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A mathematical analogy between constant vorticity flows without surface tension and irrotational flows with surface tension

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A mathematical analogy is elucidated between two physically distinct classes of two-dimensional-free boundary problems in inviscid fluid mechanics. One class involves free surface flows with constant vorticity but without surface tension; the other comprises free surface flows without vorticity but with surface tension. The results provide a rationale for several surprising observations in the recent literature that the same free surface profiles can arise in problems that are physically distinct. It is shown here, in several contexts, how two classes of physical problems amount to solving the same mathematical problem. Solutions to those mathematical problems are examined in detail. The analogy has the general implication that theoretical results on the common mathematical problem are immediately relevant to both physical problems, leading to a host of new theoretical results including theorems regarding existence, uniqueness, analytic structure and non-trivial identities involving conformal maps. There are, however, some interesting caveats that are also examined here.

1. Introduction

It has been noted recently, in several contexts and by several different authors, that equilibria for free boundary problems involving constant vorticity, but with no surface tension, admit solutions for the free boundary that are identical in shape to those for equilibria in flows where no vorticity is present, but where the free boundary has an active non-zero surface tension.

One context is in the theory of water waves, where there has been renewed interest in studying water waves with vorticity. Hur & Dyachenko [1,2] found numerically that a family of gravity waves with constant vorticity, and no surface tension, admit wave profiles that approach the limiting ‘Crapper wave’ solutions, as the effect of gravity vanishes. The steady waves of Crapper [3] are a well-known class of exact solutions for irrotational water waves with surface tension, but without gravity. Hur & Vanden-Broeck [4] took matters further and provided more numerical and asymptotic evidence of this curious reappearance of Crapper’s profiles. The connection of this constant vorticity water wave problem to Crapper’s waves has since been confirmed analytically [5].

This surprising connection between rotational flows without surface tension and irrotational flows with surface tension has also been noted in other flow scenarios. Crowdy *et al.* [6] found that a class of rotating hollow vortices, entities given the designation ‘H-states’, have exactly the same shapes as a class of stationary (non-rotating) hollow vortices with non-zero surface tension on the boundary: solutions to the latter problem had been found earlier by Crowdy [7] and Wegmann & Crowdy [8]. The H-state problem of a rotating hollow vortex can be viewed as a radial-geometry analogue of the constant vorticity water wave problem. Moreover, the problem of finding a stationary, non-rotating, hollow vortex with non-zero surface tension is the radial geometry analogue of Crapper’s problem of determining the shape of steady pure capillary waves.

The intriguing fact is that, in two different geometrical arrangements—the periodic geometry or a radial geometry—two physically distinct free boundary problems admit the same free boundary profiles as equilibrium solutions.

The work of Crowdy [7], extended later by Wegmann & Crowdy [8], emerged from insights gained from a new approach to free surface Euler flows with surface tension propounded by Crowdy [9] who, in turn, was motivated by observations due to Tanveer [10]. Crowdy [9] sought to understand how Crapper’s exact solutions relate to analytical observations made by Tanveer [10] on the functional form of (the derivative of) conformal mappings to the region exterior to a translating bubble with surface tension, another example of an irrotational free surface flow with surface tension akin to Crapper’s pure capillary wave problem. Crapper [3] used hodograph variables, not geometrical conformal mapping, to derive his solutions, but it is not clear from his approach why his ansatz works. Crowdy [9] gained more analytical insight by using a conformal mapping function $z = Z(\zeta)$ from a parametric unit disc to the physical z plane and showing how a Riccati-type structure to the analytically continued boundary condition allows deductions to be made on the functional form of $Z(\zeta)$. Crowdy’s approach followed that used by Tanveer [10] to study the problem of a steadily translating bubble with surface tension, a problem for which an isolated exact solution had been found by McLeod [11] but which, generically, does not admit exact solutions. Even so, Tanveer [10] showed that the square root of the derivative of the conformal mapping from a parametric unit ζ disc to the fluid region, or $\sqrt{dZ/d\zeta}$, is meromorphic in $|\zeta| > 1$, with an infinity of simple pole singularities clustering at infinity.

The goal of the present paper is to provide further elucidation into this landscape of results in the literature as well as to present several new results on existence, uniqueness, analytic form of solution and conformal mapping identities. Its structure is as follows.

First it is shown in §2 why the functional form of solutions to the rotating hollow vortex problem studied by Crowdy *et al.* [6] is the same as that of solutions of the problem of a non-rotating hollow vortex with surface tension as found by Crowdy [7] and Wegmann & Crowdy [8]. This is done by showing how both physical problems are equivalent to the same mathematical problem, here called problem 1, that can be shown to admit rational function solutions for

a conformal mapping function. In this sense, the two physical problems are mathematically analogous and share the same rational function conformal mappings as solutions.

It is then shown in §3 how generalizations in some sense of these two physical problems lead to a generalized mathematical problem called problem 2. The first of the two generalized physical problems is the problem of a hollow vortex in simple shear, and linear strain, but without surface tension. Llewellyn Smith & Crowdy [12] have studied the problem of a single hollow vortex in an irrotational linear strain, but without shear, and found a class of analytical solutions. Zannetti *et al.* [13] have found numerical solutions to the problem of a hollow vortex in simple shear, with no additional linear strain, and without surface tension. The second physical problem is that of a steadily translating bubble in irrotational flow and with surface tension on its boundary but unlike the hollow vortex problem, there is no circulation at ∞ ; this is a problem that has been studied earlier both analytically and numerically [10,11,14,15].

A new result of our work is to show that these two physical problems are mathematically analogous. We prove existence and uniqueness theorems for problem 2 for a range of parameters. Also, the analogy between the two problems immediately implies a host of new theoretical results: one consequence is that the numerically computed solutions of Zannetti *et al.* [13] can be deduced to share the same meromorphic structure of the square root derivative of a conformal mapping function $\sqrt{dZ/d\zeta}$ as found by Tanveer [10] for a steadily translating bubble with surface tension. Even more, the extended analysis of problem 2 here reveals the surprising result that, generically, both physical problems admit conformal mapping solutions wherein the mapping $Z(\zeta)$ itself, and not just its square root derivative $\sqrt{dZ/d\zeta}$, is meromorphic in $|\zeta| > 1$ with a cluster point of simple poles at infinity. This is itself surprising because, in general, one expects logarithmic singularities of $Z(\zeta)$ at any simple poles of $\sqrt{dZ/d\zeta}$. Furthermore, consistency between two different representations of the complex velocity for the bubble problem leads to a non-trivial conformal mapping identity that transcends the bubble problem. The analogy between the two physical problems in this case has some important caveats, however, and these are set out here in detail.

2. Two physical problems with exact solutions

This section describes the two physical problems shown in figure 1. In §2a, the problem of an isolated rotating hollow vortex with no surface tension on its boundary is formulated; it is shown on the left in figure 1. Section 2b, on the other hand, formulates the different problem of an isolated non-rotating hollow vortex with a non-zero surface tension active on its boundary. The formulations in these two subsections reveal a common mathematical structure to these distinct physical problems. Both are equivalent to solving a mathematical problem, set out in §2c, where it is given the designation ‘problem 1’.

(a) A rotating hollow vortex without surface tension

The challenge is to find a single hollow vortex with unchanging shape, also referred to as an ‘H-state’ by Crowdy *et al.* [6], with non-zero circulation Γ in steady solid body rotation with angular velocity Ω_0 . It is natural to move to a frame of reference co-rotating with the vortex so that its shape is stationary in this frame with its boundary constituting a streamline. The interior of the vortex is taken to a region of constant pressure. The flow $\mathbf{u} = (u, v)$ is incompressible so a streamfunction ψ can be introduced such that

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}. \quad (2.1)$$

Exterior to the vortex, the streamfunction ψ satisfies

$$\nabla^2 \psi = -\omega = 2\Omega_0. \quad (2.2)$$

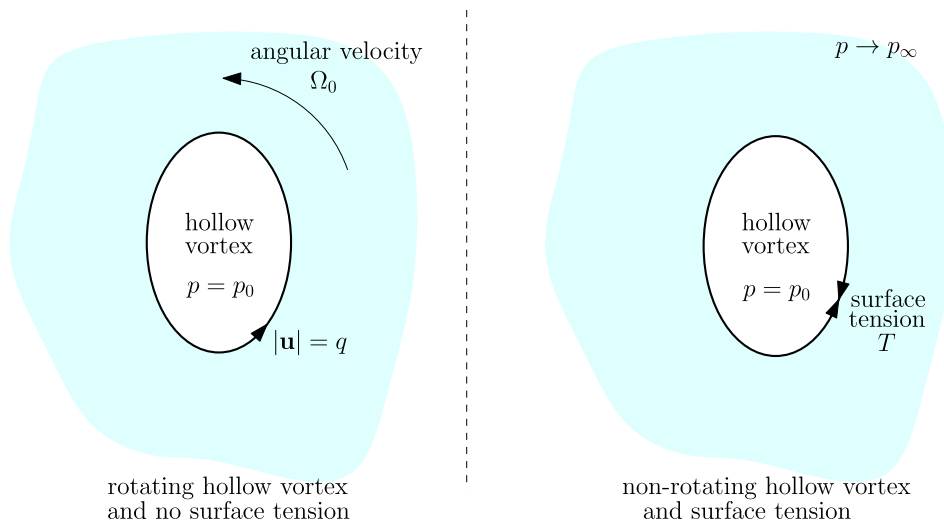


Figure 1. Two distinct physical problems whose solutions are equivalent to solving the mathematical problem 1, dependent on a non-dimensional parameter γ , and stated in §2c. For the problem on the left $\gamma = \Omega_0 L/q$; for the problem on the right $\gamma = (p_\infty - p_0)L/T$, where L is a length-scale associated with the vortex.

The kinematic condition that the vortex boundary is a streamline in the co-rotating frame, together with Bernoulli's theorem [16,17] and the condition that the pressure is continuous, implies that, on the vortex boundary,

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{u} \cdot \mathbf{t} = q, \quad (2.3)$$

where $\mathbf{n}$ is the outward normal to the vortex boundary and $\mathbf{t}$ is the tangent vector as the boundary is traversed in an anticlockwise direction. The constant q is the fluid speed around the boundary. On introducing the complex variable $z = x + iy$ the two boundary conditions (2.3) can be written in complex form as

$$u + iv = q \frac{d\mathcal{Z}}{ds}, \quad (2.4)$$

on the bubble boundary $z = \mathcal{Z}(s)$, where s is the arclength oriented counterclockwise and $d\mathcal{Z}/ds$ is the complex tangent. Equation (2.2) takes the form

$$\frac{\partial^2 \psi}{\partial z \partial \bar{z}} = \frac{\Omega_0}{2}, \quad (2.5)$$

which allows an integration with respect to z and $\bar{z}$:

$$\psi = \frac{\Omega_0}{2} z \bar{z} + \text{Im}[w(z)], \quad (2.6)$$

where the analytic function $w(z)$ is the complex potential for an irrotational flow component exterior to the vortex. Note that $w(z)$ is analytic in the fluid region except at infinity, where due to the non-zero circulation Γ at ∞ in a fixed (non-rotating) frame,

$$w(z) \sim -\frac{i\Gamma}{2\pi} \log z + o(1), \quad \text{as } |z| \rightarrow \infty. \quad (2.7)$$

From equations (2.1) and (2.6), it can be deduced that

$$u - iv = 2i \frac{\partial \psi}{\partial z} = i\Omega_0 \bar{z} + \frac{dw}{dz}. \quad (2.8)$$

It follows from equations (2.4) and (2.8) that, on the hollow vortex boundary,

$$q \frac{d\bar{\mathcal{Z}}(s)}{ds} = i\Omega_0 \overline{\mathcal{Z}(s)} + \xi_1(\mathcal{Z}(s)) \quad \text{and} \quad \xi_1(z) = \frac{dw}{dz}. \quad (2.9)$$

It is convenient to non-dimensionalize lengths using L (a length scale related to the hollow vortex area) and velocities using q , i.e.

$$\xi_1(z) = q\tilde{\xi}_1(z), \quad z = L\tilde{z} \quad \text{and} \quad s = L\tilde{s}. \quad (2.10)$$

Dropping tildes, the non-dimensional statement of the boundary condition (2.9) is

$$\frac{d\overline{\mathcal{Z}(s)}}{ds} = i\gamma \overline{\mathcal{Z}(s)} + \xi_1(\mathcal{Z}(s)) \quad \text{and} \quad \gamma = \frac{\Omega_0 L}{q}. \quad (2.11)$$

(b) A non-rotating hollow vortex with surface tension

Consider a different hollow vortex problem in which the hollow vortex is steady (i.e. not rotating) and now has non-zero surface tension T active on its boundary. This nonlinear free boundary problem was first considered by Crowdy [7], and later by Wegman & Crowdy [8]. Note that, in those earlier studies, instead of being called a hollow vortex, it was viewed as a constant-pressure bubble with non-zero circulation, which is essentially the definition of a hollow vortex [17], even if that terminology was not used there.

Let $\hat{\mathbf{u}}$ denote the two-dimensional fluid velocity and let its circulation be $\hat{\Gamma}$. The dynamic condition at the interface is that any pressure jump is balanced by surface tension $T\kappa$ where κ is the curvature,

$$p_0 - p = T\kappa. \quad (2.12)$$

According to the usual form of the steady version of Bernoulli's theorem [16,17], the pressure p in a frame of reference in which the interface is stationary is given by

$$\frac{p}{\tilde{\rho}} = H - \frac{|\hat{\mathbf{u}}|^2}{2}, \quad (2.13)$$

where H is the Bernoulli constant and $\tilde{\rho}$ is the fluid density. On evaluating equation (2.13) as $|z| \rightarrow \infty$, the Bernoulli constant can be related to the pressure at infinity p_∞ according to

$$H = \frac{p_\infty}{\tilde{\rho}}. \quad (2.14)$$

If the complex unit tangent vector on the boundary $d\mathcal{Z}/ds = e^{i\theta}$, where θ is the angle the tangent vector makes with the positive real axis, then

$$\kappa = \theta_s = -i \frac{d^2 \mathcal{Z}/ds^2}{d\mathcal{Z}/ds}. \quad (2.15)$$

Together equations (2.12)–(2.15) imply that, on the hollow vortex boundary,

$$\left(\frac{p_0 - p_\infty}{\tilde{\rho}} \right) + \frac{|\hat{\mathbf{u}}|^2}{2} = -\frac{iT}{\tilde{\rho}} \frac{d^2 \mathcal{Z}/ds^2}{d\mathcal{Z}/ds}. \quad (2.16)$$

The complex conjugate of equation (2.16) can be rearranged as

$$\left(\frac{p_0 - p_\infty}{\tilde{\rho}} \right) \frac{d\overline{\mathcal{Z}}}{ds} + \frac{1}{2} |\hat{\mathbf{u}}|^2 \frac{d\overline{\mathcal{Z}}}{ds} = \frac{iT}{\tilde{\rho}} \frac{d^2 \overline{\mathcal{Z}}}{ds^2}. \quad (2.17)$$

On multiplication by $-i$, and integration with respect to s , this boundary condition becomes

$$\frac{T}{\tilde{\rho}} \frac{d\overline{\mathcal{Z}}}{ds} = i \left(\frac{p_\infty - p_0}{\tilde{\rho}} \right) \overline{\mathcal{Z}} - \frac{1}{2} i \int^{\overline{\mathcal{Z}(s)}} |\hat{\mathbf{u}}|^2 d\overline{\mathcal{Z}}. \quad (2.18)$$

Since the flow $\mathbf{u}$ is irrotational and incompressible, there exists a complex velocity potential $\hat{w}(z)$ that is analytic in the fluid region with far-field behaviour

$$\hat{w}(z) \sim -\frac{i\hat{\Gamma}}{2\pi} \log z + o(1), \quad \text{as } |z| \rightarrow \infty \quad (2.19)$$

reflecting the non-zero circulation $\hat{\Gamma}$ of the vortex. In terms of this complex potential,

$$|\hat{\mathbf{u}}|^2 = \left| \frac{d\hat{w}}{dz} \right|^2 = \frac{d\hat{w}}{dz} \overline{\left(\frac{d\hat{w}}{dz} \right)}. \quad (2.20)$$

The streamline condition, $\text{Im}[\hat{w}] = 0$, on the hollow vortex boundary implies that

$$\frac{d\hat{w}}{dz} dz = \overline{\left(\frac{d\hat{w}}{dz} \right)} d\bar{z}, \quad (2.21)$$

hence condition (2.18), which also holds on the boundary, becomes

$$\left. \begin{aligned} \frac{2T}{\tilde{\rho}} \frac{d\bar{\mathcal{Z}}(s)}{ds} &= 2i \left(\frac{p_\infty - p_0}{\tilde{\rho}} \right) \bar{\mathcal{Z}}(s) + \xi_2(\mathcal{Z}(s)) \\ \xi_2(z) &\equiv -i \int^z \left(\frac{d\hat{w}(z')}{dz'} \right)^2 dz'. \end{aligned} \right\} \quad (2.22)$$

It is convenient to non-dimensionalize lengths with respect to L and velocities with respect to

$$v_T = \left(\frac{2T}{\tilde{\rho}L} \right)^{1/2}. \quad (2.23)$$

The non-dimensionalized form of equation (2.22) is then

$$\frac{d\bar{\mathcal{Z}}(s)}{ds} = i\gamma \bar{\mathcal{Z}}(s) + \xi_2(\mathcal{Z}(s)) \quad \text{and} \quad \gamma = \frac{2}{v_T^2} \left(\frac{p_\infty - p_0}{\tilde{\rho}} \right). \quad (2.24)$$

(c) Statement of problem 1

Determine a simple closed curve $z = \mathcal{Z}(s)$ that satisfies

$$\frac{d\mathcal{Z}(s)}{ds} = -i\gamma \mathcal{Z}(s) + \bar{\xi}(\bar{\mathcal{Z}}(s)), \quad (2.25)$$

for some a priori unknown analytic function $\xi(z)$ in the domain Ω exterior of $z = \mathcal{Z}(s)$ with

$$\xi(z) \sim O\left(\frac{1}{z}\right), \quad \text{as } z \rightarrow \infty, \quad (2.26)$$

where $\gamma \in \mathbb{R}$ is specified.

(d) Two physical problems equivalent to problem 1

On comparing the complex conjugate of equation (2.11) with equation (2.25), it is clear that the problem of a rotating hollow vortex without surface tension formulated in §2a amounts to solving problem 1 with

$$\gamma = \frac{\Omega_0 L}{q} \quad \text{and} \quad \xi(z) = \xi_1(z) = \frac{dw}{dz} \sim -\frac{i\Gamma_0}{z} \quad \text{as } |z| \rightarrow \infty, \quad \text{where } \Gamma_0 = \frac{\Gamma}{2\pi}, \quad (2.27)$$

where equation (2.7) has been used to verify the far-field condition (2.26).

Similarly, on comparing the complex conjugate of equation (2.24) with equation (2.25), the problem of a non-rotating hollow vortex with surface tension formulated in §2b amounts to solving problem 1 with

$$\left. \begin{aligned} \gamma &= \frac{2}{v_T^2} \left(\frac{p_\infty - p_0}{\tilde{\rho}} \right) \\ \xi(z) &= \xi_2(z) \sim -\frac{i\Gamma_0}{z} \quad \text{as } |z| \rightarrow \infty, \quad \text{where } \Gamma_0 = \left(\frac{\hat{\Gamma}}{2\pi} \right)^2, \end{aligned} \right\} \quad (2.28)$$

where equation (2.19) has been used in the definition (2.22) to verify the far-field condition (2.26).

As already mentioned, the rotating hollow vortex without surface tension was solved by Crowdy *et al.* [6]; similarly, the problem of a non-rotating hollow vortex with surface tension had been solved earlier by Crowdy [7] and Wegmann & Crowdy [8]. These studies made use of a conformal mapping from the interior or exterior of a unit disc, in a parametric plane, to describe the equilibrium flow domains and free boundaries in both problems. It was remarked upon by Crowdy *et al.* [6] that the solutions of the rotating hollow vortex (i.e. the H-state solutions) have the same functional form as the solutions to the quite different physical problem involving non-zero surface tension solved by Crowdy [7] and Wegmann & Crowdy [8]. This circumstance is unexpected but is explained by the observation just made that both physical problems are equivalent to the mathematical problem 1.

Solutions to problem 1 are not unique. Indeed, it is known [6–8] that problem 1 admits a family of analytical solutions expressed using conformal mappings $z = Z(\zeta)$ from the interior of a unit ζ disc to an unbounded region in a z plane with N -fold rotational symmetry where $N \geq 2$. The mappings are given by the rational functions

$$z = Z(\zeta) = -\frac{L}{\zeta} \left[1 + \frac{4N}{(N-1)^2} \frac{\zeta^N}{\zeta^N - c^N} \right], \quad 1 < c_{\text{crit}}^{(N)} < c, \quad N \geq 2. \quad (2.29)$$

In what follows, the scaling $L = 1$ is chosen. For this solution, it is readily found that

$$\left(\frac{dZ}{d\zeta} \right)^{1/2} = \frac{(N-1)c^N + (N+1)\zeta^N}{\zeta(N-1)(c^N - \zeta^N)}, \quad (2.30)$$

where γ is related to the real parameter c and integer N via

$$\gamma = \frac{(N-1)^2(c^{2N} - 1)}{2(N+1 + (N-1)c^{2N})}. \quad (2.31)$$

The limit $c \rightarrow \infty$ retrieves the circular hollow vortex of radius L . An interesting feature of these solutions is that the limiting states are found to pinch off, i.e. to lose univalence, when $c = c_{\text{crit}}^{(N)}$; these critical values, for the first few values of N , are recorded in [6]. For $c_{\text{crit}}^{(N)} < c < \infty$, the mapping (2.29) is univalent having N simple pole singularities in $|\zeta| > 1$, at the points $\{\zeta_j = c e^{2\pi i j/N} | j = 0, 1, \dots, N-1\}$. The value of γ given in equation (2.31) is forced by the requirement that $\xi(z)$ is analytic in the fluid region. This can be seen by writing equation (2.25) in terms of the mapping function (2.29) and analytically continuing $\xi(Z(\zeta))$ into the unit disc $|\zeta| < 1$. The resulting function is found to have N simple pole singularities inside the unit disc, implying simple pole singularities of $\xi(z)$ in the fluid region. By the N -fold rotational symmetry, the single condition (2.31) is enough to render all these poles removable as needed to avoid unwanted singularities in the fluid. The corresponding Γ_0 in equation (2.27) or equation (2.28) is given by

$$\Gamma_0 = \frac{(N+1)(Nc^N + N - c^N + 1)(Nc^N - N - c^N - 1)}{2(N + Nc^{2N} + 1 - c^{2N})(N-1)}. \quad (2.32)$$

Letting $K(\zeta) = -i\xi(Z(\zeta))$, it may be shown that

$$K(\zeta) = D\zeta \frac{(N-1)\zeta^N + (N+1)c^N}{(N+1)\zeta^N + (N-1)c^N}, \quad (2.33)$$

where

$$D = \frac{(N+1)((c^N + 1)N - c^N + 1)((c^N - 1)N - c^N - 1)}{(2c^{2N} + 2)N^2 + 4N - 2c^{2N} + 2}. \quad (2.34)$$

Thus $\xi(z)$, and therefore $K(\zeta)$, is completely determined. However, for the non-rotating hollow vortex, since $dw/d\zeta = i\hat{r}/2\pi\zeta$, from ξ_2 in equation (2.22),

$$K(\zeta) = -i\xi_2(Z(\zeta)) = \Gamma_0 \int_0^\zeta \frac{d\zeta}{\zeta^2 dZ/d\zeta}. \quad (2.35)$$

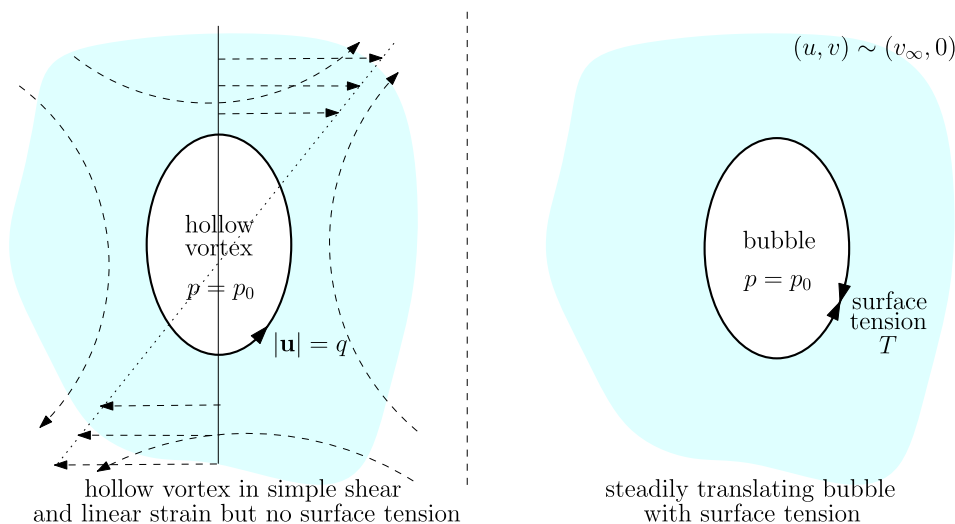


Figure 2. Two distinct physical problems whose solutions are equivalent to solving the mathematical problem 2, dependent on two non-dimensional parameters γ and b , and stated in §3c.

This separate expression for K is found to be consistent with [equation \(2.33\)](#) since from [equations \(2.33\)](#) and [\(2.29\)](#), it is found that

$$\frac{dK}{d\zeta} = \frac{\Gamma_0}{\zeta^2 dZ/d\zeta}. \quad (2.36)$$

Since problem 1 is not affected by the physical context, [equation \(2.36\)](#) may be viewed as a conformal map identity.

It is worth mentioning that Crowdy *et al.* [6] have discussed the connections between this class of free surface shapes given by rational function conformal mappings (whose derivatives are a perfect square of another rational function) and the theory of so-called double quadrature domains.

Remarkably, a similar conformal mapping identity to [equation \(2.36\)](#) can be deduced for the generalized problem 2 in the next section when parameters are constrained in a certain manner (theorem 3.11) though no closed form solution exists in that case when $\gamma \neq 0$. As with problem 1, recognizing this identity became possible only once two disparate physical problems, shown in [figure 2](#), had been connected mathematically.

3. Two physical problems with meromorphic solutions

The two physical problems described in §2a,b are not the only pair for which a mathematical analogy exists via a more general mathematical problem. In this section, we present two other free boundary problems arising in different physical contexts with a common mathematical thread. This second mathematical problem, referred to as problem 2,¹ is stated in §3c. Again, the analogy means that certain analytical results known for one problem are immediately shared by the other problem, yielding valuable new insights. However, in this case, the analogy also comes with a caveat that is studied in detail in what follows.

¹McLeod's [11] variational approach hints at both problems 1 & 2 having broader relevance beyond the ones discussed here.

(a) A hollow vortex in simple shear and linear strain

Consider a hollow vortex in a simple shear flow superposed with a linear strain so that, in the far field,

$$\mathbf{u} \sim (\dot{\alpha}y + \epsilon_x y - \epsilon_y x, \epsilon_x x + \epsilon_y y) + o(1), \quad (3.1)$$

where $\dot{\alpha}$ is the shear rate, and ϵ_x and ϵ_y are strain rates. It follows that, as $|z| \rightarrow \infty$,

$$u - iv \sim i\gamma\bar{z} - i\bar{b}z + \hat{c}_0 - \frac{i\Gamma}{2\pi z} + O(z^{-2}), \quad (3.2)$$

where

$$\gamma = \frac{\dot{\alpha}}{2} \quad \text{and} \quad b = \epsilon_x + i\epsilon_y + \frac{\dot{\alpha}}{2}. \quad (3.3)$$

Note that, at this stage, b is allowed to be complex, but we restrict to $\gamma < \frac{1}{2}$ in our analysis. The other constants $\Gamma, \hat{c}_0$ are determined as part of the solution.

The problem with $\epsilon_x = \epsilon_y = 0$, corresponding to a hollow vortex in simple shear, was considered earlier in the numerical study of Zannetti *et al.* [13] where it is assumed that the flow has uniform vorticity $-\dot{\alpha}$ everywhere exterior to the hollow vortex. The problem considered here is therefore a generalization of that problem in allowing for an additional linear strain to be imposed. Llewellyn Smith & Crowdy [12] have studied the problem of a single hollow vortex in an irrotational linear strain so the problem considered here also corresponds to a generalization of that study.

The formulation of this hollow vortex problem is also a generalization of the rotating hollow vortex problem given in §2a. Let Ω denote the fluid domain exterior to a simple closed curve $z = \mathcal{Z}(s)$ in the complex z plane. It follows from (3.2) that in the flow domain Ω exterior of the hollow vortex boundary $z = \mathcal{Z}(s)$, we must have

$$u - iv = i\gamma\bar{z} + \xi_3(z), \quad (3.4)$$

where $\xi_3(z)$ is analytic in Ω , except at ∞ , where equation (3.2) implies:

$$\xi_3(z) \sim -i\bar{b}z + \hat{c}_0 - \frac{i\Gamma}{2\pi z} + O(z^{-2}), \quad (3.5)$$

where $\hat{c}_0$ is some constant. On the hollow vortex boundary $z = \mathcal{Z}(s)$, the Bernoulli condition again implies the fluid velocity is of constant magnitude q . It follows from equations (2.4) and (2.8) that, on the hollow vortex boundary,

$$q \frac{d\bar{\mathcal{Z}}}{ds} = i\gamma\bar{\mathcal{Z}} + \xi_3(\mathcal{Z}). \quad (3.6)$$

It is convenient to non-dimensionalize lengths with respect to L and velocities with respect to q , so that equation (2.9) becomes

$$\frac{d\bar{\mathcal{Z}}(s)}{ds} = i\gamma\bar{\mathcal{Z}}(s) + \xi_3(\mathcal{Z}(s)) \quad \text{and} \quad \gamma = \frac{\dot{\alpha}L}{2q} \quad (3.7)$$

for some analytic function ξ_3 in Ω except at ∞ , where

$$\xi_3(z) \sim -i\bar{b}z + O(1) \quad \text{and} \quad b = \frac{(\epsilon_x + i\epsilon_y + \dot{\alpha}/2)L}{q}. \quad (3.8)$$

(b) A steadily translating bubble with surface tension

A problem similar to the hollow vortex problem with surface tension given in §2b will now be described. A hollow vortex with surface tension is really a constant pressure bubble having non-zero circulation around it, and with surface tension active on its boundary. If the circulation is absent, the term ‘vortex’ becomes inappropriate, and it is more accurate to call it a ‘bubble’ instead.

Consider therefore a bubble, without circulation but with surface tension on its boundary, translating steadily at speed $(-v_\infty, 0)$ in an inviscid fluid. It is natural to move to a frame of

reference co-travelling with the bubble so that, in this frame of reference, the bubble is steady and its boundary is a streamline. The complex potential $W(z)$ in this frame has the far-field form

$$W(z) \sim v_\infty z + \mathcal{O}(z^{-1}), \quad (3.9)$$

where there is no logarithmic term owing to the fact that the bubble has no circulation. It is easy to check that the formulation of §2b carries through with the Bernoulli constant H now given by

$$H = \frac{p_\infty}{\tilde{\rho}} + \frac{v_\infty^2}{2}. \quad (3.10)$$

Consequently, an equation akin to [equation \(2.22\)](#) can be derived but slightly modified to

$$\left. \begin{aligned} \frac{2T}{\tilde{\rho}} \frac{d\overline{\mathcal{Z}(s)}}{ds} &= 2i \left(\frac{p_\infty - p_0}{\tilde{\rho}} + \frac{v_\infty^2}{2} \right) \overline{\mathcal{Z}(s)} + \xi_4(\mathcal{Z}(s)) \\ \text{and} \quad \xi_4(z) &\equiv -i \int^z \left(\frac{dW(z')}{dz'} \right)^2 dz'. \end{aligned} \right\} \quad (3.11)$$

Rescaling lengths with respect to L and velocities with respect to v_T defined in [equation \(2.23\)](#), the non-dimensional version of [equation \(3.11\)](#) is

$$\frac{d\overline{\mathcal{Z}(s)}}{ds} = i\gamma \overline{\mathcal{Z}(s)} + \xi_4(\mathcal{Z}(s)), \quad (3.12)$$

where with an obvious adjustment, $\xi_4(z)$ can be written as

$$\xi_4(z) = -ibz - i \int_\infty^z \left\{ \left(\frac{dW(z')}{dz'} \right)^2 - b \right\} dz' \quad (3.13)$$

with the Weber number b and the non-dimensional Bernoulli constant γ defined as

$$b = \left(\frac{v_\infty}{v_T} \right)^2 \quad \text{and} \quad \gamma = \frac{2}{v_T^2} \left(\frac{p_\infty - p_0 + (1/2)\tilde{\rho}v_\infty^2}{\tilde{\rho}} \right). \quad (3.14)$$

In [equation \(3.13\)](#), any additive constant on the right-hand side is eliminated through translation of z . From the non-dimensional version of [equation \(3.9\)](#),

$$\frac{dW}{dz} \sim b^{1/2} + \mathcal{O}(z^{-2}), \quad (3.15)$$

hence [equation \(3.13\)](#) gives

$$\xi_4(z) \sim -ibz + \mathcal{O}(z^{-1}), \quad \text{as } z \rightarrow \infty. \quad (3.16)$$

Note that the asymptotic conditions (3.15) for $b=0$ are not equivalent to the condition (2.7) for the non-rotating hollow vortex with surface tension.

(c) Statement of problem 2

Determine a simple closed curve $z = \mathcal{Z}(s)$ that satisfies

$$\frac{d\mathcal{Z}}{ds} = -i\gamma \mathcal{Z} + \bar{\xi}(\overline{\mathcal{Z}(s)}), \quad (3.17)$$

for some a priori unknown analytic function $\xi(z)$ in the domain Ω exterior of $z = \mathcal{Z}(s)$, except at ∞ , where

$$\xi(z) \sim -ibz + \mathcal{O}(1), \quad \text{as } z \rightarrow \infty \quad (3.18)$$

and where (γ, b) is specified. A priori regularity assumptions on $\partial\Omega$, best stated in terms of a conformal mapping, are given in §3e.

(d) Two physical problems equivalent to problem 2

On comparing (the complex conjugate of) equations (3.7) and (3.8) with equations (3.17) and (3.18), it is clear that the problem of a hollow vortex in simple shear and linear strain as described in §3a reduces to problem 2 with

$$\gamma = \frac{\dot{\alpha}L}{2q}, \quad \xi(z) = \xi_3(z) \quad \text{and} \quad b = \frac{(\epsilon_x + i\epsilon_y + \dot{\alpha}L/2)}{q}. \quad (3.19)$$

Similarly, on comparing (the complex conjugate of) equations (3.12) and (3.16) with equations (3.17) and (3.18) the steadily translating bubble problem of §3b is also equivalent to problem 2 with

$$\gamma = \frac{2}{v_T^2} \left(\frac{p_\infty - p_0 + (1/2)\tilde{\rho}v_\infty^2}{\tilde{\rho}} \right), \quad \xi(z) = \xi_4(z) \quad \text{and} \quad b = \left(\frac{v_\infty}{v_T} \right)^2 \quad (3.20)$$

although, in this case, equation (3.13) must also hold whose only consequence is a constraint $b = B(\gamma)$, to be discussed later. The latter constraint does not apply to the hollow vortex in simple shear and linear strain.

Already the formulation above has unveiled a significant new result: that the problem of a hollow vortex in shear considered by Zannetti *et al.* [13] generalized to include linear strain, is mathematically analogous to the problem of a steadily translating bubble with surface tension considered by Tanveer [10], although they are not mathematically identical unless $b = B(\gamma)$. This has many previously unrecognized theoretical implications for the solution structure to the problem considered numerically by Zannetti *et al.* [13].

While problem 1 is known to admit classes of exact solutions given by rational function conformal mappings $z = Z(\zeta)$ having a finite number of simple pole singularities in $|\zeta| > 1$, much less is currently known about solutions to problem 2 from the existing literature. Tanveer [10] showed that the square root derivative, to be denoted henceforth by $Z_\zeta^{1/2}(\zeta)$, of the conformal mapping function $z = Z(\zeta)$ from the interior of a unit disc in a parametric ζ plane to the fluid region exterior to a steadily translating bubble is generically meromorphic in $|\zeta| > 1$ with an infinite number of simple pole singularities in $|\zeta| > 1$. The same analysis holds more generally for problem 2 for any (γ, b) over a certain range, that includes the case $b = \gamma$ studied earlier by Zannetti *et al.* [13]. The remainder of this paper is devoted to exploring further theoretical consequences concerning solutions to problem 2 as they pertain to the distinct physical problems described in §3a,b that include new existence and uniqueness results. One important result is that it is possible to show, for both problems, that not only $Z_\zeta^{1/2}(\zeta)$ but also the mapping $Z(\zeta)$ is generically meromorphic in $|\zeta| > 1$ with an infinity of simple pole singularities. This result—which is surprising since $Z(\zeta)$ is generically expected to have logarithmic singularities at any simple poles of $Z_\zeta^{1/2}(\zeta)$ —is new for *both* physical problems described in §3a,b. Furthermore, imposition of a scalar constraint $b = B(\gamma)$ on a solution to problem 2 results in equation (3.13) being automatically satisfied for the bubble problem. In a context more general than the bubble, this automatic satisfaction is equivalent to a conformal map identity (theorem 3.11) that would have been difficult to guess, or even directly verify, without recognizing the mathematical analogy between the two distinct physical problems.

(e) Conformal mapping representation for solutions of problem 2

As in the previous section, let $z = Z(\zeta)$ be the conformal map from the unit circle $\mathbb{D}$ to the exterior domain Ω in problem 2, whether it is the boundary of a bubble or vortex, with series representation

$$Z(\zeta) = a \left(\sum_{n=0}^{\infty} c_n \zeta^{n-1} \right), \quad \text{where } c_0 = -1,$$

for $a > 0$ that fixes the rotational degree of freedom in the Riemann mapping theorem. Furthermore, we choose length-scale L so that $a = 1$, and therefore

$$Z(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^{n-1}, \quad \text{where } c_0 = -1. \quad (3.21)$$

Contour integration shows that the area A of the region inside the curve $z = Z(s)$ is

$$A = \pi \left(1 - \sum_{m=2}^{\infty} (m-1) |c_m|^2 \right). \quad (3.22)$$

If, instead, the area of the hollow vortex or bubble is fixed to be π , say, then Z must be rescaled through multiplication by $a = \sqrt{\pi/A}$.

Define the analytic function $\bar{K}(\zeta)$ in $\mathbb{D}$ so that

$$\xi(Z(\zeta)) = i\bar{K}(\zeta). \quad (3.23)$$

Consider first the hollow vortex problem. A translational degree of freedom can be used to select c_1 in [equation \(3.21\)](#) so that, for this problem, [equation \(3.5\)](#) implies

$$\bar{K}(\zeta) \sim \bar{b}\zeta^{-1} + \left(\frac{\Gamma}{2\pi} - \bar{b}c_2 \right) \zeta + o(\zeta), \quad \text{as } \zeta \rightarrow 0. \quad (3.24)$$

Therefore, for the conjugate function K ,

$$K(\zeta^{-1}) \sim b\zeta + \left(\frac{\Gamma}{2\pi} - b\bar{c}_2 \right) \zeta^{-1} + o(\zeta^{-1}), \quad \text{as } \zeta \rightarrow \infty \quad (3.25)$$

implying a Laurent series in the form

$$K(\zeta^{-1}) = \sum_{j=0}^{\infty} k_j \zeta^{1-j}, \quad \text{where } k_0 = b, \quad k_1 = 0 \quad (3.26)$$

valid for $|\zeta| > 1$. More generally, [equation \(3.26\)](#) applies to problem 2 as may be deduced from [equations \(3.18\)](#), [\(3.21\)](#) and [\(3.23\)](#). For the hollow vortex problem, equating the coefficient of ζ^{-1} in [equations \(3.25\)](#) and [\(3.26\)](#),

$$\frac{\Gamma}{2\pi} - \bar{b}c_2 = \bar{k}_2, \quad (3.27)$$

which provides the relevant value of Γ . In §2 using exact solutions for $\gamma = 0, b \neq 0$, the parameter Γ is found to be real as required for the hollow vortex problem. This is not surprising since through a ζ -rotation, the exact solution [equation \(3.35\)](#) for complex b becomes equivalent to a solution with real b for which all series coefficients are real as is Γ . For real $\gamma \neq 0$, whether complex b can still result in real Γ is an open question.

In the analysis of problem 2 to follow, which is more general than the hollow vortex problem, we restrict to a range of real γ values and generally assume b to be complex.

For the bubble problem, although not generally for problem 2, [equation \(3.13\)](#) implies

$$\bar{K}(\zeta) = -\bar{b} \int_0^\zeta \left(\frac{(1-\zeta'^2)^2}{\zeta'^4 Z_\zeta(\zeta')} - \frac{1}{\zeta'^2} \right) d\zeta' + \frac{\bar{b}}{\zeta}. \quad (3.28)$$

When [equation \(3.28\)](#) applies, the coefficient of ζ in the Taylor series at the origin gives

$$\bar{k}_2 = \bar{b}(2 + c_2). \quad (3.29)$$

It is emphasized that [equation \(3.29\)](#) does not apply for general solutions of problem 2, only to those of its solutions for which [\(3.28\)](#) holds as for a bubble.

Using the relation between the arclength parametrization and conformal variables on $\partial\mathbb{D}$:

$$Z_s = -\frac{i\zeta Z_\zeta}{|Z_\zeta|} = -\frac{i\zeta Z_\zeta^{1/2}(\zeta)}{[\bar{Z}_\zeta(\zeta^{-1})]^{1/2}}, \quad (3.30)$$

and therefore, [equation \(3.17\)](#) (or [equation \(3.7\)](#)) implies

$$\frac{\zeta Z_\zeta^{1/2}(\zeta)}{[\bar{Z}_\zeta(\zeta^{-1})]^{1/2}} = \gamma Z + K(\zeta^{-1}). \quad (3.31)$$

(i) *A priori* regularity hypothesis on $\partial\Omega$

We seek a solution of problem 2 where the Laurent series at the origin of $Z_\zeta^{1/2}$ associated with the conformal map $Z: \mathbb{D} \rightarrow \Omega$ is absolutely convergent and non-zero in $|\zeta| \leq 1$. This is a condition on the regularity of the boundary $\partial\Omega$ allowed for in the solution to problem 2.

It is convenient to define

$$y(\zeta) = Z_\zeta^{1/2}(\zeta). \quad (3.32)$$

Then $\bar{y}(\zeta^{-1}) = \bar{Z}_\zeta^{1/2}(\zeta^{-1})$. On multiplying [equation \(3.31\)](#) by $\bar{y}(\zeta^{-1})$, it follows that

$$\zeta y(\zeta) = \gamma \bar{y} \left(\frac{1}{\zeta} \right) Z(\zeta) + \bar{y} \left(\frac{1}{\zeta} \right) K(\zeta^{-1}). \quad (3.33)$$

Remark 3.1. It is easy to show that the assumed regularity for $y = Z_\zeta^{1/2}$ implies that the series for Z_ζ , $Z(\zeta)$ and $\bar{y}(\zeta^{-1})$ are each absolutely convergent on $|\zeta| = 1$. From [equation \(3.33\)](#), the same would be true for $\bar{y}(1/\zeta)K(\zeta^{-1})$

(f) The special case $\gamma = 0$ with $b \neq 0$

The special case $\gamma = 0$ with $b \neq 0$ is important for two reasons.

First, it turns out that exact solutions can be found in this case for *both* physical problems just described and these have already been reported in the literature: Llewellyn Smith & Crowdy [12] found the exact solutions for a hollow vortex without surface tension in a pure linear straining flow with no vorticity or uniform flow component in the far-field; McLeod [11] found an exact solution for a steadily translating bubble with surface tension. Until the present work, however, it has not been appreciated that there is a mathematical connection between these apparently unrelated exact solutions to distinct physical problems.

Second, the aforementioned exact solutions, which will be rederived in a novel way in what follows, allow us to exemplify, in an explicit fashion, the main idea behind later results in the more general case of more general (γ, b) . As will be proved in the electronic supplementary material where ideas in [18–20] are found to be useful, for specified real $\gamma < \frac{1}{2}$ and generally complex b satisfying conditions in theorem 3.3, there exists a locally² unique solution to problem 2. And, rather surprisingly, while the form of $\xi(z) = \xi_3(z)$ in (3.19) for the hollow vortex problem appears not to have any ramifications, the only consequence of the different form of $\xi(z) = \xi_4(z)$ in [equation \(3.13\)](#) for the bubble problem is in constraining $b = B(\gamma)$ so as to satisfy a scalar constraint (this is [equation \(3.50\)](#) in the sequel). This difference between the two problems can be exemplified quite explicitly when $\gamma = 0$, as will now be shown.

When $\gamma = 0$, [equation \(3.33\)](#) implies

$$y(\zeta) = \zeta^{-1} \bar{y} \left(\frac{1}{\zeta} \right) K(\zeta^{-1}). \quad (3.34)$$

Since for small ζ , $y(\zeta) \sim \zeta^{-1} + O(\zeta)$ and $\bar{K}(\zeta) \sim \bar{b}\zeta^{-1} + O(\zeta)$, it follows that the right-hand side of [equation \(3.34\)](#) behaves as $b\zeta + O(\zeta^{-1})$ as $\zeta \rightarrow \infty$. Since there are no other singularities of the

²Multiple isolated solutions for the same (γ, b) have not been ruled out.

right-hand side in $|\zeta| > 1$ and the singularity in $|\zeta| \leq 1$ can only be at the origin, it follows that

$$y(\zeta) = Z_\zeta^{1/2} = \frac{1}{\zeta} + b\zeta, \quad \text{implying } Z(\zeta) = -\frac{1}{\zeta} + 2b\zeta + \frac{b^2}{3}\zeta^3. \quad (3.35)$$

It is necessary to restrict $|b| < 2\sqrt{3} - 3$ in order that $Z(\zeta)$ is univalent (see lemma 3.28). Corresponding to this exact solution, equation (3.34) implies that

$$\bar{K}(\zeta) = \frac{(\zeta^2 + b)}{\zeta(1 + b\zeta^2)}, \quad (3.36)$$

and hence from the Laurent expansion (equation (3.26)) of $K(\zeta^{-1})$ at ∞ , it follows that

$$k_2 = 1 - |b|^2. \quad (3.37)$$

Since equation (3.35) implies $c_2 = 2b$, equation (3.37) implies

$$\frac{\Gamma}{2\pi} = 1 + |b|^2, \quad (3.38)$$

confirming that Γ is real, as necessary for the solution to satisfy the physical constraints of the problem statement.

In equation (2.21) of their paper, Llewellyn Smith & Crowdy [12] report the solutions for a hollow vortex in a nonlinear strain of degree $n - 1$ to be given by conformal mappings

$$Z^{\text{LS\&C}}(\zeta) = \hat{a} \left(\frac{1}{\zeta} - \frac{2i\beta}{n-1} \zeta^{n-1} + \frac{\beta^2}{2n-1} \zeta^{2n-1} \right), \quad (3.39)$$

where β is a real parameter. It is clear that equations (3.35) and (3.39) are identical when $n = 2$ (corresponding to a far-field linear strain) and $\hat{a} = -1$ with $b = i\beta$. It is important to note that Llewellyn Smith & Crowdy [12] found the result (3.39) using free streamline theory rather than the alternative construction above where the strategy is to seek solutions to problem 2. Moreover, Llewellyn Smith & Crowdy [12] did not note that $y = Z_\zeta^{1/2}$ is a simple rational function as in equation (3.35), an interesting feature that appears to have gone unrecognized. Thus, understanding this hollow vortex problem through the lens of the mathematical problem 2 has provided new insights on an established result, as well as a novel derivation that is arguably simpler than the free streamline theory approach of Llewellyn Smith & Crowdy [12].

There is more to the story for the steadily translating bubble with surface tension, however. It turns out that there is a single additional constraint on the system deriving from the fact that $\xi(z)$ for this problem is $\xi_4(z)$ as given by equation (3.13). In this case since both equations (3.37) and (3.29) hold for some real $b > 0$, it follows that

$$b(2 + 2b) = 1 - b^2 \quad (3.40)$$

implying $b = \frac{1}{3}$ for which equation (3.35) becomes

$$Z_\zeta(\zeta) = \left(\frac{1}{\zeta} + \frac{\zeta}{3} \right)^2 \quad \text{and} \quad Z(\zeta) = -\zeta^{-1} + \frac{2}{3}\zeta + \frac{1}{27}\zeta^3. \quad (3.41)$$

This is the solution of McLeod [11] who derived it quite differently, and without consideration of the mathematical problem 2 as has been done here. From differentiation of equation (3.28),

$$\bar{K}_\zeta = -\frac{3(1 - \zeta^2)^2}{\zeta^2(\zeta^2 + 3)^2}, \quad (3.42)$$

which agrees with the derivative of equation (3.36) when $b = \frac{1}{3}$. Thus, for the bubble the only consequence of the independent expression of $\bar{K}(\zeta)$ in equation (3.28), i.e. for $\xi(z) = \xi_4(z)$ in equation (3.13) is the constraint $b = \frac{1}{3}$. Indeed, an integration of equation (3.42) with respect to ζ for $b = 1/3$ gives rise to a special case ($\gamma = 0$) of the conformal map identity equation (3.51) to follow that is valid more generally for real $\gamma < \frac{1}{2}$ when $b = B(\gamma)$.

The remainder of this article is concerned with rigorously establishing that the same paradigm elucidated above for $\gamma = 0$ and $b \neq 0$ carries over for any specified real $\gamma < \frac{1}{2}$ and in proving new existence and uniqueness results for $\gamma \neq 0$. In the latter case, more general methods must be used since exact solutions are not available even though the mapping functions, and their square-root derivatives, still turn out to have an interesting meromorphic structure.

(g) The case $\gamma \neq 0$

The main theoretical results are given in the theorems below. To the best of the authors' knowledge, the contents in theorems 3.3 and 3.11 are entirely new, while theorem 3.7 builds on the results in Tanveer [10].

Theorem 3.3 concerns existence and uniqueness of the solution to problem 2 over a range of γ, b . This includes two parts: demonstrating (i) that equation (3.33) has a unique solution and (ii) that this solution corresponds to a univalent map $Z(\zeta)$. These results are complemented by numerical investigations, presented in appendix A, outside the range where this theorem applies. In theorem 3.7, it is proven, similar to the results on an inviscid bubble established by Tanveer [10], for any solution to problem 2 with $\gamma \neq 0$, even without any constraint between b and γ , $y = Z_\zeta^{1/2}(\zeta)$ is meromorphic. The new result that $Z(\zeta)$ is itself meromorphic is also established. These results are obtained by showing that an analytically continued equation for these functions in $|\zeta| > 1$ has a Riccati form. The mathematical identity in theorem 3.11 follows from the uniqueness of the solution to problem 2 and an independent existence theorem for the bubble; together, they imply that the two expressions for K in equations (3.28) and (3.31) are consistent when $b = B(\gamma)$.

Definition 3.2.

$$\beta = \frac{b}{1-2\gamma}, \quad b_1 = |\beta|, \quad \lambda = \frac{\gamma}{1-2\gamma}, \quad \gamma_1 = |\lambda|, \quad \mu = 2\gamma_1 b_1 \left(\frac{14}{3} + 2b_1 \right) \quad (3.43)$$

and

$$\mu_v = 1 - 2b_1 - \frac{1}{3}b_1^2 - \frac{4}{9}\pi\gamma_1 b_1^2(7 + 2b_1)(4\gamma_1 b_1^3 + 14\gamma_1 b_1^2 + 3b_1 + 3). \quad (3.44)$$

Theorem 3.3. *If (b, λ) satisfies the restriction*

$$\mu < 1 \quad \text{and} \quad \mu_v > 0, \quad (3.45)$$

where μ, μ_v is defined in equations (3.43) and (3.44), there is a locally unique solution of problem 2 within the class of boundary curves $\partial\Omega$ for which the corresponding conformal map $Z: \mathbb{D} \rightarrow \Omega$ with the series representation

$$Z_\zeta^{1/2}(\zeta) = \frac{1}{\zeta} + \sum_{j=2}^{\infty} a_j \zeta^{j-1} \quad (3.46)$$

is absolutely convergent and non-zero on $|\zeta| = 1$. Furthermore, for this solution, analyticity of $Z(\zeta)$ extends to $|\zeta| = 1$, implying analytic $\partial\Omega$, and all odd coefficients $a_{2j-1} = 0 = c_{2j-1}$ for $j \geq 2$. Furthermore, the corresponding $\xi(z)$ is analytic on $\partial\Omega$. Furthermore, when b is real so are $\{a_j\}_{j=2}^{\infty}$.

The proof is given later in §3k.

Remark 3.4. This theorem is a consequence of later propositions 3.20 and 3.30 that require conditions (3.87) and (3.102), which may be checked to be equivalent to equation (3.45). For the case $b = \gamma$ (shear only), both conditions were met when $\gamma \in (-0.4375, 0.147)$, though numerical evidence presented later suggests the existence of a univalent $Z(\zeta)$ for $\gamma \in (-\infty, 0.183)$. For fixed $b \in (-\frac{1}{3}, \frac{1}{3})$, both conditions were met when $\gamma \in (-\infty, 0.037)$. This includes the inviscid bubble when the constraint $b = B(\gamma)$ holds to satisfy equation (3.50).

Remark 3.5. The proofs in both propositions 3.20 and 3.30 relied on smallness of γ_1 or of b_1 . Thus, it is not surprising that the proof did not extend to the entire range for which there is numerical evidence for the existence of a solution. However, if instead $\mathbf{a}^0$ is replaced by some empirically computed approximate solution, and the analysis of the error $\mathbf{a} - \mathbf{a}^0$ is pursued, it

would be possible to extend the proof to other ranges of (γ, b) . This approach is, however, tedious and not pursued here.

Remark 3.6. Since $K(\zeta^{-1})$ is uniquely determined through equations (3.108) and (3.111), so is $\xi(z)$ identified through the relationship $\xi(Z(\zeta)) = i\bar{K}(\zeta)$. This means that the velocity $u - iv = i\gamma\bar{Z} + \xi(z)$ for the hollow vortex and $u - iv = dW/dz$ for the bubble (the latter only with b constrained by equation (3.50)) are each uniquely determined by the solution $\mathbf{a}$ in the sequence space satisfying $\mathbf{a} = \mathcal{R}[\mathbf{a}]$.

In view of the results in §2 where two physical problems were shown to be equivalent to mathematical problem 1, which was then shown to have exact solutions given by rational function conformal mappings with a finite number of poles, it is natural to ask if problem 2 has any such solutions. This motivates the following theorem:

Theorem 3.7. *When (b, γ) is restricted by conditions of theorem 3.3, the derivative of the conformal mapping has the following meromorphic decomposition:*

$$Z_\zeta = \left(\frac{1}{\zeta} + \sum_{j=1}^{\infty} \frac{2r_j\zeta}{\zeta^2 - \zeta_j^2} \right)^2 \quad \text{and} \quad r_j = -\frac{\zeta_j}{\gamma Z_\zeta^{1/2}(\zeta_j^{-1})}, \quad (3.47)$$

where $\xi_j = \sqrt{b\gamma}\zeta_j > 0$ and for $|b\gamma j^{-1}| \ll 1$:

$$\zeta_j \sim (b\gamma)^{-1/2} \left((2j-1)\frac{\pi}{2} + \phi \right) + O(b\gamma j^{-1}) \quad \text{and} \quad \gamma r_j \sim -1 + O(b\gamma j^{-2}), \quad (3.48)$$

where the phase constant ϕ , determined in the electronic supplementary material, has the following asymptotic behaviour for $|b\gamma| \ll 1$:

$$\phi \sim -\frac{\pi}{2}\gamma \left[1 + \left(\frac{b}{(1-2\gamma)} \right)^2 + O(b\gamma) \right]. \quad (3.49)$$

Furthermore, $Z(\zeta)$ has a similar meromorphic representation implying that the residue of Z_ζ at any pole ζ_j in equation (3.47) is zero.

Remark 3.8. What is important in the proof is $y(\zeta) = Z_\zeta^{1/2}(\zeta)$ when analytically continued to $|\zeta| > 1$ satisfies $y_\zeta = q_0 + q_1y + q_2y^2$, where q_0, q_1, q_2 are analytic. A Riccati transformation leads to $w'' + gw = 0$, where g related to q_0, q_1, q_2 is analytic in $|\zeta| > 1$. The function g asymptotes to a constant as $\zeta \rightarrow \infty$, and therefore, the zeros of w , which correspond to poles of y , have a limit point at ∞ . Remarkably $Z(\zeta)$ also satisfies an equation similar to y and therefore has a similar meromorphic representation.

Remark 3.9. The arguments for a meromorphic representation of y are essentially those in Tanveer [10], though they were not presented as proofs. What is new here are (i) the asymptotic formula for phase ϕ in equation (3.49), (ii) that equation (3.48) holds even for $j = 1$ when $b\gamma \ll 1$, (iii) $Z(\zeta)$ has no log terms at $\pm\zeta_j$, and (iv) equation (3.47) is equally valid more generally for the conformal map corresponding to problem 2. Details of the proof of theorem 3.7 are set out in the electronic supplementary material.

Remark 3.10. As a supplement to theorem 3.7, numerical calculations are presented in the table below of the singularity characteristics akin to a table appearing in Tanveer [10],³ in the case of pure shear, for four different values of γ , except that the second column is now $2\phi/\pi$, instead of b . As predicted in theorem 3.7, for large j , γr_j becomes close to -1 , while $\xi_j/\pi - \phi/\pi$ approaches $j - 1/2$. Because of slow convergence ($O(j^{-1})$) in j , for $\gamma = -3.5$, we needed to choose $j = 50$ for $(\xi_j - \phi)/\pi$ to be within 0.06% of the asymptotic value $(j - 1/2)$, though the convergence of γr_j to -1 was far more rapid.

³In Tanveer [10], the table columns captioned ζ_1/π and ζ_2/π should have been ξ_1/π and ξ_2/π , respectively.

γ	$2\phi/\pi$	a	ξ_1/π	γr_1	ξ_2/π	γr_2
0.23	-0.2746674	2.198986	0.3420428	-0.9801898	1.355731	-0.9987174
-0.20	0.2388412	1.043480	0.6127640	-1.001543	1.606497	-1.000224
-0.50	0.5971030	1.153772	0.7704608	-1.010755	1.768088	-1.002022
-3.50	4.1787210	2.012300	2.113736	-1.143736	3.278167	-1.053672

Consider now the constraint

$$h(b, \gamma) := b(2 + 3\bar{a}_2) - 1 - \gamma \left(1 - \sum_{m=2}^{\infty} \bar{a}_m c_m \right) = 0, \quad (3.50)$$

satisfied by the bubble problem as shown in §3h. Theorem 3.11 addresses more generally what can be said in general when equation (3.50) is imposed on a solution to problem 2.

Theorem 3.11. *Let b, γ be restricted by the conditions of theorem 3.3. Then in the region of the parameter space corresponding to $\beta \in [0, 0.35]$, $\lambda \in [-\gamma_1, \gamma_1]$, if γ_1 is sufficiently small to satisfy $\mu_\beta^H < 2$, where μ_β^H is defined in equation (3.117), then there exists a unique $b = B(\gamma)$ so that $h(B(\gamma), \gamma) = 0$ for the function h defined in equation (3.50). Furthermore, if $b = B(\gamma)$, then the conformal map $Z(\zeta)$ corresponding to the solution of problem 2 satisfies the following identity for $|\zeta| < 1$.*

$$\frac{1}{2\pi i} \oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta' - \zeta} \left\{ \bar{Z}_\zeta^{1/2} \left(\frac{1}{\bar{\zeta}'} \right) - \gamma \bar{Z}(1/\zeta') \right\} = -b \int_0^\zeta \left(\frac{(1 - \zeta'^2)^2}{\zeta'^4 Z_\zeta(\zeta')} - \frac{1}{\zeta'^2} \right) d\zeta'. \quad (3.51)$$

The proof appears in §3l.

Remark 3.12. What is essential to the identity (3.51) is the equality of the two expressions for $K(\zeta)$ in equations (3.28) and (3.58) for the bubble problem and the mathematical equivalence of the bubble problem and the hollow vortex problem⁴ when the strain $b - \gamma$ and shear γ in the latter are constrained to satisfy $b = B(\gamma)$. We found no simple way of directly verifying equation (3.51), despite the known analytical structure of $Z(\zeta)$, which motivates presentation of this non-trivial result that emerged from mathematically unifying two seemingly different physical problems.

(h) Series solution of equation (3.33) for specified b, γ

Let the real parameter $\gamma < \frac{1}{2}$ be specified, along with the generally complex parameter b with $|b| < 1$ in equation (3.28). Further restrictions on γ, b will emerge later.

Define

$$y(\zeta) = Z_\zeta^{1/2} = \sum_{j=0}^{\infty} a_j \zeta^{j-1}, \quad \text{where } a_0 = 1, a_1 = 0. \quad (3.52)$$

Then,

$$\bar{y} \left(\frac{1}{\zeta} \right) = \sum_{j=0}^{\infty} \bar{a}_j \zeta^{-j+1}. \quad (3.53)$$

Since $Z_\zeta = y^2$, equations (3.21) and (3.52) imply

$$c_j = \frac{2a_j}{j-1} + \frac{1}{j-1} \sum_{m=2}^{j-2} a_m a_{j-m}, \quad \text{for } j \geq 2. \quad (3.54)$$

⁴More generally, the equivalence extends to any solution of problem 2 when $b = B(\gamma)$.

Using equations (3.21), (3.26), (3.52) and (3.53) in equation (3.33) and matching coefficients of ζ^j for $j \geq 0$,

$$a_j = (k_2 + b\bar{a}_2)\delta_{j,0} + b\delta_{j,2} + \gamma \sum_{m=0}^{\infty} \bar{a}_m c_{j+m}, \quad j \geq 0. \quad (3.55)$$

For $j = 0$, since $a_0 = 1, a_1 = 0, c_0 = -1$, equation (3.55) implies

$$k_2 = 1 - b\bar{a}_2 + \gamma - \gamma \sum_{m=2}^{\infty} \bar{a}_m c_m, \quad (3.56)$$

which determines k_2 in terms of the conformal mapping coefficients.

Matching negative powers of ζ in the Laurent series in equation (3.33) results in a recurrence relation for the remaining coefficients of $K(\zeta^{-1})$:

$$k_{m+2} = -b\bar{a}_{m+2} - \bar{a}_2 k_m - \sum_{j=1}^{m-1} \bar{a}_{m+2-j} k_j - \gamma \left(-\bar{a}_m + \sum_{j=2}^{\infty} c_j \bar{a}_{j+m} \right) \quad (3.57)$$

with k_2 determined from equation (3.56). Alternatively, by taking the complex conjugate of equation (3.31) on the unit circle, using analyticity of $\bar{K}(\zeta) - \bar{b}\zeta^{-1}$ in the unit disc, the Cauchy integral formula gives

$$\bar{K}(\zeta) = \frac{\bar{b}}{\zeta} + \frac{1}{2\pi i} \oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta' - \zeta} \left\{ \frac{\bar{Z}_\zeta^{1/2}(1/\zeta')}{\zeta' Z_\zeta^{1/2}(\zeta')} - \gamma \bar{Z}(1/\zeta') \right\}, \quad \text{for } |\zeta| < 1. \quad (3.58)$$

For the bubble problem, equations (3.56) and (3.29) must be consistent and hence, since $c_2 = 2a_2$, we obtain equation (3.50), where the implicit dependence on (b, γ) follows from the dependence of series coefficients (see theorem 3.3 in the sequel). For the bubble problem, equation (3.50) is a constraint between γ and b akin to that seen explicitly in §3f.

We now return to the problem of finding the conformal mapping coefficients. For $j = 1$, since $a_1 = 0$, equation (3.55) implies

$$0 = \gamma c_1 + \gamma \sum_{m=2}^{\infty} \bar{a}_m c_{1+m}, \quad (3.59)$$

which determines c_1 in terms of $\{c_n\}_{n=2}^{\infty}$ for any $\gamma \neq 0$. Note that c_1 , which encodes z -translation (equation (3.21)), does not appear in equation (3.55) for $j \geq 2$. Restricting equation (3.55) to $j \geq 2$, we obtain

$$a_j = b\delta_{j,2} + \gamma c_j + \gamma \mathcal{N}_j[\mathbf{c}, \mathbf{a}], \quad \text{where } \mathcal{N}_j[\mathbf{c}, \mathbf{a}] := \sum_{m=2}^{\infty} \bar{a}_m c_{j+m}, \quad (3.60)$$

where the following shorthand notation for sequences is used: $\mathbf{a} := (a_2, a_3, \dots)$ and $\mathbf{c} := (c_2, c_3, \dots)$. Define

$$\mathcal{M}_j[\mathbf{a}] = \frac{1}{j-1} \sum_{m=2}^{j-2} a_{j-m} a_m \quad \text{and} \quad \mathcal{T}_j[\mathbf{a}] = \frac{2}{j-1} a_j + \mathcal{M}_j[\mathbf{a}]. \quad (3.61)$$

Therefore, equation (3.54) may be written compactly as:

$$\mathbf{c} = \mathcal{T}[\mathbf{a}] = \Lambda \mathbf{a} + \mathcal{M}[\mathbf{a}], \quad \text{where } [\Lambda \mathbf{a}]_j = \frac{2}{j-1} a_j, \quad (3.62)$$

where the j component of $\mathcal{M}, \mathcal{T}$ is given by $\mathcal{M}_j, \mathcal{T}_j$. Equation (3.60) applied to $j \geq 2$: may be expressed as

$$a_j - \gamma c_j = \gamma \sum_{m=2}^{\infty} \bar{a}_m c_{j+m} + b\delta_{j,2} =: \gamma \mathcal{N}_j[\mathbf{c}, \mathbf{a}] + b\delta_{j,2}. \quad (3.63)$$

Using equations (3.62) and (3.63) for $j \geq 2$:

$$a_j = \frac{\gamma(j-1)}{j-1-2\gamma} [\mathcal{M}_j + \mathcal{N}_j] + \frac{b\delta_{j,2}}{1-2\gamma} =: \mathcal{Q}_j[\mathbf{c}, \mathbf{a}]. \quad (3.64)$$

Using equations (3.62) and (3.64), we write

$$\mathbf{a} = \mathcal{Q}[\mathcal{T}[\mathbf{a}], \mathbf{a}] =: \mathcal{R}[\mathbf{a}]. \quad (3.65)$$

Note that the exact solution considered in §3f, namely $\mathbf{a}^0 = \mathbf{a}^0 = \mathcal{R}[\mathbf{0}]$ for $\gamma = 0$, is manifest in equations (3.64) and (3.65). In this case, $\beta = b$. For the bubble problem, when the constraint (3.50) is invoked, the McLeod [11] solution $b = 1/3$ is retrieved as already seen in §3f.

It is also to be noted that for problem 1, the exact rational solutions (2.29) correspond in the sequence space to $a_j = 0 = c_j$ for $j \neq lN$ for integer $N \geq 2$, while $a_{lN} = (2N/(N-1))c^{-Nl}$, $c_{lN} = (4N/(N-1)^2)c^{-Nl}$ for $l \geq 1$ with constant $c > c_{\text{crit}}^{(N)} > 1$ as in equation (2.29). By substitution, we can verify that for $b = 0$, equations (3.62) and (3.63) are satisfied for these choices of a_j, c_j when we take γ as in equation (2.31). We note that as $c \rightarrow \infty$, $\gamma \rightarrow (N-1)/2$ and $\mathbf{a}, \mathbf{c} \rightarrow \mathbf{0}$. Therefore, for $b = 0$, $\gamma = (N-1)/2$ for any integer $N \geq 2$ is a bifurcation point from a trivial solution and the exact solution equation (2.29) corresponds to a finite amplitude extension in the sequence space. It would be interesting to study continuation of these solutions for non-zero b . For now, by placing the restriction $\gamma < \frac{1}{2}$, we preclude such solutions.

Equation (3.65) can be used as the basis of an efficient numerical calculation when the sequence $\mathbf{a}$ is truncated and a Newton iteration is employed. The details of some example calculations are given in appendix A.

In the following two subsections, we prove theorem 3.3 that applies equally to the bubble and the hollow vortex and more generally to problem 2. For the bubble problem, γ, b cannot be independently specified, but is instead constrained by equation (3.50) (theorem 3.11.)

(i) Proof of the solution to equation (3.65)

We will now prove that equation (3.65) has a unique solution for generally complex b and real $\gamma < \frac{1}{2}$ with certain restrictions stated in propositions 3.20 and 3.30.

Definition 3.13. Define

$$\mathbf{a}^0 = (\beta, 0, 0, \dots) \quad \text{and} \quad \mathbf{c}^0 = \mathcal{T}[\mathbf{a}^0] = \left(2\beta, 0, \frac{1}{3}\beta^2, 0, 0, \dots\right). \quad (3.66)$$

Accordingly, we define $Z^0(\zeta), y^0(\zeta)$ as functions in equations (3.21) and (3.52), generated by the sequences $\mathbf{c}^0, \mathbf{a}^0$, respectively. With slight abuse of notation, when it is important to display the parametric dependence on γ, b , we will use the notations $\mathcal{R}[\mathbf{a}; \gamma, b]$ and $\mathcal{R}[\mathbf{a}]$ interchangeably.

In the space of infinite sequences, define for some suitably chosen $\rho \in (0, 1]$, the weighted ℓ^1 norm

$$\|\mathbf{a}\|_\rho = \sum_{j=2}^{\infty} \rho^{-j} |a_j|. \quad (3.67)$$

The weighted sequence space will be denoted by $\mathcal{S}_\rho$. For brevity, we will use the notation $\mathcal{S} = \mathcal{S}_1$, $\|\cdot\| = \|\cdot\|_1$. Note the inclusion $\mathcal{S}_\rho \subset \mathcal{S}$. We make three simple observations for $\mathbf{a}, \mathbf{c} \in \mathcal{S}_\rho$:

$$\sum_{j=1}^{\infty} \rho^{-j} \sum_{m=2}^{\infty} |a_m| |c_{j+m}| \leq \sum_{m=2}^{\infty} \rho^{-m} |a_m| \sum_{j=1}^{\infty} \rho^{-j-m} |c_{j+m}| \leq \|\mathbf{a}\|_\rho \|\mathbf{c}\|_\rho \quad (3.68)$$

$$\sum_{j=4}^{\infty} \rho^{-j} \sum_{m=2}^{j-2} |a_m| |c_{j-m}| = \left(\sum_{k=2}^{\infty} \rho^{-k} |c_k| \right) \left(\sum_{m=2}^{\infty} \rho^{-m} |a_m| \right) \leq \|\mathbf{a}\|_\rho \|\mathbf{c}\|_\rho \quad (3.69)$$

$$\text{and} \quad \left| \sum_{m=2}^{\infty} a_m c_m \right| \leq \|\mathbf{a}\|_\infty \|\mathbf{c}\|_\rho \leq \|\mathbf{a}\|_\rho \|\mathbf{c}\|_\rho. \quad (3.70)$$

Lemma 3.14. If for given (γ, b) , $\mathbf{a} = (a_2, a_3, a_4, a_5, \dots)$ is a solution of $\mathbf{a} = \mathcal{R}[\mathbf{a}]$ then so is $\tilde{\mathbf{a}} = (a_2, -a_3, a_4, -a_5, \dots)$. Furthermore, $\bar{\mathbf{a}} = (\bar{a}_2, \bar{a}_2, \dots)$ is a solution corresponding to $(\gamma, \bar{b})$.

Proof. Noting components of the relation $\mathbf{c} = \mathcal{T}[\mathbf{a}]$:

$$c_j = \frac{a_j}{j-1} + \frac{1}{j-1} \sum_{m=2}^{j-2} a_m a_{j-m}, \quad \text{for } j \geq 2, \quad (3.71)$$

we note that replacing $\mathbf{a}$ by $\tilde{\mathbf{a}}$ in the above relation has no effect on c_j for even j , while it flips the sign of c_j for odd j , i.e. replace $\mathbf{c}$ by $\tilde{\mathbf{c}}$. The same is obviously true for $\mathcal{M}_j = 1/(j-1) \sum_{m=2}^{j-2} a_m a_{j-m}$. Furthermore, considering components of $\mathcal{N}$, we note from

$$\mathcal{N}_j[\mathbf{a}, \mathbf{c}] = \sum_{m=2}^{\infty} a_m c_{m+j} \quad (3.72)$$

that replacing $[\mathbf{a}, \mathbf{c}]$ by $[\tilde{\mathbf{a}}, \tilde{\mathbf{c}}]$ has no effect on $\mathcal{N}_j$ for even j , while it flips the sign of $\mathcal{N}_j$ for odd j . The same is true for $\mathcal{Q}_j$, which is a linear combination of $\mathcal{M}_j$, $\mathcal{Q}_j$. Therefore, replacing $\mathbf{a}$ by $\tilde{\mathbf{a}}$ switches the sign of $\mathcal{R}_j$ for odd j , while even j components remain the same. Thus, if $\mathbf{a} = \mathcal{R}[\mathbf{a}]$, $\tilde{\mathbf{a}} = \mathcal{R}[\tilde{\mathbf{a}}]$. The second part of the lemma follows from complex conjugation of $\mathcal{R}[\mathbf{a}; \gamma, b]$, resulting in $\mathcal{R}[\bar{\mathbf{a}}; \gamma, \bar{b}]$. ■

Lemma 3.15. For $\rho \in (0, 1]$,

$$\|\mathcal{N}[\mathbf{c}^1, \mathbf{a}^1] - \mathcal{N}[\mathbf{c}^2, \mathbf{a}^2]\|_{\rho} \leq \left\{ \|\mathbf{a}^1\|_{\rho} \|\mathbf{c}^1 - \mathbf{c}^2\|_{\rho} + \|\mathbf{c}^2\|_{\rho} \|\mathbf{a}^1 - \mathbf{a}^2\|_{\rho} \right\} \quad (3.73)$$

and, in particular,

$$\|\mathcal{N}[\mathbf{c}, \mathbf{a}]\|_{\rho} \leq \|\mathbf{a}\|_{\rho} \|\mathbf{c}\|_{\rho}. \quad (3.74)$$

Furthermore,

$$\|\partial_{\beta} \mathcal{N}[\mathbf{c}, \mathbf{a}]\| \leq \|\mathbf{a}\| \|\partial_{\beta} \mathbf{c}\| + \|\mathbf{c}\| \|\partial_{\beta} \mathbf{a}\|. \quad (3.75)$$

Proof. From equation (3.60),

$$\mathcal{N}_j[\mathbf{c}^1, \mathbf{a}^1] - \mathcal{N}_j[\mathbf{c}^2, \mathbf{a}^2] = \sum_{m=2}^{\infty} \bar{a}_m^1 (c_{j+m}^1 - c_{j+m}^2) + \sum_{m=2}^{\infty} c_{j+m}^2 (\bar{a}_m^1 - \bar{a}_m^2).$$

Using equation (3.68)

$$\sum_{j=1}^{\infty} \rho^{-j} |\mathcal{N}_j[\mathbf{c}^1, \mathbf{a}^1] - \mathcal{N}_j[\mathbf{c}^2, \mathbf{a}^2]| \leq \{ \|\mathbf{a}^1\|_{\rho} \|\mathbf{c}^1 - \mathbf{c}^2\|_{\rho} + \|\mathbf{c}^2\|_{\rho} \|\mathbf{a}^1 - \mathbf{a}^2\|_{\rho} \}.$$

If we take $\mathbf{a}^1, \mathbf{a}^2$ to be the $\mathbf{a}$ corresponding to β and $\beta + \Delta\beta$, respectively, and take $\Delta\beta \rightarrow 0$, then equation (3.73) implies equation (3.75). ■

Lemma 3.16. For $\rho \in (0, 1)$,

$$\|\mathcal{M}[\mathbf{a}^1] - \mathcal{M}[\mathbf{a}^2]\|_{\rho} \leq \frac{1}{3} \left(\|\mathbf{a}^1\|_{\rho} + \|\mathbf{a}^2\|_{\rho} \right) \|\mathbf{a}^1 - \mathbf{a}^2\|_{\rho} \quad (3.76)$$

and

$$\|\mathcal{T}[\mathbf{a}^1] - \mathcal{T}[\mathbf{a}^2]\|_{\rho} \leq \left\{ 2 + \frac{1}{3} \|\mathbf{a}^1\|_{\rho} + \frac{1}{3} \|\mathbf{a}^2\|_{\rho} \right\} \|\mathbf{a}^1 - \mathbf{a}^2\|_{\rho} \quad (3.77)$$

and, in particular,

$$\|\mathcal{T}[\mathbf{a}]\|_{\rho} \leq 2\|\mathbf{a}\|_{\rho} + \frac{1}{3}\|\mathbf{a}\|_{\rho}^2. \quad (3.78)$$

Furthermore,

$$\|\partial_{\beta} \mathcal{M}[\mathbf{a}]\| \leq \frac{2}{3} \|\mathbf{a}\| \|\partial_{\beta} \mathbf{a}\| \quad \text{and} \quad \|\partial_{\beta} \mathcal{T}[\mathbf{a}]\| \leq \left(2 + \frac{2}{3} \|\mathbf{a}\| \right) \|\partial_{\beta} \mathbf{a}\|. \quad (3.79)$$

Proof. From equation (3.60), for $j \geq 2$,

$$\mathcal{M}_j[\mathbf{a}^1] - \mathcal{M}_j[\mathbf{a}^2] = \frac{1}{j-1} \left\{ \sum_{m=2}^{j-2} a_m^1 (a_{j-m}^1 - a_{j-m}^2) + \sum_{m=1}^{j-1} a_{j-m}^2 (a_m^1 - a_m^2) \right\}.$$

We note that $\mathcal{M}_j = 0$ for $1 \leq j \leq 3$, and for $j \geq 4$, $1/(j-1) \leq \frac{1}{3}$. Again using the general property in equation (3.69), we have

$$\|\mathcal{M}_j[\mathbf{a}^1] - \mathcal{M}_j[\mathbf{a}^2]\|_\rho \leq \frac{1}{3} (\|\mathbf{a}^1\|_\rho + \|\mathbf{a}^2\|_\rho) \|\mathbf{a}^1 - \mathbf{a}^2\|_\rho$$

from which the statement for $\mathcal{M}$ follows. The bounds for $\mathcal{T}$ follow immediately from equation (3.61). Again, if we take $\mathbf{a}^1 = \mathbf{a}$ corresponding to β and $\mathbf{a}^2$ as $\mathbf{a}$ corresponding to $\beta + \Delta\beta$, then it is clear from equations (3.76) and (3.77) on taking the limit $\Delta\beta \rightarrow 0$, we obtain equation (3.79). ■

Corollary 3.17. For $\rho \in (0, 1]$,

$$\begin{aligned} \|\mathcal{Q}[\mathbf{c}^1, \mathbf{a}^1] - \mathcal{Q}[\mathbf{c}^2, \mathbf{a}^2]\|_\rho &\leq \gamma_1 \left\{ \left(\frac{1}{3} \|\mathbf{a}^1\|_\rho + \frac{1}{3} \|\mathbf{a}^2\|_\rho \right) \|\mathbf{a}^1 - \mathbf{a}^2\|_\rho \right. \\ &\quad \left. + \{\|\mathbf{a}^1\|_\rho \|\mathbf{c}^1 - \mathbf{c}^2\|_\rho + \|\mathbf{c}^2\|_\rho \|\mathbf{a}^1 - \mathbf{a}^2\|_\rho\} \right\}, \end{aligned} \quad (3.80)$$

and, in particular,

$$\|\mathcal{Q}[\mathbf{c}, \mathbf{a}] - \mathcal{Q}[\mathbf{0}, \mathbf{0}]\|_\rho \leq \gamma_1 \|\mathbf{a}\|_\rho \left(\frac{1}{3} + \|\mathbf{c}\|_\rho \right) \quad (3.81)$$

and

$$\|\partial_\beta \{\mathcal{Q}[\mathcal{T}[\mathbf{a}], \mathbf{a}] - \mathcal{Q}[\mathbf{0}, \mathbf{0}]\}\| \leq \gamma_1 \|\mathbf{a}\| \left\{ \left(2 + \frac{2}{3} \|\mathbf{a}\| \right) + \left(2 + \frac{1}{3} \|\mathbf{a}\| \right) \right\} \|\partial_\beta \mathbf{a}\|. \quad (3.82)$$

Proof. This follows from equation (3.62) on using lemmas 3.15 and 3.16 and noting $|(j-1)\gamma/(j-1-2\gamma)| \leq \gamma_1$ for $j \geq 2$, $\gamma < \frac{1}{2}$. Using equations (3.75), (3.79) and using equation (3.64), equation (3.82) follows. ■

Definition 3.18. Recall the definition of μ in equation (3.43). For $\rho \leq 1$, $\gamma < \frac{1}{2}$, we define

$$\mu^0 = 2\gamma_1 b_1 \left(\frac{7}{3} + \frac{2}{3} b_1 \right). \quad (3.83)$$

Lemma 3.19. $\mathcal{R}$ defined in equation (3.65) satisfies the following bounds for $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a} \in \mathcal{B}_{2b_1}$ where $\mathcal{B}_{2b_1}$ is a closed ball in $\mathcal{S}_\rho$ centred about the origin of radius $2b_1$, then

$$\|\mathcal{R}[\mathbf{a}^1] - \mathcal{R}[\mathbf{a}^2]\|_\rho \leq \mu \|\mathbf{a}^1 - \mathbf{a}^2\|_\rho \quad \text{and} \quad \|\mathcal{R}[\mathbf{0}]\|_\rho = b_1 \rho^{-2}, \quad (3.84)$$

while

$$\|\mathcal{R}[\mathbf{a}] - \mathcal{R}[\mathbf{0}]\|_\rho \leq 2\mu^0 b_1, \quad (3.85)$$

and

$$\|\partial_\beta \{\mathcal{R}[\mathbf{a}] - \mathcal{R}[\mathbf{0}]\}\| \leq 2\gamma_1 b_1 (4 + 2b_1) \|\partial_\beta \mathbf{a}\|. \quad (3.86)$$

Proof. We use the bounds on $\mathcal{Q}$ in corollary 3.17 and replace bounds involving $\mathbf{c}^1, \mathbf{c}^2, \mathbf{c}$ by those involving $\mathcal{T}[\mathbf{a}^1], \mathcal{T}[\mathbf{a}^2]$ and $\mathcal{T}[\mathbf{a}]$ from lemma 3.16. The last statement follows from equation (3.82) on using definition of $\mathcal{R}[\mathbf{a}]$ and on using $\|\mathbf{a}\|_\rho \leq 2b_1$. ■

Proposition 3.20. If μ defined in equation (3.43) satisfies

$$\mu < 1, \quad (3.87)$$

then there exists a unique solution $\mathbf{a} \in \mathcal{B}_{2b_1} \subset \mathcal{S}$ satisfying equation (3.64). Furthermore, when equation (3.87) holds, then there exists $\rho < 1$ so that $\mathbf{a} \in \mathcal{S}_\rho$ and therefore the corresponding $Z(\zeta)$ in

equation (3.21) is analytic on $|\zeta| = 1$. Furthermore, the odd components $a_{2j-1} = 0 = c_{2j-1}$ for $j \geq 1$ are each zero. Furthermore, if b is real, $\mathbf{a}, \mathbf{c}$ are each real. Furthermore, for this solution,

$$\|\partial_\beta[\mathbf{a} - \mathbf{a}^0]\| \leq \mu_\beta^a := \frac{2\gamma_1 b_1(4 + 2b_1)}{1 - 2\gamma_1 b_1(4 + 2b_1)} \quad (3.88)$$

and

$$\|\partial_\beta[\mathbf{c} - \mathbf{c}^0]\| \leq \mu_\beta^c := (2 + b_1)\mu_\beta^a. \quad (3.89)$$

Proof. Applying lemma 3.19 for $\mathbf{a} \in \mathcal{B}_{2b_1} \subset \mathcal{S}$, and using equation (3.87),

$$\|\mathcal{R}[\mathbf{a}]\| \leq \|\mathcal{R}[\mathbf{0}]\| + \|\mathcal{R}[\mathbf{a}] - \mathcal{R}[\mathbf{0}]\| \leq \left(\mu^0 + \frac{1}{2}\right) 2b_1 < 2b_1$$

since $\mu^0 < \frac{1}{2}\mu < \frac{1}{2}$. Therefore, $\mathcal{R} : \mathcal{B}_{2b_1} \rightarrow \mathcal{B}_{2b_1}$ in the space $\mathcal{S}$. Furthermore, lemma 3.19 shows that $\mathcal{R}$ is contractive when $\mu < 1$ and hence the proposition follows from the Banach contraction theorem. If $\mathbf{a} \in \mathcal{B}_{2b_1} \subset \mathcal{S}_\rho$, then

$$\|\mathcal{R}[\mathbf{a}]\|_\rho \leq \|\mathcal{R}[\mathbf{a}] - \mathcal{R}[\mathbf{0}]\|_\rho + \|\mathcal{R}[\mathbf{0}]\|_\rho \leq b_1 \rho^{-2} + 2\mu^0 b_1 < 2b_1$$

for ρ sufficiently close to 1, while $\mu < 1$ ensures contractivity as before. Since $\mathcal{S}_\rho \subset \mathcal{S}$, uniqueness of the solution in $\mathcal{B}_{2b_1} \subset \mathcal{S}$ implies $\mathbf{a} \in \mathcal{S}_\rho$. From lemma 3.14, if $\mathbf{a}$ is a solution, so is $\tilde{\mathbf{a}}$, each of which are in the same ball $\mathcal{B}_{2b_1} \in \mathcal{S}$. Uniqueness implies $\mathbf{a} = \tilde{\mathbf{a}}$ and hence $a_{2j+1} = 0$ for $j \geq 1$. Also, in lemma 3.14, it was shown that a transformation $\mathbf{a} \rightarrow \tilde{\mathbf{a}}$ results in $\mathbf{c} \rightarrow \tilde{\mathbf{c}}$, which implies $c_{2j+1} = 0$ for $j \geq 1$. Equation (3.59) implies $c_1 = 0$ as well. Furthermore, from lemma 3.14, if b is real, $\bar{\mathbf{a}}$ will also be a solution to $\mathbf{a} = \mathcal{R}[\mathbf{a}]$, which is clearly in the same ball $\mathcal{B}_{2b_1}$. Therefore, uniqueness implies $\mathbf{a} = \bar{\mathbf{a}}$, and therefore, $\mathbf{a}$ and thus $\mathbf{c}$ are each real when the conditions of the proposition hold. The statement on bounds on β derivative the observation

$$\partial_\beta[\mathbf{a} - \mathbf{a}^0] = \partial_\beta\{\mathcal{R}[\mathbf{a}] - \mathcal{R}[\mathbf{0}]\}$$

and in the estimates of the right side of equation (3.86), we use

$$\|\partial_\beta \mathbf{a}\| \leq \|\partial_\beta[\mathbf{a} - \mathbf{a}^0]\| + \|\partial_\beta \mathbf{a}^0\| = 1 + \|\partial_\beta[\mathbf{a} - \mathbf{a}^0]\|.$$

Furthermore, since

$$\|\partial_\beta[\mathbf{c} - \mathbf{c}^0]\| = \|\partial_\beta\{\mathcal{T}[\mathbf{a}] - \mathcal{T}[\mathbf{a}^0]\}\| \leq \left(2 + \frac{1}{3}\|\mathbf{a}\| + \frac{1}{3}\|\mathbf{a}^0\|\right) \partial_\beta[\mathbf{a} - \mathbf{a}^0],$$

the last estimate follows on using $\|\mathbf{a}^0\| = b_1$, $\|\mathbf{a}\| \leq 2b_1$. ■

Remark 3.21. In the case of shear only for the hollow vortex problem, $b = \gamma$, i.e. $b_1 = \gamma_1$, equation (3.87) requires $\gamma_1 < 0.307$ corresponding to $\gamma \in (-0.795, 0.190)$. Numerical results suggesting existence in a wider regime are reported in appendix A. Choosing fixed $b \in [-\frac{1}{3}, \frac{1}{3}]$, it was checked that equation (3.87) applies for $\gamma \in (-\infty, 0.139)$.

Remark 3.22. While proposition 3.20 guarantees $y(\zeta)$ and $Z(\zeta)$ generated from equations (3.52) and (3.21) through the sequence relation $\mathbf{c} = \mathcal{T}[\mathbf{a}]$, with c_1 determined by equation (3.59), satisfies equation (3.33), it does not guarantee univalence of $Z(\zeta)$ in $\mathbb{D}$ necessary for a solution to problem 2. This univalence is proved in proposition 3.30 with additional restriction on (γ, b) needed to satisfy equation (3.102) as well.

Remark 3.23. For the solution in proposition 3.20 to be relevant to the bubble problem, besides ensuring univalence of Z , we need to restrict b so that equation (3.50) is satisfied.

Lemma 3.24. When equation (3.87) is satisfied, for the solution $\mathbf{a}$ in proposition 3.20, and corresponding $\mathbf{c} = \mathcal{T}[\mathbf{a}]$,

$$\|\mathbf{a} - \mathbf{a}^0\| \leq \mu_\Delta^a := 2\mu^0 b_1 \quad \text{and} \quad \|\mathbf{c} - \mathbf{c}^0\| \leq \mu_\Delta^c := (2 + b_1)\mu_\Delta^a, \quad (3.90)$$

each of which scales as γ for small γ .

Proof. From equation (3.64), since $\mathcal{R}[\mathbf{a}] = \mathcal{Q}[\mathcal{T}[\mathbf{a}], \mathbf{a}]$, it follows that $\mathcal{R}[\mathbf{0}] = \mathbf{a}^0$ and hence

$$\mathbf{a} - \mathbf{a}^0 = \mathcal{R}[\mathbf{a}] - \mathcal{R}[\mathbf{0}], \quad (3.91)$$

from which the result follows on applying equation (3.85) in lemma 3.19 since solution $\mathbf{a} \in \mathcal{B}_{2b_1}$. Furthermore, since $\mathbf{c} = \mathcal{T}[\mathbf{a}]$, using lemma 3.16,

$$\|\mathbf{c} - \mathbf{c}^0\| \leq \left(2 + \frac{1}{3}\|\mathbf{a}^0\| + \frac{1}{3}\|\mathbf{a}\|\right) \|\mathbf{a} - \mathbf{a}^0\|, \quad (3.92)$$

from which the result for $\mathbf{c}$ follows on using $\|\mathbf{a}\| \leq 2b_1$, $\|\mathbf{a}^0\| = b_1$. ■

Remark 3.25. If we use the following iteration for $n \geq 1$,

$$\mathbf{a}^n = \mathcal{R}[\mathbf{a}^{n-1}], \quad (3.93)$$

then using bounds on $\mathcal{R}$ it is easy to prove by induction that $\|\mathbf{a} - \mathbf{a}^n\| \leq \mu_\Delta^a \mu^{n-1}$, and therefore equation (3.93) generates the asymptotic expansion for small $\gamma_1 b_1$.

(j) Univalence of $Z(\zeta)$ corresponding to solution in proposition 3.20

Lemma 3.26. Assume equation (3.87) holds. Then the conformal map $Z(\zeta)$ corresponding to solution $\mathbf{a}$ (equation (3.21)) satisfies the following bounds on $\partial\mathbb{D}$:

$$\|Z_\zeta - Z_\zeta^0\|_\infty \leq \mu_Z, \quad (3.94)$$

where

$$\mu_Z = 2\mu^0 b_1 (2 + 2b_1 + 2\mu^0 b_1). \quad (3.95)$$

Proof. Using $a_0 = 1 = a_0^0$, $a_2^0 = \beta$ and $a_{2m}^0 = 0$ for $m > 1$, it may be readily checked that

$$Z_\zeta - Z_\zeta^0 = \sum_{m=1}^{\infty} \zeta^{2m-2} \left\{ \sum_{j=1}^{m-1} (a_{2j} - a_{2j}^0)(a_{2m-2j} - a_{2m-2j}^0) + 2(a_{2m} - a_{2m}^0) + 2\beta(a_{2m-2} - a_{2m-2}^0) \right\}. \quad (3.96)$$

Therefore,

$$\|Z_\zeta - Z_\zeta^0\|_\infty \leq (2 + 2b_1)\|\mathbf{a} - \mathbf{a}^0\| + \|\mathbf{a} - \mathbf{a}^0\|^2. \quad (3.97)$$

The proof follows from lemma 3.24. ■

Definition 3.27. Define

$$m_Z^0 = \min_{\zeta_1, \zeta_2 \in \partial\mathbb{D}, \zeta_1 \neq \zeta_2} \left| \frac{Z^0(\zeta_1) - Z^0(\zeta_2)}{\zeta_1 - \zeta_2} \right|. \quad (3.98)$$

Lemma 3.28. For any β with $b_1 < 2\sqrt{3} - 3 \approx 0.4641$,

$$m_Z^0 = \left(1 - 2b_1 - \frac{1}{3}b_1^2\right) > 0. \quad (3.99)$$

Proof. We first take $\beta > 0$ real satisfying the condition above. Consider $\zeta_1 = e^{i\nu_1}$, $\zeta_2 = e^{i\nu_2} \in \partial\mathbb{D}$. With $\phi = \nu_1 + \nu_2$ and $\theta = \nu_1 - \nu_2$, it may be checked that

$$\begin{aligned} \left| \frac{Z(\zeta_1) - Z(\zeta_2)}{\zeta_1 - \zeta_2} \right|^2 &= w(\phi, \theta) = 1 + 4\beta \cos \phi + 2\beta^2 f(\theta) \cos(2\phi) \\ &\quad + 4\beta^2 + 4\beta^3 f(\theta) \cos \phi + \beta^4 f^2(\theta), \end{aligned} \quad (3.100)$$

where

$$f(\theta) = \frac{1}{3}(1 + 2\cos \theta). \quad (3.101)$$

For β as restricted above, we may check that the only critical points of $w(\theta, \phi)$ occur when $\sin \theta = 0 = \sin \phi$. For $\beta > 0$, we can check that the minimum of w occurs where $\cos \phi = -1$ and $\cos \theta =$

-1 , with $w_{\min} = (1 - 2\beta - \frac{1}{3}\beta^2)^2$. This establishes the formulae for m_Z^0 when $\beta > 0$ when $|\beta| = b_1 < 2\sqrt{3} - 3$. For complex $\beta = b_1 e^{i\theta}$, a rotation both in ζ and Z : $(\zeta, Z^0) \rightarrow (\alpha\zeta, \alpha^{-1}Z^0)$, where $\alpha = e^{-i\theta/2}$, returns the problem to the $\beta > 0$ case. ■

Remark 3.29. We can directly check that if $\beta = 2\sqrt{3} - 3$, then $Z^0(+1) = Z^0(-1)$; on the other hand, when $\beta = 3 - 2\sqrt{3}$, $Z^0(i) = Z^0(-i)$, i.e. we get loss of univalence through horizontal and vertical pinching, respectively.

Proposition 3.30. Assume (b, γ) satisfies equation (3.45). In addition, if μ_Z, m_Z^0 given by equations (3.87) and (3.98) satisfy

$$m_Z^0 - \frac{\pi}{2}\mu_Z > 0, \quad (3.102)$$

then $Z(\zeta)$ in equation (3.21) generated by the sequence $\mathbf{c} = T[\mathbf{a}]$ for the solution $\mathbf{a}$ in proposition 3.20 is univalent in $\mathbb{D}$.

Proof. From standard results in complex analysis, it is enough to show univalence of $Z(\zeta)$ on $\partial\mathbb{D}$. Take $\zeta_1 = e^{iv_1} \neq \zeta_2 = e^{iv_2} \in \partial\mathbb{D}$, where without any loss of generality, we restrict $v_2 \in [v_1 - \pi, v_2 + \pi)$. Define 2π periodic function $f(v) = Z(e^{iv}) - Z^0(e^{iv})$. We note from mean value inequality that

$$\left| \frac{f(v_1) - f(v_2)}{v_1 - v_2} \right| \leq \|f'\|_\infty = \|Z_\zeta - Z_\zeta^0\|_\infty \leq \mu_Z. \quad (3.103)$$

Therefore,

$$\left| \frac{Z(\zeta_1) - Z(\zeta_2)}{\zeta_1 - \zeta_2} \right| \geq m_Z^0 - \mu_Z \sup_{v_2 - v_1 \in [-\pi, \pi]} \left| \frac{v_1 - v_2}{2 \sin(v_1 - v_2)/2} \right| = m_Z^0 - \frac{\pi}{2}\mu_Z > 0 \quad (3.104)$$

proving univalence. ■

(k) Proof of theorem 3.3

Definition 3.31. Define $\mathcal{P}$ to be the projection applied to a Laurent series on $|\zeta| = 1$:

$$\mathcal{P} \sum_{j=-\infty}^{\infty} p_n \zeta^j = \sum_{j=1}^{\infty} p_n \zeta^j. \quad (3.105)$$

From propositions 3.20 and 3.30, $\mathbf{a}$ satisfies $\mathbf{a} = \mathcal{R}[\mathbf{a}]$, then with $\mathbf{c} = T[\mathbf{a}]$, i.e. a_j, c_j satisfy equation (3.55) for any $j \geq 2$. For brevity in summation, it is useful to define $c_0 = -1, a_0 = 1, a_1 = 0$. Multiplying each side of this equation by ζ^j and summing from $j = 1$ to ∞ and observing

$$\sum_{j=1}^{\infty} \zeta^j \left(\sum_{m=0}^{\infty} \bar{a}_m c_{j+m} \right) = \mathcal{P} \left[\sum_{j=0}^{\infty} c_j \zeta^{j-1} \sum_{k=0}^{\infty} \bar{a}_k \zeta^{1-k} \right] \quad (3.106)$$

with absolute convergence on $|\zeta| = 1$ justifying the change in order in summation. Using the series representations of $y(\zeta) = Z_\zeta^{1/2}$ and $Z(\zeta)$ from equations (3.46) and (3.21), respectively, we obtain

$$\zeta y_\zeta = \mathcal{P}[\gamma \bar{y}(\zeta^{-1})Z(\zeta)] + b\zeta^2. \quad (3.107)$$

Define the function $U(\zeta^{-1})$ analytic in $|\zeta| > 1$ through

$$U(\zeta^{-1}) = (I - \mathcal{P})[\gamma \bar{y}(\zeta^{-1})Z(\zeta)]. \quad (3.108)$$

Since $\mathbf{a}, \mathbf{c} \in \mathcal{S}$, the Laurent series for $U(\zeta^{-1})$ in $|\zeta| > 1$, is absolutely convergent on $|\zeta| = 1$; in fact, $U(\zeta^{-1})$ is analytic on $\partial\mathbb{D}$. It follows from equations (3.107) and (3.108) that

$$\zeta y_\zeta = \gamma \bar{y}(\zeta^{-1})Z(\zeta) + b\zeta^2 + U(\zeta^{-1}). \quad (3.109)$$

Since Z is univalent in $|\zeta| \leq 1$, $\bar{y}(\zeta^{-1}) \neq 0$ for $|\zeta| \geq 1$ and hence by dividing, we obtain

$$\frac{\zeta y_\zeta}{\bar{y}(\zeta^{-1})} = \gamma Z(\zeta) + K(\zeta^{-1}), \quad (3.110)$$

where

$$K(\zeta^{-1}) = \frac{b\zeta^2 + U(\zeta^{-1})}{\bar{y}(\zeta^{-1})}. \quad (3.111)$$

Since $\bar{y}(\zeta^{-1}) \sim \zeta + O(\zeta^{-1})$, $K(\zeta^{-1})$ is clearly analytic in $|\zeta| \geq 1$, except at ∞ where $K(\zeta^{-1}) \sim b\zeta + o(1)$. Hence equation (3.31) is satisfied. However, through the univalent map $Z(\zeta)$ on $|\zeta| \leq 1$, equation (3.31) is equivalent to equation (3.17) on an arclength parametrized curve when $\xi(z) = i\bar{K}(Z^{-1}(z))$, where $Z^{-1}: \Omega \rightarrow \mathbb{D}$ is the inverse of the map $Z: \mathbb{D} \rightarrow \Omega$. Thus, we have a locally unique solution of problem 2 for which the corresponding shape $\partial\Omega$ is analytic. The same is true for the function ξ on $\partial\Omega$ owing to the analyticity of $K(\zeta^{-1})$ on $|\zeta| = 1$ and consequently of $\bar{K}(\zeta)$ on $|\zeta| = 1$.

Remark 3.32. Since $\mu_v > 0$ requires $\|\mathbf{a}\| \leq 2b_1 < 1$, taking absolute value in equation (3.57) and sum over m , it is straightforward to show

$$\begin{aligned} \sum_{m=1}^{\infty} |k_{m+2}| &\leq (1 - \|\mathbf{a}\|)^{-1} \{(|b| + |k_2|)\|\mathbf{a}\| + \gamma\|\mathbf{a}\|\|\mathbf{c}\|\} \\ &\leq (1 - 2b_1)^{-1} \left\{ 2b_1(|b| + |k_2|) + 2\gamma b_1 \left(2 + \frac{2}{3}\gamma b_1 \right) \right\} \end{aligned} \quad (3.112)$$

and therefore scales with $2b_1$ for small b_1 since k_2 determined from equation (3.56) is clearly $O(1)$.

(I) The constraint (3.50) and proof of theorem 3.11

We restrict to real values of b in which case, when the conditions of theorem 3.3 are met, the relevant series are real valued on the real line and odd coefficients $c_{2j-1}, a_{2j-1}, k_{2j-1}$ vanish. Therefore, for the conjugate functions, $\bar{K}(\zeta) = K(\zeta)$, $\bar{y}(\zeta) = y(\zeta)$, $\bar{Z}(\zeta) = Z(\zeta)$. It is useful to re-express equation (3.50) in terms of β, λ :

$$0 = H(\beta, \lambda) = H_1(\beta) - H_2(\beta, \lambda), \quad (3.113)$$

where

$$H_1(\beta) = \beta(2 + 3\beta) \quad (3.114)$$

and

$$\begin{aligned} H_2(\beta, \lambda) &= 1 + \lambda(3 - 2\beta^2) - \beta(3 + 2\lambda)(a_2 - a_2^0) - \lambda\beta(c_2 - c_2^0) - \frac{1}{3}\lambda\beta^2(a_4 - a_4^0) \\ &\quad - \lambda \sum_{m=1}^{\infty} (a_{2m} - a_{2m}^0)(c_{2m} - c_{2m}^0). \end{aligned} \quad (3.115)$$

Definition 3.33. Define

$$\mu^H = b_1\mu_{\Delta}^a \left(3 + 2\gamma_1 + \frac{1}{3}\gamma_1 b_1 \right) + \gamma_1 b_1 \mu_{\Delta}^c + \gamma_1 \mu_{\Delta}^c \mu_{\Delta}^a \quad (3.116)$$

and

$$\begin{aligned} \mu_{\beta}^H &= \left(3 + 2\gamma_1 + \frac{2}{3}\gamma_1 b_1 \right) \mu_{\Delta}^a + \gamma_1 \mu_{\Delta}^c + b_1 \left(3 + 2\gamma_1 + \frac{1}{3}\gamma_1 b_1 \right) \mu_{\beta}^a + b_1 \gamma_1 \mu_{\beta}^c \\ &\quad + \gamma_1 (\mu_{\beta}^c \mu_{\Delta}^a + \mu_{\beta}^a \mu_{\Delta}^c). \end{aligned} \quad (3.117)$$

Lemma 3.34. For $\beta \in [0, \frac{1}{3}]$ the range of H_1 is $[0, 1]$.

Proof. This is obvious from noting that H_1 is an increasing function in the given interval and checking the endpoint values. ■

Lemma 3.35. For $\lambda \in [-\gamma_1, \gamma_1]$, $\beta \in [0, \frac{1}{3}]$

$$1 - 3\gamma_1 - \mu^H \leq H_2(\beta, \lambda) \leq 1 + 3\gamma_1 + \mu^H. \quad (3.118)$$

Proof. On estimating terms in H_2 and applying the bounds in lemma 3.24, it is clear that

$$|H_2(\beta, \lambda) - 1 - \lambda(3 - 2\beta^2)| \leq \mu^H, \quad (3.119)$$

and therefore, noting the range of $1 + \lambda(3 - 2\beta^2)$, the result follows. ■

Lemma 3.36. Assume the restriction $\beta \in [0, 0.35]$, $\lambda \in [-\gamma_1, \gamma_1]$. Further we assume that γ_1 is sufficiently small so that

$$\mu_\beta^H < 2, \quad (3.120)$$

then $H_\beta > 0$.

Proof. We note that

$$\begin{aligned} H_\beta - 2 - 6\beta &= 4\lambda\beta + (3 + 2\lambda)(a_2 - a_2^0) + \lambda(c_2 - c_2^0) + \frac{2}{3}\lambda\beta(a_4 - a_4^0) \\ &\quad + \beta(3 + 2\lambda)\partial_\beta(a_2 - a_2^0) + \lambda\beta\partial_\beta(c_2 - c_2^0) + \frac{1}{3}\lambda\beta^2\partial_\beta(a_4 - a_4^0) \\ &\quad + \lambda \sum_{m=1}^{\infty} \left[\partial_\beta(a_{2m} - a_{2m}^0)(c_{2m} - c_{2m}^0) + (a_{2m} - a_{2m}^0)\partial_\beta(c_{2m} - c_{2m}^0) \right]. \end{aligned} \quad (3.121)$$

From the upper bound of the right-hand side available from lemma 3.24, equations (3.88) and (3.89) in proposition 3.20, we obtain

$$|H_\beta - 2 - \beta(6 + 4\lambda)| \leq \mu_\beta^H, \quad (3.122)$$

from which the lemma follows. ■

Proposition 3.37. Assume the restriction $\beta \in [0, \beta_0]$, where $\beta_0 = 0.371$. Then for γ_1 small enough to satisfy the conditions of theorem 3.3 as well as satisfy

$$\mu_\beta^H < 2, \quad 1 - 3\gamma_1 - \mu^H \geq 0 \quad \text{and} \quad 1 + 3\gamma_1 + \mu^H \leq \beta_0(2 + 3\beta_0), \quad (3.123)$$

then for each $\lambda \in [-\gamma_1, \gamma_1]$, there exists a unique $\mathbf{B}(\lambda)$ with range in $[0, \beta_0]$ so that $H(\mathbf{B}(\lambda), \lambda) = 0$.

Proof. Since $H_1(\beta)$ is an increasing function of β whose range includes all possible values of $H_2(\beta, \lambda)$ for specified $\lambda \in [-\gamma_1, \gamma_1]$, it follows from the continuity of H in β that there will be some β for which $H_1(\beta) = H_2(\beta, \lambda)$, i.e. $H(\beta, \lambda) = 0$. Since the condition of the previous lemma applies, and $H_\beta > 0$ and hence there exists unique $\beta = \mathbf{B}(\lambda)$ so that $H(\mathbf{B}(\lambda), \lambda) = 0$. ■

Remark 3.38. It was checked that with the choice $\gamma_1 \leq 0.033$, $b_1 \leq 0.371$, all conditions of theorem 3.3 as well as the condition (3.123) for applying theorem 3.11 hold. This corresponds to $\gamma \in (-0.0353, 0.0309)$ with the range of $b = B(\gamma)$ in $[0, 0.3466]$. Numerical evidence, however, shows a wider range: for any $\gamma \in [-1, 0.1455104 \dots)$ there exists unique increasing $B(\gamma)$ with range in $[0, \frac{1}{3}]$.

Remark 3.39. Applying general theorems in Banach space to an operator $\mathcal{R}[\mathbf{a}; b, \gamma]$ with analytic dependence on b, γ , when $\mathcal{R}$ is contractive in some ball the solution to $\mathbf{a} = \mathcal{R}[\mathbf{a}; b, \gamma]$ is analytic in parameters b, γ . Therefore, this carries over to $H(\beta, \lambda)$ and smoothness of $\mathbf{B}(\lambda)$ follows.

(m) Proof of theorem 3.11

Lemma 3.36 translated to the original parameters (b, γ) implies that for given $\gamma \in (-1, \gamma_0)$, there exists a unique $B(\gamma)$ satisfying $h(B(\gamma), \gamma) = 0$ when $\mu_\beta^H < 2$. Since the bubble problem has a solution [11]⁵ and for that solution, $K(\zeta)$ is given by equation (3.28), the equality between two different expressions of K in equations (3.58) and (3.28) follows for the bubble problem. However, the solution to problem 2 for given (γ, b) satisfying conditions of theorem 1 is unique. Therefore, if we constrain $b = B(\gamma)$, this unique solution would correspond both to the bubble problem as well as

⁵We are not certain McLeod's approach extends to $\gamma \neq 0$; hence an independent proof is presented in the supplementary material.

to the hollow vortex problem with shear γ and strain $b - \gamma = B(\gamma) - \gamma$. Therefore, equation (3.51) is a conformal map identity that must be satisfied in each case.

4. Discussion

A ‘periodic version’ of the mathematical problem 1 considered in §2 can readily be formulated by direct analogy with what is done in §2. Instead of parametrizations of closed curves $z = \mathcal{Z}(s)$ those parametrizing the open curve representing a single period of a periodic wave are considered instead. It can then be shown that the following two physical problems are equivalent to this mathematical problem: that of constant vorticity periodic water waves without surface tension as studied recently by [1,2,4,5] and the classical capillary wave water problem studied by Crapper [3]. This periodic analogue of problem 1 can be shown to admit solutions for a conformal mapping that are log-rational functions (this log-rational form of these solutions were first written down by Crowdy [9] for the capillary water wave problem). Consequently, just as for the two physical problems shown here to be equivalent to problem 1 in §2, Crapper’s capillary wave water problem [3] and the problem of constant vorticity water waves without surface tension are mathematically analogous and share the log-rational functions as conformal mapping solutions. This explains the recurrence of Crapper’s wave profiles [3] found in recent numerical, asymptotic and theoretical work [1,2,4,5]. The details are omitted here, but the mathematical strategy is clear and is analogous to what has been done in §2.

Other pairs of physical problems, with different choices of far-field conditions, are likely also to be analogous in the spirit exemplified in this paper. Vanden-Broeck & Keller [21] have found numerical solutions for a two-dimensional bubble with surface tension in a linear strain, for example, and it is likely this problem is analogous to a problem of a hollow vortex, without surface tension, in some higher order background flow. The problem of flow past a bubble [21] attached to a flat plate, or more generally to corner walls, is yet another problem to consider in the current framework though physical interest of the analogous rotating hollow vortex is not clear. A study of singularities in time-evolving flows is another area of interest.

Data accessibility. The results in the main body of this paper are theoretical and any relevant data have been included in the form of tables. The data are provided in the electronic supplementary material [22].

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

Authors’ contributions. D.G.C.: conceptualization, formal analysis, funding acquisition, investigation, validation, writing—original draft and writing—review and editing; S.T.: conceptualization, formal analysis, investigation, methodology, validation, writing—original draft and writing—review and editing.

Both authors gave final approval for publication and agreed to be held accountable for the work performed therein.

Conflict of interest declaration. We declare we have no competing interests.

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Appendix A. Numerical solutions of equation (3.65)

A finite system of nonlinear algebraic equations is obtained by satisfying equation (3.64) for $2 \leq j \leq M$ and setting $a_j = 0$ for $j > M$, where $\mathcal{M}_j, \mathcal{N}_j$ are defined in equations (3.61) and (3.63). Consistency is checked by doubling M . The exact solution for $\gamma = 0$, $\mathbf{a} = (b, 0, 0, \dots)$, manifest in equation (3.64), gives an initial guess in a Newton iteration for small γ and continuation in γ employed. Results were checked with previous results for the steadily translating bubble and the hollow vortex. In particular when the left-hand side of equation (3.50) is plotted against b for a set of chosen $\gamma \in (-1, 0.1455104)$, it was found to be monotonically increasing with b with zero

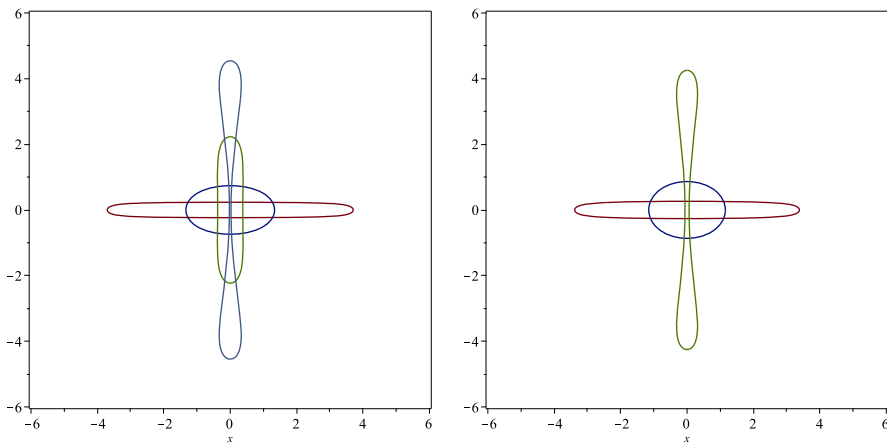


Figure 3. Left: hollow vortex shapes with shear alone for $\gamma = -3.5, -0.20, 0.20, 0.233$ corresponding to $a = 2.0123, 1.04348, 1.36545, 2.4871$, respectively. Vertical dimensions increase with increasing γ ; pinching occurs for $\gamma \in (0.233, 0.234)$. Right: hollow vortex shapes with additional strain $b - \gamma = 0.1$ for $\gamma = -3.5, -0.20, 0.18$ corresponding to $a = 1.86123, 1.0104, 2.3358$, respectively. Vertical dimensions increase with increasing γ , until there loss of univalence occurs for some $\gamma \in (0.183, 0.184)$.

crossing value, the Weber number for given Bernoulli constant in the bubble problem, agreed with prior numerical results (Vanden-Broeck & Keller [15], Shankar [14] and Tanveer [10]).

Figure 3 shows the hollow vortex case when $b = \gamma$ (pure shear) for range of γ . Each shape has been rescaled by the factor a (see line following equation (3.22)) so that the area of each rescaled vortex is π . The solutions appear to exist for indefinitely large negative γ and the nearly elliptical shapes shown for $\gamma = -3.5$ does not appear to change much as γ becomes even more negative. Figure 3 also shows the hollow vortex shapes when the strain component of the flow $b - \gamma = 0.1$ for four different γ . Positive strain $b - \gamma > 0$ elongates the hollow vortex vertically relative to no strain as may be expected physically. In this case, loss of univalence (through pinching of the blob) of the conformal map occurs very close to $\gamma = 0.183$, at a smaller value than without strain. On the other hand, when $b - \gamma < 0$, the opposite trend is visible with a tendency to impede pinching. For strain $b - \gamma = -0.1$, pinching occurs at a larger value: $\gamma \in (0.284, 0.285)$. When γ becomes large and negative, the bubble shapes become flat and elongated in the horizontal direction.

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