IMPERIAL

PhD Thesis

On Gaussian multiplicative chaos and random geometry

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A thesis presented for the degree of
Doctor of Philosophy of Imperial College London
and the
Diploma of Imperial College

Department of Mathematics

June 2025

Declaration of Authorship

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Abstract

Department of Mathematics

Doctor of Philosophy

On Gaussian multiplicative chaos and random geometry

by Federico BERTACCO

This thesis explores three interconnected topics: Gaussian multiplicative chaos (GMC), random planar maps, and random geometric objects in dimensions three and higher. The theory of GMC constructs random measures by formally exponentiating a log-correlated Gaussian field scaled by a parameter $\gamma \geq 0$. These measures undergo a phase transition at $\gamma_{\mathrm{c}} = \sqrt{2d}$, where $d \geq 1$ is the ambient dimension. The regime $\gamma < \gamma_{\mathrm{c}}$ is subcritical, $\gamma = \gamma_{\mathrm{c}}$ is critical, and $\gamma > \gamma_{\mathrm{c}}$ is supercritical. First, we investigate supercritical GMC measures. Specifically, we analyse the local structure of $\star$-scale invariant fields around their extremal points and gain a deeper understanding of the freezing phenomenon in supercritical GMC. We also establish the universality of supercritical GMC: for general convolution approximations of log-correlated Gaussian fields, the supercritical chaos measures converge to a nontrivial limit, which depends on the regularisation only through a multiplicative constant. Additionally, we rigorously prove that subcritical GMC satisfies the multifractal formalism. Next, we study random planar maps and their connection to Liouville quantum gravity (LQG), a canonical model of fractal surfaces with random geometry. We focus on the Smith embedding of planar maps, defined via discrete harmonic functions on the set of primal and dual vertices. We show that if a sequence of embedded planar maps satisfies an invariance principle assumption, then the a priori embedding is asymptotically close to an affine transformation of the Smith embedding. As a consequence, we establish the convergence of the Smith embedding of mated-CRT maps to LQG. Finally, we extend to higher dimensions certain random geometric objects originally introduced in the two-dimensional setting. Specifically, we construct the higher-dimensional analogue of the Liouville Brownian motion and analyse its spectral dimension. Furthermore, in arbitrary even dimensions, we construct the higher-dimensional analogue of the quantum cone.

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my advisor, Martin Hairer, for accepting me as his student and for his exceptional guidance and support throughout these years. I am especially grateful for the opportunity to pursue a topic beyond his immediate area of expertise and for his constant availability for discussion and collaboration. I will always carry with me the intuitive perspective and clarity of thought that characterise his mathematical insight.

During my years as a PhD student, I had the opportunity to work with several inspiring researchers. In particular, I would like to thank Ewain Gwynne and Scott Sheffield for giving me the chance to collaborate with them – I learned a great deal from this experience. A special thank you goes to Ewain for hosting me at the University of Chicago and for the many insightful mathematical discussions we had during my stay. I would also like to thank Christophe Garban for numerous stimulating conversations.

I would like to thank Ellen Powell and Pierre-François Rodriguez for agreeing to be the examiners for this thesis.

I would like to thank Luca Scarpa and Carlo Orrieri for sparking my interest in stochastic analysis and for guiding me through some of my early experiences with mathematical research. I am deeply indebted to both of them for believing in me and for their invaluable support, which played a crucial role in shaping my mathematical journey.

I would like to thank my fellow PhD colleagues for making this journey more enjoyable, with a special mention to Francesco and Maria Chiara. Thanks also to Jinwoo for the great company during the many conferences we attended together.

Above all, I would like to thank Laura for her unwavering love and support during these challenging years. Her presence has been invaluable, and I am truly grateful. I am also deeply grateful to my parents for always standing by my decisions and for their constant support throughout this journey.

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Chapter 1

Introduction

The theory of *Gaussian multiplicative chaos* (GMC) studies random measures formally defined by taking the exponential of a log-correlated Gaussian field. This theory was initially introduced by Kahane in his seminal paper [Kah85] to provide a rigorous foundation for Mandelbrot's model of energy dissipation in fully developed turbulence [Man72]. Since then, it has found applications in a wide range of fields, including mathematical finance [DRV12, FFGS22, NR18], mathematical physics [DS11, RV11], random matrix theory [BWW18], and number theory [SW20].

One of the most prominent fields in which the theory of GMC has found applications is *Liouville quantum gravity* (LQG). Introduced in the physics literature by Polyakov [Pol81] as a model for constructing a random geometry on a two-dimensional manifold, the theory of LQG was first made mathematically rigorous in works by Duplantier and Sheffield [DS11], as well as Rhodes and Vargas [RV11]. Roughly speaking, an LQG surface is a random surface whose Riemannian metric tensor is obtained by multiplying the Euclidean metric tensor by the exponential of the *Gaussian free field* (GFF). Several fundamental objects associated with this metric tensor have been rigorously defined, including the LQG area measure, the LQG metric distance, and the Liouville Brownian motion. One of the main reasons why the theory of LQG has attracted significant attention in recent years is that LQG surfaces are conjectured – and, in some cases, proved – to arise as scaling limits of random planar maps.

This introduction is structured as follows. In Section 1.1, we provide a brief overview of multiplicative cascades, which can be seen as the discrete counterpart of GMC. In Section 1.2, we introduce log-correlated Gaussian fields, which play a fundamental role in this thesis. Section 1.3 is devoted to the theory of GMC measures, where we outline some of their key properties. In Section 1.4, we present a concise introduction to the theory of LQG and its connection to random planar maps. Finally, in Section 1.5, we summarise the main results of this thesis

1.1 Multiplicative cascades

In the series of articles [Man72, Man74a, Man74b], Mandelbrot proposed a simple model, known as multiplicative cascades, to describe the dissipation of energy in turbulent fluids. Despite its simplicity, this model is of significant interest, as it can be viewed as the discrete counterpart of GMC measures.

For simplicity, we restrict our discussion to the construction of multiplicative cascades on binary trees. We begin by introducing some notation. We define the symbolic space $\mathcal{O} = \cup_{n=0}^{\infty} \mathcal{O}_n$, where $\mathcal{O}_n = \{0,1\}^n$ denotes the edges of the tree at the n-th level, with the convention that the root $\mathcal{O}_0$ is the empty-set. The

generic $\sigma \in \mathcal{O}_n$ is written as $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, and we write $\sigma|k = \sigma_1\cdots\sigma_k$ for $k \in \{1,\dots,n\}$. Every $\sigma \in \mathcal{O}_n$ naturally corresponds to a unique half-open dyadic interval I_σ of $[0,1]$. For $x \in [0,1]$ we denote by $I_n(x)$ the unique such interval that contains x. Finally, for every $x \in [0,1]$, we denote by $\sigma(x)$ the unique binary string such that $I_{\sigma(x)|n} = I_n(x)$ for all $n \in \mathbb{N}$. Consider a positive random variable X such that $\mathbb{E}[X] = 1$ and let $(X_\sigma)_{\sigma\in\mathcal{O}}$ be an i.i.d. collection of copies of X. For every $n \in \mathbb{N}$, we define on the interval $[0,1]$ the measure ρ_n given by

$$\rho_n(dx) \stackrel{\text{def}}{=} \prod_{k=1}^{n} X_{\sigma(x)|k}\,dx\,. \tag{1.1.1}$$

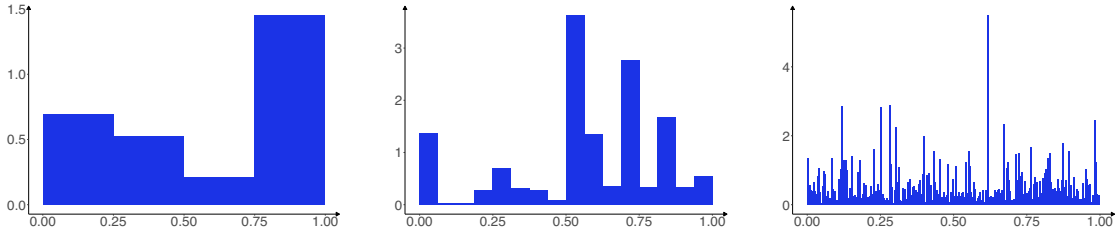


FIGURE 1.1: Densities of ρ_2, ρ_4 and ρ_6 for X a log-normal random variable.

Remark 1.1.1. We point out that the condition $\mathbb{E}[X] = 1$ is the "natural" one. Indeed, we have

$$\rho_{n+1}([0,1]) = \int_{[0,1]} \prod_{k=1}^{n+1} X_{\sigma(x)|k}\,dx = 2^{-(n+1)} \sum_{\sigma\in\mathcal{O}_{n+1}} \prod_{k=1}^{n+1} X_{\sigma|k}\,,$$

and so

$$\mathbb{E}\left[\rho_{n+1}([0,1])\right] = 2^{-n} \sum_{\sigma\in\mathcal{O}_n} \prod_{k=1}^{n} \mathbb{E}\left[X_{\sigma|k}\right] \mathbb{E}[X] = \mathbb{E}\left[\rho_n([0,1])\right] \mathbb{E}[X]\,.$$

Therefore, the condition $\mathbb{E}[X] = 1$ ensures that the expectation of the total mass is kept constant, so that it does not degenerate or explode.

The following result concerns with the almost sure existence and non-triviality of the weak limit of the sequence of random measures $(\rho_n)_{n\in\mathbb{N}}$.

Proposition 1.1.2 ([BS09, Theorem 2.1])**.** *Assume that* $\mathbb{E}[X \log_2 X] < 1$. *Then, the sequence of measures* $(\rho_n)_{n\in\mathbb{N}}$ *converges almost surely, as* $n \to \infty$, *in the space of non-negative, locally finite measures on* $[0,1]$ *equipped with the topology of vague convergence, to a non-degenerate random measure* ρ.

Many properties of multiplicative cascades are known: existence of moments, power law spectrum and multifractal behaviour. We refer to [Baroo] and references therein for further details.

Remark 1.1.3. The construction of multiplicative cascades can be easily generalised to the d-dimensional setting, with $d \geq 2$. In this case, the condition for the non-degeneracy of the limiting measure becomes $\mathbb{E}[X \log_{2^d} X] < 1$.

Example 1.1.4. We now consider the case of a multiplicative cascade associated with a log-normal random variable. More precisely, for every $\gamma \in \mathbb{R}$, we define

$$X_\gamma \stackrel{\text{def}}{=} e^{\gamma Z - \frac{1}{2}\gamma^2}\,,$$

where Z is a standard normal random variable. The multiplicative cascade ρ_γ associated with X_γ is the limit of the approximate measures $\rho_{n,\gamma}$ defined as

$$\rho_{n,\gamma}(dx) \overset{\text{def}}{=} \prod_{k=1}^{n} e^{\gamma Z_{\sigma(x)|k} - \frac{1}{2}\gamma^2} dx = e^{\gamma \sum_{k=1}^{n} Z_{\sigma(x)|k} - \frac{1}{2} n\gamma^2} dx\,, \quad \forall\, n \in \mathbb{N}\,,$$

where $(Z_\sigma)_{\sigma\in\mathcal{O}}$ is a collection of independent standard normal random variables. An easy computation shows that the non-degeneracy condition reads

$$\frac{\gamma^2}{\log 2} < 2d\,,$$

which, as we will see, is reminiscent of the non-degeneracy condition of GMC measures.

We conclude this subsection with a simple heuristic argument which emphasises the connection between multiplicative cascades and GMC measures. As a slight abuse of notation, we denote by $\rho_n(x)$ the density of the measure ρ_n at a point $x \in [0,1]$. Then, by (1.1.1) we have that

$$Y_n(x) \overset{\text{def}}{=} \log \rho_n(x) = \sum_{k=1}^{n} \log X_{\sigma(x)|n}\,, \quad x \in [0,1]\,.$$

Therefore, for large n, by the central limit theorem, the random variable Y_n is approximately Gaussian, and it is almost log-correlated. Indeed, for $x, y \in [0,1]$, letting $n_{x,y} = \max\{n \in \mathbb{N} \,:\, \sigma(x)|n = \sigma(y)|n\}$, we have

$$\mathbb{E}\left[Y_n(x)Y_n(y)\right] = \min\left(n, n_{x,y}\right)\,,$$

and so, one can easily check that

$$\mathbb{E}\left[Y_n(x)Y_n(y)\right] = \min\left(n, -\frac{1}{\log 2}\log|x-y|\right) + O(1)\,. \tag{1.1.2}$$

Hence, multiplicative cascades can be roughly seen as the exponential of log-correlated Gaussian fields, i.e., as a particular case of GMC measures.

1.2 Log-correlated Gaussian fields

Log-correlated Gaussian fields play a central role in the theory of GMC measures. For $d \geq 1$, given a domain $\mathrm{D} \subseteq \mathbb{R}^d$, a log-correlated Gaussian field X on D is formally defined as a centred Gaussian field with a covariance kernel given by

$$\mathcal{K}(x,y) \overset{\text{def}}{=} \mathbb{E}\big[\mathrm{X}(x)\mathrm{X}(y)\big] = -\log|x-y| + g(x,y)\,, \qquad \forall\, x, y \in \mathrm{D}\,, \tag{1.2.1}$$

for some continuous function $g : \overline{\mathrm{D}} \times \overline{\mathrm{D}} \to \mathbb{R}$. Since the covariance kernel $\mathcal{K}$ exhibits a logarithmic divergence along the diagonal, the random field X cannot be defined as a functional field, meaning a field indexed by points in the domain D. Instead, it is only well-defined as a distributional field, i.e., a field indexed by test functions. The existence of a Gaussian process with the given covariance structure follows directly from Kolmogorov's consistency criterion.

If $\mathrm{D} \subset \mathbb{R}^d$ is a bounded domain, the existence of such a stochastic process can also be established through a direct construction. Specifically, one can verify that $\mathcal{K}$, as defined in (4.2.1), is the kernel of a self-adjoint, symmetric, and compact operator on $L^2(\mathbb{R}^d)$. Consequently, by the spectral theorem, there exists

a non-increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ of strictly positive eigenvalues, with the corresponding eigenfunctions $(\varphi_n)_{n\in\mathbb{N}}$ forming an orthonormal basis of $\mathrm{Ker}(\mathcal{K})^\perp$ in $L^2(\mathbb{R}^d)$. This allows the field X to be defined via its Karhunen–Loève expansion:

$$\mathrm{X} = \sum_{n=1}^{\infty} \alpha_n \sqrt{\lambda_n}\varphi_n \,, \tag{1.2.2}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is a collection of i.i.d. standard normal random variables. It can be proved that the convergence of the series takes place in the space $\mathcal{H}^{-\kappa}(\mathrm{D})$, for any $\kappa > 0$ (see [JSW20, Proposition 2.3]).

Certain special types of log-correlated Gaussian fields are particularly convenient to study in the context of GMC and/or possess a canonical structure in some sense. As a result, they play a significant role in this thesis. Below, we discuss some of these fields.

Two-dimensional zero-boundary GFF An important example of a log-correlated Gaussian field is the two-dimensional zero-boundary *Gaussian free field* (GFF). Consider a bounded domain $\mathrm{D} \subset \mathbb{R}^2$ with a harmonically non-trivial boundary, meaning that a standard Brownian motion started from any point $x \in \mathrm{D}$ almost surely hits $\partial\mathrm{D}$. The zero-boundary GFF on D is the log-correlated Gaussian field whose covariance kernel is given by the Green's function $\mathcal{G}_\mathrm{D}$ of the Laplacian in D with zero-boundary conditions, i.e.,

$$\mathcal{G}_\mathrm{D}(x,y) \overset{\text{def}}{=} \begin{cases} -\log|x-y| - \widetilde{\mathcal{G}}_x(y)\,, & x,y \in \mathrm{D}\,, \\ 0\,, & x,y \notin \mathrm{D}\,, \end{cases}$$

where $\widetilde{\mathcal{G}}_x(y)$ is the harmonic extension to D of the function on $\partial\mathrm{D}$ given by $y \mapsto -\log|x-y|$. In this case, the Karhunen–Loève expansion takes the form

$$\mathrm{X} \overset{\text{def}}{=} \sum_{n=1}^{\infty} \alpha_n \frac{1}{\sqrt{\lambda_n}}\varphi_n \,, \tag{1.2.3}$$

where $(\varphi)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathrm{D})$ consisting of eigenfunctions of $-\Delta$ that vanish on $\partial\mathrm{D}$, and $(\lambda_n)_{n\in\mathbb{N}}$ are the corresponding eigenvalues. Indeed, in this case, the operator whose kernel is $\mathcal{G}_\mathrm{D}$ is exactly given by $-(\Delta/2\pi)^{-1}$. One of the key properties of the zero-boundary GFF is that it is conformally invariant (see [BP24, Section 1.11]). We refer to Figure 1.2 for a simulation of a discrete approximation of the zero-boundary GFF

The class of $\star$-scale invariant fields Another important class of log-correlated Gaussian fields is the family of $\star$-*scale invariant fields*. These are stationary, centred Gaussian fields that naturally admit an approximation by a martingale and satisfy a specific form of scale invariance known as $\star$-scale invariance (see [RV14a, Section 2.3]).

Definition 1.2.1. Consider a function $\mathfrak{K} \in \mathcal{C}^\infty(\mathbb{R}^d)$ that is radial, positive definite, satisfies $\mathfrak{K}(0) = 1$, and $|\mathfrak{K}(x)| \le C(1+|x|)^{-a}$ for some $C, a > 0$ and all $x \in \mathbb{R}^d$. Let $\overline{\mathfrak{K}} : \mathbb{R}^d \to \mathbb{R}$ denote the (unique) positive definite function whose convolution with itself equals $\mathfrak{K}$. Let ξ be a space-time white noise on $\mathbb{R}^d \times (0,\infty)$. We define the $\star$-*scale invariant field with seed covariance* $\mathfrak{K}$ as follows:

$$\mathrm{X}(\cdot) \overset{\text{def}}{=} \int_{\mathbb{R}^d}\int_0^\infty \overline{\mathfrak{K}}\big(e^r(y-\cdot)\big)e^{\frac{dr}{2}}\,\xi(dy,dr)\,. \tag{1.2.4}$$

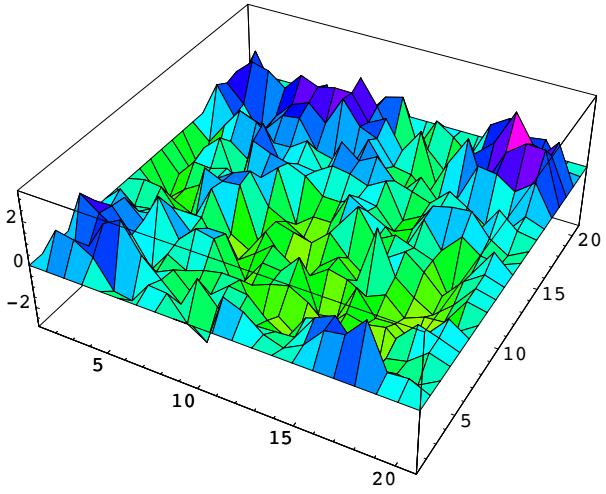


FIGURE 1.2: Simulation of a discretised version of a two-dimensional zero-boundary GFF. As the mesh-size tends to zero, this field converges to a continuum zero-boundary GFF.

Furthermore, for any $0 \leq s \leq t$, we introduce the field $\mathrm{X}_{s,t}$ on $\mathbb{R}^d$ given by

$$\mathrm{X}_{s,t}(\cdot) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \int_s^t \overline{\mathfrak{K}}\big(e^r(y-\cdot)\big) e^{\frac{dr}{2}}\, \xi(dy, dr) \;, \tag{1.2.5}$$

with the notational convention that $\mathrm{X}_t = \mathrm{X}_{0,t}$.

From its definition, it is readily seen that for all $s, t \geq 0$ and $x, y \in \mathbb{R}^d$, it holds that

$$\mathbb{E}\big[\mathrm{X}(x)\mathrm{X}(y)\big] = \int_0^\infty \mathfrak{K}\big(e^r(x-y)\big) dr \;, \qquad \mathbb{E}\big[\mathrm{X}_t(x)\mathrm{X}_s(y)\big] = \int_0^{t\wedge s} \mathfrak{K}\big(e^r(x-y)\big) dr \;. \tag{1.2.6}$$

Clearly, $\mathbb{E}[\mathrm{X}(x)^2] = \infty$, so X can only be realised as a random generalised function.

The whole space log-correlated Gaussian field Another important instance of log-correlated Gaussian field is the whole-space *log-correlated Gaussian field* (LGF). For $d \geq 2$, let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space with its usual topology, and denote by $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions, i.e., continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$. Define $\mathcal{S}_0(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ as the subspace of Schwartz functions ϕ satisfying $\widehat{\phi}(0) = 0$, where $\widehat{\phi}$ denotes the Fourier transform of ϕ. We also let $\mathcal{S}_0'(\mathbb{R}^d)$ be the topological dual of $\mathcal{S}_0(\mathbb{R}^d)$. The homogeneous Sobolev space of order $d/2$, denoted by $\mathcal{H}_d$, is the Hilbert space completion of functions $f \in \mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}_0'(\mathbb{R}^d)$ such that $\xi \mapsto |\xi|^{d/2} \widehat{f}(\xi) \in L^2(\mathbb{R}^d)$, equipped with the following inner product[1]:

$$(\varphi, \psi)_d \stackrel{\text{def}}{=} \frac{1}{2^{d-1}\pi^{d/2}\Gamma(d/2)} \int_{\mathbb{R}^d} |\omega|^d \widehat{\varphi}(\omega)\overline{\widehat{\psi}(\omega)} d\omega \;.$$

The whole-space LGF is the canonical Gaussian random variable in the Hilbert space $\mathcal{H}_d$. In other words, the whole-space LGF h is defined as follows

$$h \stackrel{\text{def}}{=} \sum_{n=1}^\infty \alpha_n \varphi_n \;, \tag{1.2.7}$$

[1]Here the choice of normalisation is such that the covariance kernel for the LGF is given by $-\log|x-y|$ rather than a multiple of it.

where α_n are i.i.d. centred normal random variables and $(\varphi_n)_{n\in\mathbb{N}}$ is an orthonormal basis for the Hilbert space $\mathcal{H}_d$. The sum converges almost surely in the space $\mathcal{S}'_0(\mathbb{R}^d)$. Alternatively, the whole-space LGF is a random element $h \in \mathcal{S}'_0(\mathbb{R}^d)$, i.e., a random distribution defined modulo a global additive constant, whose law is centred Gaussian with the following covariance:

$$\mathbb{E}\big[(h,\varphi)(h,\psi)\big] = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \log|x-y|\varphi(x)\psi(y)dxdy\ , \qquad \forall\, \phi,\psi \in \mathcal{S}_0(\mathbb{R}^d)\ .$$

Remark 1.2.2. If $d = 2$, the whole-space LGF coincides with the whole-plane GFF, as defined, for instance, in [DMS21]. However, for $d > 2$, the whole-space LGF does *not* coincide with the whole-space GFF, as the latter is no longer log-correlated.

An important feature of the whole-space LGF is its conformal invariance in every dimension $d \geq 2$. More precisely, if ϕ is a conformal automorphism of $\mathbb{R}^d$, then the law of $h \circ \phi$ coincides with the law of h (see [DRSV17, Proposition 1.2]).

1.2.1 Regularisation of log-correlated Gaussian fields

In the remainder of this thesis, we will focus on exponentiating a log-correlated Gaussian field. As discussed above, such fields are inherently rough, meaning they can only be rigorously interpreted as random generalised functions. Consequently, their exponential is, a priori, an ill-posed object. A natural way to handle this issue is to introduce a family of smooth fields that approximate the log-correlated Gaussian field under consideration, take their exponentials, and, after a suitable renormalisation, show that we can pass to the limit to remove the smoothing and obtain a well-defined, non-trivial object. In this section, we introduce the most natural ways to regularise a log-correlated Gaussian field.

Martingale approximation of a $\star$-scale invariant field For a $\star$-scale invariant field X on $\mathbb{R}^d$, as defined in (1.2.4), the most natural approximation to consider is its *martingale approximation* $(\mathrm{X}_t)_{t\geq 0}$, given in (1.2.5). This nomenclature arises from the fact that $(\mathrm{X}_t)_{t\geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, where $\mathcal{F}_t \overset{\text{def}}{=} \sigma\big(\mathrm{X}_s\ :\ s \in [0,t)\big)$. Furthermore, for each fixed $x \in \mathbb{R}^d$, the process $(\mathrm{X}_t(x))_{t\geq 0}$ has the same law as a standard Brownian motion. Moreover, as $t \to \infty$, the field X_t converges almost surely to X in the space $\mathcal{H}^{-\kappa}_{\mathrm{loc}}(\mathbb{R}^d)$ for any $\kappa > 0$. We refer to [JSW19, Proposition 4.1 (iv)] for a proof of this fact.

Convolution approximation Let X be a log-correlated Gaussian field defined on a domain $\mathrm{D} \subseteq \mathbb{R}^d$ and with covariance kernel $\mathcal{K}$ of the form (1.2.1). Consider a compactly supported smooth function $\rho : \mathbb{R}^d \to \mathbb{R}$ with unit mass. For $\varepsilon > 0$, we set $\rho_\varepsilon(\cdot) \overset{\text{def}}{=} \varepsilon^{-d}\rho(\varepsilon^{-1}\cdot)$. The convolution approximation $(\mathrm{X}_\varepsilon)_{\varepsilon>0}$ of X is defined as follows

$$\mathrm{X}_\varepsilon \overset{\text{def}}{=} \rho_\varepsilon * \mathrm{X}\ . \tag{1.2.8}$$

One can easily check that the covariance kernel of the field X_ε is given by

$$\mathbb{E}\big[\mathrm{X}_\varepsilon(x)\mathrm{X}_\varepsilon(y)\big] = \big((\rho_\varepsilon \otimes \rho_\varepsilon) * \mathcal{K}\big)(x,y)\ , \qquad \forall\, x,y \in \mathrm{D}\ . \tag{1.2.9}$$

where here we assume that $\mathcal{K}(x,y) = 0$ for $(x,y) \notin \mathrm{D}\times\mathrm{D}$.

Series truncation If X is a log-correlated Gaussian field on a bounded domain $\mathrm{D} \subseteq \mathbb{R}^d$, then it admits the series representation (1.2.2). Hence, in this setting, it is natural to consider a sequence of regularizing fields

$(\mathrm{X}_n)_{n\in\mathbb{N}}$ obtained by truncating the series (1.2.2). More precisely, for any $n \in \mathbb{N}$, we define:

$$\mathrm{X}_n \overset{\text{def}}{=} \sum_{k=1}^{n} \alpha_k \sqrt{\lambda_k}\varphi_k \,,$$

Similarly, if h is the whole-space LGF defined in (1.2.7), then we can define the sequence $(h_n)_{n\in\mathbb{N}}$ of approximating fields by letting

$$h_n \overset{\text{def}}{=} \sum_{k=1}^{n} \alpha_k \varphi_k \,,$$

Spherical average Let h be a whole-space LGF, as introduced in (1.2.7). In the LQG literature, a widely used approximation is the spherical average approximation – known as the circle average approximation in the two-dimensional setting – obtained by testing the field h against the uniform probability measure on spheres with radii shrinking to zero.

One of the key reasons why the spherical average approximation is convenient to work with is the following. Let $x \in \mathbb{R}^d$ and denote by $(h_r(x) - h_1(x))_{r>0}$ the recentred spherical average process around x. It can then be shown that, in dimension $d = 2$, the process $(h_{e^{-t}}(x) - h_1(x))_{t\in\mathbb{R}}$ has the same law as a standard two-sided Brownian motion (see [BP24, Theorem 1.59]). In any even dimension $d > 2$, it has the same law as the integral of a stationary Gaussian Markov process of order $(d-2)/2$. We refer to Chapter 6 for more details.

1.3 Gaussian multiplicative chaos

As we have seen in Section 1.1, the tree structure underlying multiplicative cascades makes them easier to analyse but also introduces certain lattice effects. For instance, multiplicative cascades satisfy an exact scaling property only for dyadic intervals and do not exhibit stationary fluctuations. This motivates the development of a continuum theory of multiplicative cascades. In his seminal 1985 paper [Kah85], Kahane developed a framework to rigorously define random measures of the form

$$\mu_\gamma(dx) \text{ “=” } e^{\gamma \mathrm{X}(x)} dx \,, \tag{1.3.1}$$

where X is a log-correlated Gaussian field defined on a domain $\mathrm{D} \subseteq \mathbb{R}^d$, with $d \geq 1$, $\gamma \geq 0$ is a real parameter, and dx denotes the Lebesgue measure on D. As we have already emphasised, the main challenge in interpreting the expression in (1.3.1) is that X is a rough Gaussian field that cannot be defined pointwise in D but only as a random distributional field. However, it is now well understood how to overcome this difficulty. Specifically, there is a standard approach that involves a sequence of steps: regularisation, renormalisation, and a limiting procedure.

It is now well-known that the behaviour of the random measure (1.3.1) undergoes a phase transition at

$$\gamma_{\mathrm{c}} \overset{\text{def}}{=} \sqrt{2d} \,.$$

Following standard convention, we call the regime $\gamma < \sqrt{2d}$ *subcritical*, the borderline case $\gamma = \sqrt{2d}$ *critical*, and the range $\gamma > \sqrt{2d}$ *supercritical*. These three different regimes differ in the normalisation needed to obtain a non-trivial limiting measure, as well as in the qualitative features of the limiting measure (see also [LRV15] for a more detailed phase diagram including complex values of γ).

1.3.1 Subcritical GMC

Consider a log-correlated Gaussian field X defined on a domain $D \subseteq \mathbb{R}^d$, with $d \geq 1$, and a covariance kernel of the form (1.2.1). For simplicity, we focus on the convolution approximation $(X_\varepsilon)_{\varepsilon>0}$ of X, as defined in (1.2.8). However, it is important to emphasise that all the results stated in this section are independent of the specific choice of approximation scheme.

We begin by recalling the following standard result regarding the existence and non-degeneracy of subcritical GMC measures.

Proposition 1.3.1 ([Ber17, Theorem 1.1]). *For $\gamma < \sqrt{2d}$, the sequence of random measures*

$$\mu_{\gamma,\varepsilon}(dx) \stackrel{\text{def}}{=} e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\varepsilon(x)^2]} dx \,, \tag{1.3.2}$$

converges in probability, as $\varepsilon \to 0$, in the space of non-negative, locally finite measures on D equipped with the topology of vague convergence, to a non-degenerate random measure μ_γ.

Remark 1.3.2. If one applies the same normalisation as in (1.3.2) for $\gamma \geq \sqrt{2d}$, the limiting measure becomes degenerate. Specifically, for all compact subsets $A \subset D$, it holds that $\lim_{\varepsilon\to 0} \mu_{\gamma,\varepsilon}(A) = 0$ in probability (see [Pow21, Lemma 2.1] for a proof of this result). We also refer to Remark 1.3.4 below for a heuristic argument explaining the emergence of the critical threshold $\gamma_c = \sqrt{2d}$.

It is a well-known fact that μ_γ assigns full measure to the set of γ-thick points of the underlying field X, which is defined as follows:

$$\mathcal{T}_\gamma = \left\{ x \in D \,:\, \lim_{\varepsilon\to 0} \frac{X_\varepsilon(x)}{|\log \varepsilon|} = \gamma \right\}. \tag{1.3.3}$$

The set $\mathcal{T}_\gamma$ consists of points $x \in D$ where the field X attains unusually large values – of the order of its variance rather than its standard deviation. It is straightforward to see that for any fixed $x \in D$, we almost surely have $x \in \mathcal{T}_0$. It is also well known (see [RV14a, Theorem 4.2]) that for $\gamma \geq 0$, the Hausdorff dimension of $\mathcal{T}_\gamma$ is given by

$$\dim_{\mathcal{H}}(\mathcal{T}_\gamma) = \left(d - \frac{1}{2}\gamma^2\right) \vee 0 \,. \tag{1.3.4}$$

Furthermore, for $\gamma > \sqrt{2d}$, we have $\mathcal{T}_\gamma = \emptyset$. This follows from the fact that the maximum of the field X_ε in a domain of order one is approximately $\sqrt{2d}|\log\varepsilon|$, up to lower-order terms. In particular, this implies that the set $\mathcal{T}_{\sqrt{2d}}$ consists of the extremal points of the field X.

Remark 1.3.3. A heuristic way to justify why points thicker than γ are assigned null measure by μ_γ is to observe that, by the Cameron–Martin theorem (see Lemma A.0.1), for $\alpha > 0$, it holds that

$$\begin{aligned}
&\mathbb{E}\Big[\mu_{\gamma,\varepsilon}\big(\{x \in D \,:\, |X_\varepsilon(x) - \gamma|\log\varepsilon|| > \alpha|\log\varepsilon|\}\big)\Big] \\
&\quad = \int_D \mathbb{E}\Big[e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\varepsilon(x)^2]} \mathbb{1}_{\{|X_\varepsilon(x) - \gamma|\log\varepsilon|| > \alpha|\log\varepsilon|\}}\Big] dx \\
&\quad = \int_D \mathbb{P}\big(|X_\varepsilon(x)| > \alpha|\log\varepsilon|\big) dx \\
&\quad = O(\varepsilon^{\alpha^2/2}) \to 0 \,, \qquad \text{as } \varepsilon \to 0 \,.
\end{aligned}$$

Sketch of the proof of Proposition 1.3.1. The proof can be divided in two different cases. The first case, known as the L^2-phase, corresponds to the regime in which $\gamma < \sqrt{d}$ and it is quite simple to treat. The second case, known as the L^1-phase, corresponds to the regime in which $\sqrt{d} \leq \gamma < \sqrt{2d}$ and it is more involved.

The L^2-phase. We fix a smooth continuous supported test function $f \in \mathcal{C}_c(\mathrm{D})$. For $\varepsilon > 0$, we have that

$$\begin{aligned}\mathbb{E}\left[\left|\int_{\mathrm{D}} f(x)\mu_{\gamma,\varepsilon}(dx)\right|^2\right] &= \int_{\mathrm{D}\times\mathrm{D}} f(x)f(y)\mathbb{E}\left[e^{\gamma(\mathrm{X}_\varepsilon(x)+\mathrm{X}_\varepsilon(y))-\frac{\gamma^2}{2}\mathbb{E}[\mathrm{X}_\varepsilon(x)^2+\mathrm{X}_\varepsilon(y)^2]}\right]dxdy \\ &= \int_{\mathrm{D}\times\mathrm{D}} f(x)f(y)e^{\gamma^2\mathbb{E}[\mathrm{X}_\varepsilon(x)\mathrm{X}_\varepsilon(y)]}dxdy \\ &\lesssim \int_{\mathrm{D}\times\mathrm{D}} |x-y|^{-\gamma^2}dxdy\,,\end{aligned}$$

so that $\mu_{\gamma,\varepsilon}(f)$ is uniformly bounded in L^2 as long as $\gamma < \sqrt{d}$. In fact, one can similarly prove that in this regime the sequence $\mu_{\gamma,\varepsilon}(f)$ is Cauchy in L^2 and hence converging. It is then standard to lift the convergence of $\mu_{\gamma,\varepsilon}(f)$ for any test function f to a convergence of the sequence of random measures $\mu_{\gamma,\varepsilon}$ with respect to the topology of vague convergence.

The L^1-phase. The above argument clearly does not apply when $\gamma \in [\sqrt{d}, \sqrt{2d})$. However, one can still establish convergence by leveraging the fact that the measure actually "lives" on the set of γ-thick points. In particular, the blow-up of the second moment is caused by points that are "thicker" than γ, but these points contribute nothing in the L^1 sense, as they are assigned zero mass[2].

More precisely, for $\alpha > 0$, one can introduce the event $\mathrm{G}_{\varepsilon,\alpha}(x) \overset{\text{def}}{=} \{\mathrm{X}_\varepsilon - \alpha|\log\varepsilon| \le 0\}$ and for a continuous compactly supported test function $f \in \mathcal{C}_c(\mathrm{D})$ consider the quantity

$$\begin{aligned}&\mathbb{E}\left[\left|\int_{\mathrm{D}} f(x)\mathbb{1}_{\mathrm{G}_{\varepsilon,\alpha}(x)}\mu_{\gamma,\varepsilon}(dx)\right|^2\right] \\ &\qquad = \int_{\mathrm{D}\times\mathrm{D}} f(x)f(y)\mathbb{E}\left[e^{\gamma(\mathrm{X}_\varepsilon(x)+\mathrm{X}_\varepsilon(y))-\frac{\gamma^2}{2}\mathbb{E}[\mathrm{X}_\varepsilon(x)^2+\mathrm{X}_\varepsilon(y)^2]}\mathbb{1}_{\mathrm{G}_{\varepsilon,\alpha}(x)}\mathbb{1}_{\mathrm{G}_{\varepsilon,\alpha}(y)}\right]dxdy\,.\end{aligned}$$

For x, $y \in \mathrm{D}$, the two processes $\mathrm{X}_{e^{-t}}(x)$ and $\mathrm{X}_{e^{-t}}(y)$ are strongly correlated for $t < |\log\varepsilon|$ and almost independent for $t > |\log\varepsilon|$. Using these facts, the Markov property of Brownian motion and Cameron–Martin theorem (see Lemma A.0.1), one gets the following uniform bound

$$\int_{\mathrm{D}\times\mathrm{D}} f(x)f(y)\mathbb{E}\left[e^{\gamma(\mathrm{X}_\varepsilon(x)+\mathrm{X}_\varepsilon(y))-\frac{\gamma^2}{2}\mathbb{E}[\mathrm{X}_\varepsilon(x)^2+\mathrm{X}_\varepsilon(y)^2]}\mathbb{1}_{\mathrm{G}_{\varepsilon,\alpha}(x)}\mathbb{1}_{\mathrm{G}_{\varepsilon,\alpha}(y)}\right]dxdy \lesssim |x-y|^{\frac{1}{2}(\alpha-2\gamma)^2-\gamma^2}\,.$$

Therefore, taking $\alpha > \gamma$ close to γ, one gets that

$$\mathbb{E}\left[\left|\int_{\mathrm{D}} f(x)\mathbb{1}_{\mathrm{G}_{\varepsilon,\alpha}(x)}dx\right|^2\right] \lesssim \int_{\mathrm{D}\times\mathrm{D}} |x-y|^{-\frac{\gamma^2}{2}+0}dxdy\,,$$

which is finite if and only if $\gamma < \sqrt{2d}$. The previous calculations have shown that the measure $\mathbb{1}_{\mathrm{G}_{\varepsilon,\alpha}}(x)\mu_{\gamma,\varepsilon}(dx)$ can be treated as in the L^2-phase, and so it has a limit in L^2. To conclude, it suffices to note that $\mu_{\gamma,\varepsilon} - \mathbb{1}_{\mathrm{G}_{\varepsilon,\alpha}}\mu_{\gamma,\varepsilon}$ converges to zero. □

For $\gamma < \sqrt{2d}$, the subcritical GMC measure μ_γ is almost surely supported on the entire domain D (see, for instance, [RV14a]). An immediate consequence of the fact that μ_γ assigns full measure to the set of γ-thick points of X and of (1.3.4) is that μ_γ is almost surely atomless and singular with respect to $\mu_{\gamma'}$ for any $\gamma \neq \gamma'$. In particular, for $\gamma \neq 0$, μ_γ is singular with respect to the Lebesgue measure. Furthermore, as γ approaches $\sqrt{2d}$, the GMC measure μ_γ is supported on a set whose Hausdorff dimension tends to 0. This implies that μ_γ

[2]This is why the regime $\gamma \in [\sqrt{d}, \sqrt{2d})$ is sometimes referred to as the "L^1-phase".

tends to cluster around the maxima of X as γ increases, until it degenerates at $\gamma = \sqrt{2d}$. This phenomenon can be observed in Figure 1.3.

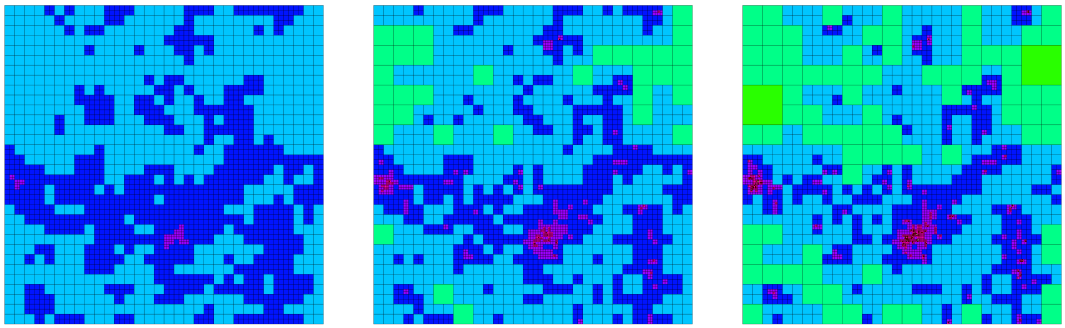


FIGURE 1.3: Consider a discrete GFF on a torus, which we view as a square with boundaries suitably identified. For $\gamma \in (0, 2)$, consider the measure obtained by exponentiating the discrete GFF multiplied by γ with respect to the counting measure on the vertices of the grid. We fix a small threshold $\delta > 0$ and proceed as follows: we divide the square into four equal sub-squares, then divide each of those into four equal sub-squares, and so on, except that we stop whenever we reach a square where the area is less than δ. In the left picture, $\gamma = 0.5$; in the middle picture, $\gamma = 1$; and in the right picture, $\gamma = 1.5$. The squares are coloured according to their Euclidean size. A similar simulation first appeared in [DS11].

It is well known that for all $\gamma < \sqrt{2d}$, the measure μ_γ admits moments of order $q < 2d/\gamma^2$, which in particular includes all $q < 0$ (see, for instance, [BP24]). Another important feature of GMC measures is their multifractal behaviour. More precisely, it can be shown that the moments of μ_γ follow a power-law scaling, with an exponent given by a nonlinear function ξ_{μ_γ}, known as the power-law spectrum, defined as

$$\xi_{\mu_\gamma}(q) := \left(d + \frac{1}{2}\gamma^2\right) q - \frac{1}{2}\gamma^2 q^2 , \qquad \forall\, q \in \mathbb{R} . \tag{1.3.5}$$

More precisely, for all $x \in D$, $r \in (0, 1]$, and $q < 2d/\gamma^2$, the following holds

$$\mathbb{E}\big[\mu_\gamma\big(B(x, r)\big)^q\big] \lesssim r^{\xi_{\mu_\gamma}(q)} .$$

The case of general reference measures We conclude this section by emphasising that, so far, we have considered GMC measures of the form (1.3.1), where the base reference measure is the Lebesgue measure on D. However, given any locally finite Borel measure ρ on D, we can also rigorously define the measure:

$$\mu_\gamma^\rho(dx) \text{ “}=\text{” } e^{\gamma X(x)} \rho(dx) ,$$

Indeed, let $\mathfrak{s}$ be the largest non-negative real number such that

$$\int_D \int_D \frac{1}{|x - y|^{\mathfrak{s}-\varepsilon}} \rho(dx)\rho(dy) < \infty . \tag{1.3.6}$$

Then, by the general theory of GMC measures (see [BP24, Section 3]), for all $\gamma < \sqrt{2\mathfrak{s}}$, the sequence of regularised measures

$$\mu_{\gamma,\varepsilon}^\rho(dx) \overset{\text{def}}{=} e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\varepsilon(x)^2]} \rho(dx)$$

converges in probability, as $\varepsilon \to 0$, to a non-degenerate random measure μ_γ^ρ in the space of non-negative, locally finite measures on D equipped with the topology of vague convergence. This fact will be used, for instance, in the construction of Liouville Brownian motion, where the measure ρ corresponds to the occupation measure of the d-dimensional Brownian motion.

1.3.2 Critical GMC

Considering the same setup as in the previous section, as noted in Remark 1.3.2, a phase transition occurs when $\gamma \geq \sqrt{2d}$. In this case, if one considers the sequence of measures defined in (1.3.2), then for any compact subset $A \subset \mathbb{R}^d$, we have $\mu_{\gamma,\varepsilon}(A) \to 0$ in probability as $\varepsilon \to 0$.

Remark 1.3.4. We outline a heuristic argument for the emergence of the critical threshold $\gamma_c = \sqrt{2d}$. Recall that for each fixed $x \in D$ and $\varepsilon > 0$, the random variable $X_\varepsilon(x)$ is centred normal with variance of order $|\log \varepsilon|$. Furthermore, recall that the maximum of $N > 0$ i.i.d. standard normal random variables is about $\sqrt{2 \log N}$. Assuming that D is a domain of order one, a rough picture for the maximum of X_ε inside D is obtained by considering ε^{-d} i.i.d. centred normal random variables with variance of order $|\log \varepsilon|$. In this case, the maximum is about $\sqrt{|\log \varepsilon|}\sqrt{2|\log \varepsilon^{-d}|} = \sqrt{2d}|\log \varepsilon|$. In particular, the contribution of each peak to the measure $\mu_{\gamma,\varepsilon}$ defined in (1.3.2) is about

$$\varepsilon^{d+\gamma^2/2} e^{\gamma\sqrt{2d}|\log \varepsilon|} = \varepsilon^{d-\gamma\sqrt{2d}+\gamma^2/2} = \varepsilon^{(\sqrt{d}-\gamma/\sqrt{2})^2} ,$$

and we see that the exponent vanishes precisely when $\gamma = \sqrt{2d}$.

The heuristic discussion above suggests that, in order to define a nontrivial limiting measure at the critical threshold $\gamma_c = \sqrt{2d}$, the sequence of approximating measures requires an additional "push" in the right direction. More precisely, we have the following result.

Proposition 1.3.5 ([Pow21, Theorem 2.5])**.** *The sequence of random measures*

$$\mu_{\gamma_c,\varepsilon}(dx) = \big(-X_\varepsilon(x) + \gamma_c|\log \varepsilon|\big) e^{\gamma_c X_\varepsilon(x) - \frac{\gamma_c^2}{2}\mathbb{E}[X_\varepsilon(x)^2]} dx \tag{1.3.7}$$

converges in probability, along any sequence $\varepsilon_n \to 0$, in the space of non-negative, locally finite measures on D *equipped with the topology of vague convergence, to a non-degenerate random measure μ_{γ_c}.*

It is worth emphasising that even in the critical regime, it is now well established that the limiting measure is independent of the specific regularisation scheme used to approximate the field X (see [JS17, Pow18, JSW19, Pow21]). The normalisation employed in (1.3.7) is referred to as the "derivative normalisation" since it can be derived by differentiating the expression on the right-hand side of (1.3.2) with respect to γ and evaluating it at $\gamma = \sqrt{2d}$.

We also highlight that the construction of critical GMC as "derivative" of subcritical GMC at $\gamma = \gamma_c$ has been made rigorous. More precisely, the following limit holds in probability in the same topology as in the above proposition,

$$\lim_{\gamma \nearrow \gamma_c} \frac{\mu_\gamma(dx)}{\gamma_c - \gamma} = 2\mu_{\gamma_c}(dx) .$$

We refer to [Pow21, Theorem 3.1] for a proof of this fact, as well as for a heuristic discussion on why the factor 2 appears on the right-hand side of the above display. We also refer to [Lac24] for a recent proof of the convergence of the critical GMC.

Remark 1.3.6. There is also an equivalent deterministic normalisation, called "Seneta–Heyde normalisation", which produces the same limiting measure μ_{γ_c} up to a deterministic multiplicative constant. More precisely, the sequence of random measures

$$\mu^{\mathrm{SH}}_{\gamma_c,\varepsilon}(dx) = \sqrt{|\log \varepsilon|} e^{\gamma_c X_\varepsilon(x) - \frac{\gamma_c^2}{2}\mathbb{E}[X_\varepsilon(x)^2]} dx \tag{1.3.8}$$

converges vaguely in probability to $\sqrt{2/\pi}\,\mu_{\gamma_c}$ as $\varepsilon \to 0$ [JSW19, Pow21].

1.3.3 Supercritical GMC

The low-temperature or supercritical regime, i.e., $\gamma > \sqrt{2d}$, is by far the least studied phase of GMC measures. In this regime, the GMC measure exhibits atomic behaviour under suitable renormalisation, with the locations and masses of the atoms dictated by the extremal statistics of the underlying log-correlated Gaussian field.

In the continuum setting, the only available mathematical result we are aware of is [MRV16]. In this article, Madaule, Rhodes, and Vargas studied a $\star$-scale invariant field X and its martingale approximation $(\mathrm{X}_t)_{t\geq 0}$, as defined in (1.2.4) and (1.2.5), and proved the convergence of the supercritical GMC measure under an appropriate normalisation. Before stating this result, we introduce the following notation.

Definition 1.3.7. For $\gamma > \gamma_c$ and a non-negative, locally finite Borel measure ν on $\mathbb{R}^d$, we let $\eta_\gamma[\nu]$ be the Poisson point measure on $\mathbb{R}^d \times (0,\infty)$ with intensity measure given by $\nu(dx) \otimes w^{-(1+\gamma_c/\gamma)}dw$. We also define the integrated atomic random measure with parameter γ and spatial intensity ν as the random purely atomic measure $\mathcal{P}_\gamma[\nu]$ on $\mathbb{R}^d$ given by

$$\mathcal{P}_\gamma[\nu](dx) \overset{\text{def}}{=} \int_0^\infty w\,\eta_\gamma[\nu](dx,dw)\ .$$

With this notation in place, the main result of [MRV16] can be stated as follows.

Proposition 1.3.8 ([MRV16, Theorem 2.2])**.** *For $\gamma > \sqrt{2d}$, the sequence of random measure*

$$\mu_{\gamma,t}(dx) = t^{\frac{3\gamma}{2\sqrt{2d}}} e^{t(\gamma/\sqrt{2}-\sqrt{d})^2} e^{\gamma \mathrm{X}_t(x) - \frac{\gamma^2}{2}t} dx\ , \tag{1.3.9}$$

converges in law in the space of non-negative, locally finite measures on $\mathbb{R}^d$, equipped with the topology of vague convergence, as $t \to \infty$ to a nontrivial, purely atomic measure μ_γ. The limiting measure μ_γ admits the following representation

$$\mu_\gamma(dx) = c_\gamma \mathcal{P}_\gamma[\mu_{\gamma_c}](dx)\ ,$$

where c_γ is a positive constant and μ_{γ_c} the critical GMC.

Remark 1.3.9. In the literature, the supercritical regime of GMC measures is often referred to as the "frozen phase". This term can be explained by the fact that the spatial landscape on which points are distributed remains frozen and corresponds to the critical GMC. The only variation occurs in the height of the atoms, which changes with the parameter $\gamma > \sqrt{2d}$.

Remark 1.3.10. The convergence in Proposition 1.3.8 indicates that the limiting measure μ_γ can be formally decomposed into two components: the location of the point masses, which is determined by an instance of the field X through the associated critical GMC measure, and an additional source of randomness that is independent of X and controls the weight of the point masses. Consequently, this implies that $\mu_{\gamma,t}$ cannot converge in probability.

A heuristic derivation We present a heuristic argument to explain the emergence of the limiting measure described in Proposition 1.3.8. Although this argument does not account for logarithmic corrections, it still provides useful insight into the origin of the limiting measure. Disregarding log-corrections, the normalised measure in (1.3.9) can be written as

$$e^{t(d-\sqrt{2d}\gamma)} e^{\gamma \mathrm{X}_t(x)} dx\ .$$

It is helpful to introduce an intermediate scale, $0 \ll s \ll t$, and, recalling (1.2.5), decompose X_t as

$$\mathrm{X}_t = \mathrm{X}_s + \mathrm{X}_{s,t} \ .$$

The first field on the right-hand side corresponds to the "large scales field" and is responsible for generating the critical GMC that determines the location of the point masses, while the second corresponds to the "small scales field" and governs the weights of the point masses. By construction, we have that

$$\mathrm{X}_s \perp \mathrm{X}_{s,t} \ , \qquad \mathrm{X}_{s,t}(\cdot) \overset{\text{law}}{=} \mathrm{X}_{t-s}(e^s \cdot) \ .$$

Since $\mathrm{X}_{s,t}$ has the same law as X_{t-s} scaled by e^s, on a domain of order one, it behaves roughly like e^{ds} independent copies of X_{t-s}[3]. We denote the collection of these independent copies by $(\mathrm{X}^i_{t-s})_{i\in[e^{ds}]}$, where $[e^{ds}] = \{1, \dots, e^{ds}\}$, and, for simplicity, we assume that $e^{ds} \in \mathbb{N}$. We recall that, for each $i \in [e^{ds}]$, the maximum $\mathbb{M}^i$ of X^i_{t-s} in a domain of order one is about $\mathbb{M}_\star = \sqrt{2d}|t-s|$. Furthermore, for all $w \in \mathbb{R}$, thanks to the Gaussian tail bound, we have that

$$\mathbb{P}\big(\mathbb{M}^i \geq \mathbb{M}_\star + w\big) \approx e^{d(t-s)} e^{-\frac{(\mathbb{M}_\star + w)^2}{2(t-s)}} \approx e^{-\sqrt{2d}w} \ . \tag{1.3.10}$$

Consequently, thanks to (1.3.10) and by a Gumbel-type argument, we have that the maximum $\max_{i\in[e^{ds}]} \mathbb{M}^i$ is of order $\sqrt{2d}|t-s| + \sqrt{d/2}s$. Furthermore, thanks again to (1.3.10), we observe that, for all $w \in \mathbb{R}$, it holds that

$$\mathbb{E}\left[\sum_{i=1}^{e^{ds}} \mathbb{1}_{\{\mathbb{M}^i - (\mathbb{M}_\star + \sqrt{d/2}s) \geq w\}}\right] = e^{ds}\mathbb{P}\big(\mathbb{M}^i \geq \mathbb{M}_\star + \sqrt{d/2}s + w\big) \approx e^{-\sqrt{2d}w} \ .$$

This fact, along with the Poisson limit theorem, implies that the collection of "residual heights" given by

$$\big\{\mathbb{M}^i - \big(\mathbb{M}_\star + \sqrt{d/2}s\big)\big\}_{i\in[e^{ds}]}$$

forms a Poisson point process on $\mathbb{R}$ with intensity $e^{-\sqrt{2d}w}dw$. Furthermore, since the correlation length of $\mathrm{X}_{s,t}$ is of order e^{-t}, the main contribution to the integral of $e^{\gamma \mathrm{X}_{s,t}(x^i)}$ comes from points within a distance of order e^{-t} from the local maximum x^i. These considerations suggest that

$$e^{t(d-\sqrt{2d}\gamma)+\sqrt{d/2}\gamma s} e^{\gamma \mathrm{X}_{s,t}(x)} dx \approx \mathcal{P}_\gamma[dx](dx) \ , \tag{1.3.11}$$

where we recall the notation introduced in Definition 1.3.7. In particular, the power-law density in the intensity of the point measure $\eta_\gamma[dx]$ arises from exponentiating γ times a Poisson point process with intensity given by $e^{-\sqrt{2d}w}dw$.

To complete the argument and arrive at the desired limiting measure, observe that, using the notation of Definition 1.3.7, for any non-negative function $f : \mathbb{R}^d \to \mathbb{R}^+_0$, we have

$$f\mathcal{P}_\gamma[\nu] \overset{\text{law}}{=} \mathcal{P}_\gamma[f^{\sqrt{2d}/\gamma}\nu] \ .$$

[3]When parsing this heuristic argument, it may be helpful to keep in mind the decomposition of the unit box shown in Figure 2.2.

Therefore, we can write

$$\begin{aligned} e^{t(d-\sqrt{2d}\gamma)}e^{\gamma \mathrm{X}_t(x)}dx &= e^{t(d-\sqrt{2d}\gamma)+\sqrt{d/2}\gamma s}e^{\gamma \mathrm{X}_{s,t}(x)}(e^{\sqrt{2d}\mathrm{X}_s(x)-ds})^{\frac{\gamma}{\sqrt{2d}}}dx \\ &\approx \mathcal{P}_\gamma[e^{\sqrt{2d}\mathrm{X}_s(x)-ds}dx](dx) \\ &\approx \mathcal{P}_\gamma[\mu_{\gamma_c}](dx)\ , \end{aligned} \tag{1.3.12}$$

which explains the origin of the limiting measure in Proposition 1.3.8.

Remark 1.3.11. We emphasise that handling the logarithmic correction in the normalised measure in (1.3.9) requires a more delicate analysis, which lies beyond the scope of the rough heuristic argument developed above. Nonetheless, the appearance of the factor $t^{\frac{3\gamma}{2\sqrt{2d}}}$ in the normalisation (1.3.9) should be clear. Indeed, assuming [Mad15, Theorem 1.1], the order of each maximum $\mathbb{M}^i$, including the second-order correction, is $\sqrt{2d}|t-s| - \frac{3}{2\sqrt{2d}}\log|t-s|$. This logarithmic correction also induces a suitable tilting of the spatial intensity in (1.3.11), which in turn leads to the derivative martingale normalisation of the critical GMC in (1.3.12).

We conclude this section by emphasising that, to the best of our knowledge, Proposition 1.3.8 is the only available result on supercritical GMC measures in the continuum. Therefore, several interesting questions remain open. For instance, it would be valuable to gain a deeper understanding of how the convergence in Proposition 1.3.8 occurs[4]. Additionally, if an arbitrary log-correlated Gaussian field is regularised via convolution approximation, does it lead to the same limiting measure? These questions will be examined in detail in Chapters 2 and 3.

1.4 Liouville quantum gravity

Over the past few decades, there has been a large amount of interest in the study of random planar maps. Since the foundational work of Polyakov in the context of bosonic string theory [Pol81], it has been believed that various types of random planar maps converge, in various topologies, to limiting random surfaces called LQG surfaces. Roughly speaking, LQG surfaces can be thought of as random two-dimensional Riemannian manifolds parametrised by a fixed underlying domain $\mathrm{D} \subseteq \mathbb{C}$ and with Riemannian metric tensor given by

$$e^{\gamma h}(dx^2 + dy^2)\ , \tag{1.4.1}$$

where $\gamma \in (0,2)$ is a real parameter, h[5] is a (variant of the) Gaussian free field on D, and dx^2+dy^2 denotes the Euclidean metric tensor. These surfaces are too rough to be Riemannian manifolds in the literal sense, but one can still define the area measure, the distance function, and the canonical diffusion associated to the metric tensor (1.4.1). As usual, in order to make rigorous sense of these objects, one should use a regularisation and limiting procedure. More precisely, assume we are given a family of continuous functions $(h_\varepsilon)_{\varepsilon>0}$ which approximate the GFF as $\varepsilon \to 0$. Then, it possible to make sense of the following objects.

1. The *LQG area measure* $\mu_{h,\gamma}$ obtained as the almost sure limit

$$\mu_{h,\gamma} \overset{\text{def}}{=} \lim_{\varepsilon\to 0} \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma h_\varepsilon(z)} d^2z\ ,$$

[4] For further details on the precise meaning of "how the convergence occurs", we refer the reader to Section 1.5.1.

[5] Although in contrast with the notation used in the previous subsections, in order to align with the notation used in the literature on LQG, we use the letter h (instead of X) to denote an instance of the Gaussian free field.

where d^2z denotes the Lebesgue measure on the domain D under consideration. We note that this is just the GMC measure on $\mathrm{D} \subseteq \mathbb{C}$ (although with a slightly different normalisation) associated with the GFF (see [DS11]).

2. The *LQG metric distance* $\mathrm{d}_{h,\gamma}$ obtained as the limit in probability of a sequence of approximating metric distances (see [DDDF20a, DFG+20, GM20, GM21b] and references therein for further details).

3. The *Liouville Brownian motion* (LBM) $\mathrm{B}_{h,\gamma}$ which is the "canonical diffusion" associated with the metric tensor (1.4.1) (see [GRV16, Ber15]). Specifically, if B is a standard two-dimensional Brownian motion independent of h, then

$$\mathrm{B}_{h,\gamma,t} \overset{\text{def}}{=} B_{\mathrm{F}^{-1}_{h,\gamma}(t)} \,, \qquad \mathrm{F}_{h,\gamma}(t) = \lim_{\varepsilon \to 0} \varepsilon^{\frac{\gamma^2}{2}} \int_0^{t \wedge \tau} e^{\gamma h_\varepsilon(B_s)} ds \,, \tag{1.4.2}$$

where τ denotes the first exit time of B from D.

Remark 1.4.1. The reason why the LBM $\mathrm{B}_{h,\gamma}$ takes the form (1.4.2) can be explained as follows. Recall that the goal is to construct the canonical diffusion associated with the metric tensor (1.4.1). Since this metric is conformally equivalent to the "flat" metric on D, it follows that, after suitable renormalisation, the LBM is formally defined as the solution of the following SDE:

$$\begin{cases} d\mathrm{B}_{h,\gamma,t} = e^{-\frac{1}{2}\gamma h(\mathrm{B}_{h,\gamma,t}) + \frac{1}{4}\gamma^2 \mathbb{E}[h(\mathrm{B}_{h,\gamma,t})^2]} d\overline{B}_t \,, & t > 0 \,, \\ \mathrm{B}_{h,\gamma,0} = x \,, & t = 0 \,, \end{cases}$$

where $\overline{B}$ is a planar Brownian motion started from $x \in \mathrm{D}$ and independent of h. Therefore, by the Dambis–Dubins–Schwartz theorem, we can express $\mathrm{B}_{h,\gamma}$ as a time change of another planar Brownian motion B, independent of h, stopped at its first exit time τ from D. More precisely, we have that

$$\mathrm{B}_{h,\gamma,t} = B_{\langle \mathrm{B}_{h,\gamma} \rangle_t} \,,$$

where $\langle \mathrm{B}_{h,\gamma} \rangle$ denotes the quadratic variation of $\mathrm{B}_{h,\gamma}$. As one can easily check, this quadratic variation is given by the left-inverse of the function $\mathrm{F}_{h,\gamma}$ as defined in (1.4.2).

An important feature of LQG surfaces is that they satisfy a suitable conformal coordinate change formula. Suppose we are given two domains $\mathrm{D}, \overline{\mathrm{D}} \subseteq \mathbb{R}^2$ and a conformal map $\phi : \overline{\mathrm{D}} \to \mathrm{D}$ between them. Let h be a GFF on D, and let

$$\overline{h} \overset{\text{def}}{=} h \circ \phi + Q \log|\phi'| \,, \qquad Q \overset{\text{def}}{=} \frac{2}{\gamma} + \frac{\gamma}{2} \,.$$

Then, it is shown in [DS11, GM21a, Ber15] that, almost surely, for any Borel measurable set $\mathrm{A} \subseteq \overline{\mathrm{D}}$ and for all $z, w \in \overline{\mathrm{D}}$, the following identities hold,

$$\mu_{h,\gamma}\big(\phi(\mathrm{A})\big) = \mu_{\overline{h},\gamma}(\mathrm{A}) \,, \qquad \mathrm{d}_{h,\gamma}\big(\phi(z), \phi(w)\big) = \mathrm{d}_{\overline{h},\gamma}(z, w) \,, \qquad \phi^{-1}\big(\mathrm{B}_{h,\gamma}\big) = \overline{\mathrm{B}}_{\overline{h},\gamma} \,,$$

where $\overline{\mathrm{B}}_{\overline{h},\gamma}$ is the LBM associated with the field $\overline{h}$ and a suitable Brownian motion $\overline{B}$ on $\overline{\mathrm{D}}$.

1.4.1 Planar maps

A planar map is a graph – multiple edges and self-loops allowed – embedded into the plane $\mathbb{C}$ (or the sphere $\mathbb{C} \cup \{\infty\}$) in such a way that no two edges cross, viewed modulo orientation-preserving homeomorphisms

$\mathbb{C} \to \mathbb{C}$ (see Figures 1.4 and 1.5). One can think of a planar map as a discrete random surface, where each face is equipped with the flat Riemannian metric coming from a polygon in $\mathbb{C}$.

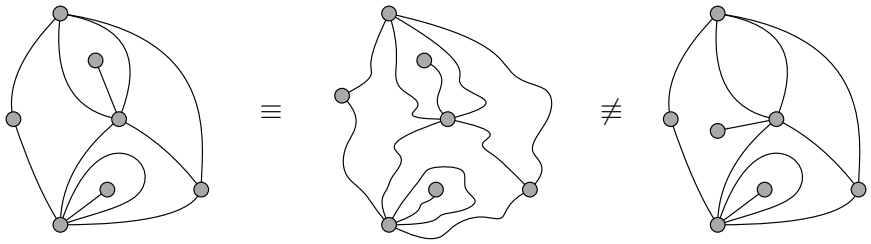


FIGURE 1.4: The first two planar maps are equivalent since there is an orientation-preserving homeomorphisms mapping one to the other. The third planar map is not equivalent to the first two.

For each $n \in \mathbb{N}$, there are only finitely many planar maps with n edges. Therefore, it makes sense to choose a map uniformly at random, or with probability proportional to the partition function of a statistical mechanics model on the map. The parameter γ in the definition of LQG surfaces is related to the type of random planar map model under consideration. For example, the case when $\gamma = \sqrt{8/3}$ describes the scaling limit of uniform planar maps, while different values of γ correspond to random planar maps sampled with probability proportional to some additional structure on the map, e.g., $\gamma = \sqrt{3}$ corresponds to the Ising model partition function.

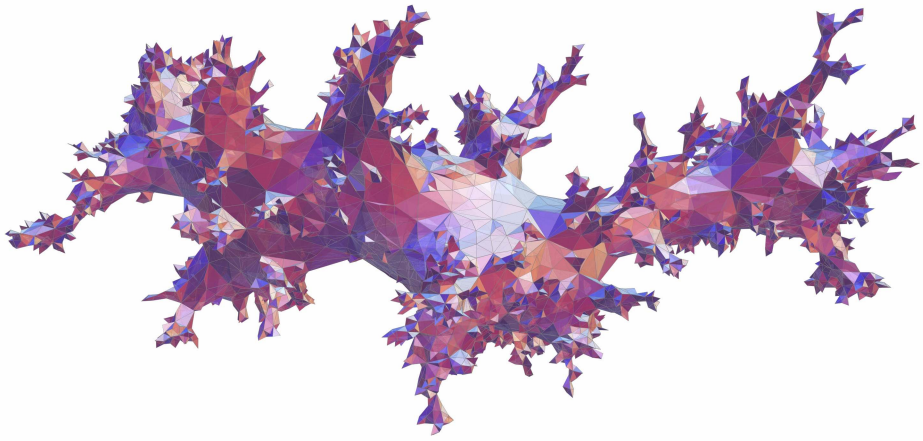


FIGURE 1.5: A large (random) planar map embedded in $\mathbb{R}^3$ using the so-called "isometric embedding". Under this embedding, all the edges of the map have approximately the same length. Credit: Nicolas Curien©.

1.4.2 LQG as scaling limit of random planar maps

There are several canonical ways in which one can study the convergence of random planar maps to LQG surfaces.

As mentioned earlier, LQG surfaces are random metric measure spaces. Similarly, a planar map can also be interpreted as a metric measure space by equipping it with the counting measure on vertices and the graph distance. Therefore, a natural approach to studying the convergence of random planar maps to LQG surfaces is through the Gromov–Hausdorff–Prokhorov distance between metric measure spaces. For instance, this approach is employed by Le Gall and Miermont in [LG13, Mie13] to prove the convergence of uniformly sampled random triangulations and quadrangulations of the sphere to the Brownian map, which is equivalent

to $\sqrt{8/3}$-LQG. The equivalence between the Brownian map and $\sqrt{8/3}$-LQG was established by Miller and Sheffield in the series of articles [MS20, MS21a, MS21b, MS21c].

An alternative formulation of the convergence of random planar maps to LQG is through discrete conformal embeddings of random planar maps. Here, a discrete conformal embedding refers to a particular way of "drawing" the map in the plane (or the sphere), which is in some sense a discrete analog of the Riemann mapping. Suppose we have a random planar map with n vertices, along with a discrete conformal embedding of the map that maps each vertex to a point in $\mathbb{C}$. This embedding creates a natural measure on the plane, with each vertex given a mass of $1/n$. In many settings, it is natural to conjecture that as n tends to infinity, the measure should converge weakly to the LQG area measure, with the parameter γ depending on the particular planar map model under consideration. Additionally, the random walk on the embedded map is expected to converge in law to the Liouville Brownian motion [Ber15, GRV16, BG22]. Several precise scaling limit conjectures for random planar maps toward LQG surfaces were formulated, e.g., in [DS11, DKRV16, She16, DMS21]. However, this very general convergence ansatz has only been rigorously proven in a few specific settings.

One of the challenges in formulating a general scaling limit result for the embedding of random planar maps is the existence of numerous discrete conformal embeddings that could be regarded as natural in some sense. We collect here some of the most commonly employed discrete conformal embeddings.

- The circle packing (see [Nac20] for a review), which represents the map as the tangency graph of a collection of non-overlapping circles.
- The Smith embedding (a.k.a. rectangle packing), which will be the focus of Chapter 5, was introduced by Brooks, Smith, Stone, and Tutte in [BSST40]. It is another popular method of embedding planar graphs, and it is defined by means of a rectangle tiling of a chosen domain, such as a cylinder, a rectangle, or a sphere. In this embedding, the vertices of the planar map correspond to horizontal segments, while the edges correspond to rectangles (see Figure 1.6). Several papers have investigated the properties of the Smith embedding of planar maps, including [BS96, Geo16, HP17, CG20].
- Other examples of discrete conformal embeddings include the Tutte embedding [GMS21], the Cardy–Smirnov embedding [HS23], and the Riemann uniformisation embedding obtained by viewing the planar map as a piecewise flat two-dimensional Riemannian manifold where the faces are identified with unit side length polygons.

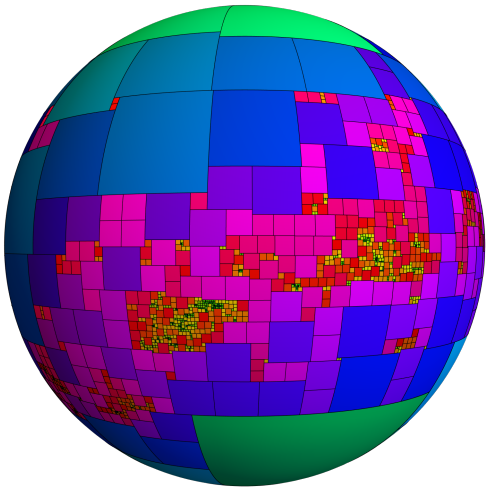


FIGURE 1.6: Simulations of the Smith embedding of a large (random) planar map into the sphere $\mathbb{S}^2$. A detailed description of the simulation's implementation is available in the GitHub repository [BGS23].

Some cases of the aforementioned conjecture, that LQG describes the scaling limit of random planar maps under discrete conformal embeddings, have been proven. For example, in [GMS21], Gwynne, Miller, and Sheffield established the convergence to LQG under the Tutte embedding for a one-parameter family of random planar maps defined using pairs of correlated Brownian motions, known as mated-CRT maps (see Chapter 5 for a definition of this family of random planar maps). Moreover, in [GMS20], the same authors proved that the Tutte embedding of the Poisson Voronoi tessellation of the Brownian disk converges to $\sqrt{8/3}$-LQG. In [HS23], Holden and Sun proved that the scaling limit of uniformly sampled triangulations under the Cardy–Smirnov embedding converges to $\sqrt{8/3}$-LQG. In Chapter 5, we provide a general convergence result for the Smith embedding of planar maps, which works whenever random walk on both the map and its dual approximate Brownian motion.

1.5 Summary of main results

In this section, we provide a brief summary of each chapter of this thesis, highlighting the main results. The discussions and results presented here are stated informally. For precise formulations, we refer the reader to the statements of the main results in the respective chapters.

The content of this thesis is as follows:

Chapter 2 is based on the preprint [BH25a], a joint work with Martin Hairer.

Chapter 3 is based on the preprint [BH25b], a joint work with Martin Hairer.

Chapter 4 is based on the paper [Ber23].

Chapter 5 is based on the paper [BGS25], a joint work with Ewain Gwynne and Scott Sheffield.

Chapter 6 is based on the preprint [BG25], a joint work with Ewain Gwynne.

1.5.1 How does the supercritical Gaussian multiplicative chaos converge?

As observed in Section 1.3, substantial research in recent years has focused on subcritical and critical GMC measures. However, in the continuum setting, supercritical GMC measures remain largely unexplored. One of the main reasons for this is that supercritical GMC measures are not measurable with respect to the underlying field. Nonetheless, this topic is of great interest due to its deep connections with the extremal theory of log-correlated Gaussian fields.

Consider a $\star$-scale invariant field X and its martingale approximation $(X_t)_{t\geq 0}$, as defined in (1.2.4) and (1.2.5). As stated in Proposition 1.3.8, Madaule, Rhodes, and Vargas showed that the sequence of random measures $(\mu_{\gamma,t})_{t\geq 0}$, introduced in (1.3.9), converges in law as $t \to \infty$ to a nontrivial purely atomic measure μ_γ. Specifically, this limiting measure can be described as follows. Let $\eta_\gamma[\mu_{\gamma_c}]$ denote the Poisson point measure on $\mathbb{R}^d \times [0,\infty]$ with intensity $\mu_{\gamma_c}(dx) \otimes w^{-(1+\gamma_c/\gamma)}dw$, where μ_{γ_c} denotes the critical GMC associated with X. If $(x_j, w_j)_{j\in\mathbb{N}} \subseteq \mathbb{R}^d \times [0,\infty]$ is an arbitrary enumeration of the atoms of $\eta_\gamma[\mu_{\gamma_c}]$, then there exists a constant $c_\gamma > 0$ such that

$$\mu_\gamma(dx) = c_\gamma \sum_{j\in\mathbb{N}} w_j \delta_{x_j} \ .$$

The main objective of Chapter 2 is to develop a deeper understanding of how this convergence occurs. A natural approach is to consider not just the limit of $\mu_{\gamma,t}$ as $t \to \infty$, but rather the limit of the measure-valued

process $(\mu_{\gamma,t+s})_{s\geq 0}$ as $t \to \infty$. As we will see below – and in more details in Chapter 2 – this procedure gives rise to a non-trivial limiting process $(\nu_{\gamma,s})_{s\geq 0}$, for which we obtain an explicit characterisation.

Remark 1.5.1. A helpful way to interpret the role of the process $(\nu_{\gamma,s})_{s\geq 0}$ is through an analogy with the CLT. Consider a collection $(X_n)_{n\in\mathbb{N}}$ of i.i.d. centred random variables with unit variance, and let $S_n \overset{\text{def}}{=} \sum_{k=1}^n X_k$. The CLT tells us that $n^{-1/2}S_n$ converges in law to a standard normal random variable as $n \to \infty$. On the linear time scale, the normalised sums quickly settle into a "stable" distribution. However, if we switch to a logarithmic time scale, we can capture how the marginals at different times evolve together. More precisely, for every $t \geq 0$, define $Y_t \overset{\text{def}}{=} \lfloor e^t \rfloor^{-1/2} S_{\lfloor e^t \rfloor}$. Then, the process $(Y_{t+s})_{s\geq 0}$ converges in the finite-dimensional sense to a stationary Ornstein–Uhlenbeck process $(U_s)_{s\geq 0}$ as $t \to \infty$. Intuitively, this follows since, for fixed $s \geq 0$, it holds that

$$Y_{t+s} = \frac{S_{\lfloor e^t \rfloor}}{\sqrt{\lfloor e^{t+s} \rfloor}} + \frac{S_{\lfloor e^{t+s} \rfloor} - S_{\lfloor e^t \rfloor}}{\sqrt{\lfloor e^{t+s} \rfloor}} \approx e^{-s/2} Y_t + (1 - e^{-s})^{1/2} Y'_t \,,$$

where Y'_t is an independent copy of Y_t. In particular, by taking the limit as $t \to \infty$ in the display above, we obtain that for all $s \geq 0$,

$$U_s = e^{-s/2} U_0 + (1 - e^{-s})^{1/2} Z \,,$$

where U_0 and Z are independent standard normal random variables. This is precisely the defining property of a stationary Ornstein–Uhlenbeck process.

A general framework For any $s, t \geq 0$, recalling (1.2.4), the following decomposition holds:

$$X_{t+s} = X_t + W_{s,t} \,, \tag{1.5.1}$$

where $W_{s,t}$ is a field independent of $\sigma(X_s \,:\, s \in [0,t))$ and such that

$$W_{s,t}(\cdot) \overset{\text{law}}{=} W_s(e^t \cdot) \qquad \text{with} \qquad W_s \overset{\text{law}}{=} X_s \,. \tag{1.5.2}$$

Therefore, to address the convergence of $(\mu_{\gamma,t+s})_{s\geq 0}$ as $t \to \infty$, we introduce a general framework that will also be instrumental in establishing the universality result for supercritical GMC in Chapter 3. Specifically, for $t \geq 0$ and $n \in \mathbb{N}$, we consider a collection of fields $(W_{1,t}, \dots, W_{n,t})$ on $(\mathbb{R}^d)^n$ independent of X_t and such that

$$\big(W_{1,t}(\cdot), \dots, W_{n,t}(\cdot)\big) \overset{\text{law}}{=} \big(W_1(e^t \cdot), \dots, W_n(e^t \cdot)\big) \,.$$

where $(W_1, \dots, W_n)$ is a collection of stationary fields on $(\mathbb{R}^d)^n$. Then, we consider the measures

$$\mu_{\gamma,t,i}(dx) \overset{\text{def}}{=} e^{\gamma W_{i,t}(x)} \mu_{\gamma,t}(dx) \,, \qquad \forall\, i \in \{1, \dots, n\} \,, \tag{1.5.3}$$

where we recall that $\mu_{\gamma,t}$ is defined as in (1.3.9). Our goal is to investigate the joint convergence in law (or more precisely the joint stable convergence) of the measure-valued vector $(\mu_{\gamma,t,1}, \dots, \mu_{\gamma,t,n})$ as $t \to \infty$.

Studying the above convergence requires a relatively long and technical detour into the analysis of extremal points of log-correlated Gaussian fields. Specifically, we need to characterise the local structure of the field around points where it attains exceptionally large values, comparable to its maximum within a domain of order one. Indeed, the strong correlation with nearby points suggests that each peak in the field comes with a cluster of high values. These clusters of high-value points are generally well-separated from each other. By selecting one of these clusters and identifying as reference point the maximum of the field inside the cluster, our goal is to describe the "shape" of the field in the vicinity of this reference point.

In order to achieve this goal, we condition the field X on two events: first, that its value at the origin is comparable to the maximum of the field within an order-one region; second, that the origin is a "mesoscopic maximum" within this region. Then, after shifting the coordinate system at this maximum (so that the value at the origin becomes 0) and after a suitable rescaling, we want to study the "shape of the field" in that region. This is the content of Theorem 2.A, which describes the law of the field governing the shape of the field X around a mesoscopic maximum.

We emphasise that in the case of dimension $d = 2$, a similar investigation has been conducted for the *discrete Gaussian free field* (DGFF) in [BL18]. However, their results rely on the fact that the DGFF is defined on the discrete grid $\mathbb{Z}^2$. There, one can condition a field on the origin being the maximum in a bounded region and then take the limit as the size of the region tends to infinity. In the continuum however, conditioning the origin to be a maximum even within a bounded region becomes problematic, as such a conditioning is already degenerate. To address this issue, we introduce a "softer" form of conditioning: we fix an arbitrary threshold $\lambda > 0$ and, instead of conditioning on the origin being a mesoscopic maximum, we condition on the value at the origin being at least as large as the nearest mesoscopic maximum minus λ. This conditioning is non-degenerate within a bounded region as long as $\lambda > 0$. Theorem 2.A proves that we can then take the limit as the region size increases to infinity, thus yielding a limiting field $\widetilde{\Upsilon}_\lambda$. Finally, we show the existence of a unique random field Ψ on $\mathbb{R}^d$ which is 0 at the origin, takes only negative values, is independent of the arbitrary threshold λ, and such that $\widetilde{\Upsilon}_\lambda$ can be expressed as a randomly shifted version of Ψ. In this sense, we can consider Ψ as the canonical field describing the shape of the field X from the perspective of a mesoscopic maximum.

Returning to our original goal of studying the joint convergence of the measures $\mu_{\gamma,t,i}$ introduced in (1.5.3), with this result in hand – though not without some technical effort – we establish in Theorem 2.C that the following joint convergence in law holds as $t \to \infty$:

$$\big(\mu_{\gamma,t,1}, \dots, \mu_{\gamma,t,n}\big) \Rightarrow \big(\mu_{\gamma,1}, \dots, \mu_{\gamma,n}\big) \,,$$

where

$$\mu_{\gamma,i}(dx) \overset{\text{def}}{=} \sum_{j\in\mathbb{N}} \mathbf{Z}_{\gamma,i,j} w_j \delta_{x_j} \,, \qquad \forall\, i \in \{1, \dots, n\} \,,$$

and $(x_j, w_j)_{j\in\mathbb{N}}$ is an arbitrary enumeration of the atoms of the Poisson point measure $\eta_\gamma[\mu_{\gamma_c}]$. Furthermore, $(\mathbf{Z}_{\gamma,i,j})_{j\in\mathbb{N}}$ is a collection of i.i.d. copies of a random variable $\mathbf{Z}_{\gamma,i}$, independent of $\eta_\gamma[\mu_{\gamma_c}]$, and defined as follows:

$$\mathbf{Z}_{\gamma,i} \overset{\text{def}}{=} a_\gamma \int_{\mathbb{R}^d} \exp\big(\gamma\big(\Psi(y) + \mathrm{W}_i(y)\big)\big) dy \,.$$

We emphasise that we do have a fairly explicit representation of the constant a_γ appearing in the above expression. For further details, we refer to the statement of Theorem 2.C.

To establish the convergence in law of the vector $(\mu_{\gamma,t,1}, \dots, \mu_{\gamma,t,n})$, we study the limit as $t \to \infty$ of its joint Laplace functional. Roughly speaking, the key observation is that, for each $i \in \{1, \dots, n\}$, the shape field associated with $\mathrm{X}_t + \mathrm{W}_{i,t}$ coincides with the "shifted shape field" $\Psi + \mathrm{W}_i$. The proof then reduces to showing that the contribution of each shifted shape field to the measure can be completely absorbed into the factor $\mathbf{Z}_{\gamma,i}$.

A measure-valued process To conclude, we now aim to close the loop by returning to where we began our discussion, i.e., the characterisation of the measure-valued process $(\nu_{\gamma,s})_{s\geq 0}$, which emerges as the limit of $(\mu_{\gamma,t+s})_{s\geq 0}$ as $t \to \infty$. Using the same notation as above, in Corollary 2.D, we show that the process

$(\nu_{\gamma,s})_{s\geq 0}$ admits the following characterisation:

$$\nu_{\gamma,s} = a_\gamma \sum_{j\in\mathbb{N}} e^{\mathbf{W}_{\gamma,s,j}} w_j \delta_{x_j} \ , \qquad \forall\, s \geq 0 \ ,$$

where $(\mathbf{W}_{\gamma,s,j})_{j\in\mathbb{N}}$ is a collection of i.i.d. copies of a random variable $\mathbf{W}_{\gamma,s}$, independent of $\eta_\gamma[\mu_{\gamma_c}]$, and defined as follows:

$$\mathbf{W}_{\gamma,s} \overset{\text{def}}{=} \log\left(\int_{\mathbb{R}^d} \exp\big(\gamma\Psi_s(x)\big)dx\right) \ , \qquad \Psi_s(\cdot) \overset{\text{def}}{=} \Psi(e^{-s}\cdot) + \big(\mathrm{W}_s(e^{-s}\cdot) - \sqrt{2d}s\big) \ ,$$

where we recall that Ψ denotes the shape field and W_s is the field introduced immediately below (1.5.1). We emphasise that the spatial locations of the point masses are fixed once and for all and do not change as s varies. The only aspect that evolves in the process $\nu_{\gamma,s}$ are the weights of the point masses, whose dynamics are governed by the process $\mathbf{W}_\gamma$. Moreover, the mass of each atom evolves independently of all the others. Intuitively, the reason why the mass of each atom evolves independently of the others is that the field $\mathrm{W}_{s,t}$ introduced in (1.5.2) becomes asymptotically, as $t \to \infty$, a white noise with finite variance.

We conclude by noting that it would be of interest to investigate the properties of the process $(\Psi_s)_{s\geq 0}$. For instance, while the process $(\nu_{\gamma,s})_{s\geq 0}$ is stationary by construction, it would be interesting to prove this property a posteriori (see Remark 2.2.20 for additional details).

1.5.2 Uniqueness of supercritical Gaussian multiplicative chaos

It is by now well known that subcritical and critical GMC measures are universal in the sense that they do not depend on the choice of regularisation used [BP24, RV14a, Sha16, Ber17]. However, to the best of our knowledge, this question remains unsettled in the case of supercritical GMC measures. Indeed, we recall that [MRV16] only considered the martingale approximation of a $\star$-scale invariant field. Therefore, in Chapter 3, we aim to investigate whether supercritical GMC measures also exhibit universality in the sense specified above. Our result confirms that this is indeed the case, up to a multiplicative constant appearing in front of the limiting measure, which depends on the choice of regularisation.

We consider a log-correlated Gaussian field X defined on a bounded domain $\mathrm{D} \subseteq \mathbb{R}^d$, with $d \geq 1$, whose covariance kernel $\mathcal{K}$ takes the form given in (1.2.1). Additionally, we introduce a mollifier $\rho : \mathbb{R}^d \to \mathbb{R}$ satisfying suitable assumptions (see (A1)–(A2) for precise conditions), and consider the convolution approximation $\mathrm{X}_{(\varepsilon)}$[6] of X as given in (1.2.8). We consider the sequence of random measures $(\mu_{\gamma,(\varepsilon)})_{\varepsilon>0}$ on D given by

$$\mu_{\gamma,(\varepsilon)}(dx) \overset{\text{def}}{=} |\log\varepsilon|^{\frac{3\gamma}{2\sqrt{2d}}}\varepsilon^{-(\gamma/\sqrt{2}-\sqrt{d})^2} e^{\gamma\mathrm{X}_{(\varepsilon)}(x)-\frac{\gamma^2}{2}\mathbb{E}[\mathrm{X}_{(\varepsilon)}(x)^2]}dx \ .$$

Then, in Theorem 3.A, we prove that there exists a finite constant $a_{\gamma,\rho} > 0$, depending on γ and the mollifier ρ, such that $\mu_{\gamma,(\varepsilon)}$ converges stably to a measure μ_γ, defined as:

$$\mu_\gamma(dx) \overset{\text{def}}{=} a_{\gamma,\rho}\mathcal{P}_\gamma[\mu_{\gamma_c}](dx) \ ,$$

where we recall that $\mathcal{P}_\gamma[\mu_{\gamma_c}]$ is integrated atomic random measure introduced in Definition 1.3.7.

[6]The smoothing parameter ε appears in brackets to distinguish the convolution approximation from the martingale approximation that will be introduced later.

Strategy of the proof The strategy to prove this result primarily revolves around reducing the proof to the already established case of the martingale approximation of $\star$-scale invariant fields. To achieve this, we make use of two key decomposition results.

The first decomposition result, stated in Proposition 3.B, asserts that there exists a $\star$-scale invariant field $X^\star$ and a constant $\mathfrak{a}_\rho \in (0,1)$ such that, by setting $t_\varepsilon = \log(\mathfrak{a}_\rho \varepsilon^{-1})$, the convolution approximation $X^\star_{(\varepsilon)}$ can be expressed (in law) as the sum of the martingale approximation $X^\star_{t_\varepsilon}$, a field W_{t_ε} satisfying $W_{t_\varepsilon}(\cdot) = W(e^{t_\varepsilon}\cdot)$ for some Gaussian field W, and a Gaussian field Z_{t_ε} with vanishing variance. In formula, we have:

$$X^\star_{(\varepsilon)} \overset{\text{law}}{=} X^\star_{t_\varepsilon} + W_{t_\varepsilon} + Z_{t_\varepsilon} \,, \tag{1.5.4}$$

where the fields on the right-hand side are all mutually independent.

Remark 1.5.2. Since the field Z_{t_ε} vanishes in the limit $\varepsilon \to 0$, the decomposition (1.5.4) roughly indicates that the convolution approximation of $X^\star$ equals its martingale approximation plus an independent field that asymptotically behaves like white noise with finite variance.

Remark 1.5.3. To the best of our knowledge, the decomposition established in Proposition 3.B is new. Generally speaking, Proposition 3.B can potentially be applied in any situation where a convergence or tightness result is known for the martingale approximation of a $\star$-scale invariant field $X^\star$, and one seeks to extend this result to the convolution approximation. For example, it would be interesting to investigate whether this result can help in extending the construction of the LQG metric in dimension $d = 2$ (see [DDG23] for an overview and further references) to any log-correlated Gaussian field X under the convolution approximation $(X_{(\varepsilon)})_{\varepsilon>0}$. In particular, one would need to verify that the tightness proof in [DDDF20b] remains valid for the sequence of fields $(X^\star_t + W_t)_{t\geq 0}$, where $X^\star_t$ and W_t are as specified above.

An important consequence of the the decomposition (1.5.4) is that it effectively places us within the framework of Theorem 2.C, from which we can deduce the convergence of the supercritical GMC constructed using the convolution approximation of the $\star$-scale invariant field $X^\star$ introduced above.

The second decomposition result is a special case of [JSW19, Theorem A], and it states that the difference between two log-correlated Gaussian fields – whose covariance kernels satisfy certain mild regularity assumptions – is a Hölder continuous Gaussian field. In our setting, let X denote the log-correlated Gaussian field introduced at the begging of this section, and let $X^\star$ be a $\star$-scale invariant field for which (1.5.4) holds. Then, thanks to [JSW19, Theorem A], we know that X and $X^\star$ can be coupled in such a way that

$$X = X^\star + L \,,$$

where L is a centred Gaussian field that is almost surely Hölder continuous. Importantly, this decomposition enables us to transfer the convergence of the supercritical GMC for a $\star$-scale invariant field under convolution approximation to the convergence of the supercritical GMC for essentially any log-correlated Gaussian field under the same regularisation.

Remark 1.5.4. We observe that, by employing a strategy similar to the one outlined above, it *should be* feasible to extend the main result of [Mad15] to the convolution approximation of any log-correlated Gaussian field. More precisely, considering the same setting as above with $D = [0,1]^d$, we claim that there exists a constant $a_\rho > 0$, depending on the mollifier ρ, such that for all $z \in \mathbb{R}$,

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\sup_{x\in[0,1]^d} X_{(\varepsilon)}(x) - \sqrt{2d}|\log\varepsilon| + \frac{3}{2\sqrt{2d}}\log|\log\varepsilon| \leq z \right) = \mathbb{E}\Big[\exp\big(-a_\rho e^{-\sqrt{2d}z}\mu_{\gamma_c}([0,1]^d)\big)\Big] \,,$$

i.e., the recentred supremum converges in law to a shifted Gumbel random variable (see Remark 3.1.13 for additional details).

Remark 1.5.5. We also highlight that it would be interesting to explore whether the result of Chapter 3 (and consequently that of Chapter 2) can be extended to the case where the supercritical GMC is constructed with respect to a reference measure other than the Lebesgue measure.

1.5.3 Multifractal analysis of Gaussian multiplicative chaos and applications

In Chapter 4, we explore the multifractal properties of subcritical GMC measures. The goal of multifractal analysis is to finely describe the heterogeneity in distribution of measures whose mass is concentrated in a highly irregular way. Specifically, for a non-negative, locally finite Borel measure ρ supported on a domain $D \subset \mathbb{R}^d$, with $d \geq 1$, the *singularity spectrum* of ρ is the function $d_\rho : [0,\infty) \to [0,\infty)$ defined by

$$d_\rho(\alpha) \stackrel{\text{def}}{=} \dim_{\mathcal{H}}\left(\left\{x \in D \,:\, \lim_{r\searrow 0} \frac{\log \mu\big(B(x,r)\big)}{\log r} = \alpha\right\}\right),$$

where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension. A measure ρ is said to be multifractal if $d_\rho(\alpha) > 0$ for multiple values of $\alpha \geq 0$ within an interval of non-empty interior.

The primary goal of multifractal analysis is to derive an expression for the singularity spectrum of ρ. To achieve this, Frisch and Parisi [PF85] introduced the notion of *multifractal formalism*, a heuristic principle that provides an explicit connection between the singularity spectrum d_ρ and the L^q-spectrum of the measure ρ, defined by

$$\tau_\rho(q) \stackrel{\text{def}}{=} \limsup_{r\searrow 0} \frac{\log \sup\Big\{\sum_{i\in I} \rho\big(B(x_i,r)\big)^q\Big\}}{-\log r}, \qquad \forall\, q \in \mathbb{R},$$

where $(B(x_i,r))_{i\in I}$ is a countable family of disjoint closed balls with radius r centred at $x_i \in D$, and the supremum is taken over all such families. We say that ρ satisfies the multifractal formalism if the following equality holds

$$d_\rho(\alpha) = \tau_\rho^*(\alpha), \qquad \forall\, \alpha \geq 0,$$

where τ_ρ^* is the Legendre transform of τ_ρ which is defined by setting

$$\tau_\rho^*(\alpha) \stackrel{\text{def}}{=} \inf_{q\in\mathbb{R}} \big\{\alpha q + \tau_\rho(q)\big\} \vee 0, \qquad \forall\, \alpha \geq 0.$$

Main results In Theorem 4.A, we show that subcritical GMC measures associated with a large class of log-correlated Gaussian fields in all dimensions satisfy the multifractal formalism. Moreover, we provide explicit expressions for both the singularity spectrum and the L^q-spectrum of these types of measures. This result was formally stated in [RV14a, Section 4], but to the best of our knowledge, it has never been rigorously proved. In order to achieve this result, we perform a careful analysis of the local mass concentration of GMC measures around thick points of the corresponding underlying field. Moreover, we provide explicit expressions for the lower singularity spectrum of the *multifractal random walk* in Theorem 4.B and the *Liouville Brownian motion* in Theorem 4.C.

1.5.4 Scaling limits of planar maps under the Smith embedding

In Section 1.4.2, we mentioned that there are several approaches to studying the scaling limit of planar maps. In Chapter 5, we focus on the scaling limits of (random) planar maps under a specific embedding known as the *Smith embedding*, first introduced by Brooks, Smith, Stone, and Tutte in [BSST40] and later generalised to infinite planar graphs by Benjamini and Schramm in [BS96].

We briefly outline the construction of the Smith embedding of a planar map and refer to Chapter 5 for precise details. Suppose we are given a doubly marked, finite, weighted[7] planar map $(\mathcal{G}, c, v_0, v_1)$, where v_0 and v_1 are the two marked vertices, and $c = (c_e)_{e\in\mathcal{E}\mathcal{G}}$ is a collection of positive unoriented weights, known as conductances. Here, $\mathcal{E}\mathcal{G}$ denotes the set of edges of $\mathcal{G}$. We also denote by $(\widehat{\mathcal{G}}, \widehat{c})$ the weighted dual planar graph associated with $(\mathcal{G}, c)$, where the conductance $\widehat{c}_{\widehat{e}}$ of a dual edge is given by the resistance of the corresponding primal edge, i.e., $\widehat{c}_{\widehat{e}} = c_e^{-1}$.

The Smith embedding of the quadruplet $(\mathcal{G}, c, v_0, v_1)$ is constructed using a tiling of a finite cylinder by rectangles – referred to as the *Smith diagram* – determined by the voltage function $\mathfrak{h}$ and its discrete harmonic conjugate $\mathfrak{w}$. The function $\mathfrak{h}$ is the unique function on the set of vertices $\mathcal{V}\mathcal{G}$ which is discrete harmonic on $\mathcal{V}\mathcal{G} \setminus \{v_0, v_1\}$ with boundary conditions given by $\mathfrak{h}(v_0) = 0$ and $\mathfrak{h}(v_1) = 1$, i.e.,

$$\mathfrak{h}(x) = \mathbb{P}_x\big(X \text{ hits } v_1 \text{ before } v_0\big)\ , \qquad \forall\, x \in \mathcal{V}\mathcal{G},$$

where $\mathbb{P}_x$ denotes the law of random walk on $(\mathcal{G}, c)$ started from x. We denote the strength of the flow induced by $\mathfrak{h}$ by setting

$$\mathfrak{n} \stackrel{\text{def}}{=} \sum_{e\in\mathcal{E}\mathcal{G}^{\uparrow}(v_0)} \nabla\mathfrak{h}(e),$$

where $\mathcal{E}\mathcal{G}^{\uparrow}(v_0)$ denotes the set of harmonically oriented edges in $\mathcal{E}\mathcal{G}$ with tails equal to v_0. The function $\mathfrak{w}$ is defined on the set of dual vertices $\mathcal{V}\widehat{\mathcal{G}}$ and satisfies the discrete Cauchy–Riemann equations associated with $\mathfrak{h}$, i.e., it is a solution of the following equation,

$$\mathfrak{w}(\widehat{e^+}) - \mathfrak{w}(\widehat{e^-}) = c_e(\mathfrak{h}(e^+) - \mathfrak{h}(e^-))\ , \qquad \forall\, e \in \mathcal{E}\mathcal{G}\ .$$

We remark that the function $\mathfrak{w}$ is only defined modulo $\mathfrak{n}\mathbb{Z}$ and modulo an additive constant that can be fixed by imposing that $\mathfrak{w}$ is equal to zero on a chosen dual vertex.

In the Smith diagram of $(\mathcal{G}, c, v_0, v_1)$, each edge $e \in \mathcal{E}\mathcal{G}$ is represented by a rectangle R_e, each vertex $x \in \mathcal{V}\mathcal{G}$ by a horizontal line segment H_x, and each dual vertex $\widehat{x} \in \mathcal{V}\widehat{\mathcal{G}}$ by a vertical line segment $\mathrm{V}_{\widehat{x}}$, as follows:

- The rectangle R_e on $\mathcal{C}_{\mathfrak{n}}$ has top and bottom heights given by $\mathfrak{h}$ at the endpoints of e and left and right widths determined by $\mathfrak{w}$ at the endpoints of the dual edge $\widehat{e}$.
- The horizontal segment H_x is the longest horizontal segment on the boundaries of all rectangles corresponding to edges incident to x.
- The vertical segment $\mathrm{V}_{\widehat{x}}$ is the longest vertical segment tangent to all rectangles corresponding to primal edges surrounding $\widehat{x}$.

We define the Smith embedding $\dot{\mathcal{S}}$ associated to the quadruplet $(\mathcal{G}, c, v_0, v_1)$ as the function from $\mathcal{V}\mathcal{G}$ to $\mathbb{R}/\mathfrak{n}\mathbb{Z} \times [0, 1]$ given by

$$\dot{\mathcal{S}}(x) \stackrel{\text{def}}{=} \mathrm{mid}(\mathrm{H}_x), \quad \forall x \in \mathcal{V}\mathcal{G},$$

[7] Here, "weighted" means that the graph is equipped with edge conductances.

where $\mathrm{mid}(\mathrm{H}_x)$ corresponds to the middle point of the horizontal line segment H_x.

Main result Let $((\mathcal{G}^n, c^n, v_0^n, v_1^n))_{n\in\mathbb{N}}$ be a sequence of doubly marked, finite, weighted planar maps, and $((\widehat{\mathcal{G}}^n, \widehat{c}^n))_{n\in\mathbb{N}}$ denote their associated weighted dual planar graphs. For each $n \in \mathbb{N}$, assume that $\mathcal{G}^n$ is given with an *a priori embedding* in the infinite cylinder $\mathcal{C}_{2\pi} \overset{\text{def}}{=} \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$, where the marked vertices v_0^n and v_1^n are mapped to $-\infty$ and $+\infty$, respectively. We also assume that the random walks on the primal map $(\mathcal{G}^n, c^n)$ and dual map $(\widehat{\mathcal{G}}^n, \widehat{c}^n)$ converge, in a suitable sense, to a Brownian motion modulo a time change as $n \to \infty$. Under these assumptions, Theorem 5.A establishes that the a priori embedding is asymptotically close, uniformly over compacts, to an affine transformation of the Smith embedding.

This result is in some ways not surprising, since it is natural to expect that if a random walk (and its dual) approximate Brownian motion, then discrete harmonic functions (and their conjugate duals) should approximate continuum harmonic functions. However, showing that this is actually true in the limit, and that the convergence is to the right continuum harmonic functions is a non-trivial task. More precisely, the particular statement we obtain is far from obvious a priori, for two main reasons:

- The invariance principle assumptions only concern the behaviour of the random walks in the *bulk* of the cylinder. We do not need any hypotheses about how the random walks behave when they get close to the marked vertices v_0^n and v_1^n.
- The width coordinate is discrete harmonic but does not admit a simple direct description in terms of the random walk on $(\widehat{\mathcal{G}}^n, \widehat{c}^n)$.

An application of the main result One application of Theorem 5.A is Theorem 5.B, where we establish the convergence of the mated-CRT map with the sphere topology to LQG under the Smith embedding. Specifically, we consider the LQG/SLE embedding of the mated-CRT map and show that the assumptions of Theorem 5.A holds under this embedding. This is essentially sufficient to conclude, since, by construction, the a priori embedding converges to LQG. Consequently, the Smith embedding also converges to LQG, as it is asymptotically close to the a priori embedding (modulo affine transformations).

We emphasise that, more broadly, Theorem 5.A reduces the problem of proving convergence to LQG under the Smith embedding for other types of random planar maps to that of finding a priori embeddings of the map and its dual, under which the vertex counting measure converges to the LQG measure and the random walk converges to Brownian motion modulo time parametrisation.

Finally, we remark that one advantage of the version of the Smith embedding described above is that its definition is particularly natural for random planar maps without boundary. This is in contrast to other embeddings under which random planar maps have been shown to converge to LQG (such as the Tutte embedding [GMS21] and the Cardy–Smirnov embedding [HS23]) which are most naturally defined for planar maps with boundary.

1.5.5 Liouville Brownian motion and quantum cones in dimension $d > 2$

As observed in Section 1.4, substantial research in recent years has focused on random geometry constructed by exponentiating the two-dimensional GFF. However, most results in this field remain confined to two dimensions and have yet to be extended to higher dimensions. This limitation primarily stems from the critical role of conformal invariance in two-dimensional results, as non-trivial conformal maps do not exist in higher dimensions. Nevertheless, recent works [Cer22, DGZ24, DSHKS24, DSHKS23] have begun exploring higher-dimensional analogues of LQG for dimensions $d > 2$.

The goal of Chapter 6 is to continue the extension of LQG to higher dimension. Roughly speaking, we aim to study a conformally flat Riemannian metric tensor on $\mathbb{R}^d$, for $d \geq 2$, of the following form,

$$e^{\gamma h}(dx_1^2 + \cdots + dx_d^2) \,, \tag{1.5.5}$$

where $\gamma \in (0, \sqrt{2d})$ is a coupling parameter, $dx_1^2 + \cdots + dx_d^2$ denotes the Euclidean metric tensor on $\mathbb{R}^d$, and h is the whole-space *log-correlated Gaussian field* (LGF) on $\mathbb{R}^d$ as defined in Section 1.2. In what follows, we denote by $\mathbf{h}$ the whole-space LGF, with the additive constant chosen so that its spherical average over the unit sphere centred at the origin is zero (see Section 6.2.2 for more details on the spherical average of the LGF).

As in the two-dimensional case, the Riemannian metric tensor in (1.5.5) remains a formal object. The most straightforward quantity to construct is the area measure $\mu_{\mathbf{h},\gamma}$, which emerges as the limit $\varepsilon \to 0$ of $\varepsilon^{\gamma^2/2} e^{\gamma \mathbf{h}_\varepsilon(x)} dx$, where $(\mathbf{h}_\varepsilon)_{\varepsilon>0}$ is a collection of regularising fields given by the spherical average approximation. Suppose that ϕ is a conformal automorphism of $\mathbb{R}^d$, then for any Borel subset $A \subseteq \mathbb{R}^d$, it holds almost surely that

$$\mu_{\mathbf{h},\gamma}\big(\phi(A)\big) = \mu_{\mathbf{h}\circ\phi + Q\log|\phi'|,\gamma}(A) \,, \qquad Q \overset{\text{def}}{=} \frac{d}{\gamma} + \frac{\gamma}{2} \,, \tag{1.5.6}$$

i.e., we can regard $(\mathbb{R}^d, \mathbf{h})$ and $(\mathbb{R}^d, \mathbf{h} \circ \phi + Q \log|\phi'|)$ as two different parameterisations of the same d-dimensional manifold, analogously to two-dimensional setting [DS11, Proposition 2.1].

The LBM and its spectral dimension The first part of Chapter 6 focuses on the canonical diffusion associated with the metric tensor (1.5.5), which is compatible with the coordinate transformation rule (1.5.6). This diffusion is the higher-dimensional analog of the *Liouville Brownian motion* (LBM) constructed in [Ber15, GRV16] and can be defined as follows.

Consider a d-dimensional Brownian motion $(\mathbf{\Omega}, \mathbf{F}, (B_t)_{t\geq 0}, (\mathbf{F}_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}^d})$, which is independent of the whole-space LGF $\mathbf{h}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $\alpha \in (0,2)$, we define the *clock process* $(\mathrm{F}_{\mathbf{h},\alpha}(t))_{t\geq 0}$ by

$$\mathrm{F}_{\mathbf{h},\alpha}(t) \overset{\text{def}}{=} \lim_{\varepsilon\to 0} \varepsilon^{\alpha^2/2} \int_0^t e^{\alpha \mathbf{h}_\varepsilon(B_s)} ds \,, \qquad \forall\, t \geq 0 \,.$$

The convergence of the limit in the right-hand side of the above display falls within the scope of the theory of subcritical GMC measures, which guarantees that, for all $x \in \mathbb{R}^d$, the clock process is $\mathbf{P}_x \otimes \mathbb{P}$-almost surely well-defined and non-trivial for all $\alpha \in (0,2)$. Then the LBM is the time-changed Brownian motion defined as follows

$$\mathrm{B}_{\mathbf{h},\alpha,t} \overset{\text{def}}{=} B_{\mathrm{F}^{-1}_{\mathbf{h},\alpha}(t)} \,, \qquad \forall\, t \geq 0 \,. \tag{1.5.7}$$

Recall that we want the LBM to be compatible with the coordinate transformation rule (1.5.6), i.e., if ϕ is a conformal automorphism of $\mathbb{R}^d$ that can be expressed as a composition of translations, dilations, and rotations, then the following should hold:

$$\big(\phi^{-1}(\mathrm{B}_{\mathbf{h},\alpha,t})\big)_{t\geq 0} = (\overline{\mathrm{B}}_{\mathbf{h}\circ\phi+Q\log|\phi'|,\alpha,t})_{t\geq 0} \,, \qquad \overline{\mathrm{B}}_{\mathbf{h}\circ\phi+Q\log|\phi'|,\alpha,t} \overset{\text{def}}{=} \overline{B}_{\overline{\mathrm{F}}^{-1}_{\mathbf{h}\circ\phi+Q\log|\phi'|,\alpha}(t)} \,,$$

with $(\overline{B}_t)_{t\geq 0} = (\phi^{-1}(B_{|\phi'|^2 t}))_{t\geq 0}$ and $\overline{\mathrm{F}}_{\mathbf{h}\circ\phi+Q\log|\phi|,\alpha}$ the associated clock process, defined with respect to the field $\mathbf{h} \circ \phi + Q\log|\phi'|$. Thanks to the fact that in every dimension $d \geq 2$, Brownian motion exhibits two-dimensional scaling, we will show in Chapter 6 that for this to hold, we need to impose that

$$\frac{2}{\alpha} + \frac{\alpha}{2} = Q \qquad \Longleftrightarrow \qquad \alpha = \alpha(Q) = \alpha(d,\gamma) \overset{\text{def}}{=} Q - \sqrt{Q^2 - 4} \,, \tag{1.5.8}$$

where Q is as defined in (1.5.6). In particular, in dimension $d = 2$, we have $\alpha = \gamma$, whereas in higher dimensions, α becomes a non-linear function of γ. Hence, the d-dimensional LBM is defined as the process introduced (1.5.7) with α related to γ as specified in (1.5.8).

So far, we have discussed the construction of LBM starting from a single point. However, to view the LBM as a diffusion process, we must ensure that the clock process $\mathrm{F}_{\mathbf{h},\alpha}$ is well-defined for all starting points $x \in \mathbb{R}^d$, $\mathbb{P}$-almost surely. The placement of "$\mathbb{P}$-almost surely" is crucial, as this introduces challenges distinct from the single-point construction. Nevertheless, we can prove that the LBM can indeed be constructed as a diffusion process for all $Q > 2$.

In particular, for $Q > \sqrt{2d}$, the transition semigroup of the LBM is absolutely continuous with respect to the GMC measure $\mu_{\mathbf{h},\alpha}$. We denote its density, called *Liouville heat kernel* (LHK), by $(\mathfrak{p}_{\mathbf{h},\alpha,t}(\cdot,\cdot))_{t\geq 0}$. Roughly speaking, the spectral dimension of LBM at $x \in \mathbb{R}^d$ is defined as follows

$$\mathfrak{d}_{\mathbf{h},\alpha}(x) \stackrel{\text{def}}{=} \lim_{t\to 0} \frac{2\log \mathfrak{p}_{\mathbf{h},\alpha,t}(x,x)}{|\log t|} \,.$$

In Theorem 6.B, we show that if $x \in \mathbb{R}^d$ is a β-thick point of the LGF for $\beta \in [0, \sqrt{2d}]$, then

$$\mathfrak{d}_{\mathbf{h},\alpha}(x) = 2 + \frac{2(d-2)}{2 + \alpha^2/2 - \alpha\beta} \,,$$

We observe that for $d = 2$, the spectral dimension is always 2, regardless of α and β (see [RV14b]).

The higher-dimensional quantum cone field In the second part of Chapter 6, we construct the higher-dimensional analogue of the quantum cone in arbitrary even dimensions $d > 2$. We recall that the two-dimensional quantum cone field – first introduced in [DMS21] – is obtained by appropriately zooming in around the origin of the whole-plane GFF with a log-singularity at the origin. The d-dimensional analogue is defined in a similar manner.

One of the main ingredient in the construction of the quantum cone in higher dimension is Theorem 6.A, where we characterise the spherical average process. Specifically, by leveraging [LSSW16], we show that in any even dimension $d > 2$, for all $x \in \mathbb{R}^d$, the recentred spherical average process $(\mathbf{h}_{e^{-t}}(x))_{t\in\mathbb{R}}$[8] has the same law as the following process:

$$\mathrm{S}_t \stackrel{\text{def}}{=} (d-2)\int_{-\infty}^{0} e^{2s}(1-e^{2s})^{\frac{d-4}{2}}\big(B_{s+t} - B_s\big)ds \,, \qquad \forall\, t \in \mathbb{R} \,.$$

where $(B_s)_{s\in\mathbb{R}}$ is a standard two-sided Brownian motion. The above representation of the spherical average process suggests that it behaves as a "smoothed-out" version of a two-sided Brownian motion (see Figure 6.1).

Remark 1.5.6. The identification of the spherical average process is restricted to even dimensions since, in odd dimensions $d > 3$, the covariance kernel of the spherical average process does not admit a "simple" representation. At a high level, this is due to the fact that the whole-space LGF is the standard Gaussian on the homogeneous Sobolev space of order $d/2$, where the inner product is defined via the non-local operator $(-\Delta)^{d/2}$. We emphasise that, similarly, in [Cer22, DSHKS24], the authors also restricted their analysis to even dimensions $d > 2$. In contrast, the construction of LBM and the tightness of approximations of the "Riemannian distance function" for (1.5.5) in [DGZ24] are not restricted to even dimensions, since these constructions are not reliant on any special exact solvability.

[8] We recall that, for $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$, the quantity $\mathbf{h}_{e^{-t}}(x)$ is defined by testing the (normalised) whole-space LGF $\mathbf{h}$ against the uniform probability measure on the sphere centred at x with radius e^{-t}.

As in the two-dimensional setting, the quantum cone field is constructed by separately specifying its averages on spheres centred at the origin and what is left when we subtract these. In particular, we will see that the homogeneous Sobolev space $\mathcal{H}_d$, introduced in Section 1.2, can be expressed as the direct sum of two subspaces. The first subspace, denoted by $\mathcal{H}_{d,\mathrm{rad}}$, is the closure of functions in $\mathcal{S}(\mathbb{R}^d)$ that are radially symmetric about the origin, viewed modulo constants. The second subspace, denoted by $\mathcal{H}_{d,\mathrm{sph}}$, consists of the closure of functions in $\mathcal{S}(\mathbb{R}^d)$ that have mean zero on all spheres centred at the origin (see Lemma 6.2.5 for additional details).

The construction of the radial part of the quantum cone proceeds as follows. Consider the whole-space LGF $\mathbf{h}$, let $d > 2$ be even, $\gamma \in (0, \sqrt{2d})$, and $\beta \in (-\infty, Q)$. For $b > 0$ a large constant, applying the coordinate change formula (1.5.6) to the field $\mathbf{h} - \beta \log|\cdot| + b$, with respect to the conformal automorphism $\mathbb{R}^d \ni x \mapsto e^{-t}x$, for some $t \in \mathbb{R}$, we obtain the field $\mathbf{h}^{t,b}$ given by

$$\mathbf{h}^{t,b} \overset{\text{def}}{=} \mathbf{h}(e^{-t}\cdot) - \beta \log|e^{-t}\cdot| - Qt + b \, .$$

Thanks to Theorem 6.A, the spherical average process $(\mathbf{h}^{t,b}_{e^{-s}}(0))_{s\in\mathbb{R}}$ admits the following representation

$$\mathbf{h}^{t,b}_{e^{-s}}(0) = \mathrm{S}_{s+t} + \beta s - (Q-\beta)t + b \, , \qquad \forall\, s \in \mathbb{R} \, .$$

In particular, by taking $s = 0$ in the previous expression, we get that $\mathbf{h}^{t,b}_1(0) = \mathrm{S}_t - (Q-\beta)t + b$. Since $\beta \in (-\infty, Q)$, the drift term is always negative and so in particular, for each $b > 0$, there exists almost surely a smallest $\sigma_b \in \mathbb{R}$ such that $\mathbf{h}^{t,b}_1(0) = 0$, i.e.,

$$\sigma_b \overset{\text{def}}{=} \inf\big\{t \in \mathbb{R} \, : \, \mathrm{S}_t - (Q-\beta)t = -b\big\} \, .$$

For each $b > 0$, we define the recentred process $(\mathrm{S}_{b,s})_{s\in\mathbb{R}}$ by letting

$$\mathrm{S}_{b,s} \overset{\text{def}}{=} \mathrm{S}_{s+\sigma_b} - \mathrm{S}_{\sigma_b} \, , \qquad \forall\, s \in \mathbb{R} \, .$$

In Theorem 6.C, we show that the process $(\mathrm{S}_{b,s})_{s\in\mathbb{R}}$ converges in total variation distance to a process $(\mathrm{S}_{\infty,s})_{s\in\mathbb{R}}$ as $b \to \infty$. This allows us to defined the the d-dimensional β-quantum cone field $\mathbf{h}^\star_{\mathrm{rad}}$ as follows:

$$\mathbf{h}^\star \overset{\text{def}}{=} \mathbf{h}^\star_{\mathrm{rad}} + \mathbf{h}^\star_{\mathrm{sph}} \, ,$$

where $\mathbf{h}^\star_{\mathrm{rad}}$ is the function that is constant equal to $\mathrm{S}_{\infty,s} + \beta s$ on each sphere $\partial B(0, e^{-s})$ for $s \in \mathbb{R}$, and $\mathbf{h}^\star_{\mathrm{sph}}$ is a field independent of $\mathbf{h}^\star_{\mathrm{rad}}$ and that has the same law of the projection of the whole-space LGF onto $\mathcal{H}_{d,\mathrm{sph}}$.

In Theorem 6.D, we state that the d-dimensional β-quantum cone field arises as the local limit of a whole-space LGF with an additional β-log singularity at a fixed point in $\mathbb{R}^d$. We emphasise that this result follows directly from our construction of the quantum cone.

Finally, in Theorem 6.E, we show that the law of the d-dimensional quantum cone $\mathbf{h}^\star$ is invariant in law modulo conformal coordinate change under shifting along the trajectories of the associated LBM. We highlight that constructing the LBM associated with $\mathbf{h}^\star$ is non-trivial, as we must ensure that it does not remain "stuck" due to the log-singularity at the origin of the quantum cone field. This possibility is ruled out at the beginning of Section 6.5.4.

Chapter 2

How does the supercritical Gaussian multiplicative chaos converge?

Abstract. In the spirit of [BL18], we study the local structure of $\star$-scale invariant fields – a class of log-correlated Gaussian fields – around their extremal points by characterising the law of the "shape" of the field's configuration near such points. As a consequence, we obtain a refined understanding of the freezing phenomenon in supercritical Gaussian multiplicative chaos.

2.1 Introduction

The theory of *Gaussian multiplicative chaos* (GMC) involves the study of random measures that can be formally expressed as

$$\mu_\gamma(dx) \text{ “=” } e^{\gamma \mathrm{X}(x)} dx \ , \tag{2.1.1}$$

where $\gamma \geq 0$ is a real parameter representing the *inverse temperature* of the model, X is a *log-correlated Gaussian field* on a domain $\mathrm{D} \subseteq \mathbb{R}^d$, and dx denotes Lebesgue measure on D. Since X only exists as a random Schwartz distribution, regularisation and renormalisation are necessary to show the existence of the measure μ_γ as defined above [Kah85, DS11, RV14a, Sha16, Ber17].

It is by now well-known that the behaviour of the random measure (2.1.1) exhibits a phase transition at

$$\gamma_{\mathrm{c}} \overset{\text{def}}{=} \sqrt{2d} \ .$$

Following standard convention, we call the regime $\gamma < \gamma_{\mathrm{c}}$ *subcritical*, the borderline case $\gamma = \gamma_{\mathrm{c}}$ *critical*, and the range $\gamma > \gamma_{\mathrm{c}}$ *supercritical*. These three different regimes differ in the normalisation needed to obtain a non-trivial limiting measure, as well as in the qualitative features of the limiting measure (see also [LRV15] for a more detailed phase diagram including complex values of γ). Notably, in the supercritical regime, the limiting random measure is *not* measurable with respect to the underlying field X and is purely *atomic*. Before delving into more details and stating our main results, we introduce the family of log-correlated Gaussian fields we will be working with.

The class of $\star$-scale invariant fields. We consider log-correlated Gaussian fields X on $\mathbb{R}^d$ with short-range correlations, which naturally admit an approximation by a martingale $(\mathrm{X}_t)_{t \geq 0}$. Here, each X_t is a smooth Gaussian field, and for every $x \in \mathbb{R}^d$, the process $(\mathrm{X}_t(x))_{t \geq 0}$ is a standard Brownian motion. Moreover, these fields satisfy a certain type of scale-invariance called $\star$-*scale invariance* [RV14a, Section 2.3]. In a nutshell,

it states that for any $s, t > 0$, the fields X_t and $\mathrm{X}_{t+s} - \mathrm{X}_t$ are independent, and that the latter is equal in law to the field X_s, spatially rescaled by a factor e^t.

The key ingredient in constructing a $\star$-scale invariant field is the so-called *seed covariance function* $\mathfrak{K} : \mathbb{R}^d \to \mathbb{R}$. Throughout this chapter, we impose a stronger set of conditions on $\mathfrak{K}$ than those given in Definition 1.2.1. Specifically, we assume that $\mathfrak{K}$ satisfies the following properties:

(K1) $\mathfrak{K}$ is positive definite, radial, and $\mathfrak{K}(0) = 1$.

(K2) $\mathfrak{K} \in \mathcal{C}^\infty(\mathbb{R}^d)$ and it is supported in $B(0,1)$.

Remark 2.1.1. The unit ball appearing in (K2) can of course be replaced by any compact subset of $\mathbb{R}^d$.

Remark 2.1.2. As a consequence of (K1)–(K2), the Fourier transform $\widehat{\mathfrak{K}}$ is a probability measure with a smooth density and admitting moments of all orders. In particular, the Hessian $D^2\mathfrak{K}(0)$ at the origin is strictly negative definite in the sense that there exists $\delta > 0$ such that, for any $v \in \mathbb{R}^d$, one has $\langle v, D^2\mathfrak{K}(0)v\rangle \le -\delta|v|^2$.

We write $\overline{\mathfrak{K}} : \mathbb{R}^d \to \mathbb{R}$ for the (unique) positive definite function such that the convolution of $\overline{\mathfrak{K}}$ with itself equals $\mathfrak{K}$ For ξ a space-time white noise on $\mathbb{R}^d \times (0,\infty)$, we consider the *$\star$-scale invariant field with seed covariance* $\mathfrak{K}$ given by

$$\mathrm{X}(\cdot) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d}\int_0^\infty \overline{\mathfrak{K}}\big(e^r(y-\cdot)\big)e^{\frac{dr}{2}}\,\xi(dy,dr)\;. \tag{2.1.2}$$

Furthermore, for $0 \le s < t$, we let $\mathrm{X}_{s,t}$ be the field on $\mathbb{R}^d$ given by

$$\mathrm{X}_{s,t}(\cdot) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d}\int_s^t \overline{\mathfrak{K}}\big(e^r(y-\cdot)\big)e^{\frac{dr}{2}}\,\xi(dy,dr)\;, \tag{2.1.3}$$

with the notational convention that $\mathrm{X}_{0,t} = \mathrm{X}_t$.

For $t \ge 0$, the fields X and X_t have the following covariance structures, for all $x, y \in \mathbb{R}^d$,

$$\mathbb{E}\big[\mathrm{X}(x)\mathrm{X}(y)\big] = \int_0^\infty \mathfrak{K}\big(e^r(x-y)\big)dr\;, \qquad \mathbb{E}\big[\mathrm{X}_t(x)\mathrm{X}_t(y)\big] = \int_0^t \mathfrak{K}\big(e^r(x-y)\big)dr\;. \tag{2.1.4}$$

Clearly $\mathbb{E}[\mathrm{X}(x)^2] = \infty$, so X can only be realised as a random Schwartz distribution. The collection of fields $(\mathrm{X}_t)_{t\ge0}$ is called the *martingale approximation* of X. Indeed, by construction, $(\mathrm{X}_t)_{t\ge0}$ is a martingale for the filtration $(\mathcal{F}_t)_{t\ge0}$ given by

$$\mathcal{F}_t \stackrel{\text{def}}{=} \sigma\big(\mathrm{X}_s\,:\,s\in[0,t)\big)\;. \tag{2.1.5}$$

Moreover, as $t \to \infty$ the field X_t converges almost surely to X in the space $\mathcal{H}^{-\kappa}_{\mathrm{loc}}(\mathbb{R}^d)$ for any $\kappa > 0$. We refer to [JSW19, Proposition 4.1 (iv)] for a proof of this fact.

The three phases of GMC measures. As previously mentioned, GMC measures exhibit three distinct phases depending on the value of the parameter $\gamma \ge 0$ in (2.1.1). Each phase is characterised by the specific form of renormalisation required to obtain a nontrivial limiting measure.

In the *subcritical regime*, i.e., when $\gamma < \gamma_c$, the sequence of random measures

$$\mu_{\gamma,t}(dx) = e^{\gamma\mathrm{X}_t(x) - \frac{\gamma^2}{2}t}dx \tag{2.1.6}$$

converges weakly in probability to a limiting positive random measure μ_γ as $t \to 0$ [Kah85, RV14a, Sha16, Ber17], which is almost surely nontrivial. It is well-known that μ_γ is almost surely non-atomic, but singular with respect to Lebesgue measure. Many further properties of these measures concerning, among others,

moments and multifractal behaviour are known [RV14a, Ber23]. An important feature of the measure μ_γ is that it is carried by the set of γ-thick points. Intuitively, a thick point is a point where the field takes an unusually large value: one where it is of the order of its variance instead of the order of its standard deviation.

When $\gamma \geq \gamma_c$ a phase transition occurs and if one considers the sequence of measures defined in (2.1.6), then for any compact subset $A \subset \mathbb{R}^d$, it holds that $\mu_{\gamma,t}(A) \to 0$ in probability as $t \to 0$. Therefore, in order to define a nontrivial limiting measure at the critical threshold γ_c, one needs to give the sequence of approximating measures an extra "push" in the right direction. More precisely, in the *critical regime*, i.e., when $\gamma = \gamma_c$, the sequence of random measures

$$\mu_{\gamma_c,t}(dx) = \big(-X_t(x) + \gamma_c\, t\big) e^{\gamma_c X_t(x) - \frac{\gamma_c^2}{2} t} dx \tag{2.1.7}$$

converges weakly in probability as $t \to \infty$ to a limiting positive random measure μ_{γ_c} [DRSV14a, DRSV14b, Pow21], which is non-atomic and has full support. The normalisation used in (2.1.7) is known as the "derivative normalisation" since it can be obtained by evaluating at $\gamma = \gamma_c$ the derivative with respect to γ of the expression on the right-hand side of (2.1.6).

In the low temperature or *supercritical regime*, i.e. $\gamma > \gamma_c$, the GMC exhibits atomic behaviour under suitable renormalisation, with the locations and masses of the atoms dictated by the extremal statistics or near maximum values of the fields X_t. In the continuum, the only available mathematical result we are aware of is [MRV16], where Madaule, Rhodes, and Vargas show that for $\gamma > \gamma_c$, the sequence of random measures

$$\mu_{\gamma,t}(dx) = t^{\frac{3\gamma}{2\sqrt{2d}}} e^{t(\gamma/\sqrt{2} - \sqrt{d})^2} e^{\gamma X_t(x) - \frac{\gamma^2}{2} t} dx\ , \tag{2.1.8}$$

converges weakly in law as $t \to \infty$ to a nontrivial purely atomic limiting measure μ_γ whose law was previously conjectured in [DRSV14a] and can be characterised explicitly in terms of the law of the critical GMC (see Proposition 1.3.8 for a precise statement).

Building on Proposition 1.3.8, the main goal of this chapter is to gain a deeper understanding of the convergence behaviour of supercritical GMC measures as $t \to \infty$. In particular, we refer to Section 1.5.1 for a brief overview of the main results presented here.

Acknowledgements. This work grew out of discussions between Martin Hairer and Christophe Garban at the 2023 SwissMAP Workshop in Mathematical Physics. We are also grateful to Michael Aizenman and Rémi Rhodes for interesting discussions on this topic.

2.2 Main results

We now state the main results of this chapter. In Section 2.2.1, we analyse the structure of a $\star$-scale invariant field around a mesoscopic maximum. Then, in Section 2.2.2, we present the result concerning the convergence of the supercritical GMC. Finally, in Section 2.2.3, we discuss the convergence of the measure-valued processes $(\mu_{\gamma,t+s})_{s\geq 0}$ as $t \to \infty$, as discussed in Section 1.5.1.

2.2.1 Local structure of extremal points

We aim to investigate the local structure around points within the domain where the field attains unusually large values, comparable to its maximum. The strong correlation with nearby points suggests that each peak in

the field comes with a cluster of high values. These clusters of high-value points are generally well-separated from each other. By selecting one of these clusters and identifying as reference point the maximum of the field inside the cluster, our goal is to describe the "shape" of the field in the vicinity of this reference point.

Remark 2.2.1. We will see that the behaviour of X_b near a mesoscopic maximum is dominated by its radial dependence. This in turn behaves like a Brownian motion, with time playing the role of the logarithm of spatial distance. It is useful to keep this analogy in mind when parsing the results in this section.

In order to make this heuristic precise, we need to introduce some notation.

Definition 2.2.2. For $b \in (0, \infty)$, we introduce the *recentering constant* $\mathfrak{m}_b$ by letting

$$\mathfrak{m}_b \stackrel{\text{def}}{=} \sqrt{2d}b - \frac{3}{2\sqrt{2d}} \log b \, . \tag{2.2.1}$$

Furthermore, we define functions $\mathfrak{h}_b : \mathbb{R}^d \to \mathbb{R}$ and $\mathfrak{a}_b : \mathbb{R}^d \to \mathbb{R}$ by

$$\mathfrak{h}_b(x) \stackrel{\text{def}}{=} \frac{1}{b} \int_0^b \mathfrak{K}(e^{-s}x) ds \, , \qquad \mathfrak{a}_b(x) \stackrel{\text{def}}{=} \int_0^b \big(1 - \mathfrak{K}(e^{-s}x)\big) ds \, , \tag{2.2.2}$$

with the definition of $\mathfrak{a}_b$ extended also to the case $b = \infty$.

In [Mad15], Madaule proved that there exists a constant $c > 0$ such that

$$\sup_{x \in B(0,1)} \mathrm{X}_b(x) - \mathfrak{m}_b - c \Rightarrow \mathrm{G} + \log \mu_{\gamma_c}\big(B(0,1)\big)$$

in law as $b \to \infty$, where G is an independent random variable with standard Gumbel distribution, and μ_{γ_c} is the critical GMC. Hence, if we aim to describe the shape of the field around a point where the value of the field is comparable to its maximum, it seems natural to condition the field X_b on achieving the value $\mathfrak{m}_b + z$ at the origin, for some fixed $z \in \mathbb{R}$, while simultaneously requiring that the origin is a mesoscopic maximum. We first zoom in around the origin by introducing the rescaled field

$$\overline{\mathrm{X}}_b(\cdot) \stackrel{\text{def}}{=} \mathrm{X}_b(e^{-b}\cdot) \, .$$

We now introduce some Gaussian fields that will play an important role in our analysis.

Definition 2.2.3. For $b \in \mathbb{R}^+ \cup \{\infty\}$, we let Φ_b be the centred Gaussian field on $\mathbb{R}^d$ such that, for all x, $y \in \mathbb{R}^d$,

$$\mathbb{E}\big[\Phi_b(x)\Phi_b(y)\big] = \mathfrak{a}_b(x) + \mathfrak{a}_b(y) - \mathfrak{a}_b(x - y) \, .$$

Remark 2.2.4. One can easily verify that the field Φ_b has the same law as the field $\overline{\mathrm{X}}_b - \overline{\mathrm{X}}_b(0)$. In particular, this implies that the map $\mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto \mathfrak{a}_b(x) + \mathfrak{a}_b(y) - \mathfrak{a}_b(x - y)$ defines a valid covariance kernel.

For $b \in \mathbb{R}^+ \cup \{\infty\}$, we let Υ_b be the Gaussian field on $\mathbb{R}^d$ given by

$$\Upsilon_b(\cdot) \stackrel{\text{def}}{=} \Phi_b(\cdot) - \sqrt{2d}\mathfrak{a}_b(\cdot) \, . \tag{2.2.3}$$

Remark 2.2.5. As one can easily check, the covariance of the field Φ_∞ resembles very much the covariance structure of the DGFF on $\mathbb{Z}^2$ pinned to zero at zero (see [BL18, Equation (2.7)]). Indeed, in this setting, the covariance takes the same form as for the field Φ_∞ with $\mathfrak{a}_\infty : \mathbb{Z}^2 \to \mathbb{R}$ given by the potential kernel of the simple symmetric random walk started from zero (see [BL18, Equation (2.8)]).

Now, going back to our previous discussion, a straightforward calculation shows that the field $\overline{\mathrm{X}}_b$ conditioned to take the value $\mathfrak{m}_b + z$ at the origin has the same law as the shifted field

$$\overline{\mathrm{X}}_b(\cdot) - \mathfrak{h}_b(\cdot)\big(\overline{\mathrm{X}}_b(0) - (\mathfrak{m}_b + z)\big) \, .$$

In particular, for every $x, y \in \mathbb{R}^d$ sufficiently close (much less than e^b) to the origin, we note that, for $b > 0$ large enough, it holds that

$$\mathbb{E}\big[\big(\overline{\mathrm{X}}_b(x) - \mathfrak{h}_b(x)\overline{\mathrm{X}}_b(0)\big)\big(\overline{\mathrm{X}}_b(y) - \mathfrak{h}_b(y)\overline{\mathrm{X}}_b(0)\big)\big] \approx \mathfrak{a}_b(x) + \mathfrak{a}_b(y) - \mathfrak{a}_b(x-y) \, ,$$

and also

$$(\mathfrak{m}_b + z) - (\mathfrak{m}_b + z)\mathfrak{h}_b(x) \approx \sqrt{2d}\mathfrak{a}_b(x) \, .$$

Therefore, the preceding computations imply that

$$\big(\overline{\mathrm{X}}_b(\cdot) - \mathfrak{h}_b(\cdot)\big(\overline{\mathrm{X}}_b(0) - (\mathfrak{m}_b + z)\big)\big) - (\mathfrak{m}_b + z) \approx \Phi_b(\cdot) - \sqrt{2d}\mathfrak{a}_b(\cdot) = \Upsilon_b(\cdot) \, .$$

Hence, recalling Definition 2.2.3, we aim at describing the limit in law as $b \to \infty$ of the field Υ_b under the condition that the origin is close to being a maximum in the ball of radius e^b centred at the origin.

Before doing so, we introduce some additional notation. For $k \geq 0$, we set

$$\mathcal{R}_k \stackrel{\text{def}}{=} \big\{\mathbf{F} : \mathcal{C}(\mathbb{R}^d) \to \mathbb{R} \, : \, \mathbf{F}(\phi) = \mathbf{F}(\psi) \text{ whenever } \phi|_{B(0,k)} = \psi|_{B(0,k)}\big\} \, . \tag{2.2.4}$$

In other words, $\mathcal{R}_k$ is the set of (measurable) mappings from $\mathcal{C}(\mathbb{R}^d)$ to $\mathbb{R}$ that depend on the values of the input function only inside $B(0,k)$. Furthermore, with a slight abuse of the usual notation, we define

$$\mathcal{C}^b_{\text{loc}}(\mathcal{C}(\mathbb{R}^d)) \stackrel{\text{def}}{=} \bigcup_{k \geq 0} \mathcal{R}_k \cap \mathcal{C}^b(\mathcal{C}(\mathbb{R}^d)) \, , \tag{2.2.5}$$

where $\mathcal{C}^b$ denotes continuous bounded functions. We can now state our first main result, where we write $\mathbb{M}_{0,b}(f)$ as a shorthand for $\sup_{|x| \leq e^b} f(x)$.

Theorem 2.A. *For each $\lambda > 0$, there exists a continuous random field $\widetilde{\Upsilon}_\lambda$ on $\mathbb{R}^d$ such that, for any function $\mathbf{F} \in \mathcal{C}^b_{\text{loc}}(\mathcal{C}(\mathbb{R}^d))$, one has*

$$\mathbb{E}\big[\mathbf{F}\big(\widetilde{\Upsilon}_\lambda\big)\big] = \lim_{b \to \infty} \mathbb{E}\big[\mathbf{F}\big(\Upsilon_b\big) \,\big|\, \mathbb{M}_{0,b}(\Upsilon_b) \leq \lambda\big] = \lim_{b \to \infty} \mathbb{E}\big[\mathbf{F}\big(\Upsilon_\infty\big) \,\big|\, \mathbb{M}_{0,b}(\Upsilon_\infty) \leq \lambda\big] \, . \tag{2.2.6}$$

We emphasise that the existence of the weak limits in (2.2.6) is part of the statement. We also observe that the conditioning on the right-hand side of (2.2.6) is singular as $b \to \infty$. More precisely, by letting

$$\alpha \stackrel{\text{def}}{=} \sqrt{2/\pi} \, ,$$

we have the following result.

Theorem 2.B. *For each $\lambda > 0$, there exists a constant $c_{\star,\lambda} > 0$, such that*

$$\lim_{b \to \infty} \sqrt{b}\, \mathbb{P}\big(\mathbb{M}_{0,b}(\Upsilon_\infty) \leq \lambda\big) = \alpha\, c_{\star,\lambda} \, .$$

Remark 2.2.6. In Section 2.4, we introduce a characterisation of the field Φ_b in terms of a stochastic integral

driven by a one-dimensional Brownian motion $(B_t)_{t\geq 0}$ (as a cartoon, think of this as being a smoothened out version of the field $x \mapsto -B_{\log|x|}$), plus an independent centred Gaussian field, see (2.4.1). In particular, we will prove a stronger version of Theorems 2.A and 2.B, where we also allow conditioning on the value of B at time b. We refer to Propositions 2.4.1 and 2.4.2 for the precise statements.

Remark 2.2.7. Although we don't have an explicit representation for the constant $c_{\star,\lambda}$, we will see in (2.4.44) below that it is given by

$$c_{\star,\lambda} = \lim_{k\to\infty} \mathbb{E}\big[B_k \mathbb{1}_{\{B_k\in[k^{1/6},k^{5/6}]\}} \mathbb{1}_{\{\mathbb{M}_{0,k}(\Upsilon_k)\leq\lambda\}}\big] \,, \tag{2.2.7}$$

with B and Υ_k related as in Remark 2.2.6. The exponents $1/6$ and $5/6$ appearing here are of course unimportant and could probably be replaced by any values in $(0,1/2)$ and $(1/2,\infty)$ respectively. We refer to Lemma 2.4.18 for a proof of the fact that $c_{\star,\lambda}\in(0,\infty)$.

We recall that the introduction of a threshold $\lambda>0$ in Theorems 2.A and 2.B is necessary due to the continuous setting in which we are working. In such a context, conditioning on the event that a field, which is zero at the origin, remains negative is ill-posed. However, it is desirable to define a "canonical" field that captures the local structure of X around an extremal point, without being arbitrarily dependent on λ.

To achieve this, we introduce below a field Ψ_λ, which is essentially just $\widetilde{\Upsilon}_\lambda$ shifted to move its maximum to the origin, but under a slightly tilted law. At first glance, it may seem contradictory to define a field that we claim is independent of λ while still denoting it as Ψ_λ. This notation arises because, from its definition, it is not immediately evident that Ψ_λ is indeed independent of the threshold λ. However, this independence (albeit in a slightly weaker sense) will be established a posteriori (see Proposition 2.2.14).

Before defining the field Ψ_λ, for all $x\in\mathbb{R}^d$, we introduce the shift operator $\tau_x : \mathcal{C}(\mathbb{R}^d)\to\mathcal{C}(\mathbb{R}^d)$ by

$$\tau_x f(\cdot) = f(\cdot+x) - f(x) \,, \qquad \forall\, f\in\mathcal{C}(\mathbb{R}^d) \,. \tag{2.2.8}$$

Furthermore, in Lemma 2.5.1 below, we define a certain uncountable set $\Lambda\subseteq\mathbb{R}^+$. Roughly speaking, Λ consists of the "good thresholds" λ for which the law of the field $\widetilde{\Upsilon}_\lambda$ exhibits some desirable properties. More precisely, these bad values are those values of λ such that $\mathbb{P}(\sup_{x\in\mathbb{R}^d}\widetilde{\Upsilon}_\lambda(x)=\lambda)>0$ or $\mathbb{P}(|\{y\in\mathbb{R}^d : \widetilde{\Upsilon}_\lambda(y)=\sup_{x\in\mathbb{R}^d}\widetilde{\Upsilon}_\lambda(x)-\lambda\}|>0)>0$ where $|\cdot|$ denotes Lebesgue measure. We emphasise that we don't expect any such bad values to exist, i.e., we expect that $\Lambda=\mathbb{R}^+$. However, since the field $\widetilde{\Upsilon}_\lambda$ is itself defined by a singular conditioning proving this fact would require additional effort. Since our main result does not require ruling out the existence of bad values, we will not investigate this fact further. In any case, we have to exclude at most countably many points, so Λ is dense.

Definition 2.2.8. For $\lambda\in\Lambda$, we let Ψ_λ be the field on $\mathbb{R}^d$ uniquely characterised by the fact that, for all $\mathbf{F}\in\mathcal{C}^b(\mathcal{C}(\mathbb{R}^d))$,

$$\mathbb{E}\big[\mathbf{F}(\Psi_\lambda)\big] \propto \mathbb{E}\left[\frac{\mathbf{F}\big(\tau_{x_\star}\widetilde{\Upsilon}_\lambda\big)e^{\sqrt{2d}\widetilde{\Upsilon}_\lambda(x_\star)}}{\int_{\mathbb{R}^d} e^{\sqrt{2d}\widetilde{\Upsilon}_\lambda(x)}\mathbb{1}_{\{\widetilde{\Upsilon}_\lambda(x)\geq\widetilde{\Upsilon}_\lambda(x_\star)-\lambda\}}dx}\right] \,, \tag{2.2.9}$$

where $x_\star=\operatorname{argmax}\{\widetilde{\Upsilon}_\lambda(x) : x\in\mathbb{R}^d\}$ and the proportionality constant is chosen in such a way that $\mathbb{E}[1]=1$.

Remark 2.2.9. The fact that the proportionality constant in the previous definition lies in $(0,\infty)$ is proved in Lemma 2.5.6.

We now state the following key "resampling property" of the field $\widetilde{\Upsilon}_\lambda$, whose proof is given in Section 2.5.2.

Proposition 2.2.10. *For each $\lambda \in \Lambda$ and for all $\mathbf{F} \in \mathcal{C}^b_{\mathrm{loc}}(\mathcal{C}(\mathbb{R}^d))$, it holds that*

$$\mathbb{E}\big[\mathbf{F}(\widetilde{\Upsilon}_\lambda)\big] = \mathbb{E}\left[\frac{\int_{\mathbb{R}^d} \mathbf{F}(\tau_x \widetilde{\Upsilon}_\lambda) e^{\sqrt{2d}\widetilde{\Upsilon}_\lambda(x)} \mathbb{1}_{\{\widetilde{\Upsilon}_\lambda(x) \geq \widetilde{\Upsilon}_\lambda(x_\star) - \lambda\}} dx}{\int_{\mathbb{R}^d} e^{\sqrt{2d}\widetilde{\Upsilon}_\lambda(x)} \mathbb{1}_{\{\widetilde{\Upsilon}_\lambda(x) \geq \widetilde{\Upsilon}_\lambda(x_\star) - \lambda\}} dx}\right], \tag{2.2.10}$$

where we recall that $x_\star = \mathrm{argmax}\{\widetilde{\Upsilon}_\lambda(x) \,:\, x \in \mathbb{R}^d\}$.

Remark 2.2.11. In fact, Proposition 2.2.10 is quite general and applies to a large class of Gaussian fields of the form $\Phi - \eta\mathfrak{a}$ where Φ is centred with $\mathbb{E}[(\Phi(x) - \Phi(y))^2] = \mathfrak{a}(x-y)$. In particular, one can replace the field $\widetilde{\Upsilon}_\lambda$ in (2.2.10) by a drifted, conditioned Brownian motion. Specifically, for $\lambda > 0$ and $\eta \geq 0$, let $(X_t^{\lambda,\eta})_{t\in\mathbb{R}}$ be a two-sided Brownian motion with drift $t \mapsto -\eta|t|$ for $t \in \mathbb{R}$, conditioned to remain below λ at all times. When $\eta > 0$ this is a condition that happens with positive probability, while the case $\eta = 0$ can be covered by a limiting procedure, yielding a two-sided three-dimensional Bessel process. Furthermore, consider the set

$$\mathrm{A}_{\lambda,\eta} \overset{\text{def}}{=} \left\{ t \in \mathbb{R} \,:\, X_t^{\lambda,\eta} \geq \sup_{s\in\mathbb{R}} X_s^{\lambda,\eta} - \lambda \right\},$$

and let $\rho_{\lambda,\eta}$ be the (random) probability measure on $\mathrm{A}_{\lambda,\eta}$ defined as follows

$$\rho_{\lambda,\eta}(dt) \overset{\text{def}}{=} \frac{e^{\eta X_t^{\lambda,\eta}}}{\int_{\mathrm{A}_{\lambda,\eta}} e^{\eta X_s^{\lambda,\eta}} ds} \mathbb{1}_{\{t\in\mathrm{A}_{\lambda,\eta}\}} dt \,.$$

Then, if $t_\star$ is a point sampled from the probability measure $\rho_{\lambda,\eta}$, we have the identity in law

$$\big(X^{\lambda,\eta}_{t+t_\star} - X^{\lambda,\eta}_{t_\star}\big)_{t\in\mathbb{R}} \overset{\text{law}}{=} \big(X_t^{\lambda,\eta}\big)_{t\in\mathbb{R}} \,,$$

which we were unable to find in the existing literature. Note that, in particular, when $\eta = 0$, the probability measure $\rho_{\lambda,0}$ is the uniform measure on $\mathrm{A}_{\lambda,0}$.

Now, returning to our setting, we observe that, thanks to Proposition 2.2.10, an alternative way to characterise Ψ_λ is by inverting (2.2.9). More precisely, we have the following result, whose proof is given in Section 2.5.2.

Proposition 2.2.12. *For any $\lambda \in \Lambda$ and for all $\mathbf{F} \in \mathcal{C}^b_{\mathrm{loc}}(\mathcal{C}(\mathbb{R}^d))$, it holds that*

$$\mathbb{E}\big[\mathbf{F}(\widetilde{\Upsilon}_\lambda)\big] \propto \mathbb{E}\left[\int_{\mathbb{R}^d} \mathbf{F}(\tau_x \Psi_\lambda) e^{\sqrt{2d}\Psi_\lambda(x)} \mathbb{1}_{\{\Psi_\lambda(x) \geq -\lambda\}} dx\right], \tag{2.2.11}$$

where the proportionality constant is chosen in such a way that $\mathbb{E}[1] = 1$.

Remark 2.2.13. We observe that the fact that the proportionality constant in the previous proposition lies in $(0,\infty)$ follows directly from Definition 2.2.8 of the field Ψ_λ, together with the fact that the proportionality constant appearing in that definition lies in $(0,\infty)$.

As claimed above, we now state a result that confirms that the field Ψ_λ, introduced in Definition 2.2.8, is indeed canonical. Specifically, we have the following result, whose proof is given in Section 2.5.3.

Proposition 2.2.14. *For any $\lambda_1, \lambda_2 \in \Lambda$, one has $\Psi_{\lambda_1} \overset{\text{law}}{=} \Psi_{\lambda_2}$.*

Since the law of Ψ_λ does not depend on λ, except possibly for a countable number of "bad" values, we just write Ψ from now on.

2.2.2 Stable convergence of supercritical GMC

We now state our main result regarding the convergence of GMC measures. Before proceeding, we fix for the remainder of this section the set $\Lambda \subseteq \mathbb{R}^+$ introduced in Lemma 2.5.1 below. We also let Ψ denote the field introduced in Definition 2.2.8, which, as noted in Proposition 2.2.14, has a law that does not depend on $\lambda \in \Lambda$.

We begin by introducing the assumptions considered in our next main theorem.

Assumption 2.2.15. For $n \in \mathbb{N}$, consider a collection of fields $(\mathrm{W}_{i,t})_{i\in[n],t\geq 0}$ on $\mathbb{R}^d$ such that:

(W1) For any $t \geq 0$, the collection of fields $(\mathrm{W}_{i,t})_{i\in[n]}$ is independent of the σ-field $\mathcal{F}_t$ defined in (2.1.5).

(W2) There exist stationary fields $(\mathrm{W}_i)_{i\in[n]}$ on $\mathbb{R}^d$ such that, for any fixed $t \geq 0$,

$$\big(\mathrm{W}_{i,t}(\cdot)\big)_{i\in[n]} \overset{\text{law}}{=} \big(\mathrm{W}_i(e^t\cdot)\big)_{i\in[n]} .$$

(W3) For $0 \leq s < t$ and for all $x, y \in \mathbb{R}^d$ such that $|x-y| > e^{-s}$, it holds that

$$\big(\mathrm{W}_{i,t}(x)\big)_{i\in[n]} \perp \big(\mathrm{W}_{i,t}(y)\big)_{i\in[n]} .$$

(W4) For all $\gamma > \sqrt{2d}$, it holds that $\sup_{x\in\mathbb{R}^d} \sum_{i=1}^n \mathbb{E}[e^{\gamma \mathrm{W}_i(x)}] < \infty$.

For $\lambda \in \Lambda$, recalling (2.2.7), we define the constant

$$a_\star \overset{\text{def}}{=} \frac{\alpha\, c_{\star,\lambda}}{\gamma\, \mathbb{E}\big[\int_{\mathbb{R}^d} e^{\sqrt{2d}\Psi(x)} \mathbb{1}_{\{\Psi(x)\geq -\lambda\}} dx\big]} \in (0,\infty) . \tag{2.2.12}$$

The subscript λ is not included in the notation $a_\star$ since it turns out that the right-hand side of (2.2.12) does not actually depend on $\lambda \in \Lambda$, as shown in Lemma 2.5.8 below. We are now ready to state our next main theorem, with the definition of stable convergence provided in Section 2.3.2.

Theorem 2.C. *Let $\gamma > \sqrt{2d}$ and consider the sequence of measures $(\mu_{\gamma,t})_{t\geq 0}$ defined in (2.1.8). For $n \in \mathbb{N}$, consider a collection of fields $(\mathrm{W}_{i,t})_{i\in[n],t\geq 0}$ satisfying (W1)–(W4). For each $t \geq 0$ and each $i \in [n]$, define the measure $\mu_{\gamma,t,i}$ by*

$$\mu_{\gamma,t,i}(dx) \overset{\text{def}}{=} e^{\gamma \mathrm{W}_{i,t}(x)} \mu_{\gamma,t}(dx) .$$

Consider the $\mathbb{R}^n$-valued random variable $\mathbf{Z}_\gamma = (\mathbf{Z}_{\gamma,i})_{i\in[n]}$ such that, for each $i \in [n]$, the i-th component $\mathbf{Z}_{\gamma,i}$ is given by

$$\mathbf{Z}_{\gamma,i} \overset{\text{def}}{=} \int_{\mathbb{R}^d} \exp\big(\gamma\big(\Psi(y) + \mathrm{W}_i(y)\big)\big) dy . \tag{2.2.13}$$

Let $(x_j, w_j)_{j\in\mathbb{N}} \subseteq \mathbb{R}^d \times [0,\infty]$ be an arbitrary enumeration of the atoms of the Poisson point measure $\eta_\gamma[\mu_{\gamma_c}]$ as introduced in Definition 1.3.7. Consider the collection of measures $(\mu_{\gamma,i})_{i\in[n]}$ constructed as follows

$$\mu_{\gamma,i}(dx) \overset{\text{def}}{=} a_\star^{\frac{\gamma}{\sqrt{2d}}} \sum_{j\in\mathbb{N}} \mathbf{Z}_{\gamma,i,j} w_j \delta_{x_j} , \qquad \forall\, i \in [n] ,$$

where $((\mathbf{Z}_{\gamma,i,j})_{i\in[n]})_{j\in\mathbb{N}}$ is a collection of i.i.d. copies of $(\mathbf{Z}_{\gamma,i})_{i\in[n]}$, independent of $\eta_\gamma[\mu_{\gamma_c}]$. Then, the sequence of measures $(\mu_{\gamma,t,i})_{i\in[n]}$ converges $\sigma(\mathrm{X})$-stably to the collection of measures $(\mu_{\gamma,i})_{i\in[n]}$ as $t \to \infty$.

Remark 2.2.16. Let $\mathbf{Z}_\gamma$ be the random variable defined by

$$\mathbf{Z}_\gamma \overset{\text{def}}{=} \int_{\mathbb{R}^d} \exp\big(\gamma\big(\Psi(y)\big)\big) dy .$$

Then, as follows from the proof of Proposition 2.2.19 below, we have $\mathbb{E}[\mathbf{Z}_\gamma^{\sqrt{2d}/\gamma}] < \infty$. In particular, this implies that $\mathbf{Z}_\gamma$ is almost surely finite.

Remark 2.2.17. Theorem 2.C is significantly more general than [MRV16, Theorem 2.2]. For instance, taking $n = 1$ and $\mathrm{W}_{1,\cdot} = 0$, we not only recover Proposition 1.3.8 but also obtain a relatively explicit representation for the multiplicative constant appearing in front of the limiting measure. Indeed, the constant c appearing in [MRV16, page 646] is given by

$$c = a_\star^{\frac{\gamma}{\sqrt{2d}}} \mathbb{E}\big[\mathbf{Z}_\gamma^{\sqrt{2d}/\gamma}\big]^{\frac{\gamma}{\sqrt{2d}}} .$$

Remark 2.2.18. We emphasise that the techniques developed in the present chapter can also be applied to the study of the recentered maximum of log-correlated Gaussian fields over a bounded region. For instance, under assumptions (K1)–(K2) on the seed covariance function, it is relatively straightforward to obtain a fairly explicit representation of the constant C^* appearing in [Mad15, Theorem 1.1]. More precisely, one has

$$C^* = \frac{\alpha\, c_{\star,\lambda}}{\sqrt{2d}\,\mathbb{E}\big[\int_{\mathbb{R}^d} e^{\sqrt{2d}\Psi(x)} \mathbb{1}_{\{\Psi(x)\geq -\lambda\}} dx\big]} .$$

As before, the quantity on the right-hand side does not depend on the choice of $\lambda \in \Lambda$.

For $\gamma > \sqrt{2d}$, we define the constant $\beta(d,\gamma)$ by letting

$$\beta(d,\gamma) \stackrel{\text{def}}{=} \frac{\Gamma(1-\sqrt{2d}/\gamma)}{\sqrt{2d}/\gamma} . \tag{2.2.14}$$

Theorem 2.C is a direct consequence of the following result, the proof of which is given in Section 2.6.4, where we compute the joint Laplace transform of the collection of measures $(\mu_{\gamma,t,i})_{i\in[n]}$. In what follows, for a measure ν on $\mathbb{R}^d$ and a function $f : \mathbb{R}^d \to \mathbb{R}$, we write $\nu(f)$ to denote the integral of f against ν.

Proposition 2.2.19. *Consider the same setting described in Theorem 2.C. Consider the mapping* $\mathrm{T}_\gamma : (\mathcal{C}_c^+(\mathbb{R}^d))^n \to \mathcal{C}_c^+(\mathbb{R}^d)$ *defined by*

$$\mathrm{T}_\gamma[f_1,\dots,f_n](\cdot) \stackrel{\text{def}}{=} \mathbb{E}\left[\left(\sum_{i=1}^n f_i(\cdot)\int_{\mathbb{R}^d} \exp\big(\gamma\big(\Psi(y)+\mathrm{W}_i(y)\big)\big)dy\right)^{\frac{\sqrt{2d}}{\gamma}}\right] . \tag{2.2.15}$$

Then, for all $(\varphi,(f_i)_{i\in[n]}) \in \mathcal{C}_c^\infty(\mathbb{R}^d)\times(\mathcal{C}_c^+(\mathbb{R}^d))^n$*, the following limit holds*

$$\lim_{t\to\infty} \mathbb{E}\left[\exp\big(i\langle \mathrm{X},\varphi\rangle\big)\prod_{i=1}^n \exp\big(-\mu_{\gamma,t,i}(f_i)\big)\right] = \mathbb{E}\Big[\exp\big(i\langle \mathrm{X},\varphi\rangle\big)\exp\big(-\widetilde{a}_\star \mu_{\gamma_c}\big(\mathrm{T}_\gamma[f_1,\dots,f_n]\big)\big)\Big] ,$$

where $\widetilde{a}_\star = \beta(d,\gamma)a_\star > 0$ *with* $a_\star$ *as defined in* (2.2.12).

2.2.3 A measure-valued process

As we have already briefly mentioned, an interesting situation where Theorem 2.C can be applied is when the admissible fields are given by the increments of the martingale approximation of the $\star$-scale invariant field X. More precisely, we note that for all $t, s \geq 0$ it holds that

$$\mathrm{X}_{t+s}(x) = \mathrm{X}_t(x) + \mathrm{W}_{s,t}(x) , \qquad \forall\, x \in \mathbb{R}^d ,$$

where $\mathrm{W}_{s,t}(\cdot) = \mathrm{W}_s(e^t\cdot)$ with $\mathrm{W}_s(\cdot)$ a centred Gaussian field independent of $\mathcal{F}_t$ and such that

$$\mathbb{E}\big[\mathrm{W}_s(x)\mathrm{W}_s(y)\big] = \int_0^s \mathfrak{K}\big(e^u(x-y)\big)du\ , \qquad \forall\, x,y\in\mathbb{R}^d\ .$$

For $\gamma > \sqrt{2d}$, the measure $\mu_{\gamma,t+s}$ can then be rewritten as

$$\mu_{\gamma,t+s}(dx) = e^{\gamma(\mathrm{W}_{s,t}(x)-\sqrt{2d}s)+ds}\big((t+s)/t\big)^{\frac{3\gamma}{2\sqrt{2d}}}\mu_{\gamma,t}(dx)\ . \tag{2.2.16}$$

Furthermore, for any finite collection of non-negative numbers $(s_1,\dots,s_n)\subseteq\mathbb{R}_0^+$, the family of fields $(\mathrm{W}_{s_i,t})_{i\in[n],t\geq 0}$ satisfies (W1)–(W4). Therefore, by Theorem 2.C and Kolmogorov's extension theorem, there exists a process $(\nu_{\gamma,s})_{s\geq 0}$ taking values in the space of non-negative, locally finite measures on $\mathbb{R}^d$ such that the collection of measures $(\nu_{\gamma,s_1},\dots,\nu_{\gamma,s_n})$ is the $\sigma(\mathrm{X})$-stable limit of $(\mu_{\gamma,t+s_1},\dots,\mu_{\gamma,t+s_n})$ as $t\to\infty$.

Let Ψ denote the field appearing in (2.2.13). For all $s\geq 0$, we define the field Ψ_s on $\mathbb{R}^d$ by letting

$$\Psi_s(\cdot) = \Psi(e^{-s}\cdot) + \big(\mathrm{W}_s(e^{-s}\cdot) - \sqrt{2d}s\big)\ . \tag{2.2.17}$$

As a consequence of (2.2.16), we then have the following corollary of Theorem 2.C.

Corollary 2.D. *For $\gamma > \sqrt{2d}$, consider the $\mathbb{R}$-valued process $(\mathbf{W}_{\gamma,s})_{s\geq 0}$ given by*

$$\mathbf{W}_{\gamma,s} \overset{\text{def}}{=} \log\left(\int_{\mathbb{R}^d}\exp\big(\gamma\Psi_s(x)\big)dx\right)\ . \tag{2.2.18}$$

Let $(x_j,w_j)_{j\in\mathbb{N}}\subseteq\mathbb{R}^d\times[0,\infty]$ be an arbitrary enumeration of the atoms of the Poisson point measure $\eta_\gamma[\mu_{\gamma_c}]$ as introduced in Definition 1.3.7. Consider the collection of measures $(\nu_{\gamma,s})_{s\geq 0}$ defined as follows

$$\nu_{\gamma,s} \overset{\text{def}}{=} a_\star^{\frac{\gamma}{\sqrt{2d}}}\sum_{j\in\mathbb{N}} e^{\mathbf{W}_{\gamma,s,j}}w_j\delta_{x_j}\ , \qquad \forall\, s\geq 0\ , \tag{2.2.19}$$

where $((\mathbf{W}_{\gamma,s,j})_{s\geq 0})_{j\in\mathbb{N}}$ is a collection of i.i.d. copies of $(\mathbf{W}_{\gamma,s})_{s\geq 0}$, independent of $\eta_\gamma[\mu_{\gamma_c}]$. Then, the collection of measures $(\mu_{\gamma,t+s})_{s\geq 0}$ converges $\sigma(\mathrm{X})$-stably in the finite dimensional sense to the collection of measures $(\nu_{\gamma,s})_{s\geq 0}$ as $t\to\infty$.

We conclude this section with a couple of remarks and conjectures.

Remark 2.2.20 (Stationarity modulo tilt)**.** The process $(\nu_{\gamma,s})_{s\geq 0}$ introduced in Corollary 2.D is stationary by construction. Nonetheless, it is desirable to provide an a posteriori justification of its stationarity. Below, we offer a brief heuristic explanation for why this property should hold.

For $s\geq 0$, write $x_s = \operatorname{argmax}\{\Psi_s(x)\,:\,x\in\mathbb{R}^d\}$, $\mathrm{Z}_s = \Psi_s(x_s)$, and $\widetilde{\Psi}_s = \Psi_s(x_s+\cdot) - \Psi_s(x_s)$. Fix a finite collection of non-negative numbers $(s_0,\dots,s_n)\subseteq\mathbb{R}_0^+$ such that $s_0\leq\ldots\leq s_n$, and consider the joint limit of $(\mu_{\gamma,t+s_i})_{i\in[n]}$ as $t\to\infty$. For all $(f_i)_{i\in[n]}\in(\mathcal{C}_c^+(\mathbb{R}^d))^n$, their joint Laplace transform is given by the expression in Proposition 2.2.19 with

$$\mathrm{T}_\gamma\big[f_1,\dots,f_n\big](\cdot) = \mathbb{E}\left[\left(\sum_{i=1}^n f_i(\cdot)\int_{\mathbb{R}^d}\exp\big(\gamma\Psi_{s_i}(y)\big)dy\right)^{\frac{\sqrt{2d}}{\gamma}}\right]$$

$$= \mathbb{E}\left[e^{\sqrt{2d}\mathrm{Z}_{s_0}} \left(\sum_{i=1}^{n} f_i(\cdot) e^{\gamma(\mathrm{Z}_{s_i} - \mathrm{Z}_{s_0})} \int_{\mathbb{R}^d} \exp\bigl(\gamma \widetilde{\Psi}_{s_i}(y)\bigr) dy \right)^{\frac{\sqrt{2d}}{\gamma}} \right] .$$

This strongly suggests that the process $(\widetilde{\Psi}_s)_{s\geq 0}$ is "stationary modulo tilt" in the sense that, for any non-decreasing sequence of non-negative times $(s_0, \dots, s_n) \subseteq \mathbb{R}_0^+$, any $t \geq 0$, and any function $\mathbf{F} \in \mathcal{C}^b((\mathcal{C}(\mathbb{R}^d))^n)$, one has

$$\mathbb{E}\bigl[e^{\sqrt{2d}\mathrm{Z}_{s_0}} \mathbf{F}(\widetilde{\Psi}_{s_1}, \dots, \widetilde{\Psi}_{s_n})\bigr] = \mathbb{E}\bigl[e^{\sqrt{2d}\mathrm{Z}_{s_0+t}} \mathbf{F}(\widetilde{\Psi}_{s_1+t}, \dots, \widetilde{\Psi}_{s_n+t})\bigr] . \tag{2.2.20}$$

Now, consider the process $(\rho_t)_{t\geq 0}$ given by

$$\rho_t \overset{\text{def}}{=} \sum_{j\in\mathbb{N}} \delta_{x_j} \otimes \delta_{w_j + \mathrm{Z}_{j,t}} \otimes \delta_{\widetilde{\Psi}_{j,t}} , \qquad \forall\, t \geq 0 , \tag{2.2.21}$$

where $(\mathrm{Z}_j, \widetilde{\Psi}_j)_{j\in\mathbb{N}}$ is a collection of i.i.d. copies of $(\mathrm{Z}, \widetilde{\Psi})$, and $(x_j, w_j)_{j\in\mathbb{N}}$ is an arbitrary enumeration of the Poisson point process with intensity $a_\star \gamma \mu_{\gamma_c}(dx) \otimes \exp(-\sqrt{2d}w)dw$, where $a_\star$ is the constant introduced in (2.2.12). Then, similarly to [RA05, Proposition 3.1], one can show that (2.2.20) implies that the process $(\rho_t)_{t\geq 0}$ is stationary.

Remark 2.2.21 (Full process convergence)**.** For $0 \ll b \ll t$, recalling (2.2.1), we define the measure $\rho_{t,b}$ on $[0,1]^d \times \mathbb{R} \times \mathcal{C}(\mathbb{R}^d)$ as follows

$$\rho_{t,b} \overset{\text{def}}{=} \sum_{x\in\mathbb{R}^d \,:\, \mathrm{X}_t(x) = \sup_{|y-x|\leq e^{b-t}} \mathrm{X}_t(y)} \delta_x \otimes \delta_{\mathrm{X}_t(x) - \mathfrak{m}_t} \otimes \delta_{\mathrm{X}_t(x + e^{-t}\cdot) - \mathrm{X}_t(x)} ,$$

which roughly speaking keeps track of the locations of the maxima, their heights, and the shape of the field around them. Then, combining the results of the present chapter with the techniques of [BL18], it should not be too hard to show, analogously to [BL18, Theorem 2.1], the convergence in law

$$\lim_{b\to\infty} \lim_{t\to\infty} \rho_{t,b} = \mathrm{PPP}\bigl(a_\star \gamma \mu_{\gamma_c}(dx) \otimes e^{-\sqrt{2d}w} dw \otimes \nu(d\phi)\bigr) ,$$

where ν denotes the law of the field Ψ on $\mathcal{C}(\mathbb{R}^d)$, and $a_\star$ is the constant introduced in (2.2.12). Furthermore, the discussion in the preceding paragraph strongly suggests that

$$\lim_{b\to\infty} \lim_{t\to\infty} \rho_{t+\cdot,b} = \rho_\cdot ,$$

where ρ is the process introduced in (2.2.21).

2.2.4 Outline

The remainder of the chapter is organised as follows. In Section 2.3, we gather some background material that will be used throughout the chapter. Section 2.4 focuses on the local structure of extremal points, and contains the proofs of Theorems 2.A and 2.B. In Section 2.5, we establish key properties of the shape field, including a suitable resampling property, and prove Propositions 2.2.12 and 2.2.14. Section 2.6 is dedicated to the proof of Theorem 2.C, which is based on Proposition 2.2.19. The proof of this proposition, in turn, relies on a key technical result, Proposition 2.6.1, whose proof is given in Section 2.7. We conclude the chapter with two appendices. In Appendix 2.A, we establish several estimates concerning the probability of a Brownian bridge

remaining above a slowly growing positive or negative curve. Finally, Appendix 2.B contains the proof of a technical lemma used in Section 2.7.

2.3 Background and preliminaries

In this section, we collect some preliminary results needed for the proof of our main theorems. In particular, in Section 2.3.1, we collect some basic and recurrent notation that will be used throughout the chapter. In Section 2.3.2, we recall some standard results related to convergence in distribution of random measures, and in particular we briefly introduce the concept of stable convergence. Finally, in Section 2.3.3, we record some properties of $\star$-scale invariant fields and their martingale approximations.

2.3.1 Basic and recurrent notation

Numbers. We write $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. We let $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{R}_0^+ = [0, \infty)$. Without specific mention, the logarithm will be taken with respect to the natural base e. For $a \in \mathbb{R}$, we use $\lfloor a \rfloor$ to represent the largest integer not greater than a. Given $n \in \mathbb{N}$, we write $[n] = \{1, \dots, n\}$ and $[n]_0 = \{0, 1, \dots, n\}$.

Subsets of Euclidean space. We consider the space $\mathbb{R}^d$ where $d \geq 1$ is a fixed dimension. We let $(e_1, \dots, e_d)$ be the orthonormal basis of $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, we write $x = (x_1, \dots, x_d)$ for its coordinates. For $r \in \mathbb{R}$ and $x \in \mathbb{R}^d$, we write $B(x, r)$ for the ball centred at x and with radius r. Furthermore, we write

$$\mathbb{B}_r(x) \stackrel{\text{def}}{=} B(x, e^r) \,, \tag{2.3.1}$$

and we simply write $\mathbb{B}_r$ for $\mathbb{B}_r(0)$. For every Lebesgue measurable set $\mathrm{D} \subseteq \mathbb{R}^d$, we denote its Lebesgue measure by $|\mathrm{D}|$.

Functions and measure spaces. We write $\mathcal{C}(\mathbb{R}^d)$ (resp. $\mathcal{C}_c(\mathbb{R}^d)$, $\mathcal{C}^b(\mathbb{R}^d)$) for the space of continuous (resp. continuous with compact support, continuous and bounded) functions from $\mathbb{R}^d$ to $\mathbb{R}$. We write $\mathcal{C}_c^+(\mathbb{R}^d)$ for the space of positive continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$ with compact support. Given a measure ν and a function f, we write $\nu(f)$ to denote the integral of f against ν.

Maxima and related sets. For a subset $\mathrm{D} \subseteq \mathbb{R}^d$, a function $f : \mathrm{D} \to \mathbb{R}$, and $\lambda > 0$ we let

$$\mathbb{M}_{\mathrm{D}}(f) \stackrel{\text{def}}{=} \sup_{x \in \mathrm{D}} f(x) \,, \qquad \mathbb{D}_{\mathrm{D}}^{\lambda}(f) \stackrel{\text{def}}{=} \left\{ x \in \mathrm{D} \,:\, f(x) \geq \mathbb{M}_{\mathrm{D}}(f) - \lambda \right\} . \tag{2.3.2}$$

For $\mathrm{D} = \mathbb{R}^d$, we simply write $\mathbb{M}(f)$ and $\mathbb{D}^{\lambda}(f)$. Additionally, if $\mathrm{D} = [0, \mathrm{R}]^d$ for some $\mathrm{R} \geq 0$, then we write $\mathbb{M}_{\mathrm{R}}(f)$ (resp. $\mathbb{D}_{\mathrm{R}}^{\lambda}(f)$) in place of $\mathbb{M}_{[0,\mathrm{R}]^d}(f)$ (resp. $\mathbb{D}_{[0,\mathrm{R}]^d}^{\lambda}(f)$). Furthermore, when it will be convenient to do so, we will use the following shorthands,

$$\mathbb{M}_{x,r}(f) = \mathbb{M}_{\mathbb{B}_r(x)}(f) \,, \qquad \mathbb{D}_{x,r}^{\lambda}(f) = \mathbb{D}_{\mathbb{B}_r(x)}^{\lambda}(f) \,, \tag{2.3.3}$$

and also, for $r_2 > r_1$, we set

$$\mathbb{M}_{x,r_2,r_1}(f) = \mathbb{M}_{\mathbb{B}_{r_2}(x) \setminus \mathbb{B}_{r_1}(x)}(f) \,, \qquad \mathbb{D}_{x,r_2,r_1}^{\lambda}(f) = \mathbb{D}_{\mathbb{B}_{r_2}(x) \setminus \mathbb{B}_{r_1}(x)}^{\lambda}(f) \,. \tag{2.3.4}$$

2.3.2 Topological preliminaries

In this subsection, we collect some results on the convergence of random measures and introduce the concept of stable convergence, which plays an important role in our main result regarding the convergence of supercritical GMC measures.

Laplace functionals. It is well known that if η is a random point measure on $\mathbb{R}^d$, then its law is uniquely characterised by its Laplace functional on the set $\mathcal{C}_c^+(\mathbb{R}^d)$:

$$\mathcal{C}_c^+(\mathbb{R}^d) \ni \varphi \mapsto \mathbb{E}\big[\exp\big(-\eta(\varphi)\big)\big] \,.$$

We also recall that, if η_ν is a Poisson point measure with intensity measure ν, then

$$\mathbb{E}\exp\big[\big(-\eta_\nu(\varphi)\big)\big] = \exp\left(-\int_{\mathbb{R}^d}\big(1-e^{-\varphi(x)}\big)\nu(dx)\right) \,.$$

Remark 2.3.1. For $\gamma > \sqrt{2d}$ and a Radon measure ν on $\mathbb{R}^d$, let $\mathcal{P}_\gamma[\nu]$ be the integrated atomic random measure with parameter γ and spatial intensity ν as specified in Defnition 1.3.7. Then, in this case, for every $\varphi \in \mathcal{C}_c^+(\mathbb{R}^d)$, it holds that

$$\mathbb{E}\big[\exp\big(-\mathcal{P}_\gamma[\nu](\varphi)\big)\big] = \exp\left(-\beta(d,\gamma)\int_{\mathbb{R}^d}\varphi(x)^{\frac{\sqrt{2d}}{\gamma}}\nu(dx)\right) \,, \tag{2.3.5}$$

where we recall the definition (2.2.14) of the constant $\beta(d,\gamma)$.

Convergence in distributions of random measures. For a locally compact Polish space $\mathcal{X}$, we let $\mathcal{M}^+(\mathcal{X})$ be the space of non-negative, locally finite measures on $\mathcal{X}$ endowed with the topology of vague convergence. We equip the space of probability distributions on $\mathcal{M}^+(\mathcal{X})$ with the topology of weak convergence. For a collection $(\nu_t)_{t\geq 0}$ of $\mathcal{M}^+(\mathcal{X})$-valued random variables, we write $\nu_t \Rightarrow \nu$ to indicate *vague convergence in distribution* as $t \to \infty$.

We record here a useful criterion to establish the vague convergence in distributions of $\mathcal{M}^+(\mathcal{X})$-valued random variables.

Lemma 2.3.2. *Let $(\nu_t)_{t\geq 0}$ be a collection of $\mathcal{M}^+(\mathcal{X})$-valued random variables and let $\nu \in \mathcal{M}^+(\mathcal{X})$. If for all $f \in \mathcal{C}_c^+(\mathcal{X})$, the following limit holds*

$$\lim_{t\to\infty}\mathbb{E}\big[\exp\big(-\nu_t(f)\big)\big] = \mathbb{E}\big[\exp\big(-\nu(f)\big)\big] \,, \tag{2.3.6}$$

then it holds that $\nu_t \Rightarrow \nu$ as $t \to \infty$.

Proof. This is an immediate consequence of the continuity theorem for Laplace transforms, see e.g. [Kal17, Theorem 4.11]. □

Remark 2.3.3. In what follows, we will be interested in the case $\mathcal{X} = [n] \times \mathbb{R}^d$ which coincides (including the topology) with the space $(\mathcal{M}^+(\mathbb{R}^d))^n$.

Given a locally compact Polish space $\mathcal{X}$ and a Hilbert space $\mathcal{Y}$ equipped with a dense subspace $\mathcal{Y}_0$, we consider probability distributions on $\mathcal{Y} \times \mathcal{M}^+(\mathcal{X})$. Similarly to before, for a sequence $(\mathrm{Y}, \nu_t)_{t\geq 0}$ of $\mathcal{Y} \times \mathcal{M}^+(\mathcal{X})$-valued random variables, we write $(\mathrm{Y}, \nu_t) \Rightarrow (\mathrm{Y}, \nu)$ to indicate *vague convergence in distribution*. We now

state the following result, which provides sufficient conditions for convergence in distribution on the space $\mathcal{Y} \times \mathcal{M}^+(\mathcal{X})$.

Lemma 2.3.4. *Let $(\mathrm{Y}, \nu_t)_{t\geq 0}$ be a sequence of $\mathcal{Y} \times \mathcal{M}^+(\mathcal{X})$-valued random variables, and let $\nu \in \mathcal{M}^+(\mathcal{X})$. If for all $(\varphi, f) \in \mathcal{Y}_0 \times \mathcal{C}_c^+(\mathcal{X})$ the following limit holds*

$$\lim_{t\to\infty} \mathbb{E}\Big[\exp\big(i\langle \mathrm{Y}, \varphi\rangle\big) \exp\big(-\nu_t(f)\big)\Big] = \mathbb{E}\Big[\exp\big(i\langle \mathrm{Y}, \varphi\rangle\big) \exp\big(-\nu(f)\big)\Big] , \tag{2.3.7}$$

then it holds that $(\mathrm{Y}, \nu_t) \Rightarrow (\mathrm{Y}, \nu)$ as $t \to \infty$.

Proof. Taking $\varphi = 0$ in (2.3.7) and using Lemma 2.3.2, we deduce that $\nu_t \Rightarrow \nu$. Consequently, the joint distribution (Y, ν_t) is tight in $\mathcal{Y} \times \mathcal{M}^+(\mathcal{X})$. Thus, the joint convergence in distribution follows if we can show the convergence of the finite-dimensional distributions. As both Y and ν_t are linear forms, the convergence of the finite-dimensional distributions can be inferred from that of the one-dimensional distributions. Therefore, it suffices to verify that, for all $(\varphi, f) \in \mathcal{Y}_0 \times \mathcal{C}_c^+(\mathcal{X})$, the following convergence holds

$$\big(\langle \mathrm{Y}, \varphi\rangle, \nu_t(f)\big) \Rightarrow \big(\langle \mathrm{Y}, \varphi\rangle, \nu(f)\big) . \tag{2.3.8}$$

Since the random variable $\nu(f)$ is almost surely non-negative, it can be readily observed that the joint convergence in distribution (2.3.8) holds if the corresponding joint Fourier–Laplace transform converges, i.e. if (2.3.7) holds. □

Remark 2.3.5. In what follows, we will be interested in the case $\mathcal{X} = [n] \times \mathbb{R}^d$ and $\mathcal{Y} = \mathcal{H}^{-\kappa}_{\mathrm{loc}}(\mathbb{R}^d)$, for some $\kappa > 0$, in which case we can take $\mathcal{Y}_0 = \mathcal{C}_c^\infty(\mathbb{R}^d)$.

Stable convergence of random measures. We now recall some facts about stable convergence of random measures. This type of convergence interpolates to some extent between convergence in law and convergence in probability. We refer to the monographs [JS03, HL15] and references therein for more details on stable convergence in a more general setting.

We consider a collection $(\nu_t)_{t\geq 0}$ of $\mathcal{M}^+(\mathcal{X})$-valued random variables defined on a common probability space $(\Omega, \mathbb{P})$ and a $\mathcal{M}^+(\mathcal{X})$-valued random variables ν defined on a possibly larger probability space. We also fix a σ-algebra Σ over Ω.

Definition 2.3.6. We say that ν_t converges Σ-stably to $\nu \in \mathcal{M}^+(\mathcal{X})$ as $t \to \infty$, if $(Z, \nu_t) \Rightarrow (Z, \nu)$ for all Σ-measurable random variables Z.

Remark 2.3.7. If Σ is the trivial σ-algebra, then this coincides with convergence in law. Conversely, if Σ is the full σ-algebra of the probability space Ω and the limiting random variable is defined on $(\Omega, \mathbb{P})$, then this corresponds to convergence in probability.

Given a Polish space $\mathcal{Y}$ and a $\mathcal{Y}$-valued random variable Y, we have the following result that characterises $\sigma(\mathrm{Y})$-stable convergence.

Lemma 2.3.8. *Consider the same setting described above. Then ν_t converges $\sigma(\mathrm{Y})$-stably to ν if and only if $(\mathrm{Y}, \nu_t) \Rightarrow (\mathrm{Y}, \nu)$ as $t \to \infty$.*

Proof. See for instance [HL15, Exercise 3.11]. □

2.3.3 Some properties of $\star$-scale invariant fields

Let X be a $\star$-scale invariant field as defined in (2.1.2) and with seed covariance function $\mathfrak{K}$ satisfying assumptions (K1)–(K2). Recalling the covariance structure (2.1.4) of X, we point out that there exists a smooth function $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that, for all $x, y \in \mathbb{R}^d$,

$$\mathbb{E}\big[\mathrm{X}(x)\mathrm{X}(y)\big] = -\log|x-y| + g(x,y)\ ,$$

or in other words, X is a log-correlated Gaussian field. Thanks to [JSW19], it is known that a partial converse is true, i.e., given a log-correlated Gaussian field X with covariance of the form $-\log|x-y| + g(x,y)$ for some function $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfying certain (weak) regularity assumptions, then X can be decomposed as $\mathrm{X} = \mathrm{X}^\star + \mathrm{L}$, where $\mathrm{X}^\star$ is a $\star$-scale invariant field, L is a centred Gaussian field with Hölder regularity, and $\mathrm{X}^\star$ and L are jointly Gaussian.

We recall that $(\mathrm{X}_t)_{t\geq 0}$ denotes the martingale approximation of X as defined in (2.1.3). For every $t \geq 0$, we recall the definition (2.1.5) of the σ-field $\mathcal{F}_t$. The following properties are straightforward to check:

1. For $0 \leq s < t$, the random field $\mathrm{X}_{s,t}$ is independent from the σ-field $\mathcal{F}_s$.
2. For any fixed $x \in \mathbb{R}^d$, the process $(\mathrm{X}_t(x))_{t\geq 0}$ has the law of a standard Brownian motion.
3. For $0 \leq s < t$, the following scaling relation holds

$$\mathrm{X}_{s,t}(\cdot) \overset{\text{law}}{=} \mathrm{X}_{t-s}(e^s\cdot)\ . \tag{2.3.9}$$

We now introduce a field that will play an import role in what follows. We recall that $\overline{\mathfrak{K}}$ is the (unique) positive definite function such that the convolution of $\overline{\mathfrak{K}}$ with itself equals $\mathfrak{K}$.

Definition 2.3.9. For ξ' a space-time white noise on $\mathbb{R}^d \times \mathbb{R}^+$, we define the field Z_∞ on $\mathbb{R}^d$ by letting,

$$\mathrm{Z}_\infty(\cdot) \overset{\text{def}}{=} \int_0^\infty \int_{\mathbb{R}^d} \big(\overline{\mathfrak{K}}(e^{-r}(\cdot - y)) - \mathfrak{K}(e^{-r}\cdot)\overline{\mathfrak{K}}(e^{-r}y)\big)e^{-\frac{dr}{2}}\,\xi'(dy,dr)\ . \tag{2.3.10}$$

Furthermore, for $0 \leq s < t$, we let $\mathrm{Z}_{s,t}$ be the field on $\mathbb{R}^d$ given by

$$\mathrm{Z}_{s,t}(\cdot) \overset{\text{def}}{=} \int_s^t \int_{\mathbb{R}^d} \big(\overline{\mathfrak{K}}(e^{-r}(\cdot - y)) - \mathfrak{K}(e^{-r}\cdot)\overline{\mathfrak{K}}(e^{-r}y)\big)e^{-\frac{dr}{2}}\,\xi'(dy,dr)\ , \tag{2.3.11}$$

with the notational convention that $\mathrm{Z}_{0,t} = \mathrm{Z}_t$.

We observe that, for any $x, y \in \mathbb{R}^d$ and $s, t \geq 0$, it holds that

$$\mathbb{E}\big[\mathrm{Z}_\infty(x)\mathrm{Z}_\infty(y)\big] = \int_0^\infty \big(\mathfrak{K}(e^{-r}(x-y)) - \mathfrak{K}(e^{-r}x)\mathfrak{K}(e^{-r}y)\big)dr\ , \tag{2.3.12}$$

$$\mathbb{E}\big[\mathrm{Z}_s(x)\mathrm{Z}_t(y)\big] = \int_0^{s\wedge t} \big(\mathfrak{K}(e^{-r}(x-y)) - \mathfrak{K}(e^{-r}x)\mathfrak{K}(e^{-r}y)\big)dr\ . \tag{2.3.13}$$

It is straightforward to check that the field $\mathrm{Z}_t(\cdot)$ converges weakly in law with respect to the local uniform topology in $\mathcal{C}(\mathbb{R}^d)$ to $\mathrm{Z}_\infty(\cdot)$ as $t \to \infty$ (see e.g. [Mad15, Proposition 2.4]).

We also record here a decomposition result for $\star$-scale invariant fields originally stated in [DRSV14a] and which can be proved by standard computation of covariances.

Lemma 2.3.10 ([DRSV14a, Lemma 16]). *For $z \in \mathbb{R}^d$, the field $(\mathrm{X}_t(x))_{t\geq 0, x\in\mathbb{R}^d}$ admits the following decomposition*

$$\mathrm{X}_t(x) = \int_0^t \mathfrak{K}(e^r(x-z))d\mathrm{X}_r(z) + \mathrm{Z}_t^z(x)\ , \qquad \forall\, t\geq 0\ , \forall\, x\in\mathbb{R}^d\ ,$$

where $(\mathrm{Z}_t^z(x))_{t\geq 0, x\in\mathbb{R}^d}$ is a centred Gaussian field independent of $(\mathrm{X}_t(z))_{t\geq 0}$ and with the following covariance structure,

$$\mathbb{E}\big[\mathrm{Z}_s^z(x)\mathrm{Z}_t^z(y)\big] = \int_0^{s\wedge t} \big(\mathfrak{K}(e^r(x-y)) - \mathfrak{K}(e^r(x-z))\mathfrak{K}(e^r(y-z))\big)dr\ , \quad \forall\, s,t\geq 0,\ \forall\, x,y\in\mathbb{R}^d\ .$$

We emphasise that for all $t\geq 0$, it holds that $\mathrm{Z}_t^0(e^{-t}\cdot) \overset{\text{law}}{=} \mathrm{Z}_t(\cdot)$, where Z_t is the field introduced in Definition 2.3.9.

2.4 Local structure of extremal points

The main goal of this section is to prove Theorems 2.A and 2.B. We emphasise that the arguments for the proofs of these two theorems are inspired by and follow similar lines to the proofs of [BL18, Theorems 2.3 and 2.4].

Before proceeding with the proofs, we outline the main ideas behind them. As motivated in Section 2.2.1, our primary goal is to study the limit as $b\to\infty$ of the law of the field $\Upsilon_b = \Phi_b - \sqrt{2d}\mathfrak{a}_b$, introduced in (2.2.3), under the condition that its supremum on the ball $\mathbb{B}_b$ is bounded above by some $\lambda > 0$. As a first step, we note in Section 2.4.2.1 that the field Φ_b admits the following "concentric decomposition":

$$\Phi_b(x) \overset{\text{law}}{=} -\int_0^b \big(1-\mathfrak{K}(e^{-s}x)\big)dB_s + \sum_{j=0}^{b-1}\mathrm{Z}_j(x)\mathbb{1}_{\{x\in\mathbb{A}_j\}} + \sum_{j=0}^{b-1}\mathrm{Z}_{j,b}(x)\mathbb{1}_{\{x\in\mathbb{A}_j\}}\ ,$$

where $(B_s)_{s\geq 0}$ is a Brownian motion and, for $j\in[b-1]_0$, Z_j and $\mathrm{Z}_{j,b}$ are defined in Definition 2.3.9 and are independent of $(B_s)_{s\geq 0}$.

The key feature of the above decomposition across annuli is that we have good control over the tails of the fields $\mathrm{Z}_j - \mathfrak{m}_j$ and $\mathrm{Z}_{j,b}$ within each annulus $\mathbb{A}_j$ (see Section 2.4.2.2). In particular, this enables us to encode the growth of the field Υ_b into the Brownian motion that drives the stochastic integral appearing in the above display. More precisely, the condition that the supremum of Υ_b over $\mathbb{B}_b$ is bounded above by $\lambda > 0$ translates into requiring this Brownian motion stay above a polylogarithmic curve (see Lemma 2.4.9). With this correspondence in place, the proof of the existence and non-degeneracy of the "shape field" follows through a series of technical steps.

This section is structured as follows. In Section 2.4.1, we introduce the precise setup and state a slightly stronger version of Theorems 2.A and 2.B. In Section 2.4.2, we explain how to convert the statement about the supremum of the field being less than λ into a condition on the driving Brownian motion to stay above a polylogarithmic curve. In Section 2.4.3, we collect some technical lemmas that are needed for the proof of the main theorems, while their proofs are contained in Section 2.4.4.

2.4.1 Setup and statement of results

Recalling Definition 2.3.9, for $b \in \mathbb{N} \cup \{\infty\}$, we introduce the field Φ_b on $\mathbb{R}^d$ given by

$$\Phi_b(\cdot) \overset{\text{def}}{=} -\int_0^b \big(1 - \mathfrak{K}(e^{-s}\cdot)\big) dB_s + \mathrm{Z}_b(\cdot) \,, \tag{2.4.1}$$

where $(B_s)_{s\geq 0}$ is a standard Brownian motion independent of the space-time white noise ξ' used in the definition of the field Z_b introduced in Definition 2.3.9. We observe that the field Φ_b defined in (2.4.1) is equal in law to the field introduced in Definition 2.2.3 (which justifies using the same notation), and we also recall that Υ_b is given by

$$\Upsilon_b = \Phi_b - \sqrt{2d}\mathfrak{a}_b \,. \tag{2.4.2}$$

We will carry our analysis in a slightly more general setting than what we specified in the introduction, namely we allow the field Υ_b to be perturbed by a suitable "well-behaved" independent field. For $b \in \mathbb{N}$, we consider an independent random field $\mathfrak{g}_b$ on $\mathbb{R}^d$, which we fix for the reminder of this section and assume to satisfy the following properties:

(G1) One has $\mathfrak{g}_b(0) = 0$ almost surely.

(G2) For all $j \in [b]_0$, there exists a constant $c > 0$ such that for all $\eta \geq 0$,

$$\mathbb{P}\big(\mathbb{M}_{0,j}(\mathfrak{g}_b) \geq \eta\big) \lesssim e^{-ce^{2(b-j)}\eta^2} \,.$$

(G3) There exist constants $c > 0$ such that for all $\eta \geq 0$,

$$\mathbb{P}\left(\sup_{i,k\in[d]} \mathbb{M}_{0,b}(\partial^2_{i,k}\mathfrak{g}_b) \geq \eta\right) \lesssim e^{-c\eta^2} \,.$$

For $b \in \mathbb{N}$, we then define the field $\Upsilon_{b,\mathfrak{g}}$ on $\mathbb{R}^d$ by

$$\Upsilon_{b,\mathfrak{g}}(\cdot) = \Upsilon_b(\cdot) + \mathfrak{g}_b(\cdot) \,. \tag{2.4.3}$$

In what follows, given $b \in \mathbb{N}$ and $x, y \in \mathbb{R}$, we use the convention that under $\mathbb{P}_{x,y,b}$, the law of $(B_s)_{s\in[0,b]}$ is that of a Brownian bridge from x to y in time b. Furthermore, $\mathbb{E}_{x,y,b}$ denotes the expectation with respect to $\mathbb{P}_{x,y,b}$.

The main goal of this section is to prove the following two propositions, which are analogous to Theorems 2.A and 2.B, but with the Brownian motion replaced by the Brownian bridge.

Proposition 2.4.1. *For each $\lambda > 0$, there exists a continuous random field $\widetilde{\Upsilon}_\lambda$ on $\mathbb{R}^d$ such that, for any function $\mathbf{F} \in \mathcal{C}^b_{\mathrm{loc}}(\mathcal{C}(\mathbb{R}^d))$, and for all $u \in [b^{1/4}, b^{3/4}]$, one has*

$$\mathbb{E}\big[\mathbf{F}\big(\widetilde{\Upsilon}_\lambda\big)\big] = \lim_{b\to\infty} \mathbb{E}_{0,u,b}\big[\mathbf{F}\big(\Upsilon_{b,\mathfrak{g}}\big) \,\big|\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\big] = \lim_{b\to\infty} \mathbb{E}\big[\mathbf{F}\big(\Upsilon_\infty\big) \,\big|\, \mathbb{M}_{0,b}(\Upsilon_\infty) \leq \lambda\big] \,. \tag{2.4.4}$$

Proposition 2.4.2. *For $\lambda > 0$, let $c_{\star,\lambda} > 0$ be the constant defined as follows*

$$c_{\star,\lambda} = \lim_{k\to\infty} \mathbb{E}\big[B_k \mathbb{1}_{\{B_k\in[k^{1/6},k^{5/6}]\}} \mathbb{1}_{\{\mathbb{M}_{0,k}(\Upsilon_k)\leq\lambda\}}\big] \,,$$

then, for all $u \in [b^{1/4}, b^{3/4}]$*, it holds that*

$$\lim_{b\to\infty} \frac{b}{u} \mathbb{P}_{0,u,b}\big(\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \le \lambda\big) = 2c_{\star,\lambda} \ .$$

2.4.2 Reduction to a Brownian motion

The main goal of this subsection is to show that the condition that the supremum of $\Upsilon_{b,\mathfrak{g}}$ over the ball $\mathbb{B}_b$ is bounded above by λ can essentially be rewritten as a condition on $(B_s)_{s\in[0,b]}$ appearing in (2.4.1). The idea is to use a suitable decomposition across annuli of the field Φ_b, which will be introduced below. The key feature of this decomposition is that, for any $j \in [b-1]$, the supremum of the field in the annulus $\mathbb{B}_{j+1} \setminus \mathbb{B}_j$ is given by the position of the driving Brownian motion at time j (modulo a sign change), plus a remainder term whose tails we have good control over (see Figure 2.1). Hence, the condition that the supremum of $\Upsilon_{b,\mathfrak{g}}$ over the ball $\mathbb{B}_b$ is bounded above by λ can be recast in terms of the requirement that this driving Brownian motion stays above some polylogarithmic curve.

2.4.2.1 Decomposition across annuli

We begin by defining the following sets

$$\mathbb{A}_0 \overset{\text{def}}{=} \mathbb{B}_1 \qquad \text{and} \qquad \mathbb{A}_j \overset{\text{def}}{=} \mathbb{B}_{j+1} \setminus \mathbb{B}_j, \quad \forall\, j \in \mathbb{N} \ .$$

With this notation in place, for all $b \in \mathbb{N} \cup \{\infty\}$ and $x \in \mathbb{B}_b$, the field Φ_b in (2.4.1) can be conveniently rewritten as follows

$$\Phi_b(x) = -\int_0^b \big(1 - \Re(e^{-s}x)\big) dB_s + \sum_{j=0}^{b-1} \mathrm{Z}_j(x) \mathbb{1}_{\{x\in\mathbb{A}_j\}} + \sum_{j=0}^{b-1} \mathrm{Z}_{j,b}(x) \mathbb{1}_{\{x\in\mathbb{A}_j\}} \ , \tag{2.4.5}$$

where the fields Z_j and $\mathrm{Z}_{j,b}$ are as in Definition 2.3.9. For every $b \in \mathbb{N} \cup \{\infty\}$ and $j \in [b-1]_0$, we aim to control the tails of the suprema of the fields Z_j and $\mathrm{Z}_{j,b}$ over $\mathbb{A}_j$. This will be the content of the next two lemmas.

2.4.2.2 Controlling the tails

Recall the definition (2.2.1) of the recentering constants $\mathfrak{m}_j$.

Lemma 2.4.3. *There exists a constant* $c > 0$ *such that for all* $j \in \mathbb{N}$*, it holds that*

$$\mathbb{P}\left(\left|\sup_{x\in\mathbb{A}_j} \mathrm{Z}_j(x) - \mathfrak{m}_j\right| \ge \eta\right) \lesssim e^{-c\eta} \ , \qquad \forall\, \eta \ge 0 \ .$$

Proof. For $j \in \mathbb{N}$ and $\eta \ge 0$, the probability in the lemma statement is equivalent to

$$\mathbb{P}\left(\left|\sup_{x\in\mathbb{B}_1\setminus\mathbb{B}_0} \mathrm{Z}_j(e^j x) - \mathfrak{m}_j\right| \ge \eta\right) . \tag{2.4.6}$$

On the other hand, since by (K2) the seed covariance function $\mathfrak{K}$ is supported in $B(0,1)$, we have that for all $x, y \in \mathbb{B}_1 \setminus \mathbb{B}_0$, it holds that

$$\mathbb{E}\big[\mathrm{Z}_j(e^j x)\mathrm{Z}_j(e^j y)\big] = \int_0^j \mathfrak{K}\big(e^s(x-y)\big)ds \,.$$

In other words, the field $\mathrm{Z}_j(e^j\cdot)$ restricted to the annulus $\mathbb{B}_1 \setminus \mathbb{B}_0$ has the same law as the martingale approximation at level j of a $\star$-scale invariant field with seed covariance kernel $\mathfrak{K}$. Hence, if the annulus $\mathbb{B}_1 \setminus \mathbb{B}_0$ in (2.4.6) is replaced by the d-dimensional unit box $[0,1]^d$, then this tightness result follows from [Aco14, Theorem 1.1]. To deduce the tightness of supremum of $(\mathrm{Z}_j(e^j\cdot))_{j\in\mathbb{N}}$ over the annulus $\mathbb{B}_1 \setminus \mathbb{B}_0$ from the one over the box $[0,1]^d$, one can simply note that $\mathbb{B}_1 \setminus \mathbb{B}_0$ contains and is contained in a box of order one. □

Regarding the field $\mathrm{Z}_{j,b}$, we have the following bound on its supremum over annuli of radii smaller than j.

Lemma 2.4.4. *There exists a constant $c > 0$ such that for $b \in \mathbb{N}\cup\{\infty\}$, $j \in [b-1]_0$, and $l \in [j]_0$, it holds that*

$$\mathbb{P}\left(\sup_{x\in\mathbb{A}_l} |\mathrm{Z}_{j,b}(x)| \geq \mathfrak{n}\right) \lesssim e^{-ce^{2(j-l)}\mathfrak{n}^2} \,, \qquad \forall\, \mathfrak{n} \geq 0 \,.$$

Proof. The result follows by a standard application of Fernique's majorizing criterion (Lemma A.0.4) and Borell-TIS inequality (Lemma A.0.3). We only detail the case $b = \infty$, as the case $b \in \mathbb{N}$ is completely analogous. Fix $j \in \mathbb{N}$ and $l \in [j]$ and note that the probability in the statement equals

$$\mathbb{P}\left(\sup_{x\in\mathbb{B}_1\setminus\mathbb{B}_0} |\mathrm{Z}_{j,\infty}(e^l x)| \geq \mathfrak{n}\right) .$$

A simple computation based on (K1) – (K2) yields that, for all $x, y \in \mathbb{B}_1 \setminus \mathbb{B}_0$, it holds that

$$\mathbb{E}\big[|\mathrm{Z}_{j,\infty}(e^l x) - \mathrm{Z}_{j,\infty}(e^l y)|^2\big] \leq 2\int_0^\infty \big(1 - \mathfrak{K}(e^{-(s+j-l)}(x-y))\big)ds \lesssim |x-y|^2 \,. \tag{2.4.7}$$

Therefore, an immediate application of Fernique's majorizing criterion (Lemma A.0.4) shows that

$$\mathbb{E}\left[\sup_{x\in\mathbb{B}_1\setminus\mathbb{B}_0} |\mathrm{Z}_{j,\infty}(e^l x)|\right] \lesssim 1 \,,$$

for some universal implicit constant. The conclusion then follows by Borell-TIS inequality (Lemma A.0.3) and thanks to the fact that

$$\sup_{x\in\mathbb{B}_1\setminus B_0} \mathbb{E}\big[\mathrm{Z}_{j,\infty}(e^l x)^2\big] = \int_0^\infty \big(1 - \mathfrak{K}(e^{-(s+j-l)}x)^2\big)ds \lesssim e^{-2(j-l)} \,,$$

where, once again, we used (K1) – (K2), and the implicit constant is independent of the quantities of interest. Finally, we remark that the cases $j = 0$ and $j \in \mathbb{N}$ with $l = 0$ can be treated similarly. □

Lemma 2.4.5. *There exists a constant $c > 0$ such that for all $b \in \mathbb{N}\cup\{\infty\}$ and $j \in [b-1]$, it holds that*

$$\sup_{x\in\mathbb{A}_j} |\sqrt{2d}\mathfrak{a}_b(x) - \mathfrak{m}_j| \leq c + \frac{3}{2\sqrt{2d}}\log(j) \,.$$

Proof. Fix $b \in \mathbb{N}$ and $j \in [b-1]$. Then, thanks to (K2), we have that, uniformly over all $x \in \mathbb{A}_j$, there exists a constant $c > 0$ such that

$$\left|\sqrt{2d}\mathfrak{a}_b(x) - \sqrt{2d}j\right| = \left|\int_j^b \big(1 - \mathfrak{K}(e^{-s}x)\big)ds\right| \leq c\,.$$

The conclusion follows by recalling the exact expression for $\mathfrak{m}_j$ given in (2.2.1). The case $b = \infty$ and $j \in \mathbb{N}$ can be treated analogously. □

2.4.2.3 Control variables

In what follows, given $j \in \mathbb{N}_0$ and a function $f : \mathbb{R}_0^+ \to \mathbb{R}$, we introduce the following notation,

$$\mathrm{Osc}_j(f) \stackrel{\text{def}}{=} \sup_{s\in[j,j+1]} f(s) - \inf_{s\in[j,j+1]} f(s)\,.$$

For $j, k \in \mathbb{N}_0$, we define

$$\Theta_k(j) \stackrel{\text{def}}{=} \big[\log\big(1 + (k \vee j)\big)\big]^2\,. \tag{2.4.8}$$

For $b \in \mathbb{N} \cup \{\infty\}$, we now introduce the control variable K_b which will play an instrumental role in our analysis.

Definition 2.4.6. For $b \in \mathbb{N} \cup \{\infty\}$, we let K_b be the smallest $k \in [b-1]$[1] such that:

(1) For each $j \in [b-1]_0$[1], it holds that $\mathrm{Osc}_j(B) \leq \Theta_k(j)$.

(2) For each $j \in [b-1]$, it holds that $|\sup_{x\in\mathbb{A}_j} \mathrm{Z}_j(x) - \mathfrak{m}_j| \leq \Theta_k(j)$.

(3) For each $j \in [b-1]_0$ and $l \in [j]_0$, it holds that $\sup_{x\in\mathbb{A}_l} |\mathrm{Z}_{j,b}(x)| \leq e^{-(j-l)/2}\Theta_k(j)$.

(4) If $b \neq \infty$, for each $j \in [b-1]_0$, it holds that $\sup_{x\in\mathbb{A}_j} |\mathfrak{g}_b(x)| \leq e^{-(b-j)/2}\Theta_k(j)$.

If no such K_b exists, then we set $\mathrm{K}_b = b$.

Remark 2.4.7. For each $b \in \mathbb{N}$, $j \in [b-1]_0$, and $l \in [j]_0$, recalling (2.4.1), we have, for all $x \in \mathbb{B}_l$,

$$\Upsilon_{b,\mathfrak{g}}(x) - \Upsilon_{j,\mathfrak{g}}(x) = -\int_j^b \big(1 - \mathfrak{K}(e^{-s}x)\big)dB_s + \mathrm{Z}_{j,b}(x)\,.$$

Now, on the event $\{\mathrm{K}_b \leq j\}$, one has

$$\sup_{x\in\mathbb{B}_l} |\mathrm{Z}_{j,b}(x)| \lesssim e^{-(j-l)/2}\big(\log j\big)^2\,,$$

as well as

$$\sup_{x\in\mathbb{B}_l}\left|\int_j^b 1 - \mathfrak{K}(e^{-s}x)dB_s\right| \lesssim e^l \sum_{i=j}^{b-1} e^{-i}\,\mathrm{Osc}_i(B) \leq e^{-(j-l)}\sum_{i=0}^{\infty} e^{-i}\Theta_j(i+j) \lesssim e^{-(j-l)}\big(\log j\big)^2\,.$$

Therefore, combining the previous two bounds, we obtain that for all $l \in [j]_0$, on the event $\{\mathrm{K}_b \leq j\}$, it holds that

$$\sup_{x\in\mathbb{B}_l} |\Upsilon_{b,\mathfrak{g}}(x) - \Upsilon_{j,\mathfrak{g}}(x)| \lesssim e^{-(j-l)/2}(\log j)^2\,. \tag{2.4.9}$$

[1] With a slight abuse of notation, if $b = \infty$, then $[b-1] = \mathbb{N}$ and $[b-1]_0 = \mathbb{N}_0$.

Thanks to Lemmas 2.4.3 and 2.4.4, we have the following result concerning the tail behaviour of the control variables.

Lemma 2.4.8. *There exists a constant $c > 0$ and $k_0 \in \mathbb{N}$ such that for $b \in \mathbb{N}$ with $b > k_0$, and $u \in [b^{1/4}, b^{3/4}]$,*

$$\mathbb{P}_{0,u,b}\big(\mathrm{K}_b = k\big) \leq e^{-c(\log k)^2} , \qquad \forall\, k \in \{k_0, \dots, b\} . \tag{2.4.10}$$

Similarly, there exists a constant $\widetilde{c} > 0$ and $\widetilde{k}_0 \in \mathbb{N}$ such that

$$\mathbb{P}\big(\mathrm{K}_\infty = k\big) \leq e^{-\widetilde{c}(\log k)^2} , \qquad \forall\, k \geq \widetilde{k}_0 . \tag{2.4.11}$$

Proof. We start with (2.4.10). Fix $b \in \mathbb{N}$, $u \in [b^{1/4}, b^{3/4}]$, and let $k \in [b]$. By Definition 2.4.6, the event $\{\mathrm{K}_b = k\}$ is contained in the union of the following events,

$$\bigcup_{j=0}^{b-1} \big\{ \mathrm{Osc}_j(B) > \Theta_{k-1}(j) \big\} , \qquad \bigcup_{j=0}^{b-1} \Big\{ \sup_{x \in \mathbb{A}_j} |\mathfrak{g}_b(x)| > e^{-(b-j)/2} \Theta_{k-1}(j) \Big\} ,$$

$$\bigcup_{j=1}^{b-1} \Big\{ \Big| \sup_{x \in \mathbb{A}_j} \mathrm{Z}_j(x) - \mathfrak{m}_j \Big| > \Theta_{k-1}(j) \Big\} , \qquad \bigcup_{j=1}^{b-1} \bigcup_{l=1}^{j} \Big\{ \sup_{x \in \mathbb{A}_l} |\mathrm{Z}_{j,b}(x)| > e^{-(j-l)/2} \Theta_{k-1}(j) \Big\} .$$

The conclusion then follows since there exists constants c_1, $c_2 > 0$ such that the probabilities of the events appearing in the unions of the above display are bounded either by $c_1 \exp(-c_2 \Theta_{k-1}(j)^2)$ (for the ones on the first line) or by $c_1 \exp(-c_2 \Theta_{k-1}(j))$ (for the ones on the second line). Summing over j then yields the desired bound.

Indeed, for the events in the union on the top-left, this follows since for all $u \in [b^{1/4}, b^{3/4}]$, the oscillation norms $\mathrm{Osc}_j(B)$ have Gaussian tails uniformly over the probability laws $\mathbb{P}_{0,u,b}$, for all $j \in [b-1]_0$. For the event in the union on the top-right this follows from (G2). Finally, regarding the events in the union on the bottom-left and bottom-right, this follows by Lemma 2.4.3 and Lemma 2.4.4, respectively. To conclude, we note that the proof of (2.4.11) proceeds in the same exact way. □

We are now ready to state and prove the following key lemma. For a diagrammatic representation related to this lemma, we refer to Figure 2.1.

Lemma 2.4.9 (Approximation by a Brownian motion)**.** *There exists a constant $\mathrm{C} > 0$ such that for all $b \in \mathbb{N} \cap \{\infty\}$ and all $j \in [b-1]$, on the event $\{\mathrm{K}_b < b\}$,*

$$\Big| \sup_{x \in \mathbb{A}_j} \Upsilon_{b,\mathfrak{g}}(x) + B_j \Big| \leq \mathrm{R}_{\mathrm{K}_b}(j) , \tag{2.4.12}$$

where $\mathrm{R}_k(j) \overset{\text{def}}{=} \mathrm{C}\big(1 + \Theta_k(j)\big)$.

Proof. Let $b \in \mathbb{N}$, $j \in [b-1]$, and $x \in \mathbb{A}_j$. Now, recalling (2.4.5), we can write

$$\Upsilon_{b,\mathfrak{g}}(x) = -B_j - \int_j^b \big(1 - \mathfrak{K}(e^{-s}x)\big) dB_s + \mathrm{Z}_j(x) + \mathrm{Z}_{j,b}(x) - \sqrt{2d}\mathfrak{a}_b(x) + \mathfrak{g}_b(x) ,$$

where we used the fact that thanks to (K2), the function $\mathfrak{K}$ is supported in $B(0,1)$. In particular, using the triangle inequality, this implies that

$$\Big|\sup_{x\in\mathbb{A}_j}\Upsilon_{b,\mathfrak{g}}(x)+B_j\Big| \le \sup_{x\in\mathbb{A}_j}\Big|\int_j^b\big(1-\mathfrak{K}(e^{-s}x)\big)dB_s\Big| + \Big|\sup_{x\in\mathbb{A}_j}\mathrm{Z}_j(x)-\mathfrak{m}_j\Big| + \sup_{x\in\mathbb{A}_j}|\mathrm{Z}_{j,b}(x)| + \sup_{x\in\mathbb{A}_j}|\sqrt{2d}\mathfrak{a}_b(x)-\mathfrak{m}_j| + \sup_{x\in\mathbb{A}_j}|\mathfrak{g}_b(x)|\,.$$

Now, on the event $\{\mathrm{K}_b<b\}$, by Definition 2.4.6 of the control variable K_b and by Lemma 2.4.5, there exists a constant $\mathrm{C}>0$ such that the last four terms on the right-hand side of the above display are bounded from above by a quantity of the form $\mathrm{C}\big(1+\Theta_{\mathrm{K}_b}(j)\big)$. Therefore, it remains to check that a similar bound also holds for the first term. Again, by Definition 2.4.6 of K_b, this follows by a simple computation. Indeed, we have that

$$\sup_{x\in\mathbb{A}_j}\Big|\int_j^b\big(1-\mathfrak{K}(e^{-s}x)\big)dB_s\Big| \lesssim e^{j+1}\sum_{m=j}^{b-1}e^{-m}\,\mathrm{Osc}_m(B) \le e^{j+1}\sum_{m=j}^{b-1}e^{-m}\Theta_{\mathrm{K}_b}(m)\,,$$

where the implicit constant is independent of everything else. The quantity on the right-hand side of the above display can be clearly bounded by a quantity of the form $\mathrm{C}(1+\Theta_{\mathrm{K}_b}(j))$, for some constant $\mathrm{C}>0$. Hence, the estimate in (2.4.12) follows. Finally, we note that if $b=\infty$, then the proof is completely analogous. □

We record here some useful inclusions that are immediate consequences of Lemma 2.4.9 and will be used several times in the remainder of this section. In particular, recalling the notation introduced in (2.3.3) and (2.3.4), and using the same notation as in the previous lemma, for each $\lambda>0$, thanks to (2.4.12), one can see that for all $b\in\mathbb{N}$ and $k\in[b-1]$, the following inclusions hold

$$\big\{\mathrm{K}_b<b\big\}\cap\big\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}})\le\lambda\big\}\subseteq\bigcap_{j=1}^{b-1}\big\{B_j\ge-\lambda-\mathrm{R}_{\mathrm{K}_b}(j)\big\}\,, \tag{2.4.13}$$

$$\big\{\mathrm{K}_b<b\big\}\cap\bigcap_{j=k}^{b-1}\big\{B_j\ge\lambda+\mathrm{R}_{\mathrm{K}_b}(j)\big\}\subseteq\big\{\mathbb{M}_{0,b,k}(\Upsilon_{b,\mathfrak{g}})\le\lambda\big\}\,. \tag{2.4.14}$$

Similar inclusions hold also for the field Υ_∞. Indeed, thanks to (2.4.12), for all $b\in\mathbb{N}$ and $k\in[b-1]$, we have that

$$\big\{\mathrm{K}_\infty<\infty\big\}\cap\big\{\mathbb{M}_{0,b}(\Upsilon_\infty)\le\lambda\big\}\subseteq\bigcap_{j=1}^{b-1}\big\{B_j\ge-\lambda-\widetilde{\mathrm{R}}_{\widetilde{\mathrm{K}}_b}(j)\big\}\,, \tag{2.4.15}$$

$$\big\{\mathrm{K}_\infty<\infty\big\}\cap\bigcap_{j=k}^{b-1}\big\{B_j\ge\lambda+\widetilde{\mathrm{R}}_{\widetilde{\mathrm{K}}_b}(j)\big\}\subseteq\big\{\mathbb{M}_{0,b,k}\Upsilon_\infty)\le\lambda\big\}\,. \tag{2.4.16}$$

2.4.3 Some technical lemmas

In this subsection we collect some technical results that are need for the proofs of Propositions 2.4.1 and 2.4.2. Before proceeding, we emphasise that all the following lemmas have an analogous counterpart in [BL18, Section 4]. We begin with the following lemma, which is a slightly augmented version of [BL18, Lemma 4.20], adapted to our setting. For its statement, given $k\in\mathbb{N}_0$, we use the notation

$$\mathcal{G}_k\overset{\text{def}}{=}\sigma\big((B_s)_{s\le k},\,\mathrm{Z}_\infty\big)\,.$$

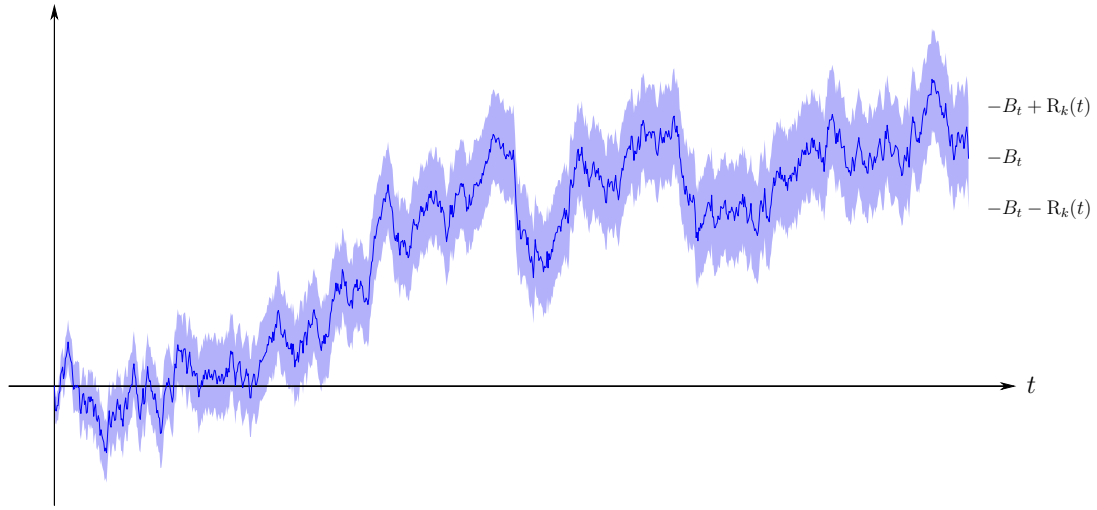


FIGURE 2.1: The blue curve represents a standard Brownian motion $-B$ run up to time b. The light blue region around the Brownian motion is the area enclosed between the curves $[0,b] \ni t \mapsto -B_t - \mathrm{R}_k(t)$ and $[0,b] \ni t \mapsto -B_t + \mathrm{R}_k(t)$ for some $0 \ll k \ll b$. Roughly speaking, Lemma 2.4.9 states that, with high probability, for any $j \in [b-1]$, the supremum of the field $\Upsilon_{b,\mathfrak{g}}(x)$ in the annulus $\mathbb{A}_j$ lies on the vertical segment at $t = j$ within the light blue region.

Lemma 2.4.10. *There exists a constant $c > 0$ such that for $\lambda > 0$, for all $b \in \mathbb{N}$ sufficiently large, $u \in [b^{1/4}, b^{3/4}]$, $k \in [b-1]$, and any event $\mathrm{A}_k \in \mathcal{G}_k$,*

$$\mathbb{P}_{0,u,b}\left(\mathrm{K}_b = k,\ \bigcap_{j=k}^{b-1}\{B_j \geq -\lambda - \mathrm{R}_k(j)\},\ \mathrm{A}_k\right) \lesssim \frac{u}{b} e^{-c(\log k)^2} \sqrt{\mathbb{P}_{0,u,b}(\mathrm{A}_k)}\,. \tag{2.4.17}$$

Similarly, there exists a constant $\widetilde{c} > 0$ such that for $\lambda > 0$, for all $b \in \mathbb{N}$ sufficiently large, $k \in [b-1]$, and any event $\mathrm{A}_k \in \mathcal{G}_k$,

$$\mathbb{P}\left(\mathrm{K}_\infty = k,\ \bigcap_{j=k}^{b-1}\{B_j \geq -\lambda - \widetilde{\mathrm{R}}_k(j)\},\ \mathrm{A}_k\right) \lesssim \frac{1}{\sqrt{b}} e^{-\widetilde{c}(\log k)^2} \sqrt{\mathbb{P}(\mathrm{A}_k)}\,. \tag{2.4.18}$$

Proof. For simplicity, and without loss of generality, we set $\lambda = 1$. The proof of this lemma follows a similar approach to the proof of [BL18, Lemma 4.20]. We start with the proof of (2.4.17). Given $b \in \mathbb{N}$ and $k \in [b-1]$, we let E_k be the event that the conditions in Definition 2.4.6 hold for all $j \in [k]$ (or $j \in [k]_0$) with $\Theta_k(j)$, but at least one of these conditions is not satisfied if $\Theta_k(j)$ is replaced by $\Theta_{k-1}(j)$. We note that $\mathrm{E}_k \in \mathcal{G}_k$ and that $\{\mathrm{K}_b = k\} \subset \mathrm{E}_k$ by definition of K_b. Hence, the probability on the left-hand side of (2.4.17) can be bounded from above by

$$\mathbb{E}_{0,u,b}\left[\mathbb{1}_{\mathrm{E}_k \cap \mathrm{A}_k} \mathbb{P}_{0,u,b}\left(\bigcap_{j=k}^{b-1}\{B_j \geq -1 - \mathrm{R}_k(j)\} \,\middle|\, \mathcal{G}_k\right)\right].$$

Now, we consider the function $\zeta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ given by

$$\zeta(s) \stackrel{\text{def}}{=} \mathrm{C}\big[1 + \log(\mathrm{C} + s)\big]^2\,,$$

where C is the constant appearing in the definition of R_k which, without any loss of generality, we can assume

to be large enough so that ζ is increasing and concave on $\mathbb{R}_0^+$. Then on the event $\{B_k = z\}$ for some $z \in \mathbb{R}$, again by possibly enlarging the constant C in the definition of ζ, thanks to the Markov property of the Brownian bridge, it holds that

$$\mathbb{P}_{0,u,b}\left(\bigcap_{j=k}^{b-1}\{B_j \geq -1 - \mathrm{R}_k(j)\}\,\middle|\,\mathcal{G}_k\right) \leq \mathbb{P}_{z,u,b-k}\left(\bigcap_{j=k}^{b-1}\{B_j \geq -\zeta(j)\}\right).$$

Now, thanks to Lemma 2.A.10, we have that

$$\mathbb{P}_{z,u,b-k}\left(\bigcap_{j=k}^{b-1}\{B_j \geq -\zeta(j)\}\right) \lesssim \mathbb{P}_{z,u,b-k}\left(\inf_{s\in[0,b-k]}\big(B_s + 2\zeta(k+s)\big) \geq 0\right), \tag{2.4.19}$$

where the implicit constant is independent of everything else. Since by definition one has $\mathrm{Osc}_j(B) \leq \Theta_k(j)$ on the event E_k, for all $j \in [k]_0$, we can assume that

$$z \in \left[-\zeta(k), (k+1)\big(\log(k+1)\big)^2\right].$$

We let $a_k \overset{\text{def}}{=} \zeta(k) \vee ((k+1)(\log(k+1))^2)$. Since the probability on the right-hand side of (2.4.19) is increasing in $z \in \mathbb{R}$, we can estimate this probability for $z \in [a_k, 3a_k]$. It follows from Proposition 2.A.5 that

$$\sup_{z\in[a_k,3a_k]} \mathbb{P}_{z,u,b-k}\left(\inf_{s\in[0,b-k]}\big(B_s + 2\zeta(k+s)\big) \geq 0\right) \lesssim a_k^2\frac{u}{b-k}. \tag{2.4.20}$$

We note that the presence of a_k^2 instead of a_k in the above expression – as one might expect from Proposition 2.A.5 – is due to the first summand in the error term (2.A.9). Furthermore, we observe that a direct application of Lemma 2.A.7 shows that the remaining summands in the error term (2.A.9) can be bounded uniformly over all $k \in [b-1]$ and $z \in [a_k, 3a_k]$.

Therefore, combining the previous considerations, so far we have proved that on the event E_k, it holds that

$$\mathbb{P}_{0,u,b}\left(\bigcap_{j=k}^{b-1}\{B_j \geq -1 - \mathrm{R}_k(j)\}\,\middle|\,\mathcal{G}_k\right) \lesssim a_k^2\frac{u}{b-k}.$$

On the other hand, thanks to Cauchy–Schwartz's inequality and arguing as in the proof of Lemma 2.4.8, we have that $\mathbb{P}(\mathrm{E}_k \cap \mathrm{A}_k) \leq e^{-c_1(\log k)^2}\sqrt{\mathbb{P}_{0,u,b}(\mathrm{A}_k)}$, for some constant $c_1 > 0$. Hence, by absorbing the factors a_k^2 and $b/(b-k)$ inside the exponential, we obtain that there exists a constant $c_2 > 0$ such that

$$\mathbb{P}_{0,u,b}\left(\mathrm{K}_b = k,\ \bigcap_{j=k}^{b-1}\{B_j \geq -1 - \mathrm{R}_k(j)\},\ \mathrm{A}_k\right) \lesssim \frac{u}{b}e^{-c_2(\log k)^2}\sqrt{\mathbb{P}_{0,u,b}(\mathrm{A}_k)},$$

from which the conclusion follows. Finally, the proof of (2.4.18) can be done similarly and hence is omitted. □

In what follows, we also require the following version of Lemma 2.4.10, where the endpoint of the Brownian bridge is constrained to be less than $b^{1/4}$.

Lemma 2.4.11. *Let $A > 0$ be fixed. There exists a constant $c > 0$ such that for $\lambda > 0$, for all $b \in \mathbb{N}$ sufficiently large, $u \in [-A, b^{1/4}]$, $k \in [b-1]$, and any event $\mathrm{A}_k \in \mathcal{G}_k$,*

$$\mathbb{P}_{0,u,b}\left(\mathrm{K}_b = k,\ \bigcap_{j=k}^{b-1}\{B_j \geq -\lambda - \mathrm{R}_k(j)\},\ \mathrm{A}_k\right) \lesssim \frac{1}{\sqrt{b}} e^{-c(\log k)^2}\sqrt{\mathbb{P}_{0,u,b}(\mathrm{A}_k)}\,.$$

Proof. The proof is almost identical to that of Lemma 2.4.8. Therefore, we only highlight the necessary changes. First, for a fixed $A > 0$, we note that (2.4.10) in Lemma 2.4.8 also holds uniformly for all $u \in [-A, b^{1/4}]$. Then, the only thing that needs to be changed is the bound (2.4.20). Indeed, Proposition 2.A.5 can only be applied for $u \in [b^{\iota}, b^{3/4}]$ for some $\iota \in (0, 1/8)$. However, we can easily overcome this issue by using monotonicity and replacing the endpoint u in the probability on the left-hand side of (2.4.20) with $u + b^{\iota}$ for some $\iota \in (0, 1/8)$. This allows us to apply Proposition 2.A.5, from which we deduce that the probability on the left-hand side of (2.4.20) is bounded above by a multiple of $a_k^2(u + b^{\iota})/(b-k)$. The conclusion then follows by proceeding with the remaining part of the argument in the proof of Lemma 2.4.8 and recalling that $u < b^{1/4}$. □

The following lemma is analogous to [BL18, Lemma 4.21] in our setting.

Lemma 2.4.12. *There exist constants $c_1,\ c_2 > 0$ such that for $\lambda > 0$, for $b \in \mathbb{N}$ large enough and $u \in [b^{1/4}, b^{3/4}]$,*

$$c_1 \frac{u}{b} \leq \mathbb{P}_{0,u,b}\big(\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\big) \leq c_2 \frac{u}{b}\,. \tag{2.4.21}$$

Similarly, there exist constants $\widetilde{c}_1, \widetilde{c}_2 > 0$ such that for $\lambda > 0$, for $b \in \mathbb{N}$ large enough,

$$\widetilde{c}_1 \frac{1}{\sqrt{b}} \leq \mathbb{P}\big(\mathbb{M}_{0,b}(\Upsilon_{\infty}) \leq \lambda\big) \leq \widetilde{c}_2 \frac{1}{\sqrt{b}}\,. \tag{2.4.22}$$

Proof. We start with the upper bound in (2.4.21). Thanks to (2.4.13), it holds that

$$\big\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\big\} \subseteq \big\{\mathrm{K}_b = b\big\} \cup \left\{\mathrm{K}_b < b,\ \bigcap_{j=1}^{b-1}\{B_j \geq -\lambda - \mathrm{R}_{\mathrm{K}_b}(j)\}\right\}.$$

Thanks to Lemmas 2.4.8 and 2.4.10, the probability of the event on the right-hand side of the above display is less than than a multiple of

$$\frac{u}{b}\sum_{k=1}^{b-1} e^{-c_1(\log k)^2} + e^{-c_2(\log b)^2}\,,$$

for some constants $c_1,\ c_2 > 0$. Therefore, the desired bound follows. We observe that the upper bound in (2.4.22) can be deduced in a similar way.

We now focus on the lower bounds. In particular, we limit ourself to study the lower bound in (2.4.21), since the one in (2.4.22) can be obtained similarly. For $k,\ b \in \mathbb{N}$ such that $k < b$, let E_k be the event that all conditions in the definition of the control variable K_b are satisfied with $\Theta_k(\cdot)$, except that we do not impose any requirements on the oscillation bounds for the Brownian motion for time indices up to and including time k. In particular, the event E_k is conditionally independent from $\sigma((B_s)_{s\leq k})$ given $\sigma(B_k)$. We have the following lower bound

$$\mathbb{P}_{0,u,b}\big(\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\big) \geq \mathbb{P}_{0,u,b}\left(\mathbb{M}_{0,k}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda,\ \mathrm{E}_k,\ \bigcap_{j=k}^{b-1}\{B_j \geq \lambda + \mathrm{R}_k(j)\}\right). \tag{2.4.23}$$

where we used the fact that, thanks to Lemma 2.4.9, on the event E_k[2], it holds that

$$\bigcap_{j=k}^{b-1}\{B_j \geq \lambda + \mathrm{R}_k(j)\} \subseteq \{\mathbb{M}_{0,b,k}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\} .$$

We also observe that thanks to (2.4.9), on the event E_k, there exists a constant $c > 0$ such that

$$\{\mathbb{M}_{0,k}(\Upsilon_{k,\mathfrak{g}}) \leq \lambda - c(\log k)^2\} \subseteq \{\mathbb{M}_{0,k}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\} .$$

Therefore, the right-hand side of (2.4.23) can be lower bounded by the following probability

$$\mathbb{P}_{0,u,b}\left(\mathbb{M}_{0,k}(\Upsilon_{k,\mathfrak{g}}) \leq \lambda - c(\log k)^2,\ \mathrm{E}_k,\ \bigcap_{j=k}^{b-1}\{B_j \geq \lambda + \mathrm{R}_k(j)\}\right) .$$

We note that the first event inside the above probability is conditionally independent from the second and third events given $\sigma(B_k)$. In particular, the conditional probability of the first event given $\{B_k = z\}$, for some $z \geq \lambda + \mathrm{R}_k(k)$, increases as z increases. Therefore, it can bounded from below by a constant $c_1(k) > 0$.

Since by definition $\{\mathrm{K}_b \leq k\} \subseteq \mathrm{E}_k$, to conclude it suffices to estimate the following probability

$$\mathbb{P}_{0,u,b}\left(\mathrm{K}_b \leq k,\ \bigcap_{j=k}^{b-1}\{B_j \geq \lambda + \mathrm{R}_k(j)\}\right) . \tag{2.4.24}$$

To this end, by using Lemma 2.4.10, we get that there exist constants $c_2, c_3 > 0$ such that

$$\mathbb{P}_{0,u,b}\left(\mathrm{K}_b > k,\ \bigcap_{j=k}^{b-1}\{B_j \geq \lambda + \mathrm{R}_k(j)\}\right) \leq c_2 e^{-c_3(\log k)^2}\frac{u}{b} ,$$

and so, the probability in (2.4.24) can be lower bounded by the following sum

$$\mathbb{P}_{0,u,b}\left(\bigcap_{j=k}^{b-1}\{B_j \geq \lambda + \mathrm{R}_k(j)\}\right) - c_2 e^{-c_3(\log k)^2}\frac{u}{b} .$$

Furthermore, by using the lower bound in (2.A.2) and Lemma 2.A.11, we get that the above probability is lower bounded by

$$\begin{aligned}
&\mathbb{P}_{0,u,b}\left(\bigcap_{j=1}^{b-1}\{B_j \geq -\lambda - \mathrm{R}_k(j)\},\ \bigcap_{j=k}^{b-1}\{B_j \geq \lambda + \mathrm{R}_k(j)\}\right)\\
&\geq \mathbb{P}_{0,u,b}\left(\bigcap_{j=1}^{b-1}\{B_j \geq 0\}\right) - \mathbb{P}_{0,u,b}\left(\bigcap_{j=1}^{b-1}\{B_j \geq -\lambda - \mathrm{R}_k(j)\},\ \bigcup_{j=k}^{b-1}\{B_j \leq \lambda + \mathrm{R}_k(j)\}\right)\\
&\geq \mathbb{P}_{0,u,b}\left(\inf_{s\in[1,b]} B_s \geq 0\right) - \mathbb{P}_{0,u,b}\left(\bigcap_{j=1}^{b-1}\{B_j \geq -\lambda - \mathrm{R}_k(j)\},\ \bigcup_{j=k}^{b-1}\{B_j < \lambda + \mathrm{R}_k(j)\}\right)\\
&\geq c_4\frac{u}{b}\left(1 - k^{-\frac{1}{16}}\right) ,
\end{aligned}$$

[2]To be precise, the conclusion of Lemma 2.4.9 holds on the event $\{\mathrm{K}_b \leq k\}$. However, as one can easily check, on the annulus $\mathbb{B}_b \setminus \mathbb{B}_k$ only the conditions on the oscillations of the Brownian motion after time k are relevant.

for some constant $c_4 > 0$. Hence, putting everything together, we showed that there exist constants $c_1(k)$, c_2, c_3, $c_4 > 0$ such that

$$\mathbb{P}_{0,u,b}\big(\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \le \lambda\big) \ge c_1(k)\frac{u}{b}\big(c_4(1-k^{-\frac{1}{16}}) - c_2 e^{-c_3(\log k)^2}\big)\,,$$

from which the claim follows. □

We finish this subsection with the following lemma.

Lemma 2.4.13. *For all $k \in \mathbb{N}$, $\varepsilon > 0$, and $\lambda > 0$, there exists $\delta \in (0,\lambda)$ such that for all $b \ge k$ sufficiently large and $\eta \in \{0,\delta\}$, it holds that*

$$\mathbb{P}_{0,u,b}\big(\mathbb{M}_{0,k}(\Upsilon_{b,\mathfrak{g}}) \ge \lambda - \delta + \eta,\ \mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \le \lambda + \eta\big) \le \varepsilon\frac{u}{b}\,. \tag{2.4.25}$$

Similarly, for all $k \in \mathbb{N}$, $\varepsilon > 0$, and $\lambda > 0$ there exists $\delta \in (0,\lambda)$ such that for all $b \ge k$ sufficiently large and $\eta \in \{0,\delta\}$, it holds that

$$\mathbb{P}\big(\mathbb{M}_{0,k}(\Upsilon_{\infty}) \ge \lambda - \delta + \eta,\ \mathbb{M}_{0,b}(\Upsilon_{\infty}) \le \lambda + \eta\big) \le \varepsilon\frac{1}{\sqrt{b}}\,. \tag{2.4.26}$$

Proof. The strategy for the proof of this lemma is quite similar to the strategy developed for the proof of Lemma 2.4.10. We will provide the proof only for the case $\eta = 0$, as the case $\eta = \delta$ is completely analogous. We fix $k \in \mathbb{N}$, $\varepsilon > 0$, and $\lambda > 0$. For $\delta \in (0,\lambda/2)$, we begin by observing that, thanks to Lemmas 2.4.9, 2.4.10, and 2.A.11, it holds that

$$\mathbb{P}_{0,u,b}\big(\mathbb{M}_{0,b,k}(\Upsilon_{b,\mathfrak{g}}) \ge \lambda - \delta\,, \mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \le \lambda\big) \lesssim k^{-\frac{1}{16}}\frac{u}{b}\,,$$

and so, if $k > b/2$ the claim follows by taking $b > 0$ large enough. Hence, from now on, we can focus on the regime $k < b/2$. Thanks to Lemma 2.4.10, by choosing $l > l_0$ for some $l_0 = l_0(\varepsilon) > k$, we can assume that we are on the event $\{\mathrm{K}_b \le l\}$ with K_b from Definition 2.4.6. Now, for $\delta \in (0,\lambda/2)$, we observe that thanks to (2.4.13),

$$\begin{aligned}\big\{\mathbb{M}_{0,k}(\Upsilon_{b,\mathfrak{g}}) \ge \lambda - \delta,\ \mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \le \lambda,\ \mathrm{K}_b \le l\big\}&\\ \subseteq \Big\{\mathbb{M}_{0,k}(\Upsilon_{b,\mathfrak{g}}) \in [\lambda-\delta,\lambda],\ \bigcap_{j=l}^{b-1}\big\{B_j \ge -\lambda - \mathrm{R}_l(j)\big\},\ \mathrm{K}_b \le l\Big\}\,.&\end{aligned} \tag{2.4.27}$$

Thanks to (2.4.9), on the event $\{\mathrm{K}_b \le l\}$, by possibly taking $l > l_1$ for some $l_1 = l_1(\varepsilon,\delta) > l_0$, we can assume that the event on the right-hand side in (2.4.27) is contained in

$$\big\{\mathbb{M}_{0,k}(\Upsilon_{l,\mathfrak{g}}) \in [\lambda - 2\delta, \lambda + 2\delta]\big\} \cap \Big\{\bigcap_{j=l}^{b-1}\big\{B_j \ge -\lambda - \mathrm{R}_l(j)\big\}\Big\}\,.$$

These two events are conditionally independent given $\sigma(B_l)$. In particular, on the event $\{B_l = z\}$ for some $z \ge -\lambda - \mathrm{R}_l(l)$, the first conditional probability is equal to

$$\mathbb{P}_{0,z,l}\big(\mathbb{M}_{0,k}(\Upsilon_{l,\mathfrak{g}}) \in [\lambda - 2\delta, \lambda + 2\delta]\big)\,.$$

Now, thanks e.g. to [PT79, Theorem 3.1], we observe that by choosing $\delta = \delta(\varepsilon, k) \in (0, \lambda/4)$ sufficiently small, it holds that

$$\sup_{z \geq -1 - \mathrm{R}_l(l)} \mathbb{P}_{0,z,l}\big(\mathbb{M}_{0,k}(\Upsilon_{l,\mathfrak{g}}) \in [\lambda - 2\delta, \lambda + 2\delta]\big) \leq \varepsilon \ . \tag{2.4.28}$$

To be precise, we cannot directly apply [PT79, Theorem 3.1] since $\Upsilon_{l,\mathfrak{g}}(0) = 0$. However, this issue can be easily overcome by noting that, by taking $\zeta = \zeta(\varepsilon) > 0$ sufficiently small, it holds that

$$\sup_{z \geq -1 - \mathrm{R}_l(l)} \mathbb{P}_{0,z,l}\left(\sup_{x \in B(0,\zeta)} \Upsilon_{l,\mathfrak{g}}(x) \geq \lambda/2\right) \leq \varepsilon \ ,$$

To obtain (2.4.28), we note that the variance of the field $\Upsilon_{l,\mathfrak{g}}$ over the annulus $\mathbb{B}_k \setminus B(0,\zeta)$ can be uniformly lower bounded by a quantity depending on ζ. Therefore, by [PT79, Theorem 3.1], the density with respect to the Lebesgue measure of the supremum of the field $\Upsilon_{l,\mathfrak{g}}$ over the annulus $\mathbb{B}_k \setminus B(0,\zeta)$ is bounded above by a constant depending on k and ζ. Hence, (2.4.28) follows by taking $\delta = \delta(\varepsilon, k) \in (0, \lambda/4)$ sufficiently small.

The conclusion then follows since by applying Lemma 2.4.10, one has that

$$\mathbb{P}_{0,u,b}\left(\bigcap_{j=l}^{b-1}\{B_j \geq -\lambda - \mathrm{R}_l(j)\}\right) \lesssim \frac{u}{b} \ .$$

The proof of the bound (2.4.26) follows a similar approach and it is therefore omitted. □

2.4.4 Asymptotic formulas

For any $\lambda > 0$, for $b, l \in \mathbb{N}$ with $b > l$, for $u \in [b^{1/4}, b^{3/4}]$, and for any function $\mathbf{F} \in \mathcal{C}^b_{\mathrm{loc}}\big(\mathcal{C}(\mathbb{R}^d)\big)$, we define the following quantities

$$\mathbb{H}_{l,\lambda}(\mathbf{F}) \overset{\text{def}}{=} \mathbb{E}\Big[\mathbf{F}(\Upsilon_l) B_l \mathbb{1}_{\{B_l \in [l^{1/6}, l^{5/6}]\}} \mathbb{1}_{\{\mathbb{M}_{0,l}(\Upsilon_l) \leq \lambda\}}\Big] \ , \tag{2.4.29}$$

$$\mathbb{H}^{0,u,b}_{l,\lambda}(\mathbf{F}) \overset{\text{def}}{=} \mathbb{E}_{0,u,b}\Big[\mathbf{F}(\Upsilon_l) B_l \mathbb{1}_{\{B_l \in [l^{1/6}, l^{5/6}]\}} \mathbb{1}_{\{\mathbb{M}_{0,l}(\Upsilon_l) \leq \lambda\}}\Big] \ . \tag{2.4.30}$$

Proposition 2.4.14. *Let $\mathbf{F} \in \mathcal{C}^b_{\mathrm{loc}}(\mathcal{C}(\mathbb{R}^d))$. For any $\varepsilon > 0$ there exists $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$, $b \geq l$ sufficiently large, $\lambda > 0$, and $u \in [b^{1/4}, b^{3/4}]$, it holds that*

$$\left|\mathbb{E}_{0,u,b}\Big[\mathbf{F}(\Upsilon_{b,\mathfrak{g}}) \mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\}}\Big] - 2\mathbb{H}_{l,\lambda}(\mathbf{F})\frac{u}{b}\right| \leq \varepsilon \frac{u}{b} \ . \tag{2.4.31}$$

Similarly, for any $\varepsilon > 0$ there exists $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$ and $b \geq l$ sufficiently large, it holds that

$$\left|\mathbb{E}\Big[\mathbf{F}(\Upsilon_\infty) \mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_\infty) \leq \lambda\}}\Big] - \mathbb{H}_{l,\lambda}(\mathbf{F})\frac{\alpha}{\sqrt{b}}\right| \leq \varepsilon \frac{1}{\sqrt{b}} \ . \tag{2.4.32}$$

Remark 2.4.15. From the proof of Proposition 2.4.14 given below, it follows that for any $\varepsilon > 0$ there exists $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$ and $b \geq l$ large enough,

$$\left|\mathbb{E}\Big[\mathbf{F}(\Upsilon_b) \mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b) \leq \lambda\}}\Big] - \mathbb{H}_{l,\lambda}(\mathbf{F})\frac{\alpha}{\sqrt{b}}\right| \leq \varepsilon \frac{u}{b} \ . \tag{2.4.33}$$

As we will see below, the proof of Theorem 2.A follows exactly the same method as the proof of Proposition 2.4.1, but using (2.4.33) instead of (2.4.31).

Before proving Proposition 2.4.14, we state two auxiliary technical lemmas.

Lemma 2.4.16. *For any $\lambda > 0$ and for all $u \in [b^{1/4}, b^{3/4}]$, it holds that*

$$\lim_{l\to\infty} \limsup_{b\to\infty} \frac{b}{u} \mathbb{P}_{0,u,b}\big(B_l \le l^{1/6},\ \mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \le \lambda\big) = 0 \,,$$

Proof. The proof follows the exact same argument as in the proof of [BL18, Lemma 5.3] (see also [BL18, Lemma 5.5] for a related statement about Brownian bridges). Specifically, it relies on Lemmas 2.4.10, 2.A.8, and 2.A.10. □

Lemma 2.4.17. *For any $\varepsilon > 0$ there exists $l_0 \in \mathbb{N}$ such that for all $l \ge l_0$, $b \ge l$ sufficiently large, $x \in [l^{1/6}, l^{5/6}]$, and $u \in [b^{1/4}, b^{3/4}]$, it holds that*

$$\left| \mathbb{P}_{0,u,b}\left(\bigcap_{j=l+1}^{b} \{B_j \ge 0\} \,\middle|\, B_l = x \right) - \frac{2xu}{b} \right| \le \varepsilon \frac{xu}{b} \,.$$

Proof. The proof follows the exact same argument as in the proof of [BL18, Lemma 5.4] (see also [BL18, Lemma 5.6] for a related statement about Brownian bridges). In particular, the lower bound follows trivially from Lemma 2.A.1, while the upper bound follows from Lemmas 2.A.5, 2.A.7, and 2.A.10. □

We now have all the necessary ingredients to prove Proposition 2.4.14.

Proof of Propositon 2.4.14. We will only prove (2.4.31), as the proof of (2.4.32) is completely analogous and, in fact, simpler.

The proof is based on a sequence of replacements that gradually convert one expectation into the other. We can and will assume that the function $\mathbf{F} \in \mathcal{C}^b_{\mathrm{loc}}(\mathcal{C}(\mathbb{R}^d))$ depends only on the value of the field inside the ball $\mathbb{B}_{k_0}$ for some fixed $k_0 > 0$. For $b \in \mathbb{N}$, $l \in [b-1]_0$, and $k \in [l]_0$, on the event $\{\mathrm{K}_b \le l\}$, arguing as in Remark 2.4.7, we have that

$$\sup_{x\in\mathbb{B}_k} |\Upsilon_{b,\mathfrak{g}}(x) - \Upsilon_l(x)| \lesssim e^{-(l-k)/2} (\log l)^2 \,. \tag{2.4.34}$$

Step 1: We start by replacing $\Upsilon_{b,\mathfrak{g}}$ by Υ_l in the argument of $\mathbf{F}$. For $k \ge k_0$, we recall that on the event $\{\mathrm{K}_b \le k\}$, both $\Upsilon_{b,\mathfrak{g}}$ and Υ_l are bounded on $\mathbb{B}_{k_0}$ by a quantity depending only on k. Therefore, for $k_0 < k < l < b$, using the uniform continuity of $\mathbf{F}$ on compacts, the bound in (2.4.34), and Lemma 2.4.10, we obtain that for any $\varepsilon > 0$, by taking $b > l$ both large enough,

$$\mathbb{E}_{0,u,b}\Big[|\mathbf{F}(\Upsilon_{b,\mathfrak{g}}) - \mathbf{F}(\Upsilon_l)| \mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}})\le\lambda\}} \mathbb{1}_{\{\mathrm{K}_b \le k\}} \Big] \le \varepsilon \frac{u}{b} \,. \tag{2.4.35}$$

Similarly, using the boundedness of $\mathbf{F}$ and Lemma 2.4.10, we also have that for any $\varepsilon > 0$, by taking $k > 0$ large enough, it holds that

$$\mathbb{E}_{0,u,b}\Big[|\mathbf{F}(\Upsilon_{b,\mathfrak{g}}) - \mathbf{F}(\Upsilon_l)| \mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}})\le\lambda\}} \mathbb{1}_{\{\mathrm{K}_b > k\}} \Big] \le \varepsilon \frac{u}{b} \,. \tag{2.4.36}$$

Hence, (2.4.35) and (2.4.36) imply that, for any $\varepsilon > 0$, by taking $b > k$ both large enough, it holds that

$$\left| \mathbb{E}_{0,u,b}\Big[\mathbf{F}(\Upsilon_{b,\mathfrak{g}}) \mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}})\le\lambda\}} \Big] - \mathbb{E}_{0,u,b}\Big[\mathbf{F}(\Upsilon_l) \mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}})\le\lambda\}} \Big] \right| \le \varepsilon \frac{u}{b} \,. \tag{2.4.37}$$

Step 2: In this step, we show that for any $\varepsilon > 0$, by taking $b > l$ both large enough, it holds that

$$\mathbb{E}_{0,u,b}\Big[\mathbf{F}(\Upsilon_l)\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}})\leq\lambda\}}\Big] \leq 2\frac{u}{b}\mathbb{H}_{l,\lambda}(\mathbf{F}) + \varepsilon\frac{u}{b}\,. \tag{2.4.38}$$

To this end, for $\delta \in (0,\lambda)$, and $k_0 < k < l < b$, we define the event $\mathrm{E}_{b,l,k,\delta}$ by letting

$$\mathrm{E}_{b,l,k,\delta} \stackrel{\text{def}}{=} \big\{\mathrm{K}_b \leq k\big\} \cap \big\{\mathbb{M}_{0,k}(\Upsilon_{b,\mathfrak{g}}) < \lambda - \delta\big\} \cap \big\{B_l \in \big[l^{1/6}, l^{5/6}\big]\big\} \cap \Big\{\bigcap_{j=k+1}^{b}\big\{B_j \geq \lambda + 2\mathrm{R}_k(j)\big\}\Big\}\,.$$

For any $\varepsilon > 0$, by Lemmas 2.4.10, 2.4.13, 2.4.16, and 2.A.11 for $l > k$ both large enough and $\delta > 0$ small enough,

$$\limsup_{b\to\infty}\frac{b}{u}\mathbb{P}_{0,u,b}\big(\mathrm{E}^c_{b,l,k,\delta},\ \mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\big) \leq \varepsilon\,.$$

Therefore, so far we proved that, for $b > l$ both large enough, it holds that

$$\mathbb{E}_{0,u,b}\Big[\mathbf{F}\big(\Upsilon_l\big)\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}})\leq\lambda\}}\Big] \leq \varepsilon\frac{u}{b} + \mathbb{E}\Big[\mathbf{F}\big(\Upsilon_l\big)\mathbb{1}_{\{\mathrm{E}_{b,l,k,\delta}\}}\Big]\,. \tag{2.4.39}$$

Now, we can choose $l \geq k$ large enough in such a way that the right-hand side of (2.4.34) is less than δ for $j \in [k]$ and less than $\mathrm{R}_k(j)$ for $j \in \{k+1,\dots,l\}$ (assuming that the constant $\mathrm{C} > 0$ in the definition of R_k in Lemma 2.4.9 is chosen large enough). In this way, on the event $\{\mathrm{K}_b \leq k\}$, we have that

$$\big\{\mathbb{M}_{0,k}(\Upsilon_b) < \lambda - \delta\big\} \subseteq \big\{\mathbb{M}_{0,k}(\Upsilon_l) < \lambda\big\}\,,$$

and also, for $j \in \{k+1,\dots l\}$,

$$\big\{B_j \geq \lambda + 2\mathrm{R}_k(j)\big\} \subseteq \big\{\mathbb{M}_{0,j+1,j}(\Upsilon_b) \leq \lambda + \mathrm{R}_k(j)\big\} \subseteq \big\{\mathbb{M}_{0,j+1,j}(\Upsilon_l) \leq \lambda\big\}\,.$$

Therefore, putting these facts together, we get that the expectation on the right-hand side of (2.4.39) is bounded above by

$$\mathbb{E}_{0,u,b}\Big[\mathbf{F}\big(\Upsilon_l\big)\mathbb{1}_{\{B_l\in[l^{1/6},l^{5/6}]\}}\mathbb{1}_{\{\mathbb{M}_{0,l}(\Upsilon_l)\leq\lambda\}}\mathbb{1}_{\{\cap_{j=l+1}^{b}\{B_j\geq 0\}\}}\Big]\,.$$

Since the field Υ_l is conditionally independent of $\sigma((B_s)_{s\geq l})$ given $\sigma(B_l)$, it follows from Lemma 2.4.17, the boundedness of $\mathbf{F}$, the absolute continuity of the law of the Brownian bridge with respect to that of the Brownian motion (see Step 4), and Lemma 2.4.12 that

$$\lim_{l\to\infty}\limsup_{b\to\infty}\bigg|\frac{b}{u}\mathbb{E}_{0,u,b}\Big[\mathbf{F}\big(\Upsilon_l\big)\mathbb{1}_{\{B_l\in[l^{1/6},l^{5/6}]\}}\mathbb{1}_{\{\mathbb{M}_{0,l}(\Upsilon_l)\leq\lambda\}}\mathbb{1}_{\{\cap_{j=l+1}^{b}\{B_j\geq 0\}\}}\Big] - 2\mathbb{H}^{0,u,b}_{l,\lambda}(\mathbf{F})\bigg| = 0\,,$$

thus obtaining (2.4.38).

Step 3: In this step, we show that for any $\varepsilon > 0$, by taking $b > l$ both large enough, it holds that

$$\mathbb{E}_{0,u,b}\Big[\mathbf{F}(\Upsilon_l)\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}})\leq\lambda\}}\Big] \geq 2\frac{u}{b}\mathbb{H}_{l,\lambda}(\mathbf{F}) - \varepsilon\frac{u}{b}\,. \tag{2.4.40}$$

To get the above inequality, we can proceed similarly to Step 2. For $k_0 < k < l < b$, we define the event $\widetilde{\mathrm{E}}_{b,l,k}$ by letting

$$\widetilde{\mathrm{E}}_{b,l,k} \stackrel{\text{def}}{=} \big\{\mathrm{K}_b \leq k\big\} \cap \big\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\big\} \cap \big\{B_l \in \big[l^{1/6}, l^{5/6}\big]\big\}\,.$$

Now, we note that, for $\delta > 0$, the following trivial lower bound holds true

$$\begin{aligned}&\mathbb{E}_{0,u,b}\Big[\mathbf{F}\big(\Upsilon_l\big)\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}})\leq\lambda\}}\Big]\\&\qquad\geq\mathbb{E}_{0,u,b}\Big[\mathbf{F}\big(\Upsilon_l\big)\mathbb{1}_{\{\widetilde{\mathrm{E}}_{b,l,k}\}}\mathbb{1}_{\{\mathbb{M}_{0,k}(\Upsilon_{b,\mathfrak{g}})\leq\lambda+\delta\}}\mathbb{1}_{\{\cap_{j=k+1}^{b}\{B_j\geq-\lambda-2\mathrm{R}_k(j)\}\}}\Big]\,.\end{aligned}\tag{2.4.41}$$

For any $\varepsilon > 0$, by Lemmas 2.4.10, 2.4.13, 2.4.16, and 2.A.11, for $k < l$ both large enough and $\delta > 0$ small enough,

$$\limsup_{b\to\infty}\frac{b}{u}\mathbb{P}_{0,u,b}\left(\widetilde{\mathrm{E}}_{b,l,k}^{c},\ \mathbb{M}_{0,k}(\Upsilon_{b,\mathfrak{g}})\leq\lambda+\delta,\ \bigcap_{j=k+1}^{b}\big\{B_j\geq-\lambda-2\mathrm{R}_k(j)\big\}\right)\leq\varepsilon\,.\tag{2.4.42}$$

As before, we can choose $l \geq k$ large enough in such a way that the right-hand side of (2.4.34) is less than δ for $j \in [k]$ and less than $\mathrm{R}_k(j)$ for $j \in \{k+1,\dots,l\}$ (assuming that the constant $\mathrm{C} > 0$ in the definition of R_k in Lemma 2.4.9 is chosen large enough). Therefore, on the event $\{\mathrm{K}_b \leq k\}$, which is contained in $\widetilde{\mathrm{E}}_{b,l,k}$, we have that

$$\big\{\mathbb{M}_{0,k}(\Upsilon_l)\leq\lambda\big\}\subseteq\big\{\mathbb{M}_{0,k}(\Upsilon_{b,\mathfrak{g}})\leq\lambda+\delta\big\}$$

and also, for $j \in \{k+1,\dots l\}$,

$$\big\{\mathbb{M}_{0,j+1,j}(\Upsilon_l)\leq\lambda\big\}\subseteq\big\{\mathbb{M}_{0,j+1,j}(\Upsilon_{b,\mathfrak{g}})\leq\lambda+\mathrm{R}_k(j)\big\}\subseteq\big\{B_j\geq-\lambda-2\mathrm{R}_k(j)\big\}\,.$$

Therefore, using (2.4.42), for $b > l$ both large enough, the right-hand side of (2.4.41) can be lower bounded by

$$\mathbb{E}_{0,u,b}\Big[\mathbf{F}\big(\Upsilon_l\big)\mathbb{1}_{\{B_l\in[l^{1/6},l^{5/6}]\}}\mathbb{1}_{\{\mathbb{M}_{0,l}(\Upsilon_l)\leq\lambda\}}\mathbb{1}_{\{\cap_{j=l+1}^{b}\{B_j\geq0\}\}}\Big]-\frac{u}{b}\varepsilon\,.$$

To obtain (2.4.40), we can apply Lemma 2.4.17 as in Step 2.

Step 4: By combining (2.4.37), (2.4.38), and (2.4.40), we have thus far proven that for any $\varepsilon > 0$, by taking $b > l$ both sufficiently large,

$$\left|\mathbb{E}_{0,u,b}\Big[\mathbf{F}(\Upsilon_{b,\mathfrak{g}})\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}})\leq\lambda\}}\Big]-2\mathbb{H}_{l,\lambda}^{0,u,b}(\mathbf{F})\frac{u}{b}\right|\leq\varepsilon\frac{u}{b}\,.$$

Therefore, to conclude, it is sufficient to show that we can replace $\mathbb{H}_{l,\lambda}^{0,u,b}(\mathbf{F})$ on the right-hand side of the previous display by $\mathbb{H}_{l,\lambda}(\mathbf{F})$. This is achieved by using the absolute continuity of the law of the Brownian bridge with respect to the law of the Brownian motion. We recall that for any $0 \leq l < b$, it holds that

$$\frac{d\mathbb{P}_{0,u,b}}{d\mathbb{P}}\Big|_{\sigma((B_s)_{s\in[0,l]})}=\sqrt{b/(b-l)}e^{\frac{u^2}{2b}-\frac{(B_l-u)^2}{2(b-l)}}\,,$$

and so, it holds that

$$\mathbb{H}_{l,\lambda}^{0,u,b}(\mathbf{F})=\sqrt{b/(b-l)}\mathbb{E}\left[\mathbf{F}(\Upsilon_l)B_l\mathbb{1}_{\{B_l\in[l^{1/6},l^{5/6}]\}}\mathbb{1}_{\{\mathbb{M}_{0,l}(\Upsilon_l)\leq1\}}e^{\frac{u^2}{2b}-\frac{(B_l-u)^2}{2(b-l)}}\right].$$

Therefore, using the boundedness of $\mathbf{F}$, the difference $|\mathbb{H}_{l,\lambda}(\mathbf{F})-\mathbb{H}_{l,\lambda}^{0,u,b}(\mathbf{F})|$ is bounded from above by a constant times

$$l^{5/6}\mathbb{E}\left[\left|1-\sqrt{b/(b-l)}e^{\frac{u^2}{2b}-\frac{(B_l-u)^2}{2(b-l)}}\right|\right]$$

which can be made arbitrarily small by taking b large enough, uniformly over $u \in [b^{1/4}, b^{3/4}]$. Hence, the desired result follows. □

Lemma 2.4.18. *There exist constants c_1, $c_2 > 0$ such that for any $\lambda > 0$ and for all $l \in \mathbb{N}$ large enough,*

$$c_1 < \mathbb{H}_{l,\lambda}(1) < c_2 \ .$$

Proof. By taking $\mathbf{F} = 1$ in (2.4.31), for each $\varepsilon > 0$, there exists $l_0 \in \mathbb{N}$ sufficiently large such that for all $l > l_0$ and for any $b > l$ sufficiently large and $u \in [b^{1/4}, b^{3/4}]$, it holds that

$$\left| \mathbb{P}_{0,u,b}\big(\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\big) - \mathbb{H}_{l,\lambda}(1)\frac{2u}{b} \right| \leq \varepsilon \frac{u}{b} \ .$$

On the other hand, from Lemma 2.4.12, we know that there exist constants c_1, $c_2 > 0$ such that for $b > 0$ large enough and $u \in [b^{1/4}, b^{3/4}]$,

$$c_1 \frac{u}{b} \leq \mathbb{P}_{0,u,b}\big(\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\big) \leq c_2 \frac{u}{b} \ , \tag{2.4.43}$$

and so the conclusion follows readily. □

We are now ready to prove Propositions 2.4.2 and 2.4.1 as well as Theorems 2.A and 2.B.

Proof of Proposition 2.4.2 and of Theorem 2.B. For the proof of Proposition 2.4.2, by taking $\mathbf{F} = 1$ in (2.4.31), for any $\varepsilon > 0$, by taking $b > l$ both large enough, it holds that

$$\mathbb{P}_{0,u,b}\big(\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\big) = \big(2\mathbb{H}_{l,\lambda}(1) + \varepsilon_b(l)\big)\frac{u}{b} \ ,$$

where $\lim_{l\to\infty} \limsup_{b\to\infty} |\varepsilon_b(l)| = 0$. By arguing in the same exact way as in the proof of [BL18, Theorem 2.4], we get that both limits in

$$\lim_{b\to\infty} \frac{b}{u}\mathbb{P}_{0,u,b}\big(\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\big) = \lim_{k\to\infty} 2\mathbb{H}_{k,\lambda}(1)$$

exist and are related as stated. Hence, the conclusion follows by letting

$$c_{\star,\lambda} \overset{\text{def}}{=} \lim_{k\to\infty} \mathbb{H}_{k,\lambda}(1) \ , \tag{2.4.44}$$

which is positive and finite thanks to Lemma 2.4.18. Finally, the proof of Theorem 2.B follows in exactly the same way by taking $\mathbf{F} = 1$ in (2.4.32). □

Proof of Proposition 2.4.1 and of Theorem 2.A. Consider a function $\mathbf{F} \in \mathcal{C}^b_{\text{loc}}(\mathcal{C}(\mathbb{R}^d))$, then by (2.4.31) from Proposition 2.4.14, the following equality holds

$$\frac{\mathbb{E}_{0,u,b}\big[\mathbf{F}(\Upsilon_{b,\mathfrak{g}})\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}})\leq\lambda\}}\big]}{\mathbb{P}_{0,u,b}\big(\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\big)} = \frac{\mathbb{H}_{l,\lambda}(\mathbf{F}) + \widetilde{\varepsilon}_b(l)}{\mathbb{H}_{l,\lambda}(1) + \varepsilon_b(l)} \ ,$$

where $\lim_{l\to\infty} \limsup_{b\to\infty} |\varepsilon_b(l)| = 0$ and $\lim_{l\to\infty} \limsup_{b\to\infty} |\widetilde{\varepsilon}_b(l)| = 0$. The same argument as in the proof of [BL18, Proposition 5.8] shows that both limits in

$$\lim_{b\to\infty} \frac{\mathbb{E}_{0,u,b}\big[\mathbf{F}(\Upsilon_{b,\mathfrak{g}})\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}})\leq\lambda\}}\big]}{\mathbb{P}_{0,u,b}\big(\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\big)} = \lim_{l\to\infty} \frac{\mathbb{H}_{l,\lambda}(\mathbf{F})}{\mathbb{H}_{l,\lambda}(1)} \tag{2.4.45}$$

exist and are related as stated. Similarly, using (2.4.33), we note that the same identity as in (2.4.45) holds if we replace the field $\Upsilon_{b,\mathfrak{g}}$ on the left-hand side with Υ_b.

Regarding the limit on the right-hand side of (2.2.6), we note that, thanks to (2.4.32), for all $\mathbf{F} \in \mathcal{C}^b_{\mathrm{loc}}(\mathcal{C}(\mathbb{R}^d))$, it holds that

$$\lim_{b\to\infty} \frac{\mathbb{E}\big[\mathbf{F}(\Upsilon_\infty)\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_\infty)\le\lambda\}}\big]}{\mathbb{P}\big(\mathbb{M}_{0,b}(\Upsilon_\infty)\le\lambda\big)} = \lim_{l\to\infty} \frac{\mathbb{H}_{l,\lambda}(\mathbf{F})}{\mathbb{H}_{l,\lambda}(1)} \,. \tag{2.4.46}$$

Since the right-hand side of the above display is equal to the right-hand side of (2.4.45), the two limits coincide.

Finally, it remains to show that there exists a continuous random field $\widetilde{\Upsilon}_\lambda$ on $\mathbb{R}^d$ such that the right-hand side of (2.4.45) equals $\mathbb{E}[\mathbf{F}(\widetilde{\Upsilon}_\lambda)]$ for any $\mathbf{F} \in \mathcal{C}^b(\mathcal{C}(\mathbb{B}_k))$. For each $k \in \mathbb{N}$ and for any $\mathbf{F} \in \mathcal{C}^b(\mathcal{C}(\mathbb{B}_k))$, the limit on the left-hand side of (2.4.45) exists. In particular, when $\mathbf{F} = 1$ this limit is 1 by definition. By [Bog07, Theorem 8.7.1], this ensures the existence of a probability measure $\widetilde{\nu}_{k,\lambda}$ on $\mathcal{C}(\mathbb{B}_k)$ such that the right-hand side of (2.4.45) coincides with $\mathbb{E}_{\widetilde{\nu}_{k,\lambda}}[\mathbf{F}(\Phi)]$, where under $\mathbb{P}_{\widetilde{\nu}_{k,\lambda}}$ the field Φ is distributed according to $\widetilde{\nu}_{k,\lambda}$. Furthermore, the collection of probability measures $(\widetilde{\nu}_{k,\lambda})_{k\in\mathbb{N}}$ is consistent, and so, by the Kolmogorov extension theorem, there exists a unique probability measure $\widetilde{\nu}_\lambda$ on $\mathcal{C}(\mathbb{R}^d)$ whose restriction to $\mathcal{C}(\mathbb{B}_k)$ coincides with $\widetilde{\nu}_{k,\lambda}$. Defining $\widetilde{\Upsilon}_\lambda$ as the continuous random field with law $\widetilde{\nu}_\lambda$ on $\mathcal{C}(\mathbb{R}^d)$, it follows that for all $\mathbf{F} \in \mathcal{C}^b_{\mathrm{loc}}(\mathcal{C}(\mathbb{R}^d))$, the right-hand side of (2.4.45) equals $\mathbb{E}[\mathbf{F}(\widetilde{\Upsilon}_\lambda)]$. □

2.4.5 Corollaries and applications

In this section, we gather some results that follow as consequences of those presented in earlier sections and that will be particularly useful later, especially in Section 2.7.

We begin with the following lemma.

Lemma 2.4.19. *There exists a constant $c > 0$ such that for $\lambda > 0$, for all $L > 0$ sufficiently large, $b > 0$ sufficiently large, and $u \in [b^{1/4}, b^{3/4}]$,*

$$\mathbb{P}_{0,u,b}\left(\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \le \lambda,\ \inf_{s\in[0,b]} B_s < -L,\ \right) \lesssim \frac{u}{b} e^{-c\sqrt{L}} \,.$$

Proof. For $L > 0$, we let $\overline{k} = \overline{k}(L)$ be the smallest $k \in [b-1]$ such that $\lambda + \mathrm{R}_k(0) \ge L/4$, where R_k is defined in the statement of Lemma 2.4.9. We note that, by possibly taking $L > 0$ large enough, we have that $\overline{k} \approx e^{c\sqrt{L}}$. As usual, we can assume that we are on the event $\{\mathrm{K}_b < \overline{k}\}$, otherwise the conclusion follows trivially by Lemma 2.4.10. In particular, on the event $\{\mathrm{K}_b < \overline{k}\}$, by Lemma 2.4.9 (see also (2.4.13)), it suffices to estimate the probability of the following event

$$\left\{\bigcap_{j=1}^{b-1}\big\{B_j \ge -\lambda - \mathrm{R}_{\overline{k}}(j)\big\},\ \inf_{s\in[0,b]} B_s < -L\right\} . \tag{2.4.47}$$

To this end, we start by noticing that thanks to Lemma 2.A.11, there exists a constant $c > 0$ such that,

$$\mathbb{P}_{0,u,b}\left(\bigcap_{j=1}^{b-1}\big\{B_j \ge -\lambda - \mathrm{R}_{\overline{k}}(j)\big\},\ \bigcup_{j=\overline{k}}^{b-1}\big\{B_j \le \lambda + \mathrm{R}_{\overline{k}}(j)\big\}\right) \lesssim \frac{u}{b} e^{-c\sqrt{L}} \,.$$

Hence, we can now assume that we are on the complement of the giant union appearing in the probability on the left-hand side of the above display, i.e., we need to estimate the probability of the following event

$$\left\{\bigcap_{j=1}^{b-1}\left\{B_j \geq -\lambda - \mathrm{R}_{\overline{k}}(j)\right\},\ \bigcap_{j=\overline{k}}^{b-1}\left\{B_j > \lambda + \mathrm{R}_{\overline{k}}(j)\right\},\ \inf_{s\in[0,b]} B_s < -L\right\}. \tag{2.4.48}$$

For each $j \in [b-1]_0$, we introduce the process $(W^j_s)_{s\in[j,j+1]}$ given by

$$W^j_s \overset{\text{def}}{=} \big((j+1)-s\big)B_j + \big(s-j\big)B_{j+1} - B_s\ , \qquad \forall\, s\in[j,j+1]\ .$$

The process W^j has the law of standard Brownian bridge indexed by times in the interval $[j,j+1]$. Moreover, the collection $(W^j)_{j\in[b-1]_0}$ is independent of the values $(B_j)_{j\in[b]_0}$. Now, we note that by definition of $\overline{k}$, for all $j\in[\overline{k}]_0$, it holds that $\lambda + \mathrm{R}_{\overline{k}}(j) \leq L/2$. In particular, this fact implies that, on the first event, the last two events in (2.4.48) are contained in the following union of events

$$\left\{\bigcup_{j=0}^{\overline{k}}\Big\{\sup_{s\in[j,j+1]}|W^j_s| \geq L/2\Big\}\right\} \cup \left\{\bigcup_{j=\overline{k}}^{b-1}\Big\{\sup_{s\in[j,j+1]}|W^j_s| \geq \lambda + \mathrm{R}_{\overline{k}}(j) + L\Big\}\right\}. \tag{2.4.49}$$

For the first event in (2.4.49), using the independence of the collection $(W^j)_{j\in[b-1]_0}$ from the values $(B_j)_{j\in[b]_0}$, the Gaussian tails of the supremum of a standard Brownian bridge, arguing as in the proof Lemma 2.4.10, there exists a constant $c>0$ such that

$$\sum_{l=0}^{\overline{k}}\mathbb{P}_{0,u,b}\left(\bigcap_{j=1}^{b-1}\left\{B_j \geq -\lambda - \mathrm{R}_{\overline{k}}(j)\right\},\ \sup_{s\in[l,l+1]}|W^l_s| \geq L/2\right) \lesssim \frac{u}{b}e^{-c\sqrt{L}}\ .$$

Finally, regarding the second event in (2.4.49), using again the independence mentioned above, and the Gaussian tails of the supremum of a standard Brownian bridge, we have that thanks to Lemma 2.4.10, there exists a constant $c>0$ such that

$$\sum_{l=\overline{k}}^{b-1}\mathbb{P}_{0,u,b}\left(\bigcap_{j=1}^{b-1}\left\{B_j \geq -\lambda - \mathrm{R}_{\overline{k}}(j)\right\},\ \sup_{s\in[l,l+1]}|W^l_s| \geq \lambda + \mathrm{R}_{\overline{k}}(l) + L\right) \lesssim \frac{u}{b}e^{-c\sqrt{L}}\ .$$

Therefore, by combining all the bounds we have established so far, the claim follows. □

We now state and prove the following results which provides a decay of the field $\Upsilon_{b,\mathfrak{g}}$ on the event that the supremum of such a field is bounded by one.

Lemma 2.4.20. *For each $\lambda>0$, for $0\leq k\leq k'\leq b-1$, and $u\in[b^{1/4},b^{3/4}]$, by choosing k large enough, it holds that*

$$\mathbb{P}_{0,u,b}\left(\bigcup_{j=k}^{k'}\Big\{\sup_{x\in\mathbb{A}_j}\Upsilon_{b,\mathfrak{g}}(x) \geq -(\log j)^2\Big\},\ \mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\right) \lesssim \frac{u}{b}k^{-\frac{1}{16}}\ , \tag{2.4.50}$$

$$\mathbb{P}\left(\bigcup_{j=k}^{k'}\Big\{\sup_{x\in\mathbb{A}_j}\Upsilon_{b,\mathfrak{g}}(x) \geq -(\log j)^2\Big\},\ \mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\right) \lesssim \frac{1}{\sqrt{b}}k^{-\frac{1}{16}}\ . \tag{2.4.51}$$

Similarly, for all $k \geq 0$ sufficiently large, one has that

$$\mathbb{P}\left(\bigcup_{j=k}^{\infty}\Big\{\sup_{x\in\mathbb{A}_j}\widetilde{\Upsilon}_\lambda(x) \geq -(\log j)^2\Big\}\right) \lesssim k^{-\frac{1}{16}} \,. \tag{2.4.52}$$

Proof. We only prove the bound for the Brownian bridge since the bound for the Brownian motion can be obtained in the same way. As usual, we can assume that we are on the event $\{\mathrm{K}_b \leq k\}$, otherwise the conclusion follows trivially from Lemma 2.4.10. For $j \in \{k, \dots, b-1\}$, thanks to Lemma 2.4.9, it holds that

$$\bigcup_{j=k}^{b-1}\Big\{\sup_{x\in\mathbb{A}_j}\Upsilon_{b,\mathfrak{g}}(x) \geq -(\log j)^2,\ \mathrm{K}_b \leq k\Big\} \subseteq \bigcup_{j=k}^{b-1}\big\{B_j \leq \lambda + 2\mathrm{R}_k(j)\big\} \,,$$

and also

$$\big\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda,\ \mathrm{K}_b \leq k\big\} \subseteq \bigcap_{j=1}^{b-1}\big\{B_j \geq -\lambda - 2\mathrm{R}_k(j)\big\} \,.$$

The claim follows since by Lemma 2.A.11 it holds that

$$\mathbb{P}_{0,u,b}\left(\bigcap_{j=1}^{b-1}\big\{B_j \geq -\lambda - 2\mathrm{R}_k(j)\big\},\ \bigcup_{j=k}^{b-1}\big\{B_j \leq \lambda + 2\mathrm{R}_k(j)\big\}\right) \lesssim \frac{u}{b}k^{-\frac{1}{16}} \,.$$

Finally, we note that the bound (2.4.52) follows directly from Theorem 2.A by taking the limit as $b \to \infty$ in (2.4.51). □

Remark 2.4.21. An immediate consequence of (2.4.52) is that there almost surely exists a (random) $k \geq 0$ such that, for all $j \geq k$, the supremum of the field $\widetilde{\Upsilon}_\lambda$ on the annulus $\mathbb{A}_j$ is less than $-(\log j)^2$.

2.4.6 Tail estimates for near-maximal level sets

In this section, we establish a key result concerning the tail behaviour of the volume of "near-maximal level sets" $|\mathbb{D}^\lambda_{0,j}(\Upsilon_{b,\mathfrak{g}})|$, for $j \leq b$, where here we recall the notation introduced in (2.3.3), as well as the definition of $\Upsilon_{b,\mathfrak{g}}$ in (2.4.3). Throughout this section, we assume that the field $\mathfrak{g}_b$ satisfies (G1) – (G3).

Lemma 2.4.22. *There exists $\delta = \delta(d) \in (0,1)$ such that for any $j_0 > 0$ sufficiently large and $b > j_0$ sufficiently large, it holds for any $\lambda > 0$, $j \in \{j_0, \dots, b\}$, $u \in [b^{1/4}, b^{3/4}]$, and $\eta \geq 0$,*

$$\mathbb{P}_{0,u,b}\big(|\mathbb{D}^\lambda_{0,j}(\Upsilon_{b,\mathfrak{g}})|^{-1} \geq \eta,\ \mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\big) \lesssim \frac{u}{b}\big(1 \wedge \eta^{-(1+\delta)}\big) \,. \tag{2.4.53}$$

Moreover, for all $\sigma \in [0, \delta)$, it holds that

$$\mathbb{E}_{0,u,b}\Big[|\mathbb{D}^\lambda_{0,j}(\Upsilon_{b,\mathfrak{g}})|^{-(1+\sigma)}\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}})\leq\lambda\}}\Big] \lesssim \frac{u}{b} \,. \tag{2.4.54}$$

Remark 2.4.23. As usual, the previous lemma admits a corresponding version for Brownian motions in place of Brownian bridges. Specifically, if we replace the conditional probability law $\mathbb{P}_{0,u,b}$ with the unconditional probability law $\mathbb{P}$, then the conclusions of the previous lemma remain valid, with the only difference that u/b is replaced by $1/\sqrt{b}$.

Before proceeding with the proof of Lemma 2.4.22, we state and prove the following auxiliary result concerning the tail behaviour of the second derivative of the field Z_b.

Lemma 2.4.24. *There exists a constant $c > 0$ such that for any $b \in \mathbb{N}$, $r \in [0, e^b]$, i, $k \in [d]$, and $\eta \geq 0$,*

$$\mathbb{P}\left(\sup_{|x| \leq r} \left| \partial^2_{i,k} \mathrm{Z}_b(x) \right| > \eta \right) \lesssim 1 \wedge r^d e^{-c\eta^2} \, . \tag{2.4.55}$$

Proof. Recall that, for x, $y \in \mathbb{B}_b$,

$$\mathbb{E}\big[\partial^2_{i,k} \mathrm{Z}_b(x) \partial^2_{i,k} \mathrm{Z}_b(y)\big] = \int_0^b \big(e^{-4s} \mathfrak{K}_{i,i,k,k}(e^{-s}(x-y)) - e^{-4s} \mathfrak{K}_{i,k}(e^{-s}x) \mathfrak{K}_{i,k}(e^{-s}y) \big) ds \, ,$$

where, given $n \in \mathbb{N}$ and $(j_1, \dots j_n) \in [d]^n$, we write $\mathfrak{K}_{j_1,\dots,j_n}$ for the n-th derivative of $\mathfrak{K}$ along the directions $e_{j_1}, \dots, e_{j_n}$. In particular, by the smoothness of $\mathfrak{K}$ and the fact that it is supported in the unit ball, it holds

$$\mathbb{E}\big[|\partial^2_{i,k} \mathrm{Z}_b(x) - \partial^2_{i,k} \mathrm{Z}_b(y)|^2 \big] \lesssim |x-y|^2 \wedge 1 \, ,$$

where the implicit constant is independent of b. It follows from Fernique's majorizing criterion (Lemma A.0.4) and Borell-TIS inequality (Lemma A.0.3) that, for any $\eta \geq 0$ and for some $c > 0$, one has the bound

$$\mathbb{P}\left(\sup_{|x-y| \leq 1} \left| \partial^2_{i,k} \mathrm{Z}_b(x) \right| > \eta \right) \lesssim e^{-c\eta^2} \, ,$$

uniformly over y. The requested estimate then follows from the union bound. □

Before proceeding with the proof of Lemma 2.4.22, we introduce some notation. For $r \geq 0$, we set

$$\mathrm{S}_r \stackrel{\text{def}}{=} \sup_{i,k \in [d],\, x \in B(0,r)} |\partial^2_{i,k} \Upsilon_{b,\mathfrak{g}}(x)| \, , \tag{2.4.56}$$

where, to simplify the notation, we have omitted the dependence of S_r on the parameter b. Furthermore, recalling the characterisation (2.4.1) of the field Φ_b, for any i, $k \in [d]$ and $x \in \mathbb{B}_b$, we have

$$\partial^2_{i,k} \Upsilon_{b,\mathfrak{g}}(x) = \int_0^b e^{-2s} \mathfrak{K}_{i,k}(e^{-s}x) dB_s + \partial^2_{i,k} \mathrm{Z}_b(x) - \sqrt{2d} \int_0^b e^{-2s} \mathfrak{K}_{i,k}(e^{-s}x) ds + \partial^2_{i,k} \mathfrak{g}_b(x), \tag{2.4.57}$$

where we have used the same notation introduced in the proof of Lemma 2.4.24.

The reason why it is useful to look at the second derivative of the field $\Upsilon_{b,\mathfrak{g}}$ is due to the following observation.

Remark 2.4.25. For any $\lambda > 0$, $\eta \geq 0$, and $0 < j \leq b$, we observe that the following implication holds

$$|\mathbb{D}^\lambda_{0,j}(\Upsilon_{b,\mathfrak{g}})|^{-1} \geq \eta \quad \Longrightarrow \quad \mathrm{S}_{e^j} > c_\lambda \eta^{2/d} \, , \tag{2.4.58}$$

for some suitable constant c_λ depending on λ and d. Indeed, if the condition on the right-hand side of the above display is not satisfied then there exists a ball of area at least η^{-1} around the maximum of $\Upsilon_{b,\mathfrak{g}}$ inside $\mathbb{B}_j$ where $\Upsilon_{b,\mathfrak{g}}$ is greater than its maximal value minus λ, which contradict the fact that $|\mathbb{D}^\lambda_{0,j}(\Upsilon_{b,\mathfrak{g}})|^{-1} \geq \eta$.

A straightforward consequence of Lemma 2.4.24 is the following result.

Lemma 2.4.26. *Let $A > 0$ be fixed. There exist constants c_1, $c_2 > 0$ such that for any $\eta \geq 0$, $b \geq j > 0$, and $u > -A$, it holds that*

$$\mathbb{P}_{0,u,b}\big(\mathrm{S}_{e^j} > \eta^{2/d}\big) \lesssim 1 \wedge e^{dj + c_1 \frac{u}{b} \eta^{2/d}} e^{-c_2 \eta^{4/d}} \, . \tag{2.4.59}$$

Proof. Consider the expression (2.4.57) for the second directional derivative of $\Upsilon_{b,\mathfrak{g}}$. We note that the deterministic term is bounded by a constant independent of everything else. Moreover, thanks to (G3), the term $\partial^2_{i,k}\mathfrak{g}_b$ has Gaussian tails uniformly over all $i, k \in [d]$ and $x \in \mathbb{B}_b$. Furthermore, we also note the field

$$\mathbb{B}_b \ni x \mapsto \int_0^b e^{-2s}\mathfrak{K}_{i,k}(e^{-s}x)dB_s - \frac{u}{b}\int_0^b e^{-2s}\mathfrak{K}_{i,k}(e^{-s}x)ds$$

has uniform Gaussian tails uniformly over $i, k \in [d]$, $x \in \mathbb{B}_b$, and over the probability laws $\mathbb{P}_{0,u,b}$, for all $b > 0$ and $u > -A$. In particular, since the integral multiplying u/b in the above display is bounded uniformly over $i, k \in [d]$ and $x \in \mathbb{B}_b$, using Lemma 2.4.24, we get that there exist constants $c_1, c_2 > 0$ such that

$$\mathbb{P}_{0,u,b}\big(\mathrm{S}_{e^j} > \eta^{2/d}\big) \lesssim 1 \wedge e^{dj + c_1\frac{u}{b}\eta^{2/d}}e^{-c_2\eta^{4/d}} ,$$

thus proving the claim. □

We are now ready for the proof of Lemma 2.4.22.

Proof of Lemma 2.4.22. For $\eta \geq 0$, we define

$$r_\eta \overset{\text{def}}{=} \lfloor \exp(\eta^{1/d}) \rfloor ,$$

so that $\log r_\eta \approx \eta^{1/d}$, and we also consider $b \geq j > 0$ and $u \in [b^{1/4}, b^{3/4}]$. We split the proof into three distinct cases.

Case 1: We begin by treating the following case:

$$\log r_\eta \geq \frac{b}{2} .$$

Let $c_\lambda > 0$ be the constant introduced in Remark 2.4.25. Then, thanks to (2.4.58) and Lemma 2.4.26, there exists a constant $c > 0$ such that

$$\mathbb{P}_{0,u,b}\big(|\mathbb{D}^\lambda_{0,j}(\Upsilon_{b,\mathfrak{g}})|^{-1} \geq \eta\big) \leq \mathbb{P}_{0,u,b}\big(\mathrm{S}_{e^j} > c_\lambda\eta^{2/d}\big) \lesssim e^{dj}e^{-c\eta^{4/d}} . \tag{2.4.60}$$

which, in the regime $\log r_\eta \geq b/2$, is trivially bounded by a constant times $(u/b)\eta^{-2}$.

Case 2: We now consider the following case:

$$\frac{j}{2} \leq \log r_\eta < \frac{b}{2} .$$

Proceeding, as above, thanks to Lemma 2.4.26, it suffices to bound the following probability

$$\mathbb{P}_{0,u,b}\big(\mathrm{S}_{e^j} > c_\lambda\eta^{2/d},\ \mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\big) .$$

Recalling Definition 2.4.6 of the control variable K_b, on the event $\mathrm{K}_b \geq \log r_\eta$, for $b \geq 0$ sufficiently large, Lemma 2.4.10 provides the following bound

$$\mathbb{P}_{0,u,b}\big(\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda,\ \mathrm{K}_b \geq \log r_\eta\big) \lesssim \frac{u}{b}e^{-c(\log\eta)^2} ,$$

for some constants $c > 0$. Hence, we can further restrict ourselves to the event $\{\mathrm{K}_b < \log r_\eta\}$. We recall that on this event, for each $l \in [b-1]_0$, it holds that $\mathrm{Osc}_l(B) \le \Theta_{\log r_\eta}(l)$. In particular, this implies that

$$\sup_{i,k\in[d],\, x\in\mathbb{B}_b} \left| \int_0^b e^{-2s} \mathfrak{K}_{i,k}(e^{-s}x) dB_s \right| \lesssim \sum_{l=0}^{b-1} e^{-2l} \,\mathrm{Osc}_l(B) \lesssim (\log \eta)^2 \,.$$

Hence, noting that the deterministic term appearing in $\partial^2_{i,k}\Upsilon_{b,\mathfrak{g}}$ is bounded by a constant independent of b, it holds that

$$\left\{ \mathrm{S}_{e^j} > c\eta^{2/d},\ \mathrm{K}_b < \log r_\eta \right\} \subseteq \left\{ \sup_{i,k\in[d],\, x\in\mathbb{B}_j} |\partial^2_{i,k}\mathrm{Z}_b(x) + \partial^2_{i,k}\mathfrak{g}_b(x)| \gtrsim \eta^{2/d} \right\} \overset{\text{def}}{=} \mathrm{E}_0 \,.$$

Now, we observe that, for $b \ge 0$ sufficiently large, thanks again to Lemma 2.4.10, it holds that

$$\mathbb{P}_{0,u,b}\big(\mathrm{K}_b < \log r_\eta,\ \mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \le \lambda,\ \mathrm{E}_0\big) \lesssim \frac{u}{b}\sqrt{\mathbb{P}(\mathrm{E}_0)} \,. \tag{2.4.61}$$

By Lemma 2.4.26 and (G3), there exist constants c_1, $c_2 > 0$ such that

$$\mathbb{P}(\mathrm{E}_0) \lesssim e^{dj - c_1\eta^{4/d}} \le e^{-c_2\eta^{4/d}} \,,$$

where the last inequality follows thanks to the fact that $j/2 \le \log r_\eta$. Combining the above bound with (2.4.61) yields the claim in this case.

Case 3: Finally, we consider the following case:

$$0 \le \log r_\eta < \frac{j}{2} \,. \tag{2.4.62}$$

Arguing as above, and using Lemma 2.4.10, by taking $b \ge 0$ sufficiently large, we can restrict ourselves to the event $\{\mathrm{K}_b < \log r_\eta\}$. We observe that, by (2.4.62) and Lemma 2.4.9,

$$\left\{ \mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \le \lambda,\ \mathrm{K}_b < \log r_\eta \right\} \subseteq \left\{ \bigcap_{j=1}^{b-1} \big\{ B_j \ge -\lambda - 2\mathrm{R}_{\mathrm{K}_b}(j) \big\},\ \mathrm{K}_b < \log r_\eta \right\} . \tag{2.4.63}$$

Now, recalling (2.4.56), we show that for suitable values of $r \in (0, r_\eta]$ and $M \ge (\log\eta)^2$, we can restrict to the event $\{\mathrm{S}_r \le M\}$. Indeed, arguing exactly as in the previous case, we have that

$$\left\{ \mathrm{S}_r > M,\ \mathrm{K}_b < \log r_\eta \right\} \subseteq \left\{ \sup_{i,k\in[d],\, x\in B(0,r)} |\partial^2_{i,k}\mathrm{Z}_b(x) + \partial^2_{i,k}\mathfrak{g}_b(x)| \gtrsim M \right\} \overset{\text{def}}{=} \mathrm{E}_1 \,.$$

By Lemma 2.4.26 and (G3), there exists a constant $a > 0$ such that

$$\mathbb{P}\big(\mathrm{E}_1\big) \lesssim r^d e^{-aM^2} \,. \tag{2.4.64}$$

By combining this with (2.4.13) and (2.4.17) of Lemma 2.4.10, we find that

$$\mathbb{P}_{0,u,b}\big(\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \le \lambda,\ \mathrm{K}_b < \log r_\eta,\ \mathrm{S}_r > M\big) \lesssim \frac{u}{b}\sqrt{\mathbb{P}(\mathrm{E}_1)} \,.$$

Hence, recalling (2.4.64), this implies that we can impose finitely many conditions of the type $\{\mathrm{S}_r \le M\}$ provided that the parameters r and M are such that $r^d e^{-aM^2} \lesssim \eta^{-\beta}$ for some $\beta > 2$. In our particular case,

we can use this to impose

$$\left\{ \mathrm{S}_{r_\eta} \le c_\lambda \eta^{\frac{2}{d}} \right\} \qquad \text{and} \qquad \left\{ \mathrm{S}_1 \le (\log \eta)^4 \right\}, \tag{2.4.65}$$

where $c_\lambda > 0$ is the same constant introduced in Remark 2.4.25. Therefore, it remains to show the bound in the statement for the event

$$\mathrm{E}_2 \overset{\text{def}}{=} \left\{ |\mathbb{D}^\lambda_{0,j}(\Upsilon_{b,\mathfrak{g}})|^{-1} \ge \eta,\ \mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \le \lambda,\ \mathrm{K}_b < \log r_\eta,\ \mathrm{S}_{r_\eta} \le c\eta^{\frac{2}{d}},\ S_1 \le (\log \eta)^4 \right\}.$$

We note now that the first condition in (2.4.65) implies that if the maximum is achieved within the ball of radius r_η, then there is a ball of area at least η^{-1} around that maximum where $\Upsilon_{b,\mathfrak{g}}$ is greater than its maximal value minus λ. This however cannot happen if $|\mathbb{D}^\lambda_{0,j}(\Upsilon_{b,\mathfrak{g}})|^{-1} \ge \eta$. In particular, since $\Upsilon_{b,\mathfrak{g}}(0) = 0$, we have shown that on the event E_2 there exists $x \in \mathbb{R}^d$ with $|x| \ge r_\eta$ such that $\Upsilon_{b,\mathfrak{g}}(x) \ge 0$. Since we are on the event $\{\mathrm{K}_b < \log r_\eta\}$, thanks to Lemma 2.4.9, there must be some $l \ge \log r_\eta$ such that $B_l \le \lambda + 2\mathrm{R}_{\log r_\eta}(l)$.

Similarly, the condition $\mathrm{S}_1 \le (\log \eta)^4$ implies that,

$$\mathrm{E}_2 \subseteq \left\{ \sup_{i\in[d]} |\partial_i \Upsilon_{b,\mathfrak{g}}(0)| \lesssim \eta^{-\frac{1}{d}} (\log \eta)^4 \right\}.$$

Otherwise, one could find a point $z \in \mathbb{R}^d$ at distance of order $\eta^{-1/d}$ from the origin such that $\Upsilon_{b,\mathfrak{g}}(z) \gtrsim \eta^{-2/d}(\log \eta)^4$, which implies that the values of $\Upsilon_{b,\mathfrak{g}}$ inside a ball of radius $\eta^{-1/d}$ around z are all positive. But since we are on the event $\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \le \lambda\}$, these points are contained in $\mathbb{D}^\lambda_{0,j}(\Upsilon_{b,\mathfrak{g}})$, leading again to a contradiction.

Now, we note that, thanks to (K2), it holds that the gradient of $\mathfrak{K}$ at the origin is zero, and so $\partial_i \Upsilon_{b,\mathfrak{g}}(0)$ is independent of B. Combining these considerations with (2.4.63), it follows that

$$\begin{aligned} \mathbb{P}_{0,u,b}(\mathrm{E}_2) \le & \mathbb{P}_{0,u,b}\left(\sup_{i\in[d]} |\partial_i \Upsilon_{b,\mathfrak{g}}(0)| \lesssim \eta^{-\frac{1}{d}} (\log \eta)^4 \right) \\ & \cdot \mathbb{P}_{0,u,b}\left(\bigcap_{l=1}^{b-1} \{B_l \ge -\lambda - 2\mathrm{R}_{\log r_\eta}(l)\},\ \bigcup_{l=\log r_\eta}^{b-1} \{B_l \le \lambda + 2\mathrm{R}_{\log r_\eta}(l)\} \right). \end{aligned} \tag{2.4.66}$$

We can now apply Lemma 2.A.11 to bound the second probability, showing that for $j_0 \ge 0$ sufficiently large, $b \ge j_0$ sufficiently large, and for any $j \in [j_0, b]$, it holds that

$$\mathbb{P}_{0,u,b}(\mathrm{E}_2) \lesssim \frac{u}{b} \eta^{-\frac{1}{16d}} \mathbb{P}_{0,u,b}\left(\sup_{i\in[d]} |\partial_i \Upsilon_{b,\mathfrak{g}}(0)| \lesssim \eta^{-\frac{1}{d}} (\log \eta)^4 \right).$$

In order to bound the remaining factor, we note that $\partial_i \Upsilon_{b,\mathfrak{g}}(0) = \partial_i \mathrm{Z}_b(0) + \partial_i \mathfrak{g}_b(0)$ and that, by (2.3.13),

$$\mathbb{E}\left[\partial_i \mathrm{Z}_b(0) \partial_j \mathrm{Z}_b(0) \right] = -\partial^2_{ij} \mathfrak{K}(0) \int_0^b e^{-2s}\, ds\,. \tag{2.4.67}$$

Since $(\partial_i \mathrm{Z}_b(0))_{i\in[d]}$ are jointly Gaussian, it follows from Remark 2.1.2 and (2.4.67) that

$$\mathbb{P}_{0,u,b}\left(\sup_{i\in[d]} |\partial_i \Upsilon_{b,\mathfrak{g}}(0)| \le \varepsilon \right) \lesssim \varepsilon^d\,, \qquad \forall\, \varepsilon \in (0,1]\,, \tag{2.4.68}$$

uniformly over $b \geq 0$ and $u \in [b^{1/4}, b^{3/4}]$. Therefore, we conclude that (2.4.6) is bounded by some constant times $(u/b)\eta^{-(1+\delta)}$ for any $\delta \in (0, 1/(16d))$.

To conclude, we note that (2.4.54) follows immediately from (2.4.53). □

We also need the following version of Lemma 2.4.22, where the end point of the Brownian bridge is taken to be less than $b^{1/4}$.

Lemma 2.4.27. *Let $A > 0$ be fixed. There exists $\delta = \delta(d) \in (0,1)$ such that for any $b \geq 0$ sufficiently large, it holds for any $\lambda > 0$, $u \in [-A, b^{1/4}]$, and $\eta \geq 0$,*

$$\mathbb{P}_{0,u,b}\Big(|\mathbb{D}^{\lambda}_{0,b}(\Upsilon_{b,\mathfrak{g}})|^{-1} \geq \eta,\ \mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}}) \leq \lambda\Big) \lesssim \frac{1}{\sqrt{b}}\eta^{-(1+\delta)}\,. \tag{2.4.69}$$

Moreover, for all $\sigma \in [0, \delta)$, it holds that

$$\mathbb{E}_{0,u,b}\Big[|\mathbb{D}^{\lambda}_{0,b}(\Upsilon_{b,\mathfrak{g}})|^{-(1+\sigma)}\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,\mathfrak{g}})\leq\lambda\}}\Big] \lesssim \frac{1}{\sqrt{b}}\,. \tag{2.4.70}$$

Proof. We do not provide a detailed proof of this lemma, as it closely follows that of Lemma 2.4.22. Instead, we outline the necessary modifications to that proof.

First, in the regime $\log r_\eta \geq b/2$, we can proceed exactly as in the proof of Lemma 2.4.22. Regarding the regime $\log r_\eta < b/2$, it suffices to use Lemma 2.4.11 in place of Lemma 2.4.10 to restrict to the event $\{\mathrm{K}_b < \log r_\eta\}$. Similarly, we obtain that the probability on the left-hand side of (2.4.6) is bounded above by a multiple of $b^{-1/2}\sqrt{\mathbb{P}(\mathrm{E}_1)}$, when $b \geq 0$ is large enough.

Hence, it remains to bound $\mathbb{P}_{0,u,b}(\mathrm{E}_2)$ appearing on the left-hand side of the first line in (2.4.66). To this end, arguing as in the proof of [BL18, Lemma 4.16] and using (2.A.2), we observe that the probability on the right-hand side of the second line in (2.4.66) can be bounded from above by

$$\mathbb{P}_{0,u,b}\Bigg(\bigcap_{j=1}^{b-1}\big\{B_j \geq -\lambda - 2\mathrm{R}_{\log r_\eta}(j)\big\}\Bigg) \lesssim \mathbb{P}_{0,u,b}\Big(\inf_{s\in[0,b]} B_s \gtrsim -(\log b)^2\Big) \lesssim b^{-5/8}\,,$$

where, to derive the last inequality, we used the fact that $u \leq b^{1/4}$. Finally, since we are in the regime $\log r_\eta < b/2$, it follows that $b^{-5/8}$ is at most a multiple of $b^{-1/2}\eta^{-1/(8d)}$. Combining this with the bound in (2.4.68) completes the proof of (2.4.69). To conclude, we note that (2.4.70) follows immediately from (2.4.69). □

Finally, we also have the following result.

Lemma 2.4.28. *For $\lambda > 0$ and for $b > 0$ sufficiently large, it holds that*

$$\mathbb{E}_{0,u,b}\big[|\mathbb{D}^{\lambda}_{0,b}(\Upsilon_{b,\mathfrak{g}})|^{-1}\big] \lesssim \begin{cases} b^{d^2}\,, & \text{if } u \in [b^{3/4}, b^{2d}]\,,\\ (u/b)^{2d}\,, & \text{if } u > b^{2d}\,. \end{cases}$$

Proof. We begin by recalling that, for $r > 0$, the quantity S_r is defined in (2.4.56). Furthermore, we also recall that, thanks to Remark 2.4.25, for all $\eta > 0$, it holds that

$$\mathbb{P}_{0,u,b}(|\mathbb{D}^{\lambda}_{0,b}(\Upsilon_{b,\mathfrak{g}})|^{-1} \geq \eta) \leq \mathbb{P}_{0,u,b}\big(\mathrm{S}_{e^b} > c_\lambda \eta^{2/d}\big)\,,$$

for some constant $c_\lambda > 0$ depending only on λ. We divide the proof into two disjoint cases.

Case 1: We begin by considering the case $u \in [b^{3/4}, b^{2d}]$. In this case, using Lemma 2.4.26, there exist constants c_1, c_2, $c_3 > 0$ such that, for any $b > 0$ sufficiently large,

$$\begin{aligned}\mathbb{E}_{0,u,b}\big[|\mathbb{D}^{\lambda}_{0,b}(\Upsilon_{b,\mathfrak{g}})|^{-1}\big] &\leq b^{d^2} + \int_{b^{d^2}}^{\infty} \mathbb{P}_{0,u,b}\big(S_{e^b} > c_\lambda \eta^{2/d}\big)\, d\eta \\ &\lesssim b^{d^2} + e^{db} \int_{b^{d^2}}^{\infty} e^{c_1 \frac{u}{b} \eta^{2/d} - c_2 \eta^{4/d}}\, d\eta \\ &\lesssim b^{d^2} + e^{db} \int_{b^{d^2}}^{\infty} e^{-c_3 \eta^{4/d}} d\eta \lesssim b^{d^2}\ .\end{aligned}$$

Case 2: We now focus on the case $u > b^{2d}$. Proceeding similarly to the previous case, we obtain that there exist constants c_1, c_2, $c_3 > 0$ such that, for any $b > 0$ sufficiently large,

$$\begin{aligned}\mathbb{E}_{0,u,b}\big[|\mathbb{D}^{\lambda}_{0,b}(\Upsilon_{b,\mathfrak{g}})|^{-1}\big] &\lesssim (u/b)^{2d} + e^{db} \int_{(u/b)^{2d}}^{\infty} e^{c_1 \frac{u}{b} \eta^{2/d} - c_2 \eta^{4/d}}\, d\eta \\ &\lesssim (u/b)^{2d} + e^{db} \int_{(u/b)^{2d}}^{\infty} e^{-c_3 \eta^{4/d}} d\eta \lesssim (u/b)^{2d}\ ,\end{aligned}$$

which completes the proof. □

2.5 Resampling property and independence from the threshold

The main goal of this section is to present some results concerning the fields $\widetilde{\Upsilon}_\lambda$ and Ψ_λ introduced in Theorem 2.A and Definition 2.2.8, respectively. In particular, in Section 2.5.1, we prove some key technical results related to the field $\widetilde{\Upsilon}_\lambda$. Then, in Section 2.5.2, we establish the "resampling property" of the field $\widetilde{\Upsilon}_\lambda$, as stated in Proposition 2.2.10, from which Proposition 2.2.12 follows directly. Finally, in Section 2.5.3, we establish the independence of certain quantities of interest from λ. In particular, we prove Proposition 2.2.14.

2.5.1 Some technical results

The main goal of this section is to establish a result concerning the limiting "shape field" Υ_λ introduced in Theorem 2.A, which will play a crucial role in the proof of some of our main theorems. Specifically, the main goal of this section is to prove the following lemma.

Lemma 2.5.1. *There exists a countable set $\Lambda^c \subseteq \mathbb{R}^+$ such that for each $\lambda, \theta \in \Lambda = \mathbb{R}^+ \setminus \Lambda^c$ it holds that*

$$\mathbb{P}\big(\mathbb{M}(\widetilde{\Upsilon}_\lambda) = \theta\big) = 0\ . \tag{2.5.1}$$

Moreover, for all $\lambda, \theta \in \Lambda$, it holds that

$$\mathbb{P}\big(|\{x \in \mathbb{R}^d\ :\ \widetilde{\Upsilon}_\lambda(x) - \mathbb{M}(\widetilde{\Upsilon}_\lambda) = -\theta\}| > 0\big) = 0\ . \tag{2.5.2}$$

Furthermore, for each $\lambda_1 \neq \lambda_2 \in \Lambda$ such that $\lambda_2 < \lambda_1$, the field $\widetilde{\Upsilon}_{\lambda_2}$ has the same law of the field $\widetilde{\Upsilon}_{\lambda_1}$ conditioned on the event $\mathbb{M}(\widetilde{\Upsilon}_{\lambda_1}) \leq \lambda_2$.

Remark 2.5.2. It follows from the proof of Lemma 2.5.1 that we may (and will) assume that, for any $\lambda, \theta \in \Lambda$, the identities (2.5.1) and (2.5.2) hold also with $\mathbb{M}(\widetilde{\Upsilon}_\lambda)$ replaced by $\mathbb{M}_{0,k}(\widetilde{\Upsilon}_\lambda)$ for some fixed $k \geq 0$.

In order to prove Lemma 2.5.1, we need to introduce some notation. We begin by defining the following space

$$\mathbb{X} \stackrel{\text{def}}{=} \left\{(b,\phi) \,:\, b \in \mathbb{R}_0^+ \cup \{\infty\},\ \phi \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}\cup\{-\infty\})\right\} .$$

We equip the space $\mathbb{X}$ with the pseudometrics $\mathrm{d}_{\mathbb{X}}$ and $\overline{\mathrm{d}}_{\mathbb{X}}$ defined as follows

$$\mathrm{d}_{\mathbb{X}}\big((b_1,\phi_1),(b_2,\phi_2)\big) \stackrel{\text{def}}{=} |e^{-b_1} - e^{-b_2}| + \sum_{k=1}^{\infty} 2^{-k}\left(1 \wedge \sup_{x\in\mathbb{B}_k\cap\mathbb{B}_{b_1\wedge b_2}} |\phi_1(x)-\phi_2(x)|\right) ,$$
$$\overline{\mathrm{d}}_{\mathbb{X}}\big((b_1,\phi_1),(b_2,\phi_2)\big) \stackrel{\text{def}}{=} |e^{-b_1} - e^{-b_2}| + \sup_{x\in\mathbb{R}^d} |\chi_{b_1}(x)e^{c\phi_1(x)} - \chi_{b_2}(x)e^{c\phi_2(x)}| ,$$

where $c = c(d) \stackrel{\text{def}}{=} \sqrt{2/d}$, and for each $b \in \mathbb{R}_0^+\cup\{\infty\}$, $\chi_b : \mathbb{R}^d \to \mathbb{R}$ is a smooth function such that $\chi_b(x) = 1$ for all $|x| \le e^b$ and $\chi_b(x) = 0$ for all $|x| \ge e^b + 1$.

For each $\lambda > 0$, we let $\mathbb{A}_\lambda \subseteq \mathbb{X}$ be the set defined as follows

$$\mathbb{A}_\lambda \stackrel{\text{def}}{=} \left\{(b,\phi) \in \mathbb{X} \,:\, \mathbb{M}_{0,b}(\phi) \le \lambda\right\} .$$

Furthermore, we define the collection of measures $(\nu_{b,\lambda})_{b>0}$ and the measure $\nu_{\infty,\lambda}$ as follows

$$\nu_{b,\lambda} \stackrel{\text{def}}{=} c_{\star,\lambda}\sqrt{b}\,\mathrm{Law}\big[\big(b,\Upsilon_b\big)\big]|_{\mathbb{A}_\lambda} \qquad \text{and} \qquad \nu_{\infty,\lambda} = \mathrm{Law}\big[\big(\infty,\widetilde{\Upsilon}_\lambda\big)\big] ,$$

where $\cdot|_{\mathbb{A}_\lambda}$ denotes the restriction to the subspace $\mathbb{A}_\lambda$, and $c_{\star,\lambda}$ is the constant introduced in Theorem 2.B.

Remark 2.5.3. The reason we introduced the "stronger" pseudometric $\overline{\mathrm{d}}_{\mathbb{X}}$ is that the boundaries of the set $\mathbb{A}_\lambda$ are *not* disjoint in $(\mathbb{X},\mathrm{d}_{\mathbb{X}})$ for different values of λ. In contrast, in $(\mathbb{X},\overline{\mathrm{d}}_{\mathbb{X}})$, we have $\overline{\partial}_{\mathbb{X}}\mathbb{A}_{\lambda_1}\cap\overline{\partial}_{\mathbb{X}}\mathbb{A}_{\lambda_2} = \emptyset$ for any $\lambda_1,\lambda_2 > 0$ with $\lambda_1 \neq \lambda_2$. This fact can be easily checked and will play an important role in what follows.

Lemma 2.5.4. *For each $\lambda > 0$, the collection of measures $(\nu_{b,\lambda})_{b>0}$ converges weakly in $(\mathbb{X},\overline{\mathrm{d}}_{\mathbb{X}})$ to $\nu_{\infty,\lambda}$ as $b \to \infty$.*

Proof. Let $(\infty,\widetilde{\Upsilon}_\lambda)$ be sampled from $\nu_{\infty,\lambda}$, and with a slight abuse of notation, let $(b,\widetilde{\Upsilon}_{b,\lambda})$ be sampled from $\nu_{b,\lambda}$. Thanks to Theorem 2.A, we know that $\nu_{b,\lambda}$ converges weakly to $\nu_{\infty,\lambda}$ in $(\mathbb{X},\mathrm{d}_{\mathbb{X}})$ as $b\to\infty$. This fact implies that we can find a coupling between the sequence $((b,\widetilde{\Upsilon}_b))_{b>0}$ and $(\infty,\widetilde{\Upsilon}_\lambda)$ such that

$$\lim_{b\to\infty} \mathbb{E}\big[\mathrm{d}_{\mathbb{X}}\big((b,\widetilde{\Upsilon}_{b,\lambda}),(\infty,\widetilde{\Upsilon}_\lambda)\big)\big] = 0 . \tag{2.5.3}$$

Therefore, to obtain the desired result, we need to check that the same limit holds if we replace the pseudometric $\mathrm{d}_{\mathbb{X}}$ in the above display with the pseudometric $\overline{\mathrm{d}}_{\mathbb{X}}$. In particular, it suffices to show that

$$\lim_{b\to\infty} \mathbb{E}\left[\sup_{x\in\mathbb{R}^d} |\chi_b(x)e^{c\widetilde{\Upsilon}_{b,\lambda}(x)} - e^{c\widetilde{\Upsilon}_\lambda(x)}|\right] = 0$$

To this end, we let $0 \le j \le b$, and we note that

$$\sup_{x\in\mathbb{R}^d} |\chi_b(x)e^{c\widetilde{\Upsilon}_{b,\lambda}(x)} - e^{c\widetilde{\Upsilon}_\lambda(x)}| \le \sup_{x\in\mathbb{B}_j} |e^{c\widetilde{\Upsilon}_{b,\lambda}(x)} - e^{c\widetilde{\Upsilon}_\lambda(x)}| + \sup_{x\in\mathbb{R}^d\setminus\mathbb{B}_j} |\chi_b(x)e^{c\widetilde{\Upsilon}_{b,\lambda}(x)} - e^{c\widetilde{\Upsilon}_\lambda(x)}| .$$

The fact that the limit as $b\to\infty$ of the expectation of the first term on the right-hand side of the above display goes to zero follows from (2.5.3). Hence, it remains to show that the expectation of the second term on the right-hand side of the above display also tends to zero as $b\to\infty$, followed by $j\to\infty$. Thanks to

Lemmas 2.4.12 and 2.4.20, it holds that

$$\lim_{j\to\infty}\lim_{b\to\infty}\mathbb{E}\left[\sup_{x\in\mathbb{B}_b\setminus\mathbb{B}_j}e^{c\widetilde{\Upsilon}_{b,\lambda}(x)}\right]=0\,,\qquad\lim_{j\to\infty}\mathbb{E}\left[\sup_{x\in\mathbb{R}^d\setminus\mathbb{B}_j}e^{c\widetilde{\Upsilon}_{\lambda}(x)}\right]=0\,.$$

Therefore, recalling that $\chi_b(x)=0$ for all $|x|\geq e^b+1$, it remains to check that

$$\lim_{b\to\infty}\mathbb{E}\left[\sup_{|x|\in[e^b,e^b+1)}e^{c\widetilde{\Upsilon}_{b,\lambda}(x)}\right]=0\,. \tag{2.5.4}$$

In particular, thanks to Lemma 2.4.12, for $b\geq 0$ large enough, by recalling the definition (2.4.2) of the field Υ_b and since, by (K2), the function $\mathfrak{K}$ is supported in $B(0,1)$, the expectation on the left-hand side of the above display is bounded above by a multiple of

$$\begin{aligned}\sqrt{b}\mathbb{E}\left[\sup_{|x|\in[e^b,e^b+1)}e^{c\Upsilon_b(x)}\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b)\leq\lambda\}}\right]&\leq\sqrt{b}\mathbb{E}\left[e^{-c(B_b+\sqrt{2d}b)}\sup_{|x|\in[e^b,e^b+1)}e^{cZ_b(x)}\right]\\&=\sqrt{b}e^{c^2b/2-\sqrt{2d}cb}\mathbb{E}\left[\sup_{|x|\in[e^b,e^b+1)}e^{cZ_b(x)}\right],\end{aligned} \tag{2.5.5}$$

and thus, we are left with the task of bounding the expectation in the final term of the above display. To achieve this, we start by noting that, by the Borell-TIS inequality (Lemma A.0.3) and Fernique's majorizing criterion (Lemma A.0.4), for any $y\in\mathbb{R}^d$ such that $B(y,1)\subseteq\mathbb{R}^d\setminus\mathbb{B}_b$, it holds that

$$\mathbb{P}\left(\sup_{x\in B(y,1)}|Z_b(x)|\geq\eta\right)\lesssim e^{-\frac{\eta^2}{2b}}\,,\qquad\forall\eta\geq 0\,.$$

Consequently, thanks to the union bound, we get that

$$\mathbb{P}\left(\sup_{|x|\in[e^b,e^b+1)}|Z_b(x)|\geq\eta\right)\lesssim e^{b(d-1)-\frac{\eta^2}{2b}}\,,\qquad\forall\eta\geq 0\,. \tag{2.5.6}$$

Next, observing that for any non-negative random variable X it holds that $\mathbb{E}[e^{cX}]=1+c\int_0^\infty e^{cx}\mathbb{P}(X>x)dx$, we can write

$$\mathbb{E}\left[\sup_{|x|\in[e^b,e^b+1)}e^{cZ_b(x)}\right]=1+c\int_0^\infty e^{c\eta}\mathbb{P}\left(\sup_{|x|\in[e^b,e^b+1)}Z_b(x)\geq\eta\right)d\eta\,.$$

In particular, for $k>0$, using the fact that $\eta^2\geq 2k\eta-k^2$, we have that

$$\begin{aligned}\int_0^\infty e^{c\eta}\left(1\wedge e^{b(d-1)-\frac{\eta^2}{2b}}\right)d\eta&\leq\int_0^k e^{c\eta}d\eta+\int_k^\infty e^{c\eta+b(d-1)-\frac{k\eta}{b}+\frac{k^2}{2b}}d\eta\\&\leq ke^{ck}+\frac{b}{k-cb}e^{ck+b(d-1)-\frac{k^2}{2b}}\,,\end{aligned}$$

provided that $k>bc$. It is natural to choose k such that $k^2/(2b)=b(d-1)$, i.e., $k=b\sqrt{2(d-1)}$ which is indeed greater than bc since $c=\sqrt{2/d}<\sqrt{2(d-1)}$. Therefore, recalling (2.5.6) and combining the previous observations, we conclude that

$$\mathbb{E}\left[\sup_{|x|\in[e^b,e^b+1)}e^{cZ_b(x)}\right]\lesssim be^{\sqrt{2(d-1)}cb}\,.$$

Therefore, by plugging the previous bound into (2.5.5), we showed that the expectation in (2.5.4) is bounded above by a multiple of

$$\mathbb{E}\left[\sup_{|x|\in[e^b,e^b+1)} e^{c\widetilde{\Upsilon}_{b,\lambda}(x)}\right] \lesssim b^{\frac{3}{2}} e^{bc(c/2+\sqrt{2(d-1)}-\sqrt{2d})} \to 0 \quad \text{as} \quad b\to\infty\,,$$

where, recalling that $c=\sqrt{2/d}$, we have used the fact that $c/2+\sqrt{2(d-1)}-\sqrt{2d}\le 0$. □

We are now ready to prove the following intermediate result.

Lemma 2.5.5. *There exists a countable set $\overline{\Lambda}^c\subseteq\mathbb{R}^+$ such that, letting $\overline{\Lambda}=\mathbb{R}^+\setminus\overline{\Lambda}^c$, for each $n\in\mathbb{N}$ and $\lambda\in\overline{\Lambda}\cap(0,n]$, it holds that $\nu_{\infty,\lambda}=\nu_{\infty,n}|_{\mathbb{A}_\lambda}$.*

Proof. Thanks to Lemma 2.5.4, we know that for each $\lambda>0$, the sequence $(\nu_{b,\lambda})_{b>0}$ converges weakly to $\nu_{\infty,\lambda}$ in $(\mathbb{X},\bar{d}_{\mathbb{X}})$. Moreover, for all $n\in\mathbb{N}$, $\lambda\in(0,n]$, and $b>0$, we have that $\nu_{b,\lambda}=\nu_{b,n}|_{\mathbb{A}_\lambda}$. We note that there exists a countable set $\overline{\Lambda}_n^c\subseteq(0,n]$ such that for all $\lambda\in(0,n]\setminus\overline{\Lambda}_n^c$, it holds that $\nu_{b,\lambda}$ converges weakly to $\nu_{\infty,n}|_{\mathbb{A}_\lambda}$ in $(\mathbb{X},\bar{d}_{\mathbb{X}})$. Indeed, this follows from the uniqueness of the weak limit and the fact that the sets $(\bar{\partial}_{\mathbb{X}}\mathbb{A}_\lambda)_{\lambda\in(0,n]}$ are disjoint, implying that $\nu_{\infty,n}$ can assign positive mass to at most countable many of them. Therefore, the conclusion follows by setting $\overline{\Lambda}^c=\bigcup_{n\in\mathbb{N}}\overline{\Lambda}_n^c$ and noting that a countable union of countable sets remains countable. □

We are now ready to prove Lemma 2.5.1

Proof of Lemma 2.5.1. We begin by observing that an immediate consequence of Lemma 2.5.5 is that for each $\lambda_1\neq\lambda_2\in\overline{\Lambda}$ such that $\lambda_2<\lambda_1$, the field $\widetilde{\Upsilon}_{\lambda_2}$ has the same law of the field $\widetilde{\Upsilon}_{\lambda_1}$ conditioned on the event $\mathbb{M}(\widetilde{\Upsilon}_{\lambda_1})\le\lambda_2$.

For $\theta>0$, we introduce the set $\mathbb{S}_{\theta,1}\subseteq\mathbb{X}$ by letting

$$\mathbb{S}_{\theta,1}\stackrel{\text{def}}{=}\left\{(\infty,\phi)\in\mathbb{X}\,:\,\mathbb{M}(\phi)=\theta\right\}.$$

Since the sets $(\mathbb{S}_{\theta,1})_{\theta>0}$ are disjoint, for all $n\in\mathbb{N}$, there exists a countable set $\overline{\Lambda}_{1,n}^c\subseteq\overline{\Lambda}$ such that for all $\theta\in\overline{\Lambda}\setminus\overline{\Lambda}_{1,n}^c$, it holds that $\nu_{\infty,n}(\mathbb{S}_{\theta,1})=0$. Indeed, if that were not the case, the measure $\nu_{\infty,n}$ would assign positive measure to uncountable many disjoint sets, which is, of course, not possible. Furthermore, thanks to Lemma 2.5.5, we know that for each $\lambda\in\overline{\Lambda}\cap(0,n]$, it holds that $\nu_{\infty,\lambda}=\nu_{\infty,n}|_{\mathbb{A}_\lambda}$. In particular, this implies that for all $\lambda,\theta\in\overline{\Lambda}\setminus\cup_{n\in\mathbb{N}}\overline{\Lambda}_{1,n}^c$, it holds that $\nu_{\infty,\lambda}(\mathbb{S}_{\theta,1})=0$.

Next, for $\theta>0$, we introduce the set $\mathbb{S}_{\theta,2}\subseteq\mathbb{X}\subseteq\overline{\mathbb{X}}$ by letting

$$\mathbb{S}_{\theta,2}\stackrel{\text{def}}{=}\left\{(\infty,\phi)\in\mathbb{X}\,:\,|\{x\in\mathbb{R}^d\,:\,\phi(x)-\mathbb{M}(\phi)=-\theta\}|>0\right\}$$

Consider the mapping $\Theta:\mathbb{X}\times\mathbb{R}^d\to\mathbb{R}$ defined as $\Theta((b,\phi),x)=\phi(x)-\mathbb{M}_{0,b}(\phi)$. For $n\in\mathbb{N}$ and a standard multivariate normal random variable $\mathcal{N}(0,I_d)$, let $\nu_{\infty,n}^\star$ be the pushforward of $\nu_{\infty,n}\otimes\mathcal{N}(0,I_d)$ under Θ, which is a measure on $\mathbb{R}_0^-$. We note that there exists a countable set $\overline{\Lambda}_{2,n}^c\subseteq\overline{\Lambda}$ such that for all $\theta\in\overline{\Lambda}\setminus\overline{\Lambda}_{2,n}^c$, it holds that $\nu_{\infty,n}^\star(\theta)=0$. This is due to the fact that the measure $\nu_{\infty,n}^\star$ can only have at most countable many point masses. In turn, this implies that for all $\theta\in\overline{\Lambda}\setminus\overline{\Lambda}_{2,n}^c$, it holds that $\nu_{\infty,n}(\mathbb{S}_{\theta,2})=0$. Now, we observe once again that thanks to Lemma 2.5.5, for each $\lambda\in\overline{\Lambda}\cap(0,n]$, it holds that $\nu_{\infty,\lambda}=\nu_{\infty,n}|_{\mathbb{A}_\lambda}$. Therefore, this implies that for all $\lambda,\theta\in\overline{\Lambda}\setminus\cup_{n\in\mathbb{N}}\overline{\Lambda}_{2,n}^c$, it holds that $\nu_{\infty,\lambda}(\mathbb{S}_{\theta,2})=0$. Finally, combining the previous discussions, the conclusion follows by setting $\Lambda\stackrel{\text{def}}{=}\overline{\Lambda}\setminus\bigcup_{n\in\mathbb{N}}(\overline{\Lambda}_{1,n}^c\cup\overline{\Lambda}_{2,n}^c)$. □

We conclude this section by proving that the proportionality constant appearing in Definition 2.2.8 lies in the interval $(0,\infty)$. Specifically, it suffices to establish the following result.

Lemma 2.5.6. *For all $\lambda \in \Lambda$, it holds that $\mathbb{E}[|\mathbb{D}^{\lambda}(\widetilde{\Upsilon}_{\lambda})|^{-1}] \in (0,\infty)$.*

Proof. The fact that $\mathbb{E}[|\mathbb{D}^{\lambda}(\widetilde{\Upsilon}_{\lambda})|^{-1}] > 0$ follows immediately from the continuity of the field $\widetilde{\Upsilon}_{\lambda}$ and from its almost sure decay at infinity, as implied by Lemma 2.4.20 (see also Remark 2.4.21). Hence, it remains to verify that $\mathbb{E}[|\mathbb{D}^{\lambda}(\widetilde{\Upsilon}_{\lambda})|^{-1}] < \infty$. To this end, fix $0 < j \leq b$. Then, using Lemma 2.4.22 (see also Remark 2.4.23) and proceeding exactly as in the proof of Lemma 2.7.12 below, one obtains that

$$\lim_{L\to\infty} \limsup_{b\to\infty} \sqrt{b}\Big|\mathbb{E}\big[|\mathbb{D}^{\lambda}_{0,j}(\Upsilon_b)|^{-1}\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b)\leq\lambda\}}\big] - \mathbb{E}\big[\big(|\mathbb{D}^{\lambda}_{0,j}(\Upsilon_b)|^{-1}\wedge L\big)\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b)\leq\lambda\}}\big]\Big| = 0\,.$$

Since $\lambda \in \Lambda$, we have that the set of discontinuities of the mapping $\mathcal{C}(\mathbb{R}^d) \ni \phi \mapsto |\mathbb{D}^{\lambda}_{0,j}(\phi)|^{-1}\wedge L$ is assigned measure zero by the law of the field $\widetilde{\Upsilon}_{\lambda}$. Therefore, we are in a position to apply Theorems 2.A and 2.B from which we can deduce that

$$\lim_{b\to\infty} \sqrt{b}\mathbb{E}\big[|\mathbb{D}^{\lambda}_{0,j}(\Upsilon_b)|^{-1}\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b)\leq\lambda\}}\big] = \alpha c_{\star,\lambda}\mathbb{E}\big[|\mathbb{D}^{\lambda}_{0,j}(\widetilde{\Upsilon}_{\lambda})|^{-1}\big]\,.$$

By invoking the monotone convergence theorem and Lemma 2.4.22 (see also Remark 2.4.23), the claim follows by taking the limit as $j\to\infty$ on both sides of the above expression. □

2.5.2 A resampling property

In what follows, for $x \in \mathbb{R}^d$, we recall the definition (2.2.8) of the shift operator τ_x. We begin with the proof of the "resampling property" stated in Proposition 2.2.10 for the field $\widetilde{\Upsilon}_{\lambda}$. We will use the notation introduced in (2.3.2), (2.3.3), and (2.3.4). Furthermore, we recall that $\Lambda \subseteq \mathbb{R}^+$ denotes the set introduced in Lemma 2.5.1.

Proof of Proposition 2.2.10. For $b \in \mathbb{N}$, we recall that $\Upsilon_b = \Phi_b - \sqrt{2d}\mathfrak{a}_b$, where the field Φ_b is such that, for all $x, y \in \mathbb{R}^d$,

$$\mathbb{E}\big[\Phi_b(x)\Phi_b(y)\big] = \mathfrak{a}_b(x) + \mathfrak{a}_b(y) - \mathfrak{a}_b(x-y)\,,$$

and where the function $\mathfrak{a}_b$ is defined as in (2.2.2). In particular, the field Φ_b is shift invariant in the sense that $\tau_x\Phi_b$ has the same law as Φ_b for every $x \in \mathbb{R}^d$. Then, by applying the Cameron–Martin theorem (see Lemma A.0.1), we have that for all $\mathbf{F} : \mathcal{C}(\mathbb{R}^d) \to \mathbb{R}$ and $y \in \mathbb{R}^d$, it holds that

$$\begin{aligned}
\mathbb{E}\big[\mathbf{F}\big(\Upsilon_b\big)\big] &= \mathbb{E}\big[\mathbf{F}\big(\Phi_b - \sqrt{2d}\mathfrak{a}_b\big)\big] \\
&= \mathbb{E}\Big[\mathbf{F}\big(\Phi_b + \sqrt{2d}\mathbb{E}[\Phi_b(\cdot)\Phi_b(y)] - \sqrt{2d}\mathfrak{a}_b\big)e^{-\sqrt{2d}\Phi_b(y) - 2d\mathfrak{a}_b(y)}\Big] \\
&= \mathbb{E}\Big[\mathbf{F}\big(\tau_{-y}\Phi_b + \sqrt{2d}\mathbb{E}[\Phi_b(\cdot)\Phi_b(y)] - \sqrt{2d}\mathfrak{a}_b\big)e^{-\sqrt{2d}\tau_{-y}\Phi_b(y) - 2d\mathfrak{a}_b(y)}\Big] \\
&= \mathbb{E}\Big[\mathbf{F}\big(\tau_{-y}\Upsilon_b + \sqrt{2d}\tau_{-y}\mathfrak{a}_b + \sqrt{2d}\mathbb{E}[\Phi_b(\cdot)\Phi_b(y)] - \sqrt{2d}\mathfrak{a}_b\big)e^{\sqrt{2d}\Phi_b(-y) - 2d\mathfrak{a}_b(y)}\Big] \\
&= \mathbb{E}\Big[\mathbf{F}\big(\tau_{-y}\Upsilon_b\big)e^{\sqrt{2d}\Upsilon_b(-y)}\Big]\,.
\end{aligned} \tag{2.5.7}$$

Now, for each $\lambda \in \Lambda$, $\mathbf{F} \in \mathcal{C}^b_{\mathrm{loc}}(\mathcal{C}(\mathbb{R}^d))$, and $0 \leq k \leq b$, we can write

$$\mathbb{E}\Big[\mathbf{F}\big(\Upsilon_b\big)\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b)\leq\lambda\}}\Big] = \mathbb{E}\left[\frac{\int_{\mathbb{B}_k}\mathbf{F}(\Upsilon_b)\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b)\leq\lambda\}}e^{\sqrt{2d}(\Upsilon_b(x)-\mathbb{M}_{x,b}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(x)\geq\mathbb{M}_{x,b}(\Upsilon_b)-\lambda\}}\,dx}{\int_{\mathbb{B}_k}e^{\sqrt{2d}(\Upsilon_b(y)-\mathbb{M}_{y,b}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(y)\geq\mathbb{M}_{y,b}(\Upsilon_b)-\lambda\}}dy}\right],$$

Therefore, using (2.5.7), the expectation on the right-hand side of the previous display is equal to

$$\int_{\mathbb{B}_k} \mathbb{E}\left[\frac{\mathbf{F}(\tau_{-x}\Upsilon_b)\mathbb{1}_{\{\mathbb{M}_{-x,b}(\Upsilon_b)-\Upsilon_b(-x)\le\lambda\}}e^{\sqrt{2d}(\Upsilon_b(-x)-\mathbb{M}_{0,b}(\Upsilon_b))}\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b)\le\lambda\}}}{\int_{\mathbb{B}_k} e^{\sqrt{2d}(\Upsilon_b(y-x)-\mathbb{M}_{y-x,b}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(y-x)\ge\mathbb{M}_{y-x,b}(\Upsilon_b)-\lambda\}}dy}\right]dx$$
$$= \mathbb{E}\left[\int_{\mathbb{B}_k}\frac{\mathbf{F}(\tau_x\Upsilon_b)e^{\sqrt{2d}(\Upsilon_b(x)-\mathbb{M}_{0,b}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(x)\ge\mathbb{M}_{x,b}(\Upsilon_b)-\lambda\}}}{\int_{\mathbb{B}_k(x)} e^{\sqrt{2d}(\Upsilon_b(y)-\mathbb{M}_{y,b}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(y)\ge\mathbb{M}_{y,b}(\Upsilon_b)-\lambda\}}dy}dx\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b)\le\lambda\}}\right]. \tag{2.5.8}$$

We observe that the integral over x inside the expectation in (2.5.8) is bounded above by a positive finite constant. Indeed, since the function $\mathbf{F}$ is bounded by assumption and the exponential terms are easily bounded, it suffices to verify that

$$\int_{\mathbb{B}_k}\frac{\mathbb{1}_{\{\Upsilon_b(x)\ge\mathbb{M}_{x,b}(\Upsilon_b)-\lambda\}}}{\int_{\mathbb{B}_k(x)}\mathbb{1}_{\{\Upsilon_b(y)\ge\mathbb{M}_{y,b}(\Upsilon_b)-\lambda\}}dy}dx<\infty\,.$$

To simplify the notation, we introduce the set $\mathrm{A}_\lambda \overset{\text{def}}{=} \{x\in\mathbb{R}^d : \Upsilon_b(x)\ge\mathbb{M}_{x,b}(\Upsilon_b)-\lambda\}$. We note that there exist a positive number σ_d that depends only on d and points $\{z_1,\dots z_{\sigma_d}\}\subseteq\mathbb{R}^d$, such that $\mathbb{B}_k\subseteq\cup_{i=1}^{\sigma_d}B(z_i,e^k/2)$. Consequently, this implies that the quantity on the left-hand side of the previous expression is bounded above by

$$\sum_{i=1}^{\sigma_d}\int_{B(z,e^k/2)}\frac{\mathbb{1}_{\{x\in\mathrm{A}_\lambda\}}}{\int_{\mathbb{B}_k(x)}\mathbb{1}_{\{y\in\mathrm{A}_\lambda\}}dy}dx\le\sum_{i=1}^{\sigma_d}\int_{B(z,e^k/2)}\frac{\mathbb{1}_{\{x\in\mathrm{A}_\lambda\}}}{\int_{B(z,e^k/2)}\mathbb{1}_{\{y\in\mathrm{A}_\lambda\}}dy}dx\le\sigma_d\,.$$

We now want to take the limit as $b\to\infty$ in the expectation in (2.5.8) by applying Theorems 2.A and 2.B. However, the function that assigns to Υ_b the quantity inside the expectation in (2.5.8) is not a function in $\mathcal{C}^b_{\mathrm{loc}}(\mathcal{C}(\mathbb{R}^d))$. In order to overcome this issue, we proceed in three steps.

Step 1: We can choose $0<k\le j\le b$ sufficiently large such that for all $x\in\mathbb{B}_k$ and all $y\in\mathbb{B}_k(x)$ the following inclusions hold

$$\mathbb{B}_j\subseteq\mathbb{B}_b\subseteq\mathbb{B}_{2b}\,,\qquad \mathbb{B}_j\subseteq\mathbb{B}_b(x)\subseteq\mathbb{B}_{2b}\,,\qquad \mathbb{B}_j\subseteq\mathbb{B}_b(y)\subseteq\mathbb{B}_{2b}\,.$$

In particular, this implies that on the event $\{\mathbb{M}_{0,2b}(\Upsilon_b)=\mathbb{M}_{0,j}(\Upsilon_b)\}$ it holds, for all $x\in\mathbb{B}_k$ and all $y\in\mathbb{B}_k(x)$, that

$$\mathbb{M}_{0,b}(\Upsilon_b)=\mathbb{M}_{x,b}(\Upsilon_b)=\mathbb{M}_{y,b}(\Upsilon_b)=\mathbb{M}_{0,j}(\Upsilon_b)\,.$$

Hence, since as we have already observed the integral over x in (2.5.8) is bounded above, we have that the difference

$$\left|\mathbb{E}\left[\int_{\mathbb{B}_k}\frac{\mathbf{F}(\tau_x\Upsilon_b)e^{\sqrt{2d}(\Upsilon_b(x)-\mathbb{M}_{0,b}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(x)\ge\mathbb{M}_{x,b}(\Upsilon_b)-\lambda\}}}{\int_{\mathbb{B}_k(x)}e^{\sqrt{2d}(\Upsilon_b(y)-\mathbb{M}_{y,b}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(y)\ge\mathbb{M}_{y,b}(\Upsilon_b)-\lambda\}}dy}dx\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b)\le\lambda\}}\right]\right.$$
$$\left.-\,\mathbb{E}\left[\int_{\mathbb{B}_k}\frac{\mathbf{F}(\tau_x\Upsilon_b)e^{\sqrt{2d}(\Upsilon_b(x)-\mathbb{M}_{0,j}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(x)\ge\mathbb{M}_{0,j}(\Upsilon_b)-\lambda\}}}{\int_{\mathbb{B}_k(x)}e^{\sqrt{2d}(\Upsilon_b(y)-\mathbb{M}_{0,j}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(y)\ge\mathbb{M}_{0,j}(\Upsilon_b)-\lambda\}}dy}dx\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b)\le\lambda\}}\right]\right|$$

is bounded by a positive constant times the following probability

$$\mathbb{P}\big(\mathbb{M}_{0,2b}(\Upsilon_b)\ne\mathbb{M}_{0,j}(\Upsilon_b),\ \mathbb{M}_{0,b}(\Upsilon_b)\le\lambda\big)\,.$$

By using Lemmas 2.4.9, 2.4.10 and a version of Lemma 2.A.11 for Brownian motions (see e.g. [BL18, Lemma 4.18]), we have that

$$\lim_{j\to\infty}\limsup_{b\to\infty}\sqrt{b}\,\mathbb{P}\big(\mathbb{M}_{0,2b}(\Upsilon_b)\neq\mathbb{M}_{0,j}(\Upsilon_b),\ \mathbb{M}_{0,b}(\Upsilon_b)\leq\lambda\big)=0\,.$$

Step 2: We note that on the event $\{\mathbb{M}_{0,j,k/2}(\Upsilon_b)<\mathbb{M}_{0,j}(\Upsilon_b)-\lambda\}$, it holds that

$$\int_{\mathbb{B}_k}\frac{\mathbf{F}(\tau_x\Upsilon_b)e^{\sqrt{2d}(\Upsilon_b(x)-\mathbb{M}_{0,j}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(x)\geq\mathbb{M}_{0,j}(\Upsilon_b)-\lambda\}}}{\int_{\mathbb{B}_k(x)}e^{\sqrt{2d}(\Upsilon_b(y)-\mathbb{M}_{0,j}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(y)\geq\mathbb{M}_{0,j}(\Upsilon_b)-\lambda\}}dy}dx$$
$$=\frac{\int_{\mathbb{B}_{k/2}}\mathbf{F}(\tau_x\Upsilon_b)e^{\sqrt{2d}(\Upsilon_b(x)-\mathbb{M}_{0,j}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(x)\geq\mathbb{M}_{0,j}(\Upsilon_b)-\lambda\}}dx}{\int_{\mathbb{B}_{k/2}}e^{\sqrt{2d}(\Upsilon_b(y)-\mathbb{M}_{0,j}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(y)\geq\mathbb{M}_{0,j}(\Upsilon_b)-\lambda\}}dy}\,.$$

Therefore using this fact, we have that the difference

$$\left|\mathbb{E}\left[\int_{\mathbb{B}_k}\frac{\mathbf{F}(\tau_x\Upsilon_b)e^{\sqrt{2d}(\Upsilon_b(x)-\mathbb{M}_{0,j}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(x)\geq\mathbb{M}_{0,j}(\Upsilon_b)-\lambda\}}}{\int_{\mathbb{B}_k(x)}e^{\sqrt{2d}(\Upsilon_b(y)-\mathbb{M}_{0,j}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(y)\geq\mathbb{M}_{0,j}(\Upsilon_b)-\lambda\}}dy}dx\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b)\leq\lambda\}}\right]\right.$$
$$\left.-\,\mathbb{E}\left[\frac{\int_{\mathbb{B}_{k/2}}\mathbf{F}(\tau_x\Upsilon_b)e^{\sqrt{2d}(\Upsilon_b(x)-\mathbb{M}_{0,j}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(x)\geq\mathbb{M}_{0,j}(\Upsilon_b)-\lambda\}}dx}{\int_{\mathbb{B}_{k/2}}e^{\sqrt{2d}(\Upsilon_b(y)-\mathbb{M}_{0,j}(\Upsilon_b))}\mathbb{1}_{\{\Upsilon_b(y)\geq\mathbb{M}_{0,j}(\Upsilon_b)-\lambda\}}dy}\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b)\leq\lambda\}}\right]\right|$$

is bounded by a finite positive constant times the following probability

$$\mathbb{P}\big(\mathbb{M}_{0,j,k/2}(\Upsilon_b)\geq\mathbb{M}_{0,j}(\Upsilon_b)-\lambda,\ \mathbb{M}_{0,b}(\Upsilon_b)\leq\lambda\big)\,.$$

Once again, by using Lemmas 2.4.9, 2.4.10 and [BL18, Lemma 4.18], we have that

$$\lim_{k\to\infty}\limsup_{b\to\infty}\sqrt{b}\,\mathbb{P}\big(\mathbb{M}_{0,j,k/2}(\Upsilon_b)\geq\mathbb{M}_{0,j}(\Upsilon_b)-\lambda,\ \mathbb{M}_{0,b}(\Upsilon_b)\leq\lambda\big)=0\,.$$

Step 3: To conclude, we observe that, since $\mathbf{F}\in\mathcal{C}^b_{\mathrm{loc}}(\mathcal{C}(\mathbb{R}^d))$, the mapping

$$\mathcal{C}(\mathbb{R}^d)\ni\phi\mapsto\frac{\int_{\mathbb{B}_{k/2}}\mathbf{F}(\tau_x\phi)e^{\sqrt{2d}(\phi(x)-\mathbb{M}_{0,j}(\phi))}\mathbb{1}_{\{\phi(x)\geq\mathbb{M}_{0,j}(\phi)-\lambda\}}dx}{\int_{\mathbb{B}_{k/2}}e^{\sqrt{2d}(\phi(y)-\mathbb{M}_{0,j}(\phi))}\mathbb{1}_{\{\phi(y)\geq\mathbb{M}_{0,j}(\phi)-\lambda\}}dy}$$

depends on the values of ϕ inside a compact subsets of $\mathbb{R}^d$. Furthermore, since $\lambda\in\Lambda$, the set of discontinuities of the above mapping is assigned measure zero by the law of the field $\widetilde{\Upsilon}_\lambda$ in $\mathcal{C}(\mathbb{R}^d)$ (see Remark 2.5.2). Hence, by combining (2.5.8) and Steps 1 and 2, by taking the limit as $b\to\infty$, and using Theorems 2.A and 2.B, we obtain that

$$\mathbb{E}\big[\mathbf{F}\big(\widetilde{\Upsilon}_\lambda\big)\big]=\lim_{k\to\infty}\lim_{j\to\infty}\mathbb{E}\left[\frac{\int_{\mathbb{B}_{k/2}}\mathbf{F}(\tau_x\widetilde{\Upsilon}_\lambda)e^{\sqrt{2d}\widetilde{\Upsilon}_\lambda(x)}\mathbb{1}_{\{\widetilde{\Upsilon}_\lambda(x)\geq\mathbb{M}_{0,j}(\widetilde{\Upsilon}_\lambda)-\lambda\}}dx}{\int_{\mathbb{B}_{k/2}}e^{\sqrt{2d}\widetilde{\Upsilon}_\lambda(y)}\mathbb{1}_{\{\widetilde{\Upsilon}_\lambda(y)\geq\mathbb{M}_{0,j}(\widetilde{\Upsilon}_\lambda)-\lambda\}}dy}\right],$$

and so the claim follows from the dominated convergence theorem and the monotone convergence theorem. □

We are now ready to prove Proposition 2.2.12.

Proof of Proposition 2.2.12. Let $\lambda\in\Lambda$ and $\mathbf{F}\in\mathcal{C}^b_{\mathrm{loc}}(\mathcal{C}(\mathbb{R}^d))$. By applying the characterisation (2.2.9) of the field Ψ_λ to the right-hand side of (2.2.11), we obtain that the claim follows if one can prove that the following

equality holds

$$\mathbb{E}\big[\mathbf{F}(\widetilde{\Upsilon}_\lambda)\big] = \mathbb{E}\left[\frac{\int_{\mathbb{R}^d} \mathbf{F}(\tau_x \widetilde{\Upsilon}_\lambda) e^{\sqrt{2d}\widetilde{\Upsilon}_\lambda(x)} \mathbb{1}_{\{\widetilde{\Upsilon}_\lambda(x) \geq \mathbb{M}(\widetilde{\Upsilon}_\lambda) - \lambda\}} dx}{\int_{\mathbb{R}^d} e^{\sqrt{2d}\widetilde{\Upsilon}_\lambda(x)} \mathbb{1}_{\{\widetilde{\Upsilon}_\lambda(x) \geq \mathbb{M}(\widetilde{\Upsilon}_\lambda) - \lambda\}} dx}\right].$$

We note that this is precisely (2.2.10), and thus the claim is established by Proposition 2.2.10. □

2.5.3 Independence from the choice of threshold

In this section, we verify the independence of certain quantities of interest from the arbitrary threshold λ. Specifically, we begin by proving Proposition 2.2.14, which establishes the independence of the law of the field Ψ_λ from $\lambda \in \Lambda$. Then, in Lemma 2.5.8, we show that the constant $a_\star$, defined in (2.2.12), is also independent from $\lambda \in \Lambda$.

We start by proving Proposition 2.2.14.

Proof of Proposition 2.2.14. Consider $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda_2 < \lambda_1$. Thanks to Lemma 2.5.1, we have that the field $\widetilde{\Upsilon}_{\lambda_2}$ has the same law as the field $\widetilde{\Upsilon}_{\lambda_1}$ conditioned on the event $\mathbb{M}(\widetilde{\Upsilon}_{\lambda_1}) \leq \lambda_2$. Therefore, for any $\mathbf{F} \in \mathcal{C}^b_{\mathrm{loc}}(\mathcal{C}(\mathbb{R}^d))$, recalling Definition 2.2.8, we can write

$$\mathbb{E}\big[\mathbf{F}(\Psi_{\lambda_2})\big] \propto \mathbb{E}\left[\frac{\mathbf{F}\big(\tau_{x_\star}\widetilde{\Upsilon}_{\lambda_1}\big) e^{\sqrt{2d}\mathbb{M}(\widetilde{\Upsilon}_{\lambda_1})}}{\int_{\mathbb{R}^d} e^{\sqrt{2d}\widetilde{\Upsilon}_{\lambda_1}(x)} \mathbb{1}_{\{\widetilde{\Upsilon}_{\lambda_1}(x) \geq \mathbb{M}(\widetilde{\Upsilon}_{\lambda_1}) - \lambda_2\}} dx} \mathbb{1}_{\{\mathbb{M}(\widetilde{\Upsilon}_{\lambda_1}) \leq \lambda_2\}}\right], \tag{2.5.9}$$

where $x_\star \in \mathbb{R}^d$ denotes the point where the field $\widetilde{\Upsilon}_{\lambda_1}$ achieves its maximum. Now, one would like to apply the resampling property (2.2.10) of the field $\widetilde{\Upsilon}_{\lambda_1}$ to the quantity on the right-hand side. However, we observe that the function that assigns to $\widetilde{\Upsilon}_{\lambda_1}$ the quantity inside the expectation in (2.5.9) is not a function in $\mathcal{C}^b_{\mathrm{loc}}(\mathcal{C}(\mathbb{R}^d))$. Indeed, this map is neither bounded nor dependent on the values of the input function inside a compact set. Moreover, note that it is also not continuous. However, since $\lambda_1, \lambda_2 \in \Lambda$, its set of discontinuities has measure zero under the law of $\widetilde{\Upsilon}_{\lambda_1}$, so this does not pose an issue.

For $k > 0$, let $x_{\star,k} \in \mathbb{B}_k$ denote the point where the field $\widetilde{\Upsilon}_{\lambda_1}$ achieves its maximum inside $\mathbb{B}_k$. Then, to overcome the issues mentioned above, we claim that the following quantity

$$\mathbb{E}\left[\frac{\mathbf{F}\big(\tau_{x_{\star,k}}\widetilde{\Upsilon}_{\lambda_1}\big) e^{\sqrt{2d}\mathbb{M}_{0,k}(\widetilde{\Upsilon}_{\lambda_1})}}{\big(\int_{\mathbb{B}_k} e^{\sqrt{2d}\widetilde{\Upsilon}_{\lambda_1}(x)} \mathbb{1}_{\{\widetilde{\Upsilon}_{\lambda_1}(x) \geq \mathbb{M}_{0,k}(\widetilde{\Upsilon}_{\lambda_1}) - \lambda_2\}} dx\big) \vee L^{-1}} \mathbb{1}_{\{\mathbb{M}_{0,k}(\widetilde{\Upsilon}_{\lambda_1}) \leq \lambda_2\}}\right] \tag{2.5.10}$$

converges to the right-hand side of (2.5.9) as $k \to \infty$ first, and then $L \to \infty$. This follows from Theorems 2.A, 2.B and Lemmas 2.4.20, 2.4.22. For brevity, we do not provide the details here, but we note that similar computations are carried out in the proofs of Lemmas 2.7.12 and 2.7.14 below.

Now, we are in a position to apply the resampling property (2.2.10) of the field $\widetilde{\Upsilon}_{\lambda_1}$ to the quantity on the right-hand side of (2.5.10) for fixed $k \geq 0$ and $L \geq 0$. Proceeding in this manner, performing some algebraic rearrangements, and then removing the cutoff parameters, it is straightforward to see that the quantity on the right-hand side of (2.5.9) coincides with $\mathbb{E}[\mathbf{F}(\Psi_{\lambda_1})]$, as desired. □

We now argue that the constant $a_\star$, defined in (2.2.12), is independent of $\lambda \in \Lambda$. To establish this, we first need to prove the following result concerning the constant $c_{\star,\lambda}$, defined in (2.2.7).

Lemma 2.5.7. *For any $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda_2 \leq \lambda_1$, it holds that*

$$c_{\star,\lambda_2} = c_{\star,\lambda_1} \mathbb{P}\big(\mathbb{M}(\widetilde{\Upsilon}_{\lambda_1}) \leq \lambda_2\big) \, .$$

Proof. Let $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda_2 \leq \lambda_1$. Recalling (2.2.7), we have that

$$c_{\star,\lambda_2} = \lim_{k\to\infty} \mathbb{E}\big[B_k \mathbb{1}_{\{B_k \in [k^{1/6}, k^{5/6}]\}} \mathbb{1}_{\{\mathbb{M}_{0,k}(\Upsilon_\infty) \leq \lambda_2\}}\big] \, .$$

Hence, for fixed $k \geq 0$, by conditioning on the value of B_k, the right-hand side of the above display admits the following representation,

$$\frac{1}{\sqrt{2\pi k}} \int_{k^{1/6}}^{k^{5/6}} u e^{-\frac{u^2}{2k}} \mathbb{P}_{0,u,k}\big(\mathbb{M}_{0,k}(\Upsilon_\infty) \leq \lambda_2, \ \mathbb{M}_{0,k}(\Upsilon_k) \leq \lambda_1\big) du \, , \tag{2.5.11}$$

where here we used the fact that $\lambda_2 < \lambda_1$. Now, by employing a standard entropic repulsion argument that have been used several times in Section 2.4 (see e.g. the proof of Lemma 2.4.20), for all $u \in [k^{1/6}, k^{5/6}]$, one has that

$$\lim_{j\to\infty} \limsup_{k\to\infty} \frac{k}{u} \mathbb{P}_{0,u,k}\big(\mathbb{M}_{0,k,j}(\Upsilon_k) > \lambda_2, \ \mathbb{M}_{0,k}(\Upsilon_k) \leq \lambda_1\big) = 0 \, .$$

This fact implies that, for $0 \leq j \leq k$, we can just focus on the following probability

$$\mathbb{P}_{0,u,k}\big(\mathbb{M}_{0,j}(\Upsilon_k) \leq \lambda_2, \ \mathbb{M}_{0,k}(\Upsilon_k) \leq \lambda_1\big) \, .$$

In particular, since $\lambda_1, \lambda_2 \in \Lambda$ (see also Remark 2.5.2), we can apply Propositions 2.4.1 and 2.4.2 with $\mathbf{F}(\Upsilon_k) = \mathbb{1}_{\{\mathbb{M}_{0,j}(\Upsilon_k) \leq \lambda_2\}}$ to obtain that

$$\begin{aligned} c_{\star,\lambda_2} &= \lim_{j\to\infty} \lim_{k\to\infty} \mathbb{P}_{0,u,k}\big(\mathbb{M}_{0,j}(\Upsilon_k) \leq \lambda_2, \ \mathbb{M}_{0,k}(\Upsilon_k) \leq \lambda_1\big) \frac{1}{\sqrt{2\pi k}} \int_{k^{1/6}}^{k^{5/6}} u e^{-\frac{u^2}{2k}} du \\ &= c_{\star,\lambda_1} \mathbb{P}\big(\mathbb{M}(\widetilde{\Upsilon}_{\lambda_1}) \leq \lambda_2\big) \, , \end{aligned}$$

from which the conclusion follows. □

We now have all the necessary ingredients to verify that the constant $a_\star$, defined in (2.2.12), is independent of $\lambda \in \Lambda$. More precisely, for $\lambda > 0$, we let

$$a_{\star,\lambda} \overset{\text{def}}{=} \frac{\alpha \, c_{\star,\lambda}}{\gamma \, \mathbb{E}\big[\int_{\mathbb{R}^d} e^{\sqrt{2d}\Psi(x)} \mathbb{1}_{\{\Psi(x) \geq -\lambda\}} dx\big]} \, ,$$

then we have the following result.

Lemma 2.5.8. *For any $\lambda_1, \lambda_2 \in \Lambda$, it holds that $a_{\star,\lambda_1} = a_{\star,\lambda_2}$.*

Proof. Let $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda_2 \leq \lambda_1$. Then, thanks to Lemma 2.5.7, we have that

$$a_{\star,\lambda_2} = \frac{\alpha \, c_{\star,\lambda_1} \mathbb{P}\big(\mathbb{M}(\widetilde{\Upsilon}_{\lambda_1}) \leq \lambda_2\big)}{\gamma \, \mathbb{E}\big[\int_{\mathbb{R}^d} e^{\sqrt{2d}\Psi(x)} \mathbb{1}_{\{\Psi(x) \geq -\lambda_2\}} dx\big]} \tag{2.5.12}$$

Now, using Proposition 2.2.12, along with the fact that, by construction, $\mathbb{M}(\Psi) = 0$, we have that

$$\mathbb{P}\big(\mathbb{M}(\widetilde{\Upsilon}_{\lambda_1}) \leq \lambda_2\big) = \frac{\mathbb{E}\big[\int_{\mathbb{R}^d} e^{\sqrt{2d}\Psi(x)} \mathbb{1}_{\{\Psi(x) \geq -\lambda_2\}} dx\big]}{\mathbb{E}\big[\int_{\mathbb{R}^d} e^{\sqrt{2d}\Psi(x)} \mathbb{1}_{\{\Psi(x) \geq -\lambda_1\}} dx\big]} \, .$$

Hence, substituting the right-hand side of the above display into the expression (2.5.12) for $a_{\star,\lambda_2}$, the conclusion follows readily. □

2.6 The joint Laplace functional

The main goal of this section is to prove Proposition 2.2.19 which is the main input for the proof of the stable convergence result Theorem 2.C. The main ingredient in the proof of this theorem is Proposition 2.6.1 below, which will be proved separately in Section 2.7. The key idea for the proof is to split the field X into two *independent scales*, namely X_s and $\mathrm{X}_{s,t}$, for $0 \leq s < t$. The former field has strong local interactions, while the latter one has a strong independence structure. In particular, as we will see, the exponential of the field $\mathrm{X}_{s,t}$ macroscopically behaves like an independent stable random measure that will integrate the exponential of the field X_s, thus generating the desired limiting measure.

This section is organised as follows. In Section 2.6.1, we introduce several definitions and state Proposition 2.6.1, which plays a fundamental role in the proof of Proposition 2.2.19. In Section 2.6.2, assuming Proposition 2.6.1, we establish the convergence of the "small scales" while conditioning on the "large large scales". In Section 2.6.3, we prove the "full" conditional convergence. Finally, in Section 2.6.4, we show how Proposition 2.2.19 follows directly from the results obtained thus far, and we also prove Theorem 2.C.

2.6.1 Setup and preliminary results

For the reminder of this section, we fix $\gamma > \sqrt{2d}$. For $n \in \mathbb{N}$, we consider a collection of fields $(\mathrm{W}_{i,t})_{i\in[n],t\geq 0}$ satisfying (W1)–(W4). We recall that we need to prove that for all $(\varphi, (f_i)_{i\in[n]}) \in \mathcal{C}_c^\infty(\mathbb{R}^d) \times (\mathcal{C}_c^+(\mathbb{R}^d))^n$, the following limit holds

$$\lim_{t\to\infty} \mathbb{E}\left[\exp\big(i\langle \mathrm{X}, \varphi\rangle\big) \prod_{i=1}^n \exp\big(-\mu_{\gamma,t,i}(f_i)\big)\right] = \mathbb{E}\Big[\exp\big(i\langle \mathrm{X}, \varphi\rangle\big) \exp\big(-\widetilde{a}_\star \mu_{\gamma_c}\big(\mathrm{T}_\gamma[f_1, \dots, f_n]\big)\big)\Big] \, ,$$

where we refer to the statement of Proposition 2.2.19 for the definitions of all the quantities involved.

To simplify the presentation, we assume that the functions $(f_i)_{i\in[n]}$ are all of the form $f_i = \theta_i \mathbb{1}_{[0,1]^d}$ for some non-negative constants $(\theta_i)_{i\in[n]} \subseteq \mathbb{R}_0^+$. The proof developed below works in the general case where $(f_i)_{i\in[n]} \in (\mathcal{C}_c^+(\mathbb{R}^d))^n$, with minor and straightforward adjustments. In particular, in the general setting, the unit box $[0,1]^d$ can be replaced by any compact set that contains the supports of all the functions $(f_i)_{i\in[n]}$.

With a slight abuse of notation, for all $t \geq 0$, it is convenient to introduce the function $\mathrm{F}_{\gamma,t} : \mathbb{R}^d \to \mathbb{R}$ and the quantity $\mathrm{T}_\gamma \geq 0$ given by

$$\mathrm{F}_{\gamma,t}(x) \stackrel{\text{def}}{=} \sum_{i=1}^n \theta_i e^{\gamma \mathrm{W}_{i,t}(x)} \, , \qquad \mathrm{T}_\gamma \stackrel{\text{def}}{=} \mathbb{E}\left[\left(\sum_{i=1}^n \theta_i \int_{\mathbb{R}^d} \exp\big(\gamma\big(\mathrm{W}_i(y) + \Psi(y)\big)\big) dy\right)^{\frac{\sqrt{2d}}{\gamma}}\right] , \tag{2.6.1}$$

where Ψ denotes the field introduced in Definition 2.2.8, which, as stated in Proposition 2.2.14, has a law independent of $\lambda \in \Lambda$, where Λ is the set introduced in Lemma 2.5.1. For this reason, we omit the subscript λ when referring to the field Ψ.

In this way, assuming that the function $(f_i)_{i\in[n]}$ are as specified above, we need to prove that the following limit holds

$$\lim_{t\to\infty} \mathbb{E}\Big[\exp\big(i\langle X, \varphi\rangle\big)\exp\big(-\mu_{\gamma,t}(F_{\gamma,t})\big)\Big] = \mathbb{E}\Big[\exp\big(i\langle X, \varphi\rangle\big)\exp\Big(-\widetilde{a}_\star T_\gamma\, \mu_{\gamma_c}\big([0,1]^d\big)\Big)\Big]\ . \tag{2.6.2}$$

In order to prove (2.6.2), we introduce a decomposition of the unit box $[0,1]^d$ as follows. For $R \geq 1$, we consider $s \geq 0$ such that $(e^s+1)(R+1)^{-1} \in \mathbb{N}$, and we write

$$[0,1]^d = \bigcup_{i\in[\mathfrak{I}]} A_i + B_{R,s}\ , \tag{2.6.3}$$

where $(A_i)_{i\in[\mathfrak{I}]}$ and $B_{R,s}$ are as described in the caption of Figure 2.2.

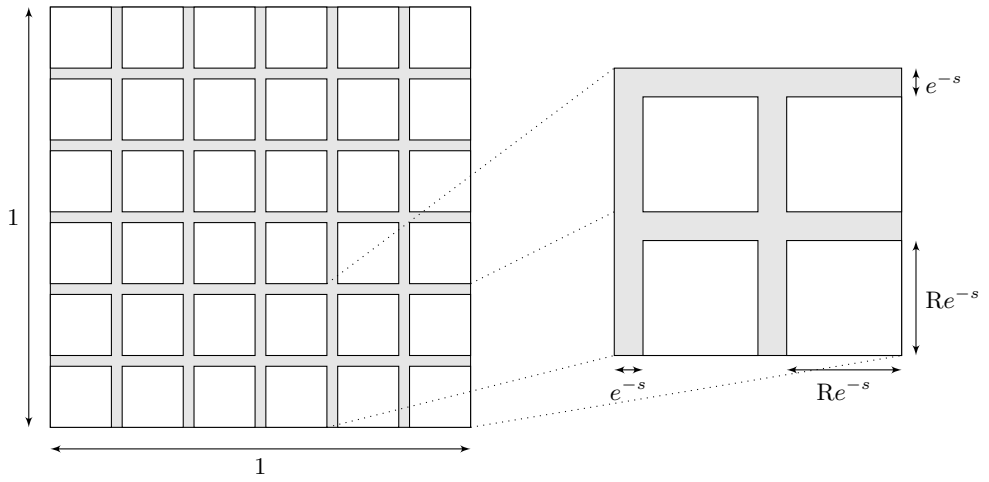


FIGURE 2.2: A diagram illustrating the decomposition of the cube $[0,1]^d$ used in the proof of Lemma 2.6.3, in the case $d = 2$. For $R \geq 1$, we suppose that along each edge of the d edges of $[0,1]^d$, there are exactly $(e^s+1)(R+1)^{-1}$ smaller closed boxes, which are drawn in white and that we enumerate by $(A_i)_{i\in[\mathfrak{I}]}$, for some $\mathfrak{I} \in \mathbb{N}$. For each $i \in [\mathfrak{I}]$, the box A_i has side length equal to Re^{-s}. The shaded region is the buffer zone $B_{R,s} = [0,1]^d \setminus \cup_{i\in[\mathfrak{I}]} A_i$. By construction, the buffer zone separates the smaller cubes by a distance of at least e^{-s}. A similar figure first appeared in [MRV16, Figure 1].

The reason why the decomposition (2.6.3) turns out to be useful is due to the short-range correlation properties of the fields X_t and W_t. Indeed, since we are assuming that the seed covariance function $\mathfrak{K}$ satisfies condition (K2), as we have observed in Section 2.3.3, for $0 \leq s < t$, it holds that $X_{s,t}(x)$ and $X_{s,t}(y)$ are independent for $|x-y| > e^{-s}$. Similarly, thanks to (W3), for $0 \leq s < t$, we have that $(W_{i,t}(x))_{i\in[n]}$ and $(W_{i,t}(y))_{i\in[n]}$ are independent for $|x-y| > e^{-s}$.

As mentioned earlier, the idea is to split the field X into two scales. Roughly speaking, we first focus on the small scales conditioned on the large scales. More precisely, we consider the sequence of measures $(\mu_{\gamma,s,t})_{0\leq s<t}$ on $\mathbb{R}^d$, defined as follows,

$$\mu_{\gamma,s,t}(dx) \overset{\text{def}}{=} (t-s)^{\frac{3\gamma}{2\sqrt{2d}}} e^{(t-s)(\gamma/\sqrt{2}-\sqrt{d})^2} e^{\gamma X_{s,t}(x) - \frac{\gamma^2}{2}(t-s)} dx\ , \tag{2.6.4}$$

The role of the large scales is played by a deterministic[3] function $\chi : \mathbb{R}^d \to \mathbb{R}$. Given a function χ, we define the measure ρ_χ by letting

$$\rho_\chi(dx) \stackrel{\text{def}}{=} \chi(x) e^{-\sqrt{2d}\chi(x)} dx \ . \tag{2.6.5}$$

We now have all the necessary ingredients to state the following key proposition, which is a generalisation of [MRV16, Proposition 3.2]. The key difference is that we allow $\mathrm{F}_{\gamma,t}$ to take the general form given in (2.6.1), rather than restricting it to be a constant. More importantly, we explicitly identify the multiplicative constant $C(\gamma)$ that appears in the statement of [MRV16, Proposition 3.2]. The proof of this proposition is postponed to Section 2.7.

Proposition 2.6.1. *Fix $\varepsilon > 0$, let F_γ and T_γ be as in (2.6.1), and consider the constant $\widetilde{a}_\star > 0$ introduced in Proposition 2.2.19. Then, for all $s > 0$ large enough and for all $t \geq s$ large enough, the following holds. For any function $\chi : [0,\mathrm{R}]^d \to \mathbb{R}$ such that*

$$\min_{x\in[0,\mathrm{R}]^d} \chi(x) \geq \frac{1}{8\sqrt{2d}} \log s \ , \quad \max_{x\in[0,\mathrm{R}]^d} \chi(x) \leq \log t \ , \quad \sup_{\substack{x,y\in[0,\mathrm{R}]^d,\\ |x-y|\leq s^{-1}}} \frac{|\chi(x)-\chi(y)|}{|x-y|^{1/3}} \leq 1 \ , \tag{2.6.6}$$

one has that,

$$\left| \mathbb{E}\left[\exp\left(- \int_{[0,\mathrm{R}]^d} \mathrm{F}_{\gamma,t}(x) e^{-\gamma\chi(x)} \mu_{\gamma,t}(dx) \right) \right] - 1 - \widetilde{a}_\star \mathrm{T}_\gamma \rho_\chi\big([0,\mathrm{R}]^d\big) \right| \leq \varepsilon \rho_\chi\big([0,\mathrm{R}]^d\big) \ .$$

Note that our definition of ρ_χ includes no analogue of the term $-\gamma^{-1}\log\theta$ appearing in [MRV16, Equation 3.11]. This is because this term can simply be absorbed into the error term by making s sufficiently large.

2.6.2 Convergence of the small scales

We are now ready to state and the prove the following result in which, roughly speaking, we compute the Laplace functional of the measure $\mu_{\gamma,s,t}$ defined in (2.6.4). Before proceeding, we introduce the following definition.

Definition 2.6.2. We say that a function $\chi : [0,1]^d \to \mathbb{R}$ is *scaled-admissible* if for all $\varepsilon > 0$, there exists $\mathrm{R}_0 \geq 1$ such that $e^{ds}\rho_\chi(\overline{B_{\mathrm{R},s}}) \leq \varepsilon$ for all $s \geq 0$ sufficiently large and for all $\mathrm{R} \geq \mathrm{R}_0$ such that $(e^s+1)(\mathrm{R}+1)^{-1} \in \mathbb{N}$.

We have the following key result.

Lemma 2.6.3. *Let $\varepsilon > 0$. Then, for $s \geq 0$ sufficiently large, for $t > s$ sufficiently large, and for any scaled-admissible function $\chi : [0,1]^d \to \mathbb{R}$ in the sense of Definition 2.6.2, such that*

$$\min_{x\in[0,1]^d} \chi(x) \geq \frac{1}{8\sqrt{2d}} \log s \ , \quad \max_{x\in[0,1]^d} \chi(x) \leq \log t \ , \quad \sup_{\substack{x,y\in[0,1]^d,\\ |x-y|\leq e^{-s}s^{-1}}} \frac{|\chi(x)-\chi(y)|}{|x-y|^{1/3}} \leq e^{s/3} \ , \tag{2.6.7}$$

[3] At this level, we are assuming that we are conditioning on the large scales. That is why, we are thinking of the large scales as being described by a deterministic function.

one has that,

$$\mathbb{E}\left[\exp\left(-e^{ds}\int_{[0,1]^d}\mathrm{F}_{\gamma,t}(x)e^{-\gamma\mathrm{X}(x)}\mu_{\gamma,s,t}(dx)\right)\right]\geq\exp\Big(-(\widetilde{a}_\star+\varepsilon)e^{ds}\mathrm{T}_\gamma\rho_\chi\big([0,1]^d\big)\Big)-\varepsilon\ ,$$
$$\mathbb{E}\left[\exp\left(-e^{ds}\int_{[0,1]^d}\mathrm{F}_{\gamma,t}(x)e^{-\gamma\mathrm{X}(x)}\mu_{\gamma,s,t}(dx)\right)\right]\leq\exp\Big(-(\widetilde{a}_\star-\varepsilon)e^{ds}\mathrm{T}_\gamma\rho_\chi\big([0,1]^d\big)\Big)+\varepsilon\ ,$$

where $\widetilde{a}_\star>0$ is the constant introduced in Proposition 2.2.19.

Proof. For $\mathrm{R}\geq 1$, $s\geq 0$ such that $(e^s+1)(\mathrm{R}+1)^{-1}\in\mathbb{N}$, we consider the decomposition of the cube $[0,1]^d$ given in (2.6.3). We also consider $t>s$ and a function $\chi:[0,1]\to\mathbb{R}^d$ satisfying conditions (2.6.6). In order to lighten the notation, we define

$$\mathbb{E}_{(2.6.8)}\overset{\text{def}}{=}\mathbb{E}\left[\exp\left(-e^{ds}\int_{[0,1]^d}\mathrm{F}_{\gamma,t}(x)e^{-\gamma\mathrm{X}(x)}\mu_{\gamma,s,t}(dx)\right)\right]\ . \tag{2.6.8}$$

We write the integral over $[0,1]^d$ appearing inside $\mathbb{E}_{(2.6.8)}$ as the following sum

$$\int_{B_{\mathrm{R},s}}\mathrm{F}_{\gamma,t}(x)e^{-\gamma\mathrm{X}(x)}\mu_{\gamma,s,t}(dx)+\int_{\cup_{i\in[\mathfrak{I}]}A_i}\mathrm{F}_{\gamma,t}(x)e^{-\gamma\mathrm{X}(x)}\mu_{\gamma,s,t}(dx)\ ,$$

and we divide the proof into three main steps. In the first step, we show that the integral over the buffer-zone can be made arbitrarily small by choosing $s>0$ large enough and $t\geq s$ large enough. In the second step, we provide, upper and lower bound for the integral over the region given by the union of the small squares. Finally, in the third step, we show how to combine everything to obtain the desired result. We remark that the proof strategy follows similar lines to the proof of [MRV16, Lemma 3.1].

Step 1: In this first step, we show that for any $\varepsilon>0$, there exist $\mathrm{R}\geq 1$ large enough such that for all $s>0$ large enough satisfying $(e^s+1)(\mathrm{R}+1)^{-1}\in\mathbb{N}$, and $t\geq s$ large enough, it holds that

$$1-\mathbb{E}\left[\exp\left(-e^{ds}\int_{B_{\mathrm{R},s}}\mathrm{F}_{\gamma,t}(x)e^{-\gamma\mathrm{X}(x)}\mu_{\gamma,s,t}(dx)\right)\right]\leq\varepsilon\ . \tag{2.6.9}$$

By making the change of variables $x\mapsto e^s x$, using the scaling relation (2.3.9), and thanks to (W1) and (W2), we obtain that the expectation on the left-hand side of (2.6.9) is equal to

$$\mathbb{E}\left[\exp\left(-\int_{e^sB_{\mathrm{R},s}}\mathrm{F}_{\gamma,t-s}(x)e^{\gamma\mathrm{X}(e^{-s}x)}\mu_{\gamma,t-s}(dx)\right)\right]\ .$$

Next, we note that there exists $\mathfrak{J}\in\mathbb{N}$ and a sequence of points $(x_j)_{j\in[\mathfrak{J}]}\subset\mathbb{R}^d$ such that

$$e^s\overline{B_{\mathrm{R},s}}=\bigcup_{j\in[\mathfrak{J}]}\big(x_j+[0,1]^d\big)\ ,\qquad\bigcap_{i=1}^{d+2}\big(x_{j_i}+[0,1]^d\big)=\emptyset,\ \ \forall\, j_1\neq\ldots\neq j_{d+2}\in[\mathfrak{J}]\ . \tag{2.6.10}$$

Therefore, using the inequality $1-\prod_{j\in[\mathfrak{J}]}a_j\leq\sum_{j\in[\mathfrak{J}]}(1-a_j)$, which is valid for $(a_j)_{j\in[\mathfrak{J}]}\subset[0,1]$, and the translation invariance of X_{t-s}, as well as (W2), we obtain that

$$1-\mathbb{E}\left[\exp\left(-\int_{e^sB_{\mathrm{R},s}}\mathrm{F}_{\gamma,t-s}(x)e^{\gamma\mathrm{X}(e^{-s}x)}\mu_{\gamma,t-s}(dx)\right)\right]$$

$$\leq 1 - \mathbb{E}\left[\exp\left(-\sum_{j\in[\mathfrak{J}]}\int_{x_j+[0,1]^d} \mathrm{F}_{\gamma,t-s}(x)e^{\gamma\chi(e^{-s}x)}\mu_{\gamma,t-s}(dx)\right)\right]$$
$$= \mathbb{E}\left[1 - \prod_{j\in[\mathfrak{J}]}\exp\left(-\int_{x_j+[0,1]^d} \mathrm{F}_{\gamma,t-s}(x)e^{\gamma\chi(e^{-s}x)}\mu_{\gamma,t-s}(dx)\right)\right]$$
$$\leq \sum_{j\in[\mathfrak{J}]}\left(1 - \mathbb{E}\left[\exp\left(-\int_{[0,1]^d} \mathrm{F}_{\gamma,t-s}(x)e^{\gamma\chi(x_j+e^{-s}x)}\mu_{\gamma,t-s}(dx)\right)\right]\right). \qquad (2.6.11)$$

Since by assumption the function χ satisfies all the conditions in (2.6.7), one can readily check that the function $\chi(x_j + e^{-s}\,\cdot)$ satisfies all the conditions in (2.6.6), for all $j \in [\mathfrak{J}]$. Therefore, by Proposition 2.6.1, for any $\varepsilon > 0$, for all $s > 0$ large enough satisfying $(e^s+1)(\mathrm{R}+1)^{-1} \in \mathbb{N}$, and for all $t \geq s$ large enough, we have that the expectation in (2.6.11) satisfies the following inequality, for all $j \in [\mathfrak{J}]$,

$$1 - \mathbb{E}\left[\exp\left(-\int_{[0,1]^d} \mathrm{F}_{\gamma,t-s}(x)e^{\gamma\chi(x_j+e^{-s}x)}\mu_{\gamma,t-s}(dx)\right)\right] \leq (\widetilde{a}_\star + \varepsilon)\mathrm{T}_\gamma e^{ds}\rho_{\chi(e^{-s}\cdot)}\big(x_j + [0,1]^d\big)\,.$$

Hence, plugging the right-hand side of the previous expression into (2.6.11), making the change of variables $x \mapsto x_j + e^{-s}x$, and using the assumptions (2.6.10) on the sequence of points $(x_j)_{j\in\mathfrak{J}}$, we obtain that

$$1 - \mathbb{E}\left[\exp\left(-\int_{e^s B_{\mathrm{R},s}} \mathrm{F}_{\gamma,t-s}(x)e^{\gamma\chi(e^{-s}x)}\mu_{\gamma,t-s}(dx)\right)\right] \leq (d+2)(\widetilde{a}_\star + \varepsilon)\mathrm{T}_\gamma e^{ds}\rho_\chi\big(\overline{B_{\mathrm{R},s}}\big)\,,$$

Since, we are assuming that χ is scaled-admissible in the sense of Definition 2.6.2, the quantity $e^{ds}\rho_\chi(\overline{B_{\mathrm{R},s}})$ can be made arbitrarily small by taking $s \geq 0$ and $\mathrm{R} \geq 1$ large enough.

Step 2: In this second step, we show that, for any $\varepsilon > 0$, there exist $\mathrm{R} \geq 1$ large enough such that for all $s > 0$ large enough satisfying $(e^s+1)(\mathrm{R}+1)^{-1} \in \mathbb{N}$, and $t \geq s$ large enough, it holds that

$$\mathbb{E}\left[\exp\left(-e^{ds}\int_{\cup_{i\in[\mathfrak{I}]}A_i} \mathrm{F}_{\gamma,t}(x)e^{-\gamma\chi(x)}\mu_{\gamma,s,t}(dx)\right)\right] \geq \exp\Big(-(\widetilde{a}_\star+\varepsilon)e^{ds}\mathrm{T}_\gamma\rho_\chi\big([0,1]^d\big)\Big)$$
$$\mathbb{E}\left[\exp\left(-e^{ds}\int_{\cup_{i\in[\mathfrak{I}]}A_i} \mathrm{F}_{\gamma,t}(x)e^{-\gamma\chi(x)}\mu_{\gamma,s,t}(dx)\right)\right] \leq \exp\Big(-(\widetilde{a}_\star-\varepsilon)e^{ds}\mathrm{T}_\gamma\rho_\chi\big([0,1]^d\big)\Big) + \varepsilon\,.$$

Since the lower bound can be obtained in a similar manner to the upper bound, we just focus on the latter. We recall that if $x, y \in [0,1]^d$ belong to two different squares in the decomposition described in Figure 2.2, then $|x-y| \geq e^{-s}$ and so $\mathrm{X}_{s,t}(x)$ and $\mathrm{X}_{s,t}(y)$ are independent. Moreover, if x, y are as above, then thanks to (W3), we also have that $(\mathrm{W}_{i,t}(x))_{i\in[n]}$ and $(\mathrm{W}_{i,t}(y))_{i\in[n]}$ are independent. Hence, using this fact, making the change of variables $x \mapsto e^s x$, and preceding in the same manner in the first step, we obtain that

$$\mathbb{E}\left[\exp\left(-e^{ds}\int_{\cup_{i\in[\mathfrak{I}]}A_i} \mathrm{F}_{\gamma,t}(x)e^{-\gamma\chi(x)}\mu_{\gamma,s,t}(dx)\right)\right]$$
$$= \prod_{i\in[\mathfrak{I}]}\mathbb{E}\left[\exp\left(-\int_{e^s A_i} \mathrm{F}_{\gamma,t-s}(x)e^{-\gamma\chi(e^{-s}x)}\mu_{\gamma,t-s}(dx)\right)\right].$$

Now, since for each $i \in [\mathfrak{I}]$, the function $\chi(e^{-s}\cdot)$ on $e^s A_i$ satisfies all the conditions stated in (2.6.6), by Proposition 2.6.1, we get that for all $s > 0$ large enough satisfying $(e^s+1)(\mathrm{R}+1)^{-1} \in \mathbb{N}$, and $t \geq s$ large

enough, it holds that

$$\mathbb{E}\left[\exp\left(-\int_{e^s A_i} \mathrm{F}_{\gamma,t-s}(x)e^{-\gamma\chi(e^{-s}x)}\mu_{\gamma,t-s}(dx)\right)\right] \leq 1-(\widetilde{a}_\star-\varepsilon)\mathrm{T}_\gamma\rho_{\chi(e^{-s}\cdot)}\big(e^s A_i\big)\ .$$

For all $i\in[\mathfrak{I}]$, using the assumptions on the function χ, we obtain that for sufficiently large $s\geq 0$, the following holds

$$(\widetilde{a}_\star-\varepsilon)\mathrm{T}_\gamma\rho_{\chi(e^{-s}\cdot)}\big(e^s A_i\big) \leq \mathrm{R}^d(\widetilde{a}_\star-\varepsilon)\mathrm{T}_\gamma\left(\sup_{z\in[\log s/(8\sqrt{2d}),\infty)} ze^{-\sqrt{2d}z}\right) < 1\ .$$

Therefore, using the inequality $\prod_{i\in[\mathfrak{I}]}(1-a_i)\leq\exp(-\sum_{i\in[\mathfrak{I}]}a_i)$, which is valid for $(a_i)_{i\in[\mathfrak{I}]}\subset[0,1]$, and making the change of variables $x\mapsto e^s x$, we get that

$$\begin{aligned}\mathbb{E}\left[\exp\left(-e^{ds}\int_{\cup_{i\in[\mathfrak{I}]}A_i}\mathrm{F}_{\gamma,t}(x)e^{-\gamma\chi(x)}\mu_{\gamma,s,t}(dx)\right)\right] &\leq \exp\Big(-(\widetilde{a}_\star-\varepsilon)e^{ds}\mathrm{T}_\gamma\rho_\chi\big(\cup_{i\in[\mathfrak{I}]}A_i\big)\Big)\\ &= \exp\Big(-(\widetilde{a}_\star-\varepsilon)e^{ds}\mathrm{T}_\gamma\big(\rho_\chi\big([0,1]^d\big)-\rho_\chi\big(B_{\mathrm{R},s}\big)\big)\Big)\\ &\leq \exp\Big(-(\widetilde{a}_\star-\varepsilon)e^{ds}\mathrm{T}_\gamma\rho_\chi\big([0,1]^d\big)\Big)+\varepsilon\ ,\end{aligned}$$

where, once again, the last inequality is due to the fact that, since by assumption χ is scaled-admissible in the sense of Definition 2.6.2, the quantity $e^{ds}\rho_\chi(\overline{B_{\mathrm{R},s}})$ can be made arbitrarily small by taking $s\geq 0$ and $\mathrm{R}\geq 1$ large enough.

Step 3: In this final step, we show how to combine the previous two steps to obtain the desired result. Thanks to decomposition (2.6.3) and to the elementary inequality $ab\geq a+b-1$ valid for $a,b\in[0,1]$, we have that

$$\begin{aligned}\mathbb{E}_{(2.6.8)} \geq \mathbb{E}\left[\exp\left(-e^{ds}\int_{\cup_{i\in[\mathfrak{I}]}A_i}\mathrm{F}_{\gamma,t}(x)e^{-\gamma\chi(x)}\mu_{\gamma,s,t}(dx)\right)\right]&\\ +\mathbb{E}\left[\exp\left(-e^{ds}\int_{B_{\mathrm{R},s}}\mathrm{F}_{\gamma,t}(x)e^{-\gamma\chi(x)}\mu_{\gamma,s,t}(dx)\right)\right]-1\ .&\end{aligned}$$

On the other hand, we also have the trivial inequality

$$\mathbb{E}_{(2.6.8)} \leq \mathbb{E}\left[\exp\left(-e^{ds}\int_{\cup_{i\in[\mathfrak{I}]}A_i}\mathrm{F}_{\gamma,t}(x)e^{-\gamma\chi(x)}\mu_{\gamma,s,t}(dx)\right)\right]\ .$$

Hence, to conclude it suffices to combine Steps 1 and 2. □

2.6.3 Convergence of the large scales

The main goal of this subsection is to compute the conditional Laplace functionals of the random measures $(\mu_{\gamma,t})_{t\geq 0}$ conditioned on the σ-field $\mathcal{F}_s$ defined in (2.1.5). Heuristically speaking, we want to get the expectations appearing in the statement of Lemma 2.6.3 to be of order one, and so we want to absorb some normalisation factor $r(s)$ into $\mu_{\gamma,s,t}$. Proceeding formally for the moment, given a function $\chi:[0,1]^d\to\mathbb{R}$ satisfying the conditions in the statement of Lemma 2.6.3, we consider the function $\widetilde{\chi}:[0,1]^d\to\mathbb{R}$ such that, for all $x\in[0,1]^d$,

$$e^{-\gamma\chi(x)} = e^{-ds}r(s)e^{-\gamma\widetilde{\chi}(x)}\ ,$$

so that, by Lemma 2.6.3 with $n = 1$, $W_{1,\cdot} = 0$, and $\theta_1 = 1$, the following approximate identity holds

$$\mathbb{E}\left[\exp\left(-\int_{[0,1]^d} r(s)e^{-\gamma\widetilde{\chi}(x)}\mu_{\gamma,s,t}(dx)\right)\right] \approx \mathbb{E}\left[\exp\left(-ce^{ds}\rho_{\widetilde{\chi}-\log(r(s))/\gamma+ds/\gamma}\left([0,1]^d\right)\right)\right] .$$

for some constant $c > 0$. Now recalling the definition (2.6.5), it is easily seen that to get something of order one on the right-hand side of the above expression, we need to choose $r(s)$ such that

$$e^{ds-\frac{\sqrt{2d}}{\gamma}ds}r(s)^{\frac{\sqrt{2d}}{\gamma}}\left|\log(e^{-ds}r(s))\right| \approx 1$$

which is achieved by setting

$$r(s) = e^{ds-\gamma s\sqrt{d/2}}s^{-\frac{\gamma}{\sqrt{2d}}} .$$

In particular, this suggests to define the measure $\widetilde{\mu}_{\gamma,s,t} \overset{\text{def}}{=} r(s)\mu_{\gamma,s,t}$ which is given by

$$\widetilde{\mu}_{\gamma,s,t}(dx) = s^{-\frac{\gamma}{\sqrt{2d}}}e^{\gamma s\frac{\sqrt{2d}}{2}}(t-s)^{\frac{3\gamma}{2\sqrt{2d}}}e^{t(\gamma/\sqrt{2}-\sqrt{d})^2}e^{\gamma X_{s,t}(x)-\frac{\gamma^2}{2}t}dx . \tag{2.6.12}$$

Given $R \geq 1$, $s \geq 0$, a function $\chi : [0,R]^d \to \mathbb{R}$, we define the measure

$$\rho_{\chi,s}(dx) \overset{\text{def}}{=} e^{-\sqrt{2d}\chi(x)}\left(\sqrt{d/2} + \frac{\chi(x)}{s} + \frac{\log s}{\sqrt{2ds}}\right)dx . \tag{2.6.13}$$

Definition 2.6.4. We say that a function $\chi : [0,1]^d \to \mathbb{R}$ is *admissible* if for all $\varepsilon > 0$, there exists $R_0 \geq 1$ such that $\rho_{\chi,s}(\overline{B_{R,s}}) \leq \varepsilon$ for all $s \geq 0$ sufficiently large and for all $R \geq R_0$ such that $(e^s+1)(R+1)^{-1} \in \mathbb{N}$.

It is immediate to check that Lemma 2.6.3 for the measure $\widetilde{\mu}_{\gamma,s,t}$ can be stated as follows.

Corollary 2.6.5. *Let $\varepsilon > 0$. Then, for $s \geq 0$ sufficiently large, for $t > s$ sufficiently large, and for any admissible function $\chi : [0,1]^d \to \mathbb{R}$ in the sense of Definition 2.6.4, such that*

$$\begin{gathered}\min_{x\in[0,1]^d}\chi(x) \geq \frac{\log s}{8\sqrt{2d}} - \frac{\sqrt{2d}s}{2} , \qquad \max_{x\in[0,1]^d}\chi(x) \leq \log t - \frac{\sqrt{2d}s}{2} - \frac{\log s}{\sqrt{2d}} , \\ \sup_{\substack{x,y\in[0,1]^d,\\ |x-y|\leq e^{-s}s^{-1}}} \frac{|\chi(x)-\chi(y)|}{|x-y|^{1/3}} \leq e^{s/3} ,\end{gathered} \tag{2.6.14}$$

one has that,

$$\mathbb{E}\left[\exp\left(-\int_{[0,1]^d} F_{\gamma,t}(x)e^{-\gamma\chi(x)}\widetilde{\mu}_{\gamma,s,t}(dx)\right)\right] \geq \exp\left(-(\widetilde{a}_\star+\varepsilon)T_\gamma\rho_{\chi,s}\left([0,1]^d\right)\right) - \varepsilon , \tag{2.6.15}$$

$$\mathbb{E}\left[\exp\left(-\int_{[0,1]^d} F_{\gamma,t}(x)e^{-\gamma\chi(x)}\widetilde{\mu}_{\gamma,s,t}(dx)\right)\right] \geq \exp\left(-(\widetilde{a}_\star-\varepsilon)T_\gamma\rho_{\chi,s}\left([0,1]^d\right)\right) + \varepsilon . \tag{2.6.16}$$

where $\widetilde{a}_\star > 0$ is the constant introduced in Proposition 2.2.19.

Remark 2.6.6. The previous lemma entails almost directly that the measure $\widetilde{\mu}_{\gamma,s,t}$ converges to an integrated atomic random measure with parameter γ and spatial intensity given by the Lebesgue measure dx, i.e, up to multiplicative constants and recalling the notation introduce in Definition 1.3.7, to $\mathcal{P}_\gamma[dx]$. To see this, it suffices to take the function χ to be of constant order. Then, taking first the limit when $t \to \infty$ and then the limit when $s \to \infty$, the "log corrections" appearing in (2.6.13) vanish, and one can read off the Laplace

functional associated to measure described above. Since we don't need this fact in the sequel of the proof, we refrain from giving further details.

In the next lemma we compute the conditional Laplace functionals of the measures $(\mu_{\gamma,t})_{t\geq 0}$.

Lemma 2.6.7. *There exists a diverging sequence $(s_n)_{n\geq 0}$ of non-negative real numbers such that, almost surely,*

$$\lim_{n\to\infty}\lim_{t\to\infty}\mathbb{E}\left[\exp\left(-\int_{[0,1]^d}\mathrm{F}_{\gamma,t}(x)\mu_{\gamma,t}(dx)\right)\middle|\mathcal{F}_{s_n}\right]=\exp\Big(-\widetilde{a}_\star\mathrm{T}_\gamma\mu_{\gamma_c}\big([0,1]^d\big)\Big)\,, \tag{2.6.17}$$

where $\widetilde{a}_\star>0$ is the constant introduced in Proposition 2.2.19, and μ_{γ_c} is the critical GMC.

Proof. In order to prove the result, we rely on Corollary 2.6.5. In particular, we want to take the function $\chi:[0,1]^d\to\mathbb{R}$ in that corollary in such a way that $\exp(-\gamma\chi(x))\widetilde{\mu}_{\gamma,s,t}(dx)=\mu_{\gamma,t}(dx)$, which forces us to take

$$\chi(\cdot)=-\mathrm{X}_s(\cdot)-\frac{\log s}{\sqrt{2d}}+\frac{\sqrt{2d}s}{2}\,. \tag{2.6.18}$$

Now, a simple computation yields that, for χ as in (2.6.18),

$$\rho_{\chi,s}(dx)=\mu_{\gamma_c,s}(dx)\,,$$

where here we recall (2.1.7). For $\varepsilon>0$, $s\geq 0$, $\mathrm{R}\geq 1$ such that $(e^s+1)(\mathrm{R}+1)^{-1}\in\mathbb{N}$, and χ as in (2.6.18), we consider the following events

$$\mathrm{E}_s^1\overset{\text{def}}{=}\left\{\min_{x\in[0,1]^d}\chi(x)\geq\frac{\log s}{8\sqrt{2d}}-\frac{\sqrt{2d}s}{2}\right\},\qquad \mathrm{E}_s^2\overset{\text{def}}{=}\left\{\max_{x\in[0,1]^d}\chi(x)\leq\frac{19\sqrt{2d}s}{2}-\frac{\log s}{\sqrt{2d}}\right\},$$

$$\mathrm{E}_s^3\overset{\text{def}}{=}\left\{\sup_{\substack{x,y\in[0,1]^d,\\ |x-y|\leq e^{-s}s^{-1}}}\frac{|\chi(x)-\chi(y)|}{|x-y|^{1/3}}\leq e^{s/3}\right\},\qquad \mathrm{E}_{s,\mathrm{R}}^4\overset{\text{def}}{=}\big\{\rho_{\chi,s}(\overline{B_{\mathrm{R},s}})\leq\varepsilon\big\}\,.$$

We also define the event $\mathrm{E}_{s,\mathrm{R}}\overset{\text{def}}{=}\cap_{i=1}^3\mathrm{E}_s^i\cap\mathrm{E}_{s,\mathrm{R}}^4$ and we note that

$$\mathbb{E}\left[\exp\left(-\int_{[0,1]^d}\mathrm{F}_{\gamma,t}(x)\mu_{\gamma,t}(dx)\right)\middle|\mathcal{F}_s\right]\geq\mathbb{E}\left[\exp\left(-\int_{[0,1]^d}\mathrm{F}_{\gamma,t}(x)\mu_{\gamma,t}(dx)\right)\mathbb{1}_{\mathrm{E}_{s,\mathrm{R}}}\middle|\mathcal{F}_s\right],$$

$$\mathbb{E}\left[\exp\left(-\int_{[0,1]^d}\mathrm{F}_{\gamma,t}(x)\mu_{\gamma,t}(dx)\right)\middle|\mathcal{F}_s\right]\leq\mathbb{E}\left[\exp\left(-\int_{[0,1]^d}\mathrm{F}_{\gamma,t}(x)\mu_{\gamma,t}(dx)\right)\mathbb{1}_{\mathrm{E}_{s,\mathrm{R}}}\middle|\mathcal{F}_s\right]+\mathbb{1}_{\mathrm{E}_{s,\mathrm{R}}^c}\,,$$

where the second inequality is simply due to the fact that the map $x\mapsto e^{-x}$ is bounded by one, for all $x\geq 0$. Moreover, we observe that, by definition, on the event $\mathrm{E}_{s,\mathrm{R}}$, the function χ defined in (2.6.18) satisfies the conditions required in the statement of Corollary 2.6.5. Therefore, for $\varepsilon>0$ as fixed above and for $s\geq 0$ sufficiently large, it holds almost surely that

$$\liminf_{t\to\infty}\mathbb{E}\left[\exp\left(-\int_{[0,1]^d}\mathrm{F}_{\gamma,t}(x)\mu_{\gamma,t}(dx)\right)\middle|\mathcal{F}_s\right]\geq\Big(\exp\Big(-(\widetilde{a}_\star+\varepsilon)\mathrm{T}_\gamma\mu_{\gamma_c,s}\big([0,1]^d\big)\Big)-\varepsilon\Big)\mathbb{1}_{\mathrm{E}_{s,\mathrm{R}}}\,,$$

$$\limsup_{t\to\infty}\mathbb{E}\left[\exp\left(-\int_{[0,1]^d}\mathrm{F}_{\gamma,t}(x)\mu_{\gamma,t}(dx)\right)\middle|\mathcal{F}_s\right]\leq\left(\exp\Big(-(\widetilde{a}_\star-\varepsilon)\mathrm{T}_\gamma\mu_{\gamma_c,s}\big([0,1]^d\big)\Big)+\varepsilon\right)\mathbb{1}_{\mathrm{E}_{s,\mathrm{R}}}+\mathbb{1}_{\mathrm{E}_{s,\mathrm{R}}^c}\,.$$

Thanks to [DRSV14a, Theorem 4], the sequence of random measure $(\mu_{\gamma_c,s})_{s\geq 0}$ converges as $s\to\infty$ almost surely to μ_{γ_c} in the topology of vague convergence. Furthermore, we can extract positive diverging sequences $(s_n)_{n\in\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}}$ such that $\mathbb{1}_{E_{s_n,R_n}}$ converges almost surely to 1 as $n\to\infty$. This follows since, thanks to [Mad15, Lemmas 3.1, 3.2, 3.3] (see also [MRV16, Lemma A.2]), we have that $\mathbb{P}(E_{s,R})$ converges to 1 as $s, R\to\infty$. In particular, this shows that the following convergences hold almost surely

$$\lim_{n\to\infty}\liminf_{t\to\infty}\mathbb{E}\left[\exp\left(-\int_{[0,1]^d}F_{\gamma,t}(x)\mu_{\gamma,t}(dx)\right)\middle|\mathcal{F}_{s_n}\right]\geq\exp\Big(-(\widetilde{a}_\star+\varepsilon)T_\gamma\mu_{\gamma_c}\big([0,1]^d\big)\Big)-\varepsilon\,,$$
$$\lim_{n\to\infty}\limsup_{t\to\infty}\mathbb{E}\left[\exp\left(-\int_{[0,1]^d}F_{\gamma,t}(x)\mu_{\gamma,t}(dx)\right)\middle|\mathcal{F}_{s_n}\right]\leq\exp\Big(-(\widetilde{a}_\star-\varepsilon)T_\gamma\mu_{\gamma_c}\big([0,1]^d\big)\Big)+\varepsilon\,.$$

Therefore, the conclusion follows by arbitrariness of $\varepsilon>0$. □

2.6.4 Stable convergence

The goal of this section is to prove Proposition 2.2.19, from which Theorem 2.C follows immediately.

Proof of Proposition 2.2.19. For $n\in\mathbb{N}$ and $\gamma>\sqrt{2d}$, we consider the collection of measures $(\mu_{\gamma,i})_{i\in[n]}$ introduced in the statement of Theorem 2.C. We recall that we need to check that, for all $(\varphi,(f_i)_{i\in[n]})\in\mathcal{C}_c^\infty(\mathbb{R}^d)\times(\mathcal{C}_c^+(\mathbb{R}^d))^n$, the following holds

$$\lim_{t\to\infty}\mathbb{E}\left[\exp\big(i\langle X,\varphi\rangle\big)\prod_{i=1}^n\exp\big(-\mu_{\gamma,t,i}(f_i)\big)\right]=\mathbb{E}\Big[\exp\big(i\langle X,\varphi\rangle\big)\exp\big(-\widetilde{a}_\star\mu_{\gamma_c}(T_\gamma)\big)\Big]\,,$$

where here, to simplify the notation, we have omitted the dependence of T_γ on $(f_i)_{i\in[n]}$, and where we recall that $\widetilde{a}_\star=\beta(d,\gamma)a_\star$ with $a_\star$ as defined in (2.2.12).

We consider the sequence $(s_n)_{n\in\mathbb{N}}$ introduced in the statement of Lemma 2.6.7. For $n\in\mathbb{N}$, we consider u, $t\geq 0$ such that $u<s_n<t$, and we note that

$$\begin{aligned}\left|\mathbb{E}\left[\exp\big(i\langle X,\varphi\rangle\big)\left(\prod_{i=1}^n\exp\big(-\mu_{\gamma,t,i}(f_i)\big)-\exp\big(-\widetilde{a}_\star\mu_{\gamma_c}(T_\gamma)\big)\right)\right]\right|&\\ \leq\left|\mathbb{E}\left[\exp\big(i\langle X_u,\varphi\rangle\big)\left(\prod_{i=1}^n\exp\big(-\mu_{\gamma,t,i}(f_i)\big)-\exp\big(-\widetilde{a}_\star\mu_{\gamma_c}(T_\gamma)\big)\right)\right]\right|&\\ +2\mathbb{E}\Big[\Big|\exp\big(i\langle X,\varphi\rangle\big)-\exp\big(i\langle X_u,\varphi\rangle\big)\Big|\Big]\,,&\end{aligned}$$

where we simply used the triangle inequality and the fact that the function $x\mapsto e^{-x}$ is bounded by 1 for $x\geq 0$. For the term appearing in the second line of the above display, we note that it is equal to

$$\left|\mathbb{E}\left[\exp\big(i\langle X_u,\varphi\rangle\big)\mathbb{E}\left[\prod_{i=1}^n\exp\big(-\mu_{\gamma,t,i}(f_i)\big)-\exp\big(-\widetilde{a}_\star\mu_{\gamma_c}(T_\gamma)\big)\,\middle|\,\mathcal{F}_{s_n}\right]\right]\right|\,,$$

where we used the fact that X_u is $\mathcal{F}_{s_n}$-measurable since by assumption $u<s_n$. On the one hand, Lemma 2.6.7 implies that the following convergence holds almost surely

$$\lim_{n\to\infty}\lim_{t\to\infty}\mathbb{E}\left[\prod_{i=1}^n\exp\big(-\mu_{\gamma,t,i}(f_i)\big)\,\middle|\,\mathcal{F}_{s_n}\right]=\exp\big(-\widetilde{a}_\star\mu_{\gamma_c}(T_\gamma)\big)\,.\tag{2.6.19}$$

On the other hand, the martingale convergence theorem implies that the following convergence holds almost surely

$$\lim_{n\to\infty} \mathbb{E}\Big[\exp\big(-\widetilde{a}_\star \mu_{\gamma_c}(\mathrm{T}_\gamma)\big) \,\Big|\, \mathcal{F}_{s_n}\Big] = \mathbb{E}\Big[\exp\big(-\widetilde{a}_\star \mu_{\gamma_c}(\mathrm{T}_\gamma)\big) \,\Big|\, \sigma(\mathrm{X})\Big] = \exp\big(-\widetilde{a}_\star \mu_{\gamma_c}(\mathrm{T}_\gamma)\big) \,. \tag{2.6.20}$$

Therefore, since the right-hand side of (2.6.19) coincides with the right-hand side of (2.6.20), by the dominated convergence theorem, we obtain that uniformly over $u \geq 0$, it holds that

$$\lim_{n\to\infty} \lim_{t\to\infty} \left| \mathbb{E}\left[\exp\big(i\langle \mathrm{X}_u, \varphi\rangle\big) \mathbb{E}\left[\prod_{i=1}^n \exp\big(-\mu_{\gamma,t,i}(f_i)\big) - \exp\big(-\widetilde{a}_\star \mu_{\gamma_c}(\mathrm{T}_\gamma)\big) \,\middle|\, \mathcal{F}_{s_n} \right] \right] \right| = 0 \,.$$

Thanks to the dominated convergence theorem and the fact that X_u converges to X as $u \to \infty$ almost surely in $\mathcal{H}^{-\kappa}(\mathbb{R}^d)$, one has

$$\lim_{u\to\infty} \mathbb{E}\Big[\Big| \exp\big(i\langle \mathrm{X}, \varphi\rangle\big) - \exp\big(i\langle \mathrm{X}_u, \varphi\rangle\big) \Big| \Big] = 0 \,,$$

and so the conclusion follows. □

We now show how the proof of Theorem 2.C follows directly.

Proof of Theorem 2.C. Thanks to Lemma 2.3.8, it suffices to show that the following joint convergence in distribution holds in $\mathcal{H}^{-\kappa}_{\mathrm{loc}}(\mathbb{R}^d) \times (\mathcal{M}^+(\mathbb{R}^d))^n$ for some $\kappa > 0$,

$$\big(\mathrm{X}, (\mu_{\gamma,t,i})_{i\in[n]}\big) \Rightarrow \big(\mathrm{X}, (\mu_{\gamma,i})_{i\in[n]}\big) \,.$$

By Lemma 2.3.4, it suffices to verify that for all $(\varphi, (f_i)_{i\in[n]}) \in \mathcal{C}^\infty_c(\mathbb{R}^d) \times (\mathcal{C}^+_c(\mathbb{R}^d))^n$, it holds that

$$\lim_{t\to\infty} \mathbb{E}\left[\exp\big(i\langle \mathrm{X}, \varphi\rangle\big) \prod_{i=1}^n \exp\big(-\mu_{\gamma,t,i}(f_i)\big) \right] = \mathbb{E}\left[\exp\big(i\langle \mathrm{X}, \varphi\rangle\big) \prod_{i=1}^n \exp\big(-\mu_{\gamma,i}(f_i)\big) \right] .$$

Furthermore, by Proposition 2.2.19, it suffices to establish that

$$\mathbb{E}\left[\exp\big(i\langle \mathrm{X}, \varphi\rangle\big) \prod_{i=1}^n \exp\big(-\mu_{\gamma,i}(f_i)\big) \right] = \mathbb{E}\Big[\exp\big(i\langle \mathrm{X}, \varphi\rangle\big) \exp\big(-\widetilde{a}_\star \mu_{\gamma_c}(\mathrm{T}_\gamma)\big) \Big] \,, \tag{2.6.21}$$

where, as before, to simplify the notation, we have omitted the dependence of T_γ on $(f_i)_{i\in[n]}$. Recalling the definition of the collection of measures $(\mu_{\gamma,i})_{i\in[n]}$ given in the statement of Theorem 2.C, one can use formula (2.3.5) to check that

$$\mathbb{E}\left[\prod_{i=1}^n \exp\big(-\mu_{\gamma,i}(f_i)\big) \,\middle|\, \sigma(\mathrm{X}) \right] = \exp\big(-\widetilde{a}_\star \mu_{\gamma_c}(\mathrm{T}_\gamma)\big) \,,$$

thus yielding (2.6.21), which completes the proof. □

2.7 Proof of Proposition 2.6.1

The main goal of this section is to prove Proposition 2.6.1, and it is structured as follows. In Section 2.7.1, we introduce some notation and state Lemmas 2.7.2 and 2.7.4, which are the two main technical lemmas used in the proof of Proposition 2.6.1. We then show how the proof of Proposition 2.6.1 follows from the two

aforementioned lemmas and Propositions 2.4.1 and 2.4.2. The remaining part of the section is then devoted to the proof of Lemmas 2.7.2 and 2.7.4. In particular, in Section 2.7.2, we prove Lemma 2.7.2, while in Section 2.7.3, we prove Lemma 2.7.4.

2.7.1 Setup and main technical lemmas

Throughout this section, we fix $\gamma > \sqrt{2d}$, $n \in \mathbb{N}$, and a collection of non-negative constants $(\theta_i)_{i\in[n]}$. For any $x \in \mathbb{R}$, $k > 0$, and $\lambda > 0$, we define the (random) function $\mathbf{F}^{\lambda}_{k,x} : \mathcal{C}(\mathbb{R}^d) \to \mathbb{R}$ by

$$\mathbf{F}^{\lambda}_{k,x}(\Phi) \stackrel{\text{def}}{=} \frac{1 - \exp\big(-e^{-\gamma x}\int_{\mathbb{B}_k} \mathrm{F}_\gamma(y) e^{\gamma\Phi(y)} dy\big)}{|\mathbb{D}^{\lambda}_{k,0}(\Phi)|} , \tag{2.7.1}$$

where $\mathrm{F}_\gamma : \mathbb{R}^d \to \mathbb{R}$ is the function given by

$$\mathrm{F}_\gamma(y) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \theta_i e^{\gamma \mathrm{W}_i(y)} . \tag{2.7.2}$$

We introduce here the main processes and fields that will be used for the reminder of this section:

- Let B' be a standard Brownian motion and R a three-dimensional Bessel process starting at zero. For any $z \geq 0$, define U_z to be a random variable uniformly distributed on the interval $[-z, 0]$, independent of all other processes. Also, let $\tau_z \stackrel{\text{def}}{=} \inf\{s \geq 0 : B'_s = \mathrm{U}_z\}$. We then define the process

$$\Gamma^z_s \stackrel{\text{def}}{=} \begin{cases} B'_s , & \text{if } s \leq \tau_z , \\ R_{s-\tau_z} + \mathrm{U}_z , & \text{if } s > \tau_z . \end{cases} \tag{2.7.3}$$

- For $b > 0$ and $z \geq 0$, we let $\mathfrak{g}^z_b$ be the field on $\mathbb{R}^d$ given by

$$\mathfrak{g}^z_b(\cdot) \stackrel{\text{def}}{=} -\int_0^\infty \big(1 - \mathfrak{K}(e^{-(s+b)}\cdot)\big) d\Gamma^z_s + \mathrm{Z}'_\infty(e^{-b}\cdot) - \sqrt{2d}\int_0^\infty \big(1 - \mathfrak{K}(e^{-(s+b)}\cdot)\big) ds , \tag{2.7.4}$$

  where Z'_∞ has the same law as the field Z_∞ defined in Definition 2.3.9, and it is independent of the process Γ^z.

- For $b > 0$, we recall that Υ_b denotes the field on $\mathbb{R}^d$ introduced in (2.4.2) and given by

$$\Upsilon_b(\cdot) \stackrel{\text{def}}{=} -\int_0^b \big(1 - \mathfrak{K}(e^{-s}\cdot)\big) dB_s + \mathrm{Z}_b(\cdot) - \sqrt{2d}\mathfrak{a}_b(\cdot) ,$$

  where Z_b is introduced in Definition 2.3.9, B is an independent Brownian motion, and $\mathfrak{a}_b$ is the function defined in (2.2.2).

Remark 2.7.1. We emphasise that the processes and fields introduced above are all assumed to be mutually independent. Additionally, given $x, y \in \mathbb{R}$ and $b > 0$, we write $\mathbb{P}_{x,y,b}$ for the probability measure under which $(B_s)_{s\in[0,b]}$ is a Brownian bridge from x to y in time b, while the other processes/fields are left unchanged. Moroever, given a function $g : \mathbb{R}^d \to \mathbb{R}$, we set $\Upsilon_{b,g} = \Upsilon_b + g$.

In what follows, for $b > 0$ and $z \geq 0$, we need to consider the field given by the sum of Υ_b and $\mathfrak{g}_b^z$. In order to lighten the notation, instead of writing the field $\mathfrak{g}_b^z$ as a subscript of Υ_b, we let

$$\Upsilon_b^z \overset{\text{def}}{=} \Upsilon_b + \mathfrak{g}_b^z \, . \tag{2.7.5}$$

We also observe that a standard Gaussian tail bound implies that, uniformly over all $z \geq 0$, the field $\mathfrak{g}_b^z$ satisfies (G1) – (G3).

For $\lambda > 0$ and A, L, $b \geq 0$, we introduce the function $\mathfrak{F}^{\lambda}_{A,L,b} : \mathbb{R} \times \mathcal{C}(\mathbb{R}^d) \to \mathbb{R}$ given by

$$\begin{aligned} \mathfrak{F}^{\lambda}_{A,L,b}(z,g) \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi b}} \int_0^{A+L} e^{\sqrt{2d}(x-L) - \frac{(x-z)^2}{2b}} \mathbb{E}_{x,z,b}\Big[\mathbf{F}^{\lambda}_{b,x-L}(\Upsilon_{b,g}) \\ \cdot \mathbb{1}_{\{\inf_{s\in[0,b]} B_s \geq 0\}} \mathbb{1}_{\{\mathbb{M}_{0,b+1,b}(\Upsilon_{b,g}) \leq x-(A+L)\}} \mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,g}) \leq \lambda\}} \Big] dx \, . \end{aligned} \tag{2.7.6}$$

Moreover, we define the constant $\mathbf{C}^{\lambda}_{A,L,b} > 0$ by letting

$$\mathbf{C}^{\lambda}_{A,L,b} \overset{\text{def}}{=} \alpha \int_0^{\infty} z \mathbb{E}\big[\mathfrak{F}^{\lambda}_{A,L,b}\big(z, \mathfrak{g}_b^z\big)\big] dz \, , \tag{2.7.7}$$

where we recall that $\alpha = \sqrt{2/\pi}$.

We are now ready to state the following key lemma whose proof is given in Section 2.7.2.

Lemma 2.7.2. *For any $\lambda > 0$, $\mathrm{R} \geq 1$, and $\varepsilon > 0$, there exist $0 \leq A < L$ sufficiently large such that there exists $b_0 > 0$ and $s_0 > 0$ sufficiently large, such that for any $s \geq s_0$ satisfying $(e^s + 1)/(\mathrm{R}+1) \in \mathbb{N}$ and any $b \geq b_0$, there exists a sufficiently large $T \geq 0$ such that for all $t \geq T$ and any function $\chi : [0,\mathrm{R}]^d \to \mathbb{R}$ satisfying the conditions in (2.6.6), it holds that*

$$\left| \mathbb{E}\left[1 - \exp\left(- \int_{[0,\mathrm{R}]^d} \mathrm{F}_{\gamma,t}(x) e^{-\gamma\chi(x)} \mu_{\gamma,t}(dx) \right) \right] - \mathbf{C}^{\lambda}_{A,L,b} \, \rho_{\chi}\big([0,\mathrm{R}]^d\big) \right| \leq \varepsilon \, \rho_{\chi}\big([0,\mathrm{R}]^d\big) \, ,$$

where we recall that the measure ρ_{χ} is defined in (2.6.5).

Remark 2.7.3. We emphasise that in the statement of Lemma 2.7.2, the conditions (2.6.6) on the function $\chi : [0,\mathrm{R}]^d \to \mathbb{R}$ depends on s.

Thanks to Lemma 2.7.2, to prove Proposition 2.6.1 it suffices to derive an explicit expression for the constant $\mathbf{C}^{\lambda}_{A,L,b}$ defined in (2.7.7) as the cutoff parameters are taken to infinity, and then show that this expression coincides with T_{γ} defined in (2.6.1), up to a multiplicative constant. To this end, for A, $L \geq 0$ and $0 \leq k < b$, recalling that $\alpha = \sqrt{2/\pi}$, we define the constant $\mathbf{C}^{\lambda,\mathrm{new}}_{A,L,k,b}$ by letting

$$\begin{aligned} \mathbf{C}^{\lambda,\mathrm{new}}_{A,L,k,b} = \alpha \int_{L/2}^{A+L} e^{\sqrt{2d}(x-L)} \int_{b^{1/4}}^{b^{3/4}} \frac{e^{-\frac{u^2}{2b}}}{\sqrt{2\pi b}}(u+x) \\ \cdot \mathbb{E}_{0,u,b}\Big[\mathbf{F}^{\lambda}_{k,x-L,L}\big(\Upsilon_b^{u+x}\big) \mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b^{u+x}) \leq \lambda\}} \Big] du dx \, , \end{aligned} \tag{2.7.8}$$

where for any $x \in \mathbb{R}$, $k > 0$, $L > 0$ and $\lambda > 0$, we define the (random) function $\mathbf{F}^{\lambda}_{k,x,L} : \mathcal{C}(\mathbb{R}^d) \to \mathbb{R}$ by

$$\mathbf{F}^{\lambda}_{k,x,L}(\Phi) \overset{\text{def}}{=} \frac{1 - \exp\big(-e^{-\gamma x} \int_{\mathbb{B}_k} \mathrm{F}_{\gamma}(y) e^{\gamma\Phi(y)} dy\big)}{|\mathbb{D}^{\lambda}_{k,0}(\Phi)| \vee L^{-1}} \, , \tag{2.7.9}$$

where we emphasise that the only difference from $\mathbf{F}^{\lambda}_{k,x}$, as defined in (2.7.1), is the presence of the maximum in the denominator. Note that, with a slight abuse of our previous notation, we have absorbed the expectation with respect to the field $\mathfrak{g}_b^{u+x}$ into the expectation $\mathbb{E}_{0,u,b}$.

We can now state the second key lemma of this section whose proof is given in Section 2.7.3.

Lemma 2.7.4. *For any $\lambda > 0$, $\varepsilon > 0$, and $A \geq 0$, there exists $L \geq 0$ sufficiently large, such that there exists $k_0 \geq 0$ for which, for all $k \geq k_0$, there exists $b_0 \geq 0$ sufficiently large such that for all $b \geq b_0$, it holds that*

$$|\mathbf{C}^{\lambda}_{A,L,b} - \mathbf{C}^{\lambda,\mathrm{new}}_{A,L,k,b}| \leq \varepsilon \ .$$

The proof of Proposition 2.6.1 follows by combining Lemmas 2.7.2 and 2.7.4.

Proof of Proposition 2.6.1. Let $\mathrm{R} \geq 1$ and $\lambda \in \Lambda$, where we recall that the set Λ is introduced in Lemma 2.5.1. For any $\varepsilon > 0$, thanks to Lemmas 2.7.2 and 2.7.4, by taking the involved parameters large enough as specified in the statements of the aforementioned lemmas, we have that

$$\left| \mathbb{E}\left[1 - \exp\left(- \int_{[0,\mathrm{R}]^d} \mathrm{F}_{\gamma,t}(x) \mu^{\chi}_{\gamma,t}(dx) \right) \right] - \mathbf{C}^{\lambda,\mathrm{new}}_{A,L,k,b} \rho_{\chi}\big([0,\mathrm{R}]^d\big) \right| \leq \varepsilon \rho_{\chi}\big([0,\mathrm{R}]^d\big) \ .$$

Hence it suffices to show that when the cutoff parameters are taken to infinity, the limit of the constant $\mathbf{C}^{\lambda,\mathrm{new}}_{A,L,k,b}$ coincides with T_{γ} defined in (2.6.1), up to a multiplicative constant. To establish that this is indeed the case, we rely on Propositions 2.4.1 and 2.4.2.

We observe that for any A, L, $k \geq 0$, $x \in [L/2, A + L]$, and $k \in \mathbb{N}$, conditional on $(\mathrm{W}_i)_{i \in [n]}$, the function $\mathbf{F}^{\lambda}_{k,x-L,L} : \mathcal{C}(\mathbb{R}^d) \to \mathbb{R}$ is continuous, bounded, depends on the values of the input field in a bounded set, and it is such that its set of discontinuities is assigned measure zero by the law of the field $\widetilde{\Upsilon}_{\lambda}$ on $\mathcal{C}(\mathbb{R}^d)$. This last fact follows since we fixed $\lambda \in \Lambda$ and from Remark 2.5.2. Furthermore, as can be easily verified using the Gaussian tail bound, the field $\mathfrak{g}^z_b$, introduced in (2.7.4), satisfies (G1)–(G3) uniformly over all $z \geq 0$. Therefore, we are in a position to apply Propositions 2.4.1 and 2.4.2, from which we deduce that for any $\varepsilon > 0$, $k \geq 0$, $b \geq 0$ sufficiently large, and $u \in [b^{1/4}, b^{3/4}]$, it holds that

$$\left| \mathbb{E}_{0,u,b}\left[\mathbf{F}^{\lambda}_{k,x-L,L}\big(\Upsilon_b^{u+x}\big) \mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b^{u+x}) \leq \lambda\}} \right] - 2c_{\star,\lambda} \frac{u}{b} \mathbb{E}\big[\mathbf{F}^{\lambda}_{k,x-L,L}\big(\widetilde{\Upsilon}_{\lambda}\big) \big] \right| \leq \varepsilon \frac{u}{b} \ . \tag{2.7.10}$$

Furthermore, thanks to Proposition 2.2.12, it holds that

$$\mathbb{E}\big[\mathbf{F}^{\lambda}_{k,x-L,L}(\widetilde{\Upsilon}_{\lambda}) \big] = \frac{\mathbb{E}\big[\int_{\mathbb{R}^d} \mathbf{F}^{\lambda}_{k,x-L,L}(\tau_z \Psi) e^{\sqrt{2d}\Psi(z)} \mathbb{1}_{\{\Psi(z) \geq -\lambda\}} dz \big]}{\mathbb{E}\big[\int_{\mathbb{R}^d} e^{\sqrt{2d}\Psi(z)} \mathbb{1}_{\{\Psi(z) \geq -\lambda\}} dz \big]} \ , \tag{2.7.11}$$

where, once again, we recall that we omit the subscript λ when writing the field Ψ, since, thanks to Proposition 2.2.14, its law does not depend on $\lambda \in \Lambda$. Therefore, recalling the definition (2.2.12) of $a_{\star}$ and performing a change of variables in the integral over x in the definition (2.7.8) of $\mathbf{C}^{\lambda,\mathrm{new}}_{A,L,k,b}$, we obtain that

$$\lim_{b \to \infty} \mathbf{C}^{\lambda,\mathrm{new}}_{A,L,k,b} = a_{\star} \gamma \int_{-L/2}^{A} e^{\sqrt{2d}x} \mathbb{E}\left[\int_{\mathbb{R}^d} \mathbf{F}^{\lambda}_{k,x,L}(\tau_z \Psi) e^{\sqrt{2d}\Psi(z)} \mathbb{1}_{\{\Psi(z) \geq -\lambda\}} dz \right] dx \ . \tag{2.7.12}$$

Recalling the definition of the function $\mathbf{F}^{\lambda}_{k,x}$ in (2.7.1) and that of $\mathbf{F}^{\lambda}_{k,x,L}$ in (2.7.9), and using the dominated convergence theorem along with the monotone convergence theorem, we observe that taking first the limit as $k \to \infty$, then as $L \to \infty$ and finally as $A \to \infty$ of the expression on the right-hand side of (2.7.12), we

conclude that $\lim_{A,L,k,b\to\infty} \mathbf{C}^{\lambda,\mathrm{new}}_{A,L,k,b}$ equals

$$a_\star \gamma \mathbb{E}\left[\int_{\mathbb{R}^d}\int_{-\infty}^{\infty} e^{\sqrt{2d}x}\left(1-\exp\left(-e^{-\gamma x}\int_{\mathbb{R}^d}\mathrm{F}_\gamma(y)e^{\gamma\tau_z\Psi(y)}dy\right)\right)\frac{e^{\sqrt{2d}\Psi(z)}\mathbb{1}_{\{\Psi(z)\geq-\lambda\}}}{|\mathbb{D}^\lambda(\tau_z\Psi)|}dxdz\right].$$

Now, recalling the definition (2.2.14) of the constant $\beta(d,\gamma)$, we note that, for all $c>0$ and any $\gamma>\sqrt{2d}$, it holds that

$$\int_{-\infty}^{\infty} e^{\sqrt{2d}x}(1-e^{-ce^{-\gamma x}})dx = \frac{c^{\frac{\sqrt{2d}}{\gamma}}\beta(d,\gamma)}{\gamma}\,.$$

Therefore, by collecting the previous considerations, using the above identity, and leveraging the independence between the fields $(\mathrm{W}_i)_{i\in[n]}$ and Ψ, as well as the stationarity of the fields $(\mathrm{W}_i)_{i\in[n]}$ implied by (W2), we obtain that

$$\begin{aligned}\lim_{A,L,k,b\to\infty}\mathbf{C}^{\lambda,\mathrm{new}}_{A,L,k,b} &= \widetilde{a}_\star\,\mathbb{E}\left[\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\mathrm{F}_\gamma(y)e^{\gamma\tau_z\Psi(y)}dy\right)^{\frac{\sqrt{2d}}{\gamma}}\frac{e^{\sqrt{2d}\Psi(z)}\mathbb{1}_{\{\Psi(z)\geq-\lambda\}}}{|\mathbb{D}^\lambda(\tau_z\Psi)|}dz\right]\\ &= \widetilde{a}_\star\,\mathbb{E}\left[\left(\int_{\mathbb{R}^d}\mathrm{F}_\gamma(y)e^{\gamma\Psi(y)}dy\right)^{\frac{\sqrt{2d}}{\gamma}}\right]\\ &= \widetilde{a}_\star\,\mathrm{T}_\gamma\,,\end{aligned}$$

where we recall that T_γ is defined in (2.6.1), and $\widetilde{a}_\star=\beta(d,\gamma)a_\star$. Hence, the desired result follows. □

2.7.2 Proof of Lemma 2.7.2

In this subsection, we prove Lemma 2.7.2 by reducing the joint Laplace transform of $\mu_{\gamma,t}$ to a more manageable quantity through a series of reduction steps, following the strategy developed in [MRV16] with some non-trivial modifications. To streamline this process, we introduce some shorthand notations. We fix for the reminder of this section $\mathrm{R}\geq 1$. For a function $\chi:[0,\mathrm{R}]^d\to\mathbb{R}$, we introduce the sequence of measures $(\mu^\chi_{\gamma,t})_{t\geq0}$ on $\mathbb{R}^d$ defined as follows,

$$\mu^\chi_{\gamma,t}(dx)\overset{\text{def}}{=}e^{-\gamma\chi(x)}\mu_{\gamma,t}(dx)\,,$$

where we recall that $\mu_{\gamma,t}$ is the regularised and normalised supercritical GMC measure as defined in (2.1.8). For $t>0$ and a function $\chi:\mathbb{R}^d\to\mathbb{R}$, we introduce the fields Y_t and Y^χ_t on $\mathbb{R}^d$, as well as the constant $\mathfrak{d}_t$ by setting

$$\mathrm{Y}_t(x)\overset{\text{def}}{=}\mathrm{X}_t(x)-\sqrt{2d}t\,,\qquad \mathrm{Y}^\chi_t(x)\overset{\text{def}}{=}\mathrm{Y}_t(x)-\chi(x)\,,\qquad \mathfrak{d}_t\overset{\text{def}}{=}-\frac{3}{2\sqrt{2d}}\log t\,, \tag{2.7.13}$$

so that the measure $\mu^\chi_{\gamma,t}$ introduced above can be written as

$$\mu^\chi_{\gamma,t}(dx)=e^{\gamma(\mathrm{Y}^\chi_t(x)-\mathfrak{d}_t)+dt}dx\,.$$

For $\lambda>0$, recalling the definition (2.6.1) of the random function $\mathrm{F}_{\gamma,t}$ and the notation introduced in (2.3.2), only for this section, we let

$$\mathbb{G}_\mathrm{R}\overset{\text{def}}{=}\exp\left(-\int_{[0,\mathrm{R}]^d}\mathrm{F}_{\gamma,t}(x)\mu^\chi_{\gamma,t}(dx)\right),\qquad \mathbb{M}_\mathrm{R}\overset{\text{def}}{=}\mathbb{M}_\mathrm{R}(\mathrm{Y}^\chi_t)\,,\qquad \mathbb{D}^\lambda_\mathrm{R}=\mathbb{D}^\lambda_\mathrm{R}(\mathrm{Y}^\chi_t)\,, \tag{2.7.14}$$

and for a subset $D \subseteq [0,R]^d$, we use $\mathbb{G}_D$ to denote the same quantity as above, but with $[0,R]^d$ replaced by D. With this notation in hand, we note that we can write

$$\mathbb{E}\big[1-\mathbb{G}_R\big] = \mathbb{E}\left[\int_{[0,R]^d} \frac{\mathbb{1}_{\{m\in\mathbb{D}_R^\lambda\}}}{|\mathbb{D}_R^\lambda|}\big(1-\mathbb{G}_R\big)dm\right], \tag{2.7.15}$$

where here we used the fact that $|\mathbb{D}_R^\lambda|$ is almost surely positive.

2.7.2.1 High value constraint

We start with the following lemma, which essentially states that only the points where the field attains sufficiently high values contribute to the integral on the right-hand side of (2.7.15).

Lemma 2.7.5. *For any $\lambda > 0$ and $\varepsilon > 0$, there exists a constant $A \geq 0$ sufficiently large such that for any $s \geq 0$ sufficiently large satisfying $(e^s+1)(R+1)^{-1} \in \mathbb{N}$, there exists $T \geq 0$ sufficiently large such that for any $t \geq T$ and $\chi : [0,R]^d \to \mathbb{R}$ satisfying the conditions in (2.6.6), it holds that*

$$\mathbb{E}\left[\int_{[0,R]^d} \frac{\mathbb{1}_{\{m\in\mathbb{D}_R^\lambda\}}\mathbb{1}_{\{Y_t^\chi(m)-\mathfrak{d}_t\leq -A\}}}{|\mathbb{D}_R^\lambda|}\big(1-\mathbb{G}_R\big)dm\right] \leq \varepsilon\rho_\chi\big([0,R]^d\big)\,. \tag{2.7.16}$$

Proof. We start by observing that on the event $\{m\in\mathbb{D}_R^\lambda,\ Y_t^\chi(m)-\mathfrak{d}_t\leq -A\}$, it holds that

$$Y_t^\chi(x)-\mathfrak{d}_t \leq -A+\lambda\,, \qquad \forall\, x\in[0,R]^d\,.$$

Therefore, using this fact, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\int_{[0,R]^d} \frac{\mathbb{1}_{\{m\in\mathbb{D}_R^\lambda\}}\mathbb{1}_{\{Y_t^\chi(m)-\mathfrak{d}_t\leq -A\}}}{|\mathbb{D}_R^\lambda|}\big(1-\mathbb{G}_R\big)dm\right]\\
&\quad\leq \mathbb{E}\left[\int_{[0,R]^d} \frac{\mathbb{1}_{\{m\in\mathbb{D}_R^\lambda\}}}{|\mathbb{D}_R^\lambda|}\left(1-\exp\left(-\int_{[0,R]^d} F_{\gamma,t}(x)\mathbb{1}_{\{Y_t^\chi(x)-\mathfrak{d}_t\leq -A+\lambda\}}\mu_{\gamma,t}^\chi(dx)\right)\right)dm\right]\\
&\quad= \mathbb{E}\left[1-\exp\left(-\int_{[0,R]^d} F_{\gamma,t}(x)\mathbb{1}_{\{Y_t^\chi(x)-\mathfrak{d}_t\leq -A+\lambda\}}\mu_{\gamma,t}^\chi(dx)\right)\right].
\end{aligned}$$

If $F_{\gamma,t}(\cdot)$ were a deterministic, t-independent function, then the conclusion would follow by a direct application of [MRV16, Proposition 4.1]. However, by following the proof of [MRV16, Proposition 4.1], the same conclusion holds also in our more general setting by using (W1) and (W4). □

For $A \geq 0$, we introduce the quantity

$$\mathbb{E}_{(2.7.17)} \overset{\text{def}}{=} \mathbb{E}\left[\int_{[0,R]^d} \frac{\mathbb{1}_{\{m\in\mathbb{D}_R^\lambda\}}\mathbb{1}_{\{Y_t^\chi(m)-\mathfrak{d}_t\geq -A\}}}{|\mathbb{D}_R^\lambda|}\big(1-\mathbb{G}_R\big)dm\right]. \tag{2.7.17}$$

The upshot of Lemma 2.7.5 is that for any $\varepsilon > 0$, we can find $A \geq 0$ sufficiently large such that for any $s \geq 0$ sufficiently large satisfying $(e^s+1)(R+1)^{-1} \in \mathbb{N}$, there exists $T \geq 0$ sufficiently large such that for any $t \geq T$ and $\chi : [0,R]^d \to \mathbb{R}$ satisfying the conditions in (2.6.6),

$$\left|\mathbb{E}\left[1-\exp\left(-\int_{[0,R]^d} F_{\gamma,t}(x)\mu_{\gamma,t}^\chi(dx)\right)\right] - \mathbb{E}_{(2.7.17)}\right| \leq \varepsilon\rho_\chi\big([0,R]^d\big)\,.$$

Therefore, in what follows we we can just focus on the expectation $\mathbb{E}_{(2.7.17)}$ for a fixed $A \geq 0$.

2.7.2.2 Path constraint

We now want to exclude the points $m \in [0,\mathrm{R}]^d$ such that $\mathrm{Y}_t^\chi(m) - \mathfrak{d}_t \geq -A$ with an unlikely path $[0,t] \ni s \mapsto \mathrm{Y}_s(m)$. To this end, for A, L, $z \geq 0$, we consider the set of functions

$$\mathbb{S}_t^{z,A,L} \overset{\text{def}}{=} \Big\{ \phi : \mathbb{R}_0^+ \to \mathbb{R} : \sup_{s\in[0,t]} \phi(s) \leq z, \sup_{s\in[t/2,t]} \phi(s) \leq z + \mathfrak{d}_t + L,\ \phi(t) \geq z + \mathfrak{d}_t - A \Big\} .$$

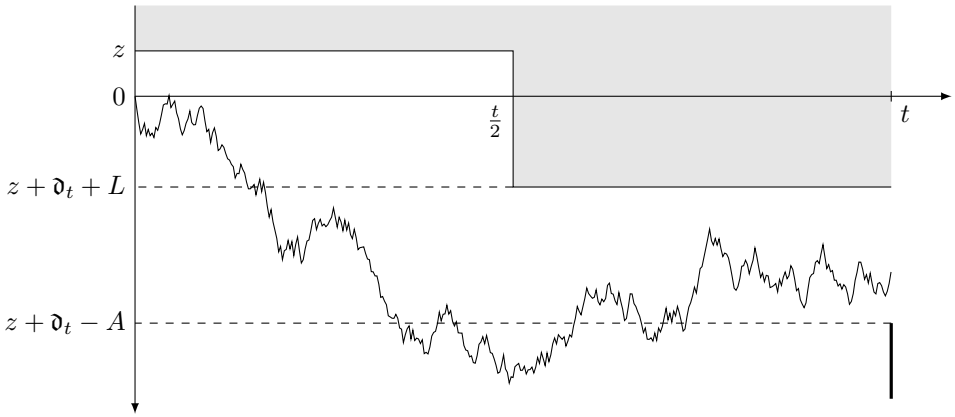


FIGURE 2.3: A typical path in $\mathbb{S}_t^{z,A,L}$.

Lemma 2.7.6 ([MRV16, Lemma 5.1])**.** *For any $\varepsilon > 0$ and $A \geq 0$, there exists $L \geq 0$ sufficiently large such that for any $s \geq 0$ sufficiently large satisfying $(e^s+1)(\mathrm{R}+1)^{-1} \in \mathbb{N}$, there exists $T \geq 0$ sufficiently large such that for any $t \geq T$ and $\chi : [0,\mathrm{R}]^d \to \mathbb{R}$ satisfying the conditions in (2.6.6), it holds that*

$$\mathbb{P}\big(\exists m \in [0,\mathrm{R}]^d \ \textit{such that}\ \mathrm{Y}_t^\chi(m) - \mathfrak{d}_t \geq -A \ \textit{and}\ \mathrm{Y}_\cdot(m) \notin \mathbb{S}_t^{\chi(m),A,L}\big) \leq \varepsilon \rho_\chi\big([0,\mathrm{R}]^d\big) .$$

Furthermore, it also holds that

$$\mathbb{P}\big(\exists m \in [0,\mathrm{R}]^d \setminus S_{\mathrm{R},t} \ \textit{such that}\ \mathrm{Y}_t^\chi(m) - \mathfrak{d}_t \geq -A\big) \leq \varepsilon \rho_\chi\big([0,\mathrm{R}]^d\big) ,$$

where $S_{\mathrm{R},t} \overset{\text{def}}{=} \big[e^{-t/2}, R - e^{-t/2}\big]^d$.

Now, for $\lambda > 0$, we define the following quantities,

$$\mathbb{E}_{(2.7.18)} \overset{\text{def}}{=} \mathbb{E}\left[\int_{S_{\mathrm{R},t}} \frac{\mathbb{1}_{\{m\in\mathbb{D}_\mathrm{R}^\lambda\}} \mathbb{1}_{\{\mathrm{Y}_\cdot(m)\in\mathbb{S}_t^{\chi(m),A,L}\}}}{|\mathbb{D}_\mathrm{R}^\lambda|} \big(1 - \mathbb{G}_\mathrm{R}\big) dm \right] , \tag{2.7.18}$$

$$\mathbb{E}_{(2.7.19)} \overset{\text{def}}{=} \mathbb{E}\left[\int_{S_{\mathrm{R},t}} \frac{\mathbb{1}_{\{m\in\mathbb{D}_\mathrm{R}^\lambda\}} \mathbb{1}_{\{\mathrm{Y}_t^\chi(m)-\mathfrak{d}_t\geq -A,\, \mathrm{Y}_\cdot(m)\notin\mathbb{S}_t^{\chi(m),A,L}\}}}{|\mathbb{D}_\mathrm{R}^\lambda|} \big(1 - \mathbb{G}_\mathrm{R}\big) dm \right] , \tag{2.7.19}$$

$$\mathbb{E}_{(2.7.20)} \overset{\text{def}}{=} \mathbb{E}\left[\int_{[0,\mathrm{R}]^d\setminus S_{\mathrm{R},t}} \frac{\mathbb{1}_{\{m\in\mathbb{D}_\mathrm{R}^\lambda\}} \mathbb{1}_{\{\mathrm{Y}_t^\chi(m)-\mathfrak{d}_t\geq -A\}}}{|\mathbb{D}_\mathrm{R}^\lambda|} \big(1 - \mathbb{G}_\mathrm{R}\big) dm \right] , \tag{2.7.20}$$

so that, using the fact that $\{\mathrm{Y}_\cdot(m) \in \mathbb{S}_t^{\chi(m),A,L}\} \subseteq \{\mathrm{Y}_t^\chi(m) - \mathfrak{d}_t \geq -A\}$, we can write

$$\mathbb{E}_{(2.7.17)} = \mathbb{E}_{(2.7.18)} + \mathbb{E}_{(2.7.19)} + \mathbb{E}_{(2.7.20)} \, .$$

Using Lemma 2.7.6, one can easily verify that that for any $\varepsilon > 0$ and $A \geq 0$, there exists $L \geq 0$ sufficiently large such that for any $s \geq 0$ sufficiently large satisfying $(e^s+1)(\mathrm{R}+1)^{-1} \in \mathbb{N}$, there exists $T \geq 0$ sufficiently large such that for any $t \geq T$ and $\chi : [0,\mathrm{R}]^d \to \mathbb{R}$ satisfying the conditions in (2.6.6),

$$\mathbb{E}_{(2.7.19)} + \mathbb{E}_{(2.7.20)} \leq \varepsilon \rho_\chi\big([0,\mathrm{R}]^d\big) \, .$$

Therefore, in what follows we can just focus on the expectation $\mathbb{E}_{(2.7.18)}$ for fixed A, $L \geq 0$.

2.7.2.3 Localisation near the maximum

For $\lambda > 0$, $0 \leq b < t/2$, and for any $m \in [0,\mathrm{R}]^d$, recalling the notation introduced in (2.3.3) and (2.3.4), we define the localised versions of the quantities in (2.7.14) by setting

$$\begin{gathered} \mathbb{G}_{m,b_t} \overset{\text{def}}{=} \exp\left(- \int_{\mathbb{B}_{b_t}(m)} \mathrm{F}_{\gamma,t}(x) \mu_{\gamma,t}^\chi(dx) \right) , \qquad \mathbb{M}_{m,b_t} \overset{\text{def}}{=} \mathbb{M}_{m,b_t}(\mathrm{Y}_t^\chi) \, , \\ \mathbb{D}_{m,b_t}^\lambda \overset{\text{def}}{=} \mathbb{D}_{m,b_t}^\lambda(\mathrm{Y}_t^\chi) \, , \end{gathered} \tag{2.7.21}$$

where $b_t \overset{\text{def}}{=} b - t$. We also introduce the following quantity

$$\mathbb{M}_{m,b_t+1,b_t} \overset{\text{def}}{=} \mathbb{M}_{m,b_t+1,b_t}(\mathrm{Y}_t^\chi) \, .$$

Furthermore, for $A \geq 0$, we define the following event

$$\mathbb{L}_{m,b_t}^A \overset{\text{def}}{=} \big\{ \exists y \in [0,\mathrm{R}]^d \setminus \mathbb{B}_{b_t}(m) \;\text{ such that }\; \mathrm{Y}_t^\chi(y) - \mathfrak{d}_t \geq -A - \lambda \big\} \, .$$

Roughly speaking, on the complement of the event $\mathbb{L}_{m,b_t}^A$, everything happens inside $\mathbb{B}_{b_t}(m)$. More precisely, on the event $\{\mathrm{Y}_t^\chi(m) - \mathfrak{d}_t \geq -A\}$ and on the complement of $\mathbb{L}_{m,b_t}^A$, the maximum of Y_t^χ over $[0,\mathrm{R}]^d$ must be attained inside $\mathbb{B}_{b_t}(m)$. Furthermore, the values of the field Y_t^χ at points in $[0,\mathrm{R}]^d \setminus \mathbb{B}_{b_t}(m)$, are more than λ away from the supremum of Y_t^χ. Consequently, we have that

$$\begin{aligned} &\mathbb{1}_{\{(\mathbb{L}_{m,b_t}^A)^c\}} \frac{\mathbb{1}_{\{m \in \mathbb{D}_{\mathrm{R}}^\lambda\}} \mathbb{1}_{\{\mathrm{Y}_t^\chi(m) - \mathfrak{d}_t \geq -A\}}}{|\mathbb{D}_{\mathrm{R}}^\lambda|} \\ &\qquad = \mathbb{1}_{\{(\mathbb{L}_{m,b_t}^A)^c\}} \frac{\mathbb{1}_{\{m \in \mathbb{D}_{m,b_t}^\lambda\}} \mathbb{1}_{\{\mathbb{M}_{m,b_t+1,b_t} - \mathfrak{d}_t < -A\}} \mathbb{1}_{\{\mathrm{Y}_t^\chi(m) - \mathfrak{d}_t \geq -A\}}}{|\mathbb{D}_{m,b_t}^\lambda|} \, . \end{aligned}$$

Remark 2.7.7. The reason why, on the right-hand side of the previous display, we included the seemingly redundant indicator function of the event $\{\mathbb{M}_{m,b_t+1,b_t} - \mathfrak{d}_t < -A\}$ is due to a technicality in the proof of Lemma 2.7.9 below. This will be better explained during the course of the proof of that lemma.

Now, by introducing the following quantities,

$$\mathbb{E}_{(2.7.22)} \overset{\text{def}}{=} \mathbb{E}\left[\int_{S_{\mathrm{R},t}} \frac{\mathbb{1}_{\{m \in \mathbb{D}_{m,b_t}^\lambda\}} \mathbb{1}_{\{\mathbb{M}_{m,b_t+1,b_t} - \mathfrak{d}_t < -A\}} \mathbb{1}_{\{\mathrm{Y}_\cdot(m) \in \mathbb{S}_t^{\chi(m),A,L}\}}}{|\mathbb{D}_{m,b_t}^\lambda|} \big(1 - \mathbb{G}_{m,b_t}\big) dm \right] , \tag{2.7.22}$$

$$\mathbb{E}_{(2.7.23)} \overset{\text{def}}{=} \mathbb{E}\left[\int_{S_{\mathrm{R},t}} \mathbb{1}_{\{\mathbb{L}^A_{m,b_t}\}} \frac{\mathbb{1}_{\{m\in\mathbb{D}^\lambda_{\mathrm{R}}\}}\mathbb{1}_{\{\mathrm{Y}_\cdot(m)\in\mathbb{S}^{\chi(m),A,L}_t\}}}{|\mathbb{D}^\lambda_{\mathrm{R}}|}\big(1-\mathbb{G}_{\mathrm{R}}\big)dm\right], \tag{2.7.23}$$

$$\mathbb{E}_{(2.7.24)} \overset{\text{def}}{=} \mathbb{E}\left[\int_{S_{\mathrm{R},t}} \mathbb{1}_{\{\mathbb{L}^A_{m,b_t}\}} \frac{\mathbb{1}_{\{m\in\mathbb{D}^\lambda_{m,b_t}\}}\mathbb{1}_{\{\mathbb{M}_{m,b_t+1,b_t}-\mathfrak{d}_t<-A\}}\mathbb{1}_{\{\mathrm{Y}_\cdot(m)\in\mathbb{S}^{\chi(m),A,L}_t\}}}{|\mathbb{D}^\lambda_{m,b_t}|} \cdot\big(1-\mathbb{G}_{m,b_t}\big)dm\right], \tag{2.7.24}$$

$$\mathbb{E}_{(2.7.25)} \overset{\text{def}}{=} \mathbb{E}\left[\int_{S_{\mathrm{R},t}} \mathbb{1}_{\{(\mathbb{L}^A_{m,b_t})^c\}} \frac{\mathbb{1}_{\{m\in\mathbb{D}^\lambda_{\mathrm{R}}\}}\mathbb{1}_{\{\mathrm{Y}_\cdot(m)\in\mathbb{S}^{\chi(m),A,L}_t\}}}{|\mathbb{D}^\lambda_{\mathrm{R}}|}\big(1-\mathbb{G}_{\mathrm{R}}\big)dm\right], \tag{2.7.25}$$

$$\mathbb{E}_{(2.7.26)} \overset{\text{def}}{=} \mathbb{E}\left[\int_{S_{\mathrm{R},t}} \mathbb{1}_{\{(\mathbb{L}^A_{m,b_t})^c\}} \frac{\mathbb{1}_{\{m\in\mathbb{D}^\lambda_{\mathrm{R}}\}}\mathbb{1}_{\{\mathrm{Y}_\cdot(m)\in\mathbb{S}^{\chi(m),A,L}_t\}}}{|\mathbb{D}^\lambda_{\mathrm{R}}|}\big(1-\mathbb{G}_{m,b_t}\big)dm\right], \tag{2.7.26}$$

one can easily check that,

$$\mathbb{E}_{(2.7.18)} = \mathbb{E}_{(2.7.22)} + \mathbb{E}_{(2.7.23)} - \mathbb{E}_{(2.7.24)} + \mathbb{E}_{(2.7.25)} - \mathbb{E}_{(2.7.26)} .$$

We will show that the fist term is dominant by separately bounding the sum of the second and third term (since both terms are positive, this dominates their difference), as well as the difference between the two last terms.

Lemma 2.7.8. *For any $\lambda > 0$ and $\varepsilon > 0$, there exists a constant $A \geq 0$ sufficiently large such that for any $s \geq 0$ sufficiently large satisfying $(e^s+1)(\mathrm{R}+1)^{-1} \in \mathbb{N}$, there exists $T \geq 0$ sufficiently large such that for any $t \geq T$, $b \geq 0$, $L \geq 0$, and $\chi : [0,\mathrm{R}]^d \to \mathbb{R}$ satisfying the conditions in (2.6.6), it holds that*

$$|\mathbb{E}_{(2.7.25)} - \mathbb{E}_{(2.7.26)}| \leq \varepsilon\rho_\chi\big([0,\mathrm{R}]^d\big) .$$

Proof. Using the elementary inequality $1-e^{-(u_1+u_2)} \leq (1-e^{-u_1})+(1-e^{-u_2})$ which is valid for u_1, $u_2 \geq 0$, we note that for all $m \in S_{\mathrm{R},t}$, it holds that

$$1-\mathbb{G}_{\mathrm{R}} \leq \big(1-\mathbb{G}_{[0,\mathrm{R}]^d\setminus\mathbb{B}_{b_t}(m)}\big)+\big(1-\mathbb{G}_{m,b_t}\big) .$$

Hence the claim is proved if we can show that there exists a constant $A \geq 0$ sufficiently large such that for any $s \geq 0$ sufficiently large satisfying $(e^s+1)(\mathrm{R}+1)^{-1} \in \mathbb{N}$, there exists $T \geq 0$ sufficiently large such that for any $t \geq T$, $b \geq 0$, $L \geq 0$, and $\chi : [0,\mathrm{R}]^d \to \mathbb{R}$ satisfying the conditions in (2.6.6), it holds that

$$\mathbb{E}\left[\int_{S_{\mathrm{R},t}} \mathbb{1}_{\{(\mathbb{L}^A_{m,b_t})^c\}} \frac{\mathbb{1}_{\{m\in\mathbb{D}^\lambda_{\mathrm{R}}\}}\mathbb{1}_{\{\mathrm{Y}_\cdot(m)\in\mathbb{S}^{\chi(m),A,L}_t\}}}{|\mathbb{D}^\lambda_{\mathrm{R}}|}\big(1-\mathbb{G}_{[0,\mathrm{R}]^d\setminus\mathbb{B}_{b_t}(m)}\big)dm\right] \leq \varepsilon\rho_\chi\big([0,\mathrm{R}]^d\big) .$$

By bringing the indicator function of the complement of the event $\mathbb{L}^A_{m,b_t}$ inside the exponential, the above expectation can be bounded from above by

$$\mathbb{E}\left[1-\exp\left(-\int_{[0,\mathrm{R}]^d} \mathrm{F}_{\gamma,t}(x)\mathbb{1}_{\{\mathrm{Y}^\chi_t(x)-\mathfrak{d}_t\leq -A-\lambda\}}\mu^\chi_{\gamma,t}(dx)\right)\right],$$

and so the conclusion follows by choosing $A \geq 0$ as in Lemma 2.7.5. □

Lemma 2.7.9. *For any $\lambda > 0$, $\varepsilon > 0$, and A, $L \geq 0$, there exist b_0, $s_0 \geq 0$ large enough such that for any $s \geq s_0$ satisfying $(e^s+1)(\mathrm{R}+1)^{-1} \in \mathbb{N}$ and $b \geq b_0$, there exists $T \geq 0$ large enough such that for any*

$t \geq T$, *and* $\chi : [0,\mathrm{R}]^d \to \mathbb{R}$ *satisfying the conditions in* (2.6.6)*, it holds that*

$$|\mathbb{E}_{(2.7.23)} + \mathbb{E}_{(2.7.24)}| \leq \varepsilon \rho_\chi\big([0,\mathrm{R}]^d\big) \ .$$

Proof. The proof of this result is similar to the proof of [Mad15, Lemmas 5.1, 5.2]. We only consider the quantity $\mathbb{E}_{(2.7.24)}$, since $\mathbb{E}_{(2.7.23)}$ can be bounded by following a similar, and in fact simpler, strategy. We introduce the lattice Λ_{b_t} by letting

$$\Lambda_{b_t} \overset{\text{def}}{=} \frac{1}{4\sqrt{d}} e^{b_t} \mathbb{Z}^d \cap [0,\mathrm{R}]^d \ .$$

Using the fact that $S_{\mathrm{R},t} \subseteq \cup_{x\in\Lambda_{t,b}} \mathbb{B}_{b_t - \log 4}(x)$, we note that the quantity inside the expectation $\mathbb{E}_{(2.7.24)}$ can be bounded from above by

$$\begin{aligned}
&\int_{S_{\mathrm{R},t}} \mathbb{1}_{\{\mathbb{L}^A_{m,b_t}\}} \frac{\mathbb{1}_{\{m\in\mathbb{D}^\lambda_{m,b_t}\}} \mathbb{1}_{\{\mathbb{M}_{m,b_t+1,b_t} - \mathfrak{d}_t < -A\}} \mathbb{1}_{\{\mathrm{Y}_\cdot(m)\in\mathbb{S}_t^{\chi(m),A,L}\}}}{|\mathbb{D}^\lambda_{m,b_t}|} dm \\
&\leq \sum_{x\in\Lambda_{b_t}} \int_{\mathbb{B}_{b_t-\log 4}(x)} \mathbb{1}_{\{\mathbb{L}^A_{m,b_t}\}} \frac{\mathbb{1}_{\{m\in\mathbb{D}^\lambda_{m,b_t}\}} \mathbb{1}_{\{\mathbb{M}_{m,b_t+1,b_t} - \mathfrak{d}_t < -A\}} \mathbb{1}_{\{\mathrm{Y}_\cdot(m)\in\mathbb{S}_t^{\chi(m),A,L}\}}}{|\mathbb{D}^\lambda_{m,b_t}|} dm \\
&\leq \sum_{x\in\Lambda_{b_t}} \mathbb{1}_{\{\mathbb{L}^A_{x,b_t-\log(4/3)}\}} \mathbb{1}_{\{\exists m\in\mathbb{B}_{b_t-\log 4}(x) \text{ such that } \mathrm{Y}_\cdot(m)\in\mathbb{S}_t^{\chi(m),A,L}\}} \ .
\end{aligned} \tag{2.7.27}$$

In order to get the last inequality, one can note that for $m \in \mathbb{B}_{b_t-\log 4}(x)$, it holds that

$$\mathbb{1}_{\{\mathbb{M}_{m,b_t+1,b_t} - \mathfrak{d}_t < -A\}} \mathbb{1}_{\{\mathrm{Y}^\chi_t(m) - \mathfrak{d}_t \geq -A\}} \leq \mathbb{1}_{\{\mathbb{M}_{m,b_t} = \mathbb{M}_{x,b_t+\log(5/4)}\}} \ ,$$

where we refer to Figure 2.4 for a diagram illustrating the sets involved.

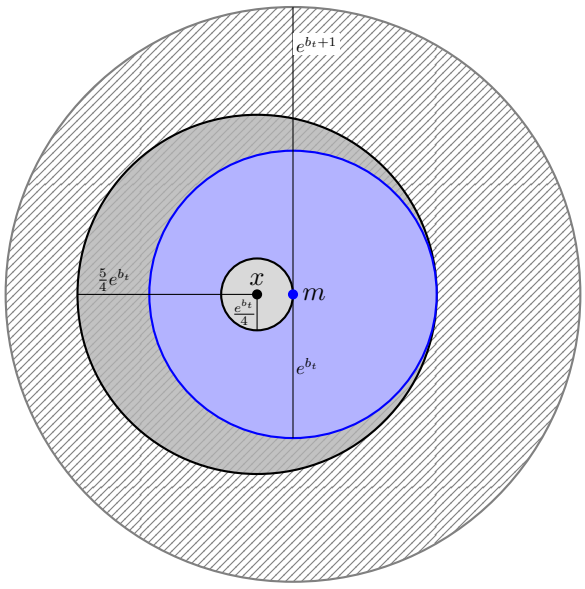


FIGURE 2.4: The point m is chosen inside the small grey ball $\mathbb{B}_{b_t-\log 4}(x)$, and the blue ball represents $\mathbb{B}_{b_t}(m)$. Since we are on the event $\{\mathrm{Y}^\chi_t(m) - \mathfrak{d}_t \geq -A\}$, on the event $\{\mathbb{M}_{m,b_t+1,b_t} - \mathfrak{d}_t < -A\}$, the supremum of the field Y^χ_t inside the large striped ball must coincide with the supremum of the field Y^χ_t inside the shaded black ball $\mathbb{B}_{b_t+\log(5/4)}(x)$.

Thus, by letting

$$\overline{\mathbb{D}}^\lambda_{x,b_t} \overset{\text{def}}{=} \big\{ y \in \mathbb{B}_{b_t-\log 4}(x) \ : \ \mathrm{Y}^\chi_t(y) \geq \mathbb{M}_{x,b_t+\log(5/4)} - \lambda \big\} \ ,$$

we have that[4]

$$\int_{\mathbb{B}_{b_t-\log 4}(x)} \frac{\mathbb{1}_{\{\mathbb{M}_{m,b_t+1,b_t}-\mathfrak{d}_t<-A\}}\mathbb{1}_{\{\mathrm{Y}^\chi_t(m)-\mathfrak{d}_t\geq -A\}}\mathbb{1}_{\{m\in\mathbb{D}^\lambda_{m,b_t}\}}}{|\mathbb{D}^\lambda_{m,b_t}|}dm \leq \int_{\mathbb{B}_{b_t-\log 4}(x)} \frac{\mathbb{1}_{\{m\in\overline{\mathbb{D}}^\lambda_{x,b_t}\}}}{|\overline{\mathbb{D}}^\lambda_{x,b_t}|}dm = 1\,.$$

Now, by following the proof of [Mad15, Lemma 5.1], one can prove that for all A, $L\geq 0$, $0\leq b<t/2$, and $x\in\Lambda_{b-t}$,

$$\mathbb{1}_{\{\mathbb{L}^A_{x,b_t-\log(4/3)}\}}\mathbb{1}_{\{\exists m\in\mathbb{B}_{b_t-\log 4}(x)\text{ such that }\mathrm{Y}_\cdot(m)\in\underline{\mathbb{S}}^{\chi(m),A,L}_t\}} \lesssim \int_{\mathbb{B}_{b_t-\log 4}(x)}\mathbb{1}_{\{\mathbb{L}^A_{m,b_t-\log 2}\}}\frac{\mathbb{1}_{\{\mathrm{Y}_\cdot(m)\in\underline{\mathbb{S}}^{\chi(m),A,L}_t\}}}{r_t(m)}dm\,, \tag{2.7.28}$$

where for A, L, $z\geq 0$, we set

$$\underline{\mathbb{S}}^{z,A,L}_t \overset{\text{def}}{=} \left\{\phi:\mathbb{R}^+_0\to\mathbb{R}\,:\,\sup_{s\in[0,t]}\phi(s)\leq z+1,\ \sup_{s\in[t/2,t]}\phi(s)\leq \mathfrak{d}_t+z+L+1,\ \phi(t)\geq \mathfrak{d}_t+z-A-1\right\},$$

and where

$$r_t(m)\overset{\text{def}}{=}\sup\left\{r\geq 0\,:\,\sup_{0\leq s\leq t,x\in B(m,r)\cap[0,\mathrm{R}]^d}|\mathrm{Y}_s(x)-\mathrm{Y}_s(m)|\leq\frac{1}{4}\right\}\wedge e^{-t}\,.$$

Hence, plugging (2.7.28) into (2.7.27), we obtain that

$$\begin{aligned}\mathbb{E}_{(2.7.24)} &\lesssim \sum_{x\in\Lambda_{t,b}}\int_{\mathbb{B}_{b_t-\log 4}(x)}\mathbb{E}\left[\mathbb{1}_{\{\mathbb{L}^A_{m,b_t-\log 2}\}}\frac{\mathbb{1}_{\{\mathrm{Y}_\cdot(m)\in\underline{\mathbb{S}}^{\chi(m),A,L}_t\}}}{r_t(m)}\right]dm\\ &\lesssim \int_{[0,\mathrm{R}]^d}\mathbb{E}\left[\mathbb{1}_{\{\mathbb{L}^A_{m,b_t-\log 2}\}}\frac{\mathbb{1}_{\{\mathrm{Y}_\cdot(m)\in\underline{\mathbb{S}}^{\chi(m),A,L}_t\}}}{r_t(m)}\right]dm\,,\end{aligned}$$

where we used the fact that $\cup_{x\in\Lambda_{b_t}}\mathbb{B}_{b_t-\log 4}(x)\subseteq[0,\mathrm{R}]^d$ and that there exists a constant $\sigma_d\in\mathbb{N}$ such that for any distinct points $x_1,\dots,x_{\sigma_d}\in\Lambda_{b-t}$, it holds that $\cap_{i\in[\sigma_d]}\mathbb{B}_{b_t-\log 4}(x_i)=\emptyset$. Finally, as one can easily check, $\mathbb{E}_{(2.7.23)}$ can be bounded by the same quantity, and so the conclusion follows from [Mad15, Lemma 5.2]. □

2.7.2.4 A renewal result

Combining Lemmas 2.7.5, 2.7.6, 2.7.8, and 2.7.9, we have shown in the previous subsection that for any $\lambda>0$ and $\varepsilon>0$, there exist $0\leq A<L$ sufficiently large such that there exists $b_0>0$ and $s_0>0$ sufficiently large, such that for any $s\geq s_0$ satisfying $(e^s+1)/(\mathrm{R}+1)\in\mathbb{N}$ and any $b\geq b_0$, there exists a sufficiently large $T\geq 0$ such that for all $t\geq T$ and any function $\chi:[0,\mathrm{R}]^d\to\mathbb{R}$ satisfying the conditions in (2.6.6), it holds that

$$\left|\mathbb{E}\left[1-\exp\left(-\int_{[0,\mathrm{R}]^d}\mathrm{F}_{\gamma,t}(x)\mu^\chi_{\gamma,t}(dx)\right)\right]-\mathbb{E}_{(2.7.22)}\right|\leq\varepsilon\rho_\chi\left([0,\mathrm{R}]^d\right)\,,$$

[4] As we mentioned in Remark 2.7.7, the reason we included the indicator function of the event $\{\mathbb{M}_{m,b_t+1,b_t}-\mathfrak{d}_t<-A\}$ is due to a technical reason. Indeed, without this indicator function, we would not be able to bound the integral by one. It is not difficult to construct a "pathological example" for a possible realisation of the field Y_t for which the integral becomes arbitrarily large.

Therefore, in what follows, we fix $\lambda > 0$ and $A, L, b \geq 0$, and we study the quantity $\mathbb{E}_{(2.7.22)}$. Given a function $g : \mathbb{R}^d \to \mathbb{R}$, we introduce the fields $\overline{\Upsilon}_b$ and $\overline{\Upsilon}_{b,g}$ on $\mathbb{R}^d$ by letting

$$\overline{\Upsilon}_b(x) \overset{\text{def}}{=} \mathrm{Y}_b(e^{-b}x) - \mathrm{Y}_b(0) \;, \qquad \overline{\Upsilon}_{b,g}(x) \overset{\text{def}}{=} \overline{\Upsilon}_b(x) + g(x) \;. \tag{2.7.29}$$

Thanks to translation invariance, the scaling relation (2.3.9), the Markov property at time $t_b \overset{\text{def}}{=} t - b$ of the process $(\mathrm{Y}_t(\cdot))_{t\geq 0}$, the assumptions (W1) and (W2), and, for $m \in S_{\mathrm{R},t}$, by applying the Cameron–Martin theorem (see Lemma A.0.1) with density $\exp(\sqrt{2d}\mathrm{Y}_{t_b}(m) + d(t-b))$ to the process $(\mathrm{Y}_s(m))_{s\in[0,t_b]}$, we can rewrite $\mathbb{E}$(2.7.22) as follows

$$\int_{S_{\mathrm{R},t}} e^{-\sqrt{2d}\chi(m)} t^{3/2} \mathbb{E}_{-\chi(m)}\Big[\mathbb{1}_{\{\sup_{s\in[0,t_b]} B_s \leq 0,\ \sup_{s\in[t/2,t_b]} B_s \leq \mathfrak{d}_t + L\}} \overline{\mathfrak{F}}^{\lambda}_{A,L,b}\big(B_{t_b} - \mathfrak{d}_t - L, \mathfrak{g}^{\chi,m}_{t,b}\big)\Big] dm \;,$$

where, under $\mathbb{P}_{-\chi(m)}$ the Brownian motion $(B_s)_{s\geq 0}$ is started from $-\chi(m)$, and recalling the definition (2.7.1) of the function $\mathbf{F}^{\lambda}_{k,x}$, the map $\overline{\mathfrak{F}}^{\lambda}_{A,L,b} : \mathbb{R} \times \mathcal{C}(\mathbb{R}^d) \to \mathbb{R}$ is given by

$$\begin{aligned}\overline{\mathfrak{F}}^{\lambda}_{A,L,b}(z,g) \overset{\text{def}}{=} e^{-\sqrt{2d}(z+L)+db}\mathbb{E}_z\Big[&\mathbb{1}_{\{\mathrm{Y}_b(0)\geq -A-L\}}\mathbb{1}_{\{\sup_{s\in[0,b]}\mathrm{Y}_s(0)\leq 0\}} \\ &\cdot \mathbb{1}_{\{\mathbb{M}_{0,b+1,b}(\overline{\Upsilon}_{b,g})+\mathrm{Y}_b(0) < -A-L\}}\mathbb{1}_{\{\mathbb{M}_{0,b}(\overline{\Upsilon}_{b,g})\leq\lambda\}}\mathbf{F}^{\lambda}_{b,-\mathrm{Y}_b(0)-L}\big(\overline{\Upsilon}_{b,g}\big)\Big] \;.\end{aligned} \tag{2.7.30}$$

where, under $\mathbb{P}_z$ the field $(\mathrm{Y}_s(x))_{s\geq 0, x\in\mathbb{R}^d}$ has the same law of $(\mathrm{Y}_s(x) + z)_{s\geq 0, x\in\mathbb{R}^d}$ under $\mathbb{P}$, and $(\mathfrak{g}^{\chi,m}_{t,b}(x))_{x\in\mathbb{R}^d}$ is an independent continuous random field given by

$$\mathfrak{g}^{\chi,m}_{t,b}(x) \overset{\text{def}}{=} \int_0^{t_b} (1 - \mathfrak{K}(e^{s-t}x))dB_s + \mathrm{Z}_{t_b}(e^{-b}x) - \sqrt{2d}\int_0^{t_b}(1-\mathfrak{K}(e^{s-t}x))ds - \big(\chi(m + e^{-t}x) - \chi(m)\big) \;,$$

where we recall once again that Z_{t_b} is the field introduced in Definition 2.3.9.

Lemma 2.7.10. *For any $\lambda > 0$ and A, L, $b \geq 0$, consider the function $\overline{\mathfrak{F}}^{\lambda}_{A,L,b} : \mathbb{R} \times \mathcal{C}(\mathbb{R}^d) \to \mathbb{R}$ defined in (2.7.30). Then, there exist two functions $\mathfrak{h} : \mathbb{R} \to \mathbb{R}^+$ and $\mathfrak{F}_* : \mathcal{C}(\mathbb{R}^d) \to \mathbb{R}^+$, possibly depending on the parameters λ, A, L, b, such that:*

(1) It holds that

$$\sup_{z\in\mathbb{R}} \mathfrak{h}(z) < \infty \qquad \text{and} \qquad \mathfrak{h}(z) = O(e^z) \quad \text{as } z \to -\infty \;.$$

(2) For any $z \in \mathbb{R}$ and $g \in C(\mathbb{R}^d)$ it holds that

$$\overline{\mathfrak{F}}^{\lambda}_{A,L,b}(z,g) \leq \mathfrak{h}(z)\mathfrak{F}_*(g) \;.$$

(3) There exists $c > 0$ such that for any $\delta \in (0,1)$ and $g \in \mathcal{C}(\mathbb{R}^d)$ satisfying

$$\sup_{x,y\in\mathbb{R}^d,\, |x-y|\leq e^b\delta} |g(x) - g(y)| \leq 1/4 \;,$$

it holds that

$$\mathfrak{F}_*(g) \leq c\delta^{-10} \;.$$

(4) There exists $c > 0$ such that for any g_1, $g_2 \in \mathcal{C}(\mathbb{R}^d)$ satisfying

$$\|g_1 - g_2\|_\infty \overset{\text{def}}{=} \sup_{x\in\mathbb{R}^d} |g_1(x) - g_2(x)| \leq 1/8 \;,$$

it holds that

$$\big|\overline{\mathfrak{F}}^{\lambda}_{A,L,b}(z,g_1)-\overline{\mathfrak{F}}^{\lambda}_{A,L,b}(z,g_2)\big|\le c\|g_1-g_2\|_{\infty}^{1/8}\mathfrak{h}(z)\mathfrak{F}_*(g_1)\ .$$

Proof. The proof can be found in Appendix 2.B. □

We are now in position to conclude the proof of Lemma 2.7.2, which as we see below follows by [Mad15, Theorem 5.6].

Proof of Lemma 2.7.2. Thanks to Lemma 2.7.10, we have that for any $\lambda>0$ and A, L, $b\ge 0$, the function $\overline{\mathfrak{F}}^{\lambda}_{A,L,b}:\mathbb{R}\times\mathcal{C}(\mathbb{R}^d)\to\mathbb{R}$ defined in (2.7.30) is "b-regular" in the sense of [Mad15]. Hence, thanks to [Mad15, Theorem 5.6], we have that for any $\varepsilon>0$, for any $s\ge 0$ sufficiently large satisfying $(e^s+1)(\mathrm{R}+1)^{-1}\in\mathbb{N}$, there exists $T\ge 0$ sufficiently large such that for any $t\ge T$ and $\chi:[0,\mathrm{R}]^d\to\mathbb{R}$ satisfying the conditions in (2.6.6), it holds that

$$|\mathbb{E}_{(2.7.22)}-\overline{\mathbf{C}}^{\lambda}_{A,L,b}\,\rho_{\chi}\big([0,\mathrm{R}]^d\big)|\le\varepsilon\rho_{\chi}\big([0,\mathrm{R}]^d\big)\ ,$$

with the constant $\overline{\mathbf{C}}^{\lambda}_{A,L,b}$ defined as

$$\overline{\mathbf{C}}^{\lambda}_{A,L,b}\overset{\text{def}}{=}\alpha\int_0^{\infty}z\mathbb{E}\big[\overline{\mathfrak{F}}^{\lambda}_{A,L,b}\big(-z,\mathfrak{g}^z_b\big)\big]dz\ ,$$

where we recall once again that $\alpha=\sqrt{2/\pi}$ and the field $\mathfrak{g}^z_b$ is defined in (2.7.4). Hence, in order to conclude, it suffices to show that for all $z\ge 0$ and $g\in\mathcal{C}(\mathbb{R}^d)$ it holds that $\overline{\mathfrak{F}}^{\lambda}_{A,L,b}(-z,g)=\mathfrak{F}^{\lambda}_{A,L,b}(z,g)$. To this end, we begin by observing that, thanks to Lemma 2.3.10, and recalling Definition 2.3.9 as well as (2.2.2), for all $y\in\mathbb{R}^d$, it holds that

$$\begin{aligned}\overline{\Upsilon}_b(y)&=-\int_0^b\big(1-\mathfrak{K}(e^{s-b}y)\big)d\mathrm{Y}_s(0)+\overline{\mathrm{Z}}^0_b(e^{-b}y)-\sqrt{2d}\int_0^b\big(1-\mathfrak{K}(e^{s-b}y)\big)ds\\&\overset{\text{law}}{=}-\int_0^b\big(1-\mathfrak{K}(e^{-s}y)\big)d\mathrm{Y}_{b-s}(0)+\mathrm{Z}_b(y)-\sqrt{2d}\mathfrak{a}_b(y)\ .\end{aligned}$$

Hence, by applying the Cameron–Martin theorem (see Lemma A.0.1) with density $\exp(\sqrt{2d}\mathrm{Y}_b(0)+db)$ to the process $(\mathrm{Y}_s(0))_{s\in[0,b]}$, using the equality in law $(B_s)_{s\in[0,b]}\overset{\text{law}}{=}(-B_s)_{s\in[0,b]}$, using the fact that $\mathbb{P}_z(B_b\in dx)=(2\pi b)^{-1/2}e^{-(x-z)^2/(2b)}dx$, and finally the fact that a time reversed Brownian bridge is still a Brownian bridge but with starting and final point swapped, we obtain that

$$\begin{aligned}\overline{\mathfrak{F}}^{\lambda}_{A,L,b}(-z,g)=\frac{1}{\sqrt{2\pi b}}\int_0^{A+L}e^{\sqrt{2d}(x-L)}e^{-\frac{(x-z)^2}{2b}}\mathbb{E}_{x,z,b}\Big[\mathbf{F}^{\lambda}_{b,x-L}\big(\Upsilon_{b,g}\big)\\\cdot\mathbb{1}_{\{\inf_{s\in[0,b]}B_s\ge 0\}}\mathbb{1}_{\{\mathbb{M}_{0,b+1,b}(\Upsilon_{b,g})<x-(A+L)\}}\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_{b,g})\le\lambda\}}\Big]dx\ .\end{aligned}$$

Therefore, the conclusion follows since the right-hand side of the previous display coincides with $\mathfrak{F}^{\lambda}_{A,L,b}(z,g)$ as desired. □

2.7.3 Proof of Lemma 2.7.4

The main goal of this section is to prove Lemma 2.7.4. As in the previous case, the proof follows a sequence of reduction steps, allowing us to transition from $\mathbf{C}^{\lambda}_{A,L,b}$ to $\mathbf{C}^{\lambda,\text{new}}_{A,L,k,b}$. Each of these reduction steps forms a lemma within this section. Recalling the definition (2.7.7) of the constant $\mathbf{C}^{\lambda}_{A,L,b}$, it is convenient to introduce

the function $\mathbf{G}^{\lambda}_{A,L,b} : \mathcal{C}(\mathbb{R}^d) \to \mathbb{R}$ given by

$$\mathbf{G}^{\lambda}_{A,L,b}(g) \overset{\text{def}}{=} \alpha \int_0^{\infty} z \mathfrak{F}^{\lambda}_{A,L,b}(z, g) dz \ . \tag{2.7.31}$$

By plugging the expression (2.7.6) for the function $\mathfrak{F}^{\lambda}_{A,L,b}$ into the right-hand side of (2.7.31), and doing a change of variables, we get that the expectation of $\mathbf{G}^{\lambda}_{A,L,b}(\mathfrak{g}^z_b)$ is equal to

$$\begin{aligned} \mathbf{G}^{\lambda}_{(2.7.32)} \overset{\text{def}}{=} \alpha \int_0^{A+L} e^{\sqrt{2d}(x-L)} \int_{-x}^{\infty} \frac{e^{-\frac{u^2}{2b}}}{\sqrt{2\pi b}} (u+x) \mathbb{E}_{0,u,b} \Big[\mathbf{F}^{\lambda}_{b,x-L}(\Upsilon^{u+x}_b) \\ \cdot \mathbb{1}_{\{\mathbb{M}_{0,b+1,b}(\Upsilon^{u+x}_b) \le x-(A+L)\}} \mathbb{1}_{\{\inf_{s\in[0,b]} B_s \ge -x\}} \mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon^{u+x}_b) \le \lambda\}} \Big] du dx \ , \end{aligned} \tag{2.7.32}$$

where we recall that $\Upsilon^z_b = \Upsilon_b + \mathfrak{g}^z_b$, where $\mathfrak{g}^z_b$ is the field introduced in (2.7.4). As before, with a slight abuse of our previous notation, we have absorbed the expectation with respect to the field $\mathfrak{g}^{u+x}_b$ into the expectation $\mathbb{E}_{0,u,b}$. In what follows, we will implicitly use the fact – previously noted in the proof of Proposition 2.6.1 – that the field $\mathfrak{g}^z_b$ satisfies (G1) – (G3) uniformly over all $z \ge 0$. In particular, all the results derived in Section 2.4 remain valid for the field Υ^z_b uniformly over all $z \ge 0$.

The goal is now to simplify the quantity $\mathbf{G}^{\lambda}_{(2.7.32)}$ into a more manageable form through a sequence of reduction steps. We begin with Lemma 2.7.11, where we show that the integral over u is concentrated around $\sqrt{b}$. Next, in Lemma 2.7.12, we establish that $\mathbf{F}^{\lambda}_{b,x-L}$ can be replaced by its truncation at a large values. Afterward, in Lemma 2.7.13, we show that the two first two indicator functions on the second line in (2.7.32) can be removed. Finally, in Lemma 2.7.14, we show that $\mathbf{F}^{\lambda}_{b,x-L}$ can be replaced by $\mathbf{F}^{\lambda}_{k,x-L}$ for some $k \ll b$.

2.7.3.1 Reduction steps

We can now begin reducing $\mathbf{G}^{\lambda}_{(2.7.32)}$ to a more manageable quantity. To this end, we start by defining

$$\begin{aligned} \mathbf{G}^{\lambda}_{(2.7.33)} \overset{\text{def}}{=} \alpha \int_0^{A+L} e^{\sqrt{2d}(x-L)} \int_{b^{1/4}}^{b^{3/4}} \frac{e^{-\frac{u^2}{2b}}}{\sqrt{2\pi b}} (u+x) \mathbb{E}_{0,u,b} \Big[\mathbf{F}^{\lambda}_{b,x-L}(\Upsilon^{u+x}_b) \\ \cdot \mathbb{1}_{\{\mathbb{M}_{0,b+1,b}(\Upsilon^{u+x}_b) \le x-(A+L)\}} \mathbb{1}_{\{\inf_{s\in[0,b]} B_s \ge -x\}} \mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon^{u+x}_b) \le \lambda\}} \Big] du dx \ , \end{aligned} \tag{2.7.33}$$

where we emphasise that the only difference between $\mathbf{G}^{\lambda}_{(2.7.33)}$ and $\mathbf{G}^{\lambda}_{(2.7.32)}$ is the domain of integration in the u variable.

Lemma 2.7.11. *For any $\lambda > 0$, $\varepsilon > 0$, and A, $L \ge 0$, there exists $b \ge 0$ sufficiently large such that*

$$|\mathbf{G}^{\lambda}_{(2.7.33)} - \mathbf{G}^{\lambda}_{(2.7.32)}| \le \varepsilon \ .$$

Proof. We begin by observing that the quantity $|\mathbf{G}^{\lambda}_{(2.7.33)} - \mathbf{G}^{\lambda}_{(2.7.32)}|$ is bounded above by a multiple of

$$\int_0^{A+L} e^{\sqrt{2d}(x-L)} \int_{-x}^{b^{1/4}} \frac{e^{-\frac{u^2}{2b}}}{\sqrt{2\pi b}} (u+x) \mathbb{E}_{0,u,b} \Big[\mathbf{F}^{\lambda}_{b,x-L}(\Upsilon^{u+x}_b) \mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon^{u+x}_b) \le \lambda\}} \Big] du dx \tag{2.7.34}$$

$$+ \int_0^{A+L} e^{\sqrt{2d}(x-L)} \int_{b^{3/4}}^{\infty} \frac{e^{-\frac{u^2}{2b}}}{\sqrt{2\pi b}} (u+x) \mathbb{E}_{0,u,b} \Big[\mathbf{F}^{\lambda}_{b,x-L}(\Upsilon^{u+x}_b) \mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon^{u+x}_b) \le \lambda\}} \Big] du dx \ . \tag{2.7.35}$$

Recalling the definition (2.7.1), we have that

$$\mathbf{F}^{\lambda}_{b,x-L}(\Upsilon^{u+x}_b) \le |\mathbb{D}^{\lambda}_{b,0}(\Upsilon^{u+x}_b)|^{-1} \ .$$

By using the above estimate, we proceed to bound (2.7.34) and (2.7.35) separately. Starting from (2.7.34), using Lemma 2.4.27, we obtain that

$$(2.7.34) \lesssim b^{-\frac{1}{2}} \int_0^{A+L} e^{\sqrt{2d}(x-L)}(b^{1/4}+x)dx \ ,$$

and the quantity on the right-hand side can be made arbitrarily small by taking $b \geq 0$ sufficiently large. Regarding (2.7.35), thanks to Lemma 2.4.28, we obtain that

$$(2.7.35) \lesssim \int_0^{A+L} e^{\sqrt{2d}(x-L)} \int_{b^{3/4}}^{\infty} \frac{e^{-\frac{u^2}{2b}}}{\sqrt{2\pi b}}(u+x)(b^{d^2}+(u/b)^{2d})dudx \ ,$$

and, as before, the quantity on the right-hand side can be made arbitrarily small by taking $b \geq 0$ sufficiently large. Hence, the claim follows. □

Thanks to the previous lemma, from now on, we can focus on $\mathbf{G}^{\lambda}_{(2.7.33)}$ instead of $\mathbf{G}^{\lambda}_{(2.7.32)}$. We now address the fact that the quantity $\mathbf{F}^{\lambda}_{b,x-L}(\Upsilon_b^{u+x})$ is not a priori bounded. The approach to overcoming this issue is quite straightforward. Specifically, recalling (2.7.9), we show that replacing $\mathbf{F}^{\lambda}_{b,x-L}(\Upsilon_b^{u+x})$ in $\mathbf{G}^{\lambda}_{(2.7.33)}$ with $\mathbf{F}^{\lambda}_{b,x-L,L}(\Upsilon_b^{u+x})$ introduces only a negligible error, provided that L is sufficiently large. More precisely, we consider

$$\begin{aligned}\mathbf{G}^{\lambda}_{(2.7.36)} \overset{\text{def}}{=} \alpha \int_{L/2}^{A+L} e^{\sqrt{2d}(x-L)} \int_{b^{1/4}}^{b^{3/4}} \frac{e^{-\frac{u^2}{2b}}}{\sqrt{2\pi b}}(u+x)\mathbb{E}_{0,u,b}\Big[\mathbf{F}^{\lambda}_{b,x-L,L}(\Upsilon_b^{u+x}) \\ \cdot \mathbb{1}_{\{\mathbb{M}_{0,b+1,b}(\Upsilon_b^{u+x})\leq x-(A+L)\}}\mathbb{1}_{\{\inf_{s\in[0,b]}B_s\geq -x\}}\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b^{u+x})\leq\lambda\}}\Big]dudx \ ,\end{aligned} \tag{2.7.36}$$

and we claim the following result.

Lemma 2.7.12. *For any $\lambda > 0$, $\varepsilon > 0$ and $A \geq 0$, there exist $L \geq 0$ sufficiently large, and $b \geq 0$ sufficiently large such that*

$$|\mathbf{G}^{\lambda}_{(2.7.36)} - \mathbf{G}^{\lambda}_{(2.7.33)}| \leq \varepsilon \ .$$

Proof. We start by addressing the fact that the integral over x from 0 to $A+L$ in $\mathbf{G}^{\lambda}_{(2.7.33)}$ can be reduced, up to a negligible error, to an integral from $L/2$ to $A+L$. To this end, for $x \in [0, A+L]$ and $u \in [b^{1/4}, b^{3/4}]$, we recall that thanks to Lemma 2.4.22, it holds that

$$\mathbb{E}_{0,u,b}\Big[\mathbf{F}^{\lambda}_{b,x-L}(\Upsilon_b^{u+x})\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b^{u+x})\leq\lambda\}}\Big] \lesssim \frac{u}{b} \ . \tag{2.7.37}$$

Using the above bound, one can easily see that the difference between $\mathbf{G}^{\lambda}_{(2.7.33)}$ and the same quantity with the integral over x from 0 to $A+L$ replaced by the integral over x from $L/2$ to $A+L$ can be made arbitrarily small by taking $L \geq 0$ sufficiently large uniformly over all $b \geq 0$.

Now, for $x \in [L/2, A+L]$ and $u \in [b^{1/4}, b^{3/4}]$, we need to bound from above the following difference,

$$\mathbb{E}_{0,u,b}\Big[|\mathbf{F}^{\lambda}_{b,x-L}(\Upsilon_b^{u+x}) - \mathbf{F}^{\lambda}_{b,x-L,L}(\Upsilon_b^{u+x})|\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b^{u+x})\leq\lambda\}}\Big] \ . \tag{2.7.38}$$

To this end, recalling the definition of the function $\mathbf{F}^{\lambda}_{k,x}$ in (2.7.1) and that of $\mathbf{F}^{\lambda}_{k,x,L}$ in (2.7.9), we note that

$$|\mathbf{F}^{\lambda}_{b,x-L}(\Upsilon_b^{u+x}) - \mathbf{F}^{\lambda}_{b,x-L,L}(\Upsilon_b^{u+x})| \leq \big(|\mathbb{D}^{\lambda}_{0,b}(\Upsilon_b^{u+x})|^{-1} - L\big)\mathbb{1}_{\{|\mathbb{D}^{\lambda}_{0,b}(\Upsilon_b^{u+x})|^{-1}\geq L\}} \ .$$

Therefore, plugging this into (2.7.38) and using Lemma 2.4.22, there exists $\delta = \delta(d) \in (0,1)$ such that

$$\begin{aligned}\mathbb{E}_{0,u,b}\Big[|\mathbf{F}^{\lambda}_{b,x-L}(\Upsilon_b^{u+x}) - \mathbf{F}^{\lambda}_{b,x-L,L}(\Upsilon_b^{u+x})|\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b^{u+x})\leq\lambda\}}\Big] \\ \leq \int_L^{\infty} \mathbb{P}_{0,u,b}\big(|\mathbb{D}^{\lambda}_{0,b}(\Upsilon_b^{u+x})|^{-1} \geq \eta,\ \mathbb{M}_{0,b}(\Upsilon_b^{u+x}) \leq \lambda\big)d\eta \\ \lesssim \frac{u}{b}\int_L^{\infty} \eta^{-(1+\delta)}d\eta \lesssim \frac{u}{b}L^{-\delta}\ .\end{aligned}$$

Therefore, the conclusion follows readily from the above bound. □

We now need to show how we can remove the first two indicator functions appearing on the second line of $\mathbf{G}^{\lambda}_{(2.7.36)}$. To this end, we consider

$$\begin{aligned}\mathbf{G}^{\lambda}_{(2.7.39)} \overset{\text{def}}{=} \alpha\int_{L/2}^{A+L} e^{\sqrt{2d}(x-L)}\int_{b^{1/4}}^{b^{3/4}} \frac{e^{-\frac{u^2}{2b}}}{\sqrt{2\pi b}}(u+x) \\ \cdot\,\mathbb{E}_{0,u,b}\Big[\mathbf{F}^{\lambda}_{b,x-L,L}(\Upsilon_b^{u+x})\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b^{u+x})\leq\lambda\}}\Big]dudx\ ,\end{aligned} \tag{2.7.39}$$

and we claim the following lemma.

Lemma 2.7.13. *For any $\lambda > 0$, $\varepsilon > 0$ and $A \geq 0$, there exist $L \geq 0$ sufficiently large, and $b \geq 0$ sufficiently large such that*

$$|\mathbf{G}^{\lambda}_{(2.7.39)} - \mathbf{G}^{\lambda}_{(2.7.36)}| \leq \varepsilon\ .$$

Proof. We divide the proof in two steps. First, we prove that the indicator function involving the Brownian motion can be removed. Then, we show that the indicator function concerning the quantity $\mathbb{M}_{0,b+1,b}(\Upsilon_b^{u+x})$ can also be disregarded.

Step 1: We start by showing how we can remove the indicator function regarding the Brownian motion. To this end, we fix $x \in [L/2, A+L]$ and $u \in [b^{1/4}, b^{3/4}]$, and we proceed to bound from above the following quantity,

$$\mathbb{E}_{0,u,b}\Big[\mathbf{F}^{\lambda}_{b,x-L,L}(\Upsilon_b^{u+x})\mathbb{1}_{\{\inf_{s\in[0,b]}B_s<-x\}}\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b^{u+x})\leq\lambda\}}\Big]\ , \tag{2.7.40}$$

which arises when considering the difference $\mathbf{G}^{\lambda}_{(2.7.39)} - \mathbf{G}^{\lambda}_{(2.7.36)}$. Since the quantity $\mathbf{F}^{\lambda}_{b,x-L,L}(\Upsilon_b^{u+x})$ is by definition bounded above by L, we can proceed to estimate the following probability

$$\mathbb{P}_{0,u,b}\left(\inf_{s\in[0,b]}B_s < -x,\ \mathbb{M}_{0,b}(\Upsilon_b^{u+x}) \leq \lambda\right)\ ,$$

for x and u as specified above. By monotonicity, since $x > L/2$, the probability in the previous display is bounded above by

$$\mathbb{P}_{0,u,b}\left(\inf_{s\in[0,b]}B_s < -L/2,\ \mathbb{M}_{0,b}(\Upsilon_b^{u+x}) \leq \lambda\right)\ .$$

This probability can be bounded by using Lemma 2.4.19, from which we deduce that is less than a constant times $e^{-c\sqrt{L}}u/b$, for some $c > 0$.

Step 2: We now show how we can remove the indicator function regarding the quantity $\mathbb{M}_{0,b+1,b}(\Upsilon_b^{u+x})$. Fix $x \in [L/2, A+L]$ and $u \in [b^{1/4}, b^{3/4}]$. Using again the fact that the quantity $\mathbf{F}^{\lambda}_{b,x-L,L}(\Upsilon_b^{u+x})$ is by definition bounded above by L, we need to estimate the following probability

$$\mathbb{P}_{0,u,b}\big(\mathbb{M}_{0,b+1,b}(\Upsilon_b^{u+x}) > x - (A+L)\big)\ .$$

Thanks to (K2), the seed covariance function $\mathfrak{K}$ is supported in $B(0,1)$ and so, recalling the decomposition of the field Φ_b in (2.4.5), we have that, for all $y \in \mathbb{R}^d$ with $|y| > e^b$,

$$\Upsilon_b^{u+x}(y) = -B_b + \mathrm{Z}_b(y) - \sqrt{2d}b + \mathfrak{g}_{b,u+x}(y) \,,$$

where we recall that Z_b is the centred Gaussian field on $\mathbb{R}^d$ introduced in Definition 2.3.9. In particular, leveraging again on the fact that $\mathfrak{K}$ is supported in $B(0,1)$, we have that for all $x, y \in \mathbb{R}^d$ with $|x|, |y| > e^b$,

$$\mathbb{E}\big[\mathrm{Z}_b(x)\mathrm{Z}_b(y)\big] = \int_0^b \mathfrak{K}(e^{-s}(x-y))ds \,.$$

In particular, this implies that the scaled field $\overline{\mathrm{Z}}_b(\cdot) \overset{\text{def}}{=} \mathrm{Z}_b(e^b\cdot)$ restricted to the annulus $\mathbb{B}_1 \setminus \mathbb{B}_0$ has the same covariance of the martingale approximation at level b of a $\star$-scale invariant field with seed covariance function $\mathfrak{K}$. Therefore, by rescaling space, the event $\{\mathbb{M}_{0,b+1,b}(\Upsilon_{b,g}) > x - (A+L)\}$ is equivalent to the event that

$$\big\{\mathbb{M}_{0,1,0}(\overline{\mathrm{Z}}_b + \mathfrak{g}_{b,u+x}(e^b\cdot)) - \sqrt{2d}b > B_b + x - (A+L)\big\} \,.$$

Therefore, on the events $\{\mathbb{M}_{0,1,0}(\mathfrak{g}_{b,u+x}(e^b\cdot)) < b^{1/4}/2\}$ and $\{B_b > b^{1/4}\}$, using the same argument as in the proof of Lemma 2.4.3, we observe that, there exists a constant $c > 0$ such that, for any $x \in [0, A+L]$ and $u \in [b^{1/4}, b^{3/4}]$,

$$\mathbb{P}\big(\mathbb{M}_{0,1,0}(\overline{\mathrm{Z}}_b) - \sqrt{2d}b > b^{1/4}/2 + x - (A+L)\big) \lesssim e^{-c(b^{1/4}-(A+L))} \,.$$

On the other hand, to treat the event $\{\mathbb{M}_{0,1,0}(\mathfrak{g}_{b,u+x}(e^b\cdot)) \geq b^{1/4}/2\}$, we can use the fact that the supremum over $\mathbb{B}_1 \setminus \mathbb{B}_0$ of $\mathfrak{g}_{b,u+x}(e^b\cdot)$ has uniform Gaussian tails, in order to obtain a similar (stronger) bound.

Finally, the conclusion follows readily by combining the bounds provided in Steps 1 and 2. □

Finally, we need to prove that the function $\mathbf{F}^{\lambda}_{b,x-L,L}$ in $\mathbf{G}^{\lambda}_{(2.7.39)}$ can be replaced by $\mathbf{F}^{\lambda}_{k,x-L,L}$ for some $0 \leq k < b$. To this end, for $0 \leq k < b$, we define

$$\begin{aligned}\mathbf{G}^{\lambda}_{(2.7.41)} \overset{\text{def}}{=} \alpha \int_{L/2}^{A+L} e^{\sqrt{2d}(x-L)} \int_{b^{1/4}}^{b^{3/4}} \frac{e^{-\frac{u^2}{2b}}}{\sqrt{2\pi b}}(u+x) \\ \cdot \mathbb{E}_{0,u,b}\Big[\mathbf{F}^{\lambda}_{k,x-L,L}(\Upsilon_b^{u+x})\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b^{u+x})\leq\lambda\}}\Big]dudx \,,\end{aligned} \tag{2.7.41}$$

and we claim the following lemma.

Lemma 2.7.14. *For any $\lambda > 0$, $\varepsilon > 0$ and $A, L \geq 0$, there exist $k \geq 0$ sufficiently large, and $b \geq k$ sufficiently large such that*

$$|\mathbf{G}^{\lambda}_{(2.7.41)} - \mathbf{G}^{\lambda}_{(2.7.39)}| \leq \varepsilon \,.$$

Proof. For $0 \leq k < b$, we define the following event

$$\mathrm{E}_{k,b} \overset{\text{def}}{=} \bigcup_{j=k}^{b-1} \Big\{ \sup_{y\in\mathbb{A}_j} \Upsilon_b^{u+x}(y) \geq -a\log(j)^2 \Big\} \,,$$

for some constant $a > 0$. By arguing as in the proof of Lemma 2.4.20, and using the fact that the function $\mathbf{F}^{\lambda}_{b,x-L,L}$ is bounded above by L, we note that for any $x \in [L/2, A+L]$, $b \geq 0$ sufficiently large, and

$u \in [b^{1/4}, b^{3/4}]$, it holds that

$$\mathbb{E}_{0,u,b}\Big[\mathbf{F}^{\lambda}_{b,x-L,L}(\Upsilon_b^{u+x})\mathbb{1}_{\{\mathrm{E}_{k,b}\}}\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b^{u+x})\leq\lambda\}}\Big] \lesssim Lk^{-\frac{1}{16}}\frac{u}{b}\,.$$

Obviously, the same bound also holds with $\mathbf{F}^{\lambda}_{b,x-L,L}(\Upsilon_b^{u+x})$ replaced by $\mathbf{F}^{\lambda}_{k,x-L,L}(\Upsilon_b^{u+x})$. Therefore, we can focus our attention on the complement of the event $\mathrm{E}_{k,b}$. Specifically, we need to estimate the following expectation

$$\mathbb{E}_{0,u,b}\Big[|\mathbf{F}^{\lambda}_{b,x-L,L}(\Upsilon_b^{u+x}) - \mathbf{F}^{\lambda}_{k,x-L,L}(\Upsilon_b^{u+x})|\mathbb{1}_{\{\mathrm{E}^c_{k,b}\}}\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b^{u+x})\leq\lambda\}}\Big]\,. \tag{2.7.42}$$

By recalling the Definition 2.4.6 of the control variable K_b, we can further restrict our attention to the event $\{\mathrm{K}_b \leq k\}$, as the bound on the complement of this event can be obtained by using Lemma 2.4.10. We recall that $\Upsilon_b^{u+x}(0) = 0$, and so on the complement of the event $\mathrm{E}_{k,b}$, it holds that $|\mathbb{D}^{\lambda}_{b,0}(\Upsilon_b^{u+x})| = |\mathbb{D}^{\lambda}_{k,0}(\Upsilon_b^{u+x})|$. Therefore, we have that

$$\begin{aligned}&|\mathbf{F}^{\lambda}_{b,x-L,L}(\Upsilon_b^{u+x}) - \mathbf{F}^{\lambda}_{k,x-L,L}(\Upsilon_b^{u+x})|\\ &\qquad\leq L\bigg(1-\exp\bigg(-e^{-\gamma(x-L)}\int_{\mathbb{B}_b\setminus\mathbb{B}_k}\mathrm{F}_\gamma(y)e^{\gamma\Upsilon_b^{u+x}(y)}dy\bigg)\bigg)\,.\end{aligned}$$

Thanks to Lemma 2.4.9, and by choosing $a > 0$ sufficiently large, we observe that there exists a constant $c_1 = c_1(a) > 0$ such that

$$\mathrm{E}^c_{k,b}\cap\big\{\mathrm{K}_b\leq k\big\} \subseteq \bigcap_{j=k}^{b-1}\big\{B_j \geq c_1(\log j)^2\big\}\cap\big\{\mathrm{K}_b\leq k\big\}\,.$$

Therefore, recalling once again the Definition 2.4.6 of the control variable K_b, on the event $\mathrm{E}^c_{k,b}\cap\{\mathrm{K}_b\leq k\}$, by using the decomposition (2.4.5) of the field Υ_b and by choosing the constant $a > 0$ sufficiently large, we obtain that there exist constants c_2, $c_3 > 0$ such that

$$\begin{aligned}\int_{\mathbb{B}_b\setminus\mathbb{B}_k}\mathrm{F}_\gamma(y)e^{\gamma\Upsilon_b^{u+x}(y)}dy &\leq \sum_{j=k}^{b-1}\int_{\mathbb{A}_j}\mathrm{F}_\gamma(y)e^{\gamma(-c_2(\log j)^2+\mathrm{Z}_j(y)-\sqrt{2d}j)}dy\\ &\lesssim \sum_{j=k}^{b-1}e^{-\gamma c_3(\log j)^2}\int_{\mathbb{B}_1\setminus\mathbb{B}_0}\mathrm{F}_\gamma(e^jy)\underbrace{e^{\gamma(\mathrm{Z}_j(e^jy)-\sqrt{2d}j-\mathfrak{d}_j)+dj}dy}_{\overline{\mu}_{\gamma,j}(dy)}\,,\end{aligned}$$

where we recall that $\mathfrak{d}_j$ is defined as in (2.7.13), and we use the notation introduced in (2.6.1). Furthermore, since thanks to (K2), the function $\mathfrak{K}$ is supported in $B(0,1)$, we observe that on $\mathbb{B}_1\setminus\mathbb{B}_0$, the measure $\overline{\mu}_{\gamma,j}$ has the same law as the measure introduced in (2.1.8) (with t replaced by j), i.e., it is a regularised and normalised supercritical GMC measure. Hence, applying the previous estimate, we deduce that the quantity in (2.7.42) is bounded above by

$$L\mathbb{E}\bigg[\bigg(1-\exp\bigg(-e^{-\gamma(x-L)}\sum_{j=k}^{\infty}e^{-\gamma c_3(\log j)^2}\int_{\mathbb{B}_1\setminus\mathbb{B}_0}\mathrm{F}_\gamma(e^jy)\overline{\mu}_j(dy)\bigg)\bigg)\mathbb{1}_{\{\mathbb{M}_{0,b}(\Upsilon_b^{u+x})\leq\lambda,\,\mathrm{K}_b\leq k\}}\bigg]\,.$$

We observe that the quantity inside the parenthesis in the previous display is independent of $(B_s)_{s\leq b}$. Therefore, using (2.4.13) and arguing as in the proof of Lemma 2.4.10, for any $k \geq 0$ sufficiently large and

for $b \geq 0$ sufficiently large, the quantity in the previous display is bounded from above by a multiple of

$$\begin{aligned}
&\frac{u}{b} L \mathbb{E}\left[1 - \exp\left(-\sum_{j=k}^{\infty} e^{-\gamma(x-L+c_3(\log j)^2)} \int_{\mathbb{B}_1 \setminus \mathbb{B}_0} \mathrm{F}_\gamma(e^j y)\overline{\mu}_{\gamma,j}(dy)\right)\right]^{\frac{1}{2}} \\
&\quad \leq \frac{u}{b} L\left(\sum_{j=k}^{\infty} \mathbb{E}\left[1 - \exp\left(-e^{-\gamma \log j} \int_{\mathbb{B}_1 \setminus \mathbb{B}_0} \mathrm{F}_\gamma(e^j y)\overline{\mu}_{\gamma,j}(dy)\right)\right]\right)^{\frac{1}{2}} \\
&\quad \lesssim \frac{u}{b} L\left(\sum_{j=k}^{\infty} j^{-\sqrt{2d}} \log j\right)^{\frac{1}{2}},
\end{aligned}$$

where here we used the elementary inequality $1 - e^{-\sum_{j=k}^{\infty} u_j} \leq \sum_{j=k}^{\infty}(1 - e^{-u_j})$ valid for $(u_j)_{j\geq k} \subset \mathbb{R}_0^+$, and the last inequality follows from [MRV16, Proposition 4.2] applied with $\chi = \log j$. Strictly speaking, we cannot directly apply [MRV16, Proposition 4.2], as the function F_γ is random. However, by following the proof of that proposition, we obtain the same conclusion in our more general setting by leveraging assumptions (W1) and (W4). Thus, the conclusion follows, as the series in the last line of the above display converges to zero as $k \to \infty$. □

We are now ready to prove Lemma 2.7.4.

Proof of Lemma 2.7.4. The claim follows by combining Lemmas 2.7.11, 2.7.12, 2.7.13, and 2.7.14. □

2.A Brownian bridge above a curve

In this appendix, we collect some estimates for the probability of a Brownian bridge staying above a positive/negative curve, and some "entropic repulsion" estimates. In particular, we are interested in results analogous to those stated in [BL18, Section 4.2]. We emphasise that in [BL18, Section 4.2], there are actually estimates for Brownian bridges but are only limited to the case where the underlying curve is symmetric about the midpoint of the lifespan of the Brownian bridge. Hence, their techniques for transferring statements about Brownian motion to statements about the Brownian bridge cannot be applied to our setting. Generally speaking, the proofs of our results are relatively simple adaptations of the proofs of the results for the Brownian motion in [BL18, Section 4.2].

2.A.1 Some preliminary results

We collet in this subsection some simple results that are used several times throughout the remaining part of this appendix. For a continuous function $\zeta : \mathbb{R}_0^+ \to \mathbb{R}$, we define the stopping time

$$\tau_\zeta \stackrel{\text{def}}{=} \inf\{s \geq 0 \,:\, B_s = \zeta(s)\}.$$

We recall that for $x, u \in \mathbb{R}$ and $b > 0$, the law of $(B_s)_{s\in[0,b]}$ under $\mathbb{P}_{x,u,b}$ is that of a Brownian bridge from x to u in time b. We start with some basic facts about Brownian bridges.

Lemma 2.A.1. *For $b > 0$ and $x, u > 0$, it holds that*

$$\mathbb{P}_{x,u,b}(\tau_0 > b) = 1 - e^{-\frac{2xu}{b}}, \tag{2.A.1}$$

and in particular, we have the following upper and lower bounds,

$$\frac{2xu}{b}\left(1-\frac{xu}{b}\right) \leq \mathbb{P}_{x,u,b}\big(\tau_0 > b\big) \leq \frac{2xu}{b}\,. \tag{2.A.2}$$

Furthermore, it holds that

$$\mathbb{P}_{x,u,b}\big(\tau_0 \in ds,\ \tau_0 \leq b\big) = \frac{bxe^{-\frac{((b-s)x+su)^2}{2bs(b-s)}}}{s^{3/2}\sqrt{2\pi b(b-s)}}\mathbb{1}_{\{s\in[0,b]\}}ds\,. \tag{2.A.3}$$

Proof. The equality (2.A.1) is standard and follows by the reflection principle. The bounds in (2.A.2) follows by the elementary inequalities $x - x^2/2 \leq 1 - e^{-x} \leq x$ which are valid for $x \geq 0$. To get (2.A.3), it suffices to note that, for $s \in [0, b]$,

$$\mathbb{P}_{x,u,b}\big(\tau_0 > s,\ \tau_0 \leq b\big) = \int_0^\infty \mathbb{P}_{x,y,s}\big(\tau_0 > s\big)\mathbb{P}_{x,u,b}\big(B_s \in dy\big)\,.$$

Since the density $\mathbb{P}_{x,u,b}\big(B_s \in dy\big)$ is the same as that of a normal random variable with mean $x + s(u-x)/b$ and variance $s(b-s)/b$ and by using the exact identity (2.A.1), the result follows by differentiation. □

Lemma 2.A.2. *For $b > 0$, consider $M_b \overset{\text{def}}{=} \inf_{r\in[0,b]} B_r$ and $T_b \overset{\text{def}}{=} \sup\{r \in [0,b] \,:\, B_r = M_b\}$. Then, for $u > 0$, it holds that*

$$\mathbb{P}_{0,u,b}(T_b \in ds,\ M_b \in dz) = \sqrt{\frac{2}{\pi}}\frac{\sqrt{b}(-z)(u-z)e^{-\frac{(bz-us)^2}{2bs(b-s)}}}{(s(b-s))^{3/2}}\mathbb{1}_{\{s\in[0,b]\}}\mathbb{1}_{\{z<0\}}dsdz\,. \tag{2.A.4}$$

Proof. The proof is inspired by the proof of [BL18, Lemma A.1]. Using the path continuity and the strong Markov property of the Brownian bridge, along with the exact identity (2.A.1), we note that for any $s \in [0, b]$ and $z \leq 0$, it holds that

$$\begin{aligned}
\mathbb{P}_{0,u,b}\big(T_b \leq s,\ M_b \leq z\big) &= \lim_{\varepsilon\to 0}\sum_{k\in\mathbb{N}_0}\int_0^s \mathbb{P}_{z-k\varepsilon,u,b-r}\big(\tau_{z-(k+1)\varepsilon} > b-r\big)\mathbb{P}_{0,u,b}\big(\tau_{z-k\varepsilon}\in dr\big)\\
&= \lim_{\varepsilon\to 0}\sum_{k\in\mathbb{N}_0}\int_0^s \mathbb{P}_{\varepsilon,u-(z-(k+1)\varepsilon),b-r}\big(\tau_0 > b-r\big)\mathbb{P}_{0,u,b}\big(\tau_{z-k\varepsilon}\in dr\big)\\
&= 2\int_{-\infty}^z\int_0^s \frac{u-w}{b-r}\mathbb{P}_{0,u,b}\big(\tau_w\in dr\big)dw\,,
\end{aligned}$$

and so the claim follows by using (2.A.3) and by differentiation. □

Lemma 2.A.3. *Let $b \geq 1$ and $x \geq 1$. For all $u \in [0, b^{3/4}]$ and $s \in [1, b]$, it holds that*

$$\mathbb{P}_{x,u,b}\big(\tau_0 > s\big) \leq \frac{2x^2}{s} + \frac{8x}{s^{1/4}}\,. \tag{2.A.5}$$

Proof. Fix $b \geq 1$, $x \geq 1$, and $u \in [0, b^{3/4}]$. As we have observed in the proof of Lemma 2.A.1, for any $s \in [1, b]$, by conditioning on the value of the Brownian bridge at times s, we can write

$$\mathbb{P}_{x,u,b}\big(\tau_0 > s\big) = \int_0^\infty \mathbb{P}_{x,y,s}\big(\tau_0 > s\big)\mathbb{P}_{x,u,b}\big(B_s \in dy\big)\,,$$

where, as one can easily check, the density $\mathbb{P}_{x,u,b}\big(B_s \in dy\big)$ is that of a normal random variable with mean $x + s(u-x)/b$ and variance $s(b-s)/b$, i.e.,

$$\mathbb{P}_{x,u,b}\big(B_s \in dy\big) = \sqrt{\frac{b}{2\pi s(b-s)}}\exp\left(-\frac{b}{2s(b-s)}\left(y - \left(x + s\frac{u-x}{b}\right)\right)^2\right)dy\,.$$

Therefore, for $s \in [1,b]$, by letting

$$r = r(s,u,b,x) \stackrel{\text{def}}{=} x + \frac{s(u-x)}{b} + \sqrt{\frac{s(b-s)}{b}}\log(e+s)\,,$$

we have that

$$\mathbb{P}_{x,u,b}\big(\tau_0 > s\big) \le \int_0^r \Big(1 - e^{-\frac{2xy}{s}}\Big)\mathbb{P}_{x,u,b}\big(B_s \in dy\big) + \int_r^\infty \mathbb{P}_{x,u,b}\big(B_s \in dy\big)\,, \tag{2.A.6}$$

where we simply bounded the integrand by one on the second half of the integration interval. By using the elementary inequality $1 - e^{-x} < x$ valid for $x \ge 0$, we have the following bound for the first integral on the right-hand side of (2.A.6),

$$\int_0^r \Big(1 - e^{-\frac{2xy}{s}}\Big)\mathbb{P}_{x,u,b}\big(B_s \in dy\big) \le \frac{2xr}{s} \le \frac{2x^2}{s} + \frac{6x}{s^{1/4}}\,.$$

where here we used that $u < b^{3/4}$ to bound u/b by $1/s^{1/4}$, and $s > 1$ to bound $\log(e+s)$ by $2s^{1/4}$. Regarding the second integral on the right-hand side of (2.A.6), we note that it is equivalent to

$$\mathbb{P}\left(\mathcal{N}\left(x + \frac{s(u-x)}{b}, \frac{s(b-s)}{b}\right) > r\right) = \mathbb{P}\big(\mathcal{N}(0,1) > \log(e+s)\big) \le \frac{2}{s^{1/4}}\,.$$

Therefore, thanks to the fact that $x \ge 1$, the claim follows by collecting the previous estimates. □

2.A.2 Brownian bridge above a positive curve

Proposition 2.A.4. *For $\iota \in (0,1/8)$, let $\zeta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuous and non-decreasing function such that $\zeta(0) > 1$ and $\zeta(s) = o(s^\iota)$ as $s \to \infty$, and let $x > \zeta(0)$. Then, for all $b \ge 1$ and $u \in [b^\iota, b^{3/4}]$, it holds that*

$$\mathbb{P}_{x,u,b}\big(\tau_\zeta > b\big) \ge (1-\delta)\frac{2xu}{b}\,,$$

where

$$\delta = 2\left(\frac{xu}{2b} + \frac{\zeta(b)}{u} + \frac{\rho(x)}{x}\right)\,,$$

where the function $\rho : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is defined as follows

$$\rho(x) = \zeta(x^4) + 2x^2\int_{x^4}^\infty \frac{\zeta(s)}{s^2}ds + 2x\int_{x^4}^\infty \frac{\zeta(s)}{s^{5/4}}ds\,. \tag{2.A.7}$$

Proof. We start by noticing that, thanks to (2.A.2), it holds that

$$\mathbb{P}_{x,u,b}\big(\tau_\zeta > b\big) = \mathbb{P}_{x,u,b}\big(\tau_0 > b\big) - \mathbb{P}_{x,u,b}\big(\tau_\zeta \le b < \tau_0\big)$$

$$\geq \frac{2xu}{b}\left(1-\frac{xu}{b}\right) - \mathbb{P}_{x,u,b}\big(\tau_\zeta \leq b < \tau_0\big)\,, \tag{2.A.8}$$

and so we can just focus on finding a suitable upper bound for the probability in (2.A.8). Using the strong Markov property of the Brownian bridge at the stopping time τ_ζ and again (2.A.2), we have that

$$\begin{aligned}\mathbb{P}_{x,u,b}\big(\tau_\zeta \leq b < \tau_0\big) &\leq \int_0^{b/2} \mathbb{P}_{x,u,b}\big(\tau_\zeta \in ds\big)\mathbb{P}_{\zeta(s),u,b-s}\big(\tau_0 > b-s\big) + \mathbb{P}_{x,\zeta(b),b/2}(\tau_0 > b/2)\\ &\leq \frac{4u}{b}\int_0^{b/2} \zeta(s)\mathbb{P}_{x,u,b}\big(\tau_\zeta \in ds\big) + \frac{4x\zeta(b)}{b}\,,\end{aligned}$$

where to obtain the bound in the first line, we observed that the probability that the first time the Brownian bridge from x to u in time b hits ζ is after time $b/2$ is bounded above by the probability that the Brownian bridge from x to $\zeta(b)$ in time $b/2$ stays above 0. We now focus on the integral in the last line of the above display. By integrating by parts, we note that

$$\begin{aligned}\int_0^{b/2} \zeta(s)\mathbb{P}_{x,u,b}\big(\tau_\zeta \in ds\big) &\leq \zeta(0) + \int_0^{b/2} \zeta'(s)\mathbb{P}_{x,u,b}\big(\tau_\zeta \geq s\big)ds\\ &\leq \zeta(x^4) + \int_{x^4}^{b/2} \zeta'(s)\mathbb{P}_{x,u,b}\big(\tau_0 \geq s\big)ds\,.\end{aligned}$$

Now, using the bound (2.A.5) and the fact that $\zeta(s) = o(s^{1/4})$ as $s \to \infty$ (since $\iota \in (0,1/8)$), we obtain that

$$\begin{aligned}\int_{x^4}^{b/2} \zeta'(s)\mathbb{P}_{x,u,b}\big(\tau_0 \geq s\big)ds &\leq x\int_{x^4}^{\infty} \zeta'(s)\left(\frac{2x}{s} + \frac{8}{s^{1/4}}\right)ds\\ &\leq 2x\int_{x^4}^{\infty} \zeta(s)\left(\frac{x}{s^2} + \frac{1}{s^{5/4}}\right)ds\,.\end{aligned}$$

Therefore, the conclusion follows by combining all the previous estimates. □

2.A.3 Brownian bridge above a negative curve

Proposition 2.A.5. *For $\iota \in (0,1/8)$, let $\zeta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuous and non-decreasing function such that $\zeta(0) > 1$ and $\zeta(s) = o(s^\iota)$ as $s \to \infty$, and let $x > \zeta(0)$. Then, for all $b \geq 1$ sufficiently large and $u \in [b^\iota, b^{3/4}]$, it holds that*

$$\mathbb{P}_{x,u,b}\big(\tau_{-\zeta} > b\big) \leq (1+\widetilde{\delta})\frac{2xu}{b}\,,$$

where

$$\widetilde{\delta} = 4\left(\frac{x}{u} + \frac{4\zeta(b)^2}{xu} + \frac{4\widetilde{\rho}(x)}{x}\right). \tag{2.A.9}$$

where, recalling the definition (2.A.7) of the function ρ, the function $\widetilde{\rho} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is defined as follows

$$\widetilde{\rho}(x) = \rho(x) + 2\frac{\zeta(x^2)^2}{x} + \int_{x^2}^{\infty} \frac{\zeta(s)^2}{s^{3/2}}ds\,. \tag{2.A.10}$$

The proof of Proposition 2.A.5 is based on the following lemma.

Lemma 2.A.6. *For $\iota \in (0,1/8)$, let $\zeta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuously differentiable and non-decreasing function such that $\zeta(0) > 1$ and $\zeta(s) = o(s^\iota)$ as $s \to \infty$. Then, for all $b \geq 1$ sufficiently large and*

$u \in [b^\iota, b^{3/4}]$, *it holds that*

$$\mathbb{P}_{0,u,b}\big(\tau_{-\zeta} > b\big) \leq 16\frac{u}{b}\left(\zeta(0) + \int_0^b \frac{\zeta(z)\zeta'(z)}{\sqrt{z}}dz + \frac{\zeta(b)^2}{u}\right).$$

Proof. Recalling the notation introduced in the statement of Lemma 2.A.2, we begin by noting that

$$\mathbb{P}_{0,u,b}\big(\tau_{-\zeta} > b\big) \leq \mathbb{P}_{0,u,b}\big(\tau_{-\zeta(0)} > b\big) + \mathbb{P}_{0,u,b}\big(-\zeta(T_b) < M_b < -\zeta(0)\big)\,. \tag{2.A.11}$$

The first probability on the right-hand side can be bounded by using (2.A.2) as follows

$$\mathbb{P}_{0,u,b}\big(\tau_{-\zeta(0)} > b\big) \leq 2\frac{\zeta(0)(\zeta(0)+u)}{b} \leq 4\frac{u}{b}\zeta(0)\,,$$

where here we assumed that $b \geq 1$ is taken large enough so that $\zeta(0)/u < 1$ uniformly over all $u \in [b^\iota, b^{3/4}]$. Concerning the second probability on the right-hand side of (2.A.11), using (2.A.4) of Lemma 2.A.2, we have that

$$\begin{aligned}
\mathbb{P}_{0,u,b}\big(-\zeta(T_b) < M_b < -\zeta(0)\big) &= \sqrt{\frac{2}{\pi}}\int_0^b \int_{-\zeta(s)}^{-\zeta(0)} \frac{\sqrt{b}(-z)(u-z)e^{-\frac{(-bz+us)^2}{2bs(b-s)}}}{(s(b-s))^{3/2}}dzds \\
&= \sqrt{\frac{2}{\pi}}\int_0^b \int_{\zeta(0)}^{\zeta(s)} \frac{\sqrt{b}z(u+z)e^{-\frac{(bz+us)^2}{2bs(b-s)}}}{(s(b-s))^{3/2}}dzds \\
&= \sqrt{\frac{2}{\pi}}\int_{\zeta(0)}^{\zeta(b)} \int_{\zeta^{-1}(z)}^b \frac{\sqrt{b}z(u+z)e^{-\frac{(bz+us)^2}{2bs(b-s)}}}{(s(b-s))^{3/2}}dsdz\,,
\end{aligned} \tag{2.A.12}$$

where here, we assumed that ζ is invertible. This is not restrictive since the general case can be obtained by a standard approximation argument. We now bound the inner integral in (2.A.12) by splitting the interval of integration around $\zeta^{-1}(z) \vee b/2$. We note that,

$$\begin{aligned}
\int_{\zeta^{-1}(z)}^{\zeta^{-1}(z)\vee b/2} \frac{\sqrt{b}z(u+z)e^{-\frac{(bz+us)^2}{2bs(b-s)}}}{(s(b-s))^{3/2}}ds &\leq \frac{2^{3/2}(u+z)z}{b}\int_{\zeta^{-1}(z)}^\infty \frac{e^{-\frac{z^2}{2s}}}{s^{3/2}}ds \\
&\leq \frac{8(u+z)}{b}\frac{z}{\sqrt{\zeta^{-1}(z)}} \\
&\leq \frac{16u}{b}\frac{z}{\sqrt{\zeta^{-1}(z)}}\,,
\end{aligned} \tag{2.A.13}$$

where in order to get the last inequality we used that fact that for $b \geq 1$ sufficiently large, it holds that $z/u < 1$, uniformly over $u \in [b^\iota, b^{3/4}]$ and $z \in [\zeta(0), \zeta(b)]$. Similarly, again by taking $b \geq 1$ sufficiently large, we have that

$$\begin{aligned}
\int_{\zeta^{-1}(z)\vee b/2}^b \frac{\sqrt{b}z(u+z)e^{-\frac{(bz+us)^2}{2bs(b-s)}}}{(s(b-s))^{3/2}}ds &\leq \int_{b/2}^b \frac{\sqrt{b}z(u+z)e^{-\frac{(bz+us)^2}{2bs(b-s)}}}{(s(b-s))^{3/2}}ds \\
&\leq \frac{2^{3/2}(u+z)z}{b}\int_0^\infty \frac{e^{-\frac{(z+u)^2}{16s}}}{s^{3/2}}ds \\
&\leq 2^{7/2}\sqrt{\pi}\frac{z}{b}\,.
\end{aligned} \tag{2.A.14}$$

Therefore, plugging the estimates (2.A.13) and (2.A.14) into (2.A.12), for $b \geq 1$ sufficiently large, it holds that

$$\begin{aligned}\mathbb{P}_{0,u,b}\big(-\zeta(T_b) < M_b < -\zeta(0)\big) &\leq \sqrt{\frac{2}{\pi}}\frac{u}{b}\int_{\zeta(0)}^{\zeta(b)}\left(\frac{16z}{\sqrt{\zeta^{-1}(z)}} + 2^{7/2}\sqrt{\pi}\frac{z}{u}\right)dz \\ &\leq 16\frac{u}{b}\left(\int_0^b \frac{\zeta(z)\zeta'(z)}{\sqrt{z}}dz + \frac{\zeta(b)^2}{u}\right).\end{aligned}$$

where to go from the first to the second line, we also performed the change of variables $z \mapsto \zeta^{-1}(z)$. □

Proof of Proposition 2.A.5. We begin by noting that, thanks to (2.A.2), it holds that

$$\mathbb{P}_{x,u,b}\big(\tau_{-\zeta} > b\big) \leq \mathbb{P}_{x,u,b}\big(\tau_0 > b\big) + \mathbb{P}_{x,u,b}\big(\tau_0 \leq b < \tau_{-\zeta}\big) \leq \frac{2xu}{b} + \mathbb{P}_{x,u,b}\big(\tau_0 \leq b < \tau_{-\zeta}\big)\,.$$

Therefore, our task is now to find a suitable upper bound for the probability on the right-hand side of the above display. To this end, using again (2.A.2), we note that

$$\begin{aligned}\mathbb{P}_{x,u,b}\big(\tau_0 \leq b < \tau_{-\zeta}\big) &\leq \mathbb{P}_{x,u,b}\big(\tau_0 \leq b/2,\ \tau_{-\zeta} > b\big) + \mathbb{P}_{x,0,b/2}\big(\tau_{-\zeta(0)} > b/2\big) \\ &\leq \mathbb{P}_{x,u,b}\big(\tau_0 \leq b/2,\ \tau_{-\zeta} > b\big) + \frac{4(x+\zeta(0))\zeta(0)}{b} \\ &\leq \mathbb{P}_{x,u,b}\big(\tau_0 \leq b/2,\ \tau_{-\zeta} > b\big) + \frac{8x^2}{b}\,,\end{aligned}$$

where to obtain the bound in the first line, we observed that the probability that the first time the Brownian bridge from x to u in time b hits 0 is after time $b/2$ is bounded above by the probability that the Brownian bridge from x to 0 in time $b/2$ stays above height $-\zeta(0)$. Therefore, from now on, we can just focus on the probability in the last line of the above display. Using the strong Markov property of the Brownian bridge, we note that

$$\mathbb{P}_{x,u,b}\big(\tau_0 \leq b/2,\ \tau_{-\zeta} > b\big) = \int_0^{b/2} \mathbb{P}_{0,u,b-s}(\tau_{-\zeta(s+\cdot)} > b-s)\mathbb{P}_{x,u,b}\big(\tau_0 \in ds\big)\,. \tag{2.A.15}$$

Now, thanks to Lemma 2.A.6, we have that, for all $s \in [0, b/2]$, it holds that

$$\mathbb{P}_{0,u,b-s}\big(\tau_{-\zeta(s+\cdot)} > b-s\big) \leq 32\frac{u}{b}\left(\zeta(s) + \int_0^b \frac{\zeta(z+s)\zeta'(z+s)}{\sqrt{z}}dz + \frac{\zeta(b)^2}{u}\right),$$

and so plugging this estimate into the right-hand side of (2.A.15), we note that

$$\begin{aligned}\mathbb{P}_{x,u,b}\big(\tau_0 \leq b/2,\ \tau_{-\zeta} > b\big) \leq 32\frac{u}{b}\Bigg(&\frac{\zeta(b)^2}{u} \\ &+ \int_0^{b/2}\zeta(s)\mathbb{P}_{x,u,b}\big(\tau_0 \in ds\big) + \int_0^{b/2}\int_0^b \frac{\zeta(z+s)\zeta'(z+s)}{\sqrt{z}}dz\mathbb{P}_{x,u,b}\big(\tau_0 \in ds\big)\Bigg)\,.\end{aligned} \tag{2.A.16}$$

Regarding the first integral in (2.A.16), we note that by proceeding in the same exact way as in the proof of Proposition 2.A.4, we get that

$$\int_0^{b/2}\zeta(s)\mathbb{P}_{x,u,b}\big(\tau_0 \in ds\big) \leq \rho(x)\,,$$

where we recall that the function ρ is defined in (2.A.7). Regarding the second integral in (2.A.16), by using (2.A.3), we note that

$$\int_0^b \int_0^{b/2} \frac{\zeta(z+s)\zeta'(z+s)}{\sqrt{z}} \frac{bxe^{-\frac{(bx+s(u-x))^2}{2bs(b-s)}}}{\sqrt{2\pi}s\sqrt{bs(b-s)}} dsdz \leq \frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \frac{\zeta(z+s)\zeta'(z+s)}{\sqrt{z}} \frac{xe^{-\frac{x^2}{2s}}}{s^{3/2}} dsdz$$
$$\leq 2\frac{\zeta(x^2)^2}{x} + \int_{x^2}^\infty \frac{\zeta(s)^2}{s^{3/2}} ds \, .$$

where to obtain the last inequality, it is sufficient to follow the calculations in the proof of [BL18, Lemma A.8]. Hence, the claim follows by gathering toghther all the previous estimates. □

Let $a, k > 1$, and consider the function $\zeta_{a,k} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ given by

$$\zeta_{a,k}(s) \stackrel{\text{def}}{=} a\big(1+\big[\log(1+k+s)\big]^2\big) \, . \tag{2.A.17}$$

Lemma 2.A.7. *Let $a, k > 1$ and consider the functions $\rho_{a,k}, \widetilde{\rho}_{a,k} : [0,\infty] \to \mathbb{R}_0^+$ defined in (2.A.7) and (2.A.10) with $\zeta = \zeta_{a,k}$ as in (2.A.17). Then, there exist constants $c_1, c_2, \widetilde{c}_1, \widetilde{c}_2, \widetilde{c}_3 > 0$, depending on a but not on k, such that*

$$\rho_{a,k}(x) \leq c_1\zeta_{a,k}(0) + c_2\big[\log(e+x^4)\big]^2 \, , \qquad \forall\, x \geq 1 \, ,$$

and

$$\widetilde{\rho}_{a,k}(x) \leq \widetilde{c}_1\zeta_{a,k}(0) + \widetilde{c}_2\frac{\zeta_{a,k}(0)^2}{x} + \widetilde{c}_3\big[\log(e+x^4)\big]^4 \, , \qquad \forall\, x \geq 1 \, .$$

Proof. For $a, k > 1$, we have that

$$\zeta'_{a,k}(s) = \frac{2a\log(1+k+s)}{1+k+s} \, , \qquad \forall s \geq 0 \, .$$

and so using the elementary inequality $\log(1+s)/(1+s) \leq e\log(e+s)/(e+s)$ and the fact that the expression on the right decreases for $s \geq 0$, we get that

$$\zeta_{a,k}(s) = \int_0^{k+s} \zeta'_{a,0}(r)dr + a = 2a\int_0^k \frac{\log(1+r)}{1+r}dr + 2a\int_k^{k+s} \frac{\log(1+r)}{1+r}dr + a$$
$$\leq \zeta_{a,k}(0) + 2ae\int_0^s \frac{\log(e+r)}{e+r}dr$$
$$\leq \zeta_{a,k}(0) + ea\big[\log(e+s)\big]^2 \, .$$

By using the above bound on $\zeta_{a,k}$ and recalling the exact expressions of the functions $\rho_{a,k}$ and $\widetilde{\rho}_{a,k}$, the claims follow by some elementary calculations. □

2.A.4 Entropic repulsion

Lemma 2.A.8. *For $a, k > 1$, let $\zeta_{a,k} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be the function defined in (2.A.17). There exists a constant $c = c(a) > 0$ such that for all $\iota \in (0, 1/8)$, $b \geq 1$ sufficiently large, $u \in [b^\iota, b^{3/4}]$, all $s \in (0, b/2]$, and all $x \geq \zeta_{a,k}(s)$,*

$$\mathbb{P}_{0,u,b}\left(\inf_{r\in[0,b]}\big(B_r + \zeta_{a,k}(r)\big) > 0, \; B_s \leq x\right) \leq c\frac{ux^2}{b\sqrt{s}} \, .$$

Proof. For $a, k > 1$ and $s \in (0, b/2]$, we let

$$A_s \overset{\text{def}}{=} \left\{ \inf_{r\in[s,b]} \big(B_r + \zeta_{a,k}(r)\big) > 0 \right\} ,$$

so that the event in the lemma statement can be written as $A_0 \cap \{B_s \le x\}$. Using the Markov property of the Brownian bridge, we note that

$$\mathbb{P}_{0,u,b}\big(A_0,\ B_s \le x\big) \le \mathbb{E}_{0,u,b}\Big[\mathbb{1}_{\{-\zeta_{a,k}(s)<B_s\le x\}} \mathbb{P}_{0,u,b}\big(A_s \big| \sigma(B_s)\big)\Big] . \tag{2.A.18}$$

On the event $\{B_s \in (-\zeta_{a,k}(s), x]\}$, the above conditional probability is maximised when $\{B_s = x\}$. Now, since $s \in (0, b/2]$ and $x \ge \zeta_{a,k}(s)$, Proposition 2.A.5 and Lemma 2.A.7 ensure that there exists a constant $c_1 = c_1(a) > 0$ such that for all $b \ge 1$ sufficiently large and $u \in [b^\iota, b^{3/4}]$, it holds that

$$\mathbb{P}_{0,u,b}\big(A_s \big| B_s = x\big) = \mathbb{P}_{x,u,b-s}\left(\inf_{r\in[0,b-s]} \big(B_r + \zeta_{a,k+s}(r)\big) > 0 \right) \le c_1 \frac{xu}{b} . \tag{2.A.19}$$

To conclude, it remains to estimate the probability of the event in the indicator function on the right-hand side of (2.A.18). Since $x \ge \zeta_{a,k}(s)$, we observe that there exists a constant $c_2 > 0$ such that

$$\mathbb{P}_{0,u,b}\big(-\zeta_{a,k}(s) < B_s \le x\big) \le \mathbb{P}_{0,u,b}\big(|B_s| \le x\big) \le c_2 \frac{x}{\sqrt{s}} .$$

Finally, putting everything together, we showed that there exist a constant $c_3 = c_3(a) > 0$ such that

$$\mathbb{P}_{0,u,b}\big(A_0,\ B_s \le x\big) \le c_3 \frac{ux^2}{b\sqrt{s}} ,$$

and so the claim follows. □

Proposition 2.A.9. *For $a,\ k > 1$, let $\zeta_{a,k} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be the function defined in (2.A.17). There exist constants $c_1 = c_1(a)$, $c_2 = c_2(a) > 0$ such that for all $\iota \in (0, 1/8)$, $b \ge 1$ sufficiently large, $u \in [b^\iota, b^{3/4}]$, and all $s \in (c_2, b/2]$,*

$$\mathbb{P}_{0,u,b}\left(\inf_{r\in[0,b]} \big(B_r + \zeta_{a,k}(r)\big) > 0,\ \inf_{r\in[s,b]} \big(B_r - \zeta_{a,k}(r)\big) < 0 \right) \le c_1 \frac{u}{b} \frac{(k+s)^{7/16}}{\sqrt{s}} .$$

Proof. For $a, k > 1$ and $s \ge 0$, we define the events

$$A_s^\pm \overset{\text{def}}{=} \left\{ \inf_{r\in[s,b]} \big(B_r \pm \zeta_{a,k}(r)\big) > 0 \right\} ,$$

so that the probability in the statement can be written as $\mathbb{P}_{0,u,b}(A_0^+ \setminus A_s^-)$. For $x \ge \zeta_{a,k}(s)$, we have that

$$\frac{\mathbb{P}_{0,u,b}(A_0^+ \setminus A_s^-)}{\mathbb{P}_{0,u,b}(A_0^+)} = 1 - \frac{\mathbb{P}_{0,u,b}(A_0^+ \cap A_s^-)}{\mathbb{P}_{0,u,b}(A_0^+)} \le 1 - \frac{\mathbb{P}_{0,u,b}(A_0^+ \cap A_s^- \cap \{B_s \ge x\})}{\mathbb{P}_{0,u,b}(A_0^+)} . \tag{2.A.20}$$

Our goal now is to find a suitable lower bound for the probability in the numerator on the right hand-side of (2.A.20). To this end, we define the event

$$A_{0,s}^+ \overset{\text{def}}{=} \left\{ \inf_{r\in[0,s]} \big(B_r + \zeta_{a,k}(r)\big) > 0 \right\} ,$$

and we note that

$$\mathbb{P}_{0,u,b}\big(A_0^+ \cap A_s^- \cap \{B_s \geq x\}\big) = \mathbb{E}_{0,u,b}\Big[\mathbb{1}_{\{A_{0,s}^+ \cap \{B_s \geq x\}\}} \mathbb{P}_{0,u,b}\big(A_s^- \,\big|\, \sigma((B_r)_{r\leq s})\big)\Big] \,.$$

For any $s \in (0, b/2]$, on the event $\{B_s \geq x\}$, thanks to Proposition 2.A.4 and Lemma 2.A.7, we have that there exists a constant $c_1 = c_1(a) > 0$ such that

$$\mathbb{P}_{0,u,b}\big(A_s^- \,\big|\, \sigma((B_r)_{r\leq s})\big) \geq \frac{2B_s u}{b-s}\left(1 - c_1\left(\frac{\zeta_{a,k+s}(0)}{x} + \frac{[\log(e+x^4)]^2}{x} + o_b(1)\right)\right) .$$

Similarly, for any $s \in (0, b/2]$, on the event $\{B_s \geq x\}$, thanks to Proposition 2.A.5 and Lemma 2.A.7, we have that there exists a constant $c_2 = c_2(a) > 0$ such that

$$\mathbb{P}_{0,u,b}\big(A_s^+ \,\big|\, \sigma((B_r)_{r\leq s})\big) \leq \frac{2B_s u}{b-s}\left(1 + c_2\left(\frac{\zeta_{a,k+s}(0)}{x} + \frac{\zeta_{a,k+s}(0)^2}{x^2} + \frac{[\log(e+x^4)]^4}{x} + o_b(1)\right)\right) .$$

Therefore, denoting by δ and $\widetilde{\delta}$ the two error terms in the previous two displays, we have that

$$\mathbb{P}_{0,u,b}\big(A_0^+ \cap A_s^- \cap \{B_s \geq x\}\big) \geq \frac{1-\delta}{1+\widetilde{\delta}} \mathbb{P}_{0,u,b}\big(A_0^+ \cap \{B_s \geq x\}\big) \,,$$

and so

$$\begin{aligned}\frac{\mathbb{P}_{0,u,b}\big(A_0^+ \cap A_s^- \cap \{B_s \geq x\}\big)}{\mathbb{P}_{0,u,b}(A_0^+)} &\geq \frac{1-\delta}{1+\widetilde{\delta}} \frac{\mathbb{P}_{0,u,b}\big(A_0^+ \cap \{B_s \geq x\}\big)}{\mathbb{P}_{0,u,b}(A_0^+)} \\ &= \frac{1-\delta}{1+\widetilde{\delta}}\left(1 - \frac{\mathbb{P}_{0,u,b}\big(A_0^+ \cap \{B_s \leq x\}\big)}{\mathbb{P}_{0,u,b}(A_0^+)}\right) . \end{aligned} \tag{2.A.21}$$

Now, by using (2.A.2), we note that, by possibly taking $b > 0$ large enough depending on k and a, it holds that

$$\begin{aligned}\mathbb{P}_{0,u,b}(A_0^+) \geq \mathbb{P}_{0,u,b}(\tau_{-\zeta_{a,k}(0)} > b) &\geq \frac{2\zeta_{a,k}(0)(u+\zeta_{a,k}(0))}{b}\left(1 - \frac{\zeta_{a,k}(0)(u+\zeta_{a,k}(0))}{b}\right) \\ &\geq \frac{\zeta_{a,k}(0)(u+\zeta_{a,k}(0))}{b} \,,\end{aligned}$$

By using this estimate along with Lemma 2.A.8, we obtain that there exists a constant $c_3 = c_3(a)$ such that, for all $x \geq \zeta_{a,k}(s)$,

$$\frac{\mathbb{P}_{0,u,b}(A_0^+ \cap \{B_s \leq x\})}{\mathbb{P}_{0,u,b}(A_0^+)} \leq c_3 \frac{ux^2}{\zeta_{a,k}(0)(u+\zeta_{a,k}(0))\sqrt{s}} \,.$$

Hence, plugging this estimate back into (2.A.21) and recalling (2.A.20), we get that

$$\frac{\mathbb{P}_{0,u,b}(A_0^+ \setminus A_s^-)}{\mathbb{P}_{0,u,b}(A_0^+)} \leq 1 - \frac{1-\delta}{1+\widetilde{\delta}}\left(1 - c_3 \frac{ux^2}{\zeta_{a,k}(0)(u+\zeta_{a,k}(0))\sqrt{s}}\right) \leq \widetilde{\delta} + \delta + c_3 \frac{ux^2}{\zeta_{a,k}(0)(u+\zeta_{a,k}(0))\sqrt{s}} \,.$$

Thanks to Proposition 2.A.5, Lemma 2.A.7, and by using the fact that $\zeta_{a,k}(s) \leq \zeta_{a,0}(s) + \zeta_{a,k}(0)$, we have that there exist a constant $c_4 = c_4(a)$ such that

$$\mathbb{P}_{0,u,b}(A_0^+) \leq c_4 \frac{\zeta_{a,k}(0)(u+\zeta_{a,k}(0))}{b} \,,$$

and so, putting everything together, we obtain that there exist constants $c_5 = c_5(a)$, $c_6 = c_6(a) > 0$ such that

$$\mathbb{P}_{0,u,b}(A_0^+ \setminus A_s^-) \leq c_5 \frac{ux^2}{b\sqrt{s}} + c_6 \frac{\zeta_{a,k}(0)(u + \zeta_{a,k}(0))}{b}(\delta + \widetilde{\delta})$$

If we choose $x = c_7(k+s)^{7/32}$, for some constant constant $c_7 = c_7(a) > 0$ for which $x > \zeta_{a,k}(s)$, one can check the first term on the right-hand side of the above expression dominates the others as soon as s is larger than some constant depending on a, and so we have that

$$\mathbb{P}_{0,u,b}(A_0^+ \setminus A_s^-) \leq c_8 \frac{u}{b} \frac{(k+s)^{7/16}}{\sqrt{s}} ,$$

for some constant $c_8 = c_8(a)$. □

2.A.5 Random walk estimates

We collect in this subsection some results that allows us to transfer the statements for Brownian bridge we obtained in the previous subsection to the case of the random walk.

Lemma 2.A.10. *Let $\zeta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a non-decreasing concave function. For each $x \in \mathbb{R}$, all $b \geq k \geq 1$ it holds that*

$$\mathbb{P}_{x,u,b}\left(\bigcap_{j=k}^{b} \{B_j > -\zeta(j)\} \,\middle|\, B_k = z \right) \leq \mathbb{P}_{z,u,b-k}\left(\inf_{s\in[0,b-k]} \big(B_s + 2\zeta(k+s)\big) > 0 \right) \prod_{j=k}^{b} \left(1 - e^{-2\zeta(j)^2}\right)^{-2} .$$

Similarly, for all $x \in \mathbb{R}$ and all $b \geq k \geq 1$, it holds that

$$\mathbb{P}_{x,u,b}\left(\bigcap_{j=k}^{b} \{B_j > -\zeta(j)\} \,\middle|\, B_k = z \right) \geq \mathbb{P}_{z,u,b-k}\left(\inf_{s\in[0,b-k]} \big(B_s + \zeta(k+s)\big) > 0 \right) .$$

Proof. The proof follows exactly the same lines as that of [BL18, Lemma 4.15]. □

Lemma 2.A.11. *For a, $k > 1$, let $\zeta_{a,k} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be the function defined in (2.A.17). There exist constants $c_1 = c_1(a)$, $c_2 = c_2(a) > 0$ such that for all $\iota \in (0, 1/8)$, $b \geq 1$ sufficiently large, $u \in [b^\iota, b^{3/4}]$, and all $k \in (c_2, b/2]$,*

$$\mathbb{P}_{0,u,b}\left(\bigcap_{j=1}^{b-1} \{B_j > -\zeta_{a,k}(j)\},\ \bigcup_{j=k}^{b-1} \{B_j < \zeta_{a,k}(j)\} \right) \leq c_1 \frac{u}{b} k^{-1/16} .$$

Proof. Thanks to Lemma 2.A.10, we have that

$$\mathbb{P}_{0,u,b}\left(\bigcap_{j=1}^{b-1} \{B_j > -\zeta_{a,k}(j)\},\ \bigcup_{j=k}^{b-1} \{B_j < \zeta_{a,k}(j)\} \right) \prod_{j=1}^{b} \left(1 - e^{-2\zeta_{a,k}(j)^2}\right)^2 \leq \mathbb{P}_{0,u,b}\left(\inf_{r\in[0,b]} \big(B_r + \zeta_{2a,k}(r)\big) > 0,\ \inf_{r\in[k,b]} \big(B_r - \zeta_{2a,k}(r)\big) < 0 \right) .$$

By observing that the product on the first line is bounded away from zero uniformly in k and b, the conclusion follows from Proposition 2.A.9 with $s = k$. □

2.B Proof of Lemma 2.7.10

In this appendix, we verify that, for any $\lambda > 0$ and $A, L, b \geq 0$, the function $\overline{\mathfrak{F}}^{\lambda}_{A,L,b} : \mathbb{R} \times \mathcal{C}(\mathbb{R}^d) \to \mathbb{R}$ defined in (2.7.30) satisfies conditions (1)–(4) of Lemma 2.7.10. To simplify the notation, we prove the case $\lambda = 1$ and note that the general case, where $\lambda > 0$, follows with straightforward modifications.

To streamline the presentation, only in this appendix, for any $g \in \mathcal{C}(\mathbb{R}^d)$ and $\eta \geq 0$, we let

$$\mathbb{D}_g(\eta) \overset{\text{def}}{=} \left\{ y \in \mathbb{B}_b \ : \ \overline{\Upsilon}_{b,g}(y) \geq \mathbb{M}_{0,b}(\overline{\Upsilon}_{b,g}) - \eta \right\} .$$

We note that if $0 \leq \eta_1 \leq \eta_2$, then $\mathbb{D}_g(\eta_1) \subseteq \mathbb{D}_g(\eta_2)$.

Proof of Lemma 2.7.10. Fix $\lambda = 1$. We begin by defining the functions $\mathfrak{h} : \mathbb{R} \to \mathbb{R}^+$ and $\mathfrak{F}_* : \mathcal{C}(\mathbb{R}^d) \to \mathbb{R}^+$ as follows

$$\mathfrak{h}(z) \overset{\text{def}}{=} e^{-\sqrt{2d}(z+L)} \mathbb{P}_{z+A+L}(\mathrm{Y}_b(0) \geq 0)^{1/2} , \qquad \mathfrak{F}_*(g) \overset{\text{def}}{=} \mathbb{E}\left[\frac{\mathbb{1}_{\{\mathbb{M}_{0,b}(\overline{\Upsilon}_{b,g}) \leq 2\}}}{e^{-8db} |\mathbb{D}_g(1/2)|^8 \wedge 1} \right]^{1/4} ,$$

We observe that (1) follows immediately from the definition of $\mathfrak{h}$, while (2) is an immediate consequence of Cauchy–Schwarz inequality. Hence, it remains to verify that (3) and (4) are satisfied.

Regarding condition (3), we fix $\delta \in (0,1)$ and $g \in C(\mathbb{R}^d)$ such that

$$\sup_{x,y \in \mathbb{R}^d, \, |x-y| \leq \delta} |g(x) - g(y)| \leq 1/4 . \tag{2.B.1}$$

We define the random radius $r \geq 0$ as the largest radius for which all of the following conditions hold:

1. There exists $x \in \mathbb{B}_b$ for which $B(x,r) \subseteq \mathbb{B}_b$ and $\overline{\Upsilon}_{b,g}(x) = \overline{\mathbb{M}}_{0,b}(\Upsilon_{b,g})$.
2. For all $y \in B(x,r)$, it holds that $\overline{\Upsilon}_{b,g}(y) \geq \overline{\mathbb{M}}_{0,b}(\Upsilon_{b,g}) - 1/2$.

For $\delta > 0$ and letting S be the volume of the unit ball, we observe that

$$\mathfrak{F}_*(g)^4 \leq S^{-8} \lceil 1/\delta \rceil^{8d} + \sum_{k = \lceil 1/\delta \rceil}^{\infty} S^{-8} (k+1)^{8d} \mathbb{E}\left[\mathbb{1}_{\{S/(k+1)^d \leq e^{-db} |\mathbb{D}_g(1/2)| < S/k^d\}} \right] .$$

For $k \in \mathbb{N}$ such that $k \geq \lceil 1/\delta \rceil$, we note that $e^{-db} |\mathbb{D}_g(1/2)| \leq S/k^d$ implies that $r \leq e^b/k < e^b \delta$. On this event, we know that there exists $z \in \mathbb{B}_b$ with $|x - z| \leq e^b/k$ such that $\overline{\Upsilon}_{b,g}(z) < \overline{\mathbb{M}}_{0,b}(\Upsilon_{b,g}) - 1/2$. In turn, this implies that

$$|\overline{\Upsilon}_{b,g}(x) - \overline{\Upsilon}_{b,g}(z)| \geq \frac{1}{2} \implies |\mathrm{Y}_b(e^{-b}x) - \mathrm{Y}_b(e^{-b}z)| \geq \frac{1}{2} - |g(x) - g(z)| .$$

In particular, recalling that we are assuming (2.B.1), for all $k \in \mathbb{N}$ such that $k \geq \lceil 1/\delta \rceil$, it holds that

$$\mathbb{P}\left(e^{-db} |\mathbb{D}_g(1/2)| \leq \frac{S}{k^d} \right) \leq \mathbb{P}\left(r \leq \frac{e^b}{k} \right) \leq \mathbb{P}\left(\sup_{x,y \in \mathbb{B}_b, \, |x-y| \leq e^b/k} \left| \mathrm{Y}_b(e^{-b}x) - \mathrm{Y}_b(e^{-b}y) \right| \geq \frac{1}{4} \right) .$$

By [Mad15, Equation (3.10)], we know that there exist constants c_1, $c_2 > 0$ such that the probability on the right-hand side in the previous display is bounded from above by $c_1 e^{-c_2 e^{-b} k}$. Hence, we have that

$$\begin{aligned}\mathfrak{F}_*(g)^4 &\leq S^{-8}\lceil 1/\delta\rceil^{8d} + \sum_{k=\lceil 1/\delta\rceil}^{\infty} S^{-8}(k+1)^{8d}\mathbb{P}\left(\sup_{x,y\in\mathbb{B}_b,\,|x-y|\leq e^b/k}\left|\mathrm{Y}_b(e^{-b}x)-\mathrm{Y}_b(e^{-b}y)\right|\geq\frac{1}{4}\right)\\ &\leq S^{-8}\lceil 1/\delta\rceil^{8d} + c_1\sum_{k=\lceil 1/\delta\rceil}^{\infty} S^{-8}(k+1)^{8d}e^{-c_2e^{-b}k} \lesssim \delta^{-10d}\,,\end{aligned}$$

where the implicit constant depends on b.

Finally, regarding condition (4), we fix g_1, $g_2 \in \mathcal{C}(\mathbb{R}^d)$ such that $\sup_{x\in\mathbb{B}_{b+2}}|g_1(x)-g_2(x)| \leq 1/8$. Then, thanks to the triangle inequality and Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned}&|\overline{\mathfrak{F}}^1_{A,L,b}(z,g_1)-\overline{\mathfrak{F}}^1_{A,L,b}(z,g_2)|\\ &\quad\leq e^{-\sqrt{2d}(z+L)+db}\mathbb{E}_z\left[\mathbb{1}_{\{\mathrm{Y}_b(0)\geq -A-L\}}\frac{|\mathbb{1}_{\{\mathbb{M}_{0,b}(\overline{\Upsilon}_{b,g_1})\leq 1\}}-\mathbb{1}_{\{\mathbb{M}_{0,b}(\overline{\Upsilon}_{b,g_2})\leq 1\}}|}{|\mathbb{D}_{g_1}(1)|}\right] &\text{(2.B.2)}\\ &\quad+ e^{-\sqrt{2d}(z+L)+db}\mathbb{E}_z\left[\mathbb{1}_{\{\mathrm{Y}_b(0)\geq -A-L\}}\mathbb{1}_{\{\mathbb{M}_{0,b}(\overline{\Upsilon}_{b,g_1})\leq 1\}}\frac{|\mathbb{1}_{\{\mathrm{E}_{g_1}\}}-\mathbb{1}_{\{\mathrm{E}_{g_2}\}}|}{|\mathbb{D}_{g_1}(1)|}\right] &\text{(2.B.3)}\\ &\quad+ e^{-\sqrt{2d}(z+L)+db}\mathbb{E}_z\left[\mathbb{1}_{\{\mathrm{Y}_b(0)\geq -A-L\}}\mathbb{1}_{\{\mathbb{M}_{0,b}(\overline{\Upsilon}_{b,g_2})\leq 1\}}\left|\frac{1}{|\mathbb{D}_{g_1}(1)|}-\frac{1}{|\mathbb{D}_{g_2}(1)|}\right|\right] &\text{(2.B.4)}\\ &\quad+ \mathfrak{h}(z)\mathfrak{F}_*(g_1)\mathbb{E}_z\left[\Delta(g_1,g_2)^4\right]^{1/4}\,, &\text{(2.B.5)}\end{aligned}$$

where, for $g \in \mathcal{C}(\mathbb{R}^d)$, we set

$$\mathrm{E}_g \overset{\text{def}}{=} \left\{\mathbb{M}_{0,b+1,b}(\overline{\Upsilon}_{b,g}) + \mathrm{Y}_b(0) < -A-L\right\}\,,$$

and also, for g_1, $g_2 \in \mathcal{C}(\mathbb{R}^d)$, we let

$$\Delta(g_1,g_2) \overset{\text{def}}{=} \exp\left(-\int_{\mathbb{B}_b}\mathrm{F}_\gamma(y)e^{\gamma(\overline{\Upsilon}_{b,g_1}(y)+\mathrm{Y}_b(0)+L)}dy\right) - \exp\left(-\int_{\mathbb{B}_b}\mathrm{F}_\gamma(y)e^{\gamma(\overline{\Upsilon}_{b,g_2}(y)+\mathrm{Y}_b(0)+L)}dy\right)\,.$$

We proceed to bound separately (2.B.2), (2.B.3), (2.B.4), and (2.B.5). We will use repeatedly the fact that, letting $\delta = \|g_1-g_2\|_\infty$, it holds that $|\mathbb{M}_{0,b}(\overline{\Upsilon}_{b,g_1}) - \mathbb{M}_{0,b}(\overline{\Upsilon}_{b,g_2})| \leq \delta$ and $|\mathbb{M}_{0,b+2,b,g_1}(\overline{\Upsilon}_{b,g_1}) - \overline{\mathbb{M}}_{0,b+2,b,g_2}(\overline{\Upsilon}_{b,g_1})| \leq \delta$.

Regarding (2.B.2), we have the following bound

$$|\mathbb{1}_{\{\mathbb{M}_{0,b}(\overline{\Upsilon}_{b,g_1})\leq 1\}} - \mathbb{1}_{\{\mathbb{M}_{0,b}(\overline{\Upsilon}_{b,g_2})\leq 1\}}| \leq \mathbb{1}_{\{\mathbb{M}_{0,b}(\overline{\Upsilon}_{b,g_1})\in[1-\delta,1+\delta]\}}\,.$$

Therefore, using [PT79, Theorem 3.1] as in the proof of Lemma 2.4.13, thanks to the Cauchy–Schwarz inequality, one obtains that

$$\text{(2.B.2)} \lesssim \|g_1-g_2\|_\infty^{1/8}\mathfrak{h}(z)\mathfrak{F}_*(g_1)\,.$$

Regarding (2.B.3), we note that

$$|\mathbb{1}_{\{\mathrm{E}_{g_1}\}} - \mathbb{1}_{\{\mathrm{E}_{g_2}\}}| \leq \mathbb{1}_{\{\mathbb{M}_{0,b+1,b}(\overline{\Upsilon}_{b,g_1})+\mathrm{Y}_b(0)+A+L\in[-\delta,\delta]\}}\,.$$

Therefore, thanks again to [PT79, Theorem 3.1] and using the Cauchy–Schwarz inequality, we have that

$$(2.B.3) \lesssim \|g_1 - g_2\|_\infty^{1/8} \mathfrak{h}(z) \mathfrak{F}_*(g_1) \, .$$

Regarding (2.B.4), we note that

$$\mathbb{1}_{\{\mathbb{M}_{0,b}(\overline{\Upsilon}_{b,g_2})\leq 1\}} \left| \frac{1}{|\mathbb{D}_{g_1}(1)|} - \frac{1}{|\mathbb{D}_{g_2}(1)|} \right| \leq \int_{\mathbb{B}_b} \frac{\mathbb{1}_{\{\mathbb{M}_{0,b}(\overline{\Upsilon}_{b,g_1})\leq 1-\delta\}}}{|\mathbb{D}_{g_1}(1/2)|^2} \mathbb{1}_{\{\mathbb{M}_{0,b}(\overline{\Upsilon}_{b,g_1}) - \overline{\Upsilon}_{b,g_1}(y)\in[1-\delta,1+\delta]\}} dy \, ,$$

where we also used the fact that $\min\{|\mathbb{D}_{g_1}(1)|, |\mathbb{D}_{g_2}(1)|\} \geq |\mathbb{D}_{g_1}(1/2)|$. Therefore, proceeding similarly to the above, we obtain that

$$(2.B.4) \lesssim \|g_1 - g_2\|_\infty^{1/8} \mathfrak{h}(z) \mathfrak{F}_*(g_1) \, .$$

To conclude, it remains to handle (2.B.5). For this term, it suffices to show that $\mathbb{E}_z[\Delta(g_1, g_2)^4]^{1/4} \lesssim \|g_1 - g_2\|_\infty^{1/8}$, which follows from some elementary computations. □

Chapter 3

Uniqueness of supercritical Gaussian multiplicative chaos

Abstract. We show that, for general convolution approximations to a large class of log-correlated Gaussian fields, the properly normalised supercritical Gaussian multiplicative chaos measures converge stably to a nontrivial limit. This limit depends on the choice of regularisation only through a multiplicative constant and can be characterised as an integrated atomic measure with a random intensity expressed in terms of the critical Gaussian multiplicative chaos.

3.1 Introduction and main results

For a domain $\mathrm{D} \subseteq \mathbb{R}^d$, with $d \geq 1$, a *Gaussian multiplicative chaos* (GMC) measure is (formally) a random measure of the form

$$\mu_\gamma(dx) \text{ “}=\text{” } e^{\gamma \mathrm{X}(x)} dx \,, \tag{3.1.1}$$

where $\gamma > 0$ is a real parameter, dx denotes the Lebesgue measure on D, and X is a log-correlated Gaussian field on D. More precisely, X is a centred Gaussian field with covariance kernel $\mathcal{K} : \mathrm{D} \times \mathrm{D} \to \mathbb{R}$ of the form

$$\mathcal{K}(x,y) \stackrel{\text{def}}{=} \mathbb{E}\big[\mathrm{X}(x)\mathrm{X}(y)\big] = -\log|x-y| + g(x,y) \,, \qquad \forall\, x,y \in \mathrm{D} \,, \tag{3.1.2}$$

for some continuous function $g : \overline{\mathrm{D}} \times \overline{\mathrm{D}} \to \mathbb{R}$. The main difficulty in defining rigorously the measure (3.1.1) is that, as one can see from (3.1.2), the covariance of X blows up along the diagonal, thus making X a rough centred Gaussian field that cannot be defined as a functional field, i.e., as a field indexed by points in the domain D, but only makes sense as a distributional field, i.e., as a linear field indexed by test functions. To overcome this difficulty, thanks to the seminal work of Kahane [Kah85], there is by now a standard roadmap which involves a regularisation, renormalisation, and limiting procedure. More precisely, one first defines a collection of continuous pointwise defined centred Gaussian fields approximating X. One then defines a sequence of properly renormalised random measures and finally passes to the limit. We refer to [RV14a, Sha16, Ber17, BP24] for more details on the topic.

It is now well-known that the behaviour of the random measure (3.1.1) undergoes a phase transition at

$$\gamma_{\mathrm{c}} \stackrel{\text{def}}{=} \sqrt{2d} \,.$$

More precisely, the regime $\gamma < \gamma_c$ is called *subcritical*, the borderline case $\gamma = \gamma_c$ is called *critical*, and the range $\gamma > \gamma_c$ is called *supercritical*. These three regimes differ both in the normalisation required to achieve a non-trivial limiting measure and in the features of the resulting measure.

In this chapter, we focus on the supercritical regime, where, to the best of our knowledge, the only existing mathematical reference in the continuum setting is [MRV16] where the authors proved the convergence of a regularised and renormalised version of (3.1.1) to a non-trivial, purely atomic random measure which is *not* measurable with respect to the underlying Gaussian field X. More precisely, [MRV16] establishes this convergence for a particular class of log-correlated Gaussian fields known as $\star$-*scale invariant fields*, using a specific approximation called the *martingale approximation*. The primary objective of this chapter is to extend this convergence result to a large class of log-correlated Gaussian fields and their convolution approximations.

3.1.1 Definitions and assumptions

Before stating our main results, we introduce some definitions. We begin by recalling the definition of the convolution approximation of a general log-correlated Gaussian field X defined on a domain $D \subseteq \mathbb{R}^d$ with a covariance kernel of the form (3.1.2). Specifically, we consider a mollifier $\rho : \mathbb{R}^d \to \mathbb{R}$ and, denoting by $\widehat{\rho}$ its Fourier transform, we assume that it satisfies the following conditions:

(A1) $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and has unit mass.

(A2) For every nonzero multi-index $j = (j_1, \dots, j_d) \in \mathbb{N}_0^d$ with $|j| \leq d-1$, it holds that $\partial^j \widehat{\rho}(0) = 0$, and $|\widehat{\rho}|$ attains a local maximum at 0.

Remark 3.1.1. The conditions in (A2) on $\widehat{\rho}$ are of a purely technical nature. The condition on the derivatives of $\widehat{\rho}$ at the origin ensures that the function $\mathbb{R}^d \ni \omega \mapsto |\omega|^{-d}(1 - |\widehat{\rho}(\omega)|^2)$ is integrable near the origin. The requirement that $|\widehat{\rho}|$ attains a local maximum at the origin guarantees that one can find $\mathfrak{a} \in (0,1)$ small enough such that $1 - |\widehat{\rho}(\mathfrak{a}\omega)|^2 \geq 0$ for all $|\omega| \leq 1$.

Remark 3.1.2. We note that although condition (A2) is crucial for our approach, it is unclear to us whether it is truly necessary for the conclusion of Theorem 3.A to hold. We also observe that in the case $d = 2$, if $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ is even, non-negative, and has unit mass, then assumption (A2) is satisfied.

Remark 3.1.3. Since the Taylor expansion of $\widehat{\rho}$ at the origin is determined by the moments of ρ, it is easy to use the same argument as in [FH20, Exercise 13.8] to construct even functions $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $\widehat{\rho}(\omega) = 1 - |\omega|^{2m} + \mathcal{O}(|\omega|^{2m+1})$, thus satisfying both of the assumptions above.

For $\varepsilon > 0$, we set $\rho_\varepsilon(\cdot) \stackrel{\text{def}}{=} \varepsilon^{-d}\rho(\varepsilon^{-1}\cdot)$. We define the convolution approximation $X_{(\varepsilon)}$ of X as follows:

$$X_{(\varepsilon)} \stackrel{\text{def}}{=} \rho_\varepsilon * X \,. \tag{3.1.3}$$

One can easily verify that the covariance kernel of the field $X_{(\varepsilon)}$ is given by

$$K^\rho_{(\varepsilon)}(x,y) \stackrel{\text{def}}{=} \mathbb{E}\big[X_{(\varepsilon)}(x)X_{(\varepsilon)}(y)\big] = \big((\rho_\varepsilon \otimes \rho_\varepsilon) * K\big)(x,y)\,, \qquad \forall\, x,y \in D\,.$$

We now introduce $\star$-scale invariant fields and their martingale approximations. The key ingredient in constructing a $\star$-scale invariant field is the so-called *seed covariance function* $\mathfrak{K} : \mathbb{R}^d \to \mathbb{R}$. Throughout this chapter, we impose a stronger set of conditions on $\mathfrak{K}$ than those given in Definition 1.2.1. Specifically, we assume that $\mathfrak{K}$ and its Fourier transform $\widehat{\mathfrak{K}}$ satisfy the following properties:

(C1) $\mathfrak{K}$ is positive definite, radial, and $\mathfrak{K}(0) = 1$.

(C2) $\mathfrak{K} \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $|\mathfrak{K}(x)| \le C(1+|x|)^{-a}$ for some $C, a > 0$ and all $x \in \mathbb{R}^d$.

(C3) $\widehat{\mathfrak{K}}$ is supported in $B(0,1)$[1], $\widehat{\mathfrak{K}}(0) \in (0,\infty)$, and $\int_{|\xi|\le|\omega|} \widehat{\mathfrak{K}}(\xi)d\xi \ge \overline{\mathfrak{a}}|\omega|^d$ for all $|\omega| < 1$ and for some $\overline{\mathfrak{a}} > 0$.

Remark 3.1.4. An example of a seed covariance function $\mathfrak{K} : \mathbb{R}^d \to \mathbb{R}$ satisfying assumptions (C1)–(C3) is given by the inverse Fourier transform of the (normalised) indicator function of the unit ball. More precisely, let $\widehat{\mathfrak{K}} : \mathbb{R}^d \to \mathbb{R}$ and $\mathfrak{K} : \mathbb{R}^d \to \mathbb{R}$ be defined as follows:

$$\widehat{\mathfrak{K}}(\omega) \stackrel{\text{def}}{=} \frac{1}{|B(0,1)|}\mathbb{1}_{\{|\omega|\le 1\}} , \qquad \mathfrak{K}(x) \stackrel{\text{def}}{=} \frac{1}{|B(0,1)|}\int_{B(0,1)} e^{i\omega\cdot x}d\omega .$$

Then, $\mathfrak{K}$ is positive definite since $\widehat{\mathfrak{K}}$ is non-negative, it is radial since the inverse Fourier transform of a radial function, and $\mathfrak{K}(0) = 1$. Additionally, $\mathfrak{K} \in \mathcal{C}^\infty(\mathbb{R}^d)$ since $\widehat{\mathfrak{K}}$ has compact support, and one can easily verify that $|\mathfrak{K}(x)| \lesssim (1+|x|)^{-(d+1)/2}$ for all $x \in \mathbb{R}^d$. The conditions in (C3) are trivially satisfied.

Remark 3.1.5. We caution the reader against interpreting the seemingly restrictive assumptions (C1)–(C3) on the seed covariance kernel as a limitation of our approach. As will become evident in the subsequent sections, it suffices to construct the supercritical GMC under the convolution approximation for a single $\star$-scale invariant field. This construction then allows us to generalise the result to essentially any log-correlated Gaussian field. In particular, one may assume throughout the remainder of this chapter that $X^\star$ is the $\star$-scale invariant field with seed covariance function as specified in Remark 3.1.4.

We write $\overline{\mathfrak{K}} : \mathbb{R}^d \to \mathbb{R}$ for the (unique) positive definite function such that the convolution of $\overline{\mathfrak{K}}$ with itself equals $\mathfrak{K}$. For ξ a space-time white noise on $\mathbb{R}^d \times \mathbb{R}^+$, we consider the *$\star$-scale invariant field with seed covariance* $\mathfrak{K}$ given by

$$X^\star(\cdot) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d}\int_0^\infty \overline{\mathfrak{K}}\big(e^r(y-\cdot)\big)e^{\frac{dr}{2}}\,\xi(dy,dr) . \tag{3.1.4}$$

Furthermore, for $t \ge 0$, we let $X_t^\star$ be the field on $\mathbb{R}^d$ given by

$$X_t^\star(\cdot) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d}\int_0^t \overline{\mathfrak{K}}\big(e^r(y-\cdot)\big)e^{\frac{dr}{2}}\,\xi(dy,dr) . \tag{3.1.5}$$

For all $x, y \in \mathbb{R}^d$ and $s, t \ge 0$, it holds that

$$\mathbb{E}\big[X^\star(x)X^\star(y)\big] = \int_0^\infty \mathfrak{K}\big(e^r(x-y)\big)dr , \qquad \mathbb{E}\big[X_s^\star(x)X_t^\star(y)\big] = \int_0^{s\wedge t} \mathfrak{K}\big(e^r(x-y)\big)dr . \tag{3.1.6}$$

The collection of fields $(X_t^\star)_{t\ge 0}$ is called the *martingale approximation* of $X^\star$. Indeed, by construction, $(X_t^\star)_{t\ge 0}$ is a martingale for the filtration $(\mathcal{F}_t)_{t\ge 0}$ given by

$$\mathcal{F}_t \stackrel{\text{def}}{=} \sigma\big(X_s^\star \;:\; s \in [0,t)\big) . \tag{3.1.7}$$

Moreover, as $t \to \infty$ the field $X_t^\star$ converges almost surely to $X^\star$ in the Sobolev space $\mathcal{H}^{-\kappa}_{\text{loc}}(\mathbb{R}^d)$, for any $\kappa > 0$. We refer to [JSW19, Proposition 4.1 (iv)] for a proof of this fact.

Remark 3.1.6. In what follows, for a $\star$-scale invariant field $X^\star$ and $t \ge 0$, we always write $X_t^\star$ for the martingale approximation of $X^\star$ at level t. For the convolution approximation, we always indicate the

[1] Note that for this to hold, the kernel $\mathfrak{K}$ must *not* be compactly supported.

"smoothing parameter" in brackets, i.e., we write $X^\star_{(\varepsilon)}$ to denote the convolution regularisation of $X^\star$ at level $\varepsilon > 0$.

3.1.2 Main results

Let X be a log-correlated Gaussian field defined on a bounded domain $D \subseteq \mathbb{R}^d$ with a covariance kernel of the form (3.1.2), and let $(X_{(\varepsilon)})_{\varepsilon>0}$ denote its convolution approximation as defined in (3.1.3). For $\gamma > \sqrt{2d}$ and $\varepsilon > 0$, we define the random measure $\mu_{\gamma,(\varepsilon)}$ on D by letting

$$\mu_{\gamma,(\varepsilon)}(dx) \stackrel{\text{def}}{=} |\log \varepsilon|^{\frac{3\gamma}{2\sqrt{2d}}} \varepsilon^{-(\gamma/\sqrt{2}-\sqrt{d})^2} e^{\gamma X_{(\varepsilon)}(x) - \frac{\gamma^2}{2}\mathbb{E}[X_{(\varepsilon)}(x)^2]} dx \ . \tag{3.1.8}$$

Before stating our main result, we introduce some additional notation.

Definition 3.1.7. For $\gamma > \sqrt{2d}$ and a non-negative, locally finite Borel measure ν on $\mathbb{R}^d$, we let $\eta_\gamma[\nu]$ be the Poisson point measure on $\mathbb{R}^d \times (0,\infty)$ with intensity measure given by $\nu(dx) \otimes w^{-(1+\gamma_c/\gamma)}dw$. We also define the integrated atomic random measure with parameter γ and spatial intensity ν as the random purely atomic measure $\mathcal{P}_\gamma[\nu]$ on $\mathbb{R}^d$ given by

$$\mathcal{P}_\gamma[\nu](dx) \stackrel{\text{def}}{=} \int_0^\infty w\, \eta_\gamma[\nu](dx, dw) \ .$$

In what follows, μ_{γ_c} denotes the critical GMC associated with X, obtained through the derivative normalisation[2] [DRSV14a, JSW19, Pow21]. Furthermore, we also introduce the measure $\overline{\mu}_{\gamma_c}$ by setting

$$\overline{\mu}_{\gamma_c}(dx) \stackrel{\text{def}}{=} e^{(d-\sqrt{d/2}\gamma)g(x,x)}\mu_{\gamma_c}(dx) \tag{3.1.9}$$

where $g : \overline{D} \times \overline{D} \to \mathbb{R}$ is the function appearing in (3.1.2).

Throughout this chapter, we denote by $\mathcal{H}^s(\mathbb{R}^d)$ the standard L^2-based Sobolev space with smoothness index $s \in \mathbb{R}$. Furthermore, given a domain $D \subseteq \mathbb{R}^d$, we define the local Sobolev space $\mathcal{H}^s_{\text{loc}}(D)$ as the space of distributions whose pairings with all test functions in $\mathcal{C}^\infty_c(D)$ belong to $\mathcal{H}^s(\mathbb{R}^d)$.

Referring to Definition 3.2.1 for the notion of stable convergence of a sequence of random measures, we are now ready to state the main result of this chapter.

Theorem 3.A. *Let* X *be a log-correlated Gaussian field on a bounded domain* $D \subseteq \mathbb{R}^d$ *with covariance kernel of the form* (3.1.2)*, where* $g \in \mathcal{H}^s_{\text{loc}}(D \times D)$ *for some* $s > d$*. Let* $\rho : \mathbb{R}^d \to \mathbb{R}$ *be a mollifier satisfying assumptions (A1)–(A2), and let* $(X_{(\varepsilon)})_{\varepsilon>0}$ *be the convolution approximation of* X *as defined in* (3.1.3)*. For* $\gamma > \sqrt{2d}$*, consider the sequence of random measures* $(\mu_{\gamma,(\varepsilon)})_{\varepsilon>0}$ *on* D *defined in* (3.1.8)*. Then, there exists a constant* $a_{\gamma,\rho} > 0$*, depending on* γ *and the mollifier* ρ*, such that* $\mu_{\gamma,(\varepsilon)}$ *converges* $\sigma(X)$*-stably to* $a_{\gamma,\rho}\mathcal{P}_\gamma[\overline{\mu}_{\gamma_c}]$ *along any sequence* $(\varepsilon_n)_{n\in\mathbb{N}}$ *converging to* 0*, where* $\overline{\mu}_{\gamma_c}$ *is the measure defined in* (3.1.9)*.*

Remark 3.1.8. The convergence in Theorem 3.A indicates that the limiting measure can be formally decomposed into two components: the location of the point masses, which is determined by an instance of the field X through the associated critical GMC, and an additional source of randomness that is independent of X and controls the weights of the point masses. Importantly, this implies that the sequence $\mu_{\gamma,(\varepsilon)}$ does not converge in probability.

[2] It is well-known that this measure does not depend on the particular choice of the approximation scheme used to define it.

Remark 3.1.9. Stable convergence has been previously used in the theory of GMC. For instance, Lacoin employed it in [Lac22] for the study of complex GMC.

The proof of Theorem 3.A relies on Theorem 2.C and on the following new decomposition result.

Proposition 3.B. *Let ρ be a mollifier satisfying assumptions (A1)–(A2), and let $X^\star$ be a $\star$-scale invariant field whose seed covariance function $\mathfrak{K}$ satisfies assumptions (C1)–(C3). Then, there exist a constant $\mathfrak{a} = \mathfrak{a}(\rho, \mathfrak{K}) \in (0,1)$ and a smooth, stationary, centred Gaussian field W, with decaying correlations[3], and independent of $\mathcal{F}_\infty$ (recall (3.1.7)), such that for any fixed $\varepsilon \in (0, \mathfrak{a})$, defining $t_\varepsilon \overset{\text{def}}{=} \log(\mathfrak{a}\varepsilon^{-1})$, the following decomposition holds*

$$X^\star_{(\varepsilon)} \overset{\text{law}}{=} X^\star_{t_\varepsilon} + W_{t_\varepsilon} + Z_{t_\varepsilon} , \tag{3.1.10}$$

where the fields on the right-hand side of (3.1.10) are all mutually independent, and for all $\varepsilon \in (0, \mathfrak{a})$, it holds that:

1. *The field W_{t_ε} is such that $W_{t_\varepsilon}(\cdot) = W(e^{t_\varepsilon}\cdot)$.*
2. *The field Z_{t_ε} is a smooth, stationary, centred Gaussian field such that $\lim_{\varepsilon\to 0} \mathbb{E}[Z_{t_\varepsilon}(\cdot)^2] = 0$.*

Remark 3.1.10. It is important to emphasise that, in Proposition 3.B, we are *not* claiming that, for any fixed $x \in \mathbb{R}^d$, the law of the process $(0,\mathfrak{a}) \ni \varepsilon \mapsto X^\star_{(\varepsilon)}(x)$ coincides with the law of the process $(0,\mathfrak{a}) \ni \varepsilon \mapsto X^\star_{t_\varepsilon}(x) + W_{t_\varepsilon}(x) + Z_{t_\varepsilon}(x)$. Rather, we are claiming that for any fixed $\varepsilon > 0$, the field $(X^\star_{(\varepsilon)}(x))_{x\in\mathbb{R}^d}$ has the same law as the field $(X^\star_{t_\varepsilon}(x) + W_{t_\varepsilon}(x) + Z_{t_\varepsilon}(x))_{x\in\mathbb{R}^d}$. This is sufficient for our purpose.

Remark 3.1.11. Since the field Z_{t_ε} vanishes in the limit $\varepsilon \to 0$, the decomposition (3.1.10) roughly indicates that the convolution approximation of $X^\star$ equals its martingale approximation plus an independent field that asymptotically behaves like white noise with finite variance.

Remark 3.1.12. Although this chapter focuses on the convolution approximation, it is worth emphasising that the conclusion of Theorem 3.A should also hold for any approximation scheme that admits a decomposition similar to (3.1.10).

To the best of our knowledge, the decomposition established in Proposition 3.B is new, and we hope that it will prove useful in other contexts as well. In particular, it would be interesting to explore whether similar techniques to those employed in the current chapter could be used to extend the main result of [Mad15] as explained in the next remark.

Remark 3.1.13. Given the close relationship between the extremal theory of log-correlated Gaussian fields and supercritical GMC, one may ask whether the techniques developed in Chapter 2, along with those in the current chapter, can be employed to generalise [Mad15, Theorem 1.1] to the convolution approximation of any log-correlated Gaussian field (see Remark 1.5.4 for a precise conjecture). The answer *should be* affirmative. Indeed, analogously to what is done in Chapter 2, one can study for any $z \in \mathbb{R}$ the following limit:

$$\lim_{t\to\infty} \mathbb{P}\left(\sup_{x\in[0,1]^d} \big(X^\star_t(x) + W_t(x)\big) - \sqrt{2d}t + \frac{3}{2\sqrt{2d}} \log t \le z \right) ,$$

where $(W_t)_{t\ge 0}$ is a collection of fields satisfying assumptions (W1)–(W4). Once this is done, one can use Propositions 3.B and 3.2.7 to transfer the result to the convolution approximation of any log-correlated Gaussian field. Note that in Proposition 3.B, the seed covariance function is required to satisfy assumptions (C1)–(C3), and is therefore not compactly supported. This issue *should be* easily overcome, similarly to what is done in Section 3.4 below, by employing Slepian's comparison lemma (see [DRSV14a, Lemma 18]).

[3] By decaying correlations, we mean that $\lim_{|x-y|\to\infty} \mathbb{E}[W(x)W(y)] = 0$.

3.1.3 Outline

The remainder of this chapter is organised as follows. In Section 3.2, we collect preliminary results that will be used throughout the chapter. In Section 3.3, we prove the main results of this work. Specifically, we begin with the proof of Theorem 3.A, followed by the proof of Proposition 3.B. In Section 3.4, we show that the convergence result in Theorem 2.C can be generalised to accommodate fields with infinite-range correlations. Finally, in Appendix 3.A, we present some results concerning moments and the multifractal spectrum of supercritical GMC measures.

Acknowledgements. We are grateful to Christophe Garban for interesting discussions during the early stages of this project.

3.2 Background and preliminaries

In this section, we gather some background material and preliminary results. More precisely, after introducing some recurring basic notations, we discuss the concept of stable convergence in Section 3.2.2 and state some of its key properties. Finally, in Section 3.2.3, we present a fundamental decomposition result for log-correlated Gaussian fields established in [JSW19].

3.2.1 Basic notation

We adopt the convention to let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. We let $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{R}_0^+ = [0, \infty)$. For a domain $\mathrm{D} \subseteq \mathbb{R}^d$, we write $\mathcal{C}(\mathrm{D})$ (resp. $\mathcal{C}_c(\mathrm{D})$) for the space of continuous (resp. continuous with compact support) functions from D to $\mathbb{R}$. We write $\mathcal{C}^\infty(\mathrm{D})$ (resp. $\mathcal{C}_c^\infty(\mathrm{D})$) for the space of smooth (resp. smooth with compact support) functions from D to $\mathbb{R}$. We write $\mathcal{C}_c^+(\mathrm{D})$ for the space of positive continuous functions from D to $\mathbb{R}$ with compact support. We let $\mathcal{M}^+(\mathrm{D})$ be the space of non-negative, locally finite measures on D. Given a measure ν and a function f, we write $\nu(f)$ to denote the integral of f against ν.

3.2.2 Topological preliminaries

We now recall some facts about stable convergence of random measures. This type of convergence, differently from the convergence in distribution, is a convergence of the sequence of random variables itself rather than of the sequence of their distributions. We refer to the monographs [JS03, HL15] and references therein for more details on stable convergence in a more general setting.

For a domain $\mathrm{D} \subseteq \mathbb{R}^d$, we equip the space $\mathcal{M}^+(\mathrm{D})$ of non-negative, locally finite measures on D with the topology of vague convergence. We equip the space of probability measures on $\mathcal{M}^+(\mathrm{D})$ with the topology of weak convergence. For a sequence $(\nu_n)_{n\in\mathbb{N}}$ of $\mathcal{M}^+(\mathrm{D})$-valued random variables, we write $\nu_n \Rightarrow \nu$ to indicate that ν_n converges *vaguely in distribution* to ν in $\mathcal{M}^+(\mathrm{D})$ as $n \to \infty$.

We consider a collection $(\nu_n)_{n\in\mathbb{N}}$ of $\mathcal{M}^+(\mathrm{D})$-valued random variables defined on a common probability space $(\Omega, \mathbb{P})$ and a $\mathcal{M}^+(\mathrm{D})$-valued random variables ν defined on a possibly larger probability space. We also fix a σ-algebra Σ over Ω.

Definition 3.2.1. We say that ν_n converges Σ-stably to $\nu \in \mathcal{M}^+(\mathrm{D})$ as $n \to \infty$, if $(Z, \nu_n) \Rightarrow (Z, \nu)$ as $n \to \infty$ for all Σ-measurable random variables Z.

Given a random variable Y defined on the same probability space as above and taking values in a Polish space $\mathcal{Y}$, we have the following result that characterises $\sigma(\mathrm{Y})$-stable convergence.

Lemma 3.2.2. *Consider the same setting described above. Then ν_n converges $\sigma(\mathrm{Y})$-stably to ν if and only if $(\mathrm{Y}, \nu_n) \Rightarrow (\mathrm{Y}, \nu)$ as $n \to \infty$.*

Proof. See for instance [HL15, Exercise 3.11]. □

We will also need the following lemma.

Lemma 3.2.3. *In the setting of Lemma 3.2.2, let Z be a random variable taking values in a Polish space $\mathcal{Z}$, and suppose that each ν_n is conditionally independent of Z given Y. If ν_n converges $\sigma(\mathrm{Y})$-stably to ν as $n \to \infty$, then it also converges $\sigma(\mathrm{Y}, \mathrm{Z})$-stably to ν as $n \to \infty$, and ν is conditionally independent of Z given Y.*

Proof. Let $\mathcal{L}(\mathrm{Z}|\mathrm{Y})$ denote the conditional law of Z given Y. This is a $\sigma(\mathrm{Y})$-measurable random variable taking values in the (Polish) space of probability measures $\mathcal{P}(\mathcal{Z})$ endowed with the weak convergence topology. Since ν_n converges $\sigma(\mathrm{Y})$-stably to ν as $n \to \infty$, it follows from [HL15, Theorem 3.17 (vii)] that the pair $(\nu_n, \mathcal{L}(\mathrm{Z}|\mathrm{Y}))$ also converges $\sigma(\mathrm{Y})$-stably to $(\nu, \mathcal{L}(\mathrm{Z}|\mathrm{Y}))$ as $n \to \infty$. In particular, it follows from [HL15, Exercise 3.11] that we have the following joint convergence in law as $n \to \infty$,

$$\big(\nu_n, \mathrm{Y}, \mathcal{L}(\mathrm{Z}|\mathrm{Y})\big) \Rightarrow \big(\nu, \mathrm{Y}, \mathcal{L}(\mathrm{Z}|\mathrm{Y})\big) \,. \tag{3.2.1}$$

To verify that ν_n converges $\sigma(\mathrm{Y}, \mathrm{Z})$-stably to ν as $n \to \infty$, it suffices, by [HL15, Exercise 3.12], to check that for any bounded continuous function $\mathrm{F} : \mathcal{M}^+(\mathrm{D}) \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$, it holds that

$$\lim_{n\to\infty} \mathbb{E}\big[\mathrm{F}(\nu_n, \mathrm{Y}, \mathrm{Z})\big] = \mathbb{E}\big[\mathrm{F}(\nu, \mathrm{Y}, \mathrm{Z})\big] \,.$$

To this end, we define the function $\overline{\mathrm{F}} : \mathcal{M}^+(\mathrm{D}) \times \mathcal{Y} \times \mathcal{P}(\mathcal{Z}) \to \mathbb{R}$ by

$$\overline{\mathrm{F}}(\rho, y, \mathcal{L}) \stackrel{\text{def}}{=} \int_{\mathcal{Z}} \mathrm{F}(\rho, y, z)\mathcal{L}(dz) \,.$$

Thanks to the conditional independence of ν_n and Z given Y, and by Fubini's theorem, we have for any $n \in \mathbb{N}$ that

$$\mathbb{E}\big[\mathrm{F}(\nu_n, \mathrm{Y}, \mathrm{Z})\big] = \mathbb{E}\left[\int_{\mathcal{Z}} \mathrm{F}(\nu_n, \mathrm{Y}, z)\mathcal{L}(\mathrm{Z}|\mathrm{Y})(dz)\right] = \mathbb{E}\big[\overline{\mathrm{F}}(\nu_n, \mathrm{Y}, \mathcal{L}(\mathrm{Z}|\mathrm{Y}))\big] \,.$$

By (3.2.1), we have that

$$\lim_{n\to\infty} \mathbb{E}\big[\overline{\mathrm{F}}(\nu_n, \mathrm{Y}, \mathcal{L}(\mathrm{Z}|\mathrm{Y}))\big] = \mathbb{E}\big[\overline{\mathrm{F}}(\nu, \mathrm{Y}, \mathcal{L}(\mathrm{Z}|\mathrm{Y}))\big] \,,$$

so that the desired result follows immediately. □

In the next lemma, we highlight an important feature of stable convergence. Specifically, the following result can be viewed as a generalisation of the classical Slutsky's theorem within the context of stable convergence.

Lemma 3.2.4. *Let $\mathrm{A} \subset \mathbb{R}^d$ be a compact set. Consider a sequence $(\nu_n)_{n\in\mathbb{N}}$ of $\mathcal{M}^+(\mathrm{A})$-valued random variables and a sequence $(\mathfrak{h}_n)_{n\in\mathbb{N}}$ of random variables taking values in $\mathcal{C}(\mathrm{A})$ (equipped with the topology of local uniform convergence). For a random variable Y, assume that ν_n converges $\sigma(\mathrm{Y})$-stably to ν, and*

$\mathfrak{h}_n$ converges in probability to a $\sigma(\mathrm{Y})$-measurable $\mathcal{C}(\mathrm{A})$-valued random variable $\mathfrak{h}$. Then it holds that $\mathfrak{h}_n\nu_n$ converges $\sigma(\mathrm{Y})$-stably to $\mathfrak{h}\nu$ as $n \to \infty$.

Proof. The proof of this result follows from [HL15, Theorem 3.18 (b)] and the continuous mapping theorem [HL15, Theorem 3.18 (c)]. □

We record here the following useful fact.

Remark 3.2.5. For $\gamma > \sqrt{2d}$ and a Radon measure ν on D, let $\mathcal{P}_\gamma[\nu]$ be the integrated atomic random measure with parameter γ and spatial intensity ν as specified in Defnition 3.1.7. Then, for every $\varphi \in \mathcal{C}_c^+(\mathrm{D})$, it holds that

$$\begin{aligned}\mathbb{E}\big[\exp\big(-\mathcal{P}_\gamma[\nu](\varphi)\big)\big] &= \mathbb{E}\left[\exp\left(-\int_{\mathrm{D}\times\mathbb{R}^+}\varphi(x)z\,\eta_\gamma[\nu](dx,dz)\right)\right]\\ &= \exp\left(-\int_{\mathrm{D}\times\mathbb{R}^+}\frac{1-e^{-\varphi(x)z}}{z^{1+\sqrt{2d}/\gamma}}\nu(dx)dz\right)\\ &= \exp\left(-\beta(d,\gamma)\int_{\mathrm{D}}\varphi(x)^{\frac{\sqrt{2d}}{\gamma}}\nu(dx)\right), \qquad (3.2.2)\end{aligned}$$

where $\beta(d,\gamma) \overset{\text{def}}{=} \Gamma(1-\sqrt{2d}/\gamma)/(\sqrt{2d}/\gamma)$.

We conclude this section with the following result which will be used in the proof of Theorem 3.A below.

Lemma 3.2.6. *For $\gamma > \sqrt{2d}$ and $\nu \in \mathcal{M}^+(\mathrm{D})$, let $\mathcal{P}_\gamma[\nu]$ be the integrated atomic random measure with parameter γ and spatial intensity ν as defined in Definition 3.1.7. Let $f \in \mathcal{C}(\mathrm{D})$ be a non-negative, continuous function. Then, one has that*

$$f\mathcal{P}_\gamma[\nu] \overset{\text{law}}{=} \mathcal{P}_\gamma\big[f^{\sqrt{2d}/\gamma}\nu\big]\,.$$

Furthermore, the same conclusion holds if $f \in \mathcal{C}(\mathrm{D})$ is a non-negative, random, continuous function independent of $\eta_\gamma[\nu]$.

Proof. The proof follows by an immediate computation (based on (3.2.2)) of the Laplace functional of the random measure $f\mathcal{P}_\gamma[\nu]$. □

3.2.3 A decomposition result for log-correlated Gaussian fields

In this short section, we recall a key decomposition result for log-correlated Gaussian fields proved in [JSW19, Theorem A]. Roughly speaking, this general theorem states that given two such fields, under some suitable mild regularity assumptions on the covariance kernels, there exists a coupling between them in such a way that their difference is given by a Hölder continuous centred Gaussian field.

In our specific setting, this result can be stated as follows.

Proposition 3.2.7. *For a domain $\mathrm{D} \subseteq \mathbb{R}^d$, let X be a log-correlated Gaussian field with covariance kernel of the form (3.1.2) with $g \in \mathcal{H}^s_{\mathrm{loc}}(\mathrm{D}\times\mathrm{D})$, for some $s > d$. Let $\mathrm{X}^\star$ be a $\star$-scale invariant field whose seed covariance satisfies (C1)–(C2). Then, for any bounded domain D' with closure satisfying $\overline{\mathrm{D}'} \subset \mathrm{D}$, we can construct copies of the fields X and $\mathrm{X}^\star$ on a common probability space in such a way that the following decomposition holds on D'*

$$\mathrm{X} = \mathrm{X}^\star + \mathrm{L}\,, \qquad (3.2.3)$$

where L *is a centred Gaussian field which is almost surely Hölder continuous on* D'*, and* $\mathrm{X}^\star$ *and* L *are jointly Gaussian.*

Proof. This result is an immediate consequence of [JSW19, Theorem A, Lemma 3.4]. In particular, the fact that the covariance of $\mathrm{X}^\star$ satisfies the assumptions of [JSW19, Theorem A] follows from [JSW19, Proposition 4.1 (vi)]. □

3.3 Proof of main results

In this section, we establish our main results. In Section 3.3.1, we first prove the uniqueness result stated in Theorem 3.A, relying on Proposition 3.B, which will be proved later in Section 3.3.2.

3.3.1 Proof of Theorem 3.A

We fix a log-correlated Gaussian field X on a bounded domain $\mathrm{D} \subseteq \mathbb{R}^d$ with covariance kernel of the form (3.1.2) with $g \in \mathcal{H}^s_{\mathrm{loc}}(\mathrm{D} \times \mathrm{D})$, for some $s > d$. Furthermore, let $\rho : \mathbb{R}^d \to \mathbb{R}$ be a mollifier satisfying assumptions (A1)–(A2) and let $(\mathrm{X}_{(\varepsilon)})_{\varepsilon>0}$ be the convolution approximation of X constructed using ρ, as defined in (3.1.3). We also fix a $\star$-scale invariant field $\mathrm{X}^\star$ with seed covariance function $\mathfrak{K}$ satisfying (C1)–(C3), and we denote by $(\mathrm{X}^\star_t)_{t\geq 0}$ its martingale approximation. We let $(\mathrm{X}^\star_{(\varepsilon)})_{\varepsilon>0}$ be the convolution approximation of $\mathrm{X}^\star$ constructed using ρ. For $\gamma > \sqrt{2d}$, we consider the collection of measures $(\mu^\star_{\gamma,(\varepsilon)})_{\varepsilon>0}$ on $\mathbb{R}^d$ defined as follows

$$\mu^\star_{\gamma,(\varepsilon)}(dx) \overset{\text{def}}{=} |\log \varepsilon|^{\frac{3\gamma}{2\sqrt{2d}}} \varepsilon^{-(\gamma/\sqrt{2}-\sqrt{d})^2} e^{\gamma \mathrm{X}^\star_{(\varepsilon)}(x) - \frac{\gamma^2}{2}\mathbb{E}[\mathrm{X}^\star_{(\varepsilon)}(x)^2]} dx \,. \tag{3.3.1}$$

Thanks to Proposition 3.2.7, we can (and will) construct copies of X and $\mathrm{X}^\star$ on a common probability space in such a way that the following decomposition holds:

$$\mathrm{X} = \mathrm{X}^\star + \mathrm{L} \,, \tag{3.3.2}$$

where L is a centred Gaussian field which is almost surely Hölder continuous. In particular, an immediate consequence of (3.3.2) is that, for all $\varepsilon > 0$, it holds that

$$\mathrm{X}_{(\varepsilon)} = \mathrm{X}^\star_{(\varepsilon)} + \mathrm{L}_{(\varepsilon)} \,, \tag{3.3.3}$$

where $\mathrm{L}_{(\varepsilon)} \overset{\text{def}}{=} \rho_\varepsilon * \mathrm{L}$. For $\varepsilon > 0$, we introduce the random function $\mathfrak{h}_{\gamma,\varepsilon} : \mathrm{D} \to \mathbb{R}$ by letting

$$\mathfrak{h}_{\gamma,(\varepsilon)}(x) \overset{\text{def}}{=} e^{\gamma \mathrm{L}_{(\varepsilon)}(x) - \frac{\gamma^2}{2}\mathbb{E}[\mathrm{X}_{(\varepsilon)}(x)^2 - \mathrm{X}^\star_{(\varepsilon)}(x)^2]} \,. \tag{3.3.4}$$

Thanks to (3.3.3), we observe that we can rewrite the measure $\mu_{\gamma,(\varepsilon)}(dx)$ defined in (3.1.8) as follows

$$\mu_{\gamma,(\varepsilon)}(dx) = \mathfrak{h}_{\gamma,(\varepsilon)} \mu^\star_{\gamma,(\varepsilon)}(dx) \,, \tag{3.3.5}$$

where the measure $\mu^\star_{\gamma,(\varepsilon)}(dx)$ is defined as in (3.3.1). Hence, to study the convergence of $\mu_{\gamma,(\varepsilon)}$, it suffices to separately analyse the convergence of $\mathfrak{h}_{\gamma,(\varepsilon)}$ and that of $\mu^\star_{\gamma,(\varepsilon)}$. In the following lemma, we focus on the latter.

Lemma 3.3.1. *Consider the setting described above. For $\gamma > \sqrt{2d}$, there exists a finite constant $a^\star_{\gamma,\rho} > 0$, depending on γ and on the mollifier ρ, such that the sequence $(\mu^\star_{\gamma,(\varepsilon)})_{\varepsilon>0}$ defined in (3.3.1) converges $\sigma(X^\star)$-stably to $a^\star_{\gamma,\rho}\mathcal{P}_\gamma[\mu^\star_{\gamma_c}]$ along any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0, where $\mu^\star_{\gamma_c}$ denotes the critical GMC associated with $X^\star$.*

Proof. We divide the proof into three main parts. In the first step, we set up the framework of the proof. In the second step, we show how the proof of the convergence of $\mu^\star_{\gamma,(\varepsilon)}$ can be reduced to the convergence of the measure $\overline{\mu}_{\gamma,t}$, defined in (3.3.7), which is essentially the normalised supercritical GMC constructed from the field $X^\star_t$ plus W_t. Finally, in the last step, by referencing Lemma 3.4.1 – stated in Section 3.4 and whose proof is based on Theorem 2.C – we establish the convergence of $\overline{\mu}_{\gamma,t}$.

Step 1: In this step, we establish the framework for the remainder of the proof and introduce several definitions. We begin by recalling that, by Proposition 3.B, there exists a constant $\mathfrak{a} = \mathfrak{a}(\rho, \mathfrak{K}) \in (0,1)$ and a smooth, stationary, centred Gaussian field W, with decaying correlations and independent of $\mathcal{F}_\infty$ (recall (3.1.7)), such that for any fixed $\varepsilon > 0$, defining $t_\varepsilon \overset{\text{def}}{=} \log(\mathfrak{a}\varepsilon^{-1})$, it holds that

$$X^\star_{(\varepsilon)} \overset{\text{law}}{=} X^\star_{t_\varepsilon} + W_{t_\varepsilon} + Z_{t_\varepsilon} \ ,$$

where we recall that all the fields on the right-hand side are mutually independent, $W_{t_\varepsilon}(\cdot) = W(e^{t_\varepsilon}\cdot)$, and Z_{t_ε} is a smooth, stationary, centred Gaussian field with vanishing variance.

We fix a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0 and we write t_n instead of t_{ε_n}. By possibly enlarging the probability space on which $X^\star$ is defined, we can (and will) assume that the following fields are also defined on the same probability space:

(S1) A copy of the field W independent of $\mathcal{F}_\infty$.

(S2) A collection of fields $(Z_{t_n})_{n\in\mathbb{N}}$, all mutually independent and independent of everything else, such that $Z_{t_n} \overset{\text{law}}{=} Z_{t_{\varepsilon_n}}$.

(S3) A collection of fields $(X^{\star,n})_{n\in\mathbb{N}}$ where each $X^{\star,n}$ has the same law as $X^\star$, and the conditional law of $X^{\star,n}$ given $X^\star_{t_n} + W_{t_n} + Z_n$ coincides with the conditional law of $X^\star$ given $X^\star_{(\varepsilon_n)}$.

We observe that (S3) guarantees that one has the almost sure identity

$$X^{\star,n}_{(\varepsilon_n)} = X^\star_{t_n} + W_{t_n} + Z_{t_n} \ .$$

For each $n \in \mathbb{N}$, we define the measure μ_{γ,t_n} by

$$\mu_{\gamma,t_n}(dx) \overset{\text{def}}{=} t_n^{\frac{3\gamma}{2\sqrt{2d}}} e^{-t_n(\gamma/\sqrt{2}-\sqrt{d})^2} e^{\gamma(X^\star_{t_n}(x)+W_{t_n}(x))-\frac{\gamma^2}{2}t_n}dx \ ,$$

Furthermore, we define the random function $\mathfrak{m}_{t_n} : \mathbb{R}^d \to \mathbb{R}$ by setting

$$\mathfrak{m}_{t_n}(x) \overset{\text{def}}{=} c_n \mathfrak{a}^{-(\gamma/\sqrt{2}-\sqrt{d})^2} e^{\gamma Z_{t_n}(x)-\frac{\gamma^2}{2}\mathbb{E}[W_{t_n}(x)^2+Z_{t_n}(x)^2]} \ , \tag{3.3.6}$$

and we define the measure $\overline{\mu}_{\gamma,t_n}$ given by

$$\overline{\mu}_{\gamma,t_n}(dx) \overset{\text{def}}{=} \mathfrak{m}_{t_n}(x)\mu_{\gamma,t_n}(dx) \ . \tag{3.3.7}$$

The constant c_n in (3.3.6) is the constant converging to 1 as $n \to \infty$ such that $\overline{\mu}_{\gamma,t_n} = \mu_{\gamma,(\varepsilon_n)}$.

Step 2: With the setup and notation introduced above, to show that $\mu^\star_{\gamma,(\varepsilon_n)}$ converges $\sigma(\mathrm{X}^\star)$-stably, thanks to Lemmas 2.3.4 and 3.2.2, it suffices to check that for all $(\varphi, f) \in \mathcal{C}^\infty_c(\mathbb{R}^d) \times \mathcal{C}^+_c(\mathbb{R}^d)$, it holds that

$$\lim_{n\to\infty} \mathbb{E}\Big[\exp\big(i\langle \mathrm{X}^\star, \varphi\rangle\big) \exp\big(-\mu^\star_{\gamma,(\varepsilon_n)}(f)\big)\Big] = \mathbb{E}\Big[\exp\big(i\langle \mathrm{X}^\star, \varphi\rangle\big) \exp\big(-\mu^\star_{\gamma}(f)\big)\Big] \,, \tag{3.3.8}$$

where we set $\mu^\star_\gamma \overset{\text{def}}{=} a^\star_{\gamma,\rho}\mathcal{P}_\gamma[\mu^\star_{\gamma_c}]$, with $a^\star_{\gamma,\rho} > 0$ the constant specified in the statement. We observe that for each $n \in \mathbb{N}$, the following equality holds thanks to (S3):

$$\mathbb{E}\Big[\exp\big(i\langle \mathrm{X}^\star, \varphi\rangle\big) \exp\big(-\mu^\star_{\gamma,(\varepsilon_n)}(f)\big)\Big] = \mathbb{E}\Big[\exp\big(i\langle \mathrm{X}^{\star,n}, \varphi\rangle\big) \exp\big(-\overline{\mu}^\star_{\gamma,t_n}(f)\big)\Big] \,.$$

We now proceed to show that

$$\lim_{n\to\infty} \Big|\mathbb{E}\Big[\exp\big(i\langle \mathrm{X}^{\star,n}, \varphi\rangle\big) \exp\big(-\overline{\mu}^\star_{\gamma,t_n}(f)\big)\Big] - \mathbb{E}\Big[\exp\big(i\langle \mathrm{X}^\star, \varphi\rangle\big) \exp\big(-\overline{\mu}^\star_{\gamma,t_n}(f)\big)\Big]\Big| = 0 \,. \tag{3.3.9}$$

The proof of (3.3.9) proceeds in three steps. First, thanks to (S3) and the Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned}\mathbb{E}\Big[\Big|\exp\big(i\langle \mathrm{X}^{\star,n}, \varphi\rangle\big) - \exp\big(i\langle \mathrm{X}^{\star,n}_{(\varepsilon_n)}, \varphi\rangle\big)\Big|\Big] &= \mathbb{E}\Big[\Big|\exp\big(i\langle \mathrm{X}^{\star}, \varphi\rangle\big) - \exp\big(i\langle \mathrm{X}^{\star}_{(\varepsilon_n)}, \varphi\rangle\big)\Big|\Big] \\ &\le \mathbb{E}\Big[\Big|\langle \mathrm{X}^\star, \varphi\rangle - \langle \mathrm{X}^\star_{(\varepsilon_n)}, \varphi\rangle\Big|^2\Big]^{1/2} \,,\end{aligned}$$

and, as one can easily check, the quantity on the second line of the above display converges to 0 as $n \to \infty$. Next, thanks again to (S3) and the Cauchy–Schwarz inequality, we have that

$$\begin{aligned}\mathbb{E}\Big[\Big|\exp\big(i\langle \mathrm{X}^{\star,n}_{(\varepsilon_n)}, \varphi\rangle\big) - \exp\big(i\langle \mathrm{X}^{\star}_{t_n}, \varphi\rangle\big)\Big|\Big] &= \mathbb{E}\Big[\Big|\exp\big(i\langle \mathrm{X}^{\star}_{t_n} + \mathrm{W}_{t_n} + \mathrm{Z}_{t_n}, \varphi\rangle\big) - \exp\big(i\langle \mathrm{X}^{\star}_{t_n}, \varphi\rangle\big)\Big|\Big] \\ &\le \mathbb{E}\Big[\Big|\langle \mathrm{W}_{t_n} + \mathrm{Z}_{t_n}, \varphi\rangle\Big|^2\Big]^{1/2} \,,\end{aligned}$$

and again the quantity on the second line of the above display converges to 0 as $n \to \infty$. This follows from the fact that $\mathrm{W}_{t_n}(\cdot) = \mathrm{W}(e^{t_n}\cdot)$, that W has decaying correlations, and from the limit $\lim_{n\to\infty} \mathbb{E}[\mathrm{Z}_{t_n}(\cdot)^2] = 0$. Finally, to complete the proof of (3.3.9), it remains to show that

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big|\exp\big(i\langle \mathrm{X}^{\star}_{t_n}, \varphi\rangle\big) - \exp\big(i\langle \mathrm{X}^{\star}, \varphi\rangle\big)\Big|\Big] = 0 \,,$$

which follows immediately from the almost sure convergence of $\mathrm{X}^\star_{t_n}$ to $\mathrm{X}^\star$ in $\mathcal{H}^{-\kappa}_{\mathrm{loc}}(\mathbb{R}^d)$ as $n \to \infty$, for any $\kappa > 0$.

Step 3: Having proved (3.3.9), to finish the proof, it remains to check that

$$\lim_{n\to\infty} \mathbb{E}\Big[\exp\big(i\langle \mathrm{X}^\star, \varphi\rangle\big) \exp\big(-\overline{\mu}^\star_{\gamma,t_n}(f)\big)\Big] = \mathbb{E}\Big[\exp\big(i\langle \mathrm{X}^\star, \varphi\rangle\big) \exp\big(-\mu^\star_{\gamma}(f)\big)\Big] \,, \tag{3.3.10}$$

which is equivalent to the fact that $\overline{\mu}^\star_{\gamma,t_n}$ converges $\sigma(\mathrm{X}^\star)$-stably to $\mu^\star_\gamma$. We recall that

$$\overline{\mu}^\star_{\gamma,t_n}(dx) = \mathfrak{m}_{t_n}(x)\mu^\star_{\gamma,t_n}(dx) \,,$$

where $\mathfrak{m}_{t_n}$ is the function defined in (3.3.6). Thanks to Lemma 3.4.1, which is stated and proved in Section 3.4 and whose proof is based on Theorem 2.C, we know that $\mu^\star_{\gamma,t_n}$ converges $\sigma(\mathrm{X}^\star)$-stably to $c^\star_{\gamma,\rho}\mathcal{P}_\gamma[\mu^\star_{\gamma_c}]$ for some finite constant $c^\star_{\gamma,\rho} > 0$, depending on γ and the mollifier ρ[4]. On the other hand, we have that $\mathbb{E}[\mathrm{W}_{t_n}(\cdot)^2]$

[4]The dependence of the constant $c^\star_{\gamma,\rho}$ on the mollifier ρ arises from the fact that the constant $c^\star_\gamma$ in Lemma 3.4.1 implicitly depends on the law of the field W.

is a finite constant since the field W is stationary. Moreover, we have that $\lim_{n\to\infty} \mathbb{E}[Z_{t_n}(\cdot)^2] = 0$ on $\mathbb{R}^d$. Hence, combining these two facts, we deduce that the random function $\mathfrak{m}_{t_n}$ converges in probability to a constant $\mathfrak{m}$ as $n \to \infty$, with respect to the topology of local uniform convergence in $\mathcal{C}(\mathbb{R}^d)$. Therefore, thanks to Lemma 3.2.4, the desired result follows with $a^\star_{\gamma,\rho} = \mathfrak{m}\, c^\star_{\gamma,\rho}$. □

With Lemma 3.3.1 established, the proof of Theorem 3.A follows from the decomposition (3.3.5) and from the properties of stable convergence.

Proof of Theorem 3.A. Consider the setting introduced at the beginning of this section, and fix a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ that converges to 0. We recall that we need to prove that $\mu_{\gamma,(\varepsilon_n)}$ converges $\sigma(X)$-stably as $n \to \infty$ to $\mu_\gamma \overset{\text{def}}{=} a_{\gamma,\rho}\mathcal{P}_\gamma[\overline{\mu}_{\gamma_c}]$. Since $\mathcal{M}^+(D)$ is endowed with the topology of vague convergence, then $\mu_{\gamma,(\varepsilon_n)}$ converges $\sigma(X)$-stably to μ_γ if and only if $\mu_{\gamma,(\varepsilon_n)}|_A$ converges $\sigma(X)$-stably to $\mu_\gamma|_A$, for all compact subsets $A \subset D$. Here, $\mu_{\gamma,(\varepsilon_n)}|_A$ (resp. $\mu_\gamma|_A$) denotes the restriction of the measures $\mu_{\gamma,(\varepsilon_n)}$ (resp. μ_γ) to A. Hence, in what follows, we fix a compact subset $A \subset D$, and in order to lighten the notation, we simply write $\mu_{\gamma,(\varepsilon_n)}$ (resp. μ_γ) instead of $\mu_{\gamma,(\varepsilon_n)}|_A$ (resp. $\mu_\gamma|_A$). Before proceeding, we recall that, thanks to (3.3.5), it suffices to show the convergence of $\mathfrak{h}_{\gamma,(\varepsilon_n)}\mu^\star_{\gamma,(\varepsilon_n)}(dx)$.

Step 1. Recalling definition (3.3.4), we note that the sequence $(\mathfrak{h}_{\gamma,(\varepsilon_n)})_{\varepsilon>0}$ can be viewed as a collection of random functions in the space $\mathcal{C}(A)$. In particular, there exists a random function $\mathfrak{h}_\gamma \in \mathcal{C}(A)$ such that $\mathfrak{h}_{\gamma,(\varepsilon_n)}$ converges as $n \to \infty$ to $\mathfrak{h}_\gamma$ in probability with respect to the topology of uniform convergence in $\mathcal{C}(A)$. Indeed, we have that $L_{(\varepsilon_n)}$ converges as $n \to \infty$ to L in probability in $\mathcal{C}(A)$, and $\mathbb{E}[X_{(\varepsilon_n)}(x)^2 - X^\star_{(\varepsilon_n)}(x)^2]$ converges uniformly in A as $n \to \infty$. This is due to the fact that the covariance kernels of both X and $X^\star$ can be written as the sum of $-\log|\cdot - \cdot|$ and some Hölder continuous functions $g, g^\star : A \times A \to \mathbb{R}$. In particular, this implies that

$$\lim_{n\to\infty} \sup_{x\in A} \mathbb{E}\big[X_{(\varepsilon_n)}(x)^2 - X^\star_{(\varepsilon_n)}(x)^2\big] = (g - g^\star)(x,x)\ .$$

Therefore, putting everything together, the random function $\mathfrak{h}_\gamma : A \to \mathbb{R}$ can be written as follows

$$\mathfrak{h}_\gamma(x) = e^{\gamma L(x) - \frac{\gamma^2}{2}(g-g^\star)(x,x)}\ . \tag{3.3.11}$$

Step 2. Thanks to Lemma 3.3.1, we know that the sequence of random measures $(\mu^\star_{\gamma,(\varepsilon_n)})_{n\in\mathbb{N}}$ converges $\sigma(X^\star)$-stably as $n \to \infty$ to $a^\star_{\gamma,\rho}\mathcal{P}_\gamma[\mu^\star_{\gamma_c}]$, where $\mu^\star_{\gamma_c}$ denotes the critical GMC associated with $X^\star$. In particular, since $\mu^\star_{\gamma,(\varepsilon_n)}$ is conditionally independent of L given $X^\star$, thanks to Lemma 3.2.3, we have that $\mu^\star_{\gamma,(\varepsilon_n)}$ converges $\sigma(X^\star, L)$-stably as $n \to \infty$ to $a^\star_{\gamma,\rho}\mathcal{P}_\gamma[\mu^\star_{\gamma_c}]$. Furthermore, thanks to the previous step, we know that $\mathfrak{h}_{\gamma,(\varepsilon_n)}$ converges as $n \to \infty$ to $\mathfrak{h}_\gamma$ in probability with respect to the topology of uniform convergence in $\mathcal{C}(A)$. Moreover, the random function $\mathfrak{h}_\gamma$ is $\sigma(X^\star, L)$-measurable, and so, thanks to Lemma 3.2.4, we have that $\mathfrak{h}_{\gamma,(\varepsilon_n)}\mu^\star_{\gamma,(\varepsilon_n)}$ converges $\sigma(X^\star, L)$-stably as $n \to \infty$ to $a^\star_{\gamma,\rho}\mathfrak{h}_\gamma\mathcal{P}_\gamma[\mu^\star_{\gamma_c}]$. Since $\sigma(X) \subseteq \sigma(X^\star, L)$, this implies that $\mathfrak{h}_{\gamma,(\varepsilon_n)}\mu^\star_{\gamma,(\varepsilon_n)}$ converges $\sigma(X)$-stably as $n \to \infty$ to $a^\star_{\gamma,\rho}\mathfrak{h}_\gamma\mathcal{P}_\gamma[\mu^\star_{\gamma_c}]$.

Step 3. To conclude, it suffices to check that there exists a constant $a_{\gamma,\rho} > 0$, depending on γ and on the mollifier ρ, such that

$$a^\star_{\gamma,\rho}\mathfrak{h}_\gamma\mathcal{P}_\gamma[\mu^\star_{\gamma_c}] = a_{\gamma,\rho}\mathcal{P}_\gamma[\overline{\mu}_{\gamma_c}]\ , \tag{3.3.12}$$

where $\overline{\mu}_{\gamma_c}$ is the measure defined in (3.1.9). To verify that (3.3.12) holds, we begin by observing that, thanks to Lemma 3.2.3, $\mathfrak{h}_\gamma$ is a random non-negative continuous function that is conditionally independent of $\mathcal{P}_\gamma[\mu^\star_{\gamma_c}]$ given $X^\star$. Therefore, thanks to Lemma 3.2.6, it holds that

$$a^\star_{\gamma,\rho}\mathfrak{h}_\gamma\mathcal{P}_\gamma\big[\mu^\star_{\gamma_c}\big] = a^\star_{\gamma,\rho}\mathcal{P}_\gamma\big[\mathfrak{h}_\gamma^{\sqrt{2d}/\gamma}\mu^\star_{\gamma_c}\big]\ .$$

On the other hand, arguing in a similar way as in Step 1 (see also the proof of [JSW19, Theorem 5.4]), we have that

$$\mu_{\gamma_c}(dx) = e^{\sqrt{2d}\,\mathrm{L}(x)-d(g-g^\star)(x,x)}\mu^\star_{\gamma_c}(dx)\ ,$$

where we recall that μ_{γ_c} (resp. $\mu^\star_{\gamma_c}$) denotes the critical GMC associated with X (resp. $\mathrm{X}^\star$). Therefore, recalling the expression (3.3.11) for the random function $\mathfrak{h}_\gamma$, it holds that

$$\mathfrak{h}_\gamma(x)^{\sqrt{2d}/\gamma}\mu^\star_{\gamma_c}(dx) = e^{(d-\sqrt{d/2}\gamma)(g-g^\star)(x,x)}\mu_{\gamma_c}(dx)\ .$$

Since by (C1) the seed covariance function $\mathfrak{K}$ is radial, it holds that the function $g^\star$ is constant along the diagonal. Hence, this allows us to conclude that there exists a finite constant $b_\star > 0$ such that

$$a^\star_{\gamma,\rho}\mathcal{P}_\gamma\big[\mathfrak{h}_\gamma^{\sqrt{2d}/\gamma}\mu^\star_{\gamma_c}\big] = a^\star_{\gamma,\rho}\mathcal{P}_\gamma\big[b_\star\overline{\mu}_{\gamma_c}\big]\ .$$

Therefore, the conclusion then follows by factoring out the constant $b_\star$ by using Lemma 3.2.6. □

3.3.2 Proof of Proposition 3.B

We now turn to the proof of Proposition 3.B.

Proof of Proposition 3.B. For all $x \in \mathbb{R}^d$, we set

$$\mathcal{K}^\rho_{(\varepsilon)}(x) \overset{\text{def}}{=} \mathbb{E}\big[\mathrm{X}^\star_{(\varepsilon)}(x)\mathrm{X}^\star_{(\varepsilon)}(0)\big]\ , \qquad \mathcal{K}_t(x) \overset{\text{def}}{=} \mathbb{E}\big[\mathrm{X}^\star_t(x)\mathrm{X}^\star_t(0)\big]\ .$$

In order to prove the result, it suffices to prove that the difference between the Fourier transforms of $\mathcal{K}^\rho_{(\varepsilon)}$ and $\mathcal{K}_t$ can be written as the sum of two non-negative functions satisfying some suitable properties.

We start by fixing a mollifier ρ satisfying assumptions (A1)–(A2). Since $\rho \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ has unit mass, we have that its Fourier transform $\widehat{\rho}$ is a smooth and rapidly decaying function satisfying $\widehat{\rho}(0) = 1$. We also fix a $\star$-scale invariant field $\mathrm{X}^\star$ on $\mathbb{R}^d$ whose seed covariance function $\mathfrak{K}$ satisfies assumptions (C1)–(C3). We recall that its Fourier transform $\widehat{\mathfrak{K}}$ is non-negative, radial, and supported in $B(0,1)$.

Step 1. We recall that, for $t \geq 0$ and $\varepsilon > 0$, we have that, for all $x \in \mathbb{R}^d$,

$$\mathcal{K}^\rho_{(\varepsilon)}(x) = \int_0^\infty \big(\rho_\varepsilon * \overline{\rho}_\varepsilon * \mathfrak{K}(e^u\cdot)\big)(x)du\ , \qquad \mathcal{K}_t(x) = \int_0^t \mathfrak{K}(e^u x)du\ ,$$

where $\overline{\rho}_\varepsilon(\cdot) \overset{\text{def}}{=} \rho_\varepsilon(-\cdot)$. We now fix a positive constant $\mathfrak{a} \in (0,1)$ to be determined later, and for all $t \geq 0$, we set $\mathcal{K}^\rho_t(x) \overset{\text{def}}{=} \mathcal{K}^\rho_{(\mathfrak{a}e^{-t})}(x)$. The Fourier transforms of $\mathcal{K}^\rho_t$ and $\mathcal{K}_t$ are given for all $\omega \in \mathbb{R}^d$ by

$$\widehat{\mathcal{K}}^\rho_t(\omega) = |\widehat{\rho}(\mathfrak{a}e^{-t}\omega)|^2\int_0^\infty \widehat{\mathfrak{K}}(e^{-u}\omega)e^{-du}du\ , \qquad \widehat{\mathcal{K}}_t(\omega) = \int_0^t \widehat{\mathfrak{K}}(e^{-u}\omega)e^{-du}du\ .$$

Now, we consider the function $\widehat{\Delta}^\rho_t : \mathbb{R}^d \to \mathbb{R}$ given by $\widehat{\Delta}^\rho_t(\omega) \overset{\text{def}}{=} \widehat{\mathcal{K}}^\rho_t(\omega) - \widehat{\mathcal{K}}_t(\omega)$. A simple computation yields that

$$\widehat{\Delta}^\rho_t(\omega) = e^{-dt}\widehat{\mathcal{K}}(e^{-t}\omega) - \big(1 - |\widehat{\rho}(\mathfrak{a}e^{-t}\omega)|^2\big)\widehat{\mathcal{K}}(\omega)\ , \qquad \forall\,\omega \in \mathbb{R}^d\ ,$$

where $\widehat{\mathcal{K}}$ denotes the Fourier transform of $\mathcal{K}$, i.e., $\widehat{\mathcal{K}} = \widehat{\mathcal{K}}_\infty$. Now, letting e_1 be the first unit vector of $\mathbb{R}^d$, we can write for all $\omega \neq 0$,

$$\widehat{\mathcal{K}}(\omega) = \int_0^\infty \widehat{\mathfrak{K}}\big(e^{-u}\omega\big)e^{-du}du = |\omega|^{-d}\int_0^{|\omega|}\widehat{\mathfrak{K}}(ue_1)u^{d-1}du = |\mathbb{S}^{d-1}|^{-1}|\omega|^{-d}\int_{|\xi|\leq|\omega|}\widehat{\mathfrak{K}}(\xi)d\xi\,,$$

where $|\mathbb{S}^{d-1}|$ denotes the $(d-1)$-Lebesgue measure of the $(d-1)$-dimensional unit sphere. Therefore, if we let $\mathcal{T}:\mathbb{R}^d\to\mathbb{R}$ be the function given by

$$\mathcal{T}(\omega) \stackrel{\text{def}}{=} \int_{|\xi|\leq|\omega|}\widehat{\mathfrak{K}}(\xi)d\xi\,, \qquad \forall\,\omega\in\mathbb{R}^d\,,$$

we can then write

$$\widehat{\mathcal{K}}(\omega) = |\mathbb{S}^{d-1}|^{-1}|\omega|^{-d}\mathcal{T}(\omega)\,, \qquad \forall\,\omega\in\mathbb{R}^d\setminus\{0\}\,.$$

Furthermore, considering that $\mathfrak{K}(0)=1$ (and so $\widehat{\mathfrak{K}}$ has unit mass) and that $\widehat{\mathfrak{K}}$ is supported in $B(0,1)$, we note that the function $\mathcal{T}$ satisfies the following properties

$$\mathcal{T}(\omega)\in[0,1]\,,\ \forall\,\omega\in\mathbb{R}^d, \qquad \mathcal{T}(\omega)=1\,,\ \forall\,|\omega|\geq 1, \qquad \lim_{|\omega|\to 0}|\omega|^{-d}\mathcal{T}(\omega) = c|\mathbb{S}^{d-1}|\,,$$

where $c\in(0,\infty)$ is such that $\widehat{\mathcal{K}}(0)=c$. With this notation in hand, we can rewrite $\widehat{\Delta}_t^\rho$ as follows

$$\widehat{\Delta}_t^\rho(\omega) = |\mathbb{S}^{d-1}|^{-1}|\omega|^{-d}\big(\mathcal{T}(e^{-t}\omega) - \big(1-|\widehat{\rho}(\mathfrak{a}e^{-t}\omega)|^2\big)\mathcal{T}(\omega)\big)\,, \qquad \forall\,\omega\in\mathbb{R}^d\setminus\{0\}\,.$$

Step 2. Now, we consider the function $\widehat{\mathcal{K}}_{\mathrm{W}}:\mathbb{R}^d\to\mathbb{R}$ given by

$$\widehat{\mathcal{K}}_{\mathrm{W}}(\omega) \stackrel{\text{def}}{=} |\mathbb{S}^{d-1}|^{-1}|\omega|^{-d}\left(\mathcal{T}(\omega) - \big(1-|\widehat{\rho}(\mathfrak{a}\omega)|^2\big)\right)\,, \qquad \forall\,\omega\in\mathbb{R}^d\setminus\{0\}\,, \tag{3.3.13}$$

which is an integrable function. In particular, we observe that the integrability at infinity follows from the rapid decay of $\widehat{\rho}$ and the fact that $\mathcal{T}(\omega)=1$ for all $|\omega|\geq 1$. The integrability near the origin follows from the properties of the function $\mathcal{T}$ listed in the previous step, and from the fact that $\widehat{\rho}$ is smooth and $\widehat{\rho}(0)=1$.

Furthermore, we claim that we can find a constant $\mathfrak{a}=\mathfrak{a}(\rho,\mathfrak{K})\in(0,1)$ such that the function $\widehat{\mathcal{K}}_{\mathrm{W}}$ is non-negative in $\mathbb{R}^d$. To see this, we need to split into two cases.

1. If $|\omega|\geq 1$, the fact that $\widehat{\mathcal{K}}_{\mathrm{W}}(\omega)$ is non-negative follows simply by noticing that $\mathcal{T}(\omega)=1$ for all $|\omega|\geq 1$. Therefore, the term in the brackets in the definition (3.3.13) of $\widehat{\mathcal{K}}_{\mathrm{W}}$ is just equal to $|\widehat{\rho}(\mathfrak{a}\omega)|^2$ which is obviously non-negative.

2. If $|\omega|<1$, we need to leverage on the constant $\mathfrak{a}\in(0,1)$. Thanks to assumption (C3), we have that $\mathcal{T}(\omega)\geq\overline{\mathfrak{a}}|\omega|^d$, for all $|\omega|<1$ and for some constant $\overline{\mathfrak{a}}>0$. On the other hand, by assumption (A2), there exists $\zeta>0$, only depending on $\widehat{\rho}$, such that

$$|\widehat{\rho}(\omega)|^2 \geq 1 - 2|\omega|^d\left(\sum_{j\in\mathbb{N}_0^d,\,|j|=d}|\partial^j\widehat{\rho}(0)|/j! + 1/10\right)\,, \quad \forall\,|\omega|<\zeta\,,$$

   Hence, putting everything together and choosing $\mathfrak{a}<\zeta$ we obtain that

$$\mathcal{T}(\omega) - \big(1-|\widehat{\rho}(\mathfrak{a}\omega)|^2\big) \geq |\omega|^d\left(\overline{\mathfrak{a}} - 2\mathfrak{a}^d\left(\sum_{j\in\mathbb{N}_0^d,\,|j|=d}|\partial^j\widehat{\rho}(0)|/j! + 1/10\right)\right)\,, \quad \forall\,|\omega|<1\,,$$

which is positive as long as we choose $\mathfrak{a} > 0$ small enough.

Finally, recalling once again that $\mathcal{T}(\omega) = 1$ for all $|\omega| \geq 1$, we observe that the rapid decay of the function $\widehat{\rho}$ implies that $\widehat{\mathcal{K}}_{\mathrm{W}}$ is also rapidly decaying.

Step 3. Now, for $\mathfrak{a} \in (0,1)$ as specified in the previous step, we introduce the function $\widehat{\mathcal{K}}_{\mathrm{Z},t} : \mathbb{R}^d \to \mathbb{R}$ defined as follows

$$\widehat{\mathcal{K}}_{\mathrm{Z},t}(\omega) = |\mathbb{S}^{d-1}|^{-1}|\omega|^{-d}\big(1 - |\widehat{\rho}(\mathfrak{a}e^{-t}\omega)|^2\big)\big(1 - \mathcal{T}(\omega)\big) \,, \qquad \forall\, \omega \in \mathbb{R}^d \setminus \{0\} \,,$$

which is an integrable function for a similar reason as explained in Step 2. We recall that $\widehat{\rho}(0) = 1$, and that, by assumption (A2), $|\widehat{\rho}|$ attains a local maximum at 0. Therefore, by taking $\mathfrak{a} \in (0,1)$ even smaller if necessary, we have that $1 - |\widehat{\rho}(\mathfrak{a}e^{-t}\omega)|^2 \geq 0$ for all $|\omega| \leq 1$ and $t \geq 0$. Moreover, since the function $\mathcal{T}$ takes values in the interval $[0,1]$, it follows that the function $\widehat{\mathcal{K}}_{\mathrm{Z},t}$ is non-negative. Furthermore, thanks to the fact that $\mathcal{T}(\omega) = 1$ for all $|\omega| > 1$, we have that the function $\widehat{\mathcal{K}}_{\mathrm{Z},t}$ is compactly supported on the ball $B(0,1)$. Finally, since $\widehat{\rho}(0) = 1$, we have that $\widehat{\mathcal{K}}_{\mathrm{Z},t}(\omega) \to 0$ as $t \to \infty$, uniformly over all $\omega \in \mathbb{R}^d$.

Step 4. For each $t \geq 0$, we define the function $\widehat{\mathcal{K}}_{\mathrm{W},t} : \mathbb{R}^d \to \mathbb{R}$ by letting $\widehat{\mathcal{K}}_{\mathrm{W},t}(\omega) = e^{-dt}\widehat{\mathcal{K}}_{\mathrm{W}}(e^{-t}\omega)$, where $\widehat{\mathcal{K}}_{\mathrm{W}}$ is the function defined in (3.3.13). Then, as one can easily verify, we can write the function $\widehat{\Delta}_t^{\rho}$ as the following sum

$$\widehat{\Delta}_t^{\rho}(\omega) = \widehat{\mathcal{K}}_{\mathrm{W},t}(\omega) + \widehat{\mathcal{K}}_{\mathrm{Z},t}(\omega) \,, \quad \forall\, \omega \in \mathbb{R}^d. \tag{3.3.14}$$

Thanks to Steps 2 and 3, both terms on the right-hand side of the above expression are non-negative for a suitable choice of the constant $\mathfrak{a} \in (0,1)$. Letting $\mathcal{K}_{\mathrm{W},t}$ be the inverse Fourier transform of $\widehat{\mathcal{K}}_{\mathrm{W},t}$, it is easily seen that $\mathcal{K}_{\mathrm{W},t}(x) = \mathcal{K}_{\mathrm{W}}(e^t x)$, where $\mathcal{K}_{\mathrm{W}}$ is the inverse Fourier transform of $\widehat{\mathcal{K}}_{\mathrm{W}}$. Moreover, $\mathcal{K}_{\mathrm{W}}$ is a smooth, decaying function since its Fourier transform is integrable and has rapid decay. Furthermore, the function $\mathcal{K}_{\mathrm{W}}$ is even. Therefore, by Bochner's theorem, it serves as the covariance kernel of a smooth, stationary, centred Gaussian field W with decaying correlations.

Similarly, we let $\mathcal{K}_{\mathrm{Z},t}$ be the inverse Fourier transform of $\widehat{\mathcal{K}}_{\mathrm{Z},t}$. We note that $\mathcal{K}_{\mathrm{Z},t}$ is smooth since its Fourier transform is compactly supported. Furthermore, since $\widehat{\mathcal{K}}_{\mathrm{Z},t}(\omega) \to 0$ as $t \to \infty$ for all $\omega \in \mathbb{R}^d$, by the dominated convergence theorem, we have that $\mathcal{K}_{\mathrm{Z},t}(0) \to 0$ as $t \to \infty$. Moreover, for each $t \geq 0$, the function $\mathcal{K}_{\mathrm{Z},t}$ is even. Hence, by Bochner's theorem once again, it is the covariance kernel of a smooth, stationary, centred Gaussian field Z_t, satisfying $\lim_{t\to\infty} \mathbb{E}[\mathrm{Z}_t(\cdot)^2] = 0$.

To conclude, we observe that the decomposition (3.1.10) follows directly from (3.3.14) by taking the inverse Fourier transform. □

3.4 The case of infinite-range correlations

The main goal of this section is to extend the convergence result in Theorem 2.C to the setting in which the fields may exhibit infinite-range correlations. To be more precise, the setting we consider is as follows. Let $\mathrm{X}^\star$ be a $\star$-scale invariant field with seed covariance function $\mathfrak{K}$ satisfying assumptions (C1) – (C2), and denote by $(\mathrm{X}_t^\star)_{t\geq 0}$ its martingale approximation. Note that in this section, we drop assumption (C3). We also consider a smooth, stationary, centered Gaussian field W with decaying correlations, independent of $\mathcal{F}_\infty$ (recall (3.1.7)). For each $t \geq 0$, we define $\mathrm{W}_t(\cdot) \overset{\text{def}}{=} \mathrm{W}(e^t\cdot)$. We define the sequence of measures $(\mu_{\gamma,t}^\star)_{t\geq 0}$ as follows

$$\mu_{\gamma,t}^\star(dx) \overset{\text{def}}{=} t^{\frac{3\gamma}{2\sqrt{2d}}} e^{t(\gamma/\sqrt{2} - \sqrt{d})^2} e^{\gamma(\mathrm{X}_t^\star(x) + \mathrm{W}_t(x)) - \frac{\gamma^2}{2}t} dx \,. \tag{3.4.1}$$

With this notation in place, we are now ready to state the main result of this section.

Lemma 3.4.1. *For $\gamma > \sqrt{2d}$, there exists a finite constant $c^\star_\gamma > 0$ such that the sequence $(\mu^\star_{\gamma,t})_{t>0}$ defined in (3.4.1) converges $\sigma(X^\star)$-stably to $c^\star_\gamma \mathcal{P}_\gamma[\mu^\star_{\gamma_c}]$ as $t \to \infty$, where $\mu^\star_{\gamma_c}$ denotes the critical GMC associated with $X^\star$.*

We begin by noting that the convergence of the measure $\mu^\star_{\gamma,t}$ would follow directly from Theorem 2.C, provided that the seed covariance function $\mathfrak{K}$ of the field $X^\star$ and the covariance of the field W are compactly supported. Therefore, the goal of this section is to extend this convergence also to the case where $\mathfrak{K}$ is not compactly supported and W satisfies the assumptions mentioned above. The strategy first involves introducing a collection of fields that approximate $X^\star$ and W with finite-range correlations. Then, by removing the cutoff and applying Kahane's convexity inequality (see Lemma A.0.2), we show how we can obtain the desired result.

The remainder of this section is organised as follows. In Section 3.4.1, we introduce suitable collections of fields with finite-range correlations that approximate $X^\star$ and W. Then, in Section 3.4.2, we show how to remove the cutoff and obtain the desired result using Kahane's convexity inequality.

3.4.1 A collection of fields with finite-range correlations

We begin by introducing the fields that will be the focus of this section. To this end, we consider a non-negative, radial function $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ whose support is contained in $B(0,1/2)$ and satisfies $\int_{\mathbb{R}^d} \varphi(x)^2 dx = 1$. We then define the function $\chi : \mathbb{R}^d \to \mathbb{R}$ by setting

$$\chi(x) \stackrel{\text{def}}{=} (\varphi * \varphi)(x) , \qquad \forall\, x \in \mathbb{R}^d .$$

Under the above assumptions on φ, it is easily seen that $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ is non-negative, radial, with support contained in $B(0,1)$, and such that $\chi(0) = 1$. Furthermore, for $\delta > 0$, we define $\chi_\delta(\cdot) \stackrel{\text{def}}{=} \chi(\delta\cdot)$, so that χ_δ converges to 1 over compact subsets of $\mathbb{R}^d$ as $\delta \to 0$. With this notation in hand, we introduce the following collections of fields.

- Let $\mathfrak{K}$ be the seed covariance function of the field $X^\star$ defined in (3.1.4). For every $\delta > 0$, we define the function $\mathfrak{K}_\delta : \mathbb{R}^d \to \mathbb{R}$ by letting, for all $x \in \mathbb{R}^d$,

  $$\mathfrak{K}_\delta(x) \stackrel{\text{def}}{=} \mathfrak{K}(x)\chi_\delta(x) ,$$

  which is a positive definite kernel as it is the product of two positive definite functions. We let $\overline{\mathfrak{K}}_\delta : \mathbb{R}^d \to \mathbb{R}$ be the function such that the convolution with itself equals $\mathfrak{K}_\delta$. Let ξ be the space-time white noise on $\mathbb{R}^d \times \mathbb{R}^+$ used in the definition (3.1.4) of $X^\star$. We define the field $X^{\star,\delta}$ by letting

  $$X^{\star,\delta}(\cdot) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \int_0^\infty \overline{\mathfrak{K}}_\delta\big(e^r(y-\cdot)\big) e^{\frac{dr}{2}}\, \xi(dy,dr) . \tag{3.4.2}$$

  Furthermore, for $0 \le s < t$, we let $X^{\star,\delta}_{s,t}$ be the field on $\mathbb{R}^d$ given by

  $$X^{\star,\delta}_{s,t}(\cdot) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \int_s^t \overline{\mathfrak{K}}_\delta\big(e^r(y-\cdot)\big) e^{\frac{dr}{2}}\, \xi(dy,dr) ,$$

  with the convention that the subscript s is dropped when $s = 0$. By definition, the seed covariance function $\mathfrak{K}_\delta$ satisfies all the conditions stated in (C1) – (C2) with the further property that $\mathfrak{K}_\delta$ has compact support.

For all $0 \leq s < t$, the field $\mathrm{X}^{\star,\delta}_{s,t}$ has the following covariance structure,

$$\mathbb{E}\big[\mathrm{X}^{\star,\delta}_{s,t}(x)\mathrm{X}^{\star,\delta}_{s,t}(y)\big] = \int_s^t \mathfrak{K}_\delta\big(e^r(x-y)\big)dr\ , \qquad \forall\, x,y \in \mathbb{R}^d\ . \tag{3.4.3}$$

For $0 \leq s < t$, we also introduce the field $\mathrm{X}^{\star,\delta,s}_t$ (note that s appears in the superscript), defined by letting

$$\mathrm{X}^{\star,s,\delta}_t(\cdot) \overset{\text{def}}{=} \mathrm{X}^\star_s(\cdot) + \mathrm{X}^{\star,\delta}_{s,t}(\cdot)\ . \tag{3.4.4}$$

We observe that, by construction, the two fields on the right-hand side of the above display are independent.

- For each $\delta > 0$, we define the field W_δ as the stationary, centred Gaussian field with covariance kernel given by
$$\mathbb{E}\big[\mathrm{W}_\delta(x)\mathrm{W}_\delta(y)\big] = \mathbb{E}\big[\mathrm{W}(x)\mathrm{W}(y)\big]\chi_\delta(x-y)\ , \qquad \forall\, x,y \in \mathbb{R}^d\ ,$$
which is still a valid covariance kernel as it is the product of two positive definite functions. Furthermore, for every $t \geq 0$, we define W^δ_t as the stationary, centred Gaussian field with covariance kernel given by
$$\mathbb{E}\big[\mathrm{W}^\delta_t(x)\mathrm{W}^\delta_t(y)\big] = \mathbb{E}\big[\mathrm{W}^\delta(e^t x)\mathrm{W}^\delta(e^t y)\big]\ , \qquad \forall\, x,y \in \mathbb{R}^d\ .$$

We now state and prove the following result which guarantees that $\mathrm{X}^{\star,\delta}_{s,t}$ and W^δ_t are "good approximation" of $\mathrm{X}^\star_{s,t}$ and W_t, respectively.

Lemma 3.4.2. *For any $\varepsilon > 0$ there exists $\delta > 0$ small enough such that, for all $x,\, y \in \mathbb{R}^d$ and $0 \leq s < t$, it holds that*

$$\mathbb{E}\big[\mathrm{X}^{\star,\delta}_{s,t}(x)\mathrm{X}^{\star,\delta}_{s,t}(y)\big] \leq \mathbb{E}\big[\mathrm{X}^\star_{s,t}(x)\mathrm{X}^\star_{s,t}(y)\big] \leq \mathbb{E}\big[\mathrm{X}^{\star,\delta}_{s,t}(x)\mathrm{X}^{\star,\delta}_{s,t}(y)\big] + \varepsilon\ , \tag{3.4.5}$$
$$\mathbb{E}\big[\mathrm{W}^\delta_t(x)\mathrm{W}^\delta_t(y)\big] \leq \mathbb{E}\big[\mathrm{W}_t(x)\mathrm{W}_t(y)\big] \leq \mathbb{E}\big[\mathrm{W}^\delta_t(x)\mathrm{W}^\delta_t(y)\big] + \varepsilon\ . \tag{3.4.6}$$

Proof. Since all the fields involved are stationary, we can, without loss of generality, set $y = 0$ for simplicity. We start by proving that the inequalities in (3.4.5) are satisfied. The fact that for all $0 \leq s < t$ and $x \in \mathbb{R}^d$ it holds that $\mathbb{E}\big[\mathrm{X}^{\star,\delta}_{s,t}(x)\mathrm{X}^{\star,\delta}_{s,t}(0)\big] \leq \mathbb{E}\big[\mathrm{X}^\star_{s,t}(x)\mathrm{X}^\star_{s,t}(0)\big]$ follows trivially from the fact that $0 \leq \chi_\delta(x) \leq 1$ for all $x \in \mathbb{R}^d$. Hence it remains to prove that for any $\varepsilon > 0$, we can find $\delta > 0$ small enough such that $\mathbb{E}\big[\mathrm{X}^\star_{s,t}(x)\mathrm{X}^\star_{s,t}(0)\big] \leq \mathbb{E}\big[\mathrm{X}^{\star,\delta}_{s,t}(x)\mathrm{X}^{\star,\delta}_{s,t}(0)\big] + \varepsilon$. Recalling (3.1.6) and (3.4.3), we note that

$$\big|\mathbb{E}\big[\mathrm{X}^\star_{s,t}(x)\mathrm{X}^\star_{s,t}(0)\big] - \mathbb{E}\big[\mathrm{X}^{\star,\delta}_{s,t}(x)\mathrm{X}^{\star,\delta}_{s,t}(0)\big]\big| \leq \int_0^\infty \big|\mathfrak{K}\big(e^r x\big) - \mathfrak{K}_\delta\big(e^r x\big)\big|dr\ .$$

At this point, the conclusion follows by applying the exact same strategy as in the proof of [MRV16, Lemma 6.2].

We now prove that the inequalities in (3.4.6) are satisfied. To this end, we note again that the inequality $\mathbb{E}\big[\mathrm{W}^\delta_t(x)\mathrm{W}^\delta_t(0)\big] \leq \mathbb{E}\big[\mathrm{W}_t(x)\mathrm{W}_t(0)\big]$ follows trivially from the fact $0 \leq \chi_\delta(x) \leq 1$ for all $x \in \mathbb{R}^d$. Hence it remains to prove that for any $\varepsilon > 0$, we can find $\delta > 0$ small enough such that $\mathbb{E}\big[\mathrm{W}_t(x)\mathrm{W}_t(0)\big] \leq \mathbb{E}\big[\mathrm{W}^\delta_t(x)\mathrm{W}^\delta_t(0)\big] + \varepsilon$. Thanks to the decay of correlations of the field W, we can find $\mathrm{R} > 0$ large enough, only depending on ε, such that $|\mathbb{E}[\mathrm{W}(x)\mathrm{W}(0)]| \leq \varepsilon$, for all $|x| > \mathrm{R}$. Hence, if $|x| > \mathrm{R}$, then for all $t \geq 0$,

$$\sup_{|x|>\mathrm{R}} \big|\mathbb{E}\big[\mathrm{W}_t(x)\mathrm{W}_t(0)\big] - \mathbb{E}\big[\mathrm{W}^\delta_t(x)\mathrm{W}^\delta_t(0)\big]\big| = \sup_{|x|>\mathrm{R}} \big|\mathbb{E}\big[\mathrm{W}_t(x)\mathrm{W}_t(0)\big]\big||1-\chi_\delta(x)| \leq \varepsilon.$$

On the other hand, uniformly over $|x| \leq \mathrm{R}$, it holds that $\chi_\delta(x) \to 1$ as $\delta \to 0$. Therefore, in this case, the conclusion follows from the fact that the covariance kernel of W is uniformly bounded on $\mathbb{R}^d$. □

3.4.2 Removing the cutoff

For each $\delta > 0$ and $0 \leq s < t$, we define the measure $\mu^{\star,s,\delta}_{\gamma,t}$ on $\mathbb{R}^d$ by letting

$$\mu^{\star,s,\delta}_{\gamma,t}(dx) = t^{\frac{3\gamma}{2\sqrt{2d}}} e^{t(\gamma/\sqrt{2}-\sqrt{d})^2} e^{\gamma(\mathrm{X}^{\star,s,\delta}_t(x)+\mathrm{W}^{\delta}_t(x))-\frac{\gamma^2}{2}\mathbb{E}[\mathrm{X}^{\star,s,\delta}_t(x)^2]}dx \ , \tag{3.4.7}$$

where we recall that the field $\mathrm{X}^{\star,s,\delta}_t$ is defined in (3.4.4). Furthermore, we recall that $\mu^{\star}_{\gamma_c}$ denotes the critical GMC measure associated with $\mathrm{X}^{\star}$. We have the following key result which is a consequence of Chapter 2.

Lemma 3.4.3. *For $\gamma > \sqrt{2d}$ and $\delta > 0$, consider the sequence of random measures $(\mu^{\star,s,\delta}_{\gamma,t})_{0\leq s<t}$ introduced in (3.4.7). Then, there exists a finite constant $c^{\star}_{\gamma,\delta} > 0$ such that $\mu^{\star,s,\delta}_{\gamma,t}$ converges $\sigma(\mathrm{X}^{\star})$-stably to $c^{\star}_{\gamma,\delta}\mathcal{P}_{\gamma}[\mu^{\star}_{\gamma_c}]$ as $t \to \infty$, followed by $s \to \infty$.*

Proof. This is a consequence of the proof of Theorem 2.C. To be precise, we recall that the field $\mathrm{X}^{\star,s,\delta}_t$ is given by the following sum

$$\mathrm{X}^{\star,s,\delta}_t = \mathrm{X}^{\star}_s + \mathrm{X}^{\star,\delta}_{s,t} \ ,$$

where the first term on the right-hand side corresponds to the "large scales field" and the second term to the "small scales field". The only subtlety is that the large scales field does not have compactly supported covariance. However, the proof of Theorem 2.C (see Section 2.6 for more details) remains valid in this more general setting. Indeed, at no point in the proof do we rely on the large scales field having compactly supported covariance, whereas it is crucial that the small scales field does. □

In what follows, in order to lighten the notation, we let

$$\mu^{\star,\delta}_{\gamma} = c^{\star}_{\gamma,\delta}\mathcal{P}_{\gamma}[\mu^{\star}_{\gamma_c}] \ , \qquad \mu^{\star}_{\gamma} = c^{\star}_{\gamma}\mathcal{P}_{\gamma}[\mu^{\star}_{\gamma_c}] \ , \tag{3.4.8}$$

for some constant $c^{\star}_{\gamma}$ (that will be identified with the limit as $\delta \to 0$ of $c^{\star}_{\gamma,\delta}$).

Now, thanks to Lemma 3.2.2 and Lemma 2.3.4, the convergence in Lemma 3.4.3 holds if and only if, for all $(\varphi, f) \in \mathcal{C}_c(\mathbb{R}^d) \times \mathcal{C}^+_c(\mathbb{R}^d)$,

$$\lim_{s\to\infty}\lim_{t\to\infty} \mathbb{E}\Big[\exp\big(i\langle \mathrm{X}^{\star}, \varphi\rangle\big)\exp\big(-\mu^{\star,s,\delta}_{\gamma,t}(f)\big)\Big] = \mathbb{E}\Big[\exp\big(i\langle \mathrm{X}^{\star}, \varphi\rangle\big)\exp\big(-\mu^{\star,\delta}_{\gamma}(f)\big)\Big] \ . \tag{3.4.9}$$

Therefore, in order to prove Lemma 3.4.1, we have to replace $\mu^{\star,s,\delta}_{\gamma,t}$ by $\mu^{\star}_{\gamma,t}$ and $\mu^{\star,\delta}_{\gamma}$ by $\mu^{\star}_{\gamma}$ in both sides of the above convergence. To do so, we divide the proof in two main lemmas. Specifically, in Lemma 3.4.4, we show that the limit of $\mu^{\star}_{\gamma,t}$ as $t \to \infty$ coincides with the limit of $\mu^{\star,\delta}_{\gamma}$ as $\delta \to 0$. Then, in Lemma 3.4.5, we compute the limit of $\mu^{\star,\delta}_{\gamma}$ as $\delta \to 0$.

Lemma 3.4.4. *For any $\gamma > \sqrt{2d}$ and for any $(\varphi, f) \in \mathcal{C}_c(\mathbb{R}^d) \times \mathcal{C}^+_c(\mathbb{R}^d)$, it holds that*

$$\lim_{t\to\infty} \mathbb{E}\Big[\exp\big(i\langle \mathrm{X}^{\star}, \varphi\rangle\big)\exp\big(-\mu^{\star}_{\gamma,t}(f)\big)\Big] = \lim_{\delta\to 0} \mathbb{E}\Big[\exp\big(i\langle \mathrm{X}^{\star}, \varphi\rangle\big)\exp\big(-\mu^{\star,\delta}_{\gamma}(f)\big)\Big] \ ,$$

where both limits exist and are nontrivial, and where we recall that $\mu^{\star}_{\gamma,t}$ is defined in (3.4.1), and $\mu^{\star,\delta}_{\gamma}$ in (3.4.8).

Proof. We fix $(\varphi, f) \in \mathcal{C}_c(\mathbb{R}^d) \times \mathcal{C}_c^+(\mathbb{R}^d)$. For $0 \leq s < t$ and $\delta > 0$, by definition of the measure $\mu_{\gamma,t}^{\star,s,\delta}$ in (3.4.7) and the field $X_t^{\star,s,\delta}$ in (3.4.4), we have that

$$\mu_{\gamma,t}^{\star,s,\delta}(f) = t^{\frac{3\gamma}{2\sqrt{2d}}} e^{t(\gamma/\sqrt{2}-\sqrt{d})^2} \int_{\mathbb{R}^d} f(x) e^{\gamma(X_s^\star(x)+X_{s,t}^{\star,\delta}(x)+W_t^\delta(x)) - \frac{\gamma^2}{2}\mathbb{E}[X_s^\star(x)^2 + X_{s,t}^{\star,\delta}(x)^2]} dx \ ,$$

where we recall that the field $X_s^\star$ and $X_{s,t}^{\star,\delta}$ are independent. We observe that, thanks to Lemma 3.4.2, for all $x, y \in \mathbb{R}^d$, it holds that

$$\mathbb{E}\big[X_{s,t}^{\star,\delta}(x) X_{s,t}^{\star,\delta}(y)\big] + \mathbb{E}\big[W_t^\delta(x) W_t^\delta(y)\big] \leq \mathbb{E}\big[X_{s,t}^\star(x) X_{s,t}^\star(y)\big] + \mathbb{E}\big[W_t(x) W_t(y)\big] \ ,$$

where we recall that the fields $X_{s,t}^{\star,\delta}$ and W_t^δ are independent. Therefore, for $0 \leq u < s < t$, we can apply Kahane's convexity inequality (Lemma A.0.2) with respect to the conditional expectation given $\mathcal{F}_s$ (recall definition (3.1.7)) with the function $\mathbb{R}^+ \ni x \mapsto e^{-x}$ to obtain that

$$\mathbb{E}\Big[\exp\big(i\langle X_u^\star, \varphi\rangle\big) \exp\big(-\mu_{\gamma,t}^{\star,s,\delta}(f)\big)\Big] \leq \mathbb{E}\Big[\exp\big(i\langle X_u^\star, \varphi\rangle\big) \exp\big(-\mu_{\gamma,t}^\star(f)\big)\Big] \ .$$

In particular, from the above inequality, we can deduce that

$$\begin{aligned} \mathbb{E}\Big[\exp\big(i\langle X^\star, \varphi\rangle\big) \exp\big(-\mu_{\gamma,t}^{\star,s,\delta}(f)\big)\Big] &\leq \mathbb{E}\Big[\exp\big(i\langle X^\star, \varphi\rangle\big) \exp\big(-\mu_{\gamma,t}^\star(f)\big)\Big] \\ &\quad + 2\mathbb{E}\Big[\Big|\exp\big(i\langle X^\star, \varphi\rangle\big) - \exp\big(i\langle X_u^\star, \varphi\rangle\big)\Big|\Big] \ . \end{aligned} \tag{3.4.10}$$

Similarly, for any $\varepsilon > 0$, thanks again to Lemma 3.4.2, we know that for $\delta > 0$ small enough and for all x, $y \in \mathbb{R}^d$, it holds that

$$\mathbb{E}\big[X_{s,t}^\star(x) X_{s,t}^\star(y)\big] + \mathbb{E}\big[W_t(x) W_t(y)\big] \leq \mathbb{E}\big[X_{s,t}^{\star,\delta}(x) X_{s,t}^{\star,\delta}(y)\big] + \mathbb{E}\big[W_t^\delta(x) W_t^\delta(y)\big] + \varepsilon \ .$$

Hence, letting $\mathcal{N}$ be a standard normal random variable and by applying Kahane's convexity inequality (Lemma A.0.2) as above, we obtain that

$$\begin{aligned} \mathbb{E}\Big[\exp\big(i\langle X^\star, \varphi\rangle\big) \exp\big(-\mu_{\gamma,t}^\star(f)\big)\Big] &\leq \mathbb{E}\Big[\exp\big(i\langle X^\star, \varphi\rangle\big) \exp\big(-e^{\gamma\sqrt{\varepsilon}\mathcal{N} - \frac{\gamma^2}{2}\varepsilon} \mu_{\gamma,t}^{\star,s,\delta}(f)\big)\Big] \\ &\quad + 2\mathbb{E}\Big[\Big|\exp\big(i\langle X^\star, \varphi\rangle\big) - \exp\big(i\langle X_u^\star, \varphi\rangle\big)\Big|\Big] \ . \end{aligned} \tag{3.4.11}$$

Now, we observe that the sequence of random measures $(\mu_{\gamma,t}^\star)_{t\geq 0}$ is tight under the topology of vague convergence, and every converging subsequence is nontrivial (see [DRSV14a, Proposition 10] whose proof is not reliant on the seed covariance function being compactly supported). In particular, by taking the limit as $t' \to \infty$ along a subsequence in (3.4.10), we obtain that

$$\lim_{t'\to\infty} \mathbb{E}\Big[\exp\big(i\langle X^\star, \varphi\rangle\big) \exp\big(-\mu_{\gamma,t'}^\star(f)\big)\Big] \geq \mathbb{E}\Big[\exp\big(i\langle X^\star, \varphi\rangle\big) \exp\big(-\mu_\gamma^{\star,\delta}(f)\big)\Big] \ , \tag{3.4.12}$$

where we also took the limits as $s \to \infty$ and $u \to \infty$ in (3.4.10) and we applied Lemma 3.4.3 along with the fact that X_u converges almost surely to X in $\mathcal{H}_{\mathrm{loc}}^{-\kappa}(\mathbb{R}^d)$ for any $\kappa > 0$.

Proceeding in the same manner, by taking the limit as $t' \to \infty$ along the same subsequence in (3.4.11), for any $\zeta > 0$, we get that

$$\lim_{t'\to\infty} \mathbb{E}\Big[\exp\big(i\langle X^\star, \varphi\rangle\big) \exp\big(-\mu_{\gamma,t'}^\star(f)\big)\Big]$$

$$\leq \mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-e^{\gamma\sqrt{\varepsilon}\mathcal{N}-\frac{\gamma^2}{2}\varepsilon}\mu_\gamma^{\star,\delta}(f)\big)\Big]$$
$$\leq \mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-(1-\zeta)\mu_\gamma^{\star,\delta}(f)\big)\Big] + \mathbb{P}\Big(e^{\gamma\sqrt{\varepsilon}\mathcal{N}-\frac{\gamma^2}{2}\varepsilon}\leq 1-\zeta\Big)\,. \tag{3.4.13}$$

In particular, by taking the $\limsup_{\delta\to 0}$ in (3.4.12), and the $\liminf_{\delta\to 0}$ followed by the $\lim_{\varepsilon\to 0}$ in (3.4.13), we obtain that

$$\limsup_{\delta\to 0}\mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-\mu_\gamma^{\star,\delta}(f)\big)\Big] \leq \lim_{t'\to\infty}\mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-\mu_{\gamma,t'}^\star(f)\big)\Big]\,,$$
$$\liminf_{\delta\to 0}\mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-\mu_\gamma^{\star,\delta}(f)\big)\Big] \geq \lim_{t'\to\infty}\mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-(1-\zeta)^{-1}\mu_{\gamma,t'}^\star(f)\big)\Big]\,.$$

Therefore, by arbitrariness of $\zeta>0$, we get that

$$\lim_{t'\to\infty}\mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-\mu_{\gamma,t'}^\star(f)\big)\Big] = \lim_{\delta\to 0}\mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-\mu_\gamma^{\star,\delta}(f)\big)\Big]\,,$$

and since the subsequence along which we took the limit was arbitrary, the desired result follows. □

Lemma 3.4.5. *For any $\gamma>\sqrt{2d}$, there exists a finite constant $c_\gamma^\star>0$ such that, for any $(\varphi,f)\in\mathcal{C}_c(\mathbb{R}^d)\times\mathcal{C}_c^+(\mathbb{R}^d)$, it holds that*

$$\lim_{\delta\to 0}\mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-\mu_\gamma^{\star,\delta}(f)\big)\Big] = \mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-\mu_\gamma^\star(f)\big)\Big]\,,$$

where we recall that the constant $c_\gamma^\star$ appears in the definition (3.4.8) of the measure $\mu_\gamma^\star$.

Proof. Recalling (3.4.8), we note that it suffices to prove that $\lim_{\delta\to 0}c_{\gamma,\delta}^\star = c_\gamma^\star$ for some positive $c_\gamma^\star>0$. This fact follows immediately from Lemma 3.4.4. □

We are finally ready to prove Lemma 3.4.1, whose proof follows immediately by combining Lemmas 3.4.4 and 3.4.5.

Proof of Lemma 3.4.1. We recall that it suffices to prove that for all $(\varphi,f)\in\mathcal{C}_c(\mathbb{R}^d)\times\mathcal{C}_c^+(\mathbb{R}^d)$,

$$\lim_{t\to\infty}\mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-\mu_{\gamma,t}^\star(f)\big)\Big] = \mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-\mu_\gamma^\star(f)\big)\Big]\,. \tag{3.4.14}$$

By using the triangle inequality, for any $t\geq 0$ and $\delta>0$, we have that

$$\Big|\mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-\mu_{\gamma,t}^\star(f)\big)\Big] - \mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-\mu_\gamma^\star(f)\big)\Big]\Big|$$
$$\leq \Big|\mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-\mu_{\gamma,t}^\star(f)\big)\Big] - \mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-\mu_\gamma^{\star,\delta}(f)\big)\Big]\Big|$$
$$+ \Big|\mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-\mu_\gamma^{\star,\delta}(f)\big)\Big] - \mathbb{E}\Big[\exp\big(i\langle X^\star,\varphi\rangle\big)\exp\big(-\mu_\gamma^\star(f)\big)\Big]\Big|\,.$$

Therefore, the conclusion follows by a direct application of Lemmas 3.4.4 and 3.4.5. □

3.A Moments of supercritical GMC

In this appendix, we gather some properties concerning the existence of moments and the multifractal spectrum of supercritical GMC measures, which follow directly from their definition. We emphasise that the results

presented in this appendix are not used anywhere in the present chapter but are recorded here for future reference.

In what follows, we consider the same setting as specified in Theorem 3.A. In particular, for $\gamma > \sqrt{2d}$, we let μ_γ denote the supercritical GMC, i.e.,

$$\mu_\gamma = a_{\gamma,\rho}\mathcal{P}_\gamma[\overline{\mu}_{\gamma_c}] \,,$$

where we recall that $\overline{\mu}_{\gamma_c}$ is the critical GMC defined in (3.1.9).

Proposition 3.A.1 (Positive moments)**.** *For $\gamma > \sqrt{2d}$ and for any $A \subset D$ non-empty, bounded and open, the random variable $\mu_\gamma(A)$ posses finite moments of order $q \in (0, \sqrt{2d}/\gamma)$.*

Proof. Let $\gamma > \sqrt{2d}$ and fix a set $A \subset D$ as in the proposition statement. We note that for every $x \geq 0$ and $q \in (0,1)$ it holds that

$$x^q = -\frac{1}{\Gamma(-q)} \int_0^\infty \big(1 - \exp(-zx)\big) \frac{dz}{z^{1+q}} \,. \tag{3.A.1}$$

Hence, for $q \in (0, \sqrt{2d}/\gamma)$, thanks to (3.A.1) and (3.2.2), there exists a constant $c > 0$ such that

$$\begin{aligned}\mathbb{E}\big[\mu_\gamma(A)^q\big] &= -\frac{1}{\Gamma(-q)} \int_0^\infty \big(1 - \mathbb{E}\big[\exp\big(-z\mu_\gamma(A)\big)\big]\big) \frac{dz}{z^{1+q}} \\ &= -\frac{1}{\Gamma(-q)} \int_0^\infty \big(1 - \mathbb{E}\big[\exp\big(-cz^{\sqrt{2d}/\gamma}\overline{\mu}_{\gamma_c}(A)\big)\big]\big) \frac{dz}{z^{1+q}} \,.\end{aligned}$$

Therefore, performing a change of variables, one obtains that

$$\mathbb{E}\big[\mu_\gamma(A)^q\big] = \frac{\gamma c^{\frac{\gamma q}{\sqrt{2d}}}\Gamma\big(-\gamma q/\sqrt{2d}\big)}{\sqrt{2d}\Gamma(-q)} \mathbb{E}\big[\overline{\mu}_{\gamma_c}(A)^{\gamma q/\sqrt{2d}}\big] \,,$$

and the result now follows thanks to the fact that $\gamma q/\sqrt{2d} < 1$ and from [Pow21, Theorem 2.11]. □

Proposition 3.A.2 (Negative moments)**.** *For $\gamma > \sqrt{2d}$ any for any $A \subset D$ non-empty, bounded and open, the random variable $\mu_\gamma(A)$ posses finite moments of every order $q < 0$.*

Proof. Let $\gamma > \sqrt{2d}$ and fix a set $A \subset D$ as in the proposition statement. We recall that for every $x > 0$ and $q > 0$ it holds that

$$\Gamma(q) = x^q \int_0^\infty \exp(-zx) \frac{dz}{z^{1-q}} \,. \tag{3.A.2}$$

Hence, if we fix $q > 0$, thanks to (3.A.2) and (3.2.2), there exists a constant $c > 0$ such that

$$\begin{aligned}\mathbb{E}\big[\mu_\gamma(A)^{-q}\big] &= \frac{1}{\Gamma(q)} \int_0^\infty \mathbb{E}\big[\exp\left(-z\mu_\gamma(A)\right)\big] \frac{dz}{z^{1-q}} \\ &= \frac{1}{\Gamma(q)} \int_0^\infty \mathbb{E}\Big[\exp\big(-cz^{\sqrt{2d}/\gamma}\overline{\mu}_{\gamma_c}(A)\big)\Big] \frac{dz}{z^{1-q}} \,.\end{aligned}$$

Therefore, performing a change of variables, one obtains that

$$\mathbb{E}\big[\mu_\gamma(A)^{-q}\big] = \frac{\gamma\Gamma(\gamma q/\sqrt{2d})}{\sqrt{2d}c^{\frac{\gamma q}{\sqrt{2d}}}\Gamma(q)} \mathbb{E}\big[\mu_{\gamma_c}(A)^{-\gamma q/\sqrt{2d}}\big] \,.$$

and the result now follows from [Pow21, Theorem 2.11]. □

Proposition 3.A.3 (Multifractal spectrum)**.** *For $\gamma > \sqrt{2d}$ and for any $q < \sqrt{2d}/\gamma$, there exists a constant $c_q > 0$ such that for any $\mathrm{A} \subset \mathrm{D}$ non-empty, bounded and open, it holds that*

$$\mathbb{E}\big[\mu_\gamma(r\mathrm{A})^q\big] \overset{r\to 0}{\asymp} c_q r^{\xi_\gamma(q)} ,$$

where $\xi_\gamma(q) = \sqrt{2d}\gamma q - \gamma^2 q^2/2$, and the implicit constant depends only on γ and A.

Proof. The result follows from the proofs of the previous two propositions and the known multifractal spectrum for critical GMC measures (see [Pow21, Theorem 2.11]). □

Chapter 4

Multifractal analysis of Gaussian multiplicative chaos and applications

Abstract. Let μ_γ be a subcritical Gaussian multiplicative chaos measure associated with a general log-correlated Gaussian field defined on a domain $\mathrm{D} \subseteq \mathbb{R}^d$, with $d \geq 1$. We find an explicit formula for its singularity spectrum by showing that μ_γ satisfies the multifractal formalism, i.e., we prove that its singularity spectrum is almost surely equal to the Legendre transform of its L^q-spectrum. Then applying this result, we compute the lower singularity spectrum of the multifractal random walk and of the Liouville Brownian motion.

4.1 Introduction and main results

The main goal of this chapter is to prove that subcritical *Gaussian multiplicative chaos* (GMC) measures associated with a large class of log-correlated Gaussian fields in all dimensions satisfy the multifractal formalism. This has been formally stated in [RV14a, Section 4], but to the best of our knowledge, it has never been proved rigorously. In order to achieve this result, we perform a careful analysis of the local mass concentration of GMC measures around thick points of the corresponding underlying field. Moreover, using this result, we provide an explicit expression for the lower singularity spectrum of the *multifractal random walk* (MRW) and of the *Liouville Brownian motion* (LBM). Before entering into the details of the main results, we briefly review the theory of GMC and of multifractal analysis of measures.

4.1.1 GMC

Given a domain $\mathrm{D} \subseteq \mathbb{R}^d$, with $d \geq 1$, the theory of GMC, originally developed by Kahane [Kah85], aims to define rigorously random measures of the form

$$\mu_\gamma(dx) \text{ “}=\text{” } e^{\gamma \mathrm{X}(x)} dx \,, \tag{4.1.1}$$

where dx denotes the Lebesgue measure, $\gamma \geq 0$ is a real parameter, and X is a log-correlated Gaussian field on D, i.e., a centred Gaussian field whose covariance kernel can be formally written as

$$\mathbb{E}[\mathrm{X}(x)\mathrm{X}(y)] = -\log|x-y| + g(x,y) \,, \qquad \forall\, x, y \in \mathrm{D} \,,$$

where $g : \mathrm{D} \times \mathrm{D} \to \mathbb{R}$ is continuous. Since the covariance kernel of X has a logarithmic divergence along the diagonal, we cannot define the field X pointwise. However, we can make rigorous sense of X by viewing it as a random Schwartz distribution. Therefore, the definition of (4.1.1) is non-trivial as, a priori, we cannot

exponentiate a random generalised function. In order to interpret (4.1.1) rigorously, we need to approximate X via a regularising procedure which involves a suitable collection of regularised random fields $(\mathrm{X}_\varepsilon)_{\varepsilon>0}$. The GMC measure associated with X is then given by the limit of the sequence of approximating measures

$$\mu_{\gamma,\varepsilon}(dx) = e^{\gamma \mathrm{X}_\varepsilon(x) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(x)^2]} dx \ .$$

As long as the parameter γ is strictly less than the critical value $\sqrt{2d}$, which is usually called the subcritical regime, it is well-known [Ber17, Kah85, RV10b] that the sequence $(\mu_{\gamma,\varepsilon})_{\varepsilon>0}$ converges weakly in probability towards a non-degenerate measure μ_γ. Moreover, it is known that μ_γ is almost surely non-atomic, but singular with respect to the Lebesgue measure. Many further properties of such measures concerning, among others, moments and multifractal behaviour are known. We refer to Section 4.2 for more details.

4.1.2 Multifractal analysis

The purpose of multifractal analysis is to finely describe the heterogeneity in distribution of measures whose mass is concentrated in a highly irregular way. For ρ a non-negative finite measure supported on a domain $\mathrm{D} \subseteq \mathbb{R}^d$, $d \geq 1$, we introduce the local Hölder exponent (or local dimension) of ρ at $x \in \mathrm{D}$ by letting

$$\dim_\rho(x) \overset{\text{def}}{=} \lim_{r\to 0} \frac{\log \rho(B(x,r))}{\log r} \ ,$$

provided the limit exists, where $B(x,r)$ denotes the closed ball centred at x with radius r. Then the irregularity on the mass concentration of ρ can be described via the dimension of the sets

$$\mathrm{E}_\rho(\alpha) \overset{\text{def}}{=} \{x \in \mathrm{D} \ : \ \dim_\rho(x) = \alpha\} \ , \qquad \forall\, \alpha \geq 0 \ .$$

In particular, we say that ρ is a multifractal measure if the sets $\mathrm{E}_\rho(\alpha)$ have positive Hausdorff dimensions for different values of $\alpha \geq 0$ belonging to an interval with non-empty interior. Hence, if ρ is a multifractal measure, then the collection of sets $(\mathrm{E}_\rho(\alpha))_{\alpha\geq 0}$ produces a decomposition of D into a family of subfractals.

The main objective of multifractal analysis is to compute the size of $\mathrm{E}_\rho(\alpha)$, i.e., to find an expression for the *singularity spectrum* of ρ, which is the function $\mathrm{d}_\rho : [0,\infty) \to [0,\infty)$ defined by

$$\mathrm{d}_\rho(\alpha) \overset{\text{def}}{=} \dim_{\mathcal{H}}(\mathrm{E}_\rho(\alpha)) \ ,$$

where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension. To this end, Frisch and Parisi [PF85] introduced the notion of *multifractal formalism*, which is a heuristic principle used to establish an explicit connection between the singularity spectrum d_ρ and the L^q-spectrum of the measure ρ. We define the L^q-spectrum of ρ by

$$\tau_\rho(q) \overset{\text{def}}{=} \limsup_{r\to 0} \frac{\log \sup\left\{\sum_{i\in I} \rho(B(x_i,r))^q\right\}}{-\log r} \ , \qquad \forall\, q \in \mathbb{R} \ ,$$

where $(B(x_i,r))_{i\in I}$ is a countable family of disjoint closed balls with radius r centred at $x_i \in \mathrm{D}$, and the supremum is taken over all such families. We say that ρ satisfies the multifractal formalism if the following equality holds

$$\mathrm{d}_\rho(\alpha) = \tau_\rho^*(\alpha) \ . \tag{4.1.2}$$

Here, τ_ρ^* refers to the Legendre–Fenchel transform of τ_ρ which is defined by setting

$$\tau_\rho^*(\alpha) \stackrel{\text{def}}{=} \inf_{q\in\mathbb{R}} \{\alpha q + \tau_\rho(q)\} \vee 0 \, .$$

We emphasise that a rigorous mathematical version of multifractal formalism was initially developed in [Ols95] and we refer to it for further details.

In order to investigate the local regularity of functions, or in our case of paths of stochastic processes, we adopt a similar approach. More precisely, if $I \subseteq \mathbb{R}$ is an interval and $f : I \to \mathbb{R}^d$, $d \geq 1$, is a given function, we define the lower local Hölder exponent (or lower local dimension) of f at $x \in I$ by setting

$$\underline{\dim}_f(x) \stackrel{\text{def}}{=} \liminf_{r\to 0} \frac{\log|f(x+r) - f(x-r)|}{\log r} \, ,$$

and we define the sets in which f has lower local Hölder exponent $\alpha \geq 0$ as

$$\underline{E}_f(\alpha) \stackrel{\text{def}}{=} \left\{x \in I \, : \, \underline{\dim}_f(x) = \alpha\right\} \, .$$

As in the case of measures, we are interested in finding an explicit expression for the lower singularity spectrum of f, which is the function $\underline{d}_f : [0,\infty) \to [0,\infty)$ defined by

$$\underline{d}_f(\alpha) \stackrel{\text{def}}{=} \dim_{\mathcal{H}}(\underline{E}_f(\alpha)) \, .$$

4.1.3 Main results

We are now ready to state the main results of this chapter.

4.1.3.1 Multifractal analysis of GMC

For $d \geq 1$, let $D \subseteq \mathbb{R}^d$ be a simply connected bounded domain. Consider a log-correlated Gaussian field X on D, and for $\gamma^2 < 2d$, let μ_γ denote the associated subcritical GMC measure. The following theorem establishes that μ_γ satisfies the multifractal formalism in the sense of (4.1.2). More precisely, we state the following result.

Theorem 4.A. *For $\gamma^2 < 2d$, the GMC measure μ_γ satisfies the multifractal formalism. More precisely, for $\alpha \geq 0$ it holds that*

$$d_{\mu_\gamma}(\alpha) = \tau^*_{\mu_\gamma}(\alpha) = \begin{cases} d - \frac{1}{2}\left(\frac{d-\alpha}{\gamma} + \frac{\gamma}{2}\right)^2 \, , & \text{if } \alpha \in \left[\left(\sqrt{d} - \frac{\gamma}{\sqrt{2}}\right)^2, \left(\sqrt{d} + \frac{\gamma}{\sqrt{2}}\right)^2\right] \, , \\ 0 \, , & \text{otherwise} \, , \end{cases} \tag{4.1.3}$$

almost surely. Moreover, the L^q-spectrum of μ_γ is given by

$$\tau_{\mu_\gamma}(q) = \begin{cases} -\xi'_{\mu_\gamma}(q_-)q \, , & \text{if } q \in (-\infty, q_-] \, , \\ d - \xi_{\mu_\gamma}(q) \, , & \text{if } q \in [q_-, q_+] \, , \\ -\xi'_{\mu_\gamma}(q_+)q \, , & \text{if } q \in [q_+, \infty) \, , \end{cases} \tag{4.1.4}$$

almost surely, where $q_\pm \stackrel{\text{def}}{=} \pm\sqrt{2d}/\gamma$ and ξ_{μ_γ} is the power law spectrum of μ_γ as defined in (4.2.5).

Remark 4.1.1. We emphasise that the range of values of $\alpha \geq 0$ for which the set $\mathrm{E}_{\mu_\gamma}(\alpha)$ is non-empty increases with γ. This is a consequence of the well known fact that when γ gets closer to the critical value $\sqrt{2d}$, the concentration of mass of μ_γ becomes more and more clustered and so it gives rise to a larger spectrum of singularities.

The strategy of the proof of Theorem 4.A is inspired by the multifractal analysis of multiplicative cascades carried out in [Bar99]. On the one hand, the upper bound for d_{μ_γ}, i.e., the inequality $\mathrm{d}_{\mu_\gamma} \leq \tau^*_{\mu_\gamma}$, follows from the general theory of the multifractal analysis of measures. On the other hand, the lower bound is more involved to prove and it is based on the following heuristic. For $\gamma^2 < 2d$, it is known that the GMC measure μ_γ is carried by the set of γ-thick points of the underlying field X which is defined by

$$\mathcal{T}_\gamma \stackrel{\text{def}}{=} \left\{ x \in \mathrm{D} \,:\, \lim_{\varepsilon \to 0} \frac{\mathrm{X}_\varepsilon(x)}{-\log \varepsilon} = \gamma \right\} , \tag{4.1.5}$$

where $(\mathrm{X}_\varepsilon)_{\varepsilon>0}$ is a suitable regularisation of X. The key to our proof is to study the local mass concentration of μ_γ around thick points of X. More precisely, following the idea of [RV14a, Theorem 4.1], we show that there exists a non-random exponent $\alpha_q \geq 0$ such that μ_γ has local dimension α_q at points in $\mathcal{T}_{q\gamma}$, for all $q^2 < 2d/\gamma^2$[1]. Consequently, this implies that μ_γ has local Hölder exponent α_q on a set of full $\mu_{q\gamma}$-measure. This fact, together with some known properties of GMC measures, is enough to prove the lower bound for d_{μ_γ}, i.e., the inequality $\mathrm{d}_{\mu_\gamma} \geq \tau^*_{\mu_\gamma}$. More precisely, as will become apparent in the course of this chapter, the strategy of the proof is to relate the singularity spectrum d_{μ_γ} and the Legendre transform of its L^q spectrum τ_{μ_γ} to a third quantity, ϕ_{μ_γ} (see (4.3.2)), referred to as the "structure function", for which we have an explicit representation.

4.1.3.2 Multifractal analysis of MRW

The MRW has been first introduced in [BDM01] as a stochastic volatility model, and it can be simply defined as follows. Fix a time $T > 0$ and let $d \geq 1$, then the d-dimensional MRW Z_γ is defined for $\gamma^2 < 2$ as

$$\mathrm{Z}_{\gamma,t} \stackrel{\text{def}}{=} B_{\mu_\gamma([0,t])} , \qquad t \in [0,T] ,$$

where μ_γ is a GMC measure on $[0,T]$ and B is an independent d-dimensional Brownian motion. In the next result, we find the relation between the lower singularity spectrum of the paths of Z_γ and the singularity spectrum of μ_γ. More precisely, we introduce for $\alpha \geq 0$ the set

$$\underline{\mathrm{E}}_{\mathrm{Z}_\gamma}(\alpha) \stackrel{\text{def}}{=} \left\{ t \in [0,T] \,:\, \underline{\dim}_{\mathrm{Z}_\gamma}(t) = \alpha \right\} ,$$

where $\underline{\dim}_{\mathrm{Z}_\gamma}(t)$ is the lower local dimension of Z_γ at $t \in [0,T]$ defined by

$$\underline{\dim}_{\mathrm{Z}_\gamma}(t) \stackrel{\text{def}}{=} \liminf_{r \to 0} \frac{\log |\mathrm{Z}_{\gamma,t+r} - \mathrm{Z}_{\gamma,t-r}|}{\log r} .$$

We want to compute the Hausdorff dimension of $\underline{\mathrm{E}}_{\mathrm{Z}_\gamma}(\alpha)$ for each $\alpha \geq 0$, i.e., we are interested in finding an explicit expression for the lower singularity spectrum $\underline{\mathrm{d}}_{\mathrm{Z}_\gamma} : [0,\infty) \to [0,\infty)$ defined by

$$\underline{\mathrm{d}}_{\mathrm{Z}_\gamma}(\alpha) \stackrel{\text{def}}{=} \dim_{\mathcal{H}}(\underline{\mathrm{E}}_{\mathrm{Z}_\gamma}(\alpha)) .$$

[1] We emphasise that α_q can be readily inferred through a heuristic scaling argument. For instance, we refer the reader to [RV14a, Equation (4.7)].

In particular, we want to find the relation between $\underline{\mathrm{d}}_{\mathrm{Z}_\gamma}(\alpha)$ and $\mathrm{d}_{\mu_\gamma}(\alpha)$. This is exactly the content of the following theorem.

Theorem 4.B. *Let $\gamma^2 < 2$ and $d \geq 1$. Then, for $\alpha \geq 0$, the lower singularity spectrum of the d-dimensional MRW Z_γ is given by*

$$\underline{\mathrm{d}}_{\mathrm{Z}_\gamma}(\alpha) = \mathrm{d}_{\mu_\gamma}(2\alpha) = \begin{cases} 1 - \frac{1}{2}\left(\frac{1-2\alpha}{\gamma} + \frac{\gamma}{2}\right)^2 , & \text{if } \alpha \in \left[\left(\frac{1}{\sqrt{2}} - \frac{\gamma}{2}\right)^2, \left(\frac{1}{\sqrt{2}} + \frac{\gamma}{2}\right)^2\right] , \\ 0 , & \text{otherwise} , \end{cases}$$

almost surely.

4.1.3.3 Multifractal analysis of LBM

The LBM has been simultaneously defined in [Ber15, GRV16] as the canonical planar diffusion associated with the Liouville quantum gravity metric tensor. In this chapter, we will consider the d-dimensional LBM, for $d \geq 2$. More precisely, let $\mathrm{D} \subseteq \mathbb{R}^d$ be a simply connected bounded domain, let B be a d-dimensional Brownian motion started inside D, and let X be an independent log-correlated Gaussian field on D. Then the LBM B_γ on D can be formally defined for $\gamma^2 < 4$ as

$$\mathrm{B}_{\gamma,t} \overset{\text{def}}{=} B_{\mathrm{F}_\gamma^{-1}(t)} , \qquad \mathrm{F}_\gamma(t) \overset{\text{def}}{=} \int_0^{t\wedge\tau} e^{\gamma \mathrm{X}(B_s) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}(B_s)^2]} ds , \qquad t \geq 0 ,$$

where τ is the first exit time of B from D. As we will see in Section 4.5, adapting the original definition of planar LBM to higher dimensions is a straightforward task. Then, we focus on computing the lower singularity spectrum of the paths of B_γ. More precisely, we are interested in the lower singularity spectrum $\underline{\mathrm{d}}_{\mathrm{B}_\gamma} : [0,\infty) \to [0,\infty)$ defined by

$$\underline{\mathrm{d}}_{\mathrm{B}_\gamma}(\alpha) \overset{\text{def}}{=} \dim_{\mathcal{H}}(\underline{\mathrm{E}}_{\mathrm{B}_\gamma}(\alpha)) ,$$

where $\underline{\mathrm{E}}_{\mathrm{B}_\gamma}(\alpha)$ denotes the set of times at which B_γ has lower local dimension equal to $\alpha \geq 0$, i.e.,

$$\underline{\mathrm{E}}_{\mathrm{B}_\gamma}(\alpha) \overset{\text{def}}{=} \left\{ t \in [0, \mathrm{F}_\gamma(\tau)] \,:\, \underline{\dim}_{\mathrm{B}_\gamma}(t) = \alpha \right\} ,$$

where

$$\underline{\dim}_{\mathrm{B}_\gamma}(t) \overset{\text{def}}{=} \liminf_{r\to 0} \frac{\log|\mathrm{B}_{\gamma,t+r} - \mathrm{B}_{\gamma,t-r}|}{\log r} .$$

Theorem 4.C. *Let $\gamma^2 < 4$ and $d \geq 2$. Then, for $\alpha \geq 0$, the lower singularity spectrum of the LBM B_γ is given by*

$$\underline{\mathrm{d}}_{\mathrm{B}_\gamma}(\alpha) = \begin{cases} 2\alpha - 2\alpha\left(\frac{2\alpha-1}{2\alpha\gamma} + \frac{1}{4}\gamma\right)^2 , & \text{if } \alpha \in \left[\left(\sqrt{2} + \frac{\gamma}{\sqrt{2}}\right)^{-2}, \left(\sqrt{2} - \frac{\gamma}{\sqrt{2}}\right)^{-2}\right] , \\ 0 , & \text{otherwise} , \end{cases}$$

almost surely.

Along the proof of the Theorem 4.C, we show that the measure ν_γ defined by $\nu_\gamma([s,t]) \overset{\text{def}}{=} \mathrm{F}_\gamma(t) - \mathrm{F}_\gamma(s)$, for $s \leq t \in [0,T]$, satisfies the multifractal formalism and we compute its singularity spectrum.

4.1.4 Outline

The reminder of this chapter is structured as follows. In Section 4.2, we collect some definitions and results that are used in the rest of the chapter. More precisely, we start by recalling the definition of log-correlated Gaussian field, and we prove a lemma concerning the fluctuations of its convolution approximation. Then we recall some known properties of GMC measures, we collect some facts about the Hausdorff dimension, and finally we state some general results on multifractal analysis of measures. In Section 4.3, we state precisely our main result and we perform its proof. Sections 4.4 and 4.5 are devoted to computing the lower singularity spectrum of the MRW and of the LBM, respectively. In Appendix 4.A, we prove Proposition 4.3.2, which is the main step of the proof of our main result. Finally, Appendix 4.B contains the proof of the finiteness of positive moments of the measure involved in the definition of the LBM.

Acknowledgements. We thank Martin Hairer for his constant support and guidance. We thank an anonymous referee for many helpful comments on an earlier version of the content of this chapter.

4.2 Preliminaries

In this section, after introducing the basic notation, we collect some definitions and results on log-correlated Gaussian fields, on the theory of GMC, on the Hausdorff dimension, and on multifractal analysis of measures.

Basic notation

We let $\mathbb{N} \stackrel{\text{def}}{=} \{1, 2, 3, \dots\}$. For $d \geq 1$, we use $\mathbb{R}^d$ to indicate the d-dimensional Euclidean space. If a and b are two quantities, we use $a \lesssim b$ to denote the statement $a \leq Cb$ for some constant $C > 0$ independent from the parameters of interest. Given a subset $\mathrm{D} \subseteq \mathbb{R}^d$, we denote by $\overline{\mathrm{D}}$ its closure and by $\partial\mathrm{D}$ its boundary.

4.2.1 Log-correlated Gaussian fields

Given a simply connected bounded domain $\mathrm{D} \subseteq \mathbb{R}^d$, with $d \geq 1$, a log-correlated Gaussian field X on D is a Gaussian field whose covariance kernel takes the form

$$\mathcal{K}(x, y) = -\log|x - y| + g(x, y) , \qquad x, y \in \mathrm{D} , \tag{4.2.1}$$

where $g \in C(\overline{\mathrm{D}} \times \overline{\mathrm{D}})$. We adopt the convention to extend the covariance kernel $\mathcal{K}$ to $\mathbb{R}^d \times \mathbb{R}^d$ by setting $\mathcal{K}(x, y) = 0$ whenever $(x, y) \notin \mathrm{D} \times \mathrm{D}$. Moreover, for (4.2.1) to be a covariance kernel, we need to require that it is symmetric and non-negative semi-definite. Since the covariance kernel $\mathcal{K}$ has a singularity on the diagonal, the field X does not make literal sense as a pointwise defined Gaussian field, but it can be rigorously interpreted as a random Schwartz distribution. This random generalised function can be characterised by the property that, for any compactly supported test function $\phi \in C_c^\infty(\mathbb{R}^d)$, the pairing (X, ϕ) produces a centred Gaussian random variable with variance

$$\mathbb{E}[(\mathrm{X}, \phi)(\mathrm{X}, \phi)] = \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x)\mathcal{K}(x, y)\phi(y)dxdy .$$

The existence of this stochastic process follows from a direct construction. Indeed, it can be easily verified that $\mathcal{K}$, as defined in (4.2.1), is the kernel of a self-adjoint Hilbert–Schmidt operator on $L^2(\mathbb{R}^d)$. In particular, this

operator is symmetric and compact, so by the spectral theorem there exist a non-increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ of strictly positive eigenvalues and corresponding eigenfunctions $(\phi_n)_{n\in\mathbb{N}}$ that form an orthonormal basis of $\mathrm{Ker}(\mathcal{K})^{\perp}$ in $L^2(\mathbb{R}^d)$. Then the field X can be defined via its Karhunen–Loève expansion

$$\mathrm{X}(x) = \sum_{n\in\mathbb{N}} \alpha_n \sqrt{\lambda_n}\phi_n \ ,$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is a collection of i.i.d. standard normal random variables. Note that the functions ϕ_n are supported on the domain D. It can be proved that the convergence of the above series takes place in the space $\mathcal{H}^{-\kappa}(\mathrm{D})$, for any $\kappa > 0$. We refer to [JSW20, Section 2] for more details.

In order to give a meaning to GMC measures, we need to define the exponential of a random Schwartz distribution. This is done through a regularisation and limiting procedure. The most general and natural way to obtain a regularisation of X is through the so-called convolution approximation. If we let $\rho \in \mathcal{C}_c^{\infty}(\mathbb{R}^d)$ be non-negative, radially symmetric, with compact support and unit mass, then the ε-convolution approximation of the field X is defined by

$$\mathrm{X}_\varepsilon := \mathrm{X} * \rho_\varepsilon \ , \tag{4.2.2}$$

where $\rho_\varepsilon(\cdot) \stackrel{\text{def}}{=} \varepsilon^{-d}\rho(\varepsilon^{-1}\cdot)$. As we have observed above, $\mathrm{X} \in \mathcal{H}^{-\kappa}(\mathbb{R}^d)$ for any $\kappa > 0$, and so the convolution $\mathrm{X} * \rho_\varepsilon$ is actually well-defined. Moreover, it is easy to check that the regularised field X_ε is a centred Gaussian field with covariance kernel given by

$$\mathcal{K}_\varepsilon(x,y) = \big((\rho_\varepsilon \otimes \rho_\varepsilon) * \mathcal{K}\big)(x,y) \ , \qquad \forall\, x, y \in \mathrm{D} \ .$$

The following properties of the convolution approximation $(\mathrm{X}_\varepsilon)_{\varepsilon>0}$ are well known (see e.g. [JSW20, Lemma 2.8]):

(P1) for Lebesgue-almost every $(x,y) \in \mathrm{D}\times\mathrm{D}$,

$$\lim_{\varepsilon\to 0} \mathcal{K}_\varepsilon(x,y) = \mathcal{K}(x,y) \ ,$$

(P2) there exists a finite constant $K > 0$ such that,

$$\sup_{0<\varepsilon'<\varepsilon} \sup_{x,y\in\mathrm{D}} |\mathbb{E}[\mathrm{X}_\varepsilon(x)\mathrm{X}_{\varepsilon'}(y)] + \log\left(|x-y|+\varepsilon\right)| < K \ ,$$

(P3) for all $\varepsilon > 0$, the map $\mathrm{D} \ni x \mapsto \mathrm{X}_\varepsilon(x)$ is almost surely continuous.

We emphasise that for particular types of log-correlated Gaussian fields there are some other natural approximations having properties (P1) – (P3). For example, if X is a two-dimensional *Gaussian Free Field* (GFF), then one can use the regularisation obtained through the circle average around each point in the domain (see [DS11]).

Before proceeding, we state and prove a lemma that allows us to control the exponential moment of the fluctuations of the convolution approximation of log-correlated Gaussian fields. This lemma is used in the proof in the proof of Proposition 4.3.2 below. We emphasise that a similar result for the two-dimensional GFF is established in [Aru20, Lemma 4.2].

Lemma 4.2.1. *For $d \geq 1$, let $\mathrm{D} \subseteq \mathbb{R}^d$ be a simply connected bounded domain. Consider a log-correlated Gaussian field* X *on* D *with covariance kernel* (4.2.1) *and let $(\mathrm{X}_\varepsilon)_{\varepsilon>0}$ be its convolution approximation, as*

defined in (4.2.2)*. Then there exists a finite constant* $C > 0$ *such that,*

$$\sup_{r\in(0,1]}\sup_{x\in \mathrm{D}_r} \mathbb{E}\left[e^{\sup_{u\in B(x,r)} \mathrm{X}_r(u)-\mathrm{X}_r(x)}\right] \leq C \ ,$$

where $\mathrm{D}_r \overset{\text{def}}{=} \{x \in \mathrm{D} \,:\, \mathrm{dist}(x, \partial \mathrm{D}) < r\}$.

Proof. Fix $r \in (0, 1]$ and $x \in \mathrm{D}_r$. Consider the centred Gaussian field $(\mathrm{X}_r^x(u))_{u\in B(x,r)}$ defined by

$$\mathrm{X}_r^x(u) \overset{\text{def}}{=} \mathrm{X}_r(u) - \mathrm{X}_r(x) \ , \qquad u \in B(x, r) \ ,$$

and the random variable Ω_r^x given by

$$\Omega_r^x \overset{\text{def}}{=} \sup_{u\in B(x,r)} \mathrm{X}_r^x(u) \ .$$

The variance of the field $(\mathrm{X}_r^x(u))_{u\in B(x,r)}$ can be uniformly bounded in $r \in (0, 1]$ and $x \in \mathrm{D}_r$. Indeed, for u, $v \in B(x, r)$, thanks to property (P2), there exists a finite constant $K > 0$ such that

$$\begin{aligned}\mathbb{E}\big[\mathrm{X}_r^x(u)\mathrm{X}_r^x(v))\big] &\leq -\log(|u - v| + r) + \log(|u - x| + r) + \log(|v - x| + r) - \log r + 4K \\ &\leq -2\log r + 2\log 2r + 4K \\ &= 2\log 2 + 4K \ , \end{aligned} \tag{4.2.3}$$

where the last inequality is due to the fact that $|u - x| \leq r$ and $|v - x| \leq r$. Thanks to property (P3), we know that the the field X_r^x is almost surely continuous and so we can apply Borell-TIS inequality (see Lemma A.0.3) to obtain that

$$\mathbb{P}\big(|\Omega_r^x - \mathbb{E}[\Omega_r^x]| > t\big) \leq 2e^{-\frac{t^2}{2\sigma_{x,r}^2}} \ , \tag{4.2.4}$$

for all $t \geq 0$, where $\sigma_{x,r}^2 \overset{\text{def}}{=} \sup_{u\in B(x,r)} \mathbb{E}\left[\mathrm{X}_r^x(u)^2\right]$. We claim that both $\sigma_{x,r}^2$ and $\mathbb{E}[\Omega_r^x]$ can be uniformly bounded in x and r. For $\sigma_{x,r}^2$, this follows easily from (4.2.3). For $\mathbb{E}[\Omega_r^x]$, we can apply Fernique's majorizing criterion (see Lemma A.0.4). Indeed, doing computation similar to the one in (4.2.3), and using the properties of the convolution approximation (see also [JSW20, Lemma 2.8]), one can easily verify that there exists a constant $A > 0$, independent of x and r, such that

$$\mathbb{E}\big[|\mathrm{X}_r^x(u) - \mathrm{X}_r^x(v)|^2\big] = \mathbb{E}\big[|\mathrm{X}_r(u) - \mathrm{X}_r(v)|^2\big] \leq A\frac{|u - v|}{r} \ , \qquad \forall\, u, v \in B(x, r) \ .$$

Therefore, applying Lemma A.0.4 to the field $(\mathrm{X}_r^x(u))_{u\in B(x,r)}$, one can readily obtain the desired uniform upper bound. Hence, from (4.2.4), it follows that there exist finite constants a_0, $a_1 > 0$, independent of x and r, such that

$$\mathbb{P}\big(\Omega_r^x > t\big) \leq 2e^{-\frac{(t-a_0)^2}{2a_1}} \ .$$

Finally, observing that for any non-negative random variable X it holds that $\mathbb{E}[e^X] = 1 + \int_0^\infty e^t \mathbb{P}(X > t)dt$, we have that

$$\mathbb{E}\left[e^{\Omega_r^x}\right] = 1 + \int_0^\infty e^t \mathbb{P}(\Omega_r^x > t)dt \leq 1 + \int_0^\infty 2e^{t-\frac{(t-a_0)^2}{2a_1}}dt < \infty \ ,$$

uniformly in x and r, which concludes the proof. □

4.2.2 Properties of GMC

We recall here the main properties of GMC measures that we need in the following sections. For $d \geq 1$, let $\mathrm{D} \subseteq \mathbb{R}^d$ be a simply connected bounded domain. Consider a log-correlated Gaussian field X defined on D with covariance kernel (4.2.1) and let $(\mathrm{X}_\varepsilon)_{\varepsilon>0}$ be its convolution approximation as defined in (4.2.2).

We start by recalling the following standard result concerning the existence and the non-degeneracy of GMC measures.

Proposition 4.2.2 ([Ber17, Theorem 1.1]). *If $\gamma^2 < 2d$, then the sequence of approximating measures*

$$\mu_{\gamma,\varepsilon}(dx) \overset{\text{def}}{=} e^{\gamma \mathrm{X}_\varepsilon(x) - \frac{1}{2}\mathbb{E}[\mathrm{X}_\varepsilon(x)^2]} dx$$

converges in probability, as $\varepsilon \to 0$, in the space of non-negative, locally finite measures on D *equipped with the topology of vague convergence, to a non-degenerate random measure μ_γ.*

For $\gamma^2 < 2d$, it can be easily verified that the random measure μ_γ is almost surely supported on the whole domain D (see [RV14a]). On the other hand, μ_γ gives full measure to the set of γ-thick points $\mathcal{T}_\gamma$, defined in (4.1.5). It is known that the Hausdorff dimension of $\mathcal{T}_\gamma$ is equal to $d - \gamma^2/2$ (see [RV14a, Theorem 4.2]). Immediate consequences of this fact are that μ_γ is almost surely atomless, and that μ_γ is singular with respect to $\mu_{\gamma'}$ for any $\gamma \neq \gamma'$. In particular, for $\gamma \neq 0$, μ_γ is singular with respect to the Lebesgue measure. We also observe that when γ^2 gets closer to $2d$, the measure μ_γ is carried by a set whose Hausdorff dimension gets closer to 0. In particular, this means that μ_γ tends to cluster as γ^2 increases, until it degenerates at $\gamma^2 = 2d$.

We have the following standard result concerning the existence of uniform bounds on positive and negative moments of GMC measures.

Proposition 4.2.3 ([RV14a, Theorems 2.11 and 2.12]). *Let $\gamma^2 < 2d$ and $q < 2d/\gamma^2$. Then for any non-empty compact set $A \subseteq \mathrm{D}$ it holds that*

$$\sup_{\varepsilon>0} \mathbb{E}\left[\left(\int_A e^{\gamma \mathrm{X}_\varepsilon(x) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(x)^2]} dx\right)^q\right] < \infty \,.$$

An important feature of GMC measures is their multifractal behaviour. More precisely, it can be shown that the moments of μ_γ have a power law behaviour in which the exponents can be expressed through a non-linear function ξ_{μ_γ}, called the power law spectrum, which is defined by

$$\xi_{\mu_\gamma}(q) := \left(d + \frac{1}{2}\gamma^2\right) q - \frac{1}{2}\gamma^2 q^2 \,, \qquad \forall\, q \in \mathbb{R} \,. \tag{4.2.5}$$

More formally, we have the following proposition.

Proposition 4.2.4 ([RV14a, Theorem 2.14]). *Let $\gamma^2 < 2d$ and $q < 2d/\gamma^2$. Then there exists a finite constant $C > 0$ such that for all $x \in \mathrm{D}$ and $r \in (0, 1]$ it holds that*

$$\mathbb{E}[\mu_\gamma(B(x,r))^q] \leq C r^{\xi_{\mu_\gamma}(q)} \,.$$

Finally, we state below a result on the local modulus of continuity of GMC measures which is an easy consequence of Proposition 4.2.4.

Proposition 4.2.5. *Let $\gamma^2 < 2d$ and $\varepsilon > 0$. Then, almost surely, there exist finite constants C_1, $C_2 > 0$ such that for all $x \in \mathrm{D}$ and $r \in (0,1]$ it holds that*

$$C_1 r^{\left(\sqrt{d}+\gamma/\sqrt{2}\right)^2+\varepsilon} \le \mu_\gamma(B(x,r)) \le C_2 r^{\left(\sqrt{d}-\gamma/\sqrt{2}\right)^2-\varepsilon}$$

Proof. A proof of the upper bound is given in [GRV16, Theorem 2.2] in the case $d = 2$. However, the proof can be easily adapted to all dimensions. The lower bound can be proved similarly. Indeed, we assume for simplicity that $\mathrm{D} = (-1,1)^d$ and let $\alpha \overset{\text{def}}{=} (\sqrt{d}+\gamma/\sqrt{2})^2$. For $n \in \mathbb{N}$, we write $\Sigma_n \overset{\text{def}}{=} 2^{-n}\mathbb{Z}^d \cap (-1,1)^d$ for the lattice $2^{-n}\mathbb{Z}^d$ restricted to the open box $(-1,1)^d$. Thanks to an union bound, Markov's inequality, and Proposition 4.2.4, it holds for each $n \in \mathbb{N}$, $\varepsilon > 0$, and $q < 0$ that

$$\begin{aligned}\mathbb{P}\left(\min_{z\in\Sigma_n} \mu_\gamma(B(z,2^{-n})) \le 2^{-n(\alpha+\varepsilon)}\right) &= \mathbb{P}\left(\max_{z\in\Sigma_n} \mu_\gamma(B(z,2^{-n}))^q \ge 2^{-nq(\alpha+\varepsilon)}\right)\\ &\le \sum_{z\in\Sigma_n} 2^{nq(\alpha+\epsilon)}\mathbb{E}[\mu_\gamma(B(z,2^{-n})^q)]\\ &\lesssim 2^{nq\varepsilon - n(\xi_{\mu_\gamma}(q)-d-\alpha q)}\,,\end{aligned}$$

where the implicit constant does not depend on n. If we chose $q = -\sqrt{2d}/\gamma$, then one can easily check that $\xi_{\mu_\gamma}(q) - d - \alpha q = 0$. Therefore, thanks to the Borel–Cantelli lemma, we have that it almost surely exists a finite constant $C > 0$ such that

$$\min_{z\in\Sigma_n} \mu_\gamma(B(z,2^{-n}) \ge C 2^{-n(\alpha+\varepsilon)}\,, \qquad \forall\, n \in \mathbb{N}\,.$$

To conclude, it is sufficient to notice that for all $x \in (-1,1)^d$ and $r \in (0,1]$, there exist $n \in \mathbb{N}$ and $z \in \Sigma_{n+1}$ such that $2^{-n} < r \le 2^{-n+1}$ and $B(z,2^{-n-1}) \subseteq B(x,r)$. □

4.2.3 Hausdorff dimension

We collect here the definition and some properties of the Hausdorff dimension that we need in the sequel. We refer to [Fal97] for further details.

Definition 4.2.6. Let $\mathrm{D} \subseteq \mathbb{R}^d$, $d \ge 1$, and $s \in \mathbb{R}$. We define the s-dimensional Hausdorff measure of D by

$$\mathcal{H}^s(\mathrm{D}) \overset{\text{def}}{=} \lim_{\delta\to 0} \mathcal{H}^s_\delta(\mathrm{D})\,,$$

where

$$\mathcal{H}^s_\delta(\mathrm{D}) \overset{\text{def}}{=} \inf\left\{\sum_{n\in\mathbb{N}} \mathrm{diam}(U_n)^s \,:\, (U_n)_{n\in\mathbb{N}} \text{ is a } \delta\text{-cover of } \mathrm{D}\right\}\,, \qquad \delta \in (0,1]\,.$$

The Hausdorff dimension of D is defined by the following equivalent formulas

$$\dim_{\mathcal{H}}(\mathrm{D}) \overset{\text{def}}{=} \inf\{s \ge 0 \,:\, \mathcal{H}^s(\mathrm{D}) = 0\} = \sup\{s \ge 0 \,:\, \mathcal{H}^s(\mathrm{D}) = \infty\}\,,$$

with the convention that $\dim_{\mathcal{H}}(\emptyset) = 0$.

Note that $0 \le \dim_{\mathcal{H}}(\mathrm{D}) \le d$ for any $\mathrm{D} \subseteq \mathbb{R}^d$, $d \ge 1$. Moreover, from the definition of Hausdorff dimension, it immediately follows that if $\mathcal{H}^s(\mathrm{D}) < \infty$, then $\dim_{\mathcal{H}}(\mathrm{D}) \le s$. Furthermore, the Hausdorff dimension enjoys monotonicity and countable stability, i.e., $\dim_{\mathcal{H}}(\mathrm{D}) \le \dim_{\mathcal{H}}(\mathrm{D}')$ for any $\mathrm{D} \subseteq \mathrm{D}'$, and $\dim_{\mathcal{H}}(\cup_{n\in\mathbb{N}}\mathrm{D}_n) = \sup_{n\in\mathbb{N}} \dim_{\mathcal{H}}(\mathrm{D}_n)$ for any collection of subsets $(\mathrm{D}_n)_{n\in\mathbb{N}}$.

Other useful properties of the Hausdorff dimension are collected in the following proposition.

Proposition 4.2.7 ([Jac18, Proposition 2.15]). *Let $f : [0,\infty) \to \mathbb{R}$ be a continuous and increasing function. Consider a set $\mathrm{D} \subseteq [0,\infty)$ and assume there exist finite constants C, R, $\alpha > 0$ such that $|f(x+r)-f(x-r)| \leq Cr^{\alpha}$ for all $r \in [0,R)$ and $x \in \mathrm{D}$. Then it holds that*

$$\dim_{\mathcal{H}}(f(\mathrm{D})) \leq \frac{1}{\alpha} \dim_{\mathcal{H}}(\mathrm{D}) \, .$$

We conclude this subsection with the following result.

Proposition 4.2.8 ([Fal97, Proposition 2.3]). *Let ρ be a non-negative finite Radon measure on $\mathbb{R}^d$ supported on a simply connected bounded domain $\mathrm{D} \subseteq \mathbb{R}^d$, $d \geq 1$. Let $E \subseteq \mathrm{D}$ such that $\rho(E) > 0$. For each $x \in E$, we let $\dim_{\rho}(x)$ to be the local dimension of ρ at x as defined in (4.2.10) below. If $\dim_{\rho}(x) = s$ for all $x \in E$, then $\dim_{\mathcal{H}}(E) = s$.*

4.2.4 Multifractal analysis of measures

We collect here some general facts about multifractal analysis of measures. Let ρ be a non-negative finite Radon measure on $\mathbb{R}^d$, $d \geq 1$, which may be random or non-random, supported on a simply connected bounded domain $\mathrm{D} \subseteq \mathbb{R}^d$.

Definition 4.2.9. The L^q-spectrum of ρ is the function $\tau_{\rho} : \mathbb{R} \to \mathbb{R}$ defined by

$$\tau_{\rho}(q) \stackrel{\text{def}}{=} \limsup_{r \to 0} \frac{\log \sup \sum_{i \in I} \rho(B(x_i,r))^q}{-\log r} \, , \qquad q \in \mathbb{R} \, , \tag{4.2.6}$$

where $(B(x_i,r))_{i \in I}$ is a countable family of disjoint closed balls with radius r centred at $x_i \in \mathrm{D}$, and the supremum is taken over all such families. Moreover, the Legendre–Fenchel transform of the L^q-spectrum τ_{ρ} is the function $\tau_{\rho}^* : [0,\infty) \to [0,\infty)$ defined by

$$\tau_{\rho}^*(\alpha) \stackrel{\text{def}}{=} \inf_{q \in \mathbb{R}} \{\alpha q + \tau_{\rho}(q)\} \vee 0 \, , \qquad \alpha \in [0,\infty) \, . \tag{4.2.7}$$

We have the following classical result concerning the L^q-spectrum.

Proposition 4.2.10 ([LN99, Section 3]). *The L^q-spectrum τ_{ρ} is a decreasing convex function with $\tau_{\rho}(1) = 0$. Moreover, $\mathrm{dom}(\tau_{\rho}) \stackrel{\text{def}}{=} \{q \in \mathbb{R} \, : \, \tau_{\rho}(q) < \infty\} = \mathbb{R}$ if and only if*

$$\limsup_{r \to 0} \frac{\log \inf_{x \in \mathrm{D}} \{\rho(B(x,r))\}}{\log r} < \infty \, .$$

We collect here some notation. For $\alpha \geq 0$, we introduce the sets $\mathrm{E}_{\rho}(\alpha)$, $\underline{\mathrm{U}}_{\rho}(\alpha)$ and $\overline{\mathrm{B}}_{\rho}(\alpha)$ defined by

$$\mathrm{E}_{\rho}(\alpha) \stackrel{\text{def}}{=} \{x \in \mathrm{D} \, : \, \dim_{\rho}(x) = \alpha\} \, , \tag{4.2.8}$$

$$\underline{\mathrm{U}}_{\rho}(\alpha) \stackrel{\text{def}}{=} \{x \in \mathrm{D} \, : \, \underline{\dim}_{\rho}(x) \leq \alpha\} \, , \qquad \overline{\mathrm{B}}_{\rho}(\alpha) \stackrel{\text{def}}{=} \{x \in \mathrm{D} \, : \, \overline{\dim}_{\rho}(x) \geq \alpha\} \, , \tag{4.2.9}$$

where

$$\dim_{\rho}(x) \stackrel{\text{def}}{=} \lim_{r \to 0} \frac{\log \rho(B(x,r))}{\log r} \, , \tag{4.2.10}$$

$$\underline{\dim}_\rho(x) \stackrel{\text{def}}{=} \liminf_{r\to 0} \frac{\log \rho(B(x,r))}{\log r} , \qquad \overline{\dim}_\rho(x) \stackrel{\text{def}}{=} \limsup_{r\to 0} \frac{\log \rho(B(x,r))}{\log r} ,$$

which are called the local dimension, the lower local dimension and the upper local dimension of ρ at x, respectively.

Definition 4.2.11. The singularity spectrum of ρ is the function $d_\rho : [0,\infty) \to [0,\infty)$ defined by

$$d_\rho(\alpha) \stackrel{\text{def}}{=} \dim_{\mathcal{H}}(E_\rho(\alpha)) , \qquad \alpha \in [0,\infty) . \tag{4.2.11}$$

We have the following fundamental result.

Proposition 4.2.12. *For $\alpha \geq 0$, it holds that* $\dim_{\mathcal{H}}(\underline{U}_\rho(\alpha) \cap \overline{B}_\rho(\alpha)) \leq \tau^*_\rho(\alpha)$.

A proof of Proposition 4.2.12, in a slightly different setting, can be found in [BMP92, Theorem 1]. However, the proof we provide here is a simple generalisation of [BJ10, Proposition A.1]. The proof is based on Besicovitch's covering theorem (see [Fal97]).

Lemma 4.2.13 (Besicovitch's covering theorem)**.** *Consider a set $E \subseteq \mathbb{R}^d$ and for each $x \in E$ fix a number $r_x > 0$ such that $\sup_{x\in E} r_x < \infty$. Then there exists an integer σ_d, depending only on the dimension d, for which there exist countable subfamilies $\mathcal{B}_1, \dots, \mathcal{B}_{\sigma_d}$ of $\{B(x, r_x) : x \in E\}$ such that $E \subseteq \cup_{i\in\{1,\dots,\sigma_d\}} \cup_{B\in\mathcal{B}_i} B$ and $\mathcal{B}_i$ is a collection of disjoint sets for each $i \in \{1, \dots, \sigma_d\}$.*

Proof of Proposition 4.2.12. Fix $\alpha \geq 0$. By definition (4.2.7) of τ^*_ρ, it is sufficient to prove that

$$\dim_{\mathcal{H}}(\underline{U}_\rho(\alpha) \cap \overline{B}_\rho(\alpha)) \leq \big\{\alpha q + \tau_\rho(q)\big\} \vee 0 , \qquad \forall\, q \in \mathbb{R} . \tag{4.2.12}$$

Without loss of generality, we can assume that $\tau_\rho(q) < \infty$, for all $q \in \mathbb{R}$. We split the proof into two parts. In the first part we prove inequality (4.2.12) for $q \geq 0$, while in the second part we prove inequality (4.2.12) for $q < 0$.

Case $q \geq 0$: Define $\overline{s}_{q,\varepsilon} \stackrel{\text{def}}{=} (\alpha+\varepsilon)q + \tau_\rho(q) + \varepsilon$, for $\varepsilon > 0$ fixed. Using the notation introduced in Subsection 4.2.3, it is sufficient to show that $\sup_{\delta\in(0,1]} \mathcal{H}_\delta^{\overline{s}_{q,\varepsilon}}(\underline{U}_\rho(\alpha) \cap \overline{B}_\rho(\alpha)) < \infty$. By definition, for every $x \in \underline{U}_\rho(\alpha) \cap \overline{B}_\rho(\alpha)$, there exists a decreasing sequence $(r_{x,n})_{n\in\mathbb{N}}$ converging to 0 such that

$$\rho(B(x, r_{x,n})) \geq r_{x,n}^{\alpha+\varepsilon} , \qquad \forall\, n \in \mathbb{N} .$$

Fix $\delta \in (0, 1]$, then for each $x \in \underline{U}_\rho(\alpha) \cap \overline{B}_\rho(\alpha)$, we choose n_x such that $r_{x,n_x} \in (0, \delta)$. For each $m \in \mathbb{N}$, we let

$$F_m \stackrel{\text{def}}{=} \big\{x \in \underline{U}_\rho(\alpha) \cap \overline{B}_\rho(\alpha) : 2^{-m} < r_{x,n_x} \leq 2^{-m+1}\big\} .$$

Thanks to the Besicovitch's covering theorem (see Lemma 4.2.13), there exists a positive integer σ_d such that for every $m \in \mathbb{N}$, we can find σ_d disjoint subsets $F_{m,1}, \dots, F_{m,\sigma_d}$ of F_m such that each $F_{m,j}$ is at most countable, the balls $B(x, r_{x,n_x})$ with centres at $x \in F_{m,j}$ are pairwise disjoint and

$$\Big(\big((B(x, r_{x,n_x}))_{x\in F_{m,j}}\big)_{j\in\{1,\dots,\sigma_d\}}\Big)_{m\in\mathbb{N}}$$

is a δ-cover of $\underline{\mathrm{U}}_\rho(\alpha)\cap\overline{\mathrm{B}}_\rho(\alpha)$. Then we have that

$$\begin{aligned}\mathcal{H}_\delta^{\overline{s}_{q,\varepsilon}}(\underline{\mathrm{U}}_\rho(\alpha)\cap\overline{\mathrm{B}}_\rho(\alpha)) &\leq \sum_{m\in\mathbb{N}}\sum_{j=1}^{\sigma_d}\sum_{x\in F_{m,j}} r_{x,n_x}^{(\alpha+\varepsilon)q+\tau_\rho(q)+\varepsilon}\\ &\leq \sum_{m\in\mathbb{N}}\sum_{j=1}^{\sigma_d}\sum_{x\in F_{m,j}} \rho(B(x,r_{x,n_x}))^q r_{x,n_x}^{\tau_\rho(q)+\varepsilon}\\ &\lesssim \sum_{m\in\mathbb{N}}\sum_{j=1}^{\sigma_d}\sum_{x\in F_{m,j}} \rho(B(x,2^{-m+1}))^q 2^{-m(\tau_\rho(q)+\varepsilon)}\,,\end{aligned}$$

where the implicit constant does not depend on m. We observe that, for all $j\in 1,\dots,\sigma_d$, the family $(B(x,2^{-m+1}))_{x\in F_{m,j}}$ can be divided into two countable families of disjoint closed balls with centres in D. Hence, by definition of τ_ρ, for m large enough, we get

$$\sum_{x\in F_{m,j}}\rho(B(x,2^{-m+1}))^q\lesssim 2^{m(\tau_\rho(q)+\varepsilon/2)}\,,$$

which implies that $\sup_{\delta\in(0,1]}\mathcal{H}_\delta^{\overline{s}_{q,\varepsilon}}(\underline{\mathrm{U}}_\rho(\alpha)\cap\overline{\mathrm{B}}_\rho(\alpha))<\infty$, and so the conclusion follows by arbitrariness of $\varepsilon>0$.

Case $q<0$**.** Define $\underline{s}_{q,\varepsilon}\overset{\text{def}}{=}(\alpha-\varepsilon)q+\tau_\rho(q)+\varepsilon$, for $\varepsilon>0$ fixed. As before, it is sufficient to show that $\sup_{\delta\in(0,1]}\mathcal{H}_\delta^{\underline{s}_{q,\varepsilon}}(\underline{\mathrm{U}}_\rho(\alpha)\cap\overline{\mathrm{B}}_\rho(\alpha))<\infty$. By definition, for every $x\in\underline{\mathrm{U}}_\rho(\alpha)\cap\overline{\mathrm{B}}_\rho(\alpha)$, there exists a decreasing sequence $(u_{x,n})_{n\in\mathbb{N}}$ converging to 0 such that

$$\rho(B(x,u_{x,n}))\leq u_{x,n}^{\alpha-\varepsilon}\,,\qquad\forall\, n\in\mathbb{N}\,.$$

Proceeding similarly to the previous case, we get

$$\mathcal{H}_\delta^{\underline{s}_{q,\varepsilon}}(\underline{\mathrm{U}}_\rho(\alpha)\cap\overline{\mathrm{B}}_\rho(\alpha))\lesssim\sum_{m\in\mathbb{N}}\sum_{j=1}^{\sigma_d}\sum_{x\in F_{m,j}}\rho(B(x,2^{-m}))^q 2^{-m(\tau_\rho(q)+\varepsilon)}\,,$$

where the implicit constant does not depend on m, and by definition of τ_ρ, for m large enough, we get

$$\sum_{x\in F_{m,j}}\rho(B(x,2^{-m}))^q\lesssim 2^{m(\tau_\rho(q)+\varepsilon/2)}\,,$$

which implies that $\sup_{\delta\in(0,1]}\mathcal{H}_\delta^{\underline{s}_{q,\varepsilon}}(\underline{\mathrm{U}}_\rho(\alpha)\cap\overline{\mathrm{B}}_\rho(\alpha))<\infty$, and so the conclusion follows once again by arbitrariness of $\varepsilon>0$. □

Since $\mathrm{E}_\rho(\alpha)\subseteq\underline{\mathrm{U}}_\rho(\alpha)\cap\overline{\mathrm{B}}_\rho(\alpha)$, an immediate consequence of Proposition 4.2.12 is the following corollary.

Corollary 4.2.14. *For $\alpha\geq 0$, it holds that $\mathrm{d}_\rho(\alpha)\leq\tau_\rho^*(\alpha)$.*

In particular, Corollary 4.2.14 implies that to prove the validity of the multifractal formalism for a non-negative Radon measure ρ, we only need to prove the bound $\mathrm{d}_\rho(\alpha)\geq\tau_\rho^*(\alpha)$ for all $\alpha\geq 0$.

4.3 Multifractal analysis of GMC

We now turn to the multifractal analysis of GMC measures, i.e., to the proof of Theorem 4.A. For $d \geq 1$, let $\mathrm{D} \subseteq \mathbb{R}^d$ be a simply connected bounded domain. Consider a log-correlated Gaussian field X on D with covariance kernel (4.2.1) and let $(\mathrm{X}_\varepsilon)_{\varepsilon>0}$ be its convolution approximation, as defined in (4.2.2). For every $\gamma^2 < 2d$, consider the GMC measure μ_γ on D associated with X, as defined in Proposition 4.2.2.

Remark 4.3.1. Since the GMC measure μ_γ is almost surely a non-negative finite Radon measure supported on a simply connected bounded domain, Proposition 4.2.10 implies that the L^q-spectrum τ_{μ_γ} is a decreasing convex function and $\tau_{\mu_\gamma}(1) = 0$. Moreover, thanks to the lower bound in Proposition 4.2.5, one can easily verify that $\mathrm{dom}(\tau_{\mu_\gamma}) = \mathbb{R}$.

We have the following key result, whose proof is deferred to Appendix 4.A.

Proposition 4.3.2. *Let $\gamma^2 < 2d$ and $q^2 < 2d/\gamma^2$. For $\mu_{q\gamma}$-almost every $x \in \mathrm{D}$, the GMC measure μ_γ satisfies*

$$\lim_{r\to 0} \frac{\log \mu_\gamma(B(x,r))}{\log r} = d + \left(\frac{1}{2} - q\right)\gamma^2 \,,$$

almost surely.

Remark 4.3.3. Roughly speaking, since for $q^2 < 2d/\gamma^2$ the measure $\mu_{q\gamma}$ is carried by the set of $q\gamma$-thick points of X, Proposition 4.3.2 implies that around such points the measure μ_γ has local dimension equal to

$$\alpha_q := d + \left(\frac{1}{2} - q\right)\gamma^2 \,. \tag{4.3.1}$$

Thanks to Corollary 4.2.14, we already know that, for $\alpha \geq 0$, it holds almost surely that $\mathrm{d}_{\mu_\gamma}(\alpha) \leq \tau^*_{\mu_\gamma}(\alpha)$. Therefore, to prove the validity of the multifractal formalism for μ_γ, it suffices to show that for $\alpha \geq 0$ the lower bound $\mathrm{d}_{\mu_\gamma}(\alpha) \geq \tau^*_{\mu_\gamma}(\alpha)$ holds almost surely. We start with the following lemma which is an immediate consequence of Proposition 4.3.2.

Lemma 4.3.4. *For $\gamma^2 < 2d$, it holds almost surely that $\mathrm{d}_{\mu_\gamma}(\alpha_1) = \alpha_1$, where α_1 is as in (4.3.1) with $q = 1$. Furthermore, almost surely, if $E \subseteq \mathrm{D}$ is such that $\dim_{\mathcal{H}}(E) < \alpha_1$, then $\mu_\gamma(E) = 0$.*

Proof. For $x \in \mathrm{E}_{\mu_\gamma}(\alpha_1)$, it obviously holds that $\dim_{\mu_\gamma}(x) = \alpha_1$. Furthermore, thanks to Proposition 4.3.2, we know that $\mu_\gamma(\mathrm{E}_{\mu_\gamma}(\alpha_1)) > 0$ almost surely. Therefore, the first part of the lemma follows from Proposition 4.2.8. Concerning the second part of the lemma, we proceed by contradiction. Consider a subset $E \subseteq \mathrm{D}$ with $\dim_{\mathcal{H}}(E) < \alpha_1$ and such that $\mu_\gamma(E) > 0$. Thanks to Proposition 4.3.2, we know that the subset $\mathrm{E}_{\mu_\gamma}(\alpha_1)$ is of full μ_γ-measure, and so we have that $\mu_\gamma(E \cap \mathrm{E}_{\mu_\gamma}(\alpha_1)) = \mu_\gamma(E) > 0$. Moreover, since for all $x \in E \cap \mathrm{E}_{\mu_\gamma}(\alpha_1)$ it holds that $\dim_{\mu_\gamma}(x) = \alpha_1$, then Proposition 4.2.8 and the monotonicity of Hausdorff dimension imply that

$$\dim_{\mathcal{H}}(E) \geq \dim_{\mathcal{H}}(E \cap \mathrm{E}_{\mu_\gamma}(\alpha_1)) = \alpha_1 \,,$$

which is clearly a contradiction. □

Now, we introduce the structure function ϕ_{μ_γ} associated with μ_γ which is defined by

$$\phi_{\mu_\gamma}(q) \overset{\text{def}}{=} d - \xi_{\mu_\gamma}(q) = \frac{1}{2}\gamma^2 q^2 - \left(d + \frac{1}{2}\gamma^2\right) q + d \,, \qquad q \in \mathbb{R} \,, \tag{4.3.2}$$

where we recall that ξ_{μ_γ} is the power law spectrum of μ_γ defined in (4.2.5). A direct computation shows that the Legendre–Fenchel transform of ϕ_{μ_γ} can be written as follows

$$\phi^*_{\mu_\gamma}(\alpha) = \left(d - \frac{1}{2}\left(\frac{d-\alpha}{\gamma} + \frac{1}{2}\gamma \right)^2 \right) \vee 0 \,, \qquad \alpha \geq 0 \,.$$

The following two lemmas provide a lower bound for $\mathrm{d}_{\mu_\gamma}(\alpha_q)$ in terms of $\phi^*_{\mu_\gamma}(q)$, for all $q^2 < 2d/\gamma^2$, and an upper bound for $\tau_{\mu_\gamma}(q)$ in terms of $\phi_{\mu_\gamma}(q)$, for all $q \in \mathbb{R}$, respectively.

Lemma 4.3.5. *Let $\gamma^2 < 2d$ and $q^2 < 2d/\gamma^2$. Then it holds almost surely that $\mathrm{d}_{\mu_\gamma}(\alpha_q) \geq \phi^*_{\mu_\gamma}(\alpha_q)$, where α_q is as defined in (4.3.1).*

Proof. On the one hand, thanks to Proposition 4.3.2, we know that the set $\mathrm{E}_{\mu_\gamma}(\alpha_q)$ is almost surely of full $\mu_{q\gamma}$-measure. On the other hand, Lemma 4.3.4 implies that $\mu_{q\gamma}$ cannot give positive measure to a set of Hausdorff dimension strictly less than $d - q^2\gamma^2/2$. Therefore, it should hold almost surely that

$$\mathrm{d}_{\mu_\gamma}(\alpha_q) \geq d - \frac{1}{2}q^2\gamma^2 = \phi^*_{\mu_\gamma}(\alpha_q) \,,$$

where the last equality can be checked directly. □

Lemma 4.3.6. *For $\gamma^2 < 2d$, it holds almost surely that $\tau_{\mu_\gamma}(q) \leq \phi_{\mu_\gamma}(q)$ for all $q \in \mathbb{R}$.*

Proof. Thanks to Remark 4.3.1, the function $\tau_{\mu_\gamma} : \mathbb{R} \to \mathbb{R}$ is convex and hence continuous. Therefore, it is sufficient to prove that for all $q \in \mathbb{R}$, it holds almost surely that $\tau_{\mu_\gamma}(q) \leq \phi_{\mu_\gamma}(q)$. For simplicity, we assume that $\mathrm{D} = (-1,1)^d$. We split the proof into two parts. In the first part we focus on $q \geq 2d/\gamma^2$, while in the second part we focus on $q < 2d/\gamma^2$.

Case $q \geq 2d/\gamma^2$. Since $2d/\gamma^2 > 1$, the conclusion in this case is trivial. Indeed, since τ_{μ_γ} is a decreasing function and $\tau_{\mu_\gamma}(1) = 0$ (see Remark 4.3.1), it holds almost surely that $\tau_{\mu_\gamma}(q) \leq 0 \leq \phi_{\mu_\gamma}(q)$, where the last inequality can be checked directly.

Case $q < 2d/\gamma^2$. For $n \in \mathbb{N}$, we write $\Sigma_n \overset{\text{def}}{=} 2^{-n-1}\mathbb{Z}^d \cap (-1,1)^d$ for the lattice $2^{-n-1}\mathbb{Z}^d$ restricted to the open box $(-1,1)^d$. Then for each $n \in \mathbb{N}$ and $\varepsilon > 0$, by Markov's inequality and Proposition 4.2.4, we have that

$$\begin{aligned} \mathbb{P}\left(\sum_{z\in\Sigma_n} \mu_\gamma(B(z,2^{-n}))^q \geq 2^{n(\phi_{\mu_\gamma}(q)+\varepsilon)} \right) &\leq 2^{-n(\phi_{\mu_\gamma}(q)+\varepsilon)} \sum_{z\in\Sigma_n} \mathbb{E}\left[\mu_\gamma(B(z,2^{-n}))^q \right] \\ &\lesssim 2^{-n(\phi_{\mu_\gamma}(q)+\varepsilon)} 2^{-n\xi_{\mu_\gamma}(q)} 2^{nd} = 2^{-n\varepsilon} \,, \end{aligned}$$

where the implicit constant does not depend on n. Hence, it holds that

$$\sum_{n\in\mathbb{N}} \mathbb{P}\left(\sum_{z\in\Sigma_n} \mu_\gamma(B(z,2^{-n}))^q \geq 2^{n(\phi_{\mu_\gamma}(q)+\varepsilon)} \right) < \infty \,,$$

and so, thanks to the Borel–Cantelli lemma, we have that it almost surely exists a finite constant $C > 0$ independent of n such that

$$\sum_{z\in\Sigma_n} \mu_\gamma(B(z,2^{-n}))^q \leq C 2^{n(\phi_{\mu_\gamma}(q)+\varepsilon)} \,, \qquad \forall\, n \in \mathbb{N} \,. \tag{4.3.3}$$

Let $(B(x_i,r))_{i\in I}$ be a countable family of disjoint closed balls with radius $r\in(0,1]$ centred at $x_i\in(-1,1)^d$, and fix $n\in\mathbb{N}$ such that $2^{-n}<r\leq 2^{-n+1}$. We observe that there exists a finite constant $\sigma_d>0$ depending only on the dimension d, such that every ball $B(x_i,r)$ is contained in the union of at most σ_d balls of the form $B(z_{ij},2^{-n+1})$, for some $z_{ij}\in\Sigma_{n-1}$, $j=1,\dots,\sigma_d$. Here, we adopt the convention that if for $i\in I$, the ball $B(x_i,r)$ is contained in $\cup_{j=1,\dots,J}B(z_{ij},2^{-n+1})$ for some $J<\sigma_d$, then $z_{ij}=z_{iJ}$ for $j=J+1,\dots,\sigma_d$. Furthermore, since the balls $(B(x_i,r))_{i\in I}$ are disjoint, we can choose the balls $(B(z_{ij},2^{-n+1}))_{i\in I,j\in\{1,\dots,\sigma_d\}}$ in such a way that for fixed $i_1\neq i_2\in I$, it holds that $z_{i_1j}\neq z_{i_2j}$ for $j=1,\dots,\sigma_d$. Now, if $q\in(0,1]$, thanks to (4.3.3) and the sub-additivity of the function $x\mapsto x^q$, it holds almost surely that

$$\sum_{i\in I}\mu_\gamma(B(x_i,r))^q\leq\sum_{j=1,..,\sigma_d}\sum_{z\in\Sigma_{n-1}}\mu_\gamma(B(z,2^{-n+1}))^q\lesssim 2^{n(\phi_{\mu_\gamma}(q)+\varepsilon)}\lesssim r^{-(\phi_{\mu_\gamma}(q)+\varepsilon)}\,, \tag{4.3.4}$$

where the implicit constant does not depend on n. Similarly, if $q\in(1,2d/\gamma^2)$, thanks to (4.3.3) and the convexity of the function $x\mapsto x^q$ for $q>1$, using Jensen's inequality, it holds almost surely that

$$\sum_{i\in I}\mu_\gamma(B(x_i,r))^q\leq\sigma_d^{q-1}\sum_{j=1,..,\sigma_d}\sum_{z\in\Sigma_{n-1}}\mu_\gamma(B(z,2^{-n+1}))^q\lesssim 2^{n(\phi_{\mu_\gamma}(q)+\varepsilon)}\lesssim r^{-(\phi_{\mu_\gamma}(q)+\varepsilon)}\,, \tag{4.3.5}$$

where the implicit constant does not depend on n. Moreover, for every ball $B(x_i,r)$, we have that $B(z,2^{-n-1})\subseteq B(x_i,r)$ for some $z\in\Sigma_{n+1}$. Therefore, if $q\leq 0$, then thanks to (4.3.3) it holds almost surely that

$$\sum_{i\in I}\mu_\gamma(B(x_i,r))^q\leq\sum_{z\in\Sigma_{n+1}}\mu_\gamma(B(z,2^{-n-1}))^q\lesssim 2^{n(\phi_{\mu_\gamma}(q)+\varepsilon)}\lesssim r^{-(\phi_{\mu_\gamma}(q)+\varepsilon)}\,, \tag{4.3.6}$$

where the implicit constant does not depend on n. Finally, the conclusion follows by taking logs in (4.3.4), (4.3.5), (4.3.6), and thanks to the arbitrariness of $\varepsilon>0$. □

Thanks to Corollary 4.2.14, Lemmas 4.3.5 and 4.3.6, we obtain that for $q^2<2d/\gamma^2$ it holds almost surely that

$$0<\phi^*_{\mu_\gamma}(\alpha_q)\leq d_{\mu_\gamma}(\alpha_q)\leq\tau^*_{\mu_\gamma}(\alpha_q)\leq\phi^*_{\mu_\gamma}(\alpha_q)\,, \tag{4.3.7}$$

which implies that $d_{\mu_\gamma}(\alpha_q)=\tau^*_{\mu_\gamma}(\alpha_q)$ and $\tau_{\mu_\gamma}(q)=\phi_{\mu_\gamma}(q)$. For $q^2<2d/\gamma^2$, we observe that the range of values of $\alpha\geq 0$ of the form α_q is equal to the interval (α_-,α_+), where

$$\alpha_-=\left(\sqrt{d}-\frac{\gamma}{\sqrt{2}}\right)^2\,,\qquad\alpha_+=\left(\sqrt{d}+\frac{\gamma}{\sqrt{2}}\right)^2\,.$$

Thanks to Corollary 4.2.14 and Lemma 4.3.6, we can easily observe that for $\alpha\notin(\alpha_-,\alpha_+)$ we have that $d_{\mu_\gamma}(\alpha)\leq\tau^*_{\mu_\gamma}(\alpha)\leq\phi^*_{\mu_\gamma}(\alpha)=0$. Therefore, these results imply that for all $\alpha\geq 0$, it holds almost surely that

$$d_{\mu_\gamma}(\alpha)=\tau^*_{\mu_\gamma}(\alpha)=\begin{cases}d-\frac{1}{2}\left(\frac{d-\alpha}{\gamma}+\frac{\gamma}{2}\right)^2\,, & \text{if } \alpha\in\left[\left(\sqrt{d}-\frac{\gamma}{\sqrt{2}}\right)^2,\left(\sqrt{d}+\frac{\gamma}{\sqrt{2}}\right)^2\right]\,,\\ 0\,, & \text{otherwise}\,,\end{cases}$$

which coincides with (4.1.3).

To conclude the proof of Theorem 4.A we need to show the validity of (4.1.4). This is exactly the objective of the subsequent lemma whose proof follows similar steps of those in [BJ10, Section 2.3].

Lemma 4.3.7. *If $\gamma^2 < 2d$, then for $q \in \mathbb{R}$ it holds almost surely that*

$$\tau_{\mu_\gamma}(q) = \begin{cases} -\xi'_{\mu_\gamma}(q_-)q \ , & \text{if } q \in (-\infty, q_-] \ , \\ d - \xi_{\mu_\gamma}(q) \ , & \text{if } q \in [q_-, q_+] \ , \\ -\xi'_{\mu_\gamma}(q_+)q \ , & \text{if } q \in [q_+, \infty) \ , \end{cases}$$

where $q_\pm = \pm\sqrt{2d}/\gamma$.

Proof. From (4.3.7), we know that $\tau_{\mu_\gamma}(q) = \phi_{\mu_\gamma}(q)$ for $q \in (q_-, q_+)$. Thanks to the continuity of the convex function τ_{μ_γ}, the above equality can be extended to the closed interval $[q_-, q_+]$. Therefore, it remains to prove that τ_{μ_γ} is derivable at q_+ and linear on $[q_+, \infty)$, and the same also for q_- and $(-\infty, q_-]$. Thanks to the equality $\tau_{\mu_\gamma}(q) = \phi_{\mu_\gamma}(q)$ on the interval $[q_-, q_+]$, we know that

$$\tau'_{\mu_\gamma}(q_+^-) = \phi'_{\mu_\gamma}(q_+) = \frac{\phi_{\mu_\gamma}(q_+)}{q_+} \ , \qquad \tau'_{\mu_\gamma}(q_-^+) = \phi'_{\mu_\gamma}(q_-) = \frac{\phi_{\mu_\gamma}(q_-)}{q_-} \ ,$$

where $\tau'_{\mu_\gamma}(q_+^-)$ (resp. $\tau'_{\mu_\gamma}(q_-^+)$) denotes the left-hand (resp. right-hand) derivative of τ_{μ_γ} at q_+ (resp. at q_-). Since the function τ_{μ_γ} is convex, we have that

$$\tau_{\mu_\gamma}(q) \geq \tau_{\mu_\gamma}(q_+) + \tau'_{\mu_\gamma}(q_+^-)(q - q_+) = \phi'_{\mu_\gamma}(q_+)q \ ,$$

and similarly $\tau_{\mu_\gamma}(q) \geq \phi'_{\mu_\gamma}(q_-)q$. For $q \geq q_+$, the reverse inequality can be obtained using the fact that $q_+/q \in (0, 1]$ and the sub-additivity of the function $x \mapsto x^{q_+/q}$. Indeed, if we consider a countable family $(B(x_i, r))_{i\in I}$ of disjoint closed balls with radius r centred at $x_i \in \mathrm{D}$, then

$$\left(\sum_{i\in I} \mu_\gamma(B(x_i, r))^q\right)^{\frac{q_+}{q}} \leq \sum_{i\in I} \mu_\gamma(B(x_i, r))^{q_+} \ .$$

Hence, taking logs and by definition of τ_{μ_γ}, we have that

$$\tau_{\mu_\gamma}(q) \leq \frac{\tau_{\mu_\gamma}(q_+)}{q_+}q = \frac{\phi_{\mu_\gamma}(q_+)}{q_+}q = \phi'_{\mu_\gamma}(q_+)q \ .$$

Therefore, for all $q \geq q_+$ it holds that $\tau_{\mu_\gamma}(q) = \phi'_{\mu_\gamma}(q_+)q$, and the same exact computation yields also that $\tau_{\mu_\gamma}(q) = \phi'_{\mu_\gamma}(q_-)q$ for all $q \leq q_-$. Since $\phi_{\mu_\gamma}(q) = d - \xi_{\mu_\gamma}(q)$, for all $q \in \mathbb{R}$, the result is proved. □

This concludes the proof of Theorem 4.A.

4.4 Multifractal analysis of MRW

Theorem 4.A gives a full characterisation of the singularity spectrum of GMC measures. A natural question that may arise is whether we can use this result to perform the multifractal analysis of objects that are built by means of these random measures. In particular, in this section, we investigate the multifractal behaviour of the *Multifractal Random Walk* (MRW).

4.4.1 Definition of the MRW

Let $T > 0$ and consider a log-correlated Gaussian field X on $[0, T]$ with covariance kernel of the from (4.2.1). For $\gamma^2 < 2$, consider the GMC measure μ_γ on $[0, T]$ associated with X, as defined in Proposition 4.2.2. In addition, for $d \geq 1$, let B be a Brownian motion on $\mathbb{R}^d$ independent from the field X.

Remark 4.4.1. As the Brownian motion B is independent from the log-correlated Gaussian field X, it is useful to decompose the underlying probability measure as follows

$$\mathbf{P} \otimes \mathbb{P} .$$

When we write "almost surely", we mean $\mathbf{P} \otimes \mathbb{P}$-almost surely. If we need to specify that a property holds almost surely with respect to the Brownian motion (resp. the field X), we write $\mathbf{P}$-almost surely (resp. $\mathbb{P}$-almost surely).

Definition 4.4.2. Fix $\gamma^2 < 2$ and $d \geq 1$. The d-dimensional MRW on $[0, T]$ associated with the field X is the process $(Z_{\gamma,t})_{t\in[0,T]}$ defined by

$$Z_{\gamma,t} \overset{\text{def}}{=} B_{\mu_\gamma([0,t])} , \qquad t \in [0, T] . \tag{4.4.1}$$

The MRW has been originally introduced in [BDM01] as a stochastic volatility model. Indeed, assuming that the Brownian motion B is one-dimensional, the MRW is used for modelling the price fluctuations of financial assets since it accounts for most of the stylised facts of financial time series. For example, it reproduces the sudden jumps and the periods of intense activity usually observed in stock prices. We refer to [BKM08] for a short overview on this topic.

4.4.2 Study of the lower singularity spectrum

The main goal of this section is to prove Theorem 4.B. For $\alpha \geq 0$, we define the set $E_{\mu_\gamma}(\alpha)$ as in (4.2.8), and the sets $\underline{U}_{\mu_\gamma}(\alpha)$, $\overline{B}_{\mu_\gamma}(\alpha)$ as in (4.2.9). Then Theorem 4.B is an easy consequence of the following lemma.

Lemma 4.4.3. *If $\gamma^2 < 2$, then for $\alpha \geq 0$, it holds $\mathbf{P}$-almost surely that*

(i) $E_{\mu_\gamma}(2\alpha) \subseteq \underline{E}_{Z_\gamma}(\alpha)$;

(ii) $\underline{E}_{Z_\gamma}(\alpha) \subseteq \underline{U}_{\mu_\gamma}(2\alpha) \cap \overline{B}_{\mu_\gamma}(2\alpha)$.

Proof. Let $(Z_{\gamma,t})_{t\in[0,T]}$ be the d-dimensional MRW as defined in (4.4.1). Thanks to well-known properties of Brownian motion, the following facts holds $\mathbf{P}$-almost surely. Fix $\varepsilon > 0$, then there exists a finite constant $C > 0$ such that, for every $t \in [0, T]$ and $r > 0$ small enough, it holds that

$$|Z_{\gamma,t+r} - Z_{\gamma,t-r}| = |B_{\mu_\gamma([0,t+r])} - B_{\mu_\gamma([0,t-r])}| \leq C\mu_\gamma(B(t, r))^{\frac{1}{2}-\varepsilon} . \tag{4.4.2}$$

Moreover, we also know that for each $\varepsilon > 0$ and $t \in [0, T]$, there exists a decreasing sequence $(r_n)_{n\in\mathbb{N}}$ converging to 0 such that

$$|Z_{\gamma,t+r_n} - Z_{\gamma,t-r_n}| = |B_{\mu_\gamma([0,t+r_n])} - B_{\mu_\gamma([0,t-r_n])}| \geq \mu_\gamma(B(t, r_n))^{\frac{1}{2}+\varepsilon} . \tag{4.4.3}$$

Proof of (i). For $\alpha \geq 0$, we fix $t \in \mathrm{E}_{\mu_\gamma}(2\alpha)$. For every $\varepsilon > 0$, we know that for $r > 0$ small enough, it holds that

$$r^{2\alpha+\varepsilon} \leq \mu_\gamma(B(t,r)) \leq r^{2\alpha-\varepsilon} \,. \tag{4.4.4}$$

Consequently, thanks to (4.4.2) and the right-hand side of (4.4.4), we get that for $r > 0$ small enough, it holds that

$$|\mathrm{Z}_{\gamma,t+r} - \mathrm{Z}_{\gamma,t-r}| \leq C\mu_\gamma(B(t,r))^{\frac{1}{2}-\varepsilon} \leq Cr^{(2\alpha-\varepsilon)\left(\frac{1}{2}-\varepsilon\right)} \,.$$

On the other hand, thanks to (4.4.3) and the left-hand side of (4.4.4), there exists a decreasing sequence $(r_n)_{n\in\mathbb{N}}$ converging to 0 such that

$$|\mathrm{Z}_{\gamma,t+r_n} - \mathrm{Z}_{\gamma,t-r_n}| \geq \mu_\gamma(B(t,r_n))^{\frac{1}{2}+\varepsilon} \geq r_n^{(2\alpha+\varepsilon)\left(\frac{1}{2}+\varepsilon\right)} \,, \qquad \forall\, n \in \mathbb{N} \,.$$

Therefore, taking logs, by arbitrariness of $\varepsilon > 0$, this implies that

$$\liminf_{r\to 0} \frac{\log|\mathrm{Z}_{\gamma,t+r} - \mathrm{Z}_{\gamma,t-r}|}{\log r} = \alpha \,.$$

which means that $t \in \underline{\mathrm{E}}_{\mathrm{Z}_\gamma}(\alpha)$, $\mathbf{P}$-almost surely.

Proof of (ii). For $\alpha \geq 0$, we fix $t \in \underline{\mathrm{E}}_{\mathrm{Z}_\gamma}(\alpha)$. Similarly as above, for each $\varepsilon > 0$, there exist decreasing sequences $(r_n)_{n\in\mathbb{N}}$, $(r'_n)_{n\in\mathbb{N}}$ both converging to 0 such that

$$r_n^{\alpha+\varepsilon} \leq |\mathrm{Z}_{\gamma,t+r_n} - \mathrm{Z}_{\gamma,t-r_n}| \leq C\mu_\gamma(B(t,r_n))^{\frac{1}{2}-\varepsilon} \,, \qquad \forall\, n \in \mathbb{N} \,,$$

where the last inequality is given by (4.4.2), and

$${r'_n}^{\alpha-\varepsilon} \geq |\mathrm{Z}_{\gamma,t+r'_n} - \mathrm{Z}_{\gamma,t-r'_n}| \geq \mu_\gamma(B(t,r'_n))^{\frac{1}{2}+\varepsilon} \,, \qquad \forall\, n \in \mathbb{N} \,,$$

where the last inequality is given by (4.4.3). By arbitrariness of $\varepsilon > 0$, these facts imply that $t \in \underline{\mathrm{U}}_{\mu_\gamma}(2\alpha) \cap \overline{\mathrm{B}}_{\mu_\gamma}(2\alpha)$, $\mathbf{P}$-almost surely. □

Finally, we show how the proof of Theorem 4.B follows immediately from Lemma 4.4.3.

Proof of Theorem 4.B. From item (i) of Lemma 4.4.3, for all $\alpha \geq 0$, we have the inequality

$$\underline{\mathrm{d}}_{\mathrm{Z}_\gamma}(\alpha) \geq \mathrm{d}_{\mu_\gamma}(2\alpha) \,,$$

almost surely. On the other hand, item (ii) of Lemma 4.4.3, Proposition 4.2.12 and Theorem 4.A imply that, for all $\alpha \geq 0$, it holds

$$\mathrm{d}_{\mathrm{Z}_\gamma}(\alpha) \leq \dim_{\mathcal{H}}\left(\underline{\mathrm{U}}_{\mu_\gamma}(2\alpha) \cap \overline{\mathrm{B}}_{\mu_\gamma}(2\alpha)\right) \leq \tau^*_{\mu_\gamma}(2\alpha) = \mathrm{d}_{\mu_\gamma}(2\alpha) \,,$$

almost surely. Since Theorem 4.A gives a full characterisation of the singularity spectrum d_{μ_γ}, this concludes the proof. □

4.5 Multifractal analysis of LBM

The main goal of this section is to study the multifractal behaviour of the *Liouville Brownian motion* (LBM). Roughly speaking, the LBM can be defined as the canonical diffusion associated with the *Liouville quantum gravity* metric tensor given by

$$e^{\gamma \mathrm{X}}(dx^2 + dy^2) \, ,$$

where X is a GFF on a domain $\mathrm{D} \subseteq \mathbb{R}^2$ and $dx^2 + dy^2$ is the Euclidean metric tensor. The LBM has been introduced simultaneously in [Ber15, GRV16], and it can be built as a time changed planar Brownian motion with respect to a GMC measure associated with the field X. We refer to the above mentioned references for more details.

In what follows, generalise the construction of the LBM in dimension $d \geq 2$. Once this is done, we consider the problem of studying the lower singularity spectrum of the d-dimensional LBM.

4.5.1 Definition of the LBM

For $d \geq 2$, consider a simply connected bounded domain $\mathrm{D} \subseteq \mathbb{R}^d$ containing the ball $B(0,1)$. Let B be a d-dimensional Brownian motion started from the origin and let τ its first exit time from D, i.e.,

$$\tau \stackrel{\text{def}}{=} \inf\{t \geq 0 \, : \, B_t \notin \mathrm{D}\} \, . \tag{4.5.1}$$

Moreover, consider a log-correlated Gaussian field X on D with covariance kernel of the form (4.2.1) and independent from the Brownian motion B. Then the d-dimensional LBM on D can be formally defined by

$$\mathrm{B}_{\gamma,t} \stackrel{\text{def}}{=} B_{\mathrm{F}_\gamma^{-1}(t)} \, , \qquad t \geq 0 \, ,$$

where the clock process F_γ is given by

$$\mathrm{F}_\gamma(t) \stackrel{\text{def}}{=} \int_0^{t \wedge \tau} e^{\gamma \mathrm{X}(B_s) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}(B_s)^2]} ds \, , \qquad t \geq 0 \, .$$

In the section, we adopt the same notation specified in Remark 4.4.1, i.e., we decompose the underlying probability measure as $\mathbf{P} \otimes \mathbb{P}$, with $\mathbf{P}$ (resp. $\mathbb{P}$) being the probability measure associated to the probability space where B (resp. X) is defined.

To give rigorous sense to the definition of the LBM, it is sufficient to show the existence of the clock process F_γ. Following [Ber15, GRV16], we proceed with a regularisation and limiting procedure. Let $(\mathrm{X}_\varepsilon)_{\varepsilon > 0}$ be the convolution approximation of the field X as defined in (4.2.2), then we have the following lemma.

Lemma 4.5.1. *If $\gamma^2 < 4$, then $\mathbf{P}$-almost surely, for all $t \geq 0$, the sequence*

$$\mathrm{F}_{\gamma,\varepsilon}(t) \stackrel{\text{def}}{=} \int_0^{t \wedge \tau} e^{\gamma \mathrm{X}_\varepsilon(B_s) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_s)^2]} ds \, , \qquad \varepsilon > 0 \, ,$$

converges in $\mathbb{P}$-probability, as $\varepsilon \to 0$, to a non-degenerate limit $\mathrm{F}_\gamma(t)$.

Proof. This fact is a standard result and it is an easy consequence of [Ber17, Theorem 1.1] (see also the last part of Section 1.3.1). We provide the details for completeness. For $t \geq 0$, let σ_t^τ be the occupation measure

between time 0 and time $t \wedge \tau$ of the Brownian motion B, so that

$$\mathrm{F}_{\gamma,\varepsilon}(t) = \int_{\mathbb{R}^d} e^{\gamma \mathrm{X}_\varepsilon(x) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(x)^2]} \sigma_t^\tau(dx) \,, \qquad \varepsilon > 0 \,.$$

Then it is sufficient to prove that $\mathbf{P}$-almost surely, for all $t \geq 0$, the measure σ_t^τ has dimension two, i.e., we need to show that for $\alpha \in (0,2)$, it holds that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^{-\alpha} \sigma_t^\tau(dx)\sigma_t^\tau(dy) < \infty \,.$$

Fix $t \geq 0$ and let σ_t be the occupation measure of the (unstopped) Brownian motion B between time 0 and time t, then it obviously holds that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^{-\alpha} \sigma_t^\tau(dx)\sigma_t^\tau(dy) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^{-\alpha} \sigma_t(dx)\sigma_t(dy) \,.$$

Therefore, taking expectation, it is enough to prove that for $\alpha \in (0,2)$, it holds that

$$\mathbf{E}\left[\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^{-\alpha} \sigma_t(dx)\sigma_t(dy)\right] = \mathbf{E}\left[\int_0^t \int_0^t |B_r - B_s|^{-\alpha} dr ds\right] < \infty \,. \tag{4.5.2}$$

Using Fubini's theorem, we have that

$$\begin{aligned}
\mathbf{E}\left[\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^{-\alpha} \sigma_t(dx)\sigma_t(dy)\right] &= \int_0^t \int_0^t \mathbf{E}\left[|B_r - B_s|^{-\alpha}\right] dr ds \\
&= \mathbf{E}\left[|B_1|^{-\alpha}\right] \int_0^t \int_0^t |r-s|^{-\frac{\alpha}{2}} dr ds \\
&\leq 2t\mathbf{E}\left[|B_1|^{-\alpha}\right] \int_0^t r^{-\frac{\alpha}{2}} dr
\end{aligned}$$

which is clearly finite if $\alpha \in (0,2)$. Therefore, this implies that for fixed $t \geq 0$, the measure σ_t^τ has dimension two, $\mathbf{P}$-almost surely. To prove that the measure σ_t^τ has dimension two for all $t \geq 0$, $\mathbf{P}$-almost surely, it is sufficient to take a sequence $(t_n)_{n\in\mathbb{N}}$ converging to infinity and noting that for $0 \leq s \leq t$, it holds that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^{-\alpha} \sigma_s^\tau(dx)\sigma_s^\tau(dy) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^{-\alpha} \sigma_t^\tau(dx)\sigma_t^\tau(dy) \,.$$

The conclusion follows thanks to [Ber17, Theorem 1.1]. □

Definition 4.5.2. Fix $\gamma^2 < 4$ and $d \geq 2$. The d-dimensional LBM on D associated with the field X is the stochastic process $(\mathrm{B}_{\gamma,t})_{t\geq 0}$ defined by

$$\mathrm{B}_{\gamma,t} \overset{\text{def}}{=} B_{\mathrm{F}_\gamma^{-1}(t)} \,, \qquad t \geq 0 \,, \tag{4.5.3}$$

where F_γ is defined as in Lemma 4.5.1.

Remark 4.5.3. We emphasise that when $d \geq 3$, Lemma 4.5.1 yields a construction of the LBM only for $\gamma^2 < 4$, and not for $\gamma^2 < 2d$ as for the case of GMC measures. Therefore, there exists a gap, for $4 \leq \gamma^2 < 2d$, where a construction of the LBM does not follow from the general theory of GMC. This is due to the well-known fact that the dimension of the occupation measure of the d-dimensional Brownian motion is at most two, for all $d \geq 2$ (see e.g. [MP10, Chapter 4]). To the best of our knowledge, if $d \geq 3$, a construction of the d-dimensional LBM for $4 \leq \gamma^2 < 2d$ has not be done yet.

Remark 4.5.4. We highlight that, in the joint work [BG25] with Ewain Gwynne – corresponding to Chapter 6 of this thesis – we identified the "correct" exponent $\alpha = \alpha(\gamma)$ that must be used in the definition of the clock process F_α of the LBM to ensure that the LBM satisfies a suitable conformal invariance. In particular, in dimension $d = 2$, we have $\alpha = \gamma$, whereas in higher dimensions, $\alpha = \alpha(\gamma)$ becomes a nonlinear function of γ, with the notable property that $\alpha^2 < 4$ for all $\gamma^2 < 2d$.

Before proceeding, we introduce some notation. For $\gamma^2 < 4$, we can define on the (random) interval $[0, \tau]$ the measure ν_γ by setting

$$\nu_\gamma([s,t]) \overset{\text{def}}{=} \mathrm{F}_\gamma(t) - \mathrm{F}_\gamma(s) \,, \qquad s \le t \in [0,\tau] \,. \tag{4.5.4}$$

It may be worth emphasizing that an immediate consequence of Lemma 4.5.1 is that the measure ν_γ can be interpreted as the weak limit of the sequence of approximating measures $(\nu_{\gamma,\varepsilon})_{\varepsilon>0}$, where

$$\nu_{\gamma,\varepsilon}(dt) \overset{\text{def}}{=} e^{\gamma \mathrm{X}_\varepsilon(B_t) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_t)^2]} \mathbb{1}_{\{\tau > t\}} dt \,, \qquad \varepsilon > 0 \,. \tag{4.5.5}$$

Similarly, we can define on the interval $[0, \mathrm{F}_\gamma(\tau)]$ the measure ν_γ^{-1} by letting

$$\nu_\gamma^{-1}([s,t]) \overset{\text{def}}{=} \mathrm{F}_\gamma^{-1}(t) - \mathrm{F}_\gamma^{-1}(s) \,, \qquad s \le t \in [0, \mathrm{F}_\gamma(\tau)] \,, \tag{4.5.6}$$

so that for all $t \in [0, \mathrm{F}_\gamma(\tau)]$, we can write $\mathrm{B}_{\gamma,t} = B_{\nu_\gamma^{-1}([0,t])}$.

Proposition 4.5.5. *For $\gamma^2 < 4$, the random function $\mathrm{F}_\gamma : [0,\tau] \to \mathbb{R}$ is almost surely continuous and strictly increasing on the interval $[0,\tau]$.*

Proof. The function F_γ is continuous and strictly increasing as long as the measure ν_γ has no atoms and it has full support on the interval $[0,\tau]$. To verify that ν_γ is almost surely atomless on $[0,\tau]$, one needs to use the power law behaviour of ν_γ (see Proposition 4.5.6 below). We refer to [Ber15, Remark 3.5] for details. The proof of the fact that ν_γ has almost surely full support on $[0,\tau]$ can be found in [GRV16, Theorem 2.7]. To be precise, in the above mentioned references the proofs are given in the planar case. However, they immediately generalise to all dimensions and hence the details are omitted. □

4.5.2 Study of the lower singularity spectrum

The main goal of this section is to prove Theorem 4.C. Since the LBM B_γ is a time changed Brownian motion, the lower singularity spectrum $\underline{\mathrm{d}}_{\mathrm{B}_\gamma}$ can be obtained through the singularity spectrum of the measure ν_γ^{-1} defined in (4.5.6), as in the case of the MRW. However, in this setting the situation is less trivial since we do not know the singularity spectrum of ν_γ^{-1} at this point. Instead of working directly with ν_γ^{-1}, we start by proving that the measure ν_γ, defined in (4.5.4), satisfies the multifractal formalism, and we find an explicit formula for its singularity spectrum. Once this is done, we can move to the proof of Theorem 4.C which consists in finding the relations between $\underline{\mathrm{d}}_{\mathrm{B}_\gamma}$, $\mathrm{d}_{\nu_\gamma^{-1}}$ and d_{ν_γ}.

4.5.2.1 Multifractal analysis of ν_γ

The main goal of this section is to prove that the measure ν_γ defined in (4.5.4) satisfies the multifractal formalism. We will follow a strategy similar to that developed in Section 4.3. As we will see, the presence of the Brownian motion creates some more technical difficulties, but the underlying ideas are the same.

We start by computing the power law spectrum of the measure ν_γ.

Proposition 4.5.6. *Let $\gamma^2 < 4$ and $q < 4/\gamma^2$. For $r \in (0,1]$, define the stopping time $\tau_{\sqrt{r}} \overset{\text{def}}{=} \inf\{t \geq 0 : B_t \notin B(0,\sqrt{r})\}$. Then there exists a finite constant $C > 0$ such that for all $r \in (0,1]$, it holds that*

$$\mathbb{E} \otimes \mathbf{E}[\nu_\gamma([0,\tau_{\sqrt{r}}])^q] \leq C r^{\xi_{\nu_\gamma}(q)} ,$$

where

$$\xi_{\nu_\gamma}(q) \overset{\text{def}}{=} \left(1 + \frac{1}{4}\gamma^2\right) q - \frac{1}{4}\gamma^2 q^2 , \qquad q \in \mathbb{R} ,$$

is the power law spectrum of ν_γ.

Proof. Thanks to Kahane's convexity inequality (see Lemma A.0.2), it is sufficient to prove the result only for the d-dimensional exactly scale invariant field $(X(x))_{x\in\mathbb{R}^d}$ defined in [RV10a]. More precisely, for $\varepsilon > 0$, we let $(X_\varepsilon(x))_{x\in\mathbb{R}^d}$ be the approximation of X as defined in [BP24, Lemma 3.19]. Then, for $\lambda \in (0,1)$, we can assume that

$$(X_{\lambda\varepsilon}(\lambda x))_{x\in B(0,1)} \overset{d}{=} (X_\varepsilon(x) + \Omega_\lambda)_{x\in B(0,1)} , \tag{4.5.7}$$

where Ω_λ is an independent centred Gaussian random variable with variance $-\log\lambda$ (see [BP24, Corollay 3.20]). Let $0 \leq \varepsilon < \sqrt{r}$, then scaling time by a factor r and using (4.5.7), we get that

$$\begin{aligned}
\nu_\gamma^\varepsilon([0,\tau_{\sqrt{r}}]) &= \int_0^{\tau_{\sqrt{r}}} e^{\gamma X_\varepsilon(B_s) - \frac{1}{2}\gamma^2\mathbb{E}[X_\varepsilon(B_s)^2]} ds \\
&= r\int_0^{\tau_{\sqrt{r}}/r} e^{\gamma X_\varepsilon(B_{rs}) - \frac{1}{2}\gamma^2\mathbb{E}[X_\varepsilon(B_{rs})^2]} ds \\
&\overset{d}{=} r\int_0^{\widetilde{\tau}} e^{\gamma X_\varepsilon(\sqrt{r}\widetilde{B}_s) - \frac{1}{2}\gamma^2\mathbb{E}[X_\varepsilon(\sqrt{r}\widetilde{B}_s)^2]} ds \\
&\overset{d}{=} r e^{\gamma\Omega_{\sqrt{r}} - \frac{1}{2}\gamma^2\mathbb{E}[\Omega_{\sqrt{r}}^2]} \int_0^{\widetilde{\tau}} e^{\gamma X_{\varepsilon/\sqrt{r}}(\widetilde{B}_s) - \frac{1}{2}\gamma^2\mathbb{E}[X_{\varepsilon/\sqrt{r}}(\widetilde{B}_s)^2]} ds ,
\end{aligned}$$

where $\widetilde{B}$ is an independent Brownian motion and $\widetilde{\tau}$ is the first exit time of $\widetilde{B}$ from $B(0,1)$. Now, raising to the power $q < 4/\gamma^2$ and taking expectations, one can easily verify that

$$\mathbb{E} \otimes \mathbf{E}[\nu_\gamma^\varepsilon([0,\tau_{\sqrt{r}}])^q] = r^{\xi(q)}\mathbb{E} \otimes \mathbf{E}\left[\left(\int_0^{\widetilde{\tau}} e^{\gamma X_{\varepsilon/\sqrt{r}}(\widetilde{B}_s) - \frac{1}{2}\gamma^2\mathbb{E}\otimes\mathbf{E}[X_{\varepsilon/\sqrt{r}}(\widetilde{B}_s)^2]} ds\right)^q\right]$$

Removing the approximation, i.e., taking the limit when $\varepsilon \to 0$, gives us the desired result provided that the quantities $\nu_\gamma^\varepsilon([0,\tau_{\sqrt{r}}])$ and $\nu_\gamma^{\varepsilon/\sqrt{r}}([0,\widetilde{\tau}])^q$ are uniformly integrable in $\varepsilon > 0$, for $q < 4/\gamma^2$. For $q \leq 0$ this fact is proved in the planar case in [GRV16, Proposition 2.12]), but the proof generalises to higher dimensions. For $q \in (0, 4/\gamma^2)$ a proof of this fact can be found in Appendix 4.B. □

We now state the main result that allows us to prove that ν_γ satisfies the multifractal formalism. We emphasise that the below result is the analogue of Proposition 4.3.2, but for the measure ν_γ.

Proposition 4.5.7. *Let $\gamma^2 < 4$ and $q^2 < 4/\gamma^2$. For $\mu_{q\gamma}$-almost every $t \in [0,T]$, the measure ν_γ satisfies*

$$\lim_{r\to 0} \frac{\log \nu_\gamma(B(t,r))}{\log r} = 1 + \left(\frac{1}{2} - q\right)\frac{\gamma^2}{2} ,$$

almost surely.

Proof. The case where the underlying field is a zero-boundary GFF on a bounded domain $\mathrm{D} \subseteq \mathbb{R}^2$ has already been addressed in [Jac18]. The proof follows similar lines to that of Proposition 4.3.2 in Appendix 4.A. Therefore, we only highlight the main modifications required for the proof to hold in the more general setting of a log-correlated Gaussian field X on a simply connected bounded domain $\mathrm{D} \subseteq \mathbb{R}^d$ with $d \geq 2$.

The main steps of the proof in [Jac18] rely on [Jac18, Lemmas 3.9 and 3.11]. We start by focusing on [Jac18, Lemma 3.9]. The only significant modification required is in [Jac18, Equation (3.19)]. Specifically, in that step, the author applies the domain Markov property of the GFF to bound the exponential moment of the fluctuations of the circle average approximation. However, for a general log-correlated Gaussian field, we can bypass the use of this property by instead employing Lemma 4.2.1 to control the fluctuations of the convolution approximation. To be more precise, we briefly introduce some notation. Let B and $\widetilde{B}$ be two independent d-dimensional Brownian motions starting from the origin, let $\mathcal{F}_t \overset{\text{def}}{=} \sigma(B_s \,:\, s \leq t)$ be the natural filtration associated with B up to time $t \geq 0$, and define the following stopping time,

$$\widetilde{\tau} \overset{\text{def}}{=} \inf\big\{s \geq 0 \,:\, \sqrt{r}\widetilde{B}_s + B_t \notin B(0,1/2)\big\} \wedge \inf\big\{s \geq 0 \,:\, \widetilde{B}_s \notin B(0,1/2)\big\} \,,$$

where $r \in (0,1]$. Then, letting $\Omega^{B_t}_{\sqrt{r}} \overset{\text{def}}{=} \inf_{x \in B(B_t,\sqrt{r})} \mathrm{X}_{\sqrt{r}}(x) - \mathrm{X}_{\sqrt{r}}(B_t)$ the following lower bound trivially holds for $s, t \in [0, \widetilde{\tau}]$,

$$\mathrm{X}_{\varepsilon\sqrt{r}}(\sqrt{r}\widetilde{B}_s + B_t) \geq (\mathrm{X}_{\varepsilon\sqrt{r}}(\sqrt{r}\widetilde{B}_s + B_t) - \mathrm{X}_{\sqrt{r}}(\sqrt{r}\widetilde{B}_s + B_t)) + \Omega^{B_t}_{\sqrt{r}} + \mathrm{X}_{\sqrt{r}}(B_t) \,.$$

Therefore, by plugging this estimate into the second line of [Jac18, Equation (3.19)], leveraging the fact that in [Jac18, Equation (3.20)] the probability is conditioned on $\mathcal{F}_t$, and following the exact same reasoning as in the proof of Proposition 4.A.1 below, we can obtain the same conclusion as [Jac18, Lemma 3.9] in our more general setting.

Finally, the modifications required in the proof of [Jac18, Lemma 3.11] are analogous and therefore omitted. □

We can now state the following proposition concerning the singularity spectrum of ν_γ.

Proposition 4.5.8. *If $\gamma^2 < 4$, then the measure ν_γ satisfies the multifractal formalism. More precisely, for $\alpha \geq 0$ it holds that*

$$\mathrm{d}_{\nu_\gamma}(\alpha) = \tau^*_{\nu_\gamma}(\alpha) = \begin{cases} 1 - \left(\frac{1-\alpha}{\gamma} + \frac{1}{4}\gamma\right)^2 \,, & \text{if } \alpha \in \left[\left(1-\frac{\gamma}{2}\right)^2, \left(1+\frac{\gamma}{2}\right)^2\right] \,, \\ 0 \,, & \text{otherwise} \,, \end{cases}$$

almost surely.

Proof. Given Propositions 4.5.6 and 4.5.7, and defining the structure function $\phi_{\nu_\gamma}(q) \overset{\text{def}}{=} 1 - \xi_{\nu_\gamma}(q)$, for all $q \in \mathbb{R}$, one can proceed similarly to Section 4.3 to prove this result. Therefore, the details are omitted. □

4.5.2.2 Conclusion of the proof

Now that we have found an explicit formula for the singularity spectrum of ν_γ, we can study the relations between the various (lower) singularity spectra. For $\alpha \geq 0$, we define the set $\mathrm{E}_{\nu_\gamma}(\alpha)$ as in (4.2.8), the sets $\underline{\mathrm{U}}_{\nu_\gamma}(\alpha)$, $\overline{\mathrm{B}}_{\nu_\gamma}(\alpha)$ as in (4.2.9), and similarly also for ν_γ^{-1}. We start with the following lemma.

Lemma 4.5.9. *If $\gamma^2 < 4$, then for $\alpha \geq 0$, it holds $\mathbf{P}$-almost surely that*

(i) $\mathrm{E}_{\nu_\gamma^{-1}}(2\alpha) \subseteq \underline{\mathrm{E}}_{\mathrm{B}_\gamma}(\alpha)$;

(ii) $\underline{\mathrm{E}}_{\mathrm{B}_\gamma}(\alpha) \subseteq \underline{\mathrm{U}}_{\nu_\gamma^{-1}}(2\alpha) \cap \overline{\mathrm{B}}_{\nu_\gamma^{-1}}(2\alpha)$.

Proof. Since for all $t \in [0,\tau]$, it holds that $\mathrm{B}_{\gamma,t} = B_{\nu_\gamma^{-1}((0,t])}$, one can use the same argument used in the proof of Lemma 4.4.3 to prove this result. □

Thanks to Lemma 4.5.9, the problem has been reduced to the computation of the Hausdorff dimensions of $\mathrm{E}_{\nu_\gamma^{-1}}(\alpha)$ and of $\underline{\mathrm{U}}_{\nu_\gamma^{-1}}(\alpha) \cap \overline{\mathrm{B}}_{\nu_\gamma^{-1}}(2\alpha)$, for $\alpha \geq 0$. We start with the following trivial lemma.

Lemma 4.5.10. *If $\gamma^2 < 4$, then for $\alpha > 0$, it holds almost surely that*

(i) $\mathrm{E}_{\nu_\gamma^{-1}}(\alpha) = \mathrm{F}_\gamma(\mathrm{E}_{\nu_\gamma}(\alpha^{-1}))$;

(ii) $\underline{\mathrm{U}}_{\nu_\gamma^{-1}}(\alpha) \cap \overline{\mathrm{B}}_{\nu_\gamma^{-1}}(\alpha) = \mathrm{F}_\gamma(\underline{\mathrm{U}}_{\nu_\gamma}(\alpha^{-1}) \cap \overline{\mathrm{B}}_{\nu_\gamma}(\alpha^{-1}))$.

Proof. The proof follows thanks to the fact that random function F_γ is almost surely continuos and strictly increasing on the interval $[0,\tau]$ (see Proposition 4.5.5). □

The following lemma is concerned with the relation between d_{ν_γ} and $\mathrm{d}_{\nu_\gamma^{-1}}$.

Lemma 4.5.11. *If $\gamma^2 < 4$, then for $\alpha > 0$, it holds almost surely that* $\mathrm{d}_{\nu_\gamma^{-1}}(\alpha) = \alpha \mathrm{d}_{\nu_\gamma}(\alpha^{-1})$.

Proof. Thanks to item (i) of Lemma 4.5.10, we know that for $\alpha > 0$, it holds almost surely that $\mathrm{E}_{\nu_\gamma^{-1}}(\alpha) = \mathrm{F}_\gamma(\mathrm{E}_{\nu_\gamma}(\alpha^{-1}))$. Fix $\varepsilon > 0$, and let $t \in \mathrm{E}_{\nu_\gamma}(\alpha^{-1})$, then we define $\delta_t > 0$ as follows

$$\delta_t \overset{\text{def}}{=} \sup\left\{\delta > 0 \,:\, |\mathrm{F}_\gamma(t+r) - \mathrm{F}_\gamma(t-r)| \leq r^{\alpha^{-1}-\varepsilon},\, \forall\, r \in (0,\delta)\right\} \wedge 1\,.$$

We define the collection of sets $(E^n_{\nu_\gamma}(\alpha^{-1}))_{n\in\mathbb{N}}$ by letting

$$\mathrm{E}^n_{\nu_\gamma}(\alpha^{-1}) \overset{\text{def}}{=} \left\{t \in \mathrm{E}_{\nu_\gamma}(\alpha^{-1}) \,:\, 2^{-n} < \delta_t \leq 2^{-n+1}\right\}\,, \qquad n \in \mathbb{N}\,.$$

Then $\mathrm{E}_{\nu_\gamma}(\alpha^{-1}) = \cup_{n\in\mathbb{N}} \mathrm{E}^n_{\nu_\gamma}(\alpha^{-1})$, and for all $t \in \mathrm{E}^n_{\nu_\gamma}(\alpha^{-1})$, it holds that $|\mathrm{F}_\gamma(t+r) - \mathrm{F}_\gamma(t-r)| \leq r^{\alpha^{-1}-\varepsilon}$ for all $r \in (0, 2^{-n})$. Thanks to Proposition 4.5.5, we know that the random function F_γ is almost surely continuous and strictly increasing. Hence, using the countable stability of the Hausdorff dimension and Proposition 4.2.7, it holds almost surely that

$$\mathrm{d}_{\nu_\gamma^{-1}}(\alpha) = \sup_{n\in\mathbb{N}} \dim_{\mathcal{H}}(\mathrm{F}_\gamma(\mathrm{E}^n_{\nu_\gamma}(\alpha^{-1}))) \leq \frac{\alpha}{1-\alpha\varepsilon} \sup_{n\in\mathbb{N}} \dim_{\mathcal{H}}(\mathrm{E}^n_{\nu_\gamma}(\alpha^{-1})) = \frac{\alpha}{1-\alpha\varepsilon}\mathrm{d}_{\nu_\gamma}(\alpha^{-1})\,.$$

The reverse inequality can be obtained similarly. Indeed, if we fix $\varepsilon > 0$ and we let $t \in \mathrm{E}_{\nu_\gamma^{-1}}(\alpha)$, then we can define $\delta_t > 0$ as follows

$$\delta_t \overset{\text{def}}{=} \sup\left\{\delta > 0 \,:\, |\mathrm{F}^{-1}_\gamma(t+r) - \mathrm{F}^{-1}_\gamma(t-r)| \leq r^{\alpha-\varepsilon},\, \forall\, r \in (0,\delta)\right\} \wedge 1\,.$$

Defining the collection of sets $(\mathrm{E}^n_{\nu_\gamma^{-1}}(\alpha))_{n\in\mathbb{N}}$ similarly as above, and proceeding in the same exact way, we obtain that it holds almost surely that

$$\mathrm{d}_{\nu_\gamma}(\alpha^{-1}) \leq \frac{1}{\alpha-\varepsilon}\mathrm{d}_{\nu_\gamma^{-1}}(\alpha)\,.$$

The conclusion follows by arbitrariness of $\varepsilon > 0$. □

We also need the following lemma which allows us to conclude the proof.

Lemma 4.5.12. *If $\gamma^2 < 4$, then for $\alpha > 0$ it holds almost surely that* $\dim_{\mathcal{H}}(\underline{U}_{\nu_\gamma^{-1}}(\alpha) \cap \overline{B}_{\nu_\gamma^{-1}}(\alpha)) \leq \alpha\tau^*_{\nu_\gamma}(\alpha^{-1})$.

Proof. Thanks to item (ii) of Lemma 4.5.10 and to the definition of $\tau^*_{\nu_\gamma}$, it is sufficient to prove that for $\alpha > 0$ it holds that

$$\dim_{\mathcal{H}}(F_\gamma(\underline{U}_{\nu_\gamma}(\alpha^{-1}) \cap \overline{B}_{\nu_\gamma}(\alpha^{-1}))) \leq q + \alpha\tau_{\nu_\gamma}(q)\,, \qquad \forall\, q \in R\,, \tag{4.5.8}$$

almost surely. We will be brief here since this result can be proved using the same procedure used in the proof of Proposition 4.2.12. We will only focus in the case $q \geq 0$. Fix $\varepsilon > 0$ and let $\overline{s}_{q,\varepsilon} \overset{\text{def}}{=} q + (\tau_{\nu_\gamma}(q) + \varepsilon)/(\alpha^{-1} - \varepsilon)$. By definition, for every $t \in \underline{U}_{\nu_\gamma}(\alpha^{-1}) \cap \overline{B}_{\nu_\gamma}(\alpha^{-1})$, there exists a decreasing sequence $(r_{t,n})_{n\in\mathbb{N}}$ converging to 0 such that

$$|F_\gamma(t + r_{t,n}) - F_\gamma(t - r_{t,n})| = \nu_\gamma(B(t, r_{t,n})) \leq r_{t,n}^{\alpha^{-1}-\varepsilon}\,, \qquad \forall\, n \in \mathbb{N}\,.$$

Fix $\delta \in (0, 1]$, then for each $t \in \underline{U}_{\nu_\gamma}(\alpha^{-1}) \cap \overline{B}_{\nu_\gamma}(\alpha^{-1})$, we choose n_t such that $r_{t,n_t} \in (0, \delta)$. For each $m \in \mathbb{N}$, we let

$$F_m \overset{\text{def}}{=} \left\{ t \in \underline{U}_{\nu_\gamma}(\alpha^{-1}) \cap \overline{B}_{\nu_\gamma}(\alpha^{-1}) \,:\, 2^{-m} < r_{t,n_t} \leq 2^{-m+1} \right\}\,.$$

Proceeding as in the proof of Proposition 4.2.12, there exists a positive integer σ_1 such that for every $m \in \mathbb{N}$, we can find σ_1 disjoint subsets $F_{m,1}, \dots, F_{m,\sigma_1}$ of F_m such that each $F_{m,j}$ is at most countable, the balls $B(t, r_{t,n_t})$ with centres at $t \in F_{m,j}$ are pairwise disjoint, and the collection of sets

$$\left(\left((F_\gamma(B(t, r_{t,n_t})))_{t\in F_{m,j}} \right)_{j\in\{1,\dots,\sigma_1\}} \right)_{m\in\mathbb{N}}$$

is a δ-cover of $F_\gamma(\underline{U}_{\nu_\gamma}(\alpha^{-1}) \cap \overline{B}_{\nu_\gamma}(\alpha^{-1}))$. Then, thanks to the definition of ν_γ and to the fact that F_γ is almost surely strictly increasing, we have that

$$\begin{aligned}
\mathcal{H}^{\overline{s}_{q,\varepsilon}}_\delta(F_\gamma(\underline{U}_{\nu_\gamma}(\alpha^{-1}) \cap \overline{B}_{\nu_\gamma}(\alpha^{-1}))) &\leq \sum_{m\in\mathbb{N}} \sum_{j=1}^{\sigma_1} \sum_{t\in F_{m,j}} |F_\gamma(t + r_{t,n_t}) - F_\gamma(t - r_{t,n_t})|^{\overline{s}_{q,\varepsilon}} \\
&\leq \sum_{m\in\mathbb{N}} \sum_{j=1}^{\sigma_1} \sum_{t\in F_{m,j}} \nu_\gamma(B(t, r_{t,n_t}))^q r_{t,n_t}^{(\tau_\mu(q)+\varepsilon)} \\
&\lesssim \sum_{m\in\mathbb{N}} \sum_{j=1}^{\sigma_1} \sum_{t\in F_{m,j}} \nu_\gamma(B(t, 2^{-m+1}))^q 2^{-m(\tau_\mu(q)+\varepsilon)}\,,
\end{aligned}$$

where the implicit constant does not depend on m. Hence, the conclusion follows as in the proof of Proposition 4.2.12. Similarly, we can prove inequality (4.5.8) for $q < 0$. □

Finally, we see how we can put everything together in order to prove Theorem 4.C.

Proof of Theorem 4.C. We start by observing that almost surely it holds that $\underline{d}_{B_\gamma}(0) = 0$ since B_γ is almost surely continuous. Thanks to item (i) of Lemma 4.5.9 and Lemma 4.5.11, for $\alpha > 0$, we have that

$$\underline{d}_{B_\gamma}(\alpha) \geq d_{\nu_\gamma^{-1}}(2\alpha) = 2\alpha d_{\nu_\gamma}((2\alpha)^{-1})\,,$$

almost surely. On the other hand, item (ii) of Lemma 4.5.9, Lemma 4.5.12 and Proposition 4.5.8 imply that, for all $\alpha > 0$, it holds that

$$\underline{\mathrm{d}}_{\mathrm{B}_\gamma}(\alpha) \leq \dim_{\mathcal{H}}\left(\underline{\mathrm{U}}_{\nu_\gamma^{-1}}(2\alpha) \cap \overline{\mathrm{B}}_{\nu_\gamma^{-1}}(2\alpha)\right) \leq 2\alpha\tau^*_{\nu_\gamma}((2\alpha)^{-1}) = 2\alpha \mathrm{d}_{\nu_\gamma}((2\alpha)^{-1}) \,,$$

almost surely. Since the singularity spectrum d_{ν_γ} is known, this concludes the proof. □

4.A Proof of Proposition 4.3.2

The proof of Proposition 4.3.2 follows similar steps of the proof in [RV14a, Theorem 4.1]. However, our proof accommodates general log-correlated Gaussian fields. Indeed, it is not enough to prove the result for an exactly stochastically scale invariant field, and then deduce the result for all log-correlated fields, by means of Kahane's convexity inequality (see Lemma A.0.2). This is due to the fact that in the proof we need to consider moments of certain integrals of the field that involve indicator functions depending on the field itself.

Since the conclusion of Proposition 4.3.2 trivially holds for $\gamma = 0$, we may assume $\gamma \neq 0$. The proof is essentially based on the following proposition and on a standard Borel–Cantelli argument.

Proposition 4.A.1. *Let $\gamma^2 < 2d$, $q^2 < 2d/\gamma^2$ and define $\alpha \overset{\text{def}}{=} d + (1/2 - q)\gamma^2$. For $\beta > 0$ and $E > 0$, there exist constants C, b, C', $b' > 0$ such that for $r \in (0,1]$ small enough it holds that*

$$\mathbb{E}\left[\mu_{q\gamma}\left(\left\{x \in \mathrm{D}_r \,:\, \mu_\gamma(B(x,r)) > E r^{\alpha-\beta}\right\}\right)\right] \leq C r^b \,, \tag{4.A.1}$$

and

$$\mathbb{E}\left[\mu_{q\gamma}\left(\left\{x \in \mathrm{D}_r \,:\, \mu_\gamma(B(x,r)) < E r^{\alpha+\beta}\right\}\right)\right] \leq C' r^{b'} \,, \tag{4.A.2}$$

where $\mathrm{D}_r \overset{\text{def}}{=} \{x \in \mathrm{D} \,:\, \mathrm{dist}(x, \partial\mathrm{D}) < r\}$.

Proof. The proof can be found in Subsection 4.A.1. □

We are now ready to prove Proposition 4.3.2.

Proof of Proposition 4.3.2. Fix $\beta > 0$ and let $\alpha = d + (1/2 - q)\gamma^2$. For $n \in \mathbb{N}$, we define the following sets

$$U_\gamma^n \overset{\text{def}}{=} \left\{x \in \mathrm{D}_{2^{-n}} \,:\, \mu_\gamma(B(x,r)) \leq r^{\alpha-\beta}, \qquad \forall\, r \in [0, 2^{-n})\right\} \,,$$

and

$$L_\gamma^n \overset{\text{def}}{=} \left\{x \in \mathrm{D}_{2^{-n}} \,:\, \mu_\gamma(B(x,r)) \geq r^{\alpha+\beta} \qquad \forall\, r \in [0, 2^{-n})\right\} \,,$$

where we recall that $\mathrm{D}_r \overset{\text{def}}{=} \{x \in \mathrm{D} \,:\, \mathrm{dist}(x, \partial\mathrm{D}) < r\}$. We claim that for every $\varepsilon > 0$, there exists almost surely a finite $N \in \mathbb{N}$ such that

$$\mu_{q\gamma}\left(\left(U_\gamma^N\right)^c\right) < \varepsilon \,, \qquad \mu_{q\gamma}\left(\left(L_\gamma^N\right)^c\right) < \varepsilon \,.$$

Thanks to Proposition 4.A.1 with $E = 2^{-(\alpha-\beta)}$ and to Markov's inequality, we know that there exist constants $C > 0$ and $b > 0$ such that for $r \in (0,1]$ small enough, it holds that

$$\mathbb{P}\left(\mu_{q\gamma}\left(\left\{x \in \mathrm{D}_r \,:\, \mu_\gamma(B(x,r)) > 2^{-(\alpha-\beta)} r^{\alpha-\beta}\right\}\right) \geq r^{\frac{b}{2}}\right)$$

$$\leq r^{-\frac{b}{2}}\mathbb{E}\left[\mu_{q\gamma}\left(\left\{x\in \mathrm{D}_r\,:\,\mu_\gamma(B(x,r))>2^{-(\alpha-\beta)}r^{\alpha-\beta}\right\}\right)\right]$$
$$\leq Cr^{\frac{b}{2}}\ .$$

For $n\in\mathbb{N}$, taking $r=2^{-n}$, thanks to the Borel–Cantelli lemma, we get that the events

$$\left(\left\{\mu_{q\gamma}\left(\left\{x\in \mathrm{D}_{2^{-n}}\,:\,\mu_\gamma(B(x,2^{-n}))>(2^{-n-1})^{\alpha-\beta}\right\}\right)\geq (2^{-n})^{\frac{b}{2}}\right\}\right)_{n\in\mathbb{N}}$$

occur only finitely often almost surely. Therefore, for all $\varepsilon>0$, there exists almost surely a finite $N\in\mathbb{N}$ such that

$$\mu_{q\gamma}\left(\bigcup_{n\geq N}\left\{x\in \mathrm{D}_{2^{-n}}\,:\,\mu_\gamma(B(x,2^{-n}))>(2^{-n-1})^{\alpha-\beta}\right\}\right)\leq\sum_{n\geq N}(2^{-n})^{\frac{b}{2}}<\varepsilon\ .$$

Let $x\in(U_\gamma^N)^c$, then there exists $r\in[0,2^{-N})$ such that $\mu_\gamma(B(x,r))>r^{\alpha-\beta}$. Since there exists $n\geq N$ such that $2^{-n-1}\leq r<2^{-n}$, we get

$$\mu_\gamma(B(x,2^{-n}))\geq\mu_\gamma(B(x,r))>r^{\alpha-\beta}\geq(2^{-n-1})^{\alpha-\beta}\ ,$$

which implies that

$$\mu_{q\gamma}\left(\left(U_\gamma^N\right)^c\right)\leq\mu_{q\gamma}\left(\bigcup_{n\geq N}\left\{x\in \mathrm{D}_{2^{-n}}\,:\,\mu_\gamma(B(x,2^{-n}))>(2^{-n-1})^{\alpha-\beta}\right\}\right)<\varepsilon\ .$$

A similar argument can be used to prove that for every $\varepsilon>0$, there exists almost surely a finite $N\in\mathbb{N}$ such that $\mu_{q\gamma}((L_\gamma^N)^c)<\varepsilon$. Therefore, introducing the sets

$$U_\gamma\overset{\text{def}}{=}\bigcup_{n\in\mathbb{N}}U_\gamma^n=\left\{x\in \mathrm{D}\,:\,\liminf_{r\to0}\frac{\log\mu_\gamma(B(x,r))}{\log r}\geq\alpha-\beta\right\}\ ,$$
$$L_\gamma\overset{\text{def}}{=}\bigcup_{n\in\mathbb{N}}L_\gamma^n=\left\{x\in \mathrm{D}\,:\,\limsup_{r\to0}\frac{\log\mu_\gamma(B(x,r))}{\log r}\leq\alpha+\beta\right\}\ ,$$

since $\varepsilon>0$ is arbitrary and U_γ^N, L_γ^N are increasing sets, we obtain that

$$\mu_{q\gamma}\left((L_\gamma\cap U_\gamma)^c\right)=0\ .$$

The conclusion then follows from the arbitrariness of $\beta>0$. □

4.A.1 Proof of Proposition 4.A.1

The objective of this subsection is to prove Proposition 4.A.1. For simplicity, but without loss of generality, we assume that $\mathrm{D}=(-1,1)^d$. The proof is based on Lemma 4.2.1 and on the following two lemmas.

Lemma 4.A.2. *For $\gamma^2<2d$ and $q^2<2d/\gamma^2$, there exist $\overline{a}\in(0,1]$ and $C>0$ such that*

$$\sup_{\varepsilon>0}\mathbb{E}\left[\left(\int_{(-1,1)^d}\frac{e^{\gamma \mathrm{X}_\varepsilon(x)-\frac{1}{2}\gamma^2\mathbb{E}[\mathrm{X}_\varepsilon(x)^2]}}{(|x|+\varepsilon)^{q\gamma^2}}dx\right)^{\overline{a}}\right]\leq C$$

Proof. The proof can be found in Subsection 4.A.2. □

Lemma 4.A.3. *For $\gamma^2 < 2d$ and $q^2 < 2d/\gamma^2$, there exists $C > 0$ such that*

$$\sup_{\varepsilon>0} \mathbb{E}\left[\left(\int_{[-1/2,1/2]^d} \frac{e^{\gamma \mathrm{X}_\varepsilon(x) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(x)^2]}}{(|x|+\varepsilon)^{q\gamma^2}} dx\right)^{-1}\right] \leq C\,.$$

Proof. The proof can be found in Subsection 4.A.2. □

Proof of Proposition 4.A.1. We split the proof into two parts. We start by proving (4.A.1) and then we move to the proof of (4.A.2). For simplicity, but without loss of generality, we consider the case $E = 1$.

Proof of (4.A.1). Let $r \in (0,1]$ and $x \in \mathrm{D}_r$, where we recall that we are assuming that $\mathrm{D} = (-1,1)^d$. Fix ε, $\varepsilon' > 0$ so that $\varepsilon' < r\varepsilon$. Then by Girsanov's theorem (see Lemma A.0.1), we have that

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\left\{\int_{B(x,r)} e^{\gamma \mathrm{X}_{r\varepsilon}(u) - \frac{1}{2}\gamma^2\mathbb{E}[\mathrm{X}_{r\varepsilon}(u)^2]} du > r^{\alpha-\beta}\right\}} e^{q\gamma \mathrm{X}_{\varepsilon'}(x) - \frac{1}{2}q^2\gamma^2\mathbb{E}[\mathrm{X}_{\varepsilon'}(x)^2]}\right] \\
&\qquad = \mathbb{P}\left(\int_{B(x,r)} e^{\gamma \mathrm{X}_{r\varepsilon}(u) + q\gamma^2\mathbb{E}[\mathrm{X}_{r\varepsilon}(u)\mathrm{X}_{\varepsilon'}(x)] - \frac{1}{2}\gamma^2\mathbb{E}[\mathrm{X}_{r\varepsilon}(u)^2]} du > r^{\alpha-\beta}\right) \\
&\qquad \leq \mathbb{P}\left(e^{\gamma^2 K|q|}\int_{B(x,r)} \frac{e^{\gamma \mathrm{X}_{r\varepsilon}(u) - \frac{1}{2}\gamma^2\mathbb{E}[\mathrm{X}_{r\varepsilon}(u)^2]}}{(|u-x|+r\varepsilon)^{q\gamma^2}} du > r^{\alpha-\beta}\right)\,,
\end{aligned} \tag{4.A.3}$$

where the last inequality is justified by the fact that, thanks to property (P2), for $x, y \in (-1,1)^d$, there exists $K > 0$ such that

$$-\log(|x-y|+r\varepsilon) - K \leq \mathbb{E}[\mathrm{X}_{r\varepsilon}(x)\mathrm{X}_{\varepsilon'}(y)] \leq -\log(|x-y|+r\varepsilon) + K\,.$$

Now, let $\Omega_r^x \overset{\text{def}}{=} \sup_{u\in B(x,r)} \mathrm{X}_r(u) - \mathrm{X}_r(x)$, then by changing variables, we see that the integral in (4.A.3) can be upper bounded as follows

$$\begin{aligned}
&\int_{B(x,r)} \frac{e^{\gamma \mathrm{X}_{r\varepsilon}(u) - \frac{1}{2}\gamma^2\mathbb{E}[\mathrm{X}_{r\varepsilon}(u)^2]}}{(|u-x|+r\varepsilon)^{q\gamma^2}} du \\
&\qquad \leq e^{\gamma(\Omega_r^x + \mathrm{X}_r(x))}\int_{B(x,r)} \frac{e^{\gamma(\mathrm{X}_{r\varepsilon}(u) - \mathrm{X}_r(u)) - \frac{1}{2}\gamma^2\mathbb{E}[\mathrm{X}_{r\varepsilon}(u)^2]}}{(|u-x|+r\varepsilon)^{q\gamma^2}} du \\
&\qquad = r^{d-q\gamma^2} e^{\gamma(\Omega_r^x + \mathrm{X}_r(x))}\int_{B(0,1)} \frac{e^{\gamma(\mathrm{X}_{r\varepsilon}(ru+x) - \mathrm{X}_r(ru+x)) - \frac{1}{2}\gamma^2\mathbb{E}[\mathrm{X}_{r\varepsilon}(ru+x)^2]}}{(|u|+\varepsilon)^{q\gamma^2}} du \\
&\qquad \leq r^{\alpha} e^{\frac{1}{2}\gamma^2 K + \gamma(\Omega_r^x + \mathrm{X}_r(x))}\int_{B(0,1)} \frac{e^{\gamma(\mathrm{X}_{r\varepsilon}(ru+x) - \mathrm{X}_r(ru+x)) + \frac{1}{2}\gamma^2\log\varepsilon}}{(|u|+\varepsilon)^{q\gamma^2}} du\,.
\end{aligned} \tag{4.A.4}$$

Substituting the last expression of (4.A.4) back into (4.A.3), we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\left\{\int_{B(x,r)} e^{\gamma \mathrm{X}_{r\varepsilon}(u) - \frac{1}{2}\gamma^2\mathbb{E}[\mathrm{X}_{r\varepsilon}(u)^2]} du > r^{\alpha-\beta}\right\}} e^{q\gamma \mathrm{X}_{\varepsilon'}(x) - \frac{1}{2}q^2\gamma^2\mathbb{E}[\mathrm{X}_{\varepsilon'}(x)^2]}\right] \\
&\quad \leq \mathbb{P}\left(e^{\gamma^2 K(1/2+|q|) + \gamma(\Omega_r^x + \mathrm{X}_r(x))}\int_{B(0,1)} \frac{e^{\gamma(\mathrm{X}_{r\varepsilon}(ru+x) - \mathrm{X}_r(ru+x)) + \frac{1}{2}\gamma^2\log\varepsilon}}{(|u|+\varepsilon)^{q\gamma^2}} du > r^{-\beta}\right) \\
&\quad \leq \mathbb{P}\left(e^{\gamma^2 K(1/2+|q|) + \gamma\Omega_r^x} > r^{-\frac{\beta}{3}}\right) + \mathbb{P}\left(e^{\gamma \mathrm{X}_r(x)} > r^{-\frac{\beta}{3}}\right) \\
&\qquad\qquad + \mathbb{P}\left(\int_{B(0,1)} \frac{e^{\gamma(\mathrm{X}_{r\varepsilon}(ru+x) - \mathrm{X}_r(ru+x)) + \frac{1}{2}\gamma^2\log\varepsilon}}{(|u|+\varepsilon)^{q\gamma^2}} du > r^{-\frac{\beta}{3}}\right)\,.
\end{aligned} \tag{4.A.5}$$

We note that all the three terms decay polynomially in r, for $r \in (0,1]$ small enough, uniformly in $x \in \mathrm{D}_r$ and ε, $\varepsilon' \in (0,1]$. Indeed, for the first term this follows from Lemma 4.2.1 and Markov's inequality. For the second term, this follows from the fact that $\mathrm{X}_r(x)$ is a centred Gaussian random variable with variance $\mathbb{E}[\mathrm{X}_r(x)^2] = -\log r + \mathcal{O}(1)$ (see [MP10, Lemma 12.9]). Finally, for the third term this follows from property (P2), Kahane's convexity inequality[2] (see Lemma A.0.2), Lemma 4.A.2, and Markov's inequality again. Therefore, for $r \in (0,1]$ small enough, integrating over $x \in \mathrm{D}_r$ the expression in (4.A.5) and putting everything together, we can see that there exist finite constants $C, b > 0$ such that for all ε, $\varepsilon' \in (0,1]$, it holds that

$$\mathbb{E}\left[\int_{\mathrm{D}_r} \mathbb{1}_{\left\{\int_{B(x,r)} e^{\gamma \mathrm{X}_{r\varepsilon}(u) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_{r\varepsilon}(u)^2]} du > r^{\alpha-\beta}\right\}} e^{q\gamma \mathrm{X}_{\varepsilon'}(x) - \frac{1}{2}q^2\gamma^2 \mathbb{E}[\mathrm{X}_{\varepsilon'}(x)^2]} dx\right] \leq C r^b \,.$$

Now, thanks to Proposition 4.2.2, we know that the sequence of measures $(\mu_{q\gamma,\varepsilon'})_{\varepsilon' \in (0,1]}$ converges to $\mu_{q\gamma}$ in probability in the topology of weak convergence of measures. Therefore, thanks to Portmanteau lemma and Fatou's lemma, we get that

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathrm{D}_r} \mathbb{1}_{\left\{\int_{B(x,r)} e^{\gamma \mathrm{X}_{r\varepsilon}(u) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_{r\varepsilon}(u)^2]} du > r^{\alpha-\beta}\right\}} \mu_{q\gamma}(dx)\right] \\
&\qquad \leq \liminf_{n\to\infty} \mathbb{E}\left[\int_{\mathrm{D}_r} \mathbb{1}_{\left\{\int_{B(x,r)} e^{\gamma \mathrm{X}_{r\varepsilon}(u) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_{r\varepsilon}(u)^2]} du > r^{\alpha-\beta}\right\}} e^{q\gamma \mathrm{X}_{\varepsilon'_n}(x) - \frac{1}{2}q^2\gamma^2 \mathbb{E}[\mathrm{X}_{\varepsilon'_n}(x)^2]} dx\right] \,,
\end{aligned}$$

for a sequence $(\varepsilon'_n)_{n\in\mathbb{N}}$ converging to zero. Therefore, since C and b are independent of ε'_n, we have that

$$\mathbb{E}\left[\int_{\mathrm{D}_r} \mathbb{1}_{\left\{\int_{B(x,r)} e^{\gamma \mathrm{X}_{r\varepsilon}(u) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_{r\varepsilon}(u)^2]} du > r^{\alpha-\beta}\right\}} \mu_{q\gamma}(dx)\right] \leq C r^b \,. \tag{4.A.6}$$

Finally, using (4.A.6) and Fatou's lemma again, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathrm{D}_r} \mathbb{1}_{\{\mu_\gamma(B(x,r)) > r^{\alpha-\beta}\}} \mu_{q\gamma}(dx)\right] \\
&\qquad \leq \liminf_{n\to\infty} \mathbb{E}\left[\int_{\mathrm{D}_r} \mathbb{1}_{\left\{\int_{B(x,r)} e^{\gamma \mathrm{X}_{r\varepsilon_n}(u) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_{r\varepsilon_n}(u)^2]} du > r^{\alpha-\beta}\right\}} \mu_{q\gamma}(dx)\right] \\
&\qquad \leq C r^b \,,
\end{aligned}$$

for a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to zero. The conclusion follows since C and b are independent also of ε_n.

Proof of (4.A.2). We shall be brief here since the argument is similar to that used in the proof of (4.A.1). Let $r \in (0,1]$ and $x \in \mathrm{D}_r$, where we recall that we are assuming that $\mathrm{D} = (-1,1)^d$. Fix ε, $\varepsilon' \in (0,1]$ so that $\varepsilon' < r\varepsilon$. Then by Girsanov's theorem (see Lemma A.0.1) and property (P2), we have

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\left\{\int_{B(x,r)} e^{\gamma \mathrm{X}_{r\varepsilon}(u) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_{r\varepsilon}(u)^2]} du < r^{\alpha+\beta}\right\}} e^{q\gamma \mathrm{X}_{\varepsilon'}(x) - \frac{1}{2}q^2\gamma^2 \mathbb{E}[\mathrm{X}_{\varepsilon'}(x)^2]}\right] \\
&\qquad \leq \mathbb{P}\left(e^{-\gamma^2 K|q|} \int_{B(x,r)} \frac{e^{\gamma \mathrm{X}_{r\varepsilon}(u) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_{r\varepsilon}(u)^2]}}{(|u-x| + r\varepsilon)^{q\gamma^2}} du < r^{\alpha+\beta}\right) \,.
\end{aligned} \tag{4.A.7}$$

Now, let $\Omega_r^x \overset{\text{def}}{=} \inf_{u\in B(x,r)} \mathrm{X}_r(u) - \mathrm{X}_r(x)$, then by proceeding in the same way as in the proof of (4.A.1), we can see that the right-hand side of (4.A.7) can be lower bounded as follows

$$\int_{B(x,r)} \frac{e^{\gamma \mathrm{X}_{r\varepsilon}(u) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_{r\varepsilon}(u)^2]}}{(|u-x| + r\varepsilon)^{q\gamma^2}} du$$

[2] More precisely, for a fixed $x \in \mathrm{D}_r$, one can readily verify that the covariance of the field $(\mathrm{X}_{r\varepsilon}(ru+x) - \mathrm{X}_r(ru+x))_{u\in B(0,1)}$ is comparable with that of the ε-regularisation of the d-dimensional exactly scale invariant log-correlated Gaussian field.

$$\geq r^{\alpha} e^{-\frac{1}{2}\gamma^2 K + \gamma(\Omega_r^x + \mathrm{X}_r(x))} \int_{B(0,1)} \frac{e^{\gamma(\mathrm{X}_{r\varepsilon}(ru+x) - \mathrm{X}_r(ru+x)) + \frac{1}{2}\gamma^2 \log \varepsilon}}{(|u|+\varepsilon)^{q\gamma^2}} du \,. \tag{4.A.8}$$

Substituting the last expression of (4.A.8) back into (4.A.7), we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\left\{\int_{B(x,r)} e^{\gamma \mathrm{X}_{r\varepsilon}(u) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_{r\varepsilon}(u)^2]} du < r^{\alpha+\beta}\right\}} e^{q\gamma \mathrm{X}_{\varepsilon'}(x) - \frac{1}{2}q^2\gamma^2 \mathbb{E}[\mathrm{X}_{\varepsilon'}(x)^2]}\right] \\
&\quad \leq \mathbb{P}\left(e^{-\gamma^2 K(1/2+|q|) + \gamma\Omega_r^x} < r^{\frac{\beta}{3}}\right) + \mathbb{P}\left(e^{\gamma \mathrm{X}_r(x)} < r^{\frac{\beta}{3}}\right) \\
&\qquad\qquad + \mathbb{P}\left(\int_{B(0,1)} \frac{e^{\gamma(\mathrm{X}_{r\varepsilon}(ru+x) - \mathrm{X}_r(ru+x)) + \frac{1}{2}\gamma^2 \log \varepsilon}}{(|u|+\varepsilon)^{q\gamma^2}} du < r^{\frac{\beta}{3}}\right) .
\end{aligned}$$

Thanks to Lemmas 4.2.1 and 4.A.3, the conclusion then follows as in the proof of (4.A.1). □

4.A.2 Proofs of Lemmas 4.A.2 and 4.A.3

We finish by proving Lemmas 4.A.2 and 4.A.3. We recall once again that, for simplicity, we are assuming that $\mathrm{D} = (-1,1)^d$.

Proof of Lemma 4.A.2. For $\varepsilon > 0$ and $a \in (0,1]$, we define the quantity

$$U_\varepsilon^a \overset{\text{def}}{=} \mathbb{E}\left[\left(\int_{(-1,1)^d} \frac{e^{\gamma \mathrm{X}_\varepsilon(x) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(x)^2]}}{(|x|+\varepsilon)^{q\gamma^2}} dx\right)^a\right] ,$$

and we need to prove that there exists $\overline{a} \in (0,1]$ such that $U_\varepsilon^{\overline{a}}$ can be uniformly bounded in $\varepsilon > 0$. We split the proof into two parts. In the first part we focus on $q \in (-\sqrt{2d}/\gamma, 0]$, while in the second part we focus on $q \in (0, \sqrt{2d}/\gamma)$.

Case $q \in (-\sqrt{2d}/\gamma, 0]$: Since $|x| + \varepsilon \leq \sqrt{d} + 1$ for all $x \in (-1,1)^d$ and $\varepsilon > 0$, we can write

$$U_\varepsilon^a \leq (\sqrt{d}+1)^{-aq\gamma^2} \mathbb{E}\left[\left(\int_{(-1,1)^d} e^{\gamma \mathrm{X}_\varepsilon(x) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(x)^2]} dx\right)^a\right] \leq C(\sqrt{d}+1)^{-aq\gamma^2} ,$$

where the existence of the constant $C > 0$, independent of $\varepsilon > 0$, follows from Jensen's inequality.

Case $q \in (0, \sqrt{2d}/\gamma)$: If $\varepsilon \in (1/2, 1]$, then we have that

$$U_\varepsilon^a \leq 2^{aq\gamma^2} \mathbb{E}\left[\left(\int_{(-1,1)^d} e^{\gamma \mathrm{X}_\varepsilon(x) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(x)^2]} dx\right)^a\right] \leq C 2^{aq\gamma^2} .$$

where the existence of the constant $C > 0$, independent of $\varepsilon \in (1/2, 1]$, follows from Jensen's inequality. We now assume that $\varepsilon \in (0, 1/2]$. Then there exists a unique $n \in \mathbb{N}$ such that $2^{-n-1} < \varepsilon \leq 2^{-n}$. For all $k \in \{0, 1, \dots, n\}$, we let $Q_k \overset{\text{def}}{=} (-2^{-k}, 2^{-k})^d$. Then by sub-additivity of the function $x \mapsto x^a$ for $a \in (0,1]$, we have that

$$\begin{aligned}
U_\varepsilon^a &\leq \mathbb{E}\left[\left(\int_{Q_n} \frac{e^{\gamma \mathrm{X}_\varepsilon(x) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(x)^2]}}{(|x|+\varepsilon)^{q\gamma^2}} dx\right)^a\right] + \sum_{k=0}^{n-1} \mathbb{E}\left[\left(\int_{Q_k \setminus Q_{k+1}} \frac{e^{\gamma \mathrm{X}_\varepsilon(x) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(x)^2]}}{(|x|+\varepsilon)^{q\gamma^2}} dx\right)^a\right] \\
&\leq 2^{na(q\gamma^2 - d)} \mathbb{E}\left[\left(\int_{(-1,1)^d} \frac{e^{\gamma \mathrm{X}_\varepsilon(2^{-n}x) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(2^{-n}x)^2]}}{(|x|+2^n\varepsilon)^{q\gamma^2}} dx\right)^a\right]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{n-1} 2^{(k+1)aq\gamma^2}\mathbb{E}\left[\left(\int_{Q_k\setminus Q_{k+1}} e^{\gamma \mathrm{X}_\varepsilon(x)-\frac{1}{2}\gamma^2\mathbb{E}[\mathrm{X}_\varepsilon(x)^2]}dx\right)^a\right]\\
&\leq 2^{na(q\gamma^2-d)}\mathbb{E}\left[\left(\int_{(-1,1)^d} e^{\gamma \mathrm{X}_\varepsilon(2^{-n}x)-\frac{1}{2}\gamma^2\mathbb{E}[\mathrm{X}_\varepsilon(2^{-n}x)^2]}dx\right)^a\right]\\
&\quad+2^{ad}\sum_{k=0}^{n-1} 2^{(k+1)a(q\gamma^2-d)}\mathbb{E}\left[\left(\int_{(-1,1)^d} e^{\gamma \mathrm{X}_\varepsilon(2^{-k}x)-\frac{1}{2}\gamma^2\mathbb{E}[\mathrm{X}_\varepsilon(2^{-k}x)^2]}dx\right)^a\right]. \qquad (4.A.9)
\end{aligned}
$$

Now let $k\in\{0,1,\dots,n\}$, then we want to estimate the following quantity

$$
\mathbb{E}\left[\left(\int_{(-1,1)^d} e^{\gamma \mathrm{X}_\varepsilon(2^{-k}x)-\frac{1}{2}\gamma^2\mathbb{E}[\mathrm{X}_\varepsilon(2^{-k}x)^2]}dx\right)^a\right].
$$

Thanks to property (P2), for $x,y\in(-1,1)^d$, there exists $K>0$ such that

$$
\mathbb{E}[\mathrm{X}_\varepsilon(2^{-k}x)\mathrm{X}_\varepsilon(2^{-k}y)]\geq -\log\left(|x-y|+2^k\varepsilon\right)+\log\left(2^k\right)-K,
$$

and therefore, if we let Z and $\Omega_{2^{-k}}$ be two real normal random variables independent from everything else with variance $2K$ and $\log 2^k$, respectively, then

$$
\mathbb{E}[(\mathrm{X}_\varepsilon(2^{-k}x)+Z)(\mathrm{X}_\varepsilon(2^{-k}y)+Z)]\geq\mathbb{E}[(\mathrm{X}_{2^k\varepsilon}(x)+\Omega_{2^{-k}})(\mathrm{X}_{2^k\varepsilon}(y)+\Omega_{2^{-k}})].
$$

Since the function $x\mapsto x^a$ for $a\in(0,1]$ is concave, thanks to Kahane's convexity inequality (see Lemma A.0.2), we have that

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_{(-1,1)^d} e^{\gamma \mathrm{X}_\varepsilon(2^{-k}x)-\frac{1}{2}\gamma^2\mathbb{E}[\mathrm{X}_\varepsilon(2^{-k}x)^2]}dx\right)^a\right]\\
&\quad=e^{\gamma^2aK(1-a)}\mathbb{E}\left[\left(\int_{(-1,1)^d} e^{\gamma\left(\mathrm{X}_\varepsilon(2^{-k}x)+Z\right)-\frac{1}{2}\gamma^2\mathbb{E}\left[\left(\mathrm{X}_\varepsilon(2^{-k}x)+Z\right)^2\right]}dx\right)^a\right]\\
&\quad\leq e^{\gamma^2aK(1-a)}\mathbb{E}\left[\left(\int_{(-1,1)^d} e^{\gamma\left(\mathrm{X}_{2^k\varepsilon}(x)+\Omega_{2^{-k}}\right)-\frac{1}{2}\gamma^2\mathbb{E}\left[\left(\mathrm{X}_{2^k\varepsilon}(x)+\Omega_{2^{-k}}\right)^2\right]}dx\right)^a\right]\\
&\quad=e^{\gamma^2aK(1-a)}\mathbb{E}\left[e^{\gamma a\Omega_{2^{-k}}-\frac{1}{2}a\gamma^2\mathbb{E}\left[\Omega_{2^{-k}}^2\right]}\right]\mathbb{E}\left[\left(\int_{(-1,1)^d} e^{\gamma \mathrm{X}_{2^k\varepsilon}(x)-\frac{1}{2}\gamma^2\mathbb{E}\left[\mathrm{X}_{2^k\varepsilon}(x)^2\right]}dx\right)^a\right]\\
&\quad\leq Ce^{\gamma^2aK(1-a)}2^{\frac{1}{2}\gamma^2ka(a-1)},
\end{aligned}
$$

where the existence of the constant $C>0$, independent of $\varepsilon\in(0,1/2]$, follows from Jensen's inequality. Hence, going back to (4.A.9), letting

$$
s_a\overset{\text{def}}{=}\left(d+\frac{1}{2}\gamma^2-q\gamma^2\right)a-\frac{1}{2}\gamma^2a^2, \qquad (4.A.10)
$$

we obtain that

$$
U_\varepsilon^a\lesssim\sum_{k=0}^{n}2^{-ks_a},
$$

where the implicit constant does not depend on n. It can be easily verified that for each $\gamma^2<2d$ and $q\in(0,\sqrt{2d}/\gamma)$ there exists $\overline{a}\in(0,1]$ such that $s_{\overline{a}}>0$. In particular, this implies that $U_\varepsilon^{\overline{a}}$ is uniformly bounded in $\varepsilon\in[0,1/2)$, which proves the result. □

Proof of Lemma 4.A.3. For every $\varepsilon > 0$, we define

$$U_\varepsilon^{-1} \overset{\text{def}}{=} \mathbb{E}\left[\left(\int_{[-1/2,1/2]^d} \frac{e^{\gamma \mathrm{X}_\varepsilon(x) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(x)^2]}}{(|x|+\varepsilon)^{q\gamma^2}} dx\right)^{-1}\right] .$$

We need to prove that U_ε^{-1} can be uniformly bounded in $\varepsilon > 0$. We split the proof into two cases.

Case $q \in [0, \sqrt{2d}/\gamma)$**.** In this case, since $|x| + \varepsilon \leq \sqrt{d/2} + 1$ for all $x \in [-1/2, 1/2]^d$ and $\varepsilon > 0$, we can write

$$\sup_{\varepsilon>0} U_\varepsilon^{-1} \leq (\sqrt{d/2}+1)^{q\gamma^2} \sup_{\varepsilon>0} \mathbb{E}\left[\left(\int_{[-1/2,1/2]^d} e^{\gamma \mathrm{X}_\varepsilon(x) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(x)^2]} dx\right)^{-1}\right] \leq C(\sqrt{d/2}+1)^{q\gamma^2} ,$$

where the existence of the constant $C > 0$, independent of $\varepsilon > 0$, follows from Proposition 4.2.3.

Case $q \in (-\sqrt{2d}/\gamma, 0)$**.** In this case, we can simply proceed as follows

$$U_\varepsilon^{-1} \leq \mathbb{E}\left[\left(\int_{B(1/4,1/8)} \frac{e^{\gamma \mathrm{X}_\varepsilon(x) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(x)^2]}}{(|x|+\varepsilon)^{q\gamma^2}} dx\right)^{-1}\right] ,$$

and since $|x| + \varepsilon \geq 1/8$ for all $x \in B(1/4, 1/8)$ and $\varepsilon > 0$, we can write

$$\sup_{\varepsilon>0} U_\varepsilon^{-1} \leq (1/8)^{-q\gamma^2} \sup_{\varepsilon>0} \mathbb{E}\left[\left(\int_{B(1/4,1/8)} e^{\gamma \mathrm{X}_\varepsilon(x) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(x)^2]} dx\right)^{-1}\right] \leq C(1/8)^{-q\gamma^2} ,$$

and also in this case the existence of the constant $C > 0$, independent of $\varepsilon > 0$, follows from Proposition 4.2.3. □

4.B Finiteness of positive moments of ν_γ

In this section, we adopt the same notation used in Section 4.5. The main goal is to prove that the moments of order $q \in (0, 4/\gamma^2)$ of the measure ν_γ, defined in (4.5.4), are uniformly bounded. The techniques used in the proof of this result are inspired from [BP24, Section 3]. Moreover, we mention that this result has been proved in the planar case in [Ber15, Subsection 3.2], but the proof does not trivially generalise to all dimensions. Before stating and proving the result, it may be worth recalling that the measure ν_γ can be interpreted as the weak limit of the sequence of approximated measures $(\nu_{\gamma,\varepsilon})_{\varepsilon>0}$ defined in (4.5.5).

Proposition 4.B.1. *Let $\gamma^2 < 4$ and $q \in (0, 4/\gamma^2)$. Then it holds that*

$$\sup_{\varepsilon>0} \mathbb{E} \otimes \mathbf{E}\left[\left(\int_0^\tau e^{\gamma \mathrm{X}_\varepsilon(B_s) - \frac{1}{2}\mathbb{E}[\mathrm{X}_\varepsilon(B_s)^2]} ds\right)^q\right] < \infty ,$$

where we recall that τ denotes the first exit time of B from D.

Proof. As in the proof of Proposition 4.5.6, thanks to Kahane's convexity inequality (see Lemma A.0.2), we can assume X to be the d-dimensional exactly scale invariant field. Moreover, without loss of generality, we can consider $q > 1$. In order to keep the notation not too cumbersome, we will give the proof only for

$q \in (1, 2 \wedge 4/\gamma^2)$. Notice that this is not restrictive if $\gamma^2 \in [2, 4)$. The case in which $\gamma^2 < 2$ and $q \in [2, 4/\gamma^2)$ is discussed at the end.

For simplicity, we consider the case $\mathrm{D} = (0,1)^d$. We fix $\varepsilon > 0$, then letting $a = q - 1 \in (0,1)$, using Fubini's theorem, Girsanov's theorem (see Lemma A.0.1), and the analogue of property (P2) for the exactly scale invariant field X_ε, we obtain that

$$\begin{aligned}
&\mathbb{E} \otimes \mathbf{E}\left[\left(\int_0^\tau e^{\gamma \mathrm{X}_\varepsilon(B_s) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_s)^2]} ds\right)^q\right] \\
&\quad = \mathbf{E}\left[\int_0^\tau \mathbb{E}\left[e^{\gamma \mathrm{X}_\varepsilon(B_s) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_s)^2]} \left(\int_0^\tau e^{\gamma \mathrm{X}_\varepsilon(B_r) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_r)^2]} dr\right)^a\right] ds\right] \\
&\quad = \mathbf{E}\left[\int_0^\tau \mathbb{E}\left[\left(\int_0^\tau e^{\gamma \mathrm{X}_\varepsilon(B_r) + \gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_s)\mathrm{X}_\varepsilon(B_r)] - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_r)^2]} dr\right)^a\right] ds\right] \\
&\quad \lesssim \mathbf{E}\left[\int_0^\tau \mathbb{E}\left[\left(\int_0^\tau \frac{e^{\gamma \mathrm{X}_\varepsilon(B_r) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_r)^2]}}{(|B_r - B_s| + \varepsilon)^{\gamma^2}} dr\right)^a\right] ds\right] . \qquad (4.B.1)
\end{aligned}$$

If $\varepsilon \in (1/2, 1]$, then thanks to the concavity of the function $x \mapsto x^a$ for $a \in (0,1)$, using Jensen's inequality, we have that

$$\mathbb{E}\left[\left(\int_0^\tau \frac{e^{\gamma \mathrm{X}_\varepsilon(B_r) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_r)^2]}}{(|B_r - B_s| + \varepsilon)^{\gamma^2}} dr\right)^a\right] \lesssim 2^{a\gamma^2} \mathbb{E}\left[\int_0^\tau e^{\gamma \mathrm{X}_\varepsilon(B_r) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_r)^2]} dr\right]^a = 2^{a\gamma^2} \tau^a .$$

Therefore, substituting this estimate into (4.B.1), we get that

$$\mathbb{E} \otimes \mathbf{E}\left[\left(\int_0^\tau e^{\gamma \mathrm{X}_\varepsilon(B_s) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_s)^2]} ds\right)^q\right] \lesssim 2^{a\gamma^2} \mathbf{E}[\tau^{1+a}] < \infty ,$$

where the last inequality follows from the fact that τ has exponentially decaying tail. We now assume that $\varepsilon \in (0, 1/2]$. Then there exists a unique $n \in \mathbb{N}$ such that $2^{-n-1} < \varepsilon \leq 2^{-n}$. For $x \in \mathbb{R}^d$ and $k \in \{0, 1, \dots, n\}$, we let $Q_k(x)$ be the open box of side length 2^{-k+1} centred at x, i.e., $Q_k(x) = (-2^{-k}, 2^{-k})^d + x$. Then by sub-additivity of the function $x \mapsto x^a$ for $a \in (0,1)$, we have that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^\tau \frac{e^{\gamma \mathrm{X}_\varepsilon(B_r) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_r)^2]}}{(|B_r - B_s| + \varepsilon)^{\gamma^2}} dr\right)^a\right] \\
&\quad \leq \mathbb{E}\left[\left(\int_0^\tau \frac{e^{\gamma \mathrm{X}_\varepsilon(B_r) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_r)^2]}}{(|B_r - B_s| + \varepsilon)^{\gamma^2}} \mathbb{1}_{\{B_r \in Q_n(B_s)\}} dr\right)^a\right] \\
&\qquad + \sum_{k=1}^n \mathbb{E}\left[\left(\int_0^\tau \frac{e^{\gamma \mathrm{X}_\varepsilon(B_r) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_r)^2]}}{(|B_r - B_s| + \varepsilon)^{\gamma^2}} \mathbb{1}_{\{B_r \in Q_{k-1}(B_s) \setminus Q_k(B_s)\}} dr\right)^a\right] \\
&\quad \leq 2^{na\gamma^2} \mathbb{E}\left[\left(\int_0^\tau \frac{e^{\gamma \mathrm{X}_\varepsilon(B_r) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_r^2]}}{(2^n|B_r - B_s| + 2^n\varepsilon)^{\gamma^2}} \mathbb{1}_{\{B_r \in Q_n(B_s)\}} dr\right)^a\right] \\
&\qquad + \sum_{k=1}^n 2^{ka\gamma^2} \mathbb{E}\left[\left(\int_0^\tau e^{\gamma \mathrm{X}_\varepsilon(B_r) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_r)^2]} \mathbb{1}_{\{B_r \in Q_{k-1}(B_s)\}} dr\right)^a\right] \\
&\quad \leq 2^{na\gamma^2} \mathbb{E}\left[\left(\int_0^\tau e^{\gamma \mathrm{X}_\varepsilon(B_r) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_r)^2]} \mathbb{1}_{\{B_r \in Q_n(B_s)\}} dr\right)^a\right] \\
&\qquad + \sum_{k=1}^n 2^{ka\gamma^2} \mathbb{E}\left[\left(\int_0^\tau e^{\gamma \mathrm{X}_\varepsilon(B_r) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_r)^2]} \mathbb{1}_{\{B_r \in Q_{k-1}(B_s)\}} dr\right)^a\right] . \qquad (4.B.2)
\end{aligned}$$

Now let $k \in \{0, 1, \dots, n\}$, then we want to estimate the following quantity

$$\mathbb{E}\left[\left(\int_0^\tau e^{\gamma \mathrm{X}_\varepsilon(B_r) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_r)^2]} \mathbb{1}_{\{B_r \in Q_k(B_s)\}} dr\right)^a\right] .$$

Using the translation and scaling invariance of the field X_ε, Jensen's inequality, and Fubini's theorem, we can proceed as follows

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^\tau e^{\gamma \mathrm{X}_\varepsilon(B_r) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_r)^2]} \mathbb{1}_{\{B_r \in Q_k(B_s)\}} dr\right)^a\right] \\
&\quad = \mathbb{E}\left[\left(\int_0^\tau e^{\gamma \mathrm{X}_\varepsilon(2^{-k}(2^k(B_r - B_s)) + B_s) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(2^{-k}(2^k(B_r - B_s)) + B_s)^2]} \mathbb{1}_{\{2^k(B_r - B_s) \in Q_0(0)\}} dr\right)^a\right] \\
&\quad = \mathbb{E}\left[e^{\gamma a \Omega_{2^{-k}} - \frac{1}{2}\gamma^2 a \mathbb{E}[\Omega_{2^{-k}}^2]} \left(\int_0^\tau e^{\gamma \mathrm{X}_{2^k \varepsilon}(2^k(B_r - B_s)) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_{2^k\varepsilon}(2^k(B_r - B_s))^2]} \mathbb{1}_{\{2^k(B_r - B_s) \in Q_0(0)\}} dr\right)^a\right] \\
&\quad \le 2^{k(\frac{1}{2}\gamma^2 a^2 - \frac{1}{2}\gamma^2 a)} \mathbb{E}\left[\int_0^\tau e^{\gamma \mathrm{X}_{2^k \varepsilon}(2^k(B_r - B_s)) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_{2^k\varepsilon}(2^k(B_r - B_s))^2]} \mathbb{1}_{\{2^k(B_r - B_s) \in Q_0(0)\}} dr\right]^a \\
&\quad = 2^{k(\frac{1}{2}\gamma^2 a^2 - \frac{1}{2}\gamma^2 a)} \left(\int_0^\tau \mathbb{1}_{\{2^k(B_r - B_s) \in Q_0(0)\}} dr\right)^a , \qquad (4.\mathrm{B}.3)
\end{aligned}$$

where we recall that $\Omega_{2^{-k}}$ denotes an independent centred Gaussian random variable with variance $\log(2^k)$. Therefore, plugging estimate (4.B.3) into (4.B.2), we obtain that

$$\mathbb{E}\left[\left(\int_0^\tau \frac{e^{\gamma \mathrm{X}_\varepsilon(B_r) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_r)^2]}}{(|B_r - B_s| + \varepsilon)^{\gamma^2}} dr\right)^a\right] \lesssim \sum_{k=0}^n 2^{k(\frac{1}{2}\gamma^2 a^2 + \frac{1}{2}\gamma^2 a)} \left(\int_0^\tau \mathbb{1}_{\{2^k(B_r - B_s) \in Q_0(0)\}} dr\right)^a .$$

Hence, integrating the previous expression over $s \in [0, \tau]$ and then taking expectation with respect to B, recalling inequality (4.B.1), we have that

$$\begin{aligned}
&\mathbb{E} \otimes \mathbf{E}\left[\left(\int_0^\tau e^{\gamma \mathrm{X}_\varepsilon(B_s) - \frac{1}{2}\gamma^2 \mathbb{E}[\mathrm{X}_\varepsilon(B_s)^2]} ds\right)^q\right] \\
&\qquad \lesssim \sum_{k=0}^n 2^{k(\frac{1}{2}\gamma^2 a^2 + \frac{1}{2}\gamma^2 a)} \mathbf{E}\left[\int_0^\tau \left(\int_0^\tau \mathbb{1}_{\{2^k(B_r - B_s) \in Q_0(0)\}} dr\right)^a ds\right] . \qquad (4.\mathrm{B}.4)
\end{aligned}$$

We now estimate the expectation with respect to B on the right-hand side of the above expression. For all $k \in \{0, 1, \dots, n\}$, applying Jensen's inequality two times, we obtain that

$$\mathbf{E}\left[\int_0^\tau \left(\int_0^\tau \mathbb{1}_{\{2^k(B_r - B_s) \in Q_0(0)\}} dr\right)^a ds\right] \le \mathbf{E}\left[\tau^{\frac{1-a}{a}} \int_0^\tau \int_0^\tau \mathbb{1}_{\{2^k(B_r - B_s) \in Q_0(0)\}} dr ds\right]^a .$$

In particular, letting $\alpha \in (0, 2)$, using Markov's inequality, Cauchy–Schwarz's inequality, and proceeding similarly to the proof of Lemma 4.5.1, we have that

$$\begin{aligned}
&\mathbf{E}\left[\tau^{\frac{1-a}{a}} \int_0^\tau \int_0^\tau \mathbb{1}_{\{2^k(B_r - B_s) \in Q_0(0)\}} dr ds\right] \\
&\qquad = \sum_{n \in \mathbb{N}} \mathbf{E}\left[\mathbb{1}_{\{n-1 \le \tau < n\}} \tau^{\frac{1-a}{a}} \int_0^\tau \int_0^\tau \mathbb{1}_{\{2^k(B_r - B_s) \in Q_0(0)\}} dr ds\right] \\
&\qquad \le \sum_{n \in \mathbb{N}} n^{\frac{1-a}{a}} \mathbf{E}\left[\mathbb{1}_{\{n-1 \le \tau < n\}} \int_0^n \int_0^n \mathbb{1}_{\{2^k(B_r - B_s) \in Q_0(0)\}} dr ds\right] \\
&\qquad \le \sum_{n \in \mathbb{N}} n^{\frac{1-a}{a}} \mathbf{P}(\tau \ge n - 1)^{\frac{1}{2}} \int_0^n \int_0^n \mathbf{P}(|B_r - B_s| < 2^{-k+1})^{\frac{1}{2}} dr ds
\end{aligned}$$

$$\leq 2^{-\alpha(k-1)}\mathbf{E}[|B_1|^{-2\alpha}]^{\frac{1}{2}} \sum_{n\in\mathbb{N}} 2n^{\frac{1-a}{a}+1}\mathbf{P}(\tau \geq n-1)^{\frac{1}{2}} \int_0^n r^{-\frac{\alpha}{2}} dr \ ,$$

and the latter series is obviously finite since we chose $\alpha \in (0,2)$ and τ has exponentially decaying tail. Therefore, going back to (4.B.4), we have that

$$\mathbb{E}\left[\left(\int_0^\tau e^{\gamma X_\varepsilon(B_s)-\frac{1}{2}\gamma^2\mathbb{E}[X_\varepsilon(B_s)^2]}ds\right)^q\right] \lesssim \sum_{k=0}^{n} 2^{ka(\frac{1}{2}\gamma^2 a+\frac{1}{2}\gamma^2-\alpha)} \ ,$$

where the implicit constant does not depend on n. To conclude it is sufficient to notice that for $a < 4/\gamma^2 - 1$, there exists $\alpha \in (0,2)$ such that $\gamma^2 a/2 + \gamma^2/2 - \alpha < 0$.

We emphasise that, if $\gamma^2 < 2$, we only showed the existence of moments of order $q \in (0,2)$. If $q \in (0, 4/\gamma^2)$, it suffices to choose $n \in \mathbb{N}$ such that $n \leq q < n+1$. Then, by applying Girsanov's theorem n times in (4.B.1), we obtain an expression similar to the last line of (4.B.1) except that we get an integral with n singularities. This expression can be treated by reproducing the same argument as above and so the details are omitted. □

Chapter 5

Scaling limits of planar maps under the Smith embedding

Abstract. The Smith embedding of a finite planar map with two marked vertices, possibly with conductances on the edges, is a way of representing the map as a tiling of a finite cylinder by rectangles. In this embedding, each edge of the planar map corresponds to a rectangle, and each vertex corresponds to a horizontal segment. Given a sequence of finite planar maps embedded in an infinite cylinder, such that the random walk on both the map and its planar dual converges to Brownian motion modulo time change, we prove that the a priori embedding is close to an affine transformation of the Smith embedding at large scales. By applying this result, we prove that the Smith embeddings of mated-CRT maps with the sphere topology converge to γ-LQG.

5.1 Introduction and main result

The main result of this chapter concerns the scaling limit of general (random) planar maps under the Smith embedding. More precisely, we consider a sequence of finite planar maps, each of which comes with an embedding into an infinite cylinder, referred to as the *a priori embedding*. Under the assumption that both the sequence of maps and their duals satisfy an invariance principle, we show that the a priori embedding is close to an affine transformation of the Smith embedding at large scales. We then apply this result to prove the convergence of the Smith embeddings of mated-CRT maps to LQG. We refer the reader to Section 1.4 for a brief introduction to LQG, planar maps, and their connection.

Throughout the chapter, we always write *weighted planar map* for a planar map with edge conductances. Moreover, throughout the chapter, we allow all of our planar maps to have multiple edges and self-loops. In order to state our main theorem, we need to introduce some notation. Given a planar graph $\mathcal{G}$, we denote the sets of vertices, edges, and faces of $\mathcal{G}$ by $\mathcal{V}\mathcal{G}$, $\mathcal{E}\mathcal{G}$, and $\mathcal{F}\mathcal{G}$, respectively. We consider a doubly marked finite weighted planar map $(\mathcal{G}, c, v_0, v_1)$, where $v_0, v_1 \in \mathcal{V}\mathcal{G}$ are the two marked vertices, and $c = \{c_e\}_{e\in\mathcal{E}\mathcal{G}}$ is a collection of positive unoriented weights called conductances. We assume that we are given a *proper embedding* of the map in the infinite cylinder $\mathcal{C}_{2\pi} \overset{\text{def}}{=} \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$ in the sense of the following definition.

Definition 5.1.1. An embedding of the quadruple $(\mathcal{G}, c, v_0, v_1)$ in the infinite cylinder $\mathcal{C}_{2\pi}$ is said to be *proper* if:

(a) the edges in $\mathcal{E}\mathcal{G}$ are continuous and do not cross;

(b) the graph $\mathcal{G}$ is connected;

(c) the two marked vertices v_0 and v_1 are mapped to $-\infty$ and $+\infty$, respectively.

We observe that, if $(\mathcal{G}, c, v_0, v_1)$ is properly embedded in $\mathcal{C}_{2\pi}$, then the set of unmarked vertices $\mathcal{V}\mathcal{G}$ is contained in $\mathcal{C}_{2\pi}$; each edge in $\mathcal{E}\mathcal{G}$ is a curve in $\mathcal{C}_{2\pi}$ that does not cross any other edge, except possibly at its endpoints; and each face in $\mathcal{F}\mathcal{G}$ is a connected component in $\mathcal{C}_{2\pi}$ of the complement of the embedded graph $\mathcal{G}$. Since the two marked vertices are mapped to $\pm\infty$, this implies that there exists an infinite face at each end of the cylinder $\mathcal{C}_{2\pi}$. In what follows, we use the convention to identify each vertex in $\mathcal{V}\mathcal{G}$ with its a priori embedding, i.e., if $x \in \mathcal{V}\mathcal{G}$ then we view x as a point in $\mathcal{C}_{2\pi}$. Furthermore, for a set $K \subset \mathcal{C}_{2\pi}$, we write

$$\mathcal{V}\mathcal{G}(K) \stackrel{\text{def}}{=} \big\{x \in \mathcal{V}\mathcal{G} : x \in K\big\}.$$

We denote by $(\widehat{\mathcal{G}}, \widehat{c})$ the dual weighted planar graph associated to $(\mathcal{G}, c)$, where the conductance $\widehat{c}_{\widehat{e}}$ of a dual edge $\widehat{e} \in \mathcal{E}\widehat{\mathcal{G}}$ is equal to the resistance of the corresponding primal edge $e \in \mathcal{E}\mathcal{G}$, i.e., we set $\widehat{c}_{\widehat{e}} \stackrel{\text{def}}{=} 1/c_e$. We assume that $(\widehat{\mathcal{G}}, \widehat{c})$ is *properly embedded* in the infinite cylinder $\mathcal{C}_{2\pi}$ in the sense of the following definition.

Definition 5.1.2. An embedding of the dual weighted planar graph $(\widehat{\mathcal{G}}, \widehat{c})$ associated to $(\mathcal{G}, c)$ in the infinite cylinder $\mathcal{C}_{2\pi}$ is said to be *proper* if:

(a) every vertex of $\widehat{\mathcal{G}}$ is contained in a face of $\mathcal{G}$;

(b) every edge e of $\mathcal{G}$ is crossed exactly at one point by a single edge $\widehat{e}$ of $\widehat{\mathcal{G}}$ which joins the two faces incident to e.

If an edge $e \in \mathcal{E}\mathcal{G}$ is oriented, the orientation of the corresponding dual edge $\widehat{e} \in \mathcal{E}\widehat{\mathcal{G}}$ can be obtained by rotating e counter-clockwise. As for the primal graph, given a set $K \subset \mathcal{C}_{2\pi}$, we write

$$\mathcal{V}\widehat{\mathcal{G}}(K) \stackrel{\text{def}}{=} \big\{\widehat{x} \in \mathcal{V}\widehat{\mathcal{G}} : \widehat{x} \in K\big\}.$$

5.1.1 Smith embedding

We are now ready to provide a somewhat informal description of the Smith embedding of a given doubly marked finite weighted planar map $(\mathcal{G}, c, v_0, v_1)$. The precise definition will be given in Section 5.2. As mentioned earlier, the Smith embedding of a planar map was first introduced by Brooks, Smith, Stone, and Tutte in [BSST40], and later generalised to infinite planar graphs by Benjamini and Schramm in [BS96].

The Smith embedding of the quadruple $(\mathcal{G}, c, v_0, v_1)$ is constructed by means of a tiling by rectangles of a finite cylinder $\mathcal{C}_\eta \stackrel{\text{def}}{=} \mathbb{R}/\eta\mathbb{Z} \times [0,1]$, where η is a positive number which depends on $(\mathcal{G}, c)$. Each vertex $x \in \mathcal{V}\mathcal{G}$ is represented by a horizontal line segment H_x, each edge $e \in \mathcal{E}\mathcal{G}$ by a rectangle R_e, and each dual vertex $\widehat{x} \in \mathcal{V}\widehat{\mathcal{G}}$ by a vertical line segment $\mathrm{V}_{\widehat{x}}$. In particular, since each edge $e \in \mathcal{E}\mathcal{G}$ corresponds to a rectangle in the tiling, we need to specify four coordinates for each edge. This is done by means of the voltage function $\mathfrak{h} : \mathcal{V}\mathcal{G} \to [0,1]$ and its discrete harmonic conjugate $\mathfrak{w} : \mathcal{V}\widehat{\mathcal{G}} \to \mathbb{R}/\eta\mathbb{Z}$.

The function $\mathfrak{h}$ is the unique function on $\mathcal{V}\mathcal{G}$ which is discrete harmonic on $\mathcal{V}\mathcal{G} \setminus \{v_0, v_1\}$ (with respect to the conductances c) with boundary conditions given by $\mathfrak{h}(v_0) = 0$ and $\mathfrak{h}(v_1) = 1$, i.e.,

$$\mathfrak{h}(x) = \mathbb{P}_x\big(X \text{ hits } v_1 \text{ before } v_0\big), \quad \forall\, x \in \mathcal{V}\mathcal{G},$$

where $\mathbb{P}_x$ denotes the law of the (weighted) random walk X^x on $(\mathcal{G}, c)$ started from x. We refer to Subsection 5.2.3 for more details.

The function $\mathfrak{w}$ is the function on $\mathcal{V}\widehat{\mathcal{G}}$ that satisfies the discrete Cauchy–Riemann equation associated to $\mathfrak{h}$, i.e., it is the function on the set of dual vertices $\mathcal{V}\widehat{\mathcal{G}}$ whose difference at the endpoints of each edge of $\widehat{\mathcal{G}}$ is equal to

the difference of $\mathfrak{h}$ at the endpoints of the corresponding primal edge times its conductance. As we will see in Subsection 5.2.4, the function $\mathfrak{w}$ is only defined modulo $\eta\mathbb{Z}$ and modulo an additive constant that can be fixed by imposing that $\mathfrak{w}$ is equal to zero on a chosen dual vertex. In particular, the choice of the additive constant of $\mathfrak{w}$ fixes the rotation of the cylinder in which the tiling takes place.

Now, we can specify the various objects involved in the definition of the Smith embedding.

- For each edge $e \in \mathcal{E}\mathcal{G}$, the rectangle R_e corresponds to the rectangle on $\mathcal{C}_\eta$ such that the height coordinates of the top and bottom sides are given by the values of $\mathfrak{h}$ at the endpoints of e, and the width coordinates of the left and right sides of R_e are given by the values of $\mathfrak{w}$ at the endpoints of the corresponding dual edge $\widehat{e}$.
- For each vertex $x \in \mathcal{V}\mathcal{G}$, the horizontal segment H_x corresponds to the maximal horizontal segment which lies on the boundaries of all the rectangles corresponding to the edges incident to x.
- For each dual vertex $\widehat{x} \in \mathcal{V}\widehat{\mathcal{G}}$, the vertical segment $\mathrm{V}_{\widehat{x}}$ corresponds to the maximal vertical segment which is tangent with all rectangles corresponding to primal edges surrounding $\widehat{x}$.

We call the map $\mathcal{S} : \mathcal{E}\mathcal{G} \cup \mathcal{V}\mathcal{G} \cup \mathcal{V}\widehat{\mathcal{G}} \to \mathcal{C}_\eta$ such that $\mathcal{S}(e) \overset{\text{def}}{=} \mathrm{R}_e$, $\mathcal{S}(x) \overset{\text{def}}{=} \mathrm{H}_x$, and $\mathcal{S}(\widehat{x}) \overset{\text{def}}{=} \mathrm{V}_{\widehat{x}}$ the *tiling map* associated to the quadruple $(\mathcal{G}, c, v_0, v_1)$. We refer to Figure 5.1 for a diagrammatic illustration of the tiling map associated to a given quadruple $(\mathcal{G}, c, v_0, v_1)$. We define the Smith embedding $\dot{\mathcal{S}}$ associated to the quadruple $(\mathcal{G}, c, v_0, v_1)$ as the function from $\mathcal{V}\mathcal{G}$ to $\mathbb{R}/\eta\mathbb{Z} \times [0,1]$ given by

$$\dot{\mathcal{S}}(x) \overset{\text{def}}{=} \mathrm{mid}(\mathrm{H}_x), \quad \forall\, x \in \mathcal{V}\mathcal{G}, \tag{5.1.1}$$

where $\mathrm{mid}(\mathrm{H}_x)$ corresponds to the middle point of the horizontal line segment H_x[1]. We refer to Subsection 5.2.5 for precise definitions.

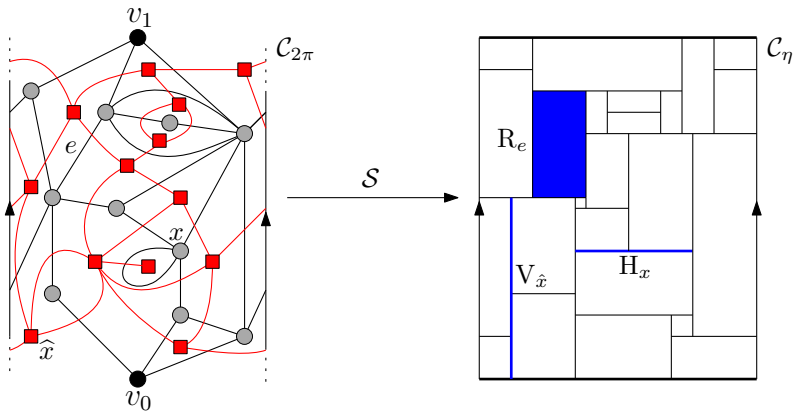


FIGURE 5.1: **Left:** A doubly marked finite weighted planar graph $(\mathcal{G}, c, v_0, v_1)$, drawn in grey, properly embedded in the infinite cylinder $\mathcal{C}_{2\pi}$ with the two marked vertices v_0 and v_1 drawn in black. Drawn in red is the corresponding weighted dual planar graph $(\widehat{\mathcal{G}}, \widehat{c})$, which is also properly embedded in $\mathcal{C}_{2\pi}$. **Right:** The Smith diagram associated to $(\mathcal{G}, c, v_0, v_1)$ constructed via the tiling map $\mathcal{S}$. The blue horizontal segment H_x corresponds to the vertex $x \in \mathcal{V}\mathcal{G}$. The blue vertical segment $\mathrm{V}_{\widehat{x}}$ corresponds to the dual vertex $\widehat{x} \in \mathcal{V}\widehat{\mathcal{G}}$. The blue rectangle R_e corresponds to the edge $e \in \mathcal{E}\mathcal{G}$. In both the left and right figure, the two vertical lines with an arrow are identified with each other.

[1]This definition of the Smith embedding is somewhat arbitrary. Indeed, for each $x \in \mathcal{V}\mathcal{G}$, one can define $\dot{\mathcal{S}}(x)$ to be any arbitrary point inside the horizontal segment H_x.

5.1.2 Main result

To state our main result, we consider a sequence of doubly marked finite weighted planar maps

$$\left\{(\mathcal{G}^n, c^n, v_0^n, v_1^n)\right\}_{n\in\mathbb{N}},$$

and the sequence of associated weighted dual planar graphs $\{(\widehat{\mathcal{G}}^n, \widehat{c}^n)\}_{n\in\mathbb{N}}$. The following assumptions are made.

(H1) (**Cylindrical embedding**) For each $n \in \mathbb{N}$, the quadruple $(\mathcal{G}^n, c^n, v_0^n, v_1^n)$ is properly embedded in the infinite cylinder $\mathcal{C}_{2\pi}$ in the sense of Definition 5.1.1. Furthermore, the associated weighted dual planar graph $(\widehat{\mathcal{G}}^n, \widehat{c}^n)$ is also properly embedded in $\mathcal{C}_{2\pi}$ in the sense of Definition 5.1.2.

(H2) (**Invariance principle on the primal graphs**) For each $n \in \mathbb{N}$, let $(\mathcal{G}^{\dagger,n}, c^{\dagger,n})$ be the lift of $(\mathcal{G}^n, c^n)$ to the universal cover of $\mathcal{C}_{2\pi}$ as specified in Subsection 5.2.2. View the embedded random walk on $(\mathcal{G}^{\dagger,n}, c^{\dagger,n})$, stopped when it traverses an unbounded edge, as a continuous curve in $\mathbb{R}^2$ obtained by piecewise linear interpolation at constant speed. For any $K > 0$ and for any $z \in \mathbb{R} \times [-K, K]$, the law of the random walk on $(\mathcal{G}^{\dagger,n}, c^{\dagger,n})$ started from the vertex $x_z^n \in \mathcal{V}\mathcal{G}^{\dagger,n}$ nearest to z weakly converges as $n \to \infty$ to the law of the Brownian motion on $\mathbb{R}^2$ started from z with respect to the local topology on curves modulo time parameterisation induced by the metric (5.2.3), uniformly over all $z \in \mathbb{R}\times[-K, K]$.

(H3) (**Invariance principle on the dual graphs**) For each $n \in \mathbb{N}$, let $(\widehat{\mathcal{G}}^{\dagger,n}, \widehat{c}^{\dagger,n})$ be the lift of $(\widehat{\mathcal{G}}^n, \widehat{c}^n)$ to the universal cover of $\mathcal{C}_{2\pi}$ as specified in Subsection 5.2.2. View the embedded random walk on $(\widehat{\mathcal{G}}^{\dagger,n}, \widehat{c}^{\dagger,n})$ as a continuous curve in $\mathbb{R}^2$ obtained by piecewise linear interpolation at constant speed. For any $K > 0$ and for any $z \in \mathbb{R} \times [-K, K]$, the law of the random walk on $(\widehat{\mathcal{G}}^{\dagger,n}, \widehat{c}^{\dagger,n})$ started from the vertex $\widehat{x}_z^n \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}$ nearest to z weakly converges as $n \to \infty$ to the law of the Brownian motion on $\mathbb{R}^2$ started from z with respect to the local topology on curves modulo time parameterisation induced by the metric (5.2.3), uniformly over all $z \in \mathbb{R} \times [-K, K]$.

Remark 5.1.3. It is natural to ask whether assumptions (H2) and (H3) are actually equivalent, i.e., if one implies the other. We don't have any strong reason to either believe or dismiss the equivalence of these two assumptions. However, we emphasise that the local structure of the dual graph can differ significantly from the primal one. For example, the Tutte embedding of a graph and its dual can look very different, particularly in cases where one allows vertices of very high degree (which corresponds to having a very large face in the dual graph). Therefore, without any restrictions on the sequences of planar maps, it might be possible to construct examples where the walk and the dual walk behave very differently.

In what follows, given a point $x \in \mathcal{C}_{2\pi}$, we write $\mathrm{Re}(x) \in [0, 2\pi)$ for its horizontal coordinate and $\mathrm{Im}(x) \in \mathbb{R}$ for its height coordinate. Similarly, if $x \in \mathcal{C}_\eta$, then $\mathrm{Re}(x) \in [0, \eta)$ denotes its horizontal coordinate and $\mathrm{Im}(x) \in [0, 1]$ denotes its height coordinate. We are now ready to state our main theorem.

Theorem 5.A. *Consider a sequence $\{(\mathcal{G}^n, c^n, v_0^n, v_1^n)\}_{n\in\mathbb{N}}$ of doubly marked finite weighted planar maps, and let $\{(\widehat{\mathcal{G}}^n, \widehat{c}^n)\}_{n\in\mathbb{N}}$ be the corresponding sequence of dual weighted planar graphs. Assume that assumptions (H1), (H2), (H3) are satisfied. For each $n \in \mathbb{N}$, let $\dot{\mathcal{S}}_n : \mathcal{V}\mathcal{G}^n \to \mathcal{C}_{\eta_n}$ denote the Smith embedding associated with the quadruple $(\mathcal{G}^n, c^n, v_0^n, v_1^n)$ as specified in (5.1.1). There exist sequences $\{c_n^{\mathfrak{h}}\}_{n\in\mathbb{N}}$, $\{b_n^{\mathfrak{h}}\}_{n\in\mathbb{N}}$, $\{b_n^{\mathfrak{w}}\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that, if we define the affine transformation $T_n : \mathcal{C}_{\eta_n} \to \mathcal{C}_{2\pi}$ by*

$$\mathrm{Re}(T_n x) \overset{\text{def}}{=} \left(\frac{2\pi}{\eta_n}\,\mathrm{Re}(x) + b_n^{\mathfrak{w}}\right) \mod(2\pi) \quad \textit{and} \quad \mathrm{Im}(T_n x) \overset{\text{def}}{=} c_n^{\mathfrak{h}}\,\mathrm{Im}(x) + b_n^{\mathfrak{h}}, \qquad \forall\, x \in \mathcal{C}_{\eta_n},$$

then, for all compact sets $K \subset \mathcal{C}_{2\pi}$, it holds that

$$\lim_{n\to\infty} \sup_{x \in \mathcal{V}\mathcal{G}^n(K)} \mathrm{d}_{2\pi}\big(T_n \dot{\mathcal{S}}_n(x), x\big) = 0,$$

where $\mathrm{d}_{2\pi}$ denotes the Euclidean distance on the cylinder $\mathcal{C}_{2\pi}$.

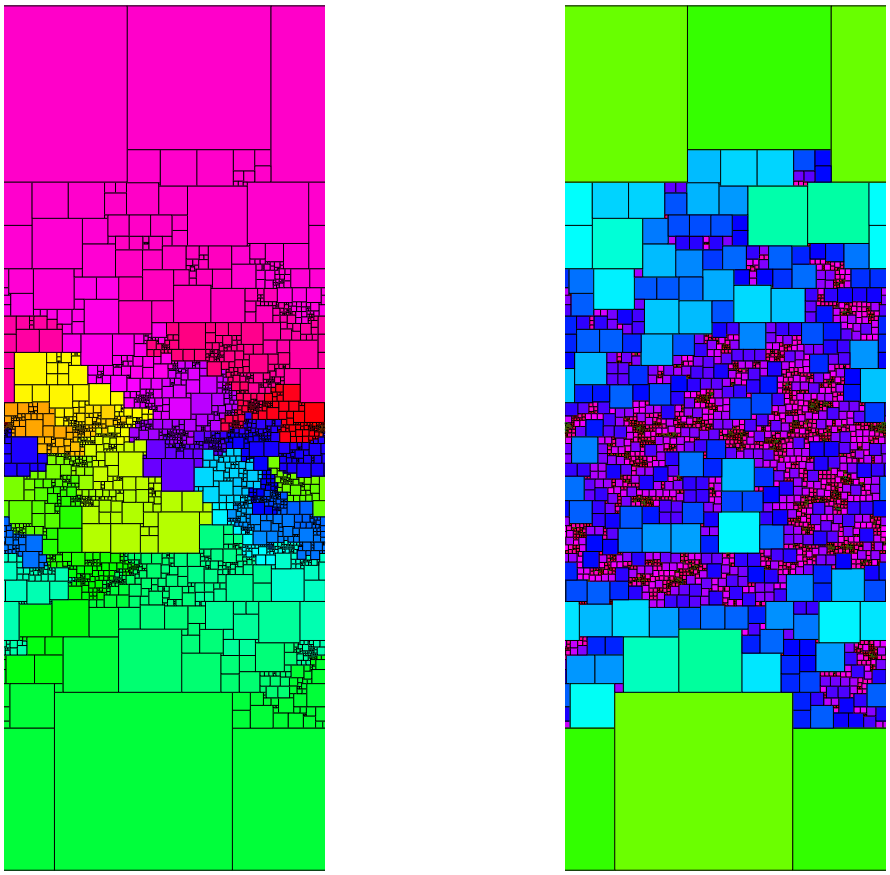


FIGURE 5.2: Consider a planar map $\mathcal{M}$ with unit edge conductances and n edges, and a distinguished spanning tree $\mathcal{T}$. Then $(\mathcal{M}, \mathcal{T})$ determines a quadrangulation $\mathcal{G}$ with n quadrilaterals obtained by replacing each edge of $\mathcal{M}$ by a quadrilateral. Moreover, the path that "snakes" between $\mathcal{T}$ and its dual crosses the edges of $\mathcal{G}$ in order. Shown here is the Smith diagram of an instance of $\mathcal{G}$ corresponding to a uniformly random $(\mathcal{M}, \mathcal{T})$ pair with two random points chosen as roots. **Left:** The squares in the figure are colored according to their order with respect to the space-filling path that snakes between $\mathcal{T}$ and its dual, i.e., based on their position in the cyclic ordering. **Right:** The squares in the figure are colored according to their Euclidean size. It is interesting to compare the figure on the right with the square subdivisions associated to the γ-LQG measure appearing, e.g., in [DS11, Figures 1, 2, 3]. The interested reader can find the Mathematica code used to generate the above simulations in the GitHub repository [BGS23]. Additionally, the repository contains a PDF file with further explanations on how these simulations were produced.

Theorem 5.A tells us that in order to say that the Smith embedding of $(\mathcal{G}^n, c^n, v_0^n, v_1^n)$ is close to a given a priori embedding (up to translation and scaling), we only need to know a certain invariance principle for random walk under the a priori embedding. This result is in some ways not surprising, since it is natural to expect that if a simple random walk (and its dual) approximate Brownian motion, then discrete harmonic functions (and their conjugate duals) should approximate continuum harmonic functions. However, showing that this is actually true in the limit, and that the convergence is to the right continuum harmonic functions, will require some new coupling tricks, which we hope may prove useful in other settings as well. More precisely, the particular statement we obtain is far from obvious a priori, for two main reasons.

- Our hypotheses only concern the *macroscopic* behaviour of the random walk on $(\mathcal{G}^n, c^n)$ in the *bulk* of the cylinder. We do not need any hypotheses about how the random walk behaves when it gets close to the marked vertices v_0^n and v_1^n. This may seem surprising at first glance since one could worry, e.g., that the structure of $(\mathcal{G}^n, c^n)$ in small neighbourhoods of v_0^n and v_1^n makes it much easier for random walk to hit v_0^n than for it to hit v_1^n, and so the height coordinate function $\mathfrak{h}^n$ is close to zero on all of $\mathcal{V}\mathcal{G}^n$. What allows us to get around this is the scaling and translation sequences $\{c_n^{\mathfrak{h}}\}_{n\in\mathbb{N}}$ and $\{b_n^{\mathfrak{h}}\}_{n\in\mathbb{N}}$. We refer to Subsection 5.4.1 for more details.
- The width coordinate $\mathfrak{w}^n$ is discrete harmonic but does not admit a simple direct description in terms of the random walk on $(\widehat{\mathcal{G}}^n, \widehat{c}^n)$. For this reason, a fair amount of work is required to get from the invariance principle for this random walk to a convergence statement for $\mathfrak{w}^n$. We refer to Subsection 5.4.2 for more details.

We remark that the Smith embedding can be very far from the identity near the ends of the cylinder: what is interesting, and perhaps surprising, is the generality in which we can show that the "bad behaviour" gets "smoothed out" in the middle of the cylinder. This is apparent in the simulations presented in Figure 5.2.

As we will discuss in the next section, one application of Theorem 5.A is the convergence of the mated-CRT map with the sphere topology to LQG under the Smith embedding. More generally, Theorem 5.A reduces the problem of proving the convergence to LQG under the Smith embeddings for other types of random planar maps to the problem of finding some a priori embeddings of the map and its dual under which the counting measure on vertices converges to the LQG measure and the random walk on the map converges to Brownian motion modulo time parametrisation.

5.1.3 Application to mated-CRT maps

Mated-CRT maps are a one-parameter family of random planar maps constructed and studied, e.g., in [GHS19, GMS19, DMS21, GMS21]. The mated-CRT maps are parameterised by a real parameter $\gamma \in (0,2)$ and are in the universality class of γ-LQG. In this chapter, we will be interested in mated-CRT maps with the sphere topology. For each $n \in \mathbb{N}$ and $\gamma \in (0,2)$, the *n-mated-CRT map with the sphere topology* is the random planar triangulation $\mathcal{G}^n$ with vertex set given by

$$\mathcal{V}\mathcal{G}^n \overset{\text{def}}{=} \frac{1}{n}\mathbb{Z} \cap (0,1],$$

and an edge set defined by means of a condition involving a pair of linear Brownian motions. More precisely, consider a two dimensional Brownian motion (L,R) with covariance matrix given by

$$\mathbb{V}\text{ar}(L_t) = \mathbb{V}\text{ar}(R_t) = |t|, \qquad \mathbb{C}\text{ov}(L_t, R_t) = -\cos\left(\frac{\pi\gamma^2}{4}\right)|t|, \tag{5.1.2}$$

and conditioned to stay in the first quadrant for one unit of time and end up at $(0,0)$, i.e., (L,R) is an excursion.

Then, two vertices $x_1, x_2 \in \mathcal{V}\mathcal{G}^n$ are connected by an edge if and only if either

$$\begin{aligned} &\max\left\{\inf_{t\in[x_1-1/n,x_1]} L_t, \inf_{t\in[x_2-1/n,x_2]} L_t\right\} \leq \inf_{t\in[x_1,x_2-1/n]} L_t, \quad \text{or} \\ &\max\left\{\inf_{t\in[x_1-1/n,x_1]} R_t, \inf_{t\in[x_2-1/n,x_2]} R_t\right\} \leq \inf_{t\in[x_1,x_2-1/n]} R_t. \end{aligned} \tag{5.1.3}$$

The vertices in $\mathcal{V}\mathcal{G}^n$ are connected by two edges if $|x_1 - x_2| \neq 1/n$ and both the conditions in (5.1.3) hold. We observe that the condition for L in (5.1.3) is equivalent to the existence of a horizontal line segment below the graph of L whose end points are of the form (t_1, L_{t_1}) and (t_2, L_{t_2}) for $t_1 \in [x_1 - 1/n, x_1]$ and $t_2 \in [x_2 - 1/n, x_2]$, and similarly for R. This allows us to give an equivalent, more geometric, version of the definition of $\mathcal{G}^n$. In particular, this procedure assigns a natural planar map structure to the mated-CRT map $\mathcal{G}^n$, under which it is a triangulation. We refer to Figure 5.3 for a diagrammatic explanation of this procedure.

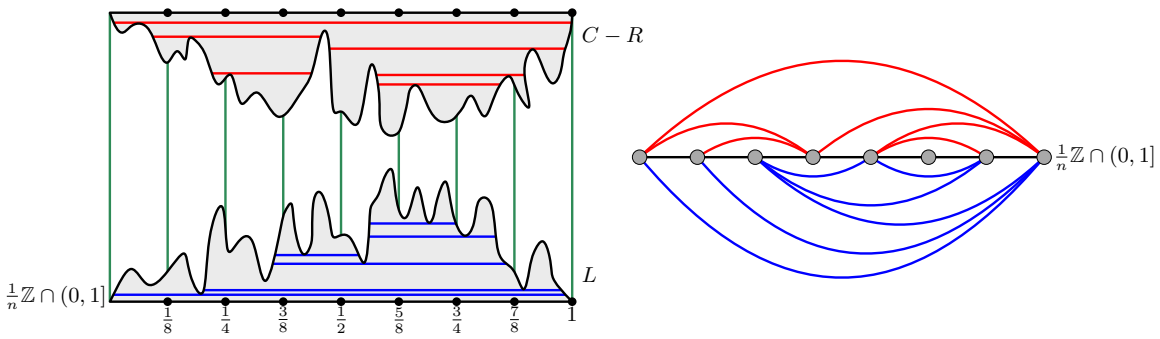


FIGURE 5.3: **Left:** A diagram showing the construction of the mated-CRT map with sphere topology and $n = 8$ vertices. To geometrically construct the mated-CRT map $\mathcal{G}^n$, we draw the graphs of L and $C - R$ with a chosen large constant $C > 0$ to ensure that the graphs do not intersect. The region between the graphs is then divided into vertical strips. Each strip corresponds to the vertex $x \in \mathcal{V}\mathcal{G}^n$ which is the horizontal coordinate of its rightmost point. Two vertices $x_1, x_2 \in \mathcal{V}\mathcal{G}^n$ are connected by an edge if and only if their respective vertical strips are connected by a horizontal line segment that is either below the graph of L or above the graph of $C - R$. For each pair of vertices for which the condition holds for L (resp. $C - R$), we have drawn the lowest (resp. highest) segment which joins the corresponding vertical strips. We note that consecutive vertices are always connected by an edge. **Right:** The graph $\mathcal{G}^n$ can be represented in the plane by connecting two vertices $x_1, x_2 \in \mathcal{V}\mathcal{G}^n$ with an arc below (resp. above) the real line if their vertical strips are connected by a horizontal segment below (resp. above) the graph of L (resp. $C - R$). Additionally, each pair of consecutive vertices in $\mathcal{V}\mathcal{G}^n$ is connected by an edge. This representation gives $\mathcal{G}^n$ a planar map structure under which it is a triangulation. A similar illustration was shown in [GMS21].

In [GMS21], Gwynne, Miller, and Sheffield proved that the Tutte embeddings of mated-CRT maps with the disk topology converge to γ-LQG. Thanks to our main theorem, we can prove an analogous result for the Smith embeddings of mated-CRT maps with the sphere topology. More precisely, for each $n \in \mathbb{N}$, pick two marked vertices $v_0^n, v_1^n \in \mathcal{V}\mathcal{G}^n$[2]. Then, we can conformally map the sphere into the infinite cylinder $\mathcal{C}_{2\pi}$ so that the marked points are mapped to $\pm\infty$, and $(\mathcal{G}^n, v_0^n, v_1^n)$[3] is properly embedded in $\mathcal{C}_{2\pi}$.

Theorem 5.B (Convergence of mated-CRT map)**.** *Fix $\gamma \in (0, 2)$ and let $\{(\mathcal{G}^n, v_0^n, v_1^n)\}_{n\in\mathbb{N}}$ be the sequence of doubly marked n-mated CRT map with the sphere topology embedded in $\mathcal{C}_{2\pi}$ as specified above. There exists a sequence of random affine transformations $\{T_n\}_{n\in\mathbb{N}}$ from $\mathcal{C}_{\eta_n}$ to $\mathcal{C}_{2\pi}$ of the form specified in the statement of Theorem 5.A such that, if we let μ_n be the push-forward with respect to the mapping $z \mapsto T_n z$ of the counting measure on the set $\dot{\mathcal{S}}_n(\mathcal{V}\mathcal{G}^n)$ scaled by $1/n$, then we have the following convergences in probability as $n \to \infty$.*

(a) On each compact subset of $\mathcal{C}_{2\pi}$, the measure μ_n weakly converges to the γ-LQG measure associated to a doubly marked unit area quantum sphere parameterised by $\mathcal{C}_{2\pi}$ in such a way that its marked points are at $\pm\infty$, as defined in [DMS21, Definition 4.21].

(b) On each compact subset of $\mathcal{C}_{2\pi}$, the image under the mapping $z \mapsto T_n z$ of the space-filling path on the Smith embedded mated-CRT map on $\mathcal{C}_{\eta_n}$ obtained from the left-right ordering of the vertices converges

[2]We refer to Section 5.5 for an explanation of how these two marked vertices are selected.
[3]Here, each edge in $\mathcal{E}\mathcal{G}^n$ has unit conductance and so we do not specify the sequence of weights c as in the general case.

uniformly with respect to the two-point compactification topology on the cylinder to space-filling SLE_κ on $\mathcal{C}_{2\pi}$, with $\kappa = 16/\gamma^2$, parameterised by γ-LQG mass.

(c) For $z \in \mathcal{C}_{2\pi}$, let $x_z^n \in \mathcal{VG}^n$ be the vertex nearest to z. The conditional law given $\mathcal{G}^n$ of the image under the mapping $z \mapsto T_n z$ of the simple random walk on the Smith-embedded mated-CRT map started from $\dot{\mathcal{S}}^n(x_z^n)$ and stopped when it hits one of the horizontal segments associated to the marked vertices weakly converges to the law of Brownian motion on $\mathcal{C}_{2\pi}$ started from z, modulo time parametrisation and uniformly over all z in a compact subset of $\mathcal{C}_{2\pi}$.

To conclude, let us point out that item (a) of Theorem 5.B solves [ABL16, Question 1] for the case of mated-CRT maps.

5.1.4 Outline

Most of the chapter is dedicated to proving Theorem 5.A, and it is organised as follows. In the first part of Section 5.2, we provide some background material on weighted planar graphs and the theory of electrical networks. We then move on to the precise construction of the tiling map and the definition of the Smith embedding in Subsection 5.2.5. As mentioned earlier, this is achieved by introducing two harmonic maps: one on the planar map itself and one on the associated dual planar map.

In Section 5.3, we establish and prove several properties of the Smith embedding. The most significant result in this section is Lemma 5.3.17, which heuristically asserts that the conditional expected horizontal winding of a Smith-embedded random walk, given the vertical coordinate, is zero. This property is crucial for proving our main theorem. To prove this intermediate result, we rely on Lemma 5.3.14, which essentially states that the conditional probability, given the vertical coordinate, that the Smith-embedded random walk hits a certain horizontal line segment is proportional to the segment's width. We note that a similar result, though without conditioning on the vertical component, was previously obtained by Georgakopoulos [Geo16, Lemma 6.2] in the context of infinite weighted planar graphs.

Section 5.4 forms the core of this chapter and contains the proof of Theorem 5.A, which is divided into two main parts. In Subsection 5.4.1, we examine the height coordinate function, and in Subsection 5.4.2, we focus on the width coordinate function. Specifically, the main result of Subsection 5.4.1 is Proposition 5.4.2, which roughly states that the height coordinate of the a priori embedding is asymptotically close to an affine transformation of the height coordinate of the Smith embedding. Similarly, the main result of Subsection 5.4.2 is Proposition 5.4.7, which states the analogous fact for the width coordinate. We refer to Subsections 5.4.1 and 5.4.2 for the proof outlines of the height and width coordinate results, respectively. Finally, in Subsection 5.4.3, we show how to combine the results for the height coordinate and width coordinate to prove Theorem 5.A.

In Section 5.5, we provide a brief introduction to the relationship between mated-CRT maps and LQG. We then prove in Subsection 5.5.2 that this family of random planar maps satisfies the assumptions of our main result. Specifically, in Subsection 5.5.3, we apply our result to show that the scaling limit of mated-CRT maps is γ-LQG, thereby proving Theorem 5.B.

Acknowledgements. Part of this work was carried out during the *Probability and Mathematical Physics* ICM satellite conference at Helsinki in Summer 2022. We thank the organisers of this conference for their hospitality.

5.2 Background and setup

5.2.1 Basic definitions

We collect here some notation that will be used throughout the chapter.

5.2.1.1 Basic notation

We write $\mathbb{N}$ for the set of positive integers and $\mathbb{N}_0$ for the set of non-negative integers. Given $n \in \mathbb{N}$ and $j \in \mathbb{N}_0$, we let $[n] \stackrel{\text{def}}{=} \{1, \dots, n\}$ and $[n]_j \stackrel{\text{def}}{=} \{j, \dots, n\}$, Furthermore, for $n \in \mathbb{N}$ and $j \in \mathbb{N}_0$, we write $[x_n]$ to denote the collection of objects $\{x_1, \dots, x_n\}$ and $[x_n]_j$ to denote the collection of objects $\{x_j, \dots, x_n\}$. If a and b are two real numbers, we write $a \lesssim b$ if there exists a constant $C > 0$, independent of the values of a or b and certain other parameters of interest, such that $a \leq Cb$, and we highlight the dependence of the implicit constants when necessary. If a and b are two real numbers depending on a variable x, we write $a = o_x(b)$ if a/b tends to 0 as $x \to \infty$. We use the convention to identify $\mathbb{R}^2$ with $\mathbb{C}$. In particular, given a point $x \in \mathbb{R}^2$, we write $\mathrm{Re}(x)$ (resp. $\mathrm{Im}(x)$) for its horizontal (resp. vertical) coordinate.

5.2.1.2 Metric on curves modulo time parametrisation

For T_1, $T_2 \in [0, \infty]$, let $P_1 : [0, T_1] \to \mathbb{R}^2$ and $P_2 : [0, T_2] \to \mathbb{R}^2$ be two continuous curves defined on possibly different time intervals. We define

$$\mathrm{d}^{\mathrm{CMP}}\big(P_1, P_2\big) \stackrel{\text{def}}{=} \inf_{\phi} \sup_{t \in [0, T_1]} \big|P_1(t) - P_2(\phi(t))\big|, \tag{5.2.1}$$

where the infimum is taken over all increasing homeomorphisms $\phi : [0, T_1] \to [0, T_2]$. In case the intervals $[0, T_1]$ and $[0, T_2]$ are not homeomorphic, we set $\mathrm{d}^{\mathrm{CMP}}(P_1, P_2) = \infty$. It is known that $\mathrm{d}^{\mathrm{CMP}}$ induces a complete metric on the set of curves viewed modulo time parametrisation (see [AB99, Lemma 2.1]).

For curves defined for infinite time, it is convenient to have a local variant of the metric $\mathrm{d}^{\mathrm{CMP}}$. Assume that $P_1 : [0, \infty) \to \mathbb{R}^2$ and $P_2 : [0, \infty) \to \mathbb{R}^2$ are two such curves. Then, for $r > 0$, let $T_{1,r}$ (resp. $T_{2,r}$) be the first exit time of P_1 (resp. P_2) from the ball $B(0, r)$ centred at 0 with radius r, or 0 if the curve starts outside $B(0, r)$. We define

$$\mathrm{d}^{\mathrm{CMP}}_{\mathrm{loc}}\big(P_1, P_2\big) \stackrel{\text{def}}{=} \int_1^\infty e^{-r}\big(1 \wedge \mathrm{d}^{\mathrm{CMP}}\left(P_1|_{[0,T_{1,r}]}, P_2|_{[0,T_{2,r}]}\right)\big)\,\mathrm{d}r. \tag{5.2.2}$$

Remark 5.2.1. We observe that given a sequence $\{P_n\}_{n \in \mathbb{N}}$ of continuous curves defined for infinite time, then $\lim_{n\to\infty} \mathrm{d}^{\mathrm{CMP}}_{\mathrm{loc}}(P_n, P) = 0$, if and only if, for Lebesgue almost every $r > 0$, P_n stopped at its first exit time from $B(0, r)$ converges to P stopped at its first exit time from $B(0, r)$ with respect to the metric (5.2.1).

We also need to introduce a slightly different metric which is more natural when considering the universal cover of the cylinder $\mathcal{C}_{2\pi}$ (see Subsection 5.2.2). In this case, considering the same setting as above, for $R > 0$, let $S_{1,R}$ (resp. $S_{2,R}$) be the first exit time of P_1 (resp. P_2) from $\mathbb{R} \times [-R, R]$, or 0 if the curve starts outside $\mathbb{R} \times [-R, R]$. We define

$$\mathrm{d}^{\mathrm{CMP}}_{\mathrm{loc,univ}}\big(P_1, P_2\big) := \int_1^\infty e^{-R}\big(1 \wedge \mathrm{d}^{\mathrm{CMP}}\left(P_1|_{[0,S_{1,R}]}, P_2|_{[0,S_{2,R}]}\right)\big)\,\mathrm{d}R. \tag{5.2.3}$$

5.2.1.3 Graph notation

Given a finite planar graph $\mathcal{G}$, besides the notation related to $\mathcal{G}$ specified in the introduction, we need to introduce some further nomenclature. In particular, in what follows, we use $e \in \mathcal{EG}$ to denote both oriented and unoriented edges. An oriented edge $e \in \mathcal{EG}$ is oriented from its tail e^- to its head e^+. Furthermore, given a vertex $x \in \mathcal{VG}$, we write $\mathcal{VG}(x)$ for the set of vertices y adjacent to x, i.e., such that there exists an edge connecting x to y. For a vertex $x \in \mathcal{VG}$, we denote by $\mathcal{EG}(x)$ the set of edges in $\mathcal{EG}$ incident to x. For a fixed orientation of the edges in $\mathcal{EG}(x)$, we let $\mathcal{EG}^{\downarrow}(x)$ (resp. $\mathcal{EG}^{\uparrow}(x)$) be the set of edges in $\mathcal{EG}(x)$ with heads (resp. tails) equal to x. Similar notation will also be used for the dual planar graph $\widehat{\mathcal{G}}$.

Metric graph. We will need to consider the metric space associated to a planar graph $\mathcal{G}$ which can be canonically built as follows. For each edge $e \in \mathcal{EG}$, we choose an arbitrary orientation of e and we let I_e be an isometric copy of the real unit interval $[0, 1]$. We define the metric space $\mathbb{G}$ associated with $\mathcal{G}$ to be the quotient of $\cup_{e\in\mathcal{EG}} I_e$ where we identify the endpoints of I_e with the vertices e^- and e^+, and we equip it with the natural path metric $\mathrm{d}^{\mathbb{G}}$. More precisely, for two points x, y lying on an edge of $\mathcal{G}$, we define $\mathrm{d}^{\mathbb{G}}(x, y)$ to be the Euclidean distance between x and y. For points x, y lying on different edges, we use the metric given by the length of the shortest path between the two points, where distances are measured along the edges using the Euclidean distance. We can also define the dual metric graph $\widehat{\mathbb{G}}$, and the associated metric $\mathrm{d}^{\widehat{\mathbb{G}}}$, in a similar way.

5.2.2 Universal cover

The concept of universal cover of a graph will play an important role in our analysis. If $\mathcal{G}$ is a graph embedded in the infinite cylinder $\mathcal{C}_{2\pi}$, then there exists a canonical way to define its lift $\mathcal{G}^\dagger$ to the universal covering space of $\mathcal{C}_{2\pi}$. More precisely, consider the universal cover $(\mathbb{R}^2, \mathrm{\sigma}_{2\pi})$ of $\mathcal{C}_{2\pi}$, where the covering map $\mathrm{\sigma}_{2\pi} : \mathbb{R}^2 \to \mathcal{C}_{2\pi}$ is defined by

$$\mathrm{\sigma}_{2\pi}(t, x) \stackrel{\text{def}}{=} \big(e^{it}, x\big), \quad \forall\, (t, x) \in \mathbb{R}^2. \tag{5.2.4}$$

Then, the lifted graph $\mathcal{G}^\dagger$ can be constructed by taking every lift of every vertex and every edge of $\mathcal{G}$ in $\mathcal{C}_{2\pi}$ to the covering space $\mathbb{R}^2$. We denote by $\mathcal{VG}^\dagger$ and $\mathcal{EG}^\dagger$ the set of vertices and edges of the lifted graph $\mathcal{G}^\dagger$, respectively. Moreover, we can also construct the lift of the dual graph $\widehat{\mathcal{G}}$ to the universal covering space $\mathbb{R}^2$ in a similar way, and we denote it by $\widehat{\mathcal{G}}^\dagger$. We adopt the following notational convention: if $x \in \mathcal{VG}$ is a vertex, then we denote by $\mathbf{x} \in \mathcal{VG}^\dagger$ a lift of x; if $e \in \mathcal{EG}$ is an edge, then we denote by $\mathbf{e} \in \mathcal{EG}^\dagger$ a lift of e; if $\widehat{x} \in \mathcal{V}\widehat{\mathcal{G}}$ is a dual vertex, then we denote by $\widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^\dagger$ a lift of $\widehat{x}$.

Moreover, if $(\mathcal{G}, c)$ is a finite weighted planar graph embedded in $\mathcal{C}_{2\pi}$, we can naturally assign to each lifted edge $\mathbf{e}$ the conductance $c^\dagger_{\mathbf{e}} \stackrel{\text{def}}{=} c_e$, and we denote by $(\mathcal{G}^\dagger, c^\dagger)$ the lifted weighted graph. By definition, the lifted graph $\mathcal{G}^\dagger$ is periodic in the sense that if $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathcal{VG}^\dagger$ are two points in $\mathbb{R}^2$ such that $\mathrm{Im}(\mathbf{x}_1) = \mathrm{Im}(\mathbf{x}_2)$ and $|\,\mathrm{Re}(\mathbf{x}_1) - \mathrm{Re}(\mathbf{x}_2)| \in \mathbb{N}_0$, then $\mathrm{\sigma}_{2\pi}(\mathbf{x}_1) = \mathrm{\sigma}_{2\pi}(\mathbf{x}_2)$. Finally, for a set $K \subset \mathbb{R}^2$, we write

$$\mathcal{VG}^\dagger(K) \stackrel{\text{def}}{=} \big\{\mathbf{x} \in \mathcal{VG}^\dagger \,:\, \mathbf{x} \in K\big\}, \qquad \mathcal{V}\widehat{\mathcal{G}}^\dagger(K) \stackrel{\text{def}}{=} \big\{\widehat{\mathbf{x}} \in \mathcal{VG}^\dagger \,:\, \widehat{\mathbf{x}} \in K\big\}.$$

Before proceeding, we recall the following simple result. An oriented path in $\mathcal{G}$ is a collection of oriented edges $e_1 \cdots e_n$ in $\mathcal{EG}$ such that $e_j^+ = e_{j+1}^-$, for all $j \in [n-1]$. Furthermore, if also $e_n^+ = e_1^-$, then $e_1 \cdots e_n$ is called an oriented loop.

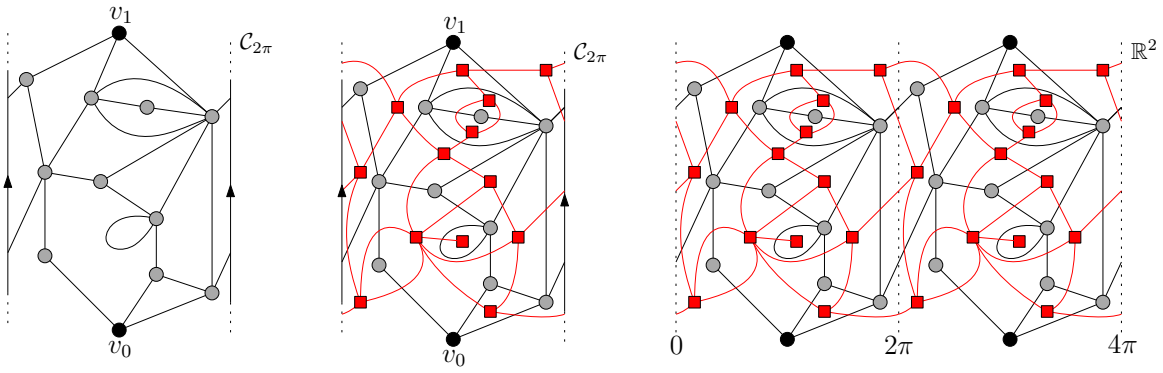


FIGURE 5.4: **Left:** A doubly marked finite weighted planar graph $(\mathcal{G}, c, v_0, v_1)$, drawn in grey, properly embedded in the infinite cylinder $\mathcal{C}_{2\pi}$ with the two marked vertices v_0 and v_1 drawn in black. **Middle:** The same doubly marked finite weighted planar graph as in the left figure together with its dual planar weighted graph $(\widehat{\mathcal{G}}, \widehat{c})$, drawn in red, properly embedded in $\mathcal{C}_{2\pi}$. **Right:** A portion of the lifted weighted graph $(\mathcal{G}^\dagger, c^\dagger)$, drawn in grey, and the associated dual lifted graph $(\widehat{\mathcal{G}}^\dagger, \widehat{c}^\dagger)$, drawn in red. In both the left and middle figure, the two vertical lines with an arrow are identified with each other.

Lemma 5.2.2. *Let $e_1 \cdots e_n$ be an oriented path in $\mathcal{G}$. Let $\mathbf{e}_1$ be a lift of e_1 to the lifted graph $\mathcal{G}^\dagger$, then there exists a unique path $\mathbf{e}_1 \cdots \mathbf{e}_n$ in $\mathcal{G}^\dagger$ such that $\mathbf{e}_j$ is a lift of e_j, for all $j \in [n]_2$.*

The main advantage of working in the universal cover of the cylinder is that we can keep track of the winding of paths.

Definition 5.2.3. Let $0 \le t_1 < t_2$, consider a path $P : [t_1, t_2] \to \mathcal{C}_{2\pi}$, and let $\mathbf{P} : [t_1, t_2] \to \mathbb{R}^2$ be a lift of P to the universal cover. We define the winding of P by letting

$$\mathrm{wind}_{2\pi}(P) \stackrel{\text{def}}{=} \frac{\mathrm{Re}(\mathbf{P}(t_2)) - \mathrm{Re}(\mathbf{P}(t_1))}{2\pi}.$$

We say that P winds around the cylinder if $|\,\mathrm{wind}_{2\pi}(P)| \ge 1$. We say that P does a noncontractible loop around the cylinder if there exist times $t_1 \le s_1 < s_2 \le t_2$ such that $P|_{[s_1,s_2]}$ winds around the cylinder and $P(s_1) = P(s_2)$.

5.2.3 Random walks and electrical networks

In this subsection, we briefly recall the main concepts in the theory of electrical networks and we refer to [LP16, Nac20] for a complete introduction. Let $(\mathcal{G}, c, v_0, v_1)$ be a doubly marked finite weighted planar graph properly embedded in the infinite cylinder $\mathcal{C}_{2\pi}$ in the sense of Definition 5.1.1 . The conductance of a vertex $x \in \mathcal{V}\mathcal{G}$ is denoted by $\pi(x)$ and it is defined to be the sum of the conductances of all the edges incident to x, i.e.,

$$\pi(x) \stackrel{\text{def}}{=} \sum_{e \in \mathcal{E}\mathcal{G}(x)} c_e, \quad \forall\, x \in \mathcal{V}\mathcal{G}.$$

Random walk. The random walk on $(\mathcal{G}, c)$ is the discrete time Markov chain $X = \{X_n\}_{n \in \mathbb{N}_0}$ with state space $\mathcal{V}\mathcal{G}$ such that, for all $n \in \mathbb{N}_0$,

$$\mathbb{P}\big(X_{n+1} = y \mid X_n = x\big) = \begin{cases} c_{xy}/\pi(x), & y \in \mathcal{V}\mathcal{G}(x), \\ 0, & \text{otherwise.} \end{cases}$$

Given a vertex $x \in \mathcal{VG}$, we write $\mathbb{P}_x$ and $\mathbb{E}_x$ for the law and expectation of X started from x. Moreover, we may write X^x in order to emphasise that the random walk X is started from the vertex $x \in \mathcal{VG}$. If the conductance on every edge of the graph is equal to one, we call the random walk in this case simple random walk. We note that we can define the the random walk $\mathbf{X}^{\mathbf{x}}$ on the lifted weighted planar graph $(\mathcal{G}^\dagger, c^\dagger)$ in the same way.

We emphasise that the random walk $\mathbf{X}^{\mathbf{x}}$ on the lifted weighted planar graph $(\mathcal{G}^\dagger, c^\dagger)$ can be canonically "descended" via the covering map to a random walk X^x on the weighted planar graph $(\mathcal{G}, c)$, where $x = \mathfrak{s}_{2\pi}(\mathbf{x})$ is its starting point.

Estimate on the total variation distance. We now state and prove an elementary lemma for general weighted planar graphs which allows to compare the total variation distance of the exit positions from a set for two random walks started from two distinct points.

Lemma 5.2.4. *Let $(\mathcal{G}, c)$ be a finite weighted planar graph and let $W \subset \mathcal{VG}$. For $x \in \mathcal{VG}$, let X^x be the random walk on $(\mathcal{G}, c)$ started from x and let τ_x be the first time that X^x hits W. Then, for $x, y \in \mathcal{VG} \setminus W$, it holds that*

$$\mathrm{d}^{\mathrm{TV}}\big(X^x_{\tau_x}, X^y_{\tau_y}\big) \leq \mathbb{P}\big(X^x|_{[0,\tau_x]} \text{ does not disconnect } y \text{ from } W\big),$$

where d^{TV} denotes the total variation distance.

Proof. The proof is a variant of [GMS22, Lemma 3.12] with the difference that one should consider a weighted spanning tree instead of a uniform spanning tree of the finite weighted planar graph $(\mathcal{G}, c)$. For the reader's convenience, we gather here a proof. The lemma is a consequence of Wilson's algorithm. Consider the weighted spanning tree $\mathcal{T}$ of the finite weighted planar graph $(\mathcal{G}, c)$, where all vertices of W are wired to a single point. We recall that the weighted spanning tree $\mathcal{T}$ is chosen randomly from among all the spanning trees with probability proportional to the product of the conductances along the edges of the tree. For $x \in \mathcal{VG}$, let L^x be the unique path in $\mathcal{T}$ from x to W. For a path P in $\mathcal{G}$, write $\mathrm{LE}(P)$ for its chronological loop erasure. By Wilson's algorithm (see [LP16, Theorem 4.1]), we can generate the union $L^x \cup L^y$ by using the following procedure.

(a) Run X^y until time τ_y and generate the loop erasure $\mathrm{LE}(X^y|_{[0,\tau^y]})$.

(b) Conditional on $X^y|_{[0,\tau_y]}$, run X^x until the first time $\widetilde{\tau}_x$ that it hits either $\mathrm{LE}(X^y|_{[0,\tau_y]})$ or W.

(c) Set $L^x \cup L^y = \mathrm{LE}(X_y|_{[0,\tau_y]}) \cup \mathrm{LE}(X^x|_{[0,\widetilde{\tau}_x]})$.

Note that $L_y = \mathrm{LE}(X^y|_{[0,\tau_y]})$ in the above procedure. Interchanging the roles of x and y in the above procedure shows that L^x and $\mathrm{LE}(X^x|_{[0,\tau_x]})$ have the same distribution. When constructing $L^x \cup L^y$ as described above, the points at which L^x and L^y hit W coincide if X^x hits $\mathrm{LE}(X^y|_{[0,\tau_y]})$ before reaching W. In particular, this occurs when $X^x|_{[0,\tau_x]}$ disconnects y from W. Thus, there exists a coupling between $\mathrm{LE}(X^x|_{[0,\tau_x]})$ and $\mathrm{LE}(X^y|_{[0,\tau_y]})$, where the probability that these two loop erasures hit W at the same point is at least $\mathbb{P}_x(X|_{[0,\tau_x]}$ disconnects y from $W)$. Now, by observing that $X^x_{\tau_x}$ corresponds to the point at which $\mathrm{LE}(X^x|_{[0,\tau_x]})$ first hit W, and similarly for y in place of x, we obtain the desired result. □

Electrical network. There exists an extremely useful correspondence between random walks and Kirchhoff's theory of electric networks. Let $(\mathcal{G}, c, v_0, v_1)$ be as above and suppose that every edge $e \in \mathcal{EG}$ is made of conducting wires with conductance equals to c_e. Connect a battery between v_1 and v_0 so that the voltage at v_1 is equal to one and the voltage at v_0 is equal to zero. Then certain currents will flow along the edges of the graph establishing the voltage at each vertex $x \in \mathcal{VG} \setminus \{v_0, v_1\}$. An immediate consequence of physical

laws is that the voltage function is harmonic on $\mathcal{VG}$ except at v_0 and v_1. More formally, we have the following definition.

Definition 5.2.5. The voltage function associated to the quadruple $(\mathcal{G}, c, v_0, v_1)$ is the unique function $\mathfrak{h} : \mathcal{VG} \to [0,1]$ such that $\mathfrak{h}(v_0) = 0$, $\mathfrak{h}(v_1) = 1$, and

$$\mathfrak{h}(x) = \frac{1}{\pi(x)} \sum_{y \in \mathcal{VG}(x)} c_{xy}\mathfrak{h}(y), \quad \forall\, x \in \mathcal{VG} \setminus \{v_0, v_1\}.$$

In view of the role that $\mathfrak{h}$ will play in the construction of the Smith embedding, we will also call $\mathfrak{h}$ the *height coordinate function*.

Given an edge $e \in \mathcal{EG}$, we say that e is *harmonically oriented* if $\mathfrak{h}(e^+) \geq \mathfrak{h}(e^-)$. In what follows, unless otherwise specified, we always consider the harmonic orientation of the edges in $\mathcal{EG}$. It is a remarkable fact that the voltage function $\mathfrak{h}$ admits a representation in terms of a random walk X on $(\mathcal{G}, c)$. More precisely, if for all $v \in \mathcal{VG}$, we define τ_v to be the first hitting time of v for X, then one can easily check that

$$\mathfrak{h}(x) = \mathbb{P}_x\big(\tau_{v_1} < \tau_{v_0}\big), \quad \forall\, x \in \mathcal{VG},$$

Moreover, since the voltage function $\mathfrak{h}$ is harmonic on $\mathcal{VG} \setminus \{v_0, v_1\}$, if X^x is a random walk on $(\mathcal{G}, c)$ started from $x \in \mathcal{VG}$ and killed upon reaching the set $\{v_0, v_1\}$, then the process $\mathfrak{h}(X^x)$ is a martingale with respect to the filtration generated by X^x.

Remark 5.2.6. We note that we can canonically lift the voltage function $\mathfrak{h}$ to the lifted weighted graph $(\mathcal{G}^\dagger, c^\dagger)$ by setting $\mathfrak{h}^\dagger : \mathcal{VG}^\dagger \to [0,1]$ as follows

$$\mathfrak{h}^\dagger(\mathbf{x}) := \mathfrak{h}(\sigma_{2\pi}(\mathbf{x})), \quad \forall\, \mathbf{x} \in \mathcal{VG}^\dagger. \tag{5.2.5}$$

Remark 5.2.7. Note that we can naturally extend the definition of the voltage function to a function on the metric graph $\mathbb{G}$ associated to $\mathcal{G}$, i.e., we can define the function $\mathfrak{h} : \mathbb{G} \to [0,1]$. This extension can be done by linearly interpolating the values at the endpoints of every edge $e \in \mathcal{EG}$. More precisely, if $x \in \mathbb{G}$ is a point lying on the harmonically oriented edge $e \in \mathcal{EG}$, then we set

$$\mathfrak{h}(x) \stackrel{\text{def}}{=} (\mathfrak{h}(e^+) - \mathfrak{h}(e^-))\mathrm{d}^{\mathbb{G}}(e^-, x) + \mathfrak{h}(e^-).$$

The flow induced by the voltage function. We finish this subsection by introducing the flow across oriented edges induced by the voltage function $\mathfrak{h}$. We denote this flow by $\nabla\mathfrak{h} : \mathcal{EG} \to \mathbb{R}$ and we define it as follows

$$\nabla\mathfrak{h}(e) \stackrel{\text{def}}{=} c_e\big(\mathfrak{h}(e^+) - \mathfrak{h}(e^-)\big), \quad \forall\, e \in \mathcal{EG}.$$

The flow $\nabla\mathfrak{h}$ satisfies the following well-known properties (see [Nac20, Section 2.2]).

(a) (*Antisymmetry*) For every oriented edge $e \in \mathcal{EG}$, it holds that

$$\nabla\mathfrak{h}(-e) = -\nabla\mathfrak{h}(e),$$

where $-e$ stands for the edge e endowed with opposite orientation.

(b) (*Kirchhoff's node law*) For all $x \in \mathcal{V}\mathcal{G} \setminus \{v_0, v_1\}$, it holds that

$$\sum_{e \in \mathcal{E}\mathcal{G}(x)} \nabla \mathfrak{h}(e) = 0, \tag{5.2.6}$$

where here the orientation of each $e \in \mathcal{E}\mathcal{G}(x)$ is fixed by letting $e^- = x$.

(c) (*Kirchhoff's cycle law*) For every directed cycle $e_1 \cdots e_n$, it holds that

$$\sum_{i=1}^{n} \frac{1}{c_{e_i}} \nabla \mathfrak{h}(e_i) = 0. \tag{5.2.7}$$

We denote the strength of the flow $\nabla \mathfrak{h}$ induced by $\mathfrak{h}$ by setting

$$\mathfrak{n} \stackrel{\text{def}}{=} \sum_{e \in \mathcal{E}\mathcal{G}^{\uparrow}(v_0)} \nabla \mathfrak{h}(e), \tag{5.2.8}$$

where $\mathcal{E}\mathcal{G}^{\uparrow}(v_0)$ denotes the set of harmonically oriented edges in $\mathcal{E}\mathcal{G}$ with tails equal to v_0. Furthermore, thanks to the harmonicity of $\mathfrak{h}$, a simple computation yields that $\mathfrak{n} = \sum_{e \in \mathcal{E}\mathcal{G}^{\downarrow}(v_1)} \nabla \mathfrak{h}(e)$, where $\mathcal{E}\mathcal{G}^{\downarrow}(v_1)$ denotes the set of harmonically oriented edges in $\mathcal{E}\mathcal{G}$ with heads equal to v_1.

5.2.4 Discrete harmonic conjugate

Let $(\mathcal{G}, c, v_0, v_1)$ be a doubly marked finite weighted planar graph properly embedded in the infinite cylinder $\mathcal{C}_{2\pi}$ according to Definition 5.1.1, and let $(\widehat{\mathcal{G}}, \widehat{c})$ be the associated weighted dual planar graph properly embedded in $\mathcal{C}_{2\pi}$ according to Definition 5.1.2. Moreover, let $\mathfrak{h} : \mathcal{V}\mathcal{G} \to [0,1]$ be the voltage function associated with $(\mathcal{G}, c, v_0, v_1)$ as defined in Definition 5.2.5. We want to define the discrete harmonic conjugate function of $\mathfrak{h}$, i.e., the function $\mathfrak{w}$ defined on the set of dual vertices $\mathcal{V}\widehat{\mathcal{G}}$ that satisfies the discrete Cauchy–Riemann equation. More formally, for every directed edge $e \in \mathcal{E}\mathcal{G}$ and its corresponding oriented dual edge $\widehat{e} \in \mathcal{E}\widehat{\mathcal{G}}$, the function $\mathfrak{w}$ should satisfy the following identity

$$\nabla \mathfrak{w}(\widehat{e}) \stackrel{\text{def}}{=} \widehat{c}_{\widehat{e}}\big(\mathfrak{w}(\widehat{e}^+) - \mathfrak{w}(\widehat{e}^-)\big) = \mathfrak{h}(e^+) - \mathfrak{h}(e^-), \tag{5.2.9}$$

where we recall that $\widehat{c}_{\widehat{e}} = 1/c_e$.

To precisely define the function $\mathfrak{w}$ specified above, it will be more convenient to work with the lifted weighted graph $(\mathcal{G}^\dagger, c^\dagger)$ and its dual $(\widehat{\mathcal{G}}^\dagger, \widehat{c}^\dagger)$. More precisely, we consider the lifted voltage function $\mathfrak{h}^\dagger : \mathcal{V}\mathcal{G}^\dagger \to [0,1]$. We fix an arbitrary vertex $\widehat{\mathbf{x}}_0 \in \mathcal{V}\widehat{\mathcal{G}}^\dagger$ on the lifted dual graph, and for every $\widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^\dagger$ we consider a directed path of lifted dual edges $\widehat{\mathbf{e}}_1 \cdots \widehat{\mathbf{e}}_n$ connecting $\widehat{\mathbf{x}}_0$ to $\widehat{\mathbf{x}}$.

Remark 5.2.8. We emphasise that the lifted dual graph $\widehat{\mathcal{G}}^\dagger$ is always connected and so we can find a path connecting $\widehat{\mathbf{x}}_0$ to any $\widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^\dagger$.

We define the function $\mathfrak{w}^\dagger : \mathcal{V}\widehat{\mathcal{G}}^\dagger \to \mathbb{R}$ by setting

$$\mathfrak{w}^\dagger(\widehat{\mathbf{x}}) := \sum_{j=1}^{n} \nabla \mathfrak{h}^\dagger(\mathbf{e}_j), \quad \forall\, \widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^\dagger. \tag{5.2.10}$$

where $\mathbf{e}_j \in \mathcal{EG}^\dagger$ is the oriented primal edge associated to $\widehat{\mathbf{e}}_j$. We call the function $\mathfrak{w}^\dagger$ defined in this way the *lifted discrete harmonic conjugate* function associated to $\mathfrak{h}^\dagger$ with base vertex $\widehat{\mathbf{x}}_0$. The following lemma guarantees that $\mathfrak{w}^\dagger$ is actually well-defined.

Lemma 5.2.9. *For all $\widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^\dagger$, the value $\mathfrak{w}^\dagger(\widehat{\mathbf{x}})$ defined in (5.2.10) does not depend on the choice of the directed path from $\widehat{\mathbf{x}}_0$ to $\widehat{\mathbf{x}}$. Moreover, for any $\widehat{\mathbf{x}}_1$, $\widehat{\mathbf{x}}_2 \in \mathcal{V}\widehat{\mathcal{G}}^\dagger$ such that $\sigma_{2\pi}(\widehat{\mathbf{x}}_1) = \sigma_{2\pi}(\widehat{\mathbf{x}}_2)$, the following relation holds*

$$\frac{\mathfrak{w}^\dagger(\widehat{\mathbf{x}}_1) - \mathfrak{w}^\dagger(\widehat{\mathbf{x}}_2)}{\eta} = \frac{\mathrm{Re}(\widehat{\mathbf{x}}_1) - \mathrm{Re}(\widehat{\mathbf{x}}_2)}{2\pi},$$

where we recall that η denotes the strength of the flow induced by $\mathfrak{h}$ as defined in (5.2.8).

Proof. For the first part of the lemma, the proof is similar to that of [BS96, Lemma 3.2]. In particular, it is sufficient to prove that for any oriented loop $\widehat{\mathbf{e}}_1 \cdots \widehat{\mathbf{e}}_n$ in $\widehat{\mathcal{G}}^\dagger$, it holds that

$$\sum_{j=1}^{n} \big(\mathfrak{w}^\dagger(\widehat{\mathbf{e}}_j^+) - \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_j^-)\big) = 0. \tag{5.2.11}$$

The key observation in [BS96, Lemma 3.2] is that every oriented loop $\widehat{\mathbf{e}}_1 \cdots \widehat{\mathbf{e}}_n$ in $\widehat{\mathcal{G}}^\dagger$ can be written as the disjoint union of simple closed loops and of paths of length two consisting of a single dual edge traversed in both directions. Here, by a simple closed path we mean that $\widehat{\mathbf{e}}_j^+ \neq \widehat{\mathbf{e}}_k^+$ for distinct $j, k \in [n]$ when $n > 2$, while when $n = 2$, we mean that $\widehat{\mathbf{e}}_1 \neq -\widehat{\mathbf{e}}_2$. Therefore, since (5.2.11) obviously holds if the path consists of a single dual edge traversed in both directions, we can assume without loss of generality that $\widehat{\mathbf{e}}_1 \cdots \widehat{\mathbf{e}}_n$ is a simple counter-clockwise oriented closed loop. Let $K \subset \mathbb{R}^2$ be the bounded connected component of $\mathbb{R}^2 \setminus \widehat{\mathbf{e}}_1 \cdots \widehat{\mathbf{e}}_n$. Then, thanks to (5.2.9), it holds that

$$\sum_{j=1}^{n} \big(\mathfrak{w}^\dagger(\widehat{\mathbf{e}}_j^+) - \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_j^-)\big) = \sum_{j=1}^{n} \nabla\mathfrak{h}^\dagger(\mathbf{e}_j) = \sum_{\mathbf{x} \in \mathcal{V}\mathcal{G}^\dagger(K)} \sum_{\mathbf{e} \in \mathcal{E}\mathcal{G}^\dagger(\mathbf{x})} \nabla\mathfrak{h}^\dagger(\mathbf{e}), \tag{5.2.12}$$

where here the orientation of each edge $\mathbf{e} \in \mathcal{E}\mathcal{G}^\dagger(\mathbf{x})$ is fixed by letting $\mathbf{e}^- = \mathbf{x}$. The second equality in (5.2.12) follows from the following argument. Fix $\mathbf{x} \in \mathcal{V}\mathcal{G}^\dagger(K)$ and consider $\mathbf{y} \in \mathcal{V}\mathcal{G}^\dagger(\mathbf{x})$. If $\mathbf{y} \notin K$, then $\mathbf{xy} = \mathbf{e}_j$ for some $j \in [n]$, while if $\mathbf{y} \in K$ then $\nabla\mathfrak{h}^\dagger(\mathbf{xy})$ cancels out with $\nabla\mathfrak{h}^\dagger(\mathbf{yx})$ thanks to the antisymmetry of $\nabla\mathfrak{h}^\dagger$. The term on the right-hand side of (5.2.12) is equal to zero thanks to the boundedness of K and Kirchhoff's node law (5.2.6).

Concerning the second part of the lemma, we can proceed as follows. Let $\widehat{\mathbf{x}}_1$, $\widehat{\mathbf{x}}_2 \in \mathcal{V}\widehat{\mathcal{G}}^\dagger$ be such that $\sigma_{2\pi}(\widehat{\mathbf{x}}_1) = \sigma_{2\pi}(\widehat{\mathbf{x}}_2)$, and let $k \in \mathbb{Z}$ be such that $k = (\mathrm{Re}(\widehat{\mathbf{x}}_2) - \mathrm{Re}(\widehat{\mathbf{x}}_1))/(2\pi)$. Then we need to prove that $\mathfrak{w}^\dagger(\widehat{\mathbf{x}}_2) - \mathfrak{w}^\dagger(\widehat{\mathbf{x}}_1) = \eta k$. In particular, thanks to the first part of the lemma, it is sufficient to consider an arbitrary directed path $\widehat{\mathbf{e}}_1 \cdots \widehat{\mathbf{e}}_n$ in $\widehat{\mathcal{G}}^\dagger$ connecting $\widehat{\mathbf{x}}_1$ to $\widehat{\mathbf{x}}_2$ and show that $\sum_{j=1}^n (\mathfrak{w}^\dagger(\widehat{\mathbf{e}}_j^+) - \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_j^-)) = \eta k$. We assume first that $k = 1$. We can choose the lifted dual edges $[\widehat{\mathbf{e}}_n]$ in such way that $\widehat{\mathbf{e}}_1 \cdots \widehat{\mathbf{e}}_n$ is a simple path oriented from left to right. Now, using an argument similar to the one used above, it is not difficult to see that $\sum_{j=1}^n \nabla\mathfrak{h}^\dagger(\mathbf{e}_j) = \eta$, and so the conclusion follows in this case. Finally, the general case can be obtained easily: we can just "glue" together, by eventually changing the orientation if k is negative, k copies of the path used in the case $k = 1$. □

From the definition (5.2.10) of the function $\mathfrak{w}^\dagger$, it follows that for every oriented edge $\mathbf{e} \in \mathcal{E}\mathcal{G}^\dagger$ and for the associated dual edge $\widehat{\mathbf{e}} \in \mathcal{E}\widehat{\mathcal{G}}^\dagger$, it holds that

$$\nabla\mathfrak{w}^\dagger(\widehat{\mathbf{e}}) \overset{\text{def}}{=} \widehat{c}_{\mathbf{e}}^\dagger(\mathfrak{w}^\dagger(\widehat{\mathbf{e}}^+) - \mathfrak{w}^\dagger(\widehat{\mathbf{e}}^-)) = \mathfrak{h}^\dagger(\mathbf{e}^+) - \mathfrak{h}^\dagger(\mathbf{e}^-),$$

i.e., $\mathfrak{w}^\dagger$ satisfies the discrete Cauchy–Riemann equation. Moreover, an immediate application of Kirchhoff's cycle law (5.2.7) implies that the function $\mathfrak{w}^\dagger$ is harmonic on $\mathcal{V}\widehat{\mathcal{G}}^\dagger$. Thanks to Lemma 5.2.9, we can define the discrete harmonic conjugate function of $\mathfrak{h}$ as follows.

Definition 5.2.10. The discrete harmonic conjugate function of $\mathfrak{h}$ with base vertex $\widehat{x}_0 \in \mathcal{V}\widehat{\mathcal{G}}$ is the unique function $\mathfrak{w} : \mathcal{V}\widehat{\mathcal{G}} \to \mathbb{R}/\eta\mathbb{Z}$ such that $\mathfrak{w}(\widehat{x}_0) = 0$ and

$$\mathfrak{w}(\widehat{x}) = \mathfrak{w}^\dagger(\widehat{\mathbf{x}}) \bmod(\eta), \quad \forall\, \widehat{x} \in \mathcal{V}\widehat{\mathcal{G}},$$

where $\mathfrak{w}^\dagger : \mathcal{V}\mathcal{G}^\dagger \to \mathbb{R}$ is the function defined in (5.2.10) with base vertex an arbitrary lift of $\widehat{x}_0$. In view of the role that $\mathfrak{w}$ will play in the construction of the Smith embedding, we will also call $\mathfrak{w}$ the *width coordinate function*.

Remark 5.2.11. As for the case of the voltage function $\mathfrak{h}$, we can naturally extend the definition of $\mathfrak{w}$ to a function from the dual metric graph $\widehat{\mathbb{G}}$, i.e., $\mathfrak{w} : \widehat{\mathbb{G}} \to \mathbb{R}/\eta\mathbb{Z}$. To be precise, if $\widehat{x} \in \widehat{\mathbb{G}}$ is a point on the edge $\widehat{e} \in \mathcal{E}\widehat{\mathcal{G}}$ and $\mathfrak{w}(\widehat{e}^+) \geq \mathfrak{w}(\widehat{e}^-)$, then we set

$$\mathfrak{w}(\widehat{x}) \stackrel{\text{def}}{=} (\mathfrak{w}(\widehat{e}^+) - \mathfrak{w}(\widehat{e}^-))\mathrm{d}^{\widehat{\mathbb{G}}}(\widehat{e}^-, \widehat{x}) + \mathfrak{w}(\widehat{e}^-).$$

5.2.5 Construction of the Smith embedding

Let $(\mathcal{G}, c, v_0, v_1)$ be a doubly marked finite weighted planar graph properly embedded in the infinite cylinder $\mathcal{C}_{2\pi}$ according to Definition 5.1.1, and let $(\widehat{\mathcal{G}}, \widehat{c})$ be the associated weighted dual planar graph properly embedded in $\mathcal{C}_{2\pi}$ according to Definition 5.1.2. The aim of this subsection is to precisely define the *Smith embedding* of $(\mathcal{G}, c, v_0, v_1)$. As we have already explained in the introduction, the Smith embedding is built in terms of a tiling of a finite cylinder with rectangles in which every edge $e \in \mathcal{E}\mathcal{G}$ corresponds to a rectangle in the tiling, every vertex $x \in \mathcal{V}\mathcal{G}$ corresponds to the maximal horizontal segment tangent with all rectangles corresponding to the edges incident to x, and every dual vertex $\widehat{x} \in \mathcal{V}\widehat{\mathcal{G}}$ corresponds to the maximal vertical segment tangent with all rectangles corresponding to primal edges surrounding $\widehat{x}$. The existence of such tiling was first proven in [BSST40] and then successively extended in [BS96].

The main objects. To precisely define the Smith embedding, it will be more convenient to work with the lifted weighted graph $(\mathcal{G}^\dagger, c^\dagger)$ and its dual $(\widehat{\mathcal{G}}^\dagger, \widehat{c}^\dagger)$. More precisely, we need to consider the lifted voltage function $\mathfrak{h}^\dagger : \mathcal{V}\mathcal{G}^\dagger \to [0,1]$ and its lifted discrete harmonic conjugate function $\mathfrak{w}^\dagger : \mathcal{V}\widehat{\mathcal{G}}^\dagger \to \mathbb{R}$. For every edge $\mathbf{e} \in \mathcal{E}\mathcal{G}^\dagger$, consider its harmonic orientation and let $\widehat{\mathbf{e}} \in \mathcal{E}\widehat{\mathcal{G}}^\dagger$ be the corresponding oriented dual edge. We define the intervals

$$\mathrm{I}_{\mathbf{e}} \stackrel{\text{def}}{=} \big[\mathfrak{w}^\dagger(\widehat{\mathbf{e}}^-), \mathfrak{w}^\dagger(\widehat{\mathbf{e}}^+)\big], \qquad \widehat{\mathrm{I}}_{\widehat{\mathbf{e}}} \stackrel{\text{def}}{=} \big[\mathfrak{h}^\dagger(\mathbf{e}^-), \mathfrak{h}^\dagger(\mathbf{e}^+)\big].$$

Then, we define the rectangle $\mathrm{R}_{\mathbf{e}}$ associated to the edge $\mathbf{e} \in \mathcal{E}\mathcal{G}^\dagger$ by letting

$$\mathrm{R}_{\mathbf{e}} \stackrel{\text{def}}{=} \mathrm{I}_{\mathbf{e}} \times \widehat{\mathrm{I}}_{\widehat{\mathbf{e}}} \subset \mathbb{R} \times [0,1], \quad \forall\, \mathbf{e} \in \mathcal{E}\mathcal{G}^\dagger. \tag{5.2.13}$$

Recalling the definition (5.2.10) of the lifted discrete harmonic conjugate function $\mathfrak{w}^\dagger$, it holds that

$$\mathfrak{w}^\dagger(\widehat{\mathbf{e}}^+) - \mathfrak{w}^\dagger(\widehat{\mathbf{e}}^-) = c_{\mathbf{e}}(\mathfrak{h}^\dagger(\mathbf{e}^+) - \mathfrak{h}^\dagger(\mathbf{e}^-)).$$

Therefore, the aspect ratio of the rectangle $\mathrm{R}_{\mathbf{e}}$ is equal to the conductance $c_{\mathbf{e}}$ of the edge $\mathbf{e} \in \mathcal{EG}^\dagger$. In particular, this implies that if an edge $\mathbf{e} \in \mathcal{EG}^\dagger$ has unit conductance, then $\mathrm{R}_{\mathbf{e}}$ is a square. For a vertex $\mathbf{x} \in \mathcal{VG}^\dagger$, we define the closed horizontal line segment $\mathrm{H}_{\mathbf{x}}$ by setting

$$\mathrm{H}_{\mathbf{x}} \stackrel{\text{def}}{=} \bigcup_{\mathbf{e} \in \mathcal{EG}^{\dagger,\downarrow}(\mathbf{x})} \mathrm{I}_{\mathbf{e}} \times \{\mathfrak{h}^\dagger(\mathbf{x})\} \subset \mathbb{R} \times [0,1], \quad \forall\, \mathbf{x} \in \mathcal{VG}^\dagger, \tag{5.2.14}$$

where $\mathcal{EG}^{\dagger,\downarrow}(\mathbf{x})$ denotes the set of harmonically oriented lifted edges with heads equal to $\mathbf{x}$. Finally, for a dual vertex $\widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^\dagger$, we define the closed vertical line segment $\mathrm{V}_{\widehat{\mathbf{x}}}$ by letting

$$\mathrm{V}_{\widehat{\mathbf{x}}} \stackrel{\text{def}}{=} \bigcup_{\widehat{\mathbf{e}} \in \mathcal{E}\widehat{\mathcal{G}}^{\dagger,\downarrow}(\widehat{\mathbf{x}})} \{\mathfrak{w}^\dagger(\widehat{\mathbf{x}})\} \times \widehat{\mathrm{I}}_{\widehat{\mathbf{e}}} \subset \mathbb{R} \times [0,1], \quad \forall\, \widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^\dagger, \tag{5.2.15}$$

where $\mathcal{E}\widehat{\mathcal{G}}^{\dagger,\downarrow}(\widehat{\mathbf{x}})$ denotes the set of harmonically oriented lifted dual edges with heads equal to $\widehat{\mathbf{x}}$. Thanks to the harmonicity of the lifted height coordinate function, we observe that in the definition of $\mathrm{H}_{\mathbf{x}}$, one can replace $\mathcal{EG}^{\dagger,\downarrow}(\mathbf{x})$ with $\mathcal{EG}^{\dagger,\uparrow}(x)$, and similarly for $\mathrm{V}_{\widehat{\mathbf{x}}}$.

Construction of the tiling. We recall that η denotes the strength of the flow induced by $\mathfrak{h}$ as defined in (5.2.8). We consider the cylinder

$$\mathcal{C}_\eta \stackrel{\text{def}}{=} \mathbb{R}/\eta\mathbb{Z} \times [0,1],$$

where $\mathbb{R}/\eta\mathbb{Z}$ denotes the circle of length η. We let $(\mathbb{R} \times [0,1], \sigma_\eta)$ be the universal cover of $\mathcal{C}_\eta$, where the covering map $\sigma_\eta : \mathbb{R} \times [0,1] \to \mathcal{C}_\eta$ is defined by

$$\sigma_\eta(t,x) \stackrel{\text{def}}{=} \left(e^{i2\pi t/\eta}, x\right), \quad \forall\, (t,x) \in \mathbb{R} \times [0,1]. \tag{5.2.16}$$

We are now ready to define the tiling map. For each $e \in \mathcal{EG}$, $x \in \mathcal{VG}$, and $\widehat{x} \in \mathcal{V}\widehat{\mathcal{G}}$, we define the following objects

$$\mathrm{R}_e \stackrel{\text{def}}{=} \sigma_\eta(\mathrm{R}_{\mathbf{e}}), \qquad \mathrm{H}_x \stackrel{\text{def}}{=} \sigma_\eta(\mathrm{H}_{\mathbf{x}}), \qquad \mathrm{V}_{\widehat{x}} \stackrel{\text{def}}{=} \sigma_\eta(\mathrm{V}_{\widehat{\mathbf{x}}}), \tag{5.2.17}$$

where $\mathbf{e} \in \mathcal{EG}^\dagger$, $\mathbf{x} \in \mathcal{VG}^\dagger$, and $\widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^\dagger$ are lifts of e, x, and $\widehat{x}$, respectively. An immediate consequence of Lemma 5.2.9 is that R_e, H_x, and $\mathrm{V}_{\widehat{x}}$ are well-defined, i.e., they do not depend on the particular choice of the lifts $\mathbf{e}$, $\mathbf{x}$ and $\widehat{\mathbf{x}}$.

The following properties are well-known (see [BS96, Theorem 3.1]).

(a) The collection of rectangles $\{\mathrm{R}_e\}_{e \in \mathcal{EG}}$ constitutes a tiling of $\mathbb{R}/\eta\mathbb{Z} \times [0,1]$, i.e., for each pair of distinct edges $e_1, e_2 \in \mathcal{EG}$, the interiors of the rectangles R_{e_1} and R_{e_2} are disjoint and $\cup_{e \in \mathcal{EG}} \mathrm{R}_e = \mathbb{R}/\eta\mathbb{Z} \times [0,1]$.

(b) For each two distinct edges $e_1, e_2 \in \mathcal{EG}$, the interiors of the vertical sides of the rectangles R_{e_1} and R_{e_2} have a non-trivial intersection only if e_1 and e_2 both lie in the boundary of some common face of $\mathcal{G}$.

(c) Two rectangles intersect along their horizontal (resp. vertical) boundaries if and only if the corresponding primal (resp. dual) edges share an endpoint.

We note that if $e \in \mathcal{EG}$ is such that no current flows through it, i.e., $\mathfrak{h}(e^-) = \mathfrak{h}(e^+)$, then the corresponding rectangle R_e is degenerate and consists only of a single point. We also remark that the existence of the aforementioned tiling was proven by Benjamini and Schramm in [BS96]. Originally, their proof was stated specifically for the case of edges with unit conductance, however, it can be readily extended to our setting.

Definition 5.2.12 (Tiling map)**.** The tiling map associated to the quadruple $(\mathcal{G}, c, v_0, v_1)$ is the map

$$\mathcal{S} : \mathcal{E}\mathcal{G} \cup \mathcal{V}\mathcal{G} \cup \mathcal{V}\widehat{\mathcal{G}} \to \mathbb{R}/\mathfrak{\eta}\mathbb{Z} \times [0,1]$$

such that

$$\mathcal{S}(e) := \mathrm{R}_e, \quad \forall\, e \in \mathcal{E}\mathcal{G}; \qquad \mathcal{S}(x) \stackrel{\text{def}}{=} \mathrm{H}_x, \quad \forall\, x \in \mathcal{V}\mathcal{G}; \qquad \mathcal{S}(\widehat{x}) := \mathrm{V}_{\widehat{x}}, \quad \forall\, \widehat{x} \in \mathcal{V}\widehat{\mathcal{G}},$$

where R_e, H_x, and $\mathrm{V}_{\widehat{x}}$ are as defined in (5.2.17). The image of the tiling map $\mathcal{S}$ is called the Smith diagram associated to $(\mathcal{G}, c, v_0, v_1)$.

We refer to Figure 5.1 for an illustration of the Smith diagram associated to a given quadruple $(\mathcal{G}, c, v_0, v_1)$ with unit conductances.

Remark 5.2.13. Since the height coordinate function $\mathfrak{h}$ can be extended to the metric graph $\mathbb{G}$, we can view each rectangle R_e of the tiling as being foliated into horizontal segments, one for each inner point of the corresponding edge $e \in \mathcal{E}\mathcal{G}$. Similarly, since the width coordinate function $\mathfrak{w}$ can be extended to the dual metric graph $\widehat{\mathbb{G}}$, we can also view each rectangle R_e of the tiling as being foliated into vertical segments, one for each inner point of the corresponding dual edge $\widehat{e} \in \mathcal{E}\widehat{\mathcal{G}}$.

It will be also convenient to introduce the lifted tiling map associated to $(\mathcal{G}, c, v_0, v_1)$ which is the map

$$\mathcal{S}^\dagger : \mathcal{E}\mathcal{G}^\dagger \cup \mathcal{V}\mathcal{G}^\dagger \cup \mathcal{V}\widehat{\mathcal{G}}^\dagger \to \mathbb{R} \times [0,1]$$

such that $\mathcal{S}^\dagger(\mathbf{x}) \stackrel{\text{def}}{=} \mathrm{H}_{\mathbf{x}}$ for each $\mathbf{x} \in \mathcal{V}\mathcal{G}^\dagger$, $\mathcal{S}^\dagger(\mathbf{e}) := \mathrm{R}_{\mathbf{e}}$ for each $\mathbf{e} \in \mathcal{E}\mathcal{G}^\dagger$, and $\mathcal{S}^\dagger(\widehat{\mathbf{x}}) := \mathrm{V}_{\widehat{\mathbf{x}}}$ for each $\widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^\dagger$. We emphasise that, since the collection $\{R_e\}_{e \in \mathcal{E}\mathcal{G}}$ forms a tiling of the cylinder $\mathbb{R}/\mathfrak{\eta}\mathbb{Z} \times [0,1]$, the collection of rectangles $\{R_{\mathbf{e}}\}_{\mathbf{e} \in \mathcal{E}\mathcal{G}^\dagger}$ forms a periodic tiling of $\mathbb{R} \times [0,1]$ of period $\mathfrak{\eta}$. We are now ready to precisely define the Smith embedding.

Definition 5.2.14 (Smith embedding)**.** The Smith embedding associated to the quadruple $(\mathcal{G}, c, v_0, v_1)$ is the function $\dot{\mathcal{S}} : \mathcal{V}\mathcal{G} \to \mathbb{R}/\mathfrak{\eta}\mathbb{Z} \times [0,1]$ such that

$$\dot{\mathcal{S}}(x) = \mathrm{mid}(\mathrm{H}_x), \quad \forall\, x \in \mathcal{V}\mathcal{G},$$

where $\mathrm{mid}(\mathrm{H}_x)$ denotes the middle point of the horizontal line segment $\mathcal{S}(x)$. Moreover, we define the lifted Smith embedding $\dot{\mathcal{S}}^\dagger : \mathcal{V}\mathcal{G}^\dagger \to \mathbb{R} \times [0,1]$ as the map that assigns to each $\mathbf{x} \in \mathcal{V}\mathcal{G}^\dagger$ the middle point of the horizontal line segment $\mathcal{S}^\dagger(\mathbf{x})$.

We emphasise once again that the choice to define the Smith embedding by picking the middle point of each horizontal line segment is rather arbitrary. Indeed, the main result of this chapter holds also if one chose an arbitrary point inside each horizontal segment. Finally, for technical reasons, we also need to introduce the following map.

Definition 5.2.15. We define the map $\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}$ that assigns to each vertex $\mathbf{x} \in \mathcal{V}\mathcal{G}^\dagger$ the random variable $\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}(\mathbf{x})$ which is uniformly distributed on the horizontal line segment $\mathcal{S}^\dagger(\mathbf{x})$.

5.3 Some properties of the Smith embedding

In this section, we collect some results that follow directly from the construction of the Smith embedding. We fix throughout this section a doubly marked finite weighted planar graph $(\mathcal{G}, c, v_0, v_1)$ properly embedded in the infinite cylinder $\mathcal{C}_{2\pi}$ according to Definition 5.1.1, and we also consider the associated weighted dual planar graph $(\widehat{\mathcal{G}}, \widehat{c})$ properly embedded in $\mathcal{C}_{2\pi}$ according to Definition 5.1.2. In what follows, we consider the metric graph $\mathbb{G}$ associated to $\mathcal{G}$, and we let $\mathfrak{h} : \mathbb{G} \to [0,1]$ be the extended height coordinate function as specified in Remark 5.2.7. Furthermore, we also consider the dual metric graph $\widehat{\mathbb{G}}$, and we let $\mathfrak{w} : \widehat{\mathbb{G}} \to \mathbb{R}/\eta\mathbb{Z}$ be the extended width coordinate function as specified in Remark 5.2.11.

5.3.1 Adding new vertices

In this subsection, we examine how declaring a finite number of interior points on certain edges of the graph as vertices affects both the height coordinate function and the random walk on the graph. Notably, as proved in Lemma 5.3.3 below, this process of adding new vertices does not alter the height coordinate function on the set of original vertices. This technique proves to be useful, and we will employ it in the proof of our main result (see, for instance, the setups for the proofs of Propositions 5.4.2 and 5.4.7 in Subsections 5.4.1 and 5.4.2, respectively).

In order to make this precise, we start with the following definition.

Definition 5.3.1. Let $W \subset \mathbb{G}$ be a finite subset of the metric graph. We define the weighted planar graph $(\mathcal{G}', c')$ associated to $(\mathcal{G}, c)$ and W as follows:

(a) The set of vertices $\mathcal{VG}'$ is given by $\mathcal{VG} \cup W$;

(b) If the interior of an edge $e \in \mathcal{EG}$ contains $n \in \mathbb{N}$ points of W, then e is split into $n+1$ new edges $[e'_{n+1}]$ according to the points in the interior of e. The edge e remains unchanged otherwise.

(c) If the interior of an edge $e \in \mathcal{EG}$ is split into $[e'_{n+1}]$ new edges, for some $n \in \mathbb{N}$, then we set

$$c'_{e'_i} \stackrel{\text{def}}{=} \frac{c_e}{\mathrm{d}^{\mathbb{G}}\big(e_i'^{,-}, e_i'^{,+}\big)}, \quad \forall\, i \in [n+1].$$

The conductance of e remain unchanged otherwise.

The weighted dual graph $(\widehat{\mathcal{G}}', \widehat{c}')$ can be naturally constructed from $(\mathcal{G}', c')$.

Remark 5.3.2. At the level of the Smith diagram, adding new vertices to the interior of some edges of the graph according to the procedure described above corresponds to horizontally dissecting the rectangles associated to such edges. More precisely, let us assume for simplicity that only one point is added to the interior of an edge e, and let e'_1 and e'_2 be the new edges in which e is split into. Suppose that $e_1'^{,-} = e^-$ and $e_2'^{,+} = e^+$. Let $\mathcal{S}'$ be the tiling map associated to new weighted graph. Then it is immediate to check that $\mathcal{S}(e) = \mathcal{S}'(e'_1) \cup \mathcal{S}'(e'_2)$. In particular the rectangles $\mathcal{S}'(e'_1)$ and $\mathcal{S}'(e'_2)$ have the same width of the rectangle $\mathcal{S}(e)$, and the height of $\mathcal{S}'(e'_1)$ is proportional to $\mathrm{d}^{\mathbb{G}}(e_1'^{,-}, e_1'^{,+})$, while that of $\mathcal{S}'(e'_2)$ is proportional to $\mathrm{d}^{\mathbb{G}}(e_2'^{,-}, e_2'^{,+})$. We refer to Figure 5.5 for a diagrammatic representation of this procedure.

Let $\mathfrak{h}' : \mathcal{VG}' \to [0,1]$ be the voltage function associated to the quadruple $(\mathcal{G}', c', v_0, v_1)$. The following lemma relates the function $\mathfrak{h}$ with $\mathfrak{h}'$.

Lemma 5.3.3. *For every $x \in \mathcal{VG}$, it holds that $\mathfrak{h}(x) = \mathfrak{h}'(x)$.*

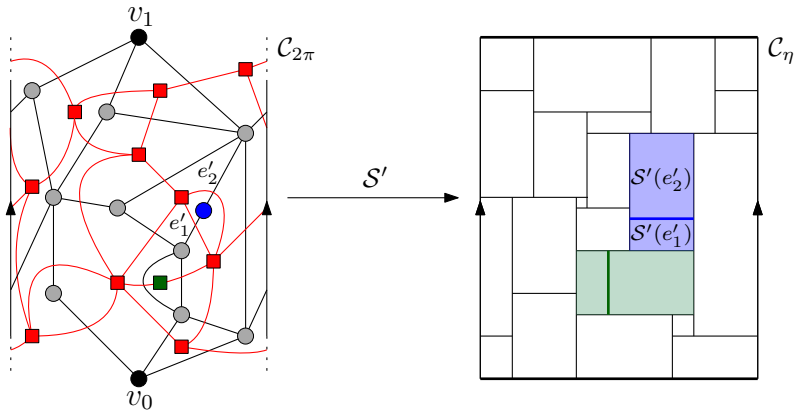


FIGURE 5.5: An edge e is split into two edges, e_1' and e_2', with conductances $c_{e_1'}$ and $c_{e_2'}$, as specified in Definition 5.3.1, by adding a new blue vertex. On the right, the corresponding original rectangle $\mathcal{S}(e)$ is split into two rectangles $\mathcal{S}'(e_1')$ and $\mathcal{S}'(e_2')$ such that $\mathcal{S}(e) = \mathcal{S}'(e_1') \cup \mathcal{S}'(e_2')$. Similarly, a dual edge is split into two dual edges with suitable conductances by adding a new green dual vertex. On the right, the corresponding original rectangle is then split into the union of two rectangles.

Proof. We denote by $K \overset{\text{def}}{=} \#\{\mathcal{V}\mathcal{G}' \setminus \mathcal{V}\mathcal{G}\}$ the number of new vertices added to the graph $\mathcal{G}'$. If $K = 0$, then the result follows immediately. Let us now assume that $K = 1$ and suppose that the edge $e \in \mathcal{E}\mathcal{G}$ is split into two new edges e_1' and e_2' with conductances $c'_{e_1'}$ and $c'_{e_2'}$ as specified in Definition 5.3.1. Then, the desired result follows immediately from the series law of electrical network (see [LP16, Section 2.3.I]). The general case follows from a simple induction argument on K. □

Lemma 5.3.4. *For $x \in \mathcal{V}\mathcal{G}$, let $X'^{,x}$ be the random walk on $(\mathcal{G}', c')$ started from x. Let $\tau_0 \overset{\text{def}}{=} 0$ and, for every $k \in \mathbb{N}_0$, we define inductively $\tau_{k+1} \overset{\text{def}}{=} \inf\{j \geq \tau_k \,:\, X'^{,x}_j \in \mathcal{V}\mathcal{G} \setminus \{X'^{,x}_{\tau_k}\}\}$. Then $\{X'^{,x}_{\tau_k}\}_{k\in\mathbb{N}}$ has the same distribution as the random walk on $(\mathcal{G}, c)$ started from x.*

Proof. As in the proof of Lemma 5.3.3, let $K \overset{\text{def}}{=} \#\{\mathcal{V}\mathcal{G}' \setminus \mathcal{V}\mathcal{G}\}$. If $K = 0$, then the result is obvious. Let us now assume that $K = 1$ and let $y \in W$ be the vertex added to $\mathcal{G}'$. We claim that the distribution of $X'^{,x}_{\tau_1}$ is equal to that of X^x_1. If all the edges in $\mathcal{E}\mathcal{G}(x)$ do not contain y, then the claim is obvious. Let us assume that there exists an edge in $\mathcal{E}\mathcal{G}(x)$ which contains y. Then, thanks to the series law of electrical network and Kirchhoff's node law, one can verify that

$$\mathbb{P}_x(X'_{\tau_1} = v) = \frac{c_{xv}}{\pi(x)}, \quad \forall\, v \in \mathcal{V}\mathcal{G}(x).$$

Therefore, thanks to the strong Markov property of $X'^{,x}$, we get that for all $k \geq 2$, it holds that

$$\mathbb{P}_x(X'_{\tau_k} = v \mid X'_{\tau_{k-1}} = w) = \begin{cases} c_{vw}/\pi(w), & v \in \mathcal{V}\mathcal{G}(w) \\ 0, & \text{otherwise,} \end{cases} \quad \forall\, k \geq 2,$$

which proves the result if $K = 1$. The general case follows from a simple induction argument on K. □

Remark 5.3.5. Given a finite set $\widehat{W} \subset \widehat{\mathbb{G}}$, following a similar procedure to the one described above, one can also construct the weighted dual graph $(\widehat{\mathcal{G}}', \widehat{c}')$ associated to $(\widehat{\mathcal{G}}, c)$ and $\widehat{W}$. In particular, results similar to the one stated above hold also for the respective dual counterparts. For example, if $\mathfrak{w}' : \mathcal{V}\widehat{\mathcal{G}}' \to \mathbb{R}/\mathfrak{y}\mathbb{Z}$ is the width coordinate function associated to the new weighted graph, then $\mathfrak{w}'$ restricted to $\mathcal{V}\widehat{\mathcal{G}}$ coincides with the original width coordinate function. Moreover, adding new dual vertices to the interior of some dual edges

corresponds to vertically dissecting the associated rectangles in the Smith diagram. We refer to Figure 5.5 for a diagrammatic representation of this procedure.

5.3.2 Periodicity

We collect here some properties of the lifted Smith diagram that are due to its periodicity. We recall that the map $\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}$ is defined in Definition 5.2.15.

Lemma 5.3.6. *Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{VG}^\dagger$ be such that $\sigma_{2\pi}(\mathbf{x}_1) = \sigma_{2\pi}(\mathbf{x}_2)$. Then, it holds almost surely that*

$$\left| \frac{\mathrm{Re}(\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}(\mathbf{x}_2)) - \mathrm{Re}(\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}(\mathbf{x}_1))}{\eta} - \frac{\mathrm{Re}(\mathbf{x}_2) - \mathrm{Re}(\mathbf{x}_1)}{2\pi} \right| \leq 1.$$

Proof. We observe that the proof of this result is an easy consequence of Lemma 5.2.9. However, it does not follow immediately from Lemma 5.2.9 since the points $\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}(\mathbf{x}_1)$ and $\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}(\mathbf{x}_2)$ are points chosen uniformly at random from the horizontal segments $\mathcal{S}^\dagger(\mathbf{x}_1)$ and $\mathcal{S}^\dagger(\mathbf{x}_2)$, respectively. We give here the details for completeness.

Let $\mathcal{EG}^{\dagger,\uparrow}(\mathbf{x}_1) = [\mathbf{e}_n^1]$ be the set of lifted harmonically oriented edges with tails equals to $\mathbf{x}_1$ ordered in such a way that $\widehat{\mathbf{e}}_1^1 \cdots \widehat{\mathbf{e}}_n^1$ forms a path in $\widehat{\mathcal{G}}^\dagger$ oriented from left to right. Define $\mathcal{EG}^{\dagger,\uparrow}(\mathbf{x}_2) = [\mathbf{e}_n^2]$ in a similar way. Then, by construction of the map $\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}$, it holds that

$$\mathrm{Re}(\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}(\mathbf{x}_1)) \in \left[\mathfrak{w}^\dagger(\widehat{\mathbf{e}}_1^{1,-}), \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_n^{1,+})\right], \qquad \mathrm{Re}(\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}(\mathbf{x}_2)) \in \left[\mathfrak{w}^\dagger(\widehat{\mathbf{e}}_1^{2,-}), \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_n^{2,+})\right].$$

Moreover, by Lemma 5.2.9, we have that

$$\mathfrak{w}^\dagger(\widehat{\mathbf{e}}_1^{1,-}) - \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_1^{2,-}) = \frac{\eta}{2\pi}\left(\mathrm{Re}(\mathbf{x}_1) - \mathrm{Re}(\mathbf{x}_2)\right), \qquad \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_n^{1,+}) - \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_n^{2,+}) = \frac{\eta}{2\pi}\left(\mathrm{Re}(\mathbf{x}_1) - \mathrm{Re}(\mathbf{x}_2)\right),$$

and also

$$\left|\mathfrak{w}^\dagger(\widehat{\mathbf{e}}_n^{1,+}) - \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_1^{1,-})\right| = \left|\mathfrak{w}^\dagger(\widehat{\mathbf{e}}_n^{2,+}) - \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_1^{2,-})\right| \leq \eta,$$

where the first equality is again due to Lemma 5.2.9, and the last inequality is due to the fact that, by construction of the Smith diagram, every horizontal line segment has width at most η. Therefore, putting together all these facts, one can readily obtain the desired result. □

In order to state and prove the next lemma of this subsection, we need to introduce some notation. Let $a \in (0,1)$, and assume that all points in the set $\mathfrak{h}^{-1}(a)$ are vertices in $\mathcal{VG}$. We define the set of harmonically oriented edges $\mathcal{EG}_a$ and the corresponding set of dual edges $\mathcal{E}\widehat{\mathcal{G}}_a$ as follows

$$\mathcal{EG}_a \overset{\text{def}}{=} \left\{e \in \mathcal{EG} \,:\, \mathfrak{h}(e^-) = a < \mathfrak{h}(e^+)\right\}, \qquad \mathcal{E}\widehat{\mathcal{G}}_a \overset{\text{def}}{=} \left\{\widehat{e} \in \mathcal{E}\widehat{\mathcal{G}} \,:\, \widehat{e} \text{ is the dual of some } e \in \mathcal{EG}_a\right\},$$

and we define the following sets of vertices

$$\mathcal{VG}_a \overset{\text{def}}{=} \left\{x \in \mathcal{VG} \,:\, x = e^- \text{ for some } e \in \mathcal{EG}_a\right\}, \qquad \mathcal{V}\widehat{\mathcal{G}}_a \overset{\text{def}}{=} \left\{\widehat{x} \in \mathcal{V}\widehat{\mathcal{G}} \,:\, \widehat{x} = \widehat{e}^- \text{ for some } \widehat{e} \in \mathcal{E}\widehat{\mathcal{G}}_a\right\},$$

i.e., $\mathcal{VG}_a$ is nothing but $\mathfrak{h}^{-1}(a)$. Furthermore, we denote by $\mathcal{EG}_a^\dagger$, $\mathcal{E}\widehat{\mathcal{G}}_a^\dagger$, $\mathcal{VG}_a^\dagger$, $\mathcal{V}\widehat{\mathcal{G}}_a^\dagger$ the lifts to the universal cover of the corresponding objects.

Now, we let $\widehat{\mathbf{x}}_0^a$ be the vertex in $\mathcal{V}\widehat{\mathcal{G}}_a^\dagger$ whose real coordinate is nearest to 0. We note that the set $\mathcal{EG}_a$ is a cutset and that the associated set of oriented dual edges $\mathcal{E}\widehat{\mathcal{G}}_a$ admits an enumeration $[\widehat{e}_n]$ such that $\widehat{e}_1 \cdots \widehat{e}_n$ forms a

counter-clockwise oriented noncontractible simple closed loop in the dual graph $\widehat{\mathcal{G}}$. In particular, every edge $\widehat{e}_i$ admits a lift $\widehat{\mathbf{e}}_i$ such that $\widehat{\mathbf{e}}_1 \cdots \widehat{\mathbf{e}}_n$ forms a simple path oriented from left to right joining $\widehat{\mathbf{x}}_0^a$ to the shifted vertex $\widehat{\mathbf{x}}_0^a + (0, 2\pi)$ (see Lemma 5.2.2). We refer to Figure 5.6 for a diagrammatic representation.

Definition 5.3.7. Letting $[\widehat{\mathbf{e}}_n]$ be as specified above, we define the set of lifted dual vertices

$$\widehat{\mathcal{P}}_a^\dagger \stackrel{\text{def}}{=} \big\{\widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}_a^\dagger \,:\, \widehat{\mathbf{x}} = \widehat{\mathbf{e}}_i^- \text{ for some } i \in [n]\big\}, \tag{5.3.1}$$

and we also define the set of lifted vertices

$$\mathcal{P}_a^\dagger \stackrel{\text{def}}{=} \big\{\mathbf{x} \in (\mathfrak{h}^\dagger)^{-1}(a) \,:\, \mathbf{x} = \mathbf{e}_i^- \text{ for some } i \in [n]\big\}, \tag{5.3.2}$$

where $[\mathbf{e}_n]$ is the set of lifted edges associated to $[\widehat{\mathbf{e}}_n]$.

Remark 5.3.8. Consider the infinite strip $\mathrm{S}_{2\pi}^a \stackrel{\text{def}}{=} \big[\mathrm{Re}(\widehat{\mathbf{x}}_0^a), \mathrm{Re}(\widehat{\mathbf{x}}_0^a) + 2\pi\big) \times \mathbb{R}$. We remark that in general the set $\widehat{\mathcal{P}}_a^\dagger$ is not fully contained in $\mathrm{S}_{2\pi}^a$. We also observe that, thanks to the fact that $\mathcal{EG}_a$ is a cutset, it holds that $\sigma_{2\pi}(\mathcal{P}_a^\dagger) = \mathfrak{h}^{-1}(a)$.

We are now ready to state and prove the next lemma of this subsection. We refer to Figure 5.6 for a diagram illustrating the various objects involved in the proof of Lemma 5.3.9.

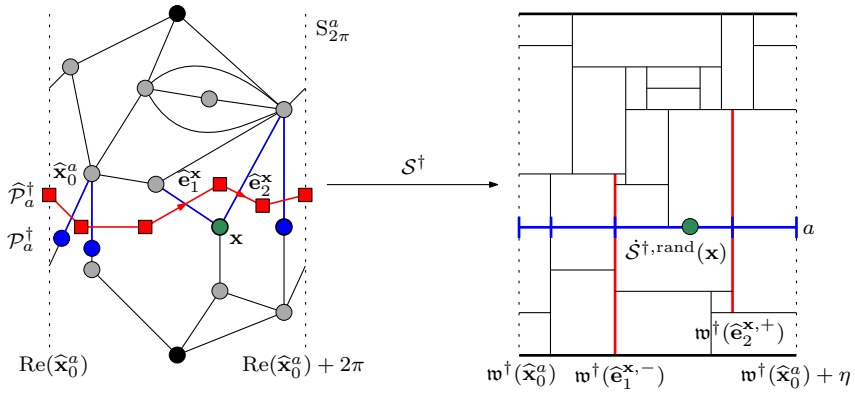


FIGURE 5.6: A diagram illustrating the proof of Lemma 5.3.9. **Left:** A portion of the lifted graph $\mathcal{G}^\dagger$ embedded in $\mathbb{R}^2$. The blue/green vertices are the vertices in the set $\mathcal{P}_a^\dagger$. The green vertex $\mathbf{x}$ is a fixed vertex in $\mathcal{P}_a^\dagger$. The red dual vertices are the vertices in $\widehat{\mathcal{P}}_a^\dagger$. The blue edges belong to the set $\mathcal{EG}_a^\dagger$, and the red dual edges belong to the set $\mathcal{E}\widehat{\mathcal{G}}_a^\dagger$. The leftmost red dual vertex corresponds to the vertex $\widehat{\mathbf{x}}_0^a$. The two red dual edges with an arrow are the edges $\widehat{\mathbf{e}}_1^{\mathbf{x}}$ and $\widehat{\mathbf{e}}_2^{\mathbf{x}}$. **Right:** A portion of the tiling of $\mathbb{R} \times [0, 1]$ constructed via the lifted tiling map $\mathcal{S}^\dagger$. The blue horizontal line corresponds to $\cup_{\mathbf{x} \in \mathcal{P}_a^\dagger} \mathcal{S}^\dagger(\mathbf{x})$. The vertical red segments correspond to $\mathcal{S}^\dagger(\widehat{\mathbf{e}}_1^{\mathbf{x},-})$ and $\mathcal{S}^\dagger(\widehat{\mathbf{e}}_2^{\mathbf{x},+})$. The green point corresponds to a possible realisation of $\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}(\mathbf{x})$.

Lemma 5.3.9. *Fix $a \in (0, 1)$ and assume that all the points in $\mathfrak{h}^{-1}(a)$ are vertices in $\mathcal{VG}$. Let $\widehat{\mathbf{x}}_0^a \in \mathcal{V}\widehat{\mathcal{G}}_a^\dagger$ be as specified above, then it holds that*

$$0 \le \mathfrak{w}^\dagger(\widehat{\mathbf{x}}) - \mathfrak{w}^\dagger(\widehat{\mathbf{x}}_0^a) \le \eta, \quad \forall \widehat{\mathbf{x}} \in \widehat{\mathcal{P}}_a^\dagger,$$

and also, it holds almost surely that

$$0 \le \mathrm{Re}(\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}(\mathbf{x})) - \mathfrak{w}^\dagger(\widehat{\mathbf{x}}_0^a) \le \eta, \quad \forall \mathbf{x} \in \mathcal{P}_a^\dagger.$$

Proof. We start by proving the first part of the lemma. We consider the set of dual edges $[\widehat{\mathbf{e}}_n]$ such that $\widehat{\mathbf{e}}_1 \cdots \widehat{\mathbf{e}}_n$ forms a simple path oriented from left to right joining $\widehat{\mathbf{x}}_0^a$ to the shifted vertex $\widehat{\mathbf{x}}_0^a + (0, 2\pi)$, as

specified before the lemma statement. Thanks to Lemma 5.2.9, we know that

$$\sum_{i=1}^{n} \nabla \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_i) = \eta,$$

and each summand in the sum is non-negative. Now, let $\widehat{\mathbf{x}} \in \widehat{\mathcal{P}}_a^\dagger$, then there exists a subpath $\widehat{\mathbf{e}}_1^{\widehat{\mathbf{x}}} \cdots \widehat{\mathbf{e}}_k^{\widehat{\mathbf{x}}}$ of $\widehat{\mathbf{e}}_1 \cdots \widehat{\mathbf{e}}_n$ which connects $\widehat{\mathbf{x}}_0^a$ to $\widehat{\mathbf{x}}$. Therefore, we have that

$$\mathfrak{w}^\dagger(\widehat{\mathbf{x}}) - \mathfrak{w}^\dagger(\widehat{\mathbf{x}}_0^a) = \sum_{i=1}^{k} \nabla \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_i^{\widehat{\mathbf{x}}}),$$

and the conclusion then follows thanks to the fact that

$$0 \leq \sum_{i=1}^{k} \nabla \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_i^{\widehat{\mathbf{x}}}) \leq \sum_{i=1}^{n} \nabla \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_i) = \eta.$$

We now prove the second part of the lemma. To this end, we fix $\mathbf{x} \in \mathcal{P}_a^\dagger$, and we let $\mathcal{EG}^{\dagger,\uparrow}(\mathbf{x}) = [\mathbf{e}_k^{\mathbf{x}}]$ be the set of lifted harmonically oriented edges with tails equal to $\mathbf{x}$ ordered in such a way that the corresponding lifted dual edges $\widehat{\mathbf{e}}_1^{\mathbf{x}} \cdots \widehat{\mathbf{e}}_k^{\mathbf{x}}$ forms a path in $\widehat{\mathcal{G}}^\dagger$ oriented from left to right (see Figure 5.6). By construction of the map $\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}$, it holds that

$$\mathrm{Re}(\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}(\mathbf{x})) \in \left[\mathfrak{w}^\dagger(\widehat{\mathbf{e}}_1^{\mathbf{x},-}), \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_k^{\mathbf{x},+})\right].$$

Finally, since both $\widehat{\mathbf{e}}_1^{\mathbf{x},-}$ and $\widehat{\mathbf{e}}_k^{\mathbf{x},+}$ belong to $\widehat{\mathcal{P}}_a^\dagger$, the conclusion follows thanks to the first part of the lemma. □

5.3.3 Hitting distribution of a horizontal line

Roughly speaking, the main goal of this subsection is to characterise the hitting distribution of a horizontal line on the Smith diagram for the Smith embedded random walk. Before precisely stating the result, we need to introduce some notation.

Definition 5.3.10. We define the map length : $\mathcal{VG} \to [0, \eta)$ as the function that assigns to each vertex $x \in \mathcal{VG}$ the length of the horizontal segment $\mathcal{S}(x)$ associated to x by the Smith embedding, i.e., we let

$$\mathrm{length}(x) \overset{\text{def}}{=} \sum_{\mathbf{e} \in \mathcal{EG}^\downarrow(x)} \nabla \mathfrak{h}(e), \quad \forall\, x \in \mathcal{VG},$$

where $\mathcal{EG}^\downarrow(x)$ denotes the set of harmonically oriented edges in $\mathcal{EG}$ with heads equal to x.

We recall that the difference of the width coordinate function between the endpoints of a dual edge is equal to the gradient of the height coordinate of the corresponding primal edge. Moreover, thanks to the fact that the height coordinate function is harmonic, it is plain to see that

$$\mathrm{length}(x) = \sum_{\mathbf{e} \in \mathcal{EG}^\uparrow(x)} \nabla \mathfrak{h}(e), \quad \forall\, x \in \mathcal{VG},$$

where $\mathcal{EG}^\uparrow(x)$ denotes the set of harmonically oriented edges in $\mathcal{EG}$ with tails equal to x. We can also naturally extend the definition of the length function to the metric graph $\mathbb{G}$. More precisely, given a point $x \in \mathbb{G} \setminus \mathcal{VG}$, if x lies in the interior of the edge $e \in \mathcal{EG}$, then we set $\mathrm{length}(x) \overset{\text{def}}{=} \nabla \mathfrak{h}(e)$.

Definition 5.3.11. Given a value $a \in (0,1)$, we define on the set $\mathfrak{h}^{-1}(a) \subset \mathbb{G}$ the measure μ_a by letting

$$\mu_a(x) \stackrel{\text{def}}{=} \frac{\text{length}(x)}{\eta}, \quad \forall\, x \in \mathfrak{h}^{-1}(a).$$

Since, by construction of the Smith diagram, it holds that $\sum_{x\in\mathfrak{h}^{-1}(a)} \text{length}(x) = \eta$, the measure μ_a is a probability measure on the set $\mathfrak{h}^{-1}(a)$.

Remark 5.3.12. We emphasise that, thanks to Remark 5.3.8, one can also view μ_a as a probability measure on the set $\mathcal{P}_a^\dagger$, where we recall that $\mathcal{P}_a^\dagger$ is the set defined in (5.3.2).

We assume throughout the whole subsection that all the points in the set $\cup_{x\in\mathcal{VG}}\mathfrak{h}^{-1}(\mathfrak{h}(x))$ are vertices in $\mathcal{VG}$. At the level of the Smith diagram, this means that for all $x \in \mathcal{VG}$, it holds that the set $\cup_{y\in\mathfrak{h}^{-1}(\mathfrak{h}(x))}\mathcal{S}(y)$ is equal to the noncontractible closed loop $\mathbb{R}/\eta\mathbb{Z} \times \{\mathfrak{h}(x)\} \subset \mathcal{C}_\eta$. We observe that, for any given finite weighted planar graph, we can always canonically construct from it another finite weighted planar graph that satisfies this assumption. Indeed, suppose that there exists a vertex $x \in \mathcal{VG}$ such that $\mathfrak{h}^{-1}(\mathfrak{h}(x)) \not\subset \mathcal{VG}$, then, using the procedure described in Subsection 5.3.1, we can declare all the points in the finite[4] set $\mathfrak{h}^{-1}(\mathfrak{h}(x)) \setminus \mathcal{VG}$ to be vertices of the graph.

We fix a value $a \in (0,1)$ such that all the points in the set $\mathfrak{h}^{-1}(a)$ are vertices in $\mathcal{VG}$, and we let X^{μ_a} be the random walk on $(\mathcal{G}, c)$ started from a point in $\mathfrak{h}^{-1}(a)$ sampled according to the probability measure μ_a. Furthermore, we let

$$\rho_0 \stackrel{\text{def}}{=} 0, \qquad \rho_n \stackrel{\text{def}}{=} \inf\left\{j \geq \rho_{n-1} \,:\, \mathfrak{h}(X_j) \neq \mathfrak{h}(X_{\rho_{n-1}})\right\}, \quad \forall n \in \mathbb{N}. \tag{5.3.3}$$

Definition 5.3.13. Let X^{μ_a} and $(\rho_n)_{n\in\mathbb{N}_0}$ be as specified above. For $N \in \mathbb{N}$, we say that a finite sequence of height coordinates $[a_N]_0 \subset (0,1)$ is *admissible* for the random walk X^{μ_a} if $a_0 = a$ and

$$\mathbb{P}_{\mu_a}\left(\{\mathfrak{h}(X_{\rho_n}) = a_n\}_{n=1}^N\right) > 0.$$

We can now state the main result of this subsection.

Lemma 5.3.14. *Let X^{μ_a} and $(\rho_n)_{n\in\mathbb{N}_0}$ be as specified above. For $N \in \mathbb{N}$, let $[a_N]_0 \subset (0,1)$ be an admissible sequence of height coordinates for the random walk X^{μ_a}. Then, for all $i \in [N]_0$, it holds that*

$$\mathbb{P}_{\mu_a}\left(X_{\rho_i} = x_i \mid \{\mathfrak{h}(X_{\rho_n}) = a_n\}_{n=1}^N\right) = \mu_{a_i}(x_i), \quad \forall\, x_i \in \mathfrak{h}^{-1}(a_i).$$

Intuitively, the proof of the above lemma goes as follows. If $i = 1$ and we only had the conditioning on the event $\mathfrak{h}(X_{\rho_1}) = a_1$, then the result would follow from a simple computation. However, since, in general, $i \neq 1$, and since we are also conditioning on the events $\{\mathfrak{h}(X_{\rho_n}) = a_n\}_{n=1}^{i-1}$ and $\{\mathfrak{h}(X_{\rho_n}) = a_n\}_{n=i+1}^N$, which represent the height coordinates of the random walk for past and future times respectively, the proof of the result is not immediate. However, the proof follows by a simple induction argument in which we show that we can "forget" about the conditioning on past and future times. Roughly speaking, the reason why we can forget about past times is due to the fact that the height component of the random walk is itself a Markov process, while the reason why we can forget about future times is due to the harmonicity of the height coordinate function. We now proceed with the proof of the lemma.

[4]Note that if $\mathfrak{h}(x) = a$ and there is a "flat" edge whose endpoints both satisfy $\mathfrak{h} = a$, then the level set $\mathfrak{h}^{-1}(a)$ is infinite, since it contains the entire continuum of points along that edge. For the remainder of the paper, whenever we apply the vertex-adding procedure, we regard $\mathfrak{h}$ as restricted to the metric graph $\mathbb{G}$ outside of any "flat" edges.

Proof of Lemma 5.3.14. The proof involves three main steps.

Step 1: We begin by showing that $\mathbb{P}_{\mu_a}(\mathfrak{h}(X_{\rho_1}) = a_1 \mid X_0 = x_0) = \mathbb{P}_{\mu_a}(\mathfrak{h}(X_{\rho_1}) = a_1)$ for all $x_0 \in \mathfrak{h}^{-1}(a)$. Since the height coordinate function is harmonic, the process $\mathfrak{h}(X^{x_0})$ is a bounded martingale. In particular, by the optional stopping theorem, we have that

$$\mathbb{E}_{x_0}\big[\mathfrak{h}(X_{\rho_1})\big] = \mathfrak{h}(x_0) = a.$$

By construction, the random walk can only reach two heights at time ρ_1: height a_1 or height $\widetilde{a}_1$ say. Therefore, we can write the previous display as follows

$$a_1 \mathbb{P}_{\mu_a}\big(\mathfrak{h}(X_{\rho_1}) = a_1 \mid X_0 = x_0\big) + \widetilde{a}_1\big(1 - \mathbb{P}_{\mu_a}\big(\mathfrak{h}(X_{\rho_1}) = a_1 \mid X_0 = x_0\big)\big) = a.$$

In particular, by rearranging the previous expression, we obtain that

$$\mathbb{P}_{\mu_a}\big(\mathfrak{h}(X_{\rho_1}) = a_1 \mid X_0 = x_0\big) = \frac{|a - \widetilde{a}_1|}{|a_1 - \widetilde{a}_1|} \overset{\text{def}}{=} p_a, \quad \forall\, x_0 \in \mathfrak{h}^{-1}(a),$$

from which the desired result follows since the right-hand side of the above expression does not depend on the particular choice of $x_0 \in \mathfrak{h}^{-1}(a)$. Now, proceeding by induction, one can prove that for any $i \in [N]_0$ and for all $x_0 \in \mathfrak{h}^{-1}(a), \ldots, x_i \in \mathfrak{h}^{-1}(a_i)$, it holds that

$$\mathbb{P}_{\mu_a}\big(\{\mathfrak{h}(X_{\rho_{i+n}}) = a_{i+n}\}_{n=1}^{N-i} \mid \{X_{\rho_j} = x_j\}_{j=0}^{i}\big) = \mathbb{P}_{\mu_a}\big(\{\mathfrak{h}(X_{\rho_{i+n}}) = a_{i+n}\}_{n=1}^{N-i} \mid \mathfrak{h}(X_{\rho_i}) = a_i\big).$$

Step 2: Thanks to the previous step and Bayes' rule, we have that $\mathbb{P}_{\mu_a}\big(X_0 = x_0 \mid \mathfrak{h}(X_{\rho_1}) = a_1\big) = \mu_a(x_0)$, for all $x_0 \in \mathfrak{h}^{-1}(a)$. In particular, using this fact, we will now prove that

$$\mathbb{P}_{\mu_a}\big(X_{\rho_1} = x_1 \mid \mathfrak{h}(X_{\rho_1}) = a_1\big) = \mu_{a_1}(x_1), \quad \forall\, x_1 \in \mathfrak{h}^{-1}(a_1).$$

To this end, without any loss of generality, we assume that $a_1 > a$. Let $\mathcal{C}_a \subseteq \mathfrak{h}^{-1}(a)$ be a connected component of $\mathfrak{h}^{-1}(a)$. Let $x_1 \in \mathfrak{h}^{-1}(a_1)$ be such that there exists at least one vertex in $\mathcal{C}_a$ connected to it. It suffices to show that

$$\mathbb{P}_{\mu_a}\big(X_{\rho_1} = x_1,\ X_0 \in \mathcal{C}_a \mid \mathfrak{h}(X_{\rho_1}) = a_1\big) = \frac{1}{\mathfrak{y}} \sum_{e \in \mathcal{E}\mathcal{G}^{\downarrow}(x_1),\, e^- \in \mathcal{C}_a} \nabla\mathfrak{h}(e).$$

Indeed, the desired result follows by summing over all connected components of $\mathfrak{h}^{-1}(a)$. Let $\mathcal{V}_{a_1} \subseteq \mathfrak{h}^{-1}(a_1)$ (resp. $\mathcal{V}_{\widetilde{a}_1} \subseteq \mathfrak{h}^{-1}(\widetilde{a}_1)$) denote the set of vertices in $\mathfrak{h}^{-1}(a_1)$ (resp. $\mathfrak{h}^{-1}(\widetilde{a}_1)$) connected to at least one vertex in $\mathcal{C}_a$. In particular, by assumption, we have $x_1 \in \mathcal{V}_{a_1}$ and $\mathcal{V}\mathcal{G}(x_0) \subseteq \mathcal{C}_a \cup \mathcal{V}_{a_1} \cup \mathcal{V}_{\widetilde{a}_1}$ for all $x_0 \in \mathcal{C}_a$. Then, we have that

$$\begin{aligned}
\mathbb{P}_{\mu_a}\big(X_{\rho_1} = x_1,\ X_0 \in \mathcal{C}_a \mid \mathfrak{h}(X_{\rho_1}) = a_1\big) &= \sum_{x_0 \in \mathcal{C}_a} \mathbb{P}_{x_0}\big(X_{\rho_1} = x_1 \mid \mathfrak{h}(X_{\rho_1}) = a_1\big)\mu_a(x_0) \\
&= \frac{1}{\mathfrak{y}} \sum_{x_0 \in \mathcal{C}_a} \mathbb{P}_{x_0}\big(X_{\rho_1} = x_1 \mid \mathfrak{h}(X_{\rho_1}) = a_1\big)\,\mathrm{length}(x_0) \\
&= \frac{1}{p_a \mathfrak{y}} \sum_{x_0 \in \mathcal{C}_a} \mathbb{P}_{x_0}\big(X_{\rho_1} = x_1\big)\,\mathrm{length}(x_0).
\end{aligned}$$

Thanks to the Markov property of the random walk, the function $\mathcal{C}_a \ni x_0 \mapsto \mathbb{P}_{x_0}(X_{\rho_1} = x_1)$ coincides with the restriction to $\mathcal{C}_a$ of the function $\mathfrak{g} : \mathcal{C}_a \cup \mathcal{V}_{a_1} \cup \mathcal{V}_{\widetilde{a}_1} \to [0,1]$ that is harmonic in $\mathcal{C}_a$ with boundary values

on $\mathcal{V}_{a_1} \cup \mathcal{V}_{\widetilde{a}_1}$ given by $\mathfrak{g}(z) = 0$ for all $z \in (\mathcal{V}_{a_1} \cup \mathcal{V}_{\widetilde{a}_1}) \setminus \{x_1\}$ and $\mathfrak{g}(x_1) = 1$. We note that

$$\begin{aligned}\sum_{x_0 \in \mathcal{C}_a} \sum_{z \in \mathcal{VG}(x_0)} c_{x_0 z}\big(\mathfrak{g}(x_0) - \mathfrak{g}(z)\big) &= \sum_{x_0 \in \mathcal{C}_a} \sum_{z \in \mathcal{VG}(x_0) \cap (\mathcal{V}_{a_1} \cup \mathcal{V}_{\widetilde{a}_1})} c_{x_0 z}\big(\mathfrak{g}(x_0) - \mathfrak{g}(z)\big) \\ &= \sum_{x_0 \in \mathcal{C}_a} \sum_{z \in \mathcal{VG}(x_0) \cap (\mathcal{V}_{a_1} \cup \mathcal{V}_{\widetilde{a}_1})} c_{x_0 z}\mathfrak{g}(x_0) - \sum_{e \in \mathcal{EG}^{\downarrow}(x_1),\, e^- \in \mathcal{C}_a} c_e,\end{aligned}$$

where the first equality holds because if x_0, $z \in \mathcal{C}_a$, then the term $c_{x_0 z}(\mathfrak{g}(x_0) - \mathfrak{g}(z))$ appears twice with opposite signs, and the second equality follows directly from the boundary conditions. Now, observe that since the function $\mathfrak{g}$ is harmonic inside $\mathcal{C}_a$, the left-hand side of the first line in the above display is equal to 0. Moreover, using the harmonicity of the function $\mathfrak{h}$, for all $x_0 \in \mathcal{C}_a$, one can check that

$$\mathrm{length}(x_0) = |a_1 - a| \sum_{z \in \mathcal{VG}(x_0) \cap \mathcal{V}_{a_1}} c_{x_0 z} = p_a |a_1 - a| \sum_{z \in \mathcal{VG}(x_0) \cap (\mathcal{V}_{a_1} \cup \mathcal{V}_{\widetilde{a}_1})} c_{x_0 z}\,.$$

Therefore, combining the previous considerations, we obtain that

$$\frac{1}{p_a \eta} \sum_{x_0 \in \mathcal{C}_a} \mathfrak{g}(x_0)\,\mathrm{length}(x_0) = \frac{|a_1 - a|}{\eta} \sum_{x_0 \in \mathcal{C}_a} \sum_{z \in \mathcal{VG}(x_0) \cap (\mathcal{V}_{a_1} \cup \mathcal{V}_{\widetilde{a}_1})} c_{x_0 z}\mathfrak{g}(x_0) = \frac{1}{\eta} \sum_{e \in \mathcal{EG}^{\downarrow}(x_1),\, e^- \in \mathcal{C}_a} \nabla\mathfrak{h}(e),$$

thus proving the claim.

Finally, proceeding again by induction, one can easily prove that, for all $i \in [N]$, it holds that

$$\mathbb{P}_{\mu_a}\big(X_{\rho_i} = x_i \mid \{\mathfrak{h}(X_{\rho_n}) = a_n\}_{n=1}^{i}\big) = \mu_{a_i}(x_i), \quad \forall\, x_i \in \mathfrak{h}^{-1}(a_i).$$

Step 3: For $i \in [N]_0$, we observe that a consequence of Step 1 is that the sequence of random variables $\{X_{\rho_j}\}_{j=0}^{i}$ is conditionally independent from $\{\mathfrak{h}(X_{\rho_{i+n}})\}_{n=1}^{N-i}$ given $\mathfrak{h}(X_{\rho_i}) = a_i$. Therefore, the conditional law of $\{X_{\rho_j}\}_{j=0}^{i}$ given $\{\mathfrak{h}(X_{\rho_n}) = a_n\}_{n=1}^{i}$ is the same as the conditional law given $\{\mathfrak{h}(X_{\rho_n}) = a_n\}_{n=1}^{N}$. Hence, this implies that

$$\mathbb{P}_{\mu_a}\big(X_{\rho_i} = x_i \mid \big\{\mathfrak{h}(X_{\rho_n}) = a_n\big\}_{n=1}^{N}\big) = \mathbb{P}_{\mu_a}\big(X_{\rho_i} = x_i \mid \big\{\mathfrak{h}(X_{\rho_n}) = a_n\big\}_{n=1}^{i}\big), \quad \forall\, x_i \in \mathfrak{h}^{-1}(a_i),$$

and so the conclusion follows from Step 2. □

5.3.4 Expected horizontal winding

Roughly speaking, the main goal of this subsection is to establish that the average winding that the Smith-embedded random walk does before hitting a given horizontal line on the Smith diagram is zero. Before precisely stating the result, we need to introduce some notation.

Definition 5.3.15. Given the random walk $\mathbf{X}^{\mathbf{x}}$ on the lifted weighted graph $(\mathcal{G}^{\dagger}, c^{\dagger})$ started from $\mathbf{x} \in \mathcal{VG}^{\dagger}$, we let

$$\mathbb{N}_0 \ni n \mapsto \dot{\mathbf{X}}^{\mathbf{x}}_n$$

be the discrete time process taking values in $\mathbb{R} \times [0, 1]$ such that, for each $n \in \mathbb{N}_0$, the conditional law of $\dot{\mathbf{X}}^{\mathbf{x}}_n$, given $\mathbf{X}^{\mathbf{x}}_n$, is equal to the law of $\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}(\mathbf{X}^{\mathbf{x}}_n)$, where we recall that $\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}$ is defined in Definition 5.2.15. We call the process $\dot{\mathbf{X}}^{\mathbf{x}}$ the *lifted Smith-embedded random walk* associated to $\mathbf{X}^{\mathbf{x}}$.

It follows from the definition of $\dot{\mathbf{X}}^{\mathbf{x}}$ that, at each time step $n \in \mathbb{N}_0$, the location of the point $\dot{\mathbf{X}}^{\mathbf{x}}_n$ is sampled uniformly at random and independently of everything else from the horizontal line segment $\mathcal{S}^\dagger(\mathbf{X}^{\mathbf{x}}_n)$. With a slight abuse of notation, we also denote by $\dot{\mathbf{X}}^{\mathbf{x}}$ the continuous time version $\{\dot{\mathbf{X}}^{\mathbf{x}}_t\}_{t\geq 0}$, where the continuous path is generated by piecewise linear interpolation at constant speed.

We assume also in this subsection that all the points in the set $\cup_{x\in\mathcal{VG}}\mathfrak{h}^{-1}(\mathfrak{h}(x))$ are vertices in $\mathcal{VG}$. Furthermore, we fix a value $a \in (0,1)$ such that all the points in the set $\mathfrak{h}^{-1}(a)$ are vertices in $\mathcal{VG}$. We denote by

$$\mathbb{N}_0 \ni n \mapsto \mathbf{X}^{\mu_a}_n$$

the random walk on the lifted weighted graph $(\mathcal{G}^\dagger, c^\dagger)$ started from a point in $\mathcal{P}^\dagger_a$ sampled according to the probability measure μ_a (see Remark 5.3.12). Moreover, we denote by $\dot{\mathbf{X}}^{\mu_a}$ the lifted Smith-embedded random walk associated to $\mathbf{X}^{\mu_a}$ as specified above.

In complete analogy with the definition of winding of a path in the infinite cylinder $\mathcal{C}_{2\pi}$, we have the following definition of winding on the cylinder $\mathcal{C}_\eta$.

Definition 5.3.16. Let $0 \leq t_1 < t_2$, consider a path $P : [t_1, t_2] \to \mathcal{C}_\eta$, and let $\mathbf{P} : [t_1, t_2] \to \mathbb{R}\times[0,1]$ be its lift to the covering space $\mathbb{R}\times[0,1]$. Then, we define the winding of P as follows

$$\mathrm{wind}_\eta(P) = \frac{\mathrm{Re}(\mathbf{P}(t_2)) - \mathrm{Re}(\mathbf{P}(t_1))}{\eta}. \tag{5.3.4}$$

In what follows, with a slight abuse of notation, if $\mathbf{P} : [t_1, t_2] \to \mathbb{R}\times[0,1]$ is a lift of a path $P : [t_1, t_2] \to \mathcal{C}_\eta$, then we may write $\mathrm{wind}_\eta(\mathbf{P})$ to denote $\mathrm{wind}_\eta(P)$.

We are now ready to state the main result of this subsection. In what follows, given the random walk $\mathbf{X}^{\mu_a}$ as specified above, we let $X^{\mu_a} = \sigma_{2\pi}(\mathbf{X}^{\mu_a})$.

Lemma 5.3.17. *Let X^{μ_a} and $\dot{\mathbf{X}}^{\mu_a}$ be as defined above, and let $(\rho_n)_{n\in\mathbb{N}_0}$ denote the sequence of stopping times associated with X^{μ_a}, as defined in (5.3.3). For $N \in \mathbb{N}$, let $[a_N]_0 \subset (0,1)$ be an admissible sequence of height coordinates for the random walk X^{μ_a} as specified in Definition 5.3.13. Then, it holds that*

$$\mathbb{E}_{\mu_a}\left[\mathrm{wind}_\eta\left(\dot{\mathbf{X}}|_{[0,\rho_N]}\right) \mid \left\{\mathfrak{h}(X_{\rho_n}) = a_n\right\}_{n=1}^N\right] = 0.$$

Heuristically speaking, the proof of the above lemma goes as follows. First, we observe that, by definition of winding, it holds that

$$\mathrm{wind}_\eta(\dot{\mathbf{X}}|_{[0,\rho_N]}) = \sum_{j=0}^{N-1} \mathrm{wind}_\eta(\dot{\mathbf{X}}|_{[\rho_j,\rho_{j+1}]}).$$

In particular, this implies that we can reduce the problem to studying the expected winding at each time step of the random walk. Suppose for a moment that $N = 1$ in the lemma statement. Then we only need to prove that the random walk started uniformly at random from height a and conditioned to hit height a_1 at its first time step has zero expected winding. The reason why this is true is due to the fact that the randomisation procedure that takes place inside each small segment ensures that the conditional hitting distribution of the segment at height a_1 is also uniformly distributed. To extend the argument to the general case $N \in \mathbb{N}$, we just need to use Lemma 5.3.14.

Proof of Lemma 5.3.17. We begin by identifying an equivalent condition under which the relation stated in the lemma holds. To this end, we observe that, thanks to the Definition 5.3.16 of winding, it holds that

$$\mathbb{E}_{\mu_a}\left[\mathrm{wind}_\eta\left(\dot{\mathbf{X}}|_{[0,\rho_N]}\right) \mid \left\{\mathfrak{h}(X_{\rho_n}) = a_n\right\}_{n=1}^N\right] = \frac{1}{\eta}\sum_{j=0}^{N-1}\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\dot{\mathbf{X}}_{\rho_{j+1}}) - \mathrm{Re}(\dot{\mathbf{X}}_{\rho_j}) \mid \left\{\mathfrak{h}(X_{\rho_n}) = a_n\right\}_{n=1}^N\right],$$

and so the claim follows if we prove that

$$\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\dot{\mathbf{X}}_{\rho_{j+1}}) \mid \left\{\mathfrak{h}(X_{\rho_n}) = a_n\right\}_{n=1}^N\right] = \mathbb{E}_{\mu_a}\left[\mathrm{Re}(\dot{\mathbf{X}}_{\rho_j}) \mid \left\{\mathfrak{h}(X_{\rho_n}) = a_n\right\}_{n=1}^N\right], \quad \forall\, j \in [N-1]_0.$$

By Lemma 5.3.14, for all $j \in [N]_0$, the conditional law of $X_{\rho_j}^{\mu_a}$ given $\{\mathfrak{h}(X_{\rho_n}^{\mu_a}) = a_n\}_{n=1}^N$ is equal to μ_{a_j}. In particular, this implies that for all $j \in [N]_0$, the conditional law of $\mathrm{Re}(\sigma_\eta(\dot{\mathbf{X}}_{\rho_j}^{\mu_a}))$ given $\{\mathfrak{h}(X_{\rho_n}^{\mu_a}) = a_n\}_{n=1}^N$ is that of a uniform random variable on the interval $[0,\eta)$. Hence, the desired result follows. □

5.4 Proof of the main result

The main goal of this section is to prove Theorem 5.A. We fix a sequence $\{(\mathcal{G}^n, c^n, v_0^n, v_1^n)\}_{n\in\mathbb{N}}$ of doubly marked finite weighted planar maps and we let $\{(\widehat{\mathcal{G}}^n, \widehat{c}^n)\}_{n\in\mathbb{N}}$ be the sequence of associated weighted dual planar graphs. We assume throughout this section that such sequences satisfy hypotheses (H1), (H2), (H3).

Before moving to the proof of the main theorem, we prove a preliminary lemma. For each $n \in \mathbb{N}$, we recall that $\mathcal{FG}^n$ denotes the set of faces of the graph $\mathcal{G}^n$. The next lemma states that, thanks to the invariance principle assumption on the sequence of weighted graphs, the maximal diameter of the faces in $\mathcal{FG}^n$ which intersect a compact subset of $\mathcal{C}_{2\pi}$ is of order $o_n(1)$.

Lemma 5.4.1. *Let $R > 0$, and, for any $n \in \mathbb{N}$, consider the set $\mathcal{FG}^n(R)$ of faces in $\mathcal{FG}^n$ that intersect $\mathbb{R}/2\pi\mathbb{Z} \times [-R, R]$. Then, for any $\varepsilon > 0$ and for any $n > n(R,\varepsilon)$ large enough, it holds that*

$$\mathrm{diam}(f) \le \varepsilon, \quad \forall\, f \in \mathcal{FG}^n(R).$$

The same result holds also for the sequence of dual graphs, i.e., with $\mathcal{F}\widehat{\mathcal{G}}^n$ in place of $\mathcal{FG}^n$.

Proof. Obviously, we can just prove the result for the sequence of primal graphs as the result for the sequence of dual graphs can be proved in the same exact way. We proceed by contradiction by assuming that there exists $\varepsilon > 0$ such that, for any $n \in \mathbb{N}$, there exists a face $f_n \in \mathcal{FG}^n(R)$ for which $\mathrm{diam}(f_n) > \varepsilon$. We notice that each set $f_n \cap \mathbb{R}/2\pi\mathbb{Z} \times [-2R, 2R]$ is compact and contains a path P_n with $\mathrm{diam}(P_n) > \varepsilon$. By compactness of the Hausdorff distance d^{H}, we can assume, by possibly taking a subsequential limit, that $\lim_{n\to\infty} \mathrm{d}^{\mathrm{H}}(P_n, P) = 0$ for some compact and connected set $P \subset \mathbb{R}/2\pi\mathbb{Z} \times [-2R, 2R]$. Now, choose a rectangle Q such that P disconnects the left and right sides of Q, or the top and bottom sides of Q. For any $n > n(R, \varepsilon, \mathrm{Q})$ large enough, also the path P_n disconnects the rectangle Q in the same way as P. Therefore, for any $x \in \mathbb{R}/2\pi\mathbb{Z} \times [-R, R]$, the random walk $X^{n,x}$ on the weighted graph $(\mathcal{G}^n, c^n)$ started from x cannot cross the rectangle Q from left to right, or from top to bottom. However, the Brownian motion B^x on $\mathcal{C}_{2\pi}$ started from x has positive probability to do so. This is in contradiction with assumption (H2), and so the desired result follows. □

We are now ready to start the proof of Theorem 5.A. We observe that this result admits a natural version on the sequence of lifted weighted graphs. Indeed, in order to prove it, we will work in the setting of the

universal cover. To be more precise, we will first study in Subsection 5.4.1 the sequence of lifted height coordinate functions, and then, in Subsection 5.4.2, we will study the sequence of lifted width coordinate functions. Once this is done, we will put everything together and we will conclude the proof of Theorem 5.A in Subsection 5.4.3.

5.4.1 Height coordinate function

For each $n \in \mathbb{N}$, consider the lifted height coordinate function

$$\mathfrak{h}_n^\dagger : \mathcal{V}\mathcal{G}^{\dagger,n} \to [0,1],$$

as defined in (5.2.5). The main goal of this subsection is to show that there exists an affine transformation of the function $\mathfrak{h}_n^\dagger$ that is asymptotically close, uniformly on lifts of compact subsets of the infinite cylinder $\mathcal{C}_{2\pi}$, to the height coordinate function $\mathbf{x} \mapsto \mathrm{Im}(\mathbf{x})$ in the a priori lifted embedded graph $\mathcal{G}^{\dagger,n}$, as $n \to \infty$. More precisely, we have the following result.

Proposition 5.4.2. *There exit two sequences $\{b_n^\mathfrak{h}\}_{n\in\mathbb{N}}$, $\{c_n^\mathfrak{h}\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that for all $R > 1$, $\delta \in (0,1)$, and for any $n > n(R,\delta)$ large enough, it holds that*

$$\left|c_n^\mathfrak{h}\mathfrak{h}_n^\dagger(\mathbf{x}) + b_n^\mathfrak{h} - \mathrm{Im}(\mathbf{x})\right| \leq \delta, \quad \forall\, \mathbf{x} \in \mathcal{V}\mathcal{G}^{\dagger,n}(\mathbb{R}\times[-R,R]).$$

The proof of this proposition is given in Subsection 5.4.1.2. As we will see, the proof will follow easily thanks to the harmonicity of the height coordinate function together with Lemma 5.4.3 below.

5.4.1.1 Setup of the proof

For each $n \in \mathbb{N}$, consider the metric graph $\mathbb{G}^n$ associated to $\mathcal{G}^n$, and let $\mathfrak{h}^n : \mathbb{G}^n \to [0,1]$ be the extended height coordinate function as specified in Remark 5.2.7. Given a value $S \in \mathbb{R}$, we define the set

$$V_S^n \overset{\text{def}}{=} \left\{x \in \mathbb{G}^n \,:\, \mathrm{Im}(x) = S\right\},$$

and we let

$$\overline{a}_S^n \overset{\text{def}}{=} \sup\left\{a \in (0,1) \,:\, \mathfrak{h}_n^{-1}(a) \cap V_S^n \neq \emptyset\right\}, \quad \underline{a}_S^n \overset{\text{def}}{=} \inf\left\{a \in (0,1) \,:\, \mathfrak{h}_n^{-1}(a) \cap V_S^n \neq \emptyset\right\}. \tag{5.4.1}$$

We fix throughout $R > 1$ and $\delta \in (0,1)$ as in the proposition statement, and we let

$$R' \overset{\text{def}}{=} \frac{R}{\delta}.$$

We consider the set

$$W_{R,\delta}^n \overset{\text{def}}{=} \left\{V_{R'}^n \cup V_{-R'}^n\right\} \cup \left\{\mathfrak{h}_n^{-1}(\overline{a}_{R'}^n) \cup \mathfrak{h}_n^{-1}(\underline{a}_{-R'}^n)\right\} \subset \mathbb{G}^n.$$

For each $n \in \mathbb{N}$, by possibly locally modifying the a priori embedding of the graph $\mathcal{G}^n$ in the infinite cylinder $\mathcal{C}_{2\pi}$, we can assume without loss of generality that each edge in $\mathcal{E}\mathcal{G}^n$ crosses the circles at height R' and $-R'$ at most finitely many times. In particular, this implies that we can assume that the set $W_{R,\delta}^n$ contains at most

finitely many points, and therefore, by Lemma 5.3.3, we can assume, without any loss of generality, that $\mathcal{VG}^n$ contains all the points in $W^n_{R,\delta}$.

In what follows, in order to lighten up notation, we adopt the following notational convention

$$\widehat{V}_S \equiv \widehat{V}^n_S, \qquad \overline{a} \equiv \overline{a}^n_{R'}, \quad \underline{a} \equiv \underline{a}^n_{-R'}.$$

Furthermore, we denote by $V^\dagger_S$ the lift to the universal cover of V_S, and we write $\mathcal{VG}^{\dagger,n}(S)$ (resp. $\mathcal{VG}^n(S)$) as a shorthand for $\mathcal{VG}^{\dagger,n}(\mathbb{R}\times[-S,S])$ (resp. $\mathcal{VG}^n(\mathbb{R}/2\pi\mathbb{Z}\times[-S,S])$). We refer to Figure 5.7 for an illustration of the sets involved in the proof of Proposition 5.4.2.

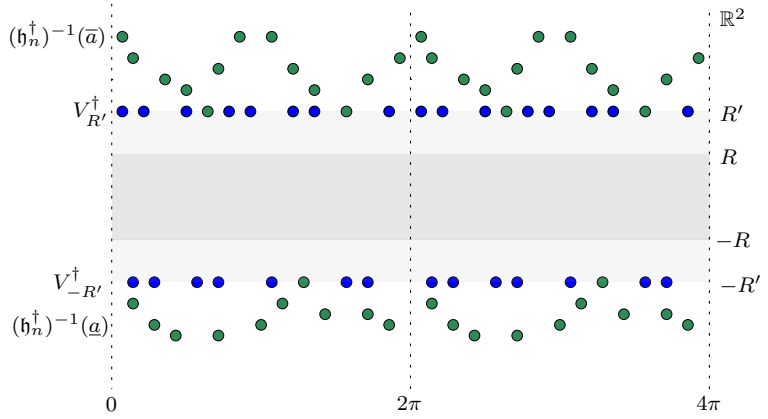


FIGURE 5.7: A diagram illustrating the sets involved in the proof of Proposition 5.4.2. The shaded dark-grey strip between heights R and $-R$ contains all the vertices in the set $\mathcal{VG}^{\dagger,n}(R)$. The shaded light-grey strip between heights R' and $-R'$ contains all the vertices in the set $\mathcal{VG}^{\dagger,n}(R')$. The blue vertices at the top (resp. bottom) are the vertices in the set $V^\dagger_{R'}$ (resp. $V^\dagger_{-R'}$). The green vertices at the top (resp. bottom) are the vertices in the set $(\mathfrak{h}^\dagger_n)^{-1}(\overline{a})$ (resp. $(\mathfrak{h}^\dagger_n)^{-1}(\underline{a})$). Note that, by definition of $\overline{a}$, the set $(\mathfrak{h}^\dagger_n)^{-1}(\overline{a})$ intersects $V^\dagger_{R'}$, and similarly, by definition of $\underline{a}$, the set $(\mathfrak{h}^\dagger_n)^{-1}(\underline{a})$ intersects $V^\dagger_{-R'}$.

Random walk notation. For $\mathbf{x}\in\mathcal{VG}^{\dagger,n}(R')$, we consider the continuous time random walk $\{\mathbf{X}^{n,\mathbf{x}}_t\}_{t\geq 0}$ on the lifted weighted graph $(\mathcal{G}^{\dagger,n}, c^{\dagger,n})$ started from $\mathbf{x}\in\mathcal{VG}^{\dagger,n}(R')$. We recall that the continuous path of this random walk is generated by piecewise linear interpolation at constant speed. We consider the following stopping times

$$\sigma_{\mathbf{x}} \overset{\text{def}}{=} \inf\{t\geq 0 \,:\, \mathbf{X}^{n,\mathbf{x}}_t \in V^\dagger_{R'}\cup V^\dagger_{-R'}\}, \quad \tau_{\mathbf{x}} \overset{\text{def}}{=} \inf\{t\geq 0 \,:\, \mathbf{X}^{n,\mathbf{x}}_t \in (\mathfrak{h}^\dagger_n)^{-1}(\overline{a})\cup(\mathfrak{h}^\dagger_n)^{-1}(\underline{a})\}, \quad (5.4.2)$$

and we observe that, thanks to the definitions of $\overline{a}$ and $\underline{a}$, it holds that $\tau_{\mathbf{x}}\geq\sigma_{\mathbf{x}}$, for all $\mathbf{x}\in\mathcal{VG}^{\dagger,n}(R')$. Looking at Figure 5.7, the stopping time $\sigma_{\mathbf{x}}$ accounts for the first time at which the random walk hits one of the blue vertices, while the stopping time $\tau_{\mathbf{x}}$ accounts for the first time at which the random walk hits one of the green vertices.

5.4.1.2 Proof of Proposition 5.4.2

We can now state a key lemma for the proof of Proposition 5.4.2. The proof of the below result is given in Subsection 5.4.1.3.

Lemma 5.4.3. *For any $n > n(R,\delta)$ large enough, there exists a real number $b'_n = b'_n(R,\delta)$ such that*

$$\left|2R'\mathbb{P}_{\mathbf{x}}\left(\mathbf{X}^n_{\tau_{\mathbf{x}}}\in(\mathfrak{h}^\dagger_n)^{-1}(\overline{a})\right) - b'_n - \mathrm{Im}(\mathbf{x})\right| \leq \delta, \quad \forall\,\mathbf{x}\in\mathcal{VG}^{\dagger,n}(R).$$

Similarly, for any $n > n(R,\delta)$ large enough, there exists a real number $b_n'' = b_n''(R,\delta)$ such that

$$\left|2R'\mathbb{P}_{\mathbf{x}}\left(\mathbf{X}^n_{\tau_{\mathbf{x}}} \in \sigma_{2\pi}^{-1}(\mathfrak{h}_n^{-1}(\underline{a}))\right) + b_n'' + \mathrm{Im}(\mathbf{x})\right| \leq \delta, \quad \forall \mathbf{x} \in \mathcal{VG}^{\dagger,n}(R).$$

Given Lemma 5.4.3, the proof of Proposition 5.4.2 follows by using the harmonicity of the height coordinate function.

Proof of Proposition 5.4.2. We divide the proof in two steps.

Step 1. In this first step we show that, for fixed $R > 1$ and $\delta \in (0,1)$, for any $n > n(R,\delta)$ large enough, we can find real numbers $b_n^{R,\delta}$ and $c_n^{R,\delta}$ such that the conclusion of the proposition holds. To this end, let $R' \overset{\text{def}}{=} R/\delta$, fix $\mathbf{x} \in \mathcal{VG}^{\dagger,n}(R)$ and let $\mathbf{X}^{n,\mathbf{x}}$ be the random walk on $(\mathcal{G}^{\dagger,n}, c^{\dagger,n})$ started from $\mathbf{x}$. Thanks to the harmonicity of the height coordinate function $\mathfrak{h}_n^\dagger$ and to the optional stopping theorem, we have that

$$\mathfrak{h}_n^\dagger(\mathbf{x}) = \overline{a}\mathbb{P}_{\mathbf{x}}\left(\mathbf{X}^n_{\tau_{\mathbf{x}}} \in (\mathfrak{h}_n^\dagger)^{-1}(\overline{a})\right) + \underline{a}\mathbb{P}_{\mathbf{x}}\left(\mathbf{X}^n_{\tau_{\mathbf{x}}} \in (\mathfrak{h}_n^\dagger)^{-1}(\underline{a})\right), \quad \forall \mathbf{x} \in \mathcal{VG}^{\dagger,n}(R).$$

Therefore, the problem has been reduced to proving that the probabilities appearing in the previous expressions are approximately affine transformations of the height coordinate function in the a priori embedding. By Lemma 5.4.3, for all $n > n(R,\delta)$ large enough, there exist real numbers $b_n' = b_n'(R,\delta)$ and $b_n'' = b_n''(R,\delta)$ for which, for all $\mathbf{x} \in \mathcal{VG}^{\dagger,n}(R)$, it holds that

$$\begin{aligned}
&\left|2R'\mathfrak{h}_n^\dagger(\mathbf{x}) - \overline{a}(b_n' + \mathrm{Im}(\mathbf{x})) + \underline{a}(b_n'' + \mathrm{Im}(\mathbf{x})\right| \\
&\quad \leq \overline{a}\left|2R'\mathbb{P}_{\mathbf{x}}\left(\mathbf{X}^n_{\tau_{\mathbf{x}}} \in (\mathfrak{h}_n^\dagger)^{-1}(\overline{a})\right) - b_n' - \mathrm{Im}(\mathbf{x})\right| + \underline{a}\left|2R'\mathbb{P}_{\mathbf{x}}\left(\mathbf{X}^n_{\tau_{\mathbf{x}}} \in (\mathfrak{h}_n^\dagger)^{-1}(\underline{a})\right) + b_n'' + \mathrm{Im}(\mathbf{x})\right| \\
&\quad \leq \left|2R'\mathbb{P}_{\mathbf{x}}\left(\mathbf{X}^n_{\tau_{\mathbf{x}}} \in (\mathfrak{h}_n^\dagger)^{-1}(\overline{a})\right) - b_n' - \mathrm{Im}(\mathbf{x})\right| + \left|2R'\mathbb{P}_{\mathbf{x}}\left(\mathbf{X}^n_{\tau_{\mathbf{x}}} \in (\mathfrak{h}_n^\dagger)^{-1}(\underline{a})\right) + b_n'' + \mathrm{Im}(\mathbf{x})\right| \\
&\quad \leq \delta.
\end{aligned}$$

Therefore, rearranging the terms in the above expression and letting

$$c_n^{R,\delta} \overset{\text{def}}{=} \frac{2R'}{|\overline{a} - \underline{a}|}, \qquad b_n^{R,\delta} \overset{\text{def}}{=} \frac{\underline{a}b_n'' - \overline{a}b_n'}{|\overline{a} - \underline{a}|},$$

we find that, for any $n > n(R,\delta)$ large enough, it holds that

$$\left|c_n^{R,\delta}\mathfrak{h}_n^\dagger(\mathbf{x}) + b_n^{R,\delta} - \mathrm{Im}(\mathbf{x})\right| \leq \delta, \quad \forall \mathbf{x} \in \mathcal{VG}^{\dagger,n}(R).$$

Step 2. In this second step, we show how we can define real sequences $\{b_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$ independent of R, δ such that the conclusion of the proposition holds. To this end, consider an increasing sequence $\{R_k\}_{k\in\mathbb{N}} \subset [1,\infty)$ and a decreasing sequence $\{\delta_k\}_{k\in\mathbb{N}} \subset (0,1)$ such that $R_k \to \infty$ and $\delta_k \to 0$, as $k \to \infty$. Then, thanks to the previous step, we know that, for all $k \in \mathbb{N}$, there exists $n_k = n_k(R_k,\delta_k) \in \mathbb{N}$ such that, for all $n > n_k$, there exist real numbers $c_n^{R_k,\delta_k}$, $b_n^{R_k,\delta_k} \in \mathbb{R}$ such that

$$\left|c_n^{R_k,\delta_k}\mathfrak{h}_n^\dagger(\mathbf{x}) + b_n^{R_k,\delta_k} - \mathrm{Im}(\mathbf{x})\right| \leq \delta_k, \quad \forall \mathbf{x} \in \mathcal{VG}^{\dagger,n}(R_k).$$

Without any loss of generality, we can assume that the sequence $\{n_k\}_{k\in\mathbb{N}}$ is increasing. Then, for all $n \in [0,n_1) \cap \mathbb{N}$, we let $c_n^\mathfrak{h} \overset{\text{def}}{=} 1$, $b_n^\mathfrak{h} \overset{\text{def}}{=} 1$, and for all $k \in \mathbb{N}$ and $n \in [n_k, n_{k+1}) \cap \mathbb{N}$, we set $c_n^\mathfrak{h} \overset{\text{def}}{=} c_n^{R_k,\delta_k}$, $b_n^\mathfrak{h} \overset{\text{def}}{=} b_n^{R_k,\delta_k}$. Therefore, if we fix $R > 1$ and $\delta \in (0,1)$, for all $n > n(R,\delta)$ large enough, it holds that

$$\left|c_n^\mathfrak{h}\mathfrak{h}_n^\dagger(\mathbf{x}) + b_n^\mathfrak{h} - \mathrm{Im}(\mathbf{x})\right| \leq \delta, \quad \forall \mathbf{x} \in \mathcal{VG}^{\dagger,n}(R),$$

which concludes the proof. □

5.4.1.3 Proof of Lemma 5.4.3

The main goal of this subsection is prove the key Lemma 5.4.3. Roughly speaking, the first estimate in Lemma 5.4.3 states that, for $n > n(R,\delta)$ large enough, the probability that the lifted random walk started inside $\mathcal{VG}^{\dagger,n}(R)$ hits the set $(\mathfrak{h}_n^\dagger)^{-1}(\overline{a})$ before hitting $(\mathfrak{h}_n^\dagger)^{-1}(\underline{a})$ depends linearly on the height coordinate of the starting point of the walk on the a priori embedding. In order to prove this result, we need to rule out the possibility that the preimage of a horizontal line on the Smith embedding has large vertical fluctuations (see Figure 5.7). To do so, we use the invariance principle assumption on the sequence of primal graphs, and more precisely we will follow the following steps.

(a) We start by proving that the probability that the lifted random walk started inside $\mathcal{VG}^{\dagger,n}(R)$ hits the set $V_{R'}^\dagger$ before hitting $V_{-R'}^\dagger$ depends linearly on the height coordinate of the starting point of the walk on the a priori embedding. This follows easily thanks to the invariance principle assumption and it is the content of Lemma 5.4.4 below.

(b) We then prove that the probability that the random walk started from $V_{-R'}^\dagger$ has probability of order $1/R'$ to hit $(\mathfrak{h}_n^\dagger)^{-1}(\overline{a})$ before hitting $(\mathfrak{h}_n^\dagger)^{-1}(\underline{a})$. Once again, this is an easy consequence of the invariance principle assumption, and it is the content of Lemma 5.4.5 below.

(c) Finally, roughly speaking, in order to conclude, we need to improve the bound on the probability appearing in the previous step from order $1/R'$ to order $o_n(1/R')$. This is done by using Lemma 5.2.4 together with the invariance principle assumption. Indeed, as it will be more clear in the proof of Lemma 5.4.3, it is sufficient to estimate the probability that a random walk started inside $\mathcal{VG}^n(R)$ does not disconnect $V_R \cup V_{-R}$ from $V_{R'} \cup V_{-R'}$ before hitting the latter set. In Lemma 5.4.6 below, we will see that this probability is of order R'/R, and this will be enough to conclude.

Before proceeding, we observe that, for all $\mathbf{x} \in \mathcal{VG}^{\dagger,n}(R)$, it holds that

$$\mathbb{P}_{\mathbf{x}}\big(\mathbf{X}^n_{\tau_{\mathbf{x}}} \in \sigma_{2\pi}^{-1}(\mathfrak{h}_n^{-1}(\underline{a}))\big) = 1 - \mathbb{P}_{\mathbf{x}}\big(\mathbf{X}^n_{\tau_{\mathbf{x}}} \in (\mathfrak{h}_n^\dagger)^{-1}(\overline{a})\big),$$

hence, from now on, we can just focus on the first estimate in the statement of Lemma 5.4.3.

We can now state and prove the technical lemmas mentioned above. We start with the following lemma where we study the probability that the lifted random walk started inside $\mathcal{VG}^{\dagger,n}(R)$ hits the set $V_{R'}^\dagger$ before hitting $V_{-R'}^\dagger$.

Lemma 5.4.4. *For any $n > n(R,\delta)$ large enough, it holds that*

$$\big|2R'\mathbb{P}_{\mathbf{x}}\big(\mathbf{X}^n_{\sigma_{\mathbf{x}}} \in V_{R'}^\dagger\big) - R' - \mathrm{Im}(\mathbf{x})\big| \le \delta, \quad \forall\, \mathbf{x} \in \mathcal{VG}^{\dagger,n}(R).$$

Proof. For $n \in \mathbb{N}$, fix a vertex $\mathbf{x} \in \mathcal{VG}^{\dagger,n}(R)$, consider a planar Brownian motion $B^{\mathbf{x}}$ started from $\mathbf{x}$, and define the stopping time

$$\sigma_{B,\mathbf{x}} \overset{\text{def}}{=} \inf\big\{t \ge 0 \,:\, \big|\mathrm{Im}(B_t^{\mathbf{x}})\big| = R'\big\}.$$

Then, by assumption (H2), for any $n > n(R,\delta)$ large enough, we have that

$$\big|\mathbb{P}_{\mathbf{x}}(\mathbf{X}^n_{\sigma_{\mathbf{x}}} \in \sigma_{2\pi}^{-1}(V_{R'})) - \mathbb{P}_{\mathbf{x}}\big(\mathrm{Im}(B_{\sigma_{B,\mathbf{x}}}) = R'\big)\big| \le \frac{\delta}{2R'}, \quad \forall\, \mathbf{x} \in \mathcal{VG}^{\dagger,n}(R).$$

Since $\mathrm{Im}(B^{\mathbf{x}})$ is just a linear Brownian motion started from $\mathrm{Im}(\mathbf{x})$, thanks to the gambler's ruin formula, it holds that

$$\mathbb{P}_{\mathbf{x}}\big(\mathrm{Im}(B_{\sigma_{B,\mathbf{x}}}) = R'\big) = \frac{R' + \mathrm{Im}(\mathbf{x})}{2R'},$$

from which the conclusion follows. □

We can now move to the second lemma which gives an estimate for the probability that the random walk started from $V^{\dagger}_{-R'}$ hits $(\mathfrak{h}^{\dagger}_n)^{-1}(\overline{a})$ before hitting $(\mathfrak{h}^{\dagger}_n)^{-1}(\underline{a})$.

Lemma 5.4.5. *For any $n > n(R,\delta)$ large enough, it holds that*

$$\mathbb{P}_{\mathbf{x}}\big(\mathbf{X}^n_{\tau_{\mathbf{x}}} \in (\mathfrak{h}^{\dagger}_n)^{-1}(\overline{a})\big) \lesssim \frac{1}{R'}, \quad \forall\, \mathbf{x} \in V^{\dagger}_{-R'},$$

where the implicit constant is independent of everything else.

Proof. We start by noticing that, for all $\mathbf{x} \in V^{\dagger}_{-R'}$, it holds that

$$\mathbb{P}_{\mathbf{x}}\big(\mathbf{X}^n_{\tau_{\mathbf{x}}} \in (\mathfrak{h}^{\dagger}_n)^{-1}(\overline{a})\big) \leq \mathbb{P}_{\mathbf{x}}\big(\sigma_{2\pi}(\mathbf{X}^n|_{[0,\tau_{\mathbf{x}}]}) \text{ does not wind around the cylinder below height } -R'\big),$$

where we recall that $\sigma_{2\pi}$ is defined in (5.2.4) and denotes the covering map of the infinite cylinder $\mathcal{C}_{2\pi}$. The above inequality is due to the fact that, if $\sigma_{2\pi}(\mathbf{X}^n|_{[0,\tau_{\mathbf{x}}]})$ winds around the cylinder below height $-R'$, then, by definition of $\underline{a}$, it has to hit the set $\mathfrak{h}_n^{-1}(\underline{a})$. We can now exploit assumption (H2) and find the corresponding upper bound for the Brownian motion. More precisely, let $B^{\mathbf{x}}$ be a planar Brownian motion started from $\mathbf{x} \in \sigma_{2\pi}^{-1}(V_{-R'})$ and define the stopping time

$$\tau_{B,\mathbf{x}} \overset{\text{def}}{=} \inf\big\{t \geq 0 \,:\, \mathrm{Im}(B^{\mathbf{x}}_t) = -2R' \text{ or } \mathrm{Im}(B^{\mathbf{x}}_t) = R'\big\}.$$

Then, for any $n > n(R,\delta)$ large enough, we have that

$$\begin{aligned}&\mathbb{P}_{\mathbf{x}}\big(\sigma_{2\pi}(\mathbf{X}^n|_{[0,\tau_{\mathbf{x}}]}) \text{ does not wind around the cylinder below height } -R'\big)\\ &\quad\leq 2\mathbb{P}_{\mathbf{x}}\big(\sigma_{2\pi}(B|_{[0,\tau_{B,\mathbf{x}}]}) \text{ does not wind around the cylinder below height } -R'\big).\end{aligned}$$

Therefore, to conclude, it is sufficient to find a uniform upper bound for the quantity on the right-hand side of the above expression. This is done in Lemma 5.A.1 from which the conclusion follows. □

In order to prove Lemma 5.4.3, we also need the following lemma which provides an estimate for the probability that a random walk started inside $\mathcal{VG}^n(R)$ disconnects $V_R \cup V_{-R}$ from $V_{R'} \cup V_{-R'}$ before hitting the latter set.

Lemma 5.4.6. *For any $n > n(R,\delta)$ large enough, it holds that*

$$\mathbb{P}_{\mathbf{x}}\big(\sigma_{2\pi}(\mathbf{X}^n|_{[0,\sigma_{\mathbf{x}}]}) \textit{ does not disconnect } V_R \cup V_{-R} \textit{ from } V_{R'} \cup V_{-R'}\big) \lesssim \frac{R}{R'}, \quad \forall\, \mathbf{x} \in \mathcal{VG}^{\dagger,n}(R),$$

where the implicit constant is independent of everything else.

Proof. For $\mathbf{x} \in \mathcal{VG}^{\dagger,n}(R)$, let $B^{\mathbf{x}}$ be a planar Brownian motion started from $\mathbf{x}$, and define the stopping time

$$\sigma_{B,\mathbf{x}} := \inf\big\{t \geq 0 \,:\, |\,\mathrm{Im}(B^{\mathbf{x}}_t)| = R'\big\}.$$

By assumption (H2), we know that for any $n > n(R,\delta)$ large enough, it holds that

$$\begin{aligned}&\mathbb{P}_{\mathbf{x}}\big(\sigma_{2\pi}(\mathbf{X}^n|_{[0,\sigma_{\mathbf{x}}]}) \text{ does not disconnect } V_R \cup V_{-R} \text{ from } V_{R'} \cup V_{-R'}\big)\\ &\quad\leq 2\mathbb{P}_{\mathbf{x}}\big(\sigma_{2\pi}(B|_{[0,\sigma_{B,\mathbf{x}}]}) \text{ does not disconnect } \mathbb{R}/2\pi\mathbb{Z}\times\{-R,R\} \text{ from } \mathbb{R}/\mathbb{Z}\times\{-R',R'\}\big).\end{aligned}$$

Therefore, it is sufficient to find a uniform upper bound for the quantity on the right-hand side of the above expression. This is the content of Lemma 5.A.2 from which the desired result follows. □

We are now ready to give a proof of Lemma 5.4.3. As we have already remarked, this will be a consequence of the previous three lemmas and of Lemma 5.2.4.

Proof of Lemma 5.4.3. We start by defining the function $\mathfrak{f}_n^\dagger : \mathcal{VG}^{\dagger,n}(R') \to \mathbb{R}$ as follows

$$\mathfrak{f}_n^\dagger(\mathbf{x}) := \mathbb{P}_{\mathbf{x}}\big(\mathbf{X}^n_{\tau_{\mathbf{x}}} \in (\mathfrak{h}_n^\dagger)^{-1}(\overline{a})\big) - \mathbb{P}_{\mathbf{x}}\big(\mathbf{X}^n_{\sigma_{\mathbf{x}}} \in V^\dagger_{R'}\big), \qquad \forall\,\mathbf{x}\in\mathcal{VG}^{\dagger,n}(R'),$$

so that, we can write

$$\mathbb{P}_{\mathbf{x}}\big(\mathbf{X}^n_{\tau_{\mathbf{x}}} \in (\mathfrak{h}_n^\dagger)^{-1}(\overline{a})\big) = \mathbb{P}_{\mathbf{x}}\big(\mathbf{X}^n_{\sigma_{\mathbf{x}}} \in V^\dagger_{R'}\big) + \mathfrak{f}_n^\dagger(\mathbf{x}), \qquad \forall\,\mathbf{x}\in\mathcal{VG}^{\dagger,n}(R'). \tag{5.4.3}$$

We now observe that, thanks to Lemma 5.4.4, for any $n > n(R,\delta)$ large enough, it holds that

$$\big|2R'\mathbb{P}_{\mathbf{x}}\big(\mathbf{X}^n_{\sigma_{\mathbf{x}}} \in V^\dagger_{R'}\big) - R' - \mathrm{Im}(\mathbf{x})\big| \leq \delta, \quad \forall\,\mathbf{x}\in\mathcal{VG}^{\dagger,n}(R). \tag{5.4.4}$$

Therefore, it is sufficient to study the function $\mathfrak{f}_n^\dagger$ appearing in (5.4.3). To this end, we consider the functions $\overline{\mathfrak{f}}_n^\dagger : \mathcal{VG}^{\dagger,n}(R') \to [0,1]$ and $\underline{\mathfrak{f}}_n^\dagger : \mathcal{VG}^{\dagger,n}(R') \to [0,1]$ defined as follows

$$\overline{\mathfrak{f}}_n^\dagger(\mathbf{x}) \overset{\text{def}}{=} \mathbb{P}_{\mathbf{x}}\big(\mathbf{X}^n_{\sigma_{\mathbf{x}}} \in V^\dagger_{-R'},\, \mathbf{X}^n_{\tau_{\mathbf{x}}} \in (\mathfrak{h}_n^\dagger)^{-1}(\overline{a})\big), \qquad \underline{\mathfrak{f}}_n^\dagger(\mathbf{x}) \overset{\text{def}}{=} \mathbb{P}_{\mathbf{x}}\big(\mathbf{X}^n_{\sigma_{\mathbf{x}}} \in V^\dagger_{R'},\, \mathbf{X}^n_{\tau_{\mathbf{x}}} \in (\mathfrak{h}_n^\dagger)^{-1}(\underline{a})\big).$$

In particular, as one can easily check, it holds that

$$\big|\mathfrak{f}_n^\dagger(\mathbf{x})\big| \leq \overline{\mathfrak{f}}_n^\dagger(\mathbf{x}) + \underline{\mathfrak{f}}_n^\dagger(\mathbf{x}), \quad \forall\,\mathbf{x}\in\mathcal{VG}^{\dagger,n}(R'),$$

and so, we can reduce the problem to the study of the functions $\overline{\mathfrak{f}}_n^\dagger$ and $\underline{\mathfrak{f}}_n^\dagger$. We will only study the function $\overline{\mathfrak{f}}_n^\dagger$ as the function $\underline{\mathfrak{f}}_n^\dagger$ can be treated similarly. Thanks to the strong Markov property of the random walk $\mathbf{X}^{n,\mathbf{x}}$, we have that

$$\overline{\mathfrak{f}}_n^\dagger(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}\big[\overline{\mathfrak{f}}_n^\dagger(\mathbf{X}^n_{\sigma_{\mathbf{x}}})\big], \quad \forall\,\mathbf{x}\in\mathcal{VG}^{\dagger,n}(R').$$

Therefore, for $\mathbf{x},\mathbf{y}\in\mathcal{VG}^{\dagger,n}(R)$, it holds that

$$\big|\overline{\mathfrak{f}}_n^\dagger(\mathbf{x}) - \overline{\mathfrak{f}}_n^\dagger(\mathbf{y})\big| \leq \sup\big\{\big|\overline{\mathfrak{f}}_n^\dagger(\mathbf{v})\big| \,:\, \mathbf{v}\in V^\dagger_{-R'}\big\}\,\mathrm{d}^{\mathrm{TV}}\big(\mathbf{X}^{n,\mathbf{x}}_{\sigma_{\mathbf{x}}}, \mathbf{X}^{n,\mathbf{y}}_{\sigma_{\mathbf{y}}}\big), \tag{5.4.5}$$

where d^{TV} denotes the total variation distance. Hence, it is sufficient to find an upper bound for the two terms on the right-hand side of (5.4.5). We treat the two factors separately.

- In order to bound the first factor, we just need to bound uniformly on $\mathbf{v}\in V^\dagger_{-R'}$ the probability that a random walk on $(\mathcal{G}^{\dagger,n}, c^{\dagger,n})$ started from $\mathbf{v}$ hits $(\mathfrak{h}_n^\dagger)^{-1}(\overline{a})$ before hitting $(\mathfrak{h}_n^\dagger)^{-1}(\underline{a})$. This is exactly the content of Lemma 5.4.5 from which we can deduce that, for all $n > n(R,\delta)$ large enough, it holds that

$$\sup\big\{\big|\overline{\mathfrak{f}}_n^\dagger(\mathbf{v})\big| \,:\, \mathbf{v}\in V^\dagger_{-R'}\big\} \lesssim \frac{1}{R'},$$

where the implicit constant is universal.

- In order to bound the second factor, we can use Lemma 5.2.4. More precisely, it is sufficient to estimate the probability that $\sigma_{2\pi}(\mathbf{X}^{n,\mathbf{x}}|_{[0,\sigma_{\mathbf{x}}]})$ disconnects $V_R \cup V_{-R}$ from $V_{R'} \cup V_{-R'}$. This is exactly the content of Lemma 5.4.6 which guarantees that, for all $n > n(R,\delta)$ large enough and for all $\mathbf{x} \in \mathcal{VG}^{\dagger,n}(R)$, it holds that

$$\mathbb{P}_{\mathbf{x}}\big(\sigma_{2\pi}(\mathbf{X}^{n}|_{[0,\sigma_{\mathbf{x}}]}) \text{ does not disconnect } V_R \cup V_{-R} \text{ from } V_{R'} \cup V_{-R'}\big) \lesssim \frac{R}{R'},$$

where the implicit constant is independent of everything else. Therefore, this fact together with Lemma 5.2.4 imply that

$$\mathrm{d}^{\mathrm{TV}}\big(\mathbf{X}^{n,\mathbf{x}}_{\sigma_{\mathbf{x}}}, \mathbf{X}^{n,\mathbf{y}}_{\sigma_{\mathbf{y}}}\big) \lesssim \frac{R}{R'}, \quad \forall\, \mathbf{x},\mathbf{y} \in \mathcal{VG}^{\dagger,n}(R).$$

Therefore, putting together the previous two bullet points and going back to (5.4.5), we get that for every $n > n(R,\delta)$ large enough, it holds that

$$\big|\bar{\mathfrak{f}}^{\dagger}_n(\mathbf{x}) - \bar{\mathfrak{f}}^{\dagger}_n(\mathbf{y})\big| \lesssim \frac{R}{R'^2}, \quad \forall\, \mathbf{x},\mathbf{y} \in \mathcal{VG}^{\dagger,n}(R),$$

Furthermore, the same uniform bound can be also obtained for the function $\underline{\mathfrak{f}}^{\dagger}_n$, but we omit the details since the argument is similar. Summing up, we obtained that, for all $n > n(R,\delta)$ large enough, it holds that

$$\big|\mathfrak{f}^{\dagger}_n(\mathbf{x}) - \mathfrak{f}^{\dagger}_n(\mathbf{y})\big| \lesssim \frac{R}{R'^2}, \quad \forall\, \mathbf{x},\, \mathbf{y} \in \mathcal{VG}^{\dagger,n}(R).$$

Hence, to conclude, we can simply proceed as follows. For every $n > n(R,\delta)$ large enough, fix an arbitrary vertex $\mathbf{y} \in \mathcal{VG}^{\dagger,n}(R)$. Then, recalling that by definition $R' = R/\delta$, it holds that

$$2R'|\mathfrak{f}^{\dagger}_n(\mathbf{x}) - \mathfrak{f}^{\dagger}_n(\mathbf{y})| \lesssim \delta, \quad \forall\, \mathbf{x} \in \mathcal{VG}^{\dagger,n}(R).$$

Therefore, thanks to (5.4.3) and estimate (5.4.4), we find that, for any $n > n(R,\delta)$ large enough, it holds that

$$\big|2R'\mathbb{P}_{\mathbf{x}}\big(\mathbf{X}^{n}_{\tau_{\mathbf{x}}} \in (\mathfrak{h}^{\dagger}_n)^{-1}(\overline{a})\big) - b'_n - \mathrm{Im}(\mathbf{x})\big| \leq \delta, \quad \forall\, \mathbf{x} \in \mathcal{VG}^{\dagger,n}(R),$$

where $b'_n \overset{\text{def}}{=} 2R'\mathfrak{f}^{\dagger}_n(\mathbf{y}) + R'$. □

5.4.2 Width coordinate function

For each $n \in \mathbb{N}$, consider the lifted width coordinate function

$$\mathfrak{w}^{\dagger}_n : \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n} \to \mathbb{R},$$

as defined in (5.2.10). The main goal of this subsection is to show that there exists an affine transformation of the function $\mathfrak{w}^{\dagger}_n$ that is asymptotically close, uniformly on lifts of compact subsets of the infinite cylinder $\mathcal{C}_{2\pi}$, to the width coordinate function $\widehat{\mathbf{x}} \mapsto \mathrm{Re}(\widehat{\mathbf{x}})$ in the a priori lifted embedded graph $\mathcal{G}^{\dagger,n}$, as $n \to \infty$. More precisely, we have the following result.

Proposition 5.4.7. *There exits a sequence* $\{b_n^{\mathfrak{w}}\}_{n\in\mathbb{N}} \subset \mathbb{R}$ *such that for all* $R > 1$, $\delta \in (0,1)$, *and for any* $n > n(R,\delta)$ *large enough, it holds that*

$$\left|\frac{2\pi}{\eta_n}\mathfrak{w}_n^\dagger(\widehat{\mathbf{x}}) + b_n^{\mathfrak{w}} - \mathrm{Re}(\widehat{\mathbf{x}})\right| \leq \delta, \quad \forall\, \widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(\mathbb{R}\times[-R,R]), \tag{5.4.6}$$

where we recall that η_n *denotes the strength of the flow induced by* $\mathfrak{h}_n$ *as defined in* (5.2.8).

The proof of this proposition is given in Subsection 5.4.2.2. As we will see, the proof is based on Lemma 5.4.9 below and the following two facts: (a) the harmonicity of the lifted width coordinate function $\mathfrak{w}_n^\dagger$; (b) the invariance principle assumption (H3) on the sequence of dual maps.

5.4.2.1 Setup of the proof

For each $n \in \mathbb{N}$, consider the metric graphs $\mathbb{G}^n$ and $\widehat{\mathbb{G}}^n$ associated to $\mathcal{G}^n$ and $\widehat{\mathcal{G}}^n$, respectively. Let $\mathfrak{h}_n : \mathbb{G}^n \to [0,1]$ be the extended height coordinate function as specified in Remark 5.2.7, and $\mathfrak{w}_n : \widehat{\mathbb{G}}^n \to \mathbb{R}$ be the extended width coordinate function as specified in Remark 5.2.11. Given a value $S \in \mathbb{R}$, we define the set

$$\widehat{V}_S^n \overset{\text{def}}{=} \big\{\widehat{x} \in \widehat{\mathbb{G}}^n \,:\, \mathrm{Im}(\widehat{x}) = S\big\}.$$

Given a value $a \in (0,1)$, we recall that the sets $\mathcal{E}\mathcal{G}_a^\dagger$, $\mathcal{E}\widehat{\mathcal{G}}_a^\dagger$, $\mathcal{V}\mathcal{G}_a^\dagger$, $\mathcal{V}\widehat{\mathcal{G}}_a^\dagger$ are all defined in Subsection 5.3.2. We also recall that $\widehat{\mathbf{x}}_0^{n,a}$ denotes the vertex in $\mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}$ whose real coordinate is nearest to 0, and we refer to (5.3.1) and (5.3.2) for the definitions of the sets $\widehat{\mathcal{P}}_a^{\dagger,n}$ and $\mathcal{P}_a^{\dagger,n}$, respectively.

We fix throughout $R > 1$ and $\delta \in (0,1)$ as in the proposition statement, and we let

$$R' \overset{\text{def}}{=} \frac{R}{\delta}.$$

For each $n \in \mathbb{N}$, we consider $\overline{a}_{R'}^n$ and $\underline{a}_{-R'}^n$ as defined in (5.4.1). Moreover, thanks to Proposition 5.4.2, for any $n > n(R,\delta)$ large enough, we can choose a value $a_{R'}^n$ such that the set $\mathfrak{h}_n^{-1}(a_{R'}^n)$ is fully contained in the infinite strip $\mathbb{R}\times[-1,1]$.

In what follows, in order to lighten up notation, we adopt the following notational convention

$$\widehat{V}_S \equiv \widehat{V}_S^n, \qquad \overline{a} \equiv \overline{a}_{R'}^n, \quad \underline{a} \equiv \underline{a}_{-R'}^n, \quad a \equiv a_{R'}^n.$$

Furthermore, we denote by $\widehat{V}_S^\dagger$ the lift to the universal cover of $\widehat{V}_S$, and we write $\mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(S)$ (resp. $\mathcal{V}\widehat{\mathcal{G}}^n(S)$) as a shorthand for $\mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(\mathbb{R}\times[-S,S])$ (resp. $\mathcal{V}\widehat{\mathcal{G}}^n(\mathbb{R}/2\pi\mathbb{Z}\times[-S,S])$).

We also adopt the convention to fix the additive constant of the lifted width coordinate function $\mathfrak{w}_n^\dagger$ by setting it to zero on the point $\widehat{\mathbf{x}}_0^{n,a}$. We observe that changing the base vertex of the lifted width coordinate function has only the effect of changing the value of $b_n^{\mathfrak{w}}$ appearing in the statement of Proposition 5.4.7.

Similarly to what we discussed in the preceding subsection, by Lemma 5.3.3 and Remark 5.3.5, we can assume, without loss of generality, that $\mathcal{V}\mathcal{G}^n$ contains all the points in the set

$$\mathfrak{h}_n^\dagger(\overline{a}) \cup \mathfrak{h}_n^{-1}(\underline{a}) \cup \mathfrak{h}_n^{-1}(a) \subset \mathbb{G}^n,$$

and that $\mathcal{V}\widehat{\mathcal{G}}^n$ contains all the points in the set

$$\widehat{V}_{R'-1} \cup \widehat{V}_{-R'+1} \subset \widehat{\mathbb{G}}^n.$$

This is allowed since, as in the case of the height coordinate function, by possibly locally modifying the a priori embedding of the dual graph $\widehat{\mathcal{G}}^n$ in $\mathcal{C}_{2\pi}$, we can assume that each edge in $\mathcal{E}\widehat{\mathcal{G}}^n$ crosses the circles at height $R'-1$ and $-R'+1$ at most finitely many times. We refer to Figure 5.8 for an illustration of the sets involved in the proof of Proposition 5.4.7.

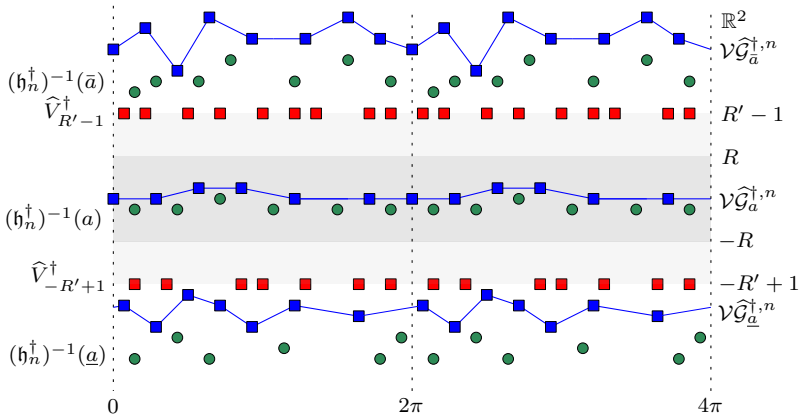


FIGURE 5.8: A diagram illustrating the sets involved in the proof of Proposition 5.4.7. The shaded dark-grey strip between heights R and $-R$ contains all the vertices in the set $\mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R)$. The shaded light-grey strip between heights $R'-1$ and $-R'+1$ contains all the vertices in the set $\mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R')$. The red dual vertices at the top (resp. bottom) are the vertices in the set $\widehat{V}^\dagger_{R'-1}$ (resp. $\widehat{V}^\dagger_{-R'+1}$). Moving from top to bottom, the blue vertices are the dual vertices in the sets $\mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}_{\overline{a}}$, $\mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}_{a}$, $\mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}_{\underline{a}}$; the blue dual edges are the edges in the sets $\mathcal{E}\widehat{\mathcal{G}}^{\dagger,n}_{\overline{a}}$, $\mathcal{E}\widehat{\mathcal{G}}^{\dagger,n}_{a}$,$\mathcal{E}\widehat{\mathcal{G}}^{\dagger,n}_{\underline{a}}$; and finally the green vertices are the vertices in the sets $(\mathfrak{h}^\dagger_n)^{-1}(\overline{a})$, $(\mathfrak{h}^\dagger_n)^{-1}(a)$, $(\mathfrak{h}^\dagger_n)^{-1}(\underline{a})$.

Remark 5.4.8. We observe that, by Proposition 5.4.2, for any $\varepsilon \in (0,1)$ and for any $n > n(R,\delta,\varepsilon)$ large enough, it holds that

$$(\mathfrak{h}^\dagger_n)^{-1}(\overline{a}) \subset \mathbb{R}\times[R'-\varepsilon, R'+\varepsilon], \qquad (\mathfrak{h}^\dagger_n)^{-1}(\underline{a}) \subset \mathbb{R}\times[-R'-\varepsilon, -R'+\varepsilon].$$

Furthermore, by definitions of $\mathcal{E}\widehat{\mathcal{G}}^{\dagger,n}_{\overline{a}}$, $\mathcal{E}\widehat{\mathcal{G}}^{\dagger,n}_{\underline{a}}$ and Lemma 5.4.1, we have that, for any $\varepsilon \in (0,1)$ and for any $n > n(R,\delta,\varepsilon)$ large enough, it holds that

$$\mathcal{E}\widehat{\mathcal{G}}^{\dagger,n}_{\overline{a}} \subset \mathbb{R}\times[R'-\varepsilon, R'+\varepsilon], \qquad \mathcal{E}\widehat{\mathcal{G}}^{\dagger,n}_{\underline{a}} \subset \mathbb{R}\times[-R'-\varepsilon, -R'+\varepsilon].$$

These facts will be of key importance in the remaining part of this subsection and they will be used several times.

Random walk notation. For $\widehat{\mathrm{x}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R'-1)$, we consider the continuous time random walk $\{\widehat{\mathbf{X}}^{n,\widehat{\mathrm{x}}}_t\}_{t\geq 0}$ on the lifted weighted dual graph $(\widehat{\mathcal{G}}^{\dagger,n}, \widehat{c}^{\dagger,n})$ started from $\widehat{\mathrm{x}}$. We recall that the continuous path of this random walk is generated by piecewise linear interpolation at constant speed. We consider the following stopping times

$$\sigma_{\widehat{\mathrm{x}}} \overset{\text{def}}{=} \inf\{t\geq 0 : \widehat{\mathbf{X}}^{n,\widehat{\mathrm{x}}}_t \in \widehat{V}^\dagger_{R'-1}\cup\widehat{V}^\dagger_{-R'+1}\}, \quad \tau_{\widehat{\mathrm{x}}} \overset{\text{def}}{=} \inf\{t\geq 0 : \widehat{\mathbf{X}}^{n,\widehat{\mathrm{x}}}_t \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}_{\overline{a}}\cup\mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}_{\underline{a}}\}. \tag{5.4.7}$$

As we will observe more precisely below, for $n > n(R,\delta)$ large enough, thanks to Proposition 5.4.2, we have that

$$\mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}_{\overline{a}} \subset \mathbb{R}\times[R'-1, R'+1], \qquad \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}_{\underline{a}} \subset \mathbb{R}\times[-R'-1, -R'+1],$$

and so, it holds that $\tau_{\widehat{\mathbf{x}}} \geq \sigma_{\widehat{\mathbf{x}}}$, for all $\widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R'-1)$. Looking at Figure 5.8, the stopping time $\sigma_{\widehat{\mathbf{x}}}$ accounts for the first time at which the random walk hits one of the red dual vertices, while the stopping time $\tau_{\widehat{\mathbf{x}}}$ accounts for the first time at which the random walk hits one of the blue dual vertices at the top or bottom of the figure.

In what follows, we also need to consider random walks on the sequence of primal lifted weighted graphs. To be precise, let a be as specified in the introduction of this section (we recall that here, a is a shorthand for $a^n_{R'}$) and consider the probability measure μ_a on the set $\mathcal{P}^{\dagger,n}_a$ (see Remark 5.3.12). We let $\mathbf{X}^{n,\mu_a}$ be the random walk on the lifted weighted graph $(\mathcal{G}^{\dagger,n}, c^{\dagger,n})$ started from a point in $\mathcal{P}^{\dagger,n}_a$ sampled according to μ_a. We also consider the associated continuous time version $\{\mathbf{X}^{n,\mu_a}_t\}_{t\geq 0}$ obtained by piecewise linear interpolation at constant speed. We define the following stopping times

$$\vartheta_+ \overset{\text{def}}{=} \inf\big\{t \geq 0 \,:\, \mathbf{X}^{n,\mu_a}_t \in (\mathfrak{h}^\dagger_n)^{-1}(\overline{a})\big\}, \quad \vartheta_- \overset{\text{def}}{=} \inf\big\{t \geq 0 \,:\, \mathbf{X}^{n,\mu_a}_t \in (\mathfrak{h}^\dagger_n)^{-1}(\underline{a})\big\}. \tag{5.4.8}$$

Furthermore, we will also need to consider the lifted Smith-embedded random walk $\{\dot{\mathbf{X}}^{n,\mu_a}_t\}_{t\geq 0}$ associated to $\mathbf{X}^{n,\mu_a}$ defined in Subsection 5.3.4.

5.4.2.2 Proof of Proposition 5.4.7

We can now state a key lemma for the proof of Proposition 5.4.7. The proof of the below result is given in Subsection 5.4.2.3.

Lemma 5.4.9. *For any $n > n(R,\delta)$ large enough, there exists a finite constant $b'_n \in \mathbb{R}$ such that*

$$\left|\mathbb{E}_{\widehat{\mathbf{x}}}\left[\frac{\mathrm{Re}(\widehat{\mathbf{X}}^n_{\tau_{\widehat{x}}})}{2\pi} - \frac{\mathfrak{w}^\dagger_n(\widehat{\mathbf{X}}^n_{\tau_{\widehat{x}}}) + b'_n}{\eta_n}\right]\right| \lesssim 1, \quad \forall\, \widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R'-1),$$

where the implicit constant is universal, and η_n denotes the strength of the flow induced by $\mathfrak{h}_n$ as defined in (5.2.8).

Given Lemma 5.4.9, the proof of Proposition 5.4.7 follows by using the harmonicity of the width coordinate function and the invariance principle assumption (H3).

Proof of Proposition 5.4.7. For $\widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R)$, let $\widehat{\mathbf{X}}^{n,\widehat{\mathbf{x}}}$ be the random walk on the lifted dual weighted graph $(\widehat{\mathcal{G}}^{\dagger,n}, \widehat{c}^{\dagger,n})$ started from $\widehat{\mathbf{x}}$. Moreover, let $B^{\widehat{\mathbf{x}}}$ be a planar Brownian motion started from $\widehat{\mathbf{x}}$, and define the stopping time

$$\tau_{B,\widehat{\mathbf{x}}} \overset{\text{def}}{=} \inf\big\{t \geq 0 \,:\, |\,\mathrm{Im}(B^{\widehat{\mathbf{x}}}_t)| = R'\big\}. \tag{5.4.9}$$

We divide the proof into several steps.

Step 1. In this first step, we show that, for any $n > n(R,\delta)$ large enough, it holds that

$$\big|\mathbb{E}_{\widehat{\mathbf{x}}}\big[\mathrm{Re}(\widehat{\mathbf{X}}^n_{\tau_{\widehat{\mathbf{x}}}})\big] - \mathrm{Re}(\widehat{\mathbf{x}})\big| \leq \delta, \tag{5.4.10}$$

where we recall that the stopping time $\tau_{\widehat{\mathbf{x}}}$ is defined in (5.4.7). As we will see below, this is an easy consequence of assumption (H3). We start by observing that, thanks to well-known properties of Brownian motion, it holds that

$$\big|\mathbb{E}_{\widehat{\mathbf{x}}}[\mathrm{Re}(B_{\tau_{B,\widehat{\mathbf{x}}}})] - \mathrm{Re}(\widehat{\mathbf{x}})\big| = 0, \quad \forall\, \widehat{\mathbf{x}} \in \mathcal{V}\mathcal{G}^{\dagger,n}(R).$$

Indeed, this follows from the fact that $|\,\mathrm{Re}(B_{\tau_{B,\widehat{\mathbf{x}}}})] - \mathrm{Re}(\widehat{\mathbf{x}})|$ has exponentially decaying tails and from the optional stopping theorem. As we observed in Remark 5.4.8, we have that, for any $\varepsilon \in (0,1)$ and for any

$n > n(R, \delta, \varepsilon)$ large enough, it holds that

$$\mathcal{E}\widehat{\mathcal{G}}^{\dagger,n}_{\overline{a}} \subset \mathbb{R} \times [R' - \varepsilon, R' + \varepsilon], \qquad \mathcal{E}\widehat{\mathcal{G}}^{\dagger,n}_{\underline{a}} \subset \mathbb{R} \times [-R' - \varepsilon, -R' + \varepsilon].$$

Therefore, from this fact and assumption (H3), we can deduce that, for any $n > n(R, \delta)$ large enough, it holds that

$$\left|\mathbb{E}_{\widehat{\mathbf{x}}}\left[\mathrm{Re}(B_{\tau_{B,\widehat{\mathbf{x}}}})\right] - \mathbb{E}_{\widehat{\mathbf{x}}}\left[\mathrm{Re}(\widehat{\mathbf{X}}^n_{\tau_{\widehat{x}}})\right]\right| \leq \delta.$$

More precisely, this fact can be obtained from assumption (H3) by arguing in the same exact way as in the proof of Lemma 5.4.11 below.

Step 2. The main goal of this step is to prove that, for any $n > n(R, \delta)$ large enough, it holds that

$$\mathbb{E}_{\widehat{\mathbf{x}}}\left[\mathfrak{w}^\dagger_n\left(\widehat{\mathbf{X}}^n_{\tau_{\widehat{\mathbf{x}}}}\right)\right] = \mathfrak{w}^\dagger_n(\widehat{\mathbf{x}}), \quad \forall\, \widehat{\mathbf{x}} \in \mathcal{V}\mathcal{G}^{\dagger,n}(R). \tag{5.4.11}$$

We start by recalling that the function $\mathfrak{w}^\dagger_n : \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n} \to \mathbb{R}$ is harmonic, and so the process $\mathfrak{w}^\dagger_n(\widehat{\mathbf{X}}^{n,\widehat{\mathbf{x}}})$ is a discrete martingale with respect to the filtration generated by $\widehat{\mathbf{X}}^{n,\widehat{\mathbf{x}}}$. Therefore, if we prove that this martingale is uniformly integrable, then the claim follows from the optional stopping theorem. To this end, it is sufficient to prove that, for $M \in \mathbb{N}$, the probability of the event $|\mathfrak{w}^\dagger_n(\widehat{\mathbf{X}}^{n,\widehat{\mathbf{x}}}_{\widehat{\tau}_{\widehat{\mathbf{x}}}}) - \mathfrak{w}^\dagger_n(\widehat{\mathbf{x}})| > MR'$ decays exponentially fast in M, uniformly in $\widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R)$ and for all $n > n(R, \delta)$ large enough. This fact can be obtained from assumption (H3) by arguing in the same exact way as in the proof of Lemma 5.4.10 below. Hence, the desired result follows.

Step 3. Consider the function $\mathfrak{f}^\dagger_n : \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R' - 1) \to \mathbb{R}$ defined as follows

$$\mathfrak{f}^\dagger_n(\widehat{\mathbf{x}}) := \mathbb{E}_{\widehat{\mathbf{x}}}\left[\frac{\mathrm{Re}(\widehat{\mathbf{X}}^n_{\tau_{\widehat{\mathbf{x}}}})}{2\pi} - \frac{\mathfrak{w}^\dagger_n(\widehat{\mathbf{X}}^n_{\tau_{\widehat{\mathbf{x}}}}) + b'_n}{\eta_n}\right], \quad \forall\, \widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R' - 1),$$

where b'_n is the same constant appearing in the statement of Lemma 5.4.9. Now, recalling the definition (5.4.7) of the stopping time $\sigma_{\widehat{\mathbf{x}}}$ and that $\sigma_{\widehat{\mathbf{x}}} \leq \tau_{\widehat{\mathbf{x}}}$, thanks to the strong Markov property of the random walk, for all $\widehat{\mathbf{x}}, \widehat{\mathbf{y}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R)$, it holds that

$$\left|\mathfrak{f}^\dagger_n(\widehat{\mathbf{x}}) - \mathfrak{f}^\dagger_n(\widehat{\mathbf{y}})\right| \leq \sup\left\{\left|\mathfrak{f}^\dagger_n(\widehat{\mathbf{v}})\right| \,:\, \widehat{\mathbf{v}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R' - 1)\right\} \mathrm{d}^{\mathrm{TV}}\left(\widehat{\mathbf{X}}^{n,\widehat{\mathbf{x}}}_{\sigma_{\widehat{\mathbf{x}}}}, \widehat{\mathbf{X}}^{n,\widehat{\mathbf{y}}}_{\sigma_{\widehat{\mathbf{y}}}}\right), \tag{5.4.12}$$

where d^{TV} denotes the total variation distance. Hence, it is sufficent to find an upper bound for the two terms on the right-hand side of (5.4.12). We treat the two factors separately.

- In order to bound the first factor, we just need to bound uniformly on $\mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R' - 1)$ the quantity $|\mathfrak{f}^\dagger_n(\mathbf{v})|$. This is exactly the content of Lemma 5.4.9 from which we can deduce that, for all $n > n(R, \delta)$ large enough, it holds that
$$\sup\left\{\left|\mathfrak{f}^\dagger_n(\widehat{\mathbf{v}})\right| \,:\, \widehat{\mathbf{v}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R' - 1)\right\} \lesssim 1,$$
where the implicit constant is independent of everything else.

- In order to bound the second factor, we can use Lemma 5.2.4. Indeed, as we have already remarked, thanks to Proposition 5.4.2 and to Lemma 5.4.1, for any $n \in \mathbb{N}$ large enough, it holds that
$$\mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}_{\overline{a}} \subset \mathbb{R} \times \left[R' - 1, R' + 1\right], \qquad \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}_{\underline{a}} \subset \mathbb{R} \times \left[-R' - 1, -R' + 1\right].$$
Therefore, it is sufficient to estimate the probability that $\sigma_{2\pi}(\widehat{\mathbf{X}}^{n,\widehat{\mathbf{x}}}|_{[0,\sigma_{\widehat{\mathbf{x}}}]})$ disconnects $\widehat{V}_R \cup \widehat{V}_{-R}$ from $\widehat{V}_{R'-1} \cup \widehat{V}_{-R'+1}$. To this end, one can argue in the same exact way as in Lemma 5.4.6 in order to prove

that, for any $n > n(R,\delta)$ large enough and for all $\widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R)$, it holds that

$$\mathbb{P}_{\widehat{\mathbf{x}}}(\sigma_{2\pi}(\widehat{\mathbf{X}}^n|_{[0,\widehat{\sigma}_{\widehat{\mathbf{x}}}]}) \text{ does not disconnect } \widehat{V}_R \cup \widehat{V}_{-R} \text{ from } \widehat{V}_{R'-1} \cup \widehat{V}_{-R'+1}) \lesssim \frac{R}{R'},$$

where the implicit constant is independent of everything else. Therefore, this fact together with Lemma 5.2.4 imply that

$$\mathrm{d}^{\mathrm{TV}}\big(\widehat{\mathbf{X}}^{n,\widehat{x}}_{\widehat{\sigma}_{\widehat{\mathbf{x}}}}, \widehat{\mathbf{X}}^{n,\widehat{\mathbf{y}}}_{\widehat{\sigma}_{\widehat{y}}}\big) \lesssim \frac{R}{R'}, \quad \forall\, \widehat{\mathbf{x}}, \widehat{\mathbf{y}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R).$$

Therefore, putting together the two bullet points above, recalling that $R' = R/\delta$, and going back to (5.4.12), we find that, for any $n > n(R,\delta)$ large enough, it holds that

$$\big|\mathfrak{f}^{\dagger}_n(\widehat{\mathbf{x}}) - \mathfrak{f}^{\dagger}_n(\widehat{\mathbf{y}})\big| \lesssim \delta, \quad \forall\, \widehat{\mathbf{x}},\, \widehat{\mathbf{y}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R), \tag{5.4.13}$$

where the implicit constant is independent of everything else.

Step 4. To conclude, for every $n > n(R,\delta)$ large enough, fix an arbitrary vertex $\widehat{\mathbf{y}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R)$. Then, thanks to (5.4.10), (5.4.11), and (5.4.13), we have that for any $n > n(R,\delta)$ large enough, it holds that

$$\left|\frac{2\pi}{\eta_n}\mathfrak{w}^{\dagger}_n(\widehat{\mathbf{x}}) + b^{R,\delta}_n - \mathrm{Re}(\widehat{\mathbf{x}})\right| \leq \delta, \quad \forall\, \widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R),$$

where $b^{R,\delta}_n \overset{\text{def}}{=} 2\pi\mathfrak{f}^{\dagger}_n(\widehat{\mathbf{y}})$. Finally, in order to conclude, we need to remove the dependence of $b^{R,\delta}_n$ from R and δ. This can be easily done by arguing in the same exact way as in the second step of the proof of Proposition 5.4.2. Therefore, the proof is concluded. □

5.4.2.3 Proof of Lemma 5.4.9

The proof of Lemma 5.4.9 is relatively long and it is based on several intermediate technical lemmas.

We recall that, for $n \in \mathbb{N}$, the map $\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}_n$, defined in Definition 5.2.15, assigns to each vertex $\mathbf{x} \in \mathcal{V}\mathcal{G}^{\dagger,n}$ the random variable $\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}_n(\mathbf{x})$, which is uniformly distributed on the horizontal segment $\mathcal{S}^{\dagger}_n(\mathbf{x})$. The proof of Lemma 5.4.9 can be basically divided into four main steps.

(a) We start by considering the continuous time lifted random walk $\mathbf{X}^{n,\mu_a}$. In Lemma 5.4.10, we prove that, for $M \in \mathbb{N}$, the conditional probability, given $\vartheta_+ < \vartheta_-$, of the event $|\mathrm{Re}(\mathbf{X}^{n,\mu_a}_{\vartheta_+}) - \mathrm{Re}(\mathbf{X}^{n,\mu_a}_0)| > MR'$ decays exponentially in M. Using this result, we then prove in Lemma 5.4.11 that the conditional expectation of $\mathrm{Re}(\mathbf{X}^{n,\mu_a}_{\vartheta_+}) - \mathrm{Re}(\mathbf{X}^{n,\mu_a}_0)$, given $\vartheta_+ < \vartheta_-$, is of order one. Both the proofs of Lemmas 5.4.10 and 5.4.11 are based on the invariance principle assumption on the sequence of primal maps.

(b) We then consider the Smith-embedded random walk $\dot{\mathbf{X}}^{n,\mu_a}$ associated with $\mathbf{X}^{n,\mu_a}$. In Lemma 5.4.12, we prove that the conditional expectation of $\mathrm{Re}(\dot{\mathbf{X}}^{n,\mu_a}_{\vartheta_+})$, given $\vartheta_+ < \vartheta_-$, is of order η_n. This result is basically a consequence of Lemma 5.3.17, which guarantees that the expected horizontal winding of the Smith-embedded random walk is equal to zero.

(c) In Lemma 5.4.14, using the relation between the maps $\dot{\mathcal{S}}^{\dagger,\mathrm{rand}}_n$ and $\mathfrak{w}^{\dagger}_n$, together with the results proved in the previous steps, we prove that there exists a finite constant $b'_n \in \mathbb{R}$ such that the values of the width coordinate function in $\mathcal{P}^{\dagger,n}_{\overline{a}}$ plus b'_n is of order η_n. The key input to prove this result comes from Lemma 5.4.13 in which we prove that the probability that the random walk $\mathbf{X}^{n,\mu_a}$ travels a large

horizontal distance in a narrow horizontal tube decays exponentially fast. The proof of this fact follows from the invariance principle assumption on the sequence of primal maps.

(d) Finally, in Lemma 5.4.15, we state the analogous dual result of Lemma 5.4.13. This fact and the periodicity of the Smith diagram will allow us to deduce Lemma 5.4.9.

Let us emphasise that all the results explained above hold also with the role of ϑ_+ and ϑ_- interchanged. In particular, the same result stated in point (c) holds with the set $\mathcal{P}^{\dagger,n}_{\overline{a}}$ replaced by $\mathcal{P}^{\dagger,n}_{\underline{a}}$.

We can now proceed to state and prove the technical lemmas mentioned above. We start with the following lemma which states that, for $M \in \mathbb{N}$, the conditional probability of the event $|\operatorname{Re}(\mathbf{X}^{n,\mu_a}_{\vartheta_+}) - \operatorname{Re}(\mathbf{X}^{n,\mu_a}_0)| > MR'$, given $\vartheta_+ < \vartheta_-$, decays exponentially in M. Heuristically speaking, this is due to the fact that, after each time that the random walk $\mathbf{X}^{n,\mu_a}$ travels horizontal distance R', there exists a positive chance of hitting the set $(\mathfrak{h}^\dagger_n)^{-1}(\overline{a})$.

Lemma 5.4.10. *There exists a universal constant $C > 0$ such that, for any $n > n(R,\delta)$ large enough, it holds that*

$$\mathbb{P}_{\mu_a}\big(|\operatorname{Re}(\mathbf{X}^n_{\vartheta_+}) - \operatorname{Re}(\mathbf{X}^n_0)| > MR' \mid \vartheta_+ < \vartheta_-\big) \lesssim \exp(-CM), \quad \forall\, M \in \mathbb{N}.$$

where the implicit constant is independent of everything else. The same conclusion holds with the role of ϑ_+ and ϑ_- interchanged.

Proof. As observed in Remark 5.4.8, for any $n > n(R,\delta)$ large enough, we can assume that

$$(\mathfrak{h}^\dagger_n)^{-1}(\overline{a}) \subset \mathbb{R}\times[R', R'+1], \quad (\mathfrak{h}^\dagger_n)^{-1}(\underline{a}) \subset \mathbb{R}\times[-R'-1, -R'], \quad (\mathfrak{h}^\dagger_n)^{-1}(a) \subset \mathbb{R}\times[-1,1]. \tag{5.4.14}$$

Now, letting $\vartheta \overset{\text{def}}{=} \vartheta_+ \wedge \vartheta_-$ and $M \in \mathbb{N}$, we can write

$$\mathbb{P}_{\mu_a}\big(|\operatorname{Re}(\mathbf{X}^n_{\vartheta_+}) - \operatorname{Re}(\mathbf{X}^n_0)| > MR' \mid \vartheta_+ < \vartheta_-\big) \leq \frac{\mathbb{P}_{\mu_a}\big(|\operatorname{Re}(\mathbf{X}^n_{\vartheta}) - \operatorname{Re}(\mathbf{X}^n_0)| > MR'\big)}{\mathbb{P}_{\mu_a}\big(\vartheta_+ < \vartheta_-\big)}.$$

We note that, thanks to assumption (H2) and (5.4.14), for any $n > n(R,\delta)$ large enough, the probability on the denominator can be lower bounded by a constant independent of everything else. Therefore, we can just focus on the probability appearing on the numerator. To this end, let $\rho_0 := 0$, and for $k \in \mathbb{N}_0$ define inductively

$$\rho_{k+1} \overset{\text{def}}{=} \inf\big\{t \geq \rho_k \,:\, \big|\operatorname{Re}(\mathbf{X}^{n,\mu_a}_t) - \operatorname{Re}(\mathbf{X}^{n,\mu_a}_{\rho_k})\big| \geq R'\big\}.$$

Moreover, for $k \in \mathbb{N}_0$, consider the event

$$A^n_k \overset{\text{def}}{=} \big\{\mathbf{X}^{n,\mu_a}|_{[\rho_k,\rho_{k+1}]} \not\subset \mathbb{R}\times\big[\operatorname{Im}(\mathbf{X}^{n,\mu_a}_{\rho_k}) - 3R', \operatorname{Im}(\mathbf{X}^{n,\mu_a}_{\rho_k}) + 3R'\big]\big\}.$$

We observe that, thanks to the strong Markov property of the random walk, the events $\{A^n_k\}_{k\in\mathbb{N}_0}$ are independent and identically distributed. Moreover, thanks to assumption (H2), to estimates (5.4.14), and to well-known properties of Brownian motion, for any $n \in \mathbb{N}$ large enough, the event A^n_0 happens with uniformly positive probability p independent of everything else. Therefore, we have that

$$\mathbb{P}_{\mu_a}\big(|\operatorname{Re}(\mathbf{X}^n_{\vartheta_+}) - \operatorname{Re}(\mathbf{X}^n_0)| > MR'\big) \leq \mathbb{P}_{\mu_a}\left(\bigcap_{i=0}^{M} \overline{A^n_i}\right) = (1-p)^M,$$

from which the desired result follows. Finally, the same argument also applies with the role of ϑ_+ and ϑ_- interchanged. □

In the next lemma, we use Lemma 5.4.10 to prove that the conditional expectation of $\mathrm{Re}(\mathbf{X}^{n,\mu_a}_{\vartheta_+}) - \mathrm{Re}(\mathbf{X}^{n,\mu_a}_0)$, given $\vartheta_+ < \vartheta_-$, is of order one.

Lemma 5.4.11. *For any $n > n(R,\delta)$ large enough, it holds that*

$$\left|\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\mathbf{X}^n_{\vartheta_+}) - \mathrm{Re}(\mathbf{X}^n_0) \mid \vartheta_+ < \vartheta_-\right]\right| \lesssim 1,$$

where the implicit constant is universal. The same result holds with the role of ϑ_+ and ϑ_- interchanged.

Proof. The proof of this result is based on assumption (H2). More precisely, let B^{μ_a} be a planar Brownian motion started from a point in $\mathcal{P}^{\dagger,n}_a$ sampled according to the probability measure μ_a, and consider the following stopping times

$$\vartheta_{B,+} \overset{\text{def}}{=} \inf\{t \geq 0 \,:\, \mathrm{Im}(B^{\mu_a}_t) = R'\}, \qquad \vartheta_{B,-} \overset{\text{def}}{=} \inf\{t \geq 0 \,:\, \mathrm{Im}(B^{\mu_a}_t) = -R'\}.$$

Since $\mathrm{Re}(B^{\mu_a})$ and $\mathrm{Im}(B^{\mu_a})$ are independent, and since the stopping times $\vartheta_{B,+}$ and $\vartheta_{B,-}$ only depend on $\mathrm{Im}(B^{\mu_a})$, thanks to well-known properties of Brownian motion, we have that

$$\mathbb{E}_{\mu_a}\left[\mathrm{Re}(B_{\vartheta_{B,+}}) - \mathrm{Re}(B_0) \mid \vartheta_{B,+} < \vartheta_{B,-}\right] = 0. \tag{5.4.15}$$

Furthermore, as observed in Remark 5.4.8, for any $\varepsilon \in (0,1)$ and for any $n > n(R,\delta,\varepsilon)$ large enough, it holds that

$$(\mathfrak{h}^\dagger_n)^{-1}(\overline{a}) \subset \mathbb{R} \times [R', R'+\varepsilon], \qquad (\mathfrak{h}^\dagger_n)^{-1}(\underline{a}) \subset \mathbb{R} \times [-R'-\varepsilon, -R'].$$

In particular, this fact and Lemma 5.4.1 imply that the set $(\mathfrak{h}^\dagger_n)^{-1}(\overline{a})$ (resp. $(\mathfrak{h}^\dagger_n)^{-1}(\underline{a})$) converges in the Hausdorff metric to the horizontal line $\mathbb{R} \times \{R'\}$ (resp. $\mathbb{R} \times \{-R'\}$). Therefore, thanks to assumption (H2), the following weak convergence of laws holds

$$\lim_{n\to\infty} \mathrm{Law}\big(\mathrm{Re}(\mathbf{X}^{n,\mu_a}_{\vartheta_+}) - \mathrm{Re}(X^{n,\mu_a}_0) \mid \vartheta_+ < \vartheta_-\big) = \mathrm{Law}\big(\mathrm{Re}(B^{\mu_a}_{\vartheta_{B,+}}) - \mathrm{Re}(B^{\mu_a}_0) \mid \vartheta_{B,+} < \vartheta_{B,-}\big).$$

Hence, thanks to Lemma 5.4.10 and Vitali's convergence theorem, for any $n > n(R,\delta)$ large enough, it holds that

$$\left|\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\mathbf{X}^n_{\vartheta_+}) - \mathrm{Re}(\mathbf{X}^n_0) \mid \vartheta_+ < \vartheta_-\right] - \mathbb{E}_{\mu_a}\left[\mathrm{Re}(B_{\vartheta_{B,+}}) - \mathrm{Re}(B_0) \mid \vartheta_{B,+} < \vartheta_{B,-}\right]\right| \leq 1. \tag{5.4.16}$$

Hence, putting together (5.4.15) and (5.4.16), we obtain the desired result. Finally, the same argument also applies with the role of ϑ_+ and ϑ_- interchanged. □

In the next lemma, we see how we can use Lemma 5.3.17 to prove that that the conditional expectation of $\mathrm{Re}(\dot{\mathbf{X}}^{n,\mu_a}_{\vartheta_+})$, given $\vartheta_+ < \vartheta_-$, is of order η_n.

Lemma 5.4.12. *For any $n \in \mathbb{N}$, it holds that*

$$\left|\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\dot{\mathbf{X}}^n_{\vartheta_+}) \mid \vartheta_+ < \vartheta_-\right]\right| \leq \eta_n,$$

and the same with the role of ϑ_+ and ϑ_- interchanged.

Proof. Since we are assuming that the base vertex $\widehat{\mathbf{x}}_0^{n,a}$ of the lifted width coordinate function $\mathfrak{w}_n^\dagger$ belongs to the set $\widehat{\mathcal{P}}_a^{\dagger,n}$, then, thanks to Lemma 5.3.9, it holds almost surely that

$$\left|\mathrm{Re}(\dot{\mathcal{S}}_n^{\dagger,\mathrm{rand}}(\mathbf{x}))\right| \leq \eta_n, \quad \forall \mathbf{x} \in \mathcal{P}_a^{\dagger,n}.$$

In particular, since the embedded random walk $\mathbf{X}^{n,\mu_a}$ is started from a point in the set $\mathcal{P}_a^{\dagger,n}$, then it holds almost surely that

$$\left|\mathrm{Re}(\dot{\mathbf{X}}_0^{n,\mu_a})\right| \leq \eta_n. \tag{5.4.17}$$

Now, we would like to apply Lemma 5.3.17 in order to conclude. However, we note that we cannot directly apply this result since, a priori, it does not hold that $\cup_{x\in\mathcal{VG}^n}\mathfrak{h}_n^{-1}(\mathfrak{h}_n(x)) \subseteq \mathcal{VG}^n$. In order to overcome this issue, we could consider the weighted graph associated to $(\mathcal{G}^n, c^n)$ and $\cup_{x\in\mathcal{VG}^n}\mathfrak{h}_n^{-1}(\mathfrak{h}_n(x))$ as specified in Definition 5.3.1. We could then apply Lemma 5.3.17 to the random walk on this new weighted graph and then transfer the result to the original weighted graph by means of Lemma 5.3.4. In order to lighten up the proof, we will assume directly that $\cup_{x\in\mathcal{VG}^n}\mathfrak{h}_n^{-1}(\mathfrak{h}_n(x)) = \mathcal{VG}^n$. Since the stopping time ϑ_+ is almost surely finite, we can proceed as follows

$$\begin{aligned}
&\mathbb{E}_{\mu_a}\left[\mathrm{wind}_{\eta_n}(\dot{\mathbf{X}}^n|_{[0,\vartheta_+]}) \mid \vartheta_+ < \vartheta_-\right] \\
&\quad= \frac{1}{\mathbb{P}_{\mu_a}(\vartheta_+ < \vartheta_-)}\mathbb{E}_{\mu_a}\left[\sum_{N\in\mathbb{N}} \mathrm{wind}_{\eta_n}(\dot{\mathbf{X}}^n|_{[0,\vartheta_+]})\mathbb{1}_{\{\vartheta_+<\vartheta_-,\vartheta_+=N\}}\right] \\
&\quad= \frac{1}{\mathbb{P}_{\mu_a}(\vartheta_+ < \vartheta_-)}\sum_{N\in\mathbb{N}} \mathbb{E}_{\mu_a}\left[\mathrm{wind}_{\eta_n}(\dot{\mathbf{X}}^n|_{[0,\vartheta_+]}) \mid \vartheta_- > N, \vartheta_+ = N\right]\mathbb{P}_{\mu_a}(\vartheta_- > N, \vartheta_+ = N).
\end{aligned} \tag{5.4.18}$$

In order to pass from the first second to the third line of the above expression, we used the fact that

$$\mathbb{E}\left[\left|\mathrm{wind}_{\eta_n}(\dot{\mathbf{X}}^{n,\mu_a}|_{[0,\vartheta_+]})\right|\right] < \infty, \tag{5.4.19}$$

and Fubini's theorem. The reason why (5.4.19) holds is an immediate consequence of Lemmas 5.3.6 and 5.4.11. In order to conclude, it is sufficient to prove that the expectation in each summand of the sum appearing in (5.4.18) is equal to zero. To this end, fix $N \in \mathbb{N}$ and consider a sequence of admissible height coordinates $[a_N]_0 \subset (0,1)$ for the random walk $X^{n,\mu_a} = \sigma_{2\pi}(\mathbf{X}^{n,\mu_a})$, as specified in Definition 5.3.13, such that $a_m > \underline{a}$ for all $m \in [N]_0$ and $a_N = \overline{a}$. Thanks, to Lemma 5.3.17 and recalling (5.3.3), we have that

$$\mathbb{E}_{\mu_a}\left[\mathrm{wind}_{\eta_n}(\dot{\mathbf{X}}^n|_{[0,\rho_N]}) \mid \{\mathfrak{h}_n(X^n_{\rho_m}) = a_m\}_{m=1}^N\right] = 0,$$

uniformly over all such sequences of height coordinates. Hence, this is sufficient to conclude that

$$\mathbb{E}_{\mu_a}\left[\mathrm{wind}_{\eta_n}(\dot{\mathbf{X}}^n|_{[0,\vartheta_+]}) \mid \vartheta_- > N, \vartheta_+ = N\right] = 0. \tag{5.4.20}$$

Therefore, the conclusion follows thanks to (5.4.17), and to the fact that (5.4.18) and (5.4.20) imply that $\mathbb{E}_{\mu_a}[\mathrm{wind}_{\eta_n}(\dot{\mathbf{X}}^n|_{[0,\vartheta_+]}) \mid \vartheta_+ < \vartheta_-] = 0$. Finally, we observe that the same argument also applies with the role of ϑ_+ and ϑ_- interchanged. □

In order to prove Lemma 5.4.9, we basically need to prove that the difference between the value of the width coordinate in the set $\widehat{\mathcal{P}}_{\overline{a}}^{\dagger,n}$ and in the set $\widehat{\mathcal{P}}_{\underline{a}}^{\dagger,n}$ is of order η_n. This fact is the content of Lemma 5.4.14 below. However, in order to prove this fact, we first need to prove that it is extremely unlikely for the random walk

$\mathbf{X}^{n,\mu_a}$ to travel a large horizontal distance in a "narrow horizontal tube". We refer to Figure 5.9 for a diagram illustrating the various objects involved in the proof of Lemma 5.4.13.

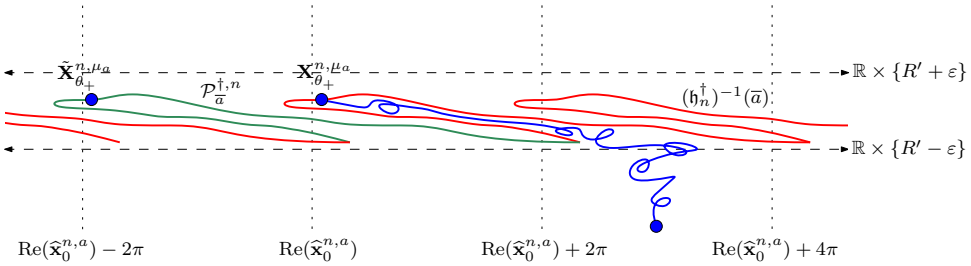


FIGURE 5.9: The vertices $(\mathfrak{h}^\dagger_n)^{-1}(\overline{a})$ are drawn for simplicity as a continuous red/green path contained in the infinite strip $\mathbb{R}\times[R'-\varepsilon,R'+\varepsilon]$. In particular, the green part of this path contains the vertices in the set $\mathcal{P}^{\dagger,n}_{\overline{a}}$. The blue vertex at the bottom of the figure is a vertex sampled from the set $\mathcal{P}^{\dagger,n}_{a}$ according to the probability measure μ_a. The blue path represents a trajectory of the random walk $\mathbf{X}^{n,\mu_a}$, and the blue point at the top-right coincides with the hitting point $\mathbf{X}^{n,\mu_a}_{\vartheta_+}$. The blue point $\widetilde{\mathbf{X}}^{n,\mu_a}_{\vartheta_+}$ at the top-left is the vertex in $\mathcal{P}^{\dagger,n}_{\overline{a}}$ such that $\sigma_{2\pi}(\widetilde{\mathbf{X}}^{n,\mu_a}_{\vartheta_+})=\sigma_{2\pi}(\mathbf{X}^{n,\mu_a}_{\vartheta_+})$. Roughly speaking, Lemma 5.4.13 says that the event depicted in the figure is extremely unlikely to happen. A consequence of this fact is that the conditional expectation of $\mathrm{Re}(\widetilde{\mathbf{X}}^{n,\mu_a}_{\vartheta_+})$, given $\vartheta_+<\vartheta_-$, is of order one.

Lemma 5.4.13. *Fix $\varepsilon\in(0,1)$, $M\in\mathbb{N}$, and define the following event*

$$\mathrm{A}^{n,+}_{M,\varepsilon}\overset{\text{def}}{=}\big\{\exists\, s,t\in[0,\vartheta_+]\,:\,\big|\mathrm{Re}(\mathbf{X}^{n,\mu_a}_t)-\mathrm{Re}(\mathbf{X}^{n,\mu_a}_s)\big|>M;\ \big|\mathrm{Im}(\mathbf{X}^{n,\mu_a}_u)\big|\in[R'-\varepsilon,R'+\varepsilon],\ \forall\, u\in[s,t]\big\}.$$

There exists a universal constant $C>0$ such that, for any $n>n(R,\delta,\varepsilon)$ large enough, it holds that

$$\mathbb{P}_{\mu_a}\big(\mathrm{A}^{n,+}_{M,\varepsilon}\mid\vartheta_+<\vartheta_-\big)\lesssim\exp(-CM/\varepsilon),\quad\forall\, M\in\mathbb{N},$$

where the implicit constant is independent of everything else. Furthermore, the same conclusion holds with the role of ϑ_+ and ϑ_- interchanged.

Proof. Fix $\varepsilon\in(0,1)$. We start by recalling that, as observed in Remark 5.4.8, for any $n>n(R,\delta,\varepsilon)$ large enough, it holds that

$$(\mathfrak{h}^\dagger_n)^{-1}(\overline{a})\subset\mathbb{R}\times[R'-\varepsilon,R'+\varepsilon].\tag{5.4.21}$$

It is easy to see that, without any loss of generality, we can assume that $\mathcal{VG}^n$ contains all the points in the set

$$\big\{x\in\mathbb{G}^n\,:\,\mathrm{Im}(x)=R'-\varepsilon\big\}.$$

Therefore, we can define the stopping time $\rho_0:=\inf\big\{t\geq0\,:\,\mathrm{Im}(\mathbf{X}^{n,\mu_a}_t)=R'-\varepsilon\big\}$, and for $k\in\mathbb{N}_0$ we define inductively

$$\widetilde{\rho}_{k+1}\overset{\text{def}}{=}\inf\big\{t\geq\rho_k\,:\,\big|\mathrm{Re}(\mathbf{X}^{n,\mu_a}_t)-\mathrm{Re}(\mathbf{X}^{n,\mu_a}_{\rho_k})\big|>M\big\},\quad\rho_{k+1}\overset{\text{def}}{=}\inf\big\{t\geq\widetilde{\rho}_{k+1}\,:\,\mathrm{Im}(\mathbf{X}^{n,\mu_a}_t)=R'-\varepsilon\big\}.$$

Moreover, for $k\in\mathbb{N}$ we consider the events

$$\mathrm{A}^{n,+}_{M,\varepsilon}(\rho_{k-1},\widetilde{\rho}_k)\overset{\text{def}}{=}\big\{\mathrm{Im}(\mathbf{X}^{n,\mu_a}|_{[\rho_{k-1},\widetilde{\rho}_k]})\subset[R'-\varepsilon,R'+\varepsilon]\big\},\quad F_k\overset{\text{def}}{=}\big\{\rho_k>\widehat{\tau}_{\widehat{\mathbf{x}}}\big\}.$$

Let K be the smallest $k \in \mathbb{N}$ such that F_k occurs. Then, we have that

$$\begin{aligned}\mathbb{P}_{\mu_a}\big(\mathrm{A}^{n,+}_{M,\varepsilon} \mid \vartheta_+ < \vartheta_-\big) &\leq \mathbb{P}_{\mu_a}\left(\bigcup_{i=1}^{K} \mathrm{A}^{n,+}_{M,\varepsilon}(\rho_{i-1}, \widetilde{\rho}_i) \mid \vartheta_+ < \vartheta_-\right) \\ &\leq \sum_{k\in\mathbb{N}} \sum_{i=1}^{k} \mathbb{P}_{\mu_a}\left(\mathrm{A}^{n,+}_{M,\varepsilon}(\rho_{i-1}, \widetilde{\rho}_i) \mid \vartheta_+ < \vartheta_-\right) \mathbb{P}_{\mu_a}\big(K = k \mid \vartheta_+ < \vartheta_-\big).\end{aligned} \tag{5.4.22}$$

Thanks to the strong Markov property of the random walk, the events $\{\mathrm{A}^{n,+}_{M,\varepsilon}(\rho_{k-1}, \widetilde{\rho}_k)\}_{k\in\mathbb{N}}$ are conditionally independent and identically distributed given $\vartheta_+ < \vartheta_-$. Now, thanks to assumption (H2) and well-known properties of Brownian motion, it is possible to prove that there exists a universal constant $C > 0$ such that for any $n \in \mathbb{N}$ large enough it holds that

$$\mathbb{P}_{\mu_a}\big(\mathrm{A}^{n,+}_{M,\varepsilon}(\rho_0, \widetilde{\rho}_1) \mid \vartheta_+ < \vartheta_-\big) \lesssim \exp\big(-CM/\varepsilon\big), \tag{5.4.23}$$

where the implicit constant is independent of everything else. More precisely, in order to obtain the above upper bound, it is sufficient to study the probability that the random walk travels horizontal distance M before exiting an infinite horizontal band of height of order ε. This can be done by proceeding similarly to the proof of Lemma 5.4.10, and so we do not detail the argument here. Now, for $k \in \mathbb{N}$, we consider the event

$$\mathrm{B}^{n,+}_{M}(\rho_{k-1}, \widetilde{\rho}_k) \overset{\text{def}}{=} \big\{\mathrm{Im}(\mathbf{X}^{n,\mu_a}|_{[\rho_{k-1},\widetilde{\rho}_k]}) \subset [-R'-1, R'+1]\big\}.$$

For the same reason explained above, we have that the events $\{\mathrm{B}^{n,+}_{M}(\rho_{k-1}, \widetilde{\rho}_k)\}_{k\in\mathbb{N}}$ are conditionally independent and identically distributed given $\vartheta_+ < \vartheta_-$. Also, using again assumption (H2) and a standard gambler's ruin estimate for Brownian motion, one can prove that

$$\mathbb{P}_{\mu_a}\big(\mathrm{B}^{n,+}_{M}(\rho_0, \widetilde{\rho}_1) \mid \vartheta_+ < \vartheta_-\big) \lesssim (M+1)^{-1}, \tag{5.4.24}$$

where the implicit constant is independent of everything else. In particular, thanks to (5.4.21), for any $k \in \mathbb{N}$, it holds that

$$\begin{aligned}\mathbb{P}_{\mu_a}\big(K = k \mid \vartheta_+ < \vartheta_-\big) &\leq \mathbb{P}_{\mu_a}\left(\bigcap_{i=1}^{k-1} \mathrm{B}^{n,+}_{M}(\rho_{i-1}, \widetilde{\rho}_i) \mid \vartheta_+ < \vartheta_-\right) \\ &= \prod_{i=1}^{k-1} \mathbb{P}_{\mu_a}\big(\mathrm{B}^{n,+}_{M}(\rho_0, \widetilde{\rho}_1) \mid \vartheta_+ < \vartheta_-\big) \\ &\lesssim (M+1)^{-(k-1)}.\end{aligned}$$

Therefore, putting together (5.4.22), (5.4.23) and (5.4.24), we obtain that there exists a universal constant $C > 0$ such that

$$\mathbb{P}_{\mu_a}\big(\mathrm{A}^{n,+}_{M,\varepsilon} \mid \vartheta_+ < \vartheta_-\big) \lesssim \exp\big(-CM/\varepsilon\big) \sum_{k\in\mathbb{N}} k(M+1)^{-(k-1)} \lesssim \exp\big(-CM/\varepsilon\big).$$

Finally, we observe that the result with the role of ϑ^+ and ϑ^- interchanged can be obtained in the same way. Therefore, this concludes the proof. □

We are now ready to prove that the difference between the value of the width coordinate in the set $\widehat{\mathcal{P}}^{\dagger,n}_{\overline{a}}$ and in the set $\widehat{\mathcal{P}}^{\dagger,n}_{\underline{a}}$ is of order η_n. We refer to Figure 5.10 for a diagram illustrating the various objects involved in

the proof of Lemma 5.4.14.

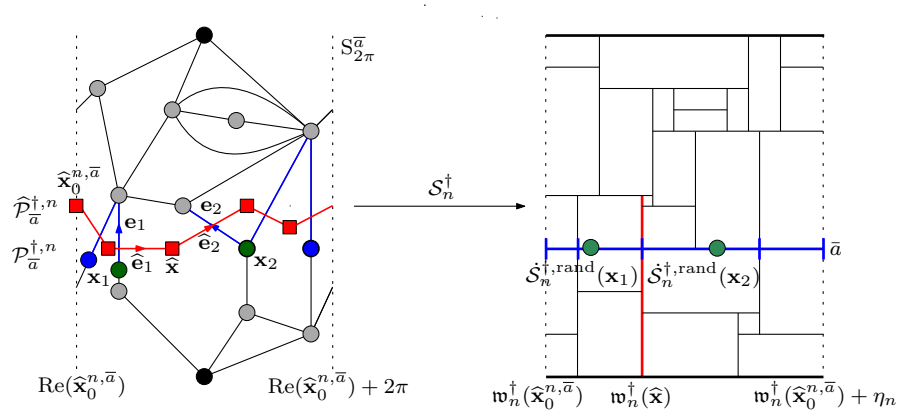


FIGURE 5.10: A diagram illustrating Step 2 of the proof of Lemma 5.4.14. **Left:** A portion of the lifted graph $\mathcal{G}^{\dagger,n}$ embedded in $\mathbb{R}^2$. The blue/green vertices are the vertices in $\mathcal{P}^{\dagger,n}_{\overline{a}}$. The blue edges are the edges in the set $\mathcal{E}\mathcal{G}^{\dagger,n}_{\overline{a}}$. The red dual edges are the edges in the set $\mathcal{E}\widehat{\mathcal{G}}^{\dagger,n}_{\overline{a}}$. The red dual vertices are the vertices in the set $\widehat{\mathcal{P}}^{\dagger,n}_{\overline{a}}$. Consider the dual vertex $\widehat{\mathbf{x}}$ in $\widehat{\mathcal{P}}^{\dagger,n}_{\overline{a}}$, then the two red dual edges with an arrow are the edges $\widehat{\mathbf{e}}_1$, $\widehat{\mathbf{e}}_2$ associated to $\widehat{\mathbf{x}}$. The blue edges with an arrow are the corresponding primal edges $\mathbf{e}_1$, $\mathbf{e}_2$. The two green points are the endpoints of $\mathbf{e}_1$, $\mathbf{e}_2$ in $(\mathfrak{h}_n^\dagger)^{-1}(\overline{a})$. **Right:** A portion of the tiling of $\mathbb{R}\times[0,1]$ constructed via the lifted tiling map $\mathcal{S}_n^\dagger$. The blue horizontal line corresponds to $\cup_{\mathbf{x}\in\mathcal{P}^{\dagger,n}_{\overline{a}}}\mathcal{S}_n^\dagger(\mathbf{x})$. The two green points are possible realisations of $\dot{\mathcal{S}}_n^{\dagger,\mathrm{rand}}(\mathbf{x}_1)$ and $\dot{\mathcal{S}}_n^{\dagger,\mathrm{rand}}(\mathbf{x}_2)$. The vertical red segment with horizontal coordinate $\mathfrak{w}_n^\dagger(\widehat{\mathbf{x}})$ corresponds to $\mathcal{S}_n^\dagger(\widehat{\mathbf{x}})$.

Lemma 5.4.14. *For any $n > n(R,\delta)$ large enough, there exists a finite constant $b'_n \in \mathbb{R}$ such that*

$$\left|\mathfrak{w}_n^\dagger(\widehat{\mathbf{x}}) + b'_n\right| \lesssim \eta_n, \quad \forall\, \widehat{\mathbf{x}} \in \widehat{\mathcal{P}}^{\dagger,n}_{\overline{a}} \cup \widehat{\mathcal{P}}^{\dagger,n}_{\underline{a}},$$

where the implicit constant is independent of everything else.

Proof. We start by letting $\vartheta \overset{\text{def}}{=} \vartheta_+ \wedge \vartheta_-$ and defining

$$b'_n \overset{\text{def}}{=} \mathbb{E}_{\mu_a}\left[\mathrm{Re}(\mathbf{X}_0^n) - \mathrm{Re}(\widetilde{\mathbf{X}}_\vartheta^n)\right].$$

We observe that b'_n is not of constant order in general. Indeed, this is due to the fact that $\mathrm{Re}(\mathbf{X}_0^{n,\mu_a})$ can be far from being of order one since the starting point of the random walk $\mathbf{X}^{n,\mu_a}$ is sampled from the set $\mathcal{P}_a^{\dagger,n}$ over which we do not have any a priori control. We will only prove the result in the case $\widehat{\mathbf{x}} \in \widehat{\mathcal{P}}^{\dagger,n}_{\overline{a}}$ since the result for $\widehat{\mathbf{x}} \in \mathcal{P}^{\dagger,n}_{\underline{a}}$ can be obtained similarly. In particular, we split the proof in two steps.

Step 1: In this first step, we claim that for any $n > n(R,\delta)$ large enough, it holds almost surely that

$$\left|\mathrm{Re}(\dot{\mathcal{S}}_n^{\dagger,\mathrm{rand}}(\mathbf{x})) + b'_n\right| \lesssim \eta_n, \quad \forall\, \mathbf{x} \in \mathcal{P}^{\dagger,n}_{\overline{a}},$$

where the implicit constant does not depend on anything else. To this end, we consider the vertex $\widetilde{\mathbf{X}}^{n,\mu_a}_{\vartheta_+} \in \mathcal{P}^{\dagger,n}_{\overline{a}}$ such that $\sigma_{2\pi}(\widetilde{\mathbf{X}}^{n,\mu_a}_{\vartheta_+}) = \sigma_{2\pi}(\mathbf{X}^{n,\mu_a}_{\vartheta_+})$. Now, applying Lemma 5.3.6, we see that

$$\left|\frac{\mathrm{Re}(\dot{\mathcal{S}}_n^{\dagger,\mathrm{rand}}(\widetilde{\mathbf{X}}^{n,\mu_a}_{\vartheta_+})) - \mathrm{Re}(\dot{\mathbf{X}}^{n,\mu_a}_{\vartheta_+})}{\eta_n} - \frac{\mathrm{Re}(\widetilde{\mathbf{X}}^{n,\mu_a}_{\vartheta_+}) - \mathrm{Re}(\mathbf{X}^{n,\mu_a}_{\vartheta_+})}{2\pi}\right| \leq 1. \tag{5.4.25}$$

Therefore, rearranging the various terms in the previous inequality and then taking the conditional expectation given $\vartheta_+ < \vartheta_-$, we get that

$$\begin{aligned}\left|\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\dot{\mathcal{S}}_n^{\dagger,\mathrm{rand}}(\widetilde{\mathbf{X}}^n_{\vartheta_+}))\mid \vartheta_+<\vartheta_-\right]+b_n'\right| &\lesssim \left|\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\dot{\mathbf{X}}^n_{\vartheta_+})\mid \vartheta_+<\vartheta_-\right]\right| \\ &\quad+\eta_n\left|\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\mathbf{X}^n_{\vartheta_+})-\mathrm{Re}(\mathbf{X}^n_0)\mid \vartheta_+<\vartheta_-\right]\right| \\ &\quad+\eta_n\left|b_n'+\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\widetilde{\mathbf{X}}^n_{\vartheta_+})-\mathrm{Re}(\mathbf{X}^n_0)\mid \vartheta_+<\vartheta_-\right]\right|.\end{aligned} \tag{5.4.26}$$

Thanks to Lemmas 5.4.11 and 5.4.12, we have that the following inequalities hold for any $n > n(R,\delta)$ large enough

$$\left|\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\dot{\mathbf{X}}^n_{\vartheta_+})\mid \vartheta_+<\vartheta_-\right]\right|\le\eta_n,\qquad \left|\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\mathbf{X}^n_{\vartheta_+})-\mathrm{Re}(\mathbf{X}^n_0)\mid \vartheta_+<\vartheta_-\right]\right|\lesssim 1. \tag{5.4.27}$$

Therefore, the claim follows if we prove that the term in the third line of (5.4.26) is of order one. More precisely, thanks to the definition of b_n', it is sufficient to prove that, for any $n > n(R,\delta)$ large enough, it holds that

$$\left|\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\widetilde{\mathbf{X}}^n_{\vartheta})\right]-\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\widetilde{\mathbf{X}}^n_{\vartheta_+})\mid \vartheta_+<\vartheta_-\right]\right|\lesssim 1.$$

Since, for any $n > n(R,\delta)$ large enough, $\mathbb{P}_{\mu_a}(\vartheta_+ < \vartheta_-)$ is of constant order, this is equivalent to prove that

$$\left|\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\widetilde{\mathbf{X}}^n_{\vartheta_+})\mid \vartheta_+<\vartheta_-\right]\right|+\left|\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\widetilde{\mathbf{X}}^n_{\vartheta_-})\mid \vartheta_-<\vartheta_+\right]\right|\lesssim 1,$$

where the implicit constant must be independent of everything else. We note that this inequality follows easily from Lemma 5.4.13 (see Figure 5.9). Hence, putting everything together and going back to (5.4.26), we obtain that

$$\left|\mathbb{E}_{\mu_a}\left[\mathrm{Re}(\dot{\mathcal{S}}_n^{\dagger,\mathrm{rand}}(\widetilde{\mathbf{X}}^n_{\vartheta_+}))\mid \vartheta_+<\vartheta_-\right]+b_n'\right|\lesssim\eta_n,$$

where the implicit constant is independent of everything else. Therefore, the desired claim follows from the previous inequality and from the fact that, thanks to Lemma 5.3.9, it holds almost surely that

$$\left|\mathrm{Re}(\dot{\mathcal{S}}_n^{\dagger,\mathrm{rand}}(\mathbf{x}_1))-\mathrm{Re}(\dot{\mathcal{S}}_n^{\dagger,\mathrm{rand}}(\mathbf{x}_2))\right|\le 2\eta_n,\quad\forall\,\mathbf{x}_1,\mathbf{x}_2\in\mathcal{P}_{\overline{a}}^{\dagger,n}$$

Step 2: In this step, we will actually prove the result in the lemma statement. To this end, we fix $\widehat{\mathbf{x}}\in\widehat{\mathcal{P}}_{\overline{a}}^{\dagger,n}$ and we consider the dual edges $\widehat{\mathbf{e}}_1,\widehat{\mathbf{e}}_2\in\mathcal{E}\widehat{\mathcal{G}}_{\overline{a}}^{\dagger,n}$ such that $\widehat{\mathbf{e}}_1^+=\widehat{\mathbf{x}}=\widehat{\mathbf{e}}_2^-$. Furthermore, we let $\mathbf{e}_1,\mathbf{e}_2\in\mathcal{E}\mathcal{G}_{\overline{a}}^{\dagger,n}$ be the corresponding primal edges, and we let $\mathbf{x}_1,\mathbf{x}_2\in\mathcal{V}\mathcal{G}^{\dagger,n}$ be the endpoints of $\mathbf{e}_1$ and $\mathbf{e}_2$ in the set $(\mathfrak{h}_n^\dagger)^{-1}(\overline{a})$ (see Figure 5.10). At this point, we need to split the proof in two different cases:

- If $\mathbf{x}_1\neq\mathbf{x}_2$, then, by construction of the Smith diagram, it holds almost surely that

$$\mathfrak{w}_n^\dagger(\widehat{\mathbf{x}})\in\left[\mathrm{Re}(\dot{\mathcal{S}}_n^{\dagger,\mathrm{rand}}(\mathbf{x}_1)),\mathrm{Re}(\dot{\mathcal{S}}_n^{\dagger,\mathrm{rand}}(\mathbf{x}_2))\right].$$

- If $\mathbf{x}_1=\mathbf{x}_2$, then it holds that

$$\mathfrak{w}_n^\dagger(\widehat{\mathbf{x}})\in\left[\min\left\{\mathrm{Re}(\mathbf{v})\,:\,\mathbf{v}\in\mathcal{S}_n^\dagger(\mathbf{x}_1)\right\},\max\left\{\mathrm{Re}(\mathbf{v})\,:\,\mathbf{v}\in\mathcal{S}_n^\dagger(\mathbf{x}_1)\right\}\right],$$

  where we recall that $\mathcal{S}_n^\dagger(\mathbf{x}_1)$ denotes the horizontal segment associated to $\mathbf{x}_1$ by the lifted tiling map.

In both cases, if $\mathbf{x}_1,\mathbf{x}_2\in\mathcal{P}_{\overline{a}}^{\dagger,n}$, then the conclusion follows from the previous step. However, in general, it could be that $\mathbf{x}_1\in\mathcal{P}_{\overline{a}}^{\dagger,n}-(2\pi,0)$ or $\mathbf{x}_2\in\mathcal{P}_{\overline{a}}^{\dagger,n}+(2\pi,0)$. In both these cases, we cannot directly appeal to

the previous step to conclude. Nevertheless, a simple application of Lemma 5.3.6 implies that the same result of the first step holds also for the vertices in $\mathcal{P}_{\underline{a}}^{\dagger,n} - (2\pi, 0)$ and in $\mathcal{P}_{\underline{a}}^{\dagger,n} + (2\pi, 0)$. Therefore, this concludes the proof. □

Before proceeding with the proof of Lemma 5.4.9, we need to state a lemma which is the dual counterpart of Lemma 5.4.13.

Lemma 5.4.15. *Fix $\varepsilon \in (0,1)$, $\widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R'-1)$, $M \in \mathbb{N}$, and define the following event*

$$\widehat{\mathrm{A}}_{M,\varepsilon}^{n,\widehat{\mathbf{x}}} \overset{\text{def}}{=} \big\{\exists\, s,t \in [0,\widehat{\tau}_{\widehat{\mathbf{x}}}] \,:\, \big|\mathrm{Re}(\widehat{\mathbf{X}}_t^{n,\widehat{\mathbf{x}}}) - \mathrm{Re}(\widehat{\mathbf{X}}_s^{n,\widehat{\mathbf{x}}})\big| > M;\ \big|\mathrm{Im}(\widehat{\mathbf{X}}_u^{n,\widehat{\mathbf{x}}})\big| \in [R'-\varepsilon, R'+\varepsilon],\ \forall\, u \in [s,t]\big\}.$$

There exists a universal constant $C > 0$ such that, for any $n > n(R,\delta,\varepsilon)$ large enough, it holds that

$$\mathbb{P}_{\widehat{\mathbf{x}}}\big(\widehat{\mathrm{A}}_{M,\varepsilon}^{n,\widehat{\mathbf{x}}}\big) \lesssim \exp(-CM/\varepsilon), \quad \forall\, M \in \mathbb{N},$$

where the implicit constant is independent of everything else.

Proof. The proof of this lemma can be done by employing a similar argument to that used in Lemma 5.4.13. □

We are now ready to prove Lemma 5.4.9, which is now an immediate consequence of the results proved above.

Proof of Lemma 5.4.9. The proof basically consists of putting together some of the previous results. More precisely, fix $\widehat{\mathbf{x}} \in \mathcal{V}\widehat{\mathcal{G}}^{\dagger,n}(R'-1)$ and let $\widetilde{\mathbf{X}}_{\widehat{\tau}_{\widehat{x}}}^{n,\widehat{\mathbf{x}}} \in \widehat{\mathcal{P}}_{\overline{a}}^{\dagger,n} \cup \widehat{\mathcal{P}}_{\underline{a}}^{\dagger,n}$ be such that $\sigma_{2\pi}(\widetilde{\mathbf{X}}_{\widehat{\tau}_{\widehat{x}}}^{n,\widehat{\mathbf{x}}}) = \sigma_{2\pi}(\widehat{\mathbf{X}}_{\widehat{\tau}_{\widehat{x}}}^{n,\widehat{\mathbf{x}}})$. Then, by Lemma 5.2.9, we have that

$$\mathbb{E}_{\widehat{\mathbf{x}}}\left[\frac{\mathrm{Re}(\widehat{\mathbf{X}}_{\widehat{\tau}_{\widehat{x}}}^{n})}{2\pi} - \frac{\mathfrak{w}_n^{\dagger}(\widehat{\mathbf{X}}_{\widehat{\tau}_{\widehat{x}}}^{n}) + b_n'}{\mathfrak{y}_n}\right] = \mathbb{E}_{\widehat{\mathbf{x}}}\left[\frac{\mathrm{Re}(\widetilde{\mathbf{X}}_{\widehat{\tau}_{\widehat{x}}}^{n})}{2\pi} - \frac{\mathfrak{w}_n^{\dagger}(\widetilde{\mathbf{X}}_{\widehat{\tau}_{\widehat{x}}}^{n}) + b_n'}{\mathfrak{y}_n}\right].$$

Therefore, the result follows thanks to Lemma 5.4.14 if we show that also $\big|\mathbb{E}_{\widehat{\mathbf{x}}}[\mathrm{Re}(\widetilde{\mathbf{X}}_{\widehat{\tau}_{\widehat{x}}}^{n})]\big| \lesssim 1$. We note that this fact is easily implied by Lemma 5.4.15, and so the proof is completed. □

5.4.3 Proof of Theorem 5.A

We are now ready to give a proof of the main theorem of this chapter. As we have already remarked, the proof of this theorem is a consequence of Propositions 5.4.2 and 5.4.7.

Proof of Theorem 5.A. Fix $R > 1$, $\delta \in (0,1)$, and consider a point $\mathbf{x} \in \mathcal{V}\mathcal{G}^{\dagger,n}(R)$. We divide the proof into three main steps.

Step 1. By definition of the Smith embedding $\dot{\mathcal{S}}_n^{\dagger}$, we have that $\mathrm{Im}(\dot{\mathcal{S}}_n^{\dagger}(\mathbf{x})) = \mathfrak{h}_n^{\dagger}(\mathbf{x})$. Hence, from Proposition 5.4.2, we know that there exist two real sequences $\{b_n^{\mathfrak{h}}\}_{n\in\mathbb{N}}$ and $\{c_n^{\mathfrak{h}}\}_{n\in\mathbb{N}}$, independent of R and δ, such that, for $n > n(R,\delta)$ large enough, it holds that

$$\big|c_n^{\mathfrak{h}}\,\mathrm{Im}(\dot{\mathcal{S}}_n^{\dagger}(\mathbf{x})) + b_n^{\mathfrak{h}} - \mathrm{Im}(\mathbf{x})\big| \le \frac{\delta}{\sqrt{2}}, \quad \forall\, \mathbf{x} \in \mathcal{V}\mathcal{G}^{\dagger,n}(\mathbb{R}\times[-R,R]).$$

Step 2. Let $\mathcal{E}\mathcal{G}^{\dagger,n,\downarrow}(\mathbf{x}) = [\mathbf{e}_k]$ be the set of harmonically oriented edges in $\mathcal{E}\mathcal{G}^{\dagger,n}$ with heads equal to $\mathbf{x}$ ordered in such a way that $\widehat{\mathbf{e}}_1 \cdots \widehat{\mathbf{e}}_k$ forms a counter-clockwise oriented path in the lifted dual graph $\widehat{\mathcal{G}}^{\dagger,n}$.

Then, by construction of the Smith embedding $\dot{\mathcal{S}}_n^\dagger$, we have that

$$\mathrm{Re}(\dot{\mathcal{S}}_n^\dagger(\mathbf{x})) \in \big[\mathfrak{w}^\dagger(\widehat{\mathbf{e}}_1^-), \mathfrak{w}^\dagger(\widehat{\mathbf{e}}_k^+)\big]. \tag{5.4.28}$$

Therefore, letting $\{b_n^{\mathfrak{w}}\}_{n\in\mathbb{N}}$ be the sequence in the statement of Proposition 5.4.7, we have that

$$\begin{aligned}&\left|\frac{2\pi}{\eta_n}\mathrm{Re}(\dot{\mathcal{S}}_n^\dagger(\mathbf{x})) + b_n^{\mathfrak{w}} - \mathrm{Re}(\mathbf{x})\right| \\ &\qquad\le \left|\frac{2\pi}{\eta_n}\mathfrak{w}_n(\widehat{\mathbf{e}}_1^-) + b_n^{\mathfrak{w}} - \mathrm{Re}(\widehat{\mathbf{e}}_1^-)\right| + \left|\mathrm{Re}(\widehat{\mathbf{e}}_1^-) - \mathrm{Re}(\mathbf{x})\right| + \frac{2\pi}{\eta_n}\left|\mathrm{Re}(\dot{\mathcal{S}}_n^\dagger(\mathbf{x})) - \mathfrak{w}_n^\dagger(\widehat{\mathbf{e}}_1)\right|.\end{aligned}$$

The first term on the right-hand side of the above expression is bounded by $\delta/(5\sqrt{2})$ thanks to Proposition 5.4.7. The second term is also bounded by $\delta/(5\sqrt{2})$ since Lemma 5.4.1 rules out the existence of macroscopic faces. Concerning the third term, recalling (5.4.28), we have that

$$\begin{aligned}&\frac{2\pi}{\eta_n}\left|\mathrm{Re}(\dot{\mathcal{S}}_n^\dagger(\mathbf{x})) - \mathfrak{w}_n^\dagger(\widehat{\mathbf{e}}_1^-)\right| \\ &\qquad\le \left|\frac{2\pi}{\eta_n}\mathfrak{w}_n^\dagger(\widehat{\mathbf{e}}_1^-) + b_n^{\mathfrak{w}} - \mathrm{Re}(\widehat{\mathbf{e}}_1^-)\right| + \left|\frac{2\pi}{\eta_n}\mathfrak{w}_n^\dagger(\widehat{\mathbf{e}}_k^+) + b_n^{\mathfrak{w}} - \mathrm{Re}(\widehat{\mathbf{e}}_k^+)\right| + \left|\mathrm{Re}(\widehat{\mathbf{e}}_k^+) - \mathrm{Re}(\widehat{\mathbf{e}}_1^-)\right|.\end{aligned}$$

The first and second term on the right-hand side of the above expression are bounded by $\delta/(5\sqrt{2})$, thanks to Proposition 5.4.7. The third term is also bounded by $\delta/(5\sqrt{2})$, for $n > n(R,\delta)$ thanks, once again, to Lemma 5.4.1. We remark that all the previous bounds obviously hold only for $n > n(R,\delta)$ large enough. Therefore, putting it all together, we obtain that, for all $n > n(R,\delta)$ large enough, it holds that

$$\left|\frac{2\pi}{\eta_n}\mathrm{Re}(\dot{\mathcal{S}}_n^\dagger(\mathbf{x})) + b_n^{\mathfrak{w}} - \mathrm{Re}(\mathbf{x})\right| \le \frac{\delta}{\sqrt{2}}, \quad \forall\, \mathbf{x} \in \mathcal{VG}^{\dagger,n}(\mathbb{R}\times[-R,R]).$$

Step 3. For each $n \in \mathbb{N}$, we define the affine transformation $T_n^\dagger : \mathbb{R}\times[0,1] \to \mathbb{R}^2$ by letting

$$\mathrm{Re}(T_n^\dagger \mathbf{x}) \overset{\text{def}}{=} \frac{2\pi}{\eta_n}\mathrm{Re}(\mathbf{x}) + b_n^{\mathfrak{w}} \quad \text{and} \quad \mathrm{Im}(T_n^\dagger \mathbf{x}) \overset{\text{def}}{=} c_n^{\mathfrak{h}}\,\mathrm{Im}(\mathbf{x}) + b_n^{\mathfrak{h}}, \qquad \forall\, \mathbf{x} \in \mathbb{R}\times[0,1],$$

Therefore, the previous two steps yield that, for any $n > n(R,\delta)$ large enough, it holds that

$$\mathrm{d}_{\mathbb{R}^2}\big(T_n^\dagger \dot{\mathcal{S}}_n^\dagger(\mathbf{x}), \mathbf{x}\big) \le \delta, \quad \forall\, \mathbf{x} \in \mathcal{VG}^{\dagger,n}(\mathbb{R}\times[-R,R]),$$

where $\mathrm{d}_{\mathbb{R}^2}$ denotes the Euclidean distance in $\mathbb{R}^2$. This is obviously equivalent to the desired result, and so the proof is completed. □

5.5 Application to mated-CRT maps

The main goal of this section is to prove Theorem 5.B. Roughly speaking, the plan is as follows. We will first introduce an a priori embedding of mated-CRT maps which is "close" to LQG. We then prove that this a priori embedding satisfies the assumptions of Theorem 5.A. Finally, we show how this allows to conclude.

5.5.1 SLE/LQG description of mated-CRT maps

We now discuss an equivalent description of mated-CRT maps in terms of SLE/LQG, which comes from the results of [DMS21]. These results imply that mated-CRT maps can be realised as cell configurations constructed from space-filling SLE_κ curves parameterised by quantum mass with respect to a certain independent LQG surface. We will not need many properties of the SLE/LQG objects involved, so we will not give detailed definitions, but we will give precise references instead.

Liouville quantum gravity surfaces. For $\gamma \in (0,2)$ and $D \subseteq \mathbb{C}$, a doubly marked γ-LQG surface is an equivalence class of quadruples (D, h, z_1, z_2) where h is a random generalised function on D (which we will always take to be an instance of some variant of the Gaussian free field), and $z_1, z_2 \in D$. Two such quadruples (D, h, z_1, z_2) and $(\widetilde{D}, \widetilde{h}, \widetilde{z_1}, \widetilde{z_2})$ are declared to be equivalent if there exists a conformal map $f : \widetilde{D} \to D$ such that

$$\widetilde{h} = h \circ f + Q \log |f'| \quad \text{and} \quad f(\widetilde{z_1}) = z_1,\ f(\widetilde{z_2}) = z_2, \quad \text{where} \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}. \tag{5.5.1}$$

For $\gamma \in (0,2)$, it is well-known that one can construct a random measure, called the γ-LQG area measure, which is formally given by $\mu_h \overset{\text{def}}{=} e^{\gamma h} \mathrm{d}^2 z$, where $d^2 z$ denotes the Lebesgue measure on D. Since h is a random generalised function, this definition does not make rigorous sense and one should proceed using a standard regularisation and limiting procedure [DS11]. The γ-LQG area measure satisfies a certain change of coordinates formula. More precisely, given two equivalent doubly marked γ-LQG surface (D, h, z_1, z_2) and $(\widetilde{D}, \widetilde{h}, \widetilde{z_1}, \widetilde{z_2})$, then it holds almost surely that $\mu_h(f(A)) = \mu_{\widetilde{h}}(A)$ for all Borel sets $A \subseteq \widetilde{D}$, where $f : \widetilde{D} \to D$ is a conformal map such that (5.5.1) holds.

In this chapter, we are interested in two different kind of doubly marked γ-LQG surfaces.

- The *doubly marked quantum sphere* $(\mathbb{C}, h, 0, \infty)$, where h is a variant of the Gaussian free field precisely defined in [DMS21, Definition 4.21]. For $\gamma \in (0,2)$, it is well-known that one can associate with the random generalised function h a random measure μ_h on $\mathbb{C}$, the γ-LQG measure, with $\mu_h(\mathbb{C}) < \infty$ (again, we will not need the precise definition here). Typically, one considers a unit-area quantum sphere, which means that we fix $\mu_h(\mathbb{C}) = 1$.
- The *0-quantum cone* $(\mathbb{C}, h^{\mathrm{c}}, 0, \infty)$, where h^{c} is a variant of the Gaussian free field precisely defined in [DMS21, Definition 4.10]. Also in this case, for $\gamma \in (0,2)$, we can associate to h^{c} the γ-LQG measure $\mu_{h^{\mathrm{c}}}$ which has infinite total mass, but it is locally finite.

Schramm–Loewner evolution. We do not need to precisely define SLE_κ, but rather it is sufficient to know that whole-plane space-filling SLE_κ, for $\kappa > 4$, is a random space-filling curve ϑ which travels from ∞ to ∞ in $\mathbb{C}$. It is a variant of SLE_κ [Schoo] which was introduced in [MS17, DMS21]. Space-filling SLE_κ for $\kappa \geq 8$ is a two-sided version of ordinary SLE_κ (which is already space-filling), whereas space-filling SLE_κ for $\kappa \in (4,8)$ can be obtained from ordinary SLE_κ by iteratively filling in the "bubbles" which the path disconnects from ∞.

Construction of the a priori embedding. An important feature of the mated-CRT map is that it comes with an a priori embedding into $\mathbb{C}$ described by SLE-decorated LQG. To explain this embedding, consider a doubly marked quantum sphere $(\mathbb{C}, h, 0, \infty)$ and, for $\gamma \in (0,2)$, consider the associated γ-LQG measure μ_h. Sample a space-filling SLE_κ curve ϑ with $\kappa = 16/\gamma^2$, independently from the random generalised function h, and reparametrise ϑ so that

$$\vartheta(0) = 0 \quad \text{and} \quad \mu_h\big(\vartheta([a,b])\big) = b - a, \quad \forall\, a, b \in \mathbb{R} \text{ with } a < b.$$

For $\gamma \in (0,2)$ and $n \in \mathbb{N}$, we define the *n-structure graph* $\mathcal{G}^n$ associated with the pair (h, ϑ) as follows. The vertex set of $\mathcal{G}^n$ is given by

$$\mathcal{V}\mathcal{G}^n \stackrel{\text{def}}{=} \frac{1}{n}\mathbb{Z} \cap (0,1].$$

Two distinct vertices $x_1, x_2 \in \mathcal{V}\mathcal{G}^n$ are connected by one edge (resp. two edges) if and only if the intersection of the corresponding cells $\vartheta([x_1 - 1/n, x_1])$ and $\vartheta([x_2 - 1/n, x_2])$ has one connected component which is not a singleton (resp. two connected components which are not singletons). We refer to Figure 5.11 for a diagrammatic construction of the SLE/LQG embedding of the mated-CRT map.

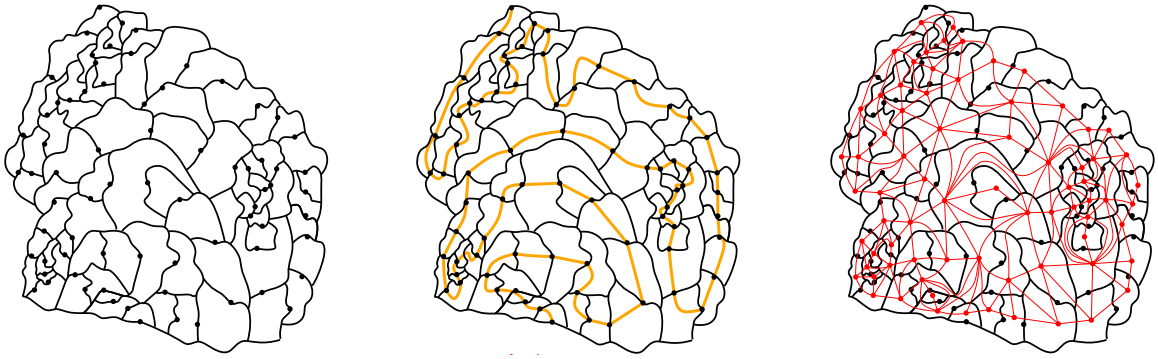


FIGURE 5.11: **Left:** A space-filling SLE_κ curve ϑ, for $\kappa \geq 8$, divided into cells $\vartheta([x - 1/n, x])$ for a collection of $x \in \mathcal{V}\mathcal{G}^n$. Each cell has μ_h-measure equal to $1/n$. **Middle:** The same as in the left figure but with a orange path showing the order in which cells are traversed by ϑ. **Right:** In each cell we drawn a red point corresponding to a vertex in $\mathcal{V}\mathcal{G}^n$. Two vertices are connected by a red edge if the corresponding cells intersect. This illustrates how the SLE/LQG embedding of the n-mated CRT map with the sphere topology is built. A similar figure has appeared in [GMS19].

When $\kappa \geq 8$, the intersection of cells $\vartheta([x_1 - 1/n, x_1]) \cap \vartheta([x_2 - 1/n, x_2])$ is always either empty or the union of one or two non-singleton connected components. However, when $\kappa \in (4,8)$ it is possible that $\vartheta([x_1 - 1/n, x_1]) \cap \vartheta([x_2 - 1/n, x_2])$ is a totally disconnected Cantor-like set, in which case x_1 and x_2 are not joint by an edge in $\mathcal{G}^n$ (see Figure 5.12).

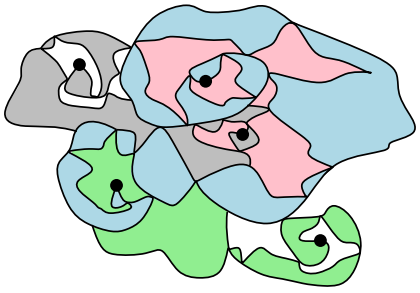


FIGURE 5.12: A typical space-filling SLE_κ curve ϑ, for $\kappa \in (4,8)$, divided into cells. The picture is slightly misleading since the set of "pinch point" where the left and right boundaries of each cell meet is actually uncountable. The black dots indicate the points where ϑ starts and finishes filling each cell. Note that the grey and green cells intersect at several points, but do not share a connected boundary arc so are not considered to be adjacent. A similar figure has appeared in [GMS21].

The following result explains the connection between the pair (h, ϑ) and the mated-CRT map $\mathcal{G}^n$, and it is a consequence of [DMS21, Theorems 1.9 and 8.18].

Proposition 5.5.1. *The family of structure graphs $\{\mathcal{G}^n\}_{n\in\mathbb{N}}$ agrees in law with the family of n-mated-CRT maps with the sphere topology defined in Subsection 5.1.3.*

The previous proposition gives us an a priori embedding of the mated-CRT map in $\mathbb{C}$ by sending each $x \in \mathcal{V}\mathcal{G}^n$ to the point $\vartheta(x) \in \mathbb{C}$. Furthermore, the graph $\mathcal{G}^n$ comes naturally with two marked vertices. Indeed, we can let $v_0^n \in \mathcal{V}\mathcal{G}^n$ (resp. $v_1^n \in \mathcal{V}\mathcal{G}^n$) to be the vertex corresponding to the cell containing 0 (resp. ∞). We also emphasise that here all the edges in $\mathcal{E}\mathcal{G}^n$ are assumed to have unit conductance.

Construction of the a priori embedding of the dual graph. The a priori SLE/LQG embedding of $\mathcal{G}^n$ also induces an a priori embedding into $\mathbb{C}$ of the associated planar dual graph $\widehat{\mathcal{G}}^n$. Indeed, each vertex of $\widehat{\mathcal{G}}^n$ is naturally identified with the set of points in $\mathbb{C}$ where three of the cells $\vartheta([x-1/n, x])$ meet[5], for $x \in \mathcal{V}\mathcal{G}^n$. The set of edges of $\widehat{\mathcal{G}}^n$ can be identified with the boundary segments of the cells which connect these vertices. To be precise, this is not an embedding when $\kappa \in (4, 8)$ since the edges can intersect (but not cross) each other (see Figure 5.12). To deal with this case, we can very slightly perturb the edges so that they do not intersect except at their endpoints. We refer to the proof of Proposition 5.5.4 for more details on how one can handle this situation.

5.5.2 Mated-CRT maps satisfy the assumptions

In this subsection, we show that the a priori embedding of mated-CRT maps with the sphere topology satisfies the assumptions of Theorem 5.A. In particular, we will show here that assumptions (H2) and (H3) are satisfied in this specific case. We recall that, for each $n \in \mathbb{N}$ and $\gamma \in (0, 2)$, the mated-CRT map $\mathcal{G}^n$ comes with two marked vertices v_0^n and v_1^n which correspond to the cell containing 0 and ∞, respectively

Proposition 5.5.2 (Invariance principle on the mated-CRT map, [GMS21, Theorem 3.4])**.** *For each $n \in \mathbb{N}$ and $\gamma \in (0, 2)$, let $\mathcal{G}^n$ be the n-mated CRT map with the sphere topology embedded in $\mathbb{C}$ under the a priori SLE/LQG embedding specified in Subsection 5.5.1. View the embedded random walk on $\mathcal{G}^n$, stopped when it hits v_1^n, as a continuous curve in $\mathbb{C}$ obtained by piecewise linear interpolation at constant speed. For each compact subset $K \subset \mathbb{C}$ and for any $z \in K$, the conditional law given the pair (h, ϑ) of the random walk on $\mathcal{G}^n$ started from the vertex $x_z^n \in \mathcal{V}\mathcal{G}^n$ nearest to z weakly converges in probability as $n \to \infty$ to the law of the Brownian motion on $\mathbb{C}$ started from z with respect to the local topology on curves modulo time parameterisation induced by the metric (5.2.2), uniformly over all $z \in K$.*

Remark 5.5.3. To be precise, in [GMS21, Theorem 3.4], the above theorem is stated for a mated-CRT map built using a quantum sphere with only one marked point. However, the quantum sphere with two marked points can be obtained from the quantum sphere with one marked point $(\mathbb{C}, h, \infty)$ by first sampling a point $z \in \mathbb{C}$ uniformly from the γ-LQG area measure μ_h, then applying a conformal map which sends z to 0 (see [DMS21, Proposition A.13]). Therefore, since Brownian motion is conformally invariant, the statement in [GMS21, Theorem 3.4] immediately implies Proposition 5.5.2 for the quantum sphere with two marked points.

The main purpose of this subsection is to prove that an analogous result holds also on the sequence of a priori SLE/LQG embedding of the dual n-mated-CRT map with the sphere topology. More precisely, we want to prove the following proposition.

Proposition 5.5.4 (Invariance principle on the dual mated-CRT map)**.** *For each $n \in \mathbb{N}$ and $\gamma \in (0, 2)$, let $\widehat{\mathcal{G}}^n$ be the dual planar graph associated to the n-mated-CRT map $\mathcal{G}^n$ with the sphere topology embedded in $\mathbb{C}$ under the a priori embedding specified in Subsection 5.5.1. View the embedded random walk on $\widehat{\mathcal{G}}^n$ as a continuous curve in $\mathbb{C}$ obtained by piecewise linear interpolation at constant speed. For each compact subset $K \subset \mathbb{C}$ and for any $z \in K$, the conditional law given the pair (h, ϑ) of the random walk on $\widehat{\mathcal{G}}^n$ started from the vertex $\widehat{x}_z^n \in \mathcal{V}\widehat{\mathcal{G}}^n$ nearest to z weakly converges as $n \to \infty$ to the law of the Brownian motion on*

[5]Note that there cannot be more than three cells meeting at a single point or we would have a face of degree greater than three.

$\mathbb{C}$ *started from* z *with respect to the local topology on curves modulo time parameterisation induced by the metric* (5.2.2)*, uniformly over all* $z \in K$.

5.5.2.1 Invariance principle on embedded lattices

In order to prove Proposition 5.5.4, we need to take a step back and state a general theorem from [GMS22] which guarantees that the random walk on certain embedded lattices has Brownian motion as a scaling limit. For the reader's convenience, we recall here the main definitions needed and in what follows we adopt notation similar to that of [GMS22].

Definition 5.5.5. An embedded lattice $\mathcal{M}$ is a graph embedded in $\mathbb{C}$ in such a way that each edge is a simple curve with zero Lebesgue measure, the edges intersect only at their endpoints, and each compact subset of $\mathbb{C}$ intersects at most finitely many edges of $\mathcal{M}$. As usual, we write $\mathcal{V}\mathcal{M}$ for the set of vertices of $\mathcal{M}$, $\mathcal{E}\mathcal{M}$ for the set of edges of $\mathcal{M}$, and $\mathcal{F}\mathcal{M}$ for the set of faces of $\mathcal{M}$.

Definition 5.5.6. For an embedded lattice $\mathcal{M}$ and $x \in \mathcal{V}\mathcal{M}$, we define the *outradius* of x by setting

$$\mathrm{Outrad}(x) \stackrel{\text{def}}{=} \mathrm{diam}\left(\bigcup_{\{H \in \mathcal{F}\mathcal{M} : x \in \partial H\}} H\right),$$

i.e., the diameter of the union of the faces with x on their boundaries. Here ∂H denotes the boundary of the face $H \in \mathcal{F}\mathcal{M}$.

For $C > 0$ and $z \in \mathbb{C}$, we write $C(\mathcal{M} - z)$ for the embedded lattice obtained by first translating everything by the amount $-z$ and then by scaling everything by the factor C. We are interested in random embedded lattices that satisfy the following assumptions.

(I1) (**Translation invariance modulo scaling**) There exists a (possibly random and $\mathcal{M}$-dependent) increasing sequence of open sets $U_j \subset \mathbb{C}$, each of which is either a square or a disk, whose union is all of $\mathbb{C}$ such that the following is true. Conditional on $\mathcal{M}$ and U_j, let z_j for $j \in \mathbb{N}$ be sampled uniformly from the Lebesgue measure on U_j. Then the shifted lattice $\mathcal{M} - z_j$ converge in law to $\mathcal{M}$ modulo scaling as $j \to \infty$, i.e., there exist random numbers $C_j > 0$ (possibly depending on $\mathcal{M}$ and z_j) such that $C_j(\mathcal{M} - z_j) \to \mathcal{M}$ in law with respect to the metric specified in [GMS22, Equation (1.6)].[6]

(I2) (**Ergodicity modulo scaling**) Every real-valued measurable function $F = F(\mathcal{M})$ which is invariant under translation and scaling, i.e., $F(C(\mathcal{M} - z)) = F(\mathcal{M})$ for each $z \in \mathbb{C}$ and $C > 0$, is almost surely equal to a deterministic constant.

(I3) (**Finite expectation**) Let $H_0 \in \mathcal{F}\mathcal{M}$ be the face of $\mathcal{M}$ containing o, then

$$\mathbb{E}\left[\sum_{x \in \mathcal{V}\mathcal{M} \cap \partial H_0} \frac{\mathrm{Outrad}(x)^2 \deg(x)}{\mathrm{Area}(H_0)}\right] < \infty,$$

where, in this case, $\deg(x)$ denotes the number of edges incident to x.

Proposition 5.5.7 ([GMS22, Theorem 1.11])**.** *Let* $\mathcal{M}$ *be an embedded lattice satisfying assumptions (I1), (I2), (I3). For* $\varepsilon > 0$*, view the embedded random walk on* $\varepsilon\mathcal{M}$ *as a continuous curve in* $\mathbb{C}$ *obtained by piecewise linear interpolation at constant speed. For each compact subset* $K \subset \mathbb{C}$ *and for any* $z \in K$*, the conditional law given* $\mathcal{M}$ *of the random walk on* $\varepsilon\mathcal{M}$ *started from the vertex* $\widehat{x}_z^\varepsilon \in \mathcal{V}(\varepsilon\mathcal{M})$ *nearest to* z *weakly converges*

[6] Several equivalent formulations of this condition are given in [GMS22, Definition 1.2].

in probability as $\varepsilon \to 0$ to the law of a Brownian motion on $\mathbb{C}$, with some deterministic, non-degenerate covariance matrix Σ, started from z with respect to the local topology on curves modulo time parameterisation induced by the metric (5.2.2), uniformly over all $z \in K$.

To be precise, the above theorem follows from the proof [GMS22, Theorem 1.11] (which gives the same statement but without the uniform rate of convergence on compact subsets) by using [GMS22, Theorem 3.10] in place of [GMS22, Theorem 1.16].

5.5.2.2 Proof of Proposition 5.5.4

The purpose of this subsection is to transfer the general result of Proposition 5.5.7 to the particular setting of Proposition 5.5.4. In order to do this, we need to proceed in two steps. First, we verify that the hypothesis of Proposition 5.5.7 are satisfied for a sequence of embedded lattices built through the 0-quantum cone. Then, we transfer the result to the sequence of n-mated CRT maps with the sphere topology by means of an absolute continuity argument.

Let $(\mathbb{C}, h^{\mathrm{c}}, 0, \infty)$ be a 0-quantum cone as specified in Subsection 5.5.1. We now define a graph associated with the 0-quantum cone in a way which is exactly analogous to the SLE/LQG description of mated-CRT maps with the sphere topology described in Section 5.5.1. For $\gamma \in (0,2)$, let ϑ be a whole-plane space-filling SLE_κ, with $\kappa = 16/\gamma^2$, sampled independently from h^{c} and then parameterised by the γ-LQG measure $\mu_{h^{\mathrm{c}}}$ in such a way that $\vartheta(0) = 0$. Let ξ be sampled uniformly from the unit interval $[0,1]$, independently from everything else, and let

$$\mathcal{H} \stackrel{\text{def}}{=} \big\{\vartheta\big([x-1,x]\big) \,:\, x \in \mathbb{Z} + \xi\big\}.$$

The reason for considering times in $\mathbb{Z}+\xi$ instead just in $\mathbb{Z}$ is to avoid making the point $0 = \vartheta(0)$ special. This is needed in order to check the translation invariance modulo scaling hypothesis (I1). We view $\mathcal{H}$ as a planar map whose vertex set is $\mathcal{H}$ itself. Two vertices H, $H' \in \mathcal{H}$ are joined by one edge (resp. two edges) if and only if H and H' intersect along one (resp. two) non-trivial connected boundary arc. (resp. arcs). For H, $H' \in \mathcal{H}$, with $H \neq H'$, we write $H \sim H'$ if they are joined by at least one edge. Given a set $A \subset \mathbb{C}$, we write

$$\mathcal{H}(A) \stackrel{\text{def}}{=} \big\{H \in \mathcal{H} \,:\, H \cap A \neq \emptyset\big\}$$

and moreover, for $H \in \mathcal{H}$, we write $\deg(H)$ for the number of cells $H' \in \mathcal{H}$ (counted with edge multiplicity) such that $H \sim H'$.

Proof of Proposition 5.5.4. For $\gamma \in (0,2)$, define the cell configuration $\mathcal{H}$ associated with a space-filling SLE_κ on a 0-quantum cone as specified above. Let $\mathcal{M}$ be the embedded defined as follows.

- The vertex set of $\mathcal{M}$ consists of points $x \in \mathbb{C}$ such that there exist three cells in $\mathcal{H}$ that meet at x and with the property that the boundary of each of such cells has a connected subset that touches x;
- The edge set of $\mathcal{M}$ consists of the boundary segments of the cells which connect the vertices.

In other words, $\mathcal{M}$ is nothing but the planar dual of $\mathcal{H}$. Moreover, by construction, $\mathcal{M}$ is an embedded lattice in the sense of Definition 5.5.5, except that embedded edges are allowed to intersect but not cross, in the case when $\gamma \in (\sqrt{2}, 2)$. To deal with this case, we can consider a different embedding of $\mathcal{M}$ in which all the vertices occupy the same position, but the edges are slightly perturbed so that they do not intersect except at their endpoints. More precisely, for each edge with touching points in its interior, we can slightly perturb it in such a way that the modification only depends on the position of the edge itself and on the positions of

the adjacent faces. In particular, this perturbation can be carried out in a translation and scaling invariant way. Therefore, since this procedure only depends on the local configuration of the lattice, if the starting cell configuration is translation invariant modulo scaling, then also the perturbed lattice is translation invariant modulo scaling.

We will now check that $\mathcal{M}$ satisfies assumptions (I1), (I2), and (I3). The translation invariance modulo scaling assumption (I1) and the ergodicity modulo scaling assumption (I2) follows from the corresponding properties for the associated cell configuration $\mathcal{H}$ as checked in [GMS21, Proposition 3.1]. Therefore, we can just focus on proving the finite expectation assumption (I3). To this end, recalling that the cell H_0 is the face of $\mathcal{M}$ containing 0, we proceed as follows

$$\begin{aligned}\sum_{x\in\mathcal{V}\mathcal{M}\cap\partial H_0}\frac{\mathrm{Outrad}(x)^2\deg(x)}{\mathrm{Area}(H_0)} &\leq 12\sum_{x\in\mathcal{V}\mathcal{M}\cap\partial H_0}\sum_{\{H\in\mathcal{H}\,:\,x\in\partial H\}}\frac{\mathrm{diam}(H)^2}{\mathrm{Area}(H_0)}\\ &\leq 48\sum_{\{H\in\mathcal{H}\,:\,H\sim H_0\}}\frac{\mathrm{diam}(H)^2}{\mathrm{Area}(H_0)}+12\frac{\mathrm{diam}(H_0)^2}{\mathrm{Area}(H_0)}\deg(H_0).\end{aligned} \tag{5.5.2}$$

In the first line of (5.5.2), we used the fact that each vertex of $\mathcal{M}$ has degree at most 3 and the inequality $(a+b+c)^2\leq 4(a^2+b^2+c^2)$. In the second line, we use that each cell $H\in\mathcal{H}$ with $H\sim H_0$ intersects H_0 along at most two disjoint connected boundary arcs (one on its left boundary and one on its right boundary), so there exist at most 4 vertices of $\mathcal{M}$ in $H\cap H_0$. Now, we notice that the second quantity in the last line of (5.5.2) has finite expectation thanks to [GMS21, Theorem 4.1]. Therefore, we can just focus our attention to the sum appearing in the last line of (5.5.2). Basically, the fact that this sum has finite expectation follows from the combination of several results in [GMS21] and [GMS22].

Let $F=F(\mathcal{H})$ denote the sum appearing in the last line of (5.5.2). We will show that $\mathbb{E}[F]<\infty$ using an ergodic theory result from [GMS22, Section 2.2]. Let $\{S_k\}_{k\in\mathbb{Z}}$ be the bi-infinite sequence of origin-containing squares of a uniform dyadic system independent from $\mathcal{H}$, as defined in [GMS22, Section 2.1]. We will not need the precise definition of these sequence here, but rather we only need to know $S_{k-1}\subset S_k$ for each $k\in\mathbb{Z}$ and $\cup_{k=1}^{\infty}S_k=\mathbb{C}$. As shown in [GMS21, Proposition 3.1], the cell configuration $\mathcal{H}$ satisfies a suitable translation invariance modulo scaling assumption, and so, we can apply [GMS22, Lemma 2.7] to $\mathcal{H}$ to find that the following is true. If we let F_z for $z\in\mathbb{C}$ be defined in the same manner as F but with the translated cell configuration $\mathcal{H}-z$ in place of $\mathcal{H}$, then it holds almost surely that

$$\mathbb{E}[F]=\lim_{k\to\infty}\frac{1}{\mathrm{Area}(S_k)}\int_{S_k}F_z dz.$$

To bound the right-hand side of the above expression, we can proceed as follows

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{\mathrm{Area}(S_k)}\int_{S_k}F_z dz &= \lim_{k\to\infty}\frac{1}{\mathrm{Area}(S_k)}\int_{S_k}\sum_{\{H\in\mathcal{H}\,:\,H\sim H_z\}}\frac{\mathrm{diam}(H)^2}{\mathrm{Area}(H_z)}dz\\ &\leq \limsup_{k\to\infty}\frac{1}{\mathrm{Area}(S_k)}\sum_{H\in\mathcal{H}(S_k)}\sum_{\{H'\in\mathcal{H}\,:\,H'\sim H\}}\mathrm{diam}(H')^2.\end{aligned} \tag{5.5.3}$$

Since the maximal size of the cells in $\mathcal{H}(S_k)$ is almost surely of strictly smaller order than the side length of S_k as $k\to\infty$ (see [GMS22, Lemma 2.9]), we find that almost surely for large enough $k\in\mathbb{N}$, each $H'\in\mathcal{H}$ with $H'\sim H$ for some $H\in\mathcal{H}(S_k)$ is contained in $\mathcal{H}(S_k(1))$, where $S_k(1)$ is the square with the same center as S_k and three times the side length of S_k. Therefore, the last line in (5.5.3) is almost surely bounded above

by

$$2 \limsup_{k\to\infty} \frac{1}{\mathrm{Area}(S_k)} \sum_{H\in\mathcal{H}(S_k(1))} \mathrm{diam}(H)^2 \deg(H),$$

which is finite by [GMS22, Lemmas 2.8 and 2.10].

Therefore, summing up, we proved that the embedded lattice $\mathcal{M}$ satisfies assumptions (I1), (I2), (I3), ad so, we can apply Proposition 5.5.7. Furthermore, we notice that, thanks to the rotational invariance of the law of (h^c, ϑ), the limiting covariance matrix Σ is given by a positive scalar multiple of the identity matrix. Hence, in order to conclude the proof of Proposition 5.5.4, we need to transfer the result to the setting of mated-CRT maps with the sphere topology. This can be done by means of absolute continuity arguments as in [GMS21, Subsection 3.3]. For the sake of brevity, we will not repeat such arguments here and we refer to [GMS21]. □

5.5.3 Convergence to LQG

In this subsection, we see how the proof of Theorem 5.B is almost an immediate consequence of the results proved above. More specifically, for $n \in \mathbb{N}$ and $\gamma \in (0,2)$, let $(\mathcal{G}^n, v_0^n, v_1^n)$ be the doubly marked n-mated-CRT map with the sphere topology under the a priori SLE/LQG embedding as specified in Subsection 5.5.1. We observe that there exists a conformal map from $\mathbb{C}$ to $\mathcal{C}_{2\pi}$ sending $\infty \mapsto +\infty$ and $0 \mapsto -\infty$. This mapping is unique up to horizontal translation and rotation. The horizontal translation can be fixed by specifying that the volume of $\mathbb{R}/2\pi\mathbb{Z} \times [0,\infty)$ under the γ-LQG measure μ_h induced by the embedding is precisely $1/2$. Furthermore, the rotation on $\mathbb{R}/2\pi\mathbb{Z}$ can be chosen uniformly at random. Therefore, using this conformal map, we can define an embedding of $(\mathcal{G}^n, v_0^n, v_1^n)$ into $\mathcal{C}_{2\pi}$ in such a way that v_0^n and v_1^n are mapped to $-\infty$ and $+\infty$, respectively. Using the same conformal map, we can also embedded the associated dual graph $\widehat{\mathcal{G}}^n$ into $\mathcal{C}_{2\pi}$. This put us exactly in the setting of Theorem 5.A from which we can deduce the following result.

Proposition 5.5.8. *Fix $\gamma \in (0,2)$ and let $\{(\mathcal{G}^n, v_0^n, v_1^n)\}_{n\in\mathbb{N}}$ be the sequence of doubly marked n-mated CRT map with the sphere topology embedded in the infinite cylinder $\mathcal{C}_{2\pi}$ as specified above. For each $n \in \mathbb{N}$, let $\dot{\mathcal{S}}_n : \mathcal{V}\mathcal{G}^n \to \mathcal{C}_{2\pi}$ be the Smith embedding associated to $(\mathcal{G}^n, v_0^n, v_1^n)$ as specified in Definition 5.2.14. There exists a sequence of random affine transformations $\{T_n\}_{n\in\mathbb{N}}$ from $\mathcal{C}_{\eta_n}$ to $\mathcal{C}_{2\pi}$ of the form specified in the statement of Theorem 5.A such that, for all compact sets $K \subset \mathcal{C}_{2\pi}$, the following convergence holds in probability*

$$\lim_{n\to\infty} \sup_{x\in\mathcal{V}\mathcal{G}^n(K)} \mathrm{d}_{2\pi}\big(T_n\dot{\mathcal{S}}_n(x), x\big) = 0,$$

where $\mathrm{d}_{2\pi}$ denotes the Euclidean distance on the cylinder $\mathcal{C}_{2\pi}$.

Proof. By construction the sequence $\{(\mathcal{G}^n, v_0^n, v_1^n)\}_{n\in\mathbb{N}}$ and the associated sequence of dual graphs $\{\widehat{\mathcal{G}}^n\}_{n\in\mathbb{N}}$ satisfy almost surely assumption (H1). Moreover, Proposition 5.5.2 guarantees the convergence in probability of the random walk on the sequence of primal graphs to Brownian motion and so assumption (H2) is satisfied. Furthermore, Proposition 5.5.4 guarantees the convergence in probability of the random walk on the sequence of dual graphs to Brownian motion and so also assumption (H3) is satisfied. Therefore, the desired result follows from Theorem 5.A. □

Using the same procedure specified at the beginning of this subsection, we can also consider the parametrisation of the unit-area quantum sphere by the infinite cylinder $\mathcal{C}_{2\pi}$. Hence, with a slight abuse of notation, we let $(\mathcal{C}_{2\pi}, h, -\infty, +\infty)$ be the unit-area quantum sphere parametrised by $\mathcal{C}_{2\pi}$ and we denote by μ_h the associated γ-LQG measure. We are now ready to prove Theorem 5.B.

Proof of Theorem 5.B. The result on the measure convergence (a) follows from Proposition 5.5.8 and the fact that ϑ is parameterised by μ_h-mass. The uniform convergence statement for curves (b) is also an immediate consequence of Proposition 5.5.8. The claimed random walk convergence (c) follows from Propositions 5.5.2 and 5.5.8. □

5.A Some standard estimates for planar Brownian motion

Throughout the whole appendix, B denotes a standard planar Brownian motion. For $\mathbf{x} \in \mathbb{R}^2$, we use the notation $B^{\mathbf{x}}$ to denote a planar Brownian motion started from $\mathbf{x}$. We recall that $\sigma_{2\pi} : \mathbb{R}^2 \to \mathcal{C}_{2\pi}$ is defined in (5.2.4) and denotes the covering map of the infinite cylinder $\mathcal{C}_{2\pi}$. In particular, if $B^{\mathbf{x}}$ is a as above, then $\sigma_{2\pi}(B^{\mathbf{x}})$ is a Brownian motion on $\mathcal{C}_{2\pi}$ started from $\sigma_{2\pi}(\mathbf{x})$.

Lemma 5.A.1. *Fix $R' > 1$. For $\mathbf{x} \in \mathbb{R}^2$, consider the following stopping times*

$$\tau \overset{\text{def}}{=} \inf\big\{t \geq 0 \,:\, \mathrm{Im}(B_t^{\mathbf{x}}) = -2R' \text{ or } \mathrm{Im}(B_t^{\mathbf{x}}) = R'\big\}.$$

Then it holds that

$$\mathbb{P}_{\mathbf{x}}\big(\sigma_{2\pi}(B|_{[0,\tau]}) \text{ does not wind around the cylinder below height } -R'\big) \lesssim \frac{1}{R'}, \quad \forall\, \mathbf{x} \in \mathbb{R} \times \{-R'\},$$

where the implicit constant is independent of everything else.

Proof. Fix $\mathbf{x} \in \mathbb{R} \times \{-R'\}$. Let $\sigma_0 := 0$, and for $k \in \mathbb{N}_0$ we define inductively

$$\tau_k \overset{\text{def}}{=} \inf\big\{t \geq \sigma_k \,:\, \mathrm{Im}(B_t^{\mathbf{x}}) = -R' - 1\big\}, \quad \sigma_{k+1} \overset{\text{def}}{=} \inf\big\{t \geq \tau_k \,:\, \mathrm{Im}(B_t^{\mathbf{x}}) = -R'\big\}.$$

Moreover, for $k \in \mathbb{N}_0$ consider the events

$$A_k \overset{\text{def}}{=} \big\{|\,\mathrm{Re}(B_{\sigma_k}^{\mathbf{x}}) - \mathrm{Re}(B_{\tau_k}^{\mathbf{x}})| \geq 1\big\}, \quad F_k \overset{\text{def}}{=} \big\{\tau_k > \tau\big\}.$$

Let K be the smallest $k \in \mathbb{N}_0$ such that F_k occurs. Then the probability that the event in the lemma statement happens is less or equal to the probability that none of the events $\{A_k\}_{k\in[K]_0}$ happen. Thanks to the strong Markov property of the Brownian motion, the events $\{A_k\}_{k\in\mathbb{N}_0}$ are independent and identically distributed. Moreover, thanks to well-known properties of Brownian motion, the event A_0 happens with uniformly positive probability p independent of R' and $\mathbf{x}$. Therefore, we obtain that

$$\mathbb{P}_{\mathbf{x}}\left(\bigcap_{i=0}^{K} \overline{A_k}\right) = \sum_{k\in\mathbb{N}_0} \mathbb{P}_{\mathbf{x}}\left(\bigcap_{i=0}^{k} \overline{A_k}\right) \mathbb{P}_{\mathbf{x}}(K = k) = \sum_{k\in\mathbb{N}_0} (1-p)^k \mathbb{P}_{\mathbf{x}}(K = k) \lesssim \frac{1}{R'},$$

for all $\mathbf{x} \in \mathbb{R} \times \{R'\}$, where the implicit constant is independent of everything else, and the last inequality follows from standard Brownian motion estimates. Hence, this concludes the proof. □

Lemma 5.A.2. *Fix $R > 1$, $R' > R$. For $\mathbf{x} \in \mathbb{R} \times [-R, R]$, let $\tau \overset{\text{def}}{=} \inf\{t \geq 0 \,:\, |\,\mathrm{Im}(B_t^{\mathbf{x}})| = R'\}$, and define the events*

$$W^+ \overset{\text{def}}{=} \big\{\sigma_{2\pi}(B^{\mathbf{x}}|_{[0,\tau]}) \text{ does a loop around the cylinder between heights } R \text{ and } R'\big\},$$
$$W^- \overset{\text{def}}{=} \big\{\sigma_{2\pi}(B^{\mathbf{x}}|_{[0,\tau]}) \text{ does a loop around the cylinder between heights } -R' \text{ and } -R\big\}.$$

Then it holds that

$$\mathbb{P}_{\mathbf{x}}\big(\overline{W^+} \cup \overline{W^-}\big) \lesssim \frac{R}{R'}, \quad \forall \mathbf{x} \in \mathbb{R} \times [-R, R],$$

where the implicit constant is independent of everything else.

Proof. Fix $\mathbf{x} \in \mathbb{R} \times [-R, R]$. It is sufficient to prove that $\mathbb{P}_{\mathbf{x}}(\overline{W^+}) \lesssim R/R'$, and the same with W^- in place of W^+. We will proceed similarly to the proof of the previous lemma. Let $\sigma_0 := \inf\{t \geq 0 : \mathrm{Im}(B_t^{\mathbf{x}}) = R\}$ and, for $k \in \mathbb{N}_0$ we define inductively

$$\tau_k \stackrel{\text{def}}{=} \inf\big\{t \geq \sigma_k : \mathrm{Im}(B_t^{\mathbf{x}}) = R + 1\big\}, \quad \sigma_{k+1} \stackrel{\text{def}}{=} \inf\big\{t \geq \tau_k : \mathrm{Im}(B_t^{\mathbf{x}}) = R\big\}.$$

Moreover, for $k \in \mathbb{N}_0$ consider the events

$$A_k \stackrel{\text{def}}{=} \big\{|\,\mathrm{Re}(B_{\sigma_k}^{\mathbf{x}}) - \mathrm{Re}(B_{\tau_k}^{\mathbf{x}})| = 1,\, \mathrm{Im}(B_{\sigma_k}^{\mathbf{x}}) = \mathrm{Im}(B_{\tau_k}^{\mathbf{x}})\big\}, \quad F_k \stackrel{\text{def}}{=} \big\{\tau_k > \tau\big\}.$$

Let K be the smallest $k \in \mathbb{N}_0$ such that F_k occurs. Then the probability that the event W^+ does not happen is less or equal to the probability that none of the events $\{A_k\}_{k \in [K]_0}$ happen. Thanks to the strong Markov property of the Brownian motion, the events $\{A_k\}_{k \in \mathbb{N}_0}$ are independent and identically distributed. Moreover, since the event A_0 happens with uniformly positive probability p independent of R and $\mathbf{x}$, we have that

$$\mathbb{P}_{\mathbf{x}}(\overline{W^+}) \leq \mathbb{P}_{\mathbf{x}}\left(\bigcap_{i=0}^{K} \overline{A_i}\right) = \sum_{k \in \mathbb{N}_0} \mathbb{P}_{\mathbf{x}}\left(\bigcap_{i=0}^{k} \overline{A_i}\right) \mathbb{P}_{\mathbf{x}}(K = k) = \sum_{k \in \mathbb{N}_0} (1-p)^k \mathbb{P}_{\mathbf{x}}(K = k) \lesssim \frac{R}{R'},$$

for all $\mathbf{x} \in \mathbb{R} \times [-R, R]$, where the implicit constant is independent of everything else, and the last inequality follows from standard Brownian motion estimates. Proceeding in a similar way, one can also prove that $\mathbb{P}_{\mathbf{x}}(\overline{W^-}) \lesssim R/R'$. Therefore, the desired result follows from the fact that $\mathbb{P}_{\mathbf{x}}(\overline{W^+} \cup \overline{W^-}) \leq \mathbb{P}_{\mathbf{x}}(\overline{W^+}) + \mathbb{P}_{\mathbf{x}}(\overline{W^-})$. □

Chapter 6

Liouville Brownian motion and quantum cones in dimension $d > 2$

Abstract. For $d > 2$ and $\gamma \in (0, \sqrt{2d})$, we study the Liouville Brownian motion associated with the whole-space log-correlated Gaussian field in $\mathbb{R}^d$. We compute its spectral dimension, i.e., the short-time asymptotics of the heat kernel along the diagonal, which, in contrast to the two-dimensional case, depends on both the coupling constant and on the thickness of the starting point. Furthermore, for even dimensions $d > 2$, we show that the spherical average of the whole-space log-correlated Gaussian field in $\mathbb{R}^d$ can be identified with the integral of a stationary Gaussian Markov process of order $(d-2)/2$. Exploiting this representation, we construct the higher-dimensional analogue of the β-quantum cone field for $\beta \in [0, Q)$, with $Q = d/\gamma + \gamma/2$. Lastly, for $\alpha = Q - \sqrt{Q^2 - 4}$, we prove that the law of the d-dimensional α-quantum cone field is invariant under shifts along the trajectories of the associated Liouville Brownian motion.

6.1 Main results

The main objective of this chapter is to extend certain random geometric objects, previously defined in two dimensions, to higher dimensions. Specifically, we first identify the spherical average process in any even dimension. Next, we investigate the spectral dimension of the higher-dimensional *Liouville Brownian motion* (LBM). Finally, we construct the quantum cone in an arbitrary even dimension and examine its interaction with the corresponding LBM. For a brief, informal discussion of these results, we refer the reader to Section 1.5.5.

Throughout the rest of the chapter, we let h be a whole-space LGF modulo an additive constant defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see Section 1.2 or Section 6.2.1 for details). We denote by $\mathbf{h}$ the whole-space LGF h with the additive constant chosen such that its spherical average over the unit sphere centred at the origin is zero. Additionally, we consider a d-dimensional Brownian motion $(\mathbf{\Omega}, \mathbf{F}, (B_t)_{t\geq 0}, (\mathbf{F}_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}^d})$, which is independent of h.

6.1.1 Identification of the spherical average in arbitrary even dimension

As shown in Section 6.2.2 below, the spherical average process of the LGF h around a point $x \in \mathbb{R}^d$, denoted by $(h_r(x))_{r>0}$, is well-defined modulo a global additive constant. Moreover, thanks to [LSSW16, Section 11.1], the process $(h_r(x))_{r\geq 0}$ is differentiable $\mathfrak{c}_d$ times, where

$$\mathfrak{c}_d \overset{\text{def}}{=} \frac{d-2}{2} . \tag{6.1.1}$$

Therefore, we can define the following stochastic processes

$$\mathrm{S}_t \overset{\text{def}}{=} h_{e^{-t}}(x) - h_1(x)\,, \qquad \mathbf{S}_t \overset{\text{def}}{=} \big(\mathrm{S}_t^{(1)}, \dots, \mathrm{S}_t^{(\mathfrak{c}_d)}\big)^\top\,, \qquad \forall\, t \in \mathbb{R}\,, \tag{6.1.2}$$

where, for $k \in \{1, \dots, \mathfrak{c}_d\}$, the process $(\mathrm{S}_t^{(k)})_{t\in\mathbb{R}}$ denotes the k-th derivative of S. We note that $(\mathrm{S}_t)_{t\in\mathbb{R}}$ is well-defined not just modulo an additive constant. It is well-known that in dimension $d = 2$, the process $(\mathrm{S}_t)_{t\in\mathbb{R}}$ has the same law of a standard two-sided Brownian motion (see e.g. [BP24, Theorem 1.59]). In our first result, for any arbitrary even[1] dimension $d > 2$, we identify the process $(\mathrm{S}_t)_{t\in\mathbb{R}}$ as the integral of a stationary Gaussian Markov process of order $\mathfrak{c}_d$. Before proceeding, we recall the following definitions.

Definition 6.1.1. Let $p \geq 1$ be an integer, and let $(X_t)_{t\in\mathbb{R}}$ be a real-valued, centred, stationary Gaussian process such that, for all $k \in [p-1]$, the k-th derivative $(X_t^{(k)})_{t\in\mathbb{R}}$ exists. We say that $(X_t)_{t\in\mathbb{R}}$ is a stationary Gaussian Markov process of order p if the $\mathbb{R}^p$-valued process $(X_t, X_t^{(1)}, \dots, X_t^{(p-1)})_{t\in\mathbb{R}}$ is a stationary Gaussian Markov process.

Definition 6.1.2. For $p \geq 1$ and $(a_i)_{i=0}^{p-1} \subset \mathbb{R}$, a $p \times p$ matrix $\mathbf{A}$ is the Frobenius companion matrix of the polynomial $p(\lambda) = \lambda^p + a_{p-1}\lambda^{p-1} + \cdots + a_1\lambda + a_0$, if it is of the following form

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{p-2} & -a_{p-1} \end{pmatrix}.$$

We observe that, by construction, the characteristic polynomial of the Frobenius companion matrix is exactly equal to $p(\cdot)$.

Theorem 6.A (Identification of the spherical average process)**.** *For $d > 2$ even and any $x \in \mathbb{R}^d$, consider the processes $(\mathrm{S}_t)_{t\in\mathbb{R}}$ and $(\mathbf{S}_t)_{t\in\mathbb{R}}$ as defined in (6.1.2). Then the multivariate process $(\mathbf{S}_t)_{t\in\mathbb{R}}$ is a stationary Gaussian Markov process and it is the unique stationary solution of the following $\mathfrak{c}_d$-dimensional Langevin equation,*

$$\mathrm{d}\mathbf{S}_t = \mathbf{A}\mathbf{S}_t\mathrm{d}t + \mathbf{b}\mathrm{d}\mathbf{B}_t\,, \tag{6.1.3}$$

where $(\mathbf{B}_t)_{t\in R}$ denotes a $\mathfrak{c}_d$-dimensional two-sided Brownian motion, $\mathbf{b}$ is the $\mathfrak{c}_d$-dimensional vector given by $\mathbf{b} = (0, \dots, 0, 2^{d/2-1}\Gamma(d/2))^\top$, and $\mathbf{A}$ is the $\mathfrak{c}_d \times \mathfrak{c}_d$ Frobenius companion matrix of the polynomial

$$p(\lambda) = \prod_{k=1}^{\mathfrak{c}_d} (\lambda + (d - 2k))\,.$$

Furthermore, the process $(\mathrm{S}_t)_{t\in\mathbb{R}}$ admits the following representation

$$\mathrm{S}_t \overset{\text{law}}{=} (d-2)\int_{-\infty}^{0} e^{2s}(1 - e^{2s})^{\frac{d-4}{2}}\big(B_{s+t} - B_s\big)ds\,, \qquad \forall\, t \in \mathbb{R}\,, \tag{6.1.4}$$

where $(B_s)_{s\in\mathbb{R}}$ is a standard two-sided Brownian motion.

[1] We refer to Remark 1.5.6 for an explanation of why some of the results in this chapter are restricted to even dimensions.

We observe that for $d > 2$ even, the representation (6.1.4) of the spherical average process indicates that $(\mathrm{S}_t)_{t\in\mathbb{R}}$ acts as a "smoothed-out" version of a two-sided Brownian motion $(B_t)_{t\in\mathbb{R}}$. Moreover, as expected, increasing the dimension d enhances the smoothness of $(\mathrm{S}_t)_{t\in\mathbb{R}}$ (see Figure 6.1).

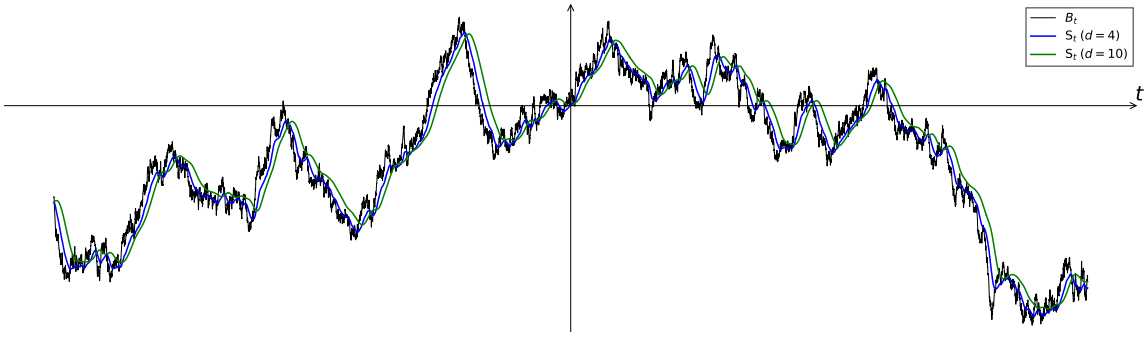


FIGURE 6.1: The blue (resp. green) curve represents a simulation of the process $(\mathrm{S}_t)_{t\in\mathbb{R}}$ in dimension $d = 4$ (resp. $d = 10$), generated according to the representation given in (6.1.4). The black curve corresponds to the driving two-sided Brownian motion $(B_t)_{t\in\mathbb{R}}$.

Remark 6.1.3. We emphasise that if $d = 4$, then (6.1.3) reduces to a one-dimensional SDE whose solution is the stationary *Ornstein–Uhlenbeck* (OU) process introduced in [UO30]. For this reason, in the literature, solutions to the SDE of the form (6.1.3) are usually called *multivariate OU* processes or also *continuous time autoregressive* processes [VVI+22].

6.1.2 Spectral dimension of the LBM in arbitrary dimension

The construction of LBM in dimensions $d > 2$ closely follows the approach used for its construction in dimension $d = 2$ [Ber15, GRV16], without introducing any significant conceptual difficulties. We also refer the reader to [DSHKS24] for a similar construction of the LBM on d-dimensional compact manifolds. For $\alpha \in (0, 2)$, we define the *clock process* $(\mathrm{F}_{\mathbf{h},\alpha}(t))_{t\geq 0}$ by

$$\mathrm{F}_{\mathbf{h},\alpha}(t) \overset{\text{def}}{=} \lim_{\varepsilon\to 0} \int_0^t \varepsilon^{\alpha^2/2} e^{\alpha \mathbf{h}_\varepsilon(B_s)} ds\,, \qquad \forall\, t \geq 0\,. \tag{6.1.5}$$

The convergence of the limit in the right-hand side of the above display falls within the scope of the theory of subcritical GMC measures, which guarantees that, for all $x \in \mathbb{R}^d$, the clock process is $\mathbf{P}_x \otimes \mathbb{P}$-almost surely well-defined and non-trivial for all $\alpha \in (0, 2)$ (see Section 6.4.1 for more details).

We then aim to define the LBM as

$$\mathrm{B}_{\mathbf{h},\alpha,t} \overset{\text{def}}{=} B_{\mathrm{F}^{-1}_{\mathbf{h},\alpha}(t)}\,, \qquad \forall\, t \geq 0\,.$$

As it is proved in Proposition 6.2.10 below, the measure $\mu_{\mathbf{h},\gamma}$ satisfies the following coordinate transformation rule,

$$\mu_{\mathbf{h},\gamma}\big(\phi(A)\big) = \mu_{\mathbf{h}\circ\phi + Q\log|\phi'|,\gamma}(A)\,, \qquad Q \overset{\text{def}}{=} \frac{d}{\gamma} + \frac{\gamma}{2}\,, \tag{6.1.6}$$

for any conformal automorphism ϕ of $\mathbb{R}^d$ and any Borel subset $A \subseteq \mathbb{R}^d$. In particular, we want the LBM to be "compatible" with (6.1.6)[2]. Specifically, as in the two-dimensional setting [Ber15, Theorem 1.4], if ϕ is a

[2]The reason it is natural for the LBM to be compatible with the coordinate change formula (6.1.6) is that, as in the two-dimensional setting, we can view $(\mathbb{R}^d, \mathbf{h})$ and $(\mathbb{R}^d, \mathbf{h}\circ\phi + Q\log|\phi'|)$ as two different parameterisations of the same d-dimensional "manifold".

conformal automorphism of $\mathbb{R}^d$ that can be expressed as a composition of translations, dilations, and rotations, then we require that

$$\left(\phi^{-1}(\mathrm{B}_{\mathbf{h},\alpha,t})\right)_{t\geq 0} \overset{\text{law}}{=} \left(\overline{\mathrm{B}}_{\mathbf{h}\circ\phi+Q\log|\phi'|,\alpha,t}\right)_{t\geq 0}\,, \quad \overline{\mathrm{B}}_{\mathbf{h}\circ\phi+Q\log|\phi'|,\alpha,t} \overset{\text{def}}{=} \overline{B}_{\overline{\mathrm{F}}^{-1}_{\mathbf{h}\circ\phi+Q\log|\phi'|,\alpha}(t)}\,, \tag{6.1.7}$$

with $(\overline{B}_t)_{t\geq 0}$ a Brownian motion on $\mathbb{R}^d$ and $\overline{\mathrm{F}}_{\mathbf{h}\circ\phi+Q\log|\phi|,\alpha}$ the associated clock process with respect to the field $\mathbf{h}\circ\phi + Q\log|\phi'|$. Thanks to the fact that in every dimension $d \geq 2$, Brownian motion exhibits two-dimensional scaling, we will show in Proposition 6.4.2 that for (6.1.7) to hold, we need to impose that

$$\frac{2}{\alpha} + \frac{\alpha}{2} = Q \qquad \Longleftrightarrow \qquad \alpha = \alpha(Q) = \alpha(d,\gamma) \overset{\text{def}}{=} Q - \sqrt{Q^2-4}\,. \tag{6.1.8}$$

In particular, in dimension $d = 2$, we have $\alpha = \gamma$, whereas in higher dimensions, α becomes a non-linear function of γ. We also introduce the critical threshold $\alpha_c = \alpha(\sqrt{2d})$ given by

$$\alpha_c \overset{\text{def}}{=} \sqrt{2d} - \sqrt{2d-4}\,. \tag{6.1.9}$$

We can now define the d-dimensional LBM associated to the whole-space LGF.

Definition 6.1.4. For $d \geq 2$ and $Q > 2$, the *Liouville Brownian motion* associated to the d-dimensional LGF $\mathbf{h}$ is the process defined as follows

$$\mathrm{B}_{\mathbf{h},\alpha,t} \overset{\text{def}}{=} B_{\mathrm{F}^{-1}_{\mathbf{h},\alpha}(t)} \qquad \forall\, t \geq 0\,,$$

where α is related to Q as specified in (6.1.8), and $\mathrm{F}_{\mathbf{h},\alpha}$ is the clock process introduced in (6.1.5).

We observe that, for all $Q > 2$, it holds that $\alpha \in (0,2)$, ensuring that the clock process is both well-defined and non-trivial. In particular, the regime $Q > \sqrt{2d}$ (equivalently, $\gamma < \sqrt{2d}$) corresponds to the higher-dimensional analogue of the *subcritical* phase of LQG. The case $Q = \sqrt{2d}$ (equivalently, $\gamma = \sqrt{2d}$) represents the higher-dimensional analogue of the *critical* phase. Meanwhile, the regime $Q \in (2, \sqrt{2d})$ corresponds to the higher-dimensional analogue of the *supercritical* phase. In this latter regime, the coupling constant γ is complex. It is expected that no area measure exists, and the associated metric exhibits infinite spikes-similar to the supercritical LQG metric in two dimensions. For further details, see [DGZ24, Problem 7.13] and [DG23a, DG23b, BGS24].

To define the LBM as a Markov process, we need to establish that, $\mathbb{P}$-almost surely, the clock process $\mathrm{F}_{\mathbf{h},\alpha}$ can be defined for all starting points $x \in \mathbb{R}^d$ of the underlying Brownian motion. The placement of "$\mathbb{P}$-almost surely" is critical here, as it presents challenges that differ fundamentally from constructing the process at a single fixed point. This fact will be explored further in Sections 6.1.4 and 6.4. As we will discuss in more detail in Section 6.1.4, in contrast to the two-dimensional case, the LBM can be constructed as a Markov process for all $Q > 2$, i.e., also in the critical and supercritical phases. However, for $Q \in (2, \sqrt{2d}]$, the LBM can only be defined for starting points outside a polar set of the d-dimensional Brownian motion.

In the subcritical phase, i.e., $Q > \sqrt{2d}$, the transition semigroup $(\mathrm{P}_{\mathbf{h},\alpha,t})_{t\geq 0}$ associated with the LBM is absolutely continuous with respect to the GMC measure $\mu_{\mathbf{h},\alpha}$. More precisely, we have the following result which is proved in Section 6.4.2.

Proposition 6.1.5. *Let $d \geq 2$ and $Q > \sqrt{2d}$. Then the Liouville semigroup $(\mathrm{P}_{\mathbf{h},\alpha,t})_{t\geq 0}$ is $\mathbb{P}$-almost surely absolutely continuous with respect to $\mu_{\mathbf{h},\alpha}$, i.e., there exists a family of jointly measurable functions*

$(\mathfrak{p}_{\mathbf{h},\alpha,t}(\cdot,\cdot))_{t\geq 0}$*, called the Liouville heat kernel (LHK), such that*

$$\mathrm{P}_{\mathbf{h},\alpha,t}[f](x) = \int_{\mathbb{R}^d} f(y)\mathfrak{p}_{\mathbf{h},\alpha,t}(x,y)\mu_{\mathbf{h},\alpha}(dy)\ , \qquad \forall\, f \in \mathcal{B}_b(\mathbb{R}^d,\mathbb{R}),\ \forall\, x \in \mathbb{R}^d,\ \forall\, t\ \geq 0\ ,$$

where $\mathcal{B}_b(\mathbb{R}^d,\mathbb{R})$ denotes the collection of bounded Borel functions in $\mathbb{R}^d$.

Remark 6.1.6. We emphasise that, a priori, for any $x \in \mathbb{R}^d$ and $t \geq 0$, we only know that $\mathfrak{p}_{\mathbf{h},\alpha,t}(x,y)$ is well-defined for $\mu_{\mathbf{h},\alpha}$-almost every $y \in \mathbb{R}^d$. This is due to the fact that $\mathfrak{p}_{\mathbf{h},\alpha,t}(x,\cdot)$ is defined as the density with respect to $\mu_{\mathbf{h},\alpha}$ of the semigroup associated with $\mathrm{B}_{\mathbf{h},\alpha,t}$ started from x.

For $x \in \mathbb{R}^d$, one would like to define the spectral dimension of the LBM at $x \in \mathbb{R}^d$ as follows,

$$\mathfrak{d}_{\mathbf{h},\alpha}(x) = \lim_{t\to 0} \frac{2\log \mathfrak{p}_{\mathbf{h},\alpha,t}(x,x)}{|\log t|}\ . \tag{6.1.10}$$

However, at present, we lack any regularity results for the LHK in arbitrary dimension $d > 2$, which means the limit on the right-hand side of (6.1.10) may not be well-defined. To address this issue, following [RV14b], instead of treating the LHK on the diagonal as a function, we consider it as a measure on $\mathbb{R}^+$, and we focus on the mass of this measure in the neighbourhood of $t = 0$. The key insight is that the mass at 0 of this measure can be determined via the behaviour of its Laplace transform. In turn, the Laplace transforms of the LHK are well-defined along the diagonal for all $x \in \mathbb{R}^d$ and can be represented as functionals of the GMC along the paths of Brownian bridges.

In order to make this rigorous, we need to introduce some notation. We use the convention that, for $x, y \in \mathbb{R}^d$ and $t > 0$, under $\mathbf{P}_{x,y,t}$, the process $(B_s)_{s\in[0,t]}$ is a d-dimensional Brownian bridge of length t from x to y[3]. We also denote by $p_t(\cdot,\cdot)$ the heat kernel of the d-dimensional Brownian motion.

Definition 6.1.7. Let $d \geq 2$ and $Q > \sqrt{2d}$. For $x \in \mathbb{R}^d$, for $\mu_{\mathbf{h},\alpha}$-almost every $y \in \mathbb{R}^d$, and for all $\chi \geq 0$, we define the following quantity:

$$\widetilde{\mathfrak{m}}_{\mathbf{h},\alpha,\chi}(x,y) \overset{\text{def}}{=} \int_0^\infty t^\chi e^{-t}\mathfrak{p}_{\mathbf{h},\alpha,t}(x,y)dt\ . \tag{6.1.11}$$

Moreover, for $x, y \in \mathbb{R}^d$ and for all $\chi \geq 0$, we introduce the following quantity:

$$\mathfrak{m}_{\mathbf{h},\alpha,\chi}(x,y) \overset{\text{def}}{=} \int_0^\infty \mathbf{E}_{x,y,t}\big[(\mathrm{F}_{\mathbf{h},\alpha}(t))^\chi e^{-\mathrm{F}_{\mathbf{h},\alpha}(t)}\big]p_t(x,y)dt\ , \tag{6.1.12}$$

with the notational convention that $\mathfrak{m}_{\mathbf{h},\alpha,\chi}(x,x) = \mathfrak{m}_{\mathbf{h},\alpha,\chi}(x)$.

A key result for the computation of the spectral dimension is that, thanks to Lemma 6.4.16 below, we have that, $\mathbb{P}$-almost surely,

$$\widetilde{\mathfrak{m}}_{\mathbf{h},\alpha,\chi}(x,y) = \mathfrak{m}_{\mathbf{h},\alpha,\chi}(x,y)\ , \qquad \forall\, x \in \mathbb{R}^d,\ \mu_{\mathbf{h},\alpha}\text{-a.e. } y \in \mathbb{R}^d\ , \tag{6.1.13}$$

where we emphasise that the quantity on the right-hand side of the above expression is well-defined for all x, $y \in \mathbb{R}^d$. We recall that our focus is on the behaviour of the quantity $\widetilde{\mathfrak{m}}_{\mathbf{h},\alpha,\chi}$ along the diagonal. However, as noted above, $\widetilde{\mathfrak{m}}_{\mathbf{h},\alpha,\chi}(x,y)$ is only well-defined for all $x \in \mathbb{R}^d$ and for $\mu_{\mathbf{h},\alpha}$-almost every $y \in \mathbb{R}^d$. In [RV14b], this issue is addressed by showing that the mapping $\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \mapsto \mathfrak{m}_{\mathbf{h},\alpha,\chi}(x,y)$ is continuous along

[3]We emphasise that, $\mathbb{P}$-almost surely, the clock process $\mathrm{F}_{\mathbf{h},\alpha}$ can be defined for all starting and end points $x, y \in \mathbb{R}^d$ and for any lifespan $t > 0$ of the underlying Brownian bridge (see Theorem 6.4.15 below).

the diagonal for all $\chi \geq 0$. Consequently, thanks to (6.1.13), $\widetilde{\mathfrak{m}}_{\mathbf{h},\alpha,\chi}(x,y)$ also admits a limit as $y \to x$ for all $x \in \mathbb{R}^d$ and for all $\chi \geq 0$.

In contrast to the two-dimensional case, in dimension $d > 2$, we expect that the quantity $\mathfrak{m}_{\mathbf{h},\alpha,\chi}(x,y)$ does not admit a limit as $y \to x$ for some $\chi \geq 0$ and $x \in \mathbb{R}^d$. Roughly speaking, this is due to the fact that as we will see below, the quantity $\mathfrak{m}_{\mathbf{h},\alpha,\chi}(x)$ is infinite (resp. finite) for $\chi \in [0,\overline{\chi})$ (resp. $\chi > \overline{\chi}$), for some $\overline{\chi}$ depending on the thickness of x (see Definition 6.1.9 below). Hence, since the set of thick points is dense, this implies that in any neighbourhood of x, there is a dense set of points where $\mathfrak{m}_{\mathbf{h},\alpha,\chi}(\cdot)$ is infinite for some $\chi > \overline{\chi}$. On the other hand, in the two-dimensional setting, the continuity of the quantity $\mathfrak{m}_{\mathbf{h},\alpha,\chi}(x,y)$ along the diagonal follows from the fact that its blow-up does not depend on the thickness of x.

The above consideration implies that we need an alternative approach to define $\widetilde{\mathfrak{m}}_{\mathbf{h},\alpha,\chi}(x,y)$ for all $x,y \in \mathbb{R}^d$. Given (6.1.13), a natural choice is to directly extend the definition of $\widetilde{\mathfrak{m}}_{\mathbf{h},\alpha,\chi}(x,y)$ to every $x,y \in \mathbb{R}^d$ by setting it equal to $\mathfrak{m}_{\mathbf{h},\alpha,\chi}(x,y)$ as defined in (6.1.12). Using this convention, we define the spectral dimension of the LBM as follows.

Definition 6.1.8. Let $d \geq 2$, $Q > \sqrt{2d}$, and $x \in \mathbb{R}^d$. If there exists $\overline{\chi} > 0$ such that

$$\mathfrak{m}_{\mathbf{h},\alpha,\chi}(x) = \infty, \quad \text{for } \chi \in [0,\overline{\chi}) \qquad \text{and} \qquad \mathfrak{m}_{\mathbf{h},\alpha,\chi}(x) < \infty, \quad \text{for } \chi > \overline{\chi}\,, \tag{6.1.14}$$

then the spectral dimension of $\mathrm{B}_{\mathbf{h},\alpha}$ at x is $\mathfrak{d}_{\mathbf{h},\alpha}(x) = 2(\overline{\chi}+1)$. Alternatively, if it holds that

$$\mathfrak{m}_{\mathbf{h},\alpha,\chi}(x) = \infty, \quad \text{for } \chi = 0 \qquad \text{and} \qquad \mathfrak{m}_{\mathbf{h},\alpha,\chi}(x) < \infty, \quad \text{for } \chi > 0\,, \tag{6.1.15}$$

then the spectral dimension of $\mathrm{B}_{\mathbf{h},\alpha}$ at x is $\mathfrak{d}_{\mathbf{h},\alpha}(x) = 2$.

Instead of computing the spectral dimension of the LBM associated to $\mathbf{h}$, we compute the spectral dimension at the origin of the LBM associated to the field $\mathbf{h} - \beta \log|\cdot|$[4], for $\beta \in (-\infty, Q)$. More precisely, we have the following result.

Theorem 6.B (Spectral dimension)**.** *Let $d \geq 2$, $Q > \sqrt{2d}$, and $\beta \in (-\infty, Q)$. It holds $\mathbb{P}$-almost surely that*

$$\mathfrak{d}_{\mathbf{h}-\beta\log|\cdot|,\alpha}(0) = 2 + \frac{2(d-2)}{2 + \alpha^2/2 - \alpha\beta}\,, \tag{6.1.16}$$

where we recall that α is related to Q as specified in (6.1.8).

We observe that for $d = 2$, the spectral dimension is equal to 2, independently of the choice of α and β. Furthermore, in dimension $d > 2$, the spectral dimension $\mathfrak{d}_{\mathbf{h}-\beta\log|\cdot|,\alpha}(0)$ coincides with the underlying dimension d when $\beta = \alpha/2$. In the limit $\alpha \to 0$ (which corresponds to the limit $\gamma \to 0$), it holds that $\mathfrak{d}_{\mathbf{h}-\beta\log|\cdot|,\alpha}(0) = d$ for all $\beta \in (-\infty, Q)$. Additionally, for a fixed dimension d and a fixed value $Q > \sqrt{2d}$, the spectral dimension is an increasing function of the parameter β.

We now discuss the implications of Theorem 6.B for the spectral dimension of the LBM around the thick points of $\mathbf{h}$. To this end, we first recall the definition of thick points. For $x \in \mathbb{R}^d$, let $(\mathbf{h}_\varepsilon(x))_{\varepsilon>0}$ denote the spherical average approximation of the whole-space LGF around x, as introduced in Section 6.2.2 below.

Definition 6.1.9. For $\beta \geq 0$, a point $x \in \mathbb{R}^d$ is said to be β-thick if the following limit holds

$$\lim_{\varepsilon \to 0} \frac{\mathbf{h}_\varepsilon(x)}{|\log \varepsilon|} = \beta\,.$$

[4]We refer to the beginning of Section 6.4.3 for an explanation of why the LBM associated with the field $\mathbf{h} - \beta\log|\cdot|$ can be constructed.

We denote the set of β-thick points of $\mathbf{h}$ by $\mathcal{T}_\beta$.

Thanks to [RV14a, Theorem 4.2], we know that the Hausdorff dimension of $\mathcal{T}_\beta$ is almost surely equal to $d - \beta^2/2$, for $\beta \in [0, \sqrt{2d}]$. Moreover, when $\beta > \sqrt{2d}$, it holds $\mathbb{P}$-almost surely that $\mathcal{T}_\beta = \emptyset$. Furthermore, for $\beta \in [0, \sqrt{2d}]$, the GMC measure $\mu_{\mathbf{h},\beta}$ [5] can be interpreted as the "uniform measure on β-thick points" (see [RV14a, Theorem 4.1]). Specifically, if x is a point sampled from this measure (restricted to some bounded open set), then near x, the field $\mathbf{h}$ locally looks like $\mathbf{h} - \beta \log|\cdot - x|$. Consequently, the behaviour of the field $\mathbf{h} - \beta \log|\cdot|$ near the origin is similar to the behaviour of $\mathbf{h}$ near a "typical" β-thick point. Therefore, thanks to Theorem 6.B, we expect that if $x \in \mathcal{T}_\beta$, then $\mathfrak{d}_{\mathbf{h},\alpha}(x) = 2 + 2(d-2)/(2 + \alpha^2/2 - \alpha\beta)$. This fact is formalised in the following remark.

Remark 6.1.10. A stronger version of Theorem 6.B would assert that $\mathbb{P}$-almost surely, for all $\beta \in [0, \sqrt{2d}]$ and for all $x \in \mathcal{T}_\beta$, it holds that $\mathfrak{d}_{\mathbf{h},\alpha}(x) = 2 + 2(d-2)/(2 + \alpha^2/2 - \alpha\beta)$. While we believe this result to be true, proving such a stronger version would involve overcoming some technical difficulties.

6.1.3 Quantum cones in arbitrary even dimension

For $d \geq 2$ even, $Q > \sqrt{2d}$, and $\beta \in (-\infty, Q)$, the β-quantum cone is, roughly speaking, the limiting field obtained by "zooming in" appropriately near the origin on a whole-space LGF with a β-log singularity at the origin. As for the two dimensional case, we will define the β-quantum cone field by specifying, separately, its averages on spheres centred at the origin, and what is left when we subtract these. The second of these components will have exactly the same law as the corresponding projection of the whole-space LGF, i.e., it is only the radial part which is different.

Consider the whole-space LGF $\mathbf{h}$. For $b > 0$ a large constant, applying the coordinate change formula to the field $\mathbf{h} - \beta \log|\cdot| + b$, with respect to the conformal automorphism $\mathbb{R}^d \ni x \mapsto e^{-t}x$, for some $t \in \mathbb{R}$, we obtain the field $\mathbf{h}^{t,b}$ given by

$$\mathbf{h}^{t,b} \stackrel{\text{def}}{=} \mathbf{h}(e^{-t}\cdot) - \beta \log|e^{-t}\cdot| - Qt + b \,. \tag{6.1.17}$$

Now, recalling Theorem 6.A, the spherical average process $(\mathbf{h}^{t,b}_{e^{-s}}(0))_{s\in\mathbb{R}}$ admits the following representation

$$\mathbf{h}^{t,b}_{e^{-s}}(0) = \mathrm{S}_{s+t} + \beta s - (Q-\beta)t + b \,, \qquad \forall\, s \in \mathbb{R} \,.$$

In particular, by taking $s = 0$ in the previous expression, we get that $\mathbf{h}^{t,b}_1(0) = \mathrm{S}_t - (Q-\beta)t + b$. Since $\beta \in (-\infty, Q)$, the drift term is always negative and so in particular, for each $b > 0$, there exists $\mathbb{P}$-almost surely a smallest $\sigma_b \in \mathbb{R}$ such that $\mathbf{h}^{t,b}_1(0) = 0$, i.e.,

$$\sigma_b \stackrel{\text{def}}{=} \inf\big\{t \in \mathbb{R} \,:\, \mathrm{S}_t - (Q-\beta)t = -b\big\} \,. \tag{6.1.18}$$

For each $b > 0$, we define the recentred process $(\mathrm{S}_{b,s})_{s\in\mathbb{R}}$ by letting

$$\mathrm{S}_{b,s} \stackrel{\text{def}}{=} \mathrm{S}_{s+\sigma_b} - \mathrm{S}_{\sigma_b} \,, \qquad \forall\, s \in \mathbb{R} \,. \tag{6.1.19}$$

The radial part of the β-quantum cone field is constructed by combining the limit of the process $(\mathrm{S}_{b,s})_{s\in\mathbb{R}}$ as $b \to \infty$ with an additional drift component.

[5] Here, we are tacitly assuming that for $\beta = \sqrt{2d}$, the measure $\mu_{\mathbf{h},\beta}$ corresponds to the critical GMC measure introduced in [DRSV14a].

Theorem 6.C (Construction of the radial part of the quantum cone)**.** *Let* $d > 2$ *be even,* $Q > \sqrt{2d}$*, and* $\beta \in (-\infty, Q)$*. For any fixed* $\mathrm{T} \in \mathbb{R}$*, the process* $(\mathrm{S}_{b,s})_{s \geq \mathrm{T}}$ *as defined in* (6.1.19) *converges in total variation distance to a process* $(\mathrm{S}_{\infty,s})_{s \geq \mathrm{T}}$ *as* $b \to \infty$*.*

In order to define the d-dimensional quantum cone, we need to observe that the homogeneous Sobolev space of order $d/2$, denote by $\mathcal{H}_d$ and introduced in Definition 6.2.1, can be written as the direct sum of two subspaces. The first subspace is the closure of functions in $\mathcal{S}(\mathbb{R}^d)$ that are radially symmetric about the origin, viewed modulo constants, and is denoted by $\mathcal{H}_{d,\mathrm{rad}}$. The second subspace consists of the closure of functions in $\mathcal{S}(\mathbb{R}^d)$ that have mean zero on all spheres centred at the origin, and it is denoted by $\mathcal{H}_{d,\mathrm{sph}}$ (see Lemma 6.2.5 for more details).

Definition 6.1.11. Let $d > 2$ be even, $\gamma \in (0, \sqrt{2d})$ and $\beta \in (-\infty, Q)$. The d-dimensional β-quantum cone field $\mathbf{h}^\star_{\mathrm{rad}}$ is given by the sum $\mathbf{h}^\star \overset{\mathrm{def}}{=} \mathbf{h}^\star_{\mathrm{rad}} + \mathbf{h}^\star_{\mathrm{sph}}$ where $\mathbf{h}^\star_{\mathrm{rad}}$ is the function that is constant equal to $\mathrm{S}_{\infty,s} + \beta s$ on each sphere $\partial B(0, e^{-s})$ for $s \in \mathbb{R}$, and $\mathbf{h}^\star_{\mathrm{sph}}$ is a field independent of $\mathbf{h}^\star_{\mathrm{rad}}$ and that has the same law of the projection of the whole-space LGF onto $\mathcal{H}_{d,\mathrm{sph}}$.

We emphasise that the above definition of the d-dimensional quantum cone field generalises the definition of the two-dimensional quantum cone introduced in [DMS21, Definition 4.10]. In particular, in what follows, for $d = 2$, we denote by $\mathbf{h}^\star$ the quantum cone field introduced in [DMS21, Definition 4.10].

We state the next result which identifies the d-dimensional β-quantum cone field as a local limit of a whole-space LGF with an additional β-log singularity at a fixed point of $\mathbb{R}^d$. This result is a straightforward consequence of our construction of the quantum cone field.

Theorem 6.D (Quantum cone as a local limit)**.** *Let* $d \geq 2$ *be even,* $Q > \sqrt{2d}$*, and* $\beta \in (-\infty, Q)$*. For* $t \in \mathbb{R}$*,* $z \in \mathbb{R}^d$*, and* $b > 0$*, consider the field* $\mathbf{h}^{t,b}$ *defined as*

$$\mathbf{h}^{t,b}(\cdot) \overset{\mathrm{def}}{=} \mathbf{h}(e^{-t} \cdot + z) - \beta \log|e^{-t} \cdot| - Qt + b \,,$$

which is obtained by applying the coordinate change formula (6.1.6) *to the field* $\mathbf{h} - \beta \log|\cdot - z| + b$ *with respect to the conformal automorphism* $\mathbb{R}^d \ni x \mapsto e^{-t}x + z$*. Furthermore, let* σ_b *be such that*

$$\sigma_b \overset{\mathrm{def}}{=} \inf\{t \in \mathbb{R} \,:\, \mathbf{h}^{t,b}_1(0) = 0\} \,. \tag{6.1.20}$$

Then, for any $R > 0$*, the field* $\mathbf{h}^{\sigma_b,b}|_{B(0,R)}$ *converges in total variation distance to* $\mathbf{h}^\star|_{B(0,R)}$ *as* $b \to \infty$*.*

Finally, we show that the law of the d-dimensional quantum cone $\mathbf{h}^\star$ is invariant in law modulo conformal coordinate change under shifting along the trajectories of the LBM. Before proceeding, we first define the LBM associated with the quantum cone field. Using the same notation as above, we recall that the process $(B_t)_{t \geq 0}$ is a d-dimensional Brownian motion independent of the quantum cone field $\mathbf{h}^\star$.

Definition 6.1.12. For $d \geq 2$ even, $Q > \sqrt{2d}$, and $\beta \in (-\infty, Q)$, the *d-dimensional Liouville Brownian motion* associated to the d-dimensional β-quantum cone field $\mathbf{h}^\star$ is the process defined as follows

$$\mathrm{B}_{\mathbf{h}^\star,\alpha,t} \overset{\mathrm{def}}{=} B_{\mathrm{F}^{-1}_{\mathbf{h}^\star,\alpha}(t)} \qquad \forall\, t \geq 0 \,,$$

where α is related to Q as specified in (6.1.8), and $\mathrm{F}_{\mathbf{h}^\star,\alpha}$ is the clock process introduced in (6.1.5) but with $\mathbf{h}$ replaced by $\mathbf{h}^\star$.

We emphasise that the well-posedness and non-triviality of the clock process $\mathrm{F}_{\mathbf{h}^\star,\alpha}$ follows from the fact that the clock process associated with the whole-space LGF $\mathbf{h}$ is well-defined, combined with the observation

that $\mathbf{h}$ and $\mathbf{h}^\star$ can be coupled together so that $\mathbf{h} - \mathbf{h}^\star$ is a continuous function (except at the origin). The only non-trivial aspect is that, due to the log-singularity at the origin of the quantum cone field, there is a possibility that $\mathrm{B}_{\mathbf{h}^\star,\alpha}$ started from the origin might become "stuck". This possibility, however, is ruled out at the beginning of Section 6.5.4.

With Definition 6.1.12 in hand, we can state our last main result.

Theorem 6.E (Invariance under shifts along the LBM)**.** *Let $d \geq 2$ be even, $Q > \sqrt{2d}$, and let α be related to Q as specified in (6.1.8). Let $\mathbf{h}^\star$ be a d-dimensional α-quantum cone field and $(\mathrm{B}_{\mathbf{h}^\star,\alpha,t})_{t\in\mathbb{R}}$ be a two-sided LBM with respect to $\mathbf{h}^\star$ started from the origin at time 0. Then, for all $t \in \mathbb{R}$, there exists a random $R > 0$ such that*

$$\big(\mathbf{h}^\star(\mathrm{B}_{\mathbf{h}^\star,\alpha,t} + R\cdot) + Q\log R, R^{-1}(\mathrm{B}_{\mathbf{h}^\star,\alpha,\cdot} - \mathrm{B}_{\mathbf{h}^\star,\alpha,t})\big) \overset{\text{law}}{=} (\mathbf{h}^\star, \mathrm{B}_{\mathbf{h}^\star,\alpha}) \,.$$

Theorem 6.E indicates that for LBM on an α-quantum cone, the environment as seen from the particle is stationary, modulo spatial scaling. This property holds only for the specific value $\alpha = Q - \sqrt{Q^2-4}$. Consequently, it appears particularly natural to consider LBM starting from a point of thickness α, where, according to Theorem 6.B, the spectral dimension equals $(4d - 2\alpha^2)/(4-\alpha^2)$.

We emphasise that, for $d = 2$, Theorem 6.E was known to experts but, to the best of our knowledge, had not been written down before.

6.1.4 Outline and main differences with the two-dimensional case

After introducing the necessary background and preliminary material in Section 6.2, we proceed in Section 6.3 to prove Theorem 6.A, which concerns the identification of the spherical average process. To the best of our knowledge, this characterisation is new. The proof relies on the expression for the covariance of the spherical average of the LGF provided in [LSSW16, Section 11.1], along with the observation that, in even dimensions $d > 2$, the vector of derivatives $(\mathbf{S}_t)_{t\in\mathbb{R}}$, as defined in (6.1.2), is a stationary Gaussian process. From this fact, we can infer that the multivariate process $(\mathbf{S}_t)_{t\in\mathbb{R}}$ must satisfy a suitable Langevin equation of the form (6.1.3). The precise identification of the coefficient matrix $\mathbf{A}$ and the diffusion vector $\mathbf{b}$ is achieved by comparing the power spectrum of the first derivative $(\mathrm{S}_t^{(1)})_{t\in\mathbb{R}}$ with that of solutions to SDEs of the form (6.1.3).

In Section 6.4, we present the construction of the d-dimensional LBM. First, we provide additional details on the construction of the LBM starting from a fixed point and prove its conformal invariance. Then, we focus on the construction of the LBM as a Markov process, building on the approach used for the two-dimensional case with minor technical adjustments. We also highlight that in [DSHKS24], the authors construct the higher-dimensional analogue of the LBM on a general d-dimensional manifold. One of the main differences in our construction is the canonical identification of the appropriate exponent $\alpha \in (0,2)$ required for the LBM to be conformally invariant in the sense of (6.1.7). Moreover, differently form [DSHKS24, Theorem 5.7], in the subcritical regime, we are able to provide an identification of the additive functional $\mathrm{F}_{\mathbf{h},\alpha}$ simultaneously from all starting points and not only for almost every point of $\mathbb{R}^d$.

Furthermore, we identify a new phase transition in the higher-dimensional setting that does not arise when $d = 2$. Indeed, as discussed in more detail in Section 6.4, to define the LBM as a proper Markov process, we need to establish that, almost surely, the clock process $\mathrm{F}_{\mathbf{h},\alpha}$ can be defined for all starting points $x \in \mathbb{R}^d$ of the underlying Brownian motion. Roughly speaking, in dimension $d > 2$, this simultaneous convergence is

possible if the measure $\mu_{\mathbf{h},\alpha}$ is regular enough to ensure that the mapping

$$B(0,R) \ni x \mapsto \int_{B(0,R)} |x-y|^{2-d} \mu_{\mathbf{h},\alpha}(dy)$$

is a continuous function of x for all $R > 0$[6]. To analyse the continuity property of the above mapping, for $Q > 2$ and $\beta \in [0, \sqrt{2d}]$, suppose that $x \in B(0,R)$ is a β-thick point of $\mathbf{h}$. Then, by using the regularity properties of the measure $\mu_{\mathbf{h},\alpha}$ (see Lemma 6.A.1), we have that

$$\int_{B(0,R)} |x-y|^{2-d} \mu_{\mathbf{h},\alpha}(dy) \approx \sum_{n \in \mathbb{N}_0} e^{-n(2+\alpha^2/2-\alpha\beta)} .$$

Therefore, in order to guarantee the convergence of the integral on the left-hand side of the above display, we need to require that

$$2 + \alpha^2/2 - \alpha\beta > 0 . \tag{6.1.21}$$

- In the subcritical case, i.e., $Q > \sqrt{2d}$, or equivalently $\alpha \in (0, \alpha_c)$, then condition (6.1.21) is satisfied for all $\beta \in [0, \sqrt{2d}]$. Hence, we can define the LBM starting simultaneously from all points of $\mathbb{R}^d$.
- In the critical case, i.e., $Q = \sqrt{2d}$, or equivalently $\alpha = \alpha_c$, the condition (6.1.21) is satisfied for all $\beta \in [0, \sqrt{2d})$.
- In the supercritical case, i.e., $Q \in (2, \sqrt{2d})$, or equivalently $\alpha \in (\alpha_c, 2)$, the condition (6.1.21) is satisfied only for $\beta \in [0, Q)$.

Recalling that the set of β-thick points $\mathcal{T}_\beta$ has Hausdorff dimension $d - \beta^2/2$, we note that for all $\beta > 2$, the set $\mathcal{T}_\beta$ has Hausdorff dimension strictly less than $d-2$. Consequently, by [MP10, Theorem 8.20], $\mathcal{T}_\beta$ is a polar set for d-dimensional Brownian motion. Thus, the LBM will never hit points of thickness greater than 2 unless it is started from one of them. Therefore, for all $Q > 2$, once the LBM is started, it will never get "stuck".

In Section 6.4.3, we prove Theorem 6.B, which establishes the spectral dimension of the LBM for $Q > \sqrt{2d}$. Unlike the two-dimensional case, where the spectral dimension is everywhere equal to two, the higher-dimensional case requires new techniques. A central component of the proof is Lemma 6.4.19, which studies how the integer moments of the conditional law given the underlying field $\mathbf{h}$ of the clock process $\mathrm{F}_{\mathbf{h}-\beta\log|\cdot|,\alpha}(t)$ scale as $t \to 0$. Proving this lemma, detailed in Appendix 6.B, requires careful control of the GMC measure of small balls centred near thick points of the LGF (see Lemma 6.A.4).

Finally, in Section 6.5, we prove Proposition 6.4.17 and construct the d-dimensional quantum cone. We recall that in the two-dimensional case the construction of the quantum cone is straightforward and relies on the fact that the spherical average process $(\mathrm{S}_t)_{t\in\mathbb{R}}$ is a two-sided Brownian motion. As a result, the fact that the recentered process $\mathrm{S}_{b,\cdot}$, defined in (6.1.19), has a limit as $b \to \infty$ is essentially a consequence of the strong Markov property of $(\mathrm{S}_t)_{t\in\mathbb{R}}$[7]. However, in even dimensions $d > 2$, the proof of Proposition 6.4.17 is non-trivial and is divided into two parts. First, by using the representation (6.1.4) of the spherical average process, we show that the collection of processes $(\mathrm{S}_{b,\cdot})_{b>0}$ is tight. Second, by leveraging the Markovianity of the multivariate process $(\mathrm{S}_t, \mathbf{S}_t)_{t\in\mathbb{R}}$ and a suitable mixing-type result, we prove that the limit as $b \to \infty$ of $(\mathrm{S}_{b,\cdot})_{b>0}$ exists with respect to the total variation distance. The section concludes with the proofs of Theorems 6.D and 6.E.

[6] This result is essentially a consequence of the theory of traces of Dirichlet forms developed in [CF12].
[7] See Remark 6.5.2 below for further details

6.1.5 Some open problems

We conclude this section by listing some problems suggested by this work.

Scaling limit of random walk on Voronoi tessellations For $d > 2$, we recall that the Voronoi tessellations of a locally finite set of points $S \subset \mathbb{R}^d$, is the partition of $\mathbb{R}^d$ into cells $(\mathcal{C}_s)_{s\in S}$ such that every point in the tile $\mathcal{C}_s$ is closer to s than any other point $s' \in S$, i.e.,

$$\mathcal{C}_s \overset{\text{def}}{=} \big\{x \in \mathbb{R}^d \,:\, |x-s| \le |x-s'| \;\; \forall\, s' \in S \setminus \{s\}\big\}.$$

For a bounded domain $D \subseteq \mathbb{R}^d$, $\gamma \in (0, \sqrt{2d})$, and each $n \in \mathbb{N}$, sample n points from the measure $\mu_{\mathbf{h},\gamma}$ restricted to D and normalised to be a probability measure. These points are then taken as the centres of the Voronoi cells. Two Voronoi cells are considered adjacent if they share a $(d-1)$-dimensional facet. We define a random walk on the adjacency graph of the Voronoi cells, with conductances given by $a(x,y) = \mathrm{Vol}_{d-1}(\mathcal{C}_x \cap \mathcal{C}_y)/|x-y|$, where Vol_{d-1} represents the $(d-1)$-dimensional Lebesgue measure. In [BRG24, Proposition 1.1 and Remark 1.2], it is shown that this random walk converges to a time-changed Brownian motion.

Inspired by [BG22], where the authors showed that the random walk on mated-CRT maps converges in law to the LBM with respect to the uniform topology, it would be interesting to establish a similar result in the setting described above.

Problem 6.1.13. Prove that the scaling limit of random walk on adjacency graph of the Voronoi cells with centres sampled from d-dimensional γ-GMC and with conductances as specified above is given by the LBM constructed with clock process $\mathrm{F}_{\mathbf{h},\alpha}$, with α related to γ as specified in (6.1.8).

We emphasise that if the points in Voronoi tessellations are sampled according to the measure $\mu_{\mathbf{h},\gamma}$, then we expect the random walk to converge to the LBM constructed with clock process $\mathrm{F}_{\mathbf{h},\alpha}$. One reason for this is that we expect that the scaling limit of the random walk should satisfy the same coordinate change rule as the measure $\mu_{\mathbf{h},\gamma}$ when we scale space, since if we preserve the measure $\mu_{\mathbf{h},\gamma}$ then we also preserve the Voronoi tessellations.

Scaling exponent inside the exponential of the LHK In [DZZ19], the authors studied the short-time asymptotic of the LHK in dimension $d = 2$ associated with a Dirichlet GFF on $[0,1]^2$. They proved that, for all $x, y \in [0,1]^2$, it holds that

$$\lim_{t\to 0} \frac{\log|\log \mathfrak{p}_{\mathbf{h},\alpha,t}(x,y)|}{|\log t|} = \frac{1}{d_\gamma - 1}\,,$$

where d_γ denotes the fractal dimension of γ-LQG. This leads us to the following problem.

Problem 6.1.14. Obtain the correct scaling exponent inside the exponential of the LHK in arbitrary dimensions $d > 2$. More precisely, for all $x, y \in \mathbb{R}^d$, compute the following limit,

$$\lim_{t\to 0} \frac{\log|\log \mathfrak{p}_{\mathbf{h},\alpha,t}(x,y)|}{|\log t|}\,.$$

Identification of the spherical average process for the quantum cone In Theorem 6.C, we established that the family of processes $(\mathrm{S}_{b,\cdot})_{b>0}$ converges in total variation distance to a unique limiting process $(\mathrm{S}_{\infty,s})_{s\in\mathbb{R}}$. This limiting process is then used to define the radial part of the d-dimensional quantum cone. However, this approach is not entirely satisfactory, as it relies on a limiting procedure.

Problem 6.1.15. Provide an explicit representation of the process $(\mathrm{S}_{\infty,s})_{s\in\mathbb{R}}$.

Odd dimensions In this chapter, we provide a characterisation of the spherical average process of the whole-space LGF and construct the d-dimensional quantum cone for even dimensions $d > 2$. In even dimensions, the fractional Laplacian $(-\Delta)^{d/2}$ is a local operator. However, in odd dimensions, it ceases to be local, leading to a series of complications. It would be interesting to extend our results to odd dimensions.

Problem 6.1.16. Provide a characterisation of the spherical average process of the whole-space LGF and/or construct the d-dimensional quantum cone for odd dimensions $d > 2$.

Acknowledgements. Part of this work was carried out during F.B.'s visit to the University of Chicago in March 2024, supported by the Doris Chen Mobility Award, and during the workshop "Two-Dimensional Random Geometry" at the Institute for Mathematical and Statistical Innovation (IMSI) in Chicago in July 2024. We thank Martin Hairer and Ahmed Bou-Rabee for helpful discussions.

6.2 Background and preliminaries

In this section, we gather some preliminary results and definitions. We begin by presenting the basic and recurrent notation that will be used throughout the chapter. In Section 6.2.1, we rigorously define the whole-space LGF, and in Section 6.2.2, we introduce its spherical average approximation. In Section 6.2.3, we introduce the GMC measure and establish some of its main properties. Finally, in Section 6.2.4, we record some results on multidimensional SDEs.

Basic and recurrent notation

We write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We let $\mathbb{R}^+ = (0, \infty)$, $\mathbb{R}^- = (-\infty, 0)$, $\mathbb{R}_0^+ = [0, \infty)$, and $\mathbb{R}_0^- = (-\infty, 0]$. Given $n \in \mathbb{N}$, we write $[n] = \{1, \ldots, n\}$ and $[n]_0 = \{0, 1, \ldots, n\}$. For $d \geq 2$, we denote by $\mathbb{S}^{d-1}$ the unit sphere centred at the origin in $\mathbb{R}^d$, and by $|\mathbb{S}^{d-1}|$ its $(d-1)$-dimensional volume. We denote by $\mathcal{B}(\mathbb{R}^d)$ the collection of Borel subsets of $\mathbb{R}^d$. We let $\mathcal{B}(\mathbb{R}^d, \mathbb{R})$ be the space of $\mathbb{R}$-valued Borel functions in $\mathbb{R}^d$. For $a, b \in \mathbb{R}$, we write $a \lesssim b$ (resp. $a \gtrsim b$), if there exists a constant $c > 0$, independent of the quantities of interest, such that $a \leq cb$ (resp. $a \geq cb$). For $k \in \mathbb{N}_0$, we write $\mathcal{C}^k_{\mathrm{loc}}(\mathbb{R})$ for the space of functions on $\mathbb{R}$ that are locally k-times continuously differentiable, equipped with the uniform local metric. We write d^{TV} for the total variation distance between two measures.

6.2.1 The whole-space log-correlated Gaussian field

For $d \geq 2$, we denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space equipped with the usual topology. The space of tempered distribution $\mathcal{S}'(\mathbb{R}^d)$ is then defined to be the space of continuous linear functionals from $\mathcal{S}(\mathbb{R}^d)$ to $\mathbb{C}$. We define the Fourier transform $\mathcal{F}[\phi]$ and inverse Fourier transform $\mathcal{F}^{-1}[\phi]$ of a Schwartz function ϕ on $\mathbb{R}^d$ by

$$\mathcal{F}[\phi](\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \phi(x) e^{-i\xi\cdot x} dx \,, \qquad \mathcal{F}^{-1}[\phi](x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \phi(\xi) e^{i\xi\cdot x} d\xi \,,$$

where the prefactor $(2\pi)^{-d/2}$ is chosen so that that it makes the Fourier transform a unitary transformation on the complex function space $L^2(\mathbb{R}^d)$. As a shorthand, we will also write $\widehat{\phi}$ in place of $\mathcal{F}[\phi]$. We define

$\mathcal{S}_0(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ to be the set of Schwartz functions ϕ such that $\widehat{\phi}(0) = 0$, or equivalently,

$$\mathcal{S}_0(\mathbb{R}^d) \stackrel{\text{def}}{=} \left\{ \phi \in \mathcal{S}(\mathbb{R}^d) \, : \, \int_{\mathbb{R}^d} \phi(x) dx = 0 \right\} .$$

We equip $\mathcal{S}_0(\mathbb{R}^d)$ with the topology inherited from $\mathcal{S}(\mathbb{R}^d)$ and denote by $\mathcal{S}'_0(\mathbb{R}^d)$ its topological dual. We note that each element of $\mathcal{S}(\mathbb{R}^d)$ can be interpreted as an element of $\mathcal{S}'_0(\mathbb{R}^d)$ by considering its equivalence class modulo global additive constant.

Definition 6.2.1. The homogeneous Sobolev space of order $d/2$, denoted by $\mathcal{H}_d$, is the Hilbert space completion of the space of functions $f \in \mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}'_0(\mathbb{R}^d)$ such that $\xi \mapsto |\xi|^{d/2} \widehat{f}(\xi) \in L^2(\mathbb{R}^d)$ with respect to the following inner product[8]

$$(f, g)_d \stackrel{\text{def}}{=} \frac{1}{2^{d-1} \pi^{d/2} \Gamma(d/2)} \int_{\mathbb{R}^d} |\xi|^d \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi \, . \tag{6.2.1}$$

We emphasise that if $d \geq 2$ is even, then the inner product (6.2.1) can be conveniently rewritten as follows

$$(f, g)_d = \big(f, (-\Delta)^{d/2} g\big) = \big((-\Delta)^{d/2} f, g\big) \, ,$$

and in particular, if $d = 2$, then the above inner product coincides with the Dirichlet inner product.

Definition 6.2.2. For $d \geq 2$, the whole-space LGF is a random element $h \in \mathcal{S}'_0(\mathbb{R}^d)$, i.e., a random distribution modulo a global additive constant, whose law is centred Gaussian with the following covariance

$$\mathbb{E}\big[(h, \phi)(h, \psi)\big] = -\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \log|x - y| \phi(x) \psi(y) dx dy \, , \qquad \forall \, \phi, \psi \in \mathcal{S}_0(\mathbb{R}^d) \, .$$

Equivalently[9], the whole-space LGF is the standard Gaussian on the Hilbert space $\mathcal{H}_d$, i.e.,

$$h = \sum_{i=1}^{\infty} \alpha_i f_i \, , \tag{6.2.2}$$

where α_i are i.i.d. centred normal random variables and $(f_i)_{i=1}^{\infty}$ is an orthonormal basis for the Hilbert space $\mathcal{H}_d$. The sum converges $\mathbb{P}$-almost surely in the space $\mathcal{S}'_0(\mathbb{R}^d)$.

An important feature of the whole-space LGF is that it is conformally invariant in every dimension $d \geq 2$.

Lemma 6.2.3 ([DRSV17, Proposition 1.2])**.** *If ϕ is a conformal automorphism of $\mathbb{R}^d$, then the law of $h \circ \phi$ agrees with the law of h.*

Remark 6.2.4. We recall that in dimension $d > 2$, Liouville's theorem asserts that every conformal automorphism of $\mathbb{R}^d$ can be expressed as a composition of translations, dilations, rotations and inversions, i.e., they are Möbius transformations. This is in contrast with what happens in dimension $d = 2$, where by the Riemann mapping theorem, all simply connected planar domains are conformally equivalent.

We conclude this section with the following decomposition result, which will play a fundamental role in constructing the d-dimensional quantum cone.

Lemma 6.2.5 (Radial decomposition)**.** *For $d \geq 2$, let $\mathcal{H}_{d,\text{rad}}(\mathbb{R}^d)$ be the subspace of $\mathcal{H}_d$ obtained as the closure of functions in $\mathcal{S}(\mathbb{R}^d)$ which are radially symmetric about the origin, viewed modulo constants. Let $\mathcal{H}_{d,\text{sph}}$ be the subspace of $\mathcal{H}_d$ obtained as the closure of functions in $\mathcal{S}(\mathbb{R}^d)$ which have mean zero about all*

[8] Here the choice of normalisation is such that the covariance kernel for the LGF is given by $-\log|x - y|$ rather than a multiple of it.
[9] We refer to [DRSV17, Proposition 1.1] for a proof of the equivalence of these two definitions.

spheres centred at the origin. Then, it holds that

$$\mathcal{H}_d = \mathcal{H}_{d,\mathrm{rad}} \oplus \mathcal{H}_{d,\mathrm{sph}} \ .$$

Proof. We start by checking the orthogonality condition. Let $f_{\mathrm{rad}} \in \mathcal{S}(\mathbb{R}^d)$ be a function defined modulo constants which is radially symmetric about the origin, and $f_{\mathrm{sph}} \in \mathcal{S}(\mathbb{R}^d)$ be a function which has mean zero about all spheres centred at the origin. We verify that the inner product $(f_{\mathrm{rad}}, f_{\mathrm{sph}})_d$ defined in (6.2.1) is equal to zero. To this end, we note that for all $\vartheta \in \mathbb{S}^{d-1}$ and $r > 0$, it holds that

$$\widehat{f}_{\mathrm{rad}}(r\vartheta) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f_{\mathrm{rad}}(x) e^{-i(r\vartheta)\cdot x} dx = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f_{\mathrm{rad}}(x) e^{-i(re_1)\cdot x} dx = \widehat{f}_{\mathrm{rad}}(re_1) \ , \tag{6.2.3}$$

where e_1 denotes the first unit vector. Moreover, for any $r > 0$, by letting $\phi_r : \mathbb{R}^d \to \mathbb{R}$ be the radial function defined as

$$\phi_r(x) = \phi_r(|x|) \overset{\mathrm{def}}{=} \int_{\mathbb{S}^{d-1}} e^{i(r\vartheta)\cdot x} d\sigma(\vartheta) \ ,$$

thanks to Fubini's theorem, it holds that

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} \overline{\widehat{f}_{\mathrm{sph}}(r\vartheta)} d\sigma(\vartheta) &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{S}^{d-1}} \left(\int_{\mathbb{R}^d} f_{\mathrm{sph}}(x) e^{i(r\vartheta)\cdot x} dx \right) d\sigma(\vartheta) \\
&= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f_{\mathrm{sph}}(x) \phi_r(|x|) dx \\
&= (2\pi)^{-\frac{d}{2}} \int_0^\infty s^{d-1} \phi_r(s) \left(\int_{\mathbb{S}^{d-1}} f_{\mathrm{sph}}(s\vartheta) d\sigma(\vartheta) \right) ds \\
&= 0 \ .
\end{aligned} \tag{6.2.4}$$

Thus, by letting $a(d)$ denote the prefactor in the definition (6.2.1) of the inner product $(\cdot,\cdot)_d$, and by applying (6.2.3) and (6.2.4), we obtain that:

$$\begin{aligned}
(f_{\mathrm{rad}}, f_{\mathrm{sph}})_d = a(d) \int_{\mathbb{R}^d} |\xi|^d \widehat{f}_{\mathrm{rad}}(\xi) \overline{\widehat{f}_{\mathrm{sph}}(\xi)} d\xi &= a(d) \int_0^\infty \int_{\mathbb{S}^{d-1}} r^{2d-1} \widehat{f}_{\mathrm{rad}}(r\vartheta) \overline{\widehat{f}_{\mathrm{sph}}(r\vartheta)} dr d\sigma(\vartheta) \\
&\overset{(6.2.3)}{=} a(d) \int_0^\infty r^{2d-1} \widehat{f}_{\mathrm{rad}}(re_1) \left(\int_{\mathbb{S}^{d-1}} \overline{\widehat{f}_{\mathrm{sph}}(r\vartheta)} d\sigma(\vartheta) \right) dr \\
&\overset{(6.2.4)}{=} 0 \ .
\end{aligned}$$

By definition of $\mathcal{H}_{d,\mathrm{rad}}(\mathbb{R}^d)$ and $\mathcal{H}_{d,\mathrm{sph}}(\mathbb{R}^d)$, these two spaces are therefore orthogonal with respect to $(\cdot,\cdot)_d$.

We now check the spanning condition. Consider a function $f \in \mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}'_0(\mathbb{R}^d)$ such that $\xi \mapsto |\xi|^{d/2} \widehat{f}(\xi) \in L^2(\mathbb{R}^d)$. Let f_{rad} be the function which on a given sphere $\partial B(0,r)$ is given by the average of f on $\partial B(0,r)$. Let $f_{\mathrm{sph}} = f - f_{\mathrm{rad}}$. Then $f = f_{\mathrm{rad}} + f_{\mathrm{sph}}$, with $f_{\mathrm{rad}} \in \mathcal{H}_{d,\mathrm{rad}}(\mathbb{R}^d)$, and $f_{\mathrm{sph}} \in \mathcal{H}_{d,\mathrm{sph}}(\mathbb{R}^d)$. Now, if $f \in \mathcal{H}_d$, there exists a collection $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$ such that $\xi \mapsto |\xi|^{d/2} \widehat{f_n}(\xi) \in L^2(\mathbb{R}^d)$ for each $n \in \mathbb{N}$, and $f = \lim_{n\to\infty} f_n$ with respect to the norm induced by the inner product $(\cdot,\cdot)_d$. Moreover, by orthogonality of $\mathcal{H}_{d,\mathrm{rad}}$ and $\mathcal{H}_{d,\mathrm{sph}}$ the sequences $(f_{n,\mathrm{rad}})_{n\in\mathbb{N}}$ and $(f_{n,\mathrm{sph}})_{n\in\mathbb{N}}$ are each Cauchy, and $\lim_{n\to\infty} f_{n,\mathrm{rad}} = f_{\mathrm{rad}} \in \mathcal{H}_{d,\mathrm{rad}}$ and $\lim_{n\to\infty} f_{n,\mathrm{sph}} = f_{\mathrm{sph}} \in \mathcal{H}_{d,\mathrm{sph}}$ with respect to the norm induced by the inner product $(\cdot,\cdot)_d$. Therefore, $f = f_{\mathrm{rad}} + f_{\mathrm{sph}}$ thus proving that $\mathcal{H}_d = \mathcal{H}_{d,\mathrm{rad}} \oplus \mathcal{H}_{d,\mathrm{sph}}$. □

6.2.2 The spherical average

In this section, we introduce the spherical average process of the whole-space LGF. Most of the material presented here is drawn from [LSSW16, Section 11]. However, due to the significance of the spherical average in this chapter, we provide a summary of the main concepts and results.

For a continuous function $f : \mathbb{R}^d \to \mathbb{R}$, we define its spherical average process around $x \in \mathbb{R}^d$ as the function $(0, \infty) \ni r \mapsto f_r(x)$ given by

$$f_r(x) \overset{\text{def}}{=} \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} f(x + r\sigma) d\sigma \, .$$

A straightforward calculation shows that for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^+)$ it holds that

$$\int_0^\infty f_r(x)\phi(r)dr = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{R}^d} f(x) \frac{\phi(|z-x|)}{|z-x|^{d-1}} dz \, .$$

This motivates the following definition.

Definition 6.2.6. For h the whole-space LGF and $x \in \mathbb{R}^d$, we define the *spherical average* around x modulo a global additive constant $h_\cdot(x)$ by letting

$$(h_\cdot(x), \phi) \overset{\text{def}}{=} |\mathbb{S}^{d-1}|^{-1} \left(h, z \mapsto \frac{\phi(|z-x|)}{|z-x|^{d-1}} \right) , \qquad \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^+) \cap \mathcal{S}_0(\mathbb{R}) \, .$$

We observe that if $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^+) \cap \mathcal{S}_0(\mathbb{R}^d)$, then the mapping $z \mapsto \phi(|z-x|)/|z-x|^{d-1}$ is in $\mathcal{S}_0(\mathbb{R}^d)$ and so the previous definition makes sense. In order to find the covariance kernel of the spherical average process, we fix $\phi, \psi \in \mathcal{C}_c^\infty(\mathbb{R}^+) \cap \mathcal{S}_0(\mathbb{R})$ and we observe that for $x_1, x_2 \in \mathbb{R}^d$, it holds that

$$\begin{aligned}
\mathbb{E}\big[(h_\cdot(x_1), \phi)(h_\cdot(x_2), \psi)\big] &= \frac{1}{|\mathbb{S}^{d-1}|^2} \mathbb{E}\left[\left(h, z \mapsto \frac{\phi(|z-x_1|)}{|z-x_1|^{d-1}} \right) \left(h, z \mapsto \frac{\psi(|z-x_2|)}{|z-x_2|^{d-1}} \right) \right] \\
&= -\frac{1}{|\mathbb{S}^{d-1}|^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \log|z_1 - z_2| \frac{\phi(|z_1-x_1|)\psi(|z_2-x_2|)}{|z_1-x_1|^{d-1}|z_2-x_2|^{d-1}} dz_1 dz_2 \\
&= -\frac{1}{|\mathbb{S}^{d-1}|^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \log|(x_1 + r_1\omega) - (x_2 + r_2\sigma)| d\sigma d\omega \right) \phi(r_1)\psi(r_2) dr_1 dr_2 \, ,
\end{aligned}$$

from which we can deduce that, for all $r_1, r_2 \in \mathbb{R}^+$,

$$\mathcal{K}_{r_1,r_2}(x_1, x_2) \overset{\text{def}}{=} \mathbb{E}\big[h_{r_1}(x_1) h_{r_2}(x_2)\big] = -\frac{1}{|\mathbb{S}^{d-1}|^2} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \log|(x_1 + r_1\omega) - (x_2 + r_2\sigma)| d\sigma d\omega \, . \tag{6.2.5}$$

In particular, if $x = x_1 = x_2 \in \mathbb{R}^d$, by a calculation based on spherical symmetries, we can further simplify the above integrals and obtain that

$$\mathcal{K}_{r_1,r_2}(x, x) = -c(d) \int_0^\pi \log\big(r_1^2 + r_2^2 - 2r_1 r_2 \cos\vartheta\big) \big(\sin\vartheta\big)^{d-2} d\vartheta \, , \qquad \forall \, r_1, r_2 \in \mathbb{R}^+ \, , \tag{6.2.6}$$

where, for $d \geq 2$, the constant $c(d) > 0$ is given by

$$c(d) \overset{\text{def}}{=} \frac{\Gamma(d/2)}{2\sqrt{\pi}\Gamma((d-1)/2)} \, .$$

We conclude this subsection with the following result regarding the regularity of the spherical average process.

Proposition 6.2.7. *For $d \geq 2$ even and $x \in \mathbb{R}^d$, there exists a modification of $(h_r(x))_{r\geq 0}$ which is differentiable $\mathfrak{c}_d$ times, where we recall (6.1.1). Moreover, the $\mathfrak{c}_d$-th derivative is δ-Hölder continuous for all $\delta \in (0, 1/2)$.*

Proof. For $d \geq 2$ even and $x \in \mathbb{R}^d$, the fact that $(h_r(x))_{r\geq 0}$ is differentiable $\mathfrak{c}_d$ times is a consequence of Theorem 6.A. We refer to [LSSW16, Proposition 11.1] for a proof (based on Kolmogorov–Chentsov continuity theorem) of the fact that that the $\mathfrak{c}_d$-th derivative is δ-Hölder continuous for all $\delta \in (0, 1/2)$. □

Remark 6.2.8. We remark that for $d \geq 2$ odd and $z \in \mathbb{R}^d$, it is expected that the spherical average process $(h_r(z))_{r\geq 0}$ is differentiable $(d-3)/2$ times.

We record here a straightforward result regarding the variance of the spherical average process, the proof of which involves a simple computation based on (6.2.6).

Lemma 6.2.9. *For any $x \in \mathbb{R}^d$, it holds that*

$$\mathbb{E}\big[\big(h_{e^{-t}}(x) - h_1(x)\big)^2\big] = |t| + 2c(d)\int_0^\pi \log\left(\frac{1 + e^{-2|t|} - 2e^{-|t|}\cos\vartheta}{2 - 2\cos(\vartheta)}\right)\big(\sin\vartheta\big)^{d-2}d\vartheta\,, \qquad \forall\, t \in \mathbb{R}\,.$$

6.2.3 Some properties of GMC measures

In what follows, for a point $x \in \mathbb{R}^d$, we denote by $(\mathbf{h}_\varepsilon(x))_{\varepsilon>0}$ the spherical average process of $\mathbf{h}$ around x, i.e., $\mathbf{h}_\varepsilon(x) = h_\varepsilon(x) - h_1(0)$. For $\gamma \in (0, \sqrt{2d})$, we define the subcritical GMC measure associated to $\mathbf{h}$ by letting

$$\mu_{\mathbf{h},\gamma}(dx) \overset{\text{def}}{=} \lim_{\varepsilon\to 0} \mu_{\mathbf{h},\gamma,\varepsilon}(dx) \overset{\text{def}}{=} \lim_{\varepsilon\to 0} \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma \mathbf{h}_\varepsilon(x)}dx\,, \tag{6.2.7}$$

where dx denotes the Lebesgue measure on $\mathbb{R}^d$. It is well-known (see e.g. [Ber17, BP24]) that the limit on the right-hand side of the above display converges in $\mathbb{P}$-probability in the space of non-negative locally finite measures on $\mathbb{R}^d$ equipped with the topology of weak convergence.

As we observed in Lemma 6.2.3, the whole-space LGF is conformally invariant, and so it is natural to inquire how this invariance property is reflected in the associated GMC measures. It turns out that these measures are *conformally covariant*, i.e., their laws are related under conformal mappings by a correction term that accounts for the domain's inflation caused by the mapping.

Proposition 6.2.10 (Coordinate change formula for $\mu_{\mathbf{h},\gamma}$)**.** *For $d > 2$ and $\gamma \in (0, \sqrt{2d})$, consider a conformal automorphism ϕ of $\mathbb{R}^d$ which can be expressed as a composition of translations, dilations and rotations. Then, it holds $\mathbb{P}$-almost surely that*

$$\mu_{\mathbf{h},\gamma}(\phi(A)) = \mu_{\mathbf{h}\circ\phi + Q\log|\phi'|,\gamma}(A)\,, \qquad \forall\, A \in \mathcal{B}(\mathbb{R}^d)\,, \tag{6.2.8}$$

where Q is defined as in (6.1.6).

Proof. Let $c > 0$ and consider the field $\mathbf{h}^c \overset{\text{def}}{=} \mathbf{h}(c\,\cdot)$. For any $\varepsilon > 0$ and $x \in \mathbb{R}^d$, we have the following relation between the spherical averages: $\mathbf{h}^c_\varepsilon(x) = \mathbf{h}_{c\varepsilon}(cx)$. For $A \in \mathcal{B}(\mathbb{R}^d)$, we have that

$$\mu_{\mathbf{h}^c,\gamma}(A) = \lim_{\varepsilon\to 0} \varepsilon^{\gamma^2/2}\int_A e^{\gamma \mathbf{h}^c_\varepsilon(x)}dx = \lim_{\varepsilon\to 0} \varepsilon^{\gamma^2/2}\int_A e^{\gamma \mathbf{h}_{c\varepsilon}(cx)}dx = \lim_{\delta\to 0} (\delta/c)^{\gamma^2/2} c^{-d}\int_{cA} e^{\gamma \mathbf{h}_\delta(x)}dx\,,$$

where in the last equality we set $\delta = c\varepsilon$ and we did the change of variables $x \mapsto x/c$. We note that the term on the right-hand side of the above display equals $c^{-d-\gamma^2/2}\mu_{\mathbf{h},\gamma}(cA)$, and so we have that

$$c^{d+\gamma^2/2}\mu_{\mathbf{h}^c,\gamma}(A) = \mu_{\mathbf{h},\gamma}(cA) \qquad \Longleftrightarrow \qquad \mu_{\mathbf{h}^c+Q\log c,\gamma}(A) = \mu_{\mathbf{h},\gamma}(cA)\ .$$

The same argument also works if instead of just a scaling we have a composition of a scaling, rotation, and translation. □

6.2.4 Some results on multidimensional SDEs

In this section, we collect some standard results on SDEs that will play a key role in our analysis, especially in Section 6.3 for the proof of Theorem 6.A.

For $p \geq 1$ and $(a_i)_{i=0}^{p-1} \subset \mathbb{R}$, we let $\mathbf{A}$ be the Frobenius companion matrix of the polynomial $p(\lambda) = \lambda^p + a_{p-1}\lambda^{p-1} + \cdots + a_1\lambda + a_0$ as specified in Definition 6.1.2. Furthermore, for some $b \in \mathbb{R}$, we consider the p-dimensional vector $\mathbf{b} = (0, 0, \ldots, 0, b)^\top$. We are interested in p-dimensional Langevin equations of the following form,

$$\mathrm{d}\mathbf{X}_t = \mathbf{A}\mathbf{X}_t\mathrm{d}t + \mathbf{b}\mathrm{d}\mathbf{B}_t\ , \tag{6.2.9}$$

where $(\mathbf{B}_t)_{t\in\mathbb{R}}$ denotes a p-dimensional two-sided Brownian motion and

$$\mathbf{X}_t = (\mathrm{X}_t, \mathrm{X}_t^{(1)}, \ldots, \mathrm{X}_t^{(p-2)}, \mathrm{X}_t^{(p-1)})^\top\ .$$

Remark 6.2.11. Roughly speaking, one should have in mind that (6.2.9) is the systems of first order SDEs associated to the following p-th order SDE,

$$\mathrm{X}_t^{(p)} + a_{p-1}\mathrm{X}_t^{(p-1)} + \cdots + a_1\mathrm{X}_t^{(1)} + a_0\mathrm{X}_t = b\xi_t\ ,$$

where $(\xi_t)_{t\in\mathbb{R}}$ is a white noise.

Lemma 6.2.12 ([KS91, Theorem 5.6.7])**.** *Consider the setting described above. If all the eigenvalues of the matrix $\mathbf{A}$ are negative real numbers, then the SDE (6.2.9) admits a unique stationary Gaussian Markov solution $(\mathbf{X}_t)_{t\in\mathbb{R}}$.*

Before stating the next result, we recall that the power spectrum of a stationary Gaussian process $(\mathrm{X}_t)_{t\in\mathbb{R}}$ is defined as the Fourier transform of its covariance kernel.

Lemma 6.2.13. *Consider the setting described above and assume that all the eigenvalues of the matrix $\mathbf{A}$ are negative real numbers. Let $(\mathbf{X}_t)_{t\in\mathbb{R}}$ be the unique stationary Gaussian Markov solution to the SDE (6.2.9). Then, the power spectrum of the first component $(\mathrm{X}_t)_{t\in\mathbb{R}}$ is given by*

$$\widehat{\mathcal{K}}(\omega) = \frac{1}{\sqrt{2\pi}}\frac{b^2}{[\sum_{k=0}^p a_k(i\omega)^k][\sum_{k=0}^p a_k(-i\omega)^k]}\ , \qquad \forall\,\omega\in\mathbb{R}\ ,$$

where $a_p = 1$. Moreover, if we assume that $(-\lambda_k)_{k=1}^p \subset \mathbb{R}^-$ are the distinct eigenvalues of the matrix $\mathbf{A}$, then the process $(\mathrm{X}_t)_{t\in\mathbb{R}}$ admits the following representations

$$\mathrm{X}_t = b\sum_{k=1}^p c_k \int_{-\infty}^t e^{-\lambda_k(t-s)}dB_s\ , \qquad \forall\, t\in\mathbb{R}\ ,$$

where, for each $k \in [p]$, we set $c_k \overset{\text{def}}{=} (\prod_{j=1,\, j\neq k}^{p}(\lambda_j - \lambda_k))^{-1}$.

Proof. These facts are standard and we refer to [PP02, Section 10.3] for a proof. □

We finish this section with a mixing-type result that will be used in the construction of the quantum cone in Section 6.5. We consider the augmented $(p+1)$-dimensional SDE of the following form,

$$d(\overline{\mathrm{X}}_t, \mathbf{X}_t)^\top = \begin{pmatrix} 0 & (1,0,\dots,0) \\ \mathbf{0} & \mathbf{A} \end{pmatrix} (\overline{\mathrm{X}}_t, \mathbf{X}_t)^\top dt + \begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix} d\mathbf{B}_t \,, \qquad (\overline{\mathrm{X}}_0, \mathbf{X}_0)^\top = \mathbf{x} \,, \tag{6.2.10}$$

where $\mathbf{A}$ and $\mathbf{b}$ are as specified above, $(\mathbf{B}_t)_{t\geq 0}$ is a $(p+1)$-dimensional Brownian motion, and $\mathbf{x} = (\overline{x}, x_0, x_1, \dots, x_{p-1}) \in \mathbb{R}^{p+1}$. In other words, $(\mathbf{X}_t)_{t\geq 0}$ is a solution to the Langevin equation (6.2.9) with initial condition at time 0 given by $(x_0, x_1, \dots, x_{p-1})$, and $(\overline{\mathrm{X}}_t)_{t\geq 0}$ is the integral of $(\mathrm{X}_t)_{t\geq 0}$ starting from $\overline{x}$.

Lemma 6.2.14. *Consider the setting described above and assume that all the eigenvalues of the matrix $\mathbf{A}$ are distinct and negative real numbers. For $\mathbf{x} = (\overline{x}, x_0, x_1, \dots, x_{p-1})$, $\mathbf{y} = (\overline{y}, y_0, \dots, y_{p-1}) \in \mathbb{R}^{p+1}$, let $(\overline{\mathrm{X}}_{\mathbf{x},t}, \mathbf{X}_{\mathbf{x},t})_{t\geq 0}$ and $(\overline{\mathrm{X}}_{\mathbf{y},t}, \mathbf{X}_{\mathbf{y},t})_{t\geq 0}$ be the solutions to the SDE (6.2.10) with initial conditions $\mathbf{x}$ and $\mathbf{y}$, respectively. Then, for any $R > 0$, it holds that*

$$\lim_{t\to\infty} \sup_{\mathbf{x},\mathbf{y}\in B(0,R)} \mathrm{d}^{\mathrm{TV}}\big[(\overline{\mathrm{X}}_{\mathbf{x},t}, \mathbf{X}_{\mathbf{x},t}), (\overline{\mathrm{X}}_{\mathbf{y},t}, \mathbf{X}_{\mathbf{y},t})\big] = 0 \,.$$

Proof. Fix $R > 0$ and let $\mathbf{x}, \mathbf{y} \in B(0,R)$. To establish the result, we use Pinsker's inequality [Tsy09, Lemma 2.5], which states that the total variation distance between two probability distributions is bounded above by the square root of their *Kullback–Leibler* (KL) divergence. Specifically, we have that

$$\mathrm{d}^{\mathrm{TV}}\big[(\overline{\mathrm{X}}_{\mathbf{x},t}, \mathbf{X}_{\mathbf{x},t}), (\overline{\mathrm{X}}_{\mathbf{y},t}, \mathbf{X}_{\mathbf{y},t})\big] \leq \sqrt{\frac{1}{2}\,\mathrm{d}_{\mathrm{KL}}\big[(\overline{\mathrm{X}}_{\mathbf{x},t}, \mathbf{X}_{\mathbf{x},t})||(\overline{\mathrm{X}}_{\mathbf{y},t}, \mathbf{X}_{\mathbf{y},t})\big]} \,. \tag{6.2.11}$$

We observe that, for each $t \geq 0$, $(\overline{\mathrm{X}}_{\mathbf{x},t}, \mathbf{X}_{\mathbf{x},t})$ and $(\overline{\mathrm{X}}_{\mathbf{y},t}, \mathbf{X}_{\mathbf{y},t})$ are multivariate Gaussians with different means but the same covariance matrix. This follows directly from the fact that

$$(\overline{\mathrm{X}}_{\mathbf{x},t}, \mathbf{X}_{\mathbf{x},t}) = e^{\overline{\mathbf{A}}t}\mathbf{x} + \int_0^t e^{\overline{\mathbf{A}}(t-s)}\,\overline{\mathbf{b}}\,dB_s \,, \qquad \forall\, t \geq 0 \,, \tag{6.2.12}$$

where $\overline{\mathbf{A}}$ and $\overline{\mathbf{b}}$ denote the augmented drift matrix and diffusion vector in the SDE (6.2.10). Naturally, a similar representation holds for $(\overline{\mathrm{X}}_{\mathbf{y},t}, \mathbf{X}_{\mathbf{y},t})_{t\geq 0}$. Therefore, the KL divergence between the law of $(\overline{\mathrm{X}}_{\mathbf{x},t}, \mathbf{X}_{\mathbf{x},t})_{t\geq 0}$ and the law of $(\overline{\mathrm{X}}_{\mathbf{y},t}, \mathbf{X}_{\mathbf{y},t})_{t\geq 0}$ is simply given by[10]

$$\mathrm{d}_{\mathrm{KL}}\big[(\overline{\mathrm{X}}_{\mathbf{x},t}, \mathbf{X}_{\mathbf{x},t})||(\overline{\mathrm{X}}_{\mathbf{y},t}, \mathbf{X}_{\mathbf{y},t})\big] = \frac{1}{2}(\mathrm{m}_{\mathbf{x},t} - \mathrm{m}_{\mathbf{y},t})\Sigma_t^{-1}(\mathrm{m}_{\mathbf{x},t} - \mathrm{m}_{\mathbf{y},t})^\top \,, \tag{6.2.13}$$

where here we set

$$\mathrm{m}_{\mathbf{x},t} \overset{\text{def}}{=} \mathbb{E}\big[(\overline{\mathrm{X}}_{\mathbf{x},t}, \mathbf{X}_{\mathbf{x},t})\big] \,, \qquad \mathrm{m}_{\mathbf{y},t} \overset{\text{def}}{=} \mathbb{E}\big[(\overline{\mathrm{X}}_{\mathbf{y},t}, \mathbf{X}_{\mathbf{y},t})\big] \,, \qquad \Sigma_t \overset{\text{def}}{=} \mathbb{C}\mathrm{ov}\big((\overline{\mathrm{X}}_{\mathbf{x},t}, \mathbf{X}_{\mathbf{x},t})\big) \,.$$

[10] We observe that the matrix $(\overline{\mathbf{b}}, \overline{\mathbf{A}}\,\overline{\mathbf{b}}, \overline{\mathbf{A}}^2\,\overline{\mathbf{b}}, \dots, \overline{\mathbf{A}}^p\,\overline{\mathbf{b}})$ has full rank, and so, thanks to Kalman rank condition (see e.g. [Cor07, Theorem 1.16]), the inverse of the covariance matrix Σ_t is well-defined for all $t > 0$.

Thanks to (6.2.11), it suffices to prove that the right-hand side of (6.2.13) converges to zero as $t \to \infty$. Using the representation (6.2.12) and applying Itô's isometry, we have that

$$\mathrm{m}_{\mathbf{x},t} = e^{\overline{\mathbf{A}}t}\mathbf{x} \,, \qquad \mathrm{m}_{\mathbf{y},t} = e^{\overline{\mathbf{A}}t}\mathbf{y} \,, \qquad \Sigma_t = \int_0^t (e^{\overline{\mathbf{A}}s}\overline{\mathbf{b}})(e^{\overline{\mathbf{A}}s}\overline{\mathbf{b}})^\top ds \,.$$

For all $t \geq 0$, due to the block-upper triangular structure of the matrix $\overline{\mathbf{A}}$, we have that

$$e^{\overline{\mathbf{A}}t} = \begin{pmatrix} 1 & \int_0^t (e^{\mathbf{A}s}e_1)^\top ds \\ \mathbf{0} & e^{\mathbf{A}t} \end{pmatrix} \,. \tag{6.2.14}$$

Now, let $\mathbf{D} = \mathrm{diag}(-\lambda_1, \ldots, -\lambda_p)$ denote the diagonal matrix of the negative eigenvalues of $\mathbf{A}$, and let $\mathbf{V}$ represent the matrix of the corresponding eigenvectors. We then have that

$$e^{\mathbf{A}t} = \mathbf{V}e^{\mathbf{D}t}\mathbf{V}^{-1} = \sum_{k=1}^{p} e^{-\lambda_k t}\mathbf{v}_k\mathbf{v}_k^{-1} \,,$$

where $\mathbf{v}_k$ and $\mathbf{v}_k^{-1}$ denote the k-th column of $\mathbf{V}$ and the k-th row of $\mathbf{V}^{-1}$, respectively. Thus, each entry of $e^{\mathbf{A}t}$ can be expressed as a linear combination of the terms $(e^{-\lambda_k t})_{k=1}^p$, which, for brevity, we denote by $\langle e^{-\lambda_k t}\rangle_{k=1}^p$. In particular, this observation implies that each entry of the vector in the top-right position of (6.2.14) is given by an $\mathcal{O}(1)$ term plus a term of the form $\langle e^{-\lambda_k t}\rangle_{k=1}^p$. Furthermore, recalling that only the last entry of $\overline{\mathbf{b}}$ is non-zero, we deduce that

$$e^{\overline{\mathbf{A}}s}\overline{\mathbf{b}} = \big(\mathcal{O}(1) + \langle e^{-\lambda_k s}\rangle_{k=1}^p, \langle e^{-\lambda_k s}\rangle_{k=1}^p, \ldots, \langle e^{-\lambda_k s}\rangle_{k=1}^p\big)^\top \,.$$

Combining these considerations, we obtain the following two facts:

(P1) The difference in means $\mathrm{m}_{\mathbf{x},t} - \mathrm{m}_{\mathbf{y},t}$ is such that its first component consists of an $\mathcal{O}(1)$ term (depending on R), while all remaining components converge to zero exponentially fast as $t \to \infty$.

(P2) The covariance matrix Σ_t is such that its top-left entry consists of an $\mathcal{O}(t)$ term plus terms that vanish as $t \to \infty$, while all the remaining entries consist of an $\mathcal{O}(1)$ term plus terms that vanish as $t \to \infty$.

One can easily check that (P2) implies that the inverse Σ_t^{-1} has elements in its first row and first column that decay to zero as $1/t$ as $t \to \infty$, while all other entries consist of $\mathcal{O}(1)$ terms. Combining these observations with (P1), we conclude that the right-hand side of (6.2.13) converges to zero as $t \to \infty$, thereby proving the claim. □

6.3 Identification of the spherical average in arbitrary even dimension

In dimension $d = 2$, it is well known that for any $x \in \mathbb{R}^d$, the process $(h_{e^{-t}}(x) - h_1(x))_{t\in\mathbb{R}}$ has the distribution of a standard two-sided Brownian motion. The aim of this section is to establish the characterisation of the spherical average process in any arbitrary even dimension $d > 2$, as stated in Theorem 6.A.

Proof of Theorem 6.A. We start by verifying that the multivariate process $(\mathbf{S}_t)_{t\in\mathbb{R}}$ defined in (6.1.4) is the stationary solution to the $\mathfrak{c}_d$-dimensional SDE of the form (6.1.3), with the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ as specified in the theorem's statement. Since, the covariance kernel of the process $(\mathrm{S}_t)_{t\in\mathbb{R}}$ is explicitly known, thanks to Lemma 6.2.13, it suffices to check that the power spectrum of the first derivative process $(\mathrm{S}_t^{(1)})_{t\in\mathbb{R}}$ coincides with that of the first component of the stationary solution of the aforementioned SDE.

We begin by noting that

$$\mathbb{E}\big[\mathrm{S}_t\mathrm{S}_s\big] = \int_0^t \int_0^s \mathbb{E}\big[\mathrm{S}_u^{(1)}\mathrm{S}_v^{(1)}\big] dv du \;, \qquad \forall\, t,\, s \in \mathbb{R} \;.$$

Hence, to compute the covariance of the first derivative $\mathrm{S}^{(1)}$, it suffices to take the second mixed derivative of the covariance kernel of S, as specified in (6.2.6). Proceeding in this manner, we obtain that

$$\mathcal{K}^{(1)}(t,s) \overset{\text{def}}{=} \mathbb{E}\big[\mathrm{S}_t^{(1)}\mathrm{S}_s^{(1)}\big] = c(d) \int_0^\pi \frac{1-\cos\vartheta\cosh|t-s|}{(\cos\vartheta-\cosh|t-s|)^2} (\sin\vartheta)^{d-2} d\vartheta \;, \qquad \forall\, s,t \in \mathbb{R} \;.$$

Since, the covariance kernel $\mathcal{K}^{(1)} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ only depends on the distance $|t-s|$, with a slight abuse of notation, we define the function $\mathcal{K}^{(1)} : \mathbb{R} \to \mathbb{R}$ by letting

$$\mathcal{K}^{(1)}(u) \overset{\text{def}}{=} c(d) \int_0^\pi \frac{1-\cos\vartheta\cosh u}{(\cos\vartheta-\cosh u)^2} (\sin\vartheta)^{d-2} d\vartheta \;, \qquad \forall\, u \in \mathbb{R} \;.$$

Our goal now is to compute the Fourier transform of $\mathcal{K}^{(1)}$. To this end, for $\vartheta \in [0,\pi]$, we define the function $\psi_\vartheta : \mathbb{R} \to \mathbb{R}$ by letting

$$\psi_\vartheta(u) \overset{\text{def}}{=} \frac{1-\cos\vartheta\cosh u}{(\cos\vartheta-\cosh u)^2} \;, \qquad \forall\, u \in \mathbb{R} \;.$$

By applying the change of variable $u \mapsto e^u$ and performing some elementary integral computations, we find that

$$\widehat{\psi}_\vartheta(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{1-\cos\vartheta\cosh u}{(\cos\vartheta-\cosh u)^2} e^{-iu\omega} du = \frac{1}{\sqrt{2\pi}} \frac{2\pi\omega}{e^{\pi\omega}-e^{-\pi\omega}} \big(e^{(\vartheta-\pi)\omega} + e^{-(\vartheta-\pi)\omega}\big) \;,$$

and so it holds that

$$\widehat{\mathcal{K}^{(1)}}(\omega) = c(d) \int_0^\pi \widehat{\psi}_\vartheta(\omega) (\sin\vartheta)^{d-2} d\vartheta = \frac{1}{\sqrt{2\pi}} \frac{c(d)2\pi\omega}{e^{\pi\omega}-e^{-\pi\omega}} \int_0^\pi \big(e^{(\vartheta-\pi)\omega} + e^{-(\vartheta-\pi)\omega}\big)(\sin\vartheta)^{d-2} d\vartheta \;.$$

Furthermore, we have the following identity

$$\int_0^\pi (e^{(\vartheta-\pi)\omega} + e^{-(\vartheta-\pi)\omega})(\sin\vartheta)^{d-2} d\vartheta = \frac{(d-2)(d-3)}{\omega^2+(d-2)^2} \int_0^\pi (e^{(\vartheta-\pi)\omega} + e^{-(\vartheta-\pi)\omega})(\sin\vartheta)^{d-4} d\vartheta \;, \tag{6.3.1}$$

and also

$$\int_0^\pi (e^{(\vartheta-\pi)\omega} + e^{-(\vartheta-\pi)\omega})(\sin\vartheta)^2 d\vartheta = \frac{2(e^{\pi\omega}-e^{-\pi\omega})}{\omega(\omega^2+4)} \;. \tag{6.3.2}$$

The identity (6.3.1) is derived using integration by parts, whereas (6.3.2) follows from some elementary integral computations. Therefore, using (6.3.1) and (6.3.2), we find that

$$\widehat{\mathcal{K}^{(1)}}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{2^{d-2}\Gamma(d/2)^2}{\prod_{k=1}^{\frac{d-2}{2}} \big(\omega^2+(d-2k)^2\big)} \;, \qquad \forall\, \omega \in \mathbb{R} \;,$$

where we used the identity $c(d)2\pi(d-2)! = 2^{d-2}\Gamma(d/2)^2$. Now, noting that we can write

$$\prod_{k=1}^{\frac{d-2}{2}} \big(\omega^2+(d-2k)^2\big) = \bigg[\prod_{k=1}^{\frac{d-2}{2}} \big((d-2k)+i\omega\big)\bigg] \bigg[\prod_{k=1}^{\frac{d-2}{2}} \big((d-2k)-i\omega\big)\bigg] \;,$$

thanks to Lemma 6.2.13, the multivariate process $(\mathbf{S}_t)_{t\in\mathbb{R}}$ is a solution to the $\mathfrak{c}_d$-dimensional SDE of the form (6.1.3) where the coefficients $(a_k)_{k=0}^{\mathfrak{c}_d}$ of the matrix $\mathbf{A}$ are chosen in such a way that its eigenvalues are

$(-\lambda_k)_{k=1}^{\mathfrak{c}_d}$ where

$$\lambda_k \stackrel{\text{def}}{=} d - 2k \,, \qquad \forall\, k \in [\mathfrak{c}_d] \,,$$

and the coefficient $b = c(d)(d-2)!\pi$. We introduce the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(s) \stackrel{\text{def}}{=} (d-2)e^{2s}(1-e^{2s})^{\frac{d-4}{2}} \,, \qquad \forall\, s \in \mathbb{R} \,.$$

Then, thanks again to Lemma 6.2.13, the process $(\mathrm{S}_t^{(1)})_{t\in\mathbb{R}}$ admits the following representation,

$$\mathrm{S}_t^{(1)} = 2^{d/2-1}\Gamma(d/2)\sum_{k=1}^{\mathfrak{c}_d} c_k \int_{-\infty}^t e^{-(d-2k)(t-s)}dB_s = \int_{-\infty}^t g(s-t)dB_s \,, \qquad \forall\, t \in \mathbb{R} \,, \tag{6.3.3}$$

where the second equality follows by recalling that $c_k = (\prod_{j=1,,j\neq k}^{\mathfrak{c}_d}(\lambda_j - \lambda_k))^{-1}$ and a straightforward computation. For $t \geq 0$, by taking the integral on both sides of (6.3.3), we get that

$$\mathrm{S}_t = \int_0^t \int_{-\infty}^u g(s-u)dB_s\, du = \int_{-\infty}^t \int_{\max(0,s)}^t g(s-u)du\, dB_s = \int_{-\infty}^0 g(s)(B_{t+s} - B_s)ds$$

where the first equality follows from stochastic Fubini's theorem, and the second equality follows from stochastic integration by parts and rearrangements. The same expression can be derived for $t < 0$, and so the claim follows. □

6.4 The LBM in arbitrary dimension

In this section, we focus on the construction of the LBM and study its main properties. First, in Section 6.4.1, we provide additional details on constructing the LBM starting from a fixed point, and we prove that if $\alpha \in (0,2)$ is chosen according to (6.1.8), then the LBM is compatible with the coordinate change rule described in Proposition 6.2.10. Next, in Section 6.4.2, we construct the d-dimensional LBM for all $Q > 2$ as a Markov process. To achieve this, we closely follow the procedure established in [GRV16], which itself builds upon [Bro70]. In particular, a key part of the construction involves potential analysis of the GMC measure $\mu_{\mathbf{h},\alpha}$. We then focus on the subcritical phase, i.e., $Q > \sqrt{2d}$, where we identify the additive functional used in the definition of the LBM and prove that its transition semigroup admits a density with respect to $\mu_{\mathbf{h},\alpha}$. Finally, in Section 6.4.3, we prove Theorem 6.B, i.e., we compute the spectral dimension of the LBM.

We recall that $(\mathbf{\Omega}, \mathbf{F}, (B_t)_{t\geq 0}, (\mathbf{F}_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}^d})$ is a d-dimensional Brownian motion, independent of the whole-space LGF h defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, for $t \geq 0$, we introduce the following occupation measures

$$\sigma(A) = \int_0^\infty \mathbb{1}_{\{B_s\in A\}}ds \,, \qquad \sigma_t(A) = \int_0^t \mathbb{1}_{\{B_s\in A\}}ds \,, \qquad \forall\, A \in \mathcal{B}(\mathbb{R}^d),\ \forall\, t \geq 0 \,. \tag{6.4.1}$$

In this section, we restrict our attention to the case $d > 2$, since the proofs of the corresponding two-dimensional results can be found in [GRV16, Ber15, RV14b]. Moreover, we assume throughout this section that the parameters $Q > 2$ and $\alpha \in (0,2)$ satisfy the relation specified in (6.1.8).

6.4.1 The LBM from a fixed point

As noted in the introduction, for a fixed point $x \in \mathbb{R}^d$ and $Q > 2$, the LBM $(\mathrm{B}_{\mathbf{h},\alpha,t})_{t\geq 0}$, as defined in Definition 6.1.4, is constructed by time-changing the standard d-dimensional Brownian motion $(B_t)_{t\geq 0}$ started from x. In particular, constructing the LBM from a fixed point is a straightforward task that falls under the framework of the theory of GMC measures.

Specifically, for all $x \in \mathbb{R}^d$, we observe that the measure σ_t defined in (6.4.1) has $\mathbf{P}_x$-almost surely *effective dimension* 2 (regardless of the ambient dimension $d \geq 2$), i.e., 2 is the largest non-negative real number $\mathfrak{s}$ such that for all $R > 0$ and all $\varepsilon > 0$,

$$\int_{B(0,R)\times B(0,R)} \frac{1}{|x-y|^{\mathfrak{s}-\varepsilon}} \sigma_t(dx)\sigma_t(dy) < \infty \ ,$$

We refer to Lemma 4.5.1 for a proof of this fact. Therefore, for $\alpha \in (0,2)$, if we define the GMC measure associated with with respect to σ as

$$\nu_{\mathbf{h},\alpha}(dx) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \varepsilon^{\frac{\alpha^2}{2}} e^{\alpha \mathbf{h}_\varepsilon(x)} \sigma(dx) \ . \tag{6.4.2}$$

it is well-known (see [Ber17, BP24, GRV16]) that for all $x \in \mathbb{R}^d$, the limit in the above expression converges $\mathbf{P}_x$-almost surely in $\mathbb{P}$-probability with respect to the topology of weak convergence in the space of non-negative, locally finite measures on $\mathbb{R}^d$. Hence, to rigorously construct the LBM starting from x, it suffices to note that the measure $\nu_{\mathbf{h},\alpha}$ is related to the clock process $\mathrm{F}_{\mathbf{h},\alpha}$, as defined in (6.1.5), through the following relationship

$$\mathrm{F}_{\mathbf{h},\alpha}(t) = \nu_{\mathbf{h},\alpha}\big([B]_t\big) \ , \qquad \forall\, t \geq 0 \ ,$$

where $[B]_t$ represents the trajectory of the Brownian motion up to time t.

In order to prove the conformal invariance of the LBM, we need the following auxiliary result.

Proposition 6.4.1 (Coordinate change formula for $\nu_{\mathbf{h},\alpha}$)**.** *Consider the same setting as in Proposition 6.2.10. For $Q > 2$, let $\alpha \in (0,2)$ be as specified in (6.1.8). Then, for all $x \in \mathbb{R}^d$, it holds $\mathbf{P}_x \otimes \mathbb{P}$-almost surely that*

$$\nu_{\mathbf{h},\alpha}(\phi(A)) = \overline{\nu}_{\mathbf{h}\circ\phi + Q\log|\phi'|,\alpha}(A) \ , \qquad \forall\, A \in \mathcal{B}(\mathbb{R}^d) \ , \tag{6.4.3}$$

where $\overline{\nu}_{\mathbf{h}\circ\phi+Q\log|\phi'|,\alpha}$ is the GMC measure associated with the field $\mathbf{h}\circ\phi + Q\log|\phi'|$, defined with respect to the occupation measure $\overline{\sigma}$ of the d-dimensional Brownian motion $(\overline{B}_t)_{t\geq 0} = (\phi^{-1}(B_{|\phi'|^2 t}))_{t\geq 0}$.

Proof. This proof is completely analogous to the proof of Proposition 6.2.10. However, we provide the details in order to highlight where the relation (6.1.8) between Q and α originates from.

For $c > 0$, define the d-dimensional Brownian motion $(\overline{B}_t)_{t\geq 0} = (c^{-1}B_{c^2 t})_{t\geq 0}$, and let $\overline{\sigma}$ be the associated occupation measure as specified in (6.4.1). We observe that, for every $d \geq 2$, the measure $\overline{\sigma}$ exhibits two-dimensional scaling, i.e., for all Borel subsets $A \subseteq \mathbb{R}^d$, it holds $\mathbf{P}_x$-almost surely that

$$\overline{\sigma}(A) = c^{-2}\sigma(cA) \ . \tag{6.4.4}$$

Now, let $\overline{\nu}_{\mathbf{h},\alpha}$ denote the GMC measure associated with the field $\mathbf{h}$, defined with respect to the measure $\overline{\sigma}$. Consider the field $\mathbf{h}^c \stackrel{\text{def}}{=} \mathbf{h}(c\,\cdot)$. For any $\varepsilon > 0$ and $x \in \mathbb{R}^d$, we have the following relation between the

spherical averages: $\mathbf{h}^c_\varepsilon(x) = \mathbf{h}_{c\varepsilon}(cx)$. Therefore, for any $A \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\overline{\nu}_{\mathbf{h}^c,\alpha}(A) = \lim_{\varepsilon\to 0} \varepsilon^{\alpha^2/2} \int_A e^{\alpha \mathbf{h}^c_\varepsilon(x)} \overline{\sigma}(dx) = \lim_{\varepsilon\to 0} \varepsilon^{\alpha^2/2} \int_A e^{\alpha \mathbf{h}_{c\varepsilon}(cx)} \overline{\sigma}(dx)$$
$$\overset{(6.4.4)}{=} \lim_{\delta\to 0} (\delta/c)^{\alpha^2/2} c^{-2} \int_{cA} e^{\alpha \mathbf{h}_\delta(x)} \sigma(dx) \ ,$$

where, in the last equality, we set $\delta = c\varepsilon$ and performed the change of variables $x \mapsto x/c$. We note that the term on the right-hand side of the above display equals $c^{-2-\alpha^2/2}\nu_{\mathbf{h},\alpha}(cA)$. Therefore, we conclude that

$$c^{2+\alpha^2/2}\overline{\nu}_{\mathbf{h}^c,\alpha}(A) = \nu_{\mathbf{h},\alpha}(cA) \qquad \Longrightarrow \qquad \overline{\nu}_{\mathbf{h}^c + Q\log c,\alpha}(A) = \nu_{\mathbf{h},\alpha}(cA) \ .$$

where we used the fact that, for $\alpha = \alpha(Q)$ as in (6.1.8), it holds that $\alpha Q = 2 + \alpha^2/2$. Finally, as for the proof of Proposition 6.2.10, we observe that the same argument also works if instead of just a scaling we have a composition of a scaling, rotation, and translation. □

6.4.1.1 Conformal invariance of the LBM

We now have all the necessary elements to prove that if $\alpha \in (0,2)$ is chosen as specified in (6.1.8), then the LBM is compatible with the coordinate change rule.

Proposition 6.4.2. *Consider the same setting as in Proposition 6.2.10. For $Q > 2$, let $\alpha \in (0,2)$ be as specified in (6.1.8). Then, for all $x \in \mathbb{R}^d$, it holds $\mathbf{P}_x \otimes \mathbb{P}$-almost surely that*

$$\big(\phi^{-1}(\mathrm{B}_{\mathbf{h},\alpha,t})\big)_{t\geq 0} = \big(\overline{\mathrm{B}}_{\mathbf{h}\circ\phi + Q\log|\phi'|,\alpha,t}\big)_{t\geq 0} \quad \textit{where} \quad \overline{\mathrm{B}}_{\mathbf{h}\circ\phi+Q\log|\phi'|,\alpha,t} \overset{\text{def}}{=} \overline{B}_{\overline{\mathrm{F}}^{-1}_{\mathbf{h}\circ\phi+Q\log|\phi'|,\alpha}(t)} \ ,$$

where $\overline{\mathrm{F}}_{\mathbf{h}\circ\phi+Q\log|\phi|,\alpha}$ is the clock process associated with the field $\mathbf{h}\circ\phi + Q\log|\phi'|$, defined with respect to the d-dimensional Brownian motion $(\overline{B}_t)_{t\geq 0} = (\phi^{-1}(B_{|\phi'|^2 t}))_{t\geq 0}$.

Proof. For $c > 0$, define the d-dimensional Brownian motion $(\overline{B}_t)_{t\geq 0} \overset{\text{def}}{=} (c^{-1}B_{c^2 t})_{t\geq 0}$ and let $\mathbf{h}^c \overset{\text{def}}{=} \mathbf{h}(c\,\cdot)$. Then, for all $t \geq 0$, it holds that

$$c^{-1}\mathrm{B}_{\mathbf{h},\alpha,t} = c^{-1}B_{\mathrm{F}^{-1}_{\mathbf{h},\alpha}(t)} = \overline{B}_{c^{-2}\mathrm{F}^{-1}_{\mathbf{h},\alpha}(t)} = \overline{B}_{\mathrm{F}^{-1}_{\mathbf{h},\alpha,c}(t)} \ ,$$

where we set $\mathrm{F}_{\mathbf{h},\alpha,c}(t) \overset{\text{def}}{=} \mathrm{F}_{\mathbf{h},\alpha}(c^2 t)$. Furthermore, for all $t \geq 0$, thanks to Proposition 6.4.1, we observe that

$$\mathrm{F}_{\mathbf{h},\alpha,c}(t) = \nu_{\mathbf{h}}([B]_{c^2 t}) \overset{(6.4.3)}{=} \nu_{\mathbf{h}^c + Q\log c,\alpha}(c^{-1}[B]_{c^2 t}) = \overline{\nu}_{\mathbf{h}^c+Q\log c,\alpha}([\overline{B}]_t) = \overline{\mathrm{F}}_{\mathbf{h}^c+Q\log c,\alpha}(t) \ ,$$

where $\overline{\nu}_{\mathbf{h}^c+Q\log c,\alpha}$ is the GMC measure associated with the field $\mathbf{h}^c + Q\log c$, defined with respect to the occupation measure of the Brownian motion $(\overline{B}_t)_{t\geq 0}$ introduced above. Finally, as for the proof of Proposition 6.4.1, we observe that the same argument also works if instead of just a scaling we have a composition of a scaling, rotation, and translation. □

Remark 6.4.3. The reason why, in Proposition 6.4.1, we only considered conformal automorphism of $\mathbb{R}^d$ which can be expressed as a composition of translations, dilations and rotations, instead of a general conformal maps, is that the Brownian motion in dimension three and higher is not conformally invariant. Indeed, consider the inversion map $\phi(z) = z/|z|^2$ and let $(B_t)_{t\geq 0}$ be a Brownian motion on $\mathbb{R}^d$ for $d > 3$. Then of course $(\phi(B_t))_{t\geq 0}$ cannot be expressed as a time-change of a Brownian motion.

6.4.2 The LBM as a Markov process

In this section we construct the d-dimensional LBM as a diffusion process for all $Q > 2$. We begin by recalling some definitions and results that will be useful in what follows.

Definition 6.4.4. A *positive continuous additive functional* (PCAF) $(\mathrm{A}_t)_{t\geq 0}$ is a $\mathbf{F}_t$-adapted continuous functional with values in $[0,\infty]$ such that there exists a set $\Lambda \in \mathbf{F}_\infty$ and a polar set $\mathrm{N} \subseteq \mathbb{R}^d$ for the d-dimensional Brownian motion for which $\mathbf{P}_x(\Lambda) = 1$, for all $x \in \mathbb{R}^d \setminus \mathrm{N}$, and

$$\mathrm{A}_{t+s}(\omega) = \mathrm{A}_t(\omega) + \mathrm{A}_s(\omega) \circ \vartheta_t(\omega) \,, \qquad \forall\, \omega \in \Lambda \,,\ \forall\, s,t \geq 0 \,,$$

where $(\vartheta)_{t\geq 0}$ denotes the family of shift mappings on Ω, i.e., $B_{t+s} = B_t \circ \vartheta_s$, for all $s, t \geq 0$. Furthermore, if $\mathrm{N} = \emptyset$, then $(\mathrm{A}_t)_{t\geq 0}$ is called a PCAF in the *strict sense*.

Definition 6.4.5. A Borel measure ρ_{A} on $\mathbb{R}^d$ is called the[11] *Revuz measure* of a PCAF $(\mathrm{A}_t)_{t\geq 0}$ if for any non-negative Borel function $f : \mathbb{R}^d \to [0,\infty]$ it holds that

$$\int_{\mathbb{R}^d} f(x)\rho_{\mathrm{A}}(dx) = \lim_{t\to 0} \frac{1}{t} \int_{\mathbb{R}^d} \mathbf{E}_x \left[\int_0^t f(B_s) d\mathrm{A}_s \right] dx \,.$$

6.4.2.1 Potential theory of the GMC measure in arbitrary dimension $d > 2$

For each $R > 0$, we let $G_R : B(0,R) \times B(0,R) \to \mathbb{R}$ be the Green's function of the Laplacian on the ball $B(0,R)$ with zero boundary conditions, i.e.,

$$-\Delta G_R(x,\cdot) = \delta(x - \cdot) \,, \qquad G_R(x,\cdot)_{|\partial B(0,R)} = 0 \,. \tag{6.4.5}$$

Definition 6.4.6. The R-potential of a positive Borel measure ρ on $\mathbb{R}^d$ is the function $\mathrm{g}_R(\rho) : B(0,R) \to \mathbb{R} \cup \{\infty\}$ defined as follows

$$\mathrm{g}_R[\rho](x) \overset{\text{def}}{=} \int_{B(0,R)} G_R(x,y)\rho(dy) \,, \qquad \forall\, x \in B(0,R) \,. \tag{6.4.6}$$

Furthermore, the proper domain of $\mathrm{g}_R(\rho)$ is the set $\mathrm{G}_R[\rho] \subseteq B(0,R)$ given by

$$\mathrm{G}_R[\rho] \overset{\text{def}}{=} \left\{ x \in B(0,R) \,:\, \mathrm{g}_R[\rho](x) < \infty \right\} .$$

It is a standard fact in potential theory that $B(0,R) \setminus \mathrm{G}_R[\rho]$ is a polar set for the d-dimensional Brownian motion (see e.g. [Bro70, page 247]).

In order to associate a PCAF to the measure $\mu_{\mathbf{h},\alpha}$, we begin with the fact such a measure does not charge polar sets. In what follows, for $R > 0$, we denote by $\mu_{\mathbf{h},\alpha}|_{B(0,R)}$ the restriction of the measure $\mu_{\mathbf{h},\alpha}$ to the ball $B(0,R)$.

Lemma 6.4.7 ([DSHKS24, Theorem 5.2])**.** *Let $d > 2$, $Q > 2$, and $R > 0$. Then, $\mathbb{P}$-almost surely, the measure $\mu_{\mathbf{h},\alpha}|_{B(0,R)}$ does not charge polar sets of the d-dimensional Brownian motion.*

Letting τ_R denote the first time the Brownian motion $(B_t)_{t\geq 0}$ exits the ball $B(0,R)$, it follows from Lemma 6.4.7 and [Bro70, Proposition 3.2] that, for all $Q > 2$, we can associate to the measure $\mu_{\mathbf{h},\alpha}|_{B(0,R)}$ a

[11] The fact that the Revuz measure of a PCAF is unique follows from the general theory of symmetric Markov processes, see e.g. [CF12, Theorem A.3.5].

unique PCAF $(\mathrm{F}_{\mathbf{h},\alpha,R}(t))_{t\geq 0}$ such that the process

$$\big(\mathrm{g}_R[\mu_{\mathbf{h},\alpha}](B_{t\wedge\tau_R}) - \mathrm{g}_R[\mu_{\mathbf{h},\alpha}](B_0) + \mathrm{F}_{\mathbf{h},\alpha,R}(t)\big)_{t\geq 0} \tag{6.4.7}$$

is a mean-zero martingale under $\mathbf{P}_x$ for all $x \in \mathrm{G}_R[\mu_{\mathbf{h},\alpha}]$.

Lemma 6.4.8. *Let $d > 2$, $Q > \sqrt{2d}$, and $R > 0$. Then, it holds $\mathbb{P}$-almost surely that,*

$$\sup_{x\in B(0,R)} \mathrm{g}_R[\mu_{\mathbf{h},\alpha}](x) < \infty\,, \tag{6.4.8}$$

or in other words $\mathrm{G}_R[\mu_{\mathbf{h},\alpha}] = B(0,R)$. Therefore, for $Q > \sqrt{2d}$, $\mathrm{F}_{\mathbf{h},\alpha,R}$ is a PCAF in the strict sense.

Proof. Fix $d > 2$, $Q > \sqrt{2d}$, and $R > 0$. By recalling that for any x, $y \in B(0,R)$, the quantity $G_R(x,y)$ is bounded by a multiple of $|x-y|^{2-d}$, it suffices to show that

$$\sup_{x\in B(0,R)} \int_{B(0,R)} |x-y|^{2-d}\mu_{\mathbf{h},\alpha}(dy) < \infty\,.$$

By using Lemma 6.A.2, for any $x \in B(0,R)$ and $\varepsilon > 0$, it holds that

$$\begin{aligned}\int_{B(0,R)} |x-y|^{2-d}\mu_{\mathbf{h},\alpha}(dy) &= \sum_{n\in\mathbb{N}_0} \int_{\{y\in B(0,R)\,:\,|x-y|\in(e^{-(n+1)}2R,e^{-n}2R]\}} |x-y|^{2-d}\mu_{\mathbf{h},\alpha}(dy)\\ &\lesssim \sum_{n\in\mathbb{N}_0} e^{-n(2-d)}\mu_{\mathbf{h},\alpha}\big(B(x,e^{-(n-1)}2R)\big)\\ &\lesssim \sum_{n\in\mathbb{N}_0} e^{-n(2+\alpha^2/2-\sqrt{2d}\alpha-\varepsilon)}\,,\end{aligned}$$

where the random implicit constants depend only on R. The desired result follows since $2+\alpha^2/2-\sqrt{2d}\alpha > 0$ for all $Q > \sqrt{2d}$ and by taking $\varepsilon > 0$ small enough. □

Remark 6.4.9. We expect that if $Q \in (2, \sqrt{2d}]$, then the set $B(0,R) \setminus \mathrm{G}_R[\mu_{\mathbf{h},\alpha}]$ is $\mathbb{P}$-almost surely *not* empty. Indeed, for $\beta \in [Q, \sqrt{2d}]$, the set $\mathcal{T}_\beta \cap B(0,R)$ is $\mathbb{P}$-almost surely non-empty, and if $x \in \mathcal{T}_\beta \cap B(0,R)$, then it should hold that $\mathrm{g}_R[\mu_{\mathbf{h},\alpha}](x) = \infty$. To see this, we recall that the field $\mathbf{h}$ around a typical point $x \in \mathcal{T}_\beta$ locally looks like $\mathbf{h} - \beta\log|\cdot - x|$. In particular, using Lemma 6.A.1 and following the same reasoning as in the proof of Lemma 6.4.8, for any $\varepsilon > 0$, we obtain that

$$\int_{B(0,R)} |x-y|^{2-d}\mu_{\mathbf{h}-\beta\log|\cdot-x|,\alpha}(dy) \gtrsim \sum_{n\in\mathbb{N}_0} e^{-n(2+\alpha^2/2-\beta\alpha+\varepsilon)}\,,$$

and since by assumption $2 + \alpha^2/2 - \beta\alpha \leq 0$, the sum on the right-hand side of the above display diverges. We also emphasise that for any $\beta \in (2, \sqrt{2d}]$, the set of β-thick points $\mathcal{T}_\beta$ has Hausdorff dimension equals to $d - \beta^2/2$, which is strictly less than $d-2$. Consequently, by [MP10, Theorem 8.20], $\mathcal{T}_\beta$ is a polar set for the standard d-dimensional Brownian motion. Combining these observations, we expect that for $Q \in (2, \sqrt{2d}]$, $\mathrm{F}_{\mathbf{h},\alpha,R}$ is a PCAF *not* in the strict sense.

6.4.2.2 Identification of the PCAF in the subcritical regime

We now limit ourself to the subcritical regime, i.e., $Q > \sqrt{2d}$, and we identify the PCAF $\mathrm{F}_{\mathbf{h},\alpha,R}$ introduced in (6.4.7). We emphasise that in [DSHKS24], the authors provide an identification of the PCAF for almost

every choice of the starting point for all $Q > 2$.

To do so, we first need to introduce some notation. In what follows, recalling (6.2.2), we define $h_{(n)} \overset{\text{def}}{=} \sum_{i=1}^{n} \alpha_i f_i$ and we let $\mathbf{h}_{(n)} \overset{\text{def}}{=} h_{(n)} - h_{(n),1}(0)$ where $h_{(n),1}(0)$ denotes the spherical average of $h_{(n)}$ over the unit sphere centred at the origin. We define the function $\mathrm{C} : \mathbb{R}^d \to \mathbb{R}$ by letting

$$\log \mathrm{C}(x) \overset{\text{def}}{=} \lim_{\varepsilon \to 0} \mathbb{E}\big[\mathbf{h}_\varepsilon(x)^2\big] + \log \varepsilon \,, \qquad \forall\, x \in \mathbb{R}^d \,, \tag{6.4.9}$$

and, for $\alpha \in (0,2)$, we introduce the measure $\mu_{\mathbf{h},\alpha,(n)}$ by letting

$$\mu_{\mathbf{h},\alpha,(n)}(dx) = e^{\alpha^2 \mathbf{h}_{(n)}(x) - \frac{1}{2}\alpha^2 \mathbb{E}[\mathbf{h}_{(n)}(x)^2] + \frac{1}{2}\gamma^2 \log \mathrm{C}(x)} dx \,. \tag{6.4.10}$$

It is a standard fact (see e.g. [BP24, Lemma 2.9]) that the measure $\mu_{\mathbf{h},\alpha}$ introduced in (6.2.7) is $\mathbb{P}$-almost surely the weak limit for $n \to \infty$ of the measure $\mu_{\mathbf{h},\alpha,(n)}$. In what follows, with a slight abuse of notation, we denote by $\mu_{\mathbf{h},\alpha,(\infty)}$ the limiting measure $\mu_{\mathbf{h},\alpha}$.

For $R > 0$, $Q > \sqrt{2d}$, and $n \in \mathbb{N}$, we observe that the mapping $B(0,R) \ni x \mapsto \mathrm{g}_R[\mu_{\mathbf{h},\alpha,(n)}](x)$ is $\mathbb{P}$-almost surely bounded and continuous. This follows easily since, for each $n \in \mathbb{N}$, the measure $\mu_{\mathbf{h},\alpha,(n)}$ is absolutely continuous with respect to the Lebesgue measure. It is well-known (see e.g. [BK92, page 467]), that we can associate to the measure $\mu_{\mathbf{h},\alpha,(n)}$ a unique PCAF $\mathrm{F}_{\mathbf{h},\alpha,(n),R}$ such that the process

$$\big(\mathrm{g}_R[\mu_{\mathbf{h},\alpha,(n)}](B_{t\wedge\tau_R}) - \mathrm{g}_R[\mu_{\mathbf{h},\alpha,(n)}](B_0) + \mathrm{F}_{\mathbf{h},\alpha,(n),R}(t)\big)_{t\geq 0}$$

is a mean-zero martingale under $\mathbf{P}_x$ for all $x \in B(0,R)$. We also emphasise that, for $n \in \mathbb{N}$, we have the following explicit representation of the PCAF $\mathrm{F}_{\mathbf{h},\alpha,(n),R}$,

$$\mathrm{F}_{\mathbf{h},\alpha,(n),R}(t) = \int_0^{t\wedge\tau_R} e^{\alpha^2 \mathbf{h}_{(n)}(B_s) - \frac{1}{2}\alpha^2 \mathbb{E}[\mathbf{h}_{(n)}(B_s)^2] + \frac{1}{2}\alpha^2 \log \mathrm{C}(B_s)} ds \,, \qquad \forall\, t \geq 0 \,.$$

For any $z \in \mathbb{R}^d$, we denote by $\mu^z_{\mathbf{h},\alpha}$ the shifted measure $\mu^z_{\mathbf{h},\alpha}(\cdot) = \mu_{\mathbf{h},\alpha}(z + \cdot)$. We have the following result which is the higher dimensional analogue of [GRV16, Proposition 2.3].

Proposition 6.4.10. *Let $d > 2$, $Q > \sqrt{2d}$, and $R > 0$. Then, it holds $\mathbb{P}$-almost surely that:*

(G1) As $n \to \infty$, it holds that $\sup_{x\in B(0,R)} |\mathrm{g}_R[\mu_{\mathbf{h},\alpha,(n)}](x) - \mathrm{g}_R[\mu_{\mathbf{h},\alpha}](x)| \to 0$.

(G2) The map $B(0,R) \ni x \mapsto \mathrm{g}_R[\mu_{\mathbf{h},\alpha}](x)$ is continuous.

(G3) For any $z_0 \in \mathbb{R}^d$, as $z \to z_0$ it holds $\sup_{x\in B(0,R)} |\mathrm{g}_R[\mu^z_{\mathbf{h},\alpha}](x) - \mathrm{g}_R[\mu^{z_0}_{\mathbf{h},\alpha}](x)| \to 0$.

Proof. We begin by proving (G1). The proof closely follows the argument presented in [GRV16, Proposition 2.3]. It is included here for completeness and to show that the techniques of [GRV16] do not heavily depend on the ambient dimension $d \geq 2$. With a slight abuse of notation, we denote the limiting measure $\mu_{\mathbf{h},\alpha}$ as $\mu_{\mathbf{h},\alpha,(\infty)}$. We consider a smooth function $\vartheta : \mathbb{R}^d \to \mathbb{R}$ such that $\vartheta(x) = 1$ for $|x| \leq 1$ and $\vartheta(x) = 0$ for $|x| \geq 2$. For $\delta > 0$, we then set $\vartheta_\delta(x) \overset{\text{def}}{=} \vartheta(x/\delta)$ and $\overline{\vartheta}_\delta(x) \overset{\text{def}}{=} 1 - \vartheta_\delta(x)$. Then, for any $n \in \mathbb{N} \cup \{\infty\}$, it holds that

$$\mathrm{g}_R[\mu_{\mathbf{h},\alpha,(n)}](x) = \underbrace{\int_{B(0,R)} G_R(x,y)\vartheta_\delta(x-y)\mu_{\mathbf{h},\alpha,(n)}(dy)}_{\mathrm{I}_{(n),\delta}(x)} + \underbrace{\int_{B(0,R)} G_R(x,y)\overline{\vartheta}_\delta(x-y)\mu_{\mathbf{h},\alpha,(n)}(dy)}_{\overline{\mathrm{I}}_{(n),\delta}(x)} \,.$$

We start by showing that the family of maps $B(0,R) \ni x \mapsto \mathrm{I}_{(n),\delta}(x)$ converges uniformly to 0 as $\delta \to 0$, i.e., we need to verify that

$$\lim_{\delta\to 0} \sup_{n\in\mathbb{N}\cup\{\infty\}} \sup_{x\in(0,R)} \int_{B(0,R)} |x-y|^{2-d}\vartheta_\delta(x-y)\mu_{\mathbf{h},\alpha,(n)}(dy) = 0 \,.$$

This holds since, by using Lemma 6.A.2, for $n \in \mathbb{N}\cup\{\infty\}$, $x \in B(0,R)$, and any $\varepsilon > 0$, it holds that

$$\begin{aligned}
&\int_{B(0,R)} |x-y|^{2-d}\vartheta_\delta(x-y)\mu_{\mathbf{h},\alpha,(n)}(dy) \\
&\qquad \le \sum_{k\ge\lfloor\log(R/(2\delta))\rfloor} \int_{\{y\in B(0,R)\,:\,|x-y|\in(e^{-(k+1)}R, e^{-k}R]\}} |x-y|^{2-d}\mu_{\mathbf{h},\alpha,(n)}(dy) \\
&\qquad \lesssim \sum_{k\ge\lfloor\log(R/(2\delta))\rfloor} e^{-k(2-d)}\mu_{\mathbf{h},\alpha,(n)}\big(B(x,e^{-(k-1)}R)\big) \\
&\qquad \lesssim \sum_{k\ge\lfloor\log(R/(2\delta))\rfloor} e^{-k(2+\alpha^2/2-\sqrt{2d}\alpha-\varepsilon)} \,,
\end{aligned}$$

where the (random) implicit constants depend only on R. For all $Q > \sqrt{2d}$ the latter series converges to 0 as $\delta \to 0$ by taking $\varepsilon > 0$ small enough. Therefore, to conclude, it suffices to show that for each fixed $\delta > 0$, the family $\bar{\mathrm{I}}_{(n),\delta}$ converges uniformly on $B(0,R)$ to $\bar{\mathrm{I}}_{(\infty),\delta}$ as $n \to \infty$. The fact that this convergence holds pointwise is a direct consequence of the fact that the measure $\mu_{\mathbf{h},\alpha,(n)}$ converges weakly to $\mu_{\mathbf{h},\alpha}$ as $n \to \infty$. Hence, it suffices to show that the family $(\bar{\mathrm{I}}_{(n),\delta})_{n\in\mathbb{N}}$ is relatively compact for the topology of uniform convergence in $B(0,R)$. This follows since for each $\delta > 0$, the mapping $B(0,R)\times B(0,R) \ni (x,y) \mapsto G_R(x,y)\bar{\vartheta}_d(x-y)$ is uniformly continuous in $B(0,R)\times B(0,R)$, the fact that $\sup_{n\in\mathbb{N}} \mu_{\mathbf{h},\alpha,(n)}(B(0,R)) < \infty$, and Arzelá–Ascoli theorem.

Finally, the proofs of (G2) and (G3) rely on similar ideas and are entirely analogous to the proofs of the corresponding properties established in [GRV16, Proposition 2.3]. Therefore, their proofs are omitted. □

Lemma 6.4.11. *Let $d > 2$, $Q > \sqrt{2d}$, and $R > 0$. Then, it holds $\mathbb{P}$-almost surely for all $x \in B(0,R)$ and $\eta > 0$ that*

$$\lim_{n\to\infty} \mathbf{P}_x\left(\sup_{t\ge 0}|\mathrm{F}_{\mathbf{h},\alpha,(n),R}(t) - \mathrm{F}_{\mathbf{h},\alpha,R}(t)| \ge \eta\right) = 0 \,.$$

Proof. This result is a consequence of (G1) in Proposition 6.4.10 and [BK92, Proposition 2.1]. □

We are finally ready to identify the PCAF of the LBM on the whole space and to state its main properties.

Theorem 6.4.12. *Let $d > 2$ and $Q > \sqrt{2d}$. Then, $\mathbb{P}$-almost surely, there exists a unique PCAF in the strict sense $(\mathrm{F}_{\mathbf{h},\alpha}(t))_{t\ge 0}$ such that, for all $R > 0$,*

$$\mathrm{F}_{\mathbf{h},\alpha}(t) = \mathrm{F}_{\mathbf{h},\alpha,R}(t) \,, \qquad \forall\, t < \tau_R \,.$$

Furthermore, the following properties hold $\mathbb{P}$-almost surely:

(B1) The Revuz measure of $\mathrm{F}_{\mathbf{h},\alpha}$ is $\mu_{\mathbf{h},\alpha}$.

(B2) For all $x \in \mathbb{R}^d$, $T > 0$, and $\eta > 0$, the following limit holds

$$\lim_{n\to\infty} \mathbf{P}_x\left(\sup_{t\le T}|\mathrm{F}_{\mathbf{h},\alpha,(n)}(t) - \mathrm{F}_{\mathbf{h},\alpha}(t)| \ge \eta\right) = 0 \,.$$

(B3) *For all $x \in \mathbb{R}^d$, it holds $\mathbf{P}_x$-almost surely that the map $[0,\infty) \ni t \mapsto \mathrm{F}_{\mathbf{h},\alpha}(t)$ is strictly increasing.*

(B4) *For all $x \in \mathbb{R}^d$, it holds $\mathbf{P}_x$-almost surely that $\lim_{t\to\infty} \mathrm{F}_{\mathbf{h},\alpha}(t) = \infty$.*

(B5) *For each $T > 0$, and for any bounded continuous function $\mathscr{F} : \mathcal{C}([0,T],\mathbb{R}_0^+) \to \mathbb{R}$, the mapping $\mathbb{R}^d \ni x \to \mathbf{E}_x[\mathscr{F}(\mathrm{F}_{\mathbf{h},\alpha}(\cdot))]$ is continous.*

The proof of Theorem 6.4.12 relies, among other things, on the following standard coupling result for d-dimensional Brownian motions which is a consequence of the strong-Markov property. Consider a Brownian motion W independent of B and defined on the same probability space. We define the collection of stopping times $(\tau_i)_{i\in\{1,\dots,d\}}$ inductively by letting

$$\tau_1 \overset{\text{def}}{=} \inf\{t \geq 0 \,:\, B_t^{(1)} = W_t^{(1)}\}\,, \qquad \tau_i \overset{\text{def}}{=} \inf\{t \geq \tau_{i-1} \,:\, B_t^{(i)} = W_t^{(i)}\}\,, \quad \forall\, i \in \{2,\dots,d\}\,,$$

where $B^{(i)}$ and $W^{(i)}$ denote the i-th coordinate of B and W, respectively. Under, $\mathbf{P}_{x,y}$ the law of (B,W) is that of two independent d-dimensional Brownian motions started from (x,y). We have the following generalisation of [GRV16, Lemma 2.9].

Lemma 6.4.13. *Consider the situation described above. Under $\mathbf{P}_{x,y}$, the process $\overline{B}$ given by*

$$\overline{B}_t \overset{\text{def}}{=} \begin{cases} (W_t^{(1)},\dots,W_t^{(d)})\,, & \text{for } t \leq \tau_1\,,\\ (B_t^{(1)},W_t^{(2)},\dots,W_t^{(d)})\,, & \text{for } t \in (\tau_1,\tau_2]\,,\\ \qquad\vdots & \\ (B_t^{(1)},\dots,B_t^{(d-1)},W_t^{(d)})\,, & \text{for } t \in (\tau_{d-1},\tau_d]\,,\\ (B_t^{(1)},\dots,B_t^{(d)})\,, & \text{for } t > \tau_d\,, \end{cases}$$

is a d-dimensional Brownian motion started from y. Furthermore, for all $\eta > 0$, it holds that

$$\lim_{\delta\to 0} \sup_{x,y\in\mathbb{R}^d,\,|x-y|\leq\delta} \mathbf{P}_{x,y}(\tau_d > \eta) = 0\,, \qquad \mathbf{P}_{x,y}(\tau_d < \infty) = 1\,.$$

Proof of Theorem 6.4.12. The proof of this result is similar to the proof of [GRV16, Theorem 2.7]. Existence and uniqueness of the PCAF are immediate consequences of the previous discussions. For (B1), we refer to the proof of [GRV16, Theorem 2.7] and references therein. For (B2), this is an immediate consequence of Lemma 6.4.11 and the fact that we can take R large enough to make $\mathbb{P}(\tau_R < T)$ arbitrarily small. Items (B3) and (B5) can be proved in the exact same way as in [GRV16, Theorem 2.7] by using Lemma 6.4.13.

It remains to prove (B4). In this case, the proof differs from [GRV16] since in that paper, the authors leverage on the exponential decay of correlation of the massive GFF. We note that it suffices to check that $\mathbb{P}$-almost surely for a fixed $x \in \mathbb{R}^d$, it holds that $\mathbf{P}_x(\lim_{t\to\infty} \mathrm{F}_{\mathbf{h},\alpha}(t) = \infty) = 1$. To get the desired result, it is then sufficient to apply Lemma 6.4.13. Therefore, we take $x = 0$ and our goal is to show that $\lim_{t\to\infty} \mathrm{F}_{\mathbf{h},\alpha}(t) = \infty$, $\mathbb{P}\otimes\mathbf{P}_0$-almost surely. To this end, thanks to (B3), it suffices to show that $\lim_{n\to\infty} \mathrm{F}_{\mathbf{h},\alpha}(\tau_n) = \infty$, $\mathbb{P}\otimes\mathbf{P}_0$-almost surely. For all $n \in \mathbb{N}$, by the scale invariance of the law of $\mathbf{h}$ modulo an additive constant, we have that $\mathbf{h}^n \overset{\text{def}}{=} \mathbf{h}(n\cdot) - \mathbf{h}_n(0) \overset{\text{law}}{=} \mathbf{h}$. Consequently, by Proposition 6.4.1, it follows that

$$\mathrm{F}_{\mathbf{h},\alpha}(\tau_n) = \nu_{\mathbf{h},\alpha}\big([B]_{\tau_n}\big) \overset{\text{law}}{=} n^{\alpha Q} e^{\alpha \mathbf{h}_n(0)} \nu_{\mathbf{h}^n,\alpha}\big(n^{-1}[B]_{\tau_n}\big)\,,$$

where we recall that the measure $\nu_{\mathbf{h},\alpha}$ is defined in (6.4.2). For any $c > 0$, using the fact that $\mathbf{h}_n(0)$ is a centred Gaussian with variance of order $\log n$ (see Lemma 6.2.9), we have that

$$\sum_{n\in\mathbb{N}} \mathbb{P}\big(n^{\alpha Q} e^{\alpha \mathbf{h}_n(0)} \le c\big) < \infty \,,$$

and so, thanks to the Borel–Cantelli lemma, we have that

$$\mathbb{P}\Big(\lim_{n\to\infty} n^{\alpha Q} e^{\alpha \mathbf{h}_n(0)} = \infty\Big) = 1 \,. \tag{6.4.11}$$

Moreover, we note that for each $n \in \mathbb{N}$, the law of $\nu_{\mathbf{h}^n,\alpha}(n^{-1}[B]_{\tau_n})$ does not depend on n, and also, thanks to (B3), for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathbb{P} \otimes \mathbf{P}_0(\nu_{\mathbf{h}^n,\alpha}(n^{-1}[B]_{\tau_n}) \ge \delta) \ge 1 - \varepsilon$, for all $n \in \mathbb{N}$. In particular, thanks to the downward continuity of measures, we have that

$$\mathbb{P} \otimes \mathbf{P}_0\bigg(\limsup_{n\to\infty}\big(\nu_{\mathbf{h}^n,\alpha}(n^{-1}[B]_{\tau_n}\big) \ge \delta\bigg) \ge \limsup_{n\to\infty} \mathbb{P} \otimes \mathbf{P}_0\big(\nu_{\mathbf{h}^n,\alpha}(n^{-1}[B]_{\tau_n}\big) \ge \delta\big) \ge 1 - \varepsilon \,. \tag{6.4.12}$$

Hence, thanks to (6.4.11) and (6.4.12) and by arbitrariness of $\varepsilon > 0$, we have that

$$\mathbb{P} \otimes \mathbf{P}_0\Big(\lim_{n\to\infty} \mathrm{F}_{\mathbf{h},\alpha}(\tau_n) = \infty\Big) = 1 \,,$$

from which the desired result follows. □

We note that the continuity of $\mathrm{F}_{\mathbf{h},\alpha}$ ensures that the LBM does not become trapped in certain regions of $\mathbb{R}^d$. Additionally, the strict monotonicity of $\mathrm{F}_{\mathbf{h},\alpha}$ guarantees that the LBM has no discontinuities, and since $\mathrm{F}_{\mathbf{h},\alpha}$ approaches infinity as $t \to \infty$, the LBM has an infinite lifetime, i.e., it will never "reach infinity" in a finite amount of time.

In what follows, we will also require the following strengthened version of the fact that the measure $\mu_{\mathbf{h},\alpha}$ is the Revuz measure associated with the PCAF $\mathrm{F}_{\mathbf{h},\alpha}$.

Lemma 6.4.14. *Let $d > 2$ and $Q > \sqrt{2d}$. Then, $\mathbb{P}$-almost surely, for all $x \in \mathbb{R}^d$ and all Borel measurable functions $\eta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $f : \mathbb{R}^d \to \mathbb{R}_0^+$ the following equality holds*

$$\mathbf{E}_x\bigg[\int_0^\infty \eta(t) f(B_t) d\mathrm{F}_{\mathbf{h},\alpha}(t)\bigg] = \int_0^\infty \int_{\mathbb{R}^d} \eta(t) f(y) p_t(x,y) \mu_{\mathbf{h},\alpha}(dy) dt \,, \tag{6.4.13}$$

where $p_t(\cdot,\cdot)$ denotes the heat kernel of the d-dimensional Brownian motion (see (6.4.18) below).

Proof. The result is proved for the two-dimensional case in [AK16, Proposition 2.5]. Given the bound on the potential of the measure $\mu_{\mathbf{h},\alpha}$ provided in Lemma 6.4.8, the proof generalises verbatim to any arbitrary dimensions $d > 2$. □

We conclude this section with the following result, which is analogous to Theorem 6.4.12, but with the Brownian motion replaced by a Brownian bridge, and which will be useful for the study of the spectral dimension. We recall that we use the convention that for x, $y \in \mathbb{R}^d$ and $t > 0$, under $\mathbf{P}_{x,y,t}$, the process $(B_s)_{s\in[0,t]}$ is a d-dimensional Brownian bridge of length t from x to y.

Theorem 6.4.15. *Let $d > 2$ and $Q > \sqrt{2d}$. Then, $\mathbb{P}$-almost surely, for all x, $y \in \mathbb{R}^d$ and $t \ge 0$, the following properties hold:*

(D1) The following limit holds

$$\lim_{n\to\infty} \mathbf{P}_{x,y,t}\left(\sup_{s\leq t}|\mathrm{F}_{\mathbf{h},\alpha,(n)}(s) - \mathrm{F}_{\mathbf{h},\alpha}(s)| \geq \eta\right) = 0 \,.$$

(D2) It holds $\mathbf{P}_{x,y,t}$-almost surely that the map $[0,t] \ni s \mapsto \mathrm{F}_{\mathbf{h},\alpha}(s)$ is strictly increasing and continuous.

(D3) For each bounded continuous function $\mathscr{F} : \mathcal{C}([0,t]) \to \mathbb{R}$, the mapping $\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \to \mathbb{E}_{x,y,t}[\mathscr{F}(\mathrm{F}_{\mathbf{h},\alpha}(\cdot))]$ is continous.

Proof. As observed in [RV14b], the proof consists in reproducing the arguments of the proof of Theorem 6.4.12 with minor modifications. □

6.4.2.3 The Liouville Heat Kernel

From the construction of the LBM and from the general theory of time changes of Markov processes, we can deduce some of its properties. First of all, for $d > 2$ and $Q > \sqrt{2d}$, the LBM $\mathrm{B}_{\mathbf{h},\alpha}$ is a diffusion process, i.e., a continuous strong Markov process, and it is transient (see [CF12, Theorems A.2.12 and 6.2.3]). Furthermore, thanks to [CF12, Theorem 6.2.1 (i)], the LBM is $\mu_{\mathbf{h},\alpha}$-symmetric, i.e., its transition semigroup $(\mathrm{P}_{\mathbf{h},\alpha,t})_{t\geq 0}$ given by

$$\mathrm{P}_{\mathbf{h},\alpha,t}[f](x) \stackrel{\text{def}}{=} \mathbf{E}_x\left[f(\mathrm{B}_{\mathbf{h},\alpha,t})\right] \,, \qquad \forall\, f \in \mathcal{B}(\mathbb{R}^d,\mathbb{R}),\ \forall\, x \in \mathbb{R}^d$$

satisfies the following equality

$$\int_{\mathbb{R}^d} \mathrm{P}_{\mathbf{h},\alpha,t}[f](x)g(x)\mu_{\mathbf{h},\alpha}(dx) = \int_{\mathbb{R}^d} \mathrm{P}_{\mathbf{h},\alpha,t}[g](x)f(x)\mu_{\mathbf{h},\alpha}(dx) \,, \qquad \forall\, f,g \in \mathcal{B}(\mathbb{R}^d,\mathbb{R}) \,. \tag{6.4.14}$$

We are now ready to prove Proposition 6.1.5 on the absolute continuity of the semigroup $\mathrm{P}_{\mathbf{h},\alpha}$ with respect to the measure $\mu_{\mathbf{h},\alpha}$.

Proof of Proposition 6.1.5. We will not provide a detailed proof of this result since it is identical to the proof of [GRV14, Theorem 2.5]. We observe that, since the semigroup $\mathrm{P}_{\mathbf{h},\alpha}$ is symmetric with respect to $\mu_{\mathbf{h},\alpha}$, by [CF12, Theorem 4.2.4], it suffices to prove that the resolvent of the LBM $\mathrm{B}_{\mathbf{h},\alpha}$ is absolutely continuous with respect to $\mu_{\mathbf{h},\alpha}$. We recall that the resolvent family $(\mathrm{R}_{\mathbf{h},\alpha,\lambda})_{\lambda>0}$ associated to the LBM is defined as follows

$$\mathrm{R}_{\mathbf{h},\alpha,\lambda}[f](x) = \int_0^\infty e^{-\lambda t}\mathrm{P}_{\mathbf{h},\alpha,t}[f](x)dt \,, \qquad \forall\, \lambda > 0 \,, \quad \forall\, f \in \mathcal{B}(\mathbb{R}^d,\mathbb{R}) \,.$$

Therefore, it suffices to prove that $\mathbb{P}$-almost surely, for all $A \in \mathcal{B}(\mathbb{R}^d)$ such that $\mu_{\mathbf{h},\alpha}(A) = 0$, it holds that $\mathrm{R}_{\mathbf{h},\alpha,\lambda}[\mathbb{1}_A](x) = 0$ for all $x \in \mathbb{R}^d$ and $\lambda > 0$. To this end, thanks to (6.4.14), we have that

$$\lambda \int_{\mathbb{R}^d} \mathrm{R}_{\mathbf{h},\alpha,\lambda}[\mathbb{1}_A](x)\mu_{\mathbf{h},\alpha}(dx) = \mu_{\mathbf{h},\alpha}(A) \,.$$

Hence, if $\mu_{\mathbf{h},\alpha}(A) = 0$, it follows that $\mathrm{R}_{\mathbf{h},\alpha,\lambda}[\mathbb{1}_A](x) = 0$ for $\mu_{\mathbf{h},\alpha}$-almost every $x \in \mathbb{R}^d$. Since the measure $\mu_{\mathbf{h},\alpha}$ has full support, this implies the existence of a dense subset of $\mathbb{R}^d$ on which the function $\mathrm{R}_{\mathbf{h},\alpha,\lambda}[\mathbb{1}_A]$ is identically zero. Therefore, the conclusion holds if the mapping $\mathbb{R}^d \ni x \mapsto \mathrm{R}_{\mathbf{h},\alpha,\lambda}[\mathbb{1}_A](x)$ is continuous. By following the proof of [GRV14, Theorem 2.4], one can show that the resolvent operator $\mathrm{R}_{\mathbf{h},\alpha,\lambda}$ maps the set of measurable bounded functions into the set of continuous bounded functions. More precisely, we observe

that the proof of [GRV14, Theorem2.4] relies on Lemma 6.4.13 and the fact that, for any fixed $R > 0$, it holds $\mathbb{P}$-almost surely that

$$\lim_{t\to 0} \sup_{x\in B(0,R)} \mathbf{E}_x\big[\mathrm{F}_{\mathbf{h},\alpha}(t)\big] = 0 \,.$$

It is easy to see that the uniform limit above holds in any dimension $d > 2$. Indeed, fix $R > 0$, take any $x \in B(0,R)$, and let $\delta \in (0,1/2)$. Then, by Lemma 6.4.14, we can write

$$\mathbf{E}_x\big[\mathrm{F}_{\mathbf{h},\alpha}(t)\big] = \int_{B(x,t^{1/2-\delta})} \int_0^t p_s(x,y)ds\,\mu_{\mathbf{h},\alpha}(dy) + \int_{B(x,t^{1/2-\delta})^c} \int_0^t p_s(x,y)ds\,\mu_{\mathbf{h},\alpha}(dy)\,. \quad (6.4.15)$$

We now proceed to show that both terms on the right-hand side of the above expression converge to 0 as $t \to 0$, uniformly for all $x \in B(0,R)$. Regarding the first term, by Lemma 6.B.2 and following the same approach as in the proof of Proposition 6.4.10, we obtain $\mathbb{P}$-almost surely that for any $\varepsilon > 0$,

$$\begin{aligned}\int_{B(x,t^{1/2-\delta})} \int_0^t p_s(x,y)ds\mu_{\mathbf{h},\alpha}(dy) &\lesssim \int_{B(x,t^{1/2-\delta})} |x-y|^{2-d}\mu_{\mathbf{h},\alpha}(dy)\\ &\lesssim \sum_{k\geq \lfloor |\log t|(1/2-\delta)\rfloor} e^{-k(2+\alpha^2/2-\sqrt{2d}\alpha-\varepsilon)}\,,\end{aligned}$$

and we note that, for all $Q > \sqrt{2d}$, by choosing $\varepsilon > 0$ sufficiently small, the term on the right-hand side of the above expression converges to 0 as $t \to 0$. Hence, it remains to control the second term on the right-hand side of (6.4.15). To this end, recalling the explicit expression of the heat kernel for the standard d-dimensional Brownian motion and applying a change of variables, we obtain that for $|x-y| \geq t^{1/2-\delta}$, there exists a constant $c > 0$ such that

$$\int_0^t p_s(x,y)ds \lesssim |x-y|^{2-d} \int_{\frac{|x-y|^2}{t}}^\infty s^{\frac{d}{2}-2}e^{-\frac{s}{2}}ds \lesssim t^{(1/2-\delta)(2-d)}e^{-c\frac{|x-y|^2}{t}}\,. \quad (6.4.16)$$

Moreover, we have that

$$\begin{aligned}\int_{B(x,t^{1/2-\delta})^c} e^{-c\frac{|x-y|^2}{t}}\mu_{\mathbf{h},\alpha}(dy) &\leq e^{-ct^{-2\delta}}\mu_{\mathbf{h},\alpha}\big(B(x,2)\big) + \int_{B(x,2)^c} e^{-c\frac{|x-y|^2}{t}}\mu_{\mathbf{h},\alpha}(dy)\\ &\lesssim e^{-ct^{-2\delta}}\mu_{\mathbf{h},\alpha}\big(B(x,2)\big) + t^d\int_{\mathbb{R}^d} e^{-c|x-y|^2}\mu_{\mathbf{h},\alpha}(dy)\,. \quad (6.4.17)\end{aligned}$$

Therefore, combining (6.4.16) and (6.4.17), we have that

$$\begin{aligned}&\int_{B(x,t^{1/2-\delta})^c} \int_0^t p_s(x,y)ds\mu_{\mathbf{h},\alpha}(dy)\\ &\qquad\lesssim t^{(1/2-\delta)(2-d)}\left(e^{-ct^{-2\delta}}\mu_{\mathbf{h},\alpha}\big(B(x,2)\big) + t^d\int_{\mathbb{R}^d} e^{-c|x-y|^2}\mu_{\mathbf{h},\alpha}(dy)\right),\end{aligned}$$

from which the claim follows since the expression on the right-hand side of the display converges to 0 as $t \to 0$, $\mathbb{P}$-almost surely. □

6.4.3 Spectral dimension of the LBM

In this section, we prove Theorem 6.B, which concerns the spectral dimension of the LBM. In particular, we focus only on the case $d > 2$ and refer to [RV14b] for the proof of Theorem 6.B in the case $d = 2$.

We begin by noting that for $Q > \sqrt{2d}$ and $\beta \in (-\infty, Q)$, the results derived in the previous section remain valid when $\mathbf{h}$ is replaced by the field $\mathbf{h} - \beta \log|\cdot|$. For instance, one can easily check that the conclusion of Lemma 6.4.8 remains valid when $\mathbf{h}$ is replaced by $\mathbf{h} - \beta \log|\cdot|$, i.e., that for all $R > 0$ it holds $\mathbb{P}$-almost surely that $\sup_{x \in B(0,R)} g_R[\mu_{\mathbf{h}-\beta\log|\cdot|,\alpha}](x) < \infty$. Indeed, in the proof of Lemma 6.4.8, it suffices to use Lemma 6.A.3 in place of Lemma 6.A.2 to obtain the desired result. Verifying that the remaining results still hold with $\mathbf{h}$ replaced by $\mathbf{h} - \beta \log|\cdot|$ reduces to similar considerations. In particular, since we showed in the previous section that the PCAF associated with $\mu_{\mathbf{h},\alpha}$ is given by $\mathrm{F}_{\mathbf{h},\alpha}$, it follows immediately from the Revuz correspondence (see Definition 6.4.5) that the PCAF associated with the measure $\mu_{\mathbf{h}-\beta\log|\cdot|,\alpha}$ is given by $\mathrm{F}_{\mathbf{h}-\beta\log|\cdot|,\alpha}$.

To simplify the notation, for a fixed $\beta \in (-\infty, Q)$, we introduce the following field,

$$\overline{\mathbf{h}} \stackrel{\text{def}}{=} \mathbf{h} - \beta \log|\cdot| \, .$$

6.4.3.1 Some preliminary results and a formal scaling argument

We denote by $p_t(\cdot,\cdot)$ the heat kernel of the d-dimensional Brownian motion given by

$$p_t(x,y) \stackrel{\text{def}}{=} (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}} \, , \qquad \forall\, t \geq 0, \ \forall\, x, y \in \mathbb{R}^d \, . \tag{6.4.18}$$

In what follows, we recall that for $x, y \in \mathbb{R}^d$ and $t > 0$, under $\mathbf{P}_{x,y,t}$, the process $(B_s)_{s\in[0,t]}$ is a d-dimensional Brownian bridge of length t from x to y. We begin with the following key lemma.

Lemma 6.4.16. *Let $d > 2$, $Q > \sqrt{2d}$, and $\beta \in (-\infty, Q)$. Then, $\mathbb{P}$-almost surely, for all continuous functions $\eta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, for all $x \in \mathbb{R}^d$, and for $\mu_{\mathbf{h},\alpha}$-almost every $y \in \mathbb{R}^d$, the following equality holds*

$$\int_0^\infty \eta(t) \mathfrak{p}_{\overline{\mathbf{h}},\alpha,t}(x,y)dt = \int_0^\infty \mathbf{E}_{x,y,t}\big[\eta(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t))\big] p_t(x,y)dt \, .$$

Proof. The proof is similar to the proof of [RV14b, Theorem 3.4] and is included here for completeness. Let $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function, and let $x \in \mathbb{R}^d$. Then, thanks to Lemma 6.4.14 with $\mathbf{h}$ replaced by the field $\overline{\mathbf{h}}$, for all continuous functions $f : \mathbb{R}^d \to \mathbb{R}_0^+$, it holds that

$$\begin{aligned}
\int_{\mathbb{R}^d} \left(\int_0^\infty \eta(t) \mathfrak{p}_{\overline{\mathbf{h}},\alpha,t}(x,y)dt \right) f(y) \mu_{\overline{\mathbf{h}},\alpha}(dy) &= \int_0^\infty \eta(t) \left(\int_{\mathbb{R}^d} f(y) \mathfrak{p}_{\overline{\mathbf{h}},\alpha,t}(x,y) \mu_{\overline{\mathbf{h}},\alpha}(dy) \right) dt \\
&= \mathbf{E}_x \left[\int_0^\infty \eta(t) f(\mathrm{B}_{\overline{\mathbf{h}},\alpha,t}) dt \right] \\
&= \mathbf{E}_x \left[\int_0^\infty \eta\big(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t)\big) f(B_t) d\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t) \right] \\
&= \mathbf{E}_x \left[\int_0^\infty \mathbf{E}_x\big[\eta\big(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t)\big) \mid \sigma(B_t)\big] f(B_t) d\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t) \right] \\
&\stackrel{(6.4.13)}{=} \int_{\mathbb{R}^d} \left(\int_0^\infty \mathbf{E}_{x,y,t}\big[\eta\big(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t)\big)\big] p_t(x,y)dt \right) f(y) \mu_{\overline{\mathbf{h}},\alpha}(dy) \, ,
\end{aligned}$$

and so the claim follows. □

We emphasise that, for $\chi \geq 0$, by choosing $\mathbb{R}^+ \ni t \mapsto \eta(t) \overset{\text{def}}{=} t^\chi e^{-t}$ in Lemma 6.4.16, we obtain precisely the relation (6.1.13). We are now in a position to state the main result of this section, from which Theorem 6.B follows directly.

Proposition 6.4.17. *Let $d > 2$, $Q > \sqrt{2d}$, $\beta \in (-\infty, Q)$, and define*

$$\overline{\chi} = \overline{\chi}(\beta) \overset{\text{def}}{=} \frac{d-2}{2 + \alpha^2/2 - \alpha\beta} \,. \tag{6.4.19}$$

Then, it holds $\mathbb{P}$-almost surely that

$$\mathfrak{m}_{\overline{\mathbf{h}},\alpha,\chi}(0) = \infty \,, \qquad \forall \chi \in [0, \overline{\chi}) \qquad \textit{and} \qquad \mathfrak{m}_{\overline{\mathbf{h}},\alpha,\chi}(0) < \infty \,, \qquad \forall \chi > \overline{\chi} \,,$$

where we recall that $\mathfrak{m}_{\overline{\mathbf{h}},\alpha,\chi}$ is defined in (6.1.12).

Before proceeding with the rigorous proof of Proposition 6.4.17, we provide a formal scaling argument to explain where the exponent $\overline{\chi}$ comes from. Recalling the definition (6.1.12) of $\mathfrak{m}_{\overline{\mathbf{h}},\alpha,\chi}(0)$, we note that, for all $\chi \geq 0$, the mapping $[0,\infty) \ni u \mapsto u^\chi e^{-u}$ is uniformly bounded by a constant $c = c(\chi)$. In particular, since the heat kernel $p_t(0,0)$ is equal to a constant times $t^{-d/2}$, which is integrable over the unbounded interval $(1,\infty)$, the part of the integral in $\mathfrak{m}_{\overline{\mathbf{h}},\alpha,\chi}(0)$ that determines the spectral dimension is the behaviour of the following quantity,

$$\int_0^1 \mathbf{E}_{0,0,t}\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t))^\chi\big] p_t(0,0) dt \,. \tag{6.4.20}$$

By the scale invariance of the law of $\mathbf{h}$ modulo an additive constant, we have that $\mathbf{h}^{\sqrt{t}} \overset{\text{def}}{=} \mathbf{h}(\sqrt{t}\,\cdot) - \mathbf{h}_{\sqrt{t}}(0) \overset{\text{law}}{=} \mathbf{h}$. In particular, using the coordinate change formula for the measure $\nu_{\mathbf{h},\alpha}$ (Proposition 6.4.1), we obtain that

$$\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t) = \nu_{\overline{\mathbf{h}},\alpha}\big([B]_t\big) \overset{\mathbf{P}\text{-law}}{=} t^{\alpha(Q-\beta)/2 + o_t(1)} \nu_{\mathbf{h}^{\sqrt{t}} - \beta \log|\cdot|,\alpha}\big([B]_1\big) \,, \tag{6.4.21}$$

Therefore, plugging (6.4.21) into (6.4.20) and using the fact that the law of $t^{-1/2}[B]_t$ does not depend on t, we obtain that

$$\int_0^1 \mathbf{E}_{0,0,t}\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t))^\chi\big] p_t(0,0) dt \approx \int_0^1 t^{\chi\alpha(Q-\beta)/2 - d/2 + o_t(1)} dt \,.$$

Since $\alpha(Q-\beta) = 2 + \alpha^2/2 - \alpha\beta$, the integral on the right-hand side of the above expression converges (resp. diverges) if $\chi > \overline{\chi}$ (resp. $\chi \in [0,\overline{\chi})$). However, there are several reasons why this scaling argument is not rigorous. For instance, while it is true that the law of the field $\mathbf{h}^{\sqrt{t}}$ does not depend on t, this is insufficient for our purposes since we need statements that hold $\mathbb{P}$-almost surely, rather than just in distribution.

6.4.3.2 Computation of the spectral dimension

We now proceed with the rigorous proof of Proposition 6.4.17. The proof is divided into two parts: in Lemma 6.4.18, we address the case $\chi \in [0,\overline{\chi})$, and in Lemma 6.4.20, we handle the case $\chi > \overline{\chi}$.

Lemma 6.4.18. *Let $d > 2$, $Q > \sqrt{2d}$, $\beta \in (-\infty, Q)$, and let $\overline{\chi}$ be as defined in (6.4.19). Then, it holds $\mathbb{P}$-almost surely that*

$$\mathfrak{m}_{\overline{\mathbf{h}},\alpha,\chi}(0) = \infty \,, \qquad \forall \chi \in [0,\overline{\chi}) \,.$$

Proof. Let $d > 2$, $Q > \sqrt{2d}$, $\beta \in (-\infty, Q)$, and consider $\chi \in [0,\overline{\chi})$. We begin by observing that

$$\int_0^\infty \mathbf{E}_{0,0,t}\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t))^\chi e^{-\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t)}\big] p_t(0,0) dt \gtrsim \int_0^\infty \mathbf{E}_{0,0,t}\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t))^\chi \mathbb{1}_{\{\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t) \leq 1\}}\big] p_t(0,0) dt \,.$$

Next, for any $t > 0$, we introduce the fields

$$\mathbf{h}^{\sqrt{t}} \overset{\text{def}}{=} \mathbf{h}(\sqrt{t}\cdot) - \mathbf{h}_{\sqrt{t}}(0) \,, \qquad \overline{\mathbf{h}}^{\sqrt{t}} \overset{\text{def}}{=} \mathbf{h}^{\sqrt{t}} - \beta \log|\cdot| \,.$$

Recall that $\mathbf{h}_{\sqrt{t}}(0)$ is centred Gaussian with variance of order $|\log\sqrt{t}|$ (Lemma 6.2.9), so it holds $\mathbb{P}$-almost surely that $\mathbf{h}_{\sqrt{t}}(0)/|\log\sqrt{t}| = o_t(1)$ as $t \to 0$. By applying the coordinate change formula for $\mathrm{F}_{\mathbf{h},\alpha}$ (Proposition 6.4.1), it follows $\mathbb{P}$-almost surely that, for any sufficiently small $\bar{t} \in (0,1)$ and for all $t \in (0,\bar{t})$,

$$\begin{aligned}
&\mathbf{E}_{0,0,t}\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t))^{\chi}\mathbb{1}_{\{\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t)\le 1\}}\big] \\
&\qquad = t^{\alpha\chi(Q-\beta)/2}e^{\alpha\chi\mathbf{h}_{\sqrt{t}}(0)}\mathbf{E}_{0,0,1}\big[(\mathrm{F}_{\overline{\mathbf{h}}^{\sqrt{t}},\alpha}(1))^{\chi}\mathbb{1}_{\{\mathrm{F}_{\overline{\mathbf{h}}^{\sqrt{t}},\alpha}(1)t^{\alpha(Q-\beta)/2}e^{\alpha\mathbf{h}_{\sqrt{t}}(0)}\le 1\}}\big] \\
&\qquad \ge t^{\chi(2+\alpha^2/2-\alpha\beta)/2+o_t(1)}\underbrace{\mathbf{E}_{0,0,1}\big[(\mathrm{F}_{\overline{\mathbf{h}}^{\sqrt{t}},\alpha}(1))^{\chi}\mathbb{1}_{\{\mathrm{F}_{\overline{\mathbf{h}}^{\sqrt{t}},\alpha}(1)\le 1\}}\big]}_{\mathrm{Z}_t} \,,
\end{aligned}$$

where to get the inequality in the second line, we used that $\beta < Q$ and the fact that $\alpha(Q-\beta) = 2+\alpha^2/2-\alpha\beta$. Hence, it holds $\mathbb{P}$-almost surely that

$$\int_0^{\infty}\mathbf{E}_{0,0,t}\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t))^{\chi}e^{-\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t)}\big]p_t(0,0)dt \gtrsim \int_0^{\bar{t}} t^{\chi(2+\alpha^2/2-\alpha\beta)/2-d/2+o_t(1)}\mathrm{Z}_t dt \,. \tag{6.4.22}$$

We next observe that by the invariance of the law of $\mathbf{h}$ modulo an additive constant, the law of Z_t does not depend on $t > 0$. Moreover, for any $\varepsilon \in (0,1)$, there exists $s_\varepsilon > 0$ such that $\mathbb{P}(\mathrm{Z}_t \ge s_\varepsilon) \ge 1-\varepsilon^2$. Then, by Markov's inequality, for all $r \in (0,1)$,

$$\mathbb{P}\left(\int_0^r \mathbb{1}_{\{\mathrm{Z}_t < s_\varepsilon\}}dt > \varepsilon r\right) \le \frac{1}{\varepsilon r}\int_0^r \mathbb{P}(\mathrm{Z}_t < s_\varepsilon)dt < \varepsilon \,.$$

By downward continuity of measures, it follows that

$$\mathbb{P}\left(\limsup_{n\to\infty}\left(\int_0^{\frac{1}{n}}\mathbb{1}_{\{\mathrm{Z}_t\ge s_\varepsilon\}}dt \ge \frac{1-\varepsilon}{n}\right)\right) \ge \limsup_{n\to\infty}\mathbb{P}\left(\int_0^{\frac{1}{n}}\mathbb{1}_{\{\mathrm{Z}_t\ge s_\varepsilon\}}dt \ge \frac{1-\varepsilon}{n}\right) \ge 1-\varepsilon \,.$$

Hence, with $\mathbb{P}$-probability at least $1-\varepsilon$, there are infinitely many $n \in \mathbb{N}$ such that

$$\int_0^{\frac{1}{n}}\mathbb{1}_{\{\mathrm{Z}_t\ge s_\varepsilon\}}dt \ge \frac{1-\varepsilon}{n} \,.$$

On that event, by choosing $n \in \mathbb{N}$ large enough so that $1/n < \bar{t}$, the right-hand side of (6.4.22) is bounded below by

$$\int_0^{\frac{1}{n}} t^{\chi(2+\alpha^2/2-\alpha\beta)/2-d/2+o_t(1)}\mathrm{Z}_t dt \ge s_\varepsilon\int_{\frac{\varepsilon}{n}}^{\frac{1}{n}} t^{\chi(2+\alpha^2/2-\alpha\beta)/2-d/2+o_t(1)}dt \to \infty \,, \qquad \text{as } n\to\infty \,,$$

where we used the fact that $-\chi(2+\alpha^2/2-\alpha\beta)/2+d/2-1 > 0$ for all $\chi \in [0,\overline{\chi})$. □

Next, we address the case $\chi > \overline{\chi}$. A crucial element in the proof of this case is the following result, which is proved in Appendix 6.B.

Lemma 6.4.19. *Let $d > 2$, $Q > \sqrt{2d}$, and $\beta \in (-\infty, Q)$. Then, it holds $\mathbb{P}$-almost surely that for any $n \in \mathbb{N}$ and for all $t \in (0,1]$,*

$$\mathbf{E}_0\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t))^n \mathbb{1}_{\{[B]_t \subseteq B(0,t^{1/2-\delta})\}}\big] \lesssim t^{n[(1/2-\delta)(2+\alpha^2/2-\alpha\beta)-\varepsilon]} , \qquad \forall\, \varepsilon,\ \delta > 0 , \tag{6.4.23}$$

for some random implicit constant.

Lemma 6.4.20. *Let $d > 2$, $Q > \sqrt{2d}$, $\beta \in (-\infty, Q)$, and let $\overline{\chi}$ be as defined in (6.4.19). Then, it holds $\mathbb{P}$-almost surely that*

$$\mathfrak{m}_{\overline{\mathbf{h}},\alpha,\chi}(0) < \infty , \qquad \forall \chi > \overline{\chi} .$$

Proof. Let $d > 2$, $Q > \sqrt{2d}$, $\beta \in (-\infty, Q)$, and consider $\chi > \overline{\chi}$. Since the function $[0,\infty) \ni u \mapsto u^\chi e^{-u}$ is uniformly bounded by a constant depending only on χ, it holds that

$$\int_1^\infty \mathbf{E}_{0,0,t}\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t))^\chi e^{-\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t)}\big] p_t(0,0)dt \lesssim \int_1^\infty t^{-d/2}dt < \infty .$$

Therefore, we can restrict our attention to the following quantity

$$\int_0^1 \mathbf{E}_{0,0,t}\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t))^\chi e^{-\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t)}\big] p_t(0,0)dt . \tag{6.4.24}$$

By the time-reversal symmetry of the Brownian bridge, the fact that for all $\chi > 0$ and $a, b > 0$ it holds that $(a+b)^\chi \lesssim a^\chi + b^\chi$ for some implicit constant depending only on χ, and the monotonicity of the mapping $\mathbb{R}_0^+ \ni t \mapsto \mathrm{F}_{\overline{\mathbf{h}},\alpha}(t)$, one obtains $\mathbb{P}$-almost surely that the term in (6.4.24) can be bounded above by a constant multiple of the following sum,

$$\int_0^1 \mathbf{E}_{0,0,t}\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t/2))^\chi e^{-\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t/2)}\big] p_t(0,0)dt + \int_0^1 \mathbf{E}_{0,0,t}\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t/2))^\chi e^{-\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t/2)}\big] p_t(0,0)dt ,$$

where we note that the two terms in the above display are equal. Next, we recall that for any $x, y \in \mathbb{R}^d$ and $s \in [0,t)$, it holds that

$$\frac{d\mathbf{P}_{x,y,t}}{d\mathbf{P}_x}|_{\mathbf{F}_s} = \left(\frac{t}{t-s}\right)^{d/2} \exp\left(\frac{|x-y|^2}{2t} - \frac{|B_s - y|^2}{2(t-s)}\right) ,$$

Therefore, invoking this absolute continuity result, we obtain that

$$\int_0^1 \mathbf{E}_{0,0,t}\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t/2))^\chi e^{-\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t/2)}\big] p_t(0,0)dt \lesssim \int_0^1 \mathbf{E}_0\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t/2))^\chi e^{-\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t/2)}\big] t^{-d/2}dt . \tag{6.4.25}$$

For any $\delta > 0$, using again the fact that for each $\chi > 0$, the function $\mathbb{R}_0^+ \ni u \mapsto u^\chi e^{-u}$ is uniformly bounded by a constant depending only on χ, we observe that the term on the right-hand side of (6.4.25) can be estimated from above by a constant multiple of

$$\int_0^1 \mathbf{E}_0\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t/2))^\chi \mathbb{1}_{\{[B]_{t/2} \subseteq B(0,(t/2)^{1/2-\delta})\}}\big] t^{-d/2}dt + \int_0^1 \mathbb{P}_0\big([B]_{t/2} \not\subseteq B(0,(t/2)^{1/2-\delta})\big) t^{-d/2}dt . \tag{6.4.26}$$

The second term in (6.4.26) is easily controlled since, by the reflection principle, for each $\delta > 0$ and some $c > 0$,

$$\mathbb{P}_0\big([B]_{t/2} \not\subseteq B(0,(t/2)^{1/2-\delta})\big) \lesssim e^{-ct^{-2\delta}} .$$

Hence, plugging this estimate into the second term in (6.4.26), we readily get that it is finite for any $\delta > 0$. For the first term in (6.4.26), let $n \in \mathbb{N}$ be such that $\chi \in [n-1, n)$. By concavity of the map $\mathbb{R}_0^+ \ni u \mapsto u^{\chi/n}$ and Jensen's inequality,

$$\mathbf{E}_0\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t/2))^{\chi}\mathbb{1}_{\{[B]_{t/2}\subseteq B(0,(t/2)^{1/2-\delta})\}}\big] \leq \mathbf{E}_0\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t/2))^{n}\mathbb{1}_{\{[B]_{t/2}\subseteq B(0,(t/2)^{1/2-\delta})\}}\big]^{\chi/n} . \quad (6.4.27)$$

Invoking Lemma 6.4.19, for any $\varepsilon > 0$, we find $\mathbb{P}$-almost surely that

$$\mathbf{E}_0\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t/2))^{\chi}\mathbb{1}_{\{[B]_{t/2}\subseteq B(0,(t/2)^{1/2-\delta})\}}\big] \lesssim t^{\chi[(1/2-\delta)(2+\alpha^2/2-\alpha\beta)-\varepsilon]} .$$

In particular, for any $\chi > \overline{\chi}$ one has $\chi(2+\alpha^2-\alpha\beta)/2 - d/2 > -1$. Hence, by choosing $\varepsilon, \delta > 0$ small enough (depending on χ, α, β), we obtain that also the first term in (6.4.26) is finite. Therefore, this concludes the proof. □

Proof of Proposition 6.4.17. The proof follows directly by combining Lemma 6.4.18 with Lemma 6.4.20. □

6.5 Quantum cones in arbitrary even dimension

In this section, we prove the results related to the quantum cone. The proof of Theorem 6.C is divided between Sections 6.5.1 and 6.5.2. In Section 6.5.1, we establish the tightness of the collection of processes $(\mathrm{S}_{b,\cdot})_{b>0}$ defined in (6.1.19). Then, in Section 6.5.2, we prove the convergence in total variation distance. In Section 6.5.3, we prove Theorem 6.D, and finally, Section 6.5.4 is dedicated to the proof of Theorem 6.E.

6.5.1 Tightness of the recentred process

Fix $d > 2$ even, $\gamma \in (0, \sqrt{2d})$, and $\beta \in (-\infty, Q)$, where we recall that Q is defined as in (6.1.6) and satisfies $Q > \sqrt{2d}$. Recalling that the collection of processes $(\mathrm{S}_{b,\cdot})_{b>0}$ is defined in (6.1.19), the main goal of this section is to prove the following tightness result.

Proposition 6.5.1. *For $d > 2$ even, the collection of processes $(\mathrm{S}_{b,\cdot})_{b>0}$ is tight in $\mathcal{C}^{\mathfrak{c}_d}_{\mathrm{loc}}(\mathbb{R})$ equipped with the local uniform metric.*

Remark 6.5.2. In dimension $d = 2$, the circle average process of the whole-plane GFF is a two-sided Brownian motion. In this case, it follows from [DMS21, Proposition 4.13] that the collection of processes $(\mathrm{S}_{b,\cdot})_{b>0}$ converges weakly as $b \to \infty$ to a process $(\widetilde{B}_t)_{t\in\mathbb{R}}$, defined as follows. Let $(B^1_t)_{t\geq 0}$ be a standard Brownian motion, and let $(B^2_t)_{t\geq 0}$ be a Brownian motion conditioned to stay above the line $t \mapsto -(Q-\beta)t$ for all $t > 0$. Then, we let

$$\widetilde{B}_t \overset{\text{def}}{=} \begin{cases} B^1_t , & \forall t \geq 0 , \\ B^2_{-t} , & \forall t < 0 . \end{cases} \quad (6.5.1)$$

The proof of Proposition 6.5.1 relies on the observation that the spherical average process can be viewed, in a loose sense, as a smoothed version of a two-sided Brownian motion (see Theorem 6.A and Figure 6.1). In particular, to prove Proposition 6.5.1, we first introduce a family of processes $(\widetilde{\mathrm{S}}_{b,\cdot})_{b>0}$ for which we can prove tightness by leveraging a suitable invariance in law. We then show that the tightness of $(\widetilde{\mathrm{S}}_{b,\cdot})_{b>0}$ implies the tightness of $(\mathrm{S}_{b,\cdot})_{b>0}$.

We introduce here some notation that will be used throughout this section. Recalling the representation (6.1.4) of the spherical average process, we introduce the function $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(s) \overset{\text{def}}{=} (d-2)e^{2s}(1-e^{2s})^{\frac{d-4}{2}} , \qquad \forall\, s \in \mathbb{R} .$$

We define the process $(\widetilde{\mathrm{S}}_t)_{t\in\mathbb{R}}$ by letting

$$\widetilde{\mathrm{S}}_t \overset{\text{def}}{=} \int_{-\infty}^{0} g(s)\big(\widetilde{B}_{s+t} - \widetilde{B}_s\big)ds , \qquad \forall\, t \in \mathbb{R} , \tag{6.5.2}$$

where $(\widetilde{B}_t)_{t\in\mathbb{R}}$ is defined as in (6.5.1). Furthermore, for $i \in [\mathfrak{c}_d]$, let $(\widetilde{\mathrm{S}}_t^{(i)})_{t\in\mathbb{R}}$ denote the i-th derivative of $(\widetilde{\mathrm{S}}_t)_{t\in\mathbb{R}}$. A straightforward computation yields that

$$\begin{aligned} \widetilde{\mathrm{S}}_t^{(i)} &= (-1)^i \int_{-\infty}^{0} g^{(i)}(s)\widetilde{B}_{s+t}ds , && \forall\, t \in \mathbb{R} ,\ \forall\, i \in [\mathfrak{c}_d - 1] , \\ \widetilde{\mathrm{S}}_t^{(\mathfrak{c}_d)} &= (-1)^{\mathfrak{c}_d+1} g^{(\mathfrak{c}_d-1)}(0)\widetilde{B}_t + (-1)^{\mathfrak{c}_d} \int_{-\infty}^{0} g^{(\mathfrak{c}_d)}(s)\widetilde{B}_{s+t}ds , && \forall\, t \in \mathbb{R} . \end{aligned} \tag{6.5.3}$$

For $b > 0$, we introduce the stopping times $\widetilde{\tau}_b$ and $\widetilde{\sigma}_b$ by letting

$$\widetilde{\tau}_b \overset{\text{def}}{=} \inf\big\{t \in \mathbb{R} : \widetilde{B}_t - (Q-\beta)t = -b\big\} , \qquad \widetilde{\sigma}_b \overset{\text{def}}{=} \inf\big\{t \in \mathbb{R} : \widetilde{\mathrm{S}}_t - (Q-\beta)t = -b\big\} . \tag{6.5.4}$$

For each $b > 0$ and $i \in [\mathfrak{c}_d]$, we also define recentred process $(\widetilde{\mathrm{S}}_{b,t})_{t\in\mathbb{R}}$ and its i-th derivative $(\widetilde{\mathrm{S}}_{b,t}^{(i)})_{t\in\mathbb{R}}$ by letting

$$\widetilde{\mathrm{S}}_{b,t} \overset{\text{def}}{=} \widetilde{\mathrm{S}}_{\widetilde{\sigma}_b+t} - \widetilde{\mathrm{S}}_{\widetilde{\sigma}_b} , \qquad \widetilde{\mathrm{S}}_{b,t}^{(i)} \overset{\text{def}}{=} \widetilde{\mathrm{S}}_{\widetilde{\sigma}_b+t}^{(i)} , \qquad \forall\, t \in \mathbb{R} . \tag{6.5.5}$$

The main reason why it is convenient to work with the process $(\widetilde{B}_t)_{t\in\mathbb{R}}$ is that it is invariant under the following random recentring

$$\big(\widetilde{B}_{\widetilde{\tau}_b+t} - \widetilde{B}_{\widetilde{\tau}_b}\big)_{t\in\mathbb{R}} \overset{\text{law}}{=} \big(\widetilde{B}_t\big)_{t\in\mathbb{R}} . \tag{6.5.6}$$

The invariance in law (6.5.6) follows directly from the definition of the process $(\widetilde{B}_t)_{t\in\mathbb{R}}$ or from [DMS21, Proposition 4.13]. Moreover, from the definition (6.5.2) of $(\widetilde{\mathrm{S}}_t)_{t\in\mathbb{R}}$ and from (6.5.3), it is readily seen that (6.5.6) implies that

$$\big(\widetilde{\mathrm{S}}_{\widetilde{\tau}_b+t}^{(i)} - \widetilde{\mathrm{S}}_{\widetilde{\tau}_b}^{(i)}\big)_{t\in\mathbb{R}} \overset{\text{law}}{=} \big(\widetilde{\mathrm{S}}_t^{(i)} - \widetilde{\mathrm{S}}_0^{(i)}\big)_{t\in\mathbb{R}} , \qquad \forall\, i \in [\mathfrak{c}_d]_0 , \tag{6.5.7}$$

where here and in what follows, with a slight abuse of notation, we use the superscript (0) to denote the process itself.

For b, $c > 0$, we define the moduli of continuity $\omega_{b,c}$, $\widetilde{\omega}_{b,c} : [0,1] \to \mathbb{R}_0^+$ of $(\mathrm{S}_{b,t})_{t\in[-c,c]}$ and $(\widetilde{\mathrm{S}}_{b,t})_{t\in[-c,c]}$, respectively, by letting

$$\omega_{b,c}(\delta) \overset{\text{def}}{=} \sum_{i=0}^{\mathfrak{c}_d} \sup_{s,t\in[-c,c],\,|t-s|\le\delta} |\mathrm{S}_{b,t}^{(i)} - \mathrm{S}_{b,s}^{(i)}| , \qquad \widetilde{\omega}_{b,c}(\delta) \overset{\text{def}}{=} \sum_{i=0}^{\mathfrak{c}_d} \sup_{s,t\in[-c,c],\,|t-s|\le\delta} |\widetilde{\mathrm{S}}_{b,t}^{(i)} - \widetilde{\mathrm{S}}_{b,s}^{(i)}| . \tag{6.5.8}$$

Similarly, for $c > 0$, we define the modulus of continuity $\widetilde{\omega}_c : [0,1] \to \mathbb{R}_0^+$ of $(\widetilde{\mathrm{S}}_t)_{t\in[-c,c]}$ by letting

$$\widetilde{\omega}_c(\delta) \overset{\text{def}}{=} \sum_{i=0}^{\mathfrak{c}_d} \sup_{s,t\in[-c,c],\,|t-s|\le\delta} |\widetilde{\mathrm{S}}_t^{(i)} - \widetilde{\mathrm{S}}_s^{(i)}| . \tag{6.5.9}$$

6.5.1.1 The collection $(\widetilde{\mathrm{S}}_{b,\cdot})_{b>0}$ is tight

We now proceed to show that the collection of processes $(\widetilde{\mathrm{S}}_{b,\cdot})_{b>0}$ is tight.

Lemma 6.5.3. *For $d > 2$ even, the collection of processes $(\widetilde{\mathrm{S}}_{b,\cdot})_{b>0}$ is tight in $\mathcal{C}^{\mathfrak{c}_d}_{\mathrm{loc}}(\mathbb{R})$ equipped with the local uniform metric.*

The proof of Lemma 6.5.3 relies on the fact that, with high probability and uniformly for $b > 0$, $\widetilde{\tau}_b$ and $\widetilde{\sigma}_b$ are close to each other.

Lemma 6.5.4. *For $b > 0$ let $\widetilde{\sigma}_b$ and $\widetilde{\tau}_b$ as defined in (6.5.4). Then, it holds that*

$$\lim_{\lambda\to\infty} \sup_{b>0} \mathbb{P}\big(|\widetilde{\sigma}_b - \widetilde{\tau}_b| \geq \lambda\big) = 0 \,.$$

Proof. For $\lambda \geq 0$, we have the following trivial inclusion

$$\big\{|\widetilde{\sigma}_b - \widetilde{\tau}_b| \geq \lambda\big\} \subseteq \big\{\widetilde{\sigma}_b - \widetilde{\tau}_b \geq \lambda\big\} \cup \big\{\widetilde{\tau}_b - \widetilde{\sigma}_b \geq \lambda\big\} \,,$$

and so we can focus on finding suitable upper bounds for the probabilities of the two events on the right-hand side of the above display.

Step 1: We start by bounding the probability of the event $\{\widetilde{\sigma}_b - \widetilde{\tau}_b \geq \lambda\}$. To this end, for any $c > 0$, we note that

$$\big\{\widetilde{\sigma}_b - \widetilde{\tau}_b \geq \lambda\big\} \subseteq \big\{|\widetilde{\mathrm{S}}_{\widetilde{\tau}_{b+c}} - \widetilde{B}_{\widetilde{\tau}_{b+c}}| \geq c\big\} \cup \big\{\widetilde{\tau}_{b+c} - \widetilde{\tau}_b \geq \lambda\big\} \,. \tag{6.5.10}$$

To see why this is true, it is more convenient to consider the complement of the given relation and prove that the intersection of the complements of the two events on the right-hand side is contained within the complement of the event on the left-hand side. Indeed, noting that

$$|\widetilde{\mathrm{S}}_{\widetilde{\tau}_{b+c}} - \widetilde{B}_{\widetilde{\tau}_{b+c}}| < c \quad \Longrightarrow \quad \widetilde{\sigma}_b < \widetilde{\tau}_{b+c} \,,$$

and combining this fact with the complement of the second event on the right-hand side of (6.5.10), we readily obtain that it must hold that $\widetilde{\sigma}_b - \widetilde{\tau}_b < \lambda$.

Hence, our task now is to bound the two probabilities appearing on the right-hand side of (6.5.10). By recalling the definition (6.5.2) of $(\widetilde{\mathrm{S}}_t)_{t\in\mathbb{R}}$, we can write

$$|\widetilde{\mathrm{S}}_{\widetilde{\tau}_{b+c}} - \widetilde{B}_{\widetilde{\tau}_{b+c}}| = \left| \int_{-\infty}^{0} g(s)(\widetilde{B}_{s+\widetilde{\tau}_{b+c}} - \widetilde{B}_{\widetilde{\tau}_{b+c}})ds - \int_{-\infty}^{0} g(s)\widetilde{B}_s ds \right| .$$

Therefore, thanks to the triangle inequality, the union bound, and the invariance in law (6.5.6), we have that

$$\mathbb{P}\big(|\widetilde{\mathrm{S}}_{\tau_{b+c}} - \widetilde{B}_{\tau_{b+c}}| \geq c\big) \lesssim \mathbb{P}\left(\left| \int_{-\infty}^{0} g(s)\widetilde{B}_s ds \right| \geq \frac{c}{2}\right) . \tag{6.5.11}$$

Moreover, using the strong Markov property of the Brownian motion, we have that

$$\mathbb{P}\big(\widetilde{\tau}_{b+c} - \widetilde{\tau}_b \geq \lambda\big) = \mathbb{P}\left(\inf_{s\in[0,\lambda]} \big(\widetilde{B}_s - (Q-\beta)s\big) \geq -c\right) \leq \mathbb{P}\big(\widetilde{B}_\lambda \geq (Q-\beta)\lambda - c\big) \lesssim e^{-a((Q-\beta)\lambda - c)^2/\lambda} \,, \tag{6.5.12}$$

for some constant $a > 0$. Therefore, choosing $c = c(\lambda) = \sqrt{\lambda}$, using the union bound, and estimates (6.5.11), (6.5.12), we readily get that

$$\lim_{\lambda\to\infty} \sup_{b>0} \mathbb{P}\big(\widetilde{\sigma}_b - \widetilde{\tau}_b \geq \lambda\big) = 0 \, .$$

Step 2: We now have to bound the probability of the event $\{\widetilde{\tau}_b - \widetilde{\sigma}_b \geq \lambda\}$. To this end, we note that, for any $c > 0$, it holds that

$$\big\{\widetilde{\tau}_b - \widetilde{\sigma}_b \geq \lambda\big\} \subseteq \Big\{\inf_{t\leq \widetilde{\tau}_{b-c}} \widetilde{\mathrm{S}}_t - (Q-\beta)t \leq -b\Big\} \cup \big\{\widetilde{\tau}_b - \widetilde{\tau}_{b-c} \geq \lambda\big\} \, . \tag{6.5.13}$$

Also here, to see why this is true, it is more convenient to consider the complement of the given relation. Indeed, noting that

$$\inf_{t\leq \widetilde{\tau}_{b-c}} \widetilde{\mathrm{S}}_t - (Q-\beta)t > -b \quad \Longrightarrow \quad \widetilde{\sigma}_b > \widetilde{\tau}_{b+c} \, ,$$

and combining this fact with the complement of the second event on the right-hand side of (6.5.13), we readily obtain that it must hold that $\widetilde{\tau}_b - \widetilde{\sigma}_b < \lambda$.

Now, writing $\widetilde{\tau}$ as a shorthand for $\widetilde{\tau}_{b-c}$, we note that the first event on the right-hand side of (6.5.13) is contained in the following union of events

$$\underbrace{\bigcup_{n\in\mathbb{N}} \Big\{ \sup_{t\in[\widetilde{\tau}-n,\widetilde{\tau}-(n-1)]} |\widetilde{\mathrm{S}}_t - \widetilde{B}_t| \geq (\log n)^2 + c/2 \Big\}}_{\mathrm{E}_1}$$
$$\cup \underbrace{\bigcup_{n\in\mathbb{N}} \Big\{ \inf_{t\in[\widetilde{\tau}-n,\widetilde{\tau}-(n-1)]} (\widetilde{B}_t - (Q-\beta)t) < (\log n)^2 - (b - c/2) \Big\}}_{\mathrm{E}_2} \, .$$

We now proceed to bound the probabilities of the events E_1 and E_2. We start by noting that, for any $n \in \mathbb{N}$, it holds that

$$\sup_{t\in[\widetilde{\tau}-n,\widetilde{\tau}-(n-1)]} |\widetilde{\mathrm{S}}_t - \widetilde{B}_t| = \sup_{t\in[-n,-(n-1)]} \Big| \int_{-\infty}^{0} g(s)(\widetilde{B}_{s+t+\widetilde{\tau}} - \widetilde{B}_{\widetilde{\tau}+t})ds - \int_{-\infty}^{0} g(s)\widetilde{B}_s ds \Big| \, .$$

By using the invariance in law (6.5.6), it holds that

$$\int_{-\infty}^{0} g(s)(\widetilde{B}_{s+t+\widetilde{\tau}} - \widetilde{B}_{\widetilde{\tau}+t})ds \overset{\text{law}}{=} \int_{-\infty}^{0} g(s)(\widetilde{B}_{s+t} - \widetilde{B}_t)ds \, , \qquad \forall\, t \in \mathbb{R} \, .$$

Therefore, applying the triangle inequality and the union bound, we obtain that the probability of the event E_1 is bounded from above, up to a finite multiplicative constant, by

$$\sum_{n\in\mathbb{N}} \mathbb{P}\Big(\sup_{t\in[-n,-(n-1)]} \Big| \int_{-\infty}^{0} g(s)(\widetilde{B}_{s+t} - \widetilde{B}_t)ds \Big| \geq (\log n)^2/2 + c/4 \Big) \lesssim \sum_{n\in\mathbb{N}} e^{-a((\log n)^2 + c)} \, , \tag{6.5.14}$$

for some constant $a > 0$ and where the implicit constant is independent of everything else. The upper bound on the right-hand side of (6.5.14) follows from the Gaussian tail bound.

On the other hand, using once again the invariance in law (6.5.6) and recalling that $\widetilde{\tau}$ is a shorthand for $\widetilde{\tau}_{b-c}$, for any $n \in \mathbb{N}$, we have that

$$\inf_{t\in[\widetilde{\tau}-n,\widetilde{\tau}-(n-1)]}(\widetilde{B}_t-(Q-\beta)t) \le (\log n)^2-(b-c/2) \overset{\text{law}}{=} \inf_{t\in[-n,-(n-1)]}(\widetilde{B}_t-(Q-\beta)t) \le (\log n)^2-c/2\,.$$

In particular, using the union bound, we obtain that the probability of the event E_2 is bounded from above, up to a finite multiplicative constant, by

$$\sum_{n\in\mathbb{N}} \mathbb{P}\left(\inf_{t\in[-n,-(n-1)]} \widetilde{B}_t \le -(Q-\beta)(n-1)+(\log n)^2-c/2\right) \lesssim \sum_{n\in\mathbb{N}} e^{-a(n+c)}\,, \tag{6.5.15}$$

for some constant $a > 0$ and where the implicit constant is independent of everything else. Also here, the upper bound on the right-hand side of (6.5.15) follows from the Gaussian tail bound.

Furthermore, proceeding exactly as in (6.5.12), we have that

$$\mathbb{P}\big(\widetilde{\tau}_b-\widetilde{\tau}_{b-c}\ge\lambda\big) \lesssim e^{-a((Q-\beta)\lambda-c)^2/\lambda}\,, \tag{6.5.16}$$

for some constant $a > 0$. Finally, the conclusion follows thanks to estimates (6.5.14), (6.5.15), and (6.5.16) by choosing $c = c(\lambda) = \sqrt{\lambda}$. □

We are now ready to prove Lemma 6.5.3.

Proof of Lemma 6.5.3. By recalling the definitions in (6.5.8) and (6.5.9), and applying the Arzelà–Ascoli theorem, to prove the tightness of $(\widetilde{\mathrm{S}}_{b,\cdot})_{b>0}$ in $\mathcal{C}^{\mathfrak{c}_d}_{\mathrm{loc}}(\mathbb{R})$ equipped with the local uniform metric, it suffices to verify that for all $c > 0$ and any $\varepsilon > 0$, it holds that

$$\lim_{\delta\to0}\limsup_{b\to\infty}\mathbb{P}\big(\widetilde{\omega}_{b,c}(\delta)>\varepsilon\big)=0\,.$$

To this end, we note that for $c > 0$ and $\widetilde{c} > c$, on the event $\{[\widetilde{\sigma}_b - c, \widetilde{\sigma}_b + c] \subseteq [\widetilde{\tau}_b - \widetilde{c}, \widetilde{\tau}_b + \widetilde{c}]\}$, it holds that

$$\begin{aligned}\widetilde{\omega}_{b,c}(\delta) &= \sum_{i=0}^{\mathfrak{c}_d}\sup_{s,t\in[\widetilde{\sigma}_b-c,\widetilde{\sigma}_b+c],\,|t-s|\le\delta}|\widetilde{\mathrm{S}}^{(i)}_t-\widetilde{\mathrm{S}}^{(i)}_s| \le \sum_{i=0}^{\mathfrak{c}_d}\sup_{s,t\in[\widetilde{\tau}_b-\widetilde{c},\widetilde{\tau}_b+\widetilde{c}],\,|t-s|\le\delta}|\widetilde{\mathrm{S}}^{(i)}_t-\widetilde{\mathrm{S}}^{(i)}_s| \\ &= \sum_{i=0}^{\mathfrak{c}_d}\sup_{s,t\in[-\widetilde{c},\widetilde{c}],\,|t-s|\le\delta}|\widetilde{\mathrm{S}}^{(i)}_{\widetilde{\tau}_b+t}-\widetilde{\mathrm{S}}^{(i)}_{\widetilde{\tau}_b+s}| \overset{\text{law}}{=} \sum_{i=0}^{\mathfrak{c}_d}\sup_{s,t\in[-\widetilde{c},\widetilde{c}],\,|t-s|\le\delta}|\widetilde{\mathrm{S}}^{(i)}_t-\widetilde{\mathrm{S}}^{(i)}_s|\,,\end{aligned}$$

where, in order to get the last equality, we used the invariance in law (6.5.7). Therefore, this fact implies that, for any $\varepsilon > 0$,

$$\mathbb{P}\big(\widetilde{\omega}_{b,c}(\delta)>\varepsilon\big) \le \mathbb{P}\big(\widetilde{\omega}_{\widetilde{c}}(\delta)>\varepsilon\big)+\mathbb{P}\big([\widetilde{\sigma}_b-c,\widetilde{\sigma}_b+c]\not\subseteq[\widetilde{\tau}_b-\widetilde{c},\widetilde{\tau}_b+\widetilde{c}]\big)\,. \tag{6.5.17}$$

The conclusion follows by first taking $\limsup_{b\to\infty}$, then $\lim_{\delta\to0}$, and finally $\limsup_{\widetilde{c}\to\infty}$ in the above expression. Indeed, since for any fixed $\widetilde{c} > 0$ it holds that $\lim_{\delta\to0}\mathbb{P}(\widetilde{\omega}_{\widetilde{c}}(\delta) > \varepsilon) = 0$, it suffices to apply Lemma 6.5.4 to obtain the desired result. □

6.5.1.2 The collection $(\mathrm{S}_{b,\cdot})_{b>0}$ is tight

We now prove that tightness of $(\widetilde{\mathrm{S}}_{b,\cdot})_{b>0}$ implies tightness of $(\mathrm{S}_{b,\cdot})_{b>0}$, from which Proposition 6.5.1 follows directly. We start by proving that the stopping times σ_b and $\widetilde{\sigma}_b$ are close to each other.

Lemma 6.5.5. *For $b > 0$ let σ_b and $\widetilde{\sigma}_b$ as defined in (6.1.18) and in (6.5.4), respectively. Then, it holds that*

$$\lim_{\lambda\to\infty} \limsup_{b\to\infty} \mathbb{P}\big(|\sigma_b - \widetilde{\sigma}_b| \geq \lambda\big) = 0 \,.$$

Proof. For $\lambda \geq 0$, we have the following trivial inclusion

$$\big\{|\sigma_b - \widetilde{\sigma}_b| \geq \lambda\big\} \subseteq \big\{\sigma_b - \widetilde{\sigma}_b \geq \lambda\big\} \cup \big\{\widetilde{\sigma}_b - \sigma_b \geq \lambda\big\} \,,$$

and so we can focus on finding suitable upper bounds for the probabilities of the two events on the right-hand side of the above display. Throughout the proof we can assume that that the two-sided Brownian motion $(B_t)_{t\in\mathbb{R}}$ used in the construction of the spherical average process $(\mathrm{S}_t)_{t\in\mathbb{R}}$ and the process $(\widetilde{B}_t)_{t\in\mathbb{R}}$ introduced in (6.5.1) are coupled in such a way that $B_t = \widetilde{B}_t$ for all $t \geq 0$.

We start by observing that, for all $s \in \mathbb{R}$, one has that

$$\mathrm{S}_s - \widetilde{\mathrm{S}}_s = \int_{-\infty}^{0} g(r-s)\big(B_r - \widetilde{B}_r\big)dr - \underbrace{\int_{-\infty}^{0} g(r)\big(B_r - \widetilde{B}_r\big)dr}_{\mathrm{D}} \,.$$

In particular, by using the exponential decay of $g(s)$ as $s \to -\infty$, it holds that

$$\lim_{t\to\infty} \mathbb{P}\left(\sup_{s\geq t} |\mathrm{S}_s - \widetilde{\mathrm{S}}_s + \mathrm{D}| > 1\right) = 0 \,. \tag{6.5.18}$$

Furthermore, thanks to the tightness of $(\widetilde{\mathrm{S}}_{b,\cdot})_{b>0}$ proved in Proposition 6.5.3, for any $c > 0$, it holds that

$$\lim_{\lambda\to\infty} \limsup_{b\to\infty} \mathbb{P}\big(\widetilde{\sigma}_{b+c} - \widetilde{\sigma}_b > \lambda\big) = 0 \qquad \text{and} \qquad \lim_{\lambda\to\infty} \limsup_{b\to\infty} \mathbb{P}\big(\widetilde{\sigma}_b - \widetilde{\sigma}_{b-c} > \lambda\big) = 0 \,. \tag{6.5.19}$$

Step 1: We start by bounding the probability of the event $\{\sigma_b - \widetilde{\sigma}_b \geq \lambda\}$. To this end, by letting $c = |\mathrm{D}| + 1$, we note that

$$\{\sigma_b - \widetilde{\sigma}_b \geq \lambda\} \subseteq \left\{\sup_{s\geq\widetilde{\sigma}_b} |\mathrm{S}_s - \widetilde{\mathrm{S}}_s + \mathrm{D}| > 1\right\} \cup \big\{\widetilde{\sigma}_{b+c} - \widetilde{\sigma}_b \geq \lambda\big\} \,. \tag{6.5.20}$$

To see why this is true, it is more convenient to consider the complement of the given relation. Indeed, we note that

$$\sup_{s\geq\widetilde{\sigma}_b} |\mathrm{S}_s - \widetilde{\mathrm{S}}_s + \mathrm{D}| \leq 1 \quad \Longrightarrow \quad \mathrm{S}_{\widetilde{\sigma}_{b+c}} - \widetilde{\mathrm{S}}_{\widetilde{\sigma}_{b+c}} \leq |\mathrm{D}| + 1 \,,$$

from which we can deduce that

$$\begin{aligned} \widetilde{\mathrm{S}}_{\widetilde{\sigma}_{b+c}} - (Q-\beta)\widetilde{\sigma}_{b+c} = -(b+c) \;&\Longrightarrow\; \mathrm{S}_{\widetilde{\sigma}_{b+c}} - (Q-\beta)\widetilde{\sigma}_{b+c} \leq -(b+c) + |\mathrm{D}| + 1 \\ &\Longrightarrow\; \mathrm{S}_{\widetilde{\sigma}_{b+c}} - (Q-\beta)\widetilde{\sigma}_{b+c} \leq -b \,, \end{aligned}$$

or, in other words, that $\sigma_b \leq \widetilde{\sigma}_{b+c}$. Combining this fact with the fact that we are on the complement of the second event on the right-hand side of (6.5.20), we readily obtain that $\sigma_b - \widetilde{\sigma}_b < \lambda$. Therefore, the desired

result follows by observing that, thanks to (6.5.18) and (6.5.19), it holds that

$$\limsup_{b\to\infty} \mathbb{P}\left(\sup_{s\geq \widetilde{\sigma}_b} |\mathrm{S}_s - \widetilde{\mathrm{S}}_s + \mathrm{D}| > 1 \right) = 0 \qquad \text{and} \qquad \lim_{\lambda\to\infty} \limsup_{b\to\infty} \mathbb{P}\big(\widetilde{\sigma}_{b+c} - \widetilde{\sigma}_b \geq \lambda\big) = 0 \,,$$

where here we implicitly used the fact that $\widetilde{\sigma}_b \to \infty$ in probability as $b \to \infty$.

Step 2: We now focus on bounding the probability of the event $\{\widetilde{\sigma}_b - \sigma_b \geq \lambda\}$. To this end, by letting $c = |\mathrm{D}| + 1$ and by fixing $T > 0$, we note that

$$\{\widetilde{\sigma}_b - \sigma_b \geq \lambda\} \subseteq \left\{ \inf_{s\in[0,T]} \big(\mathrm{S}_s - (Q-\beta)s\big) \leq -b \right\} \cup \left\{ \sup_{s\geq T} |\mathrm{S}_s - \widetilde{\mathrm{S}}_s + \mathrm{D}| > 1 \right\} \cup \big\{ \widetilde{\sigma}_b - \widetilde{\sigma}_{b-c} \geq \lambda \big\} \,. \quad (6.5.21)$$

Also here, to see why this is true, it is more convenient to consider the complement of the given relation. Indeed, we note that by proceeding analogously to the previous step, one can easily verify that on the intersection of the complement of first two events on the right-hand side of (6.5.21), it holds that

$$\inf_{t\in[0,\widetilde{\sigma}_{b-c}]} \big(\mathrm{S}_s - (Q-\beta)s\big) \geq -b \qquad \Longrightarrow \qquad \sigma_b \geq \widetilde{\sigma}_{b-c} \,.$$

Combining this fact with the fact that we are on the complement of the third event on the right-hand side of (6.5.21), we readily obtain that $\widetilde{\sigma}_b - \sigma_b < \lambda$. Therefore, the desired result follows by observing that, thanks to (6.5.18) and (6.5.19), it holds that

$$\lim_{T\to\infty} \mathbb{P}\left(\sup_{s\geq T} |\mathrm{S}_s - \widetilde{\mathrm{S}}_s + \mathrm{D}| > 1 \right) = 0 \qquad \text{and} \qquad \lim_{\lambda\to\infty} \limsup_{b\to\infty} \mathbb{P}\big(\widetilde{\sigma}_b - \widetilde{\sigma}_{b-c} \geq \lambda\big) = 0 \,,$$

and from the fact that for any fixed $T > 0$,

$$\limsup_{b\to\infty} \mathbb{P}\left(\inf_{s\in[0,T]} \big(\mathrm{S}_s - (Q-\beta)s\big) \leq -b \right) = 0 \,.$$

□

We are finally ready to prove tightness of of the collections of processes $(\mathrm{S}_{b,\cdot})_{b>0}$, i.e., to prove Proposition 6.5.1.

Proof of Proposition 6.5.1. The proof relies on similar ideas to those used in the proof of Lemma 6.5.3. Indeed, by recalling the definitions in (6.5.8) and (6.5.9), and applying the Arzelà–Ascoli theorem, to prove the tightness of $(\mathrm{S}_{b,\cdot})_{b>0}$ in $\mathcal{C}^{\mathfrak{c}_d}_{\mathrm{loc}}(\mathbb{R})$ equipped with the local uniform metric, it suffices to verify that for all $c > 0$ and any $\varepsilon > 0$, it holds that

$$\lim_{\delta\to 0} \limsup_{b\to\infty} \mathbb{P}\big(\omega_{b,c}(\delta) > \varepsilon\big) = 0 \,.$$

For $i \in [\mathfrak{c}_d]_0$, we introduce the process $(\mathrm{D}^{(i)}_t)_{t\in\mathbb{R}}$ defined as

$$\mathrm{D}^{(i)}_t \overset{\text{def}}{=} \int_{-\infty}^{0} g^{(i)}(s)\big(B_{s+t} - \widetilde{B}_{s+t}\big)ds \,, \qquad \forall\, t \in \mathbb{R} \,.$$

Proceeding similarly to the proof of Lemma 6.5.5, and by using the exponential decay of $g^{(i)}(s)$ as $s \to \infty$ for all $i \in [\mathfrak{c}_d]_0$, it is straightforward to check that, for any $\varepsilon > 0$,

$$\lim_{t\to\infty} \mathbb{P}\left(\sum_{i=0}^{\mathfrak{c}_d} \sup_{s\geq t} |\mathrm{D}_s^{(i)}| > \varepsilon \right) = 0 \,. \tag{6.5.22}$$

Now, for $b > 0$ and $\widetilde{c} > c$, on the following event

$$\left\{ \sum_{i=0}^{\mathfrak{c}_d} \sup_{s\geq \sigma_b - c} |\mathrm{D}_s^{(i)}| \leq \frac{\varepsilon}{4} \right\} \cap \left\{ [\sigma_b - c, \sigma_b + c] \subseteq [\widetilde{\sigma}_b - \widetilde{c}, \widetilde{\sigma}_b + \widetilde{c}] \right\} ,$$

one has that

$$\begin{aligned} \omega_{b,c}(\delta) &= \sum_{i=0}^{\mathfrak{c}_d} \sup_{s,t\in[\sigma_b-c,\sigma_b+c],\, |t-s|\leq\delta} |\mathrm{S}_t^{(i)} - \mathrm{S}_s^{(i)}| \leq \sum_{i=0}^{\mathfrak{c}_d} \sup_{s,t\in[\sigma_b-c,\sigma_b+c],\, |t-s|\leq\delta} |\widetilde{\mathrm{S}}_t^{(i)} - \widetilde{\mathrm{S}}_s^{(i)}| + \frac{\varepsilon}{2} \\ &\leq \sum_{i=0}^{\mathfrak{c}_d} \sup_{s,t\in[\widetilde{\sigma}_b-\widetilde{c},\widetilde{\sigma}_b+\widetilde{c}],\, |t-s|\leq\delta} |\widetilde{\mathrm{S}}_t^{(i)} - \widetilde{\mathrm{S}}_s^{(i)}| + \frac{\varepsilon}{2} = \sum_{i=0}^{\mathfrak{c}_d} \sup_{s,t\in[-\widetilde{c},\widetilde{c}],\, |t-s|\leq\delta} |\widetilde{\mathrm{S}}_{b,t}^{(i)} - \widetilde{\mathrm{S}}_{b,s}^{(i)}| + \frac{\varepsilon}{2} \,. \end{aligned}$$

In particular, this fact implies that

$$\begin{aligned} &\mathbb{P}\big(\omega_{b,c}(\delta) > \varepsilon\big) \\ &\quad \leq \mathbb{P}\big(\widetilde{\omega}_{b,\widetilde{c}}(\delta) > \varepsilon/2\big) + \mathbb{P}\left(\sum_{i=0}^{\mathfrak{c}_d} \sup_{s\geq\sigma_b-c} |\mathrm{D}_s^{(i)}| > \frac{\varepsilon}{4} \right) + \mathbb{P}\big([\widetilde{\sigma}_b - \widetilde{c}, \widetilde{\sigma}_b + \widetilde{c}] \not\subseteq [\sigma_b - c, \sigma_b + c]\big) \,. \end{aligned} \tag{6.5.23}$$

The conclusion follows by first taking $\limsup_{b\to\infty}$, then $\lim_{\delta\to 0}$, and finally $\limsup_{\widetilde{c}\to\infty}$ in the above expression. Specifically, the vanishing of the first and third probabilities on the right-hand side of (6.5.23) is a consequence of Lemma 6.5.3 and Lemma 6.5.5, respectively. The vanishing of the second probability on the right-hand side of (6.5.23) follows from (6.5.22) and the fact that $\sigma_b \to \infty$ in probability as $b \to \infty$. □

6.5.2 Convergence of the recentred process

The main goal of this section is to prove Theorem 6.C. To this end, we consider a process $(\mathrm{R}_t)_{t\in\mathbb{R}}$ with the same law of the spherical average process $(\mathrm{S}_t)_{t\in\mathbb{R}}$. For each $\mathfrak{b} > 0$, we introduce the following quantities

$$\rho_{\mathfrak{b}} \overset{\text{def}}{=} \inf\big\{ t \in \mathbb{R} \,:\, \mathrm{R}_t - (Q-\beta)t = -\mathfrak{b} \big\} \,, \qquad \mathrm{R}_{\mathfrak{b},t} \overset{\text{def}}{=} \mathrm{R}_{t+\rho_{\mathfrak{b}}} - \mathrm{R}_{\rho_{\mathfrak{b}}} \,, \quad \forall\, t \in \mathbb{R} \,.$$

For $\mathfrak{b}, b > 0$, we also introduce the following stopping time

$$\rho_{\mathfrak{b},b} \overset{\text{def}}{=} \inf\big\{ t \in \mathbb{R} \,:\, \mathrm{R}_{\mathfrak{b},t} - (Q-\beta)t = -b \big\} \,. \tag{6.5.24}$$

Lemma 6.5.6. *For any $\varepsilon > 0$, there exists $u \geq 0$ large enough such that for all $\mathfrak{b} > 0$, it holds that*

$$\mathrm{d}^{\mathrm{TV}}\big[(\mathrm{R}_{\mathfrak{b},u+t})_{t\geq 0}, (\mathrm{S}_{u+t})_{t\geq 0} \big] \leq \varepsilon \,.$$

Proof. It suffices to show that there exists $u \geq 0$ large enough such that for all $\mathfrak{b} > 0$, the conditional law of $(\mathrm{R}_{\mathfrak{b},u+t})_{t\geq 0}$ given $(\mathrm{R}_t)_{t\leq\rho_{\mathfrak{b}}}$ is close in total variation distance to the law of $(\mathrm{S}_{u+t})_{t\geq 0}$. Since $\rho_{\mathfrak{b}}$ is a stopping time for $(\mathrm{R}_t)_{t\in\mathbb{R}}$, we have that the conditional law of $(\mathrm{R}_{\mathfrak{b},u+t})_{t\geq 0}$ given $(\mathrm{R}_t)_{t\leq\rho_{\mathfrak{b}}}$ depends only on $(\mathrm{R}_{\rho_{\mathfrak{b}}}, \mathbf{R}_{\rho_{\mathfrak{b}}})$,

where $(\mathbf{R}_t)_{t\in\mathbb{R}}$ denotes the vector of derivatives of $(\mathrm{R}_t)_{t\in\mathbb{R}}$ up to order $\mathfrak{c}_d$. Moreover, recalling that $(\mathrm{R}_t)_{t\in\mathbb{R}}$ has the same law of $(\mathrm{S}_t)_{t\in\mathbb{R}}$, then thanks to Proposition 6.5.1, the collection $(\mathrm{R}_{\rho_\mathfrak{b}}, \mathbf{R}_{\rho_\mathfrak{b}})_{\mathfrak{b}>0}$ is tight in $\mathcal{C}^{\mathfrak{c}_d}_{\mathrm{loc}}(\mathbb{R})$ equipped with the local uniform metric[12]. The conclusion is then a consequence of Lemma 6.2.14. □

We are now ready to prove the desired convergence result.

Proof of Theorem 6.C. Fix $\mathrm{T} \in \mathbb{R}$. Thanks to Lemma 6.5.6, for $u \geq 0$ large enough, we can find a coupling of $(\mathrm{R}_t)_{t\in\mathbb{R}}$ and $(\mathrm{S}_t)_{t\in\mathbb{R}}$ such that, for all $\mathfrak{b} > 0$, with probability at least $1-\varepsilon$, it holds that

$$\mathrm{R}_{\mathfrak{b},u+t} = \mathrm{S}_{u+t}\ , \qquad \forall\, t \geq 0\ . \tag{6.5.25}$$

By choosing $b > 0$ sufficiently large (depending on u but not on $\mathfrak{b}$), it holds with probability at least $1-\varepsilon$ that $\rho_{\mathfrak{b},b} = \sigma_b$. Indeed, this follows since, by taking b large enough (depending on u but not on $\mathfrak{b}$), we can ensure that there is not a time $t < u$ such that $\mathrm{S}_t - (Q-\beta)t = -b$. Moreover, by the definition of $\rho_{\mathfrak{b},b}$, it holds that

$$\mathrm{S}_{\rho_{\mathfrak{b},b}} - (Q-\beta)\rho_{\mathfrak{b},b} \overset{(6.5.25)}{=} \mathrm{R}_{\mathfrak{b},\rho_{\mathfrak{b},b}} - (Q-\beta)\rho_{\mathfrak{b},b} = -b\ .$$

In particular, this fact implies that if $b > 0$ is large enough (not depending on $\mathfrak{b}$), then with probability at least $1-\varepsilon$, it holds that

$$\mathrm{R}_{\mathfrak{b},\rho_{\mathfrak{b},b}+t} = \mathrm{S}_{\sigma_b+t}\ , \qquad \forall\, t \geq \mathrm{T}\ .$$

Therefore, by noting that $\rho_{\mathfrak{b},b} + \rho_\mathfrak{b} = \rho_{b+\mathfrak{b}}$, we can use the previous identity to write

$$\mathrm{R}_{\rho_{b+\mathfrak{b}}+t} - \mathrm{R}_{\rho_{b+\mathfrak{b}}} = \mathrm{S}_{\sigma_b+t} - \mathrm{S}_{\sigma_b}\ , \qquad \forall\, t \geq \mathrm{T}\ .$$

This fact implies that the collection $((\mathrm{S}_{b,t})_{t\geq\mathrm{T}})_{b>0}$ is Cauchy in total variation distance, and so the conclusion follows. □

6.5.3 The quantum cone as a local limit

In this short section, we prove that the β-quantum cone field can be identified as a local limit of a whole-space LGF with an additional β-log singularity.

Proof of Theorem 6.D. Consider the setting described in the theorem's statement. For $t \in \mathbb{R}$, $z \in \mathbb{R}^d$, and $b > 0$, we have that

$$\mathbf{h}^{t,b}_{e^{-s}}(0) = \mathbf{h}_{e^{-(s+t)}}(z) + \beta s - (Q-\beta)t + b\ , \qquad \forall\, s \in \mathbb{R}\ .$$

Thus, by definition (6.1.20) of σ_b, and recalling the definitions in (6.1.2) and (6.1.19), we obtain that

$$\mathbf{h}^{\sigma_b,b}_{e^{-s}}(0) = \mathbf{h}_{e^{-(s+\sigma_b)}}(z) + \beta s - (Q-\beta)\sigma_b + b = \mathbf{h}_{e^{-(s+\sigma_b)}}(z) - \mathbf{h}_{e^{-\sigma_b}}(z) + \beta s \overset{\text{law}}{=} \mathrm{S}_{b,s} + \beta s\ , \qquad \forall\, s \in \mathbb{R}\ .$$

Hence, thanks to Theorem 6.C, if we take the limit as $b \to \infty$ in the previous display, we readily obtain that, for any $\mathrm{T} \in \mathbb{R}$, the process $(\mathbf{h}^{\sigma_b,b}_{e^{-s}}(0))_{s\geq\mathrm{T}}$ converges in total variation distance to the process $(\mathrm{S}_{\infty,s} + \beta s)_{s\geq\mathrm{T}}$ as $b \to \infty$. The desired result follows since the shifting and rescaling procedure we performed do not affect the law of the projection of $\mathbf{h}$ onto the subspace $\mathcal{H}_{d,\mathrm{sph}}$. □

[12] We emphasise that this is the sole instance where the tightness of $(\mathrm{S}_{b,\cdot})_{b>0}$ (and consequently of $(\mathrm{R}_{\mathfrak{b},\cdot})_{\mathfrak{b}>0}$) is used.

6.5.4 Interaction between LBM and quantum cone

The main goal of this section is to prove Theorem 6.E, i.e., to prove that the law of the d-dimensional quantum cone $\mathbf{h}^\star$ is invariant under shifting along the trajectories of the LBM.

For $d > 2$ (resp. $d = 2$) even, $Q > \sqrt{2d}$, and $\beta \in (-\infty, Q)$, let $\mathbf{h}^\star$ be the d-dimensional β-quantum cone field as specified in Definition 6.1.11 (resp. [DMS21, Definition 4.10]), and let $\mathrm{B}_{\mathbf{h}^\star,\alpha}$ be the associated LBM as introduced in Definition 6.1.12. Before proceeding, we need to rule out that $\mathrm{B}_{\mathbf{h}^\star,\alpha}$ does not stay "stuck" at zero due to the β-log singularity at the origin of $\mathbf{h}^\star$. To confirm this fact, we note that the expected time for $\mathrm{B}_{\mathbf{h}^\star,\alpha}$ to reach the boundary of the unit disk is given by $\mathbf{E}_0[\nu_{\mathbf{h}^\star,\alpha}(B(0,1))]$. Hence, it suffices to verify that it holds $\mathbb{P}$-almost surely that $\mathbf{E}_0[\nu_{\mathbf{h}^\star,\alpha}(B(0,1))] < \infty$. To this end, we need the following intermediate result related to the growth of the radial part of the quantum cone field.

Lemma 6.5.7. *For $d \geq 2$ even, $Q > \sqrt{2d}$, and $\beta \in (-\infty, Q)$, consider the collection of recentred processes $(\mathrm{S}_{b,\cdot})_{b>0}$ defined in (6.1.19). Then, it holds that*

$$\lim_{t\to\infty} \limsup_{b\to\infty} \mathbb{P}\left(\sup_{s\geq t} |\mathrm{S}_{b,s}| s^{-3/4} > 1 \right) = 0 \,,$$

where we recall that the recentred process $(\mathrm{S}_{b,s})_{s\in\mathbb{R}}$ is defined in (6.1.19).

Proof. We focus on the case $d > 2$ even, as the case $d = 2$ is trivial. For $t \geq 0$ and $b > 0$, we introduce the following events,

$$\mathrm{E}_{1,b,t} \overset{\text{def}}{=} \left\{ \sup_{s\geq t} |\mathrm{S}_{b,s} - (\widetilde{\mathrm{S}}_{s+\sigma_b} - \widetilde{\mathrm{S}}_{\sigma_b})| < 1 \right\}, \qquad \mathrm{E}_{2,b,t} \overset{\text{def}}{=} \left\{ |\sigma_b - \widetilde{\sigma}_b| < t \right\}, \qquad \mathrm{E}_{3,b,t} \overset{\text{def}}{=} \left\{ |\widetilde{\sigma}_b - \widetilde{\tau}_b| < t \right\},$$

where we recall that σ_b is defined in (6.1.18), $(\widetilde{\mathrm{S}}_s)_{s\in\mathbb{R}}$ is defined in (6.5.2), and $\widetilde{\tau}_b$ and $\widetilde{\sigma}_b$ are defined in (6.5.4). We claim that it suffices to estimate the probability of the event in the statement, restricted to the event $\cap_{i=1}^3 \mathrm{E}_{i,b,t}$. Indeed, by applying Lemma 6.5.4 and Lemma 6.5.5, and following reasoning similar to the proof of (6.5.18), it follows that

$$\lim_{t\to\infty} \limsup_{b\to\infty} \sum_{i=1}^{3} \mathbb{P}\big(\mathrm{E}_{i,b,t}^c\big) = 0 \,.$$

We start by observing that, on the event $\mathrm{E}_{1,b,t}$, it holds that

$$\sup_{s\geq t} |\mathrm{S}_{b,s}| \leq \sup_{s\geq t} |\widetilde{\mathrm{S}}_{s+\sigma_b} - \widetilde{\mathrm{S}}_{\sigma_b}| + 1 \,,$$

and so, we can focus our attention on the following event,

$$\left\{ \sup_{s\geq t} |\widetilde{\mathrm{S}}_{s+\sigma_b} - \widetilde{\mathrm{S}}_{\sigma_b}| s^{-3/4} > 1, \bigcap_{i=1}^{3} \mathrm{E}_{i,b,t} \right\} .$$

To this end, we begin by observing that, thanks to the triangle inequality,

$$\sup_{s\geq t} |\widetilde{\mathrm{S}}_{s+\sigma_b} - \widetilde{\mathrm{S}}_{\sigma_b}| \leq \sup_{s\geq t} |\widetilde{\mathrm{S}}_{s+\widetilde{\sigma}_b} - \widetilde{\mathrm{S}}_{\widetilde{\sigma}_b}| + \sup_{s\geq t} |\widetilde{\mathrm{S}}_{s+\widetilde{\sigma}_b} - \widetilde{\mathrm{S}}_{s+\sigma_b}| + |\widetilde{\mathrm{S}}_{\widetilde{\sigma}_b} - \widetilde{\mathrm{S}}_{\sigma_b}| \,. \tag{6.5.26}$$

Next, we proceed to bound each of the three terms on the right-hand side of (6.5.26) individually.

First term: By applying the triangle inequality, on the event $\mathrm{E}_{3,b,t}$, we observe that the first term on the right-hand side of (6.5.26) is bounded above by

$$\sup_{s\geq t}|\widetilde{\mathrm{S}}_{s+\widetilde{\tau}_b}-\widetilde{\mathrm{S}}_{\widetilde{\tau}_b}|+\sup_{s\geq t}\sup_{u\in[-t,t]}|\widetilde{\mathrm{S}}_{s+u+\widetilde{\tau}_b}-\widetilde{\mathrm{S}}_{s+\widetilde{\tau}_b}|+\sup_{u\in[-t,t]}|\widetilde{\mathrm{S}}_{u+\widetilde{\tau}_b}-\widetilde{\mathrm{S}}_{\widetilde{\tau}_b}|\,. \tag{6.5.27}$$

We can analyse the three terms on the right-hand side of the above display separately. Beginning with the first term, leveraging the invariance in law (6.5.7), we observe that

$$\mathbb{P}\left(\sup_{s\geq t}|\widetilde{\mathrm{S}}_{s+\widetilde{\tau}_b}-\widetilde{\mathrm{S}}_{\widetilde{\tau}_b}|s^{-3/4}>1\right)\leq\sum_{n=\lfloor t\rfloor}^{\infty}\mathbb{P}\left(\sup_{s\in[n,n+1]}|\widetilde{\mathrm{S}}_s|>n^{3/4}\right)=\sum_{n=\lfloor t\rfloor}^{\infty}\mathbb{P}\left(\sup_{s\in[n,n+1]}|\widetilde{\mathrm{S}}_s|>n^{3/4}\right).$$

By first taking $\limsup_{b\to\infty}$ and then $\lim_{t\to\infty}$ in the above inequality, we deduce that the probability on the left-hand side converges to zero. This conclusion follows directly from the Gaussian tail bound and the fact that the function g, used in the definition (6.5.2) of the process $(\widetilde{\mathrm{S}}_s)_{s\in\mathbb{R}}$, decays to zero exponentially fast as $s\to-\infty$. The remaining two terms in (6.5.27) can be handled in a similar manner.

Second term: Proceeding similarly to the first term and applying the triangle inequality, on the event $\mathrm{E}_{2,b,t}\cap\mathrm{E}_{3,b,t}$, we observe that the second term on the right-hand side of (6.5.26) is bounded above by

$$2\sup_{s\geq t}\sup_{u\in[-t,t]}|\widetilde{\mathrm{S}}_{s+u+\widetilde{\tau}_b}-\widetilde{\mathrm{S}}_{s+\widetilde{\tau}_b}|+\sup_{s\geq t}\sup_{u\in[-t,t]}\sup_{r\in[-t,t]}|\widetilde{\mathrm{S}}_{s+u+r+\widetilde{\tau}_b}-\widetilde{\mathrm{S}}_{s+u+\widetilde{\tau}_b}|\,.$$

Now, leveraging the invariance in law (6.5.7), we can control each term in the above display by following an approach analogous to that used for the first term.

Third term: Proceeding similarly to the second term and applying the triangle inequality, on the event $\mathrm{E}_{2,b,t}\cap\mathrm{E}_{3,b,t}$, we observe that the third term on the right-hand side of (6.5.26) is bounded above by

$$2\sup_{u\in[-t,t]}|\widetilde{\mathrm{S}}_{u+\widetilde{\tau}_b}-\widetilde{\mathrm{S}}_{\widetilde{\tau}_b}|+\sup_{u\in[-t,t]}\sup_{r\in[-t,t]}|\widetilde{\mathrm{S}}_{u+r+\widetilde{\tau}_b}-\widetilde{\mathrm{S}}_{u+\widetilde{\tau}_b}|\,.$$

Once again, leveraging the invariance in law (6.5.7), we can control each term in the above display by proceeding in a manner analogous to the first term. Hence, the conclusion follows. □

We now proceed to prove the following result which guarantees that $\mathbb{P}$-almost surely, the LBM $\mathrm{B}_{\mathbf{h}^\star,\alpha}$ does not stay stuck at the origin.

Lemma 6.5.8. *Let $d\geq 2$ be even, $Q>\sqrt{2d}$, $\beta\in(-\infty,Q)$, and let $\mathbf{h}^\star$ denote the law of the d-dimensional β-quantum cone. Then, it holds $\mathbb{P}$-almost surely that,*

$$\mathbf{E}_0\big[\nu_{\mathbf{h}^\star,\alpha}\big(B(0,1)\big)\big]<\infty\,.$$

Proof. We focus on the case $d>2$ even, as the case $d=2$ can be treated analogously. Proceeding as in Section 6.4 and recalling Defintion 6.1.11, we have that

$$\begin{aligned}\mathbb{E}\big[\mathbf{E}_0\big[\nu_{\mathbf{h}^\star,\alpha}\big(B(0,1)\big)\big]\,\big|\,\mathbf{h}^\star_{\mathrm{rad}}\big]&\lesssim\mathbb{E}\left[\int_{B(0,1)}|x|^{2-d}\mu_{\mathbf{h}^\star,\alpha}(dx)\,\middle|\,\mathbf{h}^\star_{\mathrm{rad}}\right]\\&=\mathbb{E}\left[\int_{B(0,1)}|x|^{2-d}e^{\alpha\mathbf{h}^\star_{\mathrm{rad}}(x)}\mu_{\mathbf{h}^\star_{\mathrm{sph}},\alpha}(dx)\,\middle|\,\mathbf{h}^\star_{\mathrm{rad}}\right]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \mathbb{E}\left[\int_{B(0,e^{-k})\setminus B(0,e^{-(k+1)})} |x|^{2-d} e^{\alpha \mathbf{h}^\star_{\mathrm{rad}}(x)} \mu_{\mathbf{h}^\star_{\mathrm{sph}},\alpha}(dx) \,\middle|\, \mathbf{h}^\star_{\mathrm{rad}} \right] \\
&\leq \sum_{k=0}^{\infty} e^{k(d-2)} e^{\alpha \sup_{t\in[k,k+1]}(\mathrm{S}_{\infty,t}+\beta t)} \mathbb{E}\big[\mu_{\mathbf{h}^\star_{\mathrm{sph}},\alpha}\big(B(0,e^{-k})\big)\big] \\
&\lesssim \sum_{k=0}^{\infty} e^{-k(2+\alpha^2/2-\alpha\beta)+\sup_{t\in[k,k+1]} \mathrm{S}_{\infty,t}} \mathbb{E}\big[\mu_{\mathbf{h}^\star_{\mathrm{sph}},\alpha}\big(B(0,1)\big)\big] \,,
\end{aligned}
$$

where the last inequality follows from the coordinate change formula Proposition 6.2.10. Therefore, the conclusion follows thanks to Lemma 6.5.7 and the fact that for all $Q > \sqrt{2d}$ and $\beta \in (-\infty, Q)$, it holds that $2 + \alpha^2/2 - \alpha\beta > 0$. □

We can now focus on the proof of Theorem 6.E. We begin with the following proposition, which can be regarded as an extended version of Theorem 6.D.

Proposition 6.5.9. *Let $d \geq 2$ be even, $Q > \sqrt{2d}$, and $\beta \in (-\infty, Q)$. For $t \in \mathbb{R}$, $z \in \mathbb{R}^d$, and $b > 0$, consider the field $\mathbf{h}^{t,b}$ introduced in the statement of Theorem 6.D and let σ_b be as defined in (6.1.20). Furthermore, let $\mathrm{B}_{\mathbf{h}-\alpha\log|\cdot-z|,\alpha,\cdot}$ be a two-sided LBM associated to the field $\mathbf{h} - \alpha\log|\cdot - z|$ started from z at time 0, and let $\mathrm{B}_{\mathbf{h}^\star,\alpha,\cdot}$ be a two-sided LBM associated to a d-dimensional α-quantum cone field $\mathbf{h}^\star$ started from the origin at time 0. Then, as $b \to \infty$, it holds that*

$$
\big(\mathbf{h}^{\sigma_b,b}, e^{\sigma_b}(\mathrm{B}_{\mathbf{h}-\alpha\log|\cdot-z|,\alpha,e^{-\alpha b}\cdot} - z)\big) \xrightarrow{\text{law}} (\mathbf{h}^\star, \mathrm{B}_{\mathbf{h}^\star,\alpha,\cdot}) \,,
$$

with respect to the product topology, where the first component converges in the space of distributions, and the second component converges in $\mathcal{C}(\mathbb{R})$ equipped with the local uniform metric.

Proof. To simplify the notation, we assume, without loss of generality, that $z = 0$, and define $\overline{\mathbf{h}} \overset{\text{def}}{=} \mathbf{h} - \alpha\log|\cdot|$. For $b > 0$, we define $(\overline{B}_t)_{t\in\mathbb{R}} \overset{\text{def}}{=} (e^{\sigma_b} B_{e^{-2\sigma_b}t})_{t\in\mathbb{R}}$ and consider the dilation $\phi_b : \mathbb{R}^d \to \mathbb{R}^d$ given by $\phi_b(x) = e^{-\sigma_b}x$. Then, by arguing similarly to the proof of Proposition 6.4.2, it holds that

$$
e^{\sigma_b}\mathrm{B}_{\overline{\mathbf{h}},\alpha,e^{-\alpha b}t} = e^{\sigma_b} B_{\mathrm{F}^{-1}_{\overline{\mathbf{h}},\alpha}(e^{-\alpha b}t)} = \overline{B}_{e^{2\sigma_b}\mathrm{F}^{-1}_{\overline{\mathbf{h}},\alpha}(e^{-\alpha b}t)} = \overline{B}_{\mathrm{F}^{-1}_{\overline{\mathbf{h}},\alpha,b}(t)} \,, \qquad \forall\, t \in \mathbb{R} \,,
$$

where we set $\mathrm{F}_{\overline{\mathbf{h}},\alpha,b}(t) \overset{\text{def}}{=} e^{\alpha b}\mathrm{F}_{\overline{\mathbf{h}},\alpha}(e^{-2\sigma_b}t)$. Furthermore, thanks to Proposition 6.4.1, for all $t \in \mathbb{R}$, we observe that

$$
\begin{aligned}
\mathrm{F}_{\overline{\mathbf{h}},\alpha,b}(t) = e^{\alpha b}\nu_{\overline{\mathbf{h}},\alpha}\big([B]_{e^{-2\sigma_b}t}\big) \overset{(6.4.3)}{=} e^{\alpha b}\nu_{\overline{\mathbf{h}}\circ\phi_b + Q\log|\phi_b'|,\alpha}\big(e^{\sigma_b}[B]_{e^{-2\sigma_b}t}\big) &= \overline{\nu}_{\mathbf{h}^{\sigma_b,b},\alpha}\big([\overline{B}]_t\big) \\
&\overset{\text{def}}{=} \overline{\mathrm{F}}_{\mathbf{h}^{\sigma_b,b},\alpha}(t) \,,
\end{aligned}
$$

where $\overline{\nu}_{\mathbf{h}^{\sigma_b,b},\alpha}$ denotes the GMC with respect to the field $\mathbf{h}^{\sigma_b,b}$ along the trajectory of the Brownian motion $(\overline{B}_t)_{t\in\mathbb{R}}$. In particular, for any $b > 0$ and $t \geq 0$, we showed that

$$
\overline{\mathrm{B}}_{\mathbf{h}^{\sigma_b,b},\alpha,t} \overset{\text{def}}{=} \overline{B}_{\overline{\mathrm{F}}^{-1}_{\mathbf{h}^{\sigma_b,b},\alpha}(t)} \quad \Longrightarrow \quad e^{\sigma_b}\mathrm{B}_{\overline{\mathbf{h}},\alpha,e^{-\alpha b}t} = \overline{\mathrm{B}}_{\mathbf{h}^{\sigma_b,b},\alpha,t} \,.
$$

Now, for any $R > 0$, consider the stopping times $\overline{\tau}_{R,+}$ and $\overline{\tau}_{R,-}$ defined as

$$
\overline{\tau}_{R,+} \overset{\text{def}}{=} \inf\big\{t \geq 0 \,:\, \overline{B}_t \notin B(0,R)\big\} \,, \qquad \overline{\tau}_{R,-} \overset{\text{def}}{=} \inf\big\{t \geq 0 \,:\, \overline{B}_{-t} \notin B(0,R)\big\}
$$

By Theorem 6.D and Brownian scaling, for any fixed $R > 0$, the joint law of $\mathbf{h}^{\sigma_b,b}|_{B(0,R)}$ and $(\overline{B}_t)_{t\in[\overline{\tau}_{R,-},\overline{\tau}_{R,+}]}$ converges in total variation distance to the joint law of $\mathbf{h}^\star|_{B(0,R)}$ and an independent two-sided Brownian motion stopped at its exit times from $B(0,R)$. Since, the LBM associated to $\mathbf{h}^{\sigma_b,b}$ (resp. $\mathbf{h}^\star$), stopped at its exit time from $B(0,R)$, is a measurable function of $\mathbf{h}^{\sigma_b,b}|_{B(0,R)}$ (resp. $\mathbf{h}^*|_{B(0,R)}$) and an independent two-sided Brownian motion stopped at its exit times from $B(0,R)$, the conclusion follows. □

We are finally ready to prove Theorem 6.E.

Proof of Theorem 6.E. Let $d \geq 2$ be even and $Q > \sqrt{2d}$. For $\mathbf{h}$ a whole-space LGF, we consider the associated two-sided LBM $(\mathrm{B}_{\mathbf{h},\alpha,t})_{t\in\mathbb{R}}$ started from the origin at time 0. Moreover, let $\tau_{\mathbf{h},\alpha,+}$ (resp. $\tau_{\mathbf{h},\alpha,-}$) be the first positive (resp. negative) time that $\mathrm{B}_{\mathbf{h},\alpha,\cdot}$ exits the ball $B(0,1)$. Let U be a point sampled uniformly at random from the interval $[\tau_{\mathbf{h},\alpha,-},\tau_{\mathbf{h},\alpha,+}]$. For $b > 0$, we let

$$\sigma_b \overset{\text{def}}{=} \inf\big\{s \in \mathbb{R} \,:\, \mathbf{h}_{e^{-s}}(\mathrm{B}_{\mathbf{h},\alpha,\mathrm{U}}) - Qs = -b\big\}\,,$$

and we consider the conformal automorphism $\phi_b(x) = e^{-\sigma_b}x + \mathrm{B}_{\mathbf{h},\alpha,\mathrm{U}}$. Thanks to [BP24, Corollary 3.17], we know that if we condition on the Brownian motion modulo time change, then the conditional law of $\mathbf{h}$ near $\mathrm{B}_{\mathbf{h},\alpha,\mathrm{U}}$ locally looks like a whole-space LGF plus an α-log singularity at $\mathrm{B}_{\mathbf{h},\alpha,\mathrm{U}}$. Therefore, by Proposition 6.5.9, we have the following joint convergence in law as $b \to \infty$,

$$\big(\mathbf{h}\circ\phi_b + Q\log|\phi_b'| + b,\, e^{\sigma_b}(\mathrm{B}_{\mathbf{h},\alpha,\mathrm{U}+e^{-\alpha b}\cdot} - \mathrm{B}_{\mathbf{h},\alpha,\mathrm{U}})\big) \xrightarrow{\text{law}} (\mathbf{h}^\star, \mathrm{B}_{\mathbf{h}^\star,\alpha})\,. \tag{6.5.28}$$

We note that for any $t \in \mathbb{R}$, the total variation distance between $(\mathbf{h}^\star, \mathrm{B}_{\mathbf{h}^\star,\alpha}, \mathrm{U} + e^{-\alpha b}t)$ and $(\mathbf{h}^\star, \mathrm{B}_{\mathbf{h}^\star,\alpha}, \mathrm{U})$ tends to 0 as $b \to \infty$. This is a consequence of the fact that the conditional law of U given $(\mathbf{h}, B)$ is that of a uniform random variable. Hence, by defining $\sigma_{b,t}$ as follows

$$\sigma_{b,t} \overset{\text{def}}{=} \inf\big\{s \in \mathbb{R} \,:\, \mathbf{h}_{e^{-s}}(\mathrm{B}_{\mathbf{h},\alpha,\mathrm{U}+e^{-\alpha b}t}) - Qs = -b\big\}\,,$$

and by considering the conformal automorphism $\phi_{b,t}(x) = e^{-\sigma_{b,t}}x + \mathrm{B}_{\mathbf{h},\alpha,\mathrm{U}+e^{-\alpha b}t}$, we get that

$$\big(\mathbf{h}\circ\phi_{b,t} + Q\log|\phi_{b,t}'| + b, e^{\sigma_{b,t}}(\mathrm{B}_{\mathbf{h},\alpha,\mathrm{U}+e^{-\alpha b}t+e^{-\alpha b}\cdot} - \mathrm{B}_{\mathbf{h},\alpha,\mathrm{U}+e^{-\alpha b}t})\big) \xrightarrow{\text{law}} (\mathbf{h}^\star, \mathrm{B}_{\mathbf{h}^\star,\alpha})\,, \tag{6.5.29}$$

and this it sufficient to conclude. Indeed, the couple on the left side of (6.5.29) is obtained from the couple on the left side of (6.5.28) by translating time for the LBM by t, translating space so that the position of the LBM at time t is mapped to the origin, and re-scaling to get the usual normalisation for the field. Hence, by (6.5.28), the joint law of the left sides of (6.5.29) and (6.5.28) converges to the joint law of $(\mathbf{h}^\star, \mathrm{B}_{\mathbf{h}^\star,\alpha})$ and the field/process pair obtained by translating time by t, translating space by $\mathrm{B}_{\mathbf{h}^*,\alpha,t}$, and re-scaling to get the usual normalisation for the field. Therefore, the conclusion follows since by (6.5.29), the second field/process pair has the same law as $(\mathbf{h}^\star, \mathrm{B}_{\mathbf{h}^\star,\alpha})$. □

6.A Some basic estimates for GMC measures

In this appendix, we collect some basic estimates for GMC measures. We recall that for each $\gamma \in (0, \sqrt{2d})$, the collection of regularised GMC measures $(\mu_{\mathbf{h},\gamma,(n)})_{n\in\mathbb{N}}$ is defined in (6.4.10). Furthermore, with a slight abuse of notation, we denote by $\mu_{\mathbf{h},\gamma,(\infty)}$ the limiting measure $\mu_{\mathbf{h},\gamma}$.

Lemma 6.A.1. *Let $d \geq 2$, $\gamma \in (0, \sqrt{2d})$, $\beta \in (-\infty, Q)$, and $x \in \mathbb{R}^d$. It holds $\mathbb{P}$-almost surely for any $r \in (0,1]$ that*

$$r^{(d+\gamma^2/2-\beta\gamma)+\varepsilon} \lesssim \mu_{\mathbf{h}-\beta\log|\cdot-x|,\gamma}\big(B(x,r)\big) \lesssim r^{(d+\gamma^2/2-\beta\gamma)-\varepsilon}\,, \qquad \forall\,\varepsilon>0\,,$$

for some random implicit constants.

Proof. The proof of this result is standard and follows from a union bound over dyadic scales, and the coordinate change formula (Proposition 6.2.10). □

Lemma 6.A.2. *Let $d \geq 2$, $\gamma \in (0, \sqrt{2d})$, and $R > 0$. It holds $\mathbb{P}$-almost surely for any $n \in \mathbb{N} \cup \{\infty\}$, all $x \in B(0,R)$, and any $r \in (0,1]$ that*

$$\mu_{\mathbf{h},\gamma,(n)}\big(B(x,r)\big) \lesssim r^{(d+\gamma^2/2-\sqrt{2d}\gamma)-\varepsilon}\,, \qquad \forall\,\varepsilon>0\,,$$

for some random implicit constant.

Proof. For $d = 2$, this fact is proved in [GRV16, Theorem 2.2], and the proof easily generalises to any dimension $d > 2$. □

Lemma 6.A.3. *Let $d \geq 2$, $\gamma \in (0, \sqrt{2d})$, $\beta \in (-\infty, Q)$, and $R > 0$. It holds $\mathbb{P}$-almost surely for all $x \in B(0,R)$, and any $r \in (0,1]$ that*

$$\mu_{\mathbf{h}-\beta\log|\cdot|,\gamma}\big(B(x,r)\big) \lesssim r^{(d+\gamma^2/2-(\beta\vee\sqrt{2d})\gamma)-\varepsilon}\,, \qquad \forall\,\varepsilon>0\,,$$

for some random implicit constant.

Proof. For $\beta \leq 0$ this is an immediate consequence of Lemma 6.A.2. We fix $\beta \in (0, Q)$ and, for simplicity, we take $R = 1$. For $n \in \mathbb{N}$, let $\Sigma_n \overset{\text{def}}{=} 2^{-n}\mathbb{Z}^d \cap B(0,1)$ be the lattice $2^{-n}\mathbb{Z}^d$ restricted to the ball $B(0,1)$. For $k \in \mathbb{N}$, we define the annulus $A_k \overset{\text{def}}{=} B(0,2^{-k}) \setminus B(0,2^{-(k+1)})$. By using the coordinate change formula (Proposition 6.2.10), we have that for all $n \geq 2$, $k \in [n-2]_0$, and any $x \in \Sigma_n \cap A_k$, it holds that

$$\begin{aligned}\mu_{\mathbf{h}-\beta\log|\cdot|,\gamma}\big(B(x,2^{-n})\big) &= 2^{-k\gamma(Q-\beta)}e^{\gamma\mathbf{h}_{2^{-k}}(0)}\mu_{\mathbf{h}^k-\beta\log|\cdot|}\big(B(2^kx,2^{k-n})\big)\\ &\lesssim 2^{-k\gamma(Q-(\beta\vee\sqrt{2d}))}e^{\gamma\mathbf{h}_{2^{-k}}(0)}\mu_{\mathbf{h}^k}\big(B(2^kx,2^{k-n})\big)\,.\end{aligned}$$

where, by the scale invariance of the law of $\mathbf{h}$ modulo additive constant, $\mathbf{h}^k(\cdot) \overset{\text{def}}{=} \mathbf{h}(2^{-k}\cdot) - \mathbf{h}_{2^{-k}}(0) \overset{\text{law}}{=} \mathbf{h}$. Moreover, the last inequality in the previous display follows from the fact that for all $n \geq 2$, $k \in [n-2]_0$, and any $x \in \Sigma_n \cap A_k$, the ball $B(2^kx, 2^{k-n})$ is at distance at most $1/4$ from the origin, and so the β-log singularity can be bounded above by a finite constant. Then, for any $\varepsilon > 0$, by letting $q = (\beta \vee \sqrt{2d})/\gamma$, which is strictly less than $2d/\gamma^2$, we have that

$$\begin{aligned}&\mathbb{P}\Big(\max_{x\in\Sigma_n\cap A_k}\mu_{\mathbf{h}-\beta\log|\cdot|,\gamma}\big(B(x,2^{-n})\big) \geq 2^{-n(d+\gamma^2/2-(\beta\vee\sqrt{2d})\gamma-\varepsilon)},\ e^{\gamma\mathbf{h}_{2^{-k}}(0)} \leq 2^{n\varepsilon/2}\Big)\\ &\qquad\lesssim \sum_{x\in\Sigma_n\cap A_k} 2^{(n-k)[q(d+\gamma^2/2-(\beta\vee\sqrt{2d})\gamma)]}2^{-qn\varepsilon/2}\mathbb{E}\big[\mu_{\mathbf{h}^k}\big(B(2^kx,2^{k-n})\big)^q\big]\\ &\qquad\leq 2^{(n-k)[q(d+\gamma^2/2-(\beta\vee\sqrt{2d})\gamma)+d]}2^{-qn\varepsilon/2}\mathbb{E}\big[\mu_{\mathbf{h}^k}\big(B(2^kx,2^{k-n})\big)^q\big]\\ &\qquad\lesssim 2^{(n-k)[d+q(d+\gamma^2/2-(\beta\vee\sqrt{2d})\gamma)-\xi_\gamma(q)]}2^{-nq\varepsilon/2} \leq 2^{-nq\varepsilon/2}\,,\end{aligned} \tag{6.A.1}$$

where we used Markov's inequality, and the fact that thanks to [BP24, Theorem 3.26], the multifractal spectrum of GMC measures $\mu_{\mathbf{h},\gamma}$ is given by $\xi_\gamma(q) \overset{\text{def}}{=} (d+\gamma^2/2)q - \gamma^2 q^2/2$. Moreover, since $\mathbf{h}_{2^{-k}}(0)$ is centred Gaussian with variance of order k (see Lemma 6.2.9), the probability that $e^{\gamma \mathbf{h}_{2^{-k}}(0)} > 2^{n\varepsilon/2}$ is summable over $n \geq 2$ and $k \in [n-2]_0$. Therefore, combining this fact with the bound in (6.A.1), we obtain that

$$\sum_{n\geq 2}\sum_{k=0}^{n-2} \mathbb{P}\Big(\max_{x\in\Sigma_n\cap A_k} \mu_{\mathbf{h}-\beta\log|\cdot|,\gamma}\big(B(x,2^{-n})\big) \geq 2^{-n(d+\gamma^2/2-(\beta\vee\sqrt{2d})\gamma-\varepsilon)}\Big) < \infty \,.$$

Hence, thanks to Borel–Cantelli lemma, we get that $\mathbb{P}$-almost surely, for all $n \geq 2$ and $k \in [n-2]_0$, it holds that

$$\max_{x\in\Sigma_n\cap A_k} \mu_{\mathbf{h}-\beta\log|\cdot|,\gamma}\big(B(x,2^{-n})\big) \lesssim 2^{-n(d+\gamma^2/2-(\beta\vee\sqrt{2d})\gamma-\varepsilon)} \,, \tag{6.A.2}$$

for some random implicit constant. Thanks to Lemma 6.A.1, we also have that $\mathbb{P}$-almost surely, for all $n \in \mathbb{N}_0$, it holds that

$$\mu_{\mathbf{h}-\beta\log|\cdot|,\gamma}\big(B(0,2^{-n})\big) \lesssim 2^{-n(d+\gamma^2/2-(\beta\vee\sqrt{2d})\gamma-\varepsilon)} \,, \tag{6.A.3}$$

Therefore, the conclusion follows by combining (6.A.2) and (6.A.3) and to the fact that for any $r \in (0,1]$, and $x \in B(0,1)$, there exist $n \in \mathbb{N}_0$ such that $2^{-(n+1)} < r \leq 2^{-n}$, and $B(x,r)$ is contained in the union of a finite numbers of balls of the form $B(z,2^{-n})$ with $z \in \Sigma_n$. □

Lemma 6.A.4. *Let $d \geq 2$ and $\gamma \in (0,\sqrt{2d})$. It holds $\mathbb{P}$-almost surely for any $s \in (0,1]$, any $r \in (0,s]$, and all $x \in B(0,s)$ that*

$$\mu_{\mathbf{h},\gamma}\big(B(x,r)\big) \lesssim s^{\sqrt{2d}\gamma-\varepsilon} r^{(d+\gamma^2/2-\sqrt{2d}\gamma)-\varepsilon} \,, \qquad \forall\, \varepsilon > 0 \,,$$

for some random implicit constant.

Proof. For $k \leq n \in \mathbb{N}_0$, by using the coordinate change formula (Proposition 6.2.10), we have that for all $x \in B(0,2^{-k})$, it holds that

$$\mu_{\mathbf{h},\gamma}\big(B(x,2^{-n})\big) = 2^{-k\gamma Q} e^{\gamma \mathbf{h}_{2^{-k}}(0)} \mu_{\mathbf{h}^k,\gamma}\big(B(2^k x, 2^{k-n})\big) \,,$$

where, by the scale invariance of the law of $\mathbf{h}$ modulo additive constant, $\mathbf{h}^k(\cdot) \overset{\text{def}}{=} \mathbf{h}(2^{-k}\cdot) - \mathbf{h}_{2^{-k}}(0) \overset{\text{law}}{=} \mathbf{h}$. We let $\Sigma_{k,n} \overset{\text{def}}{=} 2^{-n}\mathbb{Z}^d \cap B(0,2^{-k})$ be the lattice $2^{-n}\mathbb{Z}^d$ restricted to the ball $B(0,2^{-k})$. For any $\varepsilon > 0$, by letting $q = \sqrt{2d}/\gamma$, we have that

$$\begin{aligned}
&\mathbb{P}\Big(\max_{x\in\Sigma_{k,n}} \mu_{\mathbf{h},\gamma}\big(B(x,2^{-n})\big) \geq 2^{-k(\sqrt{2d}\gamma-\varepsilon)} 2^{-n(d+\gamma^2/2-\sqrt{2d}\gamma-\varepsilon)},\ e^{\gamma \mathbf{h}_{2^{-k}}(0)} \leq 2^{(k+n)\varepsilon/2}\Big) \\
&\qquad \leq \mathbb{P}\Big(\max_{x\in\Sigma_{k,n}} \mu_{\mathbf{h}^k,\gamma}\big(B(2^k x, 2^{k-n})\big) \geq 2^{(k-n)(d+\gamma^2/2-\sqrt{2d}\gamma)+(k+n)\varepsilon/2}\Big) \\
&\qquad \leq \sum_{x\in\Sigma_{k,n}} 2^{-(k+n)q\varepsilon/2} 2^{(n-k)(d+\gamma^2/2-\sqrt{2d}\gamma)+(k+n)\varepsilon/2} \mathbb{E}\big[\mu_{\mathbf{h}^k,\gamma}\big(B(2^k x, 2^{k-n})\big)^q\big] \\
&\qquad \leq 2^{-(k+n)q\varepsilon/2} 2^{(n-k)[d+q(d+\gamma^2/2-\sqrt{2d}\gamma)-\xi_\gamma(q)]} = 2^{-(k+n)q\varepsilon/2} \,,
\end{aligned} \tag{6.A.4}$$

where we used Markov's inequality, and the fact that thanks to [BP24, Theorem 3.26], the multifractal spectrum of GMC measures $\mu_{\mathbf{h},\gamma}$ is given by $\xi_\gamma(q) \overset{\text{def}}{=} (d+\gamma^2/2)q - \gamma^2 q^2/2$. Moreover, since $\mathbf{h}_{2^{-k}}(0)$ is centred Gaussian with variance of order k (see Lemma 6.2.9), the probability that $e^{\gamma \mathbf{h}_{2^{-k}}(0)} > 2^{(k+n)\varepsilon/2}$ is

summable over $k \leq n \in \mathbb{N}_0$. Therefore, combining this fact with the bound in (6.A.4), we get that

$$\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\mathbb{P}\Big(\max_{x\in\Sigma_{k,n}}\mu_{\mathbf{h},\gamma}\big(B(x,2^{-n})\big)\geq 2^{-k(\sqrt{2d}\gamma-\varepsilon)}2^{-n(d+\gamma^2/2-\sqrt{2d}\gamma-\varepsilon)}\Big)<\infty\,.$$

Hence, thanks to Borel–Cantelli lemma, we get that $\mathbb{P}$-almost surely, for all $k \leq n \in \mathbb{N}_0$, it holds that

$$\max_{x\in\Sigma_{k,n}}\mu_{\mathbf{h},\gamma}\big(B(x,2^{-n})\big)\lesssim 2^{-k(\sqrt{2d}\gamma-\varepsilon)}2^{-n(d+\gamma^2/2-\sqrt{2d}\gamma-\varepsilon)}\,,$$

for some random implicit constant. This is sufficient to conclude since for any $s \in (0,1]$, $r \in (0,s]$, and $x \in B(0,s)$, there exist $k \leq n \in \mathbb{N}_0$ such that $2^{-(k+1)} < s \leq 2^{-k}$, $2^{-(n+1)} < r \leq 2^{-(n+1)}$, and $B(x,r)$ is contained in the union of a finite numbers of balls of the form $B(z,2^{-n})$ with $z \in \Sigma_{n,k}$. □

6.B Proof of Lemma 6.4.19

Throughout this appendix, we fix $\beta \in (-\infty, Q)$, and we define the field

$$\overline{\mathbf{h}} \overset{\text{def}}{=} \mathbf{h} - \beta\log|\cdot|\,.$$

For $x_1,\dots,x_n \in \mathbb{R}^d$ and $0 < s_1 < \ldots < s_n \in \mathbb{R}_0^+$, we denote by $p_{s_1,\dots,s_n}(0,x_1,\dots,x_n)$ the transition probability density of the d-dimensional Brownian motion started from 0, i.e.,

$$\mathbf{P}_0(B_{s_1}\in dx_1,\ \dots B_{s_n}\in dx_n) = p_{s_1,\dots,s_n}(0,x_1,\dots,x_n)dx_1\cdots dx_n\,.$$

Furthermore, for $t \geq 0$ and $x \in \mathbb{R}^d$ we define the function $\mathcal{G}_t(x,\cdot) : \mathbb{R}^d \to \mathbb{R}$ by letting

$$\mathcal{G}_t(x,y) \overset{\text{def}}{=} \int_0^t p_s(x,y)ds\,. \tag{6.B.1}$$

Lemma 6.B.1. *For $d > 2$, $Q > \sqrt{2d}$, $n \in \mathbb{N}$, $t \geq 0$, and $\delta > 0$ it holds $\mathbb{P}$-almost surely that*

$$\begin{aligned}&\mathbf{E}_0\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t))^n\mathbb{1}_{\{[B]_t\subseteq B(0,t^{1/2-\delta})\}}\big]\\ &\qquad\lesssim\int_{(B(0,t^{1/2-\delta}))^n}\big(\mathcal{G}_t(0,x_1)\cdots\mathcal{G}_t(x_{n-1},x_n)\big)\mu_{\overline{\mathbf{h}},\alpha}(dx_n)\cdots\mu_{\overline{\mathbf{h}},\alpha}(dx_1)\,,\end{aligned} \tag{6.B.2}$$

where the implicit constant only depends on n.

Proof. By an immediate generalisation of Lemma 6.4.14, we observe that for all $t \geq 0$, the left-hand side of (6.B.2) is bounded above by a constant (only depending on n) times

$$\int_{(B(0,t^{1/2-\delta}))^n}\left(\int_0^t\int_{s_1}^t\cdots\int_{s_{n-1}}^t p_{s_1,\dots,s_n}(0,x_1,\dots,x_n)ds_n\cdots ds_1\right)\mu_{\overline{\mathbf{h}},\alpha}(dx_n)\cdots\mu_{\overline{\mathbf{h}},\alpha}(dx_1)\,.$$

For $0 < s_1 < \ldots < s_n$, we have that the following decomposition holds

$$p_{s_1,\dots,s_n}(0,x_1,\dots,x_n) = p_{s_1}(0,x_1)p_{s_2-s_1}(x_1,x_2)\cdots p_{s_n-s_{n-1}}(x_{n-1},x_n)\,,$$

and so, by performing the change of variables $s_j \mapsto s_j - s_{j-1}$ for all $j \in [n]$, the desired result follows readily. □

Thanks to Lemma 6.B.1, in order to prove Lemma 6.4.19, is suffices to estimate the quantity on the right-hand of (6.B.2). We start with the following lemma.

Lemma 6.B.2. *For $t \geq 0$ and $x \in \mathbb{R}^d$, consider the function $\mathcal{G}_t(x, \cdot) : \mathbb{R}^d \to \mathbb{R}$ defined in (6.B.1). Then, it holds that*

$$\mathcal{G}_t(x, y) \lesssim |y - x|^{2-d} , \qquad \forall\, x, y \in \mathbb{R}^d . \tag{6.B.3}$$

Proof. For $t \geq 0$ and x, $y \in \mathbb{R}^d$, we have that

$$\mathcal{G}_t(x, y) \leq \mathcal{G}_\infty(x, y) = \frac{\Gamma(d/2 - 1)}{2\pi^{d/2}} |y - x|^{2-d} ,$$

where the last equality in the above display follows from a simple computation (see e.g. [MP10, Theorem 3.33]). □

Now, combining (6.B.2) with (6.B.3), it suffices to estimate and get the "right asymptotic" as $t \to 0$ of the following quantity

$$\int_{B(0,t^{1/2-\delta})} |x_1|^{2-d} \cdots \int_{B(0,t^{1/2-\delta})} |x_n - x_{n-1}|^{2-d} \mu_{\overline{\mathbf{h}},\alpha}(dx_n) \cdots \mu_{\overline{\mathbf{h}},\alpha}(dx_1) .$$

To this end, we start with the following lemma.

Lemma 6.B.3. *Let $d > 2$, $Q > \sqrt{2d}$, and $\beta \in (-\infty, Q)$. It holds $\mathbb{P}$-almost surely for any $r \in (0, 1]$ and all $x \in B(0, r)$ that*

$$\int_{B(0,r)} |x - y|^{2-d} \mu_{\overline{\mathbf{h}},\alpha}(dy) \lesssim r^{(2+\alpha^2/2-\alpha\beta)-\varepsilon} , \qquad \forall\, \varepsilon > 0 , \tag{6.B.4}$$

for some random implicit constant.

Proof. For $\beta < 0$ the result is a straightforward consequence of Lemma 6.A.1. Hence, we can focus on the case $\beta \in (0, Q)$. In what follows, for $x \in \mathbb{R}^d$ and $r_2 > r_1 \geq 0$, we write $A(x, r_2, r_1)$ as a shorthand for the annulus $B(x, r_2) \setminus B(x, r_1)$. For $r \in (0, 1]$, we fix a point $x \in B(0, r)$ and we split the ball $B(0, r)$ into four different regions as depicted in Figure 6.2.

Region 1: The first region consists in the blue ball $B(0, |x|/2)$. In this case, using Lemma 6.A.1, for any $\varepsilon > 0$ small enough, we have $\mathbb{P}$-almost surely that

$$\int_{B(0,|x|/2)} |x-y|^{2-d} \mu_{\overline{\mathbf{h}},\alpha}(dy) \lesssim |x|^{2-d} \mu_{\overline{\mathbf{h}},\alpha}\big(B(0, |x|/2)\big) \lesssim |x|^{2-d} |x|^{(d+\alpha^2/2-\alpha\beta)-\varepsilon} \lesssim r^{(2+\alpha^2/2-\alpha\beta)-\varepsilon} .$$

Region 2: The second region consists in the pink ball $B(x, |x|/2)$. In this case, using the fact that all points in this region are at distance at least $|x|/2$ from 0, and using Lemma 6.A.4, for any $\varepsilon > 0$ small enough, we obtain $\mathbb{P}$-almost surely that

$$\begin{aligned} \int_{B(x,|x|/2)} |x - y|^{2-d} \mu_{\overline{\mathbf{h}},\alpha}(dy) &= \sum_{k=0}^{\infty} \int_{A(x,e^{-k}|x|/2,e^{-(k+1)}|x|/2)} |x - y|^{2-d} \mu_{\overline{\mathbf{h}},\alpha}(dy) \\ &\lesssim \sum_{k=0}^{\infty} (e^{-k}|x|)^{2-d} \mu_{\overline{\mathbf{h}},\alpha}\big(B(x, e^{-k}|x|/2)\big) \end{aligned}$$

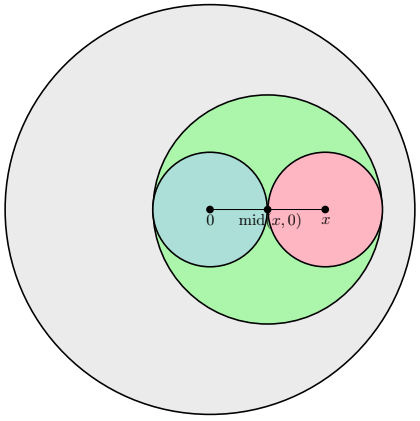


FIGURE 6.2: Subdivision of the ball $B(0,r)$ into four different regions. The first region consists in the blue ball $B(0,|x|/2)$. The second region consists in the pink ball $B(x,|x|/2)$. The third region consists in the green set $B(\text{mid}(0,x),2|x|) \setminus B(0,|x|/2) \setminus B(x,|x|/2)$. The fourth region consists in the grey set $B(0,r) \setminus B(\text{mid}(0,x),2|x|)$.

$$\begin{aligned}
&\lesssim |x|^{2-d-\alpha\beta} \sum_{k=0}^{\infty} e^{-k(2-d)} \mu_{\mathbf{h},\alpha}\big(B(x, e^{-k}|x|/2)\big) \\
&\lesssim |x|^{(2+\alpha^2/2-\alpha\beta)-\varepsilon} \sum_{k=0}^{\infty} e^{-k[(2+\alpha^2/2-\sqrt{2d}\alpha)-\varepsilon]} \\
&\lesssim |x|^{(2+\alpha^2/2-\alpha\beta)-\varepsilon} \leq r^{(2+\alpha^2/2-\alpha\beta)-\varepsilon} ,
\end{aligned}$$

where here we used the fact that $2 + \alpha^2/2 - \sqrt{2d}\alpha > 0$, for all $Q > \sqrt{2d}$.

Region 3: The third region consists in the green set $B(\text{mid}(0,x),2|x|) \setminus B(0,|x|/2) \setminus B(x,|x|/2)$. In this case, using the fact that all points in this region are at distance at least $|x|/2$ from x, and using Lemma 6.A.1, for any $\varepsilon > 0$ small enough, we have $\mathbb{P}$-almost surely that

$$\begin{aligned}
\int_{B(\text{mid}(0,x),2|x|)\setminus B(0,|x|/2)\setminus B(x,|x|/2)} |x-y|^{2-d} \mu_{\overline{\mathbf{h}},\alpha}(dy) &\lesssim |x|^{2-d} \mu_{\overline{\mathbf{h}},\alpha}\big(B(0,2|x|)\big) \\
&\lesssim |x|^{2-d} |x|^{(d+\alpha^2/2-\alpha\beta)-\varepsilon} \\
&\leq r^{(2+\alpha^2/2-\alpha\beta)-\varepsilon} .
\end{aligned}$$

Region 4: The fourth region consists in the grey set $B(0,r) \setminus B(\text{mid}(0,x),2|x|)$. In this case, using Lemma 6.A.1, we get that, for any $\varepsilon > 0$ small enough, it holds $\mathbb{P}$-almost surely that

$$\begin{aligned}
\int_{B(0,r)\setminus B(\text{mid}(0,x),2|x|)} |x-y|^{2-d} \mu_{\overline{\mathbf{h}},\alpha}(dy) &\lesssim \sum_{k=0}^{-\log(|x|/(2r))-1} \int_{A(x,2e^{-k}r,2e^{-(k+1)}r)} |x-y|^{2-d} \mu_{\overline{\mathbf{h}},\alpha}(dy) \\
&\leq \sum_{k=0}^{-\log(|x|/(2r))-1} (e^{-k}r)^{2-d} \mu_{\overline{\mathbf{h}},\alpha}\big(B(0,|x|+2e^{-k}r)\big) \\
&\lesssim \sum_{k=0}^{-\log(|x|/(2r))-1} (e^{-k}r)^{2-d} (|x|+2e^{-k}r)^{(d+\alpha^2/2-\alpha\beta)-\varepsilon} \\
&\lesssim \sum_{k=0}^{-\log(|x|/(2r))-1} (e^{-k}r)^{2-d} (e^{-k}r)^{(d+\alpha^2/2-\alpha\beta)-\varepsilon} \\
&\lesssim r^{(d+\alpha^2/2-\alpha\beta)-\varepsilon} ,
\end{aligned}$$

where to get the penultimate inequality, we used the fact that $|x| \lesssim e^{-k} r$ for all $k \leq -\log(|x|/(2r)) - 1$, and in the last inequality we used the fact that $2 + \alpha^2/2 - \alpha\beta > 0$, for all $\beta \in (-\infty, Q)$.

Therefore, the conclusion follows by the arbitrariness of $x \in B(0,r)$ and by combining the four bounds in the different regions. □

We are now ready to prove Lemma 6.4.19.

Proof of Lemma 6.4.19. Let $d \geq 2$, $Q > \sqrt{2d}$, and $\beta \in (-\infty, Q)$. For all $n \in \mathbb{N}$, $\varepsilon > 0$, $\delta > 0$, and $t \in (0,1]$, we have $\mathbb{P}$-almost surely that

$$\begin{aligned}
&\mathbf{E}_0\big[(\mathrm{F}_{\overline{\mathbf{h}},\alpha}(t))^n \mathbb{1}_{\{[B]_t \subseteq B(0,t^{1/2-\delta})\}}\big] \\
&\qquad \overset{(6.B.2)}{\lesssim} \int_{(B(0,t^{1/2-\delta}))^n} \big(\mathcal{G}_t(0,x_1)\cdots\mathcal{G}_t(x_{n-1},x_n)\big)\mu_{\overline{\mathbf{h}},\alpha}(dx_n)\cdots\mu_{\overline{\mathbf{h}},\alpha}(dx_1) \\
&\qquad \overset{(6.B.3)}{\lesssim} \int_{B(0,t^{1/2-\delta})} |x_1|^{2-d} \cdots \int_{B(0,t^{1/2-\delta})} |x_n - x_{n-1}|^{2-d} \mu_{\overline{\mathbf{h}},\alpha}(dx_n)\cdots\mu_{\overline{\mathbf{h}},\alpha}(dx_1)\,.
\end{aligned} \tag{6.B.5}$$

Thanks to Lemma 6.B.3, since x_{n-1} is sampled inside the ball $B(0,t^{1/2-\delta})$, for any $\varepsilon > 0$, the inner most integral in (6.B.5) can be bounded above $\mathbb{P}$-almost surely by

$$\sup_{x \in B(0,t^{1/2-\delta})} \int_{B(0,t^{1/2-\delta})} |x_n - x|^{2-d} \mu_{\overline{\mathbf{h}},\alpha}(dx_n) \lesssim t^{(1/2-\delta)(2+\alpha^2/2-\alpha\beta)-\varepsilon}\,,$$

for some random implicit constant. By applying the same techniques $n-1$ more times to bound the remaining integrals in (6.B.5), the desired conclusion follows. □

Appendix A

Gaussian toolbox

We collect here some well-known results on Gaussian fields that are used throughout the thesis. In all the subsequent lemmas, we let $\mathrm{D} \subseteq \mathbb{R}^d$ be a fixed domain. We begin with the well known Cameron–Martin theorem [CM44].

Lemma A.0.1 (Girsanov's theorem)**.** *Consider an almost surely continuous centred Gaussian field* $(\mathrm{X}(x))_{x\in\mathrm{D}}$ *and a Gaussian random variable* Z *which belongs to the* L^2 *closure of the vector space spanned by* $(\mathrm{X}(x))_{x\in\mathrm{D}}$*. Let* $\mathbf{F} : \mathcal{C}(\mathrm{D}) \to \mathbb{R}$ *be a bounded measurable functional. Then the following equality holds*

$$\mathbb{E}\Big[e^{\mathrm{Z}-\frac{\mathbb{E}[\mathrm{Z}^2]}{2}}\mathbf{F}\left(\mathrm{X}(\cdot)\right)\Big] = \mathbb{E}\big[\mathbf{F}\big(\mathrm{X}(\cdot)+\mathbb{E}[\mathrm{X}(\cdot)\mathrm{Z}]\big)\big]\,.$$

We now state a fundamental tool in the study of GMC measures, which is Kahane's convexity inequality. Essentially, this is an inequality that allows to compare GMC measures associated with two different fields. It can be stated as follows, and we refer to [BP24, Theorem 3.18] or [RV14a, Theorem 2.1] for a proof and additional references.

Lemma A.0.2 (Kahane's convexity inequality)**.** *Consider two almost surely continuous centred Gaussian fields* $(\mathrm{X}(x))_{x\in\mathrm{D}}$ *and* $(\mathrm{Y}(x))_{x\in\mathrm{D}}$ *satisfying*

$$\mathbb{E}\big[\mathrm{X}(x)\mathrm{X}(y)\big] \le \mathbb{E}\big[\mathrm{Y}(x)\mathrm{Y}(y)\big]\,, \qquad \forall\, x,y\in\mathrm{D}\,.$$

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}$ *be a convex function with at most polynomial growth at* 0 *and* ∞*, and let* $f : \mathrm{D} \to [0,\infty)$ *be a continuous function. Then, we have that*

$$\mathbb{E}\left[\varphi\left(\int_{\mathrm{D}} f(x)e^{\mathrm{X}(x)-\frac{1}{2}\mathbb{E}[\mathrm{X}(x)^2]}dx\right)\right] \le \mathbb{E}\left[\varphi\left(\int_{\mathrm{D}} f(x)e^{\mathrm{Y}(x)-\frac{1}{2}\mathbb{E}[\mathrm{Y}(x)^2]}dx\right)\right]\,.$$

We have the following standard concentration inequality for Gaussian fields which is known as Borell-TIS inequality. We refer to [AT07, Theorem 2.1.1] for a proof.

Lemma A.0.3 (Borell-TIS inequality)**.** *Consider an almost surely continuous centred Gaussian field* $(\mathrm{X}(x))_{x\in\mathrm{D}}$*. Then it holds that*

$$\mathbb{P}\left(\left|\sup_{x\in\mathrm{D}}\mathrm{X}(x) - \mathbb{E}\left[\sup_{x\in\mathrm{D}}\mathrm{X}(x)\right]\right| > t\right) \le 2e^{-\frac{t^2}{2\sigma_{\mathrm{D}}^2}}\,,$$

for all $t \ge 0$*, where* $\sigma_{\mathrm{D}}^2 := \sup_{x\in\mathrm{D}}\mathbb{E}[\mathrm{X}(x)^2]$.

We now state Fernique's majorizing criterion, and we refer to [AT07, Theorem 1.3.3] for a proof.

Lemma A.0.4 (Fernique's majorizing criterion)**.** *Consider an almost surely continuous centred Gaussian field* $(\mathrm{X}(x))_{x\in\mathrm{D}}$*. Consider the pseudometric on* D *defined as follows*

$$\mathrm{d}_X(x,y) := \sqrt{\mathbb{E}[|\mathrm{X}(x)-\mathrm{X}(y)|^2]}, \qquad \forall\, x,y\in\mathrm{D}\ .$$

Then there exists a universal constant C *such that for any probability measure* σ *on* D*,*

$$\mathbb{E}\left[\sup_{x\in\mathrm{D}}\mathrm{X}(x)\right] < C\sup_{x\in\mathrm{D}}\int_0^\infty \sqrt{-\log\sigma(B_{\mathrm{X}}(x,r))}dr\ ,$$

where $B_{\mathrm{X}}(x,r)\stackrel{\text{def}}{=}\{y\in\mathrm{D}\,:\,\mathrm{d}_{\mathrm{X}}(x,y)<r\}$.

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