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Applied Optics Section

STATISTICS OF COHERENT LIGHT SPECKLE PRODUCED

BY STATIONARY AND MOVING APERTURES

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ABSTRACT

Formal derivations are given for some of the first and second order statistical properties of coherent light speckle. Experimental verifications are offered for the probability density function of intensity, the autocorrelation function of intensity and the Wiener spectrum. A very large number of measurements were made to obtain statistically accurate results. The automatic scanning and digitising system which was used to make the measurements is described in the Appendix.

Consideration is given to speckle reduction techniques generally; but one method in particular - speckle reduction in the time-averaged image by moving an aperture, smaller than the pupil, in the pupil plane of the imaging lens - is selected and analysed in detail. Exact theoretical descriptions are given to the first and second order statistical properties of speckle patterns reduced in this way. The aberrations of the lens are found to have no bearing on any of the results.

The reduction technique is first considered in the context of speckle patterns formed in the far-field by scattering light from a diffuse surface. The reduction in this instance is effected by moving an aperture so as to expose different parts of the scattering surface; the time-averaging is effected by using a detector with a time-constant which is much longer than the cycle-time of the movement. For two of the second order properties, the autocorrelation function of intensity and the Wiener spectrum, experimental verification of the theory is provided, again based on a very large number of measurements.

The reduction effect is shown to have the same theoretical

description in both the far-field and the image plane schemes. It is further shown that the reduction achieved by the technique is identical to that obtained by illuminating with partially coherent light. For the rotating circular aperture with which the aforementioned verifications were obtained, the equivalent partially coherent source is found to be a quasi-monochromatic incoherent ring source.

Various types of rotating apertures are considered, and our findings enable us to say something of the likely nature of an optimum type of rotating aperture. With some aperture types it is found that a total reduction in speckle is accompanied by a total loss of resolution, but with others the loss of resolution is found to be only partial.

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CHAPTER ONEINTRODUCTIONHistorical

Towards the end of last century, the phenomenon known as 'coronas', or Fraunhofer rings as they are now referred to, had engaged the attention of many of the leading scientists of the day. In a paper,* read to the London Mathematics Society, the Hon. J.W. Strutt, later to become Lord Rayleigh, described the phenomenon in the following words: 'Coronas are formed whenever, between the eye and the source of light, a large number of small opaque bodies, regular in size but irregular in distribution, are to be found.' Rayleigh had chosen to speak on this subject so that he might correct an error in the explanation which had earlier been given to the phenomenon by Verdet.**

'Verdet,' said Rayleigh, 'comes to the conclusion that, the number of partial vibrations being n , the resultant is definitely $\sqrt{n}$ times greater than each component, provided, of course, that n is a large number.' Rayleigh disagreed with this conclusion, contending instead that the resultant amplitude would tend to no definite value. He emphasised the point as he continued: 'As to its phase, he is silent; and if asked the question, would, I think, have found it difficult to explain how a resultant vibration could have a definite amplitude in conjunction with an indeterminate phase.'

Rayleigh had done all but state that the intensity in the corona would show fluctuation, or speckling as we now call it; but he did not actually do this, and instead went on to agree with Verdet, though not for

* Rayleigh (1871).

** Verdet, *Leçons d'Optique physique*, Vol. i, p.297

the same reasons, that there would be no fluctuation. His argument was in many ways sensible: he simply did not believe that there was any light source capable of producing illumination with the required degree of coherence. In his own words, the argument was that 'the sun, or other source of light, is amply sufficient cause. (that there should be no fluctuations) Verdet shows that the disturbance from the sun cannot be considered as a system of plane waves over a space greater than the circle of $1/20$ of a millimetre diameter. Between the vibrations at two points whose mutual distance is much greater than this, there is no permanent relation of phase.'

Nevertheless, only six years after Rayleigh's remarks, Karl Exner, using only a candle flame as a light source, noted a radially fibrous structure when looking at a glass surface on which he had breathed. Exner rightly considered that the intensity fluctuations which he saw were the natural outcome when a number of identical wavetrains with random phase differences were compounded, and he correctly attributed the radial structure to the fact that many colours were present in the illumination. These observations, in the light of Rayleigh's pronouncements, might well have settled the dispute as to whether it was in the nature of coronas to show speckling; but they did not do so at that time. Indeed, it was not until some forty years later that the scientific world was to give widespread acceptance to the idea of speckling.

During the years of the First World War, von Laue (1914, 1916, 1917) produced a series of papers dealing with the scattering of light from a large number of particles. In these, he gave the form of the probability density function of intensity when the amplitude, as well as the phase, shows random variations;* and he also gave what is now recognised as the correct form for the autocorrelation function of the fluctuations. But,

* Although only credited with showing that the negative exponential form applies for the case of equal amplitudes, Rayleigh (1880), on the last page of his paper, quite clearly states that 'From this we see that the law is not altered, even if the components have different amplitudes.'

although his theory was sound and the radial structure observed by him was only due to the spectral bandwidth of his source, von Laue was confused into thinking that speckling effects could not be explained within the structure of classical wave optics. Other scientists of the day, however, were quick to correct this belief,* and at last the corona, its speckled nature in particular, was safely understood.

For the next forty years, the matter of speckling was almost totally forgotten. Save for the occasional stirring, notably by Ramachandran (1943), the world in general and the scientific world in particular had lost interest in what, perhaps, had never been regarded as more than a mere curiosity anyway. At the end of the period, however, the matter was to return in a new context, a context in which it was no longer to be regarded as a curiosity.

On its return, it was heralded by a source of light of a kind which Rayleigh could not have conceived - the LASER. In 1962, just after its invention, Rigden and Gordon observed 'that there was something very unusual about the scattered maser light'; they had noticed 'a remarkable granular or pe^pery nature not present in ordinary light.' After this 'first sighting', there followed a series of short papers on speckle, as it was soon to be termed (Oliver, 1963; Langmuir, 1963; Allen and Jones, 1963; Huntly, 1964) and in these the more obvious properties were recorded. These papers acted as precursors for a much larger number which have since appeared in the literature, but it is more appropriate to defer mention of these until later chapters.

With the invention of the laser, and its subsequent effect upon the dormant field of holography, speckling had returned in a way which demanded much more serious attention than it had been given in the context of coronas. This thesis might be considered as a small part of the response to that demand.

* A historical account of this period, and the development in the understanding of coronas generally, has been given by Hariharan (1972).

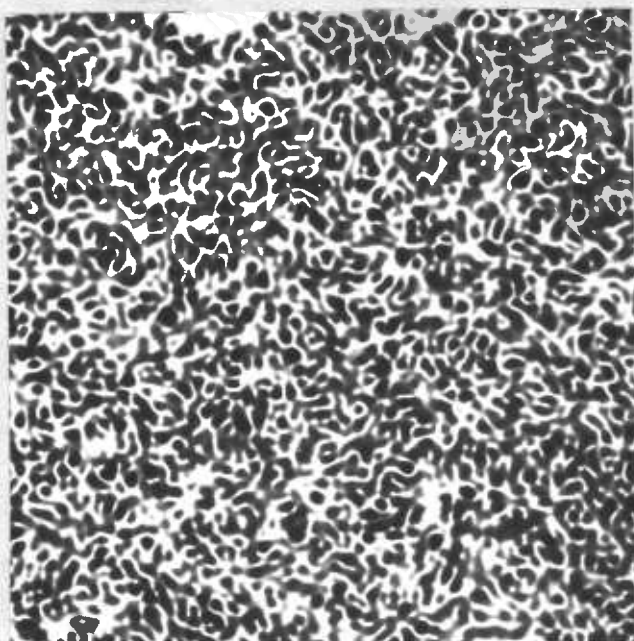
Content

There is a natural splitting of the Thesis into two main parts. The first of these encompasses Chapters Two and Three, and in these chapters consideration is given to speckle patterns as obtained through fixed apertures. In the other part, comprising Chapters Four to Seven, consideration is given to speckle patterns in which the fluctuations are reduced by using moving apertures. But before discussing the content of any of these chapters in detail, it might be profitable to give a brief indication of the type of speckle patterns which are going to be dealt with.

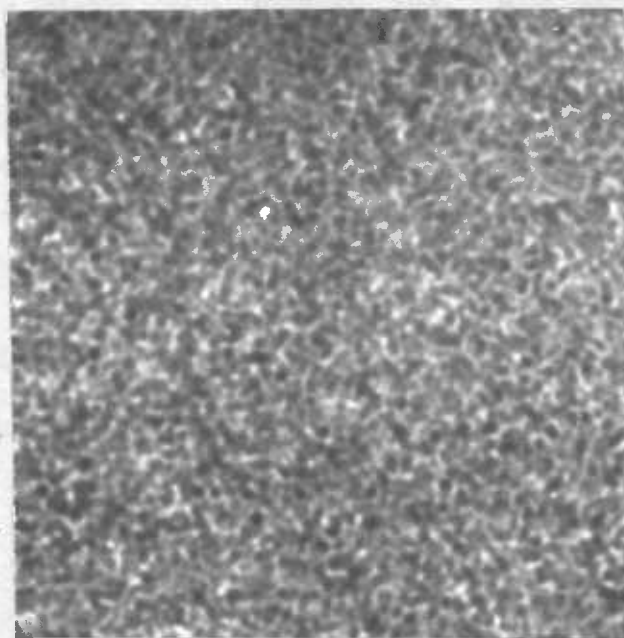
Although casual mention is given to certain other types of speckle pattern, our main interest is with only one type, and this is perhaps the simplest and most commonly occurring type of speckle pattern - namely, the pattern obtained by scattering fully coherent light from a diffuse object whose surface height fluctuations are greater than the light wavelength. In figure 1.1, a pattern formed in this way is illustrated, and this might be recognised as being of the type of pattern around which so much of the literature has been centred.

A reduced speckle pattern has also been shown in the same figure, the reduction having been obtained by recording the time-averaged pattern while moving the aperture through which the speckle was formed. This type of pattern might not seem to comply with the sort of speckle patterns which have just been defined, but, in the sense that it can be considered as the incoherent superposition of a very large number of patterns which do comply, we are justified in listing it as one of the same class.

In Chapters Two and Three, a number of the more important statistical properties are derived for speckle patterns which have been obtained through fixed-aperture systems. The first order statistics are treated in Chapter Two, with measures such as the probability density function of intensity being obtained; while in Chapter Three the second order statistics are treated, with measures such as the autocorrelation function of



(a) A speckle pattern produced in the far-field by light scattered from a fixed, circular scattering area.



(b) A reduced speckle pattern produced in the far-field by rotating a circular aperture so as to expose different areas of the scattering surface. The radius of the rotation was some three times greater than the radius of the aperture itself.

Figure 1.1

intensity being obtained.

There is almost no theoretical content of an original kind in either of these Chapters, so that a reader who is already familiar with the basic statistical descriptions, and free from doubts as to their validity, might be invited to bypass this particular part of the Thesis. However, if he does have any such doubts, then he may emerge from these Chapters with a greater degree of confidence, because in both Chapters experimental results are presented which will be found to agree very closely with the theory.

In Chapter Two, a measured form of the probability density function of intensity is shown, and in Chapter Three measured forms of the autocorrelation function of intensity and the Wiener spectrum are shown. In each case, more than 20,000 intensity measurements were made in producing the result, and this has enabled statistical accuracy at a level which had not been obtained previously. In the Appendix, a description is given to the automatic scanning and digitising system which was used to carry out the long measurement sequences.

The second part of the Thesis is entirely devoted to the reduction of speckle and, in contrast to the first part, most of the theoretical content is of an original kind. Some introductory remarks are made in Chapter Four about speckle reduction generally but, with the exception of this, interest is confined to one method in particular, viz., the reduction of speckle in the time-averaged image by moving the aperture through which the speckle is formed.

The description of the reduction technique has been given, deliberately, in these non-specific terms because there are so many different arrangements in which the technique can be applied. It might be useful at this point to indicate some of these, while, at the same time, noting that the same theoretical description will be shown to apply to all of them. The four main arrangements in which the reduction technique applies are as follows:

- (1) to speckle formed in an image when an aperture is moved in the pupil plane of the imaging lens;
- (2) in the situation identical to the above, but where the speckle is formed in the reconstructed image of a hologram while an aperture is moved in the pupil plane of the reconstructing lens;
- (3) in a Fraunhofer plane speckle pattern, produced from a trans-illuminated scattering area when an aperture is moved over its surface;
- (4) in the situation identical to (3), but where the speckle is formed in the reconstructed image of a Fraunhofer plane hologram while an aperture is moved over the surface of the hologram.

The task of analysing the reduction technique is begun in Chapter Four when case (3), of those just listed, is investigated. The second order statistics of the reduced speckle are treated in this chapter and, in particular, a method is indicated by which the autocorrelation function of intensity may be calculated. Evaluations are made for two cases of a rotating circular aperture, and in one of these the calculated autocorrelation function is compared with the measured function. The agreement between measurement and theory is found to be good, and this is somewhat reassuring in view of the surprising prediction that the 'speckles' will increase in size as the aperture begins to move.

Case (3) is again considered in Chapter Five, but in this instance the first order statistics are dealt with. Approximate and exact methods are indicated which may be used to calculate the probability density function of intensity. Some evaluations are shown for rotating circular apertures, but, because of the complexity entailed in the exact method, the approximate method only was used to produce them.

In Chapter Six, it is shown that the theory which was developed in the two preceding chapters can also be applied to the reduction of

speckle in an image - ie., to case (1). The first and second order statistics are dealt with, and it is found that the aberrations of the lens are of no consequence - although we should immediately qualify this by saying that, if the object is not uniformly illuminated and the same theory is to continue to apply, then the intensity variations over the object would have to be slow in comparison to the point spread function of the lens.

Having shown that cases (1) and (3) are both governed by the same theory, it follows that cases (2) and (4), which might be considered as holographic counterparts of the aforementioned cases, must also be governed by the same theory. If a justification of this claim is necessary, then it might suffice to say that the image-forming pencils which arise from the hologram are sampled by the moving aperture in precisely the same way as it would sample the identical image-forming pencils which would arise directly from the original object.

Chapter Six concludes with a brief discussion of speckle which is formed in an image when the object has been illuminated partially coherently. This matter arises because of an equivalence theorem, due to Welford (1971), by which the theory of imaging with a moving aperture is shown to be the same as imaging under partially coherent illumination. Some statistical results which relate to speckle formed under a certain type of partially coherent illumination are compared to some results calculated by Dainty (1970) for a closely related type of partial coherence. There is found to be sufficient similarity between the two sets of results to sustain the belief that Dainty's theoretical approach and the approach taken in this Thesis are equivalent. None-the-less, a much more detailed analysis is required before this equivalence can be claimed with any degree of confidence.

In the final chapter, Chapter Seven, an attempt is made to answer the question: What is the optimum shape of a rotating aperture? Various aperture forms are considered as we try and answer the question, and the

performance of each one is assessed in terms of the resolution afforded by it and the amount of speckle reduction which it gives. Two conclusions are reached, and these are in no way surprising. The first is that the aperture should be shaped so that, during the movement cycle, equal exposure is given to all parts of the available pupil; and the second is that the aperture, if frozen in time, should expose widely separated regions of the pupil.

CHAPTER TWOFIRST ORDER STATISTICS - FIXED APERTURES

We start off in this chapter by giving a derivation of the probability density function of the intensity in a speckle pattern. The result, which is found to take the negative exponential form, is of great importance because it contains all the information that might be required of the intensity at a point, and the intensity of course is the quantity which is normally of greatest interest. As the derivation is made, various other properties of the speckle pattern will be discovered, and not the least important of these will be the fact that the complex amplitude turns out to be a complex Gaussian process. But, although these results are important, no less important are the assumptions which must be made about the scattering surface and the scattering process.

Not all speckle patterns are endowed with the properties which have just been mentioned; indeed, those which are so endowed belong to a unique type of speckle pattern, although fortunately it is a type which occurs most often in practice. In this Thesis, we will deal only with this unique type of pattern and it is important, therefore, to discover under what conditions such patterns are produced. Because of this, the derivation - or rather the requirements and assumptions made in it - carry as much weight as any results obtained from it.

Following the derivation, consideration is given to the probability density function of intensity, as measured by an aperture which has non-infinitesimal dimensions, and which therefore 'sees' intensity as the integrated sum over an area. Approximate and exact methods of obtaining the function are discussed.

The Chapter continues with the Author's contribution in the area of first order statistics - namely, an experimental verification

of the negative exponential form. The verification was carried out by making over 20,000 recordings of intensity in a speckle pattern, using an aperture much smaller than the size of a 'speckle'.

After the experimental section, a cautionary note is struck when some remarks are made about speckle formed in images. The caution is to the effect that, unless the numerical aperture of the system is low, the statistics of the speckle will not be of the usual kind. The Chapter then concludes with some results taken from the literature which describe some special-case speckle patterns. The 'special cases' are produced by adding speckle patterns together in a variety of ways.

The Probability Density Function of Intensity - Derivation

The derivation, which is modelled closely on the derivation given by Goodman (1963), is given for a speckle pattern produced in the far-field by light scattered from a diffuse surface; it will not be shown that similar results also apply to speckle patterns produced by imaging a diffuse surface with a lens.

Coherent illumination is made to fall on a diffuse surface, and it is assumed that the scattered light is not depolarised in any way. The surface is considered as a collection of elementary scattering points, randomly distributed over the scattering surface, and randomly distributed in depth about the mean level. Suppose now that the incident illumination is represented by the complex scalar amplitude,

$$V'(x,y) = |V'(x,y)| \exp[i \theta(x,y)] \quad 2.1$$

and suppose that the amplitude over the scattering surface (Figure 2.1) just after scattering can be written

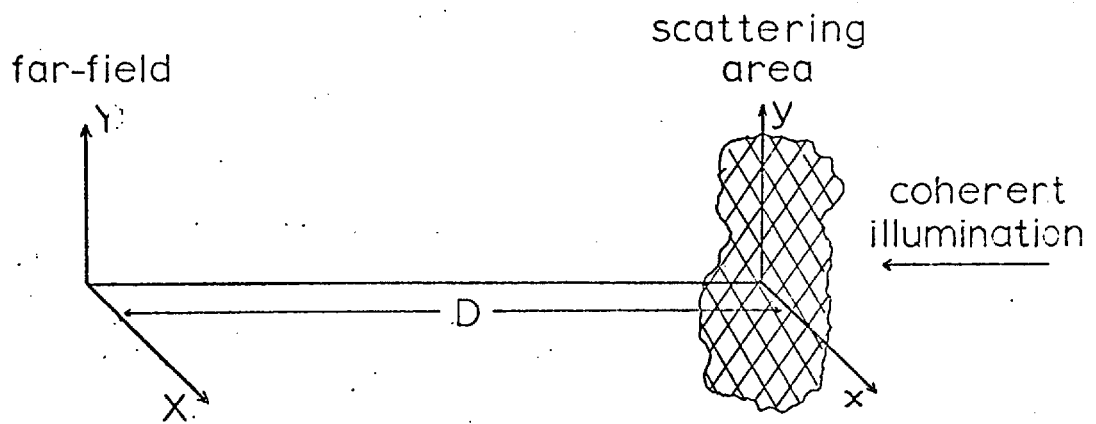


Figure 2.1 The production of a speckle pattern in the far-field,
and the coordinate systems used to describe the arrangement.

$$V'(x,y) = \sum_{k=1}^K \alpha_k \exp[i \beta_k] \delta(x-x_k) \delta(y-y_k) \quad 2.2$$

where K is the number of scatterers, α_k is the amplitude due to the k th scatterer and β_k is the phase of the scattered wave.

The following assumptions are now made about the scattering:

- (1) The scatterers are randomly distributed over the x - y plane with uniform probability.
- (2) The scattered amplitude α_k is a random variable, and all scatterers have statistically independent and identically distributed cross-sections.
- (3) The phases, β , are random variables, and each is statistically independent and uniformly distributed in the range 0 to 2π .
- (4) The scattered phase is independent of the scattered amplitude.

The second and third assumptions have been expressed in a way which, although ideal mathematically, would seem to exclude any surfaces likely to be found in practice. However, they may be relaxed somewhat by allowing correlation to exist between neighbouring scatterers, so long as the correlation distance is small compared to the total extent of the scattering surface. The assumption of uniform probability of phase in the range 0 to 2π can be shown to be a realistic assumption when the surface is rough compared to a light wavelength, and most surfaces seem to possess this property.

Making the usual far-field assumptions (Born and Wolf, 1970, Chapter 8), it can be shown that a Fourier transform relation holds between the amplitude in the far-field and the amplitude in the scattering region:

$$V(x, y) = \iint_{-\infty}^{\infty} V'(x, y) \exp\left[\frac{-2\pi i}{\lambda D} (xX + yY)\right] dx dy. \quad 2.3$$

Combining Equation 2.2 with this equation gives the complex amplitude in the far-field:

$$V(X, Y) = \sum_{k=1}^K \alpha_k \exp[i\beta_k] \exp\left[\frac{-2\pi i}{\lambda D} (x_k X + y_k Y)\right]. \quad 2.4$$

From Equation 2.4, it is evident that the far-field complex amplitude is the sum of a large number of randomly phased, random-amplitude phasors. If the number is very large, then the Central Limit Theorem (Middleton, 1960) may be used to show that $V(X, Y)$ becomes a Complex Normal Process. The real and imaginary parts of this process are identically distributed with zero means and identical variances, $\frac{\sigma_o^2}{2}$, and at any point the real and imaginary parts of the process are statistically independent. Thus, if $V(X, Y)$ is written in terms of its real and imaginary parts,

$$V(X, Y) = V_R(X, Y) + i V_I(X, Y), \quad 2.5$$

then the joint probability density function of V_R and V_I may be written

$$P(V_R, V_I) = \frac{1}{\pi \sigma_o^2} \exp\left[-\frac{(V_R^2 + V_I^2)}{\sigma_o^2}\right]. \quad 2.6$$

This expression is fundamental to the first order statistics of the type of speckle which we consider. The statistics of the intensity and the phase follow directly from it, without any further assumption being necessary. Let us now see how these quantities can be obtained.

The intensity and the phase can be written in terms of the real

and imaginary parts of the complex amplitude:

$$I = V_R^2 + V_I^2; \quad 2.7$$

$$\theta = \tan^{-1} (V_I / V_R) . \quad 2.8$$

From the inverse relations,

$$V_R = I^{\frac{1}{2}} \cos \theta \quad 2.9$$

$$\text{and } V_I = I^{\frac{1}{2}} \sin \theta, \quad 2.10$$

the Jacobian, J , may be formed:

$$J = \begin{vmatrix} \frac{\partial V_R}{\partial I} & \frac{\partial V_I}{\partial I} \\ \frac{\partial V_R}{\partial \theta} & \frac{\partial V_I}{\partial \theta} \end{vmatrix} \\ = \frac{1}{2} . \quad 2.11$$

The Joint Probability Density Function $P(I, \theta)$ is given in terms of $P(V_R, V_I)$ by the probability transform (Davenport and Root, 1958)

$$P(I, \theta) = J \cdot P(V_R, V_I) . \quad 2.12$$

By using Equations 2.6, 2.7 and 2.11, this may be written

$$P(I, \theta) = \frac{1}{2\pi\sigma_c^2} \exp \left[-\frac{I}{\sigma_c^2} \right], \quad 2.13$$

and by integrating over the appropriate variables, the probability density functions of the phase and the intensity can be obtained separately:

$$\begin{aligned}
 P(\theta) &= \frac{1}{2\pi} & 0 \leq \theta \leq 2\pi \\
 &= 0 & \text{otherwise;}
 \end{aligned}
 \tag{2.14}$$

$$\begin{aligned}
 P(I) &= \frac{1}{\sigma_0^2} \exp\left[-\frac{I}{\sigma_0^2}\right] & I \geq 0 \\
 &= 0 & \text{otherwise.}
 \end{aligned}
 \tag{2.15}$$

It follows that the phase is distributed uniformly in the range 0 to 2π and, as we had anticipated, the intensity has a negative exponential distribution. Equation 2.15 is the result which we had sought, but we may gain further insight by considering the first two moments of the intensity.

The first moment of the intensity - the mean intensity - is given by the expression

$$\begin{aligned}
 \langle I \rangle &= \int_{-\infty}^{\infty} P(I) \cdot I \cdot dI \\
 &= \sigma_0^2,
 \end{aligned}
 \tag{2.16}$$

and this result may be used to rewrite the probability density function in a more suitable form:

$$P(I) = \frac{1}{\langle I \rangle} \exp\left[-\frac{I}{\langle I \rangle}\right].
 \tag{2.17}$$

central

The second moment of the intensity - the variance - is given by the expression

$$\begin{aligned}
 \sigma^2 &= \int_{-\infty}^{\infty} P(I) \cdot I^2 \cdot dI - \langle I \rangle^2 \\
 &= \langle I \rangle^2.
 \end{aligned}
 \tag{2.18}$$

The result just obtained is an important result which tells us that the variance of the intensity is equal to the mean intensity squared or, alternatively, by taking square roots, the standard deviation of the intensity is equal to the mean intensity.

The Effect of Aperture Integration

If an attempt were made to measure the probability density function, it would be necessary to make a series of intensity measurements by sampling the speckle pattern through a very small aperture. The aperture could not of course be limitingly small and one therefore asks what effect its size will have on the measured function. Goodman (1965) gave an approximate answer to this question and we now give this our attention.

Approximate Treatment:

Goodman considered that light which entered the aperture was made up of a certain number of independent spatial cells, the intensity within any one cell being constant and uncorrelated with that in any other cell. Interpreting this as the spatial analogue of a problem dealt with by Rice (1954), Goodman then gave the probability density function as being of the Gamma Variate type:

$$P(I) = \frac{a^M I^{M-1} \exp[-aI]}{\Gamma(M)} \quad I \geq 0$$

$$= 0 \quad \text{otherwise,} \quad 2.19$$

where a and M are chosen such that the approximate distribution has the same mean and the same variance as the exact distribution. The appropriate choices of a and M are given by the equations

$$M = \frac{\langle I \rangle^2}{\sigma^2} \quad 2.20$$

$$\text{and } a = \frac{\langle I \rangle}{\sigma^2} = \frac{M}{\langle I \rangle}. \quad 2.21$$

A method for calculating the significant parameter, M , will be indicated in the next chapter.

Rice, in considering the measurement of noise power over a finite time-interval, gives a derivation of Equation 2.19, but he gives no indication as to how this approximation will compare with the true result. This aspect, however, has been treated by Mandel (1959) who, like Rice, was concerned ^{with} the time domain, and, more appropriately, in the speckle domain, by Condie (1966) and Dainty (1970), who tested the approximate formula against experiment, finding broad agreement.

Exact Treatment:

In an Appendix to his Thesis, Condie derives a condition which must be satisfied if the gamma variate approximation is to be exact. The condition which was obtained followed from an exact analysis, effected by translating into two dimensions a one-dimensional noise power problem dealt with by Slepian (1957). Condie was primarily interested in the Cumulents (Kendall, Vol. 1, 1948), which are quantities related to the probability density function; but subsequently, Dainty (1971) and Barakat (1973), following a similar approach, succeeded in producing a form for the probability density function itself.

Dainty was first to produce the exact probability density function, but due to a slight error in his analysis, his result is incorrect. Although his approach was similar to that used by Condie, he attributes the origin of the method not to Slepian, but to Kac and Siegert (1947). The error in his analysis was recently pointed out by Barakat who, at the same time, gave what would seem to be the correct result. Before giving this result, however, two new functions should first be introduced. These are:

- (1) the aperture transmission function, $h(X,Y)$,

(2) the autocorrelation function of the amplitude, $G(X,Y,X',Y')$.

$G(X,Y,X',Y')$ is a second order property which will be more fully discussed in the next chapter; for the moment, it will suffice if we give it its definition in terms of the complex amplitude $V(X,Y)$:

$$G(X,Y,X',Y') = \langle V(X,Y) \cdot V^*(X',Y') \rangle \quad 2.22$$

The probability density function, $P(I)$, measured by the aperture $h(X,Y)$ from a speckle pattern whose autocorrelation function is $G(X,Y,X',Y')$, can be shown (cf., Barakat, 1973) to be given by

$$P(I) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp[-iIt]}{\prod_{j=1}^{\infty} (1 - it\lambda_j)} dt, \quad 2.23$$

where λ_j are the eigenvalues of the Fredholm equation

$$\lambda_j \Psi(x,y) = \iint_{-\infty}^{\infty} G(x,y,x',y') h(x',y') \Psi(x',y') dx' dy', \quad 2.24$$

and where $\Psi(x,y)$ are a set of orthonormal functions (in terms of which the amplitude may be expanded) which satisfy the equation

$$\delta_{ij} = \iint h(x,y) \Psi_j(x,y) \Psi_i^*(x,y) dx dy. \quad 2.25$$

For the moment, no explanation is offered as to how this result arises. In Chapter Five, however, a very similar result will be given for the probability density function of intensity for time-averaged speckle, and in that chapter a proper derivation will be made.

The condition found by Condie for which the gamma variate solution gives an exact representation may be stated in terms of the eigenvalues λ_j : The condition requires that the λ_j should each have the same value, or alternatively the value zero. Barakat has found that this situation does occur when the aperture is much larger than the

distance over which the autocorrelation function takes significant values. In this case, there are a large number of eigenvalues all taking the same value - confluent eigenvalues - whereas the others, with the exception of just a few, are all zero. Barakat refers to this case as the 'short coherence range' case; Goodman would refer to it as a case where there are a large number of independent correlation cells.

The accuracy of Goodman's approximation has never been seriously doubted when there are a large number of independent correlation cells, so that Barakat's result is hardly surprising. It is when the number of correlation cells is small - perhaps only two or three - that doubt is introduced. In Figure 2.2, therefore, a comparison is made between the exact probability density function, calculated by Barakat (1973), and the approximate probability density function, computed using the gamma variate form. The case dealt with concerns a slit aperture and a sinc autocorrelation function. The length of the slit and the width of the autocorrelation function are such as to give, in Goodman's terminology, a small number of independent correlation cells.

The agreement between the three sets of curves is good and this would tend to support Mandel (1959) and Bedard (1967) who also considered the accuracy of the approximation, albeit in the context of a noise power problem. While it would now be dangerous to infer on the basis of this one comparison that Goodman's approximation will always be accurate, it does seem that if the autocorrelation function takes on the sort of shape likely to be found in practice then the approximation will be good.

Experimental Verifications

Two authors, * Condie (1966) and Dainty (1970),

* Ramachandran (1943), in a most interesting paper, verified that the intensity fluctuations in a corona obey the negative exponential law, but his method, although ingenious, perhaps involves too much of the 'human element'.

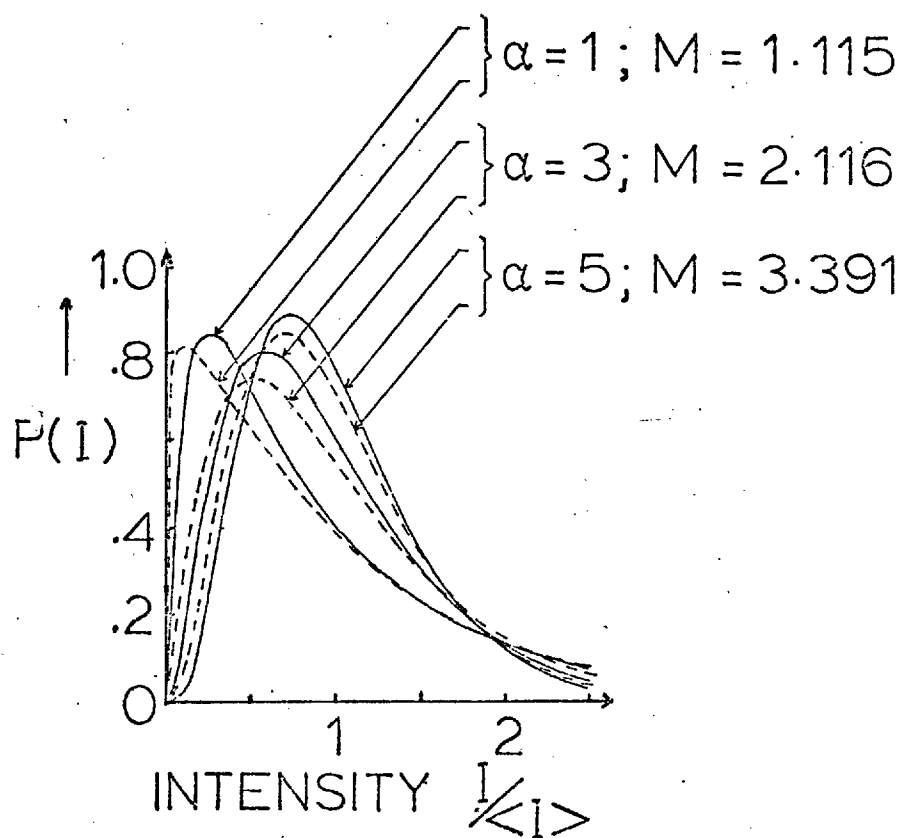


Figure 2.2 Comparisons between the exact and approximate probability density functions of intensity for a slit aperture and a sinc autocorrelation function. The exact functions are shown in firm line. α controls the correlation length and has been defined by Barakat (1973).

have presented experimental evidence to support first order statistics theory. Condie has shown sixteen cumulative distributions - integrals of the probability density function - each one based on 120 measurements; and Dainty has shown two probability density functions, each one based on about 2,400 measurements. Both authors found their results in agreement with theory, but in either case the statistical accuracy was not high.

Figure 2.3 displays a measured histogram which incorporates some 23,000 intensity measurements. The speckle pattern on which the measurements were made was produced in an arrangement of the type illustrated in Figure 2.1, and the recording was carried out automatically using the scanning and digitising system which is described in the Appendix.

The aperture through which the intensity was sampled was small in comparison to the size of an 'average speckle'. Expressed in terms of the 'k unit' - a measure introduced by Dainty (1970,1971) - its size was .07 . For a circular scattering area, the type used in the experiment, the k unit is defined by

$$k = \frac{\text{Diameter of Aperture}}{.61 \lambda D/A} \quad 2.26$$

where A is the radius of the scattering area and D is the distance to the far-field.

Because of the small size of the sampling aperture, the measured intensities would be very nearly equal to the 'point' intensities, and hence we would expect the histogram to show the negative exponential form. Drawn in firm line against the histogram, therefore, is a negative exponential reference curve, scaled so as to have the same average intensity as the measured intensities.

Except for the first bin in the histogram, to which some remarks will shortly be addressed, there would seem to be a very good agreement

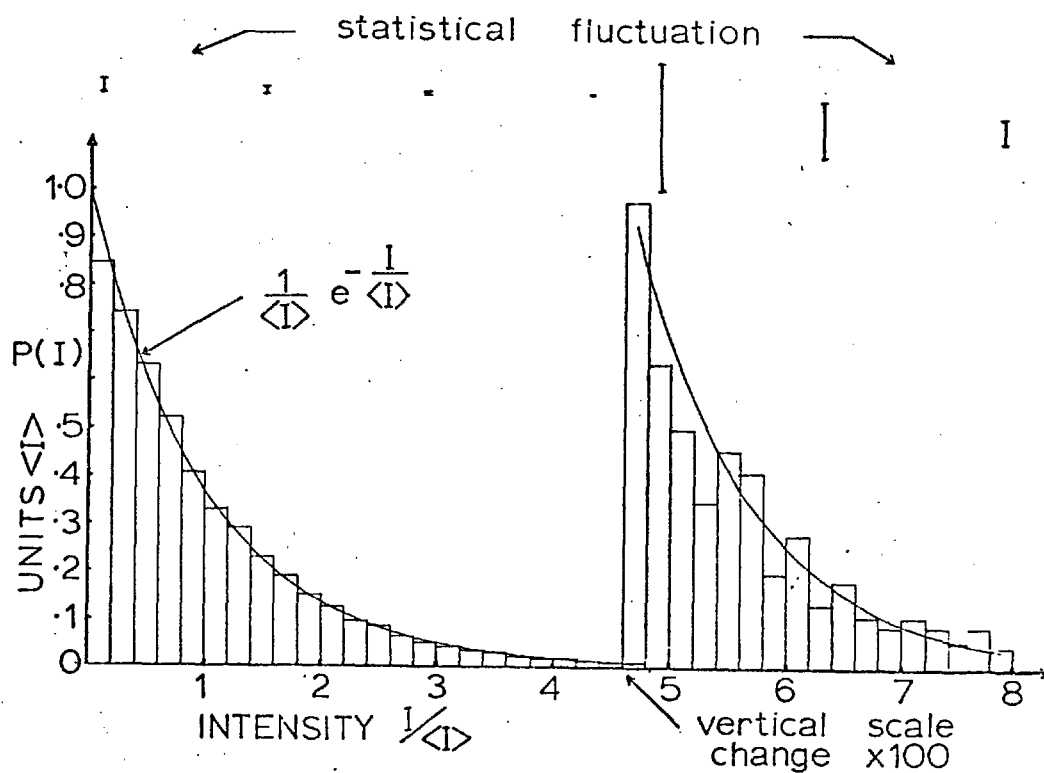


Figure 2.3 A measured histogram based on some 23,000 intensity measurements taken from a speckle pattern. Because the sampling aperture was small in relation to the speckle size, the histogram should have the same form as the negative exponential reference curve which is shown.

between theory and measurement. The error bars, which indicate the anticipated statistical fluctuation, were calculated on the basis of a Poisson distribution of bin height (Kendall, Vol. 2, 1947). Since they are drawn with a length of twice the standard deviation, the test for successful verification of the theory is that, if we centre the appropriate error bar on each bin, then on two occasions in three the bar should intersect the reference curve.

At the intensity value 4.6 (the mean intensity = 1.0), a X100 scale expansion has been made to show the behaviour beyond this point more effectively. Allowing for the fact that the error bars are proportionally longer in this region, the agreement is no less striking. Theory tells us that, of 23,000 intensity measurements distributed in a negative exponential form, about eight of them would be expected to lie beyond the intensity value 8: in fact there were nine. This result bears out the interesting prediction contained in the negative exponential form that there will be some very intense speckles.

The first bin in the histogram is some 6 per cent lower than expected and, since the standard deviation is expected to be a mere $1\frac{1}{2}$ per cent, statistical fluctuation is unlikely to be the cause. What, then, is the cause? There are three possible reasons which are now considered in turn.

Increasing Noise in the Detector:

About one third of the discrepancy is thought to be due to this cause. During the course of the measurements, it was observed that the dark current from the detector was continually increasing, probably due to a gradual rise in the room temperature. Although the dark current was regularly 'zeroed out', the net effect would inevitably be a slight upward biasing of the readings. The average level of bias was estimated at about 1/4000th of the highest intensities recorded, and this perhaps speaks for the difficulties in gaining good statistics at low brightness levels. For the bin width used, it can

readily be seen that artificially high readings produced in this way will only have an appreciable effect on the first bin.

Effect of Aperture Integration:

In choosing an aperture with k-value .07, no effects due to aperture integration were anticipated. However, in the light of Barakat's results (1973), some of which are shown in Figure 2.2, it seems that the effect of aperture integration at low brightness levels is rather more severe than Goodman's approximation would indicate. Although no specific calculations have been made, it is now thought that as much as one third of the discrepancy may be due to this effect.

While mentioning Barakat's results, it might be pointed out that they would also, in some measure, explain the rather high values found with bins 3 and 4. We need not dwell on this, however, because the disparity in this instance is only of the order of the standard deviation.

Effect of Depolarisation during Scattering:

About one quarter of a per cent of the scattered light was found to be depolarised by the scattering process. If it is assumed that a depolarised component is not correlated with the remainder, then the observed speckle pattern would be composed of two incoherently superposing patterns. The probability density function in such a situation was given by Burch (1969) as the convolution of the two separate probability density functions and we now use this result.

If the two contributing patterns have mean intensities m and $(1 - m)$, then the two separate probability density functions may be written

$$P(I) = \frac{1}{m} \exp\left[-\frac{I}{m}\right]$$

and
$$P(I) = \frac{1}{1-m} \exp\left[-\frac{I}{1-m}\right].$$

If the necessary convolution is made and the result is normalised to unity, then the combined probability density function has the form

$$P(I) = \frac{\exp\left[-\frac{I}{m}\right] - \exp\left[-\frac{I}{1-m}\right]}{1 - 2m} \quad 2.27$$

If m is given the value .0025, and this would correspond to the proportion of light which was actually depolarised in the experiment, then the probability density function would have the form shown in Figure 2.4. The function is drawn as far as the intensity value .2, and this corresponds to the width of the first bin in the histogram. Also shown, in dotted line, is a negative exponential reference curve which would be appropriate to a speckle pattern in which no depolarisation had occurred. The difference in the areas contained by the two curves would lead us to expect a lowering in the height of the first bin by about two per cent. Again, this would account for about one third of the total discrepancy.

Having explained the one serious disparity in the histogram, we can perhaps now place considerable trust in the theory governing the first order statistics of speckle. Since the basis of this theory lay in certain assumptions made about the scattering process and, in particular, the assumption about large numbers of scatterers, all these assumptions would seem to have been justified. But, as far as large numbers of scatterers, just how far can we go in claiming that this assumption will be realistic?

Speckle Patterns formed in an Image

With speckle formed in the Fraunhofer plane, it is only necessary to use a scattering area which is large compared to the lateral dimensions of the surface structure to secure the accuracy of the assumption about large numbers of scatterers. With

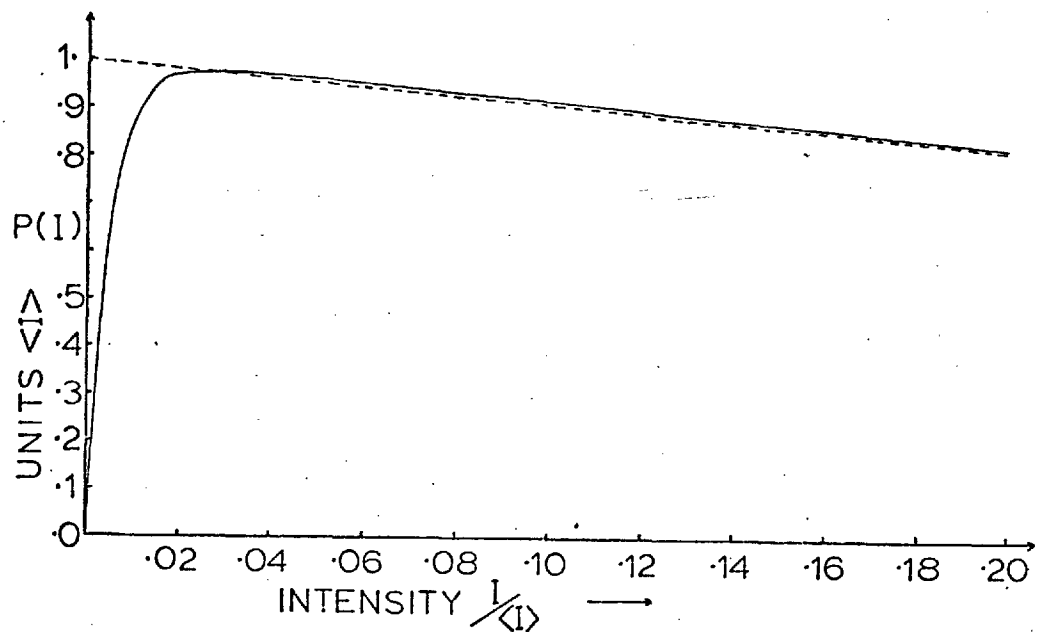


Figure 2.4 The effect of a small amount of depolarisation in the scattered light (one quarter of a per cent) on the probability density function of intensity. The effect is only significant in the low intensity regions.

speckle formed in the image of a diffuse object, however, an additional factor appears which can render this assumption totally invalid. We now devote some attention to this matter.

By referring to to Figure 2.5, we can see that those regions of a diffuse object which contribute to form the speckled image at a given point are found by centring the point spread function of the lens on the corresponding object point. If the numerical aperture of the lens is low, then the point spread function - only the central part of which is significant - will contain a sufficient number of scattering points to ensure normal, or Gaussian, statistics. But if the numerical aperture is not low, then the number of scattering centres will not be sufficient to produce normal statistics and some other statistical description would then apply.

The area contained by the central part of the point spread function of a lens (aberration free with a circular aperture) is given approximately by the quantity

$$\pi \left(\frac{.61 \lambda}{NA} \right)^2 ,$$

where NA is the numerical aperture of the lens. If we now consider that the wavefront which emerges from this area is so highly corrugated that the phase totally decorrelates within the distance λ ,^{*} then the area occupied by an effective scatterer is of the order λ^2 . The number of scatterers contributing to form the image at any particular point is given by the ratio of these areas, and this reduces approximately to the number

$$\frac{1}{(NA)^2} .$$

For a microscope used at low power, a typical numerical aperture

* It is not conceivable that decorrelation could occur in a distance smaller than λ . See, for instance, Suzuki and Hioki (1966) and Beran and Parrent (1963).

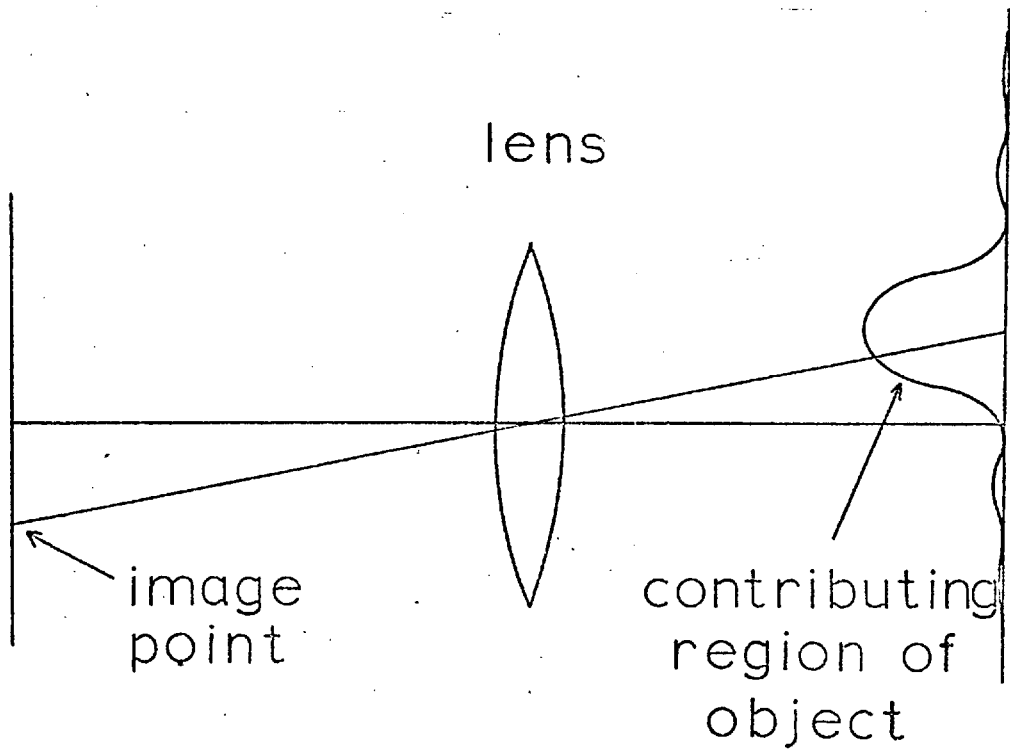


Figure 2.5 The region of a diffuse object which contributes to form the speckle at a given point in the image is found by centring the point spread function of the lens on the corresponding object point.

would be about $\cdot 15$, and this would imply about 50 scatterers contributing to any one point in the image. The type of speckle formed in this case is quite clearly not the type of interest in this Thesis.

In an experiment where a photographic lens was used to image a ground glass diffuser, some measurements were made of the variance in the speckled image. The numerical aperture in object space was about $\cdot 075$ so that, assuming that the structure of the diffusing surface was fine enough, about 200 scatterers were expected to lie within each resolution cell. When the results were analysed, the variance was found to be some 25 per cent higher than 'expected'. This finding is supported in the work of Jakeman and Pusey (1973, a and b) who, in scattering light from liquid crystals, find high variances when the number of scattering points is small.

To complete this Chapter, we now record the probability density functions for some interesting cases where light fields are added together.

Incoherent Addition of Many Speckle Patterns

We have already seen that when two speckle patterns are added together incoherently the resulting probability density function is given by the convolution of the separate probability density functions. When N patterns are incoherently added, then, by the same argument, the new probability density function will be the N -fold convolution of the separate probability density functions. If the patterns all have the same mean intensity, then this N -fold convolution will simply reduce to the gamma variate distribution; but if, on the other hand, the patterns all have different mean intensities, then the N -fold convolution could entail some difficult integration. ^{*} We may avoid this procedure, however, by using the gamma variate form to approximate the situation.

* By summing characteristic functions, the exact form is more easily obtained.

The required probability density function can be written in the usual way:

$$P(I) = \frac{a^M I^{M-1} \exp[-aI]}{\Gamma(M)} \quad 2.28$$

where M and a are defined as before in terms of the mean, $\langle I \rangle$, and the variance, σ^2 , of the distribution to be approximated;

$$M = \frac{\langle I \rangle^2}{\sigma^2} \quad \text{and} \quad a = \frac{M}{\langle I \rangle}.$$

Because of the incoherent addition, both $\langle I \rangle$ and σ^2 can be quickly evaluated. If the mean and the variance of each separate pattern are denoted by $\langle I_i \rangle$ and σ_i^2 , respectively, then

$$\langle I \rangle = \langle I_1 \rangle + \langle I_2 \rangle + \dots + \langle I_N \rangle$$

and $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_N^2.$

By now using the relations

$$\sigma_i^2 = \langle I_i \rangle^2,$$

M and a can be expressed in terms of the mean intensities of each of the N separate patterns:

$$M = \frac{(\langle I_1 \rangle + \langle I_2 \rangle + \dots + \langle I_N \rangle)^2}{\langle I_1 \rangle^2 + \langle I_2 \rangle^2 + \dots + \langle I_N \rangle^2}, \quad 2.29$$

$$\text{and } a = \frac{M}{\langle I_1 \rangle + \langle I_2 \rangle + \dots + \langle I_N \rangle}. \quad 2.30$$

If Equations 2.29 and 2.30 are now combined with Equation 2.28, the required approximation may be obtained.

Coherent Addition of Speckle Patterns

Archbold et al (1970) make casual reference to the coherent superposition of two speckle patterns, saying that 'a third speckle pattern will be produced, differing in detail, but having the same probability of brightness.' This result is intuitively obvious, but Dainty (1972) considers the matter from a more analytic standpoint. He states that since the probability density functions for the real and imaginary parts of the field are Gaussian, and the convolution of a Gaussian function is itself Gaussian, then the probability density function of the new field must also be Gaussian. From this of course it follows that the intensity will have the negative exponential distribution.

Addition of Two Partially Coherent Speckle Patterns

The partial coherence envisaged in this case would be of the type caused by lack of temporal coherence of the source. For example, if two speckle patterns are produced by splitting two beams from the same source, and the one then undergoes a longer optical path traverse than the other, then the addition will be partially coherent in the sense implied. The limiting cases of complete coherence and total incoherence will correspond to the cases where the optical path differences are much shorter and much longer, respectively, than the coherence length of the source.

Barakat (1973 b) has considered this problem and he gives the probability density function of intensity in the form

$$\begin{aligned}
 P(I) = & \frac{1}{\langle I_1 \rangle \langle I_2 \rangle (1-r)} \exp \left[\frac{-I}{\langle I_2 \rangle (1-r)} \right] \\
 & \times \int_0^I \exp \left[\frac{I'}{(1-r)} \left(\frac{1}{\langle I_2 \rangle} - \frac{1}{\langle I_1 \rangle} \right) \right] \\
 & \times I_0 \left\{ \left[\frac{(I-I') I' r}{\langle I_1 \rangle \langle I_2 \rangle} \right]^{\frac{1}{2}} \left[\frac{2}{1-r} \right] \right\} dI'.
 \end{aligned} \tag{2.31}$$

where r ($0 \leq r \leq 1$) is the correlation coefficient of the illumination, $\langle I_1 \rangle$ and $\langle I_2 \rangle$ are the mean intensities of the contributing patterns, and I_0 is the modified Bessel function of the first kind and order zero (Gray and MacRobert, 1952). When r takes the limiting values 0 and 1, the expression reduces to the forms for coherent and incoherent addition which have already been indicated.

Uniform Coherent Background Added to a Speckle Pattern

This matter has been investigated by several authors (Goodman, 1967; Archbold et al, 1970; Burch, 1970; Vilkomerson, 1971; Dainty, 1972 b). If the mean intensity of the speckle pattern is $1 - m$ and the intensity in the uniform background is m then, according to Burch, the probability density function may be written

$$P(I) = \frac{1}{1-m} \exp \left[\frac{-(I+m)}{(1-m)} \right] I_0 \left[\frac{2\sqrt{mI}}{1-m} \right], \quad 2.32$$

where I_0 is again the modified Bessel function. Schwartz (1971) identified this distribution as being of the non-central chi squared type. In Figure 2.6, the form of this distribution is shown when the background and the speckle pattern have the same brightness.

Goodman showed that the standard deviation in this case is given by

$$\sigma = \langle I_N \rangle \left[1 + \frac{2 I_D}{\langle I_N \rangle} \right]^{\frac{1}{2}}, \quad 2.33$$

where $\langle I_N \rangle$ is the mean intensity in the speckle pattern and I_D is the intensity in the uniform background. If k is defined as $I_D / \langle I_N \rangle$, then the standard deviation of the fluctuations divided by the mean intensity in the combined field has been shown by Dainty to be given by

$$\frac{\sigma}{I_D + \langle I_N \rangle} = \frac{(2k+1)^{\frac{1}{2}}}{k+1}. \quad 2.34$$

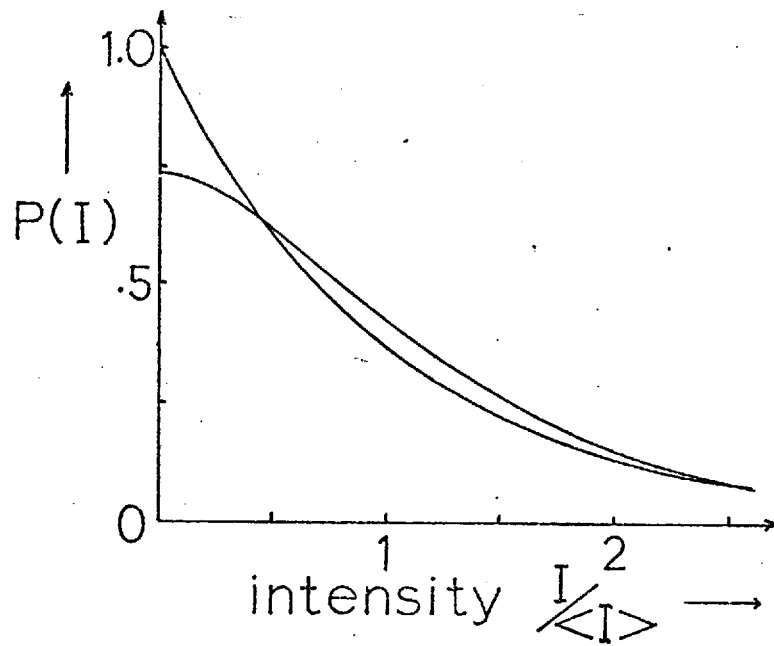


Figure 2.6 The coherent addition of a uniform background field to a speckle pattern when the mean intensity in the speckle pattern is equal to the intensity of the uniform background.

By plotting $\mathcal{C}$ against k , Dainty has illustrated the point that, even if the speckle component is much less intense than the uniform component, the fluctuations in the combined pattern will remain high.

Incoherent Uniform Background Added to a Speckle Pattern

The addition in this case has been considered by Burch (1970) and Archbold et al (1970). According to Burch, the probability density function may be written in the form

$$P(I) = \frac{1}{1-m} \exp \left[-\left(\frac{I-m}{1-m} \right) \right] , \quad 2.35$$

where $1 - m$ is the mean intensity in the speckle field and m is the intensity in the uniform background. The result is perhaps intuitively obvious.

CHAPTER THREESECOND ORDER STATISTICS - FIXED APERTURES

We began the last chapter by deriving some of the first order statistical properties of speckle. In this chapter, we shall begin in a similar manner, except that now attention will be devoted to the second order properties. The quantity which we shall derive in particular is the autocorrelation function of intensity and, again, the exposition will follow the style of Goodman (1963).

The autocorrelation function describes the average degree of correlation between the intensities at two points of a prescribed separation, but, in slightly less analytic terms, it can be considered as simply a function which indicates the 'size' of a typical speckle in the pattern. Complementary to the autocorrelation function, is a function, sometimes known as the power spectrum, but which we shall refer to as the Wiener spectrum. The term 'Wiener spectrum' was first suggested by Clark Jones (1955), and it might be considered as the optical analogue of the term power spectrum which is encountered in electrical theory. The autocorrelation function and the Wiener spectrum are rated as having equal importance in this chapter and, in consequence, any remarks which are addressed to one will tend to be followed by some corresponding remarks, addressed to the other. A Fourier transform relation exists between the autocorrelation function and the Wiener spectrum, so that knowledge of one will imply knowledge of the other.

After a discussion which ranges over some different aspects of the two functions and a discussion of those authors who have contributed to the literature in the area generally, our attention will be directed towards the problem of measurement of the two functions. If a measurement were made of either function, a difficulty would arise, similar to

that met in the last chapter, in that the measurement would be biased in some way due to the finite size of the measuring aperture. Fortunately, it is not difficult to take account of the biasing in this instance, and the chapter therefore continues by giving the general relations between the true forms of the autocorrelation function and the Wiener spectrum and those forms which are actually measured through the aperture.

The autocorrelation function at zero lag ^{*} (or the volume contained under the Wiener spectrum, since this is an equivalent measure) is simply the variance, and so the aforementioned relations enable the measured variance to be given in terms of the measuring aperture. Since the mean and the variance are all that is required to produce a gamma variate approximation for the probability density function, the result which we shall obtain is of some importance in the area of first order statistics.

The remainder of the chapter is devoted entirely to a presentation of the Author's contribution in the area of second order statistics - namely, experimental verification of the forms of the autocorrelation function and the Wiener spectrum for speckle formed in the far-field from a circular scattering area. A detailed description is given of the method which was used to obtain the results. Included in this description, are items such as the mathematical relations which were used to translate the measured quantity - intensity - into the desired functions.

The Chapter closes with a discussion of the results. The discussion includes topics such as statistical fluctuation errors, errors of a non-statistical nature and smoothing procedures, the latter being important in relation to the Wiener spectrum since this function is not ergodic.

* 'Lag' is a term taken from spectral analysis. In our context it simply means a distance measured in the speckle pattern.

The Autocorrelation Function of Intensity

The autocorrelation function, $C(\xi, \eta)$, is a function which indicates the average degree of correlation between the intensities at two points separated by the vector distance (ξ, η) . The function is formally defined by

$$C(\xi, \eta) = \langle I(x + \xi, y + \eta) I(x, y) \rangle - \langle I \rangle^2, \quad 3.1$$

where the averaging brackets denote the ensemble average.

There are several preliminary remarks which might be made about the autocorrelation function as it is displayed in Equation 3.1:

- (1) When the separation, (ξ, η) , is reduced to zero, the function will take its maximum value, and that value is clearly the variance.
- (2) For separations large enough to cause total decorrelation, the function will take the value zero.
- (3) The function has been written in a way which assumes no dependence on X or Y , ie., a stationarity assumption has been made. This is an important property which will emerge naturally in the treatment.

In deriving a more specific form for the autocorrelation function of intensity, it is convenient to begin by considering the autocorrelation function of complex amplitude, $C_V(\xi, \eta)$. A formal definition of this function is therefore given:

$$C_V(\xi, \eta) = \langle V(x + \xi, y + \eta) V^*(x, y) \rangle. \quad 3.2$$

where, again, the averaging brackets denote the ensemble average.

By making use of Equation 2.4, the above equation can be written

in more explicit terms:

$$C_V(\xi, \eta) = \sum_{k=1}^K \left\langle |\alpha_k|^2 \exp \left[\frac{2\pi i}{\lambda D} (x_k \xi + y_k \eta) \right] \right\rangle, \quad 3.3$$

where the various cross-products have been omitted since they will reduce to zero when the averaging process is performed. If the scatterers are now considered to be packed sufficiently closely on the scattering area, then the scattered intensity, $|\alpha_k|^2$, can be represented by a continuous function, which we denote by $S(x, y)$, and Equation 3.3 can then be written as an integral equation:

$$C_V(\xi, \eta) = \iint_{-\infty}^{\infty} S(x, y) \exp \left[\frac{2\pi i}{\lambda D} (x \xi + y \eta) \right] dx dy. \quad 3.4$$

The intensity function, $S(x, y)$, is proportional to the incident intensity distribution over the scattering area, so that once this distribution is known the autocorrelation function of complex amplitude is also known, apart from a multiplicative constant. The autocorrelation function of intensity is simply related to the autocorrelation function of complex amplitude. By now finding the relation, and combining it with Equation 3.4, we will produce the required result.

The right hand side of Equation 3.1 can be expanded in terms of the real and imaginary parts of the complex amplitude:

$$\begin{aligned} C(\xi, \eta) = & \left\langle V_R^2(x+\xi, y+\eta) V_R^2(x, y) \right\rangle \\ & + \left\langle V_R^2(x+\xi, y+\eta) V_I^2(x, y) \right\rangle \\ & + \left\langle V_I^2(x+\xi, y+\eta) V_R^2(x, y) \right\rangle \\ & + \left\langle V_I^2(x+\xi, y+\eta) V_I^2(x, y) \right\rangle - \langle I \rangle^2 \end{aligned} \quad 3.5$$

Because V_R and V_I are real, zero-mean, Gaussian random variables, Middleton (1960) states that the following must be true:

$$\begin{aligned} \langle V_1 V_2 V_3 V_4 \rangle = & \langle V_1 V_2 \rangle \langle V_3 V_4 \rangle + \langle V_1 V_3 \rangle \langle V_2 V_4 \rangle \\ & + \langle V_1 V_4 \rangle \langle V_2 V_3 \rangle . \end{aligned} \quad 3.6$$

If we use this relation to further expand Equation 3.5 then, by ignoring terms which average to zero and cancelling certain terms with $-\langle I \rangle^2$, we are left with the form

$$\begin{aligned} C(\xi, \eta) = & 2 \left[\langle V_R(x+\xi, y+\eta) V_R(x, y) \rangle^2 \right. \\ & \left. + \langle V_I(x+\xi, y+\eta) V_I(x, y) \rangle^2 \right] . \end{aligned} \quad 3.7$$

If a comparison is made between this equation and Equation 3.2, the autocorrelation function of intensity will be found to be just the squared modulus of the autocorrelation function of complex amplitude. By combining this result with Equation 3.4, the required autocorrelation function can be written as the squared modulus of the Fourier transform of the intensity distribution over the scattering area:

$$C(\xi, \eta) = \left| \iint_{-\infty}^{\infty} S(x, y) \exp \left[\frac{2\pi i}{\lambda D} (x\xi + y\eta) \right] dx dy \right|^2 . \quad 3.8$$

Since this equation shows no dependence on either X or Y , the autocorrelation function of intensity, as we had anticipated, is stationary.

The autocorrelation function is defined in terms of an ensemble average, and, generally, to measure such a function will require an average value to be found by considering a very large number of independent scattering areas. However, because of the stationarity property, which is common to all speckle patterns considered in this Thesis,

measurement is a much simpler matter than this.

The stationarity property allows a double-edged simplification in the measurement procedure. First, it means that we can obtain the ensemble average directly from a space average, that is, an average obtained from a sequence of measurements across a single pattern; and second, it means that the average we finally derive is appropriate to all points in the pattern. As a result, we can now think and speak in terms of space averages in this Thesis, and consider averaging brackets to mean the space average; but, in any results obtained, always the ensemble average will be implied. The ensemble average is a much more powerful average than the space average in that it applies not just to one particular scattering area, but to all scattering areas within the prescribed family.

The Autocorrelation Function for a Uniformly Illuminated, Circular Scattering Area:

For a uniformly illuminated, circular scattering area, a solution of Equation 3.8 will be identical to that obtained by calculating the point spread function for an aberration free lens with a circular aperture. This is a well known result, so that we can immediately write

$$C(\xi, \eta) = \langle I \rangle^2 \left[\frac{2 J_1 \left(\frac{2\pi A}{\lambda D} \sqrt{\xi^2 + \eta^2} \right)}{\frac{2\pi A}{\lambda D} \sqrt{\xi^2 + \eta^2}} \right]^2, \quad 3.9$$

where A is the radius of the scattering area, D is the distance to the far-field, J_1 is the first order Bessel function, and the normalization is chosen so that the value of the function at zero lag is equal to the variance (cf., Equation 2.18).

The Wiener Spectrum

The Wiener spectrum is a function which shows how the variance in the speckle pattern is distributed with spatial frequency. Goldfischer (1965), by extending a one-dimensional analysis used by Rice (1954), showed that in a Fraunhofer plane pattern the spectrum is given by the self-convolution of the intensity distribution in the scattering area. This result is confirmed in that it is essentially the same as the result given by the Wiener-Khinchin theorem (Bendat, 1958) which, for a stationary random process, states that the Wiener spectrum is the Fourier transform of the autocorrelation function.

The Wiener Spectrum for a Uniformly Illuminated, Circular Scattering Area:

According to Goldfischer, the Wiener spectrum in this case will have the same form as the optical transfer function for an aberration free lens with circular aperture; and according to the Wiener-Khinchin theorem, the spectrum will have the same form as the two-dimensional Fourier transform of Equation 3.9. Either of these representations enables the spectrum to be written in the form

$$N(u,v) = \frac{8\langle I \rangle^2}{\pi^2 t_m^2} \left[\cos^{-1} \left(\frac{\sqrt{u^2 + v^2}}{t_m} \right) - \frac{\sqrt{u^2 + v^2}}{t_m} \sqrt{1 - \frac{u^2 + v^2}{t_m^2}} \right], \quad 3.10$$

where t_m is the cut-off frequency which is given by

$$t_m = \frac{2A}{\lambda D}. \quad 3.11$$

It can easily be checked that the normalisation of Equation 2.10 is such that the volume contained under the function is equal to the variance.

In some accounts, the Wiener spectrum is shown with a delta function situated at the origin. This arises when the spectrum is produced from an autocorrelation function which has been defined so as

to omit the constant term - $\langle I \rangle^2$ (cf., Equation 3.1). However, since there is no great significance in either including or not including this term, there is no special significance in the delta function.

Autocorrelation Function and Wiener Spectrum for Speckle in an Image

So far our consideration of the autocorrelation function and the Wiener spectrum has been restricted to speckle patterns formed in the far-field. There is no real need for this restriction, however, because the treatment which we have given would also apply to speckle patterns formed in an image. In general, an expression which is appropriate to one scheme will also be appropriate to the other, subject to the condition that the angular shape of the scattering area as seen in the far-field is identical to that of the lens pupil, as seen in the image plane. This might be illustrated by saying that Equations 3.9 and 3.10 can be made appropriate to an imaging case just by redefining A to be the radius of the lens pupil and D to be the distance from the pupil plane to the image plane.

The Relation between the Autocorrelation Function and the Wiener Spectrum

It has already been stated that a two-dimensional Fourier transform relation exists between the autocorrelation function and the Wiener spectrum. A formal statement of this is now given:

$$N(u,v) = \iint C(\xi, \eta) \exp[-2\pi i(\xi u + \eta v)] d\xi d\eta. \quad 3.12$$

Isotropic Speckle Patterns:

When the limiting aperture is circular, there is no preferred direction in the speckle pattern and the pattern is then described as isotropic. For such patterns, the autocorrelation function and the Wiener spectrum are radially symmetric functions, and Bracewell

(1956) has shown that their central sections (which are sufficient to specify the functions fully) are related via the Hankel transform. If the arguments in the respective sections are r and t , where

$$t = \sqrt{u^2 + v^2} \quad \text{and} \quad r = \sqrt{\xi^2 + \eta^2} ,$$

then the transform can be written

$$N(t) = 2\pi \int_0^{\infty} C(r) J_0(2\pi tr) r dr , \quad 3.13$$

where J_0 is the zero order Bessel function of the first kind. Since the results given in this and later chapters concern only with radially symmetric functions, Equation 3.13 has more relevance in this Thesis than Equation 3.12.

Authors Concerned with Second Order Statistics

Apart from Goodman, various other authors have given accounts relating to the second order statistics of speckle. In the accounts by Skinner (1964), Goldfischer (1965), Suzuki and Hioki (1966, a and b), Tiziani (1967), and Eliasson and Mottier (1971), consideration was given to speckle formed in the Fraunhofer plane, and in each case their findings accorded with those of Goodman. Enloe (1967), Burckhardt (1970), Gabor (1970), Dainty (1970), and Lowenthal and Arsenault (1970) each considered speckle formed in an image plane, and all agreed that the autocorrelation function is in this instance given by the squared modulus of the Fourier transform of the aperture function of the lens*, which is of course a result similar to that for speckle formed in the Fraunhofer plane.

Because Goodman's results were not published, Goldfischer is often

* The term 'transmission function' might be more appropriate than 'aperture function', because the aberrations of the lens are of no significance.

accredited with being first to give the form of the autocorrelation function. Although this might not be strictly true, he was nevertheless first to give the form of the Wiener spectrum. But while Goldfischer was only concerned with the Fraunhofer plane, Enloe, Lowenthal and Arsenault, and Dainty showed that the Wiener spectrum could be similarly described in an image plane.

Enloe's result is particularly interesting because it contains a term which takes account of the finite number of scatterers within the resolution element of the lens. After giving the result, however, he observes that 'for a truly diffuse surface' this term can be ignored. The present author would have preferred him to have added the further proviso that the numerical aperture of the lens should be low.

Lowenthal and Arsenault, and Dainty obtained results in agreement with those of Enloe, but the manner of their derivation was entirely different. These authors considered that speckle in an image is formed as a result of a linear filtering of complex amplitude, followed by a square-law detection process. This approach led to a conceptually simple understanding of speckle, and Dainty put it to profitable use by giving the form for the Wiener spectrum in the case of speckle formed under partially coherent illumination. His result will be considered further in Chapter Six.

Relations between Measured and True Functions

The computation of the true autocorrelation function or the true Wiener spectrum from those functions which are actually measured by a finite aperture is carried out in broadly the same way as it would in time series analysis. However, although time series analysis has been extensively treated in the literature (Jenkins and Watts, 1969; Blackman^{and Tukey,} 1959; Bendat and Piersol, 1971), we cannot quote results which are immediately appropriate to our task because this literature deals with an inherently one-dimensional problem. Nevertheless, a relevant two-

dimensional analysis has been carried out in connection with photographic/granularity by Clark Jones (1955), and we do in fact draw on some of his results.

When the speckle pattern is seen through a finite aperture whose transmittance function $h(X,Y)$ is given by

$$\begin{aligned} h(X,Y) &= 1 && \text{inside aperture} \\ &= 0 && \text{otherwise,} \end{aligned}$$

then the measured autocorrelation function will be the double convolution of $h(X,Y)$ with the true autocorrelation function (Clark Jones). Denoting the measured autocorrelation by a suffix, we can write

$$C'(\xi, \eta) = \iiint_{-\infty}^{\infty} C(\xi - x' - x'', \eta - y' - y'') h^*(-x', -y') h(x'', y'') dx' dx'' dy' dy''. \quad 3.14$$

By either continuing to follow Clark Jones, or by making use of the convolution theorem (Bracewell, 1965), the equivalent result for the Wiener spectrum may be obtained:

$$N'(u, v) = N(u, v) |H(u, v)|^2, \quad 3.15$$

where $H(u, v)$ is the Fourier transform of the aperture function $h(X, Y)$:

$$H(u, v) = \iint_{-\infty}^{\infty} h(x, y) \exp[-2\pi i(ux + vy)] dx dy. \quad 3.16$$

Later in this chapter, some specific relations will be generated from Equations 3.14 and 3.15. These will be relevant to a slit aperture, the type used throughout this Thesis for the measurement of second order properties.

The Variance as Seen through a Finite Aperture

In the previous chapter, an approximate probability density

function was given for speckle as seen through a finite aperture (Equation 2.19), but the form given depended on our knowing the variance. Goodman (1965) gave expressions for the variance by considering the autocorrelation function; we now give equivalent expressions by considering the Wiener spectrum.

The volume contained under the measured Wiener spectrum is, by the definition of the measured spectrum, equal to the measured variance. Thus, denoting the measured variance by a suffix, we can write

$$\sigma'^2 = \iint_{-\infty}^{\infty} N'(u,v) du dv . \quad 3.17$$

By using Equation 3.15, a more useful expression is obtained for the measured variance:

$$\sigma'^2 = \iint_{-\infty}^{\infty} N(u,v) |H(u,v)|^2 du dv . \quad 3.18$$

The Variance as Seen through a Circular Aperture for a Speckle Pattern Produced from a Circular Scattering Area:

The form of the true spectrum has already been given for this case (Equation 3.10), and since it has circular symmetry we may write it in terms of its radial coordinate, t :

$$N(t) = \frac{8\langle I \rangle^2}{\pi^2 t_m^2} \left[\cos^{-1}\left(\frac{t}{t_m}\right) - \frac{t}{t_m} \sqrt{1 - \left(\frac{t}{t_m}\right)^2} \right] , \quad 3.19$$

where $t = \sqrt{u^2 + v^2}$.

The measuring aperture in this case is defined by

$$\begin{aligned} h(X,Y) &= 1 && \text{for } X^2 + Y^2 \leq a^2 \\ &= 0 && \text{otherwise ,} \end{aligned} \quad 3.20$$

where a is the radius of the measuring aperture. Since $h(X,Y)$

has a radially symmetric form, its Fourier transform $H(u,v)$ must also be radially symmetric. Whence, we may write

$$H(t) = \frac{J_1(2\pi ta)}{\pi ta}, \quad 3.21$$

where, as before, $t = \sqrt{u^2 + v^2}$, and we have normalised to unity.

If Equation 3.18 is expressed in its polar coordinate form, and Equations 3.19 and 3.21 are combined into it, then, upon writing $w = t/t_m$ and carrying out the circular integration, the measured variance can be written in the form

$$\sigma'^2 = \frac{16 \langle I \rangle^2}{\pi^3 t_m^2 a^2} \int_0^1 \left[\cos^{-1}(w) - w \sqrt{1-w^2} \right] \left[\frac{J_1(2\pi a w t_m)}{w} \right]^2 dw. \quad 3.22$$

Equation 3.22 when t_m is Expressed in k-units:

A more meaningful

relation between the measured variance and the aperture size can be obtained by expressing the cut-off frequency, t_m , in terms of k-units. The k-unit has been defined by Equation 2.26. From that definition, it follows that $t_m = .61 k / a$. Whence

$$\sigma'^2 = \frac{16 \langle I \rangle^2}{\pi^3 k^2 (.61)^2} \int_0^1 \left[\cos^{-1}(w) - w \sqrt{1-w^2} \right] \left[\frac{J_1(2\pi k w .61)}{w} \right]^2 dw. \quad 3.23$$

The dependence of the measured variance on the k value may be appreciated by referring to Dainty (1970) where the ratio $\langle I \rangle^2 / \sigma'^2 (= M)$ is plotted against k. A similar plot is shown in Figure 3.1, where M (cf., Equation 2.20) is again plotted against k. Since the behaviour for small values of k is interesting, some care has been taken to show this behaviour accurately.

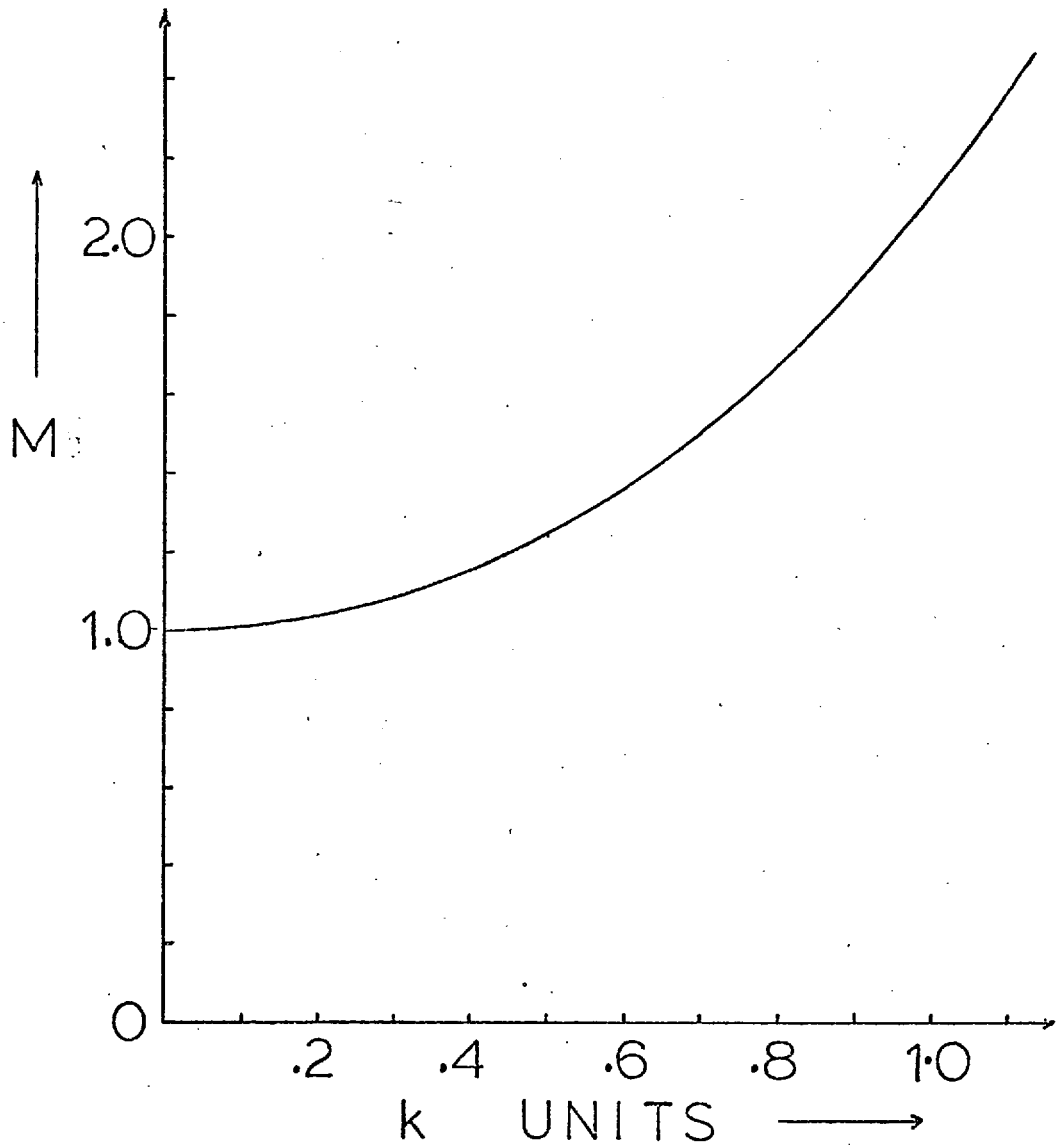


Figure 3.1 The effective number of independent speckle patterns, M , versus the aperture size expressed in k -units.

Measurement of the Second Order Statistical Properties

Until recently, there was very little experimental evidence underlying the second order statistics of speckle. Various authors had published accounts offering experimental verification of the theory, but none could be considered as entirely satisfactory.

In a paper by Goldfischer (1965), a qualitative verification was given of the form of the autocorrelation function for speckle patterns produced from variously shaped scattering areas. His method made use of two identical photographs of the speckle pattern, and these, after they had been suitably arranged in an optical system, enabled him to produce an analogue version of the autocorrelation function. His method is elegant, and the results shown by him are in many ways convincing; but to be totally convincing in this instance it is necessary to take a quantitative approach.

Such an approach was taken by two authors, namely, von Hohn (1968) and Dainty (1970). Both made measurements of the Wiener spectrum, using differing methods, and both found broad agreement with the theory. In detail, however, their results conflicted with the theory.

The present Author also adopted the quantitative approach (McKechnie, 1973), and measurements were obtained for the autocorrelation function and the Wiener spectrum. The remainder of this Chapter is now devoted to a description of the results along with an account of the method used to obtain them.

The speckle patterns upon which measurements were made were formed from uniformly illuminated, circular scattering areas, and the patterns were thus endowed with the properties of stationarity and isotropy. These two properties, as we have already indicated, allow important simplifications in the measurement procedure. The stationarity property makes it only necessary to measure space averages, and the

property of isotropy makes it only necessary to measure sections through the functions of interest.

In general, when the speckle pattern is non-stationary and non-isotropic, the autocorrelation function and the Wiener spectrum will have asymmetrical forms and will take different shapes in different regions of the pattern.* The measurement of such functions would of course be a formidable task, but fortunately it was one which did not have to be undertaken.

Although the measurement procedure which was adopted was expected to yield all relevant information, it had reduced, because of the type of patterns considered, to the comparatively simple matter of scanning across single patterns in a single arbitrary direction. The quantity which was actually measured during the scans was of course intensity; the autocorrelation function and the Wiener spectrum were subsequently calculated from the measurements.

Since a speckle pattern is an inherently random pattern, any particular estimate of a quantity derived from it, based on the average of a number of measurements, will be subject to statistical fluctuation; the fluctuation will become less, however, as more and more measurements are included in the estimate. 20,000 measurements were entailed in each of the result sets given and this has enabled statistical accuracy on a level which had not been obtained previously. As an example of this accuracy, the standard deviation on each of the points marked on the Wiener spectra is about 3 per cent. But despite their small magnitude statistical fluctuations remain the overriding source of error and because of this it is only these errors which are included in the results. Other error sources were adjudged to contribute rather less by comparison so that, although further reference is made to them in

* Such speckle patterns could be obtained by illuminating an asymmetrical scattering area with polychromatic light, but these are outside the scope of this Thesis.

the text, they are not indicated in the results.

The aperture through which the speckle pattern was observed - the aperture of the photomultiplier - was chosen in the form of a very long, very narrow slit. In choosing a slit, we followed the choice made by Clark Jones (1955) who was concerned with measuring the Wiener spectrum of photographic granularity. The slit, as we shall now see, is a shape which readily lends itself to the computation of the quantities we want from those we actually measure directly.

Relations between the True Functions and Those Measured by a Slit

To obtain measurements from the speckle pattern, the slit aperture was scanned across the pattern in a direction at right angles to its length. This is illustrated in Figure 3.2. At regular intervals, recordings were made of the integrated intensity, $I'(X,Y)$, which passed through the slit. The measured autocorrelation function, $C'(\xi,\eta)$, is defined in terms of the intensities so obtained:

$$C'(\xi,\eta) = \langle I'(x+\xi, y+\eta) I'(x,y) \rangle - \langle I \rangle^2. \quad 3.24$$

Since the scan was made in one direction only, the η dependence in Equation 3.24 may be dropped without loss of generality. Thus,

$$C'(\xi,0) = \langle I'(x+\xi, y) I'(x,y) \rangle - \langle I \rangle^2. \quad 3.25$$

The measured autocorrelation function and the measured Wiener spectrum are related by the Wiener-Khinchin theorem:

$$C'(\xi,\eta) = \iint_{-\infty}^{\infty} N'(u,v) \exp[2\pi i(\xi u + \eta v)] du dv. \quad 3.26$$

When the argument η is set to zero, this can be written

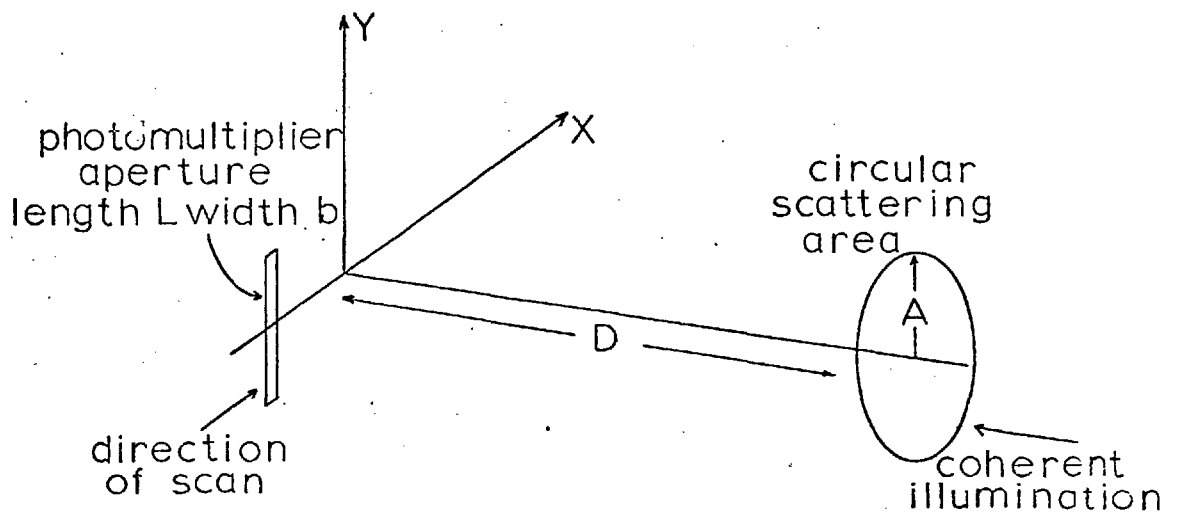


Figure 3.2 The arrangement by which the speckle pattern was scanned and measured with a slit aperture.

$$C'(\xi, 0) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} N'(u, v) dv \right] \exp(2\pi i u \xi) du, \quad 3.27$$

and by using Equation 3.15 this becomes

$$C'(\xi, 0) = \int_{-\infty}^{\infty} \left[N(u, v) |H(u, v)|^2 dv \right] \exp(2\pi i u \xi) du. \quad 3.28$$

The measured autocorrelation function, $C'(\xi, 0)$, and the measured spectrum $N'(u, 0)$, are both one-dimensional functions. The Wiener-Khinchin theorem for this case states that they must be related by the form

$$C'(\xi, 0) = \int_{-\infty}^{\infty} N'(u, 0) \exp(2\pi i u \xi) du. \quad 3.29$$

If Equation 3.29 is now compared with Equation 3.28, the relation between the measured one-dimensional spectrum and the true two-dimensional spectrum is readily seen to be

$$N'(u, 0) = \int_{-\infty}^{\infty} N(u, v) |H(u, v)|^2 dv. \quad 3.30$$

For a slit of length L and width b , the aperture function can be written as

$$\begin{aligned} h(X, Y) &= 1 \quad \begin{cases} -b/2 \leq X \leq +b/2 \\ -L/2 \leq Y \leq +L/2 \end{cases} \\ &= 0 \quad \text{otherwise;} \end{aligned} \quad 3.31$$

and the correctly normalised transform which derives from this function can be written

$$H(u, v) = \text{sinc}(bu) \cdot \text{sinc}(Lv). \quad 3.32$$

If Equations 3.30 and 3.32 are now combined, and the integration is carried out, then the true spectrum and the spectrum as measured by the slit are related in the following way:

$$N(u, 0) = \frac{L}{\text{sinc}^2(bu)} \cdot N'(u, 0), \quad 3.33$$

where the assumption has been made that the spectrum does not vary appreciably over the distance in which $\text{sinc}^2(Lv)$ takes significant values. An equivalent way of stating this last assumption would be to say that the slit is much longer than the distance over which the autocorrelation function takes significant values.

In the results, use was made of Equations 3.25 3.29 and 3.33 to calculate our realisation of the true spectrum from the measured intensities. But, to obtain our realisation of the true autocorrelation function, although in principle we need only have used Equations 3.25 and 3.14, this would not have been the simplest way to do it. What was done instead was to first generate the two-dimensional spectrum from the section and then carry out a two-dimensional Fourier transform on this. The complexity in doing this was lessened in that, since only a section through the autocorrelation function was required, the two-dimensional transform reduced to the simpler form *

$$C(\xi, 0) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} N(u, v) dv \right] \exp(2\pi i u \xi) du. \quad 3.34$$

A further measure shown in the results is the autocorrelation

*

Since $C(\xi, 0)$ and $N(u, 0)$ are both central sections, they are equivalent to the radially symmetric functions $C(r)$ and $N(t)$, and the one could therefore have been generated from the other by using the Hankel transform (cf., Equation 3.13).

function as it would be obtained by a very long slit of zero width. This particular function is included in the results because it is of some interest to know the limiting form of the measured autocorrelation function as the slit is made increasingly longer and narrower. Our realisation in this instance is got by carrying out a one-dimensional Fourier transform on the section $N(u,0)$:

$$C_L(\xi,0) = \int_{-\infty}^{\infty} N(u,0) \exp(2\pi i u \xi) du. \quad 3.35$$

Equally, we could have obtained $C_L(\xi,0)$ by taking a line integral through the true two-dimensional autocorrelation function:

$$C_L(\xi,0) = \int_{-\infty}^{\infty} C(\xi,\eta) d\eta. \quad 3.36$$

Although Equation 3.35 was used to generate our realisation of $C_L(\xi,0)$ in preference to Equation 3.36, Equation 3.36 will subsequently allow a clearer insight.

We now give the three expressions which were actually used to create the theoretical reference curves shown in the results; against them, our measured quantities can be checked for consistency with the theory.

Reference Curves Used in the Results

For the isotropic patterns considered, it is clear that the measured sections, $N(u,0)$, $C(\xi,0)$ and $C_L(\xi,0)$, will have the same forms as their radially symmetric counterparts, $N(t)$, $C(r)$ and $C_L(r)$. Indeed, the one set of functions can be produced from the other by simply interchanging the arguments, u with t and ξ with r . To be fully general, however, it is the radially symmetric functions which are now given.

The theoretical Wiener spectrum has already been given in the

required form (Equation 3.19), but for completeness in this section we reproduce the earlier result:

$$N(t) = \frac{8\langle I \rangle^2}{\pi^2 t_m^2} \left[\cos^{-1}\left(\frac{t}{t_m}\right) - \frac{t}{t_m} \sqrt{1 - \left(\frac{t}{t_m}\right)^2} \right] \quad . \quad 3.37$$

The theoretical autocorrelation function follows immediately from Equation 3.9:

$$C(r) = \langle I \rangle^2 \left[\frac{2J_1(\pi t_m r)}{\pi t_m r} \right]^2 \quad . \quad 3.38$$

The form of Equation 3.38 is identical to that of an Airy pattern and, by virtue of this, Equation 3.36 leads us to expect that $C_L(r)$ will have a form identical to that of a line spread function. With the relevant constants included, we can write this

$$C_L(r) = \frac{16\langle I \rangle^2}{\pi L t_m} \left[\frac{H_1(2\pi t_m r)}{(2\pi t_m r)^2} \right] \quad , \quad 3.39$$

where L is again the slit length and H_1 is the first order Struve function. Since the slit has zero width, width does not of course enter into the equation.

Having established the necessary theoretical framework, let us now consider the results.

Results and Discussion

As background information to the results, we first list the values taken by the system parameters:

Radius of scattering area $A = 10.33$ mm.

Distance between scatterer and far-field $R = 1307.0$ mm.

Wavelength $\lambda = .488$ microns.

Slit length	$L = 830.0$ microns.
Slit width	$b = 12.0$ microns.
Sampling interval	$= 14.0$ microns.

The sampling theorem^{eq.} (Jenkins and Watts, 1969) tells us that we can distinguish all frequencies present if we sample at twice the frequency of the highest. The highest frequency likely to be present - ie., the theoretical cut-off frequency, t_m , is 0.032 cycles/micron; or, alternatively, by taking the reciprocal, the minimum cycle distance is 30.9 microns. Since our sampling interval is less than half this value, we have indeed fulfilled the requirements of the sampling theorem. (In a separate experiment, using a much finer sampling interval, it was established that there are no frequencies present beyond the cut-off frequency.)

In Figure 3.3, we see two realisations of Wiener spectra, one corresponding to a rough diffuser and the other to a much finer diffuser. The standard deviation of the surface roughness of each diffuser is indicated; the fact that both are larger than the wavelength, λ , is necessary to comply with the assumptions made about the scatterer in the last chapter.

For each realisation, more than 20,000 measurements were made and this might suggest a very high statistical accuracy. However, although this is true in a general sense, it is a consequence which does not follow immediately because the Wiener spectrum is not ergodic. The ergodic property would apply had we found that by taking the average of more and more measurements, the variance on our estimate had reduced without limit; the reason why it does not is now briefly explained.

As we take more and more measurements, both the information obtained and the total scan length, T , increase; but, since our frequency discrimination is given by $2/T$, the additional information is spread into finer and finer frequency intervals. The net result of taking more measurements is, therefore, to make possible an estimation of the Wiener

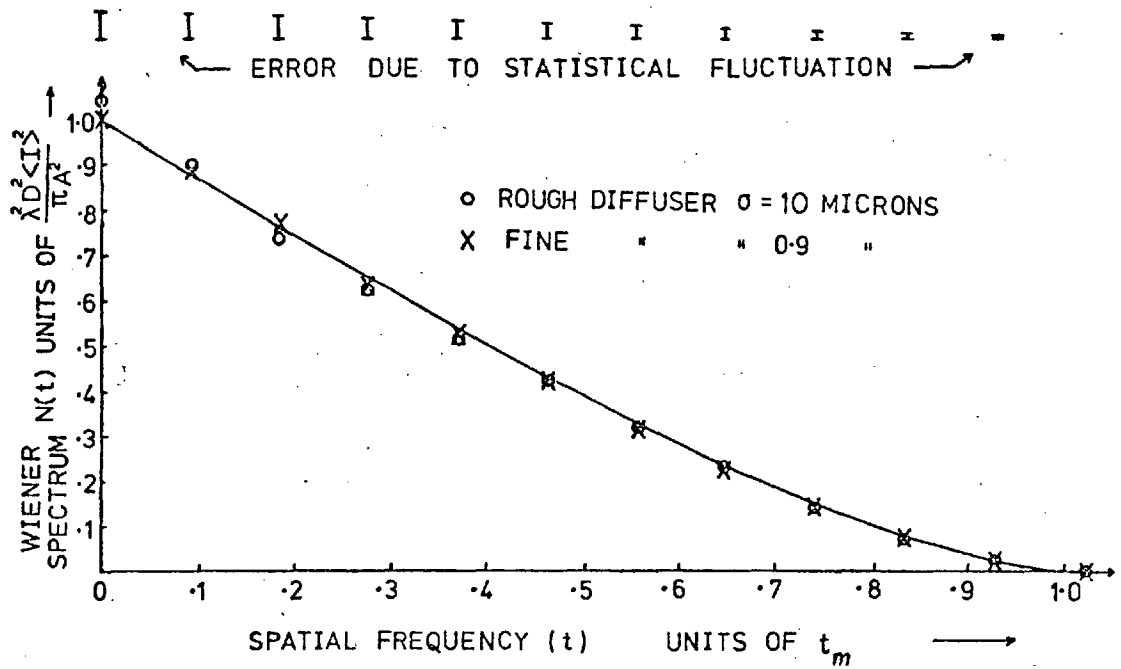


Figure 3.3 Measured Wiener spectra for a speckle pattern produced from a circular scattering area.

spectrum at narrower and narrower frequency intervals; the efficiency of each estimate in the narrowing interval, however, does not improve. To overcome this difficulty, we have to adopt smoothing procedures (Jenkins and Watts). These enable us to reduce variance - almost without limit - but at the expense of bandwidth.

The technique of smoothing can be stated broadly as one of averaging the estimates from a set of neighbouring frequency intervals and then taking this averaged value as representative of the estimate for the neighbourhood. Mathematically, this can be described as convolving the unsmoothed spectrum with some function, where clearly the width of this function will govern the degree of smoothing. A function used in this way is sometimes referred to as a spectral window.

If the spectrum is calculated, as it was in the case, by Fourier transforming the autocorrelation function, it is more convenient to carry out smoothing at the autocorrelation stage. This is done by multiplying the autocorrelation function by the transform of the spectral window. Mathematically, the two processes are equivalent so that the results are identical. The transform of which we speak is referred to as the lag window. *

The lag window can have any suitable shape and there are many in common use. Of those listed by Jenkins and Watts, the Parzen window was selected because its biasing effect ** was expected to be as small, if not smaller, than the others. The Parzen lag window is defined as follows:

* The terms 'lag window' and 'spectral window' were introduced by Blackman and Tukey (1958).

** Bias: An unavoidable side-effect of smoothing is that it introduces distortion - or bias - into the smoothed estimate. This means that the estimate obtained, although showing little of the statistical fluctuation present in the unsmoothed estimate, now takes values which are different from the true estimate. In general, a greater degree of smoothing is accompanied by a greater degree of bias.

$$\begin{aligned}
 f(r) &= 1 - 6\left(\frac{r}{W}\right)^2 + 6\left(\frac{|r|}{W}\right)^3, & |r| \leq \frac{W}{2} \\
 &= 2\left(1 - \frac{|r|}{W}\right)^3, & \frac{W}{2} \leq |r| \leq W \\
 &= 0, & |r| > W
 \end{aligned} \tag{3.40}$$

where W determines the width of the lag window. A sketch of this window is shown in Figure 3.4.

The Parzen spectral window, ie., the Fourier transform of the expression just given, can be written in the form

$$F(t) = \frac{3}{4} W \operatorname{sinc}^4\left(\frac{tW}{2}\right), \quad -\infty \leq t \leq \infty. \tag{3.41}$$

The bias introduced by the Parzen window has been shown by Jenkins and Watts to be given by

$$B(t) \approx \left(\frac{.152}{W^2}\right) \frac{d^2 N}{dN^2} + o\left(\frac{1}{W^3}\right) + \tag{3.42}$$

It would appear from Equation 3.42 that the bias introduced is proportional to the second derivative of the spectrum. Since the spectra shown in Figure 3.3 have only slow gradient changes present in them, the choice of the Parzen window would seem to be appropriate.

The width of the lag window is generally much more important than the type of window. This is reflected in Equation 3.42 where we see that bias is rather sharply dependent on window width. The width actually used was some 30 lags of the autocorrelation function and, as we shall see, the autocorrelation function at this point and beyond takes on negligible values. To have used a much narrower window would have lost valuable information and, although a finer degree of smoothing would have been achieved, this would not have compensated for the

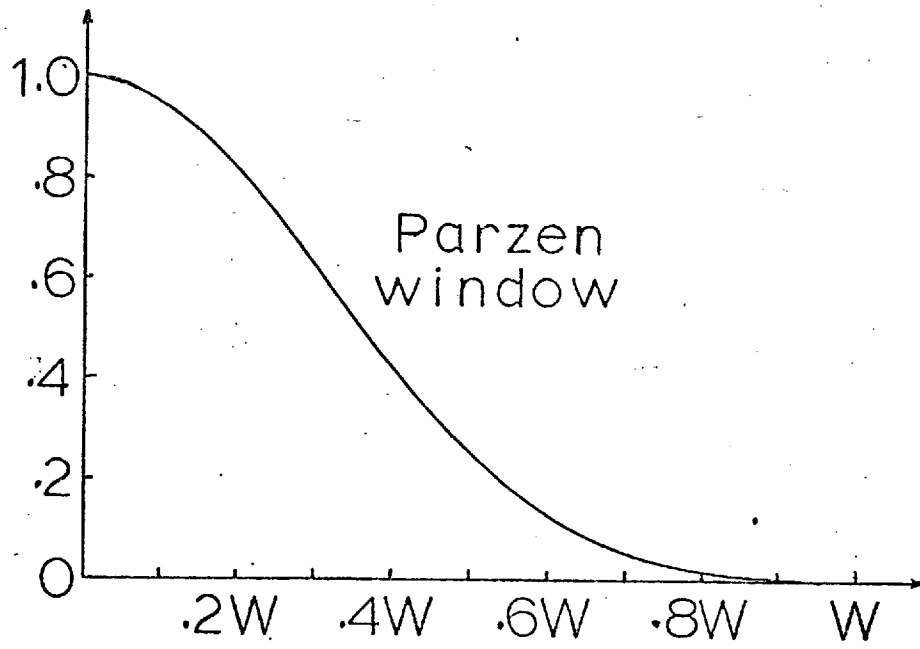


Figure 3.4 The Parzen lag window.

significant bias which would be introduced.

The lateral separation of the measured points is governed by the degree of smoothing. For the points to give independent estimates of the spectrum, they must be separated by an amount roughly equal to the width of the spectral window. To have shown points at finer intervals than we have them would have meant substantial correlation between neighbours and the additional information would then be nil.

Following Jenkins and Watts, the standard deviation of a spectral estimate $\hat{G}(t)$ can be given in terms of the spectral estimate itself:

$$\hat{G}(t) = \sqrt{\frac{.539 W}{T}} N(t), \quad 3.43$$

where W is the lag window width, T is the total scan length, and the factor .539 is associated with the Parzen window. Since W is 30 and T is 20,000, our estimate of the standard deviation is given approximately by

$$\hat{G}(t) = .03 N(t) \quad 3.44$$

The error bars are drawn with a length of twice the standard deviation, so that if we centre each one on the appropriate point below then on two occasions in three it should cut the theoretical curve. The fact that this does seem to happen must surely be taken as a verification of the theory.

Before leaving the Wiener spectrum, some final remarks should be made about errors arising from sources other than statistical fluctuation. The system parameters could all be measured with sufficiently high accuracy and, therefore, they may safely be ignored. There are, however, two effects of which something more should be said. One is due to depolarisation occurring during scattering and the other is due to a combination of the long-term instability of the laser and the non-Lambertian nature of the scattering.

The proportion of light depolarised during scattering was measured to be .25 per cent for the fine diffuser and .5 per cent for the rougher diffuser. These values can be shown to reduce the variance in the resulting patterns by .5 per cent and 1 per cent respectively. On the basis of this, we could argue that the reference curve shown in the Figure should be lowered by .5 per cent to correspond with the measurements made on the finer diffuser, and by 1 per cent for the rougher diffuser. In either case, however, the effect is not significant.

Due to a tendency of the diffuser to scatter light into a comparatively narrow, forward-pointing cone, the local average of the intensity in the speckle pattern varied from one part of the scan to the next. To minimise the effect of this, the scan was restricted to a region near the forward-pointing axis, and in this region the intensity was observed to remain constant to within 1 per cent. The combination of this effect and the long-term variations in laser power, which again were observed at about 1 per cent, would lead us to expect an additional low frequency component in the spectrum. Although smoothing acts in a way which would tend to reduce this effect, it may well have made itself apparent in the relatively poor agreement found at zero frequency for the rough diffuser.

In Figure 3.5, we see a realisation of the autocorrelation function for the finer of the two diffusers. The points indicated lie at the lag intervals at which the intensity was sampled. The theoretical reference curve was calculated from Equation 3.38.

It was not necessary to carry out smoothing in this case because the autocorrelation function, unlike the Wiener spectrum, is ergodic.* This means that as we take more and more measurements it quite naturally stabilises to its ensemble average value. The 20,000 measurements entailed in the realisation would lead us to expect statistical fluctuation of

* It does not necessarily follow that the transform of an ergodic function is itself ergodic.

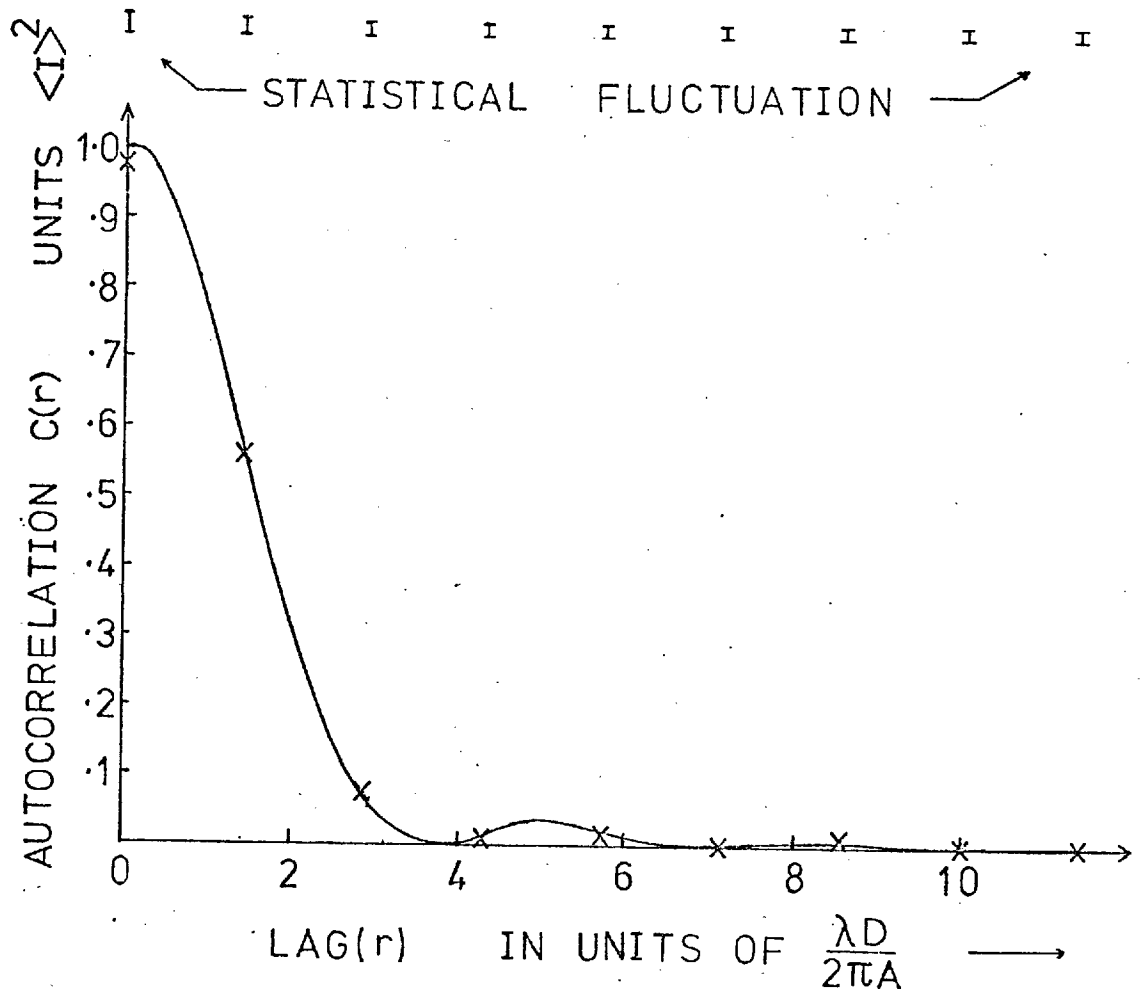


Figure 3.5 The measured autocorrelation function of intensity for a speckle pattern produced from a circular scattering area.

the amount shown. The error bars have a length of twice the standard deviation, so that again the test for successful verification of the theory is that, if we centre each error bar on the point lying immediately below it, then on two occasions in three it should cut the reference curve.

The values of the standard deviation, $\sigma(r)$, were calculated from the following expression:

$$\sigma(r) = \left[\frac{1}{T} \int_{-\infty}^{\infty} \left\{ [C(s)]^2 + C(s+r)C(s-r) \right\} ds \right]^{\frac{1}{2}}. \quad 3.45$$

This expression derives from a more general expression given by Jenkins and Watts. It will be noticed that no information is required in the expression, other than the total scan length and the form of the autocorrelation function; information about smoothing is, as we might have expected, not required by the expression.

The autocorrelation function is shown out to eight lag intervals. The value taken at the eighth is seen to be very small and this ties in with what was said earlier about the lag window width: it is clear that by ignoring the autocorrelation function after the thirtieth lag very little information is lost.

In Figure 3.6, we see a realisation of the autocorrelation function, again for the finer of the two diffusers, but this time as it would appear when measured by a very long slit of zero width. The reference curve was generated from Equation 3.39, but otherwise precisely the same comments apply as were applied to the autocorrelation function of the previous Figure.

It now only remains to say that for the type of speckle patterns considered the theory presently underlying the second order statistics is sound, at least to the accuracy we have engaged.

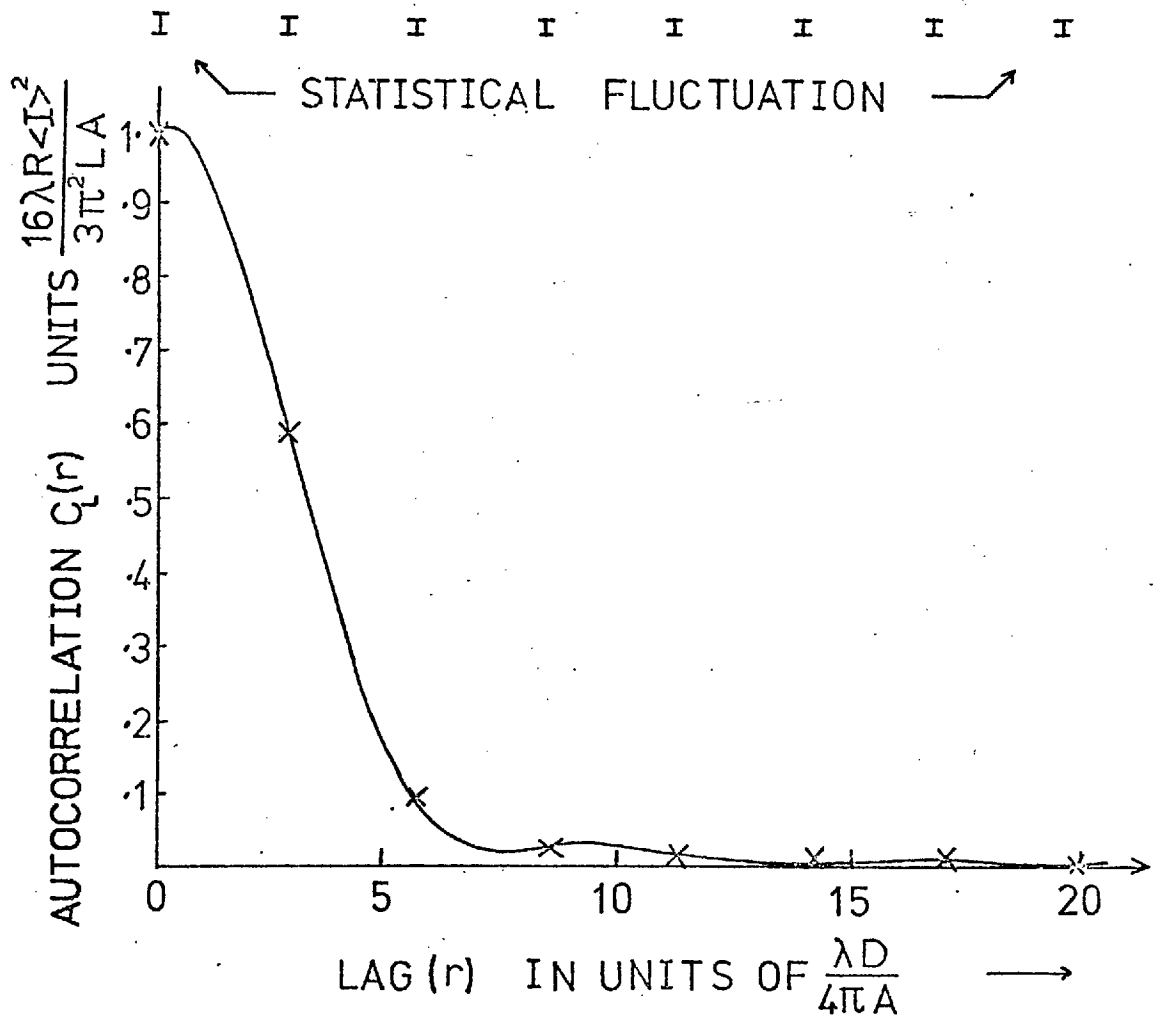


Figure 3.6 The autocorrelation function of intensity for a speckle pattern produced from a circular scattering area, as measured by a long slit of zero width.

CHAPTER FOUR

SECOND ORDER STATISTICS OF REDUCED SPECKLE IN THE FRAUNHOFER PLANE

In this Chapter and, indeed, in all the remaining chapters, we abandon our interest in speckle patterns as it was previously directed, and instead consider ways and means of, if not eliminating the speckle phenomenon, at least reducing it to within tolerable limits. The literature gives descriptions of many different procedures for enabling speckle reduction, each method having its own areas of application, its own merits and demerits, some in competition with one another and others unique in their behaviour. A brief review of some of these reduction techniques is given, by way of opening the chapter; but the larger part of the chapter, indeed, the larger part of the remainder of this Thesis, is concerned with one method in particular.

The method is due to Dainty and Welford (1971) and, as it was envisaged by these authors, it concerns speckle reduction in the time-averaged image when an aperture, smaller than the pupil, is moved in the pupil plane of the reconstruction lens. Having described the method in this way, it might now seem anomalous to state that the theory and the experimental confirmation which is given to it in this chapter, are based upon reduced speckle patterns produced in the Fraunhofer plane, the reduction being effected in the time-averaged pattern by moving an aperture over the scattering area so that different parts are exposed at different times.

The situation which we describe might not seem to connect immediately with the situation dealt with by Dainty and Welford, but it is exactly similar and in Chapter Six the equivalence will be firmly established. Anticipating the results of that chapter, the theory which

is offered is applicable to speckle reduction under four main headings:

- (1) to speckle formed in an image when an aperture is moved in the pupil plane of the imaging lens;
- (2) in the situation identical to the above, but where the speckle is formed in the reconstructed image of a hologram when an aperture is moved in the pupil plane of the reconstructing lens;
- (3) in a Fraunhofer plane speckle pattern, produced from a trans-illuminated scattering area when an aperture is moved over its surface;
- (4) in the situation identical to (3), but where the speckle is formed in the reconstructed image of a Fraunhofer plane hologram while an aperture is moved over the surface of the hologram.

For the moment, consideration is restricted to the second order statistics of the time-averaged patterns; the first order statistics will be dealt with in the next chapter. The theory is given only for the specific case of a rotating, circular aperture; but there remains sufficient generality in this case for deductions to be made about the treatment in a general case - ie., an arbitrarily shaped aperture performing some arbitrary cycle of movement.

After the theory has been given, some measured values of the autocorrelation function and the Wiener spectrum will be compared to theoretical values. A detector with a time-constant long enough to allow the aperture to make many rotations was used to provide the time-averaging, but otherwise the method used to measure the two quantities was exactly similar to the method used in the previous chapter.

In addition to the four applications given, the theory is also thought to apply to speckle reduction by illuminating with partially coherent light. If it does apply in this case, then the specific results given for the rotating, circular aperture encompass the speckle reduction

technique of Gronin and Smith (1973). The moving, coherent source they describe can be considered as a fixed, incoherent, quasi-monochromatic, ring source, and such a source produces partially coherent light of a type dealt with by our results. A fuller consideration of this matter will be given in Chapter Six.

Techniques for Reducing Speckle

Since speckle is not found with incoherent illumination, it is clear that we may reduce speckle by simply reducing the coherence of the illumination. There are two possible ways of doing this: one is to reduce the temporal coherence, and the other is to reduce the spatial coherence. Elbaum et al (1972), George and Jain (1972), and Golbach (1973) adopted the former method by using several wavelengths. The latter method, however, has been discussed rather more widely.

Arsenault and Lowenthal (1970), and Lowenthal et al (1970) reduced spatial coherence by using a single moving diffuser to produce a random phase modulation. But, although this technique is effective, an even more effective technique was used by Tiziani (1967), Schroder (1971) and Lowenthal and Joyeaux (1971). They made use of two diffusers, one moving with respect to the other, and this enabled a more rapid rate of phase modulation. Another phase-modulation technique, which involved no moving parts, was described by Kiemle and Wolff (1971). Their method made use of a liquid crystal gate in which movement of the crystals caused the necessary phase modulation.

When a non-diffusing object, for example, a transparency, is illuminated from behind by a point source and a hologram is then made of it, the reconstructed image will contain no speckle. This procedure is rarely followed, however, because the reconstructed image will suffer from other degrading artifacts, arising from dust and imperfections both on and in the hologram material and the reconstruction optics.

Rotation of the optical elements (Kirkpatrick and El-Sum, 1956) could be used to reduce the effects of the latter, but, according to Leith and Upatneiks (1964), the improvement offered by this means is not really good enough. They offer, instead, a different solution, a solution in which a diffuser is introduced into the illuminating beam.

The diffuser acts successfully in removing the noise due to the factors mentioned, and in other ways too it acts advantageously. The dynamic range in the reconstructed image is dramatically improved, and the hologram can even be broken into smaller pieces, each separate piece being capable of yielding a complete reconstruction, albeit one of reduced resolution. Unfortunately, however, the all-round improvement which has been created has been bought at significant cost - speckle.

Various authors (Upatneiks, 1967; Gerritson et al, 1968; Leith and Upatneiks, 1968; Gabor, 1970; Kato and Okino, 1973) have addressed themselves to the problem of retaining the diffused illumination while reducing the speckle produced by it. The solutions found are all rather similar, and they involve placing an amplitude and/or phase grating in or near the plane of the transparency at the time of recording the hologram. There are many photographs in the literature which show the improvement offered by these techniques and their success is indisputable. However, these methods can only be used with non-diffuse objects and their success is restricted to a single plane in the object. In any other plane, the grating will reveal itself as a confusing background feature of some sort.

When the reconstructed image is formed from a hologram of a diffuse object or, for that matter, when we form a direct image of a diffuse object, the use of phase or amplitude gratings will have no effect in reducing speckle. For this type of situation, other procedures must be adopted to effect the reduction.

Martienssen and Spiller (1967) solved the problem as it arises in holography by illuminating the object through a diffuser and then

forming a series of holograms of it, the diffuser being moved between exposures so that each hologram is given its own individual speckle pattern. At the reconstruction stage, the different holograms are successively shown during a time interval which is short in comparison to the response-time of the detector, and the effect of this is to cause an averaging of the speckle. There is no theoretical limit to the amount of speckle reduction which can be achieved by this method, and there are no degrading side-effects introduced by the process - the effect of the process is quite simply to synthesise an image which in every respect resembles an incoherent image.

With Fraunhofer holograms, the type actually considered by Martienssen and Spiller, the sequence of holograms can be moved continuously through the reconstructing beam so that accurate location of the holograms is not a critical part of the procedure. But with image plane holograms, the movements would have to be discrete and accurately located, just as in cinematography. If the highest levels of resolution were sought, inaccuracies in locating the separate holograms would ultimately limit the performance, and the limit imposed might well be considerably less than the theoretical limit of the process. A further disadvantage of the process, and this applies to both Fraunhofer and image plane holograms, is that short duration events, such as occur in bubble chambers, would be, by the nature of the process, difficult to record. The disadvantages which have been indicated have practical rather than theoretical origins, but this does not make them any less real and it seems, therefore, that alternative solutions to the problem must be found.

Alternative solutions have been given to the problem by Dainty and Welford (1971), van Ligten (1973), and Yu and Wang (1973). The solutions given by these authors are all variations on a single theme - spatial filtering - and in every case the reduction in speckle is obtained at the expense of lower resolution. The loss of resolution

is unfortunate, but it seems to be quite unavoidable. According to Gabor (1970), had we succeeded in reducing the speckle without losing resolution, we would have succeeded in breaching the 'information barrier of optical transmission'. The fact that we have so far failed is just another manifestation of the fact that 'Nature always finds a way of frustrating those who want to break through the information barrier'.

At this point it is perhaps appropriate to break off the general discussion and concentrate on the previously mentioned speckle reduction technique of Dainty and Welford. We start by considering the autocorrelation function in the time-averaged pattern.

Autocorrelation Function for a Rotating Circular Aperture

Suppose we have a scattering area, illuminated from behind, which, at any one time T_1 , has a circular part of its area exposed by an aperture (Figure 4.1). This area will give rise to a speckle pattern in the far-field, and we denote the intensity in this pattern by $I_1(X,Y)$, the suffix implying the particular time T_1 . As time progresses, and the circular aperture moves round, different parts of the scattering area will be exposed (Figure 4.2) and different speckle patterns will result.

There will be an infinite number of speckle patterns created, some correlated with one another; but, for our purposes, the situation will be described by a finite, though not necessarily small number N . To each position of the aperture, there will be associated one of the N intensity functions $I_1(X,Y)$, $I_2(X,Y)$ $I_N(X,Y)$. If the scattering area is uniformly illuminated, then each will have the same average value which we denote $\langle I \rangle$.

If the aperture is assumed to move round with constant angular velocity, then equal weighting can be given to each aperture position and, immediately, this enables an expression to be formed for the

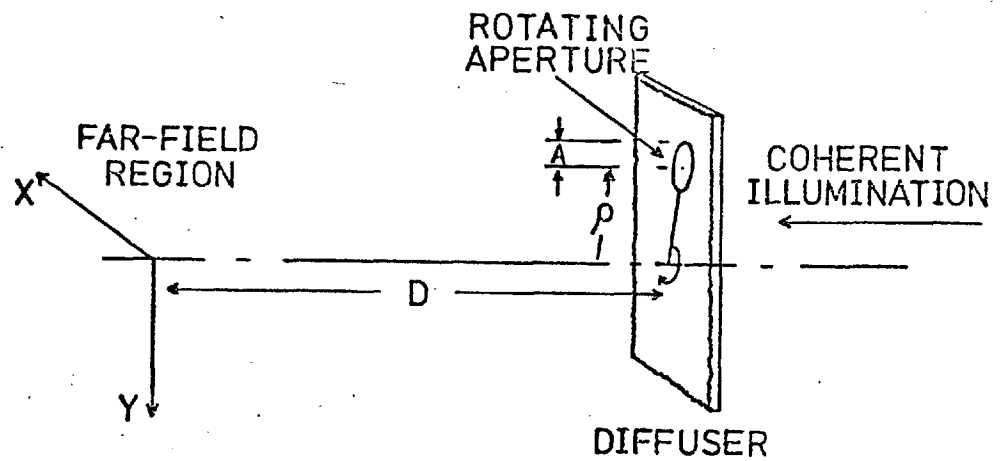


Figure 4.1 The production of a reduced speckle pattern in the far-field by rotating a circular aperture so as to expose different regions of the scattering area.

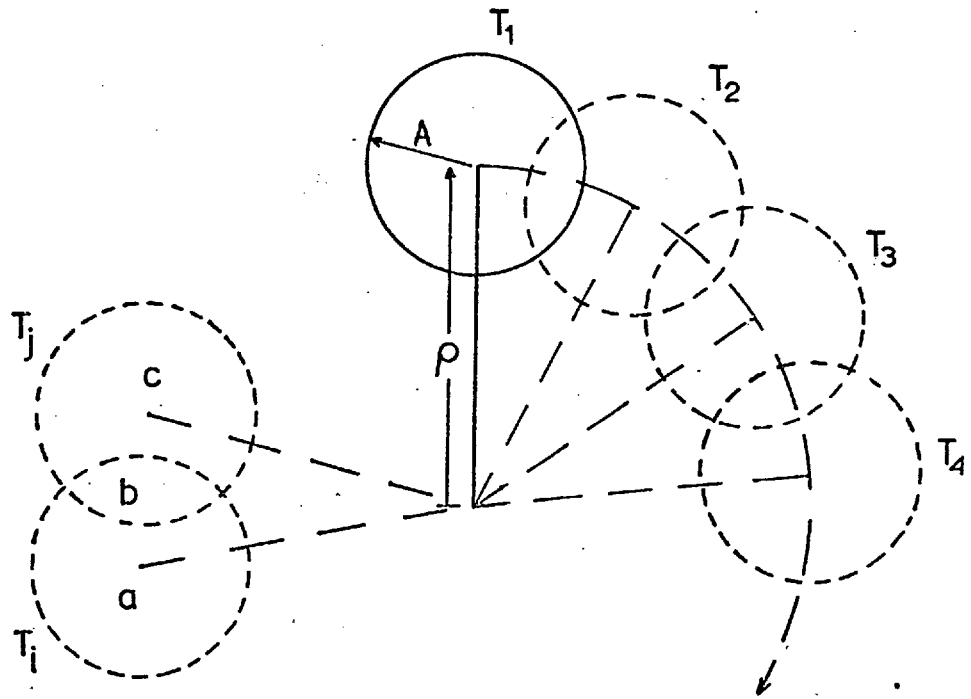


Figure 4.2 Sequential positions of the rotating aperture, and the three 'significant' areas - a, b and c .

time-averaged autocorrelation function:

$$C(\xi, \eta) = \left\langle \left[\frac{I_1(x+\xi, y+\eta) + I_2(x+\xi, y+\eta) + \dots + I_N(x+\xi, y+\eta)}{N} \right] \times \left[\frac{I_1(x, y) + I_2(x, y) + \dots + I_N(x, y)}{N} \right] \right\rangle - \langle I \rangle^2. \quad 4.1$$

By cross-multiplying terms, Equation 4.1 can be expressed in the form

$$C(\xi, \eta) = \frac{1}{N^2} \left[\left(\langle I_1(x+\xi, y+\eta) I_1(x, y) \rangle - \langle I \rangle^2 \right) + \left(\langle I_1(x+\xi, y+\eta) I_2(x, y) \rangle - \langle I \rangle^2 \right) + \dots \dots \dots \left(\langle I_N(x+\xi, y+\eta) I_N(x, y) \rangle - \langle I \rangle^2 \right) \right]. \quad 4.2$$

The first term in the above equation is the autocorrelation function for the aperture as it lies at time T_1 . There will clearly be N similar autocorrelation terms in the expansion. The most often occurring terms, however, are terms of the type

$$\langle I_i(x+\xi, y+\eta) I_j(x, y) \rangle - \langle I \rangle^2,$$

and these require some closer scrutiny.

Suppose $I_i(x+\xi, y+\eta)$ is written in terms of its associated complex amplitude $V_i(x+\xi, y+\eta)$ and suppose $I_j(x, y)$ is written similarly. We may then write

$$\langle I_i(x+\xi, y+\eta) I_j(x, y) \rangle - \langle I \rangle^2 = \langle V_i(x+\xi, y+\eta) V_i^*(x+\xi, y+\eta) V_j(x, y) V_j^*(x, y) \rangle - \langle I \rangle^2. \quad 4.3$$

We now make the usual assumption that the scattering surface is rough compared to the light wavelength. It then follows that the wavefront emerging from the diffuser will have a statistically uniform distribution of phase in the range 0 to 2π , and following Goodman (1963) the complex amplitude produced in the far-field will form a complex Gaussian process. For such a process, a theorem ^{discussed} by Reed (1962) may be used which enables Equation 4.3 to be written

$$\begin{aligned}
 \langle I_i(x+\xi, y+\eta) I_j(x, y) \rangle - \langle I \rangle^2 &= \\
 &= \left| \langle V_i(x+\xi, y+\eta) V_j^*(x, y) \rangle \right|^2 - \langle I \rangle^2 \\
 &+ \langle V_i(x+\xi, y+\eta) V_i^*(x+\xi, y+\eta) \rangle \langle V_j(x, y) V_j^*(x, y) \rangle . \\
 &= \left| \langle V_i(x+\xi, y+\eta) V_j^*(x, y) \rangle \right|^2 .
 \end{aligned} \tag{4.4}$$

The result we have used follows from a more general theorem, the exact details of which need not concern us.

Figure 4.2 shows the aperture positions corresponding to the suffixes i and j . Enclosed by these aperture positions are three significant areas, labelled a , b and c . The complex amplitude, $V_i(x+\xi, y+\eta)$, can be expressed in terms of two components, one deriving from area a and the other from area b . Expressing this mathematically,

$$V_i(x+\xi, y+\eta) = V_{ia}(x+\xi, y+\eta) + V_{ib}(x+\xi, y+\eta) ,$$

$$\text{and similarly } V_j^*(x, y) = V_{jb}^*(x, y) + V_{jc}^*(x, y) .$$

These relations allow the right-hand side of Equation 4.4 to be written

$$\left| \langle [V_{ia}(x+\xi, y+\eta) + V_{ib}(x+\xi, y+\eta)] [V_{jb}^*(x, y) + V_{jc}^*(x, y)] \rangle \right|^2$$

and this expands to yield

$$\left| \left\langle V_{ia}(x+\xi, y+\eta) V_{jb}^*(x, y) + V_{ia}(x+\xi, y+\eta) V_{jc}^*(x, y) \right. \right. \\ \left. \left. + V_{ib}(x+\xi, y+\eta) V_{jb}^*(x, y) + V_{ib}(x+\xi, y+\eta) V_{jc}^*(x, y) \right\rangle \right|^2$$

If the lateral structure size on the diffuser is small enough, then there will be no correlation between the complex amplitudes deriving from different areas on the diffuser. It then follows that the only non-zero term in the above is the term

$$\left| \left\langle V_{ib}(x+\xi, y+\eta) V_{jb}^*(x, y) \right\rangle \right|^2.$$

The complex amplitude arising at a point, from any given area of the diffuser, will remain constant with time so long as the diffuser does not move. This suggests that the suffixes i and j , which denote two different points in time, have, as far as the above term is concerned, become redundant. The term, therefore, may be more simply expressed:

$$\left| \left\langle V_b(x+\xi, y+\eta) V_b^*(x, y) \right\rangle \right|^2.$$

This form is immediately recognisable (Goodman, 1963) as the squared *autocorrelation function* of the modulus of the complex amplitude which arises from area b . Following Goodman, it can be written as the squared modulus of the Fourier transform of the intensity function which operates in area b . Hence, the typical cross-correlation term in Equation 4.2 may be written

$$\left\langle I_i(x+\xi, y+\eta) I_j(x, y) \right\rangle - \left\langle I \right\rangle^2 = \left| \iint_{-\infty}^{\infty} S(x, y) \exp[-2\pi i(\xi x + \eta y)] dx dy \right|^2. \quad 4.5$$

where $S(x, y)$ is the intensity function which, in our particular case, takes the form

$$\begin{aligned} S(x, y) &= 1 && \text{inside the overlap area} \\ &= 0 && \text{otherwise.} \end{aligned}$$

In the light of Equation 4.5, the N autocorrelation terms in Equation 4.2 do not appear to be fundamentally different from the larger number of cross-correlation terms; they merely correspond to cases where the overlap is total.

Generally speaking, an evaluation of the autocorrelation function, $C(\xi, \eta)$, would involve summing the N^2 terms contained in Equation 4.2, and each of these terms (cf., Equation 4.5) would entail a two-dimensional Fourier transform. This would indicate a very large amount of computation, but for our particular case some simplifications can be made and it might be instructive to consider these.

Simplifications in the Computation

A closer look at Equation 4.2 shows that the terms present in it occur in groups of N . Each term in a group arises from the same amount of overlap, the only difference between terms being that the overlap is located and oriented differently. The implication of this is that, if the N terms of a group were computed using Equation 4.5, each would give rise to the same result except for a rotation factor. * The effect of this is that we may account for all N terms in a group by first making a single evaluation of Equation 4.5, using any one of the terms in the group, and then carrying out an azimuthal integration of the result. Rotation of the one-dimensional function so created would produce a radially symmetric function equal in its effect to the N terms of the group.

The evaluation of Equation 4.5 may, itself, be simplified. If the evaluation is made with that term in the group whose intensity function, $S(x, y)$, has the x and y axes as axes of symmetry, and we are at liberty to choose this term, then the Fourier transform will reduce to the form

* When the Fourier transforms are first calculated, different phase shifts will be associated with each term due to the different locations of the overlaps. When the squared modulus is taken, however, these all disappear.

$$4 \int_0^{\infty} \int_0^{\infty} S(x,y) \cos(2\pi \xi x) \cos(2\pi \eta y) dx dy.$$

In addition to this, those terms which correspond to no overlap need not be calculated because the result would be zero anyway.

The autocorrelation function now appears as the sum of N radially symmetric terms, one for each different amount of overlap, and it must then, itself, be radially symmetric. This might have been expected at the outset, due to the symmetrical nature of the problem.

The Wiener spectrum

The Wiener spectrum of the time-averaged pattern can be formed directly from the autocorrelation function by carrying out a two-dimensional Fourier transform. In the particular case dealt with, however, the autocorrelation function is radially symmetric and this will imply that the Wiener spectrum is also radially symmetric. A Hankel transform (Equation 3.15) might then be more appropriate than the Fourier transform in this instance.

Experimental Verification

In support of the theory, some measured results are now compared with predictions made using Equations 4.2 and 4.5. The case dealt with concerns a circular aperture, rotating about a radius 3.15 times greater than its own radius. With respect to Figure 4.1, 3.15 refers to the ratio ρ/A which we shall henceforth refer to as the scanning ratio, R . The diffuser, which was uniformly illuminated by a laser, was made from glass, ground on one side. The grinding was to a roughness which satisfied the theoretical requirements made earlier.

In Figure 4.3, two theoretical curves are shown. One shows the autocorrelation function when the aperture is held stationary - or

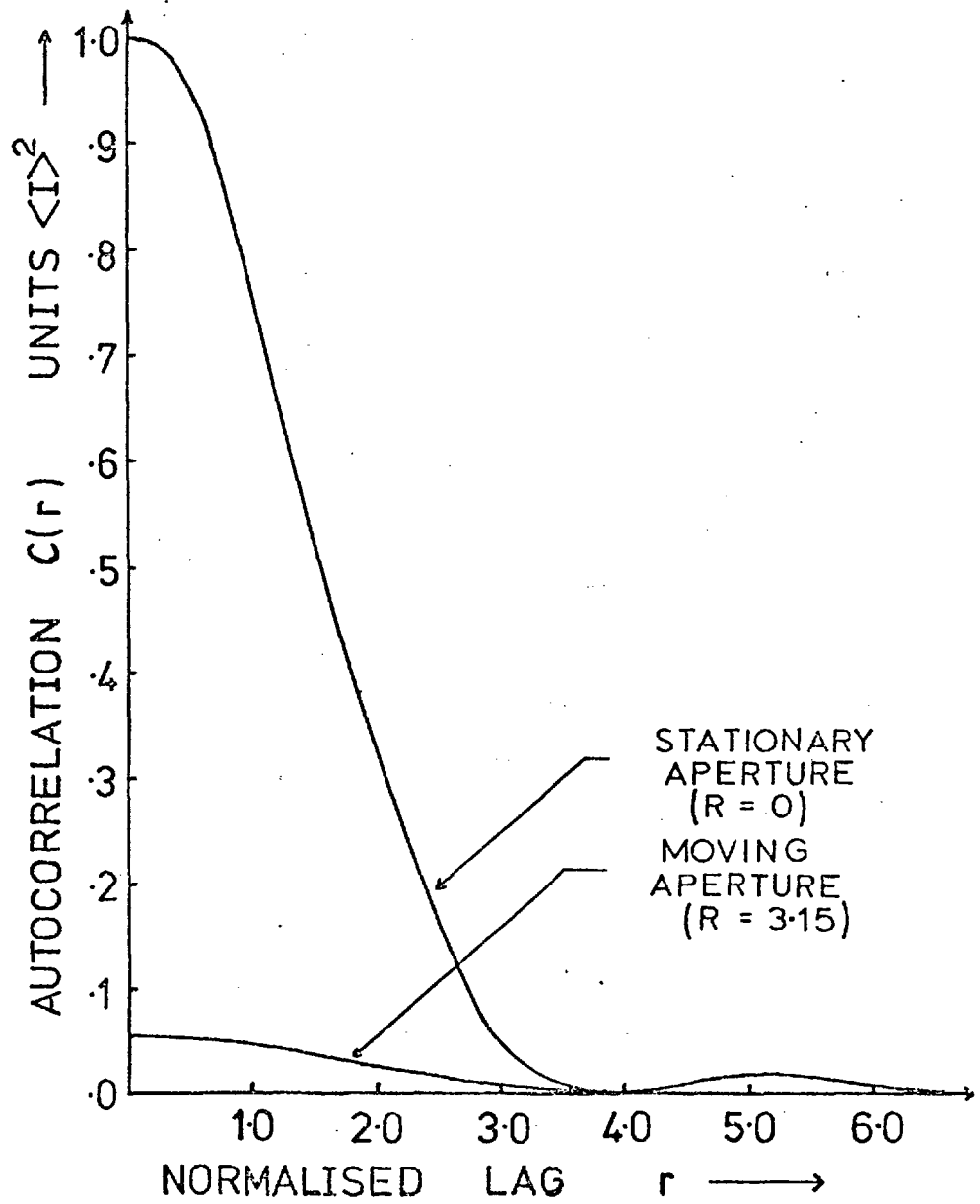


Figure 4.3 The autocorrelation function of intensity for a rotating circular aperture and, for reference, the autocorrelation function when the same aperture is held fixed.

equivalently, had R been set to zero - and the other shows the autocorrelation function when the aperture is rotating. The vertical scale in the Figure is in units $\langle I \rangle^2$, the square of the mean intensity; the horizontal scale has normalised units defined by $r = \frac{2\pi A \xi}{\lambda D}$, where ξ would be an actual distance measured in the X/Y plane. Since the value of the autocorrelation function at zero lag is simply the variance in the speckle pattern, the height difference between the two curves indicates the reduction in variance effected by the rotating aperture.

Because of its small height, details of the shape of the autocorrelation function for the rotating aperture are obscured. In Figure 4.4, therefore, the two curves are shown again, the curve for the stationary aperture (in broken line, this being the convention for reference curves in this Chapter) now drawn to a false scale so that the heights of the two curves are made equal. It is immediately apparent that the autocorrelation function for a rotating aperture is a wider function than it is for a stationary aperture, and the implication of this is that the 'speckles' will appear to increase in size as the aperture begins to move. The explanation to this behaviour is found in Equations 4.2 and 4.5 in that the autocorrelation function for a moving aperture is the average of many autocorrelation functions, each one corresponding to one of the many overlap situations illustrated in Figure 4.2. Since all of these overlapping areas are less than or equal to the stationary aperture area, their Fourier transforms - the squared moduli of which concern us - are as wide, or wider, than for the stationary aperture.

Also indicated in Figure 4.4, by crosses, are measured values of the autocorrelation function when the aperture is moving. The values shown were based on about 5,000 intensity measurements. The statistical fluctuation errors appropriate to this number are indicated in the Figure. The error bars have a length of twice the standard deviation so that, if

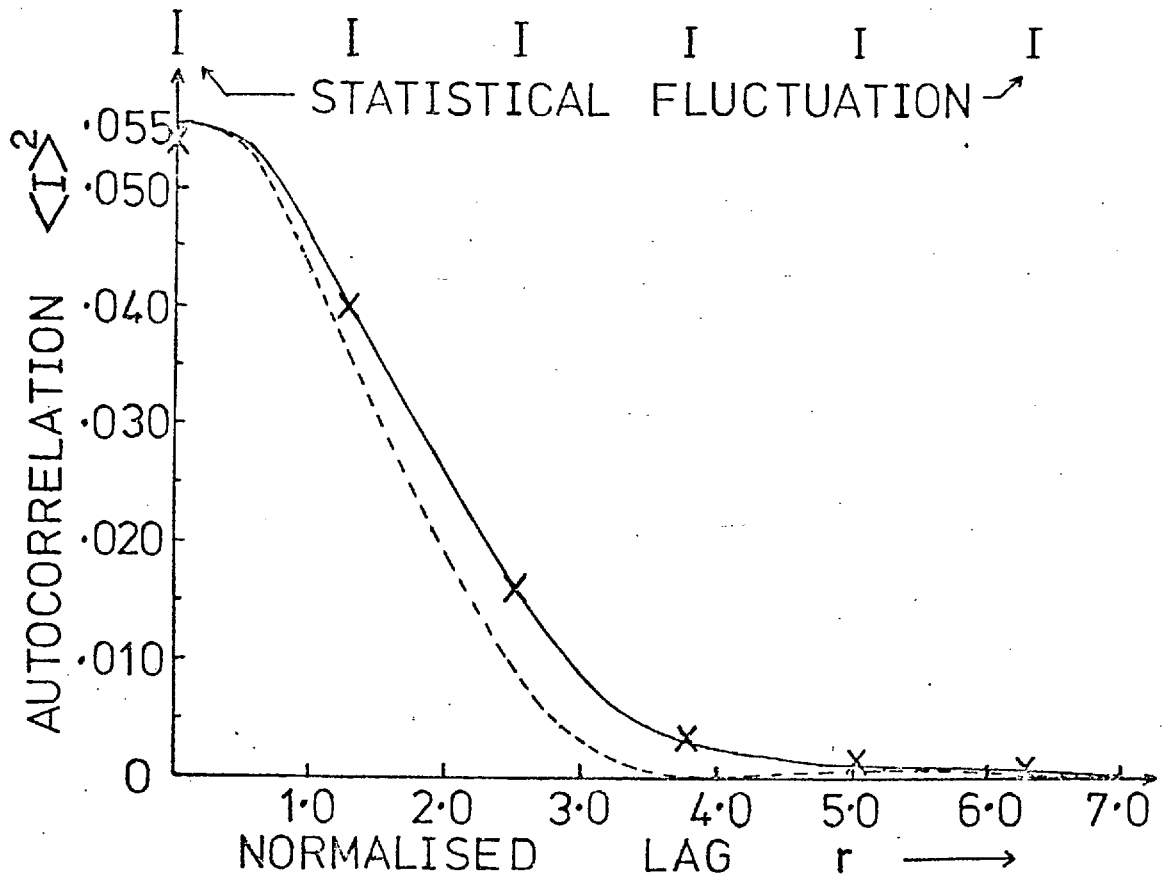


Figure 4.4 Measured and theoretical autocorrelation functions for a rotating circular aperture; $R = 3.15$. The theoretical reference curve, in broken line, is for the same aperture when held fixed; its true height would correspond to $\langle I \rangle^2$ on the scale shown.

a bar is centred on the measured value directly below it, then on two occasions in three it should cut the theoretical curve.

Figure 4.5 shows, in firm line, the Wiener spectrum for the rotating aperture case. Set against it, in broken line, is the Wiener spectrum for the same aperture when held stationary. Again, the latter curve is drawn to a false scale; its true height would be 1.0 units in terms of the scale shown. Rotation of the aperture, it seems, has a selective effect in reducing the spectrum: the higher frequencies are more severely reduced than the lower. Also indicated in the Figure, by crosses, are measured values of the spectrum for the rotating aperture. These values were generated from the measured values of the autocorrelation function of the previous figure. Again, error bars are shown which, as in Figure 4.4, have a length of twice the standard deviation.

In Figure 4.5, the cut-off frequency is found to be the same for both the stationary and the rotating apertures, and this is to be expected. The value of the cut-off frequency, t_m , is given in terms of the quantities shown in Figure 4.1:

$$t_m = \frac{2A}{\lambda D}.$$

In Figure 4.6, the theoretical autocorrelation function is shown for the case $R = 1$. Again, the curve for the stationary aperture is drawn to a false scale; the true height of this curve would be 1.0 on the scale in the Figure. There appears to be little difference between the curve for $R = 1$ in this Figure and that for $R = 3.15$ in Figure 4.4. This would suggest that most of the change in the shape of the autocorrelation function occurs as R goes from 0 (a stationary aperture situation, in effect) to 1.

As R increases beyond the value unity, although the shape of the autocorrelation function changes little, the height falls steadily. The height at zero lag - the variance - is comparatively simple to

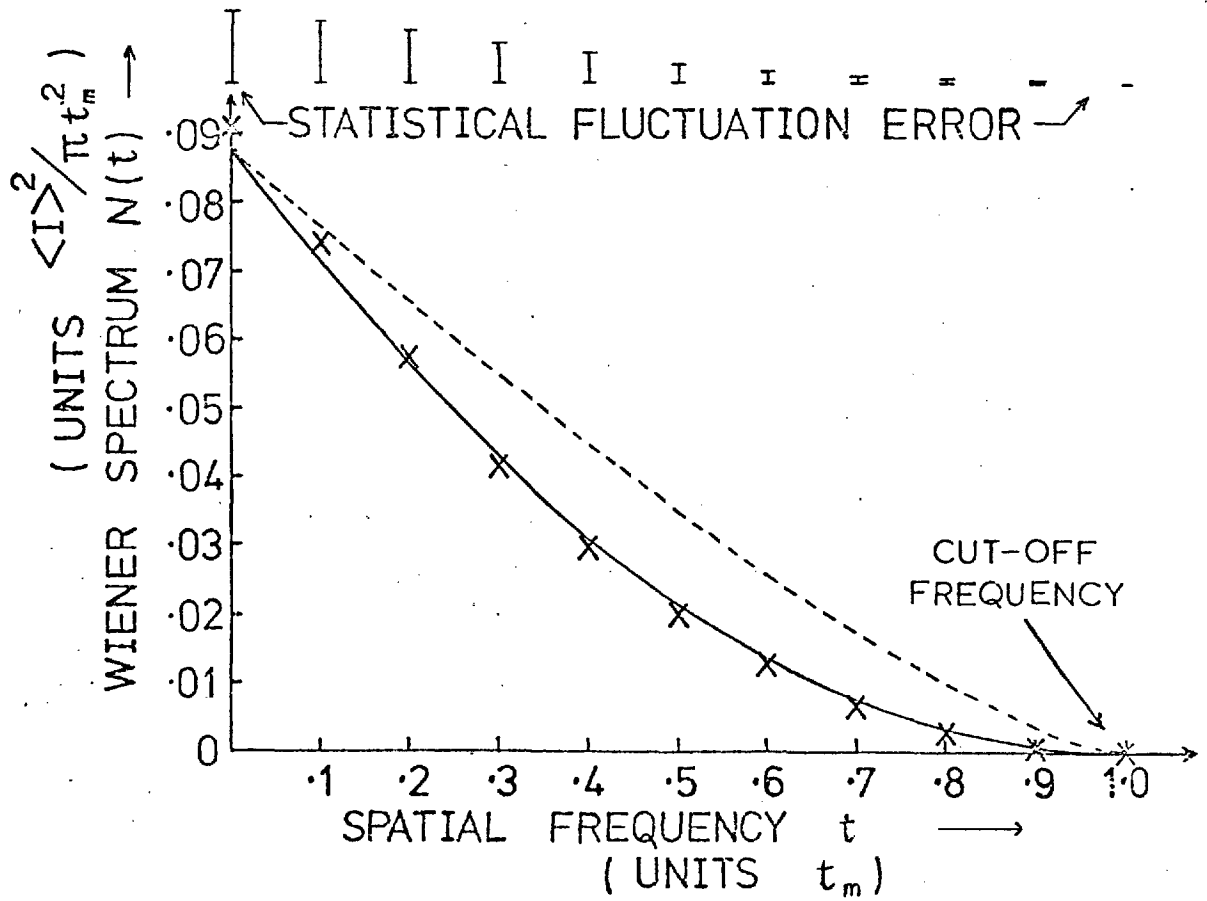


Figure 4.5 Measured and theoretical Wiener spectra for a rotating circular aperture; $R = 3.15$. The theoretical reference curve, in broken line, is for the same aperture when held fixed; its true height is several times greater than shown.

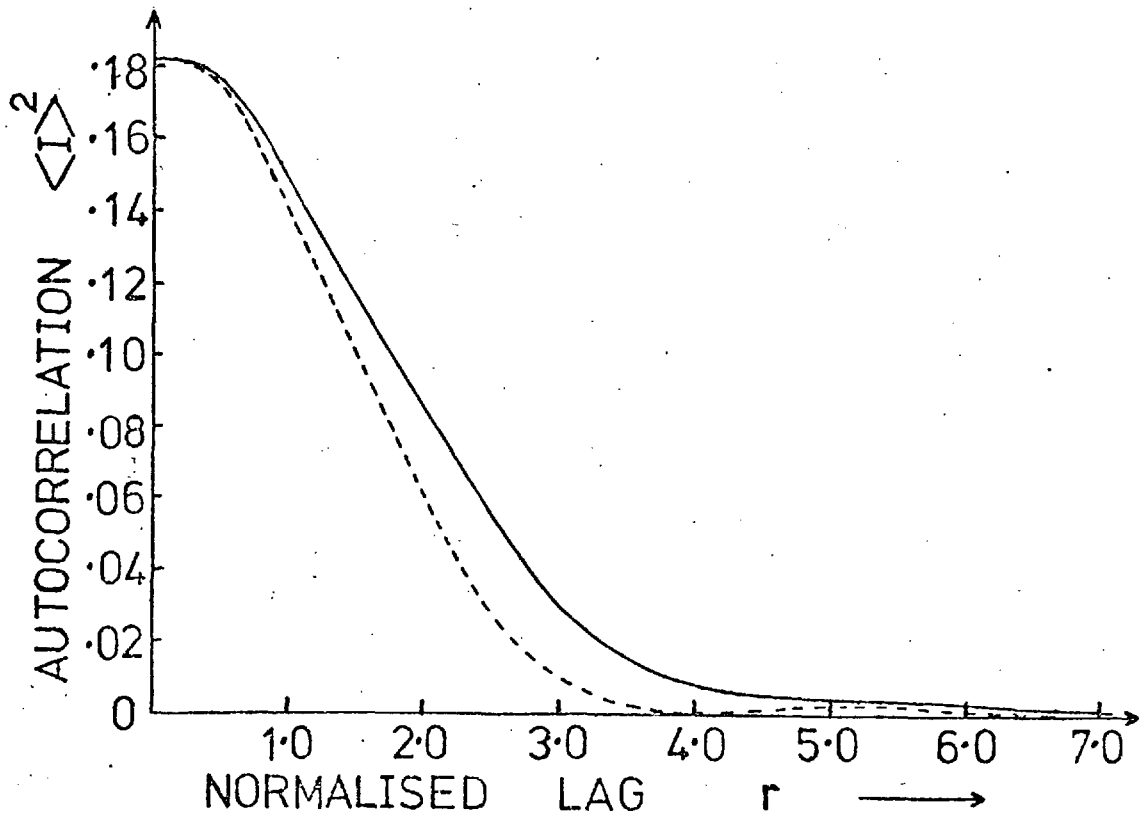


Figure 4.6 A theoretical autocorrelation function for a rotating circular aperture; $R = 1.0$. The reference curve, in broken line, is for the same aperture when held fixed; its true height would correspond to $\langle I \rangle^2$ on the scale shown.

calculate. Stated in words, it is given by the average of the squares of the areas enclosed by the N^2 overlaps, normalised by dividing by the fixed aperture area and by multiplying the result by $\langle I \rangle^2$. In Figure 4.7, the variance, calculated in this way, is plotted against the scanning ratio, R .

As R becomes larger, the variance increasingly begins to reduce as $1/R$. However, since the variance is given either by the volume contained under the Wiener spectrum or by the autocorrelation function at zero lag, both the spectrum and the autocorrelation function in this particular case must ultimately reduce in height as $1/R$. The only factor which could have avoided this situation would have been for shape changes to have occurred at large values of R , and we have seen that this does not in fact happen.

In the treatment given by Dainty and Welford (1971), the Wiener spectrum is reasoned to reduce according to the number of independent positions taken by the aperture as it moves. When R is large enough, the number will clearly begin to increase linearly with R , so that the prediction might be made that the spectrum will reduce as $1/R$. This agrees with the result just obtained, so that to a first approximation the treatment given by Dainty and Welford is correct. A fuller investigation reveals that it will be correct not just in this case, but in all cases. It is therefore appropriate to draw on their treatment to provide the useful rule: When the aperture moves, the values taken by the autocorrelation function, the Wiener spectrum and the variance are each reduced in a way that is roughly proportional to the number of independent positions taken by the aperture. Having given the rule, it should be remarked that since it is only approximate it should be used with care. It does not take into account the changes in shape of the functions concerned; it merely gives a rough order-of-magnitude estimate of the size of the effect. For a more exact appreciation, one would have to fall back on Equations 4.2 and 4.5.

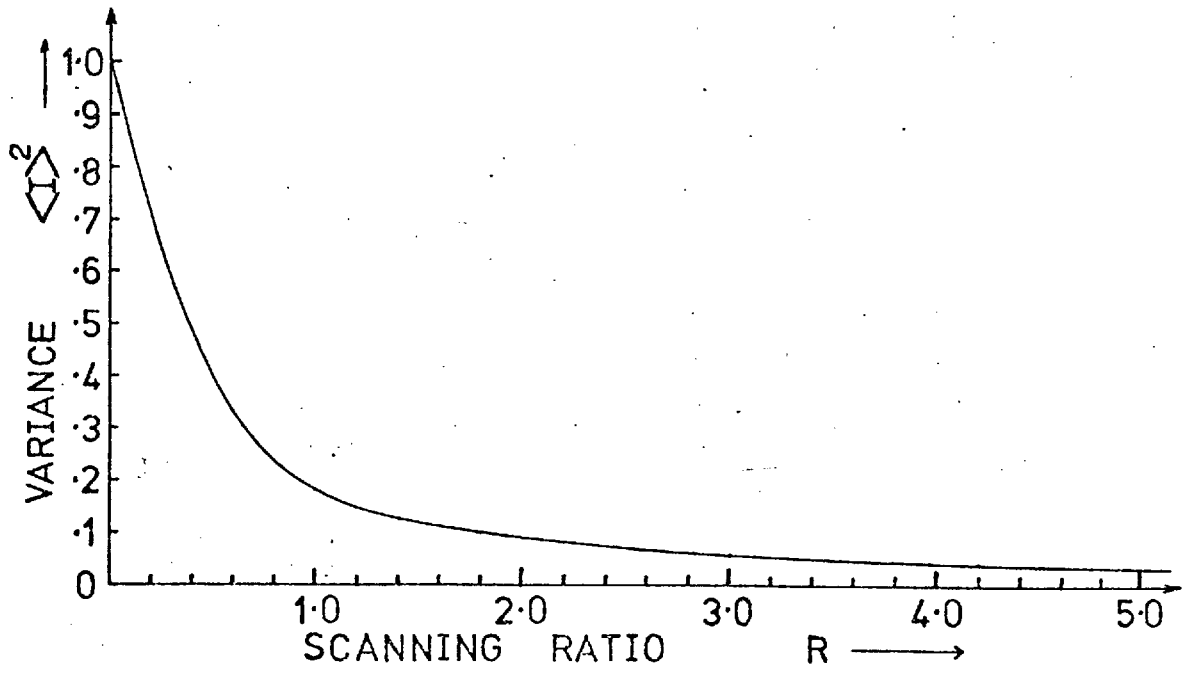


Figure 4.7

The reduction in variance as the scanning ratio increases.

Application to Partially Coherent Speckle

The justification for the claim made earlier that the results in this Chapter might be applicable to speckle reduction by using partially coherent illumination stems from a paper by Welford (1971). In his paper, he demonstrates (as a specific case of a more general theorem) the equivalence of an eccentrically rotating, circular aperture acting in a coherently illuminated optical system and the same pupil, fixed, but acting in an optical system illuminated by an incoherent, quasi-monochromatic, ring source. This result can be interpreted to show that the results which have just been given can be applied to speckle reduction by using partially coherent illumination, as produced by a ring source of the type described. To make the results appropriate, however, the following new definition must now be given to R : R is the ratio of the angle subtended by the ring source at the object to the angle subtended by the fixed pupil at the object. Further remarks on speckle reduction by employing partially coherent illumination are now deferred until Chapter Six when a more complete consideration will be given.

Loss of Resolution

We said earlier that the reduction of speckle by a spatial filtering technique is, without exception, accompanied by a lowering in resolution. The lowering incurred by the rotating, circular aperture can be quantified by comparing the widths of the point spread functions for the situation where we do employ the rotating aperture and the situation where we do not employ it. If it is assumed that the full pupil of the system is circular, and it is further assumed that the rotating aperture makes most efficient use of it, then the radius of the pupil must be given by $(\rho + A)$. Since the width of the point spread function of the system is inversely proportional to the radius of the

pupil, the factor by which resolution is lowered is given by $(\rho + A)/A$, and this may be written as $(1 + R)$.

If we refer to Figure 4.7, it will be apparent that to cause a total reduction of the speckle requires R to become infinite, and this must now imply that the resolution will fall to zero. In Chapter Seven, types of moving apertures will be considered with which total loss of resolution does not follow as a necessary consequence of total speckle reduction.

CHAPTER FIVE

FIRST ORDER STATISTICS OF REDUCED SPECKLE IN THE FRAUNHOFER PLANE

In the previous chapter, we considered the second order statistics of reduced speckle. In this Chapter, we consider the first order statistics. The treatment is again given ~~for~~ Fraunhofer speckle patterns, the sort produced in the arrangement illustrated in Figure 4.1, and again the results obtained are claimed to have the same wider relevance as those of the previous chapter. A justification will be given to this claim in the next chapter.

The specific aim in this Chapter is to produce the form of the probability density function of intensity and, in fact, two forms of solution are offered. The first solution is an approximate solution - the gamma variate approximation; the other solution, however, is exact, closely resembling the solution given in Chapter Two for the probability density function as obtained with a finite aperture.

To arrive at the exact solution, the speckle at a point is considered to arise from a Gaussian amplitude-signal, varying in time, and observed over a finite time interval by an intensity-sensitive device. This of course is precisely the problem encountered by so many authors in the context of random noise power (Kac and Siebert, 1947; Slepian, 1958; Thomas, 1969) and we therefore adopt an approach similar to theirs'. Our analysis, in fact, closely follows that given by Slepian; indeed, departure from his treatment only occurs to the extent that we treat a complex amplitude whereas he treated a real amplitude.

Before an evaluation can be made, using the exact solution, the

kernel of the homogeneous Fredholm equation must first be determined. The kernel in this instance is the time-autocorrelation function of amplitude, and this function describes the average correlation between the amplitudes at a single point in the speckle pattern when measured at two different times. A general expression will be derived for the kernel and, following this, specific forms will be generated which are appropriate to rotating circular apertures and rotating sector apertures. It will be noticeable that the forms taken by these autocorrelation functions are not similar to any of the autocorrelation functions previously encountered, so that this might be a suitable moment to emphasise the point that the time-autocorrelation function is a fundamentally different type of function and should not be confused with the previous kind.

The Approximate Solution

The gamma variate approximation for the probability density function can be given in the usual form,

$$\begin{aligned}
 P(I) &= \frac{a^M I^{M-1} \exp(-aI)}{\Gamma(M)} & I \geq 0 \\
 &= 0 & \text{otherwise,}
 \end{aligned} \tag{5.1}$$

$$\text{where } M = \frac{\langle I \rangle^2}{\sigma^2} \tag{5.2}$$

$$\text{and } a = \frac{M}{\langle I \rangle} \tag{5.3}$$

M is the effective number of independent speckle patterns which are produced during the movement cycle of the aperture. Its value depends

entirely on the variance in the reduced pattern, and we saw how to calculate this quantity in the last chapter. In Figure 5.1, M is plotted against the scanning ratio, R , for the case of the rotating, circular aperture. The curve will be recognised as a reciprocal version of the curve shown in Figure 4.7 in which the variance was plotted against R .

The gamma variate approximation, as used here, is used in the context of a one-dimensional problem, time being the dimension. It was in such a context that Bédard et al (1967) considered the accuracy of the approximation, and they showed that 'by explicit calculation for three different spectral distributions (kernels) the formula holds with good accuracy over a wide range of conditions.' It seems that we might be justified in this instance in placing a substantial degree of trust in the accuracy of the approximation.

In Figure 5.2, some probability density functions, calculated using the approximation, are shown for various rotating, circular apertures. As R increases, the probability density function narrows, as we should expect, and increasingly begins to peak around the mean intensity position. In the limit, when R has become infinite, the function would take the form of a delta function, lying exactly on the mean intensity position; by this time, the speckle would have been totally reduced.

The Exact Probability Density Function

The time-autocorrelation function

of amplitude can be written

$$C_T(x, y; t, t') = \langle V(x, y; t) V^*(x, y; t') \rangle \quad 5.4$$

In general, the autocorrelation function will have the dependence on position which we have indicated. For the type of speckle patterns

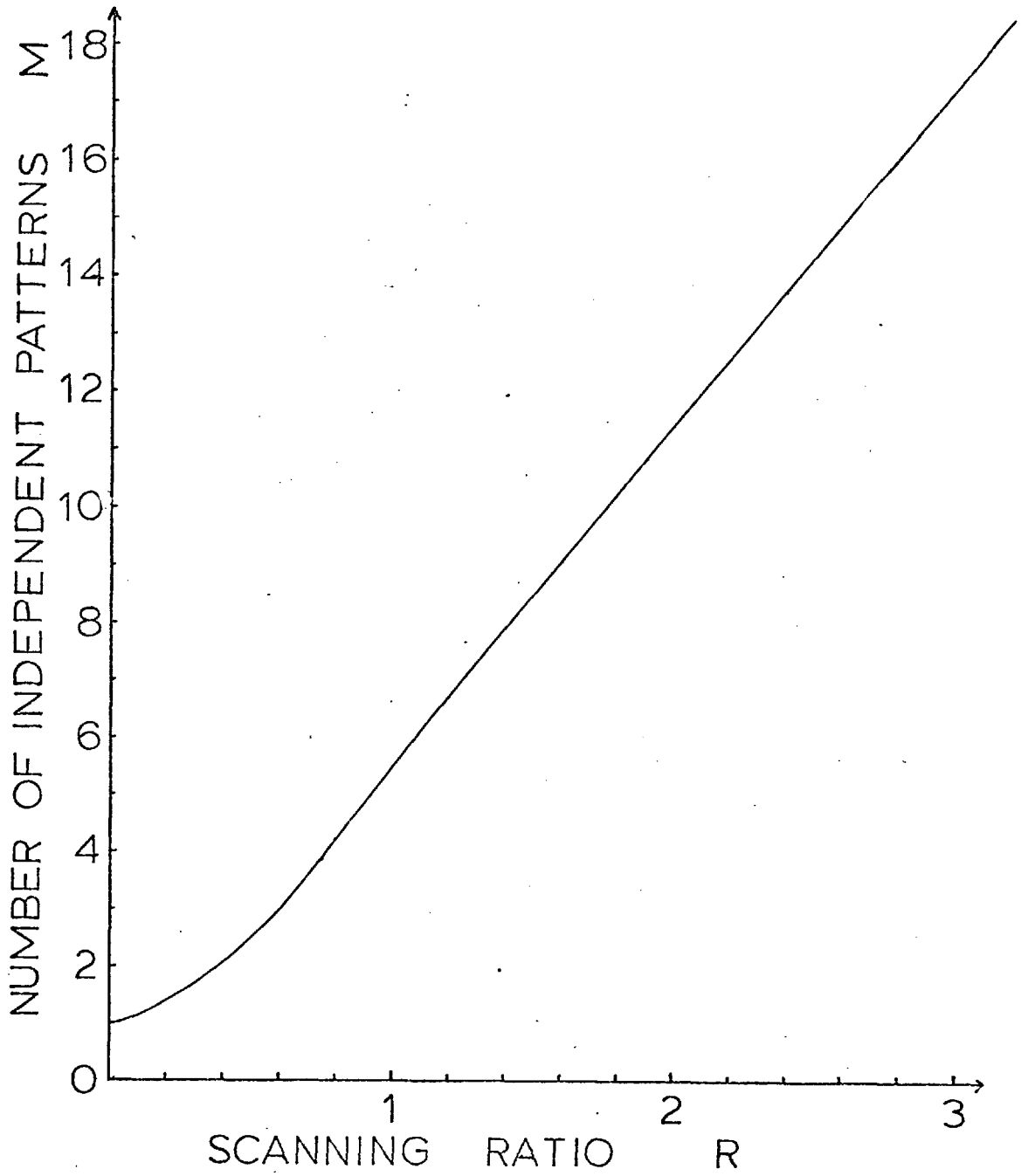


Figure 5.1 The effective number of speckle patterns produced by the rotating circular aperture, plotted as a function of the scanning ratio.

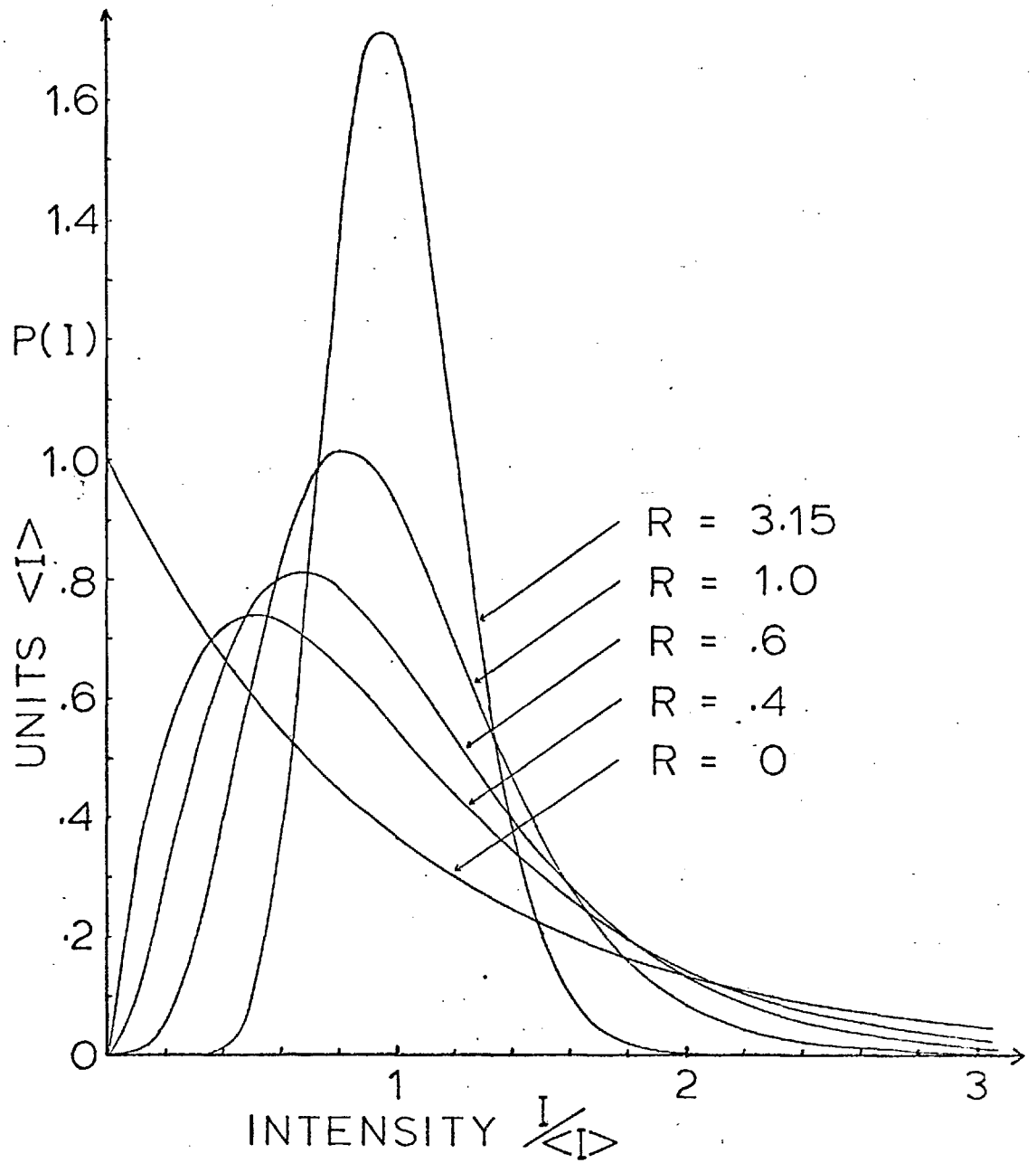


Figure 5.2 The probability density function of intensity for various rotating circular apertures, as calculated using the approximate method.

which concern us, however, there is no such dependence, and the autocorrelation function can be written more simply:

$$C_T(t, t') = \langle V(t) V^*(t') \rangle, \quad 5.5$$

where $V(t)$ and $V(t')$ are amplitudes referring to the same arbitrary point.

There is no need, as far as the treatment is concerned, to make any assumption about the autocorrelation function being stationary with time. However, since the autocorrelation function will be found to be stationary for the moving aperture situations which are specifically dealt with, it is convenient to make this stationarity assumption.

Whence, the time-autocorrelation function takes on the form

$$C_T(t-t') = \langle V(t) V^*(t') \rangle \quad 5.6$$

If the aperture is considered to move cyclically, with periodicity T_0 , and if the detecting device has a response time which is either equal to T_0 or some integer multiple thereof, * then the observed intensity, I , can be expressed in the form

$$I = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |V(t)|^2 dt. \quad 5.7$$

As the problem is now presented, we have a Gaussian complex amplitude process, a time-autocorrelation function for the process - the kernel, and a detection system which responds to the time-integrated intensity. This closely resembles the noise power problem considered by Slepian (1958), as we had previously indicated, and it is therefore

* If the response time were to be much longer than T_0 , then the restriction to integer multiples could be dropped.

convenient, in deriving the probability density function, to follow the exposition given by that author.

Let us expand the complex amplitude, $V(t)$, into the orthonormal set of functions, $\Psi_j(t)$, $j = 0, 1, 2, 3, \dots$. Whence, we may write

$$V(t) = \sum_0^{\infty} c_j \Psi_j(t) \quad |t| \leq \frac{T_0}{2}, \quad 5.8$$

where
$$c_j = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} V(t) \Psi_j^*(t) dt, \quad j = 0, 1, 2, 3, \dots \quad 5.9$$

and
$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \Psi_i(t) \Psi_j^*(t) dt = \delta_{ij}, \quad i, j = 0, 1, 2, 3, \dots \quad 5.10$$

It is convenient in this instance to choose the orthonormal set, Ψ_j , as solutions of the homogeneous Fredholm equation with $C_T(t-t')$ as kernel. That is, the Ψ 's are chosen such that

$$\lambda_j' \Psi_j(t) = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} C_T(t-t') \Psi_j(t') dt'. \quad 5.11$$

In choosing the Ψ_j to be solutions of this particular equation, we have produced a situation, as can easily be shown, where the c_j are independent/complex Gaussian variates with mean zero and variance λ_j' .

The characteristic function is the Fourier transform of the probability density function. Denoting the Characteristic function by $\Phi(\zeta)$, we can write

$$\Phi(\zeta) = \int_{-\infty}^{\infty} P(I) \exp(i \zeta I) dI, \quad 5.12$$

and this may be written in the equivalent form

$$\Phi(\zeta) = \langle \exp(i\zeta I) \rangle. \quad 5.13$$

If we now use Equations 5.8 and 5.10, the intensity, as given by Equation 5.7, can be written

$$I = \frac{1}{T_0} \sum_0^{\infty} |c_j|^2, \quad 5.14$$

and this allows another form to be given to Equation 5.13:

$$\Phi(\zeta) = \langle \exp(i\zeta \frac{1}{T_0} \sum_0^{\infty} |c_j|^2) \rangle. \quad 5.15$$

Since the c_j are independent random variables, this may be rewritten

$$\Phi(\zeta) = \prod_0^{\infty} \langle \exp(i\zeta \frac{|c_j|^2}{T_0}) \rangle. \quad 5.16$$

If a normalised set of eigenvalues, λ_j , are now defined:

$$\lambda_j = \frac{\lambda_j'}{T_0}, \quad 5.17$$

$$\text{ie.,} \quad \lambda_j = \frac{\langle |c_j|^2 \rangle}{T_0} \quad 5.18$$

then Equation 5.16 can be written as follows:

$$\Phi(\zeta) = \prod_0^{\infty} (1 - i\zeta \lambda_j)^{-1}. \quad 5.19$$

If the inverse form of Equation 5.12 is coupled with Equation 5.18,

an expression for the probability density function immediately follows:

$$P(I) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(-i\mathcal{S}I) d\mathcal{S}}{\prod_0^{\infty} (1 - i\mathcal{S}\lambda_j)} \quad 5.20$$

Equation 5.20 is the required expression for the probability density function, but whether it is of any practical use depends on whether or not we can obtain the eigenvalues. Fortunately, when the kernel is real and symmetrical - and this condition holds in cases of interest to us - numerical methods may be brought to bear on the Fredholm equation and the eigenvalues can be obtained.

In addition to the probability density function, two other interesting quantities, namely, the mean intensity and the variance, may be calculated from the eigenvalues:

$$\langle I \rangle = \sum_0^{\infty} \lambda_j \quad 5.21$$

$$\text{and} \quad \sigma^2 = \sum_0^{\infty} \lambda_j^2 \quad 5.22$$

The summation, which we notice running from zero to infinity, draws attention to the fact that for solutions to be attainable in practice the eigenvalues would have to show a reasonably fast convergence to zero.

The Time-Autocorrelation Function of Amplitude - the Kernel

If we refer back to Figure 4.2, we will see that at the two instants T_i and T_j there are two circular, partially overlapping areas indicated, and ^{it} would seem that the degree of correlation between the amplitudes at a point in the far-field, measured at the times T_i and T_j , must in some way be related to the scattering area contained

in the overlap. Let us now investigate this more closely.

The time-autocorrelation function of amplitude (Equation 5.6) can be written in terms of the times T_i and T_j as follows

$$C_T(T_i - T_j) = \langle V(T_i) V^*(T_j) \rangle. \quad 5.23$$

If we consider the amplitude at time T_i to be composed of two parts, one from area a, V_a , and the other from area b, V_b , and consider the amplitude at time T_j in a similar way, then the above equation becomes

$$C_T(T_i - T_j) = \langle (V_a + V_b)(V_b + V_c)^* \rangle \quad 5.24$$

The explicit time dependence has now disappeared, but, of course, time will still affect the situation indirectly because at different times the areas a, b and c will take different shapes and lie at different positions on the scattering surface. By decomposing the amplitude terms into their real and imaginary parts, Equation 5.24 may be expanded:

$$C_T(T_i - T_j) = \langle (V_{aR} + V_{bR} + i(V_{aI} + V_{bI}))(V_{bR} + V_{cR} - i(V_{bI} + V_{cI})) \rangle, \quad 5.25$$

and by cross-multiplication a further expansion may be made:

$$\begin{aligned} C_T(T_i - T_j) = \langle & [V_{aR} V_{bR} + V_{aR} V_{cR} + V_{bR}^2 + V_{bR} V_{cR} \\ & + V_{aI} V_{bI} + V_{aI} V_{cI} + V_{bI}^2 + V_{bI} V_{cI} \\ & + i(V_{aI} V_{bR} + V_{aI} V_{cR} + V_{bI} V_{bR} + V_{bI} V_{cR}) \\ & - i(V_{aR} V_{bI} + V_{aR} V_{cI} + V_{bR} V_{bI} + V_{bR} V_{cI})] \rangle. \quad 5.26 \end{aligned}$$

In Chapter Two, it was found that there is no correlation between the real and imaginary parts of the field produced from a

scattering area. If we now combine this result with the obvious fact that there is no correlation between the amplitudes arising from two different scattering areas, then only two terms in the above expansion will remain after the averaging has been carried out. This allows us to write the equation in the much reduced form

$$C(T_i - T_j) = \langle (V_{bR}^2 + V_{bI}^2) \rangle. \quad 5.27$$

The right hand side of this equation is interpreted as the average intensity which arises from the area of overlap, b , and this can be shown to be proportional to the integral of the intensity distribution over the overlap area. We may therefore write

$$C_T(T_i - T) = \iint_{-\infty}^{\infty} S(x, y) dx dy, \quad 5.28$$

where $S(x, y)$ is the intensity distribution in the overlap region which, as previously, is defined by

$$\begin{aligned} S(x, y) &= 1 && \text{inside overlap} \\ &= 0 && \text{otherwise.} \end{aligned}$$

With this definition of $S(x, y)$, C_T is quite clearly stationary for a circular aperture which is rotating with constant angular velocity.*

If the time difference $(T_i - T_j)$ is denoted more generally by T , then the time-autocorrelation function, correctly normalised to unity, can be expressed in the following way:

$$C_T(T) = \frac{(\text{Overlap area corresponding to the time separation } T)}{(\text{Full area of overlap})} \quad 5.29$$

*

If the intensity distribution, $S(x, y)$, is uniform as we have assumed, then the stationarity property will depend only on the rotation having constant angular velocity. The aperture shape is of no consequence.

The Kernel for a Rotating Circular Aperture

The area of overlap, normalised to unity, between two circular apertures whose centres are separated by the distance e is given by the well known form

$$\frac{2}{\pi} \left[\cos^{-1} \left(\frac{e}{2A} \right) - \frac{e}{2A} \sqrt{1 - \left(\frac{e}{2A} \right)^2} \right] .$$

where A is the radius of either aperture.

If we now refer to Figure 4.2, and consider that a full rotation of the aperture is made in time T_0 , then the angle traversed in time T is given by $2\pi T/T_0$, and consequently the separation between two aperture positions with the time difference T is given by

$$e = 2\rho \sin(\pi T/T_0)$$

where ρ , as before, is the radius on which the rotation is made.

If the above results are now combined with Equation 5.29, the required kernel is given by

$$C_T(T) = \frac{2}{\pi} \left[\cos^{-1} \left[R \sin \left(\frac{\pi T}{T_0} \right) \right] - R \sin \left(\frac{\pi T}{T_0} \right) \sqrt{1 - R^2 \sin^2 \left(\frac{\pi T}{T_0} \right)} \right] ,$$

$$\text{for } |T| \leq \frac{T_0}{\pi} \sin^{-1} \left(\frac{1}{R} \right)$$

$$= 0$$

otherwise,

5.30

where $R = \rho/A$ as before.* When R is large, it can easily be shown that the form taken by the kernel (where it is non-zero) is just the usual form for the autocorrelation of two circles. This perhaps illustrates the point that the time-autocorrelation functions of this Chapter are not similar to the autocorrelation functions of the previous chapters.

* Since Equation 5.30 is real and symmetrical, numerical techniques can be applied to obtain eigen values.

The Kernel for a Rotating Sector Aperture

For a sector aperture (Figure 7.1) there are two separate cases to be treated according to whether the sector angle, ϕ , is greater or less than π :

$$(1) \quad \underline{\phi \leq \pi}$$

$$\begin{aligned} C_T(\tau) &= 1 - \frac{2\pi\tau}{T_0\phi}, \quad |\tau| \leq \frac{\phi T_0}{2\pi} \\ &= 0 \quad \text{otherwise;} \end{aligned} \quad 5.31$$

$$(2) \quad \underline{\phi \geq \pi}$$

$$\begin{aligned} C_T(\tau) &= 1 - \frac{2\pi\tau}{T_0\phi}, \quad |\tau| \leq T_0\left(1 - \frac{\phi}{2\pi}\right) \\ &= 2\left(1 - \frac{\pi}{\phi}\right) \quad \text{otherwise} \end{aligned} \quad 5.32$$

In the former case, the kernel is a triangular-shaped function which falls to zero on either side. In the latter case, the kernel is again triangular in shape, but the fall-off on either side terminates on a constant, non-zero level.

No experimental verification has been made for the theory given in this Chapter, but reassurance can be obtained about the correctness of our general expression for the kernel (Equation 5.29). By following an argument similar to that used by Goodman (1965), it can be shown that the variance in a time-averaged pattern can be expressed in terms of the kernel:

$$\sigma^2 = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |C_T(\tau)|^2 d\tau. \quad 5.33$$

By combining the above result with Equation 5.29, it becomes apparent that the variance is given by the average of the squares of the areas enclosed by the overlaps. In the previous chapter, precisely the same result was obtained and in that chapter experimental verification was obtained.

CHAPTER SIX

REDUCED SPECKLE IN AN IMAGE AND SPECKLE IN PARTIALLY COHERENT ILLUMINATION - FIRST AND SECOND ORDER STATISTICS

In the two preceding chapters, theory was formulated for the first and second order statistics of reduced speckle in the Fraunhofer plane, and it was stated, though without proof, that the same theory was appropriate to reduced speckle in an image plane. In this Chapter it is the intention to begin by giving proof to these earlier claims.

It is convenient to treat the second order statistics before treating the first order, and so the statistics are dealt with in this rather than in the natural order. Having justified our previous claims, for aberrated as well as unaberrated optical systems, the same theory is then shown to apply to speckle reduction by using spatially partially coherent illumination. In showing that the theory can be applied in this way, we make use of a theorem which is due to Welford (1971) and this states that 'If the pupil of a coherently illuminated optical system varies in time, the time-averaged image is equivalent to that of a system with a certain fixed pupil and partially coherent illumination.'

The remainder of the Chapter is devoted to a brief discussion of speckle in partially coherent light and it should perhaps be emphasised now that the type of partial coherence envisaged would be of the type produced by an extended, quasi-monochromatic source: the discussion is in no way relevant to partial coherence in the temporal sense.

Dainty (1970) produced a form for the Wiener spectrum for speckle formed under partially coherent illumination and he gave his result in terms of an integral through the transmission cross-coefficient of the illumination/imaging system. This result, which we shall quote, does not immediately suggest itself as equivalent to the result which we shall obtain; but, since there is no reason to suspect either result as being incorrect, one presumes that an equivalence could be demonstrated. No attempt has been made to show such an equivalence, at least not in the formal sense; but an attempt is made to show a similarity between the results obtained from the two approaches. By way of doing this, a comparison is made between two evaluations, one of which was made by Dainty and the other by the present Author. The two cases deal with slightly different types of partial coherence and we do not, therefore, expect the results to be exactly similar. However, there is sufficient closeness between the two cases to lead us to expect the same form of result, and indeed this expectation is fulfilled.

Reduced Speckle in an Image - Second Order Statistics

Our purpose in this section is to show that the second order statistical properties of time-averaged speckle patterns take the same form in an image, when the aperture is moved in the pupil plane of the imaging lens, as they did in Chapter Four when we considered the Fraunhofer plane. The quantity which is actually dealt with is the autocorrelation function of intensity; it follows naturally that the same result must then hold for the Wiener spectrum.

The coordinate systems used in the analysis are shown in Figure 6.1. (X,Y) refers to the image plane and (X',Y') to the object plane, with the same numbers denoting conjugate points. (x,y) refers to the pupil plane. It should be noted that, whereas (X,Y) and (X',Y') are optical coordinates, (x,y) are coordinates which

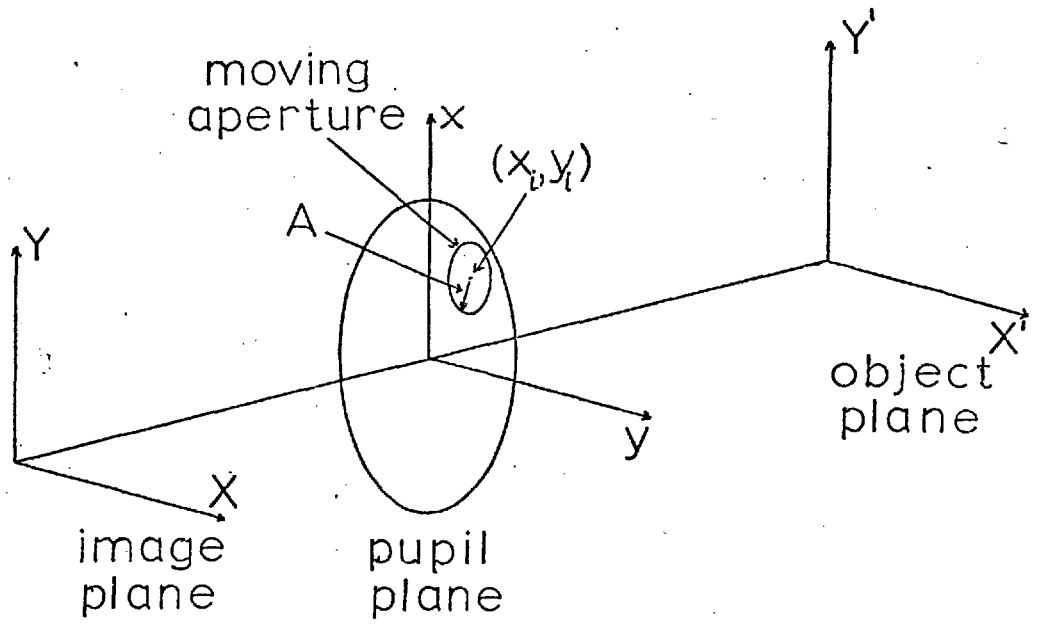


Figure 6.1 The coordinate systems used in the analysis of the statistics of reduced speckle in an image plane.

denote actual distances in the pupil plane. The object is a coherently illuminated diffuse object with surface height fluctuations which, as usual, are greater than a wavelength. The average brightness over the object is assumed to be constant, but later we shall see how this condition can be relaxed.

Analysis

Without loss of generality, we again consider a rotating, circular aperture, and associate with each of the N positions taken by it (cf., Figure 4.2) one of the N intensity functions $I_i(X,Y)$. As before, this allows us to write an expression for the time-averaged autocorrelation function (cf., Equation 4.1):

$$C(\xi, \eta) = \left\langle \left[\frac{I_1(X+\xi, Y+\eta) + I_2(X+\xi, Y+\eta) + \dots + I_N(X+\xi, Y+\eta)}{N} \right] \left[\frac{I_1(X, Y) + I_2(X, Y) + \dots + I_N(X, Y)}{N} \right] \right\rangle - \langle I \rangle^2. \quad 6.1$$

On cross-multiplication, N^2 terms arise, each of which can be written in the form

$$\langle I_i(X+\xi, Y+\eta) I_j(X, Y) \rangle - \langle I \rangle^2,$$

and, by again using Equation 4.5, this can be written

$$\left| \langle V_i(X+\xi, Y+\eta) V_j^*(X, Y) \rangle \right|^2$$

where $V_i(X,Y)$ is the complex amplitude distribution in the image when the aperture takes its i^{th} position.

The form which has just been obtained is the squared modulus of the cross-correlation function of the amplitudes which arise when the aperture lies in its i^{th} and j^{th} positions. Let us ignore the squared modulus for the moment and find an expression for the cross-correlation function by itself, in terms of the parameters of the

optical system.

When the aperture lies in its i^{th} position, that is, when its centre lies at the point (x_i, y_i) , the amplitude at the image point $(X+\xi, Y+\eta)$ may be written as the convolution of the object amplitude with the point spread function of the system, ie.,

$$V_i(X+\xi, Y+\eta) = \sum_m V(x'_m, y'_m) K_i(X+\xi-x'_m, Y+\eta-y'_m), \quad 6.2$$

where $V(x'_m, y'_m)$ is the amplitude emerging from the m^{th} scatterer in the object and K_i is the point spread function of the system when the aperture lies in its i^{th} position. Similarly, the amplitude at the image point (X, Y) , when the aperture takes its j^{th} position can be written

$$V_j(X, Y) = \sum_n V(x'_n, y'_n) K_j(X-x'_n, Y-y'_n). \quad 6.3$$

By combining Equations 6.2 and 6.3, the typical cross-correlation term can be expressed in the form

$$\begin{aligned} \langle V_i(X+\xi, Y+\eta) V_j^*(X, Y) \rangle &= \\ \langle \sum_n |V(x'_n, y'_n)|^2 K_i(X+\xi-x'_n, Y+\eta-y'_n) K_j^*(X-x'_n, Y-y'_n) \rangle, & 6.4 \end{aligned}$$

where product terms for which $m \neq n$ have been omitted since they will average to zero.

If the number of scatterers in the object (within the significant region of the point spread function) is very large, then the summation in the above equation may be replaced by an integral:

$$\begin{aligned} \langle V_i(X+\xi, Y+\eta) V_j^*(X, Y) \rangle &= \\ \iint_{-\infty}^{\infty} I(x', y') K_i(X+\xi-x', Y+\eta-y') K_j^*(X-x', Y-y') dx' dy', & 6.5 \end{aligned}$$

where $I(X', Y')$ is the intensity distribution over the object which we had provisionally assumed to be constant.

Now that Equation 6.5 has been established, it is perhaps a suitable moment to point out that we can relax our previous demand and allow $I(X', Y')$ to vary, subject to the condition that the variation is slow in relation to the size of the point spread function. If the intensity does vary in this way, then, with the exception of the scaling factor $I(X', Y')^2$, this will not materially alter the result we shall obtain for the autocorrelation function.

A dependence of any sort on X' or Y' will of course render the autocorrelation function non-stationary, with the result that a measurement of the space average would tell us nothing of the ensemble average. However, if a space average were obtained within a region where $I(X', Y')$ was roughly constant, then the space average would reflect the ensemble average for that region. Such a region might be referred to as a stationary patch and, with this terminology, the condition under which our theory will apply is that the stationary patch must be as large, or larger, than the point spread function of the system. Having made these remarks, the intensity term will now be omitted from subsequent equations.

If the wavefront aberration over the pupil of the lens is $W(x, y)$ then, when the aperture takes its i^{th} position, the aperture function can be written as

$$F(x-x_i, y-y_i) \exp(i\pi W(x, y)),$$

where F is real and, in the case of a circular aperture with radius A , defined by

$$\begin{aligned} F(x, y) &= 1 & x^2 + y^2 &\leq A^2 \\ &= 0 & \text{otherwise.} \end{aligned} \quad 6.6$$

A Fourier transform relation exists between the aperture

function and the amplitude point spread function. Thus,

$$K(x+\xi-x', y+\eta-y') = \iint_{-\infty}^{\infty} \left[F(x-x_i, y-y_i) \exp[iW(x,y)] \right] \exp\left[\frac{i}{A}[(x+\xi-x')x + (y+\eta-y')y]\right] dx dy. \quad 6.7$$

This expression allows Equation 6.5 to be written in the following way:

$$\langle V_i(x+\xi, y+\eta) V_j^*(x, y) \rangle = \iiint_{-\infty}^{\infty} K_j^*(x-x', y-y') F(x-x_i, y-y_i) \exp[iW(x,y)] \exp\left[\frac{i}{A}[(x+\xi-x')x + (y+\eta-y')y]\right] dx' dy' dx dy. \quad 6.8$$

By now using the relation

$$F(x-x_i, y-y_i) \exp[-iW(x,y)] = \iint_{-\infty}^{\infty} K_j^*(x-x', y-y') \exp\left[\frac{i}{A}[(x-x')x + (y-y')y]\right] d(x-x') d(y-y'), \quad 6.9$$

and also noticing that for any particular point, (X, Y) , we can write

$$dx' dy' = d(x-x') d(y-y'),$$

Equation 6.8 can be greatly simplified:

$$\langle V_i(x+\xi, y+\eta) V_j^*(x, y) \rangle = \iint_{-\infty}^{\infty} F(x-x_i, y-y_i) F(x-x_j, y-y_j) \exp\left[\frac{i}{A}[\xi x + \eta y]\right] dx dy. \quad 6.10$$

The noticeable feature of this equation is that the aberrations of

the system do not enter into it.

Equation 6.10 is in a suitable form for use in the next section when the first order statistics of reduced speckle in an image are treated, but for the moment the relevant quantity requires the squared modulus to be taken on each side of the Equation. Thus, we may write

$$\langle I_i(x+\xi, y+\eta) I_j(x, y) \rangle - \langle I \rangle^2 = \left| \iint_{-\infty}^{\infty} F(x-x_i, y-y_i) F(x-x_j, y-y_j) \exp\left[\frac{i}{A}[\xi x + \eta y]\right] dx dy \right|^2. \quad 6.11$$

The right-hand side of this equation can be stated in words as being the squared modulus of the Fourier transform of the area contained in the overlap for the i^{th} and j^{th} aperture positions. By now referring back to Equation 4.5, we will see that an identical result was given, and the interpretation we must then make is that the second order statistics of reduced speckle in an image are precisely the same as the second order statistics of reduced speckle in the Fraunhofer plane. *

Reduced Speckle in an Image - First Order Statistics

The problem which faces us can be stated in the same way as it was in the previous chapter: We have a ^{complex}/Gaussian amplitude signal which varies as the aperture moves and a detection system which responds to the time-averaged intensity. This of course immediately suggests the same approach as in the last chapter, but whether the result will come out to be the same will depend on the time-autocorrelation function of the amplitude - the kernel. Let us therefore derive a form for this quantity.

The time-autocorrelation function was previously defined

* This applies, of course, only if the object is well resolved, as previously indicated.

(cf., Equation 5.23) for the case where it was stationary both in the spatial and the temporal sense - that is, there was no X/Y dependence and a time-dependence that was a function of $(T_i - T_j)$ only. Because both forms of stationarity are assumed in the present treatment, * we again use Equation 5.23 to define the form of our kernel:

$$C(T_i - T_j) = \langle V(T_i) V^*(T_j) \rangle \quad 6.12$$

In the previous chapter, we derived an explicit form for the kernel (Equation 5.28) by working from first principles. In this Chapter, we choose to short-circuit the 'first principles' - as we could have done in the previous chapter - and produce an explicit form immediately by setting ξ and η to zero in Equation 6.10. Thus, we have

$$C_T(T_i - T_j) = \iint_{-\infty}^{\infty} F(x - x_i, y - y_i) F(x - x_j, y - y_j) dx dy \quad 6.13$$

The integral on the right-hand side of Equation 6.13 represents the area contained in the overlap for the i^{th} and j^{th} aperture positions. If this result is compared to Equation 5.28, it will be seen as exactly equivalent, and so we can now infer that the first order statistics of reduced speckle in an image are identical to those in the Fraunhofer plane.

Note on Aberrations

We stated previously that aberrations will not affect the statistics if the 'stationary patch' size is larger than

* If $C_T(T_i - T_j)$ is spatially stationary, then it cannot, strictly, be applied to cases where the object is not uniformly bright. However, if the X/Y dependence is slow, as was discussed previously, then the appropriate probability density function can still be produced by simply using the scaling factor $\langle I(X, Y) \rangle$. This factor is analogous to the factor $\langle I(X, Y) \rangle^2$ which occurred in the corresponding situation with the second order statistics.

the point spread function. For field-dependent aberrations, however, the point spread function will increase in size as we move away from the axis, with the result that 'tolerably slow' variations in $I(X,Y)$ for near-axial regions might not be 'slow enough' for off-axis points.

Speckle Reduction by Partially Coherent Illumination

Welford (1971) gave the general theory of the time-averaged image when a coherently illuminated object is observed through a time-varying pupil, and he showed that the image will have the same properties as an image formed by a stationary pupil of a certain shape when the object is illuminated with partially coherent light. When a diffuse object gives rise to a speckled image, then the speckle can be considered as simply the image of that object, and so we expect speckle patterns in an image to lie within the domain of Welford's theory.

Since the speckled image will look exactly the same in the two equivalent modes of imagery, the statistics which one might measure from either mode will also look exactly the same. This must then enable us to say that, if we can compute the speckle statistics for one mode, then these will also apply to speckle formed in the other mode. We have already seen how to compute the first and second order statistics of speckle produced through a moving aperture; we therefore claim that the same theory can be used to compute the statistics of speckle formed in the equivalent system, when the illumination is partially coherent. Let us now see how the equivalent system may be found. To do this, we quote the result obtained by Welford.

We continue to use the coordinates X,Y in the image plane and, in accordance with Welford, use the coordinates X_0,Y_0 and X'_0,Y'_0 in the object plane, with the same numbers denoting geometrically corresponding points. $K(X,Y;t)$ is used to denote the amplitude point spread function, produced by the time-varying pupil, and $K_1(X,Y)$ is

used to denote the amplitude point spread function for the stationary aperture in the equivalent system. If the mutual intensity of the illumination is denoted J_0 , and isoplanatism is assumed, then the following relation has been shown to be true:

$$\begin{aligned} J_0(x-x_0', y-y_0') K_1(x_0, y_0) K_1^*(x_0', y_0') = \\ \frac{1}{2T} \int_{-T}^T K(x_0, y_0; t) K^*(x_0', y_0'; t) dt. \end{aligned} \quad 6.14$$

Thus, according to Welford, 'if we choose a given physically possible form of K_1 , this equation gives the mutual intensity, J_0 , of the illumination which will simulate by partially coherent illumination the effect of the time-varying pupil.' If the van Cittert-Zernike theorem (1934) is now applied, then the required source of illumination is found to be given by an extended, quasi-monochromatic, incoherent source, whose brightness distribution is given by the inverse Fourier transform of J_0 .

Welford showed, as an example of his theory, the equivalence of a rotating, circular aperture and a quasi-monochromatic, incoherent ring source, and this allows us to claim that the results which were obtained in Chapter Four for the rotating, circular aperture must also relate to speckle, produced by illuminating with such a ring source. Shortly, a comparison will be made between an evaluation given in that chapter and an evaluation given by Dainty (1970), who would appear to be the only author to have published an account of speckle formed in partially coherent light. Before making the comparison, however, we first indicate the theoretical result which Dainty gave.

Dainty considered that the imaging of a diffuse object, illuminated by partially coherent light, was equivalent to the passage of real Gaussian noise through a real quadratic filter. By viewing the

problem in this way, he obtained the Wiener spectrum in terms of the transmission cross-coefficient of the optical system. * He gave the result in a one-dimensional form:

$$N(u) = \int_{-\infty}^{\infty} |\tau(u, u+u_1)|^2 du_1 \quad 6.15$$

where τ denotes the transmission cross-coefficient.

No evaluations of the Wiener spectrum have been made from this equation or its two-dimensional counterpart, but Dainty did make an evaluation of the zero frequency value, $N(0)$. In the case dealt with, the partially coherent illumination was produced from a circular source area and the imaging was performed by an aberration free lens with a circular aperture. The result given by Dainty was a plot of $N(0)$ against s , the ratio of the angle subtended by the source at the object to the angle subtended by the pupil at the object. A copy of Dainty's result is shown in Figure 6.2.

By making the assumption that the form of the Wiener spectrum is reasonably constant for different values of s , then the plot shown is equivalent to a plot of the variance. If we therefore make this assumption, and our findings in Chapter Four would tend to justify our making it, then we should expect to see a resemblance between the curve in this figure and the curve shown in Figure 4.7, where the variance is plotted against R (analogous to s) for a ring source.

The main features of the two curves are very similar - we do not expect exact similarity since the source types are not exactly similar - and this would support the belief that the two methods of calculation are equivalent.

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Because only real filters were considered, his result applies only to aberration free systems.

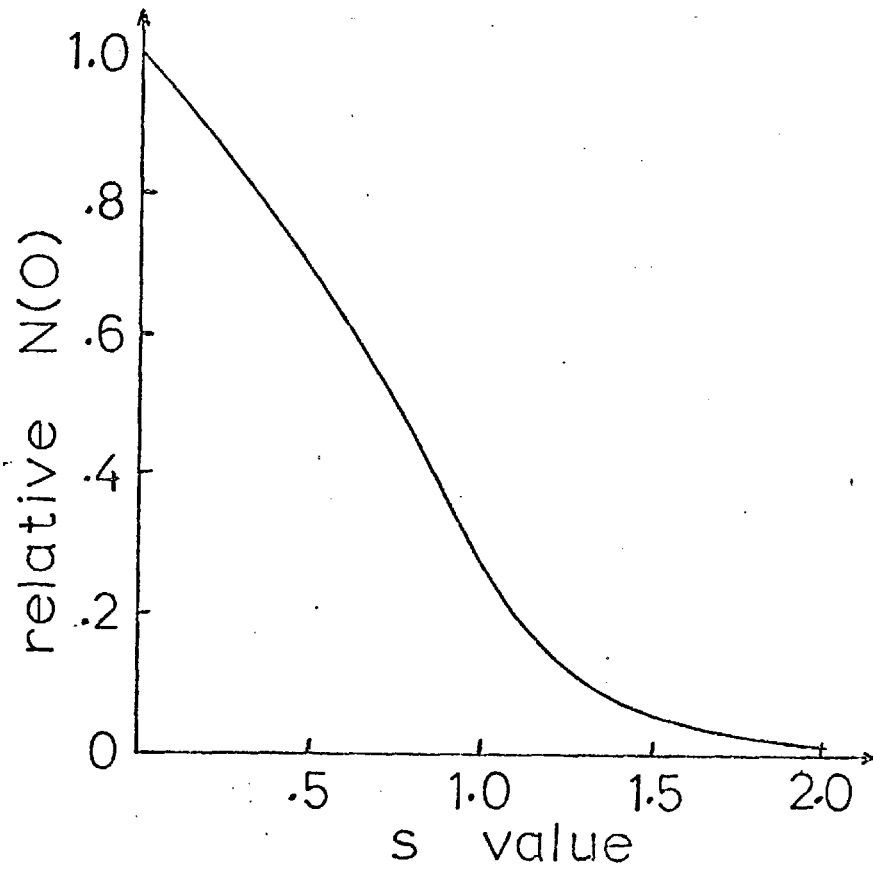


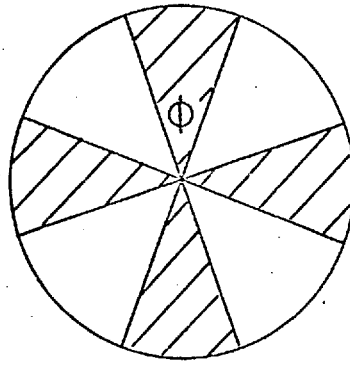
Figure 6.2 The reduction in the zero frequency value, $N(0)$, versus the partial coherence parameter, s , for a circular disc source. (Taken from Dainty, 1970)

CHAPTER SEVENTHE 'OPTIMUM' SHAPE OF A ROTATING APERTURE

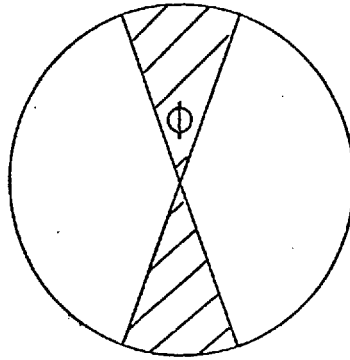
The type of improvement and the degree of improvement which a moving aperture might be required to yield could take so many different forms that a single aperture of a specified shape would not be expected to perform all of them well. We therefore make no attempt in this Chapter to discover a specific aperture shape which might be described as optimum in the sense that it would perform optimally in any given situation. In any one particular situation, of course, the prospect of finding an optimum aperture would arise, but where a general consideration is undertaken, such as in this Chapter, the inevitable result is that no single aperture is selected out.

Instead, therefore, of looking for one specific aperture type, we analyse the performance of a number of apertures, with perhaps two aims in view. The first is that we might look to the results for any suggestion that, for a particular application, one aperture type might be more suitable than another; and the second, which perhaps holds greater interest for us, is that by cross-comparing the results for different apertures we might discover pointers as to the likely nature of an optimum aperture.

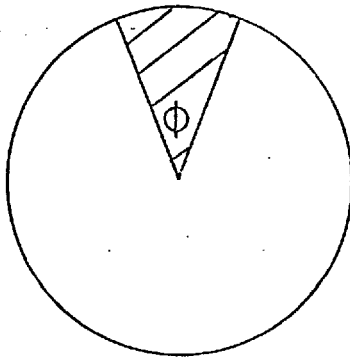
Four different types of aperture are considered. The first of these is the circular aperture, to which considerable attention has already been devoted. The remaining three are the sector aperture, the bow-tie aperture and the Maltese-cross aperture, each of which has been illustrated in Figure 7.1. The performance of the different



Maltese cross



bow-tie



sector

Figure 7.1 Three of the aperture types whose speckle reduction performance is analysed.

apertures is assessed in terms of two strategies: one, the resolution of the aperture, as expressed by the time-averaged point spread function; and the other, the degree of speckle reduction, as expressed by the autocorrelation function of intensity.

The Circular Aperture

Let us assume that the imaging lens is aberration free and has a circular pupil, so that, if we employ the whole pupil, the point spread function will be an Airy pattern. The resolution of the system, before any reduction in speckle is attempted, can be thought of in terms of the width of this Airy pattern, and so this function is used as a scheme of reference in any statements that are made about resolution.

When a small circular aperture is made to rotate within the pupil, the time-averaged point spread function will again be an Airy pattern, but it will now possess a width which is $(1+R)$ times greater than the reference pattern. * For large values of R , it is apparent that resolution will deteriorate in proportion to R . If we refer to Figure 5.1, it will be observed that the effective number of independent patterns, i.e., the factor by which the variance is reduced, is also proportional to R when R is large. Both of these tendencies are reflected in Figure 7.2 where the resolution obtained - as gauged by the width of the point spread function at half-height - is plotted against the effective number of independent patterns, M . The conclusion which must be drawn from the Figure is that to cause a total reduction in speckle will also cause a total loss of resolution.

The reduction method which we have just described is, of course,

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As before, R is defined as ρ/A , where ρ is the radius on which the scan is made and A is the radius of the moving aperture.

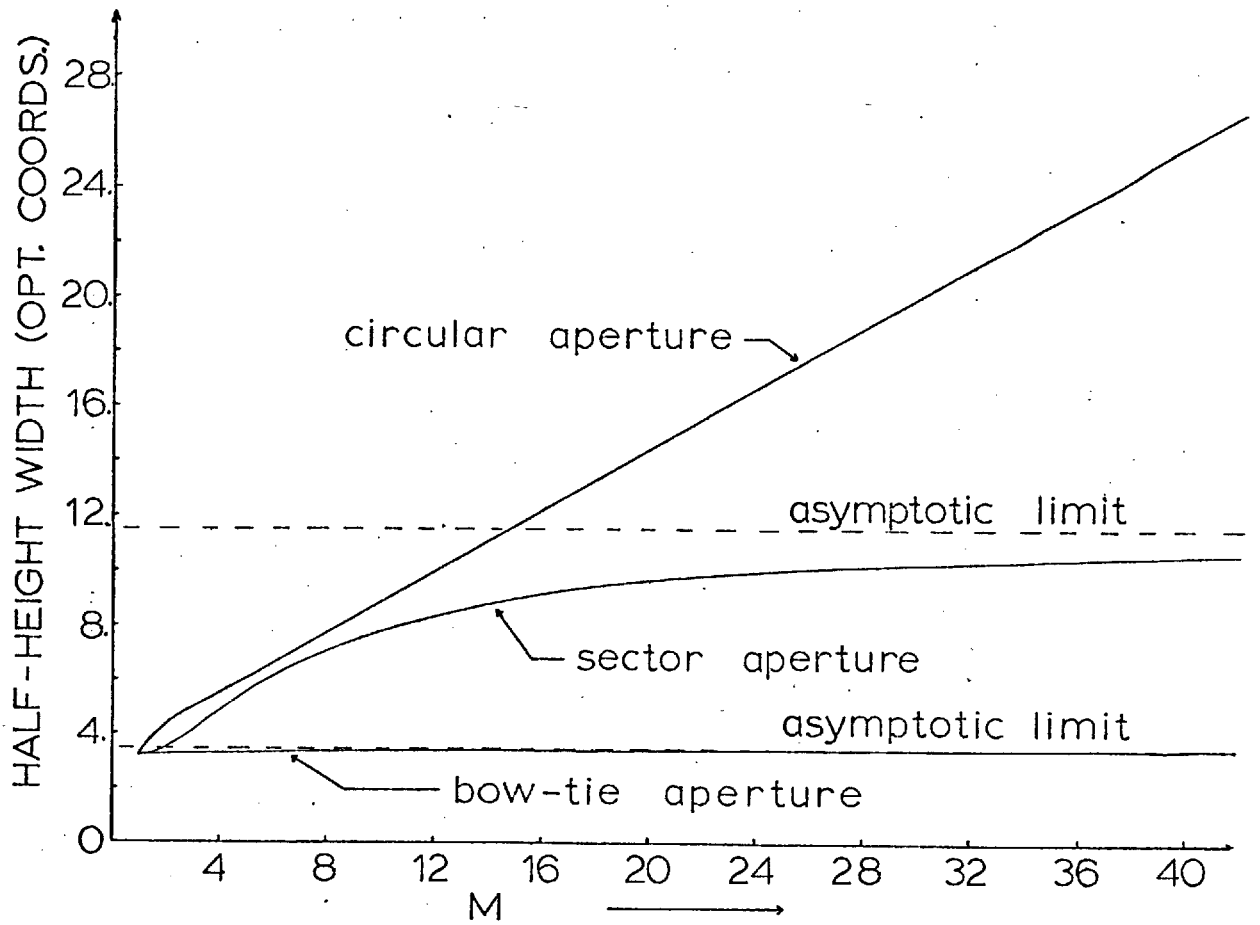


Figure 7.2

The resolution obtained from various types of rotating apertures, plotted as a function of the degree of speckle reduction.

very inefficient because, for large values of R , only a narrow annular region of the pupil is sampled. If, instead of moving the aperture in a circle, we had moved it about in some random fashion, so as to give equal exposure to all regions of the pupil, then an even greater reduction in speckle could have been obtained without any further loss in resolution. This provides us with our first pointer, namely, that an optimum aperture should move in a way which gives equal exposure to all parts of the pupil.

If the whole pupil were used in this way, then the reduction in speckle would depend on R^2 , ie., a number proportional to the ratio of the area of the whole pupil to that of the moving aperture. Although the reduction in speckle is certainly greater in this case than for a circular scan, the implication still remains that a total loss of resolution must accompany a total reduction in speckle. The question might now be asked: Is there an aperture form which will allow a total reduction in speckle without causing complete loss of resolution?

The Sector Aperture

The amplitude point spread function for an aberration free optical system with a sector aperture (Figure 7.1) has been given by Mahan et al (1964). If the real and imaginary parts of the function are denoted $V_R(r, \theta; \phi)$ and $V_I(r, \theta; \phi)$, where r and θ are polar coordinates and ϕ is the sector angle, then we may write

$$V_R(r, \theta; \phi) = \phi \left[\frac{J_1(r)}{r} \right] + 2 \sum_{n=1}^{\infty} \left[\frac{2n+1}{n(n+1)} \left[\frac{J_{2n+1}(r)}{r} \right] \sum_{m=1}^n (-1)^m \sin(m\phi) \cos(2m\theta) \right], \quad 7.1$$

$$V_I(r, \theta; \phi) = 16 \sum_{n=1}^{\infty} \left[\frac{n}{(2n-1)(2n+1)} \left[\frac{J_{2n}(r)}{r} \right] \sum_{m=1}^n (-1)^{m-1} \sin(2m-1) \frac{\phi}{2} \cos(2m-1) \theta \right], \quad 7.2$$

where J_n is the n^{th} order Bessel function.

The intensity point spread function can be computed from Equations 7.1 and 7.2 by the usual relation

$$I(r, \theta; \phi) = V_R(r, \theta; \phi)^2 + V_I(r, \theta; \phi)^2 \quad 7.3$$

If the sector aperture is made to rotate, then the observed point spread function will be the time-averaged function which we can write

$$I_T(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} I(r, \theta; \phi) d\theta. \quad 7.4$$

In Figure 7.3, $I_T(r; \phi)$ is plotted against r for several values of ϕ . For a 360° degree sector, the result is the familiar Airy pattern. For smaller sector angles, the point spread function widens as we should expect; but, rather surprisingly, it tends towards a limiting shape which is not significantly different from the 5° sector shown.

If the width at half-height is again adopted as our criterion for resolution, then there is a progressive deterioration in resolution as ϕ is reduced, and this point is illustrated in Figure 7.4. The deterioration, however, is no longer total, as it was in the case of the circular aperture.

The time-averaged autocorrelation function for the rotating

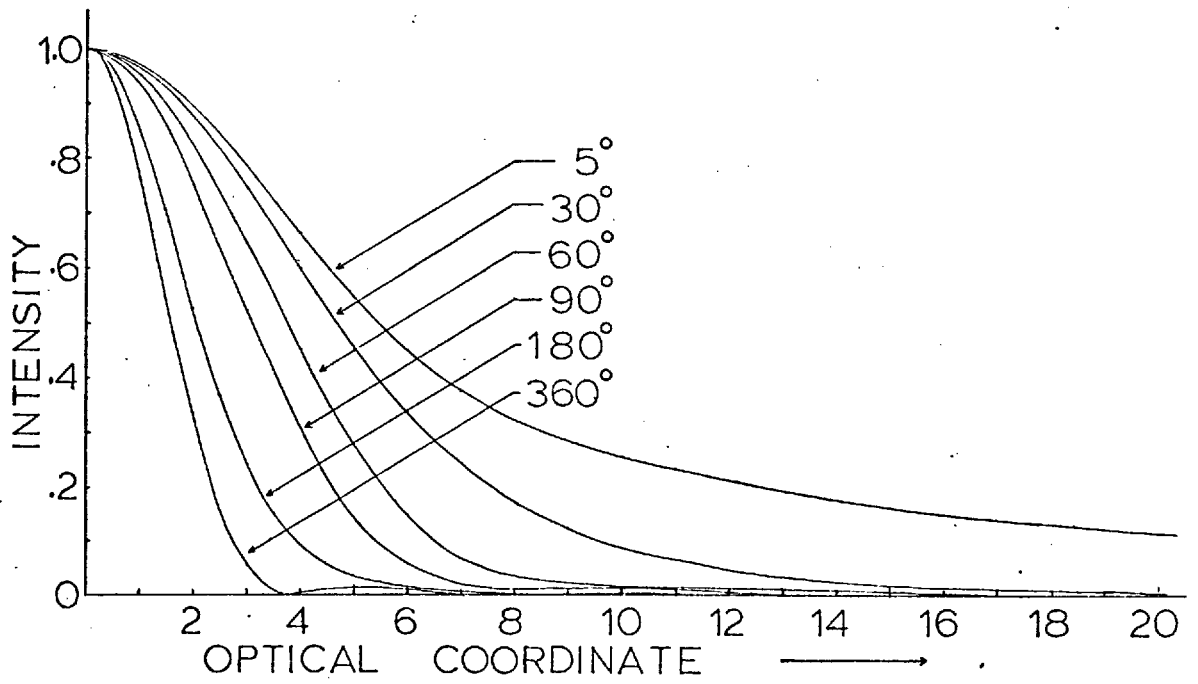


Figure 7.3 The time-averaged point spread functions for rotating sector apertures with various angles, ϕ .

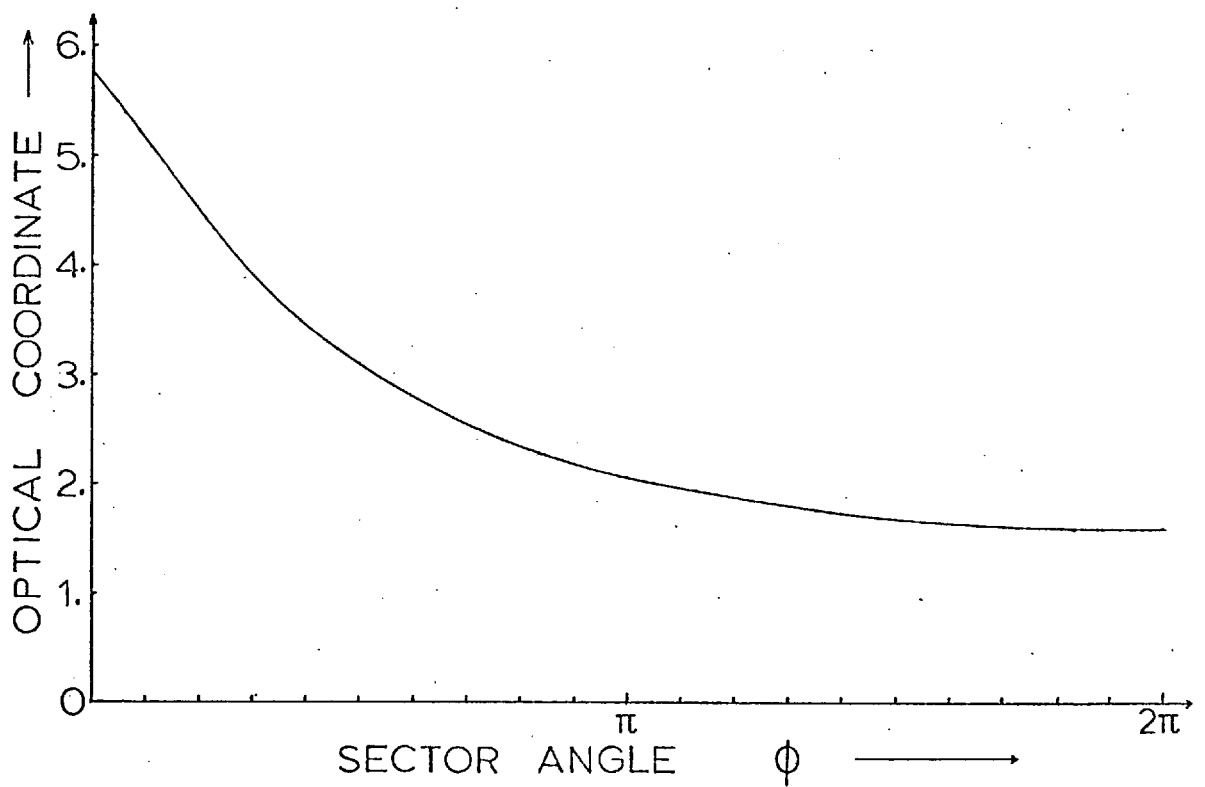


Figure 7.4 The loss of resolution as the sector angle is reduced.

The half-height width of the time-averaged point spread function is used as the resolution criterion.

sector aperture will be similar in shape to the time-averaged point spread function. Normally, however, the autocorrelation function will be a slightly wider function, the only exceptions to this rule being when the sector angle takes small values or the value 360° ; in these cases the autocorrelation function will have exactly the same shape as the point spread function. Some typical autocorrelation functions are illustrated in Figure 7.5.

The curves in Figure 7.5 are all drawn with the same height; but against each curve, the true height - the variance - is indicated by a number in brackets. These numbers were calculated from the relation (which is easily shown) connecting the variance to the sector angle:

$$\frac{\sigma^2}{\langle I \rangle^2} = \frac{\phi}{360 \times 1.5} \quad , \quad 7.5$$

where ϕ is measured in degrees. It might be an appropriate moment to again point out that the reduction in variance, although proportional to the number of times the aperture will 'fit' into the whole pupil, is certainly not exactly equal to this number.

The effective number of independent speckle patterns produced by the rotating sector, M , is the reciprocal of the variance, so that we have

$$M = \frac{360 \times 1.5}{\phi} \quad 7.6$$

M has been plotted against ϕ in Figure 7.6.

It has now been found that, whereas the deterioration in resolution does have a limit as the sector angle is reduced, there is no corresponding limit to the amount speckle reduction which can be obtained. A graphical expression of this is given in Figure 7.2, where the resolution limit is plotted against the M value which

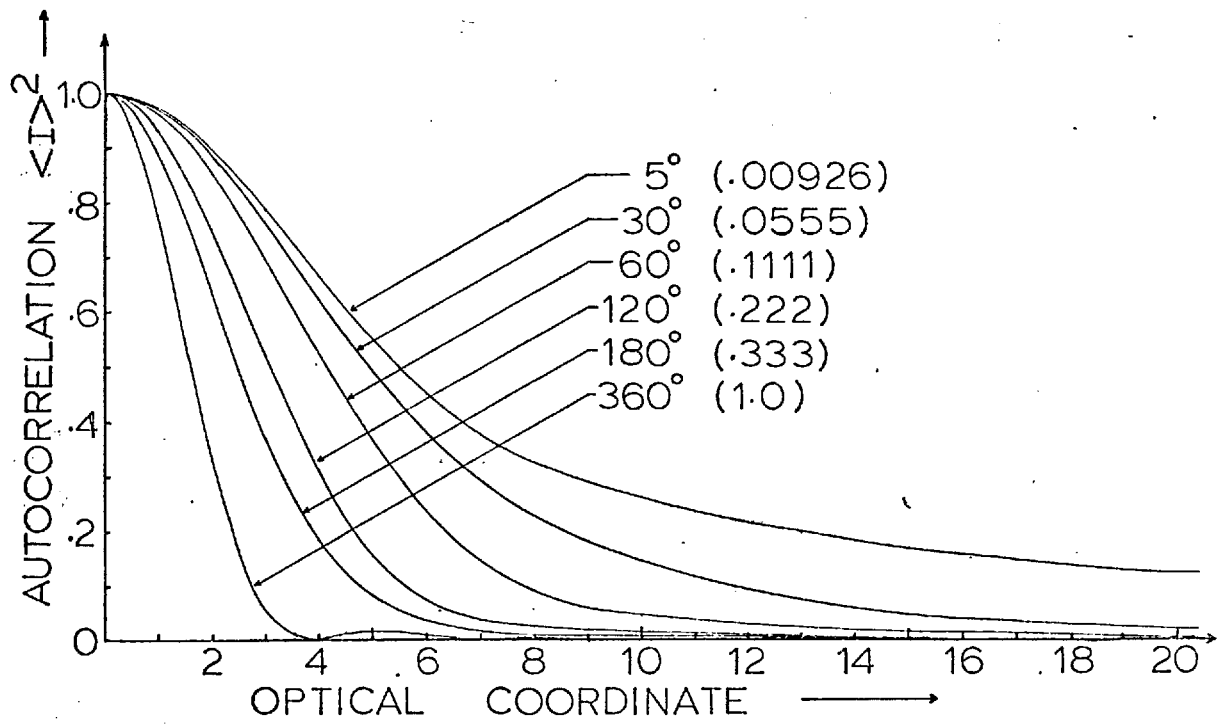


Figure 7.5 The time-averaged autocorrelation functions for various rotating sector apertures. The curves are each drawn to the same height; their true heights, in terms of the scale shown, are indicated in brackets.

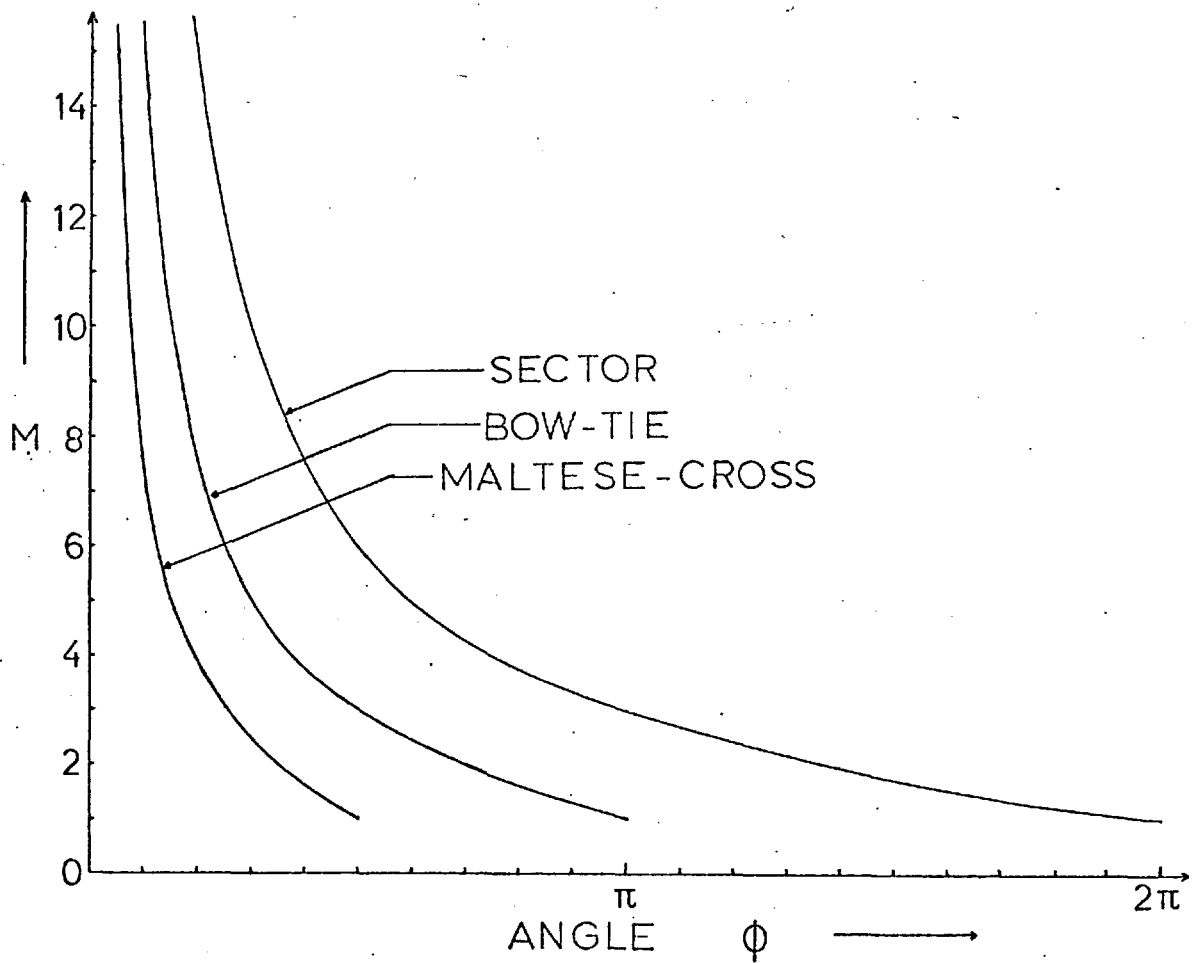


Figure 7.6 The effective number of independent speckle patterns, as a function of the aperture angle, for various types of rotating apertures.

corresponds to it. If the curve for the rotating sector aperture is compared to that for the rotating circular aperture, the noticeable improvement offered by the former underlines the importance of making full use of the available pupil area.

The Bow-tie Aperture

The amplitude point spread function for an aberration free system with a bow-tie aperture can be deduced from Equations 7.1 and 7.2. Using the previous notation, the real part of the function is found to double in magnitude,

$$V_R(r, \theta; \phi) = 2\phi \left[\frac{J_1(r)}{r} \right] + 4 \sum_{n=1}^{\infty} \left[\frac{2n+1}{n(n+1)} \left[\frac{J_{2n+1}(r)}{r} \right] \sum_{m=1}^n (-1)^m \sin(m\phi) \cos(2m\theta) \right], \quad 7.7$$

whereas the imaginary part goes to zero,

$$V_I(r, \theta; \phi) = 0. \quad 7.8$$

The intensity point spread function takes a simpler form in this case:

$$I(r, \theta; \phi) = V_R(r, \theta; \phi)^2. \quad 7.9$$

If the bow-tie aperture is made to rotate, the observed point spread function will be the time-averaged function which is given by Equation 7.4.

In Figure 7.7, the time-averaged point spread function, $I_T(r; \phi)$, is plotted against r for several values of ϕ . For a 180° bow-tie, the result is again an Airy pattern, as we should expect. For smaller angles, the function widens, although the widening is now less severe

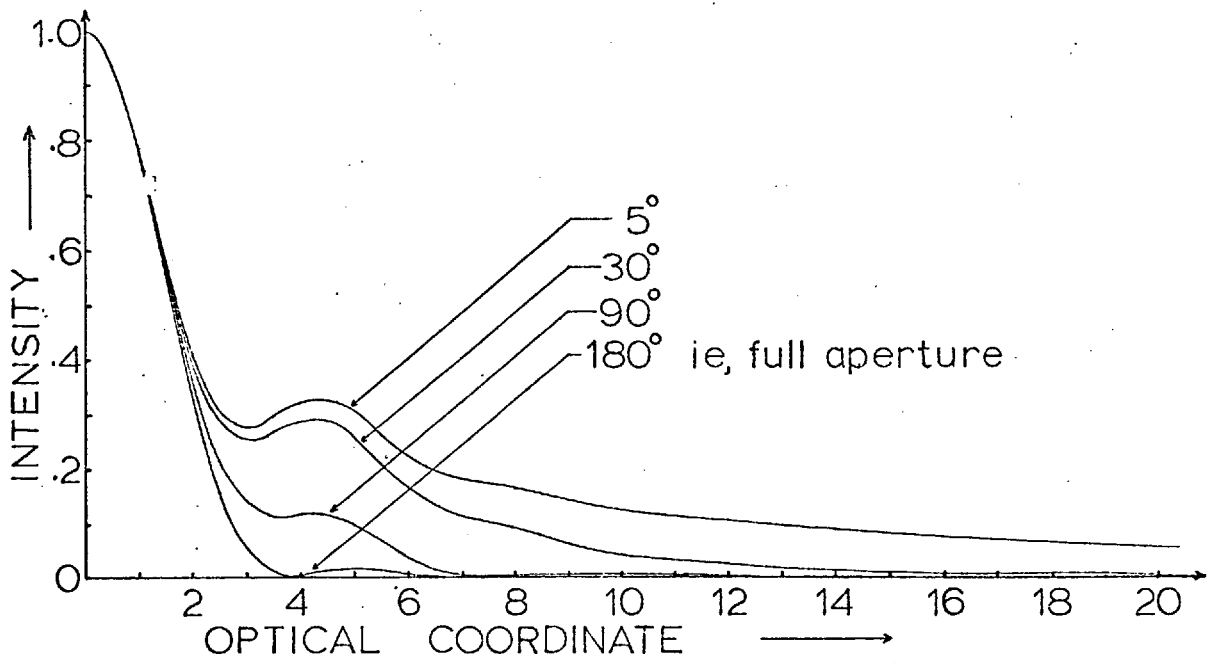


Figure 7.7 The time-averaged point spread functions for rotating bow-tie apertures of different angles, ϕ .

than it was for the sector aperture (cf., Figure 7.3). Again the point spread function takes on a limiting shape for small values of ϕ .

The half-height widths of the curves in Figure 7.7 alter only marginally as ϕ is reduced, and this would suggest that the resolution afforded by the bow-tie aperture is largely independent of the angle. In contrast, however, the amount of speckle reduction is sharply dependent on this angle, as witnessed by Figure 7.8. If we again refer to Figure 7.2, we will see how the resolution of the bow-tie aperture compares with that of other aperture forms. The noticeably improved performance of the bow-tie aperture over the sector aperture provides a further pointer, viz., that the area of an optimum aperture should be distributed over as broad a base-line as possible.

The Maltese-Cross Aperture

The amplitude point spread function for a Maltese-cross aperture can be obtained by summing the amplitudes from two bow-tie apertures which are positioned at right angles to one another. Thus,

$$V_R(r, \theta; \phi) = 4\phi \left[\frac{J_1(r)}{r} \right] + \sum_{n=1}^{\infty} \left[\frac{2n+1}{n(n+1)} \left[\frac{J_{2n+1}(r)}{r} \right] \right. \\ \left. \sum_{m=1}^n \left[1 + (-1)^m \right] \sin(m\phi) \cos(2m\theta) \right], \quad 7.10$$

$$V_I(r, \theta; \phi) = 0 \quad 7.11$$

The intensity point spread function is again given by Equation 7.9 and, when the aperture is made to rotate, the time-averaged point spread function is again given by Equation 7.4.

In Figure 7.9, $V_I(r; \phi)$ is plotted against r for several values of ϕ . As we should expect, when $\phi = 90^\circ$ the result is an Airy pattern.

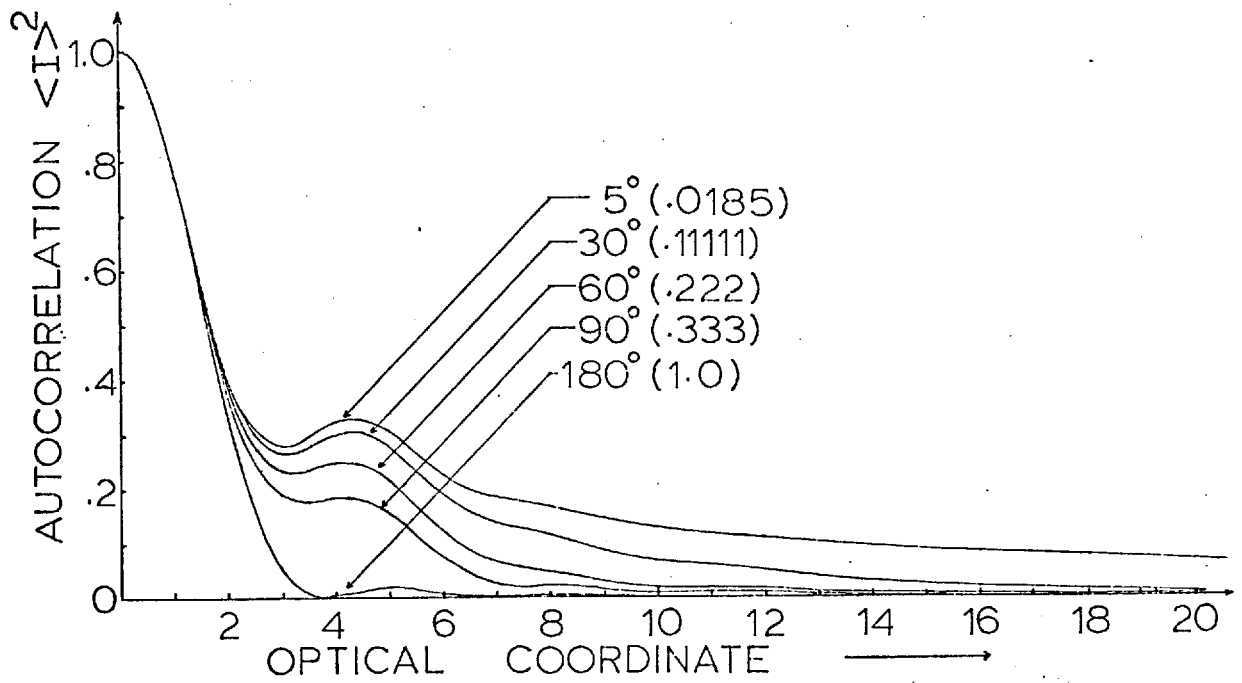


Figure 7.8 The time-averaged autocorrelation functions for various rotating bow-tie apertures. The curves are each drawn to the same height; their true heights, in terms of the scale shown, are indicated in brackets.

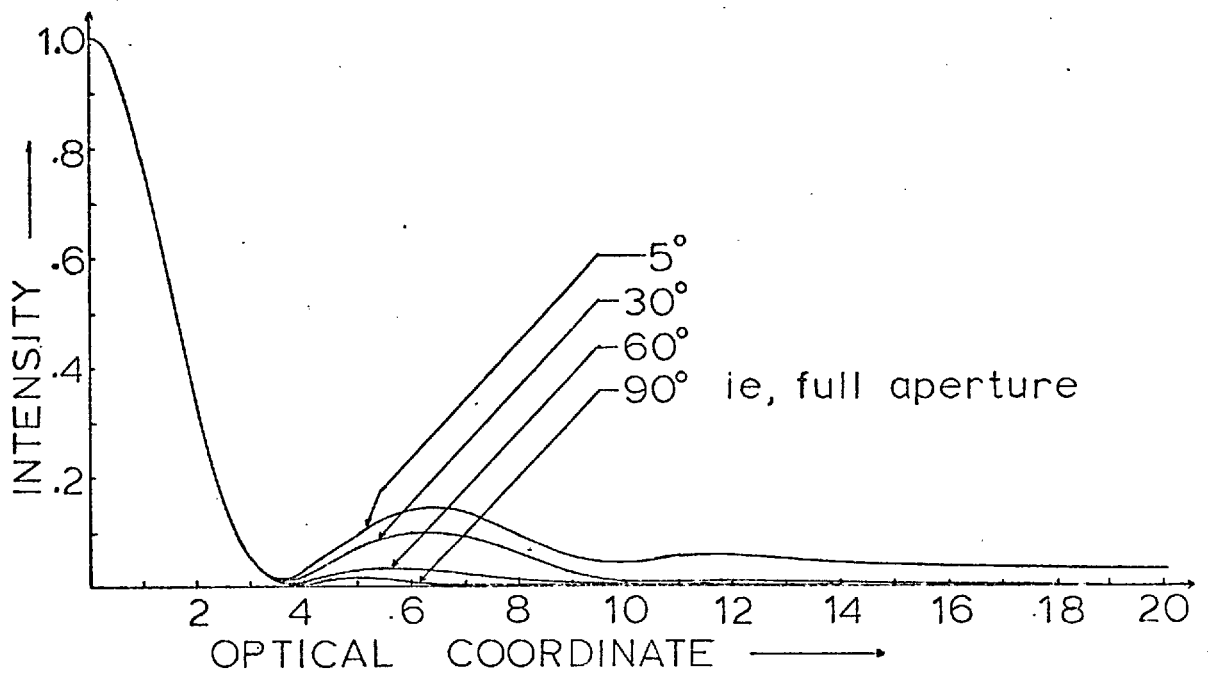


Figure 7.9 The time-averaged point spread functions for rotating Maltese-cross apertures of various angles, ϕ .

For smaller angles, the function widens until a limiting distribution is reached. The widening, however, is now less severe than it was for the other aperture forms.

The half-height widths of the point spread functions show only marginal change for different values of ϕ , and this is similar to the result found with bow-tie apertures. The side-lobes of the functions, however, are now much less pronounced, and this would suggest an enhanced resolving capacity, even though this is not indicated by our resolution criterion.

The amount of speckle reduction provided by the aperture varies in the same sort of way as it did for the bow-tie and the sector apertures. This can be seen by referring to Figure 7.6. The relevant curve in that figure was calculated from the form

$$M = \frac{90 \times 1.5}{\phi} \quad 7.12$$

Equation 7.12 differs from Equation 7.6 only by the factor 4, and this factor only arises because a Maltese-cross aperture of angle ϕ occupies four times the area of a sector aperture with the same angle.

The improved performance of the Maltese-cross aperture over the other aperture forms, the bow-tie aperture in particular, enables us to sharpen the previously given pointer by replacing the word 'base-line' by 'base area'.

Discussion

From our analysis of specific aperture forms, we have reached two conclusions about the likely form of an optimum aperture.

1. It should be shaped so that during the movement cycle equal exposure is given to all regions of the available pupil.
2. The shape should be distributed so as to cover as wide a base-area as possible.

If the above findings were put into practice, the type of aperture which might be constructed would almost certainly show similarity with the apertures used by Yu and Wang (1973). However, while these authors would be inclined to endorse our findings, Hariharan and Hegedus (1974) would almost certainly wish to refute them, their reason being that 'the use of random masks for spatial division in the cases studied gives much poorer resolution than simple subdivision of the hologram area, because of the serious loss of image contrast with the random mask.'

It seems that there is a serious conflict of opinion as to what comprises an optimum aperture. Some conflict, of course, is inevitable because an aperture which performs well in one situation might not give the same high performance in another situation. And perhaps some of the conflict is due to the fact that our resolution criterion takes no account of the reduced contrast which would almost certainly result from the long tails in the point spread functions. But, whatever the reason for the conflict, it seems clear that a closer investigation of the matter is required before any rigid statement can be made.

The investigation might have to deal in slightly more specific terms than have been dealt with in this Chapter and perhaps, in pursuing it, statistical detection techniques (Dainty, 1971) might be applied. The theoretical descriptions of time-averaged speckle patterns which have been given in the foregoing chapters would allow such an approach to be taken.

APPENDIXTHE AUTOMATIC SCANNING AND DIGITISING SYSTEM

The automatic scanning and digitising system, a diagram for which has been shown in Figure A.1, was built specifically to enable large numbers of measurements to be obtained from speckle patterns. The principle on which the system works is quite simple: A photomultiplier is made to scan across the speckle field, while, at repeated intervals, momentary pauses are made, during which the light intensity is measured and recorded.

The system is fully automatic, with two control boxes being used to govern the operation. One box controls the movement of the photo-multiplier, while the other controls the transfer of data from the digital voltmeter to the paper-tape punch.

The photomultiplier is mounted on a precision indexing table and movement is brought about by supplying logic pulses to a drive unit which, in turn, operates a stepper motor drive. The specification of this part of the system falls into four parts:-

1. For each logic pulse supplied the photomultiplier moves one micron.
2. By different settings of a dial-counter, the pulses can be supplied in sequences of up to 1,000, ie., the interval between measurements can be varied from one micron up to 1,000 microns.
3. The full travel distance of the table is some 8cms. (80,000 microns) so that if, for example, a two micron

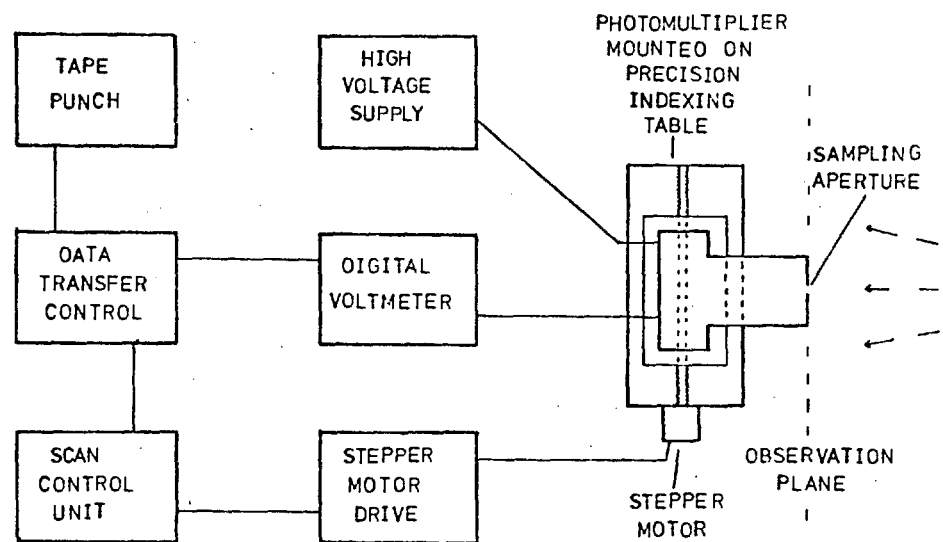


Figure A.1 A diagrammatic representation of the automatic scanning and digitising system.

sampling interval were chosen, then 40,000 recordings could be made during a single scan.

4. By throwing a switch, the direction of movement of the photomultiplier may be reversed. At either end of travel, a micro-switch is triggered which inhibits further movement, unless this is in the reverse direction.

Time allowed for recording is split into two parts. One part allows for a time-integration of the signal, and the other part allows for the actual punching process.

The integration time can be varied between .1 and 10 seconds by means of a dial potentiometer. Averaging of the signal during the period of integration is achieved by coupling one of twelve capacitors across the outputs of the photomultiplier. The capacitors cover a six decade range and the particular one chosen would couple with the output impedance to give a time constant of the same order as the integration time which was being allowed. The output impedance is governed by the range setting on the digital voltmeter.

The time allowed for punching is approximately constant. Since the tape punch handles 46 characters per second and there are 10 characters to be punched, the complete punching process takes about .2 seconds.

The features included in the data transfer control box are as follows:-

1. Punching commences when the integration time has elapsed.
2. Characters are punched sequentially. When the punch has completed a character, it supplies a 'punch ready' signal, and this is used to initiate the punching of the next character. In this way, it is not possible to punch two characters on top of one another.
3. During punching, the analogue to digital conversion is

inhibited. Normally, the conversion occurs 24 times per second.

4. The full punched record consists of 10 characters. The first gives the sign of the number; the next four give the number actually shown on the digital voltmeter display panel; the next three give the range and sensitivity settings on the digital voltmeter; and the last two, a blank and a rub-out (the blank is optional), indicate the end of record. The eight relevant characters are subsequently assembled into a number by the computer. To do this, the computer, in effect, multiplies the number displayed on the digital voltmeter by $10^{(R+S)}$, where R and S are values indicating the range and sensitivity settings (these are in decade ranges).

5. Normal functioning of the system is inhibited when either the digital voltmeter goes over-range or a tape error develops. A tape error would result if the tape ran out or became unusually tight. Display lamps indicate the condition causing the inhibit. After an error has been corrected, for example, in the case of an over-range error, by altering the range setting, normal functioning can be restarted.

6. Normally, punching of the record is triggered at completion of the integration time; but an immediate punching of the contents of the digital voltmeter can be initiated by a push-button switch. This facility might be used to re-punch a record which had been spoiled through a tape error developing.

7. Those characters in a spoiled record which had actually been punched would have to be destroyed. This could be done by means of two push-button switches: one, to backspace the tape, and the other to perform the rubbing-out. Rubbing-out involves punching out all eight holes across the tape.

8. A display on the control box keeps a running count of the number of recordings as they are made. A dial counter can be set to halt the system after a fixed number of recordings.

9. By throwing a switch on the control box, the tape punch is decoupled from the digital voltmeter and set into an 'off line' mode. In this mode we can set up and punch out any character of our choosing. The combination of holes which would give the character are set by means of a line of switches. This facility might be used to indicate on the tape the number of recordings which are to follow.

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