Bonds subject to Credit Risk: New Hedging Strategies

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Abstract

In the bond market, two categories of bonds can be distinguished: The high yield bonds and the investment grade bonds. As far as hedging is concerned, the main worry for high yield bondholders is to protect themselves against the default risk of the issuer. This is usually done using credit derivative instruments which have been playing an increasingly important role during the past few years, but which still do not form a very liquid market.

On the other hand, when hedging, investment grade bondholders are more concerned about the movements of the credit spread and default free interest rates than the pure event of default. Usually, the position in a bond subject to credit risk is hedged by shorting some gilts using the traditional duration-based hedge ratio.

The objective of this thesis (which can be separated into three parts) is to construct new theories consisting of hedging investment grade credit rated bonds using new hedge ratios and new hedging instruments.

The first part of this thesis aims to design useful tools: We derive a general formula in order to model a system of correlated processes given a correlation matrix and a volatility vector. The concept of sensitivity measure is introduced. Then, a statistical framework is built in order to accomplish the estimation of volatilities and correlations between different kinds of models (mean reverting, lognormal, etc) given the historical data.

The second part of the thesis conducts an analysis of the components of the credit spread using a probabilistic approach in order to support the idea of using new hedging instruments.

The third part of the thesis is devoted to building new hedge ratios and investigating new hedging instruments while testing them on a few investment grade bonds: First, it is shown that when hedging bonds about to mature, the traditional duration-based hedge ratio may not be the best one. Secondly, we show that by adding some stocks to the menu of hedging instruments, it is likely to give better results than when using only gilts. Moreover, the cost of hedging seems to be proportionally lower. Eventually, we show that the hedging performances are even better if we add some stock index to the menu of hedging instruments.

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Chapter 1

General Introduction

This thesis focuses on hedging investment grade bonds and some tests are conducted in the Eurosterling Corporate Bond Market which is an increasingly important market as far as number of bond issues and liquidity are concerned. Most of the trading in corporate bonds is done over the counter. A bond is said to belong to the *investment* category if its credit rating is equal or above BBB. A gilt is a fixed interest British government bond. In the whole thesis, by "risky bonds", we mean *bonds subject to credit risk.*

When hedging, investment grade bondholders are more concerned about the movements in the credit quality and of the default free interest rates than the pure event of default. Usually, the position is hedged by shorting some gilts using the traditional duration-based hedge ratio.

It is agreed that this technique is far from satisfactory, especially when the bond is a low rated one.

The objective of this thesis is to construct new theories consisting of hedging investment grade credit rated bonds using new hedge ratios and new hedging instruments.

The objective here is not to prove that the new hedge ratios will lead to a perfect hedge but rather the aim is twofold:

- Obtain improved hedging results.
- Evaluate the potentials and the limitations of these new hedging strategies.

Different correlated processes have to be modelled in this thesis: Given a cor-

relation matrix and a volatility vector, the traditional way of doing this is to apply the Cholesky method which requires us to solve a system of equations iteratively. However, we think that it is possible to give a general formula in order to model such a system of n correlated processes (we have done that for at least up to $n = 4$) and this is what the first part of **Chapter 2** aims to achieve. Possible ways of modelling bonds subject to credit risk are investigated, different conventions relative to bond prices and quotations are reviewed and finally the methodology we have chosen in order to compute sensitivity measures is carefully explained.

At some point in the thesis, it is required to make some assumptions regarding the specificity of a few processes (mean-reverting, log-normal...). It raises the question of defining a method in order to estimate the model's parameters from historical data. This is the purpose of **Chapter 3** where a statistical framework is built.

The yield to maturity of a bond subject to credit risk can be separated into a *credit free component* and a credit component called the *credit spread.* The former can easily be hedged using gilts whereas the latter is much more difficult to hedge. Therefore, **Chapter 4** conducts an analysis of the *credit spread.* The rationale for such an analysis is to show that the *credit spread* is likely to be correlated with some specific financial instruments that could be used as hedging instruments.

Chapter 5 deals with the traditional duration-based hedge ratio and its limitations as far as hedging with gilts is concerned. Therefore, in order to improve the hedging performance, two new hedge ratios are derived. In addition, the method that we think is suitable in order to measure hedging performance is explained. It is shown that when hedging bonds about to mature the traditional duration-based hedge ratio may not be the best one.

The question of adding some stocks to the menu of the hedging instruments is addressed in **Chapter 6.** The results obtained are very encouraging since it seems that hedging performances have improved along with a lower "relative cost" of hedging.

Eventually, in **Chapter 7**, it is shown that the hedging performances are even better if we add some stock index to the menu of hedging instruments.

Chapter 2

Process modelling

2.1 Introduction

The purpose of this chapter is to build a framework that is useful for the following chapters: Section[2.2] gives a general result about correlated processes. Section[2.3] aims to investigate some basic concepts relative to interest rates and bonds. Finally, section[2.4] is a summary of the methodology used in order to determine sensitivity measures.

2.2 Correlated Processes

The purpose of this section is to derive a matrix, which when multiplied by a vector of independent brownian motions gives a general system of stochastic correlated processes. Moreover this matrix is defined as a function of a correlation matrix and a volatility vector.

2.2.1 Definitions

In this chapter m, n, i and j are positive integer numbers. The time variable is represented by t . The symbol $E[.]$ stands for the expectation. In addition, the following notation will be used:

a. The symbol $\lfloor$

Two cases have to be considered: First, let U_t be an $n \times n$ dimensional time-dependent square matrix. The component which is at the intersection of the ith

row and the jth column is noted $U_t^{(i,j)}$. For any positive integer number k such that $k \leq n$, $U_{t\lfloor k}$ stands for the the matrix comprising of the first k rows and columns of U_t.

$$U_{t\lfloor k} = \left\{ U_t^{(i,j)} : i,j = 1,...,k \right\}$$

In this thesis, unless otherwise stated, the symbol $|.|$ stands for *determinant* that is to say

$$|U_t| \triangleq \det(U_t)$$

Now, let U_t be an n dimensional time-dependent vector. For any positive integer number k such that $k \leq n$, $U_{t\lfloor k}$ stands for the the vector comprising the first k elements of U_t or in other words:

$$U_{t\lfloor k} = \left\{ U_t^{(i)} : i = 1,...,k \right\}$$

b. The *minor function* $\mathsf{m}_{i,j}$

We define $\mathsf{m}_{i,j}$ as a function whose imput is a square matrix: Given any $n \times n$ dimensional time-dependent square matrix U_t , the *minor function* $\mathsf{m}_{i,j}$ applied to U_t gives a $n-1 \times n-1$ dimensional time-dependent square matrix $\mathsf{m}_{i,j}(U_t)$ which is in fact the matrix U_t from which the ith row and the jth column have been removed.

c. The φ-operator applied to a *minor function*

φ is an operator on the *minor function* $\mathsf{m}_{i,j}$ such that :

$$\varphi \circ \mathsf{m}_{i,j}(U_t) \triangleq \mathsf{m}_{i-1,j}\left(\mathsf{m}_{i,j}(U_t)\right) \triangleq \mathsf{m}_{i-1,j} \circ \mathsf{m}_{i,j}(U_t) \tag{2.2.1}$$

In other words, if the operator φ is applied to the *minor function* $\mathsf{m}_{i,j}$ (which removes the ith row and the jth column of the matrix it is applied to) it gives a new minor function $\mathsf{m}_{i-1,j}$ which removes the $i-1^{th}$ row and the jth column of the matrix obtained when applying $\mathsf{m}_{i,j}$ to the original matrix. If this operator φ is applied k *times* to a *minor function* it will be written $\varphi^{(k)}$ or equivalently:

$$\varphi^{(k)} \triangleq \underbrace{\varphi \circ \varphi \circ .. \circ \varphi.}_{k \text{ times}} \tag{2.2.2}$$

Since the above equation[2.2.2] makes sense when k is strictly positive, some conventions have to be defined for the cases when k is zero or negative:

$$\varphi^{(0)} \circ \mathsf{m}_{i,j}(U_t) \triangleq \mathsf{m}_{i,j}(U_t)$$

In other words $\varphi^{(0)}$ leaves the *minor function* unchanged whereas $\varphi^{(-1)}$ applied to the *minor function* gives the identity function:

$$\varphi^{(-1)} \circ \mathsf{m}_{i,j}(U_t) \triangleq U_t$$

and $\varphi^{(-k)}$ with $k > 1$ applied to the *minor function* gives the zero function:

$$\varphi^{(-k)} \circ \mathsf{m}_{i,j}(U_t) \triangleq 0 \quad \text{for} \quad k > 1$$

In order for the reader to become more familiar with the operator φ , an example of its use is given next.

Example of the use of φ :

Let U_t be a 5×5 matrix. The example chosen here consists in computing $|\varphi \circ \mathsf{m}_{4,2}(U_t)|$. The matrix U_t can be explicitly written as follows:

$$U_t = \begin{bmatrix} U_t^{(1,1)} & U_t^{(1,2)} & U_t^{(1,3)} & U_t^{(1,4)} & U_t^{(1,5)} \\ U_t^{(2,1)} & U_t^{(2,2)} & U_t^{(2,3)} & U_t^{(2,4)} & U_t^{(2,5)} \\ U_t^{(3,1)} & U_t^{(3,2)} & U_t^{(3,3)} & U_t^{(3,4)} & U_t^{(3,5)} \\ U_t^{(4,1)} & U_t^{(4,2)} & U_t^{(4,3)} & U_t^{(4,4)} & U_t^{(4,5)} \\ U_t^{(5,1)} & U_t^{(5,2)} & U_t^{(5,3)} & U_t^{(5,4)} & U_t^{(5,5)} \end{bmatrix}$$

By the definition of φ

$$\varphi \circ \mathsf{m}_{4,2}(U_t) = \mathsf{m}_{3,2} \circ \mathsf{m}_{4,2}(U_t) = \mathsf{m}_{3,2}(\mathsf{m}_{4,2}(U_t)) \tag{2.2.3}$$

By the definition of the *minor function*, when applying $\mathsf{m}_{4,2}$ to U_t it gives a matrix which is the matrix U_t with the 4th row and the 2nd column removed. Therefore:

$$\mathsf{m}_{4,2}\left(U_t\right) = \begin{bmatrix} U_t^{(1,1)} & U_t^{(1,3)} & U_t^{(1,4)} & U_t^{(1,5)} \\ U_t^{(2,1)} & U_t^{(2,3)} & U_t^{(2,4)} & U_t^{(2,5)} \\ U_t^{(3,1)} & U_t^{(3,3)} & U_t^{(3,4)} & U_t^{(3,5)} \\ U_t^{(5,1)} & U_t^{(5,3)} & U_t^{(5,4)} & U_t^{(5,5)} \end{bmatrix}$$

Now applying $\mathsf{m}_{3,2}$ to the matrix $\mathsf{m}_{4,2}\left(U_t\right)$ results in:

$$\mathsf{m}_{3,2}\left(\mathsf{m}_{4,2}\left(U_t\right)\right) = \begin{bmatrix} U_t^{(1,1)} & U_t^{(1,4)} & U_t^{(1,5)} \\ U_t^{(2,1)} & U_t^{(2,4)} & U_t^{(2,5)} \\ U_t^{(5,1)} & U_t^{(5,4)} & U_t^{(5,5)} \end{bmatrix}$$

And eventually refering to equation[2.2.3] ,

$$\left|\varphi \circ \mathsf{m}_{4,2}\left(U_t\right)\right| = \det \begin{bmatrix} U_t^{(1,1)} & U_t^{(1,4)} & U_t^{(1,5)} \\ U_t^{(2,1)} & U_t^{(2,4)} & U_t^{(2,5)} \\ U_t^{(5,1)} & U_t^{(5,4)} & U_t^{(5,5)} \end{bmatrix}$$

d. The δ function

δ is a function such that:

$$\delta(i,j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

This is in fact what is called the kronecker function.

e. The brownian vector $\mathbf{W_t}$

The general definition of an n dimensional brownian vector W_t is given below. Let $(\Omega, I\ , (F_t)_{t\geq 0}, P)$ be a filtered probability space. "P a.s" stands for "almost surely under the P-probability measure".

$$W_t = \begin{bmatrix} W_t^{(1)} \\ \\ W_t^{(2)} \\ \\ \vdots \\ \\ W_t^{(n)} \end{bmatrix}$$

with $W_t^{(1)}, W_t^{(2)}, ..., W_t^{(n)}$ n standard independent brownian motions. In other words using the definition given in [Lamberton & Lapeyre (1995)], $\forall\, k \in \{1, ..., n\}$, $\left(W_t^{(k)}\right)_{t\geq 0}$ is a real-valued, continuous stochastic process with independent and stationary increments. Moreover this process is said to be standard if :

$$W_0^{(k)} = 0 \quad P \text{ a.s} \qquad , \qquad E[W_t^{(k)}] = 0 \qquad \text{and} \qquad E[\left(W_t^{(k)}\right)^2] = t$$

f. The vector $\mathbf{X_t}$ representing the stochastic processes

In the same way, the n dimensional stochastic vector $\mathbf{X_t}$ is given by

$$X_t = \begin{bmatrix} X_t^{(1)} \\ \\ X_t^{(2)} \\ \\ \vdots \\ \\ X_t^{(n)} \end{bmatrix}$$

with $X_t^{(1)}, ..., X_t^{(n)}$ n stochastic processes such that

$$E[dX_t^{(k)}] = 0 \quad , \quad \forall \quad k \in \{1, ..., n\} \qquad \text{and} \qquad d\left\langle X, X^T \right\rangle_t = C_t dt$$

$C_t dt$ can be viewed as the $n \times n$ covariance matrix of the vector X_t . The general form of C_t is:

$$C_t = \begin{bmatrix} \left(\sigma_t^{(1)}\right)^2 & \sigma_t^{(1)}\sigma_t^{(2)}\rho_t^{(1,2)} & \cdots & \sigma_t^{(1)}\sigma_t^{(n)}\rho_t^{(1,n)} \\ \sigma_t^{(1)}\sigma_t^{(2)}\rho_t^{(1,2)} & \left(\sigma_t^{(2)}\right)^2 & \cdots & \sigma_t^{(2)}\sigma_t^{(n)}\rho_t^{(2,n)} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_t^{(1)}\sigma_t^{(n)}\rho_t^{(1,n)} & \sigma_t^{(2)}\sigma_t^{(n)}\rho_t^{(2,n)} & \cdots & \left(\sigma_t^{(n)}\right)^2 \end{bmatrix}$$

From this matrix it is easy to extract a correlation matrix and a volatility vector which are defined next.

g. The volatility vector σ_t

σ_t is the n dimensional volatility vector defined as:

$$\sigma_t = \begin{bmatrix} \sigma_t^{(1)} \\ \sigma_t^{(2)} \\ \vdots \\ \sigma_t^{(n)} \end{bmatrix}$$

with $\sigma_t^{(1)}, \sigma_t^{(2)}, ..., \sigma_t^{(n)}$ n time functions representing volatilities.

h. Definition of the correlation matrix Δ_t

The correlation matrix of the vector X_t is Δ_t and is defined as follows:

$$\Delta_t = \begin{bmatrix} 1 & \rho_t^{(1,2)} & \cdots & \rho_t^{(1,n)} \\ \rho_t^{(1,2)} & 1 & \cdots & \rho_t^{(2,n)} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_t^{(1,n)} & \rho_t^{(2,n)} & & 1 \end{bmatrix}$$

by convention, $\Delta_{t\lfloor 0} = 1$. Moreover, by definition

$$\forall\, i,j \in \{1,...,n\} \qquad -1 \leq \Delta_t^{(i,j)} \leq 1$$

Before giving the main result of this section (cf subsection[2.2.2]), it seems appropriate to recall an important and useful Lemma.

Lemma

Let $C_t \in \Re^{n,n}$ with $n \geq 1$ such that $C_t^T = C_t$. Let $\lambda_i\left(C_t\right)_{i=1,...,n}$ be the eigenvalues of C_t. C_t is positive definite $\Longleftrightarrow \lambda_i\left(C_t\right) > 0\ \forall i \in \{1,...,n\} \Longleftrightarrow \det\left(C_{t\lfloor k}\right) > 0$ $\forall k \in \{1,...,n\}$. A proof of this lemma can be found in [Pecastaings (1986)].

2.2.2 Main Result

Given the covariance matrix C_t (or equivalently given σ_t and Δ_t), it is possible to express the process X_t as a function of W_t under special assumptions . The relation is

$$dX_t = F\left(\Delta_t, \sigma_t\right) dW_t \tag{2.2.4}$$

with $F\left(\Delta_t, \sigma_t\right)$ the following matrix :

$$\left[F\left(\Delta_t, \sigma_t\right)\right]^{(i,j)} = \sigma_t^{(i)} \frac{\left(\prod_{k=0}^{\max(j-2,0)} \sqrt{\det \Delta_{t\lfloor k}}\right)^{\delta(i,j)}}{\sqrt{\prod_{k=0}^{j-\delta(i,j)} \det \Delta_{t\lfloor k}}} \left[\det\left(\varphi^{(i-j-1)} \circ \mathsf{m}_{i,j}(\Delta_{t\lfloor i})\right)\right]^{1-\frac{\delta(i,j)}{2}} \tag{2.2.5}$$

where all the parameters are defined in subsection[2.2.1]. The assumptions are:

$$det\left(\Delta_{t\lfloor k}\right) > 0 \quad \text{for} \quad k = 2, ..n-1 \tag{2.2.6}$$

$$det\left(\Delta_t\right) \geq 0$$

$$n = dim(X_t) \leq 4 \tag{2.2.7}$$

The assumption[2.2.7] comes from the fact that the relation[2.2.4] has been proved only up to $n = 4$. Indeed in the following parts the modelling of a number of processes bigger than 4 is not required.

The Cholesky method could have been used here but it is an iterative method (cf [Watkins (1991)]) and it is not possible to obtain a global formula like equation[2.2.5]. Moreover, since equation[2.2.5] is consistent with the fact that $det\left(\Delta_t\right)$ can be equal to 0 (meaning that the matrix C_t need not be positive definite), it appears that (for $n \leq 4$) Cholesky's theorem condition (C_t being symetric positive definite) is not a necessary condition but a sufficient one.

Proof

$I_{n\times n}$ is the n-dimensional Identity Matrix.

$$d\left\langle X, X^T\right\rangle_t = F\left(\Delta_t, \sigma_t\right) d\left\langle W, W^T\right\rangle_t F\left(\Delta_t, \sigma_t\right)^T$$

$$= F\left(\Delta_t, \sigma_t\right) F\left(\Delta_t, \sigma_t\right)^T dt \quad \text{since}$$

$$d\left\langle W, W^T\right\rangle_t = I_{n\times n} dt$$

Therefore the proof consists in showing that:

$$F\left(\Delta_t, \sigma_t\right) F\left(\Delta_t, \sigma_t\right)^T = C_t \tag{2.2.8}$$

According to the relation[2.2.5], $F\left(\Delta_t, \sigma_t\right)$ is given by:

$$\begin{bmatrix}
\sigma_t^{(1)}\sqrt{\frac{|\Delta_{t\lfloor 1}|}{|\Delta_{t\lfloor 0}|}} & 0 & 0 & 0 \\
\sigma_t^{(2)}\frac{|\mathsf{m}_{2,1}(\Delta_{t\lfloor 2})|}{\sqrt{\prod_{k=0}^{1}|\Delta_{t\lfloor k}|}} & \sigma_t^{(2)}\sqrt{\frac{|\Delta_{t\lfloor 2}|}{|\Delta_{t\lfloor 1}|}} & 0 & 0 \\
\sigma_t^{(3)}\frac{|\mathsf{m}_{2,1}\circ\mathsf{m}_{3,1}(\Delta_{t\lfloor 3})|}{\sqrt{\prod_{k=0}^{1}|\Delta_{t\lfloor k}|}} & \sigma_t^{(3)}\frac{|\mathsf{m}_{3,2}(\Delta_{t\lfloor 3})|}{\sqrt{\prod_{k=0}^{2}|\Delta_{t\lfloor k}|}} & \sigma_t^{(3)}\sqrt{\frac{|\Delta_{t\lfloor 3}|}{|\Delta_{t\lfloor 2}|}} & 0 \\
\sigma_t^{(4)}\frac{|\mathsf{m}_{2,1}\circ\mathsf{m}_{3,1}\circ\mathsf{m}_{4,1}(\Delta_{t\lfloor 4})|}{\sqrt{\prod_{k=0}^{1}|\Delta_{t\lfloor k}|}} & \sigma_t^{(4)}\frac{|\mathsf{m}_{3,2}\circ\mathsf{m}_{4,2}(\Delta_{t\lfloor 4})|}{\sqrt{\prod_{k=0}^{2}|\Delta_{t\lfloor k}|}} & \sigma_t^{(4)}\frac{|\mathsf{m}_{4,3}(\Delta_{t\lfloor 4})|}{\sqrt{\prod_{k=0}^{3}|\Delta_{t\lfloor k}|}} & \sigma_t^{(4)}\sqrt{\frac{|\Delta_{t\lfloor 4}|}{|\Delta_{t\lfloor 3}|}}
\end{bmatrix}$$

Checking the equation[2.2.8] is just a matter of basic calculus.

2.3 Interest rates

The purpose of this section is:

- Explain a possible way to model the shifts in the term structure.
- Highlight some specificities relative to bond prices.
- Review the different conventions used to determine the yield to maturity of a bond. Moreover we can specifically explain the method chosen (in this thesis) in order to compute the yield to maturity .
- Introduce the concept of sensitivity measures.
- Give an overview of the main features of interest rate modelling.

2.3.1 Term structures

Let us consider two maturity times T_0 and T_k. A change in the risk free rate term structure will in general not induce the same change in the risk free interest rate for maturity T_0 as the one in the risk free interest rate for maturity T_k (unless a parallel shift is considered which often occurs). Therefore we can write:

$$di_t(T_k) \neq di_t(T_0) \quad \text{for } k \neq 0$$

or equivalently that

$$\frac{di_t(T_k)}{di_t(T_0)} = \gamma(t, T_k, T_0) \tag{2.3.1}$$

with $\gamma(t, T_k, T_0)$ a non-zero function depending on t , T_k and T_0. Moreover, given any maturity T_j with $j \neq k$, using equation[2.3.1]:

$$\frac{di_t(T_k)}{di_t(T_j)} = \frac{\gamma(t, T_k, T_0)}{\gamma(t, T_j, T_0)} \tag{2.3.2}$$

For a given T_0 (which can be viewed as a reference), let us define $\eta_i(t, T_k)$ such that:

$$\eta_i(t, T) = \gamma(t, T, T_0)$$

and assume that $\eta_i(t, T)$ is a deterministic function of the time t and the maturity T. A shift in the risk free interest rate for a maturity T can be modelled as follows:

$$di_t(T) = \eta_i(t, T)\widetilde{di}_t \tag{2.3.3}$$

with $\widetilde{di}_t$ a stochastic process which does not depend on the maturity T but only on t. By using this modelling (cf equation[2.3.3]) we do not contradict equation[2.3.2]. By analogy to the previous discussion applied now to a shift in the credit spread term structure, a shift in the credit spread for a maturity T can be modelled as follows

$$dr_t(T) = \eta_r(t, T)\widetilde{dr}_t \tag{2.3.4}$$

with $\widetilde{dr}_t$ a stochastic process which does not depend on the maturity T but only on t.

2.3.2 Price, quotation and the multi variable model

The price of a bond can be evaluated by discounting its cash flows which is what the multi variable model (introduced in this subsection) does.

a. Definitions

We consider here a general case for the bond which is a non-zero coupon bond with N remaining cash flows.

t : current date

N : number of remaining payments

s_1 , s_2 , s_3 ,..., s_N : Values of the N remaining payments

T_1 , T_2 , T_3 ,...T_N : dates of the future cash-flows

M : number of trading days in one year

$\tau_i = \dfrac{T_i - t}{M}$: time to go until i^{th}cash flow (in years)

$B[t, (\tau_i)_{i=1,N}, (s_i)_{i=1,N}]$: price of the bond

Usually,

$$s_1 = s_2 = s_3 = ... = s_{N-1} = \text{coupon}$$

and

$$s_N = \text{coupon} + \text{principal repaid at maturity}$$

b. Quotations and price sources

Almost all bonds are quoted in *Clean Price.* However, one does not pay this price in order to acquire the bond but one has to pay what is called the *Dirty Price.* The Dirty Price is actually the quoted price plus the accrued interest. The accrued interest represent the amount of interest accumulated but not yet distributed since the last coupon payment date or since the issue date of the bond.

$$\textit{Dirty Price} = \textit{Clean Price} + \textit{Accrued Interest} \tag{2.3.5}$$

As previously mentioned, the quoted price is the clean price. However valuing a bond is sometimes not a straightforward matter depending on the liquidity of the Bond. As a matter of fact, the lower the Bond Credit Rating, the less liquid the Bond. As this thesis deals with low credit rated investment grade bonds, the problems due to pricing have to be taken into account. Since the governement bond market represents one of the most liquid markets, prices are available through real time electronic quotation systems. This data can be considered as very reliable.

As far as corporate bonds are concerned, the problems arise when bond issues do not trade frequently. The consequences of the limited trading is that it is difficult to accurately determine the current market prices and entire bond issues are often held to maturity by investors. One way of determining a reliable clean price is to subscribe to specific pricing services. As explained in [Sharpe, Alexander, Bailey], a few banks run large pricing services: "They use their bond traders to submit bid-side valuations daily for any securities that its traders regularly see traded in the market. This hand pricing by traders offers perhaps the best estimate that institutional investors can readily obtain as the true market price, or liquidation value, of their bonds."

In this thesis the clean prices are provided by *Bloomberg Generic Service* which is actually a synthesis of the prices provided by the world's largest banks.

c. The multi variable model

The multi variable model enables us to determine the price of a bond given the term structure of interest rates. It actually discounts the cash flows by taking into account the term structure of risk free interest rate and the term structure of the spread. As previously described in subsection[2.3.1], $i_t(T_k)$ represents the risk-free rate at time t over a $T_k - t$ period of time and $r_t(T_k)$ represents the credit spread rate over the same period of time. It is a multi factor model in the sense that in this model several variables govern the price process of the Bond. The discount factor $d_{t,k}$ at time t for payment s_k is :

$$d_{t,k} = \frac{1}{[1 + i_t(T_k) + r_t(T_k)]^{\tau_k}}$$

Hence the price of the bond, discounting the future cash flows, is given by:

$$B[t,(\tau_i)_{i=1,N},(s_i)_{i=1,N}] = \sum_{k=1}^{N} s_k d_{t,k} = \sum_{k=1}^{N} \frac{s_k}{[1+i_t(T_k)+r_t(T_k)]^{\tau_k}} \qquad (2.3.6)$$

In order to simplify notation we will write :

$$B_t = B[t,(\tau_i)_{i=1,N},(s_k)_{k=1,N}]$$

2.3.3 Yield to Maturity and the single variable model

In this thesis, essentially eurobonds are investigated and more precisely eurosterling bonds. In the following section it will be explained which formula the author has chosen to consider in order to extract the yield to maturity from the price of a bond. The yield to maturity quoted on the financial markets is not exactly what could be called the "true yield" or the "effective yield". Indeed, the "market yield" calculation is driven by some conventions. It follows that there are a number of different ways of calculating the redemption yields. For more details the reader can consult the European Bond Market reference book which states the following basic rules: The convention in the gilt market is to quote a redemption yield as twice the effective semi-annual interest rate. This is the reason why appropriate adjustments should be made to this yield before it can be compared with yields on the securities, which are often based on an annual yield. The standard approach to the method is to assume that, after the first coupon-date, the following coupons are paid at equal six-monthly periods. Therefore, for the investor free of tax on both income and capital gains, the effective semi-annual interest rate is found by solving the following equation for v:

$$B + a = v^{\frac{d}{182.5}}[C_1 + \frac{C}{2}(v + v^2 + ... + v^n) + Rv^n] = v^{\frac{d}{182.5}}[C_1 + \frac{C}{2}\frac{v(1-v^n)}{1-v} + Rv^n]$$

where $i = \frac{1}{v} - 1$ is the the effective semi-annual yield, giving the redemption yield of $200 * i$ per cent.

with:

B : the clean price

a : accrued interest at purchase

d : number of days to next coupon-date

C_1 : first coupon payment

C : annual rate of coupon

$n+1$: number of coupons

R : redemption proceeds

Considering Eurosterling fixed-rate bonds , the market uses the ISMA formula in determining the yield. Still according to the European Bond Market reference book, the redemption yield formula may be written as:

$$B + a = v^{f1}[C_1 + \sum_{i=1}^{n-1} \frac{C}{h} v^i] + [R + \frac{C}{h} f_2] v^{n+f_1+f_2-1}$$

with:

h : the number of coupon payments per year

n : the number of coupon payments to assumed redemption, excluding any fractional payment on redemption

f_1 : the fraction of a period from value-date to the first/next interest payment. A period is defined as the normal time between two consecutive coupon payments.

f_2 : the fraction of a period from the last normal coupon-date to the assumed redemption

It should be emphasized that this yield is not directly comparable to other yields used in the sterling market.

Having reviewed the market conventions, the method that we choose to use in order to compute the yield to maturity is explained below.

a. The single variable model

We consider a single variable model in the sense that in this model only one variable (the yield to maturity) governs the price process of the Bond. The yield

to maturity (YTM) of the Bond whose price is $B[t, (\tau_i)_{i=1,N}, (s_k)_{k=1,N}]$ is noted y_t . When dealing with simple interest rates, the discount factor $c_{t,k}$ at time t for payment s_k is:

$$c_{t,k} = \frac{1}{(1 + \tau_k y_t)}$$

However simple interest rates usually apply only to short-term instruments (money market) where only one payment is actually made. In order to value long dated coupon bonds, it makes more sense to assume discretely compounded interest rates since there is the possibility of reinvesting the coupon payments received prior to maturity. As a result the discount factor $c_{t,k}$ at time t for payment s_k is :

$$c_{t,k} = \frac{1}{(1 + y_t)^{\tau_k}}$$

Therefore the price of the bond which is actually the present value of all the expected future cashflows is given below:

$$B[t, (\tau_i)_{i=1,N}, (s_k)_{k=1,N}] = \sum_{k=1}^{N} \frac{s_k}{[1 + y_t]^{\tau_k}} \tag{2.3.7}$$

The limitation of this single variable model is that it is implicitly assumed that coupon payments can be reinvested at the yield to maturity rate over the life of the bond.

2.3.4 Sensitivity Measures

From the modelling of the bond given in equations[2.3.7] and [2.3.6], it can be understood that a bond is a function of some stochastic variables and a deterministic variable (the time t). Indeed, in the *single variable model* the stochastic variable is the yield to maturity and in in the *multi variable model* the $2 \times N$ stochastic variables are $i_t(T_k)$ and $r_t(T_k)$.

An important assumption is made here, namely that *the stochastic variables follow an Ito process.* We recall below the definition of an Ito process and Ito's Lemma (cf [Lamberton & Lapeyre (1995)]):

Ito process

Let $(\Omega, F, (F_t)_{t\geq 0}, P)$ be a filtered probability space and $(W_t)_{t\geq 0}$ an F_t-Brownian motion. $(X_t)_{0\leq t\leq T}$ is a real-valued Ito process if it can be written as

$$P \text{ a.s.} \quad \forall t \leq T \quad X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s$$

where

X_0 is F_0-measurable

$(K_t)_{0\leq t\leq T}$ and $(H_t)_{0\leq t\leq T}$ are F_t-adapted processes

$$\int_0^t |K_s|\, ds < +\infty \qquad P \text{ a.s.}$$

$$\int_0^t |H_s|^2\, dW_s < +\infty \qquad P \text{ a.s.}$$

Ito's Lemma

Let $(X_t^1, ..., X_t^n)$ be n Ito processes

$$X_t^i = X_0^i + \int_0^t K_s^i ds + \sum_{j=1}^{p} \int_0^t H_s^{i,j} dW_s^j$$

then, if f is twice differentiable with respect to x and once differentiable with respect to t, with continuous partial derivatives in (t, x)

$$f(t, X_t^1, ..., X_t^n) = f(0, X_0^1, ..., X_0^n) + \int_0^t \frac{\partial f}{\partial s} ds + \sum_{i=1}^{n} \int_0^t \frac{\partial f}{\partial x_i} dX_s^i + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{\partial^2 f}{\partial x_i x_j} d\langle X^i, X^j \rangle_s$$

The following part of this section deals with the computation of the sensitivity measures for a single and for a multi variable model. A sensitivity measure is a variable which indicates the price change for a small variation in a specific variable (yield, time...).

a. Sensitivity measures for the single variable model

A small variation of the bond price can be written as:

$$dB_t = \Psi_{1,t}dt + D_{y_t}dy_t + C_{y_t}dy_t^2 \tag{2.3.8}$$

with

$$\Psi_{1,t} = \frac{\ln(1+y_t)}{M}\sum_{k=1}^{N}\frac{s_k}{(1+y_t)^{\tau_k}} \tag{2.3.9}$$

$$D_{y_t} = -\sum_{k=1}^{N}\frac{s_k\tau_k}{[1+y_t]^{\tau_k+1}} \tag{2.3.10}$$

$$C_{y_t} = \frac{1}{2}\sum_{k=1}^{N}\frac{s_k\tau_k(\tau_k+1)}{[1+y_t]^{\tau_k+2}} \tag{2.3.11}$$

Proof

From Ito's Lemma,

$$dB_t = \sum_{k=1}^{N}\left(\frac{\partial B_t}{\partial \tau_k}\right)d\tau_k + \left(\frac{\partial B_t}{\partial y_t}\right)dy_t + \frac{1}{2}\left(\frac{\partial^2 B_t}{\partial y_t^2}\right)dy_t^2$$

As, $d\tau_k = -\dfrac{dt}{M}$, therefore

$$dB_t = \left[-\frac{1}{M}\sum_{k=1}^{N}\left(\frac{\partial B_t}{\partial \tau_k}\right)\right]dt + \left[\frac{\partial B_t}{\partial y_t}\right]dy_t + \left[\frac{1}{2}\frac{\partial^2 B_t}{\partial y_t^2}\right]dy_t^2$$

$$\Psi_{1,t} = -\frac{1}{M}\sum_{k=1}^{N}\frac{\partial B_t}{\partial \tau_k} = -\frac{1}{M}\sum_{k=1}^{N}\frac{\partial\left[\sum_{p=1}^{N}\frac{s_p}{[1+y_t]^{\tau_p}}\right]}{\partial \tau_k} = \frac{\ln(1+y_t)}{M}\sum_{k=1}^{N}\frac{s_k}{(1+y_t)^{\tau_k}}$$

Similarly, it is possible to prove that

$$D_{y_t} = \frac{\partial B_t}{\partial y_t} = -\sum_{k=1}^{N}\frac{s_k\tau_k}{[1+y_t]^{\tau_k+1}}$$

and that

$$C_{y_t} = \frac{1}{2}\frac{\partial^2 B_t}{\partial y_t^2} = \frac{1}{2}\frac{\partial D_{y,t}}{\partial y_t} = -\frac{1}{2}\frac{\partial\left(\sum_{k=1}^{N}\frac{s_k\tau_k}{[1+y_t]^{\tau_k+1}}\right)}{\partial y_t} = \frac{1}{2}\sum_{k=1}^{N}\frac{s_k\tau_k(\tau_k+1)}{[1+y_t]^{\tau_k+2}}$$

b. Sensitivity measures for the multi variable model

A small variation of the bond price can be written as:

$$dB_t = \Psi_{2,t}dt + D_{i_t}d\widetilde{i}_t + D_{r_t}d\widetilde{r}_t + C_{i_t}d\widetilde{i}_t^2 + C_{r_t}d\widetilde{r}_t^2 + C_{i_t,r_t}d\widetilde{i}_t d\widetilde{r}_t \qquad (2.3.12)$$

with

$$\Psi_{2,t} = \frac{1}{M}\ln\left(1+i_t\left(T_k\right)+r_t\left(T_k\right)\right)\sum_{k=1}^{N}\frac{s_k}{\left[1+i_t\left(T_k\right)+r_t\left(T_k\right)\right]^{\tau_k}}$$

$$D_{i_t} = -\sum_{k=1}^{N}\frac{\eta_i(t,T_k)s_k\tau_k}{\left[1+i_t\left(T_k\right)+r_t\left(T_k\right)\right]^{\tau_k+1}} \qquad D_{r_t} = -\sum_{k=1}^{N}\frac{\eta_r(t,T_k)s_k\tau_k}{\left[1+i_t\left(T_k\right)+r_t\left(T_k\right)\right]^{\tau_k+1}}$$

$$C_{i_t} = \frac{1}{2}\sum_{k=1}^{N}\frac{\eta_i^2(t,T_p)s_p\tau_p(\tau_p+1)}{\left[1+i_t\left(T_p\right)+r_t\left(T_p\right)\right]^{\tau_p+2}} \qquad C_{r_t} = \frac{1}{2}\sum_{k=1}^{N}\frac{\eta_r^2(t,T_p)s_p\tau_p(\tau_p+1)}{\left[1+i_t\left(T_p\right)+r_t\left(T_p\right)\right]^{\tau_p+2}}$$

$$C_{i_t,r_t} = \sum_{k=1}^{N}\frac{\eta_i(t,T_p)\eta_r(t,T_p)s_p\tau_p(\tau_p+1)}{\left[1+i_t\left(T_p\right)+r_t\left(T_p\right)\right]^{\tau_p+2}}$$

Proof

$$dB_t = \sum_{k=1}^{N}\left(\frac{\partial B}{\partial\tau_k}\right)d\tau_k + \sum_{k=1}^{N}\left(\frac{\partial B_t}{\partial i_t\left(T_k\right)}\right)di_t\left(T_k\right) + \sum_{k=1}^{N}\left(\frac{\partial B_t}{\partial r_t\left(T_k\right)}\right)dr_t\left(T_k\right)$$

$$+\frac{1}{2}\sum_{p=1}^{N}\sum_{q=1}^{N}\left(\frac{\partial^2 B_t}{\partial i_t\left(T_p\right)\partial i_t\left(T_q\right)}\right)di_t\left(T_p\right)di_t\left(T_q\right)+\frac{1}{2}\sum_{p=1}^{N}\sum_{q=1}^{N}\left(\frac{\partial^2 B_t}{\partial r_t\left(T_p\right)\partial r_t\left(T_q\right)}\right)dr_t\left(T_p\right)dr_t$$

$$+\sum_{p=1}^{N}\sum_{q=1}^{N}\left(\frac{\partial^2 B_t}{\partial i_t\left(T_p\right)\partial r_t\left(T_q\right)}\right)di_t\left(T_p\right)dr_t\left(T_q\right)$$

From equation[2.3.3],

$$di_t(T)=\eta_i(t,T)d\widetilde{i}_t$$

$$dr_t(T)=\eta_r(t,T)d\widetilde{r}_t$$

$$d\tau_k=-\frac{dt}{M}$$

Hence:

$$dB_t=-\frac{1}{M}\sum_{k=1}^{N}\left(\frac{\partial B_t}{\partial\tau_k}\right)dt+\sum_{k=1}^{N}\left(\frac{\partial B_t}{\partial i_t\left(T_k\right)}\right)\eta_i(t,T_k)d\widetilde{i}_t+\sum_{k=1}^{N}\left(\frac{\partial B_t}{\partial r_t\left(T_k\right)}\right)\eta_r(t,T_k)d\widetilde{r}_t$$

$$+\frac{1}{2}\sum_{p=1}^{N}\sum_{q=1}^{N}\left(\frac{\partial^2 B_t}{\partial i_t\left(T_p\right)\partial i_t\left(T_q\right)}\right)\eta_i(t,T)\eta_i(t,T)d\widetilde{i}_t^2$$

$$+\frac{1}{2}\sum_{p=1}^{N}\sum_{q=1}^{N}\left(\frac{\partial^2 B_t}{\partial r_t\left(T_p\right)\partial r_t\left(T_q\right)}\right)\eta_r(t,T)\eta_r(t,T)d\widetilde{r}_t^2$$

$$+\sum_{p=1}^{N}\sum_{q=1}^{N}\left(\frac{\partial^2 B_t}{\partial i_t\left(T_p\right)\partial r_t\left(T_q\right)}\right)\eta_i(t,T)\eta_r(t,T)d\widetilde{i}_t d\widetilde{r}_t$$

Using this representation, $\Psi_{2,t}$ which is equal to $-\frac{1}{M}\sum_{k=1}^{N}\frac{\partial B_t}{\partial\tau_k}$, can be written, after some further algebra, as:

$$\Psi_{2,t}=-\frac{1}{M}\sum_{k=1}^{N}\frac{\partial\sum_{k=1}^{N}\frac{s_p}{\left[1+i_t\left(T_p\right)+r_t\left(T_p\right)\right]^{\tau_p}}}{\partial\tau_k}$$

$$=\frac{1}{M}\ln\left(1+i_t\left(T_k\right)+r_t\left(T_k\right)\right)\sum_{k=1}^{N}\frac{s_k}{\left[1+i_t\left(T_k\right)+r_t\left(T_k\right)\right]^{\tau_k}}$$

In the same way, using the Ito's representation of dB_t , $D_{i,t}$ which is by definition equal to

$$D_{i_t} = \sum_{k=1}^{N} \frac{\partial B_t}{\partial i_t\,(T_k)} \eta_i(t, T_k)$$

can be rewritten as:

$$\sum_{k=1}^{N} \eta_i(t, T_k) \frac{\partial \sum_{p=n_{1+1}}^{N} \frac{s_p}{[1 + i_t\,(T_p) + r_t\,(T_p)]^{\tau_p}}}{\partial i_t\,(T_k)} = -\sum_{k=1}^{N} \frac{\eta_i(t, T_k) s_k \tau_k}{[1 + i_t\,(T_k) + r_t\,(T_k)]^{\tau_k + 1}}$$

Similarly,

$$D_{r_t} = \sum_{k=1}^{N} \frac{\partial B_t}{\partial r_t\,(T_k)} \eta_r(t, T_k) = -\sum_{k=1}^{N} \frac{\eta_r(t, T_k) s_k \tau_k}{[1 + i_t\,(T_k) + r_t\,(T_k)]^{\tau_k + 1}}$$

$$C_{i_t} = \frac{1}{2} \sum_{p=1}^{N} \sum_{q=1}^{N} \frac{\partial^2 B_t}{\partial i_t\,(T_p)\,\partial i_t\,(T_q)} \eta_i(t, T_p) \eta_i(t, T_q) = \frac{1}{2} \sum_{p=1}^{N} \eta_i(t, T_p) \frac{\partial}{\partial i_t\,(T_p)} \left(\sum_{q=1}^{N} \eta_i(t, T_q) \frac{\partial B_t}{\partial i_t\,(T_q)} \right)$$

$$= \frac{1}{2} \sum_{p=1}^{N} \eta_i(t, T_p) \frac{\partial D_{i,t}}{\partial i_t\,(T_p)} = \frac{1}{2} \sum_{k=1}^{N} \eta_i(t, T_p) \frac{\partial D_{i,t}}{\partial i_t\,(T_p)}$$

$$= -\frac{1}{2} \sum_{k=1}^{N} \eta_i(t, T_p) \frac{\partial \frac{\eta_i(t, T_p) s_p \tau_p}{[1 + i_t\,(T_p) + r_t\,(T_p)]^{\tau_p + 1}}}{\partial i_t\,(T_p)} = \frac{1}{2} \sum_{k=1}^{N} \frac{\eta_i^2(t, T_p) s_p \tau_p (\tau_p + 1)}{[1 + i_t\,(T_p) + r_t\,(T_p)]^{\tau_p + 2}}$$

By analogy, it is easy to compute C_{r_t} and C_{i_t, r_t}

$$C_{r_t} = \frac{1}{2} \sum_{p=1}^{N} \sum_{q=1}^{N} \frac{\partial^2 B_t}{\partial r_t\,(T_p)\,\partial r_t\,(T_q)} \eta_r(t, T_p) \eta_r(t, T_q) = \frac{1}{2} \sum_{k=1}^{N} \frac{\eta_r^2(t, T_p) s_p \tau_p (\tau_p + 1)}{[1 + i_t\,(T_p) + r_t\,(T_p)]^{\tau_p + 2}}$$

$$C_{i_t,r_t} = \sum_{p=1}^{N}\sum_{q=1}^{N} \frac{\partial^2 B_t}{\partial i_t\,(T_p)\,\partial r_t\,(T_q)} \eta_i(t,T_p)\eta_r(t,T_q) = \sum_{k=1}^{N} \frac{\eta_i(t,T_p)\eta_r(t,T_p)s_p\tau_p(\tau_p+1)}{[1+i_t\,(T_p)+r_t\,(T_p)]^{\tau_p+2}}$$

The sensitivity measures play a key role in this thesis. Moreover they can be used in order to build a framework to model the shifts in the term structures that can be found in Appendix A.

It has been explained (cf equation[2.3.6]) that the price of a bond can be expressed as the discounted value of the remaining cashflows. It is important to note that, alternatively, the price of a bond can be seen as the resut of a conditional expectation of some stochastic processes. This alternative approach is discussed in the next sub-section[2.3.5].

2.3.5 Interest rate models

Let us review here very briefly some of the main features of interest rate modelling used in order to model the default free term structure of interest rates or to price interest rate derivatives under an equivalent martingale measure. As far as credit risk is concerned, an overview of the non default-free related models is given in chapter[4]. For definitions of martingale and equivalent martingale measure, the reader is invited to refer to [Karatzas & Schreves (1991)] and [Lamberton & Lapeyre (1995)]. In this subsection, $E_Q[.\mid F_t]$ corresponds to the conditional expectation under an equivalent martingale measure Q with F_t a relevant filtration to which stochastic processes are adapted. Definition and properties of conditional expectation can be found in [Williams (1991)]. The approach consisting of pricing and hedging (arbitrage free and complete market) under an equivalent martingale probability measure has been formalised by [Harrison & Pliska (1981)] and [Harrison & Pliska (1983)]. The price $B_f(t,T)$ at time t of a default-free zero-coupon bond paying 1 at maturity T can be expressed as

$$B_f(t,T) = E_Q[e^{-\int_t^T r_s ds} \mid F_t] \tag{2.3.13}$$

with r_s representing the instantaneous risk-free interest rate process. Basically, there are two main approaches illustrated by the *whole yield curve models* and the *short term interest rate models.*

The **whole yield curve models** are based on the idea of modelling the dynamics of interest rates for all maturities,. This is usually achieved by describing the

behavior of the instantaneous spot forward rate process $f_t(T)$ as follows:

$$f_t(T) = f_0(T) + \int_0^t \mu_s^{(f)}(T)ds + \int_0^t \sigma_s^{(f)}(T)dW_s \tag{2.3.14}$$

The instantaneous spot interest rate process r_t is then derived using the fact that $r_t = f_t(t)$. Thus,

$$r_t = f_0(t) + \int_0^t \mu_s^{(f)}(t)ds + \int_0^t \sigma_s^{(f)}(t)dW_s$$

These models (see [Heath, Jarrow and Morton (1992)]) are referred to HJM models. In order to be arbitrage free, some constraints need to be imposed on the drift $\mu_s^{(f)}(T)$. In addition to the advantage of modelling the whole yield curve, they are consistent with the initial term structure and consistent with the absence of arbitrage. The main drawback of modelling directly the forward rates is that the short rate is usually not markovian which makes the conditional expectation in equation[2.3.13] extremely difficult to be calculted even with numerical methods. It is important to find some sufficient conditions under which the short rate is markovian ([Caverhill (1994)]) and to impose theoretical restrictions on the modelling of forward rates ([Bhar & Chiarella (1997)] , [Ritchken & Sankarasubramanian (1995)]).

The **short term interest rate models** are based on the idea of modelling directly the instantaneous spot interest rate process r_t such that the conditional expectation in equation[2.3.13] is represented as solutions of some stochastic differential equations and can be computed relatively easily. This is the reason why r_t is often chosen to be a Markov process. A generalisation of the short rate models is discussed in [Schmidt (1997)]. The short rate is often chosen to be a function of an Ornstein -Uhlenbeck process (*beta models*). The traditional modelling of the short rate is given by equation[2.3.15]:

$$r_t = r_0 + \int_0^t \mu_s^{(r)} ds + \int_0^t \sigma_s^{(r)} dW_s \tag{2.3.15}$$

Following [Hull (1993)], we present here the main features of the classical models: In [Vasicek (1977)], $\mu_s^{(r)} = a(b - r)$ and $\sigma_s^{(r)} = \sigma$. This is a mean reverting process that is analytically tractable in the sense that it allows to obtain a closed form solution for equation[2.3.13]. One of its drawbacks is that the short rate r_t

can become negative which is impossible in the mean reverting model given by [Cox, Ingersoll & Ross (1985)] ($\sigma_s^{(r)} = \sigma\sqrt{r_s}$). The two previous models are unlikely to be consistent with the initial term structure of interest rate. This is not the case for the two analytically tractable models proposed by [Ho & Lee (1986)] ($\mu_s^{(r)} = \theta_t$, $\sigma_s^{(r)} = \sigma$) and [Hull & White (1990)] ($\mu_s^{(r)} = (\theta_s - ar_s)$, $\sigma_s^{(r)} = \sigma$). Eventually, in order to keep the short rate positive, [Black, Derman & Toy (1990)] and [Black & Karasinski (1991)] suggest to model the logarithm of the short rate ($\mu_s^{(r)} = (\theta_t - a\ln r)$, $\sigma_s^{(r)} = \sigma$,). Although consistent with the initial term structure, these two models are not analytically tractable. For a discussion regarding the calibration of these models, the reader is invited to refer to [Hull & White (1993)].

2.4 Summary of the methodology

The methodology chosen in order to compute the yield to maturity and sensitivity measures given daily market data is as follows.

1 Determine the clean price .

2 Compute the accrued interest which is straightforward since it depends only of the coupon value, the coupon frequency and the remaining time to next coupon date.

3 Calculate the dirty price which is easily done using formula[2.3.5].

4 Solve equation[2.3.7] which gives the yield to maturity.

The sensitivity measures are then given by replacing the yield to maturity in equations[2.3.9], [2.3.10] and [2.3.11].

Having explained how to compute the yield to maturity of bonds and their sensitivity measures, some model assumptions have to be made regarding the yield to maturity processes as well as the *credit spread* processes. The need of such models is explained in chapter[5]. It raises the question of defining a method in order to assess the model's parameters from historical data, which is the purpose of the next chapter.

Chapter 3

Statistical framework

3.1 Introduction

The aim of this chapter is to create a possible framework in order to estimate some of the model's parameters from historical data. Indeed, in this thesis and especially in chapters[5,6,7] some assumptions regarding process modelling have to be made. Let x_t be the value of the stochastic process x at date t. The size of the sample that will be used to estimate some parameters (eg: variance, expectation...) is represented by the integer N.

$\widetilde{E}[.]$ stands for the Unbiased Statistical Estimator of the Expectation $E[.]$

$\widetilde{Var}[.]$ stands for the Unbiased Statistical Estimator of the Variance $Var[.]$

$\widetilde{Cov}[.,.]$ stands for the Unbiased Statistical Estimator of the Correlation $Cov[.,.]$

The processes' parameters are estimated from historical data. The bigger the size of the historical sample, the more reliable the statistical estimation. But the parameters are not constant over time. This is the reason why a trade-off has to be made. The discussion regarding the size of the historical sample is done in chapter[5] and the following ones. Moreover, it is possible to introduce filtering. The underlying idea about filtering is to put more weights on recent events than on distant events. For instance this is what is achieved when using the exponential decay:

$$\sigma^2 = \frac{1}{\sum_{t=1}^{n} \lambda^t} \sum_{t=1}^{n} \lambda^t x_t^2 \qquad \text{with} \qquad 0 < \lambda \leq 1$$

For more precise information about this, the reader is invited to refer to [Taleb (1996)]. Before starting to build the framework the following definitions are given.

3.2 Definitions

3.2.1 N-dimensional historical vector

Given a process $(x_t)_{t\geq 0}$, the N-dimensional vector which contains at time t the historical information for the previous N days is written as $\overline{x}_t$.

$$\overline{x}_t = \begin{bmatrix} x_t \\ x_{t-1} \\ x_{t-2} \\ \cdots \\ \cdots \\ x_{t-N+1} \end{bmatrix}$$

In order for the statistical estimation to make sense, the following key assumption is made: Given any current time t, it is assumed that the mean and the variance of the process are constant over the previous N days. That is to say

$$E[x_t] = E[x_{t-1}] = ... = E[x_{t-N+1}] \tag{3.2.1}$$

and

$$Var[x_t] = Var[x_{t-1}] = ... = Var[x_{t-N+1}] \tag{3.2.2}$$

3.2.2 Some specific functions

The 1-norm and the scalar product are defined below:

$$|\overline{x}_t|_1 = \sum_{k=0}^{N-1} x_{t-k} \qquad \text{and} \qquad \langle \overline{x}_t, \overline{y}_t \rangle = \sum_{k=0}^{N-1} x_{t-k} y_{t-k}$$

Let us define two functions G_N and H_N such that:

$$G_N(x_t, y_t) = \frac{1}{N(N-1)} \left[N \langle \overline{x}_t, \overline{y}_t \rangle - |\overline{x}_t|_1 |\overline{y}_t|_1 \right] \tag{3.2.3}$$

$$H_N(x_t) = \frac{|\overline{x}_t|_1}{N} \tag{3.2.4}$$

Let M be the number of trading days in a year. Let n be the number of trading days between each components of the N-dimensional vector. Then dt can be expressed as follows:

$$dt = \frac{n}{M} \tag{3.2.5}$$

Usually n is chosen to be 1.

Result

Given two stochastic processes x_t , y_t and their historical vectors $\overline{x}_t$ and $\overline{y}_t$, it follows that the estimated mean of x_t is given by $H_N(x_t)$

$$\widetilde{E}[x_t] = H_N(x_t) \tag{3.2.6}$$

The estimated variance of x_t is given by $G_N(x_t, x_t)$

$$\widetilde{Var}[x_t] = G_N(x_t, x_t) \tag{3.2.7}$$

The estimated covariance between x_t and y_t is given by $G_N(x_t, y_t)$

$$\widetilde{Cov}[x_t, y_t] = G_N(x_t, y_t) \tag{3.2.8}$$

The proof can be found in Appendix[B.1].

3.3 Estimated Drifts and Volatilities

3.3.1 Mean-reverting process

One of the models that will be tested in this thesis is:

$$dY_t = a_Y[b_Y - Y_t]\,dt + \sum_{k=1}^{q} \sigma_k^{(Y)}\, dW_t^{(k)} \tag{3.3.1}$$

More precisely this process is called the Vasicek's model (cf sub-section[2.3.5]). The volatility σ_Y is by definition

$$\sigma_Y = \sqrt{\sum_{k=1}^{q} \left(\sigma_k^{(Y)}\right)^2} \tag{3.3.2}$$

Define two new variables y_t and u_t as follows:

$$y_t = Y_t - e^{-a_Y dt} Y_{t-dt} \tag{3.3.3}$$

$$u_t = \frac{Y_t - Y_{t-dt}}{dt} \tag{3.3.4}$$

a. Estimation of a_Y and b_Y

Result

The estimated $\widetilde{a_Y}$ and $\widetilde{b_Y}$ are given by

$$\widetilde{a_Y} = -\frac{G_N(u_t, Y_t)}{G_N(Y_t, Y_t)} \quad \text{and} \quad \widetilde{b_Y} = H_N(Y_t) - \frac{G_N(Y_t, Y_t)}{G_N(u_t, Y_t)} H_N(u_t) \tag{3.3.5}$$

with u_t , H_N and G_N given by equations[3.3.4], [3.2.4] and [3.2.3] respectively.

Proof

Equation[3.3.1] can be rewritten as :

$$\frac{dY_t}{dt} = \beta + \alpha Y_t + \varepsilon_t$$

or equivalently

$$u_t = \beta + \alpha Y_t + \varepsilon_t \tag{3.3.6}$$

where

$$\alpha = -a_Y \quad , \quad \beta = a_Y b_Y \quad \text{and} \quad \varepsilon_t \text{ a random process, } N(0, \sigma_Y^2)$$

Considering that over an historical period, α and β are constant, it is then possible to estimate the two optima $\widetilde{\alpha}$ and $\widetilde{\beta}$ which minimize the expression:

$$E[(u_t - \alpha - \beta Y_t)^2]$$

then the estimated $\widetilde{a_Y}$ and $\widetilde{b_Y}$ are given by :

$$\widetilde{a_Y} = -\widetilde{\alpha} \qquad \text{and} \qquad \widetilde{b_Y} = -\frac{\widetilde{\beta}}{\widetilde{\alpha}}$$

During the previous N trading days, using equation[3.3.6], the following system can be written

$$\begin{cases} \varepsilon_t = u_t - \beta - \alpha Y_t \\ \varepsilon_{t-1} = u_{t-1} - \beta - \alpha Y_{t-1} \\ \vdots \\ \varepsilon_{t-N+1} = u_{t-N+1} - \beta - \alpha Y_{t-N+1} \end{cases} \tag{3.3.7}$$

In term of vectors, system[3.3.7] is

$$\overline{\varepsilon}_t = \overline{u}_t - \beta\overline{1} - \alpha\overline{Y}_t$$

or, equivalently

$$\overline{\varepsilon}_t = \overline{u}_t - \begin{bmatrix} \overline{Y}_t & \overline{1} \end{bmatrix} \begin{bmatrix} \alpha \\ \\ \beta \end{bmatrix}$$

The aim is to find the optimal vector $\begin{bmatrix} \widetilde{\alpha} \\ \\ \widetilde{\beta} \end{bmatrix}$ which minimises $\overline{\varepsilon}_t^T \overline{\varepsilon}_t$. This is a classical "least square problem" and the solution is

$$\begin{bmatrix} \widetilde{\alpha} \\ \\ \widetilde{\beta} \end{bmatrix} = \left(\begin{bmatrix} \overline{Y}_t^T \\ \\ \overline{1}^T \end{bmatrix} \begin{bmatrix} \overline{Y}_t & \overline{1} \end{bmatrix} \right)^{-1} \begin{bmatrix} \overline{Y}_t^T \\ \\ \overline{1}^T \end{bmatrix} \overline{u}_t$$

or equivalently

$$\begin{bmatrix} \widetilde{\alpha} \\ \\ \widetilde{\beta} \end{bmatrix} = \begin{pmatrix} \langle \overline{Y}_t, \overline{Y}_t \rangle & |\overline{Y}_t|_1 \\ \\ |\overline{Y}_t|_1 & N \end{pmatrix}^{-1} \begin{bmatrix} \overline{Y}_t^T \\ \\ \overline{1}^T \end{bmatrix} \overline{u}_t$$

After some algebra, we obtain

$$\begin{bmatrix} \widetilde{\alpha} \\ \\ \widetilde{\beta} \end{bmatrix} = \frac{1}{N\left\langle \overline{Y}_t, \overline{Y}_t \right\rangle - \left(\left|\overline{Y}_t\right|_1\right)^2} \begin{pmatrix} N\left\langle \overline{Y}_t, \overline{u}_t \right\rangle - \left|\overline{Y}_t\right|_1 \left|\overline{u}_t\right|_1 \\ \\ -\left|\overline{Y}_t\right|_1 \left\langle \overline{Y}_t, \overline{u}_t \right\rangle + \left\langle \overline{Y}_t, \overline{Y}_t \right\rangle \left|\overline{u}_t\right|_1 \end{pmatrix}$$

which is equivalent (using equation[6.4.3]) to

$$\begin{bmatrix} \widetilde{\alpha} \\ \\ \widetilde{\beta} \end{bmatrix} = \frac{1}{N(N-1)G_N\left(Y_t, Y_t\right)} \begin{pmatrix} N(N-1)G_N\left(u_t, Y_t\right) \\ \\ N(N-1)\left[H_N(u_t)G_N\left(Y_t, Y_t\right) - H_N(Y_t)G_N\left(u_t, Y_t\right)\right] \end{pmatrix}$$

and finally

$$\begin{bmatrix} \widetilde{\alpha} \\ \\ \widetilde{\beta} \end{bmatrix} = \begin{pmatrix} \dfrac{G_N\left(u_t, Y_t\right)}{G_N\left(Y_t, Y_t\right)} \\ \\ H_N(u_t) - H_N(Y_t)\dfrac{G_N\left(u_t, Y_t\right)}{G_N\left(Y_t, Y_t\right)} \end{pmatrix}$$

b. Estimation of σ_Y

Result

The estimated volatility of the mean reverting model defined in equation(3.3.1) is

$$\widetilde{\sigma_Y} = \sqrt{\frac{2\widetilde{a_Y}}{1 - e^{-2\frac{n}{M}\widetilde{a_Y}}} G_N(\widetilde{y_t}, \widetilde{y_t})} \tag{3.3.8}$$

with the functions G_N , y_t and $\widetilde{a_Y}$ defined in equations[3.2.3], [3.3.3] and [3.3.5] respectively.

Proof

It is possible to obtain a closed form expression for the process Y_t which is :

$$Y_t = Y_0 e^{-a_Y t} + b_Y(1 - e^{-a_Y t}) \; + \sum_{k=1}^{q} \sigma_k^{(Y)} e^{-a_Y t} \int_0^t e^{a_Y s} dW_s^{(k)} \tag{3.3.9}$$

The proof of this well-known result can be found in [Lamberton & Lapeyre (1995)].

Therefore, the variable y_t can be rewritten as (using equation[3.3.3]):

$$y_t = b_Y(1 - e^{-a_Y dt}) + \sum_{i=1}^{q} \sigma_i^{(Y)} e^{-a_Y t} \int_{t-dt}^{t} e^{a_Y s} dW_s^{(i)}$$

Since $E[y_t] = 0$, the variance of y_t is then:

$$Var\,[y_t] = E\left[\left(\sum_{i=1}^{q} \sigma_i^{(Y)} e^{-a_Y t} \int_{t-dt}^{t} e^{a_Y s} dW_s^{(i)}\right)\left(\sum_{j=1}^{q} \sigma_j^{(Y)} e^{-a_Y t} \int_{t-dt}^{t} e^{a_Y s} dW_s^{(j)}\right)\right]$$

$$= \sum_{i=1}^{q}\sum_{j=1}^{q} \sigma_i^{(Y)} \sigma_j^{(Y)} e^{-2a_Y t} E\left(\int_{t-dt}^{t} e^{a_Y s} dW_s^{(i)} \int_{t-dt}^{t} e^{a_Y s} dW_s^{(j)}\right)$$

Following a result from [Karatzas & Schreves (1991)] , this can be written as:

$$E\left(\int_{t-dt}^{t} e^{a_Y s} dW_s^{(i)} \int_{t-dt}^{t} e^{a_Y s} dW_s^{(j)}\right) = E\left(\int_{t-dt}^{t} e^{2a_Y s} d\left\langle W^{(i)}, W^{(j)}\right\rangle_s\right)$$

Therefore,

$$Var\,[y_t] = \sum_{i=1}^{q}\sum_{j=1}^{q} \sigma_i^{(Y)} \sigma_j^{(Y)} e^{-2a_Y t} E\left(\int_{t-dt}^{t} e^{2a_Y s} d\left\langle W^{(i)}, W^{(j)}\right\rangle_s\right)$$

Moreover, using the fact that

$$d\left\langle W^{(i)}, W^{(j)}\right\rangle_s = \begin{cases} ds & \text{for} \quad i = j \\ \\ 0 & \text{for} \quad i \neq j \end{cases}$$

and that

$$\sum_{k=1}^{q} \left(\sigma_k^{(Y)}\right)^2 = \sigma_Y^2$$

it follows that

$$Var\left[y_t\right] = E\left(\int_{t-dt}^{t} e^{2a_Y s} ds\right) e^{-2a_Y t}\sigma_Y^2 = \frac{(1 - e^{-2a_Y dt})}{2a_Y}\sigma_Y^2$$

and hence,

$$\widetilde{\sigma_Y}^2 = \frac{2\widetilde{a_Y}}{1 - e^{-2\widetilde{a_Y} dt}}\widetilde{Var}\left[\widetilde{y_t}\right]$$

3.3.2 Log-normal process

The log-normal model is defined as:

$$dX_t = \mu_X X_t dt + \sum_{k=1}^{q} \sigma_k^{(X)} X_t dW_t^{(k)}$$

The volatility σ_X of the process X_t is given by:

$$\sigma_X = \sqrt{\sum_{k=1}^{q} \left(\sigma_k^{(X)}\right)^2} \tag{3.3.10}$$

Let us define

$$x_t = \ln\left(\frac{X_t}{X_{t-dt}}\right) \tag{3.3.11}$$

a. Result

The estimated volatility $\widetilde{\sigma_X}$ from historical data is

$$\widetilde{\sigma_X} = \sqrt{\frac{M}{n} G_N(x_t, x_t)} \tag{3.3.12}$$

and the estimated drift of the process is

$$\widetilde{\mu_X} = \frac{M}{n}\left(H_N(x_t) + \frac{G_N(x_t, x_t)}{2}\right)$$

with the functions H_N and G_N defined in equations[3.2.4] and equation[3.2.3] respectively.

b. Proof

From Ito's lemma,

$$X_t = X_0 \exp\left[\left(\mu_X - \frac{\sigma_X^2}{2}\right)t + \sum_{k=1}^{q} \sigma_k^{(X)} W_t^{(k)}\right]$$

Therefore, we can write

$$x_t = \left(\mu_X - \frac{\sigma_X^2}{2}\right)dt + \sum_{k=1}^{q} \sigma_k^{(X)} \int_{t-dt}^{t} dW_s^{(k)} \tag{3.3.13}$$

From equations[3.3.13] and [3.2.5], it follows that:

$$\mu_X = \frac{\sigma_X^2}{2} + \frac{M}{n} E[x_t] \quad \text{and} \quad \widetilde{\mu_X} = \frac{\widetilde{\sigma_X}^2}{2} + \frac{M}{n} H(x_t) \tag{3.3.14}$$

Moreover,

$$Var[x_t] = E\left[\left(\sum_{i=1}^{q} \sigma_i^{(X)} \int_{t-dt}^{t} dW_s^{(i)}\right)\left(\sum_{j=1}^{q} \sigma_j^{(X)} \int_{t-dt}^{t} dW_s^{(j)}\right)\right]$$

$$= \sum_{i=1}^{q}\sum_{j=1}^{q} \sigma_i^{(X)}\sigma_j^{(X)} E\left(\int_{t-dt}^{t} dW_s^{(i)} \int_{t-dt}^{t} dW_s^{(j)}\right) = \sum_{i=1}^{q}\sum_{j=1}^{q} \sigma_i^{(X)}\sigma_j^{(X)} E\left(\int_{t-dt}^{t} d\left\langle W^{(i)}, W^{(j)}\right\rangle_s\right)$$

Using the result given by [Karatzas & Schreves (1991)] , $Var[x_t]$ is then equal to

$$\sum_{k=1}^{q} \left(\sigma_k^{(X)}\right)^2 \int_{t-dt}^{t} ds$$

Finally, it turns out that:

$$\sigma_X^2 = \frac{M}{n} Var[x_t]$$

and since $G_N(x_t, x_t) = \widetilde{Var}[x_t]$ (cf equation[3.2.7])

$$\widetilde{\sigma_X}^2 = \frac{M}{n} G_N(x_t, x_t)$$

Substituting into equation[3.3.14], we get

$$\widetilde{\mu_X} = \frac{M}{n}\left(H_N(x_t) + \frac{G_N(x_t, x_t)}{2}\right)$$

3.3.3 Normal process

The normal model is defined as :

$$dZ_t = \mu_Z dt + \sum_{k=1}^{q} \sigma_k^{(Z)} dW_t^{(k)}$$

The volatility σ_Z of the process Z_t is given by:

$$\sigma_Z = \sqrt{\sum_{k=1}^{q} \left(\sigma_k^{(Z)}\right)^2} \tag{3.3.15}$$

Let us define

$$z_t = Z_t - Z_{t-dt} \tag{3.3.16}$$

a. Result

The estimated volatility $\widetilde{\sigma_Z}$ from historical data is

$$\widetilde{\sigma_Z} = \sqrt{\frac{M}{n} G_N(z_t, z_t)} \tag{3.3.17}$$

and the estimated drift $\widetilde{\mu_Z}$ is

$$\widetilde{\mu_Z} = \frac{M}{n} H_N(z_t)$$

with the functions H_N and G_N defined in equations[3.2.4] and [3.2.3] respectively.

b. Proof

It is straightforward to show that :

$$Z_t = Z_{t-dt} + \mu_Z dt + \sum_{k=1}^{q} \sigma_k^{(Z)} dW_t^{(k)}$$

Therefore, using equation[3.3.15], z_t can be written as

$$z_t = \mu_Z dt + \sum_{k=1}^{q} \sigma_k^{(Z)} \int_{t-dt}^{t} dW_s^{(k)} \tag{3.3.18}$$

From equation[3.3.18] , it follows that:

$$\mu_Z = \frac{M}{n} E[z_t] \qquad \text{and} \qquad \widetilde{\mu_Z} = \frac{M}{n} H_N(z_t)$$

Moreover,

$$Var[z_t] = E\left[\left(\sum_{i=1}^{q} \sigma_i^{(Z)} \int_{t-dt}^{t} dW_s^{(i)}\right)\left(\sum_{j=1}^{q} \sigma_j^{(Z)} \int_{t-dt}^{t} dW_s^{(j)}\right)\right]$$

which gives (by analogy to the previous sub-section):

$$Var[z_t] = \sum_{k=1}^{q} \left(\sigma_k^{(Z)}\right)^2 \int_{t-dt}^{t} ds = \sum_{k=1}^{q} \left(\sigma_k^{(Z)}\right)^2 dt = \sigma_Z^2 dt$$

This leads to:

$$\sigma_z^2 = \frac{M}{n} Var[z_t] \qquad \text{and finally} \qquad \widetilde{\sigma}_z^2 = \frac{M}{n} G_N(z_t, z_t)$$

3.4 Estimated Correlations

3.4.1 Correlation between two mean-reverting processes

The purpose of this subsection is to determine the estimated (from historical data) correlation between two mean-reverting processes Y_t and Y_t^* . The first process Y_t is

$$dY_t = a_Y [b_Y - Y_t] dt + \sum_{i=1}^{q_1} \sigma_i^{(Y)} dW_t^{(i)} \tag{3.4.1}$$

The second process Y_t^* is

$$dY_t^* = a_{Y^*}\left[b_{Y^*} - Y_t^*\right]dt + \sum_{j=1}^{q_2}\sigma_j^{(Y^*)}dW_t^{(j)}$$

As in the previous section the volatilities can be expressed as:

$$\sigma_Y = \sqrt{\sum_{i=1}^{q_1}\left(\sigma_i^{(Y)}\right)^2} \quad \text{and} \quad \sigma_{Y^*} = \sqrt{\sum_{j=1}^{q_2}\left(\sigma_j^{(Y^*)}\right)^2}$$

The correlation ρ_{Y,Y^*} between Y_t and Y_t^* is such that

$$\rho_{Y,Y^*} = \frac{\sum_{k=1}^{\min(q_1,q_2)}\sigma_k^{(Y)}\sigma_k^{(Y^*)}}{\sigma_Y\sigma_{Y^*}}$$

By analogy to the previous section, we define:

$$y_t = Y_t - e^{-a_Y dt}Y_{t-dt} \quad \text{and} \quad y_t^* = Y_t^* - e^{-a_{Y^*}dt}Y_{t-dt}^* \tag{3.4.2}$$

a. Result

The estimated correlation between two mean reverting processes is:

$$\widetilde{\rho_{Y,Y^*}} = \frac{1}{2}\sqrt{\frac{\left(1-e^{-2\frac{n}{M}\widetilde{a_Y}}\right)\left(1-e^{-2\frac{n}{M}\widetilde{a_{Y^*}}}\right)}{\widetilde{a_Y}\widetilde{a_{Y^*}}}}\left[\frac{\widetilde{a_Y}+\widetilde{a_{Y^*}}}{\left(1-e^{-(\widetilde{a_Y}+\widetilde{a_{Y^*}})\frac{n}{M}}\right)}\right]\frac{G_N(\widetilde{y_t},\widetilde{y_t^*})}{\sqrt{G_N(\widetilde{y_t},\widetilde{y_t})G_N(\widetilde{y_t^*},\widetilde{y_t^*})}}$$

with the function G_N , y_t , y_t^* , $\widetilde{a_Y}$ and $\widetilde{a_{Y^*}}$ defined in equations[3.2.3] , [3.4.2] and [3.3.5].

b. Proof

It is possible to rewrite y_t and y_t^* as :

$$y_t = b_Y\left(1-e^{-a_Y dt}\right) + \sum_{i=1}^{q_1}\sigma_i^{(Y)}e^{-a_Y t}\int_{t-dt}^{t}e^{a_Y s}dW_s^{(i)}$$

and

$$y_t^* = b_{Y^*}\left(1 - e^{-a_{Y^*}dt}\right) + \sum_{j=1}^{q_2} \sigma_j^{(Y^*)} e^{-a_{Y^*}t} \int_{t-dt}^{t} e^{a_{Y^*}s} dW_s^{(j)}$$

Moreover,

$$cov\left[y_t, y_t^*\right] = E\left[\sum_{i=1}^{q_1} \sigma_i^{(Y)} e^{-a_Y t} \int_{t-dt}^{t} e^{a_Y s} dW_s^{(i)} \sum_{j=1}^{q_2} \sigma_j^{(Y^*)} e^{-a_{Y^*}t} \int_{t-dt}^{t} e^{a_{Y^*}s} dW_s^{(j)}\right]$$

$$= e^{-\left(a_Y + a_{Y^*}\right)t} \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \sigma_i^{(Y)} \sigma_j^{(Y^*)} E\left[\int_{t-dt}^{t} e^{\left(a_Y + a_{Y^*}\right)s} d\left\langle W^{(i)}, W^{(j)}\right\rangle_s\right]$$

$$= e^{-\left(a_Y + a_{Y^*}\right)t} \sum_{k=1}^{\min(q_1, q_2)} \sigma_k^{(Y)} \sigma_k^{(Y^*)} E\left[\int_{t-dt}^{t} e^{\left(a_Y + a_{Y^*}\right)s} d\left\langle W^{(k)}, W^{(k)}\right\rangle_s\right]$$

$$= \left[\frac{1 - e^{-\left(a_Y + a_{Y^*}\right)dt}}{a_Y + a_{Y^*}}\right] \sum_{k=1}^{\min(q_1, q_2)} \sigma_k^{(Y)} \sigma_k^{(Y^*)} = \left[\frac{1 - e^{-\left(a_Y + a_{Y^*}\right)dt}}{a_Y + a_{Y^*}}\right] \sigma_Y \sigma_{Y^*} \rho_{Y,Y^*}$$

Therefore,

$$\rho_{Y,Y^*} = \frac{a_Y + a_{Y^*}}{\left(1 - e^{-\left(a_Y + a_{Y^*}\right)dt}\right) \sigma_Y \sigma_{Y^*}} cov\left[y_t, y_t^*\right]$$

and

$$\widetilde{\rho_{Y,Y^*}} = \frac{\widetilde{a_Y} + \widetilde{a_{Y^*}}}{\left(1 - e^{-\left(\widetilde{a_Y} + \widetilde{a_{Y^*}}\right)dt}\right) \widetilde{\sigma_Y} \widetilde{\sigma_{Y^*}}} G_N(\widetilde{y_t}, \widetilde{y_t^*})$$

Replacing the estimated volatilities using equation[3.3.8], the result is then derived.

3.4.2 Correlation between two log-normal processes

The purpose is to determine the estimated correlation from historical data between two log-normal processes represented by

$$dX_t = \mu_X X_t dt + \sum_{i=1}^{q_1} \sigma_i^{(X)} X_t dW_t^{(i)} \qquad \text{and} \qquad dX_t^* = \mu_{X^*} X_t^* dt + \sum_{j=1}^{q_2} \sigma_j^{(X^*)} X_t^* dW_t^{(j)}$$

The volatilities of the processes X_t and X_t^* are respectively

$$\sigma_X = \sqrt{\sum_{i=1}^{q_1} \left(\sigma_i^{(X)}\right)^2} \qquad \text{and} \qquad \sigma_{X^*} = \sqrt{\sum_{j=1}^{q_2} \left(\sigma_j^{(X^*)}\right)^2}$$

By definition, the correlation ρ_{X,X^*} between the two processes X_t and X_t^* is

$$\rho_{X,X^*} = \frac{\sum_{k=1}^{\min(q_1,q_2)} \sigma_k^{(X)} \sigma_k^{(X^*)}}{\sigma_X \sigma_{X^*}}$$

As before two variables x_t and x_t^* are defined by

$$x_t = \ln\left(\frac{X_t}{X_{t-dt}}\right) \qquad \text{and} \qquad x_t^* = \ln\left(\frac{X_t^*}{X_{t-dt}^*}\right) \tag{3.4.3}$$

a. Result

The estimated correlation $\widetilde{\rho_{X,X^*}}$ (from historical data) is given by

$$\widetilde{\rho_{X,X^*}} = \frac{G_N\left(x_t, x_t^*\right)}{\sqrt{G_N\left(x_t, x_t\right) G_N\left(x_t^*, x_t^*\right)}}$$

with the function G_N , x_t and x_t^*defined in equation[3.2.3] and [3.4.3].

b. Proof

The processes x_t and x_t^* can be rewritten as

$$x_t = \left(\mu_X - \frac{\sigma_X^2}{2}\right) dt + \sum_{i=1}^{q_1} \sigma_i^{(X)} \int_{t-dt}^{t} dW_s^{(i)}$$

and

$$x_t^* = \left(\mu_{X^*} - \frac{\sigma_{X^*}^2}{2}\right) dt + \sum_{j=1}^{q_2} \sigma_j^{(X^*)} \int_{t-dt}^{t} dW_s^{(j)}$$

Then, using the same methodology as in sub-section[3.4.1], it is not difficult to prove

that

$$cov\left[x_t, x_t^*\right] = \rho_{X,X^*}\sigma_X\sigma_{X^*}dt$$

It follows that

$$\widetilde{\rho_{X,X^*}} = \frac{M}{n}\frac{\widetilde{cov}\left[x_t, x_t^*\right]}{\widetilde{\sigma_X}\widetilde{\sigma_{X^*}}}$$

Eventually, replacing the estimated volatilities using equation[3.3.12] the result is obtained.

3.4.3 Correlation between a mean-reverting and a log-normal process

The log-normal and the mean reverting processes are respectively

$$dX_t = \mu_X X_t dt + \sum_{i=1}^{q_1}\sigma_i^{(X)} X_t dW_t^{(i)} \qquad \text{and} \qquad dY_t = a_Y\left[b_Y - Y_t\right]dt + \sum_{j=1}^{q_2}\sigma_j^{(Y)} dW_t^{(j)}$$

Their volatilities are respectively

$$\sigma_X = \sqrt{\sum_{i=1}^{q_1}\left(\sigma_i^{(X)}\right)^2} \qquad \text{and} \qquad \sigma_Y = \sqrt{\sum_{j=1}^{q_2}\left(\sigma_j^{(Y)}\right)^2}$$

The correlation $\rho_{X,Y}$ between the two processes is

$$\rho_{X,Y} = \frac{\sum_{k=1}^{\min(q_1,q_2)}\sigma_k^{(X)}\sigma_k^{(Y)}}{\sigma_X\sigma_Y}$$

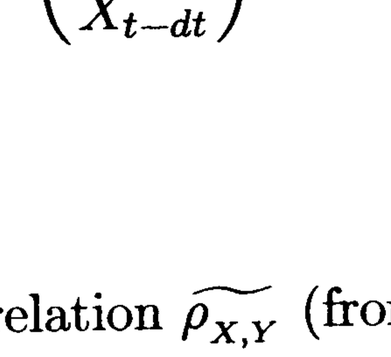
As before, x_t and y_t are defined by

$$x_t = \ln\left(\frac{X_t}{X_{t-dt}}\right) \qquad \text{and} \qquad y_t = Y_t - e^{-a_Y dt}Y_{t-dt}$$

a. Result

The estimated correlation $\widetilde{\rho_{X,Y}}$ (from historical data) between a mean-reverting and a log-normal process is

$$\widetilde{\rho_{X,Y}} = \sqrt{\frac{n}{M}}\sqrt{\frac{\widetilde{a_Y}}{2}\left(\frac{1+e^{-\frac{n}{M}\widetilde{a_Y}}}{1-e^{-\frac{n}{M}\widetilde{a_Y}}}\right)}\frac{G_N(x_t,\widetilde{y}_t)}{\sqrt{G_N(x_t,x_t)G_N(\widetilde{y}_t,\widetilde{y}_t)}}$$



b. Proof

The proof is given in Appendix[B.2].

3.4.4 Correlation between a log-normal and a normal process

The log-normal process and the normal process are respectively

$$dX_t = \mu_X X_t dt + \sum_{i=1}^{q_1} \sigma_i^{(X)} X_t dW_t^{(i)} \qquad \text{and} \qquad dZ_t = \mu_Z dt + \sum_{j=1}^{q_2} \sigma_j^{(Z)} dW_t^{(j)}$$

Their volatilities are respectively

$$\sigma_X = \sqrt{\sum_{i=1}^{q_1} \left(\sigma_i^{(X)}\right)^2} \qquad \text{and} \qquad \sigma_Z = \sqrt{\sum_{j=1}^{q_2} \left(\sigma_j^{(Z)}\right)^2}$$

The correlation $\rho_{X,Z}$ is such that

$$\rho_{X,Z} = \frac{\sum_{k=1}^{\min(q_1,q_2)} \sigma_k^{(X)} \sigma_k^{(Z)}}{\sigma_X \sigma_Z}$$

As before, x_t and z_t are designed to be

$$x_t = \ln\left(\frac{X_t}{X_{t-dt}}\right) \qquad \text{and} \qquad z_t = Z_t - Z_{t-dt}$$

a. Result

The estimated correlation $\widetilde{\rho_{X,Z}}$ between a log-normal process and a normal process is

$$\widetilde{\rho_{X,Z}} = \frac{G_N(x_t, z_t)}{\sqrt{G_N(x_t, x_t)\, G_N(z_t, z_t)}}$$

b. Proof

By analogy to the previous proofs in this chapter, it is not difficult to show that

$$cov\left[x_t, z_t\right] = \sum_{k=1}^{\min(q_1,q_2)} \sigma_k^{(X)} \sigma_k^{(Z)} dt = \rho_{X,Z} \sigma_X \sigma_Z dt$$

Therefore

$$\widetilde{\rho_{X,Z}} = \frac{M}{n} \frac{\widetilde{cov}\left[x_t, z_t\right]}{\widetilde{\sigma_X}\widetilde{\sigma_Z}}$$

3.4.5 Correlation between two normal processes

The two normal processes are

$$dZ_t = \mu_Z dt + \sum_{i=1}^{q_1} \sigma_i^{(Z)} dW_t^{(i)} \qquad \text{and} \qquad dZ_t^* = \mu_{Z^*} dt + \sum_{j=1}^{q_2} \sigma_j^{(Z^*)} dW_t^{(j)}$$

Their volatilities are respectively

$$\sigma_Z = \sqrt{\sum_{i=1}^{q_1} \left(\sigma_i^{(Z)}\right)^2} \qquad \text{and} \qquad \sigma_{Z^*} = \sqrt{\sum_{j=1}^{q_2} \left(\sigma_j^{(Z^*)}\right)^2}$$

The correlation $\rho_{X,Z}$ is such that

$$\rho_{Z,Z^*} = \frac{\sum_{k=1}^{\min(q_1,q_2)} \sigma_k^{(Z)} \sigma_k^{(Z^*)}}{\sigma_X \sigma_Z}$$

As before, x_t and z_t are designed to be

$$z_t = Z_t - Z_{t-dt} \qquad \text{and} \qquad z_t^* = Z_t^* - Z_{t-dt}^*$$

a. Result

The estimated correlation $\widetilde{\rho_{Z,Z^*}}$ between two normal processes is

$$\widetilde{\rho_{Z,Z^*}} = \frac{G_N\left(z_t, z_t^*\right)}{\sqrt{G_N\left(z_t, z_t\right) G_N\left(z_t^*, z_t^*\right)}} \tag{3.4.4}$$

b. Proof

By analogy to the previous proofs, the evidence is straightforward.

3.4.6 Correlation between a mean reverting and a normal process

The normal process and the mean reverting process are respectively

$$dZ_t = \mu_Z dt + \sum_{i=1}^{q_1} \sigma_i^{(Z)} dW_t^{(i)} \qquad \text{and} \qquad dY_t = a_Y \left[b_Y - Y_t\right] dt + \sum_{j=1}^{q_2} \sigma_j^{(Y)} dW_t^{(j)}$$

Their volatilities are respectively

$$\sigma_Z = \sqrt{\sum_{i=1}^{q_1} \left(\sigma_i^{(Z)}\right)^2} \qquad \text{and} \qquad \sigma_Y = \sqrt{\sum_{j=1}^{q_2} \left(\sigma_j^{(Y)}\right)^2}$$

The correlation $\rho_{Y,Z}$ is such that

$$\rho_{Y,Z} = \frac{\sum_{k=1}^{\min(q_1,q_2)} \sigma_k^{(Z)} \sigma_k^{(Y)}}{\sigma_Y \sigma_Z}$$

As before, y_t and z_t are designed to be

$$z_t = Z_t - Z_{t-dt} \qquad \text{and} \qquad y_t = Y_t - e^{-a_Y dt} Y_{t-dt} \tag{3.4.5}$$

a. Result

The estimated correlation $\widetilde{\rho_{Y,Z}}$ between a mean reverting process and a normal process is

$$\widetilde{\rho_{Y,Z}} = \sqrt{\frac{n}{M}} \sqrt{\frac{\widetilde{a_Y}}{2} \left(\frac{1 + e^{-\frac{n}{M}\widetilde{a_Y}}}{1 - e^{-\frac{n}{M}\widetilde{a_Y}}}\right)} \frac{G_N(z_t, \widetilde{y_t})}{\sqrt{G_N(z_t, z_t) G_N(\widetilde{y_t}, \widetilde{y_t})}} \tag{3.4.6}$$

b. Proof

By analogy to the previous proofs, the proof is straightforward.

3.5 Discussion

The motivation behind this section is to explain that there might be numerous cases when parameters (volatility, correlation) of *mean-reverting models* can be estimated using the statistical framework specific to *normal models.*

Estimations of correlations and volatilities for different models are processed from daily historical data (cf chapters[5, 6 and 7]). Given the fact that there are approximately 253 trading days in a year, equation[3.2.5] can simply be re-written as

$$dt = \frac{n}{M} = \frac{1}{253} = 3.95 * 10^{-3} \ll 1 \tag{3.5.1}$$

When dealing with a mean reverting processes Y_t, it has been previously shown that the variable y_t such that

$$y_t = Y_t - e^{-a_Y dt} Y_{t-dt} \tag{3.5.2}$$

plays a key role in the estimation of the volatility and the correlations (cf equations[3.3.3 and 3.4.5]).

For any real number x such that $|x| \ll 1$, it is possible to write the three following useful equations:

$$e^{-x} = 1 - x + o(x) \tag{3.5.3}$$

$$\sqrt{\frac{x}{2}\left(\frac{1+e^{-x}}{1-e^{-x}}\right)} = \left[\frac{2 - x + \frac{x^2}{2} + o(x^2)}{2 - x + \frac{x^2}{3} + o(x^2)}\right]^{\frac{1}{2}} = 1 + \frac{x^2}{8} + o(x^2) \tag{3.5.4}$$

$$\sqrt{\frac{2x}{1-e^{-2x}}} = \left[\frac{1}{1 - x + o(x)}\right]^{\frac{1}{2}} = 1 + \frac{x}{2} + o(x) \tag{3.5.5}$$

Since a_Y is usually less than 1, Equation[3.5.1] tends to suggest that $|\widetilde{a_Y} dt| \ll 1$. Moreover, following equation[3.5.3], it comes:

$$\widetilde{y_t} = Y_t - e^{-\widetilde{a_Y} dt} Y_{t-dt} = Y_t - \left(1 - \widetilde{a_Y} dt + o(\widetilde{a_Y} dt)\right) Y_{t-dt} \simeq Y_t - Y_{t-dt} \tag{3.5.6}$$

Moreover, referring to equations[3.5.4 and 3.5.6], the correlation estimate (see equation[3.4.6])

can be expressed as follows

$$\widetilde{\rho_{Y,Z}} = \left(1 + \frac{[\widetilde{a_Y}dt]^2}{8} + o([\widetilde{a_Y}dt]^2)\right) \frac{G_N(z_t, \widetilde{y}_t)}{\sqrt{G_N(z_t, z_t)G_N(\widetilde{y}_t, \widetilde{y}_t)}}$$

$$\simeq \frac{G_N(Z_t - Z_{t-dt}, Y_t - Y_{t-dt})}{\sqrt{G_N(Z_t - Z_{t-dt}, Z_t - Z_{t-dt})G_N(Y_t - Y_{t-dt}, Y_t - Y_{t-dt})}} \quad (3.5.7)$$

Eventually, using equation[3.5.5] and equation[3.5.6], equation[3.3.8] becomes

$$\widetilde{\sigma_Y} = \sqrt{\frac{2\widetilde{a_Y}}{1 - e^{-2\frac{n}{M}\widetilde{a_Y}}} G_N(\widetilde{y}_t, \widetilde{y}_t)} = \left(1 + \frac{\widetilde{a_Y}dt}{2} + o(\widetilde{a_Y}dt)\right) \sqrt{\frac{M}{n} G_N(\widetilde{y}_t, \widetilde{y}_t)}$$

$$\simeq \sqrt{\frac{M}{n} G_N(Y_t - Y_{t-dt}, Y_t - Y_{t-dt})} \quad (3.5.8)$$

It is now straightforward enough to notice that equations[3.5.8 and 3.5.7] are directly analogous to equations[3.3.17 and 3.4.4], respectively. In other words, it seems that it might be possible to estimate correlations and volatility of a mean reverting process by applying directly the estimation framework that has been built for normal processes. The purpose of this sub-section is indeed to investigate this possibility. In order to do so, two correlated models with correlation $\rho_{Y,Z}$ are simulated:

- A normal process Z_t (with parameters μ_Z and σ_Z) such that:

$$dZ_t = \mu_Z dt + \sigma_Z dW_t^{(1)}$$

- A mean reverting process Y_t (with parameters a_Y , b_Y, and σ_Y) such that

$$dY_t = a_Y [b_Y - Y_t] dt + \sigma_Y \left[\rho_{Y,Z} dW_t^{(1)} + \sqrt{1 - \rho_{Y,Z}^2} dW_t^{(2)}\right]$$

The method consists in showing that the two following equations[3.5.9 and 3.5.10]

are valid in practice.

$$\sqrt{\frac{M}{n} G_N(Y_t - Y_{t-dt}, Y_t - Y_{t-dt})} \simeq \sigma_Y \tag{3.5.9}$$

$$\frac{G_N(Z_t - Z_{t-dt}, Y_t - Y_{t-dt})}{\sqrt{G_N(Z_t - Z_{t-dt}, Z_t - Z_{t-dt}) G_N(Y_t - Y_{t-dt}, Y_t - Y_{t-dt})}} \simeq \rho_{Y,Z} \tag{3.5.10}$$

In discrete case, the two processes become

$$\Delta Z_t = \mu_Z \Delta t + \sigma_Z \varepsilon_1 \sqrt{\Delta t}$$

and

$$\Delta Y_t = a_Y \left[b_Y - Y_t\right] \Delta t + \sigma_Y \sqrt{\Delta t} \left[\rho_{Y,Z} \varepsilon_1 + \varepsilon_2 \sqrt{1 - \rho_{Y,Z}^2}\right]$$

The two independent standard gaussian variables (ε_1 and ε_2) are modelled (see [Lamberton & Lapeyre (1995)]) as

$$\begin{cases} \varepsilon_1 = \sqrt{-2\log(V_1)}\cos(2\pi V_2) \\ \\ \varepsilon_2 = \sqrt{-2\log(V_1)}\sin(2\pi V_2) \end{cases}$$

with V_1 and V_2 two independent uniform random variables on $[0, 1]$. The size of the data sample is chosen to be 2000. The time interval is

$$\frac{n}{M} = 3.95 * 10^{-3}$$

We have tested different parameters values for both models and it appears that we can reasonably assume (as long as $\frac{n}{M}$ is small enough) that it seems realistically possible to estimate correlations and volatility of a mean reverting process by applying directly the estimation framework that has been built for normal processes. The following table displays some of the outcomes of the simulations:

$\widetilde{\sigma_Y}$ and $\widetilde{\rho_{Y,Z}}$ represent the values for the estimates of σ_Y and $\rho_{Y,Z}$. The estimates are computed due to the statistical framework built for normal processes. Analysing the table, it seems obvious that the values obtained are similar to the theoretical ones. This is good news since this method does not require to estimate

the drift of the mean reverting process. Indeed, obtaining reliable estimates for the drift of a mean reverting process is an extremely difficult exercise in practice.

n^o of simulation	σ_Y	$\widetilde{\boldsymbol{\sigma}_{\mathbf{Y}}}$	$\rho_{Y,Z}$	$\widetilde{\boldsymbol{\rho}_{\mathbf{Y},\mathbf{Z}}}$	a_Y	b_Y	μ_Z	σ_Z	Z_0	Y_0
1	0.1	0.1007	0.25	0.2535	0.05	0.20	0.20	0.10	0.05	0.10
2	0.2	0.1994	0.25	0.2396	0.05	0.20	0.20	0.10	0.05	0.10
3	0.3	0.2978	0.25	0.2349	0.05	0.20	0.20	0.10	0.05	0.10
4	0.1	0.0986	0.75	0.7606	0.05	0.20	0.20	0.10	0.05	0.10
5	0.2	0.1975	0.75	0.7429	0.05	0.20	0.20	0.10	0.05	0.10
6	0.3	0.2982	0.75	0.7601	0.05	0.20	0.20	0.10	0.05	0.10
7	0.1	0.1018	0.25	0.2374	0.15	0.20	0.20	0.10	0.05	0.10
8	0.2	0.2034	0.25	0.2523	0.15	0.20	0.20	0.10	0.05	0.10
9	0.3	0.3016	0.25	0.2532	0.15	0.20	0.20	0.10	0.05	0.10
10	0.1	0.1002	0.75	0.7523	0.15	0.20	0.20	0.10	0.05	0.10
11	0.2	0.1983	0.75	0.7578	0.15	0.20	0.20	0.10	0.05	0.10
12	0.3	0.3063	0.75	0.7516	0.15	0.20	0.20	0.10	0.05	0.10
13	0.1	0.1003	0.25	0.2525	0.25	0.20	0.20	0.10	0.05	0.10
14	0.2	0.2014	0.25	0.2635	0.25	0.20	0.20	0.10	0.05	0.10
15	0.3	0.2905	0.25	0.2410	0.25	0.20	0.20	0.10	0.05	0.10
16	0.1	0.0982	0.75	0.7586	0.25	0.20	0.20	0.10	0.05	0.10
17	0.2	0.2017	0.75	0.7501	0.25	0.20	0.20	0.10	0.05	0.10
18	0.3	0.2995	0.75	0.7558	0.25	0.20	0.20	0.10	0.05	0.10
19	0.1	0.0990	0.25	0.2508	0.35	0.20	0.20	0.10	0.05	0.10
20	0.2	0.1966	0.25	0.2604	0.35	0.20	0.20	0.10	0.05	0.10
21	0.3	0.2964	0.25	0.2544	0.35	0.20	0.20	0.10	0.05	0.10
22	0.1	0.0997	0.75	0.7454	0.35	0.20	0.20	0.10	0.05	0.10
23	0.2	0.2003	0.75	0.7537	0.35	0.20	0.20	0.10	0.05	0.10
24	0.3	0.2940	0.75	0.7415	0.35	0.20	0.20	0.10	0.05	0.10
25	0.1	0.0976	0.25	0.2464	0.45	0.20	0.20	0.10	0.05	0.10
26	0.2	0.1965	0.25	0.2446	0.45	0.20	0.20	0.10	0.05	0.10
27	0.3	0.2971	0.25	0.2414	0.45	0.20	0.20	0.10	0.05	0.10
28	0.1	0.1013	0.75	0.7601	0.45	0.20	0.20	0.10	0.05	0.10
29	0.2	0.1998	0.75	0.7536	0.45	0.20	0.20	0.10	0.05	0.10
30	0.3	0.3061	0.75	0.7558	0.45	0.20	0.20	0.10	0.05	0.10

Chapter 4

The Credit Spread

4.1 Introduction

After having built some useful tools in chapters[2,3] and before discussing new hedging techniques in chapters[5,6,7] in order to hedge bonds subject to credit risk, it seems important at this stage of the thesis to conduct an analysis of the determinants of the *credit spread*. Indeed, the price of a non-default free bond is sensitive to the risk free interest rates and to the *credit spread*. Thus, the purpose of this chapter is to investigate the *credit spread*. The objective of this investigation is twofold:

First, it is shown that the *credit spread* is composed of a variable called the *default premium* which represents the probability of default of the issuer, or indirectly the "financial health" of the issuer. This link (between the *default premium* and the "financial health") is going to play a key role through chapters[6,7] in the sense that if we are able to find a very liquid security that is strongly dependent on the "financial health" of the issuer, this one could be used to hedge the *default premium*.

Secondly, it is emphasised that the *default premium* is not actually the principal component of the *credit spread*, the other component being the *risk premium* which does not appear to be linked as strongly to the "financial health" of the issuer.

Before starting the analysis, some credit risk related concepts and models are discussed.

4.1.1 Basic concepts

In the last few years *credit risk* has played an increasing role in the interest rate market . They enable investors to take advantage of default risk and the ability to

quantify default probabilities is therefore essential. In the case of default, bondholders will not receive the timely payment of coupons and the capital agreed originally between the parties. Once a bond has been issued the probability that the bond issuer will default is not zero. In order to compensate the investors for this risk (on the part of the issuer), a bond must offer extra remuneration which is called the *premium* or *credit spread.* Essentially this is the yield difference between a risky bond and a comparable risk-free bond which is considered to be default free.

$$Credit\ Spread = Yield\ to\ Maturity - Risk\ free\ interest\ rate$$

The ability to quantify default probabilities and then to evaluate the fair value of a bond can lead to the development of trading strategies through the ability to take advantage of pricing anomalies in the market -as will be shown later. A credit rating is an expression of the probability of default on a specific bond. It does not express the degree of loss if a default situation does occur. The two most significant rating agencies are *Moody's* and *Standard & Poor's.* These agencies measure the risk of default by assigning letter grades. A quick summary of the different credit rating category definitions is shown below:

AAA : Capacity to pay interest and repay principal is extremely strong.

AA : Capacity to pay interest and repay principal is very strong.

A : Capacity to pay interest and repay principal is strong.

BBB : Capacity to pay interest and repay principal is adequate.

BB : Less near-term vulnerability to default than other speculative issues. However, it faces ongoing uncertainties or exposure to adverse business, financial or economic conditions.

B : Greater vulnerability to default than BB, but currently has the capacity to meet interest payments and principal repayments.

CCC : Has currently identifiable vulnerability to default.

D : In default

After having highlighted the basic concepts, it seems important to review briefly the different ways of modelling credit risk in bonds and derivatives.

4.1.2 Modelling Credit Risk

Basically, there are two main approaches illustrated by the *structural models* and the *intensity based models*. In this sub-section, $E_Q[. \mid \mathcal{F}_t]$ corresponds to the conditional expectation under an equivalent martingale measure Q with $\mathcal{F}_t$ a relevant filtration to which stochastic processes are adapted.

The **structural models** are based on the value V_t of the company that has issued a defaultable zero-coupon debt whose principal is D. Implementation is not straightforward because it is difficult to observe and estimate accurately some company related variables such as the values of the liabilities (plus their level of seniority) and assets. However, *structural models* are realistic in the sense that default time is modelled as an accessible predictable stopping time. Indeed, the default is considered to occur if the value of the company hits a certain level. In other words, if the value of the company at maturity is less than the defaultable zero-coupon debt, default occurs. This is the famous approach first developed by [Merton (1974)]. The basic idea is that at maturity T of the debt the creditors will receive the payoff X such that:

$$X = \min(D, V_T)$$

Therefore, at time t the value of the defaultable debt is:

$$\begin{aligned} C(t,T) &= E_Q\left[e^{-\int_t^T r_s ds}\min(D, V_T) \mid \mathcal{F}_t\right] \qquad (4.1.1) \\ &= E_Q\left[e^{-\int_t^T r_s ds} D \mid \mathcal{F}_t\right] - E_Q\left[e^{-\int_t^T r_s ds}(D - V_T)^+ \mid \mathcal{F}_t\right] \end{aligned}$$

This is the price of a default-free bond minus a "put" option which can be viewed as an imbedded option. In addition, if V_t is modelled as a lognormal process, the traditional derivative technology allows a straightforward computation of the price of the risky debt. In the [Geske (1977)] approach, the value of the defaultable debt is analysed using a compound option framework. For a more in-depth litterature review of the *structural models*, the reader is invited to refer to [Lando (1996)].

The **intensity based models** are easier to implement than the *structural models* but they are sometimes viewed as not very realistic in the sense that default occurs in a totally unpredictable way. Indeed, in the *intensity based models*, considering a jump default process $N_t = 1_{\{\tau \leq t\}}$, the default time is modelled as an inaccessible stopping time τ with a hazard rate λ_t defined such that the following stochastic

process M_t is a martingale.

$$M_t = N_t - \int_0^t \lambda_s(1 - N_s)ds \tag{4.1.2}$$

In a general approach given by Lando, it is shown that risky debts (with maturity T) can be priced using the traditional derivative pricing technology if the spot rate r_s is replaced by a *default adjusted rate* since it is possible to write the two following equations (see [Lando (1994)] and [Lando (1996)]) :

$$\begin{aligned} C_Y(t,T) &= E_Q\left[\int_t^T Y_s 1_{\{s<\tau\}} e^{-\int_t^s r_u du} ds \quad | \quad \digamma_t\right] \\ &= E_Q\left[\int_t^T Y_s e^{-\int_t^s (r_u+\lambda_u)du} ds \quad | \quad \digamma_t\right] \end{aligned} \tag{4.1.3}$$

Equation[4.1.3] expresses the price of a derivative C_Y that pays a time-dependent rate Y provided that no default has occured.

$$\begin{aligned} C_{X,Z}(t,T) &= E_Q\left[e^{-\int_t^T r_s ds} X\ 1_{\{T<\tau\}} + e^{-\int_t^\tau r_s ds} Z_\tau \quad | \quad \digamma_t\right] \\ &= E_Q\left[e^{-\int_t^T (r_s+\lambda_s)ds} X + \int_t^T Z_s \lambda_s e^{-\int_t^s (r_u+\lambda_u)du} ds \quad | \quad \digamma_t\right] \end{aligned} \tag{4.1.4}$$

Equation[4.1.4] represents the price of a derivative $C_{X,Z}$ that pays X at time T (if no default has occured) or a recovery payment $Z\tau$ at default time τ. Indeed, most bond defaults are resolved in a way that provides bondholders with some form of recovery. One of the critical aspects of a corporate bond default is the severity of the loss incurred. The *recovery rate* (which is a key parameter in the modelling of default) usually depends on the seniority of the issue (see subsection[4.2.1]) and is defined as a percentage of par value returned to the bondholder in case of default (see [Jarrow & Turnbull (1995)]) or as a fraction of the pre-default value (see [Duffie & Singleton (1995)] and [Duffie, Schroder & Skiadas (1995)]). An interesting modelling of the recovery rate is given by [Madan & Unal (1994)] who design the default magnitude to be a random variable. The *default intensity process* can be modelled in different ways (constant, time dependent, stochastic, etc.):

For example, the default intensity is sometimes chosen to be a constant parameter (see [Jarrow & Turnbull (1995)]). By analogy to the probability transition matrix framework (discussed in section[4.2]), [Jarrow, Lando & Turnbull (1994)] model the default intensity such that the default occurs when a Markov chain reaches a special state. Also, the intensity process can be modelled as a gaussian mean reverting process (see [Davis (1997)]) or as a function of an asset value process (see [Madan & Unal (1994)]). Some models incorporate a correlation between the intensity process and the default-free interest rate ([Duffie, Schroder & Skiadas (1995)], [Lando (1994)]). As far as calibration is concerned, many models are calibrated on the term structure of the credit spread. The idea consists in establishing a mathematical relation between the credit spread and the probability of survival $p(t)$ at time t given by

$$p(t) = E_Q\left[e^{-\int_0^t \lambda_s ds}\right]$$

For a discussion regarding the calibration of these models, the reader is invited to refer to [Davis (1997)] and [Finkelstein (1997)]. In the previously discussed models, default occurs at the first jump of a poisson process. However, it is important to note that there exist some more sophisticated models where multiple defaults are allowed. For instance, [Schonbucher (1997)] has modelled the term structure of defaultable bonds in which issuers are reorganized at default, not liquidated.

4.1.3 Credit Risk Measurement

The modelling of credit risk under the risk neutral equivalent martingale measure(cf sub-section[4.1.2]) is essential when it is dealt with the pricing of defaultable bonds or credit derivatives. However, as far as credit risk measurement and analysis are concerned, a probabilistical and statistical analysis under the real world probability measure is usually conducted. For example, the use of probability transition matrices and distribution of default events is helpful in order to develop a methodology for estimating a bond portfolio distribution of default losses ([CreditRisk$^+$ (1997)]) and value-at-risk (VAR) due to changes in credit quality ([CreditMetrics (1997)]).

In the same context, the following sections deal with real world probabilities since the purpose is to estimate different kind of risks. After having reviewed the basic concepts about *credit risk*, it is now time to focus on the *credit spread* which has been previously defined. Moreover, as mentioned before, the *credit spread* is actually composed of two parts which are the *default premium* and the *risk premium*.

4.2 Default premium

The *default premium* is the premium directly associated with the risk of default. This can be defined as follows:

$$Default\ Premium = Expected\ Yield - Risk\ free\ interest\ rate$$

The magnitude of the *default premium* depends on the likely financial loss to the bondholder and the probability of default of the bond. Financial loss is a function of the *recovery rate* (cf subsection[4.2.1]) and on the *probability of survival* of the issuer (cf subsection[4.2.2]).

4.2.1 Recovery Rate

The Recovery rate is defined here as a percentage of par value returned to the bondholder in case of default. The *defaulted bond recovery rate estimates* are derived from records of bond prices that defaulted between 1970 and 1995. The table below lists the average *recovery rates* for different classes of debt.

Senior Secured	53.04%
Senior Unsecured	45.88%
Senior Subordinated	37.02%
Subordinated	29.57%
Junior Subordinated	16.4%
Pfd Stocks	6%

Sources : Moodys Investor Service Global Credit Research

4.2.2 Expected Yield of a risky bond

In order to compute the *expected yield to maturity*, the idea is to determine the right price of a risky (defaultable) bond, taking into account the probability that the timely payment of coupons and/or capital are not made. Before moving forward, it is useful to give some definitions:

- $\mathbf{R_t}$: Credit rating at time t of the company that has issued a risky bond. $R_t \in \{AAA,\ AA,\ ...,\ CCC\}$.

- $\mathbf{X}$: Class of debt. For example, it can be subordinated, senior secured, etc. This is a qualitative variable.

- $\mathbf{S(t, t_k, R_t)}$: Probability of survival at time t_k of the company whose rating at time t is R_t. This is actually the probability that no default will occur before time t_k. This is a real world probability and its computation is explained through next section.

- $\mathbf{B(t, T, R_t, X)}$: Price of a defaultable bond at time t with maturity T that has been issued by a company whose rating at time t is R_t. The class of this debt is X.

- $\mathbf{B_f(t, T)}$: Price of a default free bond at time t with maturity T

- $\mathbf{Rec(X)}$: Recovery Rate for a bond with class of debt equal to X.

- $\mathbf{i(t, t_k)}$: Risk-free interest rate at time t over the $t_k - t$ time period

- $\mathbf{N}$: Number of coupon periods during the $T - t$ period of time. N is defined such that $t_N = T$.

- $\mathbf{y_{t,T,R_t,X}}$: Yield to maturity at time t of a defaultable bond with maturity T that has been issued by a company whose rating at time t is R_t. The class of this debt is X.

- $\boldsymbol{\gamma_{t,T}}$: Yield to maturity at time t of a default free bond with maturity T

As stated in chapter[2], the price $B(t, T, R_t, X)$ of a bond can be expressed as a function of the yield to maturity (cf equation[2.3.7]), that is to say:

$$B(t, T, R_t, X) = \sum_{k=1}^{N} \frac{s_k}{[1 + y_{t,T,R_t,X}]^{\tau_k}} \tag{4.2.1}$$

With a fixed coupon frequency and a constant coupon value C and assuming a repayment price of 100, equation[4.2.1] is (just prior or after a coupon payment) then equivalent to:

$$B(t, T, R_t, X) = \sum_{k=1}^{N} \frac{100C}{[1 + y_{t,T,R_t,X}]^{k}} + \frac{100}{[1 + y_{t,T,R_t,X}]^{N}} \tag{4.2.2}$$

which can be rewritten as:

$$B(t,T,R_t,X) = \frac{100C}{y_{t,T,R_t,X}} - \frac{100C}{y_{t,T,R_t,X}(1+y_{t,T,R_t,X})^N} + \frac{100}{[1+y_{t,T,R_t,X}]^N} \quad (4.2.3)$$

The above definitions are traditionnally used when the bondholder expects that he will receive all the future coupons at the specified dates. Unfortunately this is not always the case and as the bondholder may not receive the timely payment of coupons and/or the capital agreed originally between the parties, it makes sense to compute the price of the bond, not as the price induced by the promised future payments but rather as the price induced by the expected future cashflows. Since the bond price is the sum of the discounted future payments, the expected price of the bond can be derived by discounting the expected future cashflows:

$$\begin{aligned} B(t,T,R_t,X) &= \frac{100C}{[1+i(t,t_1)]^1}S(t,t_1,R_t) + ... + \frac{100C}{[1+i(t,t_N)]^N}S(t,t_N,R_t) \\ &+ \frac{100}{[1+i(t,t_N)]^N}S(t,t_N,R_t) + \frac{100\text{Rec}(X)}{[1+i(t,t_N)]^N}\left[1-S(t,t_N,R_t)\right] \end{aligned} \quad (4.2.4)$$

or equivalently as:

$$\begin{aligned} B(t,T,R_t,X) &= \sum_{k=1}^{N}\frac{100C}{[1+i(t,t_k)]^k}S(t,t_k,R_t) + \frac{100}{[1+i(t,t_N)]^N}S(t,t_N,R_t) \\ &+ \frac{100\text{Rec}(X)}{[1+i(t,t_N)]^N}\left[1-S(t,t_N,R_t)\right] \end{aligned} \quad (4.2.5)$$

At this stage it is important to understand that equation[4.2.4] implies that in the event of default the *recovery payment* (if any) will be settled only at the maturity date of the defaultable bond. This may not be very realistic. However, as far as our purpose in this chapter is concerned, we believe that the modelling that we have chosen enables us to obtain satisfactory results.

Equalising equation[4.2.5] and equation[4.2.3], it comes out that:

$$\frac{C}{y_{t,T,R_t,X}} - \frac{C}{y_{t,T,R_t,X}(1+y_{t,T,R_t,X})^N} + \frac{1}{(1+y_{t,T,R_t,X})^N} \tag{4.2.6}$$

$$= C\sum_{k=1}^{N} \frac{S(t,t_k,R_t)}{[1+i(t,t_k)]^k} + \frac{S(t,t_N,R_t)}{[1+i(t,t_N)]^N} + \frac{[1-S(t,t_N,R_t)]\,\mathrm{Rec}(X)}{[1+i(t,t_N)]^N}$$

Hence, after solving equation[4.2.6] by trial and error, the variable $y_{t,T,R_t,X}$ can be extracted. Then the yield to maturity $y_{t,T,R_t,X}$ of a defaultable is compared to the yield to maturity $\gamma_{t,T}$ (coming fron equation[4.2.7]) of a theoretical *equivalent benchmark* which is actually a similar bond with no default risk:

$$\frac{C}{\gamma_{t,T}} - \frac{C}{(1+\gamma_{t,T})^N} + \frac{1}{(1+\gamma_{t,T})^N} = \sum_{k=1}^{N} \frac{C}{[1+i(t,t_k)]^k} + \frac{1}{[1+i(t,t_N)]^N} \tag{4.2.7}$$

The difference $y_{t,T,R_t,X} - \gamma_{t,T}$ is in fact the *default premium* which is (as previously explained) the component of the spread that depends directly on the likelihood of default of the issuer (Corporate, Sovereign...). Since equation[4.2.6] refers to the *probability of survival*, the remaining part of this subsection is devoted to the method which enables the computation of $S(t,t_k,R_t)$.

Computation of the *probability of survival*:

The basic idea is to evaluate the probability that given an initial state (e.g: a particular rating) at time t = 0 , the company will be in default at time T. Knowing this probability, it is then straigtforward to determine the *probability of survival.* To understand the method, let us define a world where a company can only be in one of the three states : $R^{(1)}$, $R^{(2)}$, $R^{(3)}$. Let us define $R^{(3)}$ as the *default state.* The next step is to evaluate the probability that a corporate in the $R^{(i)}$ -state at time t = 0 will be in the $R^{(k)}$ - state at time t = T. Assuming that probabilities of transition between each of these states are known (for a single period of time), a tree (cf figure[4.2.1]) can be drawn.

For instance, the probability that a company in state R_i at time t = 0 is in state $R^{(k)}$ at time t = 2 is:

$$P_{\left[\left(R^{(i)}\right)_{t=0} \longrightarrow \left(R^{(k)}\right)_{t=2}\right]} = p_{i1} \times p_{1k} + p_{i2} \times p_{2k} + p_{i3} \times p_{3k}$$

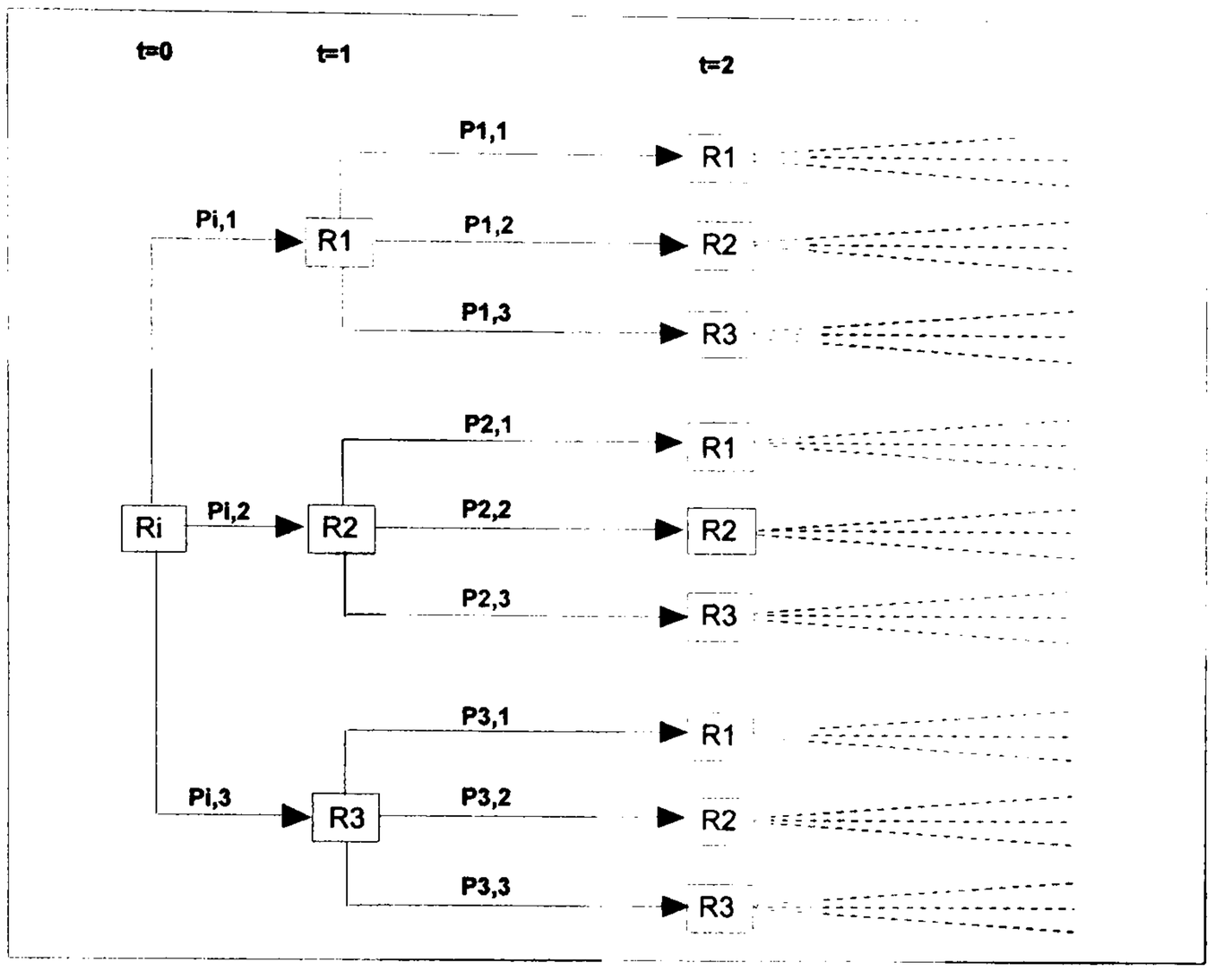


Figure 4.2.1: Multinomial tree

with

$p_{m,n}$ is the probability of moving from $R^{(m)}$ to $R^{(n)}$ in one period

This result can be generalized for m different ratings and over a time-horizon of lenght T. However the probability that a corporate moves from one rating to another rating in one-time period depends only on those two states. Actually it is assumed that this probability is not dependent on the history of the previous ratings. Moreover at any given time, the corporate is assumed to have a well defined rating. Furthermore, the number of ratings (i.e states) is finite. Therefore, we can define rating AAA as state 1, rating AA as state 2 and so on. Then it is possible to model this process by a 8-state Markov chain having the *transition matrix* T since the process is not path dependent. (cf figure[4.2.2]) :

The data which has been used come from the one-year *transition matrix*: This one displays all rating movements between rating categories from the beginning of the year through year end. The *transition matrix* that we have used is in fact the Standard & Poors historical averages (%) of credit rating movements for each year

T

Initial rating \ Rating at end of year	AAA	AA	A	BBB	BB	B	CCC	D
AAA	$P_{1,1}$	$P_{1,2}$	$P_{1,3}$	$P_{1,4}$	$P_{1,5}$	$P_{1,6}$	$P_{1,7}$	$P_{1,8}$
AA	$P_{2,1}$	$P_{2,2}$	$P_{2,3}$	$P_{2,4}$	$P_{2,5}$	$P_{2,6}$	$P_{2,7}$	$P_{2,8}$
A	$P_{3,1}$	$P_{3,2}$	$P_{3,3}$	$P_{3,4}$	$P_{3,5}$	$P_{3,6}$	$P_{3,7}$	$P_{3,8}$
BBB	$P_{4,1}$	$P_{4,2}$	$P_{4,3}$	$P_{4,4}$	$P_{4,5}$	$P_{4,6}$	$P_{4,7}$	$P_{4,8}$
BB	$P_{5,1}$	$P_{5,2}$	$P_{5,3}$	$P_{5,4}$	$P_{5,5}$	$P_{5,6}$	$P_{5,7}$	$P_{5,8}$
B	$P_{6,1}$	$P_{6,2}$	$P_{6,3}$	$P_{6,4}$	$P_{6,5}$	$P_{6,6}$	$P_{6,7}$	$P_{6,8}$
CCC	$P_{7,1}$	$P_{7,2}$	$P_{7,3}$	$P_{7,4}$	$P_{7,5}$	$P_{7,6}$	$P_{7,7}$	$P_{7,8}$

Figure 4.2.2: Transition Matrix

over the past 13 years. For illustration puposes, the matrix that we have used is given below.

%	AAA	AA	A	BBB	BB	B	CCC	D
AAA	90.83	8.27	0.74	0.06	0.11	0.00	0.00	0.00
AA	0.65	90.88	7.69	0.58	0.05	0.13	0.02	0.00
A	0.08	2.42	91.30	5.23	0.68	0.23	0.01	0.05
BBB	0.03	0.31	5.87	87.45	4.96	1.08	0.12	0.18
BB	0.02	0.12	0.64	7.7	81.07	8.39	0.98	1.08
B	0.00	0.10	0.24	0.44	6.81	82.90	3.90	5.61
CCC	0.20	0.00	0.40	1.19	2.56	11.24	61.93	22.49

From the previous matrix, it is interesting to notice that for any initial state (AAA to CCC), the rating is very likely to remain unchanged after one year. Furthermore, the higher the initial rating, the higher the probability for the rating to remain unchanged.

4.2.3 Adjusted sensitivity measures

In this subsection, notations are the same as in chapter[2]. Moreover, in order to simplify notations, $S(\tau_k)$ corresponds here to the probability of survival over a τ_k period of time (in years) and Rec is the recovery rate. To complete the discussion about sensitivity measures conducted in chapter[2], it has to be understand that sensitivity measures are generally viewed as instruments measuring the risk associ-

ated with the randomness of the risk free rate. [Bierwag & Kaufman (1988)] and [Chance (1990)] have derived some single factor measures of Duration for option-free bonds subject to default risk assuming stochastic processes governing default events or using contingent claims approach. [Fooladi , Roberts & Skinner (1997)] have made a generalization (to coupon bonds with non-flat term structure) of the two previous quoted papers. Their analysis is based on the [Jonkhart (1979)] model of the term structure consistent with default risk. In this subsection, some adjusted sensitivity measures are derived using a probabilistic approach particularly based on the transition matrix concept. The usefulness of these results as far as hedging is concerned will be discussed at some point in the next chapters especially devoted to hedging strategies. The details of the calculus are not given here since the results are derived using the same kind of methodology (Ito's lemma etc.) as the one explained in chapter[2]. As a result, a small variation in the price of a defaultable bond can be written as

$$dBt = \Psi_{1,t}^{(\text{adjusted})} dt + D_{\gamma_t}^{(\text{adjusted})} d\gamma_t + C_{\gamma_t}^{(\text{adjusted})} d\gamma_t^2 \tag{4.2.8}$$

where

$$\Psi_{1,t}^{(\text{adjusted})} = \frac{\ln(1+\gamma_t)}{M}\left[\sum_{k=1}^{N}\frac{s_k S(\tau_k)}{(1+\gamma_t)^{\tau_k}} + \frac{s_N S(\tau_N) + \text{Rec} \times (1 - S(\tau_N))}{(1+\gamma_t)^{\tau_N}}\right] \tag{4.2.9}$$

$$D_{\gamma_t}^{(\text{adjusted})} = -\sum_{k=1}^{N}\frac{s_k \tau_k S(\tau_k)}{[1+\gamma_t]^{\tau_k+1}} - \frac{\tau_N\left[s_N S(\tau_N) + \text{Rec} \times (1 - S(\tau_N))\right]}{[1+\gamma_t]^{\tau_N+1}} \tag{4.2.10}$$

and

$$C_{\gamma_t}^{(\text{adjusted})} = \frac{1}{2}\sum_{k=1}^{N}\frac{s_k \tau_k(\tau_k+1) S(\tau_k)}{[1+\gamma_t]^{\tau_k+2}} + \frac{1}{2}\frac{\tau_N(\tau_N+1)\left[s_N S(\tau_N) + \text{Rec} \times (1 - S(\tau_N))\right]}{[1+\gamma_t]^{\tau_N+2}} \tag{4.2.11}$$

Note: In order to simplify notations, $S(\tau_k)$ corresponds to the probability of survival over a τ_k period of time (in years).

4.2.4 Results

The objective of this subsection is to explain how the *theoretical default premium term structure* curves have been computed and to analyse some of the results obtained. It is assumed here that the current time is $t = 0$ and that the risk free term structure is known. In other words, keeping the same notations as the ones in sub-section[4.2.2], the annual risk-free interest rate at time 0 for the t_k time period is known for any k and is given by $i(0, t_k)$.

The first step of the computation consists in selecting a rating category. Let us assume that we want to compute the *default premium term structure* for defaultable bonds whose credit rating is R_0, now at time 0 (with $R_0 \in \{AAA,\ AA,\ ...,\ CCC\}$). Given, the credit rating R_0, the second step consists in computing the term structure of the probability of survival for a bond initially rated R_0: That is to say the values of the $S(0, t_k, R_0)_{k\in[1,N]}$. This is easily done using the transition matrix approach (cf sub-section[4.2.2]). The third step of the computation consists in selecting a particular class of debt, X, and therefore to assume an appropriate recovery rate. For instance, if it is decided to compute the *default premium term structure* for R-rated defaultable bonds which are junior subordinated, then according to sub-section[4.2.1], the recovery rate value will be chosen to be $Rec(X) = 16.4\%$. The fourth step of the computation consists in selecting a particular coupon value. For instance, if it is decided to compute the *default premium term structure* for R_0-rated defaultable bonds with coupons equal to 8% per year, then $C = 8\%$. The final step is to select a maturity T for which we want to compute the default premium. This is done by choosing the appropriate N in equation[4.2.6] and equation[4.2.7]. After replacing $S(0, t_k, R_0)_{k\in[1,N]}$, $Rec(X)$, $i(0, t_k)_{k\in[1,N]}$ and C by their respective values in equation[4.2.6] and equation[4.2.7], we are then able to compute the expected yield $y_{t,T,R_t,X}$ for the defaultable bond and the yield $\gamma_{t,T}$ for an equivalent default-free bond. The default premium for this particular maturity T is then given by $y_{t,T,R_t,X}$ minus $\gamma_{t,T}$. After selecting other maturities and applying the same methodology, the *default premium term structure* (for R_0-rated defaultable bonds with C-coupons and with a class of debt equal to X) is eventually computed.

The results obtained seem to be consistent with the common sense theory according to which: the lower the credit rating, the higher the *default premium*. Indeed, corporates with a low credit rating trade at a high yield and thus at a low price for the investor. Figure[4.2.3] displays three typical *default premium term structure* (for AAA, AA and A rated bonds) curves which are smooth and regular. The

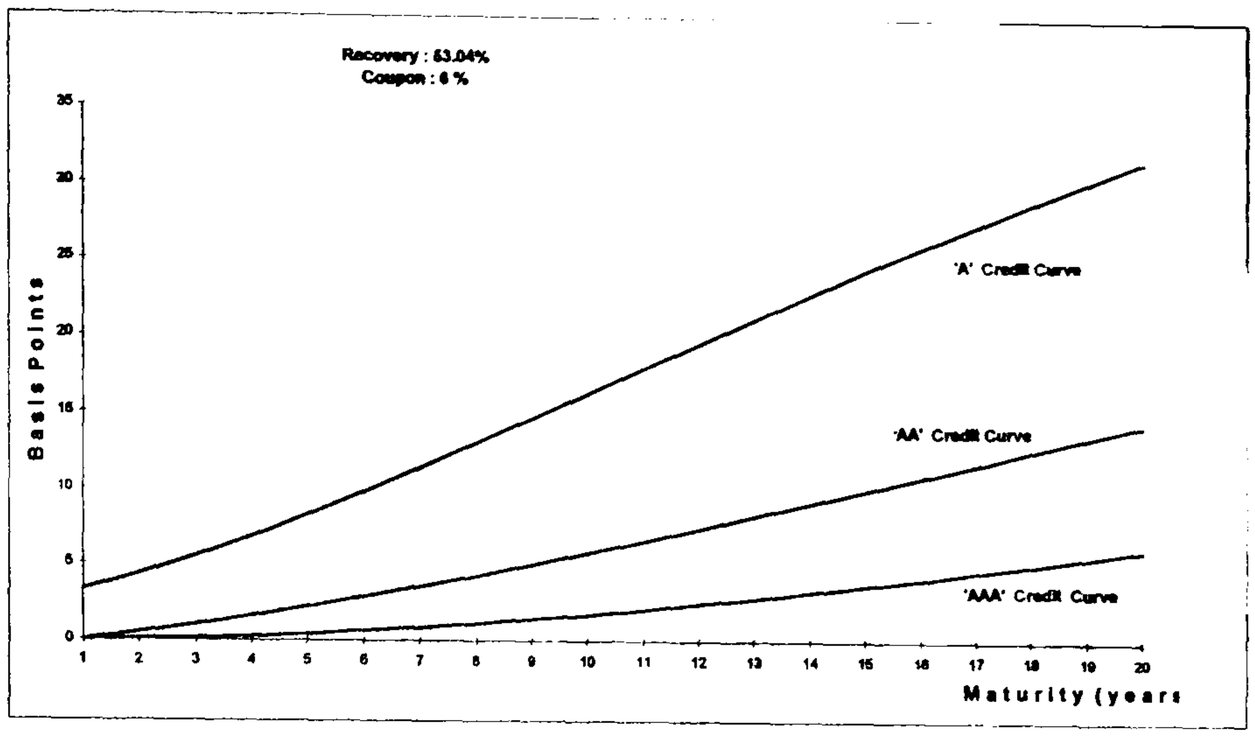


Figure 4.2.3: Theoretical default premium Curve (1)

coupon is 6% and the class of debt is chosen to be senior secured.

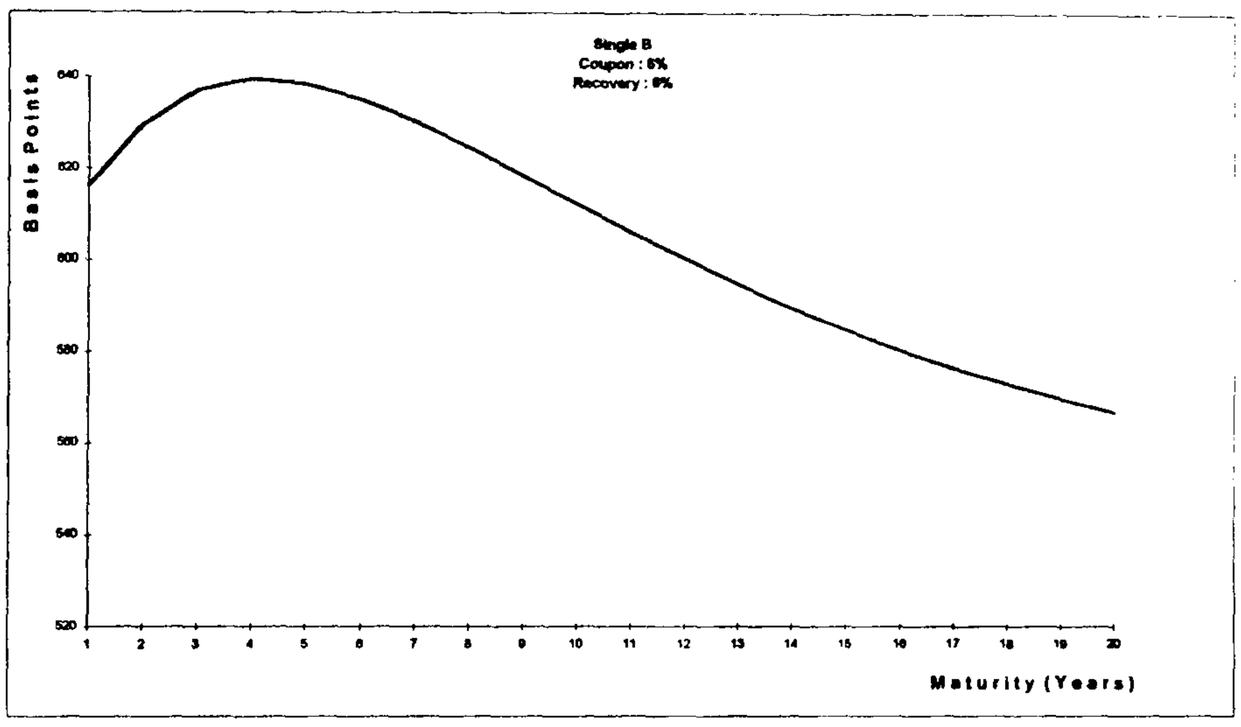


Figure 4.2.4: Theoretical default premium Curve (2)

In order to show that the *default premium term structure* is not always regularly increasing, an extreme case is investigated. The case is extreme in the sense that low rated bonds (single B) are considered and a very low recovery rate (i.e 6%) is chosen in the modelling. Figure[4.2.4] illustrates this particular *default premium term structure* (for B-rated bonds). The reason why the curve is first increasing and then decreasing can be explained by the fact that a single B corporate faces big short term risk and if it can overcome this, then the credit rating is likely to increase.

4.2.5 Further Investigation

The *transition matrix* used before is in fact the historical averages of credit rating movements for each year over the past 13 years. Instead of using the same *transition matrix* for each year, one might think that it could be interesting to consider this matrix as stochastic. This is motivated by the fact that the economy follows cycles. Therefore *transition matrices* should not be the same each year. Hence, after collecting enough data, it is possible to compute for each row vector i of the matrix, its estimated mean M_i and its estimated covariance G_i. Despite the fact that the number of data available on the market is limited, it will be assumed that the values on each row of the *transition matrix* follow a normally distributed process. Then the idea is to generate a random gaussian vector whose mean and covariance are those previously calculated. In order to do so, the normal random vector V with estimated mean M and estimated Covariance G is given by :

$$V = M + AZ$$

with A such that

$$Z = AA^T$$

and Z a vector of independent gaussian random variables with zero mean and a covariance matrix equal to the Identity. The method used in order to compute A is the Cholesky's method and is described by [Ciarlet (1990)]. Moreover each gaussian random variable z can be derived from two independent uniform random variables (U1,U2) on [0,1] :

$$z = \sqrt{-2\log(U_1)}\cos(2\pi U_2)$$

For a proof of this result, see [Lamberton & Lapeyre (1995)].

From the results obtained - figures[4.2.5] , [4.2.6] and [4.2.7]-, it is apparent that the *random default premium* curves can have different shapes depending on the *transition matrices* used. For instance figure[4.2.5] shows a particular case when "random" spread is above the "average" spread. In figure[4.2.6], "random" spread is almost the "average" spread . Eventually, figure[4.2.7] displays a situation when "average" spread is above the "random" spread . In conclusion, it is not really help-

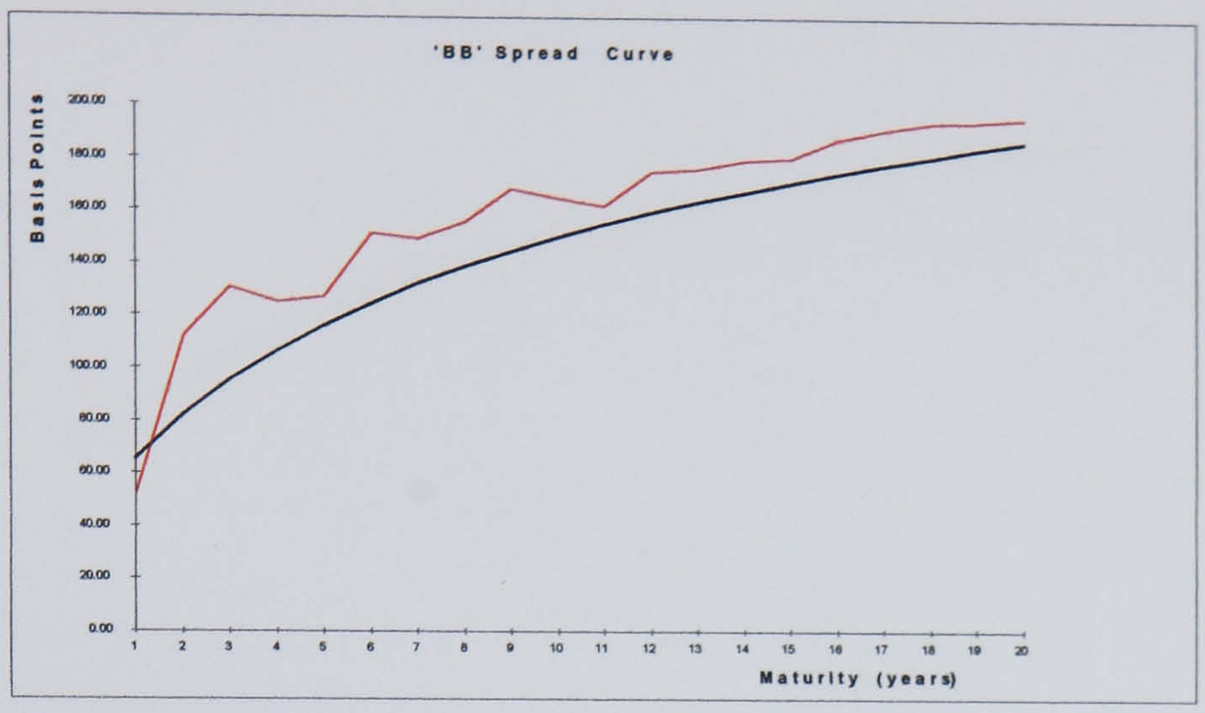


Figure 4.2.5: Stochastic simulation (1)

ful to compute *default premia* using a stochastic *transition matrix* because the results obtained are too different from one another at each random simulation. Therefore, if the use of the historical *transition matrix* does not lead to a reliable curve, it can nevertheless be considered a good enough approximation. A possible solution is to modify the average rating *transition matrix* in order to reflect an investors perception about credit quality in the future. For example it can be decided to use a *transition matrix* that includes more downgrades than the historical norm if the economy is expected to sink into a recession. After having investigated the *default premium*, the *risk premium* and the role it plays in the *credit spread* have to be focused on.

4.3 Credit Curve and Risk Premium

4.3.1 Credit Curves

The shape of the term structure is a result of the investors perception of the evolution of credit risk over time. Therefore, strange as it may seem, the actual degree of credit risk is not the principal component of the credit curve. The credit curve over gilts represents the extra yield required to earn higher return on the gilt market. Computing the credit curve is equivalent to finding the spread between the yield of a risky bond and the yield of an equivalent benchmark.

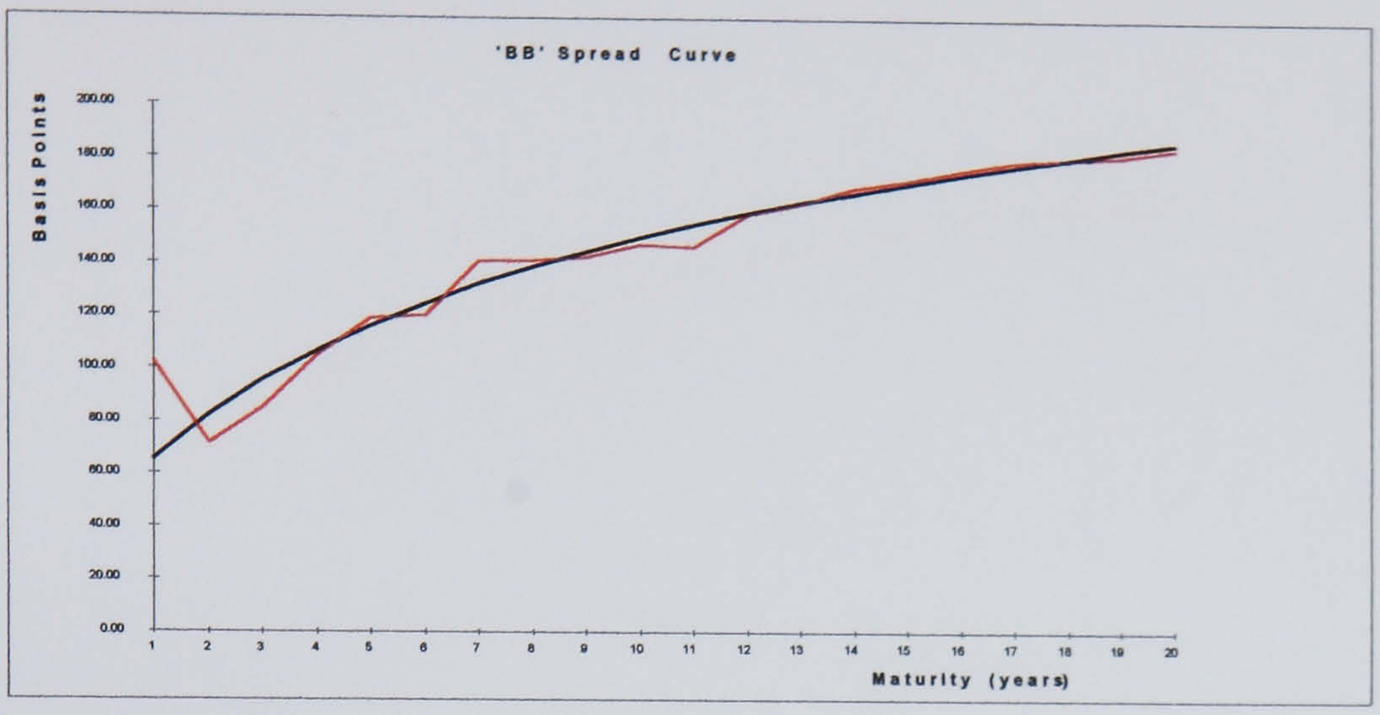


Figure 4.2.6: Stochastic simulation (2)

4.3.2 Risk Premium

The risk premium is defined as the *Yield to Maturity* minus the *Expected yield.* Or, equivalently as the *Credit Spread* minus the *Default Premium.*

After having studied a methodology for computing the *default premium*, and having computed the credit curve, it is then easy to deduce the *risk premium* associated with this credit curve. The shape and the value of the *risk premia* are very useful to investors. A precise estimation of the *risk premium* level enables investors to quantify the risks associated with this premium, and to identify market opportunities.

Basically, the *risk premium* contains three components: The first one are the *specific risks.* These are not critical since holding portfolios of securities whose risks are uncorrelated with each other leads to a reduction of the risks. The second component reflects some other risks such as those related to the taxability of a bond and its liquidity. Finally, the third component is the *systematic market risk.* This is the most important because it cannot be eliminated by portfolio diversification.

Furthermore, the *risk premium* shape tells a lot about the ability of the market to fairly price bonds since it is made up mostly of the premia for liquidity and for the perception of credit risk. For instance, according to some leading credit financial analysts, a flat risk premium means that the market fairly values the *credit risk.* Moreover, the risk premium is expected to be lower for high grade bonds than for low grade ones since liquidity tends to be greater in highly rated credits. If this is not the case, some trading strategies could be advantageously developed.

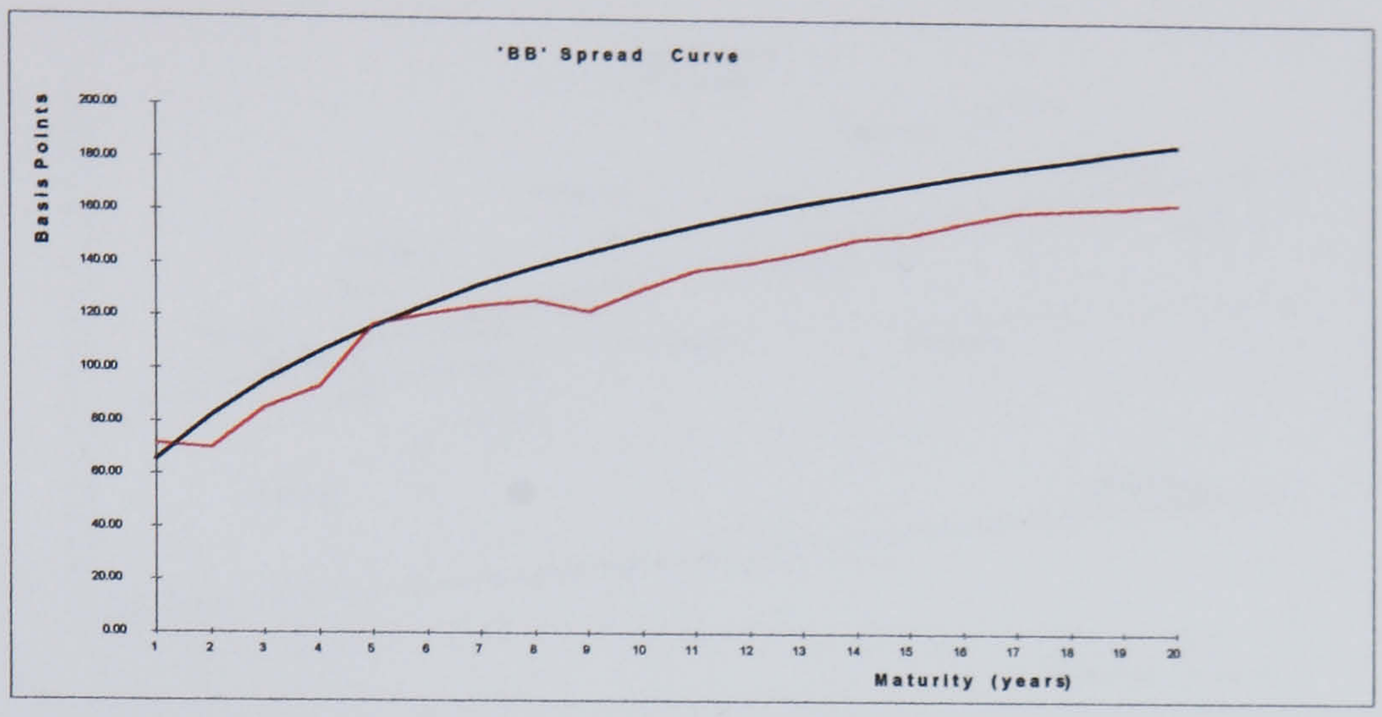


Figure 4.2.7: Stochastic simulation (3)

4.3.3 Results

Moodys ratings have been used whenever possible and bond data as of the close of business on September 24th 1997 (Sterling) and September 16th 1997 (French Franc).

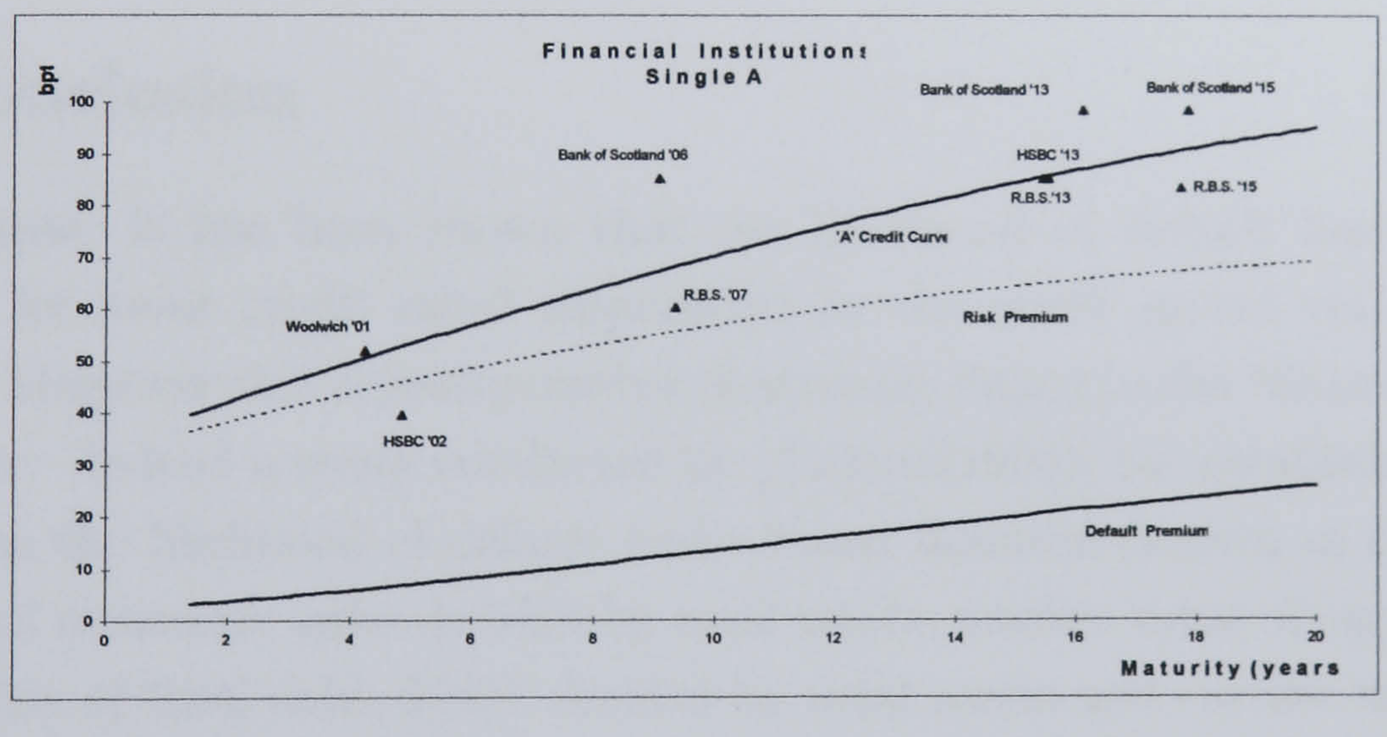


Figure 4.3.1: Euro-Sterlings: Financial Institutions

The results displayed in figures[4.3.1] , [4.3.2] , [4.3.3] and [4.3.4] confirm the two following rules: First, the higher the *credit spread*, the lower the credit rating. Second, bonds with the same credit rating but from different sectors usually do not have similar *credit spreads.*

Figures[4.3.1 and 4.3.2] display the *risk premia* for A-rated Corporate and Finan-

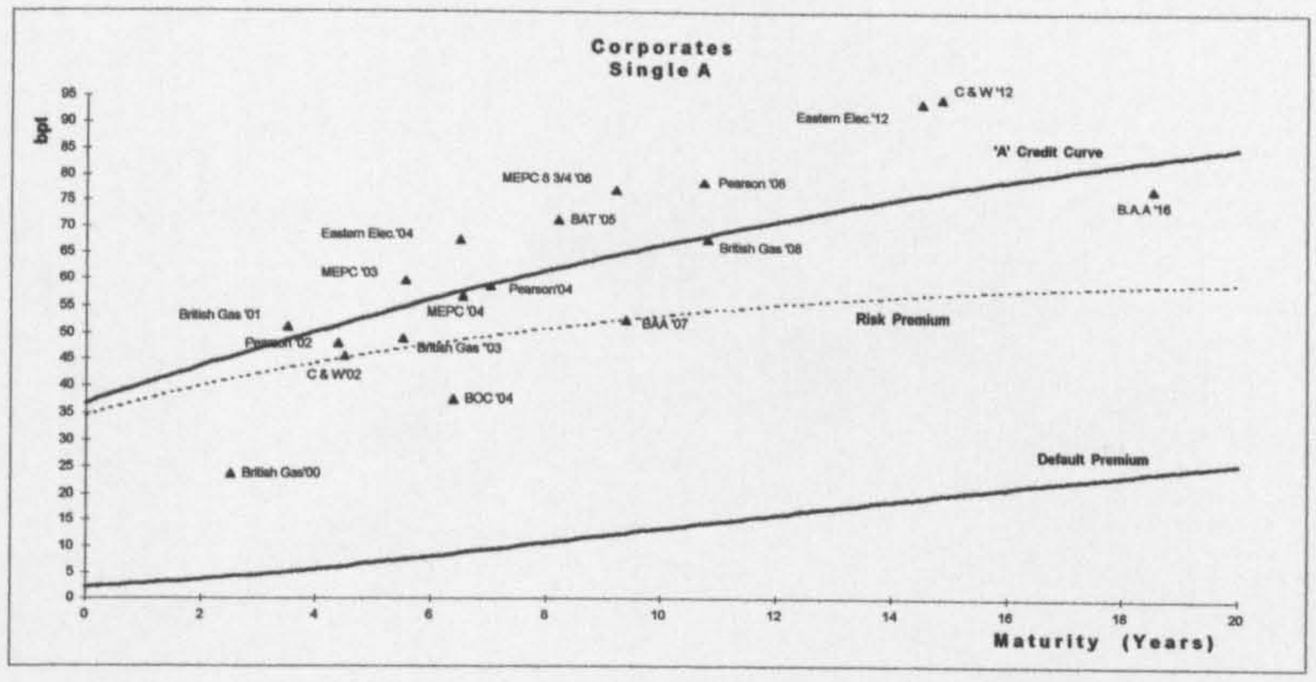


Figure 4.3.2: Euro-Sterlings: Corporates

cial Institution Eurosterling issues. The flatness of the *risk premia* tends to suggest that the sterling market fairly prices *credit risk*. On the opposite, figure[4.3.4] with its downward sloping *risk premium* curve indicates that the market is clearly misvaluing BBB-rated French Franc Bonds (Bond prices are overestimated), particularly longer dated issues.

4.4 Conclusion

In this chapter it has been shown that the likelihood of default has an impact (especially for lower credit rated corporates) on the *credit spread* via the *default premium*. Moreover this *default premium* is strongly linked to the "financial health" of the issuer: Indeed a study conducted by [Altman(1968)] has established a direct link between the likelihood of default and a linear function (known as the Z-score) of 5 financial measures: sales divided by total assets, market value of equity divided by book value of total debt, EBIT divided by total assets and current assets minus current liabilities divided by total assets. The analysis conducted in this chapter enable us to suggest in the next chapters new hedging strategies.

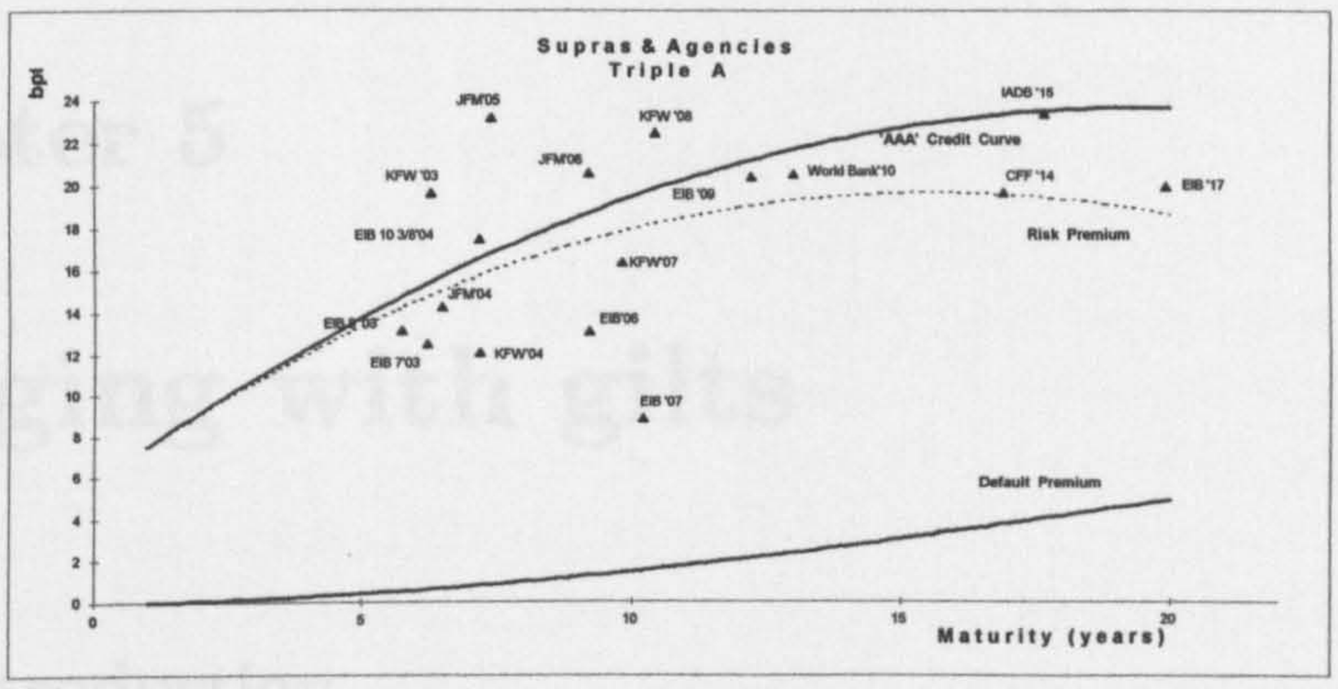


Figure 4.3.3: Euro-Sterlings: Supras

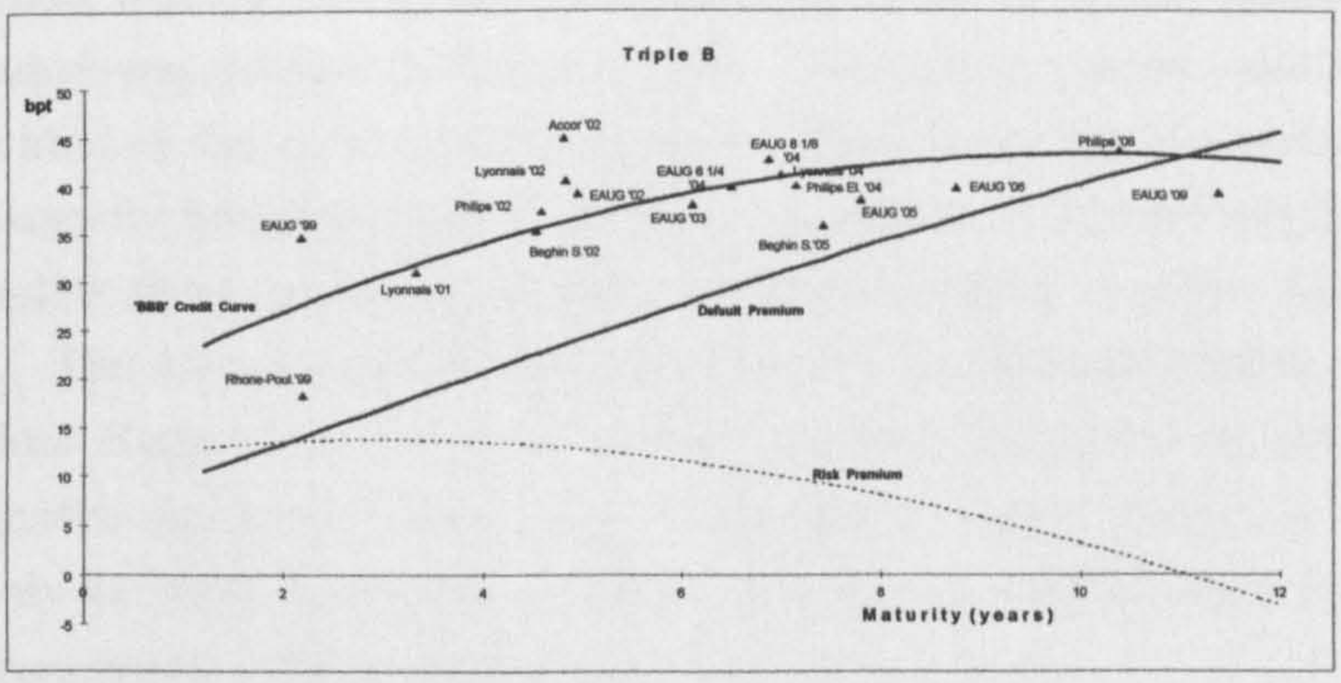


Figure 4.3.4: Euro-Francs: Corporates

Chapter 5

Hedging with gilts

5.1 Introduction

Hedging is the action which consists of taking a position in an asset designed to remove the price risk from another position in a different instrument. It has to be distinguished from the concept of *immunization* . Indeed, immunization is the action which makes the total value of a security (or a portfolio) at the end of a pre-specified investment period equal to the value expected at purchase. The objectives of both techniques (hedging and immunization) are slightly different although they cannot be achieved without using the concept of *duration* that has to be explained also. *Duration* can be viewed as the sensitivity measure of the changes in bond prices to underlying changes in discount rates. For instance equation[2.3.10] can be viewed as a kind of *duration.* One of the many studies proving the usefulness of the *duration* theory for hedging bonds is [Bierwag, Kaufman & Toevs (1983)]. However very few studies about adjusting *duration* for bonds subject to credit risk have been carried out. The matter has already been discussed in subsection[4.2.3]. Moreover, in the previous chapter we have derived some possible sensitivity measures (using a transition matrix approach) that could be applied to bonds subject to credit risk. Unfortunately we think that these *adjusted durations* are not of much help as far as hedging is concerned. The reasons are as follows:

First, [Ilmanen, McGuire & Warga (1994)] has concluded that the error induced by the use of *unadjusted duration* instead of using *default risk adjusted duration* is extremely small as far as investment corporate bonds are concerned.

Secondly, *transition matrices* are not reliable enough especially when applied to

only one bond (and not a large bond portfolio). In addition, estimating the *recovery rate* (cf subsection[4.2.1]) in case of default is an extremely difficult exercise.

Thirdly, as it is explained later, our new hedging strategy requires the *credit spread* to appear in the equations. We would not have this opportunity if we were using directly the *adjusted sensitivity measures.* Indeed, as shown in equation[4.2.8], these sensitivity measures are directly linked to the risk free rate.

The purpose of this chapter and the two following ones is to hedge (not to immunize) over a short period of time an investment grade bond subject to credit risk against risk free interest rate movements and spread movements but not against pure default. Indeed protecting the bond against the pure default risk requires a completely different strategy. Therefore, the use of credit derivatives is not adressed and as far as investment grade bonds (AAA, AA, A, BBB) are considered, the probability of default over a short period of time (hedging period) is very small. More and more financial institutions have been allowed by the Bank of International Settlement (B.I.S.) to use models developed internally in order to hedge interest rate risks (cf [Saunders (1995)] and [Skinner (1996)]). In the past all the financial institutions were required to use the *duration based hedging techniques.* This is the reason why developing new hedging techniques might be very interesting and useful at the same time. As a result, this part of the thesis aims to develop a completely new approach for hedging bonds subject to credit risk. This is a new approach in the sense that:

First, the hedge ratios will incorporate *volatilities* and *correlations* between specific financial instruments and credit spread.

Secondly, the financial hedging instruments that will be used are *gilt*, *stock* and *index.* To the author's knowledge no papers have tried to develop this theory in the past. Traditionnally, hedging bonds consists in taking a short position in the treasury bill futures or treasury bond futures (gilt in England) . In 1981 a famous paper [Kolb & Chiang (1981)] explained how to improve hedging performance using interest rate futures with a *duration based technique.* This method has proven its usefulness (as far as non-default risk bonds were involved) in order to reduce the risk associated with holding a bond[Hill & Schneeweis (1984)].

Through this thesis we will not use the bond futures market but rather the bond spot market. The reason is threefold:

First, when hedging during periods longer than a week, the cost of hedging associated with the bond futures market is not that much higher than the one

associated with the spot market which means that there is no incentive in absolutely choosing the bond futures market rather than the bond spot one.

Secondly, the author completely shares the point of view of [Skinner (1996)] which states that, when using futures, the hedging performance depends solely on which bond is actually delivered. Moreover as it is very well explained in the above quoted paper, investors who are short in futures contracts will obviously choose to deliver the cheapest bond but as a consequence of the interest rates randomness this cheapest bond might not be the same through the hedging period. Therefore it is difficult to derive a consistent hedge ratio except if a forecast of which bond will be the cheapest one to deliver is conducted. If the theory works using the spot market, there is no reasons why it should not work at all using futures (after making the necessary adjustments). [Skinner (1996)] has proved that current hedging can be improved if a corporate hedging instrument was available on the market especially during a recession-recovery stress scenario.

Our objective here is not to prove that the new hedge ratios (that are developed in this thesis) will lead to a perfect hedge but rather **the aim is to evaluate the potentials and the limitations of these new strategies hoping that in the future it can enlarge the investigation field relative to the hedging of bonds subject to credit risk.**

5.2 Rationale

By *risk free interest rate* we mean the interest rate which is not subject to credit risk. When the risk free term structure goes up, then the price of the bond goes down and the converse is true. Indeed it makes sense to short sell a certain quantity of gilts in order to hedge against movements in the credit risk free term structure. The traditional way of hedging a bond with a gilt consists of taking a short position in a certain quantity of gilts. At this stage, nothing is done to hedge against movements of the spread term structure. Basically the well known traditional hedge ratio is the *duration* of the bond to be hedged divided by the *duration* of the hedge instrument, that is to say the gilt. When deriving this hedge, the underlying assumption is that a small variation of the yield to maturity of the gilt induces approximately the same variation of the yield to maturity of the bond to be hedged. Our hedge ratio will differ from the traditional one in the sense that this assumption will not be made.

Moreover, as it is explained in section[5.3], there is practical evidence to support that the credit spread is negatively correlated to the risk free interest rate. Therefore, a quantity of gilts should be bought in order to hedge the movements in credit spread. But as it as been discussed above, in order to hedge the movements of the bond implied only by the risk free rate movements, a quantity of gilt has to be sold. As a result this is a kind of a **contradictory problem**. Therefore the purpose of our new hedge ratio is to determine the best possible trade-off: The strategy consists then in taking a smaller short position in gilts than it is usually required when using the traditional hedge ratio.

In this chapter and the next two chapters, the following notation will be used:

$\mathbf{B_t^{[Z]}}$: is the price at time t of the "risky" bond to be hedged. In order to simplify notations $B_t^{[Z]}$ will be used as well when describing the bond physically. The yield to maturity of $B_t^{[Z]}$ at time t is Z_t.

$\mathbf{B_t^{[Y]}}$: is the price at time t of the gilt used as a hedge instrument. The yield to maturity of $B_t^{[Y]}$ at time t is Y_t.

5.3 Processes

Using equation[2.2.4], modelling the *yield to maturity* of the gilt and the *yield to maturity* of the bond to be hedged is straightforward. The specific forms of the processes to be chosen will be discussed in section[5.6]. Here, a very general model is chosen in order to represent variations of the yield to maturity of the gilt $B_t^{[Y]}$. It is quite flexible in the sense that for example the drift μ_{Y_t} can be chosen to be a function of Y_t (like a mean-reverting process). The same remark can be made regarding the volatility σ_{Y_t} .

$$dY_t = \mu_{Y_t} dt + \sigma_{Y_t} dW_t^{(1)} \tag{5.3.1}$$

Since $B_t^{[Z]}$ and $B_t^{[Y]}$ do not have the same maturity, the same coupon and the same cash-flow pattern, it is not possible to perfectly determine the value of the "true" spread when computing Θ_t which is the yield to maturity of $B_t^{[Z]}$ minus the yield to maturity of $B_t^{[Y]}$.

$$\Theta_t \;=\; Yield\ To\ Maturity\ of\ B_t^{[Z]} - Yield\ To\ Maturity\ of\ B_t^{[Y]}$$

$$= Z_t - Y_t \tag{5.3.2}$$

Therefore Θ_t can be viewed as a kind of "artificial " spread. Considering the fact that $B_t^{[Y]}$ is chosen such that the maturity, the coupon and the duration are similar to those of the bond to be hedged $B_t^{[Z]}$, the artificial spread Θ_t is then a good proxy for the "true" spread. A small variation in the credit spread of the bond is approximated by equation[5.3.3].

$$d\Theta_t = \mu_{\Theta_t} dt + \sigma_{\Theta_t} \rho_t^{[Y,\Theta]} dW_t^{(1)} + \sigma_{\Theta_t} \sqrt{1 - \left(\rho_t^{[Y,\Theta]}\right)^2} dW_t^{(2)} \tag{5.3.3}$$

The correlation between the two processes described above is actually $\rho_t^{[Y,\Theta]}$. One important thing to understand in this section is the introduction of a correlation $\rho_t^{[Y,\Theta]}$ between the *credit spread* Θ_t and the *yield to maturity of the gilt* Y_t. In order to explain why it makes sense to do so, let us consider a low grade bond issued by a company. One might intuitively think that when the yield to maturity of the gilt goes up, the present value of this company's liabilities decreases so that the credit quality should increase (because the company is better off) and as a result, the *default premium* should decrease. Moreover as the company is low rated, an important part of the spread actually consists of the *default premium* as it has been seen in chapter[4]. Finally, the *credit spread* should decrease. The converse should be valid as well which means that there are good chances that $\rho_t^{[Y,\Theta]} \neq 0$ and more specifically $\rho_t^{[Y,\Theta]} < 0$. In fact the intuition seems to be correct since it is supported by some practical studies ([Duffee (1995)] and [Longstaff & Schwartz (1995)]) especially regarding the U.S. economy. One might reasonably think that the results would be similar as far as gilts and eurobonds are concerned (instead of US domestic bonds and treasury bonds). One of the main practical studies was conducted by [Duffee (1995)] who has concluded that "depending on the maturity of the corporate bond and the Treasury maturity, an increase in the Treasury yield of 100 basis points corresponds to a decline in the corporate bond spread of between 6 and 40 basis points." Moreover as previously written, this thesis deals with eurobonds which are known to be almost tax-free bonds. As a result, the correlation should be important because [Duffee (1995)] has also showed that its results have underestimated the correlation because the tax rate faced by the bondholder was important. The yield to maturity of the bond $B_t^{[Z]}$ is actually $Z_t = Y_t + \Theta_t$ and follows the process described below.

$$dZ_t = \left(\mu_{\Theta_t} + \mu_{Y_t}\right) dt + \left(\sigma_{\Theta_t} \rho_t^{[Y,\Theta]} + \sigma_{Y_t}\right) dW_t^{(1)} + \sigma_{\Theta_t} \sqrt{1 - \left(\rho_t^{[Y,\Theta]}\right)^2} dW_t^{(2)} \quad (5.3.4)$$

5.4 Hedge portfolio

5.4.1 Description

A portfolio which contains the risky bond and the gilt will be created. The aim is to determine the optimal numbers of gilts N_t that should be sold in order to hedge one unit of the risky bond. In order to short sell the gilt, the mechanism of repo is used. To have a complete overview about the *repurchase agreement* mechanism, the reader is referred to [Steiner & Pitman (1996)]. A repo consists in selling a security and making a repurchase agreement at the same time. Roughly speaking this means that at time $t = 0$, the *bond trader* will receive a bond from the *repo trader* in exchange for the dirty price of the bond. Then he is able to sell the bond to a *third participant.*

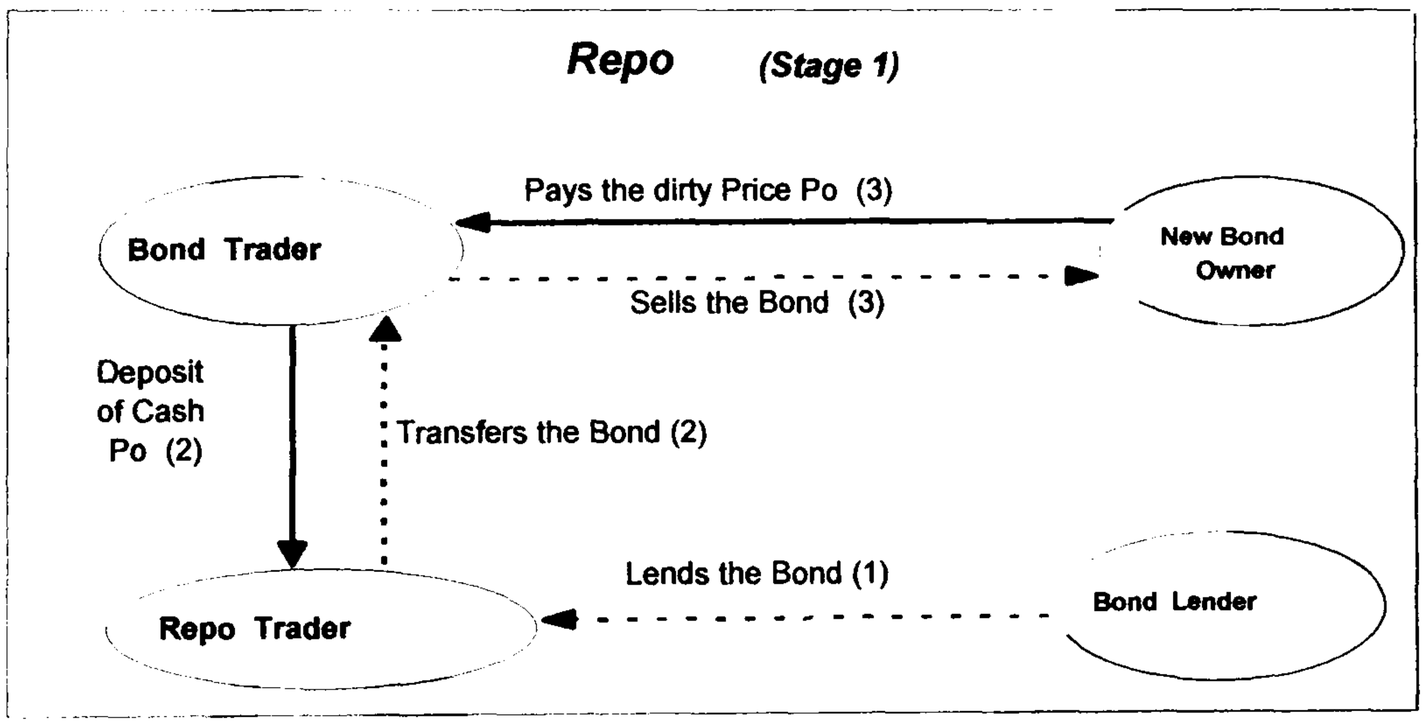


Figure 5.4.1: Repo: Step 1

If some coupons are paid during the process then the *bond lender* must receive them.

When the *bond trader* decides to close his position, two things happen: The *bond trader* buys the bond back in order to give it back to the *repo trader.* Moreover, the *repo trader* will restitute an amount of cash equal to the time zero dirty price of the bond plus some interest. Usually when a trader goes long gilts, he uses the

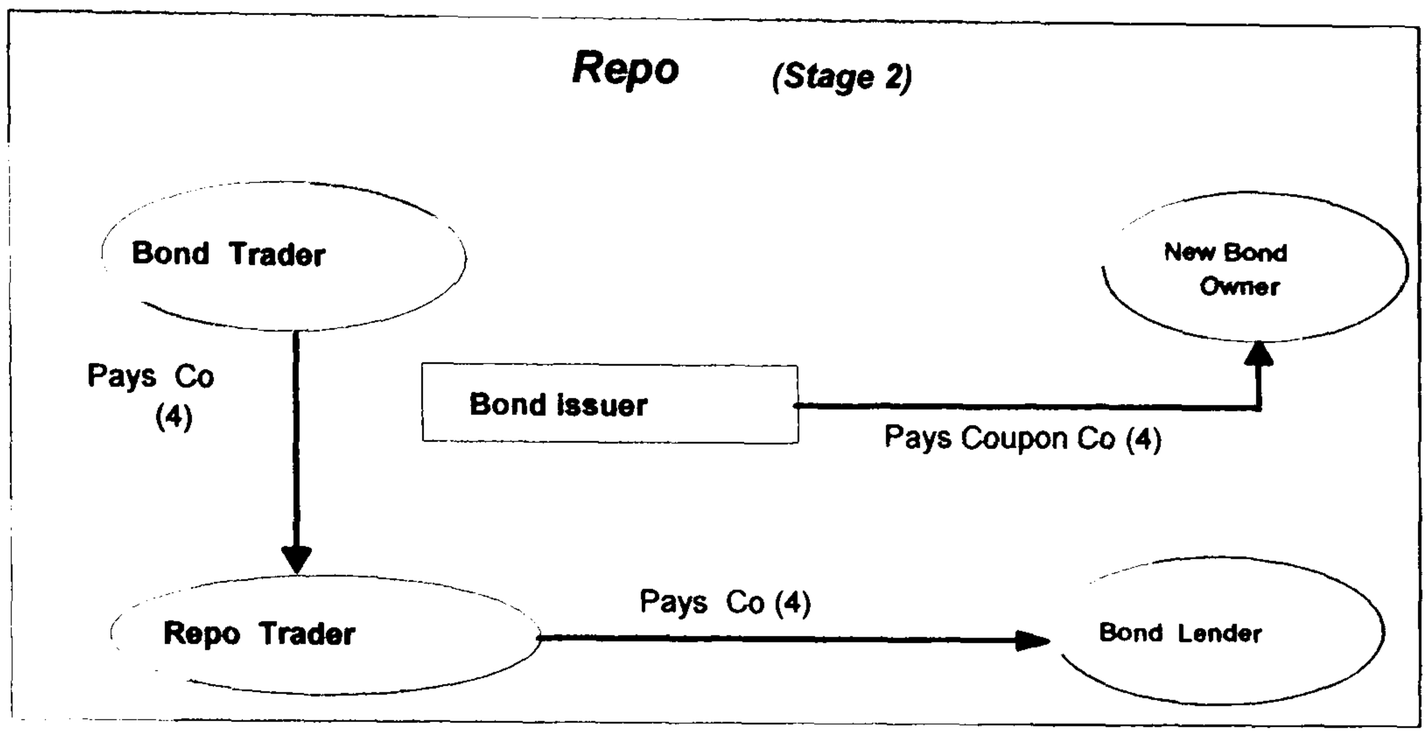


Figure 5.4.2: Repo: Step 2

term *reverse repo* rate as well. This means that he borrows an amount of cash equivalent to the dirty price of the bond from the repo trader. Then he is able to buy the bond which is kept by the *repo trader* (as a guarantee). At maturity of the term reverse repo, the bond is physically given back to the trader in exchange for an amount of cash equal to the time zero dirty price plus some interest. The same kind of mechanism exists now in order to fund a long position for a non default free eurobond but it is not very common.

The following notation will be used:

The "initial wealth of the trader" represented by $\mathbf{\Pi_t}$ is also the value of the portfolio at time t.

Given two dates t_1 and t_2 (with $t_1 < t_2$), the amount of cash at time t_2 which was equivalent to the market price of the security X at time t_1 is denoted $\mathbf{\pounds_{t_2}[X_{t_1}]}$

The bid interest rate related to the term repo at time t is $\mathbf{r_t^{[b]}}$. Moreover $\mathbf{r_t^{[f]}}$ corresponds to a risk free money market instrument rate at time t.

Eventually, the number of gilts that are sold short at time t in order to hedge the bond is $\mathbf{N_t}$.

5.4.2 Investigation

a. Result

A small variation of the initial portfolio can be written as follows:

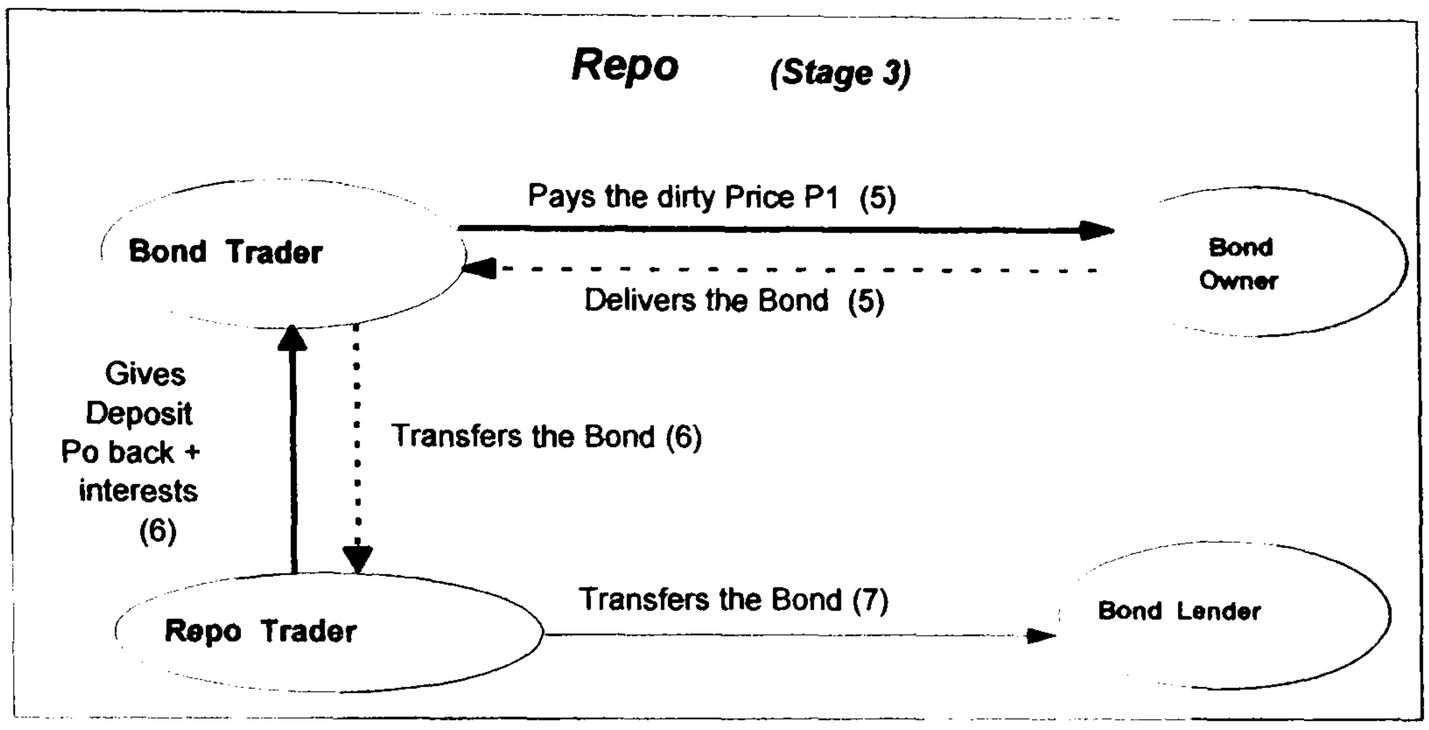


Figure 5.4.3: Repo: Step 3

$$d\Pi_t = m_t(N_t)dt + K_t(N_t, dW_t^{(1)}, dW_t^{(2)}) \tag{5.4.1}$$

$\mathbf{K_t(N_t, dW_t^{(1)}, dW_t^{(2)})}$ represents the non-deterministic and risky part of the portfolio process since it depends solely on the brownian motions. It depends as well on the variable N_t.

$K_t(N_t, dW_t^{(1)}, dW_t^{(2)})$ is equal to

$$\left[D_{Z_t}\left(\sigma_{\Theta_t}\rho_t^{[Y,\Theta]} + \sigma_{Y_t}\right) - N_t D_{Y_t}\sigma_{Y_t}\right] dW_t^{(1)} + \left[D_{Z_t}\sigma_{\Theta_t}\sqrt{1-\left(\rho_t^{[Y,\Theta]}\right)^2}\right] dW_t^{(2)}$$

The drift of the portfolio process is $\mathbf{m_t(N_t)}$ and is defined by:

$$m_t(N_t) = L_t^{(1)} + \Pi_t r_t^{[f]} + NL_t^{(2)}$$

where

$$L_t^{(1)} = \Psi_{Z_t} + D_{Z_t}\left(\mu_{\Theta_t} + \mu_{Y_t}\right) + C_{Z_t}\left[\sigma_{Y_t}^2 + \sigma_{\Theta_t}^2 + 2\sigma_{Y_t}\sigma_{\Theta_t}\rho_t^{[Y,\Theta]}\right] - B_t^{[Z]} r_t^{[f]}$$

$$L_t^{(2)} = -\Psi_{Y_t} - D_{Y_t}\mu_{Y_t} - C_{Y_t}\sigma_{Y_t}^2 + B_t^{[Y]} r_t^{[b]} \tag{5.4.2}$$

Ψ_{Z_t} and Ψ_{Y_t} are derived using equation[2.3.9]; D_{Z_t} and D_{Y_t} are given by equation[2.3.10] C_{Z_t} and C_{Y_t} by equation[2.3.11].

b. Proof

It will be assumed here that the portfolio at *time* t consists of the bond to be hedged $B_t^{[Z]}$ plus some cash. Therefore at *time* t, it is possible to separate the portfolio as follows:

$$\Pi_t = B_t^{[Z]} + \pounds_t \left[\Pi_t - B_t^{[Z]}\right] \tag{5.4.3}$$

$B_t^{[Z]}$ represents the bond to be hedged and $\pounds_t \left[\Pi_t - B_t^{[Z]}\right]$ is the extra cash available to the trader. It has to be emphasised that Π_t is the initial wealth of the trader. Indeed if the trader has no cash but only the security $B_t^{[Z]}$, then $\Pi_t = B_t^{[Z]}$ which does not contradict equation[5.4.3]. Still at *time* t, since a short position in N_t gilts $B_t^{[Y]}$ is taken, the portfolio value increases by the cash value received (due to the sale of the N_t gilts $B_t^{[Y]}$) but decreases by the same amount (N_t gilts $B_t^{[Y]}$ have been borrowed). Therefore equation(5.4.3) is equivalent to:

$$\Pi_t = B_t^{[Z]} - N_t \left(B_t^{[Y]} - \pounds_t \left[B_t^{[Y]}\right]\right) + \pounds_t \left[\Pi_t - B_t^{[Z]}\right] \tag{5.4.4}$$

At *time* $t + dt$, using equation(5.4.4) it is straightforward to write that the value of the portfolio is:

$$\Pi_{t+dt} = B_{t+dt}^{[Z]} - N_t \left(B_{t+dt}^{[Y]} - \pounds_{t+dt} \left[B_t^{[Y]}\right]\right) + \pounds_{t+dt} \left[\Pi_t - B_t^{[Z]}\right]$$

Moreover,

$$\pounds_{t+dt} \left[B_t^{[Y]}\right] = \pounds_t \left[B_t^{[Y]}\right] \left(1 + r_t^{[b]} dt\right)$$

This is the initial cash given to the repo trader plus the interest paid on it.

$$\pounds_{t+dt} \left[\Pi_t - B_t^{[Z]}\right] = \pounds_t \left[\Pi_t - B_t^{[Z]}\right] \left(1 + r_t^{[f]} dt\right)$$

This corresponds to the extra cash that was available to the trader and that has been invested at the risk free rate. The variation of the portfolio's value between t and $t + dt$ is:

$$\Pi_{t+dt} - \Pi_t = B_{t+dt}^{[Z]} - B_t^{[Z]} - N_t \left(B_{t+dt}^{[Y]} - B_t^{[Y]} - \pounds_{t+dt}\left[B_t^{[Y]}\right] + \pounds_t\left[B_t^{[Y]}\right] \right)$$

$$+\pounds_{t+dt}\left[\Pi_t - B_t^{[Z]}\right] - \pounds_t\left[\Pi_t - B_t^{[Z]}\right]$$

The above equation can equivalently be rewritten as:

$$d\Pi_t = dB_t^{[Z]} - N_t \left(dB_t^{[Y]} - \pounds_t\left[B_t^{[Y]}\right] r_t^{[b]} dt \right) + \pounds_t\left[\Pi_t - B_t^{[Z]}\right] r_t^{[f]} dt$$

As previously explained, $\pounds_{t_2}[X_{t_1}]$ is the amount of cash at time t_2 which was equivalent to the market price of the security X at time t_1. As a result, $\pounds_t[X_t] = X_t$ in term of value. It follows immediately that:

$$d\Pi_t = dB_t^{[Z]} - N_t \left(dB_t^{[Y]} - B_t^{[Y]} r_t^{[b]} dt \right) + \left(\Pi_t - B_t^{[Z]} \right) r_t^{[f]} dt \qquad (5.4.5)$$

Or equivalently,

$$d\Pi_t = dB_t^{[Z]} - N_t dB_t^{[Y]} + \left(\Pi_t r_t^{[f]} - B_t^{[Z]} r_t^{[f]} + N B_t^{[Y]} r_t^{[b]} \right) dt$$

Then $dB_t^{[Z]}$ and $dB_t^{[Y]}$ are replaced with their modellings using equation[2.3.8]. Eventually $d\Pi_t$ will be expressed as a function of dY_t, dZ_t, $dZ_t.dZ_t$ and $dY_t.dY_t$. Using equations[5.3.4 and 5.3.1] , the result is derived.

5.5 Optimal strategy

5.5.1 Objective to achieve

Let us assume that we hold a portfolio consisting of only one risky bond. The riskier the bond, the bigger the expected return of the portfolio. The purpose of hedging is to make the risk become as close to zero as possible. Therefore new securities have to be added to the initial portfolio in order to create a new portfolio and reduce the risk. As a result, the expected return of the new portfolio will be lower. A risk free portfolio should have the same rate of return as the risk free rate. Furthermore it makes sense to write that the hedge portfolio's rate of return must be *at least* as large as the risk free interest rate over the same period of time dt. A possible measure of the risk associated with the portfolio is the *variance* of its change in value over a given period of time. Hence the objective to achieve is to find the optimal position ($\widehat{N}_t$) to take in the hedge instruments in order to reduce the risk

(the variance Var) under a constraint which has been previously explained. As a consequence, the optimal quantity $\widehat{\mathbf{N}}_\mathbf{t}$ will be the one which solves the following optimisation problem:

$$\min_{N_t} \left\{ Var\left[d\Pi_t\right] : E\left[d\Pi_t \mid F_t\right] \geq \Pi_t r_t^{[f]} dt \right\} \tag{5.5.1}$$

The optimal $\widehat{N}_t$ which minimises equation[5.5.1] is the number of gilts that should be short-sold in order to hedge the long position in the risky bond. Furthermore it is obvious that the smaller dt , the more efficient the hedge but the more expensive the transaction costs since the portfolio would have to be rebalanced more often.

5.5.2 Optimal hedge ratio

a. Result

The optimal $\widehat{N}_t$ is:

$$\widehat{\mathbf{N}}_\mathbf{t}=\frac{1}{2}\left[-\frac{L_t^{(1)}}{L_t^{(2)}}-\frac{b_t}{u_t}+\frac{L_t^{(2)}}{\left|L_t^{(2)}\right|}\left|\frac{b_t}{u_t}-\frac{L_t^{(1)}}{L_t^{(2)}}\right|\right] \tag{5.5.2}$$

with $L_t^{(1)}$, $L_t^{(2)}$ given by equation[5.4.2] and

$$\mathbf{u}_t = 2D_{Y_t}^2 \sigma_{Y_t}^2$$

$$\mathbf{b}_t = -2D_{Y_t} D_{Z_t} \sigma_{Y_t} \left(\sigma_{\Theta_t} \rho_t^{[Y,\Theta]} + \sigma_{Y_t}\right)$$

b. Proof

The variance of the process portfolio process is $Var\left[d\Pi_t\right]$ which is actually (using equation(5.4.1)) equal to:

$$Var\left[K_t(N_t, dW_t^{(1)}, dW_t^{(2)})\right]$$

Some basic stochastic calculus computations give:

$$Var\left[K_t(N_t, dW_1, dW_2)\right]$$

$$= \left[\left[D_{Z_t} \left(\sigma_{\Theta_t} \rho_t^{[Y,\Theta]} + \sigma_{Y_t} \right) - N_t D_{Y_t} \sigma_{Y_t} \right]^2 + D_{Z_t}^2 \sigma_{\Theta_t}^2 \left(1 - \left(\rho_t^{[Y,\Theta]} \right)^2 \right) \right] dt$$

$$= \left[D_{Y_t}^2 \sigma_{Y_t}^2 N_t^2 - 2 D_{Y_t} D_{Z_t} \sigma_{Y_t} \left(\sigma_{\Theta_t} \rho_t^{[Y,\Theta]} + \sigma_{Y_t} \right) N_t + D_{Z_t}^2 \left(\sigma_{\Theta_t}^2 + \sigma_{Y_t}^2 + 2 \sigma_{\Theta_t} \sigma_{Y_t} \rho_t^{[Y,\Theta]} \right) \right] dt$$

Therefore $Var\left[d\Pi_t\right]$ can be expressed as a standard quadratic form:

$$Var\left[d\Pi_t\right] = \left(a_t + b_t N_t + \frac{1}{2} u_t N_t^2 \right) dt$$

where

$$a_t = D_{Z_t}^2 \left(\sigma_{\Theta_t}^2 + \sigma_{Y_t}^2 + 2 \sigma_{\Theta_t} \sigma_{Y_t} \rho_t^{[Y,\Theta]} \right)$$

$$b_t = -2 D_{Y_t} D_{Z_t} \sigma_{Y_t} \left(\sigma_{\Theta_t} \rho_t^{[Y,\Theta]} + \sigma_{Y_t} \right)$$

$$u_t = 2 D_{Y_t}^2 \sigma_{Y_t}^2 > 0$$

The optimal $\widehat{N}_t$ will be determined by solving $\begin{cases} \min\limits_{N_t} Var\left[d\Pi_t\right] \\ \text{such that } h(N_t) \leq 0 \end{cases}$

which is equivalent to $\begin{cases} \min\limits_{N_t} a_t + b_t N_t + \frac{1}{2} u_t N_t^2 \\ \text{such that } h(N_t) \leq 0 \end{cases}$

It is easy to check that $a_t + b_t N_t + \frac{1}{2} u_t N_t^2$ is a quadratic.
The constraint is equivalent to:

$$m_t(N_t) = L_t^{(1)} + \Pi_t r_t^{[f]} + N_t L_t^{(2)} \geq \Pi_t r_t^{[f]}$$

or

$$h(N_t) = -L_t^{(1)} - N_t L_t^{(2)} \leq 0$$

From the particular problem to solve, it is obvious that there will only be two possibilities concerning the general form of $h(N_t)$ depending on the value of $L_t^{(2)}$.

Let us consider α_t such that

$$\alpha_t = -\frac{L_t^{(1)}}{L_t^{(2)}}$$

Case 1 (when $L_t^{(2)} > 0$) :

$$h(N_t) = \alpha_t - N_t$$

$\alpha_t \leq N_t$

$$\widehat{N}_t = \frac{1}{2}\left[\alpha_t - \frac{b_t}{u_t} + \left|\alpha_t + \frac{b_t}{u_t}\right|\right]$$

This comes from the fact that if $\alpha_t \leq -\frac{b_t}{u_t}$ then $\widehat{N}_t = -\frac{b_t}{u_t}$ or if $-\frac{b_t}{u_t} \leq \alpha_t$ then $\widehat{N}_t = \alpha_t$

Case 2 (when $L_t^{(2)} < 0$) :

$$h(N_t) = N_t - \alpha_t$$

$N_t \leq \alpha_t$

$$\widehat{N}_t = \frac{1}{2}\left[\alpha_t - \frac{b_t}{u_t} - \left|\alpha_t + \frac{b_t}{u_t}\right|\right]$$

This comes from the fact that if $-\frac{b_t}{u_t} \leq \alpha_t$ then $\widehat{N}_t = -\frac{b_t}{u_t}$ or if $\alpha_t \leq -\frac{b_t}{u_t}$ then $\widehat{N}_t = \alpha_t$

Given the parameter values of the modelled processes, there will be only one case (among the two above cases) that will fit our optimisation problem.

So that:

$$\widehat{\mathbf{N}}_t = \frac{1}{2}\left[-\frac{L_t^{(1)}}{L_t^{(2)}} - \frac{b_t}{u_t} + \varepsilon\left(L_t^{(2)}\right)\left|\frac{b_t}{u_t} - \frac{L_t^{(1)}}{L_t^{(2)}}\right|\right] \tag{5.5.3}$$

with

$$\varepsilon\left(L_t^{(2)}\right) = \begin{cases} +1 \text{ if } L_t^{(2)} > 0 \\ -1 \text{ if } L_t^{(2)} < 0 \end{cases}$$

5.6 Implementation and results

5.6.1 Hedge ratios 1 and 2

The optimal hedge ratio is given by equation[5.5.2]. This equation has been derived in a rigorous way since it assumes that the expected return of the portfolio is at least as large as the risk-free interest rate. In practice it might be irrelevant to use this

equation (which incorporates the values of the drifts) for the following reasons:

First, equation[5.5.2] implies that the drifts of the processes (equations[5.3.3 and 5.3.1]) have to be calculated. Assuming some arbitrary models and using historical data in order to compute actual drifts (sign of the drifts and their values) will probably lead to big errors. On the other hand, computing volatilities from historical data is really more reliable and gives a good estimator of the current level of risk.

Secondly, and fortunately, it can be fully assumed that, over a short period of time, the price movements implied by the drifts can be neglected compared to those implied by the volatility:

$$m_t^2(N_t)dt << a_t + b_t N_t + \frac{1}{2} u_t N_t^2 \tag{5.6.1}$$

A sufficient condition is to prove that there exists a positive real number $K_t >> 1$ such that:

$$f_K(N_t) = a_t + b_t N_t + \frac{1}{2} u_t N_t^2 - K_t m_t^2(N_t)dt > 0 \qquad \text{with } K_t >> 1$$

The methodology to prove that such a positive real number exists (and consequently that assumption[5.6.1] holds) is as follows one: Given the bond to be hedged and the hedge instrument, the parameters (a_t, b_t, u_t, $L_t^{(1)}$, $L_t^{(2)}$ ...) are computed from historical data assuming some specific models. Replacing $m_t(N_t)$ with $L_t^{(1)} + \Pi_t r_t^{[f]} + N_t L_t^{(2)}$ (cf equation[5.4.1]), it turns out that $f_K(N_t)$ is a quadratic form. It is then not very difficult to find a positive real number K such that $K >> 1$ and such that the coefficient related to N_t^2 is positive. As a result, $f_K(N_t)$ has only one minimum which can be expressed numerically. Replacing all the parameters with their values it appears that this minimum is bigger than zero which eventually proves that the assumption[5.6.1] is correct. It seems important to us to check the assumption[5.6.1] because under certain circumstances (for instance when C_{z_t} is very big) this assumption may not hold.

As a consequence, after neglecting the effects of the drift in all the previous equations, the hedge ratio will simply become:

$$\widehat{\mathbf{N}}_t = \frac{D_{Z_t}}{D_{Y_t}} \left(1 + \frac{\sigma_{\Theta_t}}{\sigma_{Y_t}} \rho_t^{[Y,\Theta]} \right) \tag{5.6.2}$$

It is important to notice that since $\frac{D_{Z_t}}{D_{Y_t}}$ can be viewed as the traditional duration

based hedge ratio $\widehat{\mathbf{N}}_t^*$, equation[5.6.2] can be rewritten as:

$$\boxed{\widehat{\mathbf{N}}_t = \left(1 + \frac{\sigma_{\Theta_t}}{\sigma_{Y_t}} \rho_t^{[Y,\Theta]}\right) \widehat{\mathbf{N}}_t^*} \tag{5.6.3}$$

This hedge ratio will be called "*Hedge Ratio 1* ".

a. Assumptions

Since short term interest rates are usually modelled as mean reverting processes, one might think that it could be relevant to model the yield of a bond in the same way. In order to do so, the methodology explained in chapter[3] can be applied. Unfortunately the results obtained gave some unreasonable values (for the mean reverting process parameters). This means that the bond yield cannot actually be approximated by a mean reverting process. In addition, some more complex models (than the Vasiceck model) have been tested without success. For instance we tested the model:

$$dY_t = a_Y \left[b_Y - Y_t\right] dt + \sum_{k=1}^{q} Y_t^{\beta} \sigma_k^{(Y)} dW_t^{(k)}$$

$$\text{with} \quad \beta = \frac{1}{2} \quad \text{or} \quad \beta = 1$$

As a consequence, a log-normal model will be chosen. Obviously it may not be the best model to choose but the purpose of this thesis is not to find the best modelling as far as the yield is concerned but rather to investigate new techniques of hedging. The *credit spread* is modelled with a normal process.

In order to compute volatilities and correlations from historical data, a 30 day historical "window" has been chosen. That is to say that at any given time t, volatilities and correlations at this time are computed using the 30 previous daily data. Moreover it is assumed that there are 253 trading days within one year. The technique used here is fully described in chapter[3]. Due to the possible inacurracy of the modellings chosen here, only correlations with absolute values bigger than 0.5 will be taken into account. In other words, if the absolute value of a given correlation is less than 0.5, then the correlation is considered to be zero.

b. High grade investment Bonds about to mature

For very high grade bonds, the spread is almost zero. Therefore,

$$\rho_t^{[Y,\Theta]} \simeq 0$$

and $\widehat{\mathbf{N}}_\mathbf{t}$ becomes

$$\widehat{\mathbf{N}}_t^* = \frac{D_{Z_t}}{D_{Y_t}}$$

This traditional hedge ratio is derived under the assumption that $dZ_t \simeq dY_t$. In other words it is assumed that the correlation between dZ_t and dY_t is

$$\rho_t^{[Y,Z]} \simeq 1$$

However under special circumstances, from the author's point of view, if the bond to be hedged is due to mature within three years, then the correlation between the yield to maturity of this bond and the yield to maturity of the gilt is far from one. One explanation of this behavior is that the gilt and the bond do not have the same maturity (despite the fact that the gilt is selected so that its maturity is as close as possible as the one of the bond to be hedged). In fact this difference between maturites has a very small impact on the correlations when the remaining life of the two bonds is big enough but can have a very big impact when the remaining life is less than three years. This means that correlations between yields (still for a triple A bond and a gilt) are then far from one. Morover the price of the bond must converge to the *repayment price* (usually 100) . The figure[5.6.1] which displays the correlation between yields supports the above discussion.

The bond that is investigated is *UNILEVER* which is a AAA bond. The coupon frequency is annual, the coupon's value is fixed at $7\frac{3}{8}\%$. The first settlement date was June 1993 and the bond expired in December 1998. Moreover this was a non-callable bond. The corresponding gilt (the one that matches as close as possible the maturity and coupon) is a semi-anual fixed coupon at $7\frac{1}{4}\%$. The first settlement date was October 1992 and the gilt expired in March 1998.

For these reasons it seems appropriate to derive a specific hedge ratio (designed for high grade bonds whose remaining time to maturity is less than 3 years) which incorporates the correlation between the yield to maturity of the bond and the yield to maturity of the gilt. By analogy to the derivation of the first hedge ratio, and replacing equation[5.3.3] with

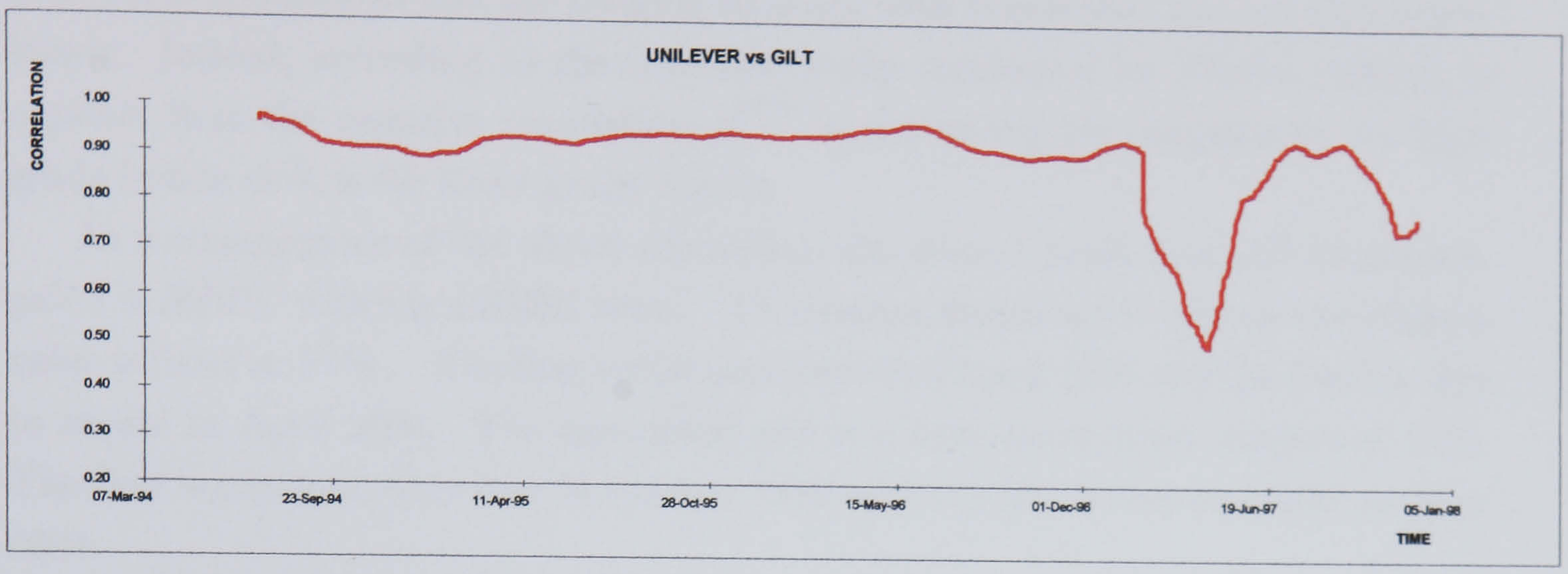


Figure 5.6.1: Correlation between yields

$$dZ_t = \mu_{Z_t} dt + \sigma_{Z_t} \rho_t^{[Y,Z]} dW_t^{(1)} + \sigma_{Z_t} \sqrt{1 - \left(\rho_t^{[Y,Z]}\right)^2} dW_t^{(2)}$$

it can be shown that the optimal hedge ratio becomes

$$\widehat{\mathbf{N}} = \frac{D_{Z_t}}{D_{Y_t}} \frac{\sigma_{Z_t}}{\sigma_{Y_t}} \rho_t^{[Y,Z]}$$

or equivalently

$$\boxed{\widehat{\mathbf{N}} = \frac{\sigma_{Z_t}}{\sigma_{Y_t}} \rho_t^{[Y,Z]} \widehat{\mathbf{N}}_t^*}$$

This hedge ratio will be called "*Hedge Ratio 2* ".

c. Choice of the Bonds to be hedged

Some investment bonds are chosen. Following the discussion regarding the high grade bonds about to mature, the *UNILEVER* bond will be tested in the hedging strategy.

The specificity of *hedge ratio 1* is that it incorporates a correlation $\rho_t^{[Y,\Theta]}$ between the spread and the yield to maturity of the gilt. A characteristic of the equation[5.6.3] is that when $\rho_t^{[Y,\Theta]} = 0$ *hedge ratio 1* becomes the same as the *traditional hedge ratio*. In order for *hedge ratio 1* to use as much as of its poten-

tialities, it is better to test the hedging strategy with bonds that are not high grade bonds. Indeed, according to the practical study conducted by [Duffee (1995)], it appears that the negative correlation $\rho_t^{[Y,\Theta]}$ is not as big (in magnitude) for high grade bonds as it is for lower grade bonds.

As a consequence of the above discussion, the second bond that will be investigated is *MEPC* which is a BBB1 bond. The coupon frequency is annual, the coupon value is fixed at $9\frac{7}{8}\%$. The first settlement date was April 1988 and the bond is due to expire in April 2004. The associated gilt is a semi-anual fixed coupon at 10%. The first settlement date was November 1985 and the gilt is due to expire in May 2004.

5.6.2 Hedging performance

a. Performance Measures

Measure 1

The hedging theory is tested using historical data but it has to be emphasised that the first concern while testing the theory must be to avoid introducing biases. Indeed it is far better, and it makes more sense, to back-test the hedging strategy with historical data than to do a simulation since it would require us to assume some specific models (to describe the bond price process for instance). In order not to get a bias, at any given time t, only historical data anterior to this time will be used when computing the hedge ratio (i.e computing volatilities and correlations of the different processes).

The performance measure will be a single variable called the RISK. Let us consider a τ day period of time. At any given time t in the past, the wealth of the trader is given by Γ_t. After τ days the wealth of the trader becomes $\Gamma_{t+\frac{\tau}{M}}$ (with M the number of trading days in one year). Therefore at time $t + \frac{\tau}{M}$ the trader will have won or lost r_t (as a percentage of the initial wealth, depending on the sign):

$$r_t = 100\left(\frac{\Gamma_{t+\frac{\tau}{M}} - \Gamma_t}{\Gamma_t}\right)$$

As a consequence, after τ days the magnitude (in absolute value) of what the trader would have lost or won as a percentage of the initial wealth is given by a single

variable called RISK:

$$RISK_t = 100 \left| \frac{\Gamma_{t+\frac{\tau}{M}} - \Gamma_t}{\Gamma_t} \right| \tag{5.6.4}$$

As a matter of fact, the purpose of hedging is to make an outcome more certain that is to say to make the variable $RISK$ as low as possible. For every single trading day it is possible to compute the RISK variable since historical data are available. This means that it is possible to know exactly what the outcomes of the securities would have been τ days after the hedge is initiated. Then it is straightforward to compare the different RISK variables obtained when hedging or not.

Measure 2

The second measure R_t that can be defined is a consequence of the first one. It is expressed as the percentage of the risk that has been hedged.

$$R_t = 100 \frac{RISK_t \text{ without hedging } - RISK_t \text{ with hedging}}{RISK_t \text{ without hedging}}$$

The bigger R_t , the better the hedging strategy. Indeed a perfect hedge would make R_t become 100%. This measure R_t will be used later.

b. Results

The life of the hedge is chosen to be 4 weeks. The figures obtained represent actually a 120 days moving average of the RISK variable.

As far as *UNILEVER* is concerned two different periods can be distinguished.

The first period (September 1994 - October 1995) : It seems that the *traditional duration based hedge ratio* performs better than *hedge ratio 2.* This can be explained by the fact that theoretically, correlations between yields are extremely high and the way they are computed from historical data assuming log-normal model is not fully satisfactory.

The second period (September 1994 - October 1995) : It seems that *hedge ratio 2* performs better than the *traditional duration based hedge ratio.* In fact the remaining life of the bond is less than three years. The correlations are then far from perfect and even if the models assumed here are not totally reliable, introducing volatilities and correlation in the hedge ratio seems to lead to better results which is an improvement.

Furthermore, *hedge ratio 2* has been tested as well on another AAA bond

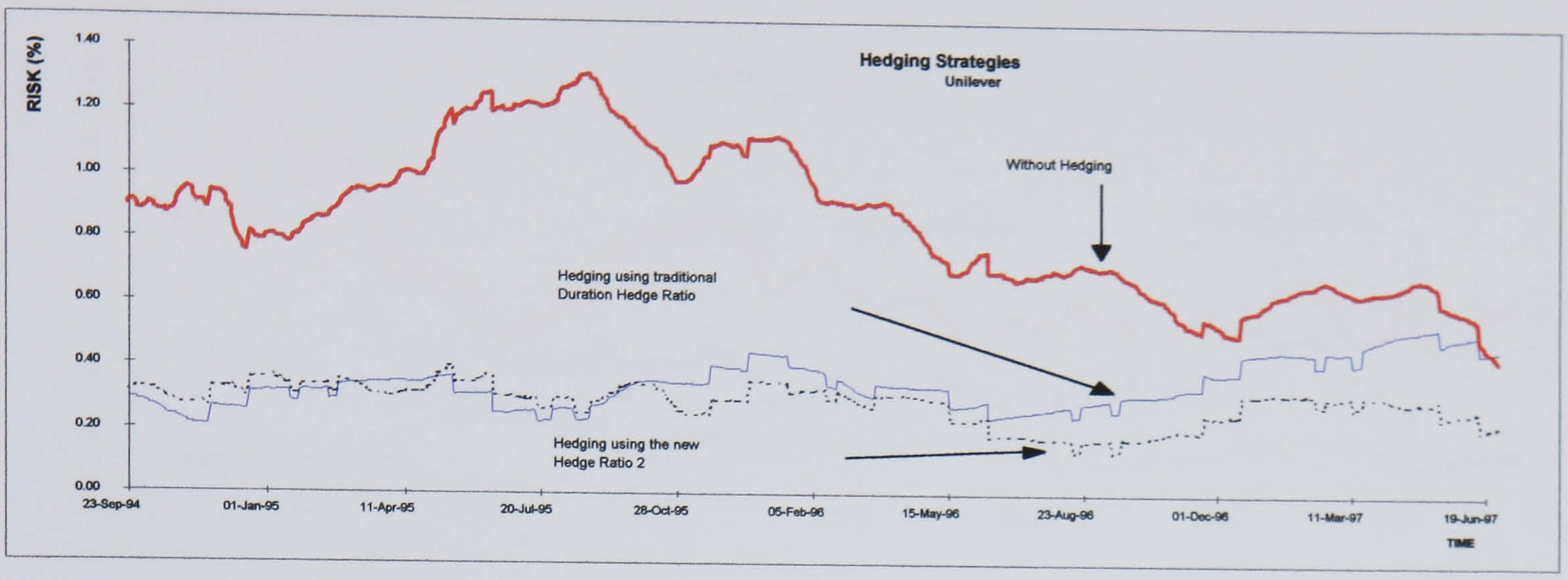


Figure 5.6.2: UNILEVER: Hedging Strategy (Hedge Ratio 2)

(*MARKS & SPENCER* $7\frac{3}{8}$) whose remaining time to maturity was less than three years at some point. The same results were obtained.

As far as *MEPC* (cf figure[5.6.3]) is concerned, *hedge ratio 1* performs worse than the *traditional duration based hedge one.* This is actually not a surprise: The *traditional hedge ratio* hedges almost completely the "risk free interest rate" part of any given bond. However, according to equation[5.6.3] and because $\rho_t^{[Y,\Theta]} < 0$, it appears that the strategy using the *hedge ratio 1* consists of taking a smaller short position in gilts than it is required when using the *traditional hedge ratio.* This is illustrated in figure[5.6.4]: With the *traditional hedge ratio* stategy, all the entire "yield to maturity component" of $B_t^{[Z]}$ is hedged by $B_t^{[Y]}$ since there is a perfect correlation as shown in figure[5.6.4]. On the other hand, with the *hedge ratio 1* strategy, a correlation between the "spread component" of $B_t^{[Z]}$ and the yield to maturity of $B_t^{[Y]}$ is taken into account. As a consequence, *hedge ratio 1* cannot entirely hedge all the "risk free interest rate" part of the the bond. However what is important to understand is that this is a trade-off because selling less gilts means that the risk associated with the "credit spread" is less important than it would have been if more gilts were sold (because of the negative correlation between the yield to maturity of the gilt and the spread). What is lost on one side is won on the other side. Moreover, it appears that with the bonds that have been tested, $\sigma_{\Theta_t} \cong \sigma_{Y_t}$. In other word, the *hedge ratio 1* strategy should theoretically be sensibly equivalent to the traditional one. In practice the two strategies are not equivalent and this can be explained by the fact that the computation of the correlation $\rho_t^{[Y,\Theta]}$ and volatilities

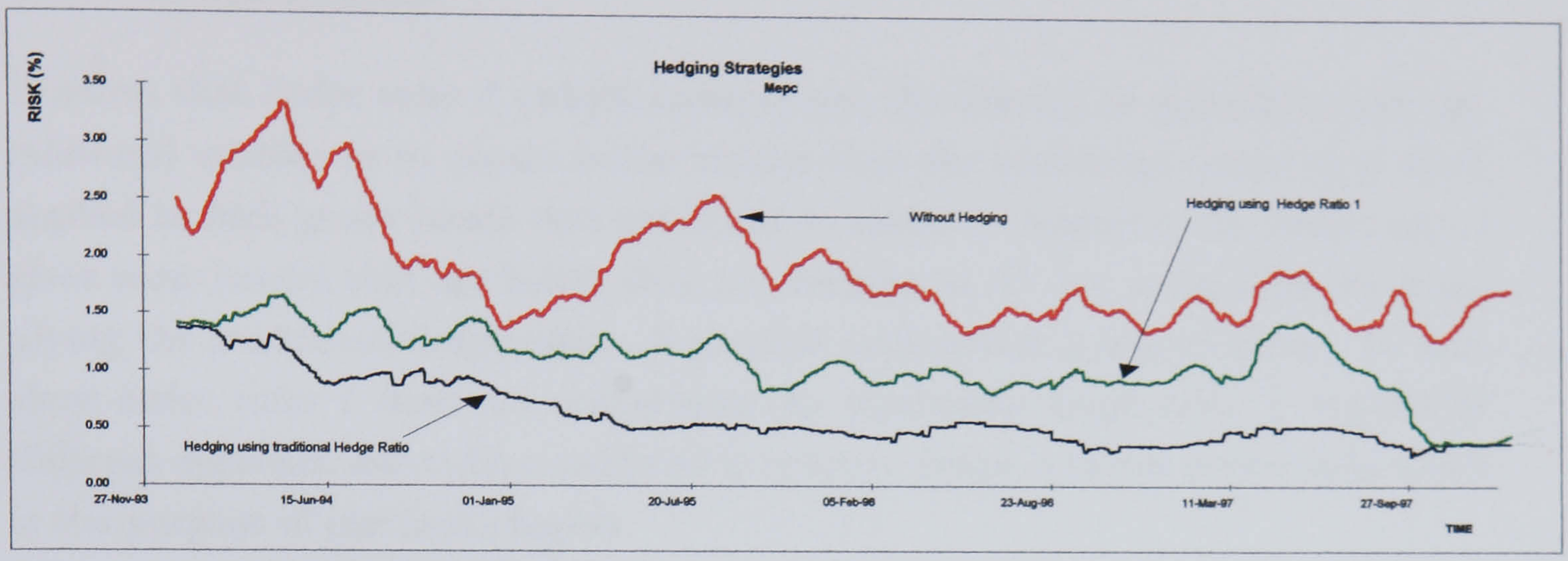


Figure 5.6.3: MEPC: Hedging Strategy (Hedge Ratio 1)

are required (for *hedge ratio 1*) and this is not very reliable since we have to assume some arbitrary models for the yield to maturity of the gilt and for the spread.

Eventually, because of this "trade-off problem" making the assumption in the calculus that

$$\rho_t^{[Y,\Theta]} = 0 \tag{5.6.5}$$

can only lead to better results as far as the hedging stategy is considered.

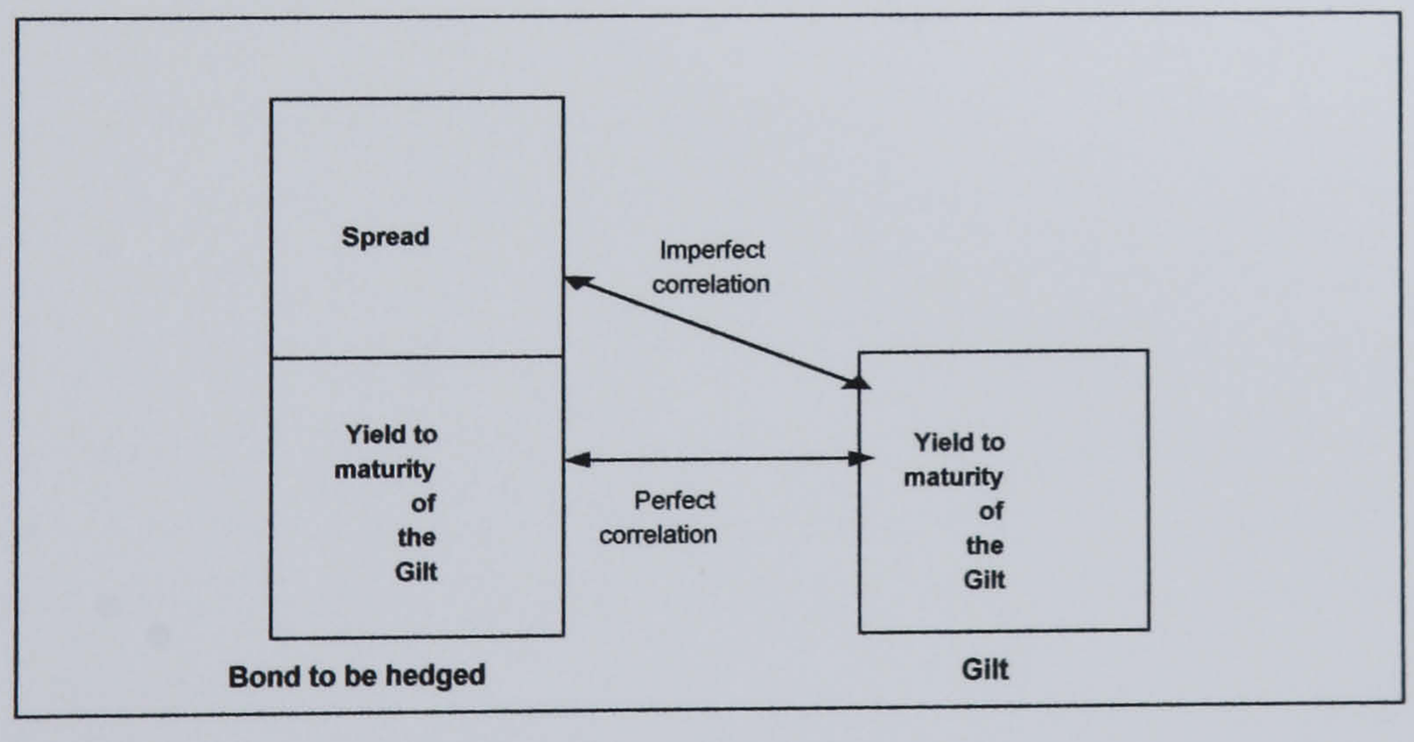


Figure 5.6.4: Trade-off problem

5.7 Conclusion

It seems that *hedge ratio 2* (which incorporates the concept of volatilities and correlations) enables us to obtain better results than the *traditional hedge ratio* when applied to high grade bonds that are about to mature. Moreover the *hedge ratio 1* gives some results that are better than not hedging at all, but worse than when applying the *traditional hedge ratio*. A possible explanation has been given. In fact, since *hedge ratio 1* does not outperform the *traditional hedge ratio*, a completely different approach has to be considered in order to devise a better hedge ratio which is the purpose of our next chapter.

Chapter 6

Hedging with gilts and stocks

6.1 Introduction

The objective of this chapter is to derive a new hedge ratio which performs better (as far as hedging is concerned) than the *traditional hedge ratio* and *hedge ratio 1*. In this chapter we add to the hedge portfolio a short position in a quantity of *stocks* of the company which issued the bond to be hedged. The rationale for using stocks is to protect the bondholder against price movements due to changes in the magnitude of the spread. The *default premium* movements can be seen as a response to new information related to the company or the country it is located in. Moreover, a change in the *credit rating* of the issuer has a direct impact on the bond price since the level of the *credit spread* is dependent on the likelihood of default as it is explained in chapter[4]. In addition, according to [Sharpe, Alexander, Bailey] a study conducted by [Fisher (1959)] has established a linear relation between the *credit spread* (for 366 bonds) and four measures: earnings variability, time without default, equity divided by debt ratio and market value of debt. The conclusion of this study was that this linear relationship explained approximately 75% of the *credit spread* variation. Therefore the *credit spread* is strongly linked to the "financial health" or equivalently to the *stock* price. Furthermore, [Weinstein (1977)], [Hand, Holthausen & Leftwich (1992)] and [Goh & Ederington (1993)] have studied the effect of bond rating changes on bond and stock prices. The conclusion is twofold:

First, bond and stock prices tend to move many months before the announcement date of the rating changes since the market is able to anticipate a downgrade or an

upgrade.

Secondly, it appears that globally *bond* and *stock* prices respond positively to upgrades and negatively to downgrades which is another fact that enables us to consider that the *credit spread* and stock prices are *negatively correlated* (especially for low credit rated corporates) .

In addition, [Weinstein (1985)] has clearly shown that investment grade corporate bonds exhibit positive stock market betas.

This was confirmed by [Cornwell & Green (1991)] who demonstrated that it was even more true for low grade bond mutual funds.

Hence considering this correlation, it appears that using *stock* as hedging instrument could possibly reduce the risk due to big movements of the *credit spread.* Therefore, short selling some *stocks* intuitively seems to be logical.

Some other *credit spread* movements are due to changes in the risk free interest rate curve since the *credit spread* is correlated to the risk free rates: The matter has already been discussed in chapter[5]. It is recalled here that by "risk free rates", we mean default free rates and by "risky bond", we mean a bond subject to credit risk.

In this chapter, the following notation is used:

$\mathbf{B_t^{[Z]}}$: is the price at time t of the "risky" bond to be hedged. In order to simplify the notations $B_t^{[Z]}$ will also be used in describing the bond physically. The yield to maturity of $B_t^{[Z]}$ at time t is Z_t.

$\mathbf{B_t^{[Y]}}$: is the price at time t of the gilt used as a hedge instrument. The yield to maturity of $B_t^{[Y]}$ at time t is Y_t.

6.2 Processes

The processes are the processes for which it is easy to find some accurate historical data. They are the *yield to maturity of the gilt*, the artificial *credit spread* and the *stock*. Obviously a correlation between the *stock* and the *credit spread* has to be introduced . Despite the fact that the other possible correlations might be very small, it is dealt with the general case. The stock considered here corresponds to the shares of the corporate which has issued the bond $B_t^{[Z]}$. The stock will be modelled as a log-normal process. Using the same notation as the one in chapter[2], we write:

$$X_t = \begin{bmatrix} X_t^{[1]} \\ X_t^{[2]} \\ X_t^{[3]} \end{bmatrix} = \begin{bmatrix} S_t \\ \Theta_t \\ Y_t \end{bmatrix} \qquad W_t = \begin{bmatrix} W_t^{(1)} \\ W_t^{(2)} \\ W_t^{(3)} \end{bmatrix} \qquad \mu_{X_t} = \begin{bmatrix} \mu_{x_t^{[1]}} \\ \mu_{x_t^{[2]}} \\ \mu_{x_t^{[3]}} \end{bmatrix} = \begin{bmatrix} \mu_{S_t} \\ \mu_{\Theta_t} \\ \mu_{Y_t} \end{bmatrix}$$

and

$$\sigma_t = \begin{bmatrix} \sigma_{x_t^{[1]}} \\ \sigma_{x_t^{[2]}} \\ \sigma_{x_t^{[3]}} \end{bmatrix} = \begin{bmatrix} \sigma_{S_t} \\ \sigma_{\Theta_t} \\ \sigma_{Y_t} \end{bmatrix} \qquad \Delta_{t,3} = \begin{bmatrix} 1 & \rho_t^{\Theta,S} & \rho_t^{Y,S} \\ \rho_t^{\Theta,S} & 1 & \rho_t^{Y,\Theta} \\ \rho_t^{Y,S} & \rho_t^{Y,\Theta} & 1 \end{bmatrix}$$

Using the above notation and equation[2.2.4], it is then possible to write:

$$dX_t^{(i)} = \mu_{x_t^{[i]}} dt + \sigma_{x_t^{[i]}} \sum_{k=1}^{3} \frac{\left(\prod_{m=0}^{\max(k-2,0)} \sqrt{\det \Delta_{t\lfloor m}} \right)^{\delta(i,k)}}{\sqrt{\prod_{m=0}^{k-\delta(i,k)} \det \Delta_{t\lfloor m}}} \left[\det \left(\varphi^{(i-k-1)} \circ \mathsf{m}_{i,k}(\Delta_{t\lfloor i}) \right) \right]^{1-\frac{\delta(i,k)}{2}} dW_t^{(k)} \tag{6.2.1}$$

and this system[6.2.1] can be decomposed into three equations, the first one being:

$$dS_t = \mu_{S_t} dt + \sigma_{S_t} dW_t^{(1)} \tag{6.2.2}$$

Equation[6.2.2] represents the *stock* process with its drift μ_{S_t} and its volatiliy σ_{S_t}.

The two other processes described below are the *credit spread* and the *yield to maturity of the gilt*. The modelling of these three processes is done in such a way that it allows some correlations between each of the different processes. The rationale for the modelling of a correlation between the *credit spread* and the *yield to maturity of the gil*t has been explained in section[5.3].

Equation[6.2.3] describes the artificial *credit spread* process with its drift μ_{Θ_t} and its volatiliy σ_{Θ_t}. It is obtained using equation[6.2.1].

$$d\Theta_t = \mu_{\Theta_t} dt + \sigma_{\Theta_t} \left[\left| \mathsf{m}_{2,1} \left(\Delta_{t\lfloor 2} \right) \right| dW_t^{(1)} + \sqrt{\left| \Delta_{t\lfloor 2} \right|} dW_t^{(2)} \right] \tag{6.2.3}$$

The rationale for the modelling of a correlation between the *stock* and the *yield to maturity of the gilt* is quite easy to understand: First, since the price of the *stock* can be traditionally viewed as the present value of all the future dividends, the correlation $\rho_t^{Y,S}$ should in theory be negative. Secondly, [Fama & French (1989)] have empirically showed that expected returns on common stocks are correlated to interest rates. However, this correlation might be positive on rare occasions when many investors decide to shift from the stock market to the bond market or vice versa. Equation[6.2.4] is the modelling of the *yield to maturity process* with the drift being μ_{Y_t} and the volatility σ_{Y_t}.

$$dY_t = \mu_{Y_t} dt + \sigma_{Y_t} \left[\left| \varphi \circ \mathsf{m}_{3,1} \left(\Delta_t \right) \right| dW_t^{(1)} + \frac{\left| \mathsf{m}_{3,2} \left(\Delta_t \right) \right|}{\sqrt{\left| \Delta_{t\lfloor 2} \right|}} dW_t^{(2)} + \sqrt{\frac{\left| \Delta_t \right|}{\left| \Delta_{t\lfloor 2} \right|}} dW_t^{(3)} \right] \tag{6.2.4}$$

Intuitively, as discussed in section[6.1] the correlation between the *stock* and the *credit spread* should be negative. It has to be understood that this might not be always the case since the *yield to maturity* of the gilt is negatively correlated to the *credit spread* (cf discussion in section[5.3]) and to the *stock* (cf discussion above).

6.3 Optimal hedge portfolio

6.3.1 Description

A portfolio which contains the *risky bond*, the *stock* and the *gilt* is created. It has to be understood that the portfolio does not actually contain some *stocks* and *gilts* but rather a short position is taken in these securities. The aim is to determine the optimal numbers of *gilts* and *shares* $\left(\text{respectively } N_t^{[1]} \text{ and } N_t^{[2]}\right)$ that should be short-sold in order to hedge one unit of the risky bond at time t.

First, let us consider a universe where the risk free interest rate term structure does not move. Any important announcement regarding the business of the company (for instance uncertainties about the sale of a subsidiary...) will induce

movements in the *credit spread* and in the *stock* such that the correlation between these movements is negative. Therefore, in order to hedge movements (not linked to risk free term structure) of the *credit spread* it seems appropriate to sell short a quantity of *stocks* of the same company.

Secondly, let us consider a universe where the only news are related to the movements of the risk free term structure. If the *yield to maturity of the gilt* moves in one direction, then the *credit spread* (cf [Duffee (1995)]) and the *stock* (cf [Fama & French (1989)]) move together in the opposite direction. This would imply that the *credit spread* and the *stock* are positively correlated. Therefore, in order to hedge movements (linked to risk free term structure) of the *credit spread* it would seem logical to buy a quantity of *stocks*. But as the *stock* is negatively correlated to the *yield to maturity of the gilt*, the quantity of *stock* that is bought will make the risk free part of the bond more difficult to be hedged. This is a **contradiction** like the one we met in the previous chapter. A discussion about how to minimise practically the impact of this contradictory problem follows.

For all of these reasons we can reasonably think that only *credit spread* movements which are not linked to risk free interest term structure shifts seem to be hedgeable (which occurs when $\rho_t^{\Theta,S} < 0$).

In order for the process of short-selling stock to be as realistic as possible, the model presented here will allow the trader to use a margin account with his broker. The mecanism is briefly explained below.

Some new notations have to be added: The interest rate on the initial margin deposit at time t is denoted by $\mathbf{r_t^{[s]}}$. The proportion of the purchase price that must come from the trader's own fund at time t is denoted by $\mathbf{x_t^{[s]}}$. All other notations are the same as those in previous sections. In order to better analyse the mechanism of short-selling, let us consider a portfolio Π_t consisting of a short position in a stock. The short-selling process can be viewed as a three-stage process described in figures[6.3.1], [6.3.2] and [6.3.3]:

It is important here to emphasize that the stock lender must receive all the dividends paid during the process.

At the end, the stock lender gets the stock back.

The mechanism explained above can be translated into equations:

The meaning of the $\pounds_t$ symbol is explained in the previous chapter.

At *time* t, it is possible to decompose the portfolio as follows:

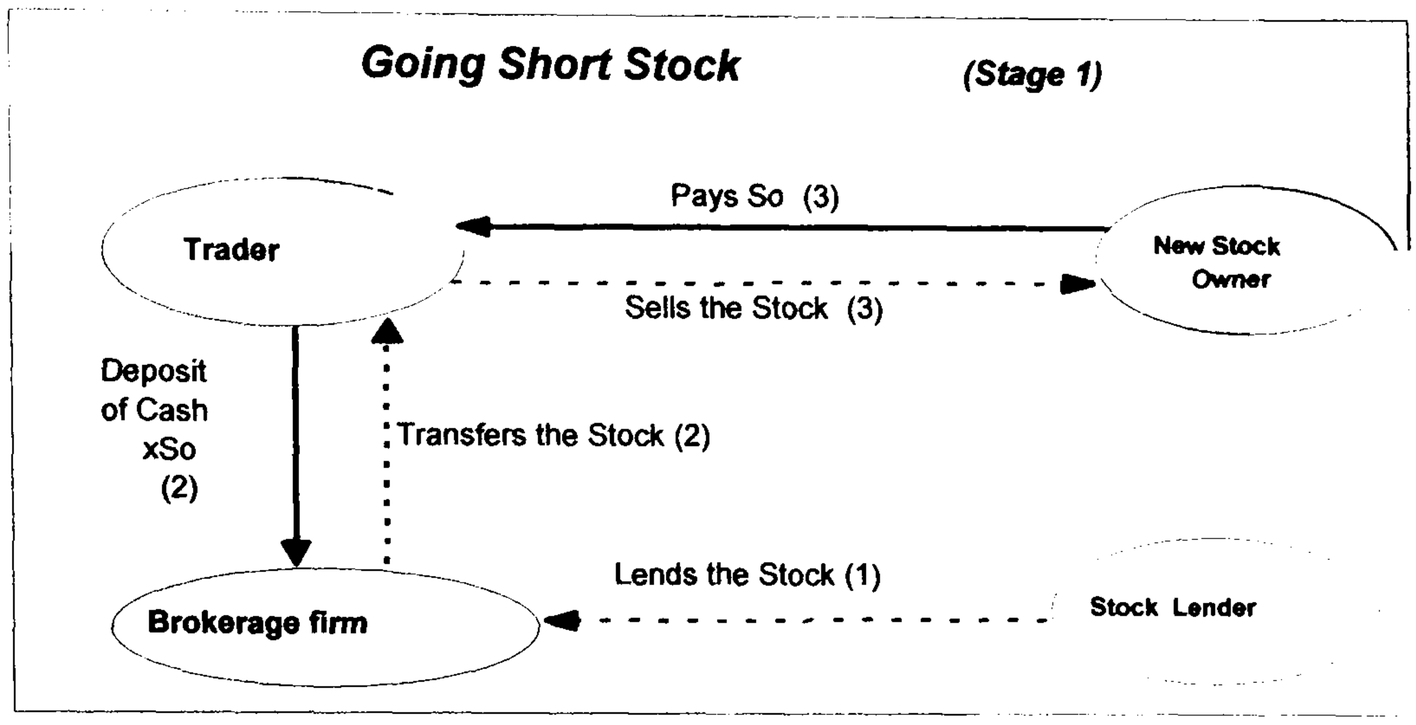


Figure 6.3.1: Going Short Stock: Step 1

$$\Pi_t = -N_t S_t + \pounds_t \left[N_t S_t\right] + \pounds_t \left[x_t^{[s]} N_t S_t\right] + \pounds_t \left[\Pi_t - x_t^{[s]} N_t S_t\right]$$

At *time* $t + dt$, the portfolio becomes

$$\Pi_{t+dt} = -N_t S_{t+dt} + \pounds_{t+dt} \left[N_t S_t\right] + \pounds_{t+dt} \left[x_t^{[s]} N_t S_t\right] + \pounds_{t+dt} \left[\Pi_t - x_t^{[s]} N_t S_t\right]$$

Moreover, as it is assumed that the trader invests the money (received from the sale of the stock) at the risk free rate,

$$\pounds_{t+dt} \left[N_t S_t\right] = \pounds_t \left[N_t S_t\right] \left(1 + r_t^{[f]} dt\right)$$

Because of the interest earned on initial margin deposit, it is possible to write

$$\pounds_{t+dt} \left[x_t^{[s]} N_t S_t\right] = \pounds_t \left[x_t^{[s]} N_t S_t\right] \left(1 + r_t^{[s]} dt\right)$$

The available cash is invested at the risk free rate. Therefore,

$$\pounds_{t+dt} \left[\Pi_t - x_t^{[s]} N_t S_t\right] = \pounds_t \left[\Pi_t - x_t^{[s]} N_t S_t\right] \left(1 + r_t^{[f]} dt\right)$$

Hence, over a short period of time dt , the variation of the portfolio is

$$d\Pi_t = -N_t dS_t + N_t S_t x_t^{[s]} r_t^{[s]} dt + \left[\Pi_t + \left(1 - x_t^{[s]}\right) N_t S_t\right] r_t^{[f]} dt \qquad (6.3.1)$$

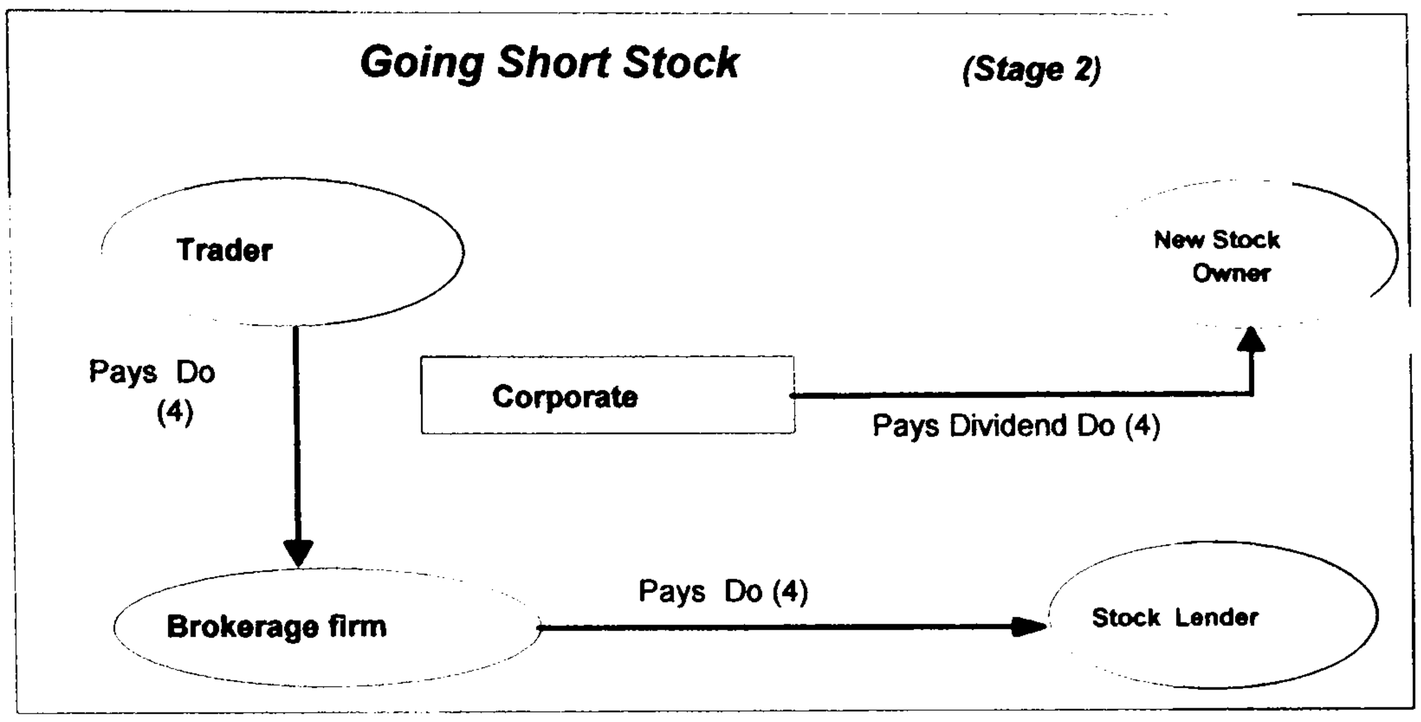


Figure 6.3.2: Going Short Stock: Step 2

6.3.2 Investigation

a. Result

A small variation in the portfolio Γ_t consisting of the bond to be hedged, some gilts and some stocks can be written as follows:

$$d\Gamma_t = m_t(N_t^{[1]}, N_t^{[2]})dt + K_t(N_t^{[1]}, N_t^{[2]}, dW_t^{(1)}, dW_t^{(2)}, dW_t^{(3)}) \tag{6.3.2}$$

with a deterministic component $\mathbf{m_t}(\mathbf{N_t^{[1]}}, \mathbf{N_t^{[2]}})$ such that

$$m_t(N_1, N_2) = L_t^{(0)} + \Gamma_t r_t^{[f]} + N_t^{[1]} L_t^{(1)} + N_t^{[2]} L_t^{(2)}$$

and a pure stochastic one $\mathbf{K_t}(\mathbf{N_t^{[1]}}, \mathbf{N_t^{[2]}}, \mathbf{dW_t^{(1)}}, \mathbf{dW_t^{(2)}}, \mathbf{dW_t^{(3)}})$ such that

$$K_t\left[(N_i)_{i=1,2}, (dW_j)_{j=1,3}\right] = \begin{bmatrix} Q_t^{[1,0]} + N_t^{[1]} Q_t^{[1,1]} + N_t^{[2]} Q_t^{[1,2]} \\ Q_t^{[2,0]} + N_t^{[1]} Q_t^{[2,1]} \\ Q_t^{[3,0]} + N_t^{[1]} Q_t^{[3,1]} \end{bmatrix}^T \begin{bmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \end{bmatrix}$$

Where

$$\mathbf{Q_t^{[1,0]}} = D_{Z_t}\left(\sigma_{Y_t} + \sigma_{\Theta_t}\right) \qquad \mathbf{Q_t^{[1,1]}} = -D_{Y_t}\sigma_{Y_t} \qquad \mathbf{Q_t^{[1,2]}} = -\sigma_{S_t}$$

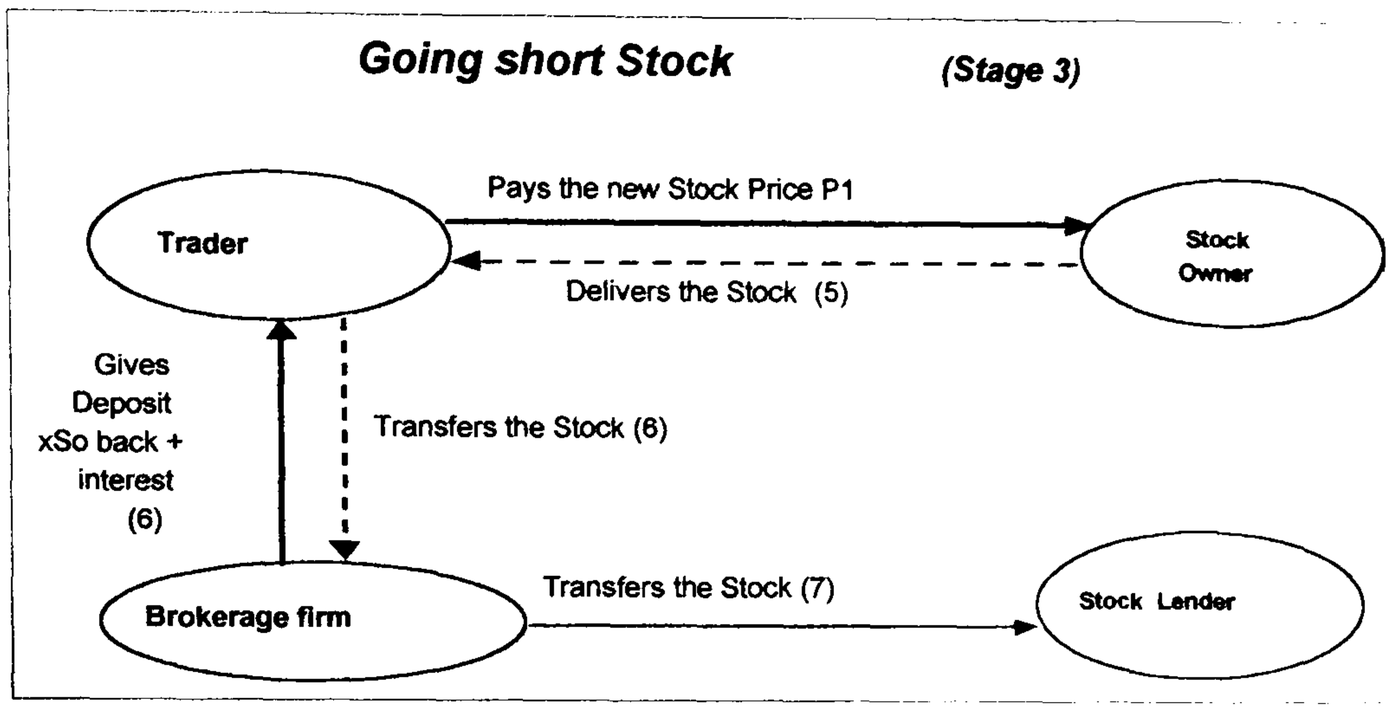


Figure 6.3.3: Going Short Stock: Step 3

$$\mathbf{Q_t^{[2,0]}} = \frac{\sigma_{Y_t}\left|M_{3,2}\left(\Delta_t\right)\right|}{\sqrt{\left|\Delta_{t\lfloor 2}\right|}} + D_{Z_t}\sigma_{\Theta_t}\sqrt{\left|\Delta_{t\lfloor 2}\right|} \qquad \mathbf{Q_t^{[2,1]}} = \frac{-D_{Y_t}\sigma_{Y_t}\left|M_{3,2}\left(\Delta_t\right)\right|}{\sqrt{\left|\Delta_{t\lfloor 2}\right|}}$$

$$\mathbf{Q_t^{[3,0]}} = D_{Z_t}\sigma_{Y_t}\sqrt{\frac{\left|\Delta_t\right|}{\left|\Delta_{t\lfloor 2}\right|}} \qquad \mathbf{Q_t^{[3,1]}} = -D_{Y_t}\sigma_{Y_t}\sqrt{\frac{\left|\Delta_t\right|}{\left|\Delta_{t\lfloor 2}\right|}}$$

$$\mathbf{L_t^{(0)}} = \Psi_{Z_t} + C_{Z_t}\left[\sigma_{Y_t}^2 + \sigma_{\Theta_t}^2 + 2\sigma_{\Theta_t}\sigma_{Y_t}\rho_t^{Y,\Theta}\right] + D_{Z_t}\left(\mu_{\Theta_t} + \mu_{Y_t}\right) - B_t^{[Y]}r_t^{[f]}$$

$$\mathbf{L_t^{(1)}} = -\Psi_{Y_t} - C_{Y_t}\sigma_{Y_t}^2 - D_{Y_t}\mu_{Y_t} + B_t^{[Y]}r_t^{[b]}$$

$$\mathbf{L_t^{(2)}} = -\mu_{S_t} + S_t\left(r_t^{[f]} + x_t^{[s]}r_t^{[s]} - x_t^{[s]}r_t^{[f]}\right)$$

b. Proof

The same methodology as in chapter[5] can be used in order to obtain $d\Gamma_t$ as a function of the different securities involved in the portfolio.

$$\begin{aligned} d\Gamma_t &= dB_t^{[Z]} - N_t^{[1]}dB_t^{[Y]} - N_t^{[2]}dS_t \\ &\quad + \left[N_t^{[1]}B_t^{[Y]}r_t^{[b]} + N_t^{[2]}S_t(r_t^{[f]} + x_t^{[s]}\left(r_t^{[s]} - r_t^{[f]}\right)) + \Gamma_t r_f - B_t^{[Z]}r_t^{[f]}\right]dt \end{aligned}$$

Then, $dB_t^{[Z]}$ and $dB_t^{[Y]}$ are replaced with their expressions using equation[2.3.8] and dS_t with expression[6.2.2]. Eventually $d\Gamma_t$ will be expressed as a function of dZ_t, dY_t and dS_t. Moreover from equation[5.3.2], it turns out that $d\Gamma$ can be expressed as a function of dY_t , $d\Theta_t$, dS_t. Replacing dY_t , $d\Theta_t$ and dS_t with their respective expressions, the result is then derived.

6.3.3 Optimal hedge ratio

a. Result

It is possible to obtain a closed form solution for the optimal $\widehat{N}_t = \begin{bmatrix} \widehat{N}_t^{[1]} \\ \\ \widehat{N}_t^{[2]} \end{bmatrix}$

The optimal $\widehat{N}_t$ is:

$$\widehat{\mathbf{N}}_t = \frac{1}{2} U_t^{-1} A_t^T \left(\frac{A_t U_t^{-1} P_t + Z_t}{A_t U_t^{-1} A_t^T} - \left| \frac{A_t U_t^{-1} P_t + Z_t}{A_t U_t^{-1} A_t^T} \right| \right) - U_t^{-1} P_t \qquad (6.3.3)$$

with

$$\mathbf{U_t} = \begin{bmatrix} D_{Y_t}^2 \sigma_{Y_t}^2 & D_{Y_t} \sigma_{S_t} \sigma_{Y_t} \rho_t^{Y,S} \\ \\ D_{Y_t} \sigma_{S_t} \sigma_{Y_t} \rho_t^{Y,S} & \sigma_{S_t}^2 \end{bmatrix} \quad \text{and} \quad \mathbf{A_t^T} = \begin{bmatrix} -L_t^{(1)} \\ \\ -L_t^{(2)} \end{bmatrix}$$

$$\mathbf{P_t} = \begin{bmatrix} -D_{Z_t} D_{Y_t} \sigma_{Y_t}^2 - D_{Z_t} D_{Y_t} \sigma_{\Theta_t} \sigma_{Y_t} \rho_t^{Y,\Theta} \\ \\ -D_{Z_t} \sigma_{Y_t} \sigma_{S_t} \rho_t^{Y,S} - D_{Z_t} \sigma_{\Theta_t} \sigma_{S_t} \rho_t^{\Theta,S} \end{bmatrix} \quad \text{and} \quad \mathbf{Z_t} = L_t^{(0)}$$

b. Proof

By definition, the variance of a small variation of the portfolio process is

$$Var\left[d\Gamma_t\right] = Var\left[K_t(N_t, dW_t)\right]$$

with

$$N_t = \begin{bmatrix} N_t^{[1]} \\ \\ N_t^{[2]} \end{bmatrix} \quad \text{and} \quad dW_t = \begin{bmatrix} dW_t^{(1)} \\ \\ dW_t^{(2)} \\ \\ dW_t^{(3)} \end{bmatrix}$$

Moreover if we define the variable G_t such that

$$G_t = \begin{bmatrix} Q_t^{[1,0]} \\ \\ Q_t^{[2,0]} \\ \\ Q_t^{[3,0]} \end{bmatrix} + \begin{bmatrix} Q_t^{[1,1]} & Q_t^{[1,2]} \\ \\ Q_t^{[2,1]} & 0 \\ \\ Q_t^{[3,1]} & 0 \end{bmatrix} N_t$$

then it is possible to write

$$Var\left[d\Gamma_t\right] = E\left[\left(G_t^T dW_t\right)\left(G_t^T dW_t\right)^T\right]$$

Moreover, since

$$E\left[dW_t \left(dW_t\right)^T\right] = I_{3\times 3}$$

it turns out that

$$Var\left[d\Gamma_t\right] = G_t^T G_t$$

After some algebra, $Var\left[d\Gamma_t\right]$ becomes equal to

$$\sum_{k=1}^{3}\left(Q_t^{[k,0]}\right)^2 + 2\begin{bmatrix} \sum_{k=1}^{3} Q_t^{[k,0]} Q_t^{[k,1]} \\ \\ Q_t^{[1,0]} Q_t^{[1,2]} \end{bmatrix}^T N_t + N_t^T \begin{bmatrix} \sum_{k=1}^{3}\left(Q_t^{[k,1]}\right)^2 & Q_t^{[1,1]} Q_t^{[1,2]} \\ \\ Q_t^{[1,1]} Q_t^{[1,2]} & \left(Q_t^{[1,2]}\right)^2 \end{bmatrix} N_t$$

Eventually,

$$Var\left[d\Gamma_t\right] = 2\left[a_t + P_t^T N_t + \frac{1}{2} N_t^T U_t N_t\right] dt$$

where

$$a_t = \frac{1}{2}\sum_{k=1}^{3}\left(Q_t^{[k,0]}\right)^2$$

$$P_t = \begin{bmatrix} \sum_{k=1}^{3} Q_t^{[k,0]} Q_t^{[k,1]} \\ \\ Q_t^{[1,0]} Q_t^{[1,2]} \end{bmatrix} \quad \text{and} \quad U_t = \begin{bmatrix} \sum_{k=1}^{3} \left(Q_t^{[k,1]}\right)^2 & Q_t^{[1,1]} Q_t^{[1,2]} \\ \\ Q_t^{[1,1]} Q_t^{[1,2]} & \left(Q_t^{[1,2]}\right)^2 \end{bmatrix}$$

After replacing the $Q_t^{[i,j]}$ by their respective expressions,

$$\mathbf{a}_t = \frac{1}{2} D_{z_t}^2 \sigma_{Y_t}^2 + \frac{1}{2} D_{z_t}^2 \sigma_{\Theta_t}^2 + D_{z_t}^2 \sigma_{\Theta_t} \sigma_{Y_t} \rho_t^{Y,\Theta}$$

$$\mathbf{P}_t = \begin{bmatrix} -D_{Z_t} D_{Y_t} \sigma_{Y_t} \left(\sigma_{Y_t} + \sigma_{\Theta_t} \rho_t^{Y,\Theta}\right) \\ \\ -D_{Z_t} \sigma_{S_t} \left(\sigma_{Y_t} \rho_t^{Y,S} + \sigma_{\Theta_t} \rho_t^{\Theta,S}\right) \end{bmatrix} \qquad \mathbf{U}_t = \begin{bmatrix} D_{Y_t}^2 \sigma_{Y_t}^2 & D_{Y_t} \sigma_{S_t} \sigma_{Y_t} \rho_t^{Y,S} \\ \\ D_{Y_t} \sigma_{S_t} \sigma_{Y_t} \rho_t^{Y,S} & \sigma_{S_t}^2 \end{bmatrix}$$

It is now essential to show that $\mathbf{U}_t$ is symmetric and positive definite. It is obvious that $\mathbf{U}_t$ is symmetric. In order to prove that it is positive definite, we use Lemma[2.2.1]. Therefore two conditions have to be checked:

The first one is that ($detU_{t\lfloor 1} > 0$) which is trivial since

$$detU_{t\lfloor 1} = D_{Y_t}^2 \sigma_{Y_t}^2 > 0$$

The second one is that ($detU_t > 0$):

$$\begin{aligned} det\,[U_t] &= D_{Y_t}^2 \sigma_{Y_t}^2 \sigma_{S_t}^2 - D_{Y_t}^2 \sigma_{S_t}^2 \sigma_{Y_t}^2 \left(\rho_t^{Y,S}\right)^2 \\ &= D_{Y_t}^2 \sigma_{Y_t}^2 \sigma_{S_t}^2 \left[1 - \left(\rho_t^{Y,S}\right)^2\right] > 0 \\ \text{since } \left|\rho_t^{Y,S}\right| &< 1 \end{aligned}$$

Therefore, U_t is a symetric and positive definite matrix.

$$\text{The optimal } \widehat{N}_t \text{ is determined by solving } \begin{cases} \min\limits_{N_t} VAR\left[d\Gamma_t\right] \\ \\ \text{such that } h(N_t) \leq 0 \end{cases}$$

$$\text{which is equivalent to } \begin{cases} \min\limits_{N_t} F(N_t) = a_t + P_t^T N_t + \frac{1}{2} N_t^T U_t N_t \\ \\ \text{such that } h(N_t) \leq 0 \end{cases}$$

The constraint is equivalent to:

$$m_t(N_t^{[1]}, N_t^{[2]}) = L_t^{(0)} + \Gamma_t r_t^{[f]} + N_t^{[1]} L_t^{(1)} + N_t^{[2]} L_t^{(2)} \geq \Gamma_t r_t^{[f]}$$

Indeed, it seems important to specify that the portfolio must earn at least the risk free interest over the period of the hedge.

$$h(N_t^{[1]}, N_t^{[2]}) = -L_t^{(0)} - N_t^{[1]} L_t^{(1)} - N_t^{[2]} L_t^{(2)}$$

$$h(N_t) = A_t N_t - L_t^{(0)}$$

with:

$$A_t^T = \begin{bmatrix} -L_t^{(1)} \\ \\ -L_t^{(2)} \end{bmatrix} \quad \text{and} \quad N_t = \begin{bmatrix} N_t^{[1]} \\ \\ N_t^{[2]} \end{bmatrix}$$

Since the number of constraints is less than the number of parameters to optimize, it is far easier to find the optimal $\widehat{\mathbf{N}}_t$ from the *Dual problem* than from the *Primal problem* directly.

$$F(N_t) = a_t + P_t^T N_t + \frac{1}{2} N_t^T U_t N_t$$

The *primal problem* is:

$$\min_{N \in \Re^n} \left\{ F(N_t) : A_t N_t \leq L_t^{(0)} \right\} \tag{6.3.4}$$

Let $Q_{\lambda,t}$ be defined as

$$Q_{\lambda,t} = -U_t^{-1}[P_t + A_t^T \lambda_t]$$

The *dual problem* is:

$$\begin{array}{c} \max \\ \lambda_t \in \Re^m \\ \lambda_t \geq 0 \end{array} \quad \left\{F(Q_{\lambda,t}) + \lambda_t^T A_t Q_{\lambda,t} - \lambda_t^T L_t^{(0)}\right\} \tag{6.3.5}$$

$$F(Q_{\lambda,t}) = a_t - \frac{1}{2}P_t^T U_t^{-1} P_t - \frac{1}{2}P_t^T U_t^{-1} A_t^T \lambda_t + \frac{1}{2}\lambda_t^T A_t U_t^{-1} P_t + \frac{1}{2}\lambda_t^T A_t U_t^{-1} A_t^T \lambda_t$$

$$F(Q_{\lambda,t}) + \lambda^T A_t Q_{\lambda,t} - \lambda_t^T L_t^{(0)} = a_t - \frac{1}{2}P_t^T U_t^{-1} P_t - \frac{1}{2}P_t^T U_t^{-1} A_t^T \lambda_t - \frac{1}{2}\lambda_t^T A_t U_t^{-1} P_t$$

$$-\frac{1}{2}\lambda_t^T A_t U_t^{-1} A_t^T \lambda_t - \lambda_t^T L_t^{(0)}$$

Moreover,

$$\max_{\lambda_t \geq 0} \left(F(Q_{\lambda,t}) + \lambda_t^T A_t Q_{\lambda,t} - \lambda_t^T L_t^{(0)}\right) = \min_{\lambda_t \geq 0} \left(-F(Q_{\lambda,t}) - \lambda_t^T A_t Q_{\lambda,t} + \lambda_t^T L_t^{(0)}\right)$$

$$= \min_{\lambda_t \geq 0} \left(-a_t + \frac{1}{2}\left[P_t^T U_t^{-1} P_t + P_t^T U_t^{-1} A_t^T \lambda_t + \lambda_t^T A_t U_t^{-1} P_t + \lambda_t^T A_t U_t^{-1} A_t^T \lambda_t + 2\lambda_t^T L_t^{(0)}\right]\right)$$

The aim is to find the optimal $\hat{\lambda} \geq 0$ such that the following expression is minimized:

$$Y(\lambda_t) = 2\left(A_t U_t^{-1} P_t + L_t^{(0)}\right)^T \lambda_t + \lambda_t^T A_t U_t^{-1} A_t^T \lambda_t \tag{6.3.6}$$

The matrix $A_t U_t^{-1} A_t^T$ is symetric positive definite since U_t^{-1} is symmetric positive definite. Indeed the fact that U_t is symmetric positive definite is equivalent to the fact that U_t^{-1} is symmetric positive definite.

Indeed,

$$\forall\, x \neq 0 \in \Re,\ x^T A_t U_t^{-1} A_t^T x = \left(A_t^T x\right)^T U_t^{-1} \left(A_t^T x\right) > 0$$

Equation[6.3.6] can be viewed as a quadratic form. Hence the λ_t which minimizes equation[6.3.6] is

$$\lambda_t = -\frac{A_t U_t^{-1} P_t + L_t^{(0)}}{A_t U_t^{-1} A_t^T}$$

But $\hat{\lambda}_t$ must be positive, therefore

$$\hat{\lambda}_t = \frac{1}{2}\left(\left|\frac{A_t U_t^{-1} P_t + L_t^{(0)}}{A_t U_t^{-1} A_t^T}\right| - \frac{A_t U_t^{-1} P_t + L_t^{(0)}}{A_t U_t^{-1} A_t^T}\right) \tag{6.3.7}$$

The optimal $\hat{N}_t$ is then given by:

$$\hat{N}_t = -U_t^{-1} P_t - U_t^{-1} A_t^T \hat{\lambda}_t$$

which eventually proves the result.

6.4 Implementation and results

6.4.1 Hedge ratio 3

Using the same arguments as those in sub-section[5.6.1], an assumption can be made regarding equation[6.3.2]: Over a short period of time, the price movements implied by the drift can be neglected compared to those implied by the volatility.

$$m_t^2(N_t)dt << a_t + P_t^T N_t + \frac{1}{2} N_t^T U_t N_t$$

Using historical data, we have checked for different securities that this assumption is valid. As a consequence, neglecting the effects of the drift in all the previous equations the hedge ratio will simply become:

$$\widehat{\mathbf{N}}_t = \begin{bmatrix} \widehat{N}_t^{[1]} \\ \\ \widehat{N}_t^{[2]} \end{bmatrix} = \frac{D_{Z_t}}{D_{Y_t}} \begin{bmatrix} 1 + \dfrac{\sigma_{\Theta_t}}{\sigma_{Y_t}} \dfrac{\rho_t^{Y,\Theta} - \rho_t^{Y,S}\rho_t^{\Theta,S}}{1-\left(\rho_t^{Y,S}\right)^2} \\ \\ D_{Y_t} \dfrac{\sigma_{\Theta_t}}{\sigma_{S_t}} \dfrac{\rho_t^{\Theta,S} - \rho_t^{Y,\Theta}\rho_t^{Y,S}}{1-\left(\rho_t^{Y,S}\right)^2} \end{bmatrix} \tag{6.4.1}$$

It is important to notice that since $\dfrac{D_{Z_t}}{D_{Y_t}}$ can be viewed as the *traditional duration based hedge ratio*, equation[6.4.1] can be rewritten as:

$$\boxed{\widehat{\mathbf{N}}_t = \begin{bmatrix} 1 + \dfrac{\sigma_{\Theta_t}}{\sigma_{Y_t}} \dfrac{\rho_t^{Y,\Theta} - \rho_t^{Y,S}\,\rho_t^{\Theta,S}}{1-\left(\rho_t^{Y,S}\right)^2} \\ \\ D_{Y_t} \dfrac{\sigma_{\Theta_t}}{\sigma_{S_t}} \dfrac{\rho_t^{\Theta,S} - \rho_t^{Y,\Theta}\,\rho_t^{Y,S}}{1-\left(\rho_t^{Y,S}\right)^2} \end{bmatrix} \widehat{\mathbf{N}}_t^*}$$

This hedge ratio will be called "*Hedge Ratio 3* ".

For very high grade bonds, $\Theta_t \simeq 0$ (since the *credit spread* is almost zero) and $\sigma_{\Theta_t} \simeq 0$. Therefore *hedge ratio 3* becomes (only for high grade bonds), the same as the *traditional duration based hedge ratio.* But for lower grade bonds, it differs in the sense that our hedge ratio incorporates the notions of volatilities and correlations.

a. Model assumptions

A log-normal model is chosen for the *stock* process. The same kind of model is used in order to describe the *yield to maturity of the gilt.* After having investigated different models, it appears that the *credit spread* is best modelled using a normal process.

All these processes are described in chapter[3] devoted to the estimation of model's parameters from historical data. When computing *hedge ratio 3*, $\rho_t^{Y,\Theta}$ is set to 0 (cf assumption[5.6.5]) and only negative $\rho_t^{\Theta,S}$ are taken into account (cf. discussion held in subsection[6.3.1]).

b. Choice of the Bonds to be hedged

Two investment grade corporate bonds are investigated. The first one is *CADBURY* which is an A2 rated bond. The coupon frequency is annual and the coupon is 8% . The first settlement date was November 1995 and the bond is due to expire in December 2000. The associated gilt is a semi-annual fixed coupon at 8%. The first settlement date was October 1994 and the gilt is due to expire in December 2000.

The second one is *BRITISH AEROSPACE* whose characteristics are described in previous chapters.

c. Transaction costs

It will be assumed here that the transaction cost is a linear function of the cash value of the securities that are sold short. Therefore, if N securities with prices p_t are sold short the transaction cost will be

$$Transaction\ cost = KNp_t \tag{6.4.2}$$

with K a positive real number. p_t can either be a bond or a stock. In reality the function is clearly not linear but as far as big transactions are concerned the relation[6.4.2] is a not bad approximation of what the true transaction cost is.

Let us consider two hedging strategies. m_1 and m_2 are two positive integers. The first strategy consists in selling short a quantity $N_t^{[1,1]}$, $N_t^{[1,2]}$, ..., $N_t^{[1,m_1]}$ of securities $p_t^{[1]}, p_t^{[2]}, ..., p_t^{[m_1]}$respectively. The second strategy consists in selling short a quantity $N_t^{[2,1]}$, $N_t^{[2,2]}$, ..., $N_t^{[2,m_2]}$ of the securities $p_t^{[1]}, p_t^{[2]}, ..., p_t^{[m_2]}$respectively.

Therefore the transaction cost induced by strategy 1 is:

$$Cost^{[1]} = K \sum_{i=1}^{m_1} N_t^{[1,i]} p_t^{[i]}$$

and the one induced by strategy 2 is :

$$Cost^{[2]} = K \sum_{i=1}^{m_2} N_t^{[2,i]} p_t^{[i]}$$

According to section[5.6.2], the variable measuring the percentage of the risk actually hedged (for strategy j) is:

$$R_t^{[j]} = 100 \frac{(RISK_t \text{ without hedging -} RISK_t \text{ with hedging using strategy j})}{RISK_t \text{ without hedging}}$$

It is then straightforward to define a transaction cost per unit of the *percentage of risk effectively hedged* which is E_t. It follows that for strategy j, the transaction cost per unit of the percentage of risk effectively hedged is:

$$E_t^{[j]} = \frac{Cost^{[j]}}{R_t^{[j]}}$$

If $E_t^{[2]} \leq E_t^{[1]}$, it means that proportionally, strategy 2 is less expensive than strategy

1 and the discount in the transaction cost is given by:

$$D_t^{[2,1]} = 100\frac{E_t^{[1]} - E_t^{[2]}}{E_t^{[1]}} = 100\left(1 - \frac{R_t^{[1]}}{R_t^{[2]}}\frac{\sum_{i=1}^{m_2} N^{[2,i]}p_t^{[i]}}{\sum_{i=1}^{m_1} N^{[1,i]}p_t^{[i]}}\right) \tag{6.4.3}$$

Our purpose is to check that the new hedging strategies created in this thesis are not too expensive (with respect to the goal that we want to achieve) compared to the traditional duration based hedge strategy. Moreover it has been decided here not to rebalance the portfolio very often. For more information relative to optimal continuous hedging strategies under transaction costs (for options), the reader is invited to refer to [Hodges & Clewlow (1997)] or [Davis, Panas & Zariphopoulou (1992)].

6.4.2 Hedging performance

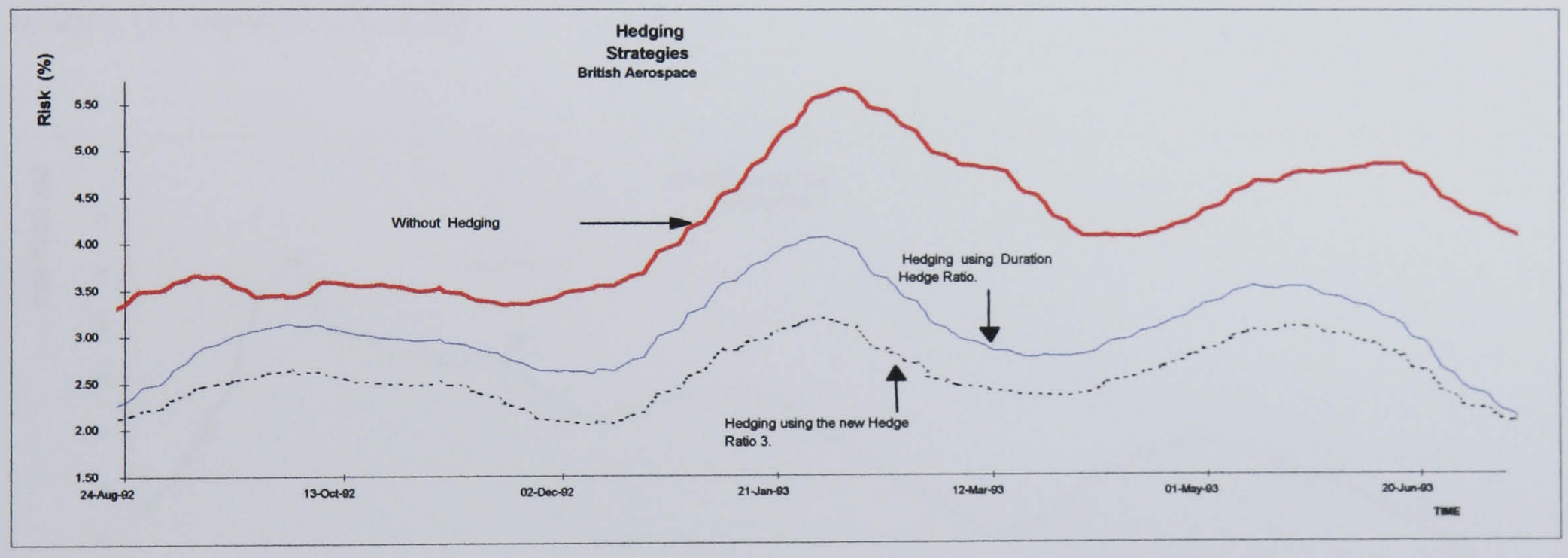


Figure 6.4.1: B. AEROSPACE: Hedging Strategy (Hedge Ratio 3)

Figure[6.4.1] shows that *hedge ratio 3* performs better than the *traditional hedge ratio*. Figure[6.4.2] shows that *hedge ratio 3* does not perform worse than the *traditional hedge ratio* and even slightly better.

6.4.3 Hedging cost

Figure[6.4.3] shows that the strategy using *hedge ratio 3* is (all proportion kept) cheaper than the one using the *traditional hedge ratio*. Indeed, it represents the discount proportionally achieved in the cost of hedging with *hedge ratio 3* relatively

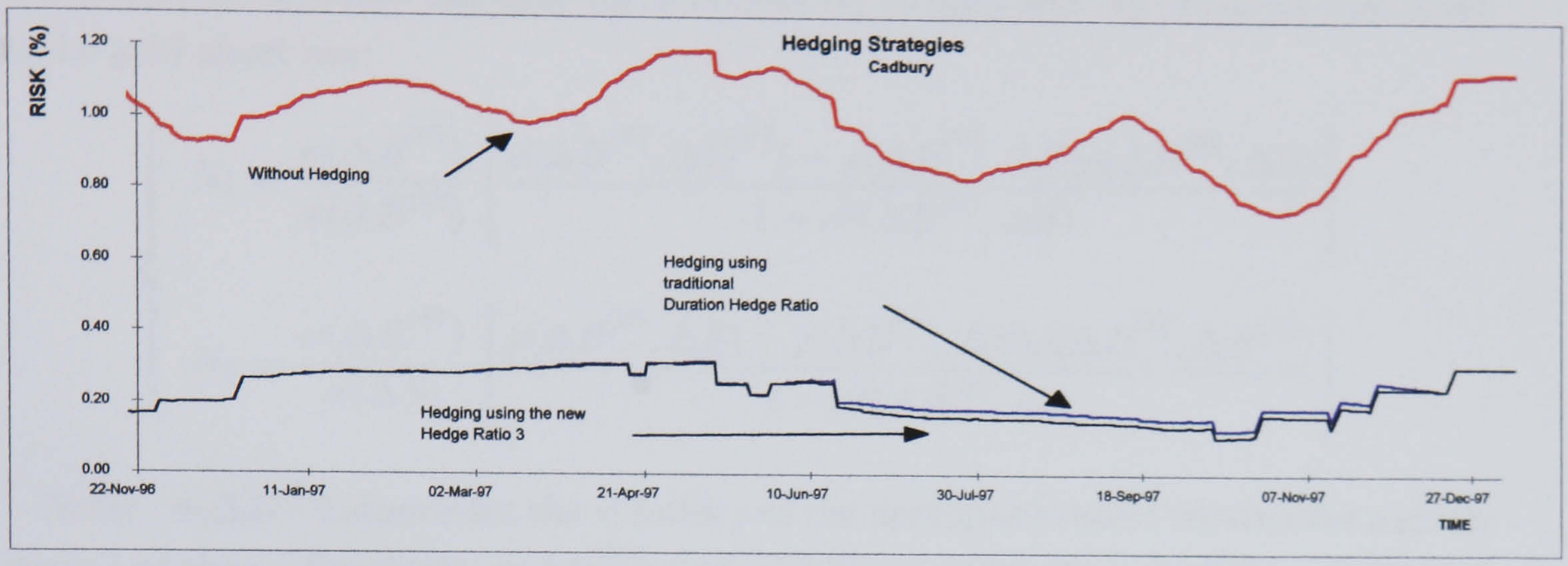


Figure 6.4.2: CADBURY: Hedging Strategy (Hedge Ratio 3)

to traditional hedge ratio. The precise meaning of "discount" has been explained before (cf equation[6.4.3]).

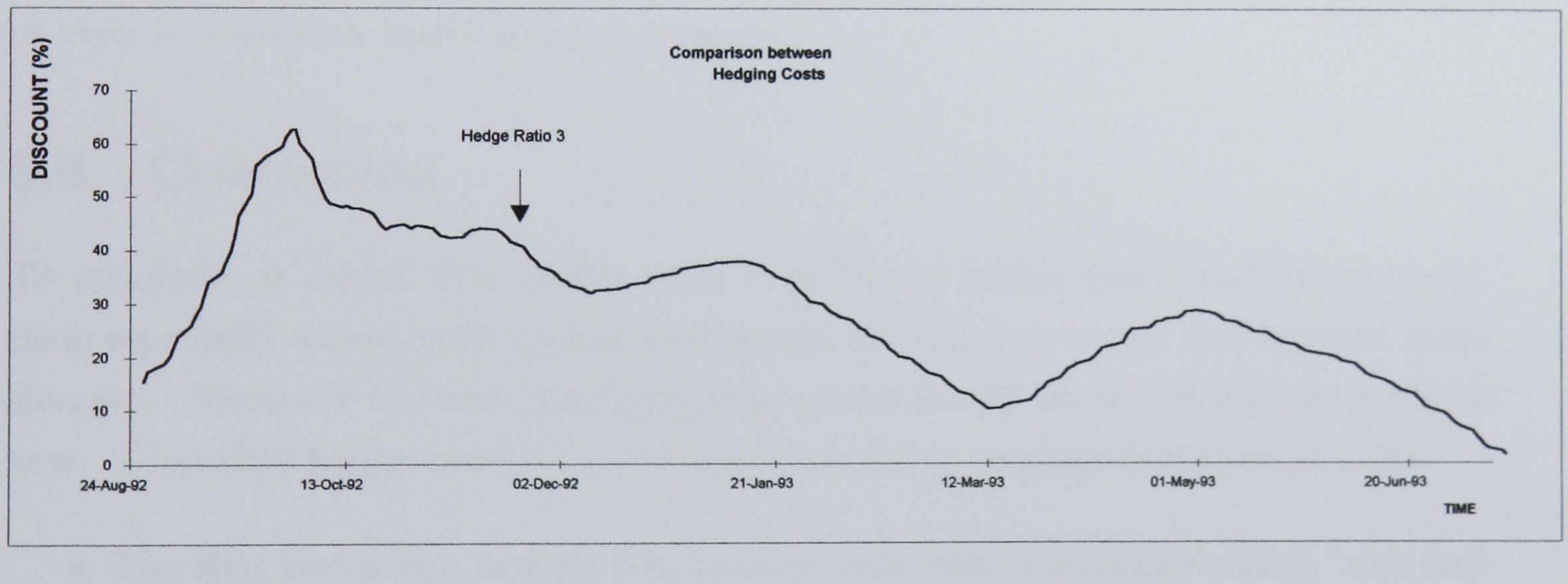


Figure 6.4.3: B. AEROSPACE: Hedging Costs comparison (1)

6.5 Minimum Variance Hedging Strategy

Hedging strategy using *hedge ratio 3* has been compared to the *Duration based* hedging strategy. Furthermore, we believe that it is worth comparing it as well to the *classical minimum variance* hedging strategy. This one is usually applied to foreign currency futures market (see [Luenberger (1998)]). Adapting this theory to

our purposes, it comes out that the numbers N_1 of gilts and N_2 of stocks that need to be sold short are:

$$\begin{cases} N_1 = \dfrac{\sigma(\Delta B^{[Z]})}{\sigma(\Delta B^{[Y]})} \left[\dfrac{\rho(\Delta B^{[Z]}, \Delta B^{[Y]}) - \rho(\Delta B^{[Y]}, \Delta S)\rho(\Delta B^{[Z]}, \Delta S)}{1 - \rho^2(\Delta B^{[Y]}, \Delta S)} \right] \\ N_2 = \dfrac{\sigma(\Delta B^{[Z]})}{\sigma(\Delta S)} \left[\dfrac{\rho(\Delta B^{[Z]}, \Delta S) - \rho(\Delta B^{[Y]}, \Delta S)\rho(\Delta B^{[Z]}, \Delta B^{[Y]})}{1 - \rho^2(\Delta B^{[Y]}, \Delta)} \right] \end{cases}$$

Note: $\sigma(\Delta B^{[Z]})$ stands for the volatility of the defaultable bond return over a given period of time. Similarly, $\sigma(\Delta B^{[Y]})$ and $\sigma(\Delta S)$ stand for the volatilities of gilt and stock returns over the same period of time. Eventually, $\rho(x, y)$ is the correlation between x and y.

We have tested this *classical minimum variance* hedging strategy using the same requirements (hedge horizon, etc ...) as the ones for other strategies. As can be viewed from numerical results and graph in Appendix[C], the *classical minimum variance* hedging strategy is not satisfactory at all compared to *hedge ratio 3* strategy or even to *Duration based* hedging strategy.

6.6 Conclusion

To conclude, it seems that *hedge ratio 3* performs better than traditional hedge ratio especially when *credit spread* movements are not due to risk free interest rates changes. Moreover the cost of hedging is proportionnally lower which is an excellent new. Therefore hedge ratio 3 can be very poweful to manage two specific risks:

- The first one is the default free interest rate risk, commonly called "risk free interest rate" risk.
- The second one is the specific risk (e.g : quality of management....) due to uncertainties about the company future financial health but not linked to uncertainties about the future risk free interest rate curve.

Chapter 7

Hedging with gilts, stocks and index

7.1 Introduction

Previously we discussed about the correlation between the *credit spread* and the *stock* price. The rationale was that when the company is in "bad health" then the *stock* price goes down and the *credit spread* widens. The converse is also true. But when the whole *stock market* follows one trend (up or down), the move in the *stock* price does not only reflect the company's "health" because it may be biased by the whole stock market trend. The price of the *index* that will be used in the hedging strategy is I_t . The *ratio of the stock price to the index* is $X_t = \frac{S_t}{I_t}$. For the reason discussed above it makes sense to think that the correlation between X_t and the *credit spread* Θ_t might be better than the one between the *stock* price S_t and the *credit spread* Θ_t. Indeed the *stock-index ratio* captures the idea of the excess return of the *stock* over the market. In order to support the intuitive idea that there should be a very good correlation between X_t and the *credit spread* Θ_t, figure[7.1.1] displays the historical data for the "British Aerospace" bond *credit spread* and the ratio of the "British Aerospace" stock divided by the value of the *FTSE Engineering Index* (since "British Aerospace" is a member of this particular index) which is a 21 stock *index*. The historical data are the daily data for a 7 year period of time (1991-1998). According to this graph, the correlation seems to be quite high. The same remark could be done if the FTSE all shares or the FTSE 100 were used instead of the FTSE Engineering Index.

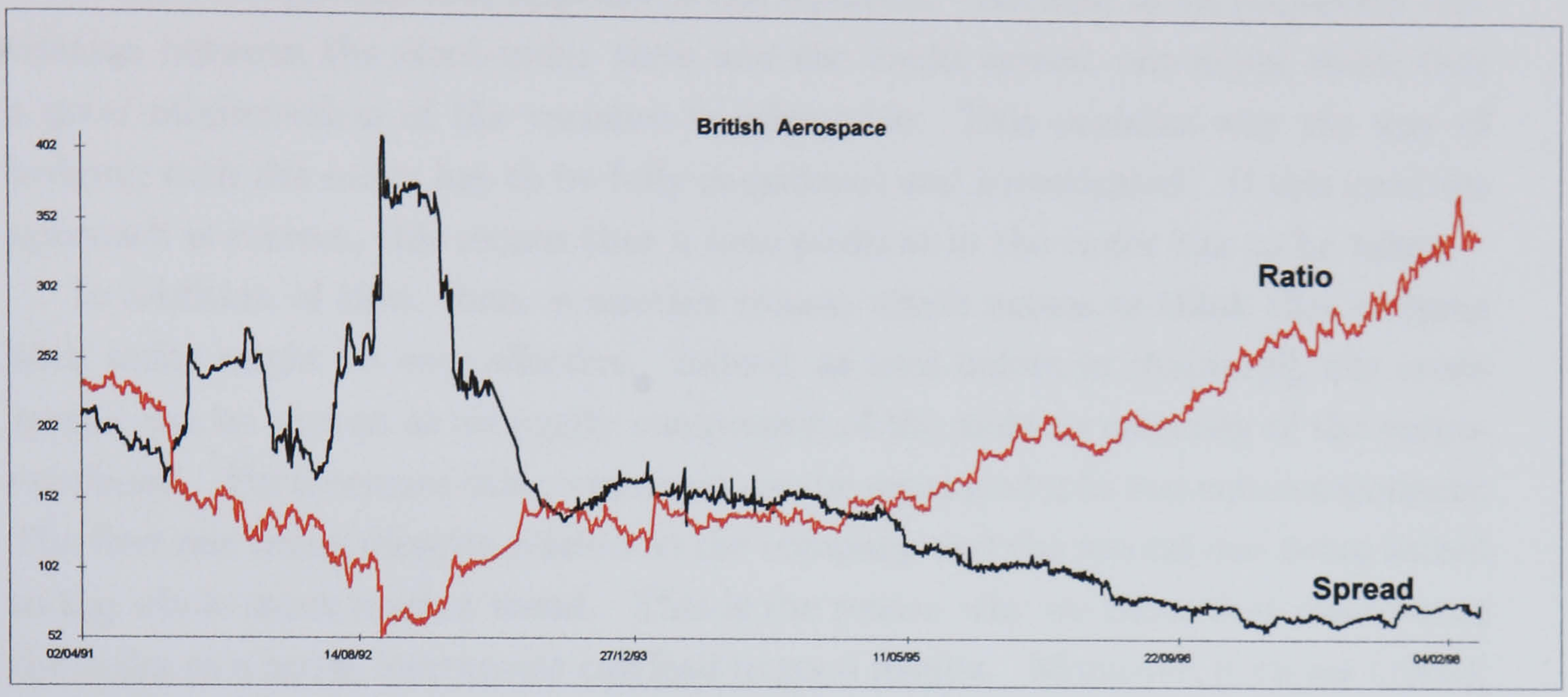


Figure 7.1.1: Correlation between Spread and Ratio

In order to be more specific about that, let us consider a portfolio Π_t containing:

- a long position in the bond to be hedged ($B_t^{[Z]}$) and in the index (I_t)
- a short position in the gilt ($B_t^{[Y]}$) and in the stock (S_t)

This portfolio can be written as

$$\Pi_t = B_t^{[Z]} - N_t^{[1]} B_t^{[Y]} - N_t^{[2]} S_t + N_t^{[3]} I_t$$

or equivalently

$$\Pi_t = B_t^{[Z]} - N_t^{[1]} B_t^{[Y]} - N_t^{[2]} X_t I_t + N_t^{[3]} I_t$$

A small variation in the portfolio's value is then

$$d\Pi_t = -N_t^{[2]} f(t) + dB_t^{[Z]} - N_t^{[1]} dB_t^{[Y]} - N_t^{[2]} I_t dX_t + \left(N_t^{[3]} - N_t^{[2]} X_t \right) dI_t$$

with $f(t) = dX_t dI_t$

It is interesting to notice that when $N_t^{[3]} = N_t^{[2]} X_t$, the process can be written as:

$$d\Pi_t = -N_t^{[2]} f(t) + dB_t^{[Z]} - N_t^{[1]} dB_t^{[Y]} - N_t^{[2]} I_t dX_t$$

so that the process dX_t appears in the equation. Because of an important correlation between the *stock-index ratio* and the *credit spread*, one might think that a good minimisation of the variance is achievable. This explains why the way of hedging with the *index* has to be fully considered and investigated. If this intuitive approach is correct, this means that a *long position* in the *index* has to be taken.

In addition of that, there is another reason which makes us think that hedging with *index* might be very effective. Indeed, as seen before in chapter[6], the *credit spread* can be viewed as an equity component of the yield to maturity of the corporate bond. Furthermore this component can be separated into two sub-components: The first one being directly related to the company and the second one being linked to the whole stock market trend. This is the reason why we think that considering the *index* as a hedge instrument can lead to good results. Moreover, [Grieves (1986)] and [Marcus & Ors (1996)] have shown that "portfolios of corporate bonds that are subject to credit risk can be more effectively hedged when *stock index futures* are included with treasury bond futures in the menu of hedging instruments". If this approach is correct, this means that a *short position* in the *index* has to be taken.

To summarize, there are two theories: The first one advocates to take a long position in the *index* whereas the second one advocates to take a short position. It is shown further which one is correct for our purposes.

In this chapter, the following notations will be used:

$\mathbf{B_t^{[Z]}}$: is the price at time t of the "risky" bond to be hedged. In order to simplify notations $B_t^{[Z]}$ will be used as well when describing the bond physically. The yield to maturity of $B_t^{[Z]}$ at time t is Z_t.

$\mathbf{B_t^{[Y]}}$: is the price at time t of the gilt used as a hedge instrument. The yield to maturity of $B_t^{[Y]}$ at time t is Y_t.

7.2 Processes

The processes that have to be modelled are the *yield to maturity of the gilt*, the artificial *credit spread*, the *stock* and the *index*. Obviously a correlation between the *stock* and the *credit spread* has to be introduced . Furthermore, as the *stock* is often a component of the *index* and since the *stock* might follow the *index* trend, a significant correlation is expected to exist between these two processes. All notations used here are the same as the ones defined in previous chapters.

$$X_t = \begin{bmatrix} X_t^{[1]} \\ X_t^{[2]} \\ X_t^{[3]} \\ X_t^{[4]} \end{bmatrix} = \begin{bmatrix} S_t \\ I_t \\ Y_t \\ \Theta_t \end{bmatrix} \qquad W_t = \begin{bmatrix} W_t^{(1)} \\ W_t^{(2)} \\ W_t^{(3)} \\ W_t^{(4)} \end{bmatrix} \qquad \mu_{X_t} = \begin{bmatrix} \mu_{X_t^{[1]}} \\ \mu_{X_t^{[2]}} \\ \mu_{X_t^{[3]}} \\ \mu_{X_t^{[4]}} \end{bmatrix} = \begin{bmatrix} \mu_{S_t} \\ \mu_{I_t} \\ \mu_{Y_t} \\ \mu_{\Theta_t} \end{bmatrix}$$

$$\sigma_{X_t} = \begin{bmatrix} \sigma_{X_t^{[1]}} \\ \sigma_{X_t^{[2]}} \\ \sigma_{X_t^{[3]}} \\ \sigma_{X_t^{[4]}} \end{bmatrix} = \begin{bmatrix} \sigma_{S_t} \\ \sigma_{I_t} \\ \sigma_{Y_t} \\ \sigma_{\Theta_t} \end{bmatrix} \quad \text{and} \quad \Delta_t = \begin{bmatrix} 1 & \rho_t^{I,S} & \rho_t^{Y,S} & \rho_t^{\Theta,S} \\ \rho_t^{I,S} & 1 & \rho_t^{Y,I} & \rho_t^{I,\Theta} \\ \rho_t^{Y,S} & \rho_t^{Y,I} & 1 & \rho_t^{Y,\Theta} \\ \rho_t^{\Theta,S} & \rho_t^{I,\Theta} & \rho_t^{Y,\Theta} & 1 \end{bmatrix}$$

Using equation[2.2.4], it is then possible to write:

$$dX_t^{(i)} = \mu_{X_t^{[i]}} dt + \sigma_{X_t^{[i]}} \sum_{k=1}^{4} \frac{\left(\prod_{m=0}^{\max(k-2,0)} \sqrt{\det \Delta_{t\lfloor m}} \right)^{\delta(i,k)}}{\sqrt{\prod_{m=0}^{k-\delta(i,k)} \det \Delta_{t\lfloor m}}} \left[\det \left(\varphi^{(i-k-1)} \circ \mathsf{m}_{i,k}(\Delta_{t\lfloor i}) \right) \right]^{1-\frac{\delta(i,k)}{2}} dW_t^{(k)} \tag{7.2.1}$$

Moreover, following equation[2.2.6], it is assumed that:

$$\det(\Delta_{t\lfloor k}) > 0 \text{ for } k \in [1,3] \tag{7.2.2}$$

As $\det(\Delta_{t\lfloor 1}) = 1 > 0$ and $\det(\Delta_{t\lfloor 2}) = 1 - \left(\rho_t^{I,S}\right)^2 > 0$, assumption[7.2.2] can be

reduced to:

$$\det(\Delta_{t\lfloor 3}) > 0 \tag{7.2.3}$$

Theoretically there are some special cases when equation[7.2.3] does not hold (for instance when two of the three correlations are extremely high and the other extremely small). However in practice, it is very rare to meet these extreme cases. Therefore, the necessary assumption[7.2.3] can be considered as a very realistic one. An explanation is given further on in the chapter.

Equation[7.2.1] can actually be decomposed into the four following equations :

$$dS_t = \mu_{S_t} dt + \sigma_{S_t} dW_t^{(1)} \tag{7.2.4}$$

The *stock* process is represented by equation[7.2.4] with volatility σ_{S_t} and drift μ_{S_t}.

$$dI_t = \mu_{I_t} dt + \sigma_{I_t} \left[\left| \mathsf{m}_{2,1}\left(\Delta_{t\lfloor 2}\right) \right| dW_t^{(1)} + \sqrt{\left|\Delta_{t\lfloor 2}\right|} dW_t^{(2)} \right] \tag{7.2.5}$$

Equation[7.2.5] is the modelling of the *index* process with drift μ_{I_t} and volatility σ_{I_t}.

$$dY_t = \mu_{Y_t} dt + \sigma_{Y_t} \left[\left| \varphi \circ \mathsf{m}_{3,1}\left(\Delta_{t\lfloor 3}\right) \right| dW_t^{(1)} + \frac{\left| \mathsf{m}_{3,2}\left(\Delta_{t\lfloor 3}\right) \right|}{\sqrt{\left|\Delta_{t\lfloor 2}\right|}} dW_t^{(2)} + \sqrt{\frac{\left|\Delta_{t\lfloor 3}\right|}{\left|\Delta_{t\lfloor 2}\right|}} dW_t^{(3)} \right] \tag{7.2.6}$$

The *yield to maturity of the gilt* is described by equation[7.2.6] with drift μ_{Y_t} and volatility σ_{Y_t}.

$$d\Theta_t = \mu_{\Theta_t} dt + \sigma_{\Theta_t} \left[\begin{array}{l} \left| \varphi^{(2)} \circ \mathsf{m}_{4,1}\left(\Delta_t\right) \right| dW_t^{(1)} + \frac{\left| \varphi \circ \mathsf{m}_{4,2}\left(\Delta_t\right) \right|}{\sqrt{\left|\Delta_{t,2}\right|}} dW_t^{(2)} \\ \\ + \frac{\left| \mathsf{m}_{4,3}\left(\Delta_t\right) \right|}{\sqrt{\left|\Delta_{t\lfloor 2}\right| \left|\Delta_{t\lfloor 3}\right|}} dW_t^{(3)} + \sqrt{\frac{\left|\Delta_t\right|}{\left|\Delta_{t\lfloor 3}\right|}} dW_t^{(4)} \end{array} \right] \tag{7.2.7}$$

Eventually, equation[7.2.7] represents the artificial *credit spread* process with volatility σ_{Θ_t} and drift μ_{Θ_t}.

7.3 Optimal hedge portfolio

7.3.1 Description

A portfolio which contains the *bond subject to credit risk*, the *gilt*, the *stock* and the *index* is created. The aim is to determine the optimal numbers of gilts, shares and index respectively ($N_t^{[1]}$, $N_t^{[2]}$, $N_t^{[3]}$) that should be bought or short-sold in order to hedge one unit of the risky bond. Without loss of generality it is assumed that a certain quantity of the *index* has to be bought. However if the hedge ratio that is derived is such that $N_t^{[3]} < 0$, then a quantity of *index* has to be sold short instead of being bought. The following notations are used in the next sections: The proportion of the purchase price that must come from the trader's own fund is $\mathbf{x}_t^{[l]}$. The interest rate on margin loan is $\mathbf{r}_t^{[l]}$. All other notations are the same as those in previous sections.

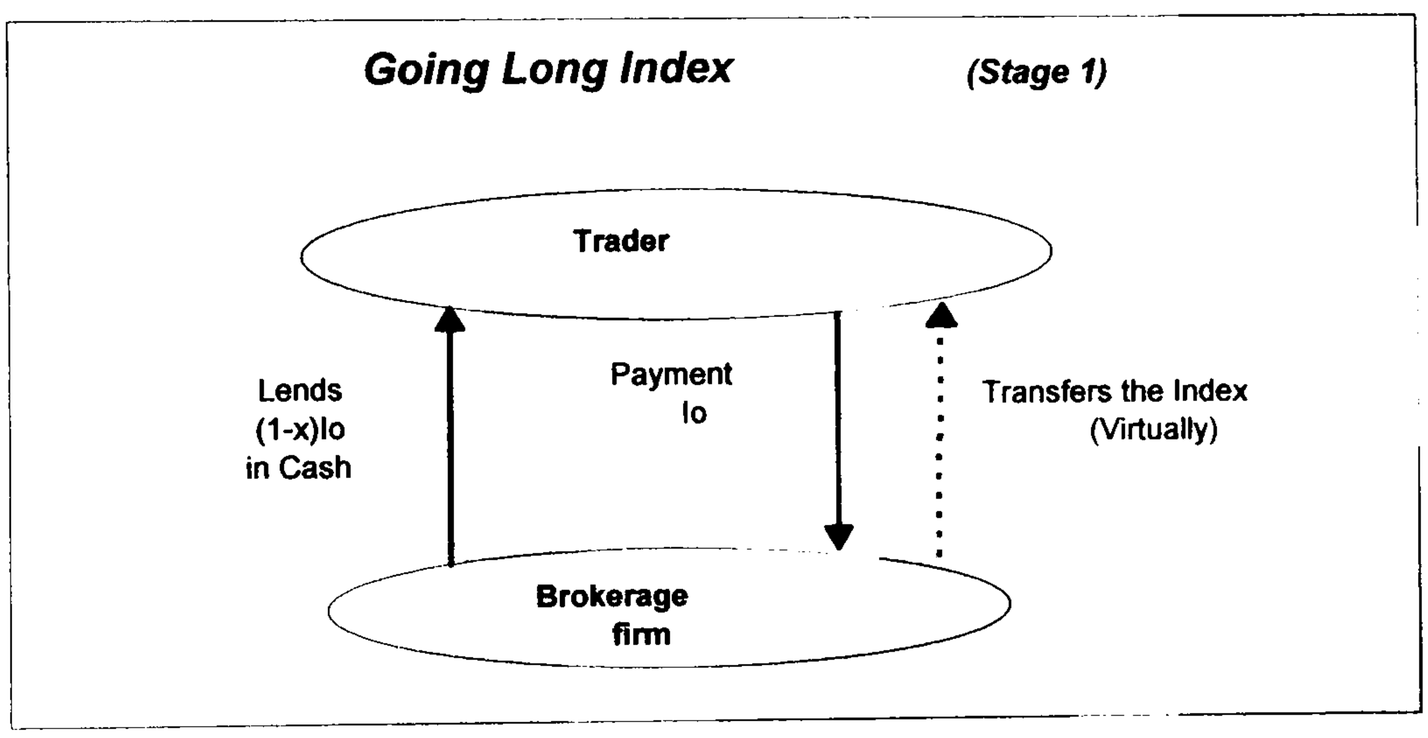


Figure 7.3.1: Going Long Index: Step 1

The process of going long index can be separated in two stages that are illustrated by figures[7.3.1 and 7.3.2]. It has to be emphasised that because of the margin loan, the leverage can be significant. At *time* t, it is possible to split a portfolio Π_t (consisting in a long position in the index) as follows:

$$\Pi_t = N_t I_t - £_t \left[N_t \left(1 - x_t^{[l]} \right) I_t \right] + £_t \left[\Pi_t - x_t^{[l]} N_t I_t \right]$$

This corresponds to the wealth of the trader at time t which becomes at *time* $t + dt$,

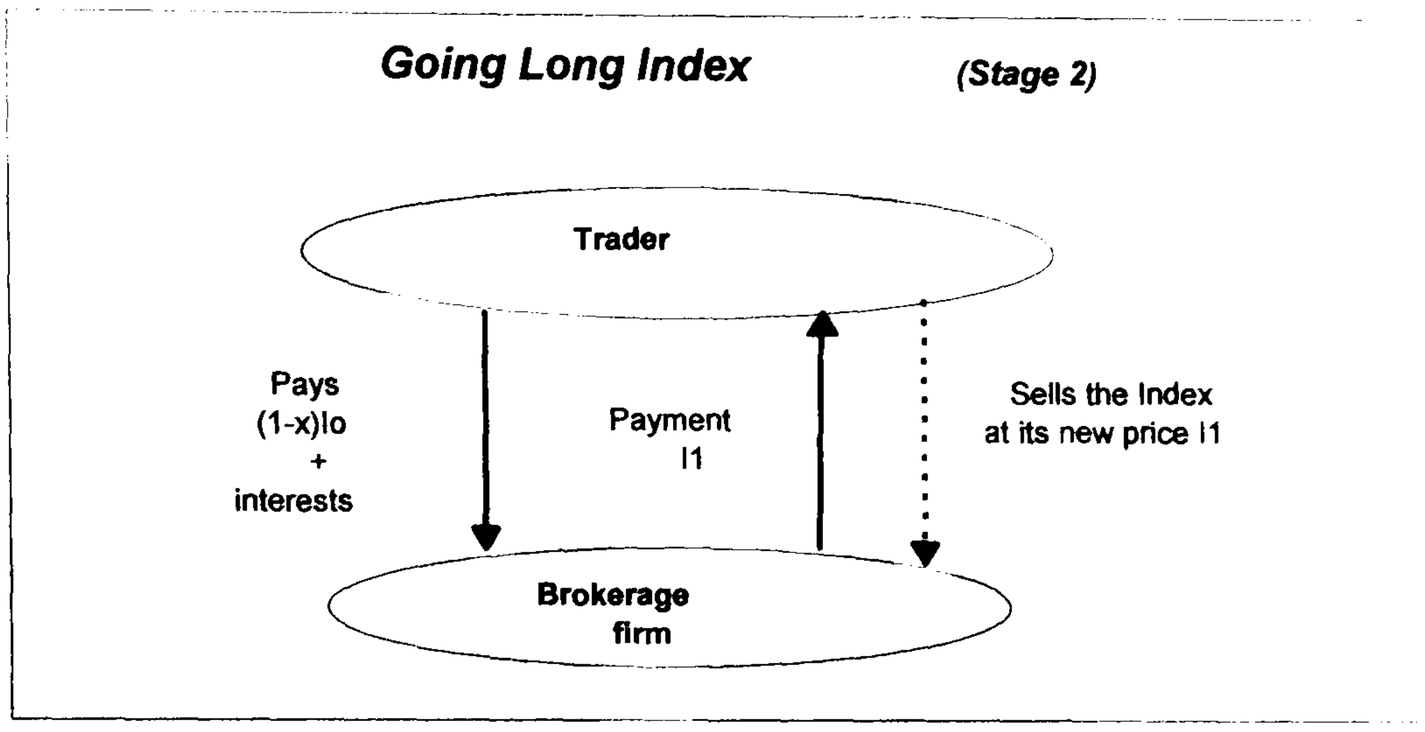


Figure 7.3.2: Going Long Index: Step 2

$$\Pi_{t+dt} = N_t I_{t+dt} - \pounds_{t+dt}\left[N_t\left(1 - x_t^{[l]}\right) I_t\right] + \pounds_{t+dt}\left[\Pi_t - x_t^{[l]} N_t I_t\right]$$

Moreover,

$$\pounds_{t+dt}\left[N_t\left(1 - x_t^{[l]}\right) I_t\right] = \pounds_t\left[N_t\left(1 - x_t^{[l]}\right) I_t\right]\left(1 + r_t^{[l]} dt\right)$$

and

$$\pounds_{t+dt}\left[\Pi_t - x_t^{[l]} N_t I_t\right] = \pounds_t\left[\Pi_t - x_t^{[l]} N_t I_t\right]\left(1 + r_t^{[f]} dt\right)$$

The variation of the portfolio's value is then:

$$d\Pi_t = N_t\left[dI_t - \left(1 - x_t^{[l]}\right) I_t r_t^{[l]} dt\right] + \left(\Pi_t - x_t^{[l]} N_t I_t\right) r_t^{[f]} dt \qquad (7.3.1)$$

7.3.2 Investigation

a. Result

If the vectors N_t and dW_t are defined as follows

$$N_t = \begin{bmatrix} N_t^{[1]} \\ N_t^{[2]} \\ N_t^{[3]} \end{bmatrix} \quad \text{and} \quad dW_t = \begin{bmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \\ dW_t^{(4)} \end{bmatrix}$$

then a small variation in the portfolio Γ_t, consisting of the bond to be hedged, some *gilts*, *stocks* and *index*, can be written as follows:

$$d\Gamma_t = m_t(N_t)dt + K_t(N_t, dW_t)$$

with a deterministic component $\mathbf{m_t(N_t)}$ such that

$$m_t(N_t) = L_t^{(0)} + \Gamma_t r_t^{[f]} + N_t^{[1]} L_t^{(1)} + N_t^{[2]} L_t^{(2)} + N_t^{[3]} L_t^{(3)}$$

and a pure stochastic component $\mathbf{K_t(N_t, dW_t)}$ such that

$$K_t(N_t, dW_t) = \begin{bmatrix} Q_t^{[1,0]} + N_t^{[1]} Q_t^{[1,1]} + N_t^{[2]} Q_t^{[1,2]} + N_t^{[3]} Q_t^{[1,3]} \\ Q_t^{[2,0]} + N_t^{[1]} Q_t^{[2,1]} + N_t^{[3]} Q_t^{[2,3]} \\ Q_t^{[3,0]} + N_t^{[1]} Q_t^{[3,1]} \\ Q_t^{[4,0]} \end{bmatrix}^T \begin{bmatrix} dW_t^{(1)} \\ dW_t^{(2)} \\ dW_t^{(3)} \\ dW_t^{(4)} \end{bmatrix} \tag{7.3.2}$$

where

$$\mathbf{Q_t^{[1,0]}} = D_{Z_t}\left(\sigma_{Y_t} + \sigma_{\Theta_t}\right) \qquad \mathbf{Q_t^{[1,1]}} = -D_{Y_t}\sigma_{Y_t} \qquad \mathbf{Q_t^{[1,2]}} = -\sigma_{S_t} \qquad \mathbf{Q_t^{[1,3]}} = \sigma_{I_t}$$

$$\mathbf{Q_t^{[2,0]}} = D_{Z_t}\sigma_{Y_t}\frac{\left|M_{3,2}\left(\Delta_{t\lfloor 3}\right)\right|}{\sqrt{\left|\Delta_{t\lfloor 2}\right|}} + D_{Z_t}\sigma_{\Theta_t}\frac{\left|\delta \circ M_{4,2}\left(\Delta_t\right)\right|}{\sqrt{\left|\Delta_{t\lfloor 2}\right|}} \qquad \mathbf{Q_t^{[2,1]}} = -D_{Y_t}\sigma_{Y_t}\frac{\left|M_{3,2}\left(\Delta_{t\lfloor 3}\right)\right|}{\sqrt{\left|\Delta_{t\lfloor 2}\right|}}$$

$$\mathbf{Q_t^{[3,0]}} = D_{Z_t}\sigma_{Y_t}\sqrt{\frac{|\Delta_{t\lfloor 3}|}{|\Delta_{t\lfloor 2}|}} + D_{Z_t}\frac{|M_{4,3}(\Delta_t)|}{\sqrt{|\Delta_{t\lfloor 2}|\,|\Delta_{t\lfloor 3}|}}$$

$$\mathbf{Q_t^{[3,1]}} = -D_{Y_t}\sigma_{Y_t}\sqrt{\frac{|\Delta_{t\lfloor 3}|}{|\Delta_{t\lfloor 2}|}}$$

$$\mathbf{Q_t^{[2,3]}} = \sigma_{I_t}\sqrt{|\Delta_{t\lfloor 2}|}$$

$$\mathbf{Q_t^{[4,0]}} = D_{Z_t}\sigma_{\Theta_t}\sqrt{\frac{|\Delta_t|}{|\Delta_{t\lfloor 3}|}}$$

$$\mathbf{L_t^{(0)}} = \Psi_{1,t}^{(c)} + D_{Z_t}\left(\mu_{Y_t} + \mu_{\Theta_t}\right) + C_{Z_t}\left(\sigma_{\Theta_t}^2 + \sigma_{Y_t}^2 + 2\rho_t^{Y,\Theta}\sigma_{\Theta_t}\sigma_{Y_t}\right) - B_t^{[Z]}r_t^{[f]}$$

$$\mathbf{L_t^{(1)}} = -\Psi_{1,t}^{(h)} - \mu_{Y_t}D_{y_t}^{[Y]} - C_{Y_t}\sigma_{Y_t}^2 + B_t^{[Y]}r_t^{[b]}$$

$$\mathbf{L_t^{(2)}} = -\mu_{S_t} + S_t(r_t^{[f]} + x_t^{[s]}r_t^{[s]} - x_t^{[s]}r_t^{[f]})$$

$$\mathbf{L_t^{(3)}} = \mu_{I_t} + I_t\left(x_t^{[l]}r_t^{[l]} - r_t^{[l]} - x_t^{[l]}r_t^{[f]}\right)$$

b. Proof

The same methodology as before can be used in order to obtain $d\Gamma_t$ as a function of the different securities involved in the portfolio. Since it is assumed that a quantity $N_t^{[3]}$ of the *index* has to be bought the process of going long (described in section[7.3.1]) is applied. However, as previously discussed, if $N_t^{[3]}$ turns out to be negative, a quantity of the *index* has to be sold short.

The value of the portfolio at time t is Γ_t :

$$\Gamma_t = B_t^{[Z]} - N_t^{[1]}\left(B_t^{[Y]} - \pounds_t\left[B_t^{[Y]}\right]\right) + N_t^{[3]}I_t - \pounds_t\left[N_t^{[3]}\left(1 - x_t^{[l]}\right)I_t\right] - N_t^{[2]}S_t$$

$$+\pounds_t\left[N_t^{[2]}S_t\right] + \pounds_t\left[x_t^{[s]}N_t^{[2]}S_t\right] + \pounds_t\left[\Gamma_t - B_t^{[Z]} - N_t^{[3]}x_t^{[l]}I_t - x_t^{[s]}N_t^{[2]}S_t\right]$$

It is straightforward to write that at time $t + dt$ the value of the same portfolio is :

$$\Gamma_{t+dt} = B_{t+dt}^{[Z]} - N_t^{[1]}\left(B_{t+dt}^{[Y]} - \pounds_{t+dt}\left[B_t^{[Y]}\right]\right) + N_t^{[3]}I_{t+dt} - \pounds_{t+dt}\left[N_t^{[3]}\left(1 - x_t^{[l]}\right)I_t\right]$$

$$-N_t^{[2]}S_{t+dt} + \pounds_{t+dt}\left[x_t^{[s]}N_t^{[2]}S_t\right] + \pounds_{t+dt}\left[\Gamma_t - B_t^{[Z]} - N_t^{[3]}x_t^{[l]}I_t + (1 - x_t^{[s]})N_t^{[2]}S_t\right]$$

After subsequent developments, the variation in the portfolio's value is expressed

as:

$$d\Gamma_t = dB_t^{[Z]} - N_t^{[1]} dB_t^{[Y]} - N_t^{[2]} dS_t + N_t^{[3]} dI_t$$

$$+ \left[N_t^{[1]} B_t^{[Y]} r_t^{[b]} + N_t^{[2]} S_t (r_t^{[f]} + x_t^{[s]} r_t^{[s]} - x_t^{[s]} r_t^{[f]}) + N_t^{[3]} I_t \left(x_t^{[l]} r_t^{[l]} - r_t^{[l]} - x_t^{[l]} r_t^{[f]} \right) \right] dt$$

$$+ \left(\Gamma_t - B_t^{[Z]} \right) r_t^{[f]} dt$$

Then, after replacing $dB_t^{[Z]}$ and $dB_t^{[Y]}$ from equation[2.3.8], $d\Gamma_t$ can be expressed as a function of dY_t, dZ_t, dS_t and dI_t. Moreover from equation[5.3.2], it turns out that $d\Gamma_t$ is actually a function of dY_t , $d\Theta_t$, dS_t. Replacing dY_t , $d\Theta_t$, dS_t , dI_t with their respective expressions, the result is easily derived.

7.3.3 Optimal hedge ratio

a. Result

It is possible to obtain a closed-form solution for the optimal $\widehat{N}_t$

$$\widehat{\mathbf{N}}_t = \frac{1}{2} U_t^{-1} A_t^T \left(\frac{L_t^{(0)} + A_t U_t^{-1} P_t}{A_t U_t^{-1} A_t^T} - \left| \frac{L_t^{(0)} + A_t U_t^{-1} P_t}{A_t U_t^{-1} A_t^T} \right| \right) - U_t^{-1} P_t \qquad (7.3.3)$$

with

$$\widehat{N}_t = \begin{bmatrix} \widehat{N}_t^{[1]} \\ \widehat{N}_t^{[2]} \\ \widehat{N}_t^{[3]} \end{bmatrix} \qquad A_t^T = \begin{bmatrix} -L_t^{(1)} \\ -L_t^{(2)} \\ -L_t^{(3)} \end{bmatrix} \qquad P_t = \begin{bmatrix} -D_{Z_t} D_{Y_t} \sigma_{Y_t}^2 - D_{Z_t} D_{Y_t} \rho_t^{Y,\Theta} \sigma_{\Theta_t} \sigma_{Y_t} \\ -D_{Z_t} \rho_t^{S,\Theta} \sigma_{S_t} \sigma_{\Theta_t} - D_{Z_t} \rho_t^{Y,S} \sigma_{S_t} \sigma_{Y_t} \\ D_{Z_t} \rho_t^{I,\Theta} \sigma_{I_t} \sigma_{\Theta_t} + D_{Z_t} \rho_t^{Y,I} \sigma_{I_t} \sigma_{Y_t} \end{bmatrix}$$

and

$$U_t = \begin{bmatrix} D_{Y_t}^2 \sigma_{Y_t}^2 & D_{Y_t} \rho_t^{Y,S} \sigma_{S_t} \sigma_{Y_t} & -D_{Y_t} \rho_t^{Y,I} \sigma_{I_t} \sigma_{Y_t} \\ D_{Y_t} \rho_t^{Y,S} \sigma_{S_t} \sigma_{Y_t} & \sigma_{S_t}^2 & -\rho_t^{I,S} \sigma_{S_t} \sigma_{I_t} \\ -D_{Y_t} \rho_t^{Y,I} \sigma_{I_t} \sigma_{Y_t} & -\rho_t^{I,S} \sigma_{S_t} \sigma_{I_t} & \sigma_{I_t}^2 \end{bmatrix}$$

b. Proof

The variance of the returns in the portfolio is

$$Var\left[d\Gamma_t\right] = Var\left[K_t(N_t, dW_t)\right]$$

Equation[7.3.2] is equivalent to $K_t = G_t^T dW_t$ where:

$$G_t = \begin{bmatrix} Q_t^{[1,0]} \\ Q_t^{[2,0]} \\ Q_t^{[3,0]} \\ Q_t^{[4,0]} \end{bmatrix} + \begin{bmatrix} Q_t^{[1,1]} & Q_t^{[1,2]} & Q_t^{[1,3]} \\ Q_t^{[2,1]} & 0 & Q_t^{[2,3]} \\ Q_t^{[3,1]} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} N_t$$

Therefore

$$Var\left[K_t(N_t, dW_t)\right] = E\left[\left(G_t^T dW_t\right)\left(G_t^T dW_t\right)^T\right] = G_t^T E\left[dW_t\left(dW_t\right)^T\right] G_t$$

Moreover,

$$E\left[dW_t\left(dW_t\right)^T\right] = I_{4\times 4} dt$$

Then it turns out that :

$$Var\left[K_t(N_t, dW_t)\right] = 2\left[a_t + P_t^T N_t + \frac{1}{2} N_t^T U_t N_t\right] dt$$

where

$$a_t = \frac{1}{2}\sum_{k=1}^{4}\left(Q_t^{[k,0]}\right)^2 \qquad P_t = \begin{bmatrix} \sum_{k=1}^{3} Q_t^{[k,0]} Q_t^{[k,1]} \\ Q_t^{[1,0]} Q_t^{[1,2]} \\ \sum_{k=1}^{2} Q_t^{[k,0]} Q_t^{[k,3]} \end{bmatrix}$$

and

$$U_t = \begin{bmatrix} \sum_{k=1}^{3} \left(Q_t^{[k,1]}\right)^2 & Q_t^{[1,1]} Q_t^{[1,2]} & \sum_{k=1}^{2} Q_t^{[k,1]} Q_t^{[k,3]} \\ Q_t^{[1,1]} Q_t^{[1,2]} & \left(Q_t^{[1,2]}\right)^2 & Q_t^{[1,2]} Q_t^{[1,3]} \\ \sum_{k=1}^{2} Q_t^{[k,1]} Q_t^{[k,3]} & Q_t^{[1,2]} Q_t^{[1,3]} & \sum_{k=1}^{2} \left(Q_t^{[k,3]}\right)^2 \end{bmatrix}$$

After replacing the $Q_t^{[i,j]}$

$$a_t = \frac{1}{2} D_{Z_t}^2 \sigma_{\Theta_t}^2 + \frac{1}{2} D_{Z_t}^2 \sigma_{Y_t}^2 + D_{Z_t}^2 \rho_t^{Y,\Theta} \sigma_{\Theta_t} \sigma_{Y_t}$$

$$P_t = \begin{bmatrix} -D_{Z_t} D_{Y_t} \sigma_{Y_t}^2 - D_{Z_t} D_{Y_t} \rho_t^{Y,\Theta} \sigma_{\Theta_t} \sigma_{Y_t} \\ -D_{Z_t} \rho_t^{S,\Theta} \sigma_{S_t} \sigma_{\Theta_t} - D_{Z_t} \rho_t^{Y,S} \sigma_{S_t} \sigma_{Y_t} \\ D_{Z_t} \rho_t^{I,\Theta} \sigma_{I_t} \sigma_{\Theta_t} + D_{Z_t} \rho_t^{Y,I} \sigma_{I_t} \sigma_{Y_t} \end{bmatrix}$$

$$U_t = \begin{bmatrix} D_{Y_t}^2 \sigma_{Y_t}^2 & D_{Y_t} \rho_t^{Y,S} \sigma_{S_t} \sigma_{Y_t} & -D_{Y_t} \rho_t^{Y,I} \sigma_{I_t} \sigma_{Y_t} \\ D_{Y_t} \rho_t^{Y,S} \sigma_{S_t} \sigma_{Y_t} & \sigma_{S_t}^2 & -\rho_t^{I,S} \sigma_{S_t} \sigma_{I_t} \\ -D_{Y_t} \rho_t^{Y,I} \sigma_{I_t} \sigma_{Y_t} & -\rho_t^{I,S} \sigma_{S_t} \sigma_{I_t} & \sigma_{I_t}^2 \end{bmatrix}$$

In order to prove that U_t is symmetric, positive definite, we use Lemma[2.2.1]. Therefore three conditions have to be checked. The first one is

$$detU_{t\lfloor 1} > 0$$

which is quite obvious. The second one is

$$detU_{t\lfloor 2} > 0$$

which is true because

$$det\left[U_{t\lfloor 2}\right] = D_{Y_t}^2 \sigma_{Y_t}^2 \sigma_{S_t}^2 \left[1 - \left(\rho_t^{Y,S}\right)^2\right] \geq 0 \quad \text{since} \quad \left|\rho_t^{Y,S}\right| < 1$$

The third one is

$$detU_{t\lfloor 3} > 0$$

Moreover, since

$$det\left[U_{t\lfloor 3}\right] = D_{Y_t} \sigma_{S_t}^2 \sigma_{I_t}^2 \sigma_{Y_t}^2 \det(\Delta_{t\lfloor 3})$$

we have to show that

$$\det(\dot{\Delta_{t\lfloor 3}}) > 0$$

Under assumptions[7.2.3] which state that $\det(\dot{\Delta_{t\lfloor 3}}) > 0$ (cf equation[7.2.2]), it turns out that $\det\left[U_{t\lfloor 3}\right] > 0$ which eventually proves that the matrix $U_{t\lfloor 3}$ is positive definite. As previously discussed in section[7.2], assumption[7.2.3] is quite realistic. Moreover it is interesting to find a sufficient condition in order $\det(\dot{\Delta_{t\lfloor 3}})$ to be strictly positive, which is explained next: The correlation (significant enough) between the *stock* and the *index* should be positive because they are likely to following the same trend (the specific stock market sector). In addition, the *stock* might be one of the components of the *index*. Therefore,

$$\rho_t^{I,S} \geq 0 \tag{7.3.4}$$

Furthermore, when the economy is expecting a recession the *yield to maturity of the gilt* increases but the *stock* price declines. A significant enough correlation between the *stock* and the *yield to maturity of the gilt* should always be negative. The same remark can be made as far as the *index* is concerned. Hence,

$$\rho_t^{Y,S} \leq 0 \tag{7.3.5}$$

$$\rho_t^{Y,I} \leq 0 \tag{7.3.6}$$

Using equations[7.3.4,7.3.5 and 7.3.6],

$$\rho_t^{I,S}\ \rho_t^{Y,I}\ \rho_t^{Y,S} \geq 0 \tag{7.3.7}$$

Keeping in mind that

$$\det(\Delta_{t\lfloor 3}) = 1 - \left(\rho_t^{I,S}\right)^2 + \left(\rho_t^{Y,S}\right)^2 + \left(\rho_t^{Y,I}\right)^2 + \rho_t^{I,S}\,\rho_t^{Y,I}\,\rho_t^{Y,S}$$

it turns out

$$\det(\Delta_{t\lfloor 3}) \geq 1 - \left(\rho_t^{I,S}\right)^2 + \left(\rho_t^{Y,S}\right)^2 + \left(\rho_t^{Y,I}\right)^2$$

Finally a sufficient condition for assumptions[7.2.3] to hold is

$$\left(\rho_t^{I,S}\right)^2 + \left(\rho_t^{Y,S}\right)^2 + \left(\rho_t^{Y,I}\right)^2 < 1 \tag{7.3.8}$$

This is a very realistic condition which enable us to assume that $\det(\Delta_{t\lfloor 3}) > 0$ and as a consequence it can be assumed that the matrix U_t is positive definite.

The optimal $\widehat{N}$ will be determined by solving $\begin{cases} \min\limits_{N} Var\left[d\Gamma_t\right] \\ \\ \text{such that } h(N_t) \leq 0 \end{cases}$

wich is equivalent to $\begin{cases} \min\limits_{N_t} H(N_t) = a_t + P_t^T N_t + \dfrac{1}{2} N_t^T U_t N_t \\ \\ \text{such that } h(N_t) \leq 0 \end{cases}$

The constraint is equivalent to:

$$m_t(N_t^{[1]}, N_t^{[2]}, N_t^{[3]}) = L_t^{(0)} + \Gamma_t r_t^{[f]} + N_t^{[1]} L_t^{(1)} + N_t^{[2]} L_t^{(2)} + N_t^{[3]} L_t^{(3)} > \Gamma_t r_t^{[f]}$$

Indeed it seems important to specify that the portfolio must earn at least the risk free interest over the period of the hedge.

$$h(N_t^{[1]}, N_t^{[2]}, N_t^{[3]}) = -L_t^{(0)} - N_t^{[1]} L_t^{(1)} - N_t^{[2]} L_t^{(2)} - N_t^{[3]} L_t^{(3)}$$

$$h(N_t) = A_t N_t - L_t^{(0)}$$

with:

$$A_t = \begin{bmatrix} -L_t^{(1)} & -L_t^{(2)} & -L_t^{(3)} \end{bmatrix} \quad \text{and} \quad N_t = \begin{bmatrix} N_t^{[1]} \\ N_t^{[2]} \\ N_t^{[3]} \end{bmatrix}$$

Since the number of constraints is less than the number of parameters to optimize, it is far easier to find the optimal $\widehat{\mathbf{N}}_t$ from the *Dual problem* than from the *Primal problem* directly.

$$H(N_t) = a_t + P_t^T N_t + \frac{1}{2} N_t^T U_t N_t$$

The *primal problem* is:

$$\min_{N_t \in \Re^3} \left\{ H(N_t) : A_t N_t \leq L_t^{(0)} \right\}$$

The *dual problem* is:

$$\begin{array}{ll} \max & \left\{ H(Q_{\lambda,t}) + \lambda_t^T A_t Q_{\lambda,t} - \lambda_t^T L_t^{(0)} \right\} \\ \lambda_t \in \Re & \\ \lambda_t \geq 0 & \end{array}$$

Then using the same methodology as the one in sub-section[6.3.3] and using the fact that U_t is symetric definite positive, the result is derived.

c. Discussion

Equation[7.3.3] gives the optimal hedge ratio when a long position in the *index* is taken. Unfortunately, after replacing the variables by the market values (for the different securities we have chosen to investigate), it appears that $\widehat{N}_t^{[3]}$ is almost always negative. In other words, this means that a short position in the *index* has to be taken. In fact the first theory explained in the introduction does not seem correct whereas the second seems valid. Therefore equation[7.3.3] determines the optimal hedge ratio only if the expression for $L_t^{(3)}$ is changed into:

$$L_t^{(3)} = -\mu_{I_t} + I_t(r_t^{[f]} + x_t^{[s]} r_t^{[s]} - x_t^{[s]} r_t^{[f]})$$

Indeed, if a short position in the *index* is taken (rather than a long one), after some algebra a different expression for $L_t^{(3)}$ is obtained.

7.4 Implementation and results

7.4.1 Hedge ratio 4

We make the assumption that the price movements implied by the drifts can be neglected compared to those implied by the volatility.

$$m_t^2(N_t)dt << a_t + P_t^T N_t + \frac{1}{2} N_t^T U_t N_t$$

The methodology to check that this assumption holds is explained in sub-section[5.6.1]. If $\widehat{\mathbf{N}}_t^*$ is the *traditional hedge ratio*, it appears that the optimal hedge ratio given by equation[7.3.3] can be written as:

$$\widehat{\mathbf{N}}_t = \begin{bmatrix} \alpha_t^{[1]} \\ \\ \alpha_t^{[2]} \\ \\ \alpha_t^{[3]} \end{bmatrix} \widehat{\mathbf{N}}_t^* \tag{7.4.1}$$

where

$$\alpha_t^{[1]} = 1 + \frac{1}{|\Delta_{t\lfloor 3}|} \frac{\sigma_{\Theta_t}}{\sigma_{Y_t}} \left(\rho_t^{Y,\Theta} \left[1 - \left(\rho_t^{I,S} \right)^2 \right] + \rho_t^{S,\Theta} \left[\rho_t^{Y,I} \rho_t^{I,S} - \rho_t^{Y,S} \right] + \rho_t^{I,\Theta} \left[\rho_t^{Y,S} \rho_t^{I,S} - \rho_t^{Y,I} \right] \right)$$

$$\alpha_t^{[2]} = \frac{D_{Y_t}}{|\Delta_{t\lfloor 3}|} \frac{\sigma_{\Theta_t}}{\sigma_{S_t}} \left(\rho_t^{Y,\Theta} \left[\rho_t^{Y,I} \rho_t^{I,S} - \rho_t^{Y,S} \right] + \rho_t^{S,\Theta} \left[1 - \left(\rho_t^{Y,I} \right)^2 \right] + \rho_t^{I,\Theta} \left[\rho_t^{Y,S} \rho_t^{Y,I} - \rho_t^{I,S} \right] \right)$$

$$\alpha_t^{[3]} = \frac{D_{Y_t}}{|\Delta_{t\lfloor 3}|} \frac{\sigma_{\Theta_t}}{\sigma_{I_t}} \left(\rho_t^{Y,\Theta} \left[\rho_t^{Y,I} - \rho_t^{Y,S} \rho_t^{I,S} \right] + \rho_t^{S,\Theta} \left[\rho_t^{I,S} - \rho_t^{Y,S} \rho_t^{Y,I} \right] - \rho_t^{I,\Theta} \left[1 - \left(\rho_t^{Y,S} \right)^2 \right] \right)$$

The optimal hedge ratio (defined by equation[7.4.1]) is called " *hedge ratio 4* ". It can be shown that when the spread Θ_t is not significant (in terms of magnitude) or is not correlated to any of the other processes (Y_t, I_t, S_t) then the optimal hedge

ratio is reduced to:

$$N_t = \begin{bmatrix} \frac{D_{Z_t}}{D_{Y_t}} \\ 0 \\ 0 \end{bmatrix}$$

which has exactly the same effect as hedging using the traditional hedge ratio. Moreover if the spread is only significantly correlated to the *yield to maturity of the gilt*, then the optimal hedge ratio is reduced to *hedge ratio 1*. Furthermore, the optimal hedge is reduced to *hedge ratio 3* whenever the *credit spread* is correlated (at a significant level) to the *yield to maturity of the gilt* and to the *stock* but not to the *index*.

a. Model assumptions

The *stock* and the *index* are modelled as log-normal processes. The *yield to maturity of the gilt* is described by a log-normal process as well. The *credit spread* process is modelled as a normal process. The processes' parameters are computed daily using a 4 week historical "window" (cf. chapter[3] for further explanations).

b. Description of the bonds to be tested

The BRITISH AEROSPACE bond has already been described in the previous chapter. A *GRANADA* bond is tested as well: Its credit rating falls in the BBB category. The coupon frequency is annual, the coupon's value is fixed at $11\frac{1}{4}\%$. The first settlement date was January 1989 and the bond is due to expire in January 2019. The corresponding gilt (the one that matches closely the maturity and coupon) is a semi-anual fixed coupon at 9%. The first settlement date was July 1992 and the gilt is due to expire in June 2012.

Moreover a *BARCLAYS* bond is investigated: It is rated AA, the coupon frequency is annual, the coupon's value is fixed at $6\frac{1}{2}\%$. The first settlement date was February 1994 and the bond is due to expire in February 2004. The corresponding gilt is a semi-anual fixed coupon at $6\frac{3}{4}\%$. The first settlement date was September 1993 and the gilt is due to expire in November 2004.

7.4.2 Hedging performance

The maturity of the hedge is chosen to be 5 weeks. The hedging stategy using *hedge ratio 4* is applied to British Aerospace, Barclays and Granada over the 1991-1998 time period. As far as Barclays, Granada are considered, it appears that the hedging performances obtained with *hedge ratio 4* are almost exactly the same as those obtained with the *traditional hedge ratio*. But for British Aerospace and especially during the 1991-1994 period, the hedging performances are far better.

Figure[7.4.1] displays the daily 120 day moving average of the variable Risk_t for different hedging strategies applied to British Aerospace.

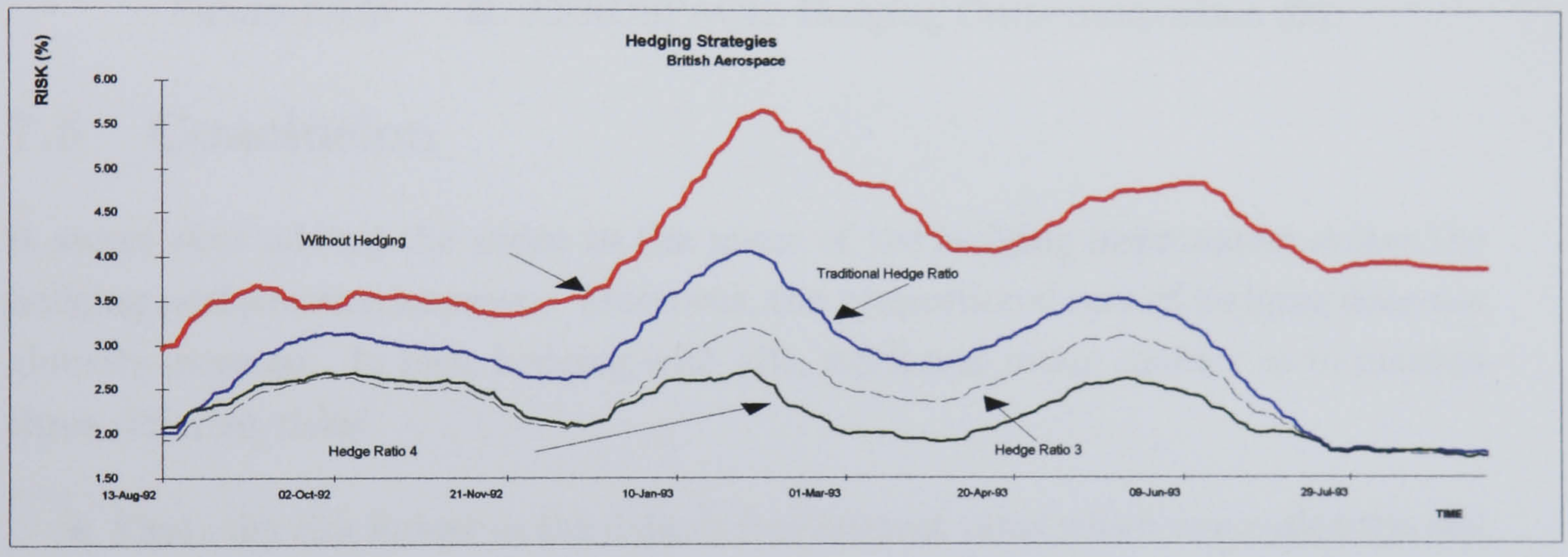


Figure 7.4.1: B. AEROSPACE: Hedging Strategy (Hedge Ratio 4)

7.4.3 Hedging cost

As it can be seen on figure[7.4.2], including the *index* in the menu of the hedging instruments does not globally seem to make the proportional cost of hedging higher than it is when hedging with stock and gilt.

Two periods have to be distinguished. During the first one (August 1992 - December 1992) the "proportional" cost of hedging is cheaper when using hedge ratio 3 (gilt + stock) than when using hedge ratio 4 (gilt + stock + index). On the other hand, during the second period (January 1993 - July 1993) the "proportional" cost of hedging is cheaper when using hedge ratio 4 than when using hedge ratio 3.

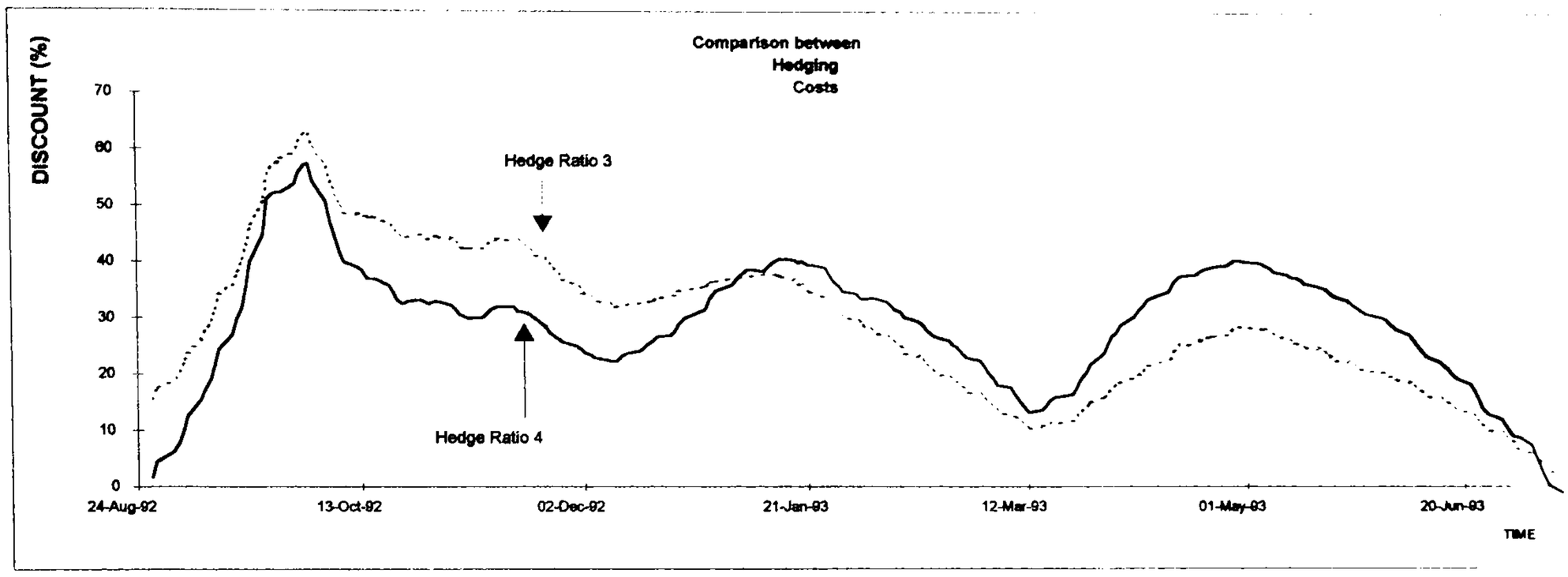


Figure 7.4.2: B. AEROSPACE: Hedging Costs comparison (2)

7.5 Conclusion

It seems that adding the *index* to the menu of the hedging instruments makes the hedging performance improve. Moreover, the proportional cost of hedging does not globally increase. In fact, hedging with gilt, stock and index enables us to manage three different risks:

- First, the risk linked to the default free interest rates which was called the risk free interest rate for reasons of simplicity .
- Secondly, the risk associated with the volatility of the "health" of the company which issued the risky bond.
- Thirdly, the risk due to movements of the whole stock market which has an impact on the bond to be hedged.

Chapter 8

General conclusion

In this thesis, a new theory of hedging investment grade bonds has been developed and improved hedging results have been obtained.

The idea of taking into account some correlations and some volatilities when constructing new hedge ratios seems to be quite useful: For example, in the case of bonds about to mature, it appears that the specific hedge ratio that has been designed for this purpose performs better than the traditional duration-based hedge ratio.

Moreover adding new instruments to the menu of the hedging instruments appears to produce quite an improvement. For example, it seems that using stocks enables us to reduce some very specific risks due to uncertainties about the future financial health of the issuer that are not linked to interest rate risks. In fact, the theory has been tested on 7 typical Eurosterling bonds over a 6 year period of time and it turns out that:

- In most cases, the new hedge ratios are equivalent to the traditional one.
- Frequently the new hedge ratios strongly outperform the traditional duration-based hedge ratio.
- On extremely rare occasions, the new hedge ratios underperform very slightly the traditional one.

This is not surprising since, according to the theory, it has been proved that the new hedge ratios cannot underperform the traditional duration-based hedge one if

the correlations and volatilities estimations are perfectly accurate. As explained in the previous chapters, when correlations are not significant enough in terms of their magnitudes, the new hedge ratios "converge" *in theory* to the traditional duration-based hedge ratio.

It has to be understood that hedging is not cost free. Therefore, there is no point of achieving better hedging if the price to pay for this kind of improvement is too high. Fortunately, as far as the bonds tested are concerned, we have shown that adding new hedging instruments has not increased the relative cost of hedging but has decreased it which, we reckon, is a very good achievement.

Therefore, the results obtained are a strong incentive to keep on investigating in this research area. This would include:

- Finding **better ways of modelling** the different processes involved and more specifically the *yield to maturity of the gilt* and the *credit spread.*

- Testing a **larger number of bonds**.

- Adapting this theory to **bond portfolios** which should not be too difficult to achieve: For example, instead of using a specific gilt, we could use a portfolio of different gilts whose duration matches the one of the portfolio to be hedged. Moreover, instead of using the stock (as the second hedging instrument), we could rather use the related business sector stock index. Finally, instead of the index (as the third hedging instrument), we could use a more global index such as the FTSE 250.

- Developing some tools in order to be able to measure accurately the value of the **liquidity component** in the risk premium. We believe that this challenge could be rewarding. Indeed, such tools would be extremely valuable since the better the understanding (qualitatif and quantitatif) of the determinants of the *risk premium* (liquidity and perception of credit risk), the easier the design of powerful hedging and trading strategies in the fast growing high yield bond market.

Appendix A

A framework to model the shifts in the term structures

Hedging and immunization stategies often implicitly assume a model of the way the yield curve changes.

The purpose of this appendix is to derive a new process that could be used in order to model shifts in the term structure of the *risk free interest rate* and of the *credit spread.*

From equation[2.3.3] and equation[2.3.4], it turns out that:

$$di_t(T) = \eta_i(t,T)d\widetilde{i}_t$$

$$dr_t(T) = \eta_r(t,T)d\widetilde{r}_t$$

It is impossible to have access directly to the processes $d\widetilde{i}_t$ and $d\widetilde{r}_t$ which are the factors governing the movements of the term structure of risk-free interest rate and the term structure of the spread. They can be viewed as "internal" processes.

On the other hand, the "external" processes are the processes for which it is possible to easily find historical data

$$dY_t = \mu_{Y_t}dt + \sigma_{Y_t}dW_t^{(1)} \tag{A.0.1}$$

$$d\Theta_t = \mu_{\Theta_t}dt + \sigma_{\Theta_t}\left[\rho_t^{\Theta,Y}dW_t^{(1)} + \overline{\rho}_t^{\Theta,Y}dW_t^{(2)}\right] \tag{A.0.2}$$

with

$$\overline{\rho}_t^{\Theta,Y} = \sqrt{1 - \left(\rho_t^{\Theta,Y}\right)^2}$$

It is important to notice that $\widetilde{di}_t$ and $\widetilde{dr}_t$ do not depend on the maturity T . But there is a possible way to calculate $\widetilde{di}_t$ and $\widetilde{dr}_t$. This will be explained in this chapter.

A.1 Risk free interest rate movements

A possible modelling is

$$\widetilde{di}_t = \mu_t^{[i]} dt + \frac{D_{y_t}^{[Y]}}{D_{i_t}^{[Y]}} \sigma_{Y_t} dW_t^{(1)}$$

with

$$\mu_t^{[I]} = \frac{\Psi_{1,t}^{[Y]} - \Psi_{2,t}^{[Y]}}{D_{i_t}^{[Y]}} + \frac{D_{y_t}^{[Y]}}{D_{i_t}^{[Y]}} + \left(\frac{C_{y_t}^{[Y]}}{D_{i_t}^{[Y]}} - \frac{C_{i_t}^{[Y]} \left(D_{y_t}^{[Y]}\right)^2}{\left(D_{i_t}^{[Y]}\right)^3} \right) \sigma_{Y_t}^2$$

Proof

The process $\widetilde{di}_t$ is chosen to be modelled as

$$\widetilde{di}_t = \mu_t^{[i]} dt + f_1(.) dW_t^{(1)} \tag{A.1.1}$$

with $\mu_t^{[i]}$ and $f_1(.)$ that have to be determined.

The trick is to write that a small variation in the price of the gilt $B_t^{[Y]}$ (in a differential sense) should be the same whatever model is considered. In other words,

$$\left[dB_t^{[Y]}\right]_{\text{single variable model}} = \left[dB_t^{[Y]}\right]_{\text{multi variable model}}$$

From equation[2.3.8],

$$\left[dB_t^{[Y]}\right]_{\text{single variable model}} = \Psi_{1,t}^{[Y]} dt + D_{y_t}^{[Y]} dY_t + C_{y_t}^{[Y]} [dY_t]^2$$

but from equation[A.0.1],

$$dY_t = \mu_{Y_t} dt + \sigma_{Y_t} dW_t^{(1)}$$

Hence :

$$\left[dB_t^{[Y]}\right]_{\text{multi variable model}} = \left\{\Psi_{1,t}^{[Y]} + \mu_{Y_t} D_{y_t}^{[Y]} + C_{y_t}^{[Y]} \sigma_{Y_t}^2\right\} dt + D_{Y_t}^{[Y]} \sigma_{Y_t} dW_t^{(1)} \quad \text{(A.1.2)}$$

From equation[2.3.12] and from the fact that as far as the Gilt is concerned, the spread is zero

$$\left[dB_t^{[Y]}\right]_{\text{multi variable model}} = \Psi_{2,t}^{[Y]} dt + D_{i_t}^{[Y]} di_t + C_{i_t}^{[Y]} di_t^2$$

From equation[A.1.1] , $\widetilde{di_t} = \mu_t^{[i]} dt + f_1(.) dW_t^{(1)}$

$$\left[dB_t^{[Y]}\right]_{\text{multi variable model}} = \left\{\Psi_{2,t}^{[Y]} + \mu_t^{[i]} D_{i_t}^{[Y]} + C_{i_t}^{[Y]} f_1^2\right\} dt + D_{i_t}^{[Y]} f_1 dW_t^{(1)} \quad \text{(A.1.3)}$$

Using equations[A.1.2] and [A.1.3], writing

$$\left[dB_t^{[Y]}\right]_{\text{single variable model}} - \left[dB_t^{[Y]}\right]_{\text{multi variable model}} = 0$$

and using the fact that

$$\left\{ \begin{array}{c} k_o dt + \sum_{i=1}^{M} k_i dW_t^{(i)} = 0 \\ \text{with } \left(dW_t^{(i)}\right)_{i=1,M} \text{ independent Brownian motions} \end{array} \right\} \Longleftrightarrow \{k_i = 0, \forall i = 0, M\}$$

we get

$$\left\{ \begin{array}{l} \Psi_{1,t}^{[Y]} - \Psi_{2,t}^{[Y]} + \mu_{Y_t} D_{y_t}^{[Y]} - \mu_t^{[i]} D_{i_t}^{[Y]} + C_{y_t}^{[Y]} \sigma_{Y_t}^2 - C_{i_t}^{[Y]} f_1^2 = 0 \\ \\ D_{y_t}^{[Y]} \sigma_{Y_t} - D_{i_t}^{[Y]} f_1 = 0 \end{array} \right. \quad \text{(A.1.4)}$$

Hence solving the two equations above, one can extract the expressions for $\mu_t^{[i]}$ and f_1.

A.2 Credit Spread movements

A possible model is

$$d\widetilde{r}_t = \mu_t^{[r]} dt + \left[\frac{D_{z_t}^{[Z]}}{D_{r_t}^{[Z]}} \left(\sigma_{Y_t} + \sigma_{\Theta_t} \rho_t^{\Theta,Y} \right) - \frac{D_{i_t}^{[Z]}}{D_{r_t}^{[Z]}} \frac{D_{y_t}^{[Y]}}{D_{i_t}^{[Y]}} \sigma_{Y_t} \right] dW_t^{(1)} + \frac{D_{z_t}^{[Z]}}{D_{r_t}^{[Z]}} \sigma_{\Theta_t} \overline{\rho}_t^{\Theta,Y} dW_t^{(2)}$$

with

$$\mu_t^{[r]} = - \left(\frac{C_{r_t}^{[Z]}}{\left(D_{r_t}^{[Z]} \right)^3} \left[D_{z_t}^{[Z]} - \frac{D_{i_t}^{[Z]} D_{y_t}^{[Y]}}{D_{i_t}^{[Y]}} \right]^2 + \frac{C_{i_t}^{[Z]} \left(D_{y_t}^{[Y]} \right)^2}{D_{r_t}^{[Z]} \left(D_{i_t}^{[Y]} \right)^2} + \frac{D_{i_t}^{[Z]} \left(D_{y_t}^{[Y]} \right)^2 C_{i_t}^{[Y]}}{D_{r_t}^{[Z]} \left(D_{i_t}^{[Y]} \right)^3} \right) \sigma_{Y_t}^2$$

$$- \left(\frac{D_{i_t}^{[Z]} C_{y_t}^{[Y]}}{D_{r_t}^{[Z]} D_{i_t}^{[Y]}} + \frac{C_{z_t}^{[Z]}}{D_{r_t}^{[Z]}} \right) \sigma_{Y_t}^2 - C_{i_t,r_t}^{[Z]} \left(\frac{D_{y_t}^{[Y]}}{D_{i_t}^{[Y]}} \frac{D_{z_t}^{[Z]}}{\left(D_{r_t}^{[Z]} \right)^2} - \frac{\left(D_{y_t}^{[Y]} \right)^2 D_{i_t}^{[Z]}}{\left(D_{i_t}^{[Y]} \right)^2 \left(D_{r_t}^{[Z]} \right)^2} \right) \sigma_{Y_t}^2$$

$$+ \left\{ 2 \frac{C_{z_t}^{[Z]}}{D_{r_t}^{[Z]}} - 2 C_{r_t}^{[Z]} \left(\frac{\left(D_{y,t}^{(c)} \right)^2}{\left(D_{r_t}^{[Z]} \right)^3} - \frac{D_{Z_t} D_{i_t}^{[Z]} D_{y_t}^{[Y]}}{\left(D_{r_t}^{[Z]} \right)^3 D_{i_t}^{[Y]}} \right) - C_{i_t,r_t}^{[Z]} \frac{D_{y_t}^{[Y]} D_{z_t}^{[Z]}}{D_{i_t}^{[Y]} \left(D_{r_t}^{[Z]} \right)^2} \right\} \sigma_{\Theta_t} \sigma_{Y_t} \rho_t^{\Theta,Y}$$

$$+ \frac{D_{z_t}^{[Z]}}{D_{r_t}^{[Z]}} \mu_{\Theta_t} + \left(\frac{D_{z_t}^{[Z]}}{D_{r_t}^{[Z]}} - \frac{D_{i_t}^{[Z]} D_{y_t}^{[Y]}}{D_{r_t}^{[Z]} D_{i_t}^{[Y]}} \right) \mu_{Y_t} + \left(\frac{C_{z_t}^{[Z]}}{D_{r_t}^{[Z]}} - \frac{C_{r_t}^{[Z]} \left(D_{z_t}^{[Z]} \right)^2}{\left(D_{r_t}^{[Z]} \right)^3} \right) \sigma_{\Theta_t}^2$$

$$+ \left(\Psi_{1,t}^{[Z]} - \Psi_{2,t}^{[Z]} \right) \left(\frac{1}{D_{r_t}^{[Z]}} - \frac{D_{i_t}^{[Z]}}{D_{r_t}^{[Z]} D_{i_t}^{[Y]}} \right)$$

Proof

The process $d\widetilde{r}_t$ is chosen to be modelled as

$$d\widetilde{r}_t = \mu_t^{[r]}(.)dt + g_1(.)dW_t^{(1)} + g_2(.)dW_t^{(2)} \qquad \text{(A.2.1)}$$

with $\mu_t^{[r]}$, $g_1(.)$ and $g_2(.)$ that have to be determined.

The trick is to write that a small variation in the price of the bond to be hedged $P_t^{(c)}$ (in a differential sense) should be the same whatever model is considered. In other words,

$$\left[dB_t^{[Z]}\right]_{\text{single variable model}} = \left[dB_t^{[Z]}\right]_{\text{multi variable model}}$$

From equation[2.3.8],

$$\left[dB_t^{[Z]}\right]_{\text{single variable model}} = \Psi_{1,t}^{[Z]}dt + D_{z_t}^{[Z]}dZ_t + C_{z_t}^{[Z]}\left[dZ_t\right]^2$$

Moreover,

$$dZ_t = d(Y_t + \Theta_t) = dY_t + d\Theta_t \tag{A.2.2}$$

Therefore, using equations[A.0.1] and [A.0.2], it turns out that

$$dZ_t = \left(\mu_{Y_t} + \mu_{\Theta_t}\right)dt + \left(\sigma_{Y_t} + \sigma_{\Theta_t}\rho_t^{\Theta,Y}\right)dW_t^{(1)} + \sigma_{\Theta_t}\overline{\rho}_t^{\Theta,Y}dW_t^{(2)}$$

Hence :

$$\left[dB_t^{[Z]}\right]_{\text{single variable model}} = \left(\Psi_{1,t}^{[Z]} + \left(\mu_{Y_t} + \mu_{\Theta_t}\right)D_{z_t}^{[Z]} + C_{z_t}^{[Z]}\left[\sigma_{Y_t}^2 + \sigma_{\Theta_t}^2 + 2\rho_t^{\Theta,Y}\sigma_{Y_t}\sigma_{\Theta_t}\right]\right)dt$$
$$+D_{z_t}^{[Z]}\left(\sigma_{Y_t} + \sigma_{t,\Theta_t}^{(\Theta)}\rho_t^{\Theta,Y}\right)dW_t^{(1)} + D_{z_t}^{[Z]}\sigma_{\Theta_t}\overline{\rho}_t^{\Theta,Y}dW_t^{(2)}$$

From equation[2.3.12]

$$\left[dB_t^{[Z]}\right]_{\text{multi variable model}} = \Psi_{2,t}^{[Z]}dt + D_{i_t}^{[Z]}di_t + D_{r_t}^{[Z]}dr_t + C_{i_t}^{[Z]}di_t^2 + C_{r_t}^{[Z]}dr_t^2 + C_{i_t,r_t}^{[Z]}di_t dr_t$$

From equations[A.1.1] and [A.2.1] ,

$$\begin{aligned} \widetilde{di_t} &= \mu_t^{[i]}(.)dt + f_1(.)dW_t^{(1)} \\ \widetilde{dr_t} &= \mu_t^{[r]}(.)dt + g_1(.)dW_t^{(1)} + g_2(.)dW_t^{(2)} \end{aligned}$$

and hence:

$$\left[dB_t^{[Z]}\right]_{\text{multi variable model}} = \Psi_{2,t}^{[Z]}dt + D_{i_t}^{[Z]}di_t + D_{r_t}^{[Z]}dr_t + C_{i_t}^{[Z]}di_t^2 + C_{r_t}^{[Z]}dr_t^2 + C_{i_t,r_t}^{[Z]}di_t dr_t$$

$$= \Psi_{2,t}^{[Z]}dt + D_{i_t}^{[Z]}\left(f_0 dt + f_1 dW_t^{(1)}\right) + D_{r_t}^{[Z]}(\mu_t^{[r]}dt + g_1 dW_t^{(1)} + g_2 dW_t^{(2)})$$

$$+C^{[Z]}_{i_t} f_1^2 dt + C^{[Z]}_{r_t}(g_1^2+g_2^2)dt + C^{[Z]}_{i_t,r_t} f_1 g_1 dt$$

$$= \left(\Psi^{[Z]}_{2,t} + \mu^{[i]}_t D^{[Z]}_{i_t} + \mu^{[r]}_t D^{[Z]}_{r_t} + C^{[Z]}_{i_t} f_1^2 + C^{[Z]}_{r_t}(g_1^2+g_2^2) + C^{[Z]}_{i_t,r_t} f_1 g_1\right) dt$$

$$+ \left(D^{[Z]}_{i_t} f_1 + D^{[Z]}_{r_t} g_1\right) dW_t^{(1)} + D^{[Z]}_{r_t} g_2 dW_t^{(2)}$$

In addition, writing

$$\left[dB_t^{[Z]}\right]_{\text{single variable model}} - \left[dB_t^{[Z]}\right]_{\text{multi variable model}} = 0 \qquad \text{(A.2.3)}$$

and using the fact that

$$\left\{ \begin{array}{c} k_o dt + \sum_{i=1}^{M} k_i dW_t^{(i)} = 0 \\ \\ \text{with } \left(dW_t^{(i)}\right)_{i=1,M} \text{ independent Brownian motions} \end{array} \right\} \Longleftrightarrow \{k_i = 0, \forall i = 0, M\}$$

a system of 3 equations is derived.

The first equation is:

$$\Psi^{[Z]}_{1,t} - \Psi^{[Z]}_{2,t} + \left(\mu_{Y_t} + \mu_{\Theta_t}\right) D_{Z_t} - \mu^{[i]}_t D^{[Z]}_{i_t} - \mu^{[r]}_t D^{[Z]}_{r_t} - C^{[Z]}_{i_t} f_1^2$$

$$+C^{[Z]}_{z_t}\left[\sigma^2_{Y_t} + \sigma^2_{\Theta_t} + 2\rho_t^{\Theta,Y}\sigma_{Y_t}\sigma_{\Theta_t}\right] - C^{[Z]}_{r_t}(g_1^2+g_2^2) - C^{[Z]}_{i_t,r_t} f_1 g_1 = 0$$

The second equation is:

$$D^{[Z]}_{z_t}\left(\sigma_{Y_t} + \sigma^{(\Theta)}_{t,\Theta_t}\rho_t^{\Theta,Y}\right) = D^{[Z]}_{i_t} f_1 + D^{[Z]}_{r_t} g_1$$

and the third one is:

$$D^{[Z]}_{z_t}\sigma_{\Theta_t}\overline{\rho}_t^{\Theta,Y} = D^{[Z]}_{r_t} g_2$$

Hence solving the three equations above, one can extract the expressions for $\mu^{[r]}_t$, g_1 and g

Appendix B

Statistical framework

B.1 Unbiased estimators

$$\widetilde{E}[x_t] = \frac{|\overline{x}_t|_1}{N} = H(x_t)$$

Moreover this estimator is unbiased since:

$$E[\widetilde{E}[x_t]] = E[\frac{|\overline{x}_t|_1}{N}] = E[\frac{1}{N}\sum_{k=1}^{N} x_{t+1-k}] = \frac{1}{N}\sum_{k=1}^{N} E[x_{t+1-k}] = \frac{1}{N}NE[x_t] = E[x_t]$$

(using assumption 3.2.1)

$$\widetilde{Var}[x_t] = \frac{1}{N-1}\left(\langle \overline{x}_t, \overline{x}_t \rangle - \frac{1}{N}|\overline{x}_t|_1^2\right) = \frac{1}{N(N-1)}\left(N\langle \overline{x}_t, \overline{x}_t \rangle - |\overline{x}_t|_1^2\right) = G(x_t, x_t)$$

The fact that this estimator is unbiased is straightforward and is a very known result:

$$E[\widetilde{Var}[x_t]] = E[\frac{1}{N-1}\left(\langle \overline{x}_t, \overline{x}_t \rangle - \frac{1}{N}|\overline{x}_t|_1^2\right)] = E[\frac{1}{N-1}\left(\sum_{k=1}^{N}(x_{t+1-k})^2 - \frac{1}{N}|\overline{x}_t|_1^2\right)]$$

$$= \frac{1}{N-1}E[\sum_{k=1}^{N}(x_{t+1-k})^2 - \frac{1}{N}|\overline{x}_t|_1^2] = \frac{1}{N(N-1)}E[N\sum_{k=1}^{N}(x_{t+1-k})^2 - |\overline{x}_t|_1^2]$$

after some manipulation,

$$= \frac{1}{N\left(N-1\right)}\left(N^2E\left[x_t^2\right] - E[\sum_{k=1}^{N} x_{t+1-k}]^2\right)$$

$$= \frac{1}{N\left(N-1\right)}\left(N^2E\left[x_t^2\right] - E[\sum_{k=1}^{N} x_{t+1-k}^2 + 2\sum_{i\neq j}^{N} x_{t+1-i}x_{t+1-j}]\right)$$

(using the assumption 3.2.1)

$$= \frac{1}{N\left(N-1\right)}\left(N^2E\left[x_t^2\right] - NE\left[x_t^2\right] - 2\sum_{i\neq j}^{N} E\left[x_{t+1-i}\right]E\left[x_{t+1-j}\right]\right)$$

$$= \frac{1}{N\left(N-1\right)}\left(N^2E\left[x_t^2\right] - NE\left[x_t^2\right] - N(N-1)E^2\left[x_t\right]\right)$$

(using the assumption 3.2.2)

$$= \frac{1}{N\left(N-1\right)}\left[N^2\left(E\left[x_t^2\right] - E^2\left[x_t\right]\right) - N\left(E\left[x_t^2\right] - E^2\left[x_t\right]\right)\right] = Var[x_t]$$

In the same way,

$$\widetilde{Cov}\left[x_t, y_t\right] = \frac{1}{N-1}\left(\langle\overline{x}_t, \overline{y}_t\rangle - \frac{1}{N}\left|\overline{x}_t\right|_1\left|\overline{y}_t\right|_1\right) = \frac{1}{N\left(N-1\right)}\left(N\left\langle\overline{x}_t, \overline{x}_t\right\rangle - \left|\overline{x}_t\right|_1\left|\overline{y}_t\right|_1\right)$$

$$= G(x_t, y_t)$$

and by analogy, it is possible to prove that $E[\widetilde{Cov}\left[x_t, y_t\right]] = Cov[x_t, y_t]$

B.2 Estimated Correlations

$$x_t = \left(\mu_X - \frac{\sigma_X^2}{2}\right) dt + \sum_{i=1}^{q_1} \sigma_i^{(X)} \int_{t-dt}^{t} dW_s^{(i)}$$

$$y_t = b_Y \left(1 - e^{-a_Y dt}\right) + \sum_{j=1}^{q_2} \sigma_j^{(Y)} e^{-a_Y t} \int_{t-dt}^{t} e^{a_Y s} dW_s^{(j)}$$

$$cov\left[x_t, y_t\right] = E\left[\sum_{i=1}^{q_1} \sigma_i^{(X)} \int_{t-dt}^{t} dW_s^{(i)} \sum_{j=1}^{q_2} \sigma_j^{(Y)} e^{-a_Y t} \int_{t-dt}^{t} e^{a_Y s} dW_s^{(j)}\right]$$

$$= e^{-a_Y t} E\left[\sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \sigma_i^{(X)} \sigma_j^{(Y)} \int_{t-dt}^{t} dW_s^{(i)} \int_{t-dt}^{t} e^{a_Y s} dW_s^{(j)}\right]$$

$$= e^{-a_Y t} \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \sigma_i^{(X)} \sigma_j^{(Y)} E\left[\int_{t-dt}^{t} e^{a_Y s} d\left\langle W^{(i)}, W^{(j)} \right\rangle_s\right]$$

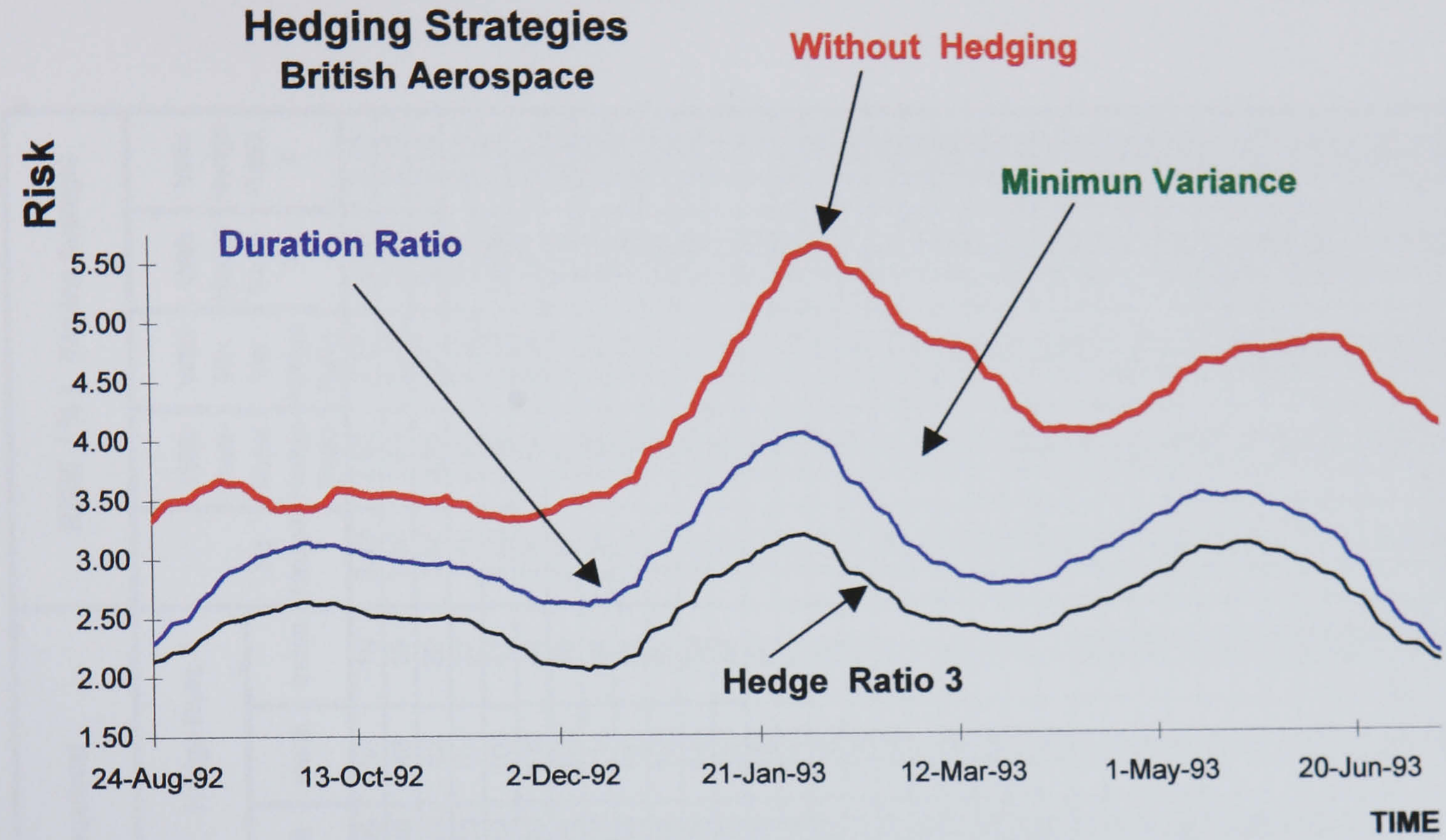
$$= e^{-a_Y t} \sum_{k=1}^{\min(q_1, q_2)} \sigma_k^{(X)} \sigma_k^{(Y)} E\left[\int_{t-dt}^{t} e^{a_Y s} ds\right] = \frac{\left[1 - e^{-a_Y dt}\right]}{a_Y} \sigma_X \sigma_Y \rho_{X,Y}$$

$$\widetilde{\rho_{X,Y}} = \frac{\widetilde{a_Y}}{\left(1 - e^{-\widetilde{a_Y} dt}\right) \widetilde{\sigma_X} \widetilde{\sigma_Y}} \widetilde{cov}\left[x_t, \widetilde{y_t}\right]$$

$$\widetilde{\rho_{X,Y}} = \frac{\widetilde{a_Y}}{\left(1 - e^{-\widetilde{a_Y} dt}\right) \widetilde{\sigma_X} \widetilde{\sigma_Y}} G(x_t, \widetilde{y_t})$$

Appendix C

Numerical Results


Hedging Strategies
British Aerospace
Without Hedging
Minimun Variance
Duration Ratio
Hedge Ratio 3
Risk
5.50
5.00
4.50
4.00
3.50
3.00
2.50
2.00
1.50
24-Aug-92
13-Oct-92
2-Dec-92
21-Jan-93
12-Mar-93
1-May-93
20-Jun-93
TIME

1 / 5

British Aerospace

Hedge horizon: 5 weeks

	BOND Dirty Price (£)	BOND Cash Flow (£)	GILT Dirty Price (£)	GILT Cash Flow (£)	GILT Yield (%)	BOND Spread Bpt	STOCK (£)	FTSE Engn. (£)	**Quantity of hedging instrument required**								**RISK (%)** (Moving Average)				
									Traditional Hedge Ratio	Minimum Variance Ratio		Hedge Ratio 3		Hedge Ratio 4			No Hedge	With Traditional Hedge Ratio	With Min. Var. Hedge Ratio	With Hedge Ratio 3	With Hedge Ratio 4
									gilt	gilt	stock	gilt	stock	gilt	stock	Index					
1-Dec-92	1002034		1025769		9.03	395	131	1042	0.76	1.01	31	0.99	40	0.99	58	18	3.43	2.63	2.27	2.09	2.16
2-Dec-92	1001130		1023205		9.06	395	127	1044	0.76	1.01	12	0.96	31	0.96	46	13	3.46	2.64	2.28	2.10	2.16
3-Dec-92	1014585		1029072		8.99	381	141	1063	0.77	1.02	142	1.15	56	1.15	82	29	3.48	2.63	2.27	2.09	2.15
4-Dec-92	1014911		1037622		8.90	390	133	1054	0.76	1.01	218	1.13	70	1.13	104	37	3.49	2.63	2.25	2.08	2.14
7-Dec-92	1015856		1036938		8.91	388	140	1064	0.76	0.94	189	1.12	71	1.12	104	41	3.50	2.62	2.24	2.08	2.12
8-Dec-92	1019312		1041245		8.86	388	139	1063	0.76	0.91	200	1.10	79	1.10	116	46	3.51	2.62	2.23	2.07	2.11
9-Dec-92	1022788		1044301		8.83	388	131	1057	0.76	0.93	188	0.76	81	0.76	120	46	3.52	2.61	2.21	2.05	2.09
10-Dec-92	1018733		1040178		8.88	389	137	1050	0.76	0.93	180	0.76	77	0.76	114	49	3.54	2.65	2.22	2.09	2.12
11-Dec-92	1024058		1045288		8.83	387	138	1048	0.76	0.91	174	0.76	82	0.76	121	52	3.54	2.64	2.23	2.08	2.11
14-Dec-92	1024694		1045844		8.82	387	138	1052	0.76	0.92	170	0.76	89	0.76	132	57	3.54	2.64	2.23	2.08	2.09
15-Dec-92	1020649		1042661		8.86	389	139	1054	0.76	0.94	175	0.76	95	0.76	140	59	3.57	2.66	2.25	2.09	2.10
16-Dec-92	1024435		1044157		8.85	387	146	1074	0.76	0.94	183	0.76	99	0.76	147	58	3.60	2.69	2.28	2.11	2.11
17-Dec-92	1026010		1045964		8.83	386	151	1084	0.76	0.93	174	0.76	110	0.76	164	66	3.63	2.72	2.30	2.13	2.12
18-Dec-92	1026336		1046074		8.84	386	165	1112	0.76	0.92	155	0.76	99	0.76	148	62	3.65	2.75	2.33	2.16	2.14
21-Dec-92	1028541		1044140		8.86	380	175	1148	0.77	0.92	171	0.77	108	0.77	163	58	3.68	2.78	2.35	2.18	2.16
22-Dec-92	976250	**59375**	1049067		8.81	377	167	1157	0.77	0.59	485	0.77	117	0.77	178	64	3.77	2.87	2.41	2.27	2.24
23-Dec-92	980635		1052443		8.77	375	160	1143	0.77	0.60	444	0.77	127	0.77	195	65	3.84	2.94	2.44	2.32	2.28
24-Dec-92	980961		1053676		8.77	375	155	1140	0.77	0.50	415	0.77	139	0.77	217	72	3.92	3.01	2.47	2.38	2.32
29-Dec-92	982262		1054543		8.76	375	159	1153	0.77	0.47	413	0.77	144	0.77	228	78	3.99	3.08	2.50	2.44	2.36
30-Dec-92	976857		1059789		8.70	390	165	1151	0.76	0.33	471	0.76	189	0.76	303	106	4.07	3.17	2.55	2.51	2.40
31-Dec-92	980523		1058596		8.73	383	163	1157	0.77	0.30	457	0.77	188	0.77	305	98	4.15	3.24	2.59	2.57	2.43
4-Jan-93	981258		1054772		8.78	377	173	1173	0.77	0.31	456	0.77	168	0.77	274	91	4.23	3.32	2.63	2.64	2.47
5-Jan-93	978253		1050339		8.83	377	176	1179	0.77	0.32	455	0.77	164	0.77	271	90	4.32	3.40	2.68	2.71	2.52
6-Jan-93	985479		1050265		8.84	367	186	1195	0.78	0.29	475	0.78	171	0.78	284	101	4.40	3.47	2.72	2.78	2.56
7-Jan-93	998305		1048012		8.87	344	192	1215	0.80	0.10	495	0.80	264	0.80	443	146	4.47	3.53	2.76	2.82	2.58
8-Jan-93	996960		1046572		8.89	344	188	1203	0.80	0.11	503	0.80	265	0.80	454	150	4.53	3.58	2.79	2.85	2.60
11-Jan-93	1013955		1039628		8.98	311	183	1172	0.83	-0.29	391	0.83	388	0.83	678	195	4.58	3.60	2.83	2.86	2.62
12-Jan-93	1011781		1037375		9.01	311	174	1157	0.83	-0.29	382	0.83	356	0.83	638	184	4.63	3.64	2.89	2.88	2.62
13-Jan-93	1010677		1039812		8.98	317	168	1155	0.83	-0.29	386	0.83	336	0.83	614	179	4.70	3.69	2.95	2.90	2.61
14-Jan-93	1008082		1040678		8.98	322	175	1169	0.82	-0.30	386	0.82	306	0.82	570	175	4.77	3.73	3.01	2.93	2.60
15-Jan-93	1009238		1042978		8.96	323	183	1170	0.82	-0.32	381	0.82	268	0.82	509	165	4.84	3.77	3.06	2.96	2.60
18-Jan-93	1010403		1038854		9.01	316	183	1172	0.83	-0.32	386	0.83	274	0.83	527	166	4.93	3.81	3.13	2.98	2.60
19-Jan-93	1004888		1039411		9.01	325	192	1170	0.82	-0.37	359	0.82	246	0.82	473	183	5.01	3.85	3.22	3.02	2.61
20-Jan-93	1005864		1038718		9.02	324	200	1180	0.82	-0.41	364	0.82	216	0.82	422	164	5.08	3.90	3.29	3.06	2.62
21-Jan-93	1008690		1044594		8.95	327	208	1183	0.82	-0.46	358	0.82	190	0.82	375	152	5.15	3.92	3.36	3.06	2.61
22-Jan-93	1023115		1045644		8.95	307	213	1187	0.84	-0.49	389	0.84	179	0.84	358	142	5.21	3.96	3.43	3.09	2.61

British Aerospace

Hedge horizon: 5 weeks

	BOND Dirty Price (£)	BOND Cash Flow (£)	GILT Dirty Price (£)	GILT Cash Flow (£)	GILT Yield (%)	BOND Spread Bpt	STOCK (£)	FTSE Engn. (£)	Quantity of hedging instrument required: Traditional Hedge Ratio gilt	Minimum Variance Ratio gilt	Minimum Variance Ratio stock	Hedge Ratio 3 gilt	Hedge Ratio 3 stock	Hedge Ratio 4 gilt	Hedge Ratio 4 stock	Hedge Ratio 4 Index	RISK (%) (Moving Average): No Hedge	With Traditional Hedge Ratio	With Min. Var. Hedge Ratio	With Hedge Ratio 3	With Hedge Ratio 4
25-Jan-93	1023930		1041200		9.00	300	225	1190	0.84	-0.44	356	0.84	132	0.84	266	117	5.29	3.99	3.52	3.12	2.63
26-Jan-93	1020506		1050197		8.90	316	226	1204	0.83	-0.60	245	0.83	104	0.83	211	90	5.37	4.03	3.62	3.16	2.66
27-Jan-93	1030232		1049514		8.91	303	225	1206	0.84	-0.76	265	0.84	80	0.84	162	68	5.43	4.04	3.71	3.16	2.67
28-Jan-93	1028717		1059440		8.79	317	233	1216	0.82	-0.69	288	0.82	51	0.82	103	44	5.51	4.06	3.80	3.19	2.68
29-Jan-93	1038452		1059250		8.80	302	251	1232	0.84	-0.74	346	0.84	21	0.84	42	20	5.57	4.06	3.88	3.19	2.69
1-Feb-93	1067268		1062926		8.76	266	271	1267	0.87	-0.66	645	0.87	4	0.87	8	4	5.60	4.08	3.93	3.21	2.71
2-Feb-93	1079443		1067863		8.71	256	271	1265	0.88	-0.61	619	0.88	0	0.88	0	0	5.62	4.05	3.97	3.19	2.68
3-Feb-93	1085419		1069050		8.69	250	284	1287	0.89	-0.69	631	0.89	0	0.89	0	0	5.65	4.04	4.02	3.17	2.65
4-Feb-93	1084494		1067736		8.71	250	275	1293	0.89	-0.71	606	0.89	0	0.89	0	0	5.66	4.02	4.06	3.16	2.62
5-Feb-93	1084410		1069096		8.71	251	273	1291	0.89	-0.76	591	0.89	0	0.89	0	0	5.67	4.00	4.10	3.14	2.58
8-Feb-93	1085565		1067472		8.73	248	264	1286	0.89	-0.78	562	0.89	0	0.89	0	0	5.63	3.95	4.12	3.10	2.54
9-Feb-93	1088390		1063969		8.77	240	258	1269	0.90	-0.80	551	0.90	5	0.90	9	5	5.59	3.87	4.11	3.03	2.47
10-Feb-93	1085616		1064836		8.76	246	257	1261	0.89	-0.82	598	0.89	17	0.89	32	17	5.53	3.79	4.10	2.96	2.40
11-Feb-93	1085632		1065082		8.76	246	260	1261	0.89	-0.85	618	0.89	28	0.89	54	28	5.48	3.71	4.09	2.90	2.34
12-Feb-93	1085957		1080822		8.59	264	255	1253	0.87	-0.21	314	0.87	45	0.87	88	45	5.45	3.66	4.07	2.87	2.32
15-Feb-93	1086282		1082318		8.57	265	248	1250	0.87	-0.23	334	0.87	71	0.87	140	72	5.43	3.63	4.07	2.86	2.31
16-Feb-93	1104838		1085695		8.54	245	241	1247	0.89	-0.15	243	0.89	100	0.89	196	98	5.40	3.58	4.04	2.83	2.28
17-Feb-93	1108724		1086872		8.53	242	243	1241	0.89	-0.15	245	0.89	115	0.89	224	111	5.36	3.53	4.01	2.80	2.26
18-Feb-93	1109889		1086498		8.53	240	248	1249	0.90	-0.08	255	0.90	138	0.90	272	130	5.32	3.47	3.98	2.77	2.23
19-Feb-93	1109585		1092238		8.48	247	259	1255	0.89	-0.12	232	0.89	136	0.89	270	143	5.27	3.42	3.95	2.74	2.21
22-Feb-93	1115750		1092795		8.47	240	258	1258	0.90	-0.16	248	0.90	151	0.90	303	161	5.21	3.37	3.91	2.70	2.17
23-Feb-93	1115135		1094601		8.45	243	250	1244	0.89	-0.24	263	0.89	168	0.89	344	177	5.14	3.31	3.87	2.67	2.15
24-Feb-93	1117361		1088918		8.52	234	262	1250	0.90	-0.17	228	0.90	189	0.90	391	206	5.08	3.26	3.84	2.64	2.13
25-Feb-93	1112057		1092594		8.48	245	264	1253	0.89	-0.16	215	0.89	177	0.89	366	222	5.03	3.18	3.80	2.57	2.06
26-Feb-93	1117702		1094904		8.46	240	260	1262	0.90	-0.17	195	0.90	181	0.90	370	213	4.98	3.13	3.78	2.54	2.03
1-Mar-93	1124897		1104211		8.36	242	258	1267	0.89	0.08	233	0.89	204	0.89	410	255	4.94	3.08	3.77	2.51	2.01
2-Mar-93	1125223		1103207		8.37	240	256	1273	0.89	0.07	223	0.89	212	0.89	421	249	4.91	3.04	3.76	2.49	1.99
3-Mar-93	1121829		1106574		8.34	250	270	1287	0.89	0.05	166	0.89	180	0.89	356	223	4.90	3.02	3.76	2.49	2.00
4-Mar-93	1126524		1107140		8.34	244	280	1292	0.89	0.04	179	0.89	174	0.89	345	225	4.86	2.98	3.75	2.47	2.00
5-Mar-93	1125289		1115690		8.25	255	273	1298	0.88	0.02	198	0.88	182	0.88	355	259	4.84	2.96	3.76	2.46	2.01
8-Mar-93	1133115		1074180	**45000**	8.22	249	274	1308	0.89	-0.08	189	0.89	179	0.89	344	253	4.83	2.94	3.77	2.47	2.00
9-Mar-93	1136250		1071616		8.25	242	269	1302	0.89	-0.08	217	0.89	178	0.89	336	237	4.82	2.92	3.77	2.46	2.01
10-Mar-93	1134726		1073743		8.23	247	277	1300	0.89	-0.09	211	0.89	142	0.89	264	201	4.81	2.90	3.77	2.45	2.01
11-Mar-93	1138181		1067740		8.30	236	277	1304	0.90	-0.08	216	0.90	137	0.90	250	182	4.81	2.87	3.77	2.43	2.01
12-Mar-93	1130067		1070979		8.27	250	272	1296	0.88	-0.10	231	0.88	120	0.88	216	144	4.81	2.84	3.77	2.41	2.00
15-Mar-93	1133832		1071226		8.27	246	275	1308	0.89	-0.11	256	0.89	112	0.89	202	135	4.77	2.84	3.76	2.42	2.01

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British Aerospace

Hedge horizon: 5 weeks

	BOND Dirty Price (£)	BOND Cash Flow (£)	GILT Dirty Price (£)	GILT Cash Flow (£)	GILT Yield (%)	BOND Spread Bpt	STOCK (£)	FTSE Engn. (£)	**Quantity of hedging instrument required**								**RISK (%)** (Moving Average)				
									Traditional Hedge Ratio	Minimum Variance Ratio		Hedge Ratio 3		Hedge Ratio 4			No Hedge	With Traditional Hedge Ratio	With Min. Var. Hedge Ratio	With Hedge Ratio 3	With Hedge Ratio 4
									gilt	gilt	stock	gilt	stock	gilt	stock	Index					
16-Mar-93	1134157		1068342		8.30	242	279	1307	0.89	-0.08	250	0.89	105	0.89	186	125	4.72	2.83	3.72	2.41	1.98
17-Mar-93	1127633		1065779		8.33	248	275	1304	0.89	-0.07	279	0.89	107	0.89	187	114	4.66	2.81	3.69	2.39	1.96
18-Mar-93	1124209		1075086		8.23	264	277	1306	0.87	-0.07	295	0.87	108	0.87	185	112	4.61	2.80	3.66	2.38	1.94
19-Mar-93	1140784		1069575		8.30	236	277	1300	0.90	-0.10	269	0.90	136	0.90	226	142	4.55	2.78	3.65	2.37	1.94
22-Mar-93	1134859		1064512		8.36	238	273	1287	0.90	-0.12	-6	0.90	123	0.90	198	94	4.52	2.78	3.64	2.37	1.94
23-Mar-93	1128935		1057568		8.44	237	275	1281	0.90	-0.12	-17	0.90	96	0.90	150	68	4.47	2.79	3.63	2.37	1.94
24-Mar-93	1124281		1059375		8.43	246	273	1276	0.89	-0.13	-60	0.89	81	0.89	123	54	4.42	2.79	3.61	2.38	1.94
25-Mar-93	1125856		1058372		8.44	243	267	1263	0.89	-0.13	-79	0.89	61	0.89	91	37	4.36	2.79	3.58	2.37	1.93
26-Mar-93	1125242		1054741		8.49	240	264	1258	0.90	-0.13	-75	0.90	44	0.90	65	27	4.29	2.78	3.55	2.36	1.92
29-Mar-93	1121507		1054048		8.50	244	262	1252	0.89	-0.13	-73	0.89	43	0.89	62	23	4.23	2.79	3.52	2.36	1.91
30-Mar-93	1120892		1056485		8.47	248	270	1259	0.89	-0.13	-73	0.89	18	0.89	26	9	4.18	2.79	3.50	2.36	1.91
31-Mar-93	1124368		1067981		8.35	257	272	1268	0.88	-0.11	-70	0.88	0	0.88	0	0	4.14	2.80	3.49	2.38	1.92
1-Apr-93	1135634		1066668		8.36	241	280	1273	0.89	-0.11	-7	0.89	0	0.89	0	0	4.11	2.81	3.48	2.38	1.92
2-Apr-93	1133459		1071157		8.32	249	278	1268	0.89	-0.12	2	0.89	9	0.89	12	4	4.09	2.81	3.47	2.39	1.91
5-Apr-93	1138784		1065774		8.38	236	268	1247	0.90	-0.13	-31	0.90	21	0.90	30	9	4.08	2.85	3.47	2.42	1.93
6-Apr-93	1135101		1069771		8.34	247	267	1239	0.89	-0.17	87	0.89	33	0.89	48	15	4.09	2.88	3.48	2.46	1.97
7-Apr-93	1139486		1073457		8.30	246	267	1235	0.89	-0.16	81	0.89	48	0.89	71	22	4.10	2.92	3.50	2.49	2.00
8-Apr-93	1143562		1078440		8.26	245	270	1245	0.89	-0.15	88	0.89	56	0.89	84	27	4.09	2.95	3.49	2.52	2.03
13-Apr-93	1147017		1083377		8.21	247	275	1263	0.89	-0.14	114	0.89	52	0.89	79	23	4.08	2.99	3.49	2.55	2.05
14-Apr-93	1152363		1076433		8.29	233	275	1266	0.90	-0.16	120	0.90	51	0.90	79	22	4.08	3.01	3.49	2.57	2.07
15-Apr-93	1144568		1073870		8.32	240	272	1268	0.90	-0.15	149	0.90	52	0.90	82	23	4.08	3.05	3.49	2.59	2.09
16-Apr-93	1144569		1063979		8.44	228	271	1272	0.91	-0.15	169	0.91	53	0.91	85	24	4.10	3.07	3.51	2.61	2.11
19-Apr-93	1144570		1066106		8.42	230	285	1275	0.91	-0.14	142	0.91	49	0.91	80	30	4.12	3.09	3.52	2.64	2.13
20-Apr-93	1139668		1065413		8.43	239	296	1281	0.90	-0.14	104	0.90	36	0.90	60	26	4.14	3.12	3.54	2.66	2.15
21-Apr-93	1139669		1063779		8.45	236	295	1293	0.90	-0.18	134	0.90	25	0.90	42	17	4.16	3.15	3.56	2.68	2.17
22-Apr-93	1139670		1061846		8.48	234	306	1318	0.90	-0.18	120	0.90	16	0.90	27	11	4.18	3.17	3.57	2.70	2.19
23-Apr-93	1139671		1060705		8.50	232	315	1319	0.91	-0.17	157	0.91	4	0.91	7	3	4.20	3.19	3.58	2.72	2.21
26-Apr-93	1131318		1061262		8.49	246	313	1312	0.89	-0.18	156	0.89	0	0.89	0	0	4.23	3.22	3.61	2.75	2.24
27-Apr-93	1131319		1065889		8.44	251	319	1319	0.89	-0.19	172	0.89	0	0.89	0	0	4.26	3.26	3.64	2.79	2.28
28-Apr-93	1131320		1058005		8.54	242	315	1316	0.90	-0.17	171	0.90	0	0.90	0	0	4.28	3.29	3.66	2.81	2.30
29-Apr-93	1131321		1047942		8.66	229	310	1313	0.91	-0.17	195	0.91	0	0.91	0	0	4.32	3.32	3.69	2.83	2.32
30-Apr-93	1131322		1050808		8.64	232	319	1327	0.91	-0.17	203	0.91	0	0.91	0	0	4.36	3.35	3.71	2.85	2.34
4-May-93	1131323		1055735		8.58	237	325	1333	0.90	-0.15	201	0.90	0	0.90	0	0	4.41	3.38	3.75	2.89	2.38
5-May-93	1131324		1056301		8.58	238	330	1322	0.90	-0.10	145	0.90	0	0.90	0	0	4.46	3.42	3.79	2.93	2.42
6-May-93	1131325		1051228		8.64	231	322	1321	0.91	-0.10	133	0.91	0	0.91	0	0	4.51	3.45	3.83	2.96	2.45
7-May-93	1131326		1052908		8.63	233	316	1316	0.91	-0.09	124	0.91	0	0.91	0	0	4.55	3.48	3.87	3.00	2.49

British Aerospace

Hedge horizon: 5 weeks

	BOND Dirty Price (£)	BOND Cash Flow (£)	GILT Dirty Price (£)	GILT Cash Flow (£)	GILT Yield (%)	BOND Spread Bpt	STOCK (£)	FTSE Engn. (£)	**Quantity of hedging instrument required**								**RISK (%)** (Moving Average)				
									Traditional Hedge Ratio	Minimum Variance Ratio		Hedge Ratio 3		Hedge Ratio 4			No Hedge	With Traditional Hedge Ratio	With Min. Var. Hedge Ratio	With Hedge Ratio 3	With Hedge Ratio 4
									gilt	gilt	stock	gilt	stock	gilt	stock	Index					
10-May-93	1131327		1053784		8.62	233	325	1320	0.91	-0.10	93	0.91	0	0.91	0	0	4.60	3.52	3.92	3.04	2.53
11-May-93	1131328		1052461		8.64	231	322	1322	0.91	-0.06	82	0.91	0	0.91	0	0	4.64	3.54	3.97	3.07	2.57
12-May-93	1131329		1055837		8.60	235	322	1338	0.90	0.06	64	0.90	0	0.90	0	0	4.67	3.55	3.97	3.07	2.56
13-May-93	1172918		1052334		8.65	189	314	1344	0.95	0.00	-243	0.95	0	0.95	0	0	4.67	3.54	4.00	3.07	2.56
14-May-93	1172919		1049324		8.69	185	311	1342	0.96	-0.07	-210	0.96	0	0.96	0	0	4.67	3.53	4.04	3.06	2.56
17-May-93	1172920		1050190		8.68	185	313	1341	0.96	-0.04	-236	0.96	0	0.96	0	0	4.66	3.51	4.07	3.06	2.56
18-May-93	1172921		1051377		8.67	186	321	1342	0.95	-0.04	-244	0.95	0	0.95	0	0	4.69	3.52	4.13	3.08	2.57
19-May-93	1172922		1056623		8.61	192	328	1345	0.95	-0.05	-252	0.95	0	0.95	0	0	4.71	3.53	4.19	3.09	2.59
20-May-93	1172923		1056560		8.62	192	328	1339	0.95	-0.05	-269	0.95	0	0.95	0	0	4.73	3.54	4.24	3.10	2.61
21-May-93	1172924		1055420		8.64	190	324	1335	0.95	-0.05	-253	0.95	0	0.95	0	0	4.76	3.54	4.30	3.11	2.62
24-May-93	1172925		1055356		8.64	189	321	1335	0.95	-0.12	-223	0.95	0	0.95	0	0	4.77	3.53	4.35	3.11	2.62
25-May-93	1172926		1055913		8.64	190	322	1330	0.95	-0.05	-307	0.95	0	0.95	0	0	4.77	3.51	4.39	3.10	2.62
26-May-93	1172927		1060539		8.59	195	322	1330	0.94	-0.01	-328	0.94	0	0.94	0	0	4.77	3.50	4.43	3.09	2.61
27-May-93	1172928		1062346		8.57	197	326	1325	0.94	-0.01	-324	0.94	0	0.94	0	0	4.76	3.47	4.44	3.07	2.59
28-May-93	1171383		1056142		8.65	191	336	1326	0.95	0.04	-352	0.95	0	0.95	0	0	4.76	3.44	4.47	3.04	2.57
1-Jun-93	1171384		1057329		8.64	192	352	1331	0.95	0.00	-300	0.95	0	0.95	0	0	4.78	3.42	4.50	3.03	2.56
2-Jun-93	1189879		1062265		8.58	177	363	1349	0.96	0.09	-193	0.96	0	0.96	0	0	4.79	3.40	4.52	3.01	2.54
3-Jun-93	1189880		1064382		8.56	179	366	1352	0.96	0.06	-188	0.96	0	0.96	0	0	4.80	3.38	4.54	2.99	2.53
4-Jun-93	1189881		1063562		8.58	177	366	1345	0.96	0.11	-192	0.96	0	0.96	0	0	4.81	3.35	4.55	2.97	2.51
7-Jun-93	1193985		1064748		8.57	174	383	1358	0.97	0.07	-171	0.97	0	0.97	0	0	4.82	3.33	4.57	2.95	2.50
8-Jun-93	1193986		1065305		8.57	175	405	1371	0.97	0.02	-150	0.97	0	0.97	0	0	4.84	3.30	4.60	2.93	2.48
9-Jun-93	1199332		1068681		8.53	173	419	1388	0.97	0.04	-125	0.97	0	0.97	0	0	4.85	3.28	4.62	2.91	2.47
10-Jun-93	1202158		1072048		8.49	174	421	1390	0.97	0.01	-99	0.97	0	0.97	0	0	4.85	3.24	4.63	2.88	2.45
11-Jun-93	1204363		1072478		8.50	171	406	1395	0.97	0.00	-89	0.97	0	0.97	0	0	4.85	3.21	4.64	2.85	2.43
14-Jun-93	1205938		1074914		8.47	172	396	1396	0.97	-0.03	-73	0.97	0	0.97	0	0	4.86	3.18	4.66	2.82	2.42
15-Jun-93	1206264		1073281		8.49	170	389	1389	0.97	-0.04	-58	0.97	0	0.97	0	0	4.85	3.14	4.67	2.79	2.39
16-Jun-93	1208800		1076027		8.46	171	397	1396	0.97	-0.01	-83	0.97	0	0.97	0	0	4.85	3.11	4.68	2.77	2.38
17-Jun-93	1208801		1081584		8.40	177	397	1392	0.96	-0.02	-75	0.96	0	0.96	0	0	4.79	3.03	4.66	2.70	2.31
18-Jun-93	1208802		1086074		8.36	181	409	1391	0.96	-0.09	-78	0.96	0	0.96	0	0	4.76	2.99	4.68	2.66	2.29
21-Jun-93	1208803		1091640		8.30	187	409	1392	0.95	-0.28	-69	0.95	0	0.95	0	0	4.72	2.93	4.70	2.62	2.27
22-Jun-93	1208804	**59375**	1092507		8.29	188	408	1402	0.95	-0.28	-62	0.95	0	0.95	0	0	4.68	2.88	4.71	2.59	2.25
23-Jun-93	1182222		1096193		8.25	157	404	1395	0.98	-0.60	44	0.98	0	0.98	0	0	4.61	2.80	4.70	2.52	2.21
24-Jun-93	1182223		1099880		8.21	161	395	1393	0.98	-0.60	44	0.98	0	0.98	0	0	4.55	2.72	4.70	2.46	2.17
25-Jun-93	1182224		1100300		8.21	161	384	1382	0.98	-0.71	38	0.98	0	0.98	0	0	4.50	2.65	4.70	2.40	2.13
28-Jun-93	1182225		1106176		8.15	167	391	1387	0.97	-0.69	36	0.97	0	0.97	0	0	4.44	2.58	4.70	2.34	2.08
29-Jun-93	1182226		1102983		8.19	163	389	1388	0.98	-0.62	46	0.98	0	0.98	0	0	4.40	2.52	4.70	2.29	2.04

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British Aerospace

Hedge horizon: 5 weeks

	BOND Dirty Price (£)	BOND Cash Flow (£)	GILT Dirty Price (£)	GILT Cash Flow (£)	GILT Yield (%)	BOND Spread Bpt	STOCK (£)	FTSE Engn. (£)	Quantity of hedging instrument required: Traditional Hedge Ratio gilt	Minimum Variance Ratio gilt	Minimum Variance Ratio stock	Hedge Ratio 3 gilt	Hedge Ratio 3 stock	Hedge Ratio 4 gilt	Hedge Ratio 4 stock	Hedge Ratio 4 Index	RISK (%) (Moving Average): No Hedge	With Traditional Hedge Ratio	With Min. Var. Hedge Ratio	With Hedge Ratio 3	With Hedge Ratio 4
30-Jun-93	1181952		1107289		8.14	169	390	1385	0.97	-0.65	41	0.97	0	0.97	0	0	4.37	2.47	4.72	2.26	2.02
1-Jul-93	1182907		1106286		8.16	167	386	1381	0.97	-0.63	44	0.97	0	0.97	0	0	4.33	2.43	4.74	2.23	2.02
2-Jul-93	1182908		1107966		8.15	168	379	1368	0.97	0.03	167	0.97	0	0.97	0	0	4.31	2.40	4.72	2.23	2.00
5-Jul-93	1182909		1109152		8.14	169	382	1362	0.97	0.05	168	0.97	0	0.97	0	0	4.29	2.38	4.70	2.22	2.02
6-Jul-93	1201433		1112839		8.10	154	386	1364	0.99	0.18	177	0.99	0	0.99	0	0	4.26	2.34	4.66	2.20	2.02
7-Jul-93	1201434		1111515		8.12	153	395	1366	0.99	0.19	176	0.99	0	0.99	0	0	4.22	2.28	4.61	2.17	2.02
8-Jul-93	1201435		1112392		8.11	153	396	1368	0.99	0.23	180	0.99	0	0.99	0	0	4.18	2.24	4.57	2.14	2.02
9-Jul-93	1201760		1114072		8.10	154	401	1373	0.99	0.23	179	0.99	0	0.99	0	0	4.15	2.20	4.53	2.11	2.02
12-Jul-93	1204585		1117748		8.06	155	415	1385	0.98	0.23	178	0.98	0	0.98	0	0	4.11	2.15	4.48	2.08	2.00
13-Jul-93	1206161		1116745		8.07	153	423	1394	0.99	0.23	180	0.99	0	0.99	0	0	4.07	2.09	4.43	2.03	1.98
14-Jul-93	1210567		1124182		7.99	157	416	1392	0.98	0.31	166	0.98	0	0.98	0	0	4.04	2.05	4.38	2.00	1.96
15-Jul-93	1212772		1126308		7.97	157	418	1395	0.98	0.33	164	0.98	0	0.98	0	0	4.01	2.02	4.33	1.97	1.95
16-Jul-93	1215908		1127048		7.97	153	415	1396	0.99	0.31	157	0.99	0	0.99	0	0	3.98	1.99	4.26	1.95	1.94
19-Jul-93	1215923		1126664		7.98	153	414	1403	0.99	0.25	163	0.99	0	0.99	0	0	3.94	1.95	4.19	1.91	1.91
20-Jul-93	1216868		1127851		7.97	153	396	1394	0.99	0.23	160	0.99	0	0.99	0	0	3.91	1.92	4.13	1.89	1.89
6-Sep-93				**45000**																	
22-Dec-93		**59375**																			

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