Fast and Slow Optimal Trading with Exogenous Information

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Abstract

We model the interaction between an investor executing trades at low frequency and a high-frequency trader as a multiperiod stochastic Stackelberg game. The high-frequency trader exploits price information more frequently and is subject to periodic inventory constraints.

We are able to explicitly compute the equilibrium strategies, in two steps. We first derive the optimal strategy of the high-frequency trader given any strategy adopted by the investor. Then, we solve the problem of the investor given the optimal strategy of the high-frequency trader, in terms of the resolvent of a Fredholm integral equation. Our results show that the high-frequency trader adopts a predatory strategy whenever the value of the trading signal is high, and follows a cooperative strategy otherwise. We also show that there is a net gain in performance for the investor from taking into account the order flow of the high-frequency trader. A U-shaped intraday pattern in trading volume is shown to arise endogenously as a result of the strategic behavior of the agents.

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1 Introduction

Modern financial markets involve a range of participants who place buy and sell orders across a wide spectrum of time scales: on one end, pension funds and mutual funds who rebalance typically on a monthly time scale while, on the other end of the spectrum, electronic market makers and high frequency trading firms who submit several thousands of orders per second (Cont, 2011), while having strict inventory constraints (U.S. Securities and Exchange Commission, 2014, p.4). Although this heterogeneity in time scales has been always present, the development of computerized trading in electronic markets has substantially widened the range of frequencies at which various market participants operate. The interaction between the flow of buy and sell orders from these different participants results in an aggregate order flow which is the superposition of components across a wide range of frequencies. The consequences of this phenomenon for market volatility, price dynamics and market stability have yet to be systematically explored.

This heterogeneity of frequencies stands in contrast with mathematical models of market microstructure which are often formulated in terms of homogeneous agents operating at a single time scale (Gârleanu and Pedersen, 2016; Evangelista and Thamsten, 2020; Voß, 2022; Neuman and Schied, 2022; Micheli et al., 2021; Casgrain and Jaimungal, 2020; Fu et al., 2021; Neuman and Voß, 2021). Yet, the repeated occurrence of 'flash crashes' (see e.g. Kirilenko et al. (2017)) demonstrates that components at different frequencies may strongly interact and possibly lead to market disruption, calling for a modeling framework which incorporates the interaction of agents operating on different time scales.

A frequent narrative, popularized by the bestseller *Flash Boys* (Lewis, 2014), is the predatory nature of HFTs towards institutional investors. Empirical evidence is less clear: while Korajczyk and Murphy (2018) and Van Kervel and Menkveld (2019) find evidence pointing to predatory behaviour of HFTs with respect to institutional investor order flow, other studies, such as Chen and Garriott (2020), do not find any

evidence of such predatory behaviour. These studies show that the strategic interaction between HFTs and institutional investors is more complex and may depend on several factors.

To investigate these phenomena, we propose a model for the dynamics of prices and order flow in a market where an institutional investor and a high-frequency trader observe a trading signal and submit buy and sell orders at different frequencies. We model the interaction between these two agents as a multiperiod stochastic game, where the institutional investor and the high-frequency trader interact through the aggregate order flow, which generates temporary and permanent impact on the asset price. The high-frequency trader exploits the trading signal at each period and is continuously subject to inventory constraints, while the institutional investor has a limited access to the signal and is only subject to a terminal inventory constraint. Given the asymmetry between the agents, it is natural to model this setting in terms of a Stackelberg game, where the HFT, the minor agent, takes advantage of the signal and the order flow generated by the institutional investor, the major agent.

Our first result describes, in Theorem 3.2, the unique optimal strategy of the high-frequency trader in response to any strategy adopted by the investor. The challenging part in establishing a Stackelberg equilibrium is to derive the strategy of the major agent i.e. the one who plays first. We develop a novel approach for this class of Stackelberg games in order to derive the major agent's optimal strategy given the optimal signal-adaptive strategy of the minor agent, using tools from the theory of integral equations. Specifically, in Theorem 3.5 we describe the optimal strategy for the major agent in terms of the resolvent of a Fredholm integral equation, thus establishing the unique multi-period Stackelberg equilibrium of the game. In Section 4 we describe properties of the solutions to the Stackelberg game and in Section 5 we derive the additional technical steps that are needed in order to obtain such explicit results directly from Theorems 3.2 and 3.5.

We propose a novel method using tools from the theory of integral equations to solve for equilibrium. This allows us to explicitly derive the equilibrium of this multi-period Stackelberg game, an atypical situation in such models which are usually intractable, see e.g. Carlin et al. (2007); Schöneborn and Schied (2009) and Roşu (2019). These analytical and computational aspects may be of independent interest and are detailed in Sections 5 and 10 and in Appendix A, where we describe the numerical method used to compute the optimal strategy of the major agent and provide the proof of its convergence.

From our theoretical results we obtain explicit expressions for the agents optimal strategies and derive interesting economic insights regarding the behaviour of high-

frequency traders and best practices for institutional investors executing large trades (see Section 4):

(i) Our results suggest that the high-frequency trader adopts a predatory strategy whenever the value of the trading signal is high, and follows a cooperative strategy otherwise. In the language of Van Kervel and Menkveld (2019), we find that HFTs "trade with the wind" when the wind is strong i.e when trading signals convey strong value of information, but "lean against the wind" otherwise. Figure 1 shows examples of such strategies.

(ii) Comparing the revenue of the investor's optimal order execution with the case where the agent is not taking into account the HFT's trading activity, we show taking the HFT's impact into account leads to a net gain in performance for the investor, as shown in Figure 5. In fact, the investor's optimal strategy displays on average a substantial improvement in performance. This contrasts with the belief that high-frequency traders order-flow can be regarded as a "noise" whose effect on low-frequency strategies tends to 'average out'.

(iii) We show that the well-known U-shaped intraday profile for trading volume arises endogenously in our model as a result of the strategic behavior of institutional investors and HFTs: the optimal trading strategy requires agents to trade more near the open and the close, as observed in Figure 6.

The emergence of predatory trading as an optimal strategy was studied by Carlin et al. (2007) in a single period setting and by Schöneborn and Schied (2009) in a two-period model. Carlin et al. (2007) studied a single-period model with two types of revenue-maximizing agents: *sellers* who start with a positive amount of assets and *competitors* who have zero initial positions. The (Nash) equilibrium strategy is described as follows: if the seller is liquidating then the competitor is first selling and later buying back her position back due to inventory constraints. In the two period model Schöneborn and Schied (2009) the seller can liquidate only in the first period, while the competitor can execute her strategy over two periods. Depending on the magnitude of price impact, there are two possible scenarios: either the competitor is buying in the first period and then selling in the second period, i.e. introducing cooperative strategies in the game (Schöneborn and Schied, 2009, Fig. 8), or doing a round trip of selling first and then closing the position all in the first period.

Our model has some critical differences with Schöneborn and Schied (2009): we assume the minor agent (HFT) is trading at a higher frequency than the major agent. This is reflected in the model via periodic inventory constraints for the HFT. The

minor agent is also reacting continuously to exogenous information while the major agent has access to the information only at the beginning of the trade. An important implication is that, in contrast to Schöneborn and Schied (2009), the minor agent's optimal portfolio is stochastic.

Another major difference with these models is the type of equilibrium. In Schöneborn and Schied (2009) an open-loop Nash equilibrium was derived, which means that all traders optimise synchronously. A more realistic representation of a heterogeneous market with participants intervening at different frequencies and latencies is to consider a *Stackelberg* equilibrium where the minor agents react to the major agent's order flow.

As neither Carlin et al. (2007) nor Schöneborn and Schied (2009) take into account exogenous trading signals, their optimal strategies are found to be deterministic. One of the main conclusions of our analysis is that this aspect has a crucial effect on the behaviour of the major agent and the minor agent, which is not captured in Carlin et al. (2007) and Schöneborn and Schied (2009).

Last but not least, despite the multiple ingredients in our model, we are able to derive explicit solutions in the multiperiod case, allowing futher analytical insights into the structure of the equilibrium strategies, in contrast to Schöneborn and Schied (2009), where only the two period model is tractable.

Roşu (2019) also studied a discrete-time model where fast traders, whose decisions depend on a market signal, trade simultaneously with slow traders, who observe a lagged version of the signal. However, in contrast to our setting, fast traders do not have different objectives nor inventory constraints, which are key ingredients in our model.

Finally, in Cartea and Sánchez-Betancourt (2022) a game between a broker who provides liquidity to an informed trader and to a noise trader was studied. In order to decouple the optimal execution problem of the informed trader from the broker's strategy, it was assumed that the informed trader's cost functional contains a log-likelihood ratio (also referred to as score function) between the probability measure of the unaffected asset price and a probability measure from class in which the asset price has an additional drift term, representing a conjectured brokers strategy. In this paper we take a different approach, in which both the institutional investor and the high-frequency trader's problems are fully coupled (see Definition 2.1).

The emergence of U-shaped patterns in trading volume is well-documented Wood et al. (1985). Admati and Pfleiderer (2015) provide a partial explanation of intraday concentration as a result of strategic interaction between informed traders and 'noise' traders. Our results sharpen the results of Admati and Pfleiderer (2015) by providing

an endogenous mechanism for the U-shaped intraday pattern in trading volume.

Outline. Section 2 defines the ingredients of the model. Our main results regarding the explicit solution to the Stackelberg game are presented in Section 3. Section 4 contains the illustrations and the financial interpretation of the main results. In Section 5 we derive rigorously the numerical scheme that we have used in order to plot the solutions in Section 4. The proofs of the results of this paper are given in Sections 6–10 as well as in Appendices A–C.

2 Trading fast and slow: a Stackelberg game

We consider a market with two participants: an investor, who is liquidating an initial amount of shares in a risky asset, and a high-frequency trader (HFT) who trades in the same asset. We model the interaction between these two agents as a Stackelberg game: in the terminology of Stackelberg games, the investor is the "major agent' and the HFT is the "minor agent".

We represent market scenarios over a horizon $T > 0$ through a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions of right continuity and completeness. The set $\mathcal{H}^2$ represents the class of all (special) semimartingales $P = (P_t)_{t\in[0,T]}$ whose canonical decomposition $P = M + A$ into a (local) martingale $M = (M_t)_{t\in[0,T]}$ and a predictable finite-variation process $A = (A_t)_{t\in[0,T]}$ satisfies

$$\mathbb{E}\left[\langle M\rangle_T\right] + \mathbb{E}\left[\left(\int_0^T |dA_s|\right)^2\right] < \infty. \tag{2.1}$$

We denote by $L^2([0,T])$ the space of square integrable functions $f : [0,T] \to \mathbb{R}$ and by $\langle\cdot,\cdot\rangle_{L^2}$ the inner product on $L^2([0,T])$, that is

$$\langle f, g\rangle_{L^2} = \int_0^T f(t)g(t)dt, \quad f, g \in L^2([0,T]),$$

and by $||\cdot||_{L^2}$ the associated norm.

Admissible strategies and price impact. The major agent has an initial holding of $q_0 \in \mathbb{R}$ shares in a risky asset. Her trading rate $\nu^0 = (\nu^0_t)_{t\in[0,T]}$ is chosen from the class of fuel-constrained *deterministic* admissible strategies $\mathcal{A}^{q_0}_M$, which is defined as

$$\mathcal{A}^{q_0}_M := \left\{\nu \in L^2([0,T]) \text{ s.t. } \int_0^T \nu_t dt = q_0\right\}. \tag{2.2}$$

Her trading rate ν^0 affects her inventory process Q^{0,ν^0} so that

$$Q_t^{0,\nu^0} = q_0 - \int_0^t \nu_s^0 ds, \quad 0 \leq t \leq T. \tag{2.3}$$

The minor agent, being a proprietary high frequency trader, is assumed to have a zero initial position in the risky asset. Her trading rate $\nu^1 = (\nu_t^1)_{t\in[0,T]}$ is chosen from a class of *adaptive* admissible strategies

$$\mathcal{A}_m := \left\{ \nu \text{ progressively measurable s.t. } \mathbb{E}\left[\int_0^T \nu_s^2 ds\right] < \infty \right\}. \tag{2.4}$$

Her trading rate ν^1 affects her inventory process Q^{1,ν^1} so that

$$Q_t^{1,\nu^1} = -\int_0^t \nu_s^1 ds, \quad 0 \leq t \leq T. \tag{2.5}$$

Throughout, we use the notation $\nu = (\nu^0, \nu^1)$ for the major agent's control ν^0 and the minor agent's control ν^1. Once ν is fixed, the visible asset mid-price P^ν satisfies

$$P_t^\nu = P_t - Y_t^\nu, \quad 0 \leq t \leq T, \tag{2.6}$$

where $P \in \mathcal{H}^2$ and where Y^ν is the permanent price impact price impact à la Almgren and Chriss (2000), which is generated by both agents and which is given by

$$Y_t^\nu = \int_0^t (\kappa_0 \nu_s^0 + \kappa_1 \nu_s^1) ds, \tag{2.7}$$

where κ_i, $i = 1, 2$, are positive constants.

Major agent: institutional investor. The major agent's execution price is affected instantaneously in an adverse manner through the presence of linear temporary price impact. The major agent's execution price is taken to be

$$S_t^{0,\nu} = P_t^\nu - \lambda_0 \nu_t^0, \tag{2.8}$$

where λ_0 is a positive constant measuring the magnitude of her temporary price impact. As a result, the major agent's cash holdings satisfy

$$X_t^{0,\nu} = x_0 + \int_0^t S_s^{0,\nu} \nu_s^0 ds, \quad 0 \leq t \leq T. \tag{2.9}$$

The major agent's objective is to optimally unwind her initial position q_0 by the trading horizon T, so to minimise her execution costs. This is equivalent to maximising the expected revenues from her liquidation, therefore, we take the major agent's performance functional to be

$$H^0\left(\nu^0;\nu^1\right) := \mathbb{E}\left[X_T^{0,\nu}\right]. \tag{2.10}$$

Note that the linear temporary price impact in (2.8) translates into quadratic trading costs with respect to the trading speed ν_t^0, in the major agent's cost functional in (2.10).

Minor agent: high-frequency trader. As in the case of the major agent, the transactions of the minor agent create temporary price impact, such that the execution price of her orders is given by

$$S_t^{1,\nu} = P_t^{\nu} - \lambda_1 \nu_t^1. \tag{2.11}$$

where λ_1 is a positive constant. Note that the temporary price impact parameter is likely to be smaller for the minor agent as HFTs can take advantage of the order-book realtime information in order to reduce their price impact.

The minor agent's cash process is given by

$$X_t^{1,\nu} = x_1 + \int_0^t S_s^{1,\nu}\nu_s^1 ds, \quad 0 \le t \le T. \tag{2.12}$$

The minor agent wishes to maximise her cash, however as an HFT, she is inclined to avoid *overnight risk*, specifically, in the form of non-zero overnight inventory. As an example, consider T to be one business week, such that $[0,T]$ can be partitioned in five disjoint and contiguous intervals of equal duration τ, where each intervals represents the market hours of each business day from Monday to Friday. Without loss of generality, in the context of our example we assume that the minor agent's intraday risk preferences are independent of the business day considered and we ignore the possibility of after-hours trading. Since the minor agent wishes to close her position by the end of each day, then as often done for terminal inventory penalties in the context of single-day liquidations, we can introduce a penalisation for non-zero inventory at the end of each day. These dynamic inventory preferences can be accounted by modelling the running inventory costs of the minor agent via a periodic function of period τ which drastically increases towards the end of each day (see e.g. (4.6)), i.e. as t approaches $\tau, 2\tau, 3\tau, 4\tau, 5\tau$ from the left. Mathematically,

in order to capture the minor agent's dynamic inventory preferences of our example and more general ones, we define the minor agent's running inventory costs in terms of a function $\phi^1 : [0,T] \to \mathbb{R}_+$ which we take to be piecewise continuous and locally bounded.

The minor agent risk-revenue functional is therefore given by

$$H^1(\nu^1;\nu^0) := \mathbb{E}\left[X_T^{1,\nu} + Q_T^{1,\nu^1}\left(P_T^\nu - \alpha Q_T^{1,\nu^1}\right) - \int_0^T \phi_t^1 \left(Q_t^{1,\nu^1}\right)^2 dt\right]. \tag{2.13}$$

The first two terms in (2.13) represent the trader's terminal wealth; that is, her final cash position, accounting for the accrued trading costs which are induced by temporary price impact and the permanent price impact of both agents as prescribed in (2.11), as well as the mark-to-market value of her terminal risky asset position. The third and fourth terms in (2.13) implement a penalty $\phi_t^1 > 0$ and $\alpha > 0$ on her running and terminal inventory, respectively. Also observe that $H^1(\nu^1;\nu^0) < \infty$ for any pair of admissible strategies $\nu^0 \in \mathcal{A}_M^{q_0}$ and $\nu^1 \in \mathcal{A}_m$.

The Stackelberg game. We formulate the competition between the major agent and the minor agent as a stochastic Stackelberg game in which the minor agent is reacting to the major agent's trading. Mathematically, the game unfolds in two steps:

(i) *Minor Agent's Problem:* for a given major agent's liquidation strategy $\nu^0 \in \mathcal{A}_M^{q_0}$, the minor agent chooses her own strategy $\nu^{1,*}(\nu^0) \in \mathcal{A}_m$ in order to maximise her objective functional H^1;

(ii) *Major Agent's Problem:* given the optimal minor agent's strategy $\nu^{1,*}$ established in (i), the major agent determines the optimal liquidation strategy $\nu^{0,*} \in \mathcal{A}_M^{q_0}$ in order to maximise her objective functional H^0.

In the context of our model, we formalise the definition of Stackelberg equilibrium as follows.

Definition 2.1 (Stackelberg equilibrium)**.** *A pair $\nu^* := (\nu^{0,*}, \nu^{1,*}(\nu^{0,*}))$ where $\nu^{0,*}$ and $\nu^{1,*}(\nu^{0,*})$ solve the major and minor agent's problems, respectively, is called a Stackelberg equilibrium.*

In order to determine the Stackelberg equilibrium of Definition 2.1, we need to perform the following two steps:

(i) Solve the minor agent's stochastic control problem for any admissible major agent strategy ν^0. This will determine the minor agent's optimal strategy $\nu^{1,*}(\nu^0)$ for any given strategy of the major agent;

(ii) Solve the major agent's optimization problem assuming the minor agent is adopting the optimal strategy $\nu^{1,*}(\nu^0)$ that we obtained in step (i). This will determine the major agent's optimal strategy $\nu^{0,*}$.

Then, by Definition 2.1, the pair $(\nu^{0,*}, \nu^{1,*}(\nu^{0,*}))$ determined through the steps just described is the Stackelberg equilibrium of the game.

3 Optimal strategies

Our main analytical result is an explicit charaterization of the unique Stackelberg equilibrium of the game. As stated in Section 2, we start by solving the minor agent's problem.

3.1 Optimal strategy of the minor agent

We denote by $L^2([0,T]^2)$ the space of measurable kernels $\mathcal{T} : [0,T]^2 \to \mathbb{R}$ such that

$$\int_0^T \int_0^T \mathcal{T}(t,s)^2 dtds < \infty. \tag{3.1}$$

Henceforth, we make the following assumption.

Assumption 3.1. *We assume that the parameters α in (2.13) and κ_1 in (2.7) are chosen such that*

$$2\alpha \geq \kappa_1.$$

Let $r^1 = (r^1_t)_{t\in[0,T]}$ be the solution to the following Riccati equation with a time varying coefficient,

$$\begin{cases} \partial_t r^1_t &= \frac{1}{\lambda_1}\phi^1_t - (r^1_t)^2, \\ r^1_T &= -\frac{2\alpha-\kappa_1}{2\lambda_1}. \end{cases} \tag{3.2}$$

Under Assumption 3.1, the solution r^1 of (3.2) exists and is unique over $[0,T]$ (see Proposition 6.5). We further define

$$\xi^{\pm}_t := e^{\pm\int_0^t r^1_z dz}, \quad 0 \leq t \leq T, \tag{3.3}$$

as well as the kernel $\mathcal{K} : [0,T]^2 \to \mathbb{R}_+$ which is given by

$$\mathcal{K}(t,s) := \xi_t^- \xi_s^+, \quad 0 \le t,s \le T. \tag{3.4}$$

Note that the kernel $\mathcal{K}$ is in $L^2([0,T]^2)$ (see Lemma 6.7). Moreover, for any $\nu^0 \in \mathcal{A}_M^{q_0}$ we define the predictable process

$$r_t^0 := \frac{1}{2\lambda_1}\mathbb{E}_t\left[\int_t^T \mathcal{K}(t,s)(dA_s - \kappa_0 \nu_s^0 ds)\right], \quad 0 \le t \le T. \tag{3.5}$$

The solution to the minor agent problem is given in the following theorem.

Theorem 3.2 (Solution to the minor agent's problem)**.** *Let $\nu^0 \in \mathcal{A}_M^{q_0}$. Under Assumption 3.1, there exists a unique optimal strategy $\nu^{1,*}(\nu^0) \in \mathcal{A}_m$ that maximizes (2.13). This strategy is given by*

$$\nu_t^{1,*} = -\left(r_t^0 + r_t^1 \int_0^t \mathcal{K}(s,t) r_s^0 ds\right), \quad 0 \le t \le T. \tag{3.6}$$

The proof of Theorem 3.2 is given in Section 6.

Remark 3.3. *In Lemma 6.2 we show that Assumption 3.1 is a sufficient condition to guarantee the strict concavity of the minor agent's functional (2.13), hence the uniqueness of the solution to the minor agent's problem.*

Remark 3.4. *The minor agent's optimal strategy in (3.6) may be written in feedback form as follows,*

$$\nu_t^{1,*} = -\left(r_t^0 + r_t^1 Q_t^{1,\nu^{1,*}}\right), \quad 0 \le t \le T.$$

3.2 Optimal strategy of the major agent

Our next step is to derive the maximiser of the major agent's objective functional (2.10), given the minor agent's optimal strategy $\nu^{1,*}$ in (3.6). As it is often the case in Stackelberg games, solving the second phase of the game is technically challenging and rarely achievable. In order to do so we make the following simplifying assumption on the signal A in (2.1). We assume that the signal process A is given by

$$A_t = \int_0^t \mu_s ds, \quad 0 \le t \le T.$$

where $\mu = (\mu_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process satisfying

$$\int_0^T \mathbb{E}[\mu_t^2]dt < \infty. \tag{3.7}$$

This is an adaptation to the present context of assumptions in previous studies Cartea and Jaimungal (2016); Lehalle and Neuman (2019) in single-agent optimal execution problems. We further denote

$$\bar{\mu}_t := \mathbb{E}[\mu_t], \quad 0 \le t \le T. \tag{3.8}$$

Next, we introduce some essential definitions related to linear operators in $L^2([0,T])$.

Definitions for linear operators in $L^2([0,T])$. For any linear operator T from $L^2([0,T])$ to $L^2([0,T])$ we define the operator norm

$$||\mathsf{T}|| := \sup\left\{||\mathsf{T}\psi||_{L^2} : \psi \in L^2([0,T]),\ ||\psi||_{L^2} \le 1\right\}, \tag{3.9}$$

and we denote by $B(L^2([0,T]))$ the space of all bounded linear operator from $L^2([0,T])$ to $L^2([0,T])$ with respect to the operator norm (3.9).

For any kernel $\mathcal{T} \in L^2([0,T]^2)$ (see (3.1)) we say that T is the integral operator generated by the kernel $\mathcal{T}$ if for any $\psi \in L^2([0,T])$,

$$(\mathsf{T}\psi)(t) = \int_0^T \mathcal{T}(t,s)\psi(s)ds, \quad 0 \le t \le T.$$

Any integral operator generated by a kernel in $L^2([0,T]^2)$ is in $B(L^2([0,T]))$ by the Cauchy-Schwarz inequality.

If T_1 and T_2 are two operators in $B(L^2([0,T]))$, then we denote by $\mathsf{T}_2\mathsf{T}_1$ the operator obtained by composing T_2 with T_1, that is for any $\psi \in L^2([0,T])$,

$$(\mathsf{T}_2\mathsf{T}_1\psi)(t) := (\mathsf{T}_2(\mathsf{T}_1\psi))(t), \quad 0 \le t \le T.$$

Special operators for our setting. Recall that $\mathcal{K}$ was defined in (3.4). We introduce the kernel $\mathcal{G} : [0,T]^2 \to \mathbb{R}_+$ defined as

$$\mathcal{G}(t,s) := \int_0^{t\wedge s} \mathcal{K}(u,t)\mathcal{K}(u,s)du, \quad 0 \le t,s \le T. \tag{3.10}$$

Note that the kernel $\mathcal{G}$ is symmetric and in $L^2([0,T]^2)$ (see Proposition 8.1). We define the operators G and S acting on any $\psi \in L^2([0,T])$ as follows,

$$(\mathsf{G}\psi)(t) := \int_0^T \mathcal{G}(t,s)\psi(s)ds, \tag{3.11}$$

$$(\mathsf{S}\psi)(t) := \frac{1}{2\lambda_0}\int_0^T \mathbb{1}_{\{s\leq t\}}\psi(s)ds + \frac{\kappa_1}{4\lambda_1\lambda_0}(\mathsf{G}\psi)(t). \tag{3.12}$$

Note that both operators G and S are in $B(L^2([0,T]))$ (see Proposition 7.4 and Lemma 7.10). Moreover, the operator G admits a spectral decomposition in terms of a sequence of positive eigenvalues $(\zeta_n)_{n\geq 1}$ and a corresponding sequence of eigenfunctions $(\psi_n)_{n\geq 1}$ in $L^2([0,T])$ (see Lemma 8.3). We define the *resolvent* kernel $\mathcal{R} : [0,T]^2 \to \mathbb{R}$ as

$$\mathcal{R}(t,s) = -\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathcal{G}(t,s) + \sum_{n\geq 1}\frac{1}{1+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\zeta_n}\left(\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\zeta_n\right)^2\psi_n(t)\psi_n(s), \tag{3.13}$$

for all $t,s \in [0,T]$ and where the sum converges uniformly and uniformly-absolutely over $[0,T]^2$, see Remark 3.10 for details. Moreover, we define the *resolvent* operator R, acting on any $\psi \in L^2([0,T])$ as follows,

$$(\mathsf{R}\psi)(t) := \psi(t) + \int_0^T \mathcal{R}(t,s)\psi(s)ds, \quad 0 \leq t \leq T. \tag{3.14}$$

The operator R is also in $B(L^2([0,T]))$, this is proved later in Proposition 7.9.

Notation. We denote by $\mathit{1}(t)$ the constant function which equals to 1 everywhere on $[0,T]$.

We are ready to state our main result regarding the solution to major's agent problem conditional on the minor agent adopting the strategy $\nu^{1,*}$ given in (3.6). Recall that $\bar{\mu}$ was defined in (3.8) and S was defined in (3.12).

Theorem 3.5 (Solution to the major agent's problem)**.** *Assume that $\nu^{1,*}$ is given by (3.6) and that Assumption 3.1 holds. Then, there exists a unique optimal strategy $\nu^{0,*} \in \mathcal{A}_M^{q_0}$ that maximizes the major agent's objective functional (2.10). It is given by*

$$\nu_t^{0,*} = \frac{\eta}{2\lambda_0}(\mathsf{R}\mathit{1})(t) + (\mathsf{R}\mathsf{S}\bar{\mu})(t), \quad 0 \leq t \leq T, \tag{3.15}$$

where

$$\eta = 2\lambda_0 \frac{q_0 - \langle \mathsf{R}\mathsf{S}\bar{\mu}, 1\rangle_{L^2}}{\langle \mathsf{R}1, 1\rangle_{L^2}}. \tag{3.16}$$

Moreover, $\nu_t^{0,*}$ *is continuous on* $[0, T]$.

The proof of Theorem 3.5 is given in Section 7. In the proof of Theorem 3.5 we also show that the constant η in (3.16) is well-defined, which is an ingredient in proving the admissibility of the optimal strategy (3.15).

The following corollary follows immediately from Theorem 3.2 and Theorem 3.5.

Corollary 3.6. *Let* $\nu^{0,*}$ *and* $\nu^{1,*}(\nu^{0,*})$ *as in Theorem 3.5 and Theorem 3.2, respectively. Then, under Assumption 3.1, the pair* $(\nu^{0,*}, \nu^{1,*}(\nu^{0,*})) \in \mathcal{A}_M^{q_0} \times \mathcal{A}_m$ *is the unique Stackelberg equilibrium in the sense of Definition 2.1.*

The following remarks discuss the result of Corollary 3.6.

Remark 3.7. *Note that* $\nu^{0,*}$ *in* (3.15) *is given in terms of the resolvent operator* R. *In Section 5 we derive a numerical scheme that approximates* $\nu^{0,*}$ *by using finite dimensional projections of* G. *The problem of computing* R *and hence* $\nu^{0,*}$ *is reduced to a finite-dimensional problem of matrix inversion. We refer to Proposition 5.4 and Theorem 5.6 for the details.*

Remark 3.8. *The most challenging step in obtaining a Stackelberg equilibrium is to derive the strategy of the player who acts first, namely the major agent. In our case we needed to develop a novel approach for deriving the optimal strategy in* (3.15), *using tools from the theory of integral equations. In Section 4 we illustrate the solutions to the Stackelberg game and in Section 5 we derive additional technical steps, which are needed in order to plot such explicit solutions directly from Theorems 3.2 and 3.5.*

Remark 3.9. *Our illustrations in Section 4 suggest that the minor agent can adopt either predatory or cooperative strategy with respect to the major agent, in each period, depending on the tradeoff between the order-flow of the major agent and the trading signal during the period (see Figure 1). This qualitative behaviour can be compared with the deterministic model of Schöneborn and Schied (2009), who showed that in the single period case the competitor is also selling and then buying her position back due to inventory constraints (see Figure 1 therein). In their two period model the seller is selling only the first period and then depending on the price impact parameters there are two possible scenarios: either the competitor is buying in the first period and then selling in the second period, i.e. introducing cooperative strategies in the game (see Figure 8 therein), or doing a round trip of selling first and then closing the position, all in the first period.*

Remark 3.10. *We remark that the sum appearing in (3.13) satisfies certain convergence properties. Define*

$$\mathfrak{R}_N(t,s) = \sum_{n=1}^{N} \frac{1}{1+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\zeta_n} \left(\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\zeta_n\right)^2 \psi_n(t)\psi_n(s), \quad t,s\in[0,T],$$
$$\mathfrak{R}^{abs}_N(t,s) = \sum_{n=1}^{N} \left|\frac{1}{1+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\zeta_n} \left(\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\zeta_n\right)^2 \psi_n(t)\psi_n(s)\right|, \quad t,s\in[0,T].$$

Then, it follows from the proof of (Porter and Stirling, 1990, Theorem 4.27) that $\mathfrak{R}_N$ converges uniformly to the sum in $\mathcal{R}$ on $(t,s)\in[0,T]^2$, and that $\mathfrak{R}^{abs}_N$ is uniformly convergent. The uniform convergence of $\mathfrak{R}^{abs}_N$ guarantees that the uniform convergence of $\mathfrak{R}_N$ is preserved even when the order of summation is changed. Therefore, as it is natural to expect, the solution $\nu^{0,*}$ in (3.15) is independent of how one enumerates the eigenvalues and the corresponding eigenfunctions of G.

4 Examples

In this section we give further describe the equilibrium strategies derived in Theorems 3.5 and 3.2. Motivated by (Lehalle and Neuman, 2019, Sec. 4), we consider the case where the signal μ in (3.7) follows an Ornstein-Uhlenbeck process,

$$d\mu_t = -\beta\mu_t dt + \sigma dW_t, \quad \mu_0 = m_0, \tag{4.1}$$

where $W=(W_t)_{t\geq 0}$ is a standard Brownian motion and β and σ are positive constants. Furthermore, we assume that M in (2.1) is given by

$$M_t = M_0 + \sigma_M \widetilde{W}_t, \tag{4.2}$$

where $\widetilde{W}$ is a standard Brownian motion independent from W, and M_0, σ_0 are positive constants. We fix the values of the price impact parameters λ_i, κ_i, the initial inventory of the major agent q_0 and the terminal penalty parameter α in (2.13) as

$$\kappa_0 = 2, \quad \kappa_1 = 2, \quad \lambda_0 = 1, \quad \lambda_1 = 1, \quad q_0 = 10, \quad \alpha = 10, \tag{4.3}$$

and the parameters of M in (4.2) and μ in (4.1),

$$m_0 = -0.5, \quad \beta = 0.1, \quad \sigma = 4, \quad M_0 = 100, \quad \sigma_M = 1. \tag{4.4}$$

Our model allows for different price impact parameters for the major and minor agents. While there is anecdotal evidence supporting the notion that HFTs can incur lower transaction costs than institutional investors, our numerical examples employ the same values for the price impact parameters for both major and minor agents.

The plots in this section are generated by using the numerical scheme which will described in detail in Section 5. We choose as a complete orthonormal basis $(a_i)_{i=1}^{\infty}$ the functions

$$a_i(t) := \begin{cases} 1/\sqrt{T} & i = 1 \\ \sqrt{2/T}\cos\left(\frac{(i-1)\pi t}{T}\right) & i = 2, 3, \dots \end{cases} \tag{4.5}$$

and such that each of corresponding degenerate kernel $\mathcal{G}_n$ defined in (5.4) represent the n^{th}-degree Fourier series approximation of the kernel $\mathcal{G}$ in (3.10). In order to strike a balance between numerical accuracy and computational efficiency, our simulations are generated by approximating the kernel $\mathcal{G}$ with the degenerate kernel $\mathcal{G}_{300}$.

The time dependence in the minor agent's inventory costs ϕ_t^1 (see (2.13)) can accommodate the setting of a liquidation carried out over several days. We take $T = k\tau$ for some positive integer k and for $\tau > 0$. Moreover, we choose the function ϕ^1 to be given by the following parametric form

$$\phi_t^1 = c_0 \left(\frac{t}{\tau} - \left\lfloor \frac{t}{\tau} \right\rfloor \right)^{c_1}, \quad 0 \le t \le T, \tag{4.6}$$

for two positive constants c_0 and c_1, which in the context of our simulations, we take to be $c_0 = 500$ and $c_1 = 15$. The function (4.6) is periodic of period τ and increases to its maximum value as t approaches $\tau,\ 2\tau,\ 3\tau, \dots,\ k\tau$ from the left, forcing the minor agent to liquidate most of her position at the end of each period. We consider a liquidation carried over a business week, from Monday to Friday, such that $T = 5$ (days) and $\tau = 1$ (day). Figure 1 illustrates three examples of a multi-day liquidation. Specifically, the top panel shows the major agent's deterministic optimal inventory (green line), deduced from (3.15), as well as three different realisations of the minor agent's optimal inventories that one obtains from (3.6) (blue, purple and red lines). The bottom panel shows the corresponding signal μ observed by the minor agent's while adopting the strategies at the top panel.

From (2.6) it follows that the price impact generated by the major agent's optimal strategy $\nu^{0,*}$ is perceived as a deterministic signal. The sell-off of shares by the major agent has the effect of pushing the price downwards, therefore, it generates opportunities which can be exploited by the minor agent. These considerations justify the fact that, as shown in (3.6), the minor agent adopts a trading strategy which tracks the "impacted" signal $\mu_t - \kappa_0 \nu_t^{0,*}$ instead of the raw market signal μ_t. Hence,

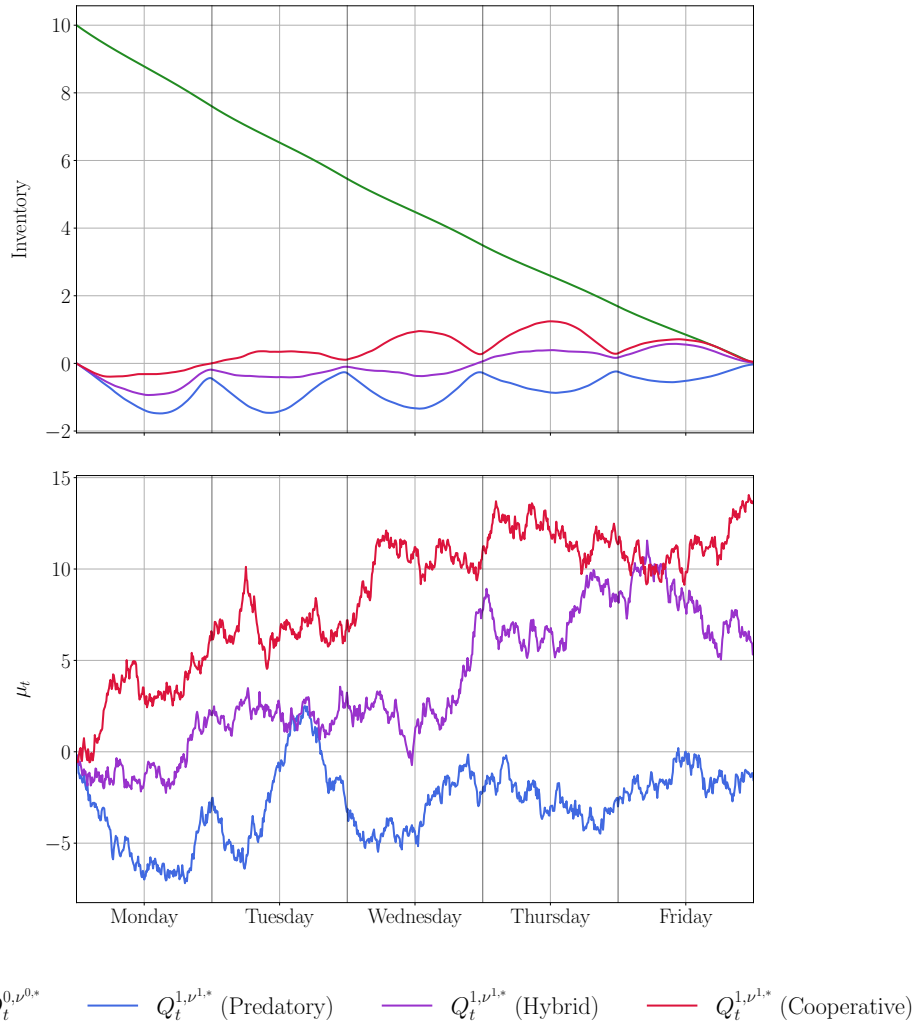


Figure 1: In the top panel, the green line represents the major agent's optimal inventory while the remaining solid lines represent the minor agent's optimal inventory when the minor agent is adopting a predatory trading style (blue line) and cooperative trading style (red line) or an hybrid of both (purple line). In the bottom panel, we show the signal μ_t corresponding to the realisations of the minor agent's inventories in the top panel.

depending on her forecast on the impacted signal $\mu_t - \kappa_0 \nu_t^{0,*}$, during each period the minor agent can decide whether to trade in the same direction of the major agent or not. This has the effect that, over the interval $[0, T]$, the observed trading style of the minor agent can be *predatory*, i.e. front running the major agent (blue line), *cooperative* (red line) or a *hybrid* of both (purple line).

To further understand several novel features of our model in the context of the multi-day liquidation we have just analysed, it is instructive to momentarily pause our discussion and consider the simpler case of a liquidation carried out over a single day. In particular, we wish to benchmark the major agent's optimal strategy in (3.15) against the strategy $\nu^{0,\mathrm{BM}}$ the major agent would use if she were unaware of the minor agent's trading activity. The strategy $\nu^{0,\mathrm{BM}}$ can be found by solving the major agent's problem with $\kappa_1 = 0$ and it is given by

$$\nu_t^{0,\mathrm{BM}} = \frac{q_0}{T} + \frac{m_0}{2\lambda_0 \beta} \left(\frac{1 - \beta T e^{-\beta t} - e^{-\beta T}}{\beta T} \right), \quad 0 \leq t \leq T. \tag{4.7}$$

Note that in the case of $m_0 = 0$ in (4.1), $\nu^{0,\mathrm{BM}}$ in (4.7) is a TWAP strategy.

We assume that the major agent wishes to liquidate his initial position over a time horizon of six hours, from 10 AM to 4 PM, hence we set $T = 6$ (hours).

In the present context, we slightly modify some of the parameters in (4.3) and (4.4): $\sigma = 1.5$, $\alpha = 50$ and $\phi_t^1 \equiv 1$. The top panel of Figure 2 shows the major agent's optimal trading rate $\nu^{0,*}$ (solid green line) and the benchmark trading rate $\nu^{0,\mathrm{BM}}$ (dashed green line). The bottom panel show 1000 realisations of minor agent's optimal trading rate $\nu^{1,*}$ (thin solid orange lines) and the cross-sectional average (thick solid brown line). We observe that the major agent's optimal strategy visibly deviates from the benchmark one in order to take into account the adverse effect of the minor agent's trading activity. We remark that since the major agent adopts a deterministic strategy, her decisions are based on the cross-sectional average of the minor agent's strategy, i.e. the solid brown line in the bottom panel of Figure 2. Initially, it is optimal to trade faster than the benchmark strategy in anticipation of the expected permanent price impact generated by the minor agent's reaction. Indeed, the early prices are more favourable to the major agent since they have not been affected yet by the extra price impact generated by presence of the minor agent. In the middle of the trading window the major agent's keeps trading but at a lower rate than the benchmark strategy. The explanation for this is that the major agent is aware that the minor agent could potentially trade in the same direction. Therefore, slowing down partially minimises the negative externality the minor agent's exerts on her via the aggregated permanent price impact. Finally, in the last section of the

trading window two factors determine the behaviour of the major agent's optimal strategy. First, the major agent must increase her trading rate to meet the terminal inventory constraint $Q_T^{0,\nu^{0,*}} = 0$. Second, the major agent is aware that, on average, the minor agent will have to close her short position at the end of the time horizon, therefore, she will have to buy shares, generating market impact and pushing the price up again. Hence, the prices at the end of the trading window are more favourable for the major agent, therefore, a substantial portion of the liquidation is postponed to the last hour. Figure 3 presents the major agent's and minor agent's inventories corresponding to the trading rates depicted in Figure 2.

Having established the major agent's and minor agent's trading patterns in the context of a single day liquidation, we turn again to the case of the multi-day liquidation presented in Figure 1. Analogously to Figure 2, the top panel of Figure 4 shows the major agent's optimal trading rate (solid green line) as well the trading rate of the benchmark strategy (dashed green line) in the context of the multiday-liquidation initially presented in Figure 1. Moreover, the bottom panel of Figure 4 presents 1000 realisations of the minor agent's trading rates (thin orange lines) as well as the cross-sectional average (solid brown line). We recover analogous trading patterns to the one observed in the single-day liquidation: the major agent's speed, when compared to the benchmark strategy, greatly increases at the beginning and at the end of each day. Moreover, on average, the minor agent acquires a short position at the beginning of each day, pushing the price down, and then, in order to meet her terminal inventory constraint at the end of each day, she pushes the price up again by buying shares. Note from Figure 1 that the predatory, cooperative and hybrid strategies share some common features. First, at the end of each day all the strategies have a very small inventory. This is because, by introducing the periodic running inventory costs of (4.6), the minor agent is strongly discouraged to hold a non-zero position at the end of each day, independently of her forecast for the impacted signal $\mu_t - \kappa_0 \nu_t^{0,*}$. Secondly, from Figure 1 we observe that the major agent is not liquidating at constant speed. Indeed, over the first day she liquidates at a speed visibly larger than, for example, the one employed over the last day. Such an intense liquidation in the first day generates an equally large alpha-signal through the corresponding price impact term $\kappa_0 \nu_t^{0,*}$. In the first day, the market impact-generated signal $\kappa_0 \nu_t^{0,*}$ is large enough to outweigh any realistic realisation of the exogenous signal μ_t, therefore, pushing the minor agent to trade in the same direction of the major agent, independently of the trading style she will adopt later on in the remaining days.

It is of practical interest to compare the financial performance of the major agent's optimal strategy $\nu^{0,*}$ against those of the benchmark strategy $\nu^{0,\mathrm{BM}}$. In the interest of

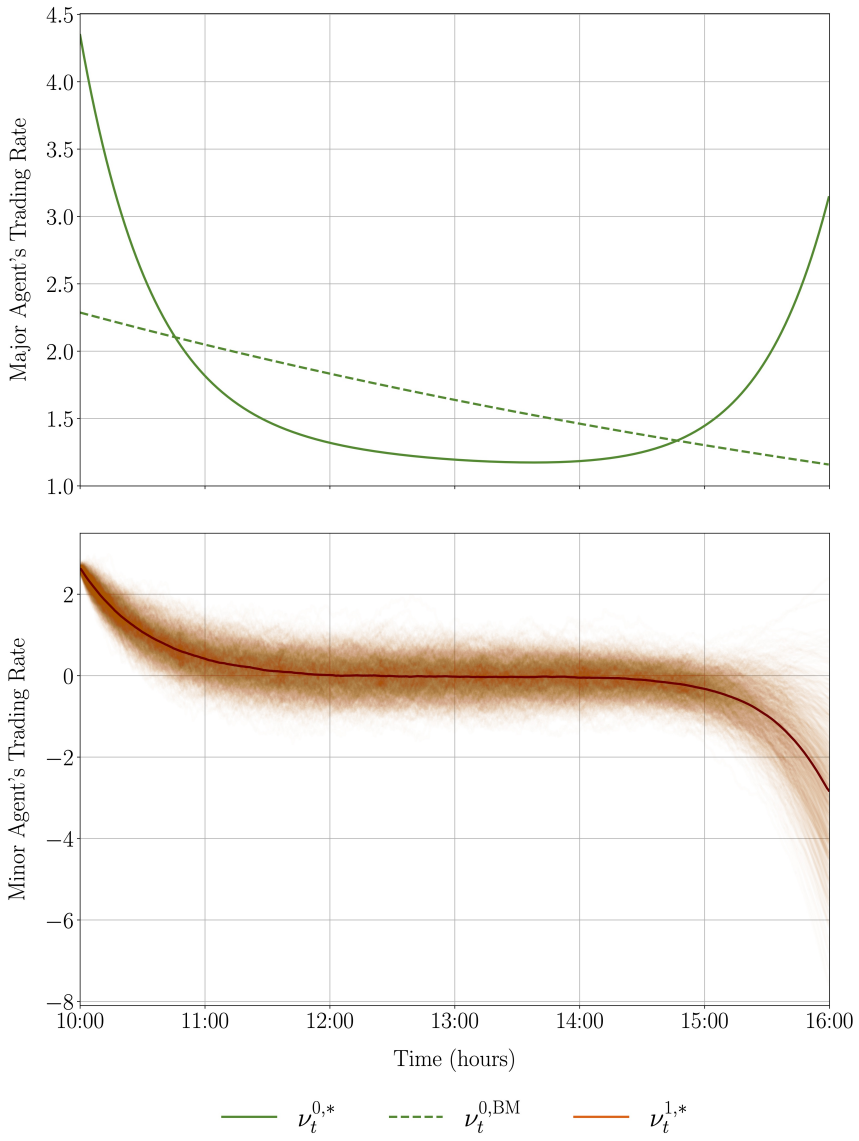


Figure 2: The major agent's and minor agent's optimal strategies in (3.15) and (3.6), respectively, for a single-day liquidation. In the top panel, the green solid line shows the major agent's optimal strategy while the dashed green line shows the benchmark strategy of (4.7). In the bottom panel, the thin orange solid lines depicts different realisations of the minor agent's optimal strategy. The brown solid line is the cross-sectional mean over the realisations.

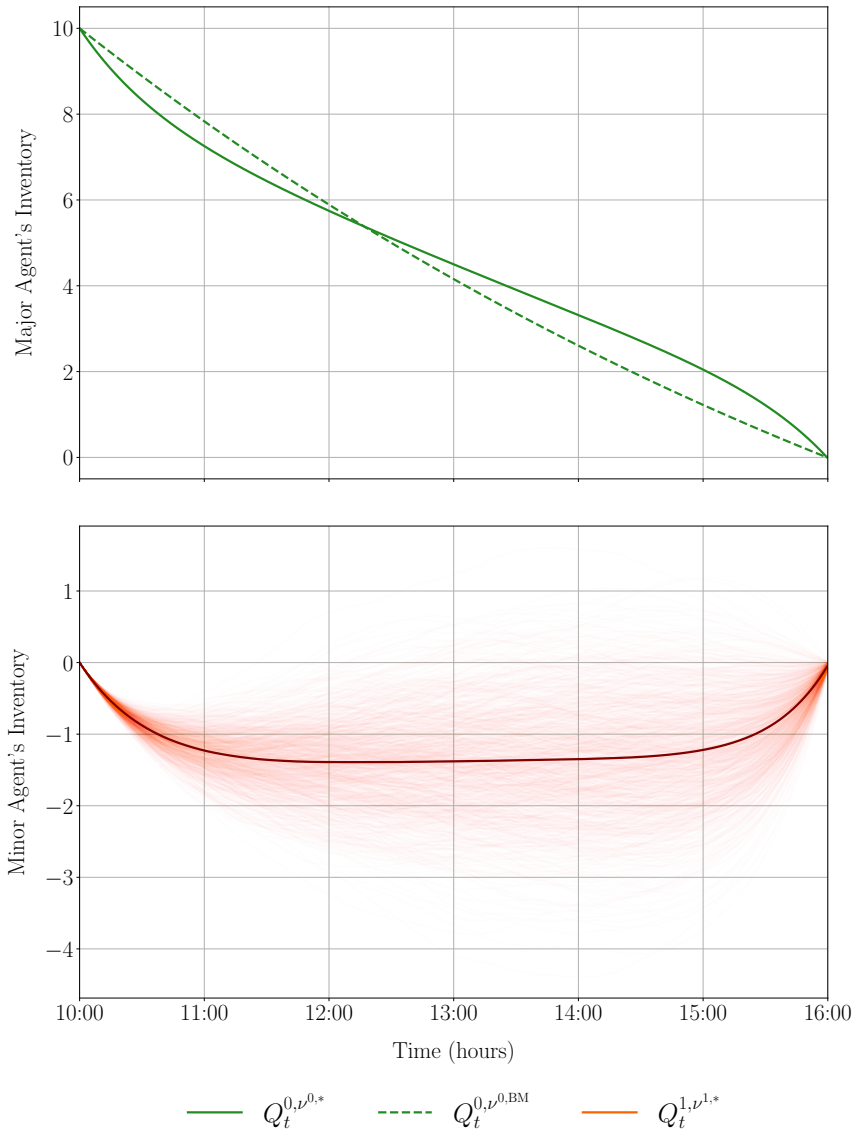


Figure 3: The major agent's and minor agent's optimal inventories corresponding to the strategies in (3.15) and (3.6), respectively, for a single-day liquidation. In the top panel, the solid green line shows the major agent's optimal inventory corresponding to the major agent's optimal strategy while the dashed green line shows the inventory corresponding to benchmark strategy in (4.7). In the bottom panel, the thin solid orange lines represent different realisations of the minor agent's optimal inventories corresponding to the strategy in (3.6) while the solid brown line is the cross-sectional mean over the realisations.

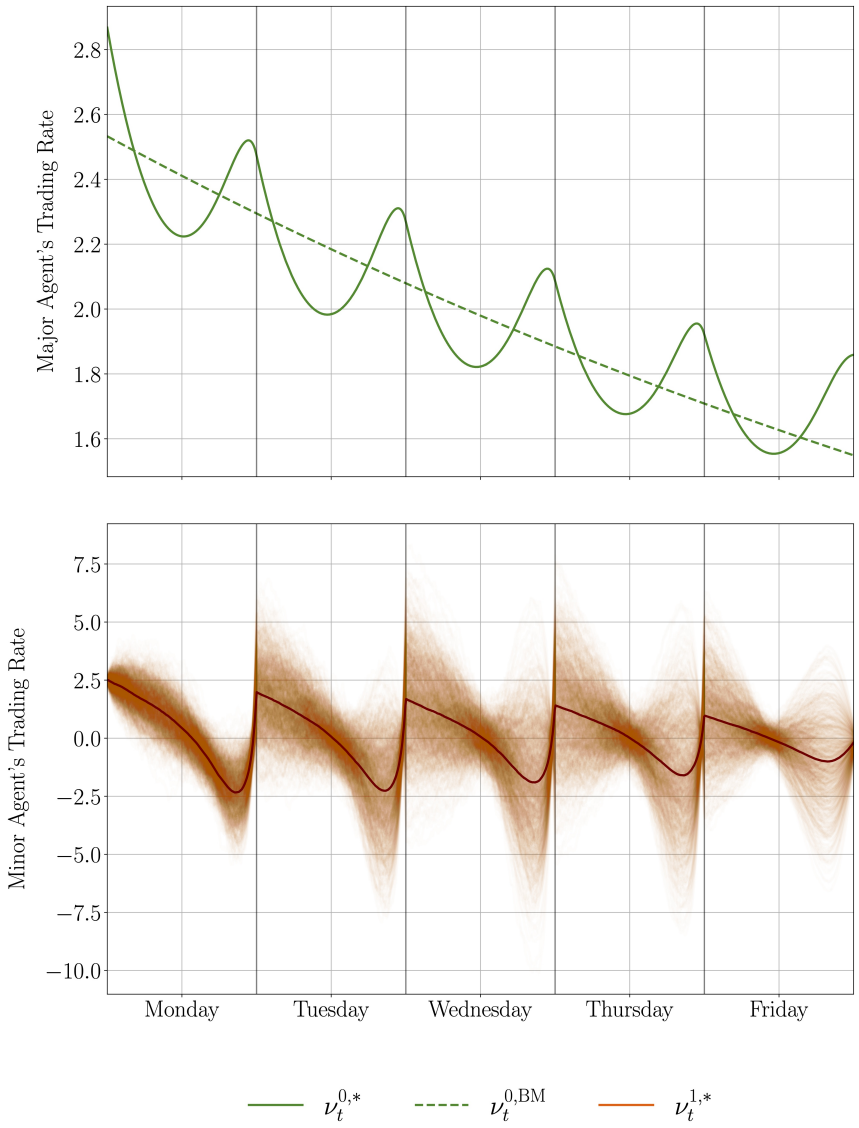


Figure 4: We present the major agent's and minor agent's optimal strategy in (3.15) and (3.6), respectively, for a multi-day liquidation. In the top panel, the green solid line shows the major agent's optimal strategy while the dashed green line shows the benchmark strategy of (4.7). In the bottom panel, the thin orange solid lines depicts different realisations of the minor agent's optimal strategy while the brown solid line represents the cross-sectional mean over the realisations.

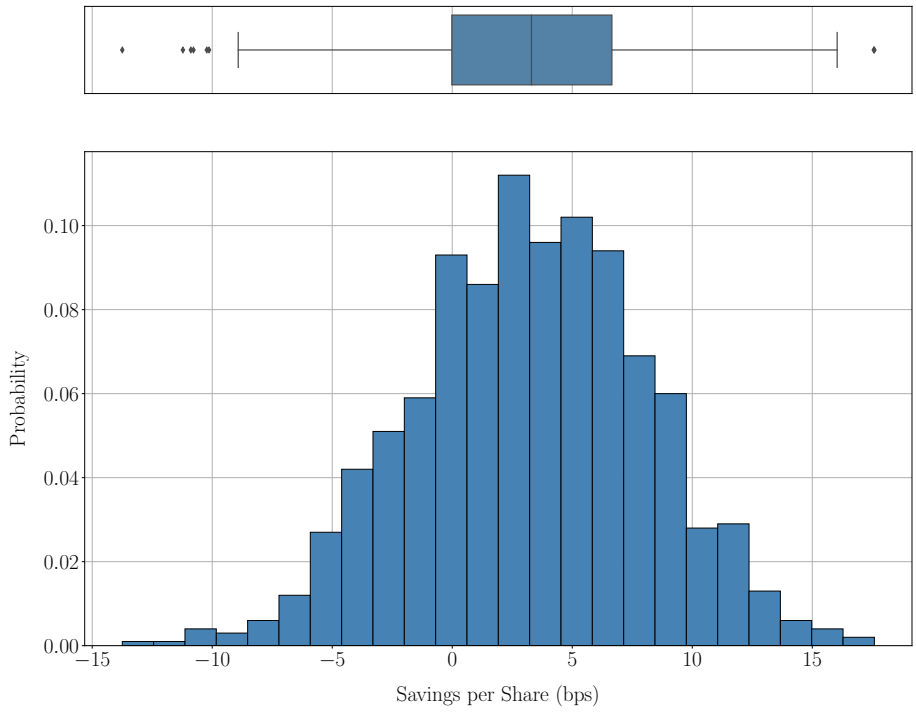


Figure 5: The savings per share, computed using (4.8), measured in bps from following the major agent's optimal strategy relative to the benchmark strategy in (4.7). The top panel shows the box-plot corresponding to the distribution in the bottom panel.

brevity, we limit ourselves to the case of the single-day liquidation presented in Figure 3. In Figure 5 we present a histogram of the empirical probability distribution of the performance of the major agent's optimal strategy in (3.15) relative to the benchmark strategy in (4.7) generated using 1,000 simulations. We compare the profit-and-loss (PnL) of the strategies in basis points (bps) through the following formula:

$$\frac{X_T^{0,\nu^{0,*}} - X_T^{0,\nu^{0,\mathrm{BM}}}}{X_T^{0,\nu^{0,\mathrm{BM}}}} \times 10^4, \tag{4.8}$$

where $X_T^{0,\nu^{0,\mathrm{BM}}}$ is the terminal cash obtained from employing the benchmark strategy $\nu^{0,\mathrm{BM}}$ and $X_T^{0,\nu^{0,*}}$ is the terminal cash obtained from employing the optimal strategy $\nu_t^{0,*}$. The mean of the distribution in Figure 5 is strictly positive, hence the major agent's optimal strategy on average outperforms the benchmark strategy.

Finally, we show that the strategic behaviour of the agents induces intraday

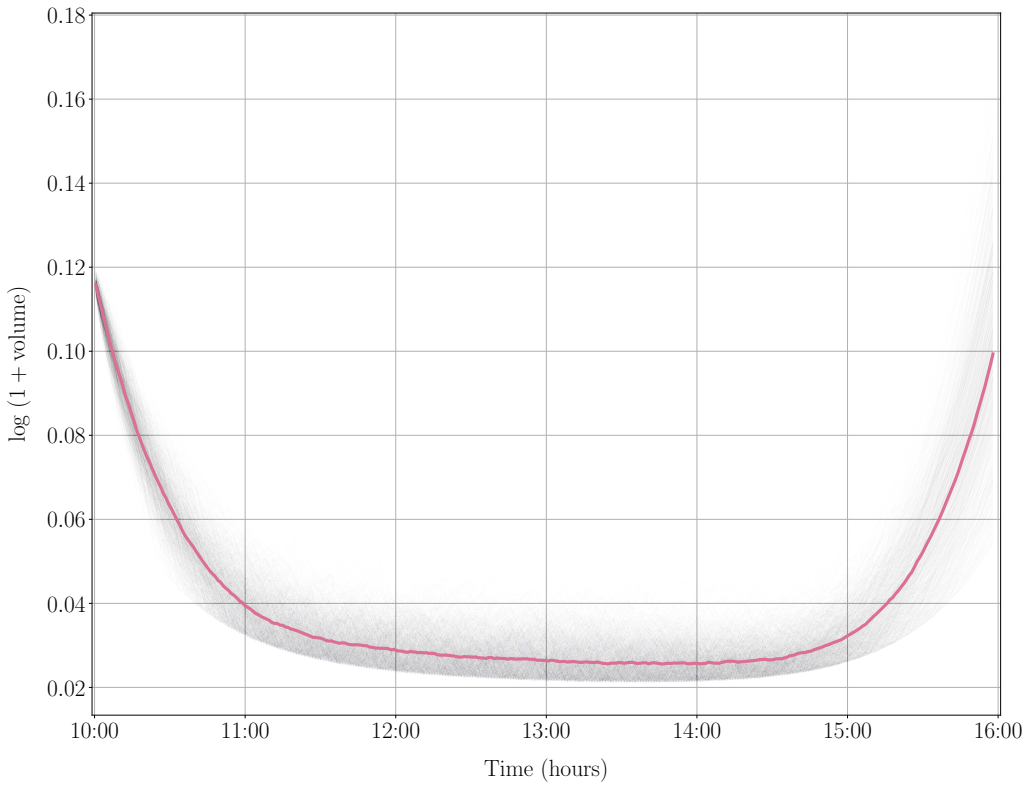


Figure 6: Intraday volume profiles. Median is displayed in pink.

patterns in trading volume. It is well-documented that intraday trading volume displays a "U-shaped" intraday profile: volume is highest at the beginning and at the end of the day (Wood et al., 1985). Figure 6 displays the intraday volume profile resulting from our model. We compute the total number of shares traded by the agents for each 1 minute bin, averaged over 1000 sample paths. Each blue line in Figure 6 represents a simulated path for the (logarithm) volume, while the magenta line is the median value of each 1-minute bin. The volume profiles in Figure 6 display a U-shaped pattern analogous, for example, to those empirically observed and reported in previous studies (Admati and Pfleiderer, 2015; Wood et al., 1985).

5 Computing the optimal strategies

Theorem 3.5 presents the unique major agent optimal strategy $\nu^{0,*}$ in closed-form. The optimal strategy $\nu^{0,*}$ is expressed in terms of the resolvent operator R, defined

in (3.13), which in turn relies on the eigenvalues $(\zeta_n)_{n\geq 1}$ and eigenfunctions $(\psi_n)_{n\geq 1}$ of the operator G defined in (3.11). In several simple cases these eigenfunctions and eigenvalues can be computed explicitly.

For example, in the case of $\phi^1 \equiv 0$, $(\zeta_n)_{n\geq 1}$ and $(\psi_n)_{n\geq 1}$ can be explicitly determined in terms of the roots of a transcendental equation (see Appendix A). Nevertheless, a closed-form representation for the eigenvalues $(\zeta_n)_{n\geq 1}$ and the eigenfunctions $(\psi_n)_{n\geq 1}$ might be unattainable when ϕ^1 is a generic non-negative piecewise continuous function. Therefore, we dedicate this section to developing a numerical scheme to compute the major agent optimal strategy $\nu^{0,*}$ which fully bypass the need of determining these eigenvalues and eigenfunctions. As a by product, such numerical scheme will also determine the Stackelberg equilibrium of Corollary 3.6.

We denote by I the identity operator on $L^2([0,T])$, that is

$$(\mathsf{I}\psi)(t) = \psi(t) \quad \text{for all } 0 \leq t \leq T,\ \psi \in L^2([0,T]). \tag{5.1}$$

As usual, the resolvent operator in (3.14) can be written as

$$\mathsf{R} = \left(\mathsf{I} + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}\right)^{-1}.$$

This is proved rigorously in Proposition 7.9. It follows that the major agent optimal strategy $\nu^{0,*}$ in (3.15) satisfies to the following integral operator equation

$$\left(\mathsf{I} + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}\right)\nu^{0,*} = \mathsf{S}\bar{\mu} + \frac{\eta}{2\lambda_0}, \tag{5.2}$$

Here, the constant η is defined as in (3.16) and the operators G and S are defined as in (3.11) and (3.12), respectively. The idea is to replace (5.2) with a sequence of approximate equations (see (5.8)) whose solutions converge to the desired optimal strategy $\nu^{0,*}$.

For the discussion that follows it is convenient to recall the definition of a finite-rank operator and that of a compact operator, which we will provide in Definition 7.2. In Proposition 7.4 we will show that the operator G is compact. Therefore, there exists a sequence of finite-rank operator $(\mathsf{G}_n)_{n\geq 1}$ in $B(L^2([0,T]))$ satisfying the approximation property

$$\lim_{n\to\infty} ||\mathsf{G}_n - \mathsf{G}|| = 0,$$

where $\|\cdot\|$ refers to the operator norm in (3.9).

In order to construct such sequence, we consider a complete orthonormal basis $(a_i)_{i=1}^{\infty}$ in $L^2([0,T])$. A possible choice of such complete orthonormal basis in $L^2([0,T])$ is given by (4.5).

Let the kernel $\mathcal{G}$ be defined as in (3.10) and let the functions $(b_i)_{i=1}^{\infty}$ be defined as

$$b_i(t) := \int_0^T \mathcal{G}(t,s)a_i(s)ds. \tag{5.3}$$

We recall the definition of a degenerate kernel (Porter and Stirling, 1990, Definition 3.1) which will be useful in the following.

Definition 5.1 (Degenerate Kernel)**.** *Let $n \geq 1$ and suppose there are finitely many functions $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ such that $a_i : [0,T] \to \mathbb{R}$ and $b_i : [0,T] \to \mathbb{R}$ for $i = 1, \ldots, n$. Assume further that $\mathcal{T} : [0,T]^2 \to \mathbb{R}$ is a kernel such that*

$$\mathcal{T}(t,s) = \sum_{i=1}^n a_i(t)b_i(s), \quad t,s \in [0,T].$$

Then, the kernel $\mathcal{T}$ is said to be degenerate.

Define the sequence of degenerate kernels $(\mathcal{G}_n)_{n\geq 1}$ as the partial sums

$$\mathcal{G}_n(t,s) := \sum_{i=1}^n a_i(t)b_i(s), \quad n \geq 1. \tag{5.4}$$

Since $\mathcal{G}$ is a kernel in $L^2([0,T]^2)$ (see Proposition 8.1) then, as shown in the proof of (Porter and Stirling, 1990, Theorem 3.4), the sequence $(\mathcal{G}_n)_{n\geq 1}$ converges to $\mathcal{G}$ in the sense

$$\lim_{n\to\infty} \int_0^T \int_0^T (\mathcal{G}(t,s) - \mathcal{G}_n(t,s))^2 dsdt = 0. \tag{5.5}$$

Given the degenerate kernels $(\mathcal{G}_n)_{n\geq 1}$ we can define a corresponding sequence of so-called finite rank integral operators $(\mathsf{G}_n)_{n\geq 1}$ as

$$(\mathsf{G}_n\psi)(t) := \int_0^T \mathcal{G}_n(t,s)\psi(s)ds, \quad \psi \in L^2([0,T]). \tag{5.6}$$

The following proposition, which is proved in Section 10, gives the convergence result for the sequence $(\mathsf{G}_n)_{n\geq 1}$.

Proposition 5.2. *Under Assumption 3.1, let $(\mathsf{G}_n)_{n\geq 1}$ be defined as in (5.6) and let G be defined as in (3.11). Then the finite rank operators G_n are in $B(L^2([0,T]))$. Moreover, we have that*

$$\lim_{n\to\infty} ||\mathsf{G}_n - \mathsf{G}|| = 0. \tag{5.7}$$

Next, we consider the following sequence of approximate equations to (5.2),

$$\left(\mathsf{I}+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}_n\right)\nu^{0,(n)}=\mathsf{S}\bar{\mu}+\frac{\eta_n}{2\lambda_0},\quad n\geq 1, \tag{5.8}$$

for a suitably defined sequence of constants $(\eta_n)_{n\geq 1}$.

Remark 5.3. *We remark that in (5.8) we continue to take the operator S to be defined in terms of G, as in (3.12), and not in terms of the sequence $(\mathsf{G}_n)_{n\geq 1}$. It is not necessary to approximate the operator S since it can be explicitly expressed in terms of the kernel $\mathcal{G}$ in (3.10), via the operator G, therefore, it can be computed explicitly via numerical integration (see also Remark 5.8).*

A solution to (5.8) exists if the inverse of the operator $\mathsf{I}+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}_n$ exists, with the candidate solution $\nu^{0,(n)}$ being given by

$$\nu^{0,(n)}=\left(\mathsf{I}+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}_n\right)^{-1}\left(\mathsf{S}\bar{\mu}+\frac{\eta_n}{2\lambda_0}\right). \tag{5.9}$$

The next result shows that for sufficiently large n, the inverse of $\mathsf{I}+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}_n$ exists. Moreover, we show that the problem of finding such inverse is reduced to the finite dimensional problem of matrix inversion.

To state our results it is convenient to introduce the sequence of matrices $(\mathbb{G}_n)_{n\geq 1}$ where $\mathbb{G}_n\in\mathbb{R}^{n\times n}$ and whose entries are defined as

$$(\mathbb{G}_n)_{ij}:=\langle a_i,b_j\rangle_{L^2} \tag{5.10}$$

for all $1\leq i,j\leq n$ and with a_i and b_j defined as in (5.3). Moreover, we will denote by I_n the n-dimensional identity matrix, that is $I_n:=\mathrm{diag}(1,\dots,1)\in\mathbb{R}^{n\times n}$.
We are now ready to state our next proposition, which is proved in Section 10.

Proposition 5.4. *Under Assumption 3.1, let $(\mathsf{G}_n)_{n\geq 1}$ be defined as in (5.6) and let $(\mathbb{G}_n)_{n\geq 1}$ be defined as in (5.10). Then, there exists $N\geq 1$ such that for all $n\geq N$ the operator $\mathsf{I}+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}_n$ and the matrix $I_n+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathbb{G}_n$ are both invertible. In particular, for all $n\geq N$ it holds that*

$$\left(\mathsf{I}+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}_n\right)^{-1}\psi=\psi-\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\sum_{i,j=1}^{n}\left(I_n+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathbb{G}_n\right)^{-1}_{i,j}\langle\psi,b_j\rangle_{L^2}\ a_i, \tag{5.11}$$

for any $\psi\in L^2([0,T])$.

Note that both operators $\mathsf{I} + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}$ and $\mathsf{I} + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}_n$ are invertible (see Proposition 7.18), nevertheless, only in the case of the latter the inverse operator can be computed via matrix inversion by exploiting the corresponding degenerate kernel decomposition, see also Remark 5.8 for additional discussion.

The next result shows that that the candidate solutions in (5.9) converge *in mean* to the optimal strategy $\nu^{0,*}$ of Theorem 3.5. Henceforth, we take the sequence of constants $(\eta_n)_{n\geq 1}$ to be defined as

$$\eta_n := 2\lambda_0 \frac{q_0 - \left\langle \left(\mathsf{I} + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}_n\right)^{-1} \mathsf{S}\bar{\mu},\, 1 \right\rangle_{L^2}}{\left\langle \left(\mathsf{I} + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}_n\right)^{-1} 1,\, 1 \right\rangle_{L^2}}, \quad n \geq 1. \tag{5.12}$$

Proposition 5.5. *Under Assumption 3.1, let $\nu^{0,*}$ and $\nu^{0,(n)}$ be defined as in (3.15) and (5.9), respectively. Then, there exists $N \geq 1$ such that for all $n \geq N$ the functions $\nu^{0,(n)}$ are well-defined and are in $L^2([0,T])$. Moreover,*

$$\lim_{n\to\infty} \left\|\nu^{0,*} - \nu^{0,(n)}\right\|_{L^2} = 0. \tag{5.13}$$

The proof of Proposition 5.5 is postponed to Section 10. Lemma 10.4 shows that for sufficiently large n, the constants η_n in (5.12) are well-defined.

In order to obtain an approximating sequence which converges *uniformly* to the optimal control $\nu^{0,*}$ we introduce the sequence of candidate functions $(\hat{\nu}^{0,(n)})_{n\geq 1}$ defined as

$$\hat{\nu}_t^{0,(n)} := -\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\left(\mathsf{G}\nu^{0,(n)}\right)(t) + (\mathsf{S}\bar{\mu})(t) + \frac{\eta_n}{2\lambda_0} \tag{5.14}$$

for all $t \in [0,T]$ and for all $n \geq 1$. Our main result for this section is the following convergence theorem.

Theorem 5.6. *Under Assumption 3.1, let $\nu^{0,*}$, $\hat{\nu}^{0,(n)}$ and $\nu^{1,*}$ be defined as in (3.15), (5.14) and (3.6), respectively. Then, there exists an $N \geq 1$ such that for all $n \geq N$ the functions $\hat{\nu}^{0,(n)}$ are in $L^2([0,T])$ and the controls $\nu^{1,*}(\hat{\nu}^{0,(n)})$ are in $\mathcal{A}_m$. Furthermore, we have that:*

(i)

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left|\nu_t^{0,*} - \hat{\nu}_t^{0,(n)}\right| = 0,$$

(ii)

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left|\nu_t^{1,*}\left(\nu^{0,*}\right) - \nu_t^{1,*}\left(\hat{\nu}^{0,(n)}\right)\right| = 0, \quad \mathbb{P}-\text{a.s.}$$

The proof of Theorem 5.6 is postponed to Section 10.

Proposition 5.4 and Theorem 5.6 show that the infinite-dimensional problem of determining the solution to (5.2), can be reduced to the finite-dimensional problem of matrix inversion.

Remark 5.7. *The proofs of the results of Proposition 5.5 and Theorem 5.6 do not rely on the existence of the orthonormal expansion in (5.4) and the corresponding convergence (5.5). Indeed, our result can be extended to any generic sequence of operators $(\mathsf{G}_n)_{n\geq 1}$ in $B(L^2([0,T]))$ satisfying the approximation property of Proposition 5.2 and which do not necessarily enjoy a integral representation of the form in (5.6).*

Remark 5.8. *The matrix entries in (5.10) must be computed numerically. The use of a numerical evaluation in (5.10) will lead to numerical errors in the entries of the matrix $I_n + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathbb{G}_n$. As shown in (Atkinson, 1997, Chapter 2.3.4), for a sufficiently accurate estimation the numerical error arising from these computations is negligible. A similar discussion applies to, among others, the numerical evaluation of the inverse of the matrix $I_n + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathbb{G}_n$ and of the integral $\mathsf{S}\bar{\mu}$. These are all elementary and well-understood convergence problems in numerical analysis and the corresponding convergence rate could be easily incorporated in the convergence results of this section. Hence, our discussion assumes that the aforementioned quantities are taken to be exact and that the corresponding numerical errors are negligible.*

Remark 5.9. *The numerical scheme we have presented has an advantage from an implementation standpoint too. Specifically, if one were to determine the major agent's optimal strategy by using the result of Theorem 3.5, she would need to mathematically determine the eigenvalues $(\zeta_n)_{n\geq 1}$ and eigenfunctions $(\psi_n)_{n\geq 1}$ each time she wishes to change the function ϕ^1, as shown, for example, in Appendix A. On the other hand, with the numerical scheme of Theorem 5.6, to achieve the same goal it is sufficient to change the expression of ϕ^1 in the numerical solver of the Riccati ODE (3.2), which usually amounts to change solely few lines of code.*

6 Proof of Theorem 3.2

We show how the Stackelberg equilibrium can be found by *backward induction*, that is by first solving the minor agent's problem and then the major agent's problem. We determine the minor agent's optimal strategy via a calculus of variations argument, as similarly done in Neuman and Voß (2022). The following results also borrow ideas from Casgrain and Jaimungal (2019).

Henceforth, we assume that $\nu^0 \in \mathcal{A}_M^{q_0}$ is a fixed major agent liquidation strategy and with a slight abuse of notation we write $H^1(\nu)$ for $H^1(\nu;\nu^0)$. We start by determining an alternative representation for the minor agent's objective.

Lemma 6.1. *The minor agent's objective H^1 in (2.13) can be alternatively represented as*

$$\begin{aligned} H^1(\nu^1) = x_1 - \mathbb{E}\Bigg[\lambda_1 \int_0^T (\nu_t^1)^2 dt + \alpha \left(Q_T^{1,\nu^1}\right)^2 + \int_0^T \phi_t^1 \left(Q_t^{1,\nu^1}\right)^2 dt \\ + \int_0^T Q_t^{1,\nu^1}(\kappa_0 \nu_t^0 dt + \kappa_1 \nu_t^1 dt - dA_t)\Bigg], \end{aligned} \tag{6.1}$$

for any $\nu^1 \in \mathcal{A}_m$.

Proof. We use (2.12), (2.11) and the Itô's product rule on $Q_T^{1,\nu^1} P_T^\nu$ to get

$$\begin{aligned} &\mathbb{E}\left[X_T^{1,\nu^1} + Q_T^{1,\nu^1} P_T^\nu\right] \\ &= x_1 + \mathbb{E}\left[\int_0^T (P_t^\nu - \lambda_1 \nu_t^1)\nu_t^1 dt + \int_0^T Q_t^{1,\nu^1} dP_t^\nu + \int_0^T P_t^\nu dQ_t^{1,\nu^1}\right], \end{aligned} \tag{6.2}$$

where we also used $Q_0^{1,\nu^1} = 0$ by (2.5). Recall that $P = M + A$. We apply (2.5), (2.6) and (2.7) to (6.2) in order to obtain

$$\begin{aligned} \mathbb{E}\left[X_T^{1,\nu^1} + Q_T^{1,\nu^1} P_T^\nu\right] &= x_1 + \mathbb{E}\left[-\lambda_1 \int_0^T (\nu_t^1)^2 dt + \int_0^T Q_t^{1,\nu^1} dP_t^\nu\right] \\ &= x_1 - \mathbb{E}\Bigg[\lambda_1 \int_0^T (\nu_t^1)^2 dt - \int_0^T Q_t^{1,\nu^1} dM_t \\ &\qquad + \int_0^T Q_t^{1,\nu^1}(\kappa_0 \nu_t^0 dt + \kappa_1 \nu_t^1 dt - dA_t)\Bigg]. \end{aligned} \tag{6.3}$$

Since $\nu^1 \in \mathcal{A}_m$ (see (2.4)), then we can drop the martingale term in (6.3) and obtain

$$\begin{aligned} \mathbb{E}\left[X_T^{1,\nu^1} + Q_T^{1,\nu^1} P_T^\nu\right] = x_1 - \mathbb{E}\Bigg[\lambda_1 \int_0^T (\nu_t^1)^2 dt \\ + \int_0^T Q_t^{1,\nu^1}(\kappa_0 \nu_t^0 dt + \kappa_1 \nu_t^1 dt - dA_t)\Bigg]. \end{aligned} \tag{6.4}$$

Substituting (6.4) in (2.13) returns (6.1). □

In the next result we use the representation of (6.1) to show that the minor agent's objective is strictly concave.

Lemma 6.2. *Under Assumption 3.1, the functional H^1 defined in (2.13) is strictly concave for $\nu^1 \in \mathcal{A}_m$.*

Proof. In order to prove that the functional H^1 is strictly concave, we must show that for any $0 < \rho < 1$ and $\nu, \omega \in \mathcal{A}_m$, such that ν and ω are $d\mathbb{P} \otimes dt$ distinguishable, it holds that

$$\mathcal{I}^1(\rho, \nu, \omega) := H^1(\rho\nu + (1-\rho)\omega) - \rho H^1(\nu) - (1-\rho)H^1(\omega) > 0. \tag{6.5}$$

It is convenient to introduce the constant $\theta := \frac{2\alpha - \kappa_1}{2}$ as well as to define the function Γ^1 as follows

$$\Gamma^1_t = \begin{pmatrix} \lambda_1 & -\theta \\ -\theta & \phi^1_t \end{pmatrix}, \quad 0 \le t \le T. \tag{6.6}$$

Note that under Assumption 3.1 it holds that $\theta \ge 0$. From (2.5) and integration by parts we get

$$\left(Q^{1,\nu^1}_T\right)^2 = -2\int_0^T Q^{1,\nu^1}_t \nu^1_t dt. \tag{6.7}$$

Using (6.7) we rewrite the minor agent's objective in (6.1) in terms of the function Γ^1 as

$$H^1(\nu^1) = x_1 - \mathbb{E}\left[\int_0^T \begin{pmatrix} \nu^1_t \\ Q^{1,\nu^1}_t \end{pmatrix}^\intercal \Gamma^1_t \begin{pmatrix} \nu^1_t \\ Q^{1,\nu^1}_t \end{pmatrix} dt + \int_0^T Q^{1,\nu^1}_t (\kappa_0 \nu^0_t dt - dA_t)\right]. \tag{6.8}$$

Note that, given the representation in (6.8), in the case of $\theta > 0$ and $\phi^1_t > 0$ for all $t \in [0,T]$ the strict concavity of $H^1(\nu^1)$ follows from (2.5) and the fact that Γ^1_t is a positive-definite matrix for all $t \in [0,T]$. In what follows we will use (6.8) to show that $H^1(\nu^1)$ is strictly concave also under the assumption that $\phi^1_t \ge 0$ and $\theta \ge 0$.

We observe that $Q^{1,\nu}$ is linear with respect to ν, that is

$$Q^{1,\rho\nu+(1-\rho)\omega}_t = \rho Q^{1,\nu}_t + (1-\rho)Q^{1,\omega}_t \quad \text{for all } \rho \in [0,1],\ \nu, \omega \in \mathcal{A}_m.$$

We substitute (6.8) in (6.5) and we use the linearity of $Q^{1,\cdot}$ to cancel out the $Q^{1,\cdot}_t(\kappa_0\nu^0_t dt - dA_t)$ terms. This yields

$$\begin{aligned}\mathcal{I}^1(\rho,\nu,\omega) = \mathbb{E}\Bigg[&\int_0^T \rho \begin{pmatrix} \nu_t \\ Q^{1,\nu}_t \end{pmatrix}^\intercal \Gamma^1_t \begin{pmatrix} \nu_t \\ Q^{1,\nu}_t \end{pmatrix} dt + (1-\rho)\begin{pmatrix} \omega_t \\ Q^{1,\omega}_t \end{pmatrix}^\intercal \Gamma^1_t \begin{pmatrix} \omega_t \\ Q^{1,\omega}_t \end{pmatrix} dt \\ &- \left(\rho\begin{pmatrix} \nu_t \\ Q^{1,\nu}_t \end{pmatrix} + (1-\rho)\begin{pmatrix} \omega_t \\ Q^{1,\omega}_t \end{pmatrix}\right)^\intercal \Gamma^1_t \left(\rho\begin{pmatrix} \nu_t \\ Q^{1,\nu}_t \end{pmatrix} + (1-\rho)\begin{pmatrix} \omega_t \\ Q^{1,\omega}_t \end{pmatrix}\right) dt\Bigg],\end{aligned}$$

where after multiplying out all the terms we get

$$\mathcal{I}^1(\rho,\nu,\omega) = \mathbb{E}\left[\int_0^T \rho(1-\rho)\left(\begin{pmatrix}\nu_t\\ Q_t^{1,\nu}\end{pmatrix}-\begin{pmatrix}\omega_t\\ Q_t^{1,\omega}\end{pmatrix}\right)^{\intercal}\Gamma_t^1\left(\begin{pmatrix}\nu_t\\ Q_t^{1,\nu}\end{pmatrix}-\begin{pmatrix}\omega_t\\ Q_t^{1,\omega}\end{pmatrix}\right)dt\right]. \quad (6.9)$$

It is convenient to introduce the function $\delta_t := \nu_t - \omega_t$ for all $t \in [0,T]$. From (2.5) it follows that $Q_t^{1,\delta} = Q_t^{1,\nu} - Q_t^{1,\omega}$ for all $t \in [0,T]$. We can rewrite (6.9) in terms of δ and $Q^{1,\delta}$ as

$$\begin{aligned}\mathcal{I}^1(\rho,\nu,\omega) = \rho(1-\rho)\Big(&\mathbb{E}\left[\int_0^T \lambda_1\delta_t^2 dt\right] + \mathbb{E}\left[\int_0^T \phi_t^1\left(Q_t^{1,\delta}\right)^2 dt\right]\\ &- \mathbb{E}\left[\int_0^T 2\theta\delta_t Q_t^{1,\delta} dt\right]\Big),\end{aligned} \quad (6.10)$$

where we have used (6.6).

Since $\phi_t^1 \geq 0$ for all $t \in [0,T]$, we have

$$\mathbb{E}\left[\int_0^T \phi_t^1\left(Q_t^{1,\delta}\right)^2 dt\right] \geq 0.$$

From (2.5) it holds that $Q_t^{1,\delta} = -\int_0^t \delta_t dt$, therefore by (6.7) we get

$$-\mathbb{E}\left[\int_0^T 2\delta_t Q_t^{1,\delta} dt\right] = \mathbb{E}\left[\left(Q_T^{1,\delta}\right)^2\right] \geq 0. \quad (6.11)$$

Finally, notice that since ν and ω are $d\mathbb{P}\otimes dt$ distinguishable then

$$\mathbb{E}\left[\int_0^T \delta_t^2 dt\right] > 0.$$

This shows that $\mathcal{I}^1(\rho,\nu,\omega) > 0$ for any $\theta \geq 0$, $0 < \rho < 1$ and any $\nu,\omega \in \mathcal{A}_m$, such that ν and ω are $d\mathbb{P}\otimes dt$ distinguishable. □

As similarly shown in Neuman and Voß (2022), a probabilistic and convex analytic calculus of variations approach can be readily applied to derive a system of coupled linear FBSDEs which characterises the unique solution to the minor agent's problem.

Since under Assumption 3.1 the map $\nu^1 \to H^1(\nu^1)$ in (6.1) is strictly concave, then it admits a unique maximiser characterised by the critical point at which the Gâteaux derivative

$$\langle \mathcal{D}H^1(\nu^1), \omega \rangle := \lim_{\epsilon \to 0} \frac{H^1(\nu^1 + \epsilon\omega) - H^1(\nu^1)}{\epsilon} \tag{6.12}$$

vanishes. In the following lemma we obtain an explicit expression for the Gâteaux derivative of H^1.

Lemma 6.3. *The Gâteaux derivative of H^1 in (6.1), in direction $\omega \in \mathcal{A}_m$ is given by*

$$\begin{aligned}\big\langle \mathcal{D}H^1(\nu^1), \omega \big\rangle = \mathbb{E}\Bigg[\int_0^T \omega_t \Big(&- 2\lambda_1 \nu_t^1 + 2\alpha Q_T^{1,\nu^1} - \kappa_1 Q_t^{1,\nu^1} + A_t \\ &+ \int_t^T \Big(2\phi_s^1 Q_s^{1,\nu^1} + \kappa_0 \nu_s^0 + \kappa_1 \nu_s^1 \Big)\, ds - A_T \Big) dt \Bigg], \end{aligned} \tag{6.13}$$

for any $\nu^1 \in \mathcal{A}_m$.

The proof of Lemma 6.3 is given in Appendix B.

From the explicit expression of the Gâteaux derivative in (6.13) we can derive a first order optimality condition. It takes the form of a coupled system of linear forward backward stochastic differential equations (FBSDE), as described in following lemma which is proved in Appendix B.

Lemma 6.4. *Under Assumption 3.1, the control $\nu^{1,*} \in \mathcal{A}_m$ is the unique maximiser to the minor agent's objective functional H^1 in (6.1) if and only if the process $(Q^{1,\nu^{1,*}}, \nu^{1,*})$ satisfies the following coupled linear FBSDE system*

$$\begin{aligned} dQ_t^{1,\nu^{1,*}} &= -\nu_t^{1,*} dt, \quad Q_0^{1,\nu^{1,*}} = 0, \\ d\nu_t^{1,*} &= \frac{1}{2\lambda_1} d\mathcal{N}_t + \frac{1}{2\lambda_1} d\mathcal{M}_t - \frac{\phi_t^1}{\lambda_1} Q_t^{1,\nu^{1,*}} dt - \frac{\kappa_0}{2\lambda_1} \nu_t^0 dt + \frac{1}{2\lambda_1} dA_t, \\ \nu_T^{1,*} &= \frac{2\alpha - \kappa_1}{2\lambda_1} Q_T^{1,\nu^{1,*}} \end{aligned} \tag{6.14}$$

$d\mathbb{P} \otimes dt$-a.e. on $\Omega \times [0,T]$ where $\mathcal{M} = (\mathcal{M}_t)_{t\in[0,T]}$ and $\mathcal{N} = (\mathcal{N}_t)_{t\in[0,T]}$ are two suitable square integrable martingales.

For the remainder of this section we focus on the derivation of the explicit solution to (6.14). We first describe the heuristics of the proof.

Heuristics for the solution to (6.14). The solution to the FBSDE system (6.14) determines the solution to the minor agent's problem. The main obstacle in solving the system (6.14) is that it presents a general time dependent coefficient ϕ_t^1. In order to solve this equation, we formulate an ansatz for the minor agent's optimal strategy $\nu^{1,*}$. Then, we demonstrate that the ansatz solution for $\nu^{1,*}$ is the unique solution to (6.14) and therefore the solution to the minor agent's problem. Due to the linear structure of the system (6.14), we make the ansatz that there are two progressively measurable processes $r^0 = (r_t^0)_{t\in[0,T]}$ and $r^1 = (r_t^1)_{t\in[0,T]}$ such that $\nu^{1,*}$ can be expressed as

$$\nu_t^{1,*} = -\left(r_t^0 + r_t^1 Q_t^{1,\nu^{1,*}}\right), \quad 0 \le t \le T. \tag{6.15}$$

We differentiate (6.15) via Itô's lemma and by using $dQ_t^{1,\nu^{1,*}} = -\nu_t^{1,*}dt$ to get

$$d\nu_t^{1,*} = -dr_t^0 - dr_t^1 Q_t^{1,\nu^{1,*}} + \nu_t^{1,*} r_t^1 dt. \tag{6.16}$$

We plug-in (6.16) into (6.14) and we arrive at

$$\begin{aligned} 0 = &\left(2\lambda_1 dr_t^1 - 2\phi_t^1 dt + 2\lambda_1 (r_t^1)^2 dt\right) Q_t^{1,\nu^{1,*}} \\ &+ \left(2\lambda_1 dr_t^0 + 2\lambda_1 r_t^1 r_t^0 dt - \kappa_0 \nu_t^0 dt + dA_t + d\mathcal{M}_t + d\mathcal{N}_t\right). \end{aligned} \tag{6.17}$$

Equation (6.17) must hold $d\mathbb{P}\otimes dt$ almost everywhere for all values $Q_t^{1,\nu^{1,*}}$. We conjecture that the terms within each of the brackets must vanish independently. The terms from (6.17) yield to two coupled differential equations for r^0 and r^1 independent of the process $Q^{1,\nu^{1,*}}$, where we determine the terminal conditions from (6.14). Specifically, the process r^1 must satisfy dt-a.e. the following non-autonomous Riccati ODE

$$\partial_t r_t^1 = \frac{1}{\lambda_1}\phi_t^1 - (r_t^1)^2, \quad r_T^1 = -\frac{2\alpha - \kappa_1}{2\lambda_1}, \tag{6.18}$$

while r^0 must satisfy the following BSDE

$$-dr_t^0 = r_t^1 r_t^0 dt - \frac{\kappa_0}{2\lambda_1}\nu_t^0 dt + \frac{1}{2\lambda_1}dA_t + \frac{1}{2\lambda_1}d\mathcal{M}_t + \frac{1}{2\lambda_1}d\mathcal{N}_t, \quad r_T^0 = 0. \tag{6.19}$$

An explicit formula for the solution of (6.18) does not exist when ϕ_t^1 is a general piecewise continuous function as in the case at hand. Nevertheless, we will prove that the solution to (6.18) exists and it is unique. Once a solution to (6.18) is found, then we can plug it to (6.19) and derive r^0.

In the following proposition, which is proved in Appendix C, we derive the existence and uniqueness of the solutions to (6.18) and (6.19).

Proposition 6.5. *Under Assumption 3.1, there exists a unique continuous function r^1 that satisfies the non-autonomous Riccati ODE (6.18) dt-a.e. on $[0,T]$. Furthermore, the BSDE (6.19) admits a closed form solution r^0 given by (3.5). Moreover,*

$$\mathbb{E}\left[\int_0^T (r_t^0)^2 dt\right] < \infty. \tag{6.20}$$

Remark 6.6. *As stated in Proposition 6.5, the function r^1 satisfies the Riccati ODE (6.18) only dt almost everywhere. This is to be expected since ϕ^1 is assumed to be piecewise continuous and the derivatives of r^1 may not exists at the points of discontinuity. Nevertheless, as we will show in the proof of Theorem 3.2, this is sufficient for our needs as we wish to solve the FBSDE system (6.14) only $d\mathbb{P} \otimes dt$ almost everywhere.*

In order to prove Theorem 3.2 we will need the following lemma, which is also proved in Appendix C.

Lemma 6.7. *Let r^1 be the unique solution of (6.18). Then, the kernel $\mathcal{K}$ is jointly continuous over $[0,T]^2$. In particular, $\mathcal{K}$ is bounded over $[0,T]^2$ and is in $L^2([0,T]^2)$.*

We are now ready to prove Theorem 3.2. In order to simplify the notation, we will often denote the process $\nu^{1,*}(\nu^0)$ by $\nu^{1,*}$.

Proof of Theorem 3.2. Let $\nu^{1,*}$ as in (3.6) with r^1 as in (6.18) and r^0 as in (3.5).

Step 1: We determine an explicit expression for $Q^{1,\nu^{1,*}}$. We argue that

$$Q_t^{1,\nu^{1,*}} = \int_0^t \mathcal{K}(s,t) r_s^0 ds. \tag{6.21}$$

This could be verified by using (3.4) so we can write (6.21) as

$$Q_t^{1,\nu^{1,*}} = \xi_t^+ \int_0^t \xi_s^- r_s^0 ds. \tag{6.22}$$

From (3.3) we note that ξ^+ satisfies the following ODE,

$$\frac{d\xi_t^+}{dt} = r_t^1 \xi_t^+ \quad 0 \le t \le T. \tag{6.23}$$

Taking the derivative in (6.22) and using (3.3), (3.4) and (6.23) we arrive at

$$\begin{aligned} dQ_t^{1,\nu^{1,*}} &= r_t^1 \left(\xi_t^+ \int_0^t \xi_s^- r_s^0 ds\right) dt + r_t^0 dt \\ &= \left(r_t^0 + r_t^1 \int_0^t \mathcal{K}(s,t) r_s^0 ds\right) dt. \end{aligned} \tag{6.24}$$

From (6.24) and (2.5) we get (6.21).

Step 2: We show that $\nu^{1,*}$ solves (6.14). Note that from (2.5) and (3.6) we can rewrite $\nu^{1,*}$ as

$$\nu_t^{1,*} = -\left(r_t^0 + r_t^1 Q_t^{1,\nu^{1,*}}\right), \quad 0 \le t \le T. \tag{6.25}$$

By plugging in (6.25) into (6.14) we conclude that it is enough to prove that (6.17) holds. Since r_t^0 satisfies (6.19) and r_t^1 satisfies (6.18) dt-a.e. on $[0,T]$, then (6.17) holds $d\mathbb{P}\otimes dt$ almost everywhere. Next, using the terminal conditions of r^0 and r^1 from (6.19) and (6.18) we deduce from (6.25) that the terminal condition in (6.14) is satisfied. Recall that the forward component in (6.14) is satisfied by (2.5). Therefore, $(Q^{1,\nu^{1,*}}, \nu^{1,*})$ solve the system (6.14), $d\mathbb{P}\otimes dt$ almost everywhere.

Step 3: We show that $\nu^{1,*} \in \mathcal{A}_m$. From (2.4) it follows that we need to verify that

$$\mathbb{E}\left[\int_0^T (\nu_t^{1,*})^2 dt\right] < \infty. \tag{6.26}$$

Form (6.21), Proposition 6.5, Lemma 6.7 and Cauchy-Schwartz inequality we get

$$\begin{aligned}\mathbb{E}\left[\int_0^T \left(Q_t^{1,\nu^{1,*}}\right)^2 dt\right] &\le \mathbb{E}\left[\int_0^T \left(\int_0^t \mathcal{K}(s,t)^2 ds\right)\left(\int_0^t (r_s^0)^2 ds\right) dt\right] \\ &< \infty.\end{aligned} \tag{6.27}$$

From Proposition 6.5 it follows that r^1 is continuous hence bounded on $[0,T]$ and that r^0 is square-integrable. Using this, (6.25) and Cauchy-Schwartz inequality gives

$$\begin{aligned}\mathbb{E}\left[\int_0^T (\nu_t^{1,*})^2 dt\right] &\le 2\mathbb{E}\left[\int_0^T (r_t^0)^2 dt\right] + 2\mathbb{E}\left[\int_0^T (r_t^1)^2 \left(Q_t^{1,\nu^{1,*}}\right)^2 dt\right] \\ &< \infty.\end{aligned} \tag{6.28}$$

Therefore, $\nu^{1,*}$ is admissible and it solves (6.14), hence by Lemma (6.4) it is the unique maximiser to the minor agent's objective functional H^1 in (6.1). □

7 Proof of Theorem 3.5

In this section we derive the major agent's optimal strategy via a calculus of variation argument. We start be defining operators which are essential to our proofs. Then we derive an equivalent representation the major agent's objective H^0 which is more convenient to our method of proof. Throughout this section we assume that Assumption 3.1 holds and that the minor agent is adopting the strategy $\nu^{1,*}$ from Theorem 3.2. Henceforth, with a slight abuse of notation we write $H^0(\nu^0)$ for $H^0(\nu^0, \nu^{1,*}(\nu^0))$.

Essential definitions of $L^2([0,T])$ operators. We denote by $\mathcal{T}^*$ the adjoint kernel of $\mathcal{T}$ for $\langle\cdot,\cdot\rangle_{L^2}$, that is

$$\mathcal{T}^*(t,s) := \mathcal{T}(s,t), \quad s,t \in [0,T], \tag{7.1}$$

and by T^* the corresponding adjoint integral operator.

We define the kernel $\mathcal{K}_1 : [0,T]^2 \to \mathbb{R}_+$ as

$$\mathcal{K}_1(t,s) := \mathcal{K}(s,t)\mathbb{1}_{\{s\le t\}}, \quad s,t \in [0,T], \tag{7.2}$$

where $\mathcal{K}$ is given in (3.4). We let K_1 to be the integral operator generated by the kernel $\mathcal{K}_1$, that is

$$(\mathsf{K}_1\psi)(t) := \int_0^T \mathcal{K}_1(t,s)\psi(s)ds, \quad t \in [0,T],\ \psi \in L^2([0,T]), \tag{7.3}$$

The following lemma, which is proved in Section 8, outlines some useful properties of K_1. Recall that the class of operators $B(L^2([0,T]))$ was defined after (3.9).

Lemma 7.1. *The operator K_1 is in $B(L^2([0,T]))$. Moreover, $\mathsf{K}_1^* \in B(L^2([0,T]))$ is given by*

$$(\mathsf{K}_1^*\psi)(t) = \int_0^T \mathcal{K}(t,s)\psi(s)\mathbb{1}_{\{t\le s\}}ds, \quad t \in [0,T],\ \psi \in L^2([0,T]). \tag{7.4}$$

We recall the definition of a compact operator from Porter and Stirling (1990), (see Definition 3.2 therein).

Definition 7.2. *An operator $\mathsf{T} : L^2([0,T]) \to L^2([0,T])$ is said to have finite rank if its image $\{\mathsf{T}\psi : \psi \in L^2([0,T])\}$ has finite dimension. An operator $\mathsf{L} \in B(L^2([0,T]))$ is said to be compact if there is a sequence $(\mathsf{L}_n)_{n\ge1}$ of finite-rank operators in $B(L^2([0,T]))$ such that $||\mathsf{L}_n - \mathsf{L}|| \to 0$ as $n \to \infty$.*

A particularly important result that we will use states that any operator generated by a kernel in $L^2([0,T]^2)$ is compact (see Theorem 3.4 in Porter and Stirling (1990)). We remark that not all operators in $B(L^2([0,T]))$ are compact operators as pointed out in example 3.6 in Porter and Stirling (1990).

Next we define non-negative and positive operators in $B(L^2([0,T]))$ as in (Porter and Stirling, 1990, Definition 6.1).

Definition 7.3. *Let $\mathsf{T} \in B(L^2([0,T]))$ be a self-adjoint operator. T is said to be non-negative if and only if $\langle \mathsf{T}\psi, \psi\rangle_{L^2} \geq 0$ for all $\psi \in L^2([0,T])$. It is said to be positive if and only if $\langle \mathsf{T}\psi, \psi\rangle_{L^2} > 0$ for all $\psi \neq 0$ in $L^2([0,T])$. If there a positive constant m for which $\langle \mathsf{T}\psi, \psi\rangle_{L^2} \geq m||\psi||^2_{L^2}$ for all $\psi \in L^2([0,T])$, then T is said to be positive and bounded below.*

Our next result outlines several important properties of the operator G in (3.11). Recall that K_1 was defined in (7.3).

Proposition 7.4. *Let G be defined as in (3.11). Then $\mathsf{G} \in B(L^2([0,T]))$ is a positive, compact and self-adjoint operator. Moreover, it satisfies*

$$(\mathsf{G}\psi)(t) = (\mathsf{K}_1\mathsf{K}_1^*\psi)(t), \quad \text{for all } t \in [0,T],\ \psi \in L^2([0,T]). \tag{7.5}$$

The proof of Proposition 7.4 is given in Section 8.

In the following lemma we determine an alternative representation for the major agent's objective functional.

Lemma 7.5. *Let H^0 be the major agent's objective functional in (2.10). Then, for any $\nu^0 \in \mathcal{A}_M^{q_0}$ it holds that*

$$\begin{aligned} H^0(\nu^0) = x_0 + M_0 q_0 - \kappa_0 \frac{q_0^2}{2} - \frac{\kappa_1\kappa_0}{2\lambda_1}\int_0^T \nu_t^0(\mathsf{G}\nu^0)(t)dt \\ + \int_0^T \left(\frac{\kappa_1}{2\lambda_1}\nu_t^0(\mathsf{G}\bar{\mu})(t) + Q_t^{0,\nu^0}\bar{\mu}_t\right)dt - \lambda_0\int_0^T (\nu_t^0)^2 dt. \end{aligned} \tag{7.6}$$

The proof of Lemma 7.5 is postponed to Section 9.

The following proposition establishes the uniqueness of the maximiser of H^0.

Proposition 7.6. *There exists at most one admissible maximiser to the major agent's objective functional H^0 in (2.10).*

Proof. To show the desired result, it is sufficient to show that the objective functional H^0 is strictly concave over $\mathcal{A}_M^{q_0}$.

Notice that since $\nu^{1,*} \in \mathcal{A}_m$, $\nu^0 \in \mathcal{A}_M^{q_0}$ and μ satisfies (3.7), then H^0 is finite. Therefore, in order to show that the objective H^0 is strictly concave over $\mathcal{A}_M^{q_0}$ we must verify that

$$\mathcal{I}^0(\rho,\nu,\omega) := H^0(\rho\nu + (1-\rho)\omega) - \rho H^0(\nu) - (1-\rho)H^0(\omega) > 0 \tag{7.7}$$

for any $\rho \in (0,1)$ and for any dt-distinguishable $\nu, \omega \in \mathcal{A}_M^{q_0}$.

We now fix such ρ and ν, ω. From (2.3) and (3.11) it follows that $Q^{0,\nu}$ and $\mathsf{G}\nu$ are linear in ν. Then from (7.6) and (7.7) we get

$$\begin{aligned}\mathcal{I}^0(\rho,\nu,\omega) = \rho(1-\rho)\lambda_0 \int_0^T (\nu_t - \omega_t)^2 dt \\ + \rho(1-\rho)\frac{\kappa_1\kappa_0}{2\lambda_1}\int_0^T (\nu_t - \omega_t)(\mathsf{G}(\nu-\omega))(t)dt.\end{aligned} \tag{7.8}$$

Note that we are only left to show that the second term in the right-hand side of (7.8) is strictly positive. Since G is a positive operator by Proposition 7.4 and ν and ω are dt-distinguishable, then the result follows. □

As similarly done in Section 6, we can apply a convex analytic calculus of variation approach to derive a first order optimality conditions which characterizes the unique solution to the major agent's problem.

Since the map $\nu^0 \to H^0(\nu^0)$ in (2.10) is strictly concave, then H^0 admits a unique maximiser characterised by the critical point at which the Gâteaux derivative

$$\langle \mathcal{D}H^0(\nu^0), \omega\rangle = \lim_{\epsilon\to 0}\frac{H^0(\nu^0+\epsilon\omega) - H^0(\nu^0)}{\epsilon} \tag{7.9}$$

vanishes. We remind the reader that the minor agent optimal strategy $\nu^{1,*}$ was fixed to be the one in (3.6), therefore, making the major agent's objective functional H^0 only a function of the major agent control ν^0. In the next lemma we explicitly compute the first-order Gâteaux derivative of H^0. Recall that $\mathcal{A}_M^{q_0}$ was defined in (2.2).

Lemma 7.7. *The Gâteaux derivative of H^0 in (7.6), in a direction $\omega \in \mathcal{A}_M^0$ is given by*

$$\langle \mathcal{D}H^0(\nu^0), \omega\rangle = \int_0^T \omega_t\left(-2\lambda_0\nu_t^0 - \frac{\kappa_1\kappa_0}{\lambda_1}(\mathsf{G}\nu^0)(t) + \frac{\kappa_1}{2\lambda_1}(\mathsf{G}\bar{\mu})(t) - \int_t^T \bar{\mu}_s ds\right)dt, \tag{7.10}$$

for any $\nu^0 \in \mathcal{A}_M^{q_0}$.

Proof. Let $\epsilon > 0$, $\nu^0 \in \mathcal{A}_M^{q_0}$ and $\omega \in \mathcal{A}_M^0$. From (7.6) we get

$$\begin{aligned} &H^0(\nu^0 + \epsilon\omega) - H^0(\nu^0) \\ &= \epsilon\Bigg(\int_0^T \omega_t \left(-2\lambda_0 \nu_t^0 + \frac{\kappa_1}{2\lambda_1}(\mathsf{G}\bar{\mu})(t) \right) dt - \frac{\kappa_1\kappa_0}{2\lambda_1} \int_0^T \omega_t (\mathsf{G}\nu^0)(t) dt \\ &\quad - \frac{\kappa_1\kappa_0}{2\lambda_1} \int_0^T \nu_t^0 (\mathsf{G}\omega)(t) dt - \int_0^T \bar{\mu}_t \left(\int_0^t \omega_s ds \right) dt \Bigg) \\ &\quad + \epsilon^2 \Bigg(- \lambda_0 \int_0^T \omega_s^2 ds - \frac{\kappa_1\kappa_0}{2\lambda_1} \int_0^T \omega_t (\mathsf{G}\omega)(t) dt \Bigg). \end{aligned} \tag{7.11}$$

Notice that all the terms in (7.11) are finite as a consequence of $\omega \in \mathcal{A}_M^0$, $\nu^0 \in \mathcal{A}_M^{q_0}$, μ satisfying (3.7) and $\mathsf{G} \in B(L^2([0,T]))$ as shown in Proposition 7.4. It follows that

$$\begin{aligned} \langle \mathcal{D}H^0(\nu^0), \omega \rangle = \int_0^T \omega_t \Big(&- 2\lambda_0 \nu_t^0 - \frac{\kappa_1\kappa_0}{2\lambda_1} (\mathsf{G}\nu^0)(t) + \frac{\kappa_1}{2\lambda_1} (\mathsf{G}\bar{\mu})(t) \Big) dt \\ &- \frac{\kappa_1\kappa_0}{2\lambda_1} \int_0^T \nu_t^0 (\mathsf{G}\omega)(t) dt - \int_0^T (\bar{\mu}_t) \left(\int_0^t \omega_s ds \right) dt. \end{aligned} \tag{7.12}$$

Proposition 7.4 shows that $\mathsf{G} \in B(L^2([0,T]))$. Since μ satisfies (3.7), then $\bar{\mu}$ is in $L^2([0,T])$ by Jensen's inequality. Since $\omega \in \mathcal{A}_M^0$ and $\bar{\mu} \in L^2([0,T])$, then we can apply Fubini's theorem to obtain

$$\int_0^T (\bar{\mu}_t) \left(\int_0^t \omega_s ds \right) dt = \int_0^T \omega_s \int_s^T \bar{\mu}_t dt ds. \tag{7.13}$$

As claimed in Proposition 7.4, G is self-adjoint, therefore, it holds that

$$\begin{aligned} \int_0^T \nu_t^0 (\mathsf{G}\omega)(t) dt &= \langle \nu^0, (\mathsf{G}\omega) \rangle_{L^2} \\ &= \langle \omega, (\mathsf{G}\nu^0) \rangle_{L^2} \\ &= \int_0^T \omega_t (\mathsf{G}\nu^0)(t) dt. \end{aligned} \tag{7.14}$$

Finally, we use (7.13) and (7.14) in (7.12) to get (7.10).

□

Recall the definition of the operator S in (3.12). In the following lemma we derive an optimality condition that takes the form of an integral equation.

Proposition 7.8. *A strategy $\nu^{0,*} \in \mathcal{A}_M^{q_0}$ maximises the major agent's performance functional (2.10) if there exists a constant $\eta \in \mathbb{R}$ such that,*

$$\nu_t^{0,*} + \frac{\kappa_1 \kappa_0}{2\lambda_0 \lambda_1} (\mathsf{G}\nu^{0,*})(t) = (\mathsf{S}\bar{\mu})(t) + \frac{\eta}{2\lambda_0}, \quad \text{for all } 0 \leq t \leq T. \tag{7.15}$$

Proof. In the proof of Proposition 7.6 we have shown that the objective functional H^0 is strictly concave over $\mathcal{A}_M^{q_0}$. Therefore, by Proposition 2.1 in Chapter 2 in Ekeland and Témam (1999) if an admissible strategy $\nu^{0,*}$ satisfies

$$\langle \mathcal{D}H^0(\nu^{0,*}), \omega \rangle = 0, \quad \text{for all } \omega \in \mathcal{A}_M^0, \tag{7.16}$$

then it is the maximiser of H^0.

Recalling the result of Lemma 7.7, we plug (7.15) into (7.10) to get

$$\begin{aligned} \langle \mathcal{D}H^0(\nu^{0,*}), \omega \rangle &= -\int_0^T \omega_t \left(2\lambda_0 (\mathsf{S}\bar{\mu})(t) + \eta - \frac{\kappa_1}{2\lambda_1} (\mathsf{G}\bar{\mu})(t) + \int_t^T \bar{\mu}_s ds \right) dt \\ &= -\left(\int_0^T \bar{\mu}_s ds + \eta \right) \int_0^T \omega_t dt \end{aligned} \tag{7.17}$$

where we used (3.12) in the second equality. Recall that $\omega \in \mathcal{A}_M^0$, then from (2.2) we get (7.16). □

The following proposition, which is proved in Section 8, derives some properties of the operator R in (3.14) that are crucial to the proof of Theorem 3.5.

Proposition 7.9. *Let R be defined as in (3.14) and G be defined as in (3.11). Then the following holds:*

(i) *The inverse operator of $\mathsf{I} + \frac{\kappa_1 \kappa_0}{2\lambda_0 \lambda_1} \mathsf{G}$ exists and it satisfies*

$$\mathsf{R} = \left(\mathsf{I} + \frac{\kappa_1 \kappa_0}{2\lambda_0 \lambda_1} \mathsf{G} \right)^{-1}. \tag{7.18}$$

(ii) *R is positive, bounded from below and is in $B(L^2([0,T]))$.*

(iii) *If $f \in C([0,T])$, then $(\mathsf{R}f) \in C([0,T])$.*

The following lemma derives some essential properties of the operator S in (3.12).

Lemma 7.10. *Let* S *be as in* (3.12). *Then* S *is in* $B(L^2([0,T]))$ *and for any* $\psi \in L^2([0,T])$,$(\mathsf{S}\psi)(t)$ *is continuous on* $0 \le t \le T$.

Proof. Proposition 7.4 proves that G is in $B(L^2([0,T]))$, hence it follows from (3.12) that also S is in $B(L^2([0,T]))$.

Next, let $\psi \in L^2([0,T])$. From Proposition 9.4 it follows that $\mathsf{G}\psi$ is continuously differentiable on $[0,T]$, hence by (3.12) the continuity of $(\mathsf{S}\psi)(\cdot)$ follows. □

We are now ready to prove Theorem 3.5.

Proof of Theorem 3.5. Recall that $\nu^{0,*}$ and η were defined in (3.15) and (3.16), respectively. We split the proof into the following steps.

Step 1. We show that $(\nu^{0,*}, \eta)$ is the unique solution to (7.15). Recall that R was defined in (3.14). From (3.7) and Lemma 7.10 it follows that $\mathsf{S}\bar{\mu}(\cdot)$ is in $L^2([0,T])$. Then, from Proposition 7.9 we get that the unique solution to (7.15) is given by

$$\begin{aligned} \nu_t^{0,*} &= \left(\left(\mathsf{I} + \frac{\kappa_1 \kappa_0}{2\lambda_0 \lambda_1} \mathsf{G} \right)^{-1} \left(\frac{\eta}{2\lambda_0} + \mathsf{S}\bar{\mu} \right) \right)(t) \\ &= \frac{\eta}{2\lambda_0} (\mathsf{R}1)(t) + (\mathsf{R}\mathsf{S}\bar{\mu})(t), \quad 0 \le t \le T. \end{aligned} \tag{7.19}$$

Step 2. We verify that $\nu^{0,*}$ satisfies the fuel constraint in (2.2). Note that we need to show that $\langle \nu^{0,*}, 1 \rangle_{L^2} = q_0$. From (3.15) and (3.16) we get

$$\begin{aligned} \langle \nu^{0,*}, 1 \rangle_{L^2} &= \frac{\eta}{2\lambda_0} \langle \mathsf{R}1\,, 1 \rangle_{L^2} + \langle \mathsf{R}\mathsf{S}\bar{\mu}, 1 \rangle_{L^2} \\ &= \left(\frac{q_0 - \langle \mathsf{R}\mathsf{S}\bar{\mu}, 1 \rangle_{L^2}}{\langle \mathsf{R}1\,, 1 \rangle_{L^2}} \right) \langle \mathsf{R}1\,, 1 \rangle_{L^2} + \langle \mathsf{R}\mathsf{S}\bar{\mu}, 1 \rangle_{L^2} \\ &= q_0. \end{aligned} \tag{7.20}$$

Note that by Proposition 7.9(ii) the operator R is positive and bounded from below. Therefore the denominator in (3.16) is strictly positive and the constant η is well-defined.

Step 3. We verify that $\nu^{0,*}$ is in $L^2([0,T])$. From Lemma 7.10 it follows that $(\mathsf{S}\bar{\mu})(t)$ is continuous on $[0,T]$. Proposition 7.9(iii) and (7.19) then imply that $\nu_t^{0,*}$ is continuous and therefore bounded on $[0,T]$. This concludes the proof. □

8 Proofs of Lemma 7.1 and Propositions 7.4 and 7.9

Proof of Lemma 7.1. Recall that kernels $\mathcal{K}$ and $\mathcal{K}_1$ were defined in (3.4) and (7.2) respectively. Let ψ be in $L^2([0,T])$. From (7.1) we get

$$\mathcal{K}_1^*(t,s) = \mathcal{K}(t,s)\mathbb{1}_{\{t\leq s\}}, \quad \text{for all } 0 \leq t,s \leq T, \tag{8.1}$$

which verifies (7.4). From Lemma 6.7 we get that $\mathcal{K}$ is in $L^2([0,T]^2)$. Then by (7.2) and (8.1) also $\mathcal{K}_1^*$ and $\mathcal{K}_1$ are in $L^2([0,T]^2)$, therefore, K_1 and K_1^* are in $B(L^2([0,T]))$. □

The following lemma describes several properties of the kernel $\mathcal{G}$ in (3.10) which are essential to the proof of Proposition 7.9.

Lemma 8.1. *Let $\mathcal{G}$ as in (3.10). Then, $\mathcal{G}$ is symmetric and jointly continuous on $[0,T]^2$. Moreover, $\mathcal{G}$ is in $L^2([0,T]^2)$.*

Proof. From (3.10) it follows that $\mathcal{G}$ is symmetric. Recall that ξ^+ and ξ^- were defined as in (3.3). Note that ξ^+ and ξ^- are continuous on $[0,T]$ and that ξ^- belongs to $L^2([0,T])$. It follows that $t \mapsto \int_0^t (\xi_u^-)^2 du$ is continuous on $[0,T]$. From (3.4) and (3.10) it follows that the kernel $\mathcal{G}$ can be rewritten as follows,

$$\mathcal{G}(t,s) = \xi_t^+ \xi_s^+ \int_0^{t\wedge s} (\xi_u^-)^2 du, \quad \text{for all } 0 \leq t,s \leq T.$$

Therefore, $\mathcal{G}(t,s)$ is jointly continuous on $[0,T]^2$, hence it is in $L^2([0,T]^2)$. □

The following lemma is needed for the proof of Proposition 7.4.

Lemma 8.2. *Let K_1 be defined as in (7.3) and let $\psi \in L^2([0,T])$. If $(\mathsf{K}_1\psi)(t) = 0$ for all $0 \leq t \leq T$, then $\psi(t) = 0$ a.e. on $[0,T]$.*

Proof. Let $\psi \in L^2([0,T])$ and assume that $(\mathsf{K}_1\psi)(t) = 0$ for all $0 \leq t \leq T$. From (3.4), (7.2) and (7.3) we get

$$(\mathsf{K}_1\psi)(t) = \xi_t^+ \left(\int_0^t \xi_s^- \psi(s) ds \right) = 0, \quad \text{for all } 0 \leq t \leq T. \tag{8.2}$$

From (3.3) it follows that $\xi_t^\pm > 0$ for all $0 \leq t \leq T$, therefore

$$\int_0^t \xi_s^- \psi(s) ds = 0, \quad \text{for all } 0 \leq t \leq T. \tag{8.3}$$

Since $\xi_t^{\pm}$ are also continuous for all $t \in [0,T]$ and $\psi \in L^2([0,T])$, then $\xi^-\psi$ is integrable. Then from Lebesgue differentiation theorem and (8.3) we conclude that

$$\xi_t^- \psi(t) = 0, \quad dt\text{-a.e. on } [0,T].$$

But again since $\xi_t^- > 0$ on $[0,T]$ we get that

$$\psi(t) = 0, \quad dt\text{-a.e. on } [0,T],$$

which proves the result. □

Proof of Proposition 7.4. Let $\psi \in L^2([0,T])$. From (3.10) we get

$$\begin{aligned}(\mathsf{G}\psi)(t) &= \int_0^T \mathcal{G}(t,u)\psi(u)du \\ &= \int_0^T \psi(u) \int_0^{t\wedge u} \mathcal{K}(s,t)\mathcal{K}(s,u)dsdu \\ &= \int_0^T \int_0^T \mathcal{K}(s,t)\mathcal{K}(s,u)\psi(u)\mathbb{1}_{\{s\le t\}}\mathbb{1}_{\{s\le u\}}duds.\end{aligned}$$

Together with (7.2) and (8.1) it follows that

$$\begin{aligned}(\mathsf{G}\psi)(t) &= \int_0^T \mathcal{K}_1(t,s) \int_0^T \mathcal{K}_1^*(s,u)\psi(u)duds \\ &= (\mathsf{K}_1\mathsf{K}_1^*\psi)(t), \quad \text{for all } 0 \le t \le T,\end{aligned} \tag{8.4}$$

which proves (7.5). Since by Lemma 7.1, $\mathsf{K}_1, \mathsf{K}_1^* \in B(L^2([0,T]))$, a straightforward application of the Cauchy-Schwartz inequality and (8.4) shows that $\mathsf{G} \in B(L^2([0,T]))$. Note that (8.4) also implies that G is self-adjoint, that is

$$\begin{aligned}\mathsf{G}^* &= (\mathsf{K}_1\mathsf{K}_1^*)^* \\ &= (\mathsf{K}_1^*)^*\mathsf{K}_1^* \\ &= \mathsf{G}.\end{aligned} \tag{8.5}$$

Next, using (3.11) we prove that G is compact (see Definition 7.2). From Lemma 8.1 it follows that $\mathcal{G}$ is in $L^2([0,T]^2)$. Then the result follows from Theorem 3.4 in Porter and Stirling (1990), which shows that any integral operator generated by a kernel in $L^2([0,T]^2)$ is compact.

Finally, we prove that G is a positive operator in the sense of Definition 7.3. We have shown that $\mathsf{G} = \mathsf{K}_1\mathsf{K}_1^*$. Since for any $\psi \in L^2([0,T])$ we have

$$\langle \mathsf{G}\psi, \psi\rangle = \langle \mathsf{K}_1\mathsf{K}_1^*\psi, \psi\rangle = ||\mathsf{K}_1\psi||^2 \geq 0,$$

it follows that G is non-negative. By Lemma 8.2 we have $(\mathsf{K}_1\psi)(t) = 0$ for all $t \in [0,T]$ only for $\psi = 0$ a.e. on $[0,T]$. Therefore, G is positive. □

In the following we present a sequence of results which are essential to the proof of Proposition 7.9. Using the results of Proposition 7.4 we are in a position to fully characterise the spectral properties of the integral operator G.

Lemma 8.3 (Spectral Decomposition of G)**.** *Let G be defined as in (3.11). Then G has a sequence of positive eigenvalues $(\zeta_n)_{n\geq 1}$ and a corresponding orthonormal sequence $(\psi_n)_{n\geq 1}$ of eigenfunctions in $L^2([0,T])$, such that for each $\varphi \in L^2([0,T])$, we have that*

$$\mathsf{G}\varphi = \sum_{n\geq 1} \zeta_n \langle \varphi, \psi_n\rangle_{L^2}\psi_n.$$

Moreover, for all $N \geq 1$, define $\mathsf{G}_N, \mathsf{G}_N^{abs} \in B(L^2([0,T]))$ by

$$\begin{aligned} \mathsf{G}_N\varphi &= \sum_{n=1}^{N} \zeta_n \langle \varphi, \psi_n\rangle_{L^2}\psi_n, \\ \mathsf{G}_N^{abs}\varphi &= \sum_{n=1}^{N} \left|\zeta_n \langle \varphi, \psi_n\rangle_{L^2}\psi_n\right|. \end{aligned} \tag{8.6}$$

Then, $\mathsf{G}_N\varphi$ converges uniformly to $\mathsf{G}\varphi$,

$$\lim_{N\to\infty} \sup_{t\in[0,T]} |(\mathsf{G}_N\varphi)(t) - (\mathsf{G}\varphi)(t)| = 0, \tag{8.7}$$

and $\mathsf{G}_N^{abs}\varphi$ is uniformly convergent, that is, there exists a function $\Phi \in L^2([0,T])$ such that

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \left|(\mathsf{G}_N^{abs}\varphi)(t) - \Phi(t)\right| = 0. \tag{8.8}$$

Proof. From Proposition 7.4 it follows that G is a self-adjoint compact operator in $B(L^2([0,T]))$. Therefore, from Theorem 4.15 of Porter and Stirling (1990) there is a sequence $(\zeta_n)_{n\geq 1}$ of non-zero eigenvalues of G and a corresponding orthonormal sequence $(\psi_n)_{n\geq 1}$ of eigenfunctions in $L^2([0,T])$ such that $\mathsf{G}\psi = \sum_{n\geq 1} \zeta_n \langle \psi, \psi_n\rangle_{L^2}\psi_n$.

Moreover, the operator G_N converges to G in mean, i.e. $||\mathsf{G} - \mathsf{G}_N|| \to 0$ as $N \to \infty$. From Proposition 7.4 it follows that G is positive and self-adjoint, hence from Lemma 6.1 in Porter and Stirling (1990) we get that all of its eigenvalues $(\zeta_n)_{n\geq 1}$ are positive. Since $\mathcal{G}$ is continuous and symmetric by Proposition 8.1, then from Theorem 4.22 of Porter and Stirling (1990) it follows that $\mathsf{G}_N\psi$ and $\mathsf{G}_N^{\text{abs}}$ satisfy the convergence in (8.7) and (8.8). □

Remark 8.4. *In Appendix A we provide an example for the spectral decomposition of* G *in Lemma 8.3.*

Lemma 8.5. *Let* G *be defined as in* (3.11)*. Then, the operator* $\mathsf{I} + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}$ *is positive and bounded from below in the sense of Definition 7.3.*

Proof. Let $\psi \in L^2([0,T])$. Since G is positive by Proposition 7.4 and $\kappa_0, \kappa_1, \lambda_0, \lambda_1 > 0$, we get that

$$\left\langle \left(\mathsf{I} + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}\right)\psi, \psi \right\rangle_{L^2} \geq ||\psi||^2_{L^2}. \tag{8.9}$$

Therefore, $\mathsf{I} + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}$ is positive and bounded from below. □

Recall that we assume that constants $\lambda_0, \lambda_1, \kappa_1$ and κ_0 are strictly positive and that we proved in Lemma 8.3 that the eigenvalues of G are all positive. The following lemma is therefore an easy corollary.

Lemma 8.6. *Let* $\zeta^* = -\frac{2\lambda_0\lambda_1}{\kappa_1\kappa_0}$ *and let* G *be defined as in* (3.11)*. Then,* ζ^* *is not an eigenvalue of the integral operator* G*.*

We recall Theorem 4.27 of Porter and Stirling (1990) which will be useful in the proof of Proposition 7.9.

Proposition 8.7 ((Porter and Stirling, 1990, Theorem 4.27))**.** *Let* $f \in L^2([0,T])$*. Suppose that* $\mathcal{T} : [0,T]^2 \to \mathbb{R}$ *is a continuous symmetric kernel and that* T *is the integral operator generated by* $\mathcal{T}$*. Let* $(\mu_n)_{n\geq 1}$ *and* $(\varphi_n)_{n\geq 1}$ *be the sequence of eigenvalues and eigenfunctions of the operator* T*. Moreover, suppose that* $\frac{1}{\lambda}$ *is not an eigenvalue of* T*. Then the unique solution to the integral equation*

$$\psi(t) - \lambda \int_0^T \mathcal{T}(t,s)\psi(s)dt = f(t), \quad t \in [0,T], \tag{8.10}$$

is given by

$$\psi(t) = f(t) + \int_0^T \widetilde{\mathcal{R}}(t,s)f(s)ds, \quad t \in [0,T],$$

where

$$\widetilde{\mathcal{R}}(t,s) = \lambda\mathcal{T}(t,s) + \sum_{n\geq 1} \frac{\lambda\mu_n}{1-\lambda\mu_n}\varphi_n(t)\varphi_n(s),$$

is jointly continuous on $[0,T]^2$.

We are now ready to prove Proposition 7.9.

Proof of Proposition 7.9. (i) Note that the operator G and the corresponding kernel $\mathcal{G}$ satisfy the assumptions of Proposition 8.7. Specifically, from Lemma 8.6 it follows that $-\frac{2\lambda_0\lambda_1}{\kappa_1\kappa_0}$ is not an eigenvalue of G. Moreover, in Proposition 8.1 we have shown that $\mathcal{G}$ is continuous and symmetric on $[0,T]^2$. Therefore, we can apply the result of Proposition 8.7 to the following integral equation,

$$\psi(t) + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\int_0^T \mathcal{G}(t,s)\psi(s)ds = f(t), \quad t\in[0,T], \tag{8.11}$$

and determine that the unique solution to (8.11) is given by

$$\psi(t) = f(t) + \int_0^T \mathcal{R}(t,s)f(s)ds, \tag{8.12}$$

with $\mathcal{R}$ as in (3.13). Moreover, it follows from Proposition 8.7 that the kernel $\mathcal{R}$ is jointly continuous on $[0,T]^2$.

Next, we show that the inverse of $\mathsf{I}+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}$ is given by R. Since by Lemma 8.6, $-\frac{2\lambda_0\lambda_1}{\kappa_1\kappa_0}$ is not an eigenvalue of G, the operator $\left(\mathsf{I}+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}\right)^{-1}$ exists. Let ψ be the solution of (8.11). Since $\mathsf{I}+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}$ is invertible it follows from (8.11) that ψ can be written as follows,

$$\psi = \left(\mathsf{I}+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}\right)^{-1} f. \tag{8.13}$$

On the other hand, from (3.14) and (8.12) we have that

$$\psi = \mathsf{R}f. \tag{8.14}$$

Therefore, by comparing (8.13) and (8.14) we find that

$$\left(\mathsf{I}+\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}\right)^{-1} f = \mathsf{R}f. \tag{8.15}$$

Since (8.15) holds for any $f\in L^2([0,T])$, (i) follows.

(ii) From Proposition 7.4 it follows that G is a compact operator. Since G is a compact and the inverse of $\mathsf{I} + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}$ exists by (i), we get from the remark below the proof of Theorem 3.3 of Porter and Stirling (1990), that the inverse of $\mathsf{I} + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}$ is in $B(L^2([0,T]))$. From (7.18) it follows that R is also in $B(L^2([0,T]))$. Recall that Lemma 8.5 shows that $\mathsf{I} + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}$ is positive and bounded from below, hence from Lemma 6.2 of Porter and Stirling (1990) it follows that its inverse is positive and bounded from below. From (7.18) we conclude that R is positive and bounded from below in the sense of Definition 7.3.

(iii) Assume that $f \in C([0,T])$ then (iii) follows from (8.14) and since $\mathcal{R}$ is jointly continuous on $[0,T]^2$ by (i). □

9 Proof of Lemma 7.5

Throughout this section we assume Assumption 3.1 such that minor agent's optimal control $\nu^{1,*}$ is well-defined. Before proving Lemma 7.5 we prove several intermediate results.

Lemma 9.1. *Let r^0 be defined as in (3.5) and K_1 be defined as in (7.3). Then the following holds for any $\nu^0 \in \mathcal{A}_M^{q_0}$,*

$$\mathbb{E}[r_t^0(\nu^0)] = \frac{1}{2\lambda_1}(\mathsf{K}_1^*\bar{\mu})(t) - \frac{\kappa_0}{2\lambda_1}(\mathsf{K}_1^*\nu^0)(t), \quad \text{for all } 0 \le t \le T, \tag{9.1}$$

Moreover $(\mathbb{E}[r_t^0(\nu^0)])_{t\in[0,T]}$ is in $L^2([0,T])$.

Proof. Let $\nu^0 \in \mathcal{A}_M^{q_0}$ and recall that $\mathcal{K}$ was defined in (3.4). From Lemma 6.7, (3.7) and (3.8) it follows that the conditions of Fubini's theorem are satisfied and we get that

$$\mathbb{E}\left[\int_t^T \mathcal{K}(t,s)\mu_s ds\right] = \int_t^T \mathcal{K}(t,s)\bar{\mu}_s ds, \quad \text{for all } 0 \le t \le T. \tag{9.2}$$

Using (3.5), (9.2) and the tower property we get

$$\begin{aligned}\mathbb{E}[r_t^0(\nu^0)] &= \frac{1}{2\lambda_1}\mathbb{E}\left[\mathbb{E}_t\left[\int_t^T \mathcal{K}(t,s)(\mu_s - \kappa_0\nu_s^0)ds\right]\right] \\ &= \frac{1}{2\lambda_1}\int_0^T \mathcal{K}(t,s)(\bar{\mu}_s - \kappa_0\nu_s^0)\mathbb{1}_{\{t\le s\}}ds.\end{aligned} \tag{9.3}$$

Using the expression for K_1^* from (7.4) in (9.3) we arrive at (9.1).

From Lemma 7.1 it follows that the operators K_1 and K_1^* are in $B(L^2([0,T]))$. By assumption $\nu^0, \bar{\mu} \in L^2([0,T])$, then from (9.1) it follows that $(\mathbb{E}[r_t^0(\nu^0)])_{t\in[0,T]}$ is in $L^2([0,T])$. □

Lemma 9.2. *Let $\nu^{1,*}$ be defined as in (3.6). Then for any $\nu^0 \in \mathcal{A}_M^{q_0}$ the following holds*

$$\mathbb{E}[\nu_t^{1,*}(\nu^0)] = \frac{\kappa_0}{2\lambda_1}[(\mathsf{K}_1^*\nu^0)(t) + r_t^1(\mathsf{G}\nu^0)(t)] - \frac{1}{2\lambda_1}[(\mathsf{K}_1^*\bar{\mu})(t) + r_t^1(\mathsf{G}\bar{\mu})(t)],$$

for all $0 \le t \le T$.

Proof. Lemmas 9.1 and 6.7 prove that $\mathbb{E}\,[r_\cdot^0(\nu^0)] \in L^2([0,T])$ and $\mathcal{K} \in L^2([0,T]^2)$, respectively. Then from (3.6) and Fubini's theorem it follows that

$$\mathbb{E}[\nu_t^{1,*}(\nu^0)] = -\mathbb{E}\left[r_t^0(\nu^0)\right] - r_t^1 \int_0^t \mathcal{K}(s,t)\mathbb{E}\left[r_s^0(\nu^0)\right] ds.$$

Together with (9.1) we get

$$\begin{aligned}\mathbb{E}[\nu_t^{1,*}(\nu^0)] = &-\frac{1}{2\lambda_1}(\mathsf{K}_1^*\bar{\mu})(t) + \frac{\kappa_0}{2\lambda_1}(\mathsf{K}_1^*\nu^0)(t) \\ &- r_t^1 \int_0^t \mathcal{K}(s,t)\left(\frac{1}{2\lambda_1}(\mathsf{K}_1^*\bar{\mu})(s) - \frac{\kappa_0}{2\lambda_1}(\mathsf{K}_1^*\nu^0)(s)\right) ds.\end{aligned} \tag{9.4}$$

By using (7.2), (7.3) and (7.5) in (9.4) we get the result. □

The following lemma simply follows from (2.2) and integration by parts hence we omit the proof.

Lemma 9.3. *Let M be a martingale as in (2.1). Then for any $\nu^0 \in \mathcal{A}_M^{q_0}$, we have*

$$\mathbb{E}\left[\int_0^T M_t \nu_t^0 dt\right] = M_0 q_0.$$

In the following proposition we derive an operator differential equation which is satisfied by the operator G in (3.11).

Proposition 9.4. *For any $\psi \in L^2([0,T])$ the operator G satisfies the following differential equation,*

$$\frac{d}{dt}(\mathsf{G}\psi)(t) = r_t^1(\mathsf{G}\psi)(t) + (\mathsf{K}_1^*\psi)(t), \quad (\mathsf{G}\psi)(0) = 0, \quad 0 \le t \le T. \tag{9.5}$$

Moreover, $(\mathsf{G}\psi)(t)$ is continuously differentiable on $[0,T]$.

Proof. Let $\psi \in L^2([0,T])$. From Proposition 8.1 it follows that $\mathcal{G}$ is jointly continuous on $[0,T]^2$ hence by (3.11) we get that $(\mathsf{G}\psi)(t)$ is continuous on $[0,T]$.

Note that by (3.3) we have

$$\frac{d\xi^{\pm}}{dt} = \pm r_t^1 \xi_t^{\pm}, \quad \text{for all } 0 \le t \le T. \tag{9.6}$$

Since by Proposition 6.5 r^1 is continuous over $[0,T]$, it follows from (9.6) that $\xi^{\pm}$ are continuously differentiable on $[0,T]$. From (8.2), (8.4) and (9.6) we get that

$$\begin{aligned}
\frac{d}{dt}(\mathsf{G}\psi)(t) &= \frac{d}{dt}\left(\xi_t^+ \int_0^t \xi_s^- (\mathsf{K}_1^*\psi)(s)ds\right) \\
&= r_t^1 \left(\xi_t^+ \int_0^t \xi_s^- (\mathsf{K}_1^*\psi)(s)ds\right) + (\mathsf{K}_1^*\psi)(t) \\
&= r_t^1 (\mathsf{G}\psi)(t) + (\mathsf{K}_1^*\psi)(t).
\end{aligned}$$

Since by (7.5) that the operator G can be represented in terms of K_1 and K_1^* and Lemma 6.7 shows that $\mathcal{K}$ is jointly continuous on $[0,T]^2$, it follows from (7.3) and (7.4) that $(\mathsf{G}\psi)(t)$ and $(\mathsf{K}_1^*\psi)(t)$ are continuous on $[0,T]$. Since we have show that r_t^1 is also continuous, it follows that $\frac{d}{dt}(\mathsf{G}\psi)$ is continuous over $[0,T]$. Finally, note that

$$(\mathsf{K}_1\psi)(0) = \int_0^T \mathcal{K}(s,0)\mathbb{1}_{\{s\le 0\}}\psi(s)ds = 0.$$

From (7.5) we have $(\mathsf{G}\psi)(t) = (\mathsf{K}_1\mathsf{K}_1^*\psi)(t)$. This proves that $(\mathsf{G}\psi)(0) = 0$ and completes the proof. □

Now we are ready to prove Lemma 7.5.

Proof of Lemma 7.5. Let $\nu^0 \in \mathcal{A}_M^{q_0}$. Recall that minor agent's strategy is assumed to be $\nu^{1,*}$ in (3.6). We define

$$Z_t^{\nu} = Y_t^{\nu} - \int_0^t \mu_s ds, \quad 0 \le t \le T. \tag{9.7}$$

Note that from (2.7) it follows that $Z_0^{\nu} = 0$.

Using (2.9) and (2.8) we get

$$\mathbb{E}\left[X_T^{0,\nu^0}\right] = x_0 + \mathbb{E}\left[\int_0^T (P_t^{\nu} - \lambda_0 \nu_t^0)\nu_t^0 dt\right].$$

Together with (2.6), (2.3) and (9.7) we arrive at

$$\mathbb{E}\left[X_T^{0,\nu^0}\right] = x_0 + \mathbb{E}\left[\int_0^T M_t \nu_t^0 dt\right] + \mathbb{E}\left[\int_0^T Z_t^\nu dQ_t^{0,\nu^0}\right] - \int_0^T \lambda_0 (\nu_t^0)^2 dt. \tag{9.8}$$

Recall that $Z_0^\nu = 0$ and $Q_T^{0,\nu^0} = 0$. Using integration by parts, (2.7), (9.7) and Fubini's theorem we obtain

$$\begin{aligned}\mathbb{E}\left[\int_0^T Z_t^\nu dQ_t^{0,\nu^0}\right] &= -\mathbb{E}\left[\int_0^T Q_t^{0,\nu^0} dZ_t^\nu\right] \\ &= -\int_0^T Q_t^{0,\nu}\left(\kappa_0 \nu_t^0 + \kappa_1 \mathbb{E}\left[\nu_t^{1,*}(\nu^0)\right] - \bar{\mu}_t\right) dt.\end{aligned} \tag{9.9}$$

Moreover, it follows from (2.3) that

$$\int_0^T Q_t^{0,\nu^0} \nu_t^0 dt = \frac{q_0^2}{2}. \tag{9.10}$$

By substituting (9.3), (9.9) and (9.10) into (9.8) we get

$$\mathbb{E}\left[X_T^{0,\nu^0}\right] = x_0 + M_0 q_0 - \kappa_0 \frac{q_0^2}{2} - \int_0^T Q_t^{0,\nu^0}(\kappa_1 \mathbb{E}[\nu_t^{1,*}(\nu^0)] - \bar{\mu}_t) dt - \int_0^T \lambda_0 (\nu_t^0)^2 dt. \tag{9.11}$$

Notice that from Proposition 9.4 and Lemma 9.2 we have

$$\mathbb{E}[\nu_t^{1,*}(\nu^0)] = \frac{\kappa_0}{2\lambda_1}\frac{d}{dt}(\mathsf{G}\nu^0)(t) - \frac{1}{2\lambda_1}\frac{d}{dt}(\mathsf{G}\bar{\mu})(t), \quad \text{for all } 0 \le t \le T. \tag{9.12}$$

Plugging in (9.12) into (9.11) gives

$$\begin{aligned}\mathbb{E}\left[X_T^{0,\nu^0}\right] = x_0 + M_0 q_0 - \kappa_0 \frac{q_0^2}{2} - \frac{\kappa_1 \kappa_0}{2\lambda_1} \int_0^T Q_t^{0,\nu^0} \frac{d}{dt}(\mathsf{G}\nu^0)(t) dt \\ + \int_0^T Q_t^{0,\nu^0}\left(\frac{\kappa_1}{2\lambda_1}\frac{d}{dt}(\mathsf{G}\bar{\mu})(t) + \bar{\mu}_t\right) dt - \lambda_0 \int_0^T (\nu_t^0)^2 dt.\end{aligned} \tag{9.13}$$

Since by (3.7) and (3.8), $\bar{\mu} \in L^2([0,T])$ and by (2.2) also $\nu^0 \in L^2([0,T])$, we get that from Proposition 9.4 that $(\mathsf{G}\bar{\mu})(0) = (\mathsf{G}\nu^0)(0) = 0$. Then by additional integration by parts and recalling that $Q_T^{0,\nu^0} = 0$ we get

$$\int_0^T Q_t^{0,\nu^0} \frac{d}{dt}(\mathsf{G}\nu^0)(t) dt = \int_0^T \nu_t^0 (\mathsf{G}\nu^0)(t) dt, \tag{9.14}$$

as well as

$$\int_0^T Q_t^{0,\nu^0} \frac{d}{dt}(\mathsf{G}\bar{\mu})(t)dt = \int_0^T \nu_t^0(\mathsf{G}\bar{\mu})(t)dt. \tag{9.15}$$

Hence, by plugging in (9.14) and (9.15) into (9.13) we obtain

$$\begin{aligned}\mathbb{E}\left[X_T^{0,\nu^0}\right] &= x_0 + M_0 q_0 - \kappa_0 \frac{q_0^2}{2} - \frac{\kappa_1\kappa_0}{2\lambda_1}\int_0^T \nu_t^0(\mathsf{G}\nu^0)(t)dt \\ &\quad + \int_0^T \left(\frac{\kappa_1}{2\lambda_1}\nu_t^0(\mathsf{G}\bar{\mu})(t) + Q_t^{0,\nu^0}\bar{\mu}_t\right)dt - \lambda_0\int_0^T (\nu_t^0)^2 dt,\end{aligned} \tag{9.16}$$

which, together with (2.10), proves the result. □

10 Convergence of the numerical scheme

we now discuss the convergence of the numerical scheme proposed in Section 5 under Assumption 3.1. Our first goal is to prove Proposition 5.2, but before getting to the proof we introduce an auxiliary lemma.

Lemma 10.1. *Let $(\mathcal{G}_n)_{n\geq 1}$ be defined as in (5.4). Then, $\mathcal{G}_n$ is in $L^2([0,T]^2)$ for any $n \geq 1$.*

Proof. Recall that G was defined in (3.11). From Proposition 7.4 it follows that G is an operator in $B(L^2([0,T]))$. Recall $(a_i)_{n\geq 1}$ is a complete orthonormal basis in $L^2([0,T])$, hence from (5.3) we get that $(b_i)_{n\geq 1}$ are in $L^2([0,T])$. We therefore get from (5.4) that

$$\begin{aligned}\int_0^T\int_0^T \mathcal{G}_n(t,s)^2 dsdt &\leq n\sum_{i=1}^n \left(\int_0^T a_i^2(t)dt\right)\left(\int_0^T b_i^2(s)ds\right) \\ &< \infty,\end{aligned} \tag{10.1}$$

and the result follows. □

Proof of Proposition 5.2. The result follows directly from Lemma 10.1 and (5.5). □

Before we prove Proposition 5.4 we introduce the following theorem from Atkinson (1997).

Theorem 10.2 ((Atkinson, 1997, Theorem 2.1.1)). *Let* G *be in* $B(L^2([0,T]))$ *and let* $\lambda \in \mathbb{R}$. *Assume that* $\mathsf{I} - \lambda\mathsf{G}$ *is invertible on* $L^2([0,T])$. *Furthermore, assume that* $(\mathsf{G}_n)_{n\geq 1}$ *is a sequence of operators in* $B(L^2([0,T]))$ *with*

$$\lim_{n\to\infty} ||\mathsf{G} - \mathsf{G}_n|| = 0.$$

Then the following holds:

(i) *there exists an* $N \geq 1$ *such that for all* $n \geq N$ *the operators* $(\mathsf{I} - \lambda\mathsf{G}_n)^{-1}$ *exists and are in* $B(L^2([0,T]))$;

(iii) $(\mathsf{I} - \lambda\mathsf{G}_n)^{-1}$ *converges to* $(\mathsf{I} - \lambda\mathsf{G})^{-1}$ *in* $B(L^2([0,T]))$, *that is*

$$\lim_{n\to\infty} \left|\left|(\mathsf{I} - \lambda\mathsf{G}_n)^{-1} - (\mathsf{I} - \lambda\mathsf{G})^{-1}\right|\right| = 0;$$

(iii) $\left|\left|(\mathsf{I} - \lambda\mathsf{G}_n)^{-1}\right|\right|$ *converges to* $\left|\left|(\mathsf{I} - \lambda\mathsf{G})^{-1}\right|\right|$, *that is*

$$\lim_{n\to\infty} \left|\left|(\mathsf{I} - \lambda\mathsf{G}_n)^{-1}\right|\right| = \left|\left|(\mathsf{I} - \lambda\mathsf{G})^{-1}\right|\right|.$$

We define

$$\mathsf{R}_n := \left(\mathsf{I} + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}_n\right)^{-1}, \quad n \geq 1. \tag{10.2}$$

Proof of Proposition 5.4. Recall that $(\mathsf{G}_n)_{n\geq 1}$ was defined in (5.6) and that G was defined as in (3.11). From Propositions 5.2, 7.4 and 7.9 it follows that the assumptions of Theorem 10.2 hold, hence there exists an $N \geq 1$ such that for all $n \geq N$ the operators $\mathsf{I} + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}_n$ are invertible. Since for any $n \geq N$ the corresponding kernels $\mathcal{G}_n$ in (5.4) are degenerate, hence it follows from Theorem 2.1.2 of Atkinson (1997) that the matrices $I_n + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathbb{G}_n$ are invertible (recall (5.10) for the definition of $\mathbb{G}_n$).

Let $g, \psi \in L^2([0,T])$ and define

$$\gamma_i = -\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\sum_{j=1}^{n}\left(I_n + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathbb{G}_n\right)^{-1}_{ij}\langle\psi, b_j\rangle_{L^2}, \quad i = 1,\ldots,n,$$

for any $n \geq N$. As shown in Chapter 3 of Porter and Stirling (1990) (see equations (3.5) – (3.7) therein) the unique solution to

$$\left(\mathsf{I} + \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\mathsf{G}_n\right) g = \psi,$$

is given by

$$g(t) = \psi(t) + \sum_{i=1}^{n} \gamma_i a_{i,n}(t), \quad \text{for all } 0 \leq t \leq T, \quad n \geq N,$$

and (5.11) follows. □

Before proving Proposition 5.5, we need to present two intermediate lemmas.

Lemma 10.3. *Let* R *be defined as in* (3.14) *and let* $(\mathsf{R}_n)_{n\geq 1}$ *be defined as in* (10.2). *Then the following holds:*

(i)

$$\liminf_n \langle \mathsf{R}_n 1\,,\, 1\rangle_{L^2} > 0,$$

(ii)

$$\lim_{n\to\infty} \frac{1}{\langle \mathsf{R}_n 1\,,\, 1\rangle_{L^2}} = \frac{1}{\langle \mathsf{R} 1\,,\, 1\rangle_{L^2}}.$$

Proof. (i) We have shown in the proof of Proposition 5.4 that the assumptions of Theorem 10.2 hold, hence there exists an $N \geq 1$ such that for all $n \geq N$ the operators R_n exist. From the Cauchy-Schwarz inequality and since $\|1\|_{L^2} = T$ we get

$$\begin{aligned} |\langle \mathsf{R}_n 1\,,\, 1\rangle_{L^2} - \langle \mathsf{R} 1\,,\, 1\rangle_{L^2}| &\leq \|\mathsf{R}_n - \mathsf{R}\| \|1\|_{L^2}^2 \\ &\leq \|\mathsf{R}_n - \mathsf{R}\| T^2. \end{aligned} \tag{10.3}$$

From Proposition 7.9(ii) it follows that the operator R is bounded from below, therefore, by Definition 7.3, there exists $\varepsilon > 0$ such that

$$\langle \mathsf{R} 1\,,\, 1\rangle_{L^2} > \varepsilon. \tag{10.4}$$

From Theorem 10.2(iii) we get that there exists an $N_1 \geq N$ such that for all $n \geq N_1$ we have that

$$||\mathsf{R}_n - \mathsf{R}|| < \varepsilon T^{-2}/2. \tag{10.5}$$

From (10.3)–(10.5) we get (i).

(ii) follows directly from (10.3), (10.4) and (i). □

Next, we prove the convergence of the sequence of constants $(\eta_n)_{n\geq 1}$ from (5.12).

Lemma 10.4. *Let η and η_n be defined as in (3.16) and (5.12), respectively. Then, there exists $N \geq 1$ such that for all $n \geq N$ the constants η_n are well-defined. Moreover,*

$$\lim_{n\to\infty} \eta_n = \eta. \tag{10.6}$$

Proof of Lemma 10.4. From Lemma 10.3(i), (3.16) and (10.2) if follows that for all n sufficiently large η_n is well defined. The same claim holds for η by (5.12), (10.4) and Proposition 7.9(i). From Cauchy-Schwarz inequality we get

$$|\langle \mathsf{R}_n \mathsf{S}\bar{\mu}, 1\rangle_{L^2} - \langle \mathsf{R}\mathsf{S}\bar{\mu}, 1\rangle_{L^2}| \leq \|\mathsf{R} - \mathsf{R}_n\| \|\mathsf{S}\bar{\mu}\|_{L^2} \|1\|_{L^2},$$

and together with Theorem 10.2(iii) and Lemma 7.10 it follows that

$$\lim_{n\to\infty} |\langle \mathsf{R}_n \mathsf{S}\bar{\mu}, 1\rangle_{L^2} - \langle \mathsf{R}\mathsf{S}\bar{\mu}, 1\rangle_{L^2}| = 0. \tag{10.7}$$

From (3.16) and (5.12) we have

$$|\eta_n - \eta| = \left| \frac{\langle \mathsf{R}_n \mathsf{S}\bar{\mu}, 1\rangle_{L^2}}{\langle \mathsf{R}_n 1, 1\rangle_{L^2}} - \frac{\langle \mathsf{R}\mathsf{S}\bar{\mu}, 1\rangle_{L^2}}{\langle \mathsf{R}1, 1\rangle_{L^2}} \right|, \tag{10.8}$$

hence (10.6) follows from Lemma 10.3 and (10.7). □

We are now ready to prove Proposition 5.5.

Proof of Proposition 5.5. From (3.15), (5.9) and (10.2) we have

$$\nu^{0,*} - \nu^{0,(n)} = (\mathsf{R} - \mathsf{R}_n)(\mathsf{S}\bar{\mu}) + \frac{1}{2\lambda_0}(\eta \mathsf{R}1 - \eta_n \mathsf{R}_n 1).$$

It follows that

$$\begin{aligned} ||\nu^{0,*} - \nu^{0,(n)}||_{L^2} \leq ||(\mathsf{R} - \mathsf{R}_n)(\mathsf{S}\bar{\mu})||_{L^2} + \frac{\eta}{2\lambda_0} ||(\mathsf{R} - \mathsf{R}_n)\, 1||_{L^2} \\ + \frac{1}{2\lambda_0} |\eta_n - \eta| \|\mathsf{R}_n 1\|_{L^2}. \end{aligned}$$

Hence by Lemma 10.4, Theorem 10.2(iii) and following similar lines as in the proof of (10.7) we get

$$\lim_{n\to\infty} \left|\left|\nu^{0,*} - \nu^{0,(n)}\right|\right|_{L^2} = 0.$$

□

Finally, we we prove Theorem 5.6.

Proof of Theorem 5.6. Throughout the proof we consider n's large enough such that the results of Lemma 10.4 and Proposition 5.5 hold, even if it is not stated explicitly.

(i) From (5.14) and (7.15) we get

$$\begin{aligned}\nu_t^{0,*} - \hat{\nu}_t^{0,(n)} &= -\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\left(\mathsf{G}\left(\nu^{0,*} - \nu^{0,(n)}\right)\right)(t) + \frac{\eta - \eta_n}{2\lambda_0} \\ &= -\frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\int_0^T \mathcal{G}(t,s)\left(\nu_s^{0,*} - \nu_s^{0,(n)}\right) ds + \frac{\eta - \eta_n}{2\lambda_0},\end{aligned}$$

where we used (3.11) in the second equality.

From Cauchy-Schwarz inequality we get for all $0 \le t \le T$ and n sufficiently large,

$$\left|\nu_t^{0,*} - \hat{\nu}_t^{0,(n)}\right| \le \frac{\kappa_1\kappa_0}{2\lambda_0\lambda_1}\left(\int_0^T |\mathcal{G}(t,s)|^2 ds\right)^{1/2} \left|\left|\nu^{0,*} - \nu^{0,(n)}\right|\right|_{L^2} + \frac{|\eta - \eta_n|}{2\lambda_0}.$$

Hence, from Proposition 8.1, Lemma 10.4 and Proposition 5.5, we get (i).

(ii) From Proposition 6.5 it follows that r^1 is bounded over $[0,T]$ and from Lemma 6.7 we get that the kernel $\mathcal{K}$ is bounded over $[0,T]^2$. Together with (3.6) we get that there exists a constant $C > 0$ such that for all $0 \le t \le T$ and n sufficiently large we have

$$\begin{aligned}\left|\nu_t^{1,*}\left(\nu^{0,*}\right) - \nu_t^{1,*}\left(\hat{\nu}^{0,(n)}\right)\right| &\le \left|r_t^0\left(\hat{\nu}^{0,(n)}\right) - r_t^0\left(\nu^{0,*}\right)\right| \\ &\quad + C\left|\int_0^t \left(r_s^0\left(\hat{\nu}^{0,(n)}\right) - r_s^0\left(\nu^{0,*}\right)\right) ds\right|.\end{aligned} \tag{10.9}$$

By plugging in (3.5) into (10.9) and observing that the stochastic part in the right hand side of (3.5) cancels, we conclude that

$$\begin{aligned}\left|\nu_t^{1,*}\left(\nu^{0,*}\right) - \nu_t^{1,*}\left(\hat{\nu}^{0,(n)}\right)\right| &\le C_1\int_t^T \left|\hat{\nu}_s^{0,(n)} - \nu_s^{0,*}\right| ds \\ &\quad + C_2\int_0^t\left(\int_s^T \left|\hat{\nu}_r^{0,(n)} - \nu_r^{0,*}\right| dr\right) ds,\end{aligned} \tag{10.10}$$

for some constants $C_1, C_2 > 0$ independent from n and t. Then (ii) follows from (10.10) and (i). □

A An Example of Spectral Decomposition of G

In this section we give an example of the spectral decomposition of G in Lemma 8.3 for the case where $\phi^1 = 0$. We continue to assume that Assumption 3.1 holds.

Lemma A.1. *Let $\psi \in C([0,T])$ and recall that K_1 be defined as in (7.3). Then, K_1 satisfies for $0 \le t \le T$,*

$$\frac{d}{dt}(\mathsf{K}_1^*\psi)(t) = -r_t^1(\mathsf{K}_1^*\psi)(t) - \psi(t), \quad (\mathsf{K}_1^*\psi)(T) = 0.$$

In particular, $(\mathsf{K}_1^*\psi)(t)$ is continuously differentiable on $[0,T]$.

Proof. The proof follows the same lines of Proposition 9.4 hence we just give an outline. We take a derivative of $\mathsf{K}_1^*\psi$ with respect to time using (7.4), (3.3) and (3.4) to get

$$\begin{aligned}\frac{d}{dt}(\mathsf{K}_1^*\psi)(t) &= \frac{d}{dt}\left(\xi_t^- \int_t^T \xi_s^+ \psi(s) ds\right) \\ &= -r_t^1(\mathsf{K}_1^*\psi)(t) - \psi(t), \quad \text{for all } 0 \le t \le T.\end{aligned}$$

Note that from (7.4) it follows that

$$(\mathsf{K}_1^*\psi)(T) = 0.$$

□

Proposition A.2. *Let G be defined as in (3.11) and assume that $\phi^1 \equiv 0$. Let $(z_n)_{n=1}^\infty$ be the increasing sequence of real positive roots of the following equation,*

$$\cot(z) = -\frac{(2\alpha - \kappa_1)}{\lambda_1}\frac{T}{z}. \tag{A.1}$$

Then the eigenvalues $(\zeta_n)_{n=1}^\infty$ and the eigenfunctions $(\psi_n)_{n=1}^\infty$ of G are given by

$$\psi_n(t) = \frac{2}{\sqrt{\zeta_n}} \frac{\sin\left(\frac{t}{\sqrt{\zeta_n}}\right)}{\sqrt{\frac{2T}{\sqrt{\zeta_n}} - \sin\left(\frac{2T}{\sqrt{\zeta_n}}\right)}}, \qquad \zeta_n = \frac{T^2}{z_n^2}. \tag{A.2}$$

Proof. We first show that the eigenvalues $(\zeta_n)_{n=1}^\infty$ and eigenfunctions $(\psi_n)_{n=1}^\infty$ are arising from solutions to an ODE. Then we show that the solutions of the ODE can be determined in terms of the roots to (A.1).

Let ζ be an eigenvalue of G and ψ the corresponding eigenfunction, i.e. ζ and ψ satisfy

$$(\mathsf{G}\psi)(t) = \zeta\psi(t), \quad 0 \le t \le T. \tag{A.3}$$

From Lemma 8.3 it follows that $\zeta > 0$ and $\psi \in L^2([0,T])$. Proposition 9.4 shows that $(\mathsf{G}\psi)(t)$ is continuously differentiable over $[0,T]$, therefore, it follows from (A.3) that $\psi(t)$ is continuously differentiable over $[0,T]$. We take a derivative on both sides of (A.3) to obtain that (ζ,ψ) must satisfy

$$\frac{d}{dt}(\mathsf{G}\psi)(t) = \zeta\psi'(t), \quad \text{for all } 0 \le t \le T. \tag{A.4}$$

Proposition 9.4 shows that $(\mathsf{G}\psi)$ is the solution to (9.5), therefore we can substitute (9.5) in (A.4) to get that ψ must satisfy

$$r_t^1(\mathsf{G}\psi)(t) + (\mathsf{K}_1^*\psi)(t) = \zeta\psi'(t), \quad \text{for all } 0 \le t \le T. \tag{A.5}$$

Proposition 6.5 proves that r^1 is the solution to (6.18). When $\phi^1 \equiv 0$ it can be computed explicitly as follows,

$$r_t^1 = \frac{2\alpha - \kappa_1}{(t-T)(2\alpha-\kappa_1) - 2\lambda_1}, \quad t \in [0,T].$$

Note that under Assumption 3.1, r^1 is continuously differentiable on $[0,T]$.

Since we have proved that $\psi(t)$ is continuous, it follows from Proposition A.1 that $(\mathsf{K}_1^*\psi)(t)$ is continuously differentiable on $[0,T]$. We take a derivative on both sides of (A.5) to get

$$\frac{dr_t^1}{dt}(\mathsf{G}\psi)(t) + r_t^1\frac{d}{dt}(\mathsf{G}\psi)(t) + \frac{d}{dt}(\mathsf{K}_1^*\psi)(t) = \zeta\psi''(t),$$

and then use (9.5) to show that ψ satisfies

$$\frac{dr_t^1}{dt}(\mathsf{G}\psi)(t) + (r_t^1)^2(\mathsf{G}\psi)(t) + r_t^1(\mathsf{K}_1^*\psi)(t) + \frac{d}{dt}(\mathsf{K}_1^*\psi)(t) = \zeta\psi''(t), \quad 0 \le t \le T. \tag{A.6}$$

By applying (6.18), (A.5) and (A.1) to (A.6) we get that ψ must satisfy

$$-\psi(t) = \zeta\psi''(t), \quad 0 \le t \le T. \tag{A.7}$$

Recall that $\zeta > 0$, hence it follows from (A.3) and Proposition 9.4 that ψ satisfies the initial condition $\psi(0) = 0$. The terminal condition $\psi'(T) = -\left(\frac{2\alpha-\kappa_1}{\lambda_1}\right)\psi(T)$ follows

by combining (A.5) with (6.18), (A.3) and $(\mathsf{K}_1^*\psi)(T) = 0$ (see Proposition A.1). It follows that (ζ, ψ) satisfy

$$\psi''(t) = -\frac{1}{\zeta}\psi(t), \quad 0 < t < T, \quad \psi(0) = 0, \quad \psi'(T) = -\left(\frac{2\alpha - \kappa_1}{\lambda_1}\right)\psi(T). \tag{A.8}$$

We show that (A.1) has an infinite number of positive roots. To see this, note that since $2\alpha - \kappa_1 \geq 0$ by Assumption 3.1, then for any $n \geq 1$,

$$\lim_{z \to (n-1)\pi^+} \cot(z) + \frac{2\alpha - \kappa_1}{\lambda_1}\frac{T}{z} = +\infty,$$
$$\lim_{z \to n\pi^-} \cot(z) + \frac{2\alpha - \kappa_1}{\lambda_1}\frac{T}{z} = -\infty.$$

Since $\cot(z) + \frac{2\alpha-\kappa_1}{\lambda_1}\frac{T}{z}$ is continuous over the intervals $((n-1)\pi, n\pi)$ for any $n \geq 1$, then it follows by the intermediate value theorem that (A.1) has a root in the interval $((n-1)\pi, n\pi)$ for any $n \geq 1$.

Next we identify ζ_n as in (A.2). Let z_n be the n^{th} positive root of (A.1) and let ζ_n, ψ_n be defined as in (A.2). First, note that since $z_n > 0$, then

$$2z_n - \sin(2z_n) > 0. \tag{A.9}$$

From (A.2) and (A.9) it follows that

$$\frac{2T}{\sqrt{\zeta_n}} - \sin\left(\frac{2T}{\sqrt{\zeta_n}}\right) > 0,$$

therefore, the function $\psi_n(t)$ is well-defined for all $t \in [0, T]$ and $||\psi_n||_{L^2} = 1$. Using the following identity which arises from (A.8),

$$\cos(z_n) = -\frac{(2\alpha - \kappa_1)}{\lambda_1}\frac{T}{z_n}\sin(z_n),$$

it is easy to verify that ψ_n in (A.2) solves (A.8) with $\zeta = \zeta_n$ for any $n \geq 1$. This completes the proof. □

B Proof of Lemmas 6.3 and 6.4

Proof of Lemma 6.3. Let $\epsilon > 0$ and $\nu^1, \omega \in \mathcal{A}_m$. We note that from (2.5) it follows that

$$Q_t^{1,\nu^1+\epsilon\omega} = Q_t^{1,\nu^1} - \epsilon\int_0^t \omega_s ds, \quad \text{for all } 0 \leq t \leq T. \tag{B.1}$$

We use the alternative representation of H^1 in (6.1) and (B.1) to get

$$\begin{aligned} H^1(\nu^1+\epsilon\omega)-H^1(\nu^1) &= \epsilon\mathbb{E}\bigg[\int_0^T \omega_t(-2\lambda_1\nu_t^1+2\alpha Q_T^{1,\nu^1}-\kappa_1 Q_t^{1,\nu^1})dt \\ &\quad+\int_0^T\bigg(\int_0^t\omega_s ds\bigg)(2\phi_t^1 Q_t^{1,\nu^1}dt+\kappa_0\nu_t^0 dt+\kappa_1\nu_t^1 dt-dA_t)\bigg] \\ &\quad+\epsilon^2\mathbb{E}\bigg[-\lambda_1\int_0^T\omega_s^2 ds-\alpha\bigg(\int_0^T\omega_s ds\bigg)^2 \\ &\quad-\int_0^T\phi_t^1\bigg(\int_0^t\omega_s ds\bigg)^2 dt+\kappa_1\int_0^T\omega_t\bigg(\int_0^t\omega_s ds\bigg)dt\bigg]. \end{aligned} \tag{B.2}$$

From (6.12) and (B.2) we get

$$\begin{aligned} \langle\mathcal{D}H^1(\nu^1),\omega\rangle &= \mathbb{E}\bigg[\int_0^T \omega_t(-2\lambda_1\nu_t^1+2\alpha Q_T^{1,\nu^1}-\kappa_1 Q_t^{1,\nu^1})dt \\ &\quad+\int_0^T\bigg(\int_0^t\omega_s ds\bigg)(2\phi_t^1 Q_t^{1,\nu^1}dt+\kappa_0\nu_t^0 dt+\kappa_1\nu_t^1 dt-dA_t)\bigg]. \end{aligned} \tag{B.3}$$

Since $\nu^1,\omega\in\mathcal{A}_m$, $\nu^0\in\mathcal{A}_M^{q_0}$ and $\mathbb{E}[(\int_0^T|dA_t|)^2]<\infty$, then we use Fubini's theorem in (B.3) to get

$$\begin{aligned} \langle\mathcal{D}H^1(\nu^1),\omega\rangle &= \mathbb{E}\bigg[\int_0^T\omega_t\Big(-2\lambda_1\nu_t^1+2\alpha Q_T^{1,\nu^1}-\kappa_1 Q_t^{1,\nu^1}+A_t \\ &\quad+\int_t^T\Big(2\phi_s^1 Q_s^{1,\nu^1}+\kappa_0\nu_s^0+\kappa_1\nu_s^1\Big)ds-A_T\Big)dt\bigg], \end{aligned}$$

which concludes the result. □

Proof of Lemma 6.4. In Lemma 6.2 we have shown that under Assumption 3.1, the functional $H^1(\nu^1)$ is strictly concave of any $\nu^1\in\mathcal{A}_m$. Therefore, we may apply Proposition 2.1 of (Ekeland and Témam, 1999, Chapter II) to obtain that

$$\langle\mathcal{D}H^1(\nu^{1,*}),\omega\rangle=0 \quad \text{for all } \omega\in\mathcal{A}_m \iff \nu^{1,*}=\arg\sup_{\nu\in\mathcal{A}_m} H^1(\nu). \tag{B.4}$$

The strict concavity of H^1 guarantees that the optimiser $\nu^{1,*}$ is unique.

Necessity: We assume that

$$\nu^{1,*} = \arg \sup_{\nu \in \mathcal{A}_m} H^1(\nu).$$

Then, (B.4) and (6.13) imply that for all $\omega \in \mathcal{A}_m$ it holds

$$\langle \mathcal{D}H^1(\nu^{1,*}), \omega \rangle = \mathbb{E}\Bigg[\int_0^T \omega_t \Big(-2\lambda_1 \nu_t^{1,*} + 2\alpha Q_T^{1,\nu^{1,*}} - \kappa_1 Q_t^{1,\nu^{1,*}} + A_t + \int_t^T \Big(2\phi_s^1 Q_s^{1,\nu^{1,*}} + \kappa_0 \nu_s^0 + \kappa_1 \nu_s^{1,*} \Big) \, ds - A_T \Big) dt \Bigg] = 0.$$

By applying the optional projection theorem we get

$$\mathbb{E}\Bigg[\int_0^T \omega_t \Bigg(-2\lambda_1 \nu_t^{1,*} + \mathbb{E}_t \Big[2\alpha Q_T^{1,\nu^{1,*}} - A_T \Big] - \kappa_1 Q_t^{1,\nu^{1,*}} + A_t + \mathbb{E}_t \Bigg[\int_t^T \Big(2\phi_s^1 Q_s^{1,\nu^{1,*}} + \kappa_0 \nu_s^0 + \kappa_1 \nu_s^{1,*} \Big) \, ds \Bigg] \Bigg) dt \Bigg] = 0, \quad \text{(B.5)}$$

As (B.5) holds for all $\omega \in \mathcal{A}_m$ we deduce the following first order condition holds

$$-2\lambda_1 \nu_t^{1,*} + \mathbb{E}_t \Big[2\alpha Q_T^{1,\nu^{1,*}} - A_T \Big] - \kappa_1 Q_t^{1,\nu^{1,*}} + A_t + \mathbb{E}_t \Bigg[\int_t^T \Big(2\phi_s^1 Q_s^{1,\nu^{1,*}} + \kappa_0 \nu_s^0 + \kappa_1 \nu_s^{1,*} \Big) \, ds \Bigg] = 0, \quad \text{(B.6)}$$

$d\mathbb{P} \otimes dt$-a.e. on $\Omega \times [0,T]$.

We define the following martingales

$$\begin{aligned} \mathcal{M}_t &:= \mathbb{E}_t \Bigg[\int_0^T \Big(2\phi_s^1 Q_s^{1,\nu^{1,*}} + \kappa_0 \nu_s^0 + \kappa_1 \nu_s^{1,*} \Big) \, ds \Bigg], \\ \mathcal{N}_t &:= \mathbb{E}_t \Big[2\alpha Q_T^{1,\nu^{1,*}} - A_T \Big]. \end{aligned} \quad \text{(B.7)}$$

Note that $\mathcal{M}$ and $\mathcal{N}$ are square-integrable since $\mathbb{E}[(\int_0^T |dA_t|)^2] < \infty$, $\nu^{1,*}, \omega \in \mathcal{A}_m$ and $\nu^0 \in \mathcal{A}_M^{q_0}$.

We plug $\mathcal{M}$ and $\mathcal{N}$ in (B.6) to get

$$-2\lambda_1 \nu_t^{1,*} + \mathcal{N}_t - \kappa_1 Q_t^{1,\nu^{1,*}} + A_t + \mathcal{M}_t - \int_0^t \Big(2\phi_s^1 Q_s^{1,\nu^{1,*}} + \kappa_0 \nu_s^0 + \kappa_1 \nu_s^{1,*} \Big) \, ds = 0, \quad \text{(B.8)}$$

From (2.5) and (B.8) it follows that $\nu^{1,*}$ solves the following BSDE

$$\begin{cases} d\nu_t^{1,*} & = \frac{1}{2\lambda_1} d\mathcal{N}_t + \frac{1}{2\lambda_1} d\mathcal{M}_t - \frac{1}{\lambda_1}\phi_t^1 Q_t^{1,\nu^{1,*}} dt - \frac{\kappa_0}{2\lambda_1}\nu_t^0 dt + \frac{1}{2\lambda_1} dA_t, \\ \nu_T^{1,*} & = \frac{2\alpha-\kappa_1}{2\lambda_1} Q_T^{1,\nu^{1,*}}, \end{cases}$$

this gives (6.14).

Sufficiency: Assume that $(Q^{1,\nu^{1,*}}, \nu^{1,*})$ solves (6.14) $d\mathbb{P}\otimes dt$-a.e. and that $\nu^{1,*} \in \mathcal{A}_m$. We will show that $\langle \mathcal{D}H^1(\nu^{1,*}), \omega\rangle$ vanishes for all $\omega \in \mathcal{A}_m$ which, once combined with (B.4), implies that $\nu^{1,*}$ is the solution to the minor agent's problem. Since $(Q^{1,\nu^{1,*}}, \nu^{1,*})$ solves (6.14) we get that

$$\begin{aligned} 2\lambda_1\nu_t^{1,*} &= \mathbb{E}_t\left[(2\alpha-\kappa_1) Q_T^{1,\nu^{1,*}}\right] - \mathbb{E}_t\left[\int_t^T dA_s\right] + \mathbb{E}_t\left[\int_t^T \left(\kappa_0\nu_s^0 + 2\phi_t^1 Q_s^{1,\nu^{1,*}}\right) ds\right] \\ &= \mathbb{E}_t\left[2\alpha Q_T^{1,\nu^{1,*}} - A_T\right] - \kappa_1 Q_t^{1,\nu^{1,*}} + A_t \\ &\quad + \mathbb{E}_t\left[\int_t^T \left(\kappa_0\nu_s^0 + 2\phi_t^1 Q_s^{1,\nu^{1,*}} + \kappa_1\nu_s^{1,*}\right) ds\right], \quad d\mathbb{P}\otimes dt - \text{a.e.}, \end{aligned} \tag{B.9}$$

where we used (2.5) in the second equality. Hence, $\nu_t^{1,*}$ satisfies (B.6), therefore the left-hand side of (B.4) hold. □

C Proof of Proposition 6.5 and Lemma 6.7

Before we prove Lemma 6.7 we introduce the following lemma.

Lemma C.1. *Under Assumption 3.1, the Riccati equation (6.18) has a unique continuous solution.*

Proof. Let $\hat{r}^1$ be the solution to the following equation

$$\begin{cases} \partial_t \hat{r}_t^1 & = -\frac{1}{\lambda_1}\phi_t^1 + (\hat{r}_t^1)^2, \quad 0 \le t \le T, \\ \hat{r}_T^1 & = \frac{2\alpha-\kappa_1}{2\lambda_1}. \end{cases} \tag{C.1}$$

Since, $\hat{r}_T^1 \ge 0$ and ϕ^1 is a piecewise continuous, locally bounded non-negative function over $[0,T]$, then by Theorem 2.1 of Wonham (1968) there exists a unique solution $\hat{r}^1$, which is absolutely continuous on $[0,T]$ (see also Theorem 3.5 in Freiling (2002)

for a more recent reference). As stated by Wonham (1968), the function $\hat{r}^1$ satisfies (C.1) only dt almost everywhere. Note that by taking $r_t^1 = -\hat{r}_t^1$, it follows that r^1 is an absolutely continuous solution to (6.18). Uniqueness of the solution to (6.18) then follows by the uniqueness for (C.1). □

Proof of Lemma 6.7. Lemma C.1 proves that r^1 is continuous on $[0,T]$. Then, from (3.3) it follows that the functions $\xi_t^{\pm}$ are continuous. Therefore, from (3.4) we get that $\mathcal{K}$ is jointly continuous on $[0,T]^2$ hence it $\mathcal{K}$ is bounded on $[0,T]^2$ and

$$\int_0^T \int_0^T |\mathcal{K}(t,s)|^2 \, ds \, dt < \infty, \quad \text{so} \qquad \mathcal{K} \in L^2([0,T]^2).$$

□

Now we are ready to prove Proposition 6.5.

Proof of Proposition 6.5. In Lemma C.1 we have established that (6.18) has a unique continuous solution r^1. We prove the rest of the claims in the following two steps.

Step 1. We show that r_t^0 given by (3.5) solves the BSDE (6.19); Note that since r^1 is continuous, the function ξ^+ in (3.3) is the unique solution of the ODE

$$\frac{d\xi_t^+}{dt} = r_t^1 \xi_t^+, \quad \xi_0^+ = 1. \tag{C.2}$$

Since r_t^1 is continuous on $[0,T]$, it holds that

$$\int_0^T (\xi_t^+)^2 dt < \infty. \tag{C.3}$$

Since ξ_t^+ satisfies (C.3), then the process

$$\xi_t^+ r_t^0 := \frac{1}{2\lambda_1} \mathbb{E}_t \left[\int_t^T \xi_s^+ (dA_s - \kappa_0 \nu_s^0 ds) \right], \quad 0 \le t \le T, \tag{C.4}$$

is the unique strong solution to the following linear BSDE

$$\begin{cases} d(\xi_t^+ r_t^0) & = \frac{\xi_t^+}{2\lambda_1}(\kappa_0 \nu_t^0 dt - dA_t) - \frac{1}{2\lambda_1}\xi_t^+ d\mathcal{M}_t - \frac{1}{2\lambda_1}\xi_t^+ d\mathcal{N}_t, \\ \xi_T^+ r_T^0 & = 0. \end{cases} \tag{C.5}$$

We multiply both sides in (C.4) by ξ_t^- from (3.3) and use the identity $\xi_t^- \xi_t^+ = 1$. By doing so, we can obtain an expression for r_t^0, that is

$$r_t^0 = \frac{1}{2\lambda_1} \mathbb{E}_t \left[\int_t^T \xi_t^- \xi_s^+ (dA_s - \kappa_0 \nu_s^0 ds) \right]. \tag{C.6}$$

We now show that r^0 from (C.6) is the solution to (6.19). From (C.5) and Itô's product rule we get

$$d\xi_t^+ r_t^0 + \xi_t^+ dr_t^0 = \frac{\xi_t^+}{2\lambda_1}(\kappa_0 \nu_t^0 dt - dA_t) - \frac{1}{2\lambda_1}\xi_t^+ d\mathcal{M}_t - \frac{1}{2\lambda_1}\xi_t^+ d\mathcal{N}_t. \tag{C.7}$$

Next, we use (C.2) and (C.7) to get

$$\xi_t^+ \left(r_t^1 r_t^0 dt + dr_t^0 - \frac{1}{2\lambda_1}(\kappa_0 \nu_t^0 dt - dA_t) + \frac{1}{2\lambda_1} d\mathcal{M}_t + \frac{1}{2\lambda_1} d\mathcal{N}_t \right) = 0. \tag{C.8}$$

Since $\xi_t^+ > 0$ for all $t \in [0, T]$, we have

$$r_t^1 r_t^0 dt + dr_t^0 - \frac{1}{2\lambda_1}(\kappa_0 \nu_t^0 dt - dA_t) + \frac{1}{2\lambda_1} d\mathcal{M}_t + \frac{1}{2\lambda_1} d\mathcal{N}_t = 0.$$

with terminal condition $r_T^0 = 0$ which follows from (C.6). By comparing this with (6.19), it follows that r_t^0 is the solution to the BSDE (6.19). Recall the definition of $\mathcal{K}$ in (3.4). We substitute the expression for $\mathcal{K}(t, s)$ from (3.4) into (C.6) from which it follows that r_t^0 given by (3.5) solves (6.19).

Step 2. We show (6.20). From (3.4) it follows that $\mathcal{K}(t, s) > 0$ for all $t, s \in [0, T]^2$. Moreover by Lemma 6.7, $\mathcal{K}$ is bounded on $[0, T]^2$. Since A is of bounded variation and $\mathcal{K}$ is bounded, then by using the conditional Jensen's inequality and the tower property we get

$$\begin{aligned} \sup_{t\in[0,T]} \mathbb{E}\left[\left(\mathbb{E}_t \left[\int_t^T \mathcal{K}(t, s) dA_s \right] \right)^2 \right] &\leq C\mathbb{E}\left[\left(\int_0^T |dA_s| \right)^2 \right] \\ &< \infty, \end{aligned} \tag{C.9}$$

where we used (2.1) in the last inequality. Similarly, we can obtain that

$$\begin{aligned} \sup_{t\in[0,T]} \mathbb{E}\left[\left(\mathbb{E}_t \left[\int_t^T \mathcal{K}(t, s) \nu_s^0 ds \right] \right)^2 \right] &\leq C\mathbb{E}\left[\int_0^T (\nu_s^0)^2 ds \right] \\ &< \infty, \end{aligned} \tag{C.10}$$

where we used the fact that $\nu^0 \in \mathcal{A}_M^{q_0}$ and (2.2). From (3.5), (C.9) and (C.10) we get

$$\sup_{t\in[0,T]} \mathbb{E}[(r_t^0)^2] < \infty,$$

and (6.20) follows. □

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