Imperial College London, Department of Mathematics

Local phenomena in random dynamical systems: bifurcations, synchronisation, and quasi-stationary dynamics

A thesis presented for the degree of Doctor of Philosophy of Imperial College London by

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Declaration

I declare that this thesis is entirely my own work and that where any material could be construed as the work of others, it is fully cited and referenced, and/or with appropriate acknowledgement given.

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Publications

The material in Sections 3.1, 3.2, 3.3 and 3.4 of this thesis is joint work with Doan Thai Son, Jeroen Lamb and Martin Rasmussen, and has been submitted for publication in the paper [38].

The material in Chapter 4 is joint work with Jeroen Lamb and Martin Rasmussen, and has been submitted for publication in the paper [45].

A paper comprising the main results in Chapter 6 is currently in preparation.

Abstract

We consider several related topics in the bifurcation theory of random dynamical systems: synchronisation by noise, noise-induced chaos, qualitative changes of finite-time behaviour and stability of systems surviving in a bounded domain.

Firstly, we study the dynamics of a two-dimensional ordinary differential equation exhibiting a Hopf bifurcation subject to additive white noise. Depending on the deterministic Hopf bifurcation parameter and a phase-amplitude coupling parameter called shear, three dynamical phases can be identified: a random attractor with uniform synchronisation of trajectories, a random attractor with non-uniform synchronisation of trajectories and a random attractor without synchronisation of trajectories. We prove the existence of the first two phases which both exhibit a random equilibrium with negative top Lyapunov exponent but differ in terms of finite-time and uniform stability properties. We provide numerical results in support of the existence of the third phase which is characterised by a so-called random strange attractor with positive top Lyapunov exponent implying chaotic behaviour.

Secondly, we reduce the model of the Hopf bifurcation to its linear components and study the dynamics of a stochastically driven limit cycle on the cylinder. In this case, we can prove the existence of a bifurcation from an attractive random equilibrium to a random strange attractor, indicated by a change of sign of the top Lyapunov exponent. By establishing the existence of a random strange attractor for a model with white noise, we extend results by Qiudong Wang and Lai-Sang Young on periodically kicked limit cycles to the stochastic context. Furthermore, we discuss a characterisation of the invariant measures associated with the random strange attractor and deduce positive measure-theoretic entropy for the random system.

Finally, we study the bifurcation behaviour of unbounded noise systems in bounded domains, exhibiting the local character of random bifurcations which are usually hidden in the global analysis. The systems are analysed by being conditioned to trajectories which do not hit the boundary of the domain for asymptotically long times. The notion of a stationary distribution is replaced by the concept of a quasi-stationary distribution and the average limiting behaviour can be described by a so-called quasi-ergodic distribution. Based on the well-explored stochastic analysis of such distributions, we develop a dynamical stability theory for stochastic differential equations within this context. Most notably,

we define conditioned average Lyapunov exponents and demonstrate that they measure the typical stability behaviour of surviving trajectories. We analyse typical examples of random bifurcation theory within this environment, in particular the Hopf bifurcation with additive noise, with reference to whom we also study (numerically) a spectrum of conditioned Lyapunov exponents. Furthermore, we discuss relations to dynamical systems with holes.

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Chapter 1

Introduction

Random dynamical systems are systems whose evolution in time is governed by dynamical laws exhibiting randomness. This means that the paths of the system follow a certain probability distribution. Many physical systems in, for example, climate science, neuroscience, finance or quantum mechanics are modelled as dynamical systems with noise, in particular as stochastic (partial) differential equations, and can be understood as random dynamical systems.

Very often the mathematical analysis focusses on the statistics of trajectories for different noise realisations (see e.g. [79] for an overview), and does not consider the dynamical aspects of the system. In contrast, the theory of random dynamical systems as coined by the works of Ludwig Arnold and his co-workers in the 1980s and 1990s and manifested in Arnold's book *Random Dynamical Systems* [2] compares trajectories with different initial conditions but driven by the same noise. A random dynamical system in this sense consists of a model of the time-dependent noise formalised as a a dynamical system θ on the probability space, and a model of the dynamics on the state space formalised as a *cocycle* φ over θ.

A central question associated with this point of view concerns the asymptotic behaviour of typical trajectories. As trajectories of random dynamical systems depend on the noise realisation, one does not a priori expect any convergent behaviour of individual trajectories to a fixed attractor. An alternative view point that circumvents this problem and often yields convergence is to consider, for a fixed noise realisation in the past, the flow of a set of initial conditions from time $t = -T$ to a fixed endpoint in time, say $t = 0$, and then take the (pullback) limit $T \to \infty$. If trajectories of initial conditions converge under this procedure to some set, then this set is called a *random pullback attractor*, or simply *random attractor*.

Bifurcations in random dynamical systems

The nature of such a random attractor is an important topic of this thesis since its properties express the typical asymptotic behaviour of the system. In many situations there is a spectrum of exponential asymptotic growth rates, the *Lyapunov exponents*. If the sum of the Lyapunov exponents is negative, we can speak of a dissipative system and we usually encounter a random attractor in such a situation. The sign of the largest (or top) Lyapunov exponent, λ_1, determines if two nearby trajectories converge or separate from each other. By that, the sign of λ_1 also provides information about the random attractor. If λ_1 is negative, we typically observe the convergence of trajectories, a phenomenon labelled *synchronisation*. Synchronisation is associated with the random attractor being an attractive random equilibrium, i.e. for each noise realisation the random attractor is simply a point. In forward time, the attractive random equilibrium constitutes a trajectory attracting all other trajectories.

Positivity of λ_1 implies sensitivity of initial conditions and is thereby associated with chaotic behaviour. The sensitivity of initial conditions means that any two trajectories starting arbitrarily close to each other will separate at a certain point of time. In other words, even the smallest error in the initial conditions leads to a considerably large error in the future. An attractor which exhibits such chaotic behaviour is often called *strange* due to its fractal-like shape (cf. e.g. [70, 90]). *Random strange attractors* have recently raised increased interest in the applied community, in particular in climate science. For example, Checkroun et al. [26] have studied numerically the random attractors of a stochastically forced Lorenz model and a stochastic model for the El Niño-Southern oscillation, detecting chaotic behaviour.

We illustrate the scenarios of synchronisation and chaos with reference to the main example from chapter 3. We consider the two-dimensional stochastic differential equation

$$\begin{aligned} \mathrm{d}x &= (\alpha x - \beta y - (ax - by)(x^2 + y^2))\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^1\,, \\ \mathrm{d}y &= (\alpha y + \beta x - (bx + ay)(x^2 + y^2))\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^2\,, \end{aligned} \tag{1.0.1}$$

where $\sigma \geq 0$ represents the strength of the noise, $\alpha \in \mathbb{R}$ is a parameter equal to the real part of eigenvalues of the linearisation of the vector field at $(0,0)$, $b \in \mathbb{R}$ represents shear strength (amplitude-phase coupling parameter when writing the deterministic part of (1.0.1) in polar coordinates), $a > 0$, $\beta \in \mathbb{R}$, and W_t^1, W_t^2 denote independent one-dimensional Brownian motions.

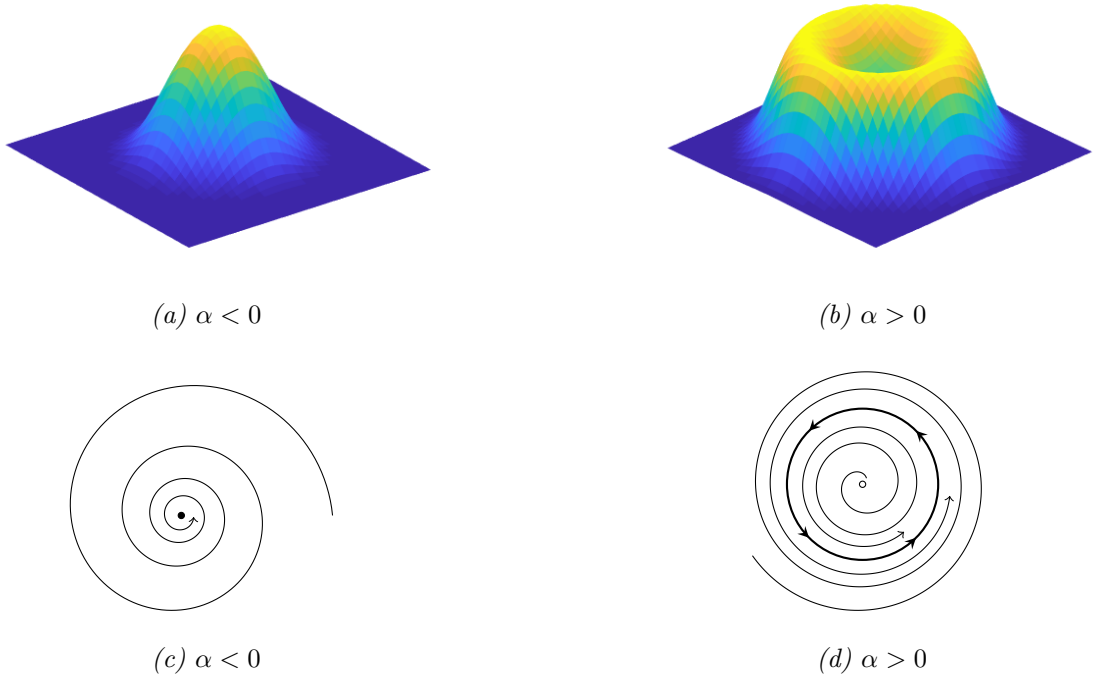
In the absence of noise ($\sigma = 0$), the stochastic differential equation (1.0.1) is a normal form for the supercritical Hopf bifurcation: when $\alpha \leq 0$ the system has a globally attracting equilibrium at $(x, y) = (0, 0)$ which is exponentially stable until $\alpha = 0$ and, when $\alpha > 0$, the system has a limit cycle at $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = \alpha/a\}$ which is globally attracting on $\mathbb{R}^2 \setminus \{0\}$.

In the presence of noise ($\sigma \neq 0$), statistical information about the (one point) dynamics of (1.0.1) can be described by the Fokker–Planck equation for the Markov process and its stationary density. In this case, the stationary density can be calculated analytically, yielding

$$\frac{2\sqrt{2a}}{\sqrt{\pi}\sigma \operatorname{erfc}(-\alpha/\sqrt{2a\sigma^2})} \exp\left(\frac{2\alpha(x^2+y^2) - a(x^2+y^2)^2}{2\sigma^2}\right). \tag{1.0.2}$$

We note in particular that this density does not depend on the shear parameter b.

We observe a clear relation between the stationary measures in the presence of noise ($\sigma > 0$) and the attractors in the deterministic limit: the stationary density is maximal on attractors of the deterministic limit dynamics and (locally) minimal on its repellers, see Figure 1.1.



(a) $\alpha < 0$ *(b)* $\alpha > 0$

(c) $\alpha < 0$ *(d)* $\alpha > 0$

Figure 1.1: Shape of the stationary density of (1.0.1) with noise and corresponding phase portraits of the deterministic limit. The qualitative features only depend on the sign of the linear stability parameter α. Figures (a) and (b) present the shapes of the stationary densities in the presence of noise. (a) is characterised by a unique maximum at the origin and (b) by a local minimum at the origin surrounded by a circle of maxima when $\alpha > 0$. Figures (c) and (d) show phase portraits in the deterministic limit $\sigma = 0$ displaying an attracting equilibrium if $\alpha < 0$ and an attracting limit cycle if $\alpha > 0$, precisely where stationary densities have their maxima.

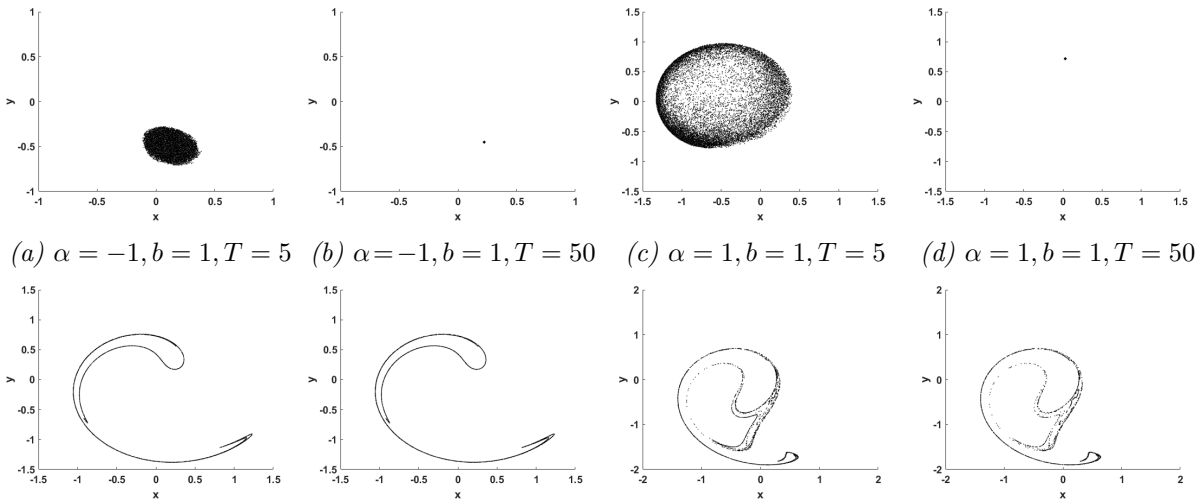
From Figure 1.1, it is natural to propose that the stochastic differential equation (1.0.1) has a bifurcation at $\alpha = 0$, represented by the qualitative change of the shape of the stationary density. Such kind of bifurcation is called a *phenomenological bifurcation*, cf. [2].

In this work, we consider the system (1.0.1) with noise from a random dynamical systems point of view: with a canonical model for the noise, (1.0.1) can be represented as

a dynamical system that is driven by a random signal.

The underlying Markov process only models probabilistically a single time-series. Therefore, the stationary density (1.0.2) provides certain statistics about the dynamics of (1.0.1), but cannot capture many relevant dynamical properties, such as a comparison of the trajectories of nearby initial conditions (with the same noise).

We compare trajectories by investigating the pullback dynamics of (1.0.1). Figure 1.2 provides some numerical examples.[1] We observe the two different behaviours described above: either all initial conditions converge to a fixed point, see (a)–(d), or all initial conditions converge to a rather complicated object, see (e)–(h). The former is indicative of synchronisation while the latter points to a random strange attractor.



(a) $\alpha = -1, b = 1, T = 5$ *(b)* $\alpha = -1, b = 1, T = 50$ *(c)* $\alpha = 1, b = 1, T = 5$ *(d)* $\alpha = 1, b = 1, T = 50$

(e) $\alpha = -1, b = 20, T = 5$ *(f)* $\alpha = -1, b = 20, T = 50$ *(g)* $\alpha = 1, b = 8, T = 5$ *(h)* $\alpha = 1, b = 8, T = 50$

Figure 1.2: Pullback dynamics of (1.0.1) *with* $\sigma = \beta = a = 1$ *for initial conditions chosen in approximation of the stationary density. In (a)–(d), in the presence of small shear, we observe synchronisation, i.e. pullback convergence of all trajectories to a single point, irrespective of the linear stability at the origin. In (e)–(h), in the presence of sufficiently large shear, there is no synchronisation but pullback convergence to a more complicated object (random strange attractor), again irrespective of the linear stability at the origin.*

Each of these two scenarios, i.e. of negative and positive top Lyapunov exponent λ_1, corresponds with a specific invariant random measure supported on the random attractor. In case of synchronisation, the random measure is a random Dirac measure concentrating all mass on the random equilibrium. In case of a random strange attractor the measures usually have a density on a subset of positive dimension, characterising the non-synchronising behaviour. In the present work, this property is called *SRB*-property, named after Sinai, Ruelle and Bowen, in accordance with, for example, [70, 78, 89].

Synchronisation and chaos are the two typical phenomena encountered in random

[1] The simulations in this thesis are based on an explicit Euler–Maruyama integration of the stochastic differential equation (1.0.1), usually with time step size 10^{-3}. When we compute Lyapunov exponents, we use an explicit second-order Runge–Kutta method for integrating the variational equation.

dynamical systems. Figure 1.2 already illustrates a central topic of this thesis: the *bifurcation*, i.e. parameter-dependent change, from one to the other. In general, the term bifurcation denotes a change of qualitative behaviour of a system depending on one or multiple parameters, often measured as a loss of topological equivalence at the bifurcation point. Our main tool to discover bifurcations will be the determination or estimation of the top Lyapunov exponent indicating the dominating asymptotic stability behaviour. This will help to understand the nature of the invariant random measure supported on the associated random attractor for relevant examples as model (1.0.1).

There is an increased interest in such bifurcation phenomena from an applied point of view. Specific laser dynamics constitute an important example for such bifurcations. Wieczorek [90] has conducted numerical bifurcation studies for stochastically forced laser models exploring transitions from synchronisation to chaos. He has also shown similar phenomena for coupled lasers with his co-workers in [18, 17]. In the context of oceanography, stochastic Hopf bifurcation has been discussed in [36].

Beyond such applications there is a genuine mathematical interest and motivation for studying the stability of random systems. In dissipative as well as conservative systems, proving chaotic behaviour has turned out to be a very challenging and rarely resolved problem. The classical example for this problem is the *standard map*, an area-preserving mapping of the two-torus which is characterised by expansion on large regions of the state space and small islands of contraction. Positivity of the first Lyapunov exponent has not been shown analytically, even if these critical regions of contraction tend to zero, and the difficulties have been quantified in [40]. Lai-Sang Young and her co-workers have shown recently in [19] that adding a tiny bit of noise to the system allows for averaging arguments over a stationary measure of the induced Markov chain. As long as the stationary measure allocates just a small amount of mass to the regions of contraction, the existence of a positive Lyapunov exponent can be shown. This turns out to hold true for a large class of maps, also in dissipative systems.

In this work, we will prove a similar result for a random dynamical system induced by a stochastic differential equation. The result lines up with the research program suggested in [92]: Young expressed the hope that if the geometry of a random map or stochastic flow suggests a positive Lyapunov exponent, then this is actually the case. The underlying philosophy is that a system with expansions on a large enough portion of its phase space can overcome tendencies to form sinks as long as the randomness is strong enough. The present thesis will add evidence for this conjecture.

Historical context

In addition to Arnold's approach to random dynamical systems, there are two other closely related schools whose work will be discussed and used in this thesis. Firstly, it was discovered around 1980 by Baxendale, Bismut, Elworthy, Kunita and others (see e.g.

[7, 16, 41, 63]) that solutions of stochastic differential equations are not solely stochastic processes but also give rise to a flow of diffeomorphisms. The geometrical properties of such stochastic flows, in particular on manifolds, were then further investigated by Baxendale [8, 9], Baxendale and Stroock [14], Caverhill [22], Caverhill and Elworthy [23], Elworthy [42, 43, 44] and others (see references therein). Le Jan [66] and Baxendale [10] have written seminal papers on the nature of invariant measures for stochastic flows. Baxendale's approach concerning the two-point motion will be particularly helpful to decide whether the invariant random measure is a Dirac measure or not.

The second school has studied the *smooth ergodic theory* or *Pesin's* theory of random diffeomorphisms. Kifer [60] defines the notion of *measure-theoretic* or *metric entropy* for random transformations and shows Ruelle's inequality which says that the sum of the positive Lyapunov exponents bounds metric entropy from above. Ledrappier & Young [70] show Pesin's entropy formula, i.e. they turn Ruelle's inequality into an equality for absolutely continuous stationary measures. Furthermore, they show the SRB-property of the sample measures using techniques from the deterministic case [67, 68]. In addition to that, they prove a dimension formula [69] for the sample measures of random transformations. A good overview with some generalisations has been given in Liu's and Quiang's book on the smooth ergodic theory of random dynamical systems [74]. We will consider more recent generalisations in chapter 5.

The work in hand will discuss approaches from these schools whenever they seem relevant for the bifurcation theory of random dynamical systems. Earlier attempts to develop such a theory beyond simple one-dimensional examples (see [2, Chapter 9.3]) have mainly been led by Ludwig Arnold and co-workers [2, 5, 84, 85]. Referring to work by Baxendale (see e.g. [11]), they coined the notions of the earlier mentioned phenomenological (or "P") bifurcations and *dynamical* (or "D") bifurcations for random dynamical systems induced by stochastic differential equations.

A P-Bifurcation describes a change in the shape of the stationary distribution associated with the stochastic process. Since the stationary distribution is solely a measure of the average behaviour of one typical trajectory, this kind of bifurcation is associated with the *one-point motion*. From a dynamical point of view, we are interested in the behaviour of multiple trajectories for the same noise realisation, described by the *two-point motion*. This leads to the concept of a D-Bifurcation which takes place when a new invariant measure ν bifurcates from an already given one, μ, indicated by the change of sign of one Lyapunov exponent with respect to μ.

The definition of a D-bifurcation was motivated within the context of the Duffing–van der Pol oscillator with multiplicative white noise, being considered as the main toy example for a stochastic Hopf bifurcation. Few rigorous results have been obtained for such examples and most studies have only led to conjectures based on numerical observations [57]. Our work among others [1, 21, 39, 65, 93] suggests that the two concepts of P- and

D-bifurcation do not capture the intricacies of bifurcations in random dynamical systems. This will become apparent in the following.

Main results

We recall model (1.0.1). The choice of additive noise is motivated by the idea that the driving noise is an external perturbation of the system of arbitrary intensity. Furthermore, the system can be seen as a two-dimensional extension of the pitchfork bifurcation with additive noise as discussed by Crauel and Flandoli [32]. They showed that the random attractor is a random equilibrium for all parameter values, as opposed to the splitting into several equilibria at the deterministic bifurcation point, and claimed that the bifurcation is "destroyed" by additive noise. We will see how this point of view translates into the setting of the Hopf bifurcation scenario, but can also be challenged similarly to [21].

Recall that the real parameter b in model (1.0.1) induces shear: if $b \neq 0$, the phase velocity depends on the radius. The stability properties of the stochastic system (1.0.1) for small noise limits and small shear have already been studied in [35] as an example of a non-Hamiltonian system perturbed by noise. The authors also conjectured asymptotic instabilities represented by a positive top Lyapunov exponent based on numerical investigations (see also [90]). They didn't prove implications for the associated random dynamical system in terms of its random attractor and invariant measure. The main topic of Chapter 3 is to prove rigorous statements about these random objects for model (1.0.1) and add new insights for the conjecture on the positive Lyapunov exponent.

In the first part of Chapter 3 we show the following assertions, expressed in detail in Theorem 3.1.1, Theorem 3.1.2 and Theorem 3.1.3.

Theorem A. *The stochastic differential equation* (1.0.1) *induces a random dynamical system and possesses a random attractor for all choices of parameters which is a random equilibrium if the top Lyapunov exponent λ_1 is negative. In this case, we have synchronisation of almost all trajectories from all initial points.*

Fixing all other parameters, there is a $\kappa > 0$ such that for $|b| \leq \kappa$ the top Lyapunov exponent is negative and synchronisation occurs.

Summarising, we show that for small enough shear the Hopf bifurcation is destroyed in the sense of Crauel and Flandoli since synchronisation happens for all values of the bifurcation parameter α.

In the second part of Chapter 3, we address a differentiation between two types of synchronisation that may arise. Synchronisation may be *uniform*, so that trajectories are guaranteed to approximate each other bounded by upper estimates that are independent of the noise realisation, or *non-uniform*, when such uniform upper estimates do not exist. In the latter case, the time it takes for two trajectories to converge up to a certain given

margin is bounded for any fixed noise realisation, but assessed over all noise realisations these bounds have no maximum.

It turns out that the uniformity of the synchronisation is related to the distribution of *finite-time Lyapunov exponents*, reflecting the average derivatives along trajectories for finite time. The (unique) top Lyapunov exponent of an attractor is associated with the limit of the distribution of finite-time Lyapunov exponents as the time over which derivatives are averaged goes to infinity. Importantly, while this distribution converges to a Dirac measure concentrated in the top Lyapunov exponent, the support of this distribution typically converges to a wider range. If this range is contained entirely within the negative real axis, synchronisation is uniform. But it may also happen that the top Lyapunov is negative while the limit of the support of finite-time Lyapunov exponents extends into the positive half line, which results in non-uniform synchronisation. In Figure 1.3, these scenarios are illustrated with numerical computations. It is natural to find an interface with non-uniform synchronisation in the phase diagram between uniform synchronisation regions and regions without synchronisation. For a schematic sketch of the corresponding phase diagram regions for (1.0.1), see Figure 1.4.

Theorem 3.1.6 fixes small shear and formalises the transition from uniform to non-uniform synchronisation in dependence on the linear stability parameter α while Theorem 3.1.7 fixes α and shows the occurrence of unbounded positive finite-time Lyapunov exponents for sufficiently large shear. We summarise these results in the following theorem:

Theorem B. *Assume that $|b| \leq a$ and $\alpha \in \mathbb{R}_0^- \cup B_\varepsilon(0)$ for some $\varepsilon > 0$. Then there exists a random equilibrium for the random dynamical system induced by* (1.0.1) *and*

(i) the random equilibrium changes from being globally uniformly attractive to being not even locally uniformly attractive at $\alpha = 0$,

(ii) at $\alpha = 0$ we observe the appearance of positive finite-time Lyapunov exponents,

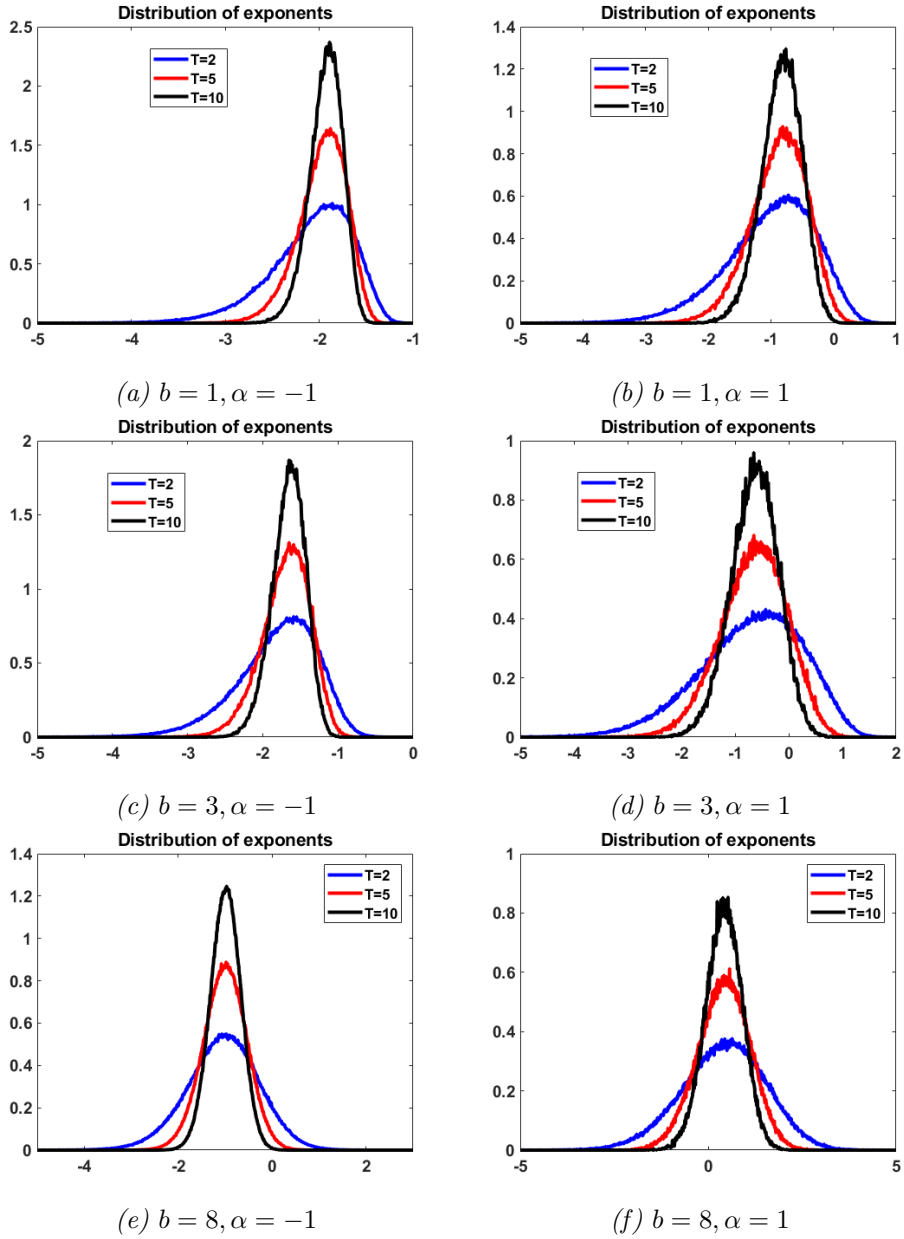
(iii) the dichotomy spectrum Σ of the linearisation around the random equilibrium is

$$\Sigma = [-\infty, \alpha]\,,$$

which implies loss of hyperbolicity at the bifurcation point $\alpha = 0$.

If we fix $\alpha \in \mathbb{R}$ and $\sigma > 0$, then for any $b > 2a > 0$ the system exhibits arbitrarily large finite-time Lyapunov exponents with positive probability for all initial points.

The bifurcation in the random equilibrium at $\alpha = 0$ is an example of the philosophy expressed in [21] which contradicts the conception that additive noise "destroys" the pitchfork bifurcation [32].



(a) $b = 1, \alpha = -1$ (b) $b = 1, \alpha = 1$

(c) $b = 3, \alpha = -1$ (d) $b = 3, \alpha = 1$

(e) $b = 8, \alpha = -1$ (f) $b = 8, \alpha = 1$

Figure 1.3: Distribution of finite-time ($T > 0$) Lyapunov exponents of (3.0.1) with $a = \beta = \sigma = 1$, $\alpha \in \{-1, 1\}$, $T \in \{2, 5, 10\}$ and $b \in \{1, 3, 8\}$, illustrating the type of distributions in phases (I) uniform synchronisation (a), (c); (II) non-uniform synchronisation (b), (d), (e); and (III) absence of synchronisation (f).

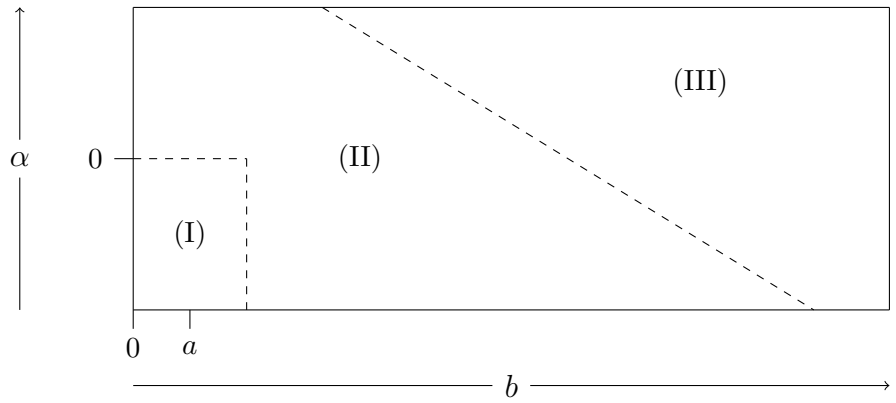


Figure 1.4: For a, β, σ fixed, we partition the (b, α)-parameter space associated with (3.0.1) *schematically into three parts with different stability behaviour. Region (I) represents uniform synchronisation, only possible for non-positive α and small b. In region (II), we observe non-uniform synchronisation, i.e. finite-time instabilities occur, but the asymptotic behaviour is exponentially stable for almost all trajectories. (The border between (I) and (II) is described in Theorem B.) Region (III) exhibits a positive top Lyapunov exponent and the absence of synchronisation since the shear is large enough for locally unstable behaviour to prevail (cf. Conjecture 3.1.5).*

The existence of arbitrarily large positive finite-time Lyapunov exponents is the first analytical contribution to the problem of *shear-induced chaos* (a term coined by Lin & Young [73]) for model (1.0.1). Most rigorous results on Lyapunov exponents (and random dynamical systems) are obtained for one-dimensional SDEs, in which case the analysis of Lyapunov exponents is significantly simplified due to the fact that all derivatives commute. In general it is very difficult to obtain lower bounds for the top Lyapunov exponent in multiple dimensions due to the subadditivity property of matrices, cf. [92]. Therefore, the analytical proof of positive Lyapunov exponents for noisy systems has been achieved only in certain special cases such as for equilibria [53], in simple time-discrete models as in [72] or under special circumstances that allow for the use of stochastic averaging [12, 13]. In Conjecture 3.1.5 we suspect a positive top Lyapunov exponent for large enough shear levels for any given value of α which would imply the existence of a random strange attractor. Figure 1.2 shows the shape of such an attractor as obtained in a numerical simulation. Based on numerical evidence, we conjecture this scenario also for negative α, which is remarkable in view of the literature usually associating shear-induced chaos with perturbed limit cycles and not equilibria.

In the third part of Chapter 3 we report about approaches to prove the conjecture hoping that this might inspire new attempts to tackle this difficult problem. The last part of the chapter discusses numerical results for other Hopf bifurcation systems with additive noise discussed in the literature [3, 84, 85], mainly concerning synchronisation by noise as there is nothing like a shear force present in the models. Note that model (1.0.1) is exemplary in the following sense: firstly, it discusses the typical phenomenon of random systems to exhibit a transition between synchronisation and chaos. Secondly, the normal

form is locally equivalent to that of a generic deterministic Hopf bifurcation and, hence, at least for small noise, we can expect other examples of Hopf bifurcation to feature similar behaviour.

The main technical challenge we address in Chapter 4 is to establish the existence of a positive top Lyapunov exponent and associated random strange attractor which remains a conjecture for model (1.0.1). We consider the following model of a stochastically driven limit cycle

$$\begin{aligned} \mathrm{d}y &= -\alpha y\,\mathrm{d}t + \sigma f(\vartheta)\circ \mathrm{d}W_t^1\,, \\ \mathrm{d}\vartheta &= (1+by)\,\mathrm{d}t\ , \end{aligned} \tag{1.0.3}$$

where $(y,\vartheta)\in\mathbb{R}\times\mathbb{S}^1$ are cylindrical amplitude-phase coordinates, and W_t^1 denotes one-dimensional Brownian motion entering the equation as noise of Stratonovich type. In the absence of noise ($\sigma=0$), the ODE (1.0.3) has a globally attracting limit cycle at $y=0$ if $\alpha>0$. In the presence of noise ($\sigma\neq 0$), the amplitude is driven by phase-dependent noise. This corresponds with model (1.0.1) written in polar coordinates where the noise is also no longer solely additive. The real parameter b induces shear: if $b\neq 0$, the phase velocity $\frac{\mathrm{d}\vartheta}{\mathrm{d}t}$ depends on the amplitude y. The stable limit cycle turns into a random attractor if $\sigma\neq 0$.

With a particular choice of the amplitude-phase coupling f we obtain the following bifurcation result summarising Theorem 4.0.1 and Corollary 4.0.2. We show at the end of Chapter 4 that the result stays robust under perturbations of f that smoothen this function. As long as $b,\sigma\neq 0$, the amplitude variable y can be rescaled so that the shear parameter becomes equal to 1 and the effective noise-amplitude becomes σb. Hence, the result below also holds with the roles of σ and b interchanged.

Theorem C. *Consider the SDE (1.0.3), where $f:\mathbb{S}^1\simeq[0,1)\to\mathbb{R}$ is continuous and piecewise linear with constant absolute value of the derivative almost everywhere. Then there is $0<c_0\approx 0.2823$ such that for all $\alpha>0$ and $b\neq 0$ the number $\sigma_0(\alpha,b)=\frac{\alpha^{3/2}}{c_0^{1/2}|b|}>0$ is the unique value of σ where the top Lyapunov exponent $\lambda_1(\alpha,b,\sigma)$ of (1.0.3) changes its sign. In more detail, we have*

$$\lambda_1(\alpha,b,\sigma)\begin{cases} <0 & \text{if } 0<\sigma<\sigma_0(\alpha,b)\,,\\ =0 & \text{if } \sigma=\sigma_0(\alpha,b)\,,\\ >0 & \text{if } \sigma>\sigma_0(\alpha,b)\,. \end{cases}$$

If $0<\sigma<\sigma_0(\alpha,b)$, the random point attractor of (1.0.3) is an attracting random equilibrium. If $\sigma>\sigma_0(\alpha,b)$ the random point attractor of the system is a random strange attractor.

The behaviour of $\lambda_1(\alpha,b,\sigma)$ and $\sigma_0(\alpha,b)$ is illustrated in Figure 4.1 and Figure 4.4. Theorem C confirms numerical results by Lin & Young [73] for a very similar model.

The mechanism whereby a combination of shear and noise causes stretching and folding, leading to a positive Lyapunov exponent, is usually referred to as shear-induced chaos, as already mentioned in the context of model (1.0.1). Wang & Young [88, 89] and Ott & Stenlund [78] have demonstrated analytically the validity of this mechanism in the case of periodically kicked limit cycles, including probabilistic characterizations of the dynamics. An analytical proof of shear-induced chaos in the stochastic setting, as presented in this chapter, had remained an open problem. We will see that the choice of f is crucial to establish rigorous lower bounds on the top Lyapunov exponent λ_1.

Similarly to Figure 1.2, Figure 4.3 displays a random strange attractor for model (1.0.3) when the parameters are chosen such that $\lambda_1 > 0$. In one-dimensional SDEs, negative Lyapunov exponents and attractive random equilibria prevail [32]. Random strange attractors can only arise in dimension two and higher and up to now, relatively little research has been devoted to such attractors. In contrast, the existence of attractive random equilibria has been studied well, also in higher dimensions [10, 48, 66, 76, 77].

Chapter 5 gives a more detailed description of random strange attractors. First we show that the sample measures are not supported on a singleton if $\lambda_1 > 0$. We follow work by Baxendale and Stroock [14, 10] extending their results to the non-compact case. Furthermore, we introduce metric entropy for random diffeomorphisms, following [60, 74, 15], and link these discrete-time systems with random dynamical systems generated by stochastic differential equations via the time-one maps. As a direct consequence of [15, Theorem 9.1] we can show Pesin's formula, i.e. the equality of metric entropy and the sum of positive Lyapunov exponents, for systems derived from stochastic differential equations also on non-compact state space and deduce that the random dynamical system induced by model (1.0.3) has positive entropy for large enough shear or noise respectively, i.e. if $\lambda_1 > 0$. The same would hold true for model (1.0.1), once positive Lyapunov exponents can be established. The following statement summarises Theorem 5.2.9 and Corollary 5.2.10.

Theorem D. *Let (θ, φ) be a random dynamical system induced by a stochastic differential equation with sufficiently smooth coefficients and an absolutely continuous stationary probability measure ρ satisfying the mild integrability condition*

$$\int_{\mathbb{R}^d} (\ln(|x|+1))^{1/2} \, \rho(\mathrm{d}x) < \infty \,.$$

Then the discrete time random dynamical system $\mathcal{X}^+$ associated with (θ, φ) satisfies Pesin's formula

$$h_\rho\left(\mathcal{X}^+\right) = \int_{\mathbb{R}^d} \sum_i \lambda_i(x)^+ m_i(x) \rho(\mathrm{d}x) \,,$$

where $h_\rho(\mathcal{X}^+)$ denotes the metric entropy, λ_i^+ the positive Lyapunov exponents and m_i its multiplicities.

As a consequence, if $\mathcal{X}^+$ denotes the discrete-time random dynamical system induced by model (1.0.3) and $\sigma > \sigma_0(\alpha, b)$, then $h_\rho(\mathcal{X}^+) > 0$.

Chapter 5 also contains a short treatise on SRB measures for random systems. The invariant sample measures are called SRB measures if they are absolutely continuous with respect to the Riemannian measure on fibres of unstable manifolds. Hence, if the sample measures can be shown to be SRB measures, they are supported on non-singular subsets of the closures of unstable manifolds, potentially even on the whole set. This gives a strong characterisation of strange attractors. In [70] the SRB property is shown for sample measures of discrete-time systems and systems induced by stochastic differential equations on compact manifolds, in case the stationary measure is absolutely continuous. It is beyond the scope of this work to show the results for the non-compact case as there are too many technical intricacies to be considered. We have to be content with the intuition that if the maps/flows and their derivatives satisfy uniform bounds, the results are applicable to the non-compact case.

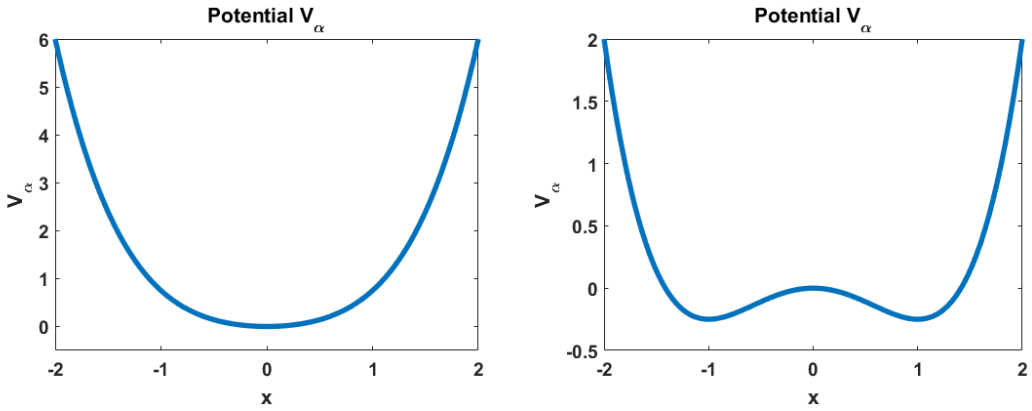
Note that most of the work presented in Chapter 5 is not original. Nevertheless, we consider it relevant to embed the results of Chapters 3 and 4 into the context of smooth ergodic theory in order to give a more thorough description of random strange attractors. In addition to that, we see it as a contribution to linking the Arnold school and the school represented by Kifer, Ledrappier and Young.

A possible objection to stochastic bifurcation theory working with unbounded white noise models concerns the local nature of the deterministic bifurcations considered, as opposed to the global nature of the noise which explores the whole state space. The stochastic models of Hopf bifurcation and attracting limit cycles in Chapters 3 and 4 exhibit local asymptotic stability for certain parameter regimes due to global stability properties. This fact can obscure bifurcation behaviour still present in the stochastic case, as we show in Theorem B. To make this issue even clearer, consider the one-dimensional pitchfork example with additive white noise

$$\mathrm{d}X_t = f_\alpha(X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\,, \quad X_0 = x \in \mathbb{R}\,,$$

where $\alpha \in \mathbb{R}$, $\sigma > 0$ and the drift is the derivative of a potential

$$f_\alpha(x) = -\partial_x V_\alpha(x)\,, \quad \text{with } V_\alpha(x) = -\frac{\alpha}{2}x^2 + \frac{1}{4}x^4\,.$$



(a) $\alpha = -1$ *(b)* $\alpha = 1$

Figure 1.5: Graphs of the potential $V_\alpha(x) = -\frac{\alpha}{2}x^2 + \frac{1}{4}x^4$ *for* $\alpha = -1$ *and* $\alpha = 1$.

We depict the potential for $\alpha = -1$ and $\alpha = 1$ in Figure 1.5. Observe that the bifurcation at $\alpha = 0$ implies the appearance of a local instability around the origin whereas the potential stays globally stable outside a neighbourhood of 0. Without noise, the bifurcation entails a change of the attractor from $\{0\}$ for $\alpha \leq 0$ to $[-\sqrt{\alpha}, \sqrt{\alpha}]$ for $\alpha > 0$. As already mentioned above, the bifurcation is "destroyed" in the presence of noise in the sense that the random attractor is a random equilibrium for all $\sigma > 0, \alpha \in \mathbb{R}$ [32]. The white noise lets the system explore the whole state space and the global stability leads to a negative Lyapunov exponent

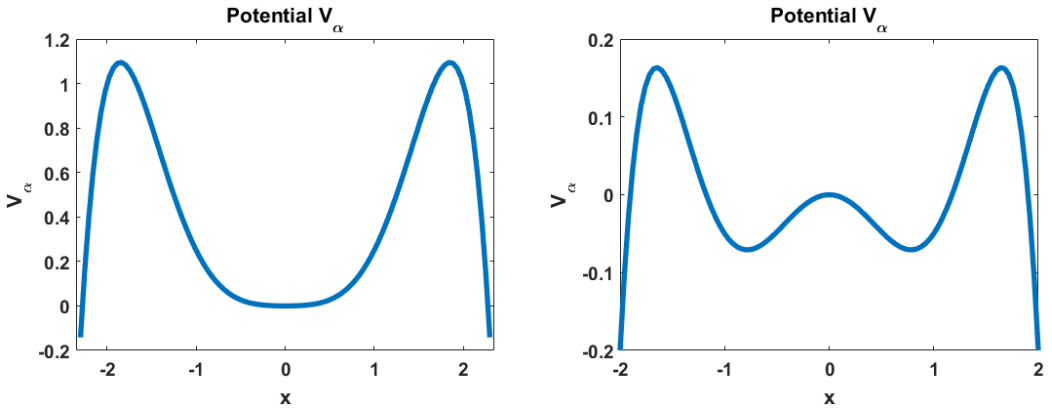
$$\lambda = \int_{\mathbb{R}} -V''_\alpha p_{\alpha,\sigma}(x)\,\mathrm{d}x$$

for all parameter values, where $p_{\alpha,\sigma}(x)$ denotes the density of the stationary distribution.

Also in this case, a random bifurcation takes place at $\alpha = 0$ if we consider the dichotomy spectrum Σ which accounts for local instabilities. According to [21] it is given by

$$\Sigma = [-\infty, \alpha] \quad \text{for all } \alpha \in \mathbb{R}\,.$$

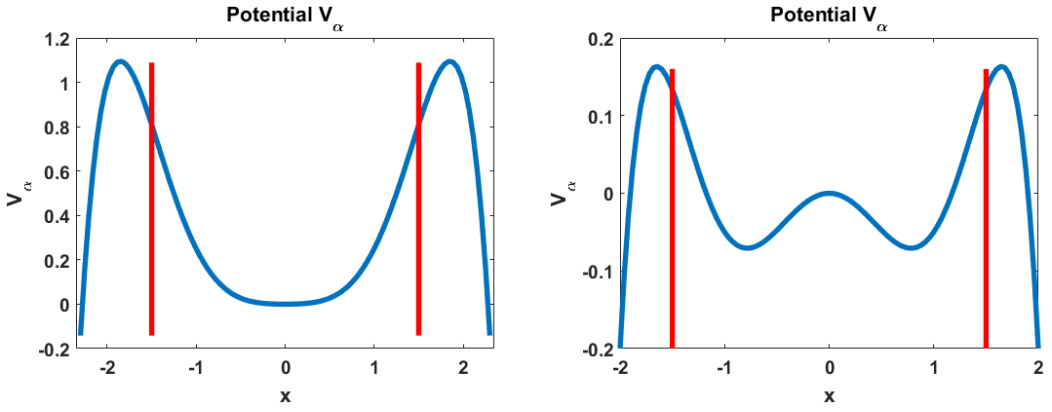
However, the dichotomy spectrum still contains a measure of global stability by covering $\mathbb{R}_0^-$ and, in general, it is not as directly interpretable as the sign of a Lyapunov exponent. The question is what kind of analysis can most accurately describe a local stochastic bifurcation, in particular if the system does not exhibit global stability outside a critical neighbourhood. If we leave the setting of a normal form, such a problem naturally arises, as can be seen in Figure 1.6: the potential $\tilde{V}_\alpha$ still induces a bifurcation at $\alpha = 0$ but the system is globally unstable outside a neighbourhood of the origin.



(a) $\alpha = -0.1$ (b) $\alpha = 0.5$

Figure 1.6: Graphs of the potential $\tilde{V}_\alpha(x) = -\frac{\alpha}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{20}x^6$ *for* $\alpha = -0.1$ *and* $\alpha = 0.5$.

Our approach to tackle this issue is to embed stochastic bifurcation theory into the context of Markov processes that induce a random dynamical system evolving in a domain with forbidden states constituting a trap. The process is said to be killed when it hits the trap and it is assumed that this happens almost surely at a finite hitting time T. In Chapter 6, we investigate the behaviour of the process before being killed, asking what happens when one conditions the process to survive for asymptotically long times.



(a) $\alpha = -0.1$ (b) $\alpha = 0.5$

Figure 1.7: Graphs of the potential $\tilde{V}_\alpha(x) = -\frac{\alpha}{2}x^2 + \frac{1}{4}x^4 - \frac{1}{20}x^6$ *for* $\alpha = -0.1$ *and* $\alpha = 0.5$, *supplemented by red lines at* $x = -1.5, 1.5$ *demarcating a stable area of the state space.*

In Figure 1.7 one can see exactly the same graphs of $\tilde{V}_\alpha$ as in Figure 1.6, now supplemented by two red lines at $x = -1.5, 1.5$. These lines indicate the boundary of a domain within which the system behaves similarly to the normal form and the global instability can be disregarded. By making such a boundary a forbidden state, we can restrict the analysis to trajectories surviving within a neighbourhood of the origin which is relevant for the bifurcation behaviour.

The topic of conditioned processes goes back to the pioneering work by Yaglom in

1947 [91] but in recent years new ideas have been developed (see [28, 75] for excellent overviews). Due to the loss of mass by absorption at the boundary, the existence of a stationary distribution is impossible and, therefore, stationarity is replaced by quasi-stationarity. A *quasi-stationary distribution* preserves mass along the process conditioned on survival. Analogously, ergodicity is a problematic concept in this context. Nonetheless, given a unique quasi-stationary distribution for a Markov process $(X_t)_{t\geq 0}$ on a state space E, one can derive the existence of a *quasi-ergodic distribution* m, characterised by

$$\lim_{t\to\infty} \mathbb{E}_x \left(\frac{1}{t} \int_0^t f(X_s)\, \mathrm{d}s \middle| T > t \right) = \int_E f \, \mathrm{d}m$$

for all measurable and bounded observables f and initial points x. Using recent work on this topic by Villemonais, Champagnat, He and others [25, 24, 50], we are able to show the following statement for systems induced by stochastic differential equations

$$\mathrm{d}X_t = f(X_t)\, \mathrm{d}t + \sigma \, \mathrm{d}W_t \,, \;\; X_0 = x \in E \,, \tag{1.0.4}$$

where f is continuously differentiable and $E \subset \mathbb{R}^d$ is a bounded domain:

Theorem E. *Let (θ, φ) be the random dynamical system with absorption at the boundary corresponding to the Markov process $(X_t)_{t\geq 0}$ solving equation (1.0.4). If the generator for the projective bundle of $(X_t)_{t\geq 0}$ and its derivative process is hypoelliptic, then for all $v \in \mathbb{R}^d \setminus \{0\}$ and $x \in E$ the conditioned expectation of the finite–time Lyapunov exponents converges to the same real number λ, i.e.*

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}_x \left[\ln \frac{\|\mathrm{D}\varphi(t,\cdot,x)v\|}{\|v\|} \middle| T > t \right] .$$

We call λ the conditioned average Lyapunov exponent.

The proof (see Proposition 6.2.3) relies on the fact that the finite-time Lyapunov exponents can be expressed as the time averages of a functional. These time averages conditioned on survival converge to an integral with respect to the quasi-ergodic distribution.

Furthermore, we can show that λ, which at first sight only gives the limit of expected values, has the strongest possible dynamical meaning for the setting with killing at the boundary, where convergence almost surely is not feasible: we prove convergence in probability of the finite-time Lyapunov exponents

$$\lambda_v(t,\cdot,x) := \frac{1}{t} \ln \frac{\|\mathrm{D}\varphi(t,\cdot,x)v\|}{\|v\|} \,.$$

Theorem F. *Let $\lambda_v(t,\cdot,x)$ denote the finite-time Lyapunov exponents associated with*

equation (1.0.4). *Then for all* $\varepsilon > 0$ *we have*

$$\lim_{t\to\infty} \mathbb{P}_x \left(|\lambda_v(t,\cdot,x) - \lambda| \geq \varepsilon \middle| T > t \right) = 0$$

uniformly over all $x \in E$, $v \in \mathbb{S}^{d-1}$. *This means that the finite-time Lyapunov exponents of the surviving trajectories converge to its assemble average in probability.*

In short, the number λ is a measure of typical asymptotic stability/instability of the surviving trajectories. In Section 6.6 we deploy the change of sign of λ as an indicator for a stochastic bifurcation in the pitchfork and the Hopf example. We further discuss the possibility of a Lyapunov spectrum for systems with killing and conduct numerical experiments that indicate its existence. However, a proof seems out of reach.

Moreover, Chapter 6 contains the proof of the following local synchronisation theorem (see Theorem 6.3.1) which characterises the situation of a negative conditioned average Lyapunov exponent λ.

Theorem G. *If* $\lambda < 0$ *there is exponentially fast local synchronisation of trajectories in discrete time with arbitrarily high probability, i.e. for all* $0 < \rho < 1$ *and* $\lambda_\varepsilon \in (\lambda, 0)$ *there is an* α_x *such that*

$$\lim_{n\to\infty} \mathbb{P}_x \left(\frac{1}{n} \ln \|\varphi(n,\cdot,x) - \varphi(n,\cdot,y)\| \leq \lambda_\varepsilon \;\; \textit{for all } y \in \overline{B}_{\alpha_x}(x) \middle| T > n \right) > 1 - \rho\,.$$

This is another indication of the dynamical relevance of λ.

We further define the dichotomy spectrum for the situation with killing at the boundary and show that it consists of a finite number $n \in \{1, \ldots, d\}$ of closed intervals (Theorem 6.5.8), where d is the dimension of the space. In addition to that, we prove a classical relation to the spectrum of finite-time Lyapunov exponents (Theorem 6.5.9). We also determine the dichotomy spectrum for the pitchfork problem with killing.

Relating quasi-stationary and quasi-ergodic measures to invariant random measures of the killed random dynamical system turns out to be difficult. Leaving out the past of the system allows for a correspondence with conditionally invariant measures of the associated open system on the skew-product space (Propositions 6.4.2 and 6.4.4), but anything beyond that result remains to be explored.

Structure of the thesis

To summarise, this work consists of three major parts. In Chapter 3, we discuss a model of Hopf bifurcation with shear and additive noise and prove synchronisation of all trajectories for small enough shear. We detect bifurcation behaviour in terms of uniform and finite-time stability and show that for large enough shear arbitrarily large finite-time instabilities occur. It remains a conjecture to show positive Lyapunov exponents for that problem.

In Chapter 4, we define a model of a stochastically driven limit cycle for which we are able to show analytically the existence of positive Lyapunov exponents. By that we establish one of the first examples for proving a transition from synchronisation to chaos in dissipative stochastic systems, a behaviour that is expected to be typical for stochastic bifurcations. In Chapter 6, we develop a theory of measuring asymptotic stability for random dynamical systems conditioned on survival in a bounded domain. The motivation is to explore the local bifurcation behaviour, that can be obscured by unbounded noise, also for the stochastic case.

Chapter 5 mainly summarises known ergodic theory to describe chaotic attractors seen in Chapter 4 and conjectured in Chapter 3. We start with Chapter 2, which introduces random dynamical systems and important concepts such as random attractors, invariant measures and multiplicative ergodic theory.

Chapter 2

Random dynamical systems

We work in the framework of random dynamical systems. A continuous-time random dynamical system on a topological state space X consists of

(i) a model of the noise on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, formalised as a measurable flow $(\theta_t)_{t\in\mathbb{R}}$ of $\mathbb{P}$-preserving transformations $\theta_t : \Omega \to \Omega$,

(ii) a model of the dynamics on X perturbed by noise formalised as a *cocycle* φ over θ.

In technical detail, the definition of a random dynamical system is given as follows [2, Definition 1.1.2].

Definition 2.0.1 (Random dynamical system)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X be a topological space. A random dynamical system (RDS) is a pair of mappings (θ, φ).

- The $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable mapping $\theta : \mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \theta_t\omega$, is a *metric dynamical system*, i.e.

  (i) $\theta_0 = \mathrm{id}$ and $\theta_{t+s} = \theta_t \circ \theta_s$ for $t, s \in \mathbb{R}$,

  (ii) $\mathbb{P}(A) = \mathbb{P}(\theta_t A)$ for all $A \in \mathcal{F}$ and $t \in \mathbb{R}$.

- The $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping $\varphi : \mathbb{R} \times \Omega \times X \to X, (t, \omega, x) \mapsto \varphi(t, \omega, x)$, is a cocycle over θ, i.e.

  $$\varphi(0, \omega) = \mathrm{id} \quad \text{and} \quad \varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s\omega, \varphi(s, \omega, \cdot)) \quad \text{for all } \omega \in \Omega \text{ and } t, s \in \mathbb{R}\,.$$

The random dynamical system (θ, φ) is called continuous if $(t, x) \mapsto \varphi(t, \omega, x)$ is continuous for every $\omega \in \Omega$. We still speak of a random dynamical system, if its cocycle is only defined in forward time, i.e. if the mapping φ is only defined on $\mathbb{R}_0^+ \times \Omega \times X$. We will make it noticeable whenever this is the case.

In the following, the metric dynamical system $(\theta_t)_{t\in\mathbb{R}}$ is often even ergodic, i.e. any $A \in \mathcal{F}$ with $\theta_t^{-1}A = A$ for all $t \in \mathbb{R}$ satisfies $\mathbb{P}(A) \in \{0, 1\}$. Note that we define θ in

two-sided time whereas φ can be restricted to one-sided time. This is motivated by the fact that a large part of this thesis will deal with random dynamical systems generated by stochastic differential equations, in particular in Chapters 3 and 4. However, we will restrict θ to one-sided time, considering the system as a *one-sided random dynamical system*, for special problems in Chapter 6, and in Chapter 5 where we additionally consider random systems in discrete time.

Further, note that the trajectories of the RDS might explode in finite time. In this case one can consider it as a *local* random dynamical system (as opposed to the *global* random dynamical system from Definition 2.0.1) being defined only for times bounded by some random explosion times $\tau^-(\omega, x)$ and $\tau^+(\omega, x)$. We will consider local RDS in more detail in the context of Chapter 6.

Throughout this thesis, we will investigate random dynamical systems induced by stochastic differential equations. Hence, we are interested in random dynamical systems adapted to a suitable filtration and of white noise type. Following [48], we make the following definition:

Definition 2.0.2 (White noise RDS)**.** Let (θ, φ) be a random dynamical system over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on a topological space X where φ is defined in forward time. Let $(\mathcal{F}_s^t)_{-\infty \le s \le t \le \infty}$ be a family of sub-σ-algebras of $\mathcal{F}$ such that

(i) $\mathcal{F}_t^u \subset \mathcal{F}_s^v$ for all $s \le t \le u \le v$,

(ii) $\mathcal{F}_s^t$ is independent from $\mathcal{F}_u^v$ for all $s \le t \le u \le v$,

(iii) $\theta_r^{-1}(\mathcal{F}_s^t) = \mathcal{F}_{s+r}^{t+r}$ for all $s \le t$, $r \in \mathbb{R}$,

(iv) $\varphi(t, \cdot, x)$ is $\mathcal{F}_0^t$-measurable for all $t \ge 0$ and $x \in X$.

Furthermore we denote by $\mathcal{F}_{-\infty}^t$ the smallest *sigma*-algebra containing all $\mathcal{F}_s^t$, $s \le t$, and by $\mathcal{F}_t^\infty$ the smallest *sigma*-algebra containing all $\mathcal{F}_t^u$, $t \le u$. Then (θ, φ) is called a *white noise (filtered) random dynamical system.*

Consider a stochastic differential equation (SDE)

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t)dW_t,\ X_0 \in \mathbb{R}^d\,, \tag{2.0.1}$$

where (W_t) denotes some r-dimensional standard Brownian motion, the drift $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz continuous vector field and the diffusion coefficient $g : \mathbb{R}^d \to \mathbb{R}^{d\times r}$ a Lipschitz continuous matrix-valued map. If in addition f satisfies a bounded growth condition, as for example a one-sided Lipschitz condition, then by [37] there is a white noise random dynamical system (θ, φ) associated to the diffusion process solving (2.0.1). The probabilistic setting is as follows: We set $\Omega = C_0(\mathbb{R}, \mathbb{R}^r)$, i.e. the space of all continuous functions $\omega : \mathbb{R} \to \mathbb{R}^r$ satisfying that $\omega(0) = 0 \in \mathbb{R}^r$. If we endow Ω with the compact

open topology given by the complete metric

$$\kappa(\omega,\widehat{\omega}) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|\omega-\widehat{\omega}\|_n}{1+\|\omega-\widehat{\omega}\|_n}, \quad \|\omega-\widehat{\omega}\|_n := \sup_{|t|\leq n} \|\omega(t)-\widehat{\omega}(t)\|\,,$$

we can set $\mathcal{F}=\mathcal{B}(\Omega)$, the Borel-sigma algebra on (Ω,κ). There exists a probability measure $\mathbb{P}$ on $(\Omega,\mathcal{F})$ called *Wiener measure* such that the r processes $(W_t^1),\ldots,(W_t^r)$ defined by $(W_t^1(\omega),\ldots,W_t^r(\omega))^{\mathrm{T}} := \omega(t)$ for $\omega\in\Omega$ are independent one-dimensional Brownian motions. Furthermore, we define the sub-σ-algebra $\mathcal{F}_s^t$ as the σ-algebra generated by $\omega(u)-\omega(v)$ for $s\leq v\leq u\leq t$. The ergodic metric dynamical system $(\theta_t)_{t\in\mathbb{R}}$ on $(\Omega,\mathcal{F},\mathbb{P})$ is given by the shift maps

$$\theta_t : \Omega\to\Omega, \quad (\theta_t\omega)(s) = \omega(s+t)-\omega(t)\,.$$

Indeed, these maps form an ergodic flow preserving the probability $\mathbb{P}$, see e.g. [2].

In chapter 3, we are not able to work with a one-sided Lipschitz condition. Instead, we will use a transformation into a random differential equation to show that the respective stochastic differential equation induces a random dynamical system.

2.1 Invariant measures

Let (θ,φ) be a random dynamical system with the cocycle φ being defined on one-or two-sided time $\mathbb{T}\in\{\mathbb{R}_0^+,\mathbb{R}\}$. Then the system generates a skew product flow, i.e. a family of maps $(\Theta_t)_{t\in\mathbb{T}}$ from $\Omega\times X$ to itself such that for all $t\in\mathbb{T}$ and $\omega\in\Omega, x\in X$

$$\Theta_t(\omega,x) = (\theta_t\omega,\varphi(t,\omega,x))\,.$$

The notion of an invariant measure for the random dynamical system is given via the invariance with respect to the skew product flow, see e.g. [2, Definition 1.4.1]. We denote by $T\mu$ the push forward of a measure μ by a map T, i.e. $T\mu(\cdot)=\mu(T^{-1}(\cdot))$.

Definition 2.1.1 (Invariant measure)**.** A probability measure μ on $\Omega\times X$ is invariant for the random dynamical system (θ,φ) if

(i) $\Theta_t\mu=\mu$ for all $t\in\mathbb{T}$,

(ii) the marginal of μ on Ω is $\mathbb{P}$, i.e. μ can be factorised uniquely into $\mu(\mathrm{d}\omega,\mathrm{d}x) = \mu_\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)$ where $\omega\mapsto\mu_\omega$ is a *random measure* (or *disintegration* or *sample measure*) on X, i.e. μ_ω is a probability measure on X for $\mathbb{P}$-a.a. $\omega\in\Omega$ and $\omega\mapsto\mu_\omega(B)$ is measurable for all $B\in\mathcal{B}(X)$.

The marginal of μ on the probability space is demanded to be $\mathbb{P}$ since we assume the model of the noise to be fixed. Note that the invariance of μ is equivalent to the invariance

of the random measure $\omega \mapsto \mu_\omega$ on the state space X in the sense that

$$\varphi(t,\omega,\cdot)\mu_\omega = \mu_{\theta_t\omega} \quad \mathbb{P}\text{-a.s. for all } t \in \mathbb{T}\,. \tag{2.1.1}$$

For white noise random dynamical systems (θ,φ), in particular random dynamical systems induced by a stochastic differential equation, there is a one-to-one correspondence between certain invariant random measures and stationary measures of the associated stochastic process, first observed in [30]. In more detail, we can define a Markov semigroup $(P_t)_{t\geq 0}$ by setting

$$P_t f(x) = \mathbb{E}(f(\varphi(t,\cdot,x))$$

for all measurable and bounded functions $f : X \to \mathbb{R}$. If $\omega \mapsto \mu_\omega$ is a $\mathcal{F}^0_{-\infty}$-measurable invariant random measure in the sense of (2.1.1), also called *Markov measure*, then

$$\rho(\cdot) = \mathbb{E}[\mu_\omega(\cdot)] = \int_\Omega \mu_\omega(\cdot)\mathbb{P}(d\omega)$$

turns out to be an invariant measure for the Markov semigroup $(P_t)_{t\geq 0}$, often also called stationary measure for the associated process. If ρ is an invariant measure for the Markov semigroup, then

$$\mu_\omega = \lim_{t\to\infty} \varphi(t,\theta_{-t}\omega,\cdot)\rho$$

exists $\mathbb{P}$-a.s. and is an $\mathcal{F}^0_{-\infty}$-measurable invariant random measure.

We observe similarly to [10] that, in the situation of μ and ρ corresponding in the way described above,

$$\mathbb{E}[\mu_\omega(\cdot)|\mathcal{F}^\infty_0] = \mathbb{E}[\mu_\omega(\cdot)] = \rho(\cdot)\,,$$

and, hence,

$$\mathbb{E}[\mu(\cdot)|\mathcal{F}^\infty_0] = (\mathbb{P}\times\rho)(\cdot)\,.$$

Therefore the probability measure $\mathbb{P}\times\rho$ is invariant for $(\Theta_t)_{t\geq 0}$ on $(\Omega\times X, \mathcal{F}^\infty_0 \times \mathcal{B}(X))$. In words, the product measure with marginals $\mathbb{P}$ and ρ is invariant for the random dynamical system restricted to one-sided path space. We will discuss a similar relation for quasi-stationary and quasi-ergodic measures in Chapter 6.

2.2 Random attractors

Let (θ,φ) be a white noise random dynamical system on a metric space (X,d). We give the definition of a random attractor of (θ,φ) with respect to tempered sets.

A random variable $R : \Omega \to \mathbb{R}$ is called *tempered* if

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \ln^+ R(\theta_t\omega) = 0 \quad \text{for almost all } \omega \in \Omega\,,$$

see also [2, p. 164]. A set $D \in \mathcal{F} \otimes \mathcal{B}(X)$ is called *tempered* if there exists a tempered random variable R such that

$$D(\omega) \subset B_{R(\omega)}(0) \quad \text{for almost all } \omega \in \Omega\,,$$

where $D(\omega) := \{x \in X : (\omega, x) \in D\}$. D is called compact if $D(\omega) \subset X$ is compact for almost all $\omega \in \Omega$. Denote by $\mathcal{D}$ the set of all compact tempered sets $D \in \mathcal{F} \otimes \mathcal{B}(X)$. We now define the notion of a random attractor with respect to $\mathcal{D}$, see also [61, Definition 14.3].

Definition 2.2.1 (Random attractor)**.** A set $A \in \mathcal{D}$ is called a *random attractor* (with respect to $\mathcal{D}$) if the following two properties are satisfied.

(i) A is φ-invariant, i.e.

$$\varphi(t, \omega)A(\omega) = A(\theta_t\omega) \quad \text{for all } t \geq 0 \text{ and almost all } \omega \in \Omega\,.$$

(ii) For all $D \in \mathcal{D}$, we have

$$\lim_{t\to\infty} \operatorname{dist}\big(\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega), A(\omega)\big) = 0 \quad \text{for almost all } \omega \in \Omega\,,$$

where $\operatorname{dist}(E, F) := \sup_{x\in E} \inf_{y\in F} d(x, y)$.

The set A is called a *weak random attractor* if it satisfies the latter property with almost sure convergence replaced by convergence in probability. We call A a *(weak) random point attractor* if it satisfies the properties above with tempered random sets D replaced by single points $y \in X$ in (ii). A (weak) random point attractor is said to be *minimal* if it is contained in each (weak) random point attractor.

Remark 2.2.2. Note that we require that the random attractor is measurable with respect to $\mathcal{F} \otimes \mathcal{B}(X)$, in contrast to a weaker statement normally used in the literature (see also [33, Remark 4]).

Remark 2.2.3. Property (ii) is sometimes demanded only for compact subsets $B \subset X$ as for example in [48]. Note that any random attractor according to our definition is a random attractor according to this weaker definition.

The existence of random attractors is proved via so-called absorbing sets. A set $B \in \mathcal{D}$ is called an *absorbing set* if for almost all $\omega \in \Omega$ and any $D \in \mathcal{D}$, there exists a $T > 0$ such that

$$\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega) \subset B(\omega) \quad \text{for all } t \geq T\,.$$

A proof of the following theorem can be found in [49, Theorem 3.5].

Theorem 2.2.4 (Existence of random attractors)**.** *Suppose that (θ, φ) is a continuous random dynamical system with an absorbing set B. Then there exists a unique random attractor A, given by*

$$A(\omega) := \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \varphi(t, \theta_{-t}\omega) B(\theta_{-t}\omega)} \quad \text{for almost all } \omega \in \Omega.$$

Furthermore, $\omega \mapsto A(\omega)$ is measurable with respect to $\mathcal{F}^0_{-\infty}$, i.e. the past of the system.

Remark 2.2.5. Naturally, random attractors are related to invariant probability measures of a random dynamical system (θ, φ). It follows directly from [31, Proposition 4.5] that, if the fibres of a random attractor A, i.e. $\omega \mapsto A(\omega)$, are measurable with respect to $\mathcal{F}^0_{-\infty}$, there is an invariant measure μ for (θ, φ) such that $\omega \mapsto \mu_\omega$ is measurable with respect to $\mathcal{F}^0_{-\infty}$, i.e. is a Markov measure, and satisfies $\mu_\omega(A(\omega)) = 1$ for almost all $\omega \in \Omega$. In particular, if there exists a unique invariant probability measure ρ for the Markov semi-group $(P_t)_{t \geq 0}$, then the invariant Markov measure, supported on A, is unique by the one-to-one correspondence explained above. Additionally, if the Markov semi-group is strongly mixing, i.e.

$$P_t f(x) \xrightarrow{t \to \infty} \int_X f(y) \rho(\mathrm{d}y) \quad \text{for all continuous and bounded } f : X \to \mathbb{R} \text{ and } x \in X\,,$$

then the set $\tilde{A} \in \mathcal{F} \times \mathbb{B}(X)$, given by $\tilde{A}(\omega) = \operatorname{supp} \mu_\omega \subset A(\omega)$ for almost all $\omega \in \Omega$, is a minimal weak random point attractor according to [48, Proposition 2.20].

2.3 Spectral theories of random dynamical systems

Lyapunov exponents

Fundamental for stochastic bifurcation theory is Oseledets' Multiplicative Ergodic Theorem, which implies the existence of Lyapunov exponents describing stability properties of a differentiable random dynamical system.

The random dynamical system (θ, φ) is called C^k if $\varphi(t, \omega, \cdot) \in C^k$ for all $t \in \mathbb{T}$ and $\omega \in \Omega$, where again $\mathbb{T} \in \{\mathbb{R}, \mathbb{R}^+_0\}$. Let's assume that X is a smooth d-dimensional manifold and that (θ, φ) is C^1. The *linearisation* or *derivative* $\mathrm{D}\varphi(t, \omega, x)$ of $\varphi(t, \omega, \cdot)$ at $x \in X$ is a linear map from the tangent space T_x to the tangent space $T_{\varphi(t,\omega,x)}$. If $X = \mathbb{R}^d$, the linearisation is simply the Jacobian $d \times d$ matrix

$$\mathrm{D}\varphi(t, \omega, x) = \frac{\partial \varphi(t, \omega, x)}{\partial x}\,.$$

Differentiating the equation

$$\varphi(t + s, \omega, x) = \varphi(t, \theta_s \omega, \varphi(s, \omega, x))$$

on both sides and applying the chain rule to the right hand side yields

$$\mathrm{D}\varphi(t+s,\omega,x) = \mathrm{D}\varphi(t,\theta_s\omega,\varphi(s,\omega,x))\mathrm{D}\varphi(s,\omega,x) = \mathrm{D}\varphi(t,\Theta_s(\omega,x))\mathrm{D}\varphi(s,\omega,x)\,,$$

i.e. the cocycle property of $\mathrm{D}\varphi$ with respect to the skew product flow $(\Theta_t)_{t\in\mathbb{T}}$.

Let us now assume that the random dynamical system possesses an invariant measure μ. In case $X = \mathbb{R}^d$, this implies that $(\Theta, \mathrm{D}\varphi)$ is a (potentially one-sided) random dynamical system with linear cocycle $\mathrm{D}\varphi$ over the metric dynamical system $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X), (\Theta_t)_{t\in\mathbb{T}})$, see e.g. [2, Proposition 4.2.1]. Generally, we have that $\mathrm{D}\varphi$ is a linear bundle random dynamical system on the tangent bundle TX (see [2, Definition 1.9.3, Proposition 4.25]).

In large parts of this work, we will be concerned with a stochastic differential equation in Stratonovich form

$$\mathrm{d}X_t = f_0(X_t)\mathrm{d}t + \sum_{i=1}^{m} f_j(X_t) \circ \mathrm{d}W_t^j \tag{2.3.1}$$

where W_t^j are independent real valued Brownian motions, f_0 is a C^1 vector field and $f_1, \ldots, f_m$ are C^2 vector fields satisfying bounded growth conditions, as e.g. (global) Lipschitz continuity, in all derivatives to guarantee the existence of a (global) random dynamical system for φ and $\mathrm{D}\varphi$. We write the equation in Stratonovich form when differentiation is concerned as the classical rules of calculus are preserved. If $X = \mathbb{R}^d$, we can apply the conversion formula to the Itô integral (see Appendix A.1) to obtain the situation of (2.0.1).

According to [6], the derivative $\mathrm{D}\varphi(t,\omega,x)$ applied to an initial condition $v_0 \in T_xX \cong \mathbb{R}^d$ solves uniquely the variational equation

$$\mathrm{d}v = \mathrm{D}f_0(\varphi(t,\omega,x))v\,\mathrm{d}t + \sum_{j=1}^{m} \mathrm{D}f_j(\varphi(t,\omega,x))v \circ \mathrm{d}W_t^j\,, \quad v(0) = v_0 \in T_xX\,. \tag{2.3.2}$$

In case the derivative can be written as a matrix, as for example for $X = \mathbb{R}^d$, the Jacobian $\mathrm{D}\varphi(t,\omega,x)$ satisfies Liouville's equation

$$\begin{aligned}\det \mathrm{D}\varphi(t,\omega,x) = \exp\Bigg(&\int_0^t \operatorname{trace} \mathrm{D}f_0(\varphi(s,\omega,x))\mathrm{d}s \\ &+ \sum_{j=1}^{m} \int_0^t \operatorname{trace} \mathrm{D}f_j(\varphi(s,\omega,x)) \circ \mathrm{d}W_s^j\Bigg).\end{aligned} \tag{2.3.3}$$

We summarise the different versions of the Multiplicative Ergodic Theorem for differentiable random dynamical systems in one-sided and two-sided time in the following theorem [2, Theorem 3.4.1, Theorem 3.4.11, Theorem 4.2.6], containing all the properties we will need in the following.

Theorem 2.3.1 (Multiplicative Ergodic Theorem)**.** *(a) Suppose the C^1-random dynamical system (θ, φ), where φ is defined in forward time, has an ergodic invariant measure ν and satisfies the integrability condition*

$$\sup_{0\le t\le 1} \ln^+ \|\mathrm{D}\varphi(t,\omega,x)\| \in L^1(\nu).$$

Then there exist a Θ-invariant set $\Delta \subset \Omega \times X$ with $\nu(\Delta) = 1$, a number $1 \le p \le d$ and real numbers $\lambda_1 > \cdots > \lambda_p$, the Lyapunov exponents with respect to ν, such that for all $0 \neq v \in T_xX \cong \mathbb{R}^d$ and $(\omega, x) \in \Delta$

$$\lambda(\omega, x, v) := \lim_{t\to\infty} \frac{1}{t} \ln \|\mathrm{D}\varphi(t,\omega,x)v\| \in \{\lambda_p, \ldots, \lambda_1\}.$$

Furthermore, the tangent space $T_xX \cong \mathbb{R}^d$ admits a filtration

$$\mathbb{R}^d = V_1(\omega, x) \supsetneq V_2(\omega, x) \supsetneq \cdots \supsetneq V_p(\omega, x) \supsetneq V_{p+1}(\omega, x) = \{0\},$$

for all $(\omega, x) \in \Delta$ such that

$$\lambda(\omega, x, v) = \lambda_i \iff v \in V_i(\omega, x) \setminus V_{i+1}(\omega, x) \quad \text{for all } i \in \{1, \ldots, p\}.$$

In case the derivative can be written as a matrix, we have for all $(\omega, x) \in \Delta$

$$\lim_{t\to\infty} \frac{1}{t} \ln \det \mathrm{D}\varphi(t,\omega,x) = \sum_{i=1}^{p} d_i\lambda_i, \tag{2.3.4}$$

where d_i is the multiplicity of the Lyapunov exponent λ_i and $\sum_{i=1}^p d_i = d$.

(b) If the cocycle φ is defined in two-sided time and satisfies the above integrability condition also in backwards time, there exists the Oseledets splitting

$$\mathbb{R}^d = E_1(\omega, x) \oplus \cdots \oplus E_p(\omega, x)$$

of the tangent space into random subspaces $E_i(\omega, x)$, the Oseledets spaces, for all $(\omega, x) \in \Delta$. These have the following properties for all $(\omega, x) \in \Delta$:

(i) The Oseledets spaces are invariant under the derivative flow, i.e. for all $t \in \mathbb{R}$

$$\mathrm{D}\varphi(t,\omega,x)E_i(\omega, x) = E_i(\Theta_t(\omega, x)),$$

(ii)

$$\lim_{t\to\pm\infty} \frac{1}{t} \ln \|\mathrm{D}\varphi(t,\omega,x)v\| = \lambda_i \iff v \in E_i(\omega, x) \setminus \{0\} \quad \text{for all } i \in \{1, \ldots, p\},$$

(iii) The dimension equals the multiplicity of the associated Lyapunov exponent, i.e.

$$\dim E_i(\omega, x) = d_i \,.$$

The Furstenberg–Khasminskii formula

The standard method for deriving an explicit formula of the largest Lyapunov exponent λ_1 is given by the *Furstenberg-Khasminskii formula* which will play a crucial role in Chapters 4 and 6, and also partly in Chapter 3. We give a short introduction based on [53]. For more detailed discussions we refer to [2, 4].

Consider the linear Stratonovich equation

$$\mathrm{d}Y_t = A_0 Y_t \,\mathrm{d}t + \sum_{j=1}^{m} A_j Y_t \circ \mathrm{d}W_t^j \,, \quad Y_0 = v \in \mathbb{R}^d \,, \tag{2.3.5}$$

where $A_0, \ldots, A_m \in \mathbb{R}^{d\times d}$ and $W^1, \ldots, W^m$ are independent Wiener processes in two-sided time. For keeping things simple, we let $A_0, \ldots, A_m \in \mathbb{R}^{d\times d}$ not depend on an underlying random system. Thus, equation (2.3.5) induces a linear cocycle Φ over the family of shifts $(\theta_t)_{t\in\mathbb{R}}$ on the Wiener space Ω, as opposed to the previous section where the probability space was $\Omega \times X$ for some manifold X and the skew product flow $(\Theta_t)_{t\in\mathbb{T}}$ replaced $(\theta_t)_{t\in\mathbb{R}}$. We will see in the course of this work that the calculations below are still applicable in such situations.

We introduce the change of variables $r_t = \|Y_t\|$ and $s_t = Y_t/r_t$, so that s_t lies on the unit sphere $\mathbb{S}^{d-1}$. The stochastic differential equation in polar coordinates is given by

$$\mathrm{d}s_t = (A_0 s_t - \langle s_t, A_0 s_t\rangle s_t)\,\mathrm{d}t + \sum_{j=1}^{m} (A_j s_t - \langle s_t, A_j s_t\rangle s_t) \circ \mathrm{d}W_t^j \,,$$

and

$$\mathrm{d}r_t = \langle s_t, A_0 s_t\rangle r_t \,\mathrm{d}t + \sum_{j=1}^{m} \langle s_t, A_j s_t\rangle r_t \circ \mathrm{d}W_t^j \,,$$

Since Stratonovich integration obeys the classical chain rule we can write

$$r_t = r_0 \exp\left(\int_0^t \langle s_\tau, A_0 s_\tau\rangle \mathrm{d}\tau + \sum_{j=1}^{m} \int_0^t \langle s_\tau, A_j s_\tau\rangle \circ \mathrm{d}W_\tau^j \right) .$$

Using the Itô-Stratonovich conversion we obtain

$$r_t = r_0 \exp\left(\int_0^t \left[h_{A_0}(s_\tau) + \sum_{j=1}^{m} k_{A_j}(s_\tau) \right] \mathrm{d}\tau + \sum_{j=1}^{m} \int_0^t \langle s_\tau, A_j s_\tau\rangle \,\mathrm{d}W_\tau^j \right) , \tag{2.3.6}$$

where

$$h_A(s) = \langle s, As\rangle\,,$$
$$k_A(s) = \frac{1}{2}\langle (A+A^*)s, As\rangle - \langle s, As\rangle^2\,.$$

It is well known that the Itô integrals in (2.3.6) are of order $(\sqrt{t})$ for large t. Hence, we can conclude that

$$\frac{1}{t}\ln r_t = \frac{1}{t}\int_0^t \left[h_{A_0}(s_\tau) + \sum_{j=1}^m k_{A_j}(s_\tau)\right]\mathrm{d}\tau + \mathcal{O}(t^{-1/2})\,. \tag{2.3.7}$$

We define

$$g_A(s) = As - \langle s, As\rangle s \quad \text{for } A \in \mathbb{R}^{d\times d}, s \in \mathbb{S}^{d-1}\,,$$

and denote by $\mathfrak{L}(g_{A_0},\dots,g_{A_m})(s)$ the Lie algebra generated by these vector fields at s. We impose the classical Hörmander condition on the hypoellpticity of these vector fields driving the dynamics of s_t:

$$\dim \mathfrak{L}(g_{A_0},\dots,g_{A_m})(s) = d-1 \;\text{ for all } s \in \mathbb{S}^{d-1}\,. \tag{2.3.8}$$

Note that the objects in Theorem 2.3.1 only depend on $\omega \in \Omega$ in our situation. According to [52], condition 2.3.8 guarantees that the distribution of the Oseledets space $E_i(\omega)$ in Theorem 2.3.1 (b) possesses a smooth density for any $i \in \{1,\dots,p\}$. Therefore any initial point $v \in \mathbb{R}^d \setminus \{0\}$ has almost surely a non-vanishing component in $E_1(\omega)$ (and $v \in V_1(\omega)\setminus V_2(\omega)$ almost surely in the situation of Theorem 2.3.1 (a)), and, hence,

$$\lambda_1 = \lim_{t\to\infty}\frac{1}{t}\ln r_t \;\text{ almost surely.}$$

The hypoellipticity condition (2.3.8) further implies irreducibility of the Markov semigroup induced by $(s_t)_{t\geq 0}$ on $\mathbb{S}^{d-1}$, and since the unit sphere is a compact manifold, we can conclude that $(s_t)_{t\geq 0}$ possesses a unique stationary probability measure with smooth density p. The density p solves the stationary Fokker-Planck equation

$$\mathcal{L}^* p = 0\,,$$

where

$$\mathcal{L} = g_{A_0} + \frac{1}{2}\sum_{j=1}^m g_{A_j}^2$$

is the generator of $(s_t)_{t\geq 0}$ in Hörmander notation and $\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$. By

Birkhoff's Ergodic Theorem we observe that

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\ln r_t &= \lim_{t\to\infty}\frac{1}{t}\int_0^t\left[h_{A_0}(s_\tau)+\sum_{j=1}^m k_{A_j}(s_\tau)\right]\mathrm{d}\tau\\ &= \int_{\mathbb{S}^{d-1}}\left[h_{A_0}(s)+\sum_{j=1}^m k_{A_j}(s)\right]p(s)\,\mathrm{d}s\,.\end{aligned}$$

The Furstenberg–Khasminskii formula for the top Lyapunov exponent is therefore given by

$$\lambda_1 = \int_{\mathbb{S}^{d-1}}\left[h_{A_0}(s)+\sum_{j=1}^m k_{A_j}(s)\right]p(s)\,\mathrm{d}s\,. \tag{2.3.9}$$

The dichotomy spectrum

In Chapters 3 and 6 we will discuss properties of the dichotomy spectrum for the linearisation of a random dynamical system. The dichotomy spectrum will turn out to capture aspects of uniform and finite-time stability which cannot be covered by the Lyapunov spectrum.

Similarly to the previous sections, consider a linear random dynamical system (θ,φ) on state space $\mathbb{R}^d$, i.e. $\varphi(t,\omega,\cdot)$ is a linear map for all $t\in\mathbb{R}$ and $\omega\in\Omega$. In this situation, there exists a matrix valued function $\Phi:\mathbb{R}\times\Omega\to\mathbb{R}^{d\times d}$ such that $\Phi(t,\omega)v=\varphi(t,\omega,v)$ for all $t\in\mathbb{R},\omega\in\Omega,v\in\mathbb{R}^d$. In the following we will simply write (θ,Φ) for such a linear RDS. An invariant projector of a linear random dynamical system (θ,Φ) is a measurable function $P:\Omega\to\mathbb{R}^{d\times d}$ with

$$P(\omega)=P(\omega)^2 \quad\text{and}\quad P(\theta_t\omega)\Phi(t,\omega)=\Phi(t,\omega)P(\omega) \quad \text{for all } t\in\mathbb{R} \text{ and } \omega\in\Omega\,.$$

Definition 2.3.2 (Exponential dichotomy)**.** Let (θ,Φ) be a linear random dynamical system and let $\gamma\in\mathbb{R}$ and $P_\gamma:\Omega\to\mathbb{R}^{d\times d}$ be an invariant projector of (θ,Φ). Then (θ,Φ) is said to admit an exponential dichotomy with growth rate $\gamma\in\mathbb{R}$, constants $\alpha>0$ and $K\geq 1$ and projector P_γ if for almost all $\omega\in\Omega$, one has

$$\begin{aligned}\|\Phi(t,\omega)P_\gamma(\omega)\| &\leq Ke^{(\gamma-\alpha)t} \quad \text{for all } t\geq 0\,,\\ \|\Phi(t,\omega)(\mathrm{Id}-P_\gamma(\omega))\| &\leq Ke^{(\gamma+\alpha)t} \quad \text{for all } t\leq 0\,.\end{aligned}$$

For the definition of the dichotomy spectrum, it is crucial for which growth rates a linear random dynamical system (θ,Φ) admits an exponential dichotomy. We additionally define that (θ,Φ) admits an exponential dichotomy with growth rate ∞ if there exists a $\gamma\in\mathbb{R}$ such that (θ,Φ) admits an exponential dichotomy with growth rate γ and projector $P_\gamma=\mathrm{Id}$. Analogously, (θ,Φ) admits an exponential dichotomy with growth rate $-\infty$ if there exists a $\gamma\in\mathbb{R}$ such that (θ,Φ) admits an exponential dichotomy with growth rate

γ and projector $P_\gamma = 0$. We write $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$.

Definition 2.3.3 (Dichotomy spectrum [21])**.** Consider the linear random dynamical system (θ, Φ). Then the dichotomy spectrum is defined by

$$\Sigma := \left\{\gamma \in \overline{\mathbb{R}} : (\theta, \Phi) \text{ does not admit an exponential dichotomy with growth rate } \gamma\right\}.$$

The corresponding resolvent set is defined by $\rho := \overline{\mathbb{R}} \setminus \Sigma$.

We are now prepared to present our results on problems of stochastic bifurcation theory. In the following chapter, we start with investigating Hopf bifurcations with additive noise.

Chapter 3

Hopf bifurcation with additive noise

We consider the following stochastic differential equations on $\mathbb{R}^2$ with purely external noise of intensity $\sigma \geq 0$,

$$\begin{aligned} \mathrm{d}x &= (\alpha x - \beta y - (ax - by)(x^2 + y^2))\mathrm{d}t + \sigma \mathrm{d}W_t^1, \\ \mathrm{d}y &= (\alpha y + \beta x - (bx + ay)(x^2 + y^2))\mathrm{d}t + \sigma \mathrm{d}W_t^2, \end{aligned} \tag{3.0.1}$$

where $a > 0$, $\alpha, \beta, b \in \mathbb{R}$ and W_t^1, W_t^2 denote independent one-dimensional Brownian motions. Recall that in the absence of noise ($\sigma = 0$), the ODE (3.0.1) is the normal form of a supercritical Hopf bifurcation. This means that for $\alpha \leq 0$ the system has a globally attracting equilibrium at $(x, y) = (0, 0)$ which is exponentially stable for $\alpha < 0$, and for $\alpha > 0$ the system has a limit cycle at $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = \alpha/a\}$, globally attracting on $\mathbb{R}^2 \setminus \{0\}$. In the presence of noise ($\sigma \neq 0$), the stable equilibrium or limit cycle respectively turns into a random attractor whose existence is proven and whose nature is explored in this chapter. Further recall that the real parameter b induces shear: if $b \neq 0$, the phase velocity depends on the radius. We investigate the possibility of shear-induced chaos in the noisy environment of this model.

In Theorem 3.1.1 we prove that the stochastic differential equation (3.0.1) induces a random dynamical system and possesses a random attractor for all choices of parameters. Using results from [48], we show in Theorem 3.1.2 that a negative top Lyapunov exponent implies the random pullback attractor to be a random equilibrium and we prove the exponentially fast synchronisation of almost all trajectories from all initial points. Theorem 3.1.3 gives an explicit upper bound of shear intensity, depending on the other parameters, for obtaining a negative top Lyapunov exponent, extending results in [35] to global parameter space. In Conjecture 3.1.5 we suspect a positive top Lyapunov exponent for large enough shear given any value of α, which would imply the existence of a random strange attractor.

The second part of the chapter focuses on parameter-dependent changes of finite-time and uniform attractivity, partially indicated by the dichotomy spectrum of the linearisa-

tion along trajectories. Theorem 3.1.6 shows the existence of a bifurcation from uniform synchronisation (phase (I) in Figure 1.4) to non-uniform synchronisation (phase (II) in Figure 1.4) in the case of small shear. We prove that the random equilibrium changes from being globally uniformly attractive to being not even locally uniformly attractive at $\alpha = 0$. Further, at $\alpha = 0$, we observe the appearance of positive finite-time Lyapunov exponents. This is associated with a loss of hyperbolicity in the dichotomy spectrum Σ along the random equilibrium. The described bifurcation in the random equilibrium at $\alpha = 0$ shows that the synchronisation by noise does not entirely "destroy" the bifurcation.

Theorem 3.1.7 shows the existence of unbounded positive finite-time Lyapunov exponents for sufficiently large shear and any value of α (phase (II) in Figure 1.4). As explained in the Introduction, this is the first analytical contribution to the problem of shear-induced chaos for model (3.0.1), while a proof of Conjecture 3.1.5 concerning phase (III) in Figure 1.4 still seems out of reach. In Section 3.5 we report about two approaches to prove the conjecture. The first tries to approximate the joint stationary distribution on the projective bundle, comprising the original system variables and the linearised system on the tangent space. Unfortunately we cannot find the right parameter scaling to find an adequate approximation. The second approach uses a finite-element method and aims for a computer-assisted proof. The missing ingredient is a rigorous argument for choosing boundary conditions on a sufficiently large domain. Still, we hope that these ideas might inspire new attempts to tackle this difficult problem.

Eventually, we discuss numerical investigations into other examples of stochastic Hopf bifurcation, namely the Duffing van der Pol system with additive noise and the stochastic Brusselator. These examples contain no shear forces and serve purely as further examples of synchronisation by additive noise.

The remainder of this chapter is organized as follows. Section 3.1 introduces briefly the technical framework and formulates the main results of this chapter. Section 3.2 is dedicated to a detailed of proof of Theorem 3.1.1 showing the existence of a random attractor for all parameters. In Section 3.3 we prove Theorems 3.1.2 and 3.1.3 and show some statistical properties of the random equilibrium. Section 3.4 contains the proofs of Theorems 3.1.6 and 3.1.7 showing various aspects of the random bifurcations in α and b. In Section 3.5 we discuss possible paths for proving Conjecture 3.1.5. Conclusively, Section 3.6 presents two further examples of synchronisation by noise in stochastic Hopf bifurcation.

3.1 Statement of the main results

The stochastic differential equation (3.0.1) can be rewritten as

$$\mathrm{d}Z_t = f(Z_t)\mathrm{d}t + \sigma \mathrm{d}W_t\,, \tag{3.1.1}$$

where $Z_t = (x_t, y_t)^\top$ and $W_t = (W_t^1, W_t^2)^\top$, and the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f(Z) := \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} Z - (x^2 + y^2) \begin{pmatrix} a & -b \\ b & a \end{pmatrix} Z \,.$$

To investigate sample path properties of the solutions of (3.0.1), it is convenient to work with the canonical sample path space of Brownian motions. Let $\Omega = C_0(\mathbb{R}, \mathbb{R}^2)$ be the space of all continuous functions $\omega : \mathbb{R} \to \mathbb{R}^2$ satisfying that $\omega(0) = 0$. We endow Ω with the compact open topology and denote by $\mathcal{F} = \mathcal{B}(\Omega)$ the Borel σ-algebra on Ω.

It is well known that there exits the so-called *Wiener probability measure* $\mathbb{P}$ on $(\Omega, \mathcal{F})$ which ensures that the two process $(W_t^1)_{t\in\mathbb{R}}$ and $(W_t^2)_{t\in\mathbb{R}}$, defined by $(W_t^1(\omega), W_t^2(\omega))^\top := \omega(t)$ for $\omega \in \Omega$, are independent one-dimensional Brownian motions. We define the sub σ-algebra $\mathcal{F}_{s,t}$ as the σ-algebra generated by $\omega(u) - \omega(v)$ for $s \le v \le u \le t$. For each $t \in \mathbb{R}$, we define the shift map $\theta_t : \Omega \to \Omega$ by

$$(\theta_t\omega)(s) = \omega(s + t) - \omega(t) \quad \text{for all } \ s \in \mathbb{R} \,.$$

It is well known that $(\theta_t)_{t\in\mathbb{R}}$ is an ergodic flow preserving the probability measure $\mathbb{P}$, see e.g. [2]. Thus, $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is an ergodic dynamical system.

3.1.1 Generation of a random dynamical system with a random attractor

Given $\omega \in \Omega$, an initial value $Z \in \mathbb{R}^2$ and $T > 0$, we say that a continuous function $\varphi(\cdot, \omega, Z) : [0, T] \to \mathbb{R}^2$ solves the stochastic differential equation (3.0.1) if it satisfies the integral equation

$$\varphi(t, \omega, Z) = Z + \int_0^t f(\varphi(s, \omega, Z))\, \mathrm{d}s + \sigma\omega(t) \quad \text{for all } \ t \in [0, T] \,.$$

The first result in this chapter concerns global existence of solutions of (3.0.1) for almost every sample path, implying that the solutions do not blow up in forward time. We show that the solutions of (3.0.1) generate a random dynamical system (θ, φ), i.e. that the $(\mathcal{B}(\mathbb{R}_0^+) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^2),\ \mathcal{B}(\mathbb{R}^2))$-measurable mapping $\varphi : \mathbb{R}_0^+ \times \Omega \times \mathbb{R}^2 \to \mathbb{R}^2, (t, \omega, x) \mapsto \varphi(t, \omega, x)$, is a cocycle over θ, which means that $\varphi(0, \omega, \cdot) \equiv \mathrm{Id}$ and

$$\varphi(t + s, \omega, x) = \varphi(t, \theta_s\omega, \varphi(s, \omega, x)) \quad \text{for all } \ \omega \in \Omega, x \in \mathbb{R}^2 \text{ and } t, s \ge 0 \,.$$

In addition to the generation of a random dynamical system, the following theorem addresses also the existence of a random attractor.

Theorem 3.1.1 (Generation of a random dynamical system with a random attractor)**.** *For the stochastic differential equation* (3.0.1)*, there exists a θ-invariant $\mathcal{F}$-measurable set $\widehat{\Omega} \subset \Omega$ of full probability such that the following statements hold.*

(i) For all $\omega \in \widehat{\Omega}$ *and* $Z \in \mathbb{R}^2$*, the stochastic differential equation* (3.0.1) *admits a unique solution* $\varphi(\cdot, \omega, Z)$ *such that* φ *forms a cocycle for a random dynamical system on* $(\widehat{\Omega}, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$.

(ii) There exists a random attractor $A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^2)$ *of the random dynamical system* (θ, φ) *such that* $\omega \mapsto A(\omega)$ *is measurable with respect to* $\mathcal{F}^0_{-\infty}$*, i.e. the past of the system.*

Since the difference of the spaces Ω and $\widehat{\Omega}$ is a set of measure zero, we identify both in the following.

3.1.2 Negativity of top Lyapunov exponent and synchronisation

The following results concern the asymptotic behaviour of trajectories, in particular their stability properties. This will give information about the structure of the random attractor A associated with the stochastic differential equation (3.0.1).

To analyse asymptotic stability, we study the linearisation $\Phi(t, \omega, Z) := \mathrm{D}_x\varphi(t, \omega, Z)$. A direct computation yields that $\Phi(0, \omega, Z) = \mathrm{Id}$ and

$$\dot{\Phi}(t, \omega, Z) = \mathrm{D}f(\varphi(t, \omega, Z))\Phi(t, \omega, Z)\,. \tag{3.1.2}$$

It is easy to observe that Φ is a linear cocycle over the skew product flow $(\Theta_t)_{t\in\mathbb{R}^+_0}$ on $\Omega \times \mathbb{R}^2$, defined by

$$\Theta_t(\omega, Z) := (\theta_t\omega, \varphi(t, \omega, Z))\,.$$

In fact, (Θ, Φ) is a linear random dynamical system, where the ergodic dynamical system $(\theta_t)_{t\in\mathbb{R}}$ is replaced by $(\Theta_t)_{t\in\mathbb{R}^+_0}$ (see Section 2.3). We obtain an ergodic probability measure for the skew product flow $(\Theta_t)_{t\in\mathbb{R}^+_0}$ by using the fact that there exists a one-to-one correspondence between the stationary measure ρ for the Markov semigroup associated to (3.1.1) and a certain invariant measure of $(\Theta_t)_{t\in\mathbb{R}^+_0}$.

In more detail, recall from (1.0.2) that the density of the unique stationary distribution ρ reads as

$$p(x, y) = K_{a,\alpha,\sigma} \exp\left(\frac{2\alpha(x^2 + y^2) - a(x^2 + y^2)^2}{2\sigma^2}\right) \quad \text{for all } (x, y) \in \mathbb{R}^2\,, \tag{3.1.3}$$

where $K_{a,\alpha,\sigma} > 0$ is the normalisation constant and is given by

$$K_{a,\alpha,\sigma} = \frac{2\sqrt{2a}}{\sqrt{\pi}\sigma\,\mathrm{erfc}\left(-\frac{\alpha}{\sqrt{2a\sigma^2}}\right)}\,.$$

Recall from Chapter 2 that the stationary measure ρ gives rise to an invariant measure μ

for $(\Theta_t)_{t\in\mathbb{R}_0^+}$ on $\Omega\times\mathbb{R}^2$ in the following sense: the push-forward limit

$$\mu_\omega := \lim_{t\to\infty} \varphi(t,\theta_{-t}\omega)\rho$$

exists for almost all $\omega\in\Omega$ and is an $\mathcal{F}^0_{-\infty}$-measurable random measure, i.e. $\omega\mapsto\mu_\omega(B)$ is $\mathcal{F}^0_{-\infty}$-measurable for any $B\in\mathcal{B}(\mathbb{R}^2)$. This defines a *Markov measure* μ on $(\Omega\times\mathbb{R}^2,\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^2))$ via

$$\mu(C) := \int_\Omega \mu_\omega(C_\omega)\,\mathbb{P}(\mathrm{d}\omega) \quad \text{for all } C\in\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^2)\,,$$

where $C_\omega := \{Z\in\mathbb{R}^2 : (\omega,Z)\in C\}$. μ is invariant under $(\Theta_t)_{t\in\mathbb{R}_0^+}$ (see e.g. [30]). Reversely, the stationary measure ρ is given by

$$\rho(B) = \int_\Omega \mu_\omega(B)\,\mathbb{P}(\mathrm{d}\omega) \quad \text{for all } B\in\mathcal{B}(\mathbb{R}^2)\,. \tag{3.1.4}$$

The uniqueness of the stationary measure ρ with density $p(x,y)$ implies that the invariant measure μ is ergodic. We will see in Proposition 3.3.1 that the linear system Φ defined in (3.1.2) satisfies the integrability condition

$$\sup_{0\le t\le 1} \ln^+\|\Phi(t,\omega,Z)\| \in L^1(\mu)\,.$$

Therefore, we can apply Oseledets' Multiplicative Ergodic Theorem (see Theorem 2.3.1) to obtain the Lyapunov spectrum of the linear random dynamical system (Θ,Φ). In particular, the top Lyapunov exponent is given by

$$\lambda_{\mathrm{top}} := \lim_{t\to\infty}\frac{1}{t}\ln\|\Phi(t,\omega,Z)\| \quad \text{for } \mu\text{-almost all } (\omega,Z)\in\Omega\times\mathbb{R}^2\,. \tag{3.1.5}$$

The top Lyapunov exponent allows to characterise synchronisation for the random dynamical system generated by (3.0.1), which means that for all $Z_1,Z_2\in\mathbb{R}^2$, we have

$$\lim_{t\to\infty}\|\varphi(t,\omega,Z_1)-\varphi(t,\omega,Z_2)\| = 0 \quad \text{for almost all } \omega\in\Omega\,.$$

Theorem 3.1.2 (Existence of random equilibrium and synchronisation of trajectories)**.** *Suppose that $\lambda_{\mathrm{top}}<0$. Then the random attractor A for the stochastic differential equation (3.0.1) is given by a random equilibrium, i.e. $A(\omega)$ is a singleton for almost all $\omega\in\Omega$. In addition, the the stochastic differential equation (3.0.1) admits exponentially fast synchronisation, i.e. for all $Z_1,Z_2\in\mathbb{R}^2$, we have*

$$\limsup_{t\to\infty}\frac{1}{t}\ln\|\varphi(t,\omega,Z_1)-\varphi(t,\omega,Z_2)\| < 0 \quad \textit{for almost all } \omega\in\Omega\,.$$

We now aim to determine the region of parameters for which $\lambda_{\mathrm{top}}<0$. In [35], analytical results are obtained that show that λ_{top} is negative in certain regions of the

parameters space, in particular when shear is small. The following theorem extends this result to a larger region in the parameter space.

Theorem 3.1.3 (Small shear implies synchronisation)**.** *For each a, α, β, σ, let*

$$\kappa := a\sqrt{\frac{\pi K_{a,\alpha,\sigma}\sigma^2}{\alpha + \pi K_{a,\alpha,\sigma}\sigma^2}\left(\frac{\pi K_{a,\alpha,\sigma}\sigma^2}{\alpha + \pi K_{a,\alpha,\sigma}\sigma^2} + 2\right)}.$$

Then the top Lyapunov exponent λ_{top} is negative if $|b| \leq \kappa$.

Remark 3.1.4. (i) Note that $K_{a,0,\sigma} = \frac{2\sqrt{2a}}{\sqrt{\pi}\sigma}$, and Theorem 3.1.3 then implies that $\lambda_{\text{top}} < 0$ provided that $|b| < \sqrt{3}a$ and α is sufficiently small. This special case is considered in [35, Proposition 4.1].
(ii) For fixed a and α, we have

$$\lim_{\sigma\to\infty} \pi K_{a,\alpha,\sigma}\sigma^2 = \lim_{\sigma\to\infty} \frac{2\sqrt{2\pi a}\sigma}{\operatorname{erfc}\left(-\frac{\alpha}{\sqrt{2a\sigma^2}}\right)} = \infty\,.$$

Therefore, by Theorem 3.1.3 we have $\lambda_{\text{top}} < 0$ provided that $|b| < \sqrt{3}a$ and the noise intensity σ is sufficiently large.

Numerical evidence from [35] and Figure 3.1 suggest that large shear leads to positive top Lyapunov exponent. Unfortunately, we are not able to prove this analytically and formulate this in the following conjecture. Section 3.5 discusses approaches to prove the conjecture. We refer to Chapter 4 and [45] where positivity of the top Lyapunov exponent is analytically established for a two-dimensional system that admits large shear.

Conjecture 3.1.5 (Large shear induces chaos)**.** *Consider the random dynamical system induced by the stochastic differential equation* (3.0.1)*, and fix $a > 0$ and $\beta \in \mathbb{R}$. Then there exists a function $C : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ such that if*

$$b \geq C(\alpha, \sigma)\,,$$

then the top Lyapunov exponent λ_{top} is positive.

The random attractor A is a random strange attractor in this situation, as illustrated in Figure 1.2 (e)–(h). We discuss the characterisation of such a strange attractor in Chapter 5 of this thesis: the sample measures μ_ω are atomless almost surely, exhibit the SRB-property and are associated with positive metric entropy.

3.1.3 Qualitative changes in the finite-time behaviour indicated by the dichotomy spectrum

The final two main results concern the qualitative changes in the finite-time behaviour. If shear is small, then these changes occur at the deterministic Hopf bifurcation point $\alpha = 0$,

since the maximal finite-time Lyapunov exponents are equal to α. If the shear is increased, then there is a transition to unbounded maximal finite-time Lyapunov exponents. We also link these phenomena to qualitative changes in the dichotomy spectrum [21], which is introduced in Section 3.1.6.

Under the assumption of small shear, the following result describes a random bifurcation that corresponds to the deterministic Hopf bifurcation. The notions of uniform and finite-time attractivity are given precisely in Section 3.4.1.

Theorem 3.1.6 (Bifurcation for small shear)**.** *Consider the stochastic differential equation (3.0.1) with $|b| < a$. Then the random attractor A is given by an attracting random equilibrium for all $\alpha \leq 0$ and all $\alpha > 0$ in a neighbourhood of 0. We observe the following bifurcation at $\alpha = 0$:*

(i) For $\alpha < 0$, the random equilibrium is globally uniformly attractive, but for $\alpha > 0$, the random equilibrium is not even locally uniformly attractive.

(ii) Let $\Phi(t, \omega) := \mathrm{D}\varphi(t, \omega, A(\omega))$ denote the linearised random dynamical system along the random equilibrium for fixed α. Then the dichotomy spectrum Σ of Φ is given by

$$\Sigma = [-\infty, \alpha]\,,$$

i.e. hyperbolicity is lost at $\alpha = 0$.

(iii) For $\alpha < 0$, the random equilibrium is finite-time attractive, whereas for $\alpha > 0$, it is not finite-time attractive.

The last rigorous result of the chapter concerns the impact of shear on finite-time Lyapunov exponents. It implies a bifurcation of the spectrum of finite-time Lyapunov exponents for some critical value of shear $b^* \in [a, 2a]$.

Theorem 3.1.7 (Shear intensity as bifurcation parameter)**.** *Let a, b, σ satisfy $b > 2a > 0$ and $\sigma \neq 0$. Then for any $z \in \mathbb{R}^2$, the finite-time Lyapunov exponents of solutions starting in z can be arbitrarily large and arbitrarily small with positive probability. More precisely, there exists a $T > 0$ such that for all $t \in (0, T]$, we have*

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{\|v\|=1} \frac{1}{t} \ln \|\mathrm{D}\varphi(t, \omega, z)v\| = \infty \quad \textit{and} \quad \operatorname*{ess\,inf}_{\omega\in\Omega} \inf_{\|v\|=1} \frac{1}{t} \ln \|\mathrm{D}\varphi(t, \omega, z)v\| = -\infty.$$

3.2 Generation of RDS and existence of a random attractor

We prove Theorem 3.1.1 in this section by following methods developed in [55, 56]. We conjugate the SDE (3.0.1) to a random differential equation via a suitable transformation using an Ornstein–Uhlenbeck process, so that we need to prove the existence of the random dynamical system and its random attractor for the corresponding random differential

equation. An advantage in working with random differential equations (in comparison to *stochastic* differential equations) is that we can work with sample path estimates of solutions.

For $c > 0$, consider the stochastic differential equation

$$\mathrm{d}Z = -cZ\mathrm{d}t + \mathrm{d}W_t\,, \tag{3.2.1}$$

where $Z \in \mathbb{R}^2$. Define the random variable $Z^* := \int_{-\infty}^0 e^{cs}\mathrm{d}W_s$. Then $t \mapsto Z^*(\theta_t\omega)$ solves (3.2.1), i.e.

$$Z^*(\theta_t\omega) = Z^*(\omega) - c\int_0^t Z^*(\theta_s\omega)\,\mathrm{d}s + \omega(t)\,. \tag{3.2.2}$$

By replacing Ω with a measurable subset $\widehat{\Omega} \subset \Omega$ of full probability that is invariant under θ, there exist two random variables K and κ such that

$$|Z^*(\theta_t\omega)|^2 \le K(\omega) + \kappa(\omega)\ln(1+|t|) \quad \text{for all } t \in \mathbb{R} \text{ and } \omega \in \widehat{\Omega}\,, \tag{3.2.3}$$

see [58]. We define the map $T : \widehat{\Omega} \times \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\omega, Z) := Z + \sigma Z^*(\omega)$. Under the change of variable $Z \mapsto T(\omega, Z)$, the SDE (3.0.1) is transformed into the random differential equation

$$\dot{Z} = g(\theta_t\omega, Z)\,, \tag{3.2.4}$$

where $g(\omega, Z) := f\left(T(\omega, Z)\right) + c\sigma Z^*(\omega)$. We show later in Lemma 3.2.2 and the proof of Theorem A that the solution $\Psi(t, \omega, Z)$ of this random differential equation,

$$\Psi(t, \omega, Z) = Z + \int_0^t g(\theta_s\omega, \Psi(s, \omega, Z))\,\mathrm{d}s\,,$$

exists for all $t \ge 0$ and forms a random dynamical system. The following lemma holds using this fact.

Lemma 3.2.1. *The following statements hold.*

(i) The random dynamical system $\varphi : \mathbb{R}_0^+ \times \widehat{\Omega} \times \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$\varphi(t, \omega, Z) := T(\theta_t\omega, \Psi(t, \omega, T(\omega)^{-1}Z))\,, \tag{3.2.5}$$

is generated by the stochastic differential equation (3.0.1)*.*

(ii) If the random dynamical system Ψ has a random attractor, then also the random dynamical system φ has a random attractor.

Proof. (i) From (3.2.5) and the definition of T, we have

$$\varphi(t, \omega, Z) = \Psi(t, \omega, Z - \sigma Z^*(\omega)) + \sigma Z^*(\theta_t\omega)\,,$$

which together with the fact that Ψ is a solution of (3.2.4) implies that

$$\begin{aligned}\varphi(t,\omega,Z) &= Z - \sigma Z^*(\omega) + \int_0^t g(\theta_s\omega, \Psi(s,\omega,Z-\sigma Z^*(\omega)))\,\mathrm{d}s + \sigma Z^*(\theta_t\omega)\\ &= Z + \int_0^t g(\theta_s\omega, T(\theta_s\omega)^{-1}(\varphi(s,\omega,Z)))\,\mathrm{d}s + \sigma(Z^*(\theta_t\omega) - Z^*(\omega))\,.\end{aligned}$$

Thus, using (3.2.2), we obtain that

$$\begin{aligned}\varphi(t,\omega,Z) &= Z + \int_0^t g(\theta_s\omega, T(\theta_s\omega)^{-1}(\varphi(s,\omega,Z))) - \sigma c Z^*(\theta_s\omega)\,\mathrm{d}s + \sigma\omega(t)\\ &= Z + \int_0^t f(\varphi(s,\omega,Z))\,\mathrm{d}s + \sigma\omega(t)\,,\end{aligned}$$

which completes the proof of this part.
(ii) This follows from the definition of a random attractor and the fact that the shifted term in the transformation $T(\omega,Z)$, namely $Z^*(\theta_t\omega)$, is tempered. □

We show that the Euclidean norm of the solutions of (3.2.4) is bounded by the growth of the corresponding solutions of the scalar equation

$$\dot{\zeta} = \gamma_t(\omega) - \sqrt{a}\zeta\,, \tag{3.2.6}$$

where the stochastic process $(\gamma_t)_{t\in\mathbb{R}}$ is chosen appropriately. Note that for each initial value $\zeta_0 \in \mathbb{R}$, the explicit solution of (3.2.6) is given by

$$\zeta(t,\omega,\zeta_0) = e^{-\sqrt{a}t}\zeta_0 + \int_0^t e^{-\sqrt{a}(t-s)}\gamma_s(\omega)\,\mathrm{d}s\,. \tag{3.2.7}$$

Lemma 3.2.2. *There exists a tempered stochastic processes $(\gamma_t)_{t\in\mathbb{R}}$, i.e.*

$$\lim_{t\to\pm\infty}\frac{|\gamma_t(\omega)|}{e^{\varepsilon|t|}} = 0 \quad \textit{for all } \varepsilon > 0 \textit{ and } \omega \in \widehat{\Omega}\,, \tag{3.2.8}$$

such that for $Z \in \mathbb{R}^2$, we have

$$\|\Psi(t,\omega,Z)\|^2 \le 2\zeta(t,\omega,\|Z\|^2)\,, \tag{3.2.9}$$

which implies that the solution $\Psi(t,\omega,Z)$ exists for all $t \ge 0$.

Proof. By replacing Z with $(x,y)^\top$ and Z^* with $(x^*,y^*)^\top$, we rewrite (3.2.4) as

$$\begin{aligned}\begin{pmatrix}\dot{x}_t\\ \dot{y}_t\end{pmatrix} &= \begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}\begin{pmatrix}x_t+\sigma x^*(\theta_t\omega)\\ y_t+\sigma y^*(\theta_t\omega)\end{pmatrix} + c\sigma\begin{pmatrix}x^*(\theta_t\omega)\\ y^*(\theta_t\omega)\end{pmatrix}\\ &\quad - \left\|\begin{pmatrix}x_t+\sigma x^*(\theta_t\omega)\\ y_t+\sigma y^*(\theta_t\omega)\end{pmatrix}\right\|^2\begin{pmatrix}a & -b\\ b & a\end{pmatrix}\begin{pmatrix}x_t+\sigma x^*(\theta_t\omega)\\ y_t+\sigma y^*(\theta_t\omega)\end{pmatrix}.\end{aligned}$$

Let $r_t := \frac{1}{2}(x_t^2+y_t^2)$. Then a direct computation yields that

$$\begin{aligned}\dot{r}_t &= x_t\dot{x}_t + y_t\dot{y}_t\\ &= 2\alpha r_t + \sigma x^*(\theta_t\omega)((\alpha+c)x_t+\beta y_t) - \sigma y^*(\theta_t\omega)(\beta x_t-(\alpha+c)y_t)\\ &\quad - \left\|\begin{pmatrix}x_t+\sigma x^*(\theta_t\omega)\\ y_t+\sigma y^*(\theta_t\omega)\end{pmatrix}\right\|^2\left(2ar_t+\sigma x^*(\theta_t\omega)(ax_t+by_t)-\sigma y^*(\theta_t\omega)(bx_t-ay_t)\right).\end{aligned}$$

Note that $\max\{(\alpha+c)x_t+\beta y_t, \beta x_t-(\alpha+c)y_t\} \le \sqrt{((\alpha+c)^2+\beta^2)2r_t}$. Thus,

$$\begin{aligned}&|x^*(\theta_t\omega)((\alpha+c)x_t+\beta y_t) - y^*(\theta_t\omega)(\beta x_t-(\alpha+c)y_t)|\\ &\le \sqrt{((\alpha+c)^2+\beta^2)2r_t}\,(|x^*(\theta_t\omega)|+|y^*(\theta_t\omega)|)\\ &\le 2\sqrt{((\alpha+c)^2+\beta^2)r_t}\,\|Z^*(\theta_t\omega)\|.\end{aligned} \tag{3.2.10}$$

On the other hand, we have

$$\left\|\begin{pmatrix}x_t+\sigma x^*(\theta_t\omega)\\ y_t+\sigma y^*(\theta_t\omega)\end{pmatrix}\right\|^2 = 2r_t+\sigma^2\|Z^*(\theta_t\omega)\|^2 + 2\sigma x^*(\theta_t\omega)x_t + 2\sigma y^*(\theta_t\omega)y_t,$$

which together with the fact that $|x^*(\theta_t\omega)x_t + y^*(\theta_t\omega)y_t| \le \|Z^*(\theta_t\omega)\|\sqrt{2r_t}$ implies that

$$\left|\left\|\begin{pmatrix}x_t+\sigma x^*(\theta_t\omega)\\ y_t+\sigma y^*(\theta_t\omega)\end{pmatrix}\right\|^2 - 2r_t - \sigma^2\|Z^*(\theta_t\omega)\|^2\right| \le 2\sigma\|Z^*(\theta_t\omega)\|\sqrt{2r_t}.$$

Consequently,

$$ar_t\left\|\begin{pmatrix}x_t+\sigma x^*(\theta_t\omega)\\ y_t+\sigma y^*(\theta_t\omega)\end{pmatrix}\right\|^2 \ge 2ar_t^2 - 2^{\frac{3}{2}}a\sigma\|Z^*(\theta_t\omega)\|r_t^{\frac{3}{2}} + a\sigma^2\|Z^*(\theta_t\omega)\|^2 r_t, \tag{3.2.11}$$

and from the fact that

$$|x^*(\theta_t\omega)(ax_t+by_t) - y^*(\theta_t\omega)(bx_t-ay_t)| \le \sqrt{a^2+b^2}\|Z^*(\theta_t\omega)\|\sqrt{2r_t},$$

we derive that

$$\begin{aligned}&\left|\sigma x^*(\theta_t\omega)(ax_t+by_t)-\sigma y^*(\theta_t\omega)(bx_t-ay_t)\right|\left\|\begin{pmatrix}x_t+\sigma x^*(\theta_t\omega)\\ y_t+\sigma y^*(\theta_t\omega)\end{pmatrix}\right\|^2\\ &\leq 2^{\frac{3}{2}}\sigma\sqrt{a^2+b^2}\|Z^*(\theta_t\omega)\|r_t^{\frac{3}{2}}+\sigma^2\sqrt{2(a^2+b^2)}\|Z^*(\theta_t\omega)\|^2 r_t\\ &+\sigma^3\sqrt{2(a^2+b^2)}\|Z^*(\theta_t\omega)\|^3\sqrt{r_t}\,.\end{aligned}\tag{3.2.12}$$

Using (3.2.10), (3.2.11), (3.2.12) and a comparison argument, we obtain that

$$\frac{1}{2}\|\Psi(t,\omega,Z)\|^2\leq\tilde{\zeta}(t,\omega,\|Z\|^2)\quad\text{for all }\ t\geq 0\text{ and }Z\in\mathbb{R}^2\setminus\{0\}\,,$$

where $t\mapsto\tilde{\zeta}(t,\omega,\|Z\|^2)=\tilde{\zeta}_t$ is the solution of the following scalar differential equation

$$\dot{\tilde{\zeta}}_t=a_t(\omega)\tilde{\zeta}_t^{\frac{1}{2}}+b_t(\omega)\tilde{\zeta}_t+c_t(\omega)\tilde{\zeta}_t^{\frac{3}{2}}-4a\tilde{\zeta}_t^2\,,$$

with initial condition $\tilde{\zeta}_0=\|Z\|^2$. Here the functions a_t,b_t,c_t are defined by

$$\begin{aligned}a_t(\omega)&:=2\sigma\sqrt{(\alpha+c)^2+\beta^2}\|Z^*(\theta_t\omega)\|+\sqrt{2}\sigma^3\sqrt{a^2+b^2}\|Z^*(\theta_t\omega)\|^2\,,\\ b_t(\omega)&:=2\alpha+4\sigma^2\sqrt{a^2+b^2}\|Z^*(\theta_t\omega)\|^2-2a\sigma^2\|Z^*(\theta_t\omega)\|^2\,,\\ c_t(\omega)&:=\left(2^{3/2}\sqrt{a^2+b^2}\sigma+2^{5/2}a\sigma\right)\|Z^*(\theta_t\omega)\|\,.\end{aligned}$$

From temperedness of $Z^*(\theta_t\omega)$, all stochastic processes $(a_t)_{t\in\mathbb{R}}$, $(b_t)_{t\in\mathbb{R}}$ and $(c_t)_{t\in\mathbb{R}}$ are also tempered. Note that

$$\begin{aligned}a\tilde{\zeta}_t^2+\sqrt[3]{\frac{a_t(\omega)^4}{4^4a}}&\geq|a_t(\omega)|\tilde{\zeta}_t^{\frac{1}{2}}\,,\\ a\tilde{\zeta}_t^2+\frac{b_t(\omega)^2}{2a}&\geq|b_t(\omega)\tilde{\zeta}_t|\,,\\ a\tilde{\zeta}_t^2+\frac{3^3c_t(\omega)^4}{4^4a^3}&\geq|c_t(\omega)\tilde{\zeta}_t^{\frac{3}{2}}|\,.\end{aligned}$$

Therefore,

$$a_t(\omega)\tilde{\zeta}_t^{\frac{1}{2}}+b_t(\omega)\tilde{\zeta}_t+c_t(\omega)\tilde{\zeta}_t^{\frac{3}{2}}-4a\tilde{\zeta}_t^2\leq\sqrt[3]{\frac{a_t(\omega)^4}{4^4a}}+\frac{b_t(\omega)^2}{2a}+\frac{3^3c_t(\omega)^4}{4^4a^3}-a\tilde{\zeta}_t^2\leq\gamma_t(\omega)-\sqrt{a}\tilde{\zeta}_t\,,$$

where

$$\gamma_t(\omega):=\frac{1}{4}+\sqrt[3]{\frac{a_t(\omega)^4}{4^4a}}+\frac{b_t(\omega)^2}{2a}+\frac{3^3c_t(\omega)^4}{4^4a^3}$$

is tempered. Hence, using a comparison argument, the solution ζ of (3.2.6) satisfies (3.2.9), which finishes the proof of this lemma. □

Proof of Theorem 3.1.1. (i) According to [2], there exists a local random dynamical sys-

tem generated by solutions of (3.2.4). Due to Lemma 3.2.2, the solution $\Psi(t,\omega,Z)$ exists for all $t \geq 0$. Hence, this proves the fact that we assumed to prove Lemma 3.2.1. Lemma 3.2.1 (i) completes the proof of (i).
(ii) Let $D \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$ be tempered. Then there exists a tempered random variable $R : \widehat{\Omega} \to \mathbb{R}^+$ such that $D(\omega) \subset B_{R(\omega)}(0)$. By Lemma 3.2.2, for all $Z \in D(\theta_{-t}\omega)$, we have

$$\|\Psi(t,\theta_{-t}\omega,Z)\|^2 \leq 2\zeta(t,\theta_{-t}\omega,R(\theta_{-t}\omega)) \leq 2e^{-\sqrt{a}t}R(\theta_{-t}\omega) + 2\int_{-t}^{0} e^{\sqrt{a}s}\gamma_s(\omega)\,\mathrm{d}s\,,$$

where we use (3.2.7) to obtain the last inequality. Since $(\gamma_t)_{t\in\mathbb{R}}$ is tempered, the integral $\int_{-\infty}^{0} e^{\sqrt{a}s}\gamma_s(\omega)\,\mathrm{d}s$ exists. On the other hand, since R is tempered, it follows that $\lim_{t\to\infty} e^{-\sqrt{a}t}R(\theta_{-t}\omega) = 0$. Define $r(\omega) := \sqrt{1 + 2\int_{-\infty}^{0} e^{\sqrt{a}s}\gamma_s(\omega)\,\mathrm{d}s}$. Thus, for each $\omega \in \widehat{\Omega}$, there exists $T > 0$ such that

$$\Psi(t,\theta_{-t}\omega,D(\theta_{-t}\omega)) \subset B_{r(\omega)}(0) \quad \text{for all } t \geq T\,.$$

This means that $B_{r(\omega)}(0)$ is an absorbing set. Applying Theorem 2.2.4 completes the proof. □

3.3 Synchronisation

We prove in this section that the system (3.0.1) admits synchronisation if the top Lyapunov exponent is negative (Theorem B), and we show that small shear implies negativity of the top Lyapunov exponent and thus synchronisation (Theorem C). In addition, we show that the system satisfies the integrability condition of the Multiplicative Ergodic Theorem, and we prove that the sum of the two Lyapunov exponents is always negative.

3.3.1 Negativity of the sum of the Lyapunov exponents

Recall that $\Phi : \mathbb{R}_0^+ \times \Omega \times \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ is the linear random dynamical system satisfying $\Phi(0,\omega,Z) = \mathrm{id}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,\omega,Z) = \mathrm{D}f(\varphi(t,\omega,Z)))\Phi(t,\omega,Z)\,.$$

We show that Φ satisfies the integrability condition of Multiplicative Ergodic Theorem with respect to the measure μ and also show that the sum of the Lyapunov exponents of Φ is always negative.

Proposition 3.3.1. *The following statements hold.*

(i) Let $\lambda^+ : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\lambda^+(Z) := \max_{\|r\|=1} \langle \mathrm{D}f(Z)r, r\rangle \,. \tag{3.3.1}$$

Then for almost all $\omega \in \Omega$ and all $Z \in \mathbb{R}^2$, we have

$$\|\Phi(t,\omega,Z)\| \le \exp\left(\int_0^t \lambda^+(\varphi(s,\omega,Z))\,\mathrm{d}s\right) \quad \text{for all } t \ge 0\,, \tag{3.3.2}$$

and the linear random dynamical system Φ satisfies the integrability condition of the Multiplicative Ergodic Theorem.

(ii) Let λ_Σ be the sum of the two Lyapunov exponents of the linear random dynamical system Φ. Then $\lambda_\Sigma < 0$ and the disintegrations of the Markov measure μ are singular with respect to the Lebesgue measure on $\mathbb{R}^2$.

Proof. (i) Let $v \in \mathbb{R}^2 \setminus \{0\}$ be arbitrary. By definition of Φ, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi(t,\omega,Z)v\|^2 &= 2\big\langle \mathrm{D}f(\varphi(t,\omega,Z))\Phi(t,\omega,Z)v, \Phi(t,\omega,Z)v\big\rangle \\
&= 2\left\langle \mathrm{D}f(\varphi(t,\omega,Z))\frac{\Phi(t,\omega,Z)v}{\|\Phi(t,\omega,Z)v\|}, \frac{\Phi(t,\omega,Z)v}{\|\Phi(t,\omega,Z)v\|}\right\rangle \|\Phi(t,\omega,Z)v\|^2 \\
&\le 2\lambda^+(\varphi(t,\omega,Z))\|\Phi(t,\omega,Z)v\|^2.
\end{aligned}$$

This implies that

$$\|\Phi(t,\omega,Z)v\|^2 \le \|v\|^2 \exp\left(2\int_0^t \lambda^+(\varphi(s,\omega,Z))\mathrm{d}s\right). \tag{3.3.3}$$

Since v is arbitrary, (3.3.2) is proved. Using (3.3.2), we obtain that

$$\sup_{0\le t\le 1} \ln^+ \|\Phi(t,\omega,Z)\| \le \int_0^1 |\lambda^+(\varphi(s,\omega,Z))|\,\mathrm{d}s\,,$$

which implies that

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}^2} \sup_{0\le t\le 1} \ln^+ \|\Phi(t,\omega,Z)\|\ \mu(\mathrm{d}\omega,\mathrm{d}Z) &\le \int_{\Omega\times\mathbb{R}^2}\int_0^1 |\lambda^+(\varphi(s,\omega,Z))|\ \mathrm{d}s\,\mu(\mathrm{d}\omega,\mathrm{d}Z) \\
&= \int_0^1 \int_{\Omega\times\mathbb{R}^2} |\lambda^+(\varphi(s,\omega,Z))|\ \mu(\mathrm{d}\omega,\mathrm{d}Z)\,\mathrm{d}s \\
&= \int_{\mathbb{R}^2} |\lambda^+(Z)|\,\rho(\mathrm{d}Z)\,,
\end{aligned} \tag{3.3.4}$$

where in the last equality, we use the fact that the skew product flow $\Theta_s(\omega,Z) = (\theta_s\omega, \varphi(s,\omega,Z))$ preserves the probability measure μ. By definition of λ^+ and the ex-

plicit form of $\mathrm{D}f$ given by

$$\mathrm{D}f(Z) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} - \begin{pmatrix} 3ax^2 + 2bxy + ay^2 & bx^2 + 2axy + 3by^2 \\ -3bx^2 - 2axy - by^2 & ax^2 - 2bxy + 3ay^2 \end{pmatrix},$$

it follows that

$$|\lambda^+(Z)| \leq |\alpha| + 6(|a| + |b|)(x^2 + y^2) \quad \text{for all } Z = (x, y)^\top \in \mathbb{R}^2 .$$

Together with (3.3.4), this implies that

$$\int_{\Omega \times \mathbb{R}^2} \sup_{0 \leq t \leq 1} \ln^+ \|\Phi(t, \omega, Z)\| \, \mu(\mathrm{d}\omega, \mathrm{d}Z) \leq |\alpha| + 6(|a| + |b|) \int_{\mathbb{R}^2} (x^2 + y^2) p(x, y) \, \mathrm{d}x \, \mathrm{d}y ,$$

where $p(x, y)$ is given as in (3.1.3). Thus, the linear random dynamical system Φ satisfies the integrability condition of the Multiplicative Ergodic Theorem.

(ii) Due to $\lambda_\Sigma = \lim_{t \to \infty} \frac{1}{t} \ln \det \Phi(t, \omega, Z)$, the sum of the two Lyapunov exponents of the linear random dynamical system generated by (3.1.2) reads as

$$\lambda_\Sigma = 2\alpha - 4a \int \int (x^2 + y^2) p(x, y) \, \mathrm{d}x \, \mathrm{d}y .$$

Using the explicit formula for $p(x, y)$ from (3.1.3), we obtain that

$$\lambda_\Sigma = 2\alpha - 4a \frac{\int \int (x^2 + y^2) \exp\left(\frac{2\alpha(x^2+y^2) - a(x^2+y^2)^2}{2\sigma^2} \right) \mathrm{d}x \, \mathrm{d}y}{\int \int \exp\left(\frac{2\alpha(x^2+y^2) - a(x^2+y^2)^2}{2\sigma^2} \right) \mathrm{d}x \, \mathrm{d}y} .$$

Applying the change of variables $x = \sigma r \sin \phi$, $y = \sigma r \cos \phi$ the previous integral yields that

$$\lambda_\Sigma = 2\alpha - 4a\sigma^2 \frac{\int_0^\infty r^3 \exp\left(\frac{2\alpha r^2 - a\sigma^2 r^4}{2} \right) \mathrm{d}r}{\int_0^\infty r \exp\left(\frac{2\alpha r^2 - a\sigma^2 r^4}{2} \right) \mathrm{d}r} .$$

A further change of variable $r^2 \mapsto r$ gives that

$$\lambda_\Sigma = 2\alpha - 4a\sigma^2 \frac{\int_0^\infty r \exp\left(\frac{2\alpha r - a\sigma^2 r^2}{2} \right) \mathrm{d}r}{\int_0^\infty \exp\left(\frac{2\alpha r - a\sigma^2 r^2}{2} \right) \mathrm{d}r} ,$$

which proves that $\lambda_\Sigma < 0$ if $\alpha \leq 0$. We also show this for $\alpha > 0$ now. Using the change

of variable $\sqrt{a}|\sigma|r - \frac{\alpha}{\sqrt{a}|\sigma|} \mapsto r$, we obtain that

$$\begin{aligned}\lambda_\Sigma &= -2\alpha - 4\sqrt{a}|\sigma| \frac{\int_{-\frac{\alpha}{|\sigma|\sqrt{a}}}^{\infty} r \exp\left(-\frac{r^2}{2}\right) \mathrm{d}r}{\int_{-\frac{\alpha}{|\sigma|\sqrt{a}}}^{\infty} \exp\left(-\frac{r^2}{2}\right) \mathrm{d}r} \\ &= -2\alpha - 4\sqrt{a}|\sigma| \frac{\exp\left(-\frac{\alpha^2}{2a\sigma^2}\right)}{\int_{-\frac{\alpha}{|\sigma|\sqrt{a}}}^{\infty} \exp\left(-\frac{r^2}{2}\right) \mathrm{d}r}\,,\end{aligned}$$

which shows that $\lambda_\Sigma < 0$ for $\alpha > 0$. As a consequence, using [66, Proposition 1] and [10, Theorem 4.15], the disintegration of the Markov measure μ is singular with respect to ρ if $\lambda_\Sigma < 0$. The fact that ρ is equivalent to the Lebesgue measure finishes the proof of this proposition. □

3.3.2 Negative top Lyapunov exponent implies synchronisation

The aim of this subsection is to prove synchronisation of the random dynamical system generated by (3.0.1) when its top Lyapunov exponent is negative. Our proof consists of two ingredients. The first ingredient is a result from [48] that implies that the fibres of the random attractor are singletons. The second ingredient is the stable manifolds theorem, which we use to verify that this random attractor is also attractive in forward time.

We make use of the following sufficient conditions for the collapse of a random attractor [48, Theorem 2.14].

Theorem 3.3.2 (Collapse of the random attractor)**.** *We assume that a random dynamical system (θ, φ) is*

(i) asymptotically stable *on a fixed non-empty open set $U \subset \mathbb{R}^2$, in the sense that there exists a sequence $t_n \to \infty$ such that*

$$\mathbb{P}\left(\omega \in \Omega : \lim_{n\to\infty} \operatorname{diam}(\varphi(t_n, \omega, U)) = 0\right) > 0\,.$$

(ii) swift transitive*, i.e. for all $x, y \in \mathbb{R}^2$ and $r > 0$, there exists a $t > 0$ such that*

$$\mathbb{P}\big(\omega \in \Omega : \varphi(t, \omega, B_r(x)) \subset B_{2r}(y)\big) > 0\,.$$

(iii) contracting on large sets*, i.e. for all $R > 0$, there exist $y \in \mathbb{R}^2$ and $t > 0$ such that*

$$\mathbb{P}\left(\omega \in \Omega : \operatorname{diam}(\varphi(t, \omega, B_R(y))) \le \tfrac{R}{4}\right) > 0\,.$$

Suppose further that (θ, φ) has a random attractor A with $\mathcal{F}^0_{-\infty}$-measurable fibres. Then $A(\omega)$ is a singleton $\mathbb{P}$-almost surely.

We use this result for the following proposition.

Proposition 3.3.3. *Suppose that the top Lyapunov exponent λ_{top} of the random dynamical system generated by (3.0.1) is negative. Then the fibres of the random attractor are singletons, given by a $\mathcal{F}^0_{-\infty}$-measurable map $A : \Omega \to \mathbb{R}^2$. Furthermore, the following statements hold:*

(i) A is a random equilibrium of φ, i.e.

$$\varphi(t, \omega, A(\omega)) = A(\theta_t \omega) \quad \text{for all } t \geq 0 \text{ and almost all } \omega \in \Omega\,.$$

(ii) The random equilibrium is distributed according to the stationary density $(x, y) \mapsto p(x, y)$, see (3.1.3). More precisely,

$$\mathbb{P}\big(\{\omega \in \Omega : A(\omega) \in C\}\big) = \int_C p(x, y)\,\mathrm{d}x\,\mathrm{d}y \quad \text{for all } C \in \mathcal{B}(\mathbb{R}^2)\,.$$

(iii) The top Lyapunov exponent of the linearisation along the random equilibrium A,

$$\dot{\xi} = \mathrm{D}f(A(\theta_t \omega))\xi\,,$$

is equal to λ_{top}.

Proof. In the first part of the proof, we show that the random dynamical system φ generated by (3.0.1) fulfils the assumptions (i), (ii), and (iii) of Theorem 3.3.2. Note that (i) follows from the negativity of the top Lyapunov exponent (see [48, Lemma 4.1 and Corollary 4.4]), and swift transitivity holds for our system according to [48, Proposition 4.9]. Hence, it remains to show contraction on large sets for φ. By definition of f, we have that

$$\begin{aligned}\big\langle f(x) - f(y), x - y\big\rangle \leq \big(\alpha - a\tfrac{1}{2}(\|x\|^2 + \|y\|^2)\big)\,\|x - y\|^2 \\ + b(x_1 y_2 - y_1 x_2)\,\big(2\langle x - y, y\rangle + \|x - y\|^2\big)\,.\end{aligned}$$

Fix $r > 0$, and consider $B_r(z)$, where $z = (R, 0)$ for some $R > 0$ to be chosen large enough. For any $x, y \in B_r(z)$, observe that

$$(x_1 y_2 - y_1 x_2)\langle x - y, y\rangle \leq r\|y\|\|x - y\|^2 + r^2\|x - y\|^2$$

and

$$(x_1 y_2 - y_1 x_2)\|x - y\|^2 \leq 2\|x - y\|^2\|y\|\|x - y\|\,.$$

This implies that for all $x, y \in B_r(z)$,

$$\begin{aligned}\langle f(x) - f(y), x - y\rangle &\leq \|x-y\|^2 \left(\alpha - a\tfrac{1}{2}(\|x\|^2 + \|y\|^2) + 2b(r\|y\| + r^2 + 2\|y\|r)\right) \\ &< K\|x-y\|^2\end{aligned}$$

for some $K < 0$ if R is big enough (due to the quadratic terms, K has negative sign). This property is called monotonicity on large sets, which implies contraction on large sets due to [48, Proposition 3.10].

We now prove the statements (i), (ii) and (iii) of the proposition.

(i) This follows immediately from the definition of a random attractor (see Section 2.2).

(ii) Note that $\omega \mapsto A(\omega)$ is measurable with respect to $\mathcal{F}^0_{-\infty}$, and thus, $\mu_\omega := \delta_{A(\omega)}$ defines a Markov measure. The invariance of μ_ω follows directly from (i). Hence, $\{\mu_\omega\}_{\omega\in\Omega}$ is the disintegration of the ergodic invariant measure μ associated with the ergodic stationary measure ρ, and we obtain from (3.1.4) that for all $C \in \mathcal{B}(\mathbb{R}^2)$

$$\mathbb{P}(\{\omega \in \Omega : A(\omega) \in C\}) = \int_\Omega \mu_\omega(C)\, \mathbb{P}(\mathrm{d}\omega) = \rho(U) = \int_C p(x,y)\, \mathrm{d}x\, \mathrm{d}y\,.$$

(iii) According to the Multiplicative Ergodic Theorem, the existence of the Lyapunov spectrum holds for a set $M \subset \Omega \times \mathbb{R}^2$ of full μ-measure. We observe that the set

$$D = \bigcup_{\omega\in\Omega} \{(\omega, A(\omega))\}$$

has full μ-measure, since

$$\mu(D) = \int_\Omega \mu_\omega(\{A(\omega)\})\, \mathbb{P}(\mathrm{d}\omega) = \int_\Omega \delta_{A(\omega)}(\{A(\omega)\})\, \mathbb{P}(\mathrm{d}\omega) = 1\,.$$

Hence, $\mu(M \cap D) = 1$. Since the Oseledets space associated with the second Lyapunov exponent has zero Lebesgue measure for any $(\omega, x) \in M \cap D$, the claim follows. □

Finally, we prove Theorem 3.1.2.

Proof of Theorem 3.1.2. The existence of the attracting random equilibrium $A : \Omega \to \mathbb{R}^2$ has been shown in Proposition 3.3.3. Define $\psi : \mathbb{R}^+_0 \times \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\psi(t, \omega, x) = \varphi(t, \omega, A(\omega) + x) - \varphi(t, \omega, A(\omega)).$$

Obviously, $\psi(t, \omega, 0) = 0$ and $\psi(t, \omega, x)$ is the solution of the random differential equation

$$\dot{\xi} = \mathrm{D}f(A(\theta_t\omega))\xi + R(t, \omega, \xi), \tag{3.3.5}$$

where

$$R(t,\omega,\xi) := f(A(\theta_t\omega)+\xi) - f(A(\theta_t\omega)) - \mathrm{D}f(A(\theta_t\omega))\xi\,.$$

Note that for $R \equiv 0$, the top Lyapunov exponent of the homogeneous equation (3.3.5) is negative. Using the stable manifold theorem [2, Theorems 7.5.5 and 7.5.16], there exists $r(\omega) > 0$ such that for almost all $\omega \in \Omega$ and $x \in B_{r(\omega)}(0)$, one has

$$\lim_{t\to\infty} e^{-\frac{\lambda_{\mathrm{top}}}{2}t}\|\psi(t,\omega,x)\| = \lim_{t\to\infty} e^{-\frac{\lambda_{\mathrm{top}}}{2}t}\|\varphi(t,\omega,x+A(\omega)) - A(\theta_t\omega)\| = 0\,. \tag{3.3.6}$$

Choose and fix an arbitrary initial value $x \in \mathbb{R}^2$, and define

$$V := \left\{\omega\in\Omega : \limsup_{t\to\infty} e^{-\frac{\lambda_{\mathrm{top}}}{2}t}\|\varphi(t,\omega,x) - A(\theta_t\omega)\| = 0\right\}.$$

It remains to show that $\mathbb{P}(V) = 1$. For each $n \in \mathbb{N}$, we define

$$\Omega_n := \big\{\omega\in\Omega : \varphi(t,\theta_{-t}\omega,x) \in B_{r(\omega)}(A(\omega)) \text{ for all } t \geq n\big\}\,.$$

Note that $(\Omega_n)_{n\in\mathbb{N}}$ is an increasing sequence of measurable sets. By virtue of Proposition 3.3.3, the random equilibrium a is the random attractor of φ, which implies that $\lim_{n\to\infty}\mathbb{P}(\Omega_n) = 1$. From the definition of Ω_n, we derive that $\varphi(n,\theta_{-n}\omega,x) \in B_{r(\omega)}(A(\omega))$ for all $\omega \in \Omega_n$. Together with (3.3.6), this implies that for all $\omega \in \Omega_n$, one has

$$\begin{aligned} 0 &= \limsup_{t\to\infty} e^{-\frac{\lambda_{\mathrm{top}}}{2}t}\|\varphi(t,\omega,\varphi(n,\theta_{-n}\omega,x)) - A(\theta_t\omega)\| \\ &= \limsup_{t\to\infty} e^{-\frac{\lambda_{\mathrm{top}}}{2}t}\|\varphi(t+n,\theta_{-n}\omega,x) - A(\theta_t\omega)\|\,. \end{aligned}$$

Consequently, $\theta_{-n}\Omega_n \subset V$, and thus, $\mathbb{P}(V) = 1$, which finishes the proof. □

3.3.3 Small shear implies synchronisation

We prove Theorem 3.1.3 in this subsection, which says that small shear implies negativity of the top Lyapunov exponent. The main ingredient for the proof of Theorem 3.1.3 is the inequality in Proposition 3.3.1(i).

We first need the following estimate on the function λ^+ defined as in (3.3.1).

Lemma 3.3.4. *For any $Z = (x,y)^\top \in \mathbb{R}^2$, we have*

$$\lambda^+(Z) \leq \alpha + \big(\sqrt{a^2+b^2} - 2a\big)(x^2+y^2)\,,$$

and equality holds if and only if $xy = 0$.

Proof. Using the following explicit form of $\mathrm{D}f(Z)$,

$$\mathrm{D}f(Z) = \begin{pmatrix} \alpha - ay^2 - 3ax^2 - 2byx & -\beta - 2axy - bx^2 - 3by^2 \\ \beta - 2axy + by^2 + 3bx^2 & \alpha - ax^2 - 3ay^2 + 2byx \end{pmatrix},$$

we obtain for any $r \in \mathbb{R}^2$ with $\|r\| = 1$ that

$$\begin{aligned} \langle \mathrm{D}f(x,y)r, r\rangle &= r_1^2(\alpha - ay^2 - 3ax^2) + r_1r_2(-\beta - 2axy) + r_1r_2(\beta - 2axy) \\ &\quad + r_2^2(\alpha - ax^2 - 3ay^2) - 2byxr_1^2 + 2byxr_2^2 + r_1r_2(2bx^2 - 2by^2) \\ &= \alpha - a(x^2+y^2) + 2b(r_1r_2x^2 - r_1r_2y^2 + (r_2^2 - r_1^2)xy) - 2a(r_1x + r_2y)^2 . \end{aligned}$$

Since $r_1^2 + r_2^2 = 1$, it is possible to write that $r_1 = \sin\phi$ and $r_2 = \cos\phi$ for some $\phi \in [0, 2\pi)$. Thus, a simple calculation yields that

$$\begin{aligned} \langle \mathrm{D}f(x,y)r, r\rangle &= \alpha - 2a(x^2+y^2) + (ax^2 + bxy - ay^2)\cos 2\phi + (bx^2 - 2axy - by^2)\sin 2\phi \\ &\le \alpha - 2a(x^2+y^2) + \sqrt{(ax^2 + 2bxy - ay^2)^2 + (bx^2 - 2axy - by^2)^2} \\ &= \alpha - 2a(x^2+y^2) + \sqrt{(a^2+b^2)(x^2-y^2)^2 + 4b^2x^2y^2} \\ &\le \alpha - 2a(x^2+y^2) + \sqrt{a^2+b^2}(x^2+y^2) , \end{aligned}$$

which completes the proof. □

Proof of Theorem 3.1.3. From inequality (3.3.2), we derive that

$$\lambda_{\mathrm{top}} = \limsup_{t\to\infty} \frac{1}{t} \ln \|\Phi(t,\omega,s)\| \le \lim_{t\to\infty} \frac{1}{t} \int_0^t \lambda^+(\varphi(s,\omega,x))\,\mathrm{d}s .$$

Note that the skew product flow $\Theta_s(\omega, Z) = (\theta_s\omega, \varphi(s,\omega,Z))$ preserves the probability measure μ, and λ^+ is integrable. By using Birkhoff's Ergodic Theorem, we obtain that

$$\lambda_{\mathrm{top}} \le \int_{\mathbb{R}^2} \lambda^+(x,y)p(x,y)\,\mathrm{d}x\,\mathrm{d}y ,$$

where the density function p is as in (3.1.3). Thus, by virtue of Lemma 3.3.4, we arrive at

$$\lambda_{\mathrm{top}} < \alpha + \left(\sqrt{a^2+b^2} - 2a\right) \int_{\mathbb{R}^2} (x^2+y^2)p(x,y)\,\mathrm{d}x\,\mathrm{d}y .$$

Inserting the explicit form of the density function p in the preceding inequality gives that

$$\lambda_{\mathrm{top}} < \alpha + (\sqrt{a^2+b^2} - 2a)K \int_{\mathbb{R}^2} (x^2+y^2) \exp\left(\frac{2\alpha(x^2+y^2) - a(x^2+y^2)^2}{2\sigma^2}\right) \mathrm{d}x\,\mathrm{d}y , \quad (3.3.7)$$

with the normalization constant $K = \frac{2\sqrt{2a}}{\sqrt{\pi}\sigma\,\mathrm{erfc}\left(-\alpha/\sqrt{2a\sigma^2}\right)}$. Using polar coordinates, we

obtain that

$$\begin{aligned} & K \int_{\mathbb{R}^2} \left(\alpha - a(x^2+y^2) \exp\left(\frac{2\alpha(x^2+y^2) - a(x^2+y^2)^2}{2\sigma^2} \right) \right) \mathrm{d}x \, \mathrm{d}y \\ &= 2\pi K \int_0^\infty (\alpha - ar^2) r \exp\left(\frac{2\alpha r^2 - ar^4}{2\sigma^2} \right) \mathrm{d}r \\ &= -\pi\sigma^2 K \,. \end{aligned}$$

This implies

$$\int_{\mathbb{R}^2} (x^2+y^2) \exp\left(\frac{2\alpha(x^2+y^2) - a(x^2+y^2)^2}{2\sigma^2} \right) \mathrm{d}x \, \mathrm{d}y = \frac{\alpha}{Ka} + \frac{\pi\sigma^2}{a} \,,$$

which together with (3.3.7) implies that

$$\lambda_{\mathrm{top}} < \alpha + \left(\sqrt{a^2+b^2} - 2a \right) \left(\frac{\alpha}{a} + \frac{\pi K \sigma^2}{a} \right) .$$

Consequently,

$$\lambda_{\mathrm{top}} < -\pi K\sigma^2 + \left(\sqrt{1 + \frac{b^2}{a^2}} - 1 \right) (\alpha + \pi K \sigma^2) \,. \tag{3.3.8}$$

Note that by definition of K it is easy to see that $\alpha + \pi K\sigma^2 > 0$. Therefore, for all $|b| \leq \kappa_b$, we have

$$\lambda_{\mathrm{top}} < -\pi K\sigma^2 + \left(\sqrt{1 + \frac{\kappa_b^2}{a^2}} - 1 \right) (\alpha + \pi K \sigma^2) = 0 \,,$$

which completes the proof of this theorem. □

3.4 Random Hopf bifurcation

We analyse random bifurcations for the stochastic differential equation (3.0.1) in this section, which captures qualitative changes in the the asymptotic as well as the finite-time behaviour.

We first need the following preparatory proposition.

Proposition 3.4.1. *Consider* (3.0.1) *such that* $|b| \leq \kappa_b$. *Then for any* $y \in \mathbb{R}^2$, $\varepsilon > 0$ *and* $T \geq 0$, *there exists a set* $E \in \mathcal{F}_{-\infty}^T$ *with* $\mathbb{P}(E) > 0$ *such that*

$$A(\theta_s\omega) \in B_\varepsilon(y) \quad \textit{for all } s \in [0,T] \textit{ and } \omega \in E \,,$$

where $\{A(\omega)\}$ *is the unique random equilibrium for* (3.0.1) *from Proposition 3.3.3.*

Proof. Let $\varepsilon > 0$ and $T \geq 0$. Since $\Omega = \bigcup_{x \in \mathbb{Q}^2} \{\omega \in \Omega \,:\, A(\omega) \in B_{\varepsilon/4}(x)\}$, there exists an

$x \in \mathbb{R}^2$ such that

$$A_0 := \{\omega \in \Omega \,:\, A(\omega) \in B_{\varepsilon/4}(x)\}$$

has positive measure. From [48, Proposition 3.10] we know that there exists $t_0 > 0$ such that

$$B_0 := \left\{\omega \in \Omega \,:\, \varphi_{t_0}(\omega, x') \in B_{\varepsilon/2}(y) \text{ for all } x' \in B_{\varepsilon/4}(x)\right\}$$

has positive measure. Since θ is measure preserving, the two sets

$$\begin{aligned} A_1 &:= \theta_{t_0} A_0 = \{\omega \in \Omega \,:\, A(\theta_{-t_0}\omega) \in B_{\varepsilon/4}(x)\}\,, \\ B_1 &:= \theta_{t_0} B_0 = \left\{\omega \in \Omega \,:\, \varphi_{t_0}(\theta_{-t_0}\omega, x') \in B_{\varepsilon/2}(y) \text{ for all } x' \in B_{\varepsilon/4}(x)\right\} \end{aligned}$$

have positive measure. Due to the Markov property of the random dynamical system, we observe that B_1 and A_1 are independent, and hence, $\mathbb{P}(B_1 \cap A_1) > 0$. Thus, the set

$$E_0 = \{\omega \in \Omega \,:\, A(\omega) \in B_{\varepsilon/2}(y)\} \supset A_1 \cap B_1$$

has positive measure and clearly lies in $\mathcal{F}^0_{-\infty}$. Fix $\omega \in E_0$. Similarly to the proof of [48, Proposition 3.10], define

$$h(t) := -\frac{t f(A(\omega))}{\sigma} \quad \text{for all } \; t \in [0, T]\,,$$

where f denotes the vector field of the drift in (3.1.1). We write $\varphi(t, g, z)$, $t \in [0, T]$, for the solution of (3.1.1) with initial condition z and path $g \in C_0^T := \{\bar{g} \in C([0, T], \mathbb{R}^2) \,:\, \bar{g}(0) = 0\}$. We can infer that $\varphi(t, h, A(\omega)) = A(\omega)$ for all $t \in [0, T]$. Recall that the map $g \mapsto \varphi(\cdot, g, z)$ is continuous from C_0^T to $C([0, T], \mathbb{R}^2)$ with respect to the supremum norm $\|\cdot\|_\infty$. Hence, there is a $\delta > 0$ such that for all $g \in C_\delta := \{\bar{g} \in C_0^T \,:\, \|\bar{g} - h\| \le \delta\}$, we have

$$\|\varphi(t, g, A(\omega)) - \varphi(t, h, A(\omega))\| < \varepsilon/2 \quad \text{for all } \; t \in [0, T]\,.$$

Since the set $E_+ := \{\omega \,:\, \omega\big|_{[0,T]} \in C_\delta\}$ has positive measure and is independent of E_0, the set $E = E_0 \cap E_+ \in \mathcal{F}^T_{-\infty}$ has positive measure and satisfies

$$A(\theta_t \omega) \in B_\varepsilon(y) \quad \text{for all } \; t \in [0, T] \text{ and } \omega \in E\,,$$

by the above construction. □

3.4.1 Bifurcation for small shear

In this subsection, we consider the stochastic differential equation (3.0.1) with small enough shear such there exists a random equilibrium for α close to zero. We prove in Theorem 3.4.2 that the random equilibrium $A : \Omega \to \mathbb{R}^2$ loses uniform attractivity at the deterministic bifurcation point $\alpha = 0$. On the other hand, we will observe a loss

of hyperbolicity at the bifurcation point in the dichotomy spectrum associated with the random equilibrium. Moreover, we can show that $A : \Omega \to \mathbb{R}^2$ is finite-time attractive before, but not after the bifurcation point, indicated by a transition from zero to positive probability of positive finite-time Lyapunov exponents.

We call the random attractor A *locally uniformly attractive* if there exists a $\delta > 0$ such that

$$\lim_{t\to\infty} \sup_{x\in B_\delta(0)} \operatorname{ess\,sup}_{\omega\in\Omega} \|\varphi(t,\omega,A(\omega)+x) - A(\theta_t\omega)\| = 0\,.$$

We call it *globally uniformly attractive* if the above holds for any $\delta > 0$.

Theorem 3.4.2. *Consider the stochastic differential equation* (3.0.1) *such that there is a unique attracting random equilibrium $A : \Omega \to \mathbb{R}^2$ (see Proposition 3.3.3 and Theorem 3.1.3). Then for $\alpha < 0$ and $|b| \le a$, the random attractor $A : \Omega \to \mathbb{R}^2$ is globally uniformly attractive. Furthermore, for all pairs of initial conditions $U, V \in \mathbb{R}^2$, we have*

$$\|\varphi(t,\omega,U) - \varphi(t,\omega,V)\| \le e^{\alpha t}\|U - V\| \quad \text{for all } t \ge 0\,.$$

For $\alpha > 0$, the random attractor $A : \Omega \to \mathbb{R}^2$ is not even locally uniformly attractive.

Proof. Fix $\alpha < 0$, and choose arbitrary $U, V \in \mathbb{R}^2$, $\omega \in \Omega$. Define

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} := \varphi(t,\omega,U) \quad \text{and} \quad \begin{pmatrix} \widehat{x}_t \\ \widehat{y}_t \end{pmatrix} := \varphi(t,\omega,V)\,.$$

From (3.0.1), we derive that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_t - \widehat{x}_t \\ y_t - \widehat{y}_t \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} x_t - \widehat{x}_t \\ y_t - \widehat{y}_t \end{pmatrix} - \\ (x_t^2 + y_t^2)\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} x_t \\ y_t \end{pmatrix} + (\widehat{x}_t^2 + \widehat{y}_t^2)\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} \widehat{x}_t \\ \widehat{y}_t \end{pmatrix}\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\begin{pmatrix} x_t - \widehat{x}_t \\ y_t - \widehat{y}_t \end{pmatrix}\right\|^2 &= (x_t - \widehat{x}_t)\frac{\mathrm{d}}{\mathrm{d}t}(x_t - \widehat{x}_t) + (y_t - \widehat{y}_t)\frac{\mathrm{d}}{\mathrm{d}t}(y_t - \widehat{y}_t) \\ &= \alpha\left\|\begin{pmatrix} x_t - \widehat{x}_t \\ y_t - \widehat{y}_t \end{pmatrix}\right\|^2 - R(x_t,\widehat{x}_t,y_t,\widehat{y}_t)\,,\end{aligned}$$

where

$$R(x_t,y_t,\widehat{x}_t,\widehat{y}_t) := a\left(r_t^2 + \widehat{r}_t^2 - (x_t\widehat{x}_t + y_t\widehat{y}_t)(r_t + \widehat{r}_t)\right) + b(x_t\widehat{y}_t - \widehat{x}_t y_t)(r_t - \widehat{r}_t)$$

with $r_t := x_t^2 + y_t^2$ and $\widehat{r}_t := \widehat{x}_t^2 + \widehat{y}_t^2$. To show global uniform attractivity, it is sufficient to

establish that $R(x_t, y_t, \widehat{x}_t, \widehat{y}_t) \geq 0$. From the inequality $(|xy| + |uv|)^2 \leq (x^2 + u^2)(y^2 + v^2)$, we derive that

$$\begin{aligned}
&|(x_t\widehat{x}_t + y_t\widehat{y}_t)(r_t + \widehat{r}_t)| + |(x_t\widehat{y}_t - \widehat{x}_t y_t)(r_t - \widehat{r}_t)| \\
&\leq \sqrt{(x_t\widehat{x}_t + y_t\widehat{y}_t)^2 + (x_t\widehat{y}_t - \widehat{x}_t y_t)^2}\sqrt{(r_t + \widehat{r}_t)^2 + (r_t - \widehat{r}_t)^2} \\
&= \sqrt{2r_t\widehat{r}_t(r_t^2 + \widehat{r}_t^2)} \\
&\leq r_t^2 + \widehat{r}_t^2 \,.
\end{aligned}$$

Together with the fact that $|b| \leq a$, this implies that $R(x_t, y_t, \widehat{x}_t, \widehat{y}_t) \geq 0$, which establishes global uniform attractivity for $\alpha < 0$.

We assume now that $\alpha > 0$. Suppose to the contrary that there exists $\delta > 0$ such that

$$\lim_{t\to\infty} \sup_{x\in B_\delta(0)} \operatorname{ess\,sup}_{\omega\in\Omega} \|\varphi(t, \omega, A(\omega) + x) - A(\theta_t\omega)\| = 0 \,.$$

This implies that there exists an $N \in \mathbb{N}$ such that for all $t > N$, we have

$$\sup_{x\in B_\delta(0)} \operatorname{ess\,sup}_{\omega\in\Omega} \|\varphi(t, \omega, A(\omega) + x) - A(\theta_t\omega)\| < \tfrac{1}{4}\sqrt{\tfrac{\alpha}{a}} \,.$$

Due to Proposition 3.4.1, there exists a positive measure set $E_0 \in \mathcal{F}_{-\infty}^0$ such that $A(\omega) \in B_{\delta/4}(0)$ for all $\omega \in E_0$. Let $\phi(\cdot, x_0)$ denote the solution of the deterministic equation (3.0.1) for $\sigma = 0$ with initial condition $x(0) = x_0$. Then there exists a $T > N$ such that

$$\|\phi(T, (\pm\tfrac{1}{4}\delta, 0))\| > \tfrac{1}{2}\sqrt{\tfrac{\alpha}{a}} \,,$$

and at the same time

$$\|\phi(T, (\tfrac{1}{4}\delta, 0)) - \phi(T, (-\tfrac{1}{4}\delta, 0))\| > \sqrt{\tfrac{\alpha}{a}} \,.$$

Recall from the proof of Proposition 3.4.1 that $\omega \mapsto \varphi(\cdot, \omega, x)$ is continuous from C_0^T to $C([0, T], \mathbb{R}^2)$ with respect to the supremum norm. This implies that there exists an $\varepsilon > 0$ such that for all $\omega \in E_\varepsilon = \{\omega \in \Omega : \sup_{t\in[0,T]} \|\omega(t)\| < \varepsilon\} \in \mathcal{F}_0^T$, we obtain

$$\begin{aligned}
\|\phi(T, (\tfrac{1}{4}\delta, 0)) - \varphi(T, \omega, (\tfrac{1}{4}\delta, 0))\| &< \frac{1}{4}\sqrt{\frac{\alpha}{a}} \\
\text{and} \quad \|\phi(T, (-\frac{1}{4}\delta, 0)) - \varphi(T, \omega, (-\frac{1}{4}\delta, 0))\| &< \frac{1}{4}\sqrt{\frac{\alpha}{a}} \,.
\end{aligned}$$

This implies that

$$\|\varphi(T, \omega, (\tfrac{1}{4}\delta, 0)) - \varphi(T, \omega, (-\tfrac{1}{4}\delta, 0))\| > \tfrac{1}{2}\sqrt{\tfrac{\alpha}{a}} \,.$$

Since E_ε and E_0 are independent sets of positive measure, we get that $\mathbb{P}(E) > 0$ where

$E = E_\varepsilon \cap E_0$. However, for all $\omega \in E$, we conclude

$$\begin{aligned}&\sup_{x\in B_\delta(0)} \|\varphi(t,\omega,A(\omega)+x) - A(\theta_t\omega)\| \\ &\geq \max\left\{\left\|\varphi(t,\omega,(\tfrac{1}{4}\delta,0)) - A(\theta_t\omega)\right\|,\ \left\|\varphi(t,\omega,(-\tfrac{1}{4}\delta,0)) - A(\theta_t\omega)\right\|\right\} > \tfrac{1}{4}\sqrt{\tfrac{\alpha}{a}}\,,\end{aligned}$$

which contradicts our assumption. □

Theorem 3.4.3. *Consider the stochastic differential equation* (3.0.1) *such that there exists a unique attracting random equilibrium $A : \Omega \to \mathbb{R}^2$ (see Proposition 3.3.3 and Theorem 3.1.3). Let $\Phi(t,\omega) := \mathrm{D}\varphi(t,\omega,A(\omega))$ denote the linearised random system along the random equilibrium. Then for $|b| < a$ and $\alpha \in \mathbb{R}$ small enough such that the random equilibrium $A : \Omega \to \mathbb{R}^2$ exists, the dichotomy spectrum Σ of Φ is given by*

$$\Sigma = [-\infty, \alpha]\,.$$

Proof. Recall from Proposition 3.3.1 that we have

$$\|\mathrm{D}\varphi(t,\omega,x)\| \leq \exp\left(\int_0^t \lambda^+(\varphi(s,\omega,x))\mathrm{d}s\right).$$

Since Lemma 3.3.4 implies that $\lambda^+(x) \leq \alpha - (a-|b|)\|x\|^2$, we have

$$\|\Phi(t,\omega)\| \leq \exp\left(\int_0^t (\alpha - (a-|b|)\|A(\theta_s\omega))\|^2)\mathrm{d}s\right). \tag{3.4.1}$$

Similarly, with $\lambda^-(x) := \min_{\|r\|=1}\langle \mathrm{D}f(x)r, r\rangle$, we have

$$\|\mathrm{D}\varphi(t,\omega,x)\| \geq \exp\left(\int_0^t \lambda^-(\varphi(s,\omega,x))\mathrm{d}s\right).$$

It is easy to see that $\lambda^-(x) \geq \alpha - 4a\|x\|^2$, which implies

$$\|\Phi(t,\omega)\| \geq \exp\left(\int_0^t (\alpha - 4a\|A(\theta_s\omega))\|^2)\mathrm{d}s\right). \tag{3.4.2}$$

From (3.4.1) we can deduce immediately that for almost all $\omega \in \Omega$, we have

$$\|\Phi(t,\omega)\| \leq e^{\alpha|t|} \quad \text{for all } t \in \mathbb{R}\,.$$

This implies that $\Sigma \subset (-\infty, \alpha]$.

We now show that $(-\infty,\alpha] \subset \Sigma$. Choose $\gamma \in (-\infty,\alpha]$, and suppose to the contrary that Φ admits an exponential dichotomy with growth rate γ with an invariant projector P_γ and constants $K, \varepsilon > 0$ (see Section 2.3). We consider the following three cases (note that the rank of the invariant projector does not depend on ω, see [21]):

(i) $P_\gamma \equiv \mathrm{id}$. This means that for almost all $\omega \in \Omega$,

$$\|\Phi(t,\omega)\| \leq K e^{(\gamma-\varepsilon)t} \quad \text{for all } t \geq 0\,.$$

Fix $T > 0$ such that $e^{\frac{1}{5}\varepsilon T} > K$. According to Proposition 3.4.1, there exists a positive measure set E such that for all $\omega \in E$ and $s \in [0,T]$, we have $A(\theta_s\omega) \in B_{\sqrt{\varepsilon/(5a)}}(0)$. We derive from (3.4.2) that for such $\omega \in E$, we have

$$\|\Phi(T,\omega)\| \geq e^{T(\alpha-\frac{4}{5}\varepsilon)} \geq K e^{(\gamma-\varepsilon)T}\,.$$

This contradicts the assumption.

(ii) $\operatorname{rk} P_\gamma \equiv 1$. The argument is the same as in the previous case, since our estimates do not depend on the tangent vector v, but hold for the norm $\|\Phi(t,\omega)\|$.

(iii) $P_\gamma \equiv 0$. This means that for almost all $\omega \in \Omega$, we have

$$\|\Phi(t,\omega)\| \geq \frac{1}{K} e^{(\gamma+\varepsilon)t} \quad \text{for all } t \geq 0\,.$$

Together with (3.4.1), this implies that

$$\frac{\ln K + (\alpha-\varepsilon-\gamma)t}{a-|b|} \geq \int_0^t \|A(\theta_s\omega)\|^2 \mathrm{d}s\,.$$

Choose some $T > 1$ and $y \in \mathbb{R}^2$ such that

$$\|y\|^2 > 4 \max\left\{\frac{\ln K}{a-|b|}, \frac{\alpha-\varepsilon-\gamma}{a-|b|}\right\}.$$

Take $\delta < \frac{\|y\|}{2}$. Then by Proposition 3.4.1, there exists a set $E \in \mathcal{F}_{-\infty}^T$ such that

$$A(\theta_s\omega) \in B_\delta(y) \quad \text{for all } s \in [0,T] \text{ and } \omega \in E\,.$$

This implies

$$\int_0^t \|A(\theta_s\omega)\|^2 \mathrm{d}s > T\frac{\|y\|^2}{4} > \frac{\ln K + (\alpha-\varepsilon-\gamma)T}{a-|b|}\,,$$

which is a contradiction.

This finishes the proof of this theorem. □

We demonstrate now that the change of sign in the dichotomy spectrum is mirrored by finite-time properties of the system. To see this, consider a compact time interval $[0,T]$ and the corresponding *finite-time top Lyapunov exponents* associated with the attractive

random equilibrium $A : \Omega \to \mathbb{R}^2$, given by

$$\lambda^{T,\omega} := \sup_{\|v\|=1} \frac{1}{T} \ln \|\Phi(T,\omega)v\| \quad \text{for all } \omega \in \Omega\,.$$

From Proposition 3.3.3 (iii), we obviously have $\lambda_{\text{top}} = \lim_{T\to\infty} \lambda^{T,\omega}$ almost surely, where λ_{top} is the top Lyapunov exponent of (3.0.1).

Proposition 3.4.4. *Consider the stochastic differential equation* (3.0.1) *with* $|b| < a$ *and* $\alpha \in \mathbb{R}$ *such that there exists a unique attractive random equilibrium* $A : \Omega \to \mathbb{R}^2$*. The following statements hold.*

(i) For $\alpha < 0$*, we have* $\lambda^{T,\omega} \leq \alpha < 0$ *for all* $\omega \in \Omega$*, which means that the random attractor* $A : \Omega \to \mathbb{R}^2$ *is* finite-time attractive*.*

(ii) For $\alpha > 0$*, we have* $\mathbb{P}\big(\omega \in \Omega : \lambda^{T,\omega} > 0\big) > 0$*, which means that the random attractor* $A : \Omega \to \mathbb{R}^2$ *is not finite-time attractive.*

Proof. (i) Recall from (3.4.1) that

$$\|\Phi(t,\omega)\| \leq \exp\left(\int_0^t \big(\alpha - (a-|b|)\|A(\theta_s\omega))\|^2\big)\,\mathrm{d}s\right),$$

which implies that

$$\lambda^{T,\omega} \leq \frac{1}{T}\int_0^T \big(\alpha - (a-|b|)\|A(\theta_s\omega))\|^2\big)\,\mathrm{d}s \leq \alpha < 0\,.$$

(ii) Recall from (3.4.2) that

$$\|\Phi(t,\omega)\| \geq \exp\left(\int_0^t \big(\alpha - 4a\|A(\theta_s\omega))\|^2\big)\,\mathrm{d}s\right).$$

Choose $\varepsilon := \sqrt{\frac{\alpha}{5a}} > 0$. According to Proposition 3.4.1, there exists a set $E \in \mathcal{F}_{-\infty}^T$ of positive measure such that $A(\theta_s\omega) \in B_\varepsilon(0)$ for all $s \in [0,T]$ and $\omega \in E$. Then

$$\lambda^{T,\omega} \geq \frac{1}{T}\int_0^T \big(\alpha - 4a\|A(\theta_s\omega))\|^2\big)\,\mathrm{d}s \geq \alpha - \frac{4\alpha}{5} = \frac{\alpha}{5} > 0 \quad \text{for all } \omega \in E\,.$$

This shows the claim. □

Proof of Theorem 3.1.6. The claims follow from Theorem 3.4.2, Theorem 3.4.3 and Proposition 3.4.4. □

The proofs of Theorem 3.4.3 and Proposition 3.4.4 explain in detail how the change of finite-time attractivity is connected to the loss of hyperbolicity in the dichotomy spectrum.

Due to [21, Theorem 4.5], we obtain

$$\lim_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \lambda^{T,\omega} = \lim_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{\|v\|=1} \frac{1}{T} \ln \|\Phi(T,\omega)v\| = \sup \Sigma\,.$$

A similar statement holds for the infimum of the dichotomy spectrum. This means that the finite-time Lyapunov exponents are, at least asymptotically, supported on the dichotomy spectrum, and having positive values in the spectrum implies that, at least asymptotically, we can observe positive finite-time Lyapunov exponents.

3.4.2 Shear intensity as bifurcation parameter

We now do not assume the existence of an attractive random equilibrium, and we aim at proving Theorem 3.1.7 in this subsection. We first show a statement that corresponds to Proposition 3.4.1 in this more general context.

Proposition 3.4.5. *Let (θ,φ) be the random dynamical system generated by* (3.0.1)*, and let $x, y \in \mathbb{R}^2$, $\varepsilon > 0$ and $T > 0$. Then for any $t_0 \in (0,T]$, there exists a set $E \in \mathcal{F}$ with $\mathbb{P}(E) > 0$ such that*

$$\varphi(s,\omega,x) \in B_\varepsilon(y) \quad \text{for all } s \in [t_0,T] \text{ and } \omega \in E\,.$$

Proof. Similarly as in the proof of [48, Proposition 3.10], fix $t_0 \in (0,T]$ and define

$$\psi(t) := x + \frac{t}{t_0}(y-x) \quad \text{for all } t \in [0,t_0]\,,$$

and

$$h(t) := \frac{1}{\sigma}\left(\psi(t) - x - \int_0^t f(\psi(s))\,\mathrm{d}s\right) \quad \text{for all } t \in [0,T]\,,$$

where f denotes the vector field of the drift in (3.0.1). As in the proof of Proposition 3.4.1, we write $\varphi(t,g,z)$ for the solution of (3.0.1) with initial condition z and path $g \in C_0^T$. We can infer that $\varphi(t,h,x) = \psi(t)$ for all $t \in [0,t_0]$, and in particular $\varphi(t_0,h,x) = y$. Recall that the map $g \mapsto \varphi(\cdot,g,z)$ is continuous from C_0^T to $C([0,t_0],\mathbb{R}^2)$ with respect to the supremum norm $\|\cdot\|_\infty$. This implies that there exists a $\delta > 0$ such that for all $g \in C_\delta := \{b \in C_0^T : \|b-h\| \le \delta\}$, we have

$$\|\varphi(t,g,x) - \varphi(t,h,x)\| < \tfrac{1}{2}\varepsilon \quad \text{for all } t \in [0,t_0]\,.$$

Hence, we have established that there is a positive measure set $E_1 := \{\omega \in \Omega : \omega|_{[0,t_0]} \in C_\delta\}$ such that for all $\omega \in E_1$, we have $\varphi(t_0,\omega,x) \in B_{\varepsilon/2}(y)$.

Similar to this argument, one can construct a set E_2 of positive measure that is independent from E_1 (by the Markov property) such that for all $\omega \in E := E_1 \cap E_2$, we

have

$$\varphi(t,\omega,x) \in B_\varepsilon(y) \quad \text{for all } t \in [t_0, T]\,.$$

This finishes the proof of this proposition. □

Proof of Theorem 3.1.7. For $(\omega, z) \in \Omega \times \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, the linear random dynamical system $t \mapsto \Phi(t,\omega,z)$ is solution of the variational equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,\omega,z) = \mathrm{D}f(\varphi(t,\omega,z))\Phi(t,\omega,z), \quad \text{where } \Phi(0,\omega,z) = \mathrm{Id}\ .$$

Define $s_t(\omega,z,v) := \frac{\Phi(t,\omega,z)v}{\|\Phi(t,\omega,z)v\|}$ and observe that for $v \in \mathbb{R}^2 \setminus \{0\}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi(t,\omega,z)v\|^2 &= 2\left\langle \mathrm{D}f(\varphi(t,\omega,z))\Phi(t,\omega,z)v, \Phi(t,\omega,z)v\right\rangle \\ &= 2\left\langle \mathrm{D}f(\varphi_t(\omega,z))s_t(\omega,z,v), s_t(\omega,z,v)\right\rangle \|\Phi(t,\omega,z)v\|^2\,.\end{aligned}$$

Let $\mu > 0$, and let $z' = (w,w) \in \mathbb{R}^2$ be such that $\frac{b-2a}{2}\|z'\|^2 = (b-2a)w^2 \geq \mu$ and $w > 1$. Note that

$$\mathrm{D}f(x,y) = \begin{pmatrix} \alpha - ay^2 - 3ax^2 - 2byx & -\beta - 2axy - bx^2 - 3by^2 \\ \beta - 2axy + by^2 + 3bx^2 & \alpha - ax^2 - 3ay^2 + 2byx \end{pmatrix}.$$

With $\tilde{r} = (0,1)$, we get

$$\langle \mathrm{D}f(z')\tilde{r}, \tilde{r}\rangle = \alpha + 2(b-2a)w^2 \geq \alpha + 2\mu\,.$$

Let

$$\varepsilon = \min\left\{1, \frac{1}{16}\frac{b-2a}{bw}, \frac{\sqrt{b-2a}}{4a}\right\} \quad \text{and} \quad \delta = \frac{1}{8}\frac{b-2a}{4b}\,.$$

Then by Proposition 3.4.5, there is a positive measure set $E_1 \subset \Omega$ such that for all $\omega \in E_1$

$$\varphi(t,\omega,z') \in B_\varepsilon(z') \quad \text{for all } t \in [0,1]\,.$$

This implies that the coefficients of $\mathrm{D}f(\varphi(t,\omega,z'))$ are bounded uniformly in $\omega \in E_1$ for $t \in [0,1]$. Because Φ is continuous, there is a $T \in (0,1]$ such that

$$\|s_t(\omega,z',\tilde{r}) - \tilde{r}\| = \left\| \frac{\Phi(t,\omega,z')\tilde{r}}{\|\Phi(t,\omega,z')\tilde{r}\|} - \tilde{r}\right\| < \delta \quad \text{for all } t \in [0,T] \text{ and } \omega \in E_1$$

Note that we obtain for any $r \in \mathbb{R}^2$ with $\|r\| = 1$

$$\begin{aligned}\langle \mathrm{D}f(x,y)r, r\rangle &= r_1^2(\alpha - ay^2 - 3ax^2) + r_1r_2(-\beta - 2axy) + r_1r_2(\beta - 2axy)\\ &\quad + r_2^2(\alpha - ax^2 - 3ay^2) - 2byxr_1^2 + 2byxr_2^2 + r_1r_2(2bx^2 - 2by^2)\\ &= \alpha - a(x^2+y^2) + 2b(r_1r_2x^2 - r_1r_2y^2 + yx(r_2^2 - r_1^2)) - 2a(r_1x + r_2y)^2\,.\end{aligned}$$

This means that for all $t \in (0,T]$ and $\omega \in E_1$, we have by the choice of ε and δ above that

$$\begin{aligned}&\langle \mathrm{D}f(\varphi_t(\omega,z))s_t(\omega,z,\tilde{r}), s_t(\omega,z,\tilde{r})\rangle\\ &\geq \alpha - 2a(w+\varepsilon)^2 + 2b(w-\varepsilon)^2(1-2\delta) - 2b\delta[(w+\varepsilon)^2 - (w-\varepsilon)^2] - 2a(w+\varepsilon)^2\\ &= \alpha + (b-2a)w^2 + \big((b-2a)w^2 - 4a(2w\varepsilon+\varepsilon^2) - 4bw\varepsilon - \delta 4b(w-\varepsilon)^2 - 2\delta 4bw\varepsilon\big)\\ &\geq \alpha + \mu\,.\end{aligned}$$

Hence, we get that for all $\omega \in E_1$ and $t \in (0,T]$, the finite-time top Lyapunov exponent of trajectories starting in z' satisfies

$$\lambda^{t,\omega,z'} := \sup_{\|v\|=1} \frac{1}{t} \ln \|\Phi(t,\omega,z')v\| \geq \alpha + \mu\,.$$

Since $\mu > 0$ was arbitrary, we obtain with positive probability arbitrarily large finite-time Lyapunov exponents when starting in z'.

We now show that for any $z \in \mathbb{R}^2$ and $t_0 \in (0,T]$, the finite-time top Lyapunov exponent $\lambda^{t,\omega,z}$, $t \in [t_0,T]$, can be arbitrarily large for ω from a set of positive measure. By Proposition 3.4.5, there exists a set $E_2 \in \mathcal{F}$ with $\mathbb{P}(E_2) > 0$ such that

$$\varphi(s,\omega,z) \in B_\varepsilon(z') \quad \text{for all } \ s \in [t_0,T] \text{ and } \omega \in E_2\,,$$

where the values of $\varphi(t,\omega,z)$, $t \in [0,t_0]$, stay close to the line between z and z' (see proof of Proposition 3.4.5). Since t_0 can be chosen arbitrarily small and the solutions stay in a compact set for $t \in [0,t_0]$, we obtain with similar arguments as before that with positive probability there are arbitrarily large finite-time Lyapunov exponents.

Let $\mu^- < 0$. Then by choosing $z'' = (w,-w)$, we obtain with similar arguments as above that for some $T \in (0,1]$

$$\inf_{\|v\|=1} \frac{1}{t} \ln \|\Phi(t,\omega,z'')v\| \leq \alpha + \mu^- \quad \text{for all } \ t \in [0,T] \text{ and } \omega \text{ from a positive measure set.}$$

By using Proposition 3.4.5 again, we can then deduce that with positive probability, there are arbitrarily small finite-time Lyapunov exponents for any initial conditions. □

3.5 Chaos via shear: approaches to proving the conjecture

Numerical simulations indicate that for fixed a, β and σ, we obtain $\lambda_1 > 0$ for large enough shear intensity b. This is illustrated in Figure 3.1, where we have fixed $a = \beta = \sigma = 1$ and numerically approximated the top Lyapunov exponent λ_1 depending on α and b. The findings support Conjecture 3.1.5.

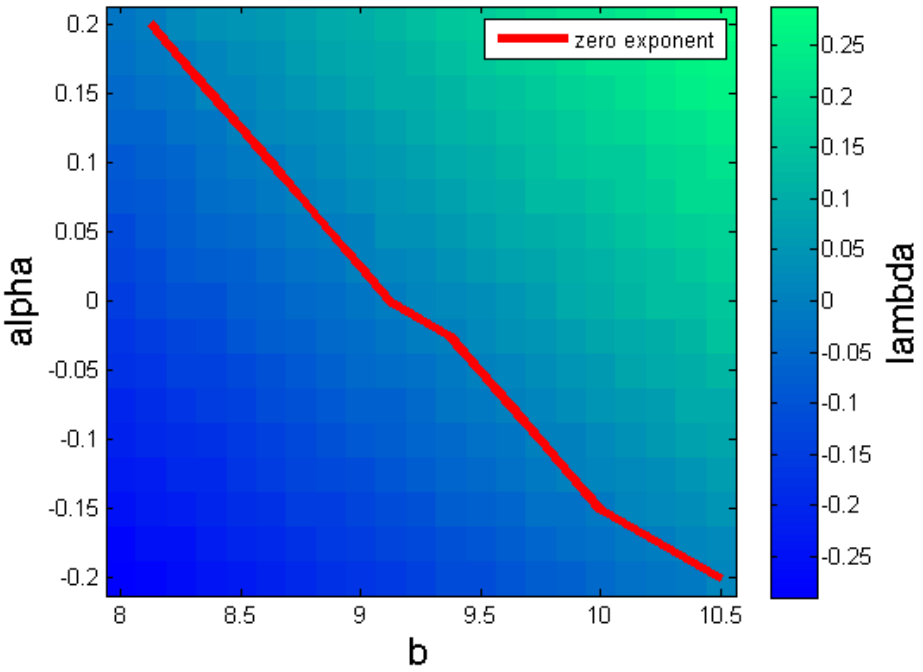


Figure 3.1: For fixed $a = \beta = \sigma = 1$ we approximate numerically the top Lyapunov exponent λ_1 for system (3.0.1) as a function of α and b. The red curve highlights the border between regions with negative and positive λ_1, corresponding to synchronisation or random strange attractor, respectively.

So far, we have made estimates on the first Lyapunov exponent λ_1 but we have not been able to find a positive lower bound for any parameter combination. This has only been possible for finite-time Lyapunov exponents as recorded in Theorem 3.1.7.

3.5.1 Formula for the top Lyapunov exponent

An alternative approach to studying the top Lyapunov exponent is derived by Deville et al. in [35], using polar coordinate transformations. We consider the polar coordinates

$$r = \sqrt{x^2 + y^2}, \ \phi = \arctan(\frac{y}{x}) .$$

Applying Itō's rule to the stochastic differential equation (3.0.1) we obtain

$$\begin{aligned} \mathrm{d}r &= \left(\alpha r - ar^3 + \frac{\sigma^2}{2r}\right)\mathrm{d}t + \sigma(\cos\phi\,\mathrm{d}W_t^1 + \sin\phi\,\mathrm{d}W_t^2), \\ \mathrm{d}\phi &= (\beta + br^2)\,\mathrm{d}t + \frac{\sigma}{r}(-\sin\phi\,\mathrm{d}W_t^1 + \cos\phi\,\mathrm{d}W_t^2). \end{aligned} \tag{3.5.1}$$

This form illustrates the role of the parameter b inducing a shear force: if $b > 0$, the phase velocity $\frac{\mathrm{d}\phi}{\mathrm{d}t}$ depends on the amplitude r. Since Gaussian random vectors are invariant under orthogonal transformations, we can define the independent Wiener processes

$$\begin{aligned} \mathrm{d}W_r &= \cos\phi\,\mathrm{d}W_t^1 + \sin\phi\,\mathrm{d}W_t^2, \\ \mathrm{d}W_\phi &= -\sin\phi\,\mathrm{d}W_t^1 + \cos\phi\,\mathrm{d}W_t^2. \end{aligned}$$

Hence, the Markov process solving

$$\mathrm{d}r = \left(\alpha r - ar^3 + \frac{\sigma^2}{2r}\right)\mathrm{d}t + \sigma\mathrm{d}W_r(t), \tag{3.5.2}$$

$$\mathrm{d}\phi = [\beta + br^2]\mathrm{d}t + \frac{\sigma}{r}\mathrm{d}W_\phi(t) \tag{3.5.3}$$

corresponds with (3.0.1) in terms of the Itô integral. However, the pathwise properties of the two processes seen as random dynamical systems are different. In (3.5.1), the radial components of the trajectories depend on ϕ which appears in the diffusion term. This is not the case in (3.5.2). The distinction is often not made clear in the literature. However, for the sake of the following computations the usage of equations (3.5.2) and (3.5.3) is justified as we will be only interested in the stationary distribution and the moments of the process. The same holds for the variational equation associated with system (3.0.1) written in polar coordinates ρ and θ. Setting $\chi_0 = \arccos(\frac{b}{\sqrt{a^2+b^2}})$, it reads

$$\begin{aligned} \mathrm{d}\rho &= \rho\{\alpha - 2a(x^2+y^2) + \sqrt{a^2+b^2}[(x^2-y^2)\sin(2\theta-\chi_0) \\ &\quad - 2xy\cos(2\theta-\chi_0)]\}\mathrm{d}t \\ &= \rho\{\alpha - 2ar^2 + r^2\sqrt{a^2+b^2}\sin(2\theta-\chi_0-2\phi)\}\mathrm{d}t, \end{aligned} \tag{3.5.4}$$

$$\begin{aligned} \mathrm{d}\theta &= \{\beta + 2b(x^2+y^2) + \sqrt{a^2+b^2}[(x^2-y^2)\cos(2\theta-\chi_0) \\ &\quad + 2xy\sin(2\theta-\chi_0)]\}\mathrm{d}t \\ &= \{\beta + 2br^2 + r^2\sqrt{a^2+b^2}\cos(2\theta-\chi_0-2\phi)\}\mathrm{d}t. \end{aligned} \tag{3.5.5}$$

We introduce the angle $\psi = 2\theta - \chi_0 - 2\phi$ for the sake of calculating the top Lyapunov exponent just in dependence on two variables, r and ψ. The stochastic differential equation solved by ψ can be obtained from (3.5.5) and (3.5.3). The radial process in the original

nonlinear system is still given by (3.5.2). We end up with the two processes given by

$$\mathrm{d}r = \left(\alpha r - ar^3 + \frac{\sigma^2}{2r}\right)\mathrm{d}t + \sigma \mathrm{d}W_r(t)\,, \tag{3.5.6}$$

$$\mathrm{d}\psi = 2[b + \sqrt{a^2+b^2}\cos(\psi)]r^2\mathrm{d}t - \frac{2\sigma}{r}\mathrm{d}W_\phi(t)\,. \tag{3.5.7}$$

By solving equation (3.5.4) and using ergodicity, we obtain the following formula for the top Lyapunov exponent as a function of r and ψ,

$$\lambda_1 = \alpha - 2a\mathbb{E}[r^2] + \sqrt{a^2+b^2}\mathbb{E}[r^2\sin(\psi)], \tag{3.5.8}$$

where the expectation has to be taken with respect to a joint distribution with density $\tilde{p}(r,\psi)$. This is essentially the Furstenberg-Khasminskii formula (2.3.9) from chapter 2, with the addition that another variable, the r-variable, is present, expressing the dependency of the linearisation on the nonlinear process.

Note that existence and uniqueness of $\tilde{p}(r,\psi)$ are not clear a priori due to the singularity of $\frac{2\sigma}{r}$ at $r=0$. However, bounded away from $r=0$ and $r=\infty$ the equations are uniformly elliptic. So we assume the existence and uniqueness of $\tilde{p}(r,\psi)$ for now, hoping to justify the assumption by an explicit solution or approximation a posteriori. First observe that the r-variable is independent of ψ such that we get the marginal distribution $\nu(r) := \tilde{p}(r\big|\psi) = \int_0^{2\pi}\tilde{p}(r,\psi)\mathrm{d}\psi$ by solving the stationary Fokker-Planck equation associated with (3.5.6). The Fokker-Planck operator is given by

$$\mathcal{L}_r^*\cdot = -\frac{\partial}{\partial r}\left(\alpha r - ar^3 + \frac{\sigma^2}{2r}\cdot\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2}\cdot\,. \tag{3.5.9}$$

The unique solution of the associated stationary equation reads

$$\nu(r) = K_1 r\exp\left(-\frac{a}{2\sigma^2}\left[r^2 - \frac{\alpha}{a}\right]^2\right)\,, \tag{3.5.10}$$

where $K_1 = K$ is the same normalisation constant as in (3.1.3).

Since the process in the ψ-variable depends on r, such a calculation is not possible for the ψ-variable. This means that the joint density $\tilde{p}(r,\psi)$ can generally not be written as a product. Hence, the difficult task is to find an approximation of $\tilde{p}(r,\psi)$ which allows to make an estimate for $\mathbb{E}[r^2\sin(\psi)]$.

3.5.2 Studying the stationary Fokker-Planck equation

First of all, we make the following observation: if we fix $r = r_0 \in (0,\infty)$ in equation 3.5.7, we can compute the unique stationary density solving the stationary Fokker-Planck equa-

tion for the operator

$$\mathcal{L}_\psi^* \cdot = -\frac{\partial}{\partial \psi}\left(2[b+\sqrt{a^2+b^2}\cos(\psi)]r_0^2 \cdot\right) + \frac{2\sigma^2}{r_0^2}\frac{\partial^2}{\partial \psi^2}\cdot . \tag{3.5.11}$$

The density is given by (cf. [35, Proposition 3.1])

$$w_{r_0}(\psi) = K_2 \int_\psi^{\psi+2\pi} \exp\{-D_0(r_0)[\phi-\psi] - D_c(r_0)[\sin(\phi)-\sin(\psi)]\}\mathrm{d}\phi, \tag{3.5.12}$$

where $D_0(r) = \frac{r^4 b}{\sigma^2}$, $D_c(r) = \frac{r^4}{\sigma^2}\sqrt{a^2+b^2}$ and K_2 is the normalisation constant.
By the independence of W_r and W_ϕ, we can write the Fokker-Planck-Operator for the joint density evolution of r and ψ as

$$\mathcal{L}_{r,\psi}^* = \mathcal{L}_r^* + \mathcal{L}_\psi^*,$$

where $\mathcal{L}_r^*$ and $\mathcal{L}_\psi^*$ are given in 3.5.9 and 3.5.11 respectively. The stationary Fokker-Planck equation in both variables reads

$$\begin{aligned} 0 = \mathcal{L}_{r,\psi}^* \tilde{p} = &-\frac{\partial}{\partial r}\left[\left(\alpha r - a r^3 + \frac{\sigma^2}{2r}\right)\tilde{p}\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial r^2}\tilde{p} \\ &- \frac{\partial}{\partial \psi}\left(2[b+\sqrt{a^2+b^2}\cos(\psi)]r^2\tilde{p}\right) + \frac{2\sigma^2}{r^2}\frac{\partial^2}{\partial \psi^2}\tilde{p}. \end{aligned} \tag{3.5.13}$$

A promising rescaling looks as follows: first we set $r = \tilde{r}\sqrt{\alpha/a}$, then we multiply by a/α and set $\varepsilon = \sigma^2 a^2/\alpha^2$ and we obtain

$$\begin{aligned} 0 &= -\frac{\partial}{\partial \tilde{r}}\left[\left(\alpha(\tilde{r}-\tilde{r}^3) + \frac{a\sigma^2}{2\alpha}\right)\tilde{p}\right] + \frac{a\sigma^2}{2\alpha}\frac{\partial^2}{\partial \tilde{r}^2}\tilde{p} \\ &\quad - \frac{\partial}{\partial \psi}\left(2[b+\sqrt{a^2+b^2}\cos(\psi)]\frac{\alpha}{a}\tilde{r}^2\tilde{p}\right) + \frac{2a\sigma^2}{\alpha\tilde{r}^2}\frac{\partial^2}{\partial \psi^2}\tilde{p} \\ &= -\frac{\partial}{\partial \tilde{r}}\left[\left(a(\tilde{r}-\tilde{r}^3) + \frac{\varepsilon}{2}\right)\tilde{p}\right] + \frac{\varepsilon}{2}\frac{\partial^2}{\partial \tilde{r}^2}\tilde{p} \\ &\quad - \frac{\partial}{\partial \psi}\left(2[b+\sqrt{a^2+b^2}\cos(\psi)]\tilde{r}^2\tilde{p}\right) + \frac{2\varepsilon}{\tilde{r}^2}\frac{\partial^2}{\partial \psi^2}\tilde{p}\,. \end{aligned} \tag{3.5.14}$$

Deville et al. [35] have used an expansion around the deterministic equilibria

$$\tilde{r} = 1 + \varepsilon^\gamma \eta, \quad \psi = \arccos\left(-\frac{b}{\sqrt{a^2+b^2}}\right) + \varepsilon^\gamma \lambda$$

which seems the most natural approach. Letting $\varepsilon \to 0$, they approximate the joint density as a product $\tilde{p}(r,\psi) = q(r)m(\psi)$ and obtain $\lambda_1 = C\varepsilon + \mathcal{O}(\varepsilon^2)$ with $C < 0$.

We want to find a scaling limit of the parameters, i.e. a choice of ε, such that λ_1

turns out to be positive. In order to abbreviate notation in the following, we write the equations (3.5.6) and (3.5.7) as

$$\begin{aligned} \mathrm{d}r &= f_{\alpha,a,\sigma}(r)\mathrm{d}t + \sigma \mathrm{d}W_r(t), \\ \mathrm{d}\psi &= r^2 g_{a,b}(\psi)\mathrm{d}t + \frac{2\sigma}{r}\mathrm{d}W_\phi(t). \end{aligned}$$

Write $\tilde{p}(r,\psi) = m(r,\psi)\nu(r)$ hoping that we can use our knowledge about ν and w_r. Using $\mathcal{L}_r^*\nu = 0$ we obtain

$$\begin{aligned} 0 &= \mathcal{L}_{r,\psi}^*\tilde{p}(r,\psi) = (\mathcal{L}_r^* + \mathcal{L}_\psi^*)m(r,\psi)\nu(r) \\ &= -f_{\alpha,a,\sigma}(r)\nu(r)\frac{\partial}{\partial r}m(r,\psi) + \frac{1}{2}\sigma^2\nu(r)\frac{\partial^2}{\partial r^2}m(r,\psi) + \sigma^2\frac{\partial}{\partial r}m(r,\psi)\frac{\partial}{\partial r}\nu(r) \\ &- \nu(r)\frac{\partial}{\partial \psi}(r^2 g_{a,b}(\psi)m(r,\psi)) + \frac{2\sigma^2}{r^2}\nu(r)\frac{\partial^2}{\partial \psi^2}m(r,\psi)\,, \end{aligned}$$

and, hence,

$$\begin{aligned} 0 = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial r^2}m(r,\psi) + \frac{\sigma^2}{2}\left(\frac{\partial}{\partial r}\nu(r)/\nu(r)\right)\frac{\partial}{\partial r}m(r,\psi) \\ - \frac{\partial}{\partial \psi}(r^2 g_{a,b}(\psi)m(r,\psi)) + \frac{2\sigma^2}{r^2}\frac{\partial^2}{\partial \psi^2}m(r,\psi)\,. \end{aligned}$$

The most promising choice of a small parameter ε for this separation ansatz is $\varepsilon = \sigma = \alpha$. We can rewrite the equation as

$$0 = \mathcal{L}_\varepsilon m(r,\psi) + \mathcal{L}_\psi^* m(r,\psi)\,, \tag{3.5.15}$$

where

$$\mathcal{L}_\varepsilon \cdot = \frac{1}{2}\varepsilon^2\frac{\partial^2}{\partial r^2}\cdot + \frac{\varepsilon^2}{2}\left(\frac{\partial}{\partial r}\nu(r)/\nu(r)\right)\frac{\partial}{\partial r}\cdot$$

and

$$\mathcal{L}_\psi^* \cdot = -\frac{\partial}{\partial \psi}\left(2[b + \sqrt{a^2+b^2}\cos(\psi)]r^2\cdot\right) + \frac{2\alpha^2}{r^2}\frac{\partial^2}{\partial \psi^2}\cdot\ .$$

Observe from an easy calculation that

$$\mathbb{E}[r^2] = \frac{\alpha}{a} + \sigma\sqrt{\frac{2}{\pi a}}\frac{e^{-\alpha^2/2a\sigma^2}}{\mathrm{erfc}(-\alpha/\sqrt{2a\sigma^2})}. \tag{3.5.16}$$

A possible strategy from here looks as follows: we consider only values of r^2 from a neighbourhood $B_{R\varepsilon}(\mathbb{E}[r^2]) \cap \mathbb{R}_0^+$ for some large $R > 0$ such that $r \approx \varepsilon^{1/2}$. Then $\left(\frac{\partial}{\partial r}\nu(r)/\nu(r)\right) \approx \varepsilon^{-1/2}$. If the second derivative of $w_r(\psi)$ in r grows slower than ε^{-2} and the first derivative slower than $\varepsilon^{-3/2}$ as $\varepsilon = \alpha = \sigma \to 0$, we can deduce from (3.5.15) that $w_r(\psi) \approx m(r,\psi)$ for $r^2 \in B_{R\varepsilon}(\mathbb{E}[r^2]) \cap \mathbb{R}_0^+$.

Therefore, if we were able to show these properties for the derivatives of $w_r(\psi)$, we would have a way to prove Conjecture 3.1.5. For in the case of $w_r(\psi) \approx m(r,\psi)$ for $r^2 \in B_{R\varepsilon}(\mathbb{E}[r^2]) \cap \mathbb{R}_0^+$, we can make the necessary estimates on the quantity

$$\begin{aligned}\mathbb{E}[r^2 \sin\psi] &= \int_0^\infty \int_0^{2\pi} r^2 \sin\psi \tilde{p}(r,\psi) d\psi dr \\ &\geq \int_0^{\sqrt{\alpha/a+R\varepsilon}} \left(\int_0^{2\pi} \sin\psi w_r(\psi) d\psi \right) r^2 \nu(r) dr \\ &+ \int_{\sqrt{\alpha/a+R\varepsilon}}^\infty \int_0^{2\pi} r^2 \sin\psi p(r,\psi) d\psi dr \,.\end{aligned}$$

Lower bounds for these two integrals would then imply that for small $\alpha = \sigma$ there is a large b such that

$$\lambda_1 = \sqrt{a^2+b^2}\mathbb{E}[r^2 \sin(\psi)] + \alpha - 2a\mathbb{E}[r^2] > 0\,.$$

3.5.3 Using the finite element method

Another possible strategy for proving Conjecture 3.1.5 consists in a computer-assisted approach. Using the finite element method, we try to approximate $\tilde{p}(r,\psi)$ with rigorous error bounds. On that account, we need to truncate the domain in the r-variable at a constant $C > 0$ and bound it away from 0 by a small number $\delta > 0$ due to the singularity at $r = 0$. We define a bounded domain

$$\Omega := [\delta, C] \times [0, 2\pi]\,.$$

Due to the exponential decay in the r-variable, we can restrict the analysis to Ω if C is taken large enough such that the impact of truncation is negligible.

We have the periodic boundary condition in ψ

$$\tilde{p}(r,0) = \tilde{p}(r,2\pi) \quad \text{for all } r \in [0,\infty).$$

The only boundary condition in r is given by

$$\int_0^C \int_0^{2\pi} \tilde{p}(r,\psi)\,\mathrm{d}\psi\mathrm{d}r = 1.$$

As we must define boundary conditions in a finite element method approach, Dirichlet boundary conditions in r seem most natural. Considering $\nu(0) = 0$ and the exponential decay of $\nu(r)$, we choose $\varepsilon_0 > 0$ small and

$$\tilde{p}(\delta,\psi) = \tilde{p}(C,\psi) = \varepsilon_0 \quad \text{for all } \psi \in [0,2\pi].$$

We rewrite the stationary Fokker-Planck equation (3.5.13) as follows:

$$\begin{aligned} 0 = L^*_{r,\psi}\tilde{p}(r,\psi) = &\left[-\alpha + 3ar^2 + \frac{\sigma^2}{2r^2} + 2r^2\sqrt{a^2+b^2}\sin(\psi)\right]\tilde{p}(r,\psi) \\ &- \left(\alpha r - ar^3 + \frac{\sigma^2}{2r}, 2[b+\sqrt{a^2+b^2}\cos(\psi)]r^2\right)(\nabla\tilde{p}(r,\psi))^\top \\ &+ \nabla\left(\left(\frac{\sigma^2}{2}, \frac{2\sigma^2}{r^2}\right)(\nabla\tilde{p}(r,\psi))^\top\right). \end{aligned}$$

From there we can transform the equation into the following variational problem over an appropriate mesh: find $u \in V$ such that for all $v \in \hat{V}$

$$\int_\Omega A(x)u(x)v(x)\mathrm{d}x - \int_\Omega \langle w_1(x), \nabla u(x)\rangle v(x)\mathrm{d}x - \int_\Omega \langle B(x)\nabla u(x), \nabla v(x)\rangle \mathrm{d}x = 0\,, \quad (3.5.17)$$

where

$$\begin{aligned} B(x) &= \begin{pmatrix} \frac{\sigma^2}{2} & 0 \\ 0 & \frac{2\sigma^2}{x_1^2} \end{pmatrix}, \\ A(x) &= -\alpha + 3ax_1^2 + \frac{\sigma^2}{2x_1^2} + 2x_1^2\sqrt{a^2+b^2}\sin(x_2), \\ w_1(x) &= \left(\alpha x_1 - ax_1^3 + \frac{\sigma^2}{2x_1}, 2[b+\sqrt{a^2+b^2}\cos(x_2)]x_1^2\right). \end{aligned}$$

and the function spaces V and $\hat{V}$ are given by

$$\begin{aligned} V &= \{v \in H^1(\Omega) : u(x_1,0) = u(x_1,2\pi) \ \text{ for all } x_1 \in [\delta, C]\}\,, \\ \hat{V} &= \{v \in H^1(\Omega) : v = \varepsilon_0 \text{ on } \partial\Omega\}. \end{aligned}$$

Note that $H^1(\Omega) \subset L^1(\Omega)$. Hence, normalising the solution u of the above problem gives the approximation $\tilde{u} = u/\|u\|_{L^1}$ of the solution $\tilde{p}$ of (3.5.13).

We have used the Python-based tool FEniCS to solve the weak form (3.5.17) on a mesh over the domain Ω and taken the normalised function $\tilde{u}$ as approximation to $\tilde{p}$. For fixed values of the parameters, formula (3.5.8), with variables from the domain Ω, can be seen as a functional h on the Sobolev space $H^1(\Omega)$, which reads as

$$h(f) = \alpha - 2a\int_\Omega x_1^2 f(x)\mathrm{d}x + \sqrt{a^2+b^2}\int_\Omega x_1^2\sin(x_2)f(x)\mathrm{d}x\,,$$

for $f \in H^1(\Omega)$. Therefore we expect

$$\lambda_1 \approx h(\tilde{u})\,,$$

if C is taken large and δ small enough. Using the FEniCS tool, we obtain rigorous error

bonds for the numerical approximation of $h(\tilde{u})$ depending on the mesh size. Our results for λ_1 derived from this procedure confirm the results displayed in Figure 3.1.

The last step to making this approach a computer-assisted proof consists in a justification of the boundary-conditions in the r-variable. If we can find rigorous error bounds here, we can indeed prove Conjecture 3.1.5.

3.6 Duffing-van der Pol equation and stochastic Brusselator

We want to mention two other examples of Hopf bifurcation systems driven by additive noise which have been investigated in the literature.

Duffing-van der Pol equation

First, we consider the Duffing-van der Pol oscillator which has been studied as a model example for stochastic Hopf bifurcation by Arnold et al. (see e.g. [5]). They mainly looked at the case with multiplicative noise but Schenk-Hoppé [84, 85] has also treated the additive noise problem. The Duffing van der Pol oscillator with additive noise is given by the equation

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + \sigma \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{d}W_t, \quad f(x) = \begin{pmatrix} x_2 \\ \alpha x_1 + \beta x_2 - x_1^3 - x_1^2 x_2, \end{pmatrix} \tag{3.6.1}$$

where W_t denotes Brownian motion, α and β are real parameters and $\sigma \geq 0$. The deterministic system ($\sigma = 0$) exhibits a Hopf bifurcation at $\beta = 0$ if $\alpha < 0$ is fixed.

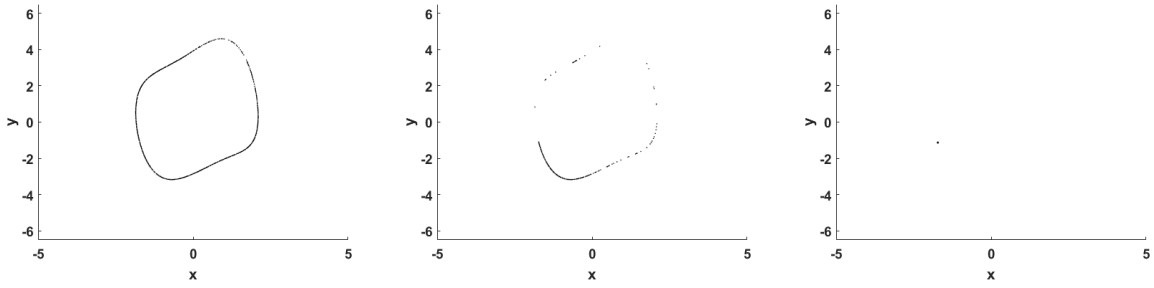
Numerical investigations by Schenk-Hoppé [84] as well as our own investigations suggest a similar picture as for the normal form. The invariant Markov measure $\omega \mapsto \mu_\omega$ seems to be a random Dirac measure supported on the random point attractor which is a singleton. This implies pullback attraction of trajectories to a singleton for almost every initial value. However, the backward dynamics for small enough noise intensity suggest another invariant Dirac measure $\omega \mapsto \nu_\omega$, measurable with respect to the future $\mathcal{F}_0^\infty$. For almost all $\omega \in \Omega$, the support of the measure lies in the interior of a disc $A(\omega)$ whose boundary $\partial A(\omega)$ contains μ_ω. The domain of attraction of ν_ω backward in time is then $\operatorname{int} A(\omega)$ whereas the orbits starting outside the disc $A(\omega)$ explode. Hence, the $A(\omega)$ are the fibres of the global random attractor $A \in \mathcal{F} \times \mathcal{B}(\mathbb{R}^2)$. The random circle $\omega \mapsto \partial A(\omega)$ is believed to be invariant in this scenario. Consequently, there has to be another random Dirac measure $\omega \mapsto \nu'(\omega)$ supported on the random circle and attracting backwards in time.

We conjecture that the numerical observations backward in time are delusive and lead away from the fact that A actually has singleton fibres. The stability backward in time is just observable for small enough noise intensities. Hence, the explanation suggesting

itself is that for small, almost bounded noise the time-reversed equation has an unstable limit cycle around a stable steady state. Only for a very large noise realisation this state is pushed to or outside the unstable limit cycle. Thus, explosion will take place with very small but positive probability.

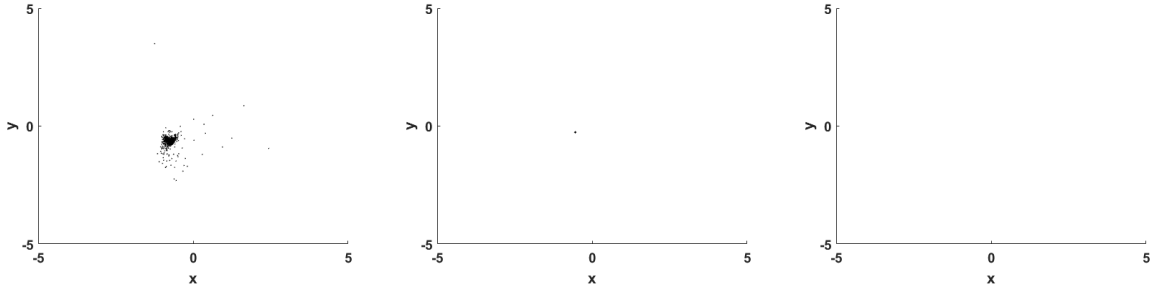
In Figure 3.2, we fix $\alpha = -1$, $\sigma = 1$ and $\beta = 1$ in model (3.6.1) and demonstrate synchronisation of the pullback dynamics. As before, we approximate the stationary probability measure by one typical trajectory and take its distribution as initial points for simulating the pullback dynamics. We observe convergence to a singleton, similarly to Figures 1.2a–1.2d. However, we remark that the synchronisation time is much longer than in the normal form example.

In Figure 3.3 we depict the backward dynamics of model (3.6.1) for the same parameters as in Figure 3.2. Starting with a grid of 10000 points, we observe convergence to a single trajectory up to time $t = -40$ but the explosion of the trajectory before time $t = -100$. This is consistent with our conjecture contradicting the scenario suggested by Schenk-Hoppé.



(a) $t = 20$ *(b)* $t = 50$ *(c)* $t = 150$

Figure 3.2: We choose $\alpha = -1$, $\sigma = 1$ *and* $\beta = 1$ *in model* (3.6.1) *and approximate numerically the random point attractor* $A(\omega)$ *by simulating the pullback dynamics for times* $t = 20, 50, 150$ *and one particular noise realisation* ω*, starting from an approximation of the stationary distribution.*



(a) $t = -10$ *(b)* $t = -40$ *(c)* $t = -100$

Figure 3.3: We choose $\alpha = -1$, $\sigma = 1$ *and* $\beta = 1$ *in model* (3.6.1) *and simulate the backward dynamics up to times* $t = -10, -40, -100$ *for one particular noise realisation* ω*, starting from a uniform mesh of* 10000 *points on* $[-5, 5] \times [-5, 5]$*. We observe convergence to a single trajectory up to time* $t = -40$ *but the explosion of the trajectory for* $t = -100$*.*

The conjecture of set attraction to singletons is backed up by the numerical observation of a negative top Lyapunov exponent λ_1. Analytically it is very hard to show the negativity of λ_1 since we do not know the invariant probability measure explicitly. Further, contraction of large sets is not as easy to show as in the normal form case since the vector field f does not exhibit nice monotonicity properties. One would need to find another way than using Theorem 3.3.2 in order to prove that the random attractor A is a singleton $\mathbb{P}$-a.s..

Stochastic Brusselator

The other example is the stochastic Brusselator studied by Arnold et al. in [3]: let $\alpha, \beta > 0$, W_t standard Brownian motion and $\sigma \geq 0$ the noise intensity. The stochastic Brusselator is given in the Itô sense by

$$\begin{aligned} \mathrm{d}X_t &= (\alpha - (\beta + 1)X_t + X_t^2 Y_t + \frac{\sigma^2}{2} X_t)\mathrm{d}t - \sigma X_t \mathrm{d}W_t, \\ \mathrm{d}Y_t &= (\beta X_t - X_t^2 Y_t - \frac{\sigma^2}{2} X_t)\mathrm{d}t + \sigma X_t \mathrm{d}W_t. \end{aligned} \tag{3.6.2}$$

Note that the white noise is chosen to be one-dimensional in this case. The noise is additive in the sense that it makes the deterministic equilibrium disappear. In the deterministic case, i.e. if $\sigma = 0$, the dynamical system exhibits a Hopf bifurcation at $\beta = \alpha^2 + 1$ for bifurcation parameter β and fixed $\alpha > 0$.

There are no proven results about stationary densities or attractors for this system. However, numerical evidence [3] indicates the following assertion concerning the dynamical behaviour of the stochastic Brusselator:

Conjecture 3.6.1. *The RDS generated by the SDE (3.6.2) has for all parameter values $\alpha, \beta > 0$ and $\sigma > 0$ a unique invariant probability measure μ which corresponds to a unique invariant distribution ρ of the Markov semigroup. Moreover, $\omega \mapsto \mu_\omega$ is almost surely a random Dirac measure, i.e. $\mu_\omega = \delta_{\xi(\omega)}$ for almost all $\omega \in \Omega$ where $\xi : \Omega \to \mathbb{R}^2$ is a random equilibrium of the RDS. The system is exponentially stable in the sense that its top Lyapunov exponent λ_1 is negative. Furthermore, the fibres of the global random attractor of the random dynamical system are given by $A(\omega) := \{\xi(\omega)\}$ almost surely.*

The fact that $\omega \mapsto \mu_\omega$ is a random Dirac measure has been indicated by the same numerical experiments as discussed for the normal form (3.0.1) and the Duffing-van der Pol oscillator (3.6.1). The conjecture that, indeed, $A(\omega) := \{\xi(\omega)\}$ are the fibres of the global random attractor for the stochastic Brusselator is deduced from the additional numerical findings that for any non-random initial value the backward explosion time is finite $\mathbb{P}$-a.s..

3.7 Discussion

We have considered the dynamics of a two-dimensional ordinary differential equation exhibiting a Hopf bifurcation subject to additive white noise and have identified three dynamical phases (recall Figure 1.4): (I) a random attractor with uniform synchronisation of trajectories, (II) a random attractor with non-uniform synchronisation of trajectories and (III) a random attractor without synchronisation of trajectories. The random attractors in phases (I) and (II) are random equilibrium points with negative top Lyapunov exponent while in phase (III) there is a random strange attractor with positive top Lyapunov exponent.

We have analysed the occurrence of the different dynamical phases as a function of the linear stability of the origin (deterministic Hopf bifurcation parameter α) and shear (amplitude-phase coupling parameter b). We have shown that small shear implies synchronisation for any value of the deterministic bifurcation parameter α and obtained that synchronisation cannot be uniform for positive α or in the presence of sufficiently strong shear. We have provided numerical results in support of a conjecture that irrespective of the linear stability of the origin, there is a critical strength of the shear at which the system dynamics loses synchronisation and enters phase (III). We have further dedicated one section to possible methods of proving the conjecture that have not been completely successful yet. The next chapter discusses the simplified model of a perturbed limit cycle for which we are able to show the positivity of the top Lyapunov exponent for large enough shear.

Chapter 4

Bifurcation of a stochastically driven limit cycle

We consider the following model of a stochastically driven limit cycle

$$\begin{aligned} \mathrm{d}y &= -\alpha y\,\mathrm{d}t + \sigma f(\vartheta)\circ \mathrm{d}W_t^1\,, \\ \mathrm{d}\vartheta &= (1+by)\,\mathrm{d}t\ , \end{aligned} \tag{4.0.1}$$

where $(y,\vartheta)\in\mathbb{R}\times\mathbb{S}^1$ are cylindrical amplitude-phase coordinates, and W_t^1 denotes one-dimensional Brownian motion entering the equation as noise of Stratonovich type. In the absence of noise ($\sigma=0$), the ODE (4.0.1) has a globally attracting limit cycle at $y=0$ if $\alpha>0$. In the presence of noise ($\sigma\neq 0$), the amplitude is driven by phase-dependent noise. The real parameter b induces shear: if $b\neq 0$, the phase velocity $\frac{\mathrm{d}\vartheta}{\mathrm{d}t}$ depends on the amplitude y.

The stable limit cycle turns into a random attractor if $\sigma\neq 0$. The main question we address in this chapter concerns the nature of this random attractor. The crucial quantity is the sign of the first Lyapunov exponent λ_1 with respect to the invariant measure associated to the random attractor. To facilitate the analysis, we choose $f:\mathbb{S}^1\simeq[0,1)\to\mathbb{R}$ to be continuous and piecewise linear with constant absolute value of the derivative almost everywhere. The simplest example is given by

$$f(\vartheta)=\begin{cases}\vartheta & \text{if } \vartheta\leq\frac{1}{2}\,,\\ (1-\vartheta) & \text{if } \vartheta\geq\frac{1}{2}\,.\end{cases} \tag{4.0.2}$$

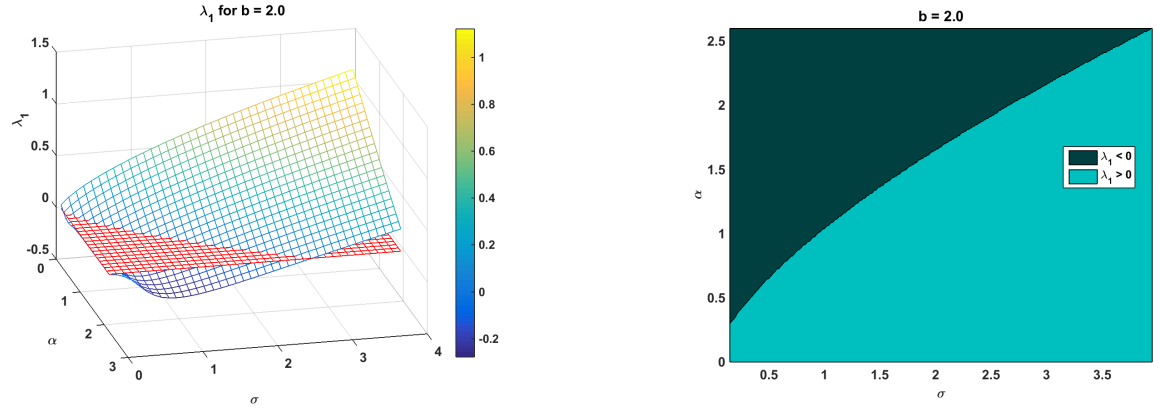
With this choice of the amplitude-phase coupling we obtain the following bifurcation result. We will show that the result stays robust under perturbations of f that smoothen this function.

Theorem 4.0.1 (Change of sign of λ_1)**.** *Consider the SDE (4.0.1) with f given by (4.0.2). Then there is $0<c_0\approx 0.2823$ such that for all $\alpha>0$ and $b\neq 0$ the number $\sigma_0(\alpha,b)=$*

$\frac{\alpha^{3/2}}{c_0^{1/2}|b|} > 0$ *is the unique value of* σ *where the top Lyapunov exponent* $\lambda_1(\alpha, b, \sigma)$ *of (4.0.1) changes its sign. In more detail, we have*

$$\lambda_1(\alpha, b, \sigma) \begin{cases} < 0 & \text{if } 0 < \sigma < \sigma_0(\alpha, b)\,, \\ = 0 & \text{if } \sigma = \sigma_0(\alpha, b)\,, \\ > 0 & \text{if } \sigma > \sigma_0(\alpha, b)\,. \end{cases}$$

As long as $b, \sigma \neq 0$, the amplitude variable y can be rescaled so that the shear parameter becomes equal to 1 and the effective noise-amplitude becomes σb. Hence, the above result also holds with the roles of σ and b interchanged. The fact that $\sigma_0(\alpha, b)$ is an increasing function of α is illustrated in Figure 4.1. Figure 4.1a depicts λ_1 as a function of α and σ for fixed $b = 2$. Figure 4.1b displays the corresponding areas of positive and negative top Lyapunov exponent in the (σ, α)-parameter space where $\lambda_1 = 0$ along the curve $\{(\sigma_0(\alpha, 2), \alpha)\}$ separating the two areas. The curve is increasing away from $(0, 0)$. If $\sigma = 0$, we clearly have $\lambda_1 = 0$ for all $\alpha > 0$. The case $\alpha = 0$ is obviously not of any interest in our model.



(a) *(b)*

Figure 4.1: In Figure 4.1a we depict the top Lyapunov exponent λ_1*, computed using (4.1.14), as a function of* α *and* σ *for fixed* $b = 2$*. The red mesh demarcates the level* $\lambda_1 = 0$*. Figure 4.1b shows the corresponding areas of positive and negative* λ_1 *in the* (σ, α)*-parameter space being separated by the curve* $\{(\sigma_0(\alpha, 2), \alpha)\}$ *which increases away from* $(0, 0)$*. The picture does not display* $\sigma = 0$*: in this case,* $\lambda_1 = 0$ *for all* $\alpha > 0$*.*

If the top Lyapunov exponent is negative, it turns out that the (weak) random point attractor is an attracting random equilibrium, i.e. its fibres are singletons almost surely. It is a direct consequence that, in this case, the random dynamical system generated by (4.0.1) exhibits *weak synchronisation*, which means that for all $Z_1, Z_2 \in \mathbb{R} \times \mathbb{S}^1$, we have

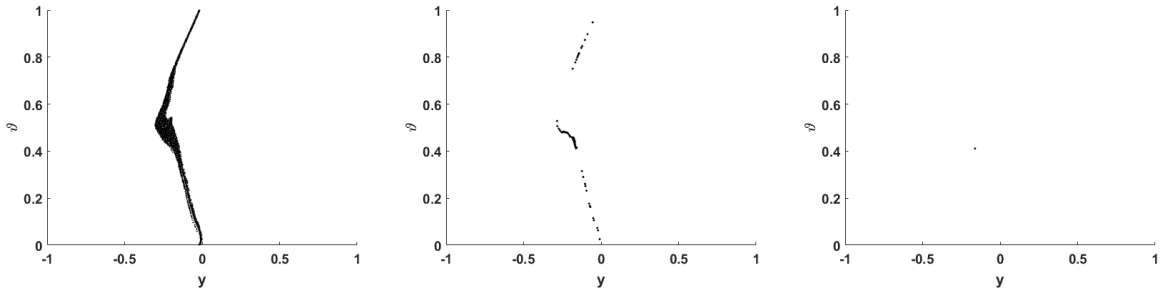
$$\lim_{t\to\infty} d(\varphi(t, \omega, Z_1), \varphi(t, \omega, Z_2)) = 0 \quad \text{in probability.} \tag{4.0.3}$$

As before, the random attractor associated with a positive top Lyapunov exponents is

referred to as random strange attractor. We will give a more thorough description of such an attractor in Chapter 5. Summarising, Theorem 4.0.1 implies the following dynamical characterisation of the bifurcation:

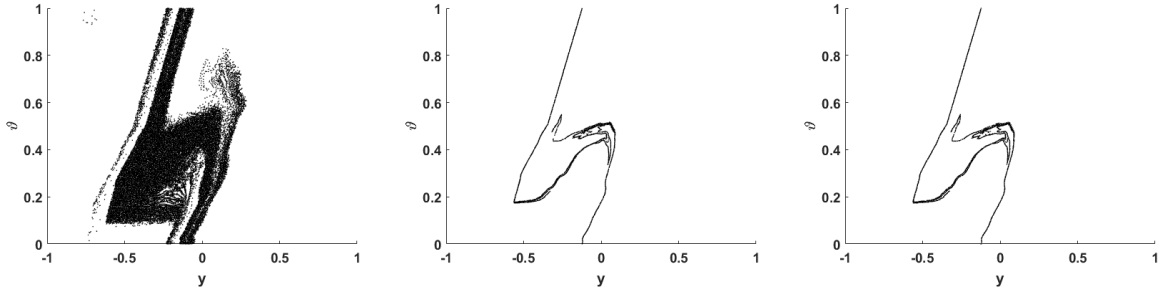
Corollary 4.0.2. *If $0 < \sigma < \sigma_0(\alpha, b)$, the (minimal weak) random point attractor of* (4.0.1) *is an attracting random equilibrium and* (4.0.1) *exhibits weak synchronisation. If $\sigma > \sigma_0(\alpha, b)$ the (minimal weak) random point attractor of system* (4.0.1) *is a random strange attractor (and not an attracting random equilibrium).*

We illustrate the shape of the random attractor for $\lambda_1 < 0$ and $\lambda_1 > 0$. In Figure 4.2 we choose the parameters such that $\lambda_1 < 0$ and observe convergence of trajectories to a singleton attractor in accordance with Corollary 4.0.2. Figure 4.3 displays the chaotic attractor in case $\lambda_1 > 0$, also in consistence with Corollary 4.0.2. In Section 4.2.2 we give an explanation of the geometric mechanism that leads to such a strange attractor.



(a) $t = 1$ *(b) $t = 10$* *(c) $t = 20$*

Figure 4.2: We choose $\alpha = 3$, $\sigma = 1$ and $b = 3$ in model (4.0.1) *and approximate numerically the random point attractor $\tilde{A}(\omega) = \operatorname{supp} \mu_\omega$, as in Figure 4.3, by simulating the pullback dynamics for times $t = -1, -10, -20$ and one particular noise realisation ω. The parameters are chosen such that we know $\lambda_1 < 0$ from formula* (4.1.14).



(a) $t = 1$ *(b) $t = 50$* *(c) $t = 100$*

Figure 4.3: We set $\sigma = 2, \alpha = 1.5, b = 3$ in model (4.0.1) *and approximate numerically the support of the invariant Markov measure $\mu_\omega = \lim_{t\to\infty} \varphi(t, \theta_{-t}\omega)\rho$ by simulating the pullback dynamics from times -1, -50 and -100 on. The parameters are chosen such that $\lambda_1 > 0$ according to formula* (4.1.14).

The main technical challenge we address in this chapter is to establish the existence of positive top Lyapunov exponents which remained a conjecture for model (3.0.1) in the

previous chapter. We have explained in the Introduction why this is usually a tremendously difficult task. In our setting, the choice of f in (4.0.2) is crucial to establish rigorous lower bounds on the top Lyapunov exponent λ_1.

The remainder of the chapter is organized as follows. Section 4.1 provides the analysis of Lyapunov exponents for our model: Subsection 4.1.1 introduces the model on the cylinder within the framework of random dynamical systems and establishes the necessary theoretical concepts. Subsection 4.1.2 discusses the Furstenberg–Khasminskii formula for the top Lyapunov exponent and in Subsection 4.1.3, we derive a formula for the top Lyapunov exponent λ_1 in the case of the specific choice of the diffusion map f. The main result concerning the change of sign of λ_1 is proven in Section 4.2 and its consequences are discussed. We give illustrations of λ_1 in dependence on the parameters, confirm a scaling conjecture by Lin & Young and explain why the results stay robust under smooth perturbations of f. We conclude with a short summary of the results in Section 4.3.

4.1 Analysis of the top Lyapunov exponent

4.1.1 The model as a random dynamical system

Consider the stochastic differential equation of Stratonovich type (4.0.1). We investigate this model in the framework of random dynamical systems as introduced in Chapter 2. The probability space Ω is $C_0(\mathbb{R},\mathbb{R}) = \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra and $\mathbb{P}$ is the two-sided Wiener measure. $\mathcal{F}_s^t$ is the σ-algebra generated by $\xi_u - \xi_v$ for $-\infty \le s \le v \le u \le t \le \infty$, where $\xi_s : \Omega \to \mathbb{R}$ is defined as $\xi_s(\omega) = \omega(s)$, and θ_t is the ergodic shift

$$(\theta_t\omega)(s) = \omega(s+t) - \omega(t)\,.$$

For each $\omega \in \Omega$ and the initial value $Z \in \mathbb{R} \times \mathbb{S}^1$, a continuous function $\varphi(\cdot,\omega,Z) : [0,T] \to \mathbb{R}\times\mathbb{S}^1$, where $T > 0$, is called a solution of (4.0.1) if it satisfies the corresponding integral equation. We assume that $f : [0,1] \to \mathbb{R}$ is a Lipschitz continuous function with $f(0) = f(1)$, which is differentiable everywhere except for finitely many points as in (4.0.2), and that the three parameters fulfil $\alpha > 0$, $\sigma > 0$ and $b \neq 0$. Note that the equation reads the same in Itô form according to the Itô–Stratonovich conversion formula (cf. Appendix A.1).

Since the drift and diffusion coefficients are globally Lipschitz continuous, the stochastic differential equation (4.0.1) generates a continuous random dynamical system (θ,φ) which is forward complete, i.e. the solutions do not blow up. The random dynamical system (θ,φ) is also a skew product flow $\Theta = (\theta,\varphi)$, which is a measurable dynamical system on the extended phase space $\Omega\times X$ (see chapter 2). The skew product flow Θ possesses an ergodic invariant Markov measure μ which is associated to the unique invariant measure (also called stationary measure) for the corresponding Markov semigroup (see chapter 2).

Furthermore, the random dynamical system induced by (4.0.1) has a random attractor $A \in \mathcal{F} \times \mathcal{B}(\mathbb{R} \times \mathbb{S}^1)$, since the diffusion is bounded and the drift is strongly dissipative. The disintegrations μ_ω of the ergodic invariant measure μ are supported on the fibres $A(\omega)$ of this random attractor as explained in Remark 2.2.5. We summarize these assertions in the following proposition:

Proposition 4.1.1. *For all $\alpha, \sigma > 0, b \neq 0$, $Z \in \mathbb{R} \times \mathbb{S}^1$ and almost all $\omega \in \Omega$ there exists a unique solution $\varphi(t, \omega, Z)$ of (4.0.1) starting from Z at time $t = 0$. Furthermore the following statements hold:*

(i) (θ, φ) is a random dynamical system where φ is a cocycle over the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}}$.

(ii) The random dynamical system (θ, φ) has a unique invariant ergodic probability measure μ with $\mathcal{F}^0_{-\infty}$-measurable disintegrations.

(iii) There exists a random attractor $A \in \mathcal{F} \times \mathcal{B}(\mathbb{R} \times \mathbb{S}^1)$ with respect to all compact tempered random measurable sets which is also measurable with respect to $\mathcal{F}^0_{-\infty}$, i.e. the past of the system.

Proof. (i) Since g and f are globally Lipschitz continuous Condition (A1) in [37] is clearly satisfied and the claim follows.

(ii) We need to show that the stationary Fokker-Planck equation corresponding with (4.0.1) has a unique positive normalized solution. In our model, this is not directly clear as the degeneracy of the noise implies that the Fokker-Planck operator is not strictly elliptic. However, the SDE (4.0.1) satisfies Hörmander's hypoellipticity condition almost everywhere as long as $f(\theta) \neq 0$ almost everywhere since the noise spreads out in the ϑ-variable via the term by. In more detail, if we denote the vector fields

$$X_0 = \alpha y \partial_y + (1 + by)\partial_\vartheta$$

and

$$X_1 = \sigma f(\vartheta)\partial_y \,,$$

we calculate

$$[X_0, X_1] = -\sigma b f(\vartheta)\partial_\vartheta + (-\alpha\sigma f(\vartheta) + \sigma(1 + by)\partial_\vartheta f(\vartheta))\partial_y \,.$$

Therefore we, indeed, have

$$\operatorname{span}\{X_0, [X_0, X_1]\} = T\mathbb{R}^2 \,.$$

Existence of a stationary distribution with positive density follows as in Lin & Young [73]. Uniqueness and ergodicity follow from [71][Theorem 2.7]. Hence, the claim is a

direct consequence of the one-to-one correspondence between stationary measures of the stochastic process and invariant Markov measures of the random dynamical system, as explained in Section 2.1.

(iii) Similarly to Section 3.2, we can transform the stochastic differential equation into a random differential equation by using the stationary Ornstein-Uhlenbeck process $(Z^*(\theta_t\cdot))_{t\in\mathbb{R}}$. In this case, the suitable transformation reads

$$T : \Omega \times (\mathbb{R} \times \mathbb{S}^1) \to (\mathbb{R} \times \mathbb{S}^1)\,, \quad T(\omega,(y,\vartheta)) = (y + \sigma f(\vartheta)Z^*(\omega),\vartheta)\,.$$

Again if $\Psi : \mathbb{R}_0^+ \times \Omega \times (\mathbb{R} \times \mathbb{S}^1) \to \mathbb{R} \times \mathbb{S}^1$ solves the corresponding random differential equation and possesses a random attractor, the random dynamical system given by

$$\varphi(t,\omega,Z) := T(\theta_t\omega, \Psi(t,\omega,T(\omega)^{-1}Z))$$

solves (4.0.1) and has a random attractor (see [58, Lemma 2.4 and Theorem 2.5]. The fact that Ψ satisfies these properties can be seen similarly to Lemma 3.2.2 but even more simply in this case due to the lack of nonlinearities. □

Recall Oseledets' Multiplicative Ergodic Theorem from chapter 2, which implies the existence of Lyapunov exponents describing stability properties of a differentiable random dynamical system. In the situation of the Stratonovich SDE

$$\mathrm{d}X_t = f_0(X_t)\mathrm{d}t + \sum_{i=1}^{m} f_j(X_t) \circ \mathrm{d}W_t^j$$

on a smooth manifold X, the Jacobian $\mathrm{D}\varphi(t,\omega,x)$ with respect to the third variable of the cocycle $\varphi(t,\omega,x)$ is a linear cocycle over the skew product flow $\Theta = (\theta,\varphi)$. The Jacobian $\mathrm{D}\varphi(t,\omega,x)$ applied to an initial condition $v_0 \in T_xX$ solves uniquely the variational equation on $T_xX \cong \mathbb{R}^d$, given by

$$\mathrm{d}v = \mathrm{D}f_0(\varphi(t,\omega)x)v\,\mathrm{d}t + \sum_{j=1}^{m} \mathrm{D}f_j(\varphi(t,\omega)x)v \circ \mathrm{d}W_t^j\,, \quad \text{where } v \in T_xX\,. \tag{4.1.1}$$

The random dynamical system (θ,φ) induced by (4.0.1) has an ergodic invariant measure μ as shown in Proposition 4.1.1. We further see in the next section that it satisfies the integrability condition

$$\sup_{0\le t\le 1} \ln^+ \|\mathrm{D}\varphi(t,\omega,x)\| \in L^1(\mu).$$

So the Multiplicative Ergodic Theorem guarantees the existence of a Θ-forward invariant set $\Delta \subset \Omega \times \mathbb{R} \times \mathbb{S}^1$ with $\mu(\Delta) = 1$ and the Lyapunov exponents $\lambda_1 > \cdots > \lambda_p$ with respect to μ where $p \le 2$. Denote by d_i the multiplicity of the Lypaunov exponent λ_i.

The tangent space $T_xX \cong \mathbb{R}^d$ admits a filtration

$$\mathbb{R}^d = V_1(\omega, x) \supsetneq \cdots \supsetneq V_{p+1}(\omega, x) = \{0\}\,,$$

such that for all $(\omega, x) \in \Delta, 0 \neq v \in T_xX \cong \mathbb{R}^d$, the Lyapunov exponent $\lambda(\omega, x, v)$ defined by

$$\lambda(\omega, x, v) = \lim_{t\to\infty} \frac{1}{t} \ln \|\mathrm{D}\varphi(t, \omega, x)v\|$$

exists and

$$\lambda(\omega, x, v) = \lambda_i \quad \Longleftrightarrow \quad v \in V_i(\omega, x) \setminus V_{i+1}(\omega, x) \quad \text{for all } i \in \{1, \ldots, p\}\,.$$

Further recall that for all $(\omega, x) \in \Delta$

$$\lim_{t\to\infty} \frac{1}{t} \ln \det \mathrm{D}\varphi(t, \omega, x) = \sum_{i=1}^{p} d_i\lambda_i\,. \tag{4.1.2}$$

4.1.2 The Furstenberg–Khasminskii formula

In the following, we calculate the top Lyapunov exponent λ_1 for the random dynamical system induced by (4.0.1). We consider the corresponding variational equation describing the flow on the tangent space $T_x(\mathbb{R} \times \mathbb{S}^1) \cong \mathbb{R}^2$ along trajectories of (4.0.1). We assume that $f \in C^1$ for now, although we consider a more general case later. The variational equation (2.3.2) reads as

$$\mathrm{d}v = \underbrace{\begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix}}_{=:A} v\,\mathrm{d}t + \underbrace{\begin{pmatrix} 0 & \sigma f'(\vartheta) \\ 0 & 0 \end{pmatrix}}_{=:B} v \circ \mathrm{d}W_t^1\,. \tag{4.1.3}$$

We omit the (t, ω)-dependence of ϑ and B. Note that the integrability condition of the Multiplicative Ergodic Theorem is clearly satisfied since all coefficients are bounded. We immediately make the following observation:

Proposition 4.1.2. *Let λ_Σ be the sum of the Lyapunov exponents of the random dynamical system induced by* (4.0.1) *which is, according to formula 4.1.2, given by*

$$\lambda_\Sigma := \lim_{t\to\infty} \frac{1}{t} \ln \det \mathrm{D}\varphi(t, \omega, x) \quad \textit{for all } (\omega, x) \in \Delta\,.$$

Then, $\lambda_\Sigma = -\alpha < 0$ for all $\sigma \neq 0$. Consequently, the Markov measure $\omega \mapsto \mu_\omega$ is singular with respect to the Lebesgue measure on $\mathbb{R} \times \mathbb{S}^1$.

Proof. It is immediate from 4.1.3 that $\lambda_\Sigma = -\alpha < 0$. Using [66, Proposition 1] and [10, Theorem 4.15], we obtain the singularity of the Markov measure. □

Because of the linearity of (4.1.3), we introduce the change of variables $r = \|v\|$ and $s = v/r$, so that s lies on the unit circle. Its dynamics are given by

$$\begin{aligned} \mathrm{d}s &= (As - \langle s, As\rangle s)\,\mathrm{d}t + (Bs - \langle s, Bs\rangle s) \circ \mathrm{d}W_t^1 \\ &= \begin{pmatrix} -\alpha s_1 - s_1(-\alpha s_1^2 + b s_1 s_2) \\ b s_1 - s_2(-\alpha s_1^2 + b s_1 s_2) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \sigma f'(\vartheta) s_2 - s_1 \sigma f'(\vartheta) s_1 s_2 \\ -s_2 \sigma f'(\vartheta) s_1 s_2 \end{pmatrix} \circ \mathrm{d}W_t^1 \,. \end{aligned}$$

Following Section 2.3 we use the Itô-Stratonovich formula to observe similarly to (2.3.7) that

$$\frac{1}{t} \ln r_t = \frac{1}{t} \int_0^t \left[h_A(s_\tau) + k_B(s_\tau) \right] \mathrm{d}\tau + \mathcal{O}(t^{-1/2}) \,, \tag{4.1.4}$$

where the functions h_A and k_B are given by

$$\begin{aligned} h_A(s) &= \langle s, As\rangle = -\alpha s_1^2 + b s_1 s_2 \,, \\ k_B(s) &= \frac{1}{2} \langle (B + B^*)\, s, Bs\rangle - \langle s, Bs\rangle^2 = \frac{1}{2} \sigma^2 f'(\vartheta)^2 s_2^2 - \sigma^2 f'(\vartheta)^2 s_1^2 s_2^2 \,. \end{aligned}$$

The Furstenberg–Khasminskii formula for the top Lyapunov exponent (see chapter 2) is given by

$$\lambda_1 = \int_{\mathbb{R}} \int_{[0,1]} \int_{\mathbb{S}^1} (h_A(s) + k_B(s))\, \rho(\mathrm{d}s, \mathrm{d}\vartheta, \mathrm{d}y), \tag{4.1.5}$$

where ρ is the joint invariant measure for the diffusion s on the unit circle and the processes ϑ and y induced by (4.0.1). Similarly to the calculations in [53], we change variables to $s = (\cos\phi, \sin\phi)$. Note that the functions h_A and k_B are π-periodic, which implies that the formula (4.1.5) for the top Lyapunov exponent reads as

$$\begin{aligned} \lambda_1 = \int_{\mathbb{R}\times[0,1]\times[0,\pi]} \Big(& -\alpha \cos^2\phi + b \cos\phi \sin\phi \\ & + \frac{1}{2} \sigma^2 f'(\vartheta)^2 \sin^2\phi (1 - 2\cos^2\phi) \Big) \tilde{\rho}(\mathrm{d}\phi, \mathrm{d}\vartheta, \mathrm{d}y), \end{aligned} \tag{4.1.6}$$

where $\tilde{\rho}$ denotes the corresponding image measure of ρ. The SDE determining the dynamics of $\phi \in [0, \pi)$ reads as

$$\mathrm{d}\phi = -\frac{1}{\sin\phi} \mathrm{d}s_1 = (\alpha \cos\phi \sin\phi + b \cos^2\phi)\mathrm{d}t - \sigma f'(\vartheta) \sin^2\phi \circ \mathrm{d}W_t^1 \,, \tag{4.1.7}$$

where we denote

$$c(\phi, \vartheta) = \sigma f'(\vartheta) \sin^2\phi \quad \text{and} \quad d(\phi) = \alpha \cos\phi \sin\phi + b \cos^2\phi \,. \tag{4.1.8}$$

The integrand in (4.1.4) and thereby in (4.1.6) only depends on ϕ and not on ϑ and y if $f'(\vartheta)^2$ is constant, and in addition to that, the dependence on ϑ in the Fokker–Planck

equation for ϕ, is restricted to $f'(\vartheta)^2$. This means that the calculation of λ_1 becomes much simpler if $f'(\vartheta)^2$ is constant, an observation that we exploit in the following.

4.1.3 Explicit formula for the top Lyapunov exponent

We continue the analysis of the top Lyapunov exponent under the assumption that $f : [0,1] \to \mathbb{R}$ is given by (4.0.2). Importantly, $f'(\vartheta)^2$ is constant in this special case and our results hold in fact for every continuous and piecewise linear f with constant absolute value of the derivative almost everywhere.

The map is not differentiable at $\frac{1}{2}$ and 0, and we verify that does not cause any problems. We need the following results to justify the variational equation defining $\mathrm{D}\varphi$:

Lemma 4.1.3. *Let $W : \mathbb{R}_0^+ \times \Omega \to \mathbb{R}$ denote the canonical real-valued Wiener process, and let $X : \mathbb{R}_0^+ \times \Omega \to [0,1]$ be a stochastic process adapted to the natural filtration of the Wiener process. Furthermore, suppose there exists a measurable set $A \subset [0,1]$ such that*

$$\mathbb{P}\big(\{\omega \in \Omega : \textstyle\int_0^t \mathbb{1}_{\{X_u \in A\}} \, \mathrm{d}u = 0\}\big) = 1 \quad \textit{for all } t > 0\,, \tag{4.1.9}$$

i.e. A is visited only on a measure zero set with full probability. Consider a measurable function $g : [0,1] \to [0,1]$ such that $g = 0$ on $[0,1] \setminus A$. Then

$$\int_0^t g(X_u)\,\mathrm{d}W_u = 0 \quad \textit{almost surely for all } t > 0\,.$$

Proof. The statement follows directly from Itô's isometry

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t g(X_u)\mathrm{d}W_u\right)^2\right] &= \mathbb{E}\left[\int_0^t g(X_u)^2\mathrm{d}u\right] \\ &= \mathbb{E}\left[\int_0^t \left(g(X_u)^2\mathbb{1}_{\{X_u\in A\}} + g(X_u)^2\mathbb{1}_{\{X_u\in[0,1]\setminus A\}}\right)\mathrm{d}u\right] = 0\,,\end{aligned}$$

where the last equality follows immediately from (4.1.9) and $g = 0$ on $[0,1] \setminus A$. We conclude

$$\left(\int_0^t g(X_u)\,\mathrm{d}W_u\right)^2 = 0 \quad \text{almost surely}$$

due to nonnegativity, and the claim follows. □

Proposition 4.1.4. *Let f' denote the weak derivative of f as given by (4.0.2). Then the choice of representative of f' by determining $f'(\frac{1}{2})$ and $f'(0)$ does not affect the solution to the variational equation (4.1.3).*

Proof. First, we show that

$$\mathbb{P}\big(\{\omega \in \Omega : \textstyle\int_0^t \mathbb{1}_{\{\vartheta_u = 1/2\}} \, \mathrm{d}u = 0\}\big) = 1 \quad \text{for all } t > 0$$

by assuming the contrary to obtain a contradiction. As ϑ is a continuously differentiable process, this implies that $\vartheta_u = \frac{1}{2}$ for $u \in [t^*, t^* + \varepsilon]$ for some $t^* \in (0, t)$ and $\varepsilon > 0$ with positive probability. This leads to $y(u) = -\frac{1}{b} \mod 1$ for $u \in (t^*, t^* + \varepsilon)$ with positive probability. However, this implies that the continuous process y_u for $u \in (t^*, t^* + \varepsilon)$ given by

$$\mathrm{d}y = -\alpha y \,\mathrm{d}u + \sigma \,\mathrm{d}W_u$$

is constant with positive probability. This contradicts its definition as an Ornstein–Uhlenbeck process. The same reasoning obviously holds for $\theta = 0$.

Let $f_1' = f_2' = f'$ on $(0,1) \setminus \{\frac{1}{2}\}$ and assign arbitrary values at $\frac{1}{2}$ and 0. Define

$$\begin{aligned}
\mathrm{d}v &= \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} v \,\mathrm{d}t + \begin{pmatrix} 0 & \sigma f_1'(\vartheta) \\ 0 & 0 \end{pmatrix} v \circ \mathrm{d}W_t^1 \,, \\
dw &= \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} w \,\mathrm{d}t + \begin{pmatrix} 0 & \sigma f_2'(\vartheta) \\ 0 & 0 \end{pmatrix} w \circ \mathrm{d}W_t^1 \,.
\end{aligned}$$

We apply Lemma 4.1.3 by choosing $X_u = \vartheta_u$ and $g(\vartheta_u) = f_1'(\vartheta_u) - f_2'(\vartheta_u)$ to conclude that

$$\int_0^t f_1'(\vartheta_u) \,\mathrm{d}W_u = \int_0^t f_2'(\vartheta_u) \,\mathrm{d}W_u \quad \text{almost surely.}$$

As we do not have an Itô–Stratonovich correction in this case, we can infer that $v_t = w_t$ almost surely for all $t > 0$. □

We view f' in the weak sense, disregarding the points $\frac{1}{2}$ and 0, and we define $f'(\vartheta) = \operatorname{sign}(\frac{1}{2} - \vartheta)$, where

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0 \,, \\ -1 & \text{if } x < 0 \,. \end{cases}$$

By Proposition 4.1.4, $\mathrm{D}\varphi(t, \omega, x)$ does not depend on the choice of $f'(\frac{1}{2})$, so the variational equation (4.1.3) becomes

$$\mathrm{d}v = \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} v \,\mathrm{d}t + \begin{pmatrix} 0 & \sigma \operatorname{sign}(\frac{1}{2} - \vartheta_t) \\ 0 & 0 \end{pmatrix} v \circ \mathrm{d}W_t^1 \,. \tag{4.1.10}$$

We derive the following formula for the first Lyapunov exponent in this case:

Proposition 4.1.5. *The top Lyapunov exponent of system (4.0.1) with f as defined in (4.0.2) is given by*

$$\lambda_1 = \int_0^\pi q(\phi) p(\phi) \,\mathrm{d}\phi \,, \tag{4.1.11}$$

where $q(\phi) := -\alpha \cos^2 \phi + b \cos \phi \sin \phi + \frac{1}{2}\sigma^2 (1 - 2\cos^2 \phi) \sin^2 \phi$, and $p(\phi)$ is the solution of the stationary Fokker–Planck equation $\mathcal{L}^* p = 0$. $\mathcal{L}^*$ is the formal L^2-adjoint of the

generator $\mathcal{L}$, which is given by

$$\mathcal{L}g(\phi) = \left(d(\phi) + \frac{1}{2}\tilde{c}(\phi)\tilde{c}'(\phi) \right) g'(\phi) + \frac{1}{2}\tilde{c}^2(\phi)g''(\phi)\,, \tag{4.1.12}$$

where $d = d(\phi)$ is defined as in (4.1.8)*, and $\tilde{c}(\phi) := \sigma \sin^2 \phi$.*

Proof. Note that in our special case, the function c from (4.1.8) reads as

$$c(\phi, \vartheta) = \sigma \operatorname{sign}(\tfrac{1}{2} - \vartheta) \sin^2 \phi\,,$$

which implies that both $c(\phi,\vartheta)c'(\phi,\vartheta)$ and $c^2(\phi,\vartheta)$ do not depend on ϑ and read as $\tilde{c}\tilde{c}'$ and $\tilde{c}^2$, respectively. Consider the SDE for the process $\phi(t)$ in Itô form

$$\mathrm{d}\phi = r(\phi)\mathrm{d}t + c(\phi,\vartheta)\mathrm{d}W_t\,,$$

where

$$r(\phi) = d(\phi) + \frac{1}{2}c(\phi,\vartheta)c'(\phi,\vartheta) = d(\phi) + \frac{1}{2}\tilde{c}(\phi)\tilde{c}'(\phi)\,.$$

As the coefficients of the SDE are smooth in ϕ, we consider the kinetic equation for the probability density function of the process $\phi(t)$ (cf. [87])

$$\frac{\partial p(\psi,t)}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \psi^n}[a_n(\psi,t)p(\psi,t)]\,,$$

where

$$a_n(\psi,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\mathbb{E}\left[(\phi(t+\Delta t) - \phi(t))^n | \phi(t) = \psi\right] \quad \text{for all } n \in \mathrm{N}\,.$$

Pick Δt small, denote $\Delta W_t = W(t+\Delta) - W(t)$ and recall that $\mathbb{E}[\Delta W_t] = 0$ and $\mathbb{E}[(\Delta W_t)^2] = \Delta t$. Observe that

$$\phi(t+\Delta t) - \phi(t) = r(\phi(t))\Delta t + c(\phi(t),\vartheta(t))\Delta W_t + o(\Delta t)\,,$$

and

$$\begin{aligned}(\phi(t+\Delta t) - \phi(t))^2 = {} & r^2(\phi(t))(\Delta t)^2 + c^2(\phi(t),\vartheta(t))(\Delta W_t)^2 \\ & + 2r(\phi(t))c(\phi(t),\vartheta(t))\Delta W_t \Delta t + o(\Delta t)\,.\end{aligned}$$

Since ΔW_t is independent from $\phi(t)$ and $\vartheta(t)$, we obtain that

$$a_1(\psi,t) = r(\psi) \quad \text{and} \quad a_2(\psi,t) = \tilde{c}^2(\psi)\,.$$

We can see immediately from above that $a_n(\psi,t) = 0$ for $n \geq 3$. This proves (4.1.12), and (4.1.11) follows from (4.1.6). □

In this case, the stationary Fokker–Planck equation reduces to a linear nonautonomous ordinary differential equation for $p = p(\phi)$ defined on $[0, \pi)$ with periodic boundary conditions:

$$-\left(\frac{1}{2}\tilde{c}^2 p\right)' + \left(d + \frac{1}{2}\tilde{c}\tilde{c}'\right) p = \kappa,$$

where the constant κ has to be determined from the boundary and the normalization condition. The ordinary differential equations is given in explicit form as

$$\begin{aligned} p' &= \left(\frac{2d(\phi)}{\tilde{c}^2(\phi)} - \frac{\tilde{c}'(\phi)}{\tilde{c}(\phi)}\right) p + \frac{2\kappa}{\tilde{c}^2(\phi)} \\ &= \left(2\frac{\alpha}{\sigma^2}\frac{1}{\sin^2\phi}\tan^{-1}\phi + 2\frac{b}{\sigma^2}\frac{1}{\sin^2\phi}\tan^{-2}\phi - \frac{\tilde{c}'(\phi)}{\tilde{c}(\phi)}\right) p + \frac{2\kappa}{\tilde{c}^2(\phi)}\,. \end{aligned}$$

The solution of this equation follows from the variation of constants formula, and is given by

$$p(\phi) = \frac{G(\phi)\int_\phi^\pi \frac{2}{\tilde{c}^2(\psi)G(\psi)}\,\mathrm{d}\psi}{\int_0^\pi G(\phi)\int_\phi^\pi \frac{2}{\tilde{c}^2(\psi)G(\psi)}\,\mathrm{d}\psi\,\mathrm{d}\phi}\,, \tag{4.1.13}$$

where

$$G(\phi) = \frac{1}{\tilde{c}(\phi)}\exp\left(-\frac{1}{\sigma^2}\left[\alpha\tan^{-2}\phi + \frac{2}{3}b\tan^{-3}\phi\right]\right).$$

The derivation of a closed formula for λ_1 and λ_2 using (4.1.13) for the stationary density of the process ϕ_t closely follows Imkeller and Lederer [53]. It can also be seen as a special case of the more general formulas obtained in [54].

Theorem 4.1.6 (Formula for λ_1 and λ_2)**.** *Consider the stochastic differential equation* (4.0.1)*, where the function f is of the form* (4.0.2)*. Then the two Lyapunov exponents are given by*

$$\lambda_1(\alpha, b, \sigma) = -\frac{\alpha}{2} + \frac{b^2\sigma^2}{2}\int_0^\infty v\; m_{\sigma,b,\alpha}(v)\,\mathrm{d}v\,, \tag{4.1.14}$$

$$\lambda_2(\alpha, b, \sigma) = -\frac{\alpha}{2} - \frac{b^2\sigma^2}{2}\int_0^\infty v\; m_{\sigma,b,\alpha}(v)\,\mathrm{d}v\,. \tag{4.1.15}$$

where

$$m_{\sigma,b,\alpha}(v) = \frac{\frac{1}{\sqrt{v}}\exp\left(-\frac{\sigma^4 b^4}{6}v^3 + \frac{\alpha^2}{2}v\right)}{\int_0^\infty \frac{1}{\sqrt{u}}\exp\left(-\frac{\sigma^4 b^4}{6}u^3 + \frac{\alpha^2}{2}u\right)\mathrm{d}u}. \tag{4.1.16}$$

Proof. We define the function $g : [0, \phi) \to \mathbb{R} \cup \{\infty\}$ by

$$\begin{aligned} g(\phi) &:= -\ln\sin\phi \quad \text{for all } \phi \in (0, \pi)\,, \\ g(0) &:= \infty\,, \end{aligned}$$

and apply this function formally to the generator as given in (4.1.12):

$$\begin{aligned}\mathcal{L}g(\phi) &= \left(\alpha\cos\phi\sin\phi + b\cos^2\phi + \sigma^2\sin^3\phi\cos\phi\right)(-\tan^{-1}\phi) + \frac{1}{2}\sigma^2\sin^4\phi\frac{1}{\sin^2\phi}\\ &= -b\tan^{-1}\phi + q(\phi).\end{aligned}$$

This can be made precise by choosing suitable C^∞-functions to approximate g. Observe that

$$0 = \int_0^\pi g\ \mathcal{L}^* p\ \mathrm{d}\phi = \int_0^\pi \mathcal{L}g\ p\ \mathrm{d}\phi = \int_0^\pi \left(-b\tan^{-1}+q\right) p\ \mathrm{d}\phi\,,$$

and we conclude that

$$\lambda_1 = b\int_0^\pi \tan^{-1}(\phi)p(\phi)\mathrm{d}\phi\,.$$

Working with expression (4.1.13), we conduct a change of variables $s = \tan^{-1}\phi$ and $t = \tan^{-1}\psi$ which leads to

$$\lambda_1 = b\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{s} s\exp\left(-\frac{1}{\sigma^2}\left[\alpha(s^2-t^2)+\frac{2}{3}b(s^3-t^3)\right]\right)\mathrm{d}t\,\mathrm{d}s}{\int_{-\infty}^{\infty}\int_{-\infty}^{s}\exp\left(-\frac{1}{\sigma^2}\left[\alpha(s^2-t^2)+\frac{2}{3}b(s^3-t^3)\right]\right)\mathrm{d}t\,\mathrm{d}s}\,. \tag{4.1.17}$$

We introduce a new variable $u = s - t$, which implies that $u \in (0,\infty)$. We observe

$$\begin{aligned}\alpha s^2 - \alpha(s-u)^2 + \frac{2}{3}bs^3 - \frac{2}{3}b(s-u)^3 &= -\alpha u^2 + 2\alpha su + 2bus^2 - 2bu^2 s + \frac{2}{3}bu^3\\ &= 2bu\left(s - \frac{u-\alpha/b}{2}\right)^2 + \frac{b}{6}u^3 - \frac{1}{2}u\frac{\alpha^2}{b}\,.\end{aligned}$$

Using this expression, we modify (4.1.17) and obtain

$$\begin{aligned}\lambda_1 &= b\frac{\int_0^\infty\int_{-\infty}^{\infty} s\exp\left(-\frac{2u}{\sigma^2}\left(s-\frac{u-\alpha/b}{2}\right)^2\right)\mathrm{d}s\exp\left(-\frac{b}{6a^2}u^3+\frac{1}{2}u\frac{\alpha^2}{\sigma^2 b}\right)\mathrm{d}u}{\int_0^\infty\int_{-\infty}^{\infty}\exp\left(-\frac{2bu}{\sigma^2}\left(s-\frac{u-\alpha/b}{2}\right)^2\right)\mathrm{d}s\exp\left(-\frac{b}{6\sigma^2}u^3+\frac{1}{2}u\frac{\alpha^2}{\sigma^2 b}\right)\mathrm{d}u}\\ &= b\frac{\int_0^\infty \frac{1}{\sqrt{u}}\frac{u-\alpha/b}{2}\exp\left(-\frac{b}{6\sigma^2}u^3+\frac{1}{2}u\frac{\alpha^2}{\sigma^2 b}\right)\mathrm{d}u}{\int_0^\infty\frac{1}{\sqrt{u}}\exp\left(-\frac{b}{6\sigma^2}u^3+\frac{1}{2}u\frac{\alpha^2}{\sigma^2 b}\right)\mathrm{d}u}\\ &= -\frac{\alpha}{2} + \frac{b^2\sigma^2}{2}\frac{\int_0^\infty\frac{1}{\sqrt{v}}v\exp\left(-\frac{\sigma^4b^4}{6}v^3+\frac{\alpha^2}{2}v\right)\mathrm{d}v}{\int_0^\infty\frac{1}{\sqrt{v}}\exp\left(-\frac{\sigma^4b^4}{6}v^3+\frac{\alpha^2}{2}v\right)\mathrm{d}v}\,,\end{aligned}$$

where we have done another change of variables $v = u/(b\sigma^2)$ in the last equality, and we used well-known properties of the normal distribution. Hence, we write

$$\lambda_1 = -\frac{\alpha}{2} + \frac{b^2\sigma^2}{2}\int_0^\infty v\ m_{\sigma,b,\alpha}(v)\,\mathrm{d}v\,,$$

where $m_{\sigma,b,\alpha}(v)$ is given as in (4.1.16). From Proposition 4.1.2, we obtain that $\lambda_1 + \lambda_2 = -\alpha$, and this means that

$$\lambda_2 = -\frac{\alpha}{2} - \frac{b^2\sigma^2}{2}\int_0^\infty v\; m_{\sigma,b,\alpha}(v)\mathrm{d}v.$$

This finishes the proof of this theorem. □

4.2 The bifurcation result

4.2.1 Bifurcation from negative to positive Lyapunov exponent

We now use Theorem 4.1.6 to prove Theorem 4.0.1, which asserts that there is a bifurcation from negative to positive Lyapunov exponent for the stochastic differential equation (4.0.1). In Subsection 4.2.3 the result is shown to stay robust under perturbations of f that smoothen this function. We recall the theorem for convenience of the reader.

Theorem 4.0.1 (Change of sign of λ_1)**.** *Consider the SDE (4.0.1) with f given by (4.0.2). Then there is $0 < c_0 \approx 0.2823$ such that for all $\alpha > 0$ and $b \neq 0$ the number $\sigma_0(\alpha, b) = \frac{\alpha^{3/2}}{c_0^{1/2}|b|} > 0$ is the unique value of σ where the top Lyapunov exponent $\lambda_1(\alpha, b, \sigma)$ of (4.0.1) changes its sign. In more detail, we have*

$$\lambda_1(\alpha,b,\sigma)\begin{cases} < 0 & \text{if } 0 < \sigma < \sigma_0(\alpha,b)\,,\\ = 0 & \text{if } \sigma = \sigma_0(\alpha,b)\,,\\ > 0 & \text{if } \sigma > \sigma_0(\alpha,b)\,. \end{cases}$$

Proof. We fix $\alpha > 0$ and $b \neq 0$, and define $\gamma := b^2\sigma^2$. We obtain from (4.1.14) that

$$\lambda_1 = -\frac{\alpha}{2} + \frac{\gamma}{2}\int_0^\infty v\; m_{\gamma,\alpha}(v)\,\mathrm{d}v\,,$$

where

$$m_{\gamma,\alpha}(v) = \frac{\frac{1}{\sqrt{v}}\exp\left(-\frac{\gamma^2}{6}v^3 + \frac{\alpha^2}{2}v\right)}{\int_0^\infty \frac{1}{\sqrt{w}}\exp\left(-\frac{\gamma^2}{6}w^3 + \frac{\alpha^2}{2}w\right)\mathrm{d}w}\,.$$

We introduce another change of variables $v = \frac{\alpha}{\gamma}u$ and obtain

$$\lambda_1 = \frac{\alpha}{2}\left(\int_0^\infty u\;\tilde{m}_{\gamma,\alpha}(u)\,\mathrm{d}u - 1\right), \tag{4.2.1}$$

where

$$\tilde{m}_{\gamma,\alpha}(u) = \frac{\frac{1}{\sqrt{u}}\exp\left(-\frac{\alpha^3}{\gamma}\left[\frac{1}{6}u^3 - \frac{1}{2}u\right]\right)}{\int_0^\infty \frac{1}{\sqrt{w}}\exp\left(-\frac{\alpha^3}{\gamma}\left[\frac{1}{6}w^3 - \frac{1}{2}w\right]\right)\,\mathrm{d}w}\,.$$

Defining $c := \frac{\alpha^3}{\gamma} = \frac{\alpha^3}{b^2\sigma^2}$, we observe that $\lambda_1(\alpha, b, \sigma)$ has the same sign as the function $G : (0, \infty) \to \mathbb{R}$ given by

$$G(c) := \int_0^\infty \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right) \exp\left(-c\left[\frac{1}{6}u^3 - \frac{1}{2}u\right]\right) \mathrm{d}u\,. \tag{4.2.2}$$

Using dominated convergence, we may interchange the order of differentiation and integration and consider

$$G'(c) = \int_0^\infty h_1(u)h_2(u)\exp\left(c\,h_2(u)\right)\mathrm{d}u\,,\ h_1(u) = \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)\,,\ h_2(u) = -\frac{1}{6}u^3 + \frac{1}{2}u\,.$$

Note that h_1h_2, and thereby the integrand, has positive sign on the interval $(1, \sqrt{3})$ and negative sign on $(0, 1)$ and $(\sqrt{3}, \infty)$. Basic claculations show that we have $|h_1(1-\delta)| > h_1(1+\delta)$ and $h_2(1-\delta) > h_2(1+\delta)$ for all $\delta \in (0, \sqrt{3}-1)$. It follows that

$$G'(c) < \int_{2-\sqrt{3}}^{\sqrt{3}} h_1(u)h_2(u)\exp\left(c\,h_2(u)\right)\mathrm{d}u < 0 \quad \text{for all } c \in (0, \infty)\,.$$

Hence, G is strictly decreasing. Furthermore, we observe that $G(c) \to \infty$ as $c \searrow 0$ (using monotone convergence on $[\sqrt{3}, \infty)$) and that $G(c) \to -\infty$ as $c \to \infty$ (using similar arguments as for G' and monotone convergence on $(0, 2-\sqrt{3})$).

Combining these observations, we may conclude that there is a unique c_0 such that $G(c_0) = 0$, $G(c) > 0$ for all $c \in (0, c_0)$ and $G(c) < 0$ for all $c \in (c_0, \infty)$. This proves the claim with $\sigma_0(\alpha, b) = \frac{\alpha^{3/2}}{c_0^{1/2}|b|}$. Numerical integration gives $c_0 \approx 0.2823$. □

Remark 4.2.1. As explained in the Introduction, the same result holds if we interchange the roles of σ and b. This can be seen also directly from the proof above.

For proving the existence of a random equilibrium when $\lambda_1 < 0$, we deploy ideas from [48, Theorem 2.23] which we formulate as follows for our purposes.

Theorem 4.2.2 (Weak synchronisation)**.** *Assume that a weakly complete, white noise random dynamical system (θ, φ) with stationary measure ρ on a Polish space X has right-continuous trajectories and is strongly mixing. Further assume that it is*

a) weakly asymptotically stable *on a set $U \subset X$ with $\rho(U) > 0$, i.e. there exists a sequence $t_n \to \infty$ and a set $\mathcal{M} \subset \Omega$ of positive $\mathbb{P}$-measure such that for all $x, y \in U$*

$$1_{\mathcal{M}}(\cdot)d(\varphi(t_n, \cdot, x), \varphi(t_n, \cdot, y)) \to 0 \text{ for } n \to \infty\,,$$

in probability,

b) pointwise strongly swift transitive, *i.e. there is a time $t > 0$ such that for every*

$x^1, x^2, z \in X$,

$$\mathbb{P}\big(\omega \in \Omega : \varphi\left(t, \omega, \{x^1, y^2\}\right) \subset B(z, 2d(x^1, x^2))\big) > 0\,,$$

c) and for all $x, y \in X$

$$\liminf_{t\to\infty} d(\varphi(t,\omega,x), \varphi(t,\omega,y)) = 0 \ \textit{a.s.}\,. \tag{4.2.3}$$

Then there is a minimal weak random point attractor $\tilde{A} \in \mathcal{F} \times B(X)$ *whose fibres are singletons, given by a* $\mathcal{F}^0_{-\infty}$*-measurable map* $a : \Omega \to X$*, the random equilibrium. We have*

$$\tilde{A}(\omega) = \operatorname{supp}(\mu_\omega) = \{a(\omega)\}\ \mathbb{P}\textit{-almost surely}\,.$$

Consequently, the random dynamical system admits weak synchronisation.

Proof. For establishing the existence of the attracting random equilibrium $a : \Omega \to X$ under assumptions a)-c), see the proof of [48, Theorem 2.23] where the existence of a is already called weak synchronisation. Our notion of weak synchronisation in (4.0.3) is a direct consequence, as one can easily observe: For all $Z_1, Z_2 \in X$ and any $\varepsilon > 0$, we have

$$\lim_{t\to\infty} \mathbb{P}\left(\{\omega \in \Omega\ :\ d(\varphi(t, \theta_{-t}\omega, Z_1), \varphi(t, \theta_{-t}\omega, Z_2)) \geq \varepsilon\}\right) = 0\,.$$

Since the probability measure $\mathbb{P}$ is invariant under θ_t for all $t \in \mathbb{R}$, we can immediately conclude that

$$\lim_{t\to\infty} \mathbb{P}\left(\{\omega \in \Omega\ :\ d(\varphi(t, \omega, Z_1), \varphi(t, \omega, Z_2)) \geq \varepsilon\}\right) = 0\,,$$

which shows the claim. □

We are prepared to deduce the first part of Corollary 4.0.2. The second part follows from Section 5.1 in Chapter 5. That section is dedicated to showing that μ_ω is atomless almost surely in case of positive Lyapunov exponents. Recall from Section 4.1.1 that the RDS induced by (4.0.1) has a pullback set attractor which is one-dimensional due to the contraction in the y-direction. The minimal weak point attractor in both cases of negative and positive λ_1 is obviously a subset of the set attractor.

Proof of Corollary 4.0.2. As the system is strongly mixing by (a hypoelliptic version of) [59, Theorem 4.3], the random compact set $\tilde{A} \in \mathcal{F} \times B(\mathbb{R}\times\mathbb{S}^1)$ with fibres $\tilde{A}(\omega) = \operatorname{supp}(\mu_\omega)$ is a minimal weak point attractor of (4.0.1) by [48][Proposition 2.20 (1)].

Firstly, let $\lambda_1 < 0$. According to [48][Proposition 2.20 (2)], there are an $N \in \mathbb{N}$ and $\mathcal{F}^0_{-\infty}$-measurable random variables $a_i : \Omega \to \mathbb{R} \times \mathbb{S}^1$ for $i = 1, \ldots, N$ such that

$$\tilde{A}(\omega) = \{a_i(\omega)\ :\ i = 1, \ldots, N\}\,.$$

The fact, that $\tilde{A}(\omega)$ is a singleton almost surely, follows from a slightly modified reasoning alongside Theorem 4.2.2 and its proof in [48]: Pick $a = a_i$ and $b = a_j$ for $i, j \in \{1, \dots, N\}$ and fix some compact ball D_0 such that $\mathbb{P}(\{\tilde{A}(\cdot) \in D_0\}) > 0$. By arguments analogous to [48, Section 3.1.1], the negative top Lyapunov exponent implies the existence of some open ball $B(x_0, \varepsilon)$ on which φ is weakly asymptotically stable and therefore condition a) is satisfied. Instead of using Theorem 4.2.2 b) and c), we show directly that there is a time T^* such that

$$\mathbb{P}\left(\{\varphi(T^*, \cdot, a(\cdot)), \varphi(T^*, \cdot, b(\cdot))\} \subset B(x_0, \varepsilon)\right) > 0. \tag{4.2.4}$$

Then by using that φ is a white noise RDS, the claim that $\tilde{A}(\omega)$ is a singleton almost surely follows as in the proof of Theorem 4.2.2 in [48].

For showing (4.2.4) pick $x, y \in D_0$ and fix $\delta << \varepsilon/2$. Due to the contracting drift in the amplitude direction, there is a $T_0 > 0$ such that $\beta := |\varphi_1(T_0, \omega, x) - \varphi_1(T_0, \omega, y)| < \delta$, i.e. the distance of the trajectories in the amplitude component is smaller than δ, for the path $W_t(\omega) = 0$ on $[0, T_0]$. For $t > T_0$, the path $W_t(\omega)$ can be chosen such that $|\varphi_1(t, \omega, x) - \varphi_1(t, \omega, y)| = \beta > 0$ for $t \in [T_0, T_1]$ where T_1 is such that $\varphi_2(T_1, \omega, x) = \varphi_2(T_1, \omega, y)$, i.e. the phase components become equal. By continuity of the function, which maps paths in the canonical space to the solution, with respect to the supremum norm on $C([0, T_1], \mathbb{R})$, we deduce that

$$\mathbb{P}\left(\{\varphi(T, \cdot, x), \varphi(T, \cdot, y)\} \subset B(z, \delta)\right) > 0\,,$$

for some $z \in X$. In this situation the noise in the y-component is arbitrarily close to additive noise and, hence, there is a small time interval $[T_1, T]$ with $T = T(x, y)$ such that we can make use of [48, Proposition 3.10, Part 1] to derive that

$$\mathbb{P}\left(\{\varphi(T, \cdot, x), \varphi(T, \cdot, y)\} \subset B(x_0, \varepsilon)\right) > 0\,.$$

The compactness of D_0 then gives a uniform bound $T^* \geq T(x, y)$ for all $x, y \in D_0$. Because of the independence of $\mathcal{F}_{-\infty}^0$ and $\mathcal{F}_0^\infty$, we can deduce (4.2.4). This finishes the proof of weak synchronisation.

In the case of $\lambda_1 > 0$, we deduce with Theorem 5.1.1 below that μ_ω is atomless almost surely. The result is stated for smooth flows but the essential argument carries over as before (ignoring the point of non-differentiability). Recall from Proposition 4.1.1 and Proposition 4.1.5 that the hypoellipticity assumption (A1) from Section 5.1 is satisfied by model (4.0.1): there exists a unique stationary distribution with smooth density for the original stochastic flow and a unique stationary distribution with smooth density for the linearised flow on the unit sphere of the tangent spaces as given by (4.1.13). The assumptions on the moment Lyapunov functions are also satisfied since the variational equation for model (4.0.1) is solely dependent on the angular process, which is defined on

the compact manifold $\mathbb{S}^1$, and therefore the theory from [14] is directly applicable.

The fact, that the measures μ_ω are atomless almost surely, implies in particular that the fibres $\tilde{A}(\omega) = \text{supp}(\mu_\omega)$ of the minimal random point attractor $\tilde{A}$ are no singletons, and, hence, $\tilde{A}$ does not consist of a random equilibrium. □

Remark 4.2.3. Corollary 4.0.2 implies that the invariant random measure associated with the random attractor changes from a random Dirac measure to a random SRB measure. One might be inclined to ask if our case satisfies the definition of a D-bifurcation [2], which takes place if there is an invariant reference measure μ in a neighbourhood of the bifurcation point from which a new invariant measure ν bifurcates in the sense of weak convergence. In our case μ_ω does not necessarily stay a reference measure and the continuity assumption of the D-Bifurcation is not necessarily satisfied. So we provide an example of interesting bifurcation behaviour which does not fit the definition of Arnold suggesting that dynamical bifurcations should be understood in a broader sense than the definition of a D-bifurcation permits.

4.2.2 Numerical exploitation of the results

For the moment, the positive top Lyapunov exponent is the only characterization of chaos we give in this case as an analysis in the sense of [89] is very difficult for white noise and provable aspects of this are postponed to chapter 5. However, the geometric mechanism of shear-induced chaos can still be understood along the same lines: the white noise drives some points on the limit cycle up and some down. Due to the phase amplitude coupling b, the points with larger y-coordinates move faster in the ϑ-direction. At the same time, the dissipation force with strength α attracts the curve back to the limit cycles. This provides a mechanism for stretching and folding characteristic of chaos. The transition to chaos in the continuous time stochastic forcing is much faster than in the case of periodic kicks due to the effect of large deviations [73]. This is due to the fact that points end up in areas with arbitrarily large values of y with positive probability. Hence, not so much shear is needed to generate the described stretching and folding due to phase amplitude coupling. However, for very small shear and noise, the dissipation leads to sinks being formed between these large deviation events, and the attractor ends up to be a singleton.

In Figure 4.4, we show the top Lyapunov exponent as a function of σ for fixed b and α according to formula (4.1.14). We have used numerical integration up to machine precision to calculate λ_1. The bifurcations of the sign of λ_1 at $\sigma_0(\alpha, b)$ are clearly seen in Figures 4.4a-4.4c. Furthermore, note that $\lambda_1 \to 0$ from below for $\sigma \to 0$. In Figure 4.4d, we choose small values of b and α, but b/α large. We see no negative values of λ_1 as we would have to take values of σ too small for the numerical integration.

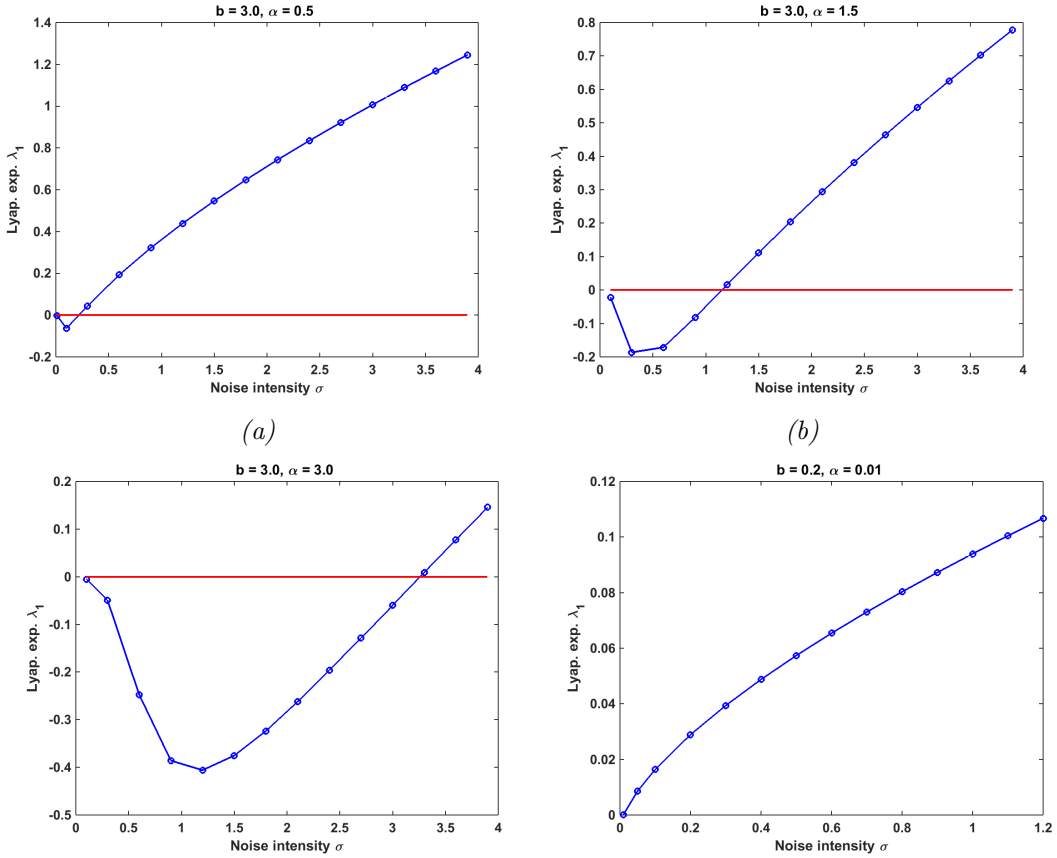


Figure 4.4: The top Lyapunov exponent λ_1 as a function of σ for fixed b and α. The dots indicate the values of λ_1 that were calculated according to (4.1.14) using numerical integration. Figures 4.4a-4.4c illustrate that $\sigma_0(\alpha, b)$ increases monotonously in α. In Figure 4.4d, b and α are small, but b/α is large. We do not see the transition to $\lambda_1 < 0$ since we would have to take values of σ too small for the numerical integration.

We have chosen the same parameter regimes as in [73], where Lin & Young investigate numerically the Lyapunov exponents of the system given by

$$\begin{aligned} \mathrm{d}y &= -\alpha y \mathrm{d}t + \sigma \sin(2\pi\vartheta) \circ \mathrm{d}W_t \,, \\ \mathrm{d}\vartheta &= (1 + by)\, \mathrm{d}t \,, \end{aligned} \tag{4.2.5}$$

taking $\vartheta \in [0, 2\pi]$. Their numerical results show exactly the same qualitative behaviour apart from a slightly different scaling due to the factor 2π. We conclude that the simpler choice of the diffusion coefficient in our case does not change the qualitative behaviour. This is not a surprise, since we can derive formula (4.1.14) for λ_1 if we choose f to be piecewise linear on the intervals $[\frac{i}{4}, \frac{i+1}{4}]$ for $i = 0, 1, 2, 3$ with $|f'|$ constant such that it represents a linear approximation of the sin-function.

Furthermore, we confirm Lin & Young's conjecture concerning a scaling property of λ_1 with respect to the parameters. For model (4.2.5) they observed numerically that,

under the transformations $\alpha \mapsto k\alpha$, $b \mapsto kb$ and $\sigma \mapsto \sqrt{k}\sigma$, λ_1 transforms approximately as $\lambda_1 \mapsto k\lambda_1$. This scaling property holds exactly for our model (4.0.1).

Proposition 4.2.4. *Consider the stochastic differential equation* (4.0.1)*, where the function f is of the form* (4.0.2)*. Then the top Lyapunov exponent λ_1 as given by* (4.1.14) *satisfies*

$$\lambda_1(k\alpha, kb, \sqrt{k}\sigma) = k\lambda_1(\alpha, b, \sigma) \qquad \text{for all } k \in \mathbb{R}^+ \setminus \{0\}\,. \tag{4.2.6}$$

Proof. First note that by the change of variables $v \mapsto \frac{v}{k^2}$, and $u \mapsto \frac{u}{k^2}$ respectively, we obtain

$$\int_0^\infty v\; m_{\sqrt{k}\sigma,kb,k\alpha}(v)\,\mathrm{d}v \;=\; \frac{\int_0^\infty v\frac{1}{\sqrt{v}}\exp\left(-\frac{\sigma^4b^4}{6}k^6v^3+\frac{\alpha^2}{2}k^2v\right)\mathrm{d}v}{\int_0^\infty \frac{1}{\sqrt{u}}\exp\left(-\frac{\sigma^4b^4}{6}k^6u^3+\frac{\alpha^2}{2}k^2u\right)\mathrm{d}u} = \frac{1}{k^2}\int_0^\infty v\; m_{\sigma,b,\alpha}(v)\,\mathrm{d}v\,.$$

Hence, we can conclude that

$$\lambda_1(\sqrt{k}\sigma, kb, k\alpha) = -k\frac{\alpha}{2} + k^3\frac{b^2\sigma^2}{2}\frac{1}{k^2}\int_0^\infty v\; m_{\sigma,b,\alpha}(v)\,\mathrm{d}v \;= k\lambda_1(\sigma, b, \alpha).$$

□

4.2.3 Robustness of the result

We can perturb f around the point of non-differentiability at $\frac{1}{2}$ and show that the sign of λ_1 is preserved under small perturbations. This demonstrates the robustness of the bifurcation behaviour. In more detail: we choose $f_\varepsilon = f$ on $[0, \frac{1}{2}-\varepsilon]\cup[\frac{1}{2}+\varepsilon, 1)$ but smooth on the whole interval with $|f_\varepsilon|, |f_\varepsilon'| \le 1$ such that $f_\varepsilon \to f$ uniformly. The existence of such f_ε follows from standard analysis. The corresponding model for f_ε is given by

$$\begin{aligned} \mathrm{d}y &= -\alpha y\,\mathrm{d}t + \sigma f_\varepsilon(\vartheta)\circ\mathrm{d}W_t^1\,,\\ \mathrm{d}\vartheta &= (1+by)\,\mathrm{d}t\,. \end{aligned} \tag{4.2.7}$$

Let λ_1^ε denote the top Lyapunov exponent for the random dynamical system $(\theta, \varphi_\varepsilon)$ induced by (4.2.7). Then indeed $\lambda_1^\varepsilon \to \lambda_1$ as $\varepsilon \to 0$, which we sketch in the following: denote by $\tilde{\rho}_\varepsilon$ the stationary density for the process in (y, ϑ, ϕ) where ϕ is given as in (4.1.7) for f_ε, carrying the information from the linearisation along trajectories:

$$\mathrm{d}\phi = (\alpha\cos\phi\sin\phi + b\cos^2\phi)\mathrm{d}t - \sigma f_\varepsilon'(\vartheta)\sin^2\phi\circ\mathrm{d}W_t^1\,.$$

Existence and uniqueness of $\tilde{\rho}_\varepsilon$ are guaranteed by similar considerations as in Section 4.1. We denote the corresponding Fokker–Planck operators (cf. Appendix A.2) for the systems in (y, ϑ, ϕ) with the diffusion coefficients f and f_ε by $\mathcal{L}^*$ and $\mathcal{L}_\varepsilon^*$ respectively. The stationary Fokker–Planck equations read $\mathcal{L}_\varepsilon^*\tilde{\rho}_\varepsilon = 0$ and $\mathcal{L}^*\tilde{\rho} = 0$ where $\tilde{\rho}$ denotes the

stationary density for the system associated to f. Note that we use the symbols $\tilde{\rho}$ and $\tilde{\rho}_\varepsilon$ both for the measures and corresponding densities.

In the following, we denote by $D = \mathbb{R} \times [0,1) \times [0,\pi)$ the domain of (y,ϑ,ϕ). Recall from (4.1.6) that

$$\lambda_1 = \int_D q(\phi)\tilde{\rho}(y,\vartheta,\phi)\,\mathrm{d}\phi\,\mathrm{d}\vartheta\,\mathrm{d}y\,, \quad \lambda_1^\varepsilon = \int_D q_\varepsilon(\phi,\vartheta)\tilde{\rho}_\varepsilon(y,\vartheta,\phi)\,\mathrm{d}\phi\,\mathrm{d}\vartheta\,\mathrm{d}y\,,$$

where

$$\begin{aligned} q(\phi) &= -\alpha\cos^2\phi + b\cos\phi\sin\phi + \frac{1}{2}\sigma^2\sin^2\phi(1-2\cos^2\phi)\,,\\ q_\varepsilon(\phi,\vartheta) &= -\alpha\cos^2\phi + b\cos\phi\sin\phi + \frac{1}{2}\sigma^2 f_\varepsilon'(\vartheta)^2\sin^2\phi(1-2\cos^2\phi)\,. \end{aligned}$$

We define

$$\lambda_1^{\varepsilon,\varepsilon} := \int_D q(\phi)\tilde{\rho}_\varepsilon(y,\vartheta,\phi)\,\mathrm{d}\phi\,\mathrm{d}\vartheta\,\mathrm{d}y\,.$$

Observe that we obtain the following estimate for small enough $\varepsilon > 0$:

$$\begin{aligned} |\lambda_1 - \lambda_1^\varepsilon| &\le |\lambda_1 - \lambda_1^{\varepsilon,\varepsilon}| + |\lambda_1^{\varepsilon,\varepsilon} - \lambda_1^\varepsilon| \\ &\le C_1(\alpha,b,\sigma)\|\tilde{\rho} - \tilde{\rho}_\varepsilon\|_{L^1(D)} + \varepsilon C_2(\sigma)\max_D|\tilde{\rho}_\varepsilon(y,\vartheta,\phi)|\,. \end{aligned} \tag{4.2.8}$$

The densities $\tilde{\rho}_\varepsilon$ are obviously uniformly bounded in ϑ and ϕ. Furthermore, since $\sigma f_\varepsilon(\vartheta) \le \sigma$, the decay of $\tilde{\rho}_\varepsilon$ in y is at least the same as for the stationary density of the Ornstein–Uhlenbeck process

$$\mathrm{d}y = -\alpha y\mathrm{d}t + \sigma\mathrm{d}W_t\,,$$

which is given by

$$p(y) = \frac{1}{Z}\exp\left(-\frac{\alpha}{2\sigma^2}y^2\right)\,, \tag{4.2.9}$$

where Z is the normalisation constant. We deduce that there exists a constant $C > 0$ such that
$\max_D|\tilde{\rho}_\varepsilon(y,\vartheta,\phi)| < C$ for any small $\varepsilon > 0$, which implies that the second term on the right-hand side of (4.2.8) vanishes as $\varepsilon \to 0$. Hence, it remains to show that $\|\tilde{\rho} - \tilde{\rho}_\varepsilon\|_{L^1(D)} \le K\varepsilon$ for some constant $K > 0$. Since all the coefficients are bounded, we have for any sufficiently regular $\psi : D \to \mathbb{R}$,

$$\begin{aligned} &|\mathcal{L}_\varepsilon^*\psi(y,\vartheta,\phi) - \mathcal{L}^*\psi(y,\vartheta,\phi)| \\ &\le C_3(\alpha,b,\sigma)\mathbb{1}_{[\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon]}(\vartheta)\bigg(\left|\frac{\partial}{\partial\phi}\psi(y,\vartheta,\phi)\right| + \left|\frac{\partial^2}{\partial^2\phi}\psi(y,\vartheta,\phi)\right| \\ &+ \left|\frac{\partial^2}{\partial^2 y}\psi(y,\vartheta,\phi)\right| + \left|\frac{\partial^2}{\partial\phi\partial y}\psi(y,\vartheta,\phi)\right|\bigg)\,. \end{aligned} \tag{4.2.10}$$

We consider the problem in the weighted function spaces $W^{2,2}_p(D)$ to $L^2_p(D)$ where p is given by (4.2.9). We understand $\mathcal{L}^*$ and $\mathcal{L}^*_\varepsilon$ as linear bounded Fredholm operators from $W^{2,2}_p(D)$ to $L^2_p(D)$ (as they are weakly elliptic from a weighted Hilbert space to a weighted Hilbert space). Since $W^{2,2}_p$ is a Hilbert space, we can define the projection P to $\ker(\mathcal{L}^*)$ which is a one-dimensional and hence closed subspace. We can further define the linear operator $\mathcal{S} : \mathcal{L}^*(W^{2,2}_p(D)) \to W^{2,2}_p(D)$ as the pseudoinverse of $\mathcal{L}^*$, i.e. define it via $\mathcal{S}\mathcal{L}^* = \mathrm{Id} - P$. Since the range of a Fredholm operator is closed, $\mathcal{S}$ is a linear and bounded operator by the open mapping theorem. Now choose $\tilde{\rho}_\varepsilon \in \ker(L^*_\varepsilon)$ such that $P\tilde{\rho}_\varepsilon = \tilde{\rho}$ (there is a normalisation error that can be easily resolved in the limit as $\varepsilon \to 0$). We use

$$\mathcal{L}^*(\tilde{\rho} - \tilde{\rho}_\varepsilon) = \mathcal{L}^*_\varepsilon \tilde{\rho}_\varepsilon - \mathcal{L}^* \tilde{\rho}_\varepsilon \,,$$

and (4.2.10), in combination with the uniform boundedness and decay of $\tilde{\rho}_\varepsilon$ and its derivatives in all variables, to deduce

$$\begin{aligned} \|\tilde{\rho} - \tilde{\rho}_\varepsilon\|_{L^1(D)} &\le C_4 \|\tilde{\rho} - \tilde{\rho}_\varepsilon\|_{W^{2,2}_p(D)} = C_4 \|\mathcal{S}\mathcal{L}^*(\tilde{\rho} - \tilde{\rho}_\varepsilon)\|_{W^{2,2}_p(D)} \\ &\le C_4 \|\mathcal{L}^*(\tilde{\rho} - \tilde{\rho}_\varepsilon)\|_{L^2_p(D)} \|\mathcal{S}\|_{B(\mathcal{L}^*(W^{2,2}_p(D)), W^{2,2}_p(D))} \le K\varepsilon \,, \end{aligned}$$

for some constant $K > 0$.

4.3 Discussion

We have investigated systems with limit cycles on a cylinder perturbed by white noise. We were able to show a transition from negative to positive top Lyapunov exponents for fixed dissipation parameter α and big enough noise σ and/or shear b, implying a bifurcation of the random attractor from a random equilibrium to a random strange attractor. In particular, we have confirmed Lin & Young's [73] scaling conjecture

$$\lambda_1(k\alpha, kb, \sqrt{k}\sigma) = k\lambda_1(\alpha, b, \sigma) \qquad \text{for all } \; k \in \mathbb{R}^+ \setminus \{0\} \,.$$

The crucial trick for deriving an explicit formula and thereby finding parameters for which $\lambda_1 > 0$ consisted in the particular choice of the diffusion coefficient $f(\vartheta)$. However, we showed the robustness of the result under smooth perturbations, and we remark that the scaling behaviour shown for λ_1 seems to be independent from this particular choice of f.

In the case of positive Lyapunov exponents, one can typically describe the attractor using concepts from ergodic theory, such as entropy and SRB measures. This is supposed to give a more rigorous notion of chaos, and a clearer picture of the dynamics and the nature of the sample measures μ_ω. We investigate these issues in the next chapter where we also add a proof for the claim that the random attractor does not have singleton fibres if $\lambda_1 > 0$.

Chapter 5

Ergodic theory of chaotic random attractors

This chapter is dedicated to shedding more light on the character of the random (point) attractor A and the invariant random measures μ_ω supported on the fibres $A(\omega)$, in case the top Lyapunov exponent λ_1 is positive. We have called A a random strange attractor in this situation and refer to Figures 1.2 and 4.3 for getting an idea of the shape of such an attractor. In Figures 1.2e–1.2h we approximate the support of μ_ω for model (3.0.1) with high shear intensity b by computing $\varphi(T, \theta_{-T}\omega)\tilde{\rho}$ for increasing T, where $\tilde{\rho}$ is a numerical approximation of the stationary measure ρ. In Figure 4.3 we do exactly the same for model (4.0.1). In both cases, we observe one-dimensional structures that resemble Henon-like attractors.

Hence, a first thing to show is that μ_ω is almost surely not supported on a singleton if $\lambda_1 > 0$; a fact we have already claimed in Corollary 4.0.2. We will follow work by Baxendale and Stroock [14, 10] extending their results to the non-compact case. This will be the content of Section 5.1 where we show Theorem 5.1.1 which states that positive Lyapunov exponents imply atomless invariant measures, in particular in the situations of Chapters 3 and 4.

Section 5.2 concerns Pesin's formula, i.e. the equality of metric entropy and the sum of positive Lyapunov exponents, for model (3.0.1) on $\mathbb{R}^d$ and model (4.0.1) on $\mathbb{R}\times\mathbb{S}^1$. The formula was proven by Ledrappier & Young [70] for discrete-time random dynamical systems generated by randomly drawn diffeomorphisms on a compact manifold with absolutely continuous stationary measure. Biskamp [15] has proven the formula for discrete-time random dynamical systems $\mathcal{X}^+(\mathbb{R}^d, \nu)$, where ν is the law of the random diffeomorphisms on $\mathbb{R}^d$, given a class of Assumptions (A1)-(A5) and an absolutely continuous stationary (not necessarily ergodic) probability measure μ. The formula reads

$$h_\mu\left(\mathcal{X}^+(\mathbb{R}^d, \nu)\right) = \int_{\mathbb{R}^d} \sum_i \lambda_i(x)^+ m_i(x) \mu(\mathrm{d}x)\,,$$

where h_μ denotes metric entropy and $\lambda_i(x)^+$ are the positive Lyapunov exponents with multiplicities $m_i(x)$. The section gives an introduction to the concept of entropy for random dynamical systems, following [15, 60, 74], and links discrete-time systems with random dynamical systems generated by stochastic differential equations via their time-one maps. This allows to formulate Theorem 5.2.9 as a direct consequence of [15, Theorem 9.1] stating Pesin's formula for systems derived from stochastic differential equations. Finally, we prove Corollary 5.2.10 which says that the random dynamical system induced by model (4.0.1) has positive entropy for large enough shear or noise respectively, i.e. if $\lambda_1 > 0$. The same would hold true for model (3.0.1), once positive Lyapunov exponents can be established.

Section 5.3 introduces the concept of SRB measures for random systems with positive Lyapunov exponent. In short, the random measures μ_ω are called SRB measures if they are absolutely continuous with respect to the Riemannian measure on fibres of unstable manifolds. Hence, if the μ_ω can be shown to be SRB measures, their supports are non-singular subsets of the closures of unstable manifolds, perhaps even equal them. This gives a strong characterisation of the shapes we observe in Figures 1.2 and 4.3. Following [70], we formulate Theorem 5.3.4 and Corollary 5.3.5 implying the SRB property for sample measures of discrete-time systems and systems induced by stochastic differential equations on compact manifolds, in case the stationary measure μ is absolutely continuous. It is beyond the scope of this work to show the results for the non-compact case since there are a lot of technical intricacies to be taken care of. We simply refer to the intuition that if the maps/flows and their derivatives satisfy uniform bounds, the results are applicable to the non-compact case.

We concede that, too a large extent, the work presented in this chapter is not original. However, it is important to embed the results of chapters 3 and 4 into the context of ergodic theory. Furthermore, we understand this chapter as a contribution to linking the Arnold school, on which the other chapters are based, to the school represented by Kifer, Ledrappier and Young that puts more emphasis on smooth ergodic theory.

5.1 Positive Lyapunov exponents imply atomless invariant measures

We are extending the statement in [10] about positive Lyapunov exponents implying atomless invariant random measures to the non-compact setting. Let M be a connected smooth Riemannian manifold of dimension N. Similarly to (2.3.1), we consider the Stratonovich stochastic differential equation on M

$$d\xi_t(x) = X_0(\xi_t(x))\mathrm{d}t + \sum_{k=1}^{d} X_k(\xi_t(x)) \circ \mathrm{d}W_t^k\,, \quad \xi_0(x) = x\,. \tag{5.1.1}$$

Here $X_0, X_1, \ldots, X_d$ are smooth vector fields on M satisfying a one-sided Lipschitz condition guaranteeing unique solutions and $\{W_t^k : t \geq 0\}, 1 \leq k \leq d$, are independent real valued Brownian motions on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let further denote $\tilde{X}_k$ the natural lifts of the vector fields X_k on M to vector fields on SM, the unit sphere bundle in TM. For $v \in SM$ and $u \in C([0,\infty), \mathbb{R}^d)$, let $\Psi(\cdot, v, u)$ denote the curve in SM satisfying

$$\dot{\Psi}(t, v, u) = \tilde{X}_0(\Psi(t, v, u)) + \sum_{k=1}^{d} u_k(t)\tilde{X}_k(\Psi(t, v, u))$$

with $\Psi(0, v, u) = v$. Now we demand the following assumption similarly to (2.3.8) which implies that the generator of (5.1.1) is hypoelliptic and that, if there is a stationary probability measure ρ, it is unique and has a smooth density with respect to the Riemannian measure on M:

(A1) $\left(\tilde{X}_0, \ldots, \tilde{X}_d\right)(v) = T_v SM$
and $\{\Psi(t, v, u) : t \geq 0$ and $u \in C([0,\infty), \mathbb{R}^d)\}$ is dense in SM for all $v \in SM$.

Equation (5.1.1) induces a random dynamical system as before where the notations identify as $\varphi(t, \cdot, x) = \xi_t(x)$. Let μ be the invariant Markov measure of the RDS, i.e. the invariant probability measure of the skew product flow corresponding with ρ, and μ_ω its disintegrations to the state space. We denote $\bar{\mu} = \mathbb{E}[\mu_\omega \times \mu_\omega]$ which is a stationary probability measure for the two-point motion $\{(\xi_t(x), \xi_t(y)) : t \geq 0\}$ on $M \times M$. The generator of the two-point motion is denoted by $L^{(2)}$. Further, Δ denominates the diagonal in $M \times M$ and we write $\hat{M} = M \times M \setminus \Delta$.

Before we can establish the statement about atomless measures, we have to introduce the moment Lyapunov function $\tilde{\Lambda} : \mathbb{R} \to \mathbb{R}$. It is defined by

$$\tilde{\Lambda}(p) = \lim_{t\to\infty} \frac{1}{t} \ln \mathbb{E}_\mu \left| D\xi_t(x)(v) \right|^p \,,\ v \in T_x M \,, |v| \neq 0 \,.$$

If M is compact, Assumption (A1) guarantees that the above limit exists and is independent from the choice of v. Further $\tilde{\Lambda}$ is a convex analytic function of p according to results in [14]. These results are directly applicable to the situation of model (4.0.1) since the variational equation is solely dependent on the angular process which is defined on the compact manifold $\mathbb{S}^1$.

We formulate the result in the following theorem. Note that the proof contains a summary of arguments coming from [14, 10, 59] that cannot be found in such an overview otherwise. The issues with non-compactness are rather benign and are pointed out at the respective parts of the argument:

Theorem 5.1.1 (Positive λ_1 implies atomless μ_ω)**.** *Assume that the top Lyapunov exponent λ_1 for the random dynamical system induced by* (5.1.1) *is positive, that the moment*

Lyapunov function $\tilde{\Lambda}$ exists independently from $v \in TM$, that the system has a stationary measure ρ and that (A1) holds. Then

$$\bar{\mu}\left(\hat{M}\right) = 1$$

or equivalently μ_ω is atomless almost surely.

Proof. Define the map $\Phi : TM \to M \times M$ by

$$\Phi((x,v)) = (x, \exp_x(v))\,,$$

where $\exp_x : T_xM \to M$ is the exponential map. For the strategy of the proof it is essential that there exists an injectivity radius $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$, Φ is a diffeomorphism from $\{(x,v) \in TM : 0 < |v| < \delta\}$ onto $\hat{M}_\delta := \{(x,y) \in M^2 : 0 < d(x,y) < \delta\}$ via

$$(x,\theta,r) \in SM \times (0,\delta) \mapsto \Phi((x,r\theta)) \in \hat{M}_\delta\,.$$

In the original setting of [14] this follows immediately from compactness. For non-compact manifolds with positive injectivity radius, as for example the infinite cylinder or $\mathbb{R}^N$, the result carries over immediately, as we will see from the following arguments. In case of no positive injectivity radius being guaranteed, fix $\varepsilon > 0$. In order to adapt the proof to this setting, we start with a compact ball K such that $(\rho \times \rho)(K \times K) > 1 - \varepsilon/3$. Writing $\hat{K} = K \times K \setminus \Delta$, we proceed as in the proof of [10, Remark 4.12].

Observe that $\tilde{\Lambda}(0) = 0$ and $\tilde{\Lambda}'(0) = \lambda_1 > 0$. From [14], we further know that $\tilde{\Lambda}(-N) \geq 0$ and that $\tilde{\Lambda}$ is convex. Hence, there is a $p < 0$ such that $\tilde{\Lambda}(p) < 0$. Denote the injectivity radius of K by δ_0. Now, according to [14, Theorem 3.18] there is a $\delta \leq \delta_0$ and $\phi_p \in C^\infty(SM \times (0,\delta))$ such that

$$L^{(2)}\phi_p \leq \tilde{\Lambda}(p)\phi_p \text{ and } Cr^p \leq \phi_p(x,\theta,r)\,, \quad (x,\theta,r) \in SM \times (0,\delta)\,, \tag{5.1.2}$$

where $C > 0$ is some constant. By the coordinate transformation from above, we take $V : \hat{K} \to \mathbb{R}$ to be a smooth non-negative extension of ϕ_p with smooth compactly supported extension $\tilde{V}$ to $\hat{M}$. Since $p, \tilde{\Lambda}(p) < 0$ we obtain from (5.1.2) that V satisfies

$$L^{(2)}V(x,y) \to -\infty \text{ as } d(x,y) \to 0\,,$$

and

$$\int_{\hat{M}} \tilde{V}\mathrm{d}(\rho \times \rho) < \infty.$$

This allows to apply Khasminskii's pointwise estimate in the proof of [59, Theorem 3.7] to the distance of the two-point motion from the diagonal for any $(x,y) \in \hat{K}$. As the upper bound in Khasminskii's estimate [59, (3.54)] does not depend on the initial point $(x,y) \in \hat{K}$ but only on the distance of the two points, we can integrate over the stationary

distribution $\rho \times \rho$ on $\hat{M}$ and conclude that there is a $\gamma > 0$ such that

$$\frac{1}{t}\int_0^t \int_{\hat{M}} \mathbb{P}\left((\varphi(s,\cdot,x),\varphi(s,\cdot,y)) \in B_\gamma(\Delta) \cap \hat{K}\right)(\rho\times\rho)(\mathrm{d}x,\mathrm{d}y)\,\mathrm{d}s < \varepsilon/3$$

for any $t > 0$. Observe that by ergodicity

$$\frac{1}{t}\int_0^t \mathbb{P}\left((\varphi(s,\cdot,x),\varphi(s,\cdot,y)) \in \hat{M}\setminus\hat{K}\right)\mathrm{d}s \xrightarrow{t\to\infty} (\rho\times\rho)(\hat{M}\setminus\hat{K}) < \varepsilon/3$$

for $(\rho\times\rho)$-almost all $(x,y) \in K^2$. Since K^2 is compact, we can conclude that there is a $t^* > 0$ such that for all $t \geq t^*$

$$\frac{1}{t}\int_0^t \int_{\hat{K}} \mathbb{P}\left((\varphi(s,\cdot,x),\varphi(s,\cdot,y)) \in \hat{M}\setminus\hat{K}\right)(\rho\times\rho)(\mathrm{d}x,\mathrm{d}y)\,\mathrm{d}s < \varepsilon/3.$$

We define $K_1 = K^2 \cap \{(x,y) \in \hat{M} \,:\, d(x,y) \geq \gamma\}$ which is obviously compact. Let $\{P_t \,:\, t \geq 0\}$ denote the Markov semi-group acting on $C(M\times M)$ and $\{P_t^* \,:\, t \geq 0\}$ the adjoint semi-group acting on measures on $M \times M$. Observe that by the choice of K and K_1 and the above it follows for $t \geq t^*$ that

$$\begin{aligned}
&\frac{1}{t}\int_0^t \int_{\hat{M}} P_s \mathbb{1}_{\hat{M}\setminus K_1}\,\mathrm{d}(\rho\times\rho)\mathrm{d}s \\
&\qquad = \frac{1}{t}\int_0^t \int_{\hat{M}} \mathbb{P}\left((\varphi(s,\cdot,x),\varphi(s,\cdot,y)) \in B_\gamma(\Delta)\cap\hat{K}\right)(\rho\times\rho)(\mathrm{d}x,\mathrm{d}y)\,\mathrm{d}s \\
&\qquad + \frac{1}{t}\int_0^t \int_{\hat{M}} \mathbb{P}\left((\varphi(s,\cdot,x),\varphi(s,\cdot,y)) \in \hat{M}\setminus\hat{K}\right)(\rho\times\rho)(\mathrm{d}x,\mathrm{d}y)\,\mathrm{d}s \\
&\qquad < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Hence, for $t \geq t^*$ we obtain

$$\left(\frac{1}{t}\int_0^t P_s^*(\rho\times\rho)\mathrm{d}s\right)(K_1) = \frac{1}{t}\int_0^t \int_{\hat{M}} P_s\mathbb{1}_{K_1}\,\mathrm{d}(\rho\times\rho)\,\mathrm{d}s > 1-\varepsilon.$$

Since $K_1 \subset \hat{M}$ compact, we have shown that the family of probability measures

$$\left\{\frac{1}{t}\int_0^t P_s^*(\rho\times\rho)\,\mathrm{d}s\right\}_{t\geq t^*},$$

is uniformly tight on $\hat{M}$. Further note that these probability measures are supported on $\hat{M}$ due to absolute continuity of ρ. Hence, by Prokhorov's Theorem there is a probability measure ν on $\hat{M}$ and a time sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $\frac{1}{t_n}\int_0^{t_n} P_s^*(\rho\times\rho)\mathrm{d}s \to \nu$ as $n \to \infty$. We know from [10, Proposition 2.6] that $P_t^*(\rho\times\rho)$ converges weakly to $\bar{\mu}$ as $t \to \infty$. So we can conclude that $\nu = \bar{\mu}$ and therefore $\bar{\mu}(\hat{M}) = 1$ as required. □

5.2 Pesin's formula

In this section, we want to investigate what positive Lyapunov exponents imply for the entropy of the system. We will follow [15, 74, 70, 60] and try to apply their work to situations interesting for stochastic bifurcation theory.

Entropy for discrete time systems

Firstly, we formulate the statements for $\mathbb{R}^d$ and random dynamical systems in discrete time generated by composed maps $\left\{ f_\omega^n \,:\, n \geq 0, \omega \in \left(\Omega^{\mathbb{N}}, \mathcal{B}(\Omega)^{\mathbb{N}}, \nu^{\mathbb{N}}\right)\right\}$ which will be referred to as $\mathcal{X}^+(\mathbb{R}^d, \nu)$. Here, Ω denotes the set of two-times differentiable diffeomorphisms on $\mathbb{R}^d$ with the topology induced by uniform convergence on compact sets for all derivatives up to order 2. The maps are i.i.d. with law ν, and for a sequence $\omega = (f_0(\omega), f_1(\omega), \dots) \in \Omega^{\mathbb{N}}$ the compositions are given as

$$f_\omega^0 = \mathrm{id}\,, \quad f_\omega^n = f_{n-1}(\omega) \circ f_{n-2}(\omega) \circ \cdots \circ f_0(\omega)\,.$$

Later we will formulate the statements for stochastic flows that are related to discrete time random dynamical systems via their time-one maps.

Definition 5.2.1 (Stationary measure)**.** A Borel probability measure μ on $\mathbb{R}^d$ is called a stationary measure of $\mathcal{X}^+(\mathbb{R}^d, \nu)$ if

$$\mu(\cdot) = \int_\Omega \mu\left(f^{-1}(\cdot)\right) \nu(\mathrm{d}f)\,.$$

If ξ is a finite partition of a Lebesgue space $(X, \mathcal{B}, \mu)$ and $C_1, \dots, C_k$ denote the elements of ξ, we define the *entropy* of ξ with respect to μ by

$$H_\mu(\xi) = -\sum_{j=1}^{k} \mu(C_j) \ln(\mu(C_j))\,.$$

Furthermore, for two partitions ξ_1 and ξ_2 we define

$$\xi_1 \vee \xi_2 = \{A \cap B \,:\, A \in \xi_1, B \in \xi_2\}\,,$$

such that elements of $\bigvee_{i=0}^{n-1} (f_\omega^i)^{-1}\xi$ are of the form

$$\{x \,:\, x \in C_{j_0}, f_\omega x \in C_{j_1}, \dots, f_\omega^{n-1} x \in C_{j_{n-1}}\}$$

for some $(j_0, \dots, j_{n-1})$ sometimes called the address of the orbit.

Following [74] and [15] we define the entropy of a random dynamical system in the following way:

Definition and Lemma 5.2.2 (Entropy)**.** For any finite partition ξ of $\mathbb{R}^d$ and stationary measure μ of $\mathcal{X}^+(\mathbb{R}^d, \nu)$ the limit

$$h_\mu\left(\mathcal{X}^+(\mathbb{R}^d,\nu),\xi\right) := \lim_{n\to\infty}\frac{1}{n}\int_{\Omega^{\mathbb{N}}} H_\mu\left(\bigvee_{k=0}^{n-1}(f_\omega^k)^{-1}\xi\right)\nu^{\mathbb{N}}(\mathrm{d}\omega)$$

exists. The number $h_\mu\left(\mathcal{X}^+(\mathbb{R}^d,\nu),\xi\right)$ is called the entropy of $\mathcal{X}^+(\mathbb{R}^d,\nu)$ with respect to ξ. The number

$$h_\mu\left(\mathcal{X}^+(\mathbb{R}^d,\nu)\right) := \sup_\xi h_\mu\left(\mathcal{X}^+(\mathbb{R}^d,\nu),\xi\right)$$

is called the entropy of $\mathcal{X}^+(\mathbb{R}^d,\nu)$.

Consider the product spaces $\Omega^{\mathbb{N}}\times\mathbb{R}^d$ and $\Omega^{\mathbb{Z}}\times\mathbb{R}^d$. Since Ω equipped with the uniform topology on compact sets is a separable Banach space, the product σ-algebras $\mathcal{B}(\Omega)^{\mathbb{N}}\otimes\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}(\Omega)^{\mathbb{Z}}\otimes\mathcal{B}(\mathbb{R}^d)$ satisfy

$$\begin{aligned}\mathcal{B}(\Omega)^{\mathbb{N}}\otimes\mathcal{B}(\mathbb{R}^d) &= \mathcal{B}(\Omega^{\mathbb{N}}\times\mathbb{R}^d),\\ \mathcal{B}(\Omega)^{\mathbb{Z}}\otimes\mathcal{B}(\mathbb{R}^d) &= \mathcal{B}(\Omega^{\mathbb{Z}}\times\mathbb{R}^d).\end{aligned}$$

We denote the left shift operator on $\Omega^{\mathbb{N}}$ and $\Omega^{\mathbb{Z}}$ by τ, i.e.

$$f_n(\tau\omega) = f_{n+1}(\omega)$$

for all $\omega\in\Omega^{\mathbb{N}}$, $n\in\mathbb{N}$ and $\omega\in\Omega^{\mathbb{Z}}$, $n\in\mathbb{Z}$ respectively, and the associated skew product systems by

$$\begin{aligned}F &: \Omega^{\mathbb{N}}\times\mathbb{R}^d\to\Omega^{\mathbb{N}}\times\mathbb{R}^d, \quad (\omega,x)\to(\tau\omega, f_0(\omega)x),\\ G &: \Omega^{\mathbb{Z}}\times\mathbb{R}^d\to\Omega^{\mathbb{Z}}\times\mathbb{R}^d, \quad (\omega,x)\to(\tau\omega, f_0(\omega)x).\end{aligned}$$

First recall the following classical result which we already mentioned in Chapter 2 for continuous time systems.

Proposition 5.2.3. *Let μ be a probability measure on $\mathbb{R}^d$. Then μ is a stationary measure for $\mathcal{X}^+(\mathbb{R}^d,\nu)$ iff $\nu^{\mathbb{N}}\times\mu$ is an invariant measure for the one-sided skew product system F.*

Proof. See for example [60, Lemma I.2.3]. □

Furthermore, we have the following proposition which associates the invariant probability measure $\nu^{\mathbb{N}}\times\mu$ on $\Omega^{\mathbb{N}}\times\mathbb{R}^d$ to an invariant probability measure μ^* on $\Omega^{\mathbb{Z}}\times\mathbb{R}^d$.

Proposition 5.2.4. *For every stationary probability measure μ of $\mathcal{X}^+(\mathbb{R}^d,\nu)$ there exists a unique Borel probability measure μ^* on $\Omega^{\mathbb{Z}}\times\mathbb{R}^d$ such that $G\mu^*=\mu^*$ and $P\mu^*=\nu^{\mathbb{N}}\times\mu$, where P denotes the projection to the measures on $\Omega^{\mathbb{N}}\times\mathbb{R}^d$.*

Proof. See [74, Proposition I.1.2]. □

Hence, we have established a relationship between the one-sided and two-sided time system with respect to their invariant measures. We can also establish a relation in terms of entropy; for that we need to study conditional entropies with respect to the appropriate σ-algebras and their generating partitions. We define the following σ-algebras with their corresponding generating partitions. For $\Omega^{\mathbb{N}} \times \mathbb{R}^d$ we have

$$\sigma_0 := \left\{\Gamma \times \mathbb{R}^d \,:\, \Gamma \in \mathcal{B}(\Omega^{\mathbb{N}})\right\}, \quad \xi_0 := \left\{\{\omega\} \times \mathbb{R}^d \,:\, \omega \in \Omega^{\mathbb{N}}\right\},$$

and for $\Omega^{\mathbb{Z}} \times \mathbb{R}^d$ we define

$$\sigma^+ := \left\{ \left(\prod_{-\infty}^{-1} \Omega\right) \times \Gamma \times \mathbb{R}^d \,:\, \Gamma \in \mathcal{B}\left(\prod_{0}^{\infty} \Omega\right) \right\},$$
$$\xi^+ := \left\{ \left(\prod_{-\infty}^{-1} \Omega\right) \times \{\omega\} \times \mathbb{R}^d \,:\, \omega \in \left(\prod_{0}^{\infty} \Omega\right) \right\},$$

and

$$\sigma := \left\{\Gamma' \times \mathbb{R}^d \,:\, \Gamma' \in \mathcal{B}(\Omega^{\mathbb{Z}})\right\}, \quad \xi := \left\{\{\omega\} \times \mathbb{R}^d \,:\, \omega \in \Omega^{\mathbb{Z}}\right\}.$$

Generally, for a probability space $(X, \mathcal{B}, \mu)$, a σ-algebra $\mathcal{A} \subset \mathcal{B}$ and a measurable partition ζ of X we define the corresponding conditional entropy as

$$H_\mu(\zeta|\mathcal{A}) := -\int_X \sum_{c\in\zeta} \mu(C|\mathcal{A}) \ln \mu(C|\mathcal{A}) \,\mathrm{d}\mu\,.$$

Following [60, 74, Section 0.4 and Section 0.5], we obtain:

Definition and Lemma 5.2.5. Consider a measure-preserving transformation $T : X \to X$ and a σ-algebra $\mathcal{A} \subset \mathcal{B}$ with $T^{-1}\mathcal{A} \subset \mathcal{A}$. Then for any measurable partition ζ of X with $H_\mu(\zeta|\mathcal{A}) < \infty$ the limit

$$h_\mu^{\mathcal{A}}(T, \zeta) := \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{k=0}^{n-1} T^{-k}\zeta \middle| \mathcal{A}\right)$$

exists. The number $h_\mu^{\mathcal{A}}(T, \zeta)$ is called the $\mathcal{A}$-conditional entropy of T with respect to ζ. The number

$$h_\mu^{\mathcal{A}}(T) := \sup_\zeta h_\mu^{\mathcal{A}}(T, \zeta)$$

is called the $\mathcal{A}$-entropy of T.

We are now ready to state the very important theorem about the equality of entropy of the random dynamical system and the conditional entropies of the skew product systems, conditioned on the σ-algebras introduced above.

Theorem 5.2.6. *Let μ be a stationary probability measure of $\mathcal{X}^+(\mathbb{R}^d, \nu)$. Then the following equalities hold for the entropy of $\mathcal{X}^+(\mathbb{R}^d, \nu)$ and the conditional entropies of the skew product systems:*

$$h_\mu\left(\mathcal{X}^+(\mathbb{R}^d, \nu)\right) = h^{\sigma_0}_{\nu^{\mathbb{N}}\times\mu}(F) = h^{\sigma^+}_{\mu^*}(G) = h^{\sigma}_{\mu^*}(G)\,.$$

Proof. See [74, Proposition I.2.2 and I.2.3] □

One can define the entropies conditioned on the σ-algebras σ_0, σ^+ and σ also via the corresponding generating partitions ξ_0, ξ^+ and ξ. For details we refer to [15, Section 2]. It is important to state the relation between the σ-algebras and the associated partitions, since this is the key to proving Pesin's formula. It is also crucial for making the following observation: We can define

$$h_\mu(f_\omega, \zeta) := \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{k=0}^{n-1} (f^k_\omega)^{-1}\zeta\right),$$

in case the limit exists. Then according to [70, Proposition 2.1.2] we have for every partition ζ with $H_\mu(\zeta) < \infty$ that for almost all $\omega \in \Omega^{\mathbb{Z}}$

$$h_\mu(f_\omega, \zeta) = h^{\sigma}_{\mu^*}\left(G, \tilde{\zeta}\right),$$

where $\tilde{\zeta} = \{\Omega^{\mathbb{Z}} \times A : A \in \zeta\}$. In particular, we obtain

$$\sup_\zeta h_\mu(f_\omega, \zeta) = h^{\sigma}_{\mu^*}(G)\,.$$

Summarising, we observe that averaged, conditioned and fibrewise entropy in the ways defined above are all the same quantity.

Furthermore, we recall Oseledets' Multiplicative Ergodic Theorem 2.3.1 and apply it to (G, μ^*) without μ^* being necessarily ergodic, following [70, Section 2.2]. There is an Oseledets splitting

$$T_xM = E_1(\omega, x) \oplus \cdots \oplus E_{p(\omega,x)}(\omega, x)$$

such that for μ^*-a.e. (ω, x)

$$\lim_{n\to\infty} \frac{1}{n} \ln \|\mathrm{D}f^{\pm n}_\omega v\| = \pm\lambda_i(\omega, x) \;\text{ if } 0 \neq v \in E_i(\omega, x)\,. \tag{5.2.1}$$

The maps $(\omega, x) \mapsto p(\omega, x), \lambda_i(\omega, x), \dim E_i(\omega, x)$ are measurable and constant along orbits of G. In fact, there are functions $p, \lambda_i, d_i : M \to \mathbb{R}$ such that for μ^*-a.e. (ω, x)

$$p(\omega, x) = p(x),\;\; \lambda_i(\omega, x) = \lambda_i(x) \text{ and } \dim E_i(\omega, x) = d_i\,,$$

where d_i is the multiplicity of λ_i. As before, if μ^* is ergodic, these functions are constant, i.e. the x-dependence vanishes.

In [15] we find five integrability assumptions which we denote by (A1)-(A5). It is not necessary for our purposes to list these here specifically as the assumptions are satisfied for sufficiently regular stochastic flows anyway. The theorem confirming Pesin's formula in this setting reads as follows:

Theorem 5.2.7 (Pesin's formula)**.** *Let $\mathcal{X}^+(\mathbb{R}^d, \nu)$ be a random dynamical system which has an absolutely continuous stationary probability measure μ and satisfies (A1)-(A5). Then we have*

$$h_\mu\left(\mathcal{X}^+(\mathbb{R}^d, \nu)\right) = \int_{\mathbb{R}^d} \sum_i \lambda_i(x)^+ m_i(x) \mu(\mathrm{d}x)\,, \tag{5.2.2}$$

where $\lambda_i(x)^+$ are the positive Lyapunov exponents and $m_i(x)$ their multiplicities.

Proof. See [15]. □

Entropy for stochastic flows

We now make this result applicable to random dynamical systems induced by stochastic differential equations. In [15] the results are stated for general stochastic differential equations driven by semimartingales. Since all our models work with Brownian motion and time-independent vector fields, we transfer the general statements into this particular setting.

Consider the Itô SDE

$$\mathrm{d}X_t = F(X_t)\mathrm{d}t + G(X_t)\mathrm{d}W_t\,, \tag{5.2.3}$$

where $F : \mathbb{R}^d \to \mathbb{R}^d$ and $G : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are the drift and diffusion coefficients of the SDE and W_t is an m-dimensional Wiener process on the canonical filtered probability space of continuous paths $\left(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\geq 0}, \mathbb{P}\right)$ as introduced in Chapter 2. We further define the diffusion tensor

$$D_{ij}(x) = \sum_{k=1}^{m} G_{ik}(x) G_{jk}(x)\,.$$

Note that for the models in Chapter 3 and 4 Stratonovich and Itô noise are the same. In general, one has to account for the Itô-Stratonovich correction if one wants to relate the following results to results stated in Stratonovich form.

Assume that the entries of the diffusion matrix D and the drift F are in $C^{k,\delta}_{\mathrm{loc}}$ for some $k \geq 1, 0 < \delta \leq 1$ and satisfy a typical linear growth condition such that the SDE induces a C^k random dynamical system (θ, φ) in the sense of Chapter 2. We relate the system to a stochastic flow of C^k diffeomorphisms in the sense of [64] by defining the maps

$$\tilde{\varphi} : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \bar{\Omega} \times \mathbb{R}^d \to \mathbb{R}^d, \quad \tilde{\varphi}_{s,t}(\bar{\omega}, x) = \varphi(t-s, \theta_s \bar{\omega}, x)\,.$$

By this definition $\tilde{\varphi}_{s,t}(\bar{\omega},\cdot)$ is a C^k diffeomorphism for each $s,t\geq 0$ and $\bar{\omega}\in\bar{\Omega}$.

We can associate the stochastic flow $\tilde{\varphi}$ with the probability space $\left(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}}\right)$, where

$$\tilde{\Omega}=C_0\left(\mathbb{R},C(\mathbb{R}^d,\mathbb{R}^d)\right):=\left\{f\,:\,\mathbb{R}\to C(\mathbb{R}^d,\mathbb{R}^d)\,:\,f\text{ is continuous and }f(0)=0\right\}$$

is equipped with the topology of uniform convergence on compact sets and $\tilde{\mathcal{F}}:=\mathcal{B}(\tilde{\Omega})$. The measure $\tilde{\mathbb{P}}$ is defined by $\tilde{\mathbb{P}}(\tilde{\omega}(0)=\mathrm{Id})=1$ and the property that for all $n\geq 0, 0\leq t_1<t_2<\cdots<t_n$ and all $B\in\mathcal{B}(C(\mathbb{R}^d,\mathbb{R}^d))^{\otimes n}$ we have

$$\begin{aligned}&\tilde{\mathbb{P}}\left(\left(\tilde{\omega}(t_1),\tilde{\omega}(t_2)\circ\tilde{\omega}(t_1)^{-1},\ldots,\tilde{\omega}(t_n)\circ\tilde{\omega}(t_{n-1})^{-1}\right)\in B\right)\\&=\mathbb{P}\left(\left(\tilde{\varphi}_{t_0,t_1},\tilde{\varphi}_{t_1,t_2},\ldots,\tilde{\varphi}_{t_{n-1},t_n}\right)\in B\right)\,.\end{aligned}$$

Now let $k\geq 2$ and define Ω as above as the space of C^2 diffeomorphisms equipped with the uniform topology on compact sets. In this case, the measure

$$\nu(\cdot)=\mathbb{P}\{\bar{\omega}\in\bar{\Omega}\,:\,\tilde{\varphi}_{0,1}(\bar{\omega},\cdot)\in\cdot\}\tag{5.2.4}$$

on $(\Omega,\mathcal{B}(\Omega))$ and the random diffeomorphisms

$$f_0(\omega)=\tilde{\omega}(1)=\tilde{\varphi}_{0,1}(\bar{\omega},\cdot)=\varphi(1,\bar{\omega},\cdot)\tag{5.2.5}$$

generate, as before, a random dynamical system in discrete time

$$\mathcal{X}^+(\mathbb{R}^d,\nu)=\left\{f_\omega^n\,:\,n\geq 0,\omega\in\left(\Omega^{\mathbb{N}},\mathcal{B}(\Omega)^{\mathbb{N}},\nu^{\mathbb{N}}\right)\right\}\,.$$

Observe that the measure μ is stationary for this system if for any set $A\in\mathcal{B}(\mathbb{R}^d)$

$$\mu(A)=\int_{\bar{\Omega}}\mu\left((\tilde{\varphi}_{0,1}(\bar{\omega},\cdot))^{-1}(A)\right)\,\mathbb{P}(\mathrm{d}\bar{\omega})\,.$$

Let $P(t,x,\cdot)$ denote the transition probabilities associated to the stochastic differential equation. Then we make the following observation:

Lemma 5.2.8. *Any invariant probability measure ρ for the Markov semi-group associated to the stochastic differential equation* (5.2.3) *is stationary for the induced discrete time system $\mathcal{X}^+(\mathbb{R}^d,\nu)$.*

Proof. For all $A\in\mathcal{B}(\mathbb{R}^d)$ we have with Fubini that

$$\begin{aligned}\rho(A)&=\int_{\mathbb{R}^d}P(1,x,A)\,\rho(\mathrm{d}x)=\int_{\mathbb{R}^d}\int_{\bar{\Omega}}\mathbb{1}_A(\tilde{\varphi}_{0,1}(\bar{\omega},x))\,\mathbb{P}(\mathrm{d}\bar{\omega})\rho(\mathrm{d}x)\\&=\int_{\bar{\Omega}}\int_{\mathbb{R}^d}\mathbb{1}_A(\tilde{\varphi}_{0,1}(\bar{\omega},x))\,\rho(\mathrm{d}x)\mathbb{P}(\mathrm{d}\bar{\omega})=\int_{\bar{\Omega}}\rho\left((\tilde{\varphi}_{0,1}(\bar{\omega},\cdot))^{-1}(A)\right)\,\mathbb{P}(\mathrm{d}\bar{\omega})\,,\end{aligned}$$

which shows the claim. □

According to [64, Section 4.1], if D and F are $C^{k,\delta}_{\mathrm{loc}}$ for some $k \geq 1$ and $0 < \delta \leq 1$ and the correction term

$$c(x,t) := \sum_{j=1}^{d} \frac{\partial D_{\cdot,j}}{\partial x_j}(x) \tag{5.2.6}$$

is also $C^{k,\delta}_{\mathrm{loc}}$, then the backward flow $\{\tilde{\varphi}_{t,s} : 0 \leq s \leq t < \infty\}$ is also $C^{k,\delta}_{\mathrm{loc}}$.

Using the relations between stochastic flows, continuous-time random dynamical systems and discrete time random dynamical systems as explained above, we can now formulate Pesin's formula for random dynamical systems induced by stochastic differential equations:

Theorem 5.2.9 (Pesin's formula for SDEs)**.** *Let (θ, φ) be a random dynamical system induced by a stochastic differential equation with diffusion matrix D, drift F and correction term c, as given in* (5.2.6)*, all being $C^{k,\delta}_{\mathrm{loc}}$ for some $k \geq 2$. Let further be ρ an absolutely continuous stationary probability measure satisfying*

$$\int_{\mathbb{R}^d} (\ln(|x|+1)^{1/2}\, \rho(\mathrm{d}x) < \infty\,. \tag{5.2.7}$$

Then the discrete time random dynamical system $\mathcal{X}^+(\mathbb{R}^d, \nu)$ associated with (θ, φ) satisfies (A1)-(A5) and, hence,

$$h_\rho\left(\mathcal{X}^+(\mathbb{R}^d, \nu)\right) = \int_{\mathbb{R}^d} \sum_i \lambda_i(x)^+ m_i(x) \rho(\mathrm{d}x)$$

holds.

Proof. A direct consequence of [15, Theorem 9.1]. □

We are now in the situation to apply this result to the models we have discussed in the previous chapters and obtain a more profound notion of chaos.

Corollary 5.2.10. *Let $X = \mathbb{R} \times \mathbb{S}^1$ and $\mathcal{X}^+(X, \nu)$ denote the discrete-time random dynamical system induced by model (4.0.1) in the way explained above. Let further be ρ the stationary probability measure for the SDE* (4.0.1) *and f be chosen as in* (4.0.2)*. If $\sigma > \sigma_0(\alpha, b)$, then the measure-theoretic entropy of $\mathcal{X}^+(X, \nu)$ is positive, i.e.*

$$h_\rho\left(\mathcal{X}^+(X, \nu)\right) > 0\,.$$

Proof. The whole proof of Theorem 5.2.9 carries obviously over to $X \subset \mathbb{R}^d$. Condition 5.2.7 is satisfied due to the same considerations as in Section 4.2.3 concerning the boundedness by the Ornstein-Uhlenbeck process, i.e. the exponential decay of ρ in the amplitude.

The drift is smooth. The diffusion matrix D is given by

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and therefore $c(x,t) = 0$ for all x, t. Hence, all the conditions of Theorem 5.2.9 are satisfied. We further have for ρ-almost all x, $\mathbb{P}$-almost all $\bar{\omega}$ and $\nu^{\mathbb{N}}$-almost all ω that for all $0 \neq v \in X \setminus V_2(\bar{\omega}, x)$

$$\lambda_1 = \lim_{n\to\infty} \frac{1}{n} \|D_x f_\omega^n(x) v\| = \lim_{n\to\infty} \frac{1}{n} \|\mathrm{D}\varphi(n, \bar{\omega}, x) v\| > 0$$

according to Theorem 4.0.1. Hence, the claim follows. □

Remark 5.2.11. Since ρ is ergodic and λ_1 is the only positive exponent, we actually have that

$$h_\rho\left(\mathcal{X}^+(X,\nu)\right) = \lambda_1 .$$

In other words, the entropy of the random dynamical system is identical to the first Lyapunov exponent.

Remark 5.2.12. The analogous corollary can obviously be stated for system (3.0.1) once Conjecture 3.1.5 is shown. The conditions of Theorem 5.2.9 are easily satisfied in this case.

5.3 SRB measures

Another more profound notion of chaos could be given by showing the SRB-property of the random measures μ_ω, which are the disintegrations of an invariant probability measure μ^* for the two-sided skew product system, i.e. $\mu^*(\mathrm{d}x, \mathrm{d}\omega) = \mu_\omega(\mathrm{d}x)\nu^{\mathbb{Z}}(\mathrm{d}\omega)$. Let's assume we are in exactly the same setting of a discrete time random dynamical system as above with the only difference that the state space is a compact smooth manifold M, calling such a system $\mathcal{X}^+(M,\nu)$, generated by C^2 diffeomorphisms and a law ν. Let the sample measures μ_ω be associated with a stationary measure μ in the sense of Proposition 5.2.4 and write $E_i(\omega, x)$ for the Oseledets spaces corresponding with the Lyapunov exponents $\lambda_i(x)$. We follow [70] for the following definitions and results.

Definition 5.3.1. Let $(\omega, x) \in \Omega^{\mathbb{Z}} \times M$ be s.t. $\lambda_i(x) > 0$ for some i. Then the unstable manifold and the stable manifold of the skew product flow G at (ω, x) are given by

$$W^u(\omega, x) = \left\{ y \in M \,:\, \limsup_{n\to\infty} \frac{1}{n} \ln d(f_\omega^{-n} x, f_\omega^{-n} y) < 0 \right\},$$
$$W^s(\omega, x) = \left\{ y \in M \,:\, \limsup_{n\to\infty} \frac{1}{n} \ln d(f_\omega^{n} x, f_\omega^{n} y) < 0 \right\}.$$

At μ^*-a.e. (ω, x) with $\lambda_i(x) > 0$ for some i, $W^u(\omega, x)$ is a $\left(\sum_{\lambda_i>0} \dim E_i(\omega, x)\right)$-dimensional C^2 immersed submanifold of M. We set $W^u(\omega, x) = \{x\}$ if $\lambda_i(x) \leq 0$ for all i. If η is a partition of $\Omega^{\mathbb{Z}} \times M$, η_ω denotes the restriction of η to the fibre $\{\omega\} \times M$ which is a partition of M. We write $\eta_\omega(x)$ for the element of η_ω that contains x.

Definition 5.3.2. A measurable partition η of $\Omega^{\mathbb{Z}} \times M$ is called subordinate to W^u if for μ^*-a.e. (ω, x), $\eta_\omega(x) \subset W^u(\omega, x)$ and contains an open neighbourhood of x in $W^u(\omega, x)$, this neighbourhood being taken in the submanifold topology of $W^u(\omega, x)$.

Identifying σ-algebras with their generating partitions, recall that σ is the partition of $\Omega^{\mathbb{Z}} \times M$ into sets of the form $\{\omega\} \times M$. If η is a partition subordinate to W^u, μ^* disintegrates into a system of conditional measures on elements of $\eta \vee \sigma$, denoted by $\left\{\mu^{*\eta\vee\sigma}_{(\omega,x)}\right\}$. For μ^*-a.e. (ω, x) we have the identification $\mu^{*\eta\vee\sigma}_{(\omega,x)} = (\mu_\omega)^{\eta_\omega}_x$. Finally let $\lambda_{W^u(\omega,x)}$ denote the Riemannian measure on $W^u(\omega, x)$.

Definition 5.3.3 (SRB measures)**.** The sample measures μ_ω are called SRB measures or absolutely continuous conditional measures on W^u-manifolds if for every measurable partition η subordinate to W^u, $\mu^{*\eta\vee\sigma}_{(\omega,x)}$ is absolutely continuous with respect to $\lambda_{W^u(\omega,x)}$ for μ^*-a.e. (ω, x).

Ledrappier & Young [70] can then prove the following statement.

Theorem 5.3.4. *Suppose the stationary measure μ of the random dynamical system $\mathcal{X}^+(M, \nu)$ is absolutely continuous with respect to the Lebesgue measure and $\lambda_1 > 0$. Then the sample measures μ_ω are SRB measures.*

Similarly to before, we can then formulate the following corollary for stochastic differential equations. As usual for the manifold case, we use the Stratonovich integral due to its classical properties in terms of the chain rule:

Corollary 5.3.5. *Let (θ, φ) be a random dynamical system induced by a stochastic differential equation of type 5.1.1 with C^2 coefficients and stationary absolutely continuous probability distribution ρ on a compact manifold M. Let further $\lambda_1 > 0$ and $\tilde{\mu}$ be the invariant probability measure of (θ, φ) corresponding to ρ. Then the sample measures $\tilde{\mu}_{\bar{\omega}}$ are SRB measures.*

Proof. By Lemma 5.2.8, ρ is stationary for the induced discrete time system $\mathcal{X}^+(M, \nu)$. Then the claim follows immediately from Theorem 5.3.4 if we can show that $\tilde{\mu}_{\bar{\omega}}$ are the disintegrations μ_ω of the invariant measure μ^* of $\mathcal{X}^+(M, \nu)$ associated to ρ. According to [70, Proposition 1.2.3], the probability measures μ_ω are given by

$$\mu_\omega = \lim_{n\to\infty} f^n_{\tau^{-n}\omega}\rho \text{ for } \nu^{\mathbb{Z}}\text{-a.e. } \omega\,.$$

However, identifying ω and $\bar{\omega}$ via relation (5.2.5) we have seen in Section 2.1 that $\tilde{\mu}_{\bar{\omega}}$ satisfies

$$\tilde{\mu}_{\bar{\omega}} = \lim_{n\to\infty} \varphi(n, \theta_{-n}\bar{\omega}, \cdot)\rho = \lim_{n\to\infty} f^n_{\tau^{-n}\omega}\rho \text{ for } \nu^{\mathbb{Z}}\text{-a.e. } \omega\,.$$

Hence, the claim follows. □

We would like to apply this theorem to our setting and generally extend it to the non-compact case. Heuristically, this isn't a problem at all for dissipative systems with compact random attractors as for example models (3.0.1) and (4.0.1). However, the proof of Theorem 5.3.4 is technically very involved and makes a lot of references to the deterministic case [67, 68]. Hence, a rigorous proof analogous to [70] would require a complete own chapter. Checkroun et al. [26, Appendix] state a Theorem analogous to Corollary 5.3.5 for stochastic differential equations on $\mathbb{R}^d$ with global random attractors, also based on the results in [70]. However, they do not give a rigorous proof accounting for the non-compactness of the state space either.

Ignoring the mentioned technical intricacies, the picture is as follows. The chaotic random (point) attractors A_ω as depicted in Figures 1.2 and 4.3 are the support of the sample measures μ_ω and thereby non-singular subsets (maybe the same) of (as) the closures of the unstable manifolds $W^u(\omega, x)$ for all $x \in A_\omega$. The measures μ_ω are absolutely continuous with respect to Lebesgue measure on $W^u(\omega, x)$ which indicates chaotic motion on the attractor A_ω.

5.4 Discussion

The main purpose of this chapter has been to give a deeper understanding of random strange attractors from the viewpoint of ergodic theory. Firstly, we have slightly extended a result from [10] to show that positive Lyapunov exponents imply atomless invariant random measures μ_ω in situations such as given by model (4.0.1). Furthermore, we have shown that the positive Lyapunov exponent in model (4.0.1) for large enough noise or shear implies positive metric entropy. To obtain this result, we have followed [15] to state Pesin's formula for discrete-time random dynamical systems in non-compact space, and also for random dynamical systems induced by stochastic differential equations generating a discrete-time system via their time-one-maps.

Eventually, we have introduced the SRB property of measures which means absolute continuity on unstable manifolds. We have followed [70] to establish this property for the invariant random measures μ_ω of random diffeomorphisms on compact manifolds when the stationary measure μ is absolutely continuous. Furthermore, we have discussed the extension to systems induced by stochastic differential equations and given an interpretation of the SRB property for attractors as seen in Figures 1.2 and 4.3. We leave it as an open problem to transfer the very intricate proof to the non-compact setting, while com-

menting that, in principle, this should be possible for sufficiently regular and tempered systems.

A topic of this chapter has been the extension of results from non-compact to compact state space. In the following chapter, we will not need to care about such issues any longer as we will leave non-compact state space and turn to an analysis of trajectories that survive in a compact subdomain for asymptotically long times.

Chapter 6

Quasi-stationary dynamics and bifurcations

Note that model (3.0.1) operates with unbounded white noise whereas the nature of the deterministic Hopf bifurcation is local, in the sense that the bifurcation happens in a neighbourhood of the origin and $\alpha = 0$. The same problem holds for the typical example of stochastic pitchfork bifurcation where the drift f_α is given as the derivative of a potential V_α, i.e.

$$f_\alpha(x) = -\partial_x V_\alpha(x)\,, \quad \text{with } V_\alpha = -\frac{\alpha}{2}x^2 + \frac{1}{4}x^4\,. \tag{6.0.1}$$

Without noise, the bifurcation implies a change of the attractor from $\{0\}$ for $\alpha \leq 0$ to $[-\sqrt{\alpha}, \sqrt{\alpha}]$ for $\alpha > 0$. Recall that the bifurcation is "destroyed" in the presence of noise in the sense that the random attractor is a random equilibrium for all $\sigma > 0, \alpha \in \mathbb{R}$ [32]. The white noise lets the system explore the whole state space and the global stability results in a negative Lyapunov exponent. Local finite-time instabilities can be captured by the dichotomy spectrum Σ which is given by $\Sigma_\alpha = [-\infty, \alpha]$ for all $\alpha \in \mathbb{R}$. However, the dichotomy spectrum is generally not as directly interpretable as the sign of a Lyapunov exponent and still contains a measure of global stability by covering $\mathbb{R}_0^-$. The question is what kind of analysis can most accurately describe a local stochastic bifurcation, in particular if the system does not exhibit global stability outside a critical neighbourhood. If the system is not in normal form, such a problem naturally arises for pitchfork as well as Hopf bifurcations.

We tackle this problem by embedding stochastic bifurcation theory into the context of Markov processes that induce a random dynamical system and are absorbed at the boundary of a domain. The process is said to be killed when it hits the trap and it is assumed that this happens almost surely at a finite hitting time T. We investigate the asymptotic dynamics of surviving trajectories. Due to the loss of mass by absorption at the boundary, the existence of a stationary distribution is impossible and, therefore, stationarity is replaced by quasi-stationarity. A *quasi-stationary distribution* preserves

mass along the process conditioned on survival. Given a quasi-stationary distribution, one can derive the existence of a *quasi-ergodic distribution* for which expectations of time averages conditioned on survival equal the space average with respect to this distribution. We introduce these concepts and summarise important results, in particular for stochastic differential equations, following [25, 24, 28, 75], in Section 6.1.

In Section 6.2, we develop a theory of asymptotic average Lyapunov exponents for systems absorbed at the boundary of a domain. We mainly focus on stochastic differential equations with additive noise, i.e.

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\,,\ X_0 = \xi \in E\,,$$

where f is continuously differentiable and $E \subset \mathbb{R}^d$ a bounded domain. Using work by Villemonais, Champagnat, He and others [25, 24, 50], we are able to show Proposition 6.2.3 which says that the conditioned expectation of the finite-time Lyapunov exponents of a system induced by such an SDE converges to a real number λ. This *conditioned average Lyapunov exponent* is given by a Furstenberg-Khasminskii-formula, i.e. the average of a functional with respect to the quasi-ergodic distribution.

Furthermore, we can show in Theorem 6.2.8 that the finite-time Lyapunov exponents of the surviving trajectories converge to its assemble average λ in probability. Note that this gives λ the strongest possible dynamical meaning in the setting with absorption at the boundary since convergence almost surely is ruled out by the killing of almost all trajectories. The crucial ingredient for proving Theorem 6.2.8 is Lemma 6.2.7 which shows decay of correlations conditioned on survival.

Section 6.3 gives negative λ a dynamical interpretation. We prove the local synchronisation Theorem 6.3.1 which says that, if $\lambda < 0$, there is exponentially fast local synchronisation of trajectories in discrete time with arbitrarily high probability. We formulate the result for general differentiable random dynamical systems with killing and observe the implications for additive noise SDEs as a corollary.

In Section 6.4 we try to relate quasi-stationary and quasi-ergodic measures to sample measures of the killed random dynamical system. Leaving out the past of the system enables us to show Propositions 6.4.2 and 6.4.4 which establish a correspondence with conditionally invariant measures for the associated open system on the skew-product space where the hole is confined to the state space. In two-sided time it turns out to be unclear how one could relate quasi-stationary or quasi-ergodic distributions to conditionally invariant measures and in particular its sample or fibre measures. We discuss some ideas into this direction, as for example the survival process, but remain sceptical whether this is a feasible endeavour.

We further define the dichotomy spectrum for the situation with killing in Section 6.5 and prove in Theorem 6.5.9 that the essential supremum and infimum of finite-time Lyapunov exponents converge to the boundary of the dichotomy spectrum. In addition to

that, we show that the dichotomy spectrum consists of a finite number $n \in \{1, \dots, d\}$ of closed intervals (Theorem 6.5.8), where d is the dimension of the space.

In Section 6.6 we consider the examples of pitchfork and Hopf bifurcation with additive noise and approximate the quasi-ergodic distribution using a finite-difference scheme. We analyse the change of sign of λ and the corresponding bifurcation behaviour depending on the bifurcation parameter and the diameter of the domain. We also determine the dichotomy spectrum for the pitchfork problem with killing. Following considerations about the possibility of a Lyapunov spectrum in Section 6.2 we conduct numerical experiments for the Hopf example that indicate its existence. However, a proof seems out of reach.

In short, this chapter is structured as follows. Section 6.1 gives an overview over results concerning killed processes and quasi-stationary distributions which we need in the following. In Section 6.2 we prove the existence of a characteristic Lyapunov exponent λ for killed systems generated by additive noise SDEs and the fact that finite-time Lyapunov exponents converge to this quantity in conditioned probability. Section 6.3 discusses local synchronisation results for negative λ and Section 6.4 relates the developed theory of quasi-stationary dynamics to open systems on the skew product space. In Section 6.5 we transfer important results for exponential dichotomies of random dynamical systems to the situation with killing. Section 6.6 discusses the stochastic pitchfork and Hopf bifurcation in bounded domains with absorbing boundary, using the stability theory developed in the previous sections.

6.1 Quasi-stationary and quasi-ergodic distributions

6.1.1 General setting

Let $(X_t)_{t\geq 0}$ be a time-homogeneous Markov process (see e.g. [81, Definition III.1.1]) on a topological state space E with boundary ∂E and Borel σ-algebra $\mathcal{E} := \mathcal{B}(E \cup \partial E)$, where the process is associated with a family of probabilities $(\mathbb{P}_x)_{x\in E}$ on a filtered space $(\Omega, (\mathcal{F}_t)_{t\geq 0})$. We have

$$\mathbb{P}_x(X_0 = x) = 1 \quad \text{for all } x \in E \cup \partial E$$

and the transition probabilities $(\hat{P}_t)_{t\geq 0}$ are given by

$$\hat{P}_t(x, A) = \mathbb{P}_x(X_t \in A) \quad \text{for all } x \in E \cup \partial E, A \in \mathcal{E}\,.$$

The process is further associated with a semi-group of operators $(P_t)_{t\geq 0}$ given by

$$P_t f(x) = \mathbb{E}_x[f(X_t)]$$

for $f \in \tilde{\mathcal{E}} := B(E \cup \partial E)$, the measurable and bounded functions from $E \cup \partial E$ to $\mathbb{R}$.

We consider the Markov process to be absorbed at ∂E, i.e. $X_s \in \partial E$ implies $X_t = X_s$

for all $t \geq s$. This implies that the random variable

$$T := \inf\{t \geq 0, X_t \in \partial E\}$$

is a stopping time and we let $X_t = X_T$ for all $t \geq T$. We make two assumptions that cover all the problems we are interested in: firstly, we assume for all $x \in E$ that

$$T < \infty \, \mathbb{P}_x - \text{a.s.}\,, \tag{6.1.1}$$

which means that almost every trajectory hits the boundary in finite time. Secondly, we demand that for all $x \in E$ and $t \geq 0$

$$\mathbb{P}_x(T > t) > 0\,, \tag{6.1.2}$$

i.e. that for any initial condition, the probability of survival until any given time is positive. We will further mainly consider Markov processes which induce a random dynamical system but we will specify this later.

Quasi-stationary distributions

For any measure μ on E we will use the notation

$$\mathbb{P}_\mu = \int_E \mathbb{P}_x \, \mu(\mathrm{d}x)\,.$$

Almost every statement in random dynamical system theory requires a stationary measure for the underlying Markov process $(X_t)_{t\geq 0}$ which is a measure μ on E with the property that

$$\mathbb{P}_\mu(X_t \in A) = \mu(A) \quad \text{for all measurable } A \subset E\,, t \geq 0\,.$$

Note that in the situation with trapping at the boundary such a measure cannot exist: assume there was a stationary measure μ. Due to (6.1.1) there is a $t^*(\mu) > 0$ such that for all $t > t^*(\mu)$

$$1 > \int_E \mathbb{P}_x(T > t)\, \mu(\mathrm{d}x) = \mathbb{P}_\mu(X_t \in E) = \mu(E) = 1\,,$$

which is a contradiction. This leads to the following definition.

Definition 6.1.1 (QSD)**.** A *quasi-stationary distribution* (QSD) is a probability measure ν on E such that for all $t \geq 0$ and all measurable sets $B \subset E$

$$\mathbb{P}_\nu\left(X_t \in B | T > t\right) = \nu(B)\,. \tag{6.1.3}$$

Without stating any further assumptions, we can make the following well-known ob-

servation [47] about the exponentially distributed killing time of a process started with a QSD ν.

Proposition 6.1.2. *If ν is a QSD, then there exists a $\lambda_0 < 0$ such that for all $t \geq 0$*

$$\mathbb{P}_\nu(T > t) = e^{\lambda_0 t}\,,$$

that is, starting from ν, T is exponentially distributed with parameter λ_0. We call λ_0 the (exponential) survival rate and $-\lambda_0$ the (exponential) escape rate associated with the quasi-stationary distribution.

Proof. We follow [28]: from the definition of a QSD and a classical application of the Monotone-Class Theorem [81, Theorem II.3.1], we have for all measurable and bounded observables $g : E \to \mathbb{R}$ that

$$\mathbb{E}_\nu(g(X_t)\mathbb{1}_{T>t}) = \left(\int_E g\,\mathrm{d}\mu\right)\mathbb{P}_\nu(T > t) \quad \text{for all } t \geq 0\,.$$

Choosing $g(x) = \mathbb{P}_x(T > s)$ for some $s \geq 0$ we obtain

$$\mathbb{E}_\nu(\mathbb{P}_{X_t}(T > s)\mathbb{1}_{T>t}) = \mathbb{P}_\nu(T > s)\mathbb{P}_\nu(T > t) \quad \text{for all } t \geq 0\,.$$

Using the Markov property and the commutation property of conditional expectations, we deduce for all $t, s \geq 0$ that

$$\begin{aligned}\mathbb{P}_\nu(T > t + s) &= \mathbb{E}_\nu(\mathbb{1}_{T>t+s}) = \mathbb{E}_\nu\left(\mathbb{E}(\mathbb{1}_{T>t+s}|\mathcal{F}_t)\mathbb{1}_{T>t}\right)\\ &= \mathbb{E}_\nu(\mathbb{E}_{X_t}(\mathbb{1}_{T>s})\mathbb{1}_{T>t}) = \mathbb{P}_\nu(T > s)\mathbb{P}_\nu(T > t)\,.\end{aligned}$$

Since the equality $f(t+s) = f(t)f(s)$ is only satisfied by exponential functions, the claim follows. □

Champagnat and Villemonais [25] have given three equivalent conditions for exponential convergence to a quasi-stationary distribution. We restrict ourselves to formulating the weakest assumption among them, denoted by (A') in the original paper. This condition will turn out to be satisfied by the stochastic differential equations we are investigating:

Assumption (A) There exists a family of probability measures $(\nu_{x_1,x_2})_{x_1,x_2\in E}$ on E such that

(A1) there exist $t_0, c_1 > 0$ such that for all $x_1, x_2 \in E$,

$$\mathbb{P}_{x_i}(X_{t_0} \in \cdot | T > t_0) \geq c_1\nu_{x_1,x_2}(\cdot) \quad \text{for } i = 1, 2\,;$$

(A2) there exists $c_2 > 0$ such that for all $x_1, x_2 \in E$ and $t \geq 0$,

$$\mathbb{P}_{\nu_{x_1,x_2}}(T > t) \geq c_2 \sup_{x \in E} \mathbb{P}_x(T > t)\,.$$

We summarise three results from [25] in the following theorem which contains the most relevant ingredients for our further purposes. Statement (a) guarantees the existence of a QSD with exponential convergence of initial distributions. Part (b) characterizes the limit of the survival probability for an initial Dirac distribution at x divided by the survival probability under the QSD as the value at x of an eigenfunction of the generator $\mathcal{L}$ with eigenvalue λ_0 from Proposition 6.1.2. Statement (c) implies λ_0 being the largest non-zero eigenvalue of $\mathcal{L}$ and the existence of a spectral gap. We sketch the proofs of (b) and (c) as η and λ_0 will be crucial objects in this chapter. Note that Champagnat and Villemonais [25] are working in the more general setting of measurable spaces and therefore without the notion of a boundary. In their case, the role of the boundary is replaced by a cemetery state $\{\partial\}$. In the proof of the following theorem, we account for this slight technical difference which does not change anything about the statements made in this chapter.

Theorem 6.1.3 (Exponential convergence to QSD and dominant survival rate as eigenvalue of the generator)**.**

(a) Assumption (A) is equivalent to the existence of a unique quasi-stationary probability measure ν on E and two constants $C, \gamma > 0$ such that for all initial distributions μ on E

$$\|\mathbb{P}_\mu(X_t \in \cdot | T > t) - \nu(\cdot)\|_{TV} \leq Ce^{-\gamma t} \text{ for all } t \geq 0\,, \tag{6.1.4}$$

where $\|\mathbb{P} - \mathbb{Q}\|_{TV} := \sup_{A \in \mathcal{F}} |\mathbb{P}(A) - \mathbb{Q}(A)|$ denotes the total variation distance for probability measures. In words: the process starting from any initial distribution μ, in particular $\mu = \delta_x$ for $x \in E$, converges exponentially fast to the QSD.

(b) In the situation of (a), we can define a non-negative function η on $E \cup \partial E$, positive on E and vanishing on ∂E, by

$$\eta(x) := \lim_{t \to \infty} \frac{\mathbb{P}_x(T > t)}{\mathbb{P}_\nu(T > t)} = \lim_{t \to \infty} e^{-\lambda_0 t} \mathbb{P}_x(T > t)\,, \tag{6.1.5}$$

where the convergence holds uniformly in $E \cup \partial E$ and $\int \eta \, \mathrm{d}\nu = 1$.

Furthermore, η is a bounded eigenfunction of the infinitesimal generator $\mathcal{L}$ of the semi-group $(P_t)_{t \geq 0}$ on $(\tilde{\mathcal{E}}, \|\cdot\|_\infty)$ with eigenvalue λ_0, i.e.

$$\mathcal{L}\eta = \lambda_0 \eta\,,$$

where $-\lambda_0$ is the exponential escape rate as in Proposition 6.1.2.

(c) Let Assumption (A) hold and $f \in \tilde{\mathcal{E}}$ be an eigenfunction of $\mathcal{L}$ for an eigenvalue λ, being constant on ∂E. Then either

(i) $\lambda = 0$ and f is constant,

(ii) or $\lambda = \lambda_0$, $f = \left(\int f \,\mathrm{d}\nu\right)\eta$ and $f|_{\partial E} \equiv 0$,

(iii) or $\lambda \leq \lambda_0 - \gamma$, $\int f \,\mathrm{d}\nu = 0$ and $f|_{\partial E} \equiv 0$.

Proof. Part (a) is a shortened version of [25, Theorem 2.1].

Part (b) is contained in [25, Proposition 2.3]. Its proof uses the following fact: Let $\mathcal{M}_1(E)$ denote the set of probability measures on E. Then it can be shown that Assumption (A) implies that for any $\mu \in \mathcal{M}_1(E)$ the constant $c_2(\mu)$ defined by

$$c_2(\mu) := \inf_{t\geq 0, \rho\in\mathcal{M}_1(E)} \frac{\mathbb{P}_\mu(T > t)}{\mathbb{P}_\rho(T > t)} \tag{6.1.6}$$

is positive. This implies immediately that $\eta(x)$ is positive if it exists. Its existence follows from showing (by using (a)) that

$$\sup_{x\in E} |\eta_{t+s}(x) - \eta_t(x)| \leq \frac{C}{c_2(\nu)} e^{-\gamma t}\,,$$

where

$$\eta_t(x) := \frac{\mathbb{P}_x(T > t)}{\mathbb{P}_\nu(T > t)}\,.$$

That η is vanishing on ∂E follows directly from its definition.

The claim in (c) is essentially [25, Corollary 2.4]. Let $\mathcal{L}f = \lambda f$. By definition of the generator, we obtain

$$\mathbb{E}_x(f(X_t)) = P_t f(x) = e^{\lambda t} f(x)\,.$$

When $f|_{\partial E} \equiv c \neq 0$, we observe, by taking $x \in \partial E$, that $\lambda = 0$. On the other hand for any $x \in E$, the left hand side converges to c and therefore f is constant. This shows (i).

Let now $f|_{\partial E} \equiv 0$. This entails together with (a) that

$$\frac{P_t f(x)}{P_t \mathbb{1}_E(x)} = \frac{\mathbb{E}_x[f(X_t)]}{\mathbb{P}_x(X_t \in E)} = \frac{\mathbb{E}_x[f(X_t)\mathbb{1}_E(X_t)]}{\mathbb{E}_x[\mathbb{1}_E(X_t)]} = \mathbb{E}_x[f(X_t)|T > t] \xrightarrow{t\to\infty} \int f \,\mathrm{d}\nu$$

uniformly in $x \in E$ and exponentially fast. To obtain (ii), we first assume that $\int f \,\mathrm{d}\nu \neq 0$. Then we obtain from (b) and the above that for all $x \in E$

$$\frac{e^{(\lambda-\lambda_0)t} f(x)}{\eta(x)} = \frac{e^{-\lambda_0 t}\mathbb{P}_x(T > t)}{\eta(x)} \frac{P_t f(x)}{P_t \mathbb{1}_E(x)} \xrightarrow{t\to\infty} \int f \,\mathrm{d}\nu\,.$$

This implies that $\lambda = \lambda_0$ and $f(x) = \left(\int f \,\mathrm{d}\nu\right)\eta(x)$ for all $x \in E$.

We finally assume $\int f \,\mathrm{d}\nu = 0$. By definition of c_2 we deduce that for all $x \in E$

$$c_2(\nu)e^{(\gamma+\lambda-\lambda_0)t}f(x) \leq \frac{e^{\gamma t}P_t f(x)}{P_t \mathbb{1}_E(x)} \xrightarrow{t\to\infty} \int f \,\mathrm{d}\nu\,.$$

The right hand-side is bounded by (a) and, hence, we obtain $\gamma + \lambda + \lambda_0 \leq 0$, which shows (iii). □

Quasi-ergodic distributions

By a classical application of the Monotone-Class Theorem [81, Theorem II.3.1], Theorem 6.1.3 implies that for all bounded and measurable functions $h : E \to \mathbb{R}$ we have

$$\lim_{t\to\infty} \mathbb{E}_x(h(X_t)|T > t) = \int_E h(y)\nu(\mathrm{d}y) \quad \text{uniformly in } x \in E\,. \tag{6.1.7}$$

However, as we want to study ergodic quantities like Lyapunov exponents, we are interested in time averages. This motivates the following definition.

Definition 6.1.4 (QED)**.** A probability measure m on E is called *quasi-ergodic distribution* (QED) if for all $t > 0$, every bounded and measurable function $h : E \to \mathbb{R}$ and every $x \in E$, the following limit exists and satisfies

$$\lim_{t\to\infty} \mathbb{E}_x\left(\frac{1}{t}\int_0^t f(X_s)\mathrm{d}s \middle| T > t\right) = \int_E f\mathrm{d}m\,. \tag{6.1.8}$$

The next theorem tells us that in the situation of Theorem 6.1.3 the quasi-ergodic distribution m exists and is absolutely continuous with respect to the quasi-stationary distribution ν. The density is exactly the function η from Theorem 6.1.3. In the proof we follow He et al. [50] who made this observation very recently based on [25]. We give the whole proof here as we will use its techniques later on.

Theorem 6.1.5 (Existence of unique QED)**.** *Assume that the process $(X_t)_{t\geq 0}$ on $E \cup \partial E$ with killing at the boundary ∂E satisfies Assumption A. Then $(X_t)_{t\geq 0}$ has a unique quasi-ergodic distribution m, where the convergence in* (6.1.8) *is uniform over all $x \in E$ and m possesses a density*

$$m(\mathrm{d}x) = \eta(x)\nu(\mathrm{d}x)\,.$$

Proof. Observe from (6.1.5) that $\int_E m(\mathrm{d}x) = \int_E \eta(x)\nu(\mathrm{d}x) = 1$. So m is a probability measure on E. Fix $u > 0$ and define $h_u : E \to \mathbb{R}_0^+$ by

$$h_u(x) = \inf_{t\geq u}\left(e^{-\lambda_0 t}\frac{\mathbb{P}_x(T > t)}{\eta(x)}\right)\,.$$

Let $f : E \to \mathbb{R}_0^+$ be bounded and measurable. Let further be $0 < q < 1$ and $(1-q)t \geq u$.

Then we obtain for all $x \in E$ that

$$\begin{aligned}\mathbb{E}_x(f(X_{qt})|T > t) &= \frac{\mathbb{E}_x[f(X_{qt})\mathbb{1}_{\{T>t\}}]}{\mathbb{P}_x(T > t)} \\ &= \frac{\mathbb{E}_x[f(X_{qt})\mathbb{1}_{\{T>qt\}}\mathbb{P}_{X_{qt}}(T > (1-q)t)]}{\mathbb{P}_x(T > t)}\,,\end{aligned}$$

where we used the Markov property. Hence, we can infer that

$$\begin{aligned}\mathbb{E}_x(f(X_{qt})|T > t) &= \frac{e^{-\lambda_0 qt}\mathbb{E}_x[f(X_{qt})\mathbb{1}_{\{T>qt\}}e^{-\lambda_0(1-q)t}\mathbb{P}_{X_{qt}}(T > (1-q)t)]}{e^{-\lambda_0 t}\mathbb{P}_x(T > t)} \\ &\geq \frac{e^{-\lambda_0 qt}\mathbb{E}_x[f(X_{qt})\mathbb{1}_{\{T>qt\}}h_u(X_{qt})\eta(X_{qt})]}{e^{-\lambda_0 t}\mathbb{P}_x(T > t)}\,,\end{aligned}$$

According to Theorem 6.1.3 η is bounded and the convergence of the limit, via which η is defined is uniform in x. Hence, there exists a constant $C > 0$ such that for all $t \geq u$ and $x \in E$

$$|f(x)h_u(x)\eta(x)| \leq \left|f(x)e^{-\lambda_0 t}\mathbb{P}_x(T > t)\right| \leq \|f\|_\infty C\|\eta\|_\infty\,. \tag{6.1.9}$$

Thus, the function $fh_u\eta$ is bounded and obviously measurable. We observe from property (6.1.7), the definition of η and the above that uniformly over all $x \in E$

$$\begin{aligned}\liminf_{t\to\infty}\mathbb{E}_x(f(X_{qt})|T > t) &\geq \lim_{t\to\infty}\frac{e^{-\lambda_0 qt}\mathbb{E}_x[f(X_{qt})h_u(X_{qt})\eta(X_{qt})\mathbb{1}_{\{T>qt\}}]}{e^{-\lambda_0 t}\mathbb{P}_x(T > t)} \\ &= \lim_{t\to\infty}\frac{e^{-\lambda_0 qt}\mathbb{P}_x(T > qt)}{e^{-\lambda_0 t}\mathbb{P}_x(T > t)}\mathbb{E}_x[f(X_{qt})h_u(X_{qt})\eta(X_{qt})|T > qt] \\ &= \int_I f(x)h_u(x)\eta(x)\nu(\mathrm{d}x)\,.\end{aligned}$$

Due to (6.1.9) and the fact that $h_u(x) \to 1$ for all $x \in E$, we can apply the dominated convergence theorem to conclude that

$$\liminf_{t\to\infty}\mathbb{E}_x(f(X_{qt})|T > t) \geq \int_I f(x)m(\mathrm{d}x)\,.$$

Replacing f by $\|f\|_\infty - f$, we can see easily that

$$\limsup_{t\to\infty}\mathbb{E}_x(f(X_{qt})|T > t) \leq \int_I f(x)m(\mathrm{d}x)\,.$$

Therefore we have shown that for all bounded and positive functions f

$$\lim_{t\to\infty}\mathbb{E}_x(f(X_{qt})|T > t) = \int_I f(x)m(\mathrm{d}x)\,,$$

uniformly over $x \in E$. We can extend the result to arbitrary measurable and bounded

f by writing $f = f_+ - f_-$ for $f_+ := \max\{f, 0\}$, $f_- := -\min\{f, 0\}$ and using linearity. Finally, with a change of variables, Fubini and the dominated convergence theorem we obtain for all bounded and measurable functions $f : E \to \mathbb{R}$ that uniformly over $x \in E$

$$\lim_{t\to\infty} \mathbb{E}_x \left(\frac{1}{t} \int_0^t f(X_s)\, \mathrm{d}s | T > t \right) = \lim_{t\to\infty} \mathbb{E}_x \left(\int_0^1 f(X_{qt})\, \mathrm{d}q | T > t \right)$$
$$= \lim_{t\to\infty} \int_0^1 \mathbb{E}_x \left(f(X_{qt}) | T > t \right)\, \mathrm{d}q = \int_E f(x)\, m(\mathrm{d}x)\,.$$

Uniqueness is immediate from the definition of a quasi-ergodic distribution. □

The measure m has an additional meaning as we learn in [25]. We will see that we can define the Q-process (Y_t) with probabilities $(\mathbb{Q}_x)_{x\in E}$ such that for any $s \geq 0$

$$\mathbb{Q}_x((Y_u)_{0\leq u\leq s} \in \cdot) = \lim_{t\to\infty} \mathbb{P}_x((X_u)_{0\leq u\leq s} \in \cdot | T > t)\,.$$

The Q-process is also called the survival process since its finite-time distributions equal the ones of the original process $(X_t)_{t\geq 0}$ conditioned on asymptotic survival. We will come back to the potential role the Q-process could play within a random dynamical systems theory of killed processes.

The measure m turns out to be the unique invariant probability measure of the Markov semigroup associated with Y_t. We give the following short version of [25, Theorem 3.1]:

Theorem 6.1.6 (Q-process and QED)**.** *Assumption (A) implies*

(i) the existence of the Q-process: there exists a family $(\mathbb{Q}_x)_{x\in E}$ of probability measures on Ω defined by

$$\lim_{t\to\infty} \mathbb{P}_x(A | T > t) = \mathbb{Q}_x(A)$$

for all $\mathcal{F}_s$-measurable sets A for any given $s \geq 0$. The process $((Y_t)_{t\geq 0}, (\mathbb{Q}_x)_{x\in E})$ on $(\Omega, (\mathcal{F}_t)_{t\geq 0})$ is an E-valued time-homogeneous Markov process. In addition, if $(X_t)_{t\geq 0}$ is a strong Markov process under $(\mathbb{P}_x)_{x\in E}$, then so is (Y_t) under $(\mathbb{Q}_x)_{x\in E}$.

(ii) exponential ergodicity: the probability measure m on E defined by

$$m(\mathrm{d}x) = \eta(x)\nu(\mathrm{d}x)$$

is the unique invariant distribution of $((Y_t)_{t\geq 0}, (\mathbb{Q}_x)_{x\in E})$. Furthermore, there are $C_1, \gamma_1 > 0$ such that for any initial distribution μ on E we obtain

$$\|\mathbb{Q}_\mu(Y_t \in \cdot) - m(\cdot)\|_{TV} \leq C_1 e^{-\gamma_1 t} \quad \text{for all } t \geq 0\,.$$

Proof. See [25, Theorem 3.1] which we have slightly reformulated for convenience of the reader. □

6.1.2 Stochastic differential equations

Consider the Markov process $(X_t)_{t\geq 0}$ as a solution of a stochastic differential equation

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t) \circ dW_t,\ X_0 \in E\,, \tag{6.1.10}$$

in a bounded connected domain $E \subset \mathbb{R}^d$ with absorption at the boundary ∂E which is assumed to be C^2. (W_t) denotes some r-dimensional standard Brownian motion, $f : E \to \mathbb{R}^d$ a Lipschitz-continuous vector field and $g : E \to \mathbb{R}^{d\times r}$ is a differentiable, Lipschitz-continuous matrix-valued map such that gg^* is uniformly elliptic and the Itô-Stratonovich correction term is also Lipschitz continuous (see Appendix A.1). Champagnat et al. [24] can then prove a result which immediately implies the following:

Theorem 6.1.7 (QSD and QED for stochastic differential equations)**.** *If $(X_t)_{t\geq 0}$ is the solution process of the stochastic differential equation* (6.1.10) *in a bounded connected domain $E \subset \mathbb{R}^d$ with absorption at the C^2 boundary ∂E and f and g are as above, Assumption (A) is satisfied. In particular,*

(a) there is a QSD ν and $C > 0, \gamma > 0$ such that for all probability measures μ on E

$$\|\mathbb{P}_\mu(X_t \in \cdot | T > t) - \nu(\cdot)\|_{TV} \leq Ce^{-\gamma t},\ \text{for all } t \geq 0\,.$$

Furthermore, there is a subset $D \subset \mathcal{D}(\mathcal{L})$ of the domain of the generator $\mathcal{L}$ on $E \cup \partial E$ such that

$$\int \mathcal{L}f \,\mathrm{d}\nu = \lambda_0 \int f \,\mathrm{d}\nu\,,\quad \text{for all } f \in D\,,$$

i.e. ν is an eigenmeasure of $\mathcal{L}^*$ for the largest non-zero eigenvalue $\lambda_0 < 0$ of $\mathcal{L}$ with Dirichlet boundary conditions on $E \cup \partial E$. As before, $-\lambda_0$ is the exponential escape rate as given in Proposition 6.1.2.

(b) there is a QED m given by

$$m(\mathrm{d}x) = \eta(x)\nu(\mathrm{d}x)\,,$$

where

$$\eta(x) = \lim_{t\to\infty} \frac{\mathbb{P}_x(T > t)}{\mathbb{P}_\nu(T > t)} = \lim_{t\to\infty} e^{-\lambda_0 t}\mathbb{P}_x(T > t)$$

is a bounded eigenfunction of $\mathcal{L}$ for eigenvalue λ_0, as in Theorem 6.1.3 (b).

(c) the QED m is the unique invariant distribution of the Q-process $((Y_t)_{t\geq 0}, (\mathbb{Q}_x)_{x\in E})$. Furthermore, there are $C_1, \gamma_1 > 0$ such that for any initial distribution μ on E we obtain

$$\|\mathbb{Q}_\mu(Y_t \in \cdot) - m(\cdot)\|_{TV} \leq C_1 e^{-\gamma_1 t}\quad \text{for all } t \geq 0\,.$$

Proof. See [24, Theorem 3.1] for showing that the process satisfies Assumption (A). The statement about ν being an eigenmeasure of $\mathcal{L}^*$ is a direct consequence of [75, Proposition

4]. The other implications are taken from Theorem 6.1.3 and 6.1.5 applying the statements to the situation of the SDE (6.1.10). □

Let us now consider a case where ν and m can be determined as eigenfunctions of the generator $\mathcal{L}$. For that purpose we study a special case of (6.1.10), namely a stochastic differential equation of the form

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + \sigma\,\mathrm{d}W_t\,, \tag{6.1.11}$$

where $X_0 \in I$, $I = (l, r)$ for some $l, r \in \mathbb{R}$ and $f \in C^1(I) \cap C(\bar{I})$. First observe that if we consider the process on the real line and if $\exp\left(\int_{-\infty}^{x} f(y)\mathrm{d}y\right)$ is integrable, the process has a stationary measure with density

$$p(x) = \frac{1}{Z}\exp\left(\frac{2}{\sigma^2}\int_{-\infty}^{x} f(y)\mathrm{d}y\right)\,,$$

where $Z > 0$ is the normalisation constant. Following [28, Section 6.1.1] we define

$$\gamma(x) := \frac{2}{\sigma^2}\int_{l}^{x} f(y)\,\mathrm{d}y\,.$$

Furthermore, we define the measure

$$\mu(\mathrm{d}x) := \exp\left(\gamma(x)\right)\mathrm{d}x$$

on I. Consider the generator of the semigroup associated with (6.1.11)

$$\mathcal{L}\cdot = \frac{\sigma^2}{2}\partial_{xx}\cdot + f\partial_x\cdot\,, \tag{6.1.12}$$

with Dirichlet boundary conditions at $x = l$ and $x = r$, and its formal adjoint

$$\mathcal{L}^*\cdot = \frac{\sigma^2}{2}\partial_{xx}\cdot - \partial_x(f\cdot)\,. \tag{6.1.13}$$

With standard theory (see e.g. [27, Chapter 7]), we observe that $\mathcal{L}$ is self-adjoint in $L^2([l, r], \mathrm{d}\mu)$ and possesses a complete orthonormal basis of eigenfunctions in $L^2([l, r], \mathrm{d}\mu)$. We can apply Theorem 6.1.3 or deploy the well-known theory for one-dimensional second-order linear ODEs, as used in [28, Lemma 6.1], to observe the following properties of the eigenvalues $\lambda_n \neq 0$ for $n \geq 0$: Each λ_n is simple, negative and the only possible accumulation point of the set $\{\lambda_n : n \geq 1\}$ is $-\infty$. As before, we write without loss of generality

$$0 > \lambda_0 > \lambda_1 > \cdots > \lambda_n > \lambda_{n+1} > \dots\,.$$

We denote by ψ_{λ_n} the unique solution to

$$\mathcal{L}\psi = \lambda_n\psi,\ \psi(l) = \psi(r) = 0,\ \int_I \psi^2 \mathrm{d}\mu = 1,\ \psi'(l) > 0\,.$$

Note that ψ_{λ_n} is smooth for all $n \geq 0$ due to the ellipticity of $\mathcal{L}$. We further observe that $\phi_{\lambda_n}(x) = \psi_{\lambda_n}(x)\exp(\gamma(x))$ satisfies

$$\mathcal{L}^*\phi = \lambda_n\phi,\ \phi(l) = \phi(r) = 0,\ \phi'(0) > 0\,.$$

Following [28], we obtain the following formula for the killed semigroup P_t defined as an operator on bounded and measurable observables $h : I \to \mathbb{R}$ by

$$P_t h(x) = \mathbb{E}_x[h(X_t)\mathbb{1}_{\{T>t\}}]\,.$$

Note that the result is even stated for the larger space $L^2(I, \mathrm{d}\mu)$.

Proposition 6.1.8. *For all t, all $x \in I$ and all $h \in L^2(I, \mathrm{d}\mu)$,*

$$\mathbb{E}_x[h(X_t)\mathbb{1}_{\{T>t\}}] = \sum_n e^{\lambda_n t}\left(\int_I h(y)\psi_{\lambda_n}(y)\,\mu(\mathrm{d}y)\right)\psi_{\lambda_n}(x)\,.$$

Proof. See [28, Proposition 6.2]. □

From this formula we can immediately derive the following similarly to [28, Theorem 6.4]:

Theorem 6.1.9. *The unique quasi-stationary distribution ν for the process $(X_t)_{t\geq 0}$ on I derived from* (6.1.11) *with absorption at the boundary ∂I is given by*

$$\nu(\mathrm{d}x) = \frac{\psi_{\lambda_0}(x)\mu(\mathrm{d}x)}{\int \psi_{\lambda_0}(y)\mu(\mathrm{d}y)} = \frac{\phi_{\lambda_0}(x)\,\mathrm{d}x}{\int \phi_{\lambda_0}(y)\,\mathrm{d}y}\,.$$

The QED m can be written as

$$m(\mathrm{d}x) = \psi^2_{\lambda_0}(x)\mu(\mathrm{d}x)$$

and we have

$$\eta(x) = \left(\int_I \psi_{\lambda_0}(y)\mu(\mathrm{d}y)\right)\psi_{\lambda_0}(x)\,.$$

The statements from Theorem 6.1.7 about exponential convergence for all initial distributions hold for ν and m.

Proof. Obviously all assumptions are satisfied to apply Theorem 6.1.7. Hence, we know already that there is a unique QSD. Furthermore, we obtain from Proposition 6.1.8 that

for all $h \in L^2([l,r], \mathrm{d}\mu)$ and $x \in I$,

$$\lim_{t\to\infty} e^{-\lambda_0 t}\mathbb{E}_x[h(X_t)\mathbb{1}_{\{T>t\}}] = \psi_{\lambda_0}(x)\int_I h(y)\psi_{\lambda_0}(y)\,\mu(\mathrm{d}y)\,.$$

In particular, by taking $h \equiv 1$, we obtain

$$\lim_{t\to\infty} e^{-\lambda_0 t}\mathbb{P}_x(T>t) = \psi_{\lambda_0}(x)\int_I \psi_{\lambda_0} y)\,\mu(\mathrm{d}y)\,.$$

So we can deduce that for all bounded and measurable functions h and $x \in I$ we have the limit

$$\lim_{t\to\infty} \mathbb{E}_x(h(X_t)|T>t) = \int_I h(y)\,\nu(\mathrm{d}y)\,,$$

where ν is given as in the statement of the theorem. Hence, ν is the unique QSD. From the fact that η has to be proportional to ψ_{λ_0} and the normalisation condition on ψ_{λ_0}, we get the expressions for m and η. □

6.2 Lyapunov exponents and local stability

We are turning to the study of killed processes from a random dynamical systems perspective. In this section, we mainly investigate the existence of asymptotic average Lyapunov exponents and the convergence behaviour of finite-time exponents to such quantities.

Let $(X_t)_{t\geq 0}$ be a time-homogeneous Markov process on a topological state space E with absorption at the boundary ∂E, where the process possesses the family of transition probabilities $(\mathbb{P}_x)_{x\in E}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Further, let there be a random dynamical system $(\theta, \hat{\varphi})$ associated with this process such that

$$\mathbb{P}_x(X_t \in B) = \mathbb{P}(\hat{\varphi}(t,\cdot,x) \in B) \quad \text{for all } \ t \geq 0, x \in E, B \in \mathcal{B}(\bar{E})\,.$$

We encounter such a setting, for example, if the random dynamical system is induced by (6.1.10) in the canonical way. In such a situation, we can make the following definition.

Definition and Lemma 6.2.1. Let $(\theta, \hat{\varphi})$ denote the random dynamical system associated with a Markov process $(X_t)_{t\geq 0}$ on a topological state space E with absorption at the boundary ∂E. Define $\tilde{T} : \Omega \times E \to \mathbb{R}_0^+$ as

$$\tilde{T}(\omega, x) = \inf\{t > 0 \,:\, \hat{\varphi}(t,\omega,x) \in \partial E\}$$

such that for all $x \in E$ and $t \geq 0$

$$\mathbb{P}_x(T > t) = \mathbb{P}(\tilde{T}(\cdot, x) > t)\,.$$

Then together with (θ, Ω), the map $\varphi : \mathbb{R}_0^+ \times \Omega \times \bar{E} \to \bar{E}$ given by

$$\varphi(t,\omega,x) = \begin{cases} \hat{\varphi}(t,\omega,x) & \text{if } t < \tilde{T}(\omega,x)\,, \\ \hat{\varphi}(\tilde{T}(\omega,x),\omega,x) & \text{if } t \geq \tilde{T}(\omega,x)\,, \\ x & \text{if } x \in \partial E\,. \end{cases}$$

constitutes a random dynamical system.

Proof. Measurability and $\varphi(0,\omega,\cdot) = \mathrm{id}$ are clear from the definition. It remains to check the cocycle property by distinguishing different cases. Fix $(\omega, x) \in \Omega \times \bar{E}$. First note from the cocycle property of the original system $\hat{\varphi}$ that for $s < \tilde{T}(\omega, x)$ we have

$$\hat{\varphi}(\tilde{T}(\omega,x) - s, \theta_s\omega, \hat{\varphi}(s,\omega,x)) = \hat{\varphi}(\tilde{T}(\omega,x),\omega,x)\,,$$

and therefore

$$\tilde{T}(\theta_s\omega, \varphi(s,\omega,x)) = \tilde{T}(\theta_s\omega, \hat{\varphi}(s,\omega,x)) = \tilde{T}(\omega,x) - s\,. \tag{6.2.1}$$

Hence, if $t + s < \tilde{T}(\omega, x)$, it follows that

$$\varphi(t+s,\omega,x) = \hat{\varphi}(t+s,\omega,x) = \hat{\varphi}(t,\theta_s\omega,\hat{\varphi}(s,\omega,x)) = \varphi(t,\theta_s\omega,\varphi(s,\omega,x))\,.$$

Now let $t + s \geq \tilde{T}(\omega, x)$: If $t, s \geq \tilde{T}(\omega, x)$, we have by definition of φ

$$\varphi(t+s,\omega,x) = \hat{\varphi}(\tilde{T}(\omega,x),\omega,x) = \varphi(t,\theta_s\omega,\hat{\varphi}(\tilde{T}(\omega,x),\omega,x)) = \varphi(t,\theta_s\omega,\varphi(s,\omega,x))\,.$$

If w.l.o.g $s < \tilde{T}(\omega, x)$, we obtain from (6.2.1) that $t \geq \tilde{T}(\omega,x) - s \geq \tilde{T}(\theta_s\omega, \varphi(s,\omega,x))$ and therefore

$$\varphi(t+s,\omega,x) = \varphi(\tilde{T}(\omega,x),\omega,x) = \varphi(t,\theta_s\omega,\varphi(s,\omega,x))\,.$$

This concludes the proof. □

Remark 6.2.2. Note that Definition and Lemma 6.2.1 defines a local random dynamical system in the sense of [2, Definition 1.2.1] with extension to the boundary. The domain $D \subset \mathbb{R}_0^+ \times \Omega \times E$ of the local random dynamical system satisfies

$$D(\omega,x) = \{t \in \mathbb{R}_0^+ \,:\, (t,\omega,x) \in D\} = [0, \tilde{T}(\omega,x))\,,$$

and

$$D(t,\omega) = \{x \in E \,:\, (t,\omega,x) \in D\} = \{x \in E \,:\, \tilde{T}(\omega,x) > t\}\,.$$

The classical motivation to consider local random dynamical systems is the possible explosion of solutions for a stochastic or random differential equation in an unbounded state

space outside the domain D. If the eternal survival sets

$$E(\omega) = \bigcap_{t\in\mathbb{R}_0^+} D(t,\omega)$$

are non-empty almost surely, any invariant random measure has to be supported on these fibres and the formulation of a Multiplicative Ergodic Theorem is possible for such an invariant measure. However, in the situations we are interested in, i.e. stochastic differential equations with additive noise on a bounded domain $E \subset \mathbb{R}^d$, $E(\omega)$ is empty almost surely. Hence, the problem demands for a new method describing asymptotic expansion and contraction rates, using the idea of quasi-ergodic distributions.

Note that φ, as defined in Definition and Lemma 6.2.1, is not continuous at $x \in \partial E$ for all $t > 0$ and $\omega \in \Omega$, so φ isn't a continuous random dynamical system in the sense of Definition 2.0.1. However, for any x in E and $\omega \in \Omega$, the system φ is continuous in x for all $t < \tilde{T}(\omega, x)$, and even differentiable if $\hat{\varphi}$ is. In the situation of (6.1.10) this is the case if the coefficients are differentiable. We can consider the finite-time Lyapunov exponents

$$\lambda_v(t,\omega,x) = \frac{1}{t}\ln\frac{\|\mathrm{D}\varphi(t,\cdot,x)v\|}{\|v\|} \quad \text{for } t < \tilde{T}(\omega,x)\,,$$

where $\mathrm{D}\varphi = D\hat{\varphi}$ solves the variational equation corresponding to (6.1.10) given by

$$\mathrm{d}Y(t,\omega,x) = \mathrm{D}f(\varphi(t,\omega,x))Y(t,\omega,x)\,\mathrm{d}t + \sum_{j=1}^{r} Dg^j(\varphi(t,\omega,x))Y(t,\omega,x)\circ\mathrm{d}W_t^j\,, \quad (6.2.2)$$

where $Y(0,\omega,x) = \mathrm{Id}$, and g^j denotes the j-th column of g and W_t^j the j-th entry of W_t. We want to investigate the convergence behaviour of the finite-time Lyapunov exponents under conditioning to absorption at the boundary.

We restrict ourselves to problems with additive noise as this will be enough for our most relevant examples. Consider the stochastic differential equation

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\,,\ X_0 = \xi \in E\,, \quad (6.2.3)$$

where $E \subset \mathbb{R}^d$ is a bounded connected domain with C^2 boundary ∂E and $f : E \to \mathbb{R}^d$ is a continuously differentiable vector field with bounded derivative. Then all the conditions of Theorem 6.1.7 are satisfied. Hence, the quasi-stationary distribution ν on E is a limiting distribution whose density ϕ vanishes at the boundary and satisfies

$$\mathcal{L}^*\phi = \lambda_0\phi\,,$$

where $\mathcal{L}^*$ is the formal adjoint of the generator $\mathcal{L}$ which is given by

$$\mathcal{L} = f \cdot \nabla + \frac{1}{2}\sigma^2 \nabla^2 .$$

Furthermore we know that

$$\mathcal{L}\eta = \lambda_0 \eta ,$$

and that the quasi-ergodic distribution m satisfies

$$m(\mathrm{d}x) = \eta(x)\nu(\mathrm{d}x) .$$

Hence, in principle these measures can be calculated explicitly.

The Jacobian $\mathrm{D}\varphi$ of the RDS is the solution of the variational equation, which in this case reads

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t,\omega,x) = \mathrm{D}f(\varphi(t,\omega,x))Y(t,\omega,x), \quad Y(0,\omega,x) = \mathrm{Id} . \tag{6.2.4}$$

A first question to ask is if there are limits for the average finite-time Lyapunov exponents as in the classical case. That means that we want to find out if for $v \in \mathbb{R}^d \setminus \{0\}$ there are $\lambda_v \in \mathbb{R}$ such that for all $x \in E$

$$\lambda_v := \lim_{t\to\infty} \mathbb{E}\left[\lambda_v(t,\cdot,x) | \tilde{T}(\cdot,x) > t\right] = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\ln \frac{\|\mathrm{D}\varphi(t,\cdot,x)v\|}{\|v\|} \middle| \tilde{T}(\cdot,x) > t\right] . \tag{6.2.5}$$

Indeed, we can show the following modified Furstenberg-Khasminskii-formula:

Proposition 6.2.3 (Conditioned average Lyapunov exponent)**.** *Let (θ,φ) be the random dynamical system with absorption at the boundary corresponding to the Markov process $(X_t)_{t\geq 0}$ solving equation* (6.2.3)*, and let*

$$s_t = \frac{\mathrm{D}\varphi(t,\cdot,\cdot)}{\|\mathrm{D}\varphi(t,\cdot,\cdot)\|}$$

denote the induced process on the unit sphere of the tangent space. If the generator $\tilde{\mathcal{L}}$ of $(X_t, s_t)_{t\geq 0}$ is hypoelliptic, then for all $v \in \mathbb{R}^d \setminus \{0\}$ the average exponent λ_v as defined in (6.2.5) *exists and is given independently from v by*

$$\lambda_v = \lambda := \int_{\mathbb{S}^{d-1}\times E} \langle s, \mathrm{D}f(y)s\rangle \ \tilde{m}(\mathrm{d}s, \mathrm{d}y), \tag{6.2.6}$$

where $\tilde{m}$ is the quasi-ergodic joint distribution of $(X_t, s_t)_{t\geq 0}$ and the convergence is uniform over all $x \in E$ and $v \in \mathbb{R}^d \setminus \{0\}$. We call λ the conditioned average Lyapunov exponent.

Proof. Note that the angular component s_t as defined above lies on the unit sphere $\mathbb{S}^{d-1}$ and write $r_t = \|\mathrm{D}\varphi(t,\cdot,\cdot)\|$ for the radial component. The variational equation (6.2.4) in

vector polar coordinates is given by

$$\begin{aligned} \mathrm{d}s_t &= \mathrm{D}f(\varphi(t,\cdot,\cdot))s_t - \langle s_t, \mathrm{D}f(\varphi(t,\cdot,\cdot))s_t\rangle s_t \,\mathrm{d}t\,,\ s_0 \in \mathbb{S}^{d-1}\,,\\ \mathrm{d}r_t &= \langle s_t, \mathrm{D}f(\varphi(t,\cdot,\cdot))s_t\rangle r_t \,\mathrm{d}t\,,\ r_0 = 1\,. \end{aligned}$$

We obtain for all $\omega \in \Omega$, $x \in E$

$$r_t(\omega, x) = r_0 \exp\left(\int_0^t h(\varphi(\tau,\omega,x), s_\tau(\omega,x))\,\mathrm{d}\tau\right)\,,$$

where $h : E \times \mathbb{S}^{d-1} \to \mathbb{R}$ is given by

$$h(x,s) = \langle s, \mathrm{D}f(x)s\rangle\,.$$

We observe that $(X_t, s_t)_{t\geq 0}$ constitutes a skew product system on $E \times \mathbb{S}^{d-1}$ with killing at $\partial E \times \mathbb{S}^{d-1}$. To apply the theory of Section 6.1.1, we need to check that Assumption (A) is satisfied. We know that $(X_t)_{t\geq 0}$ satisfies the assumption on $E \cup \partial E$, i.e. there is a family (ν_{x_1,x_2}) that fulfils (A1) and (A2) for some constants $t_0, c_1, c_2 > 0$. Due to the hypoellipticity condition, there exists a $c_0 > 0$ and a family of probability measures (μ_{z_1,z_2}) such that for any $z_i = (x_i, s_i) \in E\times\mathbb{S}^{d-1}, i = 1, 2$ and $A \in \mathcal{B}(E)$ with $\nu_{x_1,x_2}(A) > 0$

$$\mathbb{P}_{z_i}(s_{t_0} \in \cdot | X_{t_0} \in A, T > t_0) \geq c_0 \mu_{z_1,z_2}(\cdot)\,,$$

for similar reasons as in the proof of [24, Theorem 3.1]. We define the family of probability measures

$$\tilde{\nu}_{z_1,z_2}(A\times B) = \nu_{x_1,x_2}(A)\mu_{z_1,z_2}(B) \ \text{ for all measurable } A \subset E, B \subset \mathbb{S}^{d-1}, z_1, z_2 \in E\times\mathbb{S}^{d-1}\,.$$

Since $(X_t)_{t\geq 0}$ and therefore T are independent from $(s_t)_{t\geq 0}$, we observe that for all $z_1, z_2 \in E \times \mathbb{S}^{d-1}$ and measurable $A \subset E, B \subset \mathbb{S}^{d-1}$

$$\begin{aligned} \mathbb{P}_{z_i}((X_{t_0}, s_{t_0}) \in A \times B | T > t_0) &= \frac{\mathbb{P}_{z_i}((X_{t_0}, s_{t_0}) \in A \times B, T > t_0)}{\mathbb{P}_{x_i}(T > t_0)}\\ &= \mathbb{P}_{z_i}(s_{t_0} \in B | X_{t_0} \in A, T > t_0)\mathbb{P}_{x_i}(X_{t_0} \in A | T > t_0)\\ &\geq c_0\mu_{z_1,z_2}(B)\mathbb{P}_{x_i}(X_{t_0} \in A | T > t_0) \geq c_0 c_1 \tilde{\nu}_{z_1,z_2}(A\times B)\,. \end{aligned}$$

This shows (A1). Using again the independence of the hitting time T from s_t, (A2) follows by observing that for all $z_1, z_2 \in E \times \mathbb{S}^{d-1}$

$$\mathbb{P}_{\nu_{z_1,z_2}}(T > t) = \int_{\mathbb{S}^{d-1}}\int_E \mathbb{P}_x(T > t)\,\nu_{x_1,x_2}(\mathrm{d}x)\,\mu_{z_1,z_2}(\mathrm{d}s) \geq c_2 \sup_{x\in E, s\in\mathbb{S}^{d-1}} \mathbb{P}_{x,s}(T > t)\,.$$

Theorem 6.1.3 and Theorem 6.1.5 allow us to conclude that there are a unique QSD $\tilde{\nu}$

and associated QED $\tilde{m}$ on $E \times \mathbb{S}^{d-1}$ which due to the skew product structure have ν and m as their marginals on E. Hence, by definition of a quasi-ergodic measure and the fact that h is bounded and measurable by the assumptions, we conclude that for all $v \in \mathbb{R}^d$ with $v = \|1\|$ (which is enough for the claim) and $x \in E$

$$\begin{aligned}\lambda_v &= \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_{x,s_0}[\ln r_t | T > t] = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_x[\ln r_t | T > t] \\ &= \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\int_0^t h(\varphi(\tau,\cdot,x), s_\tau(\cdot,x))\,\mathrm{d}\tau \middle| \tilde{T}(\cdot,x) > t\right] = \int h\,\mathrm{d}\tilde{m}\,,\end{aligned}$$

where the convergence is uniform according to Theorem 6.1.5. This concludes the proof of the proposition. □

Remark 6.2.4. In principle, we can extend this result to the general situation of (6.1.10). We refrain from doing this here for two reasons. Firstly, in case of a nonlinear diffusion term g, the functional that has to be added to $\langle s, \mathrm{D}f(x)s\rangle$ assumes a complicated shape including second derivatives, if the original and linearised process have interfering noise terms. We avoid the loss of clarity and comprehensibility of this situation. Secondly, the relevant examples of this thesis have additive noise terms such that the formulation of Proposition 6.2.3 suffices for our purposes.

In one dimension, the problem is reduced to considering systems on an interval $I \subset \mathbb{R}$ induced by the one-dimensional SDE (6.1.11), where $f \in C^1(I) \cap C(\bar{I})$ and f' is bounded on I. In this case the finite-time exponents are simply given by

$$\lambda(t,\omega,x) = \frac{1}{t}\ln|\mathrm{D}\varphi(t,\omega,x)| \quad \text{for } t < \tilde{T}(\omega,x)\,,$$

where $\mathrm{D}\varphi(t,\omega,x)$ solves the linear variational equation for (6.1.11)

$$\dot{v}(t,\omega,x) = f'(\varphi(t,\omega,x))v(t,\omega,x), \quad v(0,\omega,x) = 1\,, \quad \text{for all } x \in I, \omega \in \Omega.$$

So in this one-dimensional scenario we can immediately infer that

$$\lambda(t,\omega,x) = \frac{1}{t}\int_0^t f'(\varphi(s,\omega,x))\,\mathrm{d}s \quad \text{for } t < \tilde{T}(\omega,x)\,.$$

Let us write $\psi = \psi_{\lambda_0}$ from now on. We can show the following formula for the conditioned average Lyapunov exponent λ in the one-dimensional scenario in congruence with Proposition 6.2.3.

Proposition 6.2.5 (Conditioned average Lyapunov exponent in one dimension)**.** *Let (θ,φ) be the random dynamical system on $\bar{I} \subset \mathbb{R}$ induced by* (6.1.11) *with absorption at the boundary, where $f \in C^1(I) \cap C(\bar{I})$ and f' is bounded on I. Then the conditioned*

average Lyapunov exponent λ is given, independently from $x \in I$, by

$$\lambda = \lim_{t\to\infty} \mathbb{E}\left(\lambda(t,\cdot,x)|\tilde{T}(\cdot,x) > t\right) = \int_I f'(y)\, m(\mathrm{d}y) = \int_I f'(y)\psi^2(y)e^{\gamma(y)}\,\mathrm{d}y \tag{6.2.7}$$

for all $x \in I$.

Proof. The claim follows immediately from Theorem 6.1.5 by using that f' is bounded and measurable on I which implies, by definition of the QED m, that

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}\left(\lambda(t,\cdot,x)|\tilde{T}(\cdot,x) > t\right) &= \lim_{t\to\infty} \mathbb{E}\left(\frac{1}{t}\int_0^t f'(\varphi(s,\cdot,x))\,\mathrm{d}s|\tilde{T}(\cdot,x) > t\right)\\ &= \lim_{t\to\infty} \mathbb{E}_x\left(\frac{1}{t}\int_0^t f'(X_s)\,\mathrm{d}s|T > t\right) = \int_I f'(y)\, m(\mathrm{d}y)\,.\end{aligned}$$

The formula for this integral is taken from Theorem 6.1.9. □

Remark 6.2.6. To obtain a priori estimates on the sign of λ, we could try to use the fact that ψ is an eigenfunction of $\mathcal{L}$ for the eigenvalue λ_0, i.e.

$$\frac{1}{2}\sigma^2\psi''(x) + f(x)\psi'(x) = \lambda_0\psi(x)\,. \tag{6.2.8}$$

Using integration by parts, we observe that

$$\begin{aligned}\int_I f'(x)\psi^2(x)e^{\gamma(x)}\mathrm{d}x &= -\int_I f(x)\left(\frac{2}{\sigma^2}f(x)\psi^2(x)e^{\gamma(x)} + 2\psi(x)\psi'(x)e^{\gamma(x)}\right)\mathrm{d}x\\ &= \underbrace{-\frac{2}{\sigma^2}\int_I f^2(x)m(\mathrm{d}x)}_{\leq 0} \underbrace{-2\int_I f(x)\psi'(x)\psi(x)e^{\gamma(x)}\,\mathrm{d}x}_{\text{sign unclear}}\,.\end{aligned}$$

Note that the second term vanishes in case $I = \mathbb{R}$ since ψ is a constant function in this case and m is the stationary distribution. That is how one observes negativity of the Lyapunov exponent in the classical setting. In our context, the sign of the second term depends on the product $f(x)\psi'(x)$ which makes a direct a priori estimate impossible. Using (6.2.8), we can rewrite it as

$$\begin{aligned}\int_I f'(x)\psi^2(x)e^{\gamma(x)}\mathrm{d}x &= -\int_I f(x)\left(\frac{2}{\sigma^2}f(x)\psi^2(x)e^{\gamma(x)} + 2\psi(x)\psi'(x)e^{\gamma(x)}\right)\mathrm{d}x\\ &= \underbrace{-\frac{2}{\sigma^2}\int_I f^2(x)\, m(\mathrm{d}x)}_{\leq 0}\underbrace{-2\lambda_0}_{>0} + \underbrace{\sigma^2\int_I \psi''(x)\psi(x)e^{\gamma(x)}\,\mathrm{d}x}_{\text{sign unclear}}\,.\end{aligned}$$

Again, we remain with a term whose sign is unclear, this time depending on $\psi''(x)$. It appears not possible to obtain general statements about the sign of λ.

We observe that λ as defined in Proposition 6.2.3 (and given in Proposition 6.2.5 for

one dimension) is defined as a limit of conditioned expected values. In random dynamical system theory, however, we are usually interested in ω-wise asymptotic statements. Due to the killing at the boundary, the best we can hope for in this context is a convergence result of finite-time Lyapunov exponents in probability. Indeed, we are able to prove such a result in Theorem 6.2.8 which shows that this number λ actually has a dynamical meaning. Before we can prove the Theorem, we show the following Lemma about decay of correlations:

Lemma 6.2.7. *Let $(X_t)_{t\geq 0}$ be a Markov process on a topological state space E with absorption at the boundary ∂E as introduced in Section 6.1.1. Then for any measurable and bounded functions $f, g : E \to \mathbb{R}$, $0 < r < q < 1$ we have*

$$\lim_{t\to\infty} \mathbb{E}_x \left(f(X_{qt})g(X_{rt})|T > t\right) = \left(\int f \mathrm{d}m\right)\left(\int g \mathrm{d}m\right)$$

uniformly over all $x \in E$.

Proof. Let $f, g : X \to \mathbb{R}_0^+$ be measurable and bounded functions, $0 < r < q < 1$ and $x \in E$. Similarly to the proof of Theorem 6.1.5, we fix $u > 0$ and define the observable

$$h_u(x) = \inf\{e^{-\lambda_1 t}\mathbb{P}_x(T > t)/\eta(x) \,:\, t \geq u\}\,.$$

Let t be large enough such that $(q-r)t \geq u$ and $(1-q)t \geq u$. We obtain with the Markov property

$$\begin{aligned}
\mathbb{E}_x\left[f(X_{qt})g(X_{rt})|T > t\right] &= \frac{\mathbb{E}_x[g(X_{rt})f(X_{qt})\mathbb{1}_{\{T>t\}}]}{\mathbb{P}_x(T > t)} \\
&= \frac{\mathbb{E}_x[g(X_{rt})f(X_{qt})1_{\{T>qt\}}\mathbb{P}_{X_{qt}}(T > (1-q)t)]}{\mathbb{P}_x(T > t)} \\
&\geq \frac{\mathbb{E}_x[g(X_{rt})\mathbb{1}_{\{T>qt\}}e^{-\lambda_0(q-1)t}f(X_{qt})h_u(X_{qt})\eta(X_{qt})]}{\mathbb{P}_x(T > t)}\,.
\end{aligned}$$

Let us denote $\rho(x) = f(x)h_u(x)\eta(x)$ and $\tilde{\rho}(x) = g(x)h_u(x)\eta(x)$ which are positive and bounded for the same reasons as in the proof of Theorem 6.1.5. Analogously to the above, we obtain

$$\begin{aligned}
&\mathbb{E}_x\left(f(X_{qt})g(X_{rt})|T > t\right) \geq e^{-\lambda_0(q-1)t}\frac{\mathbb{E}_x\left[g(X_{rt})\mathbb{1}_{\{T>rt\}}\mathbb{E}_{X_{rt}}[\mathbb{1}_{\{T>(q-r)t\}}\rho(X_{(q-r)t})]\right]}{\mathbb{P}_x(T > t)} \\
&\geq e^{-\lambda_0(q-1)t}\mathbb{E}_x\left[\frac{g(X_{rt})h_u(X_{rt})\eta(X_{rt})\mathbb{1}_{\{T>rt\}}\mathbb{E}_{X_{rt}}[\mathbb{1}_{\{T>(q-r)t\}}\rho(X_{(q-r)t})]}{\mathbb{P}_x(T > t)\mathbb{P}_{X_{rt}}(T > t)e^{-\lambda_0 t}}\right] \\
&= \mathbb{E}_x\left[\frac{e^{-\lambda_0 rt}\tilde{\rho}(X_{rt})\mathbb{1}_{\{T>rt\}}}{\mathbb{P}_x(T > t)e^{-\lambda_0 t}}\frac{e^{-\lambda_0(q-r)t}\mathbb{E}_{X_{rt}}[\mathbb{1}_{\{T>(q-r)t\}}\rho(X_{(q-r)t})]}{\mathbb{P}_{X_{rt}}(T > t)e^{-\lambda_0 t}}\right]\,.
\end{aligned}$$

By Theorem 6.1.7 the limits of the killed semigroup are uniform in x. For the second

term in the expectation above we obtain

$$\begin{aligned}
&\left|\mathbb{1}_{\{T>rt\}}\frac{e^{-\lambda_0(q-r)t}\mathbb{E}_{X_{rt}}[\mathbb{1}_{\{T>(q-r)t\}}\rho(X_{(q-r)t})]}{\mathbb{P}_{X_{rt}}(T>t)e^{-\lambda_0 t}} - \int \rho\,\mathrm{d}\nu\right| \\
=&\left|\mathbb{1}_{\{T>rt\}}\frac{e^{-\lambda_0(q-r)t}\mathbb{P}_{X_{rt}}(T>(q-r)t)}{e^{-\lambda_0 t}\mathbb{P}_{X_{rt}}(T>t)}\mathbb{E}_{X_{rt}}[\rho(X_{(q-r)t})|T>(q-r)t] - \int \rho\,\mathrm{d}\nu\right| \\
\leq& Ce^{-\gamma t} + \|\rho\|_\infty\left|\mathbb{1}_{\{T>rt\}}\frac{e^{-\lambda_0(q-r)t}\mathbb{P}_{X_{rt}}(T>(q-r)t)}{e^{-\lambda_0 t}\mathbb{P}_{X_{rt}}(T>t)} - 1\right| \\
\leq& Ce^{-\gamma t} + \|\rho\|_\infty \sup_{x\in E}\left|\frac{e^{-\lambda_0(q-r)t}\mathbb{P}_x(T>(q-r)t)}{e^{-\lambda_0 t}\mathbb{P}_x(T>t)} - \frac{\eta(x)}{\eta(x)}\right|
\end{aligned}$$

where the second term converges to 0 as $t\to\infty$ according to Theorem 6.1.7. Hence, the second factor in the expectation converges uniformly to its limit

$$\int \rho\,\mathrm{d}\nu = \int f h_u\,\mathrm{d}m\,.$$

Therefore we observe that

$$\begin{aligned}
&\liminf_{t\to\infty}\mathbb{E}_x\left(f(X_{qt})g(X_{rt})|T>t\right) \\
&\geq \lim_{t\to\infty}\mathbb{E}_x\left[\frac{e^{-\lambda_0 rt}\tilde{\rho}(X_{rt})\mathbb{1}_{\{T>rt\}}}{\mathbb{P}_x(T>t)e^{-\lambda_0 t}}\frac{e^{-\lambda_0(q-r)t}\mathbb{E}_{X_{rt}}[\mathbb{1}_{\{T>(q-r)t\}}\rho(X_{(q-r)t})]}{\mathbb{P}_{X_{rt}}(T>t)e^{-\lambda_0 t}}\right] \\
&= \lim_{t\to\infty}\frac{\mathbb{E}_x\left[e^{-\lambda_0 rt}\tilde{\rho}(X_{rt})\mathbb{1}_{\{T>rt\}}\right]}{\mathbb{P}_x(T>t)e^{-\lambda_0 t}}\int f h_u\,\mathrm{d}m \\
&= \int g h_u\,\mathrm{d}m\int f h_u\,\mathrm{d}m\,.
\end{aligned}$$

Since h_u is uniformly bounded as seen in the proof of Theorem 6.1.5 and $h_u(x)\to 1$, we have by the dominated convergence theorem that

$$\liminf_{t\to\infty}\mathbb{E}_x\left(f(X_{qt})g(X_{rt})|T>t\right) \geq \left(\int f\mathrm{d}m\right)\left(\int g\mathrm{d}m\right).$$

Replacing $f(X_{qt})g(X_{rt})$ by

$$(\|f\|_\infty - f(X_{qt}))(\|g\|_\infty + g(X_{rt})) \quad\text{and}\quad (\|f\|_\infty + f(X_{qt}))(\|g\|_\infty - g(X_{rt}))\,,$$

we can see directly that

$$\begin{aligned}
&2\|f\|_\infty\|g\|_\infty - \limsup_{t\to\infty} \mathbb{E}_x\left(2f(X_{qt})g(X_{rt})\right)|T>t) \\
&\geq \liminf_{t\to\infty} \mathbb{E}_x\left((\|f\|_\infty - f(X_{qt}))(\|g\|_\infty + g(X_{rt}))\middle|T>t\right) \\
&+ \liminf_{t\to\infty} \mathbb{E}_x\left((\|f\|_\infty + f(X_{qt}))(\|g\|_\infty - g(X_{rt}))\middle|T>t\right) \\
&\geq 2\|f\|_\infty\|g\|_\infty - \left(\int f\mathrm{d}m\right)\left(\int g\mathrm{d}m\right).
\end{aligned}$$

Therefore, we deduce that

$$\limsup_{t\to\infty} \mathbb{E}_x\left(f(X_{qt})g(X_{rt})|T>t\right) \leq \left(\int f\mathrm{d}m\right)\left(\int g\mathrm{d}m\right).$$

So, we have shown the claim for positive measurable and bounded functions f, g. We can extend the result to arbitrary measurable and bounded f, g analogously to the proof of Theorem 6.1.5 by replacing fg with $(f_+ - f_-)(g_+ - g_-)$. Uniformity of the convergence follows for the same reasons as in Theorem 6.1.5. □

We are ready to prove the following theorem which equips the limit of expected values λ, as given in Propositions 6.2.3 and 6.2.5, with the strongest possible dynamical meaning for the setting with killing at the boundary.

Theorem 6.2.8 (Convergence in conditional probability)**.** *Let (θ, φ) be the random dynamical system with absorption at the boundary corresponding to the Markov process $(X_t)_{t\geq 0}$ solving equation* (6.2.3)*. Then for all $\varepsilon > 0$ we have*

$$\lim_{t\to\infty} \mathbb{P}\left(|\lambda_v(t,\cdot,x) - \lambda| \geq \varepsilon \middle| \tilde{T}(\cdot,x) > t\right) = 0 \tag{6.2.9}$$

uniformly over all $x \in E$, $v \in \mathbb{S}^{d-1}$. This means that the finite-time Lyapunov exponents of the surviving trajectories converge to its assemble average in probability. Note that in one dimension this reads

$$\lim_{t\to\infty} \mathbb{P}\left(|\lambda(t,\cdot,x) - \lambda| \geq \varepsilon \middle| \tilde{T}(\cdot,x) > t\right) = 0\,. \tag{6.2.10}$$

Proof. Recall from above that

$$\begin{aligned}
\lambda &= \lim_{t\to\infty} \mathbb{E}_{x,s_0}\left(\lambda_v(t,\cdot,x)|T>t\right) = \lim_{t\to\infty} \mathbb{E}\left(\lambda_v(t,\cdot,x)|\tilde{T}(\cdot,x)>t\right) \\
&= \int_{\mathbb{S}^{d-1}\times E} \langle s, \mathrm{D}f(x)s\rangle\ \tilde{m}(\mathrm{d}s,\mathrm{d}x)\,.
\end{aligned}$$

In the following, we will write $g(x,s) = \langle s, \mathrm{D}f(x)s\rangle$ and $\tilde{m}(h) := \int h \,\mathrm{d}\tilde{m}$ for any bounded and measurable function h. Note that in one dimension we have $g(x,s) = f'(x)$ and $\tilde{m} = m$. We observe that

$$\begin{aligned}
&\mathbb{P}\left(|\lambda_v(t,\cdot,x) - \lambda| \geq \varepsilon \middle| \tilde{T}(\cdot,x) > t\right) \\
&\leq \mathbb{P}\left(\left|\lambda_v(t,\cdot,x) - \mathbb{E}\left(\lambda_v(t,\cdot,x)|\tilde{T}(\cdot,x) > t\right)\right| \geq \varepsilon \middle| \tilde{T}(\cdot,x) > t\right) \\
&+ \mathbb{P}\left(\left|\lambda - \mathbb{E}\left(\lambda_v(t,\cdot,x)|\tilde{T}(\cdot,x) > t\right)\right| \geq \varepsilon \middle| \tilde{T}(\cdot,x) > t\right).
\end{aligned}$$

The term in the second line converges to zero for t to infinity by definition of λ. The first term can be estimated by Chebyshev's inequality:

$$\mathbb{P}\left(\left|\lambda_v(t,\cdot,x) - \mathbb{E}\left(\lambda_v(t,\cdot,x)|\tilde{T}(\cdot,x) > t\right)\right| \geq \varepsilon \middle| \tilde{T}(\cdot,x) > t\right) \leq \frac{\mathrm{Var}(\lambda_v(t,\cdot,x)|\tilde{T}(\cdot,x) > t)}{\varepsilon^2}.$$

This means that, in order to prove the claim, we simply need to show that

$$\lim_{t\to\infty} \mathbb{E}\left(\lambda_v(t,\cdot,x)^2|\tilde{T}(\cdot,x) > t\right) = \lim_{t\to\infty}\left[\mathbb{E}\left(\lambda_v(t,\cdot,x)|\tilde{T}(\cdot,x) > t\right)\right]^2,$$

where

$$\lim_{t\to\infty}\left[\mathbb{E}\left(\lambda_v(t,\cdot,x)|\tilde{T}(\cdot,x) > t\right)\right]^2 = \lambda^2 = \tilde{m}(g)^2.$$

Similarly to the proof of Theorem 6.1.5 we obtain with Fubini that

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}\left(\lambda_v(t,\cdot,x)^2|\tilde{T}(\cdot,x) > t\right) &= \lim_{t\to\infty} \mathbb{E}_x\left(\left(\frac{1}{t}\int_0^t g(X_\tau, s_\tau)\,\mathrm{d}\tau\right)^2 |T > t\right) \\
&= \lim_{t\to\infty} \mathbb{E}_x\left(\left(\int_0^1 g(X_{qt}, s_{qt})\,\mathrm{d}q\right)^2 |T > t\right) \\
&= \lim_{t\to\infty} \int_0^1\int_0^1 \mathbb{E}_x\left(g(X_{qt}, s_{qt})g(X_{rt}, s_{rt})|T > t\right)\mathrm{d}q\,\mathrm{d}r \\
&= \lim_{t\to\infty}\left[\int_0^1\int_0^q \mathbb{E}_x\left(g(X_{qt}, s_{qt})g(X_{rt}, s_{rt})|T > t\right)\mathrm{d}r\,\mathrm{d}q\right. \\
&\left. + \int_0^1\int_0^r \mathbb{E}_x\left(g(X_{qt}, s_{qt})g(X_{rt}, s_{rt})|T > t\right)\mathrm{d}q\,\mathrm{d}r\right].
\end{aligned}$$

It follows immediately from Lemma 6.2.7 that for $0 < r < q < 1$ (and $0 < q < r < 1$)

$$\lim_{t\to\infty} \mathbb{E}_x\left(g(X_{qt}, s_{qt})g(X_{rt}, s_{rt})|T > t\right) = \lim_{t\to\infty} \mathbb{E}_{x,s_0}\left(g(X_{qt}, s_{qt})g(X_{rt}, s_{rt})|T > t\right) = \tilde{m}(g)^2,$$

where the convergence is uniform over the initial values. Hence, by using dominated

convergence, we conclude that

$$\lim_{t\to\infty} \mathbb{E}\left(\lambda_v(t,\cdot,x)^2 | \tilde{T}(\cdot,x) > t\right) = \int_0^1 \int_0^q \tilde{m}(g)^2 \,\mathrm{d}r\,\mathrm{d}q + \int_0^1 \int_0^r \tilde{m}(g)^2 \,\mathrm{d}q\,\mathrm{d}r = \tilde{m}(g)^2\,,$$

such that the claim follows. □

Proposition 6.2.3 tells us, not surprisingly, that if we take the limit of the expectation of the tangent flow, the initial vector v on the tangent space does not matter. Figuratively speaking, we average out the geometry of the dynamics. In the classical setting, this geometry is reflected by a spectrum of Lyapunov exponents associated with a filtration or splitting of flow-invariant subspaces. Something like that is not directly obtainable in our setting since such filtrations or subspaces depend on the noise realisation ω which only has a finite survival time in our context. However, we can try to find a spectrum of Lyapunov exponents following the Furstenberg-Kesten Theorem [2, Theorem 3.3.3].

For a time $t > 0$, consider $\Phi(t,\omega,x) := \mathrm{D}\varphi(t,\omega,x) \in \mathbb{R}^{d\times d}$ for all (ω,x) such that $\varphi(s,\omega,x) \in E$ for all $0 \le s \le t$. Let

$$0 < \sigma_d(\Phi(t,\omega,x)) \le \cdots \le \sigma_1(\Phi(t,\omega,x))$$

be the singular values of $\Phi(t,\omega,x)$, i.e. the eigenvalues of $\sqrt{\Phi^*(t,\omega,x)\Phi(t,\omega,x)}$. We would like to show the following:

Conjecture 6.2.9 (Lyapunov spectrum)**.** *Let Φ be the linearised flow associated with problem* (6.2.3)*. Then there are $\sigma^i \in \mathbb{R}$ such that for all $x \in E$ we have*

$$\frac{1}{t}\mathbb{E}[\ln \sigma_i(\Phi(t,\cdot,x)) | \tilde{T}(\cdot,x) > t] \xrightarrow{t\to\infty} \sigma^i \quad \textit{for all } 1 \le i \le d\,.$$

In this case, we can define a Lyapunov spectrum of average expansion rates of the surviving assemble by denoting $\lambda_1 > \lambda_2 > \ldots \lambda_p$ for the $0 < p \le d$ different values of $\sigma^1 \ge \cdots \ge \sigma^d$.

Considering Proposition 6.2.3, the goal would be to show that there are cases where $p > 1$. We will give numerical evidence for this later.

We consider the classical setting without killing at the boundary and replace $(\omega,x) \in \Omega\times\mathbb{R}^d$ by $\omega \in \Omega$ as driving metric dynamics for ease of notation. The Furstenberg-Kesten Theorem [2, Theorem 3.3.3] uses the subadditivity of exterior powers to show convergence of the exponents expressed as singular values. We denote the k-fold exterior powers of the matrix Φ by $\Lambda^k\Phi$ for $1 \le k \le d$. Then the condition

$$\sup_{0\le t\le 1} \ln^+ \|\Phi(t,\omega)^{\pm 1}\| \in L^1(\Omega)$$

guarantees that Kingman's Subadditive Ergodic Theorem [2, Theorem 3.3.2] can be used

to show that there is a measurable map γ^k such that almost surely

$$\lim_{t\to\infty} \frac{1}{t} \ln \|\Lambda^k \Phi(t,\omega)\| = \gamma^k(\omega)\,.$$

In particular, we obtain

$$\lim_{t\to\infty} \frac{1}{t} \mathbb{E}\left[\ln \|\Lambda^k \Phi(t,\cdot)\|\right] = \mathbb{E}[\gamma^k]\,.$$

We can then define measurable $\sigma^1(\omega) \geq \sigma^2(\omega) \geq \cdots \geq \sigma^d(\omega)$ such that for all $1 \leq k \leq d$ and $\omega \in \Omega$

$$\sigma^1(\omega) + \sigma^2(\omega) + \cdots + \sigma^k(\omega) = \gamma^k(\omega)\,.$$

Recall that $\sigma_k(\omega)$ denotes the k-th largest singular value of $\Phi(t,\omega)$. By using the well-known fact for all $t > 0$ and $\omega \in \Omega$

$$\sigma_1(\omega) + \cdots + \sigma_k(\omega) = \|\Lambda^k \Phi(t,\cdot)\|\,,$$

one can deduce recursively that

$$\sigma^k(\omega) = \lim_{t\to\infty} \frac{1}{t} \ln \sigma_k(Y(t,\omega))\,.$$

In particular, we may conclude that

$$\mathbb{E}[\sigma^k(\cdot)] = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}\left[\ln \sigma_k(Y(t,\cdot))\right].$$

The different numbers $\lambda_1(\omega) > \cdots > \lambda_{p(\omega)}$ in the sequence $\sigma^1(\omega) \geq \sigma^2(\omega) \geq \cdots \geq \sigma^d(\omega)$ are the Lyapunov exponents. In an ergodic situation, the Lyapunov exponents and their number $p(\omega)$ are constant over almost all ω and we obtain that $\mathbb{E}[\lambda_k(\cdot)] = \lambda_k \in \mathbb{R}$.

The main ingredient for showing Conjecture 6.2.9 is proving the convergence of the exterior powers. As mentioned above, this is achieved in the classical setting by using the subadditivity of

$$\rho^k(\omega,t) := \ln \|\Lambda^k \Phi(t,\omega)\|$$

and then applying Kingman's Subadditive Ergodic Theorem. The subadditivity follows directly from the cocycle property

$$\Lambda^k \Phi(t+s,\omega) = \Lambda^k \Phi(t,\theta_s\omega) \Lambda^k \Phi(s,\omega).$$

Going back to our problem, we would like to show the existence of

$$\lim_{t\to\infty} \frac{1}{t} \mathbb{E}[\rho^k(\cdot,t) | T > t]$$

under mild conditions. An approach using subadditivity would need to establish that

$$\mathbb{E}[\rho^k(\cdot, t+s)|T > t+s] \leq \mathbb{E}[\rho^k(\cdot, t)|T > t] + \mathbb{E}[\rho^k(\cdot, s)|T > s]\,.$$

Unfortunately, showing this seems very difficult, if not impossible, as it is generally not clear what effect the conditioning on non-absorption has on values of $\rho^k(\mathrm{cot}, t)$. Another even more general approach would be to investigate the limit of

$$\mathbb{E}[g(\cdot, t)|T > t]\,,$$

for $g : C([0,\infty)) \times \mathbb{R}_0^+ \to \mathbb{R}$ which satisfy

$$(\omega, t) \mapsto h_t(X_{0\leq s\leq t}(\omega)) \text{ for maps } h_t : C([0,t]) \to \mathbb{R}\,.$$

But this seems an even harder problem.

The following Lemma shows that we can bound the Lyapunov spectrum, if it exists, from above and below. In more detail, we define an upper and a lower conditioned average Lyapunov exponent λ^u and λ^l by

$$\lambda^u := \limsup_{t\to\infty} \sup_{\|v\|=1} \frac{1}{t}\mathbb{E}\left[\ln \|\mathrm{D}\varphi(t,\cdot,x)v\||\tilde{T}(\cdot,x) > t\right]\,,$$

and

$$\lambda^l := \liminf_{t\to\infty} \inf_{\|v\|=1} \frac{1}{t}\mathbb{E}\left[\ln \|\mathrm{D}\varphi(t,\cdot,x)v\||\tilde{T}(\cdot,x) > t\right]\,.$$

Defining similarly to chapter 3

$$\lambda^+(x) = \max_{\|r\|=1}(\mathrm{D}f(x)r, r),\ \lambda^-(x) = \min_{\|r\|=1}(\mathrm{D}f(x)r, r)\,,$$

we find the following bounds for these quantities.

Lemma 6.2.10. *Let (θ, φ) be the random dynamical system with killing at the boundary associated to* (6.2.3)*. Then the upper and lower conditioned average Lyapunov exponents satisfy*

$$\int_X \lambda^-(x)m(\mathrm{d}x) \leq \lambda^l \leq \lambda \leq \lambda^u \leq \int_X \lambda^+(x)m(\mathrm{d}x)\,,$$

where λ is the conditioned average Lyapunov exponent given by Proposition 6.2.3.

Proof. For any $\omega \in \Omega$, $x \in E$ and $0 \neq v \in \mathbb{R}^d$ define $r_t(\omega, x, v) := \frac{\mathrm{D}\varphi(t,\omega,x)v}{\|\mathrm{D}\varphi(t,\omega,x)v\|}$. We observe

that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|D\varphi(t,\omega,x)v\|^2 &= 2\left(Df(\varphi(t,\omega,x))(t,\omega,x)v, \mathrm{D}\varphi(t,\omega,x)v\right)\\ &= 2\left(\mathrm{D}f(\varphi(t,\omega,x))r_t(\omega,x,v), r_t(\omega,x,v)\right)\|\mathrm{D}\varphi(t,\omega,x)v\|^2\\ &\leq 2\lambda^+(\varphi(t,\omega,x))\|\mathrm{D}\varphi(t,\omega,x)v\|^2.\end{aligned}$$

Analogously we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|D\varphi(t,\omega,x)v\|^2 \geq 2\lambda^-(\varphi(t,\omega,x))\|\mathrm{D}\varphi(t,\omega,x)v\|^2\,.$$

Hence, we can conclude that for all $0 \neq v \in \mathbb{R}^d$

$$\|\mathrm{D}\varphi(t,\omega,x)v\|^2 \leq \|v\|^2 \exp\left(2\int_0^t \lambda^+(\varphi(s,\omega,x))\mathrm{d}s\right) \tag{6.2.11}$$

$$\|\mathrm{D}\varphi(t,\omega,x)v\|^2 \geq \|v\|^2 \exp\left(2\int_0^t \lambda^-(\varphi(s,\omega,x))\mathrm{d}s\right)\,. \tag{6.2.12}$$

Since λ^+ and λ^- are measurable and bounded on E, we can conclude with Theorem 6.1.5 that

$$\begin{aligned}\lambda^u &\leq \limsup_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t \lambda^+(\varphi(s,\cdot,x))\mathrm{d}s|\tilde{T}(\cdot,x) > t\right]\\ &= \lim_{t\to\infty}\mathbb{E}_x\left[\frac{1}{t}\int_0^t \lambda^+(X_s)\,\mathrm{d}s|T > t\right] = \int_X \lambda^+(x)\,m(\mathrm{d}x)\,,\end{aligned}$$

and

$$\begin{aligned}\lambda^l &\geq \liminf_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t \lambda^-(\varphi(s,\cdot,x))\mathrm{d}s|\tilde{T}(\cdot,x) > t\right]\\ &= \lim_{t\to\infty}\mathbb{E}_x\left[\frac{1}{t}\int_0^t \lambda^-(X_s)\,\mathrm{d}s|T > t\right] = \int_X \lambda^-(x)\,m(\mathrm{d}x)\,.\end{aligned}$$

The fact that $\lambda^l \leq \lambda \leq \lambda^u$ follows directly from the respective definitions. □

Note that the computation of λ can be very difficult and costly in higher dimensions as we have to determine $\tilde{m}(\mathrm{d}s,\mathrm{d}x)$, the joint quasi-ergodic distribution of the original process and the derivative angular process. In some cases it will definitely be easier and cheaper to simply compute or approximate $m(\mathrm{d}x)$, the quasi-ergodic distribution of the singular process. Then the integrals $m(\lambda^-)$ and $m(\lambda^+)$ can help to reveal if λ, which can be expected to equal λ_1 in case the spectrum exists, is positive or negative respectively.

6.3 Local synchronisation for nonlinear systems

This section is dedicated to showing implications of λ having a negative sign. As one would expect, the main implication in terms of the random dynamics is the synchronisation of surviving trajectories starting close enough to each other. In the following, we prove a general statement about local synchronisation in discrete time for systems with absorption at the boundary and then show a corollary for systems induced by stochastic differential equations using the definitions and results of the previous two sections.

Theorem 6.3.1 (Local synchronisation theorem)**.** *Let (θ, φ) be a continuously differentiable RDS with killing on a bounded domain $E \subset \mathbb{R}^d$, and let there exist a $\lambda < 0$ such that uniformly over all $x \in E$ and $v \in \mathbb{S}^{d-1}$*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln\|\mathrm{D}\varphi(n,\cdot,x)v\| \le \lambda \middle| \tilde{T}(\cdot,x) > n\right] = 1\,. \tag{6.3.1}$$

Then for all $\lambda_\varepsilon \in (\lambda, 0)$, $x \in E$ and $0 < \rho < 1$, there are $\alpha_x > 0$, $0 < \beta < 1$, $K_\varepsilon > 1$ and sets $\Omega_x^n \subset \Omega$ with $\mathbb{P}_x(\Omega_x^n | T > n) > 1 - \rho$ for all $n \in \mathbb{N}$ such that we have the following:

a) For all $n \in \mathbb{N}$, $\omega \in \Omega_x^n$ and $y, y' \in \overline{B}_{\alpha_x}(x)$

$$\|\varphi(n,\omega,y) - \varphi(n,\omega,y')\| \le K_\varepsilon e^{\lambda_\varepsilon n}\|y - y'\|\,,$$

and, in particular, for all $y \in \overline{B}_{\alpha_x}(x)$

$$\|\varphi(n,\omega,x) - \varphi(n,\omega,y)\| \le \beta e^{\lambda_\varepsilon n}\,.$$

b) There is exponentially fast local synchronisation of trajectories in discrete time with arbitrarily high probability, i.e.

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{1}{n}\ln\|\varphi(n,\cdot,x) - \varphi(n,\cdot,y)\| \le \lambda_\varepsilon \ \text{ for all } y \in \overline{B}_{\alpha_x}(x) \middle| \tilde{T}(\cdot,x) > n\right) > 1 - \rho\,.$$

Proof. Let $0 < \rho < 1$ be fixed for the following. For $x \in E$ and $n \in \mathbb{N}$, let $\Omega_x^n \subset \{\omega \in \Omega : \tilde{T}(\omega, x) > n\}$ be a set with $\mathbb{P}_x(\Omega_x^n | T > n) > 1 - \rho$ whose construction is given later in (6.3.4). Furthermore we define for any $x \in E$

$$U_x := \{y \in \mathbb{R}^d : x + y \in \overline{E}\}\,.$$

For fixed $(\omega, x) \in \Omega \times E$ we define on U_x

$$Z_n((\omega,x),y) := \varphi(n,\omega,y+x) - \varphi(n,\omega,x)\,.$$

Note that in particular $Z_n((\omega,x),0)=0$ for all n. Define further

$$F_{(\omega,x)}(y) := Z_1((\omega,x),y)$$

and write

$$F^n_{(\omega,x)} = F_{\Theta^{n-1}(\omega,x)} \circ \cdots \circ F_{(\omega,x)}\,.$$

In addition we define

$$L(\omega,x) = \mathrm{D}F_{(\omega,x)}(0) = \mathrm{D}\varphi(1,\omega,x)$$

and for all $n \geq 1$

$$L_n(\omega,x) = L(\Theta_{n-1}(\omega,x))\,.$$

Similarly to [82], let $0 < \eta = -\lambda_\varepsilon/2$. Since the system is C^1 on a bounded domain, we have

$$G := \sup_{(\omega,x)} \|F_{(\omega,x)}\|_{C^1} < \infty\,.$$

Let $\delta > 0$ be given. Choose $0 < \beta < 1$ such that $G\beta e^\eta < \delta$. Take further $\kappa > 1$ such that $\kappa\beta \leq 1$ and $G\kappa\beta e^\eta \leq \delta$.

Recall that

$$\Omega^K_x \subset \{\omega \in \Omega\,:\,\tilde{T}(\omega,x) > K\}$$

and define

$$S^K(\beta) = \{y \in U_x : \|F^n_{(\omega,x)}(y)\| \leq \beta e^{\lambda_\varepsilon n} \text{ for all } 0 \leq n \leq K \text{ and } \omega \in \Omega^K_x\}\,.$$

For $y \in S^K(\kappa\beta)$ we define for all $1 \leq n \leq K$

$$L'_n(\omega,x) = \int_0^1 \mathrm{D}F_{\Theta_{n-1}(\omega,x)}(tF^{n-1}_{(\omega,x)}(y))\,\mathrm{d}t\,.$$

Observe that this choice yields for $1 \leq n \leq K$

$$L'^n(\omega,x)y = L'_n(\omega,x)\cdots L'_1(\omega,x)y = F^n_{(\omega,x)}(y)\,.$$

We deduce that for any $y \in S^K(\kappa\beta)$

$$\begin{aligned}\sup_{\omega\in\Omega^K_x}\sup_{n\leq K}\|L'_n(\omega,x) - L_n(\omega,x)\|e^{\eta n} &\leq \sup_{\omega\in\Omega^K_x}\sup_{n\leq K}\|\mathrm{D}F_{\theta_{n-1}(\omega,x)}\|\kappa\beta\exp\Big(n(\eta+\lambda_\varepsilon)-\lambda_\varepsilon\Big)\\ &\leq G\kappa\beta e^\eta < \delta\,.\end{aligned}$$

Claim: We can deduce from

$$\sup_{\omega\in\Omega^K_x}\sup_{n\leq K}\|L'_n(\omega,x) - L_n(\omega,x)\|e^{\eta n} < \delta$$

that for all $\omega \in \Omega_x^K$, $y \in S^K(\kappa\beta)$ and $1 \leq n \leq K$

$$\|L'^n(\omega, x)y\| \leq K_\varepsilon e^{n\lambda_\varepsilon}\|y\|$$

uniformly over K, for some $K_\varepsilon > 1$.

Let us assume that the claim is true for the time being and define $d_x = d(x, \partial E)/2$. Choose $\alpha_x = \min(d_x, \beta/K_\varepsilon) < \beta$. From the claim we observe that uniformly over K for all $y \in \overline{B}_{\alpha_x}(0) \cap S^K(\kappa\beta)$ and $1 \leq n \leq K$

$$\|F^n_{(\omega,x)}(y)\| \leq K_\varepsilon e^{n\lambda_\varepsilon}\alpha_x \leq \beta e^{n\lambda_\varepsilon}$$

and therefore

$$D_K(\alpha_x) := \overline{B}_{\alpha_x}(0) \cap S^K(\beta) = \overline{B}_{\alpha_x}(0) \cap S^K(\kappa\beta)\,.$$

Since the boundaries of $S^K(\beta)$ and $S^K(\kappa\beta)$ are disjoint, this implies that $\overline{B}_{\alpha_x}(0) = D_K(\alpha_x)$ for all $K > 0$ and the second statement in a) follows.

For any $y, y' \in D_K(\alpha_x)$ and $1 \leq n \leq K$ we define similarly to before

$$L'_n((\omega, x)) = \int_0^1 \mathrm{D}F_{\Theta_{n-1}(\omega,x)}\left(tF^{n-1}_{(\omega,x)}(y) + (1-t)F^{n-1}_{(\omega,x)}(y')\right)\,\mathrm{d}t\,.$$

Observe that for $1 \leq n \leq K$

$$L'^n(\omega, x)(y - y') = L'_n(\omega, x)\cdots L'_1(\omega, x)(y - y') = F^n_{(\omega,x)}(y) - F^n_{(\omega,x)}(y')\,.$$

Therefore we observe analogously to before that

$$\sup_{\omega\in\Omega_x^K}\sup_{n\leq K}\|L'_n(\omega, x) - L_n(\omega, x)\|e^{\eta n} < \delta\,,$$

which using the claim gives that for all $\omega \in \Omega_x^K$, $y, y' \in D_K(\alpha_x)$ and $1 \leq n \leq K$

$$\|F^n_{(\omega,x)}(y) - F^n_{(\omega,x)}(y')\| \leq K_\varepsilon e^{\lambda_\varepsilon n}\|y - y'\|\,.$$

Hence, the first statement in a) follows.

Now, we prove the claim above. To make the notation easier, we fix (ω, x) and write L for the Jacobian and L' for the perturbation. Take $\|\xi^{(0)}\| = 1$, define the sequence $\xi^{(n)}$ as units along L_n and write

$$L_n\xi^{(n-1)} = t^{(n)}\xi^{(n)}\,.$$

For our purposes it is enough to exercise through the one-dimensional case since the convergence in (6.3.1) is assumed to be uniform over all $v \in \mathbb{S}^{d-1}$ in the d-dimensional scenario. We refer to [82] for the full details of the multidimensional scenario which are not needed here.

If we write $L'^n u = u^{(n)}$, we see that $\sup_n \|L'_n - L_n\| e^{\eta n} < \delta$ implies

$$\left|u^{(n)}\right| \leq t^{(n)} \left|u^{(n-1)}\right| + \delta e^{-n\eta} \left|u^{(n-1)}\right| . \tag{6.3.2}$$

Since the finite-time Lyapunov exponents are bounded away from $-\infty$ according to the assumptions of the model, there is a $C > 0$ independent from (ω, x) such that for any $N \in \mathbb{N}$

$$\frac{1}{C} e^{-N\eta} \leq t^{(N)} .$$

If we fix $\nu \geq 1$ and set $U^{(\nu)} = \left|u^{(\nu)}\right|$ and for $N > \nu$

$$U^{(N)} = \left(\prod_{n=\nu+1}^{N} t^{(n)} \right) \left(\prod_{n=\nu+1}^{N} (1 + C\delta e^{-n\eta}) \right) U^{(\nu)} ,$$

we can observe with (6.3.2) that $U^{(N)} \geq \left|u^{(N)}\right|$ for all $N \geq \nu$.

Now we set $\delta = \frac{1}{C} \prod_{n=1}^{\infty} (1 - e^{-n\eta})^2$ and

$$C' = \frac{\prod_{n=1}^{\infty} (1 + C\delta e^{-n\eta})}{\prod_{n=1}^{\infty} (1 - e^{-n\eta})} \leq \prod_{n=1}^{\infty} (1 - e^{-n\eta})^{-2} = \frac{1}{C\delta} .$$

Note that δ and C' do not depend on (ω, x). It is easy to infer similarly to [82] that

$$\left|u^{(N)}\right| \leq C' \left(\prod_{n=\nu+1}^{N} t^{(n)} \right) \left(\prod_{n=\nu+1}^{N} (1 - e^{-n\eta}) \right) \left|u^{(\nu)}\right| \tag{6.3.3}$$

and

$$\left|u^{(N)}\right| \geq \left(\prod_{n=\nu+1}^{N} t^{(n)} \right) \left(\prod_{n=\nu+1}^{N} (1 - e^{-n\eta}) \right) \left|u^{(\nu)}\right| .$$

We take $\nu = 0$ from here on. Observe that the finite Lyapunov exponents satisfy

$$\lambda(N, \omega, x) = \frac{1}{N} \ln \prod_{n=1}^{N} t^{(n)} .$$

Let $\varepsilon = \lambda_\varepsilon - \lambda > 0$. By assumption, there exists an $N^* > 0$ such that for all $N \geq N^*$ we have

$$\mathbb{P}\left(\lambda(N, \cdot, x) < \lambda + \varepsilon/2 \middle| \tilde{T}(\cdot, x) > N \right) > 1 - \rho .$$

Recall that δ, C, C' do not depend on (ω, x). Now define the measurable sets

$$\Omega_x^N = \begin{cases} \{\omega \in \Omega \,:\, \tilde{T}(\omega, x) > N,\ \lambda(N, \omega, x) < \lambda + \varepsilon/2\} & \text{if } N \geq N^* , \\ \{\omega \in \Omega \,:\, \tilde{T}(\omega, x) > N\} & \text{if } N < N^* . \end{cases} \tag{6.3.4}$$

Hence, $\mathbb{P}_x(\Omega_x^n | T > n) > 1 - \rho$ for all $n \in \mathbb{N}$. We can conclude from (6.3.3) that there is $C'' > 0$ such that for all $x \in E$, $N \geq N^*$ and $\omega \in \Omega_x^N$

$$\frac{1}{N} \ln \|L'^N(\omega, x)\| \leq \frac{C''}{N} + \frac{1}{N} \ln \prod_{n=\nu+1}^{N} t^{(n)} = \frac{C''}{N} + \lambda(N, \omega, x) \,.$$

From the assumptions we have that

$$s^* := \sup_{(\omega,x)} \sup_{N \leq N^*} \lambda(N, \omega, x) < \infty \,.$$

We define

$$K_\varepsilon = \max\{e^{C''}, e^{-\lambda_\varepsilon N^*} s^*\} \,.$$

Then we obtain the statement of the claim, i.e. for all $N \in \mathbb{N}$ and $\omega \in \Omega_x^N$

$$\|L'^N(\omega, x)\| \leq K_\varepsilon e^{\lambda_\varepsilon N} \,.$$

Finally, we show statement b). We conclude from the second statement in a), and the fact that $\beta < 0$, that for all $n \in \mathbb{N}$

$$\begin{aligned} &\mathbb{P}\left(\|\varphi(n, \cdot, x) - \varphi(n, \cdot, y)\| \leq e^{\lambda_\varepsilon n} \text{ for all } y \in \overline{B}_{\alpha_x}(x) \middle| \tilde{T}(\cdot, x) > n \right) \\ &\geq \mathbb{P}\left(\Omega_x^n \middle| \tilde{T}(\cdot, x) > n \right) > 1 - \rho \,. \end{aligned}$$

Hence, we actually obtain for all $n \in \mathbb{N}$

$$\mathbb{P}\left(\frac{1}{n} \ln \|\varphi(n, \omega, x) - \varphi(, \omega, y)\| \leq \lambda_\varepsilon \text{ for all } y \in \overline{B}_{\alpha_x}(x) \middle| \tilde{T}(\cdot, x) > n \right) > 1 - \rho \,,$$

which implies the claim. □

We can immediately observe the following corollary about local synchronisation of random dynamical systems induced by solutions of stochastic differential equations with absorption at the boundary of a domain:

Corollary 6.3.2. *Let the RDS with killing induced by the SDE* (6.2.3) *have a negative conditioned asymptotic Lyapunov exponent* $\lambda < 0$. *Then for all* $x \in E$, $\lambda_\varepsilon \in (\lambda, 0)$ *and* $0 < \rho < 1$, *there is an* $\alpha_x > 0$ *such that*

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{1}{n} \ln |\varphi(n, \cdot, x) - \varphi(n, \cdot, y)| \leq \lambda_\varepsilon \text{ for all } y \in \overline{B}_{\alpha_x}(x) \middle| \tilde{T}(\cdot, x) > n \right) > 1 - \rho.$$

Proof. This is an immediate consequence of Theorem 6.3.1 in combination with Theorem 6.2.8. □

We would like to show a global synchronisation theorem as well. An obvious very weak statement is expressed in the following proposition which is simply supposed to sketch a first direction for this endeavour:

Proposition 6.3.3. *Let the RDS with killing induced by the SDE* (6.2.3) *have a negative conditioned average Lyapunov exponent* λ*. Let* $x, y \in I$*,* α_x *be the radius of stability from Theorem 6.3.1 for some* $0 < \rho < 1$ *and let there be a time* $t \geq 0$ *such that*

$$\mathbb{P}\left(|\varphi(t,\cdot,x) - \varphi(t,\cdot,y)| < \alpha_x \middle| \tilde{T}(\cdot,x) > t\right) > 0.$$

Then there is a sequence $t_n \to \infty$ *such that for any* $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(|\varphi(t_n,\cdot,x) - \varphi(t_n,\cdot,y)| < \varepsilon \middle| \tilde{T}(\cdot,x) > t_n\right) > 0.$$

Proof. This is a direct consequence of Corollary 6.3.2 in combination with the Markov property. □

In the one-dimensional case one could hope for using monotonicity arguments to show a stronger statement. This is left as future work.

6.4 Relation to pathwise random dynamics

This section is dedicated to finding further relations between Markov processes with absorption at the boundary and the induced random dynamical systems, in particular understanding the latter ones as skew product systems. It suggests itself to investigate the connection between quasi-stationary and quasi-ergodic measures and suitable associated measures for the skew product, similarly to the setting without killing as introduced in Chapter 2. Hereby, conditionally invariant measures for dynamical systems with a hole turn out to be the he right concept.

6.4.1 Conditionally invariant measures on the skew product space

Let $(X_t)_{t\geq 0}$ be a Markov process on some topological space X, defined over a filtered probability space $(\Omega^+, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and equipped with a family of maps $\theta_t : \Omega^+ \to \Omega^+$ such that

$$\theta_{t+s} = \theta_t \circ \theta_s \quad \text{for all } t, s \geq 0$$

and

$$X_{t+h}(\omega) = X_t(\theta_h \omega) \quad \text{for all } t, h \geq 0 \text{ and } \omega \in \Omega^+ .$$

Note that the existence of (θ_t) is sometimes part of the definition of a Markov process (see e.g. [86]) and that the σ-algebra generated by θ_t is independent from $\mathcal{F}_t$ for all $t \geq 0$.

Let further θ_t be $\mathbb{P}$-invariant for all $t \geq 0$ and the process induce a random dynamical system with one-sided skew product flow

$$\Theta_t : \Omega^+ \times X \to \Omega^+ \times X,\ \Theta_t(\omega, x) = (\theta_t\omega, \varphi(t, \omega, x))\,, \quad \text{for all } t \geq 0\,.$$

As before let $E \subset X$ be some open subset such that $(X_t)_{t\geq 0}$ with transition probabilities $(\mathbb{P}_x)_{x\in E}$ satisfies the assumptions of this chapter if the process is absorbed at ∂E. Note that the described setting is for example given in the situation of random dynamical systems induced by stochastic differential equations as described in Section 6.1.2, defined on the one-sided canonical path space with Wiener measure $\mathbb{P}$ projected to one-sided time and canonical one-sided shift $(\theta_t)_{t\geq 0}$.

We now consider the problem of killed diffusion from the perspective of dynamical systems with holes. We can write $E = \mathring{X} = X \setminus H$ where $H = X \setminus E$ is a closed set constituting a hole in the space X. Define $\mathring{M} = \Omega^+ \times \mathring{X} = \Omega^+ \times E$ as the product space with hole in only one factor and

$$\mathring{M}_t = \bigcap_{0\leq s\leq t} (\Theta_s)^{-1}\mathring{M}\,, \quad \mathring{\Theta}_t = \Theta_t\big|_{\mathring{M}_t}\,, \tag{6.4.1}$$

i.e. the survival set and the flow restricted to the survival set. We further define for each $\omega \in \Omega^+$

$$E_t(\omega) = \bigcap_{0\leq s\leq t} \varphi(s, \omega, \cdot)^{-1}E\,, \quad \mathring{\varphi}(t, \omega, \cdot) = \varphi(t, \omega, \cdot)\big|_{E_t}\,,$$

i.e. the ω-wise survival set and cocycle restricted to the survival set.

Definition 6.4.1 (Conditionally invariant measure)**.** A measure μ supported on $\mathring{M}$ is called a *conditionally invariant probability measure* for $(\mathring{\Theta}_t)_{t\geq 0}$ on $\Omega^+ \times E$ if

$$\frac{\mu\left(\mathring{\Theta}_t^{-1}(C)\right)}{\mu(\mathring{M}_t)} = \mu(C) \quad \text{for all } t \geq 0 \text{ and } C \in \mathcal{F} \times \mathcal{B}(E)\,.$$

See for example [29] or [34] for fundamental work on conditionally invariant probability measures. First, we can show the following result similarly to Homburg and Zmarrou [51, Lemma 5.2.].

Proposition 6.4.2. *The measure $\mathbb{P}\times\nu$ is a conditionally invariant probability measure for the one-sided skew product flow $(\mathring{\Theta}_t)_{t\geq 0}$ on $\Omega^+ \times E$ iff the measure v is a quasi-stationary distribution for $(X_t)_{t\geq 0}$ on E.*

Proof. The measure v being quasi-stationary means that for all Borel-sets $B \subset E$ we have

$$\nu(B) = \mathbb{P}_\nu\left(X_t \in B \middle| T > t\right) = \frac{\int_E \mathbb{P}(\varphi(t, \cdot, x) \in B, \tilde{T}(\cdot, x) > t)\,\nu(\mathrm{d}x)}{\int_E \mathbb{P}_x(T > t)\,\nu(\mathrm{d}x)}\,. \tag{6.4.2}$$

Let $C \subset \Omega^+ \times E$ be measurable with respect to $\mathcal{F} \times \mathcal{B}(E)$. Hence, we may assume $C = A \times B$. Let us first assume that ν is a quasi-stationary distribution for the process. We observe that

$$\frac{(\mathbb{P}\times\nu)\left(\mathring{\Theta}_t^{-1}C\right)}{(\mathbb{P}\times\nu)\mathring{M}_t} = \frac{\int_E \mathbb{P}\left(\omega : \theta_t\omega \in A, \varphi(t,\omega,x)\in B, \tilde{T}(\omega,x)>t\right)\nu(\mathrm{d}x)}{\int_E \mathbb{P}_x\left(T>t\right)\nu(\mathrm{d}x)}\,.$$

Recall that $(\varphi(t,\cdot,\cdot))_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and $\sigma(\theta_t)$ is independent from $\mathcal{F}_t$ for all $t \geq 0$ since θ_t is the one-sided shift on Ω^+. Hence, we can infer with (6.4.2) that

$$\begin{aligned}\frac{(\mathbb{P}\times\nu)\left(\mathring{\Theta}_t^{-1}C\right)}{(\mathbb{P}\times\nu)\mathring{M}_t} &= \frac{\int_E \mathbb{P}(\omega : \theta_t\omega\in A)\mathbb{P}\left(\omega :, \varphi(t,\omega,x)\in B, \tilde{T}(\omega,x)>t\right)\nu(\mathrm{d}x)}{\int_E \mathbb{P}_x\left(T>t\right)\nu(\mathrm{d}x)}\\ &= \mathbb{P}_\nu\left(X_t\in B\middle|T>t\right)\mathbb{P}(A) = \nu(B)\mathbb{P}(A) = (\mathbb{P}\times\nu)(C)\,. \qquad (6.4.3)\end{aligned}$$

As one can see directly from (6.4.3), we can show the reverse direction analogously, i.e. assuming $\mathbb{P}\times\nu$ to be conditionally invariant immediately gives us (6.4.2) and therefore the quasi-stationarity of ν. □

Recall from the proof of Theorem 6.1.5 that for any bounded and measurable function f, $x \in E$ and $0 < p < 1$

$$\lim_{t\to\infty}\mathbb{E}_x(f(X_{pt})|T>t) = \int_I f(x)\,m(\mathrm{d}x)\,.$$

This observation prompts the definition of the maps $H_t^p : \Omega^+ \times X \to \Omega^+ \times X$ by

$$H_t^p(\omega,x) = (\theta_t\omega, \varphi(pt,\omega,x)) \quad \text{for all } \omega\in\Omega^+, x\in X\,,$$

for all $t \geq 0$ and $0 < p < 1$. This means that the shift on the probability space is considered for time $t \geq 0$ whereas the mapping on the state space is only followed up to time $0 \leq pt < t$. Note that H_t^p does not satisfy the flow property but the definition is supposed to illustrate the distinction between quasi-stationarity and quasi-ergodicity. We give the following definition:

Definition 6.4.3. Let $0 < p < 1$. A *p-quasi-stationary distribution* (QSD) for the process $(X_t)_{t\geq 0}$ is a probability measure m on E such that for all $t \geq 0$ and measurable sets $B \subset E$

$$\mathbb{P}_m\left(X_{pt}\in B|T>t\right) = m(B)\,. \qquad (6.4.4)$$

As before, set $\mathring{H}_t^p = H_t^p\big|_{\mathring{M}_t}$. We can prove the following proposition analogously to Proposition 6.4.2:

Proposition 6.4.4. *For any $0 < p < 1$, the measure $\mathbb{P}\times m$ is a conditionally invariant*

probability measure for the family of one-sided skew product maps $(\mathring{H}_t^p)_{t\geq 0}$ *iff the measure* m *is a* p*-quasi-stationary distribution for* $(X_t)_{t\geq 0}$ *on* E*.*

Proof. Recall from Definition 6.4.4 that the measure m is a p-quasi-stationary probability distribution iff for all Borel-sets $B \subset E$ we have

$$m(B) = \mathbb{P}_m\left(X_{pt} \in B \middle| T > t\right) = \frac{\int_E \mathbb{P}(\varphi(pt,\cdot,x) \in B, \tilde{T}(\cdot,x) > t)\, m(\mathrm{d}x)}{\int_E \mathbb{P}_x(T>t)\, m(\mathrm{d}x)}\,. \tag{6.4.5}$$

Let $C \subset \Omega^+ \times E$ be measurable with respect to $\mathcal{F} \times \mathcal{B}(E)$. Hence, we may assume $C = A \times B$. Let us first assume that m is a p-quasi-stationary distribution for the process. We observe that

$$\frac{(\mathbb{P}\times m)\left((\mathring{H}_t^p)^{-1}C\right)}{(\mathbb{P}\times m)\mathring{M}_t} = \frac{\int_E \mathbb{P}\left(\omega\,:\,\theta_t\omega \in A, \varphi(pt,\omega,x)\in B, \tilde{T}(\omega,x)>t\right)\, m(\mathrm{d}x)}{\int_E \mathbb{P}_x\left(T>t\right)\, m(\mathrm{d}x)}\,.$$

Recall that $(\varphi(pt,\cdot,\cdot))_{t\geq 0}$ is adapted to $(\mathcal{F}_{pt})_{t\geq 0}$ and $\sigma(\theta_t)$ is independent from $\mathcal{F}_{pt}$ for all $t \geq 0$ since θ_t is the one-sided shift on Ω^+ and $p < 1$. Hence, we can infer with (6.4.5) that

$$\begin{aligned}\frac{(\mathbb{P}\times m)\left(\mathring{\Theta}_t^{-1}C\right)}{(\mathbb{P}\times m)\mathring{M}_t} &= \frac{\int_E \mathbb{P}(\omega\,:\,\theta_t\omega\in A)\mathbb{P}\left(\omega\,:,\varphi(pt,\omega,x)\in B, \tilde{T}(\omega,x)>t\right)\, m(\mathrm{d}x)}{\int_E \mathbb{P}_x\left(T>t\right)\, m(\mathrm{d}x)} \\ &= \mathbb{P}_m\left(X_{pt}\in B\middle|T>t\right)\mathbb{P}(A) = m(B)\mathbb{P}(A) = (\mathbb{P}\times m)(C)\,. \end{aligned} \tag{6.4.6}$$

As one can see directly from (6.4.6), we can show the reverse direction analogously, i.e. assuming $\mathbb{P}\times m$ to be conditionally invariant for $(\mathring{H}_t^p)_{t\geq 0}$ immediately gives us (6.4.5) and therefore that m is a p-quasi-stationary distribution for the process $(X_t)_{t\geq 0}$. □

Making Proposition 6.4.4 useful for characterising a quasi-ergodic distribution m requires p-quasi-stationarity of m for some $0 < p < 1$. We conjecture the following even stronger statement for which we have not found a proof yet. Such a proof should work similarly to [75, Proposition 1], using the same tricks as in the proof of Theorem 6.1.5.

Conjecture 6.4.5. *If* m *is a quasi-ergodic distribution for* $(X_t)_{t\geq 0}$ *on* E*, then it is also a* p*-quasi-stationary distribution for any* $0 < p < 1$*.*

Therefore we obtain as a direct consequence of Proposition 6.4.4: if the measure m *is a quasi-ergodic distribution for* $(X_t)_{t\geq 0}$ *on* E*, the measure* $\mathbb{P}\times m$ *is a conditionally invariant probability measure for the family of one-sided skew product maps* $(\mathring{H}_t^p)_{t\geq 0}$ *for any* $0 < p < 1$*.*

We have discussed the correspondences between, on the one side, quasi-stationary and quasi-ergodic distributions for the process and, on the other side, conditionally invariant measures for the skew product systems, as long as we consider everything in one-sided

time. Usually, as introduced in Chapter 2, the metric dynamical system on the probability space, underlying the random dynamical system, is considered in two-sided time. This is necessary if we want to investigate random attractors, for example.

6.4.2 Two-sided time and relation to the survival process

We now consider the problem in two-sided time, i.e. we consider $(\Omega, \mathcal{F}, (\mathcal{F}_s^t)_{s\leq t\in\mathbb{R}}, \mathbb{P})$ with the maps $(\theta_t)_{t\in\mathbb{R}}$ such that θ_t is $\mathbb{P}$-invariant for all $t \in \mathbb{R}$. A natural question to ask is whether in this situation there is also a conditionally invariant probability measure for the skew product flow corresponding with a quasi-stationary distribution ν. In more detail, we would like to keep $\mathbb{P}$ fixed as the marginal and find a conditionally invariant measure for $(\mathring{\Theta}_t)_{t\geq 0}$ of the form

$$\mu(\mathrm{d}\omega, \mathrm{d}x) = \mu_\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)\,, \tag{6.4.7}$$

where the μ_ω are measurable with respect to $\mathcal{F}^0_{-\infty}$, i.e. the past of the system, and

$$\int_\Omega \mu_\omega(\cdot)\,\mathbb{P}(\mathrm{d}\omega) = \nu(\cdot)\,,$$

analogously to the case without killing of trajectories where these measures are called Markov measures. From the previous section we know that if the μ_ω are demanded to be measurable with respect to $\mathcal{F}_0^\infty$, i.e. the future of the system, we get $\mu_\omega = \nu$ almost surely according to Proposition 6.4.2.

Let us first assume the existence of such a conditionally invariant Markov measure. Then we observe that the invariance of the sample measures requires an additional assumption.

Lemma 6.4.6. *Assume that there exists a conditionally invariant Markov probability measure μ on $\Omega \times E$ for $(\mathring{\Theta}_t)_{t\geq 0}$. Then for any $t \in \mathbb{R}^+$, its disintegrations μ_ω such that $E_t(\omega)$ is non-empty satisfy the relation*

$$\frac{\mu_\omega\left(\mathring{\varphi}(t,\omega,\cdot)^{-1}A\right)}{\mu_\omega(E_t(\omega))} = \mu_{\theta_t\omega}(A) \quad \text{for almost all } \omega \in \Omega \text{ and all } A \in \mathcal{B}(E)\,, \tag{6.4.8}$$

if and only if for all $t \geq 0$ we have

$$\mu_\omega(E_t(\omega)) = \mu(\mathring{M}_t) \quad \text{for almost all } \omega \in \Omega \text{ with } E_t(\omega) \neq \emptyset\,. \tag{6.4.9}$$

Proof. Fix $t \geq 0$. We proceed similarly to [62, Proposition 1.3.27]. Take bounded and measurable functions $f : E \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ and observe that by definition of the conditional invariance

$$\int_{\Omega\times E} f(x)g(\omega)\mu_\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega) = \frac{\int_{\Omega\times E} f(\varphi(t,\omega,x))g(\theta_t\omega)\mathbb{1}_{\mathring{M}_t}(\omega,x)\mu_\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)}{\int_\Omega \mu_\omega(E_t(\omega))\mathbb{P}(\mathrm{d}\omega)}\,.$$

Using the invariance of $\mathbb{P}$ with respect to θ_t for the change of variables $\omega \to \theta_{-t}\omega$, we can therefore write

$$\begin{aligned}\int_\Omega \left[\int_E f(x)\mu_{\theta_t\omega}(\mathrm{d}x)\right] g(\omega)\mathbb{P}(\mathrm{d}\omega) &= \int_\Omega \left[\int_E f(x)\mu_\omega(\mathrm{d}x)\right] g(\theta_{-t}\omega)\mathbb{P}(\mathrm{d}\omega) \\ &= \frac{\int_\Omega \left[\int_E f(\varphi(t,\omega,x))\mathbb{1}_{\mathring{M}_t}(\omega,x)\mu_\omega(\mathrm{d}x)\right] g(\omega)\mathbb{P}(\mathrm{d}\omega)}{\int_\Omega \mu_\omega(E_t(\omega))\mathbb{P}(\mathrm{d}\omega)}\,.\end{aligned}$$

Since this is true for any g, we conclude that for almost all $\omega \in \Omega$

$$\frac{\int_E f(\varphi(t,\omega,x))\mathbb{1}_{E_t(\omega)}(x)\mu_\omega(\mathrm{d}x)}{\int_\Omega \mu_\omega(E_t(\omega))\mathbb{P}(\mathrm{d}\omega)} = \int_E f(x)\mu_{\theta_t\omega}(\mathrm{d}x)\,.$$

Hence, equation (6.4.8) is satisfied if and only if

$$\mu_\omega(E_t(\omega)) = \int_\Omega \mu_{\omega'}(E_t(\omega'))\mathbb{P}(\mathrm{d}\omega') = \mu(\mathring{M}_t) \;\text{ for almost all } \omega \in \Omega \text{ with } E_t(\omega) \neq \emptyset\,,$$

which is equivalent to Assumption (6.4.9). □

Two questions arise. First of all, one is inclined to ask how reasonable Assumption (6.4.9) is. Secondly, one might want to investigate if ν as given in (6.4.7) can now be shown to be a quasi-stationary distribution for the associated Markov process. Both are theoretical questions that should be investigated in the future.

Let us now assume that a quasi-stationary distribution ν exists. The question is how to construct the disintegrations μ_ω such that μ is a conditionally invariant Markov probability measure. The natural analogue to the case without absorption is given by

$$\mu_\omega(\cdot) := \lim_{t\to\infty} \frac{\nu\left(\mathring{\varphi}(t,\theta_{-t}\omega,\cdot)^{-1}(\cdot)\right)}{\nu(E_t(\theta_{-t}\omega))}\,,$$

if the limit exists. However, due to our assumption of almost sure killing, the numerator and denominator both become zero in finite time for almost all ω such that this limit cannot exist for any measurable subset of E.

One idea to overcome this problem would be to investigate if the Q-process, i.e. the survival process, corresponds to a particular random dynamical system that can be studied in infinite time in the classical way. However, we explain why this does not seem feasible. Recall from Section 6.1.1 that the Q-process or survival process $(Y_t)_{t\geq 0}$ is the Markov process with state space E, filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and transition probabilities $(\mathbb{Q}_x)_{x\in E}$ such that for any $s \geq 0$

$$\mathbb{Q}_x((Y_u)_{0\leq u\leq s} \in \cdot) = \lim_{t\to\infty} \mathbb{P}_x((X_u)_{0\leq u\leq s} \in \cdot | T > t)\,.$$

Further the Markov semigroup of operators Q_t defined by $Q_t f(x) = \mathbb{E}_x(f(Y_t))$ on an ap-

propriate function space is m-invariant where m is the quasi-ergodic measure for the killed process. If there was a random dynamical system (θ, φ) associated with this Q-process, each ω-trajectory $\varphi(t, \omega, x) = Y_t(\omega)$ starting from $x \in E$ would require a completely new interpretation since the boundary is never hit. If, for example, the original process $(X_t)_{t\geq 0}$ is a solution process to a stochastic differential equation, the process $(Y_t)_{t\geq 0}$ cannot solve a corresponding stochastic differential equation because there is no hitting of the boundary for the survival process. The possible generation of a random dynamical system could come from a random differential equation with bounded noise that is solved by the process $(Y_t)_{t\geq 0}$. However, it is entirely opaque how such an equation could be derived. A completely abstract generation of a random dynamical system from the Q-process seems even less attainable.

6.5 Exponential dichotomies and the dichotomy spectrum

For the investigation of exponential dichotomies we will consider the systems in two-sided time and study the linearisation around trajectories. For this purpose, let (Θ, Φ) be a linear cocycle on $\mathbb{R}^d$ over a (locally) two-sided random dynamical system $\Theta = (\theta, \varphi)$ on $\mathbb{R}^d$. As before, we consider the system on a bounded domain $E \subset \mathbb{R}^d$ with killing at the boundary ∂E. We define $T^+ : \Omega \times E \to \mathbb{R}_0^+$ and $T^- : \Omega \times E \to \mathbb{R}_0^-$ by

$$\begin{aligned} T^+(\omega, x) &= \inf\{t > 0 \,:\, \varphi(t, \omega, x) \in \partial E\} \\ T^-(\omega, x) &= \sup\{t < 0 \,:\, \varphi(t, \omega, x) \in \partial E\}\,, \end{aligned} \tag{6.5.1}$$

i.e. the hitting times of the boundary in forward and backward time. Analogously to before, we assume that $T^+ < \infty$ and $T^- > -\infty$ $\mathbb{P}_x$-a.s. for all $x \in E$ and that for all $t \geq 0$

$$\mathbb{P}_x(T^+ > t) > 0\,, \text{ for all } t \geq 0\,, \quad \text{and} \quad \mathbb{P}_x(T^- < t) > 0\,, \text{ for all } t \leq 0\,.$$

One could ask why we do not define a random dynamical system with killing as in Definition 6.2.1. The reason is that the cocycle property is not satisfied for $t > 0$ and $s < 0$ such that $T^-(\omega, x) < t + s < T^+(\omega, x)$ but $t \geq T^+(\omega, x)$ or $s \leq T^-(\omega, x)$. Hence, we stick to the original system, considering it within the boundaries of the killing time. To avoid ambiguities, we will denote the whole system by (Θ, Φ, T^+, T^-).

First, we define invariant projectors for this setting:

Definition 6.5.1 (Invariant projector)**.** Let (Θ, Φ) be a linear cocycle with stopping times T^+, T^- as given in (6.5.1). An invariant projector P for (Θ, Φ, T^+, T^-) is a measurable function $P : \Omega \times E \to \mathbb{R}^{d\times d}$ such that for all $x \in E$ and almost all $\omega \in \Omega$

(i) $P(\omega, x) = P(\omega, x)^2$,

(ii) $P(\Theta_t(\omega, x))\Phi(t, \omega, x) = \Phi(t, \omega, x)P(\omega, x)$ for all $T^-(\omega, x) < t < T^+(\omega, x)$,

(iii) the map $(\omega, x) \mapsto \mathrm{rk}(P(\omega, x))$ is measurable,

(iv) $\mathrm{rk}(P(\omega, x))$ is constant.

We denote the null space and range of P by

$$\mathcal{N}(P) = \{(\omega, x, v) \in \Omega \times E \times \mathbb{R}^d \,:\, P(\omega, x)v = 0\}\,,$$
$$\mathcal{R}(P) = \{(\omega, x, v) \in \Omega \times E \times \mathbb{R}^d \,:\, P(\omega, x)w = v \text{ for some } w \in \mathbb{R}^d\}\,.$$

According to property (iv) in Definition 6.5.1, we can define the rank of an invariant projector P by

$$\mathrm{rk}\, P := \dim \mathcal{R}(P) := \dim \mathcal{R}(P(\omega, x)) \quad \text{for almost all } \omega \in \Omega, x \in E\,,$$

and set

$$\dim \mathcal{N}(P) := \dim \mathcal{N}(P(\omega, x)) \quad \text{for almost all } \omega \in \Omega, x \in E\,.$$

Property (iii) in Definition 6.5.1 can actually be derived from (i) and (ii) and in the case without killing the same holds for (iv). In the case with killing, however, we loose the essential argument of ergodicity to show that $\mathrm{rk}(P(\omega, x))$ is constant almost surely. We now give the following definition of an exponential dichotomy for the system with absorption at the boundary:

Definition 6.5.2 (Exponential dichotomy)**.** Let (Θ, Φ) be a linear cocycle with stopping times T^+, T^- and let $\gamma \in \mathbb{R}$ and P_γ be an invariant projector for (Θ, Φ, T^+, T^-). Then (Θ, Φ, T^+, T^-) is said to admit an exponential dichotomy with growth rate γ, constants $\alpha > 0$, $K \geq 1$ and projector P_γ if for all $x \in E$ and almost all $\omega \in \Omega$

$$\|\Phi(t, \omega, x)P_\gamma(\omega, x)\| \leq Ke^{(\gamma-\alpha)t}, \text{ for all } 0 \leq t < T^+(\omega, x)\,,$$
$$\|\Phi(t, \omega, x)(\mathrm{Id} - P_\gamma(\omega, x))\| \leq Ke^{(\gamma+\alpha)t}, \text{ for all } 0 \geq t > T^-(\omega, x)\,.$$

Similarly to before, one says that (Θ, Φ, T^+, T^-) admits an exponential dichotomy with growth rate ∞ if there exists a $\gamma \in \mathbb{R}$ such that (Θ, Φ, T^+, T^-) admits an exponential dichotomy with growth rate γ and projector $P_\gamma = \mathrm{Id}$. Analogously, (Θ, Φ, T^+, T^-) admits an exponential dichotomy with growth rate $-\infty$ if there exists a $\gamma \in \mathbb{R}$ such that (Θ, Φ, T^+, T^-) admits an exponential dichotomy with growth rate γ and projector $P_\gamma = 0$. We write $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$.

Analogously to the classical case, we make the following observation:

Lemma 6.5.3. *Suppose that (Θ, Φ, T^+, T^-) admits an exponential dichotomy with growth rate γ and projector P_γ. Then the following statements are satisfied:*

(i) If $P_\gamma = \mathrm{id}$ almost surely, then (Θ, Φ, T^+, T^-) admits an exponential dichotomy with growth rate ζ and invariant projector $P_\zeta = \mathrm{id}$ for all $\zeta > \gamma$.

(ii) If $P_\gamma = 0$ almost surely, then (Θ, Φ, T^+, T^-) admits an exponential dichotomy with growth rate ζ and invariant projector $P_\zeta = 0$ for all $\zeta < \gamma$.

Proof. Immediate consequence of the definition. □

Finally, we define the dichotomy spectrum for this setting.

Definition 6.5.4 (Dichotomy spectrum)**.** Consider (Θ, Φ, T^+, T^-), derived from the linear cocycle (Θ, Φ) with stopping times T^+, T^-. Then its dichotomy spectrum is defined by

$$\Sigma := \{\gamma \in \overline{\mathbb{R}} : (\Theta, \Phi, T^+, T^-) \text{ does not admit an exp. dichotomy with growth rate } \gamma\}.$$

The corresponding resolvent set is defined by $\rho := \overline{\mathbb{R}} \setminus \Sigma$.

In Theorem 6.5.8 we will characterise the dichotomy spectrum as a disjoint union of at least one and at most d closed intervals. The proof uses a couple of Lemmas about the resolvent set that are shown in the following, similarly to [21] and [80]. First, we show that the ranks of invariant projectors are monotonically increasing with respect to the growth rate.

Lemma 6.5.5. *Consider the resolvent set ρ of (Θ, Φ, T^+, T^-) and let $\gamma_1, \gamma_2 \in \rho \cap \mathbb{R}$ such that $\gamma_1 \leq \gamma_2$. Choose invariant projectors P_{γ_1} and P_{γ_2} for the corresponding exponential dichotomies with growth rates γ_1 and γ_2. Then we have $\operatorname{rk} P_{\gamma_1} \leq \operatorname{rk} P_{\gamma_2}$. In particular, if $\gamma_1 = \gamma_2$, then $\operatorname{rk} P_{\gamma_1} = \operatorname{rk} P_{\gamma_2}$.*

Proof. Let $K_1 \geq 1$ and $K_2 \geq 1, \alpha_2 > 0$ be the corresponding constants for the exponential dichotomies with γ_1 and γ_2 respectively. Let further t^* be large enough such that $K_2 K_1 e^{-\alpha_2 t^*} < 1$, and for fixed $x \in E$ choose ω from the set $\Omega_x = \{\omega \in \Omega : T^+(\omega, x) > t^*\}$. Now let $v(\omega, x) \in \mathcal{N}(P_{\gamma_2}(\omega, x)) \cap \mathcal{R}(P_{\gamma_1}(\omega, x))$ and assume that $v(\omega, x) \neq 0$. We observe that for all $0 < t^* < t < T^+(\omega, x)$

$$\begin{aligned}
\|v(\omega,x)\| &= \|\Phi(-t, \Theta_t(\omega,x))\Phi(t,\omega,x)(\mathrm{Id} - P_{\gamma_2}(\omega,x))v(\omega,x)\| \\
&= \|\Phi(-t, \Theta_t(\omega,x))(\mathrm{Id} - P_{\gamma_2}(\Theta_t(\omega,x)))\Phi(t,\omega,x)v(\omega,x)\| \\
&\leq K_2 e^{-(\gamma_2+\alpha_2)t}\|\Phi(t,\omega,x)v(\omega,x)\| \leq K_2 K_1 e^{-(\gamma_2+\alpha_2)t} e^{\gamma_1 t}\|v(\omega,x)\| \\
&\leq K_2 K_1 e^{-\alpha_2 t}\|v(\omega,x)\| < \|v(\omega,x)\|,
\end{aligned}$$

which is a contradiction. Hence,

$$\mathcal{N}(P_{\gamma_2}(\omega,x)) \cap \mathcal{R}(P_{\gamma_1}(\omega,x)) = \{0\} \quad \text{for all } x \in E,\ \omega \in \Omega_x.$$

Since Ω_x has positive probability and the dimensions of the ranges and null spaces of invariant projectors are constant almost surely, we can deduce that

$$0 = \dim(\mathcal{R}(P_{\gamma_1}) \cap \mathcal{N}(P_{\gamma_2})) = \operatorname{rk} P_{\gamma_1} + \dim \mathcal{N}(P_{\gamma_2}) - \dim(\mathcal{R}(P_{\gamma_1}) + \mathcal{N}(P_{\gamma_2})).$$

This is used to observe that

$$\operatorname{rk} P_{\gamma_2} = d - \dim \mathcal{N} P_{\gamma_2} = \operatorname{rk} P_{\gamma_1} + d - \dim(\mathcal{R} P_{\gamma_1} + \mathcal{N} P_{\gamma_2}) \geq \operatorname{rk} P_{\gamma_1} .$$

which shows the first statement of the lemma. The second statement is an immediate consequence. □

We proceed with showing that the resolvent set is open in $\mathbb{R}$.

Lemma 6.5.6. *Consider the resolvent set ρ of (Θ, Φ, T^+, T^-). Then for all $\gamma \in \rho \cap \mathbb{R}$, there is an $\varepsilon > 0$ such that $B_\varepsilon(\gamma) \subset \rho \cap \mathbb{R}$, which means that $\rho \cap \mathbb{R}$ is an open set. Furthermore, we have $\operatorname{rk} P_\zeta = \operatorname{rk} P_\gamma$ for all $\zeta \in B_\varepsilon(\gamma)$ and every invariant projector P_γ and P_ζ of the exponential dichotomies of (Θ, Φ, T^+, T^-) with growth rates γ and ζ, respectively.*

Proof. Let $\gamma \in \rho \cap \mathbb{R}$ and α, K be the constants for the exponential dichotomy with growth rate γ and invariant projector P_γ. Set $\varepsilon := \frac{1}{2}\alpha$ and choose any $\zeta \in B_\varepsilon(\gamma)$. Then for all $x \in E$ and almost all $\omega \in \Omega$

$$\begin{aligned} \|\Phi(t, \omega, x) P_\gamma(\omega, x)\| &\leq K e^{(\zeta - \frac{1}{2}\alpha)t}, \text{ for all } 0 \leq t < T^+(\omega, x) , \\ \|\Phi(t, \omega, x)(\operatorname{Id} - P_\gamma(\omega, x))\| &\leq K e^{(\zeta + \frac{1}{2}\alpha)t}, \text{ for all } 0 \geq t > T^-(\omega, x) . \end{aligned}$$

Hence, P_γ is an invariant projector for the exponential dichotomy with growth rate ζ and we have $\operatorname{rk} P_\zeta = \operatorname{rk} P_\gamma$ for any other such invariant projector by Lemma 6.5.5. □

The last ingredient for proving Theorem 6.5.8 is the following lemma.

Lemma 6.5.7. *Consider the resolvent set ρ of (Θ, Φ, T^+, T^-) and let $\gamma_1, \gamma_2 \in \rho \cap \mathbb{R}$ such that $\gamma_1 < \gamma_2$. Then $[\gamma_1, \gamma_2] \subset \rho$ if and only if $\operatorname{rk} P_{\gamma_1} = \operatorname{rk} P_{\gamma_2}$.*

Proof. First assume that $[\gamma_1, \gamma_2] \subset \rho$, and suppose for contradiction that $\operatorname{rk} P_{\gamma_1} \neq \operatorname{rk} P_{\gamma_2}$. Choosing invariant projectors P_γ for exponential dichotomies with growth rates $\gamma \in (\gamma_1, \gamma_2)$, we define

$$\zeta_0 := \sup\{\zeta \in [\gamma_1, \gamma_2] \,:\, \operatorname{rk} P_\zeta \neq \operatorname{rk} P_{\gamma_2}\} .$$

However, according to Lemma 6.5.6 there is an $\varepsilon > 0$ such that $\operatorname{rk} P_\zeta = \operatorname{rk} P_{\zeta_0}$ for all $\zeta \in B_\varepsilon(\zeta_0)$, contradicting the definition of ζ_0. Hence, we have shown the first implication.

Assume now that $\operatorname{rk} P_{\gamma_1} = \operatorname{rk} P_{\gamma_2}$. We have already seen in the proof of Lemma 6.5.5 that

$$\mathcal{N}(P_{\gamma_2}(\omega, x)) \cap \mathcal{R}(P_{\gamma_1}(\omega, x)) = \{0\} \quad \text{for all } x \in E , \omega \in \Omega_x ,$$

where $\Omega_x = \{\omega \in \Omega \,:\, T^+(\omega, x) > t^*\}$ has positive probability. Adjusting for ω with $T^+(\omega, x) < t^*$ by the constant K at the end of the proof, we may assume that the above holds for almost all $\omega \in \Omega$. In combination with $\operatorname{rk} P_{\gamma_1} = \operatorname{rk} P_{\gamma_2}$, this implies that we

can construct an invariant projector P with $\mathcal{N}(P) = \mathcal{N}(P_{\gamma_2})$ and $\mathcal{R}(P) = \mathcal{R}(P_{\gamma_1})$. This means that for constants $K_1, \alpha_1, K_2, \alpha_2$ we have for all $x \in E$ and almost all $\omega \in \Omega$

$$\begin{aligned}\|\Phi(t,\omega,x)P(\omega,x)\| &\leq K_1 e^{(\gamma_1-\alpha_1)t}, \text{ for all } 0 \leq t < T^+(\omega,x)\,,\\ \|\Phi(t,\omega,x)(\mathrm{Id}-P(\omega,x))\| &\leq K_2 e^{(\gamma_2+\alpha_2)t}, \text{ for all } 0 \geq t > T^-(\omega,x)\,.\end{aligned}$$

Setting $K := \max\{K_1, K_2\}$ and $\alpha := \min\{\alpha_1, \alpha_2\}$, we obtain for all $\gamma \in [\gamma_1, \gamma_2]$ that for all $x \in E$ and almost all $\omega \in \Omega$

$$\begin{aligned}\|\Phi(t,\omega,x)P(\omega,x)\| &\leq K e^{(\gamma-\alpha)t}, \text{ for all } 0 \leq t < T^+(\omega,x)\,,\\ \|\Phi(t,\omega,x)(\mathrm{Id}-P(\omega,x))\| &\leq K e^{(\gamma+\alpha)t}, \text{ for all } 0 \geq t > T^-(\omega,x)\,.\end{aligned}$$

This implies that $[\gamma_1, \gamma_2] \subset \rho$. □

Analogously to the classical case [21, Theorem 3.4], we can now prove the following statement. Clarifying the notation, we define for $a \in \mathbb{R}$

$$[-\infty, a] := (-\infty, a] \cup \{-\infty\}\,, \quad [a, \infty] := [a, \infty) \cup \{\infty\}\,,$$

and

$$[-\infty, -\infty] := \{-\infty\}\,, \quad [\infty, \infty] := \{\infty\}\,, \quad [-\infty, \infty] := \overline{\mathbb{R}}\,.$$

Theorem 6.5.8 (Characterisation of the dichotomy spectrum)**.** *Consider the linear system (Θ, Φ, T^+, T^-) with dichotomy spectrum Σ. Then there exists an $n \in \{1, \dots, d\}$ such that*

$$\Sigma = [a_1, b_1] \cup \cdots \cup [a_n, b_n]$$

for some $-\infty \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n \leq \infty$.

Proof. Since $\rho \cap \mathbb{R}$ is open according to Lemma 6.5.6, $\Sigma \cap \mathbb{R}$ is the disjoint union of closed intervals. Furthermore by Lemma 6.5.3, $(-\infty, b_1] \subset \Sigma$ implies $[-\infty, b_1] \subset \Sigma$ and $[a_n, \infty) \subset \Sigma$ implies $[a_n, \infty] \subset \Sigma$.

The fact that $1 \leq n \leq d$ is a consequence of Lemma 6.5.7 which can now be derived exactly as in the proof of [21, Theorem 3.4]. □

We will give an example of an explicit calculation of Σ in Section 6.6.

In addition, we can prove the following version of [21, Theorem 4.5] adapted to the problem of absorption at the boundary, relating the upper and lower limits of the finite-time Lyapunov exponents to the boundary of the dichotomy spectrum. In the following the essential supremum means that all $x \in E$ and almost all $\omega \in \Omega$ are considered.

Theorem 6.5.9 (Supremum and infimum of the dichotomy spectrum)**.** *Let Σ denote the dichotomy spectrum of (Θ, Φ, T^+, T^-). As before, define the finite-time Lyapunov*

exponent for each $(\omega, x) \in \Omega \times E$ *and* $v \in \mathbb{R}^d \setminus \{0\}$ *by*

$$\lambda_v(t, \omega, x) = \frac{1}{t} \ln \frac{\|\Phi(t, \omega, x)v\|}{\|v\|} \quad \textit{for} \quad 0 \leq t < T^+(\omega, x)\,,$$

Then

$$\lim_{t\to\infty} \operatorname*{ess\,sup}_{\{T^+(\omega,x)>t\}} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \lambda_v(t, \omega, x) = \sup \Sigma$$

provided that $\sup \Sigma < \infty$ *and*

$$\lim_{t\to\infty} \operatorname*{ess\,inf}_{\{T^+(\omega,x)>t\}} \inf_{v\in\mathbb{R}^d\setminus\{0\}} \lambda_v(t, \omega, x) = \inf \Sigma$$

provided that $\inf \Sigma > -\infty$.

Proof. In the following, we simply denote $\tilde{T}(\omega, x) := T^+(\omega, x)$. By definition of $\lambda_v(t, \omega, x)$ we get for all $t, s \geq 0$

$$\begin{aligned}(t+s) &\operatorname*{ess\,sup}_{\{\tilde{T}(\omega,x)>t+s\}} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \lambda_v(t+s, \omega, x) \\ &\leq t \operatorname*{ess\,sup}_{\{\tilde{T}(\omega,x)>t\}} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \lambda_v(t, \omega, x) + s \operatorname*{ess\,sup}_{\{\tilde{T}(\omega,x)>s\}} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \lambda_v(s, \omega, x)\,.\end{aligned}$$

This implies that the sequence $\left(t \operatorname{ess\,sup}_{\{\tilde{T}(\omega,x)>t\}} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \lambda_v(t, \omega, x)\right)_{t\geq 0}$ is subadditive. Hence, we obtain

$$\lim_{t\to\infty} \operatorname*{ess\,sup}_{\{\tilde{T}(\omega,x)>t\}} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \lambda_v(t, \omega, x) = \limsup_{t\to\infty} \operatorname*{ess\,sup}_{\{\tilde{T}(\omega,x)>t\}} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \lambda_v(t, \omega, x).$$

Provided $\sup < \infty$ we show that

$$\gamma := \limsup_{t\to\infty} \operatorname*{ess\,sup}_{\{\tilde{T}(\omega,x)>t\}} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \lambda_v(t, \omega, x) = \sup \Sigma.$$

Since $\sup \Sigma < \infty$, there exists a $K \geq 1$ such that for almost all $\omega \in \Omega$ and all $x \in E$, $v \in \mathbb{R}^d \setminus \{0\}$

$$\|\Phi(t, \omega, x)v\| \leq K e^{t \sup \Sigma} \|v\| \quad \text{for all } 0 \leq t < \tilde{T}(\omega, x). \tag{6.5.2}$$

Assume for contradiction that $\gamma < \sup \Sigma$. From the definition of γ, this means that there exists a $t_0 > 0$ such that for almost all $\omega \in \Omega$ and all $x \in E$ with $\tilde{T}(\omega, x) > t_0$ and all $v \in \mathbb{R}^d \setminus \{0\}$

$$\|\Phi(t, \omega, x)v\| \leq K e^{\frac{1}{2}t(\gamma + \sup \Sigma)} \|v\| \quad \text{for all } t_0 \leq t < \tilde{T}(\omega, x).$$

Hence, together with (6.5.2) we obtain for almost all $\omega \in \Omega$ and all $x \in E$, $v \in \mathbb{R}^d \setminus \{0\}$

that

$$\|\Phi(t,\omega,x)v\| \leq \hat{K}e^{\frac{1}{2}t(\gamma+\sup\Sigma)}\|v\| \quad \text{for all } 0 \leq t < \tilde{T}(\omega,x)\,,$$

where

$$\hat{K} := \max\left\{1, Ke^{\frac{t_0}{2}(\sup\Sigma-\gamma)}\right\}.$$

By the definition of Σ, this implies that $\sup\Sigma \leq \frac{1}{2}(\gamma+\sup\Sigma)$ which is a contradiction. Hence, we have established that $\gamma \geq \sup\Sigma$.

Assume now that $\gamma > \sup\Sigma$ which implies $\sup\Sigma < \infty$. Hence, by the definition of the dichotomy spectrum, there exists a $K \geq 1$ such that for almost all $\omega \in \Omega$ and all $x \in E$, $v \in \mathbb{R}^d \setminus \{0\}$

$$\|\Phi(t,\omega,x)v\| \leq Ke^{\frac{1}{2}t(\gamma+\sup\Sigma)}\|v\| \quad \text{for all } 0 \leq t < \tilde{T}(\omega,x)\,.$$

On the other hand, this yields

$$\lambda_v(t,\omega,x) \leq \frac{\ln K}{t} + \frac{1}{2}(\gamma+\sup\Sigma)$$

for all $v \in \mathbb{R}^d \setminus \{0\}$ whenever $t < \tilde{T}(\omega,x)$. Since $\frac{\ln K}{t} \to 0$ as $t \to \infty$, we conclude that

$$\gamma = \limsup_{t\to\infty} \operatorname*{ess\,sup}_{\{\tilde{T}(\omega,x)>t\}} \sup_{v\in\mathbb{R}^d\setminus\{0\}} \lambda_v(t,\omega,x) \leq \frac{1}{2}(\gamma+\sup\Sigma)\,,$$

which is again a contradiction. This proves the equality.

The other claim follows analogously. □

6.6 Bifurcations

In this section we are discussing the typical examples of pitchfork and Hopf bifurcation in the setting with absorption at the boundary and apply the results from the previous sections.

6.6.1 Pitchfork bifurcation of a killed process

Recall the toy example of the stochastic pitchfork bifurcation with drift f_α given by (6.0.1) and additive noise. We consider the problem with absorption at the boundary on an interval $I = (-c, c)$ and investigate the sign of the conditioned average Lyapunov exponent as given in Proposition 6.2.5 and the dichotomy spectrum defined in Definition 6.5.4 as indicators of bifurcations.

Recall that the generator of the associated Markov semigroup is given by

$$\mathcal{L} = \frac{\sigma^2}{2}\partial_{xx} + f_\alpha\partial_x\,.$$

The crucial quantities we need to calculate are the largest non-zero eigenvalue λ_0 of $\mathcal{L}$ and the corresponding eigenfunction ψ which is the unique solution to

$$\mathcal{L}\psi = \lambda_0\psi,\ \psi(-c) = \psi(c) = 0,\ \int_I \psi^2 \mathrm{d}\mu = 1\,, \tag{6.6.1}$$

where

$$\mu(\mathrm{d}x) := \exp\left(\gamma(x)\right)\mathrm{d}x$$

and

$$\gamma(x) := \frac{2}{\sigma^2}\int_{-c}^{x} f_\alpha(y)\,\mathrm{d}y\,.$$

Determining the eigenfunction ψ enables us to calculate λ from (6.2.7). The eigenvalue λ_0 is exactly the exponential survival rate and $-\lambda_0$ the exponential escape rate when starting with the quasi-stationary distribution. Since determining ψ and λ_0 lies beyond analytical methods, we use the following finite difference scheme to attain a numerical approximation.

We choose a grid of $N+1$ points $(x_0,\ldots,x_N)$ on $[-c,c]$, where $x_0 = -c, x_N = c$, with distance $h = 2c/N$ and calculate the vector $y = (y_0,\ldots,y_N)$ which approximates the solution ψ of (6.6.1). Due to the boundary conditions we have $y_0 = \psi(x_0) = \psi(x_N) = y_N = 0$. The central difference formula gives for all $i = 1,\ldots,N-1$

$$\psi'(x_i) = \frac{\psi(x_{i+1}) - \psi(x_{i-1})}{2h} + \mathcal{O}(h^2)\,,$$

and

$$\psi''(x_i) = \frac{\psi(x_{i+1}) - 2\psi(x_i) + \psi(x_{i-1})}{h^2} + \mathcal{O}(h^2)\,.$$

Hence, the vector $\tilde{y} = (y_1,\ldots,y_{N-1})$, normalised to $h\sum_i^{N-1} y_i^2 \exp(\gamma(x_i)) = 1$, is the eigenvector belonging to the largest eigenvalue $\tilde{\lambda}_0$ of the matrix $A = (A_{ij})_{i,j=1,\ldots,N-1}$ where for all $i,j = 1,\ldots,N-1$

$$A_{ij} = \frac{1}{2h}D^1_{ij} + \frac{\sigma^2}{2h^2}D^2_{ij}\,,$$

and

$$D^1_{ij} = \begin{cases} f_\alpha(x_i) & \text{if } j = i+1 \\ -f_\alpha(x_i) & \text{if } j = i-1\,, \\ 0 & \text{otherwise} \end{cases} \qquad D^2_{ij} = \begin{cases} -2 & \text{if } j = i \\ 1 & \text{if } |j-i| = 1\,. \\ 0 & \text{otherwise} \end{cases}$$

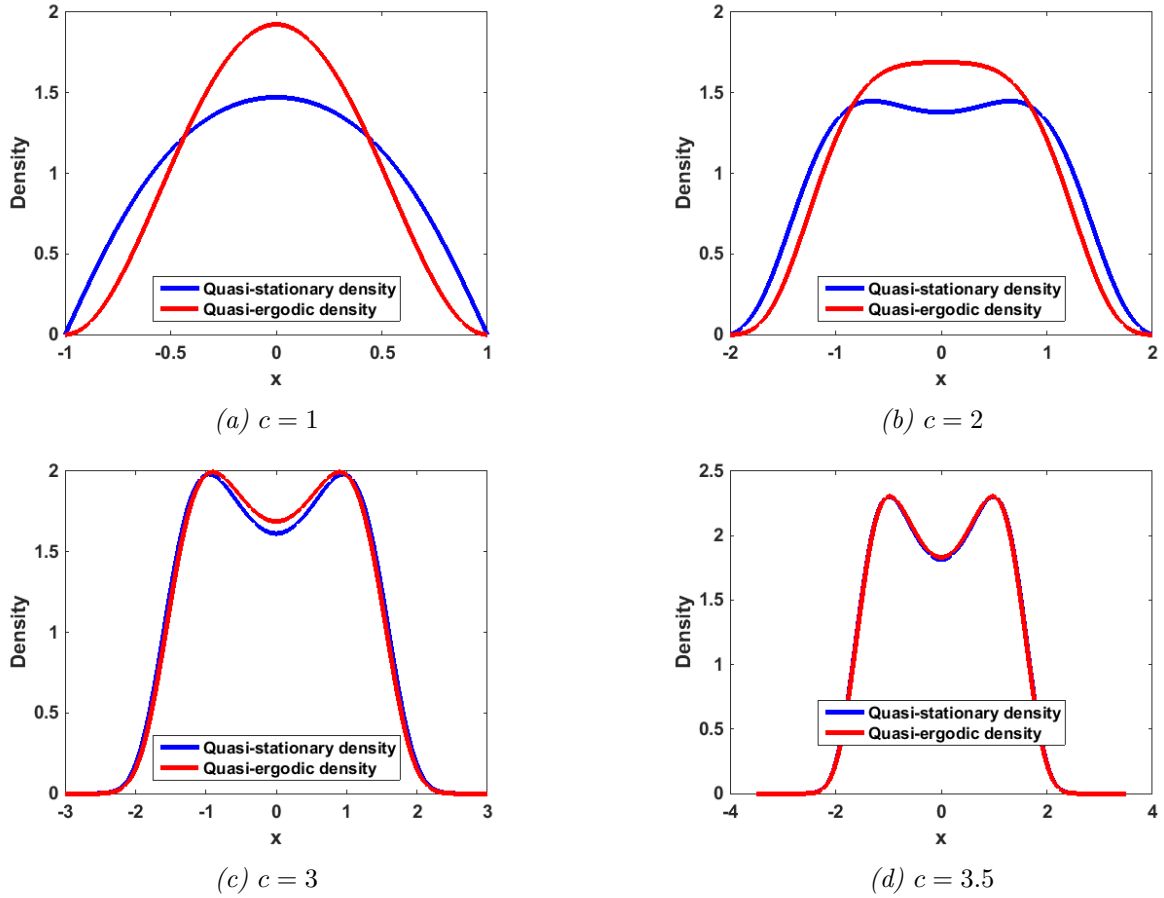
First of all, we use such a calculation to approximate the density of the quasi-stationary distribution ν and the density of the quasi-ergodic distribution m. According to Theorem 6.1.9, $\mathrm{d}\nu$ is given by

$$\nu(\mathrm{d}x) = \frac{1}{Z}\psi(x)\exp\left(\gamma(x)\right)\mathrm{d}x\,, \tag{6.6.2}$$

where $Z > 0$ is a normalisation constant, and $\mathrm{d}m$ reads as

$$m(\mathrm{d}x) = \psi^2(x) \exp\left(\gamma(x)\right) \mathrm{d}x\,. \tag{6.6.3}$$

In Fig. 6.1, we illustrate the quasi-stationary and quasi-ergodic density for fixed $\alpha = 1 = \sigma$ and different values of c. As one would expect, the two distributions converge with increasing c as, for $c = \infty$, they are both equal to the stationary distribution of the process on $\mathbb{R}$. It is interesting to observe that this convergence happens very fast. Furthermore, we can see the double-well potential is reflected more and more with increasing c. For $c = 1$, it is obviously not present due to the boundary conditions, and for $c = 2$, it is only mirrored in the quasi-stationary density but not in the quasi-ergodic density. This indicates that surviving trajectories still spend the same amount of time close to the unstable equilibrium as they do close to the two stable ones. For $c = 3$ and $c = 3.5$ the unstable equilibrium looses mass compared to the two stable ones but also the regions close to the boundary get far less quasi-stationary and quasi-ergodic mass.



(a) $c = 1$ *(b)* $c = 2$

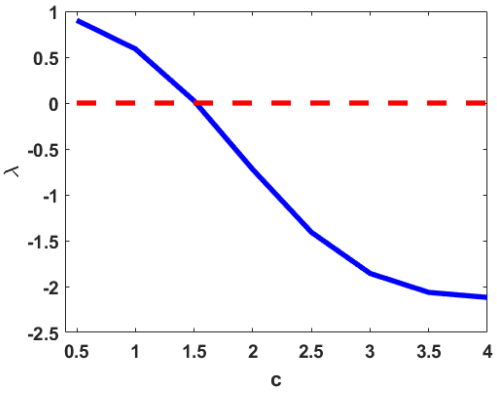
(c) $c = 3$ *(d)* $c = 3.5$

Figure 6.1: We set $\alpha = \sigma = 1$ *and compute* ψ *according to the above finite difference scheme with* $N = 100$ *grid points on* $[-c, c]$ *for different values of* c*. Due to* (6.6.2) *and* (6.6.3)*, this allows for approximations of* $\mathrm{d}m$ *and* $\mathrm{d}\nu$ *as depicted in the four pictures. We observe that with increasing* c *the double-well structure becomes more apparent in both distributions which become closer and closer.*

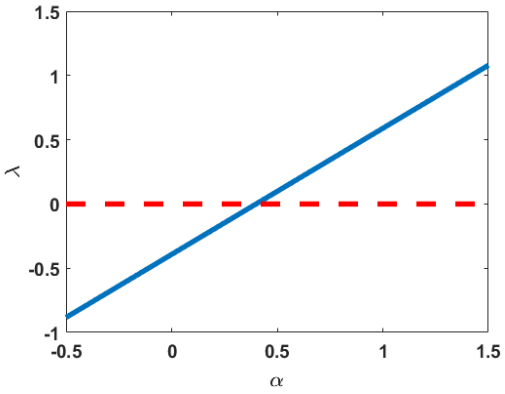
We now turn to investigating how the described changes in the quasi-ergodic distribution influence the values of the conditioned average Lyapunov exponent λ. Recall from (6.2.7) that

$$\lambda = \lim_{t\to\infty} \mathbb{E}\left(\lambda(t,\cdot,x)|\tilde{T}(\cdot,x) > t\right) = \int_I f'_\alpha(y)\, m(\mathrm{d}y) = \int_I f'_\alpha(y)\psi^2(y)e^{\gamma(y)}\,\mathrm{d}y\,.$$

We approximate the Lyapunov exponent λ for different values of the parameters α and c and observe changes of the sign of λ. In Figure 6.2a, $\alpha = 1$ is fixed and λ is computed for different values of the interval size determined by c. The change from positive to negative λ occurs for a $c^* > 1.5$, hence, in a situation where both stable equilibria are present with considerably large neighbourhoods. This means that the instabilities can still be predominant in a scenario where the regions of stability are already comparably large, and the transition from positive to negative λ is not trivial by any means. We can give Figure 6.2b the same interpretation, where we observe for fixed $c = 1$ that λ changes sign from negative to positive at some $\alpha^* < 0.5$. Note that we understand positive λ as an indication of chaotic behaviour in the quasi-ergodic limit whereas negative λ implies local synchronisation of surviving trajectories by Corollary 6.3.2.



(a) λ as a function of c



(b) λ as a function of α

Figure 6.2: In Figure 6.2a, we set $\alpha = \sigma = 1$ and compute $\lambda = \int_I f'_\alpha(y)\psi^2(y)e^{\gamma(y)}\,\mathrm{d}y$ as a function of the interval boundary size c where ψ is determined from a finite difference scheme with $N = 100$ as before. Similarly, in Figure 6.2b, we set $c = \sigma = 1$ and compute λ as a function of the bifurcation parameter α. The dashed red line demarcates the zero level in both pictures and helps to observe the change of sign of λ.

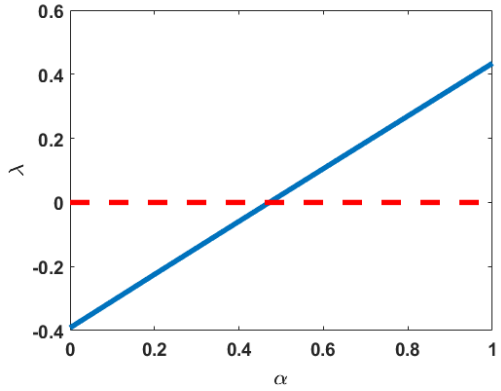
Most importantly, observe that we have reached one of the goals pointed out in the Introduction. Whereas in the global stochastic problem the Lyapunov exponent is negative for all parameter values and the deterministic bifurcation vanishes in terms of stability, we have now established non-trivial situations with positive conditioned average Lyapunov exponent λ. Recall that λ describes the typical stability behaviour of surviving trajectories since it is the limit of all finite-time Lyapunov exponents by Theorem 6.2.8. Summarising, we can capture the local bifurcation behaviour still present in the stochastic

case by showing that λ changes signs depending on bifurcation parameter α and interval size c.

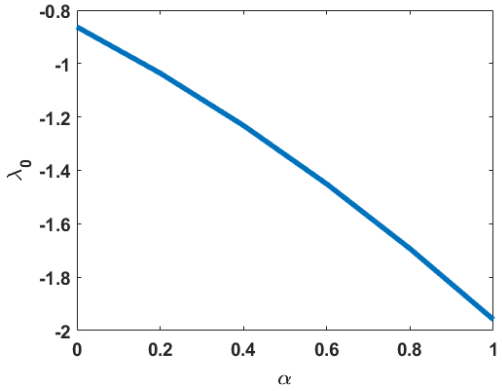
One could now argue that the cases of $\lambda > 0$ are accompanied by very high escape rates, i.e. $-\lambda_0 >> 0$. First of all, this is not necessarily true if one finds the right tuning of noise intensity σ for certain α, c such that $\lambda > 0$. In Figure 6.3 we again fix $c = 1$ and depict λ and λ_0 as functions of α. For its approximation we have used the finite difference scheme explained above with $N = 100$ grid points. We observe that even in parameter regions of positive λ, the escape, i.e. $1 - e^{\lambda_0 t}$, is moderate. Furthermore, the normal form is only supposed to describe the local behaviour and ignores possible higher order terms. As in Figures 1.6 and 1.7, the actual global formula for the drift could for example be given by

$$f_\alpha(x) = \alpha x - x^3 + \frac{3}{10}x^5 .$$

In this case, we are naturally only interested in a bounded region confined by the outer peaks of the potential since the trajectories that overcome the peaks explode. Hence, considering only surviving trajectories and their dynamical behaviour is well justified even if the escape rates might be high.



(a) λ as a function of α



(b) λ_0 as a function of α

Figure 6.3: We set $c = 1$ and $\sigma = 0.5$. In Figure 6.3a, we depict λ as a function of the bifurcation parameter α based on the same computation of the eigenfunction ψ as before. The dashed red line demarcates the zero level, helping to spot the change of sign of λ. The same finite difference scheme with $N = 100$ that approximates ψ gives an estimate of the corresponding maximal eigenvalue λ_0. Figure 6.3b shows the survival rate of trajectories λ_0 depending on α.

Finally, recall from [21] that the dichotomy spectrum for the linearisation around the random equilibrium of the random dynamical system φ_α induced by the pitchfork SDE on $\mathbb{R}$ equals $[-\infty, \alpha]$. This mirrors a loss of hyperbolicity at $\alpha = 0$. In our setting, there is no random equilibrium in the classical sense, but we have defined the dichotomy spectrum in Section 6.5 for the linear system $(\Theta_\alpha, \mathrm{D}\varphi_\alpha, T^+, T^-)$ along almost all trajectories. Here, Θ_α denotes the skew product system without killing, $\mathrm{D}\varphi_\alpha$ the Jacobian of the original random dynamical system φ_α and T^+, T^- the hitting times in forward and backward time.

The following proposition determines the dichotomy spectrum of the stochastic pitchfork problem for the setting with absorption at the boundary.

Proposition 6.6.1. *For all $\alpha \in \mathbb{R}$ the dichotomy spectrum Σ_α associated with the system $(\Theta_\alpha, \mathrm{D}\varphi_\alpha, T^+, T^-)$ on $I = (-c, c)$ satisfies*

$$\Sigma_\alpha = [\alpha - 3c^2, \alpha].$$

Proof. In the following, we denote $\Phi_\alpha := \mathrm{D}\varphi_\alpha$ and $\tilde{T}(\omega, x) := T^+(\omega, x)$. Since $f'_\alpha(x) = \alpha - 3x^2$, we clearly have

$$|\Phi_\alpha(t, \omega, x)| = \exp(\int_0^t \alpha - 3\,|\varphi_\alpha(s, \omega, x)|^2 \,\mathrm{d}s) \tag{6.6.4}$$

for all $x \in I$, $\omega \in \Omega$ and $T^-(\omega, x) < t < \tilde{T}(\omega, x)$. Hence, we observe that

$$|\Phi_\alpha(t, \omega, x)| \leq e^{\alpha t} \quad \text{for all } 0 \leq t \leq \tilde{T}(\omega, x)\,.$$

Furthermore, we have

$$|\Phi_\alpha(t, \omega, x)| \geq e^{(\alpha - 3c^2)t} \quad \text{for all } 0 \leq t \leq \tilde{T}(\omega, x)\,,$$

or equivalently

$$|\Phi_\alpha(t, \omega, x)| \leq e^{(\alpha - 3c^2)t} \quad \text{for all } 0 \geq t \geq T^-(\omega, x)\,,$$

Hence, we can immediately infer that $\Sigma_\alpha \subset [\alpha - 3c^2, \alpha]$.

We turn to showing that $[\alpha - 3c^2, \alpha] \subset \Sigma_\alpha$. Let $\gamma \in [\alpha - 3c^2, \alpha]$ be arbitrary. Suppose for contradiction that Φ_α admits an exponential dichotomy with growth rate γ with an invariant projection P_γ and $K, \varepsilon > 0$. We have to consider two cases:

(1) $P_\gamma = \mathrm{id}$: this means that for all $x \in I$ and almost all $\omega \in \Omega$

$$|\Phi_\alpha(t, \omega, x)| \leq K e^{(\gamma - \varepsilon)t} \text{ for all } 0 \leq t \leq \tilde{T}(\omega, x)\,.$$

We fix $t' > 0$ such that $e^{\frac{\varepsilon}{4}t'} > K$. Due to the nature of additive noise, one can deduce analogously to the proof of Proposition 3.4.1 that there is a positive measure set $\mathcal{A}(0)$ such that for all $\omega \in \mathcal{A}(0)$ and $s \in [0, t']$ we have $\varphi_\alpha(\omega, s, 0) \in B_{\frac{\sqrt{\varepsilon}}{2}}(0)$. We derive from (6.6.4) that for $\omega \in \mathcal{A}(0)$

$$|\Phi_\alpha(t, \omega, 0)| \geq e^{t'(\alpha - \frac{3\varepsilon}{4})} \geq K e^{(\gamma - \varepsilon)t'}\,.$$

This contradicts the assumption.

(2) $P_\gamma = 0$: in this case we have for all $x \in I$ and almost all $\omega \in \Omega$

$$|\Phi_\alpha(t,\omega,x)| \geq \frac{1}{K} e^{(\gamma+\varepsilon)t} \text{ for all } 0 \leq t \leq \tilde{T}(\omega,x)\,.$$

Together with (6.6.4), this implies that

$$\frac{\ln K}{3} + \left(c^2 - \frac{\delta}{3}\right) t = \frac{\ln K + (\alpha - \varepsilon - \gamma)t}{3} \geq \int_0^t |\varphi_\alpha(s,\omega,x)|^2 \,\mathrm{d}s\,,$$

where we have defined $\delta = \varepsilon + \gamma - \alpha + 3c^2 > 0$.

Fix some $t' > \frac{4 \ln K}{\delta}$ and take some $x \in (c - \frac{\delta}{8c}, c)$. Similarly to (1) and the proof of Proposition 3.4.1, we can deduce that there is a positive measure set $\mathcal{A}$ such that for all $\omega \in \mathcal{A}$ and $s \in [0, t']$ we have $\varphi_\alpha(\omega, s, x) \in (c - \frac{\delta}{8c}, c)$. Therefore, we obtain for all $\omega \in \mathcal{A}$

$$\int_0^{t'} |\varphi_\alpha(s,\omega,x)|^2 \,\mathrm{d}s \geq (c^2 - \frac{\delta}{4})t' > \left(c^2 - \frac{\delta}{3}\right) t' + \frac{\ln K}{3}\,.$$

This is a contradiction and finishes the proof.

□

We observe the same loss of hyperbolicity at $\alpha = 0$ as in the situation without killing at the boundary. In this sense the dichotomy spectrum still reflects the deterministic bifurcation. As we have seen in Figures 6.2b and 6.3a, the change of sign of the Lyapunov exponent λ appears at some $\alpha > 0$. This indicates a new bifurcation scenario from locally synchronising to chaotic behaviour.

6.6.2 Hopf bifurcation of a killed process

We turn to the problem of stochastic Hopf bifurcation on a bounded domain with absorption at the boundary. In the following, let $E = B_R(0)$ be an open ball of radius R around the origin and consider, similarly to Chapter 3, the model

$$dZ_t = f(Z_t)\mathrm{d}t + \sigma \,\mathrm{d}W_t\,, \quad Z_0 \in E\,, \tag{6.6.5}$$

where $Z_t = (x_t, y_t)^T \in \bar{E}$, $W_t = \begin{pmatrix} \mathrm{d}W_t^1 \\ \mathrm{d}W_t^2 \end{pmatrix}$, $\sigma > 0$ constant, and the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$f(Z) := \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} Z - (x^2 + y^2) \begin{pmatrix} a & -b \\ b & a \end{pmatrix} Z.$$

Further recall that

$$\mathrm{D}f(x,y) = \begin{pmatrix} \alpha - ay^2 - 3ax^2 - 2byx & -\beta - 2axy - bx^2 - 3by^2 \\ \beta - 2axy + by^2 + 3bx^2 & \alpha - ax^2 - 3ay^2 + 2byx \end{pmatrix},$$

and denote the Jacobian of the random dynamical system φ by $\Phi := \mathrm{D}\varphi$ which is the solution of the variational equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,\omega,x) = \mathrm{D}f(\varphi(t,\omega,x))\Phi(t,\omega,x), \quad \Phi(0,\omega,x) = \mathrm{Id}\,, \quad \text{for all } \ 0 \le t < \tilde{T}(\omega,x)\,. \tag{6.6.6}$$

First of all, we want to use this model as an exemplification of Conjecture 6.2.9. For a fixed time $t > 0$, consider $\Phi(t,\omega,x)$ for all (ω,x) such that $\varphi(s,\omega,x) \in E$ for all $0 \le s \le t$. Let

$$0 < \sigma_2(\Phi(t,\omega,x)) \le \sigma_1(\Phi(t,\omega,x))$$

be the singular values of $\Phi(t,\omega,x)$, i.e. the eigenvalues of $\sqrt{\Phi^*(t,\omega,x)\Phi(t,\omega,x)}$. We investigate if there are $\sigma^i \in \mathbb{R}$ such that for all $x \in E$ we have

$$\frac{1}{t}\mathbb{E}[\ln \sigma_i(\Phi(t,\cdot,x))|\tilde{T}(\cdot,x) > t] \xrightarrow{t\to\infty} \sigma^i \ \text{ for all } 1 \le i \le 2\,.$$

If $\sigma_1 \neq \sigma_2$, we can define a Lyapunov spectrum of average expansion rates of the surviving assemble with

$$\lambda_1 := \sigma^1 > \sigma^2 =: \lambda_2\,.$$

We approximate the singular values of $\Phi(t,\omega,x)$ numerically using the integration normalisation procedure as described in [83], based on the principle of exterior powers as explained in Section 6.2. Starting from $x \in E$, we pick M noise realisations and orthonormal vectors $v_1^0, v_2^0 \in \mathbb{R}^2$. Choosing a time step size $\mathrm{d}t$, we integrate equation 6.6.6 within a short time interval of length τ to obtain the matrix $\tilde{\Phi}_\omega$ for each noise realisation. Thereby we take trajectories of φ from an integration of equation (6.6.5) until absorption at the boundary. When a trajectory hits the boundary, we re-sample by choosing the new position uniformly from the distribution of all M trajectories at the previous time step. At time τ, we determine the vectors

$$\begin{aligned} w_1^1(\omega) &= \tilde{\Phi}_\omega v_1^0, & v_1^1(\omega) &= \frac{w_1^1(\omega)}{\|w_1^1(\omega)\|}\,,\\ w_2^1(\omega) &= \tilde{\Phi}_\omega v_2^0 - \langle \tilde{\Phi}_\omega v_2^0, v_1^1(\omega)\rangle v_1^1(\omega)\,, & v_2^1(\omega) &= \frac{w_2^1(\omega)}{\|w_2^1(\omega)\|}\,. \end{aligned}$$

We iterate this procedure K times to obtain $w_1^k(\omega)$ and $w_2^k(\omega)$ for all M ω-trajectories and $1 \le k \le K$. For large k we have for the exterior product Λ^2

$$\|\Lambda^2\Phi(k\tau,\omega,x)\| \approx \prod_{i=1}^{k} \|w_1^i(\omega)\|\|w_2^i(\omega)\|\,,$$

and hence

$$\|\sigma_1(\Phi(k\tau,\omega,x)\| \approx \prod_{i=1}^{k} \|w_1^i(\omega)\|\,,\ \ \|\sigma_2(\Phi(k\tau,\omega,x)\| \approx \prod_{i=1}^{k} \|w_2^i(\omega)\|\,.$$

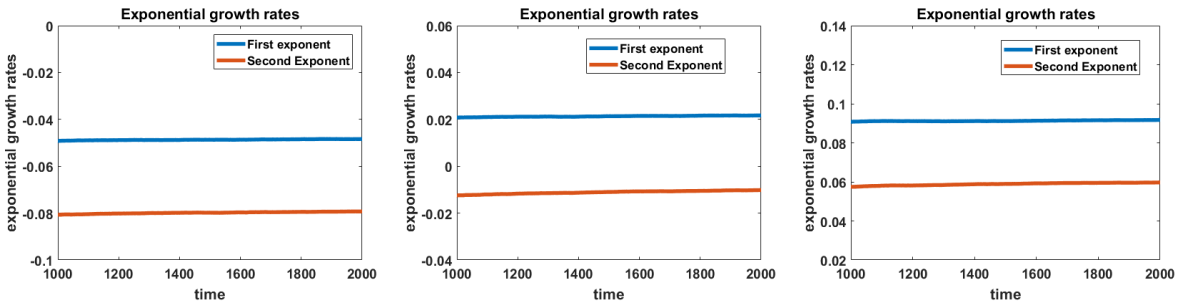
Therefore we can calculate

$$\lambda_1^k(\omega) = \frac{1}{k\tau}\sum_{i=1}^{k} \ln\|w_1^i(\omega)\|\,,\ \ \lambda_2^k(\omega) = \frac{1}{k\tau}\sum_{i=1}^{k} \ln\|w_2^i(\omega)\|\,,$$

and approximate λ_1 and λ_2 by

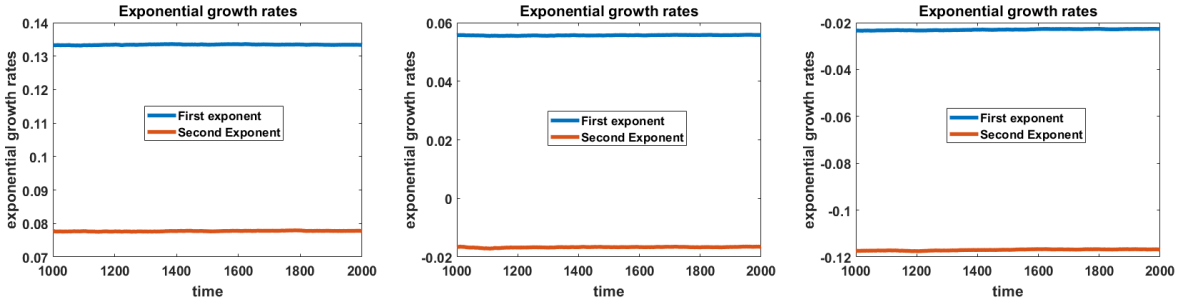
$$\lambda_1^k = \frac{1}{M}\sum_{j=1}^{M} \lambda_1^k(\omega_j)\,,\ \ \lambda_2^k = \frac{1}{M}\sum_{j=1}^{M} \lambda_2^k(\omega_j)\,. \tag{6.6.7}$$

We apply this procedure for different values of the bifurcation parameter α and radius of domain R, setting $b = 0$, $\beta = 1$ and $\sigma = 1$. We choose $\tau = 1$ and calculate λ_1^k and λ_2^k for $k = 1000, \ldots, 2000$ according to the scheme described above. In Figures 6.4 and 6.5 we fix one of the two parameters α and R and approximate the two exponents for different values of the other parameter. Firstly, note that the pictures confirm the convergence of the scheme and suggest the existence of λ_1 and λ_2 as conjectured in Conjecture 6.2.9. Secondly, observe a two-step transition of the two Lyapunov exponents from $\lambda_2 < \lambda_1 < 0$ to $0 < \lambda_2 < \lambda_1$ with $\lambda_2 < 0 < \lambda_1$ in between. Again, this indicates a bifurcation of the dynamical behaviour which does not appear in the global situation on $\mathbb{R}^2$. Recall from Theorem 3.1.3 that $\lambda_1 < 0$ for all values of α if $b = 0$.



(a) $\alpha = 0.65$ *(b)* $\alpha = 0.725$ *(c)* $\alpha = 0.8$

Figure 6.4: We set $R = 1$, $b = 0$ *and* $\sigma = \beta = 1$ *and apply the integration normalisation scheme explained above to calculate* λ_1^k *and* λ_2^k *as defined in* (6.6.7) *for* $k = 1000, \ldots, 2000$ *and different values of* α. *In Figure 6.4a, we set* $\alpha = 0.65$ *and observe* $\lambda_2^k < \lambda_1^k < 0$. *For* $\alpha = 0.725$, *Figure 6.4b displays* $\lambda_2^k < 0 < \lambda_1^k$, *and in Figure 6.4c, we observe that* $0 < \lambda_2^k < \lambda_1^k$.



(a) $R = 1.1$ *(b)* $R = 1.15$ *(c)* $R = 1.2$

Figure 6.5: We set $\alpha = 1$, $b = 0$ *and* $\sigma = \beta = 1$ *and apply the integration normalisation scheme explained above to calculate* λ_1^k *and* λ_2^k *as defined in* (6.6.7) *for* $k = 1000, \dots, 2000$ *and different values of the radius of the domain* R. *In Figure 6.5a, we set* $R = 1.1$ *and observe* $0 < \lambda_2^k < \lambda_1^k$. *For* $R = 1.15$, *Figure 6.5b displays* $\lambda_2^k < 0 < \lambda_1^k$, *and in Figure 6.5c, we observe that* $\lambda_2^k < \lambda_1^k < 0$.

Now, we include shear in our experiments. For this purpose, we could also approximate λ from Proposition 6.2.3 by a finite difference method as in the previous section. However, this would involve a three-dimensional scheme which is very costly compared to the simulations above. In case λ_1 exists in the sense of Conjecture 6.2.9, we expect to have $\lambda_1 = \lambda$ as in the classical case. So we will confine ourselves to the simulations. We now choose larger R such that the survival rate of trajectories λ_0 is high, i.e. the escape rate $-\lambda_0$ is low, and demonstrate the phenomenon of shear-induced chaos for this setting. We proceed exactly as before, choosing $\alpha = \sigma = \beta = 1$ and $R = 5$. We calculate λ_1^k and λ_2^k for $k = 2000$ and different values of shear. In Figure 6.6 we observe a transition of negative to positive first exponent close to $b = 7$. We also observe that λ_2^k increases with increasing b which is not the case in the situation without killing at the boundary of a bounded domain. In more detail recall from Proposition 3.3.1 that

$$\lambda_1 + \lambda_2 = 2\alpha - 4a \int_{\mathbb{R}^2} (x^2 + y^2) p(x, y) \, \mathrm{d}x\mathrm{d}y < 0 \,,$$

where the stationary density $p(x, y)$ does not depend on b. Hence, in this case λ_2 decreases when b, and therefore λ_1, increases. In the situation with absorption at the boundary, the value of b has an effect on the quasi-ergodic density $m(\mathrm{d}x, \mathrm{d}y)$ which is derived from the eigenvalue problem for λ_0 with Dirichlet boundary conditions, as explained in Section 6.2. It would be obviously interesting to investigate this phenomenon in more detail but this is beyond the scope of this section. In Figure 6.6 we still have $\lambda_1^k + \lambda_2^k < 0$, i.e. contraction of two-dimensional volume, but this will change with increasing b similarly to Figures 6.4 and 6.5. However, the situation of volume contraction but one-dimensional expansion is the one we have been interested in in Chapter 3 and, in particular, in Chapter 4. Hence, the picture shows that the phenomenon can be reproduced in the scenario with killing at the boundary. An important consequence is that we do not need large deviations of noise, that explores the whole state space, in order to observe shear-induced chaos.

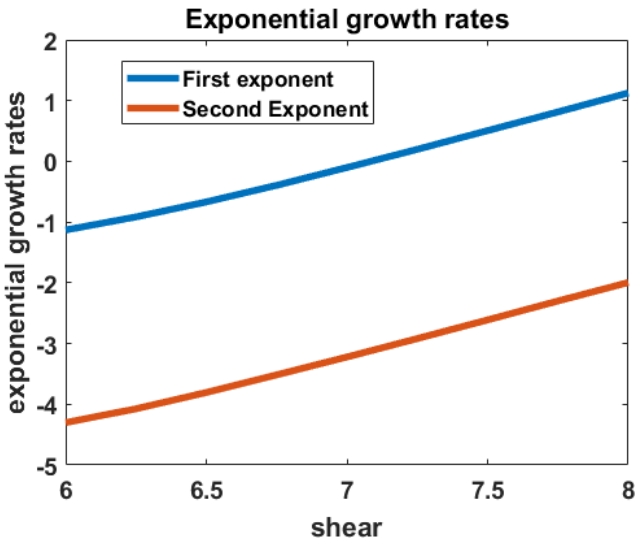


Figure 6.6: We choose $\alpha = \sigma = \beta = 1, R = 5$ and apply the integration normalisation scheme explained above to calculate λ_1^k and λ_2^k as defined in (6.6.7) for $k = 2000$ and shear $b \in [6, 8]$. We observe a transition of negative to positive λ_1^k close to $b = 7$, indicating a bifurcation from synchronising to chaotic behaviour. In addition, we note an increase of λ_2^k with increasing b which implies a decrease in volume contraction.

As mentioned before, we will have $\lambda_1 = \lambda$, the conditioned average exponent, in case λ_1 exists. Hence, we can apply Corollary 6.3.2 for the regions in parameter space that imply $\lambda_1 < 0$, given that we can trust the numerical simulations. For the shear intensities that generate $\lambda_1 > 0$, we can deduce chaotic behaviour for the surviving trajectories.

6.7 Discussion

We have integrated the problem of stochastic bifurcations into the context of processes that are absorbed at the boundary. We have thereby been able to use the well-explored concepts of quasi-stationary and quasi-ergodic distributions. The knowledge about quasi-ergodic distributions in particular has enabled us to find a formula for the conditioned average Lyapunov exponent λ for stochastic differential equations and to prove convergence of finite-time Lyapunov exponents to λ in probability, conditioned on survival. We have been able to show that negative λ implies local synchronisation, also in the situation with absorption at the boundary. It remains an open problem to show the existence of a spectrum of $0 < p \leq d$ conditioned Lyapunov exponents in d dimensions.

Furthermore, we have shown the correspondence between quasi-stationary measures ν and the conditionally invariant probability measures $\mathbb{P} \times \nu$ for the one-sided skew product flow on an open system. We have also conjectured a similar correspondence for quasi-ergodic measures. It remains an open problem to investigate such a correspondence for two-sided time, the intricacies of which we have explained.

In addition, we have defined the notion of dichotomy spectrum for the situation with

killing at the boundary. We have characterised the dichotomy spectrum as a union of disjoint closed intervals and shown its relation to finite-time Lyapunov exponents.

For the example of stochastic pitchfork bifurcation, we have determined the dichotomy spectrum which reflects the deterministic bifurcation by a loss of hyperbolicity at $\alpha = 0$. Furthermore we have approximated the quasi-ergodic distribution for different parameter regimes of the stochastic pitchfork problem, using a finite difference scheme. This allowed us to demonstrate that λ changes its sign from negative to positive at some $\alpha^* > 0$ on a bounded interval of radius $c > \alpha^*$, expressing the bifurcation phenomenon in terms of the Lyapunov exponent as opposed to the problem on $\mathbb{R}$. Similarly, we have shown bifurcations in the context of stochastic Hopf bifurcation with killing at the boundary, with and without shear. We have used a numerical scheme to compute λ_1 and λ_2, for whose existence we do not have an analytical proof yet.

As already indicated, the work presented in this chapter leaves open a couple of interesting questions. Additional problems could also comprehend the definition of other ergodic quantities, such as metric entropy, or investigating the relation to bounded noise, in particular the Hopf bifurcation with bounded noise, as for example discussed in [20].

Bibliography

[1] A. Arnaudon, A.L. De Castro, and D.D. Holm. Noise and dissipation on coadjoint orbits. *arXiv1601.02249v4 [math.DS]*, 2016.

[2] L. Arnold. *Random dynamical systems*. Springer, Berlin, 1998.

[3] L. Arnold, G. Bleckert, and K. R. Schenk-Hoppé. The stochastic Brusselator: parametric noise destroys Hopf bifurcation. In *Stochastic dynamics (Bremen, 1997)*, pages 71–92. Springer, New York, 1999.

[4] L. Arnold, E. Oeljeklaus, and E. Pardoux. Almost sure and moment stability for linear Itô equations. In *Workshop Bremen/Germany*, volume 1186 of *Lecture Notes in Mathematics*, pages 129–159. Springer, Berlin, 1986.

[5] L. Arnold, N. Sri Namachchivaya, and K. R. Schenk-Hoppé. Toward an understanding of stochastic Hopf bifurcation: a case study. *International Journal of Bifurcation and Chaos*, 6(11):1947–1975, 1996.

[6] Ludwig Arnold and Michael Scheutzow. Perfect cocycles through stochastic differential equations. *Probability Theory and Related Fields*, 101(1):65–88, 1995.

[7] P.H. Baxendale. Wiener processes on manifolds. *Poceedings of the Royal Society Edinburgh*, 87(A):127–152, 1980.

[8] P.H. Baxendale. Asymptotic behaviour of stochastic flows of diffeomorphisms: two case studies. *Probability Theory and Related Fields*, 73:51–85, 1986.

[9] P.H. Baxendale. Lyapunov exponents and relative entropy for a stochastic flow of diffeomorphisms. *Probability Theory and Related Fields*, 81:521–554, 1989.

[10] P.H. Baxendale. Statistical equilibrium and two-point motion for a stochastic flow of diffeomorphisms. In *Spatial stochastic processes*, volume 19 of *Progress in Probability*, pages 189–218. Birkhäuser, Boston, 1991.

[11] P.H. Baxendale. A stochastic Hopf bifurcation. *Probability Theory and Related Fields*, 99:581–616, 1994.

[12] P.H. Baxendale. Stochastic averaging and asymptotic behavior of the stochastic Duffing-van der Pol equation. *Stochastic Processes and their Applications*, 113:235–272, 2004.

[13] P.H. Baxendale and L. Goukasian. Lyapunov exponents for small perturbations of Hamiltonian systems. *Annals of Probability*, 30:101–134, 2002.

[14] P.H. Baxendale and D.W. Stroock. Large deviations and stochastic flows of diffeomorphisms. *Probability Theory and Related Fields*, 80:169–215, 1988.

[15] M. Biskamp. Pesin's formula for random dynamical systems on $\mathbf{R}^d$. *Journal of Dynamics and Differential Equations*, 26(1):109–142, 2014.

[16] J.-M. Bismut. A generalized formula of Itô and some other properties of stochastic flows. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 55(3):331–350, 1981.

[17] N. Blackbeard, P.S. Dutta, H. Erzgraber, and S. Wieczorek. From synchronisation to optical turbulence in laser arrays. *Physica D*, 286–287:43–58, 2014.

[18] N. Blackbeard, H. Erzgraber, and S. Wieczorek. Shear-induced bifurcations and chaos in couple-laser models. *SIAM Journal on Applied Dynamical Systems*, 10(2):469–509, 2011.

[19] A. Blumenthal, Xue.J., and L.S. Young. Lyapunov exponents for random perturbations of some area-preserving maps including the standard map. *Annals of Mathematics*, 185:1–26, 2017.

[20] R.T. Botts, A.J. Homburg, and T.R. Young. The Hopf bifurcation with bounded noise. *Discrete and Continuous Dynamical Systems. Series A*, 32(8):2997–3007, 2012.

[21] M. Callaway, T.S. Doan, J.S.W. Lamb, and M. Rasmussen. The dichotomy spectrum for random dynamical systems and pitchfork bifurcations with additive noise. *arXiv1310.6166v1[math.DS]*, 2016. to appear in: Annales de l'Institut Henri Poincaré, Probabilités et Statistiques.

[22] A. Caverhill. A nonrandom Lyapunov spectrum for nonlinear stochastic dynamical systems. *Stochastics*, 17:253–287, 1986.

[23] A. Caverhill and K.D. Elworthy. Lyapunov exponents for a stochastic analogue of the geodesic flow. *Transactions of the American Mathematical Society*, 295:85–105, 1986.

[24] N. Champagnat, K.A. Coulibaly-Pasquier, and D. Villemonais. Exponential convergence to quasi-stationary distribution for multi-dimensional diffusion processes. *arXiv:1603.07909[math.PR]*, 2016.

[25] N. Champagnat and D. Villemonais. Exponential convergence to quasi-stationary distribution and Q-process. *Probability Theory and Related Fields*, 164(1-2):243–283, 2016.

[26] M.D. Chekroun, E. Simonnet, and M. Ghil. Stochastic climate dynamics: random attractors and time-dependent invariant measures. *Physica D*, 240:1685–1700, 2011.

[27] E.A. Coddington and N. Levinson. *Theory of ordinary differential equations*. McGraw-Hill Book Company, New York, 1955.

[28] P. Collet, S. Martinez, and J.S. Martin. *Quasi-stationary distributions*. Probability and its Applications. Springer, Berlin, 2013.

[29] P. Collet, S. Martinez, and V. Maume-Deschamps. On the existence of conditionally invariant probability measures in dynamical systems. *Nonlinearity*, 13:1263–1274, 2000.

[30] H. Crauel. Markov measures for random dynamical systems. *Stochastics and Stochastics Reports*, 37(3):153–173, 1991.

[31] H. Crauel and F. Flandoli. Attractors for random dynamical systems. *Probability Theory and Related Fields*, 100:365–393, 1994.

[32] H. Crauel and F. Flandoli. Additive noise destroys a pitchfork bifurcation. *Journal of Dynamics and Differential Equations*, 10(2):259–274, 1996.

[33] H. Crauel and P.E. Kloeden. Nonautonomous and random attractors. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 117(3):173–206, 2015.

[34] M. Demers and L.-S. Young. Escape rates and conditionally invariant measures. *Nonlinearity*, 19:377–397, 2006.

[35] L. Deville, N. Sri Namachchivaya, and Z. Rapti. Stability of a stochastic two-dimensional non-Hamiltonian system. *SIAM Journal on Applied Mathematics*, 71(4):1458–1475, 2011.

[36] H.A. Dijkstra, L.M. Frankcombe, and H.S. Von der Heydt. A stochastic dynamical systems view of the atlantic multidecadal oscillation. *Philos Trans A Math Phys Eng Sci.*, 366:2545–2560, 2008.

[37] G. Dimitroff and M. Scheutzow. Attractors and expansion for Brownian flows. *Electonical Journal of Probability*, 16(42):1193–1213, 2011.

[38] T.S. Doan, M. Engel, J.S.W. Lamb, and M. Rasmussen. Hopf bifurcation with additive noise. *arXiv:1710.09649v1 [math.DS]*, 2017.

[39] T.S. Doan, M. Rasmussen, and P.E. Kloeden. The mean-square dichotomy spectrum and a bifurcation to a mean-square attractor. *Discrete and Continuous Dynamical Systems. Series B*, 20(3):875–887, 2015.

[40] P. Duarte. Plenty of elliptic islands for the standard family of area preserving maps. *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, 11(4):359–409, 1994.

[41] K.D. Elworthy. Stochastic dynamical systems and their flows. In *Stochastic analysis (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1978)*, pages 79–95. Academic Press, New York, 1978.

[42] K.D. Elworthy. Geometric aspects of diffusions on manifolds. In *École d'Été de Probabilités de Saint-Flour XV–XVII, 1985–87*, volume 1362 of *Lecture Notes in Mathematics*, pages 277–425. Springer, New York, 1988.

[43] K.D. Elworthy. Stochastic flows on Riemannian manifolds. In *Diffusion processes and related problems in analysis, Vol. II (Charlotte, NC, 1990)*, volume 27 of *Progress in Probability*, pages 37–72. Birkhäuser, Boston, 1992.

[44] K.D. Elworthy. Stochastic differential geometry. *Bulletin des Sciences Mathématiques*, 117:7–28, 1993.

[45] M. Engel, J.S.W. Lamb, and M. Rasmussen. Bifurcation analysis of a stochastically driven limit cycle. *arXiv1606.01137v1[math.DS]*, 2016.

[46] L. C. Evans. *An introduction to stochastic differential equations*. American Mathematical Society, Providence, RI, 2013.

[47] P. A. Ferrari, H. Kesten, S. Martnez, and P. Picco. Existence of quasi-stationary distributions. A renewal dynamical approach. *Annals of Probability*, 23(2):501–521, 1995.

[48] F. Flandoli, B. Gess, and M. Scheutzow. Synchronization by noise. *Probability Theory and Related Fields*, 168(3–4):511–556, 2017.

[49] F. Flandoli and B. Schmalfuss. Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative white noise. *Stochastics and Stochastics Reports*, 59(1-2):21–45, 1996.

[50] G. He, H. Zhang, and Y. Zhu. On the quasi-ergodic distribution of absorbing markov processes. *arXiv1611.01002v1[math.PR]*, 2016.

[51] A.J. Homburg and H. Zmarrou. Bifurcations of stationary measures of random diffeomorphisms. *Ergodic Theory and Dynamical Systems*, 27:1651–1692, 2007.

[52] P. Imkeller. The smoothness of laws of random flags and Oseledets spaces of linear stochastic differential equations. *Potential Analysis*, 9(4):321–349, 1998.

[53] P. Imkeller and C. Lederer. An explicit description of the Lyapunov exponents of the noisy damped harmonic oscillator. *Dynamics and Stability of Systems*, 14(4):385–405, 1999.

[54] P. Imkeller and C. Lederer. Some formulas for Lyapunov exponents and rotation numbers in two dimensions and the stability of the harmonic oscillator and the inverted pendulum. *Dynamical Systems*, 16(1):29–61, 2001.

[55] P. Imkeller and C. Lederer. The cohomology of stochastic and random differential equations, and local linearization of stochastic flows. *Stochastics and Dynamics*, 2(2):131–159, 2002.

[56] P. Imkeller and B. Schmalfuss. The conjugacy of stochastic and random differential equations and the existence of global attractors. *Journal of Dynamics and Differential Equations*, 13(2):215–249, 2001.

[57] H. Keller and G. Ochs. Numerical approximation of random attractors. In *Stochastic dynamics (Bremen, 1997)*, pages 93–115. Springer, New York, 1999.

[58] H. Keller and B Schmalfuss. Attractors for stochastic differential equations with nontrivial noise. *Bul. Acad. Ştiinţe Repub. Mold. Mat.*, 26(1):43–54, 1998.

[59] R. Khasminskii. *Stochastic stability of differential equations*, volume 66 of *Stochastic Modelling and Applied Probability*. Springer, Heidelberg, second edition, 2012. With contributions by G. N. Milstein and M. B. Nevelson.

[60] Y. Kifer. *Ergodic theory of random transformations*, volume 10 of *Progress in Probability and Statistics*. Birkhäuser, Boston, 1986.

[61] P.E. Kloeden and M. Rasmussen. *Nonautonomous dynamical systems*, volume 176 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2011.

[62] S. Kuksin and A. Shirikyan. *Mathematics of two-dimensional turbulence*, volume 194 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2012.

[63] H. Kunita. Stochastic differential equations and stochastic flows of diffeomorphisms. In *École d'été de probabilités de Saint-Flour, XII—1982*, volume 1097 of *Lecture Notes in Mathematics*, pages 143–303. Springer, Berlin, 1984.

[64] H. Kunita. *Stochastic flows and stochastic differential equations*, volume 24 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[65] J.S.W. Lamb, M. Rasmussen, and C.S. Rodrigues. Topological bifurcations of minimal invariant sets for set-valued dynamical systems. *Proceedings of the American Mathematical Society*, 143(9):3927–3937, 2015.

[66] Y. Le Jan. Équilibre statistique pour les produits de difféomorphismes aléatoires indépendants. *Annales de l'Institut Henri Poincaré. Probabilités et Statistique*, 23(1):111–120, 1987.

[67] F. Ledrappier and L.-S. Young. The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin's entropy formula. *Annals of Mathematics. Second Series*, 122(3):509–539, 1985.

[68] F. Ledrappier and L.-S. Young. The metric entropy of diffeomorphisms. II. Relations between entropy, exponents and dimension. *Annals of Mathematics. Second Series*, 122(3):540–574, 1985.

[69] F. Ledrappier and L.-S. Young. Dimension formula for random transformations. *Communications in Mathematical Physics*, 117(4):529–548, 1988.

[70] F. Ledrappier and L.-S. Young. Entropy formula for random transformations. *Probability Theory and Related Fields*, 80(2):217–240, 1988.

[71] T. Lelievre, M. Rousset, and G. Stoltz. *Free energy computations. A mathematical perspective*. Imperial College Press, London, 2010.

[72] Z. Lian and M. Stenlund. Positive Lyapunov exponent by a random perturbation. *Dynamical Systems*, 27(2):239–252, 2012.

[73] K. K. Lin and L.-S. Young. Shear-induced chaos. *Nonlinearity*, 21(5):899–922, 2008.

[74] P.D. Liu and M. Qian. *Smooth ergodic theory of random dynamical systems*, volume 1606 of *Lecture Notes in Mathematics*. Springer, Berlin, 1995.

[75] S. Méléard and D.. Villemonais. Quasi-stationary distributions and population processes. *Probability Surveys*, 9:340–410, 2012.

[76] J. Newman. Necessary and sufficient conditions for stable synchronisation in random dyncamical systems. *arXiv1408.5599v3[math.PR]*, 2015.

[77] J. Newman. Synchronisation of almost all trajectories of a random dynamical system. *arXiv1511.08831v2[math.PR]*, 2015.

[78] W. Ott and M. Stenlund. From limit cycles to strange attractors. *Communications in Mathematical Physics*, 296(1):215–249, 2010.

[79] G. A. Pavliotis. *Stochastic processes and applications*, volume 60 of *Texts in Applied Mathematics*. Springer, New York, 2014. Diffusion processes, the Fokker-Planck and Langevin equations.

[80] M. Rasmussen. Finite-time attractivity and bifurcation for nonautonomous differential equations. *Differential Equations and Dynamical Systems*, 18(1-2):57–78, 2010.

[81] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales. Vol. 1.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Foundations, Reprint of the second (1994) edition.

[82] D. Ruelle. Ergodic theory of differentiable dynamical systems. *Institut des Hautes Études Scientifiques. Publications Mathématiques*, 50:27–58, 1979.

[83] M. Sandri. Numerical calculation of Lyapunov exponents. *The Mathematica Journal*, 6(3):78–84, 1996.

[84] K. R. Schenk-Hoppé. Bifurcation scenarios of the noisy Duffing-van der Pol oscillator. *Nonlinear Dynamics*, 11(3):255–274, 1996.

[85] K. R. Schenk-Hoppé. Deterministic and stochastic Duffing-van der Pol oscillators are non-explosive. *Zeitschrift für Angewandte Mathematik und Physik*, 47(5):740–759, 1996.

[86] M. Scheutzow. Stochstic processes II. Lecture Notes, TU Berlin, 2013.

[87] T.T. Soong. *Random differential equations in science and engineering*, volume 103 of *Mathematics in Science and Engineering*. Academic Press, New York and London, 1973.

[88] Q. Wang and L.-S. Young. From invariant curves to strange attractors. *Communications in Mathematical Physics*, 225(2):275–304, 2002.

[89] Q. Wang and L.-S. Young. Strange attractors in periodically-kicked limit cycles and Hopf bifurcations. *Communications in Mathematical Physics*, 240(3):509–529, 2003.

[90] S. Wieczorek. Stochastic bifurcation in noise-driven lasers and Hopf oscillators. *Physical Review E*, 79(3):1–10, 2009.

[91] A.M. Yaglom. Certain limit theorems of the theory of branching processes. *Doklady Akademii Nauk SSSR*, 56:795–798, 1947.

[92] L.-S. Young. Chaotic phenomena in three settings: large, noisy and out of equilibrium. *Nonlinearity*, 21(11):245–252, 2008.

[93] H. Zmarrou and A.J. Homburg. Bifurcations of stationary measures of random diffeomorphisms. *Ergodic Theory and Dynamical Systems*, 27(5):1651–1692, 2007.

Appendix

A.1 Multidimensional conversion formula from Stratonovich to Itô integral

Consider the Stratonovich SDE

$$\mathrm{d}X_t = \overline{F}(t, X_t)\mathrm{d}t + G(t, X_t) \circ \mathrm{d}W_t \,,$$

where $\overline{F} : \mathbb{R}^d \to \mathbb{R}^d$ is called the drift of the SDE, $G : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ the diffusion of the SDE and W_t is an m-dimensional Wiener process. In accordance with [46], the equation has the same solutions as the Itô SDE

$$\mathrm{d}X_t = F(t, X_t)\mathrm{d}t + G(t, X_t)\mathrm{d}W_t \,,$$

where

$$\overline{F}_i(t, X_t) = F_i(t, X_t) - \frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{m} G_{jk}(t, X_t)\frac{\partial G_{ik}}{\partial X_j}(t, X_t) \,, \quad i = 1, \ldots, d \,.$$

A.2 The Fokker–Planck equation

Consider the Itô SDE

$$\mathrm{d}X_t = F(t, X_t)\mathrm{d}t + G(t, X_t)\mathrm{d}W_t \,,$$

where $F : \mathbb{R}^d \to \mathbb{R}^d$ is called the drift of the SDE and $G : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ the diffusion of the SDE and W_t is an m-dimensional Wiener process. The so called Fokker–Planck equation describes the evolution of the density of the process $(X_t)_{t\geq 0}$ under sufficient (classical or Sobolev) regularity of the coefficients:

$$\frac{\partial p(t, x)}{\partial t} = -\sum_{i=1}^{d}\frac{\partial}{\partial x_i}[F_i(t, x)p(x, t)] + \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial x_i \partial x_j}[D_{ij}(x, t)p(x, t)] \,,$$

with diffusion tensor

$$D_{ij}(x,t) = \sum_{k=1}^{m} G_{ik}(x,t) G_{jk}(x,t)\,.$$