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RESEARCH

Energy dissipation bounds for flows in pipes and past bodies of general geometry

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Abstract The problem considered can in general be described as the problem of proving rigorously that as the Reynolds number of a fluid flow tends to infinity, the drag coefficient remains bounded. It has been proven for flows in domains of certain, rather restrictive, set of geometries, but not in general. The drag is the sum of the pressure drag and friction drag. In essence, this paper gives a proof that the friction drag coefficient does remain bounded. Rather nontrivial implications of this result are discussed.

Keywords Turbulence · Energy dissipation · Bounds for time averages · Background flow method

1 Introduction

Rigorous results about turbulent flows are rare. This paper discusses the problem that in generic terms can be described as the problem of proving rigorously that as the Reynolds number (Re) of a fluid flow tends to infinity, the drag coefficient remains bounded. For example, it has been rigorously proven that as $Re \to \infty$ in the flow past a flat plate parallel to the main flow direction the drag coefficient $C_D \le 21 \times (1+\sqrt{2})^3 + O(Re^{-1/2})$ [25]. For internal flows the friction coefficient is a counterpart of the drag coefficient for the flows past bodies. It has been rigorously proven that for a flow in a plane channel $C_f \le 27(1+4\sqrt{2}/Re)^2/(32\sqrt{2})$ [7]. More results can be found in [8–12, 15–18, 20, 23, 24, 26–32, 37, 39], and even this list is probably not complete.

So far, the majority of the results of this nature were obtained by the background flow method [7, 14]. In a few exceptions when they were obtained differently they could also be obtained by this method, see [3, 19]. And so far, the background flow method results with a finite bound for arbitrary large Reynolds number were obtained only for the flows in the domains of a certain shape. Namely, the wall shape has to be such that it allows a solid body occupying the entire flow domain to move in it. For example, for a plane channel a solid bar filling the channel can move along it. A solid circular cylinder can fully fill in and still be rotated inside a circular cavity. The infinitely-thin flat plate can cut through a solid body of butter. A particularly exotic case is that of the flow through a helical pipe, with the corresponding solid motion being that of a corkscrew drilling into a cork [23].

Note that it is possible to change the problem formulation in such a way that obtaining a bound will become possible. One example is the case considered in [40], where the background flow method was used to obtain a bound on the long-time average of the energy dissipation rate when the long-time average of the turbulent kinetic energy is known. However, the existing

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rigorous bounds for the latter are such that the bound for the energy dissipation rate, which is directly related to the drag coefficient, tends to infinity with the Reynolds number.

Situations when the background flow method does not allow to obtain a finite bound for arbitrary Reynolds number are known even for domains that admit a solid-body motion inside [15]. In what follows we will, however, concentrate on the case when the shape of the wall is the major obstacle. This obstacle is well-known, see for example [21–23].

After stating a certain result, we will argue that the difficulty encountered goes well beyond the limitations of the background flow method.

2 Setup

2.1 Governing equations and problem formulation

We consider a flow of viscous incompressible fluid in an infinite pipe of an arbitrary smooth shape periodic in the direction along the pipe (Fig. 1). The flow is driven by a constant body force, such as, for example, gravity, directed along the pipe. The coordinate x is chosen to be along the pipe. The flow velocity $\mathbf{u}$ and pressure p are assumed to be periodic in x with the same period as the pipe wall. Then the governing equations and boundary and periodicity conditions are

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \nu\nabla^2\mathbf{u} + g\mathbf{e}_x, \tag{1}$$

$$\begin{aligned} &\nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}|_{\text{wall}} = 0, \\ &\mathbf{u}(x+L) = \mathbf{u}(x), \quad p(x+L) = p(x). \end{aligned} \tag{2}$$

The equations are written in a non-dimensional form, with $1/Re$ denoted ν for compactness. The equations can be also considered to be in a dimensional form, with p being the pressure divided by density, and ν being the kinematic viscosity.

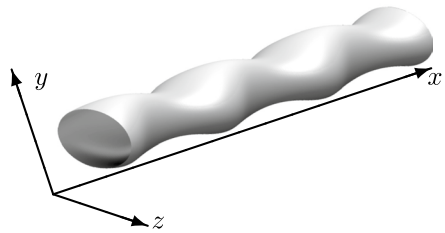


Fig. 1 The pipe of general periodic shape and the coordinate system

As it is well known, the existence of regular solutions of (1–2) has not been proved. The following results apply only to those solutions that exist and are regular enough for the validity of the operations we will perform.

The case of the flow driven by a non-zero mean pressure gradient can be reduced to (1–2) by the standard expedient of modifying the pressure, that is by replacing p with $\tilde{p} = p + gx$, using the periodicity condition $\tilde{p}(x+L) = \tilde{p}(x) - gL$ and dropping $g\mathbf{e}_x$.

A pipe shape of period L is also of period nL with any integer $n > 0$. For this reason the same proof applies for u and p of period nL with any integer $n > 0$, effectively giving a result for arbitrary long pipe.

Another typical formulation, when, instead of prescribing the driving pressure gradient and seeking the mean flow rate, the flow rate is prescribed, can be reduced to the above formulation by rescaling.

We define the inner product and the norm as

$$\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle_\Omega = \int_\Omega \mathbf{u}\cdot\mathbf{v}\, d\Omega, \quad ||\mathbf{u}||^2 = \langle \mathbf{u}, \mathbf{u}\rangle,$$

where Ω is the flow domain between two planes $x = 0$ and $x = L$, that is one period of the pipe. We will also use $\langle \mathbf{u}, \mathbf{v}\rangle_{\Omega_\delta}$ and $||\mathbf{u}||_{\Omega_\delta}$ given by the same formulae but with the integrals taken over a region Ω_δ defined later.

Energy conservation law follows from (1–2) as

$$\frac{1}{2}\frac{d||\mathbf{u}||^2}{dt} = -\nu||\nabla\mathbf{u}||^2 + \langle \mathbf{u}, g\mathbf{e}_x\rangle.$$

Energy is added to the flow by the body force per unit volume $g\mathbf{e}_x$ with the rate $\langle \mathbf{u}, g\mathbf{e}_x\rangle$ and dissipated into heat with the rate $D[\mathbf{u}] = \nu||\nabla\mathbf{u}||^2$.

The infinite-time average of any function of time $a(t)$ will be denoted by an overline and defined as

$$\overline{a(t)} = \lim_{t\to T}\frac{1}{T}\int_0^T a(t)\, dt.$$

The quantity of interest for the present study is the infinite-time average of the energy dissipation rate $\overline{D} = \overline{\nu||\nabla\mathbf{u}||^2}$.

Laminar time-independent solution of (1–2) exists for any ν. For small enough ν (that is large enough Re) turbulent, that is time-dependent and chaotic, flows are also observed. The average energy dissipation rate in laminar and turbulent flow regimes differ. For a given driving force the energy dissipation in a turbulent regime is lower than in a laminar flow. This might seem counter-intuitive, as turbulence is usually associated with higher losses. However, losses are proportional to the product of velocity and force. If the velocity is fixed and the flow goes from laminar to turbulent regime, the force grows, and so do the losses. If, however, the force is fixed then as the flow goes from laminar to turbulent, the velocity decreases, as do the losses. Equations (1–2) correspond to this second formulation. In this case information of the behaviour of turbulent flows can be obtained by considering the smallest possible energy dissipation, or a lower bound to it. In layman terms, the question addressed here is whether the turbulence is getting "infinitely strong" as the Reynolds number tends to infinity, that is whether for a fixed driving force turbulence will bring the flow infinitely close to a standstill as $Re \to \infty$. Observations appear to indicate that it does not, but a rigorous proof is desirable.

In precise terms, the problem considered here is to prove that there exists a lower bound $D_L > 0$ and a constant c such that $\nu\overline{||\nabla \mathbf{u}||^2} > D_L$ for all $\nu > c$, with D_L and c the same for all solutions of (1–2). A constructive proof allowing to find D_L and c is also desirable.

2.2 Auxiliary function method

The auxiliary functional method [5, 13] is based on the observation that under mild conditions the infinite-time average of the Lie derivative of any functional of the phase state of a dynamical system equals zero. For (1–2) the Lie derivative of any functional $V[\mathbf{u}]$ is a functional $\mathcal{L}V[\mathbf{u}]$ given by

$$\begin{aligned}\mathcal{L}V[\mathbf{u}] &= \left\langle \frac{\delta V}{\delta \mathbf{u}}, \text{RHS of (1)} \right\rangle \\ &= \left\langle \frac{\delta V}{\delta \mathbf{u}}, -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \nu\nabla^2\mathbf{u} + g\mathbf{e}_x \right\rangle,\end{aligned} \tag{3}$$

where $\delta V/\delta \mathbf{u}$ is the variational derivative of V. If $\mathbf{u}(t)$ is the solution of (1–2) then $dV[\mathbf{u}(t)]/dt = \mathcal{L}V[\mathbf{u}(t)]$. This leads to $\overline{\mathcal{L}V[\mathbf{u}(t)]} = 0$ if $V[\mathbf{u}(t)]$ is bounded.

Importantly, the definition of $\mathcal{L}V[\mathbf{u}]$ does not involve time t: $\mathcal{L}V[\mathbf{u}]$ is defined for a larger set of $\mathbf{u}$ and not only for the instantaneous values of the solutions of (1–2). Therefore, if an auxiliary functional $V[\mathbf{u}]$ and a constant D_L are found such that

$$D[\mathbf{u}] + \mathcal{L}V[\mathbf{u}] \geq D_L \quad \forall \mathbf{u} \in A, \tag{4}$$

where it is known *a priori* that for any solutions of interest $\mathbf{u}(t) \in A$ for any t, then $\overline{D[\mathbf{u}(t)]} \geq D_L$. Since (4) does not involve time, finding $V[\mathbf{u}]$ and D_L satisfying (4) might be easier than finding bounds for time averages directly from (1–2). Under certain conditions the auxiliary functional approach provides arbitrarily sharp bounds [36, 41]. The following analysis will rely conceptually on the general framework of the auxiliary functional method.

2.3 Background flow method bound

Background flow method was first introduced outside the auxiliary functional framework [7, 9, 10], but it can be interpreted as a special case of the auxiliary functional method, in which the auxiliary functional is sought in the form $\alpha||\mathbf{u}||^2 + \langle \mathbf{b}, \mathbf{u}\rangle$ [3, 5, 14]. The time-independent vector field $\mathbf{b}(\mathbf{x})$ is called a background flow, and $\alpha = \text{const}$ is called a balancing parameter. We will first repeat the standard background flow argument, but recast it in the auxiliary function method terms, specifically for the case considered, and in as simple form as possible. We choose $\alpha = 0$ and $\mathbf{b}$ to have the same basic properties as $\mathbf{u}$, namely $\mathbf{b}|_{\text{wall}} = 0$, $\mathbf{b}(x + L) = \mathbf{b}(x)$, and $\nabla \cdot \mathbf{b} = 0$.

Then $\delta\langle \mathbf{b}, \mathbf{u}\rangle/\delta\mathbf{u} = \mathbf{b}$, and after simple transformations involving integration by parts and using the boundary conditions on $\mathbf{b}$ and $\mathbf{u}$ the Lie derivative of the auxiliary functional $\langle \mathbf{b}, \mathbf{u}\rangle$ can be expressed as

$$\begin{aligned}\mathcal{L}\langle \mathbf{b}, \mathbf{u}\rangle &= \left\langle \mathbf{b}, -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \nu\nabla^2\mathbf{u} + g\mathbf{e}_x \right\rangle \\ &= \left\langle \mathbf{u}, \mathbf{u}\cdot(\nabla\mathbf{b} + \nabla\mathbf{b}^T) \right\rangle/2 \\ &\quad - \nu\langle \nabla\mathbf{b}, \nabla\mathbf{u}\rangle + \langle \mathbf{b}, g\mathbf{e}_x\rangle.\end{aligned} \tag{5}$$

The usual practice is to construct $\mathbf{b}$ such that[1]

$$\begin{aligned}H[\mathbf{u}] &= \left\langle \mathbf{u}, \mathbf{u}\cdot\left(\nabla\mathbf{b} + \nabla\mathbf{b}^T\right) \right\rangle/2 \\ &\quad + \nu||\nabla\mathbf{u}||^2/2 \geq 0 \quad \forall \mathbf{u}.\end{aligned} \tag{6}$$

[1] Since $H[\mathbf{u}]$ is homogeneous and quadratic in $\mathbf{u}$, establishing (6) can be reduced to a linear eigenvalue problem. For this reason (6) is often referred to as the spectral condition.

Since

$$-\nu\langle\nabla\mathbf{b},\nabla\mathbf{u}\rangle = \nu\big(||\nabla\mathbf{b}-\nabla\mathbf{u}||^2-||\nabla\mathbf{b}||^2-||\nabla\mathbf{u}||^2\big)/2,$$

(5) becomes

$$\mathcal{L}\langle\mathbf{b},\mathbf{u}\rangle = H[\mathbf{u}]+\nu||\nabla\mathbf{b}-\nabla\mathbf{u}||^2/2 - \nu||\nabla\mathbf{u}||^2+\langle\mathbf{b},g\mathbf{e}_x\rangle-\nu||\nabla\mathbf{b}||^2/2.$$

Hence, subject to (6)

$$\nu||\mathbf{u}||^2+\mathcal{L}\langle\mathbf{u},\mathbf{b}\rangle\geq\langle\mathbf{b},g\mathbf{e}_x\rangle-\frac{\nu}{2}||\nabla\mathbf{b}||^2\quad\forall\mathbf{u}. \tag{7}$$

Then by the standard argument of the auxiliary function method

$$\overline{\nu||\nabla\mathbf{u}||^2}\geq\langle\mathbf{b},g\mathbf{e}_x\rangle-\frac{\nu}{2}||\nabla\mathbf{b}||^2. \tag{8}$$

This lower bound can be made positive for a given ν by choosing small enough $\mathbf{b}$, since the negative term is quadratic in $\mathbf{b}$ while the first term is linear in $\mathbf{b}$.

The proof of the existence of the desired bound will be completed if $\mathbf{b}$ is constructed to satisfy the spectral condition (6). This is achieved by choosing $\mathbf{b}$ such that $\nabla\mathbf{b}+\nabla\mathbf{b}^T=0$ everywhere outside a thin boundary layer[2] adjacent to the wall. The boundary layer can be defined as a set of points such that their distance to the wall is less or equal to a thickness δ, that is $\Omega_\delta=\{\mathbf{x}\in\Omega:\mathrm{dist}(\mathbf{x},\mathrm{wall})\leq\delta\}$. Then, by Cauchy–Schwarz, Hölder's, Poincaré, and $||\nabla\mathbf{u}||_{\Omega_\delta}\leq||\nabla\mathbf{u}||$ inequalities,

$$\begin{aligned}&\Big|\big\langle\mathbf{u},\mathbf{u}\cdot(\nabla\mathbf{b}+\nabla\mathbf{b}^T)\big\rangle\Big|\\ &=\Big|\big\langle\mathbf{u},\mathbf{u}\cdot(\nabla\mathbf{b}+\nabla\mathbf{b}^T)\big\rangle_{\Omega_\delta}\Big|\overset{\text{Cauchy–Schwarz}}{\leq}\\ &||\mathbf{u}||_{\Omega_\delta}\Big|\Big|\mathbf{u}\cdot(\nabla\mathbf{b}+\nabla\mathbf{b}^T)\Big|\Big|\overset{\text{Hölder's}}{\leq}\\ &||\mathbf{u}||^2_{\Omega_\delta}\Big|\Big|\nabla\mathbf{b}+\nabla\mathbf{b}^T\Big|\Big|_\infty\overset{\text{Poincaré}}{\leq}\\ &C\delta^2||\nabla\mathbf{u}||^2_{\Omega_\delta}\Big|\Big|\nabla\mathbf{b}+\nabla\mathbf{b}^T\Big|\Big|_\infty\leq\\ &C\delta^2||\nabla\mathbf{u}||^2\Big|\Big|\nabla\mathbf{b}+\nabla\mathbf{b}^T\Big|\Big|_\infty,\end{aligned}$$

where for small δ the constant $C>0$ can be independent of δ. Then the spectral condition (6) will be satisfied provided that

$$\nu-C\delta^2\Big|\Big|\nabla\mathbf{b}+\nabla\mathbf{b}^T\Big|\Big|_\infty\geq 0. \tag{9}$$

As $\nu\to 0$, this can be achieved by choosing $\delta\sim\nu$ and constructing $\mathbf{b}$ such that not only $\nabla\mathbf{b}+\nabla\mathbf{b}^T=0$ outside the boundary layer but also $\nabla\mathbf{b}\sim 1/\delta\sim 1/\nu$ and small enough to satisfy (9) inside. For example, for a plane channel $\mathbf{b}$ can be chosen to be wall-parallel everywhere, constant outside the boundary layer and linearly dependent on the wall distance inside the boundary layer, as in the original work [7].

2.4 The difficulty in the case of a general wall shape

The above proof of the bound required constructing the background flow such that $\nabla\mathbf{b}+\nabla\mathbf{b}^T=0$ outside the boundary layer, while $\nabla b\sim 1/\delta$ inside. Since $(\nabla\mathbf{b}+\nabla\mathbf{b}^T)/2$ is the rate-of-strain tensor, it implies that the background flow outside the boundary layer is a solid-body motion. In the limit $\nu\to 0$ this solid-motion region has to fill in the entire flow domain. In the above proof the background flow velocity at the wall was chosen to be zero. Apart from walls of a special shape, the solid-body-motion flow would penetrate the wall, not satisfying not only no-slip but more importantly in this case not satisfying the impermeability condition. Then $\mathbf{b}$ has to adjust to the no-slip and impermeability conditions within a short distance δ between the edge of the boundary layer and the wall, and due to incompressibility this leads to $\nabla b\sim 1/\delta^2$ inside the layer, contrary to what is required. Removing incompressibility or impermeability constraints would leave the pressure term in

[2] In the terminology accepted in the bulk of the literature on turbulent flows, the domain of the background-flow 'boundary layer' roughly corresponds to the buffer and (even thinner) viscous sublayers forming only a part, albeit important, of a turbulent boundary layer, which also includes the logarithmic and outer sublayers. In the buffer sublayer the viscous and inviscid terms are of the same order of magnitude. See Sect. 4.1 for more details.

the inequality, which is hard to deal with. These difficulties are well-known [21, 23].

Kumar [23] posed the question of whether $\nabla\mathbf{b}+\nabla\mathbf{b}^T=0$ outside the boundary layer is necessary to satisfy the spectral condition. In the case when the background flow outside the boundary layer is assumed to be independent of ν and at least piecewise smooth, the answer is yes. The proof consists in the construction a counterexample: $\mathbf{u}$ such that the spectral condition does not hold as $\nu\to 0$. We construct a continuous velocity field $\mathbf{u}_c$ that is zero outside of a cuboid $0\le\hat{x}\le a$, $0\le\hat{y}\le h$, $0\le\hat{y}\le h$. Inside the cuboid the flow is organised as illustrated in Fig. 2, which shows the streamlines in cross-section $\hat{z}=\text{const}$. The velocity component in $\hat{z}$ direction is zero. The other two components vary with $\hat{z}$ proportionally to $\hat{z}(h-\hat{z})$. This velocity field is solenoidal. We will place this cuboid into the bulk of the flow, suitably orientating it by relating $(\hat{x},\hat{y},\hat{z})$ to (x, y, z). Since $\nabla\cdot\mathbf{b}=0$, the symmetric matrix $\nabla\mathbf{b}+\nabla\mathbf{b}^T$ has zero trace, so that the sum of its eigenvalues is zero. Unless all the eigenvalues are zero, which is impossible if $\nabla\mathbf{b}+\nabla\mathbf{b}^T\neq 0$, at least one of the eigenvalues is negative. This eigenvalue λ and the corresponding eigenvector $\mathbf{e}_c$ vary with (x, y, z). Since $\nabla\mathbf{b}+\nabla\mathbf{b}^T$ is assumed to be at least piecewise-smooth, we can find a position for the cuboid in the flow such that by choosing sufficiently small cuboid dimensions the variation of λ and $\mathbf{e}_c$ inside the cuboid will be negligible. Then, we orientate the cuboid so that the $\hat{x}$ axis is directed along $\mathbf{e}_c$. As a result, the velocity direction in the middle part of the cuboid is parallel to $\mathbf{e}_c$, and in the triangular regions at the sides it is perpendicular to $\mathbf{e}_c$. Since $\lambda<0$ by construction, $\mathbf{u}_c\cdot\left(\mathbf{u}_c\cdot\left(\nabla\mathbf{b}+\nabla\mathbf{b}^T\right)\right)<0$ in the middle part of the cuboid, but can be positive in the triangle areas. However, the volume of the triangle areas is independent of the length of the cuboid, while the volume of the central part is proportional to it. By taking a/h sufficiently large we can, therefore, ensure that $\left\langle\mathbf{u}_c,\mathbf{u}_c\cdot\left(\nabla\mathbf{b}+\nabla\mathbf{b}^T\right)\right\rangle<0$. This means that for sufficiently small ν the spectral condition (6) cannot be satisfied if $\mathbf{b}$ is independent of ν in the bulk of the flow.

Note that $\left\langle\mathbf{u}_c,\mathbf{u}_c\cdot\left(\nabla\mathbf{b}+\nabla\mathbf{b}^T\right)\right\rangle\sim h^2$ assuming $||\mathbf{u}_c||\sim 1$ as $h\to 0$. One can, therefore, attempt to enforce h to be small by making $\mathbf{b}$ dependent on ν, so that the characteristic distance over which $\mathbf{b}$ varies were small, too. A simple back-of-envelope estimate shows that this, however, will increase the last term in (8) too much, thus destroying the bound. While such an estimate is not a rigorous proof, overall the results of this section strongly suggest that getting the ν-independent bound by the background flow method for the case of the walls of general shape is not possible.

There are two possible reasons why the background flow methods does not provide the desired bound in the case we consider. First, it is possible that there is a solution of the governing equations such that $\nu\overline{||\nabla\mathbf{u}||^2}\to 0$ as $\nu\to 0$. Then the ν-independent strictly positive bound simply does not exist. Second, it might be that the bound exists and, then, can be obtained by the general auxiliary function method, but not by the background flow method, which is not guaranteed to give an arbitrary sharp bound. In what follows we will obtain results throwing certain light on these alternatives.

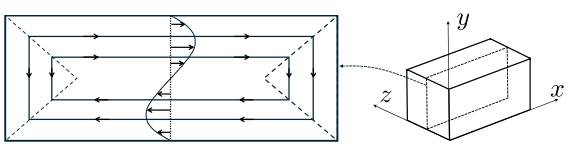


Fig. 2 Counterexample flow

3 Bound for the case of walls of general shape

3.1 Modified problem formulation

Momentum conservation requires the rate of change of fluid momentum in Ω to equal the total force exerted on it. In our notation this can be written as

$$\frac{d\langle\mathbf{e}_x,\mathbf{u}\rangle}{dt}=F_b-F_p-F_f,$$

where F_b is the total body force exerted on the fluid inside Ω, and F_p and F_f are respectively the x-components of pressure and friction forces exerted by the fluid on the wall. It is easy to check that

$$F_b=g\langle\mathbf{e}_x,\mathbf{e}_x\rangle,\quad F_p=\int_{\text{wall}}p\mathbf{e}_x\cdot\mathbf{n}\,d\sigma \quad\text{and } F_f=-\nu\int_{\text{wall}}(\mathbf{e}_x\times\boldsymbol{\omega})\cdot\mathbf{n}\,d\sigma, \tag{10}$$

where $\mathbf{n}$ is the external normal for Ω, that is a unit vector perpendicular to and directed into the wall. Averaging over infinite time gives $\overline{F_b} = \overline{F_p} + \overline{F_f}$.

In Sect. 2.3 we considered the problem of obtaining the lower bound for energy dissipation for a given F_b. In this section we will consider the same problem of finding the bound but for the mean friction force $\overline{F_f}$ given. In flows in a plane channel or a straight pipe $F_p = 0$ since the pressure force is perpendicular to the wall and hence perpendicular to $\mathbf{e}_x$. As a result $F_b = \overline{F_b} = \overline{F_f}$, that is our modified formulation fully coincides with the original one. For wall of general shape F_p might be nonzero. We should stress that the flow itself is the same in both formulations: the difference is only in what is given: the body force or the mean wall friction force.

The motivation for considering this problem will become clear later. To the best of author's knowledge this problem was never considered before, but the problems of the energy dissipation bound for flows driven by the friction force were considered [12, 16, 39]. These flows, however, differ from the flow we are considering. And in these studies the walls were flat.

3.2 Bound in terms of the mean wall friction drag

The Navier–Stokes equations (1) can be written in an equivalent form as

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla p - \nu\nabla\times\boldsymbol{\omega} + g\mathbf{e}_x, \\ \nabla\cdot\mathbf{u} &= 0, \end{aligned} \tag{11}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is the vorticity. Also, according to the Bobylev–Forsythe theorem [38], $||\nabla\mathbf{u}||^2 = ||\boldsymbol{\omega}||^2$ for $\mathbf{u}$ satisfying the boundary conditions (2). This gives an alternative expression for the energy dissipation rate.

We will again consider a thin boundary layer Ω_δ defined in Sect. 2.3, but will choose a different background flow, which will be denoted $\mathbf{w}(\mathbf{x})$. We choose it to be piecewise-differentiable, continuous, and solenoidal. We take $\mathbf{w}(\mathbf{x}) = 0$ everywhere outside the boundary layer, including its edge. At the wall we take

$$\mathbf{w}(\mathbf{x})|_{\text{wall}} = w_0\big((\mathbf{e}_x\cdot\mathbf{n})\mathbf{n} - \mathbf{e}_x\big), \tag{12}$$

thus satisfying the impermeability condition but not the no-slip condition. The value of the constant w_0 will be chosen later. Then from (11) $\mathcal{L}\langle\mathbf{w},\mathbf{u}\rangle = -\langle\mathbf{w},\mathbf{u}\cdot\nabla\mathbf{u}\rangle - \langle\mathbf{w},\nabla p\rangle - \nu\langle\mathbf{w},\nabla\times\boldsymbol{\omega}\rangle + \langle\mathbf{w},g\mathbf{e}_x\rangle$. Using the cross product rule $-\mathbf{w}\cdot(\nabla\times\boldsymbol{\omega}) = -(\nabla\times\mathbf{w})\cdot\boldsymbol{\omega} + \nabla\cdot(\mathbf{w}\times\boldsymbol{\omega})$, the boundary conditions on $\mathbf{u}$, and integration by parts gives

$$\begin{aligned} \mathcal{L}\langle\mathbf{b},\mathbf{u}\rangle = {} & \langle\mathbf{u},\mathbf{u}\cdot\nabla\mathbf{w}\rangle_{\Omega_\delta} - \nu\langle\boldsymbol{\omega},\boldsymbol{\omega}_\mathbf{w}\rangle_{\Omega_\delta} \\ & + \nu\int_{\text{wall}} (\mathbf{w}\times\boldsymbol{\omega})\cdot\mathbf{n}\,d\sigma + \langle\mathbf{w},g\mathbf{e}_x\rangle_{\Omega_\delta}, \end{aligned} \tag{13}$$

where $\boldsymbol{\omega}_\mathbf{w} = \nabla\times\mathbf{w}$ is the vorticity of the background flow and $\mathbf{n}$ is the external normal to Ω.

Substituting (12) and (10) and noticing that at the wall $(\mathbf{n}\times\boldsymbol{\omega})\cdot\mathbf{n} = 0$ (since at the wall $\boldsymbol{\omega}$ is tangent to the wall) gives $\mathcal{L}\langle\mathbf{b},\mathbf{u}\rangle = \langle\mathbf{u},\mathbf{u}\cdot\nabla\mathbf{w}\rangle_{\Omega_\delta} - \nu\langle\boldsymbol{\omega},\boldsymbol{\omega}_\mathbf{w}\rangle_{\Omega_\delta} + w_0F_f + \langle\mathbf{w},g\mathbf{e}_x\rangle_{\Omega_\delta}$. The two first terms in the right-hand side can be treated similar to how it is done in Sect. 2.3:

$$\begin{aligned} \Big|\langle\mathbf{u},\mathbf{u}\cdot\nabla\mathbf{w}\rangle_{\Omega_\delta}\Big| &\overset{\text{Cauchy–Schwarz}}{\leq} ||\mathbf{u}||_{\Omega_\delta}||\mathbf{u}\cdot\nabla\mathbf{w}||_{\Omega_\delta} \\ &\overset{\text{Hölder's}}{\leq} ||\mathbf{u}||^2_{\Omega_\delta}||\nabla\mathbf{w}||_\infty \overset{\text{Poincaré}}{\leq} P^2\delta^2||\nabla\mathbf{u}||^2_{\Omega_\delta}||\nabla\mathbf{w}||_\infty \\ &\leq P^2\delta^2||\nabla\mathbf{u}||^2||\nabla\mathbf{w}||_\infty, \end{aligned} \tag{14}$$

where $P\delta$ is the Poincaré constant for the thin boundary layer, and P can be chosen to be independent of δ for small enough δ. At the same time $-\langle\boldsymbol{\omega},\boldsymbol{\omega}_\mathbf{w}\rangle_{\Omega_\delta} = -\langle\boldsymbol{\omega},\boldsymbol{\omega}_\mathbf{w}\rangle = \big(||\boldsymbol{\omega}-\boldsymbol{\omega}_\mathbf{w}||^2 - ||\boldsymbol{\omega}_\mathbf{w}||^2 - ||\boldsymbol{\omega}||^2\big)/2$. Putting this together, adding $\nu||\nabla\mathbf{u}||^2 = \nu||\nabla\boldsymbol{\omega}||^2$ to the both sides and dropping non-negative terms gives $\nu||\nabla\mathbf{u}||^2 + \mathcal{L}\langle\mathbf{b},\mathbf{u}\rangle \geq \left(\frac{\nu}{2} - P^2\delta^2||\nabla\mathbf{w}||_\infty\right)$

$\nu||\nabla\mathbf{u}||^2 + \mathcal{L}\langle\mathbf{b},\mathbf{u}\rangle \geq \left(\frac{\nu}{2} - P^2\delta^2||\nabla\mathbf{w}||_\infty\right)||\nabla\mathbf{u}||^2 - \frac{\nu}{2}||\boldsymbol{\omega}_\mathbf{w}||^2_0F_f + \langle\mathbf{w},g\mathbf{e}_x\rangle_{\Omega_\delta}$.

Similar to (9), the next step is to choose δ and the background flow $\mathbf{w}$ in such a way that

$$\frac{\nu}{2} \geq P^2\delta^2||\nabla\mathbf{w}||_\infty. \tag{15}$$

As $\nu\to 0$, we can choose $\delta\sim\nu$, and $w_0\sim 1$. Then one can construct $\mathbf{w}$ such that $\nabla\mathbf{w}\sim\boldsymbol{\omega}_\mathbf{w}\sim w_0/\delta$ and $\langle\mathbf{w},\mathbf{e}_x\rangle = 0$. The rigorous proof that this is possible we will leave out as being quite simple. Note that one obvious candidate for $\mathbf{w}$ is the zero-flow-rate Stokes flow satisfying zero boundary condition at the edge of the boundary layer and (12) at the wall. Then (15) becomes

$$\frac{\nu}{2} \geq P^2\delta^2||\nabla \mathbf{w}||_\infty \sim P^2 \nu w_0.$$

This will therefore be satisfied for sufficiently small but finite w_0. The resulting bound is

$$\overline{\nu||\nabla \mathbf{u}||^2} \geq w_0\overline{F_f} - \frac{\nu}{2}\left|\left|\boldsymbol{\omega}_\mathbf{w}\right|\right|^2. \tag{16}$$

Since $\nabla \mathbf{w} \sim \boldsymbol{\omega}_\mathbf{w} \sim w_0/\delta \sim w_0/\nu$ and $\boldsymbol{\omega}_\mathbf{w} = 0$ outside the boundary layer, which has thickness $\delta \sim \nu$, the last term $\nu\left|\left|\boldsymbol{\omega}_\mathbf{w}\right|\right|^2 \sim w_0^2$, while the first term is linear in w_0. Hence, for sufficiently small but finite w_0 the lower bound $w_0\overline{F_f} - \nu\left|\left|\boldsymbol{\omega}_\mathbf{w}\right|\right|^2$ will be positive.

For a specific given shape it would be possible to optimise $\mathbf{w}$ at least numerically. This, however, is outside the scope of the present study. For our current purposes the proof that such a bound is obtainable is sufficient. We will discuss the implications of this result later.

3.3 Steady flow

If there is a solution of the governing equations with prescribed body force and such that $\overline{\nu||\nabla \mathbf{u}||^2} \to 0$ as $\nu \to 0$ then, naturally, proving that there exists a ν-independent lower bound of $\overline{\nu||\nabla \mathbf{u}||^2}$ is impossible. To get a glimpse into this we will consider an attempt to prove such a bound for a steady flow:

$$-\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \nu\nabla^2\mathbf{u} + g\mathbf{e}_x = 0, \tag{17}$$

$$\begin{aligned} &\nabla\cdot\mathbf{u} = 0,\ \ \mathbf{u}|_{\text{wall}} = 0, \\ &\mathbf{u}(x+L) = \mathbf{u}(x),\ p(x+L) = p(x). \end{aligned} \tag{18}$$

We will follow the same philosophy of the auxiliary functional method as for the time-dependent case. Namely, while it is not true that there is $D_L > 0$ such that $\overline{\nu||\nabla \mathbf{u}||^2} \geq D_L$ for any $\mathbf{u}$, we can try to find $\mathbf{v_u}(\mathbf{x})$ depending on $\mathbf{u}$ such that

$$\begin{aligned} &\nu||\nabla \mathbf{u}||^2 + \langle \mathbf{v_u}, -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p \\ &\quad + \nu\nabla^2\mathbf{u} + g\mathbf{e}_x\rangle \geq D_l \quad \forall \mathbf{u} \in A. \end{aligned} \tag{19}$$

Then for $\mathbf{u}$ that is a solution of (17–18) we get $\overline{\nu||\nabla \mathbf{u}||^2} \geq D_L$. Compare this with (3) and (5). There is an obvious analogy between the background flow $\mathbf{b}$ in (5) and $\mathbf{v_u}$ in (19). The difference is that $\left\langle \mathbf{v_u}, -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \nu\nabla^2\mathbf{u} + g\mathbf{e}_x\right\rangle$ need not be a Lie derivative of a functional, and that means that $\mathbf{v_u}$ can depend on $\mathbf{u}$ arbitrarily. Strictly speaking, the background flow can also be dependent on $\mathbf{u}$, but only in a very specific way ([3], Section 4(d)). One would expect that this additional freedom might make finding a bound easier. Indeed, in this case the spectral condition need not be satisfied: $\langle \mathbf{v_u}, -\mathbf{u}\cdot\nabla\mathbf{u}\rangle$ can be made non-negative for any $\mathbf{u}$ by choosing a suitable corresponding $\mathbf{v_u}$. The term $\left\langle \mathbf{v_u}, \nu\nabla^2\mathbf{u}\right\rangle$ can be dealt with in the same way as in the background flow method, and the only remaining task is to use the remaining freedom in the choice of $\mathbf{v_u}$ for obtaining a better bound. Here, however, an interesting obstacle appears. The resulting bound depends on the infimum of $\langle \mathbf{v_u}, -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + g\mathbf{e}_x\rangle$. If there are such $\mathbf{u}$ and p that

$$-\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + g\mathbf{e}_x = 0 \tag{20}$$

then no choice of $\mathbf{v_u}$ will make the infimum and, hence, the bound positive.

Note that (20) is the steady Euler equation and that, therefore, it describes a steady inviscid flow. For channels with straight walls it is easy to prove that such a flow does not exist: under the action of the body force the fluid will accelerate. However, for pipes or channels of general shape the fluid will be under the action of not only the body force but also the pressure force on the walls, and if the pressure force will be of equal magnitude and opposite direction to the body force then the above argument will not apply. Note that if such a steady inviscid flow exists, it has to have a zero flow rate. Indeed, in an invsicid steady flow there is no mechanism of energy dissipation, while the body force adds energy to the flow at the rate proportional to the product of the force and the volume flow rate, and, therefore, the latter has to be zero.

The Bernoulli equation states that along a streamline $\mathbf{u}^2/2 + p + gx = \text{const}$. In a 2D case the wall is a streamline and, since $\mathbf{u}$ and p are periodic functions of x while gx is not, it might seem that the Bernouilli equation eliminates the possibility of the inviscid flow we are considering. However, in an inviscid flow $\mathbf{u}$ is allowed to be discontinuous across streamlines, while p is continuous. Such discontinuities are the zero-viscosity limits of shear layers. Shear layers are common in separated flows.

Figure 3 shows such a hypothetical flow, with more detail given in the caption. If such a flow exists,

then the natural question is whether there is a viscous flow that tends to this inviscid flow as the Reynolds number tends to infinity. Research on high-Reynolds-number asymptotics of steady separated flows was quite extensive a few decades ago, and the problem considered in this section might ignite a new interest. To the best of author's knowledge there was only one work on the asymptotics of a steady flows in periodic pipes and channels [1]. The solution obtained in [1] is applicable only to a certain subset of possible wall shapes, namely, the 2D case in which the opposite walls of the channel are separated by a finite-thickness strip region not touching the walls. In the asymptotic solution [1] for a given driving body force the energy dissipation rate tends to infinity as $\nu \to 0$ and, thus, this solution does not provide a counterexample to the existence of a finite ν-independent lower bound. However, at high Re the solutions might be not unique.

If the flow in Fig. 3 exists and is a limit of a viscous flow as $\nu \to 0$, it is likely to provide such a counterexample. In the viscous flow there will be boundary layers near the wall and along the streamlines dividing the cells, which will allow certain non-zero flow rate with fluid snaking its way through the boundary layers. The boundary layer thickness would be of order $\sqrt{\nu}$ and the velocity of order one, so that the mass flow rate would be of order $\sqrt{\nu}$, and the rate with which the body force adds the energy to the flow will be of the same order. This rate is equal to the energy dissipation rate, $\nu||\mathbf{u}||^2$, which can be also estimated directly from the boundary layer thickness, with the same result: $\nu||\mathbf{u}||^2 \sim \sqrt{\nu}$. Since this tends to zero as $\nu \to 0$, if such asymptotic solution would be constructed, it would be a counterexample to the existence of a finite ν-independent lower bound. This has to be left for further research.

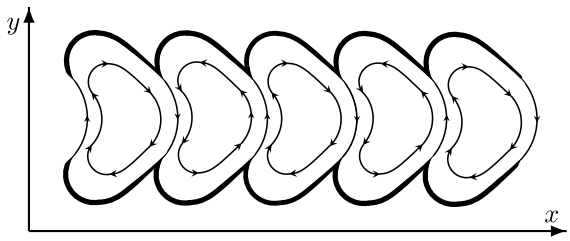


Fig. 3 Hypothetical inviscid flow. The velocity and the Bernoulli constant jump across the streamlines separating the neighbouring cells. The pressure is continuous. The body force is compensated by the curvature of the streamlines in the middle and the wall pressure. The orientation corresponds to $g < 0$, that is the body force is directed to the left

4 Discussion

4.1 Physical meaning of the boundary layers in background flows

From the viewpoint of the mainstream applied turbulent research obtaining the ν-independent lower bound on the energy dissipation rate in a turbulent flow with a given friction force can be addressed in a particularly simple way. According to the classical universality hypothesis first put forward by Ludwig Prandtl [34, 35] any high-Reynolds-number turbulent flow can be rescaled in such a way that in a certain vicinity of the wall the statistics of the flow in that vicinity will be the same for all flows regardless of the (sufficiently high) Reynolds number. This scaling corresponds to the so-called 'wall' units, in which the velocity is scaled with the 'friction velocity' $u_\tau = \sqrt{\tau/\rho}$, where τ is the time-averaged wall friction force per unit wall area, and ρ is the fluid density [33]. The lengths are scaled with $\mu/(\rho u_\tau) = \nu/u_\tau$, where μ is the dynamic viscosity. If all quantities are assumed to be nondimensional as in the present text then $\rho = 1$ and $\mu = \nu$ here. The quantities expressed in the wall units are usually denoted with superscript $+$. The universality hypothesis is well supported by experimental and numerical evidence, but can be also questioned, which is, however, far beyond the scope of the present study.

Consider now a thin near-wall layer of a thickness constant in wall units, such as for example the so-called 'buffer' layer with $y^+ \lesssim 30$, where y^+ denotes the distance to the wall.[3] By the universality hypothesis the energy dissipation rate inside this layer, if expressed in wall units, tends to a constant value as the Reynolds number tends to infinity. The total energy dissipation rate is a sum of the energy dissipation rate in the buffer layer and the energy

[3] The 'viscous' sublayer $y^+ \lesssim 5$ is sometimes not included in the buffer layer, but this does not matter in our argument.

dissipation rate in the rest of the flow. Therefore, the energy dissipation rate in the buffer layer is the lower bound for the energy dissipation rate in the entire flow.

For the flow in a plane channel the averaged wall friction is constant, and a simple substitution gives that

$$\begin{aligned}\nu\overline{||\mathbf{u}||^2_{\Omega_\delta}} &= \nu \iint\limits_{\text{wall}} \int_0^{\delta} \overline{(\nabla \mathbf{u})^2}\, dy\, dx\, dz \\ &= u_\tau^3 \iint\limits_{\text{wall}} \int_0^{\delta^+} \overline{(\nabla^+ \mathbf{u}^+)^2}\, dy^+\, dx\, dz.\end{aligned} \tag{21}$$

In a plane channel flow the total friction force (divided by density if in dimensional form)

$$\overline{F_f} = \iint_{\text{wall}} \frac{\tau}{\rho}\, dx\, dz = \iint_{\text{wall}} u_\tau^2\, dx\, dz.$$

Since in a channel flow $C = \int_0^{\delta^+} \overline{(\nabla^+ \mathbf{u}^+)^2}\, dy^+$ is independent of x and z, (21) can be rewritten as

$$\overline{\nu||\mathbf{u}||^2_{\Omega_\delta}} = C u_\tau \overline{F_f}.$$

Comparison with the bound (16) indicates that this bound is just a rigorous proof that the energy dissipation rate in the buffer layer expressed in wall units, as represented by C, remains bounded from below as the Reynolds number tends to infinity. This is a statement hardly surprising in view of Prandtl's universality hypothesis. And comparison with (12) suggests that w_0 is proportional to u_τ, which in turn indicates that the background flow of Sect. 3.2 if expressed in wall units is independent of the Reynolds number.

In the case of a pipe of a general shape the wall friction τ and the friction velocity u_τ can vary along the wall, so that the energy dissipation rate in the buffer layer will be proportional to the integral of a function of the wall friction along the wall. In contrast, the boundary condition (12) imposed on the background flow was designed to have the bound dependent on the integral of the projection of the wall friction on $\mathbf{e}_x$. However, replacing $\mathbf{e}_x$ with any other unit vector $\mathbf{e}$ will lead to a bound on energy dissipation in terms of the projection of friction on the direction of this vector. Moreover, making w_0 in (12), the thickness of the background flow boundary layer, and $\mathbf{e}$ vary along the wall can lead to a variety of bounds on the energy dissipation rate linked to given mean friction distribution. A bound can be obtained even in terms of the friction force acting on a part of the wall bounding the flow. To achieve this, one would need to make w_0 in (12) and the thickness of the background flow boundary vary along the wall becoming zero outside that part of the surface.

At intuitive level, above any patch with friction on the wall there is a vicinity in which energy dissipates. A ν-independent bound can be obtained for the rate of this energy dissipation, and such bounds are additive. In light of this it is likely that all the bounds for energy dissipation rate in wall-bounded flows so far obtained with the background flow method are bounds for the energy dissipation rate in the buffer layers, and that the very existence of such bounds directly follows from the Prandtl's universality hypothesis. Since such bounds are rigorous, they can be considered as a partial justification of this hypothesis.

For a plane channel flow the background flow $\mathbf{w}$ in our proof of the bound can be obtained from the background flow $\mathbf{b}$ of the standard proof by subtracting a constant value. A constant-value background flow is just an auxiliary-functional form of proving by momentum conservation that the total body force acting on the fluid equals the friction force acting on the channel walls. This suggests that the boundary layers of the background flow method are just masks helping to remove from consideration the parts of the flow domain outside of the buffer layers of a turbulent flow.

4.2 Physical considerations related to bounds for energy dissipation.

The proof of the bound on energy dissipation rate for a given friction force should be applicable with minor modifications to the case of the flow past a bluff body. With respect to the bound for energy dissipation rate, the difference between flows past bodies and flows inside pipes of general shape is more technical than fundamental. The following arguments can be reformulated for flows in pipes but they are more intuitive for flows past bodies, and we will continue the discussion in terms of flows past bodies.

The problem of finding a ν-independent lower bound for the time-averaged energy dissipation rate in a pipe flow under the action of a given body force is analogous to the problem of finding a ν-independent

lower bound for the energy dissipation rate in a flow past a body experiencing a given drag force. In such a flow the mean energy dissipation rate is equal to the product of the drag force and velocity at infinity. For intuitive considerations it is more convenient to consider the flow with a given velocity at infinity and unknown drag. For that case the problem in question is to prove that as the Reynolds number tends to infinity, the drag force remains bounded from above. The drag force is the sum of a friction drag and pressure drag. If it were proven that the ratio of the friction drag to the pressure drag remains bounded from below for large Reynolds numbers then the results obtained in the present study would also provide the solution of the problem in question. However, while there are no rigorous results, observations for turbulent flows and asymptotic solutions for steady flows provide a complicated picture.

For steady attached flows past bodies the standard boundary layer theory gives that the pressure and friction drag are of the same order of magnitude. Steady flows in channels with non-straight walls necessarily have separation at large enough Reynolds numbers [1]. For channels with non-straight walls the asymptotic solution given in [1] does not give an explicit answer for the pressure drag, and it also does not apply to all channels. The high-Reynolds-number asymptotics of steady separated flows past bluff bodies was found only in 2D case [2, 4, 6], and in this case the ratio of friction to pressure drag tends to zero as the Reynolds number tends to infinity. For answering the central question considered in the present study a steady solution would be of real interest if it would provide a counterexample to the existence of ν-independent bound we aim to prove. However, none of the known steady asymptotic solutions provide such a counterexample. Still, asymptotic studies might yet play a significant role in this field of research, if, for example, the hypothetical counterexample proposed in Sect. 3.3 will be constructed.

Observations and calculations of unsteady, including turbulent, flows do show that the drag coefficient remains bounded as $Re \to \infty$. But for separated flows it also does not tend to zero, while the friction drag does. That means that unless the observations for yet higher Reynolds numbers will reveal a different behaviour, the bound obtained in Sect. 3.2 cannot be used to resolve the central problem we address. At the same time even though the drag coefficient does not tend to infinity as $Re \to \infty$ in observations, there might exist unstable, and hence not observed in experiments and calculations, solutions, for which the drag coefficient does tend to infinity. This would, of course, make finding the ν-independent bound impossible without taking into account random perturbations.

If obtaining such a bound is possible in principle, observations of separated flows combined with the results of the present study draw an interesting picture. In separated flows the pressure drag is a dominant part of the total drag, and while it appears to tend to a constant as $Re \to \infty$, the friction drag decreases towards zero. The pressure drag is determined by the processes occurring in a finite vicinity of the body. However, at least in strictly two-dimensional flows at high Re the energy dissipation occurs far downstream. This is clearly illustrated by the long Karman vortex streets observed in the wakes of bluff bodies. In a 2D flow in the limit of $Re \to \infty$ there will be no energy dissipation at all, but if the flow is separated there still might be a pressure drag. One can expect that a problem of finding a bound for the drag of a body, for example a flat plate perpendicular to the velocity at infinity, in two-dimensional unsteady inviscid flow with separation points fixed at the ends of the plate might be a good prototype for the similar problem in a viscous flow. A method of finding such a bound would be of a very different nature from the approaches we discussed here.

5 Conclusions

In a flow in a pipe of any shape the driving force, such as the mean pressure gradient or, equivalently, a body force acting on the fluid, is balanced by the sum of the time-averaged pressure drag and friction drag. While proving the lower (corresponding to a turbulent flow) bound on the energy dissipation rate for a flow in a pipe of a generic shape caused by a given driving force is elusive, we found a method of deriving such a bound for a given mean friction drag. This is the central result of the present study. The method must also be applicable for flows past bodies of arbitrary shape.

We constructed a counterexample showing that if in the bulk of the flow the rate-of-strain tensor of the background flow is nonzero and independent of the Reynolds number, the spectral condition of the background flow method cannot be satisfied.

The obtained results suggest that the boundary layers of the background flows correspond to the buffer layers of near-wall turbulent flow, and the bounds obtained by the background flow methods are essentially the bounds for the energy dissipation rate in the buffer layers.

We proposed an extension of the auxiliary function method to the case of steady solutions. Applying this method to the case of a given driving force led to an interesting problem about the existence of the pipe shaped so that in that pipe a zero-flow-rate inviscid flow with a non-zero mean pressure gradient is possible. We proposed a hypothetical candidate for such a shape and a flow. In case such a flow exists, the second interesting question is whether this flow can be a limit of a steady viscous flow as the Reynolds number tends to infinity. If the answer is positive, it would be a counterexample to the possibility of proving the Reynolds-number-independent bound.

Finally, the obtained results lead to observation that in a separated flow the rate of energy dissipation is determined by the processes near the body but the energy dissipation itself occurs far downstream.

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