The Linear and Nonlinear Theory

of

Baroclinic Instability

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CONTENTS

ABSTRACT

A brief derivation of the quasi-geostrophic potential vorticity equation, that is to say the governing equation, is given for a zonal flow of baroclinic fluid including weak internal dissipative effects. This leads to an ordinary differential equation for linearised normal mode disturbances of the fourth order. In the limiting case when internal viscous and diffusive effects are ignored this equation is of the second order and for a linear, vertical shear velocity profile is the confluent hypergeometric equation.

A set of boundary conditions which are standard for baroclinic flow models is chosen. When the flow is entirely free of dissipative effects the dispersion relation thus formed recovers the models of Charney (1947), Eady (1949) and Green (1960) in appropriate limits. The number and behaviour of the zeros of this dispersion relation and the implied linear stability of the model are discussed with an attempted minimum of recourse to numerical methods.

Amplitude equations implied by weakly nonlinear stability theory are derived for some specific examples taken from the Eady model, with attention to matching these equations in the different asymptotic regimes which are involved by the inclusion of viscous effects through Ekman layers on the horizontal top and bottom boundaries alone. As has been noted elsewhere in the literature one of these amplitude equations, in common with an existing analogous result for the two-layer model of baroclinic instability, is similar to the Lorenz set but now has one member of the set in a form which is repeated an infinite number of times. The effect of truncation of the repeated modes on linear and nonlinear stability for some exact equilibrium solutions of this infinite set, which correspond to the three fixed point solutions of the Lorenz set, is studied by analytical and asymptotic methods. The type of the second bifurcation of the equations implied by nonlinear theory is classified for the specific case of the coefficients appropriate to the two-layer model of baroclinic instability.

INTRODUCTION

The motivation for the study of baroclinic flow is, in the first instance, geophysical. It is an observed feature of the atmospheres surrounding most rotating planets that within the regions of mid-latitude an atmosphere divides itself into belts which are to a close approximation azimuthal about the axis of planetary rotation. Within each belt the flow is more or less unidirectional but is characterised by the presence of large scale cyclones and anti-cyclones, that is, regions of low and high pressure, together with variations of smaller length scale which are of a generally more turbulent nature. For the Earth this type of flow may be considered to describe the westerly winds in the temperate latitudes of the northern hemisphere which carry with them often violent and unpredictable weather systems and are typified by regions of high and low pressure of the order of 1000 km across. The circulation of the oceans, although bounded by the continents, also shows large scale streaming in a statistical sense with a superimposed more random fine scale structure.

If the increased insolation of the atmosphere and oceans with approach to the equator is to be of importance in determining the pattern of such a flow then the fluid density must be a function of both the local pressure and temperature, the fluid is then said to be baroclinic. The effect of planetary rotation upon the momentum balance of the fluid away from the hydrostatic state of rest acts as so as to tilt surfaces of constant fluid density away from their hydrostatic orientation of coincidence with the potential surfaces of the gravitational field so that they tilt downwards to the North with an aggregation of cold fluid at low altitudes there and of warmer fluid at higher altitudes nearer to the Equator. A steady westerly azimuthal flow or wind is concomitant with this so that a baroclinic flow may then exist on a background density gradient which is not in a state of minimum potential energy.

It is not difficult to envisage that this basic flow pattern may be unstable with the development of wave-like pressure disturbances developing upon the basic flow and releasing the available potential energy as kinetic energy, the whole being driven by the effects of rotation and differential heating. This phenomenon, of baroclinic instability, thus comes under the purview of the theory of hydrodynamic stability but the salient point which distinguishes it from the classical problems of the subject, the stability of parallel flows, Benard convection and Taylor vortex flow is that the important or 'real' examples of baroclinic flow are observed, as are the atmosphere and oceans, to be permanently in a fluctuating or unstable state. The basic, disturbance free state of steady flow is hypothetical inasmuch as an understanding of the circulation of the atmospheres and oceans described by the above simple physical mechanism depends upon the conjecture of a 'suitable' basic flow from a class of possible basic flows, the study of its evolution in the unstable state and the correlation of the results of this theoretical evolution with atmospheric data.

In this way the problem of baroclinic instability may be regarded as one of inversion which is familiar in the geophysical sciences, in this example we may ask - given the data of an observed unstable atmosphere, to what initial or basic state does the observed unstable state correspond?

The more specific questions to be discussed here concern a basic state of unidirectional, zonal, westerly (that is West to East) flow which is prescribed to be in a state of linear vertical shear. This is the type of flow assumed in the originative work of Charney (1947) and Eady (1949); Charney's model is of a semi-infinite atmosphere incorporating the effects of variation of planetary vorticity with latitude and finite vertical scale height for the density stratification of the basic state, where both of these latter two features are sufficient for an instability to occur; Eady's model is of an atmosphere without any of these features but which may be rendered unstable by the adoption of finite vertical extent or of an upper boundary to the troposphere above which lies an infinitely stable stratosphere. As regards the relevance of this velocity profile there is much evidence in the literature to suggest that the length scales and behaviour predicted by theory are of a reasonable fit with observed phenomena. However, the stability of the zonal flow of Eady's model to non-zonal North-South perturbations of the basic velocity profile which vanish on the north and south edges of a zonal belt has been studied by McIntyre (1970). For this reason no check will be made here on the values of length scales and other physical parameters predicted by theory but I hope that such an inquiry would be entirely possible.

To this end, the first chapter contains a brief derivation of the governing equation and boundary conditions for a general formulation of the problem from the Navier-Stokes equations. This includes the effects of a small internal molecular viscosity and diffusivity which lead, in the second chapter, to an ordinary differential equation for linearised normal mode disturbances identifiable with the Orr-Sommerfeld equation for the stability of parallel flows in that it is of fourth order with polynomial coefficients and no finite singularities. In all other respects this model incorporates the contrasting features of the models of Charney and Eady, that is; an atmosphere of finite vertical extent with finite vertical scale height for the density stratification in the basic state and where the latitudinal variation of the Earth's vorticity is approximated by the linear idealisation of the β-plane effect.

The consequence of these additional terms is briefly discussed in an attempt to align the case within the extensive theory of the Orr-Sommerfeld equation, they are then dropped so that the governing differential equation is of second order. The analysis still includes the effect of viscous Ekman layers which control the flow in a way different from that of an internal viscosity by modifying the top and bottom boundary conditions for an entirely inviscid problem in the way originally proposed by Barcilon (1964). In the completely inviscid limit the model is a hybrid of the Charney and Eady models similar to that considered by Green (1960) and Garcia and Norscini (1969) but which includes a finite vertical scale height for the density stratification, it may be considered as a matching problem between the limiting cases of Charney and Eady.

After forming the appropriate dispersion relation it is found that, within the confines of linear theory, the β-plane effect has a destabilising influence on the Eady limit and that finite height has a slightly stabilising influence on the Charney limit. The parameter space associated with the stability problem is three dimensional in the completely inviscid case and it is regrettable that the most convenient choice of coordinates with which to explore the behaviour in the complex plane of the roots of the dispersion relation determining the flow stability detracts from an intuitive grasp of the physical mechanisms at play. Throughout most of parameter space the morphology of the zeros in the complex plane bears more resemblance to that of the Charney than of the Eady limit. It will transpire that the Eady limit is singular in a particular sense to be defined.

In the third and subsequent chapters the nonlinear theory of unstable flow is applied to some specific problems posed for the Eady model with the inclusion of viscous effects made manifest by Ekman layers at the top and bottom boundaries as described in the aforementioned way. Results in this field already exist, notably those of Drazin (1970 and 1972) and are paralleled by those for the simpler and more widely studied two-layer model of baroclinic instability, where the basic density and velocity profiles are step functions, which was originally due to Phillips (1954) and has been extended to the nonlinear regime in particular by Pedlosky (1970 et.seq.). The parameter space associated with linear instability of this model is again three dimensional with Cartesian coordinates corresponding to downstream wavenumber, thermal Rossby or Burger number and a viscosity parameter.

The third chapter contains an initial description of the surface of linear neutral stability in the chosen parameter space and then continues with the derivation of an equation for the weakly nonlinear evolution of a packet of linearly unstable waves when the viscosity parameter is of order one asymptotically as the supercriticality of the Burger number tends to zero at a stationary point of the neutral surface. That is, where the flow involves the interaction of a principal unstable mode with a small continuous spectrum of unstable side band perturbations with respect to the downstream wavenumber. The problem is a slight extension of part of Drazin's 1972 article.

In the fourth chapter a scheme is developed to match the viscous and completely inviscid weakly nonlinear evolution equations in the viscous and completely inviscid limits. This requires a scheme where the viscosity parameter is linked to the supercriticality as the latter tends to zero and on the application of accepted boundary conditions for the problem leads to an amplitude equation which may be expressed either as a single integrodifferential equation or as an infinite set of coupled, first order, nonlinear ordinary differential equations. Some attention has been made in this particular problem to reveal the peculiarities of the system leading to the amplitude equation and in establishing the matching of its coefficients with the existing results in the appropriate limits.

It is seen that the amplitude equations for the Eady model are of exactly the same form as those derived under similar circumstances by Pedlosky for the two-layer model, the only difference being the obvious one of differing coefficients arising from the distinct physical properties of the models.

The fifth and sixth chapters together with the notes in appendices C and E expound some of the properties of the amplitude equation of chapter four. The most important point to mention here is that the nonlinear theory predicts finite amplitude wave trains which are asymptotically steady and stable towards the viscous regime but singly periodic or vacillatory in the completely inviscid limit and that the matching amplitude equation is similar to the Lorenz set as proposed in the theory of Benard convection (Lorenz 1963). The Lorenz set and the chaotic solutions it exhibits have been the subject for much research in more recent years, the set has since been found in branches of physics other than that of fluid mechanics where the random behaviour of its solutions in phase space may be equated with the more or less random fluctuations of turbulent flow in nature.

The techniques used in the work presented here do not delve far into the abstract mathematical theory of dynamical systems and are, I hope, generally accessible since the form of the matching amplitude equation seems to beg for analytic enquiry on account of:

(i) the sensitive dependence upon initial conditions of dynamical systems, such as the Lorenz set, which possess strange attractors.

(ii) the complicated topology of the attracting structures of such systems.

(iii) the status of the amplitude equation as a system with infinitely many degrees of freedom.

Now the first two traits for the Lorenz set have been found by largely numerical investigations from many sources. At their inception Lorenz recognised these properties in his equations and was aware of both the difficulties of discrete representation and numerical integration by computer and also of the consequent need to perform integrations with the continued uprating of machine resolution. A short quantitative study of the representational errors which may be involved in applying computational methods to dynamical systems has been made by Levy (1982). It is the third feature alone which requires further care to be taken since any computation will necessarily involve truncation and thus misrepresentation of the amplitude equation itself. Pedlosky and Frenzen (1980) have integrated this form of amplitude equation numerically and, again aware of the nature of the approximations involved, found a very complicated bifurcation sequence of period doublings and chaotic solutions which are at times sensitive to machine resolution.

The seventh chapter combines the novel features of the third and fourth in its consideration of a parameter regime where the nonlinear evolution equation incorporates the interaction of a packet of linearly unstable waves centred on a principal wave number at a stationary point of the neutral surface and the type of nonlinearity which is of consequence in the matching amplitude equation of chapter four.

Some effort has been made to ensure that variables are assigned separate and appropriate symbols. The second requirement is at variance with the first so that duplication may occur between chapters but the use of suffices and introduction of each quantity with its corresponding symbol as it occurs is made within the text.

CHAPTER ONE

THE MODEL, GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

Consider a set of spherical polar coordinates (r,θ,ϕ) with origin $r = 0$ at the centre of a solid sphere of radius r_0, both rotating with constant angular velocity $\underline{\Omega}$ about the axis defined by latitude $\theta = \frac{\pi}{2}$ so that the direction of rotation is in the sense of increasing longitudinal angle ϕ. The equations of motion for a fluid of density ρ_* at pressure p_* with velocity components u_* in the zonal or West to East direction, v_* in the meridional or South to North direction and w_* in the vertical or radial direction at time t_* are

$$\frac{d\rho_*}{dt_*} + \rho_*\left\{\frac{\partial w_*}{\partial r_*} + \frac{2w_*}{r_*} + \frac{1}{r_*\cos\theta}\frac{\partial(v_*\cos\theta)}{\partial\theta} + \frac{1}{r_*\cos\theta}\frac{\partial u_*}{\partial\phi}\right\} = 0 \qquad (1)$$

representing the condition of mass continuity and

$$\frac{du_*}{dt_*} + \frac{u_*w_*}{r_*} - \frac{u_*v_*\tan\theta}{r_*} - 2\Omega\sin\theta\, v_* + 2\Omega\cos\theta\, w_* = -\frac{1}{\rho_* r_*\cos\theta}\frac{\partial p_*}{\partial\phi} + \frac{F_{\phi_*}}{\rho_*} \qquad (2)$$

$$\frac{dv_*}{dt_*} + \frac{v_*w_*}{r_*} + \frac{u_*^2\tan\theta}{r_*} + 2\Omega\sin\theta\, u_* = -\frac{1}{\rho_* r_*}\frac{\partial p_*}{\partial\theta} + \frac{F_{\theta_*}}{\rho_*} \qquad (3)$$

$$\frac{dw_*}{dt_*} - \frac{u_*^2+v_*^2}{r_*} - 2\Omega\cos\theta\, u_* = -\frac{1}{\rho_*}\frac{\partial p_*}{\partial r_*} - g + \frac{F_{r_*}}{\rho_*} \qquad (4)$$

where

$$\frac{d}{dt_*} = \frac{\partial}{\partial t_*} + \frac{u_*}{r_*\cos\theta}\frac{\partial}{\partial\phi} + \frac{v_*}{r_*}\frac{\partial}{\partial\theta} + w_*\frac{\partial}{\partial r_*} \qquad (5)$$

$(F_{\phi_*}, F_{\theta_*}, F_{r_*})$ is the body force per unit mass due to viscous effects in the fluid and g is the gravitational acceleration.

The reference frame may be transformed to a local set of longitude (x_*), latitude (y_*) and height (z_*) coordinates centred on a latitude θ_0 at the surface of the sphere by the transformation

$$\begin{aligned} x_* &= r_0 \cos\theta_0 \, \phi & & \partial_\phi = r_0 \cos\theta_0 \, \partial_{x_*} \\ y_* &= r_0(\theta - \theta_0) & \Rightarrow \quad & \partial_\theta = r_0 \partial_{y_*} \\ z_* &= r_* - r_0 & & \partial_{r_*} = \partial_{z_*} \end{aligned} \tag{6}$$

Motions of the fluid imposed by the model are ascribed horizontal length and velocity scales L and U with a height scale D which is small compared to the radius of the sphere so that the effective gravitational acceleration is constant. By geometrical considerations the vertical velocity scale is $\frac{UD}{L}$ and if the flow is subject to nonlinear effects then it develops on a time scale $\frac{L}{U}$.

Pressure and density scalings now determine more precisely the class of motions under inquiry. If the fluid velocity scale is small compared to the scale of Coriolis effects then the Rossby number

$$R_0 = \frac{U}{2\Omega L \sin\theta_0} \ll 1$$

and to leading order in an asymptotic expansion of the dependent variables of (1)-(4) in R_0 as $R_0 \to 0$ the state is one of hydrostatic equilibrium, i.e., no motion

$$\underline{u} \equiv \underline{0} \qquad \text{and} \qquad \frac{\partial p_s}{\partial z} = -\rho_s(z)\, g \tag{7}$$

where $p_s(z)$ and $\rho_s(z)$ are the $O(1)$ terms in the expansions of the pressure and density fields. The fields at the next order may be sought as a perturbation on the ambient order one state wherein the tangential pressure gradient balances Coriolis acceleration and the vertical pressure gradient balances the effective buoyancy per unit mass, implying scalings

$$p_* = p_s(z) + 2\Omega U L \sin\theta_0 \, \rho_s(z)\, p(x,y,z,t) \tag{8}$$

and

$$\rho_* = \rho_s(z)\left(1 + \frac{2\Omega U L \sin\theta_0}{g D}\, \rho(x,y,z,t)\right) \tag{9}$$

Under the assumptions of these scalings the governing equations (1)-(4), using (6)-(9) are

$$R_o\left\{\frac{du}{dt} + \frac{L}{r_*}\left[\delta uw - uv\tan\theta\right]\right\} - \frac{v\sin\theta}{\sin\theta_o} + \frac{\delta w\cos\theta}{\sin\theta_o}$$

$$= -\frac{\cos\theta_o}{\cos\theta}\,\frac{r_o}{r_*}\,\frac{\partial p}{\partial x}\cdot\frac{1}{1+R_oF\rho} + \frac{F_{\phi *}}{\rho_* 2\Omega U\sin\theta_o} \qquad (10)$$

$$R_o\left\{\frac{dv}{dt} + \frac{L}{r_*}\left[\delta vw + u^2\tan\theta\right]\right\} + \frac{u\sin\theta}{\sin\theta_o}$$

$$= -\frac{r_o}{r_*}\,\frac{\partial p}{\partial y}\cdot\frac{1}{1+R_oF\rho} + \frac{F_{\theta *}}{\rho_* 2\Omega U\sin\theta_o} \qquad (11)$$

$$(1+R_oF\rho)\left\{R_o\delta^2\frac{dw}{dt} - \frac{R_o\delta L}{r_*}\left(u^2+v^2\right) - \delta u\frac{\cos\theta}{\sin\theta_o}\right\}$$

$$= -\frac{1}{\rho_s}\,\frac{\partial}{\partial z}\left(\rho_s p\right) - \rho + \frac{\delta F_{z *}}{\rho_s 2\Omega U\sin\theta_o}$$

$$(12)$$

$$R_oF\frac{d\rho}{dt} + (1+R_oF\rho)\left\{\frac{1}{\rho_s}\,\frac{\partial}{\partial z}\left(\rho_s w\right) + \frac{2Dw}{r_*} + \frac{r_o}{r_*}\,\frac{\partial v}{\partial y} - \frac{Lv}{r_*}\tan\theta\right.$$

$$\left. + \frac{r_o}{r_*}\,\frac{\cos\theta_o}{\cos\theta}\,\frac{\partial u}{\partial x}\right\} = 0$$

$$(13)$$

in non-dimensional form where

$R_O = \dfrac{U}{2\Omega\sin\theta_O L}$ is the Rossby number, a ratio of the relative vorticity to the planetary vorticity.

$F = \dfrac{(2\Omega\sin\theta_O)^2 L^2}{gD}$ is the ratio of the rotational kinetic energy to gravitational potential energy across the flow.

$\delta = \dfrac{D}{L}$ is the aspect ratio of the length scales, so

$$\frac{r_*}{r_o} = 1 + \delta\left(\frac{L}{r_o}\right)z$$

The dimensionless parameter, β, defined by

$$\beta \equiv \frac{2\Omega\cos\theta_0}{r_0} = \frac{\partial}{\partial y_*}(2\Omega\sin\theta)\Big|_{\theta=\theta_0} \tag{14}$$

is introduced, representing the northward gradient of Coriolis acceleration at $\theta = \theta_0$. The ratio of the horizontal gradient of relative or fluid vorticity to the gradient of planetary vorticity associated with the geometry's sphericity is typically

$$\frac{U}{\beta L^2} = R_0 \cdot \frac{r_0}{L}\tan\theta_0 \tag{15}$$

The class of motions is now restricted by choosing $\tan\theta_0 = O(1)$ and $\frac{L}{r_0} = O(R_0)$ as $R_0 \to 0$ so the meridional vorticity gradients of (15) balance in the flow fields at $O(R_0)$. In taking the limit $R_0 \to 0$, termed the geostrophic limit, the dependent variables u,v,w,p,ρ are assumed to have asymptotic expansions of the form

$$u = u_0 + R_0 u_1 + R_0^2 u_2 + \ldots \tag{16}$$

etc., and the trigonometric functions in (10)-(13) when replaced by their Taylor series about the central latitude $\theta = \theta_0$ are represented asymptotically with respect to the parameter $\frac{L}{r_0} = O(R_0)$. If the dimensionless parameters δ and F are also linked to the Rossby number in the limit then substitution in (10)-(13) implies the geostrophic relations

$$u_0 = -\frac{\partial p_0}{\partial y} \tag{17}$$

$$v_0 = \frac{\partial p_0}{\partial x} \tag{18}$$

$$\rho_0 = -\frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s p_0) \tag{19}$$

$$\frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s w_0) + \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0 \tag{20}$$

at leading order. Consequently the horizontal divergence of horizontal velocity vanishes so that the continuity result (20) may be integrated to give

$$\rho_s \varpi_0 = \text{function of } x, y \text{ and } t \text{ alone.}$$

In a flow containing a rigid boundary of maximum slope $O(R_0)$ with respect to the horizontal then, for all z

$$\varpi_0 = 0 \tag{21}$$

$$\Rightarrow \qquad \varpi = R_0 \varpi_1 + R_0^2 \varpi_2 + \dots \tag{22}$$

At first order, $O(R_0)$, the governing equations are

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} - v_1 - v_0 \left(\frac{L}{r_0 R_0}\right) y \cot\theta_0 = -\frac{\partial p_1}{\partial x} - \frac{L y}{r_0 R_0} \tan\theta_0 \frac{\partial p_0}{\partial x} + \frac{F_{\phi *}}{R_0 \rho_* f_0} \tag{23}$$

$$\frac{\partial v_0}{\partial t} + u_0 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_0}{\partial y} + u_1 + u_0 \left(\frac{L}{r_0 R_0}\right) y \cot\theta_0 = -\frac{\partial p_1}{\partial y} + \frac{F_{\theta *}}{R_0 \rho_* f_0} \tag{24}$$

for horizontal momentum where $f_0 = 2\Omega \sin\theta_0$ and

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} - \frac{L}{r_0 R_0} v_0 \tan\theta_0 + \frac{L}{r_0 R_0} y \tan\theta_0 \frac{\partial u_0}{\partial x} + \frac{1}{\rho_s} \frac{\partial}{\partial z}\left(\rho_s \varpi_1\right) = 0 \tag{25}$$

for continuity where it has been assumed that the influence of internal friction may first be of importance at this order and that $\delta = o(1)$, $F = o(1)$ as $R_0 \to 0$. Formation of the equation for the vertical component of vorticity

$$\zeta_0 = \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) p_0 \tag{26}$$

upon making the Boussinesq approximation, that the density stratification is of significant consequence in determining the momentum balance of the fluid through its appearance in the gravitational or buoyancy terms of the of the body force alone. In equation (28) this implies that the molecular viscosity, μ, is effectively constant or independent of position, an approximation which may be of reasonable value in a laboratory experiment but is far more tenuous for a geophysical flow of appreciable vertical extent. A similar approximation will be made later (in equation (36)) for the constant of diffusivity of adiabatic temperature.

from (23) and (24) imply, using (17), (18) and (25)

$$\frac{d_0}{dt}(\zeta_0+\beta y) = \frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s w_1) + \hat{\underline{k}}\cdot\underline{\nabla}\wedge\left(\frac{F_{\phi *}}{R_0\rho_* U f_0}, \frac{F_{\theta *}}{R_0\rho_* U f_0}\right) \tag{27}$$

where $\frac{d_0}{dt} = \frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y}$ is the appropriate advective time derivative and $\hat{\underline{k}}$ is a unit vector parallel to the z axis.

Viscous effects in a Newtonian fluid are defined by,

$$\underline{F}_* = \nu_*\left(\underline{\nabla}^2 + \frac{1}{3}\underline{\nabla}(\underline{\nabla}.\ )\right)_* \underline{u}_* \tag{28}$$

Using the above scalings and expansions,if $\delta = o(1)$, since $w_0 = 0$

$$\frac{F_{\phi *}}{R_0\rho_* U f_0} = \frac{\nu_*}{R_0 f_0 D^2}\frac{\partial^2 u_0}{\partial z^2}, \quad \frac{F_{\theta *}}{R_0\rho_* U f_0} = \frac{\nu_*}{R_0 f_0 D^2}\frac{\partial^2 v_0}{\partial z^2} \tag{29}$$

on writing $\nu = \dfrac{\nu_*}{R_0 f_0 D^2} = O(1)$, (27) becomes

$$\frac{d_0}{dt}(\zeta_0+\beta y) = \frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s w_1) + \nu\frac{\partial^2\zeta_0}{\partial z^2} \tag{30}$$

using the geostrophic relations (17) and (18).

If $\delta = O(1)$ then (29) becomes

$$\frac{F_{\phi *}}{R_0\rho_* U f_0} = \nu\nabla_\delta^2 u_0, \quad \frac{F_{\theta *}}{R_0\rho_* U f_0} = \nu\nabla_\delta^2 v_0 \tag{31}$$

where

$$\nabla_\delta^2 = \frac{\partial^2}{\partial z^2} + \delta^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{31}$$

and the analysis preceding (27) carries through inasmuch as (30) becomes

$$\frac{d_0}{dt}(\zeta_0+\beta y) = \frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s w_1) + \nu\nabla_\delta^2\zeta_0 \tag{32}$$

but the reader is referred to Pedlosky ('79. p.327 et seq.) for the difficulty in drawing this analogy with the momentum equations (23) and (24). Equation (32) is formally the same as the vorticity equation which would have been derived in the case of a model with a flat or Cartesian geometry where Coriolis effects vary linearly in the meridional or y direction (termed the β-plane approximation) with the exception that the restriction $\delta = o(1)$ is then no longer necessary, $\delta = O(1)$ is allowed at every stage in the analysis leading to (32) and without loss of generality δ may be set equal to 1. Since the dynamics for the vertical components of vorticity, ζ_0, differs only in the representation of viscous effects and the spherical $\delta = o(1)$ case can be obtained from the Cartesian $\delta = O(1)$ case by omitting the horizontal derivatives in ∇_δ, henceforth (32) will be the form of the vorticity equation used and δ set equal to 1 and ∇_δ replaced by ∇.

To remove the geostrophic degeneracy of the problem posed by (32) the leading order vertical velocity, w_1, is related to the leading order horizontal velocity (u_0,v_0) by the thermodynamics of the system and the geostrophic relations (17)-(20). Suppose that a fluid property which is advected with the flow, $\Theta_*(\underline{x},t)$ say, is conserved but for weak diffusive effects so that the scaling of the Θ_* field is given by

$$\Theta_* = \Theta_s(z)\left(1 + R_o F\,\Theta(\underline{x},t)\right) \tag{33}$$

in accordance with the pressure and density scalings (8) and (9) where

$$\Theta(\underline{x},t) = \Theta_0 + R_o\,\Theta_1 + R_o^2\,\Theta_2 + \cdots \tag{34}$$

and suppose that the static stratified O(1) state of the flow is in equilibrium so that

$$\nabla_*^2\,\Theta_s(z) = 0 \tag{35}$$

If the effects of diffusion are weak so that Θ_* is nearly conserved with the movement of a fluid element local thermodynamic considerations (Batchelor, p.31 et seq.) suggest the law

$$\frac{d\Theta_*}{dt_*} = \kappa_* \nabla_*^2 \Theta_* \tag{36}$$

for an isotropic fluid in the absence of heating with diffusive constant κ_*. Non-dimensionalising, using the above scalings when $\delta = O(1)$, (36) becomes

$$\frac{d_0\Theta_0}{dt} + \omega_1 \cdot \frac{1}{F\Theta_s}\frac{\partial\Theta_s}{\partial z} = \frac{\kappa_*}{UL}\left(\nabla^2\Theta_0 + \frac{2}{\Theta_s}\frac{\partial\Theta_s}{\partial z}\cdot\frac{\partial\Theta_0}{\partial z}\right) \tag{37}$$

at leading order provided the stratification parameter

$$S = \frac{1}{F\Theta_s}\frac{\partial\Theta_s}{\partial z} \tag{38}$$

is O(1). The first term on the right hand side of (32) representing the small vertical stretching of a vortex filament may be evaluated in terms of the perturbation Θ_0 and ambient stratification $\Theta_s(z)$ using the geostrophic relations and independence of $\frac{\rho_s}{S}$ in x and y.

$$\frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\rho_s\omega_1\right) = \frac{\kappa_*}{UL}\cdot\frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\nabla^2\Theta_0\right) + \frac{2\kappa_*}{UL}\cdot F\cdot\frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\frac{\partial\Theta_0}{\partial z}\right)$$
$$- \frac{d_0}{dt}\left(\frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s\Theta_0}{S}\right)\right) \tag{39}$$

When Θ_0 may be related to the geostrophic stream function, p_0, (32) and (39) together define the evolution of the flow at $O(R_0)$ in differential form but such a relation depends upon choice of an equation of state.

For a liquid in which the effects of compressibility are not appreciable the equation of state may be represented simply by

$$\rho_* = \rho_{0*}\left(1 - \alpha_*\left(T_* - T_{0*}\right)\right) \tag{40}$$

where ρ_{0_*} and T_{0_*} are constant density and temperature reference values, α_* is the coefficient of thermal expansion and T_* is the temperature. The first law of thermodynamics in the absence of heating reduces to (36) where $\Theta_* = -\rho_*(\underline{x},t)$ is the density field, κ_* is the coefficient of thermal diffusivity and the geostrophic relation (19) reduces to

$$\rho_0 = -\frac{\partial p_0}{\partial z}. \qquad (41)$$

provided that

$$\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z} = O(F) = o(1) \qquad (42)$$

in which case (38) is satisfied, i.e. S is O(1). These scalings are believed to be in agreement with the observed state of the Oceans except near the pynocline, a thin layer of large density gradients at a depth of approximately 700 m. Equations (32) and (39), using (41) and (42) become

$$\frac{d_0}{dt}\left\{\zeta_0 + \beta y + \frac{\partial}{\partial z}\left(\frac{1}{S}\frac{\partial p_0}{\partial z}\right)\right\} = \kappa\frac{\partial}{\partial z}\left(\frac{1}{S}\nabla^2\frac{\partial p_0}{\partial z}\right) + \nu\nabla^2\zeta_0 \qquad (43)$$

where $\kappa = \frac{\kappa_*}{UL}$ is assumed to be O(1).

For a perfect gas the equation of state and definition of potential temperature, θ_* ;

$$p_* = \rho_* R T_* \quad , \quad \Theta_* = T_*\left(\frac{p_0}{p_*}\right)^{1-\frac{1}{\gamma}} \qquad (44)$$

imply that if in the basic state $\theta_* = \theta_s(z)$ and

$$\frac{1}{\Theta_s}\cdot\frac{\partial \Theta_s}{\partial z} = O(F) = o(1) \qquad (45)$$

then S is O(1) and

$$\frac{1}{\Theta_s}\frac{\partial \Theta_s}{\partial z} = \frac{1}{\gamma}\cdot\frac{1}{p_s}\frac{\partial p_s}{\partial z} - \frac{1}{\rho_s}\frac{\partial \rho_s}{\partial z} \qquad (46).$$

Using (8) and (9), (44) implies

$$\ln \Theta_* = \frac{1}{\gamma} \ln p_s(z) - \ln \rho_s(z) + \frac{1}{\gamma} \frac{R_o b_o^2 L^2}{p_s/\rho_s} p - R_o F \rho + O(R_o^2 F) \tag{47}$$

and suggests the sċaling

$$\Theta_* = \Theta_s(z)\left(1 + R_o F \Theta\right) \tag{48}$$

with $$\Theta = \Theta_o + R_o \Theta_1 + \ldots .$$

In agreement with (46) let

$$\ln \Theta_s(z) = \frac{1}{\gamma} \ln p_s(z) - \ln \rho_s(z)$$

then

$$\Theta_o = -\rho_o + \frac{1}{\gamma}\left(\frac{\rho_s g D}{p_s}\right) p_o$$

and from the dynamic expressions (7) and (19)

$$\Theta_o = \frac{\partial p_o}{\partial z} - p_o \cdot \frac{1}{\Theta_s} \frac{\partial \Theta_s}{\partial z}$$

so that $$\Theta_o = \frac{\partial p_o}{\partial z} \tag{49}$$

to leading order. The appropriate form of the first law of thermodynamics in the absence of heating is

$$\frac{d\Theta_*}{dt_*} = \frac{k R \Theta_*}{C_p\, p_*} \nabla_*^2 T_* \tag{50}$$

where the term on the right hand side represents the departure from the adiabatic state. If the departure from adiabatic equilibrium is small however, local considerations suggest that the flux of potential temperature is related linearly with its spatial gradient and in the case of an isotropic fluid this leads to the approximate equation

$$\frac{d\Theta_*}{dt_*} = K_* \nabla_*^2 \Theta_*$$

so that with $\Theta_* = \theta_*$ (32) and (39) using (49) become

$$\frac{d_0}{dt}\left\{ \zeta_0 + \beta y + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\frac{\partial p_0}{\partial z}\right)\right\} = \kappa \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\nabla^2\frac{\partial p_0}{\partial z}\right) + \nu\nabla^2\zeta_0 \qquad (51)$$

which is the form of the quasi-geostrophic potential vorticity equation to be used here.

For a region bounded by the latitude circles $y = \pm\frac{1}{2}$ the inviscid boundary condition is

$$\frac{\partial p_0}{\partial x} = 0 \ , \ \text{on } y = \pm\frac{1}{2} \qquad (52)$$

Taking the zonal average of (23) and using (17) there follows the additional condition that

$$\frac{\partial^2 \bar{p}_0}{\partial y\, \partial t} = 0 \ , \ \text{on } y = \pm\frac{1}{2} \qquad (53)$$

where the overbar denotes the zonal average which in terms of the local frame defined by (6) is

$$(\bar{\ \ }) \equiv \lim_{X\to\infty} \frac{1}{2X}\int_{-X}^{X} (\quad)\, dx$$

Equation (53) is also the form of the boundary condition for the Cartesian model of flow inside a rigid rectangular channel of infinite extent in the x direction if a downstream condition of periodicity or boundedness of perturbation energy is imposed.

In the case of a rigid boundary at $z = 0$ the flow is dominated locally by the balance of viscous forces with Coriolis forces and pressure gradient in an Ekman layer of thickness

$$\delta_* = (E_v)^{\frac{1}{2}} D$$

where $E_v = \dfrac{2\nu_*}{f_0 D^2}$ is the Ekman number and ν_* is the kinematic viscosity of laminar flow or an effective turbulent viscosity of non-laminar flow within the layer. It can be shown (Pedlosky 1, §6.6) that stratification of the basic state does not modify the pumping or suction influence of the boundary layer upon the main core of the flow in the limit $E_V \to 0$ provided

$$E_v = o\left(\frac{R_0}{S}\right) \qquad \text{as } R_0 \to 0.$$

Matching w_1 in (37) with the normal velocity at the edge of the Ekman layer which is proportional to the normal component of vorticity there implies, since $F = o(1)$ and $S = O(1)$

$$\frac{d_0}{dt}\left(\frac{\partial p_0}{\partial z}\right) = -\frac{SE_v^{\frac{1}{2}}}{2R_0}\left(\frac{\partial^2 p_0}{\partial x^2} + \frac{\partial^2 p_0}{\partial y^2}\right) + \kappa\nabla^2\left(\frac{\partial p_0}{\partial z}\right) \qquad (54)$$

$$\text{on } z = 0.$$

in the absence of boundary topography.

An upper boundary condition corresponding to confinement of the flow beneath a rigid flat lid at non-dimensional height $z = h$ (so that $z_* = hL = D$) will be assumed, thus

$$\frac{d_0}{dt}\left(\frac{\partial p_0}{\partial z}\right) = +\frac{SE_v^{\frac{1}{2}}}{2R_0}\left(\frac{\partial^2 p_0}{\partial x^2} + \frac{\partial^2 p_0}{\partial y^2}\right) + \kappa\nabla^2\left(\frac{\partial p_0}{\partial z}\right) \qquad (55)$$

$$\text{on } z = h.$$

In oceanic models the influence of a prescribed wind driven stress may be included, modifying the right hand side. For the atmosphere, the lid is most realistically removed and then a radiation condition is imposed in the limit $z \to \infty$. An atmosphere with different stratification and velocity profiles in the lower level or troposphere and higher level or stratosphere may be modelled (e.g. Charney '47). The overall effect of increasing the static stability of the stratosphere is to confine disturbances on the $O(R_0)$ geostrophic flow to the lower levels of the troposphere (Green, 60) and from this point of view (55) is the boundary condition imposed at the tropopause, $z = h$, below a stratosphere of infinite static stability ($S \to \infty$).

Now consider a steady state of the $O(R_0)$ geostrophic flow which is purely zonal. Writing

$$p_0 = \Psi(y, z) \qquad (56)$$

from (17) and (41) or (49) the zonal wind relationship

$$\frac{\partial u_o}{\partial z} = \begin{cases} -\dfrac{\partial \Theta_o}{\partial y} & \text{for the atmosphere} \\ -\dfrac{\partial \rho_o}{\partial y} & \text{for the oceans.} \end{cases} \tag{57}$$

connects the vertical shear of the velocity profile to meridional gradients of available potential energy in the basic flow. Note that the longitude or x independence implies that any function $\Psi(y,z)$ is automatically a solution of (51)-(53) if the diffusive terms on the right hand side of (51) may be neglected or vanish and is a solution of (54) and (55) provided

$$\frac{S E_v^{\frac{1}{2}}}{2R_o} \cdot \left.\frac{\partial^2 \Psi}{\partial y^2}\right|_{z=0,h} = 0$$

The stability of this basic, steady, near hydrostatic, zonal flow to disturbances which may be of the same order of magnitude as the basic flow but which are still subject to the conditions of the geostrophic limit may be described by a disturbance stream function $\phi(\underline{x},t)$. Putting

$$p_o = \Psi(y,z) + \phi(\underline{x},t) \tag{58}$$

in (51)-(55) the problem for $\phi(\underline{x},t)$ is

$$\left(\frac{\partial}{\partial t} + u_o\frac{\partial}{\partial x}\right) q + \frac{\partial \phi}{\partial x} \cdot \frac{\partial \Pi_o}{\partial y} + J_{x,y}(\phi, q) = \nu \nabla^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi + \frac{\kappa}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\nabla^2\frac{\partial \phi}{\partial z}\right) \tag{59}$$

$$\left(\frac{\partial}{\partial t} + u_o\frac{\partial}{\partial x}\right)\frac{\partial \phi}{\partial z} - \frac{\partial u_o}{\partial z}\cdot\frac{\partial \phi}{\partial x} + J_{x,y}\left(\phi, \frac{\partial \phi}{\partial z}\right) = -\frac{S E_v^{\frac{1}{2}}}{2R_o}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi + \kappa \nabla^2 \frac{\partial \phi}{\partial z} \tag{60}$$

on $z = 0$.

with a similar condition on z = h, and

$$\frac{\partial \phi}{\partial x} = 0 \quad , \text{on } y = \pm \frac{1}{2} \tag{61}$$

$$\frac{\partial^2 \overline{\phi}}{\partial y \partial t} = 0 \quad , \text{on } y = \pm \frac{1}{2} \tag{62}$$

where

$$q = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\frac{\partial \phi}{\partial z}\right)$$

is the potential vorticity of the disturbance field,

$$\Pi_0 = \beta y + \frac{\partial^2 \Psi}{\partial y^2} + \frac{1}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\frac{\partial \Psi}{\partial z}\right)$$

is the potential vorticity of the basic zonal flow and $J_{x,y}$ is the Jacobian operator

$$J_{x,y}(A,B) = \frac{\partial A}{\partial x}\cdot\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\cdot\frac{\partial B}{\partial x}$$

Since the vertical coordinate appears explicitly in differential form alone, the interval in z may be shifted in the case of a pair of rigid boundaries to $[-\frac{h}{2}, \frac{h}{2}]$ so that the appropriate top and bottom boundary conditions may be written as

$$\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)\frac{\partial \phi}{\partial z} - \frac{\partial u_0}{\partial z}\cdot\frac{\partial \phi}{\partial x} + J_{x,y}\left(\phi, \frac{\partial \phi}{\partial z}\right)$$
$$= \frac{2z}{h}\cdot\frac{S E_v^{\frac{1}{2}}}{2R_0}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi + \kappa \nabla^2\left(\frac{\partial \phi}{\partial z}\right) \quad \text{on } z = \pm\frac{h}{2} \tag{63}$$

The aim of the type of stability analysis applied in the following sections is to establish sufficient conditions for the instability of a basic velocity profile which is a linear vertical shear. Since the governing equations and boundary conditions for the geostrophic stream function are Galilean invariant, transformation to a frame moving with constant velocity, equal to the average of the basic steady flow over a meridional plane for example, leaves the mathematical problem unchanged. Hence writing

$$u_o = \Lambda z$$

or equivalently

$$\Psi = -\Lambda y z \tag{64}$$

where Λ is the non-dimensional shear parameter, the problem for the disturbance stream function ϕ is

$$\beta - \frac{\Lambda}{\rho_s}\partial_z\left(\frac{\rho_s}{S}\right)$$

$$\left(\frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x}\right) q + \beta \frac{\partial \phi}{\partial x} + J_{x,y}(\phi, q)$$
$$= \nu \nabla^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi + \frac{\kappa}{\rho_s}\frac{\partial}{\partial z}\left(\frac{\rho_s}{S}\nabla^2 \frac{\partial \phi}{\partial z}\right) \tag{65}$$

with boundary conditions

$$\frac{\partial \phi}{\partial x} = 0 \text{ on } y = \pm \frac{1}{2} \; ; \quad \frac{\partial^2 \bar{\phi}}{\partial y \partial t} = 0 \text{ on } y = \pm \frac{1}{2} \tag{66),(67}$$

$$\left(\frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x}\right)\frac{\partial \phi}{\partial z} - \Lambda \frac{\partial \phi}{\partial x} + J_{x,y}\left(\phi, \frac{\partial \phi}{\partial z}\right) \quad \text{on } z = \pm \frac{h}{2}$$
$$= \frac{2z}{h} \cdot \frac{S E_v^{1/2}}{2 R_o}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi + \kappa \nabla^2 \left(\frac{\partial \phi}{\partial z}\right) \tag{68}$$

The class of permissible basic, steady, geostrophic flows is large as is evident from the note preceding equation (58) and furthermore the choice of a member which is representative in geophysical application may only be established a posteriori if the dynamics of the interaction between the basic steady flow and the disturbances which may develop upon it are nonlinear. Time or space averages of an observed state of unsteady flow where the unsteadiness is caused by fluctuations which give rise to fluxes releasing potential energy stored in the basic state as kinetic energy in the observed state include the non-zero averaged values of these fluxes, the fluctuation free or basic state may however correspond to a state of vanishing flux as indeed is the case here (see e.g. Charney-Drazin '61). Hence the averaging of an observed state of the atmosphere for example does not reveal a basic state for the model upon which the observed fluctuations may have arisen through instability. The choice of a basic velocity profile which is a linear shear in the vertical coordinate follows the precedent established in the early work by Charney ('47) and Eady ('49) that the wave patterns implied by linear stability analysis are of a scale and type which is typically consistent with atmospheric data.

CHAPTER 2

TOWARDS A UNIFIED VIEW OF THE LINEAR STABILITY PROPERTIES OF BAROCLINIC FLOWS WITH SIMPLE LINEAR VERTICAL SHEAR PROFILES IN THE PRESENCE OF THE β-PLANE EFFECT

For the velocity profile $u_0(z)$ a normal mode analysis of the linearised governing equation (1.59) and boundary conditions (1.60-1.62) gives a fourth order differential equation

$$
\begin{aligned}
&(u_0-c)\left(D^2 - S(k^2+l^2) - \frac{D}{h_s}\right)\psi + \left(\beta S + \frac{Du_0}{h_s} - D^2 u_0\right)\psi \\
&+ \frac{i\kappa}{k}\left(D^2 - \frac{S_0}{\kappa}(k^2+l^2)\right)\left(D^2 - (k^2+l^2)\right)\psi - \frac{i\kappa}{k h_s}\left(D^2 - (k^2+l^2)\right)D\psi = 0
\end{aligned}
\tag{1}
$$

where $D \equiv \frac{d}{dz}$, $\frac{1}{h_s} = -\frac{1}{\rho s}\frac{\partial \rho s}{\partial z}$ and both h_s and S are assumed to be constant, for the z-dependent component of a Fourier type mode representation of the disturbance stream function ϕ

$$
\phi = \psi(z)\sin lY\left(e^{ik(x-ct)} + \text{complex conjugate.}\right) \tag{2}
$$

where
$$
Y = y + \tfrac{1}{2} \in [0, 1] \tag{3}
$$

This satisfies the sidewall boundary conditions where $\ell = m\pi$ ($m = 1,2,\ldots.$) where k and ℓ may be interpreted as real downstream and spanwise wavenumbers and $c = c_r + ic_i$ is a complex phase speed. A mode for which Im(c) is positive is termed unstable, for Im(c) < 0 is stable and for Im(c) = 0 is neutrally stable.

In the model due to Charney (47) the flow is assumed to be free from the diffusive effects of kinematic viscosity and thermal diffusion so that (1) is then of second order. Further, for a linear shear profile $u_0 = \Lambda z$, ψ must satisfy

$$
\left((\Lambda z - c)\left(D^2 - S(k^2+l^2) - \frac{D}{h_s}\right) + \beta S + \frac{\Lambda}{h_s}\right)\psi = 0 \tag{4}
$$

Eady's model ('49) is derived under the further simplifying assumptions that $\beta = \frac{1}{h_s} = 0$. Note that the meridional gradient of the potential vorticity in the basic, undisturbed state $u_0(z)$ is, from (1.17) and (1.63)

$$\frac{\partial \Pi_0}{\partial y} = \beta + \frac{Du_0}{S h_s} - \frac{D^2 u_0}{S} \qquad (5)$$

so that for a linear velocity profile with $Du_0 > 0$, corresponding to a westerly wind, this ambient vorticity gradient does not vanish unless both β and $\frac{1}{h_s}$ vanish as in Eady's case.

Equation (1) becomes the Orr Sommerfeld equation for the linear stability of parallel flow when $\nu = \kappa$ and κ is replaced by the inverse of the Reynold's number, β and $\frac{1}{h_s}$ are set to zero and S is unity. The extensive theory of asymptotic solutions of this equation for large Reynolds number or small κ in the present context as discussed by Lin ('45), developed by Wasow ('48) and others and reviewed by Reid ('65) may usefully be employed in the consideration of (1) and the 'reduced' second order equation (4). To this end consider the leading order approximation as $\kappa \to 0$ to the two of the four solutions of (1) which are given by (4). Putting

$$\psi(z) = \left(z - \frac{c}{\Lambda}\right) e^{\tilde{h} z} \cdot F\left(z - \frac{c}{\Lambda}\right) \qquad (6)$$

where z may conveniently be thought of as complex, in (4) implies

$$(\Lambda z - c)F'' + \left(2\Lambda + \left(2\tilde{h} - \frac{1}{h_s}\right)(\Lambda z - c)\right)F'$$
$$+ \left(2\tilde{h}\Lambda + \beta S + \left(\tilde{h}^2 - \frac{\tilde{h}}{h_s} - a_w^2 S\right)(\Lambda z - c)\right)F = 0$$

where $a_w^2 = k^2 + \ell^2$ so that for the exponent $\tilde{h}$ of (6) to be negative and the function F to be chosen such that the coefficient of zF is zero

$$\tilde{h}^2 - \frac{\tilde{h}}{h_s} - a_w^2 S = 0$$

then
$$\tilde{\ell} = \frac{1}{2\ell_s} - \sqrt{\left(\frac{1}{4\ell_s^2} + a_\omega^2 s\right)} < 0 \tag{7}$$

and the change of variables

$$\xi = 2\left(\frac{1}{4\ell_s^2} + a_\omega^2 s\right)^{\frac{1}{2}}\left(z - \frac{c}{\lambda}\right)$$
$$G(\xi) = F\left(z - \frac{c}{\lambda}\right) \tag{8}$$

implies

$$\xi G'' + (2-\xi)G' - aG = 0 \tag{9}$$

where the parameter a is given by

$$a = 1 - r \quad \text{and} \quad r = \frac{\beta s + \frac{\Lambda}{\ell_s}}{2\lambda\left(\frac{1}{4\ell_s^2} + a_\omega^2 s\right)^{\frac{1}{2}}} \geq 0 \tag{10}$$

Equation (9) is the confluent hypergeometric equation with one free parameter, possessing a regular singular point at $\xi = 0$ (or $z = \frac{c}{\lambda}$) and an irregular singular point of rank one at $\xi = \infty$. The parameter a is constrained to $a \in (-\infty, 1]$ so that the two power series

$$M(a,2,\xi) = \sum_{0}^{\infty} \frac{(a)_j}{(2)_j}\frac{\xi^j}{j!} \tag{11}$$

and

$$e^{\xi}\,U(2-a,2,e^{-i\pi}\xi) = \frac{1}{\Gamma(1-a)}\left\{ M(a,2,\xi)\ln(e^{-i\pi}\xi) + e^{\xi}\sum_{0}^{\infty}\frac{\Gamma(2-a+j)(e^{-i\pi}\xi)^j}{\Gamma(2-a)(j+1)!\,j!}\Big(\psi(2-a+j) - \psi(1+j) - \psi(2+j)\Big)\right\} + \frac{e^{i\pi+\xi}}{\Gamma(2-a)\xi} \tag{12}$$

form a fundamental pair for solutions to (9). The notation of the power series and that adopted for confluent hypergeometric functions follows Abramowitz and Stegun (~~'55)~~ (1964))

Power series solutions of (4) could have been found, following Tollmien ('29) by the method of Froebenius. In that context the Kummer function $M(a,2,\xi)$ is associated with a mode which is regular everywhere in the finite part of the complex z plane, i.e., for which the singularity at $z = \frac{c}{\Lambda}$ is removable, whilst the other solution has a logarithmic singularity of the type

$$\left(z - \frac{c}{\Lambda}\right) \cdot \ln\left(z - \frac{c}{\lambda}\right) \tag{13}$$

which is associated with the presence of a critical layer in the mode at the critical point or steering level $\frac{c}{\Lambda}$. At this singular point the second solution is not differentiable and in this example the vertical potential temperature gradient

$$\frac{\partial \Theta_0}{\partial z} = \frac{\partial^2 \phi}{\partial z^2} \tag{14}$$

has a simple pole. These power series solutions, which are equivalently linear combinations of (11) and (12) under the transformations (6) and (8), may be written

$$\psi_1 = \left(z - \frac{c}{\lambda}\right) - \left(\frac{\beta S}{\lambda}\right)\frac{\left(z - \frac{c}{\lambda}\right)^2}{2!} + \left(a_\omega^2 S - \frac{\beta S}{2\Lambda}\left(\frac{1}{\ell_s} - \frac{\beta S}{\lambda}\right)\right)\frac{\left(z - \frac{c}{\lambda}\right)^3}{3!} + \dots \tag{15}$$

$$\psi_2 = 1 + \left(a_\omega^2 S - \left(\frac{\beta S}{\lambda} + \frac{1}{\ell_s}\right)\left(\frac{3\beta S}{2\lambda} + \frac{1}{\ell_s}\right)\right)\frac{\left(z - \frac{c}{\lambda}\right)^2}{2!} + \dots - \left(\frac{\beta S}{\lambda} + \frac{1}{\ell_s}\right)\psi_1 \ln\left(z - \frac{c}{\lambda}\right) \tag{16}$$

The fourth order equation (1) has no singularities in the finite part of the z-plane and for this reason at least the second one of these two functions does not provide an asymptotic estimate of any solution of (1) which is uniformly valid in a neighbourhood of the critical point as $\kappa \to 0$. The limiting case when $a = 1$ is distinct in this respect and as equation (10) shows it corresponds to

$$\frac{\beta S}{\Lambda} + \frac{1}{\ell_s} = 0$$

thereby including Eady's model. The solutions (11) and (12) are then

$$M(1,2,\zeta) = \left(\frac{e^{\zeta}-1}{\zeta}\right) \tag{11a}$$

$$e^{\zeta}\,U(1,2,e^{-i\pi}\zeta) = -\frac{e^{\zeta}}{\zeta} \tag{12a}$$

and the eigenfunctions $\psi(z)$ of (2) are then

$$e^{\frac{z}{2l_s}} \sinh\left(\frac{1}{4l_s^2}+a_\omega^2 S\right)^{\frac{1}{2}} z \; , \quad e^{\frac{z}{2l_s}} \cosh\left(\frac{1}{4l_s^2}+a_\omega^2 S\right)^{\frac{1}{2}} z \tag{17}$$

which reduce to the familiar hyperbolic functions

$$\sinh\left(a_\omega^2 S\right)^{\frac{1}{2}} z \; , \quad \cosh\left(a_\omega^2 S\right)^{\frac{1}{2}} z \tag{18}$$

for the specific case of Eady's model.

There are two contending theories for the local analysis of the critical layer structure present in the linearised normal mode formulation given by equation (4). The more established of these uses the diffusive terms of the linearised theory, i.e., the higher order derivatives of ψ which are multiplied by κ or ν in (1), to remove at least some of the singular part of the eigenfunctions in a neighbourhood of the singularity which has an asymptotic thickness tending to zero as $\kappa \to 0$ while the effects of nonlinearity may be neglected. The more recent theory of Benney and Bergeron ('79) etc. assumes that the linear theory of equation (1) is no longer applicable within a neighbourhood of the critical layer and the dynamics are determined locally by a balance between nonlinear terms in the governing equation and linear advective terms. The first approach will be followed here and in particular some attention will be paid to the consistency of the theory when the critical layer is close to a rigid boundary in which case physical considerations suggest that the flow will be dominated by diffusive effects.

Analysis of the Critical Layer

The method relies upon using the asymptotic solutions (11) and (12), or equivalently (15) and (16), of equation (1) within the region of their uniformity and treating the remaining domains which in our case is the neighbourhood of the single critical point $z = \frac{c}{\Lambda}$ by the method of matched asymptotic expansions.

Wasow ('48) has investigated asymptotic representations which are power series in the asymptotic variable $(\frac{i\kappa}{k})^{1/2}$ to the solutions of an equation of the form of (1) in the limit $\kappa \to 0$, the results are valid in a domain D of the z-plane which in the present context of a monotonic wind profile $u_0(z)$ which may have at most one critical point is

$$D = \left\{ z : 0 < \epsilon \leq \left| z - \frac{c}{\Lambda} \right| \right\} \tag{19}$$

These results include:

(A) There exists a pair of solutions of (1) which to leading order as $\kappa \to 0$ are the W.K.B. type solutions

$$\psi_{3,4}^{(0)} = \left(z - \frac{c}{\Lambda}\right)^{-\frac{5}{4}} e^{\mp\left(\frac{ik\Lambda}{\kappa}\right)^{\frac{1}{2}}\left(z-\frac{c}{\Lambda}\right)^{\frac{3}{2}}} + O\left(\left(\frac{\kappa}{\Lambda k}\right)^{\frac{1}{2}}\right) \tag{20}$$

With reference to figure (1), $\psi_3^{(0)}$ is recessive in the sector S_1 and dominant elsewhere, $\psi_4^{(0)}$ is recessive in S_2 and dominant elsewhere; see Reid ('65).

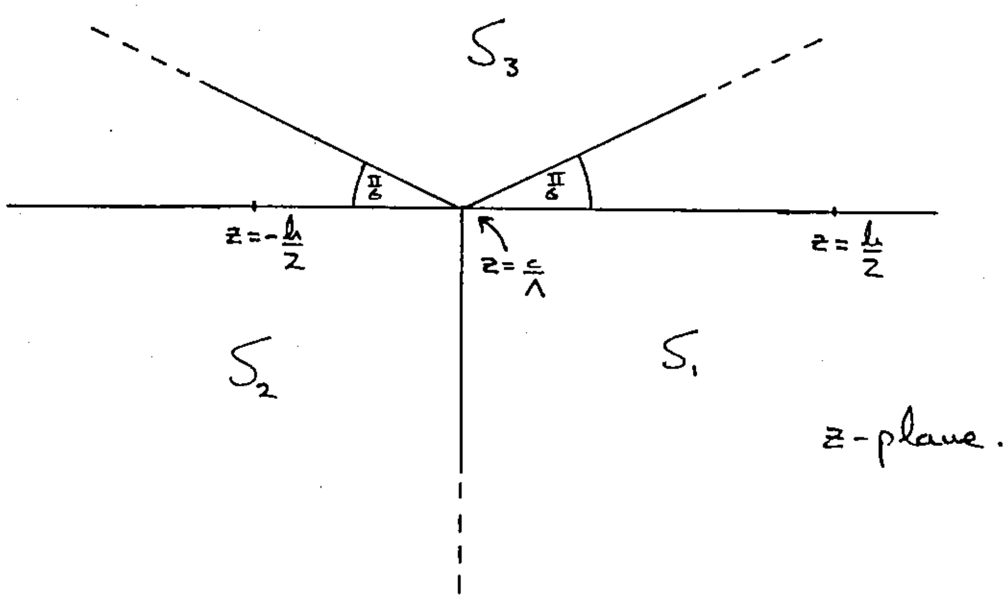


Figure 1.

An improved solution for which the domain of uniformity includes the critical point may be written in terms of Airy functions, an account appears in the same article by Reid. If κ is small then the effects of thermal diffusion may be expected to be confined to the neighbourhood of the critical point and possibly the points $z = \pm\frac{h}{2}$ associated with the top and bottom rigid boundaries, the exponential behaviour of this type of solution can not match the boundary conditions (1.63) unless exceptionally the critical point is close to a boundary point in which case one of $\psi_3{}^{(0)}$ and $\psi_4{}^{(0)}$ may be included.

(B) There exists a solution of (1) which to leading order in κ is represented by (11), or equivalently (15), and this is a uniformly valid approximation throughout D.

(C) There exists a solution of (1) which to leading order in κ is represented by (12), or equivalently (16), and this is a uniformly valid approximation in $D \setminus S_3$. In the sector S_3 the solution diverges like the pair (20).

The results also imply that the higher order terms are $O(\frac{i\kappa}{k})$. Note that the boundary points $z = \pm\frac{h}{2}$ are within D provided

$$\left| \pm\frac{l}{2} - \frac{c}{\Lambda} \right| > \epsilon \tag{21}$$

To investigate the neighbourhood of the critical point $\frac{c}{\Lambda}$ following Tollmien ('29, '47) and Holstein ('60) the 'inner region' of the asymptotic problem is defined by the transformation

$$\psi(z) = \chi(\epsilon; Z) \quad \text{where} \quad Z = \frac{(z - \frac{c}{\Lambda})}{\epsilon}, \quad \epsilon = \left(\frac{\kappa}{i\ell\Lambda}\right)^{\frac{1}{3}} \tag{22}$$

then the boundary points $z = \pm\frac{h}{2}$ become

$$Z_{\pm} = \left(\pm\frac{l}{2} - \frac{c}{\Lambda}\right)\left(\frac{\ell\Lambda}{\kappa}\right)^{\frac{1}{3}} e^{\frac{i\pi}{6}} \tag{23}$$

in the Z-plane. For a neutrally stable mode, $c_i = 0$, these points are on the line

$$\arg(Z) = \frac{\pi}{6} \text{ or } -\frac{5\pi}{6} \tag{24}$$

and values of Z corresponding to a mode with complex phase speed, $c_i \neq 0$, are such that

$$\arg(Z) = \frac{\pi}{6} + \tan^{-1}\left(\frac{c_i}{c_r - \Lambda z}\right) \tag{25}$$

with suitable choice of branches. If the critical point, at Z = 0, is such that $\frac{c_r}{\Lambda} \in (-\frac{h}{2}, \frac{h}{2})$ the critical layer is inside the flow region and the boundary points (23) are away from Z = 0 provided

$$\lim_{\left(\frac{\kappa}{k\lambda}\right)\to 0} \left|\left(\pm\frac{h}{2} - \frac{c}{\lambda}\right)\left(\frac{k\lambda}{\kappa}\right)^{\frac{1}{3}}\right| \neq 0 \tag{26}$$

i.e. this choice of inner region implies that ε of equation (21) is $(\frac{\kappa}{k\lambda})^{1/3}$. The scheme of the anti-Stokes lines of figure (1) and the boundary points are sketched in figure (2) for the neutral case, $c_i = 0$, where the sectors S_j correspond to the sectors S_j of figure (1)

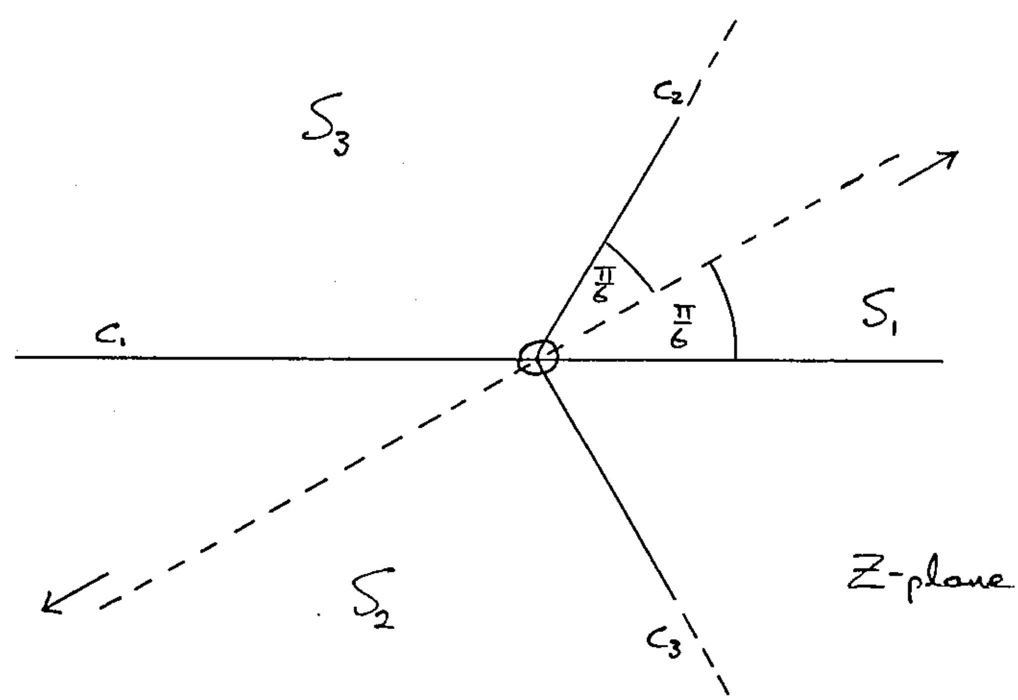


Figure 2.

and the anti-Stokes lines are the straight lines C_j dividing these sectors. The real interval $z \in [-\frac{h}{2}, \frac{h}{2}]$ transforms to a straight line shown dotted through $Z = 0$ contained in $S_1 \cup S_2$. For $c_i > 0$ the line is shifted downwards with constant gradient and is contained in $S_1 \cup S_2$ but for stable waves, $c_i < 0$, and the line is shifted upwards with constant gradient thereby intersecting the sector S_3. If the critical point, $Z = 0$, is such that

$$\frac{c_r}{\Lambda} < -\frac{h}{2} \qquad \text{or} \qquad \frac{c_r}{\Lambda} > \frac{h}{2} \tag{27}$$

then the critical layer is outside the region of the flow so that the image of the real interval $z \in [-\frac{h}{2}, \frac{h}{2}]$ in the Z-plane does not cross the critical point as c_i changes sign.

Under the transformation (22) the equation for the inner region or neighbourhood of the critical point is

$$\begin{aligned} &Z\left(D^2 - \epsilon^2 a_\omega^2 S - \frac{\epsilon D}{l_s}\right)\chi + \left(\frac{\beta S}{\Lambda} + \frac{1}{l_s}\right)\epsilon\chi \\ &\quad - \left(D^4 - \epsilon^2 a_\omega^2\left(1 + \frac{S_0}{K}\right)D^2 + \epsilon^4 a_\omega^4 \frac{S_0}{K}\right)\chi + \frac{\epsilon}{l_s}\left(D^3 - \epsilon^2 a_\omega^2 D\right)\chi \\ &\quad = 0 \end{aligned} \tag{28}$$

where $D = \frac{d}{dZ}$

Assuming an asymptotic expansion for $\chi(\epsilon;Z)$ which has the form

$$\chi(\epsilon;Z) = \chi^{(0)}(Z) + \epsilon\left(\frac{\beta S}{\Lambda} + \frac{1}{l_s}\right)\chi^{(1)}(Z) + O(\epsilon^2) \quad \text{as } \epsilon \to 0 \tag{29}$$

implies the sequence of problems

$$(D^2 - Z)D^2\chi^{(0)} = 0 \tag{30}$$

$$(D^2 - Z)D^2\chi^{(1)} = \left(1 + \frac{1}{l_s}\left(\frac{\beta S}{\Lambda} + \frac{1}{l_s}\right)^{-1}(D^2 - Z)D\right)\chi^{(0)}$$

etc. (31)

The four solutions of (30) may be expressed as

$$\chi_1^{(0)} = \epsilon Z \qquad \chi_2^{(0)} = 1$$

$$\chi_3^{(0)} = \int_{\infty_1}^{Z} dZ \int_{\infty_1}^{Z} Ai(Z)\, dZ \quad \chi_4^{(0)} = \int_{\infty_2}^{Z} dZ \int_{\infty_2}^{Z} Ai(Ze^{\frac{2i\pi}{3}})\, dZ \tag{32}$$

where Ai is Airy's function and in the notation of figure (2) $\infty_{1,2}$ denotes the end point of a path of integration which tends to the point at infinity in the sector S_1 or S_2 respectively. Asymptotic estimates of $\chi_3^{(0)}$ and $\chi_4^{(0)}$ show that they are recessive in the sectors S_1 and S_2 respectively and dominant elsewhere, i.e;

$$\chi_{3,4}^{(0)} \sim \frac{1}{2}\pi^{-\frac{1}{2}} Z^{-\frac{5}{4}} \exp\left(\mp \frac{2Z^{\frac{3}{2}}}{3}\right) \tag{33}$$

as $|Z| \to \infty$, for $|ph(Z)| < \pi$ in the case of $\chi_3^{(0)}$ and $-\frac{5\pi}{3} < ph(Z) < \frac{\pi}{3}$ in the case of $\chi_4^{(0)}$. This type of exponential behaviour as $|Z| \to \infty$ cannot be matched with the eigenfunctions (11) and (12) of equation (4) as $z \to \frac{c}{\Lambda}$, since it does in fact match with the pair (20), and so these two solutions which are called 'viscous solutions' in the theory of parallel flows may be discarded. Of the two remaining solutions $\chi_2^{(0)}$ matches the leading order (O(1)) term of the singular outer solution (16) (c.f. (12)) and $\chi_1^{(0)}$ matches the leading order $(O(z - \frac{c}{\Lambda}))$ term of the regular outer solution (15) (c.f. (11)).

At the next order (31) has a solution

$$\chi_1^{(1)} = -\frac{\epsilon Z^2}{2}\left(1 - \frac{1}{l_s}\left(\frac{\beta S}{\Lambda} + \frac{1}{l_s}\right)^{-1}\right) \tag{34}$$

which is forced by $\chi_1^{(0)}$ and this solution matches the second term in the series (15) or the second term in the regular part of the series (16), the other solution which is forced by $\chi_2^{(0)}$ is a particular integral of the equation

$$(D^2 - Z)\, D^2 \chi_2^{(1)} = 1 \tag{35}$$

which must behave like Z ln Z as $|Z| \to \infty$ in the sectors S_1 and S_2 of the Z plane containing the boundary points $Z_\pm$ to match the first term in the singular part of the series (16). The solution may be written following Reid ('65) in the form

$$\chi_2^{(1)} = \int_{\infty_{1,2}}^{Z} dZ \int_{\infty_{1,2}}^{Z} \left\{ Q_3(Z) - \frac{1}{Z} \right\} dZ + Z \ln Z + AZ + B \tag{36}$$

where

$$Q_3(Z) = 2\pi e^{\frac{i\pi}{6}} \left\{ Ai(Z) \int_{\infty_2}^{Z} Ai(Ze^{\frac{2i\pi}{3}})\, dZ - Ai(Ze^{\frac{2i\pi}{3}}) \int_{\infty_1}^{Z} Ai(Z)\, dZ \right\}$$

and $Q_3(z) = \frac{1}{Z} + O(\frac{1}{Z^4})$ for $|Z| \to \infty$ in $S_1 \cup S_2$

Matching (36) to (16) implies

$$A = \ln \epsilon \quad ; \quad B = 0 \tag{37}$$

This formulation of the inner problem implies that the regular solution of (4) may be used as a uniformly valid asymptotic solution of (1) for all c whereas the singular solution of (4) may be used provided c is such that the image in the Z-plane of the real interval $z \in [-\frac{h}{2}, \frac{h}{2}]$ lies in the domain $S_1 \cup S_2$ of the Z-plane. Returning to the z-plane of figure (1) this implies the fact long established in the theory of parallel flow instability that for all c the interval $z \in [-\frac{h}{2}, \frac{h}{2}]$ of the physical domain of the flow must be continuously distorted when used as a path of integration so that it passes beneath the singular point $z = \frac{c}{\Lambda}$ in the z-plane. Further, the leading order singularity (13) may be removed if the "corrected" eigenfunction

$$\begin{aligned} \psi_{2c} = {} & 1 + \left(a_3^2 S - \left(\frac{\beta S}{\Lambda} + \frac{1}{l_s}\right)\left(\frac{3\beta S}{2\Lambda} + \frac{1}{l_s}\right)\right) \cdot \frac{(z-\frac{c}{\Lambda})^2}{2!} + \dots \\ & - \left(\frac{\beta S}{\Lambda} + \frac{1}{l_s}\right)\left([\psi_1 - (z - \tfrac{c}{\Lambda})] \ln (z - \tfrac{c}{\Lambda}) + \left(\frac{\kappa}{i\alpha\Lambda}\right)^{\frac{1}{3}} \chi_2^{(1)}(Z) \right) \\ & + O\left(\left(\frac{\kappa}{i\alpha\Lambda}\right)^{\frac{2}{3}} \right) + O\left(\frac{i\kappa}{\alpha} \right) \end{aligned} \tag{38}$$

(provided $z \in S_1 \cup S_2$ of figure (1)) is used to replace the function ψ_2 of (16) where the terms in the first O symbol arise out of the sequence of unmatched terms of the inner problem and the second from the higher order terms of the outer problem. The remaining singularity is now as weak as

$$\left(z - \frac{c}{\lambda}\right)^2 \ln\left(z - \frac{c}{\lambda}\right) \tag{39}$$

and the inner problem must be pursued at least to order $\left(\frac{\kappa}{k\Lambda}\right)^{2/3}$ to remove it. Notice that the vertical potential temperature gradient (14) is now logarithmically divergent at the critical point and further that by virtue of the simple inviscid sidewall boundary condition the perturbation stream function ϕ of equation (2) together with its first derivative in z vanishes on the sidewalls but its higher order derivatives are singular at the critical point.

Dispersion Relation for Linearised Disturbances

The boundary conditions (1.63) have been derived under the assumption that in the neighbourhood of the boundary points $z = \pm \frac{h}{2}$ the flow is modified at leading order by viscous effects alone confined to an asymptotically thin layer which enable no-slip boundary conditions to be applied on the boundaries and the conditions (1.63) at their innermost edges. These viscous boundary layers may have the structure of an Ekman layer on a rigid surface provided only that, as in the case of a homogeneous incompressible fluid, the pressure in the layer is independent of the boundary layer coordinate

$$\tilde{z} = z / E_v^{\frac{1}{2}} \tag{40}$$

As stated previously the effect of stratification does not modify the structure provided

$$E_v < O\left(\frac{R_o}{S}\right) \qquad \text{as } R_o \to 0 \tag{41}$$

However as is indicated in the previous section the flow of a real fluid in the neighbourhood of the critical point is modified in a region of thickness

$$\left(\frac{\kappa}{k\lambda}\right)^{\frac{1}{3}} \tag{42}$$

by the effects of thermal diffusivity. If the critical layer, the position of which in relation to the flow is determined by the application of suitable top and bottom boundary conditions, overlaps a viscous boundary layer then inevitably the effects of thermal diffusion will modify the boundary layer structure. (An analogous problem for the stability of parallel flow has been studied by Smith ('79) using a triple deck analysis). When this situation, i.e;

$$\left| \pm \frac{h}{2} - \frac{c}{\Lambda} \right| = o\left(E_v^{\frac{1}{2}}\right) + o\left(\left(\frac{\kappa}{k\Lambda}\right)^{\frac{1}{3}}\right) \tag{43}$$

arises the model is inconsistent as the premise that the viscous boundary layer is an Ekman layer across which the pressure is constant contradicts the evidence that the critical layer in a real fluid is characterised by significant vertical pressure gradients.

When (43) is not satisfied the boundary and critical layers are effectively separate so that the eigenfunctions (15) and (38) describing the flow outside the boundary layers reduce to (15) and (16) where the effects of the correction at order $(\frac{\kappa}{k\Lambda})^{1/3}$ to the singular solution (16) may safely be neglected at the edge of the, now distant, Ekman layers with an error which is of order $(\frac{\kappa}{k\Lambda})$. To the same order of approximation the boundary condition (1.63) at the outer edge of the Ekman layers is

$$\left(\frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x}\right)\frac{\partial \phi}{\partial z} - \Lambda \cdot \frac{\partial \phi}{\partial x} = \frac{2z}{h}\left(\frac{SE_v^{\frac{1}{2}}}{2R_o}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi \tag{44}$$

on $z = \pm \frac{h}{2}$.

Equation (44) may be expressed in terms of $\psi(z)$ as

$$ik(\Lambda z - c)\psi'(z) - ik\Lambda\psi(z) = -\frac{2z}{h}\left(\frac{SE_v^{\frac{1}{2}}}{2R_o}\right) \cdot a_\omega^2\, \psi(z) \tag{45}$$

and using (6) and (8), equivalently

$$\begin{aligned} &\zeta^2\left(\frac{dG}{d\zeta} + \frac{\nu}{2\left(\frac{1}{4l_s^2} + a_\omega^2\zeta\right)^{\frac{1}{2}}} \cdot G\right) \\ &\qquad = \frac{2ia_\omega^2}{k\Lambda h} \cdot \left(\frac{SE_v^{\frac{1}{2}}}{2R_o}\right)\left(\frac{\zeta}{2\left(\frac{1}{4l_s^2} + a_\omega^2\zeta\right)^{\frac{1}{2}}} + \frac{c}{\Lambda}\right)\zeta \cdot G \end{aligned} \tag{46}$$

where the transformed boundary points are

$$\xi\left(-\frac{l}{2}\right) = -2\left(\frac{l}{2}+\frac{c}{\lambda}\right)\left(\frac{1}{4l_s^2}+a_\omega^2 S\right)^{\frac{1}{2}} \equiv s$$

$$\xi\left(\frac{l}{2}\right) = 2\left(\frac{l}{2}-\frac{c}{\lambda}\right)\left(\frac{1}{4l_s^2}+a_\omega^2 S\right)^{\frac{1}{2}} \equiv s+\alpha \tag{47}$$

Then putting

$$G(\xi) = A.M(a,2,\xi) + B.u(a,2,\xi) \tag{48}$$

where A, B $\in \mathbb{C}$ and both $M(a,2,\xi)$ and $u(a,2,\xi)$ are a pair of linearly independent confluent hypergeometric functions, the boundary conditions become

$$A\left\{s^2\left(M'(a,2,s)-\alpha_0 M(a,2,s)\right)+i\mu_0 s M(a,2,s)\right\}$$
$$+B\left\{s^2\left(u'(a,2,s)-\alpha_0 u(a,2,s)\right)+i\mu_0 s u(a,2,s)\right\} \tag{49}$$
$$= 0$$

and

$$A.\left\{(s+\alpha)^2\left(M'(a,2,s+\alpha)-\alpha_0 M(a,2,s+\alpha)\right)-i\mu_0(s+\alpha)M(a,2,s+\alpha)\right\}$$
$$+B\left\{(s+\alpha)^2\left(u'(a,2,s+\alpha)-\alpha_0 u(a,2,s+\alpha)\right)-i\mu_0(s+\alpha)u(a,2,s+\alpha)\right\} \tag{50}$$
$$= 0$$

where a dash denotes, e.g.;

$$M'(a,2,s+\alpha) = \left(\frac{d}{dx}M(a,2,x)\right)_{x=s+\alpha}$$

and

$$\alpha_0 = \frac{\left(\frac{1}{4l_s^2}+a_\omega^2 S\right)^{\frac{1}{2}}-\frac{1}{2l_s}}{2\left(\frac{1}{4l_s^2}+a_\omega^2 S\right)^{\frac{1}{2}}} \in \left[0,\tfrac{1}{2}\right] \tag{51}$$

when $\frac{1}{l_s} \geqslant 0$

and $\alpha_0 \to 0, \frac{1}{2}$ as $a_w{}^2 Sh_s{}^2 \to 0, \infty$ respectively;

$$\mu_0 = \frac{a_w^2}{\Lambda \ell} \left(\frac{S E_v^{\frac{1}{2}}}{2 R_0} \right) \tag{52}$$

The parameter $a \leq 1$ is given by equation (10) and

$$\begin{aligned} s &= -2 \left(\frac{1}{4 \ell_s^2} + a_w^2 S \right)^{\frac{1}{2}} \left(\frac{\ell}{2} + \frac{c}{\Lambda} \right) \\ \alpha &= 2\ell \left(\frac{1}{4 \ell_s^2} + a_w^2 S \right)^{\frac{1}{2}} > 0 \end{aligned} \tag{53}$$

The permissible normal modes are then given by the determinental condition that (49) and (50) allow non-trivial solutions of the form (48), i.e.;

$$D(a, 2, \alpha_0, \alpha, \mu_0; s) \equiv$$

$$\left(s^2 (M' - \alpha_0 M(a,2,s)) + i\mu_0 s M(a,2,s) \right)\left((s+\alpha)^2 (u' - \alpha_0 u(a,2,s+\alpha)) - i\mu_0 (s+\alpha) u(a,2,s+\alpha) \right)$$

$$-$$

$$\left(s^2 (u' - \alpha_0 u(a,2,s)) + i\mu_0 s u(a,2,s) \right)\left((s+\alpha)^2 (M' - \alpha_0 M(a,2,s+\alpha)) - i\mu_0 (s+\alpha) M(a,2,s+\alpha) \right)$$

$$= 0 \tag{54}$$

and the notation has been abbreviated by writing $M' - \alpha_0 M(a,2,s)$ for

$$\left(\frac{d}{dx} M(a,2,x) - \alpha_0 M(a,2,x) \right)_{x = s}$$

In principle, for given values of the real parameters $(a, \alpha_0, \alpha, \mu_0)$, i.e.; for a fixed point in the real four dimensional parameter space $(a, \alpha_0, \alpha, \mu_0)$, the location of the roots of the dispersion relation (54) in the complex s-plane determines the number and stability of the linear normal mode solutions. To choose a suitable function $u(a,2,s)$ from the fundamental pair (11) and (12) it is desirable to render it real along the positive real axis, hence

$$u(a,2,s) = e^s U(2-a, 2, e^{-i\pi} s) + \frac{i\pi M(a,2,s)}{\Gamma(1-a)}$$

$$u(a,2,s) = e^{s}\,\mathcal{U}(2-a,2,e^{-i\pi}s) + \frac{i\pi M(a,2,s)}{\Gamma(1-a)} \tag{55}$$

which, using Kummer's transformation, has the power series representation

$$u(a,2,s) = e^{s}\left\{\frac{1}{\Gamma(1-a)}\sum_{0}^{\infty}\frac{\Gamma(2-a+j)(e^{-i\pi}s)^{j}}{\Gamma(2-a)(j+1)!\,j!}\left(\ln s+\psi(2-a+j)-\psi(1+j)-\psi(2+j)\right) + \frac{1}{\Gamma(2-a)}\cdot\frac{1}{(e^{-i\pi}s)}\right\} \tag{56}$$

where ψ denotes the logarithmic derivative of the gamma function. Immediately, since $M(a,2,s)$ is real on the real axis, the complex roots of (54) occur in conjugate pairs unless μ_0 is non-zero.

By inspection, the function $D(a,2,\alpha_0,\alpha,\mu_0;s)$ has no poles in the finite part of the s-plane but has 2 branch points at $s = 0, -\alpha$ which arise from the logarithm in (56) when $a < 1$. The choice

$$\begin{aligned} &\mathrm{pl}(s) \in [-\pi,\pi) \\ &\mathrm{pl}(s+\alpha) \in [-\pi,\pi) \end{aligned} \tag{57}$$

implies branches of the logarithms $\ln(s)$, $\ln(s+\alpha)$ which are consistent with two branch cuts from $s = 0$ and $s = -\alpha$ to infinity along the negative real s axis. Rewriting $u(a,2,s)$ using (12) and (55) as

$$\begin{aligned} u(a,2,s) &= \frac{1}{\Gamma(1-a)}\cdot M(a,2,s)\ln s \\ &+ e^{s}\left\{\frac{1}{\Gamma(1-a)}\sum_{0}^{\infty}\frac{\Gamma(2-a+j)}{\Gamma(2-a)(j+1)!\,j!}\left(\psi(2-a+j)-\psi(1+j)-\psi(2+j)\right) + \frac{1}{\Gamma(2-a)}\cdot\frac{1}{e^{-i\pi}s}\right\} \end{aligned} \tag{58}$$

substitution in (54) shows that $D(a,\alpha_0,\alpha,\mu_0;s)$ is a continuous function of s across the line $\mathrm{Re}(s) < -\alpha$, i.e.; the branch cuts there may be removed and if

$$\ell = \{ s : -\alpha < s < 0 \} \tag{59}$$

then D is an analytic function of s in the cut plane of figure (4) and an analytic function of the real parameters $(a, \alpha_0, \alpha, \mu_0)$ for all s except at the branch points $s = 0$, $s = -\alpha$, in this cut plane. Note that equation (56) shows that since $\frac{1}{\Gamma(1-a)}$ has a simple zero at $a = 1$ the branch cut and branch points may be removed in the limit $a \to 1^-$ so that D is then analytic throughout the s-plane.

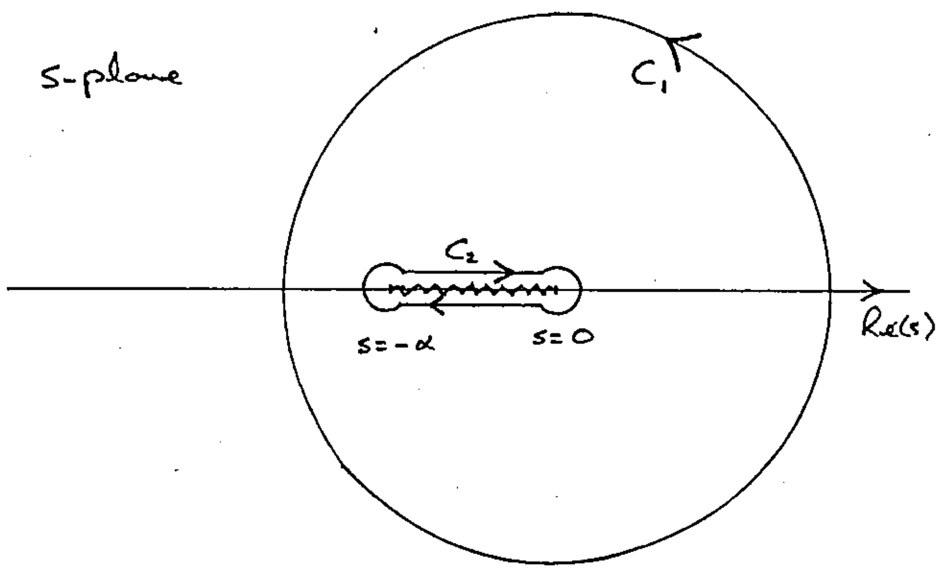


Figure 4.

<u>The number of roots of $D(a,2,\alpha_0,\alpha,\mu_0,s)$ in the s-plane</u>

The principle of the argument may be applied to any domain R_D in the s-plane where D(s) is differentiable except for a finite number of poles and then

Multiplicity of roots of D(s) - Multiplicity of poles of D(s) within R_D

$$= \frac{1}{2\pi}\left\{ \begin{array}{l} \text{change of phase of } D(s) \text{ as } s \text{ traverses} \\ \partial R_D \text{ in the positive sense.} \end{array} \right\}$$

where D(s) stands as an abbreviation for $D(a,2,\alpha_0,\alpha,\mu_0,s)$. Hence if R_D is taken to be the doubly connected domain bounded by a large circle, C_1, of radius R centre s = 0 and a closed curve C_2 surrounding ℓ in the s-plane as sketched in figure (4) then the absence of poles of D(s) implies that the limit as $R \to \infty$ and $C_2 \to \ell$ gives the number of roots of D(s) in the finite part of the s-plane, except on ℓ.

On C_1 the limit may be evaluated using the known asymptotic representations of confluent hypergeometric functions. For this purpose it is convenient to use the function U(a,2,s) which is related to the fundamental pair (11) and (12) via the connection formula

$$M(a,c,s) = \frac{\Gamma(c)}{\Gamma(c-a)} e^{i\pi a} U(a,c,s) + \frac{\Gamma(c) e^{i\pi(a-c)}}{\Gamma(a)} e^{s} U(c-a,c,e^{-i\pi}s) \quad (60)$$

where the branch is determined by $ph(-s) = -\pi$ when $ph(s) = 0$ and by continuity elsewhere. Substituting for u and M in (54) using (57) and (60) gives

$$\begin{aligned}
&\Gamma(2-a)e^{-i\pi a} D(a,2,\alpha_0,\alpha,\mu_0;s) = \\
&\quad \left\{ s^2(U' - \alpha_0 U(a,2,s)) + i\mu_0 s\, U(a,2,s) \right\} \\
&\qquad \times \left\{ (s+\alpha)^2 \left((e^{s+\alpha} U(2-a,2,e^{-i\pi}(s+\alpha)))' - \alpha_0 e^{s+\alpha} U(2-a,2,e^{-i\pi}(s+\alpha)) \right) \right. \\
&\qquad\qquad \left. - i\mu_0 (s+\alpha) e^{s+\alpha} U(2-a,2,e^{-i\pi}(s+\alpha)) \right\} \\
&\quad - \\
&\quad \left\{ (s+\alpha)^2 (U' - \alpha_0 U(a,2,s+\alpha)) - i\mu_0 (s+\alpha) U(a,2,s+\alpha) \right\} \\
&\qquad \times \left\{ s^2 \left((e^s U(2-a,2,e^{-i\pi}s))' - \alpha_0 e^s\, U(2-a,2,e^{-i\pi}s) \right) \right. \\
&\qquad\qquad \left. + i\mu_0 s e^s U(2-a,2,e^{-i\pi}s) \right\}
\end{aligned} \quad (61)$$

The asymptotic representation of U(a,2,s)

$$U(a,2,s) = s^{-a}\left(1 - \frac{a(a-1)}{s} + O(s^{-2})\right) \quad (62)$$

as $|s| \to \infty$ for $ph(s) \in (-\frac{3\pi}{2}, \frac{3\pi}{2})$ is derived from the Mellin Barnes integral representation of $U(a,c,z)$, see for example Erdelyi et al., Higher Transcendental Functions, Chapter 6. Hence the first term in curly brackets of (61) becomes

$$-(s^{2-a})\left(\alpha_0 - \frac{a[\alpha_0(a-1)-1]+i\mu_0}{s} + O(s^{-2})\right) \tag{63}$$

as $s \to \infty$ for $ph(s) \in (-\frac{3\pi}{2}, \frac{3\pi}{2})$ and since $ph(s+\alpha) \to ph(s)$ as $s \to \infty$ for finite α the third term in curly brackets of (61) becomes

$$-(s^{2-a})\left(\alpha_0 - \frac{\alpha_0[(a-2)\alpha + a(a-1)] - i\mu_0 - a}{s} + O(s^{-2})\right) \tag{64}$$

as $s \to \infty$ for $ph(s+\alpha) \in (-\frac{3\pi}{2}, \frac{3\pi}{2})$ with finite α. Using (62)

$$e^s U(2-a, 2, e^{-i\pi}s) = e^{-i\pi a + s} s^{a-2}\left(1 + \frac{(a-1)(a-2)}{s} + O(s^{-2})\right) \tag{65}$$

as $s \to \infty$ for $ph(s) \in (-\frac{\pi}{2}, \frac{5\pi}{2})$, so that in this sector of the s-plane the fourth term in curly brackets of (61) is

$$e^{-i\pi a + s} s^{a}\left(1 - \alpha_0 + \frac{(a-2)+i\mu_0}{s} + O(s^{-2})\right) \tag{66}$$

Similarly, for finite α as $s \to \infty$ for $ph(s+\alpha) \in (-\frac{\pi}{2}, \frac{5\pi}{2})$, the second term is

$$e^{-i\pi a + s + \alpha} s^{a}\left(1 - \alpha_0 + \frac{(a-2) - i\mu_0 + a\alpha(1-\alpha_0)}{s} + O(s^{-2})\right) \tag{67}$$

To evaluate $D(s)$ asymptotically as $s \to \infty$ with the choice of branches (57) it remains to find asymptotic representations of the second and fourth terms in the curly brackets of (61) which are valid throughout the sector $ph(s) \in [-\pi, -\frac{\pi}{2})$. From the analytic continuation of $U(a,2,z)$ in the z-plane

$$U(a, 2, ze^{-2\pi i}) = U(a,2,z) - \frac{2\pi i}{\Gamma(a-1)} M(a,2,z) \tag{68}$$

with (60) and the identity $\Gamma(z).\ (1-z) = \frac{\pi}{\sin \pi z}$

$$\mathcal{U}(a,2,e^{-2\pi i}z) = e^{2i\pi a}\,\mathcal{U}(a,2,z) - \frac{2\pi i e^{i\pi a}}{\Gamma(a-1)\Gamma(a)} \cdot e^{z}\,\mathcal{U}(2-a,2,e^{-i\pi}z) \tag{69}$$

A change of variables, $z = se^{2\pi i}$, then gives

$$\mathcal{U}(a,2,s) = e^{2i\pi a}\,\mathcal{U}(a,2,e^{2\pi i}s) - \frac{2\pi i e^{i\pi a}}{\Gamma(a-1)\Gamma(a)}\,e^{s}\,\mathcal{U}(2-a,2,e^{i\pi}s) \tag{70}$$

and thus

$$e^{s}\,\mathcal{U}(2-a,2,e^{-i\pi}s) = e^{-2i\pi a}.e^{s}\,\mathcal{U}(2-a,2,e^{i\pi}s) - \frac{2\pi i e^{-i\pi a}}{\Gamma(1-a)\Gamma(2-a)}\,\mathcal{U}(a,2,s) \tag{71}$$

so that (62) gives the required asymptotic representation of $e^{s}U(2-a,2,e^{-i\pi}s)$ valid in the sector

$$\left\{ ph(e^{i\pi}s) \in (-\tfrac{3\pi}{2}, \tfrac{3\pi}{2}) \right\} \cap \left\{ ph(s) \in (-\tfrac{3\pi}{2}, \tfrac{3\pi}{2}) \right\}$$

i.e.; in the sector ph(s) $\in (-\frac{3\pi}{2}, \frac{\pi}{2})$. The first term on the right hand side of (70) is

$$e^{-i\pi a}\,e^{s}\,s^{a-2}\left(1 + \frac{(a-1)(a-2)}{s} + O(s^{-2})\right) \tag{72}$$

which is formally the same as (65) except that (72) is valid for a different range of ph(s). Since the second term on the right hand side of (71) is a multiple of U(a,2,s) its contribution to D(s) is seen to vanish identically from (61). Thus the asymptotic representations (66) and (67) may be used throughout the sector phs $\in (-\frac{5\pi}{2}, \frac{5\pi}{2})$ in evaluating D(s) asymptotically.

Equations (61), (63), (64), (66) and (67) imply

$$D(a,2,\alpha_o,\alpha,\mu_o;s) = \frac{-e^s s^2}{\Gamma(2-a)}\Big\{\alpha_o(1-\alpha_o)(e^\alpha-1) + e^\alpha[a\alpha_o(1-\alpha_o)(\alpha+1-a)+a-2\alpha_o-i\mu_o] - \frac{[\alpha_o(1-\alpha_o)(a(1-a)-\alpha(a-2))+a-2\alpha_o+i\mu_o]}{s} + O(s^{-2})\Big\} \quad \text{as } s\to\infty \tag{73}$$

and the change in phase of D(s) as s moves anticlockwise once around C_1 is given by putting $s = Re^{i\theta}$ and letting θ increase through the interval $[-\pi,\pi)$. In the limit $R \to \infty$ the contribution is $4\pi\ \forall\ \alpha > 0,\ \alpha_o \in (0, \frac{1}{2}],\ \forall\mu_o$.

At this stage the effect of viscosity is completely removed from the problem by setting $\mu_o = 0$, thus the roots of D in the s-plane occur in complex conjugate pairs and $D(a,2,\alpha_o,\alpha,0,s)$ is abbreviated to $D(a,2,\alpha_o,\alpha,s)$ or D.

To evaluate the change in phase of D around the contour C_2 note that on the branch cut ℓ.

$$\begin{aligned} D(a,2,\alpha_o,\alpha;s) &= s^2(s+\alpha)^2\Big\{(M'-\alpha_o M(a,2,s))(u'-\alpha_o u(a,2,s+\alpha)) - (M'-\alpha_o M(a,2,s+\alpha))(u'-\alpha_o u(a,2,s))\Big\} \\ &= s^2(s+\alpha)^2\Big\{(M'-\alpha_o M(a,2,s))(u'-\alpha_o u(a,2,s+\alpha)) - (M'-\alpha_o M(a,2,s+\alpha)).\,Re(u'-\alpha_o u(a,2,s))\Big\} \\ &\quad \mp \frac{i\pi s^2(s+\alpha)^2}{\Gamma(1-a)}(M'-\alpha_o M(a,2,s+\alpha))(M'-\alpha_o M(a,2,s)) \end{aligned} \tag{74}$$

where the first term is real, the second term is purely imaginary arising from the logarithmic singularity in u(a,2,s) and the upper choice of sign is taken on the upper side of ℓ where ph(s)= π^-.
As $s \to -\alpha$

$$D(a,2,\alpha_o,\alpha;s) = \frac{\alpha^2}{\Gamma(2-a)}(M'-\alpha_o M(a,2,-\alpha)) + o(1) \tag{75}$$

and as $s \to 0$

$$D(a,2,\alpha_0,\alpha;s) = -\frac{\alpha^2}{\Gamma(2-a)}\left(M' - \alpha_0 M(a,2,\alpha)\right) + o(1) \tag{76}$$

both quantities are real in the limit. Away from the branch points and on the contour C_2 of figure (4), $\mathrm{Im}(D) \neq 0$ unless there exists $s \in \ell$ such that

Either
$$M' - \alpha_0 M(a,2,s+\alpha) = 0 \tag{77a}$$

or
$$M' - \alpha_0 M(a,2,s) = 0 \tag{78a}$$

In the case of (77a), for such an s,

$$R_e(D) = s^2(s+\alpha)^2\left(M' - \alpha_0 M(a,2,s)\right)\left(u' - \alpha_0 u(a,2,s+\alpha)\right) \tag{77b}$$

whereas in the case of (78a)

$$R_e(D) = -s^2(s+\alpha)^2\left(M' - \alpha_0 M(a,2,s+\alpha)\right).R_e\left(u' - \alpha_0 u(a,2,s)\right) \tag{78b}$$

Now consider the pair of functions

$$P(a,2,\alpha_0;s) \equiv M' - \alpha_0 M(a,2,s) \tag{79}$$

$$Q(a,2,\alpha_0;s) \equiv u' - \alpha_0 u(a,2,s) \tag{80}$$

First of all, their Wronskian, found by manipulating the confluent hypergeometric equation and the Wronskian of the fundamental pair (11) and (12) is given by

$$\begin{aligned} W(P,Q) &\equiv PQ' - QP' \\ &= \frac{e^s}{\Gamma(2-a)s^3}\left(2\alpha_0 - a - \alpha_0(1-\alpha_0)s\right) \end{aligned} \tag{81}$$

which is non-zero except at

$$s = \frac{2\alpha_0 - a}{\alpha_0(1-\alpha_0)} \equiv s_0(\alpha_0, a) \in \mathbb{R} \tag{82}$$

and from the power series solutions (11) and (56) as $s \to 0$

$$P(a,2,\alpha_0;s) = \frac{a}{2} - \alpha_0 + O(s) \tag{83}$$

$$Q(a,2,\alpha_0;s) = \frac{1}{\Gamma(2-a)s^2}\left(1 + o(1)\right) \tag{84}$$

Hence the roots of $P(a,2,\alpha_0;s)$ and $Q(a,2,\alpha_0;s)$ in any connected interval of the real line not containing the point $s = s_0$ are simple and interlaced, i.e.; the open interval between two consecutive roots of P or Q contains exactly one simple root of Q or P respectively. Further when $a > 2\alpha_0$, if either P or Q has a simple root on the real line to the right of $s = 0$, Q has the smallest positive root and the remaining roots of P and Q are interlaced on $s \in (0,\infty)$. When $a < 2\alpha_0$, if either P or Q has a simple root in the interval $(0,s_0)$ again Q has the smallest positive root and the remaining roots of P and Q are simple and interlaced on this interval; supposing now that either P or Q may have a root in the interval (s_0,∞) and that neither function vanishes at s_0, then the function which has the smallest root on (s_0,∞) is that one which has the largest root on $(0,s_0)$.

The point $s_0(\alpha_0,a)$ may be considered in further detail either by using the Wronskian (81) or by the change of variables

$$\phi' - \alpha_0\phi(s) = e^{\frac{s}{2}} s^{-\frac{3}{2}} \left(s - s_0(\alpha_0,a)\right)^{\frac{1}{2}} \psi(s) \tag{85}$$

where $\phi(s)$ satisfies the confluent hypergeometric equation and $\psi(s)$ satisfies the second order differential equation

$$s^2(s-s_0)^2\psi''(s) + \psi(s)\Big\{ s(s-s_0)\left[\alpha_0(\alpha_0-1)(s-s_0)^2 + (1-2\alpha_0)\left(s+\tfrac{s_0}{2}\right) - 3\right]$$
$$- \tfrac{1}{4}\left((1-2\alpha_0)s^2 - s(2+(1-2\alpha_0)s_0) + 3s_0\right)\left((1-2\alpha_0)s^2 + s(2-(1-2\alpha_0)s_0) + s_0\right)\Big\}$$
$$= 0 \tag{86}$$

Applying the method of Froebenius to the regular singular point $s = s_0$, the exponents there are $-\frac{1}{2}$ and $\frac{3}{2}$ implying that either P and Q are both non-zero there or one of them has a twice repeated root. Whilst the exponents at s_0 differ by an integer there is (obviously) no logarithmic branch point at $s = s_0$.

Consider $P(a,2,\alpha_0;s)$ alone and note that P is

$$P(a,2,\alpha_0;s) = M' - \alpha_0 M(a,2,s)$$

analytic (differentiable and free from singularity) for all a and α_0 in the finite part of the s-plane and thus as a and α_0 vary throughout their parameter ranges the number and position of its roots on the real axis vary either by

(i) Roots appearing or disappearing through the singular point at $s = \infty$

or (ii) A twice repeated root appearing or disappearing through the singular point $s = s_0$ for $\alpha_0 > 0$. Fortunately, if and only if $\alpha_0 = 0$

$$P(a,2,0;s) = M'(a,2,s) = \frac{a}{2} M(a+1,3,s) \tag{87}$$

is a (regular) confluent hypergeometric function, its real roots are all simple, may be shown to form a Sturmian chain and their loci in the (a,s) half-plane

$$a \leq 1 \quad , \quad s \in (-\infty, \infty)$$

may be sketched as in figure (5). (See Tricomi, 1, §3.5 and page 141 in particular). In the case $\alpha_0 = 0$, and when $a = -n$ $(n = 1,2,\ldots.)$, $P(-n,2,0;s)$ is a constant factor multiplied by a generalised Laguerre polynomial $L^2_{n-1}(s)$ of degree n-1, the orthogonality of this class of functions with respect to the weight function s^2e^{-s} on the real line implies that the n-1 roots of $P(-n,2,0;s)$ are simple and distributed on $s \in (0,\infty)$. Clearly, when $n = 0$, P vanishes identically in the s-plane.

Similarly $M(a,2,s)$ is related to $L^1_n(s)$ when $a = -n$ $(n = 0,1,2,\ldots.)$, all of its n roots are simple, are distributed on $(0,\infty)$ and distinct from those of its derivative $L^2_{n-1}(s)$. Hence when $\alpha_0 > 0$

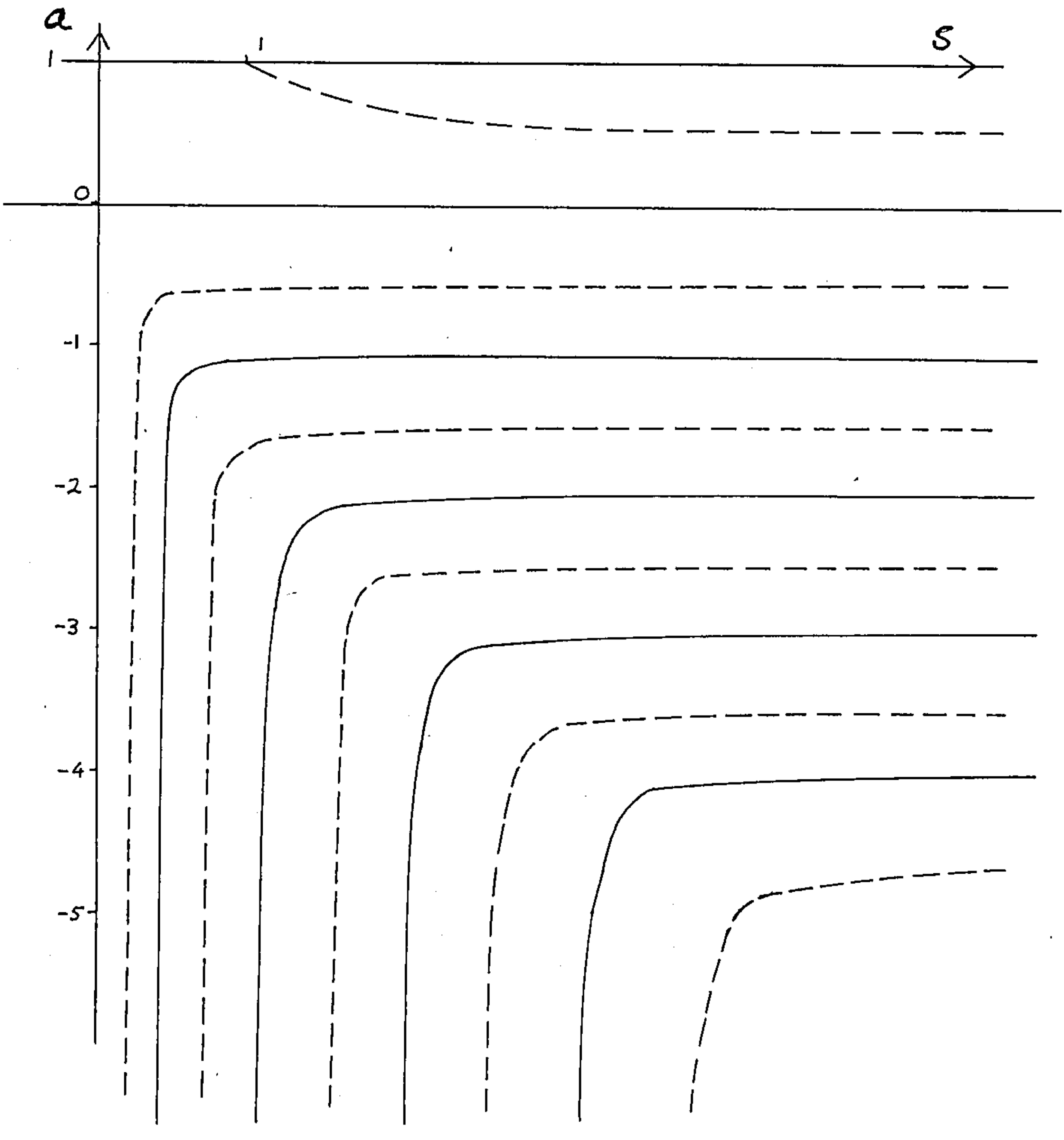


Figure 5. Contours P(a,2,0;s) = 0 for real s (solid lines), the contours Q(a,2,0;s) = 0 for real s are shown dotted. P(a,2,0;0) < 0 for a < 0; Q(a,2,0;s) $\sim \frac{1}{\Gamma(2-a)s^2} > 0$ as $s \to 0^+$; determining sgn (P), sgn (Q) throughout the plane.

$$P(-n,2,\alpha_0;s) = M' - \alpha_0 M(-n,2,s) \tag{88}$$

has at least (n-1) simple roots on $s \in (0,\infty)$ but since $P(-n,2,\alpha_0;s)$ is a polynomial of degree n with real coefficients it has a total of n roots, all on the real axis, n-2 of which must be simple. When $n = 0$, $P(-n,2,\alpha_0;s) = -\alpha_0$.

With reference to (i) above, the asymptotic representation of $P(a,2,\alpha_0;s)$ is

$$P(a,2,\alpha_0;s) = \frac{e^s s^{a-2}}{\Gamma(a)}\left((1-\alpha_0) + O(s^{-1})\right) - \frac{e^{i\pi a} s^{-a}}{\Gamma(2-a)}\left(\alpha_0 + \frac{a}{s}(1+\alpha_0(1-a)) + O(s^{-2})\right) \tag{89}$$

$$\text{as } |s| \to \infty$$

The first term is dominant as $s \to \infty$ and recessive as $s \to -\infty$, its sign is independent of α_0 but since $\frac{1}{\Gamma(a)}$ has simple zeros at $a = -n$ $(n = 0,1,\ldots.)$ this term changes sign as a passes through $a = -n$. Consequently, for all α_0, a simple root of P appears or disappears at $s = +\infty$ if and only if $a = -n$ $(n = 0,1,\ldots.)$ and with reference to figures (5) to (7) the lines $a = -n$ are therefore asymptotes to the contours $P(a,2,\alpha_0;s) = 0$ as $s \to \infty$, $\forall\alpha_0 \in [0,\frac{1}{2}]$. The second term is dominant as $s \to -\infty$ and recessive as $s \to \infty$ except when $a = -n$ $(n = 0,1,\ldots.)$ when it is dominant everywhere and P is a polynomial, its sign is independent of a for a in a neighbourhood of $a = -n$, however in the limit $\alpha_0 \to 0^+$ the sign of this term changes as $s \to \infty$ when $a < 0$ and as $s \to -\infty$ when $a > 0$. Hence the contours $P(a,2,\alpha_0;s) = 0$ asymptote to the lines $a = -n$ $(n = 0,1,\ldots.)$ from above in the (a,s) plane as $s \to \infty$ for $\alpha_0 > 0$. This implies that there must exist a value $a_0 \in (-n,-n+1)$ $(n = 1,2,\ldots.)$ of a, for all α_0, at which $P(a_0,2,\alpha_0,s)$ has a twice repeated root at $s = s_0(\alpha_0,a_0)$. A special consideration of $M(1,2,s)$ as given in (11a) shows that for $\alpha_0 > 0$

$$P(1,2,\alpha_0;s) = \frac{se^s - (1+\alpha_0 s)(e^s-1)}{s^2} \tag{90}$$

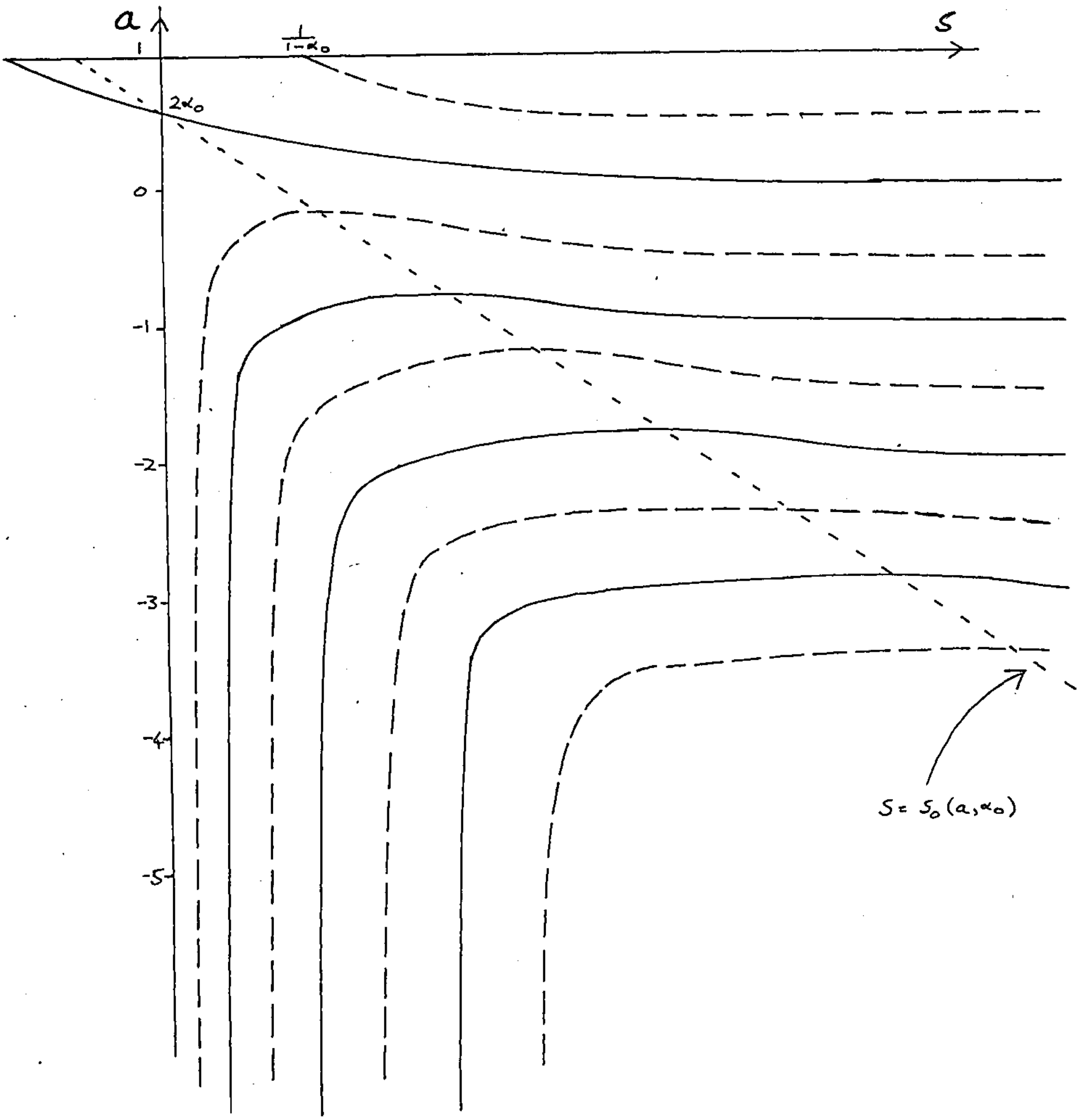


Figure 6. Contours $P(a,2,\alpha_0;s) = 0$ for a typical value of $\alpha_0 \in (0,\frac{1}{2})$, s real. The contours $Q(a,2,\alpha_0;s)$ are shown dotted.

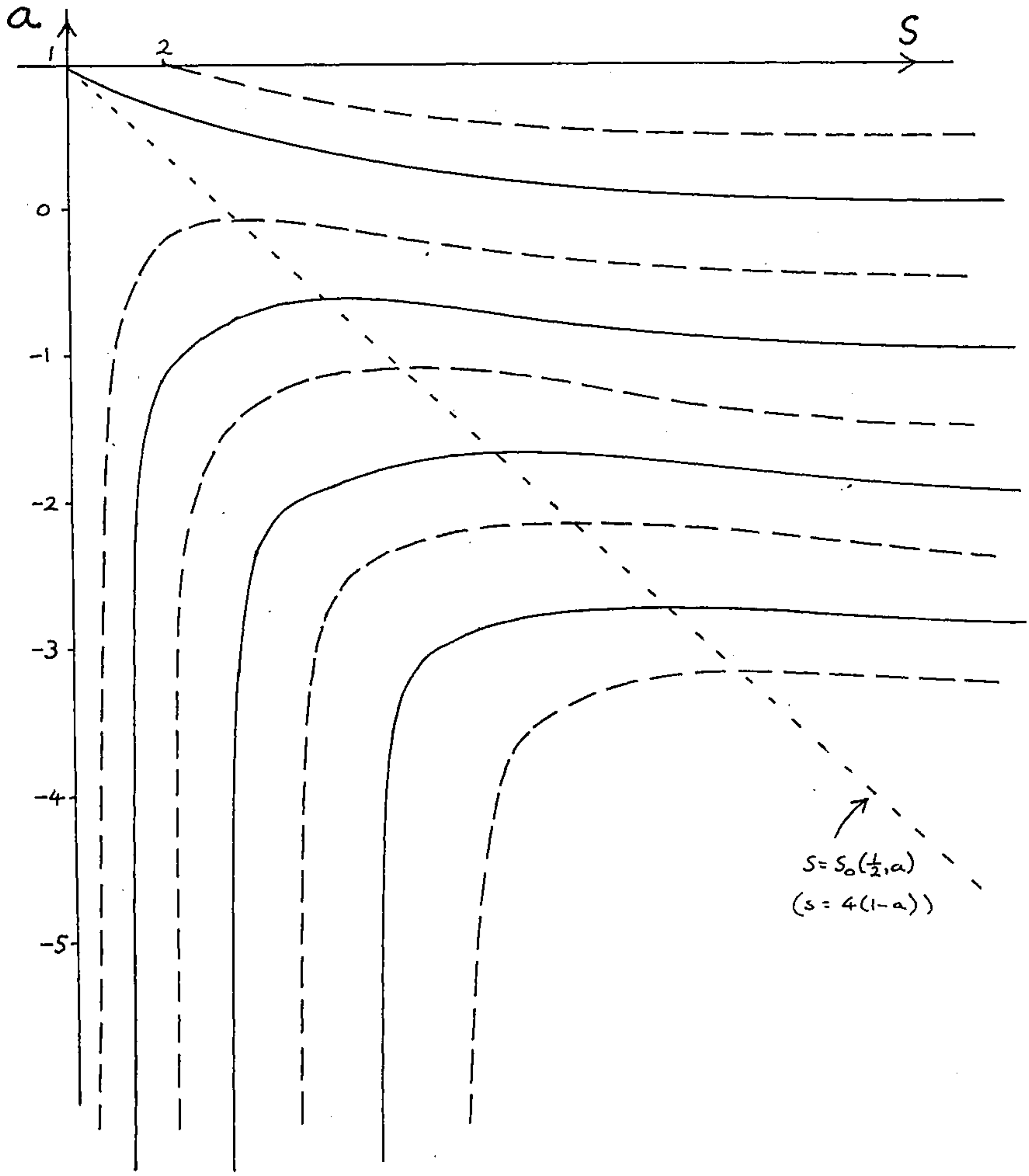


Figure 7. Contours $P(a,2,\frac{1}{2};s) = 0$ with contours $Q(a,2,\frac{1}{2};s)$ shown dotted.

having one root which moves along $(-\infty,0)$ as α_0 varies in $(0, \frac{1}{2})$ which is to the left of $s_0(\alpha_0,1)$ and both roots $\sim -\frac{1}{\alpha_0}$ to leading order as $\alpha_0 \to 0$.

The contours $P(a,2,\alpha_0;s)$ are sketched in figures (6) and (7) for a typical value of $\alpha_0 \in (0, \frac{1}{2})$ and $\alpha_0 = \frac{1}{2}$ respectively. Although some numerical integration was used in deriving quantitative detail the qualitative form of the sketches follows from the preceding discussion. Notice that the singular points of (86) coincide at $s = 0$ when $a = 2\alpha_0$, the power series solution (11) implies

$$P(a,2,\alpha_0;s) = \frac{a}{2} - \alpha_0 + \frac{a}{2}\left(\frac{a+1}{3} - \alpha_0\right)s + O(s^2) \qquad (91)$$

$$\text{as } s \to 0$$

so that $P(2\alpha_0,2,\alpha_0;0) = 0 \ \forall \alpha_0$ and the gradient in the (a,s) plane of the contour though $(2\alpha_0,0)$ is

$$-\frac{2\alpha_0}{3}(1-\alpha_0)$$

so that the contour has a simple intersection with the line $s = s_0(\alpha_0,a)$. The contours of $Q(a,2,\alpha_0;s)$ are also shown dotted in in figures (5) to (7). Their location may be inferred from the asymptotics of $u(a,2,s)$ as $s \to \infty$, the power series solution (84) at $s = 0$ which gives Q a pole of order two for all α_0 and $a \leq 1$ and the interlacing of its roots with those of P on $(0,\infty)$. With the choice of branches (57), the negative real axis is a branch cut for $Q(a,2,\alpha_0;s)$ when $a < 1$, when $a = 1$

$$Q(1,2,\alpha_0;s) = \frac{e^s}{s^2}\left(1 - s(1-\alpha_0)\right) \qquad (92)$$

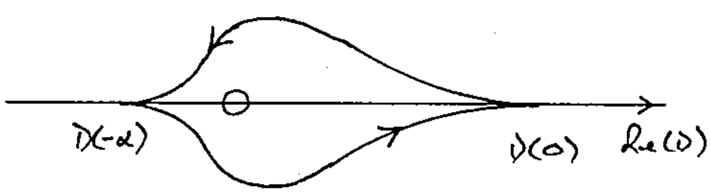
possessing one simple root at $s = \frac{1}{1-\alpha_0} \in [1,2]$.

Now return to consider the locus of $D(a,2,\alpha_0,\alpha;s)$ in the complex D-plane as s moves clockwise around the branch cut contour C_2. When $a = 1$ the disappearance of the branch cut, ℓ, implies from (71) that for $\alpha_0 \in (0, \frac{1}{2}]$ and $\alpha > 0$ there are always two roots of the dispersion relation in the finite part of the s-plane, this limit may formally be recovered throughout the analysis of this chapter and will be considered in some detail later. For $a < 1$, as s moves along the upper side of the branch cut ($ph(s) = \pi$) from $s = -\alpha$ to $s = 0$, D moves off the real axis at a point given by (75) and into the complex D-plane with an imaginary part given by (74). If and only if there is a point $s \in \ell$ where either (77a) or (78a) is satisfied then $Im(D) = 0$ there and the sign of $Re(D)$ is given by (77b) or (78b) respectively. As $s \to 0$, D tends to a point on the real axis given by (76). The locus of D as s continues to describe the lower side of the branch cut ($ph(s) = -\pi$) is simply the reflection in the real D axis of the locus for the upper side, since $\overline{D(s)} = D(\bar{s})$. Specifically, with reference to figure (6), on the upper side of the branch cut

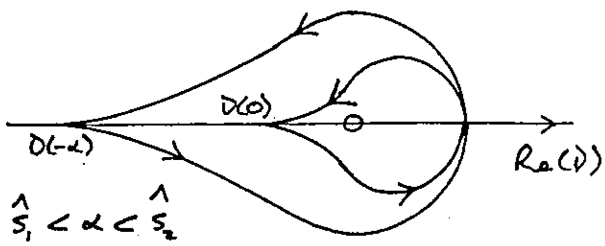
(A) For $a < 2\alpha_0$; $D(a,2,\alpha_0,\alpha,-\alpha) < 0$ and $Sgn(Im(D)) = Sgn(\frac{1}{\Gamma(1-a)}).Sgn(P(a,2,s+\alpha))$. When $Im(D) = 0$, $Sgn(Re(D)) = -Sgn(Q(a,2,s+\alpha))$. At $s = 0$, $Sgn(D(0)) = -Sgn(P(a,2,\alpha_0,\alpha))$. If the real roots of $P(a,2,\alpha_0;s)$ are arranged in ascending order

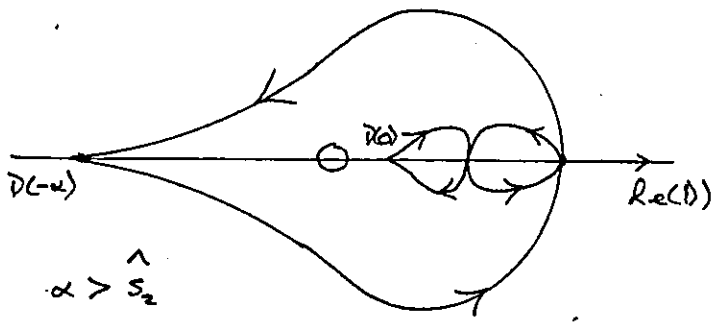
$$\hat{s}_1 < \hat{s}_2 < \ldots\ldots$$

then for a fixed value of a, $\alpha < \hat{s}_1$ implies that the locus of D is a closed curve encircling $D = 0$ once in the positive (anticlockwise) sense; $\hat{s}_1 < \alpha < \hat{s}_2$ implies that the locus of D encircles the origin twice in the positive sense, etc. If, however, for some $a \in (-n,-n+1)$ $P(a,2,\alpha_0;s)$ has $n+1$ roots, that one of the contours $P(a,2,\alpha_0;s) = 0$ which s traverses last as it moves along C_2 on the upper side of the branch cut is crossed twice and the locus of D in the D-plane encircles the origin n times in the positive sense, i.e.; for $a = -1^+$ the loci are sketched below.

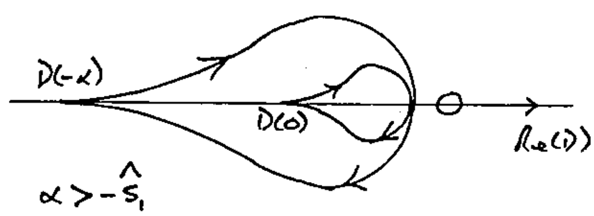


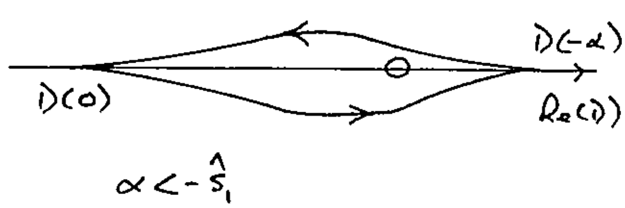
For $\alpha < \hat{s}_1$





(B) For $2\alpha_0 < a < 1$; $\mathrm{Sgn}(D(-\alpha)) = \mathrm{Sgn}(P(a,2,\alpha_0,-\alpha))$, $\mathrm{Sgn}(\mathrm{Im}(D)) = -\mathrm{Sgn}(P(a,2,\alpha_0,s))$. When $\mathrm{Im}(D) = 0$, $\mathrm{Sgn}(\mathrm{Re}(D)) = -\mathrm{Sgn}(\mathrm{Re}Q(a,2,\alpha_0;s))$. At $s = 0$, $D(0) < 0$. Since $P(a,2,s)$ has only one negative real root $\hat{s}_1 < 0$, $\alpha > -\hat{s}_1$ implies that the locus of $D(s)$ is a closed curve not encircling the origin, whereas $\alpha < -\hat{s}_1$ implies that the locus of D is a closed curve encircling the origin once in the negative or clockwise sense. The loci are typically as sketched below





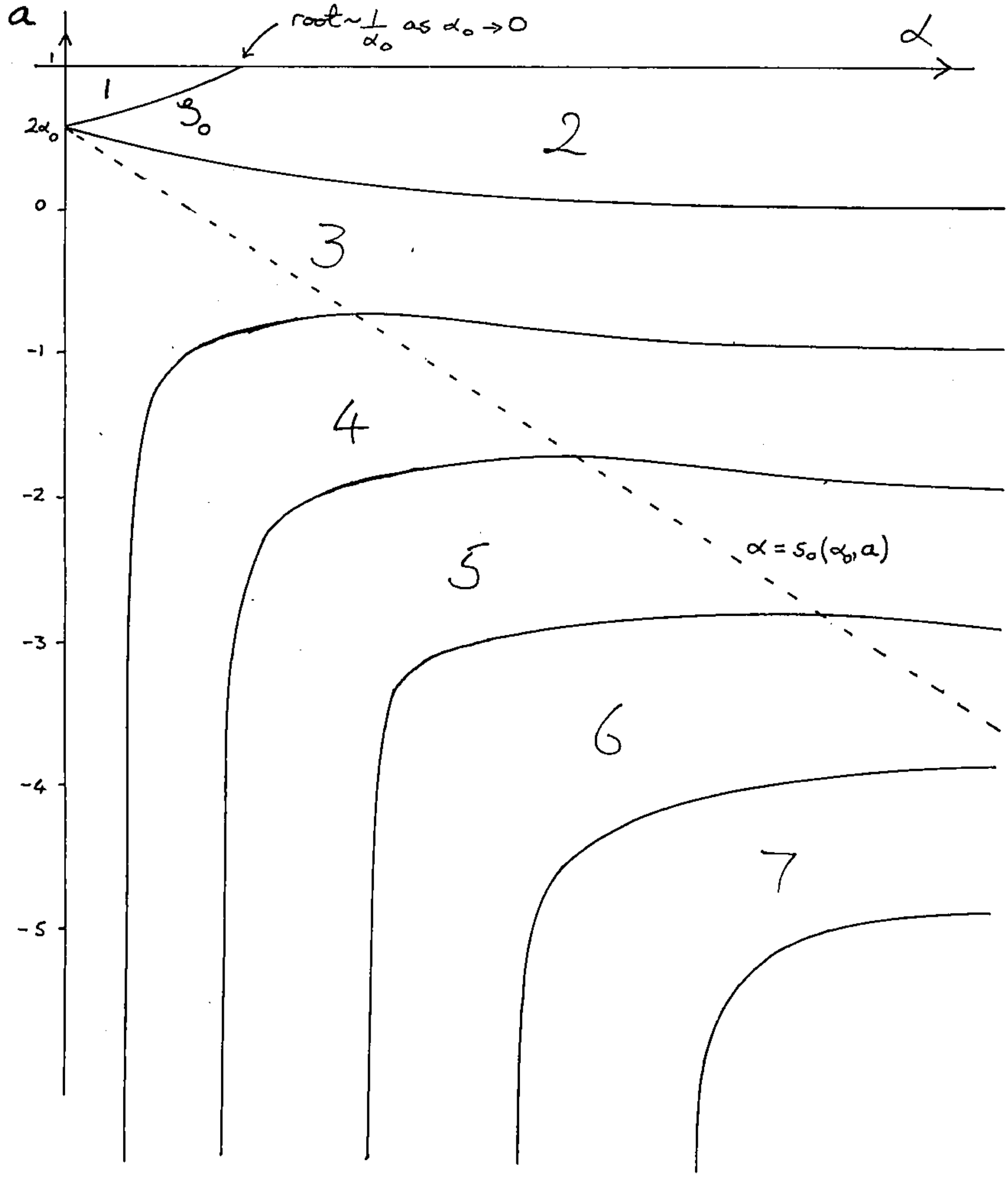


Figure 8. The number of roots of $D(a,2,\alpha_0,\alpha;s)$ for a point in the (a,α) plane for a typical value of $\alpha_0 \in (0,\frac{1}{2})$. The curves are

$$\{P(a,2,\alpha_0;\alpha) = 0 : \alpha > 0\} \cup \{P(a,2,\alpha_0;-\alpha) = 0 : \alpha > 0\}$$

Applying this type of algorithm the number of roots of the dispersion relation may be found and the results are summarised in figure (8) which shows a plane, α_0 = constant, cross-section of the three dimensional parameter space (a,α,α_0).

The Morphology of the roots of $D(a,2,\alpha_0,\alpha;s)$ in the s-plane close to the branch points.

Equation (71) shows that the change in phase of D as s encircles the point at infinity is -4π for all $a \leq 1$, $\alpha_0 \in (0, \frac{1}{2}]$ and $\alpha > 0$, this suggests that no roots enter or leave the s-plane at infinity throughout parameter space. Further, for $a < 1$ there are no roots on the branch cut $\ell = (-\alpha,0)$ since:-

For $a < 2\alpha_0$; $\mathrm{Im}(D) \neq 0$ on ℓ unless there exists $s \in \ell$ such that $P(a,2,\alpha_0;s+\alpha) = 0$ when

$$\mathrm{Re}(D) = s^2(s+\alpha)^2 P(a,2,\alpha_0;s).Q(a,2,\alpha_0;s+\alpha)$$

which from the curves $(P,Q) = 0$ of figure (6) does not vanish as $P(a,2,\alpha_0;s) < 0$ for $s \in \ell$ and $P(a,2,\alpha_0;s+\alpha)$ cannot vanish simultaneously with $Q(a,2,\alpha_0;s+\alpha)$.

For $1 < a < 2\alpha_0$; $\mathrm{Im}(D) \neq 0$ on ℓ unless there exists $s \in \ell$ such that $P(a,2,\alpha_0;s) = 0$ when

$$\mathrm{Re}\, D(a,2,\alpha_0,\alpha;s) = -s^2(s+\alpha)^2 P(a,2,\alpha_0;s+\alpha).\mathrm{Re}(u' - \alpha_0 u(a,2,s))$$

From figure (6) $P(a,2,\alpha_0;s+\alpha)$ may not vanish on ℓ. Now from the integral representation

$$U(a,2,s) = \frac{1}{\Gamma(a)} \int_0^{\infty e^{i\phi}} e^{-st} t^{a-1} (1+t)^{1-a}\, dt \tag{93}$$

which is valid when $\mathrm{Re}(a) > 0$, $-\pi < \phi < \pi$, $-\frac{\pi}{2} < \phi + \mathrm{ph}(s) < \frac{\pi}{2}$ and the contour of integration is a ray from the origin, equation (51) implies

$$\mathrm{Re}\, Q(a,2,\alpha_0;s) = \mathrm{Re} \frac{e^s}{\Gamma(2-a)} \int_0^{\infty e^{i\phi}} e^{st} t^{1-a} (1+t)^{a-1} (t + 1 - \alpha_0)\, dt \tag{94}$$

If ph(s) = π, on the upper side of the branch cut, ϕ may be set equal to zero and the right hand side is positive definite for all such s and then Re(D) < 0 along this path in the s-plane. By taking the complex conjugate of the dispersion relation there are no roots on ℓ.

Hence, when the number of roots of $D(a,2,\alpha_0,\alpha;s)$ changes it must do so through the appearance or disappearance of roots through the branch points $s = -\alpha$, $s = 0$. Equations (71) and (72) confirm that

$$s=-\alpha \text{ is a root} \iff P(a,2,\alpha_0;-\alpha)=0\text{, which for } \alpha>0 \Rightarrow a>2\alpha_0 \tag{95}$$

and

$$s=0 \text{ is a root} \iff P(a,2,\alpha_0;\alpha)=0\text{, which for } \alpha>0 \Rightarrow a<2\alpha_0 \tag{96}$$

so that on the 'critical curves', the curves of figure (8) which separate neighbouring parameter regions according to the number of roots of D, s = 0 is a root on all curves except the top left curve, $\mathcal{S}_0$, where $s = -\alpha$ is a root. For a point (a,α,α_0) moving within a region bounded by 'critical surfaces' which are the generalisation in (a,α,α_0) parameter space of the critical curves of figure (8) the analyticity properties of $D(a,2,\alpha_0,\alpha;s)$ imply that its roots move continuously in the s-plane away from the branch cut, branch points and the point at infinity.

Using the power series solutions (11) and (58) the power series representation of $D(a,2,\alpha_0,\alpha+\delta\alpha,\mu_0;s)$ about s = 0 is given in appendix A. Since D is analytic in the cut s-plane with respect to a, α_0 and α the qualitative way in which the roots of D may pass through the branch point s = 0, for example, as a point (a,α_0,α) traces a path in parameter space crossing a critical surface (96) is given by setting the series of appendix A to zero and then estimating the path of the root asymptotically as a critical surface is approached. Further, for a given point $(\tilde{a},\tilde{\alpha},\tilde{\alpha}_0)$ on a critical surface the path of the root through s = 0 in the s-plane must be a continuous function of the direction of the path through the point $(\tilde{a},\tilde{\alpha},\tilde{\alpha}_0)$ in parameter space, so for simplicity consideration may be confined mostly to paths

a = constant, α_0 = constant, α variable,

crossing the critical surfaces.

Consider first the case of (96) when also $(\tilde{a}, \tilde{\alpha}, \tilde{\alpha}_0)$ is such that

$$P'(\tilde{a}, 2, \tilde{\alpha}_0; \tilde{\alpha}) \neq 0 \tag{97}$$

then the relevant series is (A2) of appendix A which shows that $s = 0$ is a simple root of D. To first order in $\delta\alpha$ as $\delta\alpha \to 0$

$$s = -\delta\alpha \tag{98}$$

so that for small $\delta\alpha$ increasing through zero, corresponding to a point in parameter space moving from left to right across any critical curve of figure (8) apart from $\mathcal{S}_0$, the root moves through the branch point from right to left along the real s-axis. However, since there are no roots on the branch cut, for $\delta\alpha > 0$ the series must be examined at higher order. At the next order the third and sixth terms of (A2) balance implying

$$\begin{aligned} s &= -\delta\alpha + \epsilon_r \\ &= -\delta\alpha + \frac{(a-2\alpha_0)\Gamma(2-a)\, Q(\tilde{a}, 2, \tilde{\alpha}_0, \tilde{\alpha})}{2P'(\tilde{a}, 2, \tilde{\alpha}_0, \tilde{\alpha})}(\delta\alpha)^2 + o(\delta\alpha^2) \end{aligned} \tag{99}$$

which is still real and negative for $0 < \delta a \ll 1$. The lowest order imaginary part of s, ε_i say, is given by balancing the third with the tenth (lowest order logarithmic) term of (A2), i.e.;

$$\mathrm{Im}(s) = -\frac{\Gamma(2-a).(a-2\alpha_0)}{2\Gamma(1-a)}.\epsilon_r(\delta\alpha)^2.\,\rho\ell(s) \tag{100}$$

then for $\delta\alpha > 0$

$$\begin{aligned} s = &-\delta\alpha(1 + O(\delta\alpha)) \\ &- i\,\frac{(a-2\alpha_0)^2(\Gamma(2-a))^2 Q(\tilde{a}, 2, \tilde{\alpha}_0, \tilde{\alpha})}{4\Gamma(1-a)P'(\tilde{a}, 2, \tilde{\alpha}_0, \tilde{\alpha})}.\rho\ell(s)(\delta\alpha)^4.(1 + o(\delta\alpha)) \\ &\text{as } \delta\alpha \to 0 \end{aligned} \tag{101}$$

where the bracketed series are real, whereas for $\delta\alpha < 0$ the root is given to the same order by (98). Referring to figure (6) there are two cases to consider; along the upward sloping parts of the neutral curves for $\tilde{a} < 2\tilde{\alpha}_0$

$$\mathrm{Sgn}(Q(\tilde{a},2,\tilde{\alpha}_0,\tilde{a})) = -\,\mathrm{Sgn}(P(\tilde{a},2,\tilde{\alpha}_0,\tilde{a})) \tag{102}$$

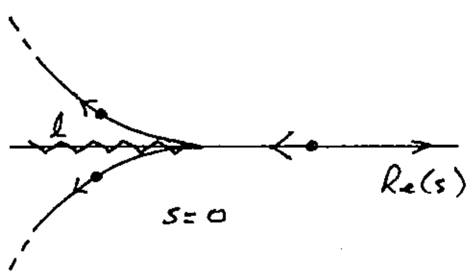
which implies $\mathrm{Sgn}(\mathrm{Im}(s)) = \mathrm{Sgn}(ph(s))$

in (101), in this case a root moves along the real s-axis dividing into two complex conjugate roots with negative real part $O(\delta\alpha)$ and imaginary part $O((\delta\alpha)^4)$ as it passes through the branch point; along the downward sloping parts of the neutral curves for $\tilde{a} < 2\tilde{\alpha}_0$

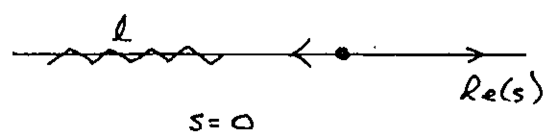
$$\mathrm{Sgn}(Q(\tilde{a},2,\tilde{\alpha}_0;\tilde{a})) = \mathrm{Sgn}(P(\tilde{a},2,\tilde{\alpha}_0;\tilde{a})) \tag{103}$$

implying $\mathrm{Sgn}(\mathrm{Im}(s)) = -\,\mathrm{Sgn}(ph(s))$

which is contradictory with the choice of branch $ph(s) \in [-\pi,\pi)$. As a point in parameter space moves from left to right across such a curve in figure (6) a root moves along the real s-axis, passes through the branch point singularity and disappears off the cut s-plane at $s = 0$.



(9a)



(9b)

Figure (9a) the arrows are in the direction of $\delta\alpha$ increasing for the situation of equation (102). Figure (9b), similarly for equation (103).

The way in which roots appear onto the cut s-plane through s = 0 differs according as

$$\tilde{\alpha} \lessgtr s_0(\tilde{\alpha}_0, \tilde{a}) = \frac{2\tilde{\alpha}_0 - \tilde{a}}{\tilde{\alpha}_0(1-\tilde{\alpha}_0)} \qquad (104)$$

The upper inequality corresponds to figure (9a) and from (53) describes a neutral mode with phase speed slightly retrograde with respect to the basic flow speed at the lower boundary, catching up with it and then changing to a pair of modes, one slightly linearly stable and one slightly linearly unstable, having equal phase speeds in advance of the boundary phase speed. For the lower inequality a new normal mode appears with neutral stability and slightly retrograde phase speed.

On the critical curves where

$$\tilde{\alpha} = s_0(\tilde{\alpha}_0, \tilde{a})$$

$$\text{in which case } P(\tilde{a}, 2, \tilde{\alpha}_0; \tilde{\alpha}) = P'(\tilde{a}, 2, \tilde{\alpha}_0; \tilde{\alpha}) = 0 \qquad (105)$$

equation (A2) shows that s = 0 is a twice repeated root of D. As $\delta\alpha$ increases through zero the path in parameter space is tangent to a critical surface so that the total number of zeros is unchanged. To first order as $\delta\alpha \to 0$ the sixth and eighth terms of (A2) balance to give

$$s = \frac{\delta\alpha\left(1 \pm i\sqrt{\left|\dfrac{\Gamma(2-a)(a-2\alpha_0)\,Q(a,2,\alpha_0;\alpha)}{P''(a,2,\alpha_0;\alpha)}\right|}\right)}{\left(1 - \dfrac{\Gamma(2-a)(a-2\alpha_0)\,Q(a,2,\alpha_0;\alpha)}{P''(a,2,\alpha_0;\alpha)}\right)} + o(\delta\alpha) \quad \text{as } \delta\alpha \to 0 \qquad (106)$$

where the right hand expression is evaluated at $(\tilde{a},\tilde{\alpha},\tilde{\alpha}_0)$. At such a critical point

$$\operatorname{Sgn} Q(a,2,\alpha_0;\alpha) = \operatorname{Sgn} P''(a,2,\alpha_0;\alpha) \qquad (107)$$

so a pair of conjugate roots move along curves which cross at $s = 0$.

To find the way in which a root appears at $s = 0$ when (105) is satisfied the expansion of $D(a,2,\alpha_0,\alpha;s)$ must be re-addressed. Consider α_0 and α fixed with a varying about the values $(\tilde{a},\tilde{\alpha},\tilde{\alpha}_0)$ of (105), using the power series (11) and (58) and the analyticity of $P(a,2,\alpha_0;s)$ and $Q(a,2,\alpha_0;s)$ with respect to a, the relevant equation is (A4) and

$$s^2 = \frac{2\frac{\partial P}{\partial a}\cdot(\delta a)}{\Gamma(2-a)(a-2\alpha_0)Q(a,2,\alpha_0;\alpha) - P''(a,2,\alpha_0;\alpha)} + o(\delta a) \quad \text{as } \delta a \to 0 \tag{108}$$

evaluated at $(\tilde{a},\tilde{\alpha},\tilde{\alpha}_0)$. At the critical points of (105) the derivative $\frac{\partial P}{\partial a}$ is of the same sign as Q and P'' so that as δa increases through zero a pair of complex conjugate roots approach the origin in the s-plane along the imaginary axis and move off along the real axis but again the logarithmic terms in the expansion imply that for $\delta a < 0$ a new root is produced. Equation (108) gives the positive real root and negative real part of a pair of complex conjugate roots whose imaginary parts

$$\pm \frac{\pi\Gamma(2-a)(a-2\alpha_0)}{2^{\frac{1}{2}}\,\Gamma(1-a)} \left| \frac{\frac{\partial P}{\partial a}\cdot(\delta a)}{\Gamma(2-a)(a-2\alpha_0)Q(a,2,\alpha_0;\alpha) - P''(a,2,\alpha_0;\alpha)} \right|^{\frac{3}{2}} \tag{109}$$

evaluated at $(\tilde{a},\tilde{\alpha}_0,\tilde{\alpha}_0)$ are found by balancing the first, third and fourth terms of (A4).

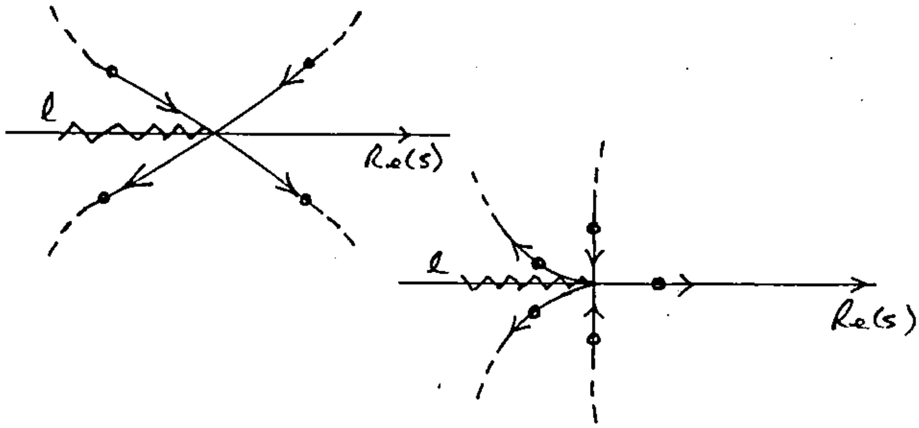


Figure (10). Arrows in direction of increasing $\delta\alpha$ for a path in parameter space tangential to the critical surface of (105), on the left. Arrows in direction of decreasing δa for a path intersecting the surface of (105), on the right.

If $2\alpha_0 < a < 1$ then from equation (95), $s = -\alpha$ is a simple root. The one root of D which occurs for values of α left of the critical curve $_0$ of figure (8) must be real, immediately implying that this region of parameter space corresponds to a neutrally stable flow which supports one neutral mode of phase speed greater than the upper boundary basic flow speed, moving along the negative real s axis from the left towards $s = -\alpha$, as the critical curve is approached. To estimate its evolution as the curve is crossed, again consider $(\tilde{a},\tilde{\alpha},\tilde{\alpha}_0)$ fixed and satisfying (95). Using the symmetry property

$$\mathcal{D}(a,2,\alpha_0,\alpha;-\alpha+s) = -\mathcal{D}(a,2,\alpha_0,-\alpha;s) \tag{110}$$

to evaluate the root in a neighbourhood of $s = -\alpha$ from the series (A1) which applies to a root in the neighbourhood of $s = 0$, some care is needed to temporarily reinterpret the branches of ph(s) and ph($s+\alpha$), since for the existing choice of (57) the function $Q(a,2,\alpha_0;s)$ has a branch cut along the negative real s-axis nullifying the procedure of Taylor expansion upon which appendix A relies. Suitably reinterpreting ph(s) and using (A2)

$$s = -\alpha + \delta\alpha + \frac{(a-2\alpha_0)\Gamma(2-a)Q(a,2,\alpha_0;-\alpha)}{2P'(a,2,\alpha_0;-\alpha)}(\delta\alpha)^2 + o(\delta\alpha^2) \text{ as } \delta\alpha \to 0 \tag{111}$$

evaluated at $(\tilde{a},\tilde{\alpha},\tilde{\alpha}_0)$. Note that the apparent ambiguity of the term $Q(\tilde{a},2,\tilde{\alpha}_0;-\tilde{\alpha})$ in this equation is removed since from (55)

$$\mathrm{Im}\left(Q(a,2,\alpha_0;s)\right) = \frac{\pm\pi\; P(a,2,\alpha_0;s)}{\Gamma(1-a)} \tag{112}$$

as ph(s) $\to \pm\pi$ which vanishes when $(a,\alpha_0,s) = (\tilde{a},\tilde{\alpha}_0,-\tilde{\alpha})$ and the real part of $Q(a,2,\alpha_0;s)$ is continuous across the branch cut there. From figure (6)

$$\mathrm{Sgn}\left(Q(\tilde{a},2,\tilde{\alpha}_0;-\tilde{\alpha})\right) = \mathrm{Sgn}\left(P'(a,2,\alpha_0;-\alpha)\right) = +1 \tag{113}$$

on the critical curve $\mathcal{S}_0$ of figure (8), so for $\delta\alpha > 0$ the imaginary parts of the pair of complex conjugate roots whose real part is given by (111) is found to be

$$\pm\,\pi\,\frac{(a-2\alpha_0)^2\left(\Gamma(2-a)\right)^2 Q(a,2,\alpha_0;-\alpha)}{4\Gamma(1-a)\,P'(a,2,\alpha_0;-\alpha)}\,.\,(\delta\alpha)^4 \tag{114}$$

and comparing (111) and (114) with (99) and (101) reveals that the behaviour of the roots in the s-plane is similar in both cases.

The Morphology of the roots of $D(a,2,\alpha_0,\alpha;s)$ and linear stability.

Having enumerated the roots of D in the s-plane throughout parameter space and estimated their behaviour asymptotically in the neighbourhood of the singular points $s = 0, -\alpha, \infty$ over small regions of the space in the neighbourhood of the critical surfaces, the stability of the flow is determined by considering the location of the remaining roots in these regions and the location of the roots at points of parameter space away from the critical surfaces. In the case $\mu_0 = 0$ the occurrence of the roots of D in complex conjugate pairs implies that the flow may be at most neutrally stable and then the number and nature of the real roots of D together with the information of the preceding section determines the type of linear stability of this inviscid flow.

The function $D(a,2,\alpha_0,\alpha;s)$ was evaluated numerically by first summing the appropriate real and imaginary parts of $u(a,2,s)$ and $M(a,2,s)$ together with their derivatives in s from the power series (11) and (56) in a neighbourhood of $s = 0$ for some chosen value of $a < 1$, then integrating the confluent hypergeometric equation along the real line using a simple forward step package for solving differential equations based on the Adams Bashforth method which outputs the values of $P(a,2,\alpha_0;s)$ and $Q(a,2,\alpha_0;s)$ for some $\alpha_0 \in (0,\frac{1}{2}]$ at each step. At each step, for sufficiently large $|s|$, these values are compared with suitable asymptotic estimates since, as may be seen from the asymptotic behaviour of the functions leading to equation (73), it is necessary for accurate evaluation of D for large $|s|$ to choose the component functions $u(a,2,s)$ of $Q(a,2,\alpha_0,s)$ so that Q behaves algebraically for large $|s|$ when $a \neq 0, -1, -2, \ldots$ i.e.; from (51) and (56), to compute

$$Q(a,2,\alpha_0;s) = \frac{-\pi \sin \pi a}{\Gamma(1-a).\Gamma(2-a)} \left(U'(a,2,s) - \alpha_0 U(a,2,s)\right)$$

and

$$P(a,2,\alpha_0;s) = M'(a,2,s) - \alpha_0 M(a,2,s)$$

for noninteger values of a and then for integer values of a to compute

$$Q(a,2,\alpha_0;s) = u'(a,2,s) - \alpha_0 u(a,2,s)$$

and

$$P(a,2,\alpha_0;s) = M'(a,2,s) - \alpha_0 M(a,2,s)$$

when Q behaves exponentially but now P behaves algebraically. Some of the resulting information was used in sketching the curves of figures (5) to (8). After adjusting the step length of integration and choosing suitable tolerances for the routine based on initial estimates of the real roots of D, the functions $P(a,2,\alpha_0;s)$ and $Q(a,2,\alpha_0;s)$ were stored and then the roots of D on the real axis estimated for some positive value of α. The programme semed to give overall accuracy in evaluating D of about six figures and was run in single precision only.

It would be a fallacy to claim that numerical searching of the (a,α,α_0) parameter space constitutes proof of the linear stability of the flow, the most that can be claimed in that respect is that it is certainly unstable immediately below and to the right of those parts of the neutral curves of figure (8) with upwards slope and is neutrally stable when D has only one root.

It is instructive to review the limiting problem formed by letting $\alpha \to \infty$ which from (53) is equivalent to removing the upper boundary at $z = \frac{h}{2}$ off to infinity in which case from (54) with $\mu_0 = 0$ using the connection formula (60) together with the asymptotic results (62) and (65)

$$\mathcal{D}(a,2,\alpha_0,\alpha;s) = \frac{s^2(s+\alpha)^2 e^{i\pi a}}{\Gamma(2-a)} \Big\{ \big(\mathcal{U}' - \alpha_0 \mathcal{U}(a,2,s)\big)\big((e^{s+\alpha}\mathcal{U})' - \alpha_0 e^{s+\alpha}\mathcal{U}(2-a,2,e^{-i\pi}(s+\alpha))\big)$$
$$- \big(\mathcal{U}' - \alpha_0 \mathcal{U}(a,2,s+\alpha)\big)\big((e^{s}\mathcal{U})' - \alpha_0 e^{s}\mathcal{U}(2-a,2,e^{-i\pi}s)\big)\Big\} \qquad (115)$$

hence as $\alpha \to \infty$ for finite s the eigenvalue problem reduces to

$$F(a,2,\alpha_0;s) \equiv s^2\big(\mathcal{U}' - \alpha_0 \mathcal{U}(a,2,s)\big) = 0 \qquad (116)$$

This is formally the same as the dispersion relation for Charney's ('47) model where there is no upper boundary confining the vertical extent of the flow and the upper boundary condition is replaced by applying the radiation condition that perturbations to the basic zonal flow vanish as $z \to \infty$. This problem has been discussed by Burger ('62)* and elaborated by Miles (64,a,b,c), the roots of (116) were enumerated by Burger and the limiting case of large α in figure (8) concurs with that result. Whilst for large but finite α in the vertically confined problem of $D(a,2,\alpha_0,\alpha;s)$ a simple root appears at $s = 0$ as a crosses the critical surfaces which asymptote to $a = -n$ $(n = 0,1,\ldots.)$ for all α_0 as $\alpha \to \infty$, in the limiting case of (116) $s = 0$ is a twice repeated root when $a = -n$ $(n = 0,1,\ldots.)$ and roots appear through the (single) branch point in a manner qualitatively similar to that of the present problem when $\tilde{\alpha} = s_0(\tilde{\alpha}_0,\tilde{a})$ as described by figure (10) and equations (108), (109). Note that in Burger ('62) the linear stability problem is solved as it is purportedly shown that for $-n < a < -n+1$ (116) has n simple roots on the positive s axis, this result is confirmed by numerical integration but I believe that it remains to be proved that

$$\mathcal{U}' - \alpha_0 \mathcal{U}(a,2,s_0(\alpha_0,a)) \neq 0 \quad \forall a < 1$$

* Kuo (1952, 1973)

either by connecting the power series solutions of (86) about the singular point $s = s_0(\alpha_0,a)$ to those at the singular point at $s = 0$ or by a method of bounding the real roots of the differential equation to the left of $s_0(\alpha_0,a)$.

However, the numerical results found suggest that the stability of the flow is consistent with those of Burger and Miles for Charney's model and with those of Garcia and Norscini ('70), as far as the computation here is of real roots alone, for the model recovered when $\alpha_0 = \frac{1}{2}$, i.e.; when the vertical scale height of the ambient density stratification is infinite. The results may be summarised by stating

(i) D has at most two complex conjugate roots

(ii) The remaining roots of D lie on the real interval $s \in [0,\infty)$ except when D has only one root, when $a > 2\alpha_0$ and the root is on $(-\infty,-\alpha]$.

(iii) The real roots of D are simple except when the pair of complex roots move onto the real axis to coalesce in a twice repeated root near $s = 0$.

Referring to figure (8), each time that part of a critical surface which is formed from the parts of the critical curves of figure (8) with upward slope is approached there will be another surface, which computation shows to be close to this part of the critical surface, on which the conjugate roots in the s-plane meet at a point s^* on $(0,\infty)$ which is closer to $s = 0$ than any other real root of D. Approaching the critical surface, one of these roots moves off to the right on $(0,\infty)$ remaining distinct from any real roots which are already there and the other root is that one which passes through the branch point $s = 0$ as in figure (9a).

The numerical results suggest that $s^*(\alpha_0,a)$ decreases monotonically towards $s = 0$ as the plane $\alpha = s_0(\alpha_0,a)$ in parameter space is approached from the plane $\alpha = 0$, recalling equations (105) to (109) and figure (10), $s^*(\alpha,a) = 0$ when $\alpha = s_0(\alpha_0,a)$. Crossing the critical surfaces at points for which $\alpha > s_0(\alpha_0,a)$ and $a < 2\alpha_0$ no more repeated roots were found and for such a region where $-n < a < -n+1$ $(n = 0,1,2,\ldots.)$, D possessed either n or $n+1$ simple roots on the positive real s axis, the change occurring as in figure (9b) when the surface is crossed. The remaining two roots are therefore complex implying that the flow is unstable at all points on the critical surfaces where $\alpha > s_0(\alpha_0,a)$, this result is supported by the more detailed numerical solutions of Garcia and Norscini for the single critical curve which asymptotes to $a = 0$

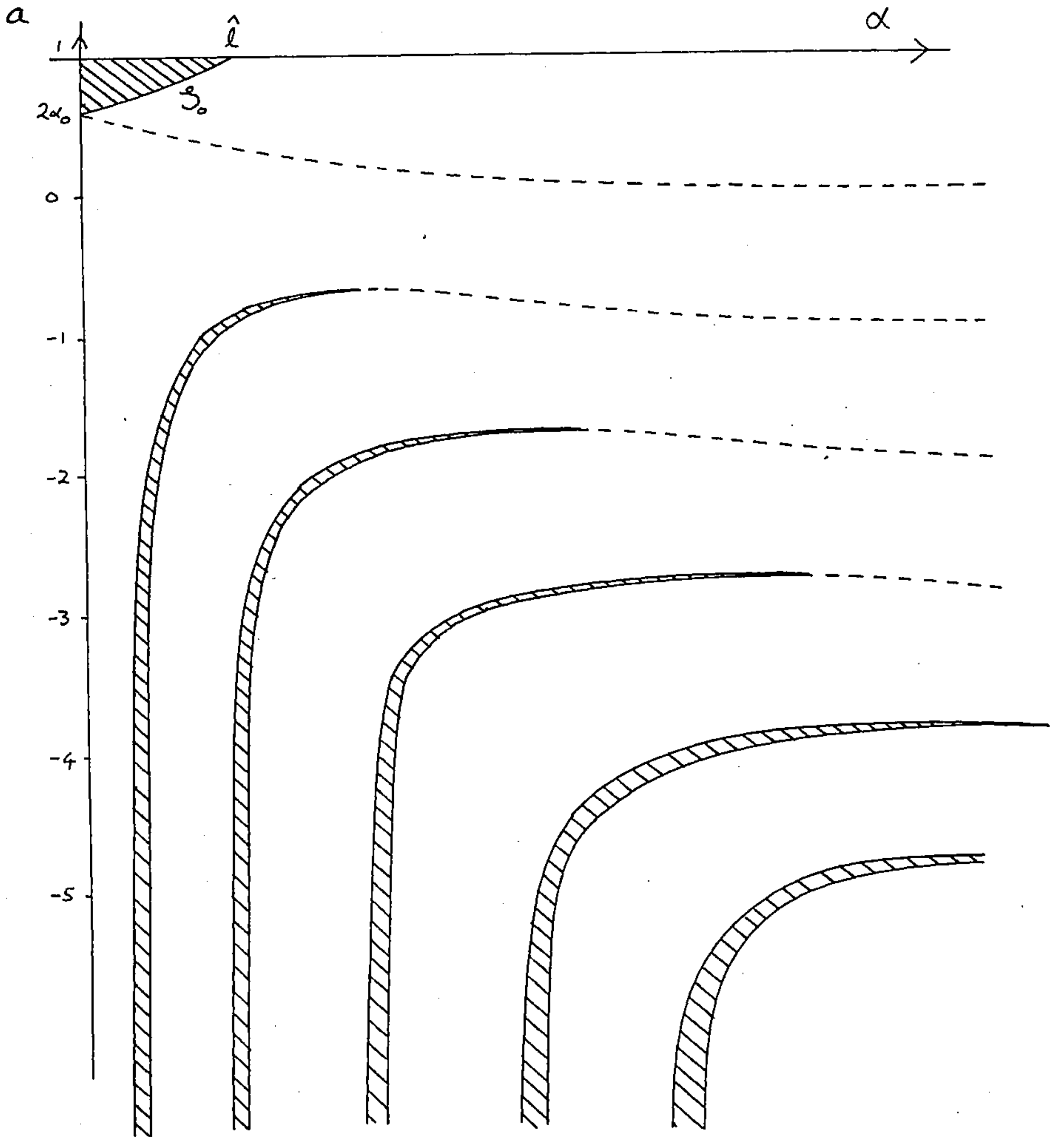


Figure 11. The flow is linearly neutrally stable in the closure of the hatched regions for $a < 1$ in the (a,α) plane for a typical value of $\alpha_0 \in (0,\frac{1}{2})$. For $\alpha < s_0(a,\alpha_0)$ the instability is strong on the left hand boundary of each hatched region and is weak on the right hand boundary.

from $a = 0^+$ when $\alpha_0 = \frac{1}{2}$ as $\alpha \to \infty$.

Figure (11) shows a plane cross-section of the parameter space α_0 = constant for a typical value of $\alpha_0 \in (0, \frac{1}{2})$ and on the basis of the above results the flow is neutrally stable in the open part of the hatched regions and marginally stable on the boundaries of these regions for $a < 1$, $\alpha > 0$. In the limiting case of Charney's model, which is on the plane $\alpha = \infty$, the flow is marginally stable on the lines $\alpha_0 \in [0, \frac{1}{2}]$, $a = -n$ $(n = 0,1,.,..)$ and unstable elsewhere. With the terminology of Miles ('64a) the onset of instability across the surface $\mathfrak{J}_0$ and across the critical surfaces for $\alpha < s_0(\alpha_0, a)$ is called a weak instability, the growth rate being $O((\delta\alpha)^4)$, whereas the onset of instability across the remaining surfaces where the flow is again marginally stable is called strong since the growth rate there will typically be $O((\delta\alpha)^{1/2})$ and the flow changes its stability through the more familiar mechanism for inviscid flows of the coalescence of real roots of the dispersion relation to form a complex conjugate pair at a point of the s-plane where the dispersion relation is analytic.

Returning to (47) and (49), the parameter regimes of figures (8) and (11) may be divided by the plane $a = 2\alpha_0$. Since

$$
\begin{aligned}
a \leq 1 \quad &\Longleftrightarrow \quad \frac{\beta S}{\Lambda} + \frac{1}{h_s} \geqslant 0 \\
a \geq 2\alpha_0 \quad &\Longleftrightarrow \quad \frac{\beta S}{\Lambda} \leqslant 0 \\
0 \leq \alpha_0 \leq \frac{1}{2} \quad &\Longleftrightarrow \quad \frac{1}{h_s} \in [0, \infty]
\end{aligned}
\tag{117}
$$

when $a < 2\alpha_0$ the regime is applicable to flow of a westerly wind with some value of $\frac{1}{h_s} \geq 0$, as both $\beta \geq 0$ and $S > 0$ by the requirements of the model as discussed in chapter one. When $a > 2\alpha_0$ the regime pertains to flow of an easterly wind $(\Lambda < 0)$ where

$$
0 > \frac{\beta S}{\Lambda} \geqslant -\frac{1}{h_s}
\tag{118}
$$

since there is no a priori restriction imposed on the sign of direction of the basic flow by the model. On the plane $a = 2\alpha_0$ for $a < 1$, β must vanish so that an f-plane model with finite vertical scale height of the ambient density stratification is recovered and figure (11) suggests that this type of flow is always linearly unstable when $\alpha > 0$. The line $a = 2\alpha_0 = 1$ corresponds to the exceptional case of Eady's model which relies upon the f-plane approximation with the further simplification that $\frac{1}{h_s} = 0$. Since $a = 1$ the branch cut in the s-plane may be removed and it is then necessary to reconsider the roots of the dispersion relation, since also $\alpha_0 = \frac{1}{2}$ equation (90) shows that the negative root of $P(1,2,\alpha_0;s)$ which behaves like $-\frac{1}{\alpha_0}$ as $\alpha_0 \to 0$ is now at $s = 0$ confirming that D has two roots in the s-plane for all $\alpha > 0$ in Eady's limit.

The limiting procedure of recovering the dispersion relation of Eady's model is regular in the limit $a \to 1^-$, $\alpha_0 \to \frac{1}{2}^-$ as is suggested by the asymptotic analysis of the behaviour of the roots in a neighbourhood of the branch points on recalling that the imaginary part of s there is multiplied by the factor $\frac{1}{\Gamma(1-a)}$ which has a simple zero at $a = 1$. The need to reconsider the stability of the flow when $a = 1$ for all $\alpha_0 \in [0, \frac{1}{2}]$ arises because the absence of the influence of the branch points and branch cut in the s-plane implies that the flow changes its overall stability as two roots of the dispersion relation change from a complex conjugate pair to a real pair through the appearance of a double root on the real axis which may prima facie lie on the interval $(-\alpha,0)$ of the branch cut. This difficulty may be highlighted by the following example:- consider a point $(\tilde{a},\tilde{\alpha}+\delta\alpha,\tilde{\alpha}_0)$ which is on the critical surface $\mathfrak{J}_0$ of figure (11) for some fixed value of $\tilde{\alpha}_0 \in (0, \frac{1}{2})$ when $\tilde{a}$ is just less than unity and $\delta\alpha = 0$, so that the point is close to $\hat{\ell} = (1,\hat{\alpha},\tilde{\alpha}_0)$ where $\hat{\alpha}$ is such that

$$P(1,2,\tilde{\alpha}_0;-\hat{\alpha}) = 0$$

As $\delta\alpha$ increases through zero a simple root splits into two and the flow changes its stability there but as $a \to 1^-$ the growth rate in this part of the unstable regime tends to zero uniformly in α_0. In the limiting case of the neighbouring point $\hat{\ell}$ in parameter space there is no branch point in the s-plane, the perturbed root passes through $s = -\alpha$ as $\delta\alpha$ increases through zero until at some $O(1)$ value of $\delta\alpha$ it coalesces with a real root, which was not present in the cut plane for $a < 1$, appearing along the real s-axis from the right and only then will the stability change.

When a = 1, using (90) and (102) in (54) with $\mu_0 = 0$, the dispersion relation becomes the quadratic in s,

$$s^2\alpha_0(1-\alpha_0)(1-e^{\alpha}) + s\left(\alpha_0(1-\alpha_0)\alpha + (1-2\alpha_0)\right)(1-e^{\alpha}) - \left((1+\alpha_0\alpha)(1-e^{\alpha}) + \alpha e^{\alpha}\right) = 0 \tag{119}$$

From the relations (53)

$$s = -\frac{\alpha}{2} - \alpha.\frac{c}{\Lambda h} \tag{120}$$

and written in terms of the complex variable $\frac{c}{\Lambda h}$ (119) becomes

$$\left(\frac{c}{\Lambda h}\right)^2\alpha_0(1-\alpha_0)\alpha^2\,\mathrm{th}\frac{\alpha}{4} - \left(\frac{c}{\Lambda h}\right)(1-2\alpha_0)\alpha\,\mathrm{th}\frac{\alpha}{4} - \left(\alpha_0(1-\alpha_0)\frac{\alpha^2}{4}\,\mathrm{th}\frac{\alpha}{4} + \mathrm{th}\frac{\alpha}{4} - \frac{\alpha}{4} - \frac{\alpha}{4}\,\mathrm{th}^2\frac{\alpha}{4}\right) = 0 \tag{121}$$

which reduces to

$$\left(\frac{c}{\Lambda h}\right)^2\cdot 4\left(\frac{\alpha}{4}\right)^2\mathrm{th}\frac{\alpha}{4} = \left(\frac{\alpha}{4} - \mathrm{th}\frac{\alpha}{4}\right)\left(\frac{\alpha}{4}\,\mathrm{th}\frac{\alpha}{4} - 1\right) \tag{122}$$

when $\alpha_0 = \frac{1}{2}$, which is the dispersion relation for Eady's ('49) model and in the notation of Drazin ('70)

$$\frac{\alpha}{4} = \frac{(a_\omega^2 S)^{\frac{1}{2}}h}{2} \quad \text{is written as} \quad qh = \frac{(a_\omega^2\beta)^{\frac{1}{2}}h}{2}$$

The stability criterion for general α_0 is then neutral stability or instability according as

$$\left(\alpha_0(1-\alpha_0)\alpha - \mathrm{th}\frac{\alpha}{4}\right)\left(\alpha_0(1-\alpha_0)\alpha\,\mathrm{th}\frac{\alpha}{4} - 1\right) \gtrless 0 \tag{123}$$

respectively, the left hand side for $\alpha_0 \in (0, \frac{1}{2}]$ has two non-negative real roots α^+ and α^{++} for which the stability is marginal

$$\frac{1}{\alpha^+} \operatorname{th} \frac{\alpha^+}{4} = \alpha_0(1-\alpha_0) \tag{124}$$

$$\alpha^{++} \operatorname{th} \frac{\alpha^{++}}{4} = \frac{1}{\alpha_0(1-\alpha_0)} \tag{125}$$

As $\alpha_0 \to \frac{1}{2}^-$; $\alpha^+ \to 0^+$ and $\alpha^{++} \to 4\bar{s}^+$ where $\bar{s}$ is the positive root of the transcendental equation

$$s \operatorname{th} s = 1 \tag{126}$$

and as $\alpha_0 \to 0^+$; α^+ and α^{++} both equal $\frac{1}{\alpha_0}$ to leading order so that in the limit $\alpha_0 = 0$ from (123) the flow is neutrally stable for all positive, finite values of α. Equations (124) and (125) imply

$$\operatorname{th} \frac{\alpha^+}{4} \cdot \operatorname{th} \frac{\alpha^{++}}{4} = \frac{\alpha^+}{\alpha^{++}} \tag{127}$$

and hence for all positive finite α^+ and α^{++}, i.e.; for all $\alpha_0 \in (0, \frac{1}{2}]$

$$\alpha^+ < \alpha^{++} \tag{128}$$

and the flow has a non-vanishing interval of instability $\alpha \in (\alpha^+, \alpha^{++})$.

Returning to the question raised in the example above, equation (90) gives the value $\hat{\alpha}$ of α at the point $\hat{\ell}$ of figure (11) as the root of the equation

$$(1-\alpha_0\alpha)(e^{\alpha}-1) = \alpha$$

or

$$\alpha_0 = \frac{4 \operatorname{th} \frac{\alpha}{4} - \alpha\left(\operatorname{th} \frac{\alpha}{4} - 1\right)^2}{4\alpha \operatorname{th} \frac{\alpha}{4}} \tag{129}$$

Inverting equation (124)

$$\alpha_0 = \frac{1}{2}\left(1 - \sqrt{\left(1 - \frac{4}{\alpha^+}\tanh\frac{\alpha^+}{4}\right)}\right) \tag{130}$$

and it may be verified that $\hat{\alpha} \leq \alpha^+$ for all $\alpha_0 \in (0, \frac{1}{2}]$ and $\hat{\alpha} = \alpha^+ = 0$ if and only if $\alpha_0 = \frac{1}{2}$. Further the roots of (119) in the s-plane are

$$s_{\pm} = -\frac{\alpha}{2} - \frac{1-2\alpha_0}{2\alpha_0(1-\alpha_0)} \pm \frac{\left(\left(\alpha_0(1-\alpha_0)\alpha - \tanh\frac{\alpha}{4}\right)\left(\alpha_0(1-\alpha_0)\alpha\tanh\frac{\alpha}{4} - 1\right)\right)^{\frac{1}{2}}}{2\alpha_0(1-\alpha_0)\left(\tanh\frac{\alpha}{4}\right)^{\frac{1}{2}}} \tag{131}$$

where the discriminant term on the right hand side for $\alpha_0 \in (0, \frac{1}{2}]$ is monotone decreasing for $\alpha \in (0,\alpha^+)$ and monotone increasing for $\alpha \in (\alpha^{++},\infty)$. Equations (130) and (131) imply that when $\alpha = \hat{\alpha}$, $s_- = -\alpha$ is a root of the disperson relation and the other root for all $\alpha_0 \in (0, \frac{1}{2}]$ is

$$s_+ = -\left(\frac{1-2\alpha_0}{\alpha_0(1-\alpha_0)}\right) = s_0(\alpha_0, 1) \leq 0 \tag{132}$$

from (82). The discussion leading to figure (6) shows that

$$-\alpha < s_0(\alpha_0, 1) < 0$$

so the root s_+ is on the interval $(-\alpha,0)$ associated with the branch cut ℓ when $a < 1$ and will disappear off the cut s-plane through the cut unless $a = 1$. Since the discriminant term of (131)

$$\sim \frac{1-2\alpha_0}{2\alpha_0(1-\alpha_0)} + o(\alpha) = -\frac{1}{2}s_0(\alpha_0,1) + o(\alpha) \quad \text{as } \alpha \to 0 \tag{133}$$

$$\sim \frac{\alpha}{2} - \frac{1}{2\alpha_0(1-\alpha_0)} + o(1) \quad \text{as } \alpha \to \infty \tag{134}$$

and has the monotone behaviour in α stated above, the root s_+ is a decreasing function of α on the interval $\alpha \in [0,\alpha^+]$ for given α_0 and

$$s_+ \in \left[-\frac{\alpha^+}{2} + \frac{s_0}{2}(\alpha_0,1),\ 0\right] \subset [-\alpha,\ 0] \tag{135}$$

while s_- is an increasing function of $\alpha \in [0,\alpha^+]$,

$$s_- \in \left[s_0(\alpha_0,1),\ -\frac{\alpha^+}{2} + \frac{s_0}{2}(\alpha_0,1)\right] \tag{136}$$

which crosses $s = -\alpha$ when $\alpha = \hat{\alpha}$. For $\alpha \in (\alpha^+,\alpha^{++})$ the flow is unstable, as the roots in the s-plane form a complex conjugate pair with real part

$$-\alpha < Re(s_\pm) = -\frac{\alpha}{2} + \frac{s_0}{2}(\alpha_0,1) < 0 \tag{137}$$

For $\alpha > \alpha^{++}$ the roots are real and distinct and from (134)

$$\begin{aligned} s_+ &= -\frac{1}{\alpha_0} + o(1) \quad \text{and} \quad s_+ \in (-\alpha, 0] \quad \text{as } \alpha \to \infty \\ s_- &= -\alpha + \frac{1}{1-\alpha_0} + o(1) \quad \text{as } \alpha \to \infty \end{aligned} \tag{138}$$

the root s_+ always has its real part in the interval $(-\alpha,0]$ and when the flow is neutrally stable represents a neutral mode with phase speed equal to that of the basic flow at some height. In contrast, the root s_- is on $(-\infty,-\alpha)$ for $\alpha < \hat{\alpha}$ and then has its real part in the interval $\ell = (-\alpha,0)$ for all $\alpha > \hat{\alpha}$.

<u>The effect of μ_0 on the roots of $D(a,2,\alpha_0,\alpha,\mu_0;s)$ in the neighbourhood of $s = 0$, $\mu_0 = 0$.</u>

In an initial attempt to include the parameter μ_0, associated with the viscously controlled Ekman layers on the top and bottom boundaries, consider the influence of μ_0 upon the neighbourhood of $s = 0$ in the s-plane which, as a branch point, has been shown to be a centre of marginal stability of the flow when $\mu_0 = 0$ and $a < 1$. Note that the limiting case of Eady's model for $\mu_0 \neq 0$ has been investigated thoroughly by Drazin ('72) and later by Moroz ('81), attention here is confined to $a < 1$, $\alpha_0 \in (0, \frac{1}{2}]$.

Setting $\delta\alpha = 0$ in equation (A1) of appendix A it follows immediately that for $\mu_0 \neq 0$, $a \leq 1$; $s = 0$ is not a root of (54), since for all $\alpha > 0$, $\alpha_0 \in [0, \frac{1}{2}]$

$$D(a,2,\alpha_0,\alpha,\mu_0;s) = -\left(\frac{1-i\mu_0}{\Gamma(2-a)}\right)\left(\alpha^2 P(a,2,\alpha_0;\alpha) - i\mu_0\alpha M(a,2,\alpha)\right) + o(s) \quad \text{as } s \to 0 \tag{139}$$

and $P(a,2,\alpha_0;\alpha)$ may not vanish simultaneously with $M(a,2,\alpha)$ from the property of confluent hypergeometric functions that they do not have repeated real roots.

To estimate the way in which the neutral normal mode associated with the root $s = 0$ is perturbed for small μ_0 equation (A1) with $\delta\alpha = 0$ may be used to give an asymptotic value as $\mu_0, s \to 0$. Suppose first of all that $s = 0$ is a simple root when $\mu_0 = 0$. Then (a, α, α_0) is such that

$$P(a,2,\alpha_0;\alpha) = 0\ ;\ P'(a,2,\alpha_0;\alpha) \neq 0$$

and the first and fourth terms of (A1) balance implying

$$s = \frac{i\mu_0 M(a,2,\alpha)}{\alpha P'(a,2,\alpha_0;\alpha)} + O(\mu_0^2) \quad \text{as } \mu_0 \to 0 \tag{140}$$

for the same point (a,α,α_0). If (a,α,α_0) changes as μ_0 increases from zero then (140) is modified by terms of first order in the change of these parameters and their contribution in (140) is real at this order, so that, in this instance, the generalisation to a path in parameter space which does not intersect the plane $\mu_0 = 0$ normally is straightforward inasmuch as the imaginary part of the perturbed root and thus the type of stability are unchanged by the direction of the path at a simple intersection.

From the discussion leading to figure (6) it may be shown that on the critical curves for which $P(a,2,\alpha_0;\alpha) = 0$ that

$$\operatorname{Sgn} M(a,2,\alpha) = \operatorname{Sgn} Q(a,2,\alpha_0;\alpha)$$

Hence at points on the critical surfaces where the flow is weakly marginally stable

$$\text{Sgn}\, M(a,2,\alpha) = -\,\text{Sgn}\, P'(a,2,\alpha_0;\alpha)$$

and the influence of the viscous terms for small μ_0 is to stabilise the mode with root $s = 0$. Conversely at the remaining points of the critical surfaces for $a < 2\alpha_0$ where $\alpha > s_0(\alpha_0,a)$, $P'(a,2,\alpha_0;\alpha)$ changes its sign implying that the influence of viscosity on the mode with root $s = 0$ is destabilising but as pointed out previously the numerical investigations show that the flow is already unstable in this region when $\mu_0 = 0$.

If $s = 0$ is a double root when $\mu_0 = 0$ so that (a,α,α_0) satisfies equation (105), again using (A1) with $\delta\alpha = 0$ for fixed (a,α,α_0) as μ_0 increases, the root is given at leading order by balancing the first $(O(\mu_0))$ term with part of the seventh and ninth $(O(s^2))$ terms and

$$\frac{i\mu_0 M(a,2,\alpha)}{\Gamma(2-a)} = \alpha s^2 \left\{ \frac{P''(a,2,\alpha_0;\alpha)}{2\Gamma(2-a)} - \frac{(a-2\alpha_0)}{2}.\, Q(a,2,\alpha_0;\alpha) \right\} \tag{141}$$

For all such points in parameter space, $a < 2\alpha_0$ and from figure (6) it may be verified that

$$\text{Sgn}\, M(a,2,\alpha) = \text{Sgn}\, Q(a,2,\alpha_0;\alpha) = \text{Sgn}\, P''(a,2,\alpha_0;\alpha)$$

there. Hence the double root for $\mu_0 = 0$ splits into a pair of roots

$$s = \left(\frac{2\mu_0\, |M(a,2,\alpha)|}{\alpha\, |P''(a,2,\alpha_0;\alpha) + (2\alpha_0-a)\Gamma(2-a) Q(a,2,\alpha_0;\alpha)|} \right)^{\frac{1}{2}} .\, e^{\frac{i\pi}{4}} \text{ or } e^{-\frac{3i\pi}{4}} \tag{142}$$

with phase difference π, which are not complex ocnjugates, the first choice corresponds to an unstable and the second to a stable mode. Again this last result holds for all values of $(a,\alpha,\alpha_0,\mu_0)$ in a neighbourhood of the critical values $(a,\alpha,\alpha_0,0)$ when $\mu_0 \neq 0$ so that the overall effect of asymptotically small viscous terms is not stabilising in this case.

CHAPTER 3

WEAKLY NONLINEAR EVOLUTION OF A LINEARLY UNSTABLE WAVE PACKET FOR EADY'S MODEL WHEN THE SQUARE ROOT OF THE EKMAN NUMBER DIVIDED BY THE ROSSBY NUMBER IS ORDER ONE WITH RESPECT TO THE SUPERCRITICALITY

The relative simplicity of the dynamics in the case of Eady's model leads to a removal of some of the analytical difficulties of the previous chapter. Equation (2.4) with $\beta = \frac{1}{h_s} = 0$ becomes

$$(\Lambda z - c)(D^2 - a_\omega^2 S)\psi(z) = 0 \tag{1}$$

governing the vertical structure of linearised quasi-geostrophic normal mode perturbations to a basic linear vertical shear flow ($u_0 = \Lambda z$) on an f-plane ($\beta = 0$) with constant density stratification in the unperturbed state ($\frac{1}{h_s} = 0$). It may be shown that the singular solutions of (1) form a continuous spectrum of modes with potential vorticity which is singular, like a Dirac delta function, at the steering level $\frac{c}{\Lambda}$ which is uniformly distributed on the real line. However as (1.63) shows the meridional gradient of the potential vorticity of the basic state vanishes in Eady's model and as a consequence the spectrum of singular solutions are found to be neutrally stable (Pedlosky (1964)) and will therefore be disregarded in the context of nonlinear evolution of disturbances in the slightly unstable state. The two regular solutions of (1) are the hyperbolic functions of (2.18)

$$\psi_1 = \sinh(a_\omega S^{\frac{1}{2}} z);\quad \psi_2 = \cosh(a_\omega S^{\frac{1}{2}} z) \tag{2),(3}$$

On substituting a linear combination in the normal mode representation for the perturbation stream function, c.f. (2.2)

$$\phi = A e^{ik(x-ct)} \sin lY \left(Q\cosh(a_\omega S^{\frac{1}{2}} z) + R\sinh(a_\omega S^{\frac{1}{2}} z)\right) \tag{4}$$

+ complex conjugate.

where an asterisk denotes complex conjugation, Q and R are complex constants and $\ell = m\pi$ $(m = 1,2,....)$ by virtue of (1.67), the remaining linearised boundary conditions (2.44) may be written

$$\begin{pmatrix} i\ell\lambda(s\cosh s - \sinh s) & -\frac{2i\ell sc}{\ell}\cosh s + \left(\frac{SE_v^{\frac{1}{2}}}{2R_o}\right)a_\omega^2 \sinh s \\ -\frac{2i\ell sc}{\ell}\sinh s + \left(\frac{SE_v^{\frac{1}{2}}}{2R_o}\right)a_\omega^2 \cosh s & i\ell\lambda(s\cosh s - \sinh s) \end{pmatrix}\begin{pmatrix} Q \\ R \end{pmatrix} = 0 \tag{5}$$

where $$s = \frac{a_\omega S^{\frac{1}{2}}\ell}{2} \tag{6}$$

which in the notation of chapter 2 is $\frac{\alpha}{4}$ when $\frac{1}{h_s} = 0$. Defining

$$\mu_o = \left(\frac{SE_v^{\frac{1}{2}}}{2\lambda R_o}\right)\frac{a_\omega^2}{\ell} \tag{7}$$

as before, (5) permits nontrivial solutions provided both

$$\frac{R}{Q} = \frac{\frac{2sc}{\lambda\ell}\tanh s + i\mu_o}{s - \tanh s} \tag{8}$$

and the dispersion relation

$$4\frac{s^2\tanh s}{\ell^2}\left(\frac{c}{\lambda}\right)^2 + \frac{2s}{\ell}\left(\frac{c}{\lambda}\right)i\mu_o(1+\tanh^2 s) - \left(\mu_o^2\tanh s + (s-\tanh s)(s\tanh s - 1)\right) = 0 \tag{9}$$

is satisfied. This compares with equation (2.122), the dispersion relation for the Eady model in the inviscid limit, when $\mu_o \to 0$ with the other parameters fixed. The loci of the roots of the dispersion relation are sketched in figure (1)

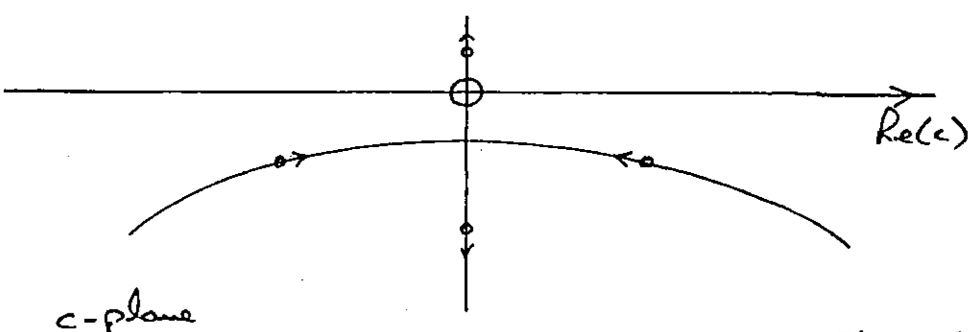


Figure (1)

Notice that since c = 0 when the flow is marginally stable (Im(c) = 0) the principle of the exchange of stabilities holds in the sense of Drazin and Reid (1981), that instability occurs as an eigenvalue passes through the origin. In the terminology of Gibbon and McGuinness (1981) the inviscid problem ($\mu_0 = 0$) is dispersively unstable, everywhere below the neutral surface in parameter space the system supports neutrally stable, dispersive travelling waves and above the surface each unstable mode is accompanied by a stable mode since the roots of the dispersion relation occur in complex conjugate pairs, whereas the viscous problem ($\mu_0 > 0$) is termed dissipatively unstable as everywhere below the neutral surface dispersive waves may exist but are stable or temporally decaying, the roots of the dispersion relation no longer occur in conjugate pairs and the flow is marginally stable when any root crosses the axis Im(c)= 0 in the c-plane.

Equation (9) determines the neutral surface by the condition

$$G(k,S,\mu) \equiv \mu_0^2\, \mathrm{th}\, s + (s - \mathrm{th}\, s)(s\, \mathrm{th}\, s - 1) = 0 \qquad (10)$$

where

$$\mu_0 = \frac{SE_v^{\frac{1}{2}}}{2\lambda R_0} \cdot \frac{k^2+\ell^2}{k} \quad (11) \qquad \text{and} \qquad S = \frac{4s^2}{\ell^2(k^2+\ell^2)} \quad (12)$$

is frequently referred to as the Burger or thermal Rossby number. If the spanwise wavenumber, ℓ, and the aspect ratio of the channel, h, are considered as given then (10) together with (11) and (12) determines a relation between the two parameters k and S and the grouping $\frac{SE_v^{\frac{1}{2}}}{2\Lambda R_0}$.
To enable comparison of this analysis of Eady's model with that of Drazin (1970 and 1972) consider the grouping

$$\frac{SE_v^{\frac{1}{2}}}{2\lambda R_0} \equiv \mu \quad \Rightarrow \quad \mu_0 = \mu\, \frac{k^2+\ell^2}{k} \qquad (13)$$

so that μ is the only parameter dependent on the viscous effects of the Ekman layers and let it vary independently of S in real three dimensional (k,S,μ) parameter space. In Drazin's work S does not appear in equation (11) because of the different choice of non-dimensionalisation for time. Solving (9) implies

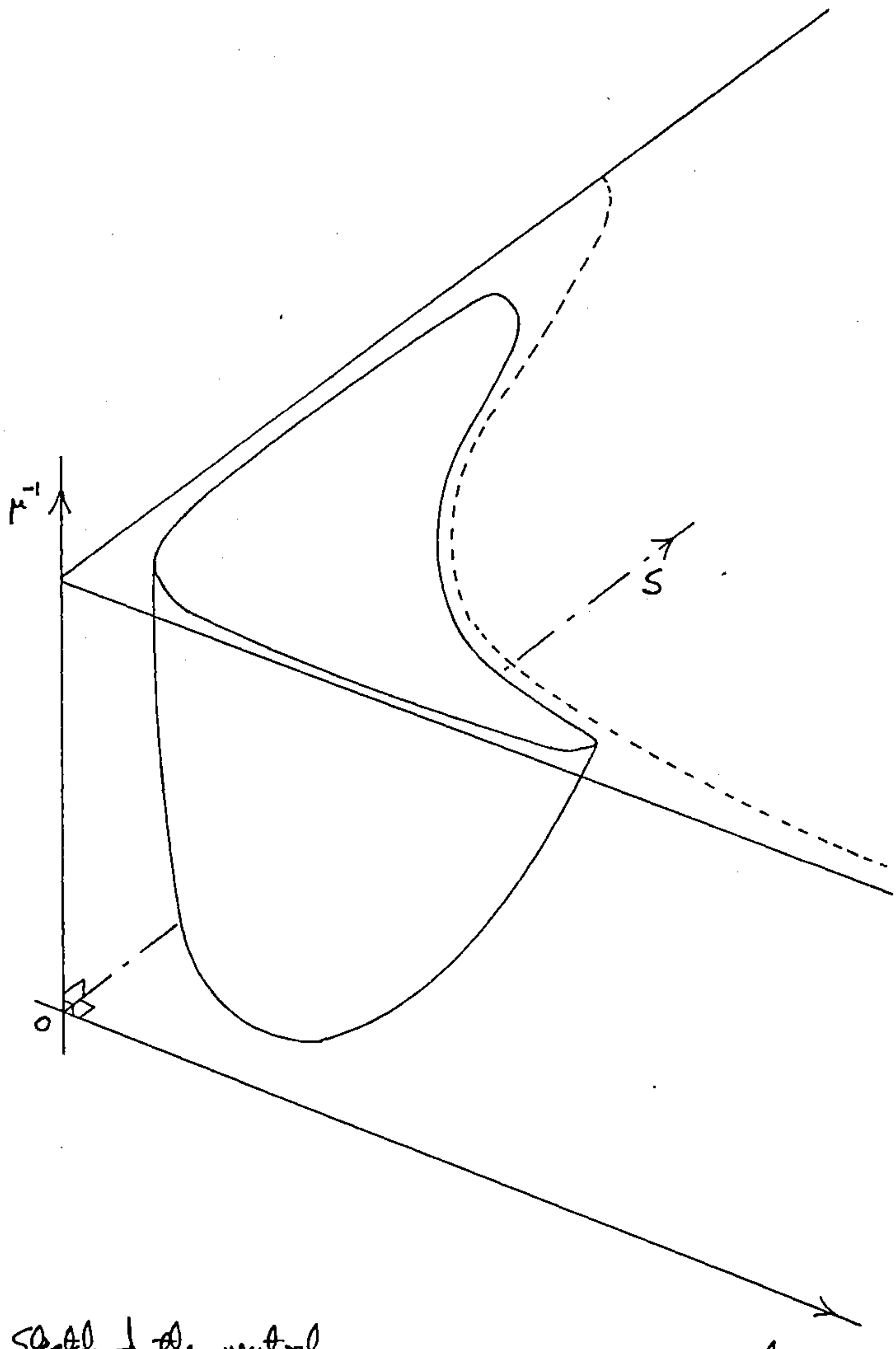


Figure (2). Sketch of the neutral surface of equation (3.10). The dotted curve is the projection of the curve of neutral stability for the inviscid problem on the plane μ = constant (See equation (4.2)). The contour is typical and shrinks to a point at (k_m, S_m, μ_m).

$$\frac{c}{\lambda} = \frac{h}{4s\mathcal{K}_s}\left(-i\mu_0(1+\mathcal{K}^2_s) \pm \sqrt{\left(-\mu_0^2(1+\mathcal{K}^2_s)^2 + 4\mathcal{K}_s\left(\mu_0^2\mathcal{K}_s + (s-\mathcal{K}_s)(s\mathcal{K}_s-1)\right)\right)}\right) \tag{14}$$

so that everywhere on the neutral surface

$$\frac{\partial(kc)}{\partial k} = \frac{-i\lambda kh}{2s\mu_0(1+\mathcal{K}^2_s)}\cdot\frac{\partial G}{\partial k}$$
$$= -\frac{i\lambda h\left(2\mu_0^2(k^2-\ell^2)\mathcal{K}_s + k^2_s\left(\mu_0^2+(s-\mathcal{K}_s)+\mathcal{K}^2_s(s\mathcal{K}_s-1)\right)\right)}{2s\mu_0(k^2+\ell^2)(1+\mathcal{K}^2_s)} \tag{15}$$

which confirms that everywhere on the neutral surface the group velocity of the marginally stable mode vanishes, provided $\mu_0 > 0$.

From the computationally assisted sketch of the neutral surface in figure (2) it may be seen that the surface has a single minimum, (k_m, S_m, μ_m), in (k, S, μ) space at which k_m, S_m, μ_m are non-zero and

$$\left.\frac{\partial G}{\partial k}\right|_m = \left.\frac{\partial G}{\partial S}\right|_m = 0 \tag{16}$$

where the suffix denotes evaluation at the minimum or critical point. Explicitly

$$G|_m = 0 \Rightarrow \quad \mu_{0_m}^2\mathcal{K}_{s_m} + (s_m - \mathcal{K}_{s_m})(s_m\mathcal{K}_{s_m} - 1) = 0 \tag{17}$$

$$\left.\frac{\partial G}{\partial k}\right|_m = 0 \Rightarrow$$
$$2(k_m^2-\ell^2)\mu_{0_m}^2\mathcal{K}_{s_m} + k_m^2 s_m\left(\mu_{0_m}^2 + s_m(s_m-\mathcal{K}_{s_m}) + \mathcal{K}^2_{s_m}(s_m\mathcal{K}_{s_m}-1)\right) = 0 \tag{18}$$

$$\left.\frac{\partial G}{\partial S}\right|_m = 0 \Rightarrow \quad \mu_{0_m}^2 + \mathcal{K}^2_{s_m}(s_m\mathcal{K}_{s_m}-1) + s_m(s_m - \mathcal{K}_{s_m}) = 0 \tag{19}$$

Equations (18) and (19) imply $k = \ell$, then on solving (17) and (19) for s equation (12) gives S for given ℓ and h. Either (17) or (19) then gives the root for μ_0 and thus μ from (13).

Consider now an attempt to follow the evolution of a packet of waves of the form (4) which are slightly unstable in the context of linear theory by examination of an asymptotically small perturbation about the critical point (k_m, S_m, μ_m) at the nose of the neutral surface, i.e.; consider the point (k_m, S_m, μ) in the unstable regime of parameter space defined by

$$0 < \frac{1}{\mu} - \frac{1}{\mu_m} = \epsilon \tag{20}$$

where the supercriticality is a (small) asymptotic parameter. In order to find an asymptotic solution of the governing equation, which is now (1.65) with $\nu = \kappa = 0$, ρ_s = constant and $\beta = 0$,

$$(\partial_t + \lambda z \partial_x)\left(\partial_z^2 + S(\partial_x^2 + \partial_y^2)\right)\phi + \frac{\partial}{\partial(x,y)}\left(\phi, (\partial_z^2 + S(\partial_x^2 + \partial_y^2))\phi\right) = 0 \tag{21}$$

and boundary conditions

$$\phi_x = \bar{\phi}_{yt} = 0 \quad \text{on } y = 0, 1 \qquad (22),(23)$$

$$(\partial_t + \lambda z \partial_x)\partial_z\phi - \lambda\partial_x\phi + \frac{\partial}{\partial(x,y)}(\phi, \partial_z\phi) = \frac{2z}{h}\mu\lambda(\partial_x^2 + \partial_y^2)\phi \quad \text{on } z = \pm\frac{h}{2} \tag{24}$$

which include the Jacobian nonlinearities for the disturbance stream function ϕ, notice first that the linear growth rate of a mode is

$$\mathrm{Re}(-ikc) = \frac{\lambda h (k_m^2 + l^2)\,\mathrm{th}\,s_m}{s_m(1 + \mathrm{th}^2 s_m)}\mu_m^2\,\epsilon \tag{25}$$

to first order, and that the locally parabolic cross-section of the neutral surface with the plane $S = S_m$ implies that the flow is unstable for a band of x-wavenumbers centred on $k = k_m$ of width $O(\epsilon^{\frac{1}{2}})$. Hence the amplitude of a packet of unstable modes varies on a slow spatial scale

$$X = \epsilon^{\frac{1}{2}} x \tag{26}$$

and on a slow temporal scale

$$\tau = \epsilon t \tag{27}$$

Since the finite spanwise extent of the flow quantises the spanwise wavenumber, ℓ, no slow scale $\varepsilon^{\frac{1}{2}}y$ will be introduced. The quadratic nonlinearity of the governing equations and boundary conditions suggests that the weakly nonlinear evolution of the packet amplitude

$$A(X,\tau) \tag{28}$$

of equation (4) is determined by a cubic nonlinearity in its governing equation so that the wave amplitude is $O(\varepsilon^{\frac{1}{2}})$. The perturbation stream function is therefore assumed to have an asymptotic expansion of the form

$$\phi(x,y,z;\epsilon,X,\tau) = \epsilon^{\frac{1}{2}}\phi_0 + \epsilon\phi_1 + \epsilon^{\frac{3}{2}}\phi_2 + \cdots \tag{29}$$

and the scalings in ε are justified a posteriori when the amplitude equation for A has been found. Re-casting equations (21) and (24) in a form suitable for finding the solution (29) via an iterated sequence of linear problems the multiple scaling transformations

$$\partial_t \to \epsilon\,\partial_\tau$$

$$\partial_x \to \partial_x + \epsilon^{\frac{1}{2}}\partial_X$$

yield, using (20),

$$\begin{aligned}\Lambda z\,\partial_x K_m \phi = {} & -\frac{\partial(\phi, K_m\phi)}{\partial(x,y)} - \epsilon^{\frac{1}{2}}\Big\{\Lambda z\,\partial_X K_m\phi + 2S_m\Lambda z\,\partial^3_{xxX}\phi \\ & \quad + 2S_m\frac{\partial(\phi,\partial^2_{xX}\phi)}{\partial(x,y)} + \frac{\partial(\phi,K_m\phi)}{\partial(X,y)}\Big\} \\ & - \epsilon\Big\{\partial_\tau K_m\phi + 3S_m\Lambda z\,\partial^3_{xXX}\phi + S_m\frac{\partial(\phi,\partial^2_X\phi)}{\partial(x,y)} \\ & \quad + 2S_m\frac{\partial(\phi,\partial^2_{xX}\phi)}{\partial(X,y)}\Big\} + O(\epsilon^{\frac{3}{2}}\phi)\end{aligned} \tag{30}$$

where

$$K_m \equiv \partial_z^2 + S_m(\partial_x^2 + \partial_y^2) \tag{31}$$

and

$$L\phi \equiv \partial_x\phi - z\partial^2_{xz}\phi + \frac{2z}{h}\mu_m(\partial_x^2+\partial_y^2)\phi$$

$$= \frac{1}{\Lambda}\frac{\partial}{\partial(x,y)}(\phi,\partial_z\phi) + \frac{2z}{h}\mu_m\left(1-(1+\epsilon\mu_m)^{-1}\right)(\partial_x^2+\partial_y^2)\phi \tag{32}$$

$$-\epsilon^{\frac{1}{2}}\left\{\partial_x\phi - z\partial^2_{xz}\phi - \frac{1}{\Lambda}\frac{\partial}{\partial(x,y)}(\phi,\partial_z\phi) + \frac{4z}{h}\mu_m(1+\epsilon\mu_m)^{-1}\partial^2_{xx}\phi\right\}$$

$$+\epsilon\left\{\frac{1}{\Lambda}\partial^2_{\tau z}\phi - \frac{2z}{h}\mu_m(1+\epsilon\mu_m)^{-1}\partial_x^2\phi\right\}$$

$$+ O(\epsilon^{\frac{3}{2}}\phi) \qquad \text{on } z = \pm\frac{h}{2}$$

Using the expansion (29) the resulting linear problems at successive orders are:

$O(\epsilon^{\frac{1}{2}})$

$$\Lambda z\partial_x K_m\phi_0 = 0$$

$\phi_{0x} = \overline{\phi}_{0y\tau} = 0$ on $Y = 0, 1$, where a bar denotes averaging on $z = \pm\frac{h}{2}$.

$$\partial_x\phi_0 - z\partial^2_{xz}\phi_0 = -\frac{2z}{h}\mu_m(\partial_x^2+\partial_y^2)\phi_0$$

which is the linear problem with the solution (4) and (8), i.e.;

$$\phi_0 = A(x,\tau)e^{ik_m x}\sin lY\left(\cosh\left(\frac{2s_m z}{h}\right) + \frac{i\mu_{0m}}{s_m - t h s_m}\sinh\left(\frac{2s_m z}{h}\right)\right) \tag{33}$$

+ complex conjugate.

$O(\epsilon)$

$$\Lambda z\partial_x K_m\phi_1 = -\frac{\partial}{\partial(x,y)}(\phi_0, K_m\phi_0) - \Lambda z\partial_x K_m\phi_0 - 2S_m\Lambda z\partial^3_{xxx}\phi_0$$

Since the solution at the previous order satisfies $K_m\phi_0 = 0$ the right hand side reduces to

$$K_m\phi_{1x} = 2S_m k_m^2 A_x e^{ik_m x}\sin lY\left(\cosh\left(\frac{2s_m z}{h}\right) + \frac{i\mu_{0m}}{s_m - t h s_m}\sinh\left(\frac{2s_m z}{h}\right)\right)$$

+ complex conjugate. (34)

using the solution for ϕ_0. The top and bottom boundary conditions are

$$L\phi_1 = -(\partial_X\phi_0 - z\partial^2_{Xz}\phi_0) - \frac{4z}{h}\mu_m\partial^2_{xX}\phi_0$$

$$+\frac{1}{\Lambda}\frac{\partial(\phi_0, \phi_{0z})}{\partial(x,y)} \quad \text{on } z = \pm\frac{h}{2}.$$

which reduces to

$$L\phi_1 = A_X e^{ik_m x}\sin lY\left\{(s_m sh\, s_m - ch\, s_m) \pm i\mu_{0m}\, ch\, s_m\right\} + c.c. \tag{35}$$

$$+ 2\mu_m k_m A_X e^{ik_m x}\sin lY\left\{\frac{\mu_{0m}\, sh\, s_m}{s_m - th\, s_m} \mp i\, ch\, s_m\right\} + c.c.$$

$$+ \frac{2 s_m k l |A|^2 \mu_{0m}}{\Lambda h(s_m - th\, s_m)} \quad \text{on } z = \pm\frac{h}{2}.$$

with the sidewall boundary conditions

$$\phi_{1x} = -\phi_{0X} = 0; \ \bar{\phi}_{1y\tau} = 0 \quad \text{on } Y = 0,1 \tag{36),(37}$$

Hence the solution for ϕ_1 contains

(i) A term independent of the downstream coordinate, x, forced by the last term of (35) arising from the lowest order nonlinear effect. This part of the solution, $\phi_1^\alpha(X,y,z,\tau)$ say, satisfies (34) and (36) automatically, to ensure that it also satisfies (37) the y derivatives of ϕ_1^α must vanish identically on the sidewalls Y = 0,1. The equation determining this part of the solution will be found at the next order, $O(\varepsilon^{3/2})$.

(ii) A 'resonant term' forced by the right hand side of (34). Prima facie there are three groups of two terms with this resonance expressed by the x, y or z coordinates in each respective group. The first group is dismissed on physical grounds since it corresponds to a train of divergently oscillating waves in the upstream-downstream direction, the second group is disregarded since it does not satisfy the sidewall boundary condition (37) for the discrete wavenumbers $l = m\pi$ $(m = 1,\ldots.)$.

The solution is sought by putting

$$\phi_1 = \tilde{\phi}_1(y,z,X,\tau)\, e^{ik_m x} + \text{complex conjugate}$$
$$+ \phi_1^{\alpha}(y,z,X,\tau) \tag{38}$$

where

$$\tilde{\phi}_1(y,z,X,\tau) = A_x \sin lY \left((A_0 z + B_1)\, \mathrm{sh}\left(\frac{2 s_m z}{h}\right) + (B_0 z + A_1)\, \mathrm{ch}\left(\frac{2 s_m z}{h}\right) \right) \tag{39}$$

and A_i, B_i are complex constants with A_1 and B_1 representing the homogeneous part of the solution. On substitution of (38) and (39) into (34) the 'resonant part' of the solution gives

$$A_0 = \frac{-i k_m S_m h}{2 s_m} \quad ; \quad B_0 = \frac{i \mu_{0m}}{s_m - \mathrm{th}\, s_m} A_0 \tag{40}$$

Equation (35) then determines the homogeneous part of the solution for ϕ_1 via

$$L \tilde{\phi}_1 = A_x \sin lY \left\{ (s_m \mathrm{sh}\, s_m - \mathrm{ch}\, s_m) \pm i \mu_{0m} \mathrm{ch}\, s_m \right\}$$
$$+ 2\mu_m k_m A_x \sin lY \left\{ \frac{\mu_{0m} \mathrm{sh}\, s_m}{s_m - \mathrm{th}\, s_m} \mp i\, \mathrm{ch}\, s_m \right\} \quad \text{on } z = \pm \frac{h}{2} \tag{41}$$

On substitution of (39) using the explicit relations (40) and the equation (17) which defines the neutral surface, the boundary condition implies

$$(k_m^2 + l^2) k_m \mu_{0m} \mathrm{th}\, s_m \left(i \mu_{0m} A_1 - (s_m - \mathrm{th}\, s_m) B_1 \right)$$
$$= (k_m^2 - l^2) \mu_{0m}^2 \mathrm{th}\, s_m + k_m^2 s_m \left(\mu_{0m}^2 + s_m (s_m - \mathrm{th}\, s_m) \right) \tag{42}$$

and

$$(k_m^2 + l^2) k_m \mu_{0m} \mathrm{th}\, s_m \left(i \mu_{0m} A_1 - (s_m - \mathrm{th}\, s_m) B_1 \right)$$
$$= -(k_m^2 - l^2) \mu_{0m}^2 \mathrm{th}\, s_m - k_m^2 s_m \mathrm{th}^2 s_m (s_m \mathrm{th}\, s_m - 1) \tag{43}$$

The two equations are equivalent or one is redundant as may be seen by comparing their left hand sides and their right hand sides with (18) which holds at the critical point (k_m, S_m, μ_m). Since neither right hand side of (42) and (43) vanishes we may choose $A_1 = 0$ in an attempt to minimise the subsequent algebra. Then after some simplification using (19)

$$B_1 = \frac{k_m s_m \,\mathrm{th}\, s_m (s_m \,\mathrm{th}\, s_m - 1)}{(k_m^2 + l^2)\mu_{0m}(s_m - \mathrm{th}\, s_m)} \tag{44}$$

whence

$$\phi_1 - \phi_1^{\alpha} = -\frac{i k_m S_m l}{2 s_m} A_x e^{i k_m x} \sin lY \left(z \,\mathrm{sh}\left(\frac{2 s_m z}{h}\right) + \frac{i \mu_{0m}}{s_m - \mathrm{th}\, s_m} z \,\mathrm{ch}\left(\frac{2 s_m z}{h}\right) \right) + B_1 A_x e^{i k_m x} \sin lY \,\mathrm{sh}\left(\frac{2 s_m z}{h}\right) + \text{complex conjugate} \tag{45}$$

$O(\varepsilon^{3/2})$

$$\begin{aligned} \Lambda z \partial_x K_m \phi_2 = & -\frac{\partial}{\partial(x,y)}(\phi_0, K_m \phi_1) - \frac{\partial}{\partial(x,y)}(\phi_1, K_m \phi_0) - \Lambda z \partial_x K_m \phi_1 \\ & - 2 S_m \Lambda z \partial^3_{xxx} \phi_1 - 2 S_m \frac{\partial}{\partial(x,y)}(\phi_0, \partial^2_{xx} \phi_0) \\ & - \frac{\partial}{\partial(x,y)}(\phi_0, K_m \phi_0) - \partial_t K_m \phi_0 - 3 S_m \Lambda z \partial^3_{xxx} \phi_0 \end{aligned} \tag{46}$$

From the equations for the scheme at $O(\varepsilon^{\frac{1}{2}})$ and $O(\varepsilon)$

$$K_m \phi_{0x} = 0$$

and $$K_m \phi_{1x} = -2 S_m \phi_{0xxx}$$

Now the solution at $O(\varepsilon^{\frac{1}{2}})$ satisfies

$$K_m \phi_0 = 0$$

and if the constant of integration at the next order also vanishes, then

$$K_m \phi_1 = -2 S_m \phi_{0xx} \tag{47}$$

so the right hand side at this order simplifies to

$$K_m \phi_{2x} = -2S_m \phi_{1xxx} - S_m \phi_{0xxx} \tag{48}$$

The condition (36) is sufficient for the terms

$$-\Lambda z \left(K_m \phi_1 + 2S_m \phi_{0xx}\right)_x - \phi_{0x}\left(K_m \phi_1 + 2S_m \phi_{0xx}\right)_y \tag{49}$$

on the right hand side of (46) to vanish everywhere but the solution $\tilde{\phi}_1 e^{ik_m x}$ has been constructed so that

$$K_m\left(\tilde{\phi}_1 e^{ik_m x} + c.c.\right) + 2S_m \partial^2_{xx} \phi_0 = 0$$

whence the terms of line (49) become

$$-\Lambda z \left(K_m \phi_1^\alpha\right)_x - \phi_{0x}\left(K_m \phi_1^\alpha\right)_y$$

Since ϕ_1^α is independent of x, the second of these terms leads to a contribution to the solution for ϕ_2 which is proportional to $x\phi_{0x}$ and becomes unbounded far upstream and downstream as $|x| \to \infty$ unless

$$K_m \phi^\alpha_{1y} = 0 \tag{50}$$

So the problem for ϕ_1^α is

$$\left(\partial_z^2 + S_m \partial_y^2\right)\phi_1^\alpha = 0$$

with

$$\phi^\alpha_{1y} = 0 \quad \text{on } Y = 0, 1 \tag{51}$$

and from (35) using (6) and (13)

$$\phi^\alpha_{1yy} = \pm \frac{8 s_m^3 l |A|^2 \sin 2lY}{S_m \Lambda k^3 (s_m - \partial_k s_m)} \qquad \text{on } z = \pm\frac{h}{2},$$

which in consideration of (51) may be integrated to give

$$\phi^\alpha_{1y} = \mp \frac{4 s_m^3 |A|^2 (\cos 2lY - 1)}{S_m \Lambda k^3 (s_m - \partial_k s_m)} \qquad \text{on } z = \pm\frac{h}{2}.$$

and then the Fourier sine series

$$\cos 2lY - 1 = -\sum_{j=0}^{\infty} a_{2j+1} \sin(2j+1)\pi Y \tag{52}$$

$$Y \in [0, 1]$$

where

$$a_{2j+1} = \frac{-2\ell^2}{(j+\frac{1}{2})\pi\left((j+\frac{1}{2})^2\pi^2-\ell^2\right)} \qquad j = 0, 1, \ldots$$

leads to the solution

$$\phi_{1y}^{\alpha} = \frac{4s_m^3|A|^2}{S_m\Lambda\ell^3(s_m-\partial_t s_m)} \sum_0^{\infty} a_{2j+1}\sin(2j+1)\pi Y \frac{\mathrm{sh}(2j+1)\pi S_m^{\frac{1}{2}}z}{\mathrm{sh}(j+\frac{1}{2})\pi S_m^{\frac{1}{2}}\ell} \quad (53)$$

With the solution at $O(\varepsilon)$ now adequately determined the remaining information for the problem at $O(\varepsilon^{3/2})$ is contained in the boundary conditions

$$\phi_{2x} = -\phi_{1X} \text{ on } Y = 0,1 \; ; \quad \bar{\phi}_{zy\tau} = 0 \text{ on } Y = 0,1 \quad (54)$$

and

$$\begin{aligned} L\phi_2 = &-(\phi_{1X} - z\,\phi_{1Xz}) + \frac{1}{\Lambda}\phi_{0\tau z} + \frac{1}{\Lambda}\frac{\partial}{\partial(x,y)}(\phi_0, \phi_{1z}) \\ &+ \frac{1}{\Lambda}\frac{\partial}{\partial(x,y)}(\phi_1, \phi_{0z}) + \frac{1}{\Lambda}\frac{\partial}{\partial(X,y)}(\phi_0, \phi_{0z}) \\ &\qquad - \frac{2z}{\ell}\mu_m\left(\partial_X^2\phi_0 + 2\partial_{xX}^2\phi_1 - \mu_m(\partial_x^2+\partial_y^2)\phi_0\right) \end{aligned} \quad (55)$$

A partial differential equation for the amplitude function $A(X,\tau)$ including the lowest order nonlinear interaction of the linearly unstable packet with the basic zonal flow is found by applying the secularity condition imposed by the linear adjoint homogeneous conditions formed from (22) to (24). The operator K_m is self-adjoint and likewise the homogeneous boundary conditions (22) and (23), however (24) leads to

$$L^{+}\phi^{*} \equiv -(\phi_x^{*} - z\,\phi_{xz}^{*}) + \frac{2z}{\ell}\mu_m(\partial_x^2+\partial_y^2)\phi^{*} = 0 \quad \text{on } z = \pm\frac{h}{2}$$

where $\dagger$ denotes the adjoint of an operator and * denotes the adjoint function. Then direct comparison of the adjoint problem with the problem at $O(\varepsilon^{\frac{1}{2}})$ shows that the adjoint solution is given by the map $x \to -x$: $\phi \to \phi^{*}$, i.e.;

$$\phi^* = e^{ik_m x}\sin lY\left(\mathrm{ch}\left(\frac{2s_m z}{h}\right) - \frac{i\mu_{om}}{s_m - \mathrm{th}\, s_m}\,\mathrm{sh}\left(\frac{2s_m z}{h}\right)\right) \qquad (56)$$

$$+ \text{ c.c.}$$

The secularity condition is then, following Drazin (1972), most simply evaluated by integrating the result

$$\int_{D_k} \phi_{2x} K_m^+ \phi_x^* \, dx\, dy\, dz = 0$$

where $D_k = \{(x,y,z) : 0 \leq x \leq \frac{2\pi}{k_m}, -\frac{1}{2} \leq y \leq \frac{1}{2}, -\frac{h}{2} \leq z \leq \frac{h}{2}\}$, successively by parts to give

$$\int_{D_k} \phi_x^* \cdot K_m \phi_{2x}\, dx\, dy\, dz + \int_0^{\frac{2\pi}{k_m}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\phi_{2x}\phi_{xz}^* - \phi_x^*\phi_{2xz}\right]_{-\frac{h}{2}}^{\frac{h}{2}} dx\, dy + \int_0^{\frac{2\pi}{k_m}}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[S_m \phi_{xy}^* \phi_{2x}\right]_{-\frac{1}{2}}^{\frac{1}{2}} dx\, dz = 0$$

where all other boundary terms vanish by the imposed periodicity in x. Further integrations by parts then give the final form

$$\int_{D_k} \phi_x^* K_m \phi_{2x}\, dx\, dy\, dz + \int_0^{\frac{2\pi}{k_m}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{\phi_x^*}{z}\cdot L\phi_2\right]_{-\frac{h}{2}}^{\frac{h}{2}} dx\, dy + \int_0^{\frac{2\pi}{k_m}}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left[S_m \phi_{xy}^* \phi_{2x}\right]_{-\frac{1}{2}}^{\frac{1}{2}} dx\, dz = 0 \qquad (57)$$

of the condition which may be evaluated from (48) and (55) using the lower order solutions for ϕ_0 and ϕ_1 together with the adjoint solution ϕ^* to give the cubicly nonlinear partial differential equation

$$A_\tau - a_2 A_{xx} = a_1 A - 4\dagger A|A|^2 \qquad (58)$$

The necessary algebra is long-winded and has been omitted, the results are:-

$$a_1 = \frac{\Lambda h\,(k_m^2 + l^2)\,\mathrm{th}\,s_m\,\mu_m^2}{s_m\,(1+\mathrm{th}^2 s_m)} \tag{59}$$

which is the (scaled) growth rate (25) of linear theory. The form of the coefficient a_2 which appears from the secularity condition is

$$\begin{aligned} a_2 = -\frac{S_m \Lambda h^3}{16 k_m \mu_{0m} s_m^3 (1+\mathrm{th}^2 s_m)(k_m^2+l^2)} \Bigg\{ & -k_m^4 s_m \left(s_m - \mathrm{th}\,s_m - \frac{\mu_{0m}^2}{(s_m - \mathrm{th}\,s_m)}\right)\times \\ & (s_m - \mathrm{th}\,s_m + s_m\,\mathrm{th}^2 s_m) - 4k_m^2 \mu_{0m}^2 \left(k_m^2 s_m (1+\mathrm{th}^2 s_m) + (k_m^2+l^2)\,\mathrm{th}\,s_m\right) \\ & - k_m^2 (k_m^2+l^2) s_m \left((3s_m + 3\,\mathrm{th}\,s_m - s_m\,\mathrm{th}^2 s_m)(s_m - \mathrm{th}\,s_m) + \frac{\mu_{0m}^2}{s_m - \mathrm{th}\,s_m}(s_m - 3\,\mathrm{th}\,s_m - 3 s_m\,\mathrm{th}^2 s_m)\right) \\ & + 2k_m^2 (k_m^2+l^2) s_m\,\mathrm{th}\,s_m \left(s_m - \mathrm{th}\,s_m - \frac{\mu_{0m}^2}{s_m - \mathrm{th}\,s_m}\right) - 2k_m^2 (k_m^2 - l^2) s_m\,\mathrm{th}^2 s_m (s_m\,\mathrm{th}\,s_m - 1) \\ & - \frac{2k_m^4 s_m^2\,\mathrm{th}\,s_m}{(s_m - \mathrm{th}\,s_m)} (s_m\,\mathrm{th}\,s_m - 1)(s_m - \mathrm{th}\,s_m - s_m\,\mathrm{th}^2 s_m) \Bigg\} \end{aligned}$$

but using the relations (17-19) defining the minimum point on the neutral surface it may be shown that

$$\begin{aligned} a_2 &= \frac{1}{2} \left.\frac{\partial^2(-ik_c)}{\partial k^2}\right|_m \\ &= \frac{\Lambda s_m}{8 S_m k_m^3 \mu_{0m} h (1+\mathrm{th}^2 s_m)} \Big\{ \mu_{0m}^2 \left(5 s_m (1+\mathrm{th}^2 s_m) + 8\,\mathrm{th}\,s_m\right) \\ &\qquad + s_m^2 \left(9 s_m (1+\mathrm{th}^2 s_m) - 16\,\mathrm{th}\,s_m\right) \Big\} \end{aligned} \tag{60}$$

which may be checked to be real and positive more readily, again the details have been omitted. Finally

$$\gamma = \frac{k_m s_m^2}{S_m \mu_{0m} l^3 (1 + \operatorname{th}^2 s_m)(s_m - \operatorname{th} s_m)} \left\{ (s_m - 2 s_m^2 \operatorname{th} s_m + s_m \operatorname{th}^2 s_m) \sum_0^\infty a_{2j+1}^2 \right.$$
$$\left. + (s_m - 2 \operatorname{th} s_m + s_m \operatorname{th}^2 s_m) \sum_0^\infty a_{2j+1}^2 (j+\tfrac{1}{2})\pi S_m^{\frac{1}{2}} l \coth (j+\tfrac{1}{2})\pi S_m^{\frac{1}{2}} l \right\} \tag{61}$$

The 'Landau constant' of equation (57) is the salient feature of this type of weakly nonlinear stability analysis. The Parseval formula associated with the Fourier sine series (52) is

$$\sum_0^\infty a_{2j+1}^2 = 3$$

which together with equation (17) defining the neutral surface may be used to rewrite (61) as

$$\gamma = \frac{k_m s_m^2 \left\{ 6\mu_{0m}^2 \operatorname{th} s_m + (s_m + s_m \operatorname{th}^2 s_m - 2 \operatorname{th} s_m) \sum_0^\infty a_{2j+1}^2 \left((j+\tfrac{1}{2})\pi S_m^{\frac{1}{2}} l \coth (j+\tfrac{1}{2})\pi S_m^{\frac{1}{2}} l - 1 \right) \right\}}{S_m \mu_{0m} l^3 (1 + \operatorname{th}^2 s_m)(s_m - \operatorname{th} s_m)} \tag{62}$$

Allowing for the factor S_m^{-1} which is introduced by the different time scaling, this is the Landau constant of Drazin (1972) for the amplitude of a single unstable mode at a point away from the nose (k_m, S_μ, μ_m) of the neutral surface when μ is $O(1)$ and fixed and the supercriticality is taken with respect to the Burger number S (or β in Drazin's notation). The terms in the expansion scheme which give rise to γ are the first two Jacobian terms in equation (55) alone; there are no nonlinear terms on the right hand side of (48) by virtue of the way in which the lower order solutions ϕ_0 and ϕ_1 have been constructed, the slow scale Jacobian term in (55) does not contribute to the final result and similarly the third term in (57) does not contribute since on the sidewalls, from (54) and (45)

$$\phi_{2x} = - \phi_{1x}^{\alpha}$$

which is independent of x.

That a_2 and γ are both real is a consequence of the validity of the principle of exchange of stabilities which holds at all points of the neutral surface (Newell 1974). Since the neutral surface is locally concave at the minimum point it then follows that a_2 is positive and γ has been computed as positive at all points of the neutral surface which correspond to integer values of k by Drazin (Drazin 1972). Assuming that γ is positive at the minimum point, equation (58) is typical of a class describing viscous instabilities which includes the important cases of Taylor-vortex flow and Bénard convection. A property of this equation has been found by Eckhaus (1965) and elaborated and clarified by Stuart and DiPrima (1978), namely that (58) possesses a set of steady solutions

$$A = A_0 e^{im_0 X} \tag{63}$$

where $$|A_0|^2 = \frac{\sigma_0}{\gamma_r} \quad \text{and} \quad \sigma_0 = a_1 - m_0^2 a_2 \tag{64}$$

corresponding to the $O(\varepsilon^{\frac{1}{2}})$ bandwidth of wavenumbers which are unstable in linear theory. Further, perturbations of the form

$$A = A_0 e^{im_0 X} + B(X,\tau)$$

to the equilibrium solution (63) are linearly stable provided that the wavenumber $k_m + \varepsilon^{\frac{1}{2}} m_0$ of the flow disturbance represented by the solution satisfies the inequality

$$\frac{k_m - k_-}{\sqrt{3}} < \varepsilon^{\frac{1}{2}} m_0 < \frac{k_+ - k_m}{\sqrt{3}}$$

when k_+ and k_- are the limiting wavenumbers for which the points $(k_+, S_m, \mu_m+\varepsilon)$ and $(k_-, S_m, \mu_m+\varepsilon)$ are on the neutral surface.

The equation (58) has been studied numerically by various authors and a summary of some results has been given in the review by Newell (1974) which suggests that when a_1, a_2 and γ are positive the solution at large times is

$$A = \pm \left(\frac{a_1}{\gamma_r}\right)^{\frac{1}{2}}$$

everywhere for a variety of initial conditions.

CHAPTER 4

WEAKLY NONLINEAR EVOLUTION OF A SINGLE LINEARLY UNSTABLE WAVE FOR EADY'S MODEL WHEN THE SQUARE ROOT OF THE EKMAN NUMBER DIVIDED BY THE ROSSBY NUMBER IS OF THE ORDER OF THE SQUARE ROOT OF THE SUPERCRITICALITY

A large proportion of the problems concerning the form of slightly unstable baroclinic flow which have appeared in the literature consider the evolution of a single wave, associated with one particular wave-number in the downstream or x-direction. Apart from the simplicity of the dynamics imposed by such a constraint the motivation for its use comes from the keenly observed results of experiments with a differentially heated rotating annulus where the finite azimuthal extent of the apparatus quantises the permissible downstream wavenumbers in the case of an ordered laminar disturbance to the basic flow. (There are several reviews of experiments using fundamentally similar apparatus, see e.g.; Buzyna, Pfeffer and Kung 1978 or Hide and Mason, 1975.) In a given experiment or at a given point in parameter space in the slightly unstable state the apparatus typically selects a single downstream wavenumber, k, which is O(1) with respect to the small groupings of the Rossby number etc. . Note that there are also regimes in parameter space close to the neutral surface (which for a continuous spectrum of k is given by equation (3.10) and sketched in figure (3.2)) where a finite number of waves of different wavenumber (k,ℓ) are linearly unstable. A problem involving the weakly nonlinear interaction of two such linearly unstable waves of equal total wavenumber with the basic flow has been studied in the inviscid limit of the Eady model by Drazin (Drazin 1970) who later (Drazin 1972) considered a similar problem for two unstable waves with equal spanwise wavenumber, ℓ, in the viscous case of the Eady model when $\mu_O = O(1)$ as the supercriticality tends to zero.

There now follows discussion of the weakly nonlinear evolution of such a single wave (where k and ℓ are O(1)) when the supercriticality is taken with respect to the Burger number (S) for given values of the viscosity parameter μ.

Recall the equation of the neutral surface is

$$G(k,S,\mu) \equiv \mu_o^2\,\mathrm{th}\,s + (s - \mathrm{th}\,s)(s\,\mathrm{th}\,s - 1) = 0 \qquad (1)$$

$$\text{where } \mu_o = \mu \cdot \frac{k^2+\ell^2}{k}$$

When $\mu_0 = 0$, the inviscid criterion for linear instability is

$$0 < S < S_0 = \frac{4s_0^2}{k^2(k^2+\ell^2)}$$

$$\text{where } s_0 > 0 \text{ is } \ni \; s_0 \,\mathrm{th}\, s_0 = 1 \tag{2}$$

which is recovered in a regular manner from (1) as $\mu_0 \to 0$. For $\mu > 0$ the neutral surface has two branches of intersection with the planes k = constant (see figure (3.2)), along the lower branch from (1) for given $O(1)$ k and ℓ where the δ suffix denotes evaluation on the neutral surface for $\mu > 0$ and the o suffix likewise for $\mu = 0$

$$s_\delta^2 \sim \frac{3\mu^2(k^2+\ell^2)^2}{k^2} + O(\mu^4)$$

$$\text{as } \mu \to 0$$

$$\Rightarrow \qquad S_\delta \sim \frac{12\mu^2(k^2+\ell^2)}{\ell^2 k^2} + O(\mu^4)$$

and $S_\delta \to 0$.
Along the upper branch

$$s_\delta \sim s_0 - \frac{\mu_0^2}{s_0(s_0^2-1)} + O(\mu^4) \tag{3}$$

$$\text{as } \mu \to 0$$

$$\Rightarrow \qquad S_\delta \sim S_0 - \frac{2S_0}{s_0^2(s_0^2-1)}\mu_0^2 + O(\mu^4) \tag{4}$$

which recovers the case of (2) in the limit $\mu \to 0$. Consideration here will be confined to the case of the upper branch. Throughout the remaining chapters it will be convenient to compare the behaviour of the Eady model with viscous effects at the top and bottom boundaries included to the 2-layer model of Phillip's ('54) which has been studied with the inclusion of viscous effects in the main part by Pedlosky (a series of references is included). A salient feature contrasting the two models arises because for the 2-layer model the basic flow and undisturbed density are independent of the vertical coordinate, z, within each layer. This independence implies that the normal mode representation of the perturbation field is also independent of z and on neglecting the effects of thermal diffusion and molecular viscosity in the interior flow (especially in the neighbourhood of the interface between different layers) equation (1.59) may be integrated in z using the boundary condition (1.60) and a similar condition on the top boundary to give a pair of equations in x, y and t, one for each of the two layers, expressing conservation of potential vorticity. The inclusion of the β-plane effect in the layer models does not change this situation inasmuch as the normal modes are still independent of z but now β appears in the dispersion relation explicitly in such a way that the inviscid limit ($\mu \to 0$) of the neutral surface is not the neutral curve of the inviscid problem ($\mu = 0$) unless $\beta = 0$, i.e.; the position of the neutral surface is destabilised in the inviscid limit when $\beta \neq 0$. This point was first made by Holopainen (1961) and has been considered in detail by Gibbon and McGuinness (1981) where it is shown that the neutral curve in the inviscid case is recovered in the inviscid limit if both the viscosity parameter and the β-plane effect are initially assumed to be of the same order of magnitude as the square root of the supercriticality. However, the content of chapter 2 shows that in the continuously sheared model (and when $\frac{1}{h_s} = 0$ as in Eady's case) the vertical structure of the perturbation field has an implicit dependence on β and the dispersion relation as yet defies analysis except in the inviscid case ($\mu_0 = 0$). In essence, the generalisation of the theory to include the β-plane effect in the 2-layer model does not appear to have an obvious counterpart in the continuously sheared and stratified models.

A further difficulty which in the context of the 2-layer model was first pointed out by Pedlosky (1971) concerns the behaviour of the linear growth rate with respect to the supercriticality over different regimes of μ. Now

$$S = \frac{4s^2}{k^2(k^2+\ell^2)} \Rightarrow S - S_\delta = \frac{2S_\delta}{s_\delta}(s - s_\delta) + O((s-s_\delta)^2) \tag{5}$$
$$\text{as } (s - s_\delta) \to 0$$

when k, ℓ and h are fixed. Then perturbing S from a point (k, S_δ, μ) on the neutral surface

$$G(k, S, \mu) = \left(\mu_0^2 + s_\delta(s_\delta - k s_\delta) + k^2 s_\delta(s_\delta k s_\delta - 1)\right)(s - s_\delta)$$
$$+ O((s-s_\delta)^2) \tag{6}$$

and from (3.14)

$$c = \frac{\Lambda h}{4 s_\delta \,\mathrm{th}\, s_\delta}\Bigg\{ -i\mu_0\left(1+\mathrm{th}^2 s_\delta\right)\left(1+O(s-s_\delta)\right)$$

$$\pm i\mu_0\left(1+\mathrm{th}^2 s_\delta\right)\left(1+O(s-s_\delta)\right)\left(1-\frac{4\,\mathrm{th}\, s_\delta}{\left(1+\mathrm{th}^2 s_\delta\right)^2}\left(\mu_0^2+s_\delta\left(s_\delta-\mathrm{th}\, s_\delta\right)+\mathrm{th}^2 s_\delta\left(s_\delta\,\mathrm{th}\, s_\delta-1\right)\right)(s-s_\delta) + O\left((s-s_\delta)^2\right)\right)^{\frac{1}{2}}\Bigg\} \qquad (7)$$

so that if μ_0 is O(1) as $(s-s_\delta) \to 0$.

$$kc = \frac{i\Lambda h k}{4 S_\delta \mu_0\left(1+\mathrm{th}^2 s_\delta\right)}\left(\mu_0^2+s_\delta\left(s_\delta-\mathrm{th}\, s_\delta\right)+\mathrm{th}^2 s_\delta\left(s_\delta\,\mathrm{th}\, s_\delta-1\right)\right)\left(S_\delta-S\right) + O(\epsilon^2) \qquad (8)$$

where $S_\delta - S = \varepsilon \ll 1$ is the supercriticality, this will be referred to as the O(1) viscous case. If $\mu_0 = o(\varepsilon^{\frac{1}{2}})$ as $\varepsilon \to 0$, then from (3)

$$s - s_\delta = s - s_0 + O\left(\mu_0^2\right)$$

$$\Rightarrow \quad s - s_\delta = s - s_0 + o(\epsilon)$$

so that from (7)

$$kc = \pm \frac{i\Lambda h k}{2}\left(\frac{s_0^2-1}{2S_0}\right)^{\frac{1}{2}}\left(S_0-S\right)^{\frac{1}{2}} + O(\epsilon) \qquad (9)$$

which will be referred to as the inviscid case. Now, the limit that $\mu \to 0$ with all other quantities fixed in (8) is incompatible with (9). Inspection of (7) shows that there is an intermediate region where $\mu = O(\varepsilon^{\frac{1}{2}})$ as $\varepsilon \to 0$ and there

$$kc = \frac{\Lambda h k}{4 s_\delta \,\mathrm{th}\, s_\delta}\Bigg\{ -i\mu_0\left(1+\mathrm{th}^2 s_\delta\right)$$

$$\pm i\mu_0\left(1+\mathrm{th}^2 s_\delta\right)\left(1-\frac{4\,\mathrm{th}\, s_\delta}{\left(1+\mathrm{th}^2 s_\delta\right)^2}\left(s_\delta\left(s_\delta-\mathrm{th}\, s_\delta\right)+\mathrm{th}^2 s_\delta\left(s_\delta\,\mathrm{th}\, s_\delta-1\right)\right)\left(\frac{s-s_\delta}{\mu_0^2}\right)\right)^{\frac{1}{2}}\Bigg\} + O(\epsilon) \qquad (10)$$

so that the growth rate in this regime is proportional to $\varepsilon^{\frac{1}{2}}$. From (10), the inviscid growth rate (9) is recovered when $\mu_0 \to 0$ with $(s-s_\delta)$ fixed and the inviscid limit of the O(1) viscous case given by (8) is recovered when $(s-s_\delta) \to 0$ with μ_0 fixed, the regime where $\mu = O(\varepsilon^{\frac{1}{2}})$ will be termed the $O(\varepsilon^{\frac{1}{2}})$ viscous case.

To investigate the weakly nonlinear dynamics of an unstable wave in the $O(\varepsilon^{\frac{1}{2}})$ viscous case put

$$\begin{aligned} S_\delta - S &= \pm \epsilon \\ \mu_0 &= \nu \epsilon^{\frac{1}{2}} \\ \tau &= \epsilon^{\frac{1}{2}} t \end{aligned} \tag{11}$$

where ν is a dimensionless variable representing the influence of the Ekman boundary layers which is strictly O(1) with respect to the supercriticality ε, and τ is the slow time scale of evolution implied by the linear growth rate (10). Since k and ℓ are O(1)

$$\begin{aligned} S_0 - S &= S_0 - S_\delta + S_\delta - S \\ &= \frac{2S_0\mu_0^2}{s_0^2(s_0^2-1)} + S_\delta - S + O(\mu^4) \\ &= \left(\frac{2S_0\nu^2}{s_0^2(s_0^2-1)} \pm 1\right)\epsilon + O(\epsilon^2) \\ &= (\sigma\nu^2 \pm 1)\epsilon + O(\epsilon^2) \end{aligned} \tag{12}$$

where

$$\sigma \equiv \frac{2S_0}{s_0^2(s_0^2-1)} \tag{13}$$

Now assume an expansion of the form

$$\phi(x,y,z,\tau;\epsilon) = \epsilon^{\frac{1}{2}}\phi_0(x,y,z,\tau) + \epsilon\phi_1 + \epsilon^{\frac{3}{2}}\phi_2 + \dots \tag{14}$$

for the disturbance stream function which will lead to a cubic nonlinearity for the amplitude function, $A(\tau)$. Note that this expansion is asymptotic (with respect to ε as $\varepsilon \to 0$) and may also be convergent (for $\varepsilon > 0$) as has been shown to be true for a similar problem of unstable flow between differentially rotating cylinders by Kirchgässner and Sorger (1969). A suitable form of the governing equations and boundary conditions (1.66) to (1.69) for the iteration sequence of linear problems at successive powers of $\varepsilon^{\frac{1}{2}}$ is

$$\Lambda z\,\partial_x K_0 \phi = -\frac{\partial}{\partial(x,y)}(\phi, K_0\phi) - \varepsilon^{\frac{1}{2}}\partial_\tau K_0\phi$$

$$+(\sigma\nu^2 \pm 1)\varepsilon\Lambda z\,\partial_x(\partial_x^2+\partial_y^2)\phi \tag{15}$$

$$+(\sigma\nu^2 \pm 1)\varepsilon\frac{\partial}{\partial(x,y)}\left(\phi, (\partial_x^2+\partial_y^2)\phi\right)$$

$$+ O(\varepsilon^{\frac{3}{2}}\phi)$$

where $K_0 \equiv \partial_z 2 + S_0(\partial_x{}^2+\partial_y{}^2)$

$$L_0\phi \equiv \phi_x - z\,\phi_{xz}$$

$$= \frac{1}{\Lambda}\frac{\partial}{\partial(x,y)}(\phi,\phi_z) + \frac{\varepsilon^{\frac{1}{2}}}{\Lambda}\phi_{\tau z}$$

$$-\nu\varepsilon^{\frac{1}{2}}z\,\frac{kh S_0}{2s_0^2}(\partial_x^2+\partial_y^2)\,\phi + O(\varepsilon^{\frac{3}{2}}\phi) \tag{16}$$

$$\text{on } z = \pm\frac{l}{2}$$

$$\phi_x = 0 \qquad \text{on } Y = 0,1 \tag{17}$$

$$\overline{\phi}_{y\tau} = 0 \qquad \text{on } Y = 0,1 \tag{18}$$

where the overbar denotes averaging with respect to x as defined in equation (1.53).

The expansion (14) yields the problems at successive order:-

$O(\varepsilon^{\frac{1}{2}})$

$$\Lambda z\,\partial_x K_0\phi_0 = 0$$

$$\phi_{0x} = \overline{\phi}_{0y\tau} = 0 \quad \text{on } Y = 0,1 \tag{19}$$

$$L_0\phi_0 = 0 \quad \text{on } z = \pm\frac{l}{2}$$

which has the solution to the linear inviscid problem of Drazin 1970, in the present notation

$$\phi_0 = \left(A e^{ilx} + A^* e^{-ilx}\right) \sin lY \,\mathrm{ch}\left(\frac{2s_0 z}{h}\right) \qquad (20)$$

where $A = A(\tau)$ is the complex wave amplitude function and

$$l = r\pi \;, \; r = 1, 2, \dots \;; \quad s_0 \,\mathrm{th}\, s_0 = 1 \;, \; s_0 > 0 \qquad (21)$$

$O(\varepsilon)$

$$\Lambda z \partial_x K_0 \phi_1 = -\frac{\partial}{\partial(x,y)}(\phi_0, K_0\phi_0) - \partial_\tau K_0 \phi_0$$

which, since the solution at previous order satisfies $K_0\phi_0 = 0$ reduces to the homogeneous equation

$$\Lambda z \partial_x K_0 \phi_1 = 0 \qquad (22)$$

and

$$\phi_{1x} = \bar{\phi}_{1y\tau} = 0 \quad \text{on } Y = 0, 1 \qquad (23),(24)$$

with

$$L_0 \phi_1 = \frac{1}{\Lambda}\frac{\partial}{\partial(x,y)}(\phi_0, \phi_{0z}) + \frac{1}{\Lambda}\phi_{0\tau z} - \nu z \frac{l h s_0}{2 s_0^2}(\partial_x^2 + \partial_y^2)\phi_0 \qquad \text{on } z = \pm \frac{h}{2}$$

which, using (20), becomes

$$L_0\phi_1 = \pm \frac{2s_0}{\Lambda h}\left(\dot{A} e^{ilx}\right) \sin lY \,\mathrm{sh}\, s_0 \pm \nu l \left(A e^{ilx}\right) \sin lY \,\mathrm{ch}\, s_0 \qquad (25)$$

+ complex conjugate on $z = \pm \frac{h}{2}$.

Note that the nonlinear term vanishes in this boundary condition, this is typical of the small $O(\varepsilon^{\frac{1}{2}})$ viscous case but not of the $O(1)$ viscous case, c.f. equation (3.35). A solution may be written

$$\phi_1 = \frac{2 i s_0}{\lambda k h (s_0^2-1)} \left(\dot{A} e^{ikx}\right) \sin lY \, \mathrm{sh}\left(\frac{2 s_0 z}{h}\right)$$

$$+ \frac{i \nu s_0}{(s_0^2-1)} \left(A e^{ikx}\right) \sin lY \, \mathrm{sh}\left(\frac{2 s_0 z}{h}\right) \qquad (26)$$

+ complex conjugate of these terms

+ $f(y,z,\tau)$

where the as yet arbitrary function $f(y,z,\tau)$ is taken to be real without loss of generality and is specified by a secularity condition at next order. The function $f(y,z,\tau)$ automatically satisfies the governing equation (22) and boundary condition (23).

$O(\varepsilon^{3/2})$

$$\lambda z \partial_x K_0 \phi_2 = -\frac{\partial}{\partial(x,y)}(\phi_0, K_0 \phi_1) - \frac{\partial}{\partial(x,y)}(\phi_1, K_0 \phi_0)$$

$$- \partial_\tau K_0 \phi_1 + (\sigma\nu^2 \pm 1)\, \lambda z \partial_x (\partial_x^2 + \partial_y^2) \phi_0$$

Again the nonlinear terms vanish on using the solutions (20) and (26) at previous orders if

$$K_0 f(y,z,\tau) = 0 \qquad (27)$$

and the equation reduces to

$$\lambda z \partial_x K_0 \phi_2 = -\lambda z i k (\sigma\nu^2 \pm 1)(k^2 + l^2) A e^{ikx} \sin lY \, \mathrm{ch}\left(\frac{2 s_0 z}{h}\right)$$

+ complex conjugate (28)

with

$$\phi_{2x} = \bar{\phi}_{2y\tau} = 0 \qquad \text{on } Y = 0, 1 \tag{29),(30}$$

The top and bottom boundary conditions are

$$L_0 \phi_2 = \frac{1}{\Lambda}\frac{\partial}{\partial(x,y)}(\phi_0, \phi_{1z}) + \frac{1}{\Lambda}\frac{\partial}{\partial(x,y)}(\phi_1, \phi_{0z})$$

$$+ \frac{1}{\Lambda}\phi_{1\tau z} - \nu z \frac{k h S_0}{2 s_0^2}(\partial_x^2 + \partial_y^2)\phi_1 \qquad \text{on } z = \pm\frac{h}{2}$$

and reduce to

$$L_0 \phi_2 = \frac{4 i s_0^2}{\Lambda^2 k h^2 (s_0^2-1)} \ddot{A} e^{ikx} \sin \ell Y \,\mathrm{ch}\, s_0 + \frac{2 i \nu s_0}{\Lambda h (s_0^2-1)} \dot{A} e^{ikx} \sin \ell Y (s_0 \,\mathrm{ch}\, s_0 + \mathrm{sh}\, s_0)$$

$$+ \frac{i k s_0 \nu^2}{(s_0^2-1)} A e^{ikx} \sin \ell Y \,\mathrm{sh}\, s_0 + \frac{ik}{\Lambda} A e^{ikx} \sin \ell Y \,\mathrm{ch}\, s_0 \, b_{yz} \tag{31}$$

$$\mp \frac{2 i k s_0}{\Lambda h} A e^{ikx} \sin \ell Y \,\mathrm{sh}\, s_0 \, b_y + \frac{1}{\Lambda} b_{\tau z} \mp \nu \frac{S_0 k h^2}{4 s_0^2} b_{yy}$$

$$+ \frac{4 s_0^2 \ell}{\Lambda^2 h^2 (s_0^2-1)} \sin 2\ell Y (\dot{|A|^2}) + \frac{4 \nu k \ell s_0^2}{\Lambda h (s_0^2-1)} \sin 2\ell Y (|A|^2) + \text{complex conjugate}$$

The x independence of the last four terms of this boundary condition leads to a secular or growing term in ϕ_2 which is proportional to xe^{ikx} unless they vanish identically. Constructing the solution ϕ_1 of the problem at previous order so that $K_0\phi_1 = 0$ implies the boundary value problem

$$(\partial_z^2 + S_0 \partial_y^2) b = 0$$

$$b_{y\tau} = 0 \quad \text{on } Y = 0, 1 \tag{32}$$

$$b_{\tau z} \mp \frac{\nu \Lambda S_0 k h^2}{4 s_0^2} b_{yy} = -\frac{4 s_0^2 \ell}{\Lambda h^2 (s_0^2-1)} \left\{ (\dot{|A|^2}) + \nu \Lambda k h |A|^2 \right\} \sin 2\ell Y \qquad \text{on } z = \pm\frac{h}{2}$$

which specifies the solution ϕ_1, however a specific solution will be postponed until later. The role of $f(y,z,\tau)$ as an x independent contribution to the perturbation stream function ϕ at order ε leads to its interpretation as a mean field (or x independent) correction to the stream function in the basic state. This is consistent with a sustained finite amplitude wave of amplitude $\hat{A} = O(\varepsilon^{\frac{1}{2}})$ drawing upon the available potential energy in the undisturbed state and depleting it at $O(\varepsilon)$ by an amount which is proportional to the wave energy $\hat{A}^2 = O(\varepsilon)$. The boundary condition (32) then simplifies to

$$L_0\phi_z = \frac{4is_0^2}{\Lambda^2 k l^2(s_0^2-1)}\ddot{A}\, e^{ilx} \sin lY \,\mathrm{ch}\, s_0 + \frac{2i\nu s_0}{\Lambda l(s_0^2-1)}\dot{A}\, e^{ilx} \sin lY(s_0 \,\mathrm{ch}\, s_0 + \mathrm{sh}\, s_0)$$

$$+ \frac{ils_0\nu^2}{(s_0^2-1)} A\, e^{ilx} \sin lY \,\mathrm{sh}\, s_0 + \frac{il}{\Lambda} A\, e^{ilx} \sin lY \,\mathrm{ch}\, s_0\left(b_{yz} \mp \frac{2}{h} b_y\right)$$

+ complex conjugate on $z = \pm \frac{h}{2}$. (33)

The secularity condition imposed by the adjoint linear homogeneous system at $O(\varepsilon^{3/2})$ is obtained as in chapter 3 by integrating the identity

$$\int_0^{\frac{2\pi}{k}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}} \phi_{2x} K_0^{+} \phi_x^{*} \, dx\, dy\, dz = 0$$

where + and * denote adjoint operators and functions. In the present context the linear problem is that of the inviscid case and is self adjoint so that

$$\phi^* = e^{ilx} \sin lY \,\mathrm{ch}\left(\frac{2s_0 z}{h}\right) \quad \text{+ complex conjugate.} \qquad (34)$$

and further, since $s_0 . \mathrm{th}(s_0) = 1$,

$$\phi^*_{xz}\Big|_{z=\pm\frac{l}{2}} = \frac{\phi^*_x}{z}\Big|_{z=\pm\frac{l}{2}}$$

then repeated integration by parts and final use of the boundary condition (29) leads to the simpler relation

$$\int_0^{\frac{2\pi}{k}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{l}{2}}^{\frac{l}{2}} \phi^*_x K_0 \phi_{2x}\, dx\, dy\, dz + \int_0^{\frac{2\pi}{k}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{\phi^*_x}{z} . L_0 \phi_2 \right]_{-\frac{l}{2}}^{\frac{l}{2}} dx\, dy = 0 \qquad (35)$$

compared with (3.57). Using the explicit forms of ϕ^*, $K_0\phi_{2x}$ and $L_0\phi_2$ in equations (34), (28) and (33) the condition implies

$$\ddot{A} + \frac{\nu \Lambda k l (s_0^2+1)}{2 s_0^2}\dot{A} \mp \frac{\Lambda^2 k^2 l^2 (s_0^2-1)}{8 s_0} A$$
$$+ \frac{\Lambda k^2 l^2 (s_0^2-1)}{2 s_0^2} A \int_0^1 \sin^2 lY \left(f_{yz} - \frac{2}{l} f_y \right)_{z=+\frac{l}{2}} dY = 0 \qquad (36)$$

where the coefficient of A in the third, linear term has been simplified using (13) and the integrand in the nonlinear term has been simplified by exploiting the fact that $f(y,z,\tau)$ is an odd function of both y and z, as can be inferred from (32).

On setting $\Lambda = h = 1$ equations (32) and (36) may be shown to be equivalent to the pair derived for the same problem by Brindley and Moroz (1980) where these two parameters were not included in the analysis.

The boundary value problem for $f(y,z,\tau)$, the function appearing in the integral term in (36) which represents the leading order nonlinear interaction of an unstable wave with the basic flow, may be considered responsible for the pecularities of the pair (32) and (36) constituting the amplitude equation. A solution to (32) may be written in terms of an eigenfunction expansion (or by Fourier's method) giving a series representation with uniform convergence in y and z to the solution, which by simple rescaling of one of the coordinate axes is seen to be a solution of Laplace's equation. So formally, setting

$$f(y,z,\tau) = \sum_{0}^{\infty} B_j(\tau) \cos (2j+1)\pi Y \, sh(2j+1)\pi S_0^{\frac{1}{2}} z \tag{37}$$

the series trivially defines a solution of the governing equation and sidewall boundary condition. Substitution in the top and bottom boundary conditions implies

$$\dot{B}_j + \frac{\nu \Lambda k l}{2 s_0^2} (j+\tfrac{1}{2})\pi S_0^{\frac{1}{2}} l \,.\, th (j+\tfrac{1}{2})\pi S_0^{\frac{1}{2}} l \,.\, B_j = \tag{38}$$

$$\frac{2 s_0^2 l}{\Lambda k (s_0^2-1)} \left((\dot{|A|^2}) + \nu \Lambda k l (|A|^2) \right) . \frac{a_{2j+1}}{(j+\frac{1}{2})\pi S_0^{\frac{1}{2}} l \,.\, ch (j+\frac{1}{2})\pi S_0^{\frac{1}{2}} l}$$

where $\{a_{2j+1}\}$ are the Fourier coefficients of $\sin 2lY$ for $Y \in [0,1]$, i.e.;

$$\sin 2lY = -\sum_{0}^{\infty} a_{2j+1} \cos (2j+1)\pi Y \tag{39}$$

where $$a_{2j+1} = \frac{2l}{(j+\frac{1}{2})^2\pi^2 - l^2}$$

$$j = 0, 1, \ldots \qquad l = r\pi \, ; \; r = 1, 2, \ldots$$

The integral of the last term in equation (36), may now be expressed as a sum, changing the order of summation and integration by the uniform convergence of the sum with respect to Y, in the form.

$$\frac{2l^2}{k} \sum_{j=0}^{\infty} B_j(\tau) \left(\frac{(j+\frac{1}{2})\pi S_0^{\frac{1}{2}} l \, ch (j+\frac{1}{2})\pi S_0^{\frac{1}{2}} l - sh (j+\frac{1}{2})\pi S_0^{\frac{1}{2}} l}{(j+\frac{1}{2})^2\pi^2 - l^2} \right) \tag{40}$$

The coefficients of the linearised amplitude equation, which reduces to the linear part of (36) alone, may easily be seen to recover the growth rates implied by equations (8) and (9) in the appropriate limits, i.e.; putting

$$A = e^{k c_i \tau}$$

in the linearised version of (36) gives

$$(kc_i)^2 + \frac{\nu\Lambda k l(s_0^2+1)}{2s_0^2}(kc_i) - \frac{\Lambda^2k^2l^2(s_0^2-1)}{8S_0} = 0 \qquad (41)$$

Keeping the supercriticality, ε, fixed and taking the limit $\nu \to 0$ the roots of (41) are

$$kc_i = \pm \frac{\Lambda k l}{2}\left(\frac{(s_0^2-1)}{2S_0}\right)^{\frac{1}{2}}$$

which is the scaled growth rate in the inviscid case given by (9). To find the viscous limit of (41) formally let $\nu \to \infty$ then to leading order the roots are

$$kc_i = \frac{\Lambda k l\, s_0^2(s_0^2-1)}{4S_0(s_0^2+1)\nu}$$

and

$$kc_i = -\frac{\nu\Lambda k l(s_0^2+1)}{2s_0^2}$$

The first root agrees with the inviscid limit, $\mu_0 \to 0$, of the growth rate in the O(1) viscous case given by equation (8) and the second root corresponds to a rapidly decaying mode of oscillation of the linear problem and originates from the other stable root of the dispersion relation in the slightly supercritical state.

In fact, to find the viscous limit of the fully nonliner amplitude equations (36) and (38) put

$$\partial_\tau = \epsilon^{\frac{1}{2}}\partial_{\tau_n} \qquad (42)$$

where $\tau_n = \varepsilon^{\frac{1}{2}}\tau$ is a slower time scale than τ which is appropriate since in the $O(\varepsilon^{\frac{1}{2}})$ viscous case the linear growth rate is $O(\varepsilon^{\frac{1}{2}})$, which is the time scale of τ whereas in the O(1) viscous case the linear growth rate is $O(\varepsilon)$, which is the slower time scale of τ_n. Next put

$$\nu = \mu_0\epsilon^{-\frac{1}{2}}$$

which is the defining relation for ν of equation (12) and keep μ_0 to be order one as ε tends to zero. Formally, equation (38) becomes

$$B_j(\tau) = \frac{4 s_0^4 l}{\Lambda k (s_0^2-1)} \cdot \frac{a_{2j+1}}{\left((j+\frac{1}{2})\pi S_0^{\frac{1}{2}} l\right)^2 \operatorname{sh}(j+\frac{1}{2})\pi S_0^{\frac{1}{2}} l} \tag{43}$$

independently of initial conditions and on substitution in (36) using (40) the limit implies

$$\dot{A} = \frac{\Lambda k l s_0^2 (s_0^2-1)}{4 S_0 (s_0^2+1)\mu_0} \cdot A - \frac{16 k l^4 s_0^4}{\mu_0 \Lambda k (s_0^2+1)} \sum_0^\infty \frac{\left((j+\frac{1}{2})\pi S_0^{\frac{1}{2}} l \coth (j+\frac{1}{2})\pi S_0^{\frac{1}{2}} l - 1\right)}{\left((j+\frac{1}{2})\pi S_0^{\frac{1}{2}} l\right)^2 \left((j+\frac{1}{2})^2\pi^2 - l^2\right)^2} A.|A|^2 \tag{44}$$

This is the same as the amplitude equation in section 3 of Drazin (1972), which describes the evolution of a single wave in the O(1) viscous case, evaluated in the limit when μ_0 (or λ there) tends to zero, i.e.; the viscous limit of the $O(\varepsilon^{\frac{1}{2}})$ nonlinear viscous problem matches the inviscid limit of the O(1) nonlinear viscous problem. In comparing (44) with the version in Drazin's work note that:

(i) a factor of 4 appearing in the nonlinear term is attributable to the amplitude function A being complex.

(ii) the parameter Λ is absent there.

(iii) the differing factors of S_0 (or β_0 in Drazin's notation) occur through the differing choice of non-dimensionalising the time.

The inviscid limit in the nonlinear case is obtained more simply by taking the limit that ν tends to zero in (36) and (38). The first order derivative vanishes in (36) and (38) integrates formally to give

$$B_j(\tau) = \frac{2 s_0^2 l}{\Lambda k (s_0^2-1)} \cdot \frac{a_{2j+1}}{(j+\frac{1}{2})\pi S_0^{\frac{1}{2}} l \operatorname{ch}(j+\frac{1}{2})\pi S_0^{\frac{1}{2}} l} \cdot \left(|A|^2 - |A(0)|^2\right) \tag{45}$$

so that in the limit

$$\ddot{A} = \frac{\Lambda^2 k^2 l^2 (s_0^2 - 1)}{8 s_0} A - 4k^2 l^4 \sum_0^\infty \frac{\left(1 - \frac{\tanh (j+\frac{1}{2})\pi s_0^{\frac{1}{2}} l}{(j+\frac{1}{2})\pi s_0^{\frac{1}{2}} l}\right)}{\left((j+\frac{1}{2})^2 \pi^2 - l^2\right)^2} A\left(|A|^2 - |A(0)|^2\right) \tag{46}$$

where A(0) is the initial wave amplitude at time t = 0. This is the same as the amplitude equation in §5 of Drazin (1970) with the same provisos (i) to (iii) above and an additional apparent difference in the form of the sum in the final term. The apparent discrepancy is caused by a different choice of eigenfunctions for the solution to the boundary problem for $f(y,z,\tau)$ at $O(\varepsilon)$ (compare equations (27), (32) and (37) here with the equations (37), (42) and (43) of that work.), however since the function $f(y,z,\tau)$ is the solution of an elliptic partial differential equation and the boundary conditions here reduce in the limit $\nu \to 0$ to those of the inviscid problem the solution is unique and the two representations must be equivalent.

Returning to the problem at hand, if the integral in (36) is replaced by the sum (40) then the resulting equation together with (38) constitutes the amplitude equation in the form of an infinite set of coupled differential equations

$$\ddot{A} + \nu \Delta_1 \dot{A} - \alpha A + A \sum_{j=0}^{\infty} \beta_j \beta_j = 0 \tag{47}$$

$$\dot{\beta}_j + \nu \Delta_{2j} \beta_j = \hat{f}_j \left((|\dot{A}|^2) + \nu \hat{\sigma} (|A|^2) \right) \tag{48}$$

$$j = 0, 1, \ldots$$

where the coefficients are

$$\Delta_1 = \frac{\Lambda k l (s_0^2 + 1)}{2 s_0^2} \; ; \quad \alpha = \frac{\Lambda^2 k^2 l^2 (s_0^2 - 1)}{8 s_0} \; ; \quad \hat{\sigma} = \lambda k l \tag{49}$$

$$\Delta_{2j} = \frac{\Lambda k l}{2 s_0^2} (j+\tfrac{1}{2})\pi s_0^{\frac{1}{2}} l \tanh (j+\tfrac{1}{2})\pi s_0^{\frac{1}{2}} l \; ;$$

$$\hat{f}_j = \frac{2 s_0^2 l}{\Lambda l (s_0^2 - 1)} \cdot \frac{a_{2j+1}}{(j+\frac{1}{2})\pi s_0^{\frac{1}{2}} l \tanh (j+\frac{1}{2})\pi s_0^{\frac{1}{2}} l}$$

$$\beta_j = \frac{\Lambda k^2 l^2 h (s_0^2 - 1)}{s_0^2} \left(\frac{(j+\frac{1}{2})\pi S_0^{\frac{1}{2}} h \, \mathrm{ch}\,(j+\frac{1}{2})\pi S_0^{\frac{1}{2}} h - \mathrm{sh}\,(j+\frac{1}{2})\pi S_0^{\frac{1}{2}} h}{\left((j+\frac{1}{2})^2 \pi^2 - l^2\right)} \right)$$

$$j = 0, 1, \ldots$$

The preceding investigation of the inviscid and O(1) viscous limits of the amplitude equations shows that the set of equations (38) may be integrated over τ giving in each instance solutions where

$B_j(\tau)$ = a function of j x a function of τ

Notice in passing that the form of the functions differs in both limits; in the inviscid case the reduced form of the amplitude equations contains an explicit dependence upon the initial condition A(0), the functions of j also differ implying that the spatial structure of the order ε zonal flow correction $f(y,z,\tau)$ differs in both limits. However, the curious point leading to the infinity of equations (47) and (48) is that a separable form of the coefficients $\{B_j\}$ with respect to j and τ in the Fourier representation (37) for the solution of the boundary value problem (32) occurs if and only if the represented solution, $f(y,z,\tau)$, is separable in space and time - in which case the integral term in (36) or the sum term in (47) has the temporal behaviour of $f(y,z,\tau)$. In the instance when the $\{B_j\}$ or $f(y,z,\tau)$ are not separable the same integral or sum term is more anfractuous, possessing the temporal behaviour of some rearrangement of the Fourier representation of $f(y,z,\tau)$ where the form of the coefficients $\{\beta_j\}$, $\{\Delta_{2j}\}$ and $\{\hat{\gamma}_j\}$ of (47) and (48) derive from the spatial dependence of $f(y,z,\tau)$.

Some cross-referencing shows that the boundary value problem for $f(y,z,\tau)$ in the $O(\varepsilon^{\frac{1}{2}})$ viscous case differs from the problems for $f(y,z,\tau)$ in the inviscid and O(1) viscous cases only insofar as the top and bottom boundary conditions in the $O(\varepsilon^{\frac{1}{2}})$ viscous case are formed from the sum of the two boundary conditions in the inviscid and O(1) viscous cases. The resulting equation (48) for $\{\beta_j(\tau)\}$ may formally be integrated over τ and reveals

$$\beta_j(\tau) = \hat{\gamma}_j \int_0^{\tau} e^{-\nu \Delta_{2j}(\tau - s) - \nu \hat{\sigma} s} \frac{d}{ds}\left(|A|^2 e^{\nu \hat{\sigma} s}\right) ds + \beta_j(0)\, e^{-\nu \Delta_{2j} \tau} \tag{50}$$

then an integration by parts gives

$$B_j(\tau) = \hat{f}_j\left(|A|^2 - |A(0)|^2 e^{-\nu\Delta_{zj}\tau}\right) + B_j(0)\,e^{-\nu\Delta_{zj}\tau}$$
$$- \nu(\Delta_{zj} - \hat{\sigma})\int_0^\tau |A(s)|^2 e^{-\nu\Delta_{zj}(\tau-s)}\,ds \tag{51}$$

A similar problem exists in the 2-layer model where the viscosity parameter

$$r = \frac{(\text{Ekman number})^{\frac{1}{2}}}{(\text{Rossby number})}$$

is proportional to the square root of the supercriticality, which is now taken with respect to the internal Froude number, and leads to a form of amplitude equation which is similar to (32) and (36) - the differences being in the form of the coefficients and the aforementioned independence of the flow quantities with respect to z in each layer - first found by Pedlosky (1971). These equations match the inviscid and $r = O(1)$ viscous problems of the 2-layer model in the appropriate limits and both of these limiting forms of the amplitude equations have the same form as the analogous equations (44) and (46) for Eady's model but of course with different coefficients. Further, with the approximation fully stated, the sidewall boundary condition (30) which implies

$$\frac{\partial f}{\partial y}(y,z,\tau) = 0 \quad \text{on } Y = 0, 1 \tag{52}$$

was relaxed in Pedlosky 1971, so that a separable solution for the boundary value problem for $f(y,z,\tau)$ (or $\Phi_{1,2}(y,\tau)$ in the different notation) exists and in the present context of Eady's model this is

$$f(y,z,\tau) = B(\tau)\sin 2lY \,\mathrm{sh}\, 2lS_0^{\frac{1}{2}} z$$

where

$$\dot{B} + \frac{\nu\Lambda k l S_0^{\frac{1}{2}} h^2}{2s_0^2}\,\mathrm{th}\, lS_0^{\frac{1}{2}} h \,.\, B$$
$$= \frac{-2s_0^2}{\Lambda h^2(s_0^2-1)S_0^{\frac{1}{2}}}\,\mathrm{sech}\, lS_0^{\frac{1}{2}} h\left(\dot{(|A|^2)} + \nu\Lambda k h(|A|^2)\right) \tag{53}$$

and the resulting amplitude equation is (36) with (53) which may be cast in the form of (47) with a single equation of type (48) or as a set of three coupled first order differential equations.

Later, Smith (Smith (1977)) re-examined the amplitude equations of Pedlosky (1971) for the 2-layer model by solving the appropriate problem for the zonal correction by Fourier's method and derived an infinite set of differential equations with exactly the same form as equations (47) and (48) where the (different) coefficients are algebraic functions of j. In the same article it is shown that relaxation of the sidewall boundary condition $\bar{\phi}_{yT} = 0$ leads to a quantitatively different system of energy conversions with a non-zero energy flux at the sidewall boundaries.

Now, the change of variables defined by

$$A = \delta X \quad ; \quad T = \omega\tau \quad ; \quad B_j = \frac{\rho\omega}{\beta_j} Z_j$$

where

$$\frac{\nu\Delta_{zj}}{\omega} = q_j \quad , \quad \sigma = \frac{\nu\hat{\sigma}}{2\omega} \quad , \quad f_j = \beta_j \hat{f}_j$$

implies (48) may be written

$$\dot{Z}_j + q_j Z_j = \frac{f_j}{2}(XY^* + X^*Y)$$

where the variable Y is introduced so that

$$\dot{X} = -\sigma(X - Y)$$

and the defining relations

$$\delta^2 = \frac{\rho\omega}{2\sigma} \quad , \quad \sigma = \frac{\rho}{\omega} \quad , \quad \Rightarrow \delta^2 = \frac{\omega^2}{2}$$

are taken. Then (47) implies

$$\dot{Y} + Y = rX - X\sum_{j=0}^{\infty} Z_j$$

where the further relations

$$\rho = \nu\Delta - \omega \quad , \quad r = 1 + \frac{\alpha}{\rho\omega}$$

are taken. This transformation of the equations, first pointed out by Dr. J.D. Gibbon, shows that the amplitude equations (47) and (48) may be written equivalently as the infinite set of coupled first order equations

$$\dot{X} = -\sigma(X-Y) \tag{54}$$

$$\dot{Y} = rX - Y - X\sum_{j=0}^{\infty} Z_j \tag{55}$$

$$\dot{Z}_j = -b_j Z_j + \frac{f_j}{2}(XY^* + X^*Y) \qquad j = 0,1,\ldots \tag{56}$$

where in terms of the underlying physical parameters

$$\sigma = s_0^2 \;, \quad \rho = \frac{\nu\lambda\,\mathrm{th}\, l}{2} \;, \quad \omega = \frac{\nu\lambda\,\mathrm{th}\, l}{2 s_0^2} \;, \quad \delta = \frac{\nu\lambda\,\mathrm{th}\, l}{2^{3/2} s_0^2}$$

$$b_j = (j+\tfrac{1}{2})\pi s_0^{1/2} l \;\mathrm{th}\,(j+\tfrac{1}{2})\pi s_0^{1/2} l \;, \quad f_j = 4\ell^2 l^4 \left(\frac{1 - \dfrac{\mathrm{th}\,(j+\frac{1}{2})\pi s_0^{1/2} l}{(j+\frac{1}{2})\pi s_0^{1/2} l}}{\left((j+\frac{1}{2})^2\pi^2 - \ell^2\right)^2} \right)$$

$$r = 1 + \frac{s_0^2(s_0^2-1)}{2 S_0 \nu^2} \tag{57}$$

The same sequence of transformations applied to (47) with (53) transforms the simplified set of amplitude equations with the sidewall boundary condition (52) relaxed to the set

$$\dot{X} = -\sigma(X-Y) \tag{57}$$

$$\dot{Y} = rX - Y - XZ \tag{58}$$

$$\dot{Z} = -bZ + \frac{f}{2}(XY^* + X^*Y) \tag{59}$$

which is formally the same as the set (54) to (56) with a single Z component and in terms of the underlying physical parameters $\sigma, \rho, \omega, \delta$ and r are as defined in (57) but now

$$\delta = l S_0^{\frac{1}{2}} h \coth\left(l S_0^{\frac{1}{2}} h\right) \quad , \quad r = k^2 l^2 \left(1 - \frac{\coth\left(l S_0^{\frac{1}{2}} h\right)}{l S_0^{\frac{1}{2}} h} \right) \qquad (60)$$

Equations (57) to (59) are the now famous Lorenz set. Their relevance to fluid mechanical systems was found by Lorenz (1963) who applied the principle to Fourier expanding the convection equations of Saltzman (1962) which describe Bénard convection between free top and bottom boundaries, and then truncating the expansions at an arbitrary number of components, there taken to be three. The equivalence of the amplitude equations for the baroclinic models in the three dimensional form of (36) and (53) with the Lorenz set was first given by Gibbon and McGuinness (1980), the generalisation of the transformation to the amplitude equations (47) and (48) giving the infinite system of Lorenz type equations (54) to (56) first appeared in Brindley and Moroz 1980.

As was mentioned at the beginning of this chapter, the two layer model may be generalised to include the β-plane effect. The effect of this generalisation which corresponds physically to the inclusion of a dispersive mechanism for a wave like disturbance in the viscous problems appears in the mathematical problem for the amplitude equations through the linear terms in A of (38) alone and renders these coefficients complex. The resulting form of the coefficients for the small viscosity problem with the relaxed form of sidewall boundary conditions has been derived by Gibbon and McGuinness (1981) leading to a set of Lorenz equations after transformation with complex coefficients appearing in some of the linear terms. The full problem in this case, retaining the sidewall boundary conditions, has been performed by Pedlosky (1982) and after transformation leads to an infinite set of Lorenz equations with complex coefficients in the form

$$\dot{X} = -\sigma(X - Y) \qquad (61)$$

$$\dot{Y} = rX - aY - X\sum_{j=0}^{\infty} Z_j \tag{62}$$

$$\dot{Z}_j = -b_j Z_j + \frac{d_j}{2}\left(XY^* + X^*Y\right) \tag{63}$$

where $r = r_1 + ir_2$ and $a = 1 - ie$ with $r_1, r_2, e \in R$.

The coefficients expressed in terms of the underlying physical parameters are given in Appendix B for both the infinite and single Z component cases.

CHAPTER 5

LINEAR STABILITY ANALYSIS OF THE EQUILIBRIUM SOLUTIONS OF THE AMPLITUDE EQUATIONS OF CHAPTER FOUR AND THE EFFECT OF MODE TRUNCATION IN THE INFINITE CASE.

The content of this chapter appeared in part in a short paper entitled 'A study of the effect of mode truncation on an exact periodic solution of an infinite set of Lorenz equations' which was written jointly with Dr. J.D. Gibbon and Dr. A.C. Fowler. I have chosen to represent it in order to elucidate some steps in the algebra and when appropriate to pay more attention to the applicability of the results in the context of baroclinic flow but hope to recapture the spirit of the more general problem under enquiry.

The set of equations (4.61) to (4.63) comprise an infinite set of coupled, first order, nonlinear differential equations. On the face of it the complexity of such a system and the way in which the infinity appears as a repetition of the equation (4.63) renders it a suitable candidate for numerical integration, however the full set may be seen to have two types of exact equilibrium solutions and the motivation of our study was to consider what information concerning these could be extracted by analytical and asymptotic methods so that some check could be made on the validity of a numerical study. The particular point of concern is that any numerical model of the infinite set involves the usual approximations of discrete representation in the integration of the dependent variables but more specifically the replacement of the infinite set by a finite or truncated set. The method of truncating equation (4.63) after some value of j and including a finite number of modes in $\{Z_j\}$ can be open to question if the inclusion of higher modes produces a radically different behaviour in the bifurcation sequence.

For such an infinite set of equations in general only numerical integration of the higher dimensional sets of equations in which more modes have been included will determine whether convergence to a particular set of transitions has occurred. This type of integration has been performed for the specific case of (4.61) to (4.63) in the form of (4.47) with (4.48) where the coefficients are for the two layer model of baroclinic instability by Pedlosky and Frenzen (1981) when the β-plane effect is excluded and the coefficients are therefore real (an example of what will be referred to as the real set) and by Pedlosky (1981 preprint) when the β-plane effect is included to give complex coefficients (and an example of what will be referred to as the complex set). To summarise the findings in these two cases; a parameter setting was chosen and then on the results of initial numerical integration the number of modes which were included was adjusted until it was numerically apparent that convergence to a particular phase portrait had occurred before resetting the parameters; in most cases approximately twelve modes were reported to have been adequate and in some instances up to thirty modes were included; in the real case the phase portraits as the viscosity parameter is increased showed a sequence of successive period doublings of limit cycle solutions, which may be related to the periodic solutions of the inviscid problem, then a regime of chaotic solutions containing 'islands' where the solutions may equilibrate to a steady state and finally a regime where all solutions ultimately become steady; in the complex case it was found that as the β effect is increased in relation to the other parameters that the 'chaotic behaviour gives way, through a sequence of period halvings, to regular steady waves.'

The case of a single Z-component as in equations (4.57) to (4.59) has been studied much more extensively since Lorenz original paper (196), there he showed that the steady solution $X = Y = Z = 0$ is linearly unstable whenever $r > 1$, which is true of all of the baroclinic models considered so far when in the slightly unstable state ($\alpha > 0$). Further, detailed numerical evidence was given in the real case of a surface (the Lorenz attractor) in three dimensional phase space in the vicinity of the pair of non-zero fixed points of the system which is attracting when these fixed points are unstable, i.e.; when $\sigma > b+1$ for a value of

$$r > \frac{\sigma(\sigma+b+3)}{(\sigma-b-1)} \qquad (1)$$

It is interesting to note that whilst the Lorenz attractor is thought of as an example of a strange attractor, to the best of my knowledge it has not been shown either way to satisfy the definition given by Smale (1967) that it is of the type satisfying Axiom A, which may loosely be interpreted as implying that it is not easily encompassed by the existing mathematical theories. It is thought that this axiom may preclude the type of attractors needed in the modelling of turbulence, see Zeeman (1980). Fowler et al. (1981) studied the complex case for a single Z-component both analytically and numerically, in contrast to the previous situation the complex single Z-component equations do not show the same transition to chaos when $\sigma > b+1$ as $r_1 = \mathrm{Re}(r)$ is increased. Instead, the origin undergoes a supercritical Hopf bifurcation when

$$r_1 > r_{1c} = 1 + \frac{(e+r_2)(e-\sigma r_2)}{(\sigma+1)^2} \tag{2}$$

to a limit cycle which is intimately related to the non-zero fixed points of the real case. This limit cycle is stable for all values of $r_1 = \mathrm{Re}(r)$ when $\sigma < b+1$ but bifurcates (subcritically) to a 2-torus when $\sigma > b+1$ at some higher value of r_1 which it is possible but long-winded to evaluate and which recovers the real case, (1), as e and r_2 tend to zero, i.e.; in the limit in which the real version of the equations is recovered from the complex case. For intermediate values of e and r_2 no further bifurcations have been found numerically but in the limit that e and r_2 tend to zero the torus undergoes a sequence of period doublings until a regime where the solutions appear to be chaotic is reached. The overall conclusion of Fowler et al. was that complexification of the coefficients turns non-zero fixed points into limit cycles, limit cycles into tori and suppresses the regime corresponding to chaotic solutions into a very small region of parameter space.

Now return to the set (4.61) to (4.63) retaining general $\{\gamma_j\}$ and $\{b_j\}$.

$$\dot{X} = -\sigma(X-Y) \tag{3}$$

$$\dot{Y} = rX - aY - X\sum_{j=1}^{\infty} Z_j \tag{4}$$

$$\dot{Z}_j = -b_j Z_j + \frac{\gamma_j}{2}(XY^* + X^*Y) \tag{5}$$

where $r = r_1 + ir_2$, $a = 1 - ie$; σ, γ_j, $b_j > 0$ $(j = 1, \infty)$; e, r_1, $r_2 \in \mathbb{C}$; $X, Y \in \mathbb{C}$; $Z_j \in \mathbb{R}$ without loss of generality. These equations have the fixed point solution $X = Y = Z_j = 0$. Unless

$$e + r_2 = 0 \tag{6}$$

there are no other fixed points. Notice that (6) is the condition under which the complex equations with e and r_2 non-zero may be transformed by a phase rotation into the equations in real form, without loss of generality it is therefore possible to assume that whenever e or r_2 is non-zero that $e + r_2 \neq 0$. A study of the linear stability of the origin shows that the stability matrix occurs in block diagonal form and the characteristic equation is

$$\prod_{j=1}^{\infty} (\lambda + b_j)\Big((\lambda+\sigma)(\lambda+a) - \sigma r\Big) = 0 \tag{7}$$

with roots

$$\lambda = -b_j \tag{8}$$

$$\lambda = \frac{1}{2}\left\{-(\sigma+a) \pm \left[(\sigma+a)^2 + 4\sigma(r-a)\right]^{\frac{1}{2}}\right\} \tag{9}$$

The value of r_1, called r_{1c}, at which the origin becomes unstable is found by setting $\mathrm{Re}(\lambda) = 0$ which from (9) implies

$$r_{1c} = 1 + \frac{(e+r_2)(e-\sigma r_2)}{(\sigma+1)^2} \tag{10}$$

which is identical to (2) for the single component complex case and now

$$\omega = \mathrm{Im}(\lambda)\Big|_{r_1 = r_{1c}} = \frac{\sigma(e+r_2)}{(\sigma+1)} \tag{11}$$

Hence in the real case when $e = r_2 = 0$ or when $e + r_2 = 0$ then $\omega = 0$ and $r_{1c} = 1$ which is the result found by Lorenz for his real equations. Since in the real case $\omega = \mathrm{Im}(\lambda) = 0$ the bifurcation is certainly not of the Hopf type. However in the complex case, with the inclusion of any number of Z components, there exists the exact limit cycle solution

$$X = A e^{i\omega t}\ ,\ Y = \sigma^{-1}(\sigma + i\omega) A e^{i\omega t} \tag{12}$$

$$Z_j = \frac{\gamma_j}{b_j} . |A|^2$$

where

$$|A|^2 = (r_1 - r_{1c}) \left(\sum_{j=1}^{\infty} \frac{\gamma_j}{b_j} \right)^{-1} \tag{13}$$

is the 'amplitude' of the limit cycle, a function of r_1, $\{\gamma_j\}$ and $\{b_j\}$, and ω and r_{1c} are given exactly by the equations (10) and (11) of linear theory. Notice first of all that this is only a valid solution if the sum in (13) converges, so that if $b_j \sim j^B$ and $\gamma_j \sim j^G$ as $j \to \infty$ then the requirement $G < B - 1$ must be satisfied. Secondly, the transition from the origin to the limit cycle solution (when it exists) is a supercritical Hopf bifurcation as (13) shows since

$$|A| \propto (r_1 - r_{1c})^{\frac{1}{2}} \tag{14}$$

Thirdly, when $e = r_2 = 0$ or $e + r_2 = 0$ then $\omega = 0$, $r_{1c} = 1$ and the solution (12) becomes a continuum of fixed points replacing the two fixed points of the set of real equations studied by Lorenz because the complex nature of X and Y, even when the coefficients are real, allows these two points to be rotated to form a closed curve of fixed points. In this case the solution may usefully be thought of as a limit cycle of zero frequency.

In the general complex case, $\omega \neq 0$ and the fortunate occurrence of the exact limit cycle solution (12) with (13) enables the investigation of its stability by transformation to a reference frame rotating with the angular frequency ω, i.e.; put

$$X = x_R e^{i\omega t}\ ,\ Y = y_R e^{i\omega t}\ ,\ Z_j = z_{R_j}\ (j = 1, \ldots) \tag{15}$$

The equations in the transformed frame are

$$\dot{x}_R = -(\sigma + i\omega) x_R + \sigma y_R \tag{16}$$

$$\dot{y}_R = r x_R - (a + i\omega) y_R - x_R \sum_{j=1}^{\infty} z_{R_j} \tag{17}$$

$$\dot{z}_{R_j} = -b_j z_{R_j} + \frac{\gamma_j}{2} (x_R^* y_R + x_R y_R^*) \tag{18}$$

Apart from the origin, which is unstable when $r_1 > r_{1c}$, the fixed points of (16) to (18) are at

$$x_R = A \;, \quad y_R = \left(1 + \frac{i\omega}{\sigma}\right) A \;, \quad z_{Rj} = \frac{d_j}{b_j} |A|^2 \tag{19}$$

where A is the amplitude of equation (13) which may now be assumed to be real since a change in phase of a complex value of A is equivalent to writing

$$\omega t \rightarrow \omega t + \text{constant}$$

in the exponential terms of equations (15) which leaves the set (16) to (18) unchanged, thereby establishing the phase invariance of the complex set (3) to (5) under

$$X, Y \longrightarrow X e^{i\phi}, Y e^{i\phi}$$

Perturbing about this solution the characteristic equation for its linear stability is

$$\mathrm{Det}\left(\begin{array}{cccc|ccc}
-\sigma N-\lambda & \sigma & 0 & 0 & 0 & \cdots & 0 \cdots \\
\rho & -L-\lambda & 0 & 0 & -A & \cdots & -A \cdots \\
0 & 0 & -\sigma N^*-\lambda & \sigma & 0 & \cdots & 0 \cdots \\
0 & 0 & \rho^* & -L^*-\lambda & -A & \cdots & -A \cdots \\
\hline
\frac{d_1 A N^*}{2} & \frac{d_1 A}{2} & \frac{d_1 A N}{2} & \frac{d_1 A}{2} & -b_1-\lambda & & 0 \\
\vdots & \vdots & \vdots & \vdots & & \ddots & \\
\frac{d_j A N^*}{2} & \frac{d_j A}{2} & \frac{d_j A N}{2} & \frac{d_j A}{2} & 0 & -b_j-\lambda & \\
\vdots & \vdots & \vdots & \vdots & & & \ddots
\end{array}\right) = 0 \tag{20}$$

where

$$L \equiv a + i\omega, \quad N \equiv 1 + \frac{i\omega}{\sigma}, \quad P \equiv r - A^2\left(\sum_{j=1}^{\infty} \gamma_j b_j^{-1}\right) = r_1 + i r_2 \tag{21}$$

and λ is the (complex) eigenvalue and from (10) and (11)

$$LN = P \tag{22}$$

The determinant (20) may be evaluated by subtracting the second column from the fourth, subtracting the conjugate of the first column from the third and using induction to give

$$0 = \lambda \left\{ \lambda\left[(\lambda+\sigma+1)^2 + q^2 \right] + A^2(\lambda+2\sigma)(\lambda+\sigma+1)\sum_{j=1}^{\infty} \frac{\gamma_j}{b_j+\lambda} \right\}. \tag{23}$$

where

$$q \equiv 2\omega - e \tag{24}$$

provided the sum converges, notice in passing the similarity of this sum to a sum of Laplace transforms, viz.

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{\gamma_j}{b_j+\lambda} &= \int_0^{\infty} \sum_{j=1}^{\infty} \gamma_j e^{-(b_j+\lambda)t}\, dt \\ &= \mathcal{L}\left\{ \sum_{j=1}^{\infty} \gamma_j e^{-b_j t} \right\} \end{aligned} \tag{25}$$

a point which will be returned to later. Three facts may be deduced immediately from (23). Firstly the $\lambda = 0$ root derives from the phase invariance of the equations which gives rise to the limit cycle. Secondly, no positive real roots in λ exist since each term in (23) is positive definite when $\lambda > 0$. Thirdly no further zero roots occur since $\Sigma \frac{\gamma_j}{b_j}$ is finite and positive. Hence, if the limit cycle is to become unstable it must be through a complex conjugate pair of roots crossing the imaginary axis from left to right.

Taking real and imaginary parts of (23) after putting $\lambda = \pm i\Omega$ where $\Omega \in \mathbb{R}$ implies that

$$\text{i)}.\ A^2.R = P \tag{28}$$

$$\text{i)}.\ A^2.S = Q \tag{29}$$

where

$$R = \sum_{j=1}^{\infty} \frac{f_j b_j}{b_j^2 + \Omega^2} \quad , \quad S = \sum_{j=1}^{\infty} \frac{f_j}{b_j^2 + \Omega^2} \tag{30}$$

and

$$D = \left(2\sigma(\sigma+1) - \Omega^2\right)^2 + \Omega^2(3\sigma+1)^2$$

$$P = -\Omega^2(3\sigma+1)\left[(\sigma+1)^2 + q_h^2 - \Omega^2\right] - 2\Omega^2(\sigma+1)\left(\Omega^2 - 2\sigma(\sigma+1)\right) \tag{31}$$

$$Q = 2\Omega^2(\sigma+1)(3\sigma+1) + \left[2\sigma(\sigma+1) - \Omega^2\right]\left[(\sigma+1)^2 + q_h^2 - \Omega^2\right]$$

or alternatively

$$P = \Omega^2\left[(\sigma-1)\left((\sigma+1)^2 + \Omega^2\right) - (3\sigma+1)q_h^2\right] \quad . \tag{32}$$

$$Q = \Omega^4 + \left[(3\sigma+1)(\sigma+1) - q_h^2\right]\Omega^2 + 2\sigma(\sigma+1)\left[(\sigma+1)^2 + q_h^2\right]$$

Now A^2, Ω^2, R, S and D are always positive, hence for a simultaneous solution of (28) and (29) to exist P and Q are necessarily positive. From the equation (32) for P this condition is not satisfied if $\sigma \leq 1$, then no roots $\Omega^2 > 0$ of (28) and (29) exist and the limit cycle is always stable for $r_1 > r_{1c}$. When $\sigma > 1$, which is true of the 2-layer and Eady models where σ is fixed and equal to 2 and $s_0^2 \approx 1.4$ respectively, the condition that P must be positive puts certain bounds on Ω. There are two cases to consider, when

$$0 < q_h^2 < \frac{(\sigma+1)^2(\sigma-1)}{(3\sigma+1)} \tag{3}$$

there are no bounds on Ω^2, but when

$$q_h^2 > \frac{(\sigma+1)^2(\sigma-1)}{(3\sigma+1)} > 0 \tag{34}$$

then any root of (28) and (29) must satisfy

$$\Omega^2 > q_h^2\frac{(3\sigma+1)}{(\sigma-1)} - (\sigma+1)^2 \tag{35}$$

The condition that Q must be positive yields no further information.

Consider now solving (28) and (29) simultaneously for Ω^2 and A^2. Since A^2 is proportional to r_1 from equation (13) consider the solutions for Ω^2 in terms of large or small A^2 with these limits being equivalent to large and small values of r_1. In the context of Benard convection r_1 is the Prandtl number and here from equation (4.57) r_1 increases as the viscosity parameter decreases with the other parameters fixed. Without resorting to the specific forms of $\{\gamma_j\}$ and $\{b_j\}$ the series R and S cannot be summed but formally consider solutions in Ω^2 of each equation separately as functions of A^2 calling them

$$\Omega_R{}^2(A^2) \quad \text{and} \quad \Omega_S{}^2(A^2) \quad . \tag{36}$$

For there to exist an instability for the limit cycle there must be a coincident solution of both equations, i.e.; there exists one or more values of A^2 such that $\Omega_R{}^2 = \Omega_S{}^2$. It may be checked that for all values of $q, \Omega_R{}^2(0) > \Omega_S{}^2(0)$ and so a sufficient condition for instability is that

$$\Omega_S{}^2(A^2) > \Omega_R{}^2(A^2) \qquad \text{as } A^2 \to \infty \tag{37}$$

and if this condition is satisfied then the curves of (36) must intersect an odd number of times. From (28) to (32) $A^2 \to \infty$ as $\Omega^2 \to \infty$ and in this limit

$$D = \Omega^4 + O(\Omega^2)$$

$$P = (\sigma-1)\Omega^4 + O(\Omega^2)$$

$$Q = \Omega^4 + O(\Omega^2)$$

and so equations (28) and (29) imply that in this limit

$$\begin{aligned} A^2 . R\left(\Omega_R^2(A^2)\right) &\sim \sigma - 1 \\ A^2 . S\left(\Omega_S^2(A^2)\right) &\sim 1 \end{aligned} \tag{38}$$

Any sufficient condition for instability will ultimately rest upon how the sums R and S behave in the limit $\Omega^2 \to \infty$ but first note that any finite truncation of the sums, assuming that they are then partial sums including the N^{th} term, gives

$$\left.\begin{aligned} R &= \frac{1}{\Omega^2}\sum_1^N \gamma_j b_j + o(\Omega^{-2}) \\ S &= \frac{1}{\Omega^2}\sum_1^N \gamma_j + o(\Omega^{-2}) \end{aligned}\right\} \text{ as } \Omega^2 \to \infty \tag{39}$$

and then the sufficient condition for instability (37) is

$$\sigma > \frac{\sum_1^N \gamma_j b_j}{\sum_1^N \gamma_j} + 1 \tag{40}$$

When a single Z component is taken in (3) to (5) then (40) reduces to $\sigma > b+1$, a result already obtained by Lorenz for his équations.

For the infinite set of equations it is necessary to deduce how the sums R and S behave as functions of Ω. In any finite truncation R and S behave like Ω^{-2} for large Ω and thus for large r_1 but may not necessarily behave like this for an infinite number of modes. In particular if R and S do not have the same leading order asymptotic behaviour for large Ω in the infinite case then the stability criterion (40) may be radically changed if it exists at all. In order to study the asymptotic behaviour of R and S for an infinite number of modes two cases which separately include the cases of the 2-layer and Eady models are considered to illustrate this point.

(i) b_j bounded as $j \to \infty$. If $\{\gamma_j\}$ and $\{b_j\}$ are such that $\Sigma\gamma_j b_j$ and $\Sigma\gamma_j$ converge, suppose that $|b_j| < B$ then

$$\frac{1}{\Omega^2} > \frac{1}{b_j^2+\Omega^2} > \frac{1}{\Omega^2}\left(1+\frac{B^2}{\Omega^2}\right)^{-1} > 0 \tag{41}$$

and since γ_j, $b_j > 0$ for all j

$$\infty > \sum_1^\infty \gamma_j b_j > \Omega^2 R = \sum_1^\infty \frac{\Omega^2}{b_j^2+\Omega^2}\cdot\gamma_j b_j > \left(1+\frac{B^2}{\Omega^2}\right)^{-1}\sum_1^\infty \gamma_j b_j > 0 \tag{42}$$

implying

$$R = \frac{\sum_1^{\infty} f_j g_j}{\Omega^2} + o(\Omega^{-2}) \qquad \text{as } \Omega \to \infty \tag{43}$$

and similarly

$$S = \frac{\sum_1^{\infty} f_j}{\Omega^2} + o(\Omega^{-2}) \qquad \text{as } \Omega \to \infty \tag{44}$$

so that the stability criterion (37) becomes (40) with the partial sums replaced by the convergent infinite sums. Hence, for this case, there is no qualitative difference between truncation at large N and the infinite system a propos the stability of the limit cycle.

The coefficients for the two layer model are in this present category:

$$f_j(a_\omega, m) = 2k^2 m^4 \pi^4 \frac{(j+\frac{1}{2})^2 \pi^2}{\left((j+\frac{1}{2})^2\pi^2 - m^2\pi^2\right)^2 \left((j+\frac{1}{2})^2\pi^2 + \frac{a_\omega^2}{4}\right)} \tag{45}$$

$$g_j(a_\omega) = 2\frac{(j+\frac{1}{2})^2\pi^2}{(j+\frac{1}{2})^2\pi^2 + \frac{a_\omega^2}{4}} \qquad \text{where } j = 0, 1, \ldots \tag{46}$$

and it is possible to evaluate the appropriate sums R and S by manipulation of Fourier series, a calculation which was first performed by Smith (1977), i.e.; put

$$i\Omega = \frac{2\mu^2}{1-\mu^2} \quad \text{where } \mu \in \mathbb{C} \tag{47}$$

$$R - i\Omega S = \sum_{j=1}^{\infty} \frac{f_j}{g_j + i\Omega} = \left(\frac{1-\mu^2}{2}\right) \cdot 2k^2 m^4 \pi \sum_{j=1}^{\infty} \frac{(j-\frac{1}{2})^2}{\left((j-\frac{1}{2})^2 - m^2\right)^2 \left((j-\frac{1}{2})^2\pi^2 + \frac{a_\omega^2 \mu^2}{4}\right)}$$

$$= \left(\frac{1-\mu^2}{2}\right) \cdot \sum_{j=1}^{\infty} f_j(a_\omega \mu, m) \tag{48}$$

$$= \frac{(1-\mu^2)\, k^2 m^4 \pi^4}{\left((a_\omega \mu)^2 + 4m^2\pi^2\right)} \left\{ 1 - \frac{4 a_\omega \mu}{\left((a_\omega \mu)^2 + 4m^2\pi^2\right)} \tanh \frac{a_\omega \mu}{2} \right\}$$

where the locus of μ in the complex μ plane as Ω varies on $[0,\infty]$ is given by

$$\left(\mu_r^2+\mu_i^2\right)^2 = \mu_r^2-\mu_i^2$$

and, explicitly, on inverting (47)

$$\mu_r = \frac{\Omega^{\frac{1}{2}}}{2(4+\Omega^2)^{\frac{1}{2}}}\left\{\left((4+\Omega^2)^{\frac{1}{2}}+2\right)^{\frac{1}{2}}+\left((4+\Omega^2)^{\frac{1}{2}}-2\right)^{\frac{1}{2}}\right\} \tag{49}$$

$$\mu_i = \frac{\Omega^{\frac{1}{2}}}{2(4+\Omega^2)^{\frac{1}{2}}}\left\{\left((4+\Omega^2)^{\frac{1}{2}}+2\right)^{\frac{1}{2}}-\left((4+\Omega^2)^{\frac{1}{2}}-2\right)^{\frac{1}{2}}\right\}$$

Smith's calculations were appropriate to the real case when $e = r_2 = \omega = 0$ and the limit cycle is a continuum of fixed points. In passing notice that in a truncated study of this case

$$R \sim \frac{1}{\Omega^2}\sum_{j=1}^{N} d_j b_j + o(\Omega^{-2}) \qquad \text{as } \Omega\to\infty$$

and $\sum_{1}^{N} \gamma_j b_j = O(a^{-4})$ as $a \to \infty$, but in the infinite case

$$R \sim \frac{1}{\Omega^2}\sum_{j=1}^{\infty} d_j b_j + o(\Omega^{-2}) \qquad \text{as } \Omega\to\infty$$

and now $\sum_{1}^{\infty} \gamma_j b_j = O(a^{-3})$ as $a \to \infty$, on manipulation of some Fourier* series. Thus writing

$$R_N = \sum_{j=1}^{N} \frac{d_j b_j}{b_j^2+\Omega^2}$$

then (50)

$$\lim_{N\to\infty}\ \lim_{a\to\infty}\ \lim_{\Omega\to\infty}(R_N) \neq \lim_{a\to\infty}\ \lim_{\Omega\to\infty}\ \lim_{N\to\infty}(R_N)$$

(ii) b_j unbounded with $b_j \sim c.j$ and c constant. In order for the limit cycle to exist $\sum^{\infty}\gamma_j b_j^{-1}$ must converge and thus necessarily $\gamma_j \sim j^{-p}$ with $p > 0$ as $j \to \infty$. Any finite truncation of the sums R and S again behaves like Ω^{-2} for large Ω and an integral estimate may be applied to the tail of each series in the infinite case. For given (small) $a_1, a_2, a_3, a_4 > 0$, there exists a value of N which is independent of Ω such that for all $j > N$

*See Appendix D.

$$1-\frac{a_1}{j} < \frac{b_j}{c\,j} < 1+\frac{a_2}{j} \qquad (51)$$

$$1-\frac{a_3}{j} < \frac{d_j}{g\,j^{-\rho}} < 1+\frac{a_4}{j}$$

Each sum is formed from a series of monotonic decreasing positive terms and so, e.g.

$$\int_{N+1}^{\infty} \frac{d_j b_j}{b_j^2+\Omega^2}\,dj < \sum_{N+1}^{\infty} \frac{d_j b_j}{b_j^2+\Omega^2} < \int_{N}^{\infty} \frac{d_j b_j}{b_j^2+\Omega^2}\,dj \qquad (52)$$

where N is sufficiently large that the integrand is an integrable function of j when j is a continuous variable and then the integrals may be estimated using (51) and

$$\int_{N+1}^{\infty} \frac{g c j^{-\rho+1}(1-\frac{a_1}{j})(1-\frac{a_3}{j})}{c^2 j^2(1+\frac{a_2}{j})^2+\Omega^2}\,dj < \sum_{N+1}^{\infty} \frac{d_j b_j}{b_j^2+\Omega^2} < \int_{N}^{\infty} \frac{g c j^{-\rho+1}(1+\frac{a_2}{j})(1+\frac{a_4}{j})}{c^2 j^2(1-\frac{a_1}{j})^2+\Omega^2}\,dj \qquad (53)$$

Putting $j_+ = j-a_1$, $j_- = j+a_2$ and then relabelling the new variables as j, there exists $a_5, a_6 > 0$ such that these integrals may be approximated

$$\int_{N+1+a_2}^{\infty} \frac{g c j^{-\rho+1}}{c^2 j^2+\Omega^2}\left(1-\frac{a_5}{j}\right)dj < \sum_{N+1}^{\infty} \frac{d_j b_j}{b_j^2+\Omega^2} < \int_{N-a_1}^{\infty} \frac{g c j^{-\rho+1}}{c^2 j^2+\Omega^2}\left(1+\frac{a_6}{j}\right)dj \qquad (54)$$

Now setting $z = j^{-1}$ in $I_N(a,\Omega) \equiv \displaystyle\int_N^{\infty} \frac{dj}{j^a(c^2 j^2+\Omega^2)}$ (55)

there follows the reduction formula

$$I_N(a,\Omega) = \frac{1}{\Omega^2(a-1)N^{a-1}} - \frac{c^2}{\Omega^2}\, I_N(a-2,\Omega) \qquad (56)$$

for $a > 1$

and

$$I_N(1,\Omega) = \frac{\ln\Omega}{\Omega^2}\left(1+o(\Omega)\right);\ I_N(0,\Omega) = \frac{\pi}{2c\Omega}\left(1+o(\Omega)\right) \qquad (57)$$

as $\Omega \to \infty$

Hence, if $p > 2$ then $R,S = O(\Omega^{-2})$ as $\Omega \to \infty$ and so again truncation does not alter the stability criterion which is (40) with the partial sums replaced by the (convergent) infinite sums. The coefficients of the Eady model given by equations (4.49) fall into this category since $p = 4$ and truncation is thus established as a procedure which will not materially affect the stability criterion.

If $0 < p \leq 2$ however then the partial sums of R and S do not converge uniformly with respect to Ω and for example when:

$$p = 2; \quad R = O\left(\frac{\ln \Omega}{\Omega^2}\right) \text{ and } S = O(\Omega^{-2}) \text{ as } \Omega \to \infty$$

$$\text{and} \tag{58}$$

$$p = 1; \quad R = O(\Omega^{-1}) \text{ and } S = O\left(\frac{\ln \Omega}{\Omega^2}\right) \text{ as } \Omega \to \infty$$

In both of these cases, using (38), for fixed $\sigma > 1$

$$\Omega_R^{\,2}(A^2) > \Omega_S^{\,2}(A^2) \qquad \text{As } A^2 \to \infty$$

and so no instability may occur as A^2 and thus r_1 tends to infinity in contrast with the finite or truncated case for which the criterion (40) always exists. Notice that for $0 < p \leq 2$ we should expect the criterion to alter since either or both of the partial sums of (40) cease to converge.

There still remains the possibility mentioned previously of multiple intersections of the $\Omega_R^{\,2}$ and $\Omega_S^{\,2}$ curves of (36). For a final instability (stability) of the limit cycle solution there would be an odd (even) number of intersections of the curves which would produce intervals or windows of instability along the r_1 axis. Some numerical checking with the coefficients of both the 2-layer and Eady models has shown no evidence of this phenomenon, there is always a single intersection of the curves for some values of r_1 when the exact criterion ((40) with $N = \infty$) is used (Although the sums R and S for the coefficients of the Eady model do not seem to be summable in closed form.) and this result is in agreement with the study by Smith (1977) for the linear stability of the non-zero fixed points in the case of real coefficients with real variables for the 2-layer model. This suggests that there is the familiar 1-1 correspondence between the bifurcation parameter r_1 and the real part of the eigenvalue λ associated with instability.

As stated our conclusions confirm by analytical methods that mode truncation is a qualitatively valid procedure in the neighbourhood of the instability of the limit cycle or continuum of fixed points for the baroclinic models. Manley and Trève (1981) and Trève (1981) applying a method of Foias and Prodi (1967) have shown that a lower bound on the number of modes in a Fourier expansion is needed in Bénard convection in order to correctly predict the critical Rayleigh number for the onset of convection in a quantitative study. We are unable to predict anything about the behaviour at other bifurcation points such as possible transitions from periodic or quasi-periodic solutions to chaos because our analysis is local - studying the behaviour of the trajectories in phase space in the vicinity of the continuum of fixed points (in the real case) or the limit cycle (in the complex case) when the viscosity parameter (μ or r in Pedlosky's notation for the 2-layer model) is of the same order as the square root of the super-criticality (ε or Δ) as this solution becomes unstable with the steady decrease of the viscosity parameter. Now this equilibrium solution is the small viscosity limit of the steady solutions of the amplitude equation formed when the viscosity parameter is order one with respect to the supercriticality ($\mu = O(1)$), this may be seen by inspection of (4.47) and (4.48) or equivalently (4.36) and (4.38) then comparison with (4.44) but recall that (4.44) is the inviscid limit of the $\mu = O(1)$ viscous problem for which the full coefficients in the required form are equations (34) and (35) of Drazin 1972 . The position of the equilibrium solution in the small viscosity $\mu = O(\varepsilon^{\frac{1}{2}})$ problem in phase space is independent of the value of the viscosity parameter ν, this follows by inspection of (4.47) and (4.48) and then (4.49), hence this solution exists for all ν (and in fact $\mu \geq O(\varepsilon^{\frac{1}{2}})$, with continuous deformation) but is not related to the centre of the dnoidal periodic solutions of the inviscid problem given by (4.46) as the latter are explicitly dependent on the initial condition $A(0)$. In the $\mu = O(1)$ viscous problems, since the coefficients of the amplitude equation in the baroclinic models are real and the equation is first order the amplitude function may be taken to be real without loss of generality and its steady, fixed point solutions are globally asymptotically stable.

The inviscid limit of the small viscosity $\mu = O(\varepsilon^{\frac{1}{2}})$ problem which is clearly recoverable from the equations (4.47) and (4.48) by letting ν tend to zero to form (4.46) is not recoverable as easily after the transformation leading to (4.54) to (4.56) is applied, see the explicit forms of the parameters which are connected through the transformation by (4.57). A different approach to the problem of examining the transitions before chaos has been taken by Pedlosky (1971 and 1972) and Smith (1977) although at that time the equivalence of the amplitude equations with a set of Lorenz type equations had not been established. The inviscid amplitude equation (4.46) has solutions which are all periodic and may be expressed in terms of Jacobian elliptic functions. When the sidewall boundary condition is relaxed so that the amplitude equations are (B10) to (B13) for the $r = O(\Delta^{\frac{1}{2}})$ small viscosity problem Pedlosky (1972) has considered the existence and stability of the limit cycle solutions for the 2-layer model with $\frac{r}{|\Delta|^{\frac{1}{2}}}$ is an asymptotically small parameter. Smith later re-examined the infinite set of equations (B1) to (B3) for the 2-layer model using the same technique and gave some numerical results in Smith and Reilly (1977). Both of these analyses are applicable when the amplitude function $A(\tau)$ is assumed real and the results are in qualitative agreement. The inviscid solutions are of two types which are referred to as the $\mathcal{E} < 0$ and $\mathcal{E} > 0$ periodic solutions when the wave amplitude is assumed to be real, the $\mathcal{E} < 0$ solutions correspond to a solution of (4.46) which is a dnoidal elliptic function and for which $A(\tau)$ is always one-signed whilst the $\mathcal{E} > 0$ solutions correspond to a solution which is a cnoidal elliptic function for which $A(\tau)$ changes sign twice during a period, the two types are separated by a single $\mathcal{E} = 0$ solution of the form $A = A_0 \operatorname{sech}(\omega_0\tau)$ which is non-periodic and where A_0 and ω_0 depend on the physical parameters (k, a_w, m). The two-timing technique applied to the $r = O(\Delta^{\frac{1}{2}})$ small viscosity equations when $\frac{r}{|\Delta|^{\frac{1}{2}}}$ is small shows that there are limit cycle solutions which arise from both types of inviscid periodic solution, the $\mathcal{E} < 0$ limit cycle exists only when the fixed point solutions are stable and it is then unstable, the $\mathcal{E} > 0$ limit cycle always exists and is stable. If $A(\tau)$ is complex then, in general, the changing of sign in the $\mathcal{E} > 0$ inviscid solution does not occur, a note to this effect appears in Appendix C. The sequence of different limit cycles found in the numerical investigation of Pedlosky and Frenzen (1980) as a result of period doublings with the increase of the viscosity parameter where the amplitude function is assumed to be real are thought to be related to the fundamental limit cycle solutions arising from the inviscid problem. A similar concept of reversible (inviscid) solutions pertaining to the irregular loopings of the trajectories about the fixed points in the case of the real Lorenz set is exploited in a short paper by Haken and Wunderlin (1977).

CHAPTER 6

WEAKLY NONLINEAR STABILITY ANALYSIS FOR THE EXACT PERIODIC LIMIT CYCLE SOLUTION OF THE COMPLEX LORENZ EQUATIONS

In pursuance of the linear stability calculation for the limit cycle solution of the infinite set of complex Lorenz type equations (5.3) to (5.5) of the previous chapter

$$\dot{X} = -\sigma(X - Y) \qquad (1)$$

$$r = r_1 + ir_2$$

$$\dot{Y} = rX - aY - X\sum_{j=1}^{\infty} Z_j \qquad a = 1 - ie \qquad (2)$$

$$\dot{Z}_j = -b_j Z_j + \frac{\gamma_j}{2}(XY^* + X^*Y) \qquad j = 1, 2, \ldots \qquad (3)$$

consider the weakly nonlinear analysis and its consequent determination of the direction of the bifurcation. The abstract theory for forming the differential equations governing the local state of bifurcation of a limit cycle solution of a set of first order differential equations which are rotationally invariant and where the limit cycle is described by an exactly sinusoidal oscillation has been given fully in §3 of Fowler, Gibbon and McGuinness (1981). The aim of the present study is to perform the calculation outlined in that reference for equations (1) to (3), i.e.; for the case of an infinite set of Z components where the coefficients $\{\gamma_j\}$ and $\{b_j\}$ have the necessary convergence properties. Note that at every stage of the calculation the finite or truncated set of equations formed by truncating (3) after some value of $j \geq 1$ and for which no convergence criteria need be imposed may be recovered by the obvious truncation in each step of the calculation as was the case for the linear stability calculation of chapter 5.

In the case of the 2-layer model of baroclinic instability (for which the appropriate equations (1) to (3) are given by (B6) to (B9) of appendix B) where the infinite sums involving the coefficients $\{\gamma_j\}$ and $\{b_j\}$ may be expressed in closed form in terms of the underlying physical parameters the type of bifurcation may be compared for a given parameter setting with and without the relaxed form of sidewall boundary condition.

First of all, the set (1) to (3) may be transformed to a frame rotating with the frequency, ω, of the limit cycle by setting

$$X = x_R e^{i\omega T}, \quad Y = y_R e^{i\omega T}, \quad Z_j = z_{Rj} \quad j = 1, \dots \tag{4}$$

and then rescaling by the limit cycle amplitude A of equation (4.13) by setting

$$x_R = Ax, \quad y_R = Ay, \quad z_{Rj} = |A|^2 z_j \quad j = 1, \dots \tag{5}$$

In the new, rotating frame the equations become

$$\begin{aligned}
\dot{x} &= -(\sigma + i\omega)x + \sigma y \\
\dot{x}^* &= -(\sigma - i\omega)x^* + \sigma y^* \\
\dot{y} &= rx - (a + i\omega)y - x|A|^2 \sum_1^\infty z_j \\
\dot{y}^* &= r^* x^* - (a^* - i\omega)y^* - x^*|A|^2 \sum_1^\infty z_j \\
\dot{z}_j &= -b_j z_j + \frac{d_j}{2} (xy^* + x^* y)
\end{aligned} \tag{6}$$

where * denotes complex conjugate and

$$a = 1 - ie, \quad \omega = \frac{\sigma(e + r_2)}{(\sigma + 1)}, \quad r = r_{1c} + |A|^2 \sum_1^\infty \frac{d_j}{b_j} \tag{7}$$

so that ω is the frequency of the limit cycle given by equation (5.11). For convenience set T = t. If (6) is re-written formally as

$$\dot{\underline{x}} = \underline{f}(\underline{x}; \mu) \tag{8}$$

where μ denotes a bifurcation parameter then rotational symmetry is established by noting that

$$\underline{\underline{R}} \cdot \underline{f}(\underline{x}; \mu) = \underline{f}(\underline{\underline{R}} \cdot \underline{x}; \mu) \tag{9}$$

when the rotation matrix $\underline{\underline{R}}$ is

$$\underline{\underline{R}}(\theta) = \begin{pmatrix} e^{i\theta} & & & & & & \\ & e^{-i\theta} & & & & 0 & \\ & & e^{i\theta} & & & & \\ & & & e^{-i\theta} & & & \\ & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \\ & & & & & & & \ddots \end{pmatrix} \tag{10}$$

Recall that there may exist a value of r_1, r_{1c}' say, with $r_{1c}' > r_{1c}$ for which the limit cycle may be linearly unstable. In the new frame the limit cycle corresponds to the fixed point

$$\underline{x}_0 = \left(1, 1, \nu, \nu^*, \frac{\gamma_1}{\delta_1}, \ldots \right) \tag{11}$$

and if the supercriticality is defined by

$$r_1 = r_{1c}' \pm \epsilon^2 \qquad\qquad (\epsilon \ll 1) \tag{12}$$

implying

$$|A|^2 = \frac{r_{1c}' - r_{1c} \pm \epsilon^2}{\sum_1^\infty \frac{\gamma_j}{\delta_j}} \tag{13}$$

then the slow time scale of evolution of perturbations to the limit cycle is

$$\tau = \epsilon^2 t \tag{14}$$

and an expansion in terms of ϵ for the position vector of the solution in the new frame may be sought in the form

$$\underline{x} = \underline{x}^{(0)} + \epsilon \underline{x}^{(1)} + \epsilon^2 \underline{x}^{(2)} + \ldots . \tag{15}$$

Substitution of the weakly perturbed state in (6) now yields the familiar sequence of linear problems at successive powers of the expansion parameter ϵ.

At first order, O(1), the equations have the continuum of fixed point solutions corresponding to the underlying limit cycle solution

$$\underline{x}^{(0)} = \underline{\underline{R}}(\phi(\tau))\,\underline{x}_0 \tag{16}$$

where the phase function, ϕ, expresses the permissible slow time scale drift of a trajectory along the direction of the limit cycle in the slightly supercritical state.,

At $O(\varepsilon)$:

The equation for $\underline{x}^{(1)}$ may be written in the notation of (8) as

$$\left(\partial_t - \underline{\underline{Df}}\right)\underline{x}^{(1)} = \underline{0} \tag{17}$$

where

$$Df_{ij} = f_{i,j}\left(\underline{\underline{R}}(\phi).\underline{x}_0\,;\mu_c\right) = \frac{\partial f_i}{\partial x_j}(\underline{x},\mu)\Bigg|_{\substack{\underline{x}=\underline{x}^{(0)}\\ \mu=\mu_c}}$$

and explicitly

$$\underline{\underline{Df}} = \left(\begin{array}{cccc|ccc}
-\sigma N & 0 & \sigma & 0 & 0 & \cdots & 0 \;\cdots \\
0 & -\sigma N^* & 0 & \sigma & 0 & \cdots & 0 \;\cdots \\
\rho & 0 & -L & 0 & -\frac{r'_{1c}-r_{1c}}{\sum \frac{d}{b}}e^{i\phi} & \cdots & -\frac{r'_{ic}-r_{ic}}{\sum \frac{d}{b}}e^{i\phi} \;\cdots \\
0 & \rho^* & 0 & -L^* & -\frac{r'_{1c}-r_{1c}}{\sum \frac{d}{b}}e^{i\phi} & \cdots & -\frac{r'_{ic}-r_{ic}}{\sum \frac{d}{b}}e^{i\phi} \;\cdots \\
\hline
\frac{d_1 N^* e^{-i\phi}}{2} & \frac{d_1 N e^{i\phi}}{2} & \frac{d_1 e^{-i\phi}}{2} & \frac{d_1 e^{i\phi}}{2} & -b_1 & & 0 \\
\vdots & \vdots & \vdots & \vdots & & \ddots & \\
\frac{d_j N^* e^{-i\phi}}{2} & \frac{d_j N e^{i\phi}}{2} & \frac{d_j e^{-i\phi}}{2} & \frac{d_j e^{i\phi}}{2} & 0 & & -b_j \\
\vdots & \vdots & \vdots & \vdots & & & \ddots
\end{array}\right) \tag{18}$$

Absorbing the part of the solution for $\underline{x}^{(1)}$ which is proportional to $\underline{u}_0 = \underline{\underline{R}}'(\alpha).\underline{x}_0$, where $\underline{u}_0$ is the right eigenvector of $\underline{\underline{Df}}$ with eigenvalue zero, i.e.; the solution to

$$\underline{\underline{Df}}.\underline{u}_0 = f_{i,j}(\underline{\underline{R}}(\phi).\underline{x}, \mu)\, u_{0j} = 0 \qquad \forall \phi, \mu$$

and which exists by virtue of the fact that the underlying limit cycle may have its phase perturbed but still be arbitrarily stable, by letting ϕ depend on ε so that

$$\phi(\tau;\epsilon) = \phi_0(\tau) + \epsilon\,\phi_1(\tau) + \ldots \qquad (19)$$

the solution of (17) with (18) may be written

$$\underline{x}^{(1)} = A(\tau)\,\overset{+}{\underline{u}}_\Omega\, e^{+i\Omega t} + A^*(\tau)\,\overset{-}{\underline{u}}_\Omega\, e^{-i\Omega t} \qquad (20)$$

where $A(\tau)$ now denotes the complex amplitude of the disturbance to the limit cycle solution. On substitution the solution (20) may be expressed as

$$\begin{aligned}
x^{(1)} &= A(\tau)\,u_{\Omega_1} e^{i\Omega t} + A^*(\tau)\,u_{\Omega_2}^* e^{-i\Omega t} \\
x^{*(1)} &= A(\tau)\,u_{\Omega_2} e^{i\Omega t} + A^*(\tau)\,u_{\Omega_1}^* e^{-i\Omega t} \\
y^{(1)} &= A(\tau)\,u_{\Omega_3} e^{i\Omega t} + A^*(\tau)\,u_{\Omega_4}^* e^{-i\Omega t} \\
y^{*(1)} &= A(\tau)\,u_{\Omega_4} e^{i\Omega t} + A^*(\tau)\,u_{\Omega_3}^* e^{-i\Omega t} \\
z_j^{(1)} &= A(\tau)\,u_{\Omega_{j+4}} e^{i\Omega t} + A^*(\tau)\,u_{\Omega_{j+4}}^* e^{-i\Omega t} \\
& \qquad\qquad j = 1, \ldots
\end{aligned} \qquad (21)$$

Then $\underline{u}_\Omega$ is such that

$$\begin{aligned}
-(\sigma N + i\Omega)\,u_{\Omega_1} + \sigma\,u_{\Omega_3} &= 0 \\
-(\sigma N^* + i\Omega)\,u_{\Omega_2} + \sigma\,u_{\Omega_4} &= 0
\end{aligned} \qquad (22)$$

$$P u_{\Omega_1} - (L+i\Omega)u_{\Omega_3} - \left(\frac{r_{1c}'-r_{1c}}{\Sigma\, \phi_j/b_j}\right)e^{i\phi}\sum_5^\infty u_{\Omega_n} = 0$$

$$P^* u_{\Omega_2} - (L^*+i\Omega)u_{\Omega_4} - \left(\frac{r_{1c}'-r_{1c}}{\Sigma\, \phi_j/b_j}\right)e^{-i\phi}\sum_5^\infty u_{\Omega_n} = 0$$

$$-(b_{n-4}+i\Omega)u_{\Omega_n} + \frac{\phi_{n-4}}{2}\left(N^* e^{-i\phi}u_{\Omega_1} + N e^{i\phi}u_{\Omega_2} + e^{-i\phi}u_{\Omega_3} + e^{i\phi}u_{\Omega_4}\right) = 0 \qquad n=5,\ldots$$

The determinental condition for a non-trivial solution, $\underline{u}_\Omega \neq \underline{0}$, to exist is simply that Ω must satisfy the characteristic equation (5.23) of the linear stability problem which may be written in the form

$$i\Omega\Bigg\{ i\Omega(\sigma N + L + i\Omega)(\sigma N^* + L^* + i\Omega) + \frac{r_{1c}'-r_{1c}}{2\sum_1^\infty \phi_j/b_j}\cdot\left(\sigma(N+N^*)+i\Omega\right)\left(\sigma(N+N^*)+L+L^*+2i\Omega\right)\cdot\sum_1^\infty \frac{\phi_j}{b_j+i\Omega}\Bigg\} = 0 \qquad (23)$$

(23)

and then a solution of (22), using (23) to eliminate the appearance of the sums R and S (as defined by equation (4.48)) is

$$u_{\Omega_1} = \sigma(\sigma N^* + L^* + i\Omega)e^{i\phi}$$

$$u_{\Omega_2} = \sigma(\sigma N + L + i\Omega)e^{-i\phi} \qquad (24)$$

$$u_{\Omega_3} = (\sigma N + i\Omega)(\sigma N^* + L^* + i\Omega)e^{i\phi}$$

$$u_{\Omega_4} = (\sigma N^* + i\Omega)(\sigma N + L + i\Omega)e^{-i\phi}$$

$$u_{\Omega_n} = \left(\sigma(N+N^*)+i\Omega\right)\left(\sigma(N+N^*)+L+L^*+2i\Omega\right)\cdot\frac{1}{2}\frac{\phi_{n-4}}{b_{n-4}+i\Omega}$$

$$n = 5,\ldots$$

$$\Rightarrow \sum_5^\infty u_{\Omega_n} = -i\Omega(\sigma N + L + i\Omega)(\sigma N^* + L^* + i\Omega)\cdot\frac{\sum_1^\infty \phi_j/b_j}{r_{1c}'-r_{1c}}$$

Note. (i) The value of Ω is found by solving the characteristic equation for linear stability (23) and that a truncation including the N^{th} Z component is recovered from the above by setting $\gamma_n = 0$ for all $n \geq N+1$ in (23) and (24) so that $\underline{u}_\Omega$ is then a vector of dimension N+4. For any given $\{\gamma_j\}$ and $\{b_j\}$ the value of Ω computed by (23) varies with N.

(ii) The first four components of $\underline{u}_\Omega$ vary on the slow time scale τ and the remaining components are associated with the components Z_j of the stable manifold $Z_j = 0$, $j \geq 1$, for the fixed point solution $\underline{x} = \underline{0}$.

At $O(\varepsilon^2)$:

The problem for $\underline{x}^{(2)}$, noting that

$$\partial_\tau \underline{x}^{(0)} = \phi_\tau \underline{\underline{R}}'(\phi).\underline{x}_0 \tag{25}$$

is

$$\left(\partial_t - \underline{\underline{A}}\right)\underline{x}^{(2)} = -\begin{pmatrix} i\phi_\tau e^{i\phi} \\ -i\phi_\tau e^{-i\phi} \\ iN\phi_\tau e^{i\phi} + \left(\dfrac{r_{1c}' - r_{1c}}{\sum_1^\infty d_j/b_j}\right) x^{(1)} \sum_1^\infty z_j^{(1)} \\ -iN^*\phi_\tau e^{-i\phi} + \left(\dfrac{r_{1c}' - r_{1c}}{\sum_1^\infty d_j/b_j}\right) x^{*(1)} \sum_1^\infty z_j^{(1)} \\ -\dfrac{d_1}{2}\left(x^{(1)}y^{*(1)} + x^{*(1)}y^{(1)}\right) \\ \vdots \\ -\dfrac{d_j}{2}\left(x^{(1)}y^{*(1)} + x^{*(1)}y^{(1)}\right) \\ \vdots \end{pmatrix} \tag{26}$$

Using the relations (24), that part of the right hand side which may be seen to be proportional to $e^{2i\Omega t}$ gives a particular solution for $\underline{x}^{(2)}$ which is proportional to $e^{2i\Omega t}$, however since $\underline{\mathcal{L}}$ has an eigenvalue equal to zero that part of the right hand side which varies on the slow time scale alone is secular. In general, the secularity condition which must be imposed upon a term in the right hand side of (26) which is proportional to $\underline{c}_1(\tau)e^{i\Omega t}$ or to $\underline{c}_0(\tau)$ i.e.; imposed in finding a solution $\underline{x}$ to the equation

$$\left(\partial_t - \underline{\mathcal{L}}\right)\underline{x} = \underline{c}_0 + \underline{c}_1 e^{i\Omega t} \tag{27}$$

which is valid for times $t = O(\varepsilon^{-2})$ is given by

$$\underline{v}_0 \cdot \underline{c}_0 = 0 \quad (28) \qquad\qquad \underline{v}_\Omega \cdot \underline{c}_\Omega = 0 \quad (29)$$

where
$$\left(\underline{\mathcal{L}}^*\right)^T \underline{v}_0^* = 0 \tag{30}$$

and
$$\left(\underline{\mathcal{L}}^*\right)^T \underline{v}_\Omega^* = -i\Omega v_\Omega^* \tag{31}$$

or equivalently where $\underline{v}_0$ and $\underline{v}_\Omega$ are the left eigenvalues of $\underline{\mathcal{L}}$ with eigenvalues 0 and $i\Omega$ respectively. Explicitly (31) becomes

$$-(\sigma N + i\Omega)v_{\Omega 1} + P v_{\Omega 3} + \frac{N^*}{2}e^{-i\phi}\sum_5^\infty d_{n-4} v_{\Omega n} = 0$$

$$-(\sigma N^* + i\Omega)v_{\Omega 2} + P^* v_{\Omega 4} + \frac{N}{2}e^{i\phi}\sum_5^\infty d_{n-4} v_{\Omega n} = 0$$

$$\sigma v_{\Omega 1} - (L + i\Omega)v_{\Omega 3} + \frac{e^{-i\phi}}{2}\sum_5^\infty d_{n-4} v_{\Omega n} = 0$$

$$\sigma v_{\Omega 2} - (L^* + i\Omega)v_{\Omega 4} + \frac{e^{i\phi}}{2}\sum_5^\infty d_{n-4} v_{\Omega n} = 0 \tag{32}$$

$$-(\delta_{n-4} + i\Omega)v_{\Omega n} - \left(\frac{r_{ic}' - r_{ic}}{\sum_j d_j/\delta_j}\right)\left(v_{\Omega 3}e^{i\phi} + v_{\Omega 4}e^{-i\phi}\right) = 0$$

$$n = 5, \dots$$

Again the determinental condition which ensures that a non-trivial solution, $\underline{v}_{\Omega} \neq \underline{0}$, exists is (23) and when this latter result is used to replace the sums R and S a solution of (32) is

$$
\begin{aligned}
v_{\Omega_1} &= (\sigma N^* + L^* + i\Omega)\left((N+N^*)L + i\Omega N^*\right) e^{-i\phi} \\
v_{\Omega_2} &= (\sigma N + L + i\Omega)\left((N+N^*)L^* + i\Omega N\right) e^{i\phi} \\
v_{\Omega_3} &= \left(\sigma(N+N^*) + i\Omega\right)(\sigma N^* + L^* + i\Omega) e^{-i\phi} \qquad (33) \\
v_{\Omega_4} &= \left(\sigma(N+N^*) + i\Omega\right)(\sigma N + L + i\Omega) e^{i\phi} \\
v_{\Omega_n} &= -\left(\frac{r_{ic}' - r_{ic}}{\sum_1^\infty d_j/\lambda_j}\right)\left(\sigma(N+N^*) + i\Omega\right)\left(\sigma(N+N^*) + L + L^* + 2i\Omega\right) \cdot \frac{1}{\lambda_{n-4} + i\Omega}
\end{aligned}
$$

$$
\Rightarrow \sum_{n=5}^{\infty} d_{n-4} v_{\Omega_n} = 2i\Omega(\sigma N + L + i\Omega)(\sigma N^* + L^* + i\Omega) \qquad n = 5, \ldots
$$

On returning to (32) and taking the limit $\Omega \to 0$ a solution of (30) is given by

$$
\begin{aligned}
v_{0_1} &= L e^{-i\phi}, \quad v_{0_2} = -L^* e^{-i\phi} \\
v_{0_3} &= \sigma e^{-i\phi}, \quad v_{0_4} = -\sigma e^{-i\phi} \qquad (34) \\
v_{0_n} &= 0 \quad n = 5, \ldots
\end{aligned}
$$

Substituting the form of the solution at previous order given by (21) in (26) the t independent vector, $\underline{c}_0(\tau)$, of (27) is

$$
\underline{c}_0(\tau) = -\begin{pmatrix}
i\phi_\tau e^{i\phi} \\
-i\phi_\tau e^{-i\phi} \\
iN\phi_\tau e^{i\phi} + \left(\frac{r_{ic}' - r_{ic}}{\sum_1^\infty d_j/\lambda_j}\right)|A|^2\left(u_{\Omega_1}\sum_5^\infty u_{\Omega_n}^* + u_{\Omega_2}^*\sum_5^\infty u_{\Omega_n}\right) \\
-iN^*\phi_\tau e^{-i\phi} + \left(\frac{r_{ic}' - r_{ic}}{\sum_1^\infty d_j/\lambda_j}\right)|A|^2\left(u_{\Omega_2}\sum_5^\infty u_{\Omega_n}^* + u_{\Omega_1}^*\sum_5^\infty u_{\Omega_n}\right) \\
-\frac{d_j}{2}|A|^2\left(u_{\Omega_1}u_{\Omega_3}^* + u_{\Omega_1}^* u_{\Omega_3} + u_{\Omega_2}u_{\Omega_4}^* + u_{\Omega_2}^* u_{\Omega_4}\right) \\
\vdots
\end{pmatrix} \qquad (35)
$$

Hence, from the secularity condition (28), the phase function $\phi(\tau)$ must satisfy

$$\left(\sigma(N+N^*)+L+L^*\right)\phi_\tau$$
$$= 2\sigma\left(\frac{r_{1c}'-r_{1c}}{\sum_{1}^{\infty} d_j/b_j}\right)|A|^2 \mathcal{J}_m\left(\sum_{s}^{\infty} u_{\Omega u}\left(u_{\Omega_1}^* e^{i\phi} - u_{\Omega_2}^* e^{-i\phi}\right)\right) \tag{36}$$

which implies

$$\phi_\tau = \beta\,|A|^2$$

(37)

with
$$\beta = q^{(2\sigma\Omega)^2}$$

where q = 2ω-e. Notice that the choice of a normalisation for $\underline{u}_\Omega$ in the solution (24) which leads to (37) from (36) implies the normalisation for $A(\tau)$ in (20).

There now exists a particular solution of (26) which, expressed as

$$\underline{x}^{(2)} = \underline{a}_0|A|^2 + \underline{a}_2^+ A^2(\tau)e^{2i\Omega t} + \underline{a}_2^- A^{*2}(\tau)e^{-2i\Omega t}$$

and substituted in (26) suggests the simpler expression

$$x^{(2)} = a_{01}|A|^2 + a_{21}A^2(\tau)e^{2i\Omega t} + a_{22}^* A^{*2}(\tau)e^{-2i\Omega t}$$

$$x^{*(2)} = a_{01}^*|A|^2 + a_{22}A^2(\tau)e^{2i\Omega t} + a_{21}^* A^{*2}(\tau)e^{-2i\Omega t}$$

(38)

$$y^{(2)} = a_{03}|A|^2 + a_{23}A^2(\tau)e^{2i\Omega t} + a_{24}^* A^{*2}(\tau)e^{-2i\Omega t}$$

$$y^{*(2)} = a_{03}^*|A|^2 + a_{24}A^2(\tau)e^{2i\Omega t} + a_{23}^* A^{*2}(\tau)e^{-2i\Omega t}$$

$$z_j^{(2)} = a_{0j+4}|A|^2 + a_{2j+4}A^2(\tau)e^{2i\Omega t} + a_{2j+4}^* A^{*2}(\tau)e^{-2i\Omega t}$$

$$j = 1, \ldots$$

The algebraic steps leading to solutions for $\underline{a}_0$ and $\underline{a}_2$ are given in appendix D.

Recall that constant solutions in $\underline{x}$, arising from the zero eigenvalue of $\underline{\underline{\mathcal{J}}}$ and which are therefore proportional to $\underline{u}_0$, have been incorporated into ϕ by the assumption of equation (19). Similarly a solution in $\underline{x}$ proportional to $\underline{u}_\Omega e^{i\Omega t}$, which corresponds to the eigen-frequency Ω of instability of $\underline{\underline{\mathcal{J}}}$, may be incorporated into $A(\tau)$ if it is assumed that $A(\tau)$ depends on ε so that

$$A(\tau;\epsilon) = A_0(\tau) + \epsilon A_1(\tau) + \cdots \qquad (39)$$

The problems for ϕ_j, A_j with $j \geq 1$ are neglected.

At $O(\varepsilon^3)$:

The problem for $\underline{x}^{(3)}$ is

$$\left(\partial_t - \underline{\underline{\mathcal{J}}}\right).\underline{x}^{(3)} = \begin{pmatrix} -\partial_\tau x^{(1)} \\ -\partial_\tau x^{*(1)} \\ -\partial_\tau y_j^{(1)} - \left(\dfrac{r_{1c}' - r_{1c}}{\sum_1^\infty d_j/b_j}\right)\left(x^{(1)}\sum_1^\infty z_j^{(2)} + x^{(2)}\sum_1^\infty z_j^{(1)}\right) \\ \qquad \mp \dfrac{e^{i\phi}}{\sum_1^\infty d_j/b_j}\cdot\sum_1^\infty z_j^{(1)} \\ -\partial_\tau y_j^{*(1)} - \left(\dfrac{r_{1c}' - r_{1c}}{\sum_1^\infty d_j/b_j}\right)\left(x^{*(1)}\sum_1^\infty z_j^{(2)} + x^{*(2)}\sum_1^\infty z_j^{(1)}\right) \\ \qquad \mp \dfrac{e^{-i\phi}}{\sum_1^\infty d_j/b_j}\cdot\sum_1^\infty z_j^{(1)} \\ -\partial_\tau z_j^{(1)} + \dfrac{d_j}{2}\left(x^{(1)}y_j^{*(2)} + x^{(2)}y_j^{*(1)} + x^{*(1)}y_j^{(2)} + x^{*(2)}y_j^{(1)}\right) \end{pmatrix} \qquad (40)$$

where the solutions at $O(\varepsilon)$ and $O(\varepsilon^2)$ of (21) and (38) may be used to evaluate the right hand side and, in addition, the result that

$$\underline{u}_\Omega = \underline{\underline{R}}(\phi).\underline{U}_\Omega \qquad (41)$$

where $\underline{U}_\Omega$ is the right eigenvector of $\underline{\underline{\mathcal{J}}}(\underline{x};\mu)$ and is independent of ϕ and thus τ (see §3 of Fowler et al. (1981)). Hence

$$\begin{aligned}\partial_\tau \underline{u}_\Omega &= \phi_\tau \underline{\underline{R}}'(\phi)\, \underline{U}_\Omega \\ &= \phi_\tau\, \underline{\underline{R}}'(0).\, \underline{\underline{R}}(\phi).\, \underline{U}_\Omega \\ &= \phi_\tau\, \underline{\underline{R}}'(0)\, \underline{u}_\Omega\end{aligned} \tag{42}$$

and thus

$$\partial_\tau \underline{u}_\Omega = \left(i\phi_\tau u_{\Omega_1}, -i\phi_\tau u_{\Omega_2}, i\phi_\tau u_{\Omega_3}, -i\phi_\tau u_{\Omega_4}, 0, \ldots\right)$$

The right hand side of (40) may thus be seen to contain terms which are proportional to both $e^{\pm 3i\Omega t}$ and $e^{\pm i\Omega t}$, the latter terms alone are secular and are given by

$$\left(\partial_t - \underline{\underline{\mathcal{J}}}\right)\underline{x}^{(3)} = \begin{pmatrix} -A_\tau u_{\Omega_1} - iA\phi_\tau u_{\Omega_1} \\ -A_\tau u_{\Omega_2} + iA\phi_\tau u_{\Omega_2} \\ \left[-A_\tau u_{\Omega_3} - iA\phi_\tau u_{\Omega_3} \mp \dfrac{e^{i\phi}\sum_s^\infty u_{\Omega_n}}{\sum_1^\infty \mathcal{S}_j b_j} A \right. \\ \left. -\left(\dfrac{r_c' - r_c}{\sum_1 \mathcal{S}_j b_j}\right)\left(u_{\Omega_1}\sum_s^\infty a_{0j} + u_{\Omega_2}^*\sum_s^\infty a_{2j} + a_{01}\sum_s^\infty u_{\Omega_j} + a_{21}\sum_s^\infty u_{\Omega_j}^*\right)A|A|^2\right] \\ \left[-A_\tau u_{\Omega_4} + iA\phi_\tau u_{\Omega_4} \mp \dfrac{e^{-i\phi}\sum_s^\infty u_{\Omega_n}}{\sum_1^\infty \mathcal{S}_j b_j} A \right. \\ \left. -\left(\dfrac{r_c' - r_c}{\sum_1 \mathcal{S}_j b_j}\right)\left(u_{\Omega_2}\sum_s^\infty a_{0j} + u_{\Omega_1}^*\sum_s^\infty a_{2j} + a_{0j}^*\sum_s^\infty u_{\Omega_j} + a_{22}\sum_s^\infty u_{\Omega_j}^*\right)A|A|^2\right] \\ -A_\tau u_{\Omega_{j+4}} + \dfrac{\mathcal{S}_j}{2}\left(u_{\Omega_1} a_{03}^* + u_{\Omega_2}^* a_{24} + u_{\Omega_2} a_{03} + u_{\Omega_1}^* a_{23} \right. \\ \left. + u_{\Omega_3}^* a_{21} + u_{\Omega_3} a_{01}^* + u_{\Omega_4} a_{01} + u_{\Omega_4}^* a_{22}\right)A|A|^2 \\ \vdots \quad j = 1, \ldots \end{pmatrix} \tag{43}$$

The secularity condition to be imposed is (29), that the right hand side contracted with $\underline{v}_\Omega$ vanishes. The amplitude equation for $A(\tau)$ is

$$
\begin{aligned}
&-A_\tau \left(v_{R_1} u_{R_1} + v_{R_2} u_{R_2} + v_{R_3} u_{R_3} + v_{R_4} u_{R_4} + \sum_5^\infty v_{R_j} u_{R_j} \right) \\
&\quad -i\beta A|A|^2 \left(v_{R_1} u_{R_1} - v_{R_2} u_{R_2} + v_{R_3} u_{R_3} - v_{R_4} u_{R_4} \right) \\
&-\left(\frac{r_{1c}' - r_{1c}}{\sum_1^\infty d_j / b_j} \right) A|A|^2 \Big\{ v_{R_3} u_{R_1} \sum_5^\infty a_{0j} + v_{R_3} u_{R_2}^* \sum_5^\infty a_{2j} + v_{R_3} a_{01} \sum_5^\infty u_{R_j} + v_{R_4} a_{22} \sum_5^\infty u_{R_j}^* \\
&\quad + v_{R_3} a_{21} \sum_5^\infty u_{R_j}^* + v_{R_4} u_{R_2} \sum_5^\infty a_{0j} + v_{R_4} u_{R_1}^* \sum_5^\infty a_{2j} + v_{R_4} a_{01}^* \sum_5^\infty u_{R_j} \Big\} \\
&+\frac{A|A|^2}{2} \sum_1^\infty d_j v_{R_{j+4}} \Big(u_{R_1} a_{03}^* + u_{R_2}^* a_{24} + u_{R_2} a_{03} + u_{R_1}^* a_{23} \\
&\quad + u_{R_4} a_{01} + u_{R_3}^* a_{21} + u_{R_3} a_{01}^* + u_{R_4}^* a_{22} \Big) \\
&\mp \frac{\sum_5^\infty u_{R_j}}{\sum_1^\infty d_j / b_j} \left(v_{R_3} e^{i\phi} + v_{R_4} e^{-i\phi} \right) A \\
&= 0
\end{aligned}
\tag{44}
$$

when $r = r_{1c}' \pm \varepsilon^2$, which because of equation (37) may be written in the form

$$A_\tau = \pm k_1 A + k_2 A|A|^2 \tag{45}$$

where k_1 and k_2 are complex coefficients which may be seen to be independent of τ and the upper choice of sign is taken in the supercritical state, i.e.; when $r_1 > r_{1c}'$.

Changing the dependent variable by putting

$$A = r e^{i\Theta} \tag{46}$$

implies

$$r_\tau = \pm Re(k_1) r + Re(k_2) r^3 \tag{47}$$

which has the solution

$$r^2(\tau) = \frac{\pm Re(k_1)\, \kappa\, e^{\pm 2Re(k_1)\tau}}{1 - Re(k_2)\, \kappa\, e^{\pm 2Re(k_1)\tau}} \tag{48}$$

where $\kappa \in \mathbb{R}^+$ is a constant of integration, then

$$\Theta_\tau = \pm Im(k_1) + Im(k_2) r^2 \tag{49}$$

and from equation (37)

$$\phi_\tau = \beta r^2 \tag{50}$$

Hence the bifurcation is of the subcritical or supercritical type according as $Re(k_2)$ is positive or negative.

In the subcritical case the perturbation to the limit cycle solution, $\underline{x}^{(1)}$, is singular in general at the finite time

$$\tau_S = \frac{\ln((\kappa . Re(k_2))^{-1})}{2\, Re(k_1)}$$

and then the above analysis does not predict the solutions of equations (1) to (3) except when $r_1 < r_{1c}$' with $Re(k_1) > 0$ and the limit cycle solution is then linearly stable exhibiting the threshold type of instability where the threshold is

$$|A_{Th}|^2 = \frac{Re(k_1)}{Re(k_2)} \tag{51}$$

If the bifurcation is of the supercritical type the solution $A(\tau)$ is bounded for all τ and for all initial conditions with

$$r^2 \sim \frac{Re(k_1)}{Re(-k_2)} \qquad \text{as } \tau \to \infty \tag{52}$$

Now the position vector (15) in the rotating frame is

$$\underline{x} = \underline{\underline{R}}(\phi(\tau)).\underline{x}_0 + \epsilon\left(A(\tau)\,\underline{u}_\Omega(\phi(\tau))e^{i\Omega t} + \begin{smallmatrix}\text{similar}\\ \text{term}\end{smallmatrix}\right) + \ldots \tag{53}$$

formed from equations (16) and (21). Recalling the result of equation (41) this may be more usefully expressed as

$$\underline{x} = \underline{\underline{R}}(\phi(\tau)).\left[\underline{x}_0 + \epsilon\left(A(\tau)\,\underline{U}_\Omega e^{i\Omega t} + \text{similar term}\right)\right] + O(\epsilon^2) \tag{54}$$

then

$$\underline{x} \sim \underline{\underline{R}}\left(\frac{\beta R_e(k_1)}{R_e(-k_2)}.\tau\right).\left\{\underline{x}_0 + \epsilon\left(\frac{R_e(k_1)}{R_e(-k_2)}\right)^{\frac{1}{2}}\underline{U}_\Omega e^{i\left(\frac{I_m(k_1).R_e(-k_2) + I_m(k_2)R_e(k_1)}{R_e(-k_2)}\tau + \Omega t\right)} + \text{similar term} + O(\epsilon^2)\right\} \tag{55}$$

So that in the rotating frame the solution tends to precess steadily around the limit cycle as $\tau \to \infty$ with asymptotic frequency on the t scale

$$\frac{\epsilon^2\beta R_e(k_1)}{R_e(-k_2)} \tag{56}$$

and has a superimposed small amplitude oscillation given by equations (47) and (48) of size proportional to $\varepsilon = \surd(r_1 - r_{1c}')$ which varies on the fast time scale, t, with frequency Ω modified by a slow drift on the slow time scale, τ, hence the solutions are doubly periodic. In the fixed frame the solution is obtained from (4) and (5) under the rotation matrix $\underline{\underline{R}}$, i.e.;

$$\underline{X} = \underline{\underline{R}}(\omega t).\underline{\underline{\Lambda}}.\underline{x} \tag{57}$$

where $\underline{\underline{\Lambda}}$ is the constant scaling matrix implied by equation (5). Since $\underline{\underline{R}}$ is a rotation matrix

$$\begin{aligned}\underline{\underline{R}}(\omega t).\underline{\underline{R}}(\phi(\tau)) &= \underline{\underline{R}}(\omega t + \phi(\tau)) \\ &\sim \underline{\underline{R}}\left(\left(\omega + \epsilon^2\frac{\beta R_e(k_1)}{R_e(-k_2)}\right)t\right)\end{aligned} \tag{58}$$

then from equation (55) the solutions in the fixed frame are again doubly periodic although now the precession of the solution around the limit cycle is on the fast time scale so that it is not separated as easily from the component frequency of the perturbation away from the limit cycle. The two frequencies may be incommensurate in general.

Explicitly, in the rotating frame, x, the first component of the position vector $\underline{x}$ is given by

$$x = e^{i\phi(\tau)}\Big(1 + \epsilon\sigma r(\tau)\big\{ (\sigma N^* + L^* + i\Omega) e^{i(\Omega t + \theta(\tau))} + (\sigma N^* + L^* - i\Omega) e^{-i(\Omega t + \theta(\tau))} \big\} \Big) + O(\epsilon^2) \tag{59}$$

showing the two periods through the precessional drift $\phi(\tau)$ about the limit cycle and the instability of the limit cycle through $\Omega t + \theta(\tau)$ where Ω is the eigenfrequency of linear instability and $\theta(\tau)$ is the correction to the eigenfrequency in finite amplitude. The $(j+4)^{th}$ component of $\underline{x}$ is z_j, given by

$$z_j = \frac{d_j}{b_j} + \epsilon r(\tau)\, R_e\Big((\sigma(N+N^*) + i\Omega)(\sigma(N+N^*) + L + L^* + 2i\Omega) \times \frac{d_j}{b_j + i\Omega} e^{i(\Omega t + \theta(\tau))} \Big) + O(\epsilon^2) \qquad j = 1, \dots \tag{60}$$

Asymptotically as $\tau \to \infty$, from (52), each z component of the solution exhibits the single period associated with the linear eigenfrequency of the limit cycle instability but again slightly corrected in finite amplitude. The absence of $\phi(\tau)$ in this case arises from the form of the rotation matrix (10) or equivalently from the orientation of the rotation associated with the limit cycle, which is in the x-y hyperplane. In the fixed frame the solutions become

$$X = \left(\frac{r_c' - r_c}{\sum_1^\infty d_j / b_j} \right)^{\frac{1}{2}} e^{i(\omega t + \phi(\tau))} \Big(1 + \epsilon\sigma r(\tau)\big((\sigma N^* + L^* + i\Omega) e^{i(\Omega t + \theta(\tau))} + (\sigma N^* + L^* - i\Omega) e^{-i(\Omega t + \theta(\tau))} \big) \Big) + O(\epsilon^2) \tag{61}$$

$$Z_j = \left(\frac{r_{1c}' - r_{1c}}{\sum_1^\infty d_j / b_j}\right) \cdot \left\{ \frac{d_j}{b_j} + \epsilon r(\tau)\, Re\left((\sigma(N+N^*)+i\Omega)(\sigma(N+N^*)+L+L^*+2i\Omega) \times \frac{d_j}{b_j+i\Omega} e^{i(\Omega t+\theta(\tau))} \right) \right\} + O(\epsilon^2) \qquad (62)$$

$$j = 1, \ldots$$

For the coefficients of the two layer model of baroclinic instability in both the infinite and single Z component cases as given by equations (B9) and (B13) of appendix B, contours of $Re(k_2)$ are sketched in the real $(k, -r_2)$ planes of figures (1) and (2) with some of the more involved steps in the algebra appearing in appendix D. Figure (2) with the coefficients of (B13) is appropriate to the case of a single Z component found by relaxation of the sidewall boundary condition (4.52) and not by truncation of the infinite set given by (B9) after the first component. The contours are surprisingly close in both instances with their differences being most pronounced for the smaller values of k or longer waves. I am at a loss for a physical interpretation of this detail.

The simple computer programmes used in these sketches also confirmed that the sufficient criterion for limit cycle instability

$$\sigma > \frac{\sum_1^\infty d_j b_j}{\sum_1^\infty d_j} + 1 \qquad (63)$$

of (5.40) is also necessary in both cases, i.e.; for all k such that (63) is not satisfied there does not exist a real root, Ω, of the characteristic equation (23) for the linear instability of this solution. Hence there is a long wave cut-off effect (for some value of $k = k_s$ say, which is given by (63) as $m\pi\sqrt{3}$ in the single component case and approximately $m\pi\sqrt{2.82}$ in the infinite case) beyond which the limit cycle solution is always linearly stable when it exists (for $r_1 > r_{1c}$) which is in accord with the findings of Pedlosky (1971) and Smith (1977) for the case of real coefficients recovered as $r_2 \to 0$. Now for all (negative) values of r_2, $Re(k_2)$ tends to infinity through positive values as $k \to k_s^+$ so that the bifurcation is subcritical in this part of the (k, r_2) plane, it is also subcritical for all positive k when $|r_2|$ is sufficiently small and this result agrees with the bifurcation in the real case, $r_2 = 0$, as studied by McLaughlin and Martin (1975). However,

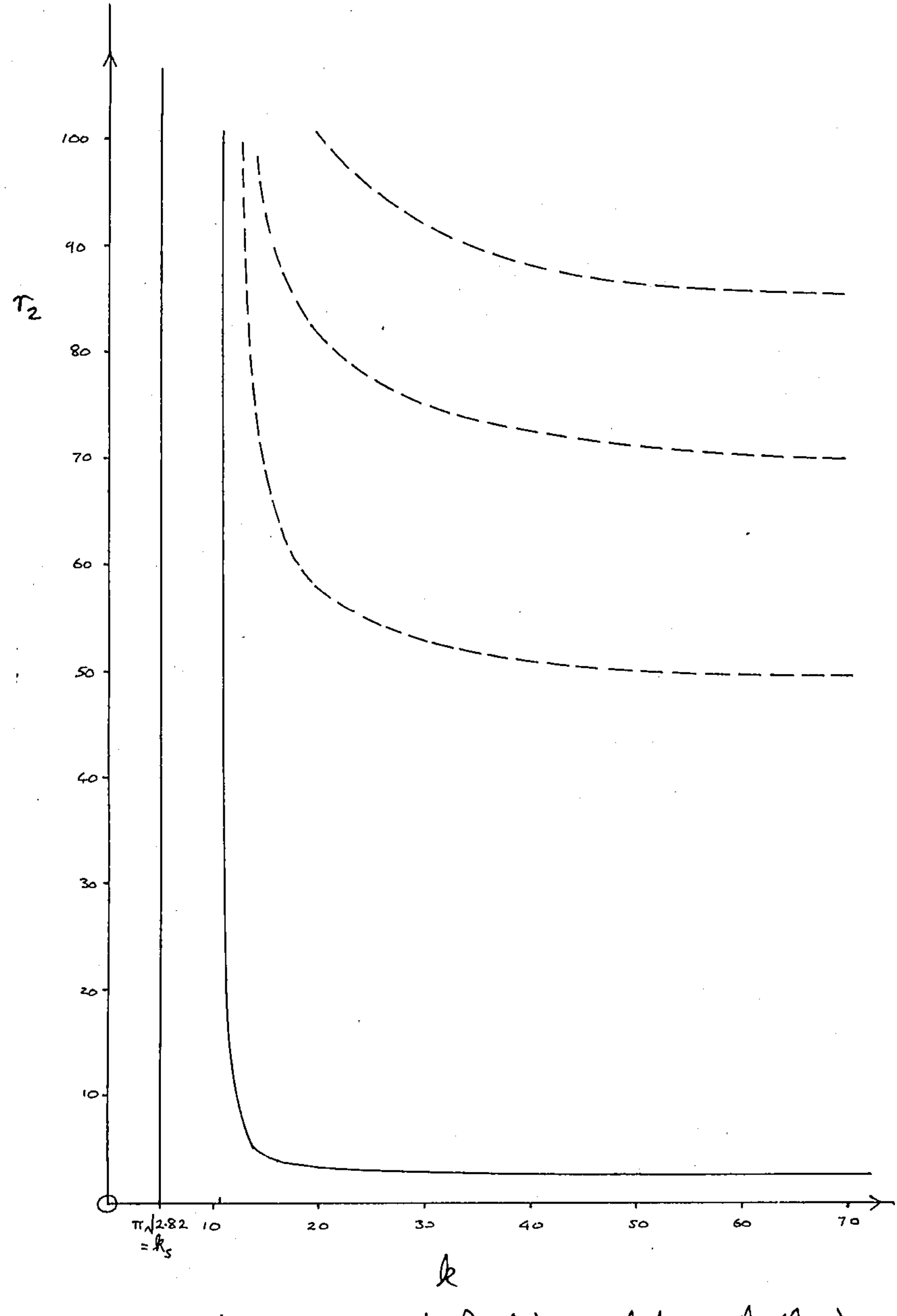


Figure (6.1). Contours of $Re(k_2)$ = constant in the (k, r_2) plane. The solid line is a cut-off boundary for instability of the limit cycle, the solid curve is the contour $Re(k_2) = 0$.

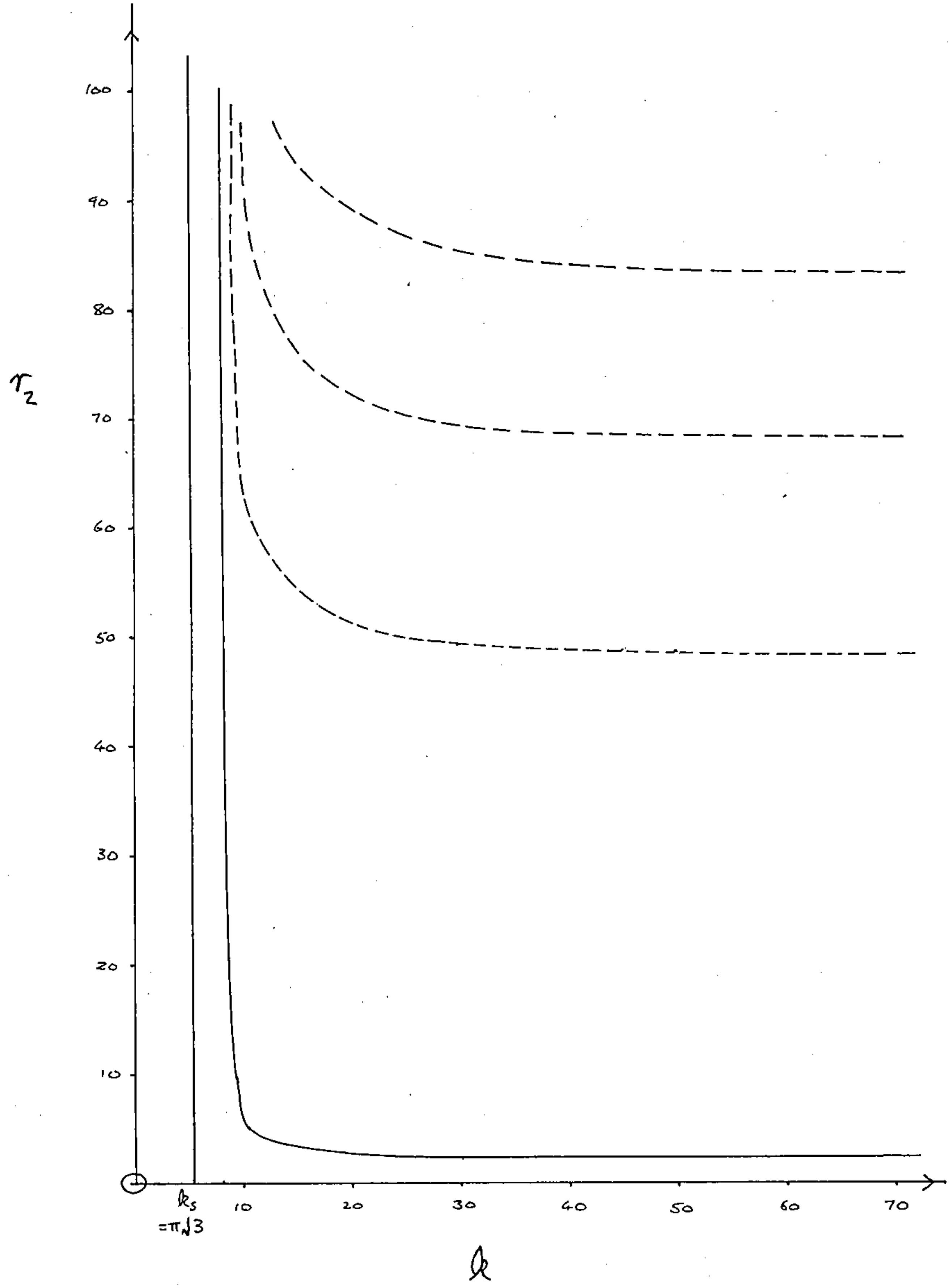


Figure (6.2). The contours and legend are as for the previous sketch but now for the case of a single Z component. (See text).

throughout most of the $(k,-r_2)$ plane $Re(k_2)$ is negative and so the bifurcation is supercritical there, occurring at some sufficiently large value of r_1.

With reference to the numerical integrations of the complex Lorenz equations with a single Z-component described in section 4 of Fowler, Gibbon and McGuinness (1982), figure (3) shows the curve r_{1c}' as a function of r_2 when $e = 3r_2$, $\sigma = 2$, $\gamma = 1$ and $b = 0.8$ (or $m = 1$, $k = \pi\sqrt{5}$ in relations B13) with the computed values of Ω and $Re(k_2)$ to the left and right along the curve; c.f. Fowler et al., figure 3. This confirms their conjecture that when $r_2 = 1$, implying $r_{1c}' = 255.67$, the phase portraits of figures (4) to (6) of that paper are showing a subcritical bifurcation to or from the limit cycle solution in the subcritical $(r_1 < r_{1c}')$ and supercritical states $(r_1 > r_{1c}')$ and thus figures (5) and (6) which are relevant to the supercritical state do indicate the presence of a second, stable, doubly periodic motion which may be similar in form to but not described by that given here in equations (59) to (62). It seems that for this particular parameter setting $Re(k_2)$ is positive at all points along the curve of figure (3) but for a smaller value of b the bifurcation is supercritical for sufficiently large $|r_2|$.

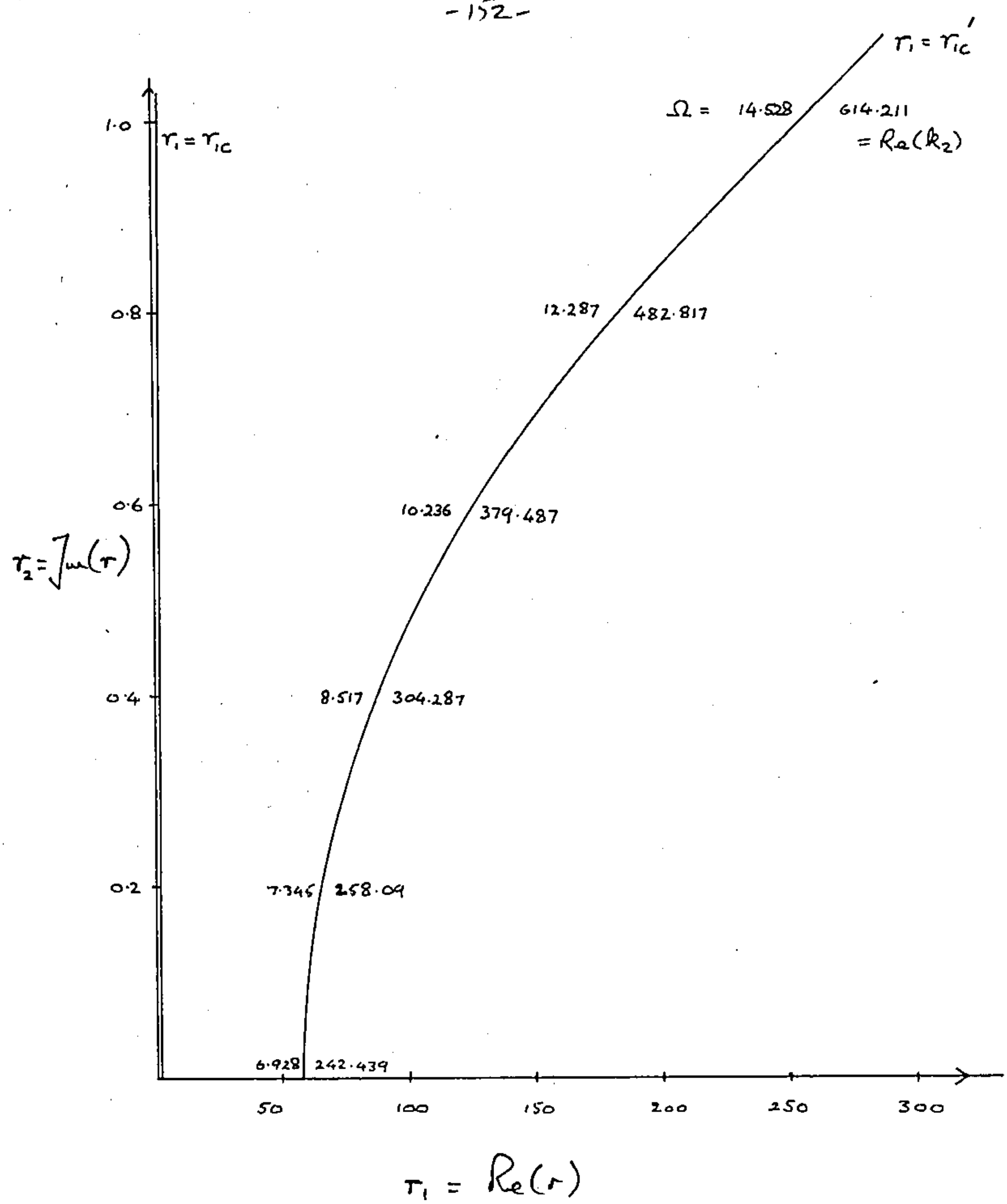


Figure (6.3); r_{1c} and r_{1c}' as functions of r_2 - see text for detail.

CHAPTER 7

WEAKLY NONLINEAR EVOLUTION OF A PACKET OF LONG WAVES FOR EADY'S MODEL WHEN THE SQUARE ROOT OF THE EKMAN NUMBER DIVIDED BY THE ROSSBY NUMBER IS OF THE ORDER OF THE SUPERCRITICALITY

The motivation for the study of the following problem is to establish an amplitude equation which, like that of chapter four, contains both a first and a second order derivative of the amplitude with respect to time in the linear terms and where the mean field correction obeys a partial differential equation similar to (4.32) but which is appropriate for the evolution of a packet of linearly unstable waves. The analogous problem for the 2-layer model excluding the β-plane effect has been studied by Pedlosky (1981) for the cases of both free and forced wave packets where the ordering of the parameters is essentially the same, i.e.; where in the different notation $r = E^{\frac{1}{2}}/\varepsilon = O(\Delta)$ as $\Delta \to 0$ and the supercriticality is taken with respect to the internal rotational Froude number, F.

It has been established in chapter three that the phase speed, c, of a normal mode disturbance

$$\phi = A e^{ik(x-ct)} \sin lY \left(Q \operatorname{ch}\left(\frac{2sz}{h}\right) + R \operatorname{sh}\left(\frac{2sz}{h}\right)\right)$$

$$+ \text{ complex conjugate} \tag{1}$$

to the basic linear shear profile $u_0 = \Lambda z$ of Eady's model is

$$c = \frac{\Lambda h}{4s \operatorname{th} s}\left\{ -i\mu_0(1+\operatorname{th}^2 s) \pm \sqrt{\left(-\mu_0^2(1+\operatorname{th}^2 s)^2 + 4\operatorname{th} s\left(\mu_0^2 \operatorname{th} s + (s-\operatorname{th} s)(s\operatorname{th} s - 1)\right)\right)}\right\} \tag{2}$$

in the present notation where

$$\mu_0 = \frac{S E_v^{\frac{1}{2}}}{2\Lambda R_0} \cdot \frac{k^2+l^2}{k} \quad \text{and} \quad S = \frac{4s^2}{h^2(k^2+l^2)} \tag{3),(4}$$

so that the surface of neutral stability is given by equation (3.10)

$$G\left(k, S, \frac{E_v^{\frac{1}{2}}}{R_0}\right) \equiv \mu_0^2 \,\mathrm{th}\, s + (s - \mathrm{th}\, s)(s\,\mathrm{th}\, s - 1) = 0 \tag{5}$$

where parameter space is now chosen to have independent coordinates k, S and $E_v^{\frac{1}{2}}/R_0$ as opposed to the previous choice in chapters three and four of k, S and μ.

The stability criterion for the inviscid problem is, from equation (5), instability if and only if

$$0 < s < s_0 \tag{6}$$

where $s_0 > 0$ is such that $s_0 \,\mathrm{th}\, s_0 = 1$. Solving (5) iteratively for small μ_0 along the upper branch of the neutral surface by applying Newton's method implies a critical value of s, s_δ, given by

$$s_\delta = s_0 - \left(\frac{S E_v^{\frac{1}{2}}}{2\lambda R_0}\right)^2 \frac{(k^2+l^2)^2}{k^2} \cdot \frac{1}{s_0(s_0^2-1)} + O(\mu_0^4) \tag{7}$$

The critical value of the Burger number, S, which is chosen as the bifurcation parameter is, from (4)

$$S_\delta = \frac{4 s_0^2}{h^2(k^2+l^2)} - \frac{32\, s_0^4}{\lambda^2 h^6 (s_0^2-1) k^2 (k^2+l^2)} \left(\frac{E_v^{\frac{1}{2}}}{R_0}\right)^2 + O(\mu_0^4) \tag{8}$$

where

$$S_0 = \frac{4 s_0^2}{h^2(k^2+l^2)}$$

is the critical value of the Burger number for the inviscid problem.

Differentiating (8) with respect to k yields the unique value of k for given $E_v^{\frac{1}{2}}/R_0$ along the upper branch

$$k_M = \frac{2 s_0 l}{h^2}\left(\frac{2}{\lambda(s_0^2-1)}\right)^{\frac{1}{2}} \left(\frac{E_v^{\frac{1}{2}}}{R_0}\right)^{\frac{1}{2}} + O\left(\frac{E_v^{\frac{1}{2}}}{R_0}\right) \tag{9}$$

where S_δ is maximised, taking the value

$$S_{\delta M} = \frac{4s_0^2}{k^2\ell^2} - \frac{4s_0^2}{\Lambda k^2\ell^4}\left(1 + \frac{8s_0^2\ell^2}{k^4(s_0^2-1)}\right)\left(\frac{E_v^{\frac{1}{2}}}{R_0}\right) + O\left(\left(\frac{E_v^{\frac{1}{2}}}{R_0}\right)^{\frac{3}{2}}\right) \tag{10}$$

Defining the supercriticality, $\varepsilon \ll 1$, by

$$S_{\delta M} - S = \epsilon$$

implies

$$\left(s - s_{\delta M}\right) = \frac{s_{\delta M}}{2S_{\delta M}}\left(S - S_{\delta M}\right) + O\left(\left(s - s_{\delta M}\right)^2\right)$$

from equation (4), hence Taylor expanding (2) about the critical point $(\dot{k}_M, S_{\delta M}, \frac{E_v^{\frac{1}{2}}}{R_O})$ the linear growth rate is the imaginary part of

$$k_c = \frac{\Lambda k}{4s_0 \ell k s_0}\left\{-i\frac{S_0\ell^2}{2\Lambda}(1+\ell k^2 s_0)\left(\frac{E_v^{\frac{1}{2}}}{R_0}\right)\right.$$

$$\left. + i\frac{S_0\ell^2}{2\Lambda}(1+\ell k^2 s_0)\left(\frac{E_v^{\frac{1}{2}}}{R_0}\right)\left(1 - \frac{8\Lambda^2 s_0(s_0 - \ell k s_0)}{S_0^3\ell^4(1+\ell k^2 s_0)^2}\cdot\frac{k_M^2\epsilon}{\left(\frac{E_v^{\frac{1}{2}}}{R_0}\right)^2}\right)^{\frac{1}{2}}\right\} \tag{11}$$

which is of order ε when $E_v^{\frac{1}{2}}/R_O$ is of order ε. So, now omitting the suffix M, set

$$\begin{aligned} S_\delta - S &= \pm\epsilon \\ \tau &= \epsilon t \\ \frac{E_v^{\frac{1}{2}}}{R_0} &= \tilde{\nu}\epsilon \end{aligned} \tag{12}$$

where the upper choice of sign is appropriate to the supercritical or linearly unstable state, then equations (8) to (10) imply that

$$S = S_0 - (\tilde{\sigma}\tilde{\nu} \pm 1)\,\epsilon + O(\epsilon^{\frac{3}{2}}) \qquad (13)$$

where
$$\tilde{\sigma} = \frac{4s_0^2}{\Lambda k^2 l^4}\left(1 + \frac{8 s_0^2 l^2}{k^4(s_0^2-1)}\right)$$

and both $\tilde{\sigma}$ and $\tilde{\nu}$ are strictly order one with respect to ε as $\varepsilon \to 0$. In the supercritical state there is a spectrum of linearly unstable downstream wavenumbers with bandwidth determined by the locally parabolic shape of the neutral surface to be of order $\varepsilon^{\frac{1}{2}}$, centred on the critical wavenumber k_M of equation (9) which is also of order $\varepsilon^{\frac{1}{2}}$. Since both the component wavenumbers of a packet of unstable waves and the packet modulation occur on the same slow scale the disturbance field depends on the downstream coordinate x through the slow spatial scale

$$X = \epsilon^{\frac{1}{2}} x \qquad (14)$$

alone. The reader is asked to compare the scalings of equations (12) to (14) with the scalings appropriate for the evolution of a single wave along the upper branch as given in chapter four by equations (4.11) and (4.12).

Adopting an expansion for the disturbance stream function, ϕ, of the form

$$\phi(X,y,z,\tau;\epsilon) = \epsilon^{\frac{1}{2}}\phi_0(X,y,z,\tau) + \epsilon\,\phi_1(X,y,z,\tau) + \cdots \qquad (15)$$

a suitable expression of the governing equation and boundary conditions, which are (1.66) to (1.69) with $\beta = \nu = \kappa = 0$, for the iteration sequence of linear problems at successive powers of $\varepsilon^{\frac{1}{2}}$ is

$$\begin{aligned}\Lambda z \partial_X (S_0\partial_y^2 + \partial_z^2)\phi = &-\frac{\partial}{\partial(X,y)}\left(\phi, (S_0\partial_y^2+\partial_z^2)\phi\right) - \epsilon^{\frac{1}{2}}\partial_\tau(S_0\partial_y^2+\partial_z^2)\phi \\ &- \epsilon\Lambda z\,\partial_X\left(S_0\partial_X^2\phi - (\tilde{\sigma}\tilde{\nu}\pm 1)\partial_y^2\phi\right) \\ &+ O(\epsilon^{\frac{3}{2}}\phi, \epsilon\phi^2)\end{aligned} \qquad (16)$$

with $$\partial_x \phi = \bar{\phi}_{y\tau} = 0 \quad \text{on} \quad Y = 0,1 \qquad (Y \equiv y + \tfrac{1}{2}) \tag{17),(18}$$

and

$$(\partial_x - z\partial^2_{xz})\phi = \frac{1}{\Lambda}\frac{\partial}{\partial(x,y)}(\phi, \partial_z\phi) + \frac{\epsilon^{\frac{1}{2}}}{\Lambda}\partial^2_{\tau z}\phi - \frac{2z}{h}\tilde{\nu}\epsilon^{\frac{1}{2}}S_0\partial_y^2\phi + O(\epsilon^{\frac{3}{2}}\phi) \quad \text{on } z = \pm\frac{h}{2} \tag{19}$$

The linear problem at leading order is now

$$\Lambda z \partial_x(S_0\partial_y^2 + \partial_z^2)\phi_0 = 0 \tag{20}$$

subject to the boundary conditions

$$\partial_x \phi_0 = \bar{\phi}_{0y\tau} = 0 \qquad \text{on } Y = 0,1 \tag{21),(22}$$

$$(\partial_x - z\partial^2_{xz})\phi_0 = 0 \qquad \text{on } z = \pm\frac{h}{2} \tag{23}$$

which has the solution

$$\phi_0 = A(X,\tau)\sin lY \cosh\left(\frac{2s_0 z}{h}\right) \tag{24}$$

of the linear inviscid problem of Drazin (1970) with k set to zero where

$$l = r\pi \qquad r = 1,2,\ldots\ldots \tag{25}$$

and $A(X,\tau)$ is the wave amplitude function which, in the absence of a fast spatial scale, x, may be assumed to be real without loss of generality. The boundary condition (22) imposes the constraint

$$\int_{-\infty}^{\infty} \frac{\partial A(X,\tau)}{\partial\tau}\, dX = 0 \tag{26}$$

which for an incipient wave may be integrated over τ to give

$$\int_{-\infty}^{\infty} A(x,\tau)\, dx = 0 \tag{27}$$

At O(ε):

$$\Lambda z \partial_x (S_0 \partial_y^2 + \partial_z^2)\, \phi_1 = -\frac{\partial}{\partial(x,y)}\left(\phi_0, (S_0\partial_y^2 + \partial_z^2)\phi_0\right) - \partial_\tau (S_0 \partial_y^2 + \partial_z^2)\, \phi_0 \tag{28}$$

With the boundary condition

$$(\partial_x - z\partial^2_{xz})\, \phi_1 = \frac{1}{\Lambda}\frac{\partial}{\partial(x,y)}(\phi_0, \partial_z \phi_0) + \frac{1}{\Lambda}\partial^2_{\tau z}\phi_0 - \frac{2z}{h}\tilde{\nu} S_0 \partial_y^2 \phi_0 \quad \text{on } z = \pm\frac{h}{2} \tag{29}$$

Since the solution at previous order satisfies

$$(S_0 \partial_y^2 + \partial_z^2)\, \phi_0 = 0$$

the governing equation (28) for ϕ_1 is homogeneous

$$\Lambda z \partial_x (S_0 \partial_y^2 + \partial_z^2)\, \phi_1 = 0 \tag{30}$$

and the boundary conditions are

$$\phi_{1x} = \bar{\Phi}_{1y\tau} = 0 \qquad \text{on } Y = 0,1 \qquad (31),(32)$$

with

$$(\partial_x - z\partial^2_{xz})\, \phi_1 = \pm\frac{2 s_0}{\Lambda h} A_\tau \sin lY \,\mathrm{sh}\, s_0 \pm \tilde{\nu} l^2 S_0 A \sin lY \,\mathrm{ch}\, s_0 \quad \text{on } z = \pm\frac{h}{2} \tag{33}$$

from (24).

Hence the solution is

$$\phi_1 = B(X,\tau)\sin lY\, \mathrm{sh}\left(\frac{2s_0 z}{L}\right) + P(y,z,\tau) \qquad (34)$$

satisfying (30) and (31) automatically and where $P(y,z,\tau)$ may be assumed to be real without loss of generality being determined as the result of a secularity condition to be imposed at the next order. The boundary condition (33) implies that the function $B(X,\tau)$ is such that

$$\partial_X B(X,\tau) = \frac{-s_0}{\Lambda L(s_0^2-1)}\left(2A_\tau + \hat{\sigma} l^2 S_0 \Lambda L A\right) \qquad (35)$$

then (32) requires

$$\int_{-\infty}^{\infty} B(X,\tau)\,dX = 0 \quad \text{and} \quad P_{y\tau} = 0 \ \text{on}\ Y = 0,1 \qquad (36)$$

At $O(\varepsilon^{3/2})$:

The governing equation is

$$\Lambda z \partial_X (S_0\partial_y^2 + \partial_z^2)\phi_2 = -\frac{\partial}{\partial(X,y)}\left(\phi_0, (S_0\partial_y^2+\partial_z^2)\phi_1\right) - \partial_\tau (S_0\partial_y^2+\partial_z^2)\phi_1$$
$$- \frac{\partial}{\partial(X,y)}\left(\phi_1, (S_0\partial_y^2+\partial_z^2)\phi_0\right) - \Lambda z \partial_X\left(S_0\partial_X^2 - (\hat{\sigma}\hat{\nu}\pm 1)\partial_y^2\right)\phi_0$$

The nonlinear terms of the right hand side vanish on using the solutions (24) and (34) at previous orders if

$$(S_0\partial_y^2 + \partial_z^2)P(y,z,\tau) = 0 \qquad (37)$$

and the equation reduces to

$$\Lambda z \partial_X (S_0\partial_y^2 + \partial_z^2)\phi_2 = -\Lambda z\left(S_0\partial_X^3 A + l^2(\hat{\sigma}\hat{\nu}\pm 1)\partial_X A\right)\sin lY\, \mathrm{ch}\left(\frac{2s_0 z}{L}\right) \qquad (38)$$

The top and bottom boundary conditions are

$$(\partial_x - z\partial^2_{xz})\phi_2 = \frac{1}{\Lambda}\frac{\partial}{\partial(x,y)}(\phi_0, \partial_z\phi_1) + \frac{1}{\Lambda}\frac{\partial}{\partial(x,y)}(\phi_1, \partial_z\phi_0) + \frac{1}{\Lambda}\partial^2_{\tau z}\phi_1$$

$$-\frac{2z}{h}\tilde{\gamma} S_0 \partial_y^2 \phi_1 \qquad \text{on } z = \pm\frac{h}{2} .$$

and may be seen to reduce to

$$(\partial_x - z\partial^2_{xz})\phi_2 = \frac{s_0 l}{\Lambda h} \sin 2lY \left(\partial_x(AB) - 2AB_x\right)$$

$$+\frac{2s_0}{\Lambda h}\sin lY \,\mathrm{cls}_0\, B_\tau + \tilde{\gamma} S_0 l^2 \sin lY \,\mathrm{sls}_0\, B$$

$$+\frac{1}{\Lambda}\sin lY\,\mathrm{cls}_0\, P_{yz}\Big|_{z=\pm\frac{h}{2}} . A_x \mp \frac{2s_0}{\Lambda h}\sin lY\,\mathrm{sls}_0\, P_y\Big|_{z=\pm\frac{h}{2}} . A_x \tag{39}$$

$$+\frac{1}{\Lambda}P_{\tau z}\Big|_{z=\pm\frac{h}{2}} \mp \tilde{\gamma} S_0 P_{yy}\Big|_{z=\pm\frac{h}{2}}$$

with the boundary conditions

$$\phi_{2x} = \overline{\phi_{2y\tau}} = 0 \qquad \text{on } Y = 0,1. \tag{40),(41}$$

Now the X independent part of the right hand side of (39) leads to the inclusion of secular terms in the solution for ϕ unless it vanishes identically, hence for a solution representing a smooth packet of unstable waves for which $|A| \to 0$ as $|X| \to \infty$, averaging the right hand side of (39) and using the results (34) leads to

$$P_{\tau z} \mp \tilde{\gamma} S_0 \Lambda P_{yy} = \frac{2s_0 l}{\Lambda h}\sin 2lY\, \overline{AB_x} \quad \text{on } z = \pm\frac{h}{2}$$

and so the boundary value problem for $P(y,z,\tau)$ is

$$(S_0\partial_y^2 + \partial_z^2)P = 0$$

$$P_{y\tau} = 0 \qquad \text{on } Y = 0,1$$

$$P_{\tau z} \mp \tilde{\gamma} S_0 \Lambda P_{yy} = \frac{-2s_0^2 l}{(\Lambda h)^2(s_0^2-1)}\left(\partial_\tau(\bar{A}^2) + \tilde{\gamma} l^2 S_0 \Lambda h \bar{A}^2\right)\sin 2lY \quad \text{on } z = \pm\frac{h}{2} \tag{42}$$

The boundary condition (39) then simplifies to

$$(\partial_x - z\partial^2_{xz})\phi_z = \frac{s_0 l}{\Lambda h}\sin 2lY\left(\partial_x(AB) - 2(AB_x - \overline{AB_x})\right)$$

$$+\frac{1}{\Lambda}A_x\cdot\sin lY\,\mathrm{ch}\,s_0\,\rho_{yz}\Big|_{z=\pm\frac{h}{2}} \mp \frac{2s_0}{\Lambda h}A_x\cdot\sin lY\,\mathrm{sh}\,s_0\,\rho_y\Big|_{z=\pm\frac{h}{2}}$$

$$+\frac{2s_0}{\Lambda h}B_\tau\sin lY\,\mathrm{ch}\,s_0 + B\hat{\nu}S_0 l^2\sin lY\,\mathrm{sh}\,s_0 \qquad (43)$$

$$\text{on } z = \pm\frac{h}{2}.$$

The secularity condition to be imposed by the adjoint linear homogeneous problem is obtained by integrating the identity

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}} \phi_{2x}\,(S_0\partial_y^2 + \partial_z^2)^\dagger\,\phi^*\,dy\,dz = 0 \qquad (44)$$

twice by parts, where † and * denote adjoint operators and functions. As was found to be the case in chapter four the linear problem is effectively inviscid and self-adjoint so that

$$\phi^* = \sin lY\,\mathrm{ch}\,\frac{2s_0 z}{h} \qquad (45)$$

Then, making use of the boundary condition (40) together with the property of ϕ^* that

$$\phi^* = 0 \text{ on } Y = 0, 1 \quad \text{and} \quad \phi^*_z = \frac{\phi^*}{z} \qquad \text{on } z = \pm\frac{h}{2}$$

(44) becomes

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}} \phi^*\,\partial_x(S_0\partial_y^2 + \partial_z^2)\,\phi_2\,dy\,dz$$

$$+\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[\frac{\phi^*}{z}(\phi_{2x} - z\,\phi_{2xz})\right]_{-\frac{h}{2}}^{\frac{h}{2}} dy = 0 \qquad (46)$$

From (38), (43), (45) and the property of $P(y,z,\tau)$ that it is an odd function of both y and z, which may be inferred from (42), the secularity condition (46) implies

$$-\left(S_0 A_{XXX} + \ell^2(\tilde{\sigma}\tilde{s} \pm 1)A_X\right)\frac{L}{4} + \frac{4 s_0}{\Lambda \ell^2} B_\tau + \frac{2\tilde{s} S_0 \ell^2}{s_0 L} B + \frac{4A_X}{\Lambda L}\int_0^1 \left(P_{yz} - \frac{2}{L}P_y\right)\Big|_{z=\frac{L}{2}} \sin^2 \ell Y \, dY = 0$$

On differentiation with respect to X and using the relation (35) between A and B the evolution equation for $A(X,\tau)$ is found to be

$$\begin{aligned} A_{\tau\tau} &+ \frac{\tilde{s}\ell^2 S_0 \Lambda L (s_0^2+1)}{2 s_0^2} A_\tau + \frac{(\tilde{s}\ell^2 S_0 \Lambda L)^2}{4 s_0^2} A \\ &+ \frac{(\Lambda \ell^2)^2 (s_0^2-1)}{32 s_0^2}\left(S_0 A_{XXXX} + \ell^2(\tilde{\sigma}\tilde{s} \pm 1)A_{XX}\right) \\ &- \frac{\Lambda \ell^2 (s_0^2-1)}{2 s_0^2} A_{XX} \int_0^1 \sin^2 \ell Y \left(P_{yz} - \frac{2}{L}P_y\right)_{z=\frac{L}{2}} dY = 0 \end{aligned} \tag{47}$$

where $P(y,z,\tau)$ is the solution to the boundary value problem (42).

The role of the function $P(y,z,\tau)$, which is of consequence in the dynamics at finite amplitude, is analogous to that of the function $f(y,z,\tau)$ of chapter four, so that it may be interpreted physically as the correction at order ε to the X independent, mean field stream function caused by depletion of the available potential energy of the basic flow by what is now the downstream average of the energy of a wave packet disturbance. The boundary value problem determining this correction may be solved by Fourier's method, setting

$$P(y,z,\tau) = \sum_{j=0}^{\infty} D_j(\tau) \cos(2j+1)\pi Y \, \mathrm{sh}(2j+1)\pi S_0^{\frac{1}{2}} z \tag{48}$$

thereby satisfying the governing equation and boundary conditions on the sidewalls automatically. The top and bottom boundary conditions then determine the evolution of the coefficients $\{D_j(\tau)\}$ in terms of $\overline{A^2}$, the downstream average of A^2.

$$\dot{D}_j(\tau) + \frac{2\tilde{\nu}\Lambda}{h}\,(j+\tfrac{1}{2})\pi S_0^{\frac{1}{2}}h\;\mathrm{th}\,(j+\tfrac{1}{2})\pi S_0^{\frac{1}{2}}h\,.\;D_j(\tau)$$

$$= \frac{s_0^2\,\ell\,h}{(\Lambda L)^2(s_0^2-1)}\cdot\frac{a_{2j+1}}{(j+\frac{1}{2})\pi S_0^{\frac{1}{2}}h\;\mathrm{ch}(j+\frac{1}{2})\pi S_0^{\frac{1}{2}}h}\left((\bar{A}^2)_\tau + \tilde{\nu}\,\ell^2 S_0\Lambda h\,\bar{A}^2\right) \tag{49}$$

where $$a_{2j+1} = \frac{2\ell}{\left((j+\frac{1}{2})^2\pi^2 - \ell^2\right)} \qquad j = 0, 1, \ldots$$

are the Fourier cosine coefficients of sin $2\ell Y$ for $Y \in [0,1]$ per equations (4.39). The integral appearing in the nonlinear term of equation (47) may now be replaced by an infinite sum, changing the order of summation and integration by appealing to the uniform convergence of the sum in equation (48) with respect to both y and z. The term is

$$\frac{2\ell^2}{h}\sum_{j=0}^{\infty} D_j(\tau)\left(\frac{(j+\frac{1}{2})\pi S_0^{\frac{1}{2}}h\;\mathrm{ch}(j+\frac{1}{2})\pi S_0^{\frac{1}{2}}h - \mathrm{sh}(j+\frac{1}{2})\pi S_0^{\frac{1}{2}}h}{(j+\frac{1}{2})^2\pi^2 - \ell^2}\right)$$

so that the amplitude equation may be written as

$$A_{\tau\tau} + \tilde{\nu}\Delta_1 A_\tau + \tilde{\nu}^2\Delta_2 A + \Delta_3 A_{xxxx} + \Delta_4 A_{xx} - A_{xx}\sum_{j=0}^{\infty}\tilde{\beta}_j D_j = 0 \tag{50}$$

with

$$D_{j\tau} + \tilde{\nu}\Delta_{5j} D_j = \tilde{d}_j\left((\bar{A}^2)_\tau + \tilde{\nu}\hat{\tilde{\sigma}}(\bar{A}^2)\right) \tag{51}$$

$$j = 0, 1, \ldots$$

where, in terms of the underlying physical parameters,

$$\Delta_1 = \frac{\ell^2 S_0\Lambda h\,(s_0^2+1)}{2s_0^2} \qquad \Delta_2 = \frac{(\ell^2 S_0\Lambda h)^2}{4s_0^2}$$

$$\Delta_3 = \frac{(\Lambda k^2)^2 S_0(s_0^2-1)}{32 s_0^2} \qquad \Delta_4 = \frac{(\Lambda k^2)^2(s_0^2-1)\ell^2(\tilde{\sigma}\tilde{\nu}\pm 1)}{32 s_0^2}$$

$$\Delta_{5j} = \frac{2\Lambda}{h}\left(j+\tfrac{1}{2}\right)\pi S_0^{\frac{1}{2}}h \,\mathrm{th}\left(j+\tfrac{1}{2}\right)\pi S_0^{\frac{1}{2}}h \qquad j = 0, 1, \ldots$$

$$\hat{\hat{\sigma}} = l^2 S_0 \Lambda h$$

$$\tilde{\gamma}_j = \frac{2 s_0^2 l^2 h}{(\Lambda h)^2 (s_0^2 - 1)} \cdot \frac{1}{\left(\left(j+\frac{1}{2}\right)^2\pi^2 - l^2\right)\left(\left(j+\frac{1}{2}\right)\pi S_0^{\frac{1}{2}}h \,\mathrm{ch}\left(j+\frac{1}{2}\right)\pi S_0^{\frac{1}{2}}h\right)}$$

$$\tilde{\beta}_j = \frac{\Lambda h l^2 (s_0^2 - 1)}{s_0^2} \cdot \frac{\left(j+\frac{1}{2}\right)\pi S_0^{\frac{1}{2}}h \,\mathrm{ch}\left(j+\frac{1}{2}\right)\pi S_0^{\frac{1}{2}}h - \mathrm{sh}\left(j+\frac{1}{2}\right)\pi S_0^{\frac{1}{2}}h}{\left(j+\frac{1}{2}\right)^2\pi^2 - l^2} \tag{52}$$

and in the linearly unstable state, all of these coefficients are real and positive.

Comparison of the evolution equation (50) and (51) for the current problem with the amplitude equation of chapter four for the case of a single wave in the form of equations (4.47) and (4.48) suggests a Fourier decomposition of the wave packet amplitude function $A(X,\tau)$. Put

$$A(X,\tau) = \sum_{k \in S} \left(A_k(\tau) e^{ikX} + A_k^*(\tau) e^{-ikX} \right) \tag{53}$$

implying

$$E(\tau) \equiv \overline{A^2(\tau)} = 2 \sum_{k \in S} |A_k(\tau)|^2$$

where the sum is taken over all discrete, order one values of k in a chosen spectrum, S, of $A(X,\tau)$ being either finite or infinite as desired. In the continuous case the sum may be replaced by a Fourier integral. Equations (50) and (51) then become the set of ordinary differential equations for the Fourier components $\{A_k\}$

$$A_{k\tau\tau} + \nu \Delta_1 A_{k\tau} - (k^2\Delta_4 - k^4\Delta_3 - \nu^2\Delta_2) A_k + k^2 \sum_{j=0}^{\infty} \tilde{\beta}_j D_j = 0 \tag{54}$$

$$D_{j\tau} + \nu \Delta_{5j} D_j = \tilde{\gamma}_j \left(\frac{dE}{d\tau} + \nu \hat{\hat{\sigma}} E \right) \qquad j = 0, 1, \ldots \tag{55}$$

where

$$E(\tau) = 2 \sum_{k \in S} |A_k|^2 \tag{56}$$

Under a transformation similar to that leading to equations (4.54) to (4.56) for each component A_k, viz.

$$A_k = \delta X_k \ , \quad T = \omega \tau \quad , \quad \nu_j = \frac{\rho\omega}{k^2 \tilde{\beta}_j} Z_j$$

where

$$b_j = \frac{\tilde{\nu}\Delta_{5j}}{\omega} \quad , \quad \sigma = \frac{\tilde{\nu}\hat{\tilde{\sigma}}}{2\omega} \quad , \quad d_j = k^2 \tilde{d}_j \tilde{\beta}_j$$

equation (55) becomes

$$\dot{Z}_j + b_j Z_j = \frac{d_j}{2} \sum_{k \in S} (X_k Y_k^* + X_k^* Y_k) \qquad \text{where } j = 0, 1, \ldots$$

by choice of the defining relations

$$\delta^2 = \frac{\rho\omega}{2\sigma} \quad , \quad \sigma = \frac{\rho}{\omega} \qquad \left(\Rightarrow \delta^2 = \frac{\omega^2}{2} \right)$$

and introduction of the dependent variables $\{Y_k\}$ defined by

$$\dot{X}_k = -\sigma (X_k - Y_k)$$

Then (54) becomes

$$\dot{Y}_k = r X_k - Y_k - X_k \sum_{j=0}^{\infty} Z_j$$

where

$$\rho = \tilde{\nu}\Delta_1 - \omega \quad \text{and} \quad r = 1 + \frac{k^2\Delta_4 - k^4\Delta_3 - \mathfrak{z}^2\Delta_2}{\rho\omega}$$

and so each Fourier component of the spectrum S is such that

$$\dot{X}_k = -\sigma (X_k - Y_k) \tag{57}$$

$$\dot{Y}_k = r X_k - Y_k - X_k \sum_{j=0}^{\infty} Z_j \tag{58}$$

$$\dot{Z}_j = -b_j Z_j + \frac{d_j}{2} \sum_{k \in S} (X_k Y_k^* + X_k^* Y_k) \tag{59}$$

$$j = 0, 1, \ldots$$

This set of equations compares in form with the Lorenz type equations (4.54) to (4.56), the symbols have been duplicated for clarity and represent different quantities in the two analogous problems, but note that the nonlinearity of equation (59) now involves the sum over all components of the Fourier spectrum S. This feature arises in the case of an unstable wave packet since, as mentioned previously, the mean field correction, $P(y,z,\tau)$, to the basic flow is now a function of the downstream average of the total energy $(\overline{A^2})$ of the packet which is the sum of the energies of its individual spectral members. Some numerical integrations of the set of equations (57) to (59) have been performed by Brindley and Moroz (unpublished) for the coefficients of a multi-layer model of baroclinic instability.

Now return to the amplitude equation in the form of equations (54) to (56). Linearising the equations to consider the initial evolution of the $\{A_k\}$ for an incipient wave in the supercritical state gives the single equation

$$A_{k\tau\tau} + \tilde{\nu}\Delta_1 A_{k\tau} - k^2 g(k) A_k = 0 \qquad (60)$$

where $$g(k) = \Delta_4 - \Delta_3 k^2 - \frac{\tilde{\nu}\Delta_2}{k^2} \quad \text{and} \quad \Delta_4 > 0 \qquad (61)$$

Then it is easy to see by comparison with the form of the amplitude equations for the 2-layer model as derived by Pedlosky (see Pedlosky, 1981, equations (2.33) and (2.34)) that the analytical conclusions of section 3 of that paper, concerning the evolution of free waves in the absence of any source of potential vorticity in the basic flow, are also valid for Eady's model. A little algebra establishes these results; the linear growth rate of each component, A_k, is

$$\omega(k) = -\frac{\tilde{\nu}\Delta_1}{2} + \frac{1}{2}\sqrt{\left((\tilde{\nu}\Delta_1)^2 + 4k^2 g(k)\right)} \qquad (62)$$

for consideration of the unstable state, so that for the temporal growth of this component necessarily

$$g(k) > 0 \qquad \text{i.e.} \qquad \Delta_4 - \Delta_3 k^2 - \frac{\tilde{\nu}\Delta_2}{k^2} > 0$$

and the unique value of k which maximises g(k), with all other parameters fixed, is

$$k_1 = \left(\frac{\tilde{\nu}^2 \Delta_2}{\Delta_3}\right)^{\frac{1}{4}} \tag{63}$$

and $g(k_1) = \max\limits_{k} g(k) = \Delta_4 - 2\tilde{\nu}(\Delta_2\Delta_3)^{\frac{1}{2}}$ there so that the growth of any incipient disturbance requires

$$\Delta_4 - 2\tilde{\nu}(\Delta_2\Delta_3)^{\frac{1}{2}} > 0, \text{ or equivalently, } \quad \tilde{\nu} < \frac{\Delta_4}{2(\Delta_2\Delta_3)^{\frac{1}{2}}} \tag{64}$$

The curve g(k) locally reproduces the geometry of the cross-section of the neutral surface (5) in the (k,S) plane and is sketched in figure (1).

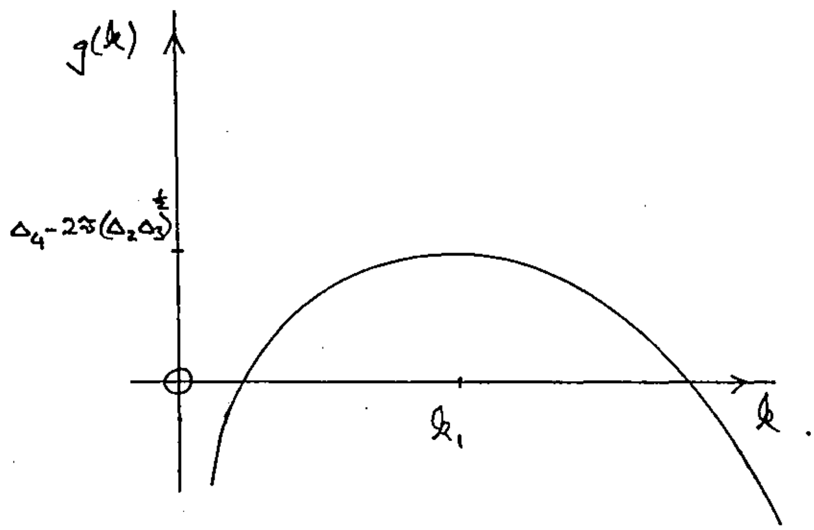


Figure (1)

However from (62), the unique value of k which maximises the linear growth rate under the necessary condition for instability (64) is the value

$$k_2 = \left(\frac{\Delta_4}{2\Delta_3}\right)^{\frac{1}{2}} \tag{65}$$

which maximises $k^2 g(k)$. It is straightforward to verify that

$$\omega(k_2) \geqslant \omega(k_1) \qquad \text{since} \quad \left(23\,\Delta_2^{\frac{1}{2}} - \frac{\Delta_4}{\Delta_3^{\frac{1}{2}}}\right)^2 \geqslant 0 \tag{66}$$

with equality if and only if

$$\operatorname*{Max}_{k}\left(g(k)\right) = 0$$

in which case $k_1 = k_2$ and the interval of unstable values of k shrinks to the single wavenumber k, which is then linearly neutrally stable.

When the inequalities (66) are strict the spectrum S may admit more than one linearly unstable member and the linearly most unstable of these, the one which will dominate all others in the initial development of a wave packet disturbance, is k_2. Note that

$$k_2 > k_1$$

so that this wave component has a shorter downstream wavelength than the wave component associated with k_1.

Now consider the possibility of evolution to an asymptotically steady state. Then from (54) and (55) in the steady state where the component A_k assumes the value $A_s(k)$

$$A_s\left(g(k) - 2\hat{\tilde{\sigma}}\sum_{j=0}^{\infty}\frac{\tilde{\beta}_j\tilde{f}_j}{\Delta_{sj}}\cdot\sum_{k\in S}A_s^2\right) = 0$$

which implies that either $A_s(k) = 0$, or

$$\sum_{k\in S}A_s^2(k) = \frac{g(k)}{2\hat{\tilde{\sigma}}\cdot\sum_{j=0}^{\infty}\frac{\tilde{\beta}_j\tilde{f}_j}{\Delta_{sj}}} \tag{67}$$

where the sum over j may be seen to converge from (52). Hence from figure (1), since there are only two values of k which correspond to a given value of $\Sigma A_s^{\,2}$, at most two components A_k may coexist if a steady state is attained. The unique value of k which maximises $\Sigma A_s^{\,2}$, and thus from equation (53) which maximises $\overline{A^2}$, in the steady state is the value k_1 which maximises $g(k)$. This last result is the analogy of

Pedlosky's result for the 2-layer model that the steady state of largest $\overline{A^2}$ consists of a single wave which is longer than the wave of maximum linear growth rate. Moreover it may be suggested that this single steady wave of largest $\overline{A^2}$ is the only steady solution of the amplitude equation which is linearly stable to perturbations throughout the spectrum S. Consider the linear stability of a steady state involving the two distinct wavenumbers κ_1 and κ_2, then in this state

$$A_s^2(\kappa_1) + A_s^2(\kappa_2) = \frac{g(\kappa_1)}{2\hat{\tilde{\sigma}} \sum_0^\infty \frac{\tilde{\beta}_j \hat{\tilde{\gamma}}_j}{\Delta_{sj}}}$$

and $A_k = 0 \quad \forall k \neq \kappa_1, \kappa_2$.
so that if the state is perturbed by setting

$$A_{\kappa_1} = A_s(\kappa_1) + a_{\kappa_1}$$

$$A_{\kappa_2} = A_s(\kappa_2) + a_{\kappa_2}$$

$$A_k = a_k \qquad \forall k \neq \kappa_1 \text{ or } \kappa_2$$

then the linear stability of the background disturbances $\{a_k : k \neq \kappa_1, \kappa_2\}$ is given by

$$\ddot{a}_k + \tilde{\nu}\Delta_1 \dot{a}_k - k^2 a_k \left(g(k) - g(\kappa_1)\right) = 0 \qquad (68)$$

and so only those components of the disturbance for which $g(k) > g(\kappa_1)$ will be unstable. The same equation (68) also determines the linear stability of $\{a_k\}$ when the steady state consists of the single wavenumber k_1, now as it is this value of k which maximises $g(k)$ the background disturbances are then all linearly stable or decaying. If it may be shown that for any disturbance

$$\partial_\tau \overline{A^2} = 2\int_{-\infty}^{\infty} A A_\tau \, dX = 0 \left(\Rightarrow \sum_{k \in S} A_k^2(\tau) = \text{constant}\right)$$

then it may be inferred that this is the only linearly stable, steady solution of the equations in finite amplitude.

The paper by Pedlosky contains a wealth of detailed numerical integrations of the evolution equation as determined for the 2-layer model and the reader is referred to §3 there for an account. Later in the same article the evolution of a disturbance towards a steady state is considered when either the supercriticality is time dependent or the flow is forced by the inclusion of a source term of potential vorticity within each layer and the results compared to the free case.

CONCLUSION

In comparing the results of the model from which the dispersion relation (2.54) of chapter two is derived with those of the existing completely inviscid models of baroclinic instability, it should be noted that the form of the second order differential equation determining the eigenfunctions of normal mode disturbances is decisive in determining the morphology of the zeros of the dispersion relation in the complex plane and thus of the stability of the flow. The model here adopts a linear shear velocity profile which is singular at infinite height and this singularity coincides with that arising from the vanishing density of the fluid there so that the governing equation is the confluent hypergeometric equation, possessing a regular singularity at the wave steering level, $z = \frac{c_r}{\Lambda h}$, and an irregular singular point of rank one at $z = \infty$ occurring through a confluence of two regular singularities there. When the upper boundary is at finite height the latter singularity is outside the physical domain of the flow but, if we accept that it is to some extent the singularities of the differential equation which govern the behaviour of instability, this is seen to be of no consequence.

The way in which the flow may change stability occurs in one of two main and usually distinct ways. In the first it becomes unstable as the phase speed of two neutral modes coincides, this is reminiscent of stability change in the Eady model but the phase speed at marginal stability here is less than the basic flow speed at the lower boundary and not equal to the average flow speed between the boundaries. In the second, the flow becomes unstable as the phase speed of a neutral mode approaches the speed of the basic flow at the lower boundary, the mode then splits into a pair of exponentially growing and decaying modes in a qualitatively similar way to the passage to instability through neutral stability for Charney's model (see Miles 1964a); although the latter is in fact formed by the coincidence of the two main instances cited above in the appropriate limit, as is explained in the text. Throughout parameter space there is at most one unstable mode. A further type of behaviour of the zeros found here which in no way affects the stability of the flow corresponds to the vanishing of a neutral mode as its phase speed approaches that of the basic flow at the lower boundary when the flow is already linearly unstable.

This last property of the model is similar to that found by Miles (1964c) for a different one. Miles noted that the singularity of the linear shear velocity profile at infinite height is unphysical for a semi-infinite atmosphere and devised a semi-infinite model with a basic state in which the velocity is bounded at all heights and only the density singularity remains at infinite height. The governing second order differential equation for this model is the hypergeometric equation, possessing three regular singular points two of which are in the finite part of the complex plane. It is found that the flow is linearly unstable throughout parameter space with one unstable mode. The growing and conjugate decaying modes remain as such for all parameter values but neutral modes may appear or disappear in the way described above and as sketched in figure (2.9b).

No attempt has been made to obtain bounds for the growth rate and phase speed of the non-neutral modes but it seems very likely that these could be obtained by the methods needed for the proof of Howard's semi-circle theorem (Howard 1961) and of Pedlosky's extension to this (Pedlosky 1964).

Another endeavour might be to consider the application of weakly nonlinear theory to the two cases of marginal instability of this model and to compare its results with the existing ones for the two-layer and Eady models where the β-plane effect may and may not be included respectively. A further problem which suggests itself in chapter two would be to include the effects of viscosity through Ekman layers upon the model and to carefully consider how the formulation of this problem may need some further modification if the zeros of the dispersion relation come close to the top and bottom boundary points.

The β-plane effect, whilst showing a destabilising influence in the linear regime, appears from the results of chapters five and six to exert a stabilising or smoothing influence upon the dynamics at finite amplitude. The inclusion of the β-plane effect for the amplitude equation of chapter four in the case of the two-layer model does add another parameter to the problem but this permits an approach to the region in parameter space corresponding to chaotic solutions in a different direction and seems to leave the problem as tractable.

For the doubly-periodic solutions found after bifurcation in chapter six there exists the problem of perturbing about a point on the solid curves of figures (6.1) and (6.2) where the coefficient $Re(k_2)$ vanishes which will require taking the iteration scheme to higher orders. A solution to this problem may possibly lead to some further unravelling of the full bifurcation sequence.

In this context note that it is not an increase of the (initial) bifurcation parameter, which is either the Burger or Froude number, alone which is responsible for the development of new periods in the stable solutions of the amplitude equation but rather the variation of the effects of viscosity through the parameter $\bar{r}$ or ν. Recall that $\bar{r}$ and ν are the strictly order one parts of the viscosity parameter in its asymptotic linking with the supercriticality for the slightly linearly unstable state. Thus the doubly-periodic nature of the solutions to the amplitude equation which are found when the exact singly-periodic limit cycle solution becomes unstable with the increase of the bifurcation parameter r_1 occur in a slightly dissimilar way to the development of a second period in Taylor vortex flow with its transition through wavy Taylor vortex flow with increasing Taylor number as described by Stuart (1971).

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APPENDIX A

From the power series solutions (11) and (35), substitution in the definition (32) with $\mu_0 \neq 0$ for $ph(s) \in (-\pi,\pi)$ and $ph(s+\alpha) \in (-\pi,\pi)$ yields

$$
\begin{aligned}
D(a,2,\alpha_0,\alpha+\delta\alpha,\mu_0;s) = &-\left(\frac{1-i\mu_0}{\Gamma(2-a)}\right)\left(\alpha^2 P(\alpha) - i\mu_0\alpha M(\alpha)\right) \\
&- \frac{i\mu_0}{\Gamma(1-a)}\left(\alpha^2 P(\alpha) - i\mu_0\alpha M(\alpha)\right) s\ln s \\
&+\left(i\mu_0\left(\alpha^2 Q(\alpha) - i\mu_0\alpha u(\alpha)\right) - \left(\frac{1+i\mu_0 a_1}{\Gamma(1-a)} + \frac{\alpha_0 - i\mu_0}{\Gamma(2-a)}\right)\left(\alpha^2 P(\alpha) - i\mu_0\alpha M(\alpha)\right)\right) s \\
&- \left(\frac{1-i\mu_0}{\Gamma(2-a)}\right)\left(\alpha^2 P'(\alpha) + 2\alpha P(\alpha) - i\mu_0\left(\alpha M'(\alpha) + M(\alpha)\right)\right)(s+\delta\alpha) \\
&- \left(\frac{a(1+i\mu_0) - 2\alpha_0}{2\Gamma(1-a)}\right)\left(\alpha^2 P(\alpha) - i\mu_0\alpha M(\alpha)\right) s^2\ln s \\
&- \frac{i\mu_0}{\Gamma(1-a)}\left(\alpha^2 P'(\alpha) + 2\alpha P(\alpha) - i\mu_0\left(\alpha M'(\alpha) + M(\alpha)\right)\right) s(s+\delta\alpha)\ln s \\
&- \Bigg(\left(\alpha^2 P(\alpha) - i\mu_0\alpha M(\alpha)\right)\left(\frac{(a-2)(a_2+1+i\mu_0 a_2) + 2(a_1+1) + 2a_1(i\mu_0 - \alpha_0)}{2\Gamma(1-a)} + \frac{2\alpha_0 - 1 - i\mu_0}{2\Gamma(2-a)}\right) \\
&\qquad - \left(\frac{a}{2}(1+i\mu_0) - \alpha_0\right)\left(\alpha^2 Q(\alpha) - i\mu_0\alpha u(\alpha)\right)\Bigg) s^2 \\
&+\Bigg(i\mu_0\left(\alpha^2 Q'(\alpha) + 2\alpha Q(\alpha) - i\mu_0\left(\alpha u'(\alpha) + u(\alpha)\right)\right) - \left(\frac{1+i\mu_0 a_1}{\Gamma(1-a)} + \frac{\alpha_0 - i\mu_0}{\Gamma(2-a)}\right) \\
&\qquad \times\left(\alpha^2 P'(\alpha) + 2\alpha P(\alpha) - i\mu_0\left(\alpha M'(\alpha) + M(\alpha)\right)\right)\Bigg) s(s+\delta\alpha) \\
&- \left(\frac{1-i\mu_0}{\Gamma(2-a)}\right)\left(\frac{\alpha^2}{2} P''(\alpha) + 2\alpha P'(\alpha) + P(\alpha) - i\mu_0\left(\frac{\alpha}{2} M''(\alpha) + M'(\alpha)\right)\right)(s+\delta\alpha)^2 \\
&- \left(\alpha^2 P(\alpha) - i\mu_0\alpha M(\alpha)\right)\left(\frac{a(a+1)(2+i\mu_0) - 6\alpha_0 a}{3!\,2!\,\Gamma(1-a)}\right) s^3\ln s \\
&- \left(\alpha^2 P(\alpha) + 2\alpha P(\alpha) - i\mu_0\left(\alpha M'(\alpha) + M(\alpha)\right)\right)\left(\frac{a(1+i\mu_0) - 2\alpha_0}{2\Gamma(1-a)}\right) s^2(s+\delta\alpha)\ln s \\
&- \frac{i\mu_0}{\Gamma(1-a)}\left(\frac{\alpha^2}{2} P''(\alpha) + 2\alpha P'(\alpha) + P(\alpha) - i\mu_0\left(\frac{\alpha}{2} M''(\alpha) + M'(\alpha)\right)\right) s(s+\delta\alpha)^2\ln s \\
&+ O\left(s^3, s^2\delta\alpha, s\delta\alpha^2, \delta\alpha^3, s^4\ln s, s^3\delta\alpha\ln s, \text{etc.}\right) \qquad \text{(A1)}
\end{aligned}
$$

When $\mu_0 = 0$

$$D(a, z, \alpha_0, \alpha+\delta\alpha, 0; s) = -\frac{\alpha^2 P(\alpha)}{\Gamma(2-a)} - \left(\frac{1}{\Gamma(1-a)} + \frac{\alpha_0}{\Gamma(2-a)}\right)\alpha^2 P(\alpha) s$$

$$-\frac{1}{\Gamma(2-a)}\left(\alpha^2 P'(\alpha) + 2\alpha P(\alpha)\right)(s+\delta\alpha)$$

$$+\frac{2\alpha_0 - a}{2\Gamma(1-a)}\alpha^2 P(\alpha) s^2 \ln s - \alpha^2 P(\alpha)\left(\frac{(a-2)(a_2+1)+2(a_1+1-a_1\alpha_0)}{2\Gamma(1-a)} + \frac{2\alpha_0 - 1}{2\Gamma(2-a)}\right)s^2$$

$$+\left(\frac{a}{2} - \alpha_0\right)\alpha^2 Q(\alpha) s^2 - \left(\frac{1}{\Gamma(1-a)} + \frac{\alpha_0}{\Gamma(2-a)}\right)\left(\alpha^2 P'(\alpha) + 2\alpha P(\alpha)\right) s(s+\delta\alpha)$$

$$-\frac{1}{\Gamma(2-a)}\left(\frac{\alpha^2}{2!}P''(\alpha) + 2\alpha P'(\alpha) + P(\alpha)\right)(s+\delta\alpha)^2 - \alpha^2 P(\alpha)\left(\frac{a(a+1)(2+i\mu_0) - 6\alpha_0 a}{3!\,2!\,\Gamma(1-a)}\right)s^3 \ln s$$

$$-\left(\alpha^2 P'(\alpha) + 2\alpha P(\alpha)\right)\left(\frac{a - 2\alpha_0}{2\Gamma(1-a)}\right) s^2(s+\delta\alpha)\ln s \qquad \text{(A2)}$$

$$+ \cdots\cdots$$

When α is such that $P(\alpha) = 0$ and further $\delta\alpha = 0$

$$D(a, z, \alpha_0, \alpha, \mu_0, s) = i\mu_0\alpha\left(\frac{1 - i\mu_0}{\Gamma(2-a)}\right)M(\alpha) - \frac{\mu_0^2 \alpha M(\alpha)}{\Gamma(1-a)} s \ln s$$

$$+\Bigg(i\mu_0\left(\alpha^2 Q(\alpha) - i\mu_0\alpha u(\alpha)\right) + \left(\frac{1+a_1 i\mu_0}{\Gamma(1-a)} + \frac{\alpha_0 - i\mu_0}{\Gamma(2-a)}\right)i\mu_0\alpha M(\alpha)$$

$$-\left(\frac{1 - i\mu_0}{\Gamma(2-a)}\right)\left(\alpha^2 P'(\alpha) - i\mu_0\left(\alpha M'(\alpha) + M(\alpha)\right)\right)\Bigg) s$$

$$+\left(i\mu_0\alpha M(\alpha)\left(\frac{a(1+i\mu_0) - 2\alpha_0}{2\Gamma(1-a)}\right) - \frac{i\mu_0}{\Gamma(1-a)}\left(\alpha^2 P'(\alpha) - i\mu_0\left(\alpha M'(\alpha) + M(\alpha)\right)\right)\right)s^2 \ln s$$

$$+\Bigg(i\mu_0\left(\alpha^2 Q'(\alpha) + 2\alpha Q(\alpha) - i\mu_0\left(\alpha u'(\alpha) + u(\alpha)\right)\right) + \left(\frac{a}{2}(1+i\mu_0) - \alpha_0\right)\left(\alpha^2 Q(\alpha) - i\mu_0\alpha u(\alpha)\right)$$

$$-\left(\frac{1+i\mu_0 a_1}{\Gamma(1-a)} + \frac{\alpha_0 - i\mu_0}{\Gamma(2-a)}\right)\left(\alpha^2 P'(\alpha) - i\mu_0\left(\alpha M'(\alpha) + M(\alpha)\right)\right) \qquad \text{(A3)}$$

$$+ i\mu_0\alpha M(\alpha)\left(\frac{(a-2)(a_2+1+i\mu_0 a_2) + 2(a_1+1) + 2a_1(i\mu_0 - \alpha_0)}{2\Gamma(1-a)} + \frac{2\alpha_0 - 1 - i\mu_0}{2\Gamma(2-a)}\right)$$

$$-\left(\frac{1 - i\mu_0}{\Gamma(2-a)}\right)\left(\frac{\alpha^2}{2!}P''(\alpha) + 2\alpha P(\alpha) - i\mu_0\left(\frac{\alpha}{2}M''(\alpha) + M'(\alpha)\right)\right)\Bigg] s^2 + \cdots$$

The following abbreviations have been used:-

$$a_1 = \psi(2-a) - \psi(1) - \psi(2)$$

$$a_2 = \psi(3-a) - \psi(2) - \psi(3)$$

where $\psi(z)$ is the psi or digamma function

$$P(\alpha) \text{ denotes } P(a,2;\alpha) = \left(M'(a,2,s) - \alpha_0 M(a,2,s)\right)\Big|_{s=\alpha}$$

$$Q(\alpha) \text{ denotes } Q(a,2,;\alpha) = \left(u'(a,2,s) - \alpha_0 u(a,2,s)\right)\Big|_{s=\alpha}$$

Similarly, when $\mu_0 = \delta\alpha = 0$ and $P(\alpha) = P'(\alpha) = 0$

$$D(a+\delta a, 2, \alpha_0, \alpha, 0; s) = \alpha^2\left(\frac{a-2\alpha_0}{2}\right) Q(a,2,\alpha_0;\alpha)\, s^2$$

$$- \frac{\alpha^2}{\Gamma(2-a)} \frac{\partial P}{\partial a}(a,2,\alpha_0,\alpha) . \delta a - \frac{\alpha^2}{2\Gamma(2-a)} P''(a,2,\alpha_0;\alpha)\, s^2$$

$$- \frac{\alpha^2(a-2\alpha_0)}{2\Gamma(1-a)} \frac{\partial P}{\partial a}(a,2,\alpha_0;\alpha) . \delta a\, s^2 \ln s \qquad (A4)$$

$$+ \ldots$$

APPENDIX B

For the 2-layer model where $r = O(\Delta^{\frac{1}{2}})$, $\beta = O(\Delta^{\frac{1}{2}})$ and $k = O(1)$ the amplitude equations for the wave amplitude $A(\tau)$ is (4.47) with (4.48), i.e.;

$$\ddot{A} + \nu\Delta_1 \dot{A} - \alpha A + A\sum_{j=0}^{\infty} \beta_j B_j = 0 \tag{B1}$$

$$\dot{B}_j + \nu\Delta_{2j} B_j = \hat{\phi}_j \left((|\dot{A}|^2) + \nu\hat{\sigma}|A|^2 \right)_{j=0,1,\cdots} \tag{B2}$$

where now Δ_1 and α are complex functions of the underlying physical parameters which, with the notation of Gibbon and McGuinness (1981) (with $r = |\Delta|^{\frac{1}{2}}\bar{r}$, $\beta = |\Delta|^{\frac{1}{2}}\bar{\beta}$, etc.), implies

$$\nu\Delta_1 = \frac{3}{2}\left(\bar{r} - \frac{i k\bar{\beta}}{a^2}\right) \qquad \alpha = \frac{k^2(\Delta u)^2}{4a^2} + \frac{5k^2\bar{\beta}^2}{9a^4} + \frac{i k\bar{r}\bar{\beta}}{a^2}$$

$$\nu\hat{\sigma} = 2\bar{r} \qquad \beta_j = \frac{2k^2(\Delta u)m^2\pi^2}{a^2}\cdot\frac{(j+\frac{1}{2})^2\pi^2}{\left((j+\frac{1}{2})^2\pi^2 - m^2\pi^2\right)} \tag{B3}$$

$$\nu\Delta_{2j} = \frac{\bar{r}(j+\frac{1}{2})^2\pi^2}{(j+\frac{1}{2})^2\pi^2 + \frac{a^2}{4}} \qquad \hat{\phi}_j = \frac{a^2m^2\pi^2}{(\Delta u)}\,\frac{1}{\left((j+\frac{1}{2})^2\pi^2 - m^2\pi^2\right)\left((j+\frac{1}{2})^2\pi^2 + \frac{a^2}{4}\right)}$$

Then, under the transformation preceding equations (4.54) to (4.56), i.e.;

$$\frac{\nu\Delta_{2j}}{\hat{\omega}} = \delta_j \qquad \sigma = \frac{\hat{\sigma}\nu}{2\hat{\omega}} \qquad \phi_j = \beta_j\hat{\phi}_j \qquad \sigma = \frac{\rho}{\hat{\omega}}$$

$$\delta^2 = \frac{\omega^2}{2} \qquad A = \delta X \qquad T = \hat{\omega}\tau \qquad B_j = \frac{\rho\hat{\omega}}{\beta_j} Z_j \tag{B4}$$

where the last two relations there are now replaced by

$$\hat{\omega} = R_e(\nu\Delta_1) - \rho \qquad r_1 = 1 + \frac{R_e(\alpha)}{\rho\hat{\omega}}$$

$$e = -\frac{I_m(\nu\Delta_1)}{\hat{\omega}} \qquad r_2 = \frac{I_m\,\alpha}{\rho\hat{\omega}} - e \tag{B5}$$

the amplitude equations for the transformed amplitude X as a function of the new time T are (4.61) to (4.63)

$$\dot{X} = -\sigma(X - Y) \tag{B6}$$

$$\dot{Y} = rX - aY - X\sum_{j=0}^{\infty} Z_j \qquad \begin{array}{l} r = r_1 + ir_2 \\ a = 1 - ie \end{array} \tag{B7}$$

$$\dot{Z}_j = -b_j Z_j + \frac{d_j}{2}(XY^* + X^*Y) \qquad j = 0, 1, \ldots \tag{B8}$$

where, in terms of the underlying physical parameters

$$\sigma = 2 \,,\ \rho = \bar{r} \,,\ \hat{\omega} = \frac{\bar{r}}{2} \,,\ e = \frac{3k\bar{\beta}}{a^2\bar{r}}$$

$$r_2 = -\frac{k\bar{\beta}}{a^2\bar{r}} \quad \Rightarrow \quad e = -3r_2 \tag{B9}$$

$$r_1 = 1 + \frac{2}{\bar{r}^2}\left(\frac{k^2(\Delta u)^2}{4a^2} + \frac{5k\bar{\beta}}{9a^2}\right)$$

$$b_j = \frac{2(j+\frac{1}{2})^2\pi^2}{\left((j+\frac{1}{2})^2\pi^2 + \frac{a^2}{4}\right)} \qquad d_j = 2k^2\omega^4\pi^2 \cdot \frac{(j+\frac{1}{2})^2}{\left((j+\frac{1}{2})^2 - \omega^2\right)^2\left((j+\frac{1}{2})^2\pi^2 + \frac{a^2}{4}\right)} \qquad j = 0, 1, \ldots$$

When the sidewall boundary condition $\Phi^{(2)}_{iy\tau} = 0$ is relaxed the amplitude equation becomes

$$\ddot{A} + \nu\Delta_1\dot{A} - \alpha A + \beta AB = 0 \tag{B10}$$

$$\dot{B} + \nu\Delta_2 B = (|\dot{A}|^2) + \nu\hat{\sigma}|A|^2 \tag{B11}$$

with a single B component where the coefficients are the same as in (B3) except that now

$$\{\beta_j\} \to \beta = \frac{2k^2\omega^4\pi^4}{(a^2 + 4\omega^2\pi^2)} \,,\ \nu\Delta_2 = \bar{r} \cdot \frac{4\omega^2\pi^2}{(a^2 + 4\omega^2\pi^2)} \,,\ \hat{d} = 1 \tag{B12}$$

and under transformation the equations are (B6) to (B8) with a single Z component and the same coefficients that appear in (B9) except that now

$$\{b_j\} \to b = \frac{8\omega^2\pi^2}{(a^2 + 4\omega^2\pi^2)} \,,\qquad \{d_j\} \to 1 \tag{B13}$$

With reference to the sets of 2-layer equations mentioned at the end of chapter 5, note that the limit $\beta \to 0$ (i.e.; recovering the real sets of equations) is a regular limit in the above.

APPENDIX C

Consider the second order differential equation

$$A_{\tau\tau} - i k_0 A_\tau = k_1 A - k_2 A|A|^2 \qquad A \in \mathbb{C} \tag{C1}$$

which is formed from the amplitude equations (B1) and (B2) in the limit $\bar{r} \to 0$, that is where the viscous effects of the Ekman layers vanishes. The constants k_0, k_1 and k_2 are real and positive and may be expressed in terms of the underlying physical parameters from (B3). Recall that k_1, as indicated by equation (4.46) of chapter four, is a function of the initial condition A(O).

Under the transformation

$$T = k_1^{\frac{1}{2}} \tau \quad , \qquad A = (k_1/k_2)^{\frac{1}{2}} X$$

(C1) becomes

$$X_{TT} - i\beta X_T = X - X|X|^2 \tag{C2}$$

where $\beta = k_0/k_1^{\frac{1}{2}} > 0$ is a measure (after non-dimensionalisation and several scalings) of the β-plane effect. Now set

$$X = r e^{i\theta}$$

where r and θ are real, then

$$\ddot{r} - r\dot{\theta}^2 + \beta r \dot{\theta} = r - r^3$$

and

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - \beta\dot{r} = 0$$

The second equation of this pair may be integrated to give

$$r^2\dot{\theta} = h + \frac{\beta r^2}{2} \tag{C3}$$

where h is a constant of integration identifiable with an angular momentum, then on substitution the first equation becomes

$$\ddot{r} = \frac{h^2}{r^3} + \left(1 - \frac{\beta^2}{4}\right) r - r^3$$

which may be integrated to give

$$\dot{r}^2 + \frac{h^2}{r^2} - \left(1 - \frac{\beta^2}{4}\right) r^2 + \frac{r^4}{2} = E \tag{C4}$$

where $\mathcal{E}$ is a constant of integration identifiable with an energy which in the limiting case $\beta = 0$, $h = 0$ denotes the same quantity as the symbol $\mathcal{E}$ of chapter five. All quantities appearing in equations (C3) and (C4) are now real so that for a non-trivial solution of (C1) to exist

$$\frac{\ell^2}{r^2} - \left(1-\frac{\beta^2}{4}\right)r^2 + \frac{r^4}{2} \leq \mathcal{E} \tag{C5}$$

Equation (C4) may be integrated to give the solution for r and θ expressed in terms of elliptic functions. Changing the variable to $u = r^2$ yields, for $h \neq 0$,

$$\int_{u_0}^{u} \frac{du}{\sqrt{\left(2\mathcal{E}u - 2\ell^2 + 2\left(1-\frac{\beta^2}{4}\right)u^2 - u^3\right)}} = \sqrt{2}\int_{T_0}^{T} dT$$

From (C5) the radicand is positive over the region of integration and considered as a cubic in u it has one real negative root and two real positive roots. Call these a, b and c with

$$a > b > 0 > c$$

the location of each root depends upon the initial conditions through h and $\mathcal{E}$ and on the underlying physical parameters. Hence the radicand may be factorised to the form

$$(a-u)(b-u)(u-c)$$

and the path of integration is contained in the interval [b,a]. Hence

$$r = \pm\sqrt{\left(\frac{b - c\kappa^2 . \mathrm{sn}^2\left(\sqrt{\frac{a-c}{2}}(T-T_0), \kappa\right)}{\mathrm{dn}^2\left(\sqrt{\frac{a-c}{2}}(T-T_0), \kappa\right)}\right)} \tag{C6}$$

(See for example, Byrd and Friedman (1953)), where κ is the modulus of the elliptic function sn such that

$$\kappa^2 = \frac{a-b}{a-c}.$$

The single period of r is then

$$T_p = 2\sqrt{\frac{2}{a-c}}\,.\,K(\kappa)$$

where $K(\kappa)$ is the complete elliptic integral of the first kind and hence $T_p \to \infty$ as $b \to c$. From (C3) it may then be shown that

$$\theta - \theta_0 = \frac{\beta T}{2} + \frac{hT}{c} + \frac{h(c-b)}{bc}\left(\frac{2}{a-c}\right)^{\frac{1}{2}} \Pi\left(\phi, \frac{c\kappa^2}{b}, \kappa\right) \qquad (C7)$$

$$\left(\text{where } \sin\phi = \left(\frac{a-c}{2}\right)^{\frac{1}{2}} T\right)$$

where Π is an incomplete elliptic integral of the third kind.

Now restrict consideration to the special case when $h = 0$. Then

$$\dot{r}^2 = \mathcal{E} + \left(1 - \frac{\beta^2}{4}\right) r^2 - \frac{r^4}{2}$$

so that either $r = A\ \mathrm{cn}(\omega(T-T_0), \kappa)$ when $\mathcal{E} > 0$ and

$$\begin{aligned} \omega^2 A^2 (1-\kappa^2) &= \mathcal{E} \\ (2\kappa^2 - 1)\,\omega^2 &= 1 - \frac{\beta^2}{4} \\ \frac{\kappa^2\omega^2}{A^2} &= \frac{1}{2} \end{aligned} \qquad (C8)$$

or $\qquad r = B\ \mathrm{dn}(\omega(T-T_0), \kappa)$ when

and β is not large enough to invalidate (C5), i.e.; $\beta < 2$ and

$$\begin{aligned} \omega^2 B^2 (1-\kappa^2) &= -\mathcal{E} \\ \omega^2 (2-\kappa^2) &= 1 - \frac{\beta^2}{4} \\ \frac{\omega^2}{B^2} &= \frac{1}{2} \end{aligned} \qquad (C9)$$

The limiting case where $\mathcal{E} = 0$, giving a separatrix solution of infinite period, may be found as a limit of either (C8) or (C9).

The topology of these solutions in phase space may be determined simply from equations (C3) and (C4). Note that although the solutions of a second order differential equation for a complex variable are generally in four dimensions the phase space is three dimensional (θ is an 'ignorable coordinate') and a representative point may be written as (Ω,r,u) by defining

$$\dot{\theta} = \Omega$$
$$u = \dot{r}$$

The integral surfaces, upon which the trajectories must lie, become

$$\Omega = \frac{\beta}{2} + \frac{h}{r^2} \tag{C10}$$

$$u^2 + \frac{h^2}{r^2} - \left(1-\frac{\beta^2}{4}\right)r^2 + \frac{r^4}{2} = \mathcal{E} \tag{C11}$$

When $h = 0$, $\Omega = \frac{\beta}{2}$ is a constant and the trajectories in the (r,u) plane are as sketched in either figure (1) or figure (2) according as β is either less than or not less than 2. As β increases from zero the two centres and trajectories of figure (1) are drawn

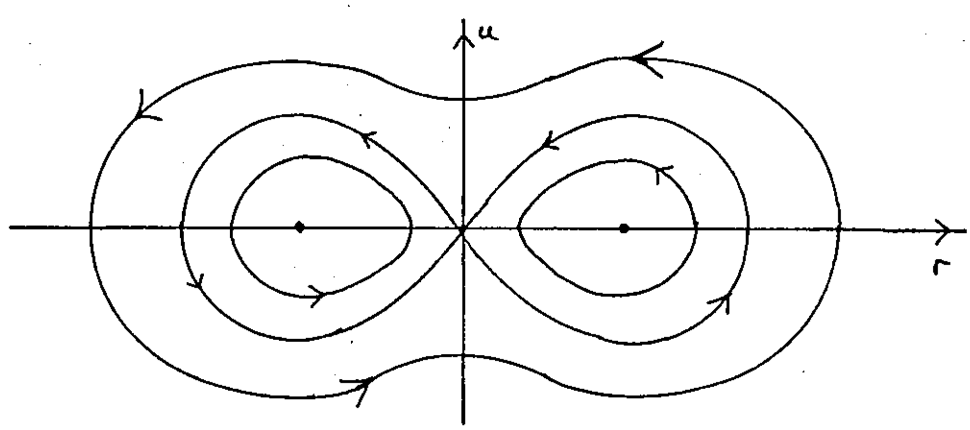

Figure (1). $\beta < 2$. The separatrix corresponds to the solution with $\mathcal{E} = 0$; inside $\mathcal{E} < 0$, outside $\mathcal{E} > 0$.

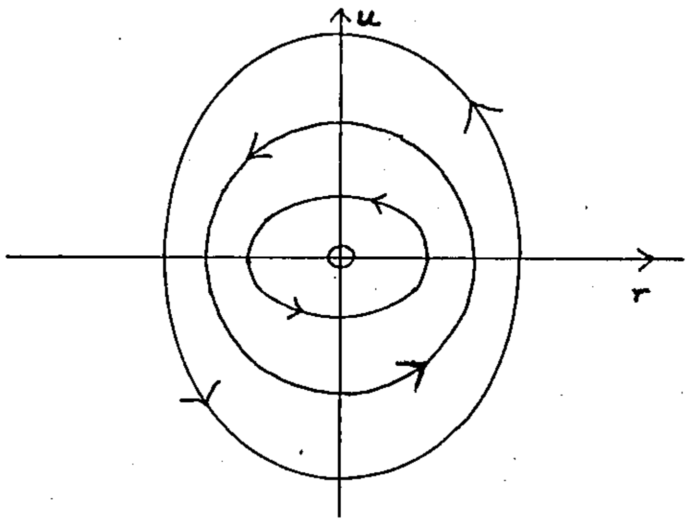

Figure (2). $\beta \geq 2$. $\mathcal{E} > 0$ for all solutions except the centre at the origin, for which $\mathcal{E} = 0$.

towards the u axis until the region of solutions where $\mathcal{E} < 0$ vanishes when $\beta = 2$ and the two centres coalesce to form a single centre at the origin. The trajectories then persist as in figure (2) for all larger values of β.

When $h \neq 0$, for each value of $\mathcal{E}$ and h such that (C5) is valid over some interval of r the surfaces of (C11) form a family of pairs of cylinders with their axes parallel to the Ω axis. These cylinders appear as pairs reflected in the plane $r = 0$ in much the same way as the cross-sectional curves of figure (1) do when $\mathcal{E} < 0$ there but now the singular term of (C11) precludes intersection of the integral surfaces with the reflecting plane for all permissible values of $\mathcal{E}$. The surfaces of (C10) have no normal component parallel to the u axis, asymptote to the planes $\Omega = \frac{\beta}{2}$ and $r = 0$ and are reflected in but do not intersect the plane $r = 0$. Hence the families of trajectories are reflected in and do not intersect the plane $r = 0$. A single trajectory for any set of values β, h and $\mathcal{E}$, which may be considered as typical when $h \neq 0$, is sketched in the half plane $r > 0$ of figure (3). As h increases with $\mathcal{E}$ fixed the cylinders and their axes move away from $r = 0$ and the other integral surfaces move away from $r = 0$ and $\Omega = \frac{\beta}{2}$. As $\mathcal{E}$ increases with h fixed the cylinders and their axes alone move, approaching $r = 0$.

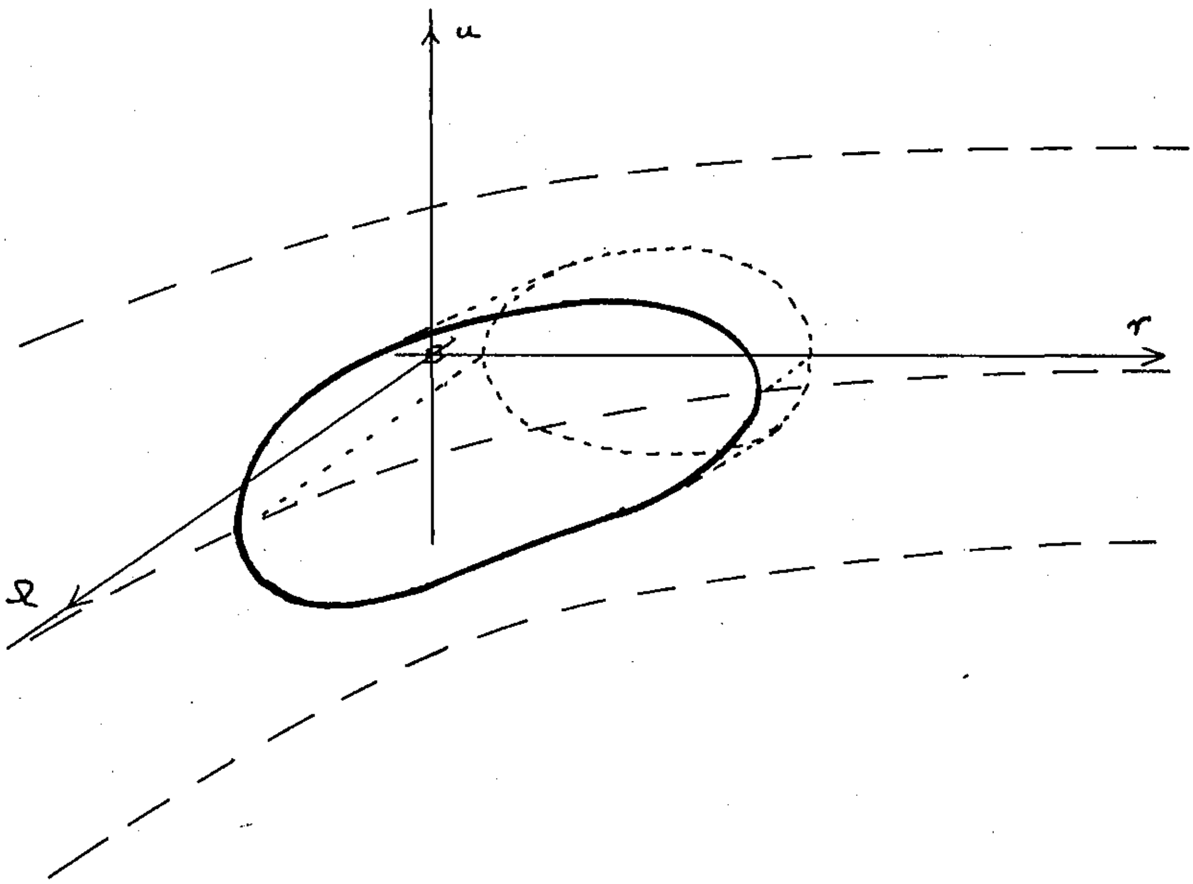

Figure (3). The solid closed curve is a typical trajectory and is accompanied by a copy reflected in the plane r = 0. The dotted lines show the corresponding integral surfaces.

The physical interpretation of $\theta(T)$ as a slowly varying downstream phase correction to a wave-like disturbance may be seen by considering the linear solutions to the nonlinear perturbation schemes of chapters three and four (Equations (3.33) and (4.20)). On the other hand, its mathematical origin is found as the generalisation to a complex amplitude function whose effect upon the coefficients in the ensuing amplitude equation may be traced through the perturbation scheme. In the limiting case when $\beta = 0$, which is of direct relevance to the amplitude equation for Eady's model since the β-plane effect there gives an apparently different modification to the stability problem, the case of a real amplitude function is recovered from the complex generalisation in the limit $h = 0$ and the coefficients k_1 and k_2 are unchanged in considering the complex case (see equation (4.46), the factor of 4 should be removed in considering the real case as a limit of the complex case by renormalising A to $\frac{A}{2}$ for complex A.)

This generalisation to the complex case for the inviscid amplitude equation (C1) can be seen to have the consequence that the position vector of a solution in phase space is a discontinuous function of the initial condition, h, at h = 0, i.e.; when h = 0 all trajectories are confined to the plane $\Omega = \frac{\beta}{2}$ and are as in figures (1) and (2); perturbation of h then distorts this integral surface in a way which is not uniform in r as $h \to 0$, with the uniformity failing at the singularity r = 0. The same type of behaviour persists at h = 0 when $\beta \neq 0$ but in this case it is clear that the amplitude function of (C1) is not purely real.

APPENDIX D

Substituting the solution at $O(\varepsilon)$, (21) in the problem at $O(\varepsilon^2)$, (26), and using (27) the right hand side of (26) the result may be expressed as

$$(\partial_t - \underline{\underline{\mathcal{J}}})\underline{x}^{(2)} = \begin{pmatrix} -i\beta e^{i\phi}|A|^2 \\ i\beta e^{-i\phi}|A|^2 \\ \left\{ -iN\beta|A|^2 e^{i\phi} - \left(\dfrac{r_{ic}'-r_{ic}}{\sum_1^\infty d_j/b_j}\right)\left(|A|^2\left(u_{R_1}\sum_5^\infty u_{Rj}^* + u_{R_2}^*\sum_5^\infty u_{Rj}\right) + A^2 e^{2i\Omega t} u_{R_1}\sum_5^\infty u_{Rj} + A^{*2} e^{-2i\Omega t} u_{R_2}^*\sum_5^\infty u_{Rj}^*\right)\right\} \\ \left\{ iN^*\beta|A|^2 e^{-i\phi} - \left(\dfrac{r_{ic}'-r_{ic}}{\sum_1^\infty d_j/b_j}\right)\left(|A|^2\left(u_{R_1}^*\sum_5^\infty u_{Rj} + u_{R_2}\sum_5^\infty u_{Rj}^*\right) + A^2 e^{2i\Omega t} u_{R_2}\sum_5^\infty u_{Rj} + A^{*2} e^{-2i\Omega t} u_{R_1}^*\sum_5^\infty u_{Rj}^*\right)\right\} \\ \dfrac{d_j}{2}\left(|A|^2\left(u_{R_1}u_{R_3}^* + u_{R_1}^*u_{R_3} + u_{R_2}u_{R_4}^* + u_{R_2}^*u_{R_4}\right) + A^2 e^{2i\Omega t}\left(u_{R_1}u_{R_4} + u_{R_2}u_{R_3}\right) + A^{*2} e^{-2i\Omega t}\left(u_{R_1}^*u_{R_4}^* + u_{R_2}^*u_{R_3}^*\right)\right) \\ j = 1, \ldots \qquad \vdots \end{pmatrix} \qquad \text{(D1)}$$

Using (38), the problem for $\underline{a}_0$ and $\underline{a}_2$ is given by

$$\sigma N a_{01} - \sigma a_{03} = -i\beta e^{-i\phi}$$

$$-P a_{01} + L a_{03} + \left(\frac{r_{ic}'-r_{ic}}{\sum_1^\infty d_j/b_j}\right) e^{i\phi}\sum_5^\infty a_{0j} = -iN\beta e^{i\phi} - \left(\frac{r_{ic}'-r_{ic}}{\sum_1^\infty d_j/b_j}\right)\left(u_{R_1}\sum_5^\infty u_{Rj}^* + u_{R_2}^*\sum_5^\infty u_{Rj}\right) \qquad \text{(D2)}$$

$$-\frac{d_{n-4}}{2}\left(N^* a_{01} e^{-i\phi} + N a_{01}^* e^{i\phi} + a_{03} e^{-i\phi} + a_{03}^* e^{i\phi}\right) + b_{n-4} a_{0n} = \frac{d_{n-4}}{2}\left(u_{R_1}u_{R_3}^* + u_{R_1}^*u_{R_3} + u_{R_2}u_{R_4}^* + u_{R_2}^*u_{R_4}\right)$$

$$n = 5, \ldots$$

i.e.; $\underline{\underline{\mathcal{D}}}\ \underline{a}_0 = -\underline{c}_0$ and

$$(\sigma N + 2i\Omega)a_{21} - \sigma a_{23} = 0$$

$$(\sigma N^* + 2i\Omega)a_{22} - \sigma a_{24} = 0$$

$$(L+2i\Omega)a_{23} - \rho a_{21} + \left(\frac{r_{1c}'-r_{1c}}{\sum_1^\infty d_j/b_j}\right)e^{i\phi}\sum_5^\infty a_{2j} = -\left(\frac{r_{1c}'-r_{1c}}{\sum_1^\infty d_j/b_j}\right)u_{R1}\sum_5^\infty u_{Rj} \qquad \text{(D3)}$$

$$(L^*+2i\Omega)a_{24} - \rho^* a_{22} + \left(\frac{r_{1c}'-r_{1c}}{\sum_1^\infty d_j/b_j}\right)e^{-i\phi}\sum_5^\infty a_{2j} = -\left(\frac{r_{1c}'-r_{1c}}{\sum_1^\infty d_j/b_j}\right)u_{R2}\sum_5^\infty u_{Rj}$$

$$(b_{n-4}+2i\Omega)a_{2n} - \frac{d_{n-4}}{2}\left(N^*e^{-i\phi}a_{21} + Ne^{i\phi}a_{22} + e^{-i\phi}a_{23} + e^{i\phi}a_{24}\right) = \frac{d_{n-4}}{2}\left(u_{R1}u_{R4} + u_{R2}u_{R3}\right)$$

i.e.; $(-\underline{\underline{\mathcal{D}}} + 2i\Omega\underline{\underline{I}}) \cdot \underline{a}_2 = \underline{c}_2$.

Note:- (D2) is not invertible since $\underline{\underline{\mathcal{D}}}$ has an eigenvalue equal to zero and so $\underline{a}_0$ is determined only up to the inclusion of arbitrary multiples of $\underline{u}_0$, which span the null space of $\underline{\underline{\mathcal{D}}}$. However, the solution to the problem for $\underline{x}$ is constructed so that at next order, $O(\varepsilon^3)$, the secularity condition which leads to the amplitude equation for $A(\tau)$ annihilates any vector in the null space of $\underline{\underline{\mathcal{D}}}$. Hence the coefficients in the amplitude equation are fully determined.

It is assumed that (D3) is invertible. This is equivalent to assuming that as the limit cycle becomes linearly unstable with the associated eigenfrequency Ω, that it does not simultaneously become unstable with an eigenfrequency of 2Ω - a detail which may be verified by solving the characteristic equation (5.23) or (6.23) in a particular example.

Then from (D2) using the first and third equations

$$a_{0n} = \frac{d_{n-4}}{2b_{n-4}}\left\{(N+N^*)\left(a_{01}e^{-i\phi} + a_{01}^*e^{i\phi}\right) + u_{R1}u_{R3}^* + u_{R1}^*u_{R3} + u_{R2}u_{R4}^* + u_{R2}^*u_{R4}\right\}$$

$$n = 5, \ldots$$

Hence from the second equation

$$a_{01}e^{-i\phi}+a_{01}^{*}e^{i\phi} = \frac{-2i(\sigma N+L)\beta}{\sigma(r_{1c}'-r_{1c})(N+N^{*})} - \frac{2e^{-i\phi}\left(u_{r1}\sum_{5}^{\infty}u_{rj}^{*}+u_{r2}\sum_{5}u_{rj}^{*}\right)}{\sum_{1}^{\infty}\frac{d_j}{b_j}(N+N^{*})}$$

$$-\left(\frac{1}{N+N^{*}}\right)\left(u_{r1}u_{r3}^{*}+u_{r1}^{*}u_{r3} + u_{r2}u_{r4}^{*}+u_{r2}^{*}u_{r4}\right)$$

where the right hand side may be seen to be real by use of the secularity condition $\underline{v}_0\cdot\underline{c}_0 = 0$ leading to (6.37). In terms of the underlying parameters of the linear problem

$$a_{01}e^{-i\phi}+a_{01}^{*}e^{i\phi} = -\left(\frac{2\sigma\Omega^2}{r_{1c}'-r_{1c}}\right).\left((\sigma+1)^2-3q^2+\Omega^2\right) - 2\sigma^2\left((\sigma+1)^2+q^2+\Omega^2\right)$$

$$a_{0n} = -\left(\frac{2\sigma\Omega^2}{r_{1c}'-r_{1c}}\right)\left((\sigma+1)^2-3q^2+\Omega^2\right)\frac{d_{n-4}}{b_{n-4}} \qquad n=5,\ldots$$

which will suffice here.

For (D3), using the first and second equations in the fifth

$$a_{2n} = \frac{d_{n-4}}{2(b_{n-4}+2i\Omega)}\left(u_{r1}u_{r4}+u_{r2}u_{r3}+\frac{(\sigma(N+N^{*})+2i\Omega)(a_{21}e^{-i\phi}+a_{22}e^{i\phi})}{\sigma}\right) \qquad n=5,\ldots$$

Then substitution in the third and fourth equations implies

$$\mathrm{Det}(2i\Omega)\,a_{21} = \left(\frac{r_{1c}'-r_{1c}}{\sum_{1}^{\infty}\frac{d_j}{b_j}}\right)\sigma\left\{\left(\frac{r_{1c}'-r_{1c}}{\sum\frac{d_j}{b_j}}\right)(\sigma(N+N^{*})+2i\Omega).\frac{1}{2}\sum_{1}^{\infty}\frac{d_j}{b_j+2i\Omega}\right.$$

$$\times\left(u_{r2}e^{2i\phi}-u_{r1}\right)\sum_{5}^{\infty}u_{rn}$$

$$\left.-2i\Omega(\sigma N^{*}+L^{*}+2i\Omega)\left(u_{r1}\sum_{5}^{\infty}u_{rn}+e^{i\phi}(u_{r1}u_{r4}+u_{r2}u_{r3}).\frac{1}{2}\sum_{1}^{\infty}\frac{d_j}{b_j+2i\Omega}\right)\right\}$$

$$\mathrm{Det}(2i\Omega)\, a_{22} = \left(\frac{r_{1c}'-r_{1c}}{\sum_1^\infty d_j/b_j}\right)\sigma \left\{ \left(\frac{r_{1c}'-r_{1c}}{\sum_1^\infty d_j/b_j}\right)(\sigma(N+N^*)+2i\Omega)\cdot\frac{1}{2}\cdot\sum_1^\infty \frac{d_j}{b_j+2i\Omega} \times (u_{x_1}e^{-2i\phi}-u_{x_2})\sum_5^\infty u_{x_n} - 2i\Omega(\sigma N+L+2i\Omega)\left(u_{x_2}\sum_5^\infty u_{x_n} + e^{-i\phi}(u_{x_1}u_{x_4}+u_{x_2}u_{x_3})\right)\cdot\frac{1}{2}\cdot\sum_1^\infty \frac{d_j}{b_j+2i\Omega}\right\}$$

where

$$\mathrm{Det}(2i\Omega) = 2i\Omega\left\{\left(\frac{r_{1c}'-r_{1c}}{\sum d_j/b_j}\right)(\sigma(N+N^*)+2i\Omega)(\sigma(N+N^*)+L+L^*+4i\Omega) \times \frac{1}{2}\sum_1^\infty \frac{d_j}{b_j+2i\Omega} + 2i\Omega(\sigma N+L+2i\Omega)(\sigma N^*+L^*+2i\Omega)\right\}$$

so that with this notation, the characteristic equation of linear stability of the limit cycle is $\mathrm{Det}(i\Omega) = 0$. Then

$$a_{23} = \frac{(\sigma N+2i\Omega)}{\sigma}a_{21}, \quad a_{24} = \frac{(\sigma N^*+2i\Omega)}{\sigma}a_{22}$$

and

$$a_{2n} = \frac{d_{n-4}}{(b_{n-4}+2i\Omega)}\cdot\frac{i\Omega}{\mathrm{Det}(2i\Omega)}\left\{2i\Omega(\sigma N+L+2i\Omega)(\sigma N^*+L^*+2i\Omega)(u_{x_1}u_{x_4}+u_{x_2}u_{x_3}) - \left(\frac{r_{1c}'-r_{1c}}{\sum_1^\infty d_j/b_j}\right)(\sigma(N+N^*)+2i\Omega)\sum_5^\infty u_{x_n}\left(u_{x_1}e^{-i\phi}(\sigma N^*+L^*+2i\Omega) + u_{x_2}e^{i\phi}(\sigma N+L+2i\Omega)\right)\right\}$$

$$n = 5, \ldots$$

so that on substitution, in terms of the underlying parameters of the linear stability problem

$$a_{2n} = \frac{d_{n-4}}{(b_{n-4}+2i\Omega)}\cdot\frac{-\sigma\Omega^2}{\mathrm{Det}(2i\Omega)}(\sigma(N+N^*)+2i\Omega)(\sigma N+L+i\Omega)(\sigma N^*+L^*+i\Omega) \times (\sigma(N+N^*)+L+L^*+3i\Omega)(\sigma(N+N^*)+L+L^*+4i\Omega)$$

$$n = 5, \ldots$$

$$a_{21} = \frac{i\sigma^2\Omega}{\mathrm{Det}(2i\Omega)}e^{i\phi}(\sigma N+L+i\Omega)(\sigma N^*+L^*+i\Omega)\left\{2i\Omega(\sigma N^*+L^*+i\Omega)(\sigma N^*+L^*+2i\Omega) - \frac{1}{2}\left(\frac{r_{1c}'-r_{1c}}{\sum_1^\infty d_j/b_j}\right)\cdot\sum_1^\infty\frac{d_j}{b_j+2i\Omega}(\sigma(N+N^*)+2i\Omega)(\sigma(N+N^*)+L+L^*+4i\Omega)\right\}$$

$$a_{22} = \frac{i\sigma^2\Omega}{Det(2i\Omega)} e^{-i\phi}(\sigma N+L+i\Omega)(\sigma N^*+L^*+i\Omega)\Bigg\{2i\Omega(\sigma N+L+i\Omega)(\sigma N+L+2i\Omega)$$

$$-\frac{1}{2}\left(\frac{r_{1c}'-r_{1c}}{\sum_1^\infty d_j/b_j}\right)\cdot\sum_1^\infty \frac{d_j}{b_j+2i\Omega}\left(\sigma(N+N^*)+2i\Omega\right)\left(\sigma(N+N^*)+L+L^*+4i\Omega\right)\Bigg\}$$

The coefficient of $A|A|^2$ which appears on the left hand side of (6.44) is

$$-\left(\frac{r_{1c}'-r_{1c}}{\sum_1^\infty d_j/b_j}\right)\cdot\left((v_{\Omega_3}u_{\Omega_1}+v_{\Omega_4}u_{\Omega_2})\sum_5^\infty a_{0j} + (v_{\Omega_3}a_{01}+v_{\Omega_4}a_{01}^*)\sum_5^\infty u_{\Omega_j}\right)$$

$$+\frac{1}{2}\sum_5^\infty d_{j-4}\, v_{\Omega_j}\left(u_{\Omega_1}a_{03}^* + u_{\Omega_2}a_{03} + u_{\Omega_4}a_{01} + u_{\Omega_3}a_{01}^*\right)$$

which on substitution for $\underline{v}_\Omega$ and $\underline{u}_\Omega$ yields

$$-\left(\frac{r_{1c}'-r_{1c}}{\sum_1^\infty d_j/b_j}\right)\cdot\left(v_{\Omega_3}u_{\Omega_1}+v_{\Omega_4}u_{\Omega_2}\right)\sum_5^\infty a_{0j}$$

$$+i\Omega(\sigma N+L+i\Omega)(\sigma N^*+L^*+i\Omega)\left(\sigma(N+N^*)+i\Omega\right)\left(\sigma(N+N^*)+L+L^*+2i\Omega\right)$$
$$\times\left(a_{01}e^{-i\phi}+a_{01}e^{i\phi}\right)$$

$$-\frac{\beta\Omega}{\sigma}(\sigma N+L+i\Omega)(\sigma N^*+L^*+i\Omega)\left(u_{\Omega_2}e^{i\phi}-u_{\Omega_1}e^{-i\phi}\right)$$

The sigma sums appearing in chapter 6 for the coefficients of the 2-layer model including a weak β-plane effect and weak diffusion.

The coefficients are given in (B9) of appendix B. The Fourier cosine series of $\sin 2m\pi x$, which is uniformly convergent on the closed interval $[0,1]$, is

$$\sin 2m\pi x = -\frac{2m}{\pi}\sum_{j=0}^{\infty}\frac{\cos 2(j+\frac{1}{2})\pi x}{(j+\frac{1}{2})^2-m^2} \qquad \text{(D4)}$$

whence

$$\sum_{j=0}^{\infty} \frac{1}{(j+\frac{1}{2})^2 - m^2} = 0$$

and from the Parseval formula associated with the series of (D4)

$$\sum_{j=0}^{\infty} \frac{1}{((j+\frac{1}{2})^2 - m^2)^2} = \frac{\pi^2}{4m^2}$$

The even function $|sh(ax)|$ for $x \in [-1,1]$ has the uniformly convergent Fourier cosine expansion

$$|sh(ax)| = (ch\,a - 1)\left\{\frac{2}{a} + 2a\sum_{j=1}^{\infty}\frac{\cos 2j\pi x}{a^2+(2j\pi)^2}\right\}$$

$$- 2a(1+ch\,a)\sum_{j=1}^{\infty}\frac{\cos(2j-1)\pi x}{a^2+(2j-1)^2\pi^2}$$

Whence

$$\sum_{j=0}^{\infty}\frac{1}{(j+\frac{1}{2})^2\pi^2 + \frac{a^2}{4}} = \frac{1}{a}\,th\,\frac{a}{2}$$

and on differentiation

$$\sum_{j=0}^{\infty}\frac{1}{((j+\frac{1}{2})^2\pi^2 + \frac{a^2}{4})^2} = \frac{1}{a^3}\left(2\,th\,\frac{a}{2} - a\,sech^2\frac{a}{2}\right)$$

Expressing each term of the required sums as a sum of terms of the above yields

$$\sum_{j=0}^{\infty} d_j = \frac{2k^2m^4\pi^4}{(a_\omega^2 + 4m^2\pi^2)}\left\{1 - \frac{4a_\omega}{(a_\omega^2+4m^2\pi^2)}\,th\,\frac{a_\omega}{2}\right\}$$

$$\sum_{j=0}^{\infty} \frac{d_j}{b_j} = \frac{k^2m^2\pi^2}{4} \qquad \text{(D5)}$$

$$\sum_{j=0}^{\infty} d_j b_j = \frac{16k^2m^4\pi^4}{(a_\omega^2 + 4m^2\pi^2)^2}\left\{m^2\pi^2 - \frac{8m^2\pi^2}{(a_\omega^2+4m^2\pi^2)}\,a_\omega\,th\,\frac{a_\omega}{2} + a_\omega\left(th\,\frac{a_\omega}{2} - \frac{a_\omega}{2}\,sech^2\frac{a_\omega}{2}\right)\right\}$$

Then using the transformation $i\Omega = \dfrac{2\mu^2}{1-\mu^2}$ implies

$$\sum_{j=0}^{\infty} \frac{d_j(a_w, m)}{(b_j(a_w, m)+i\Omega)} = \frac{(1-\mu^2)}{2} \cdot \sum_{j=0}^{\infty} d_j(a_w\mu, m) \qquad a_w^2 = k^2+m^2\pi^2$$

which concurs with (D5) in the limit $\Omega \to 0$. Further, using the same transformation and principle:

$$\sum_{j=0}^{\infty} \frac{d_j}{(b_j+i\Omega)^2} = \frac{(1-\mu^2)^2 k^2 m^4 \pi^2}{2\left((a_w\mu)^2+4m^2\pi^2\right)} \left\{ \pi^2(a_w^2+4m^2\pi^2) \right.$$

$$+ \left(4m^2\pi^2(1-3\mu^2)+(a_w\mu)^2(\mu^2-3)\right) \frac{2a_w\pi^2 \tanh\frac{a_w\mu}{2}}{\mu\left((a_w\mu)^2+4m^2\pi^2\right)}$$

$$\left. - a_w^2\pi^2(\mu^2-1)\left(1-\tanh^2\left(\frac{a_w\mu}{2}\right)\right) \right\}$$

APPENDIX E

Consider the infinite complex Lorenz type equations

$$\dot{X} = -\sigma(X - Y) \tag{E1}$$

$$\dot{Y} = rX - aY - X\sum_{j=0}^{\infty} Z_j \qquad \begin{array}{l} r = r_1 + ir_2 \\ a = 1 - ie \end{array} \tag{E2}$$

$$\dot{Z}_j = -b_j Z_j + \frac{\gamma_j}{2}(XY^* + X^*Y) \qquad j = 0, \ldots \tag{E3}$$

where γ_j and b_j are positive for all $j = 0,1,\ldots.$ and $\{b_j\}$ has a positive lower bound. Formally performing the time integration of (E3) implies

$$Z_j(t) = Z_j(0).e^{-b_j t} + \int_0^t \gamma_j e^{-b_j(t-s)} \frac{(XY^* + X^*Y)}{2}(s)\, ds$$

where, since each Z component is a time dependent coefficient in the Fourier expansion of the mean flow correction, $Z_j(0) = 0$ for all j for an incipient wave. The sum of (E2) may then be written as

$$\sum_{j=0}^{\infty} \int_0^t \gamma_j e^{-b_j(t-s)} \frac{(XY^* + X^*Y)}{2}(s)\, ds + \sum_{j=0}^{\infty} Z_j(0) e^{-b_j t}$$

Assuming that it is permissible to interchange the order of summation and integration and restricting to the case of an incipient wave (the generalisation back is straightforward) this becomes

$$\int_0^t \sum_{j=0}^{\infty} \gamma_j e^{-b_j(t-s)} . \frac{(XY^* + X^*Y)}{2}\, ds \tag{E4}$$

Alternatively, eliminating Y by use of (E1) and integrating once by parts gives

$$\frac{1}{2\sigma}\left(XX^*(t).\sum_{j=0}^{\infty}\gamma_j - XX^*(0)\sum_{j=0}^{\infty}\gamma_j e^{-b_j t}\right)$$
$$+ \int_0^t \sum_{j=0}^{\infty}\gamma_j e^{-b_j(t-s)}\left(1 - \frac{b_j}{2\sigma}\right) XX^*(s)\, ds \tag{E5}$$

Recall that this is the sequence of steps leading to equation (4.50) of chapter four.

Now define the 'delay kernel' G(t-s) by

$$G(t-s) = \sum_{j=0}^{\infty} f_j e^{-b_j(t-s)} \tag{E6}$$

then the set (E1) to (E3) may be written in the form of a single integrodifferential equation for X (see Pedlosky 1971 for a similar equation).

$$\ddot{X} + (\sigma+a)\dot{X} + \sigma(a-r)X + \frac{X}{2}\int_0^t G(t-s)\left(\frac{d}{ds}(|X|^2)+2\sigma|X|^2\right)(s)\,ds = 0 \tag{E7}$$

where the lower limit of integration is appropriate for initial conditions applied at t = 0. Now the behaviour of a solution to this type of delay equation will depend upon the form of the delay kernel G(t-s) and this in turn will depend upon the coefficients $\{\gamma_j\}$ and $\{b_j\}$. For the case of a single Z component G(t-s) is a decaying exponential function of its argument, this type of kernel is termed a weak delay kernel in the literature and some results of the effect of this and other types of kernel upon various forms of equation may be found in Cushing 1977. In attempting to analyse the function G(t-s) in the present case consider the related integral

$$J(t-s) = \int_{n_0}^{\infty} f(n)\, e^{-b(n)(t-s)}\, dn \tag{E8}$$

Note: (i) If the infinite set of discrete modes $\{Z_j(t)\}$ are replaced by the single function Z(n,t) where n is a continuous variable, i.e.; in the hypothetical case where the set (E1) to (E3) may be modelled by the set of equations

$$\dot{X} = -\sigma(X-Y)$$

$$\dot{Y} = rX - aY - X\int_{n_0}^{\infty} Z(n)\,dn$$

$$\dot{Z}(n) = -b(n)Z(n) + \frac{f(n)}{2}(XY^* + X^*Y)$$

$$n \in [n_0, \infty)$$

then the appropriate integrodifferential equation is (E7) with $G(t-s)$ replaced exactly by the related integral $J(t-s)$ of (E8) (for an incipient wave).

(ii) In the discrete case it may be possible to relate the functions $G(t-s)$ and $J(t-s)$ by use of the Euler McLaurin summation formula, e.g.:

$$\sum_{n=0}^{\infty} f_n = \int_{-1}^{\infty} f(n)\,dn - \frac{1}{2} f(-1) + \frac{1}{12} f'(-1) - \frac{1}{720} f'''(-1) + \dots \qquad \text{(E9)}$$

where it has been assumed that $f(n)$ and all its derivatives tend to zero as n tends to infinity. The validity of this result depends upon the behaviour of $f(n)$ as a function of a continuous variable and requires that the function is analytic along the path of integration. The 'boundary terms' forming the infinite series on the right hand side of (E9) are each exponential functions of $(t-s)$ in the present context of (E6).

(iii) If the variable of integration is changed in (E8) by setting $u = b(n)$, then formally

$$J(p) = \int_{b(n_0)}^{b(\infty)} \gamma(u)\,\frac{dn}{db}(u)\,.\,e^{pu}\,du$$

which it may be possible to evaluate as a Laplace transform.

In attempting to remove the boundary terms of (E9) introduce a set of modes $\{Z_{-j-1}\}$ and the coefficients $\{\gamma_{-j-1}\}$ and $\{b_{-j-1}\}$ $(j = 0,1,\dots.)$ defined such that

$$\gamma_{-j-1} = \gamma_j$$

$$b_{-j-1} = b_j \qquad\qquad j = 0,1,\dots \qquad \text{(E10)}$$

$$\dot{Z}_{-j-1} = -b_{-j-1} Z_{-j-1} + \frac{\gamma_{-j-1}}{2}(XY^* + X^*Y)$$

$$\Rightarrow \quad \dot{Z}_{-j-1} = -b_j Z_{-j-1} + \frac{\gamma_j}{2}(XY^* + X^*Y)$$

If further the initial conditions are chosen such that

$$Z_{-j-1}(0) = Z_j(0) \qquad j = 0, 1, \ldots$$

considering X and Y as functions of time, the introduced modes are such that

$$Z_{-j-1}(t) = Z_j(t) \qquad j = 0, 1, \ldots \quad \forall t$$

and these introduced modes do not alter the system of equations (E1) to (E3), but now

$$\sum_{j=0}^{\infty} Z_j(t) = \frac{1}{2}\sum_{j=0}^{\infty}\left(Z_j(t) + Z_{-j-1}(t)\right) \qquad (E11)$$

$$= \frac{1}{2}\sum_{j=-\infty}^{\infty} Z_j(t)$$

The sum of equation (E2), after a formal time integration and swapping of limits, becomes

$$\int_0^t G_2(t-s)\frac{(XY^* + X^*Y)}{2}(s)\,ds + \sum_{j=-\infty}^{\infty} Z_j(0)\,e^{-\delta_j t}$$

where now

$$G_2(t-s) = \frac{1}{2}\sum_{j=-\infty}^{\infty} \gamma_j\, e^{-\delta_j (t-s)} \qquad (E12)$$

Provided the integrand and all its derivatives with respect to n vanish as $n \to \pm\infty$ and further $\gamma(u)$ is such that the Euler McLaurin summation formula may be applied

$$G_2(t-s) = J_2(t-s) \qquad (E13)$$

where

$$J_2(t-s) = \frac{1}{2}\int_{-\infty}^{\infty} \gamma(u)\, e^{-\delta(u)(t-s)}\,du \qquad (E14)$$

For the coefficients of the 2-layer model

$$\gamma_j = \frac{2k^2 m^4 \pi^2 (j+\frac{1}{2})^2}{((j+\frac{1}{2})^2 - m^2)^2((j+\frac{1}{2})^2\pi^2 + \frac{a^2}{4})} \quad , \quad b_j = \frac{2(j+\frac{1}{2})^2\pi^2}{((j+\frac{1}{2})^2\pi^2 + \frac{a^2}{4})}$$

so that $\gamma(-n-1) = \gamma(n)$ and $b(-n-1) = b(n)$ automatically but $\gamma(n)$ as a function of a continuous variable has poles of order two at $n = \pm m - \frac{1}{2}$ so that the Euler McLaurin summation formula may not be applied. Undaunted by this, write the related integral as

$$J_{2l}(p) = k^2 m^4 \pi^2 \fint_{-\infty}^{\infty} \frac{(n+\frac{1}{2})^2 e^{-\frac{2(n+\frac{1}{2})^2\pi^2}{((n+\frac{1}{2})^2\pi^2 + \frac{a_w^2}{4})}\cdot p}}{((n+\frac{1}{2})^2 - m^2)^2((n+\frac{1}{2})^2\pi^2 + \frac{a_w^2}{4})}\, dn$$

where the crossed integral sign is used as a reminder that without a suitable consideration of the singularities of the integrand (E13) will not be valid. The change of variables

$$s = \frac{(n+\frac{1}{2})^2\pi^2}{(n+\frac{1}{2})^2\pi^2 + \frac{a^2}{4}} \quad \Rightarrow \quad s \in [0,1) \text{ for } n \in (-\infty, \infty)$$

implies that

$$J_{2l}(p) = \frac{k^2 a_w}{2\pi} \fint_0^1 \frac{s^{\frac{1}{2}}(1-s)^{\frac{1}{2}} e^{-2sp}}{(1-\rho s)^2}\, ds \qquad \text{(E15)}$$

where

$$\rho = \frac{a_w^2 + 4m^2\pi^2}{4m^2\pi^2} \in \left(\frac{5}{4}, \infty\right)$$

so that the integrand has a pole of order two along the path of integration $\in (0, \frac{4}{5})$. Define a function $F(\rho;p)$ by

$$F(\rho,p) = \int_0^1 \frac{(1-s)^{\frac{1}{2}} e^{-2sp}}{s^{\frac{1}{2}}(1-\rho s)}\, ds$$

which defines an analytic function for all p in the complex plane when $|\rho| < 1$. In fact,

$$F(\rho, p) = 2\,\Phi_1\left(\tfrac{1}{2}, 1, \tfrac{3}{2}; \rho; -2p\right) \quad \text{for } |\rho| < 1$$

where $\Phi_1(\alpha,\beta,\gamma;x;y)$ is a confluent hypergeometric function of two variables showing the confluence through y (or p in the above formula), the notation followed here is that of Erdélyi et al. 1953 but see Humbert, 1921. Then

$$\frac{\partial F}{\partial \rho} = \int_0^1 \frac{s^{\frac{1}{2}}(1-s)^{\frac{1}{2}} e^{-2sp}}{(1-\rho s)^2}\, ds \tag{E16}$$

provided $|\rho| < 1$ so that $F(\rho, p)$ converges and is differentiable. Comparison of (E15) with (E16) suggests that the delay kernel $G_{2\ell}(t-s)$, may be related in this case to a continuation of $\frac{\partial F}{\partial \rho}$ in the complex ρ plane. Note in passing that

$$\frac{\partial F}{\partial p} + \frac{2}{\rho} F = \frac{\pi}{\rho} M\left(\tfrac{1}{2}, 2, -2p\right)$$

where $M(a,c,z)$ is Kummer's function.

It may be quite possible to suggest coefficients, $\{\gamma_j\}$ and $\{b_j\}$, in equations (E1) to (E3) for which the above technique is both possible and easier to apply, giving a less complicated delay kernel. With the supposition that there exists a delay kernel which may be expressed in terms of elementary functions, it would then be possible to compare the effectiveness of different numerical integration schemes on equation (E7) and then to compare these results with numerical integrations of a truncated version of equations (E1) to (E3). The coefficients for the Eady model contain hyperbolic functions and appear to be quite intractable in the above scheme.