PROBLEMS OF PERFECT FLUIDS

IN GENERAL RELATIVITY

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ABSTRACT

The motion of matter under self-gravitational and internal pressure forces in Einstein's Theory of General Relativity is systemetically investigated. In Chapter 1 following a brief discussion of the observational tests the basic postulates of General Relativity are examined. In Chapter 2 the theory of time-like congruences and the energy-momentum tensor is described. The field equations and Bianchi identities are split into propagation equations for the kinematic quantities and their integrability conditions, and the theory of Petrov's classification of the Weyl tensor is outlined.

Following these introductory chapters, attention is restricted to shear-free normal flows. In Chapter 3 these flows are found to be of Petrov type I. It is also shown that all such non-degenerate perfect fluid fields are static and that the flow is of degenerate Petrov type if and only if the Ricci tensor of the spatial cross-sections has degenerate eigenvalues. All exact solutions of the fields equations representing a perfect fluid with degenerate Weyl tensor are found, the conformally flat ones in Chapter 3 and those of type D in Chapter 4. It is also shown in Chapter 4 that all vacuum fields admitting a shear-free normal timelike congruence are static.

Chapter 5 is devoted to perfect fluids with spherical

planar or hyperbolic symmetry. Three special cases are investigated, namely geodesic, shear-free and expansion-free flows. For the shear-free and expansion-free cases the field equations reduce to one ordinary differential equation, and for the shear-free case all uniform density solutions are obtained as well as some variable density soltuions. The spherically symmetrical case with vanishing pressure is discussed further in the Appendices where the form of the singularity and the propagation of light are considered.

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Parts of the Thesis which are Original

The following parts of the thesis are original:-

Chapter 2: Equations (2.3.19) to (2.3.22), all of Chapters 3 and 4 and Chapter 5: Sections 5.3 and 5.4, and Appendices I.2, II, III and IV.

Appendix II has been accepted for publication in Journal of Physics A, Appendix III (with Dr. G.J. Whitrow) and IV have been published in Monthly Notices of the Royal Astronimical Society, 148, 193, (1970) and Journal of Physics A, 3, 653, (1970) respectively.

Summary of Notation

The following notation will be used unless otherwise stated.

Einstein's summation convention : summation over repeated indices.

Time-like co-ordinate : x^4.

Signature of metric tensor g_{ab} : +2.

Determinant of metric tensor : g.

Range of tensor indices : lower case Latin, 1 to 4; lower case Greek, 1 to 3.

Triad indices : upper case Latin indices, 1 to 3.

Partial differentiation : $V_{a,b}$ in 4-dimensional space; $V_{\alpha|\beta}$ in 3-dimensional space.

Covariant differention : $V_{a;b}$ in 4-dimensional space; $V_{\alpha\|\beta}$ in 3-dimensional space.

Repeated differentiation : $V_{,a;b} = V_{;ab}$; $V_{|\alpha\|\beta} = V_{\|\alpha\beta}$.

Tetrad component of covariant derivative : $V_{.B}$; $V_{:B}$.

Differentiation with respect to:-

(i) proper time : $\dot{V}$.

(ii) co-ordinate time : V'.

(iii) the argument of a function of one variable :

$$f'(x) = \frac{d}{dx} f(x).$$

Square brackets denote antisymmetrisation : $V_{[ab]} = \frac{1}{2}(V_{ab} - V_{ba})$.

Round brackets denote symmetrisation : $V_{(ab)} = \frac{1}{2}(V_{ab} + V_{ba})$.

Co-ordinates : $x^1 \leftrightarrow x, X, \Theta$; $x^2 \leftrightarrow \phi$; $x^3 \leftrightarrow z, r, R, Y$;

$x^4 \leftrightarrow t, T$.

Two-dimensional metric depending only on x^1, x^2 : $d\sigma^2 ; d\Omega^2$.

Christoffel symbols : $\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left(g_{bd,c} + g_{cd,b} - g_{bc,d} \right)$.

Curvature tensor : $\frac{1}{2} R^a{}_{bcd} = \Gamma^a{}_{b[d,c]} + \Gamma^a{}_{e[c} \Gamma^e{}_{d]b}$,

so that the Ricci identity is

$$u_{a;[bc]} = \frac{1}{2} u^d R_{dabc} .$$

Ricci tensor : $R_{ab} = R^c{}_{acb}$.

Permutation tensor : $\eta_{abcd} = \eta_{[abcd]}, \eta_{1234} = (-g)^{\frac{1}{2}}$.

Duality mapping : $V^*_{ab} = \frac{1}{2} \eta_{abcd} V^{cd}$.

Bivector metric : $g_{abcd} = 2 g_{a[c} g_{d]b}$.

Units chosen so that $c = 8\pi G = 1$.

CHAPTER 1

GENERAL INTRODUCTION

1.1 General Relativity

In this thesis the motion of matter under self-gravitation and internal pressure forces will be considered. We shall accept the validity of Einstein's General Relativity theory, as it is perhaps the most widely accepted and conceptually simple theory which predicts results that are in agreement (to within experimental error) with observations. However, we make the reservation that if the oblateness of the solar isothermals measured by Dicke and Goldenburg (1967) does in fact represent a gravitational quadrapole moment it would appear that General Relativity is not consistent with experiment. It would then be necessary to adopt the scalar-tensor theory of Brans and Dicke (1961). Nevertheless it seems that the latest radar experiments (Schapiro et al., 1968; Anderson et al., 1970) are in better agreement with General Relativity than with the rival scalar-tensor theory.

It should be noted that as yet all observations have been made in the quasi-static, spherically symmet-

-rical, weak external (i.e. vacuum) gravitational field of the Sun or the Earth. Consequently the verification of the theory rests solely upon one particular solution of the field equations, namely the exterior Schwartzschild field. It must therefore remain a matter of doubt whether General Relativity is any more applicable to more general situations, e.g. to interior, strong or non-static fields, than, say, classical Newtonian theory. This objection also applies to any other relativistic gravitational theory.

However for want of a better established theory we will assume that General Relativity is valid in these more general situations and investigate the implications of this assumption. In recent months it is possible that 'black holes', one of the strong field predictions of General Relativity, have been observed. The stellar system ϵ-Auriga which appears to be binary but in which only one component is visible has been reported. The mass estimates for the invisible component for the system is in excess of fourteen solar masses and under currently accepted non-relativistic theories of stellar structure such massive bodies ought to be visible. It appears that the least unlikely explanation (Cameron, 1971, Stothers, 1971) is that these objects are black holes,

that is massive stars which have burnt out all nuclear fuel and are unable to support themselves by degeneracy pressure and so undergo catastophic and irreversible collapse. The light from these bodies is strongly red-shifted and eventually becomes trapped and cannot escapr, and that is why such bodies are called 'black holes'. However this interpretation is still uncertain and there are rival explanations (Demarque and Morris, 1971).

Another phenomenon predicted by General Relativity, namely gravitational waves, may recently have been observed by Weber (1967-70). These gravitational waves appear to originate from the Galactic Centre and as yet no scalar component has been detected. If these results are confirmed they will lead to difficulties in the scalar tensor theory in which a scalar component is to be expected, but not in General Relativity where only quadrupole and higher order radiation is to be expected.

1.2 Postulates of Relativity

I. Space-time is represented as a four-dimensional paracompact manifold M of class C^2 with a C^2 piecewise C^4 differential structure.

II. On the manifold M there is a C^1 piecewise C^3

metric with Lorenzian signature +2.

The piecewise C^4 structure is convenient for a physical theory since it allows certain discontinuities in the curvature of the space-time which correspond to discontinuities that seem physically realistic, e.g. density discontinuities and gravitational shock waves. All piecewise C^4 solutions may be arbitrarily closely approximated by C^4 or even C^∞ solutions. In this Thesis we shall consider only local problems and we may therefore assume that the structure is C^4.

For a manifold satisfying the conditions of I a metric as in II will exist if either the manifold is non-compact or if it is compact and a certain topological condition (the vanishing of the Euler invariant) holds (Hawking, 1966).

Co-ordinates may be chosen so that at a point P the metric $g_{ij} = \eta_{ij}$, where η_{ij} is the Minkowski metric. Such co-ordinates are called orthonormal. Furthermore it is always possible to find orthonormal co-ordinates for which $\Gamma^i_{jk}(P) = 0$ and such co-ordinates are called inertial. The metric allows us to classify vectors u^i into three classes :- timelike, null or space-like, if $g_{ij} u^i u^j < 0, = 0, > 0$, respectively. In terms of orthonormal co-ordinates at P the null cone

in the neighbourhood of P is given by
$(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 = 0$.

In order that the above postulates I and II may have physical significance it is necessary to relate them to physically observable quantities. We therefore invoke the following two principles:-

(a) Local Causality

Let V be a convex orthonormal co-ordinate neighbourhood of a point (event) P and let Q be a second point Q. Then P cannot be causally related to Q by effects confined to V unless there is a C^1 non-space-like curve from P to Q.

(b) General Covariance

The laws of physics should be expressible in a form that holds good for all reference frames (or co-ordinate systems), that is they should be expressible as tensor equations with all derivatives being covariant derivatives with the symmetric metric connection.

In order to specify the metric it is necessary either to know how matter determines the geometry of space-time via field equations or to impose some a priori condition on the space-time. In Special Relativity the latter course is taken and the Riemann tensor is required to vanish. This implies that there exist inertial

co-ordinates so that over a finite, i.e. non-infinitesimel, region the metric has the Minkowski form. In Special Relativity matter is described by the energy-momentum tensor T^{ab} which is a symmetric 4 by 4 tensor. The physical meaning of the components of this tensor is given by

$$T^{ab} = \left(\begin{array}{l|l} \text{pressure} & \text{energy} \\ \text{tensor} & \text{current} \\ & \text{density} \\ \hline \text{energy current density} & \text{energy} \\ & \text{density} \end{array}\right) .$$

The law of conservation of energy and momentum is given by

$$T^{ab}{}_{,\,b} = 0 . \qquad (1.2.1.)$$

In a non-inertial co-ordinate system equation (1.2.1.) becomes

$$T^{ab}{}_{;b} = 0 . \qquad (1.2.2.)$$

From this equation it may be deduced that the world-lines of small particles without internal structure (charge, spin etc.) are time-like geodesics (Hawking, 1966). Consequently we see that Special Relativity cannot describe gravitational effects since we know that the world-lines of the Planets do not even approximate to

geodesics in Minkowski space.

The existence of inertial co-ordinates at a point shows that there exist reference frames in which gravitational effects over a small enough region are negligible. This is known as the Equivalence Principle. It seems reasonable to assume that in an inertial frame the local physics of a system is adequately described by Special Relativity. In particular, local conservation of energy is given by equation (1.2.1.) and in a general frame by (1.2.2.).

In order to complete the theory it is still necessary to specify the way in which the matter determines the geometry of space-time. We shall suppose that the field equations are tensor equations of the form

$$T_{ab} = Q_{ab}(g_{cd}, g_{cd,e}, g_{cd,ef}, \ldots), \quad (1.2.3.)$$

where Q_{ab}'s are functions of the metric tensor and its derivatives only. This assumption already excludes the scalar-tensor theories of Brans and Dicke. From experience, and by analogy with Newtonian theory, we would like it to be necessary for the initial values and rates of change of the fields to determine the subsequent evolution of the system. Consequently we expect Q_{ab} to depend only on derivatives of the metric tensor up

to the second. It follows from equations (1.2.2. and 3.) that Q_{ab} has vanishing divergence. Under these conditions it may be shown that Q_{ab} must be of the form

$$Q_{ab} = \gamma G_{ab} + \Lambda g_{ab}, \qquad (1.2.4.)$$

where γ and Λ are constants and where G_{ab} is the Einstein tensor. This result has long been known if Q_{ab} is assumed to be linear in the second derivatives of the metric tensor but has been proved in the general case only very recently (Lovelock, 1971).

The constant γ may be determined by demanding, that the equations reduce to the Newtonian equations in the static weak field limit. It is found that $\gamma = 8\pi G/c^4$; where G is Newton's gravitational constant and c is the velocity of light in vacuo. We may choose units so that both c and $8\pi G$ are equal to unity. Consequently the field equations may be written in the form

$$R_{ab} - \tfrac{1}{2} R g_{ab} + \Lambda g_{ab} = T_{ab}. \qquad (1.2.5.)$$

The <u>cosmological constant</u> Λ is a new fundamental constant. An upper limit on its magnitude may be obtained from observations of the orbits of the outer planets since

a large Λ would produce observable effects on these orbits (Tolman, 1934).

The constant Λ is often set equal to zero but for completeness we shall not do this here. Instead, for mathematical convenience we shall absorb Λ into the energy momentum tensor, i.e.

$$T^{ab} = T^{ab}_{matter} - \Lambda g^{ab} . \qquad (1.2.6.)$$

The field equations become

$$R_{ab} - \tfrac{1}{2} R g_{ab} = T_{ab} . \qquad (1.2.7.)$$

Vacuum fields, i.e. $T^{ab}_{matter} = 0$, are characterised by the condition $R_{ab} = \Lambda g_{ab}$, and are consequently Einstein spaces.

It is possible in certain cases to derive equation (1.2.5.) from a Lagrangian. We consider the Lagrangian $R - 2\Lambda - L$ where L is a function of the physical fields ψ^A excluding the gravitational field. By the index A we denote any combination of tensor or spinor indices either raised or lowered. The equations of the fields are obtained (Landau and Lifschitz, 1962) by requiring

that the action integral I given by $I = \int_D (R - 2\Lambda - L)\sqrt{(-g)}\, d^4x$ is stationary for all variations of the fields ψ^A. The equations are

$$\frac{\partial(\sqrt{(-g)}L)}{\partial\psi^A} - \left(\frac{\partial(\sqrt{(-g)}L)}{\partial\psi^A{}_{;a}}\right)_{;a} = 0. \qquad (1.2.8.)$$

The gravitational field equations are obtained by varying the metric. The energy-momentum tensor is defined as the variation of $\sqrt{(-g)}L$ by the metric tensor and the variation of $(R - 2\Lambda)\sqrt{(-g)}$ can be shown to be

$$\left(R_{ab} - \tfrac{1}{2}R\, g_{ab} + \Lambda g_{ab}\right)\Delta g^{ab},$$

and consequently we obtain the field equations (1.2.5.).

Alternative field equations may be obtained by this method if we replace the Ricci invariant by another invariant, e.g. $R_{ab}R^{ab}$, $R^{abcd}R_{abcd}$ or some higher order metric invariant. By varying the metric fourth or higher order equations are obtained. These field equations were first considered by Eddington (1923) and subsequently by Stephenson (1959) and Buchdahl (1948, 1960 and 1970). It may be shown that vacuum field equations derived from these other invariants are always satisfied by solutions of the equations $R_{ab} = 0$. It is extremely difficult to work with these higher order equations in more general situations and there is no

compelling reason why they should be preferred to the usual field equations and so in this thesis we shall consider only Einstein's field equations (1.2.5.).

Equations (1.2.5.) together with the postulates I and II and the rules for physical interpretation contain essentially the whole of General Relativity. However the equations admit solutions which possess certain global properties which from a physical or philosophical point of view are undesirable. It is usual therefore to eliminate these possibilities by a priori. It is therefore usually assumed that the space-time manifold is connected and orientable and that the space-time metric is time-orientable and satisfies certain global causality conditions.

The conditions are reasonable since if the space-time were not connected it would be impossible for observers in different components to communicate with each other. If the manifold were not orientable and time-orientable then it would not be possible to distinguish globally between say a left and a right hand or between future and past. There are in fact a number of possible causality assumptions that could be imposed. Among these the most important are the chronology and causality assumptions that there are no closed future directed time-

like or null curves respectively. These two assumptions are equivalent to the statement that an observer may not travel into (communicate with) his past.

A number of such global assumptions are essential in the singularity theorems of Penrose (1965) and Hawking (1966). It should be noted that these global assumptions are also necessary in Special Relativity and other relativistic theories. Examples of flat space-times which do not satisfy one or more of the global assumptions are given by Geroch (1970). As we shall be concerned in this thesis only with local problems we shall not consider these global assumptions further.

References

Anderson, J.D. et al., 1970, Proc. 13th COSPAR Planary Meeting (Leningrad)

Brans, C., and Dicke R.H. ; 1961, Phys. Rev. 124, 925.

Buchdahl, 1948, Q. Jl. Math. Oxford Ser., 19, 150.

Buchdahl, 1960, Proc. Camb. Phil. Soc. , 56, 396.

Buchdahl, 1970, M N R A S, 150, 1.

Cameron, A.G.W., 1971, Nature, 229, 178, ibid, 231, 148.

Demarque, P., and Morris D.C. ; 1971, Nature, 230, 516.

Dicke, R.H. and Goldenburg, H.M. ; 1967, Phys. Rev. Lett. 18, 313.

Eddington, A.S. ; 1923, The Mathematical Theory of Relativity. (Cambridge Univ. Press) 141-144.

Geroch, R.P. ; 1970, Proceedings Int. Sch. Phys. 'Enrico Fermi' (Varenna, 1969).

Hawking, S.W. ; 1966, Essay submitted for Adams Prize.

Hawking, S.W. ; 1966a Phys. Rev. Lett. 17, 445.

Hawking, S.W. ; 1966b Proc. Roy. Soc. A, 294, 511.

Landau, L.D. and Lifschitz, E.M. ; 1962, The Classical Theory of Fields, (Pergammon) 307-18.

Lovelock, D. ; 1971, J.M.P. 12, 498.

Penrose, R. ; 1965, Phys. Rev. Lett. 14, 57.

Schapiro, I.I. et. al. ; 1968, Phys. Rev. Lett., 20, 1265.

Stephenson, G. ; 1959, Proc. Camb. Phil. Soc., 55, 375.

Stothers, R. ; 1971, Nature, 229, 181.

Tolman, R.C. ; 1934, Relativity, Thermodynamics and Cosmology, (Oxford Univ. Press), 191-202.

Weber, J. ; 1966, Phys. Rev. Lett. 17, 1228 ; 1967, ibid, 18 498, 1969, ibid, 22, 1320; 1970, ibid, 24, 276; ibid, 25, 180.

CHAPTER 2

MATHEMATICAL INTRODUCTION

2.1 The Kinematics of Time-like Congruences

Congruences of time-like curves play an important part in the theory of relativity since such congruences represent the world-lines of clouds of test-particles or the flow lines of a fluid. A number of kinematic quantities are defined below which describe the behaviour of the congruence.

The unit tangent vectors of the congruence form a vector field u^a and this vector field defines a $3 + 1$ splitting of space-time at each point. The projection tensor into the infinitesimal three-space orthogonal to u^a is given by

$$h_{ab} = g_{ab} + u_a u_b \,. \qquad (2.1.1.)$$

Since $u^a u_a = -1$, it follows immediately from (2.1.1.) that

$$h_{ab} u^b = 0 \,. \qquad (2.1.2.)$$

In the following a dot '' . '' will represent a covariant derivative along u^a , i.e. for any tensor T :::,

$\dot{T}^{\cdots}_{\cdots} = T^{\cdots}_{\cdots;c} u^c$. In particular $\dot{u}_a$ is the acceleration of a curve of the congruence relative to a geodesic at the same point. The acceleration vector is space-like since from the equation $u^a u_a = -1$ we obtain

$$\dot{u}^a u_a = 0. \qquad (2.1.3.)$$

We consider a curve $\lambda_0(t)$ joining two points on neighbouring curves of the congruence and construct a family of curves $\lambda_s(t)$ by moving each point on the curve $\lambda_0(t)$ a proper distance s along the curves of the congruence. The tangent vector z^a of this family of curves measures the separation of corresponding points on the two neighbouring curves of the congruence. Such a vector field z^a is said to be a connecting vector field. It follows from the definition of $\lambda_s(t)$ that the Lie derivative of z^a along u^a vanishes, i.e.

$$[u, z] = u^a{}_{;b} z^b - z^a{}_{;b} u^b = 0. \qquad (2.1.4.)$$

It follows from equations (2.1.3., 4.) that,

$$(u^a z_a)^{\cdot} = \dot{u}_a z^a,$$

and hence in general if z^a is orthogonal to u^a at one point on a curve it will not remain so unless u^a is geodesic.

We consider instead the part of z^a orthogonal to u^a defined by $z_\perp^a = h^a_b z^b$. This vector is the relative position vector in the infinitesimal three-space orthogonal to u^a of the two points joined by the curve $\lambda_s(t)$. Their relative velocity v^a is given by

$$v^a = h^a_b (z_\perp^b)^\cdot . \qquad (2.1.5.)$$

From equations (2.1.1., 4) it follows that

$$v^a = u^a_{;b} z_\perp^b = V^a_b z_\perp^b , \qquad (2.1.6.)$$

where $V_{ab} = h^c_a h^d_b u_{c;d}$. The tensor V_{ab} may be split into symmetric and skew-symmetric parts,

$$V_{ab} = \Theta_{ab} + \omega_{ab} , \qquad (2.1.7.)$$

where $\Theta_{ab} = \Theta_{(ab)}$ and $\omega_{ab} = \omega_{[ab]}$. Further Θ_{ab} may be split into an isotropic and a trace-free part,

$$\Theta_{ab} = \sigma_{ab} + \tfrac{1}{3} \Theta h_{ab} , \qquad (2.1.8.)$$

where $\sigma^a_a = 0$ and $\Theta = \Theta^a_a$. It follows from

equations (2.1.1., 3,6,7,8.) that

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \tfrac{1}{3}\theta h_{ab} - \dot{u}_a u_b, \qquad (2.1.9.)$$

where $\omega_{ab}u^b = \sigma_{ab}u^b = 0$ and $\theta = u^a{}_{;a}$.

In order to see the physical significance of the quantities defined above we will follow the analysis of Hawking (1966). A vector Y^a will be said to undergo Fermi-Walker transport along a time-like curve with unit tangent vector u^a if its Fermi-Walker derivative defined by

$$\frac{D_F Y^a}{\partial s} = \dot{Y}^a - \dot{u}^b Y_b u^a + Y^b u_b \dot{u}^a, \qquad (2.1.10.)$$

vanishes. This transport preserves scalar products and the vector u^a automatically has vanishing Fermi-Walker derivative. If the vector Y^a is orthogonal to u^a then the last term on the right-hand side of (2.1.10.) may be omitted and the vector is said to be Fermi-propagated. In this case the equation $\frac{D_F Y^a}{\partial s} = 0$, is equivalent to

$$h^a{}_b \dot{Y}^b = 0, \qquad (2.1.11.)$$

or

$$\dot{Y}_{[a} u_{b]} = 0. \qquad (2.1.12.)$$

We now introduce an orthonormal basis $(e^a_A, e^a_4 = u^a)$, where capital Latin indices run from 1 to 3, at a point and Fermi-propagate this basis along the integral curves of u^a to form an orthonormal basis at each point of the curve. We may then express a tensor field along the curve in terms of its components with respect to the basis. In particular for $Z^a_\perp$ we have

$$Z^a_\perp = Z^A e^a_A . \qquad (2.1.13.)$$

On using equations (2.1.6., 11,13.), equation (2.1.5.) becomes

$$\frac{d}{ds} Z^A = V^A{}_{;B} Z^B .$$

As this equation is linear it follows that

$$Z^A(s) = A^A{}_B(s)\, Z^B(0) , \qquad (2.1.14.)$$

where

$$\frac{d}{ds} A^A{}_B = V^A{}_{;C} A^C{}_B , \qquad (2.1.15.)$$

and

$$A^A{}_B(0) = \delta^A_B .$$

From equation (2.1.14.) we see that $A^A{}_B$ represents the change in the position vector $Z^a_\perp$ along the integral curves of u^a.

We will adopt a matrix notation in order to simplify

the following equations. The 3 x 3 matrix $\underline{A}$ may be expressed uniquely as

$$\underline{A} = \underline{U} \cdot \underline{S} , \qquad (2.1.16.)$$

where $\underline{U}^t \cdot \underline{U} = \underline{I}$, $\underline{S}^t = \underline{S}$ and $\underline{S}(0) = \underline{U}(0) = \underline{I}$. The matrix $\underline{U}$ represents the rotation that neighbouring curves have undergone with respect to the Fermi-propagated frame, $\underline{S}$ represents the separation of the curves, and $\det \underline{S} = \det \underline{A}$ is the fractional change in volume of a volume element bounded by curves in the congruence.

From equations (2.1.15., 16.) is follows that

$$\underline{V} = \frac{d}{ds}\underline{U} + \frac{d}{ds}\underline{S} ,$$

at the point $s = 0$. Since $\frac{d}{ds}\underline{U}$ is skew-symmetric and $\frac{d}{ds}\underline{S}$ symmetric we obtain:-

$$\frac{d}{ds}\underline{U} = \underline{\omega} = \tfrac{1}{2}(\underline{V} - \underline{V}^t) ,$$

$$\frac{d}{ds}\underline{S} = \underline{\Theta} = \tfrac{1}{2}(\underline{V} + \underline{V}^t) ,$$

$$\theta = \mathrm{Trace}(\underline{V}) ,$$

and

$$\underline{\sigma} = \underline{\Theta} - \tfrac{1}{3}\theta\underline{I} .$$

Hence $\underline{\omega}$, $\underline{\Theta}$ and θ represent respectively the instanteous rates of change of rotation, separation and volume at the point $s = 0$, i.e. the vorticity, expansion and

the volume expansion, respectively. The trace-free part $\underline{\sigma}$ of $\underline{\Theta}$ is the rate of distortion at constant volume and hence represents the shear of the flow.

Reverting to co-ordinate indices and writing $Z_{\perp}^{a} = Z_{\perp} e^{a}$ where $e^{a} u_{a} = 0$ and $e^{a} e_{a} = 1$, we obtain, on using equations (2.1.5.-8.), the following equations for the change in magnitude and direction of $Z_{\perp}^{a}$

$$\dot{Z}_{\perp}/Z_{\perp} = \Theta_{ab} e^{a} e^{b} = \tfrac{1}{3}\Theta + \sigma_{ab} e^{a} e^{b}, \qquad (2.1.17.)$$

$$h^{a}{}_{b} \dot{e}^{b} = \left[\omega^{a}{}_{b} + \sigma^{a}{}_{b} - \sigma_{cd} e^{c} e^{d} h^{a}_{b}\right] e^{b}. \qquad (2.1.18.)$$

We may also define the vorticity vector ω^{a} by

$$\omega_{a} = \tfrac{1}{2} \eta_{abcd} u^{b} \omega^{cd}. \qquad (2.1.19.)$$

Consequently this vector satisfies the equations

$$\omega_{ab} \omega^{b} = \omega_{a} u^{a} = 0.$$

This vector lies along the axis of rotation of the flow with respect to the Fermi-propagated frame which may be identified with the local inertial frame (Synge, 1960).

The magnitude of the vorticity, shear and acceleration are given by

$$\omega^{2} = \omega_{a} \omega^{a} = \tfrac{1}{2} \omega_{ab} \omega^{ab},$$

$$\sigma^{2} = \tfrac{1}{2} \sigma_{ab} \sigma^{ab},$$

and

$$\dot{u}^2 = \dot{u}^a u_a .$$

These quantities vanish if and only if the corresponding tensors vanish. A congruence is called geodesic if $\dot{u} = 0$. If $\omega = 0$ it is called normal, and if $\sigma = \theta = 0$ then it is said to be a rigid congruence. Futhermore, a rigid congruence for which $\dot{u}_{[a;b]} = 0$ is an isometric congruence. An isometric congruence for which $\omega \neq 0$ is called stationary whereas if $\omega = 0$ static.

We will now consider normal congruences. The equation $\omega = 0$ is equivalent to $\omega_{[ab}\, u_{c]} = 0$. It follows from (2.1.9.) that

$$u_{[a;b}\, u_{c]} = u_{[a,b}\, u_{c]} = 0 .$$

This equation implies that u_a is proportional to a gradient and hence there exist functions t and V such that

$$t_{,a} = V^{-1} u_a . \qquad (2.1.20.)$$

Thus the vector u^a is orthogonal to the hypersurface $t =$ constant. Choosing t as the x^4-coordinate, we see that the metric takes the form

$$G = P^{-2}(x^a)\, \gamma_{\mu\nu}(x^a) dx^\mu dx^\nu - V^2(x^a) dt^2 , \qquad (2.1.21.)$$

where Greek indices run from 1 to 3 and $det(Y_{\mu\nu})=1$. In this co-ordinate system the kinematic quantities are given by (Trumper, 1962),

$$\left.\begin{aligned} \sigma_{\mu\nu} &= (2P^2V)^{-1}\, Y'_{\mu\nu}, \\ \Theta &= -3(PV)^{-1}P', \\ \dot{u}_\mu &= V^{-1}V_{,\mu}, \end{aligned}\right\} \qquad (2.1.22.)$$

where a prime denotes partial differentiation with respect to t.

2.2 The Energy-Momentum Tensor

As we have seen in Chapter 1, the energy-momentum tensor is a symmetric tensor of valence two. It represents the energy density, the energy and momentum flux, and the stress of the matter. This may be seen more clearly if T_{ab} is decomposed with respect to a time-like vector field u^a. We obtain (Ehlers, 1961).

$$T_{ab} = \mu u_a u_b + 2q_{(a}u_{b)} + S_{ab}, \qquad (2.2.1.)$$

where $q_a u^a = S_{ab}u^b = S_{[ab]} = 0$. The quantities μ and q_a are the energy-density and energy-flux of the matter with respect to u^a, i.e. the energy density and flux that would be measured by an observer with 4-velocity u^a. The spatial stress tensor S_{ab} can be split into an isotropic and a trace-free part, viz.

$$S_{ab} = \pi_{ab} + ph_{ab}, \qquad (2.2.2.)$$

where $\pi^a{}_a = 0$. The conservation equations become on using equations (2.2.1., 2)

$$\dot{\mu} + (\mu+p)\theta + \pi^{ab}\sigma_{ab} + q^a{}_{;a} + \dot{u}^a q_a = 0, \qquad (2.2.3.)$$

$$h^c{}_a\left(p_{,c} + \pi_c{}^b{}_{;b} + \dot{q}_c\right) + (\mu+p)\dot{u}_a + \left(\omega_{ac} + \sigma_{ac} + \tfrac{4}{3}\theta h_{ac}\right)q^c = 0. \qquad (2.2.4.)$$

Equation (2.2.3.) is the equation of energy conservation

and (2.2.4.) the equation of momentum conservation.

The time-like eigenvector and corresponding eigenvalue of T_{ab} (if they exist) represent respectively the dynamical mean velocity u^a_D and proper energy density μ_D of the matter, i.e.

$$T^a{}_b u^b_D = -\mu u^a_D . \qquad (2.2.5.)$$

If this vector u^a_D is used in the decomposition formula (2.2.1.) it follows that $q_a = 0$.

In the present thesis attention will be confined for the most part to one particular form of energy-momentum tensor, namely that of a perfect fluid. For a perfect fluid the stress tensor S_{ab} is isotropic and so using equations (2.2.1., 2.) it follows that

$$T_{ab} = \mu u_a u_b + p h_{ab} . \qquad (2.2.6.)$$

For a perfect fluid the conservation equations (2.2.3,4.) become

$$h^a{}_b p_{,a} + (\mu + p)\dot{u}_b = 0, \qquad (2.2.7.)$$

and

$$\dot{\mu} + (\mu + p)\theta = 0. \qquad (2.2.8.)$$

In order to describe the local physics of the matter,

equations of state relating the quantities μ, p, q_a etc. are required. In particular for a perfect fluid an equation of state relating p and μ is needed. If the energy-momentum tensor (2.2.6.) is derivable from a Lagrangian function, the flow is called <u>isentropic</u> and the pressure p is a function of the energy density only, i.e. $p = p(\mu)$, (Hawking, 1966).

We list below a number of equations of state that are of particular importance. For a perfect gas, the equation of state is $\mu = \rho + (\gamma-1)^{-1} p$, $p = \beta \rho^{\gamma-1}$, where β and γ are constants with $1 \leq \gamma \leq 2$. The function ρ is the rest mass density of the gas and it satisfies the conservation equation

$$\dot{\rho} + \rho \theta = 0. \qquad (2.2.9.)$$

The temperature of the gas is defined by $p = \alpha \rho T$, where α is a constant. A special case of a perfect gas is <u>dust</u>, i.e. matter for which the equations of state are $p = 0$ and $\mu = \rho$. We note that the two conservations equations (2.2.8., 9.) are consistent and that equation (2.2.7.) implies that $\dot{u}_b = 0$. Thus the flow lines of an incoherent fluid (dust) are geodesics (Synge, 1960).

Another important case is that of <u>isotropic radiation</u>.

The equation of state is given by $p = \frac{1}{3}\mu$ and the temperature of the radiation is defined by $\mu = \epsilon T^4$, where ϵ is a constant. Finally, if the equation of state $p = -\mu$ is valid we will regard the energy-momentum tensor as representing not matter but rather a vacuum field with a non-zero cosmological constant. This interpretation is consistent, since the conservation equations (2.2.7., 8.) imply that both p and μ are constant. If we appeal to equation (1. 2.6.) and set $\Lambda = p$, we obtain

$$T^{ab}_{\text{matter}} = R^{ab} - \tfrac{1}{2} R g^{ab} + \Lambda g^{ab} = 0.$$

It may be seen from equation (2.2.8.) that only the volume expansion enters into the energy equation. Consequently, if the behaviour of Θ and the equation of state of a system are known, the complete thermal history of the system can be determined. This form of analysis has been used to obtain the thermal history of the Universe (Ellis, 1970). However, to determine the behaviour of Θ with time it is necessary to study the field equations.

2.3 The Field Equations and the Bianchi Identities

The field equations (1. 2.7.) become on using the decomposition formulae (2.2.1., 2) for the energy-momentum tensor

$$R_{ab} = \tfrac{1}{2}(\mu+3p)u_a u_b + \tfrac{1}{2}(\mu-p)h_{ab} + 2q_{(a}u_{b)} + \pi_{ab}, \quad (2.3.1.)$$

where units have been chosen so that the velocity of light c is equal to unity and the gravitational constant G is equal to $1/8\pi$. It is seldom possible to integrate the field equations as they stand except in very special cases. However in recent years a number of formalisms have been developed which enable a large number of exact solutions to be obtained. In this thesis we shall use the theory of time-like congruences summarised in section 1 of this chapter to obtain a number of exact solutions and theorems on perfect fluids. This method of analysing the field equations was developed by Ehlers (1961), Kundt and Trumper (1962) and has been used extensively by several authors to prove a number of important results on vacuum and perfect fluid fields. Essentially the method relates the kinematic quantities defined in Section 2.1 to the energy-momentum tensor by means of the field equations (2.3.1.) and the Ricci identity,

$$u_{a;[bc]} = \tfrac{1}{2}u^d R_{dabc}. \quad (2.3.2.)$$

If we multiply this equation by u^b we obtain

$$(u_{a;c})^{\cdot} - \dot{u}_{a;c} - u_{a;d}u^d{}_{;c} + R_{abcd}u^b u^d = 0 \qquad (2.3.3.)$$

On projecting by means of $h^c_a h^d_b$ and using equations (2.1.3., 6.) we find that

$$h^c_a h^d_b (V_{cd})^{\cdot} - \dot{u}_a \dot{u}_b - h^c_a h^d_b \dot{u}_{c;d} + V_{ad}V^d{}_b + R_{acbd}u^c u^d = 0. \qquad (2.3.4.)$$

This equation is essentially the propagation equation of V_{cd} along u^a, and from it the equations of propagation of the shear, vorticity and volume expansion may be obtained.

On using equations (2.1.7.-9., 23.) and (2.3.1.) we see that the trace of equation (2.3.4.) is

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 - \dot{u}^a{}_{;a} + 2(\sigma^2 - \omega^2) + \tfrac{1}{2}(\mu + 3p) = 0. \qquad (2.3.5.)$$

This is the equation of propagation of the volume expansion and is known as Raychaudhuri's equation (Raychaudhuri, 1955). From this equation we see that vorticity produces expansion whilst shear and isotropic pressure induce contraction. This effect of isotropic pressure is purely relativistic, there being no pressure

term in the corresponding Newtonian equation.

The skew-symmetric part of (2.3.4.) is the equation of propagation of vorticity. This becomes, on using equations (2.1.7., 8.),

$$h^c_a h^d_b \dot{\omega}_{cd} - h^c_a h^d_b \dot{u}_{[c;d]} + 2\sigma_{d[a}\omega^d{}_{b]} + \tfrac{2}{3}\theta\omega_{ab} = 0. \quad (2.3.6.)$$

We see that the rate of change of vorticity does not depend directly on the Riemann tensor. It is an immediate consequence of equation (2.3.6.) that, if the curves are geodesic and ω vanishes at one point on the curves, then ω vanishes everywhere along the curves.

The symmetric trace-free part of (2.3.4.) is the equation of propagation of the shear. On using equations (2.1.7., 8.) and (2.3.1., 5.) we obtain

$$h^c_a h^d_b \dot{\sigma}_{cd} - h^c_a h^d_b \dot{u}_{(c;d)} - \dot{u}_a \dot{u}_b + \omega_{ad}\omega^d{}_b + \sigma_{ad}\sigma^d{}_b$$

$$+\tfrac{2}{3}\theta\sigma_{ab} + \tfrac{1}{3}h_{ab}\left(2(\omega^2-\sigma^2) + \dot{u}^c{}_{;c}\right) - \tfrac{1}{2}\pi_{ab} + E_{ab} = 0. \quad (2.3.7.)$$

where $E_{ab} = C_{acbd}$ and C_{abcd} is the Weyl conformal curvature tensor defined by

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2\,\delta^{[a}_{[c} R^{b]}_{d]} + \tfrac{1}{3}\delta^{a}_{[c}\delta^{b}_{d]} R. \qquad (2.3.8.)$$

The tensor E_{ab} is known as the tidal force or the free gravitational field and is that part of the gravitational field not determined by the local matter distribution. It may be seen from equation (2.3.8.) that E_{ab} has the properties

$$E^{a}{}_{a} = E_{ab}u^{b} = E_{[ab]} = 0. \qquad (2.3.9.)$$

The equations (2.3.5.-7.) do not contain all the information of the Ricci identity (2.3.2.). On using the symmetry $R^{a}{}_{[bcd]} = 0$ of the Riemann tensor and equation (2.1.9.), we obtain

$$\omega_{[ab;c]} - \dot{u}_{[a;c}u_{b]} - \dot{u}_{[a}\omega_{bc]} = 0,$$

or equivalently,

$$\omega^{a}{}_{;a} = 2\omega^{a}\dot{u}_{a}. \qquad (2.3.10.)$$

If the Ricci identity (2.3.2.) is contracted and then projected with $h^{a}{}_{b}$ we obtain, on using equation (2.1.9.) and (2.3.1.),

$$h^{a}{}_{b}\left(\omega^{bc}{}_{;c} - \sigma^{bc}{}_{;c} + \tfrac{2}{3}\theta^{;b}\right) + \left(\omega^{a}{}_{b} + \sigma^{a}{}_{b}\right)\dot{u}^{b} = q_{a}. \qquad (2.3.11.)$$

On taking the dual of equation (2.3.2.) and then multiplying by u^b we obtain, on using equations (2.1.9.) and (2.3.8., 10., 11.),

$$H_{ab} = 2\dot{u}_{(a}\omega_{b)} + h_a^c h_b^d \left[\omega_{(c}{}^{e;f} + \sigma_{(c}{}^{e;f}\right]\eta_{d)gef}u^g, \quad (2.3.12.)$$

where $H_{ab} = {}^*C_{acbd}u^c u^d$ and ${}^*C_{abcd}$ is the dual of the Weyl tensor. It follows from the definition that

$$H_{[ab]} = H^a{}_a = H_{ab}u^b = 0.$$

The two tensors E_{ab} and H_{ab} contain all the information of the Weyl tensor, and together with the Ricci tensor R_{ab} completely determine the Riemann tensor R_{abcd}. The tensor E_{ab} is analogous to the Newtonian tidal force but H_{ab} has no Newtonian analogue.

The Weyl tensor is not completely arbitrary, since the Riemann tensor must satisfy the Bianchi identities,

$$R_{ab[cd;e]} = 0.$$

These equations are equivalent (Kundt and Trumper, 1962) to the equations

$$C^{abcd}{}_{;d} = J^{abc}, \quad (2.3.13)$$

where

$$J_{abc} = R_{c[a;b]} - \tfrac{1}{6}g_{c[a}R_{,b]}. \quad (2.3.14.)$$

There are twenty independent equations in (2.3.13.)
These are equivalent to the four equations

$$\left(R^{b}_{a} - \tfrac{1}{2} R\, \delta^{b}_{a}\right)_{;b} = 0,$$

and the sixteen equations below (Hawking, 1966) :-

$$h^{b}_{a} E_{bc;d} h^{cd} + 3 H_{ab} \omega^{b} + \eta_{abcd} u^{b} \sigma^{ce} H^{d}_{e} = -J_{a}^{\ cd} u_{c} u_{d} \; ; \qquad (2.3.15.)$$

$$h^{b}_{a} H_{bc;d} h^{cd} - 3 E_{ab} \omega^{b} + \eta_{abcd} u^{b} \sigma^{ce} E^{d}_{e} = -\tfrac{1}{2} \eta_{abcd} J^{cde} u^{b} u_{e} \; ; \qquad (2.3.16.)$$

$$h^{c}_{a} h^{d}_{b} \dot{E}_{cd} + h^{f}_{(a} \eta_{b)cde} H_{f}^{\ d;e} + E_{ab} \theta - 3 E^{c}_{(a} \sigma_{b)c} + h_{ab} E_{cd} \sigma^{cd}$$

$$- E^{c}_{(a} \omega_{b)c} - 2 H^{d}_{(a} \eta_{b)cde} u^{c} \dot{u}^{e} = J^{cde} u_{c} h_{d(a} h_{b)e} \; ; \qquad (2.3.17.)$$

$$h^{c}_{a} h^{d}_{b} \dot{H}_{cd} - h^{f}_{(a} \eta_{b)cde} E_{f}^{\ d;e} + H_{ab} \theta - H^{c}_{(a} \omega_{b)c} - 3 H^{c}_{\ (a} \sigma_{b)c}$$

$$+ h_{ab} H_{cd} \sigma^{cd} - 2 E^{d}_{(a} \eta_{b)cde} u^{c} \dot{u}^{e} = -\tfrac{1}{2} h_{f(a} \eta_{b)cde} u^{c} J^{def} \; . \qquad (2.3.18.)$$

These equations are similar to Maxwell's equation in electromagnetic theory. The decomposition of the Weyl tensor into the tensors E_{ab} and H_{ab} is analogous to the

decomposition of the electromagnetic field tensor F_{ab} into an electric and magnetic field and for this reason E_{ab} and H_{ab} are called respectively the electric and magnetic components of the Weyl tensor with respect to the vector field u^a. The sixteen integrability equations (2.3.15.-18.) replace the eight integrability conditions in Newtonian theory. Consequently, the general relativistic equations are more restrictive than their Newtonian analogues. For instance, it has been shown by Ellis (1967) that a shear-free flow of dust in general relativity theory is either stationary or rigid whereas in Newtonian theory there are such flows which are neither stationary or rigid.

On using the decomposition formula (2.2.1., 2.) of the energy-momentum tensor, the conservation equations (2.2.3., 4.) and the field equations (2.3.1.), the following expressions for the source terms on the right-hand sides of equations (2.3.15.-18.) are obtained:-

$$J_a^{bc} u_b u_c = -\tfrac{1}{3}\mu_{,b} h^b_a - q_c(\omega^c_a + \sigma^c_a - \dot{u}^c u_a) + h^{bc}\pi_{ab;c} - \sigma_{bc}\pi^{bc}u_a, \quad (2.3.19)$$

$$\tfrac{1}{2}\eta_{abcd} J^{cdc} u^b u_e = -(\mu+p)\omega_a - \tfrac{1}{2}\eta_{abcd}u^b\left[q^{c;d} + \pi^c_e(\omega^{ed} + \sigma^{ed})\right], \quad (2.3.20.)$$

$$J^{cde}u_c h_{d(a}h_{b)e} = -\tfrac{1}{2}(\mu+p)\sigma_{ab} - \tfrac{1}{2}\pi_{c(a}\sigma_{b)}{}^{c} + \tfrac{1}{6}\pi_{cd}\sigma^{cd}h_{ab}$$
$$+ \tfrac{1}{6}(q^c{}_{;c} + \dot{u}^c q_c)h_{ab}$$
$$-\tfrac{1}{2}\dot{\pi}_{cd}h_a^c h_b^d - \tfrac{1}{2}h^c_{(a}q_{b);c} + \tfrac{1}{2}q_c[\omega^c_{(a} + \sigma^c_{(a}]u_{b)} + \tfrac{1}{6}q_{(a}u_{b)}, \qquad (2.3.21.)$$

$$-\tfrac{1}{2}h_{f(a}\eta_{b)cde}u^c J^{def} = -q_{(a}\omega_{b)} - \tfrac{1}{2}\eta_{(b}{}^{cde}[\omega_{a)e} + \sigma_{a)e}]u_c q_d$$
$$+ \tfrac{1}{2}u_{(a}\eta_{b)}{}^{cde}u_c\pi_{df}(\omega^f_e + \sigma^f_e) + \tfrac{1}{2}\eta_{cde(b}\pi_{a)}{}^{d;e}u^c, \qquad (2.3.22.)$$

For the case of a perfect fluid, i.e. $q_a = \pi_{ab} = 0$, the source terms are greatly simplified. In fact equations (2.3.19.-22.) become

$$J_a{}^{bc}u_b u_c = -\tfrac{1}{3}\mu_{,b}h^b{}_a, \qquad (2.3.23.)$$

$$\tfrac{1}{2}\eta_{abcd}J^{cde}u^b_b u_e = -(\mu+p)\omega_a, \qquad (2.3.24.)$$

$$J^{cde}u_c h_{d(a}h_{b)e} = -\tfrac{1}{2}(\mu+p)\sigma_{ab}, \qquad (2.3.25.)$$

$$-\tfrac{1}{2}\, h_{f(a}\eta_{b)cde}\, u^c J^{def} = 0. \qquad (2.3.26.)$$

2.4 The Petrov Classification

In Section 2.3 the Weyl tensor was seen to represent the free gravitational field. It has the symmetries of the vacuum Riemann tensor, and indeed for a vacuum field the two tensors are identical. Gravitational fields have been classified by Petrov (1962) who considered the canonical forms of the Weyl tensor. There are a number of ways by which the classification may be obtained, e.g. the spinor method of Penrose (1965) and the method of principal null directions (Debever, 1959), (Bel, 1959), (Penrose 1960). For the purpose of this thesis the most convenient method is that of Jordan, Ehlers and Kundt (1960). This method of classification is summarised below.

The set of bivectors form a six-dimensional linear vector space with a metric $g_{abcd} = 2\, g_{a[c}\, g_{d]b}$ induced by g_{ab}. Any tensor T_{abcd} with the symmetries $T_{ab[cd]} = T_{[ab]cd} = T_{abcd}$ may be regarded as a linear transformation on the bivector space. In particular the transformation induced by C_{abcd} is symmetrical with respect to the metric g_{abcd}, is trace-free and commutes with the duality mapping of the bivector space defined by $\overset{*}{V}_{ab} = \tfrac{1}{2}\eta_{abcd}V^{cd}$. The bivector space is isomorphic and isometric to the complex

vector space with Euclidean metric. The isomorphism is given by

$$V_A = \overset{*}{V}_{4A} + i V_{4A},$$

where components are taken with respect to an orthonormal tetrad $(e_A^a, e_4^a = u^a)$ with u^a time-like. The transformation of $\mathcal{C}^3$ induced by C_{abcd} is given by

$$u_A = D_{AB} v_B, \qquad (2.4.1.)$$

where $D_{AB} = E_{AB} + i H_{AB}$, E_{AB} and H_{AB} being the 'electric' and 'magnetic' components with respect to u^a of the Weyl tensor.

Transformations of the tetrad, i.e. Lorentz transformations, induce transformations in the complex vector space. The transformations induced by the proper orthochronous Lorentz group are the complex proper rotations. Rotations of the space-like triad induce real rotations and space and time reflections map into its complex conjugate. The problem of classification of the Weyl tensor is therefore equivalent to that of finding the canonical forms of symmetric trace-free matrices under the group of complex rotations.

The Weyl tensor is said to be of type I, II or III if

the transformation given by $\mathcal{D}_{AB}$ has invariant subspaces of dimension one, two or three respectively. $\mathcal{D}_{AB}$ has one of the three following normal forms:-

$$\underline{\mathrm{I}}: \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1-\lambda_2 \end{pmatrix}, \quad \underline{\mathrm{II}}: \begin{pmatrix} -\frac{1}{2}\lambda+1 & i & 0 \\ i & -\frac{1}{2}\lambda-1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \underline{\mathrm{III}}: \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 1 \\ i & 1 & 0 \end{pmatrix}. \qquad (2.4.2.)$$

Type I admits two possible degenerate cases, namely type D (or Id); $\lambda_1 = \lambda_2 \neq 0$ and type Idd : $\lambda_1 = \lambda_2 = 0$. The fields of type Idd are conformally flat. Type II fields with $\lambda = 0$ are called IId or <u>null</u> fields. If the field is not of non-degenerate type I, then it is called <u>algebraically special</u>.

The orthonormal tetrads with respect to which $\mathcal{D}_{AB}$ takes its canonical form are called <u>principal tetrads</u>, and their individual vectors <u>principal vectors</u>. The eigenvalues λ_μ are invariants with respect to the proper Lorentz group and are mapped into $\bar{\lambda}_\mu$ by reflections of the tetrad. If $\lambda_\mu = \alpha_\mu + i\beta_\mu$, the α's are called the <u>eigenscalars</u>, or principal scalars, and the β's the <u>eigenpsuedoscalars</u>.

If for some tetrad the 'magnetic' component of C_{abcd}

vanishes, then $\mathcal{D}_{AB}$ is real. Since $\mathcal{D}_{AB}$ is also symmetric it may be diagonalised by means of real rotations of the tetrad, and consequently the Weyl tensor is of type I with vanishing eigenpsuedoscalars. If $(e^a_A, e^a_4 = u^a)$ is a Weyl principal tetrad then the electric component of the Weyl tensor is given by

$$E^{ab} = \sum_{A=1}^{3} \alpha_A e^a_A e^b_A , \qquad (2.4.3.)$$

where $\sum \alpha_A = 0 .$ (2.4.4)

References

Bel, L., 1959, Compt. Rend. 248, 1297, 2561.

Debever, R. 1959, Compt. Rend. 249, 1744.

Ehlers, J., 1961, Akad. Wiss. Mainz. Abh. Math. Nat. Kl. No. 11.

Ellis, G.F.R., 1970 Proc. Int. School Physics ''Enrico Fermi'' Varenna 1969.

Hawking, S., 1966, Singularities and the Geometry of Space-time, Essay submitted for the Adams Prize.

Jordan, P. Ehlers, J. and Kundt, W., 1960, Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. No. 2.

Kundt, W. and Trumper, M., 1962, Akad. Wiss. Mainz. Abh. Math. Nat. Kl. No. 12

Penrose, R. 1960, Ann. Phys. 10, 171.

Penrose, R. 1965, The Structure of Space-time, Adams Prize Essay.

Petrov, A.S. 1962, Einstein Spaces (Pergammon), pp88-126.

Raychaudhuri, A.K. 1955, Phys. Rev. 98, 1123.

Trumper, M. 1962, Zeit. Phys. 168, 55.

Synge, J.L., 1960, Relativity. The General Theory. (North Holland).

CHAPTER 3

SHEAR-FREE NORMAL CONGRUENCES

3.1 General Results

In this chapter space-times admitting a shear-free normal congruence of time-like curves will be considered and a number of results proved. When the congruence is composed of the stream-lines of a perfect fluid it will be shown that the assumptions, $\sigma = \omega = 0$, impose strong restrictions on the space-time. In fact the space-time is either static or of Petrov type I degenerate. This result is not altogether suprising, since Trumper (1962) has obtained a similar result for vacuum space-times. Furthermore, space-times admitting other special congruences have been investigated by a number of authors and these space-times were also found to be of a very special nature. For instance, Ellis (1967) considered shear-free flows of incoherent matter and showed that the space-time is either stationary or the congruence is normal. Similar results hold for null congruences. The theorem of Goldberg and Sachs (1962) states that a vacuum space-time is algebraically special if and only if it admits a shear-free null geodesic congruence.

We will investigate the restrictions that are placed on the Weyl tensor by the existence of a shear-free normal congruence of time-like curves. From equation (2.3.12.), since $\sigma_{ab} = \omega_{ab} = 0$, it follows that the equation $H_{ab} = 0$ is valid. Thus, as was shown in Section 2.4, the tensor $\mathcal{D}_{AB}$ is real and may be diagonalised by means of a rotation of the space-like triad. The following theorem is therefore valid.

Theorem 3.1

A space-time admitting a shear-free normal congruence of time-like curves with tangent vector u^a is of Petrov type I with vanishing eigenpsuedoscalers and u^a is a Weyl principal vector. If (e_A^a, u^a) is a principal tetrad, then $E^{ab} = \sum_A \alpha_A e_A^a e_A^b$.

This theorem has been proved by Ehlers (1961) with the additional assumption that either $\Theta = 0$ or that u^a is an eigenvector of the Ricci tensor, but as far as I am aware the theorem has not previously been proved without these additional assumptions.

If we now choose co-ordinates as in equation (2.1.20.) the metric takes the form (2.1.21.) with $\gamma_{\mu\nu} = \gamma_{\mu\nu}(x^1)$. In this co-ordinate system $\Gamma^4_{\mu 4} = V^{-1} V_{,\mu} = \dot{u}_\mu$ and consequently,

$$\dot{u}^a{}_{;a} = V^{-1} V^{,\mu}{}_{\mu} \qquad (3.1.1.)$$

where a single and double stroke represent respectively partial and covariant derivatives taken with the metric $H = P^{-2}\gamma_{\mu\nu}(x^1)\,dx^\mu dx^\nu$. Raychaudhuri's equation (2.3.5.) becomes on using equations (2.1.22.) and (3.1.1.),

$$V^{-1}V^{1}{}_{\mu}{}^{\mu} - 3\left(\frac{P^1}{PV}\right)^2 + 3V^{-1}\left(\frac{P^1}{PV}\right)^1 = \tfrac{1}{2}(\mu + 3p). \qquad (3.1.2.)$$

From equation (2.3.11.) it follows that

$$3q_a/2 = h_a{}^b\Theta_{,b} = \Theta_{,a} + \dot{\Theta}u_a, \qquad (3.1.3.)$$

or equivalently using equation (2.1.22.)

$$q_\mu = -\left(2P^1/PV\right)_{1\mu} . \qquad (3.1.4.)$$

If u^a is a Ricci eigenvector, i.e. $q_a = 0$, then it follows that

$$\Theta_{,[a}u_{b]} = 0 \quad ; \quad \Theta = \Theta(t) . \qquad (3.1.5.)$$

Equations (3.1.2.) and (3.1.4.) are respectively the (4,4) and (μ ,4) components of the field equations (2.3.1.). We now proceed to derive the remaining field equations.

The second fundamental form of the hypersurface t = constant is given by (see equation (A.1.1.) in Appendix I),

$$d_{\mu\nu} = \left(P^3 V\right)^{-1} P' \gamma_{\mu\nu} = -\tfrac{1}{3}\Theta h_{\mu\nu}. \qquad (3.1.6.)$$

Equation (A.1.5.) then reduces to the following expression for the Ricci tensor

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + \tfrac{1}{3}h_{\mu\nu}\left(\dot{\Theta} + \Theta^2\right) - V^{-1}V_{\|\mu\nu}, \qquad (3.1.7.)$$

where $\tilde{R}_{\mu\nu}$ is the Ricci tensor of the space cross-sections. Hence the (μ, ν) component of the field equations (2.3.1.) becomes with the aid of equation (2.1.22.)

$$\tilde{R}^{\mu}_{\nu} - V^{-1}V'^{\mu}_{\|\nu} + \left[3\left(\frac{P'}{PV}\right)^2 - V^{-1}\left(\frac{P'}{PV}\right)'\right]\delta^{\mu}_{\nu} = \tfrac{1}{2}(\mu - p)\delta^{\mu}_{\nu} + \pi^{\mu}_{\nu}. \qquad (3.1.8.)$$

In this case the vorticity propagation equation (2.3.6.) is

$$h^{c}_{a}h^{d}_{b}\,\dot{u}_{[c;d]} = \dot{u}_{[a;b]} + \ddot{u}_{[a}u_{b]} + \tfrac{1}{3}\Theta\,\dot{u}_{[a}u_{b]} = 0. \qquad (3.1.9.)$$

It is easily seen from this equation and equation (2.1.9.) that if $\left(\ddot{u}_{[a} - \tfrac{1}{3}\Theta_{,[a}\right)u_{b]} = 0,$

$$\left(\dot{u}_{[a} - \tfrac{1}{3}\Theta u_{[a}\right);_{b]} = 0.$$

Consequently, there exists a function U such that

$$\dot{u}_a - \tfrac{1}{3}\Theta u_a = U_{,a}, \qquad (3.1.10.)$$

and on defining the vector $\xi^a = e^U u^a$, it follows that

$$\xi_{(a;b)} = \left(e^U\right)^{\cdot} g_{ab} = \tfrac{1}{4} \xi^c{}_{;c}\, g_{ab} . \qquad (3.1.11.)$$

We have shown that the space-time admits a conformal Killing vector parallel to u^a . From equations (3.1.10.) and (2.1.22.) it follows that

$$\dot{u}_\mu = U_{|\mu} = V^{-1} V_{|\mu} , \quad \tfrac{1}{3}\Theta V = U' = -\frac{P'}{P} .$$

Consequently, $f(x^\mu) P^{-1} = g(t)\, V$, where f and g are functions of integration and, by a suitable redefinition of $\gamma_{\mu\nu}(x^\lambda)$ and t, the metric (2.1.21.) becomes

$$G = V^2 \left(\gamma_{\mu\nu}(x^\lambda)\, dx^\mu dx^\nu - dt^2 \right) . \qquad (3.1.12.)$$

The converse is also true. If the space-time admits a conformal Killing vector ξ^a then the unit vector u^a parallel to ξ^a satisfies the equation $\left(\dot{u}_{[a} - \tfrac{1}{3}\Theta_{|[a}\right) u_{b]} = 0$. In particular if ξ^a is a Ricci eigenvector, it follows from (3.1.5.) that $\dot{u}_{[a}\, u_{b]} = 0$, i.e. $\dot{u}_a$ is Fermi propagated along u^a. The equation $\dot{u}_{[a}\, u_{b]} = 0$ implies that

$$\left(V^{-2} V_{|\mu}\right)' = 0 ,$$

and, on integration,

$$V^{-1} = A(x^\lambda) + B(t) \,. \qquad (3.1.13.)$$

Further if $\Theta = 0$, $V' = u' = 0$ and equation (3.1.11.) becomes

$$\xi_{(a;b)} = 0 \,,$$

and consequently ξ^a is a Killing vector and the space-time is static.

The shear propagation equation (2.3.8.) becomes in the case $\sigma = \omega = 0$,

$$E_{ab} - \tfrac{1}{2}\pi_{ab} = h_a^c h_b^d \dot{u}_{(c;d)} + \dot{u}_a u_b - \tfrac{1}{3} h_{ab} \dot{u}^c{}_{;c} \,.$$

This becomes, on using equations (2.1.7.), (2.3.4., 5.) and (3.1.9.),

$$E_{ab} - \tfrac{1}{2}\pi_{ab} = R_{acbd} u^c u^d - \tfrac{1}{6}(\mu + 3p) h_{ab} \,. \qquad (3.1.14)$$

If the co-ordinate system defined in equation (2.1.20.) is used we see that the (a,4) components of all the tensors in equation (3.1.14.) vanish and consequently the Latin

indices may be replaced by Greek indices running from 1 to 3. Equation (A.1.4.) becomes on using (3.1.6., 7.)

$$R^4{}_{\alpha 4\beta} = -V^{-1}V_{\|\alpha\beta} + \tfrac{1}{3}(\dot{\Theta}+\Theta^2)h_{\alpha\beta} = R_{\alpha\beta} - \tilde{R}_{\alpha\beta} - \tfrac{2}{9}\Theta^2 h_{\alpha\beta}.$$

Equation (3.1.14.) becomes,

$$E_{\alpha\beta} - \tfrac{1}{2}\pi_{\alpha\beta} = \tilde{R}_{\alpha\beta} - R_{\alpha\beta} + \tfrac{2}{9}\Theta^2 h_{\alpha\beta} - \tfrac{1}{6}(\mu+3p)h_{\alpha\beta},$$

and on eliminating $R_{\alpha\beta}$ by means of the field equations (2.3.1.),

$$E_{\alpha\beta} + \tfrac{1}{2}\pi_{\alpha\beta} = \hat{R}_{\alpha\beta} - \tfrac{2}{3}(\mu - \tfrac{1}{3}\Theta^2)h_{\alpha\beta}. \qquad (3.1.15.)$$

Immediate consequences of this equation are

$$\hat{R} = 2(\mu - \tfrac{1}{3}\Theta^2), \qquad (3.1.16.)$$

and

$$\tilde{P}_{\alpha\beta} = E_{\alpha\beta} + \tfrac{1}{2}\pi_{\alpha\beta}, \qquad (3.1.17.)$$

where $\tilde{P}_{\alpha\beta}$ is the trace-free part of $\tilde{R}_{\alpha\beta}$. Consequently, the Ricci tensor of the space cross-sections, t = constant, has equal eigenvalues if and only if the tensor $E_{\alpha\beta} + \tfrac{1}{2}\pi_{\alpha\beta}$ has. In particular for a perfect fluid the following theorem is valid.

Theorem 3.2

The eigenvectors of the Ricci tensor of the space

cross-sections are Weyl principal vectors and the eigenvalues of the trace-free Ricci tensor are Weyl principal scalars, the eigenvalues of the Ricci tensor being

$$\alpha_A + \tfrac{2}{3}\left(\mu - \tfrac{1}{3}\Theta^2\right) \quad .$$

It follows from this theorem that the Ricci tensor has equal eigenvalues if and only if the space-time is of degenerate Petrov type.

3.2 The Integrability Conditions

From now onwards attention will be restricted to the case of a perfect fluid, i.e. $\pi_{ab} = q_a = 0$. The investigation and integration of the field equations (3.17.8.) are greatly facilitated if use is made of the integrability conditions (2.3.15.-18.). This method of analysis is equivalent to that used by a number of authors (Levi-Civita, 1917-19; Jordan, Ehlers and Kundt, 1960, and Barnes, 1971) to investigate static space-times but it is more direct. Equation (2.3.17.) with the aid of equations (2.1.21., 22.) and (2.3.25.) becomes

$$\left(E_{\mu\nu}\right)' - P^{-1}P' E_{\mu\nu} = \left(P^{-1}E_{\mu\nu}\right)' = 0 . \quad (3.2.1.)$$

Thus the tensor $P^{-1}E_{\mu\nu}$ is independent of the co-ordinate t. This result was obtained by Trumper (1962) for vacuum

space-times admitting a shear-free normal congruence.

On taking tetrad components of equations (2.3.15.-18.) with respect to a Weyl principal tetrad ($e^a_A, e^a_4 = u^a$) and making use of equations (2.3.23.-26.) and (2.1.22.) we obtain

$$E_{AD\cdot D} - \tfrac{1}{3}\mu_{\cdot A} = 0, \qquad (3.2.2.)$$

$$E_{AB\cdot 4} + E_{AB}\Theta = 0, \qquad (3.2.3.)$$

$$\delta_{F(A}\epsilon_{B)DE}E_{FD\cdot E} + 2V^{-1}\alpha_A\delta_{D(A}\epsilon_{B)DE}V_{\cdot E} = 0, \qquad (3.2.4.)$$

where a dot signifies a tetrad component of a covariant derivative. In equation (3.2.4.) there is no sum over A and ϵ_{ABC} ($= \eta_{ABC4}$) is the three-dimensional permutation symbol. On taking tetrad components of the covariant derivative of equation (2.4.3.) and suspending the summation convention we obtain

$$E_{AB\cdot K} = \alpha_{A\cdot K}\delta_{AB} + (\alpha_B - \alpha_A)\gamma_{BAK}, \qquad (3.2.5.)$$

where $\gamma_{KLM} = e_{Ka;b}e^a_L e^b_M$ are the Ricci rotation coefficients of the tetrad, K,L,M, running from 1 to 4.

Equations (3.2.2.-4.), with the aid of equations (3.2.5.), can be shown to be equivalent to the following equations:-

$$\alpha_{A\cdot A} + \sum_D (\alpha_A - \alpha_D)\gamma_{ADD} - \tfrac{1}{3}\mu_{\cdot A} = 0, \qquad (3.2.6.)$$

$$\alpha_{A\cdot 4}\,\delta_{AC} + (\alpha_C - \alpha_A)\gamma_{CA4} + \Theta\alpha_A\,\delta_{AC} = 0, \qquad (3.2.7.)$$

$$E = \gamma_{123}(\alpha_1 - \alpha_2) = \gamma_{231}(\alpha_2 - \alpha_3) = \gamma_{312}(\alpha_3 - \alpha_1), \qquad (3.2.8.)$$

$$\alpha_{A\cdot C} - \alpha_{B\cdot C} + (\alpha_C - \alpha_B)\gamma_{CBB} - (\alpha_C - \alpha_A)\gamma_{CAA} + 2V^{-1}V_{\cdot C}(\alpha_A - \alpha_B) = 0, \qquad (3.2.9.)$$

where in the last equation A,B and C are unequal. From equations (3.2.6., 9.) and (2.4.4.), it follows that

$$\alpha_{A\cdot C} + (\alpha_A - \alpha_C)\gamma_{CAA} + V^{-1}V_{\cdot C}(\alpha_A - \alpha_B) + \tfrac{1}{6}\mu_{\cdot C} = 0, \qquad (3.2.10)$$

for A,B,C $\neq$. It follows from equations (2.1.22.) and (3.2.7.) that

$$\alpha_A' - 3P^{-1}P'\alpha_A = 0,$$

and on integration

$$\alpha_A = P^3 \beta_A(x^\lambda). \qquad (3.2.11.)$$

We now prove the following theorem.

Theorem 3.3

A space-time admitting a shear-free normal congruence admits a Weyl principal tetrad such that the space-like triad is Fermi-propagated along the time-like vector, u^a. Furthermore each space-like vector of the triad is two-forming with u^a.

Proof

If the space-time is non-degenerate, i.e. if the α_A's are unequal it follows from equation (3.2.7.) that $\gamma_{AB4} = 0$. This is precisely the condition that e^a_A be Fermi-propagated along u^a. If the field is type D, we may take without loss of generality $\alpha_1 = \alpha_2$. It then follows from equations (3.2.7.-9.) that

$$\gamma_{312} = \gamma_{231} = \gamma_{134} = \gamma_{234} = \gamma_{311} - \gamma_{322} = 0. \qquad (3.2.12.)$$

These equations are preserved by the rotation of the vectors e_1^a and e_2^a:

$$\bar{e}_1^a = e_1^a \cos\theta + e_2^a \sin\theta, \qquad (3.2.13.)$$

$$\bar{e}_2^a = - e_1^a \sin\theta + e_2^a \cos\theta ,$$

where $\theta = \theta(x^\lambda, t)$, but γ_{124} and γ_{123} transform in the following way:

$$\bar{\gamma}_{124} = \gamma_{124} + \theta_{.4} ,$$

$$\bar{\gamma}_{123} = \gamma_{123} + \theta_{.3} .$$

Consequently, by a suitable choice of θ, $\bar{\gamma}_{124}$ may be set equal to zero.

For the conformally flat case it follows from equation (3.1.17.) that $\tilde{P}_{\alpha\beta} = 0$. Hence the space cross-sections are Einstein spaces of dimension 3 and are therefore of uniform curvature $K(t)$ (Eisenhart, 1949). Consequently there exist co-ordinates (x^ν) such that $h_{\mu\nu} = P^{-2} \delta_{\mu\nu}$ and the co-ordinate vectors $e_{A\mu} = P^{-1} \delta_{A\mu}$ satisfy the required conditions.

The second part of the theorem follows almost immediately. Equation (2.1.9.) implies that $\gamma_{4AB} = 0$ for $A \neq B$, and since $\gamma_{(KL)M} = 0$ it follows that $\gamma_{A[4B]} = 0$ for $A \neq B$. This is precisely the condition that e_B^a and u^a be two-forming.

In the following we will consider tetrads of theorem 3.3

and the co-ordinates of equations (2.1.20.-22.) will be used. If the vectors $n_{A\mu}$ are defined by $n_{A\mu} = P e_{A\mu}$, it follows that

$$\gamma_{AB4} = (PV)^{-1}(n_{A\mu})' e_B^{\mu} = 0 \,. \qquad (3.2.14.)$$

Hence the vectors $n_{A\mu}$ do not depend on time. The auxilary metric $\gamma_{\mu\nu}$ is given by $\gamma_{\mu\nu} = \sum_A n_{A\mu} n_{A\nu}$, and if indices of $n_{A\mu}$ are raised and lowered by means of $\gamma_{\mu\nu}$ it follows that

$$n_A{}^{\mu} = n_A{}^{\mu}(x^{\lambda}) = P^{-1} e_A{}^{\mu} \,.$$

The Ricci rotation coefficients γ_{ABC} may be expressed in terms of those of the triad ($n_A{}^{\mu}$). We obtain with the aid of equation (A.1.8.)

$$\gamma_{ABC} = P\Gamma_{ABC} + 2\delta_{C[A}P_{:B]}\,, \qquad (3.2.15.)$$

where $\Gamma_{ABC} = n_{A\mu\|\nu} n_B^{\mu} n_C^{\nu}$, a double stroke denoting a covariant derivative with respect to the metric $\gamma_{\mu\nu}$, and where $P_{:B} = P_{|\lambda} n^{\lambda}_B$. We are now able to prove the following theorem.

Theorem 3.4

A space-time admitting a shear-free normal congruence contains a Weyl tetrad of hypersurface orthogonal vectors

if it is either degenerate or if one of the space-like triad of vectors is hypersurface orthogonal.

Proof

If the space-time is non-degenerate and one of the vectors e^a_A is hypersurface orthogonal it follows from equation (3.2.8.) that $E=0$ and consequently $\gamma_{123} = \gamma_{231} = \gamma_{312} = 0$. In Theorem 3.3 it was proved that $\gamma_{A34} = \gamma_{A4B} = 0$, and hence $\gamma_{KLM}=0$ for K,L,M unequal. This is the condition that the tetrad be hypersurface orthogonal. For spaces of type D we have proved that all $\gamma_{KLM} = 0$ except possibly γ_{123} . The tetrad rotation (3.2.13.) with $\theta = \theta(x^4)$ preserves the equations $\gamma_{KLM} = 0$, for $(K,L,M) \neq (1,2,3)$ whilst γ_{123} transforms as

$$\bar{\gamma}_{123} = \gamma_{123} + \theta_{.3} ,$$

or equivalently using (3.2.15.)

$$\bar{\Gamma}_{123} = \Gamma_{123} + \theta_{.3} .$$

All terms in this equation are independant of t and so θ may be chosen so that $\bar{\Gamma}_{123}$ and $\bar{\gamma}_{123}$ are zero. For the conformally flat case the vectors defined in theorem 3.3 are also hypersurface orthogonal.

Theorem 3.4 generalises that of Jordan, Ehlers and Kundt (1960) which referred to static vacuum fields. If

the conditions of the theorem are satisfied then we may choose co-ordinates (x^ν) so that the metric $\gamma_{\mu\nu}$ takes the form

$$\gamma_{\mu\nu}dx^\mu dx^\nu = \left(e^{\gamma_1}dx^1\right)^2 + \left(e^{\gamma_2}dx^2\right)^2 + \left(e^{\gamma_3}dx^3\right)^2. \quad (3.2.16.)$$

The co-ordinate vectors are Ricci eigenvectors and the vectors n_A^μ and their rotation coefficients are given by

$$n_A^\mu = e^{-\gamma_A}\delta_A^\mu, \quad \Gamma_{ABC} = 0, \quad A,B,C \neq$$

and

$$\Gamma_{ABB} = \gamma_{B:A} = e^{-\gamma_A}\gamma_{B|a}\delta_A^a.$$

If equations (3.2.11., 15.) are used to eliminate α_A and γ_{ABC} from equations (3.2.6., 9.), the following equations are obtained

$$\beta_{A\cdot A} + \sum_D (\beta_A - \beta_D)\Gamma_{ADD} - \tfrac{1}{3}P^{-3}\mu_{:A} = 0, \quad (3.2.17.)$$

$$2P^3P_{:C}(\beta_A - \beta_B) + P^4(\beta_A - \beta_B)_{:C} + P^4(\beta_C - \beta_B)\Gamma_{CBB}$$
$$- P^4(\beta_C - \beta_A)\Gamma_{CAA} + 2V^{-1}V_{:C}P^4(\beta_A - \beta_B) = 0. \quad (3.2.18.)$$

The only terms involving t in equation (3.2.18) are P and V, and consequently if the space is non-degenerate we may deduce that

$$\left(P^{-1}P_{|\mu} + V^{-1}V_{|\mu}\right)' = 0.$$

On integrating the following expression is obtained

$$PV = f(x^\lambda)\, g(t) .$$

The metric (2.1.21.) takes the form

$$G = (V/g)^2 \left(f^{-2} \gamma_{\mu\nu} dx^\mu dx^\nu - g^2 dt^2 \right).$$

By suitable redefinitions of $\gamma_{\mu\nu}$, t and V the metric may be expressed in the form

$$G = V^2 \left(\gamma_{\mu\nu}(x^\lambda)\, dx^\mu dx^\nu - dt^2 \right). \qquad (3.2.19.)$$

We have proved that the space-time admits a conformal Killing vector. Since the Killing vector is also an eigenvector of the Ricci tensor equation (3.1.3.) is valid.

i.e. $$V^{-1} = A(x^\lambda) + B(t) . \qquad (3.2.20)$$

This result generalises that of Trumper (1962) who considered only vacuum fields.

3.3 Space-times admitting a conformal Killing vector

It has been shown in the last section that a non-degenerate space-time admits a conformal Killing vector and so it is of interest to investigate these space-times further. The analysis of this section is

applicable to the degenerate as well as the non-degenerate cases.

From equations (A.1.1.-7.) we see that the Ricci tensor $\bar{R}_{ab}$ of the metric

$$\bar{g}_{ab}\,dx^a dx^b = \gamma_{\mu\nu}\,dx^\mu dx^\nu - dt^2,$$

is given by

$$\bar{R}_{\mu\nu} = \bar{\bar{R}}_{\mu\nu}, \quad \bar{R}_{44} = \bar{R}_{4\mu} = 0, \qquad (3.3.1.)$$

where $\bar{\bar{R}}_{\mu\nu}$ is the Ricci tensor of $\gamma_{\mu\nu}$. The Ricci tensor R_{ab} of the metric $G = (A+B)^{-2}\,(\gamma_{\mu\nu}\,dx^\mu dx^\nu - dt^2)$ is related to $\bar{R}_{ab}$ in the following way [see equation (A.1.10)]

$$R_{ab} = \bar{R}_{ab} - 2(A+B)^{-1}(A+B)_{\|ab} + \tfrac{1}{2}(A+B)\left\{(A+B)^{-1}\right\}^{\|c}_{\|c}\,\bar{g}_{ab}, \qquad (3.3.2.)$$

where $\|$ represents a covariant derivative with respect to the metric $\bar{g}_{ab}$. Since $\gamma_{\mu\nu}$ is independent of t , it follows that for a, b = 1,2,3 the covariant derivatives may be replaced by covariant derivatives with respect to $\gamma_{\mu\nu}$ and for a, b = 4 they may be replaced by partial differention with respect to t .

On using the equations (3.3.1.,2.) and the field equations (2.3.1.) we may deduce that

$$\bar{\bar{R}}_{\mu\nu} - 2(A+B)^{-1}\left(B'' Y_{\mu\nu} + A_{\|\mu\nu}\right) = (A+B)^{-2}(\mu+p)\, Y_{\mu\nu}. \qquad (2.3.3.)$$

If this equation is differentiated with respect to t it is easily seen that $A_{\|\mu\nu}$ is proportional to $Y_{\mu\nu}$, unless $B' = 0$. It follows from equation (2.3.3.) that $\bar{\bar{R}}_{\mu\nu}$ is proportional to $Y_{\mu\nu}$ and consequently

$$\bar{\bar{R}}_{\mu\nu} - \tfrac{1}{3}\bar{\bar{R}}\, Y_{\mu\nu} = 0.$$

Consequently, the space-sections t = constant with metric $Y_{\mu\nu}(x^\lambda)$ are Einstein spaces and hence are of constant curvature K.

Thus the following equations are valid

$$\bar{R}^{\mu\nu}{}_{\lambda\sigma} = \bar{\bar{R}}^{\mu\nu}{}_{\lambda\sigma} = 2K\,\delta^{\mu}_{[\lambda}\delta^{\nu}_{\sigma]},$$
$$\bar{R}^{\mu}{}_{\nu} = \bar{\bar{R}}^{\mu}{}_{\nu} = 2K\,\delta^{\mu}_{\nu},$$
$$\bar{R} = \bar{\bar{R}} = 6K,$$
$$\bar{R}^{ab}{}_{c4} = \bar{R}^{a}{}_{4} = 0.$$

From these equations and (A.1.11.) it is easily verified that the metric $\bar{G} = Y_{\mu\nu}\,dx^\mu dx^\nu - dt^2$ is conformally flat. Consequently the space-time itself is conformally flat.

If however $B' = 0$ it follows that the expansion Θ of

the flow lines given by $\Theta = -3B'$, vanishes. It may be deduced from equations (3.1.10.,11.) that the conformal Killing vector is in fact a Killing vector and consequently the space-time is static. We have therefore proved the following theorem.

Theorem 3.5

If the flow-lines of a perfect fluid are parallel to a normal conformal Killing vector field, then the flow is either static or the space-time is conformally flat.

Corollary

A non-degenerate perfect fluid field in which the flow lines form a normal shear-free congruence is static.

3.4 Conformally Flat Space-times

In this Section conformally flat space-times containing a perfect fluid will be considered and the general solution of the field equations for this case will be obtained. It is not necessary to assume that the flow is shear-free and normal since the condition $C_{abcd} = 0$ implies that $\sigma = \omega = 0$. This result was first obtained by Kundt and Trumper (1962) and follows immediately from equations (2.3.16., 17., 24., 25.) since $E_{ab} = H_{ab} = 0$. It follows from equations (2.3.15., 23.) that

$$h_a{}^b \mu_{,b} = 0 , \quad \text{i.e.} \quad \mu = \mu(t). \qquad (3.4.1.)$$

Since the flow is normal and shear-free, the analysis of Sections 3.1 and 3.2 can be used. As was seen in the proof of Theorem 3.3 the space cross-sections are of uniform curvature $K(t)$ and hence 'spherical polar' co-ordinates (r, θ, ϕ) may be chosen so that (see equations (A.2., 12,24,25.) in Appendix I)

$$h_{\mu\nu}dx^{\mu}dx^{\nu} = \left\{\alpha + \beta|\underline{r}-\underline{a}|^2\right\}^{-2}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \quad (3.4.2.)$$

where α, β and $\underline{a}$ are all functions of t and where $K(t) = 4\alpha\beta$.

Two cases arise depending on whether the expansion θ vanishes or not. If $\theta \neq 0$ it follows from equations (2.1.22.) and (3.1.5.) that

$$V = -\left(3/\theta(t)\right)\frac{\alpha' + \beta'|\underline{r}-\underline{a}|^2 - 2\beta\,\underline{a}'\cdot(\underline{r}-\underline{a})}{\alpha + \beta|\underline{r}-\underline{a}|^2} = \frac{-3P'}{\theta P}. \quad (3.4.3.)$$

From equations (3.1.2., 16.) we obtain the following expressions for the density μ and pressure p of the matter:-

$$\mu = 12\beta\alpha + \tfrac{1}{3}\theta^2 = 3K + \tfrac{1}{3}\theta^2, \quad (3.4.4.)$$

$$p = -\left(3K + \tfrac{1}{3}\theta^2\right) - (\theta V)^{-1}\left(3K + \tfrac{1}{3}\theta^2\right)'. \quad (3.4.5.)$$

The solution given by equations (3.4.2.-5.) reduces to known solutions in certain special cases. For the case where $\mu + p = 0$, it follows from the field equations (2.3.1.) that the space-time is an Einstein space and being conformally flat is of constant curvature (Eisenhart, 1949). Thus, if $\mu + p = 0$, the space-time is one of the de Sitter space-times.

If the flow is isentropic i.e. $p = p(\mu)$ and $\mu + p \neq 0$ it follows from equations (2.1.22.), (2.2.8.) and (3.4.1.) that

$$p_{|\alpha} = \dot{u}_\alpha = V^{-1} V_{|\alpha} = 0. \qquad (3.4.6.)$$

Consequently the pressure is uniform and the flow is geodesic. On integration and a suitable co-ordinate transformation $\hat{t} = \hat{t}(t)$, we see that $V = 1$. We may deduce from equation (3.4.3.) that P takes the form

$$P = S^{-1}(t)\left(1 + \tfrac{1}{4} K |\underline{r} - \underline{a}|^2\right),$$

where $S = 1/\alpha$, $K = 4\beta/\alpha$ = constant, and $\underline{a}$ is a constant vector. By a suitable redefinition of the spatial co-ordinates we may set $\underline{a} = 0$, and consequently the metric takes the form

$$G = \frac{S^2(t)}{(1+\frac{1}{4}\kappa r^2)^2}\left\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right\} - dt^2. \qquad (3.4.7.)$$

Thus every conformally flat solution representing the isentropic flow of a perfect fluid is locally isometric to a Friedmann cosmological solution. This result was first proved by Kundt and Trumper (1962).

If $\underline{a}$ is a constant vector we may redefine the spatial co-ordinates so that the metric becomes

$$G = (\alpha + \beta r^2)^{-2}\left\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{9}{\theta^2}(\alpha' + \beta' r^2)^2 dt^2\right\}. \qquad (3.4.8.)$$

This metric admits a local isometry group which is isomorphic to that of the two-dimensional sphere and the space in spherically symmetrical. This solution was first obtained by Thompson and Whitrow (1967) and was used by them to investigate the collapse of spheres of a uniformly dense perfect fluid.

Alternatively, instead of the form (3.4.2.) we may choose co-ordinates so that the spatial metric has the form (see equation (A.2.24.) in Appendix I)

$$h_{\mu\nu}dx^\mu dx^\nu = \left\{A\cos\log r + B\sin\log r + (a\cos\phi + b\sin\phi)\sinh\theta + c\cosh\theta\right\}^{-2} \cdot \left(\frac{dr^2}{r^2} + d\theta^2 + \sinh^2\theta\, d\phi^2\right), \qquad (3.4.2a.)$$

or

$$h_{\mu\nu}dx^\mu dx^\nu = \left\{A + B\log r + \tfrac{1}{2}c(\theta^2+\log^2 r) + (a\cos\phi + b\sin\phi)\theta\right\}^{-2} \cdot \left(\frac{dr^2}{r^2} + d\theta^2 + \theta^2 d\phi^2\right). \qquad (3.4.2b.)$$

From equations (2.1.22.) and (3.1.5.) it follows that

$$V = \frac{-3\left\{A'\cos\log r + B'\sin\log r + c'\cosh\theta + (a'\cos\phi + b'\sin\phi)\sinh\theta\right\}}{\theta\left\{A\cos\log r + B\sin\log r + c\cosh\theta + (a\cos\phi + b\sin\phi)\sinh\theta\right\}}, \qquad (3.4.3a.)$$

or

$$V = \frac{-3\left\{A' + B'\log r + \tfrac{1}{2}c'(\theta^2+\log^2 r) + (a'\cos\phi + b'\sin\phi)\theta\right\}}{\theta\left\{A + B\log r + \tfrac{1}{2}c(\theta^2+\log^2 r) + (a\cos\phi + b\sin\phi)\theta\right\}}. \qquad (3.4.3b.)$$

From equations (3.1.2,,16) we obtain the following expressions for the density μ and pressure p

$$\mu - \tfrac{1}{3}\theta^2 = 3K = -3(A^2+B^2) - 3(a^2+b^2-c^2), \qquad (3.4.4a.)$$

or

$$\mu - \tfrac{1}{3}\theta^2 = 3K = -3B^2 + 6Ac, \qquad (3.4.4b.)$$

$$p = -\left(3K + \tfrac{1}{3}\theta^2\right) - (\theta V)^{-1}\left(3K + \tfrac{1}{3}\theta^2\right)'. \qquad (3.4.5.)$$

If in equation (3.4.2a., b.) a, b and c are proportional to $\left(A^2+B^2\right)^{\frac{1}{2}}$ or B, respectively then by means of co-ordinate transformations we may set a = b = c = o. The metric admits a local isometry group which is isomorphic to that of the two dimensional Lobatschewski or Euclidean planes and the space possesses hyperbolic, respectively planar, symmetry. These solutions are obtained by a different method in Chapter 5.

If $\theta = 0$, the above analysis after equation (3.4.2.) is not valid. Since $\theta = 0$ it follows from (2.1.22.) that $P' = 0$ and hence $\alpha' = \beta' = \underline{a}' = 0$. Consequently, the metric may be expressed as

$$G = \left(1 + \tfrac{1}{4}Kr^2\right)^{-2}\left\{dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right\} - V^2 dt^2, \quad (3.4.9.)$$

where K is a constant. Since $E_{\mu\nu} = 0$ it is easily seen from equations (3.1.8., 17) that

$$V^{|\mu}{}_{|\nu} - \tfrac{1}{3} V^{|\lambda}{}_{|\lambda} \delta^{\mu}_{\nu} = 0, \quad (3.4.10.)$$

when covariant derivatives are taken with respect to the metric

$$H = \left(1 + \tfrac{1}{4}Kr^2\right)^{-2}\left\{dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right\}.$$

From equation (A.2.20.) in Appendix I we see that

$$V = \left(1 + \tfrac{1}{4}Kr^2\right)^{-1}\left(A(t) + B(t)r^2 + \underline{r}\cdot\underline{a}(t)\right), \quad (3.4.11.)$$

where A, B and $\underline{a}$ are arbitrary functions of time. The density and pressure are given by the following expressions:-

$$\mu = 3K, \quad (3.4.12.)$$

$$p = -3K + (AK + 4B)/V. \quad (3.4.13.)$$

If $A' = B' = \underline{a}' = 0$, the line-element (3.4.9.) with

equation (3.4.11.) is the interior Schwarzschild solution in isotropic co-ordinates, whilst for $AK + 4B = 0$, i.e. $\mu + p = 0$, the solution is one of the de Sitter solutions. If $V = f(t)\,g(x^\lambda)$, the space-time is static, and by a suitable transformation of the form $\tilde{t} = \tilde{t}(t)$ and a rotation of the spatial co-ordinates θ, ϕ we see that V may be put in the form

$$V = \left(1 + \tfrac{1}{4}Kr^2\right)^{-1}\left(A + Br^2 + Cr\cos\theta\right), \qquad (3.4.14.)$$

where A, B and C are constants. This solution was found in a different co-ordinate system by Stepanyuk (1968). We now prove that this solution admits a four-parameter group of local isometries, G_4 where $G_4 = G_3 \times G_1$ and G_3 is isomorphic to the group of a two-dimensional space of constant curvature.

If $K \neq 0$, V may be rewritten as

$$V = D + \left(1 + \tfrac{1}{4}Kr^2\right)^{-1}\left\{E\left(1 - \tfrac{1}{4}Kr^2\right) + Cr\cos\theta\right\}, \qquad (3.4.15.)$$

where $2D = A + 4B/K$ and $2E = A - 4B/K$.
In the following it is convenient to use Cartesian co-ordinates defined by

$$x = x^1 = r\sin\theta\sin\phi\,, \quad y = x^2 = r\sin\theta\cos\phi\,, \quad z = x^3 = r\cos\theta. \qquad (3.4.16.)$$

The space-sections being of constant curvature admit six independent Killing vector fields given by (Robertson and Noonan, 1968)

$$\xi^{\alpha}_{A} = \left(1-\tfrac{1}{4}Kr^{2}\right)\delta^{\alpha}_{A} + \tfrac{1}{2}K x^{\alpha}\delta_{A\beta}x^{\beta}, \quad \xi^{\alpha}_{A} = -\epsilon_{ABC}\delta^{\alpha}_{B}\delta_{C\beta}x^{\beta}. \quad (3.4.17.)$$

If we consider a four-vector η^{i} such that η^{α} is a Killing vector of the space-sections and $\eta^{4} = 0$, Killing's equations $\eta_{(i;j)} = 0$ become $V_{(\alpha}\eta^{\alpha)} = 0$. On using equations (3.4.14.,17.) it is easily seen that $\eta^{i}_{A}, {}^{A=1,2,3}$ are Killing vectors of the space-time, η^{α}_{A} being given by

$$\eta^{\alpha}_{1} = \xi^{\alpha}_{1} + C^{-1}F\xi^{\alpha}_{2}, \quad \eta^{\alpha}_{2} = \xi^{\alpha}_{2} - C^{-1}F\xi^{\alpha}_{1}, \quad \eta^{\alpha}_{3} = \xi^{\alpha}_{3}, \quad (3.4.18.)$$

where $F = EK$ for $K \neq 0$ and $F = B$ for $K = 0$.

Since all three vectors η^{i}_{A} lie in the two-dimensional surface V = constant, t = constant, this surface is of constant curvature and the η^{i}_{A}'s are generators of a group of (local) isometries which is isomorphic to the isometry group two-dimensional space of constant curvature. If the Killing operators X_{A} are defined by $X_{A} = \eta^{i}_{A}\frac{\partial}{\partial x^{i}}$, it is easily seen from equations (3.4.17., 18.) that

their commutation relations are

$$[X_1, X_2] = -(K + C^{-2}F^2)X_3, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = -X_2. \qquad (3.4.19.)$$

If $K>0$ or $K=0$ and $B\neq 0$ or $K<0$ and $F^2+C^2K>0$, we may write $\bar{X}_A = (C^{-2}F^2+K)^{-\frac{1}{2}} X_A$ for $A = 1, 2$. The commutation relations become

$$[X_1, X_2] = -X_3, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = -X_2.$$

The group is isomorphic to that of the two-dimensional sphere and the space-time is spherically symmetrical.

If $K = 0$ and $B = 0$, it may be seen from equation (3.4.19.) that the group is isomorphic to that of the two-dimensional plane. The metric (3.4.9.) with (3.4.14) takes the form

$$G = dx^2 + dy^2 + dz^2 - (A + Cz)^2 dt^2. \qquad (3.4.20.)$$

This metric is flat, the transformation to Minkowski co-ordinates being given by

$$\bar{z} = (z + A/C)\cosh(Ct), \quad \bar{t} = (z + A/C)\sinh(Ct).$$

If $K<0$ and $F^2 + C^2K < 0$ then on dividing X_1 and X_2 by $(-C^{-2}F^2K^2 - K)^{1/2}$ we see that the

commutation relations become

$$[X_1, X_2] = +X_3 \qquad [X_2, X_3] = -X_1, \qquad [X_3, X_1] = -X_2.$$

The group is isomorphic to that of the Lobatschewskian plane and the space-time is hyperbolically symmetrical. If $K<0$ and $C^2 + E^2 K = 0$, the commutation relations (3.4.19.) are those of the group of the two-dimensional plane, and the space-time possesses planar symmetry.

In all three cases since the space cross sections are of constant curvature we see from equation (A.2.15.) that co-ordinates (r, θ, ϕ) may be chose so that

$$G = (\alpha - Kr^2)^{-1} dr^2 + r^2 d\Omega^2 - V^2(r) dt^2, \qquad (3.4.21.)$$

where $d\Omega^2$ is a two-dimensional metric of constant curvature α, where $\alpha = +1, 0$ or -1. On integrating equation (3.4.10.) we obtain

$$\left.\begin{aligned} V &= D + E_1 (\alpha - Kr^2)^{\frac{1}{2}} \quad \text{for} \quad K \neq 0, \\ V &= A - \tfrac{1}{4} C^2 / B + B r^2 \quad \text{for} \quad K = 0. \end{aligned}\right\} \qquad (3.4.22.)$$

If equation (3.4.21.) is expressed in isotropic co-ordinates we see from equations (A.2.17.-20.) that

$$G = F^{-2}(r)\left(dr^2 + r^2 d\Omega^2\right) - V^2(r) dt^2, \qquad (3.4.23.)$$

where

$$\left.\begin{aligned} &F=\left(1+\tfrac{1}{4}Kr^2\right),\ V=D+E_1\left(1-\tfrac{1}{4}Kr^2\right)\left(1+\tfrac{1}{4}Kr^2\right)^{-1}, && \text{for } \alpha=+1,\\ &F=\sqrt{(-K)}\,r\log r,\ V=D+E_1/\log r, && \text{for } \alpha=0,\\ &F=\sqrt{(-K)}\cos\log r,\ V=D+E_1\tan\log r, && \text{for } \alpha=-1. \end{aligned}\right\} \quad (3.4.24.)$$

In the metric (3.4.21. or 3.4.23.) co-ordinates θ, ϕ may be chosen so that the two-dimensional metric of constant curvature α takes the form

$$d\Omega^2 = d\theta^2 + f^2(\theta)\, d\phi^2,$$

where $f(\theta) = \sin\theta,\ \theta,\ \sinh\theta$, for $\alpha = +1,\ 0,\ -1$, respectively. The pressure and density are given by

$$\mu = 3K,\quad p = -3K + 2DK/V, \quad \text{for} \quad K \neq 0,$$

and

$$\mu = 0,\quad p = 4B/V \quad \text{for} \quad K = 0.$$

The isometry group G_4 is complete unless $D=0$, $B=0$ or V is constant. This follows since the metric invariant p is not constant unless these conditions are valid.

Consequently, the isometry group acts in the hypersurface p = constant, i.e. r = constant and is therefore of dimension at most six. Since $p+\mu \neq 0$ otherwise $D=0$ or $B=0$, the vector u^a is unique and hence the isotropy group is of dimension at most one and acts in the surface r = constant, t = constant. Consequently the complete isometry group is of dimension four. If V is constant but $p+\mu \neq 0$, the dimensions of the complete isometry and isotropy group are at most seven and three respectively since u^a is unique. In fact the groups are of dimension seven and three respectively since the line element (3.4.9.) becomes that of the Einstein Universe. If $p+\mu=0$, i.e. $D=0$ or $B=0$, the space-time is one of the de Sitter spaces or flat space and admits a ten-dimensional isometry group.

It may be noted from the above analysis, that if $\mu \geq 0$, i.e. $K \geq 0$, that the space is spherically symmetric. Consequently we have proved that all conformally flat static perfect fluid fields with non-negative density are spherically symmetrical.

If in equation (3.4.11.) the vector $\underline{a}(t)$ is given by $\underline{a}(t) = a(t)\ \underline{a}_0$ where $\underline{a}_0$ is a constant vector and $a(t)$ is proportional to $A(t)K - 4B(t)$ we can show

by the same argument used above that the space admits a three parameter isometry group and the solution is given by equations (3.4.21., 22.) where in equation (3.4.22.) A,B,C,D, and E_1 are arbitrary functions of t .

References

Barnes, A.; 1971, Preprint. Imperial College, London. Appendix II.

Ehlers, J.; 1961, Akad. Wiss. Mainz. Math. Nat. Kl. No. 11.

Eisenhart, L.P.;1949, Riemannian Geometry. (Princeton U.P.) p. 92-93.

Ellis, G.F.R.; 1967, Jour. Math. Phys. 8, 1171-94.

Goldberg, J.N.; Sachs, R.K.; 1962, Acta. Phys. Polon. 22,13.

Jordan, P., Ehlers, J., Kundt, W.; 1960, Akad. Wiss. Mainz. Math. Nat. Kl. No.2.

Kundt, W. and Trumper, M.; 1962, Akad. Wiss., Mainz. Math. Nat. Kl. No.12.

Levi-Civita, T.; 1917-19, Rend. Acc. Lincei,9 articles in Vols. 26-28.

Robertson, H.T., and Noonan, T.W.; 1968, Relativity and Cosmology. (W.B. Saunders) p.219-22, 325-27.

Stepanyuk, N.M.; 1968, Sov. Phys. JETP. 26, 369-70.

Thompson I.M. and Whitrow, G.J.; 1967, M.N.R.A.S. 136,207-17. 1968, M.N.R.A.S. 139,499-513.

Trumper, M.; 1962, Zeit. Phys. 168 ,55-60.

CHAPTER 4

FIELDS OF TYPE D

4.1 Introduction and the Basic Equations

In this chapter fields of type D which admit a shear-free normal congruence of flow lines will be considered and classified. The field equations are then solved for each class in turn and all solutions for fields of this type are found. In two classes the exact solutions cannot be found in closed form and one differential equation remains to be integrated. These classes consist of the general static and shear-free flows of fluid which posses spherical and related symmetries. The method of analysis is a generalisation of that used by Jordan, Ehlers and Kundt (1960) and Barnes (1971) (see Appendix II) to investigate static vacuum and static perfect fluid fields respectively.

The conservation equations (2.2.7.) becomes with the aid of equations (2.1.21., 22.)

$$p_{1\alpha} = - V^{-1} V_{1\alpha} (p+\mu) . \qquad (4.1.1.)$$

We may deduce from this equation that

$$p_{1[\alpha} V_{1\beta]} = \mu_{1[\alpha} V_{1\beta]} = 0. \qquad (4.1.2.)$$

If $V = V(t)$ it follows from equations (3.1.1., 17.) and the equation $\Pi_{\alpha\beta} = 0$ that $E_{\alpha\beta} = 0$, and consequently that the space is conformally flat. Hence $V \neq V(t)$ for fields of type D, and from equation (4.1.2.) it may be deduced that

$$p = p(V,t) \ , \quad \mu = \mu(V,t). \qquad (4.1.3.)$$

Since for type D fields $\alpha = \alpha_3 = -2\alpha_1 = -2\alpha_2$, it is easily seen that the integrability equations (3.2.10.) are equivalent to the following equations:-

$$\left.\begin{aligned} &(\alpha - \tfrac{1}{3}\mu)_{.3} + 3\alpha(\gamma_c - \log P)_{.3} = 0, \\ &(\alpha + \tfrac{1}{6}\mu)_{.c} + \tfrac{3}{2}\alpha(\gamma_3 + \log(V/P))_{.c} = 0, \\ &(\alpha - \tfrac{1}{3}\mu)_{.c} + 3\alpha(\log V)_{.c} = 0, \end{aligned}\right\} \qquad (4.1.4.)$$

where $c = 1, 2$, and the γ's are defined by equation (3.2.16.). It follows from equation (3.2.11.) that P may be chosen so that $P = \alpha^{1/3}$ whilst preserving the form of the metric (2.1.21.). With this choice of P, the integrability conditions (4.1.4.) may be written

$$\gamma_{c\cdot 3} = \tfrac{1}{9}\alpha^{-1}\mu_{\cdot 3}\,, \qquad (4.1.5.)$$

and

$$\gamma_{3\cdot c} = -2\left\{\log(\alpha^{1/3}V)\right\}_{\cdot c} = -\tfrac{2}{9}\alpha^{-1}\mu_{\cdot c}\,. \qquad (4.1.6.)$$

Since $\gamma_A = \gamma_A(x^1)$, it follows from the above two equations that

$$\left(\alpha^{-1}\mu_{1V}\right)' = 0\,, \qquad (4.1.7.)$$

$$V = \alpha^{-1/3} e^{-\frac{1}{2}\gamma_3 + Z(x^3,t)}\,, \qquad (4.1.8.)$$

$$\alpha = V^{-3}\left(\tfrac{1}{3}\int \frac{\partial\mu(V,t)}{\partial V} V^3 dV + Y(x^3,t)\right), \qquad (4.1.9.)$$

and

$$\gamma_c = V + \bar{\gamma}_c(x^1,x^2)\,, \qquad (4.1.10.)$$

where $\gamma_{13} = \frac{1}{9}\alpha^{-1}\mu_{13}$, and where Y and Z are arbitrary functions of integration. Consequently, the metric may be put in the form

$$G = \alpha^{-2/3}\left\{e^{2\gamma}d\sigma^2 + e^{2\gamma_3}(dx^3)^2 - e^{-\gamma_3+2Z}dt^2\right\}, \qquad (4.1.11.)$$

where $d\sigma^2$ is a two-dimensional metric involving x^1 and x^2 only.

The fields may be divided into four classes:-

I $\mu = \mu(t)$, i.e. $\mu_{|\alpha} = 0$.

II $\mu = \mu(x^3, t)$, i.e. $\mu_{|[\alpha} e_{3\beta]} = 0$, $\mu_{|\alpha} e_3^{\alpha} \neq 0$.

III $\mu = \mu(x^1, x^2, t)$, i.e. $\mu_{|[\alpha} e_{3\beta]} \neq 0$, $\mu_{|\alpha} e_3^{\alpha} = 0$.

IV $\mu = \mu(x^1, x^2, x^3, t)$, i.e. $\mu_{|[\alpha} e_{3\beta]} \neq 0$, $\mu_{|\alpha} e_3^{\alpha} \neq 0$.

This classification is invariant since e_3^{α} is uniquely determined whenever u^a is. The vector u^a need not be uniquely determined if $\mu + p = 0$, but this condition was shown in Section 2.2 to imply that $\mu =$ constant. Consequently all fields for which u^a is not unique belong to class I. This classification is a generalisation of that of static perfect fluid fields (Barnes 1971).

It follows from equation (2.1.22.) that the expansion Θ is given by

$$\Theta(t) = -(\alpha V)^{-1} \alpha^{1} . \qquad (4.1.11.)$$

We will now proceed to investigate the four classes defined above, starting with class I.

4.2 Fields of Class I (Uniform Density)

Since $\mu = \mu(t)$ the conservation equations (4.1.1.) may be integrated to obtain

$$p + \mu = 2A(t)/V . \qquad (4.2.1.)$$

It is easily seen from equations (4.1.8.-10.) that for this class

$$\gamma_c = \gamma_c(x^1, x^2) , \quad V = \alpha^{-1/3} \gamma^{1/3} , \quad \gamma_3 = \gamma_3(x^3) .$$

γ_3 may be set equal to zero by means of a co-ordinate transformation of the form $\tilde{x}^3 = \tilde{x}^3(x^3)$, and consequently $\gamma^{\frac{1}{3}} = e^Z$. The metric (4.1.11.) becomes

$$G = \alpha^{-2/3} \left(d\sigma^2 + (dx^3)^2 - e^{2Z} dt^2 \right) . \qquad (4.2.2.)$$

We may subdivide this class into four invariant sub-classes:- defined as follows:-

(A) the hypersurface α = constant contains the space-like eigenblade of the Weyl tensor, i.e. $\alpha = \alpha(x^3, t)$;

(B) the invariant α is constant;

(C) the hypersurface α = constant contains the time-like eigenblade of the Weyl tensor i.e. $\alpha = \alpha(x^1, x^2)$;

(D) the hypersurface contains neither eigenblade, i.e. $\alpha = \alpha(x^1, x^2, x^3, t)$.

Subclass A

The metric (4.2.2.) may be expressed as

$$G = W^2(z,t)\left(dz^2 + z^2 d\sigma^2\right) - V^2(z,t)\,dt^2, \qquad (4.2.3.)$$

by means of a co-ordinate transformation of the form $Z = Z(x^3)$. The field equations (3.1.2.,4.,8.) become on using equation (A.1.12.,13.)

$$\tfrac{1}{2}(\mu - p) = -\frac{1}{W^2}\left(\frac{1-K}{z^2} + \frac{W_{133}}{W} + \frac{2W_{13}}{Wz} + \left(\frac{W_{13}}{W} + \frac{1}{z}\right)\frac{V_{13}}{V}\right) + \frac{1}{V^2}\left(\frac{2W'^2}{W^2} + \frac{W''}{W} - \frac{W'V'}{WV}\right), \qquad (4.2.4.)$$

$$\tfrac{1}{2}(\mu - p) = -\frac{1}{W^2}\left(\frac{V_{133}}{V} - \frac{V_{13}W_{13}}{VW} + \frac{2W_{133}}{W} - \frac{2W_{13}^2}{W^2} + \frac{2W_{13}}{zW}\right) + \frac{1}{V^2}\left(\frac{2W'^2}{W^2} + \frac{W''}{W} - \frac{W'V'}{WV}\right), \qquad (4.2.5.)$$

$$\tfrac{1}{2}(\mu + 3p) = \frac{1}{W^2}\left(\frac{V_{133}}{V} + \frac{V_{13}W_{13}}{VW} + \frac{2V_{13}}{zV}\right) + \frac{3}{V^2}\left(\frac{W'V'}{WV} - \frac{W''}{W}\right), \qquad (4.2.6.)$$

$$0 = \left(\frac{3W'}{VW}\right)_{13}, \qquad (4.2.7.)$$

where $K = K(x^1, x^2)$ is the Gaussian curvature of $d\sigma^2$. Since all the functions except K in equation (4.2.4.) are independent of x^1 and x^2, it follows that K is a constant and $d\sigma^2$ is of constant curvature. By a suitable scale change of the co-ordinate z, K may be set equal to ± 1 or 0 and co-ordinates $x^1 = \theta$, $x^2 = \phi$ may be chosen so that

$d\sigma^2 = d\theta^2 + f^2(\theta)\, d\phi^2$ where $f(\theta) = \sin\theta, \theta, \sinh\theta$, for $K = +1, 0, -1$, respectively. The most important case from the physical point of view is $K = +1$. In this case the metric is spherically symmetrical. For $K = 0$ and $K = -1$ the field possesses planar and hyperbolic symmetry, respectively. In the present chapter we will consider only two special cases, namely those in which $\alpha = \alpha(z)$ and $\alpha = \alpha(t)$. The remaining fields of this subclass together with fields of class II (which are similar) will be considered in more detail in Chapter 5.

In the case $\alpha = \alpha(z)$ it follows from equation (4.1.11.) that $\theta(t) = 0$ and from equation (2.2.8.) that $\mu =$ constant. The metric (4.2.2) may be written, in terms of a curvature co-ordinate r defined by $r = \alpha^{-1/3}$, in the form

$$G = W^2(r)\, dr^2 + r^2 d\sigma^2 - V^2(r, t)\, dt^2. \qquad (4.2.8.)$$

The remaining field equations are easily integrated and one obtains the following expressions for W and V (Barnes, 1971, Appendix II)

$$W^{-2} = K - \tfrac{1}{3}\mu r^2 - 2mr^{-1}, \qquad (4.2.9.)$$

and

$$V = W\left(B(t) + A(t)\int r\left(K - \tfrac{1}{3}\mu r^2 - 2mr^{-1}\right)^{-3/2} dr\right). \qquad (4.2.10.)$$

The pressure p is given by equation (4.2.1.) and the invariant α by

$$\alpha = -2mr^{-3}. \qquad (4.2.11.)$$

If $m = 0$ then $\alpha = 0$ and the field is conformally flat. In this case V is given by

$$V = B(t)\sqrt{\left(K - \tfrac{1}{3}\mu r^2\right)} + A(t)/3\mu .$$

This solution has previously been obtained in Section 3.4.

For $K = 1$ and $m = 0$ the solution is the interior solution of Schwarzschild (1916) with variable pressure. This solution might represent a motionless Schwarzschild sphere onto which matter is accreting or being thrown off. The case $m \neq 0$ is usually ruled out as the metric becomes singular at $r = 0$. However the solution may not

be valid near the origin. A physical situation in which this would be the case is the case of an annular region of constant density surrounding a core with a different density distribution. If $A(t)$ is proportional to $B(t)$, V may be made independent of t by means of a scale change in t and the field is static. If $A = 0$, the field is a vacuum field and is the Schwarzschild exterior solution with cosmological constant or a related solution with planar or hyperbolic symmetry (Schwarzschild, 1916, Levi-Civita 1917-19).

If $\alpha = \alpha(t)$ it follows from equation (2.1.22.) that $V = V(t)$ or $\alpha' = 0$. In the former case the metric may be expressed in the form (3.1.12.) and the space-time admits a conformal Killing vector parallel to the fluid flow-vector. The analysis of Section 3.3 shows that the space-time is static (since we are considering only Types D space-times). Hence α = constant in both cases and so these fields belong to subclass B.

Subclass B

By means of scale changes in the co-ordinates the metric (4.2.2.) may be written

$$G = d\sigma^2 + dz^2 - e^{2Z(z,t)}dt^2. \qquad (4.2.12.)$$

With the aid of equations (A.1.12.,13.) it is easily seen

that the field equations (3.1.2.,8.) are

$$K(x^1, x^2) = -\tfrac{1}{2}(p - \mu),$$

$$Z_{133} + Z_{13}^2 = \tfrac{1}{2}(p - \mu) = \tfrac{1}{2}(\mu + 3p),$$

where K is the Gaussian curvature of $d\sigma^2$. It follows from these equations that $p + \mu = 0$ and $K = \mu$. Furthermore it follows from equation (A.1.14.) that $-(Z_{133} + Z_{13}^2)$ is the Gaussian curvature of $d\sigma_2^2 = dz^2 - e^{2z}dt^2$. This metric is hence of constant curvature K and co-ordinates may be chosen so that the metric (4.2.12.) becomes

$$G = I(\mu)\left(d\theta^2 + f^2(\theta)d\phi^2 + dz^2 - f^2(z)dt^2\right), \qquad (4.2.13.)$$

where $f(\theta) = \sin\theta, \theta, \sinh\theta$ and $I(\mu) = \mu^{-1}, 1, -\mu^{-1}$ for $\mu > 0, = 0$ and < 0 respectively. The invariant α is given by $\alpha = -\frac{2}{3}\mu$, and for $\mu = 0$ the space-time is flat. The other two solutions are static and possess a complete six-parameter group of local isometrics. The completeness follows from the fact that the maximal group of a Type D field is of dimension 6.

Subclasses C and D

It is convenient to consider these two cases together.

We will introduce a spatial metric $\overline{\overline{G}}$ where $\overline{\overline{G}} = \alpha^{2/3}\tilde{G}$, with $\tilde{G} = \alpha^{-2/3}\left(d\sigma^2 + (dx^3)^2\right)$, so that $\overline{\overline{G}} = d\sigma^2 + (dx^3)^2$. The Ricci tensor $\overline{\overline{R}}_{ab}$ of the metric $\overline{\overline{G}}$ is given by (see equation A.1.10).

$$\hat{\overline{\overline{R}}}_{\alpha\beta} = \overline{\overline{R}}_{\alpha\beta} + \alpha^{-1/3}\left(\alpha^{1/3}\right)_{\|\alpha\beta} - \alpha^{1/3}\left(\alpha^{-\frac{1}{3}}\right)^{\|\gamma}_{\|\gamma}\overline{\overline{g}}_{\alpha\beta}, \qquad (4.2.14.)$$

where covariant derivatives are made with the metric $\overline{\overline{G}}$. On using equations (3.1.2.) and (A.1.9) it may be deduced that the field equations (3.1.8.) are equivalent to the following equations

$$\overline{\overline{R}}_{\alpha\beta} + 2\alpha^{-1/3}\left(\alpha^{1/3}\right)_{\|\alpha\beta} - e^{-Z}\left(e^{Z}\right)_{\|\alpha\beta}$$

$$= -\left[2p\alpha^{-2/3} + 2\left\{\left(\frac{\alpha'}{3\alpha}\right)^2 - \frac{2\alpha' Z'}{3\alpha}\right\}e^{-2Z} - \alpha^{2/3}e^{-Z}\left(\alpha^{-2/3}e^{Z}\right)^{\|\gamma}_{\|\gamma}\right]\overline{\overline{g}}_{\alpha\beta}. \qquad (4.2.15.)$$

It is easily seen from equation (A.1.15.) that, owing to the special form of the metric $\overline{\overline{G}}$, second covariant derivatives involving the index 3 may be replaced by ordinary derivatives and those involving only the indices 1 and 2 may be replaced by their respective derivatives with respect to the two-dimensional metric $d\sigma^2$.

Furthermore from equation (A.1.13.) it follows that

$$\bar{\bar{R}}_{ab} = K \bar{\bar{g}}_{ab} \quad , \quad \bar{\bar{R}}_{3a} = \bar{\bar{R}}_{33} = 0 \quad , \qquad (4.2.16)$$

where a and b take the values 1 and 2 only and K is the Gaussian curvature of $d\sigma^2$.

From the equations (4.2.15.,16.) with $\alpha = a$ and $\beta = 3$ it may be deduced that $(\alpha^{1/3})_{|a3} = 0$. Hence

$$\pm \alpha^{1/3} = X(x^1, x^2, t) + Y(x^3, t) .$$

From equation (4.1.11.) it may be deduced, since $V = \alpha^{-1/3} e^Z$ that $X_{|a4} = 0$ and hence $X = X(x^1, x^2) + f(t)$. By redefining the function Y we see that

$$\pm \alpha^{1/3} = X(x^1, x^2) + Y(x^3, t) . \qquad (4.2.17.)$$

We may deduce from equation (4.2.15.-17.) with $\alpha = a$ and $\beta = b$ that $X_{\|ab}$ is proportional to $\bar{\bar{g}}_{ab}$. Consequently,

$$X_{\|ab} = \tfrac{1}{2} X^{\|c}{}_{\|c} \bar{\bar{g}}_{ab} . \qquad (4.2.18.)$$

It may be deduced from this equation that the metric $d\sigma^2$ is of the form (Jordan, Ehlers and Kundt, 1960)

$$d\sigma^2 = \left(f(x)\right)^{-1} dx^2 + f(x)\, d\phi^2 .$$

Thus the space-time metric is of the form

$$G = \left(X + Y(x^3,t)\right)^{-2}\left\{f^{-1}dx^2 + f\,d\phi^2 + (dx^3)^2 - e^{2Z(x^3,t)}dt^2\right\}. \quad (4.2.19.)$$

Subclass C

For this class $\alpha = \alpha(x^1,x^2)$ and consequently Y = constant. This constant may be set equal to zero by means of a translation of the co-ordinate X. From equations (3.1.2.,8.) it may be deduced that

$$V^{-1}V^{1a}{}_{|1a} + \tilde{R}^3{}_3 = \mu + p,$$

where a = 1, 2. On using equations (A.1.15.,16.) with $V = X^{-1}$ and the equation $V = X^{-1}e^{Z}$, it is easily seen that $\mu + p = 0$.

It is unnecessary to integrate the remaining field equations, since by means of the complex co-ordinate transformations:- $x = X^{-1}$, $\tilde{\phi} = it$ $\tilde{t} = i\phi$ the metric (4.2.19.) may be put in the form of (4.2.8.). Since $\mu + p = 0$ the field equations are mapped to the corresponding field equations of subclass A with $\mu + p = 0$. Consequently, it may be deduced from equations (4.2.1.,9.,10.) that the solution is

$$G = \left(K - \tfrac{1}{3}\mu x^2 - 2mx^{-1}\right)^{-1}dx^2 + \left(K - \tfrac{1}{3}\mu x^2 - 2mx^{-1}\right)d\phi^2 + x^2\left(dz^2 - f^2(z)\,dt^2\right), \quad (4.2.19.)$$

where $f(z) = \sin z$, z , $\sinh z$ for $K = +1, 0, -1$ respectively. The invariant α is given by $\alpha = -2m x^{-3}$. The time-like eigenblade of the Weyl tensor is of constant curvature K and the solution is static. The solution possesses a complete four-dimensional isometry group and a one-dimensional isotropy group and generalises that obtained by Jordan, Ehlers and Kundt (1960) to include vacuum solutions with non-zero cosmological constant.

Subclass D

Only three of the field equations remain to be satisfied:- the (3.3) component, the trace of equation (4.2.15.), and Raychaudhuri's equation (3.1.2). With the aid of equations (A.1.1.-7.) these three equations may be shown to be equivalent to

$$(x+y)\left(Z_{133} + Z_{13}^2 - \tfrac{1}{2} f''(x)\right) - 2Y_{133} + f'(x) = 0, \qquad (4.2.20.)$$

$$(x+y)^2\left(Z_{133} + Z_{13}^2\right) - 2(x+y)\left(Y_{133} + Y_{13}Z_{13} + \tfrac{1}{2} f'(x)\right)$$

$$+ 3\left(Y_{13}^2 + f(x)\right) = p + \left(Y_{14}^2 + 2(x+y)Y_{14}Z_{14}\right)e^{-2Z}, \qquad (4.2.21.)$$

and

$$\left(Y_{133} - Y_{13}Z_{13} - Y_{14}Z_{14}e^{-2Z}\right)(x+y) = \tfrac{1}{2}(\mu + p) - Y_{14}^2 e^{-2Z}, \qquad (4.2.22.)$$

respectively. From the equations $V=(X+Y)^{-1}e^{Z}$ and (4.2.1.,22.) it may be deduced by equating coefficients of X that

$$Y_{133}-Y_{13}Z_{13}-Y_{14}Z_{14}e^{-2Z}=A(t)e^{-Z}, \qquad (4.2.23.)$$

and

$$Y_{14}^{2}e^{-2Z}=0.$$

Consequently $Y_{14}=0$ and α is independent of t. The flow-lines are therefore rigid, and since $\Theta(t)=0$, the density μ is constant.

Equation (4.2.20.) may be rewritten as

$$f''(x)Y-2(Z_{133}+Z_{13}^{2})X=-f''x+2f'-4Y_{133}+2(Z_{133}+Z_{13}^{2})Y. \qquad (4.2.24.)$$

The right-hand side of this equation is the sum of a function of X, $h(x)$ say, and a function of x^3 and t. On differentiating equation (4.2.24.) twice with respect to X it follows that

$f^{iv}-h''=0$, and hence

$$f=ax^{3}+bx^{2}+cx+d, \qquad h=2(bx+c),$$

where a, b, c and d are constants. On equating coefficients of X in equation (4.2.24.), we obtain

$$Z_{133}+Z_{13}^{2}-3aY+b=0, \qquad 2Y_{133}-(Z_{133}+Z_{13}^{2})Y-c=0.$$

Consequently,

$$2Y_{133}=3aY^{2}-2bY+c,$$

and hence

$$Y_{13}^2 = aY^3 - bY^2 + cY + \tilde{d} = g(Y),$$

where $\tilde{d}$ is a constant.

If a co-ordinate transformation is made so that x^3 is replaced by Y, the metric becomes

$$G = (X+Y)^{-2}\left(f^{-1}dx^2 + f\,d\phi^2 + g^{-1}dY^2 - e^{2Z(Y,t)}dt^2\right), \quad (4.2.25.)$$

where $f = f(X)$ and $g = g(Y)$. Equation (4.2.23.) becomes

$$\tfrac{1}{2}g'(Y) - g(Y)\frac{\partial Z}{\partial Y} = Ae^{-Z},$$

which on integration gives

$$e^Z = \sqrt{g}\left(B(t) + A(t)\int g^{-3/2}dY\right). \quad (4.2.26)$$

On substituting equations (4.2.23.,26.) in (4.2.21.) we obtain

$$d + \tilde{d} + \tfrac{1}{3}\mu = 0,$$

and hence

$$g(X) = -f(-X) - \tfrac{1}{3}\mu. \quad (4.2.27.)$$

The invariant α is given by

$$\alpha = a(X+Y)^3,$$

and the pressure p by

$$p = -\mu + 2A(t)(X+Y)e^{-Z}. \qquad (4.2.28.)$$

If $a = o$ the space-time is conformally flat. If $a \neq o$ we may set $a = \pm 1$ and $b = o$ by means of the affine co-ordinate transformations $\tilde{X} = |a|^{1/3}(X + \frac{1}{3}b/a)$, $\tilde{Y} = |a|^{1/3}(Y - \frac{1}{3}b/a)$. The form of the equations (4.2.25.-29.) is preserved and f and α are now given by

$$f(X) = \pm X^3 + cX + d, \qquad \alpha = \pm (X+Y)^3. \qquad (4.2.29.)$$

Thus the solution for subclass D is given by equations (4.2.25.-29.) and it possesses a complete one or two-parameter group of isometries. The completeness of this group follows from the fact that there exist two functionally independent invariants:

$$\alpha = \pm (X+Y)^3, \qquad Y^{-2} Y_{ij} Y^{ij} = f(X) + g(Y),$$

where $Y = \alpha^{1/3}$. Consequently the trajectories of the group are the two-dimensional surface X = constant, Y = constant and the group is of dimension at most three. If it is of dimension three then there must

exist a one-dimensional isotropy group, but this is impossible since for a type D field the isotropy group acts in either the (x,ϕ) - or the (y,t) - surface.

This solution is a generalisation of the static vlume solution of Jordan, Ehlers and Kundt (1960) to the case of perfect fluid of uniform density with rigid normal flow-lines. If $A(t)$ is proportional to $B(t)$ then the field is static and V may be made independent of t by means of a scale change of the form $\tilde{t} = \tilde{t}(t)$. In this case the isometry group is of dimension two and abelian, otherwise the group is of dimension one. In particular if $A = 0$, i.e. $p + \mu = 0$, the field is static and $e^{Z} = \sqrt{g}$.

Vacuum Solutions

We note that in subclasses B, C, D those solutions with $p + \mu = 0$, i.e. vacuum solutions with cosmological constant, are all static. We shall now investigate fields of subclass A to see if they also possess this property. By means of co-ordinate transformations of the form $R = z\,W(z,t)$ and $T = T(z,t)$ the metric (4.2.3.) may be written as

$$G = A^2(R,T)\,dR^2 + R^2 d\sigma^2 - B^2(R,T)\,dT^2. \quad (4.2.30.)$$

In fact the most general metric possessing hyperbolic spherical or planar symmetry may be expressed in this form. We may now use the analysis of Tolman (1934) to prove that such fields with $p+\mu=0$ are static. In this proof Tolman only considers spherically symmetrical fields but in fact the relevant equations are not altered if $d\sigma^2$ has constant negative or zero curvature and so the proof generalises trivially.

Consequently we have proved the result that all type D vacuum fields admitting a time-like normal shear-free congruence of curves are static. If we recall the results of Section 3.4 and Theorem 3.5 we see that the following theorem is valid.

Theorem 4.1

A vacuum space-time which possesses spherical, hyperbolic or planar symmetry, or which admits a normal shear-free congruence of time-like curves, is static.

This theorem generalises those of Birkhoff (1927) and Trumper (1962) which applied to spherically symmetrical and non-degenerate space-times respectively. It should be noted that the theorem does not imply that any normal shear-free congruence is a static congruence but merely that among all such congruences there exists at least one that is static. It is clear from the analysis of

Jordan, Elhers and Kundt (1960) that the static congruence is unique only in the non-degenerate case and in classes A and D. In classes B and C and the conformally flat case there exist a time-like isotropy group and so the static congruence is not unique.

4.3 Fields of Classes II-IV (Non-Uniform Density)

Class II

In this class $\mu = \mu(x^3, t)$ with $\mu_{,3} \neq 0$ and from equation (4.1.3.) we may deduce that $V = V(x^3, t)$, $p = p(x^3, t)$. From equations (4.1.5.,6.) it follows on using the above result that $\alpha = \alpha(x^3, t)$, $\gamma_3 = \gamma_3(x^3)$ and $\gamma_\alpha = \gamma(x^3) + \bar{\gamma}_\alpha(x^1, x^2)$. Consequently, the metric (4.1.11.) may be expressed, after a suitable co-ordinate transformation of the form $z = z(x^3)$, in a form identical with equation (4.2.3.). The analysis used in class IA may be used to show that $d\sigma^2$ is of constant curvature. The general fields of this class will be considered in more detail in Chapter 5 and only fields for which $\alpha = \alpha(z)$ will be considered here. Since $\alpha = \alpha(z)$ it follows from equations (4.1.11.) that the expansion of the flow-lines vanishes and from equation (2.2.8.) that $\mu = \mu(z)$. In terms of curvature co-ordinates the metric takes the same form as in equation (4.2.8.) and the field equations

are easily integrated to give

$$W^{-2} = K - r^{-1}\int \mu(r)\, r^2 dr - 2mr^{-1},$$

$$\alpha = \tfrac{1}{3} r^{-3}\int \mu'(r)\, r^3 dr - 2mr^{-3}.$$

In general the remaining field equation cannot be integrated exactly. The spherically symmetrical case has been considered by a large number of authors (Tolman (1939), Bondi (1964), Harrison et. al. (1965) and references therein) and will not be considered further.

Classes III and IV

It is convenient to consider these two classes together. For both classes the condition $\mu_{1c} \neq 0$ for $c = 1,2$ is valid and for class III the condition $\mu_{13} = 0$ is also satisfied. It follows from equation (4.1.2.) that $V_{1c} \neq 0$. Since the co-ordinate lines in equation (4.1.11.) are Ricci eigendirections, it follows that $\tilde{R}^3_1 = \tilde{R}^3_2 = 0$, where $\tilde{R}_{\mu\nu}$ is the Ricci tensor of the spatial metric, and consequently from equation (3.1.8.) that $V^{13}_{11c} = 0$. On using equation (A.1.1.,6.) we may deduce that

$$\left(V_{13} e^{-\delta}\right)_{1c} - \beta_{13} V_{1c} e^{-\delta} = 0, \qquad (4.3.1.)$$

and

$$\left(\beta_{13}e^{-\delta}\right)_{1c} = 0, \qquad (4.3.2.)$$

where $c = 1, 2$ and for simplicity of writing we have written

$$e^{\beta} = \alpha^{-1/3}e^{\gamma}, \quad e^{\delta} = \alpha^{-1/3}e^{\gamma_3}, \quad V = \alpha^{-1/3}e^{-\frac{1}{2}\gamma_3 + Z}. \qquad (4.3.3.)$$

From equation (4.1.2.,6.) it follows that

$$\gamma_{3|[c}V_{|b]} = \alpha_{|[c}V_{|b]} = 0, \qquad (4.3.4.)$$

for $b, c = 1, 2$. From equation (4.3.1.) it is easily seen that $V_{|3[1}V_{|2]} = 0$ and hence $\gamma_{3|3[1}\gamma_{3|2]} = 0$ where the last '3' is omitted from the antisymmetrisation. On integrating this equation we obtain $\gamma_3 = \gamma_3\left(f(x^1, x^2), x^3\right)$. By means of the co-ordinate transformation $\tilde{x}^1 = X = f(x^1, x^2)$ and writing $z = x^3$ we see that $\gamma_3 = \gamma_3(X, z)$. It follows immediately from equations (4.3.4.) and (4.1.2.) that $V = V(X, z, t)$, $\gamma = \gamma(X, z)$, $\alpha = \alpha(X, z, t)$ and $\mu = \mu(X, z, t)$. Furthermore it may be deduced from equations (4.3.1.,2.) that

$$\beta_{13}e^{-\delta} = h(z, t), \qquad (4.3.5.)$$

$$V_{13}e^{-\delta} = h(z,t)\,V + l(z,t), \qquad (4.3.6.)$$

where h and l are functions of integration.

If the co-ordinate $x^2 = \phi$ is chosen so that the x^1- and x^2- lines are orthogonal the metric (4.1.11.) takes the form

$$G = e^{2\beta}\left(A^2(x,\phi)\,dx^2 + B^2(x,\phi)\,d\phi^2\right) + e^{2\delta}dz^2 - V^2dt^2.$$

As the metric is of type D it follows from equations (3.1.8.,15.) that $\tilde{R}^{\mu}{}_{\nu}$ and $V^{1\mu}{}_{11\nu}$ are proportional to $\delta^{\mu}{}_{\nu}$ for $\mu, \nu = 1,2$. Hence it follows that

$$A^{-3}A_{12}V_{11}e^{-2\beta} = 0,$$

and

$$\left(V_{11}e^{-2\beta}\right)_{11} - \left(\frac{A_{11}}{A} + \frac{B_{11}}{B}\right)V_{11}e^{-2\beta} = 0. \qquad (4.3.7.)$$

From these two equations it may be deduced that $A_{12} = 0$ and $B = \tilde{B}(x)\,g(\phi)$. By means of a co-ordinate transformation of the form $\tilde{\phi} = \tilde{\phi}(\phi)$ and $\tilde{x} = \tilde{x}(x)$, we may set $B = B(x) = A(x)$, and on redefining $\beta = \beta + \log A$, we see that the metric is

of the form

$$G = e^{2\beta}(dx^2 + d\phi^2) + e^{2\delta}dz^2 - V^2dt^2. \qquad (4.3.8.)$$

The equations $\tilde{R}^1_1 = \tilde{R}^2_2$ and (4.3.7.) with the aid of equation (A.1.1.,5.) are equivalent to

$$\left(V_{11}e^{-2\beta}\right)_{11} = \left(\delta_{11}e^{\delta-2\beta}\right)_{11} = 0.$$

Hence on intregation we obtain

$$V_{11}e^{-2\beta} = f(z,t), \quad \delta_{11}e^{\delta-2\beta} = g(z,t). \qquad (4.3.9.)$$

Since $f \neq 0$, for otherwise $V_{11} = 0$, it follows from equation (4.3.9.) that

$$e^{\delta} = m(z,t)V + k(z,t), \qquad (4.3.10.)$$

where $m = g/f$ and k is a function of integration.

If β is eliminated from (4.3.1.) with the aid of equations (4.1.5.,6.) we obtain

$$V_{131} - V^{-1}V_{13}V_{11} - \tfrac{1}{3}\left(\delta_{13}V_{11} + 2\delta_{11}V_{13} - 2Z_{13}V_{11}\right) - \tfrac{1}{9}\alpha^{-1}\left(\mu_{11}V_{13} - \mu_{13}V_{11}\right) = 0.$$

On eliminating μ, V_{13} and δ from this equation by using equations (4.3.3.,5.,10.) and (4.1.2.) we obtain the

following expression

$$V_{11}\left\{\tfrac{1}{3}V(2Z_{13}m - m_{13} - 3l^2 m) + \tfrac{1}{3}(2Z_{13}k - k_{13} - 6ml(k) - lk^2 V^{-1}\right\} = 0.$$

Since $V_{11} \neq 0$, it follows from this equation that

$$3l^2 m + m_{13} - 2Z_{13}m = 0, \quad k_{13} - 2Z_{13}k = 0, \quad lk^2 = 0.$$

Hence either

$$k = 0, \quad 3l^2 m - 2Z_{13}m + m_{13} = 0,$$

or

$$l = 0, \quad m_{13} = 2Z_{13}m, \quad k_{13} = 2Z_{13}k. \qquad (4.3.11.)$$

In the former case it is easily seen that equations (4.1.6.) and (4.3.10.) imply that $Y_{31} = \mu_{1} = 0$. These fields do not belong to classes III and IV but to classes I and II and consequently equations (4.3.11.) are valid. Since $l = 0$, it follows from equations (4.3.5.,6.) that $V^{-1}V_{13} = \beta_{13}$, and hence

$$V^2 = H(x,t)\,e^{2\beta}, \qquad (4.3.12.)$$

or equivalently $e^{-\gamma_3 - 2\gamma} = H(x,t)\,e^{2Z(z,t)}$. Since the left-hand side of this equation is independent of t we may deduce that $e^{Z} = C(t)\,e^{\tilde{Z}(z)}$. Consequently the metric (4.1.11.) admits a conformal Killing vector and from

Section 3.3 it follows that the field is static. We may choose the t- co-ordinate so that e^Z and all the functions in equations (4.3.5.- 12.) are independent of t . Furthermore it follows from equations (4.3.10.11) that, after a suitable scale change in the z-co-ordinate, we may write $e^{\delta} = mV + k$, where m and k are constants.

From equations (4.3.6.,10.) with $\lambda = 0$ we see that

$$V = k\left(G(x)\, n(z) - m\right)^{-1}, \qquad (4.3.13\ .)$$

where G is a function of integration and $n(z) = e^{\int k h(z)\, dz}$. From equations (4.3.9.,12 ,.13 .) it follows that

$$H(x)\, f(z) = -k\, G'(x)\, n(z),$$

and hence $H(x)$ is proportional to $G'(x)$. If the co-ordinate transformation $x = -(G(x))^{-1}$ and suitable scale changes in the other co-ordinates are made the metric may be written in the form

$$G = \left(n(z) + mx\right)^{-2}\left(F^{-1}dx^2 + F d\phi^2 + dz^2 - x^2 dt^2\right), \qquad (4.3.14\ .)$$

where $F = F(x)$.

The field equations (3.1.2.,8.) that are not identically

satisfied are found with the aid of equation (A.1.1,5,7,4.) to be:-

$$\tfrac{1}{2}(p-\mu) = \tfrac{1}{2}(n+mx)F'' + \tfrac{1}{2}(n+mx)(n-3mx)F'/x + m(2mx-n)F/x - (n+mx)n'' + 3n'^2, \qquad (4.3.15\ .)$$

$$\tfrac{1}{2}(p-\mu) = -m(n+mx)F' + m(2mx-n)F/x - 3n''(n+mx) + 3n'^2, \qquad (4.3.16\ .)$$

$$\tfrac{1}{2}(\mu+3p) = n(n+mx)F'/x - 3mnF/x - n''(n+mx) + 3n'^2. \qquad (4.3.17\ .)$$

On subtracting (4.3.16 .) from (4.3.15.) we obtain

$$4n'' + n(F''+F'/x) + m(F''x - F') = 0. \qquad (4.3.18\ .)$$

On differentiating this equation with respect to z, it may be deduced that either $n'(z) = 0$, or $F''+F'/x = 4a$. where a is a constant. In the first case the metric (4.3.14 .) is independent of z and consequently so is μ, and the field belongs to class III.

Class III

Putting $n' = 0$ in equation (4.3.18 .) we obtain

$$F''(n+mx) + (n-mx)F'/x = 0.$$

On integrating we find

$$F = b + c\left(n^2 \log x + 2mnx + \tfrac{1}{2}m^2x^2\right), \qquad (4.3._{19})$$

where b and c are constants. The pressure and density are given by

$$2p = cx^{-2}(n+mx)^3(n-mx) + 2m(m-2n/x)F,$$

$$2\mu = cx^{-2}(n+mx)^3(n+3mx) - 6m^2F,$$

and the invariant α by

$$\alpha = -\tfrac{1}{3}ncx^{-2}(n+mx)^3.$$

Consequently, for $m = o$ we obtain

$$p = \mu = \tfrac{1}{2}n^4cx^{-2}, \quad \alpha = -\tfrac{1}{3}n^4cx^{-2}, \quad F = b + n^2c\log x. \qquad (4.3._{20}.)$$

If $m \neq o$ we may set $m = \pm 1$, by means of the transformations

$$\tilde{x} = |m|x, \quad \tilde{\phi} = \phi/|m|, \quad \tilde{t} = t/|m|, \quad \tilde{F} = m^2F,$$

whilst preserving the form of the metric. Equation $(4.3._{19}.)$ becomes

$$\tilde{F} = \tilde{b} + \tilde{c}\left(n^2\log\tilde{x} \pm 2n\tilde{x} + \tfrac{1}{2}\tilde{x}^2\right), \qquad (4.3._{21}.)$$

where $\tilde{c} = m^2c$, $\tilde{b} = m^2\left(b + cn^2\log|m|\right)$.

Omitting the tildes, the pressure, density and the invariant α are given by:-

$$\left.\begin{aligned} 2p &= cx^{-2}(n\pm x)^3(n\mp x) + 2(1\mp 2n/x)F, \\ 2\mu &= cx^{-2}(n\pm x)^3(n\pm 3x) - 6F, \\ \alpha &= -\tfrac{1}{3}nc\,x^{-2}(n\pm x)^3 . \end{aligned}\right\} \qquad (4.3.22.)$$

Clearly the space is conformally flat if $nc=0$. If $nc\neq 0$ the metric admits a three-dimensional abelian local isometry group. This group is complete since the trajectories of the complete group lie in the hypersurface X = constant. The isotropy group is of dimension at most one since the metric is type D and lies in the surface X = constant, ϕ = constant. Since the vector u^a is unique as $p+\mu\neq 0$, it follows that the isotropy group is discrete and therefore the complete isometry group is of dimension three.

Class IV

In this case, since $n'(z)\neq 0$ the equation $F''+F'/x=4a$ is valid. On integration we obtain

$$F = ax^2 + b\log x + c, \qquad (4.3.23.)$$

where b and c are constants. It may be deduced from

equation (4.3.18.) that

$$4(n'' + an) = d, \qquad m(xF'' - F') = d, \tag{4.3.24.}$$

where d is a constant. On using equations (4.3.23.,24.) we obtain the equations d = o and mb = o. On integrating the first of equation (4.3.24.) with d = o we obtain

$$n = A f(z), \text{ where } f(z) = \sin(\sqrt{a}z), z, \sinh(\sqrt{-a}z), \text{ for } a > 0, = 0, < 0, \tag{4.3.25.}$$

respectively. The second constant of integration has been set equal to zero by means of a translation of z.

The invariant α is given by

$$\alpha = -\tfrac{1}{3} b n^2/x^2,$$

as can be shown by using the equations mb = o and (A.1.,12.,13.). If the space-time is not conformally flat, $b \neq o$ and and consequently m = o. Thus the metric (4.3.14.) is of the form

$$G = n^{-2}\left(F^{-1}dx^2 + F d\phi^2 + dz^2 - x^2 dt^2\right), \tag{4.3.26.}$$

where F and n are given by equations (4.3.23.,25.). From equations (4.3.15.-17.) we obtain the following expressions for the pressure p and density μ :-

$$2p = 6A^2 I(a) + bn^2/x^2, \quad 2\mu = -6A^2 I(a) + bn^2/x^2,$$

where $I(a) = |a|, 1$, for $a \neq 0$, $a=0$, respectively. The matter satisfies the equation of state $p = \mu + 6A^2 I(a)$.

A second invariant (functionally independent of α) is given by

$$Y = \tfrac{1}{4}\alpha^{-2}\alpha_{|i}\alpha^{|i} = A^2 I(a) + (b\log x + c)n^2/x^2.$$

We may use arguments similar to those used in discussing class I D to show that the complete local isometry group is of dimension two and abelian.

4.4 Summary of Chapters 3 and 4

In Chapters 3 and 4 normal, shear-free flows of a perfect fluid have been considered. All the flows were shown to be of Petrov type I and the non-degenerate fields are all static. In Section 3.4 all conformally-flat perfect fluid solutions were obtained and the space cross-sections shown to be of uniform curvature. The static solutions admit at least a four parameter group of local isometries. In Chapter 4 fields of type D are considered and the exact solutions found for all cases except class IA and II where one differential equation remains to be solved.

From the results of Chapters 3 and 4 it may be deduced that all Einstein spaces (vacuum fields with a cosmological constant) admitting a shear-free normal congruence are static. All space-times of classes IB and C are Einstein spaces. Amongst the true perfect fluid flows, i.e. $p+\mu \neq 0$, those of classes III and IV are static whilst those of class ID are rigid. Therefore only in classes IA and II can the expansion of the flow-lines be non-zero. These fields are considered further in Section 3 of Chapter 5. All rigid and static solutions of type D are displayed below in Table I. The dimensions of the complete local isometry and isotropy groups are given in columns 5 and 6 respectively. If two figures appear in column 5 the first corresponds to rigid flows and the second to static flows. If only one figure occurs the flow is static. In column 6, the S or T denotes that the isotropy group acts in the space-like or time-like eigen-surface of the Weyl tensor respectively.

TABLE I.

Class	Metric	α	Pressure and Density	5	6
IA	$G = r^2(d\theta^2 + f^2(\theta)d\phi^2) + [F(r)]^{-1}dr^2 - F(r)\left(B(t) + A(t)\int r\,F^{-3/2}dr\right)^2 dt^2$, $F(r) = K - \frac{1}{3}\mu r^2 - 2m/r$, $f(\theta) = \sin\theta, \theta, \sinh\theta$, for $K = +1, 0, -1$.	$-2m/r^3$	$p = -\mu + 2AF^{-1/2}\left(B + A\int F^{-3/2}dr\right)^{-1}$	3, 4.	1 S.
IB	$G = I(\mu)\{d\theta^2 + f^2(\theta)d\phi^2 + dz^2 - f^2(z)dt^2\}$, $f(\theta) = \sin\theta, \theta, \sinh\theta$, and $I(\mu) = \mu^{-1}, 1, -\mu^{-1}$, for $\mu > 0, = 0, < 0$.	$-\frac{2}{3}\mu$	$p = -\mu$	6.	2 S, T.
IC	$G = \left(K - \frac{1}{3}\mu x^2 - 2mx^{-1}\right)^{-1}dx^2 + \left(K - \frac{1}{3}\mu x^2 - 2mx^{-1}\right)d\phi^2 + x^2(dz^2 - f^2(z)dt^2)$, with f as in Class IA.	$-2mx^{-3}$	$p = -\mu$	4.	1 T.
ID	$G = (X+Y)^{-2}\{(f(X))^{-1}dX^2 + f(X)d\phi^2 + (g(Y))^{-1}dY^2 - e^{2Z}dt^2\}$, $f(X) = \pm X^3 + cX + d$, $g(X) = -f(-X) - \mu$, $e^Z = g^{\frac{1}{2}}(Y)\left(B(t) + A(t)\int g^{-3/2}dY\right)$.	$\pm(X+Y)^3$	$p = -\mu + 2A(X+Y)e^{-Z}$	1, 2.	0.
II	$G = r^2(d\theta^2 + f^2(\theta)d\phi^2) + \left(K - r^{-1}\int\mu(r)r^2dr - 2mr^{-1}\right)^{-1}dr^2 - e^{2U(r,t)}dt^2$, with $f(\theta)$ as in Class IA and U given by equations (4.2.4.-6).	$\frac{1}{3}r^{-3}\int\mu'(r)r^3dr$ $-2mr^{-3}$	$p = p(r, t)$, $\mu = \mu(r)$	3, 4.	1 S.
III	$G = (n+mX)^{-2}\{(F(X))^{-1}dX^2 + F(X)d\phi^2 + dz^2 - X^2dt^2\}$, with $m = \pm1, 0$, $F(X) = b + c(n^2\log X + 2mnX + \frac{1}{2}m^2X^2)$ and $X \geq 0$.	$-\frac{1}{3}ncX^{-2}(n+mX)^3$	$2p = cX^{-2}(n+mX)^3(n-mX) + 2m(n-2nX^{-1})F$, $2\mu = cX^{-2}(n+mX)^3(n+3mX) - 6m^2F$	3.	0.
IV	$G = [n(z)]^{-2}\{[F(X)]^{-1}dX^2 + F(X)d\phi^2 + dz^2 - X^2dt^2\}$, $F(X) = aX^2 + b\log X + c$, $n(z) = A\sin(\sqrt{a}z)$, Az, $A\sinh(\sqrt{(-a)}z)$, for $a > 0, = 0, < 0$.	$-\frac{1}{3}bn^2X^{-2}$	$2p = 6A^2I(a) + bn^2X^{-2}$, $2\mu = -6A^2I(a) + bn^2X^{-2}$, $I(a) = \lvert a\rvert, 1$ for $a \neq 0, = 0$.	2.	0.

References

Barnes, A., 1971, Preprint Imperial College London, Appendix II.

Birkhoff, G.D., 1927, Relativity and Modern Physics (C.U.P.) p.253.

Bondi, H., 1964, Proc. Roy. Soc. A 282, 303-17.

Harrison, B.K., Thorne, K.S. Wakano, M., and Wheeler, J.A., 1965, Gravitation Theory and Gravitational Collapse (Univ. Chicago Press).

Jordan, P., Ehlers, J., and Kundt, W., 1960, Akad. Wiss. Mainz, Math-Nat Kl. No.2.

Levi-Civita, T., 1917-19, Rend. Acc. Lincei, 9 arts in Vols 26-28

Schwarzschild, K., 1916, Sitz Preuss. Akad. Wiss 189, 424.

Tolman, R.C., 1934, Relativity, Thermodynamics and Cosmology (Clarendon Press, Oxford).

Tolman, R.C. 1939, Phys. Rev. 55, 364-73.

Trumper, M., 1962, Zeit. Phys. 168, 55-60.

CHAPTER 5

SPHERICAL, PLANAR, AND HYPERBOLIC SYMMETRY.

5.1 Introduction

In this Chapter perfect fluid fields which possess a three-dimensional isometry group that acts on a two-dimensional space-like surface will be considered. Since the group is three dimensional and acts on two-dimensional surfaces, these surfaces must be of constant curvature K (Robertson and Noonan, 1968). Consequently the field possesses spherical, planar or hyperbolic symmetry for $K = +1, 0, -1$ respectively. Co-ordinates (r, θ, ϕ, t) may be chosen so that the metric is of the form

$$G = e^{\lambda(r,t)} dr^2 + e^{\eta(r,t)} \left(d\theta^2 + f^2(\theta) d\phi^2 \right) - e^{\nu(r,t)} dt^2, \quad (5.1.1.)$$

where $f(\theta) =$ sin θ, θ, sinh θ for $K = 1, 0, -1$ respectively.

The position of a particle in the fluid is given by

$r = f(\tilde{r}, t)$, $\theta = \tilde{\theta}$, $\phi = \tilde{\phi}$ where $\tilde{r}$, $\tilde{\theta}$, $\tilde{\phi}$ are the co-ordinates of the particle at some initial time. Provided that the flow-lines do not intersect, i.e. if $f(\tilde{r}, t)$

is invertible, we may use $\tilde{r} = g(r,t)$ as the radial co-ordinate. The metric may be rediagonalised by a suitable choice of the co-ordinate t. The flow-lines of the fluid are given by $\tilde{r}$ = constant, θ = constant, ϕ = constant and the velocity of the fluid by $u^a = e^{-\nu/2}\delta^a_4$. The co-ordinates defined above are said to be <u>co-moving</u>. We have thus shown that there exist co-moving co-ordinates such that the metric is of the form (5.1.1.).

The field equations (1.2.5.) are (Thompson and Whitrow, 1967)

$$p = -\tfrac{1}{2}e^{-\nu}\left(2\eta'' - \eta'\nu' + \tfrac{3}{2}\eta'^2\right) + \tfrac{1}{2}e^{-\lambda}\left(\tfrac{1}{2}\eta_r^2 + \eta_r\nu_r\right) - Ke^{-\eta}, \quad (5.1.2.)$$

$$p = \tfrac{1}{2}e^{-\nu}\left(-\eta'' - \tfrac{1}{2}\eta'^2 + \tfrac{1}{2}\eta'\nu' - \tfrac{1}{2}\eta'\lambda' + \tfrac{1}{2}\lambda'\nu' - \lambda'' - \tfrac{1}{2}\lambda'^2\right)$$
$$+ \tfrac{1}{2}e^{-\lambda}\left(\eta_{rr} + \tfrac{1}{2}\eta_r^2 + \nu_{rr} + \tfrac{1}{2}\nu_r^2 + \tfrac{1}{2}\eta_r\nu_r - \tfrac{1}{2}\lambda_r\nu_r - \tfrac{1}{2}\lambda_r\eta_r\right), \quad (5.1.3.)$$

$$\mu = \tfrac{1}{2}e^{-\nu}\left(\tfrac{1}{2}\eta'^2 + \lambda'\eta'\right) - \tfrac{1}{2}e^{-\lambda}\left(2\eta_{rr} - \lambda_r\eta_r + \tfrac{3}{2}\eta_r^2\right) + Ke^{-\eta}, \quad (5.1.4.)$$

$$0 = \eta_r\eta' - \eta_r\lambda' + 2\eta_r' - \eta'\nu_r, \quad (5.1.5.)$$

where a prime and an ν signify differentiation with respect to t and ν respectively. The equations for $K=0,-1$ may be obtained from those with $K=+1$ by means of the substitutions: $\Theta=\tilde{\Theta}/m$, $e^{\eta}=m^2e^{\tilde{\eta}}$, $m\to\infty$; $\Theta=i\hat{\Theta}$, $e^{\eta}=-e^{\hat{\eta}}$.

From equations (2.1.21.,22.) we obtain the following expressions for the kinematic equatities:-

expansion: $$\Theta = \tfrac{1}{2}(\lambda'+2\eta')e^{-\nu/2}, \qquad (5.1.6.)$$

shear: $$\sigma^2 = \tfrac{1}{6}(\eta'-\lambda')^2 e^{-\nu}, \qquad (5.1.7.)$$

acceleration: $$\dot{u}_a = \tfrac{1}{2}\nu_\nu \delta^3_a. \qquad (5.1.8.)$$

As a consequence the conservation equations (2.2.7.,8.) become

$$p_\nu = -\tfrac{1}{2}\nu_\nu(p+\mu), \qquad (5.1.9.)$$

$$\mu' = -(\tfrac{1}{2}\lambda'+\eta')(p+\mu). \qquad (5.1.10.)$$

Following Thompson and Whitrow (1967) we define a function m by

$$8m = \eta'^2 e^{3\eta/2-\nu} + 4Ke^{\eta/2} - \eta_\nu^2 e^{3\eta/2-\lambda}. \qquad (5.1.11.)$$

It follows from equations (5.1.2.,4.,5.) that

$$4m_\nu = \mu\eta_\nu e^{3\eta/2}, \qquad (5.1.12.)$$

$$4m' = -p\eta' e^{3\eta/2}. \qquad (5.1.13.)$$

In the spherically symmetric case, the function $m(r,t)$ has been shown to represent the mass within a radius r at time t by Thompson and Whitrow (1968), Bondi (1969) and Cahill and McVittie (1969). In vacuo ($p = \mu = 0$) it is equal to the constant m in equations (4.2.9.,10.). The analysis by these authors may be extended to the cases of hyperbolic or planar symmetry but in these cases there is no clear Newtonian analogue of the mass m .

From equations (5.1.2.,3.,5.) it follows that

$$4\left[\left(\eta_{,}e^{-\lambda/2}\right)_{,}e^{\eta}\right]' + \left(1-\lambda'/\eta'\right)\left(\eta_{,}^{2}e^{-\lambda}\right)'e^{\eta+\lambda/2}$$

$$+4K\eta' e^{\lambda/2} + 2\eta' e^{-\nu/2}\left[\left(\eta'-\lambda'\right)e^{\eta+\lambda/2-\nu/2}\right]' = 0. \qquad (5.1.14.)$$

It is difficult to proceed further without making additional simplifying assumptions. We shall consider three such assumptions, namely cases where (i) the flow is geodesic, (ii) the flow is shear-free, or (iii) the expansion of the flow-lines is zero.

5.2 Perfect Fluids with Uniform Pressure

In case (i) above since the acceleration vector is zero

it follows from equation (2.2.7.) that $p = p(t)$ and $V = V(t)$. By a suitable scale change in the co-ordinate t we may set $V = 1$. From equation (5.1.5.) it may be deduced that if $\eta_r \neq 0$

$$e^{\lambda} = \tfrac{1}{4} f^{-1} \eta_r^2 e^{\eta} = f^{-1} R_r^2, \qquad (5.2.1.)$$

where $f = f(r)$ and $R = e^{\eta/2}$. The case $\eta_r = 0$ was considered by Kantowski and Sachs (1966) and Ellis (1967) and will not be considered here. From equations (5.1.2.,11.) and (5.2.1.) we obtain the equations

$$2R''R + R'^2 = f(r) - K - p R^2, \qquad (5.2.2.)$$

$$R'^2 = 2m/R + f(r) - K, \qquad (5.2.3.)$$

and

$$R'' = -m/R^2 - \tfrac{1}{2} p R. \qquad (5.2.4.)$$

Equation (5.2.4.) shows clearly that a uniform positive pressure tends to produce contraction.

If the pressure is zero, it follows from equation (5.1.13.) that $m = m(r)$. Bondi (1947) pointed out, the equations (5.1.12.) and (5.2.3.,4.) with $p = 0$ are identical to the equations for spherically symmetric motion of a

pressureless fluid in Newtonian theory. The 'radius' $R(r,t)$ however cannot be interpreted as the radial distance in General Relativity but is instead the radius of curvature of the two-surface r = constant, t = constant. The general solution of equation (5.2.3.) was obtained by Bondi (1947) for the spherically symmetrical case $K=1$ and by Ellis (1967) for the other two cases.

Since the equations are identical, we expect that many Newtonian results have analogues in General Relativity. In particular it is shown in Appendix III that the density distribution at the moment a singularity forms has a very special form. In Appendix IV the radial propagation of light in spherically symmetrical dust distributions is investigated with special reference to the escape of light from collapsing bodies in an otherwise expanding universe.

5.3 Shear Free Flows

If the shear of the flow vanishes it follows from equation (5.1.7.) that $\eta' = \lambda'$ and the metric (5.1.1.) may be put in a form identical with equation (4.2.3.). Consequently such fields belong to classes IA or II of Chapter 4 or are conformally flat. Spherically symmetrical fields of this type have been considered extensively in recent years in varying degrees of generality by McVittie

(1966), McVittie and Stabell (1967), Thompson and Whitrow (1967, 1968), Bonnor and Faulkes (1967), Nariai (1968), Vaidya (1968), Bondi (1969) and Faulkes (1970).

Since $\lambda' = \eta'$, $e^{\eta} = f^{2}(r)\, e^{\lambda}$ and after a suitable change of scale in r we may write $e^{\eta} = r^2 e^{\lambda}$. The metric is given by $G = e^{\eta}(r^{-2}dr^2 + d\sigma^2) - e^{\nu}dt^2$. On simplifying equation (5.1.14.) and integrating, we obtain

$$r^2 y_{rr} + r\, y_r - K y = b(r) y^2, \qquad (5.3.1.)$$

where $y = e^{-\eta/2}$ and b is an arbitrary function of integration. From equations (5.1.4.) and (5.3.1.) it follows that

$$\mu_r = 2\, b'(r)\, y^3 . \qquad (5.3.2.)$$

On using equations (A.1.12.,13.) we see that the invariant α is given by

$$\alpha = \tfrac{2}{3}\, b(r)\, y^3 . \qquad (5.3.3.)$$

Equation (5.1.5.) with the aid of (5.1.6.) becomes if $\eta' \neq 0$,

$$e^{\nu} = \tfrac{9}{4}\eta'^{2} / \Theta^2(t). \qquad (5.3.4.)$$

It follows from equation (5.3.3.) that the field is

conformally flat if and only if $b = 0$. Amongst this class are the uniform density spherically symmetric fields considered by Thompson and Whitrow (1967). For $b = 0$ equation (5.3.2.) is easily integrated to give:-

$$\left.\begin{aligned} y &= A r^{-1} + B r \quad , && \text{for } K=1, \\ y &= A + B \log r \quad , && \text{for } K=0, \\ y &= A \cos \log r + B \sin \log r, && \text{for } K=-1, \end{aligned}\right\} \qquad (5.3.5.)$$

where $A = A(t)$ and $B = B(t)$ are arbitrary functions of integration. From equations (5.1.4.) and (5.3.4.) we obtain the following expressions for the density in the three cases:-

$$\mu - \tfrac{1}{3}\Theta^2(t) = 12AB \quad , \quad = -3B^2, \quad = -3(A^2+B^2), \qquad (5.3.6.)$$

for $K = +1, 0, -1$, respectively.

In order to specify the system completely it is necessary to impose several boundary conditions. These are usually taken to be the configuration of the body at some initial time, and the equation of state at a given point, usually the 'centre' $r = 0$. Using these boundary conditions, Bondi (1969) has obtained a number of conditions for the collapse or the 'bounce' of a sphere

to occur. The other two cases could be analysed in a similar way but as they seem to have no physical applications we will not consider this problem here.

The remaining uniform density solutions are given by $b(r) = b =$ constant and belong to type D class IA. In this case equation (5.3.1.) may be integrated with the aid of the integrating factor $2y_r$. We obtain

$$r^2 y_r^2 = C + Ky^2 + \tfrac{2}{3} b y^3, \qquad (5.3.7.)$$

where $C = C(t)$ is an arbitrary function of integration. Equation (5.3.7.) may be integrated a second time and the solution may be written in terms of the Weierstrassian? elliptic functions $\wp(z)$. The general solution of equation (5.3.7.) is

$$y = (2b)^{-1}\left\{-1 + \tfrac{1}{3}\wp(B \pm \log r)\right\}, \qquad (5.3.8.)$$

where $B = B(t)$ is an arbitrary function of integration and the invariants (Whittaker and Watson 1969) of the elliptic function are given by

$$g_2 = \tfrac{1}{12} K^2, \qquad g_3 = -\tfrac{1}{72}\left(2Cb^2 + \tfrac{1}{3}K^3\right). \qquad (5.3.9.)$$

From equations (5.1.4.) and (5.3.4.,7.) we see that

$$\mu = \tfrac{1}{3}\Theta^2(t) - 3C, \qquad (5.3.10.)$$

and from equation (5.1.11.) we obtain

$$2m = \tfrac{2}{3}b + \tfrac{1}{3}\mu R^3.$$

The last term on the righ-hand side ot this equation is the contribution to the mass from the matter of uniform density between radii 0 and R, since

$$\tfrac{1}{3}\mu R^3 = \int_0^R \mu R^2 dR = 2\int_{m(0)}^{m(R)} dm.$$

Consequently we may interpret $m(0) = \frac{1}{3}b$ as 'point-mass' at R=0. By a 'point-mass' we mean a singularity of the same type as that at $r = 0$ in equations (4.2.8.-10.). If $C = \frac{1}{9}\Theta^2(t)$, we see from equation (5.3.10.) that $\mu = 0$ and the solution is one of the vacuum fields in Section 4.2.

These metrics are not regular at $r = 0$ and for this reason they have usually been ruled out on physical grounds. However the solution might represent an annular region surrounding a core with a different density distribution.

For certain special values of the function C the integral of equation (5.3.7.) may be written in terms of elementary functions. Since the case $K = 1$, i.e. the

spherically symmetrical cases, is the most important physically we will only consider such fields. There are however solutions in the cases $K = 0$ and $K = -1$ corresponding to these spherically symmetrical solutions.

For $C = 0$ the integral of equation (5.3.7.) is

$$y = 6b^{-1}\beta r(1-\beta r)^{-2}, \qquad (5.3.11.)$$

where $\beta = \beta(t)$ is an arbitrary function of integration. If $C = -(3b^2)^{-1}$ we obtain

$$y = (2b)^{-1}\left\{1 + 3\tan^2\left(\tfrac{1}{2}\log\beta r\right)\right\}, \qquad (5.3.12.)$$

where $\beta = \beta(t)$.

We will now consider cases where the density is not uniform. These fields belong to class II of Chapter 4. Equation (5.3.1.) may be written in the form

$$\frac{\partial^2 v}{\partial x^2} = A(x)\, v^2, \qquad (5.3.13.)$$

where $y = uv$, $x = \pm\int(ru^2)^{-1}\,dr$, $A(x) = b(r)\,u^5$, and u is a solution of the linear equation $r^2u_{rr} + ru_r - Ku = 0$. If $A(x) = \frac{3}{2}A =$ constant,

i.e. if $b(v) = \frac{3}{2} A u^{-5}$, equation (5.3.13.) may be integrated to yield

$$\left(\frac{\partial v}{\partial x}\right)^2 = C(t) - A v^3, \qquad (5.3.14.)$$

where C is an arbitrary function of integration. The general solution of equation (5.3.14.) is

$$v = -(CA^{-1})^{\frac{1}{3}} \wp\left(B(t) \pm \tfrac{1}{2}(A^2C)^{\frac{1}{6}} x\right), \qquad (5.3.15.)$$

where B is an arbitrary function of integration and the invariants of the elliptic function are $g_2 = 0$, $g_3 = -4$. If C = 0, the general integral of equation (5.3.14.) may be written in terms of elementary functions. It is

$$v = (B - D x)^{-2}, \qquad (5.3.16.)$$

where $D = \frac{1}{2}(-A)^{1/2}$.

We consider first the spherically symmetrical case. If we take $u = av + bv^{-1}$, we obtain the following equations:-

$$x = \pm (2a)^{-1}(b + av^2)^{-1}, \text{ for } a \neq 0, \quad x = \tfrac{1}{2} t^2 b^{-2}, \text{ for } a = 0, \qquad (5.3.17.)$$

$$v = (ar + br^{-1})^{-1} y , \qquad (5.3.18.)$$

$$b(r) = \tfrac{3}{2} A r^5 (ar^2 + b)^{-5} . \qquad (5.3.19.)$$

Not all these solutions are distinct, since the metric is preserved by co-ordinate transformations of the form $R = \alpha r^{\pm 1}$, where α is a constant. The left-hand side of equation (5.3.1.) is also preserved by this transformation and so solutions of this equation with $b = f(r)$ and $b = f(\alpha r^{\pm 1})$ are equivalent. There are three distinct forms of the function $b(r)$:-

$$b(r) = \tfrac{3}{2} A r^5 , \quad = \tfrac{3}{2} A (r + r^{-1})^{-5} , = \tfrac{3}{2} A (r - r^{-1})^{-5} . \qquad (5.3.20.)$$

These forms of $b(r)$ are given by setting $a = 0$, $b = 1$; $a = 1$, $b = 1$; $a = 1$, $b = -1$; respectively in equations (5.3.17.-18.), and the corresponding solutions may be obtained by substituting these equations in (5.3.15.) or (5.3.16.). The solution with $b(r) = \tfrac{3}{2} A r^5$ and $C = 0$ was first obtained by Faulkes (1970).

For $K = 0$, i.e. planar symmetry, the corresponding

solution is given by equation (5.3.15 or 16.) with

$$y = v \log v, \quad x = 1/\log v, \quad b(v) = \tfrac{1}{2} A (\log v)^{-5}. \qquad (5.3.21.)$$

For $K = -1$ the solution is given by

$$y = v \cos \log v, \quad x = \tan \log v, \quad b(v) = \tfrac{1}{2} A (\cos \log v)^{-5}. \qquad (5.3.22.)$$

It is possible to obtain further solutions of equation (5.3.13.) for different forms of the function $A(x)$. These may be found by applying a transformation of the form (5.3.17.,18.) (5.3.21.) or (5.3.22.) to equation (5.3.13.). An equation of the form (5.3.1.) results and a new solution is obtained if $b(v) =$ constant in the transformed equation.

By means of the substitution $x = \frac{1}{c} \tan \log r$, $z = c^{\frac{1}{2}} (1 + c^2 x^2)^{-\frac{1}{2}} v$, the equation $r^2 z_{rr} + r z_r + z = b z^2$ may be written as

$$\frac{\partial^2 v}{\partial x^2} = b c^{5/2} (1 + c^2 x^2)^{-5/2} v^2.$$

From equations (5.3.8.,9.) we see that the solution of this equation is

$$v = (2bc^3)^{-1} (1 + c^2 x^2)^{\frac{1}{2}} \left\{ \tfrac{1}{3} \wp(B \pm \tan^{-1}(cx)) - 1 \right\}, \qquad (5.3.23.)$$

where the invariants g_2, g_3 are given by $g_2=\frac{1}{12}$, $g_3=-\frac{1}{72}(2Cb^2c^5-\frac{1}{3})$. Consequently, we may obtain the solutions of equation (5.3.1.) for $K=+1$ and $b(r)=bc^{5/2}r^5/\{(ar^2+b)^2+c^2\}^{5/2}$ by means of the substitutions $y=(ar+br^{-1})r$, $x=(2a)^{-1}(ar^2+b)^{-1}$, and for $K=0$, $b(r)=bc^{5/2}/\{1+c^2\log^2 r\}^{5/2}$ by means of the substitutions $y=r$, $x=\log r$.

If we consider the equation $r^2z_{rr}+rz_r-z=Az^2$, and perform the transformations $r=(ar+br^{-1})^{-1}y$, $x=(2a)^{-1}(ar^2+b)^{-1}$, it follows that

$$\frac{\partial^2 v}{\partial x^2}=\frac{Av^2}{(2x)^{5/2}(1-2abx)^{5/2}}.$$

From equation (5.3.8.,9.) we may deduce that the solution of this equation is

$$v=(2A)^{-1}\{2x(1-2abx)\}^{\frac{1}{2}}\left[\tfrac{1}{3}\wp\left(B\pm\tfrac{1}{2}\log(x^{-1}-2ab)\right)-1\right], \quad (5.3.24.)$$

where $g_2=\frac{1}{12}$, $g_3=-\frac{1}{72}(2CA^2+\frac{1}{3})$.

New solutions of equation (5.3.1.) may be produced from equation (5.3.24.). By means of the substitutions $v=ry$, $x=\frac{1}{2}r^2$, we obtain the solution corresponding to $K=+1$, $b(r)=A(1-abr^2)^{-5/2}$. By the substitutions $v=y$, $x=\log r$ and $v=y\log r$, $x=1/\log r$, we obtain two new solutions of (5.3.1.)

corresponding to $K = 0$, $b(r) = 2^{-5/2}A(\log r)^{-5/2}(1 - 2ab\log r)^{-5/2}$ and $K = 0$, $b(r) = 2^{-5/2}A(\log r - 2ab)^{-5/2}$, respectively. For the third case $K = -1$, a solution corresponding to $b(r) = A(2\sin\log r)^{-5/2}(\cos\log r - 2ab\sin\log r)^{-5/2}$ can be obtained from equation (5.3.24.) by means of the substitutions $x = \tan\log r$, $y = v\cos\log r$.

The elliptic function in equation (5.3.24.) reduces to an elementary function for two special values of the function C. For $C = 0$, it follows from equations (5.3.8.,11.,24.) that

$$v = (6\alpha\beta/A)(1 - 2abx)\{a - \beta(\tfrac{1}{2}x^{-1} - ab)^{\frac{1}{2}}\}^{-2}, \qquad (5.3.25.)$$

and for $C = -(3A^2)^{-1}$ we obtain

$$v = (2A)^{-1}(2x(1 - 2abx))^{\frac{1}{2}}\{1 + 3\tan^2[\tfrac{1}{4}\log\beta(x^{-1} - 2ab)]\}, \qquad (5.3.26.)$$

where $\beta = \beta(t)$ is an arbitrary function of integration.

The solution (5.3.25.) has previously been found by Nariai (1968) and the transformation to the form obtained by Nariai is given by $x = (2R^2)^{-1}$, $y = Rv$.

In all the general solutions above there are two arbitrary functions of time. In order to specify the

system completely one needs to impose boundary conditions. These usually adopted are the specification of the initial configuration of the body and the equation of state at the centre. The equation of state at other points is then determined by the field equations, and is often not physically realistic. This is because there are five unknown functions ρ, μ, λ, η, and ν in the field equations and five equations to be satisfied (four independant field equations and the condition $\lambda' = \eta'$). We cannot therefore impose an arbitrary equation of state throughout the body since the system is already completely determined.

However the exact solutions are useful as they give some insight into the results to be expected from a numerical solution of the field equations (5.1.2.-5.) and an equation of state, and further more they could also be used to obtain estimates of the errors in numerical solutions by comparing the numerical and analytic solutions of the same problem. Disney et al. (1968) and Crampin et al. (1969) have used exact shear-free solutions of the Newtonian equations in order to estimate the growth of rounding errors etc. in their calculation scheme for solving the full Newtonian equations. This method is much more sensitive than integral methods such as conservation of mass.

5.4 Flows with Zero Expansion

From equation (5.16) we see that flows for which the expansion is zero are characterised by the condition $\lambda' + 2\eta' = 0$. On integrating this equation, we obtain

$$e^{\lambda} = \left(f(t)\, e^{\eta}\right)^{-2}, \qquad (5.4.1.)$$

where f is an arbitrary function of integration. By means of a scale change of the co-ordinate r we may set $f = 1$.

On eliminating λ from equation (5.1.5.) with the aid of the condition $\lambda' + 2\eta' = 0$, we see that

$$3\eta_{,r}\eta' + 2\eta'_{,r} - \eta' v_{,r} = 0.$$

If $\eta' \neq 0$ this equation may be integrated to give

$$e^{v} = F^{2}(t)\, \eta'^{2} e^{3\eta}, \qquad (5.4.2.)$$

where F is an arbitrary function of integration. If F is not constant we may set $F = t$ by means of the co-ordinate transformation $\tilde{t} = F(t)$.

If $\eta' = 0$ it follows from equation (5.1.7.) that the shear of the flow vanishes and consequently the field

belongs to classes IA or II of Chapter 4 with $\alpha = \alpha(r)$ or is conformally flat. These fields will not be considered further here.

From the conservation equation (5.1.10.) we see that $\mu' = 0$ and consequently $\mu = \mu(r)$. With the aid of this equation and (5.4.1.,2.) we see that equation (5.1.4.) may be written as

$$\left(\eta_{rr} + \tfrac{7}{4}\eta_r^2\right)e^{2\eta} + \tfrac{3}{4}F^{-2}e^{-3\eta} - Ke^{-\eta} = -\mu(r).$$

By means of the substitution $y = e^{\eta}$ we obtain

$$y\,y_{rr} + \tfrac{3}{4}y_r^2 + \tfrac{3}{4}F^{-2}y^{-3} - Ky^{-1} = -\mu(r). \quad (5.4.3.)$$

On differentiating this equation with respect to t and using (5.4.1.,2.) we see that equation (5.1.14.) is identically satisfied.

It is difficult to integrate equation (5.4.3.) owing to its extreme non-linearity. However, if the density μ is constant we may integrate the equation with the aid of the integrating factor $2y_r y^{\frac{1}{2}}$ to obtain the first integral

$$y_r^2 y^{3/2} - 4Ky^{1/2} - F^{-2}y^{-3/2} = C(t) - \tfrac{4}{3}\mu y^{3/2}, \quad (5.4.4.)$$

where C is an arbitrary function of integration. In order to see the physical significance of C we consider equation (5.1.11.). With the aid of equations (5.4.1.,2.) we obtain the equation

$$y^2 + y^{3/2} - F^{-2} y^{-3/2} - 4K y^{1/2} = -8m, \qquad (5.4.5.)$$

and from equation (5.4.4.) we see that

$$m = \tfrac{1}{6}\mu y^{3/2} - \tfrac{1}{8} C(t). \qquad (5.4.6.)$$

Consequently $-\frac{1}{8} C$ is the 'point-mass' at $y = 0$ since from (5.1.12.) we see that the term $\frac{1}{6}\mu y^{3/2}$ is the contribution to m of the matter lying between radii 0 and $y^{1/2}$. From equation (5.1.13.) we obtain the following expressions for the pressure,

$$p + \mu = \tfrac{1}{2} C'(t)\left(\eta' e^{3\eta/2}\right)^{-1} = \tfrac{1}{2} C' F e^{-\nu/2}. \qquad (5.4.7.)$$

It follows from this equation that vacuum fields with cosmological constant are characterised by the condition $C' = 0$. Hence, by Theorem 4.1 these fields are static. The field is the Schwarzschild exterior field or a related solution, but the co-ordinate system in

which it is expressed is a moving system.

It does not appear possible to integrate equation (5.4.4.) in terms of elementary or even elliptic functions except in very special cases. For example, if $\mu = C = K = 0$ the equation is easily integrated, but for this case $p = \mu = m = 0$ and the solution represents flat space. However it should be possible to obtain solutions numerically without too much difficulty for given forms of the function C. In such a numerical calculation it would be advantageous to use the arbitrariness of the t -coordinate to set $F(t) = t$ or some other convenient form. If F is constant, C could be set equal to some convenient prescribed function of t without loss of generality.

The solutions for K = 1 represent the collapse (explosion) of an incompressible fluid sphere. The proper volume of an element of fluid remains constant along the flow-lines and as the surface area of the sphere decreases (increases) the matter is stretched (compressed) radially. For $K = 0, -1$, the solution represents an analogous situation with planar or hyperbolic symmetry. Since the solutions do not appear to be physically realistic, this problem will not be considered any further here.

References

Bondi, H.; 1947, MNRAS, 107, 410.

Bondi, H.; 1969, ibid, 142, 333.

Bonnor, W.B. and Faulkes, M.C.; 1967, MNRAS, 137, 239.

Cahill, M.E. and McVittie, G.C.; 1970, JMP, 11, 1382-1401.

Crampin, D.J. et. al.; 1962, MNRAS, 145, 423-33.

Disney, M.J. et. al.; 1968, MNRAS, 140, 319.

Ellis, G.F.R.; 1967, JMP, 8, 1171-94.

Faulkes, M.C.; 1970, Prog. Theor. Phys., 42 1139-43.

Kantowski, R. and Sachs, R.K. ; 1966 JMP, 7, 443.

McVittie, G.C.; 1967, Ann. Inst. Henri Poincare, VI, 1.

McVittie, G.C. and Stabell, R., 1967, Ann. Inst. Henri Poincare, VII, 103-14.

Nariai, H.; 1968 Prog. Theor. Phys., 40, 1013.

Robertson, H.T. and Noonan, T.W.; 1968, Relativity and Cosmology (WB Saunders) pp 219-22, 325-27.

Thompson, I.H. and Whitrow G.J.; 1967, MNRAS, 136, 207-17.

Thompson, I.H. and Whitrow G.J.; 1968, ibid., 139, 499-513.

Whittaker, E.T., and Watson, G.N.; 1969, A Course in Modern Analysis, (Cambridge Univeristy Press) pp. 429-61.

APPENDIX I

A.1 The Riemann and Ricci Tensors

In this Section a number of formulae that are useful for calculating the Riemann and Ricci tensors are listed.

For an n-dimensional Riemannian space with metric

$$G = g_{\mu\nu}(x^k)\,dx^\mu dx^\nu \pm \left(V(x^n)\,dx^n\right)^2,$$

where $k = 1, \ldots n$ and $\mu, \nu = 1, 2, \ldots n-1$, the second fundamental form of the hypersurface, x^n = constant, is given by $D = d_{\mu\nu}\,dx^\mu dx^\nu$, where

$$d_{\mu\nu} = -(2V)^{-1} g_{\mu\nu,n} . \qquad \text{(A.1.1.)}$$

The Riemann tensor is given by

$$R^{\mu\nu}{}_{\rho\sigma} = \bar{R}^{\mu\nu}{}_{\rho\sigma} \mp 2 d^{\mu}_{[\rho} d^{\nu}_{\sigma]} , \qquad \text{(A.1.2.)}$$

$$R^{n\nu}{}_{\rho\sigma} = \mp 2V^{-1} d^{\nu}_{[\rho\,|\,|\,\sigma]} , \qquad \text{(A1.3.)}$$

$$R^{n\nu}{}_{n\lambda} = V^{-1}\left(\pm d'^{\nu}{}_{\lambda} - V^{|\nu}{}_{|\,|\lambda}\right) \mp d^{\nu}{}_{\mu} d^{\mu}_{\lambda} , \qquad \text{(A.1.4.)}$$

where $\bar{R}^{\mu\nu}{}_{\rho\sigma}$ is the Riemann tensor of the $n-1$

dimensional metric $g_{\mu\nu}$, covariant derivatives are made with $g_{\mu\nu}$ and a prime signifies partial differentiation with repect to x^h . From the last three equations the following expressions for the Ricci tensor are obtained

$$R^{\mu}_{\nu} = \bar{R}^{\mu}_{\nu} - V^{-1}\left(V'^{\mu}_{\|\nu} \mp d'^{\mu}_{\nu}\right) \mp d\, d^{\mu}_{\nu}, \qquad \text{(A.1.5.)}$$

$$R^{h}_{\mu} = \mp V^{-1}\left(d^{\nu}_{\mu} - d\,\delta^{\nu}_{\mu}\right)_{\|\nu}, \qquad \text{(A.1.6.)}$$

$$R^{h}_{h} = -V^{-1}\left(V'^{\nu}_{\|\nu} \mp d'\right) \mp d_{\mu\nu} d^{\mu\nu}. \qquad \text{(A.1.7.)}$$

If in an n-dimensional Riemannian space there exist two conformally related metrics G and $\bar{G}$ such that $V^2 \cdot \bar{G} = G$, the Riemann Christoffel symbols of the two metrics are related by the following equation

$$\Gamma^{i}_{jk} = \bar{\Gamma}^{i}_{jk} + A^{i}_{jk}, \qquad \text{(A.1.8.)}$$

where $A^{i}_{jk} = V^{-1}\left(\delta^{i}_{j} V_{,k} + \delta^{i}_{k} V_{,j} - \bar{g}_{jk} V_{,l} \bar{g}^{il}\right)$.
The Riemann tensors are related by

$$\bar{R}^{hi}_{\ \ jk} = V^2 R^{hi}_{\ \ jk} + \delta^{[h}_{[j} V^{i]}_{k]}, \qquad \text{(A.1.9.)}$$

where

$$V_{ik} = 4VV_{;ik} - 2V_{,i}V^{,l}g_{ik},$$

and covariant derivatives are made with the metric G.

The Ricci tensor is given by

$$\bar{R}_{ik} = R_{ik} + (n-2)V^{-1}V_{;ik} - \frac{1}{n-2}V^{n-2}\left(V^{2-n}\right)_{;l}^{,l}g_{ik}. \quad \text{(A.1.10.)}$$

The Weyl tensor $C^{hi}{}_{jk}$ is defined by

$$C^{hi}{}_{jk} = R^{hi}{}_{jk} - \frac{4}{n-2}\delta^{[h}_{[j}R^{i]}_{k]} + \frac{2R}{(n-1)(n-2)}\delta^{h}_{[j}\delta^{i}_{k]}. \quad \text{(A.1.11.)}$$

From equations (A.1.9.-11.) it follows that

$$\bar{C}^{h}{}_{ijk} = C^{h}{}_{ijk}.$$

For a three-dimensional Riemannian space with a metric of the form

$$G = \left(W(x^3)\,d\sigma\right)^2 + \left(V(x^3)\,dx^3\right)^2,$$

where $d\sigma^2$ is a two-dimensional metric which depends only on the co-ordinates x^1, x^2, the x^3-line is a Ricci eigen-direction and may be chosen so that the x^1- and x^2- lines are eigendirections. The Ricci tensor is given by

$$R^1_1 = R^2_2 = W^{-2}\left(K - \frac{1}{V}\left(\frac{WW_{13}}{V}\right)_{13}\right), \quad \text{(A.1.12.)}$$

$$R^3_3 = -\frac{2}{VW}\left(\frac{W_{13}}{V}\right)_{13}, \quad \text{(A.1.13.)}$$

where K is the Gaussian curvature of $d\sigma^2$.

For a two-dimensional space with metric g_{ij} the Riemann and Ricci tensors are given by

$$R^{hi}_{jk} = 2K\,\delta^h_{[j}\delta^i_{k]}\,, \quad R^i_k = K\delta^i_k\,, \quad R = 2K\,.$$

Co-ordinates x^1 and x^2 may always be chosen so that the metric takes the orthogonal form

$$G = (W\,dx^1)^2 \pm (V\,dx^2)^2\,.$$

The Gaussian curvature K is given by

$$K = -(VW)^{-1}\left[(W^{-1}V_{,1})_{,1} \pm (V^{-1}W_{,2})_{,2}\right]\,. \qquad \text{(A.1.14.)}$$

For a three-dimensional metric of the form

$$G = d\sigma^2 + V^2(x^\lambda)(dx^3)^2\,,$$

where $\lambda = 1, 2$ and $d\sigma^2$ is a two-dimensional metric involving only x^1 and x^2, the Riemann-Christoffel symbols are given by

$$\Gamma^\lambda_{\mu\nu} = \bar\Gamma^\lambda_{\mu\nu}, \quad \Gamma^3_{\lambda 3} = V^{-1}V_{,\lambda}\,, \quad \Gamma^\lambda_{33} = -VV^{,\lambda}, \qquad \text{(A.1.15.)}$$

$$\Gamma^3_{33} = \Gamma^3_{\lambda\mu} = \Gamma^\lambda_{\mu 3} = 0,$$

where the $\bar\Gamma^\lambda_{\mu\nu}$ are the Riemann-Christoffel symbols of the metric $d\sigma^2$. The Ricci tensor is given by

$$R^3_3 = -V^{-1}V^{,\lambda}{}_{,\lambda}\,, \quad R^3_\lambda = 0, \qquad \text{(A.1.16.)}$$

$$R^{\mu}{}_{\nu} = K S^{\mu}{}_{\nu} - V^{-1} V^{|\mu}{}_{|\nu} , \qquad \text{(A.1.17.)}$$

where K is the Gaussian curvature of $d\sigma^2$ and covariant derivatives are taken with respect to the metric $d\sigma^2$.

A.2 The Equation $V^{1\mu}{}_{|1\nu} - \frac{1}{3} V^{1\lambda}{}_{|1\lambda} \delta^{\mu}_{\nu} = 0$ and Spaces of Uniform Curvature.

In Section 3.4..we need to integrate the equation $V^{1\mu}{}_{|1\nu} - \frac{1}{3} V^{1\lambda}{}_{|1\lambda} \delta^{\mu}_{\nu} = 0$ where $V = V(r, \theta, \phi, t)$ and where the covariant derivatives are taken with respect to the three-dimensional metric

$$G = e^{\alpha} dr^2 + e^{\beta} \left(d\theta^2 + f^2(\theta)\, d\phi^2 \right), \qquad \text{(A.2.1.)}$$

where $f(\theta) = \sin\theta,\ \theta,\ \sinh\theta$ for $k = +1,\ 0,\ -1$ respectively, k being the Gaussian curvature of the metric $d\theta^2 + f^2(\theta)\, d\phi^2$, $\alpha = \alpha(r,t)$ and $\beta = \beta(r,t)$. Since f is of this form it follows that

$$f'^2(\theta) - f''(\theta) f(\theta) = 1. \qquad \text{(A.2.2.)}$$

The equations $V^{1\mu}{}_{|1\nu} - \frac{1}{3} V^{1\lambda}{}_{|1\lambda} \delta^{\mu}_{\nu} = 0$ are equivalent to the following equations:-

$$V^{13}{}_{|11} = \left(V_r e^{-\alpha} \right)_{\theta} - \tfrac{1}{2} \beta_r e^{-\alpha} V_{\theta} = 0, \qquad \text{(A.2.3.)}$$

$$V^{13}{}_{|12} = \left(V_r e^{-\alpha} \right)_{\phi} - \tfrac{1}{2} \beta_r e^{-\alpha} V_{\phi} = 0, \qquad \text{(A.2.4.)}$$

$$V^{11}{}_{|12} = \left(V_{\theta} e^{-\beta} \right)_{\phi} - f^{-1} f' e^{-\beta} V_{\phi} = 0, \qquad \text{(A.2.5.)}$$

$$V^{11}{}_{|11} - V^{12}{}_{|12} = e^{-\beta} \left(V_{\theta\theta} - f^{-2} V_{\phi\phi} - f^{-1} f' V_{\theta} \right) = 0, \qquad \text{(A.2.6.)}$$

$$V^{13}_{113} - V^{1\cdot}_{11\cdot} = \left(V_{rr} - \tfrac{1}{2}(\alpha_r + \beta_r)V_r\right)e^{-\alpha} - V_{\theta\theta}\,e^{-\beta} = 0. \qquad \text{(A.2.7.)}$$

On integrating equations (A.2.3.-5.) we obtain the equations

$$V_\phi = A(r,\phi,t)f(\theta) = B(\theta,\phi,t)e^{\frac{1}{2}\beta}, \quad V_\theta = C(\theta,\phi,t)e^{\frac{1}{2}\beta}, \qquad \text{(A.2.8.)}$$

where A, B and C are functions of integration.

On taking the second logarithmic derivative with respect to r and ϕ and with respect to θ and ϕ of both the first and second of equations(A.2.8.), we deduce that

$$A(r,\phi,t) = D_\phi(\phi,t)e^{\frac{1}{2}\beta}, \quad B(\theta,\phi,t) = D_\phi(\phi,t)f(\theta),$$

where $D_\cdot$ is a function of ϕ and t . Consequently,

$$V_\phi = D_\phi f(\theta)\, e^{\frac{1}{2}\beta}. \qquad \text{(A.2.9.)}$$

From equations (A.2.8.,9.) we see that $C_\phi = f'(\theta)\, D_\phi$. Hence,

$$C = Df' + E_\theta(\theta,t),$$

and

$$V = \left(Df' + E_\theta\right)e^{\frac{1}{2}\beta}, \qquad \text{(A.2.10.)}$$

where $E(\theta,t)$ is an arbitrary function. From equations (A.2.9.,10.) it follows that

$$V = \left\{ D(\phi,t) f(\theta) + E(\theta,t) \right\} e^{\frac{1}{2}\beta} + F(r,t), \qquad \text{(A.2.11.)}$$

where F is an arbitrary function of integration.

On eliminating V from equation (A.2.6.) with the aid of (A.2.11.) we obtain

$$D_{\phi\phi} + (f'^2 - f''f) D = f E_{\theta\theta} - f' E_{\theta}.$$

Noting equation (A.2.2.) we see that the left and right-hand sides of this equation are functions of ϕ, t and θ, t respectively. Consequently each is equal to a function of t say, $d(t)$. On integrating the resulting equations we obtain the equations

$$D(\phi,t) = a(t)\cos\phi + b(t)\sin\phi + d(t), \quad E(\theta,t) = -d(t) f(\theta) + c(t) g(\theta) + \hat{e}(t),$$

where a, b, c and $\hat{e}$ are arbitrary functions of integration, and $g(\theta) = \cos\theta, \frac{1}{2}\theta^2, \cosh\theta$, for $k = +1, 0, -1$, respectively. From equation (A.2.11.) it follows that

$$V = \left[(a\cos\phi + b\sin\phi) f(\theta) + c\, g(\theta) \right] e^{\frac{1}{2}\beta} + F(r,t), \qquad \text{(A.2.12.)}$$

where $\tilde{e}\, e^{\frac{1}{2}\beta}$ has been absorbed into $F(r,t)$.

From equations (A.2.7.,12.) we see that

$$e^{-\alpha}\left\{ F_{rr} - \tfrac{1}{2}(\alpha_r + \beta_r) F_r \right\} + \left(\tfrac{1}{2}\beta_{rr} - \tfrac{1}{4}\alpha_r\beta_r \right)\left\{ (a\cos\phi + b\sin\phi) f(\theta) + c\, g(\theta) \right\} e^{\frac{1}{2}\beta - \alpha}$$

$$- e^{-\frac{1}{2}\beta}\left\{ (a\cos\phi + b\sin\phi) f'' + c\, g'' \right\} = 0.$$

If at least one of a,b, or c is non-zero we obtain the equations

$$\left(\tfrac{1}{2}\beta_{rr} - \tfrac{1}{4}\alpha_r\beta_r\right)e^{\frac{1}{2}\beta-\alpha} + ke^{-\frac{1}{2}\beta} = 0, \qquad \text{(A.2.13.)}$$

$$e^{-\alpha}\left(F_{rr} - \tfrac{1}{2}(\alpha_r+\beta_r)F_r\right) - Jc\,e^{-\frac{1}{2}\beta} = 0, \qquad \text{(A.2.14.)}$$

where $J = 1, 0$ for $k = 0, \pm 1$ respectively.

From equations (A.1.12.,13.) we see that equation (A.2.13.) is satisfied if and only if the metric (A.2.1.) is of uniform curvature $K(t)$. We will now consider the case where the metric (A.2.1.) is independant of t and is of constant curvature K. We may define a curvature co-ordinate by means of the equation $R = e^{\frac{1}{2}\beta(r)}$. Equation (A.2.13.) becomes $\left(1 + \tfrac{1}{2}R\alpha_R\right)e^{-\alpha} = k$. On integrating and using equations (A.1.12.-13.) we obtain

$$e^{-\alpha} = k - KR^2, \qquad \text{(A.2.15.)}$$

On using equation (A.2.15.) to eliminate α from equation (A.2.14.) we see that

$$\left(k - KR^2\right)F_{RR} - kR^{-1}F_R = JcR^{-1}.$$

On integrating we obtain the equations

$$\left.\begin{aligned} F &= C + D(k - KR^2)^{\frac{1}{2}}, \quad \text{for } K\neq 0,\ k\neq 0, \\ F &= C + DR^2, \quad \text{for } K=0,\ k\neq 0, \\ F &= c(-2KR)^{-1} + CR + D, \quad \text{for } K<0,\ k=0, \end{aligned}\right\} \qquad (A.2.16.)$$

where C and D are functions of time.

In terms of isotropic co-ordinates, i.e. $r^2 e^{\kappa} = e^{\beta}$, we see that equation (A.2.13.) possesses the solutions

$$\left.\begin{aligned} e^{-\frac{1}{2}\beta} &= r^{-1} + \tfrac{1}{4}Kr, \quad \text{for } k=1, \\ e^{-\frac{1}{2}\beta} &= \sqrt{(-K)}\log r, \quad \text{for } k=0, \\ e^{-\frac{1}{2}\beta} &= \sqrt{(-K)}\cos\log r, \quad \text{for } k=-1, \end{aligned}\right\} \qquad (A.2.17.)$$

The second arbitrary constant of integration has been transformed away by means of a suitable dilation of the r co-ordinate. The transformation from curvature co-ordinates to isotropic co-ordinates is $R = e^{\frac{1}{2}\beta}$, and equation (A.2.12.) with the aid of (A.1.16.) becomes

$$V = \frac{[(a\cos\phi + b\sin\phi)\sin\theta + c\cos\theta]r + D(1 - \frac{1}{4}Kr^2)}{1 + \frac{1}{4}Kr^2} + C, \quad \text{for } k=1, \qquad (A.2.18.)$$

$$V = [(a\cos\phi + b\sin\phi)\sinh\theta + c\cosh\theta]\sec\log r + D\tan\log r + C, \text{ for } k=-1, \quad (A.1.19.)$$

$$V = [(a\cos\phi + b\sin\phi)\theta + \tfrac{1}{2}c\theta^2 + C](k(k)\log r)^{-1} + \tfrac{1}{2}c(-k)^{-\frac{1}{2}}\log r + D, \text{ for } k=0. \quad (A.2.20)$$

We shall now consider the most general form of the function $V = V(r, \theta, \phi, t)$ such that the metric,

$$G = V^{-2}\{dr^2 + r^2(d\theta^2 + f^2(\theta)\,d\phi^2)\},$$

is of uniform curvature $K(t)$. On using equation (A.1.10.) we see that

$$2K\delta^i_k = V^2\tilde{R}^i_k + VV^{;i}_{;lk} + (VV^{;l}_{;ll} - 2V^{;l}V_{;l})\delta^i_k, \quad (A.2.21.)$$

where $\tilde{R}^i_k$ is the Ricci tensor of the metric $dr^2 + r^2(d\theta^2 + f^2(\theta)\,d\phi^2)$ and covariant derivatives are taken with respect to this metric. From equations (A.1.12.,13.) we see that

$$\tilde{R}^1_1 = \tilde{R}^2_2 = r^{-2}(k-1), \quad \tilde{R}^3_3 = 0, \quad \tilde{R}^i_k = 0, \text{ otherwise.} \quad (A.2.22.)$$

Equations (A.2.21.,22.) are equivalent to equations (A.2.3.-6.) with $e^\beta = r^2$, $e^\alpha = 1$ and the equation

$$V^{;13}_{;13} - V^{;11}_{;11} + V(\tilde{R}^3_3 - \tilde{R}^1_1) = 0. \quad (A.2.23.)$$

Consequently, it follows that V is given by (A.2.12.),

where F satisfies $r F_{rr} - F_r + (1-k) r^{-1} F - J_e = 0$,

where $J = 1, 0$, for $k = 0, \pm 1$, respectively. The general solution of this equation is given by

$$\left.\begin{aligned} F &= A + Br^2 , && \text{for } k = +1, \\ F &= r\left[A + B\log r + \tfrac{1}{2}c(\log r)^2\right], && \text{for } k = 0, \\ F &= r\left[A \cos \log r + B \sin \log r\right], && \text{for } k = -1, \end{aligned}\right\} \quad \text{(A.2.24.)}$$

where A and B are arbitrary function of t. From equation (A.2.21.) we see that the curvature K is given by

$$\left.\begin{aligned} K &= 4AB - (a^2+b^2+c^2), && \text{for } k = +1, \\ K &= 2Ac - B^2 - (a^2+b^2), && \text{for } k = 0, \\ K &= -(A^2+B^2) - (a^2+b^2-c^2), && \text{for } k = -1. \end{aligned}\right\} \quad \text{(A.2.25.)}$$

APPENDIX II

STATIC PERFECT FLUIDS IN GENERAL RELATIVITY

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STATIC PERFECT FLUIDS IN GENERAL RELATIVITY

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All solutions of Einstein's field equations which represent a static perfect fluid are considered and a number of results for vacuum fields obtained by Ehlers and Kundt (1962) are generalised. In Sections 5 - 7 all such solutions with a degenerate Weyl tensor are found explicitly. As is the case with vacuum fields of Ehlers and Kundt all the solutions obtained are found to admit at least a two-parameter group of local isometries.

1 Introduction

In this paper I shall consider static solutions of Einstein's equations

$$R_{\alpha\beta} - \tfrac{1}{2} R\, g_{\alpha\beta} = -T_{\alpha\beta} \,, \qquad (1)$$

Greek indices running from 1 to 4, where $T_{\alpha\beta}$ is the energy-momentum tensor of a perfect fluid. Consequently,

$$T_{\alpha\beta} = (p+\rho)\, u_\alpha u_\beta - p\, g_{\alpha\beta} \,, \qquad (2)$$

where u_α, ρ and p are respectively the four-velocity, energy-density and the pressure of the fluid. If there exists a non-zero cosmological constant Λ it may be absorbed in the tensor $T_{\alpha\beta}$,

viz. $$p = p_{\text{matter}} + \Lambda \,, \quad \rho = \rho_{\text{matter}} - \Lambda \,. \qquad (3)$$

All exact solutions of the equations (1) and (2) which have a degenerate Weyl tensor will be obtained by a method that generalises that used by Levi-Civita (1917-19) and Ehlers and Kundt (1962) (referred to below as E.K.) to analyse vacuum fields.

2 Theorems on Static Space-Times

We shall appeal to several general theorems on static space-times which we list below. The proofs of these theorems are to be found in E.K.

A space-time is called static if it admits a time-like Killing vector ξ_α which is hypersurface orthogonal, i.e.

$$\xi_\alpha \xi^\alpha = - e^{2u} < 0, \quad \xi_{(\alpha;\beta)} = 0, \quad \xi_{[\alpha;\beta} \xi_{\gamma]} = 0. \qquad (4)$$

If u^α is the unit vector parallel to ξ^α it can be shown that

$$u_\alpha u^\alpha = -1 \quad , \quad u_{\alpha;\beta} = - \dot{u}_\alpha u_\beta \quad , \quad \dot{\ddot{u}}_{[\alpha} u_{\beta]} = 0 \quad , \qquad (5)$$

where a dot signifies covariant differention along u^α. Equation (5) means that the integral curves of u^α form a normal rigid congruence along which $\dot{u}_\alpha$ is Fermi-propagated (Synge, 1960).

Space-times may be classified (Petrov, 1962) by considering certain canonical forms of the Weyl tensor defined by

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + g_{\alpha\beta\epsilon[\gamma} R^{\epsilon}{}_{\delta]} + \frac{1}{6} R\, g_{\alpha\beta\gamma\delta} .$$

Tetrad components are taken with respect to an orthonormal tetrad of vectors (e^α_A, e^α_4), where e^α_4 is a time-like vector and capital Latin indices run from 1 to 3. We consider the canonical forms of the complex tensor $\mathcal{D}_{AB}$ defined by

$$\mathcal{D}_{AB} = E_{AB} + i H_{AB} = - \left(C_{A4B4} + i {}^*C_{A4B4} \right) .$$

The Weyl tensor is said to be of Petrov type I, II or III, respectively, if $\mathcal{D}_{AB}$ when regarded as a linear transformation on a three-dimensional complex vector space has invariant sub-spaces of dimension 1, 2 or 3 respectively.

It may be shown (E.K.) from equation (5) that the vector u^α is an eigenvector of $R_{\alpha\beta}$ with eigenvalue $\dot{u}^\gamma{}_{;\gamma}$. If we choose $e_4^\alpha = u^\alpha$, since $H_{\alpha\gamma} = -{}^*C_{\alpha\beta\gamma\delta}u^\beta u^\delta = 0$, it follows that u^α is a principal vector of the Weyl tensor and, furthermore, that the Weyl tensor is of type I with vanishing pseudoscalars. This follows since D_{AB} is real and may consequently be diagonalised by a rotation of the space-like triad (e_A^α).

If the Weyl tensor is non-degenerate, the principal tetrad is uniquely determined up to reflections and so u^α is unique. In the degenerate case, the vector u^α, being an eigenvector of $R_{\alpha\beta}$ and consequently of $T_{\alpha\beta}$, is unique if $p+\rho \neq 0$. If, however, $p+\rho = 0$, the vector u^α need not be unique.

In E.K. it is shown that co-ordinates may be chosen in such a way that the metric G takes the form

$$G = \tilde{g}_{ab}\, dx^a\, dx^b - e^{2u(x^a)}\, dt^2 , \qquad (6)$$

where lower case Latin indices run from 1 to 3 and $t = x^4$. In this co-ordinate system $u^\alpha = e^{-u}\delta_4^\alpha$ and the co-ordinates are unique (if u^α is unique) up to the transformations

$$\tilde{x}^a = \tilde{x}^a(x^b) \quad ; \quad \tilde{t} = at + b , \qquad (7)$$

where a and b are constants.

The field equations (1) become

$$\tilde{R}_{ab} + e^{-u}(e^u)_{\|ab} = \tfrac{1}{2}(p-\rho)\tilde{g}_{ab} , \qquad (8)$$

$$e^{-u}(e^u)^{\|a}{}_{\|a} = \tfrac{1}{2}(\rho+3p) , \qquad (9)$$

where $\tilde{R}_{ab}$ is the Ricci tensor of $\tilde{g}_{ab}$ and $\|$ signifies a covariant derivative with respect to the metric $\tilde{g}_{ab}$.

It follows from equations (8) and (9) that

$$R = -2\rho \quad . \tag{10}$$

The contracted Bianchi identities (the equations of hydrostatic equilibrium) are

$$p_{|a} = -U_{|a}(p+\rho) \quad , \tag{11}$$

where $|$ signifies ordinary differentiation. It follows from equation (11) that

$$p_{[|a}U_{|b]} \equiv \rho_{[|a}U_{|b]} = 0 \quad , \tag{12}$$

and so if U is not constant both p and ρ are functions of U only. A second consequence of equation (11) is that both p and ρ are constant when $p+\rho=0$. From equation (3), we can interpret solutions with $p+\rho=0$ as vacuum solutions with a cosmological constant.

It will be convenient to express the metric (6) in the form

$$G = e^{-2U}\left(g'_{ab}\,dx^a dx^b\right) - e^{2U}\,dt^2 \quad . \tag{6}'$$

Using equation A1 in the Appendix with $n=3$, $V=e^{-U}$, the field equations (8) and (9) become

$$R'_{ab} + 2U_{|a}U_{|b} = 2pe^{-2U}g'_{ab} \tag{8}'$$

and

$$U'^{a}{}_{||a} = \tfrac{1}{2}(\rho+3p)e^{-2U} \quad , \tag{9}'$$

where R'_{ab} is the Ricci tensor of g'_{ab} and covariant derivatives are taken with respect to g'_{ab}.

Taking tetrad components with respect to a Weyl principal tetrad $(e^{\alpha}_{A}\ , e^{\alpha}_{4}=U^{\alpha})$ and remembering that the Weyl tensor is

of type I with $H_{AB}=0$, we obtain

$$g_{Ac} = \delta_{Ac}, \quad g_{44} = -1, \quad g_{A4} = 0,$$

and (13)

$$E_{Ac} = -C_{A4c4} = \sum_{M=1}^{3} \alpha_M \delta_{MA} \delta_{Mc}, \quad \Sigma \alpha_M = 0,$$

where the α_M are the Weyl pricipal scalars.

Since $C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + g_{\alpha\beta\epsilon[\gamma} R^{\epsilon}_{\delta]} + \frac{1}{6} R g_{\alpha\beta\gamma\delta}$,

we have

$$E_{Ac} = -\left(R_{A4c4} + \tfrac{1}{2} g_{Ac} R^4_4 + \tfrac{1}{2} R_{Ac} - \tfrac{1}{6} R g_{Ac}\right).$$

Using the equation $R_{A4c4} = \widetilde{R}_{Ac} - R_{Ac}$, we obtain

$$E_{Ac} = \tfrac{1}{2} R_{Ac} - \widehat{R}_{Ac} - \tfrac{1}{2} \delta_{Ac}\left(R^4_4 - \tfrac{1}{3} R\right).$$

With the aid of equations (1), (2) and (13), we obtain

$$\widetilde{R}_{Ac} = -\tfrac{2}{3} \rho \delta_{Ac} - \sum_M \alpha_M \delta_{MA} \delta_{Mc},$$

or equivalently (14)

$$P_{Ac} = -\sum_M \alpha_M \delta_{MA} \delta_{Mc},$$

where P_{Ac} is the trace-free part of $\widetilde{R}_{Ac}$. Thus we see that a space-like triad of Weyl principal vectors form an eigen-triad of the Ricci tensor of the 3-spaces (t= constant) with eigenvalues $-\frac{2}{3}\rho - \alpha_M$.

3 Integrability Conditions

In Section 4, in order to solve the field equations (8) and (9) we shall need their integrability conditions.

Using equation (8) we obtain

$$\widetilde{R}_{a[b\|c]} = e^{-U}\left(e^{U}\right)_{\|a[b} U_{\|c]} - e^{-U}\left(e^{U}\right)_{\|a[b\,c]} + \tfrac{1}{2}(p-\rho)_{\|[c} \tilde{g}_{b]a}.$$

Eliminating second and third derivatives of U by means of equations (8), (10) and the Ricci identity, we obtain

$$L_{a[b\|c]} = -R_{a[b} U_{\|c]} + \tfrac{1}{2}(p-\rho)\tilde{g}_{a[b} U_{c]} + \tfrac{1}{2} g_{a[b} p_{\|c]} + \tfrac{1}{2} U^{\|d} \widehat{R}_{dabc},$$

where $L_{ab} = \widetilde{R}_{ab} - \frac{1}{4}\widetilde{R}\,\tilde{g}_{ab}$.

Using equation (11) and the identity valid for spaces of dimension 3, namely $\tilde{R}_{abcd} = -\eta_{abe}\left(\tilde{R}^{ef} - \frac{1}{2}\tilde{R}g^{ef}\right)\eta_{fcd}$, we obtain

$$L^{a}_{[b\|c]} = -U_{[d}\left(\delta^{a}_{[b}P^{d}_{c]} + 2\,\delta^{d}_{[c}P^{a}_{b]}\right) . \qquad (15)$$

On taking triad components of equation (15) and suspending the summation convention, it follows from equation (14) that

$$2L_{A[B\cdot C]} = -(\alpha_B-\alpha_A)\gamma_{BAC} + (\alpha_C-\alpha_A)\gamma_{CAB} + \alpha_{C\cdot B}\delta_{AC} - \alpha_{B\cdot C}\delta_{AB} - \tfrac{1}{3}\delta_{A[B}\rho_{\cdot C]},$$

and

$$2L_{A[B\cdot C]} = \delta_{AB}U_{\cdot C}(\alpha_C + 2\alpha_B) - \delta_{AC}U_{\cdot B}(\alpha_B + 2\alpha_C) ,$$

where $\gamma_{ABC} = e_{A\mu;\nu}e_B^{\mu}e_C^{\nu}$ are the Ricci rotatiom coefficients of the triad (e_A^{μ}) and where a dot signifies the triad component of a covariant derivative.

The equations above are equivalent to

$$\gamma_{123}(\alpha_1-\alpha_2) = \gamma_{231}(\alpha_2-\alpha_3) = \gamma_{312}(\alpha_3-\alpha_1) = E, \qquad (16)$$

say, and

$$\alpha_{A\cdot C} + (\alpha_A-\alpha_C)\gamma_{CAA} + U_{\cdot C}(\alpha_A-\alpha_B) + \tfrac{1}{6}\rho_{\cdot C} = 0, \text{ for } A,B,C \neq . \qquad (17)$$

Using equations (13), (17), we obtain

$$\alpha_{A\cdot A} + \sum_D (\alpha_A-\alpha_D)\gamma_{ADD} - \tfrac{1}{3}\rho_{\cdot A} = 0. \qquad (18)$$

The classification of vacuum, non-degenerate fields (Jordan, Ehlers and Kundt, 1960, referred to below as J.E.K.) can be extended immediately to non-degenerate perfect fluid fields since equation (16) is identical with equation (3.1.16) of J.E.K.

If $E=0$, then $\gamma_{123}=\gamma_{321}=\gamma_{312}=0$, and so the Weyl tetrad is hypersurface orthogonal. If, however, $E\neq 0$, only $e_4^{\alpha}=u^{\alpha}$ of the tetrad is hypersurface orthogonal.

For type D fields ($\alpha_1 = \alpha_2 = -\frac{1}{2}\alpha$) we see from equations (16) and (17) that $\gamma_{231} = \gamma_{312} = 0$ and $\gamma_{311} = \gamma_{322}$. Hence we can use the result obtained in J.E.K. that, by a suitable rotation about e_3, e_1 and e_2 can be made hypersurface orthogonal. Hence, a co-ordinate system can be chosen so that

$$G = (e^{\beta_1} dx^1)^2 + (e^{\beta_2} dx^2)^2 + (e^{\beta_3} dx^3)^2 - (e^{\mu} dt)^2 . \qquad (19)$$

In this co-ordinate system,

$$e_A^{\mu} = e^{-\beta_A} \delta_A^{\mu} , \quad \gamma_{ABC} = 0 \quad \text{for } A,B,C \neq , \qquad (20)$$

and

$$\gamma_{ABB} = \beta_{B\cdot A} = e^{-\beta_A} \beta_{B|a} \delta_A^a .$$

Conformally-flat fields will be considered in Section 7.

4 Type D Fields

These fields are characterised by the condition

$$\alpha = \alpha_3 = -2\alpha_1 = -2\alpha_2 .$$

If we choose the space-like triad so that equations (19) and (20) hold and specialise equation (17) to degenerate fields, we obtain, for $c = 1, 2$,

$$\left.\begin{aligned} (\alpha - \tfrac{1}{3}\rho)_{\cdot 3} + 3\alpha\beta_{c\cdot 3} &= 0 , \\ (\alpha + \tfrac{1}{6}\rho)_{\cdot c} + \tfrac{3}{2}(\beta_{3\cdot c} + \mu_{\cdot c}) &= 0 , \\ (\alpha - \tfrac{1}{3}\rho)_{\cdot c} + 3\alpha\,\mu_{\cdot c} &= 0 . \end{aligned}\right\} \qquad (21)$$

We note that μ cannot be constant for type D fields, since equation (8) implies that the 3-space is an Einstein space and hence of constant curvature (Eisenhart, 1949). It follows from equation (14) that $\alpha_M = 0$, $M = 1, 2, 3$, and

consequently the space is conformally flat.

On integrating the third of equation (21), we obtain

$$\alpha = e^{-3u}\left[\tfrac{1}{3}\int e^{3u}\,\rho'(u)\,du + e^{3Z(z)}\right], \qquad (22)$$

where $z = x^3$ and $Z(z)$ is an arbitrary function of integration. Eliminating $\rho_{.c}$ from the second and third of equations (21), we obtain

$$\beta_{3\cdot c} = -\alpha^{-1}\alpha_{.c} - 2u_{.c},$$

and hence $$e^{\beta_3} = \alpha^{-1}\, e^{-2u}\, f(z). \qquad (23)$$

From the first equation of (21) it follows that

$$\beta_c = \beta(x^1, x^2, z) + \bar{\beta}_c(x^1, x^2), \quad \beta_{13} = -\tfrac{1}{3}\alpha^{-1}\left(\alpha - \tfrac{1}{3}\rho\right)_{13}. \qquad (24)$$

Thus the metric takes the form

$$G = e^{2\beta}\,d\sigma^2 + f^2(z)\,\alpha^{-2}e^{-4u}\,dz^2 - e^{2u}\,dt^2, \qquad (25)$$

where $d\sigma^2$ is a two-dimensional metric form dependent only on x^1 and x^2.

The fields may be divided into four classes:-

I $\quad \rho = \text{constant}.$

II $\quad \rho = \rho(z)$, i.e. $\rho_{|[\alpha}\, e_{3\beta]} = 0$, $\rho_{|\alpha}\, e_3^{\alpha} \neq 0$.

III $\quad \rho = \rho(x^1, x^2)$, i.e. $\rho_{|[\alpha}\, e_{3\beta]} \neq 0$, $\rho_{|\alpha}\, e_3^{\alpha} = 0$.

IV $\quad \rho = \rho(x^1, x^2, z)$, i.e. $\rho_{|[\alpha}\, e_{3\beta]} \neq 0$, $\rho_{|\alpha}\, e_3^{\alpha} \neq 0$.

The classification is invariant since e_3^{α} is uniquely determined when u^{α} is, that is when $p+\rho \neq 0$. The condition $p+\rho = 0$ has already been shown to imply ρ = constant, and consequently all fields with $p+\rho = 0$ belong to class I.

5 Space-Times of Class I (Constant Density)

Using the fact that ρ is constant in equations (22) and (24) and eliminating U from equation (23) by means of equation (22), we see that the metric takes the form

$$G = \alpha^{-2/3}\left(d\sigma^2 + dz^2 - e^{2z}\,dt^2\right) . \qquad (26)$$

The function of z before the dz^2 term in the metric has been eliminated by a co-ordinate transformation of the form: $\tilde{z} = \tilde{z}(z)$. For the case of constant density, equation (11) may be integrated to obtain

$$p + \rho = 2A\,e^{-U} . \qquad (27)$$

We note that equation (26) is identical with equation (3.2.12) of J.E.K. if we put $\alpha = \pm e^{-3z}$. Hence we can follow J.E.K. in subdividing metrics of the form (26) into the four classes:-

(A) the hypersurface α =constant contains the space-like eigen-blade of the Weyl tensor, i.e. $\alpha = \alpha(z)$;

(B) the invariant α is constant;

(C) the hypersurface α =constant contains the time-like eigen-blade of the Weyl tensor, i.e. $\alpha = \alpha(x^1, x^2)$;

(D) the hypersurface, α =constant contains neither eigen-blade of the Weyl tensor, i.e. $\alpha = \alpha(x^1, x^2, z)$.

Case (A)

In this case, by means of the co-ordinate transformation $z = \pm\,\alpha^{-1/3}$, the metric can be expressed in the form

$$G = z^2\,d\sigma^2 + V^2(z)\,dz^2 - e^{2U(z)}\,dt^2 . \qquad (28)$$

On using A3 and A4 in the Appendix, with $W = z$, the field

equations become

$$-\frac{K}{z^2} + V^{-2}\left(\frac{U_{13}}{z} - \frac{V_{13}}{zV} + 1\right) = \frac{1}{2}(p-\rho) , \tag{29a}$$

$$V^{-2}\left(U_{133} + U_{13}^2 - \frac{V_{13}U_{13}}{V} - \frac{2V_{13}}{zV}\right) = \frac{1}{2}(p-\rho) , \tag{29b}$$

$$V^{-2}\left(U_{133} + U_{13}^2 - \frac{V_{13}U_{13}}{V} + \frac{2U_{13}}{z}\right) = \frac{1}{2}(\rho+3p) , \tag{29c}$$

where $K = K(x^1, x^2)$ is the Gaussian curvature of $d\sigma^2$. Since K is the only function in equation (29a) that depends on x^1 and x^2 it must be constant. By a change in the scale of z we may set $K = \pm 1$, or 0, and choose co-ordinates θ and ϕ such that

$$d\sigma^2 = d\theta^2 + f^2(\theta)\, d\phi^2 , \tag{30}$$

where $f(\theta) = \sin\theta$, θ, $\sinh\theta$ for $K = +1, 0, -1$ respectively.

From equations (29a) and (29c), we obtain

$$\frac{K}{z^2} - \frac{1}{V^2 z^2} + \frac{2V_{13}}{zV^3} = \rho ,$$

which, when integrated, gives

$$V^{-2} = K - \tfrac{1}{3}\rho z^2 - 2mz^{-1} , \tag{31}$$

where m is constant. A second consequence of equations (29a and c) is the equation,

$$V^{-2}z^{-1}\left(U_{13} + V^{-1}V_{13}\right) = \frac{1}{2}(p+\rho) .$$

On using equations (27) and (31) we obtain

$$\left(K - \tfrac{1}{3}\rho z^2 - 2mz^{-1}\right)U_{13}e^U + \left(\tfrac{1}{3}\rho z - mz^{-2}\right)e^U = Az ,$$

and, on integration,

$$e^{U} = \left(K - \tfrac{1}{3}\rho z^2 - 2mz^{-1}\right)\left[B + A\int z\left(K - \tfrac{1}{3}\rho z^2 - 2mz^{-1}\right)^{-3/2} dz\right]. \quad (32)$$

In general, equation (32) involves elliptic integrals, but for $m=0$ it becomes

$$e^{U} = C + B/\left(K - \tfrac{1}{3}\rho z^2\right) , \quad (32a)$$

whereas for $p+\rho=0$ (i.e. $A=0$) equation (32) becomes

$$e^{U} = V^{-1} . \quad (32b)$$

Using equation A3 we see that the invariant α is given by

$$\alpha = -2mz^{-3} .$$

If $m=0$, it follows that $\alpha=0$ and hence that the space is conformally flat.

The metric (28) admits a four-parameter isometry group $G_4 = G_3 \times G_1$, and G_3 which acts in the (x^1, x^2) surface contains a one-dimensional isotropy subgroup. For $K=1$, G_3 is isomorphic to the group of the sphere and the space is spherically symmetric. For $K=0$, G_3 is isomorphic to the isometry group of the plane, whereas for $K=-1$, G_3 is isomorphic to the group of the two-dimensional Lobatschewski space.

If $K=1$ and $m=0$ the solution is the interior solution of Schwarzschild (1916), and if $K=1$ and $A=0$ it is the exterior solution of Schwarzchild (1916) with cosmological constant. These solutions generalise those obtained in J.E.K. (Case A) for vacuum fields to the case of a perfect fluid of constant density.

Case (B)

The metric (26) becomes in this case

$$G = d\sigma^2 + dz^2 - e^{2Z(z)}dt^2 .$$

Using equations A3 and A4 with $V=W=1$, the field equations (8) become

$$K = -\tfrac{1}{2}(p-\rho) ,$$

and

$$Z_{133} + Z_{13}^2 = \tfrac{1}{2}(p-\rho) = \tfrac{1}{2}(\rho+3p) ,$$

where K is the Gaussian curvature of $d\sigma^2$. It follows from these equations that $p+\rho=0$ and $K=\rho$. Further from equation A2 we note that $-(Z_{133}+Z_{13}^2)$ is the Gaussian curvature of $d\sigma_1^2 = dz^2 - e^{2Z}dt^2$, and so the space is the direct product of two two-dimensional spaces of the same constant curvature. In fact the metric is given explicitly by

$$G = d\theta^2 + \sin^2(\sqrt{\rho}\,\theta)\,d\phi^2 + dz^2 - \sin^2(\sqrt{\rho}\,z)\,dt^2 \quad \text{for } \rho>0,$$

and

$$G = d\theta^2 + \sinh^2(\sqrt{(-\rho)}\,\theta)\,d\phi^2 + dz^2 - \sinh^2(\sqrt{(-\rho)}\,z)\,dt^2 \quad \text{for } \rho<0.$$

If $\rho=0$ the space-time is flat. The invariant α is given by $\alpha = -\frac{2}{3}\rho$. The above solutions all possess a six-parameter group of (local) isometries. If $\rho\neq 0$, the space-time is of type D and so the group is complete.

Cases (C) and (D)

It is convenient to consider these two cases together. We write $\alpha = \pm e^{-3\tau}$, and introduce a metric G'' by $G'' = G' e^{-(4\tau+2z)}$, i.e. $G'' = d\sigma^2 + dz^2$. On using equation A1 with $n=3$ and $V = e^{-(2\tau+z)}$, we obtain

$$R'_{ab} = R''_{ab} - e^{2\tau+z}\left(e^{-(2\tau+z)}\right)_{\|ab} + e^{-(2\tau+z)}\left(e^{2\tau+z}\right)^{\|c}_{\|c}\, g''_{ab} ,$$

where covariant derivatives are taken with respect to G'' and R'_{ab} and R''_{ab} are the Ricci tensors of G' and G'' respectively. On substituting the above equation into equation (8)', we obtain

$$R''_{ab} - 2e^{\tau}(e^{-\tau})_{\|ab} + e^{-z}(e^{z})_{\|ab} = \left[2pe^{2\tau} - e^{-(2\tau+z)}\left(e^{2\tau+z}\right)^{\|c}_{\|c}\right] g''_{ab} . \quad (33)$$

This equation differs from the equation (3.2.29) in J.E.K. by

the term involving p , and hence we may use the proof in J.E.K. to show that, in appropriate co-ordinates (X, ϕ, z, t), the metric takes the form:

$$G = (X+Y)^{-2}\left[f^{-1}dx^2 + f\,d\phi^2 + dz^2 - e^{2Z}dt^2\right] , \qquad (34)$$

where $X+Y = e^{-\tau}$, $Y = Y_{(z)}$, $Z = Z_{(z)}$ and $f = f(X)$.
In case C, since $\tau_{13} = 0$, it follows that Y is constant and, by a suitable translation of the co-ordinate X , may be put equal to zero. The metric is of the form:

$$G = X^{-2}\left[f^{-1}dx^2 + f\,d\phi^2 + dz^2 - e^{2Z}dt^2\right] . \qquad (35)$$

On using equations (8) and (9) we obtain

$$e^{-U}(e^{U})^{|\lambda}{}_{|\lambda} - \tilde{R}^3{}_3 = p+\rho \quad , \text{ where } \quad \lambda = 1,2 .$$

Using equation A5 with $V = X^{-1}$ and remembering that $e^{U} = X^{-1}e^{Z}$, we obtain $$p+\rho = 0 .$$

It is unnecessary to integrate the remaining field equations, since by means of the complex co-ordinate transformation: $x = X^{-1}$, $\tilde{\phi} = it$, $\hat{t} = i\phi$ the metric (35) and field equations may be put in a form identical with that of equations (28) and (29) with $p+\rho = 0$. With the aid of equations (31) and (32b) we obtain the following solution,

$$G = \left(K + 2mr^{-1} - \tfrac{1}{3}\rho r^2\right)^{-1}dr^2 + \left(K + 2mr^{-1} - \tfrac{1}{3}\rho r^2\right)d\phi^2 + r^2\left(d\theta^2 - f^2(\theta)\,dt^2\right) ,$$

where $f(\theta) = \sin\theta, \theta, \sinh\theta$ for $K = +1, 0, -1$ respectively. If G is to have the correct signature, $K + 2mr^{-1} - \tfrac{1}{3}\rho r^2 > 0$.

The invariant α is given by

$$\alpha = 2mr^{-3} .$$

The solution has a complete four-dimensional isometry group and a one-dimensional isotropy group. The time-like eigen-surface of the Weyl tensor (r=constant, ϕ=constant) has constant curvature K. This solution generalises that of J.E.K. (Case B) to include vacuum solutions with a non-zero cosmological constant.

In case D $Y' \neq 0$ and $X' \neq 0$. Only two of the field equations (33) remain to be integrated. If we put $a=b=3$ and use equation A5 with $V=1$ we obtain

$$Z'' + Z'^2 - 2(x+y)^{-1} Y'' = \left[2p e^{2z} - e^{-(2z+Z)} \left(e^{2z+Z} \right)^{1c}_{11c} \right],$$

and if we contract (33) we obtain

$$f'' + Z'' + Z'^2 + 2(x+y)^{-1}(Y'' + f') = 3\left[2p e^{2z} - e^{-(2z+Z)} \left(e^{2z+Z} \right)^{1c}_{11c} \right].$$

On using equation A6 with $g^{11} = (g^{22})^{-1} = f(x)$ and $g^{33} = 1$ we obtain

$$e^{-(2z+Z)} \left(e^{2z+Z} \right)^{1c}_{11c} = Z'' + Z'^2 - 2(x+y)^{-1}\left(Y'' + 2Y'Z' + f' \right) + 6(x+y)^{-2}\left(Y'^2 + f \right).$$

From the last three equations we obtain

$$\left(Z'' + Z'^2 \right)(x+y)^2 - \left(2Y'' + 2Y'Z' + f' \right)(x+y) + 3\left(Y'^2 + f \right) = p, \qquad (36a)$$

and

$$\left(Z'' + Z'^2 - \tfrac{1}{2} f'' \right)(x+y) - 2Y'' + f' = 0. \qquad (36b)$$

In addition, equation (9′) becomes, with the aid of equation A6,

$$\left(Z'' + Z'^2 \right)(x+y)^2 - \left(Y'' + 3Y'Z' + f' \right)(x+y) + 3\left(Y'^2 + f \right) = \tfrac{1}{2}(\rho + 3p). \qquad (37)$$

From equations (27), (36a) and (37) it can be seen that

$$Y'' - Y'Z' = A e^{-Z}. \qquad (38)$$

Equation (36b) is equivalent to the equation

$$f''Y - 2(Z''+Z'^2)X = -f''X + 2f' - 4Y'' + 2(Z''+Z'^2)Y. \qquad (39)$$

We note that the right-hand side of this equation is the sum of a function of X, $h(X)$ say, and a function of Z. On differentiating equation (39) twice with respect to X it follows that $f^{iv} = h'' = 0$, and hence

$$f = aX^3 + bX^2 + cX + d, \quad h = 2(bX + c), \qquad (40)$$

where a, b, c and d are constants.

If we substitute the equations (40) in equation (39) we obtain

$$Z'' + Z'^2 - 3aY + b = 0, \quad 2Y'' - (Z''+Z'^2)Y = c - bY.$$

Consequently,

$$2Y'' = 3aY^2 - 2bY + c,$$

$$Y'^2 = aY^3 - bY^2 + cY + \tilde{d} = g(Y), \qquad (41)$$

and hence

where $\tilde{d}$ is a constant.

If a co-ordinate transformation is made so that Z is replaced by Y, then the metric becomes

$$G = (X+Y)^{-2}\left[f^{-1}dX^2 + f d\phi^2 + g^{-1}dY^2 - e^{2Z}dt^2\right]. \qquad (42)$$

From equations (38) and (41) we obtain the equation:

$$\tfrac{1}{2}g'(Y) - g(Y)\frac{dZ}{dY} = Ae^{-Z},$$

which on integration gives

$$e^{Z} = \sqrt{g}\left(B + A\int\{g(s)\}^{-3/2}ds\right). \qquad (43)$$

On substituting equation (43) in equation (36), we obtain, with the aid of equation (38),

$$3(d+\tilde{d}) + \rho = 0,$$

and hence

$$f(X) = -g(-X) - \tfrac{1}{3}\rho. \qquad (44)$$

The invariant α is given by

$$\alpha = a(X+Y)^3 ,$$

and so if $a=0$ the space is conformally flat. We shall assume that $a\neq 0$, and then by affine transformations of the co-ordinates X and Y and a change of scale of the co-ordinates ϕ and t we may set $a = \pm 1$ and $b=0$ whilst preserving the form of the metric (42). The metric is given by equation (42), where $f=\pm X^3+cX+d$ and g and Z are given by equations (43) and (44). There exist two functionally independent metric invariants:-

$$\alpha = \pm(X+Y)^3 , \quad Y^{-2}Y_{;i}Y^{;i} = \pm(X^3+Y^3) + c(X+Y) - \tfrac{2}{3}\rho ,$$

where $Y = \alpha^{1/3}$. Thus the trajectories of the isometry group are the two-dimensional surfaces X =constant, Y =constant. Since the metric is type D, the isotropy group acts in either the (X,ϕ)-surface or the (Y,t)-surface and consequently must be discrete. Thus, the complete isometry group is two-dimensional. From equations (27) and (43), we see that the pressure is given by

$$p = -\rho + 2Ag^{-1/2}(X+Y)\left(B+A\int g^{-3/2}ds\right)^{-1} . \qquad (45)$$

The above solution generalises the solution C in J.E.K. to the case of a uniform density perfect fluid and contains the solutions in J.E.K. as a special case.

6 Space-Times of Classes II-IV (Variable Density)

In case II $\rho = \rho(Z)$, and from equation (12) it follows that $U = U(Z)$ and hence from equations (22)-(24) we may deduce that

$$\alpha = \alpha(Z), \quad \beta_3 = \beta_3(Z) , \quad \beta_c = \beta(Z) + \bar{\beta}_c(x^1, x^2) .$$

By means of the co-ordinate transformation $\tilde{z} = e^{\beta(z)}$ it follows that the metric (25) takes the form

$$G = z^2 d\sigma^2 + V^2(z) dz^2 - e^{2U(z)} dt^2. \qquad (46)$$

This equation is identical with equation (28) and we may use the analysis following equation (28) to show that $d\sigma^2$ is of constant curvature K, and consequently takes the form (30) and that

$$V^{-2} = K - z^{-1}\int_0^z \rho(s) s^2 ds + 2mz^{-1} . \qquad (47)$$

For this metric the invariant α is given by

$$\alpha = 2mz^{-3} + \tfrac{1}{3} z^{-3}\int_0^z \rho'(s) s^3 ds .$$

The metric (46) admits a four-parameter isometry group with a one-dimensional isotropy subgroup. For $K=1$ the space is spherically symmetrical. Spherically symmetrical fluids with non-constant density have been considered by a number of authors (Tolman (1939) Wyman (1949), Bondi (1964)) and many of their results have trivial generalisations to the other two cases ($K=0, K=-1$) obtained above.

We shall consider classes III and IV together. For these two classes $\rho_{1c} \neq 0$ for $c=1,2$ and in class III the condition $\rho_{13}=0$ is also satisfied. The metric is of the form (25)

$$G = e^{-2U}\left(e^{2V} d\sigma^2 + \alpha^{-2} e^{-2U} dz^2\right) - e^{2U} dt^2 , \qquad (25)'$$

where $\gamma = \beta + U$ and

$$\gamma_{13} = U_{13} - \tfrac{1}{3}\alpha^{-1}(\alpha - \tfrac{1}{3}\rho)_{13} . \qquad (24)'$$

With the aid of equation A7, the field equations (8)' become when $a=3$ and $b=1,2$,

Eliminating Y from this equation with the aid of equations (21) and (24)$'$ we obtain

$$2\left(U_{13b}+U_{13}U_{1b}\right)+\tfrac{2}{3}\alpha^{-1}\left(\alpha_{13}U_{1b}+U_{13}\alpha_{1b}\right)+\tfrac{1}{9}\alpha^{-1}\left(U_{1b}\rho_{13}+U_{13}\rho_{1b}\right)=0.$$

With the aid of equations (12) and (21) we see that

$$U_{13b}+2U_{13}U_{1b}+\tfrac{1}{3}\alpha^{-1}\left(\alpha_{13}U_{1b}+2\alpha_{1b}U_{13}\right)=0\ . \qquad (48)$$

One consequence of equations (12), (21), and (48) is that

$$U_{11}U_{132}=U_{12}U_{131}\ ,$$

and hence $U=U\left(f(x^1,x^2),z\right)$ and, by means of the co-ordinate transformation $\tilde{x}^1=X=f(x^1,x^2)$, we see that $U=U(X,z)$.

Equations (21) and (24)$'$ imply that

$$\alpha=\alpha(X,z)\ ,\qquad Y=Y(X,z)\ .$$

On choosing $x^2=\phi$, so that the x^1- and x^2- lines are orthogonal, we see that the metric has the form:

$$G=e^{-2U}\left[e^{2Y}\left(A^2(X,\phi)dx^2+B^2(X,\phi)d\phi^2\right)+\alpha^{-2}e^{-2U}dz^2\right]-e^{2U}dt^2.$$

The remaining field equations (8)$'$ become, on using equation A7 with $d^a{}_b=-\alpha e^{U}Y_{13}\,\delta^a_b$,

$$\alpha e^{U}\left(\alpha^{-1}e^{-U}\right)^{|a}_{|a}+2\left(Y_{133}+Y_{13}^2+Y_{13}U_{13}+U_{13}^2+\alpha^{-1}\alpha_{13}Y_{13}\right)\alpha^2e^{2U}=2\rho e^{-2U}\ , \qquad (49)$$

and

$$\tilde{R}^a{}_b+\alpha e^{U}\left(\alpha^{-1}e^{-U}\right)^{|a}_{|b}+\alpha^2e^{2U}\delta^a_b\left(Y_{133}+2Y_{13}^2+Y_{13}U_{13}+\alpha^{-1}\alpha_{13}Y_{13}\right)+2U^{|a}U_{1b}=2\rho e^{-2U}\delta^a_b\ , \qquad (50)$$

where $\tilde{R}^a{}_b$ is the Ricci tensor of $H=e^{2Y}\left(A^2dx^2+B^2d\phi^2\right)$ and $a,b=1,2$. For $a=1$ and $b=2$ equation (50) is $\left(\alpha^{-1}e^{-U}\right)^{|1}_{|2}=0$, i.e. $A_{|2}\left(\alpha^{-1}e^{-U}\right)_{|1}=0$. Consequently, either $A=A(X)$ or $\alpha e^{U}=f^{|z}(z)$. In the latter case equation (50) implies $U^{|a}U_{1b}$ is proportional to δ^a_b, and since $U_{12}=0$ it follows that

$U = U(z)$ and, from equation (12), that $\rho = \rho(z)$. This contradicts the defining equations of classes III, IV . Hence it follows that $A = A(x)$. Subtracting the equation (50) with $a = b = 2$ from that with $a = b = 1$, we see that

$$\alpha e^{U}\left[\left(\alpha^{-1}e^{-U}\right)^{11}_{11} - \left(\alpha^{-1}e^{-U}\right)^{12}_{12}\right] + 2U^{11}U_{12} = 0,$$

which implies that $B^{-1}B_{12} = f^{12}(x,z)$. Since $B = B(x,\phi)$, it follows that $B^{-1}B_{12} = f^{12}(x)$ and consequently $B = \tilde{B}(x)\,g(\phi)$. By means of co-ordinate transformations of the form $\tilde{x} = \tilde{x}(x)$, and $\tilde{\phi} = \tilde{\phi}(\phi)$, the metric may be put in the form:

$$G = e^{-2U}\left[e^{2\gamma}A^2(x)\left(dx^2 + d\phi^2\right) - \alpha^{-2}e^{-2U}dz^2\right] - e^{2U}dt^2, \qquad (51)$$

where $U = U(x,z)$, $\gamma = \gamma(x,z)$ and $\alpha = \alpha(x,z)$. In class III we have in addition $U_{13} = 0$ and so equations (24) and (48) imply that $\alpha_{13} = \gamma_{13} = 0$. The metric is of the form (51) but in this case $U = U(x)$, $\gamma = \gamma(x)$ and $\alpha = \alpha(x)$. The metric (51) admits a two-parameter abelian isometry group in class IV and a three-parameter abelian group in class III. The metric can be put in the Weyl normal form:

$$G = e^{-2U}\left[e^{2\eta}\left(dx^2 + dz^2\right) + e^{2\delta}d\phi^2\right] - e^{2U}dt^2,$$

where $U = U(x,z)$, $\eta = \eta(x,z)$ and $\delta = \delta(x,z)$, by means of co-ordinate transformations of the form $\tilde{x} = \tilde{x}(x,z)$, $\tilde{z} = \tilde{z}(x,z)$. However, it seems easier to integrate the remaining field equations if the metric is left in the form (51).

If we write $e^{\beta} = Ae^{\gamma - U}$ and $V = \alpha^{-1}e^{-2U}$ in equation (51) and use equation A7 with $d^{\mu}_{\nu} = -\beta_{13}V^{-1}\delta^{\mu}_{\nu}$ for $\mu,\nu = 1,2$, we obtain

$$\tilde{R}^{\mu}_{\nu} = e^{-2\beta}\left(\beta_{11} + V^{-1}\beta_{12}V_{11}\right)\delta^{\mu}_{\nu} + V^{-2}\left(\beta_{133} - V^{-1}\beta_{13}V_{13} + 2\beta_{13}^2\right)\delta^{\mu}_{\nu} + V^{-1}e^{-2\beta}\left(V_{122} - 2\beta_1 V_{12}\right)\delta^{\mu}_{1}\delta^{1}_{\nu},$$

$$\tilde{R}^{3}_{\mu} = V^{-1}\left(V^{-1}\beta_{13}\right)_{11}\delta^{1}_{\mu}, \qquad (52)$$

and $\tilde{R}^3_3 = V^{-1}e^{-2\beta}V_{11} + 2V^{-2}(\beta_{133} + \beta_{13}^2 - V^{-1}\beta_{13}V_{13})$.

Since the metric is of type D and since the z-direction is the distinguished Ricci eigenvector it follows that $\tilde{R}^3_\mu = 0$. Noting that $\tilde{R}^1_2 = 0$, we see that the X- and ϕ-directions are also Ricci eigenvectors and consequently $\tilde{R}^1_1 = \tilde{R}^2_2$. Using equation (52) we see that

$$V^{-1}\beta_{13} = h(z) \quad , \quad V_{11}e^{-2\beta} = g(z) \quad . \tag{53}$$

It follows from the field equations (8) that

$$(e^U)^{11}_{111} = (e^U)^{12}_{112} \quad , \quad (e^U)^{13}_{111} = 0 \, ,$$

where the covariant derivatives are taken with respect to $\tilde{g}_{ab}$. These equations imply that

$$U_{11}e^{U-2\beta} = f(z) \, , \qquad (V^{-1}U_{13}e^U)_{11} = V^{-1}\beta_{13}U_{11}e^U \, . \tag{54}$$

We note that $f \neq 0$ since $U_{11} \neq 0$ for classes III and IV. Equations (53) and (54) imply that

$$V = \frac{g}{f}e^U + k(z) \, , \qquad \frac{U_{13}e^U}{V} = h(z)e^U + l(z) \, . \tag{55}$$

If we eliminate α from equation (48) we obtain the equation

$$U_{113} - \tfrac{1}{3}V^{-1}(U_{11}V_{13} + 2U_{13}V_{11}) = 0 \, .$$

On substituting equation (55) in this equation and writing $m(z) = g/f$, we obtain the equation

$$U_{11}\left[3lk^2e^{-U} + k' + (3lm^2 + m')e^U\right] = 0 \, .$$

Since $U_{11} \neq 0$, it follows that either

$$l = k' = m' = 0 \qquad \text{or} \qquad k = l + \tfrac{1}{3}m^{-2}m' = 0 \, .$$

If $k=0$, it follows from equations (22), (55) and the definition of V that

$$m^{-1} = \tfrac{1}{3}\int \rho'(u)e^{3u}du + e^{3Z(z)} .$$

Consequently $\int \rho'(u)e^{3u}du$ is a function of z only and hence either $u=u(z)$ or ρ =constant. In either case this contradicts the equation defining classes III and IV. It follows that the equations $l=k'=m'=0$ must be valid. On substituting these equations in (54), (55) we see that

$$V = me^{u} + k \quad , \quad \beta_{13} = u_{13} = h(z)V .$$

On integrating these equations, we find that

$$e^{2\beta} = H(x)e^{2u} \quad , \quad e^{u} = k\left(G(x)\,n(z) - m\right)^{-1} ,$$

where $H(x)$ and $G(x)$ are arbitrary functions of integration and $n(z) = e^{\int k h\, dz}$. From the first of the equations (54) it follows that

$$H(x)f(z) = -kG'(x)\,n(z) ,$$

and hence $H(x)$ is proportional to $G'(x)$. If we make the co-ordinate transformation $\tilde{x} = -1/G(x)$ and then make suitable scale changes in the other co-ordinates, we may write the metric in the form

$$G = (n+mx)^{-2}\left[F^{-1}dx^2 + F\,d\phi^2 + dz^2 - x^2dt^2\right] , \qquad (56)$$

where $F=F(x)$. The field equations (8) and (9) that are not identically satisfied are

$$\tfrac{1}{2}(n+mx)F'' - \tfrac{1}{2}(n+mx)(3m-n/x)F' + m(2m-n/x)F - (n+mx)n'' + 3n'^2 = \tfrac{1}{2}(p-\rho) , \qquad (57a)$$

$$-(n+mx)\,m\,F' + (2m - n/x)\,m\,F - 3(n+mx)\,n'' + 3n'^2 = \tfrac{1}{2}(p-\rho)\,. \quad (57b)$$

and $$(m+n/x)\,n\,F' - 3mn\,x^{-1}F - (n+mx)\,n'' + 3n'^2 = \tfrac{1}{2}(\rho+3p)\,. \quad (57c)$$

On subtracting (57b) from (57a) we obtain

$$4n'' + n(F'' + F'/x) + mx(F'' - F'/x) = 0\,. \quad (58)$$

It follows, on differentiating equation (58) with respect to z, that either $n'(z) = 0$ or $F'' + F'/x = 4a$, where a is a constant.

In the first case the metric (56) is independent of z and so these cases belong to class III. The remaining solutions will be shown to belong to class IV or to be conformally flat.

In the case where $n' = 0$ equation (58) becomes

$$F''(n+mx) + (n - mx)F'/x = 0\,.$$

On integrating we find

$$F = b + c(n^2 \log_e x + 2mnx + \tfrac{1}{2}m^2x^2)\,, \quad (59)$$

where b and c are constants. From the equations (57) we obtain the following expressions for the pressure and density

$$2p = cx^{-2}(n+mx)^3(n-mx) + 2m(m - 2n/x)F\,,$$

$$2\rho = cx^{-2}(n+mx)^3(n+3mx) - 6m^2F\,.$$

The invariant α is given by

$$\alpha = -\tfrac{1}{3}nc\,x^{-2}(n+mx)^3\,.$$

Clearly the space is conformally flat if $nc = 0$. If $nc \neq 0$ the metric admits a complete three-dimensional abelian local isometry group. The completeness follows from the fact that trajectories of the complete group must be contained in the

hypersurface X =constant and so the group is of dimension at most 6. However the isotropy group is of dimension at most one since the metric is of type D and acts in the surface X =constant, ϕ =constant. Since the vector u^α is unique it follows that the isotropy group is discrete and that the complete isometry group is of dimension three.

If however $n' \neq 0$, $F'' + F'/X = 4a$. On integration we obtain the equation

$$F = aX^2 + b\log_e X + c \ , \qquad (60)$$

where b and c are constants. From equation (58) it follows that

$$4(n'' + an) + d = 0 \ , \quad m(XF'' - F') = 0 \ , \qquad (61)$$

where d is constant. Using equations (60) and (61) we see that $d=0$ and $mb=0$. On integrating the first of equations (61) we obtain $n = Af(z)$, where $f(z) = \sin(az+B)$, $z+B$, $\sinh(az+B)$ for $a>0$, $a=0$, $a<0$ respectively.

On using the equation A7 we see that the invariant α is given by

$$\alpha = -\tfrac{1}{3} b n^2 X^{-2} \ .$$

If the space-time is not conformally flat, $b \neq 0$ and consequently $m=0$. Thus the metric (56) has the form

$$G = n^{-2}\left(F^{-1}dX^2 + F d\phi^2 + dz^2 - X^2 dt^2\right)$$

where $F = F(X)$ is given by equation (60). On using equations (57) we obtain the following expressions for the pressure and energy-density of the matter:

$$\begin{aligned} 2p &= 6A^2 I(a) + b n^2 X^{-2} \ , \\ 2\rho &= -6A^2 I(a) + b n^2 X^{-2} \ , \end{aligned} \qquad (62)$$

where $I(a) = |a|, 1$ for $a \neq 0, a = 0$ respectively.

A second (functionally independent of α) invariant is given by

$$Y = \tfrac{1}{4}\alpha^{-2}\alpha_{|i}\alpha^{|i} = \left(b\log_e x + c\right)n^2 x^{-2} + A^2 I(a) .$$

Thus we may use arguments similar to those for class ID to show that the complete local isometry group is an abelian two-dimensional group. It is easily seen from equations (62) that the equation of state of the matter is $p = \rho + 6A^2 I(a)$.

7 Conformally Flat Space-Times

In this case $C_{\alpha\beta\gamma\delta} = 0$ and so from equation (13) it follows that $\alpha_M = 0$ for $M = 1,2,3$. From equation (14) we see that $P_{Ac} = 0$ and consequently the space cross-sections t =constant, are Einstein spaces of dimension three. Hence the space is of constant curvature $K = \frac{1}{3}\rho$ (Eisenhart, 1949). Consequently we may choose co-ordinates such that the metric takes the form

$$G = (1 - Kr^2)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) - V^2(r,\theta,\phi)dt^2 . \quad (63)$$

The field equations (8) and (9) become

$$V^{-1} V^{|a}{}_{|b} = \tfrac{1}{6}(\rho + 3p)\delta^a{}_b \quad , \quad (64)$$

and

$$V^{-1} V^{|a}{}_{|a} = \tfrac{1}{2}(\rho + 3p) \quad .$$

The integration of equations (64) is a long but straightforward calculation and eventually one obtains

$$V = (B\sin\phi + C\cos\phi)r\sin\theta + Dr\cos\theta + E\sqrt{(1 - Kr^2)} + A/K \quad \text{for } K \neq 0, \quad (65)$$

$$V = (B\sin\phi + C\cos\phi)r\sin\theta + Dr\cos\theta + E + \tfrac{1}{2}Ar^2 \quad \text{for } K = 0 ,$$

where A, B, C, D, E are all constants, A being given by equation (27). By means of a rotation we set $B = C = 0$ in

equation (65) whilst preserving the form of the metric (63). This solution of the field equations was obtained in a slightly different co-ordinate system by Stepanyuk (1968). It will be shown below that the solution in fact belongs to class IA with $m=0$.

In the following it will be convenient to use Cartesian co-ordinates defined by

$$x = x^1 = r\sin\theta\sin\phi \ , \quad y = x^2 = r\sin\theta\cos\phi, \quad z = x^3 = r\cos\theta. \tag{66}$$

If $K=0$, equation (63) with equations (65),(66) becomes

$$G = dx^2 + dy^2 + dz^2 - \left(Dz + \tfrac{1}{2}Ar^2 + E\right)^2 dt^2 . \tag{67}$$

By means of the co-ordinate transformation $\tilde{z} = z + D/A$, the metric may be expressed as

$$G = dx^2 + dy^2 + dz^2 - \left(\tfrac{1}{2}Ar^2 + E - D^2/A^2\right)^2 dt^2 .$$

This metric is clearly spherically symmetrical. Since $\rho = 3K = 0$ it follows from equation (27) that

$$p = 2A/\left(\tfrac{1}{2}Ar^2 + E - D^2/A^2\right)$$

The above analysis is not valid if $A=0$ but in this case the metric (67) is flat. The transformation to Minkowski co-ordinates being given by $\tilde{z} = \left(z + E/D\right)\cosh Dt$, $\tilde{t} = \left(z + E/D\right)\sinh Dt$.

If $K \neq 0$ it is convenient to rewrite the metric (63) in isotropic co-ordinates (R, θ, ϕ, t) where R is defined by $r = R\left(1 + \frac{K}{4}R^2\right)^{-1}$. We obtain

$$G = \left(1 + \tfrac{K}{4}R^2\right)^{-2}\left[dR^2 + R^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) - \left(\frac{A}{K} + E\left(1 - \tfrac{K}{4}R^2\right) + DR\cos\theta\right)^2 dt^2\right]. \tag{68}$$

The space sections being of constant curvature admit six independent Killing vector fields given by (Robertson and Noonan, 1968)

$$\xi^a_A = \left(1 - \tfrac{K}{4}R^2\right)\delta^a_A + \tfrac{1}{2}Kx^a\delta_{Ab}x^b \ , \quad \eta^a_A = -\epsilon_{ABC}\,\delta^a_B\delta_{Cb}\,x^b . \tag{69}$$

It is easily verified that the three 4-vectors η^{α}_{A} defined by

$$\eta_1^{\alpha} = D\xi_1^{\alpha} + EK\xi_2^{\alpha}\,,\quad \eta_2^{\alpha} = D\xi_2^{\alpha} - EK\xi_1^{\alpha}\,,\quad \eta_3^{\alpha} = \xi_3^{\alpha}\,,\quad \eta_4^{\alpha} = 0\,, \tag{70}$$

are Killing vectors of the space-time. Since these vectors lie in the two dimensional surface V=constant, t=constant, this surface is of constant curvature and using the analysis of Section 5, we see that the metric takes the form

$$G = (\alpha - Kr^2)^{-1}dr^2 + r^2(d\theta^2 + f^2(\theta)\,d\phi^2) - \left[\frac{A}{K} + E^2(\alpha - Kr^2)^{\frac{1}{2}}\right]dt^2, \tag{71}$$

where $f(\theta) = \sin\theta, \theta, \sinh\theta$, for $\alpha = +1, 0, -1$, respectively.

The commutation relations of the operators $X_A \left(X_A = \eta^{\alpha}_{A}\frac{\partial}{\partial x^{\alpha}}\right)$ are

$$[X_1, X_2] = -\left(D^2K + E^2K^2\right)X_3\,,\quad [X_2, X_3] = -X_1\,,\quad [X_3, X_1] = -X_2\,. \tag{72}$$

Hence if $D^2K + E^2K^2 > 0, = 0, < 0$ the group is isomorphic to that of the two-dimensional sphere, plane, Lobatschewski space, respectively and in the metric (71) $\alpha = +1, 0, -1$ respectively.

If $\rho = 3K > 0$, we not that $D^2K + E^2K^2 > 0$ unless $D = E = 0$ in which case the metric (63) is that of the Einstein Universe.

Consequently we have proved that the only positive density conformally flat static perfect fluid solutions are locally isomorphic to the interior Schwartzchild solution or one of the Einstein or de Sitter Universes.

8 Conclusions

All degenerate static perfect fluid solutions of Einstein's equations have been found. These are all found to possess at least a two-parameter group of local isometries. The solutions obtained are displayed below in Table I. The dimensions of the complete isometry and isotropy groups are given in columns 5 and 6 respectively. In column 6 the letter S or T denotes that the isotropy group acts in the space-like or time-like eigen-surface of the Weyl tensor respectively.

Appendix

For the reader's convenience we list below a number of useful formulae taken from J.E.K.

If in an n-dimensional pseudo-Riemannian manifold there exist two conformally related metrics G and $\bar{G}$ where

$$G = V^2 \bar{G} \quad , \text{ then}$$

$$\bar{R}_{ik} = R_{ik} - (n-2) V^{-1} V_{\|ik} + (n-2)^{-1} V^{n-2} \left(V^{2-n}\right)^{\|l}_{\|l} g_{ik} \, , \qquad (A1)$$

where covariant derivatives are taken with respect to G.

For a two-dimensional metric of the form

$$d\sigma^2 = \left(W dx^1\right)^2 \pm \left(V dx^2\right)^2 \, ,$$

the following equations are valid:-

$$R^i_k = -K \delta^i_k \, , \quad R^{ij}_{\ \ kl} = -2K \delta^i_{[k} \delta^j_{l]} \, ,$$

where $$K = -\frac{1}{VW}\left[\left(\frac{V_1}{W}\right)_{12} \pm \left(\frac{W_2}{V}\right)_{12}\right] . \qquad (A2)$$

For a three-dimensional metric of the form

$$G = W^2(x^3)\, d\sigma^2 + \left(V(x^3)\, dx^3\right)^2 \, ,$$

where $d\sigma^2$ is a two-dimensional metric involving x^1 and x^2 only, we can choose x^1 and x^2 so that the co-ordinate lines are Ricci eigendirections. Further the equations,

$$R^1_1 = R^2_2 = \frac{1}{W^2}\left[-K + \frac{1}{V}\left(\frac{W W_3}{V}\right)_3\right] , \quad R^3_3 = \frac{2}{VW}\left(\frac{W_3}{V}\right)_3 , \quad R^{\mu}_{\nu=0} \quad (A3)$$

for $\mu \neq \nu$,

where K is the Gaussian curvature of $d\sigma^2$, are valid.

For this metric

$$\Gamma^i_{3j} = \frac{W_3}{W}\delta^i_j \, , \quad \Gamma^3_{i3} = \Gamma^i_{33} = 0 \, , \quad \Gamma^3_{33} = \frac{V_3}{V} \, , \text{ for } i,j = 1,2 \, ,$$

and if $F = F(x^3)$ then

$$F^{\|i}_{\|j} = \frac{W_3}{V^2 W} F' \delta^i_j \, , \quad F^{\|3}_{\|i} = 0 \, , \quad F^{\|3}_{\|3} = V^{-2}\left(F'' - V^{-1} F' V_3\right) . \qquad (A4)$$

For a metric of the form

$$G = d\sigma^2 + \left(V(x^i)\,dx^3\right)^2 \quad , \quad i,j = 1,2 \ ,$$

the Ricci tensor is given by

$$R^i_j = V^{-1}V^{|i}_{\|j} - K\delta^i_j \ , \ R^i_3 = 0 \ , \ R^3_3 = V^{-1}V^{|i}_{\|i} \ . \qquad (A5)$$

For this metric we note that, since $\Gamma^a_{3b} = \Gamma^3_{ab} = 0$ for $a,b = 1,2,3$, $U_{\|i;j} = U_{;ij}$, where a semi-colon denotes a covariant derivative made with $d\sigma^2$.

For a metric of the form $G = g_{\mu\nu}dx^\mu dx^\nu$ where $\mu,\nu = 1,2,\ldots n$, and $det(g_{\mu\nu}) = 1$, it follows that $V^{|\mu}_{\|\mu} = \left(V_{|\mu}g^{\mu\lambda}\right)_{|\lambda}$, and if $g_{\mu\nu} = 0$ for $\mu \neq \nu$,

$$V^{|\mu}_{\|\mu} = \sum_{\mu=1}^{n}\left(V_{|\mu}g^{\mu\mu}\right)_{|\mu} \ . \qquad (A6)$$

Finally for a metric of the form

$$G = g_{\mu\nu}(x^k)\,dx^\mu dx^\nu + \left(V(x^k)\,dx^n\right)^2 ,$$

where $\mu,\nu = 1,2,\ldots\ldots,n-1$ and $k = 1,2,\ldots\ldots\ldots,n$, the second fundamental form $D = d_{\mu\nu}dx^\mu dx^\nu$ of the hypersurfaces x^n =constant is given by $d_{\mu\nu} = -\frac{1}{2}V^{-1}\frac{\partial}{\partial x^n}(g_{\mu\nu})$. The Ricci tensor of this metric is given by

$$\begin{aligned} R^\mu_\nu &= \bar{R}^\mu_\nu + V^{-1}\left(V^{|\mu}_{\|\nu} - \dot{d}^\mu_\nu\right) + dd^\mu_\nu \ , \\ R^n_\mu &= V^{-1}\left(d^\nu_{\mu} - d\delta^\nu_\mu\right)_{\|\nu} \ , \\ R^n_n &= V^{-1}\left(V^{|\nu}_{\|\nu} - \dot{d}\right) + d_{\mu\nu}d^{\mu\nu} \ , \end{aligned} \qquad (A7)$$

and

where a dot signifies $\frac{\partial}{\partial x^n}$, $\bar{R}^\mu_\nu$ is the Ricci tensor of $g_{\mu\nu}$ and where covariant derivatives are taken with respect to $g_{\mu\nu}$.

References

Bondi, H.; 1964 Proc. Roy. Soc. A, 282, 303-17.

Ehlers, J. and Kundt, W.; 1962 Gravitation. An Introduction to Current Research. Ed L Witten (John Wiley), 49-101.

Eisenhart, L.P.; 1949 Riemannian Geometry (Princeton U P) p92.

Jordan, P., Ehlers, J., and Kunolt W.; 1960 Akad. Wiss. Mainz Abh. Math-Kl. Jahrg 1960, No. 2, 21-124.

Levi-Civita, T.; 1917-19 Rend. Acc. Lincei, 9 Articles in Vols. 26-28.

Petrov, A.S.; 1962 Einstein Spaces, (Pergamon), 88-126.

Schwarzschild, K.; 1916 Sitz. d Preuss Akad. Wiss. 7, 189.

Stepanyuk, N.M.; 1968 Sov. Phys. JETP, 26, 369-70.

Synge, J.L.; 1960 Relativity. The General Theory (North Holland) 123-32.

Tolman, R.C.; 1939 Phys. Rev., 55, 364-73.

Wyman, M.; 1949 Phys. Rev., 75, 1930-5.

Robertson, H.P., and Noonan, T.W.; 1968 Relativity and Cosmology, (W.B. Saunders) 325-7.

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TABLE I.

Class	Metric	α	Pressure and Density	r	s
IA	$G = r^2(d\theta^2 + f^2(\theta)\,d\phi^2) + [F(r)]^{-1}dr^2 - F(r)\left(B + A\int r\,F^{-3/2}\,dr\right)^2 dt^2$, $F(r) = \alpha - \frac{1}{3}\rho r^2 - 2m/r$, $f(\theta) = \sin\theta,\ \theta,\ \sinh\theta$ for $\alpha = +1, 0, -1$.	$-\frac{2m}{r^3}$	$p = -\rho + 2AF^{-\frac{1}{2}}\left(B + A\int r\,F^{-3/2}dr\right)^{-1}$	4	1S
IB	$G = d\theta^2 + h^2(\theta)\,d\phi^2 + dz^2 - h^2(z)\,dt^2$, $h(\theta) = \sin(\sqrt{\rho}\,\theta)$, $\rho > 0$; $h(\theta) = \sinh(\sqrt{(}-\rho)\,\theta)$, for $\rho < 0$.	$-\frac{2}{3}\rho$	$p = -\rho$	6	2ST
IC	$G = [F(r)]^{-1}dr^2 + F(r)\,d\phi^2 + r^2(d\theta^2 - f^2(\theta)\,dt^2)$, F and f as in IA	$-\frac{2m}{r^3}$	$p = -\rho$	4	1T
ID	$G = (x+y)^{-2}\left([f(x)]^{-1}dx^2 + f(x)\,d\phi^2 + [g(y)]^{-1}dy^2 + e^{2z}\,dt^2\right)$, $f(x) = \pm x^3 + cx + d$, $g(x) = -f(-x) - \rho$, $e^z = g^{\frac{1}{2}}(y)\left(B + A\int g^{-3/2}dy\right)$.	$\pm(x+y)^3$	$p = -\rho + 2A(x+y)e^{-z}$	2	0
II	$G = r^2(d\theta^2 + f^2(\theta)\,d\phi^2) + \left(\alpha - r^{-1}\int\rho(r)r^2dr - 2mr^{-1}\right)^{-1}dr^2 - e^{2U(r)}dt^2$, $f(\theta)$ as in class IA, U given by equations (29).	$\frac{1}{3}r^{-3}\int\rho'(r)r^3dr - 2mr^{-3}$	$p = p(r)$, $\rho = \rho(r)$	4	1S
III	$G = (n+x)^{-2}\left([F(x)]^{-1}dx^2 + F(x)\,d\phi^2 + dz^2 - x^2dt^2\right)$, $F(x) = b + c(n^2\log x + 2nx + \frac{1}{2}x^2)$.	$-\frac{1}{3}cx^{-2}(n+x)^3$	$2p = cx^{-2}(n+x)^3(n-x) - 2(n - 2/x)F(x)$; $2\rho = cx^{-2}(n+x)^3(n+3x) - 6F(x)$	3	0
IV	$G = [n(z)]^{-2}\left([F(x)]^{-1}dx^2 + F(x)\,d\phi^2 + dz^2 - x^2dt^2\right)$, $F = ax^2 + b\log x + c$, $n(z) = A\sin(\sqrt{a}z + B)$, $Az + B$, $A\sinh(\sqrt{(}-a)z + B)$ for $a > 0, = 0, < 0$.	$-\frac{1}{3}bn^2(z)x^{-2}$.	$2p = 6A^2I(a) - bn^2(z)x^{-2}$; $2\rho = -6A^2I(a) - bn^2(z)x^{-2}$; $I(a) = \lvert a\rvert$ for $a \neq 0$, [illegible] $a = 0$	2	0

APPENDIX III

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ON THE DYNAMICS OF SELF-GRAVITATING SPHERES

A. Barnes and G. J. Whitrow

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SUMMARY

It is shown that the analytical results obtained by M. V. Penston for the Newtonian pressure-free collapse of a spherically symmetrical sphere of gas can be extended to general relativity.

Penston (1969) has recently considered the free-fall of matter collapsing under its self-gravitational attraction and has obtained analytical solutions for the final stages. He has assumed that all the matter is initially at rest and that it collapses spherically symmetrically under Newtonian gravitation and negligible pressure. The object of the present note is to point out that the same solutions are obtained when Newtonian theory is replaced by general relativity.

The problem of zero-pressure spherically symmetrical collapse in general relativity was considered by Datt (1938), and also by Oppenheimer & Snyder (1939) who used an internal metric with co-moving coordinates that can be expressed in terms of Penston's notation in the form

$$ds^2 = -e^{\lambda}\,da^2 - r^2\,d\Omega^2 + e^{\nu}\,dt^2, \tag{1}$$

where r, λ, ν are functions of a, t, $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$, and units are chosen so that c is unity. Prime symbols will be used to denote differentiation with respect to a and dots with respect to t. It follows, for example, from equation (14) of Thompson & Whitrow (1967), that the vanishing of the pressure implies that $\nu' = 0$. Consequently, ν is a function of time only, so that by a suitable choice of time-measurement we can arrange that $e^{\nu} = 1$. From equation (10) of Thompson & Whitrow (1967), it is easily seen that we can write

$$e^{\lambda} = r'^2/f(a), \tag{2}$$

where $f(a)$ is an arbitrary function of integration. From equation (7) of the same paper, we deduce that

$$\dot{r}^2 - \frac{2GM(a)}{r} = f(a) - 1, \tag{3}$$

and hence that

$$\ddot{r} = -\frac{GM(a)}{r^2}, \tag{4}$$

where $M(a)$ is another arbitrary function of integration, and for comparison with Newtonian theory we have introduced G explicitly. From equation (9) of the same paper, it can be shown that

$$4\pi\rho r^2 r' = M'(a). \tag{5}$$

The function $M(a)$ can be interpreted as the mass within coordinate radius a by the following argument based on consideration of the external metric. We first note

that, if $\rho = 0$ for $a > a_s$, then, from equation (5), it follows that $M(a) = M(a_s)$ for $a > a_s$. If we next introduce variables R, T defined by

$$R = r(a, t), \tag{6}$$

and

$$dT = -\frac{\sqrt{\left\{1 - \frac{1 - 2GM(a_s)}{f(a)}\right\}}}{1 - 2GM(a_s)/R}\, dR + \frac{dt}{\sqrt{f(a)}}, \tag{7}$$

the minus sign arising before the first term because the system is contracting, the metric (1) can be written in the form

$$ds^2 = \frac{-dR^2}{1 - 2GM(a_s)/R} - R^2\, d\Omega^2 + (1 - 2GM(a_s)/R)\, dT^2. \tag{8}$$

Equation (3) ensures that the right-hand side of (7) is a perfect differential, and hence (8) can be interpreted as the Schwarzschild external metric. Consequently, we can regard $M(a)$ as the mass within coordinate radius a.

Equation (4) is the same as equation (1) in Penston's paper. Initially all the matter is assumed to be at rest, and if the co-moving radial coordinate a is identified with the initial value of r we obtain the same parametric solution as is given by equation (2) of Penston's paper, namely

$$\left.\begin{aligned} r &= a \cos^2 \theta, \\ t &= \left\{\frac{8\pi}{3} G\bar{\rho}(a)\right\}^{-1/2} (\theta + \tfrac{1}{2} \sin 2\theta), \end{aligned}\right\} \tag{9}$$

where $\bar{\rho}(a)$ is the mean density (inside a) defined by

$$M(a) = \frac{4\pi}{3} a^3 \bar{\rho}(a).$$

From equations (9), it is easily seen that, if collisions between neighbouring shells do not occur, $\bar{\rho}(a)$ must be a decreasing function of a. As Penston states, if the density has a smooth maximum at the centre, its value for small a would be given by a formula of the form

$$\bar{\rho}(a) = \rho(0) \left(1 - \frac{a^2}{A^2}\right),$$

where A is a constant. He shows that after a time

$$t_f = \sqrt{\left(\frac{3\pi}{32G\rho_0}\right)},$$

where $\rho_0 = \bar{\rho}(0) = \rho(0)$, a singularity will develop at the centre of the form

$$\rho \propto r^{-12/7},$$

and we now see that this result also holds in general relativity. In fact, all the results in Sections 2–4 of Penston's paper can be regarded as valid in general relativity.

This extension of the results of Penston's investigation to general relativity is analogous to the formal equivalence, discovered by Milne & McCrea (1934), of pressure-free world-models in general relativity and in Newtonian theory. In these

systems, however, the density ρ is uniform throughout, being a function of the time variable only. The common feature of Milne's and McCrea's investigation and of the present work is the zero-pressure condition.

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REFERENCES

Datt, B., 1938. *Z. Phys.*, **108,** 314.
Milne, E. A. & McCrea, W. H., 1934, *Q. J. Math.* (*Oxford*), **5,** 73.
Oppenheimer, J. R. & Snyder, H., 1939. *Phys. Rev.*, **56,** 455.
Penston, M. V., 1969. *Mon. Not. R. astr. Soc.*, **144,** 425.
Thompson, I. H. & Whitrow, G. J., 1967, *Mon. Not. R. astr. Soc.*, **136,** 207.

APPENDIX IV

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On gravitational collapse against a cosmological background

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Abstract. The propagation of light in a spherically symmetric dust distribution is investigated. A sufficient condition for light to escape from a collapsing region to an expanding region of the system is obtained. This corrects the 1966 result of Raychaudhuri. It is also shown that, contrary to the conclusion drawn by Som in 1968, no event-horizon other than that associated with the Schwarzschild radius occurs. These results are illustrated by some numerical calculations for particular models.

1. Introduction

The gravitational frequency shift of light radially transmitted from a static source to a static observer in the Schwarzschild exterior field has long been known (Schwarzschild 1916). Synge (1966) obtained the corresponding result for non-radial light waves. More recently the frequency shift of light radiated from a pressure-free collapsing sphere has been considered by Faulkner *et al.* (1964), and also by Banerjee (1966 a), who obtained an analytic formula. The generalization of these results for non-radial light rays was given by Jaffe (1969).

The additional effects of cosmological expansion and of the gravitational field of matter between the source and the observer have been considered by Raychaudhuri (1966) and by Banerjee (1966 b) and by Som (1968). If there are no discontinuities in the density, Raychaudhuri concluded that light from the body would be trapped as soon as collapse sets in, but this will be shown to be incorrect. Instead, so long as light is not transmitted through any matter that has contracted to a radius $R < 2M(r)$, where $M(r)$ is the gravitational mass of matter within the co-moving coordinate radius r, it will escape. This result holds even if there are discontinuities in density. We note that the 2-surfaces (R = constant $< 2M(r)$, r = constant) are closed trapped surfaces (Penrose 1965). Thus it appears that no essentially new features occur when the expansion of the universe is included in the analysis.

2. The equations governing a pressure-free system

In general relativity the metric of a spherically symmetric system may be written (Landau and Lifshitz 1962) in terms of co-moving coordinates r, θ, ϕ and t in the form

$$ds^2 = dt^2 - (R'/\Gamma)^2\,dr^2 - R^2(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{1}$$

with

$$\dot{R}^2 = \Gamma^2 - 1 + 2M(r)/R \tag{2}$$

and

$$M' = 4\pi\rho R^2 R' \tag{3}$$

where Γ and M are arbitrary functions of r, ρ is the energy density of matter and prime and dot symbols denote differentiations with respect to r and t respectively.

Differentiating equation (2) with respect to t we obtain

$$\ddot{R} = -M/R^2. \tag{4}$$

Equations (2)–(4) are identical in form with the Newtonian equations and so we interpret M as the active gravitational mass within coordinate radius r. We will consider only matter with positive gravitational mass since no matter with negative mass has yet been observed.

The proper distance s from the centre to radius r is a strictly increasing function of r and, since s is given by

$$s = \int_0^r (R'/\Gamma)\,\mathrm{d}r$$

we see that $R'/\Gamma \geqslant 0$. We deduce from (1) that Γ is the derivative of R with respect to proper distance in the radial direction, and so where $\Gamma < 0$ the gravitational field is so strong that it begins to close the space: i.e. on hypersurfaces of constant t, the area of the 2-surfaces given by r constant decreases as r increases.

We will assume that $r = 0$ is a point, i.e. $R(0, t) = 0$, and from (2) we see that $M(0) = 0$ and $\Gamma(0) = 1$.

To simplify the analysis below, we write $\Gamma^2 - 1 = f$. Integration of equation (2) then yields

$$t = \frac{2M\{(x+x^2)^{1/2} - \sinh^{-1}(x)^{1/2}\}}{f^{3/2}} \qquad \text{for } f > 0 \tag{5}$$

$$t = \frac{2R^{3/2}}{3(2M)^{1/2}} \qquad \text{for } f = 0 \tag{6}$$

$$t = \frac{2M\{\sin^{-1}(x)^{1/2} - (x-x^2)^{1/2}\}}{(-f)^{3/2}} \qquad \text{for } f < 0 \text{ and } \dot{R} > 0 \tag{7a}$$

$$t = \frac{2M\{\pi - \sin^{-1}(x)^{1/2} + (x-x^2)^{1/2}\}}{(-f)^{3/2}} \qquad \text{for } f < 0 \text{ and } \dot{R} < 0 \tag{7b}$$

where $x = |f|R/2M$ and $\sin^{-1}$ takes its principal value. We have chosen the arbitrary function of integration so that the system originates from a singularity at time $t = 0$, i.e. $R(r, 0) = 0$ for all r.

When M is constant, we see from equation (3) that $\rho = 0$, and hence by Birkhoff's theorem that (1) must be the exterior Schwarzschild form. In fact, the coordinate transformations

$$\mathrm{d}T = \mathrm{d}t/\Gamma + \dot{R}\,\mathrm{d}r/(1-2M/R)\Gamma$$

and

$$R_{\mathrm{Sch}} = R(r, t)$$

transform (1) to the usual Schwarzschild form.

The equations (1)–(7) given above are equivalent to those of Raychaudhuri (1966), except that he used (7*a*) for both positive and negative $\dot{R}$ which is incorrect.

We note from (7) that matter with $f < 0$ expands to a maximum radius $2M/(-f)$ at time $T = \pi M/(-f)^{3/2}$ and then contracts to a singularity at time $2T$. We will assume that after this time the matter coalesces to form a point-mass at $R = 0$. From (6) we see that matter with $f = 0$ expands to infinity with $R = (9Mt^2/2)^{1/3}$, whilst from (5) it follows that shells with $f > 0$ expand to infinity with $R \sim f^{1/2}t$ for large t.

The particular system described by the model depends, of course, on f (or Γ) and M. For instance, if $M = \beta r^3$ and $f = kr^2$, we obtain one of the Friedmann universes, whereas Raychaudhuri's example, namely $M = \beta r^3$, $f = -r^2\{2\exp(-r/r_0)-1\}$, represents a universe that for large r is asymptotically a negatively curved Friedmann universe containing a region ($r < r_0 \ln 2$) that ultimately contracts.

3. Restrictions on the solution

Γ and M are not completely arbitrary since the first and second fundamental forms of any hypersurface, r = constant = b, must be the same whether the hypersurface is regarded as embedded in the region $r < b$ or $r > b$. This ensures that a system of admissible coordinates, i.e. one in which the metric tensor is C^1 and piecewise C^3, exists (Robson 1968, Misner and Sharp 1964). The continuity of the first fundamental form implies that $R(r, t)$, and hence $\dot{R}$, are continuous. Consequently, from (2) we see that Γ^2 and M are continuous. The continuity of the second fundamental form ψ, where

$$\psi = R\Gamma(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)$$

shows that Γ itself is continuous. Continuity of M implies that surface distributions of mass do not exist.

We note that $R'/\Gamma \geqslant 0$ cannot hold for all t if R' changes sign, and so we must find conditions on M and f that ensure that this does not occur. We need the following relations:

$$R' = \left(\frac{M'}{M} - \frac{f'}{f}\right)R - \dot{R}t\left(\frac{M'}{M} - \frac{3f'}{2f}\right) \qquad \text{for } f \neq 0 \tag{9a}$$

and

$$R' = \frac{M'R}{3M} + \frac{f'R^2}{10M} \qquad \text{for } f = 0 \tag{9b}$$

which follow from (2), (5), (6) and (7); and

$$\dot{R}t = 2R/3 \qquad \text{for } f = 0 \tag{10a}$$

$$\dot{R}t < 2R/3 \qquad \text{for } f < 0 \tag{10b}$$

and

$$2R/3 < \dot{R}t < R \qquad \text{for } f > 0 \tag{10c}$$

which follow from (2), (4) and (6).

It will now be shown that $R' > 0$ if and only if

$$M' \geqslant 0 \quad \text{and} \quad \frac{M'}{M} - \frac{3f'}{2f} \geqslant 0 \qquad \text{for } f < 0 \tag{11a}$$

$$M' \geqslant 0 \quad \text{and} \quad f' \geqslant 0 \qquad \text{for } f \geqslant 0 \tag{11b}$$

but where f' and M' are not simultaneously zero.

That these conditions are sufficient for $f = 0$ can be seen immediately from (9*b*). For $f > 0$, using (9*a*) and (10*c*) we have

$$R' > (\tfrac{3}{2}\dot{R}t - R)f'/f > 0.$$

For $f < 0$, a similar argument, using (9*a*) and (10*b*), shows that the conditions (11*a*) are sufficient. The condition $M' \geqslant 0$ is necessary since from (9), and the fact that

$\dot{R}t \sim \frac{2}{3}R$ for small t, we see that $R' \sim M'R/3M$. For $f \geqslant 0$ the necessity of the second condition $f' \geqslant 0$ follows from the asymptotic behaviour of R' for large t, namely $R' \sim f't/2f^{1/2}$ for $f > 0$ and $R' \sim f'R^2/10M$ for $f = 0$. If $f < 0$, we note that collapse back to a singularity occurs after time $T = 2\pi M/(-f)^{3/2}$ and, since $\mathrm{d}T/\mathrm{d}r \geqslant 0$ (for otherwise R would decrease outwards at some stage of the collapse sufficiently near the final singularity), the second of the conditions (11a) follows directly.

To obtain necessary and sufficient conditions for $R' < 0$ we need only reverse the inequalities in (11).

If the conditions (11) do not hold then at some stage of the motion collisions take place between neighbouring shells and we can no longer use co-moving coordinates. We could try to follow the motion after this point using non-co-moving coordinates, but from (3), we note that if $M' \neq 0$, $R' = 0$ implies that ρ is infinite, and so we would expect the zero-pressure assumption to be violated. We also note that the conditions (11) imply that ρ is never negative at any stage of the motion.

We now consider the situation on hypersurfaces with $\Gamma = 0$. If Γ^2 has a zero of order n and the metric (1) is non-singular, then R' must have a (permanent) zero of order $\frac{1}{2}n$ and if ρ is to be non-singular then so must M'. From equation (9a) for consistency we must have $n \geqslant 2$. However, so long as these conditions are satisfied there is nothing singular about the hypersurface $\Gamma = 0$ and it cannot be an 'impenetrable barrier' as Bondi (1947) stated. For example, in the positively curved Friedmann universe we have $\Gamma = \cos r$, $f = -\sin^2 r$, $M = \beta \sin^3 r$ and $R = S(t) \sin r$ but the hypersurface $r = \pi/2$ is in no way singular.

4. The propagation of light

We consider only light emitted radially, i.e. $\theta = \phi = \text{constant}$. For light emitted outwards, that is r increasing along the ray, we have from (1), putting $\mathrm{d}s = 0$,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\Gamma}{R'} \tag{12}$$

and using (2),

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \Gamma + \dot{R} = \Gamma \pm \left(\Gamma^2 - 1 + \frac{2M}{R}\right)^{1/2} \tag{13}$$

the positive sign arising when $\dot{R} > 0$ and negative sign when $\dot{R} < 0$.

We now restrict attention to systems with $f < 0$ for $r < a$ and $f \geqslant 0$ for $r \geqslant a$, where a is a constant. This class comprises systems which contain a region ($r < a$) that ultimately collapses, surrounded by a region which expands to infinity. We will also assume that M and f are bounded, except possibly as r tends to infinity. This rules out infinite masses in finite regions of spacetime, and is also a restriction on the coordinate systems we use, since it rules out those systems that cover the whole of spacetime with a finite range of values of r.

Consider first the case where $\Gamma < 0$ for some $r < a$. Since $\Gamma \geqslant 1$ for $r \geqslant a$, it follows that there is a neck $\Gamma = 0$ separating regions with $\Gamma < 0$ from the expanding background. It follows immediately from (13) that, if a ray from the region $r < a$ reaches the neck after collapse has set in, then it can never escape to the expanding region since $\mathrm{d}R/\mathrm{d}t < 0$. Whether it escapes if it reaches the neck whilst it is expanding will depend on the details of the particular model.

If $\Gamma > 0$ and $\dot{R} < 0$, we see from (13) that $\mathrm{d}R/\mathrm{d}t \leqslant 0$ whenever $2M/R \geqslant 1$. Since $M' \geqslant 0$, it follows that $\mathrm{d}(2M/R)/\mathrm{d}t \geqslant 0$, and so if at any point on the ray the condition $2M/R \geqslant 1$ holds, then it will continue to hold along the ray.

We have seen that the light cannot escape from collapsing regions once they have passed within their own Schwarzschild radius.

However if, for $r < a$, $2M/R < \alpha < 1$ along the ray, we have from (13)

$$\frac{\mathrm{d}R}{\mathrm{d}t} > \Gamma - (\Gamma^2 - 1 + \alpha)^{1/2} > 1 - \alpha^{1/2} > 0$$

but for the shell $r = a$ ($f(a) = 0$) it follows from (2) and (16) that

$$\dot{R}(a, t) = (4M/3t)^{1/3} \to 0 \qquad \text{as} \qquad t \to \infty.$$

Thus the light ray must eventually overtake the shell $r = a$ and reach the expanding region. The condition $2M/R < \alpha < 1$ is not very useful in practice, since, in order

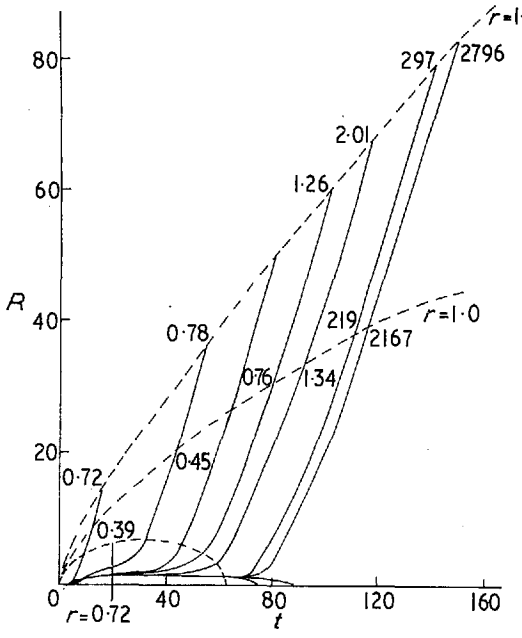


Figure 1. The results of numerical integrations for light rays in Raychaudhuri's model ($f = -r^2\{2\exp(-r/r_0) - 1\}$ and $M = r^3$ where $r_0 = 1/\ln 2$ and hence $f(1) = 0$) are shown. For small r a singularity forms after a time $t = 6{\cdot}283185$ but light emitted from the centre is not trapped until after $t = 6{\cdot}283183$. At $r = 0{\cdot}72$ the respective values are 63·38 and 62·86. The broken lines represent the history of shells with r constant and the numbers given at the intersections are the values of the red-shift z of light from $r = 0{\cdot}72$ and received by an observer moving with the matter.

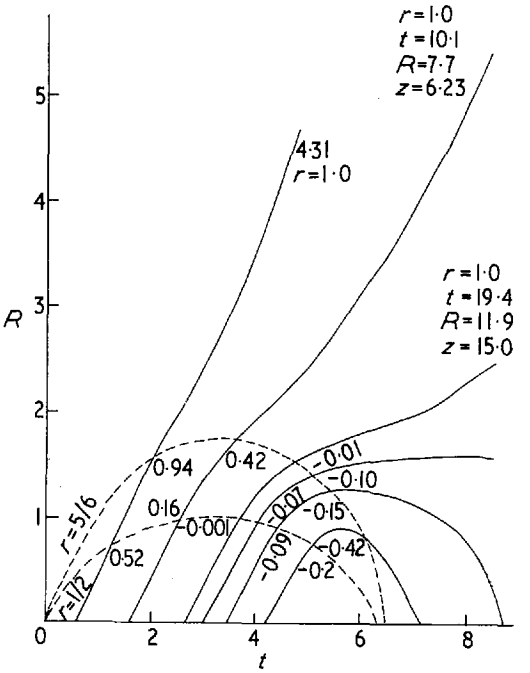


Figure 2. As for figure 1 except that here $f = -r^2 + r^{20}$. This model has a smaller minimum value of $f(-0{\cdot}854$ compared with $-0{\cdot}116$ in Raychaudhuri's example) and light from the centre is trapped after time $t = 3{\cdot}04$. This is earlier than the point of maximum expansion which occurs after a time $t = 3{\cdot}141593$. The red-shift values given refer to light emitted from the centre.

to see if it is satisfied, one would need to integrate (numerically) either (12) or (13). However, a sufficient condition for a ray to escape is clearly $2M(a)/R_{\rm em} < 1$ when the suffix 'em' denotes the value at the point of emission of the light ray. This last result seems to contradict that of Raychaudhuri (1966) who states that a ray would be trapped if it met collapsing matter at any point along its path. We illustrate this point with numerical integrations (see figures 1–3) for several models including that

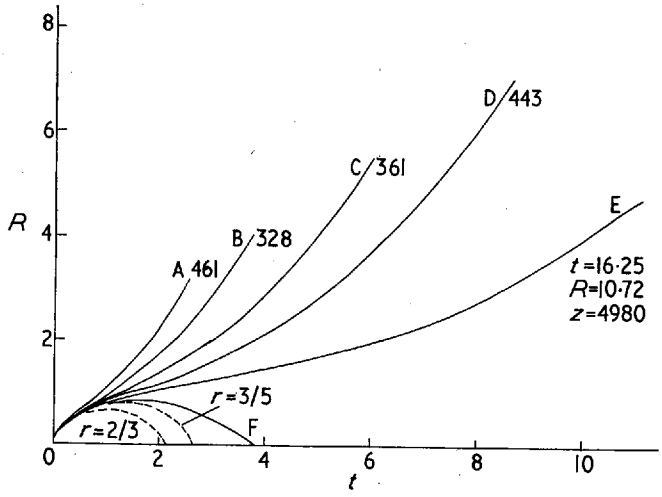


Figure 3. This model contains a region with $R' < 0$ and a neck at $r = 2/3$. Light emitted from shells with r less than about 0·55 can never escape and the cut-off at the neck occurs at about time $t = 0{\cdot}36$. Results are shown for several rays emitted from $r = 0{\cdot}6$. Rays emitted from this radius after a time $t = 0{\cdot}026$ are trapped. The rays labelled by A, B, C, D, E and F are emitted at times 0·001, 0·004, 0·011, 0·019, 0·024 and 0·044 respectively. The explicit forms of M and F are:

$f = -\sin^2 3\pi r/2$, $M = \sin^3 3\pi r/2$, for $r \leqslant 1/3$

$f = -(2+\sin 3\pi r/2)/3$, $M = (2-\cos 3\pi r)/3$, for $1/3 < r \leqslant 1/2$

$f = -(2-\cos 3\pi r/2)/3$, $M = (2-\cos 3\pi r)/3$, for $1/2 < r \leqslant 2/3$

and

$f = \cos 3\pi r/2$, $M = (2-\cos 3\pi r)/3 + (r-2/3)^3$, for $2/3 < r \leqslant 1$.

given by Raychaudhuri. A ray emitted from a shell that is expanding may eventually meet collapsing matter which is within its Schwarzschild radius and hence the ray becomes trapped. For an example of this latter phenomenon one need only consider a section of a positively curved Friedmann universe separated from the expanding universe by a region of empty space.

We now consider the question of whether an event-horizon other than that associated with the Schwarzschild radius develops. From (9*a*), (10*c*) and (12), we obtain for $f > 0$

$$\frac{\mathrm{d}r}{\mathrm{d}t} > \frac{(1+f)^{1/2}}{(M'/3M+f'/2f)R}. \tag{14}$$

From (5) we have

$$\frac{f^{1/2}t}{R} = \left(1+\frac{1}{x}\right)^{1/2} - \frac{\sinh^{-1} x^{1/2}}{x} \tag{15}$$

and so, unless $x \to 0$ along the ray, there exists a positive β such that

$$R < \beta f^{1/2} t.$$

If, however, $x \to 0$ along the ray, by expanding (15) as a power series in x, it follows that for some $\beta_1 > 0$ and sufficiently large t

$$R < \beta_1 M^{1/2} t^{2/3}.$$

From (14) we have

$$\int_{r_{\rm em}}^{r} \frac{(M'/3M + f'/2f) M^{1/2}\, \mathrm{d}r}{(1+f)^{1/2}} > \int_{t_{\rm em}}^{t} \frac{\mathrm{d}t}{\beta_1 t^{2/3}}$$

and since the right-hand integral tends to infinity as $t \to \infty$ and the integrand on the left-hand side is non-zero and is bounded, except possibly for large r, it follows that $r \to \infty$ as $t \to \infty$. A similar argument proves that $r \to \infty$ as $t \to \infty$ if $R < \beta f^{1/2} t$. Thus no event-horizon develops in the region $r > a$. This seems to contradict a result of Som (1968) who considered the special case where the collapsing body is separated from the background by a region of empty space. We merely note than an event-horizon would occur if a positive cosmological constant were introduced.

A further possibility remains: a ray may approach asymptotically the coordinate radius a. For such a ray $\mathrm{d}R/\mathrm{d}t$ must tend to zero, and so it follows from (12) that $R \to 2M(a)$ along the ray in such a way that for large t the ray is passing through collapsing matter. We now show there is only one such ray (for given θ, ϕ and $r_{\rm em}$) and rays which leave at later times cannot escape, whereas those emitted earlier escape in a finite time.

If there were two such rays then all rays emitted at intermediate times would have this property, otherwise they would cross one or other of the two rays and the future null direction at this event would not be well-defined. If we label two such rays by I and II, II being later than I, we have $r_{\rm I}(t) > r_{\rm II}(t)$ and hence (as $R' > 0$) $R_{\rm I}(t) > R_{\rm II}(t)$. Since for both rays $R(t) \to 2M(a)$ as $t \to \infty$, it follows that eventually

$$\mathrm{d}R_{\rm I}(t)/\mathrm{d}t < \mathrm{d}R_{\rm II}(t)/\mathrm{d}t$$

and since this holds for all rays between I and II, then $(\mathrm{d}R/\mathrm{d}t)' < 0$.

However, as $t \to \infty$, $r_{\rm I}(t) \to a$, $R_{\rm I}(t) \to 2M(a)$ and $f\{r(t)\} \to 0$ and so eventually $-2M/fR > 1$. Now, from (9a) and (11a), it follows that by using $\dot{R} < 0$ we obtain

$$(-f)\left(\frac{2M}{-fR}\right)' = \left(\frac{2M}{R}\right)' + \frac{2Mf'}{R(-f)} < 0$$

and so

$$\left(\frac{2M}{R} + f\right)' < 0.$$

Using (13) we have

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)' = \{(1+f)^{1/2}\}' - \left\{\left(\frac{f+2M}{R}\right)^{1/2}\right\}' > 0$$

and since this contradicts the condition above, there can be only one such ray. In the particular case of a collapsing body surrounded by a region of empty space, this ray is emitted as the body passes through its Schwarzschild radius.

5. The frequency shift

If u^a is the four-velocity of the matter and k^a the four-momentum of the radiation, then the fractional shift in frequency z of radiation emitted from the matter and received by an observer moving with the matter is given by

$$1+z = \frac{(k^a u_a)_{\text{emitted}}}{(k^a u_a)_{\text{observed}}} = \frac{k^4{}_{\text{emitted}}}{k^4{}_{\text{observed}}}.$$

Since k^a is given by

$$k^a k_a = 0, \qquad k^a{}_{;b}\, k^b = 0$$

we obtain

$$k^{4\prime} \pm (k^4 R'/\Gamma)^{\cdot} = 0 \tag{16a}$$

and so

$$\frac{\mathrm{d}k^4}{\mathrm{d}r} = \mp \frac{k^4 \dot{R}'}{\Gamma} \tag{16b}$$

with a negative sign for an outgoing ray and a positive sign for an incoming ray.

Equation (16) may be integrated analytically in two special cases. The first case is the class of isotropic cosmological models where $k^4 = 1/S(t)$ with $R = rS(t)$. The second case is where $M' = 0$ (empty space) and here we have

$$k^4 = \frac{\Gamma - \dot{R}}{1-2M/R} = \frac{1}{(1-2M/R)^{1/2}} \left(\frac{\Gamma-\dot{R}}{\Gamma+\dot{R}}\right)^{1/2}. \tag{17}$$

The first term on the right-hand side of (17) is the usual Schwarzschild red-shift factor, whereas the second is due to the motion. Equation (17) is most easily found by transforming from the usual Schwarzschild coordinates to co-moving coordinates by equation (8).

In general, however, it is difficult to integrate (16) and one must resort to numerical integration. Equation (12) was integrated numerically and the red shift estimated using

$$1+z = \tau_{\text{observed}}/\tau_{\text{emitted}} \tag{18}$$

where $\tau_{(r)}$ is the time interval between the arrival of two rays at coordinate radius r. The time interval between emission of the rays is assumed small.

The results for various forms of f and M are given in figures 1–3.

6. Conclusions

A general pressure-free system containing both an expanding region and a region that ultimately collapses has been considered. It has been found that light rays emitted from matter in the collapsing region which meet a closed trapped surface in the collapsing region eventually fall back into a singularity and so an event-horizon develops and at later times no light rays can pass from the collapsing to the expanding region. If, however, a ray does not meet a closed trapped surface then, except for the limiting case considered in § 4, it must escape and furthermore it eventually overtakes all expanding matter and so no event-horizons develop in the expanding region. These results correct those of Raychaudhuri (1966) and Som (1968).

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References

BANERJEE, A., 1966 a, *Phys. Rev.*, **150,** 1086–8.
—— 1966 b, *Proc. Phys. Soc.*, **91,** 794-7.
BONDI, H., 1947, *Mon. Not. R. Astr. Soc.*, **107,** 410–25.
FAULKNER, J., HOYLE, F., and NARLIKAR, J. V., 1964, *Astrophys. J.*, **140,** 1100–5.
JAFFE, J., 1969, *Ann. Phys.*, **55,** 374–90.
LANDAU, L. D., and LIFSHITZ, E. M., 1962, *Classical Theory of Fields* (Oxford: Pergamon).
MISNER, C. W., and SHARP, D. H., 1964, *Phys. Rev.*, **136,** B571–6.
PENROSE, R., 1965, *Phys. Rev. Lett.*, **14,** 57–9.
RAYCHAUDHURI, A. K., 1966, *Proc. Phys. Soc.*, **88,** 545–7.
ROBSON, E. T., 1968, *Ph.D. Thesis*, University of Newcastle.
SCHWARZSCHILD, K., 1916, *Sber. Preuss. Akad. Wiss.*, **7,** 189.
SOM, M. M., 1968, *J. Phys.* A: *Gen. Phys.*, **1,** 393–8.
SYNGE, J. L., 1966, *Mon. Not. R. Astr. Soc.*, **131,** 463–6.