

APPLICATION OF
FADDEEV'S THEORY TO THREE-NUCLEON PROBLEMS
(A NUMERICAL SOLUTION)

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by

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ABSTRACT

The formalism, developed by S.P.Y. Lee, whereby the once-iterated Faddeev equation for a system of three particles is formulated according to the group $SU(3)$, is subjected to a first numerical test. We distinguish between a system consisting of three identical bodies and one composed of two neutrons and one proton. Both systems are assumed to be governed by interacting pairs, the interaction being given by a simple central potential. Three such potentials have been considered: Yukawa's, Yamaguchi's and the exponential type of interaction. The system of identical particles provides the basis for testing the behaviour of the computed binding energy as a function of the coupling constant in the potential. The system also allows the assessment of the error arising from the approximation of the integrals by weighted sums. Spin and isospin are subsequently included for all three types of potentials. A number of partial-wave amplitudes are calculated and plotted.

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PREFACE

This work is in essence the second part of a twin-project, suggested by Dr. S. Hochberg and handed to S.P.Y. Lee and myself in 1968. The former set out to develop a new formalism whereby the problem of systems composed of three nucleons, and indeed of many nucleons as well, is approached from an entirely new angle within the framework of Faddeev's theory. In this work he succeeded brilliantly, as shown by the wide interest it received ~~and will receive~~^{has since}. Lee's work is exact and general, except for the latter part, where the formalism was modified to suggest ways of ascertaining, numerically, the validity of the theory.

By contrast, this work is strongly oriented towards the numerical solution of a complex physical problem and thus emphasizes the practical aspects of computerized research. The pure physicist will undoubtedly find Lee's work more rewarding, whereas this work may be considered the necessary 'bread-and-butter' compendium, designed to give computational evidence for the theory's validity.

Numerical verification of new formalisms by means of high speed electronic computers, however, constitutes an ever increasing part in today's research and in the author's view, this part is often not being dealt with in sufficient detail. We have therefore decided, in spite of this work's dependence upon Lee's work, to keep it as self-contained as possible and to present a detailed account of the computational methods involved together with a full description and documentation of the programs, where these are non-standard.

I wish to thank Dr.Hochberg for suggesting the problem and for his constant support in the course of this investigation. I also wish to express my gratitude to Dr.Lee for discussing his formalism with me and for his many helpful suggestions.

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I N T R O D U C T I O N

Prior to Lee's recent work⁽¹⁾, various attempts were made to solve Faddeev's equations⁽²⁾ for a three-particle system. The main difficulty encountered, when setting up such a formalism lies in reducing the number of variables, such that reliable numerical tests may be performed without exceeding the capacity of presently available high-speed electronic computers. Various restrictions had to be introduced, such as separable non-local potentials⁽³⁾ in order to reduce the number of variables involved. Osborn^(1,5) eventually succeeded in obtaining a set of equations with two integrating variables, via application of Omnes' method of symmetrical angular momentum reduction and restricting himself to the case of three spinless particles, interacting in pairs through a central Yukawa potential.

The new formalism, which is numerically tested in this investigation, is based on solving Faddeev's equations in the $SU(3)$ representation of three-particle states. Using Dragt's^(1,6) discussion of such states, Lee then proceeds to construct a set of irreducible

representations of these states on a five-dimensional sphere S_5 in a differential form, with each state being characterized by the eigenvalues of the two Casimir operators of the Lie algebra of $SU(3)$. The eigenfunctions are then 'reordered' in such a way as to have simple properties under S_3 , the group of permutations of three objects, for certain values of the total angular momentum of the system. Fourier-Transformation of the eigenfunctions then provides the basis for writing Faddeev's equations in the $SU(3)$ representation.

For the sake of self-consistency, this thesis includes a summary of Lee's work; this summary, which is presented in Chapter I, is by no means intended to form a substitute for the reading of the original work, it is rather to be thought of as an introduction into its concepts and the nature of the variables involved. It is hoped that as an introduction it will be sufficiently detailed, in order to clarify the purpose of this investigation.

The next chapter deals with some of the preliminaries to the actual numerical verification of Lee's formalism. It includes a list of the equations to be solved, followed by a discussion of the three types of pair-interactions

we have chosen and the determination of the shape of these potentials.

Chapter III presents a detailed account of the layout of the computations. It covers the methods used to obtain values for the binding energies of the two systems under consideration, that consisting of three bosons and the one consisting of three fermions. This is followed by a description of the way the shapes of the partial wave-amplitudes were obtained for the bound-state.

In the last chapter we present a list of all the relevant results, together with a discussion of their meaning and some remarks concerning the errors arising out of the computational approximations.

CHAPTER I SUMMARY OF LEE'S GENERAL FORMALISM

This chapter summarizes the concepts, whereby Faddeev's equations for a system of three particles are solved in the representation of the group $SU(3)$. We have sought to verify this formalism numerically for the special case of the three-particle bound-state, and this summary is presented with this particular aspect in mind.

The first section briefly deals with the formal origins of Faddeev's theory⁽²⁾ (see also ref.^s(1,2)). Section I.2 outlines the three major steps leading to the formal solution of the iterated Faddeev Equation for the idealized case of the three-boson bound-state. The last section introduces spin-isospin, again with a view towards obtaining the equations for the three-body bound-state.

SECTION I.1 Faddeev's Theory and Iterated Equation

When one attempts to extend the well-tried Lippmann-Schwinger formalism ^(A) for the description of two-particle systems to the case of three particles, one runs up against fundamental difficulties, arising basically out of the fact that whereas the two-body system is characterized by one initial state ϕ_i and only one possible final state ϕ_f , a three-body system has four possible final states to each given initial state. Mathematically speaking we are faced with the difficulty of having to eliminate the arbitrariness in the choice of the resolving Green's function in the LS-formalism. This arbitrariness leads to the integral equations not having unique solutions, unless additional assumptions are brought in, thus curtailing the basic advantage of the LS-formalism. Another way of stating this difficulty is that the application of the two-body concepts to a system of three particles leads to the appearance of δ -functions in the kernel of the integral equations defeating the classical methods of solution such as iteration.

(+1) Abbreviated to LS-formalism in the following

Faddeev's work is based on the introduction of a modification of the LS-resolvent. This new resolvent arises out of applying the Green's function method to the Schroedinger Equation for a system of three particles, and it has the form:

$$G^+(z) = G_0^+(z) - G_0^+(z) V G^+(z) \quad (\text{I.1.1})$$

where

$$G_0^+(z) = (H_0 - z^+)^{-1} \quad (\text{I.1.2})$$

and

$$V = V_{23} + V_{31} + V_{12} \quad (\text{I.1.2i})$$

with the quantities

H_0 = the total free Hamiltonian

$z^+ = E + \delta i$; E = total energy of the system

V_{ij} = pair-interaction operator between the particles i and j .

Faddeev introduces the concept of a generalized T-Matrix in the modified resolvent, which takes the form

$$G(z) = G_0(z) - G_0(z) T(z) G_0(z) \quad (I.1.3)$$

with

$$T(z) = V - V G_0(z) T(z) \quad (I.1.3i)$$

Eqn. (I.1.3) is then rearranged in order to eliminate terms such as $V_{ij} G_0(z) V_{ij}$, which appear upon iteration of eqn. (I.1.3) and give rise to the δ -functions in the kernel, as mentioned above. (See also Chapter III of ref. (2)). The object of this rearrangement is to obtain integral equations of the Fredholm type and $T(z)$ is therefore expressed as a sum of three terms

$$T(z) = T^{(1)}(z) + T^{(2)}(z) + T^{(3)}(z) \quad (I.1.3ii)$$

where $T^{(k)}(z)$ is given by

$$T^{(k)}(z) = V_{ij} - V_{ij} G_0(z) T(z) ; k, i, j = 1, 2, 3 \quad (I.1.3iii)$$

$k \neq i \neq j$

The components in eqn.(I.1.3) satisfy the matrix equation

$$\begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} - \begin{bmatrix} 0 & T_1 & T_1 \\ T_2 & 0 & T_2 \\ T_3 & T_3 & 0 \end{bmatrix} G_0(z) \begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix} \quad (\text{I.1.4})$$

where by $T_i(z)$ we denote the transition operator belonging to the two-particle subsystem (j,k) with $i, j, k = 1, 2, 3$ and $i \neq j \neq k$.

Equation (I.1.4) is a coupled set of integral equations of the Fredholm type and lies at the root of the successful formal solution of the three-body problem. Regrouping the resolvent $G(z)$ in a fashion similar to eqn.(I.1.3ii) such that

$$G(z) = G_0(z) + \sum_{l=1}^3 G^{(l)}(z) \quad (\text{I.1.5})$$

four sets of coupled integral equations can be derived for a system of three particles, each of which can be combined in general matrix notation to take

the form

$$\begin{bmatrix} |\Psi^{(1)}\rangle \\ |\Psi^{(2)}\rangle \\ |\Psi^{(3)}\rangle \end{bmatrix} = \begin{bmatrix} |\Psi_0^{(1)}\rangle \\ |\Psi_0^{(2)}\rangle \\ |\Psi_0^{(3)}\rangle \end{bmatrix} - \begin{bmatrix} 0 & G_0 T_1 & G_0 T_1 \\ G_0 T_2 & 0 & G_0 T_2 \\ G_0 T_3 & G_0 T_3 & 0 \end{bmatrix} \begin{bmatrix} |\Psi^{(1)}\rangle \\ |\Psi^{(2)}\rangle \\ |\Psi^{(3)}\rangle \end{bmatrix} \quad (\text{I.1.6})$$

$$\text{with } |\Psi\rangle = |\Psi_0\rangle + \sum_{i=1}^3 |\Psi^{(i)}\rangle$$

representing an interacting three-body state. (For further details see Section III.2 of ref.(2)).

Equation (I.1.6) is in essence a LS-equation in operator form. The intrinsic physical difference between the two- and three-particle systems is reflected in the fact that for the latter the equation is of higher 'dimension'. The former possesses a scalar kernel, whereas eqn.(I.1.6), while retaining the basic simplicity of the LS-equation, has a kernel consisting of a matrix operator designed to correlate the given initial state with the four possible final states. This kernel moreover conforms to the wider definition of a Fredholm equation in that it is bounded and square-integrable.

In momentum space the two-body transition operators typically are of the form

$$\langle \underline{k} | T_i(z) | \underline{k}' \rangle = \delta(\underline{\xi}_i - \underline{\xi}'_i) \langle \underline{\eta}_i | t_i(z - \xi_i^2) | \underline{\eta}'_i \rangle \quad (\text{I.1.6i})$$

where ξ_i and η_i are the usual centre-of-mass relative momentum coordinates. Due to the δ -function in eqn. (I.1.6i) the kernel in (I.1.6) is still not completely continuous in the sense that solutions can be obtained by the usual iterative methods. It has been shown ^(1,33) that for particular types of pair-interactions, such as Yukawa's, the squared matrix operator is completely continuous for all z ; Lee therefore suggests using the once-iterated form of eqn. (I.1.6) thus obtaining a kernel, to which the Fredholm-theory applies in its stricter sense. The iterated equation is

$$|\tilde{\Psi}^{(i)}\rangle = |\tilde{\Psi}_0^{(i)}\rangle + \sum_{j \neq i} K^{(i,j)}(z) |\tilde{\Psi}^{(j)}\rangle \quad (\text{I.1.7})$$

L.S.5

with redefined 'asymptotic' states

$$|\tilde{\Psi}_0^{(i)}\rangle = |\tilde{\Psi}^{(i)}\rangle - \sum_{j \neq i} G_0 T_i |\tilde{\Psi}_0^{(j)}\rangle \quad (\text{I.1.7i})$$

L.S.6

The kernel $K^{(i,j)}(z)$ now has the form

$$K(z) =$$

$$\begin{bmatrix} G_0 T_1 G_0 T_2 + G_0 T_1 G_0 T_3 & G_0 T_1 G_0 T_3 & G_0 T_1 G_0 T_2 \\ G_0 T_2 G_0 T_3 & G_0 T_2 G_0 T_1 + G_0 T_2 G_0 T_3 & G_0 T_2 G_0 T_1 \\ G_0 T_3 G_0 T_2 & G_0 T_3 G_0 T_1 & G_0 T_3 G_0 T_1 + G_0 T_3 G_0 T_2 \end{bmatrix} \quad \begin{matrix} 1.1.8 \\ (2.5.7) \end{matrix}$$

Equation (I.1.7) forms the basis of Lee's work.

Since this investigation deals only with three-particle boundstates, the inhomogeneous part of the equation will be omitted henceforth.

SECTION I.2 Formal Solution of the Iterated Equation
for a Three-Boson Bound-State.

The problem of applying equations (I.1.7) and (I.1.8) to a system of three nuclear particles, Bosons or Fermions, ~~boils down~~ ^{reduces to} to the necessity of finding a suitable general representation for the states of such a system. Until Lee's work, attempts had been made to classify such states in various ways and to solve Faddeev's Equations in the framework of those representations. All these theories incorporated the restrictive assumption that the particles interact in pairs through separable, non-local potentials, such as the one proposed by Yamaguchi⁽³⁾. The advantage of this restriction lay in the fact that part of the kernel could be solved in closed form, thus reducing the number of variables involved. Apart from the obvious disadvantage of such a restriction, there was also the considerable difficulty of generalizing such formalisms to include spin-isospin for the treatment of the more realistic system consisting of three Fermions.

Moreover, all these methods were stretched to the limit in order to produce results of limited significance only. In numerical terms the solving methods were intricate: integral equations with variable limits, requiring large computers with large word-lengths to carry as many digits as possible in order to keep rounding errors on a tolerable level. We do not propose to have ^{solved} all the problems arising from a numerical approximation to the solution of integral equations, but Lee's new formulation of the problem leaving one with a one-variable integral equation intrinsically reduces the number of error sources. The numerical effort is still considerable, but computational errors now become more controlled and predictable to the extent that the numerical content of the theory can be better assessed.

I.2A) Classification of Three-Particle States within the Algebra of $SU(3)$.

We start by quoting some well-known group-theoretical facts about rotations in three dimensions.

The description of such rotations requires three ^{non-}independent commuting operators, e.g. J_x , J_y and J_z . Each operator is understood to represent an infinitesimal rotation about its respective axis. These operators obey the well-known relations (where for convenience we have the indices 1,2,3 for x,y,z respectively).

$$[J_1, J_m] = i \pi \epsilon_{1m}^n J_n \quad 1,m,n = 1,2,3 \quad (\text{I.2A,1})$$

with

$$\epsilon_{1m}^n = \begin{cases} 1 & \text{for even permutations of } 1,m,n \\ 0 & \text{for any two or all three indices equal} \\ -1 & \text{for odd permutations of } 1,m,n \end{cases}$$

The three above operators form the group denoted by O_3 . Eqn. (I.2A.1) defines the algebra of this group, with ϵ_{1m}^n being the structure constant. This group is a compact, semi-simple Lie group^(1,15).

An important property of an algebra is its rank, defined as the number of operators which can be simultaneously diagonalized. In terms of physics it is the number of group-elements which can be observed

simultaneously as measurable quantities. The rank of O_3 is one and it is customary to diagonalize J_3 . The rank is also equal to the number of Casimir operators, which can be formed from the algebra of a group. These operators commute with every element of the group, a condition which in the case of O_3 is satisfied by the square of J , the total angular momentum operator:

$$[J^2, J_i] = 0 \quad i = 1, 2, 3 \quad (I.2A.1i)$$

J^2 is therefore a C-operator of O_3 . Since the algebra of O_3 allows for only one operator to be diagonalized, there is only one C-operator. Having determined the algebra of a group, the next step consists of finding irreducible representations for the group. It can be shown that the number of labels required to specify an I.R. is equal to the rank of the group. In the case of O_3 we therefore need one label, usually chosen in accordance with the eigenvalues of the C-operator J^2 , i.e. $j(j+1)$. A particular component in the I.R. requires $\frac{1}{2}(n-r)$ labels, where n denotes the number of group parameters and r stands for the rank. In O_3 $n = 3$ and $r = 1$,

hence 1 label. Thus the specification of an angular momentum state requires altogether 2 labels, normally denoted by j and j_z where j_z is the eigenvalue label of the operator J_z . Knowing the values for j and j_z one can then construct a complete set of states $|j, j_z\rangle$, upon which the group-operators act.

We shall now proceed to look at three-particle states and their transformations. Altogether 12 quantities are involved in the specification of such a state: 9 position (or momentum) quantities, 2 for spin and isospin and, for completeness, one for hypercharge. If for the moment we disregard the last three, we are left with real 9-dimensional state vectors, transformation of which are effected by 9×9 matrix operators.

By imposing the condition that $\underline{R}^2 = \sum_{i=1}^3 \underline{r}_i^2$ (or $\underline{p}^2 = \sum_{i=1}^3 \underline{p}_i^2$) remain invariant and furthermore shifting our considerations to the CM-frame with the new vectors

$$\begin{aligned}\underline{r}^{(1)} &= \frac{1}{\sqrt{2}}(\underline{r}_2 - \underline{r}_1) \\ \underline{r}^{(2)} &= \frac{1}{\sqrt{6}}(2\underline{r}_3 - \underline{r}_2 - \underline{r}_1) \\ \underline{r}^{(3)} &= 0\end{aligned}\tag{I.2A.2}$$

we obtain the 6-dimensional state vectors $\underline{r} = (\underline{r}^{(1)}, \underline{r}^{(2)})$ and the required transformations are the elements of the group O_6 (orthogonal six-dimensional transformations). The algebra of this group is given by the 15 real antisymmetric matrices R_{ij} obeying the relations

$$\begin{aligned} [R_{ij}, R_{jk}] &= R_{ik} \\ R_{ij} &= -R_{ji} \\ [R_{ij}, R_{mn}] &= 0 \quad \text{for } i \neq j \neq m \neq n \end{aligned} \quad (I.2A.3)$$

Group theory tells us that O_6 is reducible and one therefore has to decide how to effect the reduction. In view of the fact that later on we wish to include spin-isospin, it seems natural to select those elements of O_6 which treat the three particles symmetrically. Dragt^(1,6) shows that there exists a subalgebra L_1 to L_0 (the algebra of O_6) which has this property. We also note here that a linear operator S can be constructed, which commutes with all elements of L_1 and which can be interpreted as a C-operator providing one label of those required to specify an irreducible representation (I.R.) of the system.

Let us now realize L_0 and L_1 as an algebra of hermitean operators acting on three-particle state vectors. The hermitean analogue of eqn.(I.2A.3) is

$$[\Lambda_{jk}, \Lambda_{lm}] = 0 \quad i, j \neq k \neq l \neq m$$

$$[\Lambda_{jk}, \Lambda_{kl}] = -i\Lambda_{jl}$$

$$\Lambda_{ji} = -\Lambda_{ij} \quad (I.2A.4)$$

k.2.10

with $\Lambda_{jk} = r_j p_k - r_k p_j$, $j, k = 1, \dots, 6$, being a set of operators defined by

$$[\Lambda_{jk}, \underline{r}] = i\hbar R_{jk} \underline{r} \quad (I.2A.4i)$$

The quadratic operator

$$\Lambda^2 = \frac{1}{2} \sum_{i,j} \Lambda_{ij}^2 \quad (I.2A.4ii)$$

is found to commute with all the Λ_{ij} and may therefore

be interpreted as a quadratic C-operator of the algebra

L_0 , whose eigenvalues supply a further label for an I.R.

This operator is associated with the 'Grand Total

Angular Momentum' of the three-particle system. The

algebra L_i turns out to be that of the group $U(3)$,

with the nine elements

$$J_{ij} = R_{ij} + R_{i+3,j+3} \quad ; \quad i, j \leq 3, i \neq j$$

$$K_{ij} = R_{i,j+3} - R_{i+3,j} \quad ; \quad i, j \leq 3 \quad (\text{I.2A.4iii})$$

Extracting the operator S, which is of the form

$$S = \frac{1}{2} \sum K_{ij} \quad (\text{I.2A.4iv})$$

from L_1 leads to the algebra L_2 with 8 elements.

L_2 is shown to be isomorphic to $SU(3)$, the algebra of traceless 3×3 hermitean matrices. L_2 is irreducible and of rank two, with the two C-operators Λ^2 and S. An irreducible representation of $SU(3)$, in analogy to what was said at the beginning of this section, therefore requires two labels, chosen to be the eigenvalue-labels of the two C-operators.

Since $r = 2$ and $n = 8$, each component within an I.R. requires another 3 labels. If the group $SU(3)$ is chosen to be decomposed in terms of the sub-group $SO(3)$ (the unimodular O_3), then two of the labels are provided by the eigenvalues of the three-dimensional rotation generators J^2 and J_z , discussed previously. The last remaining label belongs to possibly another cubic operator, but need not be considered further, since it

deals with degeneracy of the states at higher energies.

Thus a three-particle state is specified by altogether 5 labels. These labels are related to a specific decomposition of $SU(3)$ suited to meet the requirements of dealing with the three particles in a symmetric fashion. The symmetry aspect has been taken into account by the reduction of O_6 via its subgroup S_3 , the group of permutations on three objects. The operator S indeed is related to all elements of S_3 and commutes with each one.

I.2B) Construction of Irreducible Representations of $SU(3)$ States.

In the previous section we have shown that an I.R. of $SU(3)$ is specified by two labels, chosen to be the eigenvalue-labels of the two Casimir-operators Λ^2 and S . These two labels shall be denoted by λ and η . Since these operators are related to the concept of infinitesimal generators, it is natural to seek their representation in some differential form, and more specifically in such a form that they show

the operators of the sub-group $SO(3)$ explicitly.

We recall eqn.(I.2A.2) and note that we are dealing with six-component vectors $\underline{r} = (\underline{r}^{(1)}, \underline{r}^{(2)})$ (or alternatively $\underline{p} = (\underline{p}^{(1)}, \underline{p}^{(2)})$). Based on the work of previous authors^(1,13), Lee chooses a set of equivalent variables, which we shall simply state here without derivation:

$$\underline{r} = (r, \psi, \phi, \alpha, \beta, \gamma)$$

where the variables range as follows

$$0 \leq r \leq \infty$$

$$0 \leq \psi \leq \frac{\pi}{4}$$

$$0 \leq \phi \leq 2\pi$$

} Dalitz-Fabbri Coordinates

(I.2B.1)

$$0 \leq \alpha \leq 2\pi$$

$$0 \leq \beta \leq \pi$$

$$0 \leq \gamma \leq 2\pi$$

} Euler Angles

The first three of these variables represent the 'internal' coordinates of a triangular figure with one particle at each vertex. The Euler angles then describe the orientation of this figure in space. The reader is recommended to consult Chapter 3 of Lee's thesis for a full understanding of these

coordinates.

The differential form of the operators Λ^2 and S must be such that their eigenfunctions, i.e. the I.R. of $SU(3)$, lie on S_5 , the manifold of a five-dimensional sphere, parameterized by the set (I.2B.1). In this set Lee obtains the following expressions for these two operators:

$$\begin{aligned} \Lambda^2 = & - \left(\frac{\partial^2}{\partial \psi^2} + \frac{4 \cos 4\psi}{\sin^4 \psi} \frac{\partial}{\partial \psi} + \frac{4}{\cos^2 2\psi} \frac{\partial^2}{\partial \varphi^2} \right) \\ & + \left[\frac{2L^2}{\sin^2 \psi} + \left(\frac{1}{\cos^2 2\psi} - \frac{2}{\sin^2 2\psi} \right) L_w^2 + 4i\pi \frac{\sin 2\psi}{\cos^2 2\psi} L_w \frac{\partial}{\partial \varphi} \right. \\ & \left. + \frac{\cos 2\psi}{\sin^2 2\psi} (L_+^2 - L_-^2) \right] \end{aligned} \quad (\text{I.2B.2})$$

and

$$S = 2i\pi \frac{\partial}{\partial \varphi} \quad (\text{I.2B.2i})$$

In eqn.(I.2B.2) $\underline{L} = (L_{\underline{u}}, L_{\underline{v}}, L_{\underline{w}})$ is the angular momentum operator with respect to the orthogonal, body-fixed (triangular figure) frame, where $\underline{u}$ and $\underline{v}$ coincide with the two principle axes of inertia;

furthermore

$$L_{\underline{w}} = -i\hbar \frac{\partial}{\partial \gamma}$$

and

$$L_{\pm} = L_{\underline{u}} \pm iL_{\underline{v}} = i\hbar e^{\mp i\gamma} \left(\frac{1}{\sin \beta} \frac{\partial}{\partial \alpha} - i \frac{\partial}{\partial \beta} \mp \cot \beta \frac{\partial}{\partial \gamma} \right)$$

Λ^2 can be identified with the Laplace-Beltrami^(1,7) operator on S_5 and is also the angular part of the six-dimensional Laplace operator ∇_6^2 . The eigenfunctions of Λ^2 are the surface harmonics $S_{\lambda}^{\nu}(\hat{\underline{x}})$ on S_5 of degree λ , i.e.

$$\Lambda^2 S_{\lambda}^{\nu}(\hat{\underline{x}}) = \lambda(\lambda + 4) \hbar^2 S_{\lambda}^{\nu}(\hat{\underline{x}}) \quad (\text{I.2B.3})$$

where we have used the notational abbreviations:

$$\hat{\underline{x}} \hat{=} (\psi, \varphi, \alpha, \beta, \gamma)$$

$\nu \hat{=}$ the remaining set of labels for the harmonics.

These harmonics are also eigenfunctions of the S-operator

$$S S_{\lambda}^{\mu, \nu}(\hat{\underline{x}}) = 2\mu\hbar S_{\lambda}^{\mu, \nu}(\hat{\underline{x}}) \quad (\text{I.2B.4})$$

For given λ , the range of μ is

$$-\frac{\lambda}{2}, -\frac{\lambda}{2} + 1, \dots, \frac{\lambda}{2} - 1, \frac{\lambda}{2}$$

The remaining set of labels now numbers at most 3.

If in accordance with the decomposition of $SU(3)$ in terms of the subgroup $SO(3)$ we impose the condition that the surface harmonics also be eigenfunctions of the operators L^2 and L_z (for the time being we consider only integer values of j and j_z), i.e. that $S_{\lambda}^{\mu, \nu}(\hat{r})$ satisfy the equations

$$L^2 S_{\lambda}^{\mu, LM}(\hat{r}) = L(L+1) \hbar^2 S_{\lambda}^{\mu, LM}(\hat{r}) \quad (I.2B.5)$$

$$L_z S_{\lambda}^{\mu, LM}(\hat{r}) = M \hbar S_{\lambda}^{\mu, LM}(\hat{r}) \quad (I.2B.6)$$

we find these harmonics to be of the form

$$S_{\lambda}^{\mu, LM}(\hat{r}) = \sum_K G_{\lambda \mu}^{LMK}(\psi, \varphi) D_{MK}^L(R) \quad (I.2B.7)$$

where $D_{MK}^L(R) = e^{iM\alpha} d_{MK}^L(\beta) e^{iK\gamma}$, the rotational

matrix^(1,12)

$$\text{and } G_{\lambda\mu}^{\text{LMK}}(\psi, \varphi) = N_{\Omega} g_{\lambda\mu}^{\text{LM,K}}(\psi) e^{-i\mu\varphi}$$

with N being the normalization factor arising out of normalising $S_{\lambda}^{\mu, \text{LM}}(\hat{r})$ to unity. For small values of λ as well as for low angular momentum the third label of the set may be disregarded because the system can be shown to be multiplicity-free.

In principle the functions $g_{\lambda\mu}^{\text{LM,K}}(\psi)$ can be determined from eqn.s (I.2B.3) and (I.2B.7) but for general L the solution of the coupled differential equations proves to be very difficult. Lee lists the major properties of these functions but, since we are treating only ~~bound~~^S states, all we need to consider is the case when $L = 0$ (the M-label can be dropped as the functions do not depend on it at all). For this case, eqn.(I.2B.7) reduces to one equation of the form

$$S_{\lambda\mu}(\hat{r}) = N_{\Omega} e^{-i\mu\varphi} (\cos 2\psi)^{\frac{1}{2}(\frac{\lambda}{2} - |\mu|)} \begin{matrix} |\mu| \\ P \end{matrix} \begin{matrix} |\mu| \\ \mu \end{matrix}, 0 (1 - 2\cos^2 2\psi) \quad (\text{I.2B.8})$$

by means of the Jacobi polynomials $P_1^{n,m}(x)$.

Having obtained an I.R. of $SU(3)$ in the form of a complete set of functions $S_{\lambda}^{\mu, LM}(\hat{\underline{r}})$ on S_5 in configuration space, we note that in momentum space the angular part remains exactly the same. We can now construct the general normalized states in both spaces. Using Dirac's ket-and-bra notation and again denoting the remaining set of labels by λ and ν , we obtain the states as

$$\langle \underline{r} | k^2, \lambda \nu \rangle = \frac{1}{\sqrt{2}} \frac{J_{\lambda+2}(kr)}{r^2} S_{\lambda}^{\nu}(\hat{\underline{r}}) \quad (I.2B.9)$$

$$\langle \underline{k} | k^2, \lambda \nu \rangle = \frac{\sqrt{2}}{k'^2} \frac{\delta(k'^2 - k^2)}{k'^2} S_{\lambda}^{\nu}(\hat{\underline{k}})$$

where $\underline{k}$ stands for the momentum space counterparts of the five angular variables in configuration space.

In view of discussing symmetry properties, Lee reorders the set of functions $S_{\lambda}^{\nu}(\hat{\underline{r}})$ making use of the adjoint representation of the I.R. of $SU(3)$. The rearranged set, denoted by $S_{\lambda\nu}(\hat{\underline{r}})$ greatly simplifies the specific equations for the boundstate, as we shall see later on. For the rest of this discussion, we shall adhere to this notation.

The set of states (I.2B.9) being complete, a general interacting state can be represented as

$$\langle \underline{k} \mid \psi \rangle = \sum_{\lambda v} \frac{x_{\lambda v}(k)}{k^2} S_{\lambda v}(\hat{\underline{k}}) \quad \text{-(I.2B.10)}$$

in momentum space, where $x_{\lambda v}(k) = \sqrt{2} \langle k^2, \lambda v \mid \psi \rangle$ is the generalized partial wave amplitude.

The configuration space analogue is given by

$$\langle \underline{r} \mid \psi \rangle = \sum_{\lambda v} \frac{u_{\lambda v}(v)}{r^2} S_{\lambda v}(\hat{\underline{r}}) \quad \text{(I.2B.10i)}$$

The two amplitudes are related by

$$x_{\lambda v}(k) = \int J_{\lambda+2}(kr) u_{\lambda v}(r) r dr \quad \text{(I.2B.10ii)}$$

with equal normalization

$$\int |x_{\lambda v}(k)|^2 k dk = \int |u_{\lambda v}(r)|^2 r dr \quad \text{(I.2B.10iii)}$$

I.2C) Formulation of Faddeev's Equations in the SU(3) Representation.

Having found a complete set of three-particle states $|k^2, \lambda \nu\rangle$ via the I.R. of SU(3), we now wish to apply this to Faddeev's theory. With a special view towards the application of Omnes' angular momentum reduction, we introduce an intermediate set of states. This set arises out of a representation of the system, in which the total energy $\mathfrak{u} = (\underline{k}_1^2, \underline{k}_2^2, \underline{k}_3^2)$, the total angular momentum $\underline{L}$ and its components M and K about the space- and body-fixed axes respectively are diagonalized. Such a state shall be denoted by $|\mathfrak{u}, L M K\rangle$. We note here that this set of states is completely equivalent to the set denoted by $|\underline{k}^2, \lambda \nu\rangle$; it is also equivalent to the momentum space counterpart of the state $|\underline{r}_i\rangle = |\underline{r}_i^{(1)}, \underline{r}_i^{(2)}\rangle$, which is denoted by $|\underline{k}_i\rangle = |\underline{\xi}_i, \underline{\eta}_i\rangle$ ($\underline{\xi}_i$ and $\underline{\eta}_i$ are in no way connected with the variables in eqn.(I.1.6i), p.17). The index i in the latter set refers to the three possible choices of relative coordinates; the index i indicates that $\underline{r}^{(2)}$ and $\underline{\eta}$ are placed along $\underline{r}_i$ and $\underline{k}_i$ respectively.

The reason behind the choice of the intermediate set $|u, L M K\rangle$ becomes clear when we consider Faddeev's iterated equation (I.1.7) and the calculation of the kernel elements $\langle k^2, \lambda \nu | K^{(i,j)}(z) | k'^2, \lambda' \nu' \rangle$. The functions $S_{\lambda \nu}(\underline{k})$ were constructed in such a way that their angular dependence on α, β and γ was separated out. With the help of Omnes' angular momentum reduction and the intermediate set, we can obtain the elements of $K^{(i,j)}(z)$ with the $SO(3)$ part again separated out. Thus we can deal with three of the six variables separately.

In the $SU(3)$ representation the homogeneous part of eqn.(I.1.7) reads:

$$\begin{aligned} & \langle k^2, \lambda \nu | \Psi^{(i)} \rangle \\ &= \sum_{j=1}^3 \sum_{\lambda' \nu'} \int_0^\infty \langle k^2, \lambda \nu | K^{(i,j)} | k'^2, \lambda' \nu' \rangle 2k' dk' \langle k'^2, \lambda' \nu' | \Psi^{(j)} \rangle; \end{aligned}$$

with $i = 1, 2, 3$, (I.2C.1)

which is a coupled one-variable integral equation in k' .

We shall omit the formal details arising out of the above remarks concerning the intermediate set of states and we merely quote the expression, which results for the kernel-elements $\langle k^2, \lambda_\nu | K^{(i,j)}(t) | k'^2, \lambda'_{\nu'} \rangle$ after application of Omnes' angular momentum reduction:

$$\begin{aligned} & \langle k^2, \lambda_\nu | K^{(i,j)} | k'^2, \lambda'_{\nu'} \rangle \\ &= \frac{1}{32\sqrt{3}(2L+1)} \sum_{K, K'} \int [\langle k^2, \psi_\varphi, LMK | K^{(i,j)}(t) | k'^2, \psi'_{\varphi'}, LMK' \rangle \\ & \quad * G_{\lambda_\mu j z_\kappa}^{*L, K}(\psi^*, \varphi^*) G_{\lambda'_\mu j z'_\kappa}^{L, K'}(\psi', \varphi') k^2 k'^2] d\Delta \end{aligned} \quad (I.2C.2)$$

$$\text{with } d\Delta = \cos 2\psi^* \cos 2\psi' d(\cos 2\psi^*) d\varphi^* d\varphi' \quad (I.2C.2i)$$

The labelling of the functions $G_{\lambda_\mu j z_\kappa}^{L, K}(\psi, \varphi)$ has been chosen in accordance with the rearrangement of the surface harmonics (see remarks preceeding eqn. (I.2B.10)).

Dealing with a boundstate of three spinless particles implies first of all $L = M = K = 0$; furthermore the total symmetry of such states with respect to all

permutations of the particles implies that the labels λ and μ (rearranged into two groups denoted by μ_j and z , $j = 1, 2, 3$) appear only with the following values:

$$\lambda = 0, 4, 6, 8 \text{ and even integers}$$

$$z = 2 \quad (\text{this label will be omitted in future})$$

$$\mu_j = \mu_3 = 3n, \quad n = 0, 1, 2, 3, \dots$$

The surface harmonics then reduce from eqn. (I.2B.8) to

$$S_{\lambda\mu_3}(\underline{k}) = G_{\lambda\mu_3}(\psi, \varphi)$$

$$= \left[\frac{\lambda+2}{3} \right]^{\frac{1}{2}} \cos \mu_3 \varphi (\cos 2\psi)^{\mu_3} P^{\mu_3, 0}_{\frac{1}{2}(\frac{1}{2}\lambda - \mu_3)} (1 - 2\cos^2 \psi)$$

$$\times \begin{cases} 1 & \text{for } \mu_3 \neq 0 \\ \sqrt{\frac{1}{2}} & \text{for } \mu_3 = 0 \end{cases}$$

The factor $(\frac{1}{2})^{\frac{1}{2}}$ for $\mu_3 = 0$ is needed to compensate for the double contribution of $S_{\lambda\mu_3}$, since $S_{\lambda\mu_3} = S_{\lambda-\mu_3}$ for $\mu_3 = 0$ (see ref. (1), eqn. (4.35i)).

The eqn. (I.2C.2) now reads

$$\begin{aligned}
 \langle k^2, \lambda \mu_3 | K^{(i,j)}(z) | k'^2, \lambda' \mu'_3 \rangle &= \frac{[(\lambda+2)(\lambda'+2)]^{\frac{1}{2}}}{6\sqrt{3} \pi} k^2 k'^2 \\
 &\times \int [\langle k^2, \psi^* \varphi^* | K^{(i,j)}(z) | k'^2, \psi'' \varphi'' \rangle \cos \mu_3 \varphi^* \cos \mu'_3 \varphi'' \\
 &\times (\cos 2\psi^*)^{\mu_3} (\cos 2\psi'')^{\mu'_3} \times P^{\mu_3, 0}_{\frac{1}{2}(\frac{\lambda}{2} - \mu_3)} (1 - 2\cos^2 2\psi^*) \\
 &\times P^{\mu'_3, 0}_{\frac{1}{2}(\frac{\lambda'}{2} - \mu'_3)} (1 - 2\cos^2 2\psi'')] \times d\Delta \\
 &\times \begin{cases} 1 & \text{for } \mu_3, \mu'_3 \neq 0 \\ \sqrt{\frac{1}{2}} & \text{for either } \mu_3 \text{ or } \mu'_3 = 0 \\ \frac{1}{2} & \text{for } \mu_3 = \mu'_3 = 0 \end{cases}
 \end{aligned}$$

The elements $\langle k^2, \psi^* \varphi^* | K^{(i,j)} | k'^2, \psi'' \varphi'' \rangle$ in the integrand consist of typical terms such as $\langle k^2, \psi \varphi | G_0 T_r G_0 T_l | k'^2, \psi'' \varphi'' \rangle$.

The transformation properties of the subgroup S_3 on a system described internally by the Dalitz-coordinates coupled with the fact that μ is restricted to μ_3 leads to the particularly useful relation

$$\langle k^2, \lambda_\mu | G_0 T_1 G_0 T_1 | k'^2, \lambda' \mu'_3 \rangle = \langle k^2, \lambda_{\mu_3} | G_0 T_1 G_0 T_2 | k'^2, \lambda' \mu'_3 \rangle \quad (\text{I.2C.4i})$$

The complete equivalence of the sets $|k^2, \lambda_\nu\rangle$ and $|\underline{w}, 1mk\rangle$ allows the r.h.s. of eqn. (I.2C.4i) to be represented in the form

$$\begin{aligned} \langle \underline{w} | G_0 T_1 G_0 T_2 | \underline{w}' \rangle &= \frac{2\pi^2}{\omega_1 + \omega_2 + \omega_3 - z} \frac{1}{(\omega_1 \omega_2')^{\frac{1}{2}}} \\ &\quad (\sqrt{\omega_1} + \sqrt{\omega_2'})^2 \\ &\quad \times \frac{\int_1^t (\omega_1 \omega_2 \omega_3, \omega_1 \omega_2' \omega_3''; z - \frac{3}{2}\omega_1) \cdot t(\omega_2' \omega_3'' \omega_1, \omega_2' \omega_3' \omega_1'; -\frac{3}{2}\omega_2') d\omega_3''}{(\omega_1 + \omega_2' - z) + \omega_3''} \\ &\quad (\sqrt{\omega_1} - \sqrt{\omega_2'})^2 \end{aligned}$$

where $t(\underline{w}, \underline{w}')$ is the two-body transition operator (see eqn. (I.1.6i) in the notation

$$\langle \underline{\eta} | t_i(z - \underline{\xi}_i'^2) | \underline{\eta}'_i \rangle = t_1(\underline{\eta}_i, \underline{\eta}'_i; z - \underline{\xi}_i'^2)$$

with the arguments

$$\begin{aligned} \underline{\xi}_i &= \left(\frac{3}{2\omega_1}\right)^{1/2} \\ \underline{\eta}_i &= \sqrt{\frac{1}{2}(2\omega_i + 2\omega_k - \omega_j)}; \quad i, j, k = 1, 2, 3 \\ &\quad i \neq j \neq k \end{aligned} \quad (\text{I.2C.5i})$$

The formulation of the equations to be tested numerically is thus complete, except for the introduction of a small restriction designed to facilitate the computational process: It can be shown that for a system of three spinless particles interacting in pairs only small values of λ , λ' need to be considered. We chose $\lambda = 0, 4$, noting that the next higher value contributes only a few percent of the partial wave amplitude $\chi_{\lambda\gamma}(k)$ with $\lambda, \mu = 0$. The implementation of this restriction leads us to the following coupled set of equations:

$$\begin{aligned} &\langle k^2, \lambda\mu_3 | \Psi_s \rangle \\ &= \sum_{\lambda'\mu'_3} \int \langle k^2, \lambda\mu_3 | G_e T_1 G_o T_2 | k'^2, \lambda'\mu'_3 \rangle dk'^2 \langle k'^2, \lambda'\mu'_3 | \Psi_s \rangle \end{aligned} \quad (\text{I.2C.6})$$

where by $|\Psi_s\rangle$ we denote the totally symmetric state.

We have hitherto considered spinless particles only. In the next section we shall therefore briefly introduce the concepts of enlarging the theory to include spin-isospin.

SECTION I.3 Inclusion of Spin-Isospin.

A physical triton is composed of three possible ${}^2S_{\frac{1}{2}}$, three ${}^2P_{\frac{1}{2}}$, the ${}^4P_{\frac{1}{2}}$ and the three ${}^4D_{\frac{1}{2}}$ states. Of these the ${}^2S_{\frac{1}{2}}$ states dominate the boundstate wavefunction to the extent that for the rest of this discussion the other states may be disregarded, since their contribution is minimal. The ${}^2S_{\frac{1}{2}}$ states, with $L = 0$ and $S = J = \frac{1}{2}$, consist of

- a) a state, which is fully symmetric with respect to interchange of the space coordinates of all three nucleons;
- b) a state, which is asymmetric with respect to interchange of coordinates of any two particles;
- c) a state of mixed symmetry.

Lee constructs a complete set of three-particle spin-isospin states with given total spin-isospin (S, I) and respective z-components (S_z, I_z) and subject to the condition that they have definite symmetry properties under transformations of S_3 . The group theoretical approach to the classification is similar to the method used in configuration space and we shall not go into the details, but only sum up the relevant results of his analysis.

The method rests on constructing a spin-space for a system of three nucleons in the form of a multispinor of rank three, represented as

$$F_{i_1 i_2 i_3} = x_{i_1}^{(1)} x_{i_2}^{(2)} x_{i_3}^{(3)} \quad (\text{I.3.1})$$

where $x_{i_k}^{(k)}$ is the spinor representation of the k -th nucleon ($k = 1, 2, 3$) and the spin state of a nucleon with $S = \frac{1}{2}$ is given in the form of a two-component spinor $\underline{x} = (x_1, x_2)$. If this two-component spinor is required to form a spin-space for the representation of the unimodular group of unitary transformations $SU(2)$ and if an I.R. of this group is sought in the 2^3 dimensional tensor-space spanned by eqn. (I.3.1), this tensor has to be resolved into components, which transform irreducibly under the group. This is shown to be equivalent to resolving it into tensors, which have definite symmetry properties with respect to S_3 . By means of Pierce's resolution of the identity element into four operators Lee obtains basis functions for an I.R. of S_3 and subsequently decomposes the tensor (I.3.1) into four components: one totally symmetric, one totally asymmetric (which vanishes when dealing with a triton) and two of mixed symmetry.

Since the totally symmetric state has $J = \frac{3}{2}$ and the mixed symmetry states have $J = \frac{1}{2}$ the triton states are carried by components of the mixed symmetry tensors.

Lee derives the two properly normalized three-particle spin-space basis functions:

$$\Theta_1 = \sqrt{\frac{1}{2}} \left(F \begin{array}{|c|c|} \hline i_1 & i_2 \\ \hline i_3 & \\ \hline \end{array} + 2 F \begin{array}{|c|c|} \hline i_1 & i_3 \\ \hline i_2 & \\ \hline \end{array} \right) \quad (I.3.2)$$

$$\Theta_2 = \sqrt{\frac{3}{2}} F \begin{array}{|c|c|} \hline i_1 & i_2 \\ \hline i_3 & \\ \hline \end{array}$$

In obtaining these, use has been made of the Young tableaux to denote the symmetry properties; the components on the r. h. sides are tensors generated by the two mixed-symmetry operators (arising from the Pierce resolution) acting on the tensor $F_{i_1 i_2 i_3}$ in eqn.(I.3.1).

In isospin-space the analogous argument applies, and the resulting basis functions are denoted by

$$\Pi_1 = \sqrt{\frac{1}{2}} \left(G \begin{array}{|c|c|} \hline i_1 & i_2 \\ \hline i_3 & \\ \hline \end{array} + 2 G \begin{array}{|c|c|} \hline i_1 & i_3 \\ \hline i_2 & \\ \hline \end{array} \right) \quad (I.3.3i)$$

$$\Pi_2 = \sqrt{\frac{3}{2}} G \begin{array}{|c|c|} \hline i_1 & i_2 \\ \hline i_3 & \\ \hline \end{array} \quad (\text{I.3.3ii})$$

The combined spin-isospin-space is spanned by the inner product of the two representations. The properly normalized basis function for a system with $S = S_z = \frac{1}{2}$ and $I = -I_z = \frac{1}{2}$ are listed below:

$$\begin{aligned} Z_s &= \sqrt{\frac{1}{2}} (\Theta_1 \Pi_1 + \Theta_2 \Pi_2) \\ Z_a &= \sqrt{\frac{1}{2}} (\Theta_1 \Pi_2 - \Theta_2 \Pi_1) \\ Z_1 &= \sqrt{\frac{1}{2}} (\Theta_1 \Pi_2 + \Theta_2 \Pi_1) \\ Z_2 &= \sqrt{\frac{1}{2}} (\Theta_1 \Pi_1 - \Theta_2 \Pi_2) \end{aligned} \quad \left. \vphantom{\begin{aligned} Z_s \\ Z_a \\ Z_1 \\ Z_2 \end{aligned}} \right\} (\text{I.3.4})$$

where Z_s , Z_a , Z_1 and Z_2 denote the symmetric, asymmetric and the two mixed-symmetry states respectively (for notational convenience the indices a, s, 1, 2 will be substituted by (1), (2), (3), (4) resp.ly, from here onwards).

The most general three-nucleon state will be of the form

$$|\Psi\rangle = \sum_{v=1}^4 |\Psi(v)\rangle Z(v) \quad (\text{I.3.5})$$

where the choice of the spatial kets $|\Psi(v)\rangle$ conforms to

Pauli's principle, whereby the complete ket is to be fully asymmetric, i.e.

$$\begin{aligned}
 |\Psi(1)\rangle &\equiv |\Psi_a\rangle && \text{asymmetric spatial ket} \\
 |\Psi(2)\rangle &\equiv |\Psi_s\rangle && \text{symmetric spatial ket} \\
 |\Psi(3)\rangle, |\Psi(4)\rangle &&& \text{spatial kets of mixed symmetry} \\
 &\equiv |\Psi_1\rangle, \equiv |\Psi_2\rangle
 \end{aligned}$$

With the symmetry properties of these kets already known, one may deduce that their expansion in terms of SU(3) states involves only those states with the same respective symmetry properties. Since $L = 0$, the states required for such an expansion are $|k^2, \lambda\mu_j \nu\rangle$, with the labels μ, ν referring to the rearrangement of the states in terms of the transformations under S_3 (see page 34). In the following we list the four types of expansions:

$$\begin{aligned}
 |\Psi_a\rangle &= \sum_{\lambda\mu_3} \int |k^2, \lambda\mu_3 1\rangle dk^2 \langle k^2, \lambda\mu_3 1 | \Psi_a\rangle \\
 |\Psi_s\rangle &= \sum_{\lambda\mu_3} \int |k^2, \lambda\mu_3 2\rangle dk^2 \langle k^2, \lambda\mu_3 2 | \Psi_s\rangle \\
 |\Psi_1\rangle &= \sum_{\lambda\mu_j} \int |k^2, \lambda\mu_j 1\rangle dk^2 \langle k^2, \lambda\mu_j 1 | \Psi_1\rangle \\
 |\Psi_2\rangle &= \sum_{\lambda\mu_j} \int |k^2, \lambda\mu_j 2\rangle dk^2 \langle k^2, \lambda\mu_j 2 | \Psi_2\rangle
 \end{aligned}
 \tag{I.3.6}$$

j $\neq$ 3

With a special view towards using Faddeev's equations for the state $|\psi\rangle$ the expansion coefficients of eqn.(I.3.6) are decomposed into three components, e.g.

$$\langle k^2, \lambda\mu_3 | \psi_a \rangle = \sum_{j=1}^3 \langle k^2, \lambda\mu_3 | \psi_a^{(j)} \rangle \quad (\text{I.3.6i})$$

with the kets $|\psi^{(i)}(v)\rangle$ redefined to

$$|\psi(v)\rangle = \sum_{i=1}^3 |\psi^{(i)}(v)\rangle \quad (\text{I.3.6ii})$$

With this decomposition the general interacting state is once again given by

$$|\psi\rangle = \sum_{i=1}^3 |\psi^{(i)}\rangle \quad (\text{I.3.6iii})$$

and the spin-generalized Faddeev Equations read

$$|\psi^{(i)}\rangle = -G_0(z)T_j(|\psi^{(i)}\rangle - |\psi^{(k)}\rangle) \quad (\text{I.3.7})$$

$$i, j, k = 1, 2, 3; \quad i \neq j \neq k$$

This is formally the same as the homogeneous equation of eqn.(I.1.6), with T_i now being given as a combination of a projection operator $P_{i,\alpha}(jk)$ for the spin-isospin

state of the two-nucleon subsystem (jk) and $T_{i,\alpha}$, the transition operator of that state:

$$T_i = \sum_{\alpha=1}^4 P_{i,\alpha}(jk) T_{i,\alpha} \quad (I.3.8)$$

The $P_{i,\alpha}$ are given by:

$$P_{i,1} = P_{\sigma}^{+} P_{\tau}^{+} ; \quad P_{i,2} = P_{\sigma}^{+} P_{\tau}^{-} ; \quad P_{i,3} = P_{\sigma}^{+} P_{\tau}^{+} ; \quad P_{i,4} = P_{\sigma}^{+} P_{\tau}^{-} \quad (I.3.8i)$$

$$\text{with } P_{\sigma,\tau}^{+}(jk) = \frac{1}{2} (1 \pm (jk)_{\sigma,\tau})$$

Equation (I.3.7) operates in spin-isospin space as well as in abstract Hilbert space. Since we are interested in the spin-isospin states, we extract these to obtain

$$|\Psi^{(i)}(\nu)\rangle = -G_0(z) \left[\sum_{\alpha=1}^4 Z(\nu) P_{i,\alpha} Z(\nu') T_{i,\alpha} \right] (|\Psi^{(i)}(\nu')\rangle + |\Psi^{(k)}(\nu')\rangle) \quad (I.3.9)$$

where use has been made of eqn.(I.3.5) as it applies to each component of eqn.(I.3.6iii), together with the orthogonality of the basis states $Z(\nu)$. The transformation properties of these basis states under S_3 provide a matrix representation for the projection operators $P_{i,\alpha}$ in spin-isospin space, such that the

elements $Z(v) P_{i,\alpha} Z(v')$ may be evaluated.

Lee proceeds to show that the transition operators $T_{i,1}$ and $T_{i,4}$ are null operators and that therefore the components $|\psi_a^{(i)}\rangle$ for $i = 1, 2, 3$ as well as $|\psi_2^{(3)}\rangle$ vanish identically.

After rearranging the rows and columns and iterating the integral equation, the final homogenous equation is obtained in the form

$$\begin{pmatrix} \underline{|\psi_s\rangle} \\ \underline{|\psi_1\rangle} \\ \underline{|\psi_2\rangle} \end{pmatrix} = \begin{pmatrix} K_{ss} & K_{s1} & K_{s2} \\ K_{1s} & K_{11} & K_{12} \\ K_{2s} & K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} \underline{|\psi_s\rangle} \\ \underline{|\psi_1\rangle} \\ \underline{|\psi_2\rangle} \end{pmatrix} \quad (I.3.10)$$

where

$$\underline{|\psi_s\rangle} = \begin{pmatrix} |\psi_s^{(1)}\rangle \\ |\psi_s^{(2)}\rangle \\ |\psi_s^{(3)}\rangle \end{pmatrix} = \text{symmetric component}$$

and the two mixed-symmetry components

$$\underline{|\psi_1\rangle} = \begin{pmatrix} |\psi_1^{(1)}\rangle \\ |\psi_1^{(2)}\rangle \\ |\psi_1^{(3)}\rangle \end{pmatrix}$$

$$\underline{|\Psi_2\rangle} = \begin{pmatrix} |\Psi_2^{(1)}\rangle \\ |\Psi_2^{(2)}\rangle \end{pmatrix}$$

The explicit form of the matrix elements in eqn.(I.3.10) are to be found through relation (7.26) and Appendix 8 of ref.(1) and will not be listed here.

The numerical solution of (I.3.10) presents a major problem in itself. Since in this thesis we do not attempt to obtain the full solution, we shall leave the discussion at this stage and return to it later on, when we shall seek a solution in the light of special assumptions.

CHAPTER II

FIRST STEP TOWARDS NUMERICAL SOLUTION

The solution of boundstate equations for a system of three bosons, and later on three fermions, depends to a large extent upon the choice of pair-interactions, through which the system is supposed to interact. Before attempting the solution, we must therefore determine the exact shape of the potentials under consideration. This chapter deals with this aspect of the numerical verification.

Before discussing our choice of potentials, we shall list the explicit equations arising from the formalism discussed in the previous chapter. This will be followed by a section introducing the potentials, of which there are three: the two local potentials, exponential and Yukawa's, and a non-local separable potential, proposed by Yamaguchi⁽³⁾. The last section deals with determining the shapes of these potentials for the triplet as well as for the singlet case.

Section II.1 List of Equations to be solved, Bosons
and Fermions.

Bosons

Recalling the remarks made in the last section of the previous chapter, the set of coupled equations (I.2C.6) reduces to

$$\begin{aligned} & \langle k^2, 00 | \Psi_S \rangle \\ &= 4 \sum_{\lambda'=0,4} \int_0^\infty \langle k^2, 00 | G_0 T_1 G_0 T_2 | k'^2, \lambda' 0 \rangle dk'^2 \langle k'^2, \lambda' 0 | \Psi_S \rangle \end{aligned}$$

$$\begin{aligned} & \langle k^2, 40 | \Psi_S \rangle \\ &= 4 \sum_{-\lambda'=0,4} \int_0^\infty \langle k^2, 40 | G_0 T_1 G_0 T_2 | k'^2, \lambda' 0 \rangle dk'^2 \langle k'^2, \lambda' 0 | \Psi_S \rangle \end{aligned}$$

(II.1.1)

(as $\lambda, \lambda' = 0, 4$ the only eigenvalue of S to be considered is $\mu = \mu_3 = 0$)

For convenience we rewrite this coupled set in operator form

$$\begin{pmatrix} \chi_{0,0} \\ \chi_{4,0} \end{pmatrix} = \begin{pmatrix} K_{00} & K_{04} \\ K_{40} & K_{44} \end{pmatrix} \begin{pmatrix} \chi_{0,0} \\ \chi_{4,0} \end{pmatrix} \quad (\text{II.1.2})$$

where the $\chi_{\lambda\lambda}, (k)$ arise out of eqn. (I.2B.10).

Condensing the notation even further results in

$$(1 - \bar{K}) \bar{\chi} = 0 \quad , \quad (\text{II.1.3})$$

$\bar{K}$ stands for the matrix of integral operators $K_{\lambda,\lambda}$, whose action becomes clear from considering eqn. (II.1.1). If for the time being we restrict ourselves to obtaining values for the binding energy E_b , our task consists of finding that value of the total energy E , for which

$$\det |1 - \bar{K}| = 0 \quad (\text{II.1.4})$$

The problem of obtaining the boundstate-wavefunction is equivalent to solving the eigenvalue problem

$$\bar{\chi} = \bar{K} \times \bar{\psi} \quad (\text{II.1.5})$$

The set of variables involved is $|k^2, \psi, \varphi\rangle$ together with the primed set; alternatively we use the set $|\omega_1, \omega_2, \omega_3\rangle$, related to the first one by the relations

$$\begin{aligned}\omega_1 &= k_1^2 = \frac{1}{3} k^2 [1 + \cos 2\psi \cos(\varphi - \frac{2}{3}\pi)] \\ \omega_2 &= k_2^2 = \frac{1}{3} k^2 [1 + \cos 2\psi \cos(\varphi + \frac{2}{3}\pi)] \\ \omega_3 &= k_3^2 = \frac{1}{3} k^2 [1 + \cos 2\psi \cos \varphi] \quad (\text{II.1.6})\end{aligned}$$

Fermions

As for the case of bosons the consideration of the three nucleon boundstate is also restricted to $\lambda, \lambda' = 0, 4$. This leads to a set of coupled equations involving altogether 11 amplitudes, as can be verified from eqn.(I.3.10). Since in the context of this thesis we are satisfied with obtaining only a few results suitable for comparison with the boson case, we make an additional restriction: since the triton boundstate exists predominantly in the symmetric state, we shall only consider $\langle k^2, \lambda \mu_3 z | \psi_s \rangle$ to compute the boundstate energy and later on the wavefunction in the zeroth approximation, i.e. we solve only for the homogeneous equation of $\langle k^2, \lambda \mu_3 z | \psi_s \rangle$ in eqn.(I.3.10). The spatial transformation properties

are the same as for bosons (see eqn.(I.2C.4i), for example) and the resulting integral equation for this partial wave amplitude is:

$$\begin{aligned}
 & \langle k^2, \lambda \mu_3^2 | \psi_s \rangle \\
 &= \frac{1}{2} \sum_{\lambda' \mu_3'} \int \langle k^2, \lambda \mu_3^2 | (G_0^T T_{1,2} G_0^T T_{2,2} + 3G_0^T T_{1,2} G_0^T T_{2,3} \\
 & \quad + 3G_0^T T_{1,3} G_0^T T_{2,2} + G_0^T T_{1,3} G_0^T T_{2,3}) \\
 & \quad | k'^2, \lambda' \mu_3'^2 \rangle dk'^2 \langle k'^2, \lambda' \mu_3'^2 | \psi_s \rangle
 \end{aligned} \tag{II.1.7}$$

This equation in fact reduces to eqn.(I.2C.6) for vanishing spin-space, when the two-particle transition operators $T_{i,2}$ and $T_{i,3}$ merge into one, T_i .

Once provisions have been made to deal with the two distinct two-particle transitions, the solution of eqn.(II.1.7) is obtained with virtually the same computational setup as for solving eqn.(I.2C.6).

Section II.2

Pair-interactions and Potentials

Since the formalism outlined so far is based on the assumption that the three-body system interacts through pair-potentials only, it is clear that in order to solve eqn.s(II.1.1) and (II.1.7) we require solutions of the two-body partial-wave amplitudes $t(\eta, \eta'; s)$, where s stands for the total CM-energy of the pair under consideration (see eqn.(I.2C.5i)).

We use the Lippmann-Schwinger-Equation as provided by Lovelace⁽⁴⁾ in the form of a principal-value integral, with $l = 0$:

$$t_0(\eta, \eta'; s) = (2\pi)^{\frac{3}{2}} V_0(\eta, \eta') - (2\pi)^{\frac{5}{2}} P \int_0^{\infty} dq^2 q \frac{V_0(\eta, q) t_0(q, \eta'; s)}{q^2 - s} \quad (\text{II.2.1})$$

The use of this expression is subject to the two conditions

- a) that the potential used is symmetric in η and η' , and
- b) that η and η' remain on the real axis.

Both conditions are satisfied in our case.

Comparison of Lovelace's original equation (2.26) (which

we may equate to t_0 , as we remain on the real axis) with eqn.(II.2.1) reveals differing factors. This arises out of the Fourier transform convention adopted by Lee

$$\psi(p) = (2\pi)^{-\frac{3}{2}} \int_{-\infty}^{+\infty} d\underline{x} e^{i p \underline{x}} \phi(\underline{x}) \quad (\text{II.2.1i})$$

as opposed to Lovelace's

$$\psi_L(p) = \int_{-\infty}^{+\infty} d\underline{x} e^{i p \underline{x}} \phi(\underline{x}) \quad (\text{II.2.1ii})$$

When dealing with positive energies, Lovelace derives the useful relation between the real quantity $A_1(\sqrt{s}, \sqrt{s}; s)$ and the phase-shift δ_1 on the energy shell, of the form ($l = 0$):

$$A_0(\sqrt{s}, \sqrt{s}; s) = - \frac{1}{2\pi^2 \sqrt{s} \cot \delta_0} \quad (\text{II.2.2})$$

We note here that in computational terms A_0 is the same as t_0 , but it must be remembered that for positive energies s the amplitudes are not given by eqn.(II.2.1); in this case the amplitudes are obtained via the off-shell unitarity relation and Lovelace's subsequent decomposition of the

amplitude equation into its real and imaginary parts. He derives a coupled set of equations for these parts, which involves the real principal-value equation for A_1 . For the purpose of this discussion we do not need the explicit values of the two-body amplitudes involving positive energies. As we shall see later~~on~~, all we need are a few values for the scattering phase-shifts at low energies (≤ 10 MeV). The first two terms of the expansion for $\sqrt{s} \cot \delta_0$

$$\sqrt{s} \cot \delta_0 = -\frac{1}{a} + \frac{r_0}{2} + \dots \quad (\text{II.2.2i})$$

then provide a means of obtaining the exact shapes of the two-parameter pair-potentials via a fitting process, where the scattering length a and the effective range r_0 are experimental values.

We have chosen to test the formalism under the influence of the following three pair-interactions:

- A) The local potential of the Yukawa type⁽⁵⁾,
- B) the exponential potential⁽⁶⁾, and
- C) the non-local, separable potential proposed by Yamaguchi⁽³⁾.

II.A) The Yukawa Potential

In configuration space Yukawa's potential takes the form

$$V_r(r) = G_{Yu} \frac{e^{-\gamma_{Yu} r}}{r} \quad (\text{II.2A.1})$$

where r denotes the relative position vector of the two particles involved. G_{Yu} and γ_{Yu} are two constant parameters to be fixed.

We require the partial-wave potential in momentum space. Fourier-transforming eqn.(II.2A.1) yields

$$V_p(p, p') = \bar{G}_{Yu} [(p - p')^2 + \gamma_{Yu}^2]^{-1} \quad (\text{II.2A.2})$$

with $\frac{\bar{G}_{Yu}}{G_{Yu}} = \text{constant}$.

We rewrite this in the form

$$V_p(p, p') = \bar{\bar{G}}_{Yu} \frac{1}{pp'} \times \frac{1}{t - \cos(p, p')} ; \quad \frac{\bar{\bar{G}}_{Yu}}{\bar{G}_{Yu}} = \text{const.} \quad (\text{II.2A.3})$$

where $t = \frac{p^2 + p'^2 + \gamma_{Yu}^2}{2pp'}$.

A comparison of the expansion for V_p and the expression $1/[t - \cos(p, p')]$, both being listed below

$$V_p(p, p') = \sum_1 V_1(p, p') (2l+1) P_l[\cos(p, p')] \quad (\text{II.2A.3i})$$

$$\frac{1}{t - \cos(p, p')} = \sum_1 Q_1(t) (2l+1) P_l[\cos(p, p')] \quad (\text{II.2A.3ii})$$

with $l = 0, 1, 2, 3, \dots$,

P_l = Legendre Polynomials of the 1st kind

Q_l = " " " " 2nd "

yields by analogy the partial-wave potential as

$$V_l(p, p') = \bar{G}_{Yu} \frac{1}{pp'} \times Q_l\left(\frac{p^2 + p'^2 + \gamma_{Yu}^2}{2pp'}\right) \quad (\text{II.2A.4})$$

Since $l = 0$ we have

$$Q_0(t) = \frac{1}{2} \log\left(\frac{t+1}{t-1}\right) \quad (\text{II.2A.4i})$$

and hence

$$V_0(p, p') = \bar{G}_{Yu} \frac{1}{pp'} \log\left[\frac{(p+p')^2 + \gamma_{Yu}^2}{(p-p')^2 + \gamma_{Yu}^2}\right] \quad (\text{II.2A.5})$$

The coupling factors $\bar{G}_{Yu}$ and G_{Yu} are related through

$$\bar{G}_{Yu} = \frac{1}{2} \frac{G_{Yu}}{(2\pi)^{7/2}}$$

The two constants, the coupling constant G_{Yu} and the range-related constant γ_{Yu} , have separate values for the triplet and singlet case, and these will be established in the next section.

II.2B) The Exponential Potential

This is another local non-separable potential,
whose form in configuration space is:

$$V_r(r) = G_{Ex} e^{-\gamma_{Ex} r} \quad (II.2B.1)$$

In momentum space this becomes

$$V_p(p, p') = \bar{G}_{Ex} \frac{\gamma_{Ex}}{((p - p')^2 + \gamma^2)^2} \quad (II.2B.2)$$

We may express this in the form

$$V_p(p, p') = - \bar{G}_{Ex} \frac{\gamma_{Ex}}{p^2 p'^2} \times \frac{\partial}{\partial t} \left(\frac{1}{t - \cos(p, p')} \right) \quad (II.2B.3)$$

where t stands for the same expression as in eqn.(II.2A.3).

We proceed along the same line as before and we obtain:

$$V_1(p, p') = - \bar{G}_{Ex} \frac{\gamma_{Ex}}{(pp')^2} \times \frac{\partial}{\partial t} Q_1(t); \quad 1 = 0, 1, 2, \dots \quad (II.2B.4)$$

With $l = 0$ this becomes

$$V_0(p, p') = \bar{G}_{Ex} \frac{\gamma_{Ex}}{(p^2 - p'^2)^2 + \gamma_{Ex}^2 (p^2 + p'^2) + \gamma_{Ex}^4}$$

where $\bar{G}_{Ex} = 4 \frac{G_{Ya}}{(2\pi)^{7/2}}$

The two 'shape'-parameters still to be fixed are then

G_{Ex} and γ_{Ex}

II.2C) The Yamaguchi Potential

This type of non-local separable potential was proposed by Yamaguchi in 1954 in dealing with the t -nucleon problem. It is given by

$$V_0(p, p') = G_{Ya} \frac{1}{((p^2 + \gamma_{Ya}^2) (p'^2 + \gamma_{Ya}^2))} \quad (\text{II.2C.1})$$

The separability of this potential makes the Lippmann-Schwinger equations particularly easy to solve exactly in closed form. Due to its mathematical simplicity it has been widely applied to the three-body system^(1,3), and we include this potential in our investigation to see how the results based on this type of interaction

compare with those arising from the local potentials.

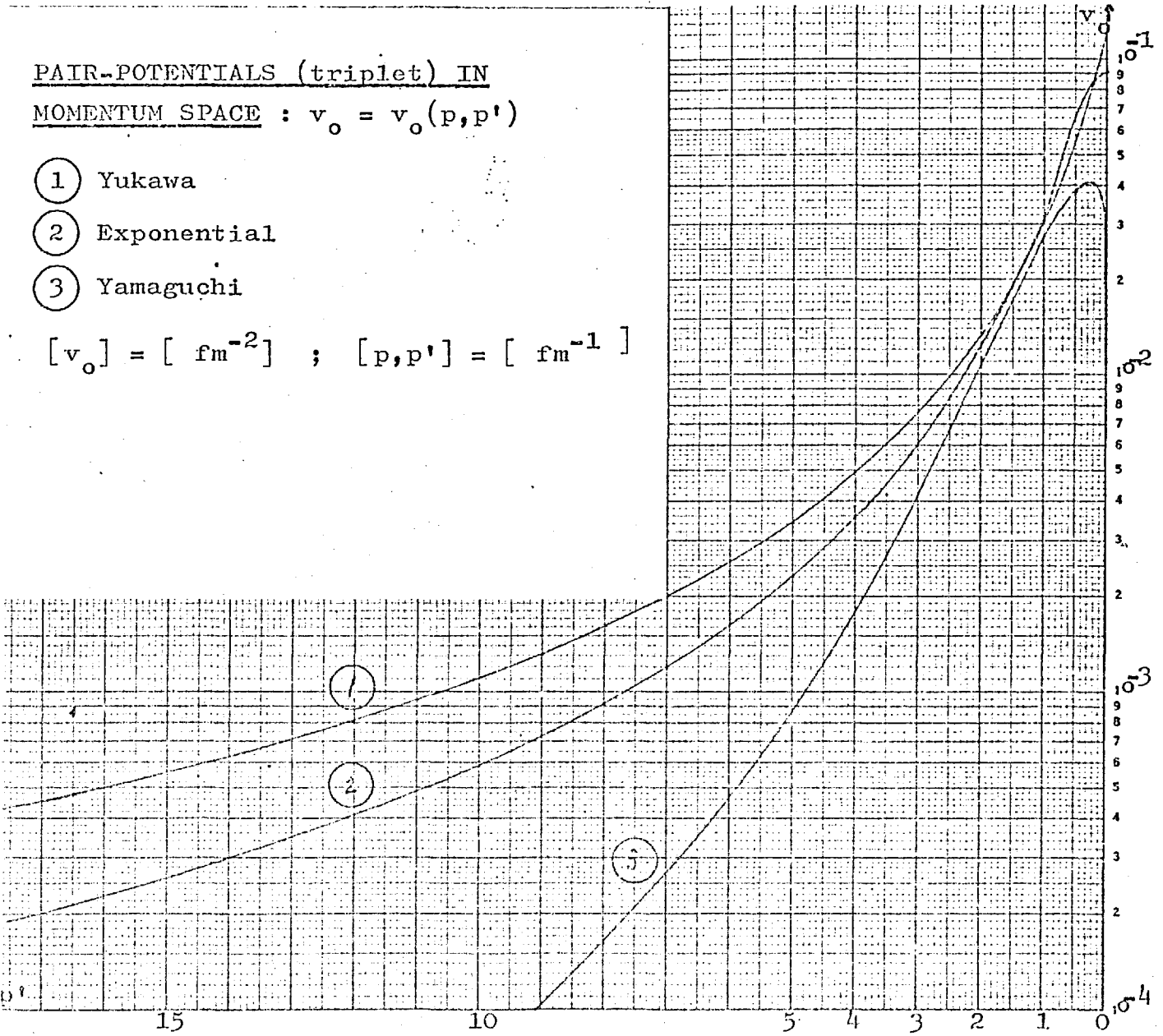
Of these three potentials only the Yukawa interaction can be justified on physical grounds. In momentum space, however, it presents a computational difficulty in that it has a comparatively long 'tail'; the exponential potential on the other hand dies off rapidly with increasing momenta and it therefore does not require so many points along the p-axis to yield the same accuracy as the Yukawa potential. This is the main reason for considering this type of pair-interaction. The convergence properties of the Yamaguchi potential are even stronger than those of the other two. The diagram on the following page shows the shapes of the three potentials on the same scale.

DIAGRAM 1

PAIR-POTENTIALS (triplet) IN
MOMENTUM SPACE : $v_o = v_o(p, p')$

- ① Yukawa
- ② Exponential
- ③ Yamaguchi

$$[v_o] = [\text{fm}^{-2}] \quad ; \quad [p, p'] = [\text{fm}^{-1}]$$



Section II.3 Determination of Pair Interaction Parameters

Having established the momentum-space representation of our three two-parameter pair-potentials, we must now fix the shape of each of these, both for the triplet and singlet interaction (in talking of the 'shape' of a potential, we adhere to the terminology of Preston⁽⁶⁾). Each potential requiring two parameters, we need altogether 12 constants. In principle, the parameters of a particular potential are obtained from a fit to the low-energy scattering data of either the singlet or the triplet state. Since most of these constants are known, our task consists of confirming these results and to modify the constants, where due to our particular Fourier-transform convention and our definition of relative vectors they differ from the values in the literature. In particular, we are mainly concerned with the coupling factors G , since the range - parameter γ is affected only by our differing relative vector definition, and this in a simple way.

To make matters easier, we distinguish two cases in our computational approach:

A) Experimental evidence suggests that the deuteron

bound state may, for most practical purposes, be taken as existing in the triplet state only. We have used this fact to obtain all the triplet-state parameters, i.e. we look for those parameters of a particular potential, which reproduce the correct binding energy. The results are later checked against the scattering data.

B) Since the deuteron singlet state is unbound, the above approach, of course, fails for the singlet parameters. These will therefore have to be obtained via eqn.s (II.2.2) and (II.2.2i).

At this stage we introduce our energy units. Since we shall specify the energy values in units of (fermi)⁻², we require the correct conversion factor from MeV-units to (fm)⁻². This conversion factor is obtained with the non-relativistic kinetic energy relation

$$E = \frac{1}{2} \eta v^2 \quad \text{with } v = \frac{\hbar}{\eta} (\underline{k}_1 - \underline{k}_2) \quad (\text{II.3.1})$$

We recall our relative vector definition

$$\underline{p}_{12} = \sqrt{\frac{1}{2}} (\underline{p}_1 - \underline{p}_2) \quad ; \quad \underline{r}_{12} = \sqrt{\frac{1}{2}} (\underline{r}_1 - \underline{r}_2) \quad (\text{II.3.1i})$$

as opposed to the usual convention

$$\underline{p}'_{12} = \frac{1}{2} (\underline{p}_1 - \underline{p}_2) \quad ; \quad \underline{r}'_{12} = \underline{r}_1 - \underline{r}_2 \quad (\text{II.3.1ii})$$

With the reduced mass η being given by $\eta = \frac{1}{2}(m_{\text{proton}} + m_{\text{neutron}})$ our conversion factor becomes

$$1 \text{ (fm)}^{-2} = 20.77443 \text{ MeV} \quad (\text{II.3.2})$$

as opposed to the usual

$$1 \text{ (fm)}^{-2} = 41.49486 \text{ MeV} \quad (\text{II.3.2i})$$

arising from the convention (II.3.1ii).

The effect of our relative vector definition on the range-related parameters of the potentials consists simply of a multiplying factor of $\sqrt{2}$, and we can list these parameters as follows:

TABLE 1.

	triplet	singlet
γ_{Yu}	0.8952	1.347
γ_{Ex}	1.6348	1.6348
γ_{Ya}	2.0489	2.0489

In units
of fm^{-1}

The following two subsections describe the way in which the corresponding coupling constants are obtained.

II.3.A) Triplet State Constants

Returning to eqn.(II.2.1) we may drop the inhomogeneous term since we are dealing with the deuteron bound state only. The kernel is nonsingular throughout the whole range of integration, since $s < 0$, and the equation has completely continuous solutions.

In this particular context we are not seeking the solutions $t_0(k, k(;s))$. We may express eqn.(II.2.1) in operator form as follows:

$$(1 - G\bar{K}) T_0 = 0 \quad (II.3A.1)$$

Here G denotes the coupling constant, $\bar{K}$ an integral operator in matrix form, whose elements are given by numerically approximating the integration, i.e.

$$\bar{K} = \frac{(2\pi)^{\frac{5}{2}}}{G} \int_0^{\infty} dq^2 q \frac{V_0(\eta, q)}{q^2 - s} \quad (II.3A.1i)$$

with elements

$$(\bar{K})_{i,j} = \frac{(2\pi)^{\frac{5}{2}}}{G} q_j \frac{v_0(\eta_i, q_j)}{q_j - s} w_j \quad (\text{II.3A.1ii})$$

w_j denoting the weight factors associated with the numerical quadrature.

We may now rephrase the problem: We seek the value of G in the potential v_0 for which, with s set equal to the deuteron binding energy, eqn. (II.3A.1) is satisfied, i.e. for which

$$\det |1 - G\bar{K}| = 0 \quad (\text{II.3A.2})$$

The following remarks concerning the numerical quadrature apply to the whole of this work. In an effort to minimize the error arising from performing all integrations numerically, we have chosen the Gaussian quadrature method, which optimizes the choice of abscissae and their corresponding weights for a given interval. There are, of course, many powerful methods for integrating numerically⁽⁸⁾, but we have found the following to offer high accuracy while being the least laborious. Furthermore, the particular method quoted here offers a high degree of adaptability to varied ranges of integration.

A standard Gaussian range is $-1 \leq x \leq +1$. The abscissae and weights are tabulated up to large numbers

of intervals⁽⁹⁾. A simple mapping process, described by Osborne⁽¹⁰⁾, enables one to use these tabulated values for different ranges of integration. The numerical integral of a given integral

$$I = \int_a^b Y(x) dx \quad (\text{II.3A.3})$$

is performed by means of a weighted summation

$$I = \lim_{N \rightarrow \infty} \sum_{i=1}^N Y(x_i) w_i \quad (\text{II.3A.4})$$

where x_i and w_i are the abscissae and weights, respectively. Now, by means of a transformation of the integrating variable, any region $a \leq x \leq b$ can be mapped into the range $-1 \leq u \leq +1$, provided that the mapping process does not introduce discontinuities into the integrand. Our specific requirements are met by the following two cases:

a) $|a|, |b| \neq \infty$:

$$u = 2(x-a)/(b-a) - 1$$

$$x = \frac{1}{2}(u+1)(b-a) + a$$

$$dx = \frac{1}{2}(b-a)du$$

therefore

$$I = \int_a^b Y(x) dx$$

$$\begin{aligned}
 &= \int_{-1}^{+1} Y'(u)(b-a)du \\
 &= \lim_{N \rightarrow \infty} \sum_{i=1}^N Y'(u_i)w_i \\
 I &= \lim_{N \rightarrow \infty} \sum_{i=1}^N Y(x_i)(b-a)w_i \quad (\text{II.3A.5})
 \end{aligned}$$

b) $a = 0 ; b = + \infty :$

$$u = (x-1)/(x+1)$$

$$x = (1+u)/(1-u)$$

$$dx = 2/(1-u)^2 du$$

$$= \frac{1}{2}(x+1)^2 du$$

and hence

$$I = \lim_{N \rightarrow \infty} \frac{1}{2} \sum_{i=1}^N Y(x_i)(x_i+1)^2 w_i \quad (\text{II.3A.6})$$

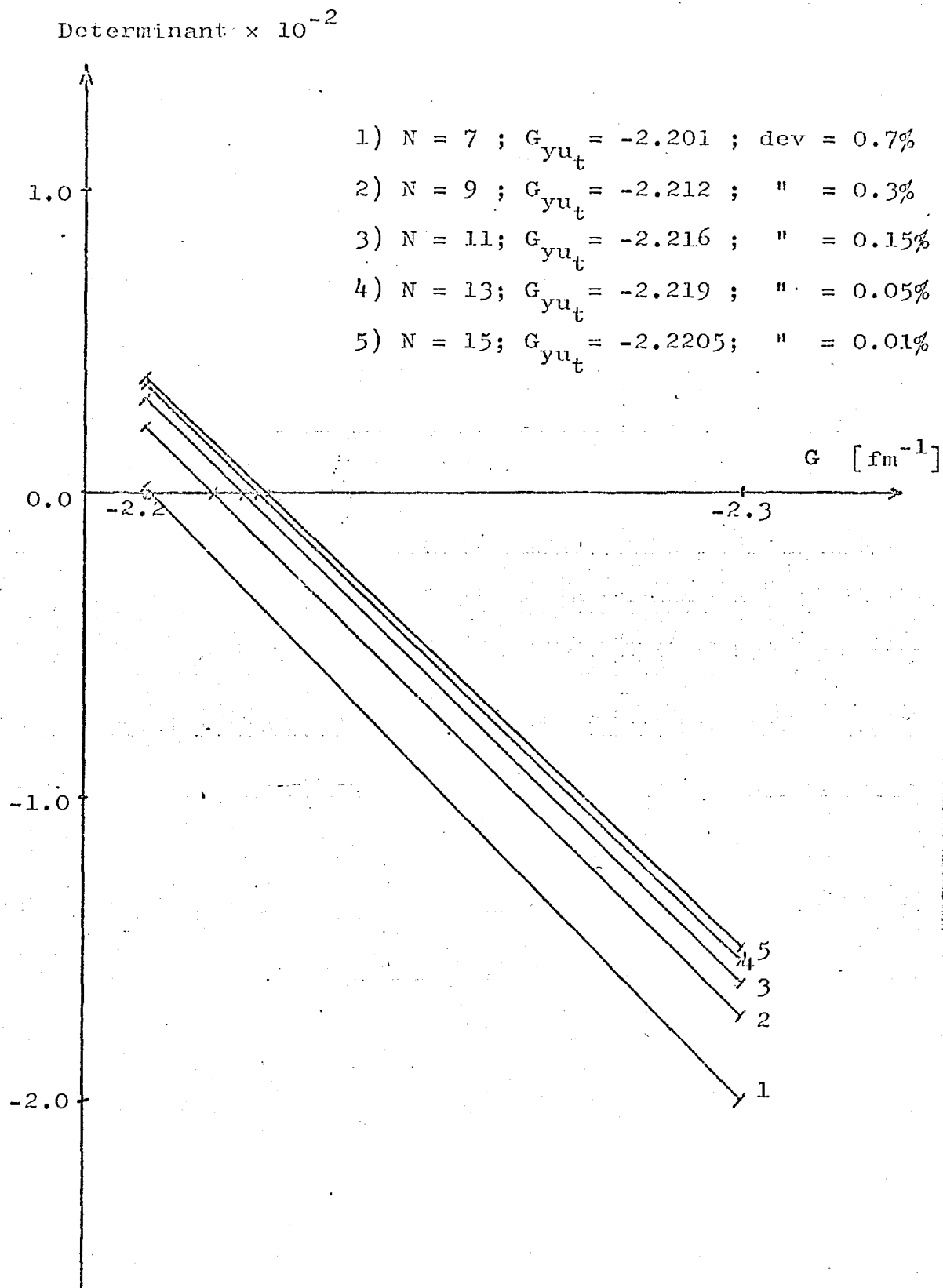
Case b) applies to the numerical approximation of eqn.(II.2A.1i). The computation relating to eqn. (II.2A.2) is performed by first fixing N, and then obtaining values for the function Det(G) in the

region, where this function has a root. The precise value of G corresponding to this root is then obtained by fitting a polynomial; in practice, we have varied G in sufficiently small steps around where the root was to be expected, to warrant a straight-line fit.

This process was repeated for further values of N , in order to obtain G as a function of the number of intervals. Taking $N = 7, 9, 11, 13$ and 15 proved sufficient to extrapolate the convergence value of G for $N \rightarrow \infty$ to an accuracy of $< 0.05\%$. The following two diagrams have been included to give an example of the way G_{∞} was obtained.

DIAGRAM 2

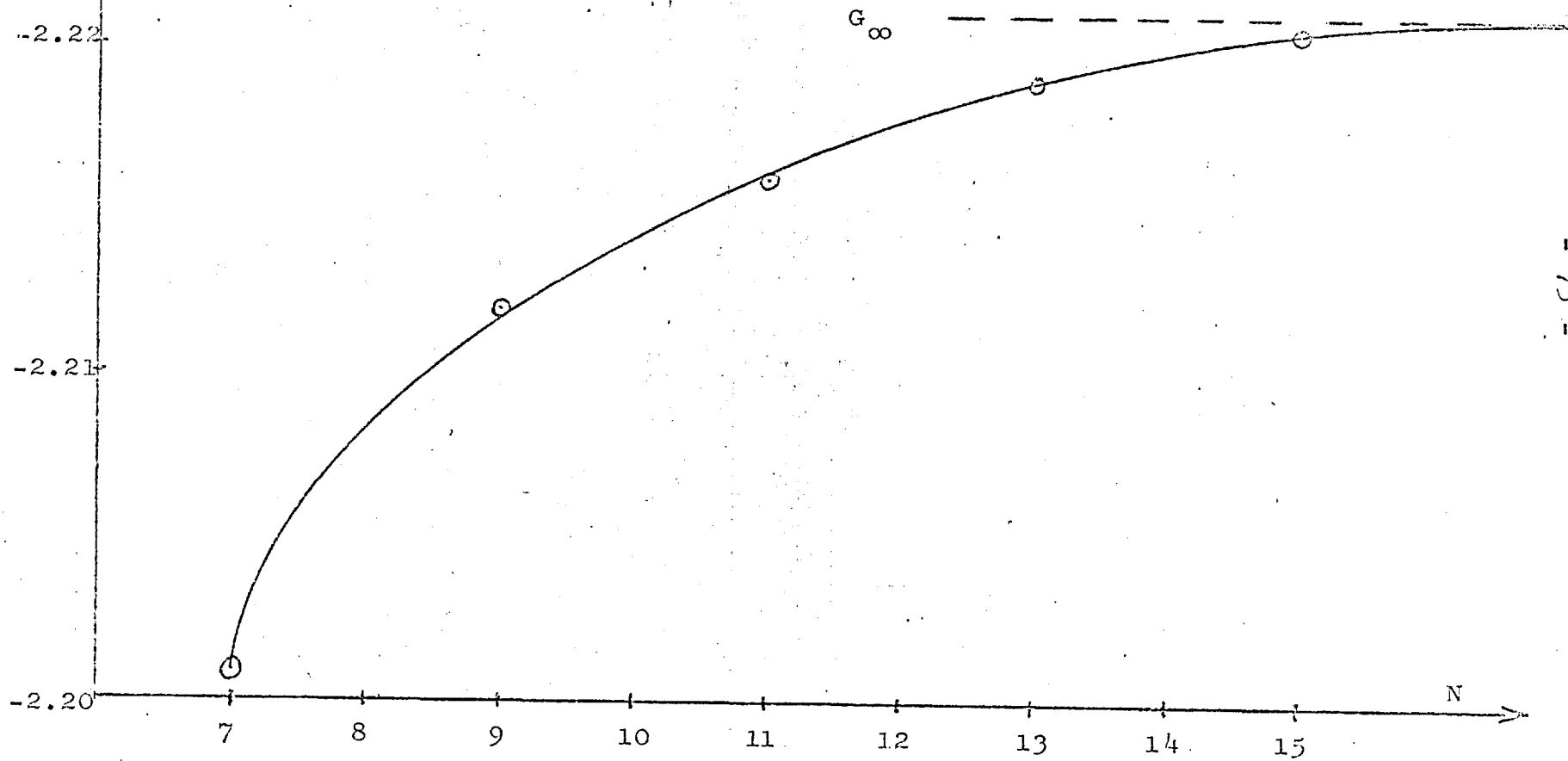
YUKAWA POTENTIAL COUPLING CONSTANT (TRIPLET)



$G \text{ [fm}^{-1}\text{]}$

DIAGRAM 3

Convergence of G_{yu_t}



The numerical values obtained for the triplet constants will be listed together with all the other two-body interaction parameters at the end of the next section.

II.3B) Singlet State Constants

The computation of the singlet state coupling constants is distinct from the triplet case in that it involves positive energies. We need to compute scattering phase shifts to perform the fit according to eqn.(II.2.2i). With the values of s in eqn. (II.2.1) now lying on the positive real axis, we come up against a weak singularity⁽¹¹⁾ in the kernel of this equation at $q^2 = s$. Dyson et.al.⁽¹²⁾ describe a method of overcoming this difficulty by introducing a modification in the kernel. This modification consists of an additive term in the numerator of the kernel thus leading to the following equation:

$$u_0(\eta, \eta'; s) = (2\pi)^{\frac{3}{2}} v_0(\eta, \eta') - (2\pi)^{\frac{5}{2}} P_0 \int_0^\infty dq^2 q \frac{v_0(\eta, q) - \sqrt{\frac{s}{q}} v_0(\eta, \sqrt{s})}{q^2 - s} u_0(q, \eta'; s) \quad (\text{II.3B.1})$$

One can easily verify the existence of the kernel at the point $q^2 = s$, since the derivative of v_0 with respect to the integrating variable q^2 exists at this point. The relation between $t_0(\eta, \eta'; s)$ and $u_0(\eta, \eta'; s)$ is given by

$$t_0(\eta, \eta'; s) = \frac{u_0(\eta, \eta'; s)}{-1 + 2\pi \frac{v_0(\eta, \sqrt{s})}{v_0(\eta, \eta')} P \int_0^\infty dq^2 \frac{u_0(q, \eta'; s)}{q^2 - s}} \quad (\text{II.3B.2})$$

The principal-value integration can be made more amenable to numerical computation by applying the same concept of integrand-modification once again:

$$\begin{aligned} & P \int_0^\infty \frac{u_0(q, \eta'; s)}{q^2 - s} dq^2 \\ &= (s+1) [u_0(\sqrt{s}, \eta'; s)] P \int_0^\infty \frac{dq^2}{(q^2 - s)(q^2 + 1)} \\ &+ \int_0^\infty \frac{u_0(q, \eta'; s) - \frac{s+1}{q^2 + 1} u_0(\sqrt{s}, \eta'; s)}{q^2 - s} dq^2 \end{aligned} \quad (\text{II.3B.2i})$$

The first term on the r.h.s. of eqn. (II.3B.2i) reduces to

$$\begin{aligned}
 & -(s+1)[u_o(Vs, \eta'; s)] P \int_0^{\infty} \frac{dq^2}{(q^2-s)(q^2+1)} \\
 & = -u_o(Vs, \eta'; s) \log s + \\
 & \int_0^{\infty} dq^2 \frac{u_o(q, \eta'; s) - \frac{s+1}{q^2+1} u_o(Vs, \eta'; s)}{q^2 - s} \quad (II.3B.2ii)
 \end{aligned}$$

since

$$\begin{aligned}
 & P \int_0^{\infty} \frac{dq^2}{(q^2-s)(q^2+1)} \\
 & = \lim_{\epsilon \rightarrow 0} \left[\frac{1}{s+1} \left\{ \left[\log \frac{q^2-s}{q^2+1} \right]_0^{s-\epsilon} + \left[\log \frac{q^2-s}{q^2+1} \right]_{s+\epsilon}^{\infty} \right\} \right] \\
 & = - \frac{\log s}{1+s}
 \end{aligned}$$

It is again easily verifiable that the integrand on the r.h.s. of eqn. (II.3B.2ii) exists at the point $q^2 = s$.

Thus all the integrations involved in obtaining $t_o(\eta, \eta'; s)$ are now non-singular.

In solving eqn.(II.3B.1) we employ the method of successive approximations, often associated with the name of Picard:

A standard approach towards solving an integral equation of the Fredholme type

$$y(x) = f(x) + G \int_a^b K(x,z)y(z)dz \quad (\text{II.3B.3})$$

using electronic computers is to approximate the equation by a set of simultaneous linear equations, which we denote by the operator equation below

$$\left(\mathbf{1} - G \times \bar{K} \right) Y = F \quad (\text{II.3B.3i})$$

The solution is obtained in the form of the vector Y.

The method of successive approximations requires the evaluation of the series

$$Y_n = F + \sum_{k=1}^n G^k \bar{K}^k F \quad (\text{II.3B.3ii})$$

Provided the kernel is square-integrable, i.e. bounded in the interval (a,b), the sequence Y_n converges uniformly, i.e.

$$y(x) = \lim_{n \rightarrow \infty} Y_n$$

for all values of x, which lie inside the interval (a,b).

This condition is met by the kernel of eqn.(II.3B.1).

The advantage of this particular method of solution in our case consists of being able to stipulate the relative accuracy of our solution explicitly, i.e. we require the iterative process to continue until the relative accuracy reaches

$$\frac{|y_n - y|}{|y_n|} \leq \epsilon$$

where ϵ is fed to the process.

Based on the 15-point Gaussian quadrature formula, we have found a relative error of $\epsilon \leq 10^{-4}$ to be fully sufficient for a good approximation to the solution $u_0(\eta, \eta'; s)$; the convergence properties proved to be reasonable, with an average of 15 iterations (i.e. $n=15$ in eqn.s(II.2B.3ii-iii)) required to achieve the stated accuracy.

The actual fit according to eqn.(II.2.2) and (II.2.2i) is based on the values for the singlet scattering length and effective range⁽³⁾

$$a_s = -23.69/\sqrt{2} = -16.8 \text{ fm}^{-1}$$

$$r_{os} = 2.151/\sqrt{2} = 1.532 \text{ fm}$$

where the factor of $(2)^{-\frac{1}{2}}$ arises out of our definition for relative vectors eqn.(II.3.1i). We have chosen the fit to involve five values for the total

two-particle CM-energy s , which for computational convenience were made to coincide with the lower five of the fifteen values for q^2 obtained from the Gaussian points via eqn.(II.3A.6). These values are:

$$\begin{aligned} s_1 &= 0.00604 \text{ fm}^{-2} \\ s_2 &= 0.03238 \text{ fm}^{-2} \\ s_3 &= 0.08213 \text{ fm}^{-2} \\ s_4 &= 0.15981 \text{ fm}^{-2} \\ s_5 &= 0.27310 \text{ fm}^{-2} \end{aligned} \quad (\text{II.3B.4})$$

These energy values are low enough to allow the expansion of $\eta \cot \delta_0$ to be cut off after the first two terms, an approximation, which as is commonly known, is valid up to $s \leq 10 \text{ MeV} \approx 0.5 \text{ fm}^{-2}$.

With the correct two-body interaction parameters, the on-shell value of the amplitude $t_0(\sqrt{s}, \sqrt{s}; s)$ must satisfy the relation

$$-\frac{1}{2\pi^2 t_s(\sqrt{s}, \sqrt{s}; s)} = -\frac{1}{a_s} + \frac{1}{2} r_{os} s \quad (\text{II.3B.5})$$

Diagram 4 shows the l.h.s. of this relation plotted against the energy for the singlet Yamaguchi interaction.

As a check both of the method as well as the triplet constants, we have performed the fit with the triplet scattering data as well and have found our constants to give results with an error of well below 1%. For the singlet states we have found coupling constants, which together with the range-related constants on p.68, reproduced the scattering parameters a_s and r_{os} to within 1% of the values quoted above. This comparatively large error is to a certain extent due to the arbitrariness, which exists in the fixing of a_s and r_{os} ; in the literature these values vary somewhat partly because the experimental information concerning the singlet state is still rather poor and partly to suit the particular requirements of the authors. Our results lie within the range of those quoted by various authors (see for example ref.(3) and (7)).

Table 3 presents the layout of the program for fixing the two-particle interaction constants. This program has two main branches: the first, 3a, is entered, when the total energy s is negative, which implies that triplet constants are being sought; the second, 3b, is entered when s is positive. In Table 2 we list all the pair-potential parameters obtained by us and used in our computations later on.

DIAGRAM 4

YAMAGUCHI POTENTIAL COUPLING CONSTANT

(SINGLET)

$\sqrt{s} \cot \delta_0$
 $\times 10^{-2}$

$$G_{ya_s} = -12.963 \text{ fm}^{-6}$$

$$\gamma_{ya_s} = 2.0489 \text{ fm}^{-1}$$

$$\tan \alpha = \frac{1}{2} r_{os} = 0.762 \text{ fm} \quad \underline{r_{os} = 1.523 \text{ fm}}$$

$$(a_s)^{-1} = -0.0596 \text{ fm}^{-1}$$

$$\underline{a_s = -16.8 \text{ fm}}$$

6.0

7.0

8.0

5.0

0.1

0.2

$s \text{ [fm}^{-2}\text{]}$

○

α

TABLE 2

'SHAPE'-PARAMETERS FOR TWO-NUCLEON INTERACTIONS

SINGLET STATE

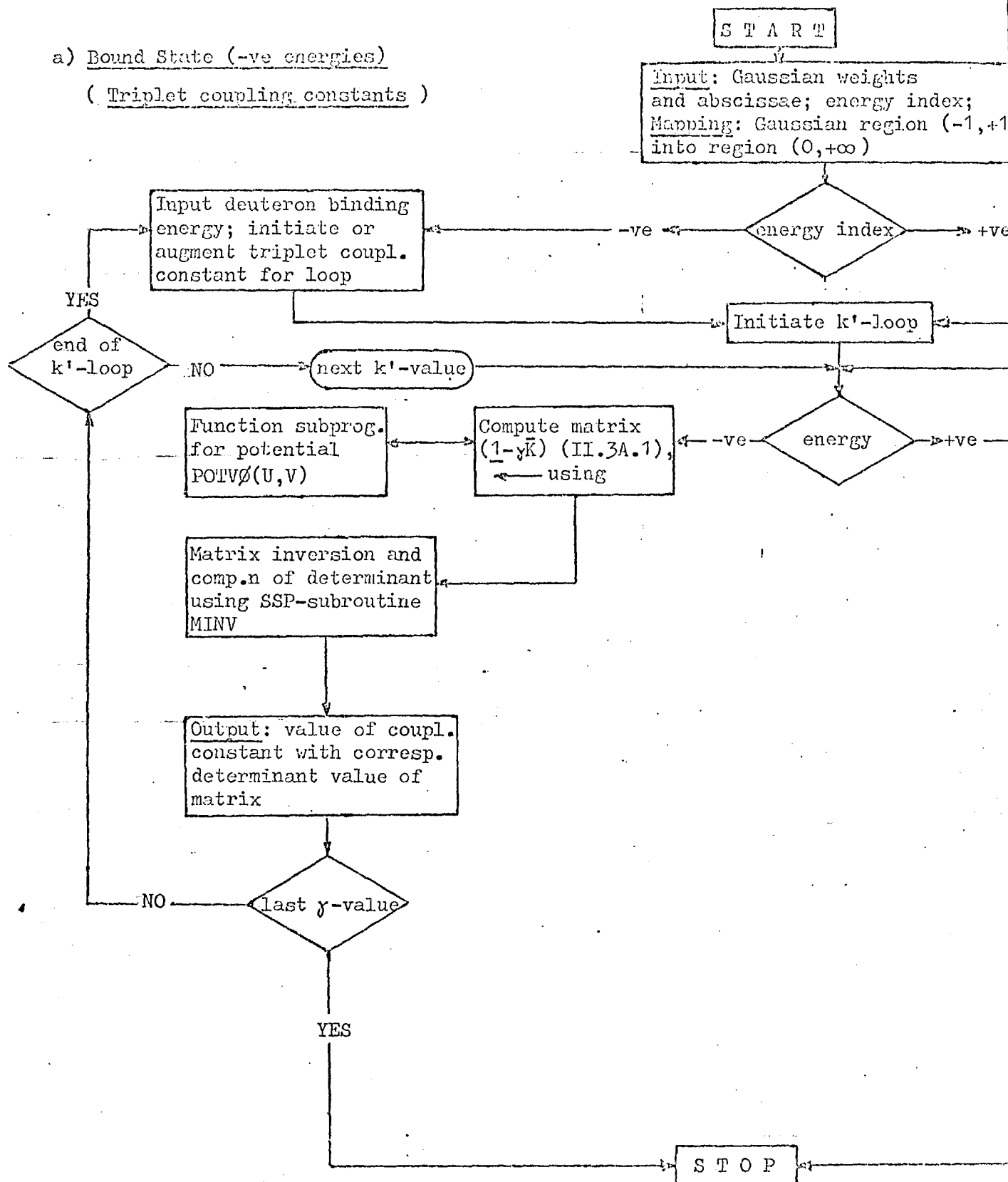
	Yukawa	Exponential	Yamaguchi
Coupl.const. G_s	-2.134 fm^{-4}	-3.529 fm^{-5}	-12.963 fm^{-6}
Range param. γ_s	1.347 fm^{-1}	1.6348 fm^{-1}	2.0489 fm^{-1}

TRIPLET STATE

	Yukawa	Exponential	Yamaguchi
Coupl.const. G_t	-2.221 fm^{-4}	-6.0124 fm^{-5}	-18.46 fm^{-6}
Range param. γ_t	0.8952 fm^{-1}	1.6348 fm^{-1}	2.0489 fm^{-1}

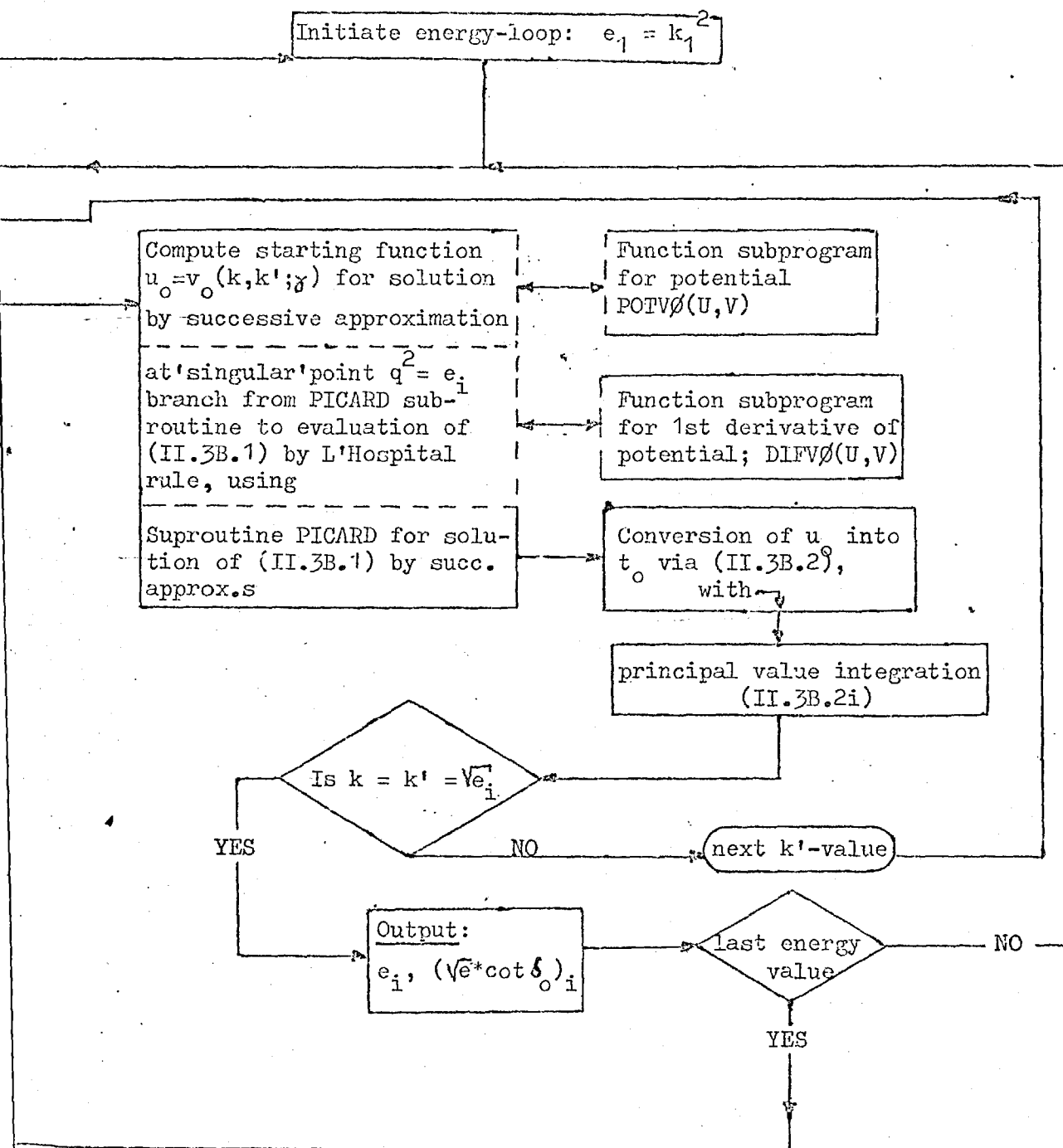
TABLE 3: FLOW-CHART FOR COMPUTATION OF COUPLING CONSTANTS

a) Bound State (-ve energies)
(Triplet coupling constants)



(Table 3 continued)

b) Scattering (+ve energies)
(Singlet coupling constants)



CHAPTER III DESCRIPTION OF NUMERICAL METHODS AND
LAYOUT OF THE COMPUTATIONS

Having determined the nature and shape of the three types of pair-interactions we wish to consider in the context of this investigation, we now turn to the problem of performing computations on the basis of the expressions given by eqn.s (II.1.1) and (II.1.7), referring to systems composed of three bosons and three fermions, respectively. Both cases are, for the time being, subject to the approximations of restricting the number of eigenvalues λ and $\mu_3: \lambda, \lambda' = 0, 4; \mu_3 \mu_3' = 0$. The extension to higher eigenvalues follows quite naturally from the ensuing discussions in this chapter.

The overall purpose of our computations is twofold:

- i) We wish to obtain values for the total energy of the three particle systems, such that eqn. (II.1.3), or alternatively its spin-isospin counterpart is satisfied. These values will be compared with the experimental value for the triton binding energy. For the boson case, we shall furthermore investigate the behaviour of the computed binding energy as a function of the coupling constant in the pair-interactions.

As for the spin-isospin system the computed binding energy will serve for comparison purposes, and the coupling constants of the subsystems will not be varied.

2) We also seek the approximate shapes of the bound-state wave amplitudes. For the boson system this entails solving for $\bar{X}$ in eqn.(II.1.3) and obtaining the shape via relation (I.2B.10). When dealing with three nucleons, we shall solve only for the symmetric state, in accordance with the remarks made prior to eqn.(II.1.7).

In computational terms the differences between the boson and the fermion case are not fundamental. We shall therefore treat the former in detail and append the necessary alterations, when dealing with a system of three nucleons.

The setting up and designing of a program dealing with a complex problem is very often influenced by the logistical configuration of the computational means at the programmer's disposal. It seems often true that while there may exist alternative ways of programming the problem, which make more and better use of peripheral facilities, the programmer may be compelled

to adopt a less efficient approach in terms of computer utilization, simply because the desired degree of efficiency may require more direct access to the machine than is available. We suffer from this handicap, even though not in any fundamental way. We know of at least one way in which the computational structure described in the following could be improved in terms of central processor running time (CDC 6600 configuration at the University of London Computing Centre). However, for a number of reasons we have not yet made full use of the theoretically available peripheral facilities of the above configuration; one reason is that all our calculations performed on the CDC 6600 are input through a 200 Terminal link, stationed at Chelsea College of Science and Technology, thus somewhat limiting the degree of access. We shall, however, mention where the optimization could be applied. It should be noted that this does not alter the fundamental numerical and computational approach to the problem.

Section III.1 Computation of Binding Energies for
three Bosons and three Fermions

In accordance with our introductory remarks, the emphasis in the following sections lies on discussing the computation of eqn.s(II.1.1-5). The present section in particular deals with eqn.(II.1.4).

III.1A) Treatment of Variables

The calculation of each element of the matrix in eqn.(II.1.3), whose determinant is to be computed, involves basically a fivefold integration, with only the innermost integration having variable limits (see also eqn.s(I.2C.4-5)). Considering the equations (I.2C.6) and (II.1.2) the matrix-operator $\bar{K}$ can be thought of as consisting of a number of submatrices, each of which belongs to a particular combination of the eigenvalue-set λ, λ' . Thus for example, with $\lambda, \lambda' = 0, 4$ there are 4 such submatrices, as in eqn.(II.1.2); the choice of $\lambda, \lambda' = 0, 4, 6$ leads to 9 submatrices.

The number of elements in each submatrix of $\bar{K}$ is the same and is determined by the number of nodes c chosen for the numerical quadrature over the variable

k'^2 in the range $0 \leq k'^2 \leq +\infty$. The choice of this number is of paramount importance to the entire program, for reasons which will become clear soon. Let us, for the time being, assume that the integration range of each variable we are dealing with be subdivided into the same number of intervals and let us denote this number by N . Obviously, the larger this number the smaller the error resulting from numerically approximating the integrations. The volume of the computation, however, is such that the feasibility of the calculations becomes a major problem, in view of the required central processor running time. In fact we have the approximate relation

$$\text{CP-time} = \text{Constant} * N^6 \quad (\text{III.1A.1})$$

On the very fast CDC 6600 a typical value for the CP-time, using $N = 5$, is 10 minutes to compute five values for the determinant (II.1.4) corresponding to five (negative) values for the total energy of the system. Doubling N thus increases cp-time by a factor of 64, i.e. the same calculation requires more than 10 hours. This is clearly impractical, and we therefore have to limit ourselves to low numbers of intervals in the numerical quadratures.

In order to retain a maximum of precision here, we resort to the previously discussed Gaussian quadrature with the mapping technique (see eqn.s(II.3A.3-6) which minimizes the error arising from the approximation of the integrals.

Each element of the matrix $\bar{K}$ is given by an integration of a corresponding matrix element $\langle k'^2, \psi'^* \varphi'^* | K^{(i,j)} | k^2, \psi \varphi \rangle$ multiplied by other terms (see eqn.(I.2C.4)) over a 4-dimensional 'rectangular box'. The element in the integrand is essentially the transition amplitude for one internal three-body configuration to another, in terms of either the Dalitz-related coordinates (k^2, φ, ψ) or the Omnes set $(\omega_1, \omega_2, \omega_3)$.

The matrix element in the integrand of eqn.II.2C.4) is itself the result of an integration with variable limits, as given by eqn.(I.2C.5). The integrand of the latter brings us back to the two-body subsystems, discussed in the previous chapter. It requires the solution of eqn.(II.2.1) for two sets of variables $(\eta, \eta'; s)$, which, for simplicity, we shall denote by t_1 and t_2 from left to right (the fact that we are dealing with orbital angular momentum $l=0$ is left understood).

The two-body energies s_1 and s_2 are both clearly negative in the context of our calculations and eqn.(II.2.1) therefore is non-singular at all times. The arguments η, η' are given as functions of $|\underline{w}\rangle$ and $|\underline{w}'\rangle$ as follows:

$$\left. \begin{aligned} \eta_1 &= \eta_1(\omega_1, \omega_2, \omega_3) \\ \eta'_1 &= \eta'_1(\omega_1, \omega'_2, \omega''_3) \end{aligned} \right\} \text{ for } t_1 \quad (\text{III.1A.2})$$

$$\text{with } s_1 = (E - \frac{3}{2} \omega_1)$$

and

$$\left. \begin{aligned} \eta_2 &= \eta_2(\omega'_2, \omega''_3, \omega_1) \\ \eta'_2 &= \eta'_2(\omega'_2, \omega'_3, \omega'_1) \end{aligned} \right\} \text{ for } t_2 \quad (\text{III.1A.3})$$

$$\text{with } s_2 = (E - \frac{3}{2} \omega'_2)$$

The explicit functional relations are obtained via

$$\eta_i^2 = \frac{1}{2}(2\omega_i + 2\omega_k - \omega_j) ; i, j, k = 1, 2, 3 \quad (\text{III.1A.4})$$

$$i \neq j \neq k$$

III.1B) Sequence of Computation

For the sake of clarity we shall explicitly list the expressions to be computed:

With $\lambda, \lambda' = 0, 4$ the matrix, whose determinant is to vanish for a particular value E_b of the total energy is

$$(1 - \bar{K}) = 1 - \begin{pmatrix} \bar{K}_{0,0} & \bar{K}_{0,4} \\ \bar{K}_{4,0} & \bar{K}_{4,4} \end{pmatrix} \quad (\text{III.1B.1})$$

The elements of the submatrices $\bar{K}_{\lambda, \lambda'}$ are derived from equation (I.2C.4) and are given by

$$(\bar{K}_{0,0})_{i,j} = \frac{8}{3\sqrt{3}\pi} k_i^2 k_j^2 \int_0^1 \int_0^1 \int_{-\frac{1}{3}\pi}^{\frac{2}{3}\pi} \int_{-\frac{2}{3}\pi}^{\frac{1}{3}\pi} \langle k_i^2, \psi^\pi \varphi^\pi | K(E) | k_j^2, \psi'' \varphi'' \rangle \\ \times \cos 2\psi^\pi \cos 2\psi'' d(\cos 2\psi^\pi) d(\cos 2\psi'') d\varphi^\pi d\varphi'' \quad (\text{III.1B.2})$$

$$(\bar{K}_{4,0})_{i,j} = \frac{8}{3\pi} k_i^2 k_j^2 \int_0^1 \int_0^1 \int_{-\frac{1}{3}\pi}^{\frac{2}{3}\pi} \int_{-\frac{2}{3}\pi}^{\frac{1}{3}\pi} \langle k_i^2, \psi^\pi \varphi^\pi | K(E) | k_j^2, \psi'' \varphi'' \rangle \\ \times (1 - 2\cos^2 2\psi'') \times \cos 2\psi^\pi \cos 2\psi'' d(\cos 2\psi^\pi) d(\cos 2\psi'') d\varphi^\pi d\varphi'' \quad (\text{III.1B.3})$$

$$(\bar{K}_{4,0})_{i',j'} = \frac{8}{3\pi} k_i^2 k_j^2 \int_0^1 \int_0^1 \int_{-\frac{1}{3}\pi}^{\frac{2}{3}\pi} \int_{-\frac{2}{3}\pi}^{\frac{1}{3}\pi} \langle k_i^2, \psi^{\frac{2}{3}} \varphi^{\frac{2}{3}} | K(E) | k_j^2, \psi^{\frac{2}{3}} \varphi^{\frac{2}{3}} \rangle$$

$$\times (1 - 2\cos^2 \psi^{\frac{2}{3}}) \times \cos 2\psi^{\frac{2}{3}} \cos 2\psi^{\frac{2}{3}} d(\cos 2\psi^{\frac{2}{3}}) d(\cos 2\psi^{\frac{2}{3}}) d\varphi^{\frac{2}{3}} d\varphi^{\frac{2}{3}}$$

(III.1B.4)

$$(\bar{K}_{4,4})_{i',j'} = \frac{8}{\sqrt{3}\pi} k_i^2 k_j^2 \int_0^1 \int_0^1 \int_{-\frac{1}{3}\pi}^{\frac{2}{3}\pi} \int_{-\frac{2}{3}\pi}^{\frac{1}{3}\pi} \langle k_i^2, \psi^* \varphi^* | K(E) | k_j^2, \psi^* \varphi^* \rangle$$

$$\times (1 - 2\cos^2 \psi^*) \times (1 - 2\cos^2 \psi^*) \cos 2\psi^* \cos 2\psi^* d(\cos 2\psi^*)$$

$$\times d(\cos 2\psi^*) d\varphi^* d\varphi^*$$

(III.1B.5)

with the following index conventions relating to the position of a particular element in the matrix-array:

$$i = 1, 2, \dots, NK ; \quad j = 1, 2, \dots, NK ;$$

$$i' = (1+NK); (2+NK), \dots, 2NK ; \quad j' = (1+NK), (2+NK), \dots, 2NK .$$

It is clear that $k_i^2 = k_i^2$, and $k_j^2 = k_j^2$; furthermore, for notational convenience we have expressed the elements $\langle k_i^2, \psi^* \varphi^* | K^{(i,j)}(E) | k_j^2, \psi^* \varphi^* \rangle$ in the integrand of eqn. (I.2C.4) as $\langle k_i^2, \psi^* \varphi^* | K(E) | k_j^2, \psi^* \varphi^* \rangle$.

The elements in the integrands of the above four equations do not depend on the eigenvalues λ and μ_3 and they are thus the same for each of the sub-matrices $\bar{K}_{\lambda, \lambda}$.

This suggests a division of the computation into two parts:

Part 1 : Computation of the elements

$\langle k^2, \psi^* \varphi^* | K(E) | k'^2, \psi^* \varphi^* \rangle$ and storage of the results on magnetic tape or disc.

Part 2 : Retrieval of the results from Part 1 to compute equations (III.1B.2-5) to fill the matrix array for eqn.(III.1B.1) and subsequent calculation of the determinant value (II.1.4).

Part 1 is by far the most time consuming of the two, as we shall see. Once the complete set of the elements are computed and stored by this part, we can use this data to perform the computations under Part 2. The sets of eigenvalues λ, λ' and μ_3, μ'_3 can be varied to investigate the higher order contributions. The second part generally runs 1/50th of the cp-time taken up by Part 1, a fact which makes the above subdivision compulsory. This investigation is very much based on the set $\lambda, \lambda' = 0, 4$, with the occasional inclusion of 6, which proves to be quite sufficient (see remarks following eqn.(I.2C.5i)).

PART 1 The Elements $\langle k^2, \psi^* \varphi^* | K(E) | k'^2, \psi'' \varphi'' \rangle$

Since these elements are a function of altogether six variables, defined either by the set $(k^2, k'^2, \psi^*, \psi'', \varphi^*, \varphi'')$ or $(\omega_1, \omega_2, \omega_3, \omega'_1, \omega'_2, \omega'_3)$, consideration must be given to the order in which these variables are to be processed. Restricting ourselves in this consideration to the former set, and recalling the ranges of these variables, we must fix the number of nodes for each range. The examination of the two-body potentials in momentum space reveals that the Yukawa potential for example has a comparatively long tail and that therefore the number of nodes required for k^2 and k'^2 will probably be greater than that for the other variables, which have well-defined ranges. We denote the former by NK , while choosing an equal number of nodes N for all the other variables. Based on this particular choice, it is clear that for one value of the total three-body energy we must compute $NK^2 \times N^4$ matrix-elements, which we shall henceforth denote briefly by $K(E)_{NK,N}$. Here we obtain a first impression of the differences in computational volume for the various choices of the set NK, N):

(NK, N)	Number of $K(E)_{NK, N}$
$(5, 5)$	15 625
$(7, 7)$	117 649
$(10, 10)$	1 000 000
$(15, 15)$	2 025 000
$(8, 5)$	40 000
$(10, 5)$	62 500

Bearing in mind that at present a magnetic tape (800 b.p.i., 7 track, 2400 ft. long), written in full accuracy mode on a machine such as the CDC 6600 with 60-bit words will hold an absolute maximum of 2 736 000 numbers (without taking the fact into account that in practice the information is written onto the tape in the form of blocks with a block-separating $\frac{3}{4}$ " of tape) one realizes a further reason for limiting the number of nodes, apart from the running-time aspect.

The order in which the variable-combinations in $K(E)_{NK, N}$ are altered depends on the order of integration in the eqn.s(III.1B.2-5), i.e. the processing order of this data in Part 2.

Consistency here insures continuous tape-reading in Part 2, thus saving time and reducing tape-wear to a minimum. We have chosen the following order:

OUTERMOST INNERMOST

$$k^2 \rightarrow k'^2 \rightarrow \psi^* \rightarrow \varphi^* \rightarrow \psi'' \rightarrow \varphi''$$

The computation of an element $K(E)_{NK,N}$ involves the solution of the two-body amplitudes t_1 and t_2 with arguments given by eqn.(III.1A.2-4); the order in which the Omnes variables occur follows from the above order and eqn.(II.1.6). With these solutions the integration (I.2C.5) can be performed. This is done numerically by again subdividing the range of the integrating variable ω_3''

$$(\sqrt{\omega_1} - \sqrt{\omega_2^i})^2 \leq \omega_3'' \leq (\sqrt{\omega_1} + \sqrt{\omega_2^i})^2 \quad (\text{III.1B.6})$$

into a number of intervals, which we have chosen to be NK , by means of mapping the Gaussian abscissae into the above range via eqn.(II.3A.5) giving

$$\omega_{3_i}'' = 2\sqrt{\omega_1\omega_2^i} u_i + \omega_1 + \omega_2^i \quad (\text{III.1B.6i})$$

as nodes along the abscissa and

$$w_{\omega_{3i}}'' = 2\sqrt{\omega_1 \omega_2} w_i \quad (\text{III.1B.6ii})$$

The range of the index i , i.e. the number of nodes used in eqn.(I.2C.5) does not, of course, affect the number of elements $K(E)_{NK,N}$ and can therefore be chosen as large as the accuracy requires. Actual computations have shown, however, that the ultimate results remain practically unaffected by doubling the number of nodes, while considerably increasing the cp-time of the calculations. The choice of $1 \leq i \leq NK$ has proved quite sufficient.

In calculating eqn.(I.2C.4) for each $K(E)_{NK,N}$ we attempted the direct approach, i.e. first solving $t(\eta, \eta'; s)$ $2NK$ times with the various argument combinations, computing the weighted sum of the integrand and multiplying the sum by the appropriate factor. This method quickly turned out to be impossible in terms of cp-time. An alternative method of making the solutions of the two-body equation available for the integration in eqn.(I.2C.4) exploits the symmetry of t_0 with respect to interchange of its arguments η, η' .

A numerical solution of eqn.(II.2.1) usually appears as an NK-component vector (assuming NK to be the number of nodes along the integrating variable); each component represents an approximation of the two-body amplitude corresponding to a particular set of arguments $(\eta_i, \eta'; s)$. The second of these arguments is supplied to the iteration procedure as a parameter, thus in purely numerical terms being on the same footing as the parameter s . The iteration is performed over η ; variation of the 'parameter' η' e.g. in the form η'_j with $j = 1, 2, \dots, NK$, then leads to a matrix-array containing the required set of approximate solutions of eqn.(II.2.1). Looking at the integrand of eqn.(I.2C.4) we see that the integrating variable ω_3'' occurs once in the string of quantities determining and once in those forming the 'parameter'-variable η' , whereas we would wish ω_3'' to occur only in the formation of η in order to exploit the interpolatory properties of Picard's iteration method. This difficulty is resolved by realizing that the principle of invariance with respect to time-reversal must hold and that eqn.(II.2.1) must be invariant with respect to interchange of η and η' [remark a) following eqn.(II.2.1)].

We can state that

$$\begin{aligned} t_1(\omega_1, \omega_2, \omega_3; \omega_1, \omega_2', \omega_3''; E - \frac{3}{2}\omega_1) \\ = t_1(\omega_1, \omega_2', \omega_3''; \omega_1, \omega_2, \omega_3; E - \frac{3}{2}\omega_1) \end{aligned} \quad (\text{III.1B.7})$$

The functional relations (III.1A.2-3) are modified to

$$\begin{aligned} \eta_1 &= \eta_1(\omega_1, \omega_2', \omega_3'') \\ \eta_1' &= \eta_1'(\omega_1, \omega_2, \omega_3) \end{aligned} \quad (\text{III.1B.8})$$

$$\begin{aligned} \eta_2 &= \eta_2(\omega_2', \omega_3'', \omega_1) \\ \eta_2' &= \eta_2'(\omega_1', \omega_2', \omega_3') \end{aligned} \quad (\text{III.1B.9})$$

with s_1 and s_2 remaining unaffected.

We can compute all the solutions of t_1 and t_2 belonging to the $NK \times N^2$ values of their respective 'parameter'-variables η_1' and η_2' arising from all the possible combinations of $\omega_1, \omega_2, \omega_3$ and $\omega_1', \omega_2', \omega_3'$ respectively, and store all these NK-point tabular functions in the order, in which the values of η_1' and η_2' occur; the total number of solutions to be stored is therefore $2NK \times N^2$. This part of the computation can be done quite separately. When

the particular values of t_1 and t_2 are required for the evaluation of eqn. (I.2C.4), one only has to retrieve the two tabular solutions and interpolate their function values for the arguments η_1 and η_2 respectively, using the same Picard routine applied for obtaining the solutions in the first place.

Interpolation does not require any iteration, but only one summation of the now easily obtainable and weighted values of the integrand in eqn. (II.2.1), whereas a full solution of the two-body equation involves 15 to 20 summations. Since the integrand of eqn. (I.2C.4) involves a total of $2NK^2 \times N^5$ function values of $t_0(\eta, \eta'; s)$ the amount of computing time saved in this way becomes obvious.

In order to increase the accuracy of the NK-point solutions for t_1 and t_2 we have chosen to obtain these in two steps. The first consists of computing tabular solutions on a higher number of points, say M, followed by the interpolation of the NK-point solutions from the corresponding M-point sets. This leads to a slight increase of the computational volume, which, however, is justified, because it improves the solutions for t_1 and t_2 , especially for low values of NK.

PART 2: The matrix $\bar{K}_{\lambda,\lambda'}$

Having thus obtained $NK^2 \times N^4$ values for the elements $K(E)_{NK,N}$ and storing them in the prescribed order, the computation of the elements $\bar{K}_{\lambda,\lambda'}$ is very straightforward. Each element is the result of one of the four-fold integrations (III.1B.2-5). The numerical procedure here involves the usual mapping of the Gaussian data into the ranges of the four integrating variables:

$$\begin{aligned}
 1) \quad \psi^* \text{ and } \psi'' : \quad & 0 \leq \cos 2\psi \leq 1 \\
 & (\cos 2\psi)_i = \frac{1}{2}(u_i + 1) \\
 & w_{\psi_i} = \frac{1}{2}w_i \quad (\text{III.1B.10})
 \end{aligned}$$

$$\begin{aligned}
 2) \quad \varphi^* : \quad & -\frac{1}{3}\pi \leq \varphi^* \leq \frac{2}{3}\pi \\
 & \varphi_i^* = \frac{1}{2}\pi(u_i + 1) - \frac{1}{3}\pi \\
 & w_{\varphi_i^*} = \frac{1}{2}\pi w_i \quad (\text{III.1B.11})
 \end{aligned}$$

$$\begin{aligned}
 3) \quad \varphi'' : \quad & -\frac{2}{3}\pi \leq \varphi'' \leq \frac{1}{3}\pi \\
 & \varphi_i'' = \frac{1}{2}\pi(u_i + 1) - \frac{2}{3}\pi \\
 & w_{\varphi_i''} = \frac{1}{2}\pi w_i \quad (\text{III.1B.12})
 \end{aligned}$$

After performing the integration and filling the matrix array according to the index order prescribed by eqn.(III.1B.5i) the program is diverted to a standard

sub-program⁽¹³⁾, which computes the inverse of the matrix together with its determinant. The method of inversion employed by this routine is the usual Gauss-Jordan one, with the value of the determinant resulting from the product of the pivot elements. In order to keep rounding errors negligible, we used this routine in the optional double-precision mode on the CDC 6600, thus maintaining 30-digit accuracy throughout the inversion.

The outline of our computations is thus complete, except for the inclusion of spin-isospin. The basic layout remains the same, but the volume of the computations is roughly doubled, since in addition to the triplet-state transition amplitudes of the two-body subsystems, we now need the singlet-state amplitudes. This leads to the following modifications in the program:

Spin-Mod. Part 1 : Returning to eqn.(II.1.7), one

realizes that due to the two types of transition amplitudes there are four operators of the kind

$G_0 T_{i,\alpha} G_0 T_{j,\beta}$ to be dealt with as opposed to only one,

$G_0 T_1 G_0 T_2$, in the case of a system composed of three

bosons. In computational terms this amounts to very little. It simply means that along with calculating the set of N-point triplet amplitudes we now have to

calculate the corresponding singlet amplitudes; this is done via the same computational procedure, only that now the pair-potential subprogram is accessed with the singlet-state parameters of Chapter II. The two types of solutions are then retrieved for interpolation when computing the elements $\bar{K}(E)_{NK,N}^{(s)}$, where the index (s) refers to spin-isospin, and combined in the integrand of eqn.(I.2C.5) in the order prescribed by the operators in eqn.(II.1.7).

Spin-Mod. Part 2 : The amendment in Part 2 is even simpler. It merely consists of altering the multiplying constants appearing in eqn.s (III.1B.2-5) by a factor of 0.125, otherwise this part of the computation remains exactly as in the case of three bosons.

The program outlined in this section leads to a value of the determinant of the matrix (III.1B.1) corresponding to one particular value of the total energy E of the system. In order to obtain the value E_b , for which this determinant vanishes, the program must be rerun for a number of energies, usually five, in order to allow a curve to be fitted to the functional dependence $\det = \det(E)$, which is then solved for the roots.

Returning to the remarks made at the beginning of this chapter, regarding the problem of reducing the cp-time, the following suggestion can be made:

The function subprogram evaluating the pair-potentials lie at the very centre of the program and are therefore accessed many times (more than 10^6 times with $NK = N = 5$ and $M = 10$). It is therefore crucial to make these subprograms as efficient as possible. The values are clearly independent of the energy parameter E . One could therefore store all, or at least a number of, these potential-values on magnetic tape or disc during the first energy-run and instruct the program to retrieve this information in the subsequent runs, thus saving computing time. This suggestion has in fact been implemented by Lee with our program on the CDC 3600 configuration at McMaster University, Ontario, Canada, and the results suggest that cp-time can be reduced by a factor of three for a full five-energy run. This method, however, greatly increases the amount of time used by the necessary peripheral equipment, such as tape-drives, a fact which with our access facilities to the computer configuration of the University of London has an adverse effect on the job-turnaround time.

Therefore, for the sake of speeding up our investigation, we did not make use of this obviously advantageous method of saving computer time.

For a full documentation of the program outlined in this section the reader is referred to the appendix.

III.1C) Peripheral Program

As mentioned above, the output of the program discussed so far consists of a small number of values for the determinant of the matrix (III.1B.1) as a function of the total energy E of the three-particle system. The range of E should be chosen near the point where the determinant-function changes sign in order to facilitate the exact determination of the point E_b where the determinant vanishes.

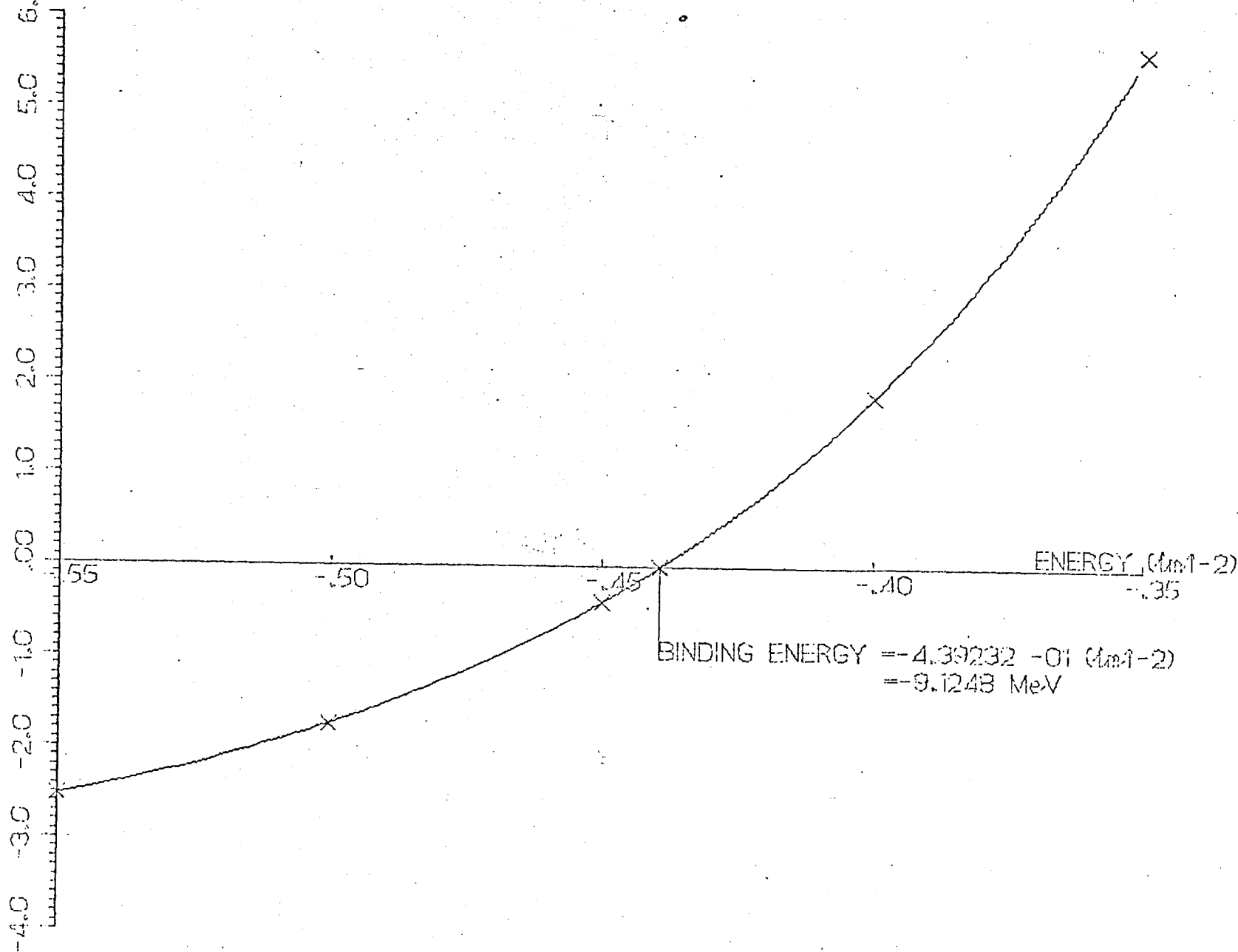
To find this root E_b it is necessary to fit a curve with determinable functional dependence on E to the tabular values of the determinant. In order to exert more control over the fitting process, it was decided to separate this part of the calculations from the rest.

Good functional approximations and curve fitting are rather dependent on the data and visual inspection of particular fits by the user. We have tried various

fitting methods, such as approximation by exponential functions or Chebyshev polynomials, but since most methods applied with the help of computers ~~boil down~~ ^{reduce} to combinations of straightforward polynomial approximations, we have finally resorted to using direct polynomial fits. The routine used is one supplied by the Chelsea College Computing Centre and this proved to be best suited for the data at hand. In the vicinity of the point E_b , the determinant-function seems well-behaved, and with five values at our disposal we fitted fourth order polynomials, which fully satisfied the accuracy requirements in obtaining the value E_b .

The procedure consisted essentially of plotting the data received for the determinant versus the energy axis by means of a digital plotter connected to the Elliott 503 computer at Chelsea College; the approximating polynomial was then determined and also plotted, and then input into a root-finding routine. The resulting root, lying inside the given range of E was selected and marked on the graph as the binding energy E_b . All this is performed automatically, while still allowing us to vary the fitting criteria. A sample of this process is given overleaf with $NK = N = 5$ and $M = 20$.

DETERMINANT * 1.0 +00



Section III.2 Determining the Shape of the Bound-State Wavefunctions

In the previous chapter the method was described with which we obtain values for the binding energy E_b for a system governed by a particular type and shape of pair-interaction. Once this value has been established, we return to eqn.(II.1.3) and the problem of determining the shape of the corresponding wavefunction. The purpose of this is twofold: first, we wish to ascertain that the formalism does in fact yield bound-state wavefunctions, which die off rapidly as the momentum increases beyond certain values; secondly, this calculations provides a check for the accuracy and correctness of determining the value E_b . As to the latter, we can say that if the determination of the binding energy is reasonably accurate, the matrix $\bar{K}$ in eqn.(II.1.3) must have one eigenvalue lying close to unity. It must be emphasized however that this check does not provide an estimate of the error arising from the numerical methods employed in determining E_b , it merely examines the consistency within the computation.

With the value of E_b at hand we proceed as follows:

a) Since we require the matrix $\bar{K}$, the value E_b is fed into the programs PART 1 and PART 2. A logical switch indicates to Part 2 that the matrix is not to be inverted, but retained for input into a program, which determines the eigenvalues and eigenvectors of the matrix. Otherwise, the two programs proceed in exactly the same way as before. The resulting matrix is of the size $2NK \times 2NK$ with $\lambda, \lambda' = 0, 4$ or $3NK \times 3NK$ for $\lambda, \lambda' = 0, 4, 6$.

b) In determining the eigenvalues and eigenvectors of $\bar{K}$, we have made use of a package of ALGOL-procedures⁽¹⁴⁾ made available by Chelsea College and we have converted these to a form acceptable to the ALGOL-compiler in the CDC 6600, in order to improve the accuracy of this computation. This package essentially performs the following operations on a given square matrix:

It converts the matrix into an upper Hessenberg matrix according to Householder's method⁽¹⁵⁾; the eigenvalues are subsequently computed and stored. The package then proceeds to calculate the eigenvectors by means of inverse iteration, each step

involving Gaussian elimination with partial pivoting. The program requires, apart from the input matrix, a number of tolerance parameters adjusted for maximum accuracy and efficiency.

c) Stage b) provides us with $2NK$ or $3NK$ eigenvalues and corresponding eigenvectors for the values of $\lambda = 0, 4$ or $\lambda = 0, 4, 6$. From these we select the eigenvector belonging to the eigenvalue which lies closest to one; the selected partial wave amplitudes $\chi_{\lambda\nu}(k)$ are subsequently normalized according to

$$C_N \sum_{\lambda, \nu} \int |\chi_{\lambda\nu}(k)|^2 k \, dk = 1 \quad (\text{III.2.1})$$

where C_N is a normalization factor.

The amplitudes $\chi_{\lambda\nu}(k)$ are then plotted against the square of the momentum on a digital plotter.

In the case of three fermions we proceed in exactly the same manner, having already restricted ourselves to the zeroth approximation of the symmetric state (see remarks on p.): the plots will show the amplitudes $\chi_{0,0}(k)_s$, $\chi_{4,0}(k)_s$ and $\chi_{6,3}(k)_s$.

CHAPTER IV. RESULTS AND CONCLUSION

This chapter presents the set of computational results obtained in this first numerical assessment of Lee's formalism. The guidelines for seeking the results are set out in the introductory remarks to chapter III. Accordingly, we begin by giving the values for the binding energy computed for a system of three identical particles interacting in pairs through one of the three central potentials described and established in chapter II. The coupling constants have also been varied to obtain the behaviour of the binding energy as a function of these constants.

The inclusion of spin and isospin turns the system into one of three fermions, for which comparative binding energy values were computed, subject to the restrictions mentioned in section 1 of chapter II.

Both systems are associated with partial-wave amplitudes in their bound-state and these are presented in section 2.

This is followed by some concluding remarks.

Section IV.1 Computed Binding Energies

The set of note-parameters (see section III.1B) pertaining to the results given in this and the following section is made up as follows:

The number of nodes NK over the range $0 \leq k^2 \leq \infty$ is 8 unless stated otherwise;

the number of nodes N is 5 for each of the following ranges: $0 \leq \cos 2\psi^{*,''} \leq +1$

$$-\frac{1}{3}\pi \leq \varphi^* \leq +\frac{2}{3}\pi$$

$$-\frac{2}{3}\pi \leq \varphi'' \leq +\frac{1}{3}\pi$$

A) BOSONS

With this distribution of nodes we obtain values for the binding energy, depending on the type of interaction used and the number of irreducible representations included, listed below:

No. of I.R.s	Binding Energy E_b [MeV]		
	Yukawa	Exponential	Yamaguchi
2	-9.0609	-18.068	-11.456
3	-9.8814	-18.309	-12.640

These figures would indicate that Yukawa's potential leads to the best result, while the exponential interaction gives rise to considerable overbinding, with Yamaguchi's potential yielding a value between the other two. The discrepancy between these figures and the pattern of results we would expect on the basis of the characteristics of the potentials as shown in diagram 1 in chapter II is obvious:

The Yukawa potential is stronger than the other two, a fact, which must of necessity lead to a deeper bond than the ones produced by the exponential potential and Yamaguchi's interaction. Also, the contribution arising from including a third irreducible representation was predicted to be $\leq 1\%$ of the value obtained with $\lambda, \lambda' = 0, 4$ only; the exponential potential satisfies this estimate near enough, whereas the result yielded by Yukawa's interaction shows an additional contribution nearer 9 %.

All this suggests that the error induced by the numerical approximations is still appreciable. Undoubtedly the main error source lies within the numerical evaluation of the integrals. The ranges involving angular variables such as φ and ψ as well as ω_3'' have been approximated by 5 weighted points, as an increase in

the number of nodes yielded a negligible improvement of the results.

However, when the infinite range of the momentum variable is subdivided, it leads to an effective cutoff in the potential, and in this case it becomes a function of the number of Gaussian nodes NK ; for example, with $NK = 8$ the cutoff lies at $k^2 = 49.4 \text{ fm}^{-2}$, whereas with $NK = 13$ it is pushed up to 125.4 fm^{-2} . From diagram 1 it becomes obvious that the potential most affected by cutoff is Yukawa's the exponential and Yamaguchi's potential are, as demonstrated by the results below, 'short' enough to be adequately covered by an 8-point integration grid.

Thus, in order to obtain a measure for the effect of the cutoff on the potentials, we have increased NK in steps. The following table lists the results obtained with $NK = 11$ and $NK = 13$, with the figures for $NK = 8$ included for comparison.

TABLE 4

NK	No. of I.Rs.	Binding Energy E_b [MeV]		
		Yukawa	Exponential	Yamaguchi
8	2	-9.0609	-18.068	-11.456
	3	-9.8814	-18.309	-12.640
11	2	-30.667	-16.916	-11.514
	3	-31.441	-17.157	-12.722
13	2	-31.611	-16.737	-11.448
	3	-32.297	-16.982	-12.656

The table shows a sharp increase of the binding energy for Yukawa's interaction between $NK=8$ and $NK=11$, evening out as the integration grid is refined further. As expected, the exponential potential is affected to a much lesser extent. The results for Yamaguchi's potential remain fairly stable throughout.

Further refinement of the Gaussian integration grid amounting to $NK \rightarrow \infty$ leads to the following convergence values for E_b :

TABLE 5

No. of I.Rs.	Binding Energy E_b [MeV]		
	Yukawa	Exponential	Yamaguchi
2	-31.65 ± 0.2	-16.713 ± 0.05	-11.473 ± 0.021
3	-32.35 ± 0.2	-16.975 ± 0.05	-12.673 ± 0.025

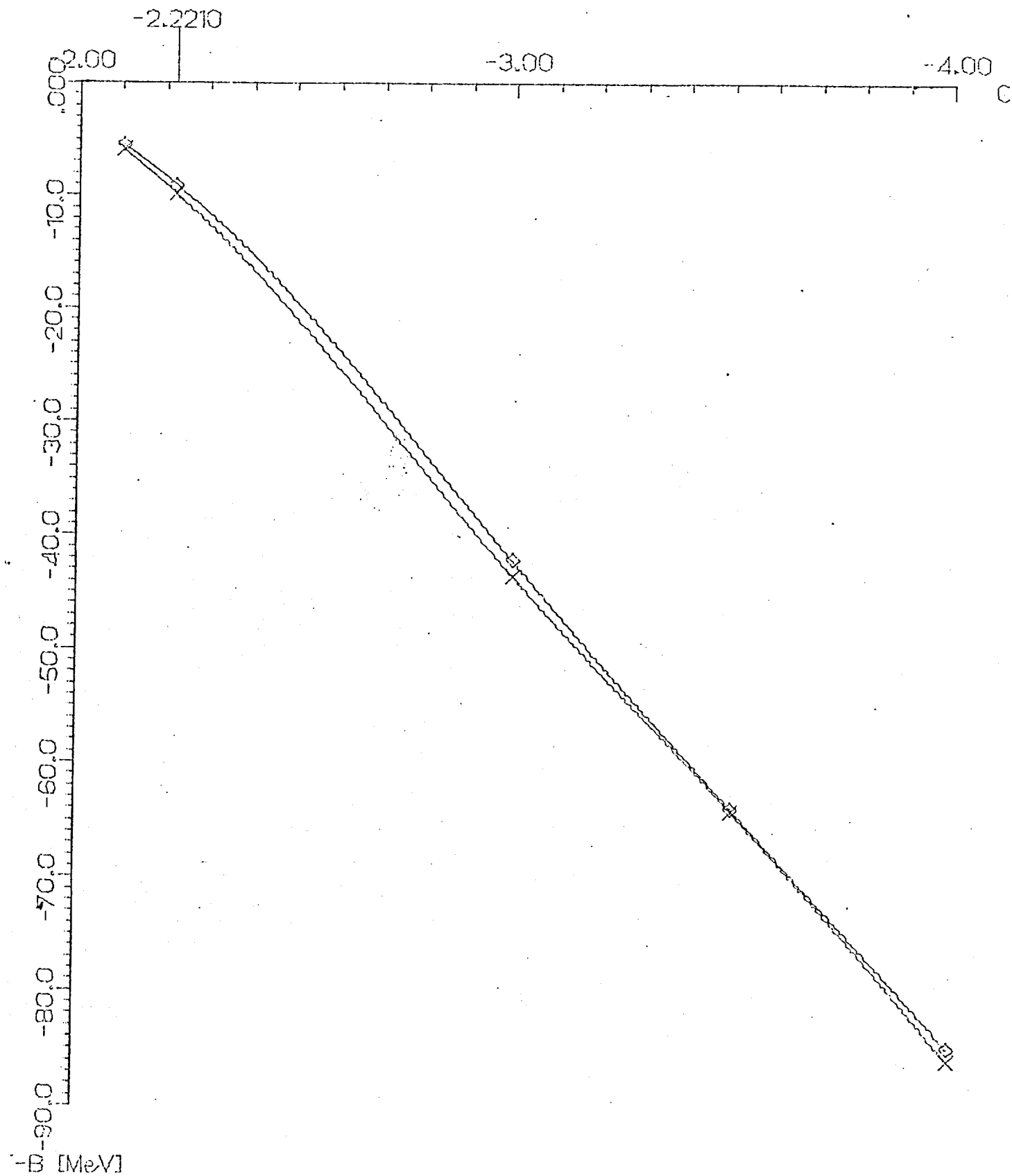
Thus treating the triton as a system of three identical particles leads to overbinding in all three potentials. But the actual values of E_b are of secondary importance here, because only the inclusion of spin-isospin will allow a realistic assessment of the computed binding energy as compared with the experimental value of -8.48 MeV. at this point we merely note the pattern, which emerges from the above values: Yukawa's interaction yields the strongest bond, followed by the exponential potential, and Yamaguchi's interaction giving the weakest bound-state. This is what one would expect considering the relative strengths of the potentials under consideration.

Before we proceed to look at the results obtained with the inclusion of spin and isospin, we list our findings regarding the behaviour of the binding energy

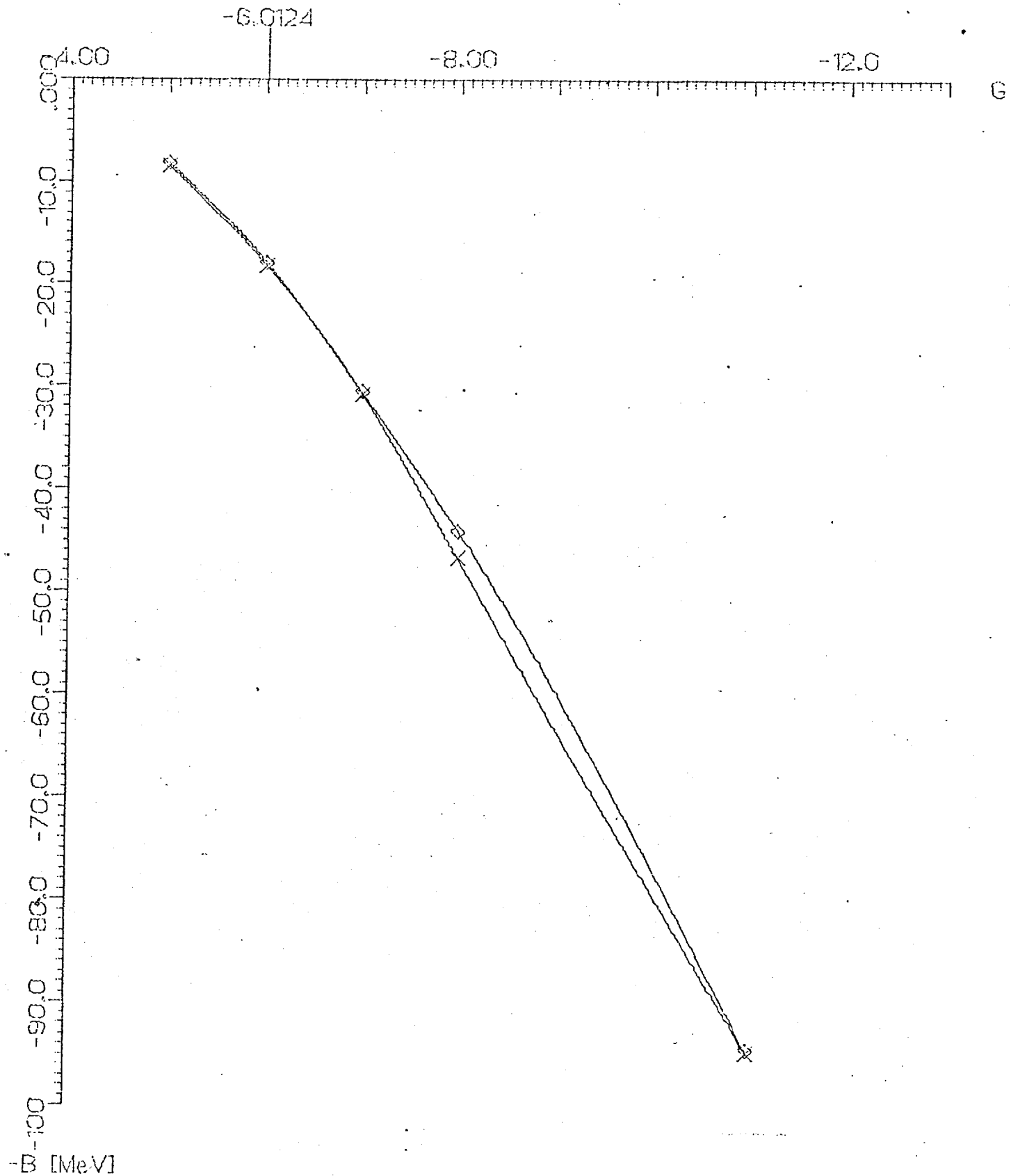
E_b as the depth of the potential is increased. This has been the subject of some conflicting evidence in the past. Osborn⁽¹⁰⁾ observed a so-called 'collapse phenomenon', when plotting E_b against the coupling constant G of the pair-potential. He found E_b to drop very sharply after a certain point along the G -axis. This observation was made for several types of potentials. Malfliet and Tjion⁽¹⁶⁾ on the other hand observed no such sudden drop. Our results are in agreement with the latter findings, as can be seen from the diagrams on the following pages: Each diagram shows two curves, one for the case, in which we include only $\lambda = 0,4$ (—◇—◇—) and the other with $\lambda = 0,4,6$ (—x—x—).

None of the three diagrams displays any sudden change in the gradient of the curve. The behaviour of the function $E_b(G)$ varies slightly from one type of potential to the other, but the essential characteristics remain the same for all three : a continuous increase of the absolute value of the binding energy as the depth of the interaction is increased.

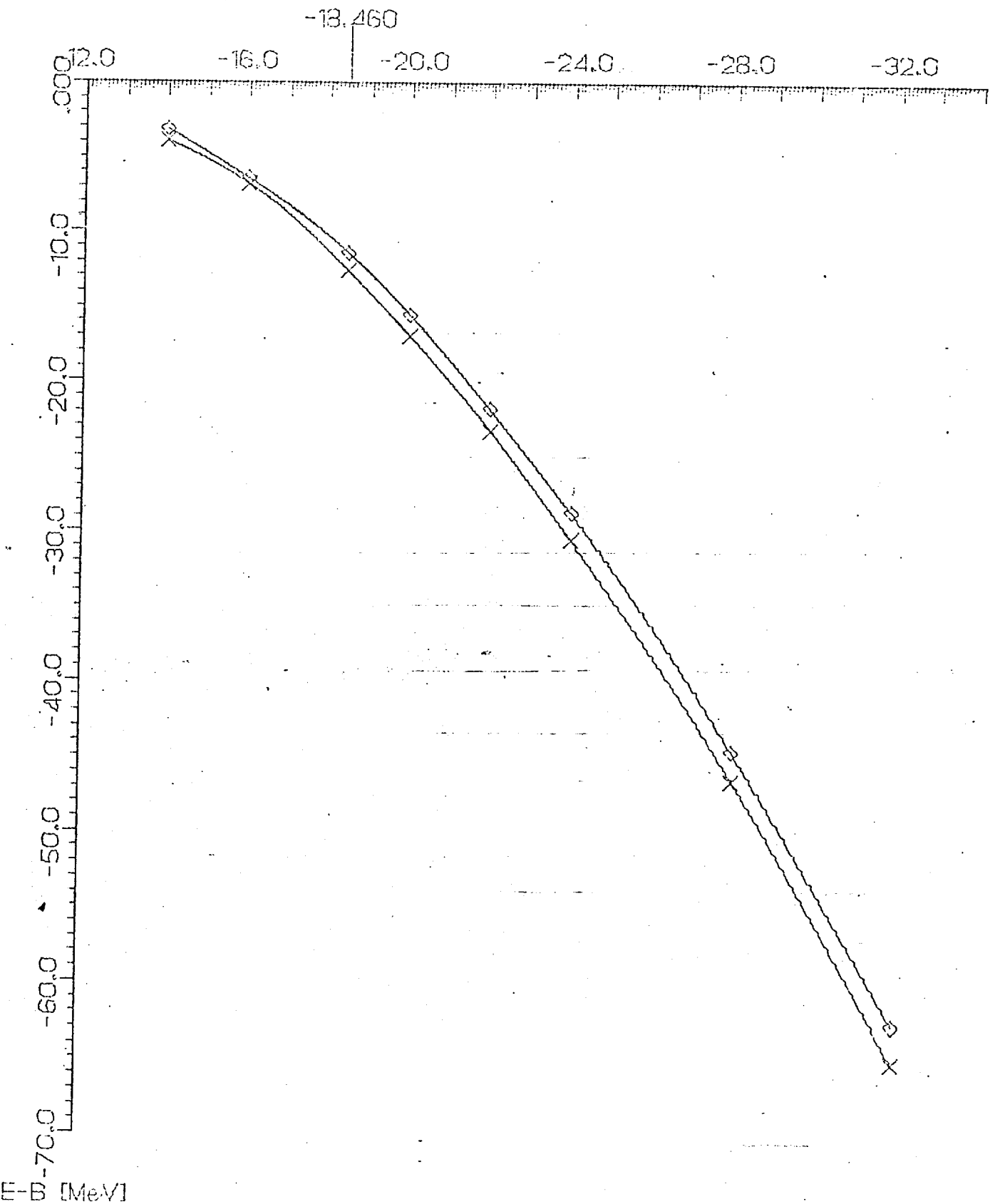
COMPUTED BINDING ENERGY E-B AS A FUNCTION OF THE COUPL.CONSTANT G
POTENTIAL : Y U K A W A



COMPUTED BINDING ENERGY E-B AS A FUNCTION OF THE COUPL. CONSTANT G
POTENTIAL : E X P O N E N T I A L



COMPUTED BINDING ENERGY E-B AS A FUNCTION OF THE COUPL. CONSTANT G
POTENTIAL : Y A M A G U C H I



B) FERMIONS

In taking spin and isospin into account we used the singlet potentials, whose shape was determined in chapter II and whose parameters are listed in table 2 page 84. The results we obtained with $NK = 8$ and $N = 5$ are as follows:

TABLE 6

No. of I.Rs.	Binding Energy E_b [MeV]		
	Yukawa	Exponential	Yamaguchi
2	-3.339	-6.920	-5.983
3	-3.995	-7.155	-6.816

Here, as opposed to the case of three identical particles, we have underbinding throughout. Yet we observe very much the same pattern of numerical distortion of the Yukawa result as before. This is indeed to be expected, because of the considerably shorter range of this potential for the singlet case compared to the triplet one. The latter therefore remains the dominating interaction and we can resort to the previous sub-section for the correction factor. The result for the

exponential potential is also given a slight adjustment based on the same argument. The final adjusted values for E_b with the inclusion of spin-isospin are as follows:

TABLE 7

NO. of I.Rs.	Binding Energy E_b [MeV]		
	Yukawa	Exponential	Yamaguchi
2	-11.663	-6.404	-5.983
3	-13.037	-6.635	-6.816

Section IV.2 Bound-State Partial-Wave Amplitudes

In this section we present only a limited but representative section of the partial-wave amplitudes, which we have computed as a function of the momenta for the various bound-states. The methods employed are those described in chapter III, section 2.

Each function is given in the form of 8 tabular points; these points have been 'smoothed' into a curve by means of a technique, which uses cubic spline functions⁽¹⁹⁾. The (momentum)²-axis is subdivided in such

a way as to accentuate the peaks of the amplitudes; this is done by way of mapping the values for k^2 arising from the Gaussian 8-point abscissae into a region of fixed width (determined by the size of the digital on-line plotter) via

$$x_i = \frac{k_i^2 - 1}{k_i^2 + 1} + 1 ; \quad i = 1, 2, \dots, 8 .$$

The diagrams of the amplitudes appear in the following order:

Boson-Case

- a) Yukawa, $G = -2.221$, $\lambda = 0, 4$
- b) " " , $\lambda = 0, 4, 6$
- c) Exponential, $G = -6.0124$, $\lambda = 0, 4$
- d) " " , $\lambda = 0, 4, 6$
- e) Yamaguchi, $G = -18.46$, $\lambda = 0, 4$
- f) " " , $\lambda = 0, 4, 6$

This is followed by two diagrams showing the amplitudes of the bound-state, where the depth of the Yamaguchi potential has been increased to -24.00 and subsequently to -28.00 units:

- g) Yamaguchi, $G = -24.00$, $\lambda = 0, 4, 6$
- f) " , $G = -28.00$, "

Next we list the diagrams showing the partial-wave amplitudes for the totally symmetric three-fermion bound-state:

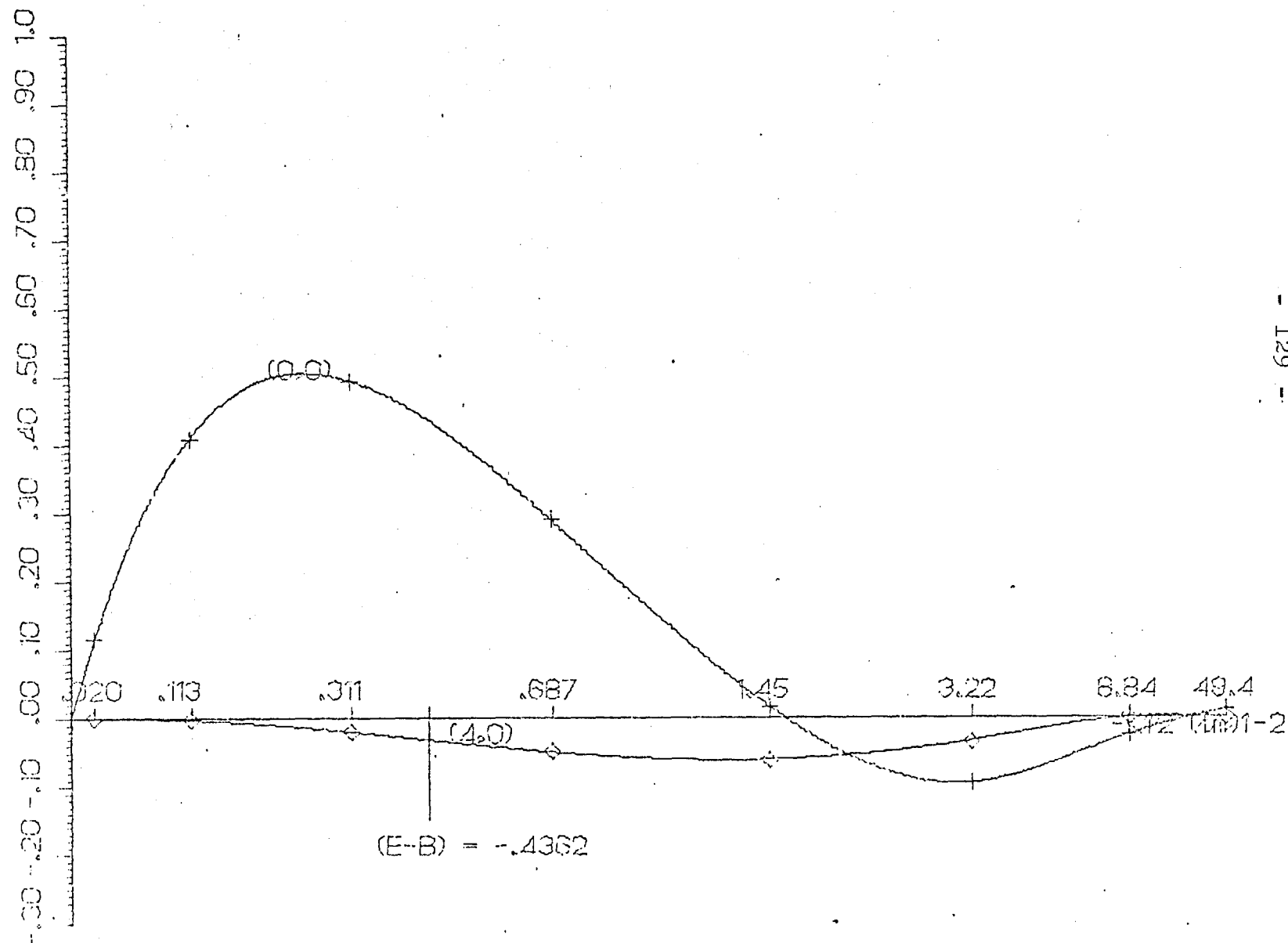
FERMION-Case

- i) Yukawa, $\lambda = 0, 4$
- j) " , $\lambda = 0, 4, 6$
- k) Exponential, $\lambda = 0, 4$
- l) " , $\lambda = 0, 4, 6$
- m) Yamaguchi, $\lambda = 0, 4$
- n) " , $\lambda = 0, 4, 6$

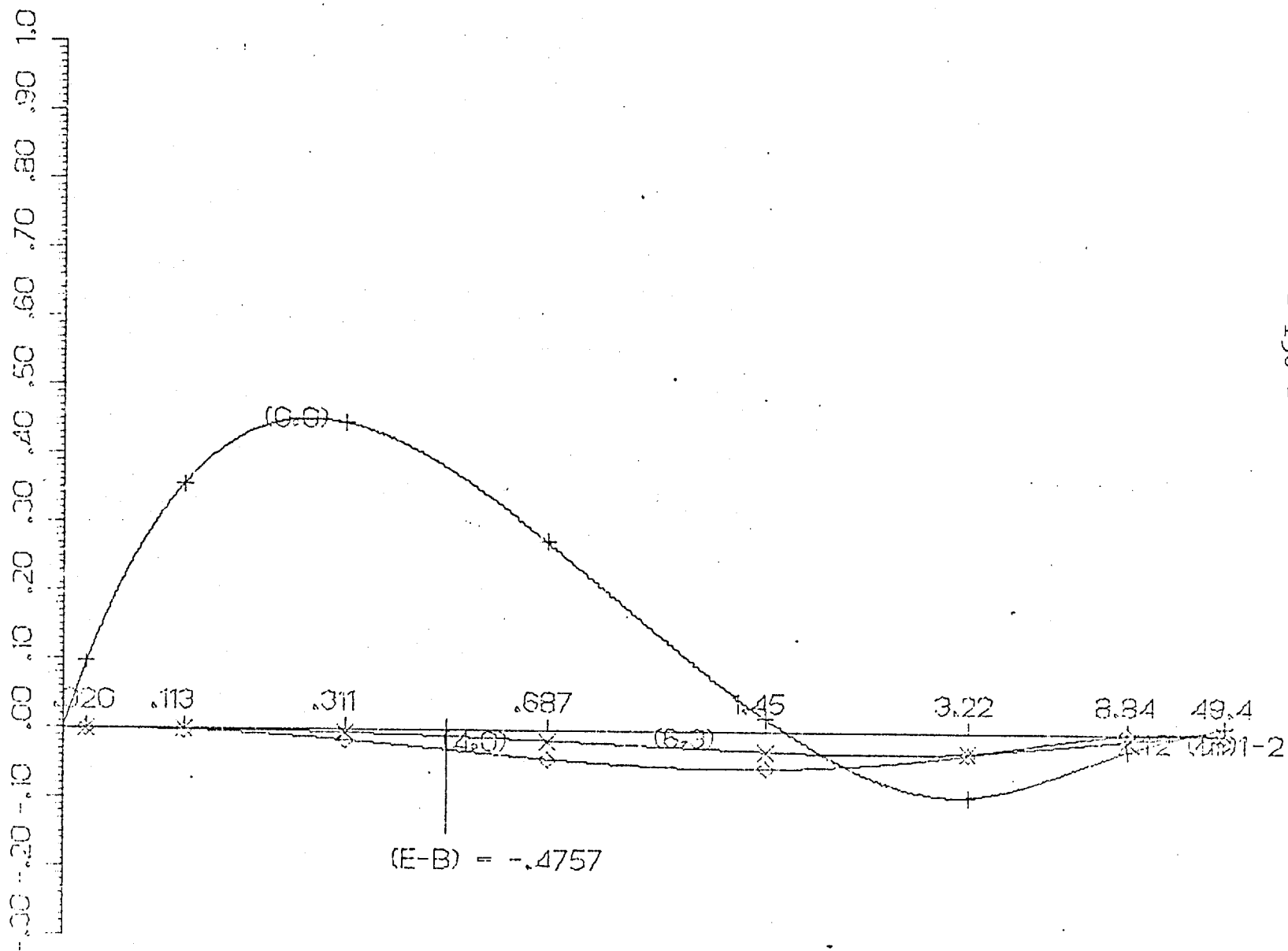
The spline-smoothing process does not, of course, render the precise shape of the amplitude functions. It should also be noted that these functions are derived from the 8-point computations; this, while it affects the numerical value of the binding energy, does not alter the basic characteristics of the amplitudes except absolute values. We do in fact have the momentum states with the highest probability occurring near the energy E_b . We note here that the bound-state energy E_b has been plotted along the positive k^2 -axis, even though E_b carries a negative sign; we have left it understood that the abscissae of the graphs refer to the imaginary momenta $i.k$ of the bound-states.

At the beginning of section III.2 we have indicated that the computation of the wave-amplitudes lends itself to an appraisal of the overall machine-dependent accuracy of the computations. We have checked the computed eigenvalues of the matrix $\bar{K}$ for each of the 14 functions, plotted on the following pages, and we have found that the mean deviation lies in the region of $\leq 1\%$. Considering the volume of the computations, this error appears acceptable, especially in view of the fact that it includes any error arising from the calculation of E_b by means of polynomial fit.

PARTIAL-WAVE AMPLITUDES $\chi(0,0)$ AND $\chi(4,0)$
 BOSONS, YUKAWA, $GT = -2.221$, $MU_{12}-T = 0.80138$

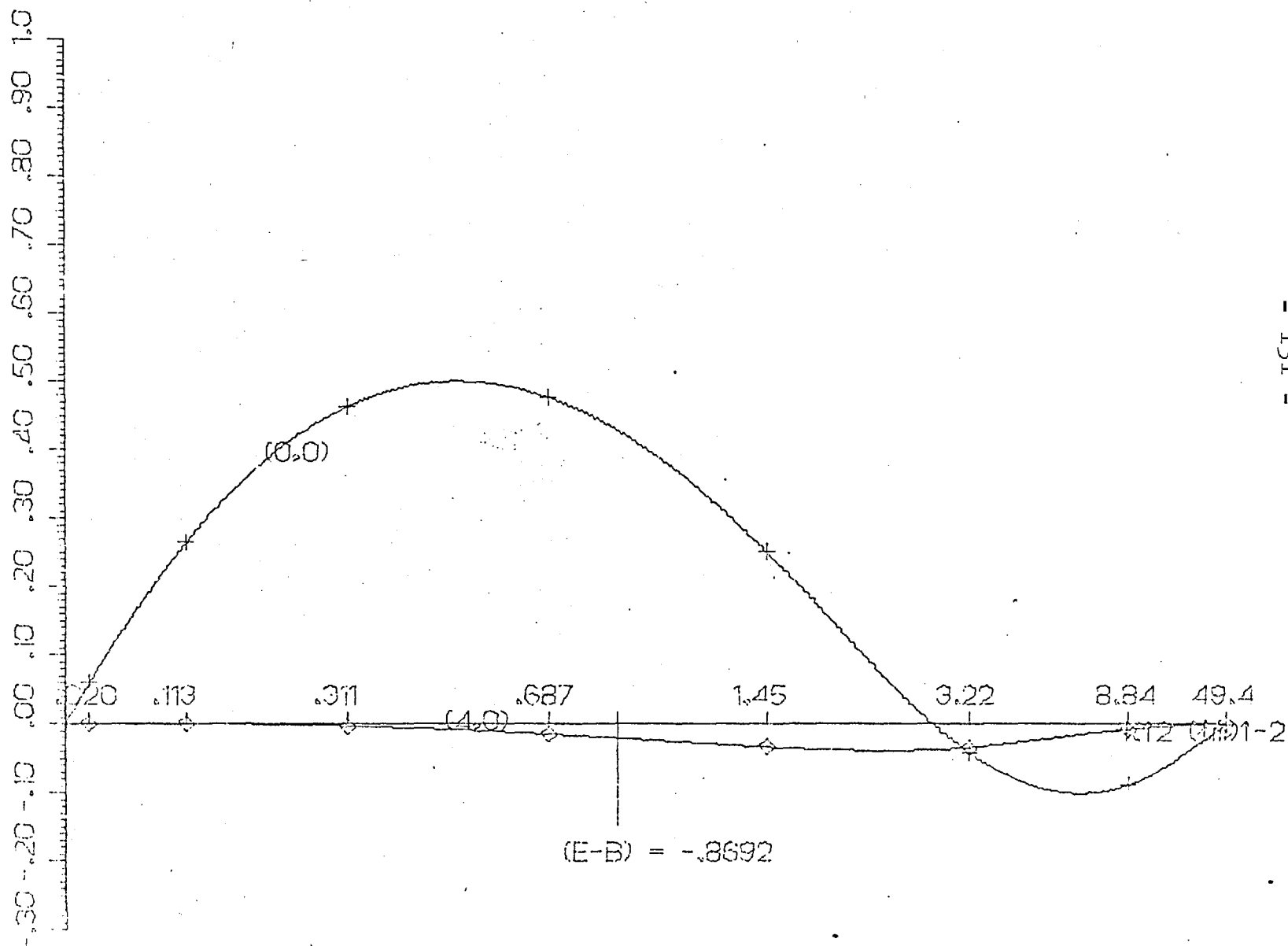


PARTIAL-WAVE AMPLITUDES $\text{CHI}(0,0)$ AND $\text{CHI}(4,0)$ AND $\text{CHI}(6,0)$
 BOSONS, YUKAWA, $\text{GT} = -2.221$, $\text{MU}12\text{-T} = 0.80138$, 3 IRR. REP.

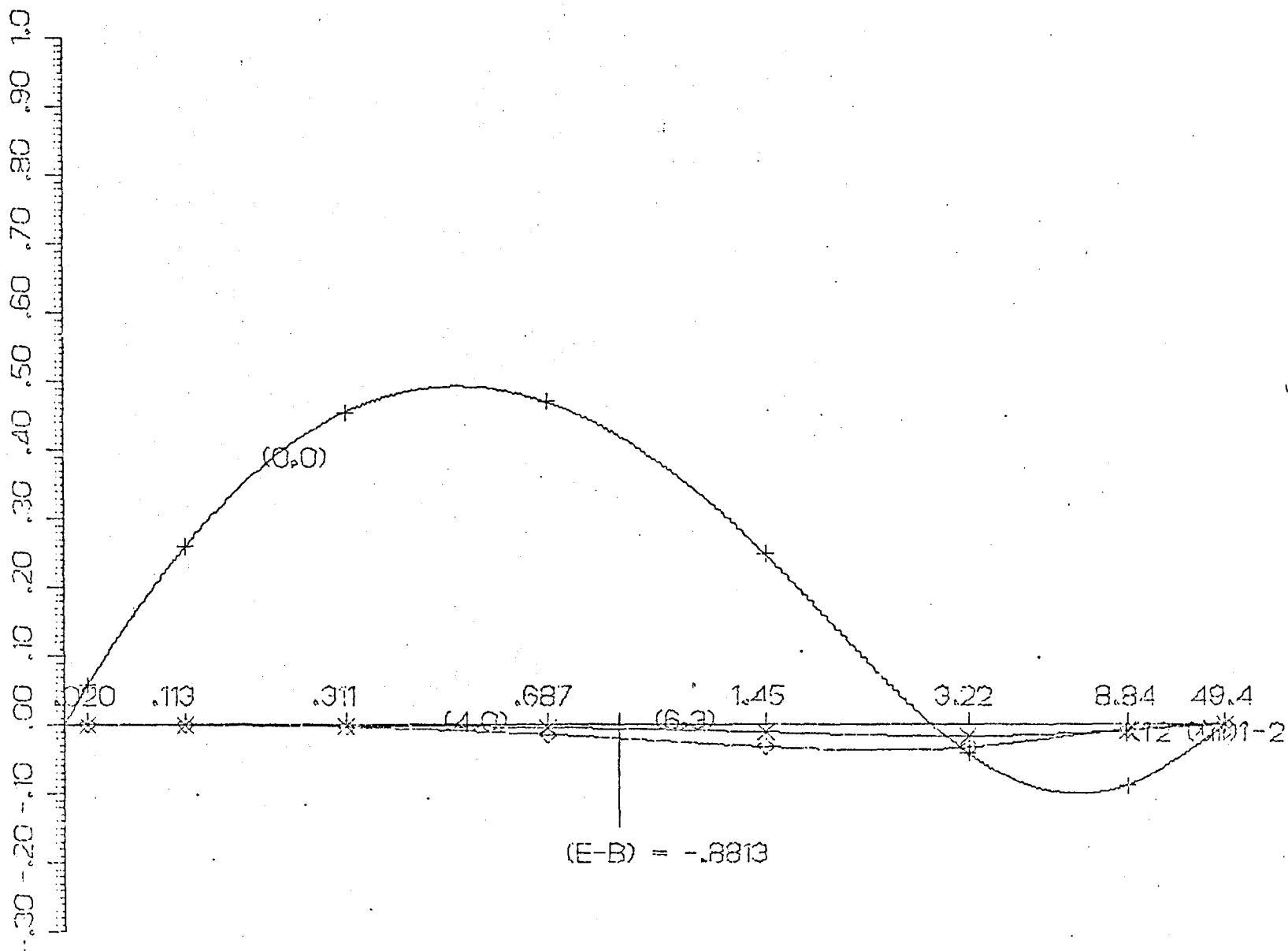


PARTIAL-WAVE AMPLITUDES $\chi(0,0)$ AND $\chi(4,0)$
 BOSONS, EXPONENTIAL, $GT = -6.0124$, $MU12 = 2.67257$

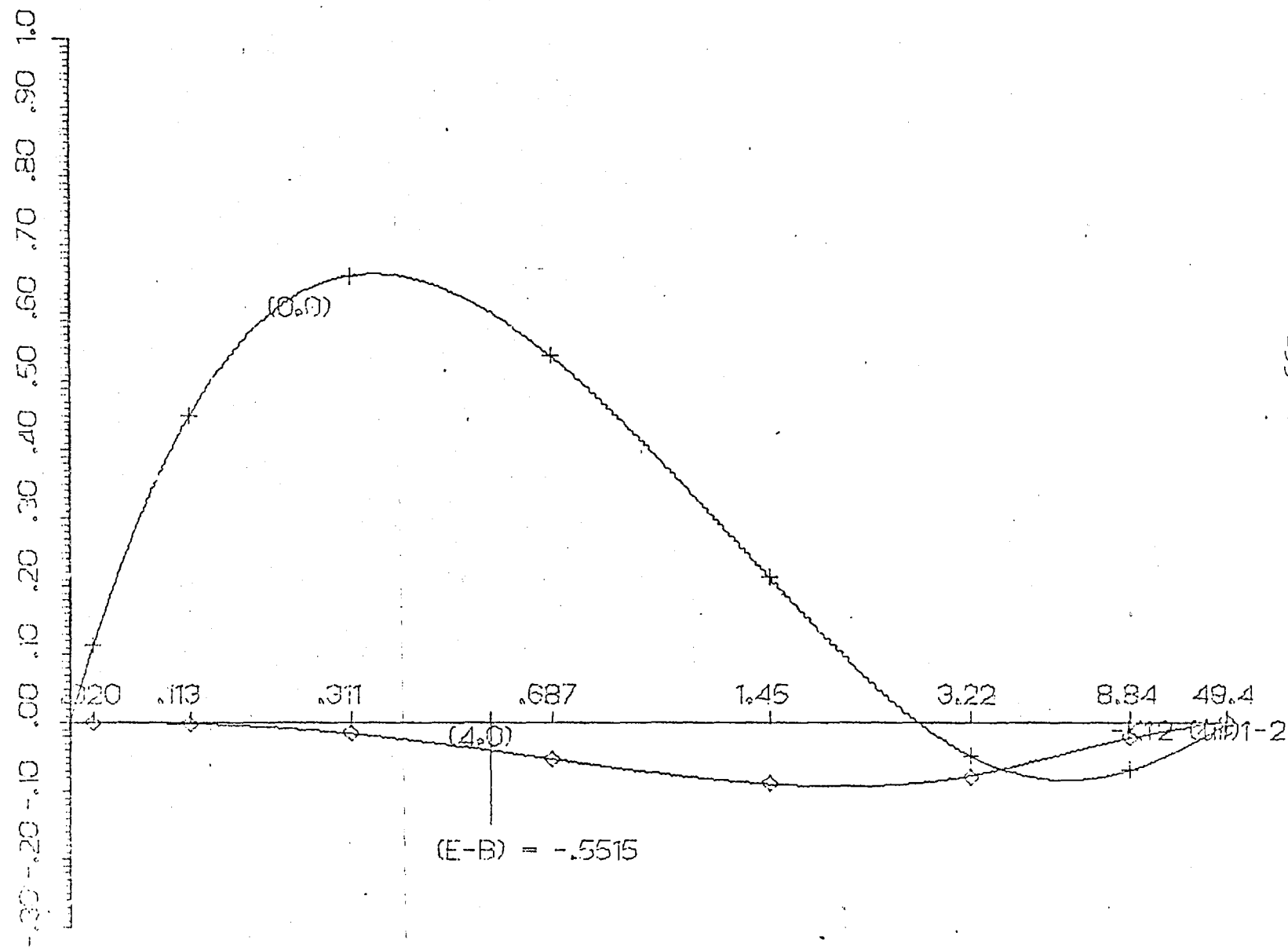
- 131 -



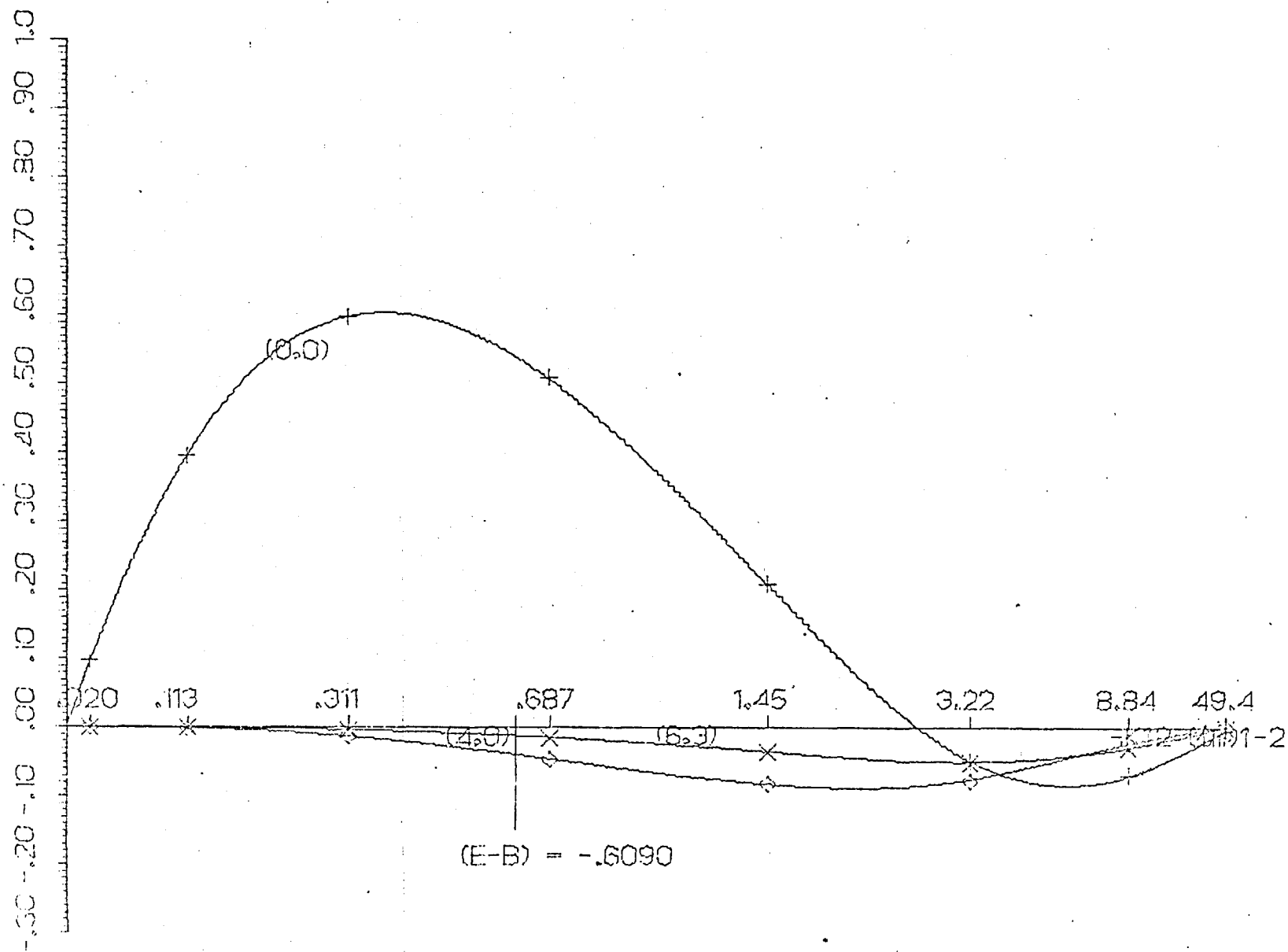
PARTIAL-WAVE AMPLITUDES $\text{CHI}(0,0)$ AND $\text{CHI}(4,0)$ AND $\text{CHI}(6,3)$
 BOSONS, EXPONENTIAL, $\text{GT} = -6.0124$, $\text{MU12} = 2.67257$, 3 I.R.



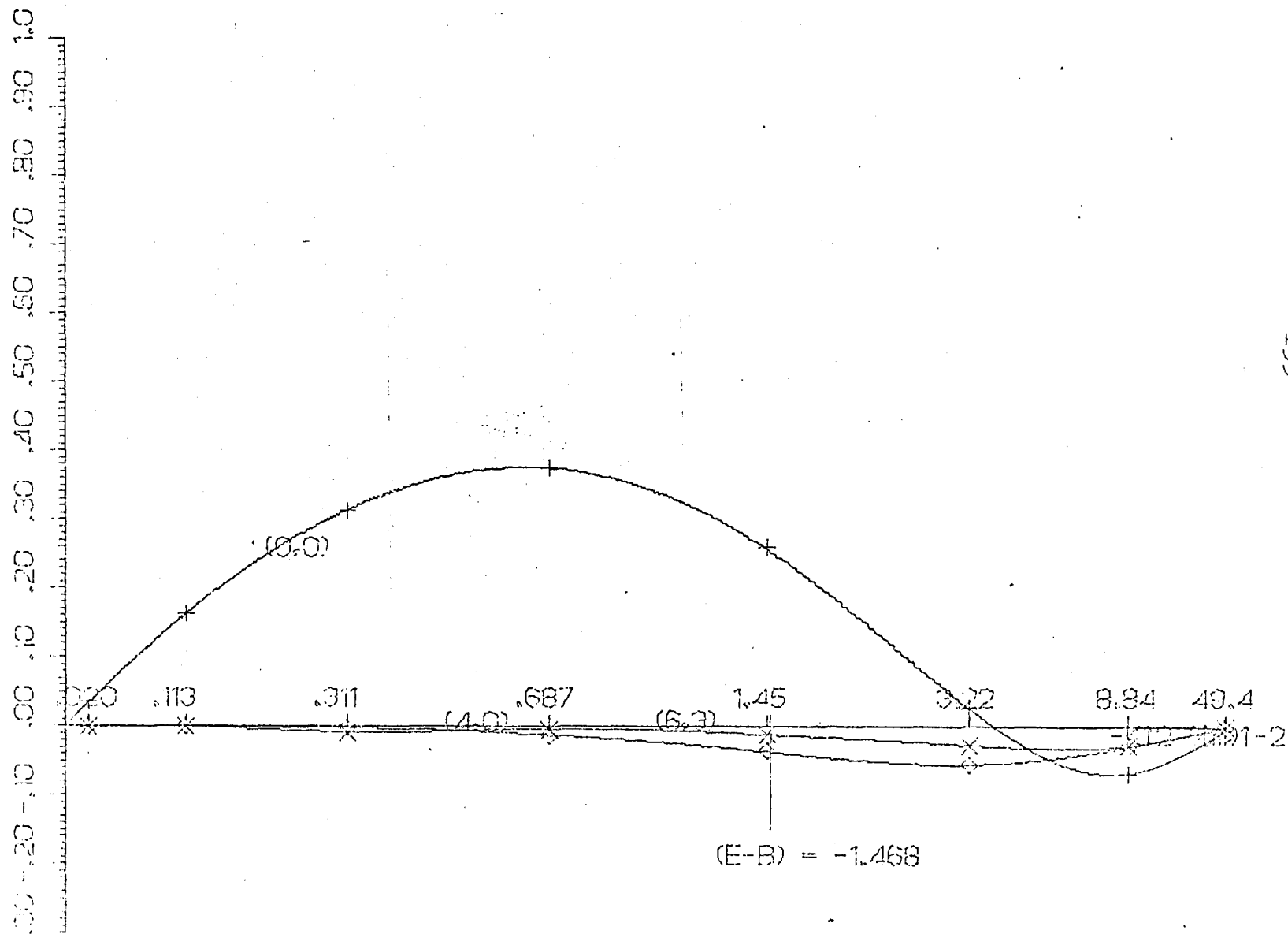
PARTIAL-WAVE AMPLITUDES $\chi(0,0)$ AND $\chi(4,0)$
 BOSONS, YAMAGUCHI, $GT = -18.46$, $MU12 = 4.198$



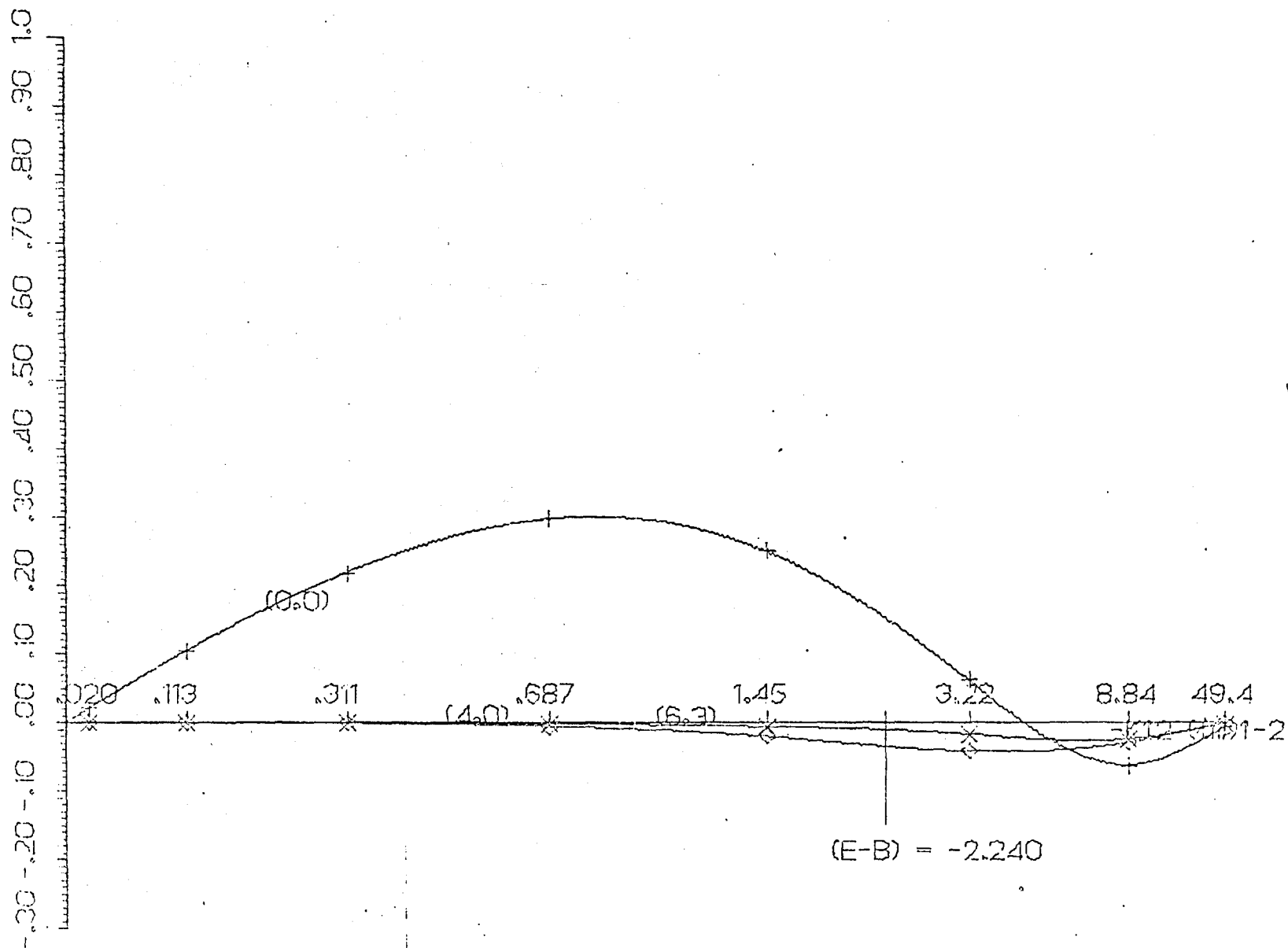
PARTIAL-WAVE AMPLITUDES $\text{CHI}(0,0)$ AND $\text{CHI}(4,0)$ AND $\text{CHI}(6,3)$
 BOSONS, YAMAGUCHI, $\text{GT} = -18.46$, $\text{MU12} = 4.198$



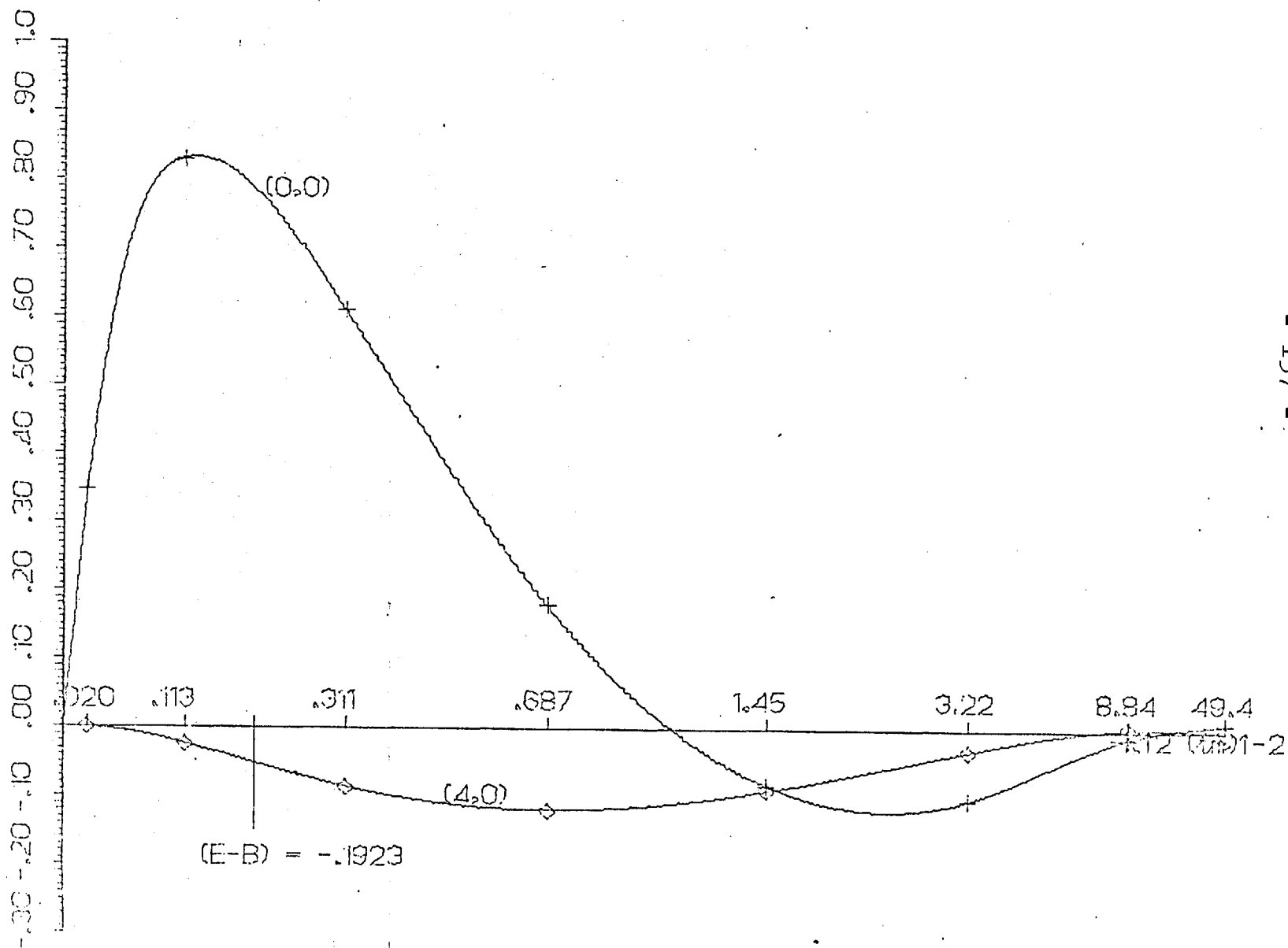
PARTIAL-WAVE AMPLITUDES $\chi(0,0)$ AND $\chi(4,0)$ AND $\chi(6,0)$
 BOSONS, YAMAGUCHI, $GT = -24.0$, $MU12-T = 4.198$, 3 IRR. REP.



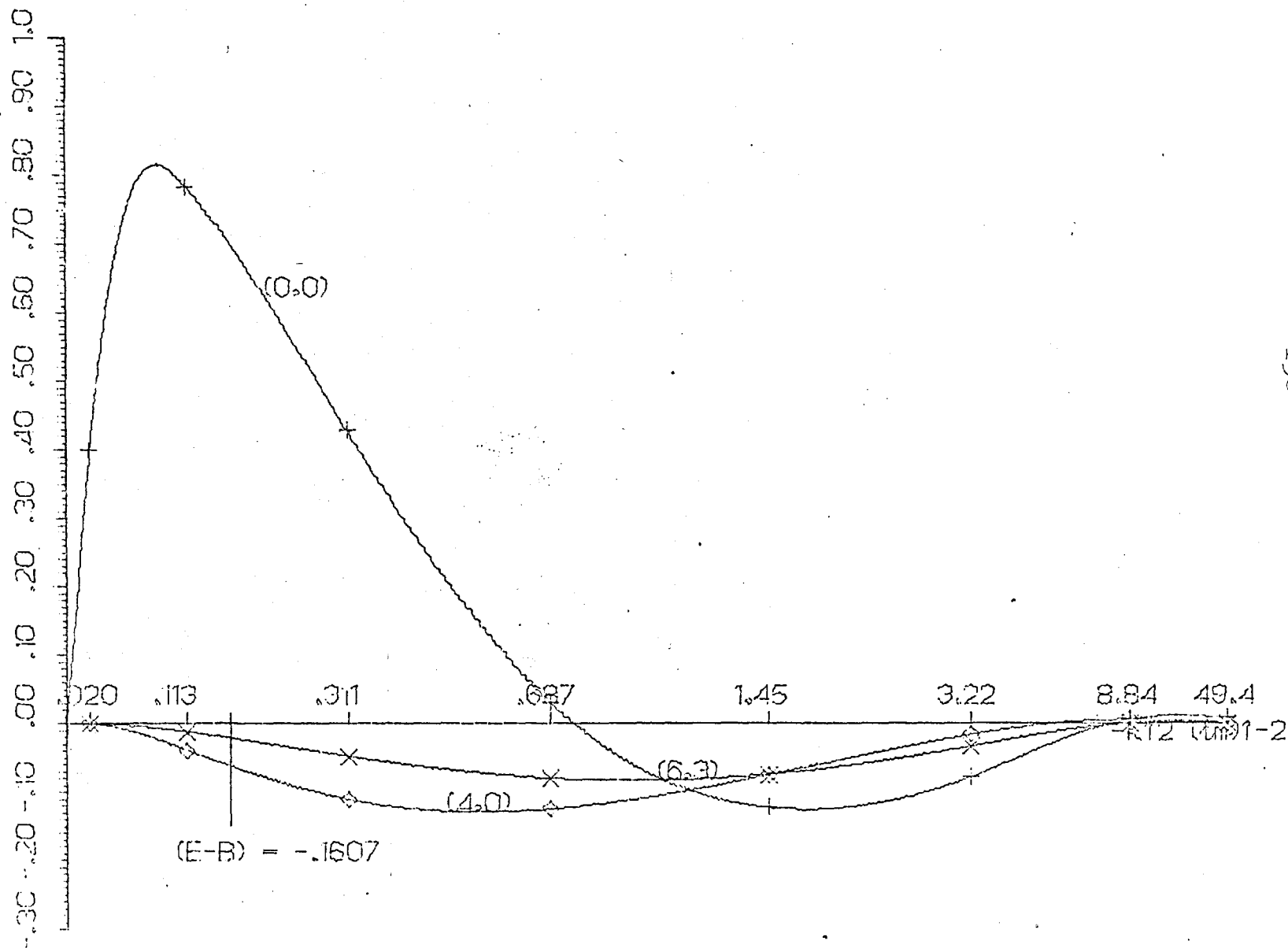
PARTIAL-WAVE AMPLITUDES $\text{CHI}(0,0)$ AND $\text{CHI}(4,0)$ AND $\text{CHI}(6,3)$
 BOSONS, YAMAGUCHI, $\text{GT} = -28.0$, $\text{MU}12 = 4.198$



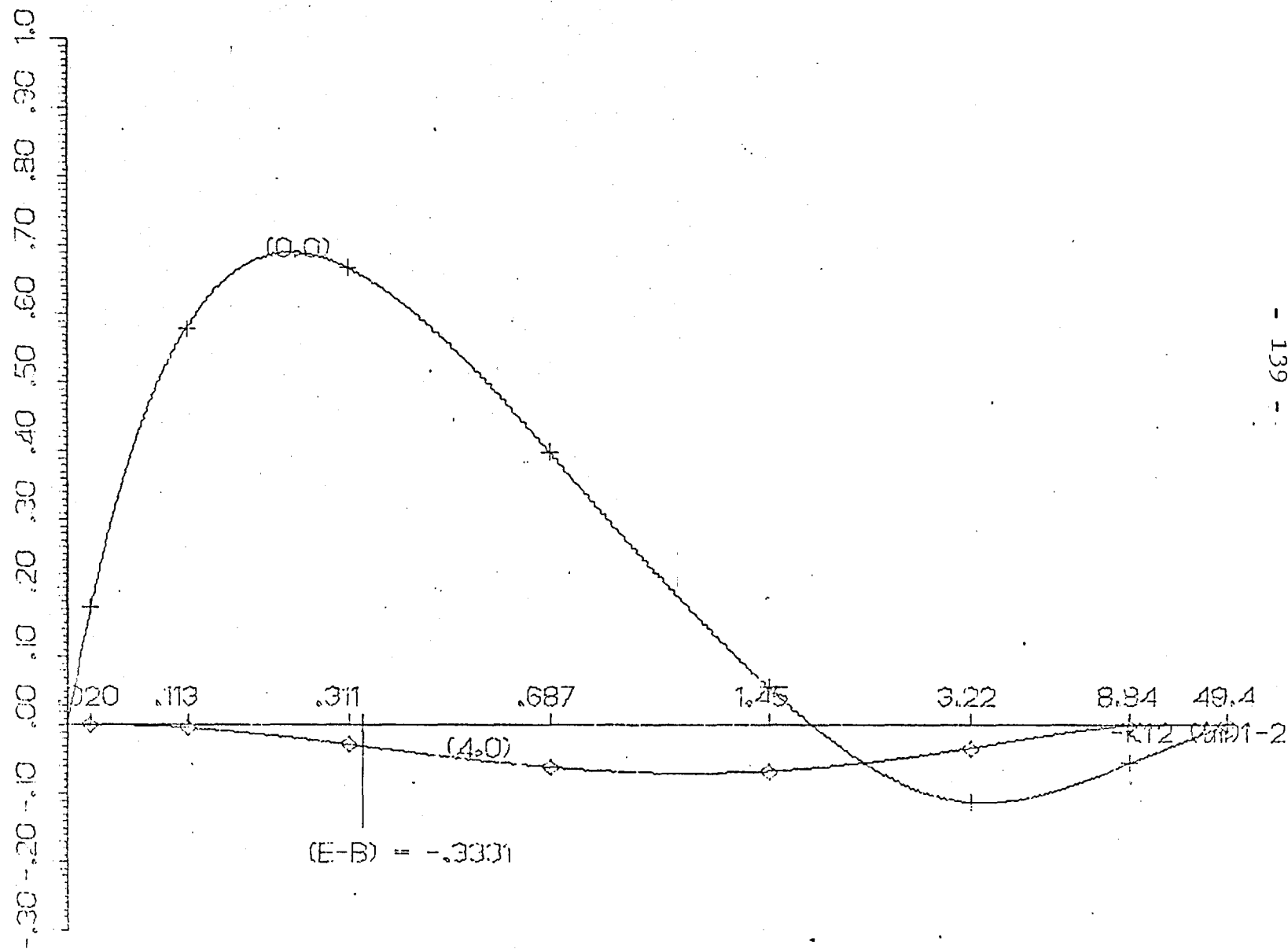
PARTIAL-WAVE AMPLITUDES $\chi(0,0)$ AND $\chi(4,0)$
 FERMIONS, YUKAWA, $GT = -2.221$, $GS = -2.134$, $MU12-T = 0.80136$, $MU12-S = 1.8144$



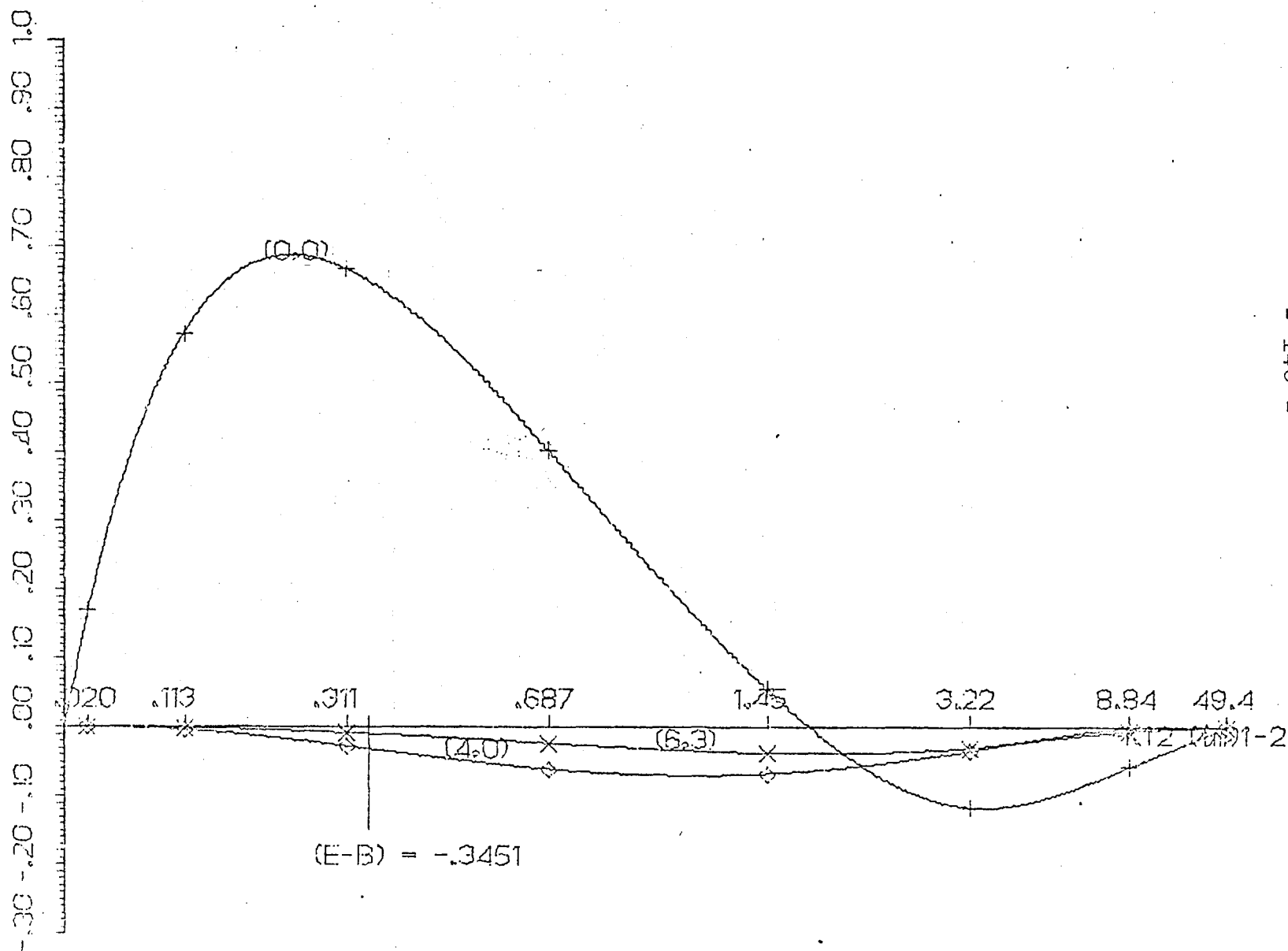
PARTIAL-WAVE AMPLITUDES $\chi(0,0)$ AND $\chi(4,0)$ AND $\chi(6,3)$
 FERMIONS, YUKAWA, $G_T = -2.221$, $G_S = -2.134$, $\mu_{12} - T = 0.80138$, $\mu_{12} - S = 1.8144$



PARTIAL-WAVE AMPLITUDES $\chi(0,0)$ AND $\chi(4,0)$
 FERMIONS, EXPONENTIAL, $GT = -6.0124$, $GS = -3.529$, $MU12-S = MU12-T = 2.67257$

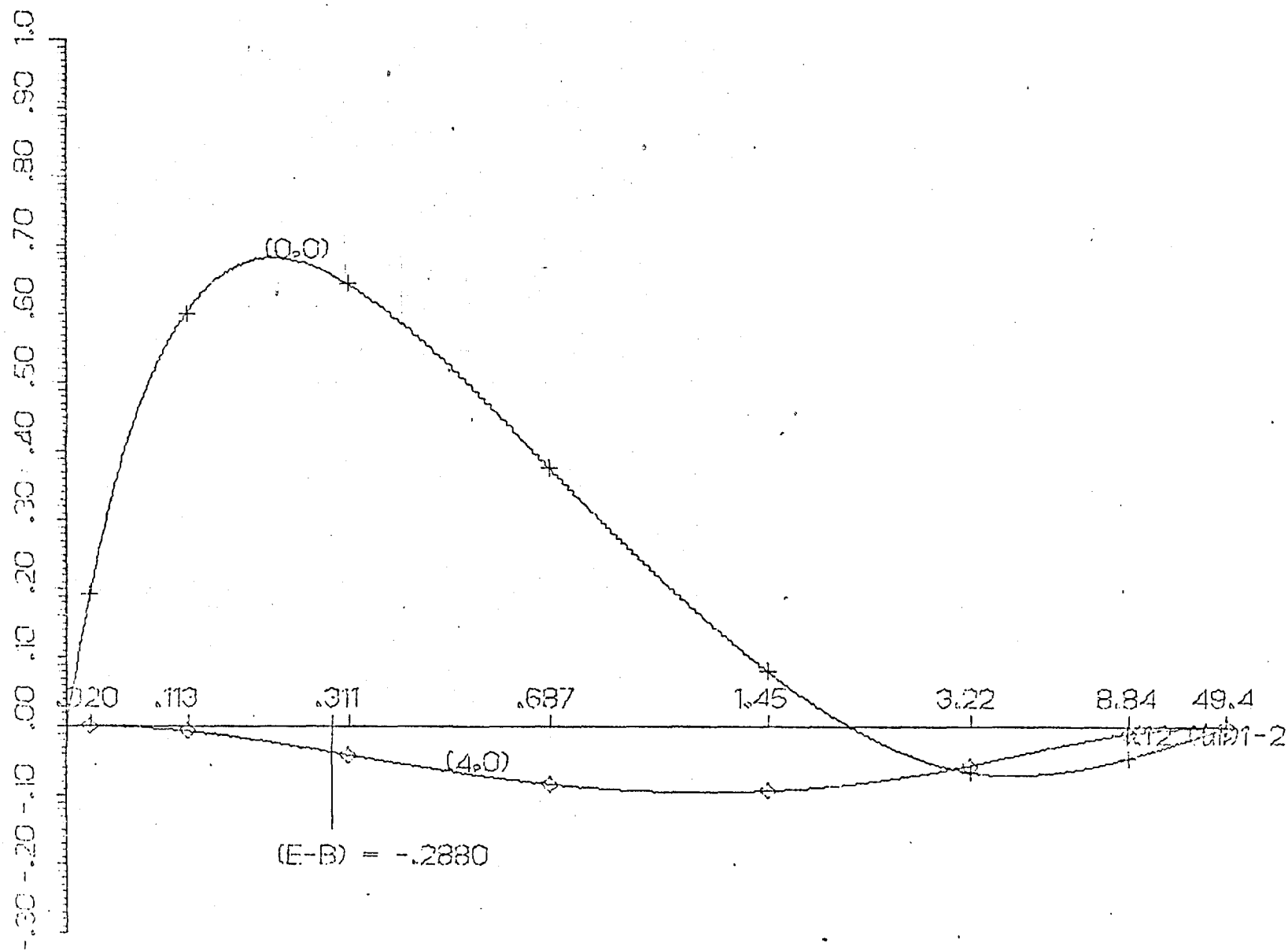


PARTIAL-WAVE AMPLITUDES $\chi(0,0)$ AND $\chi(4,0)$ AND $\chi(6,3)$
 FERMIONS, EXPONENTIAL, $GT = -6.0124$, $GS = -3.529$, $MU12-T = MU12-S = 4.198$

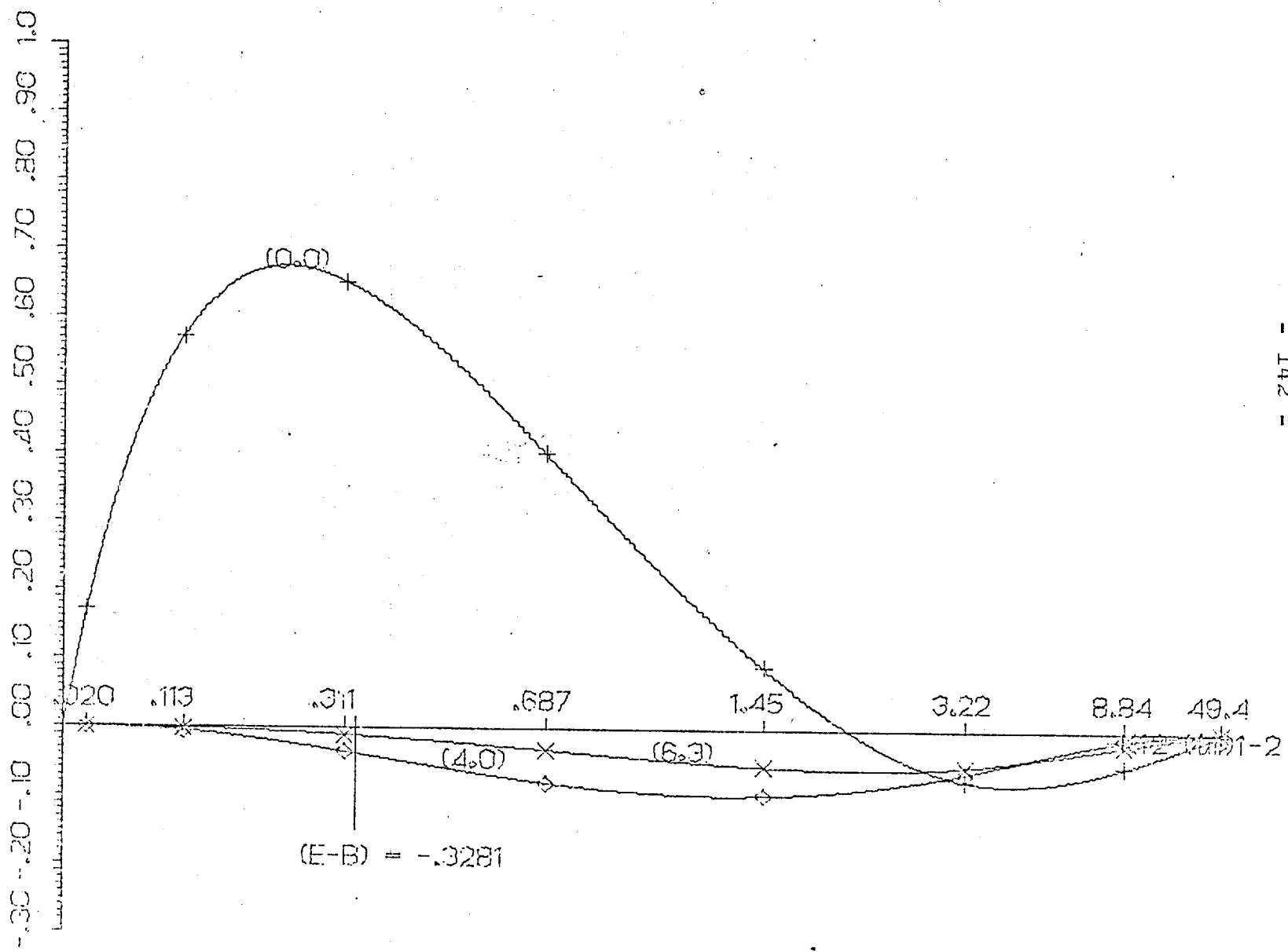


PARTIAL-WAVE AMPLITUDES $\chi(0,0)$ AND $\chi(4,0)$

FERMIONS, $N_K=8, N=5, M=10, YAMAGUCHI, GT=-18.46, GS=-12.963, MU12-S=MU12-T=4.198$



PARTIAL-WAVE AMPLITUDES $\text{CHI}(0,0)$ AND $\text{CHI}(4,0)$ AND $\text{CHI}(6,3)$
 FERMIONS, YAMAGUCHI, $\text{GT}=-18.46$, $\text{GS}=-12.963$, $\text{MU12-S}=\text{MU12-T}=4.198$



CONCLUSION

In this work we have undertaken to provide a numerical verification of Lee's formalism for dealing with three-particle systems and in particular the triton bound-state. We have tested the theory with three different central potentials, two local and one non-local. From the numerous computations in this investigation the following conclusions may be drawn regarding the characteristics of the potentials within this formalism:

Yukawa's Interaction

This potential leads to extreme overbinding in the case of three identical particles. The inclusion of spin-isospin lowers the value for the computed binding energy E_b considerably, yet the system still remains too strongly bound. From a purely numerical viewpoint the potential in its triplet form creates difficulties due to its long tail in momentum space, a fact, which necessitates more points to be chosen along the momentum axis in order to keep cutoff errors on a tolerable level. The singlet counterpart is weaker, and shorter by a factor of approximately $e^{-2.2r}$.

The Yukawa potential on its own therefore proves too strong to be satisfactory as an approximation of the nuclear pair forces within the triton.

Exponential Potential

This local interaction of shorter range is covered quite adequately by an 8-point Gaussian grid, as can be seen from the small variations in the results, when the number of points NK is increased. The final adjustment ^{me} is in the range of a few percent only. One may thus take these values for E_b to be more representative of what the theory is able to predict with a central pair-force only. In the case of three identical particles we again obtain an overbound system, but at about half the strength of what Yukawa's potential yields. The inclusion of spin and isospin in this case, however, returns a value, which lies below the experimental one. We believe this to be mainly due to the uncertainties in determining the shape parameters of the singlet potentials, as we pointed out in chapter II. Recent calculations ⁽¹⁷⁾ performed with an exponential interaction, amongst others, gave results, which lie much closer to the experimental value of -8.48 MeV;

the shape parameters have been taken from the literature and they are not consistent with the low energy scattering data we have used in this work.

Yamaguchi's Potential

This is the only non-local type of potential we have used. It is of very short range and there is practically no residual force beyond the cutoff introduced by the 8-point quadrature formula. The values for the computed binding energy are the lowest of all three cases.

A seeming contradiction between the theory's prediction and the actual results lies in the fact that the inclusion of one further irreducible representation leads to a consistent increase in the absolute value of E_b by more than 10 % as opposed to less than 1 % as expected. This potential is very short and according to Badalyan et.al.^(1,10) the contribution of the higher I.Rs. as measured against the 'zeroth' wavefunction $u_{o,o}(r)$ is not, as given in eqn.(5.38) of ref.(1), but rather

$$u_{\lambda,\mu_j}(r) \simeq \frac{1}{\lambda^{3/2}} u_{o,o}(r)$$

for $r \ll \frac{1}{2}\lambda a$, where a is the scattering length.

Petrov et.al.⁽¹⁹⁾ shows that in configuration space the Yamaguchi potential has a radial dependence in the form of a series in n : For $n = 1$ the interaction corresponds to Yukawa's type, and for $n = 2$ to an exponential potential. If we compare our results with those given by Petrov, we see that our value for E_b including spin-isospin and using Yukawa's interaction lies well below Petrov's (we have used $r_{os} = 2.151$ fm), while our results obtained with Yamaguchi's potential for three identical particles compare very well with those listed by Petrov, where spin and isospin are included.

We feel that this first numerical evaluation shows the validity of Lee's ^{formalism} ~~^~~ Nonethelless, further tests seem necessary; in particular, the nuclear forces should be refined to include tensor forces. As the radial term one would recommend Yukawa's interaction, despite the numerical awkwardness involved. The other types of radial terms have no real physical motivation and are weak. Petrov's finding suggest that the inclusion of tensor terms in the nuclear forces is likely to lead to a con-

siderable improvement of the results for the binding energy.

It is hoped that in continuing this line of research it will be possible to implement the suggestions made in the introduction to chapter III in order to reduce the amount of computer time used.

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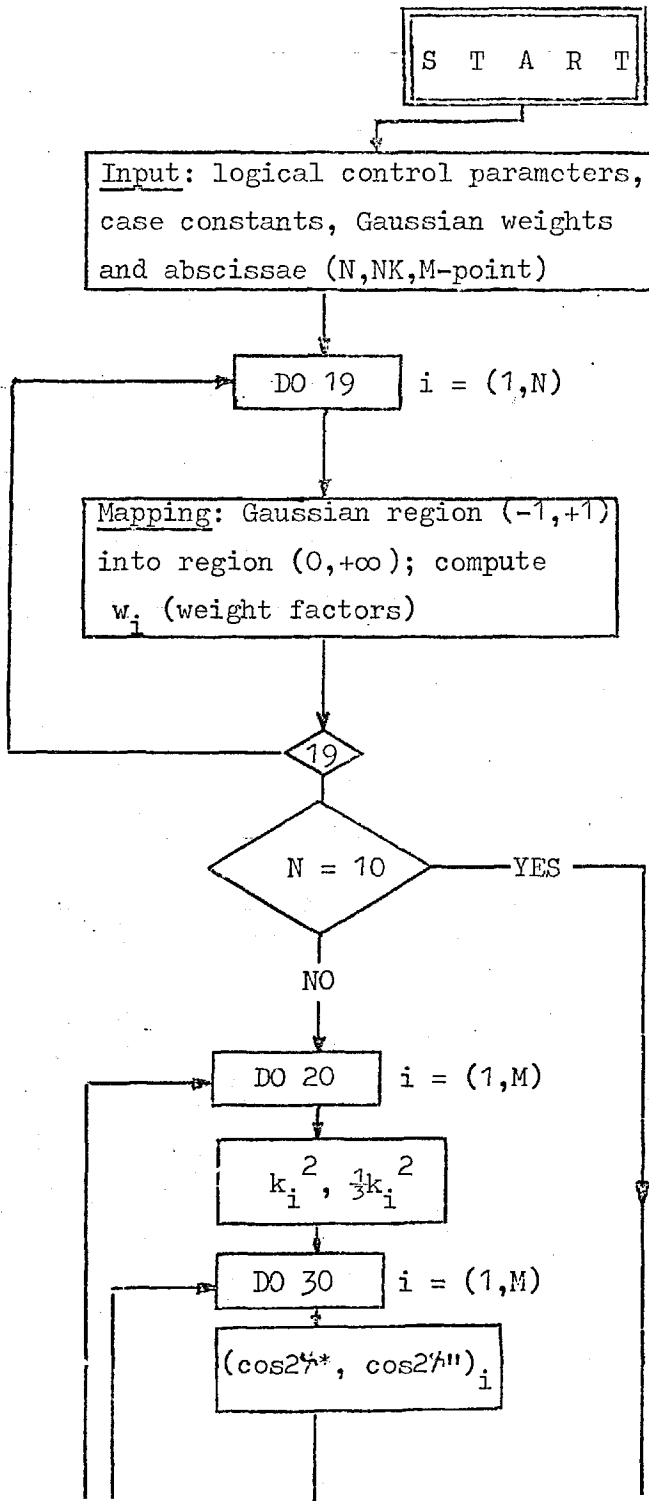
APPENDIX: PROGRAM DOCUMENTATION

This appendix is divided into three parts:

- 1) Flow-Chart of the program structure for calculating three-body bound-states. Our notation included 'i' as the general indication of a loop-index, whose range in each case is given in brackets, e.g.
 $i = (1, N)$; where the upper and lower limits of a loop depend upon computed quantities during execution, this is indicated by the letters 'iv', followed by the variable names and/or numbers, which determine the size of the loop, i.e. the difference (upper l.) - (lower l.), for example $iv = (NK * NK^2)$. The comments contain references to listed variable names, where this is felt necessary; the number of a particular name is also given in brackets.
- 2) Selective List of Variable Names.
- 3) Program Listing. This is a listing of the program as stored on magnetic tape. The arrays are dimensioned for $NK = 8$ and $N = 5$. Of LAM6 only the main body is listed, the rest being the same as for PART 2.

1) FLOW-CHART OF THE PROGRAM STRUCTURE

P A R T 1



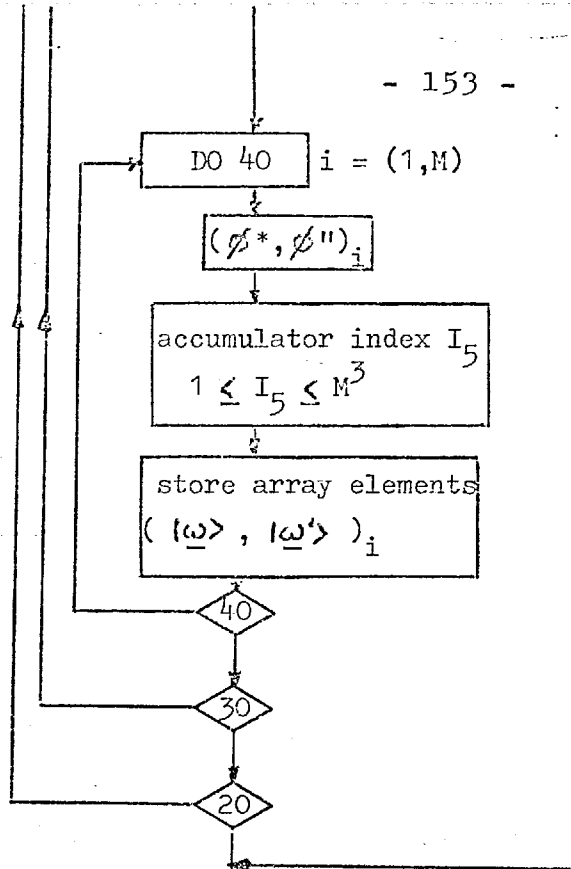
Comments and References

(1) ÷ (21)

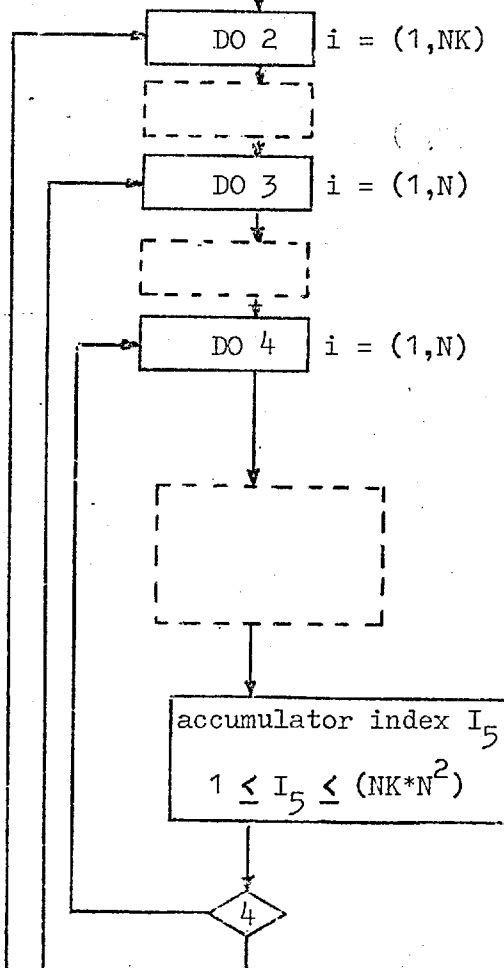
see p. 45 ff. ;

(1) ÷ (3) Non-Input Arrays

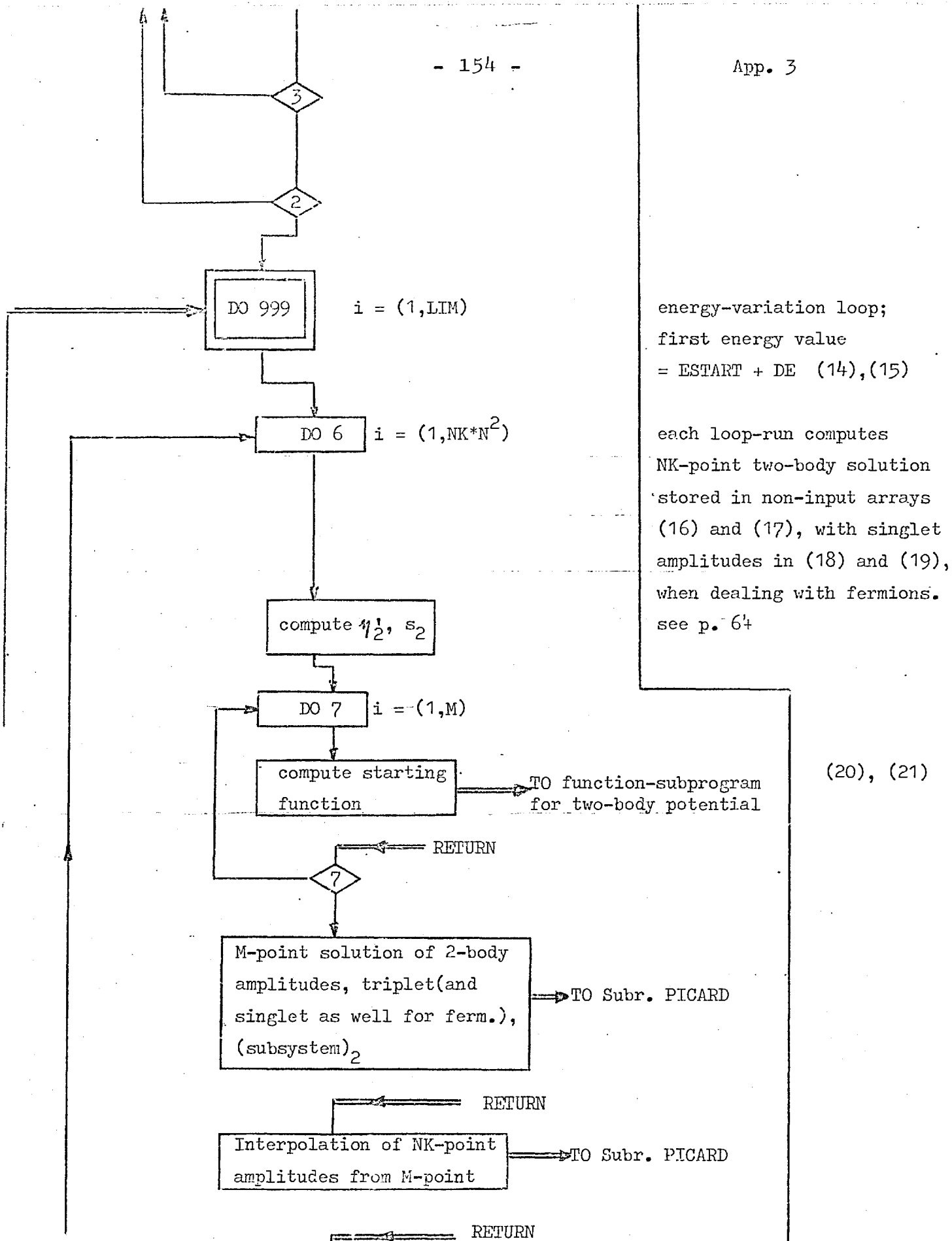
(10) ÷ (15)



location index for arrays
(10) to (15)



(4) ÷ (9) Non-Input Arrays

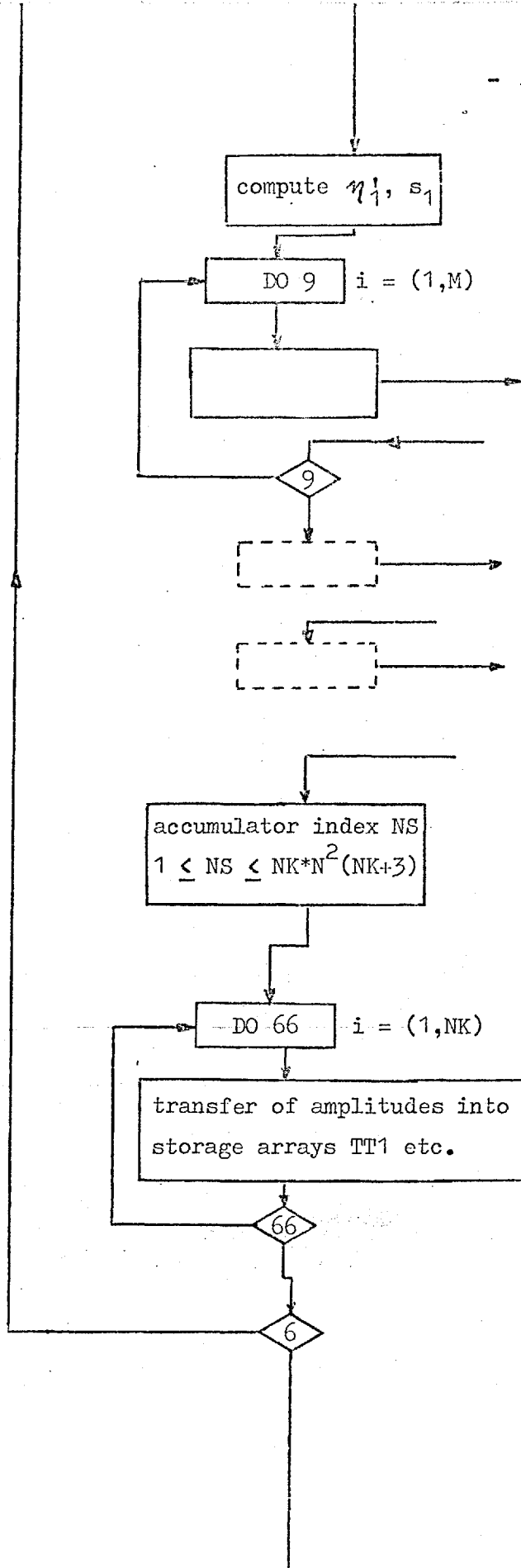


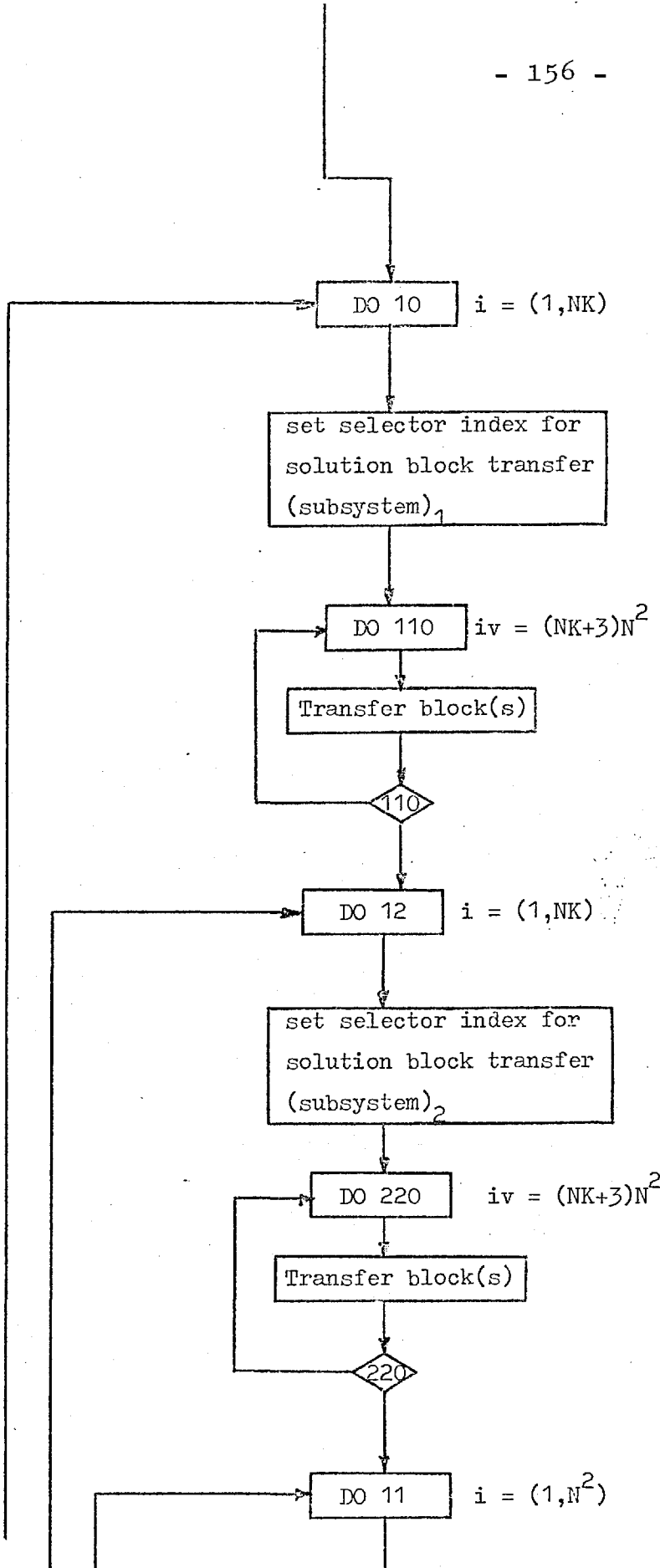
see p. 64

computation of amplitudes
for (subsystem)₁ as for
(subsystem)₂

location index for storing
amplitudes in arrays (28) +
(31), together with respec-
tive set of arguments

End of preliminary compu-
tations: two-body amplitude
belonging to all the re-
quired sets of Omnes varia-





bles now available for
subsequent computation of
elements $K(E)_{NK,N}$ (p.60ff)

k^2 -loop; see variable
processing order p.62

index for transfer of block
($NK \cdot N^2$) of NK -point amplitu-
de solutions from (28),(30)
into (32),(34) resp.ly

solution block transfer-loop

k'^2 -loop

(29),(31) into (33),(35)
respectively

loop for processing γ^* and
 ϕ^* variables; N^2 combinatio.

thereof for each set (k^2, k'^2)

loop for recovering particular two-body solution; (16), (18)

for interpolation in loop 14, see below

loop for processing γ " and ϕ " variables; analogue to loop 11

recovering solution; analogue to loop 111; (17), (19)

recuperate set $|\omega\rangle_i$

DO 111 $i = (1, NK)$

recovery of individual solution

111

Set parameter and argument in t_1 , singlet and triplet for fermions

Set file processing index

DO 13 $i = (1, N^2)$

recuperate set $|\omega'\rangle_i$

DO 131 $i = (1, NK)$

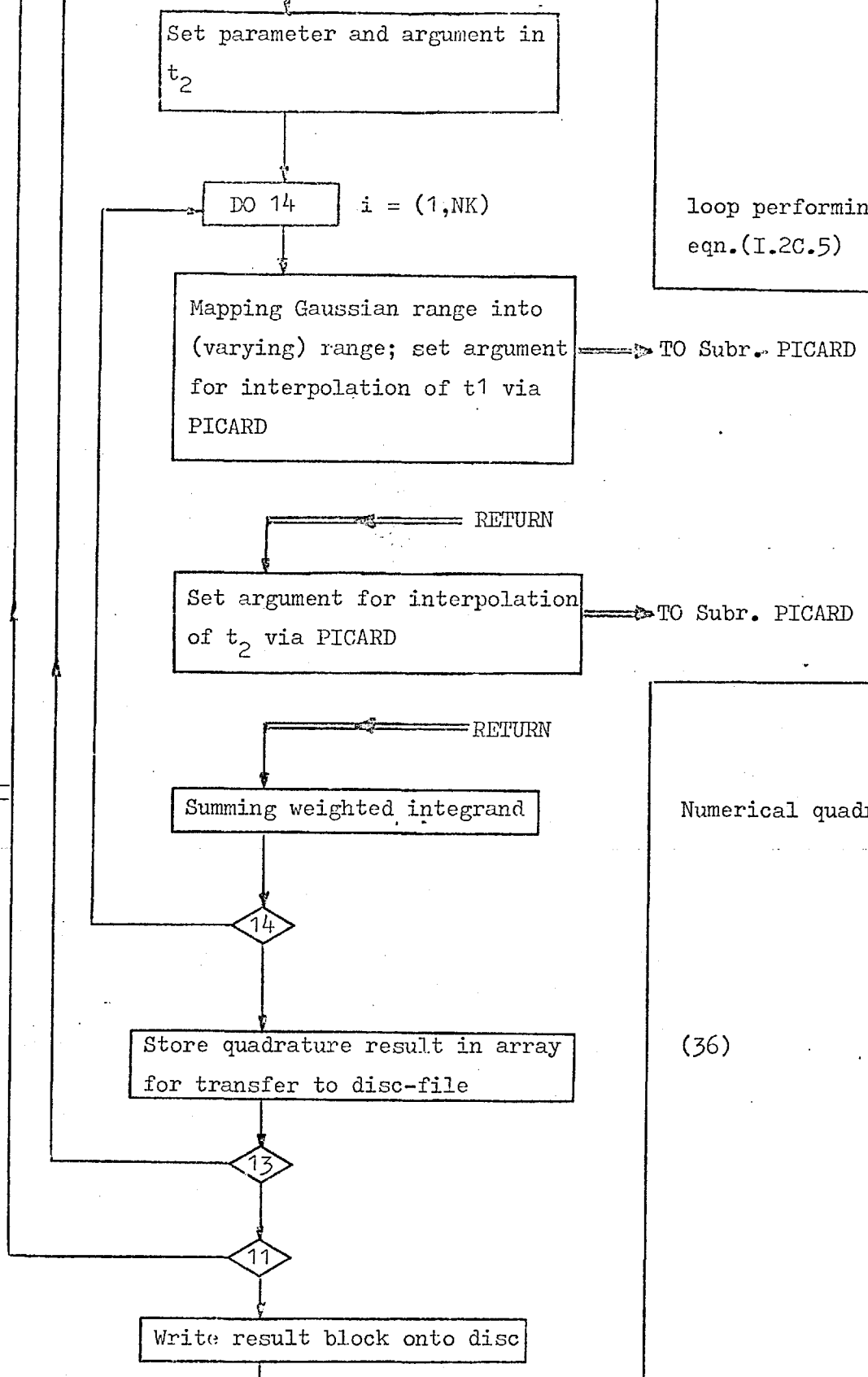
recovery of individual solution

131

10

12

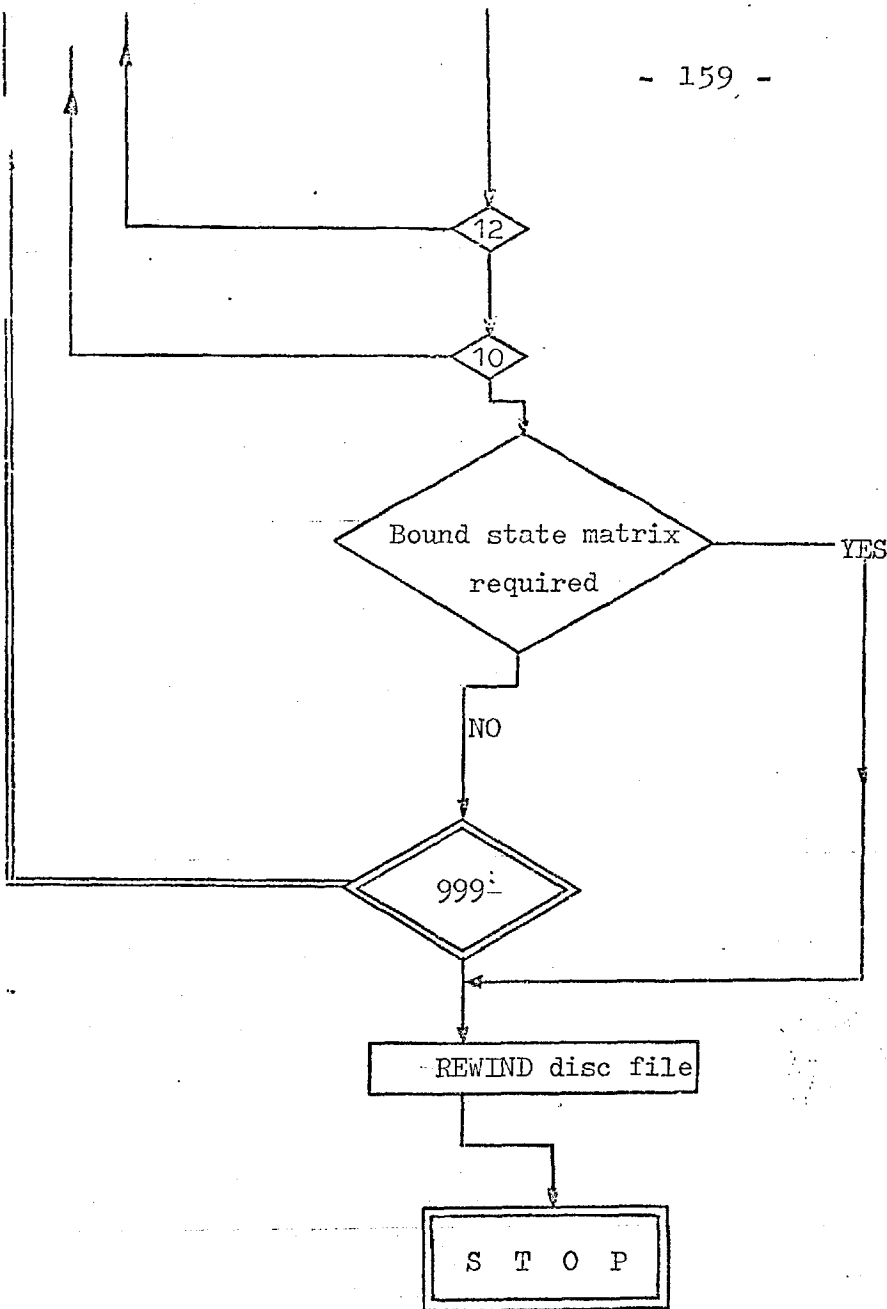
11



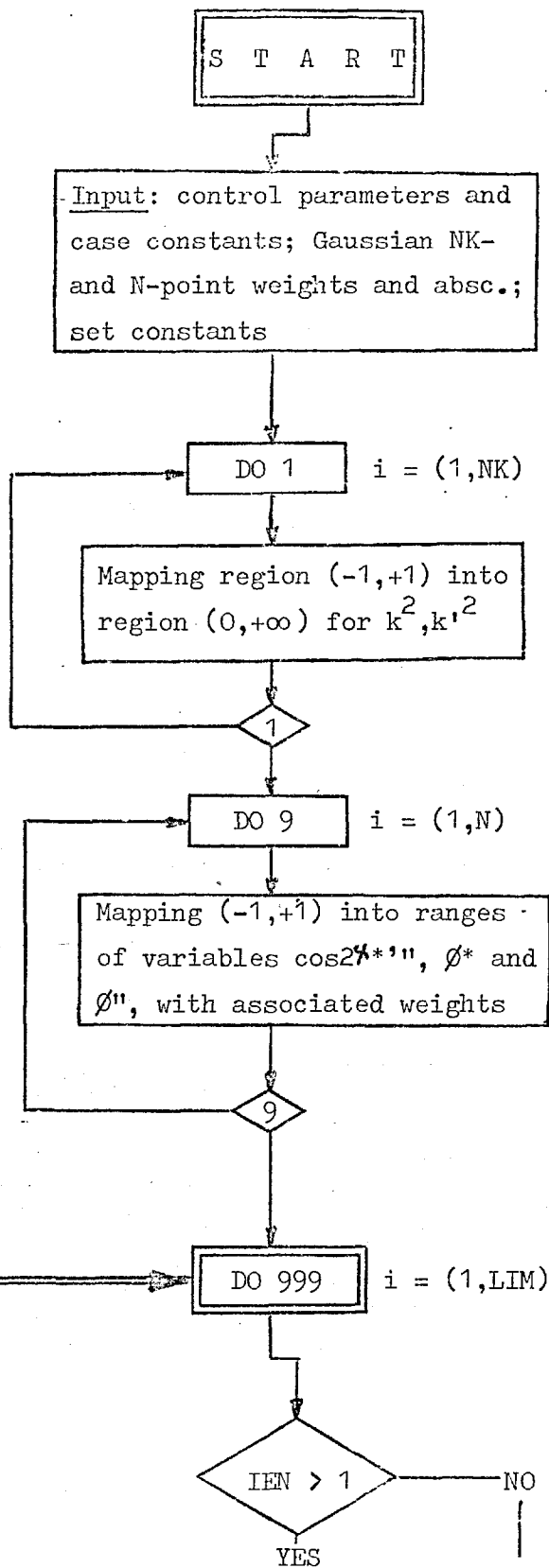
loop performing integration
eqn.(I.2C.5)

Numerical quadrature

(36)

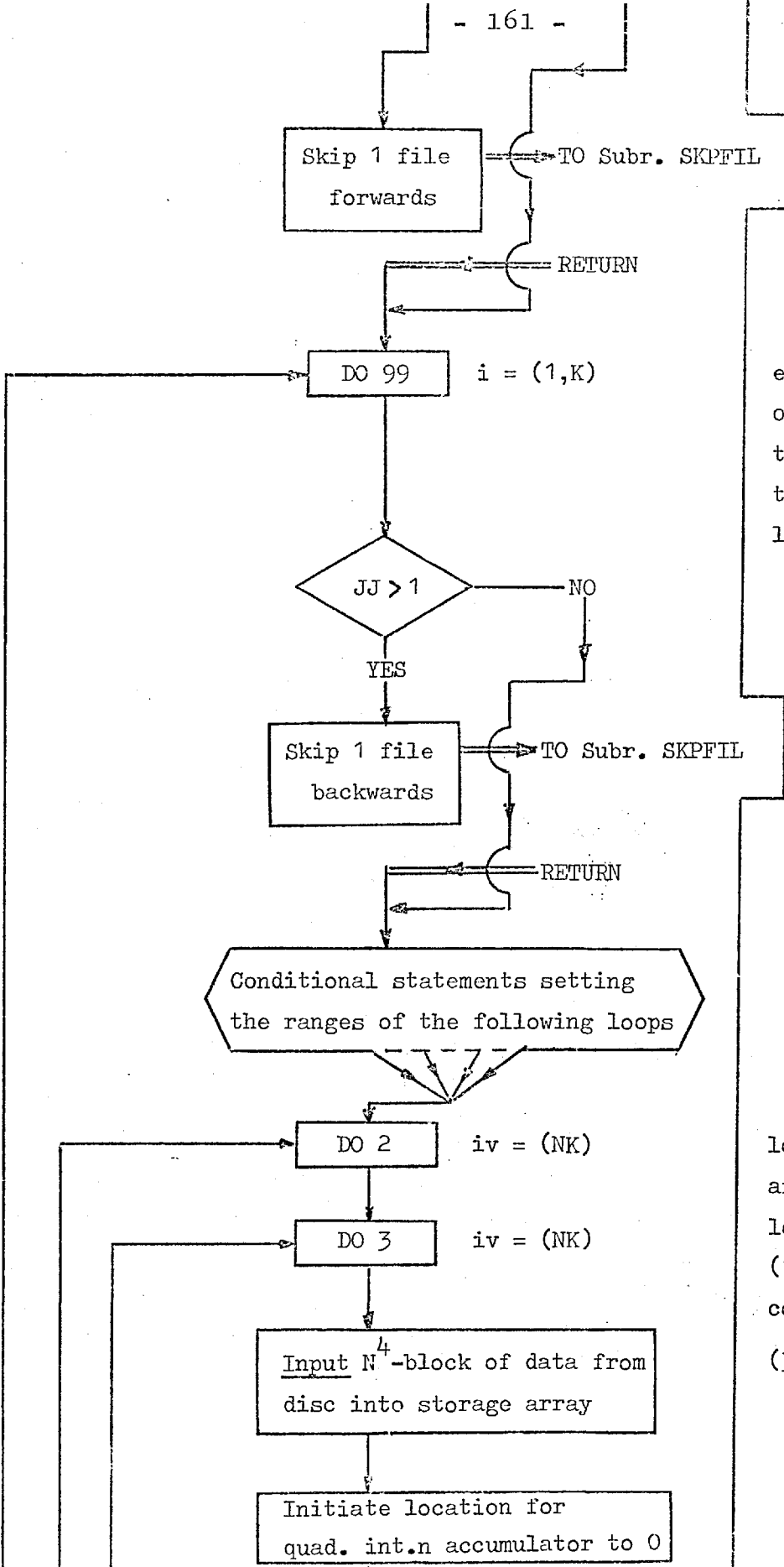


Comments and References



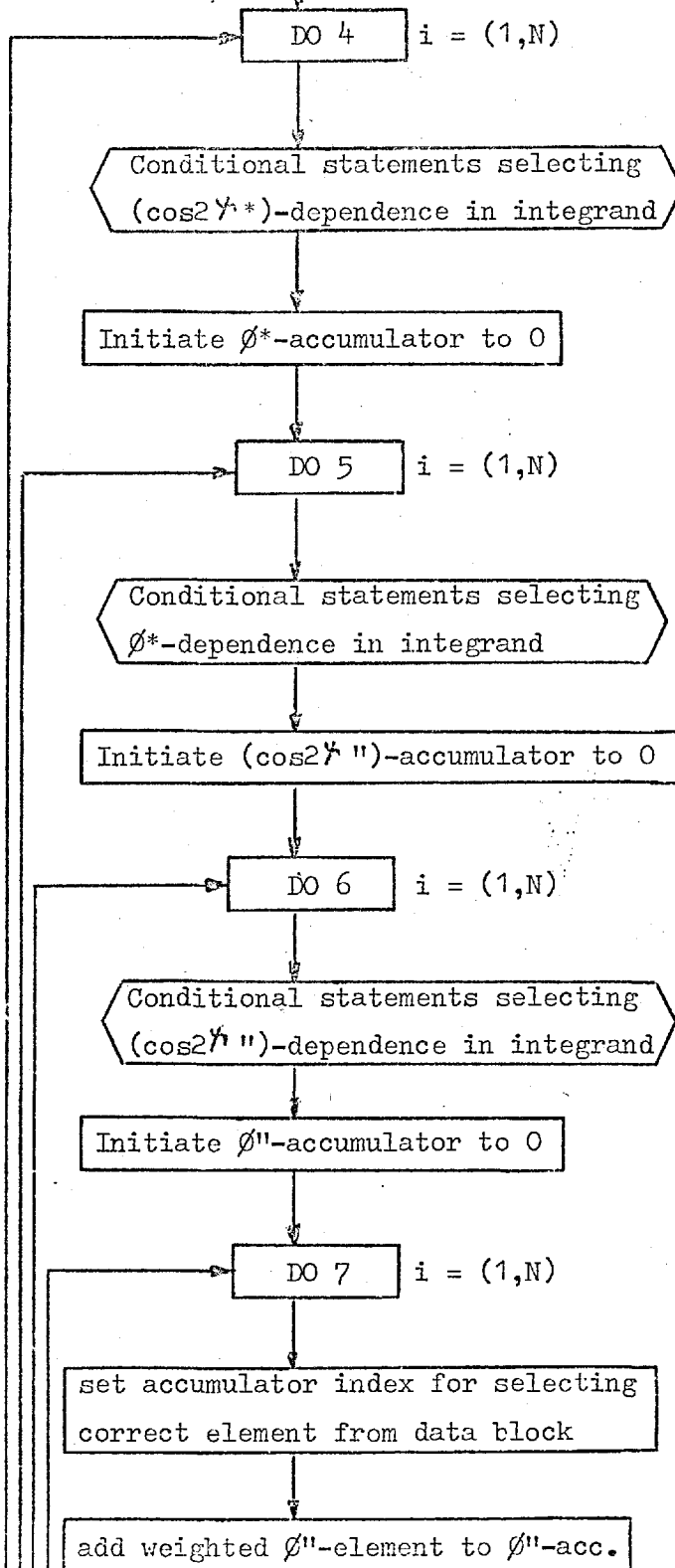
(2) ÷ (7)

loop-index = IEN; LIM = no.
of energy values to be pro-
cessed = no. of files on
disc from PART I



eqn.(III.1B.1); K = no.
of submatrices = 4 with
two irred. repres.s = 9 wit
three irred. repres.s;
loop-index = JJ; p.58ff

loops 2 and 3 fill matrix
array (1) PART II; each
loop has a total range of
(1, K^2N) since the array
comprises $K \cdot NK^2$ locations
(36) non-input PART I



(cos2Y*)-loop

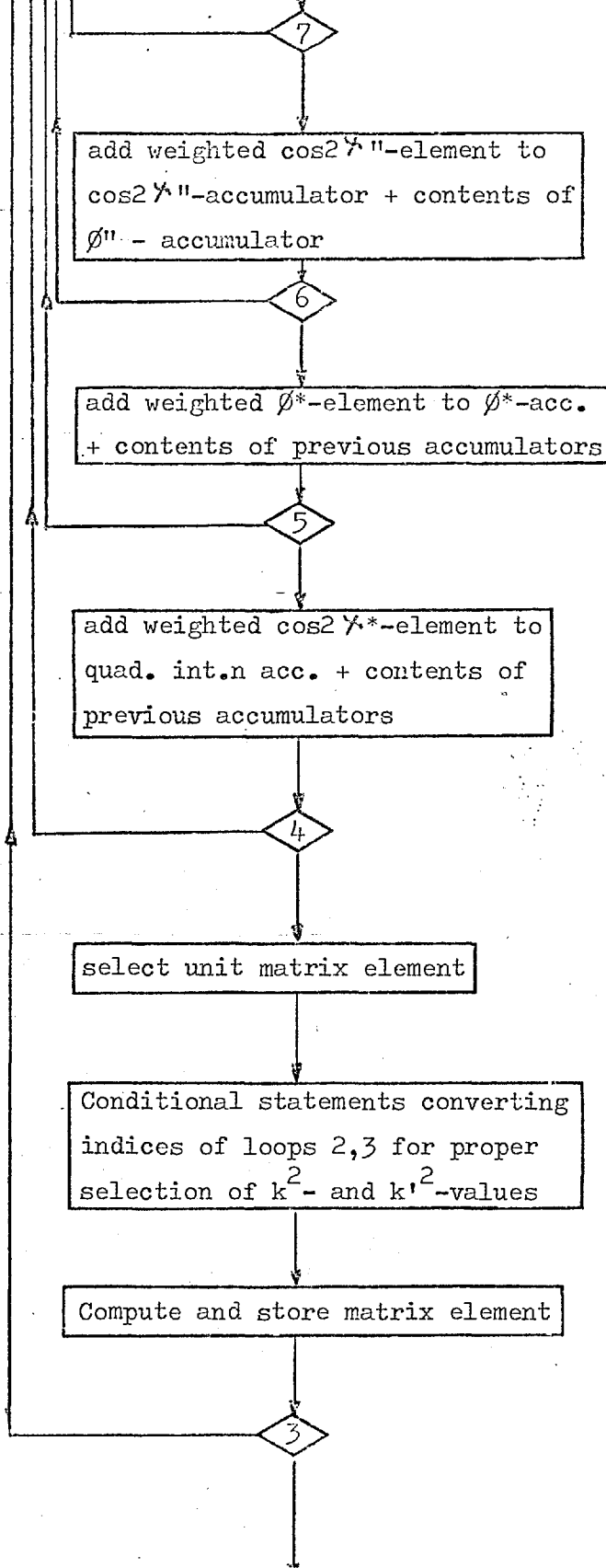
p.59

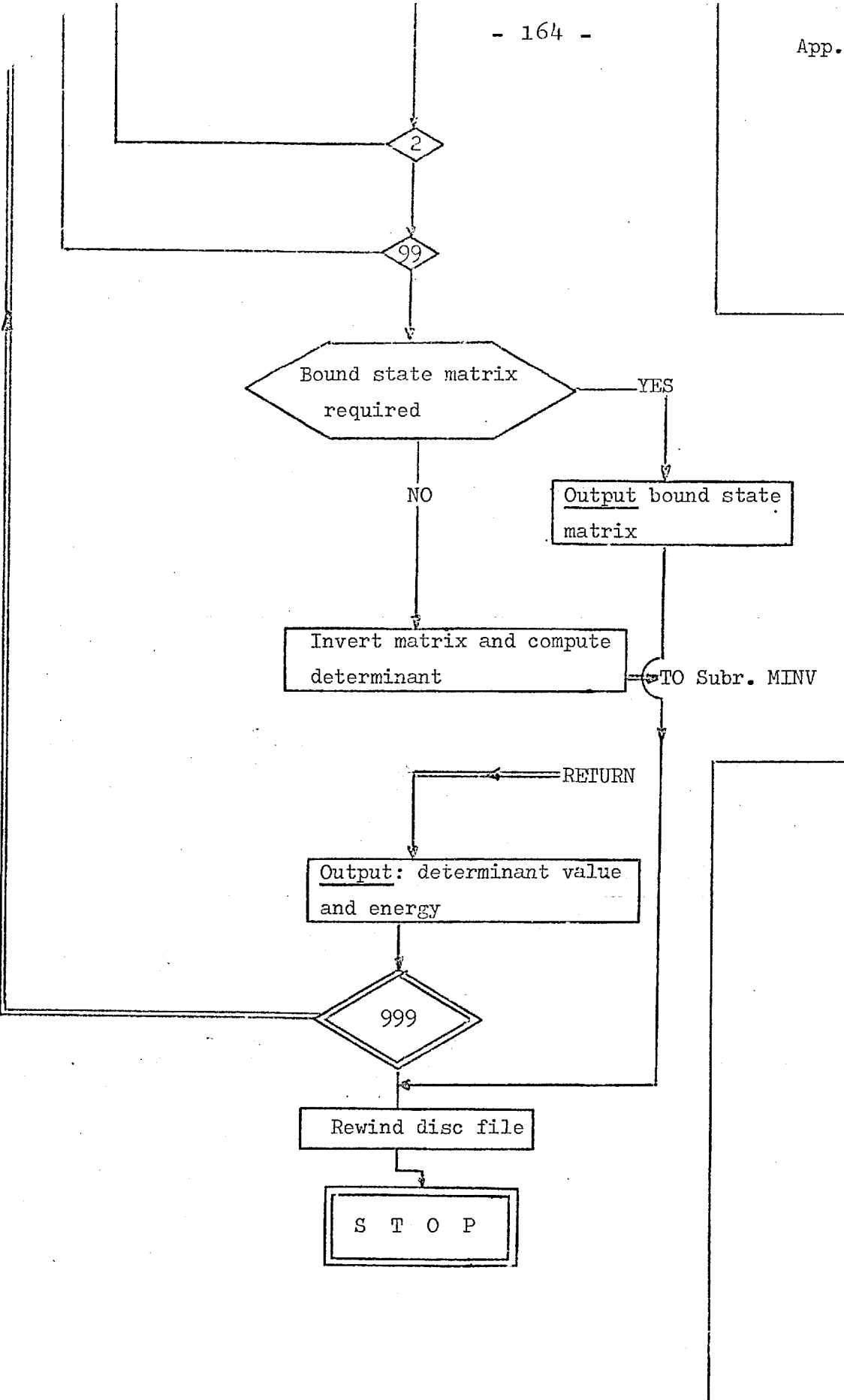
Ø*-loop

cos2Y"-loop

Ø"-loop

(36) PART I; index = JY





SELECTIVE LIST OF VARIABLES

Below we list, with comments and references, all the input variables, arrays and arguments of subroutine-calls, occurring in the program.

At present, it consists of three decks, named PART 1, PART 2 and LAM6 respectively. The two latter decks differ in the number of irreducible representations, i.e. values of λ , λ' included, which are 0 and 4 for PART 2 and 0, 4 and 6 for LAM6. The coding differences are marginal and self-evident and are not included in this list; they follow from the program-listing included in this appendix.

Variables fitted with * are used with the same meaning in all three parts. The array dimensions are stated in brackets and given in terms of the constants NK, N and M.

PART 1Input Variables

No.	Variable	Type	Remarks	Ref.
1	SPIN *	logical	Set to 'TRUE' for spin-isospin, 'FALSE' otherwise.	
2	YUK	"	Switch variable for selection of a particular two-body potential; see data sheet.	
3	YAM	"		
4	EXPO	"		
5	BSMAT *	"	'TRUE' if bound-state matrix $\bar{K}$ is required, 'FALSE' otherwise.	Sn.III.2
6	N *	integer	Number of weight factors and abscissae for Dalitz-variables $\psi^*(\psi), \varphi^*, \varphi$	p.61ff
7	NK *	"	Same as N, but for variables k^2, k'^2	
8	LIM *	"	The number of total energy values E to be processed (usually 1÷5) for det=det(E)	p.68
9	GS	real	Singlet-state coupling constant, pair-potent.	Table 4
10	GT	"	Triplet-state coupling constant, pair-potent.	Table 4
11	AMS	"	Singlet-state range parameter, pair-potential (squared)	Table 4

No.	Variable	Type	Remarks	Ref.
12	AMT	real	Triplet-state range parameter, pair-potential (squared).	Table 4
13	EB *	"	Value of bound-state energy, input only if BSMAT='TRUE', 0.0 otherwise.	
14	ESTART *	"	Starting value of E for bound-state scanning (first value of energy E + DE).	
15	DE *	"	Energy interval.	
16	WT(N) *	real array	Input of Gaussian N-point weight factors.	p.65
17	Y(N)	" "	Input of Gaussian N-point abscissae.	
18	WK(NK)	" "	Input of Gaussian NK-point weight factors.	
19	YK(NK)	" "	Input of Gaussian NK-point abscissae.	
20	WTO(M)	" "	Input of Gaussian M-point weight factors (usually M=10)	
21	YO(M)	" "	Input of Gaussian M-point abscissae.	

Non-Input Arrays

No.	Variable	Type	Remarks	Ref.
1	W(NK) *	real	Weight factors, mapped from Gaussian range $(-1,+1)$ into k^2 -range $(0, +\infty)$	p.47
2	X(NK) *	"	Abscissae corresponding to W(NK)	p.47
3	SX(NK)	"	Square-root (X)	
4	W1(NK*N ²)	"	Set of arrays, containing all possible values of the Omnes-arguments $\omega_1, \omega_2, \omega_3$ arising from the Dalitz-set (k^2, ψ^*, φ^*)	11.1.6
5	W2(NK*N ²)	"		
6	W3(NK*N ²)	"		
7	WP1(NK*N ²)	"	Equivalent to arrays 4,5,6 for the arguments $\omega_1^!, \omega_2^!, \omega_3^!$ arising from $(k^{!2}, \psi^!, \varphi^!)$	
8	WP2(NK*N ²)	"		
9	WP3(NK*N ²)	"		
10	W10(M ³)	"	Equivalent to arrays 4,5,6 for M-point Gaussian data, $M \geq NK \geq N$, for computation of pair-amplitudes for higher accuracy.	p.65
11	W20(M ³)	"		
12	W30(M ³)	"		
13	WP10(M ³)	"	Equivalent to arrays 7,8,9 for M-point Gaussian.	
14	WP20(M ³)	"		
15	WP30(M ³)	"		
16	T1(NK)	"	Array for storing triplet-state $t_1(\eta_1, \eta_1^!; s)$ (NK-component vector).	p.64ff
17	T2(NK)	"	Same as array 16, but for $t_2(\eta_2, \eta_2^!; s)$	

No.	Variable	Type	Remarks	Ref.
18	S1(NK)	real	Same as array 16, but for singlet-state.	p.67
19	S2(NK)	"	Same as array 17, but for singlet-state.	
20	STFNT(M)	"	Array for input of starting values to iteration routine for solving t_1 and t_2 , triplet-case only.	p.50
21	STFNS(M)	"	Same as array 20, but for singlet-case.	
22	SX1(1)	"	Input argument $\eta_1 = \eta_1(\omega_1, \omega_2^!, \omega_3^!)$ for interpolation of t_1	p.63ff
23	SX2(1)	"	Same as array 22, argument $\eta_2 = \eta_2(\omega_2^!, \omega_3^!, \omega_1)$ for t_2	
24	PF1(1)	"	Return interpolated t_1 -value (triplet).	
25	PF2(1)	"	Return interpolated t_2 value (triplet).	
26	PG1(1)	"	Return interpolated t_1 value (singlet).	
27	PG2(1)	"	Return interpolated t_2 value (singlet).	
28	TT1(NK(NK+3)*N ²)		Storage of NK*N ²) NK-point triplet $t_1(\eta_1, \eta_1^!; s)$, each solution being stored with its respective <u>parameters</u> $\omega_1^!, \omega_2^!, \omega_3^!$ for $\eta_1^!$	

No.	Variable	Type	Remarks	Ref.
29	TT2(NK(NK+3)*N ²)		Same as array 28, but for $t_2(\eta_2, \eta'_2; s)$	
30	TS1(NK(NK+3)*N ²)	}	Same as arrays 27, 28 but for singlet-states.	
31	TS2(NK(NK+3)*N ²)			
32	T1W((NK+3)*N ²)		Intermediate storage area for the N ² NK-point solutions of t_1 , belonging to a particular set (k^2, k'^2) , triplet-state.	
33	T2W((NK+3)*N ²)		Same as array 32, but for t_2	
34	S1W((NK+3)*N ²)	}	Same as arrays 32,33 but for singlet state, respectively.	
35	S2W((NK+3)*N ²)			
36	TINT(N ⁴)		Storage area for the n ⁴ elements $\langle k^2, \psi^* \varphi^* K(E) k'^2, \psi'' \varphi'' \rangle$ belonging to a particular set of k^2, k'^2 , for transfer to and from mag-tape.	p.60ff

CALL of Pair-Potential Functions $v_o(k, k')$

CALL: <Name of potential> (U,V,CF,AMU)

Name either YUKAWA, YAMAG or EXPNTL

1	U	Δ	k
2	V	Δ	k'
3	CF	Δ	coupling constant * appropriate factors
4	AMU	Δ	(range-parameter) ²

CALL of Picard-Routine, performing iterative solution or
interpolation of pair-amplitudes.

CALL: PICARD(PF,WF,ZI,WP,IT,E,P1,IN,NF,ZP,GF,ANU,YK,YG,EP)

1	PF(1)	dummy-dimensioned(dd) output array for three O/P options: 1) NK-point solution of pair-amplitude function. 2) M-point solution of pair-amplitude function. 3) single result of interpolation.
2	WP(1)	dd-array for input of weight factors.
3	ZI(1)	dd-array for input of abscissae in iteration process.
4	WF(1)	dd-array for input of starting function (see arrays 20 and 21, PART 1).
5	IT	option indicator (1 for iteration, 0 for interpolation).
6	E	real-input of two-body subsystem energy s
7	P1	input (real) of 'parameter' value η
8	IN	integer number of points to be interpolated
9	NF	integer number of intervals in integration range
10	ZP(1)	input of argument, for which amplitude function is to be interpolated
11	GF	input (real) of pair-potential coupling constant * factors

12	ANU	input (real) of pair-potential (range parameter) ²
13	YK	logical selectors of pair-potential ('TRUE' or 'FALSE')
14	YG	
15	EP	

PART 2 (LAM 6)

(The list below contains only those declared variables not mentioned in the list of PART 1.)

No.	Variable	Type	Remarks	Ref.
1	AM(2NK,2NK)	real array	two-dimensional array for matrix $\bar{K}_{\lambda,\lambda}$, dimensions = (3NK,3NK) in LAM6	p.66ff
2	PSI(N)	real array	storage area for N arguments $\cos\psi^*$ (same as $\cos\psi''$)	III.1B.10
3	WPSI(N)	real array	weight factors belonging to PSI(N)	
4	PHI(N)	real array	arguments φ^*	III.1B.11
5	WPHI(N)	real array	weight factors belonging to PHI(N)	
6	PHJ(N)	real array	arguments φ''	III.1B.12
7	WPHJ(N)	real array	weight factors belonging to PHJ(N)	

No.	Variable	Type	Remarks	Ref.
8	YT(N)	real array	array corresponding to array 17 PART 1	
9	NU(2NK)	integer array	working space for inversion-routine MINV	
10	NV(2NK)	integer array	dimensions=(3NK) in LAM6	

CALL of SSP Subroutine MINV

CALL: MINV(A,N,D,L,M)

1	A	<u>A</u>	input array for AM(,), double precision
2	N	<u>A</u>	2NK for PART 2, 3NK for LAM6
3	D	<u>A</u>	returned value of determinant, double precision
4	L	<u>A</u>	NU()
5	M	<u>A</u>	NV()

CALL of SKPFIL-Routine; this routine searches for a specified file on mag-tape, by skipping end-of-file marks, forwards and backwards.

CALL: SKPFIL(IFIL,IUNIT)

1	IFIL		integer number of e.o.f.-marks to be skipped, +ve for forwards, -ve for backwards
2	IUNIT		symbolic tape unit number (chosen as 7 in this program)

PROGRAM PART1(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE7)

C
C
C***** PART 1 : COMPUTATION OF MATRIX ELEMENTS K(E)-N *****

C
C
C
C THE FOLLOWING DIMENSIONS APPLY TO THE SPECIFIC CHOICE OF
C PARAMETERS

C NK = 8 , N = 5 , M = 10 DENOTED BY (8,5,10).

C MODIFICATION OF THIS SET ENTAILS ALTERING THE ARRAY DIMENSIONS.
C TYPICAL SETS AND CORRESPONDING DIMENSIONS ARE LISTED BELOW.

C
C (8,5,10) : 8 5 10 200 275 1000 2200 625
C
C (5,5,10) : 5 5 10 125 200 1000 1000 625
C
C (5,5,20) : 5 5 20 125 200 8000 1000 625
C
C (7,7,10) : 7 7 10 343 490 1000 3430 2401
C
C (9,9,10) : 9 9 10 729 972 1000 7280 6561
C
C

DIMENSION WT(5),W(8),Y(5),X(8),SX(8),W1(200),W2(200),W3(200),T1(8) MOOPT1
1,T2(8),S1(8),STFNT(20),STFNS(20),PF1(1),S2(8),PF2(1),PG1(1),PG2(1) PART1
2,SX1(1),SX2(1),T1NT(625), T1W(275),T2W(275),S1W(275),S2W(275),W10(PART1
310),W0(10),Y0(10),X0(10),SX0(10),W10(1000),W20(1000),W30(1000),T10 PART1
4(10),T20(10),WP1(200),WP2(200),WP3(200),WP10(1000),WP20(1000),WP30 PART1
5(1000),T11(2200),T12(2200),TS1(2200),TS2(2200),YK(8),WK(8) PART1

00002
00003
00004
00005
00006
00007
00008
00009
00010
00011
00012
00013
00014
00015
00016
00017
00018
00019
00020
00021
00022
00023
00024
00025
00001
00027
00028
00029
00030
00031

```

C      LOGICAL SPIN,YUK,YAM,BSMAT,EXPO
C
C      COMMON SPI5,SPI3,SPI7
C      READ(5,1002) SPIN,YUK,YAM,EXPO,BSMAT
C      READ(5,1006) N,NK,LIM,GS,GT,AMS,AMT,EB,ESTART,DE
C
C      *WRITE(6,9090)
C      9090 FORMAT(1H///,50X,*DATE OF SUBMITTING JOB:*,//,55X,
C      1 * 16 DECEMBER 1970*,//)
C
C      G = 10
C      NK2 = N*N*NK
C      NP3 = NK + 3
C      N2 = N**2
C      N3 = N**3
C      N4 = N2**2
C      TWOPI = 2*3.14159265358979
C      T3 = TWOPI/3.
C      T4 = TWOPI/4.
C      T6 = T3/2.

```

```

PART1 00032
PART1 00033
PART1 00034
PART1 00035
PART1 00036
PART1 00037
MODPT1 00002
MODPT1 00003
MODPT1 00004
MODPT1 00005
MODPT1 00006
MODPT1 00007
MODPT1 00008
MODPT1 00009
MODPT1 00010
PART1 00038
PART1 00039
PART1 00040
PART1 00041
PART1 00042
PART1 00043
PART1 00044
PART1 00045
PART1 00046
PART1 00047

```

```

SPI3 = SQRT(TWOPI)**3
SPI5 = (SQRT(TWOPI)**5)
SPI7 = (SQRT(TWOPI)**7)
TUPI3 = TWOPI**3
IF(YUK) CFT = GT/(2.*SPI7)
IF(YUK) CFS = GS/(2.*SPI7)
IF(YAK) CFT = GT/TUPI3
IF(YAK) CFS = GS/TUPI3
IF(EXPO) CFT = GT*4.0/SPI7
IF(EXPO) CFS = 4.0*GS/SPI7
IP4 = 2.*(TWOPI**2)
READ(5,1000) (WT(N1),N1=1,N)
READ(5,1000) (Y(L1),L1=1,N)
READ(5,1000) (WK(NN1),NN1=1,NK)
READ(5,1000) (YK(NN2),NN2=1,NK)
READ(5,1000) (WTO(M10),M10=1,M)
READ(5,1000) (Y0(M20),M20=1,M)
DO 19 M3=1,M
X0(M3) = (1.+Y0(M3))/(1.-Y0(M3))
SX0(M3) = SQRT(X0(M3))
19 W0(M3) = .5*WTO(M3)*((1.+X0(M3))**2)
WRITE(6,1019) (K2,WTO(K2),Y0(K2),X0(K2),K2=1,M)

```

```

C
C**** COMPUTATION OF 10-POINT ARGUMENTS ,FOR HIGHER PRECISION *****
MSG = M**2
DO 20 I20=1,M
P = X0(I20)
P3 = P/3.
DO 30 I30=1,M
PSI = (Y0(I30) + 1.)/2.

```

```

PART1 00048
PART1 00049
PART1 00050
PART1 00051
PART1 00052
PART1 00053
PART1 00054
PART1 00055
PART1 00056
PART1 00057
PART1 00058
PART1 00059
PART1 00060
PART1 00061
PART1 00062
PART1 00063
PART1 00064
PART1 00065
PART1 00066
PART1 00067
PART1 00068
PART1 00069
PART1 00070
PART1 00071
PART1 00072
PART1 00073
PART1 00074
PART1 00075
PART1 00076
PART1 00077

```

```

DO 40 I40=1,M
PHI = T4*(Y0(I40)+1.) - T6
PHJ = T4*(Y0(I40)+1.) - T3
AM = PHI - T3
AP = PHI + T3
BM = PHJ - T3
BP = PHJ + T3
I50 = I40 + (I30-1)*M + (I20-1)*MS0
W10(I50) = P3*(1.+PSI*COS(AM))
W20(I50) = P3*(1.+PSI*COS(AP))
W30(I50) = P3*(1.+PSI*COS(PHI))

C
WP10(I50) = P3*(1.+PSI*COS(BM))
WP20(I50) = P3*(1.+PSI*COS(BP))
WP30(I50) = P3*(1.+PSI*COS(PHJ))
40 CONTINUE
30 CONTINUE
20 CONTINUE
190 DO 1 I = 1,NK
X(I) = (1. + YK(I))/(1. - YK(I))
SX(I) = SORT(X(I))
1 W(I) = WK(I)*((1.+X(I))**2)*.5
WRITE(5,1019)(K1,WK(K1),YK(K1),X(K1),K1=1,NK)
C***** COMPUTATION OF W1, W2, W3 (N POINTS) *****
DO 2 I2=1,NK

```

```

PART1 00078
PART1 00079
PART1 00080
PART1 00081
PART1 00082
PART1 00083
PART1 00084
PART1 00085
PART1 00086
PART1 00087
PART1 00088
PART1 00089
PART1 00090
PART1 00091
PART1 00092
PART1 00093
PART1 00094
PART1 00095
PART1 00096
PART1 00097
PART1 00098
PART1 00099
PART1 00100
PART1 00101
PART1 00102

```

```

P = X(I2)
P3 = P/3.
DO 3 I3=1,N
PSI = (Y(I3)+1.)/2.
DO 4 I4=1,N
PHI = T4*(Y(I4)+1.) - T6
PHJ = T4*(Y(I4)+1.) - T3
AM = PHI - T3
AP = PHI + T3
BM = PHJ - T3
BP = PHJ + T3
IS = I4+(I3-1)*N+(I2-1)*N2
W1(I5) = P3*(1.+PSI*COS(AM))
W2(I5) = P3*(1.+PSI*COS(AP))
W3(I5) = P3*(1.+PSI*COS(PHI))
C
WP1(I5) = P3*(1.+PSI*COS(BM))
WP2(I5) = P3*(1.+PSI*COS(BP))
WP3(I5) = P3*(1.+PSI*COS(PHJ))
1 CONTINUE
3 CONTINUE
2 CONTINUE
C
C
C
C ***** ENERGY VARIATION LOOP, IF BSMAT FALSE *****
C
ENGY = ESTART
DO 999 IEN = 1,LIM
C

```

```

PART1 00103
PART1 00104
PART1 00105
PART1 00106
PART1 00107
PART1 00108
PART1 00109
PART1 00110
PART1 00111
PART1 00112
PART1 00113
PART1 00114
PART1 00115
PART1 00116
PART1 00117
PART1 00118
PART1 00119
PART1 00120
PART1 00121
PART1 00122
PART1 00123
PART1 00124
PART1 00125
PART1 00126
PART1 00127
PART1 00128
PART1 00129
PART1 00130
PART1 00131
PART1 00132

```

```

      ENGY = ENGY + DE
C
      IF (BSMAT) ENGY = EB
C
C***** SOLUTIONS OF T-1 AND T-2 *****
C
      DO 6 I6=1,NK2
      ETA2 = WP1(I6) + WP3(I6) - .5*WP2(I6)
      SET = SQRT(ETA2)
      EN2 = ENGY - 1.5*WP2(I6)
C***** STARTING FUNCTION FOR PICARD-LOOP *****
      DO 7 I7=1,M
      S = SX0(I7)
      IF (YUK) VLT = YUKAWA(S,SET,CFT,AMT)
      IF (YAM) VLT = YAMAG(S,SET,CFT,AMT)
      IF (EXPO) VLT = EXPNTL(S,SET,CFT,AMT)
      IF (SPIN.AND.YUK) VLS = YUKAWA(S,SET,CFS,AMS)
      IF (SPIN.AND.YAM) VLS = YAMAG(S,SET,CFS,AMS)
      IF (SPIN.AND.EXPO) VLS = EXPNTL(S,SET,CFS,AMS)
      STFNT(I7) = SPI3*VLT
      IF (SPIN) STFNS(I7) = VLS*SPI3
7 CONTINUE
C
      CALL PICARD(T20,STFNT,SX0,W0,1,EN2,SET,M,M,SX0,CFT,AMT,YUK,YAM,EXP
10)

```

```

PART1 00133
PART1 00134
PART1 00135
PART1 00136
PART1 00137
PART1 00138
PART1 00139
PART1 00140
PART1 00141
PART1 00142
PART1 00143
PART1 00144
PART1 00145
PART1 00146
PART1 00147
PART1 00148
PART1 00149
PART1 00150
PART1 00151
PART1 00152
PART1 00153
PART1 00154
PART1 00155
PART1 00156
PART1 00157

```

CALL PICARD(T2,T20,SX0,W0,0,EN2,SET,NK,M,SX,CFT,AMT,YUK,YAM,EXPO)	PART1	00158
IF(.NOT.SPIN) GO TO 60	PART1	00159
CALL PICARD(T20,STFNS,SX0,W0,1,EN2,SET,M,M,SX0,CFS,AMS,YUK,YAM,EXP	PART1	00160
10)	PART1	00161
CALL PICARD(S2,T20,SX0,W0,0,EN2,SET,NK,M,SX,CFS,AMS,YUK,YAM,EXPO)	PART1	00162
C	PART1	00163
60 ETA1 = W2(I6) + W3(I6) - .5*W1(I6)	PART1	00164
SET = SQRT(ETA1)	PART1	00165
EN1 = ENGY - 1.5*W1(I6)	PART1	00166
C***** STARTING FUNCTION FOR PICARD LOOP *****	PART1	00167
DO 9 I9=1,M	PART1	00168
S = SX0(I9)	PART1	00169
IF(YUK) VLT = YUKAWA(S,SET,CFT,AMT)	PART1	00170
IF(YAM) VLT = YAMAG(S,SET,CFT,AMT)	PART1	00171
IF(EXPO) VLT = EXPNTL(S,SET,CFT,AMT)	PART1	00172
IF(SPIN.AND.YUK) VLS = YUKAWA(S,SET,CFS,AMS)	PART1	00173
IF(SPIN.AND.YAM) VLS = YAMAG(S,SET,CFS,AMS)	PART1	00174
IF(SPIN.AND.EXPO) VLS = EXPNTL(S,SET,CFS,AMS)	PART1	00175
STFNT(I9) = SPI3*VLT	PART1	00176
IF(SPIN) STFNS(I9) = VLS*SPI3	PART1	00177
9 CONTINUE	PART1	00178
C	PART1	00179
CALL PICARD(T10,STFNT,SX0,W0,1,EN1,SET,M,M,SX0,CFT,AMT,YUK,YAM,EXP	PART1	00180
10)	PART1	00181
CALL PICARD(T1,T10,SX0,W0,0,EN1,SET,NK,M,SX,CFT,AMT,YUK,YAM,EXPO)	PART1	00182
IF(.NOT.SPIN) GO TO 61	PART1	00183
CALL PICARD(T10,STFNS,SX0,W0,1,EN1,SET,M,M,SX0,CFS,AMS,YUK,YAM,EXP	PART1	00184
10)	PART1	00185
CALL PICARD(S1,T10,SX0,W0,0,EN1,SET,NK,M,SX,CFS,AMS,YUK,YAM,EXPO)	PART1	00186
C	PART1	00187

61 NS = 1 + (I6- 1)*NP3

NS1 = NS + 1

NS2 = NS + 2

TT1(NS) = W1(I6)

TT1(NS1) = W2(I6)

TT1(NS2) = W3(I6)

C

TT2(NS) = WP2(I6)

TT2(NS1) = WP3(I6)

TT2(NS2) = WP1(I6)

DO 65 I66=1,NK

J66 = I66 + NS2

TT1(J66) = T1(I66)

TT2(J66) = T2(I66)

IF(.NOT.SPIN) GO TO 66

TS1(J66) = S1(I66)

TS2(J66) = S2(I66)

66 CONTINUE

6 CONTINUE

C
C
C
C
C
C

PART1 00188
PART1 00189
PART1 00190
PART1 00191
PART1 00192
PART1 00193
PART1 00194
PART1 00195
PART1 00196
PART1 00197
PART1 00198
PART1 00199
PART1 00200
PART1 00201
PART1 00202
PART1 00203
PART1 00204
PART1 00205
PART1 00206
PART1 00207
PART1 00208
PART1 00209
PART1 00210
PART1 00211
PART1 00212

C***** CREATION OF TAPE-FILE FOR N+6 VALUES OF K(E)-N *****

C
C
C

ITN = NK2+ 3*N2

C***** LOOP FOR THE N BLOCKS OF T-1 DATA *****

DO 10 I10 = 1,NK

C***** TRANSFER OF T-1 BLOCK *****

IS1 = 1 + (I10-1)*ITN

IF1 = I10*ITN

ITIN = (I10-1)*ITN

DO 110 I110=IS1,IF1

J110 = I110 - ITIN

IF(.NOT.SPIN) GO TO 110

SW(J110) = TS1(I110)

110 TT1(J110) = TT1(I110)

C
C***** LOOP FOR THE N BLOCKS OF T-2 DATA *****

DO 12 I12=1,NK

C***** TRANSFER OF T-2 BLOCK *****

IS2 = 1 + (I12-1)*ITN

IF2 = I12*ITN

JITN = (I12-1)*ITN

DO 220 I220=IS2,IF2

J220 = I220 - JITN

IF(.NOT.SPIN) GO TO 220

SW(J220) = TS2(I220)

220 TT2(J220) = TT2(I220)

C

C***** OUTER T-1 LOOP FOR PROCESSING *****

PART1 00213
PART1 00214
PART1 00215
PART1 00216
PART1 00217
PART1 00218
PART1 00219
PART1 00220
PART1 00221
PART1 00222
PART1 00223
PART1 00224
PART1 00225
PART1 00226
PART1 00227
PART1 00228
PART1 00229
PART1 00230
PART1 00231
PART1 00232
PART1 00233
PART1 00234
PART1 00235
PART1 00236
PART1 00237
PART1 00238
PART1 00239
PART1 00240
PART1 00241
PART1 00242

```

DO 11 I11=1,N2
NIS = 1 + (I11-1)*NP3
NIS1 = NIS + 1
NIS2 = NIS + 2
C***** TRANSFER OF SINGLE SOLUTIONS OF T-1 *****
O1 = T1W(NIS)
O2 = T1W(NIS1)
O3 = T1W(NIS2)
DO 111 I111=1,NK
J111 = I111 + NIS2
T1(I111) = T1W(J111)
IF(.NOT.SPIN) GO TO 111
S1(I111) = S1W(J111)
111 CONTINUE
C
EAT = O1 + O2 + O3
EN1 = ENGY - 1.5*O1
SE = O2 + O3 - .5*O1
ETA1 = SQRT(SE)
C
NZI11 = (I11-1)*N2
C
C***** INNER T-2 LOOP FOR PROCESSING *****
DO 13 I13=1,N2
NJS = 1 + (I13-1)*NP3

```

```

PART1 00243
PART1 00244
PART1 00245
PART1 00246
PART1 00247
PART1 00248
PART1 00249
PART1 00250
PART1 00251
PART1 00252
PART1 00253
PART1 00254
PART1 00255
PART1 00256
PART1 00257
PART1 00258
PART1 00259
PART1 00260
PART1 00261
PART1 00262
PART1 00263
PART1 00264
PART1 00265
PART1 00266
PART1 00267

```

```

      NJS1 = NJS + 1
      NJS2 = NJS + 2
C***** TRANSFER OF SINGLE T-2 SOLUTION *****
      O2P = T2W(NJS)
      O3P = T2W(NJS1)
      O1P = T2W(NJS2)
      DO 131 I131=1,NK
      J131 = I131 + NJS2
      T2(I131) = T2W(J131)
      IF(.NOT.SPIN) GO TO 131
      S2(I131) = S2W(J131)
131 CONTINUE
C
      EN2 = ENGY - 1.5*O2P
      SE = O1P + O3P - .5*O2P
      E1A2 = SQRT(SE)
      ST12 = 2.*SQRT(O1*O2P)
      O12P = O1 + O2P
      O2P1 = O2P - .5*O1
      O21P = O1 - .5*O2P
C***** INTEGRATION AND PICARD INTERPOLATION *****
      TT = .0
      DO 14 I14=1,NK
      OLP = ST12*YK(I14) + O12P
      E1DP = ODP + O2P1
      E2DP = ODP + O21P
      SX1(1) = SQRT(E1DP)
      SX2(1) = SQRT(E2DP)
C
      CALL PICARD(PF1,T1,SX,W ,0,EN1,ETA1,1,NK,SX1,CFT,AMT,YUK,YAM,EXPU)

```

```

PART1 00268
PART1 00269
PART1 00270
PART1 00271
PART1 00272
PART1 00273
PART1 00274
PART1 00275
PART1 00276
PART1 00277
PART1 00278
PART1 00279
PART1 00280
PART1 00281
PART1 00282
PART1 00283
PART1 00284
PART1 00285
PART1 00286
PART1 00287
PART1 00288
PART1 00289
PART1 00290
PART1 00291
PART1 00292
PART1 00293
PART1 00294
PART1 00295
PART1 00296
PART1 00297

```

CALL PICARD(PF2,T2,SX,W ,0,EN2,ETA2,1,NK,SX2,CFT,AMT,YUK,YAM,EXPO)	PART1	00298
IF(.NOT.SPIN) GO TO 140	PART1	00299
CALL PICARD(PG1,S1,SX,W ,0,EN1,ETA1,1,NK,SX1,CFS,AMS,YUK,YAM,EXPO)	PART1	00300
CALL PICARD(PG2,S2,SX,W ,0,EN2,ETA2,1,NK,SX2,CFS,AMS,YUK,YAM,EXPO)	PART1	00301
TT = TT + WK(I14)*(PF1(1)*(PF2(1)+3.*PG2(1))+PG1(1)*(3.*PF2(1) +	PART1	00302
1PG2(1)))/(012P+ODP-ENGY)	PART1	00303
IF(SPIN) GO TO 14	PART1	00304
C	PART1	00305
140 TT = TT + WK(I14)*PF1(1)*PF2(1)/(012P + ODP - ENGY)	PART1	00306
14 CONTINUE	PART1	00307
TT = TT*2.*IP4/(RK1-ENGY)	PART1	00308
C	PART1	00309
IX1 = I13 + N2I11	PART1	00310
TINT(IX1) = TT	PART1	00311
13 CONTINUE	PART1	00312
11 CONTINUE	PART1	00313
C	PART1	00314
WRITE(7) (TINT(IX2),IX2=1,N4)	PART1	00315
C	PART1	00316
12 CONTINUE	PART1	00317
10 CONTINUE	PART1	00318
IF(SPIN.AND.YUK.AND.IEN.EQ.1) WRITE(6,1001) GS,AMS,GT,AMT	PART1	00319
IF(.NOT.SPIN.AND.YUK.AND.IEN.EQ.1) WRITE(6,1005) GT, AMT	PART1	00320
IF(SPIN.AND.YAM.AND.IEN.EQ.1) WRITE(6,1007) GS,AMS,GT,AMT	PART1	00321
IF(.NOT.SPIN.AND.YAM.AND.IEN.EQ.1) WRITE(6,1008) GT,AMT	PART1	00322

IF (SPIN.AND.EXPO.AND.IEN.EQ.1) WRITE(6,1010) GS,AMS,GT,AMT	PART1	00323
IF (.NOT.SPIN.AND.EXPO.AND.IEN.EQ.1) WRITE(6,1011) GT,AMT	PART1	00324
C	PART1	00325
END FILE 7	PART1	00326
WRITE(6,1004) IEN,ENGY	PART1	00327
IF(BSMAT) GO TO 909	PART1	00328
999 CONTINUE	PART1	00329
C	PART1	00330
909 END FILE 7	PART1	00331
REWIND 7	PART1	00332
C	PART1	00333
C	PART1	00334
1000 FORMAT(SF14.12)	PART1	00335
1001 FORMAT(1X////20X,* 3 FERMIONS, YUKAWA S-WAVE POTENTIAL *,//20X,	PART1	00336
1* SINGLET COUPL. CONSTANT = *,F10.5,5X,*SINGLET MU ² = *,F10.7/	PART1	00337
220X,* TRIPLET COUPL. CONSTANT = *,F10.5,5X,* TRIPLET MU ² = *,	PART1	00338
3F10.7///)	PART1	00339
1002 FORMAT(SL10)	PART1	00340
1004 FORMAT(1X//20X,10HFILE NO. =,2X,I2,5X,*ENERGY =*,2X,E15.8//)	PART1	00341
1005 FORMAT(1X////20X,* 3 BOSONS, YUKAWA S-WAVE POTENTIAL *,//20X,	PART1	00342
1* COUPLING CONSTANT (TRIPLET) = *,F10.5,5X,* MU ² = *,F10.7///)	PART1	00343
1006 FORMAT(12,13,15,4F10.5,F12.8,2F5.2)	PART1	00344
1007 FORMAT(1X////20X,* 3 FERMIONS, YAMAGOUCHI S-WAVE POTENTIAL *,//	PART1	00345
120X,* SINGLET COUPL. CONSTANT = *,F10.5,5X,* SINGLET MU ² = *,	PART1	00346
2F10.7/20X,* TRIPLET COUPL. CONSTANT = *,F10.5,5X,* TRIPLET MU ² =	PART1	00347
3 *,F10.7///)	PART1	00348
1008 FORMAT(1X////20X,* 3 BOSONS, YAMAGOUCHI S-WAVE POTENTIAL *,	PART1	00349
1//20X,* COUPLING CONSTANT (TRIPLET) = *,F10.5,5X,* MU ² = *,	PART1	00350
2F10.7///)	PART1	00351
1010 FORMAT(1X////20X,* 3 FERMIONS, EXPONENTIAL S-WAVE POTENTIAL *,//	PART1	00352
120X,* SINGLET COUPL. CONSTANT = *,F10.5,5X,* SINGLET MU ² = *,	PART1	00353
2F10.7/20X,* TRIPLET COUPL. CONSTANT = *,F10.5,5X,* TRIPLET MU ² =	PART1	00354
3 *,F10.7///)	PART1	00355
1011 FORMAT(1X////20X,* 3 BOSONS, EXPONENTIAL S-WAVE POTENTIAL *,	PART1	00356
1//20X,* COUPLING CONSTANT (TRIPLET) = *,F10.5,5X,* MU ² = *,	PART1	00357
2F10.7///)	PART1	00358
1014 FORMAT(1X/10X,14HWEIGHT-FACTOR ,I2,2X,F10.7,10X,15HGAUSS-ABSCISSA	PART1	00359
1,2X,F10.7,10X,9HABSCISSA ,2X,E14.7)	PART1	00360
STOP	PART1	00361
END	PART1	00362

```

C
C
FUNCTION YAMAG(U,V,CF,AMU)
COMMON SPI5,SPI3,SPI7
C ***** YAMAGUCHI POTENTIAL, S-WAVE ONLY *****
C
YAMAG = CF/((U**2+AMU)*(V**2+AMU))
C
RETURN
END

```

```

PART1 00363
PART1 00364
PART1 00365
PART1 00366
PART1 00367
PART1 00368
PART1 00369
PART1 00370
PART1 00371
PART1 00372
PART1 00373

```

```

C
C
FUNCTION YUKAWA(U,V,CF,AMU)
COMMON SPI5,SPI3,SPI7
C ***** YUKAWA POTENTIALS, S-WAVE ONLY *****
C
YUKAWA = CF/(U*V)*ALOG{(((U+V)**2)+AMU)/(((U-V)**2)+AMU)}
C
RETURN
END

```

```

PART1 00374
PART1 00375
PART1 00376
PART1 00377
PART1 00378
PART1 00379
PART1 00380
PART1 00381
PART1 00382
PART1 00383
PART1 00384

```

```

C
C
FUNCTION EXPNTL(U,V,CF,AMU)
COMMON SPI5,SPI3,SPI7
C ***** EXPONENTIAL POTENTIAL, S-WAVE ONLY *****
C
UU = U**2
VV = V**2
EXPNTL = CF*SQRT(AMU)/((UU-VV)**2+2.*AMU*(UU+VV)+AMU**2)
RETURN
END

```

```

PART1 00385
PART1 00386
PART1 00387
PART1 00388
PART1 00389
PART1 00390
PART1 00391
PART1 00392
PART1 00393
PART1 00394
PART1 00395
PART1 00396

```

```

P2 = ZI(K)
IF(YK) VX = YUKAWA(PX,P2,GF,ANU)
IF(YG) VX = YAMAG(PX,P2,GF,ANU)
IF(EP) VX = EXPNTL(PX,P2,GF,ANU)
PF(J) = PF(J) -SP15*WP(K)*P2*VX/(P2**2 - E) * CF(K)
C
2 CONTINUE
C
PF(J) = PF(J) + VL1*SP13
C
1 CONTINUE
IF(IT.LT.1) RETURN
IC = 0
DO 3 L=1,IN
IF(ABS((PF(L)-CF(L))/PF(L)).LE.10.E-3) IC = IC + 1
CF(L) = PF(L)
3 CONTINUE
IF(IC.LT.IN) GO TO 99
RETURN
END

```

```

PART1 00422
PART1 00423
PART1 00424
PART1 00425
PART1 00426
PART1 00427
PART1 00428
PART1 00429
PART1 00430
PART1 00431
PART1 00432
PART1 00433
PART1 00434
PART1 00435
PART1 00436
PART1 00437
PART1 00438
PART1 00439
PART1 00440
PART1 00441

```

```

C
C
SUBROUTINE PICARD(PF,WF,ZI,WP,IT,E,P1,IN,NF,ZP,GF,ANU,YK,YG,EP)
DIMENSION PF(1),WF(1),ZI(1),ZP(1),WP(1),CF(50)
LOGICAL YK,YG,EP
COMMON SPI5,SPI3,SPI7
C
C***** THIS ROUTINE PERFORMS PICARD'S ITERATION SOLUTION*****
C
C
      DO 4 M=1,NF
4  CF(M) = WF(M)
C
      ICOUNT = 0
99  ICOUNT = ICOUNT + 1
      IF(ICOUNT.GT.30) RETURN
C
      DO 1 J=1,IN
      RJ = J
      PX = ZP(J)
      IF(YK) VL1 = YUKAWA(PX,P1,GF,ANU)
      IF(YG) VL1 = YAMAG(PX,P1,GF,ANU)
      IF(EP) VL1 = EXPNTL(PX,P1,GF,ANU)
      PF(J) = .0
      DO 2 K=1,NF

```

```

PART1 00397
PART1 00398
PART1 00399
PART1 00400
PART1 00401
PART1 00402
PART1 00403
PART1 00404
PART1 00405
PART1 00406
PART1 00407
PART1 00408
PART1 00409
PART1 00410
PART1 00411
PART1 00412
PART1 00413
PART1 00414
PART1 00415
PART1 00416
PART1 00417
PART1 00418
PART1 00419
PART1 00420
PART1 00421

```

PROGRAM PART2(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE7,TAPE8)	PART2	00002
C	PART2	00003
C	PART2	00004
C***** PART2 COMPUTES MATRIX ELEMENTS OF MOMENTUM VARIABLES ,	PART2	00005
C K-PRIME^2 AND K^2, VIA QUADRUPOLE INTEGRATIONS *****	PART2	00006
C	PART2	00007
C	PART2	00008
DOUBLE PRECISION AM,DET	PART2	00009
C	PART2	00010
C ***** SEE REMARKS PART1 FOR DIMENSIONS *****	PART2	00011
DIMENSION AM(16,16),WT(5),WK(8),WPSI(5),WPHI(5),X(8),PSI(5),	PART2	00012
IPHI(5),NU(16),NV(16),Y(8),TINT(625),YT(5),PHJ(5),DISMAT(256)	PART2	00013
LOGICAL SPIN,BSMAT	PART2	00014
READ(5,1002) N,NK,SPIN,BSMAT	PART2	00015
READ(5,1006) ESTART,EB,DE,LIM	PART2	00016
C	PART2	00017
N2 = N*N	PART2	00018
N3 = N2*N	PART2	00019
N4 = N3*N	PART2	00020
M2 = NK*2	PART2	00021
TWOPI = 2*3.14159265358979	PART2	00022
T3 = TWOPI/6.0	PART2	00023
SPI = TWOPI/2.	PART2	00024
COM = SQRT(2./(SPI**3))	PART2	00025
S3PI = SPI*1.73205080756888	PART2	00026
F00 = 8./(3.*S3PI)	PART2	00027
F40 = 8./(3.*SPI)	PART2	00028
F44 = 8./S3PI	PART2	00029
READ(5,1000) (WT(N1),N1=1,N)	PART2	00030
READ(5,1000) (YT(NN2),NN2=1,N)	PART2	00031

```

      READ(5,1000) (WK(NN1),NN1=1,NK)
      READ(5,1000) (Y(M3),M3=1,NK)
C
C***** COMPUTATION OF WEIGHTS AND ABSCISSAE *****
      DO 1 I=1,NK
        X(I) = (1.+Y(I))/(1.-Y(I))
1      WK(I) = .5*WK(I)*((1.+X(I))**2)
      DO 9 I9=1,N
        WPSI(I9) = .5*WT(I9)
        BPHI(I9) = WT(I9)*TWOPI*.25
        PHI(I9) = .25*TWOPI*(YT(I9)+1.) - T3
        PHJ(I9) = PHI(I9)-T3
9      PSI(I9) = .5*(YT(I9)+1.)
C
      WRITE(6,1004) (I1,X(I1),Y(I1),WK(I1),I1=1,NK)
      WRITE(6,1005) (I2,WT(I2),YT(I2),PSI(I2),PHI(I2),PHJ(I2),I2=1,N)
C
      DELT = .0
C
      DO 8 I8 = 1,42
        NU(I8) = 0
        NV(I8) = 0
8      CONTINUE
C
C

```

```

PART2 00032
PART2 00033
PART2 00034
PART2 00035
PART2 00036
PART2 00037
PART2 00038
PART2 00039
PART2 00040
PART2 00041
PART2 00042
PART2 00043
PART2 00044
PART2 00045
PART2 00046
PART2 00047
PART2 00048
PART2 00049
PART2 00050
PART2 00051
PART2 00052
PART2 00053
PART2 00054
PART2 00055
PART2 00056

```

PART2

CDC 6600 FTN V3.0-P221 OPT=2 04/03/71 11.59.21.

```

REWIND 7
ENGY = ESTART
DO 999 IEN = 1,LIM
ENGY = ENGY + DE
IF (ESMAT) ENGY = EB
IF (IEN.GT.1) CALL SKPFIL(1,7)

```

C
C
C
C
C

C***** QUADRUPOLE INTEGRATION *****

```

C
C      JJ = 0
99  IS = 1
C      JS = 1
      IF (JJ.EQ.1) JS = NK + 1
      IF (JJ.EQ.2) IS = NK + 1
      IF (JJ.EQ.3) IS = NK + 1
      IF (JJ.EQ.3) JS = IS
      IF = IS + NK - 1
      JF = JS + NK - 1
      DO 2 J2=IS,IF
      DO 3 J3=JS,JF

```

C

REAL(7) (TINT(JX),JX=1,N4)

C

C***** PSI-STAR LOOP *****

```

      RINT = .0
      DO 4 J4=1,N
      IF (J2.LE.NK) SIS = PSI(J4)
      IF (J2.GT.NK) SIS = (1.-2.*(PSI(J4)**2))*PSI(J4)

```

```

PART2 00057
PART2 00058
PART2 00059
PART2 00060
PART2 00061
PART2 00062
PART2 00063
PART2 00064
PART2 00065
PART2 00066
PART2 00067
PART2 00068
PART2 00069
PART2 00070
PART2 00071
PART2 00072
PART2 00073
PART2 00074
PART2 00075
PART2 00076
PART2 00077
PART2 00078
PART2 00079
PART2 00080
PART2 00081
PART2 00082
PART2 00083
PART2 00084
PART2 00085
PART2 00086

```

```

C***** PHI-STAR LOOP *****
      R5 = .0
      DO 5 J5=1,N
C***** PSI-PRIME LOOP *****
      R6 = .0
      DO 6 J6=1,N
      IF (J3.LE.NK) SIP = PSI(J6)
      IF (J3.GT.NK) SIP = (1.-2.*(PSI(J6)**2))*PSI(J6)
C***** PHI-PRIME LOOP *****
      R7 = .0
      DO 7 J7=1,N
C
      JY = J7+(J6-1)*N+(J5-1)*N2+(J4-1)*N3
      TT = TINT(JY)
C
      7 R7 = R7 + SIP*SIS*TT*WPHI(J7)
      R6 = R6 + R7*WPSI(J6)
      6 CONTINUE
      R5 = R5 + R6*WPHI(J5)
      5 CONTINUE
      RINT = RINT + R5*WPSI(J4)
      4 CONTINUE
C
C***** MATRIX ALLOCATION *****
C

```

```

PART2 00087
PART2 00088
PART2 00089
PART2 00090
PART2 00091
PART2 00092
PART2 00093
PART2 00094
PART2 00095
PART2 00096
PART2 00097
PART2 00098
PART2 00099
PART2 00100
PART2 00101
PART2 00102
PART2 00103
PART2 00104
PART2 00105
PART2 00106
PART2 00107
PART2 00108
PART2 00109
PART2 00110
PART2 00111

```

PART2

CDC 6600 FTN V3.0-P221 OPT=2 04/03/71 11.59.21.

IF (J2.LE.NK.AND.J3.LE.NK) F = F00	PART2	00112
IF (J2.GT.NK.OR.J3.GT.NK) F = F40	PART2	00113
IF (J2.GT.NK.AND.J3.GT.NK) F = F44	PART2	00114
J22 = J2	PART2	00115
J33 = J3	PART2	00116
IF (J3.GT.NK) J33 = J3 - NK	PART2	00117
IF (J2.GT.NK) J22 = J2 - NK	PART2	00118
C***** MATRIX ELEMENT *****	PART2	00119
CRONEK = 0.0	PART2	00120
IF (J2.EQ.J3) CRONEK = 1.	PART2	00121
IF (BSMAT.AND..NOT.SPIN) AM(J2,J3)=F*RINT*X(J22)*X(J33)*WK(J33)	PART2	00122
IF (.NOT.BSMAT.AND..NOT.SPIN) AM(J2,J3) = CRONEK - F*RINT *	PART2	00123
1 X(J22)*X(J33)*WK(J33)	PART2	00124
IF (BSMAT.AND.SPIN) AM(J2,J3) = .125*F*RINT*X(J22)*X(J33)*WK(J33)	PART2	00125
IF (.NOT.BSMAT.AND.SPIN) AM(J2,J3) = CRONEK - .125*F*	PART2	00126
1 RINT*X(J22)*X(J33)*WK(J33)	PART2	00127
C	PART2	00128
3 CONTINUE	PART2	00129
2 CONTINUE	PART2	00130
JJ = JJ+1	PART2	00131
IF (JJ.LI.4.AND.IEN.EQ.1) REWIND 7	PART2	00132
IF (JJ.LI.4.AND.IEN.GT.1) CALL SKPFIL(-1,7)	PART2	00133
IF (JJ.LI.4) GO TO 99	PART2	00134
C	PART2	00135
C	PART2	00136
IF (BSMAT) 900,909	PART2	00137
900 IF (SPIN) WRITE(6,1007)	PART2	00138
IF (.NOT.SPIN) WRITE(6,1008)	PART2	00139
WRITE(6,1003) ENGY	PART2	00140
C	PART2	00141

DO 90 I99=1,M2	PART2	00142
DO 90 J99=1,M2	PART2	00143
L99 = J99 + M2*(I99-1)	PART2	00144
90 DISMAT(L99) = AM(I99,J99)	PART2	00145
REWIND 8	PART2	00146
MM2 = M2*M2	PART2	00147
WRITE(8) (DISMAT(K99),K99=1,MM2)	PART2	00148
C	PART2	00149
GO TO 901	PART2	00150
909 IF(IEN.EQ.1.AND.SPIN) WRITE(6,1009)	PART2	00151
IF(IEN.EQ.1.AND..NOT.SPIN) WRITE(6,1010)	PART2	00152
C	PART2	00153
CALL MINV(AM,M2,DET,NU,NV)	PART2	00154
C	PART2	00155
WRITE(6,1001) ENGY, DET	PART2	00156
C	PART2	00157
999 CONTINUE	PART2	00158
C	PART2	00159
901 REWIND 7	PART2	00160
C	PART2	00161
C	PART2	00162
1000 FORMAT(3F14.12)	PART2	00163
1001 FORMAT(1X////10X,*ENERGY = *,1X,E15.8,5X,*DETERMINANT = *,1X,D15.8	PART2	00164
1)	PART2	00165
1002 FORMAT(2I2.2L10)	PART2	00166
1003 FORMAT(1X////10X.8HENERGY =,1X,E15.8)	PART2	00167
1004 FORMAT(1X,10X,12,4X,* K ² =*,2X,E15.8,4X,* GAUSS ABSCISSA =*,2X,	PART2	00168
1E15.8,4X,*HEIGHT FACTOR =*,2X,E15.8/)	PART2	00169
1005 FORMAT(1X,12,3X,*G-WEIGHT=*,E15.8,3X,*G-ABSC.=*,E15.8,3X,*PSI =*,	PART2	00170
1E15.8,3X,*PHI=*,E15.8,3X,*PHJ=*,E15.8/)	PART2	00171
1006 FORMAT(F5.2,E15.8,F5.2,I10)	PART2	00172
1007 FORMAT(1X////20X,* BOUND STATE MATRIX, 3 NUCLEON, SPIN-ISOSPIN *,	PART2	00173
1///)	PART2	00174
1008 FORMAT(1X////20X,* BOUND STATE MATRIX, 3 NUCLEONS, BOSONS *,///)	PART2	00175
1009 FORMAT(1X////20X,*DETERMINANT VALUES FOR LOCATING BOUND STATE ENER	PART2	00176
16Y, 3 FERMIONS*///)	PART2	00177
1010 FORMAT(1X////20X,* DETERMINANT VALUES FOR LOCATING BOUND STATE ENR	PART2	00178
16Y, 3 BOSONS*///)	PART2	00179
STOP	PART2	00180
END	PART2	00181

```

SUBROUTINE MINV(A,N,D,L,M)
DOUBLE PRECISION A,D,BIGA,HOLD
DIMENSION A(1),L(1),M(1)

.....

SEARCH FOR LARGEST ELEMENT

D=1.0
NK=-N
DO 10 K=1,N
  NK=NK+N
  L(K)=A
  M(K)=K
  KK=NK+K
  BIGA=A(KK)
DO 20 J=K,N
  IZ=K*(J-1)
DO 20 I=K,N
  IJ=IZ+I
10 IF (DABS(BIGA)-DABS(A(IJ))) 15,20,20
15 BIGA=A(IJ)
  L(K)=I
  M(K)=J
20 CONTINUE

INTERCHANGE ROWS

J=L(K)

```

C

```
SUBROUTINE MINV(A,N,D,L,M)
DOUBLE PRECISION A,D,BIGA,HOLD
DIMENSION A(1),L(1),M(1)
```

C

C

5

C

C

SEARCH FOR LARGEST ELEMENT

$$D = 2.0$$
$$NK = -13$$
$$00 \leq 0 \quad K=1, N$$
$$N_k = i\gamma_k + i\eta$$
$$L(K) = K$$
$$M(K) = K$$
$$KK = PK + E$$
$$B16A=4(KK)$$
$$00 \quad 20 \quad J = K + N$$
$$IZ = N^*(J-1)$$
$$00 \leq 0 \quad T = K + N$$
$$I_j = I_0 + 1$$

```
10 IF (ABS(BIGA)-DABS(A(IJ))) 15,20,20
```

17. $B(A) = A(I)$

$$L(0) = 1$$
$$M(\beta) = J$$

20 CONTINUE

6.

6.

C

INTERCHANGE ROWS

$$J = L(K)$$

PART2	00182
PART2	00183
PART2	00184
PART2	00185
PART2	00186
PART2	00187
PART2	00188
PART2	00189
PART2	00190
PART2	00191
PART2	00192
PART2	00193
PART2	00194
PART2	00195
PART2	00196
PART2	00197
PART2	00198
PART2	00199
PART2	00200
PART2	00201
PART2	00202
PART2	00203
PART2	00204
PART2	00205
PART2	00206
PART2	00207
PART2	00208
PART2	00209
PART2	00210
PART2	00211
PART2	00212

IF (J-K) 35,35,25

25 KI=K-N

DO 30 I=1,N

KI=KI+N

HOLD=-A(KI)

J1=KI-K+J

A(KI)=A(J1)

30 A(J1)=HOLD

C
C
C

INTERCHANGE COLUMNS

35 I=M(K)

IF (I-K) 45,45,38

38 JP=N*(I-1)

DO 40 J=1,N

JK=NK+J

J1=JP+J

HOLD=-A(JK)

A(JK)=A(J1)

40 A(J1)=HOLD

C
C
C
C

DIVIDE COLUMN BY MINUS PIVOT (VALUE OF PIVOT ELEMENT IS
CONTAINED IN BIGA)

PART2 00213
PART2 00214
PART2 00215
PART2 00216
PART2 00217
PART2 00218
PART2 00219
PART2 00220
PART2 00221
PART2 00222
PART2 00223
PART2 00224
PART2 00225
PART2 00226
PART2 00227
PART2 00228
PART2 00229
PART2 00230
PART2 00231
PART2 00232
PART2 00233
PART2 00234
PART2 00235
PART2 00236

45 IF (BIGA) 48,46,48

46 D=0.0

RETURN

48 DO 55 I=1,N

IF (I-K) 50,55,50

50 IK=IK+I

A(IK)=A(IK)/(-BIGA)

55 CONTINUE

C

C

REDUCE MATRIX

C

DO 65 I=1,N

IK=IK+I

IJ=I-N

DO 65 J=1,N

IJ=IJ+N

IF (I-K) 60,65,60

60 IF (J-K) 62,65,62

62 KJ=IJ-I+K

A(IJ)=A(IK)*A(KJ)+A(IJ)

65 CONTINUE

C

C

DIVIDE ROW BY PIVOT

C

KJ=K-N

DO 75 J=1,N

KJ=KJ+N

IF (J-K) 70,75,70

70 A(KJ)=A(KJ)/BIGA

75 CONTINUE

C

PART2 00237

PART2 00238

PART2 00239

PART2 00240

PART2 00241

PART2 00242

PART2 00243

PART2 00244

PART2 00245

PART2 00246

PART2 00247

PART2 00248

PART2 00249

PART2 00250

PART2 00251

PART2 00252

PART2 00253

PART2 00254

PART2 00255

PART2 00256

PART2 00257

PART2 00258

PART2 00259

PART2 00260

PART2 00261

PART2 00262

PART2 00263

PART2 00264

PART2 00265

PART2 00266

PART2 00267

```

C      PRODUCT OF PIVOTS
C
C      D=D*BIGA
C
C      REPLACE PIVOT BY RECIPROCAL
C
C      A(KK)=1.0/BIGA
80  CONTINUE
C
C      FINAL ROW AND COLUMN INTERCHANGE
C
C      K=N
100  K=(K-1)
      IF(K) 150,150,105
105  I=L(K)
      IF(I-K) 120,120,106
106  JQ=N*(K-1)
      JR=N*(I-1)
      DO 110 J=1,N
      JK=JQ+J
      HOLD=A(JK)
      JI=JR+J
      A(JK)=-A(JI)
110  A(JI) =HOLD

120  J=M(K)
      IF(J-K) 100,100,125
125  KI=K-N
      DO 130 I=1,N
      KI=KI+N
      HOLD=A(KI)
      JI=KI-K+J
      A(KI)=-A(JI)
130  A(JI) =HOLD
      GO TO 100
150  RETURN
      END

```

```

PART2 00268
PART2 00269
PART2 00270
PART2 00271
PART2 00272
PART2 00273
PART2 00274
PART2 00275
PART2 00276
PART2 00277
PART2 00278
PART2 00279
PART2 00280
PART2 00281
PART2 00282
PART2 00283
PART2 00284
PART2 00285
PART2 00286
PART2 00287
PART2 00288
PART2 00289
PART2 00290
PART2 00291

PART2 00292
PART2 00293
PART2 00294
PART2 00295
PART2 00296
PART2 00297
PART2 00298
PART2 00299
PART2 00300
PART2 00301
PART2 00302
PART2 00303

```

SUBROUTINE SKPFIL(IFIL,IUNIT)

IF(IFIL) 20,90,10

10 I = 0

11 I = I + 1

13 READ(IUNIT) T

IF(EOF(IUNIT))12,13

IF(EOF,IUNIT) 12,13

12 IF(IFIL-1) 90,90,11

20 I = 0

23 DO 2 J2=1,2

2 BACKSPACE IUNIT

READ(IUNIT) T

IF(EOF(IUNIT))22,23

IF(EOF,IUNIT) 22,23

22 I = I + 1

IF(I+IFIL) 23,90,90

90 RETURN

END

PART2 00304
PART2 00305
PART2 00306
PART2 00307
PART2 00308
PART2 00309
PART2 00310
PART2 00311
PART2 00312
PART2 00313
PART2 00314
PART2 00315
PART2 00316
PART2 00317
PART2 00318
PART2 00319
PART2 00320
PART2 00321
PART2 00322
PART2 00323
PART2 00324
PART2 00325
PART2 00326
PART2 00327

```

C      PROGRAM LAM6(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE7,TAPE8)
C**** PROGRAM COMPUTING AND INVERTING MATRIX, LAMBDA = 6 *****
C      DIMENSION AM(24,24),WT(5),WK(8),WPSI(5),WPHI(5),X(8),PSI(5),
1PHI(5),NU(24),NV(24),Y(8),TINT(625),YT(5),PHJ(5),DISMAT(576)
      LOGICAL SPIN,BSMAT
      DOUBLE PRECISION AM,DET
C
C      READ(5,1002) N,NK,SPIN,BSMAT
      READ(5,1006) ESTART,EB,DE,LIM
      N2 = N*N
      N3 = N2*N
      M2 = NK*2
      N4 = N3*N
C
      L3 = 3*NK
      TWOPI = 2*3.14159265358979
      T3 = TWOPI/6.0
      S2 = 1.4142135623731
      P1 = TWOPI/2.
      S3PI = P1*SQRT(3.)
      F00 = 8./(3.*S3PI)
      F40 = 8./(3.*P1)
      F44 = 8./S3PI
      F063 = 16.*S2/(3.*S3PI)
      F463 = 16.*S2/(3.*P1)
      F036 = 64./(3.*S3PI)
C
      READ(5,1000) (WT(N1),N1=1,N)
      READ(5,1000) (YT(M3),M3=1,N)

```

```

LAM6 00002
LAM6 00003
LAM6 00004
LAM6 00005
LAM6 00006
LAM6 00007
LAM6 00008
LAM6 00009
LAM6 00010
LAM6 00011
LAM6 00012
LAM6 00013
LAM6 00014
LAM6 00015
LAM6 00016
LAM6 00017
LAM6 00018
LAM6 00019
LAM6 00020
LAM6 00021
LAM6 00022
LAM6 00023
LAM6 00024
LAM6 00025
LAM6 00026
LAM6 00027
LAM6 00028
LAM6 00029
LAM6 00030
LAM6 00031
LAM6 00032
LAM6 00033

```

```

      READ(5,1000) (WK(NN1),NN1=1,NK)
      READ(5,1000) (Y(NN2),NN2=1,NK)
C
C***** COMPUTATION OF WEIGHTS AND ABSCISSAE *****
      DO 1 I=1,NK
        X(I) = (1.+Y(I))/(1.-Y(I))
        1 WK(I) = .5*WK(I)*((1.+X(I))**2)
      DO 9 I9=1,N
        PSI(I9) = (YT(I9)+1.)/2.
        PHI(I9) = .25*TWOPI*(YT(I9)+1.) - T3
        PHJ(I9) = PHI(I9) - T3
        WPSI(I9) = .5*WT(I9)
        9 WPHI(I9) = .25*TWOPI*WT(I9)
C
      WRITE(6,1004) (I1,X(I1),Y(I1),WK(I1),I1=1,NK)
      WRITE(6,1005) (I2,WT(I2),YT(I2),PSI(I2),PHI(I2),PHJ(I2),I2=1,N)
C
      DET = .0
      DO 8 I8=1,L3
        NU(I8) = 0
        NV(I8) = 0
      8 CONTINUE
C

```

```

LAM6 00034
LAM6 00035
LAM6 00036
LAM6 00037
LAM6 00038
LAM6 00039
LAM6 00040
LAM6 00041
LAM6 00042
LAM6 00043
LAM6 00044
LAM6 00045
LAM6 00046
LAM6 00047
LAM6 00048
LAM6 00049
LAM6 00050
LAM6 00051
LAM6 00052
LAM6 00053
LAM6 00054
LAM6 00055
LAM6 00056

```

```

C
REWIND 7
WRITE(6,1011)
C
C
C***** ENERGY VARIATION LOOP *****
ENGX = ESTART
DO 999 IEN = 1,LIM
ENGX = ENGX + DE
IF(BSMAT) ENGX = EB
IF(IEN.GT.1) CALL SKPFIL(1,7)
C
C
C***** QUADRUPOLE INTEGRATION *****
C
DO 99 JJ=1,9
IF(JJ.GT.1.AND.IEN.EQ.1) REWIND 7
IF(JJ.GT.1.AND.IEN.GT.1) CALL SKPFIL(-1,7)
IS = 1
JS = 1
IF(JJ.EQ.1) F = F00
IF(JJ.EQ.2) JS=NK+1
IF(JJ.EQ.2) F = F40
IF(JJ.EQ.3) JS = M2+1
IF(JJ.EQ.3) F = F063
IF(JJ.EQ.4) IS=NK+1
IF(JJ.EQ.4) F=F40
IF(JJ.EQ.5) IS = NK+1
IF(JJ.EQ.5) JS = NK+1
IF(JJ.EQ.5) F = F44
IF(JJ.EQ.6) IS=NK+1

```

```

LAM6 00057
LAM6 00058
LAM6 00059
LAM6 00060
LAM6 00061
LAM6 00062
LAM6 00063
LAM6 00064
LAM6 00065
LAM6 00066
LAM6 00067
LAM6 00068
LAM6 00069
LAM6 00070
LAM6 00071
LAM6 00072
LAM6 00073
LAM6 00074
LAM6 00075
LAM6 00076
LAM6 00077
LAM6 00078
LAM6 00079
LAM6 00080
LAM6 00081
LAM6 00082
LAM6 00083
LAM6 00084
LAM6 00085
LAM6 00086
LAM6 00087
LAM6 00088

```

```

IF (JJ.EQ.6) JS=M2+1
IF (JJ.EQ.6) F=F463
IF (JJ.EQ.7) IS=M2+1
IF (JJ.EQ.7) F = F063
IF (JJ.EQ.8) IS=M2+1
IF (JJ.EQ.8) JS=NK+1
IF (JJ.EQ.8) F=F463
IF (JJ.EQ.9) IS=M2+1
IF (JJ.EQ.9) JS=M2+1
IF (JJ.EQ.9) F=F636
J2F= IS +NK-1
JF = JS +NK-1
DO 2 J2=IS,J2F
DO 3 J3=JS,JF

```

```

C READ(7) (TINT(JX),JX=1,N4)

```

```

C ***** PSI-STAR LOOP *****

```

```

PINT = .0
DO 4 J4=1,N
IF (JJ.LE.3) SIS = PSI(J4)
IF (JJ.GE.4.AND.JJ.LE.6) SIS = PSI(J4)*(1.-2.*(PSI(J4)**2))
IF (JJ.GT.6) SIS = PSI(J4)**4

```

```

LAM6 00089
LAM6 00090
LAM6 00091
LAM6 00092
LAM6 00093
LAM6 00094
LAM6 00095
LAM6 00096
LAM6 00097
LAM6 00098
LAM6 00099
LAM6 00100
LAM6 00101
LAM6 00102
LAM6 00103
LAM6 00104
LAM6 00105
LAM6 00106
LAM6 00107
LAM6 00108
LAM6 00109
LAM6 00110
LAM6 00111

```

C***** PHI-STAR LOOP *****

R5 = .0

DO 5 J5=1,N

FIS = 1.

IF (JJ.GT.6) FIS = COS(3.*PHI(J5))

C***** PSI-PRIME LOOP *****

R6 = .0

DO 6 J6=1,N

IF (JJ.EQ.1.OR.JJ.EQ.4.OR.JJ.EQ.7) SIP = PSI(J6)

IF (JJ.EQ.2.OR.JJ.EQ.5.OR.JJ.EQ.8) SIP = PSI(J6)*(1.-2.*(PSI(J6)**2
1))

IF (JJ.EQ.3.OR.JJ.EQ.6.OR.JJ.EQ.9) SIP = PSI(J6)**4

C***** PHI-PRIME LOOP *****

R7 = .0

DO 7 J7=1,N

FIP = 1.

IF (JJ.EQ.3.OR.JJ.EQ.6.OR.JJ.EQ.9) FIP = COS(3.*PHI(J7))

C
JY = J7 + (J6-1)*N + (J5-1)*N2 + (J4-1)*N3
TI = TIN1(JY)

7 R7 = R7 + SIP*SIS*FIP*FIS*TI*WPHI(J7)

C
R6 = R6 + R7*WPSI(J6)

6 CONTINUE

R5 = R5 + R6*WPHI(J5)

5 CONTINUE

RINT = RINT + R5*WPSI(J4)

4 CONTINUE

C
C***** MATRIX ALLOCATION *****

C
CRONER = .0

LAM6 00112
LAM6 00113
LAM6 00114
LAM6 00115
LAM6 00116
LAM6 00117
LAM6 00118
LAM6 00119
LAM6 00120
LAM6 00121
LAM6 00122
LAM6 00123
LAM6 00124
LAM6 00125
LAM6 00126
LAM6 00127
LAM6 00128
LAM6 00129
LAM6 00130
LAM6 00131
LAM6 00132
LAM6 00133
LAM6 00134
LAM6 00135
LAM6 00136
LAM6 00137
LAM6 00138
LAM6 00139
LAM6 00140
LAM6 00141
LAM6 00142
LAM6 00143
LAM6 00144

```

J22 = J2
J33 = J3
IF (J3.GT.NK) J33 = J3 - NK
IF (J2.GT.NK) J22 = J2 - NK
IF (J3.GT.M2) J33 = J3 - M2
IF (J2.GT.M2) J22 = J2 - M2
C***** MATRIX ELEMENT *****
IF (J2.EQ.J3) CRONEK = 1.
IF (BSMAT.AND..NOT.SPIN) AM(J2,J3)=F*RINT*X(J22)*X(J33)*WK(J33)
IF (.NOT.BSMAT.AND..NOT.SPIN) AM(J2,J3) = CRONEK - F*RINT *
1 X(J22)*X(J33)*WK(J33)
IF (BSMAT.AND.SPIN) AM(J2,J3) = .125*F*RINT*X(J22)*X(J33)*WK(J33)
IF (.NOT.BSMAT.AND.SPIN) AM(J2,J3) = CRONEK - .125*F*
1 RINT*X(J22)*X(J33)*WK(J33)
C
3 CONTINUE
2 CONTINUE
99 CONTINUE
IF (BSMAT) 900,909
900 IF (SPIN) WRITE(6,1007)
IF (.NOT.SPIN) WRITE(6,1008)
WRITE(6,1003) ENGY

```

```

LAM6 00145
LAM6 00146
LAM6 00147
LAM6 00148
LAM6 00149
LAM6 00150
LAM6 00151
LAM6 00152
LAM6 00153
LAM6 00154
LAM6 00155
LAM6 00156
LAM6 00157
LAM6 00158
LAM6 00159
LAM6 00160
LAM6 00161
LAM6 00162
LAM6 00163
LAM6 00164
LAM6 00165
LAM6 00166

```

```

C
DO 90 I99=1,L3
DO 90 J99=1,L3
L99 = J99 + L3*(I99-1)
90 DISMAT(L99) = AM(I99,J99)
REWIND 8
LL3=L3*L3
WRITE(8) (DISMAT(K99),K99=1,LL3)

C
GO TO 901

999 IF (IEN.EQ.1.AND.SPIN) WRITE(6,1009)
IF (IEN.EQ.1.AND..NOT.SPIN) WRITE(6,1010)

C
C
CALL MINV(AM,L3*DET,NU,NV)
WRITE(6,1001) ENGY,DET

C
C
999 CONTINUE

C
C
901 REWIND 7

C
C
1000 FORMAT(5F14.12)
1001 FORMAT(1X////10X,*ENERGY = *,1X,E15.8,5X,*DETERMINANT = *,1X,D15.8

```

```

LAM6 00167
LAM6 00168
LAM6 00169
LAM6 00170
LAM6 00171
LAM6 00172
LAM6 00173
LAM6 00174
LAM6 00175
LAM6 00176
LAM6 00177
LAM6 00178
LAM6 00179
LAM6 00180
LAM6 00181
LAM6 00182
LAM6 00183
LAM6 00184
LAM6 00185
LAM6 00186
LAM6 00187
LAM6 00188
LAM6 00189
LAM6 00190
LAM6 00191
LAM6 00192

```

1)	LAMB	00193
1002 FORMAT(2I2,2L10)	LAMB	00194
1003 FORMAT(1X///30X,8HENERGY =,1X,E15.8//)	LAMB	00195
1004 FORMAT(1X,10X,I2,4X,* K ² =*,2X,E15.8,4X,* GAUSS ABSCISSA =*,2X,	LAMB	00196
1E15.8,4X,*WEIGHT FACTOR =*,2X,E15.8/)	LAMB	00197
1005 FORMAT(1X,I2,3X,*G-WEIGHT=*,E15.8,3X,*G-ABSC.=*,E15.8,3X,*PSI =*,	LAMB	00198
1E15.8,3X,*PHI=*,E15.8,3X,*PHJ=*,E15.8/)	LAMB	00199
1006 FORMAT(F5.2,E15.8,F5.2,I10)	LAMB	00200
1007 FORMAT(1X///20X,* BOUND STATE MATRIX, 3 NUCLEON, SPIN-ISOSPIN *,	LAMB	00201
1//)	LAMB	00202
1008 FORMAT(1X///20X,* BOUND STATE MATRIX, 3 NUCLEONS, BOSONS *,///)	LAMB	00203
1009 FORMAT(1X///20X,*DETERMINANT VALUES FOR LOCATING BOUND STATE ENER	LAMB	00204
IGY, 3 FERMIONS*,///)	LAMB	00205
1010 FORMAT(1X///20X,* DETERMINANT VALUES FOR LOCATING BOUND STATE ENE	LAMB	00206
IRGY, 3 BOSONS*///)	LAMB	00207
1011 FORMAT(1X///,20X,*GRAND TOTAL ANG. MOM. EIG. VALUES , LAMBDA =	LAMB	00208
10, 4 AND 6 *,///)	LAMB	00209
STOP	LAMB	00210
END	LAMB	00211