to 1} A = 0, \quad \lim_{v\to 1} B = \frac{p}{2|\bar{x}-\bar{t}|} \tag{5.A.10}$$

so B diverges when $\bar{x} = \bar{t}$ in the limit $v \to 1$. The metric is therefore only well-defined for $\bar{x} \neq \bar{t}$, and we need to work harder to find an expression for all $\bar{t}$, $\bar{x}$.

Consider the coordinate transformation $T(v) : (\bar{t}, \bar{x}, \bar{y}, \bar{z}) \to (t', x', y', z')$ defined by

$$x' - vt' = \bar{x} - v\bar{t} \tag{5.A.11}$$

$$x' + vt' = \bar{x} + v\bar{t} - 4p\ln\left[\sqrt{(\bar{x}-v\bar{t})^2 + (1-v^2)} - (\bar{x}-\bar{t})\right]. \tag{5.A.12}$$

This leaves $\bar{y}$, $\bar{z}$, and A invariant. As we are interested in the $v \to 1$ limit, we will drop all terms in the metric that vanish in that limit. Under the transformation $T(v)$, only the $-\mathrm{d}\bar{t}^2 + \mathrm{d}\bar{x}^2$ and $(\mathrm{d}\bar{t} - v\mathrm{d}\bar{x})^2$ change. The coefficient of the former (*i.e.* g_{00} and g_{11}) is

$$\lim_{v\to 1}(1+A)^4 = 1, \tag{5.A.13}$$

and the coefficient of the latter is

$$\lim_{v\to 1}\frac{1}{1-v^2}\left[(1+A)^4 - \left(\frac{1-A}{1+A}\right)^2\right] = \lim_{v\to 1}\frac{1}{1-v^2}\left(8A - 2A^2 + \dots\right) = 8B, \tag{5.A.14}$$

i.e. neither diverge or vanish in the limit $v \to 1$. When transforming the metric, we can therefore concentrate on the terms that vanish in the limit. It can then be seen that

$$\mathrm{d}x' - v\mathrm{d}t' = \mathrm{d}\bar{x} - v\mathrm{d}\bar{t} \tag{5.A.15a}$$

$$\mathrm{d}x' + v\mathrm{d}t' = \mathrm{d}\bar{x} + v\mathrm{d}\bar{t} + \frac{4p(\mathrm{d}\bar{x} - \mathrm{d}\bar{t})}{\left[(\bar{x}-v\bar{t})^2 + (1-v^2)\right]^{1/2}}. \tag{5.A.15b}$$

Therefore, up to terms that vanish in the ultra-relativistic limit.

$$-\mathrm{d}\bar{t}^2 + \mathrm{d}\bar{x}^2 = -\mathrm{d}t'^2 + dx'^2 - \frac{4p(\mathrm{d}t' - \mathrm{d}x')^2}{\left[(x'-vt')^2 + (1-v^2)\right]^{1/2}} \tag{5.A.16}$$

$$8B(\mathrm{d}\bar{t} - \mathrm{d}\bar{x})^2 = \frac{4p(\mathrm{d}t' - \mathrm{d}x')^2}{\left[(x'-vt')^2 + (1-v^2)R^2\right]^{1/2}}. \tag{5.A.17}$$

Further, it can be shown that

$$\lim_{v\to 1}\left\{\left[(x'-vt')^2 + (1-v^2)\right]^{-1/2} - \left[(x'-vt')^2 + (1-v^2)R^2\right]^{-1/2}\right\} = 2\delta(x'-t')\ln(R). \tag{5.A.18}$$

Putting everything together, as $v \to 1$, the metric in the (t', x', y', z') coordinates becomes

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}t'^2 + \mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2 \\ &\quad + 4p\left\{\frac{1}{[(x'-vt')^2+(1-v^2)]^{1/2}} - \frac{1}{[(x'-vt')^2+(1-v^2)R^2]^{1/2}}\right\}(\mathrm{d}t'-\mathrm{d}x')^2 \\ &= -\mathrm{d}t'^2 + \mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2 + 8p\delta(x'-t')\ln(y'^2+z'^2)(\mathrm{d}t'-\mathrm{d}x')^2. \end{aligned}$$

Writing $u = t - x$, we can see that this is precisely in the form of a pp-wave metric in Brinkmann coordinates:

$$\mathrm{d}s^2 = -\mathrm{d}u\,\mathrm{d}v + \mathrm{d}x_i\mathrm{d}x^i + 8p\delta(u)\ln(x^2)\mathrm{d}u^2. \tag{5.A.19}$$

This is in fact a shock wave solution, even though it looks slightly different to the solution we found previously. This is because we are working in $D = 4$ dimensions, where the Green's function is logarithmic instead of a power law.

5.B Lightcone Gauge

To demonstrate that lightcone gauge is good, we will briefly review perturbations to pp-waves in pure GR. To understand the procedure, we will try to compute the equations of motion for the pp-wave perturbations first in general relativity.

The background equation of motion is just Einstein's equation. In vacuum, this is

$$R_{\mu\nu} = 0. \tag{5.B.1}$$

From (5.1.13), we know that the only non-trivial component of this is

$$R_{uu} = -\frac{1}{2}\partial_i\partial^i F = 0, \tag{5.B.2}$$

which is a simple condition on F in order for the metric to be an exact solution to Einstein's equations.

The equations of motion for metric perturbations $h = g - \bar{g}$, are

$$\delta R_{\mu\nu} = 0, \tag{5.B.3}$$

where

$$-2\delta R_{\mu\nu} = -2g^{\rho\sigma}\nabla_\rho\nabla_{(\mu}h_{\nu)\sigma} + \Box h_{\mu\nu} + \nabla_\mu\nabla_\nu h. \tag{5.B.4}$$

We claim that lightcone gauge, where

$$h_{v\mu} = 0, \tag{5.B.5}$$

is appropriate. Covariant derivatives of $h_{v\mu}$ vanish since $\Gamma^\alpha{}_{\beta v} = 0$ and we can always replace v-components of covariant derivatives with partial derivatives. Denoting the scalar Laplacian by

$$\tilde{\Box}\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi = g^{\mu\nu}\partial_\mu\partial_\nu\phi = \left(-F\partial_v^2 + 2\partial_u\partial_v + \partial_i\partial^i\right)\phi \tag{5.B.6}$$

then the equations of motion are

$$-2\delta R_{uu} = \tilde{\Box} h_{uu} - 2\partial_u \left(\partial_v h_{uu} + \partial^i h_{ui}\right) - (\partial^i\partial^j F) h_{ij} + (\partial^i F)\partial_v h_{ui} - (\partial^j F)\partial^i h_{ij} + \nabla_u \nabla_u h \tag{5.B.7a}$$

$$-2\delta R_{uv} = -\partial_v \left(\partial_v h_{uu} + \partial^i h_{ui}\right) + \nabla_u \nabla_v h \tag{5.B.7b}$$

$$-2\delta R_{ui} = \tilde{\Box} h_{ui} + (\partial^j F)\partial_v h_{ij} - \partial_i \left(\partial_v h_{uu} + \partial^j h_{uj}\right) - \partial_u \left(\partial_v h_{ui} + \partial^j h_{ij}\right) + \nabla_u \nabla_i h \tag{5.B.7c}$$

$$-2\delta R_{vv} = \nabla_v \nabla_v h \tag{5.B.7d}$$

$$-2\delta R_{vi} = -\partial_v \left(\partial_v h_{ui} + \partial^j h_{ij}\right) + \nabla_v \nabla_v h \tag{5.B.7e}$$

$$-2\delta R_{ij} = \tilde{\Box} h_{ij} - 2\partial_{(i|} \left(\partial_v h_{|j)u} + \partial^m h_{|j)m}\right) + \nabla_i \nabla_j h. \tag{5.B.7f}$$

The $\delta R_{v\mu} = 0$ equations are constraints, which we can solve by taking

$$\partial_\mu h^\mu{}_\nu = 0, \quad h = h^i{}_i = 0. \tag{5.B.8}$$

We are left with the following dynamical equations of motion:

$$-2\delta R_{uu} = \tilde{\Box} h_{uu} - (\partial^i\partial^j F) h_{ij} + 2(\partial^i F)\partial_v h_{ui} = 0 \tag{5.B.9}$$

$$-2\delta R_{ui} = \tilde{\Box} h_{ui} + (\partial^j F)\partial_v h_{ij} = 0 \tag{5.B.10}$$

$$-2\delta R_{ij} = \tilde{\Box} h_{ij} = 0. \tag{5.B.11}$$

Now, note that

$$\partial^j \tilde{\Box} h_{ij} = \tilde{\Box}(\partial^j h_{ij}) - (\partial^j F)\partial_v^2 h_{ij} = -\tilde{\Box}(\partial_v h_{ui}) - (\partial^j F)\partial_v^2 h_{ij} = -\partial_v \left[\tilde{\Box} h_{ui} + (\partial^j F)\partial_v h_{ij}\right] \tag{5.B.12}$$

and therefore

$$\begin{aligned}\partial^i\partial^j \tilde{\Box} h_{ij} =& -\partial_v \left[\tilde{\Box}(\partial^i h_{ui}) - (\partial^i F)\partial_v^2 h_{ui} + (\partial^i\partial^j F)\partial_v h_{ij} + (\partial^j F)\partial_v \partial^i h_{ij}\right] \\ =& -\partial_v \left[-\tilde{\Box}(\partial_v h_{uu}) - (\partial^i F)\partial_v^2 h_{ui} + (\partial^i\partial^j F)\partial_v h_{ij} - (\partial^j F)\partial_v^2 h_{uj}\right] \\ =& \partial_v^2 \left[\tilde{\Box}(h_{uu}) - (\partial^i\partial^j F) h_{ij} + 2(\partial^i F)\partial_v h_{ui}\right].\end{aligned} \tag{5.B.13}$$

We can decompose the perturbations in terms of (Fourier) eigenmodes $e^{ip_v v}$ of the null Killing vector l, so that $\partial_v \to ip_v$. As long as $p_v \neq 0$, *i.e.* no null modes are present, we can formally invert the differential operator ∂_v and the only equation we need to solve is therefore

$$\tilde{\Box} h_{ij} = 0 \tag{5.B.14}$$

and all other equations follow from it.

This is essentially decoupled, and all master equations will be of this form. We will see that higher-order corrections will break this degeneracy.

5.C EFT validity bound on wavefunction spread

In Section 3.4.4, we discuss how a generic initial quantum state has a non-zero spread σ in transverse space and thus cannot be perfectly balanced at an unstable equilibrium point. As a state in a low-energy EFT, this spread is in fact also bounded below by Λ^{-1}. To demonstrate this, we must bound a Lorentz-invariant quantity dependent on σ (as we did for $H(u,r)$, k_v and b in Section 5.2.3). The spread in momentum space is inversely related to the spread in real space, $|\Delta\mathbf{k}| \sim \sigma^{-1}$, so we would like to bound the inner product of two opposing wave vectors $\mathbf{k}^{\pm} = \mathbf{k^0} \pm \Delta\mathbf{k}$, where $\mathbf{k^0}$ is some central momentum. However, to construct a scalar quantity, they must first be embedded in D-vectors on the full D-dimensional pp-wave spacetime

$$\left(k_\alpha^{\pm}\right) = \left(k_u^{\pm}, k_v^{\pm}, \mathbf{k}^{\pm}\right) , \tag{5.C.1}$$

with the condition that $k^{\pm}$ is null due to the GR equations of motion,

$$g^{\alpha\beta}k_\alpha^{\pm}k_\beta^{\pm} = 0 \quad \Longrightarrow \quad k_u^{\pm} = \frac{F(k_v^{\pm})^2 - \mathbf{k}^{\pm 2}}{2k_v^{\pm}}. \tag{5.C.2}$$

For a fixed "mass" $k_v^+ = k_v^- = k_v$, the inner product reduces to

$$k^+ \cdot k^- = -2\Delta\mathbf{k}^2. \tag{5.C.3}$$

With the Lorentz-invariant bound $k^+ \cdot k^- \ll \Lambda^2$, we get the desired result $|\Delta\mathbf{k}| \ll \Lambda$, or $\sigma\Lambda \gg 1$.

5.D Higher-dimension EFT field equations

In section 5.2.3, we show that control over the higher-dimension EFT action (5.2.31) amounts to exactly the regime of validity found by the generic arguments. In this appendix, we provide the field equations for that action and also identify the tensor-type master variable and its associated master equation.

The background equation is

$$\mathcal{E}_{\alpha\beta} \coloneqq G_{\alpha\beta} + 2\frac{c_{\text{GB}}}{\Lambda^2}B_{\alpha\beta} + 2\frac{c_{\text{R3}}}{\Lambda^4}C_{\alpha\beta} + 2\frac{c_{\text{R4}}}{\Lambda^6}D_{\alpha\beta} = 0, \tag{5.D.1}$$

where

$$B_{\alpha\beta} = 4R_{\sigma\alpha\beta\rho}R^{\sigma\rho} + 2R_{\alpha}{}^{\sigma\rho\kappa}R_{\beta\sigma\rho\kappa} - 4R_{\alpha\sigma}R_{\beta}{}^{\sigma} + 2RR_{\alpha\beta} - \frac{1}{2}R^2_{\text{GB}}\,g_{\alpha\beta}, \tag{5.D.2a}$$

$$C_{\alpha\beta} = -\frac{1}{2}g_{\alpha\beta}(R^3) + 3(R^3)_{\alpha\sigma\beta}{}^{\sigma} + 6\nabla_\sigma\nabla_\rho(R^2)_{\alpha}{}^{\sigma}{}_{\beta}{}^{\rho}, \tag{5.D.2b}$$

$$D_{\alpha\beta} = -\frac{1}{2}g_{\alpha\beta}(R^4) + 4(R^4)_{\alpha\sigma\beta}{}^{\sigma} + 8\nabla_\sigma\nabla_\rho(R^3)_{\alpha}{}^{\sigma}{}_{\beta}{}^{\rho}, \tag{5.D.2c}$$

and

$$(R^n)_{\alpha\beta\sigma\rho} = R_{\alpha\beta\gamma\delta}R^{\gamma\delta\bullet\bullet}\ldots R^{\bullet\bullet\kappa\lambda}R_{\kappa\lambda\sigma\rho}, \tag{5.D.3a}$$

$$(R^n) = (R^n)_{\alpha\beta}{}^{\alpha\beta}. \tag{5.D.3b}$$

The vacuum pp-wave metric is again a solution to (5.D.1). The perturbations of these tensors are

$$\begin{aligned}\delta B_{\mu\nu} =& -4R_{(\mu}{}^{\alpha}{}_{\nu)}{}^{\beta}\nabla_\lambda\nabla_\beta h_\alpha{}^\lambda + 2R_\mu{}^\alpha{}_\nu{}^\beta\left(\Box h_{\alpha\beta} + \nabla_\beta\nabla_\alpha h\right)\\ &+ 4R_{(\mu|\alpha\beta\rho}\nabla^\rho\nabla_{|\nu)}h^{\alpha\beta} - 4R_{(\mu|\rho\alpha\beta}\nabla^\rho\nabla^\beta h_{|\nu)}{}^\alpha\\ &+ 2g_{\mu\nu}R_{\alpha\rho\beta\sigma}\nabla^\sigma\nabla^\rho h^{\alpha\beta}\end{aligned} \tag{5.D.4}$$

$$\begin{aligned}\delta C_{\mu\nu} =& 12R_{(\mu|\rho\alpha\beta}\nabla^\rho\Box\nabla^\beta h_{|\nu)}{}^\alpha - 12R_{(\mu|\rho\alpha\beta}\nabla^\rho\nabla^\beta\nabla_\sigma\nabla_{|\nu)}h^{\sigma\alpha}\\ &- 12\nabla_\sigma R_{(\mu|\rho\alpha\beta}\nabla^\rho\nabla^\beta\nabla_{|\nu)}h^{\alpha\sigma} + 12\nabla_\sigma R_{(\mu|\rho\alpha\beta}\nabla^\rho\nabla^\sigma\nabla^\beta h_{|\nu)}{}^\alpha\\ &+ 3\left(R_\mu{}^{\rho\lambda\alpha}R_{\nu\lambda}{}^{\sigma\beta} + R_\mu{}^{\lambda\rho\alpha}R_\nu{}^\sigma{}_\lambda{}^\beta\right)\nabla_\rho\nabla_\sigma h_{\alpha\beta}\end{aligned} \tag{5.D.5}$$

$$\begin{aligned}\delta D_{\mu\nu} =& -16R_{(\mu|\rho\beta\lambda}R_{|\nu)\sigma\alpha\kappa}\nabla^\sigma\nabla^\rho\nabla^\lambda\nabla^\kappa h^{\alpha\beta} - 32\nabla_\sigma R_{(\mu|\rho\beta\lambda}R_{|\nu)}{}^\sigma{}_{\alpha\kappa}\nabla^\rho\nabla^{(\kappa}\nabla^{\lambda)}h^{\alpha\beta}\\ &- 16\nabla^\sigma R_{(\mu|}{}^\rho{}_{\alpha\kappa}\nabla_\rho R_{|\nu)\sigma\beta\lambda}\nabla^\lambda\nabla^\kappa h^{\alpha\beta}.\end{aligned} \tag{5.D.6}$$

In lightcone gauge, the presence of the R^3-operator modifies the v-component perturbation equations

$$-2\delta\mathcal{E}_{vv} = \partial_v^2 h^i{}_i, \tag{5.D.7a}$$

$$-2\delta\mathcal{E}_{vu} = -\partial^i\left(\partial_v h^v{}_i + \partial_j h^j{}_i + 12\frac{c_{\rm R3}}{\Lambda^4}\left[(\partial_i\partial_j\partial_k H)\partial_v^2 h^{kj} + (\partial_i\partial_j H)\partial_v^2\partial_k h^{kj} + (\partial_i\partial_j H)\partial_v^3 h^{vj}\right]\right), \tag{5.D.7b}$$

$$-2\delta\mathcal{E}_{vi} = \partial_v\left(\partial_v h^v{}_i + \partial_j h^j{}_i + 12\frac{c_{\rm R3}}{\Lambda^4}\left[(\partial_i\partial_j\partial_k H)\partial_v^2 h^{kj} + (\partial_i\partial_j H)\partial_v^2\partial_k h^{kj} + (\partial_i\partial_j H)\partial_v^3 h^{vj}\right]\right), \tag{5.D.7c}$$

such that, while the traceless condition $h^i{}_i = 0$ is unchanged, the second constraint equation (5.2.11) becomes

$$\partial_\mu h^\mu{}_i = -12\frac{c_{\rm R3}}{\Lambda^4}\partial_v^2\left(\partial_k\left[(\partial_i\partial_j H)h^{kj}\right] + (\partial_i\partial_j H)\partial_v h^{vj}\right). \tag{5.D.8}$$

The $\delta\mathcal{E}_{ij}$ is also modified by both new operators

$$\begin{aligned}-2\delta\mathcal{E}_{ij} =& \tilde{\Box}h_{ij} - \partial_i(\partial_v h_{ju} + \partial^k h_{jk}) - \partial_j(\partial_v h_{iu} + \partial^k h_{ik}) + \partial_i\partial_j h^k{}_k - \delta_{ij}\tilde{\Box}h^k{}_k\\ &+ \delta_{ij}\partial^k(\partial_v h_{ku} + \partial^l h_{kl}) + \delta_{ij}\partial_v(\partial_v h_{uu} + \partial^k h_{ku})\\ &+ \frac{4c_{\rm GB}}{\Lambda^2}\partial_v^2(\partial_i\partial_j H h^k{}_k - 2\partial_{(i}\partial^k H h_{j)k} + \delta_{ij}\partial^k\partial^l H h_{kl})\\ &-24\frac{c_{\rm R3}}{\Lambda^4}\partial_v^2\Big[\partial_{(i}\partial^k H\tilde{\Box}h_{j)k} + \partial_u\partial_{(i|}\partial^k H\partial_v h_{|j)k}\\ &\qquad -\partial_{(i}\partial^k H\partial_{j)}(\partial_v h_{ku} + \partial^l h_{kl}) - \partial_{(i}\partial^k\partial^l H(\partial_{j)}h_{kl} - \partial_k h_{jl})\Big]\\ &+ 16\frac{c_{\rm R4}}{\Lambda^6}\partial_i\partial^k H\partial_j\partial^l H\partial_v^4 h_{kl}\,.\end{aligned} \tag{5.D.9}$$

The final constraint will again come from the trace of this equation. However, the process is complicated by the presence of the R^3-operator. In EGB theory, the second constraint (5.2.11)

allowed us to exactly remove the h_{iu} components, which appeared in the ij-equation. But now, the h_{iu} components also appear on the right hand side of (5.D.8). As a result, they may only be replaced *perturbatively* in inverse powers of the cut-off inside $\delta\mathcal{E}_{ij}$. Up to corrections of $\mathcal{O}(\Lambda^{-8})$, the third constraint equation is

$$\begin{aligned}\partial_\mu h^{\mu v} = &-\frac{4(D-4)}{D-2}\frac{c_{\text{GB}}}{\Lambda^2}(\partial_i\partial_j H)\partial_v h^{ij}\\ &-\frac{16}{D-2}\frac{c_{\text{R3}}}{\Lambda^6}(\partial_i\partial_k h)(\partial_j\partial^k H)\partial_v^3 h^{ij}\\ &+12\frac{c_{\text{R3}}}{\Lambda^4}\partial_v\bigg(-\frac{2}{D-2}\left[(\partial_i\partial_j H)\tilde{\Box}h^{ij}+(\partial_i\partial_j\partial_\mu H)\partial^\mu h^{ij}\right]+(\partial_i\partial_j\partial_k H)\partial^i h^{jk}+(\partial_i\partial_j H)\partial^i\partial_\mu h^{\mu j}\bigg).\end{aligned} \tag{5.D.10}$$

As before, we see that the dynamical degrees of freedom are contained within the metric perturbations in the transverse directions h_{ij}. Applying the three constraint equations, their equation of motion becomes (up to corrections of $\mathcal{O}(\Lambda^{-8})$):

$$\tilde{\Box}h_{ij}-8\frac{c_{\text{GB}}}{\Lambda^2}\partial_v^2 X_{ij}-24\frac{c_{\text{R3}}}{\Lambda^4}\partial_v^2 Y_{ij}-16\frac{c_{\text{R4}}}{\Lambda^6}\partial_v^4 Z_{ij}=0, \tag{5.D.11}$$

where

$$X_{ij}=(\partial_m\partial_{(i}H)h_{j)}{}^m-\frac{g_{ij}}{D-2}(\partial_m\partial_n H)h^{mn} \tag{5.D.12a}$$

$$\begin{aligned}Y_{ij}=&-(\partial_m\partial_{(i}H)\tilde{\Box}h_{j)}{}^m+\frac{\delta_{ij}}{d}(\partial_m\partial_n H)\tilde{\Box}h^{mn}\\ &-(\partial_\mu\partial_m\partial_{(i}H)\partial^\mu h_{j)}{}^m+\frac{\delta_{ij}}{d}(\partial_\mu\partial_m\partial_n H)\partial^\mu h^{mn}\\ &+(\partial_i\partial_j\partial_m\partial_n H)h^{mn}+(\partial_i\partial_j\partial_k H)\partial_\mu h^{\mu k},\end{aligned} \tag{5.D.12b}$$

$$Z_{ij}=-(\partial_i\partial_m H)(\partial_j\partial_n H)h^{mn}+\frac{\delta_{ij}}{d}(\partial_k\partial_m H)(\partial^k\partial_n H)h^{mn}. \tag{5.D.12c}$$

5.D.1 Tensor modes

The perturbation equations (5.D.11) couple all components of h_{ij} in a non-trivial fashion when $c_{\text{R3}}\neq 0$. Even in the point source case, the identification of master variable used in section 5.2.2 for EGB theory no longer works when the R^3-operator is included. For illustrative purposes, we consider just the spherically symmetric point source case. Due to the symmetry, we may perform a SVT decomposition described in appendix A.2, with each type of perturbation (scalar, vector or tensor) evolving independently of the other types. The simplest of them are the tensor modes, which are parameterised by

$$h_{rr}=h_{r\bar{i}}=0,\quad h_{\bar{i}\bar{j}}=r^2\Phi_T\mathbb{T}_{\bar{i}\bar{j}}, \tag{5.D.13}$$

where $\mathbb{T}_{\bar{i}\bar{j}}$ is a tensor spherical harmonic on the $(d-1)$-sphere and satisfies

$$\left(\hat{\Delta}_{d-1}+\kappa_T^2\right)\mathbb{T}_{\bar{i}\bar{j}}=0, \tag{5.D.14a}$$

$$\mathbb{T}^{\bar{i}}{}_{\bar{i}}=0,\quad \hat{D}^{\bar{i}}\mathbb{T}_{\bar{i}\bar{j}}=0. \tag{5.D.14b}$$

We used κ_T^2 to represent the eigenvalue of $\mathbb{T}_{\bar{i}\bar{j}}$ instead of the traditional k^2 to avoid any possible confusion with the momentum of the perturbations $(k_u, k_v, \mathbf{k})$ used elsewhere. There is one tensor mode Φ_T for each tensor spherical harmonic on the $(d-1)$-sphere. They all share a master equation, which can be derived from the $(\bar{i},\bar{j})$-components (with $\bar{i}\neq\bar{j}$) of (5.D.11)

$$\begin{aligned} 0 = \tilde{\Box}\Phi_T - \frac{\kappa_T^2+2}{r^2}\Phi_T - 8\frac{c_{\rm GB}}{\Lambda^2}\frac{\partial_r H}{r}\partial_v^2\Phi_T + 24\frac{c_{\rm R3}}{\Lambda^4}\partial_v^2\left[d\frac{\partial_r H}{r^2}\left(\partial_r\Phi_T + \frac{\Phi_T}{r}\right)\right. \\ \left. -\frac{\partial_u\partial_r H}{r}\partial_v\Phi_T\right] + 16\frac{c_{\rm R4}-12c_{\rm GB}c_{\rm R3}}{\Lambda^6}\left(\frac{\partial_r H}{r}\right)^2\partial_v^4\Phi_T. \end{aligned} \tag{5.D.15}$$

In arriving at this equation, the $\tilde{\Box}h_{\bar{i}\bar{j}}$ terms present in $Y_{\bar{i}\bar{j}}$ have been replaced perturbatively using the lower-order equation of motion, giving rise to the cross term $\propto c_{\rm GB}c_{\rm R3}$. Once again, this equation is valid up to corrections of order $\mathcal{O}(\Lambda^{-8})$.

5.E Master variables in balancing source background

Identifying the master variables for the metric perturbations on the point-source background was a straightforward procedure described above equation (5.2.15). When a second source is introduced (to "balance" the first, as discussed in Section 5.4), spherical symmetry is lost and the previous decomposition is no longer suitable. Without loss of generality, assume the two sources are aligned along the z-axis. Then, we can write the transverse-space metric in cylindrical coordinates as:

$$\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}z^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega_{d-2}^2. \tag{5.E.1}$$

For this appendix only, γ_{ab} will refer to the metric on the $(d-2)$-dimensional sphere. In these coordinates, the harmonic condition on the metric function is

$$\frac{\partial^2}{\partial z^2}H(u,r,z) = -\frac{\partial^2}{\partial r^2}H(u,r,z) - (d-2)\frac{1}{r}\frac{\partial}{\partial r}H(u,r,z). \tag{5.E.2}$$

While most of the components of the metric perturbations — h_{rr}, h_{rz}, h_{zz}, $h_{r\bar{i}}$, $h_{z\bar{i}}$ and the diagonal components of $h_{\bar{i}\bar{j}}$ — are coupled in a complicated way via (5.2.13), it happens that the off-diagonal components of $h_{\bar{i}\bar{j}}$ are completely decoupled from the rest and evolve independently. These account for $(D-5)(D-4)/2$ of the total $D(D-3)/2$ propagating modes. Their equation of motion is

$$\tilde{\Box}h_{\bar{i}\bar{j}} - 8\frac{c_{\rm GB}}{\Lambda^2}\frac{\partial_r H}{r}\partial_v^2 h_{\bar{i}\bar{j}} = 0, \quad \bar{i}\neq\bar{j}\,, \tag{5.E.3}$$

which has the same format as the equations for the modes in the spherically-symmetric background (5.2.15).

5.F Expansion of the Wigner-Smith time delay

The expressions (5.5.30) give the parameter dependence of the first three orders of the Wigner-Smith time delay expanded in 1) S-matrix perturbation theory, 2) small Gaussian width, and 3) early times. Four dimensionless numbers $\{a^{(2)}, \tilde{a}^{(2)}, a^{(3)}, \tilde{a}^{(3)}\}$ feature in these equations and are

given below in terms of integrals, where $\hat{\mathbf{b}} = \mathbf{b}/b$ is the unit vector in the direction of $\mathbf{b}$.

$$
\begin{aligned}
a^{(2)} = & -2\left(-\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}\right)^2 \int_0^1 \mathrm{d}w_1\,\mathrm{d}w_2 \Theta(w_1-w_2) \int \frac{\mathrm{d}^d p_1}{(2\pi)^d}\frac{\mathrm{d}^d p_2}{(2\pi)^d} \\
& \frac{e^{i\mathbf{p}_1\cdot\hat{\mathbf{b}}}+\Theta_{\text{bal}}e^{-i\mathbf{p}_1\cdot\hat{\mathbf{b}}}}{\mathbf{p}_1^2}\frac{e^{i\mathbf{p}_2\cdot\hat{\mathbf{b}}}+\Theta_{\text{bal}}e^{-i\mathbf{p}_2\cdot\hat{\mathbf{b}}}}{\mathbf{p}_2^2}\left(\mathbf{p}_1+\mathbf{p}_2\right)^2(w_1-w_2)\mathbf{p}_1\cdot\mathbf{p}_2\,,
\end{aligned}
\tag{5.F.1a}
$$

$$
\begin{aligned}
\tilde{a}^{(2)} = & -2\left(-\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}\right)^2 \int_0^1 \mathrm{d}w_1\,\mathrm{d}w_2 \Theta(w_1-w_2) \int \frac{\mathrm{d}^d p_1}{(2\pi)^d}\frac{\mathrm{d}^d p_2}{(2\pi)^d} \\
& \frac{e^{i\mathbf{p}_1\cdot\hat{\mathbf{b}}}+\Theta_{\text{bal}}e^{-i\mathbf{p}_1\cdot\hat{\mathbf{b}}}}{\mathbf{p}_1^2}\frac{e^{i\mathbf{p}_2\cdot\hat{\mathbf{b}}}+\Theta_{\text{bal}}e^{-i\mathbf{p}_2\cdot\hat{\mathbf{b}}}}{\mathbf{p}_2^2}\left(w_1\mathbf{p}_1+w_2\mathbf{p}_2\right)^2(w_1-w_2)\mathbf{p}_1\cdot\mathbf{p}_2\,,
\end{aligned}
\tag{5.F.1b}
$$

$$
\begin{aligned}
a^{(3)} = & 2\left(-\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}\right)^3 \int_0^1 \mathrm{d}w_1\,\mathrm{d}w_2\,\mathrm{d}w_3 \int \frac{\mathrm{d}^d p_1}{(2\pi)^d}\frac{\mathrm{d}^d p_2}{(2\pi)^d}\frac{\mathrm{d}^d p_3}{(2\pi)^d} \\
& \frac{e^{i\mathbf{p}_1\cdot\hat{\mathbf{b}}}+\Theta_{\text{bal}}e^{-i\mathbf{p}_1\cdot\hat{\mathbf{b}}}}{\mathbf{p}_1^2}\frac{e^{i\mathbf{p}_2\cdot\hat{\mathbf{b}}}+\Theta_{\text{bal}}e^{-i\mathbf{p}_2\cdot\hat{\mathbf{b}}}}{\mathbf{p}_2^2}\frac{e^{i\mathbf{p}_3\cdot\hat{\mathbf{b}}}+\Theta_{\text{bal}}e^{-i\mathbf{p}_3\cdot\hat{\mathbf{b}}}}{\mathbf{p}_3^2}\left(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3\right)^2 \\
& \left(2\Theta(w_1-w_2)\Theta(w_2-w_3)-\Theta(w_2-w_1)\Theta(w_2-w_3)\right)\,,
\end{aligned}
\tag{5.F.1c}
$$

$$
\begin{aligned}
\tilde{a}^{(3)} = & 2\left(-\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}\right)^3 \int_0^1 \mathrm{d}w_1\,\mathrm{d}w_2\,\mathrm{d}w_3 \int \frac{\mathrm{d}^d p_1}{(2\pi)^d}\frac{\mathrm{d}^d p_2}{(2\pi)^d}\frac{\mathrm{d}^d p_3}{(2\pi)^d} \\
& \frac{e^{i\mathbf{p}_1\cdot\hat{\mathbf{b}}}+\Theta_{\text{bal}}e^{-i\mathbf{p}_1\cdot\hat{\mathbf{b}}}}{\mathbf{p}_1^2}\frac{e^{i\mathbf{p}_2\cdot\hat{\mathbf{b}}}+\Theta_{\text{bal}}e^{-i\mathbf{p}_2\cdot\hat{\mathbf{b}}}}{\mathbf{p}_2^2}\frac{e^{i\mathbf{p}_3\cdot\hat{\mathbf{b}}}+\Theta_{\text{bal}}e^{-i\mathbf{p}_3\cdot\hat{\mathbf{b}}}}{\mathbf{p}_3^2}\left(w_1\mathbf{p}_1+w_2\mathbf{p}_2+w_3\mathbf{p}_3\right)^2 \\
& \left(2\Theta(w_1-w_2)\Theta(w_2-w_3)-\Theta(w_2-w_1)\Theta(w_2-w_3)\right)\,.
\end{aligned}
\tag{5.F.1d}
$$

Concluding Remarks

In part I of this dissertation, we focused on causality constraints on gravitational effective field theories. Borrowing intuition from field theory on curved spacetime, we argued in favour of using infrared causality (the absence of resolvably negative corrections to the GR time delay) as a useful criterion to constrain higher-derivative corrections to Einstein gravity. This was further confirmed by comparison to expectations from gravitational positivity bounds for the Galileon and prototypical Goldstone EFT on curved spacetime as toy models. Based on this, we then looked at gravitational wave scattering in black hole and pp-wave background geometries in Einstein-Gauss-Bonnet gravity (the leading-order effective field theory in five and higher spacetime dimensions), and derived bounds on its coupling that are consistent with known positivity/bootstrap bounds [168]. Scattering on pp-waves in particular served as an illustrative example to rule out stronger bounds which circumvent the resolvability criterion using stacked or balanced configurations of sources. This is a resolution to the apparent violation of causality entirely within the regime of validity of the low-energy theory without appealing to additional degrees of freedom, such as an infinite tower of massive higher-spin states.

There are several possible future directions for this.

- We have restricted ourselves to an effective field theory describing only gravitational degrees of freedom. However, we have seen from explicit constructions, that UV completions often involve other massless degrees of freedom such as photons or neutrinos. These are naturally also of interest to phenomenological applications such as cosmology [257, 258]. There are also massive fields that we might wish to keep within the description of our low-energy physics. Including all of these will lead to many more operators that the causality bounds can probe.

- When we considered scattering off pp-waves in order to obtain sharper bounds on the Wilson coefficients in effective field theory, it was ultimately the trade-off between the validity of the EFT and validity of the JWKB approximation which meant that we could not trust our expressions for the time delay sufficiently to accumulate time delay from sufficiently many scattering events. Given a fixed background geometry, we cannot improve the EFT regime of validity, but the validity of the JWKB approximation is in principle a technical limitation. Whether pp-waves are able to circumvent the resolvability criterion can be definitely answered if an exact solution was known to the JWKB problem. In the meantime, we can include higher-order corrections to the JWKB approximation to improve the

accuracy of our expressions for the time delay.

- Morally speaking, a background off of which we scatter gravitational perturbations corresponds to choice of Hilbert space that we probe microcausality on. Taking the perspective that a higher-derivative operator consistent with UV physics should respect infrared causality on all backgrounds, we obtain the strongest bounds on the respective Wilson coefficients by considering as many different backgrounds as possible. The solutions we have considered so far, static and spherically symmetric vacuum solutions and pp-waves, have a lot of symmetry — generalising this other spacetimes therefore comes with the obvious technical difficulties of spacetimes over which we have less analytical control. It is however to be expected that spacetimes with less symmetry are also able to lead to better bounds. For instance, we obtain no non-trivial bounds for our example, Einstein-Gauss-Bonnet gravity, when we consider gravitational wave scattering in flat space. Different backgrounds can also be constructed to study bounds on different operators. In particular, it is possible to construct backgrounds, for which the effect of operators that are naïvely leading-order in the derivative expansion is suppressed. As long as the derivative expansion itself is still under control, this means that we can find non-suppressed bounds on higher-order operators. Work in this direction was done in [180, 181].

- Infrared causality itself refers only to the EFT corrections to the GR background geometry, and is not sensitive to the asymptotic structure of spacetime. In principle, this should allow one to consider causality properties of spacetimes which are not asymptotically flat. These should be tightly linked to causality conditions in cosmology discussed in [259, 260].

Appendix A

Staticity and Spherical Symmetry

In this appendix, we briefly review the aspects of gravitational physics and scattering relevant in spherical symmetry. We review the static and spherically symmetric Ansatz for the background metric in appendix A.1, and discuss the decomposition of metric perturbations on top of these backgrounds in terms of spherical harmonics in appendix A.2. Relatedly, we also review scattering off spherical potentials in classical scattering problems in appendix A.3.

A.1 Background Ansatz

The Ansatz for a static and spherically symmetric spacetime in $D \geq 4$ dimensions is

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + \frac{1}{B(r)}\mathrm{d}r^2 + C(r)r^2\mathrm{d}\Omega_{D-2}^2, \tag{A.1.1}$$

where the metric on the $(D-2)$-Sphere S^{D-2} is

$$\mathrm{d}\Omega_{D-2}^2 = \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \sum_{i=2}^{D-1}\left(\prod_{j=2}^{i-1}\sin^2(\chi_j)\right)\mathrm{d}\chi_i^2. \tag{A.1.2}$$

A.1.1 Curvature

First note that the connections with indices in purely the orbit space or base space are the same as the connections on the orbit space or base space. The non-zero components of $\Gamma^a{}_{bc}$, *i.e.* with indices in the orbit space, are

$$\Gamma^0{}_{01} = \frac{1}{2}\frac{A'}{A}, \quad \Gamma^1{}_{00} = \frac{1}{2}A'B, \quad \Gamma^1{}_{11} = -\frac{1}{2}\frac{B'}{B}. \tag{A.1.3}$$

For the Christoffel symbols with indices mixed between base and orbit space, the only non-zero ones are

$$\Gamma^1{}_{ij} = -\frac{B}{r}g_{ij} \tag{A.1.4a}$$

$$\Gamma^{i}{}_{1j} = \frac{1}{r}\delta^{i}_{j}. \tag{A.1.4b}$$

Using the Cartan structure equations, we find the non-zero components of the Riemann tensor to be

$$R_{0101} = \sqrt{\frac{A}{B}}\partial_r\left[\sqrt{B}\partial_r(\sqrt{A})\right] \tag{A.1.5a}$$

$$R_{0i0j} = \frac{1}{r}B\sqrt{\frac{A}{C}}\partial_r(\sqrt{A})\partial_r(\sqrt{C}r)g_{ij} \tag{A.1.5b}$$

$$R_{1i1j} = -\frac{1}{r}\sqrt{\frac{1}{BC}}\partial_r\left[\sqrt{B}\partial_r(\sqrt{C}r)\right]g_{ij} \tag{A.1.5c}$$

$$R_{ijkl} = \frac{1}{Cr^2}\left[1 - B\left(\partial_r(\sqrt{C}r)\right)^2\right](g_{ik}g_{jl} - g_{il}g_{jk}). \tag{A.1.5d}$$

The non-zero components of the Ricci tensor are

$$R_{00} = \sqrt{AB}\partial_r\left[\sqrt{B}\partial_r(\sqrt{A})\right] + (D-2)\sqrt{\frac{A}{C}}B\frac{1}{r}\partial_r(\sqrt{A})\partial_r(\sqrt{C}r) \tag{A.1.6a}$$

$$R_{11} = -\frac{1}{\sqrt{AB}}\partial_r\left[\sqrt{B}\partial_r(\sqrt{A})\right] - (D-2)\frac{1}{\sqrt{BC}}\frac{1}{r}\partial_r\left[\sqrt{B}\partial_r(\sqrt{C}r)\right] \tag{A.1.6b}$$

$$R_{ij} = \Big[-Br\sqrt{\frac{C}{A}}\partial_r(\sqrt{A})\partial_r(\sqrt{C}r) - \sqrt{BC}r\partial_r\left[\sqrt{B}\partial_r(\sqrt{C}r)\right] \tag{A.1.6c}$$

$$+ (D-3)\left(1 - \left[\sqrt{B}\partial_r(\sqrt{C}r)\right]^2\right)\Big]\gamma_{ij} \tag{A.1.6d}$$

and the Ricci scalar is

$$R = -2\sqrt{\frac{B}{A}}\partial_r\left[\sqrt{B}\partial_r(\sqrt{A})\right] + \frac{1}{Cr^2}(D-3)(D-2)\left(1 - \left[\sqrt{B}\partial_r(\sqrt{C}r)\right]^2\right) \tag{A.1.7a}$$

$$-2(D-2)\sqrt{\frac{1}{AC}}B\frac{1}{r}\partial_r(\sqrt{A})\partial_r(\sqrt{C}r) - 2(D-2)\sqrt{\frac{B}{C}}\frac{1}{r}\partial_r\left[\sqrt{B}\partial_r(\sqrt{C}r)\right]. \tag{A.1.7b}$$

When $A = B$, the components of the Riemann tensor simplify to the following

$$R_{abcd} = -\frac{1}{2}A''(g_{ac}g_{bd} - g_{ad}g_{bc}) \tag{A.1.8a}$$

$$R_{aibj} = -\frac{1}{2r}A'g_{ab}g_{ij} \tag{A.1.8b}$$

$$R_{ijkl} = \frac{1}{r^2}(1-A)(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{A.1.8c}$$

A.2 Spherical Decomposition

In this appendix, we will describe the decomposition of tensors, vectors, and scalars in terms of eigentensors on the sphere.

A.2.1 Spherical Harmonics

As we will be dealing with spherically symmetric spacetimes, we can decompose the perturbations of interest using spherical harmonics. It will be helpful to review some basic properties of spherical harmonics.

Consider p-dimensional Euclidean space $\mathbb{R}^d$. A polynomial $P_n(x_1, ..., x_d)$ is said to be homogeneous of degree n in the d variables $x_1, ..., x_d$ if it satisfies the following scaling relation

$$P_n(tx_1, ..., tx_d) = t^n P_n(x_1, ..., x_d). \tag{A.2.1}$$

A polynomial $Q(x_1, ..., x_d)$ is said to be harmonic if

$$\Delta Q = 0, \tag{A.2.2}$$

where the Laplacian is defined to be

$$\Delta = \partial_i \partial^i = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j \right), \tag{A.2.3}$$

where g is the metric on $\mathbb{R}^d$ and $i, j \in \{1, .., d\}$.

Now, consider embedding S^{d-1} into $\mathbb{R}^d$. We can then define $\Delta_{S^{d-1}}$ as the Laplace operator on the $(d-1)$-sphere using the induced metric γ. Using spherical coordinates, and letting $m, n \in \{2, ..., d\}$, we find that

$$\Delta = \partial_r^2 + \frac{d-1}{r} \partial_r + \frac{1}{r^2} \Delta_{S^{d-1}}. \tag{A.2.4}$$

A spherical harmonic $Y_n(x)$ in S^{d-1} of degree n a harmonic homogeneous polynomial of degree n. In other words, it is the restriction of some harmonic homogeneous polynomial $H_n(X)$ in $\mathbb{R}^d$ to S^{d-1}, *i.e.*

$$Y_n(x) = H_n(x) \tag{A.2.5}$$

for $\vec{x} \in S^{d-1}$ and $\vec{x} = \vec{X}/r$. Using the properties described above

$$0 = \Delta H_n(X) = \Delta H_n(rx) = r^{n-2} \left[n(n+d-1) Y_n(x) + \Delta_{S^{d-1}} Y_n(x) \right]. \tag{A.2.6}$$

A.2.2 Master variables for metric perturbations

In this appendix, we summarise the procedure laid out in [243] for identifying the propagating degrees of freedom (the "master variables") of metric perturbations.

Following the notation of [243], we write the background spacetime as

$$\mathrm{d}s^2 = \eta_{ab}(y) \mathrm{d}y^a \mathrm{d}y^b + r^2 \gamma_{ij} \mathrm{d}z^i \mathrm{d}z^j, \tag{A.2.7}$$

where η_{ab} is the Lorentzian metric of the 2-dimensional orbit spacetime:

$$\eta_{ab}(y)\mathrm{d}y^a\mathrm{d}y^b = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2, \tag{A.2.8}$$

and γ_{ij} is the metric on the unit $(D-2)$-sphere S^{D-2}. The metric perturbations follow the same index conventions as the background metric. Furthermore, we denote the covariant derivative with respect to γ_{ij} on S^{D-2} by $\hat{D}_j$, and the associated Laplace-Beltrami operator by $\hat{\Delta}_{D-2} \equiv \gamma^{ij}\hat{D}_i\hat{D}_j$. The covariant derivative on the orbit spacetime is denoted by D_a, and the associated Laplace-Beltrami operator by $\Box_2 \equiv \eta^{ab}D_aD_b$. The Laplace-Beltrami operator on the full spacetime manifold is denoted simply by $\Box \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$.

It is worth noting that there are different conventions in the literature on what defines scalar, vector, or tensor modes [242, 261]. Mathematically speaking, we can say anything with no, one, or two indices is a scalar, vector or tensor. However, we could for example build something tensorial out of scalars by using derivatives or metrics. Physically, these would still count as scalar modes.

Tensor modes

The tensor-type metric perturbations are expanded in terms of tensor spherical harmonics $\mathbb{T}_{ij}$ on S^{D-2}, which satisfy

$$\left(\hat{\Delta}_{D-2} + k_T^2\right)\mathbb{T}_{ij} = 0, \tag{A.2.9a}$$

$$\mathbb{T}^i{}_i = 0, \quad \hat{D}_j\mathbb{T}^j{}_i = 0, \tag{A.2.9b}$$

where k_T^2 is the eigenvalue of $\hat{\Delta}_{D-2}$ acting on the tensor $\mathbb{T}_{ij}$, and takes discrete values:

$$k_T^2 = \ell(\ell + D - 3) - 2, \quad \ell = 1, 2, \ldots. \tag{A.2.10}$$

All mode numbers (*e.g.* ℓ) that could label $\mathbb{T}_{ij}$ have been suppressed. Expressions for the symmetric tensor spherical harmonics on the $(D-2)$-sphere in terms of the mode number ℓ can be found in [262] but are not relevant for our purposes. For each such tensor, the tensor-type metric perturbations can be written at each ℓ as

$$h_{ab} = 0, \quad h_{ai} = 0, \quad h_{ij} = 2r^2 H_T\mathbb{T}_{ij}. \tag{A.2.11}$$

It is clear that the only tensor-mode freedom is in the function $H_T \equiv H_T(t, r)$.

Vector modes

The vector-type metric perturbations are expanded in terms of vector spherical harmonics $\mathbb{V}_i$ on S^{D-2}, which satisfy

$$\left(\hat{\Delta}_{D-2} + k_V^2\right)\mathbb{V}_i = 0, \tag{A.2.12a}$$

$$\hat{D}_j\mathbb{V}^j = 0, \tag{A.2.12b}$$

where k_V^2 is the eigenvalue of $\hat{\Delta}_{D-2}$ acting on the vector $\mathbb{V}_i$ and takes discrete values[1]:

$$k_V^2 = \ell(\ell + D - 3) - 1, \quad \ell = 1, 2, \ldots. \tag{A.2.13}$$

All mode numbers (*e.g.* ℓ) that could label $\mathbb{V}_i$ have been suppressed. For each vector, the vector-type metric perturbations can be written at each ℓ as

$$h_{ab} = 0, \quad h_{ai} = r f_a \mathbb{V}_i, \quad h_{ij} = 2r^2 H_T \mathbb{V}_{ij}, \tag{A.2.14}$$

where

$$\mathbb{V}_{ij} = -\frac{1}{2k_V}\left(\hat{D}_i \mathbb{V}_j + \hat{D}_j \mathbb{V}_i\right). \tag{A.2.15}$$

The gauge-invariant combination of the variables is

$$F_a = f_a + \frac{r}{k_V} D_a H_T. \tag{A.2.16}$$

Scalar modes

The scalar-type metric perturbations are expanded in terms of scalar spherical harmonics $\mathbb{S}$ on S^{D-2}, which satisfy

$$\left(\hat{\Delta}_{D-2} + k_S^2\right)\mathbb{S} = 0, \tag{A.2.17}$$

where k_S^2 is the eigenvalue of $\hat{\Delta}_{D-2}$ acting on the scalar $\mathbb{S}$ and takes discrete values[2]:

$$k_S^2 = \ell(\ell + D - 3), \quad l = 0, 1, 2, \ldots. \tag{A.2.18}$$

All mode numbers (*e.g.* ℓ) that could label $\mathbb{S}$ have been suppressed, as before. For each ℓ, the scalar-type metric perturbations can be written as

$$h_{ab} = f_{ab}\mathbb{S}, \quad h_{ai} = r f_a \mathbb{S}_i, \quad h_{ij} = 2r^2\left(H_\ell \gamma_{ij}\mathbb{S} + H_T \mathbb{S}_{ij}\right), \tag{A.2.19}$$

where

$$\mathbb{S}_i = -\frac{1}{k_S}\hat{D}_i \mathbb{S}, \tag{A.2.20a}$$

$$\mathbb{S}_{ij} = \frac{1}{k_S^2}\hat{D}_i\hat{D}_j\mathbb{S} + \frac{1}{D-2}\gamma_{ij}\mathbb{S}. \tag{A.2.20b}$$

The gauge-invariant combination of the variables is

$$F = H_\ell + \frac{1}{D-2}H_T + \frac{1}{r}D^a r X_a, \tag{A.2.21a}$$

$$F_{ab} = f_{ab} + D_a X_b + D_b X_a, \tag{A.2.21b}$$

[1]The $\ell = 1$ vector harmonic corresponds to rotational perturbations of the BH and not a dynamical degree of freedom.

[2]The $\ell = 0$ scalar harmonic corresponds to a shift in the BH mass, while the $\ell = 1$ scalar harmonic turns out to be pure gauge [243], so neither are dynamical.

$$X_a = \frac{r}{k_S}\left(f_a + \frac{r}{k_S} D_a H_T\right). \tag{A.2.21c}$$

A.3 Scattering from a Spherically Symmetric Potential

The fact the equations of motion for the different propagating degree of freedom are wave-like equations allows us to use classic results from scattering theory. In particular, we will to derive the partial wave amplitudes, phase shifts, and time delays in our spherically symmetric system. In this section, we review basic results from scattering in spherically symmetric set-ups. This discussion follows [263, 264].

Consider a spherically symmetric system in $D > 3$ to allow for centrifugal terms. Take, for instance, the Schrödinger equation with a central potential $V(r)$ in $D = 4$. Separating variables in the spatial wavefunction

$$\psi(r, \theta, \phi) = R(r) Y_{\ell,m}(\theta, \phi), \tag{A.3.1}$$

and writing the radial part of the wave equation as $R(r) = u(r)/r$, we see that the Schrödinger equation reduces to

$$\frac{d^2u}{dr^2} - \left(V(r) + \frac{\ell(\ell+1)}{r^2}\right) u = -k^2 u, \tag{A.3.2}$$

where ℓ is the angular momentum quantum number, and k is the momentum. Equations of this form are also familiar from Newtonian orbital mechanics, and are the ones we will be concentrating on.

At very large r, both the potential and the centrifugal contribution vanish. This is known as the radiation zone. The general solution here is simply

$$u(r) = Ae^{ikr} + Be^{-ikr}, \tag{A.3.3}$$

for some constants A and B. In this non-relativistic setting, we will take the first term to be the outgoing and the second term to be the incoming wave. At large r, we can therefore say that the scattered wave is

$$R(r) \sim \frac{e^{ikr}}{r}. \tag{A.3.4}$$

Assuming the potential is reasonably localised, we can find a regime at large r, where the centrifugal contribution dominates over the potential. In this regime, the radial equation becomes

$$\frac{d^2u}{dr^2} - \frac{\ell(\ell+1)}{r^2} u = -k^2 u, \tag{A.3.5}$$

and the general solution is a linear combination of spherical Hankel functions, which are linear combinations of the spherical Bessel functions (not to be confused with Bessel functions) with complex coefficients

$$R(r) = A h_\ell^{(1)}(kr) + B h_\ell^{(2)}(kr), \tag{A.3.6}$$

for some constants A and B. Since $h_\ell^{(1)} \sim (-1)^\ell e^{ix}$ and $h_\ell^{(2)} \sim (-1)^\ell e^{-ix}$ as $r \to \infty$, we are

interested in the spherical Hankel function of the first kind $h_\ell^{(1)}$ for the scattered, *i.e.* outgoing, wave. Fixing the direction the incident plane wave is coming from to be the z direction, the full spatial part of the wavefunction becomes

$$\psi(r,\theta,\phi) = A\left(e^{ikz} + \sum_{\ell,m} C_{\ell,m} h_\ell^{(1)}(kr) Y_{\ell,m}(\theta,\phi)\right) \tag{A.3.7}$$

for some constants $C_{\ell,m}$ that determine the precise linear combination of Hankel functions.

A.3.1 Partial Wave Amplitudes

By spherical symmetry, there cannot be any ϕ dependence, so only the $m = 0$ mode survives. Recalling that

$$Y_{\ell,0}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos(\theta), \tag{A.3.8}$$

where P_ℓ are the Legendre polynomials, and defining the partial wave amplitudes a_ℓ via

$$C_{\ell,0} = i^{\ell+1} k\sqrt{4\pi(2\ell+1)}\, a_\ell, \tag{A.3.9}$$

then the spatial wave function becomes

$$\begin{aligned}\psi(r,\theta,\phi) &= A\left[e^{ikz} + k\sum_\ell i^{\ell+1}(2\ell+1) a_\ell h_\ell^{(1)}(kr) P_\ell(\cos\theta)\right] \\ &= A\sum_\ell i^\ell (2\ell+1)\left[j_\ell(kr) + ika_\ell h_\ell^{(1)}(kr)\right] P_\ell(\cos\theta),\end{aligned} \tag{A.3.10}$$

where we have used Rayleigh's formula in the second line, and j_ℓ is the spherical Bessel function of the first kind. Asymptotically, at $r \to \infty$

$$\psi(r,\theta,\phi) \sim A\left(e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right) \tag{A.3.11}$$

as expected on physical grounds. Here,

$$f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) a_\ell P_\ell(\cos\theta) \tag{A.3.12}$$

is known as the scattering amplitude, from which we can evaluate the cross section.

A.3.2 Phase Shifts and Time Delays

In the set-up above, an incident wave contains no mode with angular momentum around the "z" axis by spherical symmetry. However, we see from Rayleigh's equation

$$e^{ikz} = A\sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta). \tag{A.3.13}$$

that it does contains modes known as partial waves labelled by ℓ. Their angular momenta are all conserved, so we do not expect there to be mixing of modes with different ℓ. We can therefore look at each partial wave labelled by ℓ separately. In particular, asymptotically at $r \to \infty$

$$\psi_0^{(l)} = A\, i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta) \sim A \frac{2\ell+1}{2ikr} \left(e^{ikr} - (-1)^\ell e^{-ikr} \right) P_\ell(\cos\theta), \tag{A.3.14}$$

which describes outgoing and incoming waves respectively. In the presence of a potential, the incoming wave remains unchanged, while the scattered outgoing wave can be different. However, even in the presence of a potential, each partial wave must scatter with no change in the amplitude due to unitarity — only the phases can shift. Therefore, as $r \to \infty$

$$\psi^{(\ell)} \sim A \frac{2\ell+1}{2ikr} \left(e^{i(kr+2\delta_\ell)} - (-1)^\ell e^{-ikr} \right) P_\ell(\cos\theta), \tag{A.3.15}$$

where δ_ℓ is the ℓ-dependent phase shift, and the factor of 2 is by convention, and is clearer when this is written in terms of $\sin$. This is equivalent to the statement that the partial wave S-matrix is $S_\ell = e^{i2\delta_\ell}$.

Using the previous formula involving partial wave amplitudes, as $r \to \infty$

$$\psi^{(\ell)} \sim A \left[\frac{2\ell+1}{2ikr} \left(e^{ikr} - (-1)^\ell e^{-ikr} \right) + \frac{2\ell+1}{r} a_\ell e^{ikr} \right] P_\ell(\cos\theta), \tag{A.3.16}$$

so by comparison

$$a_\ell = \frac{1}{2ik} \left(e^{i2\delta_\ell} - 1 \right) = \frac{1}{k} e^{i\delta_\ell} \sin(\delta_\ell). \tag{A.3.17}$$

The time difference between the incoming and outgoing waves will be the difference of the coefficients of t sitting in the exponents (cf. (A.3.15)). Taylor expanding the phase shift, we then see that the time delay at fixed impact parameter and at fixed ℓ are defined as

$$\Delta T_b = 2 \frac{\partial \delta_\ell}{\partial \omega}\bigg|_b, \quad \Delta T_\ell = 2 \frac{\partial \delta_\ell}{\partial \omega}\bigg|_\ell \tag{A.3.18}$$

respectively. The former coincides with the classical Shapiro time delay in general relativity, while the latter is more meaningful in quantum mechanics and coincides with the usual definition of the Eisenbud-Wigner time delay. It is extremely important to distinguish between these two.

Part II

Extremal Near-Horizon Geometries

Chapter 6

Introduction

In this part of the thesis we shall be interested in extremal black holes.

The surprising feature of extremal black holes that we shall be concerned with in the first part lies in (or rather on) their horizons. Horizons of Schwarzschild black holes in $D \geq 4$, and in fact due to uniqueness all asymptotically flat (AF) and stationary black holes in $D = 4$, have smooth horizons. However, the expectation that this persists when any of these conditions is relaxed, *i.e.* for solutions in higher-dimensions or less symmetry, turns out to be wrong [265, 266]. The simplest and most famous counterexample to this is the Majumdar-Papapetrou solution, describing static configurations of extremal black holes, which has horizons that have finite regularity class [267–269].

Recently, by studying particular deformations of extremal Anti-de Sitter (AdS) Reissner-Nordström (RN) black holes, the authors of [184, 185] established that this feature is generic to "almost all" AdS black holes. These deformations are particular stationary metric perturbations that exhibit critical scaling of the form

$$h \sim \rho^{\gamma} \tag{6.0.1}$$

in an orthonormal frame. The horizon is located at $\rho = 0$, so curvature invariants diverge on the horizon for modes with $\gamma < 0$. Further, in [270], the authors identified a marginal deformation of AF RN black holes in $D = 5$ which opens up the possibility for UV sensitivity — the GR contribution to γ vanishes and four-derivative corrections seemingly lead to negative corrections to γ for Wilson coefficients satisfying the WGC. The fact that EFT corrections alone determine the sign of γ and therefore the presence of singularities on the horizon was suggested to signal a breakdown of EFT and an amplification of UV physics — a sentiment reiterated in [271].

In chapter 7, we revisit these deformations to the near-horizon region of extremal black holes and study a scalar toy model related to deformations to extremal charged black branes. We also consider a toy model of a spectator scalar field with EFT corrections, anticipating the discussion in the following chapter on UV sensitivity. In chapter 8 we extend the analysis of deformed horizons in the presence of UV corrections to AdS RN black holes and systematically study

the effect of UV physics on marginal deformations thereof. In a theory of gravity with higher-derivative corrections, the effect of said corrections on perturbations resums into the exponent of (6.0.1). These take the form of

$$\gamma = \gamma_{\text{GR}} + \gamma_{\text{EFT}}, \tag{6.0.2}$$

such that γ reduces to γ_{GR} in the absence of higher-derivative corrections. Curvature invariants blow up on the horizon when $\gamma < 0$, so $\gamma_{\text{crit}} := 0$ constitutes a marginal case or critical value for singular behaviour [1]. When the GR contribution is marginal, *i.e.* for $\gamma_{\text{GR}} = 0$, the EFT contribution dominates if non-zero, and the system is said to exhibit UV sensitivity. It turns out that for extremal AdS RN black holes, there exists an infinite number of deformations to the near-horizon geometry for which this is exactly the case.

Consider *e.g.* Einstein-Maxwell theory with zero cosmological constant in $D = 5$ described by the following EFT-corrected action

$$S_{\text{IR}} = \int \mathrm{d}^5x\sqrt{-g}\left[\frac{1}{2\kappa}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c\frac{\kappa}{\Lambda^2}\left(F_{\mu\nu}F^{\mu\nu}\right)^2\right]. \tag{6.0.3}$$

From the EFT perspective, this theory is of course highly tuned, but the correction itself is generic. Perturbatively in the Wilson coefficient c, the theory admits AF, static, and spherically symmetric background solutions with $\mathrm{AdS}_2\times S^3$ near-horizon geometry in the extremal limit. On top of these, perturbations take the form of scaling solutions in (6.0.1) with exponents (6.0.2), where

$$\gamma_{\text{GR}} = \frac{1}{2}\left(-1 + |\ell - 1|\right), \quad \gamma_{\text{EFT}} = -\frac{c}{\Lambda^2 r_H^2}\frac{6\ell(\ell+2)}{\ell - 1}. \tag{6.0.4}$$

As anticipated, the $\ell = 2$ is a marginal deformation for which

$$\hat{\gamma} := \gamma\big|_{\gamma_{\text{GR}}=0} = -48\frac{c}{\Lambda^2 r_H^2}. \tag{6.0.5}$$

This is negative when $c > 0$, which is the sign consistent with positivity bounds and the WGC, and *vice versa*. In other words, within our truncated theory, we find that "consistent" low-energy physics is associated with a definite sign for the correction $\hat{\gamma}$.

It is tempting to view this sensitivity to UV physics as signalling a *breakdown of EFT*. This occurs when curvature invariants constructed within the deformed geometry become large. Since the EFT is only constructed (and hence valid) to first order in the metric perturbations, we should only consider curvature invariants to that order. For an EFT with a Lorentz-invariant energy cut-off Λ on an extremal black hole background with horizon radius r_H, we show that this breakdown occurs when the metric perturbations reach

$$h \sim h_0\rho^\gamma \sim (\Lambda r_H)^2 \tag{6.0.6}$$

in some orthonormal frame. For the EFT to be valid even at the level of the background, we also require that $\Lambda r_H \gg 1$. On the other hand, metric perturbation theory breaks itself down when the quadratic action in perturbations no longer dominates over higher-order terms in the action.

[1] For a discussion on different critical values of γ, see section 7.2.4.

We confirm that this happens when

$$h \sim h_0 \rho^\gamma \sim 1, \tag{6.0.7}$$

as expected. Evidently metric perturbation theory breaks down first — in other words, we lose control over metric perturbation theory long before the EFT is out of control. In [185], this is shown to signal an instability of the IR fixed point and suggests that RG flow leads to a different IR theory likely with less symmetry. It is however notable that, in the marginal case, the point at which metric perturbation theory breaks down is determined by the EFT. Whether or not metric perturbation theory is reliable therefore depends on the EFT — from the perspective of an observer (in the bulk), this is a form of UV sensitivity.

The following set-up is illustrative. Consider the action

$$S_{\text{UV}} = \int \mathrm{d}^5 x \sqrt{-g} \left[\frac{1}{2\kappa} R - \frac{1}{4} e^{\alpha\phi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\nabla\phi)^2 - \frac{1}{2} m^2 \phi^2 \right] \tag{6.0.8}$$

describing Einstein-Maxwell theory coupled to a massive dilaton ϕ with m in $D = 5$. It is a partial UV completion of the effective action (6.0.3) — more precisely, when the dilaton is heavy relative to the background curvature, *i.e.* $m^2 r_H^2 \gg 1$, this is the tree-level effective action we obtain from integrating out said massive dilaton in (6.0.8), with $c = 1/32$ and $\Lambda^2 = \kappa m^2/\alpha^2$.

The theory (6.0.8) is renormalisable and hence perfectly well-behaved as a UV theory — it is described by a two-derivative action which universally arises as the bosonic sector in string theory and its supergravity [272–274]. Perturbatively in α this also admits $\mathrm{AdS}_2 \times S^3$ Robinson-Bertotti background solutions [275, 276], on top of which perturbations take precisely the form of the scaling solutions in (6.0.1) with scaling exponent

$$\gamma = \frac{1}{2}\left(-1 + |\ell - 1|\right) - \frac{\alpha^2}{32 m^2 r_H^2 \kappa} \frac{6\ell(\ell+2)}{\ell - 1} \frac{1 - \frac{4\ell(3\ell-5)}{(\ell+1) m^2 r_H^2}}{1 + \frac{4\ell}{m^2 r_H^2}}. \tag{6.0.9}$$

The leading term at $\mathcal{O}(\alpha^0)$ is what we would obtain as the uncorrected γ_{GR} in pure Einstein-Maxwell theory, so the $\mathcal{O}(\alpha^2)$-term is a correction to this. Further, let us make the trivial observation that, depending on the parameters of the theory (α, m) and the solution (ℓ, r_H), the exponent γ can take either sign. As discussed, $\gamma < 0$ leads to metric perturbations blowing up at the horizon, rendering metric perturbation theory out of control and signalling that the near-horizon will be replaced by a solution with less symmetry.

The leading-order contribution itself vanishes when $\ell = 2$, in which case the marginal scaling takes the form

$$\hat{\gamma} = -48 \frac{\alpha^2}{32 m^2 r_H^2 \kappa} \frac{1 - \frac{8}{3 m^2 r_H^2}}{1 + \frac{8}{m^2 r_H^2}}. \tag{6.0.10}$$

As expected, this agrees with (6.0.5) in the limit where $m r_H \to \infty$ and with appropriate identification of c and Λ. Once again, this can also take either sign — it is positive when $0 < m^2 r_H^2 < 8/3$, but negative otherwise. Whether or not metric perturbation theory blows up is therefore sensitive to the precise mass of the dilaton. Having $\hat{\gamma}$ positive however automatically

takes us outside the regime of validity of the low-energy theory. We see that metric perturbation theory already breaks down in the UV, and the EFT in fact captures this effect perfectly. If it was the EFT itself that breaks down, we would have expected that details of the UV completion are necessary and sufficient to cure this. Since the singularity persists even in the UV, this is certainly not associated to a breakdown of EFT.

For deformations of AF RN black holes, the combination of Wilson coefficients for four-derivative EFT corrections entering the AF extremal charge-to-mass ratio also enters the scaling exponent $\hat{\gamma}$ in the marginal case. Therefore, Wilson coefficients satisfying the AF WGC lead to worse behaviour on the horizons of marginal deformations to AF extremal RN black holes. There is an infinite tower of marginal deformations of extremal RN black holes in AdS, with different combinations of Wilson coefficients entering the marginal scaling exponent $\hat{\gamma}$. On a speculative basis, but informed by examples, we conjecture that EFTs with consistent UV completions also lead to worse behaviour on the horizons of these marginally deformed AdS RN geometries. We explore the implications of this so-called near-horizon negativity, along the lines of other criteria and bounds within the EFT consistency program [164, 180, 181, 277].

In chapter 9, we turn to dynamics of matter fields in the near-horizon geometry of undeformed extremal black holes. Moving beyond the stationary deformations, the dynamical perturbations to near-extremal black holes are described by JT gravity when we dimensionally reduce onto the AdS_2 -factor — this is a special case of the $2D$ dilaton gravity models from section 1.2.2. It is described, up to boundary terms, by the action

$$S_{\mathrm{JT}}[g,\Phi] = \int_M \mathrm{d}^2x\sqrt{-g}\Phi\left(\frac{R}{2}-\Lambda\right) \tag{6.0.11}$$

which admits AdS_2 as a background solution. AdS_2 holography is subtle, because we can only consistently describe the ground states of the system. The boundary is one-dimensional, so conformal symmetry is associated to the Hamiltonian of the field theory vanishing. It can be parameterised by a single function $t(u)$, but is invariant under reparameterisations of it — these are an asymptotic symmetry of AdS_2 . To study non-trivial dynamics, we turn to nearly-AdS_2/nearly-CFT holography, where leading-order corrections away from AdS_2 are taken into account. Time reparameterisation invariance is broken spontaneously and explicitly to the $\mathrm{SL}(2)$ subgroup. EFT principles dictate that the dynamics of the boundary are described to leading-order by the Schwarzian action for the boundary time $t(u)$, with the Schwarzian derivative being the lowest-order derivative operator invariant under the residual symmetry. This is a field theory which is amenable to perturbative QFT.

We are interested in studying boundary correlation functions of Goldstone EFT (1.1.20) in the bulk with contributions from the Schwarzian modes. The reason for this is twofold. As we already seen, the $(\partial\phi)^4$-operator is the prototypical example of an operator subject to non-trivial restrictions from positivity and causality constraints. On the other hand, there are interesting interpretations of the $(\partial\phi)^4$-operator itself in AdS_2 . First of all, it can be thought of as the $T\overline{T}$-deformation of free massless scalar field — these deformations have found a lot of interest in recent years as exactly solvable, integrable, but irrelevant deformations to QFTs. These are

known to generate square-root structures in brane actions, which also famously give rise to the Goldstone EFT. Interestingly, these can be thought of as coupling to two-dimensional topological massive gravity, in which case the Wilson coefficient of the Goldstone EFT interaction is related to the mass squared of the graviton. The fact that positivity bounds prefer a sign of Wilson coefficient is on one-to-one correspondence with positivity of the graviton mass squared.

It is interesting to see what implications different signs of the Wilson coefficient have on the observables within the boundary theory itself. To that end, we present preliminary results for matter boundary correlation functions. In the matter sector, the holographic dictionary can be used to determine the connected generating functional

$$W_{\text{matter}}[\chi_r, t] = \sum_{m=2}^{\infty} \left(\prod_{i=1}^{m} \int \mathrm{d}t_i \chi_r(t_i) \right) \left\langle \prod_{j=1}^{m} \mathcal{O}_\chi(t_j) \right\rangle_{\text{matter}}, \tag{6.0.12}$$

where χ_r act as sources in the boundary theory. We determine the 2- and 4-point function up to 1-loop, making use of the fact that bulk loop integrals are just one-dimensional. The generating function also includes an integral over the dynamical gravitational degrees of freedom, in this case the boundary time t, so generally

$$Z[\chi_r] = \int \mathcal{D}t \, e^{-I_{\text{Schw}}[t] - W_{\text{matter}}[\chi_r, t]}, \tag{6.0.13}$$

so that

$$\left\langle \prod_{j=1}^{m} \mathcal{O}_\chi(t_j) \right\rangle_{\text{grav.}} = \int \mathcal{D}t e^{-I_{\text{Schw}}} \left\langle \prod_{j=1}^{m} \mathcal{O}_\chi(t_j) \right\rangle_{\text{matter}}. \tag{6.0.14}$$

We evaluate leading-order corrections away from the gravitational saddlepoint.

Chapter 7

Deformations of Near-Horizon Geometries

In this chapter, we revisit deformations of extremal charged black holes and discuss similar deformations to black branes. We start in section 7.1 by introducing the set-up by reviewing background solutions and the decoupling of perturbations thereof in pure Einstein-Maxwell theory. We solve for these deformations explicitly and discuss some of their properties in section 7.2. By starting in the near-horizon region from the out-set, we generalise to deformations to the near-horizon region of extremal black branes and discuss a holographic interpretation.

Details on the decompositions of perturbations on the sphere and AdS_2 are given in appendices 7.A and 7.B respectively. In appendix 7.C, we collect results on geodesics and tidal forces in the background and its deformation.

7.1 Reissner-Nordström Black Holes

Consider Einstein-Maxwell theory in $D = d + 2$ dimensions with cosmological constant Λ, described by the following action

$$S_{\text{EM}} = \int \mathrm{d}^D x \sqrt{-g} \left[\frac{1}{2\kappa} \left(R - 2\Lambda \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] . \tag{7.1.1}$$

In terms of the Planck mass M_{Pl}, $1/\kappa = M_{\text{Pl}}^d$. For convenience, we will concentrate on asymptotically AdS spacetimes and write

$$\Lambda = -\frac{(D-1)(D-2)}{2L^2}, \tag{7.1.2}$$

where L denotes the AdS radius. The AF and dS cases can be reached via the limit $L \to \infty$ or analytic continuation $L^2 \mapsto -L^2$ respectively.

The background equations of motion associated to this are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \quad T_{\mu\nu} = F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu} \tag{7.1.3}$$

and

$$\nabla^{\mu}F_{\mu\nu} = 0. \tag{7.1.4}$$

7.1.1 Background Solution

For convenience, we restrict ourselves to static and spherically symmetric, and electrically charged background solutions, which corresponds to the following Ansatz

$$\bar{g}_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = -A(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{B(r)} + r^2\mathrm{d}\Omega_d^2, \tag{7.1.5a}$$

$$\bar{F} = \Psi'(r)\mathrm{d}t \wedge \mathrm{d}r, \tag{7.1.5b}$$

where $\mathrm{d}\Omega_d^2$ is the round metric on S^d.

The solution to the Einstein-Maxwell equations (7.1.3) and (7.1.4) of this form is the AdS RN solution, and it is given by $A(r) = B(r) = f(r)$, for

$$f(r) := 1 + \frac{r^2}{L^2} - \frac{M}{r^{D-3}} + \frac{Q^2}{r^{2(D-3)}}, \tag{7.1.6a}$$

$$\Psi(r) = \frac{q}{r^{D-3}}, \qquad Q^2 = \frac{(D-3)q^2\kappa}{D-2}. \tag{7.1.6b}$$

7.1.2 Perturbations

To study perturbations on top of the D-dimensional RN backgrounds, we consider small perturbations $\sqrt{\kappa}h := g - \bar{g}$ and $\delta F := F - \bar{F}$ around the background solutions and linearise the equations of motion (7.1.3) and (7.1.4) in these. Following [278], it is natural to decompose perturbations with respect to the background spherical symmetry in terms of *tensor, vector, and scalar harmonics* — a discussion of this can be found in 7.A Particular linear combinations of the gauge-invariant variables known as "master variables", which mix the gravitational and electromagnetic (EM) perturbations, obey Schrödinger-like "master equations"

$$\Box_2\Phi_M - U_M\Phi_M = 0, \quad \Box_2 \cdot := -\frac{1}{f}\partial_t^2 \cdot + \partial_r(f\partial_r \cdot), \tag{7.1.7}$$

where $M \in \{T, V+, V-, S+, S-\}$ labels tensor, EM and gravitational vector, and EM and gravitational scalar modes respectively. Further, $U_M(r)$ is a mode-dependent effective potential containing the angular dependence, although the details of the effective potential are irrelevant in the upcoming discussions and we will drop the subscript M for now. See appendix 7.A for more details.

We shall be interested in the near-horizon region, where Schwarzschild-like coordinates are not

suitable, so it is useful to employ ingoing coordinates (v, r, x^i) with

$$v := t + r_*, \quad \mathrm{d}r_* := \frac{\mathrm{d}r}{f}. \tag{7.1.8}$$

The perturbations now obey the master equation (7.1.7) with

$$\Box_2 \cdot = 2\partial_v \partial_r \cdot + \partial_r (f \partial_r \cdot). \tag{7.1.9}$$

Due to staticity of the background, we can use the Killing vector $k = \partial/\partial v$ to decompose the perturbations in terms of its eigenfunctions $\Phi_M(t, r) = e^{-i\omega v}\phi_M(r)$, so that (7.1.7) reduces to

$$f\phi_M'' + (f' - i\omega)\phi_M' - U_M \phi_M = 0. \tag{7.1.10}$$

This is the equation we wish to solve.

7.1.3 The Extremal Limit

RN spacetime has two real horizons $r_\pm$ which are implicitly defined by $f(r_\pm) = 0$, while the degenerate extremal horizon r_H is implicitly defined by $f(r_H) = 0$ and $f'(r_H) = 0$ simultaneously. For AF RN black holes in D dimensions, this is explicitly given by

$$r_\pm^{D-3} = \frac{M}{2} - \sqrt{\left(\frac{M}{2}\right)^2 - Q^2} \tag{7.1.11}$$

so the extremal limit is reached when

$$|Q| = \frac{M}{2}. \tag{7.1.12}$$

It will often be convenient to repackage dependence on the AdS length scale L into

$$\sigma = 1 + \frac{(D-2)(D-1)}{(D-3)^2}\varsigma, \quad \varsigma = \frac{r_H^2}{L^2}. \tag{7.1.13}$$

Hence the limit $\sigma \to 1$ is degenerate in that it corresponds to both $r_H \to 0$ and $L \to \infty$.

We are interested in the deformations from [184, 185, 270, 271] — these are *stationary* perturbations of *extremal* black holes within the *near-horizon* limit. It turns out that the extremal limit is in tension with both the stationary limit and the near-horizon expansion, and the order in which these expansions are performed is subtle. To see this, our prescription will be to first consider the general case of non-extremal and non-stationary perturbations outside the near-horizon limit, and then take at most two limits consecutively to see any non-commutativity.

The Stationary Limit

By stationary perturbations, we mean solutions to (7.1.10) with $\omega = 0$ [1]

[1] *Stationary* scalar and vector perturbations generically have off-diagonal components, so generic stationary metric perturbations are not *static* — only pure stationary pure tensor perturbations are static.

	$\omega \neq 0$	$\omega = 0$
sub.	$r_\pm$ reg. ∞ irreg.	$r_\pm$ reg. ∞ irreg.
ext.	r_H irreg. ∞ irreg.	r_H reg. ∞ irreg.

Table 7.1: Classification of singular points of the master equation (7.1.10) in different limits. This suggests that one should take the stationary limit first.

As an ODE, the master equation (7.1.10) has singular points at the horizon(s) $r = r_\pm$ (or $r = r_H$ in the extremal limit) and at $r \to \infty$, and the classification into regular and irregular singular points depends on whether the stationary and/or extremal limits are taken. For non-stationary and sub-extremal perturbations, the two horizons are regular singular points, which remain regular in the stationary limit and become confluent and irregular in the extremal limit. In the simultaneous stationary and extremal limit, the degenerate horizon once again becomes regular. In all cases, $r \to \infty$ is an irregular singular point. This is summarised in table 7.1. The equation of motion for non-stationary extremal perturbations has two irregular singular points, so the solutions differ drastically from the other cases. This suggests that the stationary limit should be taken *before* the extremal limit.

The stationary solutions, for which we set $\omega = 0$ from the out-set, are not solutions that can be reached in the naïve limit as $\omega \to 0$ for finite-ω excitations. We should therefore think of these solutions as *deformations* of the background solution which are not dynamically excitable with finite energy.

The Near-Horizon Limit

In the sub-extremal case, it is convenient to parameterise

$$r_\pm = r_H(1 \pm \varepsilon), \quad \rho = r - r_+, \tag{7.1.14}$$

where r_H is the location of the extremal horizon. The near-horizon and near-extremal limits correspond to expansions in ε and ρ/r_H respectively. Since the location of the outer horizon is affected by the extremal limit, these will not be independent.

Consecutive expansions are unambiguous *e.g.* for the metric function f(r) itself. However, the standard form of the master equation (7.1.10) also involves factors of f^{-1}, which is not uniformly convergent in ε and ρ — consecutive expansions therefore do not commute. To avoid this ambiguity, our prescription will be to expand only $f(r)$ (and not its inverse) within the equations of motion, and then carefully take the respective limits of the solutions.

Physically, this corresponds to taking the perspective that extremal black holes should be seen as extremal limit of non-extremal black holes physics. More precisely, the near-horizon dynamics of *extremal* black hole perturbations are to be understood as the extremal limit of the near-horizon dynamics of *non-extremal* black hole perturbations. For singular geometries, one motivation for this is the criterion established in [279]. At its core, this states that zero-temperature singular so-

lutions are admissible if they can be reached as the zero-temperature limit of finite-temperature non-singular solutions. The physical motivation for this is that for such geometries, the would-be zero-temperature naked singularities would be shielded by the horizon of a regular geometry in the finite-temperature case.

7.2 Stationary Deformations

With the subtleties from the previous section 7.1.3 in mind, let us now solve (7.1.10) starting with the sub-extremal case in the near-horizon region. The extremal limit of these solutions are indeed found to be of the form (6.0.1), and we discuss the role of the near-horizon geometry in fixing this.

7.2.1 Sub-Extremal Deformations

In general, we can write

$$f(r) = f'(r_+)\rho + \frac{1}{2}f''(r_+)\rho^2 + \dots \tag{7.2.1}$$

within the near-horizon limit. Notably $f'(r_+) = \mathcal{O}(\varepsilon)$ and vanishes in the extremal limit, so let us define $\beta = \frac{2f'(r_+)}{\varepsilon f''(r_+)}$ which is finite in the extremal limit. In the near-horizon region, the master equation (7.1.10) can be approximated by

$$\left(\rho^2 + \varepsilon\beta\rho\right)\phi'' + \left(2\rho + \varepsilon\beta\right)\phi' - U_*\phi = 0, \tag{7.2.2}$$

where $U_* := 2U(r = r_+)/f''(r_+)$. This is the hypergeometric differential equation in disguise, and the solutions are Legendre's functions

$$\phi = AP_\alpha(z) + BQ_\alpha(z), \tag{7.2.3}$$

with integration constants A and B, and

$$\alpha = \frac{1}{2}\left(-1 + \sqrt{1 + 4U_*}\right), \quad z = 1 + \frac{2\rho}{\varepsilon\beta}. \tag{7.2.4}$$

The combination ρ/ε in the expression for z is symptom of the tension between the extremal and near-horizon expansions. In the strictly sub-extremal case, the near-horizon limit corresponds to $z \to 1$, and the non-extremal solution (7.2.3) takes the form

$$\phi \sim A\left[1 + \frac{1}{2}\alpha(\alpha-1)(z-1) + \dots\right] + B\left[c - \frac{1}{2}\log(z-1) + \dots\right], \tag{7.2.5}$$

for some constant c involving Digamma functions. Finite boundary conditions at the horizon require $B = 0$, and the remaining solution is regular on the horizon. We will discard the former branch of solutions even in the extremal case, regardless of behaviour at the boundaries, as they would not arise from regular non-extremal solutions.

7.2.2 Extremal Limit

The horizon ceases to be analytic in the extremal limit as z becomes large. Setting $B = 0$ from the out-set for finite boundary conditions at the horizon in the sub-extremal case, the asymptotic expansion of the solution (7.2.3) in ε is

$$\phi \sim A \left[\frac{\Gamma(-1-2\tilde{\alpha})}{\Gamma(-\tilde{\alpha})^2} \left(\frac{\beta\varepsilon}{\rho}\right)^{1+\tilde{\alpha}} + \cdots + \frac{\Gamma(1+2\tilde{\alpha})}{\Gamma(\tilde{\alpha})^2} \left(\frac{\beta\varepsilon}{\rho}\right)^{-\tilde{\alpha}} + \ldots \right], \tag{7.2.6}$$

where $\tilde{\alpha}$ is the extremal limit of α. The subleading terms come in the form of a power series in ε/ρ, so even for $\tilde{\alpha} \in \mathbb{C}$ we can simply rescale the integration constants by the most negative real power of ε respectively in order to obtain finite solutions in the extremal limit. What remains is

$$\phi = C\rho^{\tilde{\alpha}}, \tag{7.2.7}$$

where C is some constant. Had we not set $B = 0$ from the out-set, the other solution with exponent $-1-\tilde{\alpha}$ could have also survived. Note that these are precisely the deformations from [184, 270] of the form (6.0.1), obtained by directly solving the extremal limit of the master equation (7.1.10)

$$\rho^2\phi'' + 2\rho\phi' - \tilde{U}_*\phi = 0 \tag{7.2.8}$$

with $\tilde{U}_*$ the extremal limit of U_*. In terms of their notation, the scaling exponents are

$$\gamma_\pm = -\frac{1}{2} \pm \left(\tilde{\alpha} + \frac{1}{2}\right) = \frac{1}{2}\left(-1 \pm \sqrt{1+4\tilde{U}_*}\right). \tag{7.2.9}$$

We can understand the critical scaling of the near-horizon solutions from a holographic perspective [280]. The near-horizon limit of the extremal AdS RN metric

$$\mathrm{d}s^2 = \frac{2}{f''(r_H)} \left[-\rho^2 \left(\frac{f''(r_H)}{2}\mathrm{d}t\right)^2 + \frac{\mathrm{d}\rho^2}{\rho^2} \right] + r_H^2 \mathrm{d}\Omega_d^2 \tag{7.2.10}$$

is the Robinson-Bertotti solution, *i.e.* $\mathrm{AdS}_2 \times S^d$ with AdS_2 length scale $L_2 = 2/f''(r_H)$. We expect this AdS_2 -factor to be generic (under suitable assumptions) to the near-horizon geometry of extremal black holes even in theories with other matter fields and higher-derivative corrections [281, 282] — see [283] for a review on this. This means that we expect to find such critical scaling behaviour of stationary perturbations in the near-horizon limit even in such more general theories.

If we are only interested in finding the scaling dimensions themselves (which is the case), it is more straightforward to follow the procedure outlined in [184]. By using the isometries of the near-horizon geometry, the problem is reduced to finding $\mathrm{AdS}_2 \times S^d$ backgrounds and decomposing the perturbations with respect to the enhanced isometries thereof — details for this are given in appendix 7.B. The resulting perturbations equations are easier to decouple and will just be algebraic (quadratic) in γ.

7.2.3 Black Branes

We just saw that near-horizon deformations of extremal black holes are most easily studied by solving for stationary perturbations of the Robinson-Bertotti geometry $\mathrm{AdS}_2 \times S^d$. Higher-dimensional AdS spacetimes arise within the near-horizon geometry of black branes, so by starting within the near-horizon limit, we can generalise the discussion above to deformations of extremal black p-branes. We discuss some properties of these deformations, which also apply to the deformations of extremal black holes from before.

Consider Einstein gravity with zero cosmological constant minimally coupled to a $(p+1)$-form gauge field $A_{(p+1)}$ via the action

$$S = \int \mathrm{d}^D x \sqrt{-g} \left[\frac{1}{2\kappa} R - \frac{1}{2(p+2)!} F_{(p+1)} \wedge \star F_{(p+1)} \right], \tag{7.2.11}$$

for $F_{(p+2)} = \mathrm{d}A_{(p+1)}$. This admits $\mathrm{AdS}_{p+2} \times S^{d-p}$ Freund-Rubin spacetime, which is described by the metric

$$\mathrm{d}s^2 = \rho^2 \mathrm{d}x_a \mathrm{d}x^a + \frac{\mathrm{d}\rho^2}{\rho^2} + \mathrm{d}\Omega^2_{d-p}, \quad a \in \{0, \ldots, p\}, \tag{7.2.12}$$

as background solutions [284], where we have set the AdS_{p+2} length scale to unity for convenience.

As a special case, these Freund-Rubin geometries include the Robinson-Bertotti solution $\mathrm{AdS}_2 \times S^d$. As in the special case, these solutions also arise (up to a conformal factor) as the near-horizon geometries of extremal electrically charged black p-branes

$$\mathrm{d}s^2 = H^{-4\frac{d-p-1}{d\Delta}} \eta_{ab} \mathrm{d}x^a \mathrm{d}x^b + H^{4\frac{p+1}{d\Delta}} \left(\mathrm{d}\hat{\rho}^2 + \hat{\rho}^2 \mathrm{d}\Omega^2_{d-p} \right), \quad H = 1 + \left(\frac{r_0}{\hat{\rho}} \right)^{d-p-1}, \tag{7.2.13}$$

where $\Delta = 4/n$ for $\frac{1}{2^n}$-BPS black branes [285–287] [2].

The equation of motion for a test scalar field takes the form

$$-\rho^{-2} \partial_a \partial^a \phi + \rho^2 \partial_\rho^2 \phi + (p+2) \rho \partial_\rho \phi - \tilde{U}_* \phi = 0 \tag{7.2.14}$$

for some effective potential U. In the near-horizon limit, perturbations to these extremal black branes see the Freund-Rubin spacetime (7.2.12), so the equation of motion governing them takes the same form. The most natural extension to the *stationary* perturbations of black holes from section 7.2 are modes which satisfy the massless Klein-Gordon equation on the brane — these are plane waves localised to the brane with null wave vector, or more explicitly

$$\phi(x^a, x^i) \sim \phi(x^i)\, e^{i k_a x^a}, \quad k_a k^a = 0. \tag{7.2.15}$$

Indeed for $p = 0$, this reduces to $\omega = k^0 = 0$, and the modes are stationary by definition. Thus

[2] The M2- and M5- or F1- and NS5-branes in $D = 11$ and $D = 10$ supergravity respectively are well-known examples of $\frac{1}{2}$-BPS brane solutions [273, 274, 288].

we find the analogue of (7.2.8) generalised for black p-branes:

$$\rho^2\phi'' + (p+2)\rho\phi' - \tilde{U}_*\phi = 0. \tag{7.2.16}$$

The solutions to this are nonetheless scaling solutions of the form (6.0.1) now with

$$\gamma_\pm = \frac{p+1}{2}\left(-1 \pm \sqrt{1 + \frac{4}{(p+1)^2}\tilde{U}_*}\right) \tag{7.2.17}$$

instead of (7.2.9).

7.2.4 Singularities

A priori, there is no reason why $1 + 4\tilde{U}_*/(p+1)^2 \geq 0$, so we have $\gamma_\pm \in \mathbb{C}$ in principle. Splitting $\gamma_\pm = \gamma_R \pm i\gamma_I$ with $\gamma_R, \gamma_I \in \mathbb{R}$, the solution can be written as

$$\phi = c_R\rho^{\gamma_R}\cos(\gamma_I \log\rho) + c_I\rho^{\gamma_R}\sin(\gamma_I\log\rho) \tag{7.2.18}$$

for some constants $c_R, c_I \in \mathbb{R}$. When $\gamma_I \neq 0$ we always have $\gamma_R = -\frac{p+1}{2}$, and we see that a non-zero imaginary part leads to only a slightly worse singularity on the horizon in a manner which is not sensitive to the sign or magnitude of it. The qualitative nature of the horizon is therefore effectively determined solely by γ_R, the lower bound for which is $\gamma_{\text{min}} = -\frac{p+1}{2}$.

Now, recall that the field ϕ with near-horizon scaling in (7.2.7) was derived from the equations of motion governing perturbations propagating on extremal RN background geometries. In real terms, this translates to the following scaling for the perturbed metric and Weyl tensors in Schwarzschild-like coordinates:

$$h_{..} \sim \rho^\gamma, \quad \delta C_{....} \sim \rho^{\gamma-2}. \tag{7.2.19}$$

Either by changing coordinates $(t,\rho) \mapsto (t,z)$ with $z = 1/\rho$ or going into an orthonormal frame, one sees that all scalar invariants take the form

$$S = \rho^{n\gamma}, \quad n \in \mathbb{N}^+, \tag{7.2.20}$$

even when these are constructed with covariant derivatives acting on tensors. Given a causal geodesic with tangent vector u, normalised as $u^2 \in \{-1, 0\}$, one can construct a parallel-propagated orthonormal frame by taking $\hat{e}_0{}^\mu = u^\mu$, so that the component of the Riemann tensor entering the geodesic deviation equation takes the form

$$\hat{R}^1{}_{010} = \left(u^2\right)^2 R_{0101} \supset \left(u^2\right)^2 \rho^{\gamma-2}. \tag{7.2.21}$$

We therefore see that a *timelike* scalar polynomial (s.p.) or parallel-propagated (p.p.) singularity, in the sense of [160], occurs when $\gamma < 0$ or $\gamma < 2$ respectively. For details such as explicit construction of the vielbeine, see appendix 7.C.

In physical terms, p.p. singularities cause divergent tidal forces for point-particle observers — in other words, they lead to a breakdown of the worldline EFT, as was shown in [271]. We can think of these as an artefact of treating the observer as a point-particle, or more concretely the geodesic/geometric optics approximation. In the following, we will take the perspective that $\gamma_{\text{crit}} = 0$ is the more interesting critical value that divides regular from singular behaviour on the horizon. This is achieved when $\tilde{U}_* = 0$ irrespective of p, so we conclude that deformations of extremal black branes qualitatively do not differ much from ones of extremal black holes. It should be noted that other choices of γ_{crit} are also interesting, and there will be deformations corresponding to those.

At first thought, the presence of either type of singularity on the extremal horizon might be concerning, and prompt a dismissal of the solution. However, (7.2.5) shows that all of these singular solutions arise as non-extremal solutions that are regular at the horizon, so they are "good" and hence admissible singular solutions in the sense of [279]. For AdS RN black holes, [185] takes the perspective that these signal an RG instability in the IR of the dual field theory.

7.2.5 Holography

From the perspective of AdS_{p+2} holography, the critical exponents (7.2.17) can be understood as the negative scaling dimension of the operator dual to a scalar field with (effective) mass squared $\tilde{U}_*$. The solution regular on the horizon, with scaling exponent γ_+, is dual to an operator with scaling dimension

$$\Delta_+ := p + 1 + \gamma_+ \tag{7.2.22}$$

with *alternative boundary conditions* [289].

Recall that for the Einstein-Hilbert action supplied with the canonical boundary term (with which we associate Dirichlet boundary conditions at the conformal boundary), the bulk scalar field can have fall-off $(1/\rho)^{\Delta_+}$ and finite action when we find ourselves above the Breitenlohner-Freedman (BF) bound $m^2 \geq m^2_{BF} = -(p+1)^2/4$. However, there exists an alternative boundary term (with which we associate von Neumann boundary conditions at the conformal boundary), for which the scalar field can also have fall-off $(1/\rho)^{\Delta_-}$ with $\Delta_- := p + 1 - \Delta_+$ as long as we are in the BF *window*

$$-\frac{(p+1)^2}{4} \leq m^2 \leq -\frac{(p+1)^2}{4} + 1 \quad \longrightarrow \quad \frac{p-1}{2} \leq \Delta_- \leq \frac{p+1}{2}. \tag{7.2.23}$$

The lower bound on Δ_- is corresponds to the unitarity bound for scalar fields in AdS_{p+2} — see *e.g.* [290, 291] for details [3]. The lower bound on $\gamma_+ = -\Delta_- > -\frac{p+1}{2}$ hence corresponds to the upper bound on Δ_-, which ensures finite boundary action for the bulk fields. This fact motivates our conjecture in section 8.4.1.

Holography is helpful in shedding on light on (near-)extremal black holes, as their thermodynamic properties can be studied directly from the dynamics of the near-horizon perturbations.

[3] The statement that the same bulk Lagrangian can give rise to different theories depending on the boundary term typically goes under the name of standard/alternative quantisation in the context of holography.

We will discuss this in chapter 9.

7.A Spherical Symmetry Decomposition

In this appendix, we briefly summarise the spherical decomposition of the EM and gravitational perturbations in Einstein-Maxwell theory (7.1.1), most closely following [278] and building on [242, 243]. We focus on scalar perturbations as only they can lead to marginal deformations. Explicit expressions for the effective potentials are complicated and of secondary relevance, hence omitted.

7.A.1 Tensor Modes

The tensor harmonics $\mathbb{T}_{ij}$ satisfy

$$(\hat{\Delta} + k_T^2)\mathbb{T}_{ij} = 0, \quad \mathbb{T}^i{}_i = 0, \quad \hat{D}_i \mathbb{T}^i{}_j = 0, \tag{7.A.1}$$

where

$$k_T^2 = \ell(\ell + D - 3) - 2, \quad \ell = 1, 2, \dots \quad . \tag{7.A.2}$$

We parameterise our tensor metric and field strength perturbations as

$$\delta g_{ab} = 0, \quad \delta g_{ai} = 0, \quad \delta g_{ij} = 2r^2 H_T \mathbb{T}_{ij}, \tag{7.A.3a}$$

$$\delta F_{ab} = 0, \quad \delta F_{ai} = 0, \quad \delta F_{ij} = 0. \tag{7.A.3b}$$

The perturbation H_T is already gauge-invariant, and the tensor mode master variable satisfying the master equation (7.1.7) is defined to be

$$\Phi_T = r^{(D-2)/2} H_T. \tag{7.A.4}$$

7.A.2 Vector Modes

The vector harmonics $\mathbb{V}_i$ satisfy

$$(\hat{\Delta} + k_V^2)\mathbb{V}_i = 0, \quad \hat{D}_i \mathbb{V}^i = 0, \tag{7.A.5}$$

and we define from it vector-derived tensor harmonics as

$$\mathbb{V}_{ij} = -\frac{1}{k_V} \hat{D}_{(i} \mathbb{V}_{j)}, \tag{7.A.6}$$

where

$$k_V^2 = \ell(\ell + D - 3) - 1, \quad \ell = 1, 2, \dots \quad . \tag{7.A.7}$$

However, it can be shown that $\ell = 1$ corresponds to $\mathbb{V}_{ij}$ vanishing, and therefore needs to treated carefully.

We parameterise our vector metric and field strength perturbations as

$$\delta g_{ab} = 0, \quad \delta g_{ai} = r f_a \mathbb{V}_i, \quad \delta g_{ij} = 2r^2 H_T \mathbb{V}_{ij}, \tag{7.A.8a}$$

$$\delta F_{ab} = 0, \quad \delta F_{ai} = D_a \mathcal{A} \mathbb{V}_i, \quad \delta F_{ij} = 2\mathcal{A}\hat{D}_{[i}\mathbb{V}_{j]}, \tag{7.A.8b}$$

and define the gauge-invariant variables via

$$r^{D-3}\left(f_a + \frac{r}{k_V} D_a H_T\right) = g_{ab}\epsilon^{bc} D_c \left(\Omega + \frac{2(D-3)\kappa q}{m_V}\mathcal{A}\right). \tag{7.A.9}$$

There are now two distinct vector mode master variables satisfying the master equation (7.1.7), and they are linear combinations of the gauge-invariant EM and gravitational perturbations

$$\Phi_V^{\pm} = a_{\pm}\Omega + b_{\pm}\mathcal{A}, \tag{7.A.10}$$

for some constants $a_\pm$ and $b_\pm$. In the decoupling limit, $a_+ = 0$ and $b_- = 0$, so we refer to Φ_V^+ and Φ_V^- as the EM and gravitational vector modes respectively. Only the EM $\ell = 1$ mode is dynamical, while the gravitational counterpart corresponds to adding rotation to the background solution.

7.A.3 Scalar Perturbations

The scalar harmonic $\mathbb{S}$ satisfies

$$(\hat{\Delta} + k_S^2)\mathbb{S} = 0, \tag{7.A.11}$$

and we define from it scalar-derived vector and tensor harmonics:

$$\mathbb{S}_i = -\frac{1}{k_S}\hat{D}_i\mathbb{S}, \quad \mathbb{S}_{ij} = \frac{1}{k_S^2}\hat{D}_i\hat{D}_j\mathbb{S} + \frac{1}{D-2}\gamma_{ij}\mathbb{S}, \tag{7.A.12}$$

where

$$k_S^2 = \ell(\ell + D - 3), \quad \ell = 0, 1, \ldots \quad . \tag{7.A.13}$$

However, $\ell = 0$ has constant eigenmodes and therefore corresponds to just a shift in the background parameters. The $\ell = 1$ mode corresponds to $\mathbb{S}_{ij}$ vanishing, and therefore needs to be treated carefully.

We parameterise our scalar metric and field strength perturbations as

$$\delta g_{ab} = f_{ab}\mathbb{S}, \quad \delta g_{ai} = r f_a \mathbb{S}_i, \quad \delta g_{ij} = 2r^2\left(H_L\gamma_{ij}\mathbb{S} + H_T\mathbb{S}_{ij}\right), \tag{7.A.14a}$$

$$\delta F_{ab} = -\frac{1}{k_S}\epsilon_{ab}\mathbb{S}D_c B^c, \quad \delta F_{ai} = \epsilon_{ab}\mathbb{S}_i B^b, \quad \delta F_{ij} = 0, \tag{7.A.14b}$$

and define the gauge-invariant variables by

$$F = H_L + \frac{1}{D-2}H_T + \frac{1}{r}D_a r X^a \tag{7.A.15a}$$

$$F_{ab} = f_{ab} + 2D_{(a}X_{b)} \tag{7.A.15b}$$

with

$$X^a = g^{ab}\frac{r}{k_S}\left(f_b + \frac{r}{k_S}D_b H_T\right) \tag{7.A.16a}$$

$$B^b = \left(r\mathcal{E}^b + k_S E_0 X^b\right). \tag{7.A.16b}$$

There are two master variables satisfying the master equation (7.1.7), the EM and gravitational scalar modes Φ_S^+ and Φ_S^- respectively, that are related to the gauge-invariant variables above via

$$F = a_-\Phi_S^- + b_-\partial_r\Phi_S^- + a_+\Phi_S^+ + b_+\partial_r\Phi_S^+, \tag{7.A.17a}$$

$$\mathcal{E}_0 = c_-\Phi_S^- + c_+\Phi_S^+, \tag{7.A.17b}$$

$$\mathcal{E}_1 = \tilde{a}_-\Phi_S^- + \tilde{b}_-\partial_r\Phi_S^- + \tilde{a}_+\Phi_S^+ + \tilde{b}_+\partial_r\Phi_S^+, \tag{7.A.17c}$$

for some constants $a_\pm$, $b_\pm$, $\tilde{a}_\pm$, $\tilde{b}_\pm$, and $c_\pm$. For $\ell = 1$, only the EM scalar mode is gauge-invariant and dynamical.

7.B Near-Horizon Isometries Decomposition

In this appendix, we will briefly outline how to compute the critical exponents by starting within the near-horizon limit, following [184].

7.B.1 Background

First consider the background. In ingoing coordinates (v, r, x^i) with $\mathrm{d}v = \sqrt{A/B}\,\mathrm{d}t + \mathrm{d}r/B$, the most general static and spherically symmetric Ansatz (7.1.5) becomes

$$\mathrm{d}s^2 = -B\mathrm{d}v^2 + 2\mathrm{d}v\,\mathrm{d}r + r^2\mathrm{d}\Omega_d^2, \tag{7.B.1a}$$

$$F = \sqrt{\frac{B}{A}}\Psi'(r)\,\mathrm{d}v\wedge\mathrm{d}r, \tag{7.B.1b}$$

so in the near-horizon limit

$$\mathrm{d}s^2 = -B_{\mathrm{NH}}\rho^2\mathrm{d}v^2 + 2\mathrm{d}v\,\mathrm{d}\rho + r_H^2\mathrm{d}\Omega_d^2, \tag{7.B.2a}$$

$$F = \sqrt{\frac{B_{\mathrm{NH}}}{A_{\mathrm{NH}}}}F_{\mathrm{NH}}\mathrm{d}v\wedge\mathrm{d}\rho, \tag{7.B.2b}$$

where $A_{\mathrm{NH}} = \frac{1}{2}A''(r_H)$, $B_{\mathrm{NH}} = \frac{1}{2}B''(r_H)$, and $F_{\mathrm{NH}} = \Psi'(r_H)$ are constants. This is the form of the $\mathrm{AdS}_2 \times S^d$ Ansatz we will use. It is given in Gaussian null coordinates, which are a natural and well-defined set of coordinates near null surfaces.

7.B.2 Perturbations

Using the isometries of the AdS_2 factor in the near-horizon geometry, we can decompose the metric and field strength perturbations in the scalar sector as

$$\delta g = \rho^\gamma \Bigg[\sqrt{\frac{A_{\text{NH}}}{B_{\text{NH}}^3}} f_{00}\, \mathbb{S} \left(\sqrt{A_{\text{NH}} B_{\text{NH}}}\rho^2\, \mathrm{d}v^2 + 2\mathrm{d}v\, \mathrm{d}\rho + \frac{1}{\sqrt{A_{\text{NH}} B_{\text{NH}}}} \rho^{-2} \mathrm{d}\rho^2 \right) \tag{7.B.3a}$$

$$+ 2 f_0\, \mathbb{S}_i\, \mathrm{d}x^i \left(\rho\, \mathrm{d}v + \frac{1}{\sqrt{A_{\text{NH}} B_{\text{NH}}}} \rho^{-1} \mathrm{d}\rho \right) + (h_L \gamma_{ij} \mathbb{S} + h_T \mathbb{S}_{ij}) \mathrm{d}x^i \mathrm{d}x^j \Bigg] \tag{7.B.3b}$$

$$\delta F = \rho^\gamma \left[a\, \mathbb{S}\, \mathrm{d}v \wedge \mathrm{d}\rho + \rho\, e_0\, \mathbb{S}_i\, \mathrm{d}v \wedge \mathrm{d}x^i + \rho^{-1} e_1\, \mathbb{S}_i\, \mathrm{d}\rho \wedge \mathrm{d}x^i \right], \tag{7.B.3c}$$

where f_{00}, f_0, h_L, h_T, a, e_0, and e_1 are constants. In order for δF to be closed, we also need

$$a = -\frac{\gamma+1}{k_S} e_0. \tag{7.B.4}$$

7.B.3 Explicit Solution in $D = 5$

For concreteness, let us give explicit expressions for the perturbations in Einstein-Maxwell theory (7.1.1) for $D = 5$. The near-horizon Ansatz solves the background equations of motion for

$$A_{\text{NH}} = 2F_{\text{NH}}^2 \kappa \frac{\sigma}{1+2\sigma}, \quad B_{\text{NH}} = \frac{4\sigma}{r_H^2} \tag{7.B.5}$$

and the perturbation equations are solved by

$$h_L = 0 \tag{7.B.6a}$$

$$f_{00} = \frac{h_T}{6 r_H^4 (\gamma+1)(\gamma+2)} \left(16 - 3k_S^2 + 28\gamma(\gamma+1)\sigma \right) \tag{7.B.6b}$$

$$f_0 = \frac{h_T}{6 k_S r_H^2} \left(-\frac{k_S^2}{\gamma+1} - 12\gamma\sigma \right) \tag{7.B.6c}$$

$$a = -\frac{\gamma+1}{k_S} e_0 = \frac{k_S}{r_H^2 \gamma} e_1 = \frac{h_T}{4\sqrt{2}} \frac{4 - k_S^2 + 4\gamma(\gamma+1)\sigma}{\kappa r_H^3 \sqrt{1+2\sigma}} \tag{7.B.6d}$$

up to gauge transformations.

7.C Tidal Forces

Let us derive the tidal force felt by an observer crossing the horizon of such a deformed geometry.

7.C.1 Geodesics in Spherical Symmetry

Let us first review the geodesics on the background.

Consider a spherically symmetric spacetime, *i.e.* with metric

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_d^2. \tag{7.C.1}$$

Let u be the tangent vector to a (affinely parameterised) geodesic, so the appropriate Lagrangian is $L = -g_{\mu\nu}u^\mu u^\nu$. From the Euler-Lagrange equation for θ, we see that we can pick $\theta = \pi/2$ without loss of generality. Further, the spacetime has two Killing vectors

$$k = \frac{\partial}{\partial t}, \quad m = \frac{\partial}{\partial \phi}, \tag{7.C.2}$$

which are associated to the conserved quantities

$$E = -k \cdot u = f\dot{t}, \quad h = m \cdot u = r^2 \sin(\theta)^2 \dot{\phi}, \tag{7.C.3}$$

which are the energy and the angular momentum. We can therefore write

$$u = \left(\frac{E}{f}, \dot{r}, 0, \frac{h}{r^2} \right). \tag{7.C.4}$$

Let us further restrict ourselves to an orbit with no angular momentum, *i.e.* $h = 0$. In order for the geodesic to be properly normalised,

$$\sigma = u^2 = -\frac{E^2}{f} + \frac{\dot{r}^2}{f} \tag{7.C.5}$$

so that for a timelike geodesic ($\sigma = -1$),

$$\frac{\dot{r}^2}{2} = \frac{E^2 + \sigma f(r)}{2}, \tag{7.C.6}$$

which is the equation of motion we want.

7.C.2 Local Inertial Frame

The (inverse) orbit space vielbein basis satisfying $e_0^\mu = u^\mu$ amongst the other conditions mentioned is

$$\begin{aligned} \hat{e}_0{}^\mu &= \left(\frac{E}{f}, -\sqrt{E^2 + \sigma f}, \mathbf{0} \right) \\ \hat{e}_1{}^\mu &= \left(-\frac{\sqrt{E^2 + \sigma f}}{f}, E, \mathbf{0} \right), \end{aligned} \tag{7.C.7}$$

or in terms of the orbit space vielbeine themselves

$$e^0 = E\,\mathrm{d}t + \frac{\sqrt{E^2 + \sigma f}}{f}\mathrm{d}t, \quad e^1 = \sqrt{E^2 + \sigma f}\mathrm{d}t + \frac{E}{f}\mathrm{d}r. \tag{7.C.8}$$

This can be inverted to give

$$\sigma \mathrm{d}r = \sqrt{E^2 + \sigma f} e^0 - E e^1, \quad \sigma \mathrm{d}t = -\frac{E}{f} e^0 + \frac{\sqrt{E^2 + \sigma f}}{f} e^1, \tag{7.C.9}$$

which is notably degenerate for null geodesics. Using Cartan structure equations, the only non-vanishing orbit space torsion-free connection coefficient is

$$\omega^1{}_0 = \frac{f'}{2}\mathrm{d}t + \frac{f'}{2f}\frac{E}{\sqrt{E^2+\sigma f}}\mathrm{d}r. \tag{7.C.10}$$

The base space vielbeine are the usual ones in spherical symmetry, and while generically $\omega_{ai} \neq 0$ (for $a, b, \cdots \in \{0, 1\}$ and $i, j, \cdots \in \{2, \ldots, D-1\}$) because the orbit space vielbeine are non-diagonal, $\omega^i{}_a \sim e^i$ and therefore only have spacetime components in the base space directions. For example, in $D = 4$ spacetime dimensions

$$\sigma \mathrm{d}e^3 = \sin\theta \mathrm{d}r \wedge \mathrm{d}\phi + r\cos\theta \mathrm{d}\theta \wedge \mathrm{d}\phi = -\frac{\sqrt{E^2+\sigma f}}{r}e^3 \wedge e^0 + \frac{E}{r}e^3 e^1 - \frac{\cot\theta}{r}e^3 \wedge e^2 \tag{7.C.11}$$

so that

$$\begin{aligned}
\sigma\omega^3{}_0 &= \frac{\sqrt{E^2+\sigma f}}{r}e^3 = \sqrt{E^2+\sigma f}\sin\theta \mathrm{d}\phi \\
\sigma\omega^3{}_1 &= -\frac{E}{r}e^3 = E\sin\theta \mathrm{d}\phi \\
\sigma\omega^3{}_2 &= \frac{\cot\theta}{r}e^3 = \cos\theta \mathrm{d}\phi.
\end{aligned} \tag{7.C.12}$$

7.C.3 Tidal Forces

However, the claim in [270] is that the tidal forces diverge for $\gamma < 2$. Indeed, we will see that the divergent components of the Riemann tensor enter the geodesic deviation equation.

Geodesic Deviation on Extremal Horizons

Given a congruence of geodesics with tangents u^μ and deviation vectors η^μ, the geodesic deviation equation reads

$$\frac{D^2\eta^\mu}{D\lambda^2} = -R^\mu{}_{\alpha\nu\beta}u^\alpha u^\beta \eta^\nu. \tag{7.C.13}$$

To see this, we follow [292] and switch to a (inverse) tetrad basis $\hat{e}_a{}^\mu$ of radially free-falling observers — in other words, a tetrad basis which is parallely propagated: $D\hat{e}_a{}^\mu/D\lambda = 0$. More concretely, this can be done by picking

$$\hat{e}_0{}^\mu = u^\mu, \tag{7.C.14}$$

where u^μ is the unit tangent vector to the geodesic. Since $u \cdot \eta = 0$, we also have $\hat{\eta}_0 = 0$.

Since the tetrad basis is parallely propagated, the geodesic deviation equation simplifies to

$$\frac{D^2\hat{\eta}^a}{D\lambda^2} = -\hat{R}^a{}_{0b0}\hat{\eta}^b. \tag{7.C.15}$$

The radial component is the one that describes the radial force an observer would feel as they cross the horizon along a null geodesic — this is therefore the component of physical interest.

The relevant equation is

$$\frac{D^2\hat{\eta}^1}{D\lambda^2} = -\hat{R}^1{}_{010}\hat{\eta}^1, \tag{7.C.16}$$

so the component of interest, in tetrad basis, is

$$\hat{R}^1{}_{010} = \left(\hat{e}_0{}^0\hat{e}_1{}^1 - \hat{e}_0{}^1\hat{e}_1{}^0\right)^2 R_{0101} = \sigma^2 R_{0101}. \tag{7.C.17}$$

Since $R \supset \bar{R} + \delta R$, we see that for $\delta R_{0\rho 0\rho} \sim \rho^{\gamma-2}$, the right-hand side of this diverges for $\gamma < 2$ for timelike geodesics. This is the radial force felt by observers following geodesics crossing the horizon, and is therefore argued to be physical. The tidal forces do not diverge in the angular directions, so scalar contractions of this (contracted with just $\hat{\eta}$) also inherit the tidal force singularity from the radial component. It is therefore physical.

Chapter 8

Extremal AdS RN and the UV

We now turn to near-horizon deformations of extremal AdS RN black holes in the EFT of gravity. In particular, we will closely examine the critical exponents in GR and identify a tower of marginal deformations subject to UV sensitivity in section 8.1, before discussing UV corrections in extended versions of the two examples in the introduction in section 8.2. In section 8.3 we show that this UV sensitivity is associated to a breakdown of metric perturbation theory rather than EFT, and conclude by presenting a speculative conjecture about constraints on EFT corrections in section 8.4.

Explicit expressions for the critical exponents for deformations to the near-horizon region of extremal Reissner-Nordström black holes are given in appendix 8.A. In appendix 8.B, we give expressions for higher-order terms in the expansion of the Einstein-Hilbert in metric perturbations.

8.1 Critical Exponents

The critical exponents for the tensor, vector, and scalar deformations to extremal AdS RN in pure Einstein-Maxwell theory (7.1.1) are given in (8.A.1). They are derived from (7.2.9), with the effective potentials derived from the spherical decomposition described in appendix 7.A and explicitly given in [278]. These are functions of the horizon size r_H and the multipole moments $\ell \in \{2, 3, \dots\}$.

The tensor, vectors and EM scalar modes all have positive-definite critical exponents, so they are not associated to s.p. singularities on the horizon and do not allow for marginal cases — these are of no interest to us. However, the critical exponent for gravitational scalar modes can become arbitrarily small in magnitude and even negative, which is associated to s.p. singularities on the horizon — this is exacerbated for larger black holes and in higher dimensions. More specifically, in $D = 4$, the $\ell = 2$ and $\ell = 3$ modes are always p.p. singular, and the larger the black hole is, the more modes with $\ell > 3$ are also p.p. singular. Deformations of extremal AdS RN in $D \geq 5$ are always s.p. singular for $2 \leq \ell \leq D - 3$ and for sufficiently large black holes with $\ell > D - 3$.

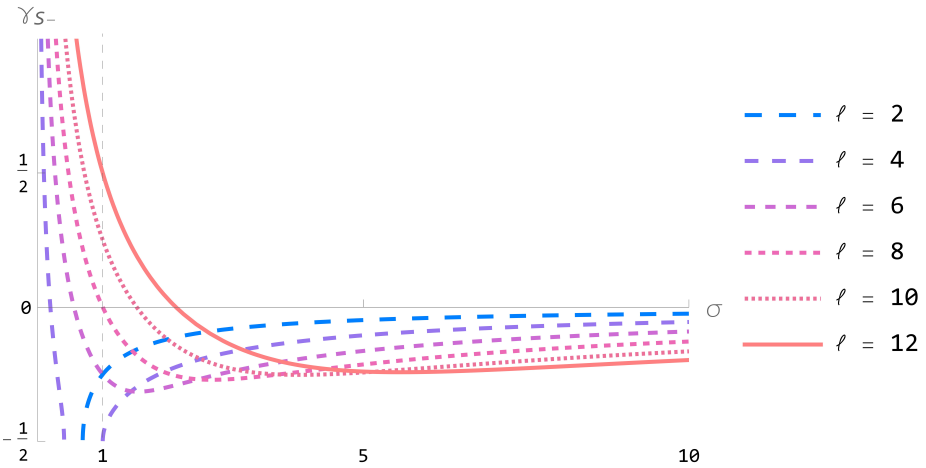


Figure 8.1: The (more regular) scaling dimension γ_+ for gravitational scalar modes in $D = 11$ for various ℓ.

In particular, the scaling is marginal, *i.e.* $\gamma_{S-} = 0$, and hence vulnerable to UV corrections when

$$\sigma = -\frac{2}{D-4} + \frac{D-2}{2(D-4)(D-3)^2}\ell(\ell+D-3). \tag{8.1.1}$$

This can be achieved for $\ell = 2$ in $D = 5$ or $\ell \geq D-3$ in $D > 5$. As an example, figure 8.1 shows γ_{S-} for several ℓ-modes in $D = 11$. In the full non-linear theory, generically all modes will be excited, so deformations to extremal AdS RN black holes in $D = 4$ and $D > 4$ will generically be p.p.- and s.p. singular respectively. This means that deformations to extremal AdS RN black holes are generically singular [184, 185].

Analytic continuation to $\Lambda > 0$ is achieved by extending the domain of σ from $[1, \infty)$ to $[0, 1)$. Modes with $\ell > D-3$ on backgrounds with sufficiently large black holes are still singular, but for $\ell \leq D-3$ only some black holes are.

8.1.1 AF Limit

The AF limit is given by taking $\sigma \to 1$ and is degenerate with the small black hole limit. As before, the critical exponents are positive-definite for the tensors, vectors, and EM scalars for all physical modes, while for the gravitational scalar modes

$$\gamma_{S-} = -\frac{1}{2} + \frac{|\ell - \frac{D-3}{2}|}{D-3}. \tag{8.1.2}$$

Similar to AdS RN, this can be arbitrarily close to zero and even negative. In $D = 4$, we find that $\gamma_{S-} \in \mathbb{N}^+$, so the perturbations are smooth on the horizon. However, in higher dimensions γ_{S-} is generally fractional, indicating non-smoothness of the horizon. The second type of s.p. or p.p. singular modes, *i.e.* ones with $\ell > 3$ in $D = 4$ or $\ell > D-3$ in higher dimensions, is absent. We therefore only find negative γ_{S-} and hence an s.p. singularity for $\ell < D-3$ going down to $\gamma = -1/2$, and the marginal $\gamma_{S-} = 0$ for $\ell = D-3$. This confirms that there indeed exist perturbations to extremal black holes in $D > 5$ that are s.p. singular at the horizon, and

we expect that non-linear solutions will generally be s.p. singular at the horizon. Remarkably, deformations to AF RN in $D = 5$ remain marginal even fully non-linearly.

8.1.2 Boundary Conditions

A stationary perturbation of a RN black hole with appropriate boundary conditions at the horizon and asymptotic infinity constitutes a linearisation of a black hole solution that is close to the original RN geometry, and should be subject to uniqueness theorems [293, 294]. More explicitly, the effective potentials and their so-called S-deformations given in [278] can be used to show that extremal modes that are finite at the horizon are inconsistent with the right fall-off conditions at asymptotic infinity, establishing perturbative uniqueness.

This means that additional sources able to change the asymptotic boundary conditions are required to support near-horizon modes with the desired near-horizon scaling and fall-off conditions at infinity. We can understand this from the extremal limit of the sub-extremal solution (7.2.6) — scaling in ε will generically be different at asymptotic infinity, so the extremal limit will produce non-normalisable or trivial modes in the absence of external sources. Physically, these deformations should therefore be thought of as sourced by tidal deformations.

An explicit realisation of this are multi-black hole solutions, which are known exact solutions in Einstein-Maxwell theory. In practice a convenient way of realising this is to simply impose Dirichlet boundary conditions (BCs) at finite distance, which is particularly natural in asymptotically AdS spaces.

Multi-Black Hole Solutions

The Majumdar-Papapetrou solution is an illustrative example of an explicit non-linear realisation of the deformations above [295]. This is an exact solution to the Einstein-Maxwell equations (7.1.3) and (7.1.4) given by the metric

$$\mathrm{d}s^2 = -H^{-2}\mathrm{d}t^2 + H^{\frac{2}{D-3}}\eta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad H(\mathbf{x}) = 1 + \sum_{i=1}^{N}\frac{M_i}{|\mathbf{x}-\mathbf{x}_i|^{D-3}}, \tag{8.1.3}$$

and describes a static spacetime with N extremal black holes with masses M_i and horizons at $\mathbf{x}_i$. By linearising this exact solution around the horizon of any individual black hole, we can see how it is tidally deformed by the other ones.

For convenience, let us pick $\mathbf{x}_1 = \mathbf{0}$ as the reference. Denoting $r_i = |\mathbf{x}_i|$ and $\hat{\rho} = |\mathbf{x}|$, and defining θ_i by $\mathbf{x}\cdot\mathbf{x}_i = \hat{\rho}\, r_i \cos\theta_i$, we can write

$$H = \frac{\hat{\rho}^{D-3}+M}{\hat{\rho}^{D-3}}\left[1+g(\hat{\rho})\right], \quad g(\hat{\rho}) = \frac{1}{\hat{\rho}^{D-3}+M}\sum_{i=1}^{N-1}\sum_{j=0}^{\infty} m_{ij}(\theta_i)\left(\frac{\hat{\rho}}{r_i}\right)^{D-3+j} \tag{8.1.4}$$

where $m_{ij}(\theta_i) = M_i C_j^{\left(\frac{D-3}{2}\right)}(\cos\theta_i)$ and $C_j^{(\alpha)}$ are Gegenbauer polynomials. The natural Schwarzschild-

like radial coordinate is $r = \hat{\rho} H^{1/(D-3)}$, with which the metric takes the form

$$\mathrm{d}s^2 = -f\left(1 - \frac{g}{\sqrt{f}}\right)^2 \mathrm{d}t^2 + \left(1 - \frac{M}{r^{D-3}}\frac{g}{\sqrt{f}} + \frac{1}{2}\hat{\rho} H g'\right)^{-2} \frac{\mathrm{d}r^2}{f} + r^2 \mathrm{d}\Omega_d^2. \tag{8.1.5}$$

We see that perturbations to the background metric scale as

$$h \sim \hat{\rho}^{D-3+j-1}, \quad j \in \mathbb{N}. \tag{8.1.6}$$

The near-horizon coordinate identified previously is $\rho = r - r_H \sim \hat{\rho}^{D-3}$, so this corresponds to the near-horizon scaling (6.0.1) with scaling exponent

$$\gamma = 1 + \frac{j-1}{D-3}. \tag{8.1.7}$$

However, $j \in \{0, 1\}$ are gauge modes related to the mono- and dipole, *i.e.* our choice of reference black hole and near-horizon coordinate. Identifying $j-1 = \ell$, we see that these are the EM scalar modes in (8.A.1). In $D > 4$ this is not generally an integer, which is merely a manifestation of the fact that horizons of multi-black hole solutions in $D = 4$ and $D > 4$ are smooth and generically non-smooth respectively [265–269].

8.2 Corrections from the UV

A notable feature of the gravitational scalar modes is that for every $\ell = 2$ in $D = 5$ and $\ell \geq D-3$ in $D > 5$, there exists a r_H/L via σ for which the near-horizon scaling is marginal, *i.e.* $\gamma_{S-} = 0$ [1]. In particular, we see from (8.1.2) that the $(\ell, \sigma) = (D-3, 1)$-deformation of AdS RN black holes is always marginal, while for higher ℓ it is just a matter of finding a sufficiently large σ. This opens up an intriguing possibility for UV sensitivity — it was first noticed in [270] for AF-RN backgrounds, although a morally similar example was discussed in *e.g.* [296].

What do we actually mean by UV sensitivity? We generally expect UV completions of GR to correct the action (7.1.1) by higher-derivative terms. The effect of these is to change the effective potential (and possibly the effective propagation speed) of decoupled modes in (7.1.10), which show up as a correction to the near-horizon scaling (6.0.1) that can be *resummed* into the critical scaling exponents into the form of (6.0.2). Schematically, given some higher-derivative operators $\mathcal{O}$ with mass dimension $[\mathcal{O}]$ supplied with Wilson coefficients $c_{\mathcal{O}}$, we expect

$$S = S_{\text{EM}} + \int \mathrm{d}^D x \sqrt{-g} \sum_{\mathcal{O}} \frac{c_{\mathcal{O}}}{\Lambda^{[\mathcal{O}]-D}} \mathcal{O} \quad \longrightarrow \quad \gamma_{\text{EFT}} = \sum_{\mathcal{O}} c_{\mathcal{O}} \gamma_{\mathcal{O}}. \tag{8.2.1}$$

Let us take $\hat{\gamma}$ to denote the scaling exponent in the marginal case, *i.e.* when $\gamma_{\text{GR}} = 0$. The existence of singularities on the horizon of the deformed geometry depends on is very sensitive to the sign of this quantity $\hat{\gamma}$ regardless of how small its magnitude is. This in turn depends on the signs of the $c_{\mathcal{O}}$, which encode the details of the UV completion — this is a form of UV

[1] Once again, we take marginal deformations to have $\gamma = 0$ because we are interested in s.p. singularities (or the lack thereof). However, other choices of γ_{crit} are also possible, and it remains true that deformations for some particular (ℓ, σ) will be marginal with respect to that γ_{crit}.

sensitivity and is in tension with what we would expect from cherished EFT principles such as decoupling and separation of scales.

Of course, for UV physics to cause a qualitatively distinct effect, we have to excite individual marginal ℓ-modes only with completely regular modes — this is difficult to achieve non-linearly. We already saw that non-linear deformations of AdS RN in $D \geq 5$ and AF RN in $D > 5$ generically suffer from s.p. singularities on their horizons, so there is no UV sensitivity since the EFT corrections do not change the qualitative behaviour. However, no deformation of AF RN in $D = 5$ is s.p. singular, and the lowest harmonic $\ell = 2$ is just marginal. In this case, we therefore do not expect deformations in GR to be s.p. singular even fully non-linearly, and EFT corrections can cause an s.p. singularity via the marginal mode depending on the sign of the Wilson coefficient. This means that deformations in $D = 5$ are truly marginal even non-linearly, so we will focus on the $D = 5$ case from now on unless specified otherwise.

As a concrete example of corrections to the critical scaling exponents coming from modifications within the IR and the UV, let us revisit the example from the introduction in more detail. For these, it is convenient to make full use of the near-horizon geometry symmetries following the procedure in appendix 7.B.

8.2.1 IR Modification: EFT Correction

We take the action to be

$$S = S_{\text{EM}} + \frac{\kappa c}{\Lambda^2} \int \mathrm{d}^5 x \sqrt{-g} (F_{\mu\nu} F^{\mu\nu})^2, \tag{8.2.2}$$

where S_{EM} is given in (7.1.1). It is a specific yet generic correction that one would expect in a generic low-energy EFT, which is parameterised by the Wilson coefficient c.

The theory admits spherically symmetric, static, and electrically charged background solutions of the form (7.1.5) with metric functions and electrostatic potential

$$A(r) = B(r) = f(r) - \frac{c}{\Lambda^2} \frac{12 Q^4}{r^{10}}, \quad \Psi(r) = \frac{q}{r^2} - 16 c \frac{\kappa}{\Lambda^5} \frac{q^3}{r^8}, \tag{8.2.3}$$

and $f(r)$ is given in (7.1.6). The extremal limit is reached when

$$\frac{Q^2}{r_H^4} = 1 + 2\varsigma + c \frac{48}{\Lambda^2 r_H^2} (1 + 2\varsigma)^2, \quad \frac{M}{2 r_H^2} = 1 + \frac{3}{2}\varsigma + c \frac{18}{\Lambda^2 r_H^2} (1 + 2\varsigma)^2 \tag{8.2.4}$$

and, particularly, in the AF limit

$$\frac{2|Q|}{M} = 1 + \frac{6}{\Lambda^2 r_H^2} c. \tag{8.2.5}$$

The fact that the charge-to-mass ratio at which extremality is reached also receives corrections is responsible for bounds from the WGC [155].

Stationary perturbations of these exhibit critical scaling in the near-horizon region, for which the critical exponents can be found directly by using the background Ansatz and decomposition of perturbations in appendix 7.B. For the gravitational scalar modes, these take the form of (6.0.2),

with $\gamma_{\rm GR} = \gamma_{S-}$ once again given in (8.A.1), and

$$\gamma_{\rm EFT} = \frac{c}{\Lambda^2 r_H^2} \Delta\, \delta\gamma_7, \tag{8.2.6}$$

with Δ and $\delta\gamma_7$ given around (8.A.9) respectively. In $D = 5$ the GR contribution vanishes when $\sigma = -2 + 3k_S^2/8$, where $k_S^2 = \ell(\ell+2)$, and

$$\hat{\gamma} = -\frac{c}{\Lambda^2 r_H^2} \frac{72 k_S^2 (k_S^2-4)^2}{15 k_S^4 - 128 k_S^2 + 256}. \tag{8.2.7}$$

This is negative-definite for all $\ell \geq 2$. In the AF limit, the marginal mode corresponds to $\ell = 2$, in which case the expressions above reduce to the ones given in the introduction.

8.2.2 Partial UV Completion: Tower of Massive Spin-0

Next, consider Einstein-Maxwell-Dilaton theory in D dimensions, with N scalars with masses m_i for $i \in \{1, \dots N\}$, described by the following action [2]

$$S_{\rm EMD} = \int \mathrm{d}^D x \sqrt{-g} \left[\frac{1}{2\kappa}(R - 2\Lambda) + \sum_{i=1}^{N} \left(-\frac{1}{4N} e^{\alpha\phi_i} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \nabla_\mu \phi_i \nabla^\mu \phi_i - \frac{1}{2} m_i^2 \phi_i^2 \right) \right]. \tag{8.2.8}$$

We want to treat α perturbatively, and have chosen our normalisation such as to reproduce the Einstein-Maxwell action (7.1.1) as $\alpha \to 0$ and the example (6.0.8) from the introduction for $N = 1$.

Our claim is that this is actually a partial UV completion to Einstein-Maxwell theory with higher-derivative corrections when the dilaton masses m_i are very large relative to the background curvature. In this case, we can ignore the kinetic term for the dilaton so, treating α perturbatively, the individual equations of motion for the dilatons become

$$m_i^2 \phi_i = -\frac{1}{4N} \alpha F_{\mu\nu} F^{\mu\nu} \left[1 + \mathcal{O}\left(\frac{\alpha^2}{m_i^2} \right) \right]. \tag{8.2.9}$$

At energies $E \ll m_i$, we can find the tree-level effective action by evaluating the full action on the saddle point for the dilaton, *i.e.* its classical equations of motion, to give

$$S_{\rm EMD} \supset \int \mathrm{d}^D x \sqrt{-g} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha^2}{32 N^2 m_{\rm eff}^2} (F_{\mu\nu} F^{\mu\nu})^2 + \dots \right], \tag{8.2.10}$$

where we have defined the effective mass $m_{\rm eff}$ via

$$\frac{1}{m_{\rm eff}^2} = \sum_i \frac{1}{m_i^2}. \tag{8.2.11}$$

We identify this truncated effective theory with the EFT-corrected theory (8.2.2) with Wilson coefficient $c = 1/32$ and the Lorentz-invariant cut-off $\Lambda^2 = N^2 \kappa m_{\rm eff}^2/\alpha^2$. More generally, the

[2] A coupling of the form $\mathcal{L} \supset F^2 e^{\alpha \sum_i \phi_i}$ would have been more natural, but the equations of motion for the dilatons are less straightforward to decouple.

low-energy dynamics of the massive dilatons are captured by an $F^{2(n+1)}$-term at order $\mathcal{O}(\Lambda^{-2n})$.

Let us consider $D = 5$ again from now on. The full theory admits $\mathrm{AdS}_2 \times S^3$ background solutions given in (8.A.14), and we can use the decomposition (7.B.3) to find the critical scaling exponents. These take the form of

$$\gamma = \gamma_{\text{GR}} + \frac{\alpha^2}{N^2 r_H^2 m_{\text{eff}}^2 \kappa} \delta\gamma, \tag{8.2.12}$$

with γ_{GR} as before and $\delta\gamma$ given in (8.A.15) respectively. When the GR contribution vanishes, we now find

$$\hat{\gamma} = -\frac{\alpha^2}{N^2 r_H^2 m_{\text{eff}}^2 \kappa} \frac{3k_S^2(k_S^2-4)^2\left[2r_H^2 m_{\text{eff}}^2 \hat{\chi} - 1\right]}{4\left(15k_S^4 - 128k_S^2 + 256\right)}, \quad \hat{\chi} = \sum_{i=1}^{N} \frac{2}{r_H^2 m_i^2 + k_S^2} \tag{8.2.13}$$

which is non-negative when

$$m_{\text{eff}}^2 r_H^2 \hat{\chi} \leq \frac{1}{2}. \tag{8.2.14}$$

At first thought, it seems like $\hat{\gamma}$ does not have a preferred sign from the perspective of the UV theory. However, the dilaton masses are not arbitrary when we think of (8.2.10) as a partial UV completion for the F^4-correction. In order to remain within the regime of validity, we require the dilaton masses to be large relative to the background curvature, and the leading-order term is

$$\hat{\gamma} = -\frac{\alpha^2}{N^2 r_H^2 m_{\text{eff}}^2 \kappa} \frac{9k_S^2(k_S^2-4)^2}{4(15k_S^4 - 128k_S^2 + 256)} + \mathcal{O}\left(m_i^{-2}\right). \tag{8.2.15}$$

This matches the expectation from the F^4-contribution in (8.2.7), and is negative. Deformations can only have positive $\hat{\gamma}$ when subleading corrections become sizable. The marginal case in the AF limit corresponds to $\ell = 2$, and reproduces the example from the introduction.

From discussions around the species bound *e.g.* in [297], we expect that a UV cut-off can be lowered when there is a large number of light states in the UV, and we might therefore expect a breakdown of EFT and positive $\hat{\gamma}$ even at low energies in the large-N limit. In this spirit, let us briefly comment on different mass distributions m_i in the large-N limit. For m_n with explicit scaling in N, we want $m_n = \mathcal{O}(N)$ at the very least — these cancel in $m_{\text{eff}}^2\hat{\chi}$ and therefore do not affect the sign of $\hat{\gamma}$. When $m_n = o(N)$, there are different distributions we could consider. In the simplest case when all the masses are equal, *i.e.* $m_n = m$, we find that

$$m_{\text{eff}}^2 \hat{\chi} = \frac{2m^2}{m^2 r_H^2 + k_S^2}, \tag{8.2.16}$$

which does not scale with N. Similarly, if *e.g.* we had a tower of successively more massive states $m_n = nm$, as one would expect from string theory, we would have

$$m_{\text{eff}} = m H_{N,2}^{-1/2} \sim m\left[\pi^2/6 - 1/N + \mathcal{O}(N^{-2})\right], \quad \hat{\chi} = \frac{1}{k_S}\left[-1 + \frac{k_S\pi}{mr_H}\coth\left(\frac{k_S\pi}{mr_H}\right)\right] + \mathcal{O}\left(N^{-1}\right), \tag{8.2.17}$$

where $H_{n,m}$ is the n-th generalised harmonic number of order m. In both of these examples, the

subleading corrections remain subdominant in N and hence have no chance of flipping the sign of $\hat{\gamma}$ by overwhelming the leading-order contribution.

8.3 Breakdown of Breakdowns

In the previous section, we saw that for every harmonic ℓ there exists a black hole with tuned size r_H/L deformations of the near-horizon geometry to which are marginal. The sign of $\hat{\gamma}$ depends on the signs of the Wilson coefficients, so it seems that the presence of curvature singularities on the horizon of deformed extremal black holes is strongly sensitive to the details of the UV theory. This suggests that there is a breakdown of EFT near the horizon — or does it?

One hint from our discussion in section 8.1 is that singular critical scaling near the horizon is already present in GR, where we understand this to signal a breakdown of metric perturbation theory near the horizon. This suggests that the apparent breakdown of EFT above is better understood as metric perturbations going out of control. This section aims to address this, by determining when metric perturbation theory and EFT break down respectively. It will indeed turn out that metric perturbation theory will have gone out of control by the time EFT breaks down.

8.3.1 Effective Field Theory

Effective field theory breaks down when curvature invariants of the geometry become large — this is when the asymptotic expansion in higher-derivative interactions goes out of control. For this, we have to consider both curvature invariants of the unperturbed and the deformed geometry.

At the level of the unperturbed background geometry, this means that the EFT regime of validity is schematically defined by

$$\left(\frac{\nabla}{\Lambda}\right)^a \left(\frac{\text{Rie}}{\Lambda^2}\right)^b \ll 1, \tag{8.3.1}$$

as $a, b \to \infty$. For the sake of concreteness, we can consider a curvature invariant evaluated on the horizon such as

$$(R^n) = R^{\mu_1}{}_{\mu_2} R^{\mu_2}{}_{\mu_3} \dots R^{\mu_n}{}_{\mu_1} \sim \frac{1}{r_H^{2n}}, \tag{8.3.2}$$

which leads to the intuitive bound $\Lambda r_H \gg 1$ on the background curvature. There are no other scales in the system, so this is the strongest bound we expect to find on the background.

As we are working to first order in metric perturbations, we will consider curvature invariants perturbed to first order only, and the EFT expansion is under control when, schematically

$$\delta\left[\left(\frac{\nabla}{\Lambda}\right)^a \left(\frac{\text{Rie}}{\Lambda^2}\right)^b\right] \sim h \left(\frac{\nabla}{\Lambda}\right)^p \left(\frac{\text{Rie}}{\Lambda^2}\right)^q \ll 1 \tag{8.3.3}$$

as $a, b \to \infty$ or $p, q \to \infty$, with $p + q + 1 = a + b$. We expect this to lead to a bound on the metric perturbation $h \sim h_0 \rho^\gamma$, but in order to ensure that actual gauge-invariant contractions do not

conspire to vanish we once again evaluate these on-shell of the explicit solutions (7.B.6).

The most obvious invariant to check is

$$\delta\left[(R^n)\right] \sim \frac{h_0\rho^\gamma}{r_H^{2n}}. \tag{8.3.4}$$

As $n \to \infty$, we just recover the bound on the background quantities, and we see that the strongest bound on h is obtained for small n. However, terms with small n are not required to be subdominant for the EFT expansion, seen as an asymptotic expansion, to be under control. To obtain the strongest bound on h, we need to consider operators that vanish on the undeformed background, such as

$$\tilde{R} := R - \bar{R}, \tag{8.3.5}$$

where $\bar{R}$ is the background value for the Ricci scalar. These vanish at the level of the background, and it turns out that the leading-order contribution of $\tilde{R}^n$ on the deformed geometry is

$$\delta(\tilde{R}^n) \sim \left[\gamma(\gamma+1)h_0\rho^\gamma r_H^2\right]^n. \tag{8.3.6}$$

From the limit $n \to \infty$, we see that we require generally require

$$h \sim h_0\rho^\gamma \ll (\Lambda r_H)^2 \tag{8.3.7}$$

in order for EFT expansion to remain under control. In the marginal case, $\gamma \sim \frac{1}{\Lambda^2 r_H^2}$ and we require

$$h \sim h_0\rho^\gamma \ll (\Lambda r_H)^4. \tag{8.3.8}$$

8.3.2 Metric Perturbation Theory

A precise discussion on the expansion of the Einstein-Maxwell action (7.1.1) in perturbations can be found in appendix 8.B. Schematically, this takes the form

$$S[h] \sim \int d^Dx\sqrt{-g}\, h\, \partial^2\left[h + h^2 + \dots\right]. \tag{8.3.9}$$

The quadratic action in perturbations is the part responsible for linearised perturbation theory. It is clear that perturbation theory breaks down when the action, as an asymptotic expansion in perturbations, is no longer well-defined. This occurs when higher-order terms in the action dominate, so a naïve guess for when control over perturbations is lost is

$$h \sim h_0\rho^\gamma \sim 1 \tag{8.3.10}$$

in an orthonormal frame.

We should be concerned that the actual coefficients of individual terms in the actual action for perturbations (8.B.1) might vanish — in the worst case, all higher-order terms vanish and the quadratic term is trivially under control. To concretely check this is not the case, let us explicitly evaluate the Einstein-Maxwell action at cubic and quartic order in perturbations, given in (8.B.2)

and (8.B.3) respectively, on-shell for the explicit solutions (7.B.6) in $D = 5$. The expressions are complicated, but we indeed find that they are non-vanishing. By treating the spherical harmonics as $\mathcal{O}(1)$ again, we can estimate the order of magnitude of the integrands as

$$\mathcal{L}^{(3)}_{\text{on-shell}} \sim \frac{h_T^3 \rho^{3\gamma}}{r_H^8}, \quad \mathcal{L}^{(4)}_{\text{on-shell}} \sim \frac{h_T^4 \rho^{4\gamma}}{r_H^{10}}. \tag{8.3.11}$$

We identify the amplitude as $\phi_0 \sim h_T/r_H^2$, and we see that as long as $\phi_0 \rho^\gamma \ll 1$ we also have $S^{(4)}_{\text{on-shell}} \ll S^{(3)}_{\text{on-shell}}$, confirming our earlier naïve estimate.

The threshold for a breakdown of metric perturbation theory will have already been reached when the EFT is out of control. Finding singular behaviour on the horizon hence only indicates a breakdown of metric perturbation theory. This is supported by the examples in section 8.2, where the deformations suffer from singularities on their horizons in both the IR theory (8.2.2) and its UV completion (8.2.8). Within the UV theory itself, there is no EFT that could break down, and if breakdown of EFT were the cause of the divergences in the IR, we would have expected details of the UV physics to be sufficient to resolve them. The fact that this effect persists even in the UV therefore suggests that the EFT managed to accurately capture the UV physics. In this sense, finding negative $\hat{\gamma}$ is a feature rather than a bug of the EFT. This is consistent with the holographic perspective provided in [185], which suggests that the true near-horizon geometry will be replaced by one with less symmetry.

8.4 UV Sensitivity

In the previous section we established that the UV sensitivity of marginal deformations is associated with metric perturbation theory going out of control rather than a breakdown of EFT, as one might initially think. However, precisely when and if metric perturbation breaks down is still sensitive to the signs of Wilson coefficients, and one might hope to get interesting insights on UV physics from this.

8.4.1 Near-Horizon Negativity

In addition to corrections coming from the F^4-term, we also derive the corrections to the critical exponents for deformations of AdS RN due to all other four-derivative EFT corrections to the Einstein-Maxwell action in appendix 8.A.2. It was previously noted in [270] that these lead to deformations of AF RN with negative marginal scaling exponent $\hat{\gamma} < 0$ when the Wilson coefficients respect the WGC and *vice versa*. More specifically, the combination of Wilson coefficients

$$c_0 = \frac{1}{2}c_1 + \frac{11}{2}c_2 + \frac{31}{2}c_3 + 3c_4 + 6(c_5 + c_6) + 9(2c_7 + c_8) \tag{8.4.1}$$

enters both the extremal charge-to-mass ratio (8.2.5) and $\hat{\gamma}$ in (8.2.7) within the AF limit, with the sign satisfying the WGC and related entropy bounds leading to a negative contribution to $\hat{\gamma}$ [155, 161].

Making such a connection in AdS is more subtle: There are proposals for the WGC in AdS,

but the status of a definite statement is unclear, so we shall stick to discussing the AF WGC [298]. The effective combination of c_i that enters $\hat{\gamma}$ is now a function of L, which reduces to the combination entering the WGC bound in the limit $L \to \infty$. Requiring the AF WGC therefore no longer determines the sign of $\hat{\gamma}$! It is however notable that there is a link at the level of individual contributions. All Wilson coefficients enter $\hat{\gamma}$ in (8.2.7) of deformed extremal AdS RN with a negative coefficient and the AF extremal charge-to-mass ratio (8.2.5) with a positive coefficient. Individual contributions pushing towards satisfying the WGC therefore cause worse behaviour on the horizons of the deformed extremal AdS RN black holes.

Evidently, Wilson coefficients consistent with healthy UV completions on AF backgrounds seem to be associated to worse behaviour of deformed extremal black holes. Let us take this at face value and think about the full parameter space of operators that lead to worse behaved deformations on the horizon. As is familiar from positivity and causality bounds, we allow for violations due to gravitational interactions that should vanish in the decoupling limit, so concretely

$$\hat{\gamma} \lesssim +\mathcal{O}(\kappa)\,. \tag{8.4.2}$$

We dub this *near-horizon negativity*. Crude physical motivation behind this is that, from the perspective of AdS_2 holography, corrections pushing us towards more negative γ (such as the ones from known partial UV completions or with Wilson coefficients obeying known bounds) are precisely the ones trying to move further away from the unitarity bound on the AdS_2 scaling dimension Δ. This is however notably in stark contrast with the fact that the lower bound $\gamma > -1/2$ ensures finite action for the bulk fields in AdS_2 . The fact that the scaling dimensions γ contain non-trivial information about physical UV completions should not come as a surprise. The effective potential for perturbations around backgrounds, in different limits, is also bounded by infrared and quasi-normal mode causality [1, 2, 277] through the scattering time delay and quasi-normal mode frequencies respectively. This clearly suggests some physical information about consistent UV completions is however non-trivially retained in the effective potential, and the scaling dimensions are derived from precisely this.

There is a more direct perspective connecting black holes in AdS to the AF WGC. We previously thought of AF RN black holes as AdS RN black holes in the limit where $L \to \infty$, but we should proceed with caution as L is not dimensionless. The most obvious scale that this should be compared to is the curvature of the black hole set by r_H — in this case, the AF limit is controlled by the dimensionless ratio r_H/L. This is what we have been implicitly thinking of when stating that the AF limit is given by sending $\sigma \to 1$. Since the L-dependence of the scaling exponents is only in terms of r_H/L via σ, only the $\ell = 2$ harmonic is marginal. However, in more complicated and realistic configurations it is conceivable that there exists another length scale R that we can compare the AdS radius L to. When $R < r_H$, σ is not fixed and we can take an AF limit where the tower of marginal modes survives — in this case, we essentially identify asymptotically AdS spacetimes with sufficiently large AdS lengthscale as AF. In other words, if we do not distinguish between actually AF spacetimes and asymptotically AdS spacetimes with AdS radius large relative to some intermediate length scale, we might expect that contributions from UV corrections are qualitatively similar.

8.4.2 Constraints on Four-Derivative Corrections

Let us now study the consequences near-horizon negativity has on the four-derivative corrections to the Einstein-Maxwell action (8.A.5). For this we must determine the EFT corrections to the scaling exponent. There is however one subtlety which is not encountered in the AF story. As mentioned in appendix 8.A.2, the generic EFT corrections from (8.A.5) modify the background solution in such a way that the asymptotic AdS length scale is corrected. It is however only meaningful to compare scaling exponents for theories with the same asymptotics. To account for this, we must either work in a form of the EFT action for which there are no corrections to the asymptotic cosmological constant or account for the redefinition of the asymptotic observables when the EFT corrections are included. The latter is most easily done by performing from the outset the field redefinitions that put the EFT corrections into the minimal Weyl tensor form

$$S_{\text{EFT}} = \int \mathrm{d}^5x\sqrt{-g}\left[\frac{\tilde{c}_3}{\kappa\Lambda^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{\tilde{c}_6}{\Lambda^2}C_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + \frac{\kappa\tilde{c}_7}{\Lambda^2}(F^2)^2 + \frac{\kappa\tilde{c}_8}{\Lambda^2}F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\sigma F_\sigma{}^\mu\right], \tag{8.4.3}$$

with $\tilde{c}_i$ defined in (8.A.12) of .

Either way, the corrections to the critical scaling exponents in this theory can then be determined from the general expressions give in appendix 8.A.2. From around (8.A.9) it can be shown that, the marginal critical exponent takes the form of

$$\hat{\gamma} = -\frac{1}{\Lambda^2 r_H^2}\frac{64k_S^2}{256 - 128k_S^2 + 15k_S^4}\tilde{c}_0, \tag{8.4.4}$$

and

$$\tilde{c}_0 = \frac{1}{16}c_0\left(k_S^2-4\right)^2 + \frac{k_S^2-8}{64}\left[\left(72-21k_S^2\right)c_3 + \left(k_S^2-4\right)c_6\right]. \tag{8.4.5}$$

Since $\tilde{c}_0$ enters $\hat{\gamma}$ with a negative coefficient, near-horizon negativity would require $\tilde{c}_0 > 0$. The AF limit corresponds to $k_S^2 = 8$ and reproduces the original WGC bound

$$\tilde{c}_0(\ell = 2) = c_0 > 0\,. \tag{8.4.6}$$

The general bound $\tilde{c}_0 > 0$ on AdS RN will give an infinite tower of additional bounds. Concretely, the new bounds beyond the AF WGC come from $\ell > 2$.

In particular we note that if we assume near-horizon negativity is valid in the limit $k_S \to \infty$, *i.e.* for large BHs, then we infer

$$\hat{c}_0 = \lim_{\ell\to\infty}\frac{\tilde{c}_0(\ell)}{k_S^4} = \frac{1}{64}\left(4c_0 - 21c_3 + c_6\right). \tag{8.4.7}$$

This conjectured bound is independent of the AF bound on c_0.

8.A Critical Exponents for Extremal Reissner-Nordström

In this appendix, we collect the critical exponents of the near-horizon scaling solutions (7.2.7) in GR and its IR and UV modifications considered in the main text. We only state the more regular power γ_+ in (7.2.9), as the other branch of the solutions can be switched off by finite BCs at the horizon in the sub-extremal case.

8.A.1 Pure GR

Let us first consider pure GR. We can use the effective potentials in [278] to find that for the tensor, vector, and scalar modes respectively

$$\gamma_T = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\ell(\ell+D-3)}{\sigma}}, \tag{8.A.1a}$$

$$\gamma_{V\pm} = -\frac{1}{2} + \frac{1}{2}\sqrt{5 + \frac{4(\ell+1)(\ell+D-4)}{(D-3)^2\sigma} \pm 4\sqrt{1 + \frac{2(\ell+1)(\ell+D-4)[1+(D-3)\sigma]}{(D-3)^3\sigma^2}}}, \tag{8.A.1b}$$

$$\gamma_{S\pm} = -\frac{1}{2} + \frac{1}{2}\sqrt{5 + \frac{4\ell(\ell+D-3)}{(D-3)^2\sigma} \pm 4\sqrt{1 + \frac{4\ell(\ell+D-3)[1+(D-3)\sigma)]}{(D-3)^2(D-2)\sigma^2}}}. \tag{8.A.1c}$$

Note that

$$\gamma_M > 0, \quad M \in \{T, V+, V-, S+\} \tag{8.A.2}$$

for all $\sigma \in [0, \infty)$ and ℓ.

This is however not the case for the gravitational scalar modes γ_{S-}. In $D = 4$ we find that $\gamma_{S-} \leq 2$ for

$$\sigma \geq \frac{\ell(\ell+1)}{4}\left(1 - \sqrt{\frac{\ell^2+\ell+4}{3\ell(\ell+1)}}\right). \tag{8.A.3}$$

For $\ell \geq 3$, the right-hand side of the inequality takes values in $[1, \infty)$ so this is never achieved for $\Lambda > 0$. In $D \geq 5$ this is worse, and we even find $\gamma_{S-} \leq 0$ if

$$\sigma \geq \frac{(D-2)k_S^2 - 4(D-3)^2}{2(D-4)(D-3)^2} \tag{8.A.4}$$

for $\ell > D-3$ in $\sigma \in [0, \infty)$. For modes with $\ell \leq D-3$ this also holds for all $\sigma \in [1, \infty)$ and some values of σ when $\Lambda > 0$. This is illustrated for $D = 11$ in figure 8.1.

Further, note that $\gamma_{S-} \geq -1/2$ and this is saturated for $\ell = (D-3)/2$ at $\sigma = 1$. As is shown in figure 8.2, this is possible only in odd D, although in even D, $\ell = \lfloor (D-3)/2 \rceil$ will achieve $\gamma = -\frac{1}{2} + \frac{1}{2(D-3)}$. An exception to this is $D = 5$, where the minimum is not attained since $\ell = 1$ is not physical for gravitational scalar modes.

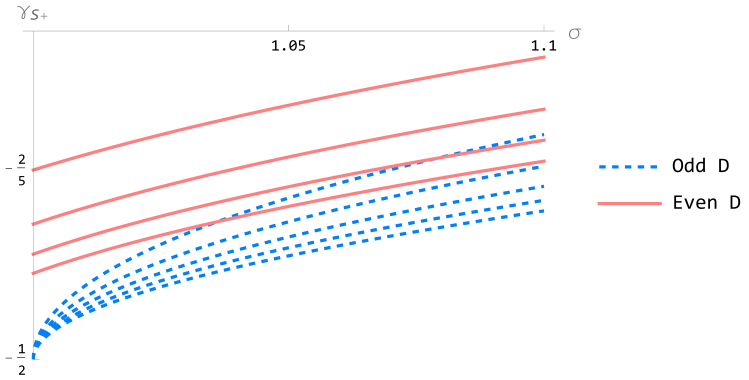


Figure 8.2: The (more regular) scaling dimension γ_+ for the most singular mode with $\ell = \lfloor (D-3)/2 \rceil$ in various dimensions D. For odd D, the minimum $\gamma = -1/2$ is attained at $\sigma = 1$.

8.A.2 IR: EFT Correction

In this section, we will derive the critical exponents of gravitational scalar perturbations in Einstein-Maxwell theory with EFT corrections. We shall focus on the leading four-derivative corrections given by the action

$$\begin{aligned} S_{\text{EFT}} = \int \mathrm{d}^5x\sqrt{-g}\Big[& d_1 R^2 + d_2 R_{\mu\nu}R^{\mu\nu} + d_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + d_4 RF^2 \\ & + d_5 R_{\mu\nu}F^{\mu}{}_{\lambda}F^{\nu\lambda} + d_6 R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + d_7 (F^2)^2 + d_8 F_{\mu}{}^{\nu}F_{\nu}{}^{\rho}F_{\rho}{}^{\sigma}F_{\sigma}{}^{\mu}\Big]. \end{aligned} \tag{8.A.5}$$

The d_i have been taken to be dimensionful for notational convenience, but with Lorentz-invariant cut-off Λ we can write these in terms of dimensionless couplings c_i as

$$d_{1,2,3} = \frac{c_{1,2,3}}{\kappa\Lambda^2}, \quad d_{4,5,6} = \frac{c_{4,5,6}}{\Lambda^2}, \quad d_{7,8} = \frac{\kappa c_{7,8}}{\Lambda^2}. \tag{8.A.6}$$

To find the critical scaling exponents, we follow the procedure in appendix 7.B and start directly within the extremal near-horizon geometry. For simplicity, we restrict ourselves to $D = 5$.

In particular, the Ansatz (7.B.2) satisfies the background equations of motion for

$$\begin{aligned} A_{\text{NH}} = \frac{2\kappa F_{\text{NH}}^2}{1+2\sigma}\Big\{ & \sigma + \frac{1}{r_H^2\Lambda^2}\frac{1}{2(1+2\sigma)}\big[2(-3+4\sigma)(-1+4\sigma+8\sigma^2)c_1 + 2(1-4\sigma+16\sigma^3)c_2 \\ & + 2(1-4\sigma+32\sigma^3)c_3 + 4(1-4\sigma+24\sigma^3)c_4 + 4\sigma(1+2\sigma)(-1+6\sigma)(c_5+2c_6) \\ & + 4(1+2\sigma)^2(-1+4\sigma)(2c_7+c_8)\big]\Big\} \end{aligned} \tag{8.A.7a}$$

$$\begin{aligned} B_{\text{NH}} = \frac{4}{r_H^2}\Big\{ & \sigma + \frac{1}{r_H^2\Lambda^2}\big[2(3-4\sigma)c_1 + 2(c_2+c_3) + 4(1-4\sigma^2)c_4 - 4\sigma(1+2\sigma)(c_5+2c_6) \\ & - 4(1+2\sigma)^2(2c_7+c_8)\big]\Big\}. \end{aligned} \tag{8.A.7b}$$

Using the decomposition in appendix 7.B.2, we find that perturbations on top of this background

take the form of the scaling solutions (6.0.1) with critical exponents in the form of (6.0.2). More specifically, we find that

$$\gamma_{\rm EFT} = \frac{1}{\Lambda^2 r_H^2}\Delta\sum_{i=1}^{8} c_i\delta\gamma_i, \tag{8.A.8}$$

where

$$\delta\gamma_1 = -576(z-2)z(4\sigma-3)\sigma - 16k_S^2\left[(-8\sigma^2-40\sigma+27)z-20\sigma+5\right] - 16k_S^4(1+2\sigma) \tag{8.A.9a}$$

$$\begin{aligned}\delta\gamma_2 =&576(z-2)z(1+2\sigma+4\sigma^2)\sigma - 16k_S^2\left[(56\sigma^2+28\sigma+9)z+48\sigma^2+80\sigma+31\right]\\ &+40k_S^4(1+2\sigma)\end{aligned} \tag{8.A.9b}$$

$$\begin{aligned}\delta\gamma_3 =&576(z-2)z(1+4\sigma+16\sigma^4)\sigma + 16k_S^2\left[-(256\sigma^2+20\sigma+9)z-144\sigma^2-280\sigma-47\right]\\ &+64k_S^4(-1+7\sigma)\end{aligned} \tag{8.A.9c}$$

$$\delta\gamma_4 = -1152(z-2)z(-1+4\sigma^2)\sigma + 96k_S^2(1+2\sigma)\left[(10\sigma-3)z+2\sigma+3\right] - 96k_S^4(1+2\sigma) \tag{8.A.9d}$$

$$\begin{aligned}\delta\gamma_5 =&576(z-2)z\sigma^2\left[(z-6)\sigma-2\right] - 48k_S^2\left[9z^2\sigma^2-12z\sigma(1+4\sigma)-4(\sigma^2+6\sigma+2)\right]\\ &-36k_S^4\left[(-3z+8)\sigma+2\right] - 9k_S^6\end{aligned} \tag{8.A.9e}$$

$$\begin{aligned}\delta\gamma_6 =&2304(z-2)z\left[(9z-14)+2\right]\sigma^2 - 192k_S^2\left[(9z^2-32z-7)\sigma^2-(z+19)\sigma-2\right]\\ &+48k_S^4\left[(9z-14)\sigma+2\right] - 36k_S^6\end{aligned} \tag{8.A.9f}$$

$$\begin{aligned}\delta\gamma_7 =&-2304(z-2)z(1+2\sigma^2)\sigma + 192k_S^2(1+2\sigma)\left[3z(1+6\sigma)+2\sigma+13\right]\\ &-576k_S^4(1+2\sigma)\\ =&\,2\delta\gamma_8,\end{aligned} \tag{8.A.9g}$$

and

$$\begin{aligned}\Delta =&\frac{1}{3}(\gamma_{\rm GR}+1)(\gamma_{\rm GR}+2)\big\{-48(1+\gamma_{\rm GR})^2(2+\gamma_{\rm GR})^2(-1+2\gamma_{\rm GR})\sigma^2\\ &+8k_S^2\left[(6\gamma_{\rm GR}^2+10\gamma_{\rm GR}+3)\sigma-4\gamma_{\rm GR}-6\right]+3k_S^4(2\gamma_{\rm GR}+3)\big\}^{-1}.\end{aligned} \tag{8.A.10}$$

We have further defined

$$z = \gamma_{\rm GR}^2+\gamma_{\rm GR} = 1+\frac{k_S^2}{4\sigma} - \sqrt{1+\frac{1+2\sigma}{3\sigma^2}k_S^2} \tag{8.A.11}$$

for convenience. The AF limit is given by sending $\sigma \to 1$ and reproduces the expressions in [270].

The expressions above are not manifestly invariant under field redefinitions, because the asymptotics of the full solution are modified by the EFT corrections. A basis of linear combinations of Wilson coefficients invariant under field redefinitions is $\{c_3, c_6, \tilde{c}_7, \tilde{c}_8\}$, with

$$\tilde{c}_7 = c_7 + \frac{1}{36}c_1 - \frac{7}{36}c_2 - \frac{19}{72}c_3 + \frac{1}{6}c_4 - \frac{1}{6}c_5 - \frac{1}{4}c_6 \tag{8.A.12a}$$

$$\tilde{c}_8 = c_8 + c_2 + \frac{4}{3}c_3 + c_5 + \frac{4}{3}c_6\,. \tag{8.A.12b}$$

To connect with [155], it is also convenient to use $\{c_0, c_3, c_6, c_9\}$ as a basis, with

$$c_9 = c_2 + c_5 + c_8. \tag{8.A.13}$$

8.A.3 UV: Einstein-Maxwell-Dilaton

In this section, we derive the critical exponents in Einstein-Maxwell-Dilaton theory with action (8.2.8), treating everything perturbatively in α. We are only interested in the critical exponents, so we follow the procedure in appendix 7.B and start directly within the extremal near-horizon geometry.

For simplicity, we restrict ourselves to $D = 5$. Then, the Ansatz (7.B.2) satisfies the background equations of motion for

$$\phi_i = \frac{1}{N}\frac{\alpha B_{\text{NH}} F_{\text{NH}}^2}{2m_i^2}, \tag{8.A.14a}$$

$$A_{\text{NH}} = \frac{F_{\text{NH}}^2}{2}\left[\frac{4\kappa}{3}\frac{1+3\varsigma}{1+2\varsigma} + \frac{1}{N^2}\frac{3\alpha^2}{r_H^2 m_{\text{eff}}^2}(1+4\varsigma)\right], \quad B_{\text{NH}} = \frac{4}{r_H^2}(1+3\varsigma) - \frac{1}{N^2}\frac{9\alpha^2}{\kappa m_{\text{eff}}^2 r_H^4}(1+2\varsigma)^2\,. \tag{8.A.14b}$$

Since the background for the scalar comes in at $\mathcal{O}(\alpha)$, the corrections in the perturbation equations will come in at $\mathcal{O}(\alpha^2)$. Thus the first non-trivial corrections to the critical exponents will come in at $\mathcal{O}(\alpha^2)$.

In the end, we are only interested in the gravitational scalar modes, as these are the ones that exhibit UV sensitivity. The decomposition in appendix 7.B.2 then renders the perturbation equations algebraic in the constants, and the critical exponent takes the form (8.2.12) with γ_{GR} as given in (8.A.1) and

$$\frac{1}{\Delta}\delta\gamma = -9k_S^2 m_{\text{eff}}^2 r_H^2(1+2\sigma)(-4z\sigma-4+k_S^2)\chi + 6(1+2\sigma)^2\left[-12(z-2)z\sigma + k_S^2(1+3z)\right], \tag{8.A.15}$$

where

$$\chi = \sum_{i=1}^{N}\frac{2}{r_H^2 m_i^2 + k_S^2 - 4z\sigma}. \tag{8.A.16}$$

8.B Einstein-Maxwell Action Expansion

An expansion of the Einstein-Maxwell action (7.1.1) in the metric and field strength perturbations h and δF takes the form

$$S_{\text{EM}}[g, F] = \sum_{N=0}^{\infty} S^{(N)}[\bar{g}, h; \bar{F}, \delta F] \tag{8.B.1}$$

where $S^{(N)}$ has N factors of the metric or field strength perturbations in total. When evaluated on the background, $S^{(0)}$ is just some constant and $S^{(1)}$ is a boundary contribution at most. Furthermore, when evaluated on-shell of perturbations, the quadratic action $S^{(2)}$ is also only a boundary contribution.

8.B.1 Explicit Expressions

The explicit form of the cubic and quartic action is

$$\begin{aligned}\frac{1}{3!\kappa^{1/2}}S^{(3)}[h,\delta F] = \int \mathrm{d}^D x\sqrt{-g}\Big\{ & h^{\mu\nu}\Big[h^{\rho\sigma}\nabla_\sigma\nabla_{[\nu}h_{\rho]\mu} + h_\mu{}^\rho\left(\nabla^\sigma\nabla_{[\rho}h_{\sigma]\nu} + \nabla_\rho\nabla_{[\sigma}h_{\nu]}{}^\sigma\right) + \nabla_\nu h_\mu{}^\rho\nabla_{[\sigma}h_{\rho]}{}^\sigma \\ & + \nabla_{[\nu}h_{\rho]\mu}\nabla_\sigma h^{\rho\sigma} + \nabla_\nu h^\sigma{}_{[\rho}\nabla_{\sigma]}h_\mu{}^\rho + \frac{1}{2}h\nabla_\rho h_\mu{}^\rho\nabla_\sigma h_\nu{}^\sigma - \frac{1}{2}\nabla_\rho h^\sigma{}_{[\nu}\nabla^\rho h_{\sigma]\mu} - \frac{1}{2}\nabla^\rho h^\sigma{}_\nu\nabla_{[\rho}h_{\sigma]\mu} \\ & - \frac{1}{4}\nabla_\sigma h_{\nu\rho}\nabla^\sigma h_\mu{}^\rho - \frac{3}{8}\nabla_\mu h^{\rho\sigma}\nabla_\nu h_{\rho\sigma} + \frac{1}{8}\nabla_\mu h\nabla_\nu h - \frac{1}{2}h_\mu{}^\rho h_\nu{}^\sigma\bar{R}_{\rho\sigma} + \frac{1}{2}\delta F^\rho{}_\nu\,\delta F_{\rho\mu}\Big] \\ & - \kappa^{1/2}\delta F^{\mu\nu}\left[h^{\rho\sigma}\bar{F}_{\mu\rho}h_{\nu\sigma} + \frac{1}{2}\bar{F}^{\rho\sigma}h_{\mu\rho}h_{\nu\sigma}\right] + \kappa\bar{F}^{\mu\nu}h_{\sigma\lambda}h_\nu{}^\lambda\bar{F}^\rho{}_{(\mu}h_{\sigma)\rho}\Big\}\end{aligned} \tag{8.B.2}$$

and

$$\begin{aligned}\frac{1}{4!\kappa}S^{(4)}[h,\delta F] = \int d^D x\sqrt{-g}\Big\{ & h^{\mu\nu}\Big[h_\mu{}^\rho h_\nu{}^\sigma\left(-\nabla^\lambda\nabla_{[\sigma}h_{\lambda]\rho} - \nabla_\sigma\nabla_{[\lambda}h_{\rho]}{}^\lambda + \frac{1}{2}h_\rho{}^\lambda\bar{R}_{\sigma\lambda} + \frac{1}{2}h^{\kappa\lambda}\bar{R}_{\rho\kappa\sigma\lambda}\right) \\ & + h_\mu{}^\rho h^{\sigma\lambda}\left(-\nabla_\lambda\nabla_{[\rho}h_{\sigma]\nu} - \nabla_\rho\nabla_{[\lambda}h_{\nu]\sigma} + \frac{1}{2}h_\sigma{}^\kappa\bar{R}_{\nu\rho\kappa\lambda} + \frac{1}{2}h_\nu{}^\kappa\bar{R}_{\rho\lambda\sigma\kappa}\right) \\ & + h_\mu{}^\rho\Big(-2\nabla^\lambda h_{\sigma[\lambda}\nabla_{\rho]}h_\nu{}^\sigma + \nabla_{(\rho}h\nabla_{\sigma)}h_\nu{}^\sigma - \nabla^\sigma h_{\nu[\sigma}\nabla^\lambda h_{\rho]\lambda} - \nabla_\rho h^{\sigma\lambda}\nabla_{[\lambda}h_{\nu]\sigma} \\ & - \frac{1}{2}\nabla_{(\lambda}h_{\sigma)\rho}\nabla^\lambda h_\nu{}^\sigma - \frac{1}{4}\nabla_\nu h^{(\sigma}{}_\lambda\nabla_\rho h^{\lambda)}{}_\sigma - \frac{1}{4}\nabla_\sigma h\nabla^\sigma h_{\nu\rho}\Big) \\ & + h^{\rho\sigma}\Big(-2\nabla_\nu h_{\mu[\sigma}\nabla_{\lambda]}h_\rho{}^\lambda + \nabla_\nu h_{\mu(\lambda|}\nabla_\sigma h_{|\rho)}{}^\lambda + \nabla_{(\sigma|}h_\rho{}^\lambda\nabla_{|\lambda)}h_{\mu\nu} - \nabla_\sigma h_\nu{}^\lambda\nabla_{[\lambda}h_{\rho]\mu} \\ & + \frac{1}{2}\nabla_{[\sigma}h_{\nu]\lambda}\nabla_\rho h_\mu{}^\lambda - \frac{1}{4}\nabla_\lambda h_{\rho[\sigma}\nabla^\lambda h_{\mu]\nu} + \frac{1}{4}\nabla_\lambda h_{\nu\sigma}\nabla^\lambda h_{\mu\rho} - \frac{1}{4}\nabla_\rho h_{\mu\nu}\nabla_\sigma h\Big) \\ & - \frac{1}{2}h_{\rho\mu}\bar{F}^\rho{}_\sigma\,\bar{F}_\nu{}^\sigma - h_{\rho\sigma}\bar{F}_\mu{}^\rho\bar{F}_\nu{}^\sigma\Big] - \kappa^{1/2}\delta F^{\mu\nu}\left[\bar{F}^{\rho\sigma}h_\sigma{}^\lambda h_{\mu[\lambda}h_{\rho]\lambda} + \bar{F}^\rho{}_\mu h_\rho{}^\lambda h_{\sigma\lambda}h_\nu{}^\sigma\right] \\ & - \frac{1}{2}\kappa\bar{F}^{\mu\nu}\left[h_\sigma{}^\lambda h_{\kappa\lambda}\left(\bar{F}^{\rho\sigma}h_{\mu\rho}h_\nu{}^\kappa + \bar{F}_\mu{}^\rho h_\nu{}^\sigma h_\rho{}^\kappa\right) + \frac{1}{2}\bar{F}^{\rho\sigma}h_\mu{}^\kappa h_\nu{}^\lambda h_{\rho\kappa}h_{\sigma\lambda}\right]\Big\}\end{aligned} \tag{8.B.3}$$

respectively, with $\bar{R}$ being the Riemann tensor evaluated on the background solution.

Chapter 9

AdS_2 Holography

In this chapter, we will study the boundary correlation functions of dynamical bulk matter fields on the AdS_2 factor within the near-horizon geometry of these extremal black holes. This is motivated by the many facets the Goldstone EFT (1.1.20) has in two dimensions — it can be thought of as a free theory coupled to topological massive gravity or a deformed by the $T\overline{T}$ operator. The causality constraints on the coupling of the Goldstone operator are well understood from the perspective of both massive gravity and the $T\overline{T}$ deformation, and our aim is to understand this from the perspective of boundary observables. To this end, we will present preliminary results for holographic boundary correlation functions.

In section 9.1, we start by reviewing JT gravity and solve for its bulk and boundary dynamics — we mostly follow [97, 299, 300] but also found [301] useful. In section 9.2, we study holographic boundary correlation functions for a free scalar field and the leading-order Goldstone EFT (1.1.20) up to 1-loop in bulk matter fields and Schwarzian modes.

A brief review of the Schwarzian derivative is given in 9.A. Background on how this set-up can be reinterpreted in terms of $T\overline{T}$ deformations and coupling to topological massive gravity is given in appendix 9.B.

9.1 JT Gravity

In the preceding chapters, we were interested in stationary deformations to extremal black holes in the near-horizon limit. We will now study honest dynamical perturbations in the near-horizon limit of near-extremal black holes, which are described by JT gravity. This can be done by spherical reduction, *i.e.* by focusing on the s-waves in the gravitational sector which are entirely independent of the angular directions. The JT gravity action (1.2.29) is obtained by using an appropriate spherically symmetric (warped) product metric.

In this section, we will solve the bulk equations of motion, find these to be largely trivial, and perform a boundary reduction of the action.

9.1.1 Classical Bulk Solutions

Let us start by finding the classical bulk solutions (essentially the saddle point) of the dilaton-gravity system, by solving the classical equations.

Gravity: Bulk AdS_2

The action is linear in the dilaton, so its equation of motion simply imposes

$$R = 2\Lambda = -2/L^2 \tag{9.1.1}$$

which we choose to negative, with L defining the AdS length scale. We will set $L = 1$ from now on. All two-dimensional manifolds are conformally flat, so this means our bulk spacetime is just AdS_2 [1]. We can write its metric in different coordinates as

$$\text{Poincaré: } \mathrm{d}s^2 = \frac{-\mathrm{d}t^2 + \mathrm{d}z^2}{z^2} = -\frac{4\mathrm{d}u\mathrm{d}v}{(u-v)^2}, \quad u = t - z, v = t + z \tag{9.1.2a}$$

$$\text{Global: } \mathrm{d}s^2 = -\frac{4\mathrm{d}U\,\mathrm{d}V}{\sin(U-V)^2} = \frac{-\mathrm{d}T^2 + \mathrm{d}Z^2}{\sin^2(Z)}, \quad U = \frac{T-Z}{2} = \arctan u, V = \frac{T+Z}{2} = \arctan v \tag{9.1.2b}$$

$$\text{Rindler: } \mathrm{d}s^2 = \mathrm{d}\rho^2 - \sinh^2(\rho)\mathrm{d}\tau^2, \quad t = \coth\rho\, e^{\tau}, z = \mathrm{csch}\rho\, e^{\tau} \tag{9.1.2c}$$

which cover different areas of the space. The isometry group of AdS_2 is $\mathrm{SO}(1,2) \simeq \mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2$, *i.e.* the Möbius transformations [2]. We will therefore want any observables to be invariant under this these transformations. One can verify that, indeed the metric in Poincaré coordinates is invariant under $(u,v) \mapsto (\frac{au+b}{cu+d}, \frac{av+b}{cv+d})$ for $ad - bc = 1$. This means that an equivalent coordinate transformation to the Rindler patch is

$$t \pm z = \frac{a(\pm 1 + \cosh\rho)e^{\tau/2} + b\sinh\rho\, e^{-\tau/2}}{c(\pm 1 + \cosh\rho)e^{\tau/2} + d\sinh\rho\, e^{-\tau/2}}. \tag{9.1.3}$$

In particular, choosing $(a,b,c,d) = \frac{1}{\sqrt{2}}(1,-1,1,1)$, has the nice property that near the boundary ($z \to 0$ or $\rho \to \infty$), we get $t \sim \tanh(\tau/2)$. We will mostly be interested in Euclidean signature, which we reach by defining Euclidean time $t_E = it_L$. We get similar expressions for the metric in the different coordinates, which all now cover the entire hyperbolic disk $\mathbb{H}^2$.

When considering the field theory just on AdS_2 , we will find that the dilaton and (if present) bulk fields diverge near the boundary. This is unproblematic — in the context of holography these correspond to UV divergences in the CFT, which we know how to handle[3]. We should

[1] In two dimensions, the Riemann tensor is fully specified by the Ricci scalar and one can always find a conformal transformation to make this vanish.

[2] Note that $\mathrm{SO}(1,2)$ has trivial centre while $\mathrm{SL}(2,\mathbb{R})$ does not.

[3] In holographic renormalisation, we think of the radial direction of AdS as an energy scale in the CFT. For example, empty AdS corresponds to an RG flow trajectory that stays at the CFT. Turning on some irrelevant deformation in the CFT means that our trajectory gets deflected as we flow from towards the UV in the CFT, *i.e.* the boundary of AdS, in an uncontrolled manner. This is precisely what the non-normalisable modes correspond to on the AdS side. We know how to treat this with a UV cutoff and counterterms on the CFT side, which corresponds to an IR cutoff on the gravity side.

therefore study cutoff EAdS_2 as a cutout from the full space defined by a boundary parameterised by the Poincaré coordinates (t, z). Let u be a boundary time (no relation to the lightcone coordinate), so that $t = t(u)$ and $z = z(u)$. We fix the length of this boundary by demanding that its line element is given by

$$\mathrm{d}s\big|_{\text{boundary}} = \frac{1}{\epsilon}\mathrm{d}u \tag{9.1.4}$$

for some UV cutoff $\epsilon \to 0^+$. We can write this is as

$$\mathrm{d}s^2\big|_{\text{boundary}} = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}u}\frac{\mathrm{d}x^\nu}{\mathrm{d}u}\mathrm{d}u^2 \tag{9.1.5}$$

and recognise that it is equivalent to demanding that metric induced on the boundary (parameterised by just u) is fixed to be

$$h = g\big|_{\text{boundary}} = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}u}\frac{\mathrm{d}x^\nu}{\mathrm{d}u} = \frac{t'(u)^2 + z'(u)^2}{z(u)^2} = \frac{1}{\epsilon^2}. \tag{9.1.6}$$

this uniquely fixes $z(u) = \epsilon t'(u) + \mathcal{O}(\epsilon^2)$ in terms of $t(u)$, so there is only a single dynamical boundary gravitational degree of freedom.

The original Einstein-Hilbert action enjoys full diffeomorphism invariance, and is in fact identical for all configurations. It includes arbitrary reparameterisations of the boundary time $u \mapsto f(u)$, that give rise to zero modes in the action. Allowing for cutouts means that the symmetry group is explicitly broken down to only a subgroup of these transformations — some will actually change the shape of the cutout while others, the asymptotic diffeomorphisms (e.g. rotations and translations) leave them invariant. These are precisely the $\mathrm{PSL}(2, \mathbb{R})$ transformations. In the limit where $\epsilon \to 0^+$, $z \to 0$, and we see that the dynamical mode $t(u)$ transforms as

$$t(u) \mapsto \frac{at(u) + b}{ct(u) + d}, \quad ad - bc = 1. \tag{9.1.7}$$

It is sometimes said that the $t(u)$-modes are the Goldstone modes for the corresponding breaking of the symmetry[4].

Dilaton

The equations of motion for the dilaton are obtained by varying the JT gravity action with respect to the metric. In the presence of matter with stress-tensor $T_{\mu\nu}$ this is just

$$T_{\mu\nu} = \nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}(\Box + \Lambda)\Phi, \tag{9.1.8}$$

or in Euclidean Poincaré coordinates

$$T_{tt} = -\left(\frac{1}{z}\partial_z + \partial_z^2 + \frac{\Lambda}{z^2}\right)\Phi \tag{9.1.9a}$$

$$T_{tz} = \left(\frac{1}{z} + \partial_z\right)\partial_t\Phi \tag{9.1.9b}$$

[4]The Mermin-Wagner theorem states that there are no Goldstone modes in one or two dimensions. This is not a spontaneous breaking of symmetry.

$$T_{zz} = -\left(-\frac{1}{z}\partial_z + \partial_t^2 + \frac{\Lambda}{z^2}\right)\Phi. \tag{9.1.9c}$$

Let us solve this in the absence of matter, with $\Lambda = -1$. The (t,z)-component tells us that

$$\Phi(t,z) = \frac{f(t)}{z} + g(z) \tag{9.1.10}$$

for some arbitrary functions $f(t)$ and $g(z)$. Plugging this Ansatz into the other components gives us

$$\frac{1}{z^2}g(z) - \frac{1}{z}g'(z) - g''(z) = 0, \quad -\frac{1}{z^2}g(z) - \frac{1}{z}g'(z) + \frac{1}{z}f''(z) = 0 \tag{9.1.11}$$

which can be solved by

$$f(t) = \frac{b}{z^2}t^2 + bt + d, \quad g(z) = \frac{a}{z} + cz \tag{9.1.12}$$

for some arbitrary constants a, b, c, and d. Putting everything together, and relabelling $a+d \to a$, we find that

$$\Phi = \frac{a + bt + c(t^2+z^2)}{z} = \frac{\tilde{a} + \tilde{b}(u+v) + \tilde{c}uv}{u-v} \tag{9.1.13}$$

where $(\tilde{a},\tilde{b},\tilde{c}) = (-2a, -b+2c)$. Further, note that the Möbius transformation $(u,v) \mapsto (\frac{Au+B}{Cu+D}, \frac{Av+V}{Cv+D})$ with $C = -\frac{1}{2B}$ and $D = \frac{1}{2A}$ (such that $AD - BC = 1$ and $AD + BC = 0$) sends

$$\frac{u+v}{u-v} \to \frac{1}{AB}\frac{B^2 - A^2uv}{u-v}, \tag{9.1.14}$$

effectively shifting $(\tilde{a},\tilde{b},\tilde{c}) \mapsto (\tilde{a} + B/A, 0, \tilde{c} - A/B)$. This means we can always effectively set $b = 0$.

This solution has some interesting properties. First, we note that it indeed diverges as we approach the boundary $z = 0$. For holographic renormalisation, we therefore set the boundary condition for the dilaton as

$$\Phi\big|_{\text{bdry}} = \lim_{z\to 0}\Phi \sim \frac{1}{\epsilon}\frac{a + ct^2(u)}{t'(u)} = \frac{\Phi_r}{\epsilon} \tag{9.1.15}$$

where Φ_r is the renormalised field. Furthermore, note that $\zeta^\mu = \epsilon^{\mu\nu}\partial_\nu\Phi$ satisfies

$$2\nabla_{(\mu}\zeta_{\nu)} = 2\epsilon_{\rho(\nu}\nabla^\rho\nabla_{\mu)}\Phi = 2\epsilon_{\rho(\nu}\delta^\rho_{\mu)}(\Box + \Lambda)\Phi = 2\epsilon_{(\mu\nu)}(\Box + \Lambda)\Phi = 0 \tag{9.1.16}$$

by the equations of motion. In other words ζ is a Killing of the dilaton-gravity system — this is not a boring observation because higher-genus surfaces do not admit Killing vectors.

9.1.2 Boundary Dynamics

We now proceed to derive the action that governs the dynamics of the boundary modes and then study them.

Boundary Schwarzian Action

We have seen that the classical bulk equations of motion fix the bulk metric to be AdS_2 while forcing the dilaton to blow up at the boundary. On this saddle point, we are left with the following Euclidean action

$$I_{\mathrm{JT}} = -\int_{\partial M} \mathrm{d}x\sqrt{h}\,\Phi(K-1) = -\int_{\partial M} \mathrm{d}s\,\Phi(K-1) = -\int_{\partial M} \frac{\mathrm{d}u}{\epsilon}\frac{\Phi_r}{\epsilon}(K-1). \tag{9.1.17}$$

Let us now compute this. First note that the unit normalised tangent and normal vector fields to the curve $(t(u), z(u))$ are given by

$$t^\mu = \frac{z}{\sqrt{t'^2+z'^2}}\begin{pmatrix} t'(u) \\ z'(u)\end{pmatrix}, \quad n^\mu = \frac{z}{\sqrt{t'^2+z'^2}}\begin{pmatrix} -z'(u) \\ t'(u)\end{pmatrix}. \tag{9.1.18}$$

Then, using the fact that the only non-zero Christoffel symbols of EAdS_2 in Poincaré coordinates are $-\Gamma^t{}_{zt} = \Gamma^z{}_{tt} = -\Gamma^z{}_{zz} = 1/z$, we find that the trace of the extrinsic curvature induced on the boundary curve is [5] [6]

$$K = h^{uu}K_{uu} = \left(g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}\right)^{-1}\frac{dx^\alpha}{du}\frac{dx^\beta}{du}K_{\alpha\beta} = \frac{t'\left(t'^2+z'^2+zz''\right)-zz't''}{\left(t'^2+u'^2\right)^{3/2}}. \tag{9.1.19}$$

Near the AdS_2 boundary, $z = \epsilon t' + \mathcal{O}(\epsilon^2)$,

$$K = 1 + \epsilon^2(St)(u) + \mathcal{O}(\epsilon^4) \tag{9.1.20}$$

where the Schwarzian derivative is given by

$$(Sf)(z) = -\frac{1}{2}\left(\frac{f''}{f'}\right)^2 + \left(\frac{f''}{f'}\right)' = \frac{-3f''^2+2f'f'''}{2f'^2}. \tag{9.1.21}$$

Details on the properties of the Schwarzian derivative are presented in 9.A. This means that dynamics of the boundary degrees of freedom $t(u)$ is governed by the boundary Schwarzian action

$$I_{\mathrm{Schw}} = -\int_{\partial M} \mathrm{d}u\,\Phi_r(u)(St)(u). \tag{9.1.22}$$

The bulk and boundary equations of motion for the dilaton are equivalent to each other, so the solution for the renormalised dilaton field is simply the boundary limit of the bulk dilaton solution. For convenience, we redefine our notion of boundary time with $d\tilde{u} = \frac{\bar{\Phi}}{\Phi_r}du$, under which,

$$(St)(u) = \tilde{u}'^2(St)(\tilde{u}) + (S\tilde{u})(u) \tag{9.1.23}$$

[5] The surface is one-dimensional, so the tensor only has one component.
[6] This calculation looks deceptively simple!

so that

$$\begin{aligned} I_{\text{Schw}} &= -\int_{\partial M} \mathrm{d}\tilde{u} u' \, \Phi_r(u)(St)(u) \\ &= -\int_{\partial M} \mathrm{d}\tilde{u} \left[\tilde{u}' \Phi_r(u)(St)(\tilde{u}) + \tilde{u}'^{-1} \, \Phi_r(u)(S\tilde{u})(u)\right] \\ &= -\int_{\partial M} \mathrm{d}\tilde{u} \left[\tilde{u}' \Phi_r(u)(St)(\tilde{u}) + \tilde{u}'^{-1} \, \Phi_r(u) \frac{\Phi_r'^2 - 2\Phi_r'' \Phi_r}{2\Phi_r^{-2}}\right] \\ &= -\int_{\partial M} \mathrm{d}\tilde{u} \left[\bar{\Phi}(St)(\tilde{u}) + \frac{1}{2\bar{\Phi}} \left(\Phi_r'^2 - 2\Phi_r'' \Phi_r\right)\right]. \end{aligned} \tag{9.1.24}$$

We need to fix a boundary condition for the dilaton Φ_r to be constant [7], in which case the above action reduces to

$$I_{\text{Schw}} = -\bar{\Phi} \int_{\partial M} \mathrm{d}\tilde{u} (St)(\tilde{u}). \tag{9.1.25}$$

Fixing the asymptotics of the dilaton like this leaves freedom only in the boundary time parameter, so we will now turn to study the dynamics of these.

Saddle Point and Fluctuations

We can minimally couple the Schwarzian action to matter in the following hybrid way

$$I = -\bar{\Phi}_r \int \mathrm{d}u \, (St)(u) + \int \mathrm{d}t \mathrm{d}z \, \mathcal{L}_m(\phi, \partial_t \phi) = -\bar{\Phi}_r \int \mathrm{d}u \, (St)(u) + \int \mathrm{d}u \mathrm{d}z \, t' \mathcal{L}_m(\phi, \partial_t \phi). \tag{9.1.26}$$

The pure matter variation is

$$\delta I \subset \int \mathrm{d}t \mathrm{d}z \left(\frac{\partial \mathcal{L}_m}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}_m}{\partial \partial_t \phi} \delta \partial_t \phi\right) = \int \mathrm{d}t \mathrm{d}z \left(\frac{\partial \mathcal{L}_m}{\partial \phi} - \partial_t \frac{\partial \mathcal{L}_m}{\partial \partial_t \phi}\right) \delta\phi \tag{9.1.27}$$

which just gives the standard Euler-Lagrange variation

$$\frac{\partial \mathcal{L}_m}{\partial \phi} - \partial_t \frac{\partial \mathcal{L}_m}{\partial \partial_t \phi} = 0. \tag{9.1.28}$$

However, varying with respect to gravity in terms of the boundary parameter u gives the following [8]

$$\begin{aligned} \delta I &= \int \mathrm{d}u \left(-\bar{\Phi} \partial_t (St)(u) \delta t + \int \mathrm{d}z \left[\delta t' \mathcal{L}_m + t' \frac{\partial \mathcal{L}_m}{\partial t} \delta t\right]\right) \\ &= \int \mathrm{d}u \left(-\bar{\Phi} \frac{1}{t'} \partial_u (St)(u) \delta t + \int \mathrm{d}z \left[\delta t' \mathcal{L}_m + t' \left(\frac{\partial \mathcal{L}_m}{\partial \phi} \partial_t \phi + \frac{\partial \mathcal{L}_m}{\partial \partial_t \phi} \partial_t^2 \phi\right) \delta t\right]\right) \\ &= \int \mathrm{d}u \left(-\bar{\Phi} \frac{[(St)(u)]'}{t'} \delta t + \int \mathrm{d}z \left[\delta t' \mathcal{L}_m + t' \partial_t \left(\frac{\partial \mathcal{L}_m}{\partial \partial_t \phi} \partial_t \phi\right) \delta t\right]\right) \\ &= \int \mathrm{d}u \left(-\bar{\Phi} \frac{[(St)(u)]'}{t'} \delta t + \int \mathrm{d}z \, \delta t' \left[\mathcal{L}_m - \frac{\partial \mathcal{L}_m}{\partial \partial_t \phi} \partial_t \phi\right]\right) \\ &= \int \mathrm{d}u \, \delta t \left(-\bar{\Phi} \frac{[(St)(u)]'}{t'} - \partial_u \int \mathrm{d}z \left[\mathcal{L}_m - \frac{\partial \mathcal{L}_m}{\partial \partial_t \phi} \partial_t \phi\right]\right). \end{aligned} \tag{9.1.29}$$

[7] From a higher-dimensional perspective, the dilaton value at the boundary corresponds to the radial the area of fixed radial hypersurfaces, so it is natural to fix this.

[8] This is, once again, deceptively simple-looking!

This leads to the classical equations of motion

$$\bar{\Phi}\frac{[(St)(u)]'}{t'(u)} = -\partial_u \int \mathrm{d}z \left(\mathcal{L}_m - \frac{\partial \mathcal{L}_m}{\partial \partial_t \phi}\partial_t \phi\right) = t' \frac{\mathrm{d}H}{\mathrm{d}t} \tag{9.1.30}$$

for the boundary Schwarzian mode, where $H = \int \mathrm{d}z\, T_{tt}$ is the Hamiltonian of the matter sector.

We can solve this in the absence of matter, when $H = 0$. The solutions to this are functions with constant Schwarzian derivative, but we do not want to consider the trivial cases of the constant function and $\mathrm{PSL}(2,\mathbb{R})$ (gauge) transformations. To find the non-trivial solutions, we switch to Euclidean Rindler time near the boundary $t(u) = i\tanh\frac{-i\tau(u)}{2} = \tan\frac{\tau(u)}{2}$, with which

$$(S[t\circ\tau])(u) = \tau'^2 (St)(\tau) + (S\tau)(u) = (S\tau)(u) + \frac{1}{2}\tau'^2. \tag{9.1.31}$$

We therefore see that Rindler time τ linear in boundary time is a non-trivial solution to the equations of motion. Since we identify $\tau \sim \tau + 2\pi$ in Euclidean signature, we take this solution to be

$$\tau(u) = \frac{2\pi}{\beta}u \tag{9.1.32}$$

so that $u \sim u + \beta$ is also identified. However, for simplicity we will set $\beta = 2\pi$ from now on. Note that this is however a fourth-order differential equation, and the solutions are not generally known.

We can also go beyond the semiclassical regime and study gravitational dynamics by considering fluctuations around the saddle point

$$\tau(u) = u + \varepsilon(u). \tag{9.1.33}$$

Tree-level gravitational effects are described by linearised fluctuations, so we want to expand the action to quadratic order. Once again, switching to Rindler time

$$(St)(u) = (S\tau)(u) + \frac{1}{2}\tau'(u)^2 = \frac{1}{2} + \epsilon\left(\varepsilon' + \varepsilon''\right) + \frac{1}{2}\varepsilon^2\left(\varepsilon^2 - 3\varepsilon''^2 - 2\varepsilon'\varepsilon'''\right) + \mathcal{O}\left(\varepsilon^3\right) \tag{9.1.34}$$

and dropping constant and/or total derivative terms

$$I^{(2)}_{\text{Schw}} = -\frac{1}{2}\bar{\Phi}\int_{\partial M} \mathrm{d}u\,(\varepsilon'^2 - \varepsilon''^2). \tag{9.1.35}$$

The boundary ∂M is parameterised by $u \sim u + 2\pi \in [0, 2\pi)$, so we can expand in terms of Fourier modes

$$\varepsilon(u) = \sum_{n\in\mathbb{Z}} e^{-inu}\varepsilon_n, \quad \varepsilon_n = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}u\, e^{inu}\varepsilon(u) \tag{9.1.36}$$

which diagonalises the quadratic action

$$I^{(2)}_{\text{Schw}} = -\frac{1}{2}\bar{\Phi}\sum_{n\in\mathbb{Z}}(n^2 - n^4)\varepsilon_n\varepsilon_{-n} = -\frac{1}{2}\bar{\Phi}\sum_{n\in\mathbb{Z}} n^2(1-n^2)\varepsilon_n\varepsilon_{-n}. \tag{9.1.37}$$

From the definition of the 1PI effective action in Lorentzian signature, we find that in Euclidean

signature $[G^{(2)}]^{-1} = i\Gamma^{(2)} \to -\Gamma^{(2)}$, so we should be able to read off the two-point function from $S \sim \Gamma$ at tree-level. However, we need to mod out the $\mathrm{PSL}(2,\mathbb{R})$ gauge redundancies in the path integral. These give rise to the zero modes in the action, with $n = -1, 0, +1$ (or ε_{-1}, ε_0, and ε_1) corresponding precisely to the three parameters/generators of the $\mathrm{PSL}(2,\mathbb{R})$ group. We therefore end up with the following expression

$$\langle \varepsilon(u)\varepsilon(0)\rangle = \sum_{n\neq 0,\pm 1} e^{-inu} \langle \varepsilon_n \varepsilon_{-n}\rangle = \frac{1}{\bar{\Phi}} \sum_{n\neq 0,\pm 1} \frac{e^{-inu}}{n^2(1-n^2)} \tag{9.1.38}$$

for the coordinate-space two-point function. To evaluate this, we note that

$$f(s) = \frac{1}{e^{-2\pi i s} - 1} \frac{e^{-isu}}{s^2(1-s^2)} \tag{9.1.39}$$

has poles at $s \in \mathbb{Z}$ with the desired residue. In particular

$$\oint_{\mathcal{C}} ds\, f(s) = \sum_{n\neq 0,\pm 1} \frac{e^{-inu}}{n^2(1-n^2)} \tag{9.1.40}$$

for some contour $\mathcal{C}$ picking up only poles at $n \neq 0, \pm 1 \in \mathbb{Z}$. We can take $\mathcal{C}$ to be the union of a counter-clockwise circular contour $\mathcal{C}_1$ encompassing all $s \in \mathbb{Z}$ and another clockwise contour $\mathcal{C}_2$ only encircling $s = -1, 0, +1$. It turns out that the integral over $\mathcal{C}_2$ vanishes. To see this, we break $\mathcal{C}_2$ into two pieces $\mathcal{C}_2^{\pm}$ in the upper and lower half planes. For large $|s|$ with $s = x + iy$,

$$f(s) \sim \frac{e^{-is(u-2\pi)}}{s^2(1-s^2)} \sim \frac{e^{-ix(u-2\pi)+y(u-2\pi)}}{s^2(1-s^2)} \tag{9.1.41}$$

and since the original expression is symmetric under $u \to -u$, we can pick $u > 0 \sim 2\pi$ or $u < 0 \sim 2\pi$ for $\mathcal{C}_2^-$ or $\mathcal{C}_2^+$ respectively. At large $|s|$, the integrals over both vanish, and we are just left with a contour integral over $\mathcal{C}_1$ which picks up (minus) the residues at $s = -1, 0, +1$ so that

$$\langle \varepsilon(u)\varepsilon(0)\rangle = \frac{1}{\bar{\Phi}} \left[(u-\pi)\sin(u) + \frac{5}{2}\cos(u) - \frac{(u-\pi)^2}{2} + 1 + \frac{\pi^2}{6} \right], \quad u > 0. \tag{9.1.42}$$

By symmetry, this expression holds for $u < 0$ when we just replace $u \mapsto |u|$, so that generally

$$\langle \varepsilon(u_i)\varepsilon(u_j)\rangle = \frac{1}{\bar{\Phi}} \left[(|u_{ij}| - \pi)\sin(|u_{ij}|) + \frac{5}{2}\cos(|u_{ij}|) - \frac{(|u_{ij}|-\pi)^2}{2} + 1 + \frac{\pi^2}{6} \right] \tag{9.1.43}$$

where $u_{ij} = u_i - u_j$. It is also useful to note that

$$\langle \varepsilon'(u_i)\varepsilon(u_j)\rangle = \mathrm{sgn}(u_{ij}) \frac{\partial}{\partial |u_{ij}|} \langle \varepsilon(u_i)\varepsilon(u_j)\rangle \tag{9.1.44}$$

$$\langle \varepsilon'(u_i)\varepsilon'(u_j)\rangle = -\frac{\partial^2}{\partial |u_{ij}|^2} \langle \varepsilon(u_i)\varepsilon(u_j)\rangle \,. \tag{9.1.45}$$

9.2 Matter Coupling

Let us couple the theory to some matter and study its boundary dynamics with gravity. This can be done using the usual holographic dictionary, where the on-shell bulk partition function is identified with the boundary field theory generating functional [94].

Take the matter sector to be described by the Euclidean action $I_m[\chi]$ for some (collection of) field(s) χ. To find boundary correlation functions, we only need to take functional derivatives of the boundary partition function $Z[\chi_r]$ with respect to χ_r. On the bulk side, we integrate out the dilaton and then matter (in this order!) to find

$$Z[\chi_r] \sim \int_{\Phi_r,\chi_r} \mathcal{D}g\,\mathcal{D}\Phi\,\mathcal{D}\chi\, e^{-I_{\text{JT}}[g,\Phi]-I_m[\chi,g]} \sim \int \mathcal{D}t\, e^{-I_{\text{Schw}}[t]} Z_m[\chi_r,t]. \tag{9.2.1}$$

In the saddle point approximation on the gravitational side, this reduces to

$$Z_{\text{Saddle}}[\chi_r] \sim e^{-I^{\text{on-shell}}_{\text{Schw}}[t]} Z_m[\chi_r,t] \tag{9.2.2}$$

evaluated at the combined saddle. Note that, even at tree-level, the saddle of $t(u)$ will also depend on the matter sector and is therefore non-trivial. However, in a regime where the gravitational dynamics are dominant, we can ignore the backreaction from the matter sector and determine the saddle for $t(u)$ independently[9].

9.2.1 Free Theory

Take, for instance, the free theory

$$I_m = -\frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\left(g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + m^2\chi^2\right). \tag{9.2.3}$$

Pure Matter Sector

We can find $Z_m[\chi_r,t]$ exactly. Solving the bulk equations of motion asymptotically near the boundary, we find

$$\chi(t,z) = z^{1-\Delta}\tilde{\chi}_r(t) + \dots \tag{9.2.4}$$

where ellipses denote asymptotically subleading terms as $z \to 0$, and

$$\Delta = \frac{1}{2} + \sqrt{\frac{1}{4}+m^2} = \frac{1}{2}\left(1+\sqrt{1+4m^2}\right). \tag{9.2.5}$$

In our putative dual theory we think of χ_r as sources for operators with scaling dimension Δ. The holographic dictionary (at tree-level) tells us that the boundary field theory partition

[9] From a higher-dimensional perspective, the Schwarzian action comes with extra factors of M_{Pl} so for sufficiently slowly growing scaling dimensions, this is satisfied.

function will be determined by the on-shell bulk action. In particular, the quadratic part is[10]

$$\log Z_m[\chi_r, t] \supset D \int \mathrm{d}t\, \mathrm{d}t' \frac{\tilde{\chi}_r(t)\tilde{\chi}_r(t')}{|t-t'|^{2\Delta}} \tag{9.2.6}$$

where

$$D = \frac{(\Delta - \frac{1}{2})\Gamma(\Delta)}{\sqrt{\pi}\Gamma(\Delta - \frac{1}{2})} = \frac{1}{2} c_\Delta\big|_{d=1} \tag{9.2.7}$$

and

$$c_\Delta = (2\Delta - d)\frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma\left(\Delta - \frac{d}{2}\right)} \tag{9.2.8}$$

is the correct normalisation for scalar two-point functions when $\Delta = d/2 + k$ and $k \in \mathbb{Z}$[94].

When we consider the free matter theory, this is in fact the only term in the effective action $W_m = -\log Z_m$, but there will be other contributions to the connected correlation functions when we account for gravitational fluctuations. As discussed previously, these are contained in the AdS_2 boundary and are therefore hidden in the (t, z)-dependence. Defining $\chi_r(u) = t'(u)^{1-\Delta}\tilde{\chi}_r(t)$ such that the asymptotic boundary condition becomes

$$\chi(t, z) = z^{1-\Delta}\tilde{\chi}_r(t) + \cdots = [\epsilon t'(u)]^{1-\Delta}\tilde{\chi}_r(t) + \cdots = \epsilon^{1-\Delta}\chi_r(u) + \ldots, \tag{9.2.9}$$

we see that

$$\log Z_m[\chi_r, t] \supset D \int \mathrm{d}u\, \mathrm{d}u' \left[\frac{t'(u)t'(u')}{(t(u) - t(u'))^2}\right]^\Delta \chi_r(u)\chi_r(u') \tag{9.2.10}$$

with everything evaluated on the saddle.

Boundary Correlation Functions

In the regime where the gravitational dynamics are dominant, there are no matter self-interactions via gravity and the operator $\mathcal{O}_\chi$ dual to χ_r is essentially free, and the only non-vanishing correlation function is the 2-point function at tree-level

$$\langle \mathcal{O}_\chi(u_1)\mathcal{O}_\chi(u_2)\rangle \sim \left[\frac{t'(u_1)t'(u_2)}{(t(u_1) - t(u_2))^2}\right]^\Delta \tag{9.2.11}$$

with $t(u)$ on the saddle of the purely Schwarzian theory.

We can capture gravitational loops by expanding around the Schwarzian saddle point

$$t(u) = \tan\frac{u + \varepsilon(u)}{2} \tag{9.2.12}$$

which gives

$$\left[\frac{t'(u_1)t'(u_2)}{(t(u_1) - t(u_2))^2}\right]^\Delta = \frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}}\left[1 + \mathcal{B}(u_1, u_2) + \mathcal{C}(u_1, u_2) + \mathcal{O}(\varepsilon^3)\right] \tag{9.2.13}$$

[10] This is slightly subtle. The boundary is not located at fixed $z = \epsilon$, so we need to evaluate the boundary term with respect to the vector n^μ that is normal to the wiggling boundary curve $(t(u), z(u))$. The corrections are subleading in the UV cut-off ϵ.

where $u_{ij} = u_i - u_j$ and the higher-loop corrections are captured in

$$\mathcal{B}(u_1,u_2) = \Delta\left(\varepsilon'(u_1)+\varepsilon'(u_2) - \frac{\varepsilon(u_1)-\varepsilon(u_2)}{\tan\frac{u_{12}}{2}}\right) \tag{9.2.14}$$

$$\begin{aligned}\mathcal{C}(u_1,u_2) = \frac{\Delta}{(2\sin\frac{u_{12}}{2})^2}\Big[&(1+\Delta+\Delta\cos u_{12})\big[\varepsilon(u_1)-\varepsilon(u_2)\big]^2 \\ &- 2\Delta\sin u_{12}\big[\varepsilon(u_1)-\varepsilon(u_2)\big]\big[\varepsilon'(u_1)+\varepsilon'(u_2)\big] \\ &- (1-\cos u_{12})\Big((1-\Delta)\big[(\varepsilon'(u_1)^2+\varepsilon'(u_2)^2)\big] - 2\Delta\varepsilon'(u_1)\varepsilon'(u_2)\Big)\Big].\end{aligned} \tag{9.2.15}$$

We can then simply use our previous result for $\langle\epsilon(u_1)\epsilon(u_2)\rangle$ for Wick contractions to evaluate corrections due to the Schwarzian. In particular, expanding up to quadratic order in fluctuations, we find that the Wilson effective action takes the form

$$\begin{aligned}\log Z[\chi_r] \sim \int \mathcal{D}t\, e^{-I_{\text{Schw}}[t]}\Bigg[&D\left(\prod_{i=1}^{2}\int \mathrm{d}u_i\,\chi_r(u_i)\right)\frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}}\left[1+\mathcal{C}(u_1,u_2)\right] \\ &+\frac{D^2}{2}\left(\prod_{i=1}^{4}\int \mathrm{d}u_i\,\chi_r(u_i)\right)\frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}\left(2\sin\frac{u_{34}}{2}\right)^{2\Delta}}\mathcal{B}(u_1,u_2)\mathcal{B}(u_3,u_4) \\ &+\dots\Bigg].\end{aligned} \tag{9.2.16}$$

This means that

$$\langle\mathcal{O}_\chi(u_1)\mathcal{O}_\chi(u_2)\rangle \sim \frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}}\left[1+\langle\mathcal{C}(u_1,u_2)\rangle\right] \tag{9.2.17}$$

and

$$\langle\prod_{i=1}^{4}\mathcal{O}_\chi(u_i)\rangle \sim \frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}\left(2\sin\frac{u_{34}}{2}\right)^{2\Delta}}\langle\mathcal{B}(u_1,u_2)\mathcal{B}(u_3,u_4)\rangle \tag{9.2.18}$$

which we will evaluate shortly.

9.2.2 Example: Goldstone EFT

We have previously seen that there are interesting non-trivial constraints on the Goldstone EFT. Let us therefore study the boundary correlation functions in this theory, and take

$$\mathcal{L}_\lambda[h;\chi] = -\frac{1}{2}h^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \frac{\lambda}{4}\left(h^{\mu\nu}\partial_\mu\chi\partial_\nu\chi\right)^2 \tag{9.2.19}$$

This is actually an interesting set-up from the perspective of AdS_2 holography — the $(\partial\phi)^4$ interaction typical for Goldstone EFTs is also related to $T\overline{T}$-flows and coupling to massive gravity in $D=2$. This is perhaps not surprising — we saw that the Goldstone EFT above describe transverse fluctuations of branes embedded into higher dimensions. The actions for these typically have square-root structure, which are also associated with $T\overline{T}$-deformations (via brane actions) and massive gravity. Details on this connection are reviewed in appendix 9.B.

Recall that, in holographic renormalisation, we can find correlation functions of operators dual to the boundary fields by expanding the perturbative on-shell action asymptotically. For this, we

solve the bulk fields perturbatively

$$\chi = \chi_0 + \lambda\chi_1 + \dots \tag{9.2.20}$$

with

$$\chi_0(z,x) = \int \mathrm{d}x' K_\Delta(z,x;x')\bar{\chi}(x') \tag{9.2.21a}$$

$$\chi_1(z,x) = \int \mathrm{d}z'dx' G_\Delta(z,x;z',x')h^{\mu\nu}\nabla_\mu\left(X_0\nabla_\nu\chi\right)(z',x') \tag{9.2.21b}$$

where $\bar{\chi}$ is the boundary value of the bulk field χ, and for $d = \Delta = 1$ the bulk-to-boundary and bulk-to-bulk propagators are respectively given by [11]

$$K(z,x;x') = \frac{1}{\pi}\frac{z}{z^2+(x-x')^2}, \tag{9.2.26a}$$

$$D(z,x;z',x') = \frac{1}{2\pi}\mathrm{arctanh}(\xi) \tag{9.2.26b}$$

where the chordal distance is defined as

$$\xi(z,x;z',x') = \frac{2zz'}{z^2+z'^2+(x-x')^2}. \tag{9.2.27}$$

Two-Point Function

First, let us consider the one-loop correction to the two-point correlator

$$\begin{aligned}\langle\mathcal{O}_\chi(x_1)\mathcal{O}_\chi(x_2)\rangle \supset\ & \frac{\lambda}{4}\int \mathrm{d}z\mathrm{d}x\sqrt{h}\left(h^{\mu\nu}h^{\rho\sigma}+\text{perm.}\right)\partial_\mu K(z,x;x_1)\partial_\nu K(z,x;x_2)\partial_\rho\partial_\sigma D_\Delta(z,x;z,x)\\ =\ & \frac{\lambda}{4}\int \mathrm{d}z\mathrm{d}x\frac{1}{z^2}\Big[\partial_\mu K(z,x;x_1)\partial^\mu K(z,x;x_2)\Box D(z,x;z,x)\\ & +\partial^\mu K(z,x;x_1)\partial^\nu K(z,x;x_2)\partial_\mu\partial_\nu D(z,x;z,x)\Big].\end{aligned} \tag{9.2.28}$$

[11] Just for completeness, we also note that in $D = d+1$ bulk dimensions, the bulk-to-boundary and bulk-to-bulk propagators are

$$K_\Delta(z,\mathbf{x};\mathbf{x}') = C_\Delta\left(\frac{z}{z^2+|\mathbf{x}-\mathbf{x}'|^2}\right)^\Delta \tag{9.2.22a}$$

$$D_\Delta(z,\mathbf{x};z',\mathbf{x}') = \frac{2^{-\Delta}C_\Delta}{2\Delta-d}\xi^\Delta F\left(\frac{\Delta}{2},\frac{\Delta}{2}+\frac{1}{2};\Delta-\frac{d}{2}+1;\xi^2\right) \tag{9.2.22b}$$

with normalisation constant

$$C_\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta-\frac{d}{2})} \tag{9.2.23}$$

and the chordal distance given by

$$\xi(z,\mathbf{x};z',\mathbf{x}') = \frac{2zz'}{z^2+z'^2+|\mathbf{x}-\mathbf{x}'|^2}. \tag{9.2.24}$$

It is instructive to note that

$$K(z,\mathbf{x};\mathbf{x}') \sim \lim_{z'\to 0}\frac{1}{\sqrt{\gamma}}n^i\partial_i D_\Delta(z,\mathbf{x};z',\mathbf{x}') \sim z^{d-\Delta}\delta^{(d)}(\mathbf{x}-\mathbf{x}'). \tag{9.2.25}$$

However, as expected, this is UV divergent! In particular,

$$D_\Delta(z,x;z,x) = D_\Delta(\xi=1) \tag{9.2.29}$$

diverges and needs to be regulated by replacing $\xi \mapsto \frac{\xi}{1+\kappa}$ for some small κ [302]. All derivatives of this vanish, and we find that the one-loop correction to the two-point function vanishes.

This means that the scalar field does not receive any corrections to its mass (in the bulk) or anomalous dimension (in the boundary). Physically, we could have anticipated this, because the bulk scalar field enjoys a shift symmetry, which a correction to the mass would violate.

Four-Point Function

Given the quartic interaction, the other non-trivial correlation function we can consider is the four-point function at tree-level. When the gravitational modes are decoupled, we can just use Witten diagrams to determine this

$$\begin{aligned}\langle \mathcal{O}_\chi(x_1)\mathcal{O}_\chi(x_2)\mathcal{O}_\chi(x_3)\mathcal{O}_\chi(x_4)\rangle &\supset \frac{\lambda}{4}4!\int \mathrm{d}z\mathrm{d}x\sqrt{h}h^{(\mu_1\mu_2}h^{\mu_3\mu_4)}\prod_{i=1}^{4}\left(\frac{\partial}{\partial x^{\mu_i}}K(z,x;x_i)\right)\\ &\supset \frac{\lambda}{4}\int \mathrm{d}z\mathrm{d}x\frac{1}{z^2}\big[\partial_\mu K(z,x;x_1)\partial^\mu K(z,x;x_2)\partial_\nu K(z,x;x_3)\partial^\nu K(z,x;x_4)\\ &\quad + \text{perm.}\big].\end{aligned} \tag{9.2.30}$$

Before proceeding, it is worth pointing out some of the peculiarities for correlation functions in $d=1$ CFTs, which are reminiscent of $d=2$ QFTs [303]. First of all, ordering of the (space-)time points is important, and we cannot reach different channels or operator position orderings via crossing. We therefore need to *a priori* fix an ordering of the x_i, *e.g.* $x_i \leq x_{i+1}\forall i$. Further, conformal transformations can be used to reduce n-point correlation functions to functions of $n-3$ independent cross ratios

$$u_i = \frac{x_{1,i}x_{n-1,n}}{x_{i,n}x_{1,n-1}} \tag{9.2.31}$$

where $x_{ij} = x_i - x_j$. For $n=4$, this means that the correlation functions will depend on a single cross ratio[12]

$$u_2 = \frac{x_{12}x_{34}}{x_{24}x_{13}}. \tag{9.2.32}$$

For the ordering $x_1 \leq x_2 \leq x_3 \leq x_4$ this takes values in $u_2 \in [0,1]$.

As pointed out in [304], the boundary being 1-dimensional also means that we can use complex methods to actually the integrals over said boundary. Then, in terms of this $u = u_2$ and the auxiliary variable $v = 1-u$, we find that

$$\langle \mathcal{O}_\chi(x_1)\mathcal{O}_\chi(x_2)\mathcal{O}_\chi(x_3)\mathcal{O}_\chi(x_4)\rangle \supset -\frac{\lambda}{4\pi^3}\frac{f(u,v)}{x_{13}^2x_{24}^2}, \tag{9.2.33a}$$

$$f(u,v) = \left(\frac{1-uv}{uv}\right)^2 + \frac{1}{2}\frac{2u^4-u^3-u+2}{u^3}\log(v) + \frac{1}{2}\frac{2v^4-v^3-v+2}{v^3}\log(u) \tag{9.2.33b}$$

[12] This is similar to how scattering in $d=2$ only depends on one Mandelstam variable.

which is manifestly symmetric under both crossing: $(u,v) \mapsto (v,u)$, and braiding: $(u,v) \mapsto (-u/v, 1/v)$.

However, we are actually interested in the interaction of these boundary matter degrees of freedom with the boundary Schwarzian modes. Relabelling $x_i \mapsto t_i = t(u_i)$ and restoring dynamical gravity[13], we find that the contribution to the matter sector effective action due to the interaction term is

$$\begin{aligned}\log Z_m[\chi_r, t] &\supset -\frac{1}{4!}\frac{\lambda}{4\pi^3}\left(\prod_{i=1}^{4}\int \mathrm{d}t_i \tilde{\chi}_r(t_i)\right)\frac{f(u,v)}{t_{13}^2 t_{24}^2} \\ &= -\frac{1}{4!}\frac{\lambda}{4\pi^3}\left(\prod_{i=1}^{4}\int \mathrm{d}u_i \chi_r(u_i) t'(u_i)\right)\frac{u^2 f(u,v)}{t_{12}^2 t_{34}^2}.\end{aligned} \tag{9.2.34}$$

Now, the full matter effective action is

$$\begin{aligned}Z_m[\chi_r, t] = 1 &+ \left(\prod_{i=1}^{2}\int \mathrm{d}u_i\, \chi_r(u_i)\right) D\frac{\prod_{i=1}^{2} t'(u_i)}{\left[t(u_1)-t(u_2)\right]^2} \\ &+ \left(\prod_{i=1}^{4}\int \mathrm{d}u_i\, \chi_r(u_i)\right)\left(\frac{D^2}{2} - \frac{1}{4!}\frac{\lambda}{4\pi^3}u^2 f(u,v)\right)\frac{\prod_{i=1}^{4} t'(u_i)}{\left[t(u_1)-t(u_2)\right]^2\left[t(u_3)-t(u_4)\right]^2} \\ &+ \dots\end{aligned} \tag{9.2.35}$$

To determine the corrections due to gravitational loops, we want to expand up to quadratic order in fluctuations around the Schwarzian saddle. First, we can write the cross ratio as

$$u = \frac{\left[t(u_1)-t(u_2)\right]\left[t(u_3)-t(u_4)\right]}{\left[t(u_1)-t(u_3)\right]\left[t(u_2)-t(u_4)\right]} = \alpha(u_i) + \beta(u_i) + \gamma(u_i) \tag{9.2.36}$$

where

$$\alpha(u_i) = \frac{\sin\left(\frac{u_{12}}{2}\right)\sin\left(\frac{u_{34}}{2}\right)}{\sin\left(\frac{u_{13}}{2}\right)\sin\left(\frac{u_{24}}{2}\right)} \tag{9.2.37}$$

$$\begin{aligned}\beta(u_i) = \frac{1}{8\left(\sin\frac{u_{13}}{2}\right)^2\left(\sin\frac{u_{24}}{2}\right)^2}\big(&\epsilon(u_1)\left[\sin u_{23} + \sin u_{34} + \sin u_{42}\right] - \epsilon(u_2)\left[\sin u_{34} + \sin u_{41} + \sin u_{13}\right] \\ &+\epsilon(u_3)\left[\sin u_{41} + \sin u_{12} + \sin u_{24}\right] - \epsilon(u_4)\left[\sin u_{12} + \sin u_{23} + \sin u_{31}\right]\big)\end{aligned} \tag{9.2.38}$$

[13] Note that u and the u_i are only indirectly related.

$$\gamma(u_i) = \frac{1}{4\left(\sin\frac{u_{13}}{2}\right)^2\left(\sin\frac{u_{24}}{2}\right)^2}\Bigg\{\left[\cos\left(\frac{u_{12}+u_{34}}{2}\right) - \cos\left(\frac{u_{13}}{2}\right)\cos\left(\frac{u_{24}}{2}\right)\right]$$
$$\times\left(\frac{\sin\frac{u_{24}}{2}}{\sin\frac{u_{13}}{2}}\epsilon(u_1)\epsilon(u_3) + \frac{\sin\frac{u_{13}}{2}}{\sin\frac{u_{24}}{2}}\epsilon(u_2)\epsilon(u_4)\right)$$
$$+\left(\sin\frac{u_{34}}{2}\right)^2\epsilon(u_1)\epsilon(u_2) + \left(\sin\frac{u_{12}}{2}\right)^2\epsilon(u_3)\epsilon(u_4) - \left(\sin\frac{u_{14}}{2}\right)^2\epsilon(u_2)\epsilon(u_3) - \left(\sin\frac{u_{23}}{2}\right)^2\epsilon(u_1)\epsilon(u_4)$$
$$+\cot\left(\frac{u_{13}}{2}\right)\sin\left(\frac{u_{23}}{2}\right)\sin\left(\frac{u_{34}}{2}\right)\sin\left(\frac{u_{42}}{2}\right)\epsilon(u_1)^2 - \cot\left(\frac{u_{24}}{2}\right)\sin\left(\frac{u_{34}}{2}\right)\sin\left(\frac{u_{41}}{2}\right)\sin\left(\frac{u_{13}}{2}\right)\epsilon(u_2)^2$$
$$+\cot\left(\frac{u_{31}}{2}\right)\sin\left(\frac{u_{41}}{2}\right)\sin\left(\frac{u_{12}}{2}\right)\sin\left(\frac{u_{24}}{2}\right)\epsilon(u_3)^2 - \cot\left(\frac{u_{42}}{2}\right)\sin\left(\frac{u_{12}}{2}\right)\sin\left(\frac{u_{23}}{2}\right)\sin\left(\frac{u_{31}}{2}\right)\epsilon(u_4)^2\Bigg\}. \tag{9.2.39}$$

We can then use this to expand

$$f(u,v) = X(u_i) + Y(u_i) + Z(u_i) \tag{9.2.40}$$

where

$$X(u_i) = \left(1 - \frac{1}{\alpha\tilde{\alpha}}\right)^2 + \frac{2-\tilde{\alpha}-\tilde{\alpha}^3+2\tilde{\alpha}^4}{2\tilde{\alpha}^3}\log(\alpha) + \frac{2-\alpha-\alpha^3+2\alpha^4}{2\alpha^3}\log(\tilde{\alpha}) \tag{9.2.41}$$

$$\begin{aligned} Y(u_i) &= \left(-\frac{(\alpha-\tilde{\alpha})(-1+\alpha\tilde{\alpha})(6+\alpha\tilde{\alpha})}{2\alpha^3\tilde{\alpha}^3} - \frac{-3+\tilde{\alpha}+\tilde{\alpha}^4}{\tilde{\alpha}^4}\log(\alpha) + \frac{-3+\alpha+\alpha^4}{\alpha^4}\log(\tilde{\alpha})\right)\beta \\ &= Y^{(\beta)}(u_i)\beta \end{aligned} \tag{9.2.42}$$

$$\begin{aligned} Z(u_i) &= \frac{1}{\alpha^4\tilde{\alpha}^4}\left(3\left[8+\alpha\tilde{\alpha}(-30+11\alpha\tilde{\alpha})\right]\beta^2 - 2(\alpha-\tilde{\alpha})\alpha\tilde{\alpha}(-1+\alpha\tilde{\alpha})(6+\alpha\tilde{\alpha})\gamma\right) \\ &\quad -\frac{3(-4+\tilde{\alpha})\beta^2 + 2\tilde{\alpha}(-3+\tilde{\alpha}+\tilde{\alpha}^4)\gamma}{2\tilde{\alpha}^5}\log(\alpha) + \frac{3(-4+\alpha)\beta^2+2\alpha(-3+\alpha+\alpha^4)\gamma}{2\alpha^5}\log(\tilde{\alpha}) \\ &= Z^{(\beta)}(u_i)\beta^2 + Z^{(\gamma)}(u_i)\gamma \end{aligned} \tag{9.2.43}$$

and we have denoted $\tilde{\alpha} = 1-\alpha$. Putting everything together, and suppressing the u_i-dependence of our auxiliary variables, we can finally write our matter sector partition function as

$$\begin{aligned} Z_m[\chi_r, t] &= 1 + \left(\prod_{i=1}^{2}\int \mathrm{d}u_i\,\chi_r(u_i)\right) D\frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^2}\left[1+\mathcal{B}(u_1,u_2)+\mathcal{C}(u_1,u_2)\right] \\ &\quad + \left(\prod_{i=1}^{4}\int \mathrm{d}u_i\,\chi_r(u_i)\right)\left[\frac{D^2}{2} - \frac{1}{4!}\frac{\lambda}{4\pi^3}\left[\alpha(u_i)+\beta(u_i)+\gamma(u_i)\right]^2\left(X(u_i)+Y(u_i)+Z(u_i)\right)\right] \\ &\qquad \times \frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^2\left(2\sin\frac{u_{34}}{2}\right)^2}\left[1+\mathcal{B}(u_1,u_2)+\mathcal{C}(u_1,u_2)\right]\left[1+\mathcal{B}(u_3,u_4)+\mathcal{C}(u_3,u_4)\right] \\ &\quad + \dots \end{aligned} \tag{9.2.44}$$

which we could have, in principle, expanded further. To find the full partition function, we need to evaluate the path integral over the Schwarzian modes, which we can do perturbatively. Since

$\langle\varepsilon\rangle = 0$, this just leaves us with the even orders

$$
\begin{aligned}
Z[\chi_r, t] &= \langle Z_m[\chi_r, t]\rangle \\
&= 1 + \left(\prod_{i=1}^{2}\int \mathrm{d}u_i\,\chi_r(u_i)\right) D\frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^2}\left[1 + \langle\mathcal{C}(u_1,u_2)\rangle\right] \\
&+ \left(\prod_{i=1}^{4}\int \mathrm{d}u_i\,\chi_r(u_i)\right)\frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^2\left(2\sin\frac{u_{34}}{2}\right)^2} \\
&\quad\times\Bigg[\left(\frac{D^2}{2} - \frac{1}{4!}\frac{\lambda}{4\pi^3}X\alpha^2\right)\left(1 + \langle\mathcal{C}(u_1,u_2)\rangle + \langle\mathcal{C}(u_3,u_4)\rangle + \langle\mathcal{B}(u_1,u_2)\mathcal{B}(u_3,u_4)\rangle\right) \\
&\qquad - \frac{1}{4!}\frac{\lambda}{4\pi^3}\left\langle\left(Y\alpha^2 + 2X\alpha\beta\right)\left(\mathcal{B}(u_1,u_2) + \mathcal{B}(u_3,u_4)\right)\right\rangle - \frac{1}{4!}\frac{\lambda}{4\pi^3}\left\langle Z\alpha^2 + 2\alpha\beta Y + X\left(\beta^2 + 2\alpha\gamma\right)\right\rangle\Bigg] \\
&+ \dots \\
&= 1 + \frac{1}{2!}\left(\prod_{i=1}^{2}\int \mathrm{d}u_i\,\chi_r(u_i)\right)\frac{1}{\pi}\frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^2}\left[1 + \langle\mathcal{C}(u_1,u_2)\rangle\right] \\
&+ \frac{1}{4!}\left(\prod_{i=1}^{4}\int \mathrm{d}u_i\,\chi_r(u_i)\right)\frac{1}{\pi^2}\frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^2\left(2\sin\frac{u_{34}}{2}\right)^2} \\
&\quad\times\Bigg[\left(3 - \tilde{\lambda}X\alpha^2\right)\left(1 + \langle\mathcal{C}(u_1,u_2)\rangle + \langle\mathcal{C}(u_3,u_4)\rangle + \langle\mathcal{B}(u_1,u_2)\mathcal{B}(u_3,u_4)\rangle\right) \\
&\qquad - \tilde{\lambda}(\alpha^2 Y^{(\beta)} + 2\alpha X)\left\langle\beta\left(\mathcal{B}(u_1,u_2) + \mathcal{B}(u_3,u_4)\right)\right\rangle \\
&\quad - \tilde{\lambda}(\alpha^2 Z^{(\beta)} + 2\alpha Y^{(\beta)} + X)\left\langle\beta^2\right\rangle - \tilde{\lambda}(\alpha^2 Z^{(\gamma)} + 2\alpha X)\left\langle\gamma\right\rangle\Bigg] + \dots \quad ,
\end{aligned}
\tag{9.2.45}
$$

where we have defined $\tilde{\lambda} = \lambda/4\pi$. After exponentiating, we find

$$
\begin{aligned}
\log Z[\chi_r, t] &= \frac{1}{2!}\left(\prod_{i=1}^{2}\int \mathrm{d}u_i\,\chi_r(u_i)\right)\frac{1}{\pi}\frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^2}\left[1 + \langle\mathcal{C}(u_1,u_2)\rangle\right] \\
&+ \frac{1}{4!}\left(\prod_{i=1}^{4}\int \mathrm{d}u_i\,\chi_r(u_i)\right)\frac{1}{\pi^2}\frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^2\left(2\sin\frac{u_{34}}{2}\right)^2} \\
&\quad\times\Bigg[\left(3 - \tilde{\lambda}X\alpha^2\right)\left\langle\mathcal{B}(u_1,u_2)\mathcal{B}(u_3,u_4)\right\rangle - \tilde{\lambda}X\alpha^2(1 + \langle\mathcal{C}(u_1,u_2)\rangle + \langle\mathcal{C}(u_3,u_4)\rangle) \\
&\qquad - \tilde{\lambda}(\alpha^2 Y^{(\beta)} + 2\alpha X)\left\langle\beta\left(\mathcal{B}(u_1,u_2) + \mathcal{B}(u_3,u_4)\right)\right\rangle - \tilde{\lambda}(\alpha^2 Z^{(\beta)} + 2\alpha Y^{(\beta)} + X)\left\langle\beta^2\right\rangle \\
&\quad - \tilde{\lambda}(\alpha^2 Z^{(\gamma)} + 2\alpha X)\left\langle\gamma\right\rangle\Bigg] + \dots \quad ,
\end{aligned}
\tag{9.2.46}
$$

Therefore, the connected 4-point function with correction from the $T\bar{T}$-deformation is

$$\begin{aligned}\langle\prod_{i=1}^{4}\mathcal{O}_\chi(u_i)\rangle = &\frac{1}{\pi^2}\frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^2\left(2\sin\frac{u_{34}}{2}\right)^2}\\ &\Bigg[\left(3-\tilde{\lambda}X\alpha^2\right)\langle\mathcal{B}(u_1,u_2)\mathcal{B}(u_3,u_4)\rangle - \tilde{\lambda}X\alpha^2(1+\langle\mathcal{C}(u_1,u_2)\rangle+\langle\mathcal{C}(u_3,u_4)\rangle)\\ &-\tilde{\lambda}(\alpha^2Y^{(\beta)}+2\alpha X)\langle\beta\left(\mathcal{B}(u_1,u_2)+\mathcal{B}(u_3,u_4)\right)\rangle - \tilde{\lambda}(\alpha^2Z^{(\beta)}+2\alpha Y^{(\beta)}+X)\langle\beta^2\rangle\\ &-\tilde{\lambda}(\alpha^2Z^{(\gamma)}+2\alpha X)\langle\gamma\rangle\Bigg].\end{aligned} \tag{9.2.47}$$

9.A The Schwarzian Derivative

From the definition given above, we can immediately verify that the chain rule for the Schwarzian derivative is

$$(S[f\circ g])(z) = g'^2(Sf)[g(z)] + (Sg)(z). \tag{9.A.1}$$

We can also use the following function of two variables

$$F(z,w) = \log\left(\frac{f(z)-f(w)}{z-w}\right) \tag{9.A.2}$$

as an alternative way to define the Schwarzian derivative, as

$$(Sf)(w) = 6\frac{\partial^2F(z,w)}{\partial z\partial w}\bigg|_{z=w} = 6\lim_{z\to w}\left(\frac{f'(z)f'(w)}{(f(z)-f(w))^2}-\frac{1}{(z-w)^2}\right). \tag{9.A.3}$$

Using this, it is easy to see that for $M(z)=\frac{az+b}{cz+d}$,

$$\begin{aligned}(SM)(w) =&6\lim_{z\to w}\left(\frac{\frac{ad-bc}{(cz+d)^2}\frac{ad-bc}{(cw+d)^2}}{\left(\frac{az+b}{cz+d}-\frac{aw+b}{cw+d}\right)^2}-\frac{1}{(z-w)^2}\right)\\ =&6\lim_{z\to w}\left(\frac{(ad-bc)^2}{\left[(az+b)(cw+d)-(aw+b)(cz+d)\right]^2}-\frac{1}{(z-w)^2}\right)\\ =&6\lim_{z\to w}\left(\frac{(ad-bc)^2}{\left[(ad-bc)(z-w)\right]^2}-\frac{1}{(z-w)^2}\right) = 0\end{aligned} \tag{9.A.4}$$

so the Schwarzian derivatives vanishes for $\mathrm{PSL}(2,\mathbb{R})$ transformations. Now, $(Sf)(z)=0$ is a third-order differential equation, so three initial conditions are required to find the unique solution. These map precisely into the four parameters (subject to one constraint) of $\mathrm{PSL}(2,\mathbb{R})$ via

$$(M(0),M'(0),M''(0)) = \left(\frac{b}{d},\frac{1}{d^2},-2\frac{c}{d^3}\right), \tag{9.A.5}$$

so that $f=M$ is the unique solution to $(Sf)(z)=0$.

Using the composition law for the Schwarzian derivative, we then find that for two functions

$f(z)$ and $g(z)$ related by a Möbius transformation $f = M \circ g$,

$$(Sf)(z) = (S[M \circ g])(z) = g'^2(SM)(z) + (Sg)(z) = (Sg)(z), \tag{9.A.6}$$

so their Schwarzian derivatives are equal. Conversely, two functions with equal Schwarzian derivative must be related by a Möbius transformation. In other words, it is $\mathrm{PSL}(2,\mathbb{R})$ invariant!

9.B $T\overline{T}$ Deformations and Massive Gravity

We claimed that the leading-order Goldstone EFT interaction $\mathcal{L} \supset (\partial\chi)^4$ can be generated by a $T\overline{T}$ deformation or equivalently coupling to two-dimensional (topological) massive gravity. In this appendix, we will show this explicitly.

9.B.1 $T\overline{T}$ Deformation

Let us start by reviewing $T\overline{T}$ deformations briefly.

$T\overline{T}$-Operator

Given some Lagrangian density $\mathcal{L}$ of some fields collectively denoted as ϕ on flat space, the stress tensor is defined to be

$$T_{\mu\nu} = \partial_\mu\phi\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} - \eta_{\mu\nu}\mathcal{L} \tag{9.B.1}$$

and satisfies

$$\partial_\mu T^{\mu\nu} = 0 \tag{9.B.2}$$

on-shell. In complex coordinates $z = x^1 + ix^2$ we have $g = \mathrm{d}z\mathrm{d}\bar{z}$, and we define

$$T = -2\pi T_{zz} = -\frac{\pi}{2}\left(T_{11} - T_{22} - 2iT_{12}\right) \tag{9.B.3a}$$

$$\bar{T} = -2\pi T_{\bar{z}\bar{z}} = -\frac{\pi}{2}\left(T_{11} - T_{22} + 2iT_{12}\right) \tag{9.B.3b}$$

$$\Theta = 2\pi T_{z\bar{z}} = \frac{\pi}{2}\left(T_{11} + T_{22}\right) \tag{9.B.3c}$$

so that stress-energy conservation is

$$\bar{\partial}T = \partial\Theta, \quad \partial\bar{T} = \bar{\partial}\Theta. \tag{9.B.4}$$

The Zamolodchikov operator is defined as

$$(T\bar{T})(z) = \lim_{z\to w}(T\bar{T})(z,w) = \lim_{z\to w}\left[T(z)\bar{T}(w) - \Theta(z)\Theta(w)\right] \tag{9.B.5}$$

but it is not obvious that this limit is well-defined [186]. To show that it is, note that

$$\begin{aligned}\partial_z\left[T(z)\bar{T}(w)-\Theta(z)\Theta(w)\right]=&\partial_z T(z)\bar{T}(w)-\partial_z\Theta(z)\Theta(w)\\=&\partial_z T(z)\bar{T}(w)-\bar{\partial}_z T(z)\Theta(w)\\=&\partial_z T(z)\bar{T}(w)-\bar{\partial}_z T(z)\Theta(w)+T(z)\left[\partial_w\bar{T}(w)-\bar{\partial}_w\Theta(w)\right]\\=&(\partial_z+\partial_w)\big(T(z)\bar{T}(w)\big)-(\bar{\partial}_z+\bar{\partial}_w)\big(T(z)\Theta(w)\big)\end{aligned}\tag{9.B.6}$$

and similarly

$$\begin{aligned}\bar{\partial}_z\left[T(z)\bar{T}(w)-\Theta(z)\Theta(w)\right]=&\bar{\partial}_z T(z)\bar{T}(w)-\bar{\partial}_z\Theta(z)\Theta(w)\\=&\partial_z\Theta(z)\bar{T}(w)-\bar{\partial}_z\Theta(z)\Theta(w)\\=&\partial_z\Theta(z)\bar{T}(w)-\bar{\partial}_z\Theta(z)\Theta(w)+\Theta(z)\left[\partial_w\bar{T}(w)-\bar{\partial}_w\Theta(w)\right]\\=&(\partial_z+\partial_w)\big(T(z)\bar{T}(w)\big)-(\bar{\partial}_z+\bar{\partial}_w)\big(\Theta(z)\Theta(w)\big).\end{aligned}\tag{9.B.7}$$

Now, for any function f

$$(\partial_z+\partial_w)f(z-w)=f'(z-w)-f'(z-w)=0\tag{9.B.8}$$

so, OPE expanding

$$T(z)\bar{T}(w)=\sum_i A^i(z-w)\mathcal{O}(w),\quad T(z)\Theta(w)=\sum_i B^i(z-w)\mathcal{O}(w),\quad \Theta(z)\Theta(w)=\sum_i C^i(z-w)\mathcal{O}(w),\tag{9.B.9}$$

we find that

$$\begin{aligned}\partial_z(T\bar{T})(z,w)&=\sum_i(\partial_z+\partial_w)\left[A^i(z-w)\mathcal{O}_i(w)\right]-(\bar{\partial}_z+\bar{\partial}_w)\left[B^i(z-w)\mathcal{O}_i(w)\right]\\&=\sum_i A^i(z-w)\partial_w\mathcal{O}_i(w)-B^i(z-w)\bar{\partial}_w\mathcal{O}_i(w)\end{aligned}\tag{9.B.10}$$

and similarly

$$\bar{\partial}_z(T\bar{T})(z,w)=\sum_i A^i(z-w)\partial_w\mathcal{O}_i(w)-C^i(z-w)\bar{\partial}_w\mathcal{O}_i(w).\tag{9.B.11}$$

This means that in the OPE of $T\bar{T}$

$$(T\bar{T})(z,w)=\sum_i D^i(z-w)\mathcal{O}(w)\tag{9.B.12}$$

either $\mathcal{O}_i$ are all derivatives themselves or come with constant coefficients[14]. We can therefore write

$$(T\bar{T})(z,w)=\mathcal{O}_{T\bar{T}}(w)+\text{derivative terms}\tag{9.B.13}$$

and so

$$(T\bar{T})(z)=:\mathcal{O}_{T\bar{T}}(w):\tag{9.B.14}$$

[14] This argument is slick!

is a well-defined operator.

It was pointed out in [305] that the $T\bar{T}$-deformation can be equivalently written in terms of the determinant

$$\det T = \frac{1}{2}\epsilon_{\alpha\mu}\epsilon_{\beta\nu}T^{\alpha\beta}T^{\mu\nu} = T_{11}T_{22} - T_{12}^2 = -\frac{1}{\pi^2}(T\bar{T} - \Theta^2) \tag{9.B.15}$$

which we will use to define our theory[15].

Deformation by the $T\overline{T}$-operator

Consider a sequence of theories defined by actions S_λ labelled by some parameter $\lambda \in \mathbb{R}$. To each theory, we can associate a stress tensor T with which we can compute the $(T\bar{T})$-operator at λ. We then define $S_{\lambda+\delta\lambda}$ by adding the operator

$$\delta S_\lambda = \delta\lambda \int \mathrm{d}^2 z\, \mathcal{O}_\lambda = -\frac{\delta\lambda}{\pi^2}\int \mathrm{d}^2 z\, (T\bar{T}) \tag{9.B.16}$$

evaluated at each λ. This defines the $T\bar{T}$-deformed theory.

Alternatively, we can think of this sequence of theories as a trajectory S_λ in theory space obeying the flow equation

$$\frac{dS_\lambda}{d\lambda} = -\frac{1}{\pi^2}\int d^2 z\, (T\bar{T}). \tag{9.B.17}$$

We note that the prefactors have been chosen for later convenience. This deformation is special because the operator exists in any QFT, even without special symmetries or integrability [306, 307].

Connection to Goldstone EFT

Now consider a free massless scalar field in the matter sector. Using the shorthand $X = h^{\mu\nu}\partial_\mu\chi\partial_\nu\chi$, we take the Lagrangian to be just given by the kinetic term

$$\mathcal{L}_m[h;\chi] = -\frac{1}{2}X. \tag{9.B.18}$$

The stress tensor is given by

$$T_{\mu\nu} = h_{\mu\nu}\mathcal{L} - 2\frac{\partial\mathcal{L}}{\partial X}\partial_\mu\chi\partial_\nu\chi \tag{9.B.19}$$

so one can show that

$$\det T^a{}_\mu = \frac{1}{z^2}\left(1 - X\frac{\partial}{\partial X}\right)\mathcal{L}^2. \tag{9.B.20}$$

For covariance, we require the deformed action to take the form

$$S_\lambda = \int d^2x\sqrt{-h}\mathcal{L} \tag{9.B.21}$$

[15] In [305], T and $\bar{T}$ are defined without factors of $\pm 2\pi$, and the deformation is defined as $\frac{dS}{d\lambda} = 4\int d^2x \det T$. Furthermore d^2z is defined to be without the Jacobian from switching between Cartesian and complex coordinates, as opposed to the convention in *e.g.* Polchinski. This is therefore different by a factor 4.

resulting in the following flow equation

$$\frac{\partial \mathcal{L}_\lambda}{\partial \lambda} = \left(1 - X\frac{\partial}{\partial X}\right)\mathcal{L}^2. \tag{9.B.22}$$

We can solve this perturbatively with the initial condition $\mathcal{L}_{\lambda=0} = -\frac{1}{2}X$, but from this we can guess that $\mathcal{L}_\lambda = \frac{1}{\lambda}f(\lambda X)$, giving us

$$\mathcal{L}_\lambda[h;\chi] = \frac{1}{2\lambda}\left(\sqrt{1-2\lambda X}-1\right) = -\frac{1}{2}h^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \frac{\lambda}{4}\left(h^{\mu\nu}\partial_\mu\chi\partial_\nu\chi\right)^2 + \mathcal{O}(\lambda^2) \tag{9.B.23}$$

which defines our theory — this is reminiscent of the DBI action (1.1.21). We see that, to leading in the $T\overline{T}$-flow, the deformed theory reproduces the Goldstone EFT.

9.B.2 Massive Gravity

We also claimed that the leading-order Goldstone EFT interaction $\mathcal{L} \supset (\partial\chi)^4$ can be generated by coupling to massive gravity in $2D$. We will show this explicitly by showing equivalence to a $T\bar{T}$-deformation.

Massive Gravity in $D = 2$

Denoting the dynamical and reference metric by g and h, coupling to massive gravity amounts to adding

$$S_{\text{dRGT}} = \frac{1}{2}\int \mathrm{d}^2x\left(\sqrt{-g}R[g] + \frac{1}{2}m^2\alpha\mathcal{L}^{\text{mass}}\right) \tag{9.B.24}$$

where the only term in $D = 2$ is

$$\begin{aligned}\mathcal{L}^{\text{mass}} &= \alpha\sqrt{-g}\mathcal{L}_2\big(\mathcal{K}[g,h]\big) = 2\alpha\sqrt{-g}\left(\text{Tr}[\mathcal{K}]^2 - \text{Tr}[\mathcal{K}^2]\right)\\ &= \tilde{\alpha}\sqrt{-h}\mathcal{L}_2\big(\mathcal{K}[h,g]\big)\end{aligned} \tag{9.B.25}$$

and

$$K^\mu{}_\nu[g,h] = \delta^\mu{}_\nu - \left(\sqrt{g^{-1}h}\right)^\mu{}_\nu. \tag{9.B.26}$$

Connection to $T\overline{T}$ Deformations

It turns out that the $T\bar{T}$-deformation is equivalent to coupling the original seed theory living on the background h to $2D$ massive gravity with dynamical metric g and reference metric h, with $\alpha = 1$ and $m^2 = 1/\lambda$ [187].

This is easier to see in the first-order formulation. Let e and f be the zweibeine associated to the metrics g and h. Then, the $T\bar{T}$-deformation of the seed theory $S_0[h]$ defined on the background h is, suppressing the dependence on the matter sector

$$S_{T\bar{T}}[e,f] = S_0[e] + S_{\text{dRGT}}[e,f], \tag{9.B.27}$$

where[16]

$$S_{\text{dRGT}} = \frac{1}{2\lambda}\int d^2x\,\epsilon_{ab}(e-f)^a\wedge(e-f)^b = \frac{1}{2\lambda}\int d^2x\,\epsilon_{ab}\epsilon^{\mu\nu}(e-f)^a{}_\mu(e-f)^b{}_\nu. \tag{9.B.28}$$

First note that the stress-tensor with respect to the reference f is defined by

$$\det(f)T_a{}^\mu[e,f] = \frac{\delta S_\lambda[e,f]}{\delta f^a{}_\mu} = -\frac{1}{\lambda}\epsilon^{\mu\nu}\epsilon_{ab}(e-f)^b{}_\nu \tag{9.B.29}$$

which we can invert to give

$$e^a{}_\mu - f^a{}_\mu = -\lambda\epsilon^{ab}\epsilon_{\mu\nu}\det(f)T_b{}^\nu[e,f]. \tag{9.B.30}$$

Next note that the equation of motion for e is

$$0 = \frac{\delta S_\lambda[e,f]}{\delta e^a{}_\mu} = \frac{\delta S_0[e]}{\delta e^a{}_\mu} + \frac{1}{\lambda}\epsilon^{\mu\nu}\epsilon_{ab}(e-f)^b{}_\nu \tag{9.B.31}$$

and let us denote its solutions by e_*. Then, on-shell

$$\det(f)T_a{}^\mu[e_*,f] = -\frac{1}{\lambda}\epsilon^{\mu\nu}\epsilon^{ab}(e_*-f)^b{}_\nu = \left.\frac{\delta S_0[e]}{\delta e^a{}_\mu}\right|_{e=e_*} = \det(e_*)T_{0,a}{}^\mu[e_*]. \tag{9.B.32}$$

The indices are raised/lowered with different metrics g_* and h on either side, so using $\det(g) = \det(e)^2$, we find that the determinant of this equation is

$$\det T^a{}_\mu[e_*,f] = \det T^{0,a}{}_\mu[e_*]. \tag{9.B.33}$$

We can then finally check that

$$\begin{aligned}
\frac{dS_{T\bar{T}}[e_*,f]}{d\lambda} &= \int \mathrm{d}^2x\left(\left.\frac{\delta S_{T\bar{T}}[e,f]}{\delta e^a{}_\mu}\right|_{e=e_*}\frac{de_*{}^a{}_\mu}{d\lambda} - \frac{1}{2\lambda^2}\epsilon_{ab}(e_*-f)^a\wedge(e_*-f)^b\right)\\
&= -\frac{1}{2\lambda^2}\epsilon_{ab}\epsilon^{\mu\nu}\int \mathrm{d}^2x\left(-\lambda\epsilon_{\mu\alpha}\epsilon^{ac}\det(f)T_c{}^\alpha[e_*,f]\right)\left(-\lambda\epsilon_{\nu\beta}\epsilon^{bd}\det(f)T_d{}^\beta[e_*,f]\right)\\
&= -\frac{1}{2}\epsilon^{ab}\epsilon_{\mu\nu}\int \mathrm{d}^2x\,\det(f)^2T_a{}^\mu[e_*,f]T_b{}^\nu[e_*,f]\\
&= -\int \mathrm{d}^2x\,\det T^a{}_\mu[e_*,f]
\end{aligned} \tag{9.B.34}$$

where $\det T = \frac{1}{2}\epsilon^{\mu\nu}\epsilon_{ab}T^a{}_\mu T^b{}_\nu$. This is the generalisation of the $T\bar{T}$ flow equation to curved spacetime, and we can identify $S_\lambda[f] = S_{T\bar{T}}[e_*,f]$.

[16]Note that ϵ here is the one used in the definition of the wedge product of vielbeine and the determinant. It does not transform as a tensor, but as a tensor density, and we cannot just raise/lower its indices!

Concluding Remarks

In part II of this dissertation, we focused on extremal black holes.

First, we considered stationary deformations to the near-horizon region of extremal black holes. We reviewed how these arise from the extremal limit of regular non-extremal solutions and generalised the analysis to extremal black branes. The near-horizon geometry contains an AdS factor, which essentially fixes the scaling of deformations within the near-horizon geometry. These present an interesting window into high-energy physics because they are UV sensitive — metric perturbation theory breaks down sooner for certain values of Wilson coefficients. We clarified that this is not associated to a breakdown of the derivative expansion by constructing an explicit UV completion, in which the problem persists. From examples, we then speculatively conjecture that effective theories derived from consistent UV completions always lead to worse behaviour of stationary deformations near the horizon — we refer to this as near-horizon negativity.

There are several future directions for this.

- A straightforward generalisation to this is to study the full gravitational deformations for extremal black branes.
- Other possible generalisations are stationary deformations for other types of extremal black holes. From the perspective of putting constraints on Wilson coefficients of higher-derivative operators, these constitute different backgrounds we can use to probe the parameter space of Wilson coefficients.
- In AdS_2 , the isometries fix both the scaling exponents of the stationary deformations to the near-horizon geometry and the frequencies of dynamical perturbations or scalar fields in the near-horizon geometry through the Killing vector $v\partial_v - \rho\partial_\rho$. The latter can be used to understand the Aretakis instability — the fact that derivatives of certain conserved quantities transverse to the horizon grow polynomially along the horizon [308]. The scalar toy model on the extremal black brane background exhibited qualitatively similar behaviour to the scalar toy model and honest gravitational perturbations to the extremal black hole. This suggests that we might be able to understand the Aretakis instability for extremal black branes similarly. This is work in progress.
- As it stands, near-horizon negativity does not stand on particularly firm footing as a conjecture. It is mainly based on observations about specific examples, and we only have

rough holographic arguments for why it should hold. It would be interesting to see if stronger arguments could be made — a reasonable guess is that this could be related to the averaged null energy conditions, focusing theorem, or some expectation value of stress tensors in the CFT [309–312].

In chapter 9, we considered dynamical fluctuations within the near-horizon geometry of extremal black holes — the leading-order corrections from the exact AdS_2 -factor therein are described by JT gravity. Specifically, we studied a bulk Goldstone EFT and used AdS_2 -holography to present preliminary results for the boundary four-point correlation with 1-loop corrections from the matter fields and the Schwarzian modes. Goldstone EFT is the prototypical example of an EFT subject to various constraints, and set-up is particularly interesting for its connections to $T\overline{T}$ deformations and coupling to $2D$ massive gravity, which suggest a preferred sign for the Wilson coefficient of the Goldstone EFT for reasons unrelated to the typical causality and analyticity arguments.

- As mentioned previously, the reason we are interested in the four-point function is because it can be used to extract observables quantifying the chaos of the system. It would be interesting to see whether there is a qualitative difference distinguishing different signs of the Wilson coefficient from this perspective.

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