Tube-based robust Economic Model Predictive Control on dissipative systems with generalized optimal regimes of operation

Zihang Dong[1] and David Angeli[1,2]

Abstract— **This paper presents a tube-based robust economic MPC controller for discrete-time nonlinear systems that are perturbed by disturbance inputs. The proposed algorithm minimizes a modified economic objective function which considers the worst cost within a tube around the solution of the associated nominal system. Recursive feasibility and an a-priori upper bound to the closed-loop asymptotic average performance are ensured. Thanks to the use of dissipativity of the nominal system with a suitable supply rate, the closed-loop system under the proposed controller is shown to be asymptotically stable, in the sense that it is driven to an optimal robust invariant set. Finally, some illustrative examples, optimally operated at qualitatively different regimes, are addressed and the performances by using our new controller and those in the literature are compared.**

I. INTRODUCTION

In recent years, Economic Model Predictive Control (EMPC) has emerged as a popular control tool, coherently taking the economic efficiency of nonlinear systems into consideration by adopting cost functionals which are not necessarily positive definite with respect to any equilibrium point: refer to [1], [2], [3] and references therein on EMPC for deterministic systems. In spite of this, the implementation mechanism of EMPC is the same as that of traditional MPC, that is, both adopt the receding horizon approach to achieve the system's closed-loop state (or output) feedback control, although directly optimizing the economic performance may bring more sophisticated stability issues.

Within the literature on EMPC, some approaches require the introduction of strong duality or dissipativity assumptions to convert the economic cost function to positive definite functions, in combination with the terminal equality or inequality constraints so that recursive feasibility and closed-loop stability are established. On the other hand, in [3], an EMPC controller without terminal constraints is presented and analyzed on the basis of certain turnpike and suitable controllability properties. Although a simpler optimization problem, by omitting terminal constraints, might result in a larger feasible set, recursive feasibility is not straightforward and a degradation of guaranteed average performance may occur.

While traditional EMPC approaches require exact knowledge of the plants' dynamics, many practical application scenarios are inherently affected by uncertainties. In this respect, several ways of designing robust EMPC controllers have been recently proposed. In [4], a min-max EMPC approach for linear systems is discussed which transforms the underlying optimization problem to a second-order cone program to improve computational efficiency of its solution. In [5] and [6], the authors use the concept of robust invariant set and convert the standard tube-based MPC scheme to an EMPC setting with integrated economic cost function. Another stream of research, which considers the stochastic information of the disturbance, includes chance-constrained stochastic EMPC [7] and Multi-Stage EMPC [8].

To the best of our knowledge, there are few works on robust EMPC design which achieve closed-loop asymptotic stability for nonlinear systems with general optimal regimes of operation and a robust bound of the long run average cost. In particular, [5] and [6] present an asymptotic average performance bound and stability results for systems with optimal steady-state operation under some additional constraints on the storage function. Motivated by this approach, the present paper shows: (i) it is in fact possible to design a tube-based robust EMPC algorithm by constructing the optimization problem using the initial nominal sequences and a weighting function on nominal initial state. There is no need to have additional constraints based on the storage function limiting the selection of the nominal initial state. (ii) Recursive feasibility and closed-loop asymptotic stability hold not only for steady-state operation but also for optimal periodic operation, provided that dissipativity with respect to nominal dynamics is fulfilled with a suitable supply rate. (iii) The bound on asymptotic average performance, in the case of optimal periodic operation, depends on whether the components of the optimal robust invariant set intersect.

The reminder of this note is organized as follows. Section II reviews the problem setup on EMPC and demonstrates the motivation of our proposed algorithm. Section III addresses the robust EMPC algorithm with its formulation in Section III-A and recursive feasibility analysis in Section III-B. The asymptotic stability and average performance of the closed-loop system by using this EMPC controller are analyzed in Section IV. Finally, simulative examples using the proposed EMPC algorithm are presented in Section V and Section VI concludes this paper.

Notation: Let symbols $\mathbb{R}$ and $\mathbb{I}$ denote the sets of real numbers and integers, respectively. $\mathbb{R}_{\geq a}$ ($\mathbb{I}_{\geq a}$) denotes the real numbers (integers) greater or equal to a and $\mathbb{I}_{[a,b]}$ denotes the integers $\{a, a+1, \cdots, b\}$ for some $a \in \mathbb{I}$ and $b \in \mathbb{I}_{\geq a}$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathscr{K}$, if it is zero at zero and strictly increasing. It is of *class* $\mathscr{K}_\infty$ if in

[1]Zihang Dong and David Angeli are with Department of Electrical and Electronic Engineering, Imperial College London, SW7 2AZ, UK. zihang.dong14@imperial.ac.uk, d.angeli@imperial.ac.uk

[2]David Angeli is also with the Dipartimento di Ingegneria dell'Informazione, Università di Firenze, Italy

addition $\alpha(s) \to +\infty$ as $s \to +\infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a *class* $\mathscr{KL}$ function if $\beta(\cdot,t)$ is a *class* $\mathscr{K}$ function for every fixed $t \in \mathbb{I}_{\geq 0}$, and $\beta(s,\cdot)$ is strictly decreasing with $\beta(s,t) \to 0$ as $t \to +\infty$ for every fixed $s > 0$. For sets $X, Y \in \mathbb{R}^n$, the Minkovski set addition is defined by $X \oplus Y := \{x+y \in \mathbb{R}^n : x \in X, y \in Y\}$, the Pontryagin set difference is defined as $X \ominus Y := \{z \in \mathbb{R}^n : z+y \in X, \forall y \in Y\}$. The Euclidean norm of x is $|x|$. The distance of a point $x \in \mathbb{R}^n$ to a set Π is denoted as $|x|_\Pi := \min_{y \in \Pi} |x-y|$.

II. PRELIMINARIES AND SETUP

We consider finite dimensional discrete-time nonlinear control systems described by difference equations

$$x_{t+1} = f(x_t, u_t, \omega_t), \tag{1}$$

with $t \in \mathbb{I}_{\geq 0}$, state variable $x_t \in \mathbb{X} \subseteq \mathbb{R}^n$, control input $u_t \in \mathbb{U} \subseteq \mathbb{R}^m$, unknown but bounded disturbance $\omega_t \in \mathbb{W} \subseteq \mathbb{R}^q$ and continuous state-transition map $f : \mathbb{X} \times \mathbb{U} \times \mathbb{W} \to \mathbb{X}$. We assume the set $\mathbb{W}$ is compact, with the origin as an interior.

The state and input of the system must fulfill the pointwise-in-time state and input constraints

$$(x_t, u_t) \in \mathbb{Z}, \ \forall t \in \mathbb{I}_{\geq 0}, \tag{2}$$

for some compact set $\mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$.

Our objective is to enhance the profitability by minimizing the economic costs accumulated in the long term system operation

$$\sum_t \ell(x_t, u_t), \tag{3}$$

where $\ell(x,u) : \mathbb{Z} \to \mathbb{R}$ is the economic stage cost, which may take arbitrary form coherently with the EMPC setup (see [2]), and need not be positive definite with respect to any equilibrium state.

A. Invariant error sets

As customary in robust MPC, we introduce the nominal system, associated with system (1), that is

$$z_{t+1} = f(z_t, v_t, 0), \tag{4}$$

where $z_t \in \mathbb{X}$ and $v_t \in \mathbb{U}$ are the nominal state and input at time t, respectively. Then, the resulting state error is $e_t = x_t - z_t$, which tracks the deviation between the real system (1) and the nominal system (4). To derive bounds on the error, we apply the control policy

$$u_t = \mu(v_t, x_t, z_t) \tag{5}$$

to the real system (1), such that the state error at subsequent time instant $t+1$ fulfills

$$e_{t+1} = f(x_t, \mu(v_t, x_t, z_t), \omega_t) - f(z_t, v_t, 0). \tag{6}$$

Before proceeding, we recall the definition of robust invariant set from [5], [9], and [10]

***Definition** 1:* A set $\Omega \subseteq \mathbb{R}^n$ is robust positively invariant (RPI) for the error system (6) if and only if for all $x_t, z_t \in \mathbb{X}$ with $e_t = x_t - z_t \in \Omega$, all $v_t \in \mathbb{U}$, and all $\omega_t \in \mathbb{W}$, it holds $e_{t+1} \in \Omega$ and $(x_t, \mu(v_t, x_t, z_t)) \in \mathbb{Z}$.

***Remark** 1:* The feedback control law $u_t = \mu(v_t, x_t, z_t)$ ensures that state trajectories of the real system are always inside the robust invariant set Ω around the nominal state trajectory generated by system (4), regardless of the disturbance realization.

***Assumption** 1:* There exists a robust positively invariant set Ω as in Definition 1 for error system (6) under state feedback control policy (5).

For linear systems with additive disturbance, the error system (6) is able to be simplified and there is a broad literature in order to compute some robust invariant sets, e.g., [9] and [11]. However, determining a robust invariant set for general nonlinear systems is a difficult task.

In order to guarantee feasibility of the real system, we should consider tightened constraints for the nominal system

$$(z_t, v_t) \in \overline{\mathbb{Z}}, \ \forall t \in \mathbb{I}_{\geq 0}, \tag{7}$$

where $\overline{\mathbb{Z}} = \{(z,v) | (x, \mu(v,x,z)) \in \mathbb{Z}, \forall x \in \{z\} \oplus \Omega\}$, and the projection of set $\overline{\mathbb{Z}}$ on the state space is denoted by $\overline{\mathbb{X}}$.

B. Optimal regimes of operation

As indicated by the motivating example in [5], simply transferring the design procedure of tube-based tracking MPC to an EMPC setup may be sub-optimal, we define a modified robust cost function as

$$\tilde{\ell}(z,v) = \max_{x \in \{z\} \oplus \Omega} \ell(x, \mu(v,x,z)), \tag{8}$$

where Ω is the associated robust invariant set for the error system (6). This modification takes the worst cost into account and it provides an upper bound of the real system cost regardless of the uncertainty realizations. The new robust optimal steady-state operation is

$$(z_s, v_s) = \underset{(z,v) \in \overline{\mathbb{Z}}, z = f(z,v,0)}{\arg\min} \ \tilde{\ell}(z,v). \tag{9}$$

While optimal steady-state operation is often plausible, in general one might need to identify more complex optimal operation regimes, for nominal system (4) with respect to the modified robust cost function (8). For instance, any fixed integer P, one might define the best feasible $P-$periodic solution fulfilling

$$\begin{aligned} &z_{s,i+1} = f(z_{s,i}, v_{s,i}, 0) \ \forall i \in \mathbb{I}_{[0,P-2]} \\ &z_{s,0} = f(z_{s,P-1}, v_{s,P-1}, 0) \\ &(z_{s,i}, v_{s,i}) \in \overline{\mathbb{Z}} \ \forall i \in \mathbb{I}_{[0,P-1]} \end{aligned} \tag{10}$$

where $f(\cdot,\cdot,0)$ is the dynamic function in (4), and the optimal average performance is

$$\tilde{\ell}^*_{av} = \frac{\sum_{i=0}^{P-1} \tilde{\ell}(z_{s,i}, v_{s,i})}{P}. \tag{11}$$

For convenience, the set of states visited along the solution is explicitly denoted as

$$\Pi = \{z_{s,0}, z_{s,1}, \cdots, z_{s,i}, \cdots z_{s,P-1}\}, \ \forall i \in \mathbb{I}_{[0,P-1]}, \tag{12}$$

and the corresponding tube is denoted by $\bigcup_{z \in \Pi} [\{z\} \oplus \Omega]$ which is referred to as the optimal robust invariant set in later discussions.

Next, we define the optimal operation at a set Π for the nominal dynamics.

***Definition** 2:* The nominal system (4) is optimally operated at a set Π with respect to the modified cost function $\tilde{\ell}$, if for any solution of (4) such that $(z_t, v_t) \in \overline{\mathbb{Z}}, \forall t \in \mathbb{I}_{\geq 0}$, it holds

$$\liminf_{T\to+\infty} \frac{\sum_{t=0}^{T-1} \tilde{\ell}(z_t, v_t)}{T} \geq \tilde{\ell}^*_{av}, \tag{13}$$

where $\tilde{\ell}^*_{av}$ is the average modified cost of the optimal robust invariant set as in (11).

According to this definition, the nominal system, operating at Π, provides the best economic cost compared to the long-run average modified cost along any other feasible nominal dynamics. If this set is a singleton ($P = 1$), system (4) is optimally operated at steady-state (similar to Definition 4 in [6] in terms of average integral cost). Otherwise, the system is optimally operated at some P-periodic solution.

The sufficiency of dissipativity for a nominal system to be optimally operated at steady-state is obtained in [2], and the necessity, under a mild additional controllability assumption, is proved in [12]. We recall the notion of dissipativity as given in Definition 4.1 in [2], and also introduce a definition of strict dissipativity with respect to a set.

***Definition** 3:* A discrete time system is dissipative with a supply rate $s : \overline{\mathbb{Z}} \to \mathbb{R}$ if there is a continuous storage function $\lambda : \overline{\mathbb{X}} \to \mathbb{R}$ such that:

$$\lambda(f(z, v, 0)) - \lambda(z) \leq s(z, v) \tag{14}$$

for all $(z, v) \in \overline{\mathbb{Z}}$. If, in addition, a positive definite function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ [1] exists such that:

$$\lambda(f(z, v, 0)) - \lambda(z) \leq -\rho(|z|_\Pi) + s(z, v), \tag{15}$$

then the system is strictly dissipative with respect to a set Π.

The generalized notions of dissipativity and controlled dissipativity proposed in [13] are used to derive upper and lower bounds on the optimal asymptotic average performance without reference to any underlying regime of operation and only depending upon the specific form of the supply. For stability analysis, the supply rate $s(z, v) = \tilde{\ell}(z, v) - \tilde{\ell}^*_{av}$ will be adopted for the dissipation inequality.

In [6], a more restrictive dissipativity statement called Ω-robust dissipativity, which takes the supremum over all possible nominal initial states at the next sampling time, is adopted to establish Ω_∞-robustly optimal steady-state operation, in order to cope with the lack of connection between the nominal initial states at two consecutive time instants. However, the nominal dissipativity as in Definition 3, which provides a sufficient condition for system (4) to be optimally operated at the set Π, is enough for the purpose of closed-loop stability analysis. Moreover, as shown in the analysis in Section IV-B, the closed-loop nominal sequence will typically follow the nominal dynamics after finite number of steps because of its convergence (as a result of strict dissipativity (15)).

[1] A continuous function $\rho(\cdot)$ is positive definite if $\rho(0) = 0$, and $\rho(r) > 0, \forall r > 0$.

C. Motivation

Authors of [5] have proposed an algorithm which controls both the disturbed and nominal closed-loop system by using a specifically defined integral stage cost. The resulting closed-loop asymptotic average performance, stability and optimal steady-state operation are analyzed in detail. Following the same approaches in the context, it can be found that these properties, regarding both optimal steady-state operation and periodic operation, still hold if we change the stage cost to the form of (8). Moreover, in order to possibly afford some performance improvement, they introduced the nominal initial state as an optimization variable. While this provides an additional degree of freedom, this technique alone prevents convergence of nominal sequences. To overcome this issue, the authors in [5] state that the nominal initial state can be used as an optimization variable provided that asymptotic convergence is ensured by imposing an additional constraint, viz.

$$\lambda(z_t(0)) \leq \lambda(z_{t-1}(1)). \tag{16}$$

However, this constraint may restrict the selection of the nominal initial state. Therefore, we propose an alternative approach which considers the nominal initial state as an optimization variable at every sampling time, without the need of the additional constraint for stability purpose, so that the closed-loop economic cost in the transient phase might be improved.

III. THE ROBUST ECONOMIC MPC ALGORITHM

This section is to propose a new recursive feasible EMPC algorithm which employs the modified economic cost function together with a terminal region and a penalty function.

A. Robust Economic MPC formulation

Notice that the nominal (optimal) initial states at two consecutive time instants, in general, do not fulfill the nominal dynamic (4). Given the measured system state x, let us denote the set of possible nominal states by

$$\mathscr{A}(x) := \{z \in \overline{\mathbb{X}} \,|\, x \in \{z\} \oplus \Omega\}. \tag{17}$$

Then, for a fixed prediction horizon $N \in \mathbb{I}_{\geq 0}$, the finite horizon economic optimization problem is formulated as

$$\mathbb{P}_N(x_t): \quad \min_{\boldsymbol{z}_t, \boldsymbol{v}_t} V_N(\boldsymbol{z}_t, \boldsymbol{v}_t) \tag{18a}$$

$$s.t. \quad z_t(0) \in \mathscr{A}(x_t), \tag{18b}$$

$$z_t(k+1) = f(z_t(k), v_t(k), 0), \tag{18c}$$

$$(z_t(k), v_t(k)) \in \overline{\mathbb{Z}}\ \forall k \in \mathbb{I}_{[0,N-1]}, \tag{18d}$$

$$z_t(N) \in \overline{\mathbb{X}}_f, \tag{18e}$$

where

$$V_N(\boldsymbol{z}_t, \boldsymbol{v}_t) = \lambda(z_t(0)) + \sum_{k=0}^{N-1} \tilde{\ell}(z_t(k), v_t(k)) + V_f(z_t(N)) \tag{19}$$

is the economic objective function and the set $\overline{\mathbb{X}}_f$ is the tightened terminal region.

The storage function $\lambda(\cdot)$ appears in the objective function and it can be seen as a weighting function for the nominal initial state. Compared with constraint (16), this initial weighting function is more natural and will not reduce the feasible region. When the inequality $\tilde{\ell}(z,v) \geq \tilde{\ell}(z_s,v_s), \forall (z,v) \in \overline{\mathbb{Z}}$ holds, the supply rate $s(z,v) = \tilde{\ell}(z,v) - \tilde{\ell}(z_s,v_s)$ is positive semidefinite and any constant function may be regarded as a storage function fulfilling the disspativity inequality. In this situation, the weighting function is unnecessary and the constraint (16) is automatically satisfied, so that these two methods boil down to the same optimization problem.

Then, we make the following assumptions on feasibility, robust invariance and terminal stability.

***Assumption** 2:* There exists a terminal set $\mathbb{X}_f \subseteq \mathbb{X}$ with associated set $\overline{\mathbb{X}}_f := \{x \in \overline{\mathbb{X}} \,|\, \{x\} \oplus \Omega \subseteq \mathbb{X}_f\}$ that contains the support of nominal optimal operation Π in its interior, and a control policy $\kappa_f : \overline{\mathbb{X}}_f \to \overline{\mathbb{U}}$ such that for all $z \in \overline{\mathbb{X}}_f$
(i) $(z, \kappa_f(z)) \in \overline{\mathbb{Z}}$
(ii) $f(z, \kappa_f(z), 0) \in \overline{\mathbb{X}}_f$
(iii) $V_f(f(z, \kappa_f(z), 0)) - V_f(z) \leq \tilde{\ell}^*_{av} - \tilde{\ell}(z, \kappa_f(z))$.

***Remark** 2:* Conditions (i) and (ii) are standard requirements in economic MPC framework. Condition (iii) on terminal cost can be regarded as the controlled dissipativity with supply rate $s(z,v) = \tilde{\ell}^*_{av} - \tilde{\ell}(z,v)$.

The decision variables of this optimization problem are

$$\boldsymbol{z}_t = [z_t(0), z_t(1), \cdots, z_t(N)],\ \boldsymbol{v}_t = [v_t(0), v_t(1), \cdots, v_t(N-1)]. \tag{20}$$

Once the optimization problem $\mathbb{P}_N(x_t)$ admits a feasible solution, it is denoted by

$$\boldsymbol{z}^*_t = [z^*_t(0), z^*_t(1), \cdots, z^*_t(N)],\ \boldsymbol{v}^*_t = [v^*_t(0), \cdots, v^*_t(N-1)], \tag{21}$$

and the corresponding optimal value function is

$$V^*_N(x_t) = V_N(\boldsymbol{z}^*_t, \boldsymbol{v}^*_t). \tag{22}$$

Let us define the admissible set $\mathscr{Z}_N$ for a fixed state x as

$$\begin{aligned}\mathscr{Z}_N(x) := \{(\boldsymbol{z}, \boldsymbol{v}) \,|\, & z(0) \in \mathscr{A}(x), z(k+1) = f(z(k), v(k), 0) \\ & z(N) \in \overline{\mathbb{X}}_f, (z(k), v(k)) \in \overline{\mathbb{Z}}, \forall k \in \mathbb{I}_{[0,N-1]}\}.\end{aligned} \tag{23}$$

and the collection of admissible states x,

$$\mathscr{X}_N = \{x \in \mathbb{R}^n \,|\, \exists (\boldsymbol{z}, \boldsymbol{v}) \in \mathscr{Z}_N(x)\}. \tag{24}$$

Notice that the terminal set fulfills $\mathbb{X}_f \subseteq \mathscr{X}_N$, the set $\mathscr{X}_N$ is non-empty and hence the optimization problem $\mathbb{P}_N(x)$ has at least one feasible solution for any $x \in \mathbb{X}_f$.

Following the manner of feedback algorithm in EMPC, the optimal control implemented to the system (1) is

$$\kappa_N(x_t) := \mu(v^*_t(0), x_t, z^*_t(0)), \tag{25}$$

and the resulting closed-loop dynamic is

$$x_{t+1} = f(x_t, \kappa_N(x_t), \omega_t). \tag{26}$$

Notice that, $v^*_t(0)$ and $z^*_t(0)$ might be non-unique, in which case, we take any pair that achieves the optimum.

B. Recursive feasibility

Note that, provided that the optimization problem $\mathbb{P}_N(\cdot)$ is feasible, the optimal nominal state-input pair $(z^*_t(0), v^*_t(0))$ is restricted by the tightened constraint (7). The rationale is that true state and input solutions, in the usual spirit of tube MPC, will then fulfill

$$(x_t, \mu(v^*_t(0), x_t, z^*_t(0))) \in \mathbb{Z}, \forall t \in \mathbb{I}_{\geq 0}, \tag{27}$$

whatever the disturbance signal is until time t. Therefore, the real system's solutions fulfill pointwise-in-time constraints.

To this end, we are ready to claim the recursive feasibility of the optimization problem proposed in Section III-A.

***Proposition** 1:* Let Assumptions 1 and 2 hold. Then, for any initial state $x \in \mathscr{X}_N$, the EMPC optimization problem $\mathbb{P}_N(\cdot)$ is recursively feasible.

Proof: The proof for recursive feasibility follows the standard argumentation in robust MPC (see i.e. [5]). It is worth to be noted that by constraint (18b), we have $x_t - z^*_t(0) \in \Omega$. Then, due to Assumption 1, for all $\omega_t \in \mathbb{W}$, it holds $x_{t+1} - z^*_t(1) = f(x_t, \kappa_N(x_t), \omega_t) - f(z^*_t(0), v^*_t(0), 0) \in \Omega$, which implies $z^*_t(1) \in \mathscr{A}(x_{t+1})$. ■

IV. ASYMPTOTIC PERFORMANCE AND STABILITY ANALYSIS

This section is dedicated to the analysis of closed-loop behaviors of the system using the proposed EMPC controller. We start by proving the stability of the optimal robust invariant set, which is inferred from the asymptotic convergence of the nominal state sequence. Then, an upper bound of the asymptotic average performance, regardless of the disturbance realizations is derived.

A. Stability analysis

This section explores the asymptotic stability of the closed-loop system under EMPC control actions. Before the statement of asymptotic stability, we define an explicit notation for the terminal control policy and make some assumptions.

***Definition** 4:* The terminal control policy is defined as the set valued function $\kappa_f : \overline{\mathbb{X}}_f \to \overline{\mathbb{U}}$ which is the solution of the following optimization problem

$$\min_{v \in \overline{\mathbb{U}}, f(z,v,0) \in \overline{\mathbb{X}}_f} V_f(f(z,v,0)) + \tilde{\ell}(z,v). \tag{28}$$

Note that if the $\kappa_f(\cdot)$ exists as in Assumption 2, this Definition 4 provides an appropriate approach to compute the terminal control policy.

***Assumption** 3:* There exists a continuous storage function $\lambda(\cdot)$ such that the nominal system (4) is strictly dissipative with supply rates $\tilde{\ell}(z,v) - \tilde{\ell}^*_{av}$, i.e.,

$$\lambda(f(z,v,0)) - \lambda(z) \leq \tilde{\ell}(z,v) - \tilde{\ell}^*_{av} - \alpha_1(|z|_\Pi), \tag{29}$$

where $\alpha_1(\cdot)$ is a *class* $\mathscr{K}$ function.

***Assumption** 4:* The storage function and the terminal cost function satisfy $\lambda(z) + V_f(z) = 0$ for all $z \in \Pi$ and $\lambda(z) + V_f(z) > 0$ for all $z \notin \Pi$.

Now, we are ready to state the first main result in this note

Theorem 1: Let Assumptions 1, 2, 3 and 4 hold, under the application of economic MPC feedback control policy (25), the set $\bigcup_{z\in\Pi}[\{z\}\oplus\Omega]$ is asymptotically stable for the closed-loop system (26) with region of attraction $\mathscr{X}_N$.

Proof: The idea of this proof is to follow the usual way to prove asymptotic stability of a closed-loop system under an EMPC controller with the aid of dissipativity assumption. With the notion of rotated cost functions, the optimal rotated objective function is shown to be a Lyapunov function. The details are omitted for brevity, but readers may refer to Theorem 15 in [14]. ■

B. Closed-loop asymptotic performance

Another main goal of EMPC is the guaranteed closed-loop performance. Before stating the main result, we make the following assumptions.

Assumption 5: For any $z_{s,i}$, $z_{s,j}\in\Pi$ where $i\neq j\in\mathbb{I}_{[0,P-1]}$, the corresponding components in optimal robust invariant sets $\{z_{s,i}\}\oplus\Omega$ and $\{z_{s,j}\}\oplus\Omega$ are disjoint.

Assumption 6: For any disturbance realization, there exists a finite time t such that the closed-loop system (26) enters the optimal robust invariant set, i.e., $x_t\in\bigcup_{z\in\Pi}[\{z\}\oplus\Omega]$.

Now, we state the second main result in this note.

Theorem 2: Suppose Assumptions 1, 2, 5 and 6 are satisfied, the asymptotic average performance of the closed-loop system (26) is no worse than $\tilde{\ell}^*_{av}$, regardless of $\omega(\cdot)\in\mathbb{W}^\infty$, i.e.,

$$\limsup_{T\to+\infty}\frac{\sum_{t=0}^{T-1}\ell(x_t,\kappa_N(x_t))}{T}\leq\tilde{\ell}^*_{av}. \tag{30}$$

Proof: The proof of Theorem 2 is similar to that of Theorem 18 in [14]. One should note that, at the first time $x_t\in\bigcup_{z\in\Pi}[\{z\}\oplus\Omega]$, we denote the phase of the optimal solution by $i(t)\in\mathbb{I}_{[0-P-1]}$. Then, the one step ahead predicted nominal state is $z^*_t(1)=z_{s,i(t)+1}$. Because of Assumption 5, the real system state at time $t+1$ satisfies $x_{t+1}\in\{z_{s,i(t)+1}\}\oplus\Omega$. Then, the optimal nominal initial state is $z^*_{t+1}(0)=z_{s,i(t)+1}=z^*_t(1)$ which in turn yields $\lambda(z^*_{t+1}(0))=\lambda(z^*_t(1))$. ■

Remark 3: This upper bound of the asymptotic performance is not guaranteed if the components of the optimal robust invariant sets intersect. In this case, the phase of the optimal nominal solution might "reset" at each sampling time. Therefore, the closed-loop nominal sequence may not be synchronized with the optimal periodic operation. The asymptotic performance bound is more conservative than the average cost and it is still bounded by $\limsup_{T\to+\infty}\frac{\sum_{t=0}^{T-1}\ell(x_t,\kappa_N(x_t))}{T}\leq\max_{i\in\mathbb{I}_{[0,P-1]}}\tilde{\ell}(z_{s,i},v_{s,i})$.

V. EXAMPLES

Throughout the examples, we compare the economic performance by adopting three methods: fixed nominal initial state with dynamic update according to nominal system model (FNS), variable nominal initial state with additional constraint (VAC) using (16), and variable nominal initial state with initial weighting function (VWF).

A. Linear System: Robust optimal steady-state operation

We revisit the scalar example in [5],

$$x_{t+1}=0.9x_t+u_t+\omega_t,$$

constrained by $\mathbb{Z}=\{(x,u)\in\mathbb{R}^2\,|\,|x|\leq 10,|u|\leq 1\}$ and $\mathbb{W}=\{\omega\in\mathbb{R}\,|\,|\omega|\leq 0.1\}$. We also choose control policy $u_t=\mu(v_t,x_t,z_t)=v_t+K(x_t-z_t)$ with $K=-0.9$ and the resulting RPI set is $\Omega=\mathbb{W}$. Then, the tightened constraints for nominal system is $\overline{\mathbb{Z}}=\mathbb{Z}\ominus(\Omega\times K\Omega)=\{(z,v)\in\mathbb{R}^2\,|\,|z|\leq 9.9,|v|\leq 0.91\}$, that is, $\overline{\mathbb{X}}=\{z\in\mathbb{R}\,|\,|z|\leq 9.9\}$. The economic cost function is defined as $\ell(x,u)=\ell_1(x)+\ell_2(u)$, where

$$\ell_1(x)=\begin{cases}x^2+0.1x & x<-0.05\\ 0.225x-0.0363 & -0.05\leq x<0.15\\ x^2-0.3x+0.02 & x\geq 0.15\end{cases}$$

$$\ell_2(u)=\begin{cases}u^2+1.18u+0.0981 & u<-0.59\\ -0.25 & -0.59\leq u<-0.41\\ u^2+0.82u-0.0819 & u\geq -0.41\end{cases}$$

and the corresponding modified cost function is

$$\tilde{\ell}(z,v)=z^2+v^2-z+f(z,v,0)=z^2+v^2-0.1z+v.$$

One can find a possible storage function $\lambda(z)=z$ fulfilling the dissipativity. According to (9), the robust optimal equilibrium is $(z_s,v_s)=(0,0)$, with the best modified robust cost $\tilde{\ell}^*_{av}=\tilde{\ell}(z_s,v_s)=0$. By using terminal control policy $\kappa_f(z)=-Kv$, the terminal region can be defined as $\overline{\mathbb{X}}_f=\overline{\mathbb{X}}$ and the terminal penalty function is $V_f(z)=\tilde{\ell}(z,\kappa_f(z))$ which fulfills controlled dissipativity.

TABLE I compares the accumulated transient economic stage cost performance under various initial conditions and 1000 randomly generated disturbance realizations by using the three approaches. Generally, the two methods changing the nominal initial state as a variable lead to less accumulated cost during the transient steps compared with the approach using the nominal closed-loop system. Moreover, it indicates that our proposed controller VWF performs better in more than 70%. In particular, the FNS and VWF have the same results at $x_0=0$, which cost less in 723 out of 1000 simulations.

	FNS(%)	VAC(%)	VWF(%)
x_0 = -9	0	0	100
x_0 = -5	2	0	99.8
x_0 = 0	72.3	27.7	72.3
x_0 = 5	0	27.4	72.6
x_0 = 9	0	27.2	72.8

TABLE I: Percentage of best performance achieved in 1000 simulations with random disturbances in the case of robust optimal steady-state operation.

B. Linear System: Robust optimal periodic operation

We reconsider the linear system perturbed by the disturbance in Section V-A. Now, the point-wise-in-time constraint is $\mathbb{Z}=\{(x,u)\in\mathbb{R}^2\,|\,|x|\leq 10,0.1\leq|u|\leq 1\}$ and cost function is

$$\ell(x,u)=\begin{cases}(x+0.1)^2 & x<0\\ (x-0.1)^2 & x\geq 0.\end{cases}$$

By using the same control policy with $K=-0.9$, the RPI set is $\Omega=\mathbb{W}$ and the tightened constraint is $\overline{\mathbb{Z}}=\mathbb{Z}\ominus(\Omega\times K\Omega)=\{(z,v)\in\mathbb{R}^2\,|\,|z|\leq 9.9, 0.19\leq|v|\leq 0.91\}$. The modified cost function is

$$\tilde{\ell}(z,v)=\begin{cases}0.01 & -0.1\leq z<0.1\\ z^2 & \text{otherwise,}\end{cases}$$

and the resulting robust optimal periodic solution is

$$(z_{s,0},v_{s,0})=(-0.1,0.19),\ (z_{s,1},v_{s,1})=(0.1,-0.19).$$

with average economic cost $\tilde{\ell}^*_{av}=\frac{\tilde{\ell}(z_{s,0},v_{s,0})+\tilde{\ell}(z_{s,1},v_{s,1})}{2}=0.01$. Using the same terminal control policy and terminal region as in the previous example, the terminal cost function in this case is

$$V_f(z)=\begin{cases}0.01 & -0.1\leq z<0.1\\ z^2 & \text{otherwise.}\end{cases}$$

To fulfill the dissipativity inequality with supply rate $s(z,v)=\tilde{\ell}(z,v)-\tilde{\ell}^*_{av}$, the storage function can be chosen as $\lambda(z)=c$ where $c\in\mathbb{R}$ is a constant real number or $\lambda(z)=-\tilde{\ell}(z,v)$.

In the first case, constant storage function causes two methods VWF and VAC are equivalent, the simulation results with initial condition $x_0=0.3$ and prediction horizon $N=20$ are shown in Fig.1, in which the real system switches between the intervals $[-0.2,0]$ and $[0,0.2]$ which are essentially non-overlapping if the single point 0 is removed. The asymptotic average performance is lower than and approaching to the cost $\tilde{\ell}^*_{av}$ asymptotically for the random disturbance realization and the worst scenario, respectively.

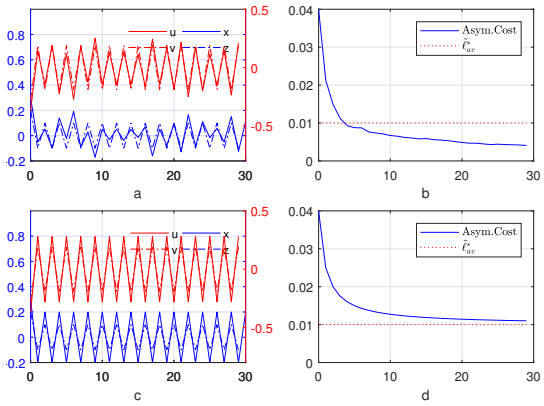


Fig. 1: Random disturbance: a. Closed-loop states and inputs. b. Asymptotic average cost. Worst disturbance realization: c. Closed-loop states and inputs. d. Asymptotic average cost.

When the storage function is $\lambda(z)=-\tilde{\ell}(z,v)$, the best performing controller depends on the initial condition and the disturbance. Simulated results are in TABLE II. It can be seen that the advantage of using VWF is obvious if the initial condition is far away from the origin.

	FNS(%)	VAC(%)	VWF(%)
x_0 = -9	0	0	100
x_0 = -5	0	0	100
x_0 = -1	28.5	31.6	39.9
x_0 = 0	25.6	36.9	37.5
x_0 = 1	30.8	31.3	37.9
x_0 = 5	0	0	100
x_0 = 9	0	0	100

TABLE II: Percentage of best performance achieved in 1000 simulations with random disturbances in the case of robust optimal periodic operation.

VI. CONCLUSIONS

In conclusion, we presented a tube-based robust economic MPC algorithm for nonlinear systems subject to disturbances. By considering the worst cost of the tube along the artificial nominal system and a weighting function on nominal initial state, the robustness against the disturbances is guaranteed in our EMPC design. Within this paper, constraint tightening is used to prove recursive feasibility and a bound of the closed-loop asymptotic average performance is derived. Moreover, the stability of the optimal robust invariant set is inferred from the asymptotic stability of the nominal state sequence.

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