

OPTIMAL STOCHASTIC CONTROL OF CONTINUOUS PROCESSES
- SOME APPROXIMATIONS APPLICABLE TO CHEMICAL
ENGINEERING PROBLEMS

A thesis Submitted for the
Degree of Doctor of Philosophy
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by

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To my Parents

**ALL MISSING
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A B S T R A C T

This thesis represents an investigation into the conditions under which optimal control designs, based on deterministic process models, may be applied to stochastic process systems. The stochastic systems investigated show non-linear behaviour in states, controls, and physically realisable noise processes. These noise processes are assumed to be small. Two deterministic models are considered: the second-order model of Joffe and Sargent⁽¹⁾, and the model obtained by setting all noise processes in the stochastic model to their average values.

The principal theoretical results are as follows:-

- (i) When the process state is measurable at all times without error, designs based on the models give a very good approximation to the optimal stochastic performance. The deterioration is of the same order as the model error.
- (ii) When the state is measurable at discrete times, a controller based on discrete updates of open-loop control designs is considered. The effect of the sampling interval is determined.
- (iii) When only noisy measurements are available, which are non-linear functions of the process state, Separation Principle control designs are considered. Whilst the Separation Principle is known to be only strictly applicable to the linear-quadratic-Gaussian control problem, it is shown that good results can be expected in the non-linear case.

These results may be regarded as special cases of a general theorem concerning the design of controllers using models.

Two examples are used to illustrate the theoretical results: a start-up problem for a continuous stirred-tank reactor, and the operation of a batch reactor with decaying catalyst.

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C H A P T E R O N E

INTRODUCTION AND PROBLEM DEFINITION

1.1 Introduction

The main criticism⁽²⁾ of most optimal control studies done on chemical process systems to date is that two of the assumptions on which they are based are invalid. The assumptions in question are:-

- (i) that an accurate deterministic model for the process is available;
- and
- (ii) that all states in the model may be observed without error.

Relaxation of these assumptions inevitably results in a stochastic optimal control problem. Whereas the theory of deterministic optimal control is well-developed, the theory of stochastic optimal control is still in its infancy. Apart from closed-form solutions to the stochastic analogue of the familiar linear-quadratic control problem, little of practical value is known. Since many chemical process systems possess severe non-linear features, stochastic control theory as it stands at present can give little guidance to process control system design.

However, some work^{(3), (4)} has been done on stochastic process control problems, based on the assumption that noise processes appearing in the system are small. Control designs have used a Separation Principle approach, which is exact for the linear-quadratic problem, but no theoretical justification for this approximation has been given for non-linear stochastic systems. It is hoped that some theoretical foundation for future applications is provided by the work in this thesis.

In the remainder of this chapter, the control problem to be considered is formulated mathematically and discussed, and the previous work on stochastic chemical processes is briefly reviewed.

Chapter two is mainly concerned with a special case of the general problem: when the system state vector may be measured at all times without error. It turns out that the general problem may be analysed within the framework of this particular case. A continuous feedback control scheme is proposed which, it is shown, gives a very good approximation to the optimal stochastic controller provided that the noise is small. The performance of certain feedforward and discrete feedback schemes is related to the performance of the optimal stochastic controller. A general theorem concerning the design of controllers using models is presented and the above results are

established as special cases of that theorem.

Despite the simplicity of the proofs of the results of this chapter, none of them appear to have been presented before.

Chapter three gives examples illustrating the theoretical results of the previous chapter. A Monte Carlo technique is used to analyse the performance of the various control schemes put forward in chapter two on two simulated chemical process systems. The results appear to supply confirmation of the theory.

The general problem of optimal stochastic control based on noisy process measurements is considered in chapter four. It is shown that a Separation Principle control scheme gives very good control performance, the approximation to the optimal stochastic controller being limited only by the quality of the state estimates and by the error in the model used to design the controller. A particular estimation scheme, based on the extended Kalman filtering algorithm, is proposed and is shown to have the bounded error characteristic required by the approximate Separation Principle. Neither the approximate Separation Principle nor the proof of the bound on the error of the filtering algorithm appear to have been presented previously. An example is presented to illustrate the theoretical results.

Chapter five summarises the main contributions of this work.

1.2 Problem Statement and Discussion

The problem to be considered in this work may be formulated as follows:-

P 1 Given

$$\text{System} \quad \frac{d}{dt} x = f(t, x, u(t), \xi(t)) : t \in [0, t_f] \quad (1.2.1)$$

$$\text{Initial condition} \quad x(0) \in X_0$$

$$\text{Observations} \quad y(t_i) = g[t_i, x(t_i), n(t_i)] \quad (1.2.2)$$

$$\text{Admissible controls} \quad u(t) \in U$$

$$u(t) = h(t, Y_t) \quad (1.2.3)$$

$$Y_t \triangleq \{y(t_i) : t_i \in [0, t]\}$$

$$\text{Performance index} \quad J(t, Y_t) = E\{c^T x(t_f) | (t, Y_t)\} \quad (1.2.4)$$

Problem

$$\min_h J(t, Y_t) \quad \text{for } t \in [0, t_f] \quad (1.2.5)$$

where $x(t)$ is an n -vector of states
 X_0 is a set of admissible initial conditions
 $u(t)$ is an m -vector of controls
 U is a set of admissible control values
 $\xi(t)$ is an r -vector stochastic process
 $y(t)$ is a p -vector of observations
 $\eta(t)$ is a q -vector stochastic process
 and c is a given n -vector

The initial condition to eqn. (1.2.1) will be assumed to be a random vector of known mean and covariance. The means and covariances of the noise processes $\xi(t)$ and $\eta(t)$ will also be assumed to be known.

Although many chemical process systems are modelled by partial differential equations⁽⁵⁾, we confine our attention to lumped parameter systems for the sake of simplicity. This is a restriction of the theory, but in the past useful results have been obtained using lumped approximations to distributed processes^{(4),(6),(7)}. Also, many of the numerical techniques for solving partial differential equations are based on parameter lumping.

Our formulation includes parameter uncertainties since they may be included as system states with no time variation i.e.

$$\frac{d}{dt} x_j = 0 \quad : \quad x_j(0) \text{ unknown} \quad (1.2.6)$$

may be used to represent unknown parameters in the model.

Our objective (1.2.5) is the most commonly used in stochastic control theory. Others, such as maximising the probability of a given event⁽⁸⁾ or minimising the maximum possible excursion from a desired condition⁽⁹⁾, have been put forward. The former approach appears to be limited to cases where analytical expressions for the conditional probability distribution function of the state are available, or else this function may be characterised by a few parameters. For the non-linear system, eqn. (1.2.1), no such expression is at present available. The latter formulation is more generally applicable, since the problem may be solved using deterministic methods; no probabilistic concepts are used. The approach has been shown to be of some use in extreme cases⁽⁹⁾, and recently equivalences have been established between the minimax approach and the approach adopted here for a special class of problems⁽¹⁰⁾ viz. linear systems perturbed by white Gaussian noise, and a performance index expressible as an exponential function of a quadratic form in state and control.

Savage⁽¹¹⁾ has derived an expected utility theorem which, roughly speaking, states that a decision maker conforming with certain consistency postulates (and with the rules of logic) will attempt to minimise the expected value of his performance index conditional on any information he may receive. If one accepts Savage's postulates, then the expected utility theorem provides sufficient justification for (1.2.5).

1.3 Related Work in Chemical Engineering

Previous works in the chemical engineering literature associated with problem P 1 may be divided into three types of contribution. First, some authors have concerned themselves with the dynamic behaviour of stochastic chemical process systems. Second, there has been some work done on the on-line estimation of states and parameters in systems like eqn. (1.2.1) from noisy measurements like eqn. (1.2.2). Third, there have been a few specific control problems solved which may be fitted into the general framework of problem P 1.

1.3.1 Dynamics of Stochastic Chemical Processes

The pioneering works in this field appear to be those of Aris and Amundson⁽¹²⁾ and of Katz⁽¹³⁾. Aris and Amundson considered the behaviour of a stirred tank reactor with external cooling in which the exothermic, second-order reaction $2A \rightarrow B$ occurs. Perturbations in feed concentration of A, reactant flowrate and inlet temperature were treated using a linearised analysis based on the spectral methods described by Laning and Battin⁽¹⁴⁾. Katz applied similar techniques to cascades of perfect mixers. It is both difficult and dangerous to draw general conclusions from these two studies, but they seem to show that the effect of the process is to damp external disturbances to some extent.

The problem of non-ideal mixing in a stirred vessel has been treated by several authors using stochastic analysis. Václavek⁽¹⁵⁾ has fitted experimental data from a non-ideal continuous mixer using a stochastic linear operator; King⁽¹⁶⁾ has treated simulated data for a similar system using a stochastic transfer function approach. Both authors found that the influence of random processes in the system was important.

Pell and Aris⁽¹⁷⁾ considered the effect of random fluctuations on the choice of the optimal running time for a batch reactor operating the reaction $A \rightarrow B \rightarrow C$, where B is the required product. By considering fluctuations in rate constants (as opposed to reactor temperature) they were able to use the linear theory of Ariaratnam and Graefe⁽¹⁸⁻²⁰⁾. Pell and Aris showed that the optimal running time depended upon the intensity of the random

fluctuations, but that the dependence was not very strong. However, quite large degradations in mean performance were predicted. Their theoretical results were confirmed by simulation studies.

Recently, Berryman and Himmelblau have published a series of papers⁽²¹⁻²³⁾ studying the effect of random inputs and parameters on chemical engineering unit operations. Their approach throughout has been based on Monte Carlo simulation techniques.

In their first paper⁽²¹⁾, Berryman and Himmelblau described their simulation procedure in detail and applied it to nominally isothermal reactor systems. They pointed out that care must be taken to ensure that any noise processes incorporated into an initially deterministic model are physically meaningful: dimensionless groups may cause difficulties if more than one quantity in the group has a random component. Unfortunately, this kind of consideration often makes it difficult to generalise the results of a particular Monte Carlo simulation study. For the particular parameters chosen by Berryman and Himmelblau, the following conclusions may be drawn. For stirred tank reactors operating at steady state, the effect of disturbances in flowrate, inlet concentration and rate constant was considerably damped, and negligible differences between mean performance of the stochastic system and the performance of the corresponding deterministic system were found. The outlet concentration variance was found to increase directly with reactor volume, and to vary inversely with feed flowrate. For plug flow reactors, similar mean performance was observed, but the disturbances were found to be amplified by the process, the amplification being greater the longer was the residence time. The implication of these results in the choice of safety factors in reactor design was discussed by the authors.

More results for the above reactor systems are reported in Berryman and Himmelblau's second paper⁽²²⁾. They tend to confirm the earlier conclusions, although small deviations of mean response from the corresponding deterministic response were observed for the plug flow reactor for some values of the system parameters.

For the heat exchanger example considered in their latest paper⁽²³⁾, Berryman and Himmelblau again found little difference between the mean response of the stochastic system and the response of the corresponding deterministic system, but the variances of the outputs were such as to give quite large oversize factors for the exchanger tube length.

1.3.2 Application of On-Line Estimation Techniques to Chemical Process Systems

The application of sequential estimation techniques to chemical process systems has been the subject of several recent feasibility studies. Gavallas and Seinfeld⁽⁶⁾ considered the application of these techniques to reactors experiencing catalyst decay. Although this type of system is usually modelled by partial differential equations, Gavallas and Seinfeld suggested a lumped parameter approximation and the application of the second-order filters of Detchmendy and Sridhar⁽²⁴⁾ for continuous measurements and of Cox⁽²⁵⁾ for discrete measurements. Their proposal is based on the earlier work of Seinfeld⁽²⁶⁾ who derived a second-order filter for distributed parameter systems, the computational requirements for which became impracticable for any realistic application. Gavallas and Seinfeld presented a simple example showing the application of their approximate filtering algorithms.

Coggan and Noton⁽²⁷⁾ advocated the use of linearised filtering techniques in non-linear process application with discrete measurements. Prediction was achieved using a non-linear dynamic model, and discrete updates were made using linear theory. Their examples contained all the features commonly encountered in chemical processes, viz. strong non-linearities, intermittent and inaccurate measurements, large unmeasured random disturbances, unknown drifting parameters, unknown transport lags and unknown initial values. The application of the filter to the examples was fairly successful in all cases, although performance was not consistently good. There were applications where convergence of the filter to incorrect state estimates was observed; this being generally the result of mutual error compensation effects. Clearly, the success of a linearised filtering application is not guaranteed.

Goldmann and Sargent⁽⁷⁾ presented two examples of the application of filtering techniques to deterministic processes with noise corrupted measurements. They advocated the use of a linearised theory both for prediction and for updates. In the first example, a binary distillation column at steady state, it was demonstrated that drifts in instrument readings may be effectively dealt with using linear theory. In the second, a tubular reactor with decaying catalyst, it was shown that drastic model simplification is possible without deterioration in filter performance, and that model parameters may be estimated using a linear filter.

Wells⁽²⁸⁾ successfully applied an extended Kalman filter to a non-linear reactor problem. Prediction was afforded by the non-linear process

dynamics, and Kalman updates at discrete times were used.

From all these examples it seems reasonable to assume that an estimation procedure for chemical process systems can be devised in many cases. The theory of chapter four relies upon this assumption.

1.3.3 Control of Stochastic Chemical Process Systems

In an early series of two papers⁽²⁹⁾⁽³⁰⁾, Chien and Aris considered the control of a batch reactor with uncertain parameters. Deterministic studies⁽³¹⁾⁽³²⁾ had shown that to achieve a desired conversion in minimum time, the optimal heat removal policy often consists of an initial period of adiabatic operation. Chien and Aris addressed themselves to the question of whether measurements taken during this adiabatic operation could be used to obtain estimates of the model parameters. These estimates would then be used to compute the remainder of the optimal control policy. In the first paper⁽²⁹⁾, Chien and Aris proposed an estimation procedure based on the collection of an observation record which was fitted using non-linear least squares techniques. The method was tested using simulated data. The second paper⁽³⁰⁾ was concerned with the optimal control, and with several near-optimal control schemes which could be implemented in a practical situation.

A similar problem was considered by Seinfeld⁽³⁾. An extended Kalman filter was used to generate state and parameter estimates which were used in a feedback control law designed to maximise the instantaneous speed of approach of the system to a desired state. The example considered by Seinfeld was a start-up problem for a continuous stirred tank reactor with a first-order exothermic reaction. Control of the overall heat transfer coefficient between the reactor and a coolant was assumed to be possible. Temperature measurements corrupted by random noise were available; process noise was assumed to be negligible. The feedback control, which was found to be bang-bang in character, was found as a function of the reactor temperature, and of some parameters in the process model. Improvements were observed over the response without the filter when the parameters were known, the filter being used to give a better estimate of the reactor temperature, but parameter estimation using the filter was not very good, hence the full problem could not be said to have been solved satisfactorily.

Joffe and Sargent⁽⁴⁾ considered a specific example of the general problem P 1: an irreversible, exothermic reaction $A \rightarrow B \rightarrow C$ conducted in an adiabatic tubular reactor. The feed is pure A and B is the desired product. The reaction is catalytic; the catalyst decays in time, causing the necessity

of periodic shut-down periods for regeneration. The inlet temperature of the reactant is controllable by means of an external heat exchanger. The temperature at ten equidistant points along the bed may be measured, but these measurements are subject to random error. Other sources of error are the imperfect inlet temperature control, and the uncertainty of the catalyst regeneration process. The problem of finding a feedback control policy for the inlet temperature based on the temperature measurements, so as to maximise the time-average yield of B, was considered by Joffe and Sargent. The optimal shut-down time was also considered as a control variable.

Joffe and Sargent's solution was based on two approximations. First, the distributed parameter model of the reactor was replaced by a simpler lumped parameter model. This approximation is based on the earlier work of Goldmann and Sargent⁽⁷⁾. Second, a Separation Principle approach to the optimal control was adopted. The temperature measurements were used as inputs to an extended Kalman filter, the outputs of which were state and parameter estimates used in a feedback controller obtained by deterministic methods. The resulting control scheme was shown by Monte Carlo simulation techniques to have good performance characteristics. Little deterioration from the predicted deterministic performance was observed.

In addition to these examples, some theoretical work has been done on problems of type P 1. For linear distributed parameter systems with perfect observation of the state function, Denn⁽³³⁾ has derived an optimal control scheme using variational arguments. This has been extended to the case of noisy measurements made at a discrete set of points by Pell and Aris⁽³⁴⁾. Joffe and Sargent⁽¹⁾ have presented some theoretical results for systems like eqn. (1.2.1) when the noise is small (see chapter two).

From the above discussion, it can be seen that most of the previous work on stochastic chemical engineering systems has been concentrated on specific examples; only recently has the search for some theoretical foundation been attempted. It is the aim of this work to provide some theoretical background which, it is hoped, may guide future applications of optimal control theory in a stochastic environment.

C H A P T E R T W O

OPTIMAL CONTROL BASED ON COMPLETE OBSERVATIONS OF THE STATE - THEORY

2.1 Scope

In this chapter we consider a special case of the general problem already stated:-

P2 Given

$$\text{System} \quad \frac{d}{dt} x = f[t, x, u(t), \xi(t)] : t \in [0, t_f] \quad (2.1.1)$$

$$\text{Initial condition} \quad x(0) \in X_0$$

$$\text{Observations} \quad y(t) = x(t) \quad \text{for } \forall t \in [0, t_f] \quad (2.1.2)$$

$$\text{Admissible controls} \quad u(t) \in U$$

$$u(t) = h(t, X_t) \quad (2.1.3)$$

$$X_t \triangleq \{x(\tau) : \forall \tau \in [0, t]\}$$

$$\text{Performance} \quad J(t, X_t) = E\{c^T x(t_f) | (t, X_t)\} \quad (2.1.4)$$

Problem

$$\min J(t, X_t) \quad \text{for } \forall t \in [0, t_f] \quad (2.1.5)$$

2.2 Previous Work

We review the progress of other workers towards a solution of P2. It turns out that this review may be conveniently divided into three parts. First, we consider attempts at a full solution to problems similar to P2. Then, because the work described in that subsection is for a different system model than eqn. (2.1.1), it will be necessary to discuss modelling using stochastic differential equations. The third subsection will consider the solution of P2 when the disturbances are small in some sense. The treatment of this section is largely based on the review of Fleming⁽³⁵⁾.

2.2.1 Solution of the Full Problem

For deterministic systems, the current state vector has the nice property that knowledge of it and the future controls is sufficient to predict all

future states. For stochastic systems, the analogous property is that the current state forms a sufficient statistic for the prediction of future behaviour⁽³⁶⁾. This property is displayed by the class of stochastic processes called Markov processes, and it is well known that a large class of sample-continuous Markov processes may be modelled by stochastic differential equations⁽³⁷⁾:-

$$dx(t) = f[t, x(t), u(t)]dt + \sigma[t, x(t)]dw(t) \quad (2.2.1)$$

Definition of the Brownian motion process $w(t)$ and a fuller exposition of the properties of eqn. (2.2.1) will be deferred to the next subsection.

Most of the work in optimal stochastic control has been directed towards systems of the type shown in eqn. (2.2.1) because of the analogy of the Markov property to the properties of the state of a deterministic system. Some results are given by Fleming⁽³⁵⁾, and are obtained by dynamic programming arguments.

First, we note that, for eqn. (2.2.1), only Markov control policies need be considered (see, for instance, Fleming⁽³⁸⁾). That is, the optimal control is a function of the current event only; no further improvement is obtained by using more complex feedback policies. Given this fact, one may formally apply the basic dynamic programming idea of Bellman⁽³⁹⁾. A 'value function' of the observed data is defined:-

$$\phi[t, x(t)] = \min_h [E\{c^T x(t_f) | (t, x(t))\}] \quad (2.2.2)$$

A non-linear parabolic partial differential equation of second order, the Bellman-Hamilton-Jacobi equation, may be derived for ϕ :-

$$\frac{\partial \phi}{\partial t} + \min_{u \in U} [A\{t, x(t), u\}\phi] = 0 \quad (2.2.3)$$

$$\begin{aligned} \text{where } A\{t, x(t), u\} = & \sum_{i=1}^n \sum_{j=1}^n a_{ij}\{t, x(t)\} \frac{\partial^2}{\partial x_i \partial x_j} + \\ & + \sum_{i=1}^n f_i[t, x(t), u] \frac{\partial}{\partial x_i} \end{aligned} \quad (2.2.4)$$

$$\text{and } a_{ij} = \sum_{k=1}^P \sigma_{ik} \sigma_{jk}$$

with boundary condition:-

$$\phi(t, x) = c^T x \quad (2.2.5)$$

for (t, x) on the boundary of the set of observable data (which should, for complete solution, be the attainable set for eqn. (2.2.1)). Now J in

eqn. (2.1.4) is minimised by setting

$$u = g[t, x(t)] \quad (2.2.6)$$

and the problem is solved, at least in the formal sense.

Unfortunately, the solution as postulated is beset with both theoretical and practical difficulties. From the theoretical point of view, the derivation of eqn. (2.2.3) is based on the assumption of the existence of the requisite partial derivatives, which may not exist everywhere. For the case when the operator A in eqn. (2.2.4) is uniformly elliptic, i.e. it satisfies

$$b ||\lambda||^2 \leq \sum_{i=1}^n \sum_{j=1}^n a_{ij}[t, x(t)] \lambda_i \lambda_j \quad (2.2.7)$$

for some $b > 0$ and all $t \in [0, t_f]$, $x \in E^n$ and $\lambda \in E^n$, a sufficiently smooth solution to eqn. (2.2.3) can be proved to exist, and the dynamic programming argument is put on a rigorous basis. However, many systems do not possess this property. We shall return to this problem when we discuss the modelling of systems like eqn. (2.1.1) by eqn. (2.2.1), but we note here that some recent work by Rishel⁽⁴⁰⁾ suggests that assumption (2.2.7) may be unduly restrictive. He considered a special case of eqn. (2.2.1):-

$$\begin{aligned} dx_1(t) &= f_1[t, x_1(t), x_2(t)] dt \\ dx_2(t) &= f_2[t, x_1(t), x_2(t), u(t)] dt + \sigma[t, x_2(t)] dw(t) \end{aligned} \quad (2.2.8)$$

Fleming⁽³⁸⁾ had proved that a generalised solution (in the sense of the theory of Schwartz distributions⁽⁴¹⁾) exists for eqn. (2.2.3) whenever the underlying dynamical system is of the type shown in eqn. (2.2.8). Rishel showed that if the probability density function $p(t, x_1, x_2)$ of the uncontrolled system

$$\begin{aligned} dx_1(t) &= f_1[t, x_1(t), x_2(t)] dt \\ dx_2(t) &= \sigma[t, x_2(t)] dw(t) \end{aligned} \quad (2.2.9)$$

obeys

$$\int_B \int_h^{t_f} [p(t, x_1, x_2)]^\alpha dt dx_1 dx_2 < \infty \quad (2.2.10)$$

for some $\alpha > 2$ and $0 < h < t_f$ then the unique generalised solution to eqn. (2.2.3) is the value function of eqn. (2.2.2). The set B is any compact set in state space over which the solution to the control problem

is required. The disadvantage of this result appears to be that, in general, there is no a priori way of checking that condition (2.2.10) is satisfied.

One may wish to ignore the theoretical problems and proceed heuristically, solving eqn. (2.2.3) and hoping that the solution may furnish some information about the optimal stochastic controller. This approach was taken, to some extent, by Kushner and Kleinman⁽⁴²⁾. They considered the autonomous system:-

$$dx(t) = f[x(t), u(t)]dt + \sigma[x(t)]dw(t) \quad (2.2.11)$$

where the matrix σ need not be of full rank. The equation for the value function becomes elliptic for this type of system:-

$$\min_{u \in U} [A(x, u)\phi] = 0 \quad (2.2.12)$$

$$\text{with} \quad \phi(x) = c^T x \quad (2.2.13)$$

for x on the boundary of the region over which the solution is desired. The operator A is defined as before. An approximate solution to eqn. (2.2.12) is obtained by finite difference methods. The problem is either to locate the attainable set in state space, in which case the correct boundary condition, eqn. (2.2.13), may be applied, or to find appropriate boundary conditions for a subset of the attainable set. An example given by Kushner and Kleinman shows that the form of the control law is greatly influenced by these rather arbitrary extra boundary conditions. One is also restricted to systems with a state space of low dimension by the storage requirements for the computational solution of the finite difference equations.

A different approach to the optimum problem P2 has been given by Shapiro and Mon^{(43), (44)}. Again, they considered systems with the Markov property, modelled by eqn. (2.2.1). It is well known that, under suitable conditions, the probability density function for the process generated by eqn. (2.2.1) conditioned on initial data satisfies the Fokker-Planck equation (see Wong⁽⁴⁵⁾):-

$$\frac{\partial p}{\partial t} + \sum_{i=1}^n \frac{\partial}{\partial x_i} (f_i p) - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij} p) = 0 \quad (2.2.14)$$

and the approach of Shapiro and Mon consists of expressing the solution of eqn. (2.2.14) as an infinite series. The time-varying coefficients in the expansion become the states in a new deterministic optimum problem,

the optimal controls for which are the same as those for the original problem.

Two difficulties arise from this approach. First, we note that the state space for the new control problem is infinite dimensional, making numerical solution of the problem as it stands infeasible. This difficulty is circumvented by some extra results of Shapiro and Mon⁽⁴⁴⁾. They have shown that applying the limit of the sequence of controls obtained by solving finite dimensional approximations to the infinite dimensional problem gives the same performance as applying the optimal control for the latter problem. Some examples show that convergence can be quite rapid. Typically, only the five dimensional problem could be considered. However, it is necessary to solve several of the subproblems in order to satisfy oneself that convergence has been obtained.

The second difficulty with Shapiro and Mon's method is that a form of control law must be postulated a priori, and their algorithm used to find the coefficients in that law. This is a way of introducing feedback into a problem which may then be solved by feedforward methods. This approach has been suggested by several authors (see, for instance, (1), (46), 47)) but of course it is suboptimal. However, Shapiro and Mon's ideas do have the advantage that a solution is provided by solving deterministic problems for which many efficient algorithms are known⁽⁴⁷⁻⁴⁹⁾. We shall find that the idea of deterministic models for the performance of stochastic systems is particularly useful when the noise is small.

To summarise, there has been a lot of work done on systems of the type shown in eqn. (2.2.1). A dynamic programming argument leads to a partial differential equation, the solution of which may yield the optimal control as a side condition. However, theoretical results are not quite complete enough to justify total confidence in this approach for every system. From the practical point of view, an algorithm for the autonomous case has been proposed and tested, but often the true solution is masked by the effects of the arbitrary extra conditions which have to be imposed to make the computations practicable. Rather than transform the problem into a deterministic problem in partial differential equations, Shapiro and Mon's approach consists of converting to a deterministic optimal control problem. This seems a promising line of attack, but the feedforward nature of the solution to the resulting problem has not been overcome completely satisfactorily.

The adaption of the above methods to the type of system used in P2 is discussed in the next subsection.

2.2.2 Modelling Using Stochastic Differential Equations

In the previous subsection, the stochastic differential equation (2.2.1) was introduced with little discussion. In this subsection, its properties are looked at in more detail, with particular reference to its use in the modelling of real processes. A good introduction to the material of this section may be found in Jazwinski's book⁽⁵⁰⁾, particularly in chapters three and four, on which this section is largely based. For a complete mathematical treatment, Doob's book⁽³⁷⁾ should be consulted.

We begin by considering the Brownian motion process, $w(t)$. A Brownian motion process is defined by:-

- (i) $w(t)$ has stationary independent increments, i.e. if we consider any finite set of times $\{t_i : t_i < t_{i+1}\}$, the random vectors $w(t_1) - w(t_2); w(t_2) - w(t_3); \dots w(t_n) - w(t_{n-1})$ are independent, and further, the distribution of $w(t + h) - w(\tau + h)$ is the same as that for $w(t) - w(\tau)$ for every $t > \tau$ and every $h > 0$;
- (ii) for every t , $w(t)$ is normally distributed;
- (iii) for every τ , $E\{w(\tau)\} = 0$;
- (iv) $\Pr\{w(0) = 0\} = 1$

From our point of view, the most important properties of the process $w(t)$ are its continuity and differentiability properties. It can be shown ((37) p. 393) that almost all sample functions of the Brownian motion process are continuous (i.e. $w(t)$ is continuous with probability one), but that they are nowhere differentiable functions of time. Further, they are of unbounded variation.

Given these properties, one might ask how the solution to eqn. (2.2.1) is defined. That is, how does one evaluate $\int_0^t dw(t)$ when $w(t)$ is of unbounded variation? The answer is that a special definition is required. Quite a number of definitions have been proposed⁽³⁵⁾, but two are more popular than the others at present.

The most commonly used definition is due to $\hat{I}$ to. We may recall the definition of a Riemann integral:-

$$\int_{t_0}^{t_f} \sigma(t) dt = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \sigma(t'_i)(t_{i+1} - t_i) \quad (2.2.16)$$

provided that the limit exists. Here t' is any time in the interval $[t_i, t_{i+1})$. However, for a stochastic integral, t' is constrained to a particular point in the given interval. The Ito stochastic integral is defined by:-

$$\int_{t_0}^{t_f} \sigma[t, x(t)] dw(t) = \text{l.i.m.}_{n \rightarrow \infty} \sum_{i=0}^{n-1} \sigma[t_i, x(t_i)] [w(t_{i+1}) - w(t_i)] \quad (2.2.17)$$

i.e. the lower bound of the interval is used. The second definition is due to Stratonovich. It is less general than the Ito definition, since the kernel function may only be an explicit function of $w(t)$, but has the advantage that it may be formally manipulated by the rules for Riemann integrals. Despite this, most of the work in stochastic control theory uses the Ito integral, because it has many useful mathematical properties not possessed by the Stratonovich integral (see (37), chapter nine). As an illustration of this, it is interesting to note that Stratonovich's approach in the paper putting forward his integral⁽⁵¹⁾ is first to derive the relation between his integral and Ito's and then to use results for the Ito integral to derive the properties of the solutions of Stratonovich stochastic differential equations. In fact, Jazwinski ((50), p.140) goes so far as to say ". . . all the results concerning the Stratonovich integral are proved using Ito's theory. Of itself, the Stratonovich integral does not offer any additional physical insight". It is clear from the above discussion that eqn.(2.2.1) is only a formal representation of the equation:-

$$\begin{aligned} x(t) = x(0) &+ \int_0^t f[\tau, x(\tau), u(\tau)] d\tau + \\ &+ \int_0^t \sigma[\tau, x(\tau)] dw(\tau) \end{aligned} \quad (2.2.18)$$

The first integral may be evaluated as a Riemann integral, and the second is usually defined as an Ito stochastic integral. We now consider the properties of the process $x(t)$.

Under assumptions as to the smoothness of f and σ with respect to their arguments, which are analogous to the assumptions utilised for ordinary differential equations, it can be shown (see (37), p.277) that eqn.(2.2.18) defines a unique process bounded in mean square. The process is unique in

the sense that only one realisation of $x(t)$ corresponds to a given realisation of $w(t)$. Also, almost all the sample functions of the $x(t)$ process are continuous. However, they do not possess a derivative in the normal sense, because of the erratic properties of the Brownian motion process.

Having defined the process $w(t)$ and listed the salient properties of $x(t)$, the solution of eqn. (2.2.1), we may now turn to a discussion of the use of eqn. (2.2.1) in the modelling of processes which may be described by ordinary differential equations like eqn. (2.1.1). We would like to be able to convert eqn. (2.1.1) to an equivalent form like eqn. (2.2.1), and perhaps the simplest way of doing this is to model $\xi(t)$ as a continuous Markov process, as suggested by Fleming⁽³⁵⁾ and by Joffe and Sargent⁽¹⁾. By adjoining the $\xi(t)$ vector to $x(t)$, we arrive at an extended state vector with the Markov property. This attractively simple idea has two major drawbacks. First, by increasing the dimension of the state vector, the system has been rendered only partially observable. Hence, the methods described in section 2.2 for the solution of the optimal control problem are inapplicable. Second, even if the extra states could be measured, there is still the problem that the augmented system will certainly be nonuniformly elliptic, hence preventing the rigorous use of dynamic programming methods.

Wong and Zakai⁽⁵²⁾ considered the modelling of scalar systems by scalar stochastic differential equations. Their system model is a special case of eqn. (2.1.1):-

$$dx(t) = f[t, x(t)]dt + g[t, x(t)]d\xi(t) \quad (2.2.19)$$

where the process $\xi(t)$ is a polygonal approximation to a Brownian motion process $w(t)$ given by:-

$$\xi(t) = w(t_i) + \frac{w(t_{i+1}) - w(t_i)}{t_{i+1} - t_i} (t - t_i); t_i \leq t \leq t_{i+1} \quad (2.2.20)$$

$$\text{Let } \rho = \max_i (t_{i+1} - t_i) \quad (2.2.21)$$

Wong and Zakai showed that the limit in mean square of the solutions to eqn. (2.2.19) as $\rho \rightarrow 0$ satisfies the $\hat{\text{Ito}}$ equation:-

$$dx(t) = \left\{ f[t, x(t)] + \frac{1}{2} g[t, x(t)] \frac{\partial g}{\partial x} [t, x(t)] \right\} dt + g[t, x(t)] dw(t) \quad (2.2.22)$$

or the Stratonovich equation:-

$$dx(t) = f[t, x(t)]dt + g[t, x(t)]dw(t) \quad (2.2.23)$$

under conditions which guarantee the existence and uniqueness of the solutions to these equations. There are two points to be noticed from these results. First, we note that eqns. (2.2.22) and (2.2.23) are different, showing that the $\hat{I}$ to and Stratonovich integrals give different answers for the same kernel function. This is not too important, since in modelling one is only concerned with defining differentials such that the integrals are the same. Second, we note that the Stratonovich differential equation has an analogous form to each equation in the sequence, whereas for the $\hat{I}$ to definition an extra term has appeared in the limit. This is a manifestation of the correspondence already mentioned between Riemann integrals and Stratonovich stochastic integrals.

Wong and Zakai's results have been extended to the vector case by Clark⁽³⁵⁾, and Cumming has shown how to use them in numerical simulation⁽⁵⁴⁾ and control design⁽⁵⁵⁾. His arguments are on a less rigorous basis than those of Clark and Wong and Zakai, but as a result are applicable to a wider class of systems. Specifically, the noise process $d\xi(t)$ in eqn. (2.2.19), or its vector generalisation, may be any large band width process, i.e. any process with a large range of frequency components. The modelling stochastic differential equations are of the same form as eqn. (2.2.22) and eqn. (2.2.23). In ref. 55, Cumming showed that care must be taken in the interpretation of the results of a stochastic control study. In general, if the $\hat{I}$ to calculus is used, the control law for the stochastic differential equations is not the same as that for the original system. Thus, not only is a transformation needed to convert a 'physical' system to an $\hat{I}$ to-type system, but also a correction to the optimal controller may be required before it is applicable to the real system.

In this section we have discussed the properties of the stochastic differential equation (2.2.1). From the modelling viewpoint, in general extra states are required to render a system of the type shown in eqn. (2.1.1) Markov. This causes the new state to be unobservable. If the physical system is driven by a wide bandwidth noise process appearing linearly in the right hand side of the governing differential equation (2.1.1), then it may be modelled by a stochastic differential equation in a state space of the same dimension as that of the original system. Some care must be taken in applying the solution of the resulting optimal control problem to the physical process.

2.2.3 Optimal Control for Small Disturbances

In the hope of simplifying the problems in optimal stochastic control, several authors have made some assumption bounding the size of the disturbances. This approach seems reasonable, since in most practical situations large disturbances could not be tolerated.

Stratonovich, in an early series of papers^{(56),(57)}, considered the optimal control of a single state linear system subject to white noise disturbances. The objective of the controller was to cause the system to reproduce some desired signal as closely as possible. In the first paper⁽⁵⁶⁾, Stratonovich showed that the optimal control is bang-bang, and in the second⁽⁵⁷⁾, he used a successive approximation algorithm to solve the Bellman-Hamilton-Jacobi equation in order to obtain the switching points in state space. At each stage in the approximation procedure, a first order linear partial differential equation is to be solved, and it was assumed that the procedure is convergent since the noise variance is small.

Kushner⁽⁵⁸⁾ considered a non-linear system subject to disturbances appearing linearly in the right hand sides of the governing differential equations. The disturbances were assumed to be in the form of impulses occurring at several known times. A correction to the deterministic optimal control was made as a power series in the deviation of the actual state from its deterministic optimal value. This deviation may be expressed as a power series in the noise, the coefficients in which were shown to satisfy an infinite set of ordinary differential equations. By truncating the power series for the state deviation and for the control correction so that they incorporate terms only up to second order, the solution was made computationally more tractable. However, there is still a large number of equations to be solved, especially for a state space of large dimension. Also, the number of equations increases as the number of impulses to which the system is subjected is increased; for a continuous noise process Kushner's approach thus appears to be infeasible.

In a recent paper, Fleming⁽⁵⁹⁾ considered the effects of applying to the stochastic system, eqn. (2.2.1), controls which are optimal for the corresponding deterministic system:-

$$dx(t) = f[t, x(t), u(t)] dt \quad (2.2.24)$$

His results are limited to the case when the function f is linear in $u(t)$ i.e.:-

$$f[t, x(t), u(t)] = A[t, x(t)] + B[t, x(t)]u(t) \quad (2.2.25)$$

and also σ must be a constant matrix. By expressing the solution of the Bellman-Hamilton-Jacobi equation as a power series in a small parameter ϵ (ϵ^2 giving the order of magnitude of the noise variance) using singular perturbation methods, Fleming was able to show that the deterioration in performance caused by applying the optimal deterministic policy as opposed to the optimal stochastic policy is of order ϵ^4 . The optimal deterministic policy is feedback in the sense that controls at every point in event space are determined by solving a feedforward optimal control problem, using the current event as the initial condition for eqn. (2.2.24). If, instead of applying this policy, one applies the feedforward optimal control policy based on the initial event only, then Fleming showed that the deterioration to be expected is of order ϵ^2 . Thus, with little loss of accuracy, the optimal stochastic control problem is replaced by a deterministic problem, when the noise is small enough.

One would like to be able to reduce the more general problem P2 to a deterministic one, and some work has been done in this area by Joffe and Sargent⁽¹⁾. They considered a system of the type shown in eqn. (2.1.1), and made various assumptions bounding the disturbances, $\xi(t)$. As in Fleming's work, there is a parameter ϵ which, roughly speaking, bounds the noise magnitude (i.e. ϵ^2 is a measure of the noise variance). Also, the noise is not a Brownian motion process, but has a finite autocorrelation bounded by:-

$$\int_0^t ||E\{\delta\xi(t)\delta\xi^T(t')\}|| dt \leq t_f \epsilon^3 \quad (2.2.26)$$

for all $t' \in [0, t_f]$. Bounds were placed on some third order moments of the noise as well. With their assumptions, Joffe and Sargent were able to show that the expected value of $x(t)$ conditioned on $x(0)$ is well approximated by

$$\frac{d}{dt} x^* = f[t, x^*, u(t), \xi^*(t)] + z(t)$$

$$\text{where } x^*(t) = E\{x(t) | x(0)\}$$

$$\xi^*(t) = E\{\xi(t)\}$$

$$z^k(t) = \frac{1}{2} \text{trace} [f_{\xi\xi}^k Q(t)]$$

$$Q(t) = E[\{\xi(t) - \xi^*(t)\}\{\xi(t) - \xi^*(t)\}^T] \quad (2.2.27)$$

(The subscripts on f refer to partial differentiation with respect to the subscript). In fact, the error in the approximation is of order ϵ^3 . Joffe and Sargent suggested that their model may be used to choose parameters optimally in a fixed control law, i.e. postulate

$$u(t) = h[t, x(t), \alpha(t)] \quad (2.2.28)$$

say, and use deterministic methods to choose the optimal parameter set, $\alpha(t)$, for the model, eqn. (2.2.27). Joffe and Sargent have applied this method with some success to an example^{(4), (60)}. However, it would be useful to know how much better one could do if one could find the optimal stochastic control policy, and if possible to avoid having to guess the form of the function h in eqn. (2.2.28). In section 2.4, we show that a very good choice for h is the optimal deterministic policy for the model, eqn. (2.2.27). Our results are analogous to those of Fleming for Markov processes.

To summarise, several authors have considered the problem of controlling stochastic systems subject to small disturbances. In general, their approach has been to look at the optimal control for the corresponding deterministic system, and either to find a suitable correction procedure (in the case of Stratonovich and Kushner) or to show that this control may be sufficient (in the case of Fleming). A promising line of attack for the general problem P2 has been given by Joffe and Sargent, based on a deterministic model for the average behaviour of eqn. (2.1.1). This line will be pursued in the next three sections.

2.3 A General Theorem on Optimal Stochastic Control Using Models

We have seen that a common approach to the control of stochastic systems with small noise has been to consider the optimal controller for a related deterministic system. This is the approach that will be adopted here. In this section, we present a rather general theorem on the computation of stochastic controls using models which approximate the system dynamic behaviour. In the following section, it is shown that the second-order model of Joffe and Sargent satisfies the conditions of the theorem, and hence can be used for the design of feedback controllers when the noise is small. In section 2.5, feedforward and discrete feedback controllers designed using the second-order model are discussed.

We begin by considering the design of feedback controllers for an arbitrary stochastic system using a stochastic model which approximates the system behaviour. The system control problem is:-

$$\text{System} \quad \frac{dx}{dt} = f_s(t, x, u_s(t), \xi_s(t)) : t \in [0, t_f] \quad (2.3.1)$$

$$\text{Measurements } y(t) = g_s(t, x(t), \eta_s(t)) \quad (2.3.2)$$

Problem Find a control policy

$$\begin{aligned} u_s(t) &= h_s(t, Y_s(t)) \\ Y_s(t) &= \{y(\tau) : \tau \in [0, t]\} \end{aligned} \quad (2.3.3)$$

which minimises

$$J_s = E \{c^T x(t_f)\} \quad (2.3.4)$$

We will consider also a model problem:-

$$\text{Model} \quad \frac{dx}{dt} = f_m(t, x, u_m(t), \xi_m(t)) : t \in [0, t_f] \quad (2.3.5)$$

$$\text{Measurements } y(t) = g_m(t, x(t), \eta_m(t)) \quad (2.3.6)$$

Problem Find a control policy

$$u_m(t) = h_m(t, Y_m(t)) \quad (2.3.7)$$

which minimises

$$J = E \{c^T x(t_f)\} \quad (2.3.8)$$

We assume that the solutions to these two problems have been found. We may apply both of the resulting control laws to the system and to the model, obtaining different performance indices as shown in Table 2.1

TABLE 2.1 Performance Indices Resulting from the Application of
Control Laws to the System and to the Model

<u>Control Law</u>	<u>System</u>	<u>Model</u>
$h_s(t, Y(t))$	J_{ss}	J_{ms}
$h_m(t, Y(t))$	J_{sm}	J_{mm}

From the optimality properties of the control laws,

$$J_{sm} \geq J_{ss} \quad ; \quad J_{ms} \geq J_{mm} \quad (2.3.9)$$

We now specify the properties which we expect of the model in order that it should approximate the behaviour of the system:-

$$J_{ms} - J_{ss} = K_s \delta \quad (2.3.10)$$

$$J_{mm} - J_{sm} = K_m \delta \quad (2.3.11)$$

for some $\delta > 0$ with $K_s, K_m < \infty$. Thus, the model is expected to provide an $O(\delta)$ prediction of the performance of the system under both the control laws shown in Table 2.1. A sufficient condition for this to be true is that the model should give such a prediction for all admissible controls. From eqn. (2.3.10), (2.3.11),

$$(J_{ms} - J_{mm}) + (J_{sm} - J_{ss}) = (K_s - K_m)\delta \quad (2.3.12.)$$

Now, we see from eqn. (2.3.9) that the left hand side of eqn. (2.3.12) is the sum of two non-negative quantities. Hence,

$$0 \leq J_{ms} - J_{mm} \leq (K_s - K_m)\delta \quad (2.3.13)$$

$$0 \leq J_{sm} - J_{ss} \leq (K_s - K_m)\delta \quad (2.3.14)$$

Thus, from eqn. (2.3.14), the performance index for the system with the model control law is an $O(\delta)$ approximation to the performance index for the system with the system control law. That is, the solution to the model control problem is a good approximation to the solution of the system problem. Also, from eqn. (2.3.10), (2.3.11), (2.3.13) and (2.3.14):-

$$K_m \delta \leq J_{mm} - J_{ss} \leq K_s \delta \quad (2.3.15)$$

showing that the performance index for the model problem is a good approximation to that for the system problem. Thus, we have proved:-

Theorem 2.3.1

If we have a system control problem given by eqns. (2.3.1) - (2.3.4) and a model of the system which satisfies eqns. (2.3.10) - (2.3.11), then the solution of the model control problem given by eqns. (2.3.5) - (2.3.8) is a good approximation to the solution of the system control problem, in the sense that eqn. (2.3.14) is satisfied.

Summarising, we see that a good approximation to the optimal stochastic controller for a system can be computed, provided a suitable model of the system can be found. The order of the approximation is fixed by the ability of the model to reproduce the system behaviour. In the next section, we present a deterministic model which has the required approximation properties provided that the system is perturbed by small noise.

2.4 Optimal Stochastic Control with Small Noise : A Deterministic Model

Having established theorem 2.3.1, the problem of optimal stochastic control may be reduced to the problem of finding a suitably accurate model, for which the best control policy is easily computed. Clearly, it would be advantageous to have a deterministic model for this purpose, since the theory of deterministic optimal control is well developed. For the case of small noise, the approach of Joffe and Sargent⁽¹⁾ appears to be promising.

We begin by reviewing Joffe and Sargent's model, and then we show that it retains its approximation properties for more general feedback control policies than those envisaged by Joffe and Sargent.

2.4.1 The Second-order Approximation of Joffe and Sargent

The work reviewed in this section may be found in a paper by Joffe and Sargent⁽¹⁾. They consider a stochastic system of the form:-

$$\frac{dx}{dt} = f(t, x, u(t), \xi(t)) : t \in [t_0, t_f] \quad (2.4.1)$$

where $x(t) \in E^n$ is a vector of state variables,

$u(t) \in U \subseteq E^m$ is a vector of control variables,

$\xi(t) \in V \subseteq E^r$ is a vector-valued stochastic process,

and $f(t, x, u, \xi)$ is an n -dimensional vector-valued function

of its arguments. U and V are given sets, and it is assumed that V is convex. The initial condition $x(t_0)$ is a random variable, taking values from some given bounded set $X(t_0) \subseteq E^n$. The functions $u(t)$ and $\xi(t)$ are Lebesgue-integrable functions of t , and hence $\xi(t)$ may not be a Brownian motion process.

By defining various properties for $f(t, x, u, \xi)$ it is well known that a unique solution can be guaranteed for eqn. (2.4.1) on the interval $[t_0, t_f]$ corresponding to a given initial condition $x(t_0) \in E^n$ and given functions $u(t)$ and $\xi(t)$. A sufficient set of such conditions, holding for all $t \in [t_0, t_f]$, $x, x^* \in E^n$, $u \in U$ and $\xi \in V$, is:-

- (i) For each set of values of x, u, ξ , $f(t, x, u, \xi)$ is a measurable function of t .
- (ii) For each fixed t , $f(t, x, u, \xi)$ is continuous in x, u, ξ .
- (iii) There exists a function $S(t)$ summable on $[t_0, t_f]$ and a function $\phi(z)$, positive and continuous for $z \geq 0$ but not summable on $[0, \infty)$, such that

$$||f(t, x, u, \xi)|| \leq S(t)/\phi(||x||)$$

for each t, x, u , and ξ .

(iv) There exists a function $M(t)$ summable on $[t_0, t_f]$ such that

$$|(x-x^*)^T(f(t, x, u, \xi) - f(t, x^*, u, \xi))| \leq M(t) ||x-x^*||^2$$

The aim of Joffe and Sargent is to produce a model which approximates the expected behaviour of the system, eqn. (2.4.1), provided that certain boundedness and continuity assumptions are satisfied. We now give those conditions. If we define the set $X(t_f)$ as the convex hull of the set of points $x(t)$ attainable from any allowed initial state $x(t_0)$ using allowed functions $u(t)$ and $\xi(t)$, then the following condition is assumed to hold for all, $t \in [t_0, t_f], x, x^* \in X(t_f), u \in U$ and $\xi, \xi^* \in V$:-

(v) For each t and u the second partial derivatives of the elements of $f(t, x, u, \xi)$ with respect to x and ξ exist, and there is a constant $L < \infty$ such that all these derivatives satisfy Lipschitz conditions of type

$$||f_{x\xi}^k(t, x, u, \xi) - f_{x\xi}^k(t, x^*, u, \xi)|| \leq L ||x-x^*||$$

$$||f_{x\xi}^k(t, x, u, \xi) - f_{x\xi}^k(t, x, u, \xi^*)|| \leq L ||\xi-\xi^*||$$

(we follow Joffe and Sargent's notation: subscripts refer to differentiation variables, superscripts to elements of vectors)

(vi) If we define,

$$\xi^*(t) = E\{\xi(t)\} \quad ; \quad \delta\xi(t) = \xi(t) - \xi^*(t)$$

$$x_0^* = E\{x(t_0)\} \quad ; \quad \delta x_0 = x(t_0) - x_0^*$$

then the following inequalities are assumed to hold for some

$$\epsilon < \infty \text{ and } A < \infty:-$$

$$\int_{t_0}^{t_f} E\{||\delta\xi(t)||^3\} dt \leq T\epsilon^3 \quad ; \quad E\{||\delta x_0||^3\} \leq A^3\epsilon^3$$

$$\int_{t_0}^{t_f} ||E\{\delta\xi(t)\delta\xi^T(t^1)\}|| dt \leq T\epsilon^3 \quad ; \quad ||E\{\delta x_0\delta\xi^T(t^1)\}|| \leq A\epsilon^3$$

$$\int_{t_0}^{t_f} E\{||\delta\xi(t)|| ||\delta\xi(t^1)||^2\} dt \leq T\epsilon^3 \quad ; \quad E\{||\delta x_0|| ||\delta\xi(t^1)||^2\} \leq A\epsilon^3$$

(2.4.2)

$$\int_{t_0}^{t_f} E\{||\delta\xi(t)||^2 ||\delta\xi(t')||\} dt \leq T\epsilon^3 ; E\{||\delta x_0||^2 ||\delta\xi(t^1)||\} \leq A^2 \epsilon^3$$

where $T = t_f - t_0$ and $t' \in [t_0, t_f]$.

Under the above conditions, Joffe and Sargent prove the following:-

Theorem 2.4.1

If the system given by eqn. (2.4.1) satisfies conditions (i) - (vi), then a good approximation to $E\{x(t)\}$ is given by $x^*(t)$, the solution of:-

$$\frac{dx^*}{dt} = f(t, x^*, u(t), \xi^*(t)) + z(t), x^*(t_0) = x_0^*$$

$$\frac{dP}{dt} = f_x P + P f_x^T, P(t_0) = E\{\delta x_0 \delta x_0^T\}$$

$$\text{where } z^k(t) = \frac{1}{2} \text{trace} [f_{xx}^k P(t) + f_{\xi\xi}^k Q(t)] \quad (2.4.3)$$

$$\text{and } Q(t) = E\{\delta\xi(t)\delta\xi^T(t)\}$$

(Where arguments of functions are not specified, the functions are to be evaluated at $(t, x^*(t), u(t), \xi^*(t))$.)

The approximation is good in the sense that

$$||E\{x(t)\} - x^*(t)|| \leq K\epsilon^3 \quad (2.4.4)$$

for some

$$K < \infty$$

The proof of this theorem is rather involved and will not be repeated here. The argument is similar to that used in section 2.4.2, where a related result is proved. The following corollary (due to Joffe and Sargent) is immediately obvious.

Corollary 2.4.2

If $x(t_0)$ is known, then $P(t) = P(t_0) = 0$, and eqn. (2.4.3) reduces to

$$\frac{dx^*}{dt} = f(t, x^*, u(t), \xi^*(t)) + z(t), x^*(t_0) = x(t_0)$$

(2.4.5)

$$\text{where } z^k(t) = \frac{1}{2} \text{trace} [f_{\xi\xi}^k Q(t)]$$

From theorem 2.4.1, we see that we have found a model satisfying the conditions of theorem 2.3.1 for feedforward controls. This result is formalised in the next corollary.

Corollary 2.4.3

Given the control problem

$$\text{System} \quad \frac{dx}{dt} = f(t, x, u(t), \xi(t)), x(t_0) \in X(t_0), t \in [t_0, t_f] \quad (2.4.6)$$

$$\text{Performance} \quad J_1 = E\{c^T x(t_f)\} \quad (2.4.7)$$

Problem : Find $\{u(t) : t \in [t_0, t_f]\}$ so as to minimise J_1

and assuming that eqn. (2.4.6) satisfies the conditions of theorem 2.4.1, then the solution of the model problem:-

$$\text{System} \quad \frac{dx^*}{dt} = f(t, x^*, u(t), \xi^*(t)) + z(t) : x(t_0) = x_0^* \\ t \in [t_0, t_f]$$

$$\text{Performance} \quad J_2 = c^T x^*(t_f)$$

Problem : Find $\{u(t) : t \in [t_0, t_f]\}$ to minimise J_2

is a good approximation to the solution of the original problem. The approximation is good in the sense that the value of J_1 obtained using the model optimal control is within $O(\epsilon^3)$ of the optimal value of J_1 .

We may also establish a result for the case of optimisation among a class of feedback controls depending on the current state using corollary 2.4.2. If we substitute

$$u(t) = h(t, x(t)) \quad (2.4.8)$$

into eqn. (2.4.1), we obtain

$$\frac{dx}{dt} = f(t, x, h(t, x), \xi(t)) \quad (2.4.9)$$

Provided $f(t, x, u, \xi)$ and $h(t, x)$ obey certain smoothness properties, a model of the form of eqn. (2.4.5) will have $O(\epsilon^3)$ approximation properties. Then, theorem 2.3.1 may be applied to obtain:-

Corollary 2.4.4

Let the system given by eqn. (2.4.1) satisfy the conditions of theorem 2.4.1 and also the following:-

- (vii) For each t , the second partial derivatives of the elements of $f(t, x, u, \xi)$ with respect to x, u , and ξ exist, and satisfy conditions of type

$$||f_{u\xi}^k(t, x, u, \xi) - f_{u\xi}^k(t, x, u^*, \xi)|| \leq L ||u - u^*|| \quad (2.4.10)$$

for all $u, u^* \in U$.

Consider the following problem for a system given by eqn. (2.4.1)

Measurements $y(t) = x(t)$ for all $t \in [t_0, t_f]$

Performance $J_1 = E\{c^T x(t_f)\}$

Problem Minimise J_1 in the class of controls given by eqn.

(2.4.8) where the function $h(t, x)$ has the property:-

(viii) For each t , $h(t, x)$ is twice differentiable with respect to x and the second derivatives satisfy Lipschitz conditions of the form

$$||h_{xx}(t, x) - h_{xx}(t, x^*)|| \leq L ||x - x^*|| \quad (2.4.11)$$

for all $x, x^* \in X(t_f)$.

Then, the solution of the model control problem, with a system given by eqn. (2.4.5), and performance given by

$$J_2 = c^T x^*(t_f)$$

where the problem is to minimise J_2 in the class of feedback controls

$$u(t) = h(t, x^*(t)) \quad (2.4.12)$$

with $h(t, x)$ satisfying condition (viii) is a good approximation to the solution of the system control problem. The approximation is good in the sense that the application of the optimal model control law to the system gives performance which is within $o(\epsilon^3)$ of the minimum value of J_1 for controls given by eqn. (2.4.8.).

From the above corollary, we see that we may solve a feedback control problem for a small noise stochastic system quite accurately by solving a deterministic feedback control problem. Since the model, eqn. (2.4.5), is deterministic, we would hope to solve the model problem by solving a sequence of feed-forward control problems for different values of $(t_0, x^*(t_0))$. However, we have no guarantee that the resulting control law satisfies condition (viii). We would therefore like to relax assumption (viii) in corollary 2.4.4. This would enable us to consider a wider class of control laws, and hence hopefully to obtain better performance from the resulting control system. The related assumption (vii) does not appear to be a severe restriction for most systems in chemical engineering.

Another problem concerned with corollary 2.4.4 is that we only consider control laws based on current state feedback. Since we are unable to guarantee Markov properties for eqn. (2.4.1) it would seem better to consider controls which are functional on all past values of the state vector, i.e.

$$\begin{aligned} u(t) &= h(t, X_t) \\ X_t &= \{x(\tau) : \tau \in [0, t]\} \end{aligned} \tag{2.4.13}$$

To summarise, we have given a model which has the properties required by theorem 2.3.1 for a rather general system perturbed by small noise, controlled by feedforward controls. By restricting the class of control laws to those with certain smoothness properties, we have extended this model to include feedback controls depending on the current state. Two problems have been pointed out:-

- (a) The smoothness properties required of the feedback control laws are rather restrictive. It would be useful to relax them somewhat.

and,

- (b) Because the system given by eqn. (2.4.1) is non-Markov in general, we would like to extend the class of control laws to include those functional on the history of the state vector.

Problem (b) is dealt with in the next section, together with some further discussion of problem (a).

2.4.2 An Extension to Joffe and Sargent's Model to Include Non-Markov Control Policies

One problem remaining at the end of the last section was that the optimal control for the stochastic system, eqn. (2.4.1) should be of the form:-

$$\begin{aligned} u(t) &= h(t, X_t) \\ X_t &= \{x(\tau) : \tau \in [0, t]\} \end{aligned} \tag{2.4.14}$$

Corollary 2.4.4 shows that feedback controls based on the current state could be designed using Joffe and Sargent's model. In this section, we extend the model to allow controllers of the more general form shown in eqn. (2.4.14).

By substituting eqn. (2.4.14) in eqn. (2.4.1) we obtain:-

$$\frac{dx}{dt} = f(t, x, h(t, X_t), \xi(t)) \quad (2.4.15)$$

This is a delay-differential equation for $x(t)$. The theory of such equations is reviewed in a book by Oguztoreli⁽⁶¹⁾, which will be heavily drawn upon throughout this section. Through an abuse of notation, we will represent eqn. (2.4.15) in the form:-

$$\frac{dx}{dt} = f(t, X_t, \xi(t)) : t \in [0, t_f] \quad (2.4.16)$$

and we will assume that

$$X_t = \{x(\tau) : \tau \in [t-t_f, t]\} \quad (2.4.17)$$

$$X_0 = 0 \quad (2.4.18)$$

This specialises $h(t, X_t)$, (eqn. (2.4.14)), but not to any great extent. We shall make various assumptions about $f(t, X_t, \xi)$ and the process $\xi(t)$ analogous to those of Joffe and Sargent. We employ the following norms:-

For a vector, $x \in E^n$,

$$||x|| = \max_k |x^k|$$

For a continuous function, X_t ,

$$||X_t|| = \sup_{\tau \in [0, t]} ||x(\tau)||$$

For a matrix, A ,

$$||A|| = \max_{||x|| \leq 1} ||Ax||$$

First, to ensure existence and uniqueness of solutions, we assume the following five properties.

- (i) The functional $f(t, X_t, \xi)$ is defined and bounded for $t \in [0, t_f]$ for all $\xi(t) \in V \subset E^r$ and for all functions $x(t)$ continuous on $[0, t_f]$.
- (ii) $f(t, X_t, \xi)$ is Lebesgue integrable in t for all fixed functions $x(t)$ and $\xi(t)$ and continuous in X_t and ξ for each fixed t .

- (iii) There exists a Lebesgue integrable function $m(t)$ for $t \in [0, t_f]$ such that

$$||f(t, X_t, \xi(t))|| \leq m(t)$$

uniformly with respect to functions $x(t)$ and $\xi(t)$.

- (iv) Define $\chi(\rho) = \sup ||f(t, X_t, \xi(t)) - f(t, Y_t, \xi(t))||$ for $t \in [0, t_f]$, $||x(t) - y(t)|| \leq \rho$. Then the integral

$$\int_0^{\infty} \frac{d\rho}{\chi(\rho)}$$

diverges.

- (v) The functional $f(t, X_t, \xi)$ is differentiable with respect to the components of X_t and ξ in the functional and in the ordinary sense respectively.

Under the above conditions ((61), p.153), a unique solution to eqn. (2.4.16), with initial condition given by eqn. (2.4.18), can be shown to exist for each admissible function, $\xi(t)$. This solution is absolutely continuous and can be shown to be uniformly bounded on the interval $[0, t_f]$.

Following Joffe and Sargent, we introduce the set $X(t_f)$ as the convex hull of the set of all points $x(t)$ attainable from the initial condition in the interval $[0, t_f]$ with admissible functions $\xi(t)$. Since all solutions are bounded on $[0, t_f]$ it follows that $X(t_f)$ is a bounded set. We make the following further assumption about $f(t, X_t, \xi)$. This condition is assumed to hold for all $t \in [0, t_f]$, all absolutely continuous functions $x(t), x^*(t)$ taking values in $X(t_f)$ and all $\xi, \xi^* \in V$.

- (vi) For each t , the second partial derivatives of the elements of $f(t, X_t, \xi)$ with respect to X_t (in the functional sense) and with respect to ξ (in the ordinary sense) exist, and there is a constant $L < \infty$ such that these derivatives satisfy Lipschitz conditions of type:-

$$\begin{aligned} ||f_{X\xi}^k(\tau, X_t, \xi) - f_{X\xi}^k(\tau, X_t^*, \xi)|| &\leq L ||X_t - X_t^*|| \\ ||f_{X\xi}^k(\tau, X_t, \xi) - f_{X\xi}^k(\tau, X_t, \xi^*)|| &\leq L ||\xi - \xi^*|| \end{aligned} \tag{2.4.19}$$

for all $\tau \in [0, t]$.

From eqn. (2.4.19) and the boundedness of the sets $X(t_f)$ and V , it may be shown that:-

$$\begin{aligned} ||f_X(t, X, \xi)|| &\leq C_1 \quad ; \quad ||f_\xi(t, X, \xi)|| \leq C_2 \\ ||f_{XX}^k(t_1, t_2, X, \xi)|| &\leq C_3 \quad ; \quad ||f_{X\xi}^k(t, X, \xi)|| \leq C_3 \end{aligned} \quad (2.4.20)$$

and
$$||f_{\xi\xi}^k(t, X, \xi)|| \leq C_3$$

and thus that:-

$$\begin{aligned} ||f(t, X_t, \xi) - f(t, X_t, \xi^*)|| &\leq C_2 ||\xi - \xi^*|| \\ ||f(t, X_t, \xi) - f(t, X_t^*, \xi)|| &\leq C_1 ||X_t - X_t^*|| \\ ||f_X(t, X_t, \xi) - f_X(t, X_t^*, \xi)|| &\leq C_3 ||X_t - X_t^*|| \end{aligned} \quad (2.4.21)$$

We shall make use of different bounds on the process $\xi(t)$ to those of Joffe and Sargent.

(vii) The following inequalities are assumed to hold for all $t, t' \in [0, t_f]$ and some $\epsilon < \infty$:-

$$\begin{aligned} E\{||\delta\xi(t)||\} &\leq \epsilon \\ ||E\{\delta\xi(t)\delta\xi^T(t')\}|| &\leq \epsilon^3 \\ E\{||\delta\xi(t)||^2 ||\delta\xi(t')||\} &\leq \epsilon^3 \end{aligned} \quad (2.4.22)$$

The following definition will be of use in the sequel ((61), p.14).

Definition Let $\phi(t)$ be a continuous function defined for all $t \in [-t_f, 0]$ and consider the equation

$$\frac{dx}{dt} = f(t, X_t) \quad : \quad t \in [0, t_f] \quad (2.4.23)$$

A function $x_\epsilon(t) = x_\epsilon(t, \phi)$ is said to be an ϵ -approximate solution to eqn. (2.4.23) with initial data $\phi(t)$ if

- (a) $x_\epsilon(t)$ is absolutely continuous on $[-t_f, t_f]$.
- (b) $x_\epsilon(t) = \phi(t)$ for $t \in [-t_f, 0]$.
- (c) $x_\epsilon(t)$ is differentiable on the interval $[0, t_f]$ except at most for a countable set S , and satisfies

$$||\frac{dx_\epsilon}{dt} - f(t, X_{\epsilon t})|| \leq \epsilon \quad \text{for } t \in [0, t_f] - S. \quad (2.4.24)$$

Oguztoreli ((61),p.18) gives the following comparison theorem for two ϵ -approximate solutions to eqn. (2.4.23).

Lemma 2.4.5

Suppose that the functional $f(t, X_t)$ is locally Lipschitzian with respect to X_t with the Lipschitz constant L^* . Let $x_{\epsilon_1}(t, \phi)$ and $y_{\epsilon_2}(t, \psi)$ be ϵ_1 - and ϵ_2 -approximate solutions to eqn. (2.4.23) with initial functions $\phi(t)$ and $\psi(t)$ respectively. Then,

$$||x_{\epsilon_1}(t, \phi) - y_{\epsilon_2}(t, \psi)|| \leq r_0 e^{L^*t} - \frac{\epsilon_1 + \epsilon_2}{L^*} (e^{L^*t} - 1)$$

$$\text{where } ||\phi(t) - \psi(t)|| \leq r_0$$

We may specialise lemma 2.4.5 to the case where $f(t, X_t)$ is linear in X_t and $\phi(t) = \psi(t) = 0$. The exact solution to eqn. (2.4.23) is then

$$x(t) = 0 \text{ for all } t \in [0, t_f]$$

So, if eqn. (2.4.24) is satisfied for some function $x_{\epsilon}(t, 0)$ and $f(t, X_t)$ is linear in X_t we have

$$||x_{\epsilon}(t, 0)|| \leq \frac{\epsilon}{L^*} (e^{L^*t} - 1) \quad (2.4.25)$$

We shall only utilise the specialised form of the lemma, given in eqn. (2.4.25).

We are now in a position to state and prove an extension to corollary 2.4.2, which allows the model of Joffe and Sargent to be used to design non-Markov controls.

Theorem 2.4.6

For a system:-

$$\frac{dx}{dt} = f(t, X_t, \xi(t)) : X_0 = 0 \quad (2.4.26)$$

which satisfies conditions (i) - (vii), the model:-

$$\frac{dx^*}{dt} = f(t, X_t^*, \xi^*(t)) + z(t) : X_0^* = 0 \quad (2.4.27)$$

$$\text{where } z^k(t) = \frac{1}{2} \text{trace} [f_{\xi\xi} Q(t)]$$

has the property that

$$||E\{x(t)\} - x^*(t)|| \leq K\epsilon^3 \quad (2.4.28)$$

for some $K < \infty$.

Proof

$$\text{Define } \delta x(t) = x(t) - x^*(t) \quad (2.4.29)$$

We begin by establishing a crude bound on $E\{||\delta x(t)||\}$ which will be of some use later.

From eqns. (2.4.26), (2.4.27) and (2.4.29) we have:-

$$\frac{d}{dt}\delta x(t) = f(t, X_t, \xi(t)) - f(t, X_t^*, \xi^*(t)) - z(t) \quad (2.4.30)$$

$$\begin{aligned} &= f(t, X_t, \xi(t)) - f(t, X_t, \xi^*(t)) + \\ &\quad + f(t, X_t, \xi^*(t)) - f(t, X_t^*, \xi^*(t)) - z(t) \end{aligned} \quad (2.4.31)$$

Taking norms, using the triangle inequality and eqns.(2.4.21), we may obtain from eqn.(2.4.31)

$$\frac{d}{dt}||\delta x(t)|| \leq C_1||\delta X_t|| + C_2||\delta \xi(t)|| + ||z(t)|| \quad (2.4.32)$$

Because of the definition of $||\delta X_t||$ we may distinguish two cases:

(a) $||\delta X_t|| = ||\delta x(t)||$ and (b) $||\delta X_t|| = ||\delta x(t')||$ for some $t' \in [0, t)$

For case (a), applying Gronwall's Lemma⁽¹⁾ to eqn.(2.4.32)

$$||\delta x(t)|| \leq e^{C_1 t} \int_0^t e^{-C_1 \tau} \{C_2||\delta \xi(\tau)|| + ||z(\tau)||\} d\tau$$

Similarly, for case (b),

$$||\delta x(t)|| \leq ||\delta x(t')||\{1 + C_1(t-t')\} + \int_{t'}^t \{C_2||\delta \xi(\tau)|| + ||z(\tau)||\} d\tau$$

Now, for any realisation of the noise, there is a time interval near $t = 0$ where case (a) applies. Following this, case (b) and case (a) will alternate depending on the particular noise realisation. For any fixed division of $[0, t_f]$ into intervals where case (a) or case (b) applies, the two relations above may be combined with eqn.(2.4.22) to show

$$E\{||\delta x(t)||\} \leq K\epsilon \text{ for some } K < \infty. \quad (2.4.33)$$

By averaging this relation over all possible divisions of $[0, t_f]$ we can show that eqn.(2.4.33) holds in general.

We now consider the k th of eqns. (2.4.30):-

$$\frac{d}{dt}\delta x^k(t) = f^k(t, X_t, \xi(t)) - f^k(t, X_t^*, \xi^*(t)) - z^k(t)$$

From the assumed differentiability properties of $f(t, X_t, \xi)$ we may make a Taylor series expansion ((61), p.47):-

$$\begin{aligned}
 \frac{d}{dt}\delta x^k(t) &= \int_0^t f_X(t_1, X_t^*, \xi^*(t))\delta x(t_1)dt_1 + \\
 &+ f_\xi^k(t, X_t^*, \xi^*(t))\delta \xi(t) + \\
 &+ \frac{1}{2} \int_0^t \int_0^t \delta x^T(t_1) f_{XX}^k(t_1, t_2, X_t^*, \xi^*(t))\delta x(t_2)dt_1 dt_2 + \\
 &+ \frac{1}{2} \delta \xi^T(t) f_{\xi\xi}^k(t, X_t^*, \xi^*(t))\delta \xi(t) + \\
 &+ \delta \xi^T(t) \int_0^t f_{\xi X}^k(t_1, X_t^*, \xi^*(t))\delta x(t_1)dt_1 - \\
 &- z^k(t) + \eta^k(t)
 \end{aligned} \tag{2.4.34}$$

where $|\eta^k(t)| \leq \frac{1}{6}L (||\delta x(t)|| + ||\delta \xi(t)||)^3$

From eqn. (2.4.30) we may also obtain a lower order expansion:-

$$\begin{aligned}
 \frac{d}{dt}\delta x(t) &= \int_0^t f_X(t_1, X_t^*, \xi^*(t))\delta x(t_1)dt_1 + f_\xi(t, X_t^*, \xi^*(t))\delta \xi(t) \\
 &- z(t) + v(t)
 \end{aligned} \tag{2.4.35}$$

where $||v(t)|| \leq \frac{1}{2}C_3 (||\delta x(t)|| + ||\delta \xi(t)||)^2$

By multiplying eqn. (2.4.35) by $\delta \xi^T(t')$ for some fixed $t' \in [0, t_f]$ we may obtain:-

$$\begin{aligned}
 \frac{d}{dt}\delta x(t)\delta \xi^T(t') &= \int_0^t f_X\delta x(t_1)\delta \xi^T(t')dt_1 + \\
 &+ f_\xi\delta \xi(t)\delta \xi^T(t') - z(t)\delta \xi^T(t') \\
 &+ v(t)\delta \xi^T(t')
 \end{aligned}$$

Taking expectations and norms, using the triangle and Schwartz inequalities:-

$$\begin{aligned}
 &||\frac{d}{dt}E\{\delta x(t)\delta \xi^T(t')\} - \int_0^t f_X E\{\delta x(t_1)\delta \xi^T(t')\}dt_1|| \leq \\
 &\leq ||f_\xi|| ||E\{\delta \xi(t)\delta \xi^T(t')\}|| + ||E\{z(t)\delta \xi^T(t')\}|| + \\
 &+ ||E\{v(t)\delta \xi^T(t')\}|| \\
 &\leq C_2\epsilon^3 + C_3\epsilon^3 + L\epsilon^3
 \end{aligned}$$

where we have obtained the last line by using eqns. (2.4.21), (2.4.22).

Applying lemma 2.4.5, we obtain

$$||E\{\delta x(t)\delta \xi^T(t')\}|| \leq C_5 \epsilon^3 \quad (2.4.36)$$

where

$$C_5 = \frac{C_2 + C_3 + L}{C_1} (e^{C_1 t_f} - 1)$$

By multiplying eqn. (2.4.35) by $\delta x^T(t')$ and using arguments similar to the above we may obtain:-

$$||E\{\delta x(t)\delta x^T(t')\}|| \leq C_6 \epsilon^3 \quad (2.4.37)$$

We return to eqn. (2.4.34) and take expectations and norms, using eqns. (2.4.21), (2.4.22), (2.4.36) and (2.4.37), to obtain

$$\begin{aligned} & ||\frac{d}{dt}E\{\delta x(t)\} - \int_0^t f_X E\{\delta x(t_1)\} dt_1|| \leq \\ & \leq \frac{t_f^2}{2} C_3 C_6 \epsilon^3 + t_f C_3 C_5 \epsilon^3 + \frac{L}{6} \epsilon^3 \end{aligned}$$

Applying lemma 2.4.5 we have the required result:-

$$||E\{\delta x(t)\}|| \leq K \epsilon^3$$

where

$$K = \frac{t_f^2 C_3 C_6 / 2 + t_f C_3 C_5 + L/6}{C_1} (e^{C_1 t_f} - 1)$$

Q.E.D.

Consider now eqn. (2.4.15). Provided $f(t, x, u, \xi)$ and $h(t, X_t)$ are sufficiently smooth to satisfy the conditions of theorem 2.4.6, the model:-

$$\frac{dx^*}{dt} = f(t, x^*, h(t, X_t^*), \xi^*(t)) + z(t) \quad (2.4.38)$$

is a good approximation to the expected behaviour of eqn. (2.4.15). Now, eqn. (2.4.38) is merely eqn. (2.4.5) with the substitution

$$u(t) = h(t, X_t^*)$$

We have therefore shown that the second-order model retains its approximation properties for non-Markov control policies. Thus, from theorem 2.3.1, we have immediately:-

Corollary 2.4.7

Let the system

$$\frac{dx}{dt} = f(t, x, u(t), \xi(t)), t \in [0, t_f]$$

be given, together with a class of admissible control laws

$$u(t) = h(t, X_t)$$

Let the function $f(t, x, u, \xi)$ and the functional $h(t, X_t)$ satisfy enough conditions to ensure that the equations

$$\frac{dx}{dt} = f(t, x, h(t, X_t), \xi(t))$$

satisfy conditions (i) - (vi) of this section. Further, let the process $\xi(t)$ satisfy condition (vii). Then, consider the problem

$$J_1 = E\{c^T x(t_f)\} : \min_h J_1$$

A good approximation to the solution of the above problem is the solution of the model problem

$$\frac{dx^*}{dt} = f(t, x^*, u(t), \xi^*(t)) + z(t)$$

$$J_2 = c^T x^*(t_f) : \min_{h(t, X_t^*)} J_2$$

The approximation is good in the sense that the model optimal control applied to the system gives performance which is within $O(\epsilon^3)$ of the optimal stochastic controller.

The above theory extends the results of the previous section to non-Markov controllers and hence overcomes problem (b) of that section. However, we still have quite severe smoothness requirements for the admissible control laws. Before considering this problem further, we first note that the solution of the model control problem is simpler than corollary 2.4.7 would suggest. This is because the model is deterministic, so the Principle of Optimality⁽³⁹⁾ tells us that the optimal control within the class of feedback controls on the state history gives the same performance as the optimal control based on current state feedback. Thus, the solution to the model problem is a control law depending only on the current state. This leads to the interesting conclusion that for non-Markov systems obeying the assumptions of corollary 2.4.7, a feedback control law based only on the current state vector is a good approximation to the optimal stochastic controller. Such a system could be said to be 'almost Markov'.

However, we must be careful that the control laws found satisfy enough properties to allow the application of corollary 2.4.7. Alternatively, we

would like to generalise our results to include piecewise - continuous control laws. The first difficulty with such an extension is to guarantee that solutions of the system equations exist. There has been some work done on ordinary differential equations with piecewise - continuous right-hand sides. In a paper by Filippov⁽⁷⁹⁾, a new definition of a solution is introduced, and existence and uniqueness properties of the solutions are established. Two distinct cases are discussed by Filippov.

- (a) The solution passes straight through all surfaces of discontinuity in state space. In this case, a solution can be constructed in pieces almost everywhere, each piece obeying an equation which satisfies the continuity assumptions of section 2.4.1. Hence we would expect corollary 2.4.4 to apply.
- (b) The solution, being found at some time on a discontinuity surface in state space, cannot leave it. This type of situation could arise in relay control systems. In this case, the solution may be approximated arbitrarily well by the solution of an equation with a continuous right-hand side, which changes rapidly in a narrow strip near the discontinuity. Thus, by applying a limiting argument to a sequence of such continuous right-hand sides, we would expect to be able to prove results like corollary 2.4.4.

The above arguments have not been presented rigorously, since they are concerned only with the case of current state feedback, and we are more interested in non-Markov control laws. Unfortunately, the theory of delay-differential equations with discontinuous right-hand sides is not well developed, so little can be said about this case.

To summarise, we have established results about the optimal controller for a rather general stochastic system perturbed by small noise, if the state of the system can be observed at all times. We have presented a deterministic model, the optimal controller for which gives a good approximation to the optimal stochastic controller provided that certain smoothness assumptions are satisfied. Since the model is deterministic, the controller can be found by well established and efficient methods.

2.5 Feedforward and Discrete Feedback Control Policies

In this section we present some results relating the performance of a system under discrete feedback control to that under the optimal stochastic controller with continuous state feedback. The assumption is made throughout

that the system under consideration has enough properties to ensure that the results of section 2.4 apply. We specialise the admissible noise processes further than in that section. We assume that $\xi(t)$ is a bounded piecewise constant process, i.e. there exists a finite sequence of times $\{t_0, t_1, \dots, t_k\}$ with $t_0 < t_1 < \dots < t_k$ and $t_0 = 0, t_k = t_f$ such that

$$\xi(t) = \sum_{i=1}^k I_i(t) \xi_i$$

$$\begin{aligned} \text{where} \quad I_i(t) &= 1 \quad \text{for } t \in [t_{i-1}, t_i) \\ &= 0 \quad \text{otherwise} \end{aligned} \quad (2.5.1)$$

and the ξ_i are a sequence of independent random vectors with

$$E(\xi_i) = 0$$

$$E(\xi_i \xi_j) = \delta_{ij} Q_i ; \delta_{ij} \text{ being the Kronecker delta} \quad (2.5.2)$$

By assuming

$$t_{i+1} - t_i \leq \epsilon \quad (2.5.3)$$

for all i , and

$$||\xi(t)|| \leq \epsilon \quad (2.5.4)$$

with probability one for all $t \in [0, t_f]$, we are assured that the process $\xi(t)$ has the smallness properties required in section 2.4. A useful lemma can be proved (Joffe⁽²⁷⁾).

Lemma 2.5.1

Let $u(t)$ be an arbitrary admissible control. By applying this control to the system:-

$$\frac{dx}{dt} = f(t, x, u(t), \xi(t)) : x(0) \text{ given} \quad (2.5.5)$$

let a realisation, $x(t)$, of the state process be obtained. Applying the control to the model

$$\frac{dx^*}{dt} = f(t, x^*, u(t), \xi^*(t)) + z(t) = f^*(t, x^*, u(t)) : x^*(0) = x(0) \quad (2.5.6)$$

a trajectory $x^*(t)$ is obtained. Then,

$$||x(t) - x^*(t)|| \leq K t_f e^{Lt_f} \epsilon \quad \text{w.p.1} \quad (2.5.7)$$

for some $K, L < \infty$ and all $t \in [0, t_f]$.

Suppose that the optimal feedforward policy for eqn. (2.5.6) in order to minimise

$$J = c^T x^*(t_f)$$

given initial conditions (t_o, x_o^*) is

$$u(t) = h(t, t_o, x_o^*)$$

In the following we will assume that $h(t, t_o, x)$ is Lipschitz continuous in x , i.e.

$$||h(t, t_o, x) - h(t, t_o, x^*)|| \leq K_1 ||x - x^*|| \quad (2.5.8)$$

for some $K_1 < \infty$. From corollary 2.4.7, we may consider

$$u(t) = h(t, t, x(t)) \quad (2.5.9)$$

as a good approximation to the optimal stochastic controller for eqn. (2.5.5) with performance index

$$J = E\{c^T x(t_f)\} \quad (2.5.10)$$

By applying eqn. (2.5.9) to the system, eqn. (2.5.5), we obtain a realisation of the process $x(t)$. By lemma 2.5.1, this realisation is well approximated by the solution, $x_1^*(t)$, of

$$\frac{dx_1^*}{dt} = f^*(t, x_1^*, h(t, t, x(t))) : x_1^*(o) = x(o) \quad (2.5.11)$$

Let $y(t)$ be an arbitrary process satisfying

$$||x(t) - y(t)|| \leq K_2 \delta \quad \text{w.p.1} \quad (2.5.12)$$

and consider applying the control

$$u(t) = h(t, t, y(t)) \quad (2.5.13)$$

to the system. The performance is well approximated by the solution, $x_2^*(t)$, of

$$\frac{dx_2^*}{dt} = f^*(t, x_2^*, h(t, t, y(t))) : x_2^*(o) = x(o) \quad (2.5.14)$$

Thus, by comparing $x_1^*(t)$ and $x_2^*(t)$ we are comparing the performance of the two control laws given by eqns. (2.5.9) and (2.5.13).

We now prove:-

Theorem 2.5.2

$$||x_2^*(t_f) - x_1^*(t_f)|| \leq K_3 \delta$$

for some

$$K_3 < \infty.$$

Proof

Define $\delta x^*(t) = x_1^*(t) - x_2^*(t)$ (2.5.15)

From eqns. (2.5.11), (2.5.14), (2.5.15)

$$\begin{aligned} \frac{d}{dt} x^* &= f^*(t, x_1^*, h(t, t, x(t))) - f^*(t, x_2^*, h(t, t, y(t))) \\ &= f^*(t, x_1^*, h(t, t, x(t))) - f^*(t, x_1^*, h(t, t, y(t))) + \\ &\quad + f^*(t, x_1^*, h(t, t, y(t))) - f^*(t, x_2^*, h(t, t, y(t))) \end{aligned} \quad (2.5.16)$$

Now, from the assumed properties of $f(t, x, u, \xi)$, $f^*(t, x, u)$ is guaranteed to be Lipschitzian in x and u , i.e. there exist scalars $K_4, K_5 < \infty$ such that

$$||f^*(t, x, u) - f^*(t, x^*, u)|| \leq K_4 ||x - x^*|| \quad (2.5.17)$$

$$||f^*(t, x, u) - f^*(t, x, u^*)|| \leq K_5 ||u - u^*||$$

Taking norms of eqn. (2.5.16) and using eqn. (2.5.8), (2.5.12), and (2.5.17)

$$\frac{d}{dt} ||\delta x^*(t)|| \leq K_5 K_1 K_2 \delta + K_4 ||\delta x^*(t)|| \quad (2.5.18)$$

Gronwall's lemma⁽¹⁾ may be applied to eqn. (2.5.18) to give

$$\begin{aligned} ||\delta x^*(t)|| &\leq \exp \{K_4 t_f\} t_f K_5 K_1 K_2 \delta \\ &= K_3 \delta \end{aligned}$$

where

$$K_3 = K_1 K_2 K_5 t_f \exp (K_4 t_f)$$

Q.E.D.

We may now establish results concerning feedforward and discrete feedback controls as special cases of theorem 2.5.2. The optimal feedforward control for the model is given by

$$u(t) = h(t, o, x(o)) \quad (2.5.19)$$

Substituting this control into eqn. (2.5.6) gives:-

$$\frac{dx^*}{dt} = f^*(t, x^*, h(t, o, x(o))) : x^*(o) = x(o)$$

or since this equation is deterministic:-

$$\frac{dx^*}{dt} = f^*(t, x^*, h(t, t, x^*(t))) : x^*(o) = x(o) \quad (2.5.20)$$

From lemma 2.5.1,

$$||x(t) - x^*(t)|| \leq K t_f e^{L t_f} \epsilon \quad \text{w.p.1} \quad (2.5.21)$$

where $x(t)$ is any solution of

$$\frac{dx}{dt} = f(t, x, h(t, t, x(t)), \xi(t)) : x(0) \text{ given} \quad (2.5.22)$$

We have thus produced a process $x^*(t)$ which satisfies eqn. (2.5.12). Hence the optimal feedforward control for the model gives performance which is within $O(\epsilon)$ of the optimal stochastic controller based on perfect observation of the state.

For the case of discrete feedback control, based on measurements of the state at discrete times Δt apart, we define a process $y(t)$ by

$$\begin{aligned} \frac{dy}{dt} &= f^*(t, y, h(t, t, y)) : y(0) = x(0), \\ y(n\Delta t) &= x(n\Delta t), n=1, 2, \dots, N \end{aligned}$$

That is, we reset $y(t)$ to $x(t)$ at discrete times when we are allowed to measure $x(t)$. Otherwise, we compute $y(t)$ using the second-order model. From lemma 2.5.1, it is clear that

$$||y(t) - x(t)|| \leq K \Delta t e^{L \Delta t} \epsilon \quad \text{w.p.1}$$

and hence that the discrete feedback control policy based on updating the optimal feedforward control policy for the model as a new state vector is observed, gives performance which is within $O(\Delta t \cdot \epsilon)$ of the optimal stochastic policy based on continuous observation of the state.

C H A P T E R T H R E E

OPTIMAL CONTROL BASED ON COMPLETE OBSERVATIONS

OF THE STATE - EXAMPLES

3.1 Scope

In this chapter, two examples are presented which illustrate the results of chapter two. The first is a simple start-up problem for a continuous stirred tank reactor. It is hoped that this example illustrates the theory in a qualitative way, without resort to complex models. As a result of this, the applicability of the model and results to any real process may be questioned.

Our second example is concerned with the batch reactor system studied by Szepe and Levenspiel⁽¹⁶⁾. It may justifiably be said that this example is also rather 'academic'. It is unfortunate that, in order to give meaningful comparisons of different control schemes, many simulations must be performed. This constrains our illustrative examples to those for which an analytical solution for the control law is available; repeated on-line optimisation calculations being infeasible in terms of computation time when Monte Carlo techniques are to be used. Of course, in industrial applications, on-line optimisation may well be feasible⁽⁶²⁾, so that more complex applications than the ones presented here may be contemplated. The aim of the examples is to give a qualitative picture of the kind of behaviour to be expected, rather than to guide a specific application of the theory.

3.2 Start-up of a Continuous Stirred Tank Reactor

3.2.1 Problem Description

We consider a continuous stirred tank reactor in which we propose to run the reaction



The kinetics are assumed to be first-order in the concentration of species A. For example, the hydrolysis of acetic anhydride might be considered. (In fact, the kinetic parameters used in this study are based on the data of Eldridge and Piret⁽⁶³⁾ for that reaction.) We assume that, once started up, the reactor will be run isothermally at steady state. The start-up

procedure consists of filling the reactor with solvent, establishing the required operating flowrate and temperature, and then gradually introducing species A into the reactor feed stream, maintaining a constant flow and temperature. We concern ourselves with the last stage of the procedure, and attempt to find the optimal feed concentration policy in order to get as close as possible to steady state in a fixed time period.

We may write down a material balance for species A in the reactor in terms of perturbations of the inlet and outlet concentrations from their desired steady state values:-

$$\frac{dx_1}{dt} = \frac{1}{\tau} (u - x_1) - kx_1 : t \in [0, t_f] \quad (3.2.2)$$

where x_1 is the perturbation of the outlet concentration of A from its desired value,

u is the perturbation of the inlet concentration of A from its desired value,

τ is the reactor time constant,

and k is the reaction rate constant.

In order to make the solution to the problem as simple as possible, we take for our performance index the quadratic form:-

$$J = x_1^2(t_f) + \alpha \int_0^{t_f} u^2(t) dt \quad (3.2.3)$$

The first term accounts for the final deviation from the desired steady state, whilst the second is included to account for constraints on the control action. The parameter α may be used for tuning of the controller. To apply the results of chapter two, our performance index must be expressed as a linear function of the final state of the system, so another state, $x_2(t)$, defined by

$$\frac{dx_2}{dt} = \alpha u^2(t) + 2x_1(t) \frac{dx_1}{dt} : x_2(0) = x_1^2(0) \quad (3.2.4)$$

is introduced. Clearly,

$$J = x_2(t_f) \quad (3.2.5)$$

So far, we have made no mention of any noise in the process description. We will assume that perfect temperature control is impossible; the reactor temperature is subject to random fluctuations about its desired value, causing fluctuations in the rate constant:-

$$k = k_o \exp (-\beta/T) \quad (3.2.6)$$

$$T = T^* + \xi(t) \quad (3.2.7)$$

where T is the actual reactor temperature

T^* is the desired reactor temperature

and $\xi(t)$ is a zero mean physically realisable stochastic process.

The rate constant is assumed to show the usual Arrhenius dependence on temperature.

We now have a stochastic control problem, and we must reformulate our performance index slightly. If we observe the state x at time t , then our objective is to minimise

$$J^*(t, x) = E\{x_1^2(t_f) + \alpha \int_0^t u^2(s) ds | x(t)\} \quad (3.2.8)$$

i.e. the expected value of $x_2(t_f)$ conditioned on $x(t)$.

3.2.2 Method of Solution

We will be interested in the various approximations to the optimal policy proposed in chapter two. Since the system under consideration is not of the Markov type, it is not possible to determine the true optimal policy. Our comparisons will have to be more intuitively based.

The first step in the solution to our problem is to formulate the second-order approximation for the system of eqns. (3.2.2) and 3.2.4):-

$$\frac{dx_1^*}{dt} = ax_1^* + bu(t) : x_1^*(t_o) \text{ given}$$

$$\frac{dx_2^*}{dt} = \alpha u^2(t) + 2x_1^* [ax_1^* + bu(t)] : x_2^*(t_o) = [x_1^*(t_o)]^2$$

where $a = -\frac{1}{\tau} - k^*$ (3.2.9)

$$k^* = k(T^*) \left\{ 1 + \frac{q\beta}{2(T^*)^3} \left(\frac{\beta}{T^*} - 2 \right) \right\}$$

$$b = 1/\tau$$

$$q = E\{\xi^2\}, \text{ assumed constant}$$

and $x_1^*(t)$, $x_2^*(t)$ are approximations to the conditional expectations of the process state variables. According to the results of chapter two, a good approximation to the optimal stochastic policy for the system (eqn. 3.2.2), (3.2.4)) is the optimal deterministic policy for the second-order model (eqn. (3.2.9)). Since the optimal control problem for the second-order

model is of the linear-quadratic type, a solution may be easily obtained by Pontryagin's Principle⁽⁶⁴⁾:-

$$u(t) = \frac{b}{\alpha} \left(\frac{p(t)}{2} - 1 \right) x_1^*(t) \quad (3.2.10)$$

where $p(t)$ obeys a Riccati equation:-

$$\frac{d}{dt}p = 4a - \frac{2b^2}{\alpha} + 2p\left(\frac{b^2}{\alpha} - a\right) - \frac{b^2}{2} \frac{p^2}{\alpha} : p(t_f) = 0 \quad (3.2.11)$$

The solution to eqn. (3.2.11) is:-

$$p(t) = \frac{\sqrt{2\alpha}}{b} \left\{ c-d \left[\frac{1+f \exp\{2w(t-t_f)\}}{1-f \exp\{2w(t-t_f)\}} \right] \right\}$$

for $c^2 > \frac{2b^2}{\alpha} - 4a$

and

$$p(t) = \frac{\sqrt{2\alpha}}{b} \left[c-d \tan\{w(t_f-t) + \tan^{-1}(-c/d)\} \right]$$

for $c^2 < \frac{2b^2}{\alpha} - 4a$ (3.2.12)

where

$$c = \sqrt{\frac{2}{\alpha}} \left(b - \frac{a}{b} \sqrt{2\alpha} \right)$$

$$d = \sqrt{|c^2 + 4a - 2b^2/\alpha|}$$

$$f = (c-d)/(c+d)$$

and $w = bd/\sqrt{2\alpha}$

3.2.3 Simulation Procedure and Results

In order to demonstrate the theory of chapter two, some Monte Carlo simulations of the reactor were performed, using the various control schemes suggested in that chapter. These control policies are implemented by replacing $x_1^*(t)$ in eqn. (3.2.10) by the appropriate state variable. Thus, for continuous feedback control:-

$$u(t) = \gamma(t)x_1(t) \quad (3.2.13)$$

where $\gamma(t) = \frac{b}{\alpha} \left(\frac{p(t)}{2} - 1 \right)$ (3.2.14)

and $x_1(t)$ is the state of the actual process, eqn. (3.2.2). For feedforward control:-

$$u(t) = \gamma(t)x_1^*(t) \quad (3.2.15)$$

whilst for discrete updates of the feedforward policy, eqn. (3.2.15) is used, but the initial condition to eqn. (3.2.9) is updated at each measurement time.

We have studied some properties of the distributions of the final states of the process, and how they vary according to which type of control is chosen. These properties were estimated using a Monte Carlo simulation approach. For the purposes of simulation, eqns. (3.2.2) and 3.2.4) were integrated numerically using a fourth-order Runge-Kutta procedure; the noise was assumed to be constant over each integration step. The random variable in each step was realised using a standard random number generator, giving a zero mean rectangularly distributed variable over $[-w, w]$. The variance of this distribution can be shown to be $w^2/3$. All computations were carried out on a CDC 6400 machine in single precision, giving about thirteen significant figures.

The following parameter values were chosen for the study:-

$$\begin{aligned} \tau &= 5.0 \text{ min.} \\ t_f &= 2.5 \text{ min.} \\ x_1(0) &= -0.173 \\ \alpha &= 10^{-3} \\ k_o &= 0.38 \exp(31.9) \text{ min}^{-1} \\ \beta &= 10,000 \text{ K} \\ T^* &= 313 \text{ K} \end{aligned}$$

$$\text{Step length for integration} = 0.0625 \text{ min}$$

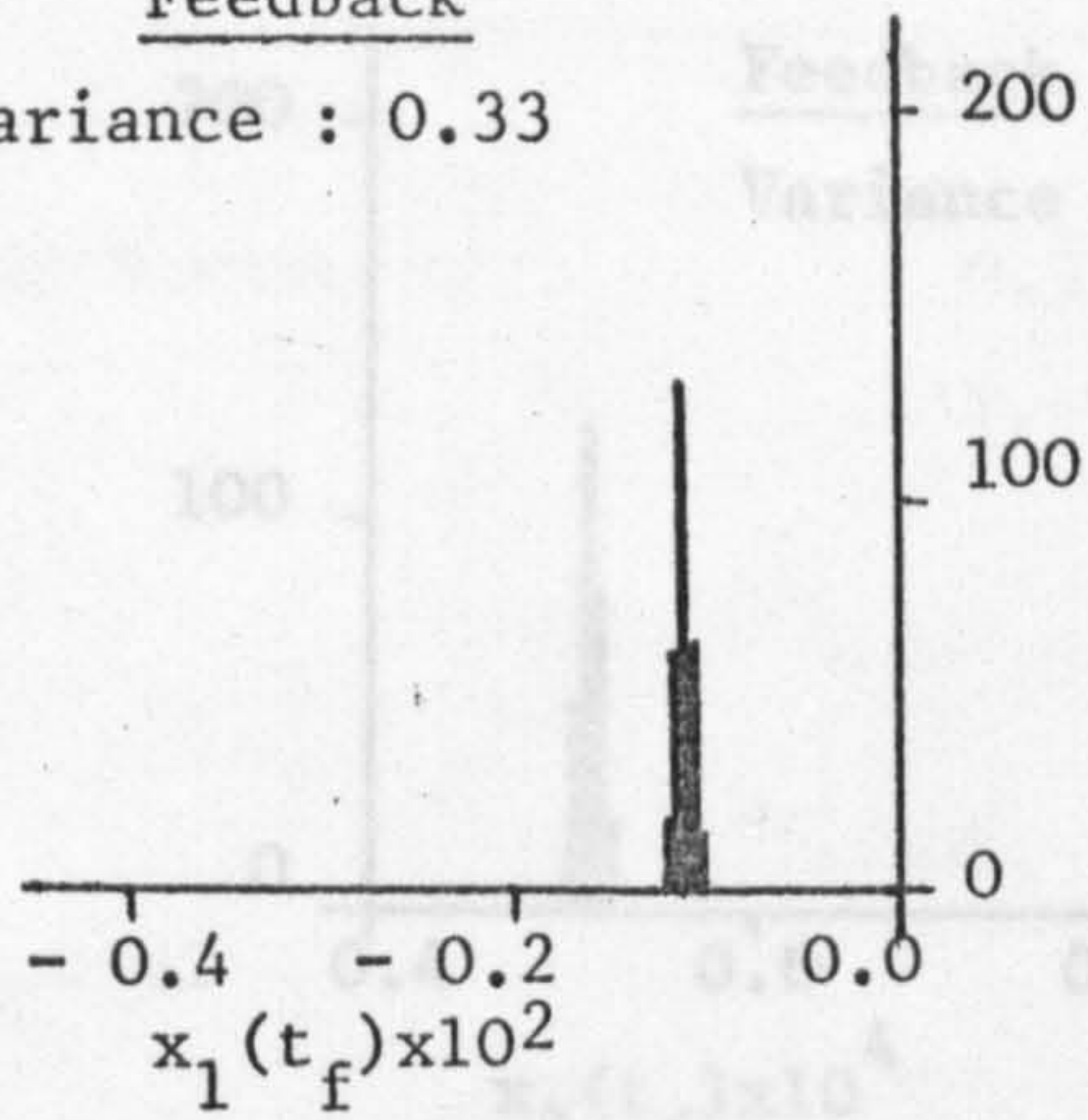
$$\text{Number of simulations per run} = 300$$

In Figs. 3.1a and 3.1b we compare the effectiveness of feedforward and feedback control schemes. The figures give histograms for the two final state variables $x_1(t_f)$ (in Fig. 3.1a) and $x_2(t_f)$ (in Fig. 3.1b). The most obvious feature of these figures is the tighter control on the final state $x_1(t_f)$ which feedback has achieved. This is reflected in the estimated variance of this variable which is shown in Table 3.1 for both feedback (FB) and feedforward (FF) control. The noise variances correspond to maximum temperature fluctuations as follows: 0.33 corresponds to ± 1 deg K, 1.33 to ± 2 , 5.33 to ± 4 , and 8.33 to ± 5 . The numbers after the commas give the exponent of ten, e.g. .116,-9 is equivalent to 0.116×10^{-9} . As is to be expected, the variance of $x(t_f)$ increases with the noise variance.

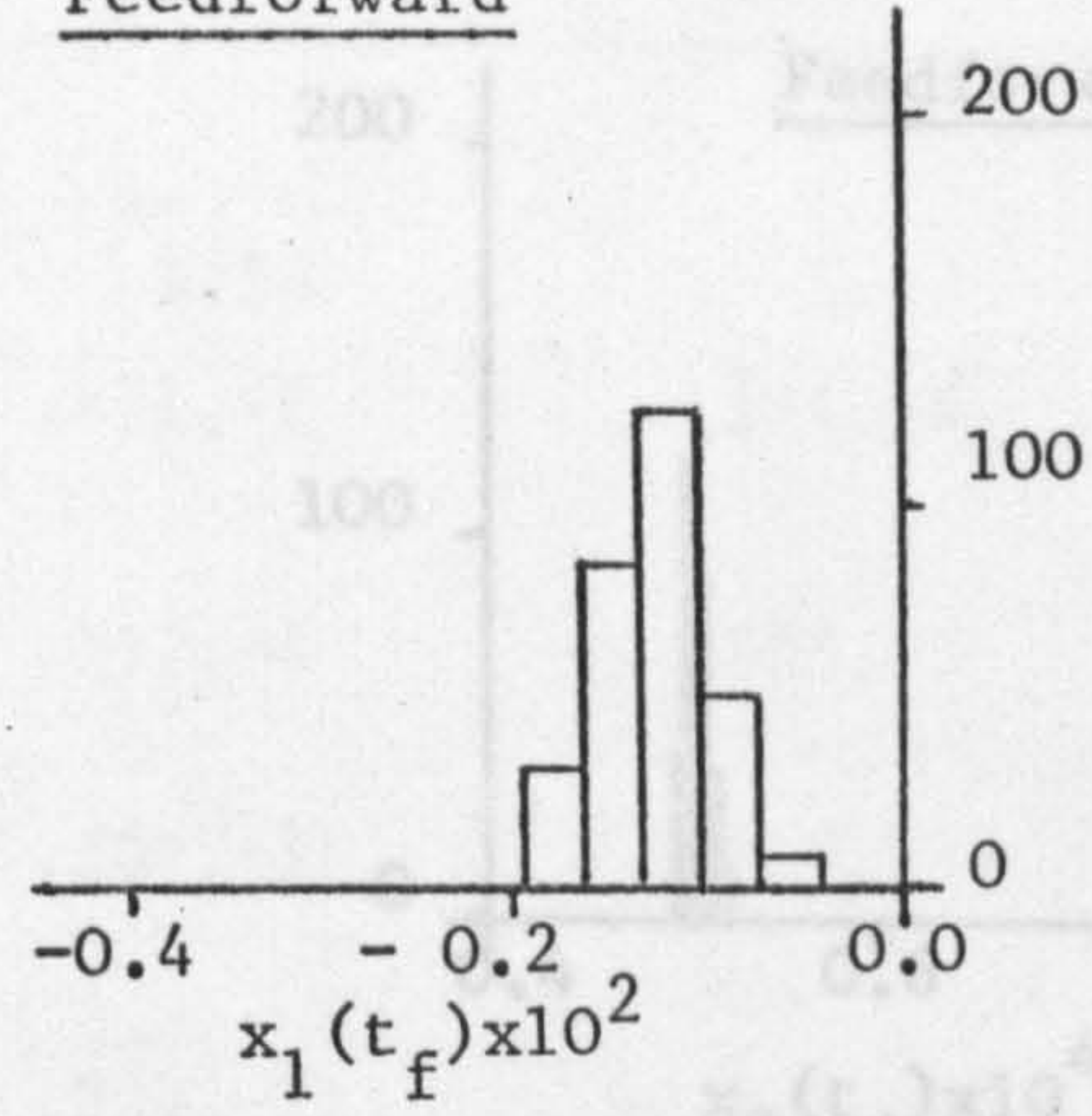
FIGURE 3.1a

Histograms for $x_1(t_f)$

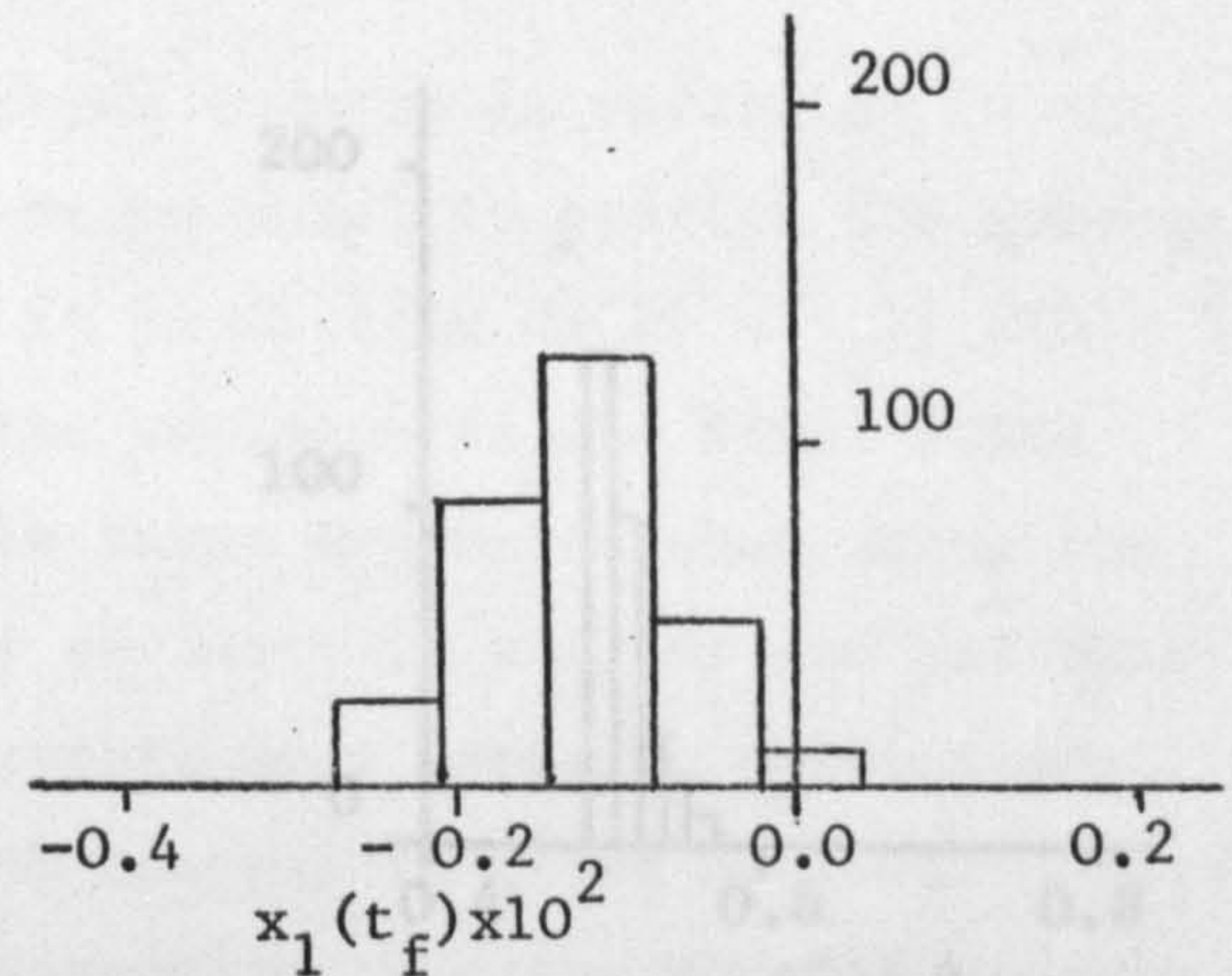
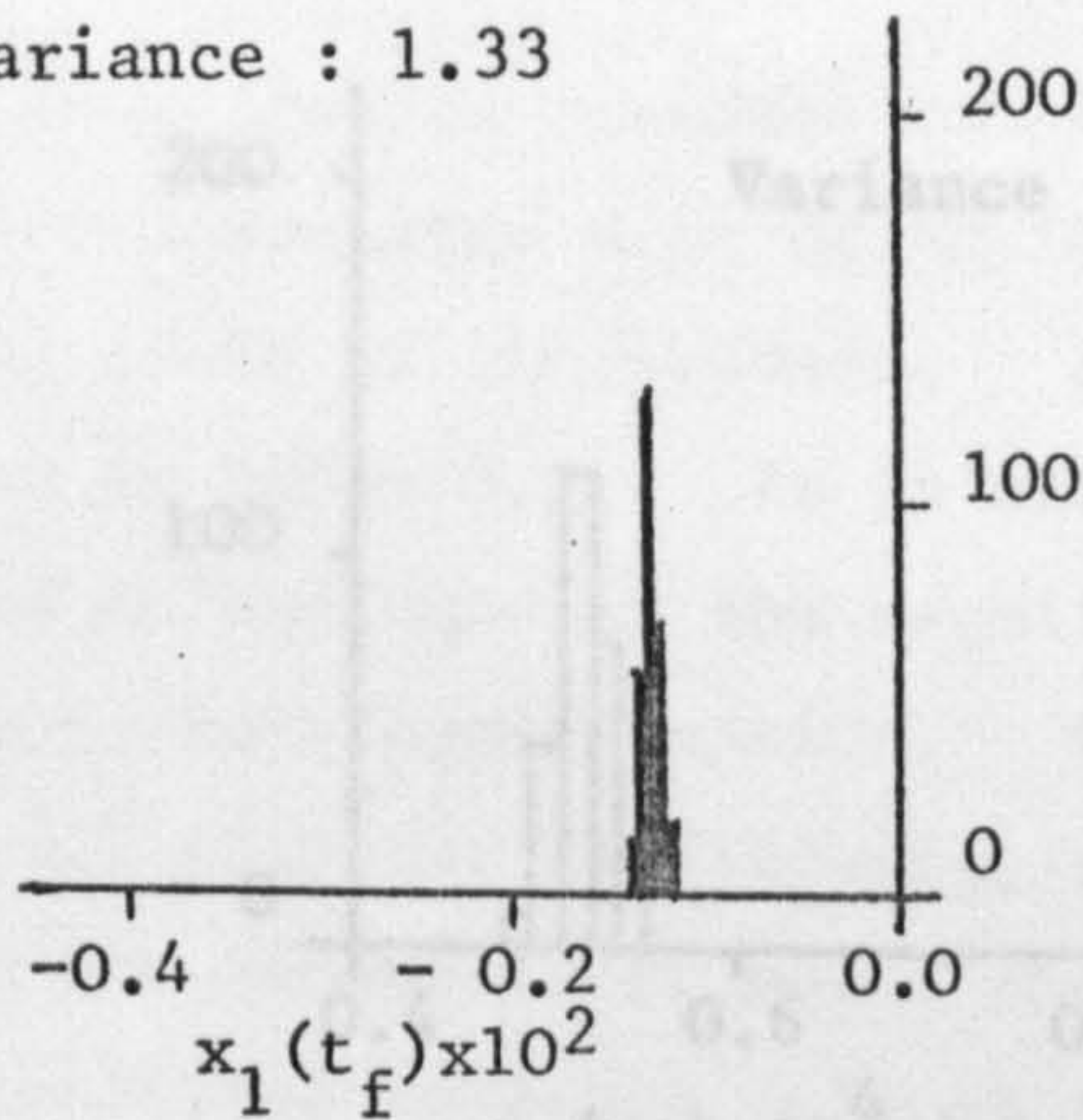
Feedback
Variance : 0.33



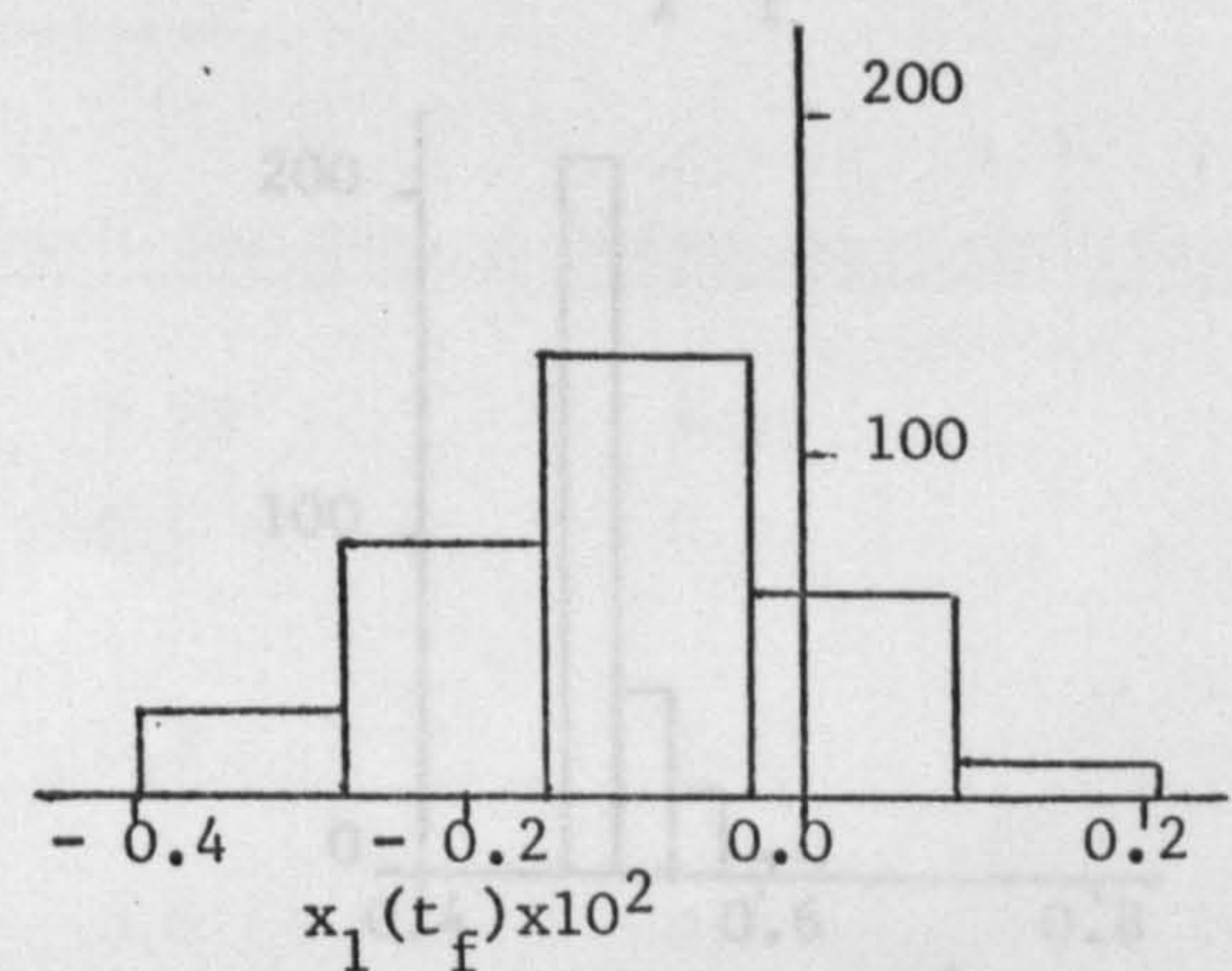
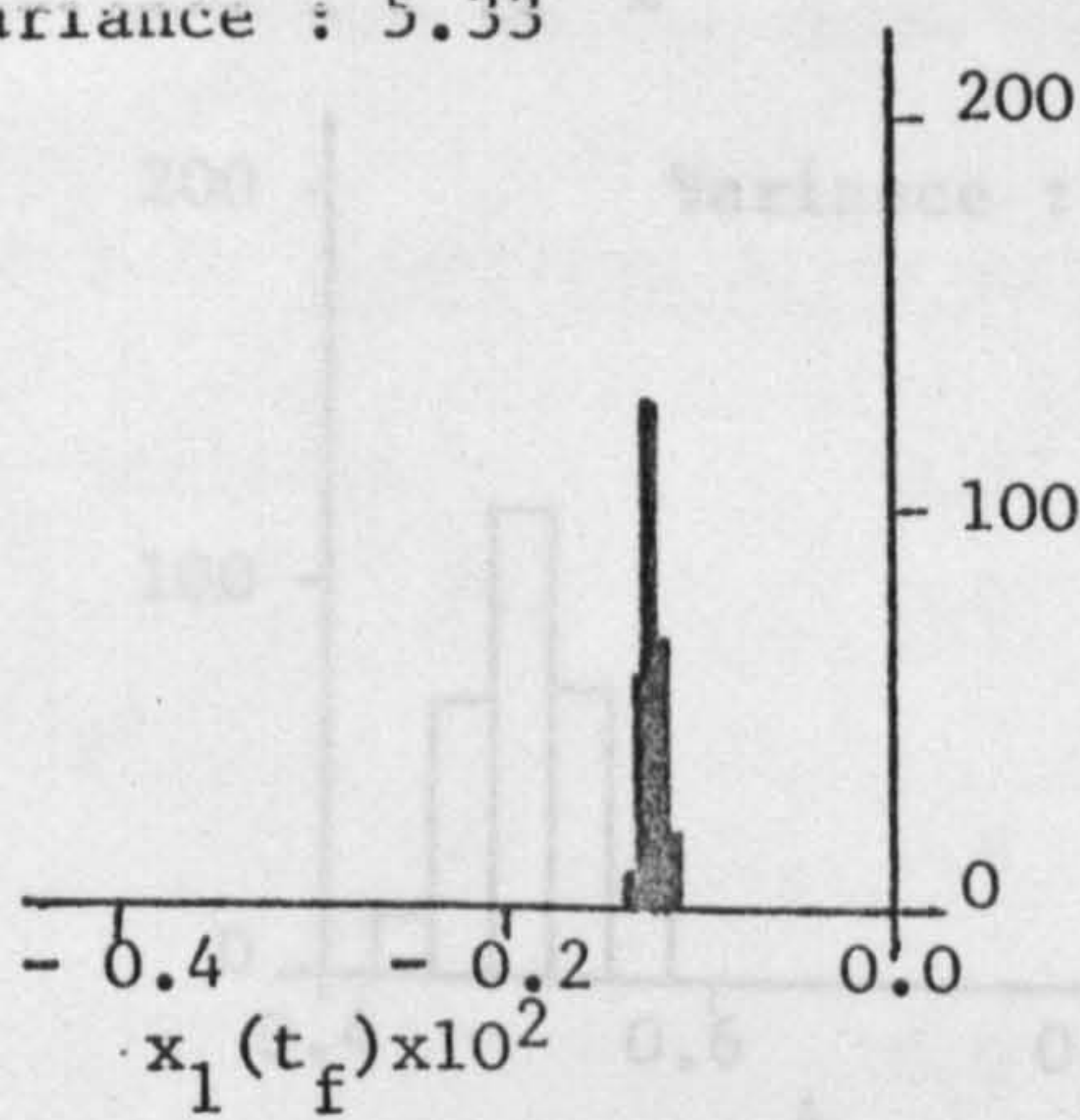
Feedforward



Variance : 1.33



Variance : 5.33



Variance : 8.33

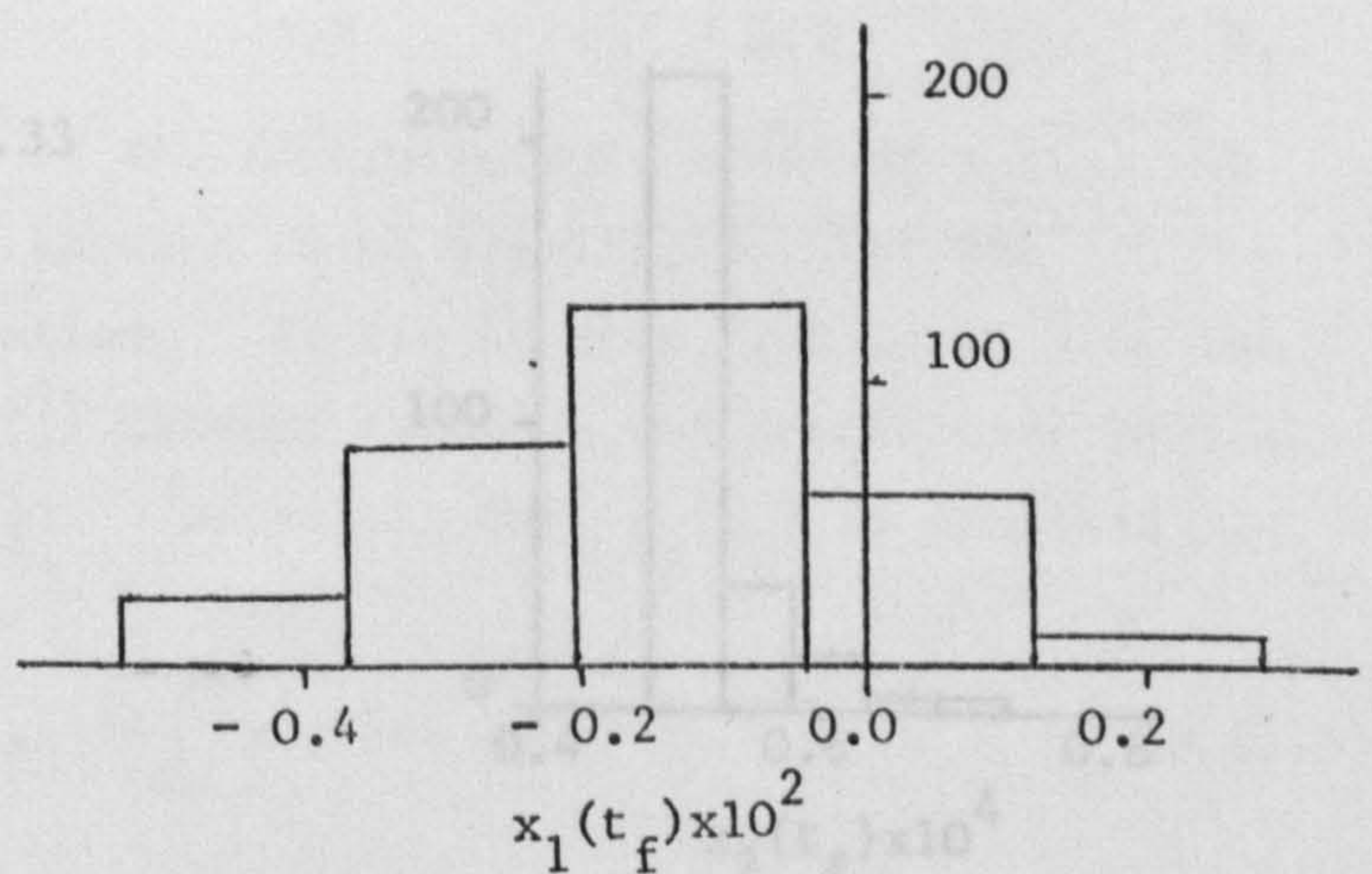
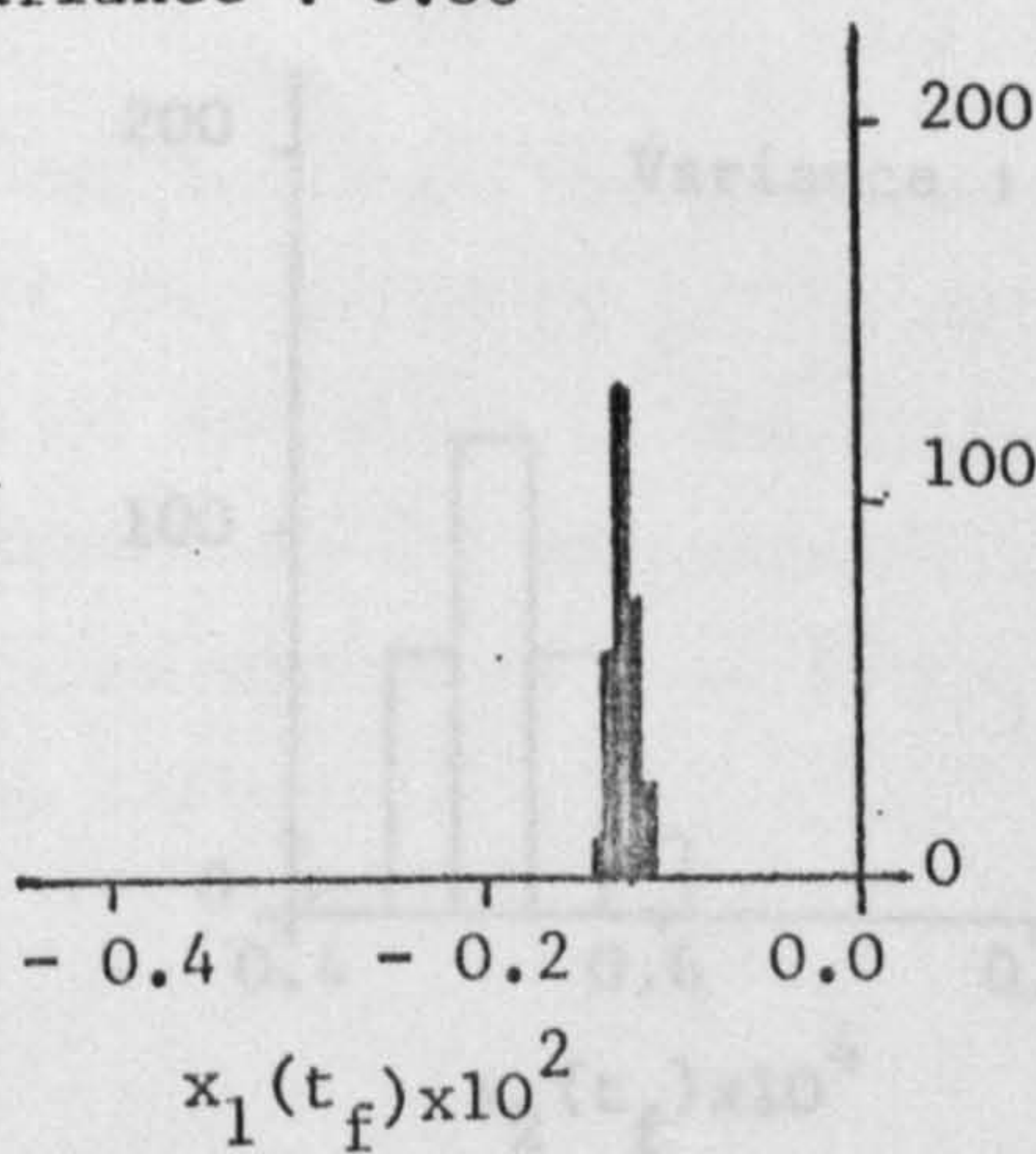


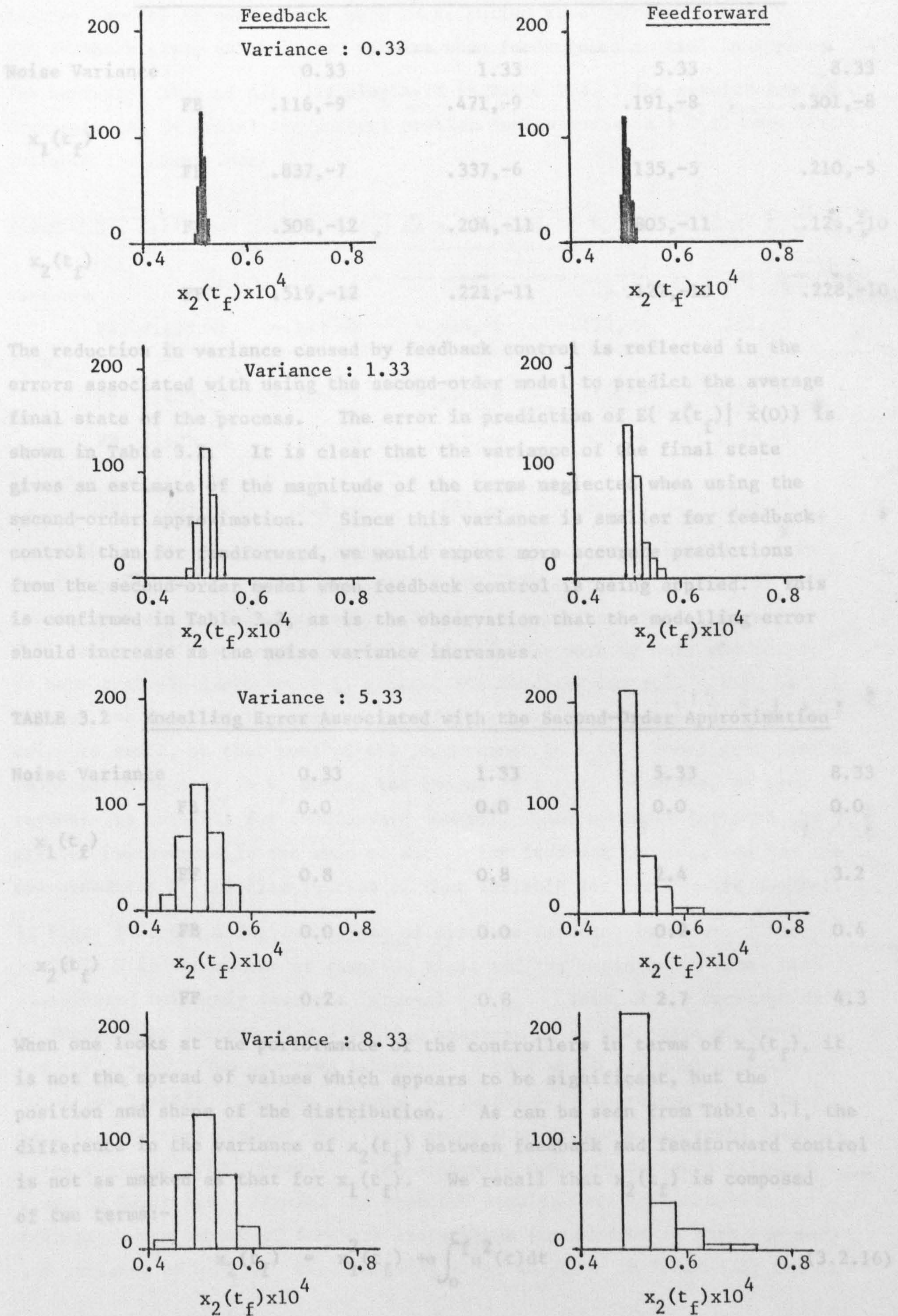
FIGURE 3.1b Histograms for $x_2(t_f)$ 

TABLE 3.1 Estimated Variance of $x(t_f)$ as a Function of Noise Variance

Noise Variance		0.33	1.33	5.33	8.33
$x_1(t_f)$	FB	.116,-9	.471,-9	.191,-8	.301,-8
	FF	.837,-7	.337,-6	.135,-5	.210,-5
$x_2(t_f)$	FB	.508,-12	.204,-11	.805,-11	.124,-10
	FF	.519,-12	.221,-11	.122,-10	.228,-10

The reduction in variance caused by feedback control is reflected in the errors associated with using the second-order model to predict the average final state of the process. The error in prediction of $E\{x(t_f) | x(0)\}$ is shown in Table 3.2. It is clear that the variance of the final state gives an estimate of the magnitude of the terms neglected when using the second-order approximation. Since this variance is smaller for feedback control than for feedforward, we would expect more accurate predictions from the second-order model when feedback control is being applied. This is confirmed in Table 3.2, as is the observation that the modelling error should increase as the noise variance increases.

TABLE 3.2 Modelling Error Associated with the Second-Order Approximation

Noise Variance		0.33	1.33	5.33	8.33
$x_1(t_f)$	FB	0.0	0.0	0.0	0.0
	FF	0.8	0.8	2.4	3.2
$x_2(t_f)$	FB	0.0	0.0	0.2	0.4
	FF	0.2	0.8	2.7	4.3

When one looks at the performance of the controllers in terms of $x_2(t_f)$, it is not the spread of values which appears to be significant, but the position and shape of the distribution. As can be seen from Table 3.1, the difference in the variance of $x_2(t_f)$ between feedback and feedforward control is not as marked as that for $x_1(t_f)$. We recall that $x_2(t_f)$ is composed of two terms:-

$$x_2(t_f) = x_1^2(t_f) + \alpha \int_0^{t_f} u^2(t) dt \tag{3.2.16}$$

Any difference in the contribution of the first term to the variance due to tighter control is compensated by a contribution from the second term in the feedback case, which is not present when feedforward control is applied. The estimated mean of $x(t_f)$ is displayed in Table 3.3. The results for the corresponding deterministic control problem (noise variance = 0.0) have been included for comparison.

TABLE 3.3 Estimated Mean of $x(t_f)$ as a Function of Noise Variance

Variance	0.0	0.33	1.33	5.33	8.33
$x_1(t_f)$	FB -.123,-2	-.123,-2	-.123,-2	-.122,-2	-.121,-2
	FF -.123,-2	-.124,-2	-.124,-2	-.125,-2	-.125,-2
$x_2(t_f)$	FB .523,-4	.522,-4	.519,-4	.507,-4	.498,-4
	FF .523,-4	.523,-4	.523,-4	.520,-4	.518,-4

The addition of noise in the reactor temperature causes an improvement in reactor performance; as the noise is increased, $x_2(t_f)$ decreases for both feedforward and feedback control. Similar results have been observed by Pell and Aris⁽¹⁷⁾ and appear to be related to the work of Horn and Bailey⁽⁶⁵⁾. We note that the improvement is greater for feedback control. This is because the expected improvement in $x_1(t_f)$ resulting from the addition of noise is small, so that most of the improvement in $x_2(t_f)$ comes from control response to changes in x_1 during the course of a run. However, no such response is possible for feedforward control, which accounts for both the greater improvement in the mean of $x_2(t_f)$ for feedback control, and for the one-sidedness of the distribution of that variable for feedforward control.

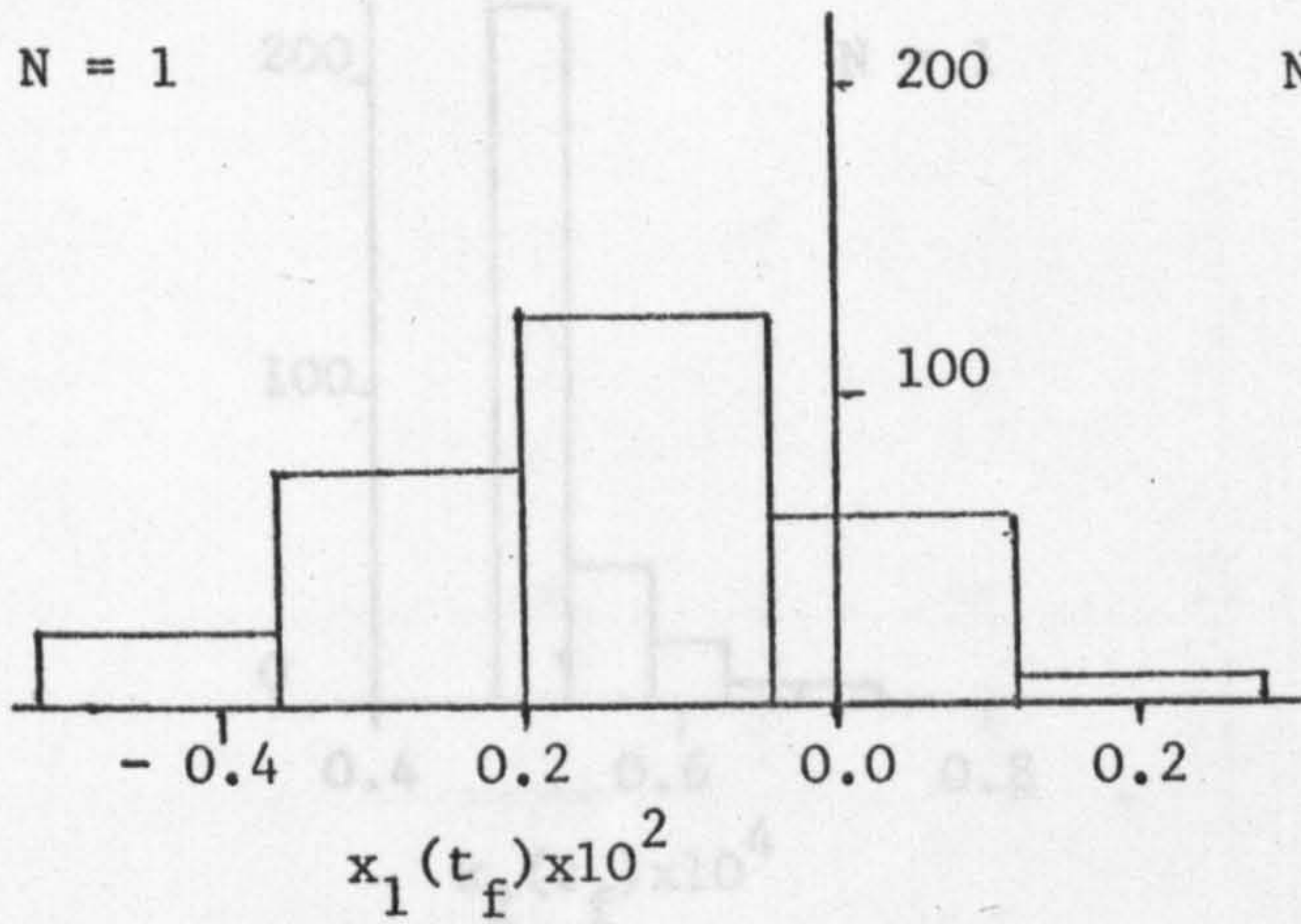
In Figs. 3.2a and 3.2b, the effect of discrete feedback is shown. The variable N is the number of sampling times and the measurement times were distributed uniformly over the interval $[0, t_f]$. Thus, $N = 1$ corresponds to feedforward control, $N = 2$ implies measurement of the state at $t = 0$ and $t_f/2$, etc. The observed behaviour is that which would be expected on intuitive grounds. As the 'amount' of feedback is increased, the control on $x_1(t_f)$ becomes tighter, but there comes a point where increased feedback gives negligible improvement. The estimated mean and variance data are given in Table 3.4. Again, the expected results have been observed.

Increasing the amount of feedback leads to an improvement in both the mean and variance of $x(t_f)$. The modelling error is shown in Table 3.5, and is

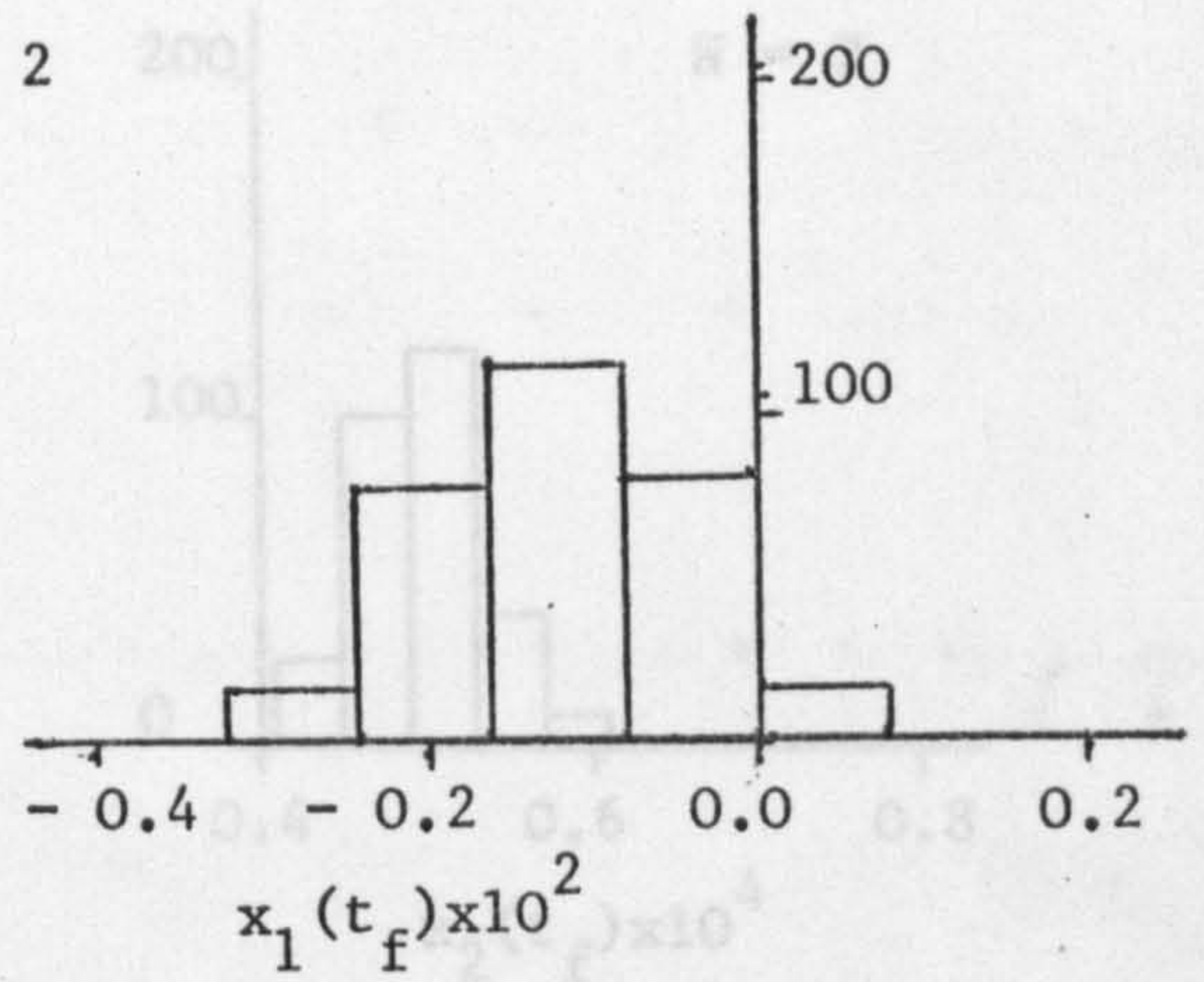
FIGURE 3.2a

Histograms for $x_1(t_f)$ -Variance : 8.33

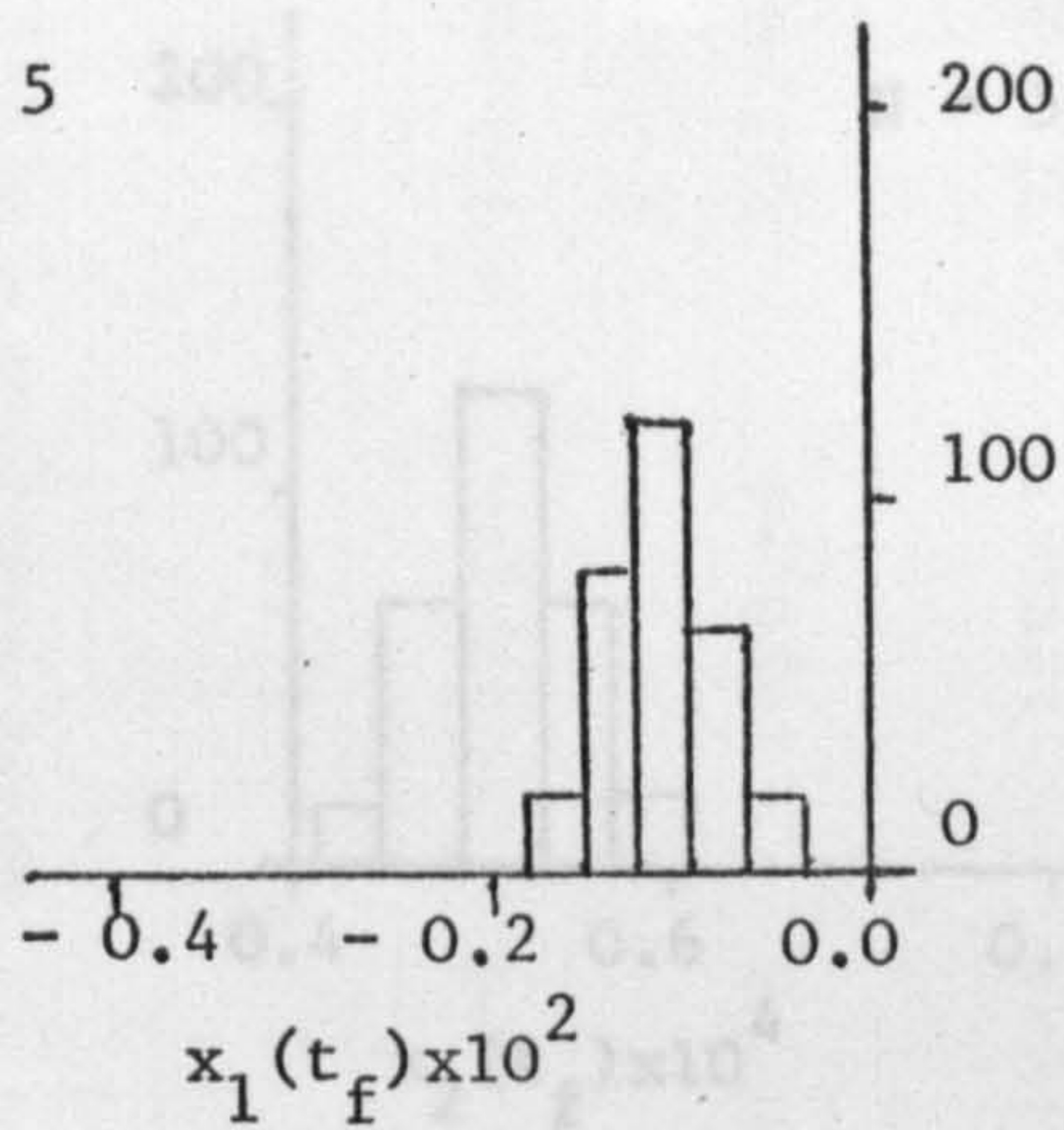
N = 1



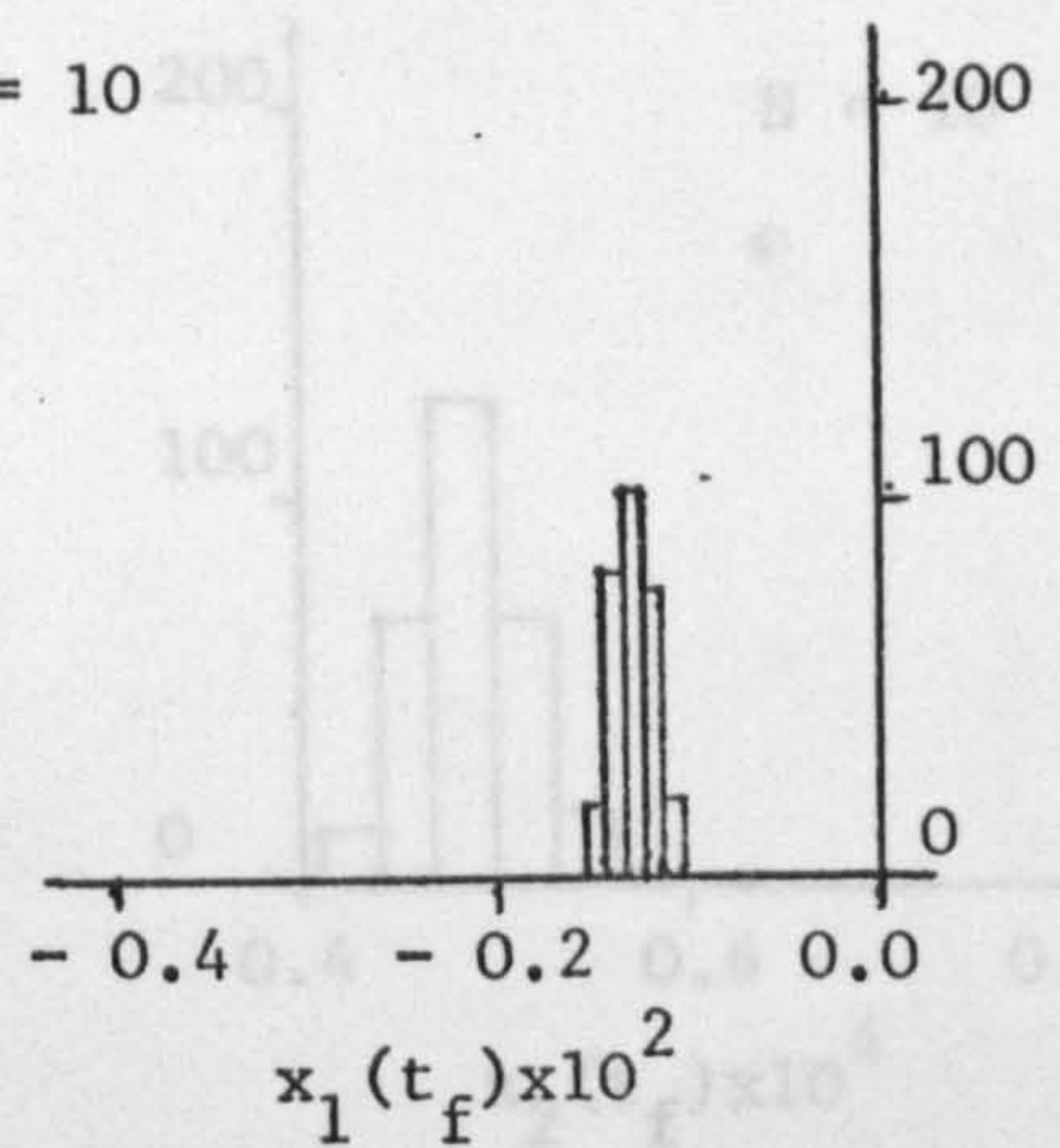
N = 2



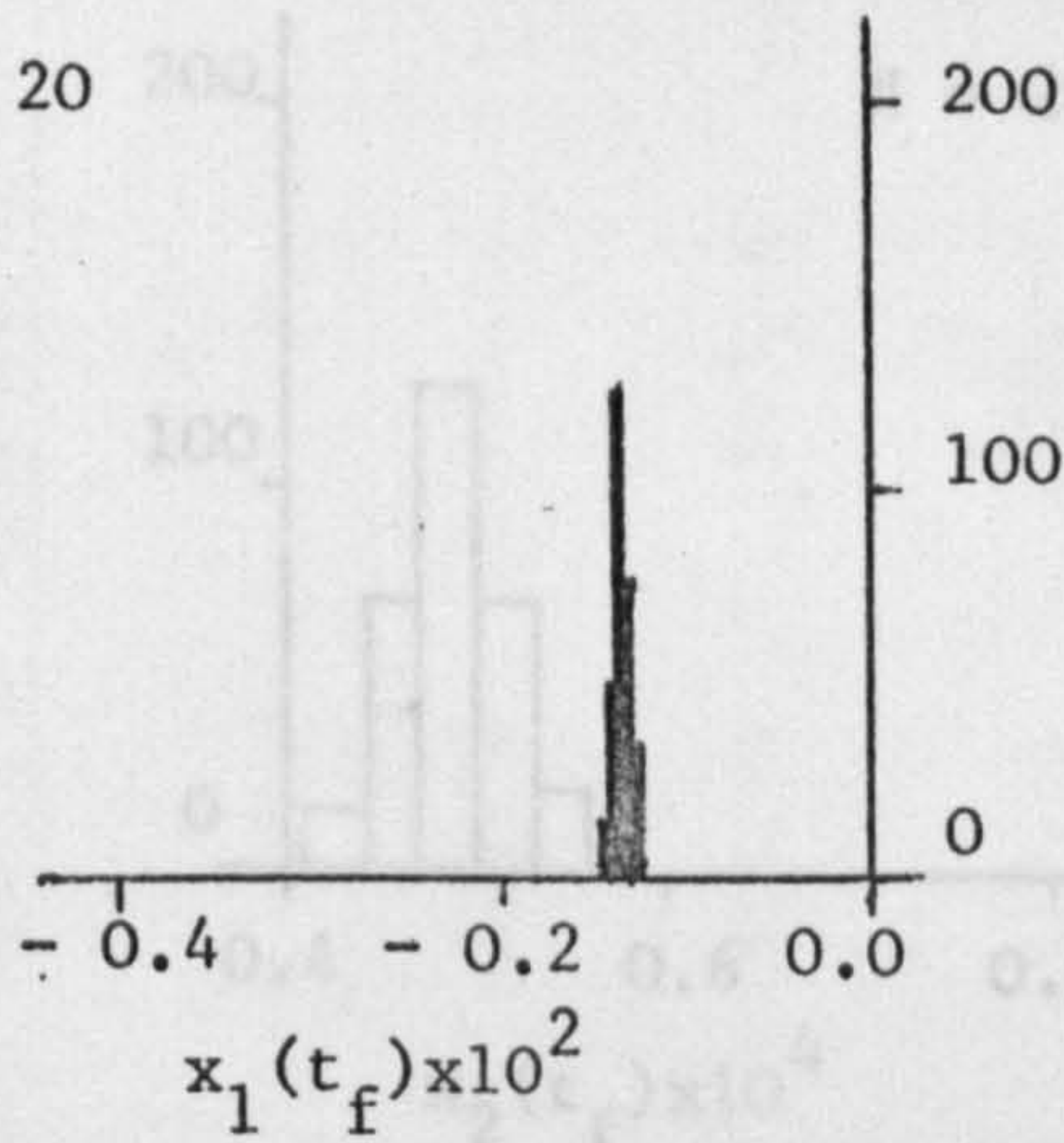
N = 5



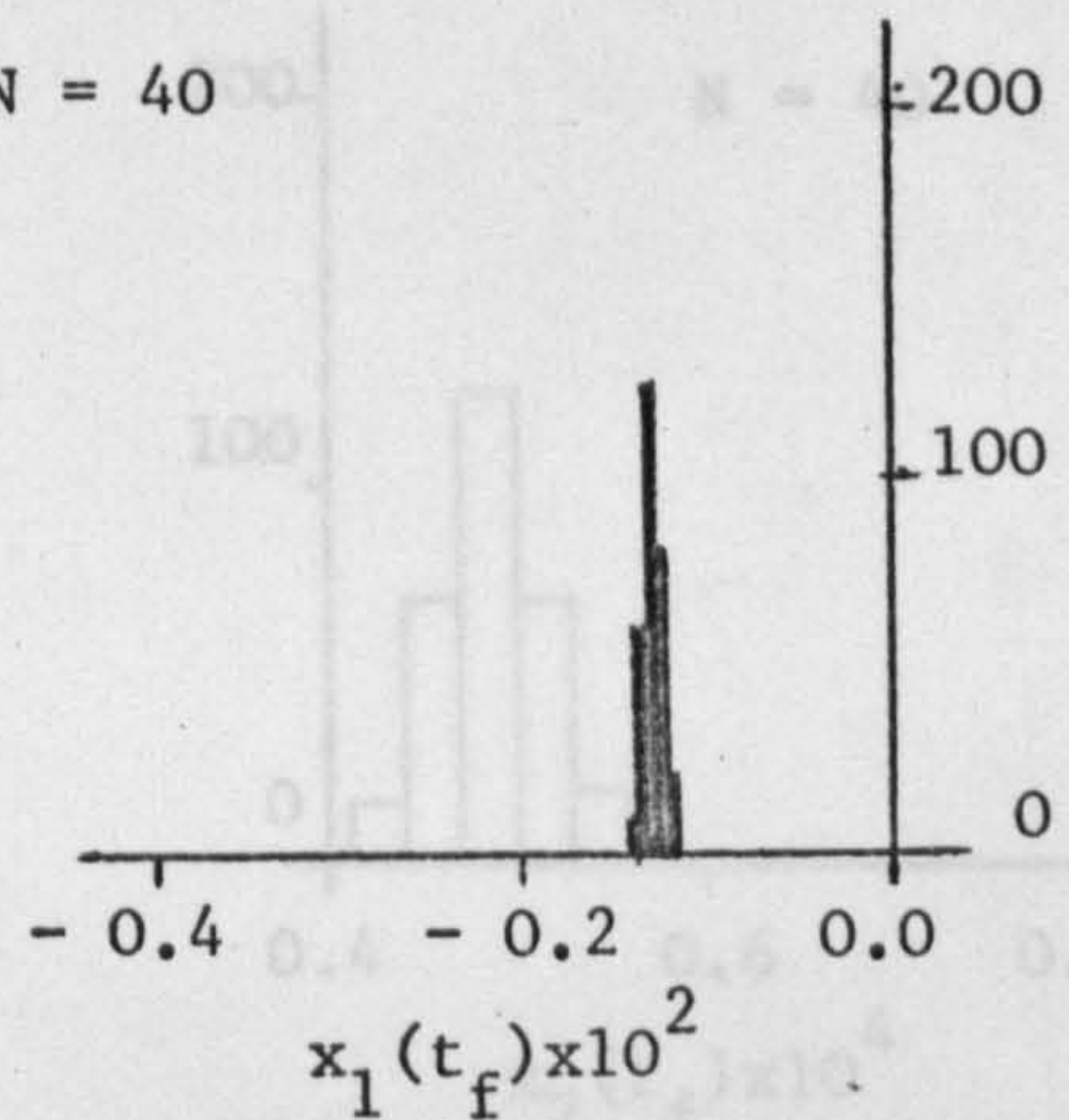
N = 10



N = 20



N = 40



N = ∞

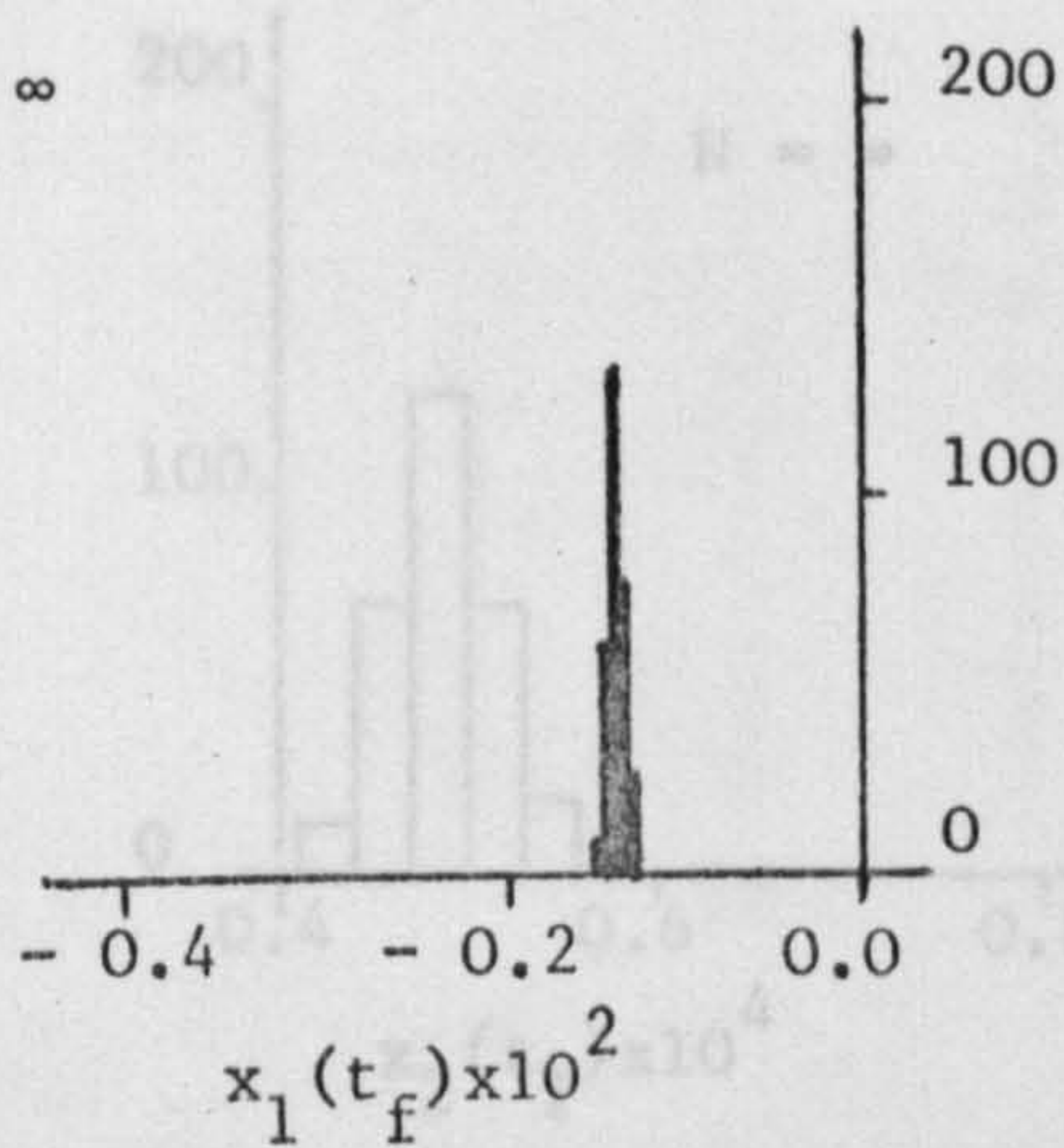


FIGURE 3.2b

Histograms for $x_2(t_f)$ - Variance : 8.33

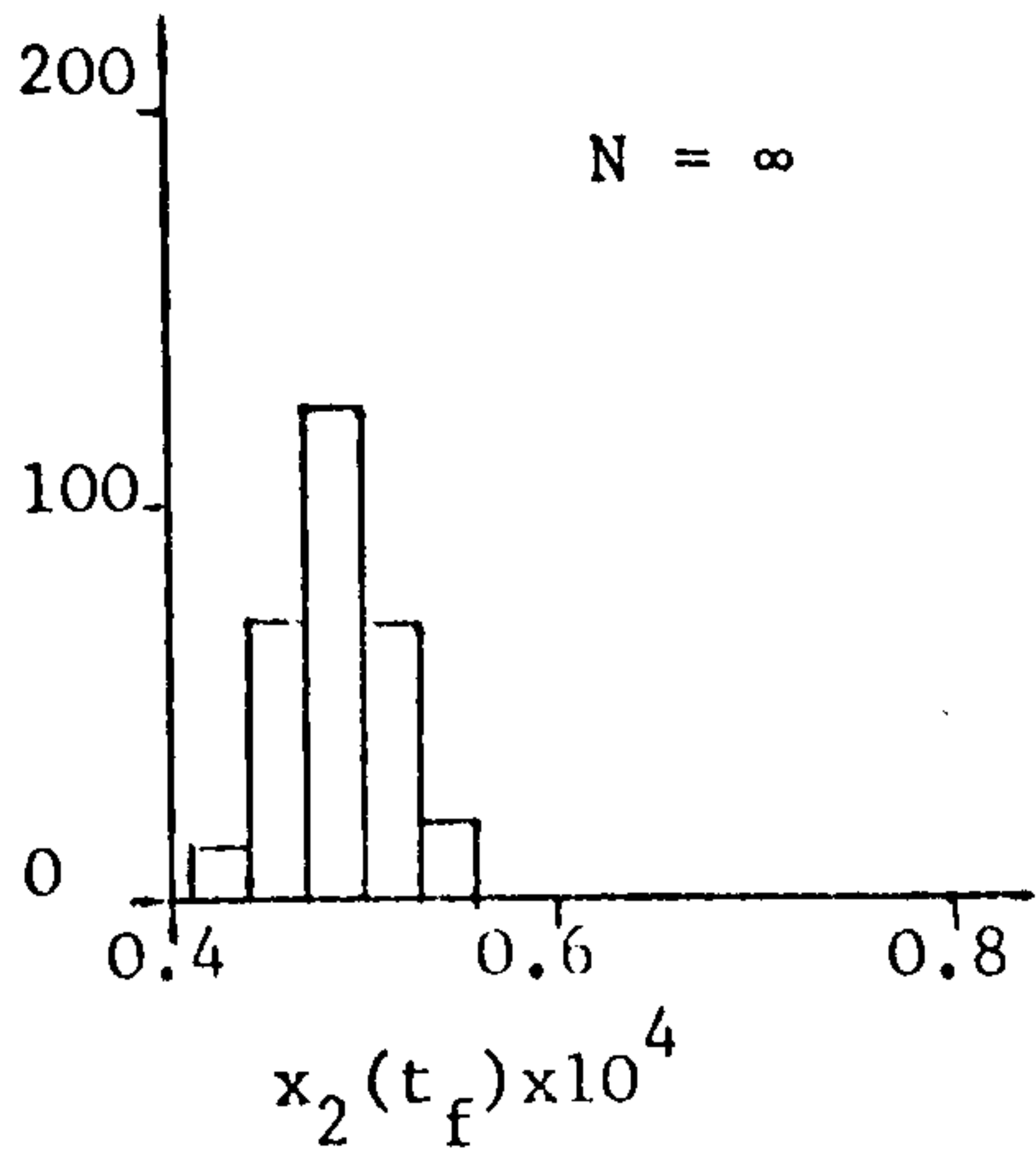
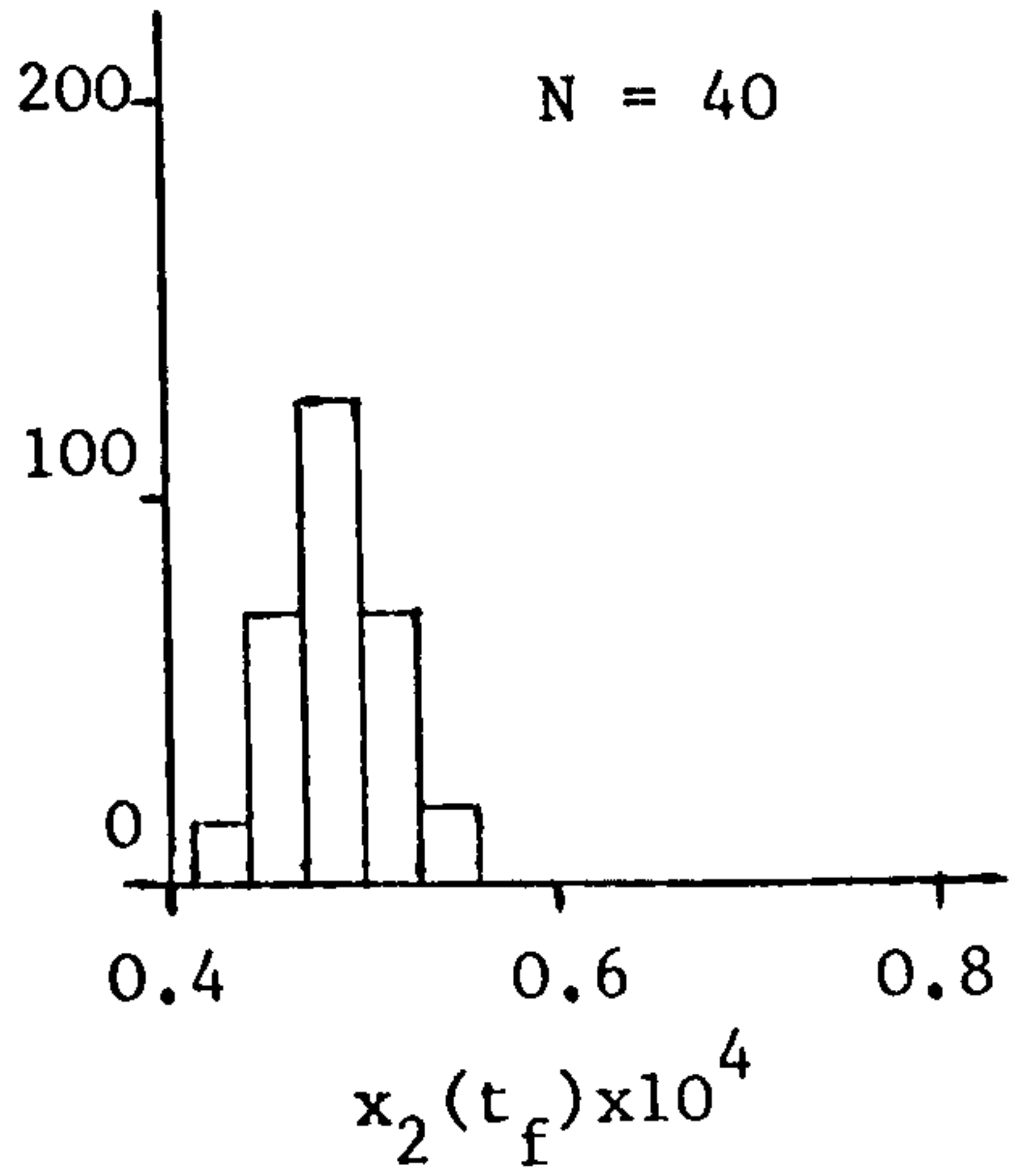
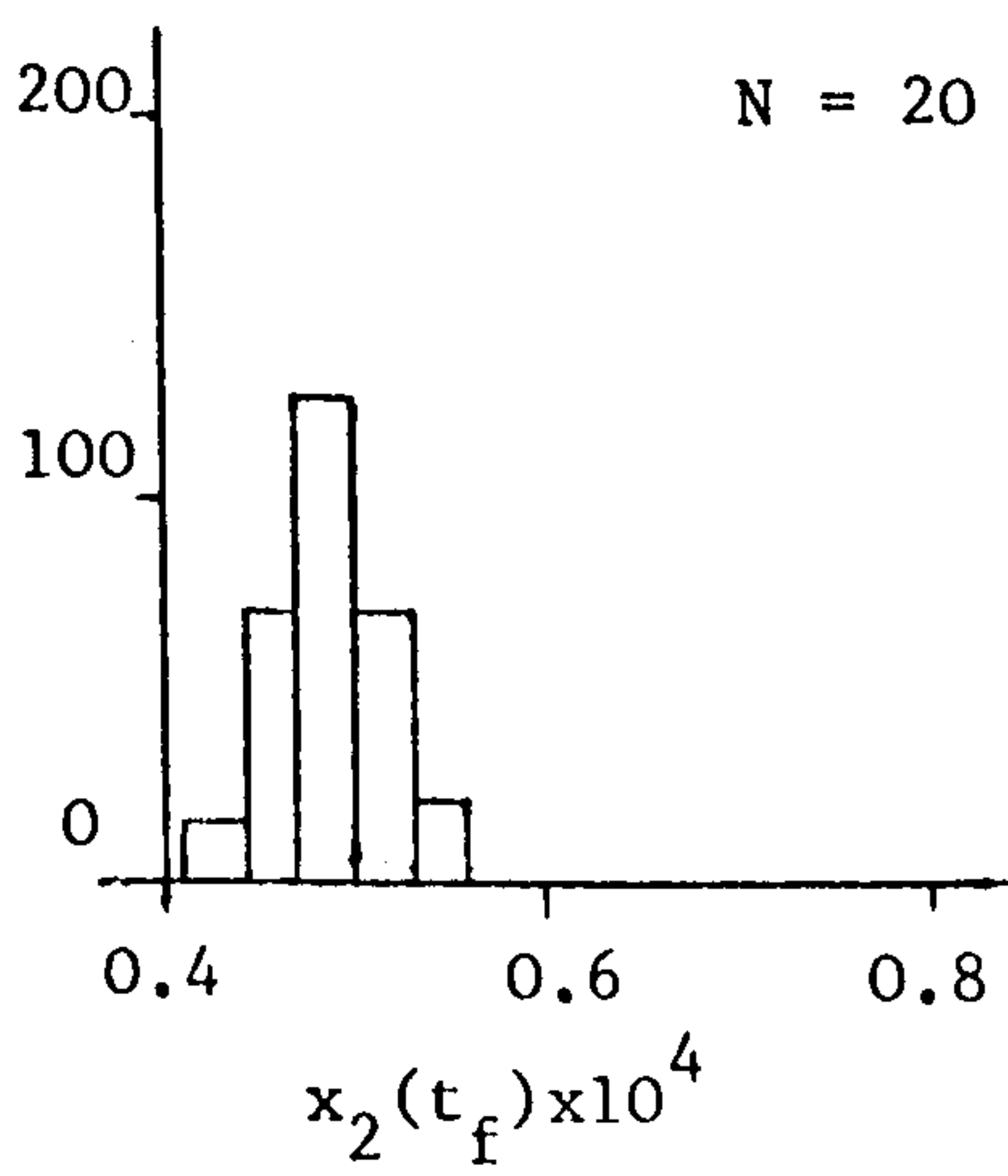
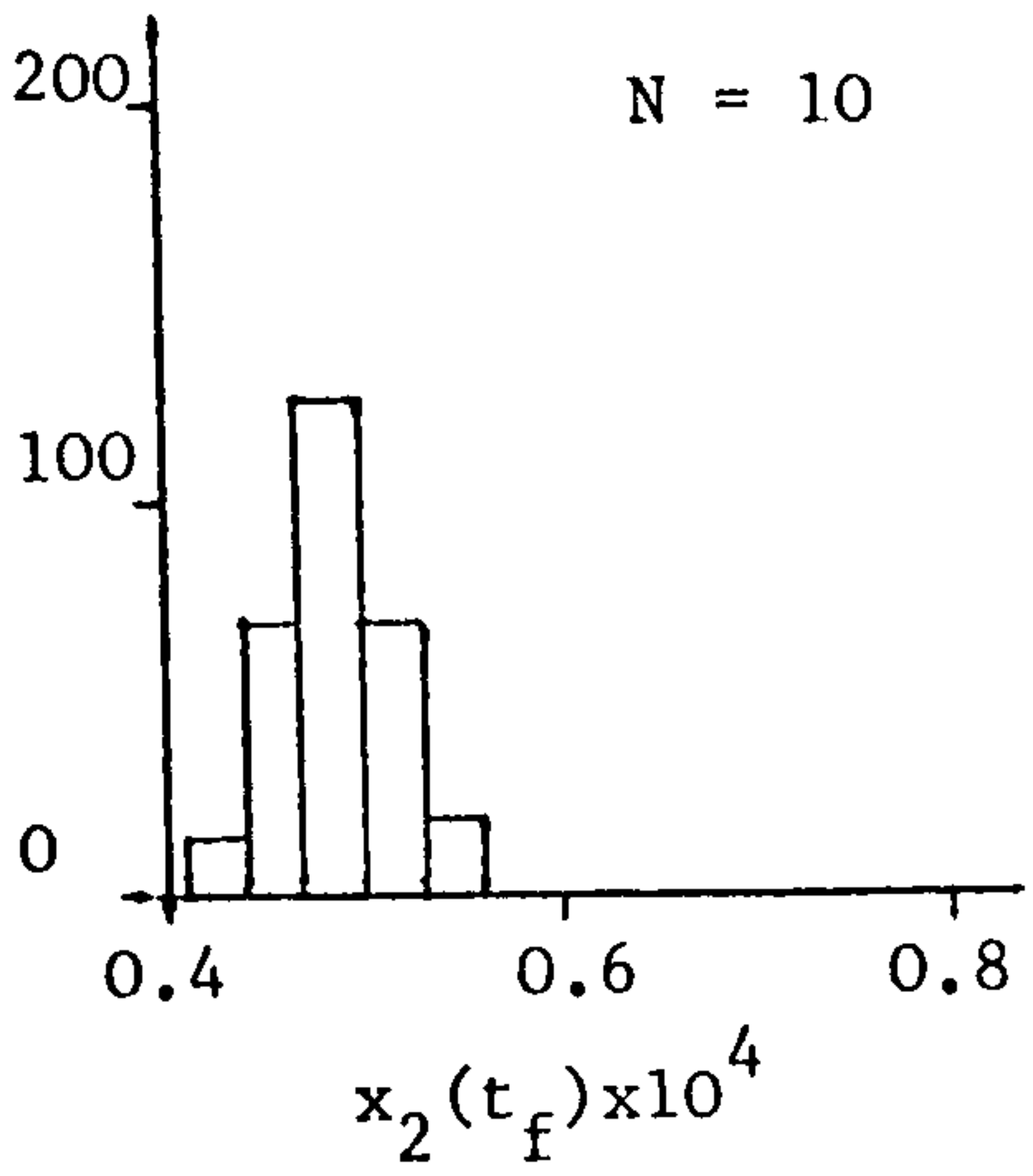
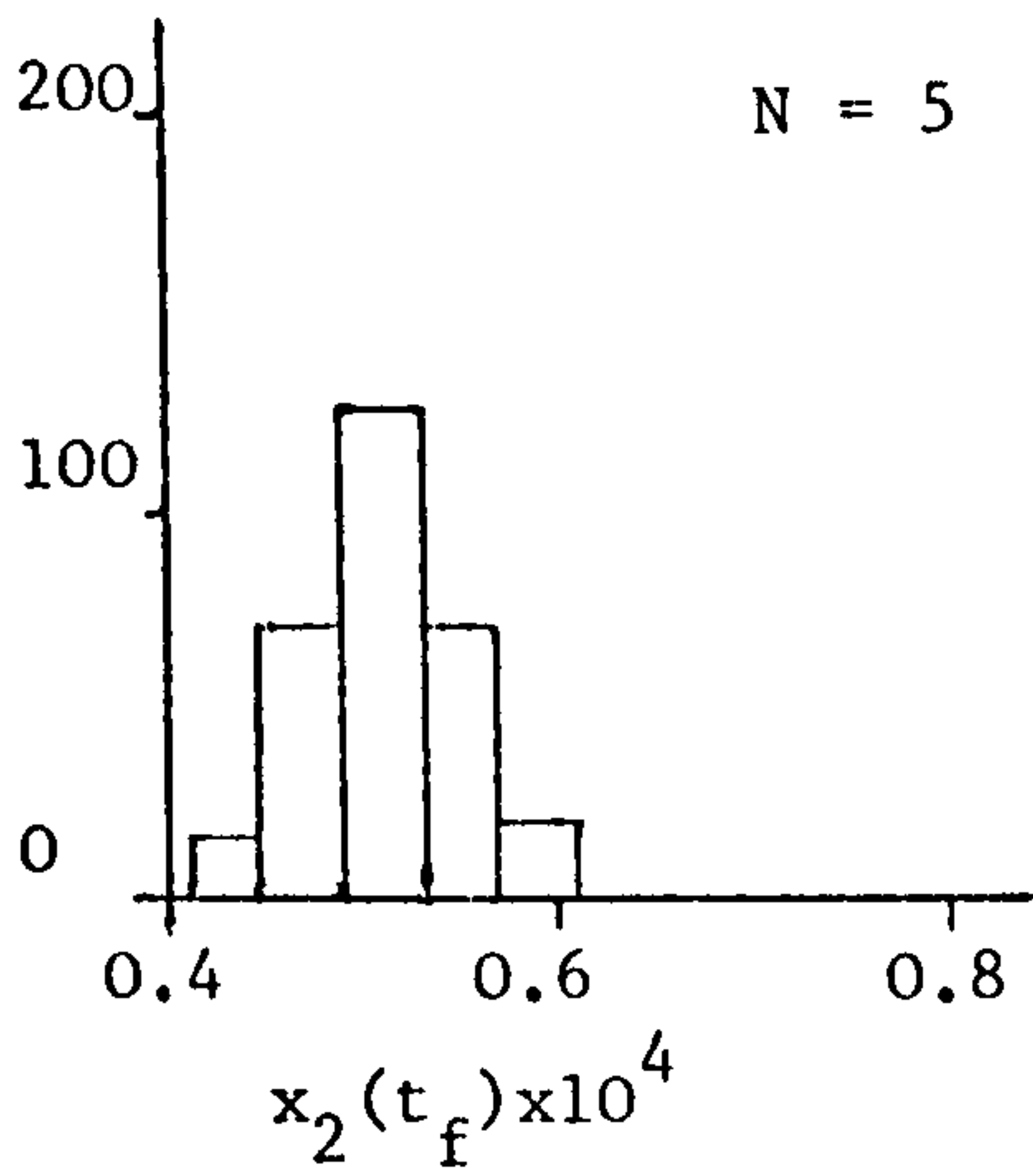
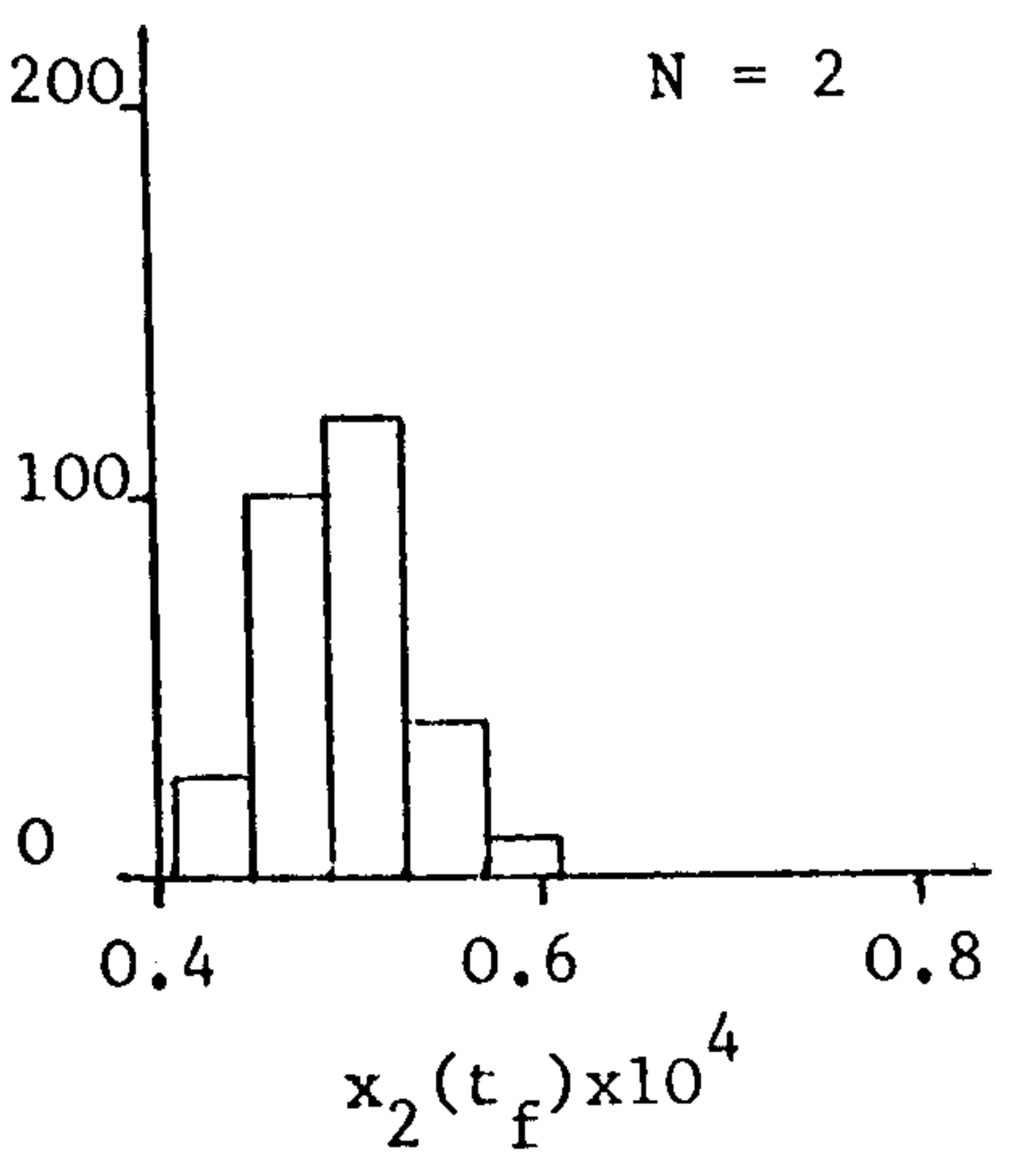
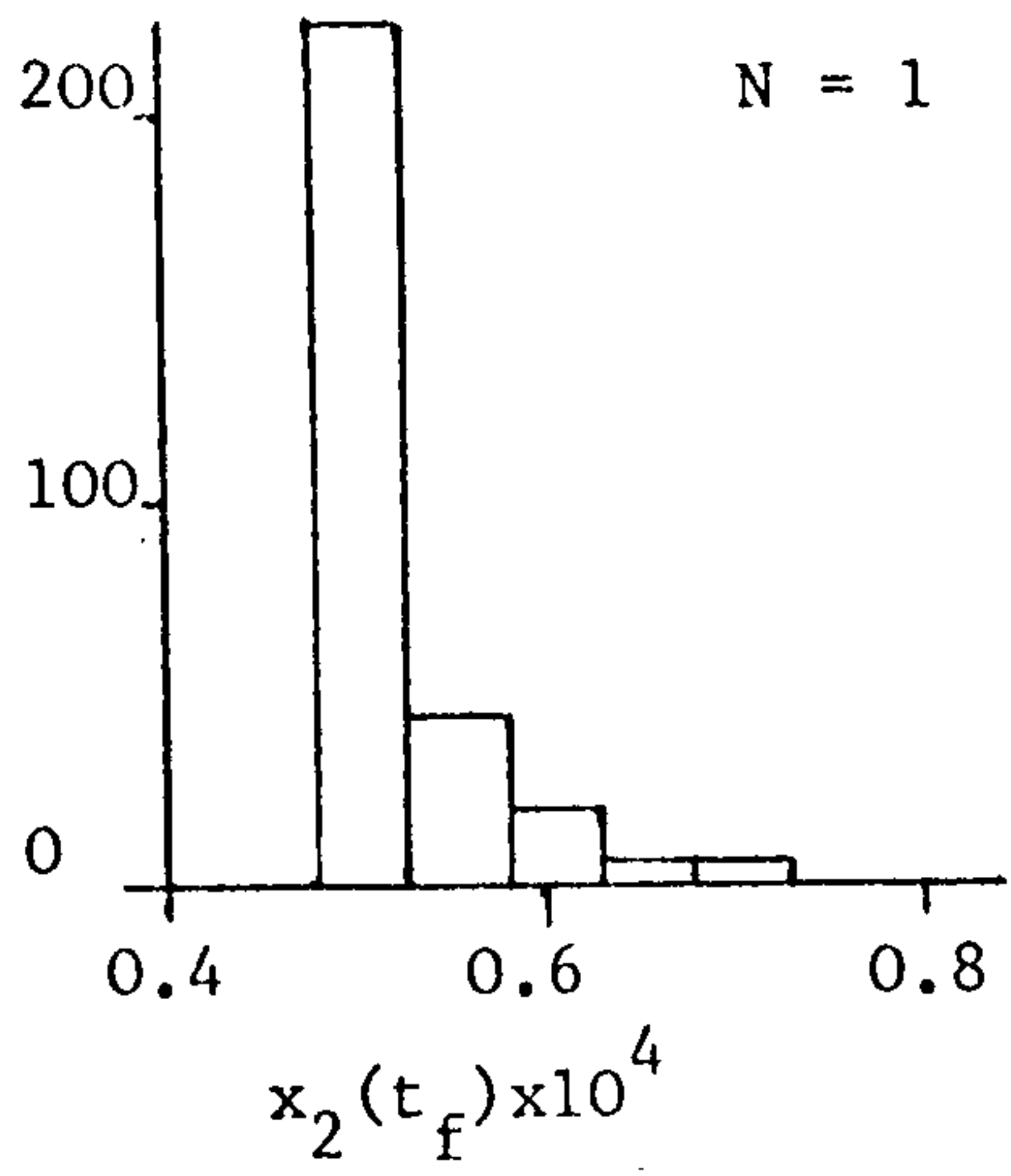


TABLE 3.4 Discrete Feedback Control

<u>Estimated mean and Variance of $x(t_f)$</u>		(Noise variance = 8.33)						
N		1	2	5	10	20	40	∞
Mean of $x_1(t_f)$		-.125,-2	-.124,-2	-.121,-2	-.121,-2	-.121,-2	-.121,-2	-.121,-2
Mean of $x_2(t_f)$		.518,-4	.503,-4	.499,-4	.498,-4	.498,-4	.498,-4	.498,-4
Variance of $x_1(t_f)$		.210,-5	.520,-6	.559,-7	.129,-7	.527,-8	.349,-8	.301,-8
Variance of $x_2(t_f)$		.228,-10	.135,-10	.125,-10	.124,-10	.124,-10	.124,-10	.124,-10

consistent with the variance data of Table 3.4, and with arguments given above.

TABLE 3.5 Discrete Feedback Control
Modelling Error in Second-Order Approximation
 (Noise variance = 8.33)

N		1	2	5	10	20	40
Modelling	x_1	3.2	2.4	0.8	0.0	0.0	0.0
error (%)	x_2	4.3	1.4	0.6	0.4	0.4	0.4

We have now seen demonstrations of all the results of chapter two. The difference between the $O(\epsilon^3)$ bound for feedback control and the $O(\epsilon)$ bound for feedforward control is clearly shown in Fig. 3.1, whilst Fig. 3.2 shows the effect of the sampling interval Δt in reducing $O(\epsilon)$ to $O(\epsilon^3)$. The $O(\epsilon^3)$ bound for the difference between the optimal stochastic performance and the performance of the optimal deterministic policy for the second-order system cannot be so clearly demonstrated. The small spread of final state values and the proximity of the mean performance to the optimal deterministic performance give some indication of the degree of approximation.

A practical point associated with the implementation of the control design method proposed is its sensitivity to the estimated noise variance used in the second-order approximation. So far, we have used the exact noise variance for control design, but we may consider the variation of n in the equation:-

$$q_{\text{design}} = n q_{\text{actual}} \tag{3.2.17}$$

The effect of n on controller performance for the case of feedback control is shown in Table 3.6. We see that control performance is insensitive to the noise variance used for design. The prediction error in the second-order approximation is shown in Table 3.7. The errors are of the same order as those for feedforward control using the correct variance. However, it should be remembered that, in contrast to the feedforward case, these errors represent an upper bound on the errors in the information used by the controller. Most of the predictions are based on a shorter 'time to go' than that used in Table 3.7. It is the constant updating of predictions which leads to good controller performance.

TABLE 3.6 Continuous Feedback Control
Estimated Mean and Variance of $x(t_f)$ as a Function of n
(Noise Variance = 8.33)

n	0	1	2
Mean of $x_1(t_f)$	-.119,-2	-.121,-2	-.124,-2
Mean of $x_2(t_f)$	.498,-4	.498,-4	.498,-4
Variance of $x_2(t_f)$	.289,-8	.301,-8	.314,-8
Variance of $x_2(t_f)$	.122,-10	.124,-10	.127,-10

We may contrast the above results with those for feedforward control, shown in Tables 3.8, 3.9.

TABLE 3.7 Continuous Feedback Control
Error in Prediction of Average Final State using Design Variance
Value (Noise variance = 8.33)

n	0	1	2
Prediction x_1	3.4	0.0	4.0
Error (%) x_2	5.0	0.4	5.4

TABLE 3.8 Feedforward Control
Estimated Mean and Variance of $x(t_f)$ as a Function of n
(Noise variance = 8.33)

n	0	1	2
Mean of $x_1(t_f)$	-.179,-3	-.125,-2	-.231,-2
Mean of $x_2(t_f)$	.529,-4	.518,-4	.531,-4
Variance of $x_1(t_f)$	.207,-5	.210,-5	.214,-5
Variance of $x_2(t_f)$	.891,-11	.228,-10	.562,-10

TABLE 3.9 Feedforward Control
Error in Prediction of Average Final State using Design Variance
Value (Noise variance = 8.33)

n	0	1	2
Prediction x_1	574.0	3.2	48.5
error (%) x_2	1.1	4.3	11.3

The variance of $x(t_f)$ is seen to be insensitive to design error. This is hardly surprising, since feedforward control can have little effect on the final state variance anyway. However, the mean performance is strongly affected by errors in control design. If one is only concerned with mean performance, then, for this example, it seems that a deterministic design ($n = 0$) is best- as n is increased the average control performance deteriorates. It looks as if n is a valuable tuning parameter for the feedforward controller. However, a feedforward control design should not be based solely on the predictions of the second-order model as Table 3.9 shows. Even if the noise variance is accurately known, these predictions can be subject to non-negligible errors (see Table 3.10).

TABLE 3.10 Feedforward Control
Error in Prediction of Average Final State using Actual Noise
Variance Value (= 8.33)

n		0	1	2
Prediction	x_1	23.5	3.2	1.7
error (%)	x_2	4.0	4.3	4.3

3.2.4 Conclusions

The conclusions one draws from the above study depend upon the type of control one is interested in designing. If large deviations from desired conditions are not undesirable, then it is clear that the complication of feedback control is not necessary. Indeed, this control may be too tight, giving less desirable mean performance as a trade-off against a lower final state variance. However, if consistency is required from the process, then feedforward control is inadequate. This does not mean that continuous feedback is essential: sufficiently good results may be obtained by incorporating a few updates of a feedforward policy into the controller.

The use of the second-order model as a design tool to predict the average performance of a process can be advantageous in terms of computation time, but, for feedforward control, quite large errors in prediction were observed. This must throw some doubt on the design method proposed by Joffe and Sargent⁽¹⁾, except for very small disturbances.

3.3 Control of a Batch Reactor Subject to Catalyst Deactivation

3.3.1 Problem Description and Solution Method

We consider a special case of the control problem treated by Szepe and Levenspiel⁽⁶⁶⁾. We have a perfectly mixed batch reactor in which the reaction



is to take place. The rate of production of B per mole of reaction mixture is given by

$$r_B = k_1 x_2 x_1^2 \quad (3.3.2)$$

where x_1 is the mole fraction of A in the mixture

x_2 is the catalyst activity

and k_1 is a rate constant, showing the usual Arrhenius temperature dependence:-

$$k_1 = \exp(\beta_1 - \beta_2/T) \quad (3.3.3)$$

where T is the reactor temperature.

The catalyst decay rate is governed by the following equation:-

$$\frac{dx_2}{dt} = -k_2 x_2 \quad (3.3.4)$$

where k_2 is a rate constant satisfying

$$k_2 = \exp(\beta_3 - \beta_4/T) \quad (3.3.5)$$

A mass balance on component A yields:-

$$\frac{dx_1}{dt} = -k_1 x_2 x_1^2 \quad (3.3.6)$$

The (deterministic) control problem considered by Szepe and Levenspiel is as follows. Given the initial catalyst activity, x_{20} , the initial mole fraction of A, x_{10} , a specified final activity, x_{2f} , and reaction time, t_f , find the temperature policy which minimises $x_1(t_f)$ (i.e. which maximises the conversion of A). The final activity and reaction time are assumed to be chosen optimally, taking account of catalyst regeneration costs, running costs, etc. Szepe and Levenspiel obtained the following solution to this problem using variational calculus:-

$$T(t) = \frac{1}{A + B \ln(c - t/t_f)} \quad (3.3.7)$$

where

$$\begin{aligned}
 A &= \frac{1}{\beta_4} \left[\ln \left(\frac{\beta_4 t_f}{\beta_2} \right) + \beta_3 \right] \\
 B &= 1/\beta_4 \\
 C &= \frac{1}{1 - (x_{2f}/x_{20})^{\beta_4/\beta_2}}
 \end{aligned}
 \tag{3.3.8}$$

If we consider that the actual reactor temperature is the desired temperature perturbed by an additive noise process, and that the regeneration and initial reactor filling operations are subject to uncertainty causing uncertainties in x_{10} and x_{20} , then we may formulate a stochastic control problem analogous to the problem of Szepe and Levenspiel. Given the expected initial catalyst activity, x_{20}^* , the expected initial mole fraction of A, x_{10}^* , and the reaction time, t_f , find the control policy, $u(t)$, which minimises $E\{x_1(t_f) \mid \text{all information up to } t\}$ subject to the constraint that $E\{x_2(t_f) \mid \text{all information up to } t\}$ be equal to a specified value x_{2f}^* . In this chapter we will assume that $x_1(t)$ and $x_2(t)$ may be measured exactly at each t during the operation of the reactor. In chapter four we shall relax this somewhat unrealistic assumption. For the present, at any rate, we have a stochastic optimal control problem with perfect observation of the state, and the results of chapter two tell us that a good approximation to the optimal control law is to use the feedback policy for the deterministic system obtained by setting all noise terms to their expected values. (Note that we are not using the second-order model here, so that the approximation is $O(\epsilon^2)$ as opposed to $O(\epsilon^3)$). We may obtain the optimal feedback law for this system from eqns. (3.3.7), (3.3.8) by making the following transformations:-

Let τ be the current time, then set

$$\begin{aligned}
 t &= 0 \\
 x_{20} &= x_2(\tau) \\
 t_f &= t_f - \tau
 \end{aligned}
 \tag{3.3.9}$$

in eqns. (3.3.7), (3.3.8). This yields the following control to be applied at time t when the observed catalyst activity is $x_2(t)$

$$u(t) = \frac{1}{A^* + B \ln C^*}
 \tag{3.3.10}$$

where

$$A^* = \frac{1}{\beta_4} \left[\ln \left\{ \frac{\beta_4}{\beta_2} (t_f - t) \right\} + \beta_3 \right] \quad (3.3.11)$$

$$C^* = \frac{1}{1 - [x_{2f}/x_2(t)]^{\beta_4/\beta_2}}$$

3.3.2 Simulation Procedure and Results

We wish to simulate the following stochastic system:-

$$\begin{aligned} \frac{dx_1}{dt} &= -k_1 x_2 x_1^2 & : & \quad x_1(0) = x_{10}^* + \chi_1 \\ \frac{dx_2}{dt} &= -k_2 x_2 & : & \quad x_2(0) = x_{20}^* + \chi_2 \end{aligned} \quad (3.3.12)$$

with

$$\begin{aligned} k_1 &= \exp(\beta_1 - \beta_2/T) \\ k_2 &= \exp(\beta_3 - \beta_4/T) \end{aligned} \quad (3.3.13)$$

and

$$T(t) = u(t) + \xi(t) \quad (3.3.14)$$

We are interested in applying similar types of control to the simulated system as those considered in section 3.2. Feedforward control may be obtained from eqns. (3.3.7), (3.3.8) with x_{20} replaced by x_{20}^* . Eqns. (3.3.7), (3.3.8) may also be used for discrete feedback control, the parameters A and C being updated whenever a new state measurement becomes available. For continuous feedback control, eqns. (3.3.10), (3.3.11) may be used, except when t approaches t_f , when the control problem becomes ill-posed. To overcome this difficulty, a feedforward control scheme was applied over the last integration step.

A Monte Carlo simulation approach was adopted to estimate the mean and variance of the final state of the system. The state equations, eqns. (3.3.12), were integrated numerically using a fourth-order Runge-Kutta procedure. Realisations of the random variables, χ_1 and χ_2 , and of the random process, $\xi(t)$, were generated using a random number generator giving a zero-mean rectangularly distributed variable. The random process $\xi(t)$ was assumed to be piecewise constant over the Runge-Kutta integration steps. Single precision arithmetic on a CDC 6400 computer was used to perform all calculations.

The following parameters were used in this study:-

$$\begin{aligned}
 t_f &= 8.0 \text{ hour} \\
 x_{10}^* &= x_{20}^* = 0.95 \\
 x_{2f}^* &= 0.1 \\
 \beta_1 &= 34.3 \\
 \beta_2 &= 12000.0 \text{ K} \\
 \beta_3 &= 42.6 \\
 \beta_4 &= 15000.0 \text{ K}
 \end{aligned}$$

Step length for integration = 0.4 hour

Number of simulations per run = 300

The deterministic optimal control, and the corresponding state trajectories for this parameter set are shown in Fig. 3.3. A monotonically increasing reactor temperature is observed, as predicted by Szepe and Levenspiel. We note that the chosen value of x_{2f}^* is in the attainable set for the deterministic system.

A comparison between feedforward and continuous feedback control is made in Table 3.11 for a range of values of b_I (the maximum deviation of the initial states from their expected values), and in Table 3.12 for a range of values of b_p (the maximum value of $\xi(t)$). In contrast to the last example, we can see that the addition of process noise causes a deterioration in mean performance (the deterministic value of $x_1(t_f)$ being 0.402). However, the magnitude of the deterioration does not appear to be very significant for

TABLE 3.11 Comparison of Feedforward (FF) and Feedback (FB) Control Schemes
($b_p = 2.0 \text{ K}$)

b_I		0.00	0.05	0.10
Mean of $x_1(t_f)$	FF	0.403	0.404	0.405
	FB	0.403	0.404	0.405
Variance of $x_1(t_f)$	FF	0.193,-5	0.798,-4	0.316,-3
	FB	0.901,-6	0.864,-4	0.344,-3
Mean of $x_2(t_f)$	FF	0.985,-1	0.984,-1	0.983,-1
	FB	0.100	0.100	0.100
Variance of $x_2(t_f)$	FF	0.872,-4	0.987,-4	0.128,-3
	FB	0.397,-4	0.396,-4	0.395,-4

the noise levels considered here. In terms of $x_1(t_f)$ there appears to be little to choose between feedforward and feedback control schemes. In fact, the performance of the feedback controller is seen to be slightly worse than

an open loop scheme with respect to both the mean and the variance of $x_1(t_f)$. The feedback scheme clearly does better in controlling $x_2(t_f)$ as Fig. 3.4 shows. The mean value of this variable is seen to satisfy the desired final state constraint, for all cases except the most noisy, and the variance of $x_2(t_f)$ is approximately halved when feedback is introduced. It is interesting to note that the variance of $x_2(t_f)$ is completely insensitive to the value of b_I when feedback control is used.

TABLE 3.12 Comparison of Feedforward (FF) and Feedback (FB) Control Schemes
($b_I = 0.05$)

$b_p(K)$		0.0	2.0	5.0
Mean of $x_1(t_f)$	FF	0.404	0.404	0.405
	FB	0.404	0.404	0.406
Variance of $x_1(t_f)$	FF	0.781,-4	0.798,-4	0.885,-4
	FB	0.846,-4	0.864,-4	0.925,-4
Mean of $x_2(t_f)$	FF	0.998,-1	0.984,-1	0.898,-1
	FB	0.100	0.100	0.995,-1
Variance of $x_2(t_f)$	FF	0.931,-5	0.987,-4	0.482,-3
	FB	0.274,-10	0.396,-4	0.207,-3

The effect of feedback is to continually reset the control so as to achieve $E\{x_2(t_f)\} = 0.1$, and this will be possible provided the value 0.1 remains within the attainable set for the deterministic system, since modelling error is so small. The variance of $x_2(t_f)$ is caused mostly by the process noise to which it can be seen to be very sensitive (see Table 3.12).

The effect of discrete feedback is shown in Table 3.13. The variable N represents the number of state observations per run ($N = 1$ corresponds to one at $t = t_f/2$ etc.). Again $x_1(t_f)$ is seen to be insensitive to the type of control applied. For $x_2(t_f)$, we see that increased feedback is beneficial, both in terms of mean performance and in terms of variance.

3.3.3 Conclusions

The example demonstrates the effect of imposing final state constraints. A feedback control scheme is seen to be better able to satisfy such a constraint, and the variance of the constrained variable also shows improvement over an open-loop controller. However, for this example, the performance index, $E\{x_1(t_f)\}$, shows little variation over a whole range of different control schemes, and the introduction of feedback increases the variance of this variable. This appears to be the trade-off for better control on $x_2(t_f)$.

FIGURE 3.3 Optimal Deterministic Trajectory for the Chosen Parameter Set

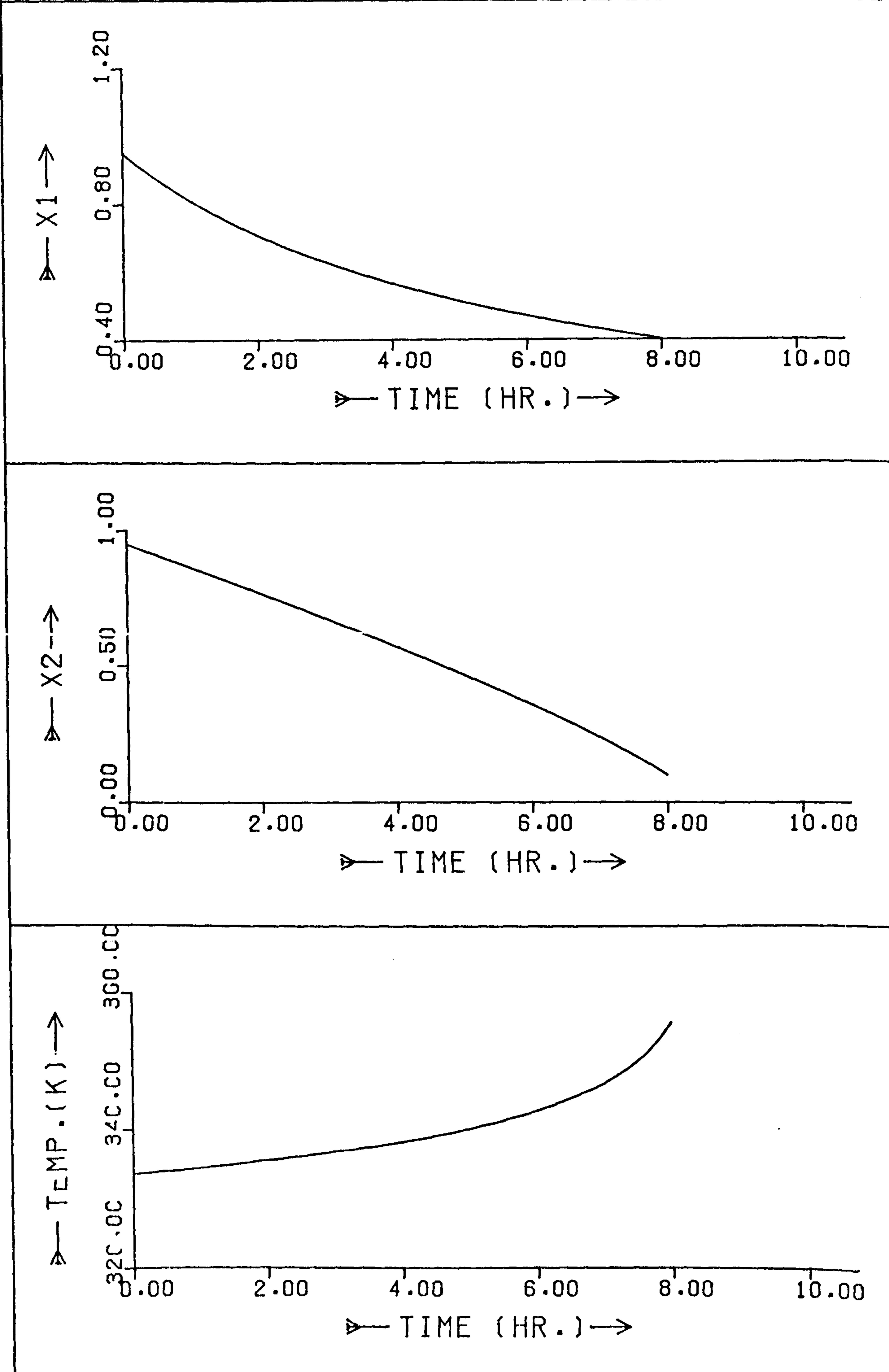


FIGURE 3.4 Histograms for $x_2(t_f)$: $b_p = 2.0$; $b_I = 0.05$

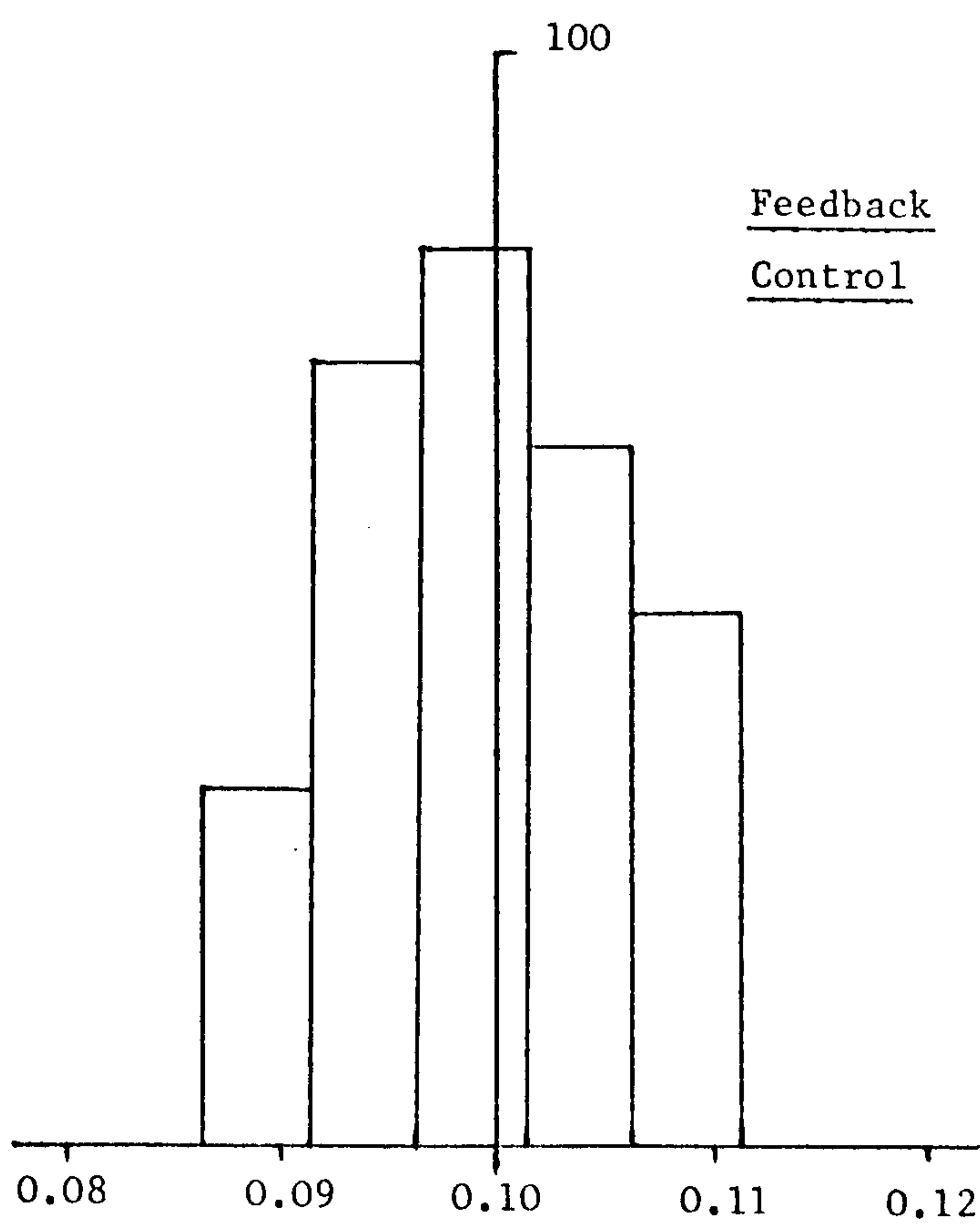
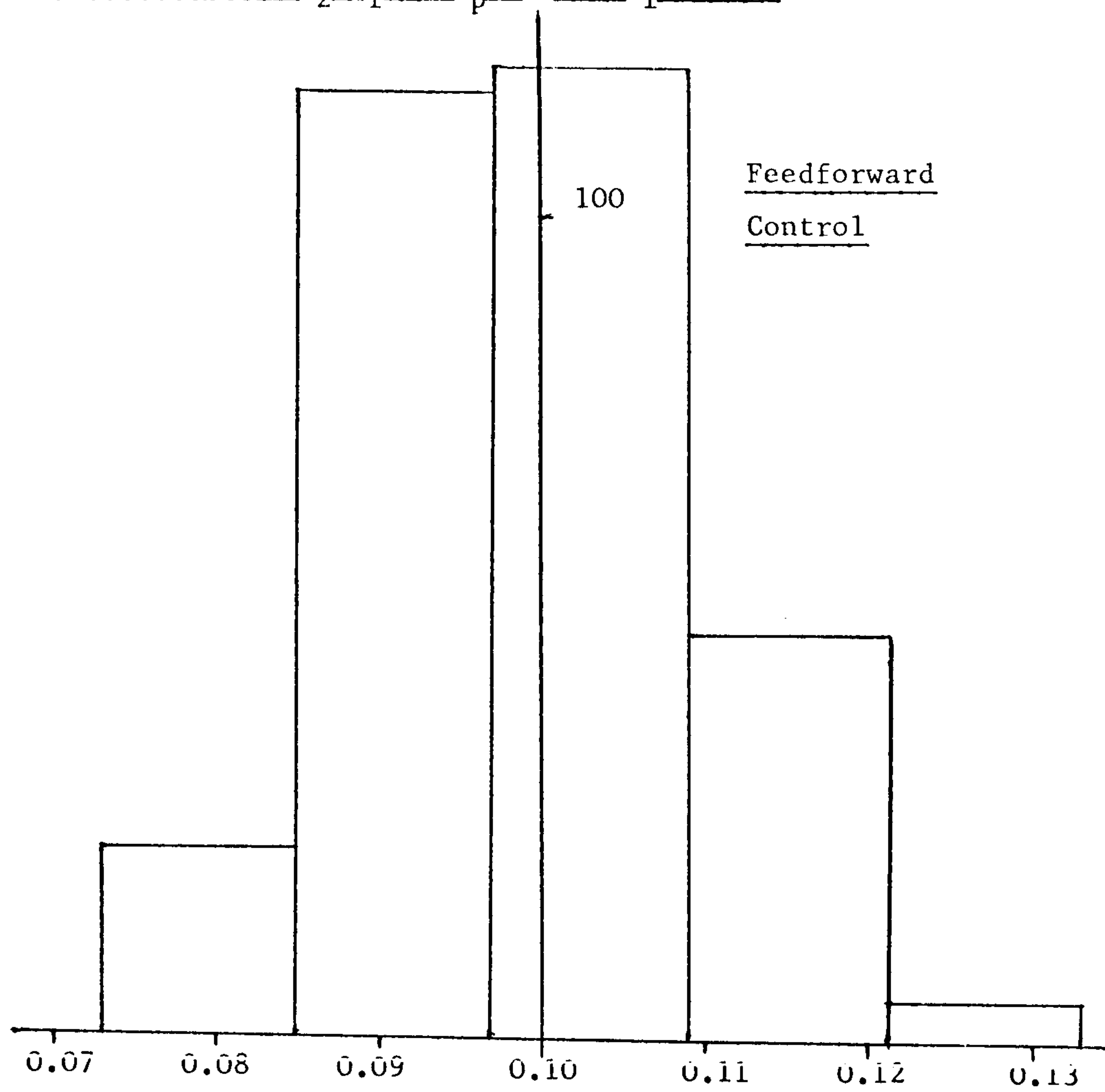


TABLE 3.13 Comparison of Different Discrete Feedback Control Schemes ($b_p = 2.0, b_I = 0.05$)

N	0	1	2	5	10	20	∞
Mean of $x_1(t_f)$	0.404	0.404	0.404	0.404	0.404	0.404	0.404
Variance of $x_1(t_f)$	0.798,-4	0.862,-4	0.864,-4	0.877,-4	0.870,-4	0.864,-4	0.864,-4
Mean of $x_2(t_f)$	0.984,-1	0.985,-1	0.991,-1	0.995,-1	0.100	0.100	0.100
Variance of $x_2(t_f)$	0.987,-4	0.869,-4	0.818,-4	0.736,-4	0.575,-4	0.396,-4	0.396,-4

C H A P T E R F O U R

OPTIMAL CONTROL BASED ON NOISY MEASUREMENTS

4.1 Scope

In this chapter we return to the general problem formulated in chapter one:-

P1. Given

$$\text{System} \quad \frac{d}{dt} x = f[t, x, u(t), \xi(t)] : t \in [0, t_f] \quad (4.1.1)$$

$$\text{Initial condition} \quad x(0) \in X_0$$

$$\text{Observations} \quad y(t_i) = g[t_i, x(t_i), \eta(t_i)] \quad (4.1.2)$$

$$\text{Admissible controls} \quad u(t) \in U$$

$$u(t) = h(t, Y_t) \quad (4.1.3)$$

$$Y_t = \{y(t_i) : t_i \in [0, t]\}$$

$$\text{Performance index} \quad J(t, Y_t) = E\{c^T x(t_f) | (t, Y_t)\} \quad (4.1.4)$$

Problem

$$\min_h J(t, Y_t) \quad \text{for } \forall t \in [0, t_f]$$

4.2 Previous Work

As in the case where the state is perfectly observable, previous work on problem P1 has been concentrated on Markov models:-

$$\begin{aligned} dx(t) &= f(t, x, y, u)dt + \sigma_1(t, x, y)dw_1(t) \\ dy(t) &= g(t, x, y, u)dt + \sigma_2(t, x, y)dw_2(t) \end{aligned} \quad (4.2.1)$$

and a dynamic programming approach has usually been adopted. However, the application of dynamic programming cannot be as formal as in the case of perfect state observation, since past measurements may no longer be ignored. This renders the closed loop system for the partial observations case non-Markovian, and it is the Markov property which forms the basis of the principle of optimality for stochastic control problems.

Rishel⁽⁶⁷⁾ has made a study of this difficulty, and proves an analogue of Bellman's principle of optimality under very general conditions: the controlled processes are not required to be Markovian, and the control laws are allowed to be functional on incomplete measurements of the system. For each admissible control, u , Rishel defines a value function, $V_u(t)$, as the

minimum over all admissible controls which agree with u up to time t of the remaining contribution to the performance index. Under the assumption that the class of admissible controls is 'relatively complete', a kind of continuity condition of the performance index over the admissible controls, an analogue of Bellman's principle of optimality is established. The intuitive idea behind the principle is that it is always better to use immediately an optimal control than to use some other control for a short time and then to use an optimal control. Thus, a principle of optimality may be derived for the non-Markovian case, but it should be noted that the value function for this case is control-dependent. This is related to the fact that, in contrast to the perfectly observable case where the current observation is sufficient to predict future behaviour, prediction in the partially observable case is aided by a knowledge of the past controls. The prediction from a measurement record, Y_t , with control, u_1 , will not necessarily be the same as that for Y_t with control, u_2 , even though u_1 and u_2 may agree after time t . We may contrast this with the case where $x(t)$ is precisely known, when the past controls are irrelevant because of the Markov property of the system. Recently, Davis and Varaiya⁽⁶⁸⁾ have established Hamilton-Jacobi-type conditions for the partially observable problem, but, as they themselves admit, their results are not altogether satisfactory because of the fact that every control has a different value function.

Mortensen⁽⁶⁹⁾ considers a special case of eqn. (4.2.1):-

$$dx(t) = f(t, x, u)dt + dw_1(t) \quad (4.2.2)$$

$$dy(t) = g(t, x)dt + dw_2(t) \quad (4.2.3)$$

and formulates the control problem in a slightly different form than is usual. Realising that knowledge of the conditional probability density of the state given the observations is sufficient for the prediction of future behaviour, Mortensen recasts the original finite-dimensional problem into an infinite-dimensional state space by taking the conditional probability density as the system state. This approach is analogous to that of Shapiro and Mon^(43,44) for the perfectly observable case. A value function, V , is defined in the usual way, and it is shown that V satisfies:-

$$\begin{aligned} 0 = & \frac{\partial V}{\partial t} - \min_{u(t, \Gamma(t, x))} \left\{ \int_{-\infty}^{\infty} \frac{\delta V}{\delta \Gamma} \cdot \frac{\partial}{\partial x} [f(t, x, u) \Gamma(t, x)] dx \right\} + \\ & + \int_{-\infty}^{\infty} \frac{\delta V}{\delta \Gamma} \left\{ \frac{1}{2} \frac{\partial^2 \Gamma}{\partial x^2} + \Gamma(t, x) g(t, x) \hat{g}(t) \right\} dx + \\ & + \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\delta^2 V}{\delta \Gamma(x_1) \delta \Gamma(x_2)} \Gamma(t, x_1) g(t, x_1) \Gamma(t, x_2) g(t, x_2) dx_1 dx_2 \end{aligned} \quad (4.2.4)$$

where $\Gamma(t,x)$ is the (unnormalised) probability density function for $x(t)$

$$\text{and} \quad \hat{g}(t) = \frac{\int_{-\infty}^{\infty} g(t,x) \Gamma(t,x) dx}{\int_{-\infty}^{\infty} \Gamma(t,x) dx} \quad (4.2.5)$$

The symbol $\delta(\cdot)$ refers to Fréchet differentiation. (For full details see Mortensen's paper⁽⁶⁹⁾. It should be noted that only the scalar case is displayed here.) The solution of eqn. (4.2.4) yields an optimal control law based on the state variable, $\Gamma(t,x)$, and it remains to give a procedure for computing this function from a measurement record, Y_t . Of course, this calculation must be done in real-time. It can be shown that Γ satisfies a stochastic partial differential equation:-

$$\begin{aligned} d\Gamma(t,x) = & \left\{ -\frac{\partial}{\partial x} [f(t,x,u)\Gamma(t,x)] + \frac{1}{2} \frac{\partial^2 \Gamma}{\partial x^2} \right\} dt + \\ & + \Gamma(t,x)g(t,x)dy(t) \end{aligned} \quad (4.2.6)$$

with initial condition

$$\Gamma(0,x) = p_0(x) \quad (4.2.7)$$

where $p_0(x)$ is the (given) probability density of the initial state. A block diagram for the closed loop system is shown in Fig. 4.1. Clearly, even if eqn. (4.2.4) could be solved for the optimal control law, the on-line computational demands of Mortensen's scheme are prohibitive.

The interesting feature of Mortensen's approach is that his control scheme is a two-stage process. First, the measurements are used to generate an estimate of the system state, which is then used in a precomputed control law. It is true that, in Mortensen's case, a transformation of the state space is necessary in order to achieve this structure. An interesting question is: under what circumstances may the optimal controller be expected to be of a similar structure, but without the necessity of a state space transformation? The type of closed loop system envisaged is shown in Fig. 4.2. The most commonly quoted result of this type is the "classical" Separation Theorem, first proved by Joseph and Tou⁽⁷⁰⁾ for discrete systems, and by Potter⁽⁷¹⁾ for continuous systems. For a system that is linear in state and control perturbed by an additive white Gaussian noise process, with observations that are linear in the state with additive white Gaussian noise, and a quadratic performance index, not only may the optimal control be shown to be dependent on the best state estimate conditional on the measurements,

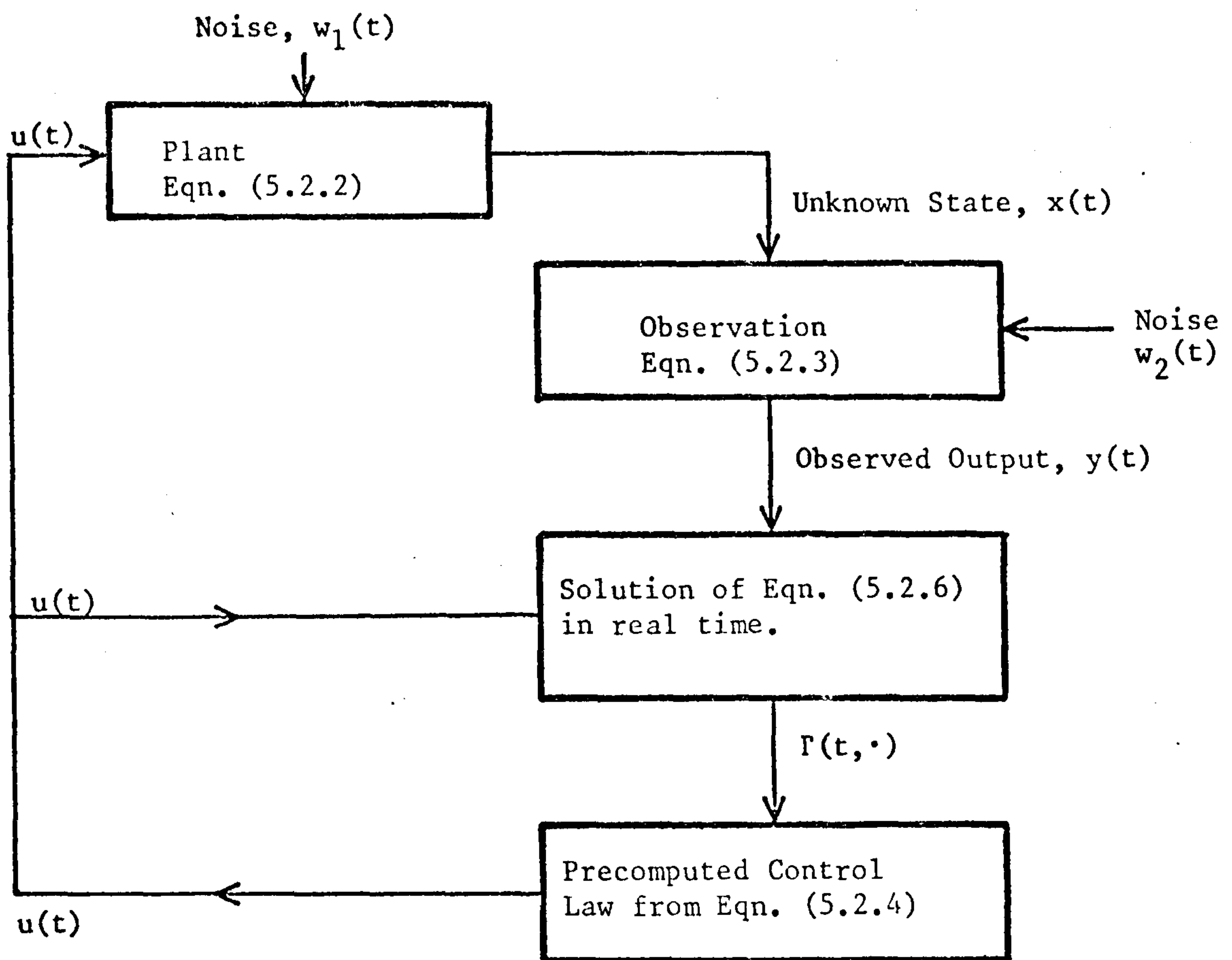


Figure 4.1 Black Diagram of Control Scheme Suggested by Mortensen

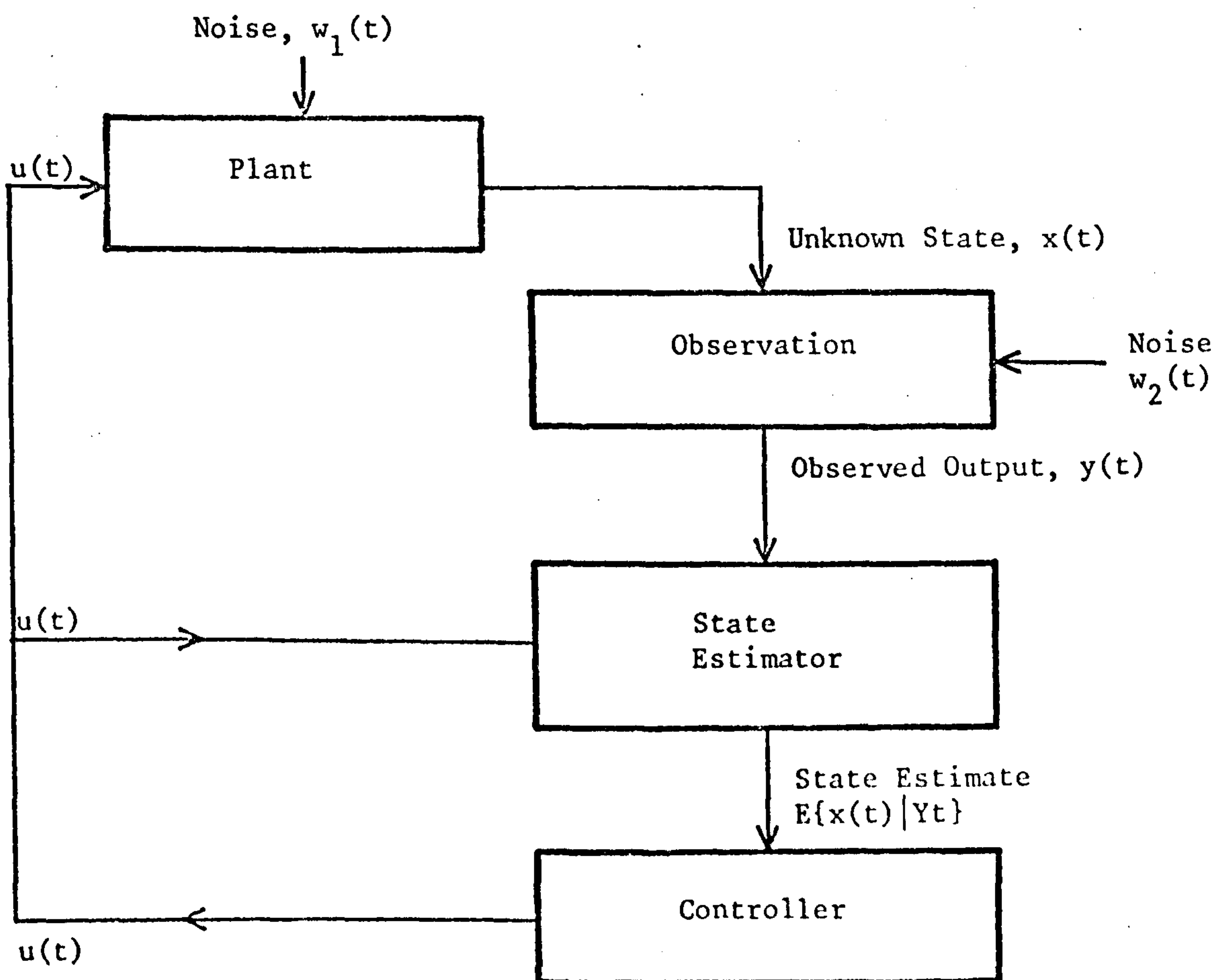


Figure 4.2 Controller Structure Suggested by the Separation Principle

but also the control law is the same as that for the deterministic system obtained by setting the noise processes in the original system to zero. The conditional state estimate may be generated by the use of the Kalman-Bucy filter⁽⁷²⁾.

Some extensions of the Separation Principle to account for non-linearities have been carried out. Wonham⁽⁷³⁾ considers a system non-linear in the control. The estimation function for this case may still be performed by the Kalman filter, but the control design must take the noise statistics into account. In the case where a non-linear observation function is admitted⁽⁷⁴⁾, estimation must be accomplished by the use of Kushner's infinite dimensional filter⁽⁷⁵⁾.

An interesting review of the fortuitous circumstances behind the separation principle is given by Tse⁽⁷⁶⁾. The basic property of linear systems responsible for the application of the Principle is that only first-order conditional moments of the state are control-dependent. In particular, the error covariance representing the performance of the estimation procedure is not influenced by the control at all. However, as Fel'dbaum notes⁽⁷⁷⁾, the control has a dual function in a general non-linear situation:-

(a) to achieve a certain performance objective;

and

(b) to excite certain modes and to regulate signal-to-noise ratio so as to help estimation performance.

Thus, we may expect control and estimation procedures to be coupled together in general. In spite of this, there have been several successful applications of Separation Principle control designs to non-linear processes, particularly in the field of aerospace systems⁽⁷⁸⁾. In the chemical engineering literature, the works of Seinfeld⁽³⁾ and Joffe and Sargent⁽⁴⁾ may be cited. The applications usually rely on the noise processes being small. Hence, estimation errors will be small hopefully, and the effect of the control on these errors may be negligible. It is the purpose of the next section to display the kind of conditions sufficient for this approach to be valid for problem P 1, and to give estimates of the performance deterioration to be expected when a Separation Principle approach is used rather than the true optimal control.

4.3 An Approximate Separation Principle for Small Disturbances

Consider the system:-

$$\frac{dx}{dt} = f[t, x, u(t), \xi(t)] : x(0) = x_0^* + \chi \quad (4.3.1)$$

where χ and $\xi(t)$ are noise processes satisfying the conditions of Corollary 2.4.7. Then that theorem states that, if the state, $x(t)$, were exactly measurable for all t , a good approximation to the optimal stochastic control policy is to use a continuously updated feedforward policy for the second-order approximation appropriate to eqn. (4.3.1). Now suppose that we are unable to perfectly observe the state but instead we have devised an estimation scheme based on noisy measurements:-

$$y(t) = g[t, x(t), \eta(t)] \quad (4.3.2)$$

which is guaranteed to give an estimate, $x^*(t)$, which is an $O(\delta)$ approximation to the true state, i.e.

$$\|x(t) - x^*(t)\| \leq K\delta \quad \text{for all } t \in [0, t_f] \quad (4.3.3)$$

Then we can show that using this estimate instead of the true state in a continuously updated feedforward control scheme gives an $o(\delta)$ approximation to the optimal stochastic policy if the state could be perfectly observed. We have already met a special case of this result in Section 2.5, in which it was shown that the state of the second-order model with arbitrary feedforward control could be guaranteed to be within $o(\epsilon)$ of the state of the system with the same control, provided a hard bound is placed on the noise. Hence, since the feedforward control for the (deterministic) model is also the updated feedforward control for that model, feedforward controls optimised for the model give performance on the system which is within $o(\epsilon)$ of the optimal stochastic policy if the state were perfectly observable.

In actual fact, the new result follows directly from Theorem 2.5.2. The usefulness of the result relies on finding processes satisfying eqn. (4.3.3). In this section, a particular estimation scheme will be presented which satisfies eqn. (4.3.3) with $\delta = \epsilon$.

The system, eqn. (4.3.1), with noise satisfying the assumptions of Section 2.5 (i.e. noise with a hard bound), is to be observed from measurements given by eqn. (4.3.2) made at discrete times during the control of the system. The measurement noise process, $\eta(t)$, satisfies

$$\|\eta(t) - \eta^*(t)\| \leq \epsilon \quad \text{w.p.1} \quad (4.3.4)$$

where $\eta^*(t) = E\{\eta(t)\}$ (4.3.5)

The following estimation scheme is proposed:-

Given $x^*(t_{k-1}|t_{k-1}), P(t_{k-1}|t_{k-1})$
 Prediction $\frac{d}{dt} x^* = f[t, x^*, u(t), \xi^*(t)]$
 to t_k (4.3.6)
 or $\frac{d}{dt} x^* = f[t, x^*, u(t), \xi^*(t)] + z(t)$

$$\begin{aligned} x^*(t_{k-1}) &= x^*(t_{k-1}|t_{k-1}) \\ x^*(t_k) &= x^*(t_k|t_{k-1}) \\ \frac{d}{dt} P &= f_x P + P f_x^T \\ P(t_{k-1}) &= P(t_{k-1}|t_{k-1}) \end{aligned} \quad (4.3.7)$$

$$\begin{aligned} P(t_k) &= P(t_k|t_{k-1}) \\ z^k(t) &= \frac{1}{2} \text{trace} [f_{xx} P(t) + f_{\xi\xi} Q(t)] \end{aligned} \quad (4.3.8)$$

Update at t_k : $P(t_k|t_k) = P(t_k|t_{k-1}) - P(t_k|t_{k-1}) g_x^T x$ (4.3.9)

$$\begin{aligned} x & [g_x P(t_k|t_{k-1}) g_x^T + R(t_k)]^{-1} g_x P(t_k|t_{k-1}) \\ x^*(t_k|t_k) &= x^*(t_k|t_{k-1}) + P(t_k|t_k) g_x^T R^{-1}(t_k) x \\ x & \{y(t_k) - g[t_k, x^*(t_k|t_{k-1}), \eta^*(t_k)]\} \end{aligned} \quad (4.3.10)$$

where $R(t_k) = E\{\delta\eta\delta\eta^T(t_k)\}$

Thus, we have prediction of the state by either the deterministic model obtained by setting the process noise to zero, or by the second-order model. Prediction of the covariance of the estimate is by the second-order equation of Joffe and Sargent, and a least-squares updating procedure of the type proposed by Joffe⁽⁶⁰⁾ is used whenever a measurement is made. We have a least-squares formulation of the extended Kalman filter, with one essential difference: eqn. (4.3.7). The usual covariance prediction equation is⁽⁶⁰⁾

$$\frac{d}{dt} P = f_x P + P f_x^T + f_{\xi} Q(t) f_{\xi}^T \quad (4.3.11)$$

i.e. an extra term is incorporated to account for the process noise. However, this equation is derived for the case of white Gaussian noise, and the extra term arises from the special properties of the $\hat{\text{Ito}}$ calculus used in the derivation of eqn. (4.3.11), in particular from the $\hat{\text{Ito}}$ differential rule. This rule gives the differential equation satisfied by a function of the solution of another stochastic differential equation. Let $\phi[x(t)]$ be a scalar-valued, twice continuously differentiable function of $x(t)$ where $x(t)$ satisfies

$$dx(t) = f[x(t)]dt + \sigma[x(t)]dw(t) \quad (4.3.12)$$

where $w(t)$ is a Brownian motion process. Then, the $\hat{\text{Ito}}$ differential rule states that ϕ satisfies

$$d\phi(t) = \phi_x^T dx(t) + \frac{1}{2} \text{trace } \sigma Q(t) \sigma^T \phi_{xx} dt \quad (4.3.13)$$

where $Q(t)$ is the covariance function for the process $w(t)$. If the ordinary rules of calculus were applicable then the equation

$$d\phi(t) = \phi_x^T dx(t) \quad (4.3.14)$$

would be expected. The extra term in eqn. (4.3.13) results in an extra term in eqn. (4.3.11) which is not present if the ordinary rules of calculus were applicable.

The above discussion gives a theoretical justification for the use of eqn. (4.3.7) in preference to eqn. (4.3.11). From the practical point of view, Joffe⁽⁶⁰⁾, who used eqn. (4.3.11) with the estimation scheme proposed above found that the performance of the scheme was completely insensitive to the values chosen for $Q(t)$, so that its incorporation even as an extra tuning parameter does not seem worthwhile.

For the estimation procedure given by eqns. (4.3.6) - (4.3.10) applied to a system described by eqns. (4.3.1) and (4.3.2) we can show:-

Theorem 4.1

Let $\xi(t)$ have the properties given in Section 2.5. (i.e. hard bounded noise). Further, let $\eta(t)$ obey eqn. (4.3.4) and let

$$\|x(0) - x^*(0)\| \leq \epsilon \quad \text{w.p.1} \quad (4.3.15)$$

Then,

$$\|x(t) - x^*(t)\| \leq K\epsilon \quad \text{w.p.1 for some } K < \infty \quad (4.3.16)$$

provided that $\|g_x\| \leq M_g$ for $M_g \in [0, \infty)$, and that g is differentiable with respect to η .

Proof

The result is proved by induction. We assume that $x^*(t_{k-1}|t_{k-1})$ and $P(t_{k-1}|t_{k-1})$ are such that

$$\begin{aligned} ||x(t_{k-1}) - x^*(t_{k-1}|t_{k-1})|| &\leq K_{k-1}\epsilon \\ ||P(t_{k-1})|| &\leq K_{k-1}\epsilon^2 \end{aligned} \quad (4.3.17)$$

for some finite, positive K_{k-1} . Then, from a result similar to Lemma 2.5.3,

$$||x(t) - x^*(t|t_{k-1})|| \leq K_{k-1}\epsilon$$

in particular

$$||x(t_k) - x^*(t_k|t_{k-1})|| \leq K_k K_{k-1}\epsilon \quad (4.3.18)$$

$$\text{also, } ||P(t_k|t_{k-1})|| \leq K_k K_{k-1}\epsilon^2 \quad (4.3.19)$$

Since eqn. (4.3.7) is a linear equation. Thus, the prediction step of the algorithm gives an $o(\epsilon)$ approximation to the actual system state.

For the update, from eqn. (4.3.10)

$$||x^*(t_k|t_k) - x^*(t_k|t_{k-1})|| \leq \frac{||P(t_k|t_k)|| ||g_x||}{||R(t_k)||} ||\delta y(t_k)|| \quad (4.3.20)$$

$$\text{where } \delta y(t_k) = y(t_k) - g[t_k, x^*(t_k|t_{k-1}), \eta^*(t_k)] \quad (4.3.21)$$

From eqns. (4.3.21), (4.3.2) and the assumed differentiability properties of g ,

$$\begin{aligned} ||\delta y(t_k)|| &\leq ||g[t_k, x(t_k), \eta(t_k)] - g[t_k, x^*(t_k|t_{k-1}), \eta^*(t_k)]|| \\ &\leq M_g ||x(t_k) - x^*(t_k|t_{k-1})|| + L_2 ||\delta \eta(t_k)|| \\ ||\delta y(t_k)|| &\leq (M_g K_{k|k-1} + L_1)\epsilon \end{aligned} \quad (4.3.22)$$

From eqns. (4.3.20), (4.3.22) and (4.3.4)

$$||x^*(t_k|t_k) - x^*(t_k|t_{k-1})|| \leq \frac{||P(t_k|t_k)|| M_g (M_g K_{k|k-1} + L_1)\epsilon}{K_\eta \epsilon^2} \quad (4.3.23)$$

From eqn. (4.3.9)

$$||P(t_k|t_k)|| \leq ||P(t_k|t_{k-1})|| \alpha \quad (4.3.24)$$

where $\alpha = || I - g_x^T \{ g_x^T P(t_k | t_{k-1}) g_x^T + R(t_k) \}^{-1} g_x^T P(t_k | t_{k-1}) ||$

$$\alpha \leq 1 + \frac{||g_x||^2 K_{k|k-1} \epsilon^2}{||g_x^T P(t_k | t_{k-1}) g_x^T + R(t_k)||} \quad (4.3.25)$$

Use has been made here of eqn. (4.3.19).

Consider the denominator in eqn. (4.3.25)

$$0 < ||g_x^T P(t_k | t_{k-1}) g_x^T + R(t_k)|| \leq ||g_x||^2 K_{k|k-1} \epsilon^2 + K_\eta \epsilon^2$$

$$\text{Hence,} \quad \alpha \leq 1 + K_\alpha \quad (4.3.26)$$

for some finite positive K_α .

From eqns. (4.3.24), (4.3.26)

$$||P(t_k | t_k)|| \leq (1 + K_\alpha) K_{k|k-1} \epsilon^2 \quad (4.3.27)$$

From eqns. (4.3.23), (4.3.27)

$$||x^*(t_k | t_k) - x^*(t_k | t_{k-1})|| \leq \frac{(1+K_\alpha) K_{k|k-1} M_g (M_g K_{k|k-1} + L_1) \epsilon}{K_\eta} \quad (4.3.28)$$

Hence, from eqns. (4.3.18), (4.3.27), (4.3.28)

$$\begin{aligned} ||x(t_k) - x^*(t_k | t_k)|| &\leq K_k \epsilon \\ ||P(t_k | t_k)|| &\leq K_k \epsilon^2 \end{aligned} \quad (4.3.29)$$

Thus, eqn. (4.3.16) will be satisfied over the interval $(t_{k-1}, t_k]$ if it is satisfied at t_{k-1} . Now, from eqn. (4.3.15), eqns. (4.3.17) are satisfied at $t = 0$. Therefore, the theorem holds for all $t \geq 0$.

Q.E.D.

4.4 An Example

We return to the example of Section 3.3, but instead of assuming that both components of the state vector may be measured without error, we assume that only x_1 may be measured. Further, these measurements may be made only at discrete times, and are corrupted by random error. We now have the following system:-

$$\begin{aligned} \frac{dx_1}{dt} &= -k_1 x_2 x_1^2 & : & & x_1(0) &= x_1^*(0) + \chi_1 \\ \frac{dx_2}{dt} &= -k_2 x_2 & : & & x_2(0) &= x_2^*(0) + \chi_2 \end{aligned} \quad (4.4.1)$$

$$\begin{aligned} \text{with} \quad k_1 &= \exp(\beta_1 - \beta_2/T) \\ k_2 &= \exp(\beta_3 - \beta_4/T) \end{aligned} \quad (4.4.2)$$

$$\text{and} \quad T(t) = u(t) + \xi(t)$$

$$y(t_i) = x_1(t_i) + \eta(t_i) \text{ for } t_i \in \tau \quad [t_o, t_f] \quad (4.4.3)$$

From the result of the previous section, a good approximation to the optimal stochastic control for this system with performance index:-

$$J(t, Y_t) = E\{x_1(t_f) | (t, Y_t)\} \quad (4.4.4)$$

and final state constraint:-

$$E\{x_2(t_f) | (t, Y_t)\} = x_{2f}^* \quad (4.4.5)$$

is to apply a Separation Principle type of control policy. State estimates generated by an on-line estimation technique are used in a control law calculated for the deterministic system obtained by setting any process noise term to zero. The order of the approximation to the optimal stochastic controller is fixed by the error in the state estimates. It is bounded above by $o(\epsilon)$ where ϵ is a bound on the system noise processes if prediction is afforded by the deterministic equations, and a least squares update is used whenever a measurement is made.

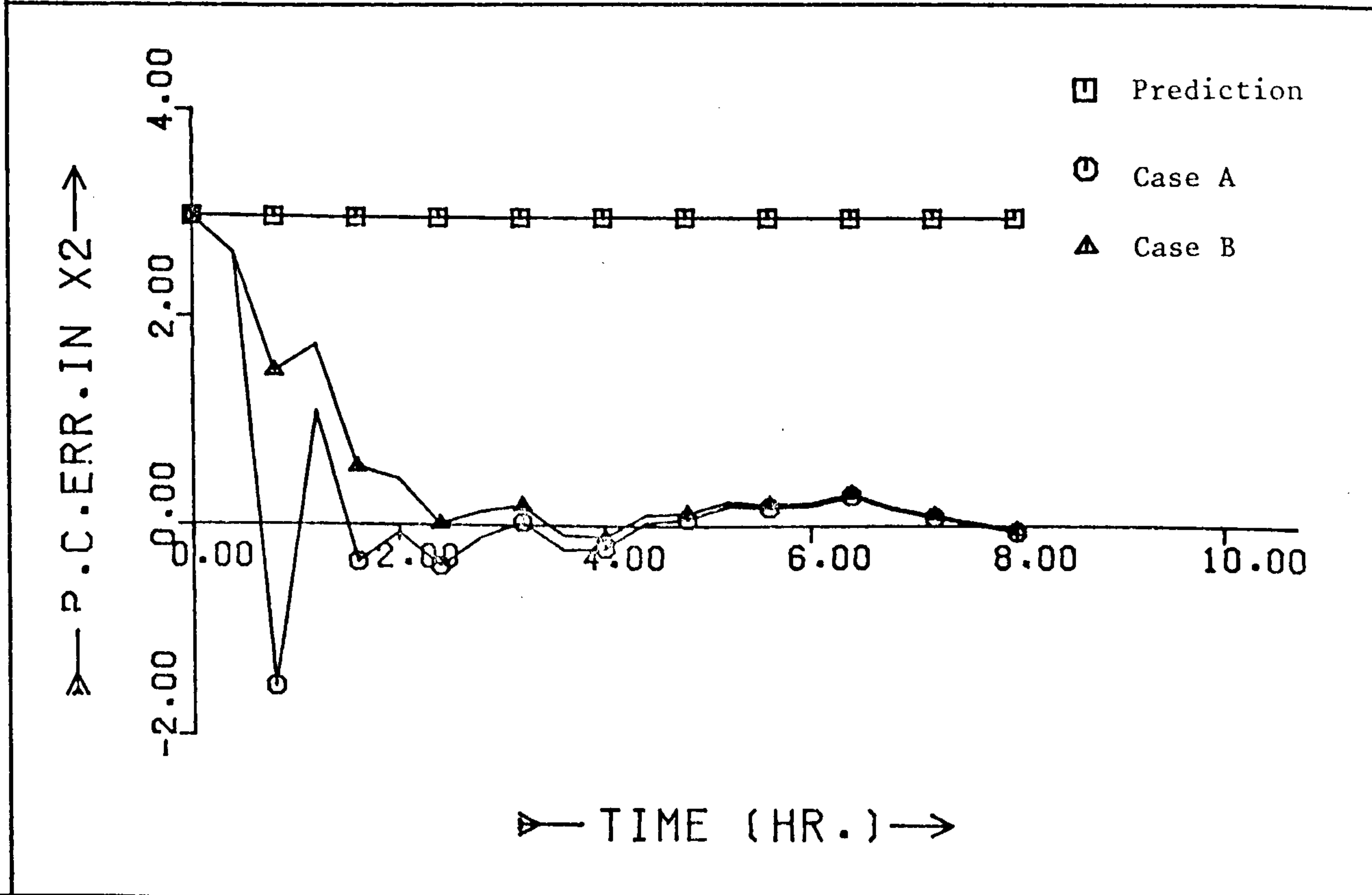
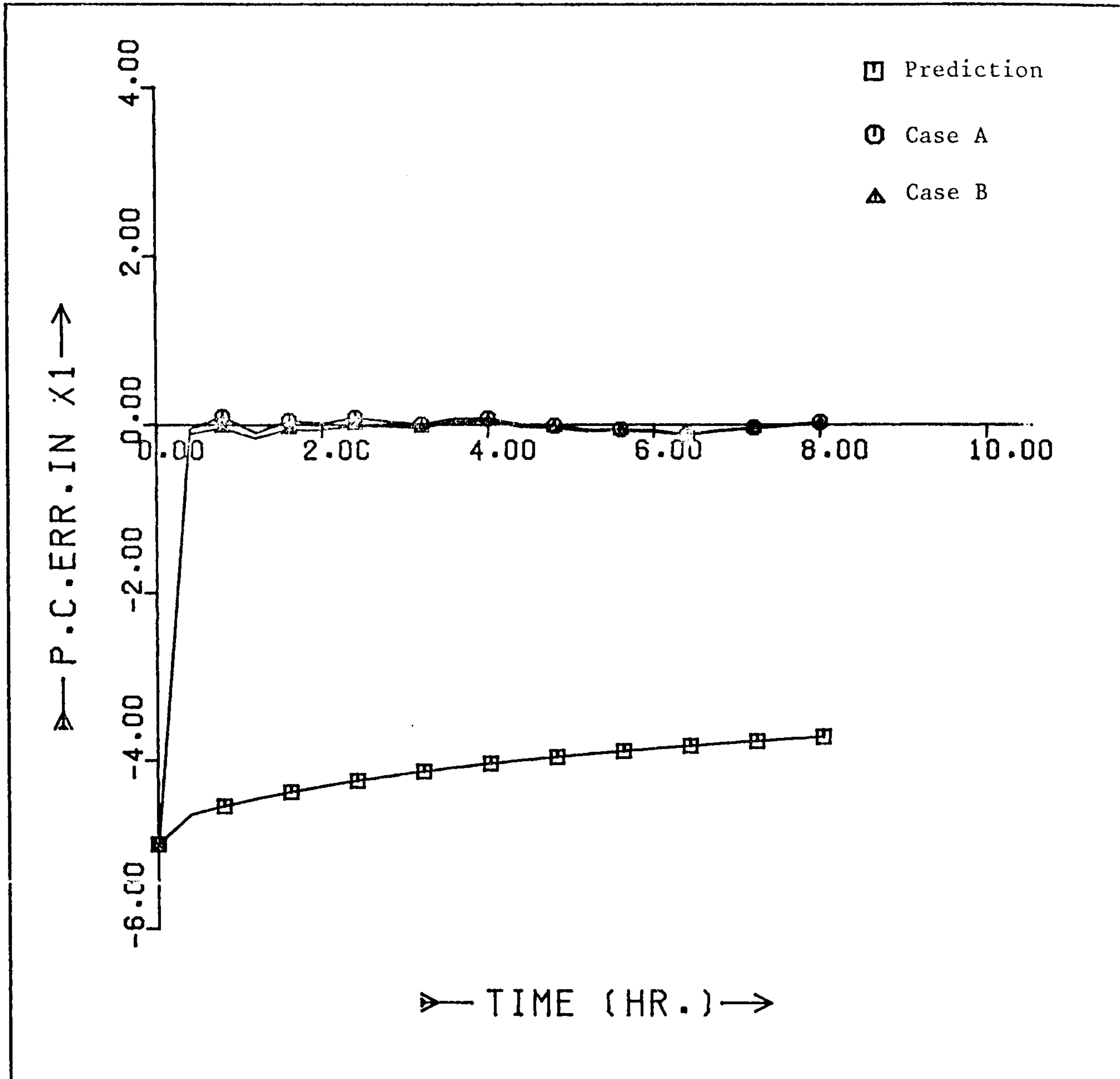
In order to make direct comparisons with the results of Section 3.3, the same parameter values are used here. We assume that measurements may be made at intervals of 0.4 hours.

There has been a lot of previous work (e.g. (1), (2)) on the effect of the values of the parameters $P(o)$ and $R(t)$ on the performance of filters, and so this point will not be considered in great detail. However, it does seem worthwhile to present results for a few different choices of these parameters to confirm that filter performance is quite insensitive to them. We confine ourselves at present to the case where the effects of process noise may be neglected. Fig. 4.3 shows the performance of the filter on a typical batch of simulated data for two different choices of $P(o)$ and $R(t)$. The noise was bounded as follows:-

$$|\xi(t)| \leq b_p \quad ; \quad |\chi_i| \leq b_I \quad i=1,2 \quad ; \quad |\eta| \leq b_M$$

$$\text{with} \quad b_p = 0.0 \quad ; \quad b_I = 0.05 \quad ; \quad \text{and } b_M = 0.002.$$

FIGURE 4.3 Filter Performance on a Typical Batch - $b_p = 0.0$



Thus, the true values of $P(o)$ and $R(t)$ are:-

$$P(o) = \begin{bmatrix} .833,-3 & 0.0 \\ 0.0 & .833,-3 \end{bmatrix} \quad ;R(t) = .133,-5$$

The chosen parameter sets for the two cases shown in Fig. 4.3 are as follows:-

Case A $P(o) = \begin{bmatrix} .100,-2 & 0.0 \\ 0.0 & .100,-2 \end{bmatrix} \quad ;R(t) = .100,-5$

Case B $P(o) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \quad ;R(t) = .100,-1$

The results for prediction only with no updating of state estimates to account for new measurements are given for comparison. There is seen to be little difference in filter performance between Case A where the filter parameters are almost exact, and Case B where they are grossly overestimated. This lack of difference is reflected in the performance of the overall control scheme for the two cases, shown in Table 4.1.

TABLE 4.1 Actual Final States for Typical Run

	$x_1(t_f)$	$x_2(t_f)$
Prediction only	.971,-1	.419
Case A	.100	.419
Case B	.100	.419

However, it would be unwise to deduce too much from one simulation of the control system: the results of three hundred Monte Carlo simulations are presented in Table 4.2. The results for feedforward control are included for comparison. The incorporation of the filter has produced better performance in terms of $x_2(t_f)$ at the expense of greater spread of values of $x_1(t_f)$. Mean performance of $x_1(t_f)$ has been slightly improved. The different filter parameters cause little variation in controller performance, as was expected from earlier observations.

TABLE 4.2 Performance of Control Scheme over 300 Monte Carlo Simulations,
 $b_p = 0.0$

	Filtering		
	Feedforward	Case A	Case B
Mean of $x_1(t_f)$	0.404	0.402	0.402
Variance of $x_1(t_f)$	.781,-4	.807,-4	.805,-4
Mean of $x_2(t_f)$	.998,-1	.100	.100
Variance of $x_2(t_f)$	.931,-5	.448,-7	.461,-7

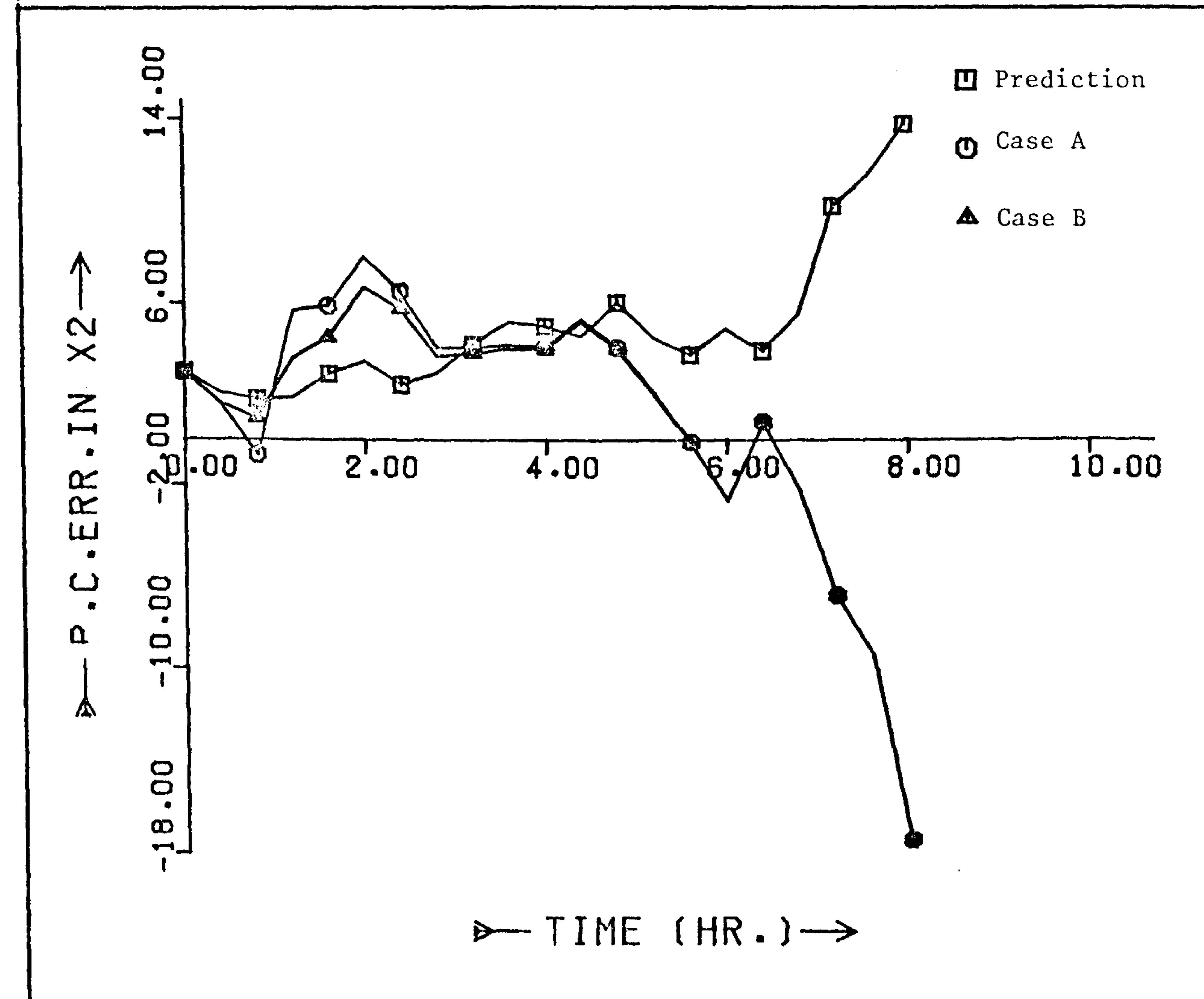
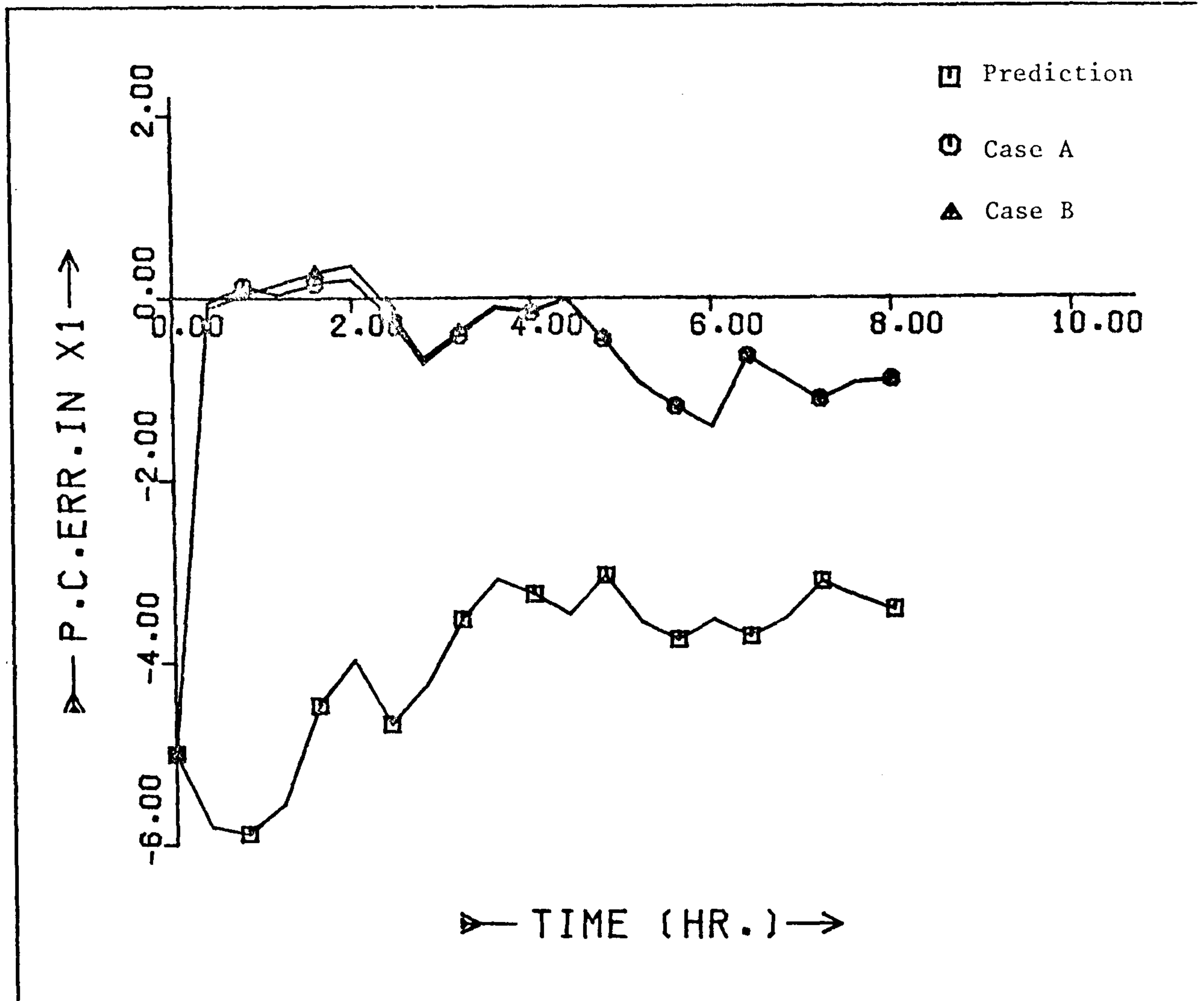
The results of Table 4.2 may also be compared with those of Table 3.17, where it was assumed that the state could be perfectly observed. The relevant part of Table 3.17 is repeated here as Table 4.3.

TABLE 4.3 Results for Perfect State Feedback - $b_p = 0.0$

Mean of $x_1(t_f)$	0.404
Variance of $x_1(t_f)$	0.846,-4
Mean of $x_2(t_f)$	0.100
Variance of $x_2(t_f)$	0.274,-10

It should be remembered that these figures represent a very good approximation to the performance of the optimal stochastic controller for the case of perfect state observation. It is clear that very good control results from the application of the Separation Principle approach to the particular problem considered here. This was to be expected from the theory of the previous section once a scheme for giving good state estimates had been found. We now consider the effect of process noise on the performance of the filter. As in Chapter two, the process noise was assumed to be piecewise constant over the integration procedure step length, which was 0.4 hours. The performance of the filter over a typical batch of data, with $b_p = 2.0$ K, is shown in Fig. 4.4. Any slight differences in the performance of the filter with different parameter sets is completely masked by the inability of the filter to obtain convergent state estimates. This failure is reflected in the control performance figures for the batch, shown in Table 4.4, and in the performance over three hundred Monte Carlo simulations shown in Table 4.5.

FIGURE 4.4 Filter Performance on a Typical Batch - $b_p = 2.0$



Negligible differences between the results for feedforward control and for feedback control can be seen. However, this is a demonstration of the $o(\epsilon)$ bound predicted for Separation Principle control schemes in Section 4.3, since even if the filter fails to converge, performance can be expected from the overall control scheme comparable to that for feedforward control.

TABLE 4.4 Actual Final States for Typical Run, $b_p = 2.0$

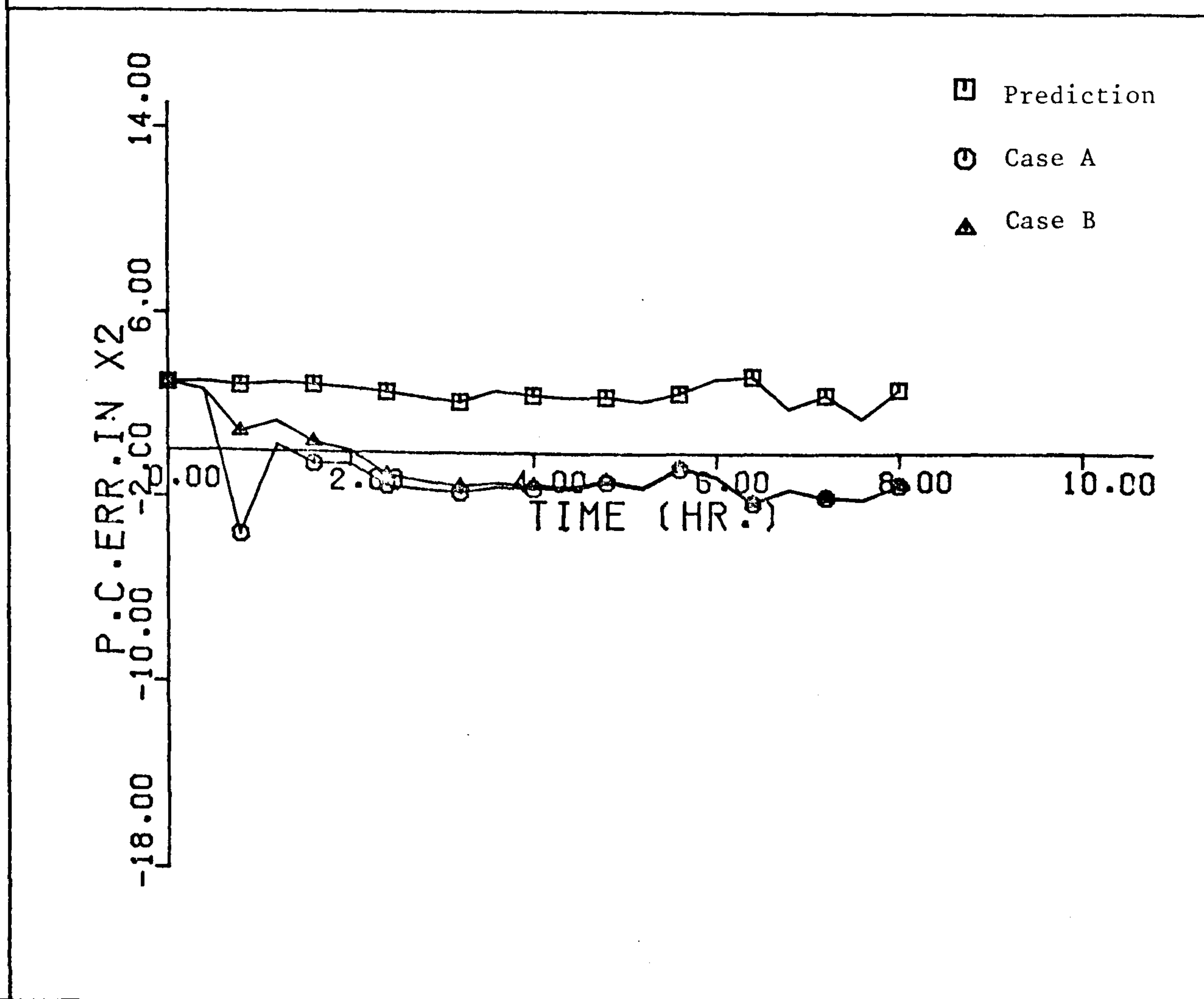
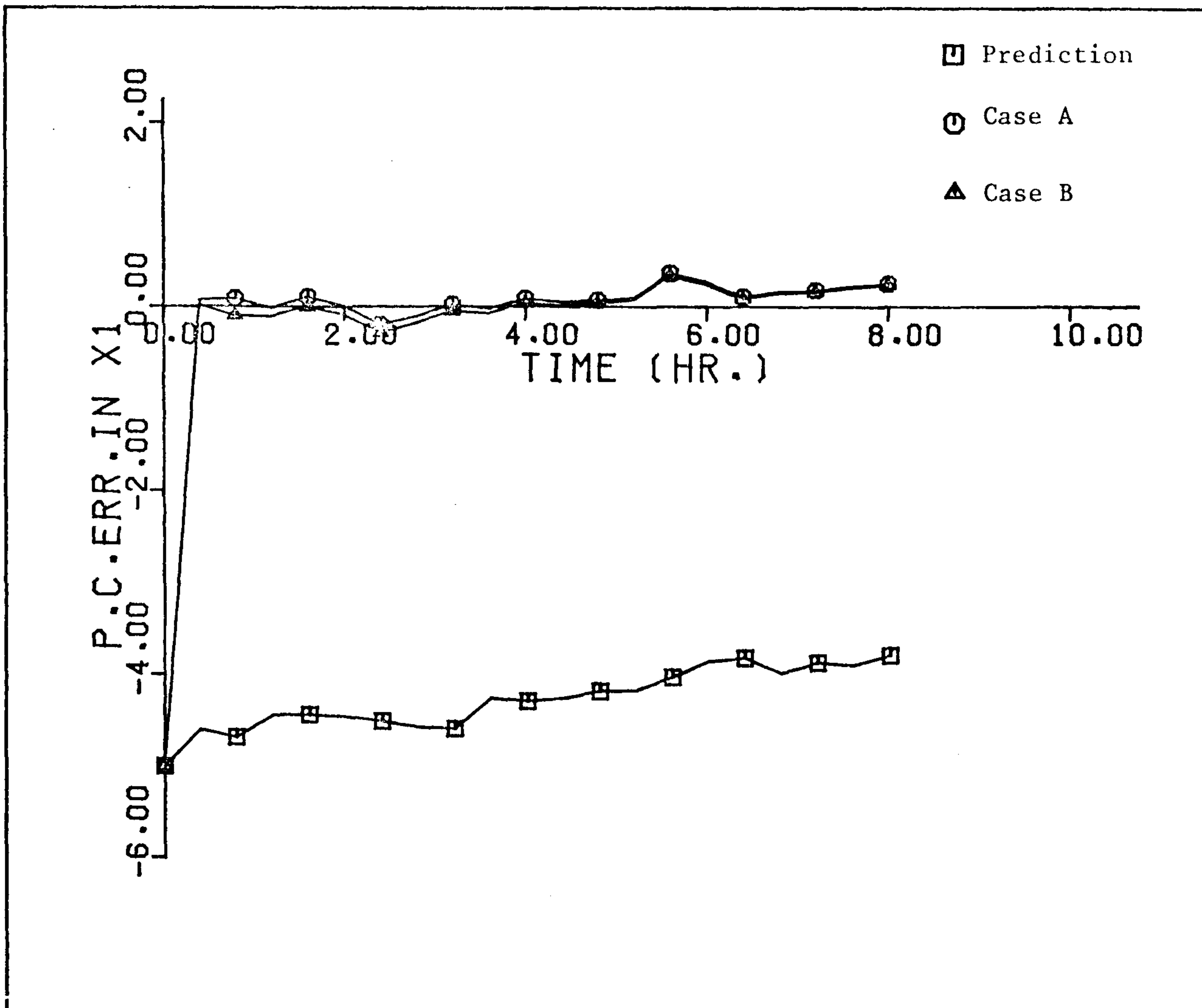
	$x_1(t_f)$	$x_2(t_f)$
Prediction only	0.417	0.880,-1
Case A	0.423	0.121
Case B	0.423	0.121

TABLE 4.5 Performance of Control Scheme over 300 Monte Carlo Simulations,
 $b_p = 2.0$

	Filtering		
	Feedforward	Case A	Case B
Mean of $x_1(t_f)$	0.404	0.402	0.402
Variance of $x_1(t_f)$	0.798,-4	0.822,-4	0.787,-4
Mean of $x_2(t_f)$	0.984,-1	0.984,-1	0.986,-1
Variance of $x_2(t_f)$	0.987,-4	0.110,-3	0.105,-3

The reason for the failure of the filter appears to be connected with the low noise process frequency. Because this noise switches only at measurement times, the prediction steps of the filter have a bias for which the updates cannot compensate. The results of filtering a batch of data with process noise of a much higher frequency are shown in Fig. 4.5. In this case the noise is switched every 0.025 hours. It can be seen that filter performance is much improved.

FIGURE 4.5 Filter Performance on a Typical Batch - High Frequency Noise



C H A P T E R F I V E

CONCLUSIONS

In this thesis, stochastic optimal control problems for a rather general lumped parameter system are considered: the system equations are non-linear in states, controls, and noise processes. Observation of the system can be performed in various ways: by no measurements; by noisy measurements which are non-linear functions of the system states; by accurate measurement of the system state at discrete times; or by continuous accurate measurement of the state.

In Chapter two, a stochastic control problem with continuous observation of the state is considered. A review of the available theory indicates that there are problems associated with the exact solution of such a problem. Since the theory is based on systems with the Markov property, it is first necessary to find a stochastic differential equation having similar properties to that of the original system. Having done that, the methods of dynamic programming may be applied to obtain the Bellman-Hamilton-Jacobi equation for the control problem. This partial differential equation must be solved on the attainable set in state space of the system, whence the optimal controller is obtained. The computational requirements of this approach appear to be prohibitive, particularly for large systems. Thus, there appears to be little hope at present of solving the general problem.

However, many workers have found it worthwhile to look at problems where the noise processes entering the system are small. For this class of problem it might be expected that methods based on deterministic models would give good performance. For the general non-linear model considered in this thesis, it is shown in Chapter two that this is indeed the case. By making assumptions bounding various expected values of the noise processes, it is possible to derive a deterministic model which adequately predicts the expected performance of the system under various control policies: the Joffe-Sargent model. A general theorem (theorem 2.3.1) comparing the performance on the system of the optimal control for a model with that for the system is derived. This theorem is used to establish that the optimal feedback controller for the Joffe-Sargent model is a good approximation to the system optimal controller, whenever continuous observation of the state is feasible.

Apart from the small-noise assumptions inherent in the Joffe-Sargent model, it is also necessary to make some smoothness assumptions about the optimal stochastic control law. These assumptions are restrictive, but there appears to be no way at present of relaxing them using available mathematics. The derivation of the Joffe-Sargent model relies on the theory of delay-differential equations, and current existence theory assumes continuity of the right hand side functions of the equations. Once this problem has been overcome, there is still the fact that the Joffe-Sargent model is derived using Taylor series so that differentiability assumptions must be made to allow this.

Having established that the optimal control for the Joffe-Sargent model is a good approximation to the optimal control for the system, the performance of other control policies can be predicted by comparing their performance with that of the optimal model controller on the model. In section 2.5, feedforward and discrete feedback control schemes are examined in this way. A comparison theorem (theorem 2.5.2) for two control schemes, one based on perfect observation of the state, the other based on some process which well approximates the state is proved by comparing the Joffe-Sargent models for the controlled systems. By placing extra bounds on the noise processes entering the system, it is possible to find suitable approximating processes for the cases of feedforward control, and of discrete feedback control. In this way, the effect of using these control policies rather than the optimal stochastic controller is predicted.

Two examples illustrating the theoretical results of the previous chapter are presented in Chapter three. They appear to provide confirmation of the theory.

In Chapter four, the problem of optimal stochastic control based on noisy measurements is considered. A review of available theory reveals that the most useful result from the practical point of view is the Separation Principle. Unfortunately, the principle is applicable only to linear systems, and does not appear to generalise to the non-linear problem considered in this thesis. However, for small noise, theorem 2.5.2 may be regarded as an approximate Separation Principle. From that theorem, it can be said that a Separation Principle type control scheme will perform adequately, provided that the output from the estimator approximates the actual state of the system well at all times. The problem is to guarantee the performance of the estimator. It is shown in Chapter four that an estimation scheme based on the extended Kalman filter can be guaranteed to perform adequately, provided that the

process and measurement noise and the initial state uncertainty are small and bounded. Thus, in that situation, a Separation Principle type control scheme based on the extended Kalman filter can be expected to perform adequately. However, the simple analysis used does not show that one does better by using the measurements than by ignoring them. Further, the assumptions involved in the derivation of the result are rather stringent. Clearly, more work is required to establish better bounds on the performance of the estimator, under less restrictive assumptions about the noise processes. Also, theorem 2.5.2 could usefully be generalised by replacing the requirement that the approximating process be close to the actual state always by something less stringent.

In summary, the thesis has been concerned with optimal stochastic control of a non-linear lumped parameter system perturbed by noise processes which are assumed to be small. The performance of various approximations to the optimal stochastic controller based on a deterministic model and various different measurement schemes has been considered. Orders of magnitude for performance deterioration resulting from using the approximations rather than the optimal controller have been established theoretically and verified using examples. Further theoretical work should be aimed at relaxing the assumptions used in the derivation of the theory. From the practical point of view, testing of the approximate control schemes on real processes would be useful.

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