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Passivity-based control with state observer for a class of underactuated mechanical systems with series-elastic actuator ⋆

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Abstract: This work investigates the passivity-based control of a class of underactuated mechanical systems with series-elastic actuator for which the actuator states are not measured. The main contribution is a new design of the integral interconnection-and-damping assignment passivity-based control that employs a dynamic extension to estimate the actuator states. Numerical simulations demonstrate the effectiveness of the new controller.

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Keywords: Passivity-based control, integral action, actuator dynamics, nonlinear systems.

1. INTRODUCTION

A mechanical system is said to be *underactuated* if its input matrix is rank-deficient. Among the various approaches that have been proposed in the literature for these systems, the *Interconnection and Damping Assignment Passivity-based Control* (IDA-PBC) methodology is particularly advantageous since it offers a physical interpretation of the control action in terms of energy and dissipation, see Ortega et al. (2002). IDA-PBC has subsequently been extended to account for actuator dynamics, including DC electric motors in Gandarilla et al. (2017), fluidic actuation in Franco and Astolfi (2022), and high-order actuator dynamics in Franco and Astolfi (2023). Besides IDA-PBC, the actuator dynamics can also be accounted for with the backstepping methodology, see Krstic et al. (1995). However, this approach does not provide the same physical interpretation of the control action in terms of system energy as IDA-PBC, which is convenient for analysis purposes. In general, the aforementioned works require the actuator states to be measured, which might not always be possible in engineering practice. Common examples include DC motors, where the armature current might not be measured, or robotic systems with harmonic drives, where the joint position and the motor position are represented by different state variables.

Previous works have shown that IDA-PBC can be implemented even if full state feedback is not available. For instance, an implementation of IDA-PBC without velocity feedback was proposed for mechanical systems with constant inertia matrix by employing the *dirty derivative* of the position, see Ortega et al. (2002). In addition, a passive momentum observer was proposed in Ferguson and McLean (2023) for a class of mechanical systems, and was incorporated in a control-by-interconnection scheme. A dynamic extension was employed in Borja et al. (2023) to replace velocity feedback in IDA-PBC for the case of potential energy shaping. Finally, in our prior work we have proposed an IDA-PBC design with dynamic extension for a class of underactuated mechanical systems with first-order actuator dynamics and uncertain coupling, see Franco and Borja (2024). The former result borrowed the idea of dynamic extension from the *integral* IDA-PBC (iIDA-PBC) methodology, which however aims to reject additive disturbances, see Donaire et al. (2017); Ferguson et al. (2017, 2019). Recent extensions of iIDA-PBC have explored a wider class of disturbances while still relying on full-state feedback, see Franco (2023); Franco et al. (2024a,b). In summary, to the best of the authors' knowledge, an implementation of IDA-PBC for systems with high-order actuator dynamics, where the actuator states are not measured, is still unavailable.

This work proposes a new design of IDA-PBC with dynamic extensions for a class of underactuated mechanical systems with series-elastic actuator (SEA) and unmeasured actuator states. Stability conditions are discussed, and the effectiveness of the controller is demonstrated via simulations on an inertia-wheel pendulum with SEA.

Notation: function arguments are omitted in equations and are provided in the inline text on first use.

2. PROBLEM SETTING

2.1 Overview of IDA-PBC

The dynamics of an underactuated mechanical system with n DOFs and control input $u \in \mathbb{R}^m$ applied through the input matrix $G(q) \in \mathbb{R}^{n\times m}$, where $\operatorname{rank}(G) = m < n$ for all $q \in \mathbb{R}^n$, is described in port-Hamiltonian form as

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & I\\ -I & 0\end{bmatrix}\begin{bmatrix}\nabla_q H\\ \nabla_p H\end{bmatrix} + \begin{bmatrix}0\\ G\end{bmatrix} u. \tag{1}$$

The system states are the position $q \in \mathbb{R}^n$ and the momenta $p = M(q)\dot{q} \in \mathbb{R}^n$. The total mechanical energy of the system,

$$H(q,p) = \frac{1}{2}p^\top M(q)^{-1}p + \Omega(q), \tag{2}$$

is the sum of the potential energy $\Omega(q)$ and the kinetic energy $T(q,p) = \frac{1}{2}p^\top M(q)^{-1}p$, where the inertia matrix

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is $M(q) = M(q)^\top \succ 0$. The remaining terms in (1) are the identity matrix I, the vector of partial derivatives of H with respect to q, $\nabla_q H$, and the vector of partial derivatives of H with respect to p, $\nabla_p H$. The passive output of (1) is $y = G^\top \nabla_p H$. The controller design aims at stabilizing the prescribed equilibrium $(q,p) = (q^\star, 0)$. This goal is achieved by using the IDA-PBC control law

$$\begin{aligned} u_{IDA} = G^\dagger \left(\nabla_q H - M_d M^{-1} \nabla_q H_d + J_2 \nabla_p H_d\right) \\ - K_v G^\top \nabla_p H_d, \end{aligned} \tag{3}$$

where $H_d(q,p) = \frac{1}{2} p^\top M_d(q)^{-1} p + \Omega_d(q) > 0$ and $G^\dagger = \left(G^\top G\right)^{-1} G^\top$, see Ortega et al. (2002). The design parameters in (3) are the potential energy $\Omega_d(q)$, the inertia matrix $M_d(q) = M_d^\top(q) \succ 0$, the matrix $J_2(q,p) = -J_2^\top(q,p)$, and the constant matrix $K_v = K_v^\top \succ 0$. To achieve the regulation goal, the potential energy Ω_d should admit a strict minimum in $q^\star$ hence verifying the conditions $\Omega_d(q) \geq 0, \forall q \in \mathbb{R}^n$, $\nabla_q \Omega_d(q^\star) = 0$ and $\nabla_q^2 \Omega_d(q^\star) \succ 0$. The control law (3) exists provided that $M_d(q)$ and $\Omega_d(q)$ verify for all $(q,p) \in \mathbb{R}^{2n}$ the partial differential equations (PDEs), also known as matching equations,

$$G^\perp \left(\nabla_q T - M_d M^{-1} \nabla_q T_d + J_2 M_d^{-1} p\right) = 0, \tag{4}$$

$$G^\perp \left(\nabla_q \Omega - M_d M^{-1} \nabla_q \Omega_d\right) = 0, \tag{5}$$

where $G^\perp$ is defined such that $G^\perp G = 0$ and $\operatorname{rank}\left(G^\perp\right) = n - m$, and $T_d = \frac{1}{2} p^\top M_d(q)^{-1} p$. The resulting closed-loop dynamics is

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1} M_d \\ -M_d M^{-1} & J_2 - G K_v G^\top \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix}, \tag{6}$$

where $y_d = G^\top \nabla_p H_d$ is a passive output of (6). Computing the time derivative of H_d along the trajectories of the closed-loop system (6) yields

$$\dot{H}_d = -\left(\nabla_p H_d\right)^\top \left(G K_v G^\top\right) \nabla_p H_d \leq 0. \tag{7}$$

Thus the equilibrium $(q,p) = (q^\star, 0)$ is asymptotically stable for all $K_v \succ 0$ provided that $y_d = G^\top \nabla_p H_d$ is detectable, that is $y_d \to 0 \implies (q,p) \to (q^\star, 0)$, see Ortega et al. (2002). Moreover, the stability properties are global if $H_d(q,p)$ is radially unbounded, see Khalil (2002). The iIDA-PBC methodology builds on the IDA-PBC (3) with a dynamic extension in order to compensate matched constant additive disturbances, see Donaire et al. (2017).

2.2 System class

This section defines a class of underactuated mechanical systems with $n = 2$ and constant G. The complete system dynamics can be represented in port-Hamiltonian form as

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\theta} \\ \dot{\omega} \end{bmatrix} &= \begin{bmatrix} 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{J} \\ 0 & 0 & -\frac{1}{J} & -\frac{K_e^2}{R_a J^2} \end{bmatrix} \begin{bmatrix} \nabla_q W \\ \nabla_p W \\ \nabla_\theta W \\ \nabla_\omega W \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K_e}{R_a J} u \end{bmatrix}, \\ W &= H + \frac{1}{2} K_a (G^\top q - \theta)^2 + \frac{1}{2} J \omega^2, \end{aligned} \tag{8}$$

where $\theta, \omega \in \mathbb{R}$ denote the angular position and velocity of the SEA, $R_a, K_e, J \in \mathbb{R}$ denote the armature resistance, the torque constant and the inertia of the DC motor driving the SEA. The armature inductance is assumed to be negligible. The spring stiffness of the SEA is $K_a \in \mathbb{R}$ and represents the coupling between the DC motor and the mechanical system, see Section 4. The following assumptions are introduced for control purposes.

Assumption 1 (Energy shaping). The system has $n = 2$ and constant G. The PDEs (4) and (5) are solvable with $M_d(q), J_2(q,p)$ and $\Omega_d(q)$. The equilibrium $q^\star$ is assignable, that is $G^\perp \nabla_q \Omega(q^\star) = 0$, and it is a strict minimum of Ω_d, that is $\Omega_d(q) \geq 0, \forall q \in \mathbb{R}^n$, $\Omega_d(q^\star) = 0, \nabla_q \Omega_d(q^\star) = 0, \nabla_q^2 \Omega_d(q^\star) \succ 0$. The output $y_d = G^\top M_d^{-1} p$ of (6) is detectable. The model parameters are known, and the system states (q,p) are measured.

This assumption summarizes the requirements of the IDA-PBC methodology, see Ortega et al. (2002). The assumption on n and G is satisfied by various mechanical systems, including the Acrobot system, the disk-on-disk system, the ball-on-beam system, and soft robotic manipulators, see Ryalat and Laila (2016); Arpenti et al. (2024). For the inertia-wheel pendulum, the PDEs (4) and (5) are solvable analytically with a constant M_d and $J_2 = 0$, and the prescribed equilibrium $q^\star = (0, q_2^\star)$ corresponds to the upright pendulum with an arbitrary position of the wheel, see Section 4.

Assumption 2 (Actuator dynamics). The SEA has second-order dynamics, where the states θ and ω are not measured. The parameters R_a, K_e, J and K_a are known.

This assumption is verified by a limited yet relevant class of actuators that exhibit second-order linear dynamics, which includes SEA and harmonic drives. The presence of the actuator dynamics prevents the use of the standard IDA-PBC control (3) since system (8) is not affine.

3. CONTROLLER DESIGN

The control law is designed for system (8), where an IDA-PBC controller (3) already exists for the mechanical subsystem (1), to achieve the regulation goal $q = q^\star$ and the closed-loop dynamics

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\theta} \\ \dot{\omega} \\ \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{bmatrix} = \begin{bmatrix} 0 & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ -S_{12}^\top & -S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ -S_{13}^\top & -S_{23}^\top & -S_{33} & S_{34} & S_{35} & S_{36} \\ -S_{14}^\top & -S_{24}^\top & -S_{34}^\top & -S_{44} & S_{45} & S_{46} \\ -S_{15}^\top & -S_{25}^\top & -S_{35}^\top & -S_{45}^\top & -S_{55} & S_{56} \\ -S_{16}^\top & -S_{26}^\top & -S_{36}^\top & -S_{46}^\top & -S_{65} & -S_{66} \end{bmatrix} \begin{bmatrix} \nabla_q W_d \\ \nabla_p W_d \\ \nabla_\theta W_d \\ \nabla_\omega W_d \\ \nabla_{\zeta_1} W_d \\ \nabla_{\zeta_2} W_d \end{bmatrix}. \tag{9}$$

The storage function $W_d(q,p,\theta,\omega,\zeta_1,\zeta_2)$ is defined as

$$\begin{aligned} W_d = H_d + \frac{1}{2} \varsigma_1^\top \varsigma_1 + \frac{1}{2} K_1 \left(\zeta_1 + \alpha G^\top p - \theta\right)^2 \\ + \frac{1}{2} \varsigma_2^\top \varsigma_2 + \frac{1}{2} K_2 \left(\zeta_2 + \beta G^\top p - \omega\right)^2, \end{aligned} \tag{10}$$

where α and β are scalar constants, ζ_1, ζ_2 are dynamic extensions that estimate θ and ω, and ς_1, ς_2 are defined as

$$\begin{aligned} \varsigma_1 = G^\dagger \left(-\nabla_q H + M_d M^{-1} \nabla_q H_d\right) + K_v G^\top \nabla_p H_d \\ - G^\dagger J_2 \nabla_p H_d - K_a \left(G^\top q - \zeta_1 - \alpha G^\top p\right), \end{aligned} \tag{11a}$$

$$\begin{aligned} \varsigma_2 = S_{13}^\top \nabla_q H_d + S_{23}^\top \nabla_p H_d + \left(\zeta_2 + \beta G^\top p\right) \\ + K_{I1} \left(\nabla_{\zeta_1} \varsigma_1\right) \varsigma_1. \end{aligned} \tag{11b}$$

The terms S_{ij} are defined so that the open-loop dynamics (8) matches the closed-loop dynamics (9), that is

$$
\begin{aligned}
&S_{12} \triangleq M^{-1}M_d, \quad S_{13} \triangleq S_{12}(G\alpha - \nabla_p\varsigma_1(\nabla_{\zeta_1}\varsigma_1)^{-1}),\\
&S_{14} \triangleq S_{16} + S_{12}G\beta, \quad S_{15} \triangleq -S_{12}\nabla_p\varsigma_1(\nabla_{\zeta_1}\varsigma_1)^{-1},\\
&S_{16} \triangleq (-S_{12}\nabla_p\varsigma_2 - S_{15}\nabla_{\zeta_1}\varsigma_2)(\nabla_{\zeta_2}\varsigma_2)^{-1}\\
&S_{22} \triangleq GK_vG^\top - J_2, \quad S_{24} \triangleq S_{26} - S_{22}G\beta,\\
&S_{23} \triangleq S_{25} - S_{22}G\alpha + GK_a/K_1,\\
&S_{25} \triangleq G\left(1 + G^\dagger S_{12}^\top\nabla_q\varsigma_1 + G^\dagger S_{22}\nabla_p\varsigma_1\right)(\nabla_{\zeta_1}\varsigma_1)^{-1}\\
&\quad + G^{\perp^\top}\left(G^\otimes S_{12}^\top\nabla_q\varsigma_1 + G^\otimes S_{22}\nabla_p\varsigma_1\right)(\nabla_{\zeta_1}\varsigma_1)^{-1},\\
&S_{26} \triangleq (S_{12}^\top\nabla_q\varsigma_2 + S_{22}\nabla_p\varsigma_2 - S_{25}\nabla_{\zeta_1}\varsigma_2)(\nabla_{\zeta_2}\varsigma_2)^{-1},\\
&S_{33} \triangleq -S_{35} + S_{23}^\top G\alpha, \; S_{34} \triangleq S_{36} - S_{23}^\top G\beta + 1/K_2,\\
&S_{35} \triangleq (S_{13}^\top\nabla_q\varsigma_1 + S_{23}^\top\nabla_p\varsigma_1)(\nabla_{\zeta_1}\varsigma_1)^{-1} - K_{I1},\\
&S_{36} \triangleq (1 + S_{13}^\top\nabla_q\varsigma_2 + S_{23}^\top\nabla_p\varsigma_2 - S_{35}\nabla_{\zeta_1}\varsigma_2)(\nabla_{\zeta_2}\varsigma_2)^{-1},\\
&S_{44} \triangleq K_{I2} - K_4, \quad S_{45} \triangleq -S_{34}^\top + S_{24}^\top G\alpha + K_a/(K_1J),\\
&S_{46} \triangleq -S_{44} + S_{24}^\top G\beta + K_e^2/(JR_aK_2),\\
&S_{55} \triangleq S_{35}^\top - S_{25}^\top G\alpha, \quad S_{56} \triangleq -S_{45}^\top + S_{25}^\top G\beta,\\
&S_{65} \triangleq S_{36}^\top - S_{26}^\top G\alpha, \quad S_{66} \triangleq S_{46}^\top - S_{26}^\top G\beta,
\end{aligned}
\tag{12}
$$

where $G^\otimes \triangleq (G^\perp G^{\perp^\top})^{-1}G^\perp$, while it follows from (11a) and (11b) that $\nabla_{\zeta_1}\varsigma_1 = K_a, \nabla_{\zeta_2}\varsigma_2 = 1, \nabla_{\zeta_1}\varsigma_2 = K_{I1}K_a^2$. The control input is

$$
\begin{aligned}
u = &\frac{R_a}{K_e}\left(\frac{K_e^2}{R_a}(\zeta_2 + \beta G^\top p) + K_a(\zeta_1 + \alpha G^\top p - G^\top q)\right)\\
&- \frac{JR_a}{K_e}\left(S_{14}^\top\nabla_q W_d + S_{24}^\top\left(\nabla_p H_d + (\nabla_p\varsigma_1)\varsigma_1 + (\nabla_p\varsigma_2)\varsigma_2\right)\right)\\
&+ \frac{JR_a}{K_e}\left(S_{45}\left((\nabla_{\zeta_1}\varsigma_1)\varsigma_1 + (\nabla_{\zeta_1}\varsigma_2)\varsigma_2\right) + S_{46}(\nabla_{\zeta_2}\varsigma)\varsigma_2\right),
\end{aligned}
\tag{13}
$$

where S_{14}, S_{24}, S_{45} and S_{46} are given in (12), ς_1, ς_2 are given in (11a) and (11b), and the time derivatives of the dynamic extensions ζ_1 and ζ_2 are

$$
\begin{aligned}
\dot\zeta_1 = &-S_{55}\left((\nabla_{\zeta_1}\varsigma_1)\varsigma_1 + (\nabla_{\zeta_1}\varsigma_2)\varsigma_2\right) + S_{56}(\nabla_{\zeta_2}\varsigma)\varsigma_2 \quad (14a)\\
&- S_{15}^\top\nabla_q W_d - S_{25}^\top\left(\nabla_p H_d + (\nabla_p\varsigma_1)\varsigma_1 + (\nabla_p\varsigma_2)\varsigma_2\right),\\
\dot\zeta_2 = &-S_{65}\left((\nabla_{\zeta_1}\varsigma_1)\varsigma_1 + (\nabla_{\zeta_1}\varsigma_2)\varsigma_2\right) - S_{66}(\nabla_{\zeta_2}\varsigma)\varsigma_2 \quad (14b)\\
&- S_{16}^\top\nabla_q W_d - S_{26}^\top\left(\nabla_p H_d + (\nabla_p\varsigma_1)\varsigma_1 + (\nabla_p\varsigma_2)\varsigma_2\right).
\end{aligned}
$$

Proposition 1 (Target closed-loop dynamics). The system (8) with *Assumptions 1–2* in closed-loop with the control law (13) and the time derivatives of the dynamic extensions (14a) and (14b), yields (9) with the storage function (10) and the parameters (12). The proof is given in Appendix A.

Proposition 2 (Stability conditions). Consider the system (8) that satisfies *Assumptions 1–2* in closed-loop with the control law (13), where the time derivatives of the dynamic extensions are given in (14a) and (14b). Define the scalar constants $K_v > 0, \alpha > 0, \beta > 0, K_1 > 0, K_2 > 0, K_{I1} > 0, K_{I2} > 0, K_4 > 0$ such that $\Xi_1 > 0, \Xi_2 > 0, \Xi_4 > 0, \Xi_5 > 0$ and $\Xi_4\Xi_5 > \Xi_3^2$ where

$$
\begin{aligned}
\Xi_1 = &K_{I1} + G^\top\left(K_a/K_1 + 1/K_a\right)(G\alpha - \nabla_p\varsigma_1/K_a)\\
&- (G\alpha - \nabla_p\varsigma_1/K_a)^\top GK_vG^\top(G\alpha - \nabla_p\varsigma_1/K_a),\\
\Xi_2 = &K_{I2} - K_4,\\
\Xi_3 = &\frac{1}{2}\left(\frac{1}{K_2} - \frac{K_a}{K_1J} - \frac{K_a}{K_1}\beta G^\top G + 2G^\top S_{22}G\beta\alpha\right),\\
\Xi_4 = &(\nabla_p\varsigma_1)^\top G\frac{K_v}{K_a^2}G^\top(\nabla_p\varsigma_1) - GG^\top\alpha/K_a - K_{I1}\\
&+ \left(1/K_a^2 + 1/K_1 - 2K_vG^\top G\alpha/K_a\right)G^\top\nabla_p\varsigma_1,\\
\Xi_5 = &K_4 - K_{I2} + K_e^2/(JR_aK_2) - G^\top GK_vGG^\top\beta^2.
\end{aligned}
\tag{15}
$$

Then the following claims hold.

C1. The equilibrium point of the closed-loop system (9), that is $(q, p, \theta, \omega, \zeta_1, \zeta_2) = \left(q^\star, 0, G^\top q^\star, 0, G^\top q^\star, 0\right)$, is locally asymptotically stable.

C2. Stability properties are global if $H_d(q, p)$ is radially unbounded.

The proof is given in Appendix B.

Remark 1. The proposed controller design builds upon the IDA-PBC (3) employing the same matrices M_d, J_2 and scalar function Ω_d. For systems with constant inertia matrix, such as the inertia-wheel pendulum, the PDE (4) is solvable with a constant M_d and with $J_2 = 0$. Thus $\nabla_p\varsigma_1 = K_vM_d^{-1}G - K_a\alpha G$, which substituted into (15) yields constant matrices. Consequently, the conditions $\Xi_1 > 0, \Xi_2 > 0, \Xi_4 > 0, \Xi_5 > 0$ and $\Xi_4\Xi_5 > \Xi_3^2$ can be evaluated globally in advance, see Section 4. Finally, $(\zeta_1 + \alpha G^\top p)$ and $(\zeta_2 + \beta G^\top p)$ can be interpreted as reduced-order observers for θ and ω respectively.

4. SIMULATION RESULTS

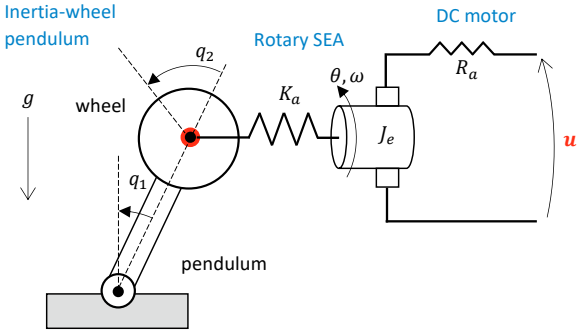


Fig. 1. Schematic of the inertia-wheel pendulum actuated by a SEA, with key model parameters.

The inertia-wheel pendulum with series-elastic actuator is depicted in Figure 1. The conventional inertia-wheel pendulum system consists of an unactuated planar inverted pendulum with an actuated wheel attached to its extremity, see Ryalat and Laila (2016). The system has two degrees-of-freedom; the angular position of the pendulum q_1 and the angular position of the wheel q_2. Its dynamics in port-Hamiltonian form is given by (1) with $\Omega = a_3(\cos(q_1) + 1)$, the input matrix is $G^\top = [0\ 1]$, while $M = \begin{bmatrix} a_1 + a_2 & a_2 \\ a_2 & a_2 \end{bmatrix}$ is the inertia matrix. The terms a_1, a_2, a_3 are constant parameters depending on the size of

the pendulum and of the wheel. The control goal consists in reaching the prescribed position $(q_1, q_2) = (0, q_2^\star)$. The IDA-PBC controller (3) for the inertia-wheel pendulum is

$$u = \gamma_2 \sin(q_1) + K_p\gamma_3(q_2 - q_2^\star + \gamma_0\epsilon_0(a_1+a_2)q_1) - K_v\frac{(-m_2p_1 + m_1p_2)}{a_1a_2(m_1m_3 - m_2^2)},$$

where $\gamma_0 = \frac{1}{a_2(m_2-m_1)}$, $\gamma_2 = -a_3\gamma_0a_2(m_2 - m_3)$, and $\gamma_3 = -\epsilon_0(a_1+a_2)\gamma_0a_2(m_2-m_3) - (-m_2a_2 + m_3(a_1+a_2))$, see Ryalat and Laila (2016). The closed-loop dynamics of the inertia-wheel pendulum is given by (6) where

$$\Omega_d = -a_3\gamma_0\cos(q_1) + \frac{1}{2}k_p(q_2 - q_2^\star + \gamma_0\epsilon_0(a_1+a_2)q_1)^2,$$
$$M_d = a_1a_2\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}, \quad J_2 = 0,$$

with $m_2 = \frac{m_1a_2}{a_1+a_2} + \epsilon_0$ and the constant tuning parameters $K_p, K_v, m_1, m_3, \epsilon_0$. The control input for the complete system is computed with (13). The terms (15) yield

$$\begin{aligned} \Xi_1 &= \frac{K_v(K_a^2 + K_1)m_1\sigma - K_v^3m_1^2K_1}{K_1K_a^2\sigma^2}, \\ \sigma &= a_2(a_2 - a_1)(m_1m_3 - m_2^2), \quad \Xi_2 = K_{I2} - K_4, \\ \Xi_3 &= \frac{JK_a(1 + 2K_2K_v\alpha\beta) - K_2(1 + J\beta)}{JK_2K_a}, \\ \Xi_4 &= (K_a\alpha)/K_1 - K_v\alpha^2 - \Xi_1, \\ \Xi_5 &= K_4 - K_{I2} + K_e^2/(JK_2R_a) - K_v\beta^2. \end{aligned} \tag{16}$$

Numerical simulations have been performed in MATLAB using an ODE23t solver with the model parameters $a_1 = 0.0124, a_2 = 0.0025, a_3 = 0.4446$, and $R_a = 0.01, K_e = 10, K_a = 1, J = 0.1$, for illustrative purposes. The IDA-PBC parameters have been set as $K_p = 10, m_1 = 0.4, m_3 = 5, \epsilon_0 = 1, K_v = 0.0002$ while $\beta = 2, K_1 = 0.4, K_2 = 0.5, K_{I2} = 2, K_4 = 1$. In addition, three sets of the remaining tuning parameters specific to the controller (13) have been employed for comparison purposes: $\alpha = 40, K_{I1} = 100$ indicated as 'Tuning A', or $\alpha = 80, K_{I1} = 200$ indicated as 'Tuning B', or $\alpha = 160, K_{I1} = 400$ indicated as 'Tuning C'. All values verify the stability conditions of *Proposition 2* with the expressions given in (16). The initial conditions are $(q_1, q_2, \dot{q}_1, \dot{q}_2, I_a, \zeta_1, \zeta_2) = (0.5, 0, 0, 0, 0, 0, 0)$ while $q_2^\star = 2$. The system response is shown in Figure 2: the new controller (13) correctly achieves the regulation goal while ζ_1, ζ_2 converge to $\theta = q_2^\star$ and $\omega = 0$ at the equilibrium. Larger values of K_{I1}, α yield a shorter transient at the cost of higher control effort. Instead, employing the IDA-PBC controller (3) results in instability since the actuator dynamics is neglected.

5. CONCLUSIONS

In this work, a new passivity-based controller has been presented for a class of underactuated mechanical systems with SEA. The controller extends previous results on IDA-PBC by proposing a new design of the dynamic extensions that removes the need to measure the actuator states. Simulation results on an inertia-wheel pendulum with SEA indicate that the proposed controller is effective. Future work will aim to extend the results to a broader class of systems, including more complex actuator dynamics.

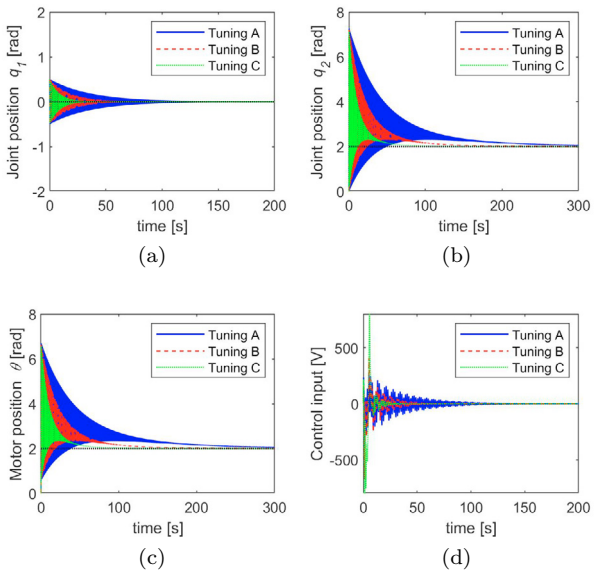


Fig. 2. Simulation results for inertia-wheel pendulum with SEA using the new controller (13) and different tuning parameters: (a) pendulum position q_1; (b) wheel position q_2; (c) motor position θ; (d) control input u.

REFERENCES

Arpenti, P., Franco, E., and Donaire, A. (2024). Integral passivity-based control of an underactuated hydraulic soft manipulator with uncertain nonlinear stiffness. In *IFAC-PapersOnLine*, volume 58, 13–18. Elsevier.

Borja, P., Chan-Zheng, C., and Scherpen, J.M. (2023). Stabilization of Physical Systems via Saturated Controllers With Partial State Measurements. *IEEE Transactions on Control Systems Technology*, 31, 2405–2419.

Donaire, A., Romero, J.G., Ortega, R., and Siciliano, B. (2017). Robust IDA-PBC for underactuated mechanical systems subject to matched disturbances. *International Journal of Robust and Nonlinear Control*, 27(6), 1000–1016.

Ferguson, J., Donaire, A., and Middleton, R. (2017). Integral control of port-Hamiltonian systems: Nonpassive outputs without coordinate transformation. *IEEE Transactions on Automatic Control*, 62(11), 5947–5953.

Ferguson, J., Donaire, A., Ortega, R., and Middleton, R. (2019). Matched disturbance rejection for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 65(4), 1710–1715.

Ferguson, J. and McLean, K. (2023). Passive momentum observer for nonholonomic systems. In *IFAC-PapersOnLine*, volume 56, 373–378. Elsevier B.V.

Franco, E. (2023). Integral passivity-based control of underactuated mechanical systems with actuator dynamics and constant disturbances. *International Journal of Robust and Nonlinear Control*, 33, 10024–10045.

Franco, E., Arpenti, P., and Donaire, A. (2024a). Integral passivity-based control of underactuated mechanical systems with state-dependent matched disturbances. *International Journal of Robust and Nonlinear Control*, 34, 3565–3585.

Franco, E., Arpenti, P., Donaire, A., and Ruggiero, F. (2024b). Integral IDA-PBC for Underactuated Mechanical Systems Subject to Matched and Unmatched

Disturbances. *IEEE Control Systems Letters*, 8, 568–573.

Franco, E. and Astolfi, A. (2022). Energy shaping control of underactuated mechanical systems with fluidic actuation. *International Journal of Robust and Nonlinear Control*, 10011–10028.

Franco, E. and Astolfi, A. (2023). Energy shaping control of a class of underactuated mechanical systems with high-order actuator dynamics. *European Journal of Control*, 72, 100828.

Franco, E. and Borja, P. (2024). Integral IDA-PBC of Underactuated Mechanical Systems with Actuator Dynamics and Uncertain Coupling. In *IFAC-PapersOnLine*, volume 58, 19–24. Elsevier.

Gandarilla, I., Santibáñez, V., and Sandoval, J. (2017). Control of a self-balancing robot driven by DC motors via IDA-PBC. In *ACM International Conference Proceeding Series*, 13–17. Association for Computing Machinery, New York, New York, USA.

Khalil, H.K. (2002). *Nonlinear systems*. Prentice-Hall, New Jersey, third edition.

Krstic, M., Kanellakopoulos, I., and Kokotovic, P. (1995). *Nonlinear and Adaptive Control Design*. Adaptive and Learning systems for signal processing, communications and control. Wiley.

Ortega, R., Spong, M., Gomez-Estern, F., and Blankenstein, G. (2002). Stabilization of a class of underactuated mechanical systems via interconnection and damping assignment. *IEEE Transactions on Automatic Control*, 47(8), 1218–1233.

Ryalat, M. and Laila, D.S. (2016). A simplified IDA-PBC design for underactuated mechanical systems with applications. *European Journal of Control*, 27, 1–16. doi:10.1016/j.ejcon.2015.12.001.

Appendix A. PROOF OF PROPOSITION 1

Computing the partial derivatives of $W_d(q,p,\theta,\omega,\zeta_1,\zeta_2)$ from (10) yields

$$\begin{aligned}
\nabla_q W_d &= \nabla_q \Omega_d + \frac{1}{2}\nabla_q\left(p^\top M_d^{-1} p\right) + (\nabla_q \varsigma_1)\varsigma_1 + (\nabla_q \varsigma_2)\varsigma_2,\\
\nabla_p W_d &= M_d^{-1} p + (\nabla_p \varsigma_1)\varsigma_1 + K_1 \alpha G\left(\zeta_1 + \alpha G^\top p - \theta\right)\\
&\quad + (\nabla_p \varsigma_2)\varsigma_2 + K_2 \beta G\left(\zeta_2 + \beta G^\top p - \omega\right),\\
\nabla_\theta W_d &= -K_1\left(\zeta_1 + \alpha G^\top p - \theta\right),\\
\nabla_\omega W_d &= -K_2\left(\zeta_2 + \beta G^\top p - \omega\right),\\
\nabla_{\zeta_1} W_d &= K_1\left(\zeta_1 + \alpha G^\top p - \theta\right) + (\nabla_{\zeta_1}\varsigma_1)\varsigma_1 + (\nabla_{\zeta_1}\varsigma_2)\varsigma_2,\\
\nabla_{\zeta_2} W_d &= K_2\left(\zeta_2 + \beta G^\top p - \omega\right) + (\nabla_{\zeta_2}\varsigma_2)\varsigma_2.
\end{aligned} \tag{A.1}$$

Equating the corresponding rows of (8) and of (9) yields the matching equations

$$\begin{aligned}
M^{-1}p &= S_{12}\nabla_p W_d + S_{13}\nabla_\theta W_d + S_{14}\nabla_\omega W_d \qquad \text{(A.2a)}\\
&\quad + S_{15}\nabla_{\zeta_1} W_d + S_{16}\nabla_{\zeta_2} W_d,\\
-\nabla_q\Omega &- \frac{1}{2}\nabla_q\left(p^\top M^{-1} p\right) - K_a G(G^\top q - \theta) = \qquad \text{(A.2b)}\\
&\quad - S_{12}^\top \nabla_q W_d - S_{22}\nabla_p W_d + S_{23}\nabla_\theta W_d\\
&\quad + S_{24}\nabla_\omega W_d + S_{25}\nabla_{\zeta_1} W_d + S_{26}\nabla_{\zeta_2} W_d,\\
\omega &= -S_{13}^\top \nabla_q W_d - S_{23}^\top \nabla_p W_d - S_{33}\nabla_\theta W_d \qquad \text{(A.2c)}\\
&\quad + S_{34}\nabla_\omega W_d + S_{35}\nabla_{\zeta_1} W_d + S_{36}\nabla_{\zeta_2} W_d,\\
\frac{K_a}{J}(G^\top q - \theta) &- \frac{K_e^2}{R_a J}\omega + \frac{K_e}{R_a J}u = \qquad \text{(A.2d)}\\
&\quad - S_{14}^\top \nabla_q W_d - S_{24}^\top \nabla_p W_d - S_{34}^\top \nabla_\theta W_d\\
&\quad - S_{44}\nabla_\omega W_d + S_{45}\nabla_{\zeta_1} W_d + S_{46}\nabla_{\zeta_2} W_d,\\
\dot{\zeta}_1 &= -S_{15}^\top \nabla_q W_d - S_{25}^\top \nabla_p W_d - S_{35}^\top \nabla_\theta W_d \qquad \text{(A.2e)}\\
&\quad - S_{45}^\top \nabla_\omega W_d - S_{55}\nabla_{\zeta_1} W_d + S_{56}\nabla_{\zeta_2} W_d,\\
\dot{\zeta}_2 &= -S_{16}^\top \nabla_q W_d - S_{26}^\top \nabla_p W_d - S_{36}^\top \nabla_\theta W_d \qquad \text{(A.2f)}\\
&\quad - S_{46}^\top \nabla_\omega W_d - S_{65}\nabla_{\zeta_1} W_d - S_{66}\nabla_{\zeta_2} W_d.
\end{aligned}$$

Step 1. Substituting $S_{13}, S_{14}, S_{15}, S_{16}$ from (12) and the partial derivatives of W_d from (A.1) into (A.2a) yields

$$\begin{aligned}
M^{-1}p &= S_{12}(M_d^{-1}p + (\nabla_p\varsigma_1)\varsigma_1 + K_1\alpha G\left(\zeta_1 + \alpha G^\top p - \theta\right))\\
&\quad + S_{12}((\nabla_p\varsigma_2)\varsigma_2 + K_2\beta G\left(\zeta_2 + \beta G^\top p - \omega\right))\\
&\quad - S_{12}(G\alpha - \nabla_p\varsigma_1(\nabla_{\zeta_1}\varsigma_1)^{-1})K_1\left(\zeta_1 + \alpha G^\top p - \theta\right)\\
&\quad - (S_{16} + S_{12}G\beta)K_2\left(\zeta_2 + \beta G^\top p - \omega\right)\\
&\quad + (-S_{12}\nabla_p\varsigma_1(\nabla_{\zeta_1}\varsigma_1)^{-1})K_1\left(\zeta_1 + \alpha G^\top p - \theta\right)\\
&\quad + \underbrace{(-S_{12}\nabla_p\varsigma_1(\nabla_{\zeta_1}\varsigma_1)^{-1})}_{S_{15}}((\nabla_{\zeta_1}\varsigma_1)\varsigma_1 + (\nabla_{\zeta_1}\varsigma_2)\varsigma_2)\\
&\quad + \underbrace{(-S_{12}\nabla_p\varsigma_2 - S_{15}\nabla_{\zeta_1}\varsigma_2)(\nabla_{\zeta_2}\varsigma_2)^{-1}}_{S_{16}}(\nabla_{\zeta_2}\varsigma_2)\varsigma_2\\
&\quad + S_{16}K_2\left(\zeta_2 + \beta G^\top p - \omega\right).
\end{aligned}$$

Refactoring terms cancels $\zeta_1, \zeta_2, \varsigma_1, \varsigma_2, \theta, \omega$. Substituting S_{12} yields $M^{-1}p = M^{-1}M_d M_d^{-1}p$, which holds for all $M_d \succ 0$.

Step 2. Substituting S_{23}, S_{24} from (12) and the partial derivatives of W_d from (A.1) into (A.2b) yields

$$\begin{aligned}
&-\nabla_q\Omega - \frac{1}{2}\nabla_q\left(p^\top M^{-1}p\right) - K_a G(G^\top q - \theta) =\\
&- S_{12}^\top\left(\nabla_q\Omega_d + \frac{1}{2}\nabla_q\left(p^\top M_d^{-1}p\right) + (\nabla_q\varsigma_1)\varsigma_1 + (\nabla_q\varsigma_2)\varsigma_2\right)\\
&- S_{22}(M_d^{-1}p + (\nabla_p\varsigma_1)\varsigma_1 + K_1\alpha G\left(\zeta_1 + \alpha G^\top p - \theta\right))\\
&- S_{22}((\nabla_p\varsigma_2)\varsigma_2 + K_2\beta G\left(\zeta_2 + \beta G^\top p - \omega\right))\\
&- (S_{25} - S_{22}G\alpha + GK_a/K_1)K_1\left(\zeta_1 + \alpha G^\top p - \theta\right)\\
&- (S_{26} - S_{22}G\beta)K_2\left(\zeta_2 + \beta G^\top p - \omega\right)\\
&+ S_{25}(K_1\left(\zeta_1 + \alpha G^\top p - \theta\right) + (\nabla_{\zeta_1}\varsigma_1)\varsigma_1 + (\nabla_{\zeta_1}\varsigma_2)\varsigma_2)\\
&+ S_{26}(K_2\left(\zeta_2 + \beta G^\top p - \omega\right) + (\nabla_{\zeta_2}\varsigma_2)\varsigma_2).
\end{aligned}$$

Refactoring terms in the former expression cancels θ, ω, ζ_2. Substituting S_{25}, S_{26} yields

$$\begin{aligned}
&-\nabla_q\Omega - \frac{1}{2}\nabla_q\left(p^\top M^{-1}p\right) - K_a G(G^\top q - \zeta_1 - \alpha G^\top p) =\\
&- S_{12}^\top(\nabla_q\Omega_d + \frac{1}{2}\nabla_q\left(p^\top M_d^{-1}p\right) + (\nabla_q\varsigma_1)\varsigma_1 + (\nabla_q\varsigma_2)\varsigma_2)\\
&- S_{22}(M_d^{-1}p + (\nabla_p\varsigma_1)\varsigma_1 + (\nabla_p\varsigma_2)\varsigma_2)\\
&+ G\left(1 + G^\dagger S_{12}^\top\nabla_q\varsigma_1 + G^\dagger S_{22}\nabla_p\varsigma_1\right)\varsigma_1\\
&\quad + G^{\perp\top}\left(G^\otimes S_{12}^\top\nabla_q\varsigma_1 + G^\otimes S_{22}\nabla_p\varsigma_1\right)\varsigma_1\\
&+ S_{25}(\nabla_{\zeta_1}\varsigma_2)\varsigma_2 + (S_{12}^\top\nabla_q\varsigma_2 + S_{22}\nabla_p\varsigma_2 - S_{25}\nabla_{\zeta_1}\varsigma_2)\varsigma_2.
\end{aligned}$$

Refactoring terms in the former expression cancels ς_2, while multiplying it by $G^\dagger$ yields ς_1 in (11a) on both sides of the equation. Substituting S_{12}, S_{22} and multiplying the former expression by $G^\perp$ yields the sum of the PDEs (4) and (5), which are verified by *Assumption 1.*

Step 3. Substituting $S_{33}, S_{34}, S_{35}, S_{36}$ from (12) and the partial derivatives of W_d from (A.1) into (A.2c) yields

$$\begin{aligned}\omega = & -S_{13}^\top(\nabla_q H_d + (\nabla_q \varsigma_1)\varsigma_1 + (\nabla_q \varsigma_2)\varsigma_2)\\ & - S_{23}^\top\left(M_d^{-1}p + (\nabla_p \varsigma_1)\varsigma_1 + K_1\alpha G\left(\zeta_1 + \alpha G^\top p - \theta\right)\right)\\ & - S_{23}^\top\left((\nabla_p \varsigma_2)\varsigma_2 + K_2\beta G\left(\zeta_2 + \beta G^\top p - \omega\right)\right)\\ & + (-S_{35} + S_{23}^\top G\alpha)K_1\left(\zeta_1 + \alpha G^\top p - \theta\right)\\ & - \left(S_{36} - S_{23}^\top G\beta + \frac{1}{K_2}\right)K_2\left(\zeta_2 + \beta G^\top p - \omega\right)\\ & + \underbrace{\left(\left(S_{13}^\top \nabla_q \varsigma_1 + S_{23}^\top \nabla_p \varsigma_1\right)\left(\nabla_{\zeta_1}\varsigma_1\right)^{-1} - K_{I1}\right)}_{S_{35}}(\nabla_{\zeta_1}\varsigma_1)\varsigma_1\\ & + S_{35}(K_1\left(\zeta_1 + \alpha G^\top p - \theta\right) + (\nabla_{\zeta_1}\varsigma_2)\varsigma_2)\\ & + \underbrace{\left(1 + S_{13}^\top \nabla_q \varsigma_2 + S_{23}^\top \nabla_p \varsigma_2 - S_{35}\nabla_{\zeta_1}\varsigma_2\right)}_{S_{36}\nabla_{\zeta_2}\varsigma_2}\varsigma_2\\ & + S_{36}K_2\left(\zeta_2 + \beta G^\top p - \omega\right),\end{aligned}$$

Refactoring the former expression cancels θ and ω yielding ς_2 in (11b) on both sides of the equation.

Step 4. Substituting S_{45}, S_{46} from (12) and the partial derivatives of W_d from (A.1) into (A.2d) yields

$$\begin{aligned}& \frac{K_a}{J}(G^\top q - \theta) - \frac{K_e^2}{R_a J}\omega + \frac{K_e}{R_a J}u =\\ & - S_{14}^\top(\nabla_q H_d + (\nabla_q \varsigma_1)\varsigma_1 + (\nabla_q \varsigma_2)\varsigma_2)\\ & - S_{24}^\top\left(M_d^{-1}p + (\nabla_p \varsigma_1)\varsigma_1 + K_1\alpha G\left(\zeta_1 + \alpha G^\top p - \theta\right)\right)\\ & - S_{24}^\top\left((\nabla_p \varsigma_2)\varsigma_2 + K_2\beta G\left(\zeta_2 + \beta G^\top p - \omega\right)\right)\\ & + S_{34}^\top K_1\left(\zeta_1 + \alpha G^\top p - \theta\right)\\ & + S_{44}K_2\left(\zeta_2 + \beta G^\top p - \omega\right)\\ & + \underbrace{\left(-S_{34}^\top + S_{24}^\top G\alpha + \frac{K_a}{K_1 J}\right)}_{S_{45}}K_1\left(\zeta_1 + \alpha G^\top p - \theta\right)\\ & + \underbrace{\left(-S_{44} + S_{24}^\top G\beta + \frac{K_e^2}{JR_aK_2}\right)}_{S_{46}}K_2\left(\zeta_2 + \beta G^\top p - \omega\right)\\ & + S_{45}((\nabla_{\zeta_1}\varsigma_1)\varsigma_1 + (\nabla_{\zeta_1}\varsigma_2)\varsigma_2) + S_{46}(\nabla_{\zeta_2}\varsigma_2)\varsigma_2,\end{aligned}$$

Refactoring terms cancels θ and ω yielding (13).

Step 5. Substituting S_{54}, S_{55}, S_{56} from (12) and the partial derivatives of W_d from (A.1) into (A.2e) yields

$$\begin{aligned}\dot\zeta_1 = & -S_{15}^\top(\nabla_q H_d + (\nabla_q \varsigma_1)\varsigma_1 + (\nabla_q \varsigma_2)\varsigma_2)\\ & - S_{25}^\top\left(M_d^{-1}p + (\nabla_p \varsigma_1)\varsigma_1 + K_1\alpha G\left(\zeta_1 + \alpha G^\top p - \theta\right)\right)\\ & - S_{25}^\top\left((\nabla_p \varsigma_2)\varsigma_2 + K_2\beta G\left(\zeta_2 + \beta G^\top p - \omega\right)\right)\\ & + S_{35}^\top K_1\left(\zeta_1 + \alpha G^\top p - \theta\right) + S_{45}^\top K_2\left(\zeta_2 + \beta G^\top p - \omega\right)\\ & -(S_{35}^\top - S_{25}^\top G\alpha)K_1\left(\zeta_1 + \alpha G^\top p - \theta\right)\\ & -(S_{35}^\top - S_{25}^\top G\alpha)((\nabla_{\zeta_1}\varsigma_1)\varsigma_1 + (\nabla_{\zeta_1}\varsigma_2)\varsigma_2)\\ & + (-S_{45}^\top + S_{25}^\top G\beta)(K_2\left(\zeta_2 + \beta G^\top p - \omega\right) + (\nabla_{\zeta_2}\varsigma_2)\varsigma_2),\end{aligned}$$

Refactoring terms cancels θ and ω yielding (14a).

Step 6. Substituting S_{65}, S_{66} from (12) and the partial derivatives of W_d from (A.1) into (A.2f) yields

$$\begin{aligned}\dot\zeta_2 = & -S_{16}^\top(\nabla_q H_d + (\nabla_q \varsigma_1)\varsigma_1 + (\nabla_q \varsigma_2)\varsigma_2)\\ & - S_{26}^\top\left(M_d^{-1}p + (\nabla_p \varsigma_1)\varsigma_1 + K_1\alpha G\left(\zeta_1 + \alpha G^\top p - \theta\right)\right)\\ & - S_{26}^\top\left((\nabla_p \varsigma_2)\varsigma_2 + K_2\beta G\left(\zeta_2 + \beta G^\top p - \omega\right)\right)\\ & + S_{36}^\top K_1\left(\zeta_1 + \alpha G^\top p - \theta\right) + S_{46}^\top K_2\left(\zeta_2 + \beta G^\top p - \omega\right)\\ & -(S_{36}^\top - S_{26}^\top G\alpha)K_1\left(\zeta_1 + \alpha G^\top p - \theta\right)\\ & -(S_{36}^\top - S_{26}^\top G\alpha)((\nabla_{\zeta_1}\varsigma_1)\varsigma_1 + (\nabla_{\zeta_1}\varsigma_2)\varsigma_2)\\ & - (S_{46}^\top - S_{26}^\top G\beta)(K_2\left(\zeta_2 + \beta G^\top p - \omega\right) + (\nabla_{\zeta_2}\varsigma_2)\varsigma_2),\end{aligned}$$

Refactoring terms cancels θ and ω yielding (14b) □

Appendix B. PROOF OF PROPOSITION 2

The storage function $W_d(q, p, \theta, \omega, \zeta_1, \zeta_2)$ defined in (10) is positive definite by design. Computing its time derivative along the trajectories of the closed-loop system (9) yields

$$\begin{aligned}\dot W_d = & -(\nabla_p W_d)^\top S_{22}\nabla_p W_d - (\nabla_\theta W_d)^\top S_{33}\nabla_\theta W_d\\ & - (\nabla_\omega W_d)^\top S_{44}\nabla_\omega W_d - (\nabla_{\zeta_1} W_d)^\top S_{55}\nabla_{\zeta_1} W_d\\ & + (\nabla_{\zeta_1} W_d)^\top S_{56}\nabla_{\zeta_2} W_d - (\nabla_{\zeta_2} W_d)^\top S_{65}\nabla_{\zeta_1} W_d\\ & - (\nabla_{\zeta_2} W_d)^\top S_{66}\nabla_{\zeta_2} W_d.\end{aligned} \tag{B.1}$$

Substituting the matrices S_{ij} form (12) into (B.1) while omitting skew-symmetric terms, and recalling that $\nabla_{\zeta_1}\varsigma_1 = K_a, \nabla_{\zeta_2}\varsigma_2 = 1, \nabla_{\zeta_1}\varsigma_2 = K_{I1}K_a^2$ yields

$$\begin{aligned}\dot W_d \le & -(\nabla_p W_d)^\top (GK_v G^\top)\nabla_p W_d\\ & - (\nabla_\theta W_d)^\top \Xi_1 \nabla_\theta W_d - (\nabla_\omega W_d)^\top \Xi_2 \nabla_\omega W_d\\ & - [\nabla_{\zeta_1} W_d\ \nabla_{\zeta_2} W_d]\begin{bmatrix}\Xi_4 & \Xi_3\\ \Xi_3 & \Xi_5\end{bmatrix}\begin{bmatrix}\nabla_{\zeta_1} W_d\\ \nabla_{\zeta_2} W_d\end{bmatrix},\end{aligned} \tag{B.2}$$

where $\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5, \Xi_6$ are scalars defined in (15). It follows from (12) and (15) that $\Xi_1 > 0$ and $\Xi_2 > 0$ for sufficiently large $K_{I1} > 0$ and $K_{I2} > K_4 > 0$. In addition, $\Xi_4 > 0$ for a sufficiently large $K_v > 0$ and a sufficiently small $\alpha > 0$, while $\Xi_5 > 0$ for a sufficiently small $\beta > 0$ and for $K_2 > 0$ such that $K_4 + K_e^2/(JR_aK_2) > K_{I2}$. Finally, $\Xi_4\Xi_5 > \Xi_3^2$ for sufficiently small α, β.

C1. It follows from (B.2) that $\dot W_d \le 0$, thus the equilibrium is stable and all states are bounded. Invoking LaSalle's theorem (see Theorem 3.4 in Khalil (2002)), it follows that the trajectories of (9) converge asymptotically to the invariant set $G^\top\nabla_p W_d = 0 \cap \nabla_{\zeta_1} W_d = 0 \cap \nabla_{\zeta_2} W_d = 0 \cap \nabla_\theta W_d = 0 \cap \nabla_\omega W_d = 0$. It follows then from (A.1) that the invariant set corresponds to $G^\top M_d^{-1}p = 0, \varsigma_1 = 0, \varsigma_2 = 0$. Computing $\dot p$ from (9) within this set yields $\nabla_q \Omega_d = 0$, since $y_d = G^\top M_d^{-1}p$ is detectable by *Assumption 1*, see Ortega et al. (2002). Since $\nabla_q \Omega_d(q^\star) = 0$ by design, the set corresponds to $\zeta_1 = \theta = G^\top q^\star$ and $\zeta_2 = \omega = 0$. Thus, the equilibrium point $(q, p, \theta, \omega, \zeta_1, \zeta_2) = \left(q^\star, 0, G^\top q^\star, 0, G^\top q^\star, 0\right)$ is locally asymptotically stable.

C2. To prove the global claim it is sufficient to show that W_d is radially unbounded, see Khalil (2002). To this end, note that if $(q, p) \to \infty \implies H_d \to \infty$, then it follows from (10) that $(q, p) \to \infty \implies W_d \to \infty$. In addition, $\theta, \omega \to \infty \implies W_d \to \infty$ unless $\zeta_1, \zeta_2 \to \infty$. Finally, $\zeta_1, \zeta_2 \to \infty \implies \varsigma_1, \varsigma_2 \to \infty$ unless either $q, p \to \infty$, while $\varsigma_1, \varsigma_2 \to \infty \implies W_d \to \infty$. Thus W_d is radially unbounded and the stability properties are global □