THE CALCULATION AND REDUCTION OF ELECTRICAL NETWORK SENSITIVITY

A thesis presented by

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Abstract

This thesis presents new and efficient methods of sensitivity analysis based on the exploitation of the inverse of the nodal admittance matrix of the circuit. Using these methods, a small-change sensitivity of any given order may be obtained by simple calculation, provided that all the sensitivity coefficients of lower order have been found previously. Also described is a method whereby the effect on a circuit output of any large element change may be simply predicted from a knowledge of the inverse nodal admittance matrix (large-change sensitivity). The large-change sensitivity technique is also applicable to the inverse problem of tolerancing whereby the change in element value must be found for a given change in output. An attempt at directly generating low sensitivity D.C. networks by constraining branch voltages is also described. Finally the methods of small-change sensitivity are applied in an optimisation algorithm which reduces a complex cost function. This cost function is comprised of the weighted squared modulus of the difference between actual and desired output and the weighted squared moduli of the element sensitivities.

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Statement of Originality

The contributions listed below are original as far as the author is aware; any results which are not original to the author are acknowledged by references in the text.

1) The method of exploiting the inverse of the nodal admittance matrix to find the first order sensitivities and the alternative derivation of the adjoint network.
2) The extension of the above method to the calculation of higher order sensitivities.
3) The 'substitution current source' method for calculating large-change sensitivities and element tolerances.
4) The extension of the large-change sensitivity method to the case of calculating the effect on the output of a circuit of multiple element changes.
5) The direct generation of low sensitivity D.C. networks by constraining the branch voltages of the high sensitivity elements to assume lower values.
6) Application of the small-change sensitivity methods in an optimisation algorithm to minimise output error and element sensitivity. This algorithm has proved to be very efficient in its use of computer time.

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Glossary of Commonly Used Terms and Definitions

i_b	branch current
v_b	branch voltage
Y_b	branch admittance
g_m	voltage controlled current source
v_b'	controlling voltage of a controlled current source
e_F	independent voltage source
I_s	independent current source
I_F	current in voltage controlled current source
λ	scalar position parameter
S_x^i	sensitivity of the i^{th} output to an element x
$[Y_b]$	diagonal matrix of branch admittances
$[Y_N]$	nodal admittance matrix
$[i_b]$	vector of branch currents
$[I_s]$	vector of independent current sources
$[J]$	another vector of independent current sources
$[A]$	reduced incidence matrix
$[v_N]$	vector of nodal voltages
$[g_m]$	diagonal matrix of controlled current sources
$[I_N]$	vector of nodal forcing currents
z_{ij}	the ij^{th} element of the inverted nodal admittance matrix
D_i	the i^{th} component of a search direction vector $[D]$

χ, ϕ and ε denote elements and their admittances

The 'active nodes' in a circuit are all the nodes excluding the reference node.

The 'sensitivity' of an element is the normalised change in its circuit output for a normalised minute change in the

element value.

An element 'differential' is the differential of the output of a circuit with respect to this element.

All circuits are assumed to be analysed on the nodal admittance basis unless otherwise stated.

The output of a circuit is assumed to be the voltage between node 2 and the reference node unless otherwise stated.

Chapter 1

Introduction

The design of an electrical circuit is a process which relies very heavily on the past experience of a designer. At present it does not appear to be practical, in the financial sense at least, to give a desired network function to a computer and then to get a suitable circuit back. The aim of the research reported here has been to provide the designer with sufficient information about an existing circuit so that he may then take a decision on the suitability of this circuit for the proposed application. Assuming that the circuit is unsatisfactory in some respect, it may then be optimised in the computer so that the resulting circuit will meet the required specifications. Although most of the results presented in this thesis may be obtained by more circuitous routes, it is essential to find and use the quickest methods of calculating the desired information. Otherwise computer aided circuit design, and in particular circuit optimisation, will remain the sole province of research institutes where financial considerations are ignored.

When a circuit design consists of a few elements then a bench test is usually sufficient to determine its suitability. However, for a large circuit, it is very costly in time for a designer to alter each of the constituent elements in turn and determine the effect on the circuit output. Also it is usually impossible for a designer to find a device, such as a transistor, which only varies in

one of its parameters from an original device. Therefore some form of model which consists solely of ideal elements must be found for each element in the desired circuit so that the circuit may be simulated on a computer. This is especially necessary in the case of integrated circuits since once a design has been finalised and the necessary diffusion masks constructed, it is then very expensive to alter these masks. Once an initial design has been completed, this design should be simulated to find the various factors affecting the performance of the circuit. One factor which determines the quality of the circuit is the effect on its output of a small change in any element value. Two measures are used to represent this effect and these are the differential of the output quantity with respect to the element under consideration and the element sensitivity. The element sensitivity is defined as the normalised change in output for a normalised minute change in element value.

Once a satisfactory circuit design has been achieved it is usually considered necessary to tolerance the consistuent element values. The tolerancing of elements is the process whereby the accuracy, to which elements must be made, is determined. It is pointless, and expensive, to construct all the elements to say one percent of their nominal value when a value twenty percent either side of the nominal may be more than sufficient for most of the elements. While element sensitivity provides a reasonable guide to the required element tolerances, care must be exercised in the use of these figures since element sensitivity is only a small change measure.

Historical Background:

The field of sensitivity and its implication for circuit design was immediately apparent to early workers in circuit synthesis. Some of the early classical synthesis techniques[1-4] produced circuits which, though having the desired responses when all the circuit elements were at nominal value, in practice usually exhibited an immittance function differing widely from that required. This was due to the circuit elements having spreads around their nominal values. Due to the required precision in element values, these synthesis techniques, although the only means available for satisfying complex specified immittance functions, are of limited value today owing to the expense of constructing the resulting circuit design. None of the classical methods such as Brune[2] and Darlington[3] syntheses were capable of handling non-reciprocal elements and hence work in non-reciprocal circuits proceeded on a more or less empirical basis. Here, the most important factor was the designer's judgement and knowledge of previous circuits and their responses. To avoid this empirical approach, much of the early work in the active synthesis field was devoted to breaking down complex functions so that these functions could be synthesised as a cascade connection of circuit blocks each having a much simpler immittance function.[7,8,15]

As inductance is the most difficult type of element to construct so that the actual and ideal behaviour correspond, much effort was devoted to the developement of inductorless circuits. Two lines of approach were followed, the first and most obvious was the developement of 'gyrator'

circuits and the second was the developement of active RC circuits.[6,9,1 ,12] The ideal gyrator[11] is a two port passive device having inversion properties between its two ports so that if one port is terminated by a capacitor, the other port then has the properties of an inductor. At the present time no really satisfactory general class of inductorless active filter has been found to replace the inductor/capacitor filter generated by the improved classical techniques. This field now appears as if it may be taken over by digital filters; for example, the British Post Office is actively considering changing from linear passive circuits to integrated digital filters for channel seperation on their trunk cables.[14] In the subsequent pages linear active circuits -such as linear amplifiers- designed on the semi-empirical basis are the type of circuits to be considered.

The advent of integrated circuit techniques has changed the philosophy of design. Formerly, active devices were bulky and required relatively high power for operation. Although the advent of the transistor did much to alleviate this problem the most important consideration was that both transistors and valves were by far the most expensive items in an active circuit. Over the last decade, however, integrated circuit technology has advanced so rapidly that transistors and resistors may now be fabricated with equal ease. Hence the cost of these two devices is approximately equal and when very high or low values of resistance are called for the cost of a transistor is far below that of this required resistance. However the

advent of this new technology has produced even greater need for low sensitivity circuits due to the difficulty in fabricating devices with values close to the nominal (low tolerances). Despite the difficulty in obtaining nominal values for any component it is economically feasible to obtain two or more elements which are accurately matched provided these elements are fabricated in close proximity.[5] Therefore circuits designed for integration do not attempt to minimise the number of devices but rely on a profusion of elements, both active and passive. These elements may have values which differ considerably from the nominal value (high tolerance). This is in direct contrast to discrete element design where the number of active devices was minimised and the passive elements had far more precise values thar the active devices.

The possibility of an initially designed circuit functioning as desired is small unless the design problem is trivial. In order to test a circuit a designer must either build the circuit or else simulate it on a computer. Since a knowledge of the element sensitivities is usually required, if only for element tolerancing, a computer simulation study should be made for the economic reasons discussed earlier. The element sensitivities also give the designer an idea of the direction in which circuit elements should be altered in order to minimise a difference between the actual and desired circuit output. Early methods of calculating element sensitivity merely transfered, to the computer, the practical approach of changing an element value and observing the new output (see, for example, Cradwick[13]). This is extremely costly in computer time

since the analysis of a circuit of reasonable size is an extremely tedious operation. Therefore, if the circuit is subjected to an optimisation process which depends on a knowledge of the element sensitivities or differentials, the number of circuit analyses may well run into the thousands. Therefore the general purpose optimisation programs, which usually rely on perturbation methods for finding derivatives, are very costly in computer time.[22] Also, numerical differentiation is unreliable since it relies on the subtraction of nearly equal numbers.

The first published attempt to reduce the computational effort involved in sensitivity analysis was presented by J.V. Leeds.[16] In this paper a method of calculating the differentials of all possible transfer functions with respect to one particular element of the network was described. The method relies on the definition of an 'auxiliary' network which is identical to the original network except that all the initial independent sources are set to zero. This auxiliary network is then forced by a source in the branch of interest. The original mathematics demanded that the source be either a current or voltage source depending on which element type was being considered. However the possibility of either a current or voltage source is not an extra complication due to the ease of transformation between current and voltage sources providing an impedance is present; this condition being met in these circumstances. The differentials of the desired responses are then given by the values of the currents and voltages in the auxiliary network corresponding to the desired responses in the original network.

In the following year, a joint paper by J.V. Leeds and G.I. Ugron[17] enlarged the auxiliary network concept. This extension presented a method for the efficient calculation of the differentials of one immittance function with respect to all the network elements, but for reciprocal networks only. For ease of discussion it is now assumed that both original and auxiliary circuits are being considered on a nodal admittance basis. Consider placing a unit current source across the port of the auxiliary network corresponding to the output port of the original network. Then the voltage appearing across any element is that which would appear at the output of the auxiliary network if the current source had been placed across this element. This is because the auxiliary network is reciprocal since the original network has been restricted to the reciprocal set. Therefore the auxiliary network is analysed with a unit current source across the output port. The differentials of all the elements are then given by the product of the element voltages in the original and auxiliary network (see Chapter 2 for a rigorous derivation and explanation of this result).

S.W. Director, in a thesis submitted to the University of California in July 1968,[18] presented the concept of the 'mutually reciprocal' or 'adjoint' network. This concept was a re-discovery of the work on inter-reciprocal networks carried out in 1957 by J.L. Bordewijk.[23] Amongst other useful properties, the 'adjoint' network provided for the extension of the Leeds and Ugron result to non-reciprocal networks. The whole topic of sensitivity calculation and inter-reciprocity is dealt with in far more detail in Chapter 2.

Summary:

Chapter 2 presents a unified mathematical approach linking the work of Leeds, Leeds and Ugron and Director.[16,17,18] The approach is based on the differentiation of the standard branch relationship rather than an approach using Tellegen's theorem.[33,18] Finally in this chapter, a method of calculating first order sensitivities is proposed which, although slower than the 'adjoint' network approach, will lead to saving in computer time if further information is required about the circuit.[20] Chapter 3 then extends the proposed method to the calculation of any order differential of the output with respect to any combination of circuit elements.

Large change sensitivity and element tolerancing is the main topic of Chapter 4. Very efficient means of calculating the effect on the circuit output of any large change in an element have been found by exploiting the inverse of the nodal admittance matrix. The technique is also applicable to the reverse procedure of tolerancing where the element variation corresponding to a fixed change in output must be found. Two different methods are then developed whereby the effect on the circuit output may be calculated for multiple element change. In the first method the amount of calculation may be greater than that involved in a new analysis of the complete circuit if the number of variable elements exceeds the number of 'active' nodes in the circuit. In the second case, however, the amount of computation necessary to find the change in output will never exceed that which is necessary for a new analysis of the circuit.

Chapter 5 describes an attempt at the direct generation of low sensitivity networks. Difficulties in retaining positive element values, as well as the restriction of the technique to D.C. circuits, were the principal reasons behind the cessation of this investigation. As the differential of a circuit output with respect to any element in the circuit is given by the product of the element voltages in the original and adjoint circuits,[18] a sensitivity reduction was sought by forcing a reduction in the voltage across the element in question. The method used to force this reduction in element voltage was an automated form of the 'constrained embedding' technique described by E.B. Kozemchak and M.A. Murray-Lasso.[29] Examples are presented which show that although the method can produce a spectacular sensitivity reduction, it is more normal for the technique to produce negative elements.

The results of chapters 2 and 3 are used in chapter 6 to implement a very simple, but highly effective optimisation algorithm. This algorithm optimises circuits so as to reduce element sensitivity as well as the difference between actual and desired output. Each iteration searches along the direction of steepest descent which is found by the analytic techniques of chapters 2 and 3. The minimum along each search direction is usually found from only two function evaluations, one at the start of the iteration, and the second one at a carefully chosen position along the search direction. Based on the assumption of quadratic behaviour, the position of the search direction minimum is predicted. Experimental justification of the validity

of the quadratic assumption is presented. The efficiency of the algorithm is illustrated by the rapid rate of convergence of the chosen examples. From these examples it appears that a reduction in branch sensitivity is a difficult process to achieve. This is because the algorithm quickly reduces the difference between desired and actual output while effective reduction of element sensitivity is continued for far more iterations. Future research problems and possible solutions are the subject of Chapter 7.

Chapter 2

First Order Sensitivities

In the field of circuit design, one of the prime considerations is 'will the circuit work?'. Laboratory built models usually function correctly whereas problems often arise when a circuit goes into production or field service. This situation can usually be ascribed to components changing their values slightly due to environmental change, or else to a new batch of components having slightly different values from the preceding batch. Therefore any new circuit design must be examined to find how dependent the output is on the values of all its constituent elements. When the circuit under examination is a reasonable size it is impractical, and sometimes impossible, to alter each element in turn and examine the change in output. Therefore an efficient mathematical means is required for calculating the effect, on the output of the circuit, of a small change in each individual element taken one at a time.

First order sensitivity is, as previously defined, the ratio of the normalised change in the response of a network to the normalised change in any element in the network. To facilitate future discussion it will be assumed that any circuit is being considered on the nodal admittance basis unless stated to the contrary. Consider a circuit composed of a number of standard branches, each having the form shown in figure 2.1 and described by

$$i_b = Y_b (v_b - e_F) - I_s + I_F \qquad - - - 2.1$$

where i_b and v_b are the branch current and branch voltage

respectively, I_s and e_F are independent current and voltage sources and I_F is a voltage controlled current source. Equation 2.2 is obtained by forming all the branch equations into the matrix form

$$[i_b] = [Y_b][v_b - e_F] - [I_s] + [I_F] \qquad - - - 2.2$$

where $[Y_b]$ is a diagonal matrix and the remaining quantities are column vectors. On pre-multiplying equation 2.2 by the reduced incidence matrix, $[A]$, (see, for example, ref. 24), the left hand side of the equation goes to zero as shown in equation 2.3,

$$[A][i_b] = [0] = [A][Y_b][v_b - e_F] - [A][I_s] + [A][I_F] \quad - - 2.3$$

This leads to an expanded version of the nodal analysis equation

$$[A][Y_b][A]^t[v_N] + [A][g_m][A]^t[v_N] = [A][I_s] + [A][Y_b][e_F] \quad - - 2.4.$$

In this equation $[v_N]$ is the vector of nodal voltages, the term for the controlled current sources has been expressed explicitly as $[g_m][A]^t[v_N]$ for ease of reference later on, and $[g_m]$ is the non-diagonal matrix of mutual conductances. If a comparision with the usual compact form of the expression for nodal analysis,

$$[Y_N][v_N] = [I_N] \qquad - - - 2.5$$

is made, it is seen that

$$[Y_N][v_N] = [A][Y_b][A]^t[v_N] + [A][g_m][A]^t[v_N] \qquad - - - 2.6$$

and that $$[I_N] = [A][I_s] + [A][Y_b][e_F] \qquad - - - 2.7.$$

Matrix equation 2.4 may be solved by one of a variety

of techniques to yield all or some of the nodal voltages, depending on the method chosen. The only methods used in the following work are the Gaussian method for the solution of linear equations and the Gauss-Jordan complete elimination method for the inversion of a matrix (see Appendix A for a discussion of these techniques).

Derivation of First Order Sensitivity:

Consider equation 2.1 rewritten with the term for the controlled current source expressed in terms of the mutual conductance factor, g_m, and the controlling voltage, v_b'.

$$i_b = Y_b (v_b - e_F) + g_m v_b' - I_s \qquad - - - 2.8$$

Differentiation of this branch equation with respect to a dummy variable, χ, provides the starting point of all the known efficient methods of sensitivity analysis[16,18,20]. The element sensitivities may, of course, be found by the perturbation method. In this method each individual element value is altered by a few percent and the circuit is re-analysed to give the change in output. However this method is subject to numerical error, as well as being very expensive in computer time since each element change requires a separate analysis.

The differentiation of equation 2.8 has three possible results depending on which element χ represents. If χ is not an element of the branch under consideration, that is $\chi \neq Y_b$ and $\chi \neq g_m$, then

$$\frac{\partial i_b}{\partial \chi} = Y_b \frac{\partial v_b}{\partial \chi} + g_m \frac{\partial v_b'}{\partial \chi} \qquad - - - 2.9.$$

However if χ is the branch admittance ($\chi = Y_b$) then the differential is

$$\frac{\partial i_b}{\partial \chi} = Y_b \frac{\partial v_b}{\partial \chi} + (v_b - e_F) + g_m \frac{\partial v_b'}{\partial \chi} \qquad - - - 2.10.$$

The last alternative is when χ is the branch mutual conductance ($\chi = g_m$), then

$$\frac{\partial i_b}{\partial \chi} = Y_b \frac{\partial v_b}{\partial \chi} + g_m \frac{\partial v_b'}{\partial \chi} + v_b'' \qquad - - - 2.11$$

as $\frac{\partial Y_b}{\partial \chi} = 1$ if $Y_b = \chi$ and $\frac{\partial g_m}{\partial \chi} = 1$ if $g_m = \chi$.

Use of the Kronecker-Delta function,

$$\delta_{ij} = 1 \quad \text{if } i=j$$

$$\delta_{ij} = 0 \quad \text{if } i \neq j$$

allows equations 2.9 to 2.11 to be expressed far more compactly as

$$\frac{\partial i_b}{\partial \chi} = Y_b \frac{\partial v_b}{\partial \chi} + g_m \frac{\partial v_b'}{\partial \chi} + \delta_{\chi Y_b} (v_b - e_F) + \delta_{\chi g_m} v_b' \qquad - - - 2.12$$

Pre-multiplication of the generalised matrix form of this equation by the reduced incidence matrix, $[A]$, gives

$$[A][Y_b][A]^t\left[\frac{\partial v_N}{\partial \chi}\right] + [A][g_m]\left[\frac{\partial v_b}{\partial \chi}\right] = -[A]\left[\delta_{\chi Y_b}(v_b - e_F) + \delta_{\chi g_m} v_b'\right] \qquad - - 2.13$$

which may also be written as

$$[A][Y_b][A]^t\left[\frac{\partial v_N}{\partial \chi}\right] + [A][g_m][A]^t\left[\frac{\partial v_N}{\partial \chi}\right] = -[A]\left[\delta_{\chi Y_b}(v_b - e_F) + \delta_{\chi g_m} v_b'\right] \qquad - - 2.14.$$

The left hand sides of this equation and equation 2.4 are identical in form. This observation means that equation 2.14 may be interpreted as a new network, topologically identical to the original and with the same element values, from which the voltage differentials of the original network may be found. These differentials, with respect to an element χ, are equal to the nodal voltages of the new network when this network is driven by

a suitably valued current source across the same element χ. As shown by the right side of equation 2.14, this one current source, of value $v_b - e_F$, across χ is the only independent source in the new network. The new network, shown in figure 2.2, is the auxiliary network proposed by J.V. Leeds.[16] Hence, by analysing one extra circuit, the sensitivities of all possible transfer functions to one chosen element can be found. However, what is normally required is the sensitivity of one particular output to all the elements, so no gain in computational efficiency over the perturbation method has been obtained. Where this technique does have an advantage compared with perturbation is that the value of the differentials are far more accurate. This is because the process does not rely on the subtraction of two large numbers of near equal value.

If the class of networks under consideration is limited to the reciprocal set, then, using the principal of reciprocity, the auxiliary network may be used to find all the element sensitivities with respect to one nodal voltage. Consider forcing the auxiliary network across the output port by a unti current source (figure 2.3c). The voltage, ψ, appearing across any element, χ, is identically equal to the voltage that would appear across the output if the current source had been placed across the element in question (see figure 2.3b). This because the auxiliary network must be reciprocal since the original circuit is defined as being reciprocal. Therefore if the auxiliary network is analysed when the output port is being forced by a unit current source, then the sensitivity of the original output voltage to any element may easily be found.

The differential with respect to any element is the negative product of the original element voltage and the auxiliary element voltage. This was the result derived by J.V. Leeds and G.I. Ugron[17] in 1967.

The restriction of the Leeds and Ugron technique to reciprocal networks was lifted by the introduction of the 'mutually reciprocal" or 'adjoint' circuit[18,23]. This network exhibits a form of reciprocity with the original circuit as shown in figure 2.4. If the port of the adjoint network corresponding to the output port of the original circuit is forced by a current source, then, the voltage appearing at the port corresponding to the input port of the original circuit is identically equal to the output voltage of the original circuit if the two forcing currents are equal. Therefore this adjoint network is analysed with a unit current source forcing the corresponding output port and the branch voltages are found. These branch voltages correspond to the output voltages that would have been obtained if the original circuit had been forced by a unit current source across each branch in turn and the circuit re-analysed. The differential of the output voltage of the original circuit with respect to any element is the negative product of the original element voltage and the adjoint circuit element voltage. Therefore the sensitivities of one network response to all elements are found by the analysis of two circuits only, the original and adjoint networks.

When the original network is reciprocal then the adjoint, auxiliary and original networks are identical except for the independent sources which are set to zero in both

the auxiliary and adjoint networks. The adjoint network provides a conceptually simple circuit idea for the understanding of element sensitivities and optimisation gradient directions (see reference 19) which does not rely heavily on an understanding of the properties of circuit matrices. Also the adjoint circuit is applicable in time domain analysis where time has the fascinating property of 'running' backwards for the analysis of the adjoint circuit[19].

However the approach taken by the author was to consider equation 2,14 more closely. Rewriting this equation in a more compact form

$$[Y_N]\left[\frac{\delta V_N}{\delta \chi}\right] = -[A][\delta_{\chi Y_b}(v_b - e_F) + \delta_{\chi g_m} v_b'] \qquad - - - 2.15$$

it is seen that if $[v_N]$ is substituted for $\left[\frac{\delta V_N}{\delta \chi}\right]$ the left hand side is equal to the left hand side of equation 2.5. Since equation 2.5 is the expression for the original analysis of the circuit it is therefore unnecessary to do two complete circuit analyses. If equation 2.5 is solved by inverting the nodal admittance matrix, $[Y_N]$, this one inversion will not only solve the original network, but also supply all the element sensitivities. The sensitivities are calculated by utilising a suitable driving vector, $[D]$, equal to the right hand side of equation 2.15.

$$\text{i.e. } [D] = -[A][\delta_{\chi Y_b}(v_b - e_F) + \delta_{\chi g_m} v_b'] \qquad - - - 2.16$$

This vector, $[D]$, has, at the most, two entries for any circuit element. These entries are in rows i and j corresponding to the nodes to which the branch is connected (see figure 2.1). If the element, χ, which the output is

being differentiated with respect to, is an admittance, the value of the entry in the j^{th} row is the value, $v_b - e_F$, of the voltage developed across this admittance. However, if χ is a voltage controlled current source, then the value of the j^{th} entry in $[D]$ is the value of the controlling voltage, v_b'. The i^{th} entry in $[D]$ is merely the negative of the j^{th} entry.

This method of sensitivity analysis is slightly slower than the adjoint network approach since the inversion process is about two and a half times slower than a direct solution of the circuit equations (see Appendix A). However, far more information is known about the circuit if a full inversion of the nodal admittance matrix equation is available. If the circuit equations are solved by a simultaneous equation routine, then the output impedance and reverse transfer admittance are not known whereas these quantities are immediately available from the inverted circuit matrix. If higher order differentials or the effect of large element changes are required (see chapters 3 and 4), then an inversion of the circuit matrix is by far the most efficient technique to employ.

Derivation of the 'Adjoint Network':

As illustrated in figure 2.4, the 'adjoint' or 'mutually reciprocal' network has a form of reciprocal relationship with the original network. Consider a non-reciprocal network forced by a single current source, I_F, between nodes k and l with response voltage appearing between nodes m and n. Therefore

$$V_{mn} = (z_{mk} - z_{nk})I_F - (z_{ml} - z_{nl})I_F \qquad - - - 2.17$$

where z_{ij} are the elements of the inverted admittance matrix $[Y_N]^{-1}$ (see equation 2.5). The desired property for the adjoint network is that when the current source, I_F, is placed between nodes m and n the voltage appearing between nodes k and l is identically equal to the voltage, V_{mn}, of the original network. Therefore

$$V_{mn} = V_{kl} = (z'_{km} - z'_{lm})I_F - (z'_{kn} - z'_{ln})I_F \quad - - - 2.18$$

where z'_{ij} are elements of the inverted admittance matrix for the adjoint network. However the voltage across nodes k and l in the adjoint network must equal the response of the original circuit, V_{mn}, when any pair of nodes k and l are chosen for the forcing current, I_F. Therefore

$$V_{mn} = V_{kl} = (z'_{km} - z'_{lm})I_F - (z'_{kn} - z'_{ln})I_F \quad - - - 2.19$$

for all k and l and by equating the right hand sides of equations 2.17 and 2.19

$$(z_{mk}-z_{nk}) - (z_{ml}-z_{nl}) = (z'_{km}-z'_{lm}) - (z'_{kn}-z'_{ln}) \quad - - 2.20$$

results after the cancellation of I_F. Rearrangement of the right hand side of this equation gives

$$(z_{mk}-z_{nk}) - (z_{ml}-z_{nl}) = (z'_{km}-z'_{kn}) - (z'_{lm}-z'_{ln}) \quad - - 2.21$$

for all l and k. Careful examination of the indices shows that the indices of the right hand side of equation 2.21 are the reversed indices of the left hand side. Hence the adjoint network has the transposed nodal admittance matrix of the original circuit.

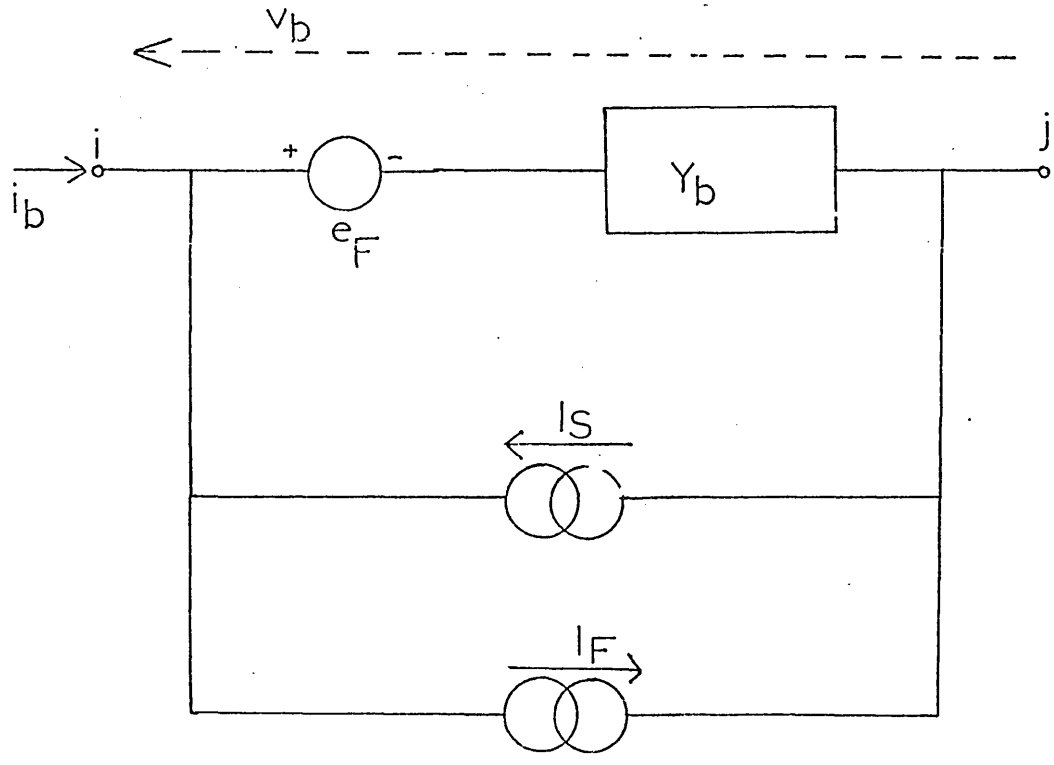
This knowledge is sufficient in itself to analyse the adjoint network. However it is conceptually useful

to derive the network which has this admittance matrix. Since two terminal elements have symmetrical entries about the main diagonal in any admittance matrix, these elements are identical in the original and adjoint networks. Now consider a voltage controlled source between nodes i and j, controlled by the voltage between nodes k and l. This source will have four entries in the original circuit's admittance matrix which are in rows i and j and in columns k and l. Transposition of the admittance matrix means that now these entries are in the k^{th} and l^{th} rows and the i^{th} and j^{th} columns. Hence, in the adjoint network, the controlled source is connected to the k^{th} and l^{th} nodes and controlled by the voltage between the i^{th} and j^{th} nodes. This means that the adjoint network is formed by keeping all two terminal elements the same but controlled current sources have their controlling and controlled branches interchanged.

Examples:

The circuits designated as 5 and 6, shown in figures 2.5 and 2.6 respectively, were analysed by inverting their circuit admittance matrices. All the element sensitivities were then found by exploiting the inverse admittance matrices. From the results tabulated in figure 2.7 it may be seen that circuit 6 could be said to be better than circuit 5. This is because circuit 5 has four branches with higher sensitivities than the highest branch sensitivity in circuit 6. The sensitivities of the source branches, R2 and R13 in circuits 5 and 6 respectively, have been ignored since the inclusion of these branches is only a mathematical device for simulating a voltage source; both these branches

have sensitivities very close to one. In order to demonstrate the triviality of the sensitivity calculations in comparision with the actual analysis of the circuit, the times for the various parts of the calculation are also given in figure 2.7. These times refer to the use of the program on a Digital PDP 15 computer having 16K core store, the clock being started immediately before the formation of the nodal admittance matrix, $[Y_N]$. The times given do not include the time involved in printing out which is lengthy. Since the real-time clock in the machine is run from the mains frequency, the times are only accurate to within twenty milliseconds. The PDP 15 computer on which this program was run has an extended arithmetic unit (EAE) but all real number multiplications and divisions are done by software and hence are relatively time consuming.



$$i_b = Y_b(v_b - e_F) - I_S + I_F$$

$$v_b = v_i - v_j = v_{ij}$$

$$I_F = g_m v_{kl}$$

A Standard Network Branch

Figure 2.1

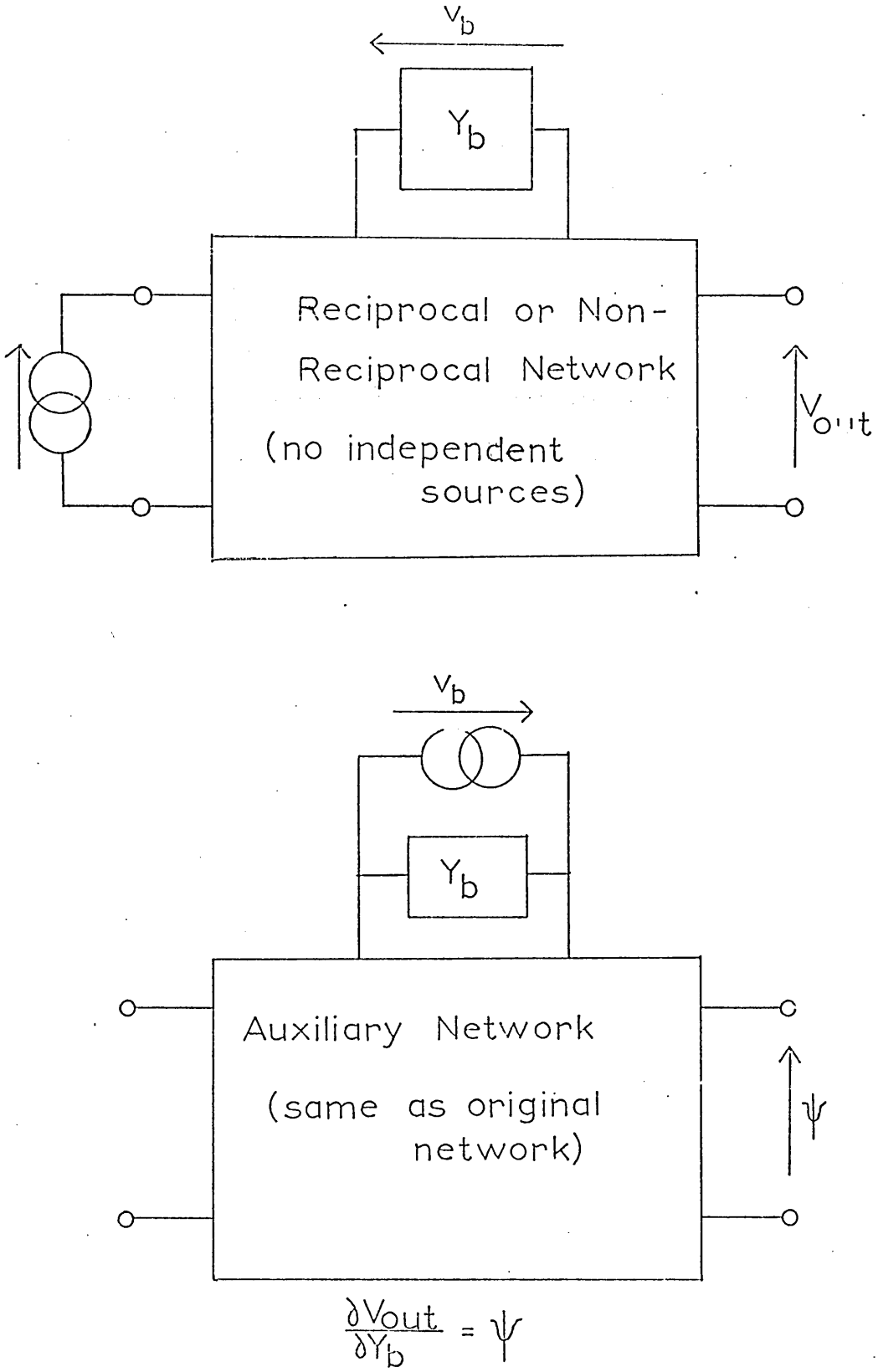
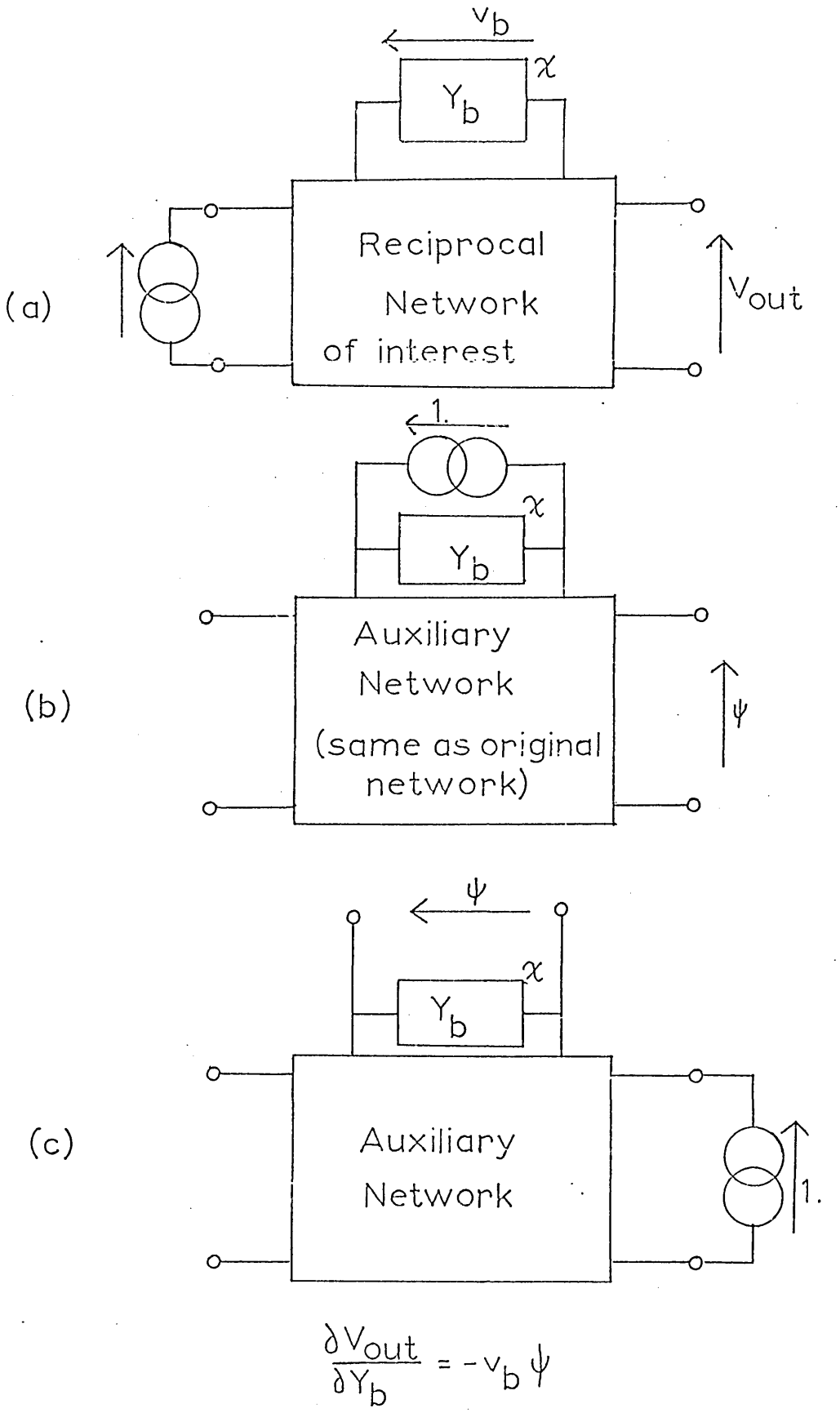


Illustration of the Auxiliary Network

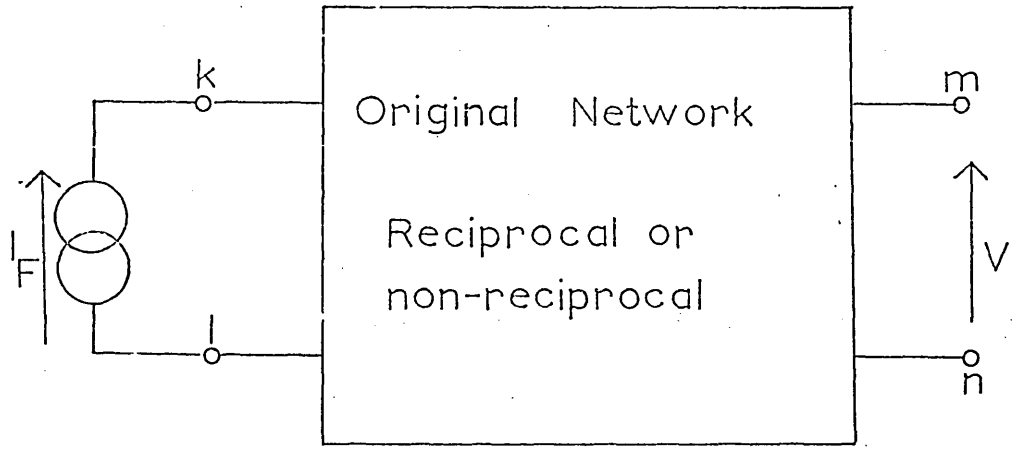
Figure 2.2

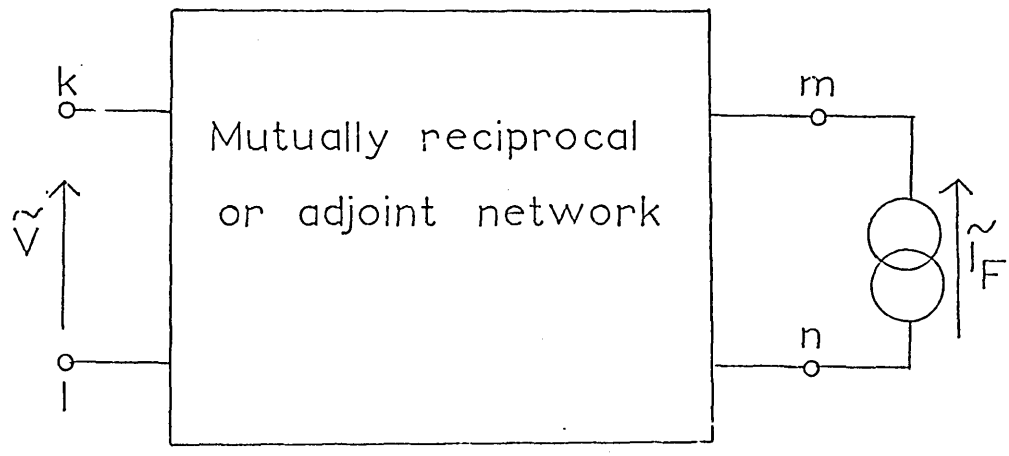


$$\frac{\partial V_{out}}{\partial Y_b} = -v_b \psi$$

Illustration of J.V. Leeds and G.I. Ugron's result for Reciprocal Networks.

Figure 2.3

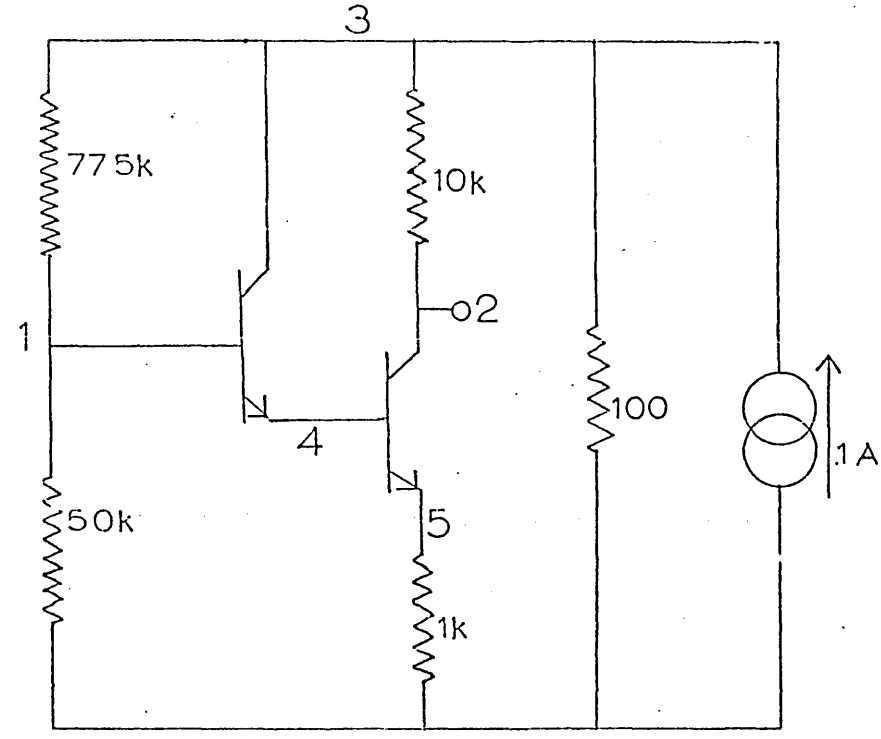




$$\frac{V}{I_F} = \frac{\tilde{V}}{\tilde{I}_F}$$

Example illustrating the mutual reciprocal properties between original and adjoint networks.

Figure 2.4



Circuit 5

modelled by

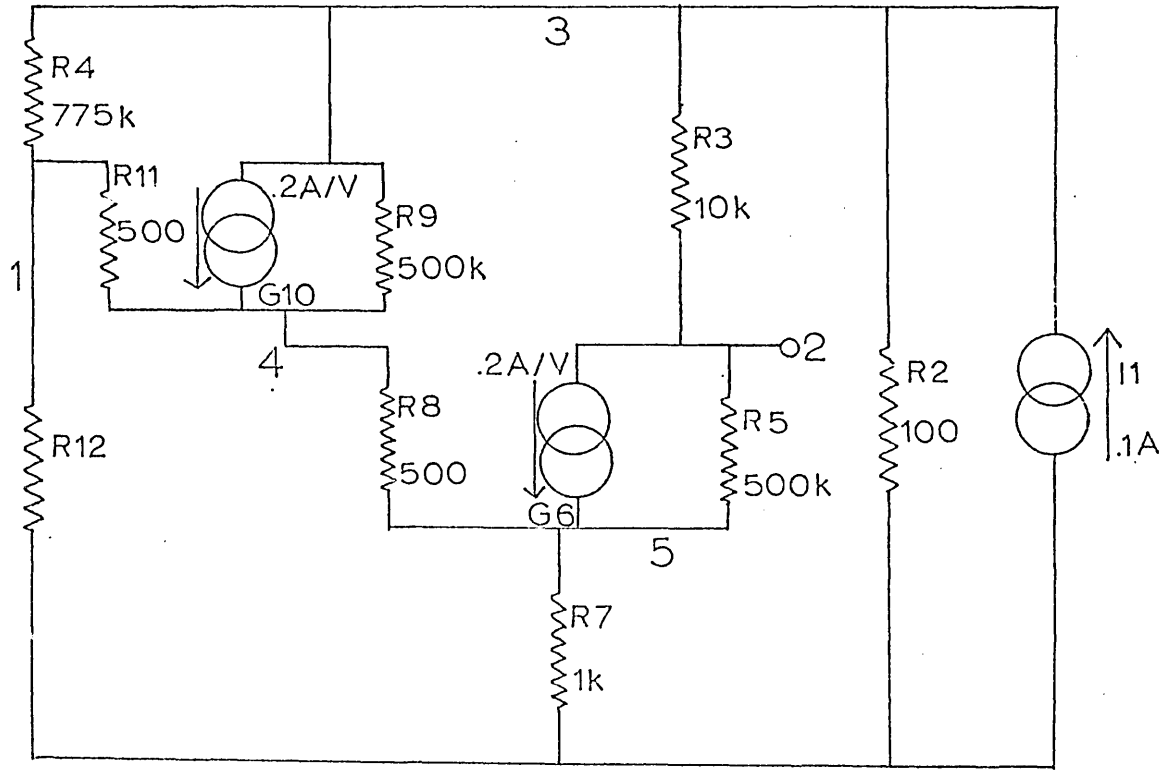
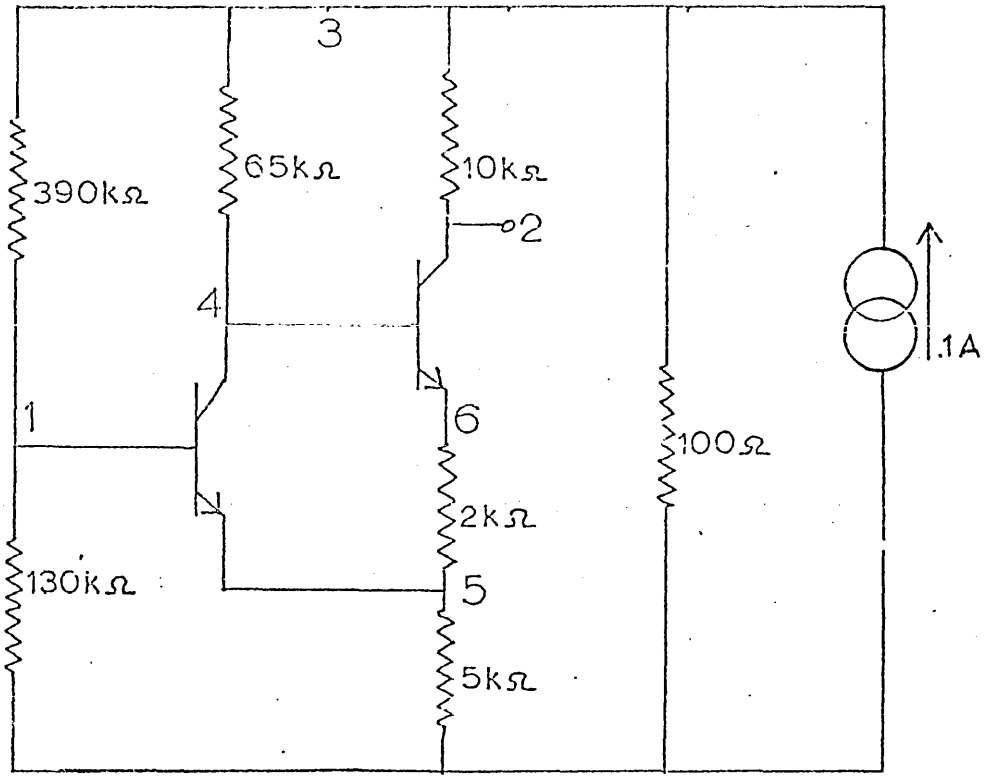


Figure 2.5



Circuit 6 modelled

by

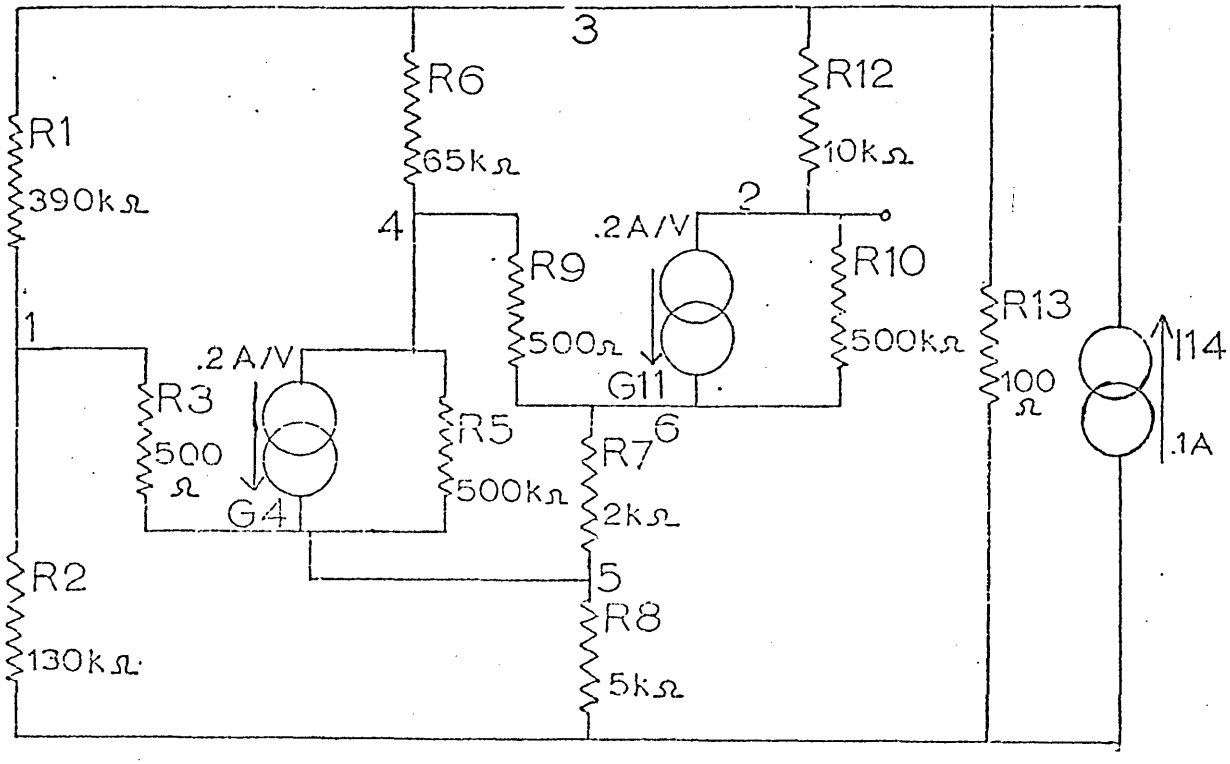


Figure 2.6

Nodal Voltage Table

Node Number	Circuit 5 voltage	Circuit 6 voltage
1	.608	2.39
2	3.94	6.23
3	9.94	9.95
4	.608	3.14
5	.605	2.39
6	N/A	3.14

Time taken for analysis

Circuit 5 = .64 secs

Circuit 6 = 1.04 secs

Sensitivity of Node 2 voltage with respect to all branches

Branch No.	Circuit 5 Sensitivity	Circuit 6 Sensitivit
1	0.	.697
2	.994 (Source Resis.)	-.661
3	-1.52	-.038
4	1.42	-.038
5	$.33 \times 10^{-3}$	$.57 \times 10^{-3}$
6	-.029	-.215
7	1.51	-.024
8	-.022	.806
9	.022	$.16 \times 10^{-2}$
10	.015	$-.28 \times 10^{-4}$
11	.015	$.17 \times 10^{-2}$
12	-1.42	-.598
13	N/A	.995 (Source Resis
14	N/A	0.

Additional time taken for sensitivity calculations

Circuit 5 = .06 secs

Circuit 6 = .08 secs

Figure 2.7

Chapter 3

Higher Order Sensitivities

The better general purpose optimisation algorithms rely not only on deriving the first order differentials of the cost function with respect to the system parameters, but also on finding an approximation of the Hessian* matrix.[25] Many different techniques have been proposed for finding an approximation to the Hessian matrix, one of the better known methods being that described by Fletcher and Powell in 1963.[26] It will be shown in this chapter that the complete Hessian matrix may be determined analytically at a computational cost comparable to that of a single circuit analysis by following the methods developed in the previous chapter.

Consider equations 2.9, 2.10 and 2.11, rewritten below as equations 3.1, 3.2 and 3.3 respectively, which describe the differentiation of a branch current i_b with respect to any element χ.

$$\frac{\delta i_b}{\delta \chi} = Y_b \frac{\delta v_b}{\delta \chi} + g_m^b \frac{\delta v_b'}{\delta \chi} \qquad - - - \ 3.1$$

---

* The Hessian matrix, $[H]$, of a function F is defined as the matrix consisting of the second order differentials of F with respect to all the elements comprising the function, F. That is, if

$$F = f(x_1, x_2 \ldots\ldots x_n)$$

then $$[H] = \frac{d^2F}{dx_j dx_i}$$

where i and j are the row and column indices respectively.

for $\chi \neq Y_b$ and $\chi \neq g_m^b$,

$$\frac{\partial i_b}{\partial \chi} = Y_b \frac{\partial v_b}{\partial \chi} + (v_b - e_F) + g_m^b \frac{\partial v_b'}{\partial \chi} \qquad - - - 3.2$$

for $\chi = Y_b$ and

$$\frac{\partial i_b}{\partial \chi} = Y_b \frac{\partial v_b}{\partial \chi} + g_m^b \frac{\partial v_b'}{\partial \chi} + v_b' \qquad - - - 3.3$$

for $\gamma = g_m^b$

Further differentiation of these equations with respect to an element, ϕ, of the network leads to variety of results, all of which are systematically listed below in equations 3.4 to 3.12. Differentiation of equation 3.1 with respect to ϕ when ϕ is not an element of the branch gives

$$\frac{\partial^2 i_b}{\partial \phi \partial \chi} = Y_b \frac{\partial^2 v_b}{\partial \phi \partial \chi} + g_m^b \frac{\partial^2 v_b'}{\partial \phi \partial \chi} \qquad - - - 3.4$$

for $\chi \neq Y_b$, $\chi \neq g_m^b$, $\phi \neq Y_b$ and $\phi \neq g_m^b$.

However when ϕ is the branch admittance then

$$\frac{\partial^2 i_b}{\partial \phi \partial \chi} = Y_b \frac{\partial^2 v_b}{\partial \phi \partial \chi} + \frac{\partial v_b}{\partial \chi} + g_m^b \frac{\partial^2 v_b'}{\partial \phi \partial \chi} \qquad - - - 3.5$$

for $\chi \neq Y_b$, $\chi \neq g_m^b$ and $\phi = Y_b$.

The last possibility for the differentiation of equation 3.1 is when ϕ is the branch mutual conductance. That is

$$\frac{\partial^2 i_b}{\partial \phi \partial \chi} = Y_b \frac{\partial^2 v_b}{\partial \phi \partial \chi} + g_m^b \frac{\partial^2 v_b'}{\partial \phi \partial \chi} + \frac{\partial v_b'}{\partial \chi} \qquad - - - 3.6$$

for $\chi \neq Y_b$, $\chi \neq g_m^b$ and $\phi = g_m^b$.

Similarly differentiation of equation 3.2 with respect to ϕ gives

$$\frac{\partial^2 i_b}{\partial \phi \partial \chi} = Y_b \frac{\partial^2 v_b}{\partial \phi \partial \chi} + \frac{\partial v_b}{\partial \phi} + g_m^b \frac{\partial^2 v_b'}{\partial \phi \partial \chi} \qquad - - - 3.7$$

for $\chi = Y_b$, $\phi \neq Y_b$ and $\phi \neq g_m^b$

$$\frac{\delta^2 i_b}{\delta\phi\delta\chi} = Y_b \frac{\delta^2 v_b}{\delta\phi\delta\chi} + \frac{\delta v_b}{\delta\chi} + \frac{\delta v_b}{\delta\phi} + g_m^b \frac{\delta^2 v_b'}{\delta\phi\delta\chi} \quad - - - 3.8$$

for $\chi = \phi = Y_b$ and

$$\frac{\delta^2 i_b}{\delta\phi\delta\chi} = Y_b \frac{\delta^2 v_b}{\delta\phi\delta\chi} + \frac{\delta v_b}{\delta\phi} + g_m^b \frac{\delta^2 v_b'}{\delta\phi\delta\chi} + \frac{\delta v_b'}{\delta\chi} \quad - - - 3.9$$

for $\chi = Y_b$ and $\phi = g_m^b$.

Also differentiation of equation 3.3 with respect to ϕ gives

$$\frac{\delta^2 i_b}{\delta\phi\delta\chi} = Y_b \frac{\delta^2 v_b}{\delta\phi\delta\chi} + g_m^b \frac{\delta^2 v_b'}{\delta\phi\delta\chi} + \frac{\delta v_b'}{\delta\phi} \quad - - - 3.10$$

for $\chi = g_m^b$, $\phi \neq Y_b$ and $\phi \neq g_m^b$,

$$\frac{\delta^2 i_b}{\delta\phi\delta\chi} = Y_b \frac{\delta^2 v_b}{\delta\phi\delta\chi} + \frac{\delta v_b}{\delta\chi} + g_m^b \frac{\delta^2 v_b'}{\delta\phi\delta\chi} + \frac{\delta v_b'}{\delta\phi} \quad - - - 3.11$$

for $\chi = g_m^b$ and $\phi = Y_b$ and

$$\frac{\delta^2 i_b}{\delta\phi\delta\chi} = Y_b \frac{\delta^2 v_b}{\delta\phi\delta\chi} + g_m^b \frac{\delta^2 v_b'}{\delta\phi\delta\chi} + \frac{\delta v'b}{\delta\chi} + \frac{\delta v_b'}{\delta\phi} \quad - - - 3.12$$

for $\chi = \phi = g_m^b$.

These equations are expressed concisely in equation 3.13 by the use of the Kronecker Delta function, δ_{ij}, defined in the previous chapter.

$$\frac{\delta^2 i_b}{\delta\phi\delta\chi} = Y_b \frac{\delta^2 v_b}{\delta\phi\delta\chi} + g_m^b \frac{\delta^2 v_b'}{\delta\phi\delta\chi} + \delta_{\phi,Y_b} \frac{\delta v_b}{\delta\chi} + \delta_{\chi,Y_b} \frac{\delta v_b}{\delta\phi} \quad \ldots.$$

$$\ldots. + \delta_{\phi,g_m^b} \frac{\delta v_b'}{\delta\chi} + \delta_{\chi,g_m^b} \frac{\delta v_b'}{\delta\phi} \quad - - 3.13.$$

Premultiplication of the matrix form of equation 3.13 by the reduced incidence matrix, $[A]$, yields, on rearrangement,

$$[Y_N]\left[\frac{\delta^2 v_N}{\delta\phi\delta\chi}\right] = -[A]\left[\delta_{\phi,Y_b} \frac{\delta v_b}{\delta\chi} + \delta_{\chi,Y_b} \frac{\delta v_b}{\delta\phi} + \delta_{\phi,g_m^b} \frac{\delta v_b'}{\delta\chi} \ldots\right.$$

$$\left. \ldots + \delta_{\chi,g_m^b} \frac{\delta v_b'}{\delta\phi}\right] - 3.14$$

which is identical in form to equation 2.15. The right hand side of this equation may be expressed in a more compact form as

$$\left[Y_N\right]\left[\frac{\partial^2 v_N}{\partial\phi\partial\chi}\right] = -\left[A\right]\left\{\left[\frac{dY}{d\phi}\right]\left[\frac{\partial v_b}{\partial\chi}\right] + \left[\frac{dY}{d\chi}\right]\left[\frac{\partial v_b}{\partial\phi}\right]\right\} \qquad - - - 3.15$$

where $\left[Y\right] = \left[Y_b\right] + \left[g_m\right]$, both of which have been defined in Chapter 2. This is because

$$\frac{dY_b}{df} = 1 \quad \text{if } f = Y_b$$
$$= 0 \quad \text{if } f \neq Y_b$$

and

$$\frac{dg_m^b}{df} = 1 \quad \text{if } f = g_m^b$$
$$= 0 \quad \text{if } f \neq g_m^b$$

Thus the second order differentials of any nodal voltage with respect to any two elements in the circuit may be found very simply once the nodal admittance matrix, $\left[Y_N\right]$, has been inverted. All that is required in addition to the inverted admittance matrix is the Jacobian matrix, $\left[J\right]$. This Jacobian matrix is defined as

$$\left[J\right] = \left[\frac{\partial v_b^i}{\partial f_j}\right] \qquad - - - 3.16$$

where i and j are the row and column indices respectively, f is any element of the $\left[Y\right]$ matrix and v_b^i is the branch voltage of the i^{th} element. In practice it is more convenient to generate the elements of the Jacobian matrix as required utilising the technique discussed in the previous chapter and identical in form to the above derivation. This is because the available core store on most computers is at a premium compared with the minimal increase in the amount of computer time required to evaluate the elements of the Jacobian matrix as they are needed.

Once the second order derivatives have been obtained there is still an additional term to be calculated if the differentials of the sensitivities are required. Since the sensitivity, S_χ^i, of an output i with respect to any element, χ, of the network is defined as

$$S_\chi^i = \frac{\chi}{V_i} \frac{\partial V_i}{\partial \chi} \qquad \text{- - - 3.17,}$$

the differential of the sensitivity factor, S_χ^i, with respect to any element, ϕ, in the network is given by

$$\frac{\partial S^i}{\partial \phi} = \frac{\partial^2 V_i}{\partial \phi \partial \chi} \frac{\chi}{V_i} - \frac{\chi}{V_i^2} \frac{\partial V_i}{\partial \chi} \frac{\partial V_i}{\partial \phi} \qquad \text{- - - 3.18}$$

for $\chi \neq \phi$

$$\text{or} \quad \frac{\partial S^i}{\partial \phi} = \frac{\partial^2 V_i}{\partial \phi \partial \chi} \frac{\chi}{V_i} - \frac{\chi}{V_i^2} \frac{\partial V_i}{\partial \chi} \frac{\partial V_i}{\partial \phi} + \frac{1}{V_i} \frac{\partial V_i}{\partial \chi} \qquad \text{- - - 3.19}$$

when $\chi = \phi$

These equations may then be condensed by the use of the now familiar Kronecker-Delta function as shown in equation 3.20.

$$\frac{\partial S^i}{\partial \phi} = \frac{\partial^2 V_i}{\partial \phi \partial \chi} \frac{\chi}{V_i} - \frac{\chi}{V_i^2} \frac{\partial V_i}{\partial \chi} \frac{\partial V_i}{\partial \phi} + \delta_{\chi,\phi} \frac{\partial V_i}{\partial \chi} \frac{1}{V_i} \qquad \text{- - - 3.20}$$

Hence, using equation 3.20, differentials of the sensitivities may be simply evaluated from the values of the first and second order circuit output differentials.

Higher Order Differentials:

Consider equation 3.15 differentiated with respect to a further element, ε, of the network. Only two terms result from the differentiation of the right hand side of this equation since the terms $[A]$, $\left[\frac{dY}{d\phi}\right]$ and $\left[\frac{dY}{d\chi}\right]$ contain

ones and zeros only. Therefore differentiation of these terms result in the null matrix $[0]$ and hence

$$\frac{\partial}{\partial \varepsilon}\left(-[A]\left\{\left[\frac{dY}{d\phi}\right]\left[\frac{\partial V_b}{\partial \chi}\right] + \left[\frac{dY}{d\chi}\right]\left[\frac{\partial V_b}{\partial \phi}\right]\right\}\right)$$

$$= -[A]\left\{\left[\frac{dY}{d\phi}\right]\left[\frac{\partial^2 V_b}{\partial \varepsilon \partial \chi}\right] + \left[\frac{dY}{d\chi}\right]\left[\frac{\partial^2 V_b}{\partial \varepsilon \partial \phi}\right]\right\} \quad - - - 3.21.$$

Now consider differentiation of the left hand side of equation 3.15 with respect to ε. This process is more easily carried out if this section of the equation is expanded by use of the two substitutions

$$[Y_N] = [A][Y][A]^t \quad - - - 3.22$$

where $[Y] = [Y_b] + [g_m^b]$ as previously defined

and $$\frac{\partial^2 V_b}{\partial \phi \partial \chi} = [A]^t\left[\frac{\partial^2 V_N}{\partial \phi \partial \chi}\right] \quad - - - 3.23$$

since $[V_b] = [A]^t[V_N]$.

Therefore $$\frac{\partial}{\partial \varepsilon}\left([Y_N]\left[\frac{\partial^2 V_N}{\partial \phi \partial \chi}\right]\right) = \frac{\partial}{\partial \varepsilon}\left([A][Y]\left[\frac{\partial^2 V_b}{\partial \phi \partial \chi}\right]\right) \quad - - - 3.24$$

giving $$\frac{\partial}{\partial \varepsilon}\left([Y_N]\left[\frac{\partial^2 V_N}{\partial \phi \partial \chi}\right]\right) = [A][Y]\left[\frac{\partial^3 V_b}{\partial \varepsilon \partial \phi \partial \chi}\right] + [A]\left[\frac{dY}{d\varepsilon}\right]\left[\frac{\partial^2 V_b}{\partial \phi \partial \chi}\right] \quad - 3.25$$

Use of the substitution given in equation 3.24 then leads to

$$[Y_N]\left[\frac{\partial^3 V_N}{\partial \varepsilon \partial \phi \partial \chi}\right] = -[A]\left\{\left[\frac{dY}{d\phi}\right]\left[\frac{\partial^2 V_b}{\partial \varepsilon \partial \chi}\right] + \left[\frac{dY}{d\chi}\right]\left[\frac{\partial^2 V_b}{\partial \varepsilon \partial \phi}\right] + \left[\frac{dY}{d\varepsilon}\right]\left[\frac{\partial^2 V_b}{\partial \phi \partial \chi}\right]\right\} - 3.26.$$

Note that there is no restriction on the choice of χ, ϕ and ε as long as they are elements of the original network. Equation 3.26 may be generalised to give the n^{th} order differentials of the nodal voltage vector, $[V_N]$. This generalisation is expressed by

$$\left[Y_N\right]\left[\frac{\partial^n V_N}{\prod_{i=1}^{n}\partial x_i}\right] = -\left[A\right]\sum_{i=1}^{n}\left[\frac{dY}{dx_i}\right]\left[\frac{\partial^{n-1} V_b}{\prod_{j=1}^{n}\partial x_j}_{j\neq i}\right] \qquad - - - \ 3.27$$

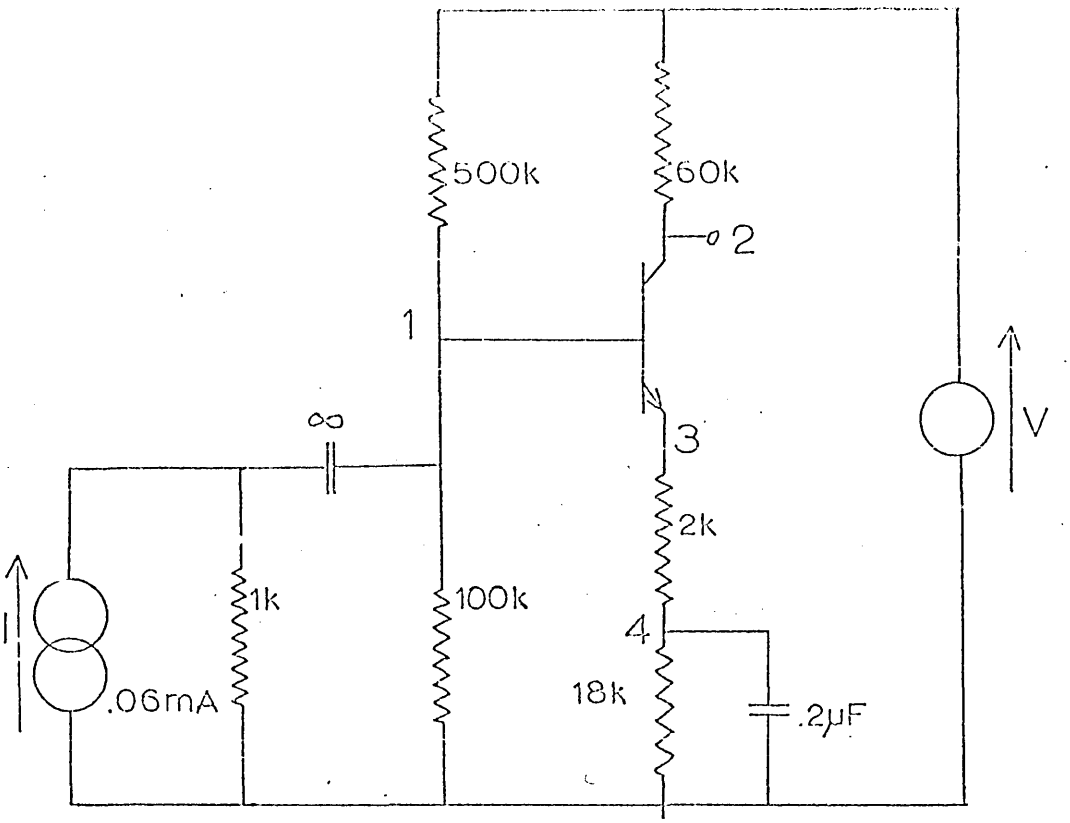
where x_i is any element of the network.

Therefore to find the n^{th} order differentials of the nodal voltages the inverse of the nodal admittance matrix Y_N and all the lower differentials are required. The terms on the right hand side of equation 3.27 are extremely easy to calculate as $\left[\frac{dY}{dx_i}\right]$ has only one unity entry, the remaining entries being zero. This unity entry is in the row corresponding to the branch number, the column index being determined by whether the element, x_i, is a two terminal element or controlled current source. If x_i is a two terminal element then the entry falls on the main diagonal of $\left[\frac{dY}{dx_i}\right]$ while if x_i is a voltage controlled current source then the column index is that associated with the branch across which the controlling voltage is developed. Multiplication of the two matrices $\left[\frac{dY}{dx_i}\right]$ and $\left[\frac{\partial^{n-1} V_b}{\prod_{j=1}^{n}\partial x_j}_{j\neq i}\right]$ then gives a comumn vector having only one entry in the row corresponding to the branch number of the element x_i. The value of this entry is the $n-1^{th}$ differential of the controlling branch voltage. Pre-multiplication of this resultant vector by the reduced incidence matrix $\left[A\right]$ produces another vector of reduced size having a maximum of two possible entries. This vector has a size equal to the number of 'active' network nodes and the entries are equal in magnitude but opposite in sign. The magnitude of the entries is equal to the $n-1^{th}$ differential of the controlling branch voltage.

Therefore the calculation of each term of the right hand side of equation 3.27 is a simple look-up procedure of two quantities. These quantities are the nodal connections of the circuit element, x_i, and the n-1th differential of the controlling branch voltage of x_i with respect to the remaining differands. In the case of a two terminal element, x_i, the controlling branch voltage is the voltage expressed across the element x_i.

Examples:

The second order element normalised differentials, $y\frac{\delta}{\delta y}\left(x\frac{\delta V_2}{\delta x}\right)$, have been calculated for circuits 8 and 6, shown respectively in figures 3.1 and 3.5. Figure 3.2 shows the results of two analyses of circuit 8, the first analysis is at 1 kHz while the second is at D.C.. Since the real time clock in the PDP 15 computer is synchronised to the mains, all timed operations were repeated ten times to achieve a reasonable timing accuracy. The second order differential matrices for 1 kHz and D.C. are presented in figures 3.3 and 3.5 respectively. As these matrices are symmetrical the calculation time may be reduced by a factor of $(n^2 + n)/2n^2$ where n is the number of branches excluding independent sources. Therefore the time of calculation of the complete second order matrix is of the same order of magnitude as the analysis time of the original circuit. Figures 3.6, 3.7 and 3.8 present the analysis, first order sensitivities and the second order element normalised differentials for circuit 6.



Circuit 8 modelled

by

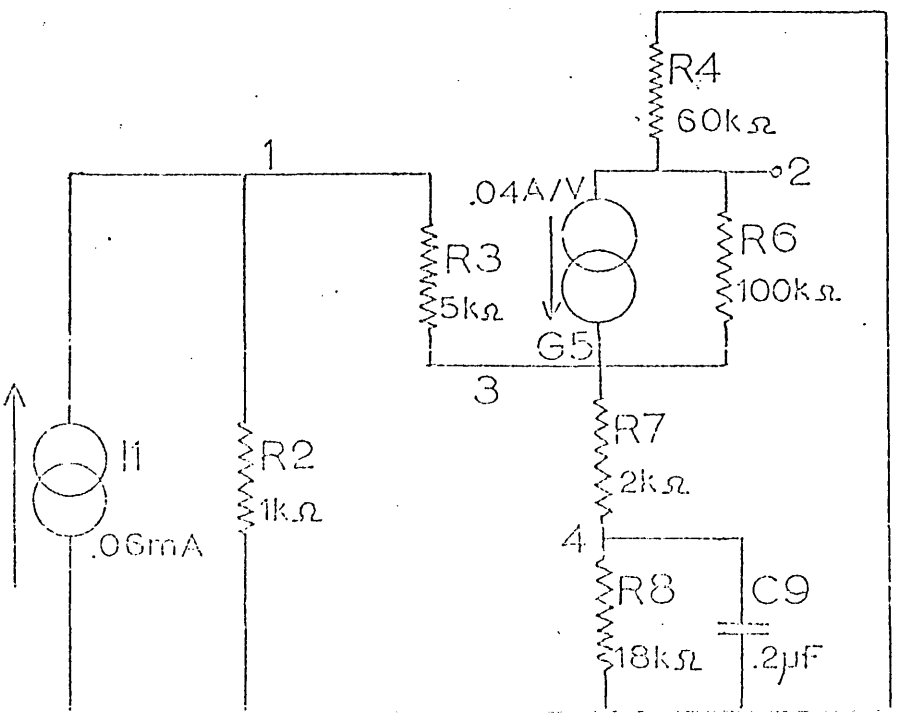


Figure 3.1

BRANCH	NODES		TYPE	VALUE	CONTROLLING NODES	
1	1	0	I	0.600E-04		
2	1	0	R	0.100E+04		
3	1	3	R	0.500E+04		
4	2	0	R	0.600E+05		
5	2	3	G	0.400E-01	1	3
6	2	3	R	0.100E+06		
7	3	4	R	0.200E+04		
8	4	0	R	0.180E+05		
9	4	0	C	0.200E-06		

FINISHED

Y

FREQUENCY = 1E3

NODE	REAL VOLT	IMAG VOLT
1	0.598E-01	-0.761E-04
2	-0.150E+01	-0.571E+00
3	0.588E-01	-0.456E-03
4	0.850E-02	-0.196E-01

TIME = 3.72 SECS

BRANCH	REAL SEN	IMAG SEN
1	0.000E+00	0.000E+00
2	0.997E+00	-0.127E-02
3	0.113E-01	0.117E-02
4	0.990E+00	-0.283E-02
5	0.281E-01	0.751E-02
6	0.108E-01	0.279E-02
7	-0.838E+00	-0.320E+00
8	0.142E-01	0.689E-02
9	0.156E+00	-0.320E+00

FREQUENCY = 0

NODE	REAL VOLT	IMAG VOLT
1	0.600E-01	0.000E+00
2	-0.178E+00	0.000E+00
3	0.598E-01	0.000E+00
4	0.539E-01	0.000E+00

TIME = 3.44 SECS

BRANCH	REAL SEN	IMAG SEN
1	0.000E+00	0.000E+00
2	0.100E+01	0.000E+00
3	0.936E-02	0.000E+00
4	0.996E+00	0.000E+00
5	0.116E-01	0.000E+00
6	0.516E-02	0.000E+00
7	-0.999E-01	0.000E+00
8	-0.899E+00	0.000E+00
9	0.000E+00	0.000E+00

Analyses of Circuit 8 at Frequencies of 1 kHz and 0 Hz. Times given are actual also.

Figure 3.2

```
ROW  1
  0.000E+00 0.000E+00   0.000E+00 0.000E+00   0.000E+00 0.000E+00
  0.000E+00 0.000E+00   0.000E+00 0.000E+00   0.000E+00 0.000E+00
  0.000E+00 0.000E+00   0.000E+00 0.000E+00   0.000E+00 0.000E+00

ROW  2
  0.000E+00 0.000E+00  -0.148E+01-0.559E+00  -0.204E-01-0.119E-01
 -0.148E+01-0.555E+00  -0.419E-01-0.308E-01  -0.162E-01-0.117E-01
  0.107E+01 0.948E+00  -0.172E-01-0.182E-01  -0.412E+00 0.389E+00

ROW  3
  0.000E+00 0.000E+00  -0.204E-01-0.119E-01   0.158E-01 0.795E-02
 -0.219E-01-0.110E-01   0.154E-01 0.747E-02   0.592E-02 0.281E-02
  0.130E-01 0.163E-01  -0.203E-03-0.305E-03  -0.690E-02 0.459E-02

ROW  4
  0.000E+00 0.000E+00  -0.148E+01-0.555E+00  -0.219E-01-0.110E-01
 -0.145E+01-0.541E+00  -0.511E-01-0.367E-01  -0.283E-01-0.202E-01
  0.106E+01 0.934E+00  -0.171E-01-0.180E-01  -0.406E+00 0.386E+00

ROW  5
  0.000E+00 0.000E+00  -0.419E-01-0.308E-01   0.154E-01 0.747E-02
 -0.511E-01-0.367E-01   0.361E-01 0.252E-01   0.139E-01 0.954E-02
  0.368E-01 0.619E-01  -0.548E-03-0.114E-02  -0.258E-01 0.124E-01

ROW  6
  0.000E+00 0.000E+00  -0.162E-01-0.117E-01   0.592E-02 0.281E-02
 -0.283E-01-0.202E-01   0.139E-01 0.954E-02   0.143E-01 0.100E-01
  0.140E-01 0.234E-01  -0.209E-03-0.431E-03  -0.974E-02 0.472E-02

ROW  7
  0.000E+00 0.000E+00   0.107E+01 0.948E+00   0.130E-01 0.163E-01
  0.106E+01 0.934E+00   0.368E-01 0.619E-01   0.140E-01 0.234E-01
 -0.114E+00-0.133E+01   0.172E-01 0.419E-01   0.947E+00-0.389E+00

ROW  8
  0.000E+00 0.000E+00  -0.172E-01-0.182E-01  -0.203E-03-0.305E-03
 -0.171E-01-0.180E-01  -0.548E-03-0.114E-02  -0.209E-03-0.431E-03
  0.172E-01 0.419E-01   0.153E-01 0.191E-01   0.157E-01 0.436E-01

ROW  9
  0.000E+00 0.000E+00  -0.412E+00 0.389E+00  -0.690E-02 0.459E-02
 -0.406E+00 0.386E+00  -0.258E-01 0.124E-01  -0.974E-02 0.472E-02
  0.947E+00-0.389E+00   0.157E-01 0.436E-01   0.570E+00 0.360E-01
TIME =11.10 SECS
```

Matrix of Element Normalised 2 nd Order Differentials at 1 kHz. for Circuit 8. Time given is actual x10. The boxed entry is the real and imaginary parts of $R_3 \frac{d}{dR_3} \left(R_2 \frac{dV}{dR_2} 2 \right)$.

Figure 3.3

ROW 1

0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00

ROW 2

0.000E+00	0.000E+00	-0.178E+00	0.000E+00	-0.174E-02	0.000E+00
-0.177E+00	0.000E+00	-0.214E-02	0.000E+00	-0.952E-03	0.000E+00
0.178E-01	0.000E+00	0.160E+00	0.000E+00	0.000E+00	0.000E+00

ROW 3

0.000E+00	0.000E+00	-0.174E-02	0.000E+00	0.164E-02	0.000E+00
-0.221E-02	0.000E+00	0.163E-02	0.000E+00	0.727E-03	0.000E+00
0.155E-03	0.000E+00	0.140E-02	0.000E+00	0.000E+00	0.000E+00

ROW 4

0.000E+00	0.000E+00	-0.177E+00	0.000E+00	-0.221E-02	0.000E+00
-0.176E+00	0.000E+00	-0.273E-02	0.000E+00	-0.160E-02	0.000E+00
0.177E-01	0.000E+00	0.159E+00	0.000E+00	0.000E+00	0.000E+00

ROW 5

0.000E+00	0.000E+00	-0.214E-02	0.000E+00	0.163E-02	0.000E+00
-0.273E-02	0.000E+00	0.202E-02	0.000E+00	0.899E-03	0.000E+00
0.230E-03	0.000E+00	0.207E-02	0.000E+00	0.000E+00	0.000E+00

ROW 6

0.000E+00	0.000E+00	-0.952E-03	0.000E+00	0.727E-03	0.000E+00
-0.160E-02	0.000E+00	0.899E-03	0.000E+00	0.910E-03	0.000E+00
0.895E-04	0.000E+00	0.805E-03	0.000E+00	0.000E+00	0.000E+00

ROW 7

0.000E+00	0.000E+00	0.178E-01	0.000E+00	0.155E-03	0.000E+00
0.177E-01	0.000E+00	0.230E-03	0.000E+00	0.895E-04	0.000E+00
0.142E-01	0.000E+00	-0.319E-01	0.000E+00	0.000E+00	0.000E+00

ROW 8

0.000E+00	0.000E+00	0.160E+00	0.000E+00	0.140E-02	0.000E+00
0.159E+00	0.000E+00	0.207E-02	0.000E+00	0.805E-03	0.000E+00
-0.319E-01	0.000E+00	-0.127E+00	0.000E+00	0.000E+00	0.000E+00

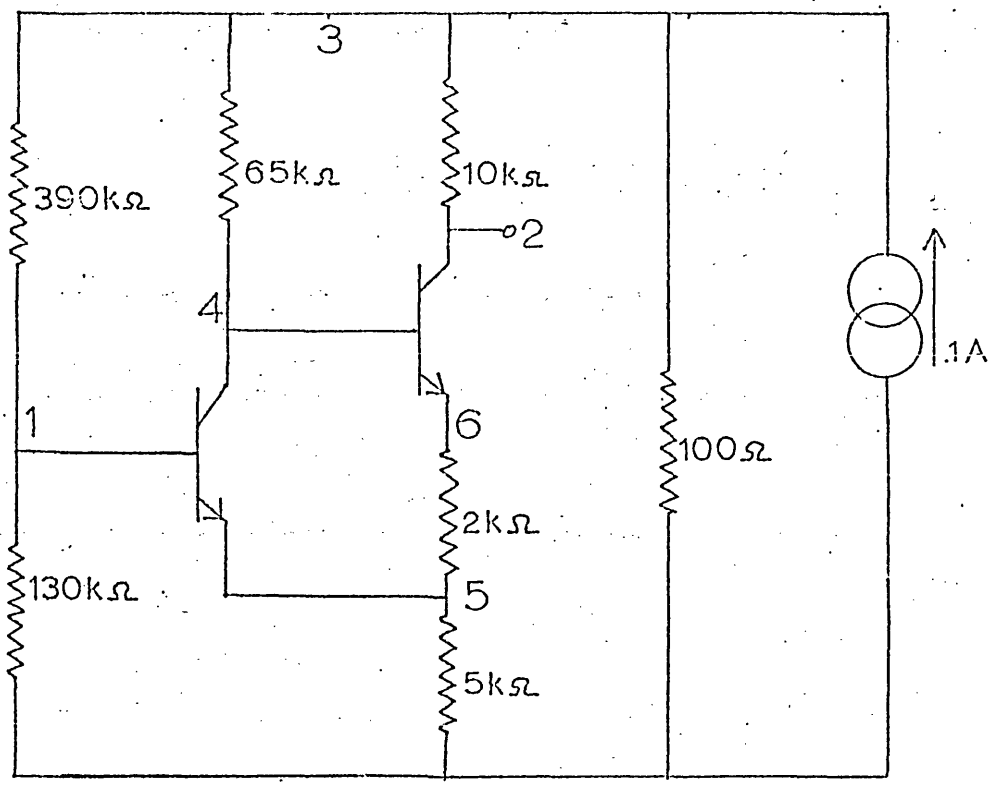
ROW 9

0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00
0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00	0.000E+00

TIME = 9.24 SECS

Matrix of Element Normalised 2nd Order Differentials at 0 Hz. for Circuit 8. Time given is actual x10. The boxed entry is the real and imaginary parts of $R_3 \frac{d}{dR_3} (R_2 \frac{dV_2}{dR_2})$.

Figure 3.4



Circuit 6 modelled

by

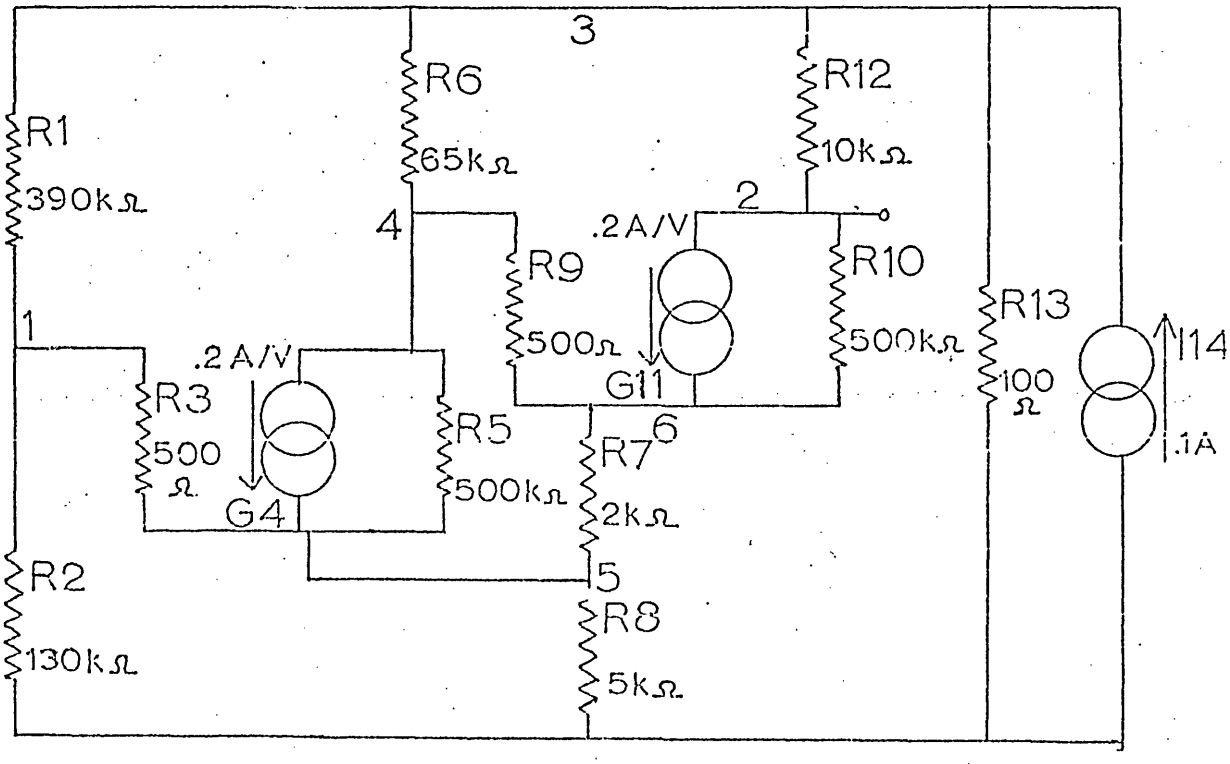


Figure 3.5

BRANCH	NODES		TYPE	VALUE	CONTROLLING NODES	
1	3	1	R	0.390E+06		
2	1	0	R	0.130E+06		
3	1	5	R	0.500E+03		
4	4	5	G	0.200E+00	1	5
5	5	4	R	0.500E+06		
6	3	4	R	0.650E+05		
7	6	5	R	0.200E+04		
8	5	0	R	0.500E+04		
9	4	6	R	0.500E+03		
10	2	6	R	0.500E+06		
11	2	6	G	0.200E+00	4	6
12	2	3	R	0.100E+05		
13	3	0	R	0.100E+03		
14	3	0	I	0.100E+00		

FINISHED

Y

FREQUENCY = 0

NODE	REAL VOLT	IMAG VOLT
1	0.239E+01	0.000E+00
2	0.623E+01	0.000E+00
3	0.995E+01	0.000E+00
4	0.314E+01	0.000E+00
5	0.239E+01	0.000E+00
6	0.314E+01	0.000E+00

TIME =10.32 SECS

BRANCH	REAL SEN	IMAG SEN
1	0.697E+00	0.000E+00
2	-0.661E+00	0.000E+00
3	-0.375E-01	0.000E+00
4	-0.376E-01	0.000E+00
5	0.570E-03	0.000E+00
6	-0.215E+00	0.000E+00
7	-0.243E-01	0.000E+00
8	0.806E+00	0.000E+00
9	0.162E-02	0.000E+00
10	-0.283E-04	0.000E+00
11	0.168E-02	0.000E+00
12	-0.598E+00	0.000E+00
13	0.995E+00	0.000E+00
14	0.000E+00	0.000E+00

Analysis of Circuit 6 at 0 Hz. Time given is actual x10.

Figure 3.6

ROW 1

-0.208E+01	0.000E+00	0.225E+01	0.000E+00	-0.175E+00	0.000E+00
-0.175E+00	0.000E+00	-0.322E-02	0.000E+00	-0.623E+00	0.000E+00
0.167E+00	0.000E+00	-0.401E+01	0.000E+00	-0.100E-01	0.000E+00
-0.446E-03	0.000E+00	-0.105E-01	0.000E+00	0.430E+01	0.000E+00
0.434E+01	0.000E+00	0.000E+00	0.000E+00		

ROW 2

0.225E+01	0.000E+00	-0.219E+01	0.000E+00	-0.598E-01	0.000E+00
-0.593E-01	0.000E+00	0.648E-02	0.000E+00	0.342E+00	0.000E+00
-0.189E+00	0.000E+00	0.386E+01	0.000E+00	0.181E-01	0.000E+00
0.276E-03	0.000E+00	0.186E-01	0.000E+00	-0.408E+01	0.000E+00
-0.412E+01	0.000E+00	0.000E+00	0.000E+00		

ROW 3

-0.175E+00	0.000E+00	-0.598E-01	0.000E+00	0.245E+00	0.000E+00
0.246E+00	0.000E+00	-0.340E-02	0.000E+00	0.293E+00	0.000E+00
0.234E-01	0.000E+00	0.153E+00	0.000E+00	-0.847E-02	0.000E+00
0.177E-03	0.000E+00	-0.852E-02	0.000E+00	-0.231E+00	0.000E+00
-0.233E+00	0.000E+00	0.000E+00	0.000E+00		

ROW 4

-0.176E+00	0.000E+00	-0.593E-01	0.000E+00	0.246E+00	0.000E+00
0.247E+00	0.000E+00	-0.342E-02	0.000E+00	0.294E+00	0.000E+00
0.235E-01	0.000E+00	0.154E+00	0.000E+00	-0.851E-02	0.000E+00
0.178E-03	0.000E+00	-0.856E-02	0.000E+00	-0.232E+00	0.000E+00
-0.235E+00	0.000E+00	0.000E+00	0.000E+00		

ROW 5

-0.322E-02	0.000E+00	0.648E-02	0.000E+00	-0.340E-02	0.000E+00
-0.342E-02	0.000E+00	-0.356E-02	0.000E+00	0.832E-03	0.000E+00
0.383E-02	0.000E+00	-0.791E-02	0.000E+00	-0.541E-04	0.000E+00
0.564E-06	0.000E+00	-0.634E-04	0.000E+00	0.352E-02	0.000E+00
0.355E-02	0.000E+00	0.000E+00	0.000E+00		

ROW 6

-0.623E+00	0.000E+00	0.342E+00	0.000E+00	0.293E+00	0.000E+00
0.294E+00	0.000E+00	0.832E-03	0.000E+00	0.149E+01	0.000E+00
0.102E+00	0.000E+00	0.207E-01	0.000E+00	0.118E-02	0.000E+00
0.149E-03	0.000E+00	0.935E-03	0.000E+00	-0.134E+01	0.000E+00
-0.133E+01	0.000E+00	0.000E+00	0.000E+00		

ROW 7

0.167E+00	0.000E+00	-0.189E+00	0.000E+00	0.234E-01	0.000E+00
0.235E-01	0.000E+00	0.383E-02	0.000E+00	0.102E+00	0.000E+00
-0.164E+00	0.000E+00	0.231E+00	0.000E+00	0.235E-02	0.000E+00
0.222E-04	0.000E+00	0.238E-02	0.000E+00	-0.151E+00	0.000E+00
-0.151E+00	0.000E+00	0.000E+00	0.000E+00		

Matrix of the Element Normalised 2 nd Order Differentials for Circuit 6 at 0 Hz. (continued in Figure 3.8).

Figure 3.7

```
ROW  8
 -0.401E+01 0.000E+00   0.386E+01 0.000E+00   0.153E+00 0.000E+00
  0.154E+00 0.000E+00  -0.791E-02 0.000E+00   0.207E-01 0.000E+00
  0.231E+00 0.000E+00  -0.506E+01 0.000E+00  -0.220E-01 0.000E+00
 -0.209E-03 0.000E+00  -0.226E-01 0.000E+00   0.497E+01 0.000E+00
  0.502E+01 0.000E+00   0.000E+00 0.000E+00

ROW  9
 -0.100E-01 0.000E+00   0.181E-01 0.000E+00  -0.847E-02 0.000E+00
 -0.851E-02 0.000E+00  -0.541E-04 0.000E+00   0.118E-02 0.000E+00
  0.235E-02 0.000E+00  -0.220E-01 0.000E+00  -0.102E-01 0.000E+00
  0.169E-03 0.000E+00  -0.102E-01 0.000E+00   0.102E-01 0.000E+00
  0.101E-01 0.000E+00   0.000E+00 0.000E+00

ROW 10
 -0.446E-03 0.000E+00   0.276E-03 0.000E+00   0.177E-03 0.000E+00
  0.178E-03 0.000E+00   0.564E-06 0.000E+00   0.149E-03 0.000E+00
  0.222E-04 0.000E+00  -0.209E-03 0.000E+00   0.169E-03 0.000E+00
  0.176E-03 0.000E+00   0.176E-03 0.000E+00   0.378E-04 0.000E+00
 -0.176E-03 0.000E+00   0.000E+00 0.000E+00

ROW 11
 -0.105E-01 0.000E+00   0.186E-01 0.000E+00  -0.852E-02 0.000E+00
 -0.856E-02 0.000E+00  -0.634E-04 0.000E+00   0.935E-03 0.000E+00
  0.238E-02 0.000E+00  -0.226E-01 0.000E+00  -0.102E-01 0.000E+00
  0.176E-03 0.000E+00  -0.105E-01 0.000E+00   0.106E-01 0.000E+00
  0.105E-01 0.000E+00   0.000E+00 0.000E+00

ROW 12
  0.430E+01 0.000E+00  -0.408E+01 0.000E+00  -0.231E+00 0.000E+00
 -0.232E+00 0.000E+00   0.352E-02 0.000E+00  -0.134E+01 0.000E+00
 -0.151E+00 0.000E+00   0.497E+01 0.000E+00   0.102E-01 0.000E+00
  0.378E-04 0.000E+00   0.106E-01 0.000E+00  -0.372E+01 0.000E+00
 -0.370E+01 0.000E+00   0.000E+00 0.000E+00

ROW 13
  0.434E+01 0.000E+00  -0.412E+01 0.000E+00  -0.233E+00 0.000E+00
 -0.235E+00 0.000E+00   0.355E-02 0.000E+00  -0.133E+01 0.000E+00
 -0.151E+00 0.000E+00   0.502E+01 0.000E+00   0.101E-01 0.000E+00
 -0.176E-03 0.000E+00   0.105E-01 0.000E+00  -0.370E+01 0.000E+00
  0.613E+01 0.000E+00   0.000E+00 0.000E+00

ROW 14
  0.000E+00 0.000E+00   0.000E+00 0.000E+00   0.000E+00 0.000E+00
  0.000E+00 0.000E+00   0.000E+00 0.000E+00   0.000E+00 0.000E+00
  0.000E+00 0.000E+00   0.000E+00 0.000E+00   0.000E+00 0.000E+00
  0.000E+00 0.000E+00   0.000E+00 0.000E+00   0.000E+00 0.000E+00
  0.000E+00 0.000E+00   0.000E+00 0.000E+00
TIME =26.12 SECS
```

Continuation of the Element Normalised 2nd Order Differentials for Circuit 6. Time given is actual x10. The boxed entry is the real and imaginary parts of $R_3 \frac{d}{dR_3} \left(R_{13} \frac{dV_2}{dR_{13}} \right)$.

Figure 3.8

Chapter 4

Large-Change Sensitivity

Efficient methods for the calculation of small-change sensitivity have been developed in the previous chapters. Often, however, the effect of large element changes on the output of the circuit is of interest to the designer. In this situation the small-change sensitivity values are inapplicable and hence an efficient means is required to predict the effect of large element value changes. Another aspect of circuit design which must be considered by the designer is the tolerancing of circuit elements. Tolerancing is the reverse procedure to the large-change sensitivity calculation in that the range of element values to give a specified change in circuit output is now required. This chapter developes a technique which will not only efficiently calculate the large-change sensitivity, but also predict the range of element values which will cause any given alteration in the output.

Consider a two port network containing a two terminal element ϕ connected to nodes i and j. Now create a new port across this element ϕ (see figure 4.1). The voltage, V_{ij}, across the element ϕ is the voltage which appears across the extracted port. Unless the network is modified (by inserting an ammeter say), the current in ϕ may only be found from the knowledge of the admittance y_i of ϕ and the voltage V_{ij} (Ohm's Law). If a current source, I_{ij}, is now connected across ϕ by means of the extracted port, (see figure 4.2), both the output voltage, V_{out}, and the voltage

across ϕ, V_{ij}, are changed. Now consider the terminal relationship of ϕ combined with the current source I_{ij}. Since the voltage across the combined element pair is the now modified V_{ij}, then the total current I_T in this pair is the sum of the source current I_{ij} and the current i_ϕ in the element ϕ. This current, i_ϕ, is equal to the voltage V_{ij} across the element multiplied by the admittance y_i of ϕ. Since ideal elements, excluding independent sources, have linear voltage current relationships intersecting the origin of the axes (that is, V=ZI) then the voltage, V_{ij}, and current I_T fully describe a simulated admittance Y_s. This is because two points are sufficient to define any linear function, one point being V_{ij} and I_T while the origin provides the other (see figures 4.3 to 4.5). Thus, the combination of the current source I_{ij} and the element ϕ simulates a new element admittance.

If z_{ij} is defined as a general term of the inverted nodal admittance matrix, that is

$$[Y_N]^{-1} = [z_{ij}] \qquad - - - 4.1$$

then the output voltage, V_{out}, of the original circuit is

$$V_{out} = \sum_k (z_{lk} - z_{mk})\, J_k \qquad - - - 4.2$$

where l and m are the output voltage nodes and J_k is the k^{th} row of the nodal current forcing vector, $[J]$. This vector $[J]$ is formed by premultiplying the vector form of the forcing currents by the reduced incidence matrix, $[A]$ (see equation 2.3). The original voltage V_{ij} across the

element, ϕ, is given by

$$V_{ij} = \sum_k (z_{ik} - z_{jk}) J_k \qquad - - - 4.3$$

Now if the current source I_{ij} is connected across the element ϕ by means of the extracted port, then the output voltage is now

$$V_{out} = \sum_k (z_{1k} - z_{mk}) J_k + (z_{1i} - z_{1j} - z_{mi} + z_{mj}) I_{ij} \qquad - 4.4$$

and the new element voltage V_{ij} is given by

$$V_{ij} = \sum_k (z_{ik} - z_{jk}) J_k + (z_{ii} - z_{ij} - z_{ji} + z_{jj}) I_{ij} \qquad - - 4.5.$$

This then enables the total current, I_T, flowing in the simulated admittance Y_s to be found as

$$I_T = V_{ij} y_i - I_{ij} \qquad - - - 4.6.$$

Examination of equation 4.5 shows that all the terms are constant except two and these constant terms depend only on the original values of the network elements. Therefore this equation may be expressed more meaningfully as

$$V_{ij} = C_1 + C_2 I_{ij} \qquad - - - 4.7$$

where $C_1 = \sum_k (z_{ik} - z_{jk}) J_k$

and $C_2 = z_{ii} - z_{ij} - z_{ji} + z_{jj}$

Substitution of equation 4.7 into equation 4.6 then gives

$$I_T = (C_1 + C_2 I_{ij}) y_i - I_{ij} \qquad - - - 4.8.$$

The simulated element admittance, Y_s, is given by

$$Y_s = \frac{I_T}{V_{ij}} \qquad \text{- - - 4.9}$$

which on substitution of equation 4.8 and 4.7 may then be expressed as

$$Y_s = \frac{(C_1+C_2I_{ij})y_i - I_{ij}}{C_1 + C_2I_{ij}} \qquad \text{- - - 4.10.}$$

Rearrangement of this equation then clearly shows the bilinear relationship between the simulated admittance Y_s and the 'simulating current source' I_{ij}.

$$Y_s = \frac{C_1y_i + (C_2y_i - 1)I_{ij}}{C_1 + C_2I_{ij}} \qquad \text{- - - 4.11.}$$

Therefore the required value of the current source I_{ij} to give any simulated admittance Y_s is given by

$$I_{ij} = \frac{C_1(y_i - Y_s)}{-(C_2y_i - 1) + C_2Y_s} \qquad \text{- - - 4.12.}$$

The implications of this result will be discussed after the next section.

Extension to Voltage Controlled Current Sources:

Consider a network containing a voltage controlled current source, forced by a current source I_s and having a response voltage, V_{out}, between nodes 1 and m. Now connect an accessible port across the controlled current source, $\mathcal{E}$, as shown in figure 4.6. The output of the network, V_{out}, and the controlling voltage V_{ij} of the controlled source are given by

$$V_{out} = \sum_k (z_{1k}-z_{mk})J_k \qquad \text{- - - 4.13}$$

$$V_{ij} = \sum_k (z_{ik}-z_{jk})J_k \qquad \text{- - - 4.14}$$

where z_{ij} and J_k have been defined in equations 4.1 and 4.2.

Now connect a current source, I_{kn}, across the extracted port as shown by the dashed lines in figure 4.6. This additional current source now changes the values of V_{out} and V_{ij} to

$$V_{out} = \sum_k (z_{lk}-z_{mk})J_k + (z_{lk}-z_{ln}-z_{mk}+z_{mn})I_{kn} \quad --- 4.15$$

$$V_{ij} = \sum_k (z_{ik}-z_{jk})J_k + (z_{ik}-z_{in}-z_{jk}+z_{jn})I_{kn} \quad --- 4.16$$

respectively. If the controlled current source, $\mathcal{E}$, and the inserted current source, I_{kn}, are considered as one voltage controlled current source then the value, g_{ms}, of this element is given by

$$g_{ms} = \frac{-I_{kn} + g_m V_{ij}}{V_{ij}} \quad --- 4.17$$

where g_m is the initial value of the controlled source. As in the two terminal case the voltage V_{ij} appearing across the controlling branch of $\mathcal{E}$ may be expressed solely in terms of constants and the current source I_{kn}, since all the remaining terms of equation 4.16 are constant. That is

$$V_{ij} = K_1 + K_2 I_{kn} \quad --- 4.18$$

where $K_1 = \sum_k (z_{ik}-z_{jk})J_k$

and $K_2 = (z_{ik}-z_{in}-z_{jk}+z_{jn})$

Substituting this equation into equation 4.17 and rearranging, a bilinear equation, 4.19, is obtained relating the new value of the controlled current source, g_{ms}, and the current source, I_{kn}.

$$g_{ms} = \frac{K_1 g_m + (g_m K_2 - 1) I_{kn}}{K_1 + K_2 I_{kn}} \qquad - - - 4.19$$

Therefore the value of the current source, I_{kn}, necessary to simulate a new value, g_{ms}, of the controlled current source, $\mathcal{E}$, is given by

$$I_{kn} = \frac{K_1 (g_m - g_{ms})}{-(K_2 g_m - 1) + K_2 g_{ms}} \qquad - - - 4.20.$$

Comparision of this equation with equation 4.12 shows that the expressions relating two terminal element and controlled current source admittances to their respective simulating current sources are identical in form.

Application:

Providing the initial analysis of the circuit is performed by inverting the nodal admittance matrix, then, after the evaluation of two simple constants involving terms of the inverted admittance matrix, the effect of a large element change on the output may be simply calculated. Not only is it possible to calculate the large-change sensitivity but also the inverse problem of finding element values for a given output change (tolerancing) is solved. Another factor not immediately apparent from the mathematics is that, using this method, the designer is not restricted to changes in element values only. This means that the output may also be predicted if the element type is changed from, for example, a resistance to an inductance.

If the effect on the output of a network of a change in a certain admittance is required, then after calculating the constants C_1 and C_2, the procedure is to calculate the value of the 'simulating current source', I_{ij}, from the

relevant equation. The output for the desired admittance change is then found by substituting the value of I_{ij} into equation 4.4. When, however, the change in output is known and the associated change in branch admittance must be found, then the necessary value of I_{ij} is found from equation 4.4. The branch admittance necessary to give this output is then found from equation 4.11 or equation 4.19 depending whether the admittance is a two terminal element or a controlled current source. Note that in this case the predicted element admittance will, in general, be a complex quantity consisting of both conductance and susceptance.

The computational effort involved is calculating the constants C_1 and C_2 and evaluating the two equations relevant to two vector multiplications plus a few multiplications, divisions and additions. Assuming that the circuit has been analysed by inverting the nodal admittance matrix, the calculation time for this method is infinitesimal compared with solving the circuit equations.* The only other direct way to calculate element values necessary to provide a given output is to calculate the bilinear relationship between an element admittance and the output.[32] This relationship has to be calculated for each element in turn by analysing the circuit and so this method is grossly inefficient compared with the proposed method. Note that the above equations apply to the case when only one element at any given time has a simulated value. Multiple element

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* This method would realise a considerable saving of computer time in the tolerancing applications described by Butler.[31]

simulation is dealt with later on in this chapter.

Special Cases:

Consider the ideal network modelling Circuit 9, both of which are shown in figure 4.7. In this network branches 9 to 13 have no effect on the response due to the forcing source, branch 15. A cursory examination of the circuit shows that the voltages of nodes 2, 4, 5 and 6 are identically equal irrespective of the values assigned to the admittances of branches 9 to 13. Therefore the constant, C_1, defined in equation 4.7 is zero for branches 9 to 13 (note that C_1 is the original branch voltage of the branch in question). If equation 4.12 is then employed to find the value of the current source necessary to simulate a new admittance in, for example, branch 9, then the value of the corresponding simulating current source, I_{ij}, is found to be zero. A little consideration of the circuit shows that this result is the only logical answer since the value of the admittance of branch 9 is immaterial to any output voltage and therefore the value of the simulating current source, I_{ij}, must be zero.

The case of element tolerancing is where some care must be exercised for circuits similar to that shown in figure 4.7. For tolerancing, the values of I_{ij} are given by substitution of the specified output voltages into equation 4.4. In the case of branch 9 in circuit 9, if the output quantity is the voltage of node 6 then finite values for the current source, I_{ij}, will be obtained. However substitution of these values of I_{ij} into equation 4.11 to find the value of the admittance being simulated

gives invalid results. This is because the constant, C_1, for branch 9 is zero and substitution of $C_1 = 0$ into equation 4.11 gives

$$Y_s = \frac{(C_2 y_i - 1) I_{ij}}{C_2 I_{ij}} \qquad \text{- - - 4.20.}$$

Since I_{ij} is a common factor of the right hand side of equation 4.20, this equation states that the value of the simulated admittance, Y_s, is independent of the value of the current source I_{ij}. Also the value of the simulation admittance given by equation 4.20 is not equal to the original admittance, y_i. Therefore in any automatic scheme utilising this reverse procedure to find the element tolerances from knowledge of the output tolerances, a check must be made on the value of the constant C_1. If the value of C_1 is zero, then this branch is of no use in the circuit and hence may be removed by suitable means, normally by short circuiting.

Multiple Element Simulation:

Often it is necessary, such as in a worst case study, for a designer to vary some or all of the values of the elements constituting a network. However the amount of variation is not usually known and hence several different variations must be tried. Normally each trial would involve a circuit analysis to discover the effect of these element variations. When the number of trials runs into the hundreds, such as in a computerised circuit optimisation scheme, the amount of calculation involved is prodigious. Therefore any scheme which reduces the amount of calculation necessary to determine the effect on the output of a circuit

is worth-while using.

Consider a circuit such as that shown in figure 4.1. Let there be already n elements in the network which have new admittance values simulated by current sources. Therefore the output voltage of the circuit is

$$V_{out} = \sum_k (z_{1k}-z_{mk})J_k + \sum_k (z_{1k}-z_{mk})I_k \qquad - - - 4.21$$

where $[I_k] = [A][I_{ij}]$, and $[I_{ij}]$ and $[A]$ are the vector of simulating currents and the reduced incidence matrix respectively. If the change, ΔV_{out}, of the output voltage rather than the actual output voltage is expressed in terms of z_{ij}, J_k and I_k then

$$\Delta V_{out} = \sum_k (z_{1k}-z_{mk})I_k \qquad - - - 4.22$$

since the original output voltage was given by $\sum_k (z_{1k}-z_{mk})J_k$.

The voltage V_{ij} across the element ϕ is given by

$$V_{ij} = \sum_k (z_{ik}-z_{jk})J_k + \sum_k (z_{ik}-z_{jk})I_k \qquad - - - 4.23$$

which may be written as

$$V_{ij} = C_1 + \sum_k (z_{ik} - z_{jk})I_k \qquad - - - 4.24$$

where $C_1 = \sum_k (z_{ik}-z_{jk})J_k$

This substitution is made since the value of C_1 (the original branch voltage) is only dependent on the initial values of the circuit. Equation 4.24 then shows that if any one of the simulating current sources has its value changed, then the values of all the simulated admittances are changed since, in general, all the branch voltages change. Figure 4.8 demonstrates the effect of the changing simulated admittance

value with changing element pair voltage. Therefore if more than one element is having its value changed by a current source, then the addition or changing of any circuit element affects the value of all the elements having simulated values. Hence, in any scheme using the simulating current source technique for multiple element variation, a constant check and adjustment of all the simulating current sources must be made for any element change.

Techniques for Handling Multiple Element Simulation:

Case 1

As the previous section has shown, the alteration of any circuit element alters the values of all the simulated elements. Therefore each time a change is made in any desired combination of elements then all the values of the simulating current sources must be changed. In this section the relationship between element values and their simulating current sources is derived. This relationship then enables multiple element value simulation to be successfully achieved.

Let the required admittance value of an element ϕ be Y_s, obtained by the parallel connection of ϕ and a suitably valued current source, I_{ij}. Then, by substitution of equation 4.6 in equation 4.9 and dividing through by V_{ij},

$$Y_s = y_i - \frac{I_{ij}}{V_{ij}} \qquad - - - 4.25$$

is obtained where y_i is the original admittance value of the chosen element ϕ. However V_{ij} may be expressed in matrix form as

$$[V_{ij}] = [D][z_{ij}]\{[A][J_s] + [E][I_{ij}]\} \quad --- 4.26$$

where $[D]$ and $[E]$ are topological matrices consisting of 1, -1 and 0. The matrix $[D]$ describes the nodal connections of the branches across which the various voltages in the vector $[V_{ij}]$ are developed. If the first element in $[V_{ij}]$ is the voltage difference between nodes i and j then the only non-zero entries in the first row of $[D]$ are 1 and -1 in columns i and j respectively. The matrix $[E]$, which is an abbreviated version of the reduced incidence matrix $[A]$, adds into the forcing current vector $[I_N]$ (see equation 2.5) the contributions due to the current vector $[I_{ij}]$. Rearrangement of equation 4.25 and expression in matrix form gives

$$[I_{ij}] = [y_i - Y_s][V_{ij}] \quad --- 4.27$$

where $[y_i - Y_s]$ is a diagonal matrix. Substitution of equation 4.26 then gives

$$[I_{ij}] = [y_i - Y_s][D][z_{ij}]\{[A][J_s] + [E][I_{ij}]\} \quad --- 4.28$$

This equation allows the value of the simulated current source vector $[I_{ij}]$ to be expressed explicitly as

$$\{[I] - [y_i - Y_s][D][z_{ij}][E]\}[I_{ij}] = [y_i - Y_s][D][z_{ij}][A][J_s] \quad - 4.29$$

where $[I]$ is the unit matrix.

Despite the rather formidable appearance of equation 4.29 the vector $[I_{ij}]$ is very simply evaluated. First consider the right hand side; the product of the last three terms, $[z_{ij}]$, $[A]$ and $[J_s]$ yields the original nodal voltage vector $[v_N]$. Therefore premultiplication of $[v_N]$ by the matrix $[D]$ gives the initial branch voltages of the

branches whose values are being changed. Evaluation of the expression

$$[y_i - Y_s][D][z_{ij}][A][J_s]$$

may therefore be made by a single pass through the list of elements whose values are being altered. This is achieved by multiplying the branch voltage by the difference in admittance between the original and new values. The left hand side of equation 4.29 is more difficult to interpret although equally simple to implement. Consider a general ij term of the product $[D][z_{ij}][E]$ expressed in the summation as

$$[D][z_{ij}][E] = \sum_k \sum_l D_{ik} z_{kl} E_{lj} \qquad - - - \ 4.30.$$

From the previous definitions of $[D]$ and $[E]$ the i^{th} row of $[D]$ contains entries in the columns representing the nodes on which the i^{th} branch of the modified branch list is incident. The j^{th} column of $[E]$ contains entries in the rows corresponding to the nodes on which the j^{th} branch is incident.

Providing that neither the i^{th} or j^{th} branch impinges on the ground node the general term for the product shown in equation 4.30 consists of the summation of four terms in the inverse admittance matrix $[z_{ij}]$. The term $[y_i - Y_s]$ multiplies each row of the above product by the corresponding row entry in $[y_i - Y_s]$, there being only one entry since this matrix is diagonal. Therefore each term of the matrix representing the left hand side of equation 4.29 may also be evaluated by a single pass.

The result of equation 4.29 is a set of simultaneous

equations in the simulating current sources $[I_{ij}]$. This set of equations may then be solved by any of the standard techniques to give the simulating current vector $[I_{ij}]$. Once this vector, $[I_{ij}]$, is known the change in output of the circuit is given by the vector multiplication expressed in equation 4.22 since $[I_k] = [A][I_{ij}]$.

The evaluation of the various terms in equation 4.29 may be made for approximately the same amount of computation as forming the nodal admittance matrix. Hence this technique presents a saving in computational effort if the number of variable elements, which determines the size of the vector $[I_{ij}]$, is smaller than the number of 'active' nodes in the circuit; the comparision being made between alteration of the circuit elements followed by re-analysis of the circuit and the proposed technique.

Case 2:

It has just been shown that if the number of variable branches exceeds the number of active nodes then the solution of equation 4.29 increases rather than decreases the amount of computation necessary to find the effect on the circuit output. However another approach can be taken which achieves a reduction in equation size and hence a reduction in the amount of computation necessary. As in the previous method this technique only applies in certain cases but has the advantage that the size of the equations found will never exceed the size of the nodal admittance equations.

Consider the branch voltage vector $[v_b]$ expressed as

$$[v_b] = [A]^t [z_{ij}] [A] \{[J_s] + [I_s]\} \qquad - - - 4.31$$

where $[I_s]$ is a vector of length NB (number of branches) and contains the entries corresponding to the various simulating currents. The vector $[I_s]$ may be written as

$$[I_s] = [y_i - Y_s]^d [v_b] \qquad - - - 4.32$$

where $[y_i - Y_s]^d$ is of size NB by NB and is a diagonal matrix containing some zeros on the main diagonal. Note that the matrix $[y_i - Y_s]^d$ is larger than the matrix $[y_i - Y_s]$ defined in equation 4.27. Substitution of equation 4.31 in equation 4.32 gives

$$[I_s] = [y_i - Y_s]^d [A]^t [z_{ij}] [A] \{[J_s] + [I_s]\} \qquad - - - 4.33.$$

Premultiplication of equation 4.33 by the reduced incidence matrix $[A]$ and some rearrangement leads to equation 4.34.

$$\{[I] - [A][y_i - Y_s]^d [A]^t [z_{ij}]\}[A][I_s] = [A][y_i - Y_s]^d [A]^t [z_{ij}][A][J_s] \qquad - 4.34$$

However $[I_k] = [A][I_s]$ and so $\qquad - - - 4.35$

$$\{[I] - [A][y_i - Y_s]^d [A]^t [z_{ij}]\}[I_k] = [A][y_i - Y_s]^d [A]^t [z_{ij}][A][J_s] \qquad - 4.36$$

where $[I_k]$ is the nodal form of the simulating currents.

On first appearances no saving in computation is possible since the size of this matrix equation is identical to that of the nodal admittance equations. However, any node which does not have a simulated element connected to it has a zero entry in the vector $[I_k]$. Therefore equation 4.36 is a consistent but redundant set of equations. The redundant equations may be removed by ignoring the rows and columns associated with the nodes which do not have any simulated elements connected to them.

The matrix equation 4.36 is as simple as equation 4.29 to set up. First consider the right hand side of equation 4.36, the last four terms of which, when multiplied out, give the original branch voltage vector $[v_b]$. Premultiplication by $[y_i - Y_s]^d$ is a simple product of the branch voltage and the difference between the desired and original branch admittance. The final multiplication by $[A]$ then inserts this product into the rows corresponding to the nodal connections of the branch being considered. Therefore the right hand side of equation 4.36 may be evaluated by a single pass through the element list. Now consider the term $[A][y_i - Y_s]^d[A]^t$. Since this is identical in form to the expression for the nodal admittance matrix $[Y_N]$ (see equation 2.6) each branch has four possible entries in the resulting matrix. Since this matrix can be formed by a single pass through the element branch list the left hand side of equation 4.38 can be evaluated by a further matrix multiplication and one matrix subtraction. After the superfluous rows and columns have been removed, these equations are then solved by one of the standard methods.

Examples:

Two branches chosen at random in circuits 3 and 7, shown in figures 4.9 and 4.10 respectively, are altered one at a time by the simulating current source method. To provide a check on the accuracy of this method the actual output voltage was calculated by altering the chosen element and re-solving the circuit equations. Figures 4.11 and 4.12 show the results for circuits 3 and 7 respectively. The only deviation noticeable between the actual and predicted

output voltage is for branch 8 in circuit 3 but this error is slight even at a 75% increase in element value. These graphs were obtained directly from a PDP 15 computer driving a X-Y plotter via a digital to analogue convertor.

Conclusion:

This chapter has developed a technique whereby the output change for any element change, and, vice versa, may be predicted very efficiently providing the initial circuit is analysed by an inversion of the nodal admittance matrix. The increased computational effort of inversion compared with an equation solving routine is more than offset if the effect of two or more element changes are required. Catastrophic changes in element values may also be studied by this technique.

Equations have also been developed that will find the change in output for multiple element changes. Providing certain simple conditions exist, these methods for multiple element changes are computationally more efficient than changing the actual element admittances and re-analysing the circuit.

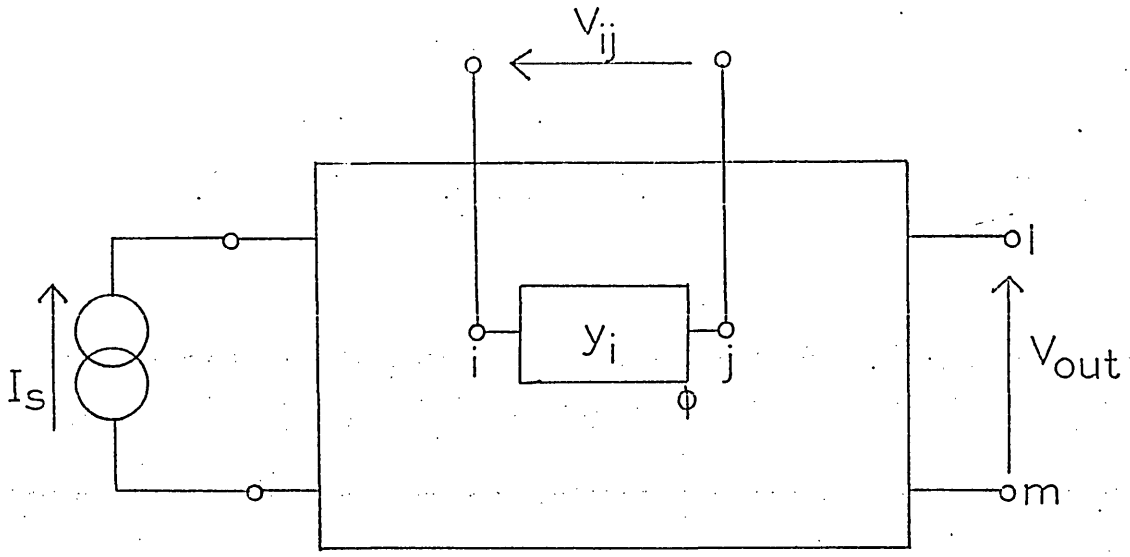


Figure 4.1

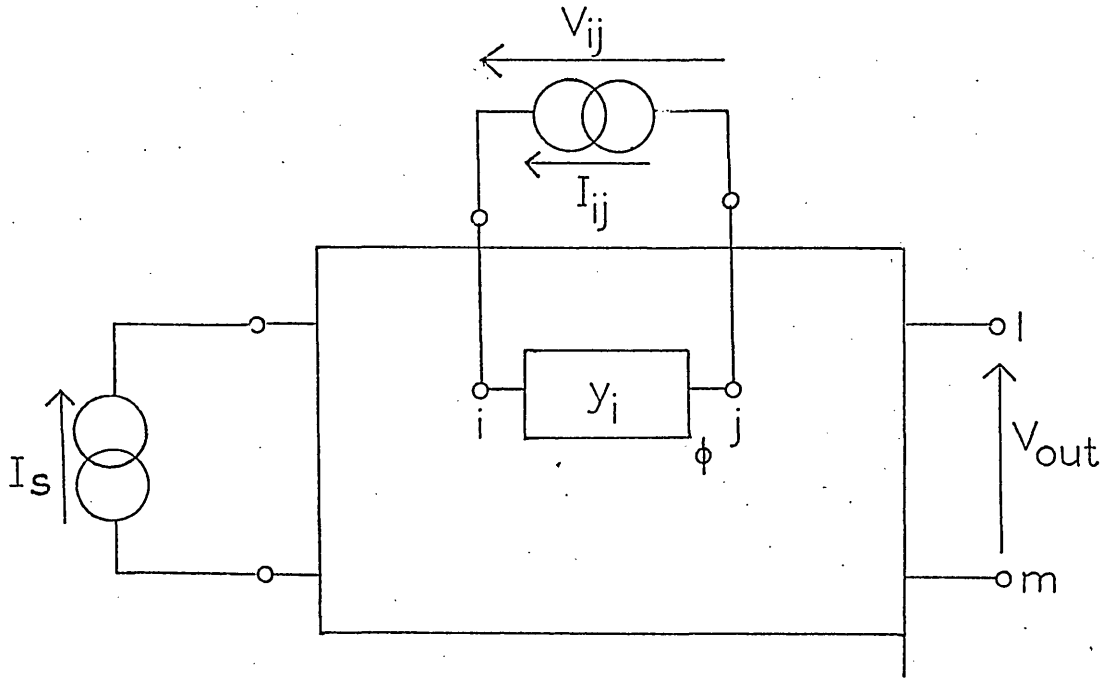


Figure 4.2

Illustration of the connection of a current source I_{ij} across any element ϕ in a network.

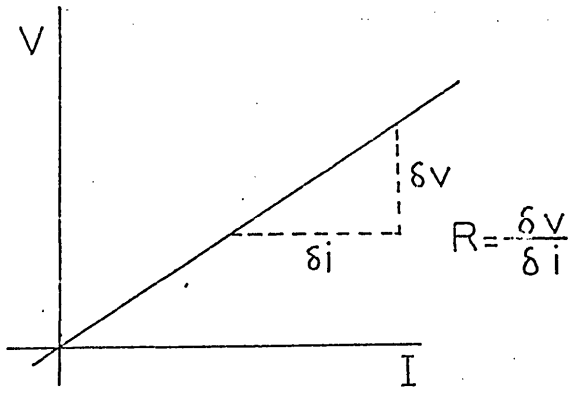


Figure 4.3 Voltage current relationship of a resistor of R ohms.

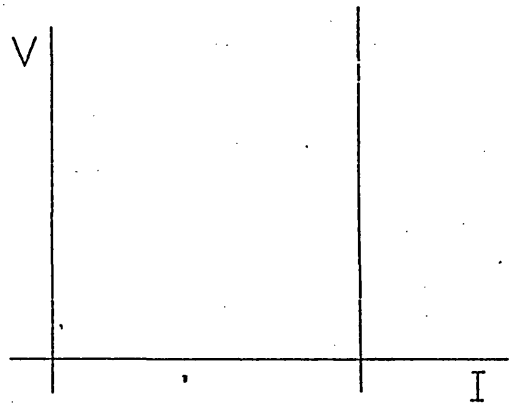


Figure 4.4 Voltage current relationship of a current source.

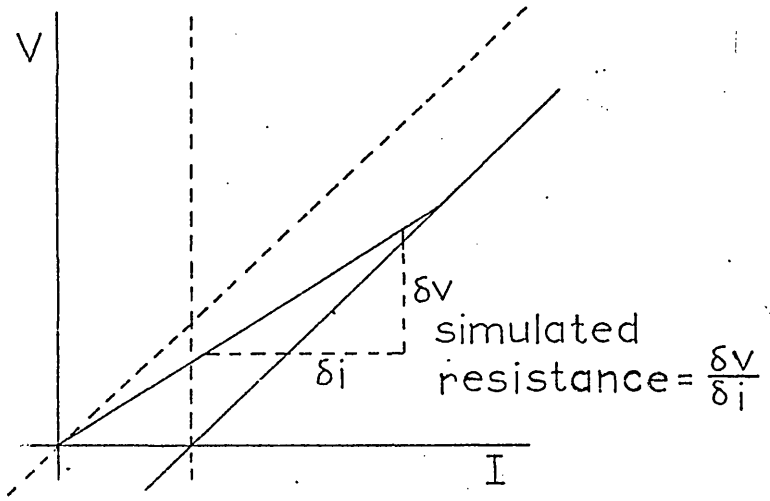


Figure 4.5 Voltage current relationship of a combined resistor and current source.

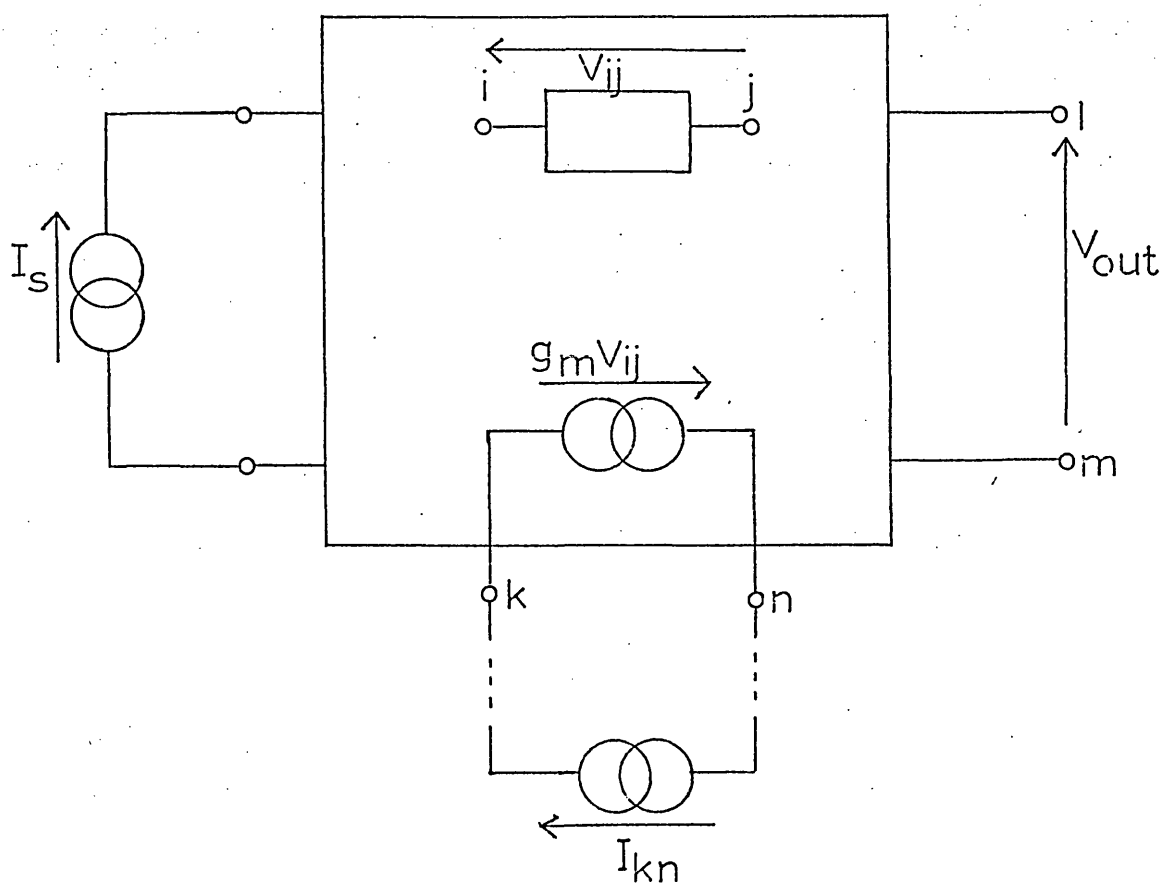
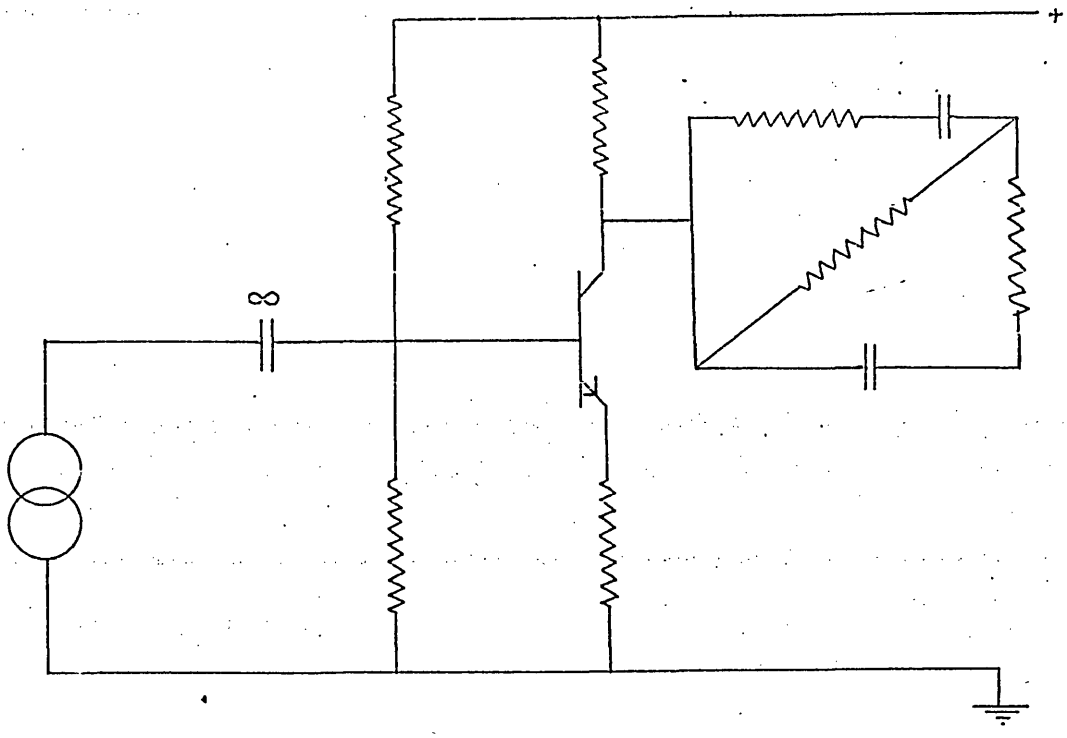


Illustration of changing the value of a controlled current source by the parallel connection of a current source I_{kn}.

Figure 4.6



Circuit 9
modelled by

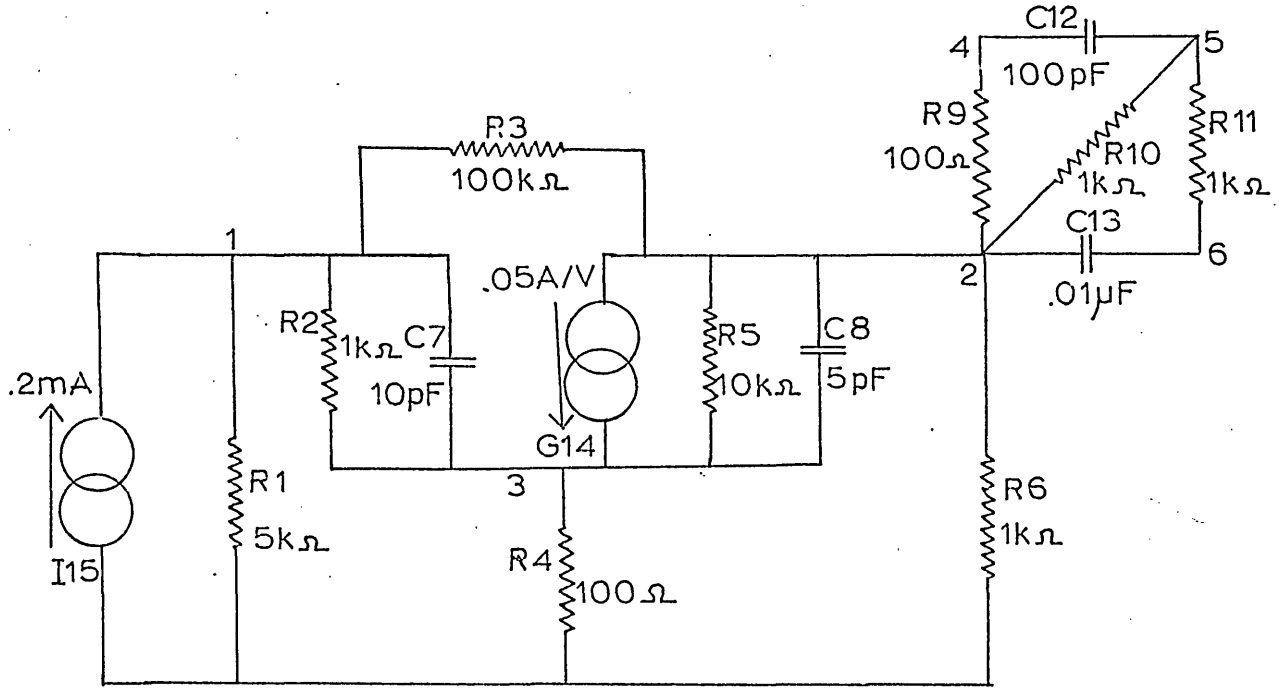


Figure 4.7

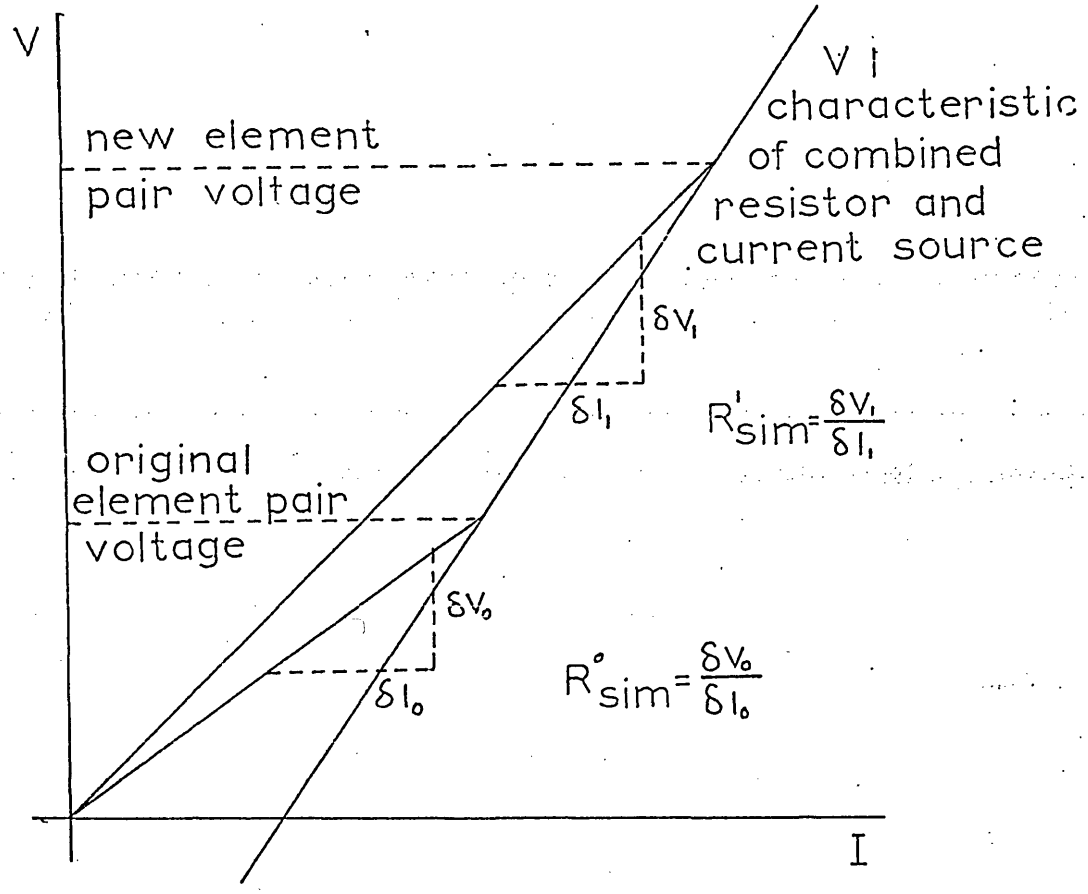
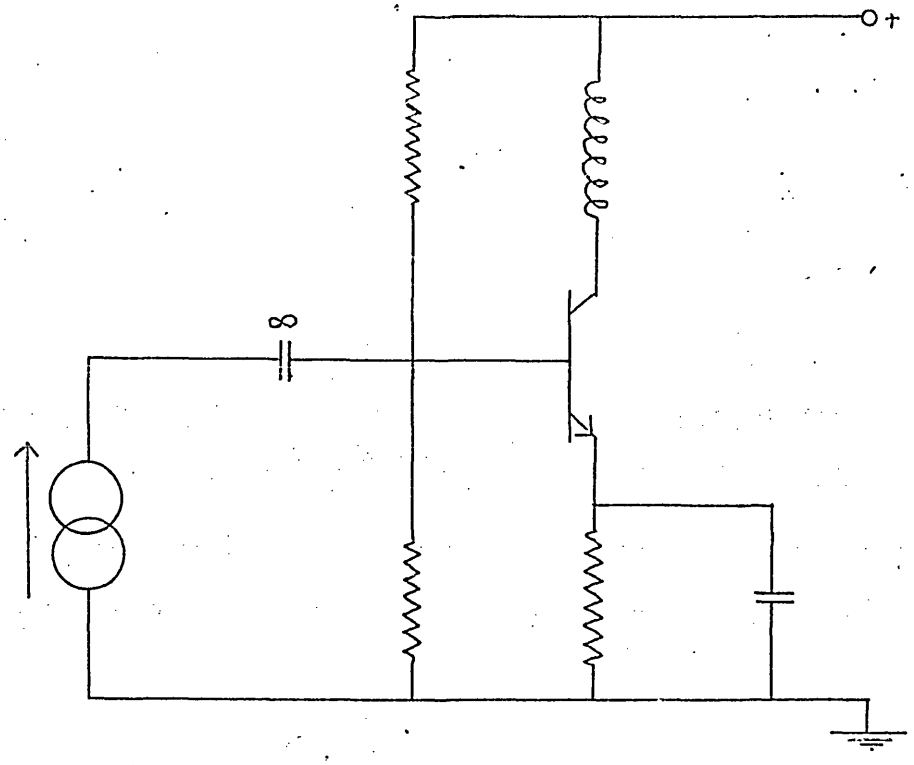


Illustration of the effect of changing simulated resistance with changing voltage across the combined element pair.

Figure 4.8



Circuit 3
modelled by

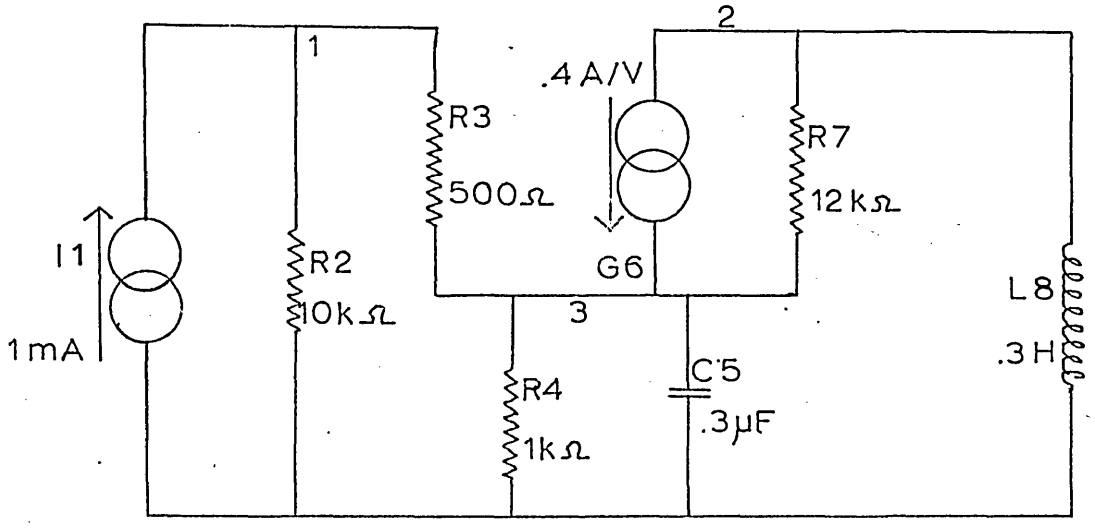
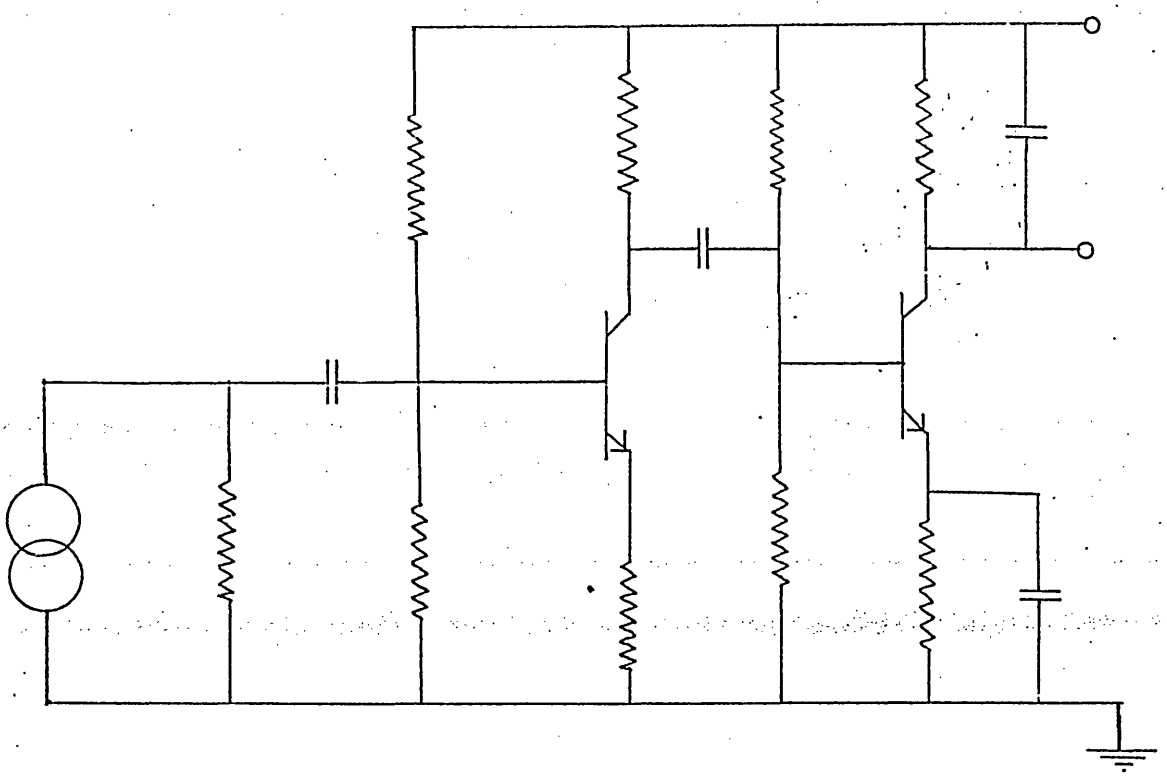


Figure 4.9



Circuit 7
modelled by

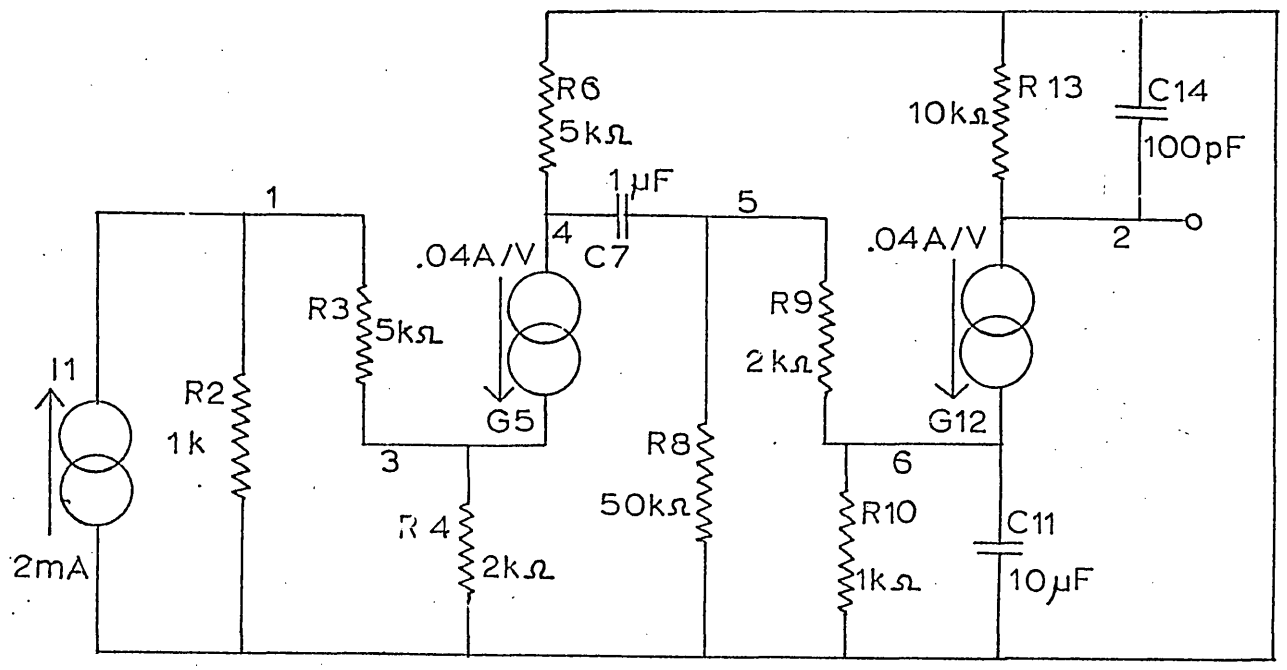
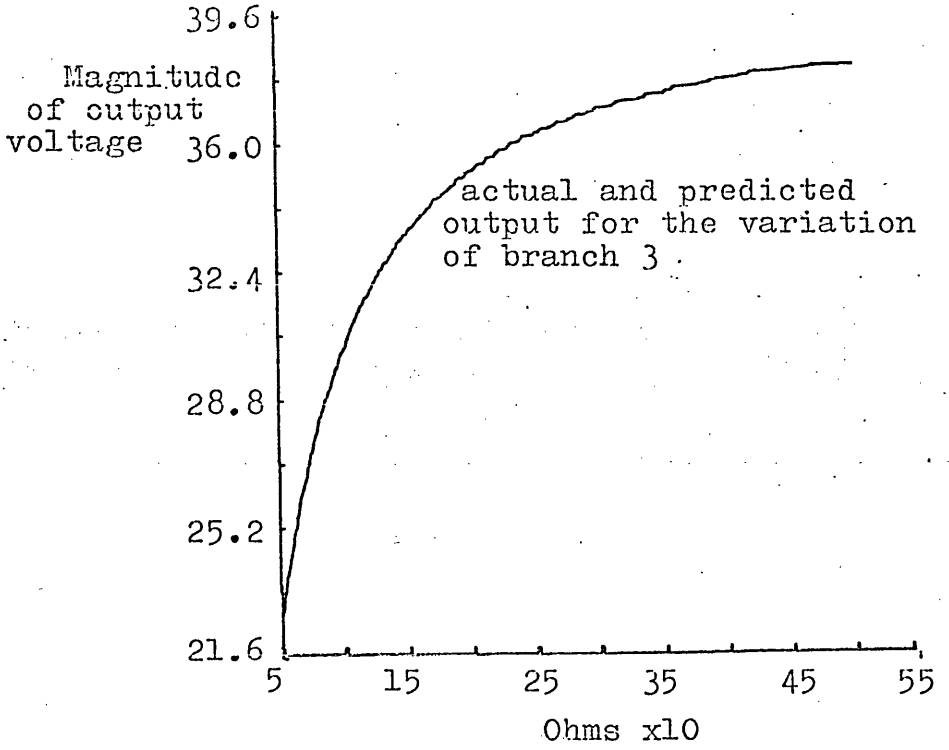
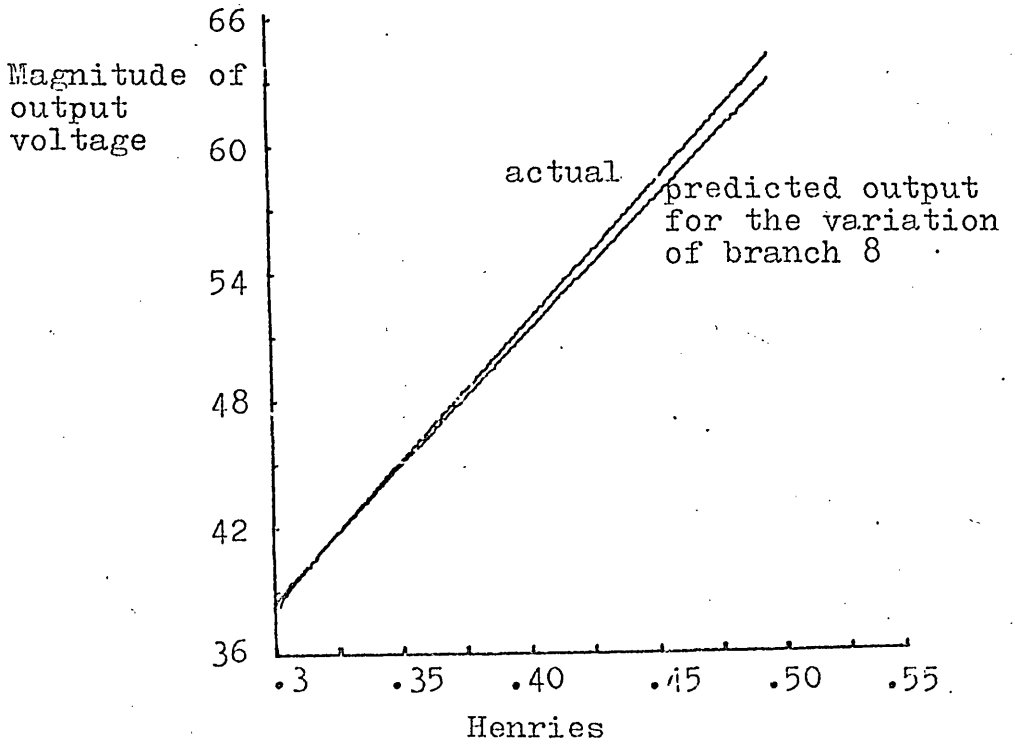


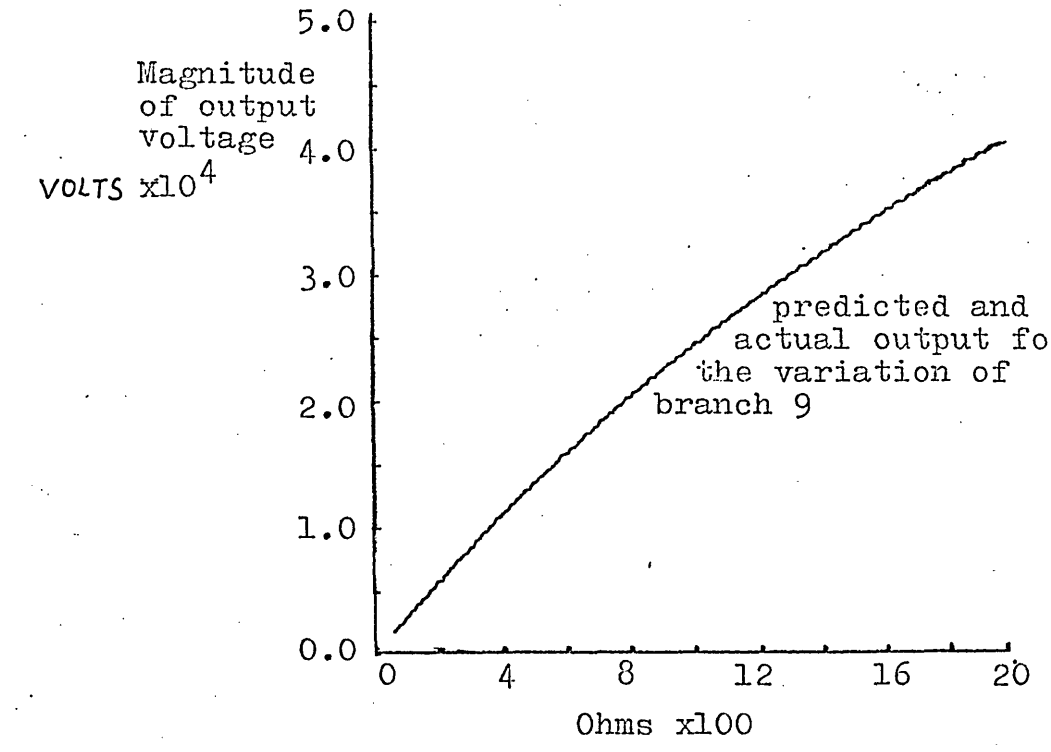
Figure 4.10

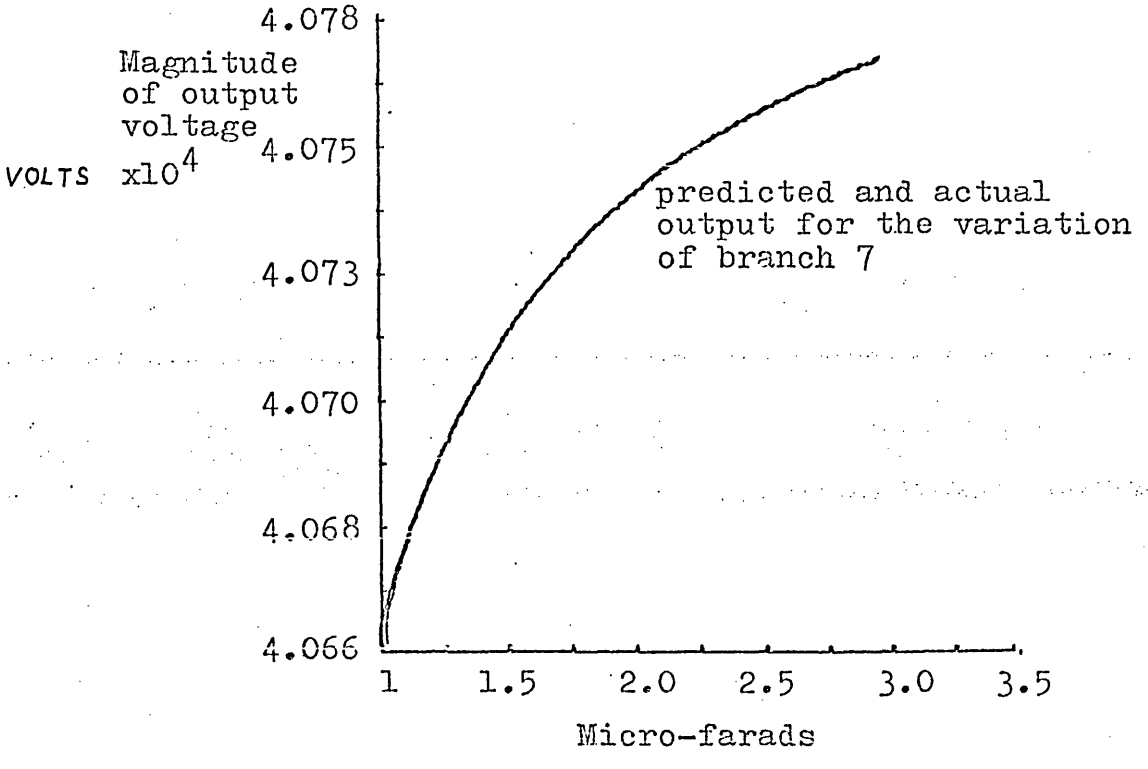




Predicted and Actual Output Variation due to the variation in two branches in Circuit 3. The predicted output was found by the Large-Change Sensitivity method.

Figure 4.11





Predicted and Actual Output Variation due to the variation in two branches in Circuit 7. The predicted output was found by the Large-Change Sensitivity method.

Figure 4.12

Chapter 5

Element Sensitivity Reduction by Constrained Embedding

This chapter describes an attempt to minimise high element sensitivities by reducing the voltages expressed across these elements. A reduction in the branch voltage of high sensitivity elements was sought since an element differential is the product of its branch voltage and another quantity. The method used to reduce the branch voltages of these critical elements was an automated form of the 'constrained embedding' approach formulated by E.B. Kozemchak and M.A. Murray-Lasso.[29] It is possible using this technique to force certain voltages or currents to take either a fixed value or to lie within a certain range of values. However sufficient degrees of freedom must be added to the network in the form of arbitrary branches so that the network equations are solvable. This technique proved to be intractable due to the difficulty in preventing the formation of negative elements, especially in the case of non-reciprocal networks.

The object of utilising this approach was to reduce the sensitivities of critical elements while accepting an increase in the sensitivity of the low sensitivity elements. Ultimately it was hoped that the highest sensitivity would be in the same order of magnitude as the lowest sensitivity. However this result was only achieved on very simple reciprocal networks such as two resistors and a current source. Although the statistical correlation between the maximum element sensitivity and the maximum branch voltage is non-existent there does appear to be some correlation between

an element branch voltage and its sensitivity. The reason for this possible correlation is apparent if equation 2.15, which gives the nodal voltage differentials with respect to any element change, is examined. Inspection of this equation shows that the differential of any nodal voltage, v_l, in the network with respect to an element ϕ is given by the product of two quantities. One of these quantities is the controlling voltage,* v_b, of the element while the other is the difference between two terms in the inverted nodal admittance matrix $[z_{ij}]$. These two terms are in the row l corresponding to the nodal voltage in question while the column indices are specified by the nodes to which the element ϕ is connected. That is

$$\frac{\delta v_l}{\delta \phi} = - (z_{li} - z_{lj})\, v_b^{\phi} \qquad - - - 5.1$$

where the l^{th} nodal voltage is the output voltage and i and j are the nodal connections of the element ϕ.** Since the sensitivity of any element is defined as the normalised change in the output voltage with respect to the normalised element change, then

$$S_{\phi}^{v_l} = \frac{\delta v_l}{v_l} \cdot \frac{\phi}{\delta \phi} \qquad - - - 5.2$$

$$= - \frac{\phi}{v_l} (z_{li} - z_{lj}) v_b^{\phi} \qquad - - - 5.3$$

On casual examination the sensitivity of the element ϕ

---

* The controlling voltage of a two terminal element is its own branch voltage.

** Note that the term $(z_{li} - z_{lj})$ is also the controlling branch voltage of the element in question in the adjoint network.

appears to be linearly related to the controlling voltage of ϕ when the output quantity, v_1, is kept constant. However, if the circuit is altered in such a manner that the output voltage is held constant while the branch voltage of the element ϕ is reduced, then, in general, the values of the inverse admittance matrix $[z_{ij}]$, as well as the impedance of ϕ, are changed. Hence the sensitivity of ϕ is not linearly proportional to its controlling voltage, v_b. The relationship, however was still regarded as sufficient justification to attempt the reduction of high element sensitivities by reducing the branch voltages of these elements by the constrained embedding approach. In all the successful examples which have been run, there have been no cases in which the sensitivity of an element has been increased significantly by decreasing its branch voltage, although some of the examples have not produced a significant sensitivity change either way.

Possible Constraints:

Consider the nodal admittance equation

$$[Y_N][v_N] = [I_N] \qquad - - - 5.4$$

where $[Y_N]$ has a size N by N, N being the number of 'active'* nodes in the circuit. There are now several different types of conditions which may be required in the circuit. These are:

1. A specific voltage across an element;
2. A specific current through an element;

---

* The 'active' nodes are all the nodes in the circuit excluding the reference (ground) node.

3. A specific voltage between two nodes.

On a mathematical basis the terminal relationships of any element may be considered as a function of three variables v_ϕ, i_ϕ and z_ϕ, only two of which are independent; there is, however, no restriction of which variables are chosen to be independent. Now consider the three possible conditions in turn. If the element value under consideration is fixed, then the first two conditions are identical since if two variables are fixed, then the third variable is determined. Note that the third condition is a special case of the first condition with the element ϕ having an infinite impedance.

Realisation of the Various Conditions:

Case 1: Consider choosing a value, v_F, for the voltage expressed across an element ϕ connected between nodes i and j. That is,

$$v_i - v_j = v_F \qquad - - - 5.5.$$

Expressed in matrix form this is

$$[D][v_N] = [v_F] \qquad - - - 5.6$$

where $[D]$ is a topological matrix describing the constraint connections and $[v_F]$ is a column vector of the voltage constraints. In the above example the matrix $[D]$ is a row vector containing +1 and -1 in columns i and j respectively. Since there are only N variables, equation 5.6 combined with equation 5.4 gives rise to an inconsistent or redundant set of equations. For a general value of v_F, these equations are inconsistent and hence the system requires

a further degree of freedom. An extra degree of freedom may be obtained if the impedance of ϕ is allowed to assume an arbitrary value. When this is the case the entry due to the impedance of the element ϕ does not appear in the left hand side of equation 5.4, but appears as an arbitrary current source, i_a, in the right hand side of the equation. That is

$$\begin{aligned}
&y_{11}v_1 + y_{12}v_2 + \cdots + y_{1N}v_N = i_1 \\
&y_{i1}v_1 + y_{i2}v_2 + \cdots + y_{iN}v_N = i_i + i_a \\
&y_{j1}v_1 + y_{j2}v_2 + \cdots + y_{jN}v_N = i_j - i_a \\
&y_{N1}v_1 + y_{N2}v_2 + \cdots + y_{NN}v_N = i_N
\end{aligned} \qquad \text{-- 5.7}$$

This current source may now be eliminated by the addition of rows i and j leaving the row size of the equation 5.7 reduced by one.[30] These equations, when combined with equation 5.6, produce a consistent set of equations which may be solved for the nodal voltages by one of the standard techniques. The value of the arbitrary current source is then found by back substitution into the i^{th} or j^{th} row of equation 5.7. Knowledge of i_a and branch voltage, $v_i - v_j$, allows the value of the impedance or ϕ to be calculated.

This approach may be generalised to deal with more than one constraint providing a sufficient number of degrees of freedom are added to the system. The necessary degrees of freedom may be added to the system by the introduction of a number of suitably chosen arbitrary elements. These elements must be chosen in such a way that no loop of arbitrary elements exists. Any arbitrary element which completes a loop of these elements does not contribute an

extra degree of freedom to the circuit. This is because the two nodes on which this loop-closing branch impinges have already had their appropiate Kirchhoff current law equations summed together and hence the row dimension of equation 5.7 is not reduced by the introduction of this branch. Care must also be taken in choosing the voltage constraints since if a loop of voltage constraints is chosen, then one of these voltage constraints must be either inconsistent or redundant.

Despite the obvious simplicity, the technique has a pitfall due to a tendency towards the formation of negative elements. One of the major reasons for negative elements appears to be the ill-advised choice of an arbitrary element when the voltage constraint is expressed between a pair of nodes across which no element is directly connected. The necessity for the choice of a voltage constraint across nodes which do not have an element directly connected arises from the desire to maintain the output voltage at a constant value. If the output voltage is not held constant but allowed to vary at will when applying the voltage constraints, any attempt to change the circuit by a different process to regain the original transfer function usually results in another circuit as undesirable as the original.

Case 2:

Let the current in an element ϕ be required to take on a specified value. Whether or not the impedance of ϕ is specified there is no entry in the left hand side of equation 5.4 attributed to the conductance of ϕ. This is because the specification of the impedance of ϕ in the admittance matrix would result in a current proportional

to the voltage across ϕ, thus disallowing the possibility of a specified current in this branch. Therefore the matrix equation 5.4 assumes the form of equation 5.7 except that the term i_a is a specific quantity in this case. That is, the element ϕ is replaced by a current source of the required value. Providing sufficient degrees of freedom have been introduced, the network equations may then be solved by one of the standard techniques. The value of the impedance of ϕ is then determined from the values of the specified current and the computed branch voltage of ϕ. When both the current through ϕ and the impedance of the element ϕ are specified then this is identical to the first case except that the element may not be chosen to be the arbitrary element. In this case the problem of choosing a suitable element to assume an arbitrary value arises.

Multiple Constraints:

The feasibility of multiple constraints has already been mentioned in previous sections. In such a situation, however, checks must be made on the consistency of the voltage constraints and also on whether the choice of arbitrary elements gives rise to sufficient degrees of freedom.

Consider fixing the voltages across three elements in a network. These elements are connected to node pairs i j, j k and k i. Let the desired voltages across these elements be v_{F1}, v_{F2} and v_{F3} respectively. Therefore

$$v_i - v_j = v_{F1}$$

$$v_j - v_k = v_{F2} \qquad - - - 5.8$$

and $$v_k - v_i = v_{F3}$$

However if the first two of these equation are summed together the expression

$$v_i - v_k = v_{F1} + v_{F2} \qquad - - - 5.9$$

is obtained. Since the third constraint in equation 5.8 states that $v_k - v_i = v_{F3}$, this set of equations is either redundant or inconsistent. Therefore a check must be made on whether a voltage constraint completes a loop of voltage constraints.

Now consider letting these same three elements take on arbitrary impedance values. Let the currents through these elements be represented by i_{a1}, i_{a2} and i_{a3}. The modified nodal admittance equations incorporating the desired conditions are then

$$\begin{aligned}
&y_{11}v_1 + - - - - + y_{1N}v_N = i_1 \\
&y_{i1}v_1 + - - - - - - - - - = i_i + i_{a1} - i_{a3} \\
&y_{j1}v_1 + - - - - - - - - - = i_j - i_{a1} + i_{a2} \qquad - - 5.10 \\
&y_{k1}v_1 + - - - - - - - - - = i_k - i_{a2} + i_{a3} \\
&y_{N1}v_1 + - - - - - - - - - = i_N
\end{aligned}$$

The addition of rows i, j and k then remove these arbitrary current sources and hence only two additional degrees of freedom have been introduced to the network.* This is

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* Note that this is a rather obscure proof of the Blakesley Transformation. This transformation states that a current source I between two nodes i and j may be replaced by two current sources, each of value I, connecting nodes i to k and k to j; the node k being any arbitrary node.

exactly the same effect as if only i_{a1} and i_{a2} were introduced into the circuit. Therefore the addition of an arbitrary current source which completes a loop of such elements does not add an extra degree of freedom to the system.

Once the system equation has been set up and solved, the problem of determining the resistance values of the arbitrary elements arises. In the case where there is a simgle arbitrary element no difficulty is encountered. Consider, however, the case of equation 5.10. Here there are three arbitrary current sources connected between three nodes i, j and k. Back substitution of the values of $[v_N]$ into the equation 5,10 gives three equations in the arbitrary current sources i_{a1}, i_{a2} and i_{a3}. Examination of these equations shows that only two are independent and hence one of the arbitrary currents must be specified. Assuming that the current i_{a3} is specified, the relevant part of equation 5.10 is

$$\begin{aligned} K_1 &= i_i' + i_{a1} \\ K_2 &= i_j + i_{a2} - i_{a1} \\ K_3 &= i_k' - i_{a2} \end{aligned} \qquad \text{- - - 5.11}$$

where $i_i' = i_i - i_{a3}$, $i_k' = i_k + i_{a3}$ and the constants K_1 to K_3 represent the multiplication of the nodal voltage vector $[v_N]$ by the appropiate rows in the nodal admittance matrix $[Y_N]$. The value of either i_{a1} or i_{a2} is given by the subtraction of i_i' from K_1 or K_3 from i_k' respectively. Therefore the second line in equation 5.11 is not needed. Now assume that the third line in equation 5.11 does not exist in such a simple form (this being the more usual case). Hence only the value of i_{a1} may be found by a direct substitution

of terms. The simplest procedure to find the value of the current source i_{a2} is that of back substitution. Since the current i_{a1} through and the voltage across the first arbitrary element is now known, the resistance of this element is determined and its effect may be added into the nodal admittance equations. This modifies the constants K_1 to K_3 in equation 5.11 and since the term i_{a1} has now been absorbed into the left hand side of the equation, the value of i_{a2} is now found by a single subtraction.

Thus, when the system equation incorporating the various required voltage constraints and arbitrary current sources is being formed, a count must be made of how many undetermined current sources are connected to each node. After solving the system equation, a node must be found which only has one undetermined current source incident on it. This condition then enables the value of the current source connected to the node to be solved which then determines the resistance of this element. After updating the nodal admittance matrix with this element value, the nodes to which this element is connected have their arbitrary element count reduced by one.

In the examples given at the end of this chapter the number of constraints and the maximum voltage reduction are free parameters chosen by the designer. Another necessary provision is the ability to ignore any of the branches according to the designer's choice which also means that these branch resistances remain unaltered. A branch usually ignored is the source branch necessary to simulate a voltage source such as branch 2 in circuit 8 (see figure 5.1). This is because if the impedance of this branch is low

compared with the input impedance of the circuit, then the sensitivity of this branch will be close to unity. Any attempt to lower the voltage across this branch results in a lowering of the branch impedance. This then only raises the sensitivity of the source branch even closer to unity

The constraint voltages for the branches being modified are determined by multiplying the original branch voltages by one minus a factor derived from the branch sensitivities. This factor is equal to a constant, set by the designer, multiplied by the ratio of the sensitivity of the branch under consideration to the highest sensitivity of all the branches being considered. That is

$$v_{Fi} = v_{bi} \left(1 - \frac{S_i}{S_{max}} w_d \right) \qquad - - - 5.12$$

where v_{Fi} and v_{bi} are the constraint and branch voltages of an element i respectively, S_i and S_{max} are the sensitivities of the branch i and the maximum sensitivity branch respectively and w_d is a designer set constant.

Negative Elements:

One of the main causes of the formation of negative elements has its basis in the common desire to maintain the circuit output constant. An extra degree of freedom must clearly be introduced to allow for this constraint, but at what location in the circuit? If a high sensitivity element is allowed to assume an arbitrary value then the whole object of reducing the voltage across high sensitivity elements is defeated. The approach adopted in the following examples has been to allow the most sensitive branch which is not having its branch voltage reduced to assume an

arbitrary value. Often, however, this element does not have sufficient influence to compensate for the changes in output caused by constraining the branch voltages of the more sensitive elements, while all the arbitrary elements remain positive.

One possible solution to this problem is to allow more than one of the lower sensitivity elements to take on arbitrary values and then to specify the requisite number of current values by some other means (see equation 5.11). However some form of optimisation must then be used to find a set of current values which provide a positive set of element values, assuming that such a set exists. Optimisation then means the introduction of an iterative process with all the attendant drawbacks. This is one of the principal reasons why this method is undesirable; the primary object of the investigation reported in this chapter was to provide means of finding networks having low element sensitivities.

Another technique which was tried, but with very little success, was to replace the negative elements with voltage controlled current sources. Since the transistor is the simplest device capable of providing an approximation to a voltage controlled current source, only three-terminal controlled sources were allowed. Once the allowed transistor type (PNP or NPN) has been specified by the designer, the current flow direction in the element determines the relative emitter collector connections. Providing a 'base' node can be found which has the requisite potential difference with respect to the 'emitter' node this method is successful. However, such a node is not usually found and hence a redistribution of the voltage pattern by the constrained

embedding approach is necessary to create a suitable nodal potential. This then has a strong tendency to produce one or more negative elements in the rest of the network. Any attempt to deal with these elements by the same method usually results in yet more such negative elements with the process eventually settling down to a limit cycle. In a third approach that was tried, the output was permitted to vary, but was then restored to its former level by means of an iterative process. This reduced the number of negative elements, but when the second iterative stage of the process found the element values necessary to bring the output back to its former level no significant improvement was obtained.

In view of the difficulties encountered, further research on this topic was considered to be fruitless.

Examples:

Using some of the techniques described above, attempts have been made to reduce the high sensitivity elements of circuits 6 and 8. These circuits, which are shown in figures 5.1 and 5.2 respectively, have their output voltage expressed between node 2 and ground. The user specifies the number of constraints desired, whether the output is to be held constant, the fractional reduction in the branch voltage of the most sensitive branch to be considered and the branches to be ignored, if any. In order to provide an insight into what is happening in the circuit, the fractional branch voltage reduction is subdivided by twenty nine.* Then the constraint severity is incremented twenty nine times from zero up to the user specified figure. At each value of the constraint, the sensitivity of the most

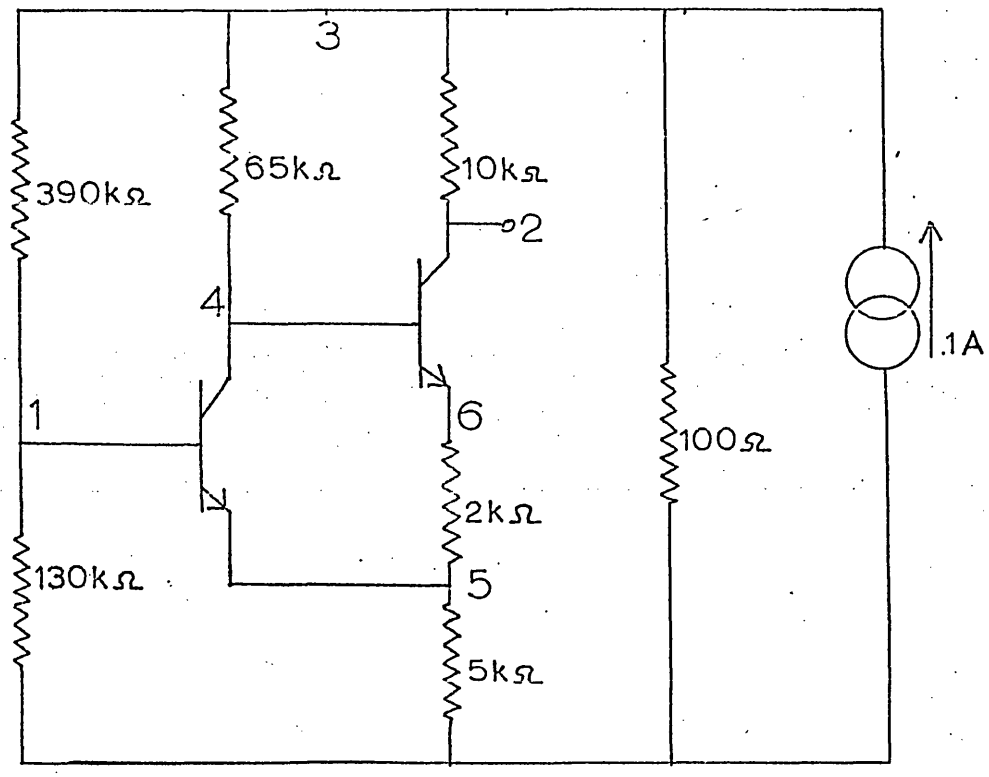
* arbitrary number to provide graph points

sensitive element, its resistance value and the output voltage are stored. These values are then plotted out against the constraint severity.

Figure 5.3 and the top graph of figure 5.4 show the results of trying to reduce high element sensitivity in circuit 8 by this technique. Reducing the voltage across branch 4 decreases the resistance of this branch while slightly increasing its sensitivity. Due to the position of branch 4 it is impossible to maintain the output voltage and decrease the voltage of branch four since the output voltage is expressed across this branch. The bottom graph of figure 5.4 and the graphs in figure 5.5 show some of the problems that negative resistance values cause when the technique has been applied to circuit 6. However, if the output voltage had been fixed, as shown in figure 5.6, the process produces a very high sensitivity reduction. This set of constraints only produce negative elements at a fractional branch voltage reduction figure of .45 (not shown) when, as usual, the sensitivity figures increase rapidly.

The top graph of figure 5.7 shows the effect of numerical noise in the output voltage for the case when the output voltage has supposedly been held constant. Figure 5.8 and the bottom graph in figure 5.7 show the case of three constraints applied to circuit 6. Once again a negative resistance value immediately appears and leads to unacceptable sensitivity figures. These examples show that although the process can produce spectacular sensitivity reductions, the method is plagued by the formation of negative resistance in the branches of interest. The fact that this

method is only applicable to D.C. networks is also a very severe restriction.



Circuit 6 modelled

by

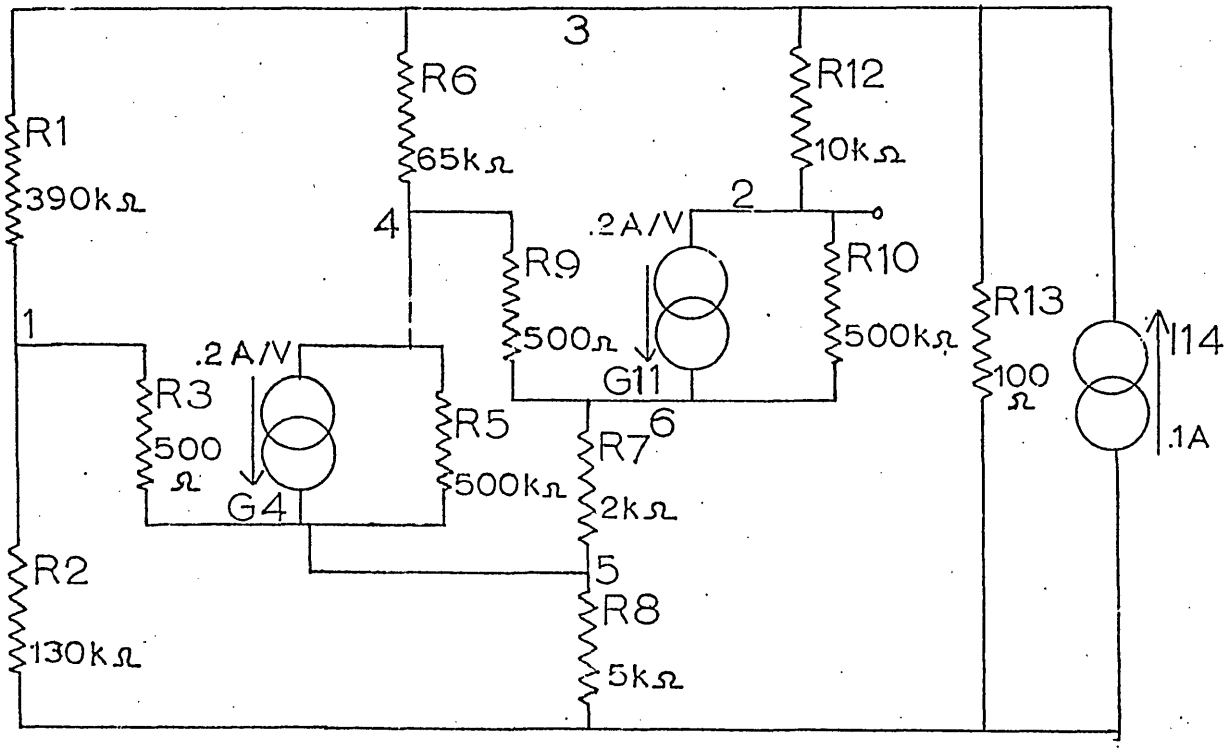
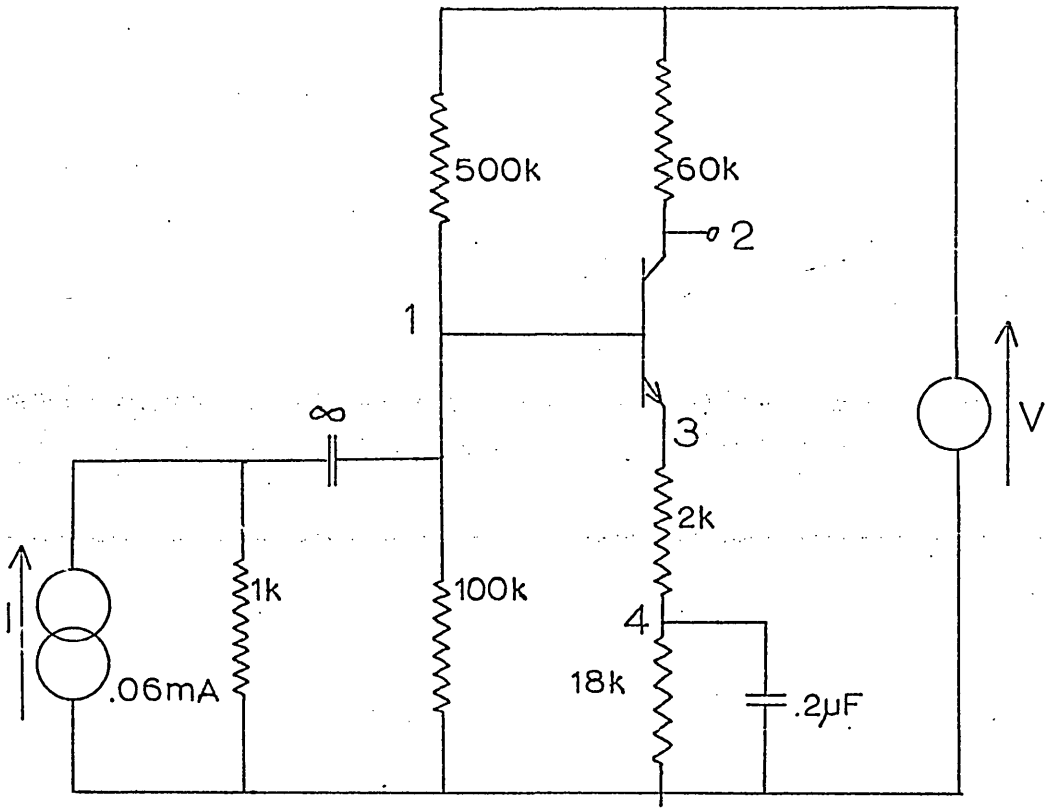


Figure 5.1



Circuit 8 modelled

by

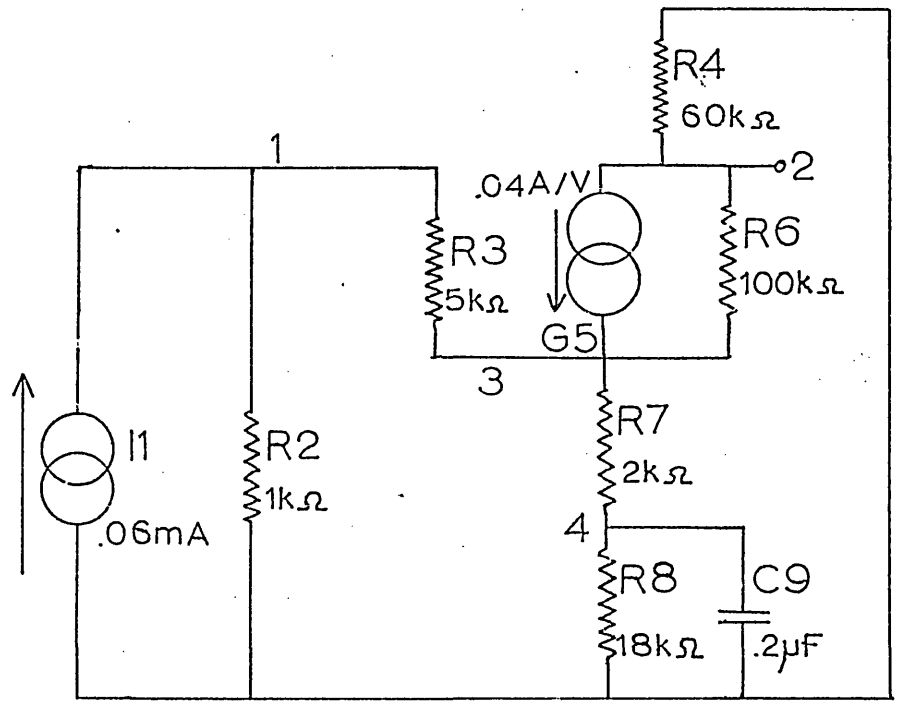
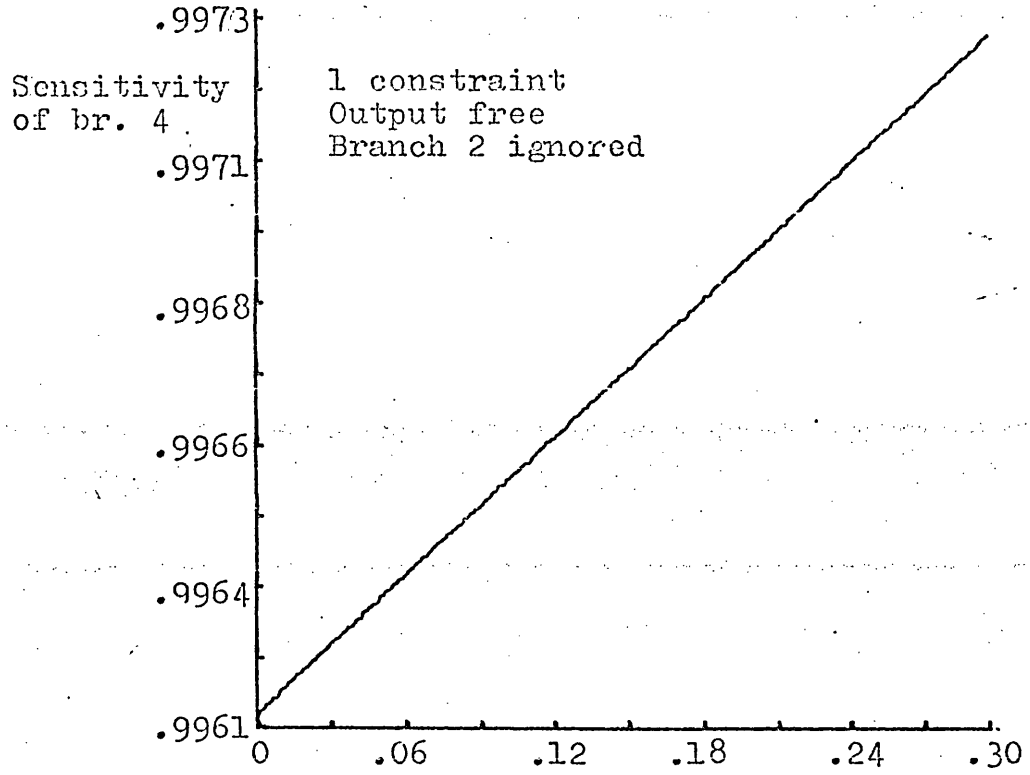
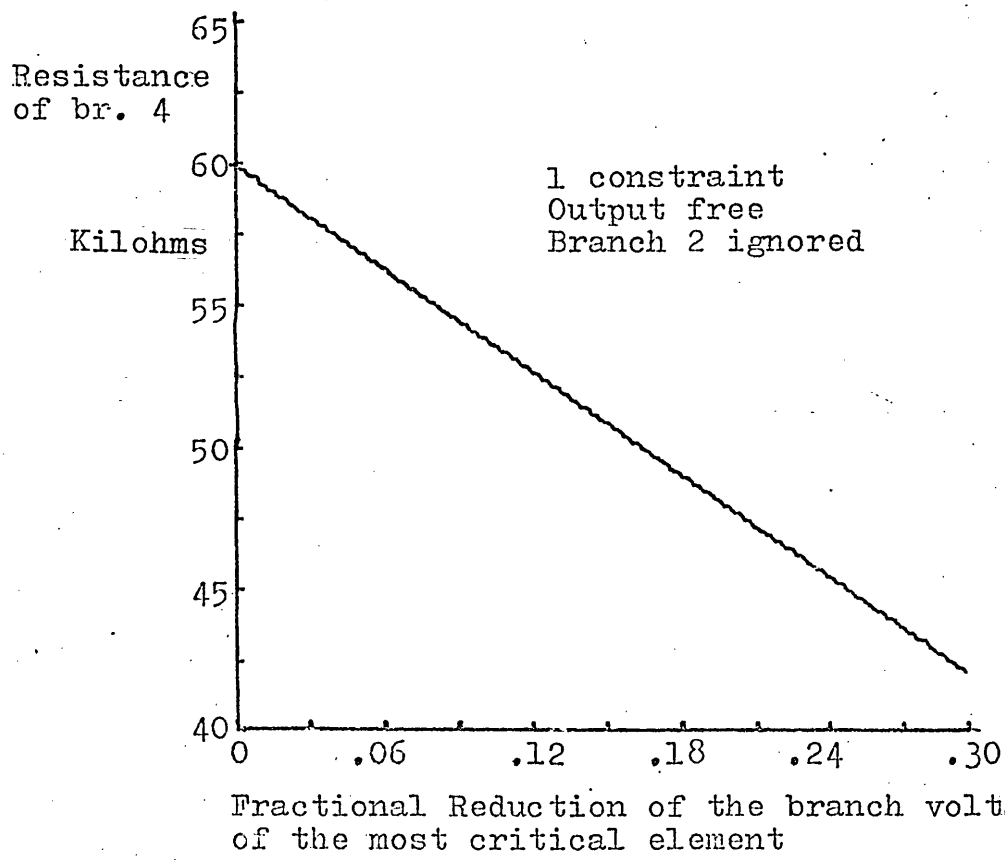


Figure 5.2

Sensitivity of branch 4 against constraint severity



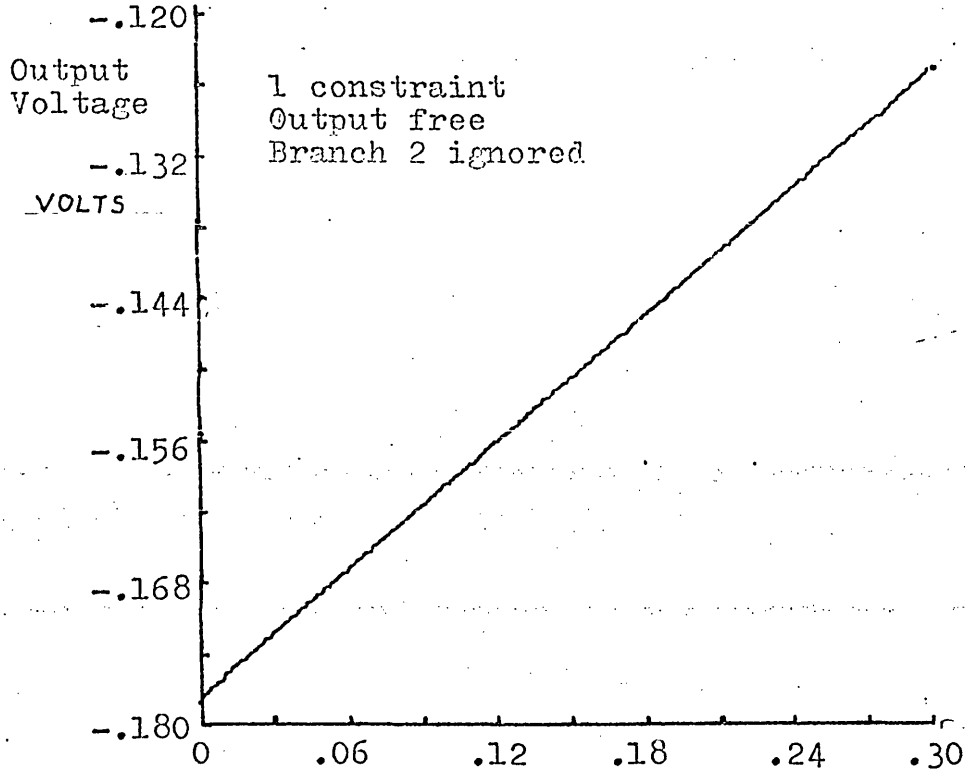
Resistance of branch 4 against constraint severity



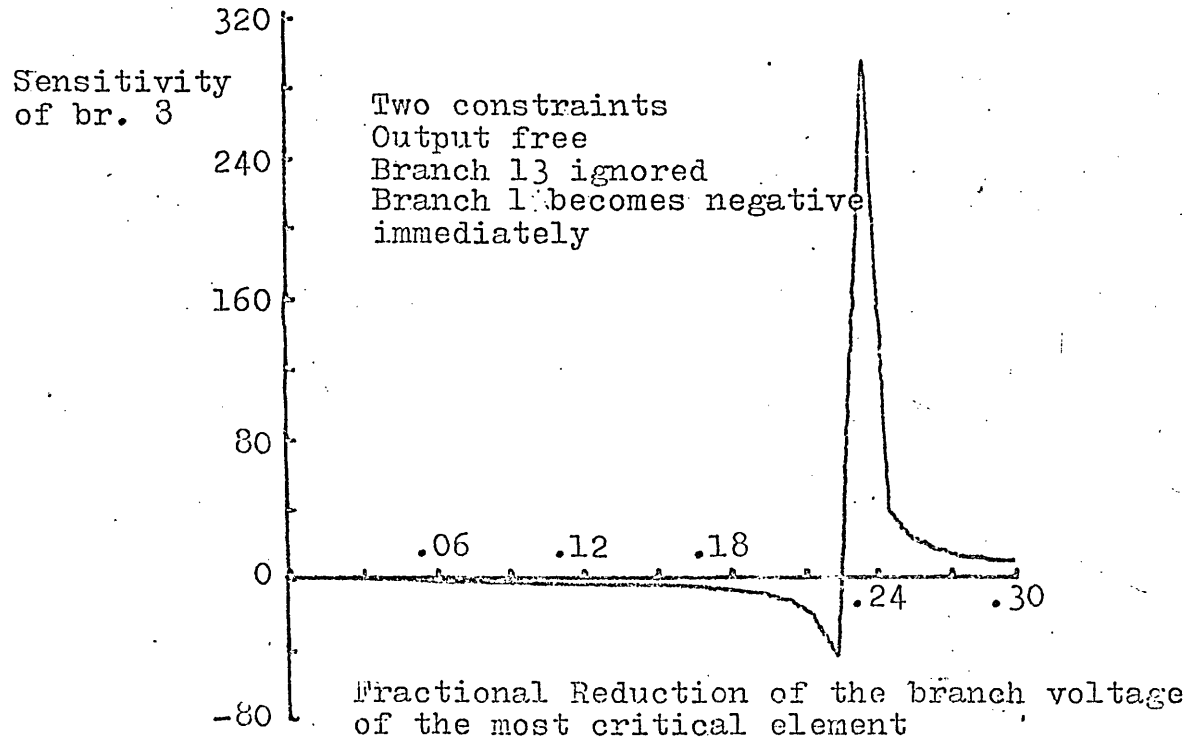
Resistance and Sensitivity of branch 4 plotted against constraint severity for circuit 8.

Figure 5.3

Output voltage of circuit 8 against constraint severity



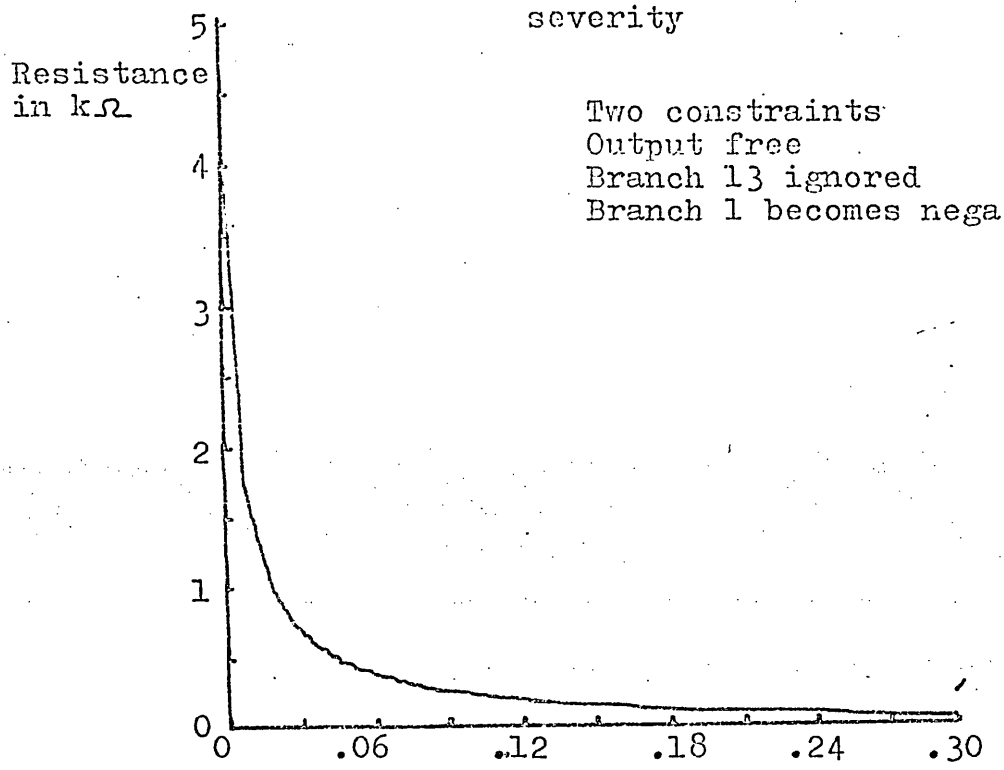
Sensitivity of branch 8 against constraint severity



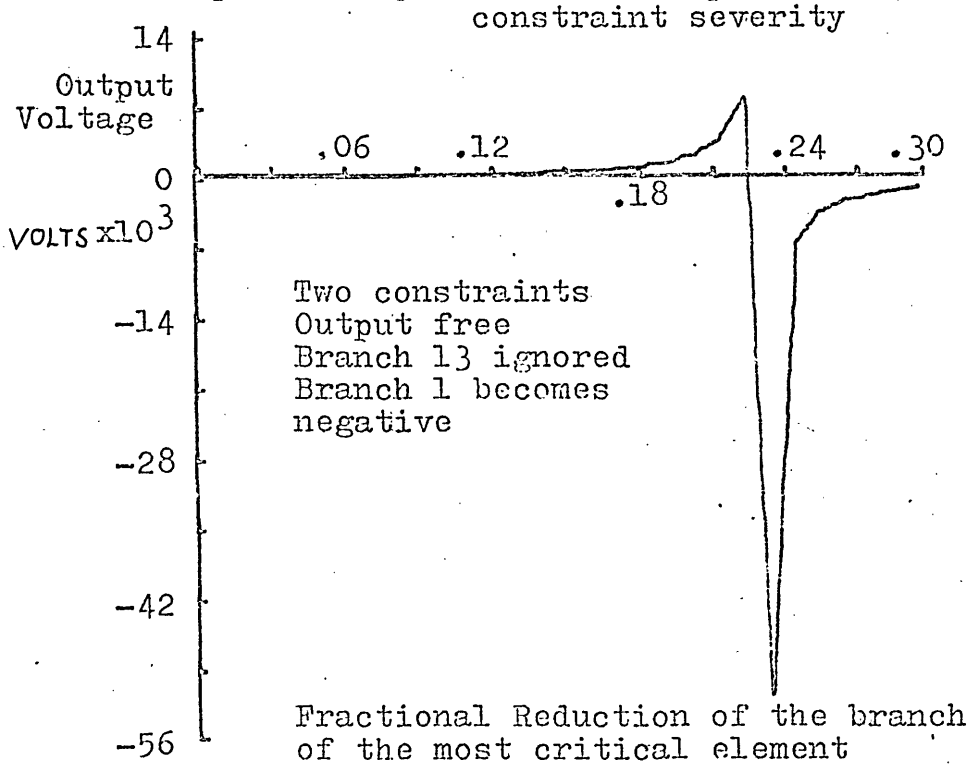
The top graph shows the variation of the output voltage of circuit 8 against constraint severity while the bottom graph shows the variation of the sensitivity of branch 8 against constraint severity for circuit 6.

Figure 5.4

Resistance of branch 8 plotted against constraint severity



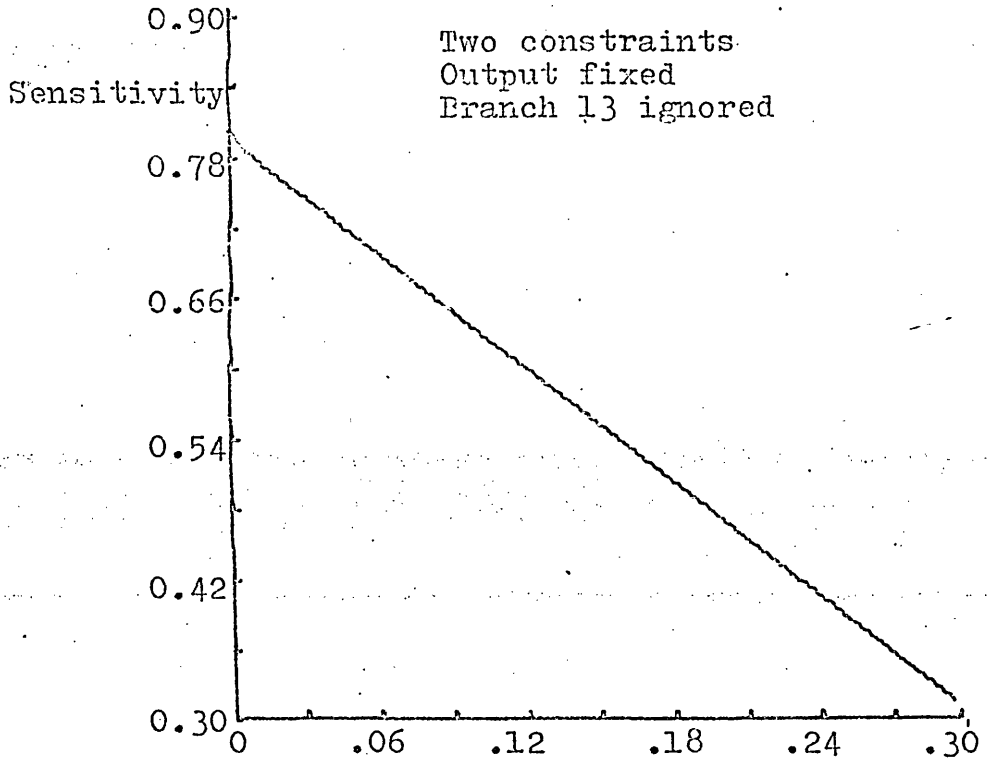
Output voltage of circuit 6 plotted against constraint severity



Resistance of branch 8 and the output voltage of circuit 6 is shown plotted against constraint severity for two constraints. Branch 1 goes negative immediately and the circuit exhibits peculiar behaviour for any degree of constraint.

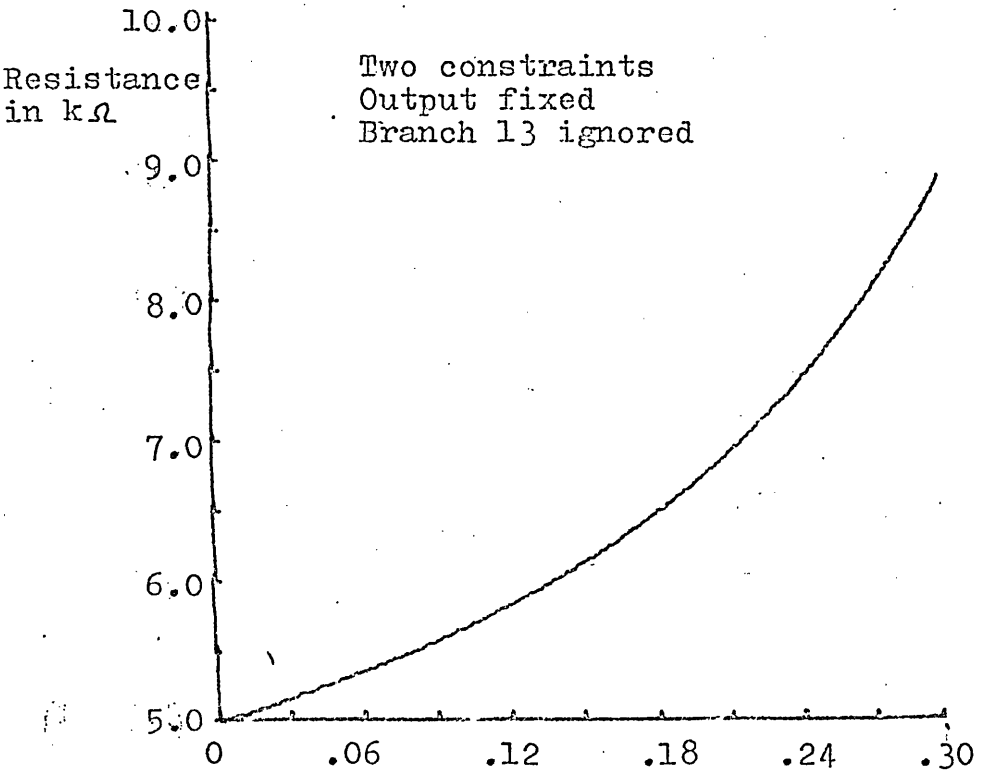
Figure 5.5

Sensitivity of branch 8 against constraint severity



Fractional Reduction of branch voltage of critical element

Resistance of branch 8 against constraint severity

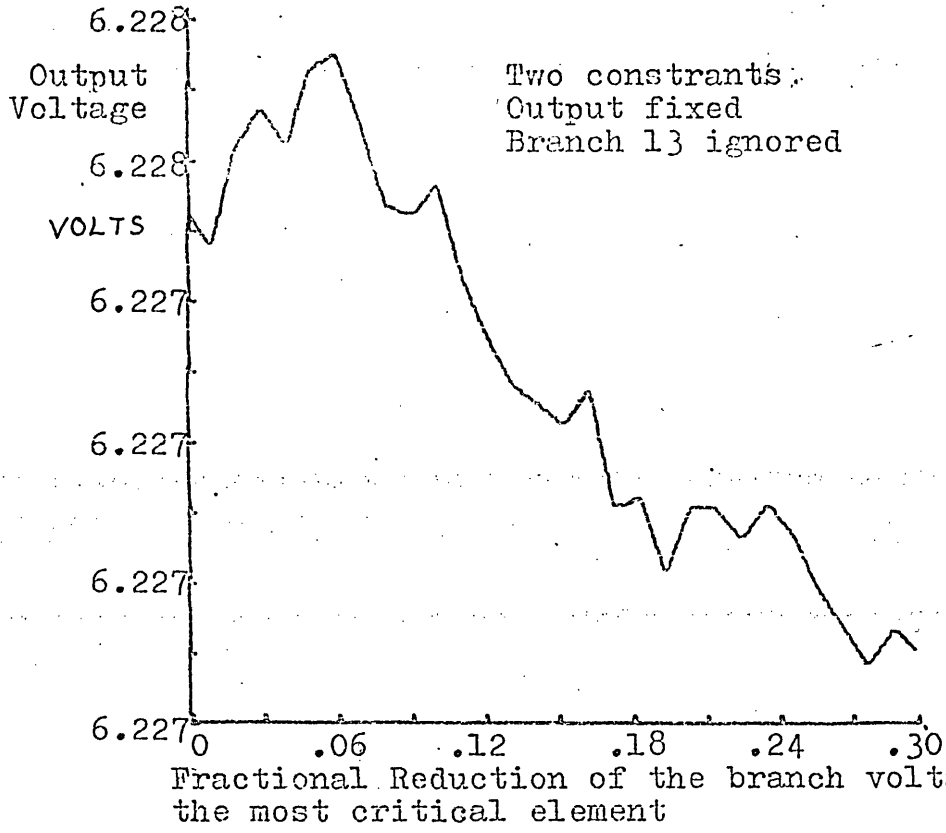


Fractional Reduction of branch voltage of critical element

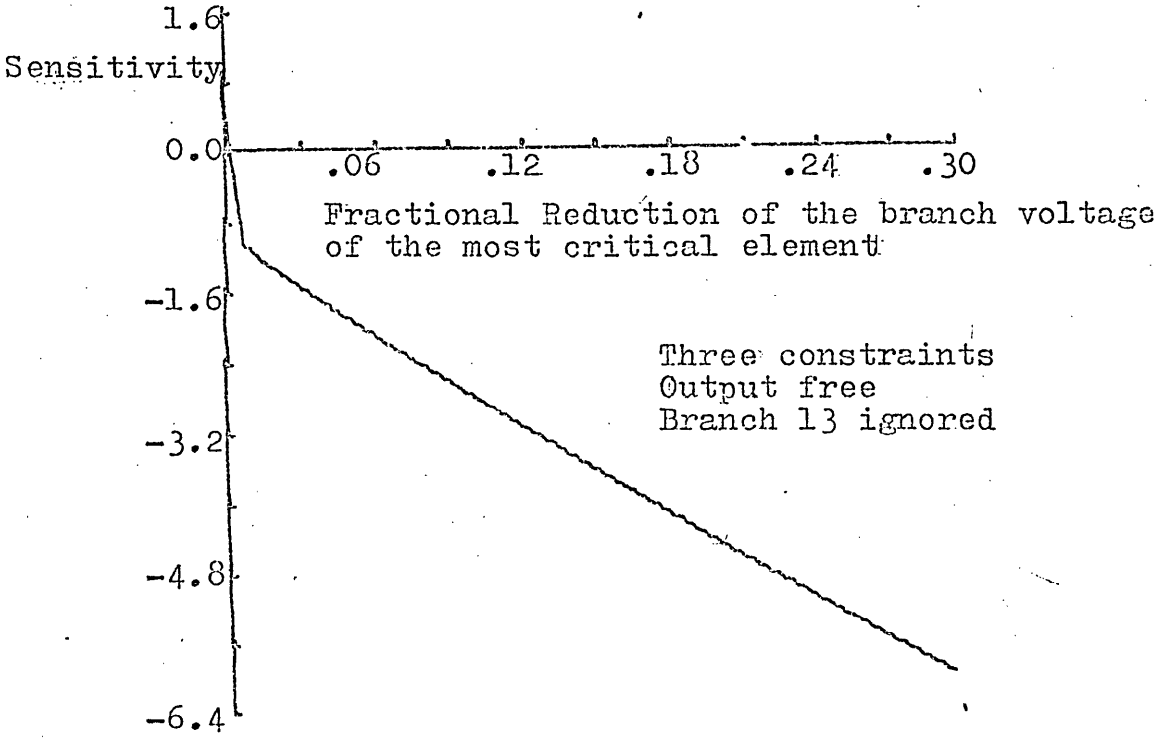
Graphs showing the sensitivity and resistance value of branch 8, the most sensitive branch, against the fractional reduction of this branch voltage.

Figure 5.6

Output voltage of Circuit 6 plotted against constraint severity.

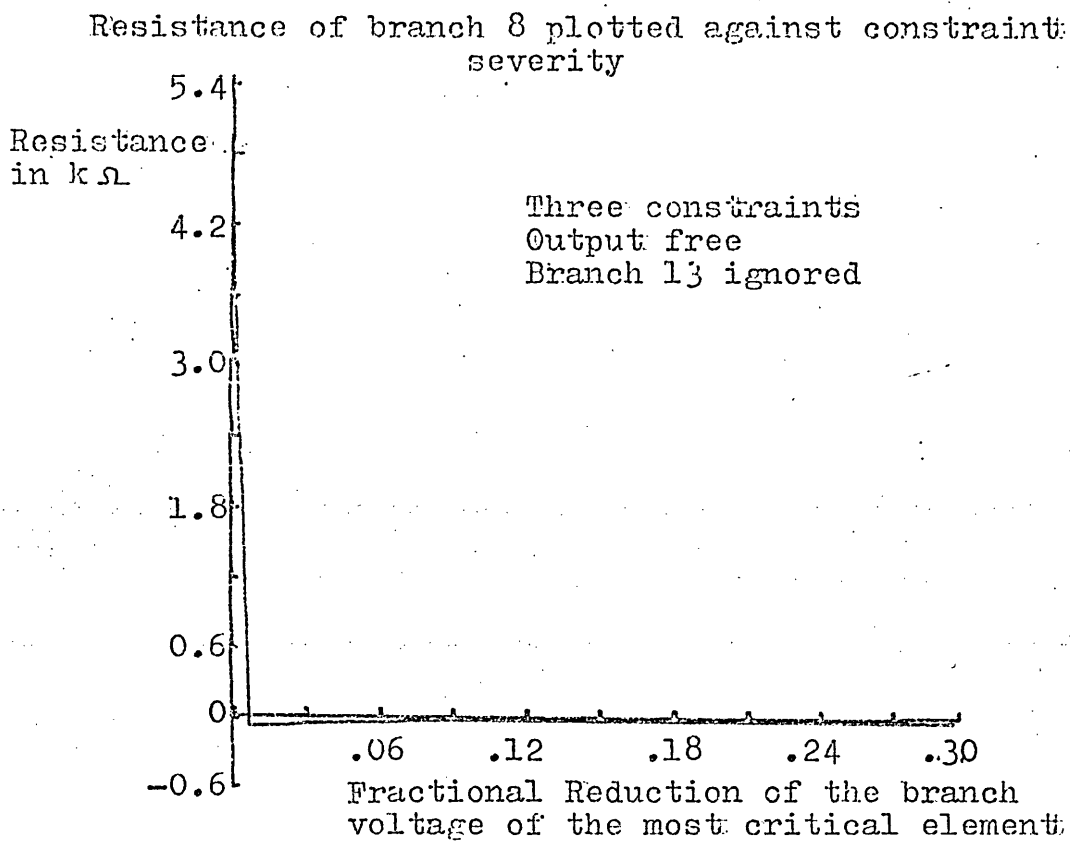


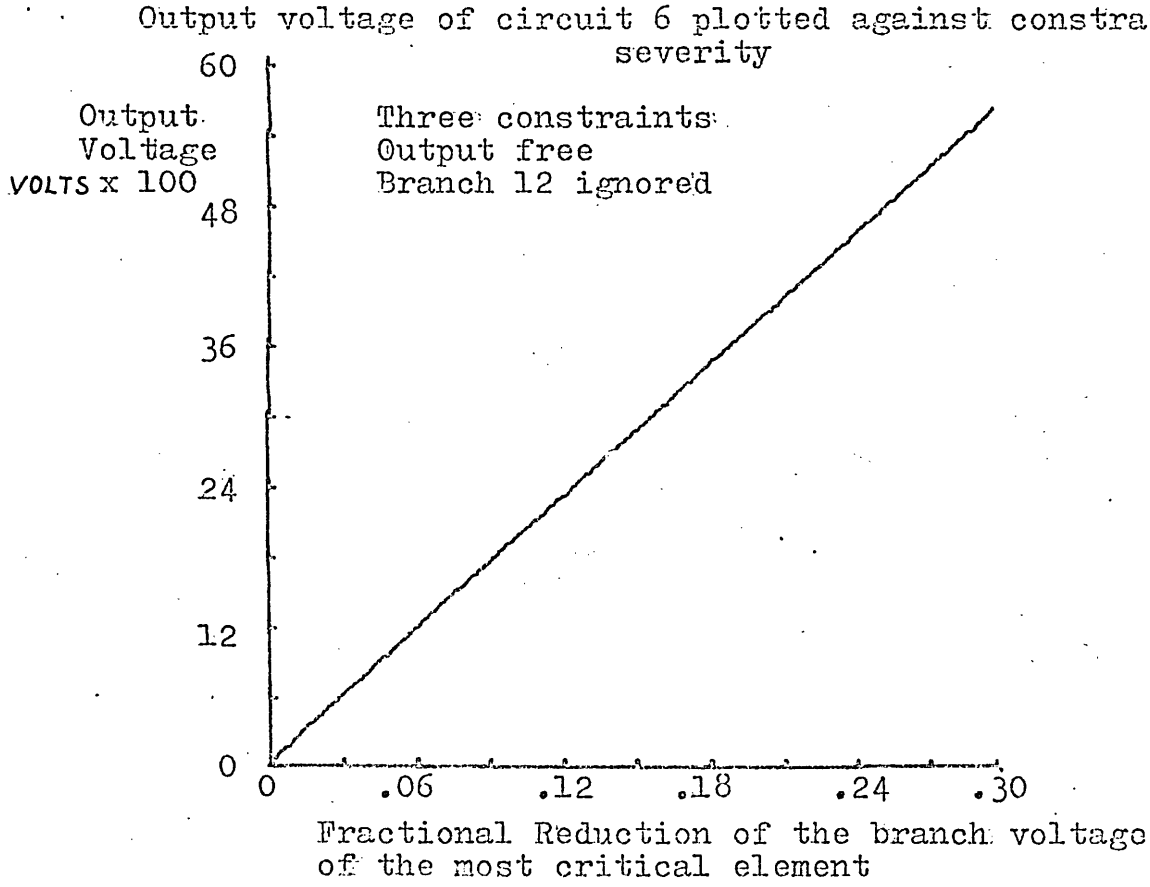
Sensitivity of branch 8 plotted against constraint severity



The upper graph shows the small variation in output voltage due to numerical error as the constraint severity is increased. Sensitivity against constraint severity is shown in the lower graph. Due to the formation of negative resistance in branch 8 this example is unsuccessful.

Figure 5.7





Resistance of branch 8, the most critical element, and the output voltage of circuit 6 plotted against constraint severity for three constraints.

Figure 5.8

Chapter 6

Optimisation

The initial design of a circuit usually will not meet the required specifications and must therefore be subjected to some form of optimisation. In this chapter an algorithm is proposed which alters circuit elements in such a way as to reduce (1) the high element sensitivities and (2) the discrepancy between the desired and actual circuit output. This algorithm uses the techniques developed in chapters 2 and 3 to calculate efficiently the first and second order differentials of the circuit output.

A problem in any optimisation scheme is the choice of a suitable cost function. This is because a vast number of alternatives exist as the cost function must accurately reflect the effect of a large number of parameters. In most circuits the element sensitivities have been found to lie between zero and unity and hence a suitable normalising factor for the output error must be found so that the cost function is relatively circuit independent. For most types of optimisation problem the output is of the same order of magnitude as the required output and so the output may be normalised to either the actual output or the required output. In order to avoid any anomalies the normalising factor, which is fixed at the start of the optimisation, is taken as the greater of the moduli of either the actual or desired output. The cost function, CF, used in the examples in this chapter is the sum of two components; one is the squared modulus of the output error normalised by the factor discussed above, and the

other is the sum of all the squared moduli of the element sensitivities. That is

$$CF = \frac{(V-\hat{V})(V-\hat{V})^*}{A^2} w_d + \sum_i S_i S_i^* \qquad - - - 6.1$$

where CF is the cost function, $\hat{V}$ is the desired response, V is the actual response, S_i the i^{th} branch sensitivity, A is the greater of $|V|$ and $|\hat{V}|$ and w_d is an output error weighting factor specified by the designer.

Optimisation of this cost function could be carried out by a standard software package which does not utilise analytically calculated differentials. It would, however, require a large number of function evaluations -circuit and sensitivity analyses in this case- which are very time consuming. An alternative, and far more efficient approach, is to use the methods developed in chapters 2 and 3 to calculate the cost function and its differentials. These differentials are 'analytically' calculated by the numeric substitution of the circuit variables into the relevant equations in chapters 2 and 3. This method of calculation of the differentials avoids the numerical error of perturbation techniques, a point of considerable importance in the prediction of search direction optima.

Steepest Descent Algorithm:

Since the so called 'steepest descent' algorithm is very simple conceptually, it will be used to demonstrate some of the advantages of using analytically calculated differentials. This algorithm searches along the direction of steepest descent until the cost function minimum has been found; the direction of steepest descent being that which is defined by altering the system parameters in

proportion to their first order differentials. At each search direction minimum the new direction of steepest descent is found and the procedure is repeated. When the first order differentials are sufficiently close to zero, the process is terminated. In general, the steepest descent algorithm is very effective when the parameters are some way from their optimum values but rapidly decreases in efficiency when approaching an optimum. This is due to the differentials becoming very small and hence inaccurate if perturbation techniques are being used to find these differentials. However analytically calculated differentials avoid this type of problem and so the steepest descent algorithm becomes far more useful.

General Considerations:

Clearly, no element values may be allowed to become negative during the course of optimisation. However care must be exercised in the method used for restricting element values to remain positive, constrained optimisation being generally a more 'thorny' problem than unconstrained optimisation. This is due to the discontinuities in the parameter values which tend to give rise to bizarre mathematical behaviour in the function space. To prevent 'brick wall' discontinuities continuing from one iteration to the next, element changes were limited to a predetermined proportion of their value at the start of a search direction. By restricting this proportionality factor to be less than one, the element values will always remain positive and never reach zero. In the examples shown at the end of this chapter this proportionality factor limiting the maximum element change was set to .8 by the program. In order to

allow the output information to be circuit independent, an internal scaling factor was used to normalise the search direction vector. This normalising factor is equal to the magnitude of the largest component of the search direction vector. The i^{th} element value along the search direction is given by

$$x_i = x_i^o \left(1 - \lambda \frac{D_i}{A^o} \right) \qquad - - - 6.2$$

where x_i^o and x_i are the element values at the start and at any position along the search direction respectively, A^o is the scaling factor, λ is the position factor along the search direction and D_i is the i^{th} component of the search direction vector.

Cost Function Differentials:

The cost function differentials are required for two reasons. These reasons are:

1) the need to define the search direction and;
2) the differentials are used in a technique for predicting each search direction minimum.

Consider the cost function differentiated with respect to an admittance y_i.

$$\frac{\delta CF}{\delta y_i} \equiv \frac{w_d}{A^2} \frac{\delta}{\delta y_i} (V-\hat{V})(V-\hat{V})^* + \sum_j \left(S_j \frac{\delta S_j^*}{\delta y_i} + S_j^* \frac{\delta S_j}{\delta y_i} \right) \qquad - - - 6.3$$

Since the differential of the output sensitivity of element j with respect to the admittance y_i is

$$\frac{\delta S_j}{\delta y_i} = \frac{\delta}{\delta y_i} \left(\frac{y_j}{V} * \frac{\delta V}{\delta y_j} \right) \qquad - - - 6.4,$$

which may be expanded as

$$\frac{\delta S_j}{\delta y_i} = \frac{y_i}{V} \left(\frac{\delta^2 V}{\delta y_i \delta y_j} - \frac{1}{V} \frac{\delta V}{\delta y_i} \frac{\delta V}{\delta y_j} + \delta_{ij} \frac{1}{y_i} \frac{\delta V}{\delta y_j} \right) \qquad - - 6.5,$$

this term may be evaluated by the formulae of chapters 2 and 3. The differentials of the complex conjugate terms are easily found since the term y_i is either purely real or imaginary and hence for any complex quantity t

$$\frac{dt^*}{dy_i} = (-1)^n \left(\frac{dt}{dy_i}\right)^* \qquad \text{- - - 6.6}$$

where n = 2 if y_i is real

and n = 1 if y_i is imaginary.

Therefore equation 6.3 is simply evaluated using the techniques developed in previous chapters. Although only the first differential of the cost function is being found, the expression involves second order differential terms of the output voltage.

Normalisation:

If the differentials of the cost function are normalised with respect to the element admittances, a symmetrical equation in the element admittances is obtained. As a result, the amount of computation necessary to evaluate the summation part of equation 6.3 is approximately halved (see equation 6.14). Also, normalised differentials apply not only to the admittance plane, but, with a sign change, to the impedance plane as well. Hence the programming in the optimisation part of the algorithm is considerably simplified as element values rather than admittance values may then be altered.

Consider the partial expansion of equation 6.3 as

$$\frac{\delta CF}{\delta y_i} = \frac{w_d}{A^2}\left\{(V-\hat{V})^* \frac{\delta V}{\delta y_i} + (V-\hat{V})\frac{\delta V^*}{\delta y_i}\right\} + \sum_j \left(S_j \frac{\delta S_j^*}{\delta y_i} + S_j^* \frac{\delta S_j}{\delta y_i}\right) \quad - 6.7$$

Now for ease of analysis let

$$\frac{\partial P}{\partial y_i} = \tilde{V}^* \frac{\partial V}{\partial y_i} + \tilde{V} \frac{\partial V^*}{\partial y_i} \qquad - - - 6.8$$

where $\tilde{V} = V - \hat{V}$ and P is a dummy variable defined by this equation. Substitution of equation 6.6 gives

$$\frac{\partial P}{\partial y_i} = \tilde{V}^* \frac{\partial V}{\partial y_i} + \tilde{V} (-1)^n \left(\frac{\partial V}{\partial y_i} \right)^* \qquad - - - 6.9$$

where $n = 1$ if y_i is imaginary

and $n = 2$ if y_i is real.

Expanding $\tilde{V}$ as A + jB and $\frac{\partial V}{\partial y_i}$ as C + jD in equation 6.9, equation 6.10 is obtained.

$$\frac{\partial P}{\partial y_i} = (A-jB)(C+jD) + (A+jB)(-1)^n(C-jD) \qquad - - - 6.10$$

$$= 2 (AC + BD) \text{ for } n = 2 \qquad - - - 6.11a$$

$$= 2j(DA - BC) \text{ for } n = 1 \qquad - - - 6.11b$$

Now normalise this differential by multiplying through by y_i

i.e. $$y_i \frac{\partial P}{\partial y_i} = 2G (AC + BD) \quad \text{for } y_i = G \qquad - - - 6.12a$$

and $$y_i \frac{\partial P}{\partial y_i} = 2S (BC - DA) \quad \text{for } y_i = jS \qquad - - - 6.12b.$$

This same result is also obtained if the real parts and imaginary parts of $\tilde{V}$ and $y_i \frac{\partial V}{\partial y_i}$ are multiplied, summed and then multiplied by 2.

i.e. $$y_i \frac{\partial P}{\partial y_i} = 2 \, Re(\tilde{V}) Re\left(y_i \frac{\partial V}{\partial y_i}\right) + Im(\tilde{V}) Im\left(y_i \frac{\partial V}{\partial y_i}\right) \qquad - - - 6.13$$

This normalisation is especially useful for calculating the differentials of the sensitivity terms since $\frac{\partial S}{\partial y_i}j$ is not symmetrical in the admittance terms (see equation 6.5) whereas

$$y_i \frac{\partial S_j}{\partial y_i} = \frac{y_i y_j}{V} \frac{\partial^2 V}{\partial y_i \partial y_j} - \frac{y_i y_j}{V^2} \frac{\partial V}{\partial y_i} \frac{\partial V}{\partial y_j} + \delta_{ij} \frac{y_i}{V} \frac{\partial V}{\partial y_i} \qquad - - 6.14$$

is. Therefore only the $(n^2 + n)/2n^2$ fraction of the terms

in $y_i \frac{\partial S_j}{\partial y_i}$ need to be calculated which results in a substantial saving of computer time. Note that care must be taken in the case of resistors or inductors since the normalised differential changes sign in the transformation between admittance and element values.

Function Minimisation along a Search Direction:

Consider minimising the cost function along a direction vector $[D]$. The value of the i^{th} parameter, x_i at any position along the search direction is given by equation 6.2, rewritten below as equation 6.15

$$x_i = x_i^o \left(1 - \lambda \frac{D_i}{A^o} \right) \qquad - - - 6.15$$

where x_i and x_i^o are the present and initial values of x_i, λ is the scalar position parameter, A^o the circuit scaling factor and D_i is the i^{th} component of the direction vector. The change of independent variable from y_i to x_i is to emphasize that this variable is not necessarily an admittance. This is important as the normalised differentials permit the element values rather than the element admittances to be used in the optimisation algorithm.

Differentiation of equation 6.15 with respect to λ is then a necessary stage in deriving the expression for the variation of the cost function with λ.

$$\frac{dx_i}{d\lambda} = -x_i^o \frac{D_i}{A^o} \qquad - - - 6.16$$

However the differentiation of the cost function with respect to λ may be expressed as

$$\frac{d\,CF}{d\lambda} = \sum_i \frac{\partial CF}{\partial x_i} \frac{dx_i}{d\lambda} \qquad - - - 6.17$$

Therefore, by the substitution of equation 6.16 in equation 6.17, equation 6.18 is obtained

$$\frac{dCF}{d\lambda} = -\frac{1}{A^{\circ}} \sum_i \frac{\partial CF}{\partial x_i} x_i D_i \qquad - - - 6.18$$

which may easily be evaluated since the cost function differentials with respect to any variable, x_i are known. Note that in the steepest descent algorithm the i^{th} search direction component, D_i, is $\frac{\partial CF}{\partial x_i}$.

Now change the parameter values along the search direction by choosing a suitable value, ε, for λ (see equation 6.15). Re-analysis of the circuit to find the new cost function and its differentials then gives an estimate of the second order differential of the cost function with respect to λ as

$$\frac{d^2CF}{d\lambda^2} \doteq \frac{\frac{dCF}{d\lambda}\Big|_{\lambda=\varepsilon} - \frac{dCF}{d\lambda}\Big|_{\lambda=0}}{\varepsilon} \qquad - - - 6.19.$$

If the cost function is represented as a quadratic function plus an error term, i.e.

$$CF = K_1 + K_2 + K_3\lambda^2 + \xi \qquad - - - 6.20$$

where ξ is the error term and K_1, K_2 and K_3 are arbitrary constants, then the differential of the cost function with respect to λ is

$$\frac{dCF}{d\lambda} \doteq K_2 + 2K_3\lambda \qquad - - - 6.21.$$

Hence the second differential of the cost function with respect to λ is given by

$$\frac{d^2CF}{d\lambda^2} \doteq 2K_3 \qquad - - - 6.22.$$

However the values of $\frac{dCF}{d\lambda}$ and $\frac{d^2CF}{d\lambda^2}$ at $\lambda=0$ are known and

therefore the values of the constants K_2 and K_3 are also known. Since a maximum, minimum or saddle point occurs when $\frac{dCF}{d\lambda} = 0$, the value, λ_{opt}, associated with the minimum of the cost function along the search direction is given by

$$\lambda_{opt} \doteq - \frac{K_2}{2K_3} \qquad - - - 6.23.$$

$$\text{Since } 2K_3 \doteq \frac{d^2CF}{d\lambda^2} \doteq \frac{\left.\frac{dCF}{d\lambda}\right|_{\lambda=\varepsilon} - \left.\frac{dCF}{d\lambda}\right|_{\lambda=0}}{\varepsilon} \qquad - - - 6.24$$

$$\text{and} \quad K_2 \doteq \left.\frac{dCF}{d\lambda}\right|_{\lambda=0} \qquad - - - 6.25$$

$$\text{then} \quad \lambda_{opt} \doteq - \frac{\varepsilon \left.\frac{dCF}{d\lambda}\right|_{\lambda=0}}{\left.\frac{dCF}{d\lambda}\right|_{\lambda=\varepsilon} - \left.\frac{dCF}{d\lambda}\right|_{\lambda=0}} \qquad - - - 6.26.$$

The initial value for ε is specified by the designer but each subsequent ε is set equal to the λ_{opt} value for the previous search direction. Note that the value of ε does not need to be small since if the behaviour of the cost function is quadratic, then the second order differential is constant. If the behaviour deviates from quadratic, then a chord value of the second order differential will give a better result in the prediction of the search direction minimum than will the true value of this differential at the start of the search direction. This is because ε is usually a reasonable estimate of the value, λ_{opt}, of λ which minimises the cost function along the search direction. Therefore the chord value of the second order differential of the cost function with respect to λ over this interval will give a very good estimate of λ_{opt} for both cases.

Algorithm:

After the initial circuit information has been fed

into the computer and the circuit analysed for both response voltages and sensitivities, the initial optimisation parameters are then required. These initial parameters are:

1) the number of iterations;
2) the required output, $\hat{V}$;
3) the parameter alteration weights, if required (these weights modify the search direction)*;
4) special sensitivity weights, if required;*
5) the output error weighting and;
6) the initial value of ε.

The algorithm now consists of eight distinct stages per iteration. These stages are:

1) Analysis of the circuit by inversion of the nodal admittance matrix;
2) Evaluation of the cost function and its first order differentials;
3) Alteration of the element values along the direction of steepest descent by setting λ in equation 6.15 equal to ε (see preceding section);
4) Re-analysis of the circuit with the temporary element values found by the previous stage;
5) Re-evaluation of the cost function and its first order differentials for the temporary element values;
6) Estimation of the value, λ_{opt}, which minimises the cost function along the search direction;
7) Subjection of the estimated value of λ_{opt} to three simple tests to check the validity of the quadratic convergence assumption (see the section entitled Strategy);

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* This is discussed in a subsequent section in this chapter

8) Alteration of the circuit element values from the values held at the start of the search direction using the value of λ_{opt} in equation 6.15.

Experimental Verification of the Quadratic Assumption:

Circuit 2, shown in figure 6.1, was the subject of a series of one dimensional line searches. At the start of each search direction the cost function, CF, and its first order differentials were evaluated to define the search direction. The distance to be searched was then subdivided by forty and the cost function was evaluated at each of these points. Then, using the first and last points of the distance searched, the value, λ_{opt}, of λ to minimise the cost function was predicted by the use of equation 6.26. The initial search interval was provided by the user but each subsequent interval was determined as the product of the preceding interval's predicted λ_{opt} and a factor of 1.2. Each search direction was started from the true minimum of the cost function found along the previous direction to ensure continuity in the cost function values.

The results for the first ten search directions are shown in figures 6.2 to 6.6. As shown on the graphs, the desired output was -1.2 - j22.8 at a frequency of 1kHz, which is the same output voltage as the initial circuit has at this frequency. Note that for the top graph in figure 6.3 no minimum was reached as the search interval, specified from the previous search interval, was too small. All the predicted line minima are very close to the true minima which is a justification of the proposed method. The cost

function is reduced from 2.67 to 2.295 by these ten iterations and another twenty iterations will reduce the cost function to 1.54 (see the top graph in figure 6.7). Due to the low output error weighting factor the output changes from $-1.2 - j22.8$ to $-5.8 - j19.8$ after these ten line searches.

Computational Effort:

Circuit 2 was analysed by inverting the nodal admittance matrix since the inverted admittance matrix is needed to evaluate the cost function, CF, and its first order differentials. The time taken to analyse the circuit by this technique was .76 seconds while the evaluation of the cost function and its differentials took another 1.88 seconds. If the nodal admittance matrix was not inverted, then two circuit analyses would have to be made just to find the element sensitivities (see chapter 2). This in itself would take about 67% of the time needed to analyse the circuit by inverting the admittance matrix. However the problem still arises of how to find the first order differentials of the cost function. Suppose that perturbation techniques were now used to find these differentials, then, if one solution of the circuit's simultaneous equations is taken as a basic unit of time, two extra time units are needed for each differential. Therefore, for circuit 2 which has eleven branches excluding the driving source, twenty four time units are needed to analyse and evaluate the cost function and its first order differentials. However, by inverting the circuit matrix, the circuit analysis and evaluation of the cost function and its differentials will take only ten time units.

A factor which is very important on a small word size

computer such as a PDP 15 is that the perturbation technique only gives an estimate of the differentials. Imprecision in the cost function differentials disallows the possibility of predicting the search direction minima with any degree of accuracy. The amount of time needed for the evaluation of the cost function and its differentials is proportional to a factor of $\frac{1}{2}n^2$ where n is the number of branches. Since the time taken for the circuit analysis is approximately proportional to the 2.5 power of the circuit node size, the circuit analysis takes a proportionally longer time than the evaluation of the cost function and its differentials for increasing circuit size. This factor was observed in the examples run.

Strategy:

The value of λ_{opt}, predicted from equation 6.26, must be subjected to three simple tests to check the validity of the predicted minima. These tests are:

1) does the value of λ_{opt} exceed .8, if so set ε to .8;
2) if λ_{opt} is negative, is the value of the cost function at $\lambda=\varepsilon$ greater than the cost function at the start of the search direction, if so divide ε by ten and repeat iteration stages 3 to 7 (see Algorithm);
3) if λ_{opt} is negative, does the value of the cost function at the start of the search direction exceed the value of the cost function at $\lambda=\varepsilon$, if so set λ_{opt} equal to ε and proceed.

Test 1 limits the change in any element to eighty percent of its value at the start of each iteration. This prevents the element values from going negative as well as

increasing very rapidly. The situations described in the last two test conditions are rarely encountered but are necessary to ensure stability in the algorithm. If the algorithm converges very rapidly, then the step size from a previous successful iteration may be far too large for the next iteration and hence will cause the quadratic assumption to break down. This is self correcting providing the predicted value of λ_{opt} is positive. However, if λ_{opt} is negative, then the second test will apply since the value of the cost function at $\lambda=\varepsilon$ will normally be greater than the cost function at $\lambda=0$. In this case the value of ε is divided by ten and the prediction process for λ_{opt} repeated.

The third test only comes into operation when the circuit being optimised is a long way from its optimum values. Very occasionally the cost function versus λ surface is convex rather than concave and so the negative value of λ_{opt} is predicting the position of a maximum rather than a minimum. Therefore λ_{opt} is assumed to be equal to ε and the algorithm is continued.

Optimisation Examples:

The proposed algorithm was used to optimise several circuits, the results of which are shown in diagrams 6.7 to 6.19. In figure 6.7 the two graphs show the result of optimising circuit 2, shown in figure 6.1, for the initial conditions shown on the top graph of figure 6.7. After decreasing the cost function smoothly for the first eighteen iterations there is a sharp increase in the rate of reduction of the cost function. This sharp increase in the cost function reduction, which is accompanied by a corresponding

increase in the step length along each search direction, is maintained until the cost function reduction levels off at the twenty fifth iteration. The final element values, all of which have been allowed to vary, are given in the circuit diagram in figure 6.8.

Figure 6.9 shows the results of another optimisation of circuit 2. In this case, however, the desired output is now different from the initial output of this circuit and an output error weighting factor of four has been introduced. As shown in the top graphs of figures 6.9 and 6.10, the initial reduction in the cost function is due to the reduction in output error only. The element values resulting from this optimisation are given in the circuit diagram in figure 6.10.

Circuit 7, which is shown in figure 6.11, was now optimised by the proposed algorithm. The results of this optimisation are shown in figures 6.12 and 6.13. Unlike circuit 2, the cost function converged very rapidly to near its final value. This rapid convergence of the cost function is more typical of the limited number of circuits which have been optimised than the more complex behaviour of circuit 2.

In the above examples all the elements have been allowed to vary. This is suitable for a theoretical investigation but in a practical problem it is often required to hold some of the network elements constant. Hence provision has been made for the designer to be able to fix the value of certain elements such transistor parameters, for example branches 5, 8 and 9 in circuit 2

(see figure 6.1) by altering the element weights. Mathematically this is achieved by setting the relevant components of the search direction vector $[D_i]$ to zero (see equation 6.15).

Circuit 2 was subjected to a two stage optimisation process with branches 2, 5, 8 and 9 held constant. The first stage consisted of specifying the initial circuit output as the desired output but only applying a low output error weighting factor of unity. This weighting factor was then increased to five for the second stage of the optimisation. Figures 6.14 to 6.17 show the relevant quantities of both stages of this optimisation. A phenomena worthy of further investigation is that, on the 25^{th} iteration of the first stage, a program limited step size is accompanied by an increase in cost function and an even greater step reduction in the sum of the squared sensitivities (see the graphs in figure 6.14 and the top graph of figure 6.16). From the graphs in figure 6.16 it is seen that the second stage of the process rapidly increased the sum of the squared sensitivities although the final result was half way between the initial and final sensitivity figures for the first stage.

Another desirable feature in an optimisation scheme of this type is the ability to weight the contribution, to the cost function, of each element sensitivity. A common set of elements which are required to have low sensitivities are the elements comprising a transistor model. This is because these parameters have a strong tendency to vary widely from one device to another. Inclusion of an element sensitivity weighting factor

does not present any mathematical difficulties in either the evaluation of the cost function or its differentials. Since these weighting factors are set by the designer they are independent of the element values. Therefore equation 6.3, which expresses the differentials of the cost function, is modified to

$$\frac{\partial CF}{\partial y_i} = \frac{w_d}{A^2} \frac{\partial}{\partial y_i} \left\{ (V-V)(V-V)^* \right\} + \sum_j w_j^s \left(S_j \frac{\partial S_j^*}{\partial y_i} + S_j^* \frac{\partial S_j}{\partial y_i} \right) \quad - 6.27$$

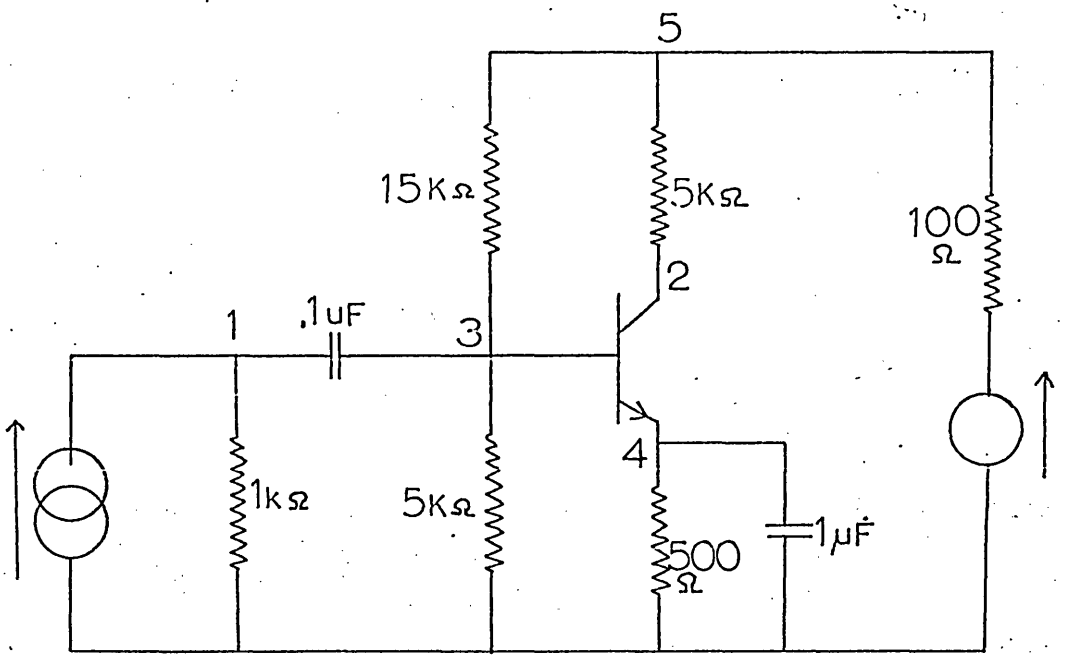
where w_j^s is the sensitivity weighting factor on the j^{th} element.

Circuit 2 has been optimised holding branches 2, 5, 8 and 9 constant and with sensitivity weights of zero on branch 2 and ten on branches 5, 8 and 9. The cost function and step size versus iteration are shown in figure 6.18 while the sum of the squared sensitivities multiplied by their weights and the resultant circuit are shown in figure 6.19. Figure 6.20 shows the tabulated node voltages and element sensitivities for the original and final circuits, as well as providing an example of the format of an on-line console session with the algorithm. The underlining of certain data shows which information has been supplied to the program by the designer. This algorithm was implemented on a PDP 15 computer and the times mentioned in this chapter refer to this machine.

Conclusions:

The proposed algorithm has been shown to be very effective in reducing the cost function specified in equation 6.1. As each search direction usually only requires two function and derivative evaluations, the method is very

efficient in its use of computer time. Hence small realistic circuits such as Circuits 2 and 7 may be optimised on slow computers such as a PDP 15; each iteration for Circuit 2 takes approximately six seconds of computer time. The process may be speeded up if the elements whose values are required to remain constant do not have their cost function differentials calculated. As noted earlier, the reduction in the output error contribution to the cost function is swiftly obtained while effective sensitivity reduction continues for considerably longer. A very interesting point is that the optimisation of Circuit 2 requires far more iterations than that required for Circuit 7 but Circuit 7 is a more complex circuit than 2. One possible explanation is that Circuit 7 was a commerically designed circuit from a tape recorder whereas Circuit 2 was an untested design by the author. In the author's opinion, the stability and convergence rates obtained with this proposed algorithm justify the extension of the program to handle a number of frequency points.



CIRCUIT 2

modelled by

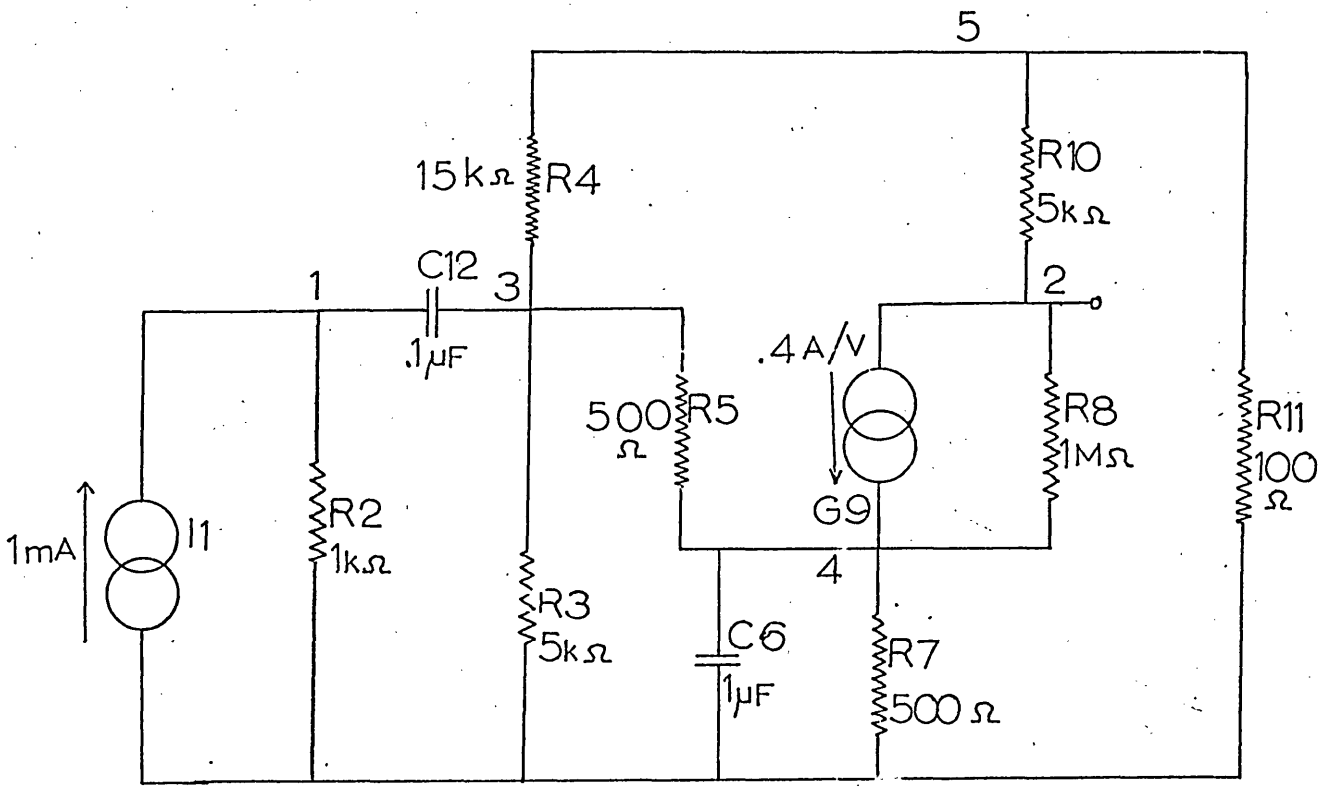
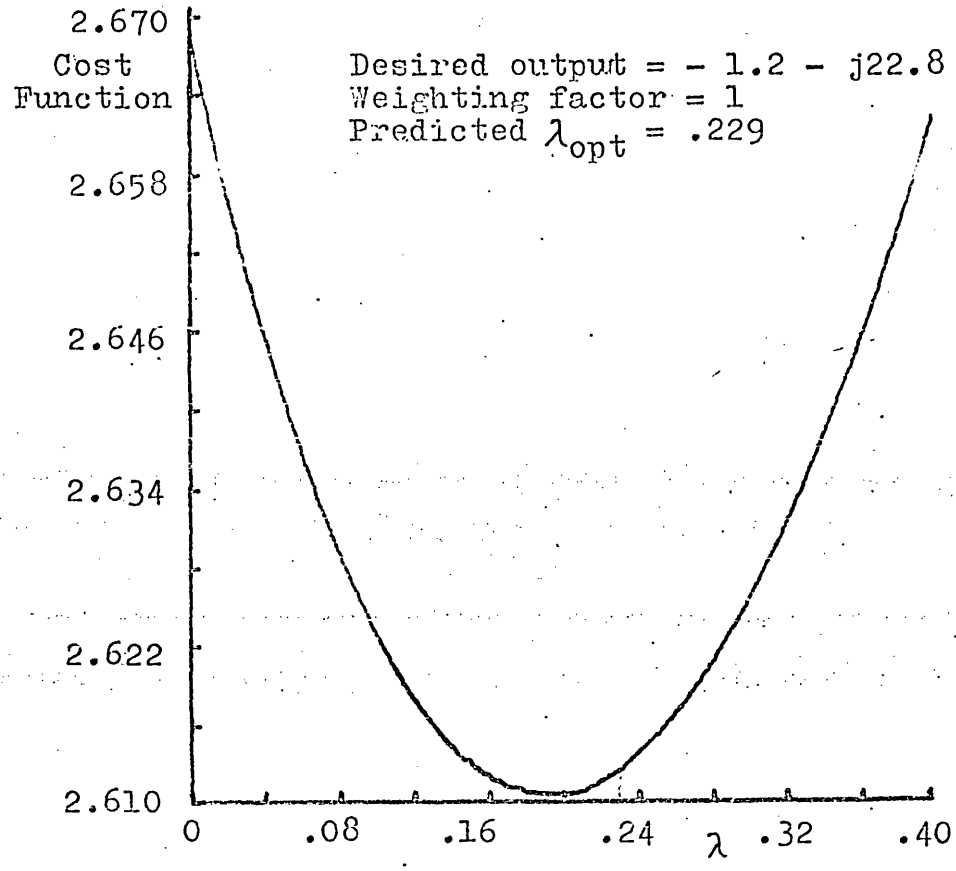
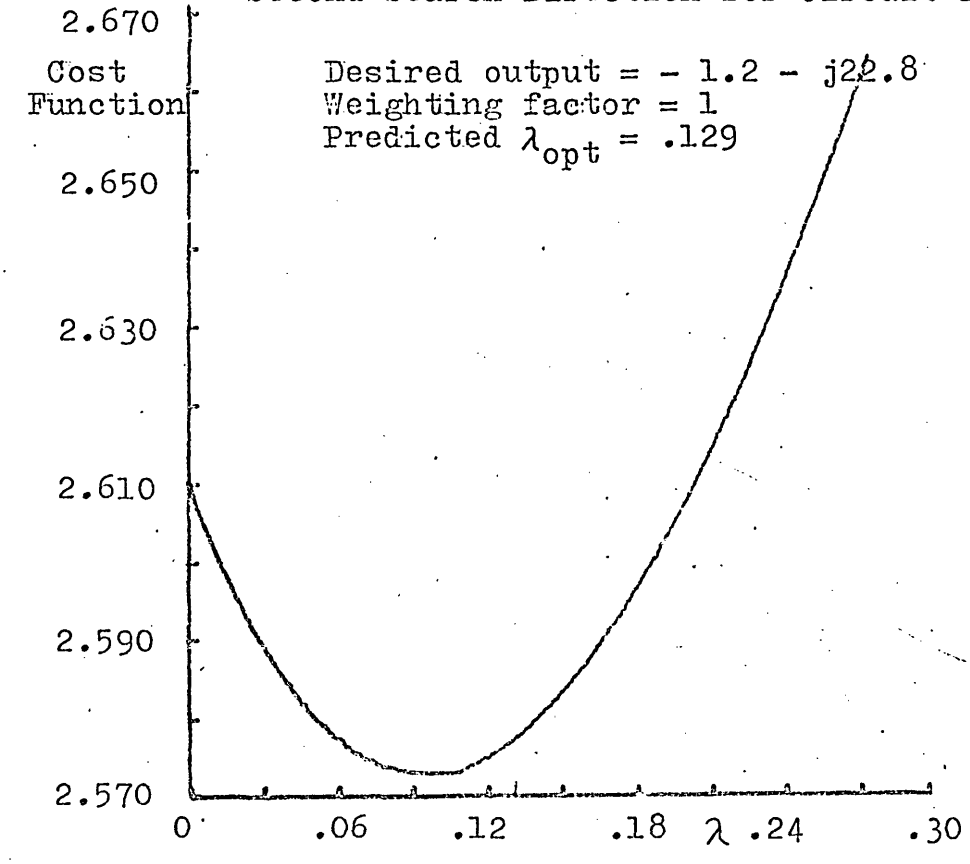


Figure 6.1

First Search Direction for Circuit 2



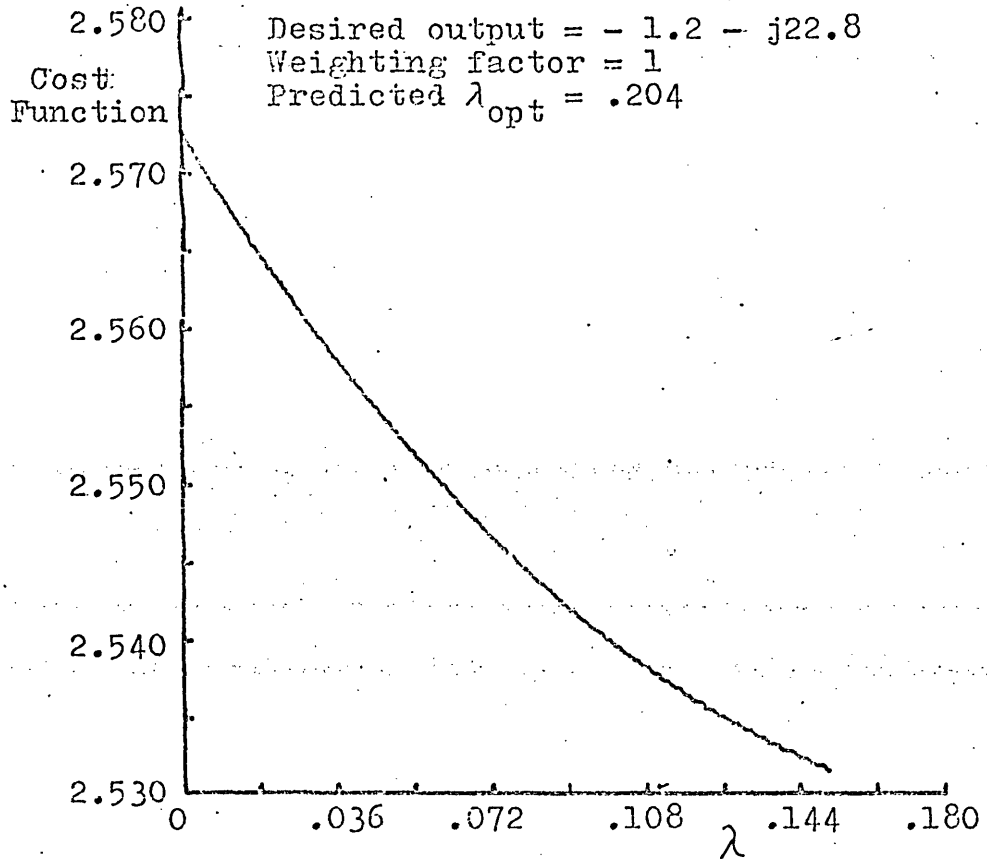
Second Search Direction for Circuit 2



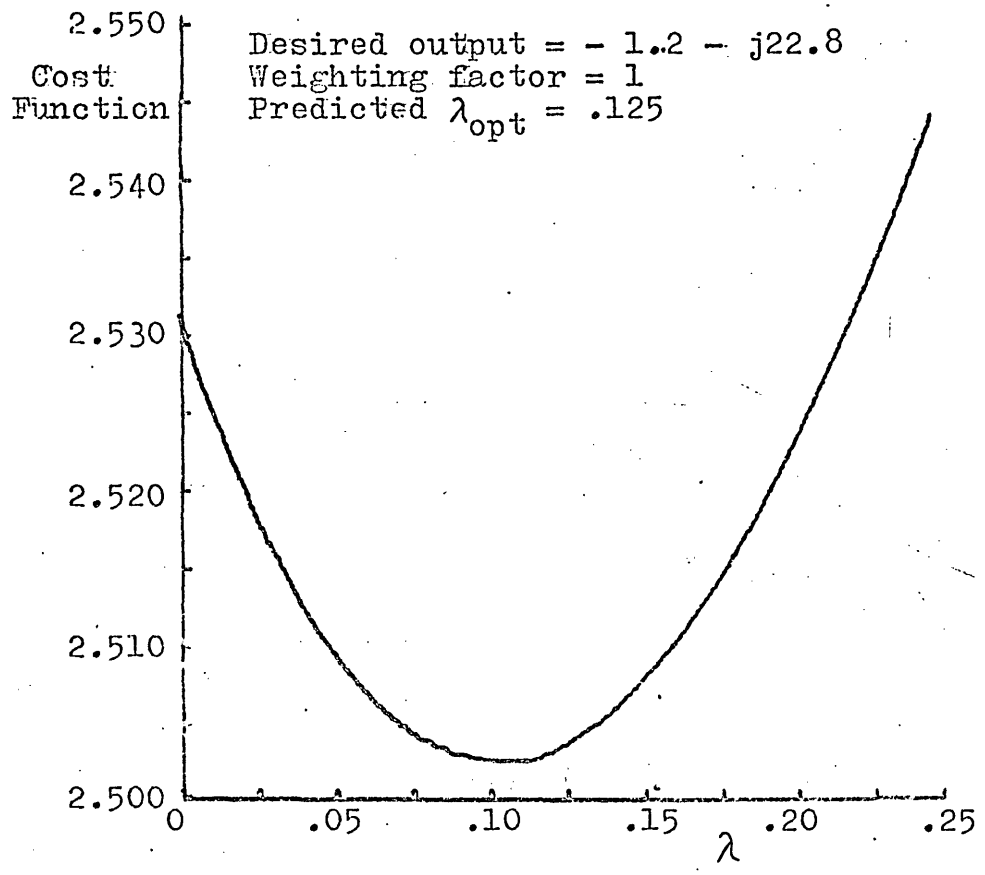
Representation of the Cost Function Surface for the First and Second Search Directions in the optimisation of Circuit 2

Figure 6.2

Third Search Direction for Circuit 2



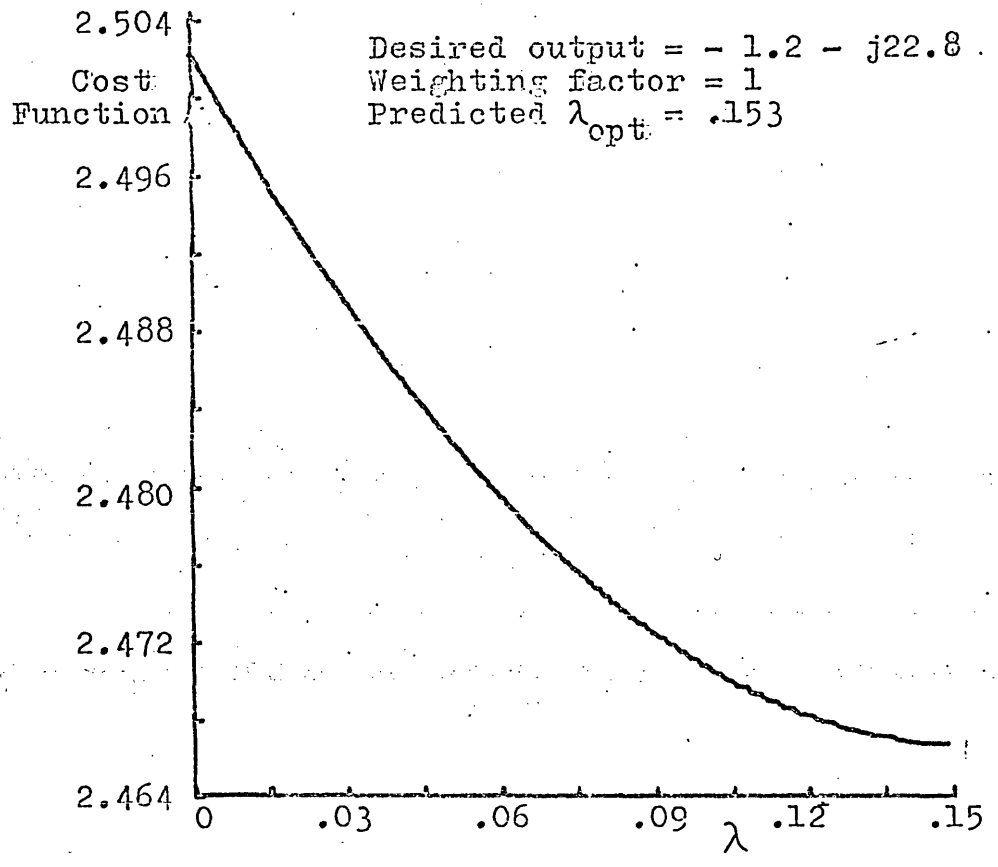
Fourth Search Direction for Circuit 2



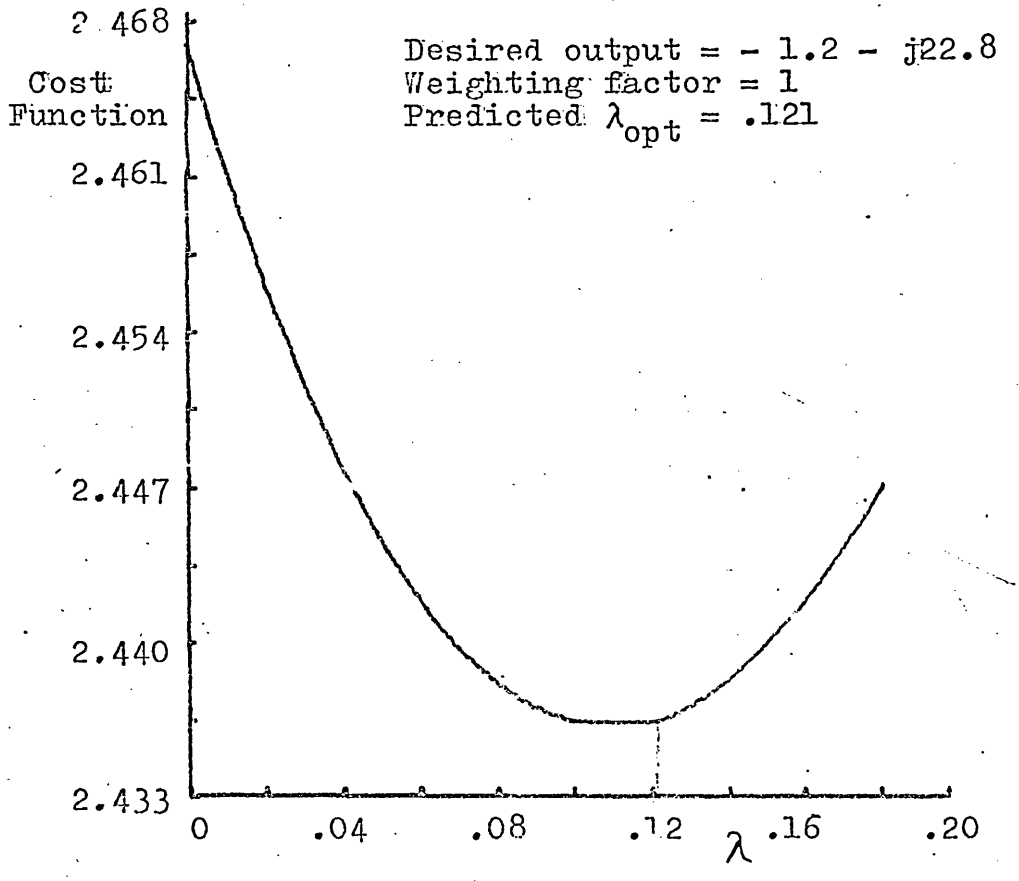
Representation of the Cost Function Surface for the Third and Fourth Search Directions in the optimisation of Circuit 2.

Figure 6.3

Fifth Search Direction for Circuit 2



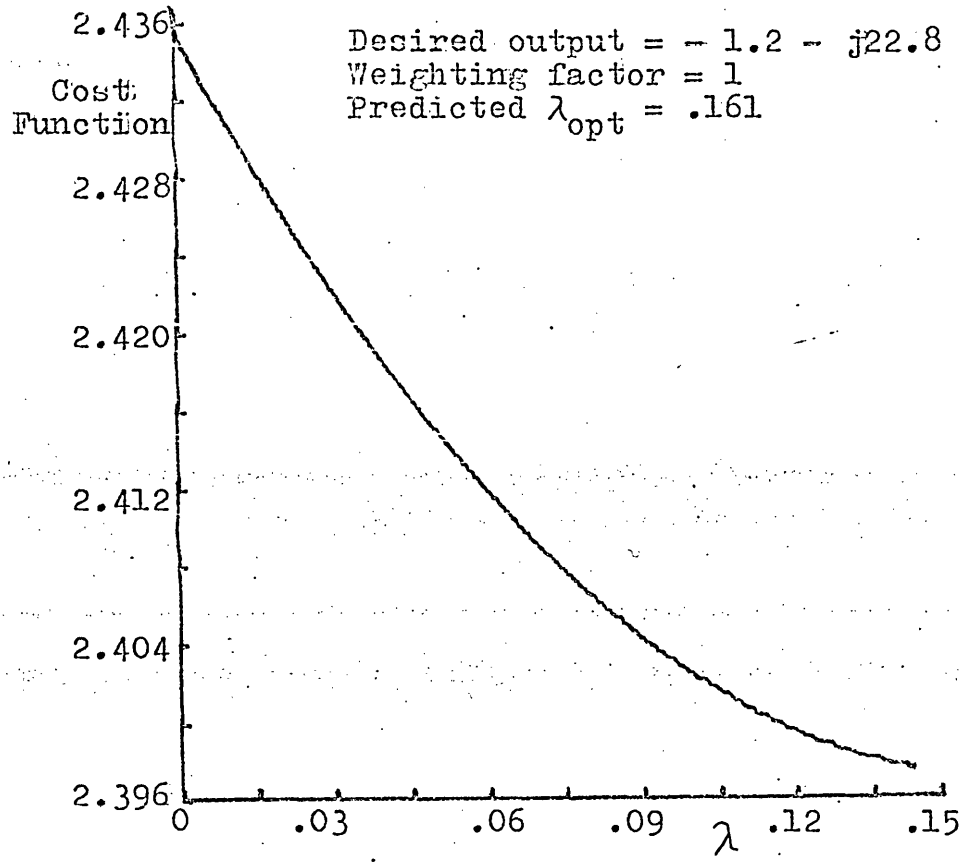
Sixth Search Direction for Circuit 2



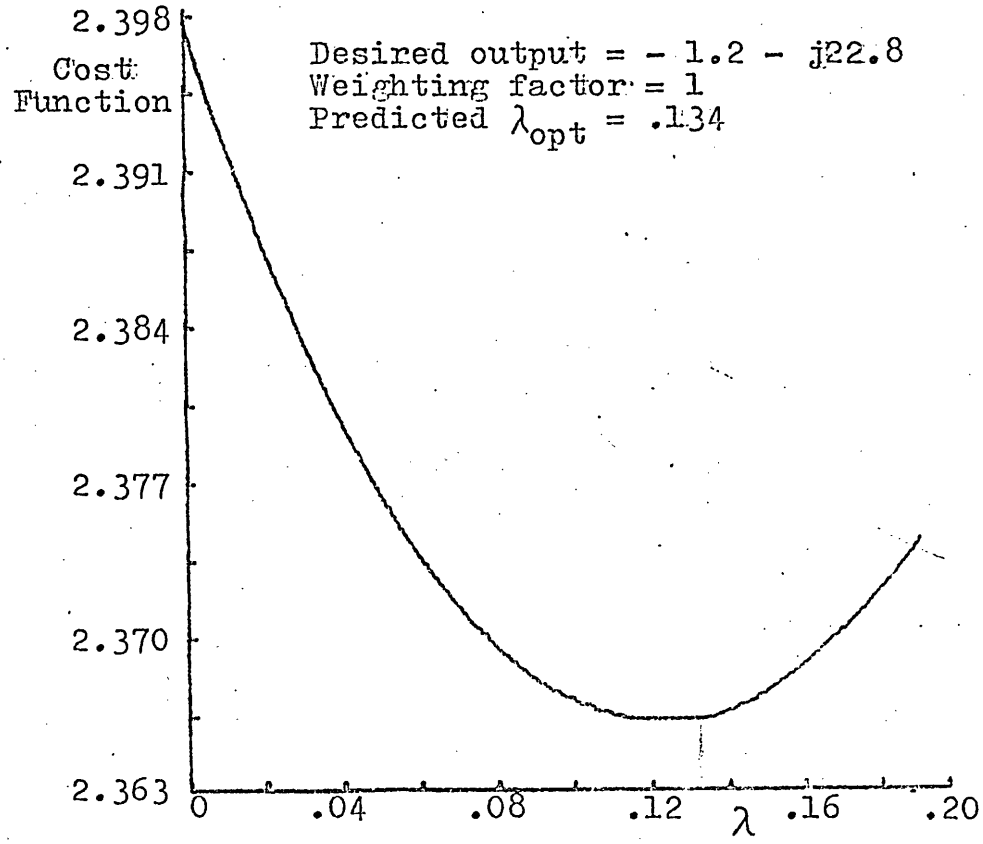
Representation of the Cost Function Surface for the Fifth and Sixth Search Directions in the optimisation of Circuit 2

Figure 6.4

Seventh Search Direction for Circuit 2



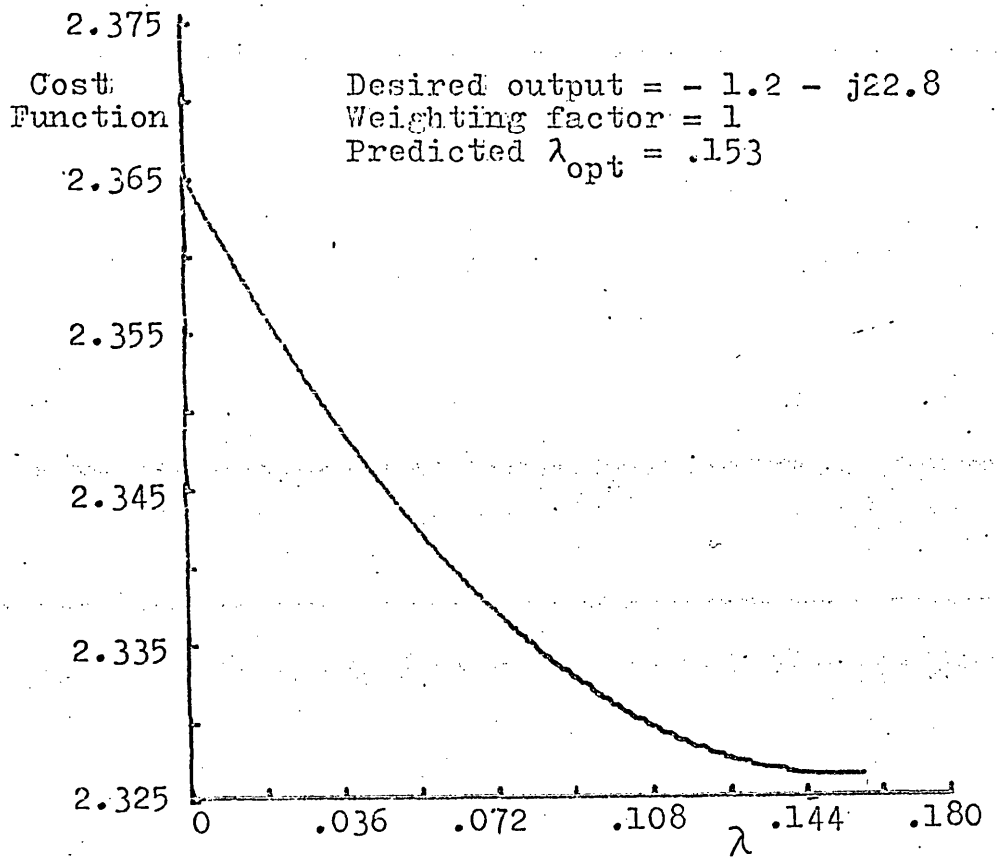
Eighth Search Direction for Circuit 2



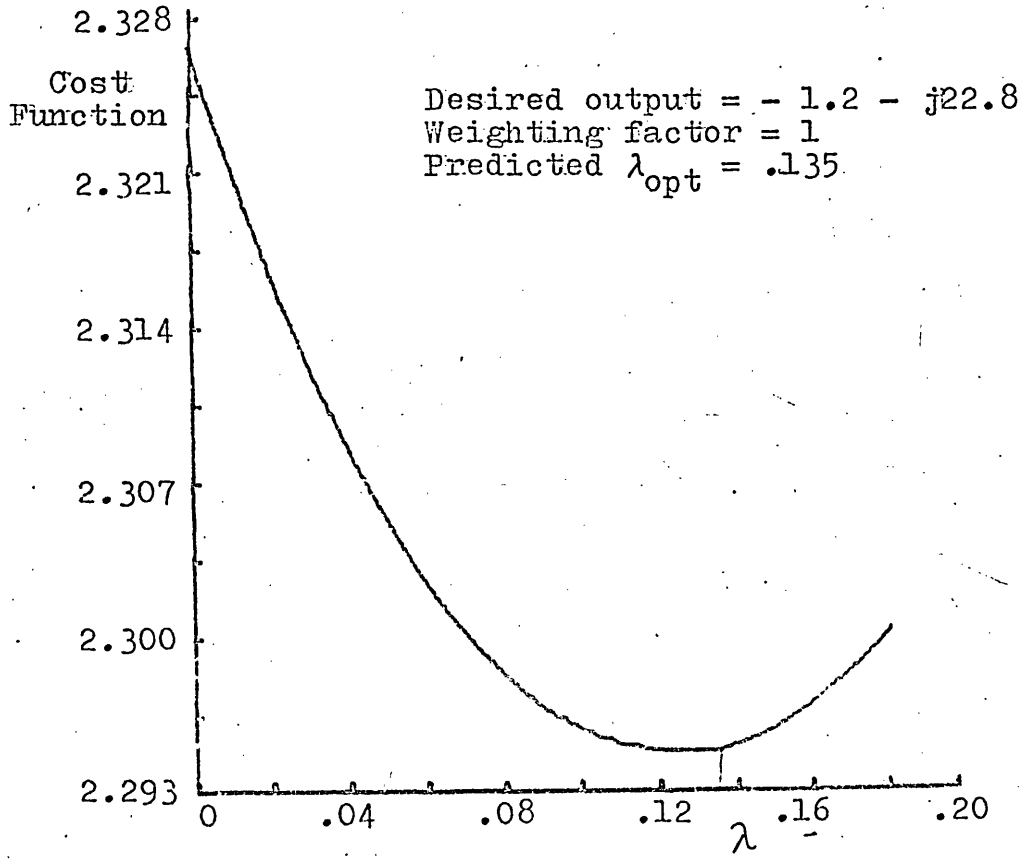
Representation of the Cost Function Surface for the Seventh and Eighth Search Directions in the optimisation of Circuit 2

Figure 6.5

Ninth Search Direction for Circuit 2



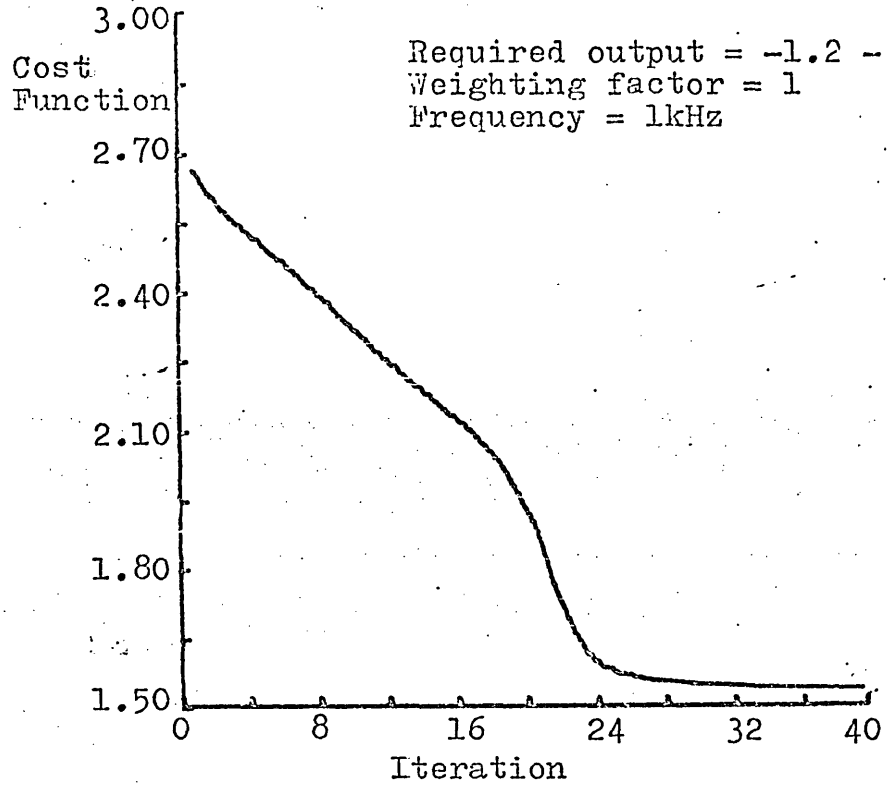
Tenth Search Direction for Circuit 2



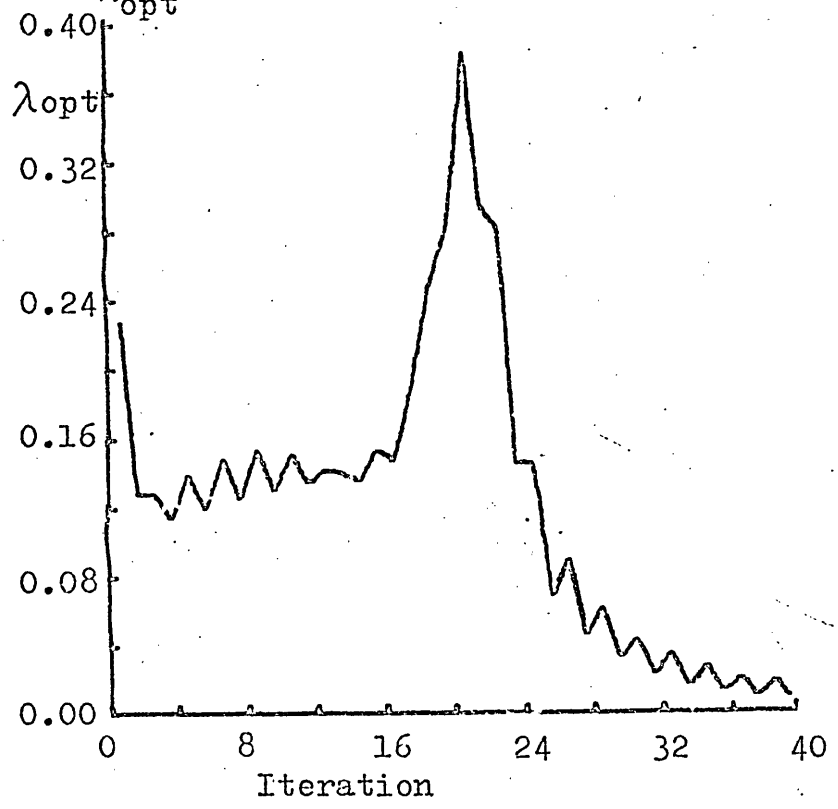
Representation of the Cost Function Surface for the Ninth and Tenth Search Directions in the optimisation of Circuit 2.

Figure 6.6

Cost Function versus Iteration for Circuit 2

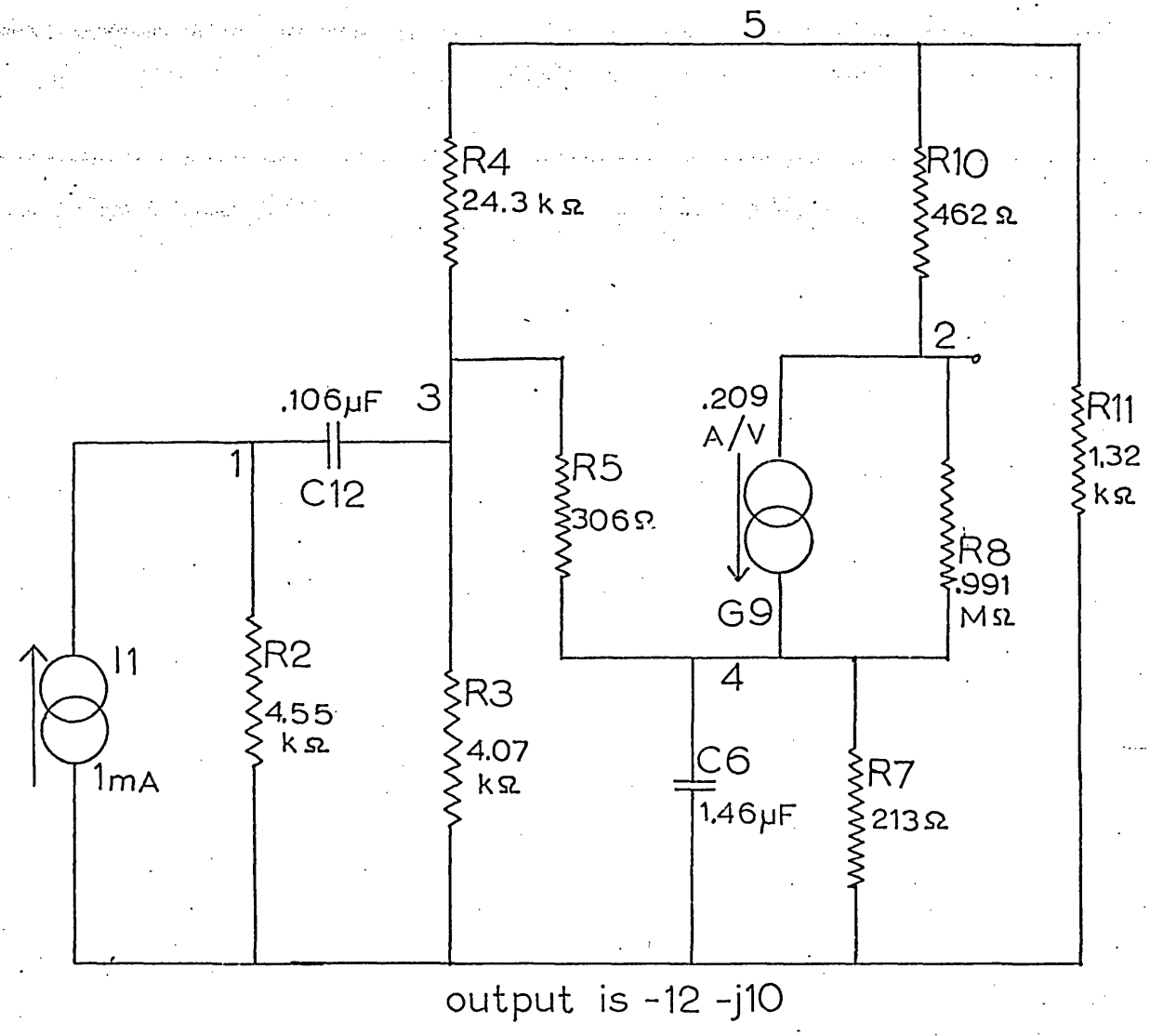


Predicted λ_{opt} versus Iteration for Circuit 2



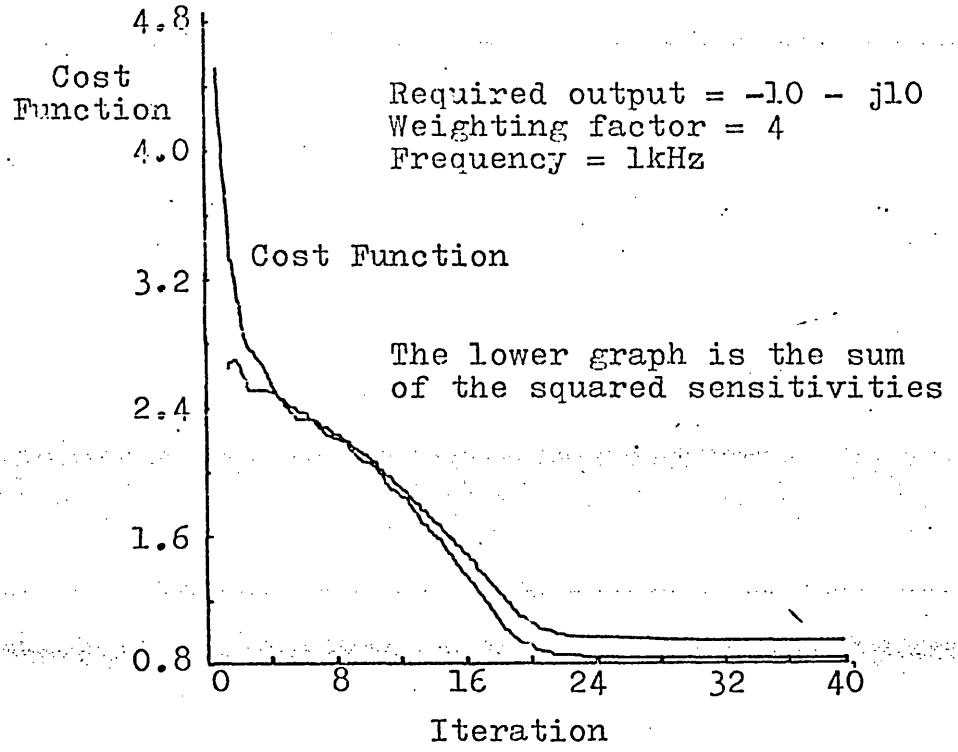
Cost Function and Optimum Step Length plotted against Iteration for Steepest Descent Optimisation of Circuit 2

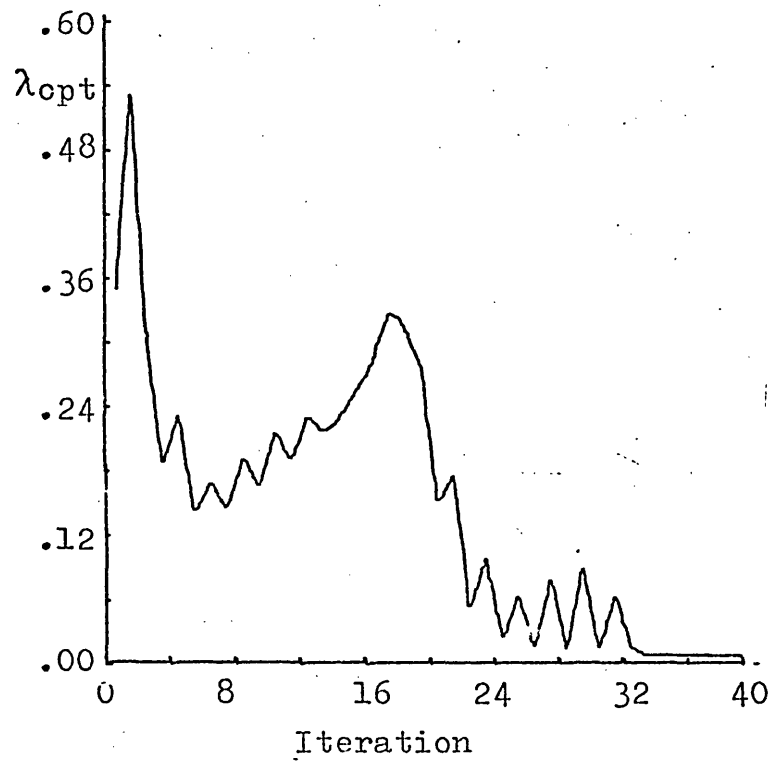
Figure 6.7



CIRCUIT 2 optimised by Steepest Descent Algorithm

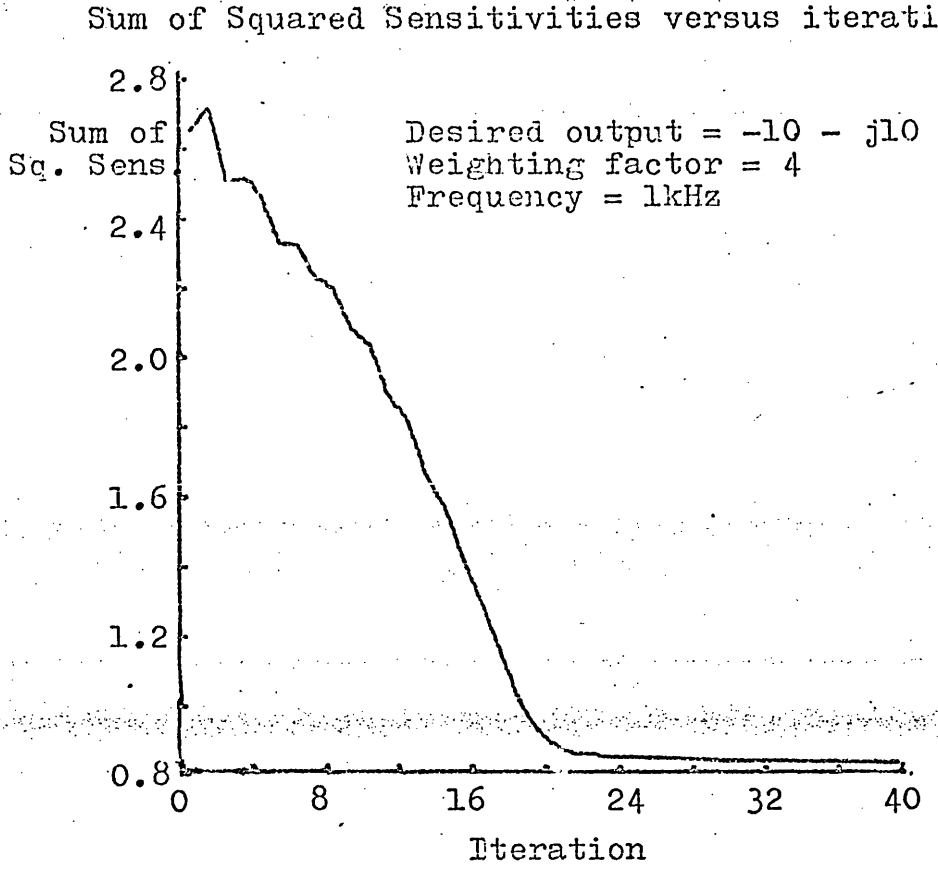
Figure 6.8



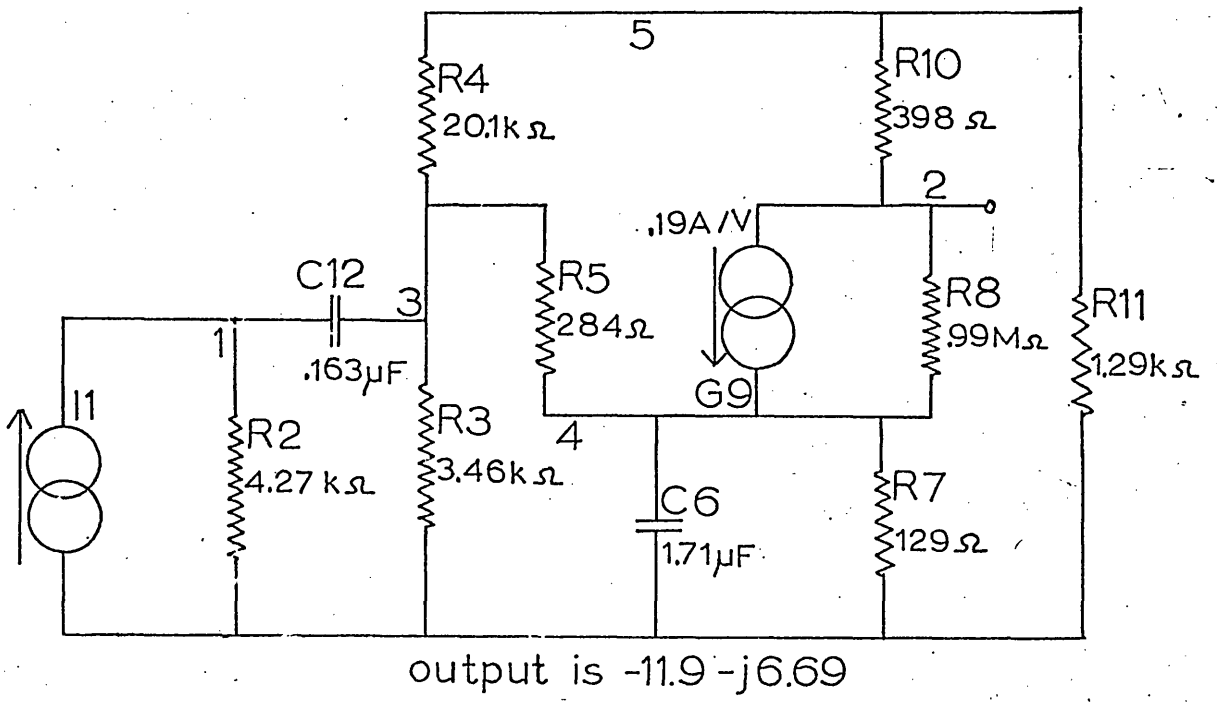


Cost Function, as well as the sum of squared sensitivities, against iteration has been shown in the top graph. The bottom graph shows the variation of optimum step length with iteration. These graphs are for the optimisation of Circuit 2.

Figure 6.9

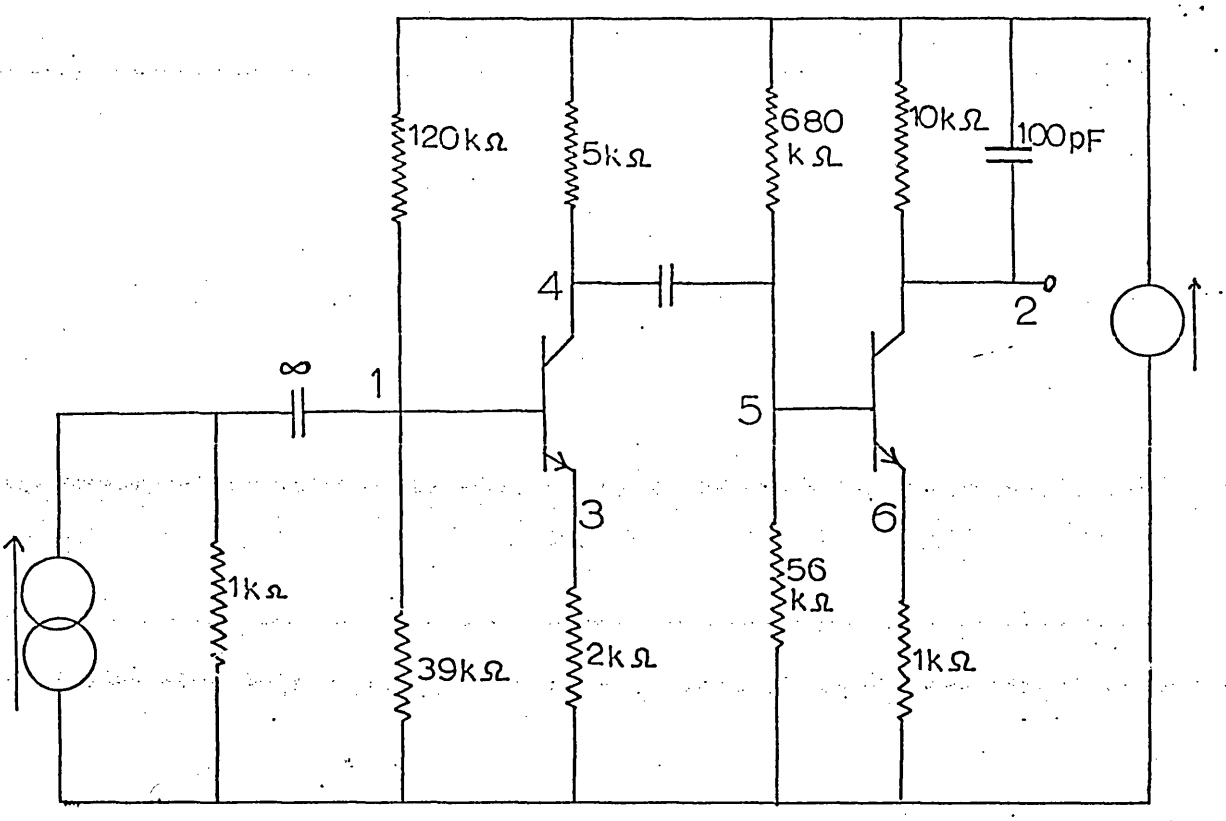


The sum of the squared sensitivities is shown plotted against iteration for the optimisation of Circuit 2 at a frequency of 1kHz.



Resultant circuit from optimisation of Circuit 2 at a frequency of 1kHz. Desired output is -10 -j10. Weighting factor = 4.

Figure 6.10



CIRCUIT 7

modelled by

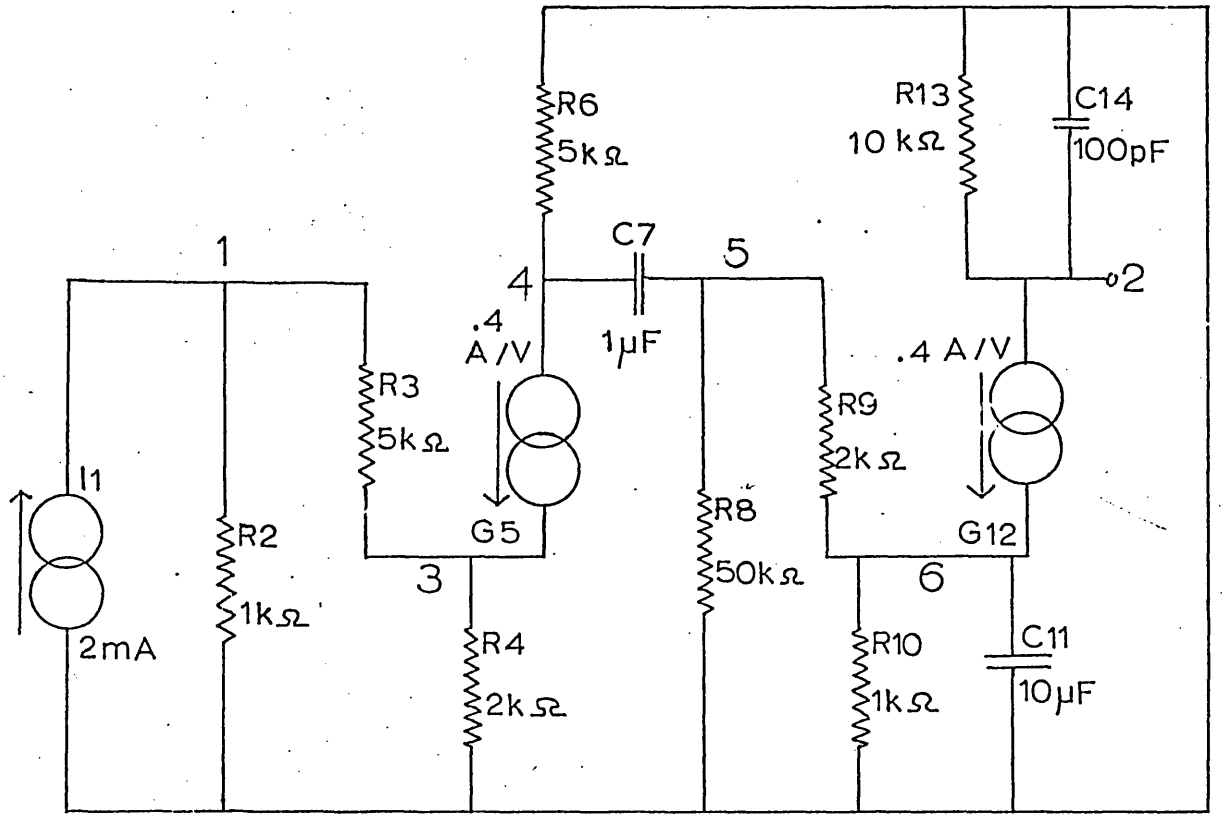
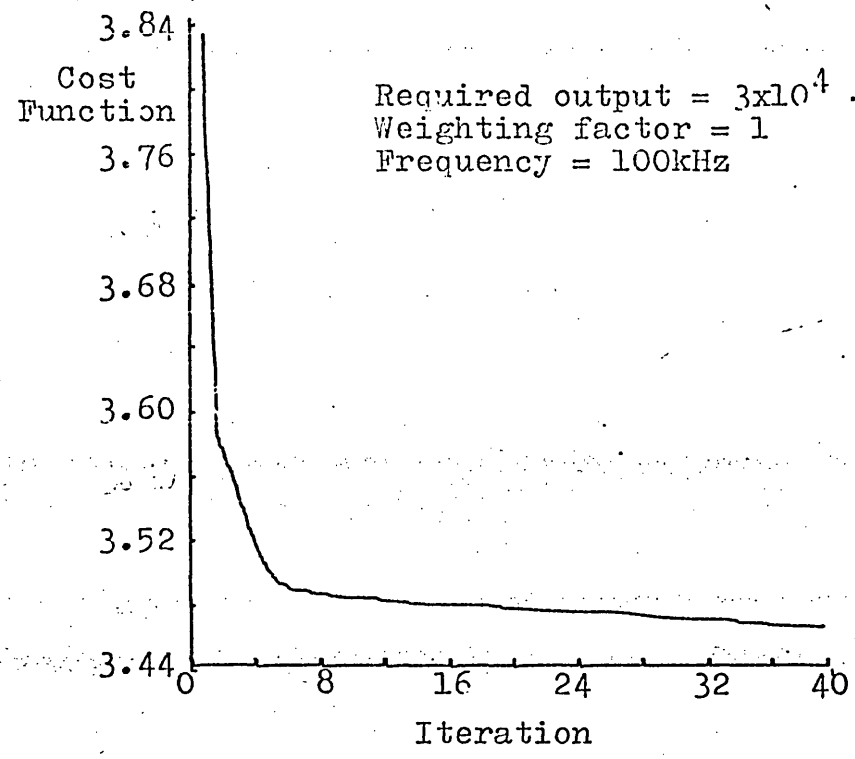
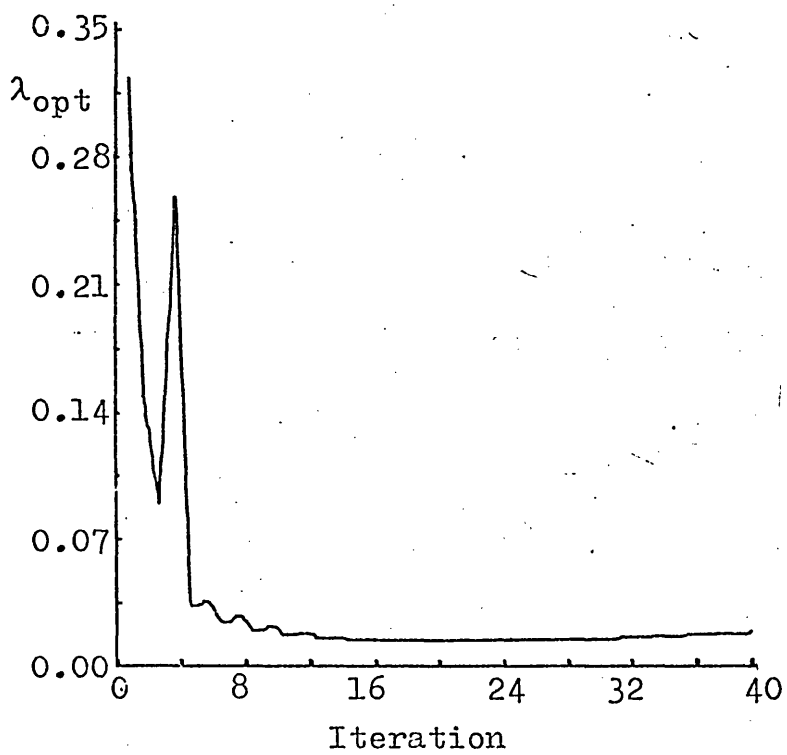


Figure 6.11

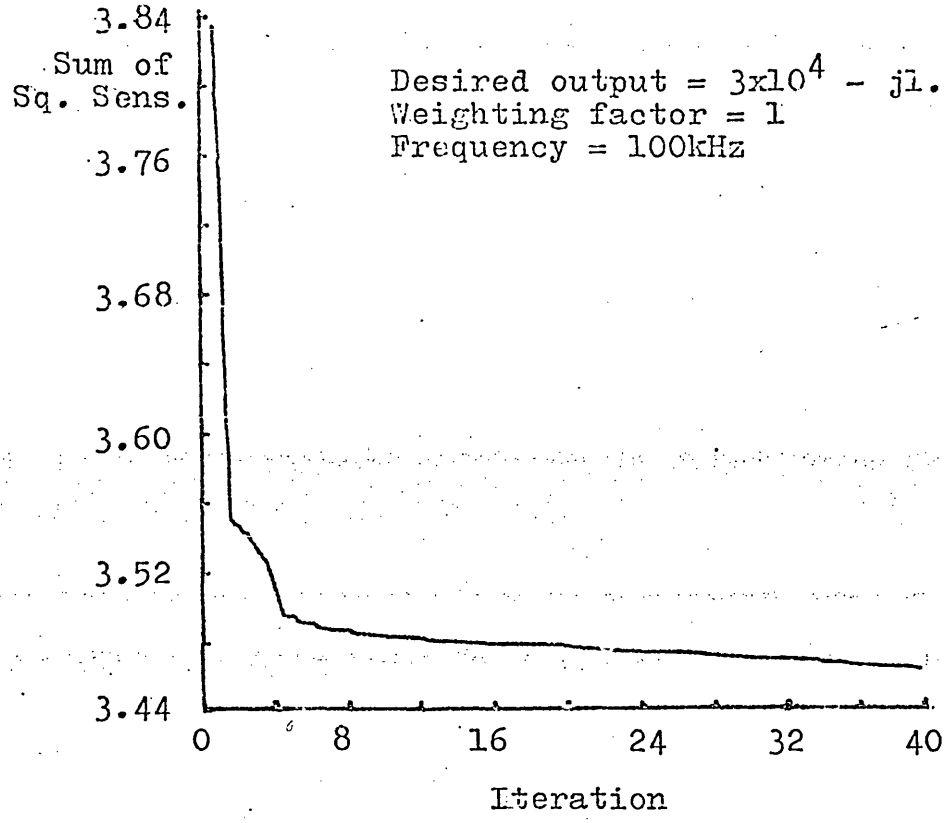




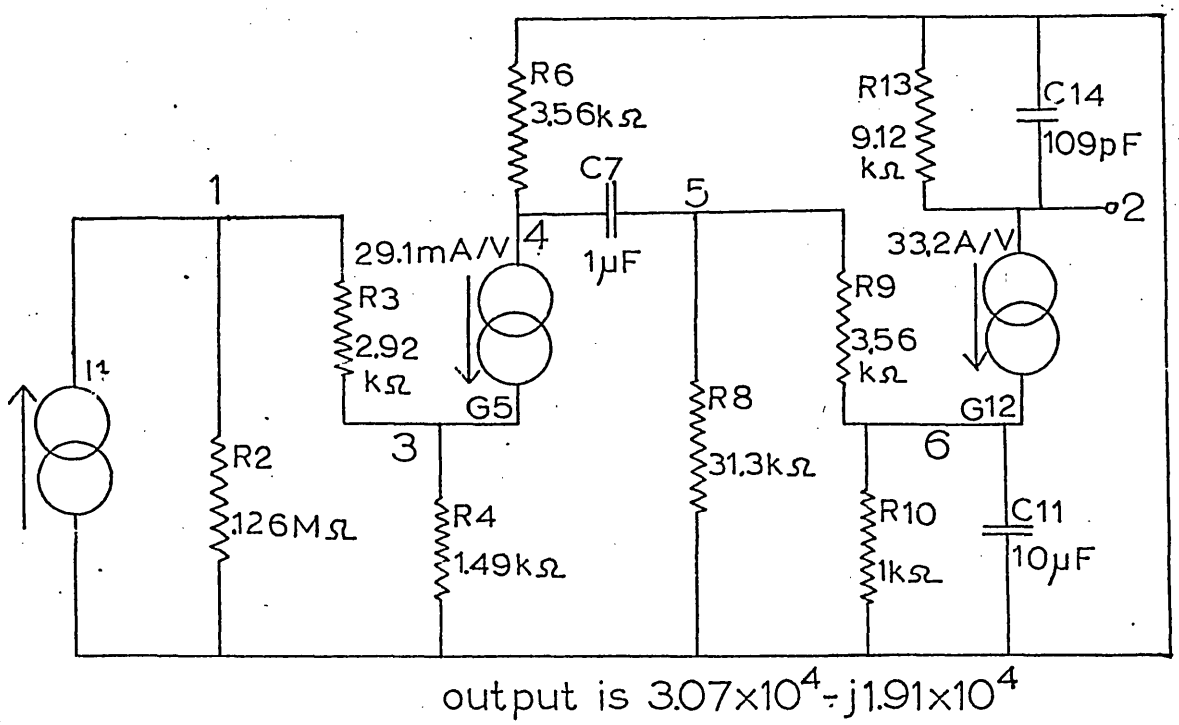
Cost Function and Optimum Step Length plotted against Iteration for the Steepest Descent Optimisation of Circuit 7

Figure 6.12

Sum of squared sensitivities versus iteration.

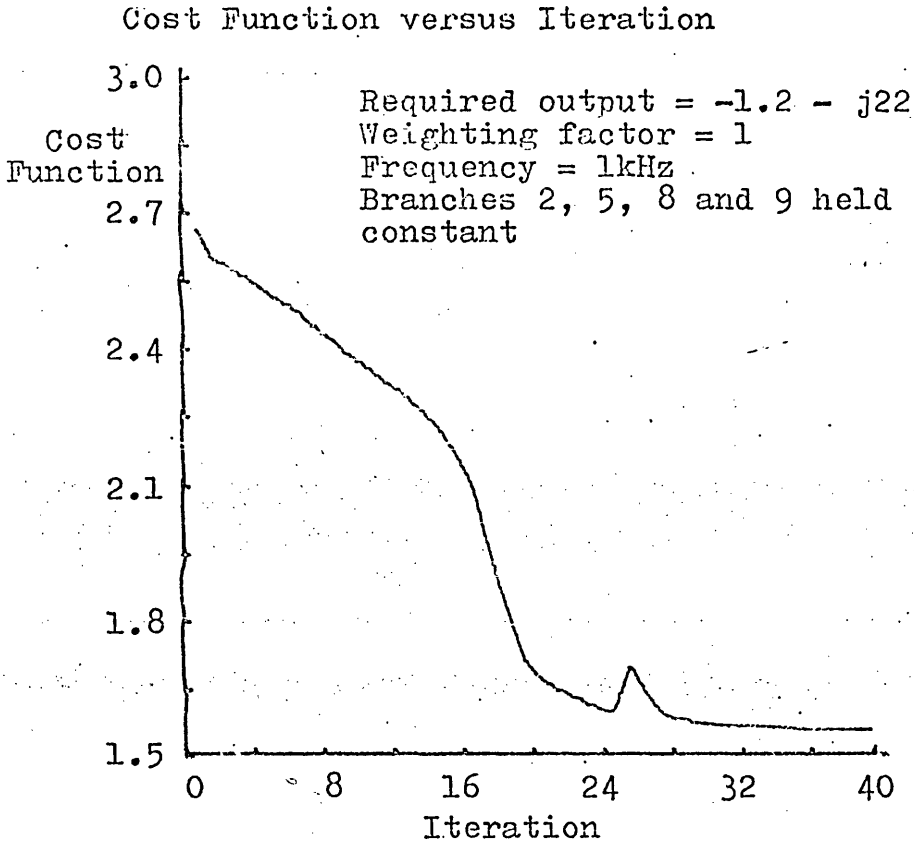


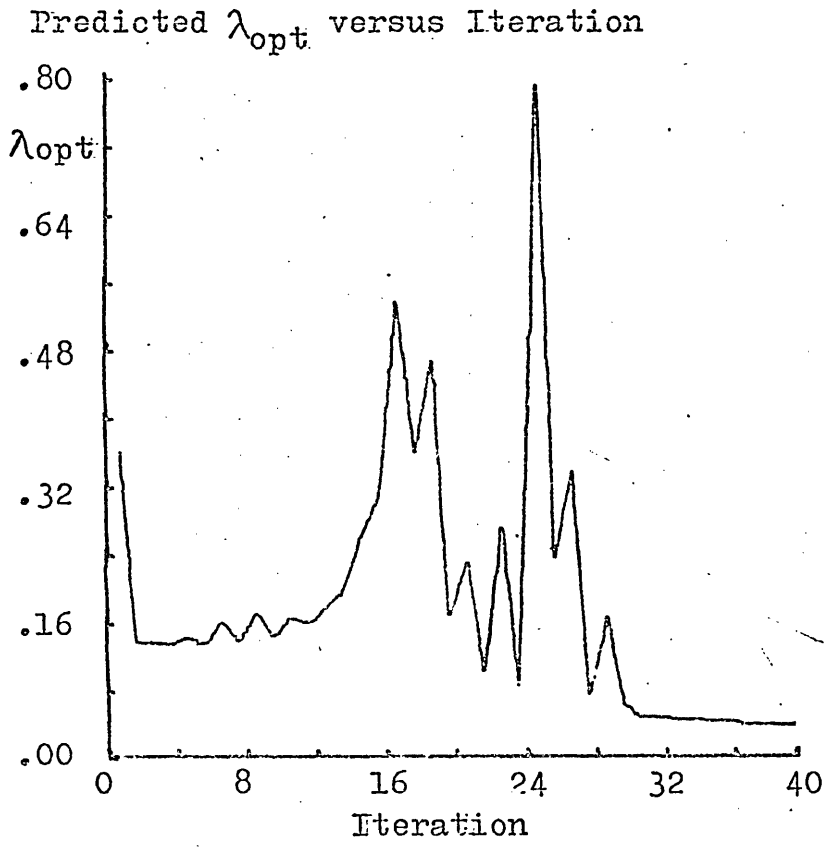
The sum of the squared sensitivities is shown plotted against iteration for the optimisation of Circuit 7 at a frequency of 100kHz.



Resultant circuit from optimisation of Circuit 7 at a frequency of 100kHz. Desired output is $3\times10^4-j1.87\times10^4$. Weighting factor =1.

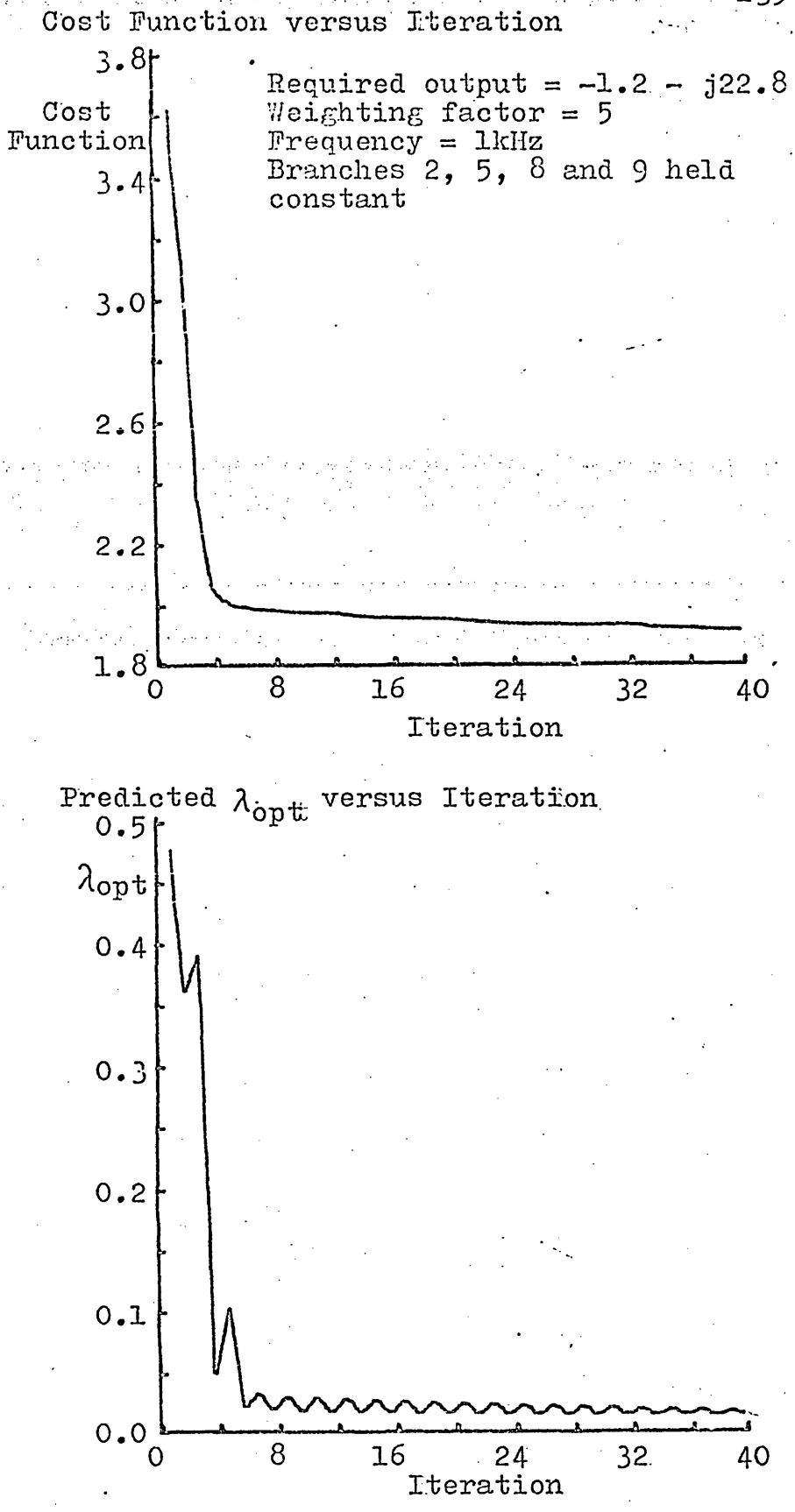
Figure 6.13





Cost Function and Optimum Step Length plotted against Iteration for Steepest Descent Optimisation of Circuit 2

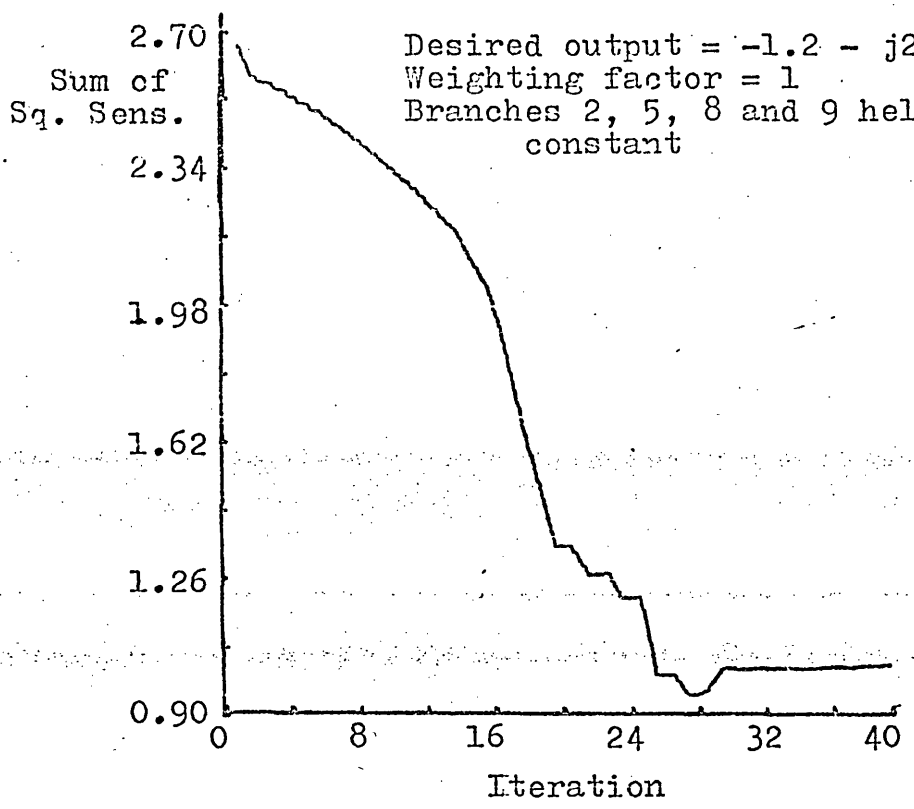
Figure 6.14



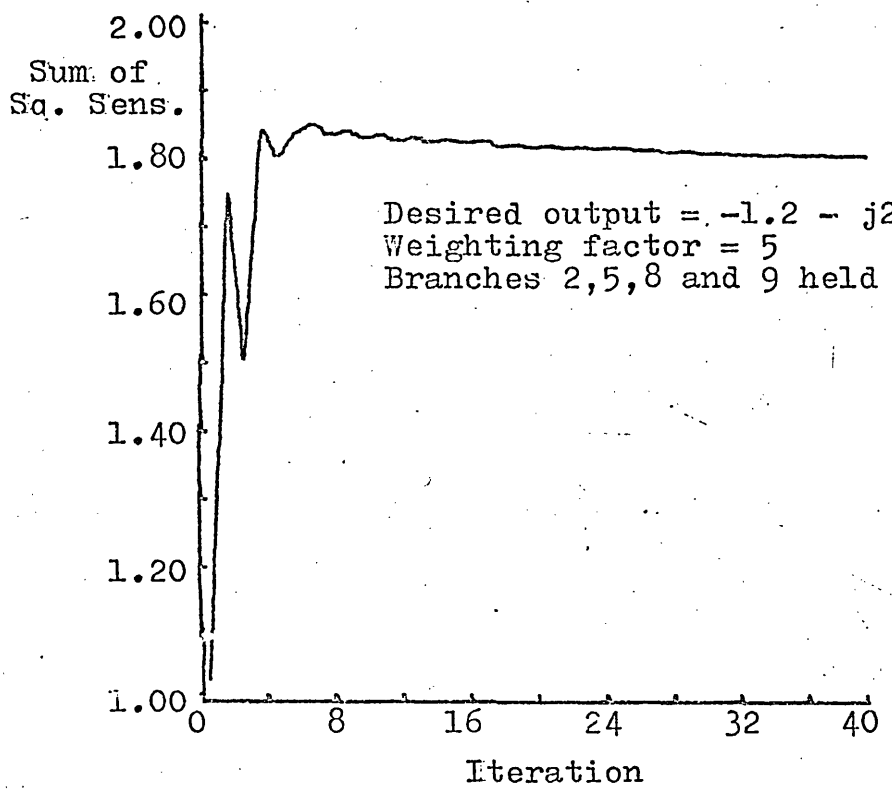
Cost Function and Optimum Step Length plotted against Iteration for continuation of Steepest Descent Optimisation of Circuit 2

Figure 6.15

Sum of Squared Sensitivities versus Iteration

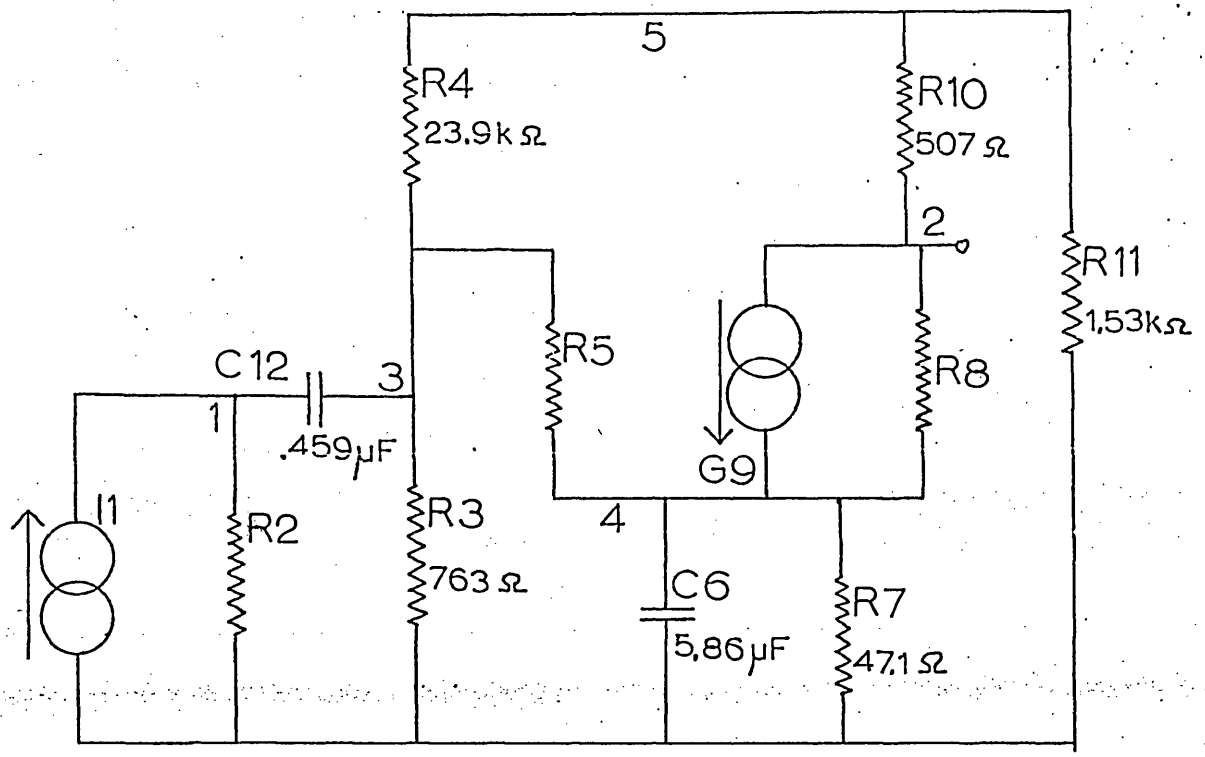


Sum of Squared Sensitivities versus Iteration



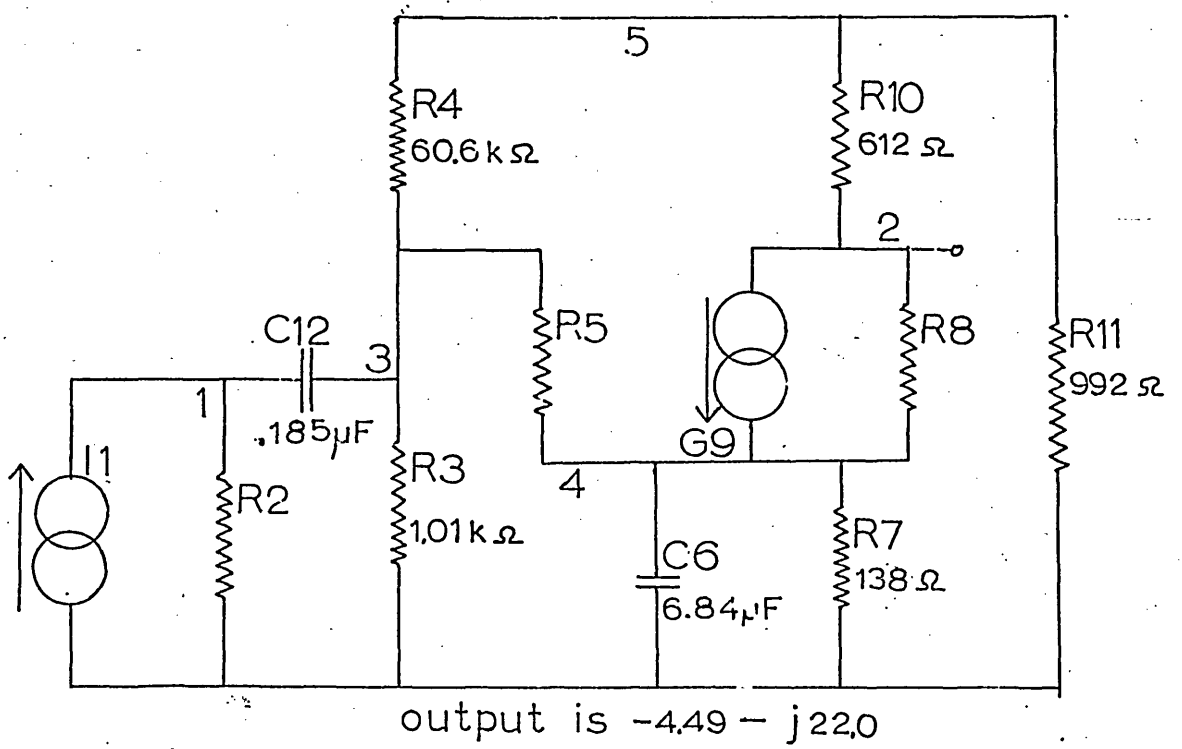
The sum of the squared sensitivities is shown against iteration for the optimisation of Circuit 2 at a frequency of 1kHz. For the second graph the output weighting factor has been increased and the process continued.

Figure 6.16



output is $-12.4 - j10.8$

Resultant circuit from optimisation of Circuit 2 at a frequency of 1kHz. Branches 2,5,8 and 9 held constant. Desired output is $-1.2 - j22.8$. Weighting factor = 1

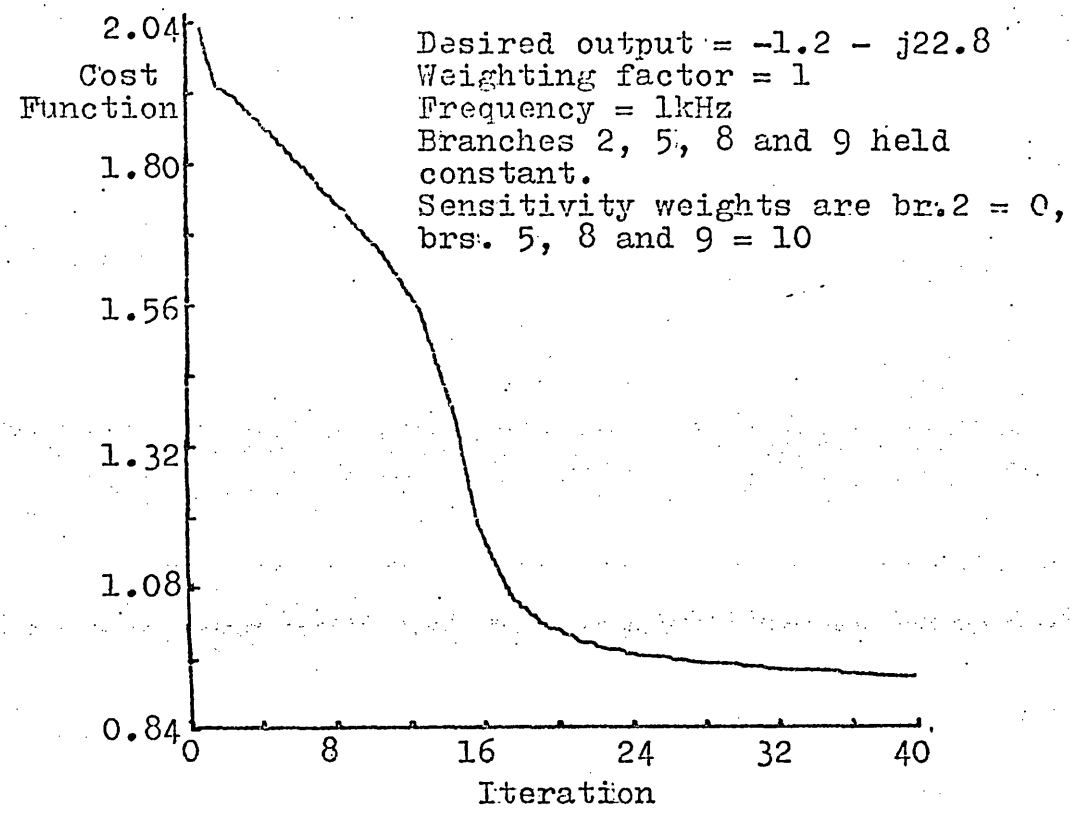


output is $-4.49 - j22.0$

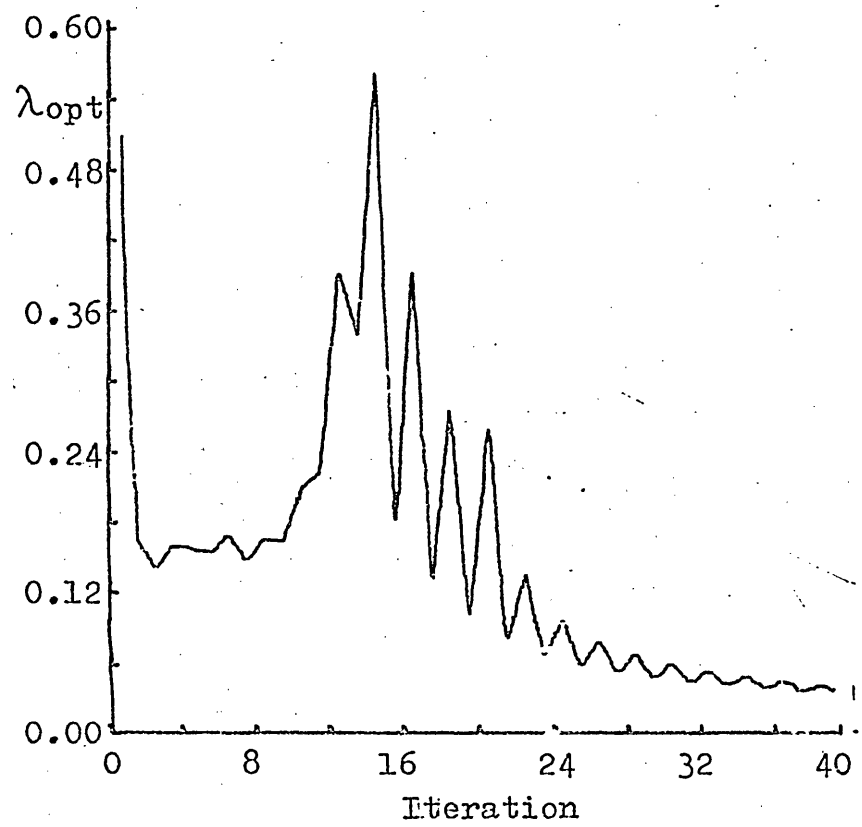
Resultant circuit from optimisation of top circuit at a frequency of 1kHz. Branches 2,5,8 and 9 held constant. Desired output is $-1.2 - j22.8$. Weighting factor = 5.

Figure 6.17

Cost Function versus Iteration for Circuit 2



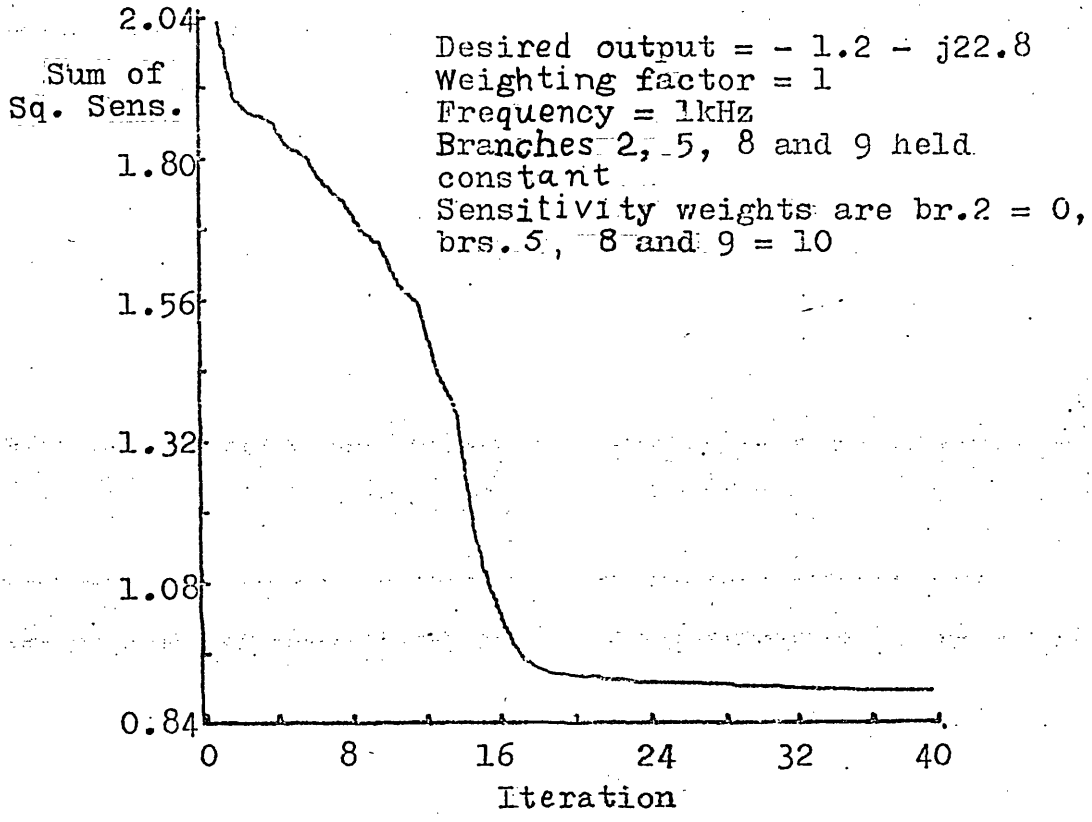
Optimum Step Size versus Iteration



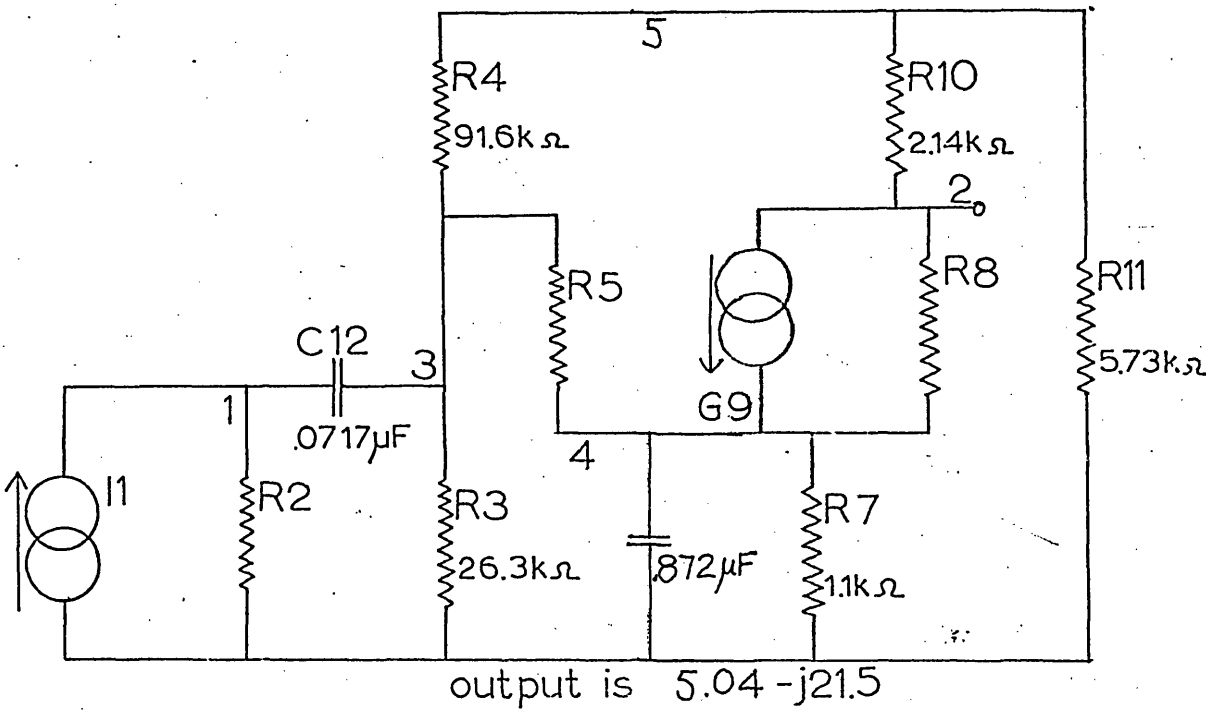
Graphs showing the optimisation progress of Circuit 2 when branches 2,5,8 and 9 have been held constant. Sensitivity weights of 10 are applied to branches 5, 8 and 9 while branch 2 has a sensitivity weight of zero.

Figure 6.18

Sum of the squared sensitivities (with weights) versus iteration



Graph showing the variation of the sum of the squared sensitivities (with weights) with iteration.



Circuit 2 optimised at a frequency of 1kHz with an output error weighting factor of 1. Branches 2,5,8 and 9 held constant. Sensitivity weights of 10 applied to branches 5, 8 and 9, zero weight to branch 2. Desired output is -1.2 -j22.8.

Figure 6.19

```
NUMBER OF ITERATIONS = 40

REQUIRED OUTPUT (COMPLEX) = -1.2  -22.8

ALTER THE ELEMENT WEIGHTS(Y OR N)

Y
BRANCH NO. 2 WEIGHT = 0 BRANCH NO. 5 WEIGHT = 0
BRANCH NO. 8 WEIGHT = 0 BRANCH NO. 9 WEIGHT = 0
BRANCH NO. 0

ALTER THE ELEMENT SENSITIVITIES WEIGHTS(Y OR N)

Y
BRANCH NO. 2 WEIGHT = 0 BRANCH NO. 5 WEIGHT = 10
BRANCH NO. 8 WEIGHT = 10 BRANCH NO. 8 WEIGHT = 10
BRANCH NO. 0
```

Note that a branch number of zero terminates a sequence.

```
WEIGHTING FACTOR (INTEGER) = 1

MAXIMIM ELEMENT VARIATION = .4
```

Original Circuit — Optimised Circuit

Nodal Voltages

NODE	REAL VOLT	IMAG VOLT	NODE	REAL VOLT	IMAG VOLT
1	0.821E+00	-0.101E+00	1	0.931E+00	-0.181E+00
2	-0.120E+01	-0.228E+02	2	-0.504E+01	-0.215E+02
3	0.661E+00	0.183E+00	3	0.530E+00	-0.285E-01
4	0.660E+00	0.172E+00	4	0.528E+00	-0.356E-01
5	-0.192E-01	-0.443E+00	5	-0.360E+01	-0.154E+02

Branch Sensitivities

BRANCH	REAL SEN	IMAG SEN	BRANCH	REAL SEN	IMAG SEN
1	0.000E+00	0.000E+00	1	0.000E+00	0.000E+00
2	0.821E+00	-0.101E+00	2	0.931E+00	-0.181E+00
3	0.191E+00	-0.174E+00	3	0.177E-01	-0.458E-01
4	0.111E+00	-0.304E-01	4	0.435E+00	0.621E-01
5	0.415E-01	0.203E-01	5	0.380E-01	0.634E-02
6	0.843E+00	0.207E+00	6	0.551E+00	-0.383E-02
7	-0.658E-01	0.268E+00	7	0.635E-03	0.915E-01
8	0.246E-03	0.168E-03	8	0.307E-03	0.960E-04
9	0.470E-01	0.342E-01	9	0.406E-01	0.137E-01
10	0.980E+00	-0.355E-03	10	0.284E+00	-0.426E-03
11	-0.287E-01	-0.271E-01	11	0.286E+00	-0.755E-01
12	0.161E+00	-0.284E+00	12	0.401E+00	-0.152E+00

Example of a on-line console session for the optimisation of Circuit 2. Also given are the original and final nodal voltages and branch sensitivities to enable a comparision to be made between original and optimised circuits.

Figure 6.20

Chapter 7

Future Research Topics

Efficient methods of calculating and utilising electrical network sensitivity have been developed in the preceding chapters. The amount of computer time saved, as well as the accuracy obtained, make the use of these methods highly desirable in any serious computer aided circuit design work. Also the use of these techniques enables effective circuit design to be performed on samll computers such as a Digital Corporation PDP 15.

The accuracy obtained in the calculation of element differentials by the proposed methods is essential if optimisation algorithms which depend on first order differentials are to be used successfully when approaching local or global optima. This is because first order differentials tend to zero when approaching an optimum and hence perturbation techniques for differential calculation fail. As shown in chapter 6, the steepest descent algorithm, when using an estimate of each search direction optimum based on a quadratic behaviour assumption, converges very rapidly (less than twenty iterations) even for the complex cost function used in this chapter. The proposed algorithm may be further speeded up by only calculating the differentials of the elements which are allowed to alter their admittance values.

A restriction in nodal analysis is the inability to allow the presence of ideal voltage sources in the circuit under analysis. The usual procedure for dealing with these sources is to insert a suitable source resistance and then to convert the combined voltage source and resistance to a

parallel connection of a current source and source resistance (Norton Transformation). However this technique means the introduction of another element if there is no suitable impedance present to act as a source impedance. This is inconvenient for several reasons which are:

1) Another element must be added and;
2) A suitable value for the branch impedance must be chosen so that the impedance value is big enough not to cause numerical difficulties while being small enough to simulate voltage source behaviour.

If the possibility of a hybrid matrix in place of the nodal admittance matrix is accepted, then, using some of the methods discussed in chapter 5, it is possible to introduce a voltage source directly into the system equations. This is achieved by first assuming that an arbitrary current source is connected between the nodes i and j where the voltage source is required (see equation 5.7). As in chapter 5 this arbitrary current source is removed by the addition of the two rows representing the Kirchhoff current law for the nodes i and j. The extra equation necessary to make the system determinant is then provided by specifying the desired voltage relationship as in equation 5.5. This procedure has been successfully used in the analysis of circuits containing voltage sources.

However no work on sensitivity calculation using this hybrid matrix has been carried out. Theoretically there seems to be no reason why the approach used in equation 2.15 (see chapter 2) should not hold. Obviously more 'book-keeping' must be carried out since the evaluation of the differential of the hybrid matrix with respect to circuit elements is not quite so simple. This 'book-keeping' would

only consist of an extra vector which specifies the row to which each row of the hybrid admittance matrix has been added, if any. Also the checks discussed in chapter 5 under the heading of Multiple Constraints would have to be performed if more than one voltage source was required in the circuit.

The contents of chapters 4 and 6 provide the greatest scope for further research. In chapter 6 the most obvious line for future work to follow is the extension of the proposed optimisation algorithm to handle a frequency range rather than a single frequency point. This is easily achieved by selecting frequency points and their appropiate weights over the desered range. The weighting factors would express the relative importance of their associated frequency to the desired response function. A composite cost function formed form the weighted sum of the individual cost functions at each frequency appears to be the simplest, and perhaps the most relevant, cost function to apply. Verfication of the quadratic convergence using the methods proposed in chapter 6 would have to be performed for the various possible search directions. One of these search directions would be the weighted average of the search directions at each individual frequency point. Another possibility which might well be considered is the steepest descent direction at the frequency point which has the highest contribution to the cost function. This would be a form of mini-maximum optimisation. Intepretation and understanding of the circuits obtained by these optimisation schemes opens up a new host of possible research projects. This type of research is needed to determine if there is a

empirical basis of design whereby low sensitivity networks could be directly formulated.

The large-change sensitivity technique developed in chapter 4 opens up exciting possibilities in tolerance analysis, a topic of growing interest,[31] by the simplicity and speed of the method. Also this technique can be used to achieve great savings in computation if a large circuit has to be optimised with only a few of its elements allowed to vary. As seen in chapter 4 (Multiple Element Simulation) the output voltage and the branch voltages of a modified circuit may be found by the analysis of a set of equations equal in size to the number of variable elements. Hence, if a twenty node circuit was under study and the output variation was required with the simultaneous variation of four elements, the simulating current source method would achieve a saving of around fifty. This comparision is made between the time needed to solve the complete circuit equations and that needed to solve the set of equations given by the large-change sensitivity technique. Sensitivities of all the elements in the perturbed network can be found iff a further set of simulating current vectors are found for the adjoint network. However the simulating current technique will not give higher order sensitivities analytically and hence this technique could not be used in the evaluation of the cost function defined in chapter 6. This is because this cost function depends on a knowledge of the second order differentials. Therefore, if second order sensitivities are required in a circuit having simulated element values, perturbation techniques with all their disadvantages must be introduced. Hence other techniques for 'wrapping up'

unchanging parts of a circuit should be examined when higher order differentials of the variable elements are required so that the methods of chapter 3 are applicable.

Appendix A

Time Comparision between Equation solving and Matrix Inverting Routines

A comparision has been made between the times required to solve any given network by a simultaneous equation method and by a matrix inversion method. The simultaneous equation routine is based on the maximal pivot, Gaussian reduction technique where back substitution is used to determine the value of the dependent vector*. Inversion of the original matrix is obtained by reducing this matrix to a unit matrix while, at the same time, performing the identical operations on an initially unit matrix (Gauss-Jordan).*

Since the time of analysis of any circuit, which includes the time for forming the relevant matrices, is of interest rather than the actual time for solving the circuit equations, times for analysing circuits of varying complexity have been plotted against circuit node size in figure A.1. The top graph shows the times taken using the simultaneous equation routine while the bottom graph shows the times taken using the inverting routine. Although the equation solving routine is theoretically more accurate than the matrix inversion technique, little difference between the two methods has been noted in practice. This is most likely due to the fact that circuit equations give

---

* see for example R.W. Hamming, 'Numerical Methods for Scientist and Engineers' published by McGraw-Hill in 1962

a diagonally predominant admittance matrix except when the circuit is in or near resonance. The equation solving routine is theoretically more accurate since the matrix is reduced using the largest element found in the unused rows and columns as the pivot. In the inversion routine, however, the largest unused main diagonal element is used as the pivot for the next reduction. Once the original matrix has been reduced to triangular form, all the remaining non-main diagonal terms are removed by the subtraction of the requisite multiples of the relevant rows starting with the bottom row and working up.

In figure A.2 the logarithms of analysis time and node size have been plotted against each other. The top graph refers to the simultaneous equation routine, while the bottom graph is for the inversion routine. Assuming that the time of analysis may be expressed as

$$T = CN^{\lambda} \qquad \text{- - - A.1}$$

where C and λ are constants, T is the time of analysis and N is the circuit node size, then

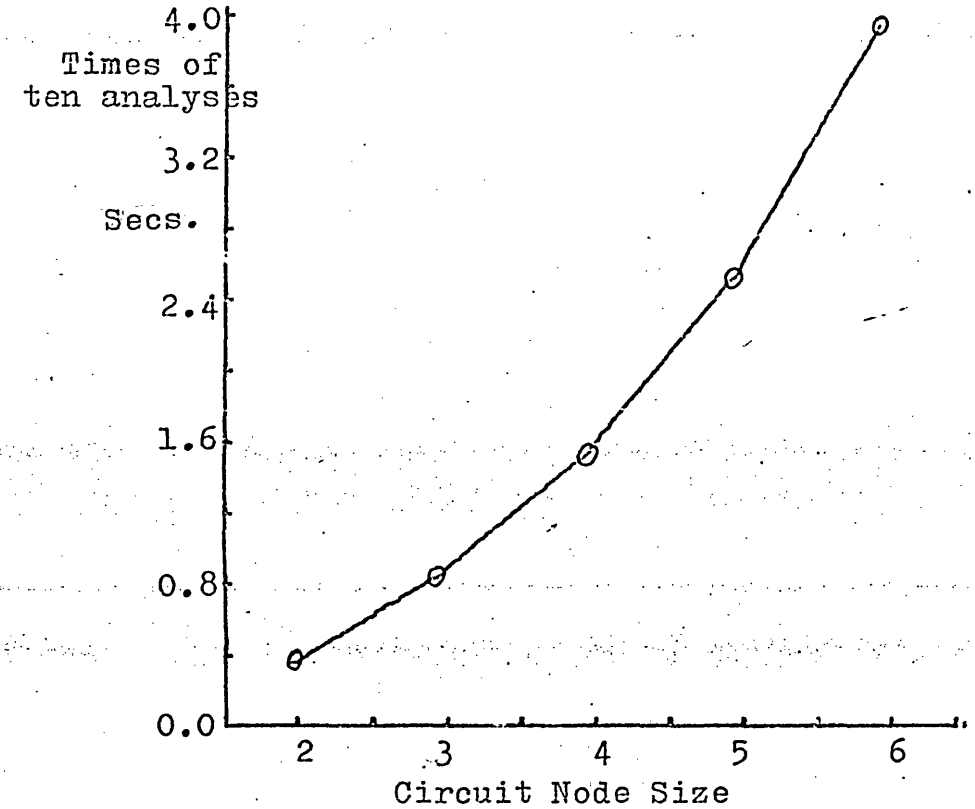
$$\log_{10}T = \log_{10}C + \lambda \log_{10}N \qquad \text{- - - A.2.}$$

Therefore the slope of the graph of $\log_{10}T$ against $\log_{10}N$ gives the value of λ.

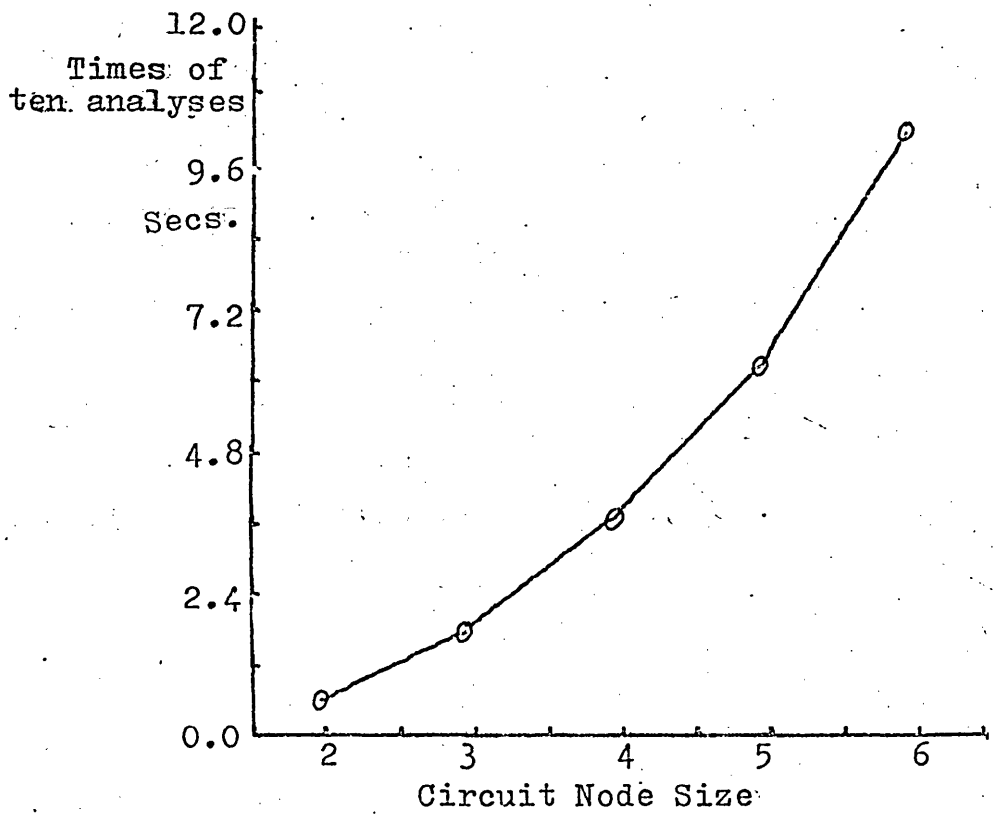
From the slopes of the graphs given in figure A.2 the inversion routine analysis time is dependent on the 2.42 power of the circuit node size while the analysis time of the simultaneous equation routine is only dependent on the 2.24 power. This difference in the power of the circuit node size is relatively small and hence for small

and medium size circuits a factor of 2.5 in the difference of analysis times between the two methods may be assumed. These timings were obtained when the programs were run on a PDP 15 computer which had an extended arithmetic unit. On this machine all floating point number operations have to be performed using the provided software. As multiplication and division form the majority of the operations in any form of simultaneous equation solving routine, the value of λ should not vary greatly from machine to machine. However the constant C will be highly machine dependent.

Time of ten analyses using a simultaneous equation routine versus Circuit Node Size



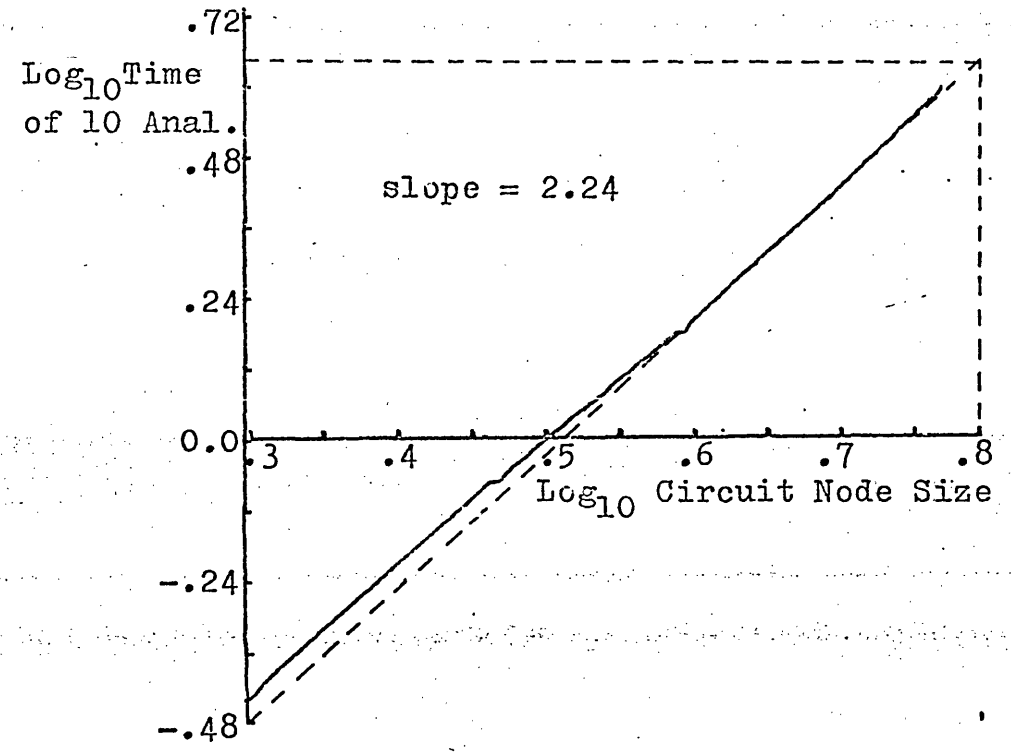
Time of ten analyses using a matrix inverting routine versus Circuit Node Size



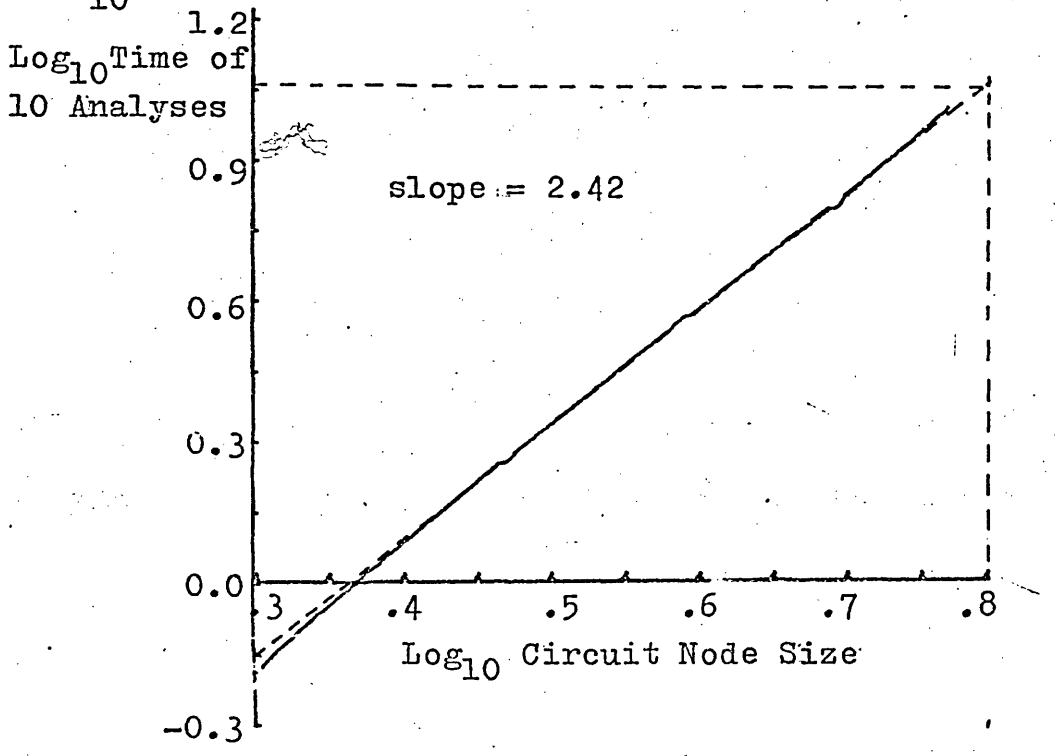
Graphs showing the times taken for analysing various circuits using either a simultaneous equation solving routine or a matrix inverting routine. The computer used was a PDP15 with an extended arithmetic unit.

Figure A.1

Log_{10} of Time of ten analyses using a simultaneous equation routine versus Log_{10} of Circuit Node Size



Log_{10} of Time of ten analyses using a matrix inverting routine versus Log_{10} of Circuit Node Size



Graphs showing the logarithm of the time of ten analyses of various circuits plotted against the logarithm of the node size of these circuits. The power of the circuit size which the analysis time is proportional to is given by the slope of these graphs.

Figure A.2

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