

Wage Rigidity and Retirement in Optimal Portfolio Choice

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Abstract

We study an agent's lifecycle portfolio choice problem with stochastic labor income, borrowing constraints and a finite retirement date. Similarly to [7], wages evolve in a path-dependent way, but the presence of a finite retirement time leads to a novel, two-stage infinite dimensional stochastic optimal control problem with explicit optimal controls in feedback form. We find an explicit solution to the associated Hamilton-Jacobi-Bellman (HJB) equation, which is an infinite dimensional PDE of parabolic type. The identification of the optimal feedbacks is delicate due to the presence of time-dependent state constraints, which appear to be new in the infinite dimensional stochastic control literature. The explicit solution allows us to study the properties of optimal strategies and discuss their implications for portfolio choice. As opposed to models with Markovian dynamics, path dependency can now modulate the hedging demand arising from the implicit holding of risky assets in human capital, leading to richer asset allocation predictions consistent with wage rigidity and the agents learning about their earning potential.

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1 Introduction

We study the lifecycle portfolio problem of an agent receiving stochastic labor income and facing borrowing constraints as well as a finite retirement time. Labor income is affected by economic shocks with some delay, which may characterize the stickyness of wages documented at least since [33]. Several studies have shown how discrete time Autoregressive Moving Average Processes (ARMA) can match the empirical evidence on earnings in a number of settings (e.g., [36],[30], [40], [38]), but the introduction of path dependency in continuous time portfolio choice problems has traditionally been challenging and analytical solutions have remained elusive (e.g., [4,18]). Contributions [6,7] exploited the fact that under certain conditions Stochastic Delay Differential Equations (SDDEs) can be understood as the weak limits of discrete time ARMA processes (e.g., [16,35,45,47]) to introduce realistic labor income features in continuous time portfolio choice problems while retaining analytical tractability.

Similarly, [52,53] used SDDEs to model cointegrated asset dynamics and succeeded in solving continuous time mean-variance problems with and without pre-commitment, respectively.

The most tractable SDDEs are the linear ones, which are used by [7] (see also [15]) to model log-labor income y with a delay term appearing in the drift. The delay term weighs the past path of y over a bounded time window $[-d, 0]$ with a Lebesgue-square integrable function $\phi : [-d, 0] \rightarrow \mathbb{R}$ (see [6] for the case of a delay appearing also in the volatility term). As the agent can borrow against future labor income, the budget constraint requires that total wealth must be non-negative, where total wealth is the sum of financial wealth and the current market value of future labor income (human capital). Although contribution [6] does not focus on optimization, it provides an explicit formula for human capital when it is driven by an SDDE, thus opening the way to solving explicitly the infinite horizon portfolio problem studied in [7]. There, the authors consider power utility and solve an optimal portfolio and consumption choice problem in which agents are allowed to borrow against future labor income, as in [18]. The solution strategy follows the dynamic programming approach. As such an approach applies to Markovian

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settings, the key step is to rewrite the problem extending the state space to include both current wealth/labor income and the past trajectory of labor income as state variables. This leads us to study an infinite dimensional HJB equation, which can be seen as an infinite dimensional version of the classical HJB for the Merton problem with labor income. A similar approach is used in [5], which does not consider retirement but allows the weight appearing in the delay term to be a general Radon measure taking values in an uncertainty set. We must note that such type of problems could be treated also with the Maximum Principle approach, as done in different contexts, for example, [34], [54] [50], [10] [20]. The novelty of this paper is the introduction of a finite retirement date, which leads to two main mathematical problems. First, conversion of the borrowing constraint into a Markovian form is considerably more challenging than in the infinite horizon case considered in [7]. Second, the infinite dimensional stochastic control problem now features discontinuous dynamics and time dependent state constraints. As far as we know, the treatment of such family of problems appears to be new in the literature. As in [7], the key is the derivation of human capital in explicit form, so that the constraint on total wealth can be properly addressed. The result is more involved than the one offered in [7], but reveals novel insights into how the market value of human capital trades off the past contribution of labor income against the shrinking time to retirement. This has considerable bearing for the optimal controls, which are time varying not only in recognition of the residual time to retirement, but also because of the interplay between the past and future contribution of labor income to human capital and hence total wealth. In line with [48], a hedging demand arises in the allocation to the risky asset because of the market risk channelled by risky wages (e.g., [9]), but its structure now supports a wider range of empirical predictions than found in the extant literature, as hedging demand is shaped by the joint effect of path dependent wages and the residual time to retirement. The problem considered in this paper is amenable to interesting extensions and ramifications. A first direction is the consideration of tighter borrowing constraints, which even in the case of fully spanned labor income would prevent the agent from perfectly hedging labor income risk; see [19] and references therein. A second direction is the case in which labor market participation is endogenous, which can be captured by a controlled labor income process (see [42] for a general semimartingale setting) or by making the retirement date τ_R a stopping time controlled by the agent (as in [17,18] in a Markovian environment); see [11,2] for the case in which both continuous labor supply and irreversible retirement are endogeneous. These extensions are an open problem within our setting and are left for future research.

The paper is organized as follows. In the next Section, we introduce the model and associated expected utility maximization problem. Section 3 then deals with the valuation of human capital and the Markovian representation of the borrowing constraint. This follows from a non trivial analysis, part of which is relegated to the Appendix. We then rewrite the path dependent state equation into a Markovian stochastic differential equation (SDE) in infinite dimension. This allows us to reformulate the original portfolio problem in a fully Markovian way. Section 4 is dedicated to the decoupling of the Markovian problem into a two-stage optimal control problem. In particular, we consider in sequence the problems faced by the agent before retirement (finite horizon) and after retirement (infinite horizon). In Sections 5

and 6, we apply the Dynamic Programming (DP) approach to each of the two problems separately. The infinite horizon problem follows the results of [7]; see Section 5. The finite horizon problem, which is fully solved in Section 6, is more challenging, as the associated infinite dimensional HJB equation is of parabolic type and with time dependent state constraints. By making an educated guess, however, we can find an explicit solution whose verification requires some effort. We obtain the optimal controls in explicit feedback form, which is used to discuss the properties of the optimal strategies. Finally, Section 7 offers a summary of the main results and their financial implications.

2 Assumptions and problem statement

On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{P})$, consider the price of a riskless bond S_0 and an adapted, vector valued process S representing the price evolution of n risky assets, $S = (S_1, \dots, S_n)^\top$, with dynamics given by

$$\begin{cases} dS_0(s) = S_0(s)rds, & S_0(0) = 1; \\ dS(s) = \text{diag}(S(s))(\mu ds + \sigma dZ(s)), & S(0) \in \mathbb{R}_+^n, \end{cases} \quad (1)$$

where we assume the following.

Hypothesis 1 (i) Z is an n -dimensional (standard) Brownian motion and the filtration $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is the augmentation of the one generated by Z ; (ii) μ is $\mathbb{R}^n$ -valued and $\sigma \in \mathbb{R}^{n \times n}$ is invertible.

An agent is endowed with initial wealth $w \geq 0$ and receives wages at labor income rate y until retirement or death, whichever occurs first. The death time τ_δ is modeled as an exponential random variable with parameter $\delta > 0$, independent of Z . The retirement date is fixed and set equal to $\tau_R \in \mathbb{R}_+$. The reference filtration is the minimal enlargement $\mathbb{G} := (\mathcal{G}_t)_{t \geq 0}$ of $\mathbb{F}$ satisfying the usual conditions and making the death indicator process $D := (\mathbb{I}_{[0, \tau_\delta]}(t))_t$ adapted. Equivalently, we are considering the minimal enlargement of $\mathbb{F}$ making τ_δ a stopping time. Each sigma-field $\mathcal{G}_t$ can be shown to be given by (e.g., [1, Proposition 1.12]):

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma_g(D_u : u \leq t)$$

where $\mathcal{A} \vee \mathcal{B}$ indicates the sigma algebra generated by $\mathcal{A}, \mathcal{B}$ and $\sigma_g(\cdot)$ denotes the sigma-field generated by the random variable in brackets. By [1, Proposition 2.11-(b)], we know that for every $\mathbb{G}$ -predictable process $\tilde{A}$ we can find a process A which is $\mathbb{F}$ -predictable and satisfies, $\mathbb{P}$ -almost surely, the following condition:

$$A(s, \omega) = \tilde{A}(s, \omega) \quad \forall s \in [0, \tau_\delta(\omega)]. \quad (2)$$

The processes A and $\tilde{A}$ are therefore indistinguishable up to τ_δ . We refer to A as to the pre-death version of $\tilde{A}$. From now on, we will work with $\mathbb{F}$ -predictable, pre-death versions of all processes considered, including state variables and controls.

The agent consumes at rate $c(\cdot) \geq 0$ and can invest in the riskless and risky assets. We denote by the vector $\theta(\cdot) \in \mathbb{R}^n$ the wealth amounts allocated to the risky assets. The agent can also purchase life insurance to reach a bequest target $\tilde{B}(\tau_\delta)$ at death. In the pre-death world this simply means that the agent chooses a control $B(\cdot) \geq 0$. The pre-death insurance premium paid by the agent is $\delta(B(t) - W(t))$. As in [18], we interpret a negative face value $B(t) - W(t) < 0$ as a term life annuity trading wealth at death for a positive income flow $\delta(W(t) - B(t))$ while living. The controls c, θ , and B are for now assumed to belong to the following set:

$$\Pi_0 := \{\mathbb{F}\text{-predictable triplets } ((c(\cdot), B(\cdot)), \theta(\cdot)) \text{ in } L_{loc}^1(\Omega \times \mathbb{R}_+; \mathbb{R}_+^2) \times L_{loc}^2(\Omega \times \mathbb{R}_+; \mathbb{R}^n)\}, \quad (3)$$

where $f \in L_{loc}^p(\Omega \times \mathbb{R}_+; U)$ stands for the integrability condition

$$\mathbb{E}\left[\int_0^T |f(t)|^p dt\right] < +\infty$$

being satisfied for all $T > 0$. We denote by W the agent's financial wealth and by y the labor income rate process. As opposed to standard bilinear SDEs (e.g., [18]), we assume here the process y to follow a bilinear SDDE with delay in the drift. For fixed memory time $d > 0$, the delay term weights the realization of labor income within the time window $[t-d, t]$ by a time independent function $\phi \in L^2(-d, 0; \mathbb{R})$. The dynamics of the state variable pair (W, y) is then given by:

$$\begin{cases} dW(s) = [W(s)r + \theta(s)^\top(\mu - r\mathbf{1}) + (1 - R(s))y(s) - c(s) - \delta(B(s) - W(s))]ds + \theta(s)^\top \sigma dZ(s), \\ dy(s) = [y(s)\mu_y + \int_{-d}^0 \phi(\zeta)y(\zeta + s)d\zeta]ds + y(s)\sigma_y^\top dZ(s), \\ W(0) = w, \quad y(0) = x_0, \quad y(\zeta) = x_1(\zeta) \text{ for } \zeta \in [-d, 0]. \end{cases} \quad (4)$$

with $s \geq 0$, $\mu_y, w, x_0 \in \mathbb{R}$, $\sigma_y \in \mathbb{R}^n$, and where the triplets $(c(\cdot), B(\cdot), \theta(\cdot))$ are as in (3), $\mathbf{1} = (1, \dots, 1)^\top$ denotes the unitary vector in $\mathbb{R}^n$ and $x_1(\cdot)$ lives in $L^2(-d, 0; \mathbb{R})$. The function $R(s) := \mathbb{I}_{\{\tau_R \leq s\}}$ represents a retirement indicator, where $\tau_R \in \mathbb{R}_{++}$ denotes a deterministic retirement date. This realistic feature is absent in [7], which only considers the case without retirement, i.e., $\tau_R = +\infty$ and $R(s) = 0$ a.e.. Existence and uniqueness of a strong solution to the SDDE for y is ensured by Theorem I.1 and Remark I.3(iv) in [41]. Existence and uniqueness of a strong solution to the SDE for W then follows, for example, from [32, Chapter 5.6.C]. As is clear from dynamics (4), in our baseline model labor income is perfectly instantaneously correlated with the risky assets. This is the benchmark that will be used to develop the solution of the portfolio choice problem. We study the problem of maximizing the agent's expected utility of lifetime consumption and bequest:

$$\mathbb{E}\left[\int_0^{\tau_\delta} e^{-\rho s} \left(\frac{(K^{R(s)}\tilde{c}(s))^{1-\gamma}}{1-\gamma} + \frac{(k\tilde{B}(\tau_\delta))^{1-\gamma}}{1-\gamma} \right) ds\right],$$

which can be reduced to the following one after exploiting the fact that the death time is exponentially distributed:

$$\mathbb{E}\left[\int_0^{+\infty} e^{-(\rho+\delta)s} \left(\frac{(K^{R(s)}c(s))^{1-\gamma}}{1-\gamma} + \delta \frac{(kB(s))^{1-\gamma}}{1-\gamma} \right) ds\right]. \quad (5)$$

Optimization is carried out over all triplets (c, θ, B) taken as in (3) and satisfying a suitable state constraint introduced in (9) further below. In the above, the parameter $K > 1$ allows the utility from consumption to differ before and after time τ_R , whereas $k > 0$ measures the intensity of preference for leaving a bequest. The parameter γ is the relative risk aversion coefficient, which is assumed to satisfy $\gamma \in (0, 1) \cup (1, +\infty)$. Finally, the parameter $\rho > 0$ denotes the agent's subjective discount rate.

In the reference financial market (1), the pre-death state-price density of the agent is unique and is given by:

$$\begin{cases} d\xi(s) = -\xi(s)(r + \delta)ds - \xi(s)\kappa^\top dZ(s), \\ \xi(0) = 1. \end{cases} \quad (6)$$

where κ is the market price of risk and is defined as follows (e.g., [32]):

$$\kappa := (\sigma)^\top(\mu - r\mathbf{1}). \quad (7)$$

Let us denote now by H the current market value of future wages, or human capital:

$$H(s) := \xi^{-1}(s)\mathbb{E}\left(\int_s^{+\infty} \xi(u)(1 - R(u))y(u)du \middle| \mathcal{F}_s\right). \quad (8)$$

The agent's budget constraint is then assumed to satisfy:

$$W(s) + H(s) \geq 0, \quad \forall s \in [0, \infty). \quad (9)$$

The above means that human capital, in addition to financial wealth, can be pledged as collateral to support investment and consumption. Note that the agent cannot default

on his/her debt obligations upon death, as B is assumed to be nonnegative. Constraint (9) was introduced as a no-borrowing-without-repayment condition in [18]. Its impact on portfolio choice was explored numerically in the context of mean-reverting labor income dynamics chosen as a proxy for sticky wages. Let us denote by $W^{w,x}(s; c, B, \theta)$ and $y^x(s)$ the solutions of system (4) at time $s \geq 0$, and by $H^x(s)$ the corresponding human capital defined in (8), where we emphasize the dependence of these quantities on the initial conditions (w, x) and strategies (c, B, θ) , where relevant. We can then define the set of admissible controls, as follows:

$$\Pi(w, x) := \{ (c(\cdot), B(\cdot), \theta(\cdot)) \in \Pi_0 \text{ such that:}$$

$$W^{w,x}(s; c, B, \theta) + H^x(s) \geq 0 \quad \forall s \geq 0 \}. \quad (10)$$

Our problem is then to maximize the functional given in (5) over all controls in $\Pi(w, x)$. Let us now introduce the effective discount rate for labor income (e.g., [18]), which is assumed to be strictly positive:

$$\beta := r + \delta - \mu_y + \sigma_y^\top \kappa > 0. \quad (11)$$

Consider also the following assumption, which is the analog of Merton's condition for the maximization problem to be well-posed and will be recalled explicitly whenever needed:

Hypothesis 2 *The model parameters satisfy the inequality $\rho + \delta - (1 - \gamma)(r + \delta + \frac{\kappa^\top \kappa}{2\gamma}) > 0$.*

Remark 3 *In paper [7], a transversality condition is required on account of the infinite horizon problem; this imposes an additional restriction on β , see [7, Hypothesis 2.4-(i)] and Remark 12 further below. Hypothesis 2, on the other hand, is required here to ensure that the value function is finite and coincides with [7, Hypothesis 2.4-(ii)], as it depends only on the target functional and the wealth equations; see Remark 21 for further discussion.*

Our problem can be summarized as follows:

Problem 4 *Given initial wealth w , initial income and income history pair $x = (x_0, x_1)$, and retirement date τ_R , with associated indicator function $R(s) := \mathbb{I}_{\{\tau_R \leq s\}}$, choose consumption, bequest, and investment triplets $(c(\cdot), B(\cdot), \theta(\cdot))$ so as to maximize expected utility (5) under state equation (4) and subject to the borrowing constraint (9).*

3 Rewriting the state equation and the constraint

In Section 3.1, we briefly recall how to rewrite the labor income equation as a Markovian SDE in the Delfour-Mitter space M_2 , which is an infinite dimensional Hilbert space. In Subsection 3.2, we derive the explicit expression for the agent's human capital and rewrite the constraint in M_2 .

3.1 Reformulation of the SDDE for labor income.

We follow the main ideas of [7, Subsection 3.1], but consider the generic initial time $t \geq 0$ instead of time zero. The state equation for the labor income y is an SDDE, which means that y is not Markovian and the dynamic programming principle does not apply in its standard formulation. As usual, it is convenient to reformulate the problem in an infinite dimensional Hilbert space, which takes into account both the present and past values of the states, thus ensuring the Markov property of the states (see, for example, [49] and [12], or [14, Section 0.2] and [22, Section 2.6.8]).

To be precise, we introduce the Delfour-Mitter Hilbert space M_2 (see e.g. [3, Part II, Chapter 4]):

$$M_2 := \mathbb{R} \times L^2(-d, 0; \mathbb{R}),$$

with inner product, for $x = (x_0, x_1), y = (y_0, y_1) \in M_2$, defined as $\langle x, y \rangle_{M_2} := x_0 y_0 + \langle x_1, y_1 \rangle_{L^2}$, where

$$\langle x_1, y_1 \rangle_{L^2} := \int_{-d}^0 x_1(\zeta) y_1(\zeta) d\zeta.$$

For ease of notation, we will drop below the subscript $L^2 = L^2(-d, 0; \mathbb{R})$ from the inner product of such space, writing simply $\langle x_1, y_1 \rangle$. To embed the state y of the original problem in the space M_2 we introduce the linear operators A (unbounded) and C (bounded). Setting $\mathcal{D}(A) := \{(x_0, x_1) \in M_2 : x_1(\cdot) \in \mathcal{W}^{1,2}([-d, 0]; \mathbb{R}), x_0 = x_1(0)\}$, the operator $A : \mathcal{D}(A) \subset M_2 \rightarrow M_2$ is defined as

$$A(x_0, x_1) := \left(\mu_y x_0 + \int_{-d}^0 \phi(\zeta) x_1(\zeta) d\zeta, \frac{dx_1}{d\zeta} \right), \quad (12)$$

with μ_y, ϕ appearing in equation (4). The operator $C : M_2 \rightarrow \mathcal{L}(\mathbb{R}^n, M_2)$ is bounded and defined as

$$C(x_0, x_1) := (x_0 \sigma_y^\top, 0),$$

where σ_y shows up in (4). Proposition A.27 in [14] shows that A generates a strongly continuous semigroup in M_2 . Consider, for $t \geq 0$, $x \in M_2$, the following stochastic differential equation in M_2 :

$$\begin{cases} dX(s) = AX(s)ds + CX(s)dZ(s) \\ X(t) = x \in M_2. \end{cases} \quad (13)$$

The equivalence between the equation for y in (4) and the above (13) is formalized in the following Proposition.

Proposition 5 *For every $t \geq 0$ and $x = (x_0, x_1) \in M_2$ the infinite dimensional equation (13) has a unique mild (and weak) solution with almost surely continuous trajectories, denoted by $X^x(\cdot; t) = (X_0^x(\cdot; t), X_1^x(\cdot; t))$.*

Moreover the equation for y in (4), possibly with initial time $t \geq 0$, has a unique mild solution, in a strong probabilistic sense, for every $x \in M_2$, which we denote by $y^x(\cdot; t)$. Finally, for every $s \geq 0$ and $\zeta \in [-d, 0]$,

$$(X_0^x(s; t), X_1^x(s; t)(\zeta)) = (y^x(s; t), y^x(s + \zeta; t)). \quad (14)$$

PROOF. Existence and uniqueness of a continuous mild solution in M_2 of (13) and the equivalence between mild and weak solutions follow from [14, Theorem 6.7] (see also [14, Theorem 7.2]). In the same book, p. 160-161, the reader can find the precise definitions of mild and weak solution in this case. The equivalence property has been proven in Theorems 3.1-3.9 and Remark 3.7 in [12] (see also [25], [21]). $\square$

We will denote the mild solution of (13) with $X^x(\cdot)$ to underline its dependence on the initial condition. When no confusion is possible we may simply write $X(\cdot)$. Note that X now is Markovian. We now provide a standard, yet useful, result about the adjoint operator of A .

Proposition 6 *The adjoint operator of $(A, \mathcal{D}(A))$, i.e. $A^* : \mathcal{D}(A^*) \subset M_2 \rightarrow M_2$ is given by*

$$\begin{aligned} \mathcal{D}(A^*) &:= \{(z_0, z_1) \in M_2 : z_1(\cdot) \in \mathcal{W}^{1,2}(-d, 0; \mathbb{R}), z_1(-d) = 0\}, \\ A^*(z_0, z_1) &:= (\mu_y z_0 + z_1(0), -z_1' + z_0 \phi). \end{aligned} \quad (15)$$

PROOF. See [7, Proposition 3.1].

Remark 7 *The main reason for rewriting our state delay equation as a standard SDE in an infinite dimensional Hilbert space comes from the need of applying the Dynamic Programming (DP) method. Indeed DP works only when the state dynamics are Markovian. To make our model Markovian we need to write it in an infinite dimensional setting. In some special cases (see e.g. [41]) such "Markovianization" can be done without resorting to infinite dimensional spaces, but this is not our case.*

Remark 8 *In our model the delay weight function ϕ belongs to $L^2(-d, 0; \mathbb{R})$. This is a restriction as in some applied models the delay is pointwise which means that ϕ is a combination of Dirac deltas. Indeed this case can be treated with the same ideas used here but at a price of some technical complications, in particular concerning the operator A . This case is considered in [5, Section 3] in an infinite horizon case, but the approach used there can be transferred to our setting without substantial changes.*

3.2 A formula for human capital

After the preliminary Lemma 9, in Proposition 10 we derive an explicit expression for the agent's human capital in the infinite dimensional framework introduced in the previous Section. The expression extends the result provided by [6] to a finite horizon setting. We then use it in Corollary 11 to suitably rewrite the constraint (9).

Lemma 9 *Let $\phi \in L^2(-d, 0; \mathbb{R})$ and $d, \tau_R > 0$, $\beta \in \mathbb{R}$. Consider the system (here $(t, \zeta) \in [0, \infty) \times [-d, 0]$)*

$$\begin{cases} g(t) = \mathbb{I}_{\{t < \tau_R\}} \int_t^{\tau_R} e^{-\beta(\tau-t)} (h(\tau, 0) + 1) d\tau, \\ h(t, \zeta) = \mathbb{I}_{\{t < \tau_R\}} \cdot \\ \int_{(\zeta+t-\tau_R) \vee (-d)}^{\zeta} e^{-(r+\delta)(\zeta-\tau)} g(t + \zeta - \tau) \phi(\tau) d\tau. \end{cases} \quad (16)$$

We have the following results:

- (i) *The system (16) admits a unique solution, i.e., a pair (g, h) , with $g \in C(\mathbb{R}_+; \mathbb{R})$ and $h \in C(\mathbb{R}_+ \times [-d, 0]; \mathbb{R})$, satisfying (16). Moreover g and h depend continuously on $\phi \in L^2(-d, 0; \mathbb{R})$. If ϕ is a.e. positive, then also (g, h) is positive.*
- (ii) *Assume now that $\phi \in C^1([-d, 0]; \mathbb{R})$ with $\phi(-d) = 0$. Then the solution (g, h) of (16) belongs to $C^1([0, \tau_R]; \mathbb{R}) \times C^1([0, \tau_R] \times [-d, 0]; \mathbb{R})$ and satisfies the system*

$$\begin{cases} g'(t) - \beta g(t) + h(t, 0) + 1 = 0, & t \in [0, \tau_R) \\ -(r + \delta)h(t, \zeta) + \frac{\partial h(t, \zeta)}{\partial t} - \frac{\partial h(t, \zeta)}{\partial \zeta} + g(t)\phi(\zeta) = 0, & t \in (0, \tau_R), \zeta \in (-d, 0) \\ g(t) = 0, & t \geq \tau_R \\ h(t, \zeta) = 0, & t \geq \tau_R, \zeta \in [-d, 0] \\ h(t, -d) = 0, & t \in [0, +\infty). \end{cases} \quad (17)$$

- (iii) *If $\phi \in L^2(-d, 0; \mathbb{R})$ we have $g \in C^1([0, \tau_R]; \mathbb{R})$ and the map $[0, \tau_R] \rightarrow L^2(-d, 0; \mathbb{R})$, $t \mapsto h(t, \cdot)$ belongs to $C^1([0, \tau_R]; L^2(-d, 0; \mathbb{R}))$. From now on we will write $h(t)$ for $h(t, \cdot)$.*
- (iv) *It holds that $(g(t), h(t)) \in \mathcal{D}(A^*)$ for any $t \geq 0$. Moreover $(g, h) \in L^\infty(0, \tau_R; \mathcal{D}(A^*))$, that is*

$$\sup_{t \in [0, \tau_R)} \|(g(t), h(t))\|_{\mathcal{D}(A^*)} < \infty.$$

Finally, the couple (g, h) is a mild solution (see e.g. [22, Definition 1.119] for the definition) and also a strict solution (see [3, Definition 3.1, p.129]) of the Cauchy problem in $L^2(-d, 0; \mathbb{R})$:

$$(g'(t), h'(t)) = (-A^*(g(t), h(t)) + ((\beta + \mu_y)g(t) - 1, (r + \delta)h(t))), \quad (t \in [0, \tau_R)) \text{ with zero final condition at } t = \tau_R. \quad (18)$$

- (v) *The function $r \mapsto g'(r)$ has a discontinuity in $r = \tau_R$.*

PROOF. We postpone the long and technical proof of this result to the Appendix.

Proposition 10 *Let $x \in M_2$ and let y be the corresponding solution of the second equation of (4). The market value of Human Capital defined in (8) admits the following representation:*

$$H(s) = g(s)y(s) + \int_{-d}^0 h(s, \zeta)y(s + \zeta) d\zeta, \quad s \in [0, \infty), \mathbb{P}\text{-a.s.}, \quad (19)$$

where the function $(g, h) \in C([0, +\infty); \mathbb{R}_+) \times C([0, +\infty) \times [-d, 0]; \mathbb{R}_+)$ is the unique solution of (16). Let now X be the solution of equation (13) with initial datum x . Then we have for all $s \in [0, \infty)$, $\mathbb{P}\text{-a.s.},$

$$H(s) = \langle (g(s), h(s)), X(s) \rangle_{M_2} = g(s)X_0(s) + \langle h(s), X_1(s) \rangle. \quad (20)$$

PROOF. Thanks to Proposition 5 the second equation in (4) and equation (13) with initial datum $t = 0$, are equivalent and we have the identification

$(X_0(s), X_1(s)(\zeta)) = (y(s), y(s + \zeta))$, $s \geq 0$, $\zeta \in [-d, 0)$, as stated in (14). It is then immediate to see that the formulations for the Human Capital (19) and (20) are equivalent.

We now prove that representation (20) holds. This is obvious for $s \geq \tau_R$. When $s \in [0, \tau_R)$ we apply Itô's formula [22, Proposition 1.165] to compute the differential of the process $\langle (g(s), h(s)), (X_0(s), X_1(s)) \rangle_{M_2}$, thus obtaining, for $s \in [0, \tau_R)$, thanks to Lemma 9-(iii).

$$\begin{aligned} d\langle (g(s), h(s)), (X_0(s), X_1(s)) \rangle_{M_2} \\ = \langle (g'(s), h'(s)), (X_0(s), X_1(s)) \rangle_{M_2} ds \\ + \langle A^*(g(s), h(s)), (X_0(s), X_1(s)) \rangle_{M_2} ds + g(s)X_0(s)\sigma_y^\top dZ(s). \end{aligned} \quad (21)$$

By Lemma 9-(iv) the pair (g, h) is the unique solution to (18). Thus we can rewrite (21) as

$$d\langle (g(s), h(s)), X(s) \rangle_{M_2} = g(s)X_0(s)\sigma_y^\top dZ(s) + [(\beta + \mu_y)g(s) - 1]X_0(s) + (r + \delta)\langle h(s), X_1(s) \rangle ds. \quad (22)$$

Let us now define $\bar{H}(s) := \xi(s)(g(s)X_0(s) + \langle h(s), X_1(s) \rangle) = \xi(s)\langle (g(s), h(s)), X(s) \rangle_{M_2}$. Recalling that SDE (6) satisfied by the pre-death state-price density ξ , we can apply the standard Itô formula to $\bar{H}$ (seen as the product of two one-dimensional processes) and use (22) to obtain:

$$\begin{aligned} d\bar{H}(s) &= d\left[\xi(s)\langle (g(s), h(s)), X(s) \rangle_{M_2}\right] \\ &= -\xi(s)X_0(s)ds + \left[-\bar{H}(s)\kappa^\top + \xi(s)g(s)X_0(s)\sigma_y^\top\right]dZ(s), \end{aligned} \quad (23)$$

where we used (11). Let us now fix $0 \leq s \leq \tau_R$. From (23) we infer, integrating on $[s, \tau_R]$, the following result:

$$\begin{aligned} \bar{H}(\tau_R) - \bar{H}(s) &= -\int_s^{\tau_R} \xi(u)X_0(u)du + \\ &+ \int_s^{\tau_R} \left[-\bar{H}(u)\kappa^\top + \xi(u)g(u)X_0(u)\sigma_y^\top\right]dZ(u). \end{aligned} \quad (24)$$

Now observe that g and h are bounded and ξ and X are p -integrable processes for any $p \geq 1$, see e.g. [22, Theorem 1.130]. Hence, we can take the conditional mean in (24) and, using the fact that the stochastic integral appearing there is a martingale starting at zero, we obtain

$$\mathbb{E}[\bar{H}(\tau_R)|\mathcal{F}_s] - \bar{H}(s) = -\mathbb{E}\left[\int_s^{\tau_R} \xi(u)X_0(u)du \mid \mathcal{F}_s\right]. \quad (25)$$

Since, by the definition of g and h , $(g(\tau_R), h(\tau_R)) = 0 \in M_2$ (see equation (17)) we get, using (23),

$$\bar{H}(\tau_R) = 0, \quad \mathbb{P} - a.s..$$

Therefore (25) becomes

$$\bar{H}(s) = \mathbb{E}\left[\int_s^{\tau_R} \xi(u)X_0(u)du \mid \mathcal{F}_s\right]. \quad (26)$$

Equations (8), (23), and (26) then yield the desired result.

As a direct consequence of Proposition 10 we rewrite the constraint given in (9) in a more convenient way.

Corollary 11 *For any $s \in [0, +\infty)$, the constraint (9) can be equivalently rewritten as*

$$W(s) + g(s)X_0(s) + \langle h(s), X_1(s) \rangle \geq 0. \quad (27)$$

Remark 12 *The above Corollary 11 is the finite retirement version of [6, Theorem 2.1] (see also [7, Proposition 3.3]). As we have finite retirement, here we do not need to care about the long run behavior of labor income. This means that we do not need additional restrictions on β and ϕ (e.g., [7, Hypothesis 2.4-(i)]).*

On the other hand, the finite retirement time makes the rewriting of the constraint much more complicated, as the functions g and h now depend on time; this is the reason why we need to prove the long technical Lemma 9.

We conclude this Subsection with a remark on the financial meaning of the key functions g and h .

Remark 13 *To better understand the role of the functions g and h appearing in (16), let us first rewrite g by distinguishing two different components, g_1 and g_2 :*

$$g(t) = \mathbb{I}_{\{t < \tau_R\}} \int_t^{\tau_R} e^{-\beta(\tau-t)} d\tau + \quad (28)$$

$$+ \mathbb{I}_{\{t < \tau_R\}} \int_t^{\tau_R} e^{-\beta(\tau-t)} h(\tau, 0) d\tau = g_1(t) + g_2(t).$$

The function g is essentially an annuity factor capturing the market value of a stream of unitary wages to be received until retirement. Ignoring the delay term in income dynamics, the cumulative discounted value of each unit of labor income to be received until retirement is given by

$$g_1(t) = \mathbb{I}_{\{t < \tau_R\}} \int_t^{\tau_R} e^{-\beta(\tau-t)} d\tau = \frac{1 - e^{-\beta(\tau_R-t)^+}}{\beta}, \quad (29)$$

which is nothing else than the usual annuity term appearing in models without delay (e.g., [18]). In our context, however, each unit of labor income received also affects the wages to be received in the future. Such effect is precisely captured by the second component g_2 , as illustrated in the next simple example.

Assume that $d > 1$ and that $\phi(-1) = a\delta_{-1}$ for some $a > 0$ (here δ_{-1} denotes the Dirac delta in the point -1). Then the wage realizations influence labor income with a delay of one time unit. Their cumulative effect is then given by the following equation:

$$g_2(t) = \int_t^{t \vee (\tau_R - 1)} e^{-\beta(\tau-t)} \left(e^{-(r+\delta)} ag(\tau+1) \right) d\tau. \quad (30)$$

The integral above provides the cumulative market value of the delayed contribution of each unit of income received before retirement. At each time $\tau \in [t, \tau_R - 1]$, any unit of income received is weighted by the term a and received with certainty in case of survival with a delay of one unit of time, thus delivering a time- $(\tau+1)$ annuity value of $ag(\tau+1)$. The time- τ value to the agent of such a contribution is therefore $e^{-(r+\delta)} ag(\tau+1)$, and its time- t value is then obtained by applying the standard discount factor. For each time $\tau \in (\tau_R - 1, \tau_R]$, the delayed contribution has no time to materialize before retirement and hence is not accounted for in g_2 .

4 Solving the optimization problem: preliminary work

In Section 4.1 we rewrite the problem exposed in Section 2 using the results introduced in Section 3 and letting the initial time t vary. We then study in Section 4.3 the time evolution of admissible paths and in section 4.2 we sketch the idea of how we will solve the problem.

4.1 Statement of Problem 4 rewritten

For $t \geq 0$ and $(w, x) \in \mathcal{H} := \mathbb{R} \times M_2$, we rewrite (4) with initial time t and unknown (W, X) as:

$$\begin{cases} dW(s) = [(r + \delta)W(s) + \theta^\top(s)(\mu - r\mathbf{1}) + \\ (1 - R(s))X_0(s) - c(s) - \delta B(s)] ds + \theta^\top(s)\sigma dZ(s), \\ dX(s) = AX(s) ds + CX(s) dZ(s), \\ W(t) = w, \quad X_0(t) = x_0, \\ X_1(t)(\zeta) = x_1(\zeta) \text{ for } \zeta \in [-d, 0), \end{cases} \quad (31)$$

By Proposition 5 the above admits a unique strong solution (W, X) corresponding to the unique solution (W, y) of (4) when the initial time is t . We denote such solution at time $s \geq 0$ by $(W^{w,x}(s; t, c, B, \theta), X^x(s; t))$. With this notation

we emphasize the dependence of the solutions on the present time t , the initial time 0, the initial conditions $(w_0, y) \in \mathcal{H}$ and the admissible controls $(c(\cdot), B(\cdot), \theta(\cdot))$. For readability, we will often denote a triplet of controls $(c(\cdot), B(\cdot), \theta(\cdot))$ as $\pi(\cdot)$, as well as use the shorthand notation $(W(s; \pi), X(s))$, in which dependence on initial time and conditions is subsumed. We take the following set of admissible controls

$$\begin{aligned} \Pi(t, w, x) &:= \left\{ \mathbb{F}\text{-predictable } \pi(\cdot) = (c(\cdot), B(\cdot), \theta(\cdot)) \text{ in } \right. \\ &L_{loc}^1(\Omega \times [t, +\infty), \mathbb{R}_+^2) \times L_{loc}^2(\Omega \times [t, +\infty), \mathbb{R}^n) \text{ and s. t.} \\ &\left. W^{w,x}(s; t, \pi) + g(s)X_0^x(s; t) + \langle h(s), X_1^x(s; t) \rangle \geq 0 \text{ } s \geq t \right\}. \end{aligned} \quad (32)$$

Thanks to the results proved in Section 3, the above set coincides with the one of (10) for $t = 0$.

The objective functional $J(t, w, x; \pi)$ is (compare with the one of (5))

$$\mathbb{E} \left[\int_t^{+\infty} e^{-(\rho+\delta)s} \left(\frac{(K^{R(s)}c(s))^{1-\gamma}}{1-\gamma} + \delta \frac{(kB(s))^{1-\gamma}}{1-\gamma} \right) ds \right] \quad (33)$$

For simplicity, here we write J as a function of π , although it only depends on the control triplet through c and B , and not through θ . Notice that the functional J may take the value $-\infty$ when $\gamma > 1$ and both $c(\cdot)$ and $B(\cdot)$ are identically zero. On the other hand, it cannot take the value $+\infty$ when $\gamma \in (0, 1)$, when Hypothesis 2 holds. Our target Problem (4) is then reformulated as follows:

Problem 14 For fixed initial state $(w, x) \in \mathcal{H}$ such that $w + g(0)x_0 + \langle h(0, \cdot), x_1 \rangle \geq 0$, find $\bar{\pi} \in \Pi(0, w, x)$ such that

$$J(0, w, x; \bar{\pi}) = \sup_{\pi \in \Pi(0, w, x)} J(0, w, x; \pi).$$

4.2 Solving Problem 14: solution strategy

To solve Problem 14, we use a two-stages optimal control technique solving Problems 19 and 22 below, in that order. Before stating them, let us fix some notation allowing us to rewrite the total wealth of the agent and the constraint condition in terms of the function Γ defined as follows.

Definition 15 We define the map $\Gamma : \mathbb{R}_+ \times \mathcal{H} \rightarrow \mathbb{R}$ as

$$\Gamma(t, w, x) := w + g(t)x_0 + \langle h(t), x_1 \rangle. \quad (34)$$

It is also convenient to introduce the following subsets of $\mathbb{R}_+ \times \mathcal{H}$: for $0 \leq s < r \leq \infty$:

$$\mathcal{H}_+^{s,r} := \{(t, w, x) \in [s, r] \times \mathcal{H} : \Gamma(t, w, x) \geq 0\} \quad (35)$$

and

$$\mathcal{H}_{++}^{s,r} := \{(t, w, x) \in [s, r] \times \mathcal{H} : \Gamma(t, w, x) > 0\}.$$

In the case $0 \leq s < \infty$, by $\mathcal{H}_+^{s,\infty}$ we mean the subset $\{(t, w, x) \in [s, \infty) \times \mathcal{H} : \Gamma(t, w, x) \geq 0\}$. The case $\mathcal{H}_{++}^{s,\infty}$ is analogous.

We now define the value function $V : \mathcal{H}_+^{0,\infty} \rightarrow \mathbb{R}$ as ¹

$$V(t, w, x) := \sup_{\pi \in \Pi(t, w, x)} J(t, w, x; \pi). \quad (36)$$

We note that at time τ_R the dynamics of our problem changes. In particular, when $t \geq \tau_R$, $(g(t), h(t)) \equiv 0$, hence $\Pi(t, w, x)$ does not depend on x : in this case we will denote it simply by $\Pi(t, w)$. If we assume that the Dynamic Programming Principle (DPP) holds (see e.g. [22, Section 2.3]) we can rewrite the value function V as follows.

¹ Note that at this point we allow V to take the values $\pm\infty$.

- For any $t \in [\tau_R, \infty)$, $V(t, w, x) = V(t, w) =$ and the common value is

$$\sup_{\pi \in \Pi(t, w)} \mathbb{E} \left[\int_t^\infty e^{-(\rho+\delta)s} \left(\frac{(Kc(s))^{1-\gamma}}{1-\gamma} + \delta \frac{(kB(s))^{1-\gamma}}{1-\gamma} \right) ds \right].$$

- For any $t \in [0, \tau_R]$, $V(t, w, x)$ equals:

$$\sup_{\pi \in \Pi(t, w, x)} \mathbb{E} \left[\int_t^{\tau_R} e^{-(\rho+\delta)s} \left(\frac{c(s)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB(s))^{1-\gamma}}{1-\gamma} \right) ds + V(\tau_R, W^{w,x}(\tau_R; t, \pi), X^x(\tau_R; t)) \right]. \quad (37)$$

Note that we cannot say, at this stage, if DPP holds, as we do not even know whether V is finite. However, we will overcome this difficulty by proceeding as follows.

- **Step (I)** First, we solve problem (37) finding its value function V^{ih} (which must be equal to V by construction) and the optimal strategies when $t \geq \tau_R$. We call this ‘Problem 3’ and solve it in next Section (Section 5).
- **Step (II)** Second, we consider problem (37) with the final payoff given by the solution found at Step (I). We find again its value function V^{fh} and its optimal strategies. We call this ‘Problem 4’ and solve it in Section 6.
- **Step (III)** Observe that $V^{fh} = V$ and then conclude at the end of Section 6.

Remark 16 The strategy we follow to solve the optimization Problem 14 relies on finding an explicit solution of the HJB and prove that this is equal to the value function. We employ the two-stages optimal control technique and solve, in this order, Problems 19 and 22 since the HJB equations associated to these problems are smooth in time. Smoothness in time is instead lost if we consider the HJB equation associated to Problem 14. Intuitively this is evident since in τ_R the dynamics of the state abruptly changes. More formally, as one can see from (78), the value function associated to Problem 14 will be a function of the total wealth of the agent. In particular it will be a function of g (see (16)). Since, by Lemma 9 the map $r \mapsto g'(r)$ has a discontinuity in $r = \tau_R$, it is clear that smoothness in time will be lost by approaching directly Problem 14 via a DP approach.

4.3 The evolution of admissible paths

Fix the initial condition $(t, w, x) \in \mathcal{H}_+^{0,\infty}$ for system (31) and $\pi \in \Pi(t, w, x)$. Let $(W^{w,x}(\cdot; t, \pi), X^x(\cdot; t))$ be the corresponding solution. In the following we will often use the shorthand notation

$$\bar{\Gamma}(s) := \Gamma(s, W^{w,x}(s; t, \pi), X^x(s; t)), \quad s \geq t, \quad (38)$$

and we will denote by τ_t the first exit time of the process $[t, +\infty) \times \Omega \rightarrow \mathbb{R}_+ \times \mathcal{H}$, $s \rightarrow (s, W^{w,x}(s; t, \pi), X^x(s; t))$ from $\mathcal{H}_{++}^{0,\infty}$, that is

$$\begin{aligned} \tau_t &:= \inf \{s \geq t : (s, W^{w,x}(s; t, \pi), X^x(s; t)) \notin \mathcal{H}_{++}^{0,\infty}\} \\ &= \inf \{s \geq t : (s, W^{w,x}(s; t, \pi), X^x(s; t)) \in \partial \mathcal{H}_{++}^{0,\infty}\} \\ &= \inf \{s \geq t : \Gamma(s, W^{w,x}(s; t, \pi), X^x(s; t)) = 0\}. \end{aligned} \quad (39)$$

Lemma 17 Fix the initial condition $(t, w, x) \in \mathcal{H}_+^{0,\infty}$ for system (31) and $\pi \in \Pi_0$. Let $(W^{w,x}(\cdot; t, \pi), X^x(\cdot; t))$ be the corresponding solution. Then we have the following (where we write $X_0(s)$ for $X_0^x(s; t)$).

- (i) The process $\bar{\Gamma}(s)$ in (38) satisfies, for $s \geq t$, the SDE

$$\begin{aligned} d\bar{\Gamma}(s) &= [(r + \delta)\bar{\Gamma}(s) - c(s) - \delta B(s) \\ &+ (\theta^\top(s)\sigma + g(s)X_0(s)\sigma_y^\top) \kappa] ds \\ &+ [\theta^\top(s)\sigma + g(s)X_0(s)\sigma_y^\top] dZ(s). \end{aligned} \quad (40)$$

(ii) Assume that $\Gamma(t, w, x) = 0$, i.e. that $\bar{\Gamma}(t) = 0$. Then for every $s \geq t$ it must be $\bar{\Gamma}(s) = 0$, $\mathbb{P}$ -a.s., and

$$c(s, \omega) = 0, \quad B(s, \omega) = 0, \quad \theta^\top(s) \sigma + g(s) X_0(s) \sigma_y^\top = 0, \quad (41)$$

must hold $ds \otimes \mathbb{P}$ -a.e. in $[t, \infty) \times \Omega$. Let now $\bar{\Gamma}(t) > 0$. Then, for every $s \geq t$, $\mathbb{I}_{\{\tau_t < s\}} \bar{\Gamma}(s) = 0$, $\mathbb{P}$ -a.s., and

$$\mathbb{I}_{\{\tau_t < s\}}(\omega) c(s, \omega) = 0, \quad \mathbb{I}_{\{\tau_t < s\}}(\omega) B(s, \omega) = 0, \\ \mathbb{I}_{\{\tau_t < s\}} \left[\theta^\top(s) \sigma + g(s) X_0(s) \sigma_y^\top \right] = 0,$$

$ds \otimes \mathbb{P}$ -a.e. in $[t, \infty) \times \Omega$.

PROOF. (i). We take $t < \tau_R$ as the case $t \geq \tau_R$ is immediate from the argument below.

Since the function g' has a discontinuity in τ_R (and thus we can not apply Itô's formula directly on the time interval $[t, +\infty)$), we consider separately the case $s \in [t, \tau_R)$ and $s \in [\tau_R, +\infty)$.

When $s \in [\tau_R, +\infty)$, we have $d\bar{\Gamma}(s) = dW(s)$ and by the first equation in (4), the identity (40) immediately follows, recalling that $g \equiv 0$ on $[\tau_R, +\infty)$.

When $s \in [0, \tau_R)$, we have that

$$d\bar{\Gamma}(s) = dW(s) + d(g(s)X_0(s) + \langle h(s), X_1(s) \rangle),$$

and proceeding as in the proof of Proposition 10 we can write

$$d\langle (g(s), h(s)), X(s) \rangle = \\ = [(\beta + \mu_y)g(s) - 1]X_0(s) + (r + \delta)\langle h(s), X_1(s) \rangle] ds \\ + g(s)X_0(s)\sigma_y^\top dZ(s).$$

Using the dynamics of W given by the first equation in (31) and (7) we get (40). This proves (i).

(ii). This proof, given (i) above, is completely analogous to the proof of [7, Lemma 4.5(ii)] and we omit it for brevity.

Remark 18 We note that the temporal dynamics of the total wealth process in (40) and of admissible controls in (41) abruptly changes at $t = \tau_R$. In particular, for $t \geq \tau_R$, the time evolution of the total wealth process and of the admissible strategies are as in the classical Merton problem. For $t \in [0, \tau_R]$ we recover instead [7, Lemma 4.9].

5 Post-retirement problem: an infinite horizon problem.

We start by Step (I) above dealing with the case $t \in [\tau_R, \infty)$. Note that, in this case, the dynamic of W does not depend on X and the functions g and h are null. Thus the constraint in (27) and consequently the set of admissible controls do not depend on X : the value function depends just on the state variable W . We then have to solve the following:

Problem 19 Let $t \in [\tau_R, +\infty)$ and $w \in \mathbb{R}_+$. Consider the state equation

$$\begin{cases} dW(s) = [(r + \delta)W(s) + \theta^\top(s)(\mu - r\mathbf{1}) - c(s) - \delta B(s)] ds + \\ \quad + \theta^\top(s) \sigma dZ(s), \\ W(t) = w. \end{cases} \quad (42)$$

Find the strategy $\bar{\pi} \in \Pi(t, w)$ that maximizes the objective functional

$$J^{ih}(t, w; \pi) := \mathbb{E} \left[\int_t^{+\infty} e^{-(\rho + \delta)s} \left(\frac{(Kc(s))^{1-\gamma}}{1-\gamma} + \delta \frac{(kB(s))^{1-\gamma}}{1-\gamma} \right) ds \right] \quad (43)$$

recalling that, in this case, the constraint is simply

$$W^w(s; t, \pi) \geq 0 \quad \text{for all } s \geq t.$$

The value function $V^{ih} : [\tau_R, +\infty) \times \mathbb{R}_+ \rightarrow \bar{\mathbb{R}}$ is defined as ²

$$V^{ih}(t, w) := \sup_{\pi \in \Pi(t, w)} J^{ih}(t, w; \pi). \quad (44)$$

Note that, by construction, it must be that

$$V(t, w, x) = V^{ih}(t, w), \quad \forall (t, w, x) \in [\tau_R, +\infty) \times \mathcal{H}.$$

This problem can be easily solved by appealing to [7, Theorem 5.1], as follows.

Proposition 20 (Solution of Problem 19) Consider the infinite-horizon optimization Problem 19 under Hypothesis 2.

- The value function $V^{ih} : [\tau_R, \infty) \times \mathbb{R}_+ \rightarrow \bar{\mathbb{R}}$ is given by

$$V^{ih}(t, w) = e^{-(\rho + \delta)t} \hat{\eta}^\gamma \frac{w^{1-\gamma}}{1-\gamma}, \quad (45)$$

where

$$\hat{\eta} := (K^{-b} + \delta k^{-b})\nu, \quad b = 1 - \frac{1}{\gamma}, \quad (46)$$

$$\nu := \frac{\gamma}{\rho + \delta - (1 - \gamma)(r + \delta + \frac{\kappa^\top \kappa}{2\gamma})}.$$

- The optimal strategies in feedback form are (here $s \geq t$)

$$c^*(s) := K^{-b} \hat{\eta}^{-1} W^*(s), \quad B^*(s) := k^{-b} \hat{\eta}^{-1} W^*(s),$$

$$\theta^*(s) := \frac{W^*(s)}{\gamma} (\sigma^\top)^{-1} \kappa, \quad (47)$$

where the optimal wealth process W^* satisfies $W(t) = w$ and has dynamics (here $s \geq t$)

$$dW^*(s) = \\ = W^*(s) \left(r + \delta + \frac{1}{\gamma} \kappa^\top \kappa - \nu^{-1} \right) ds + \frac{W^*(s)}{\gamma} \kappa^\top dZ(s),$$

PROOF. For the proof of the result we appeal to [7, Theorem 5.1] fixing $(X_0, X_1) \equiv 0$, for example setting the initial condition for X at 0, i.e. $x = 0 \in M_2$ in their notation. When we apply this result we just have to pay attention to the different initial time of the problem. This is not a big issue since the problem is autonomous and it is sufficient to rescale time. We thus obtain that the value function associated with Problem 19 is as in (45).

Remark 21 Problem 19 is a variant of the classical infinite horizon Merton problem taking into account the availability of life insurance. Solution to Problem 19 can also be found by adapting classical arguments (see e.g. [46]). Note, in particular, that one has to assume $\nu > 0$, as required in Hypothesis 2, in order for the Merton problem to have a solution. If $\nu < 0$, for example, one can prove that an investor can achieve infinite utility by delaying consumption when $\gamma \in (0, 1)$ (see [46, Section 1.6 and Proposition 1.3] and [7, Remark 2.5]). For the case $\gamma > 1$, we refer to [23], in which it is proved, in a related deterministic case, that $\nu < 0$ implies $V^{ih}(t, w) = -\infty$.

6 Pre-retirement problem: a finite horizon problem

From the above Proposition 20 we know that $V(\tau_R, w, x) = V^{ih}(\tau_R, w)$ is given by (45). Hence, we can now maximize the functional in (37) with $V(\tau_R, w, x)$ as terminal datum:

² Note that we allow at this point V^{ih} to take the values $\pm\infty$.

Problem 22 Let $t \in [0, \tau_R]$, and $(w, x) \in \mathcal{H}$ such that $\Gamma(t, w, x) \geq 0$. Take the state equation

$$\begin{cases} dW(s) = \\ \left[W(s)r + \theta^\top(s)(\mu - r\mathbf{1}) + X_0(s) - c(s) + \delta(W(s) - B(s)) \right] ds \\ + \theta^\top(s)\sigma dZ(s), \\ dX(s) = AX(s)ds + CX(s)dZ(s), \\ W(t) = w, \quad X(t) = x. \end{cases} \quad (48)$$

Find the strategy $\bar{\pi} \in \Pi(t, w, x)$ maximizing the objective:

$$\begin{aligned} J^{fh}(t, w, x; \pi) := \\ \mathbb{E} \left[\int_t^{\tau_R} e^{-(\rho+\delta)s} \left(\frac{(c(s))^{1-\gamma}}{1-\gamma} + \delta \frac{(kB(s))^{1-\gamma}}{1-\gamma} \right) ds + \right. \\ \left. + e^{-(\rho+\delta)\tau_R} \frac{\hat{\eta}^\gamma (W^{w,x}(\tau_R; t, \pi))^{1-\gamma}}{1-\gamma} \right]. \end{aligned}$$

Here, the constraint is:

$$W^{w,x}(s; t, \bar{\pi}) + g(s)X_0^x(s; t) + \langle h(s), X_1^x(s; t) \rangle \geq 0 \quad \forall s \in [t, \tau_R].$$

Remark 23 We observe that the above problem does not change if we truncate the admissible trajectories at time τ_R , i.e., if we take as set of admissible controls the following one, which is the standard one for our finite horizon problem:

$$\Pi^{\tau_R}(t, w, x) :=$$

$$\left\{ \mathbb{F}\text{-predictable } \pi(\cdot) \in L^1(\Omega \times [t, \tau_R], \mathbb{R}_+^2) \times L^2(\Omega \times [t, \tau_R]; \mathbb{R}^n); \right.$$

$$\left. \text{such that: } W^{w,x}(s; t, \pi) + g(s)X_0^x(s; t) + \langle h(s), X_1^x(s; t) \rangle \geq 0 \right. \\ \left. \forall s \in [t, \tau_R] \right\}. \quad (49)$$

We will keep the less standard choice of the set of admissible control because it allows us to simplify the notation.

We define the value function $V^{fh} : \mathcal{H}_{++}^{0, \tau_R} \rightarrow \mathbb{R}$ as

$$V^{fh}(t, w, x) := \sup_{\pi \in \Pi(t, w, x)} J^{fh}(t, w, x; \pi). \quad (50)$$

To write the associated HJB equation we rewrite system (48) in the unknowns (W, X) as an equation in a single unknown $\mathcal{X}$. We introduce two linear operators on $\mathcal{H} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$: the unbounded operator $\mathcal{B}$ with values in $\mathcal{H}$ given by

$$D(\mathcal{B}) = \mathbb{R} \times D(A) \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n,$$

$$\mathcal{B}(w, x, c, B, \theta) := \left((r + \delta)w + \theta^\top(\mu - r\mathbf{1}) + x_0 - c - \delta B, Ax \right),$$

and the bounded operator $\mathcal{S}$ with values in $L(\mathbb{R}^n; \mathcal{H})$:

$$\mathcal{S}(w, x, c, B, \theta) := \left[z \rightarrow \left(\theta^\top \sigma z, (Cx)z \right) \right].$$

It is not difficult to check that, for every fixed $(w, x, c, B, \theta) \in \mathcal{H} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$, the adjoint of $\mathcal{S}(w, x, c, B, \theta)$ is the map $\mathcal{S}(w, x, c, B, \theta)^* \in L(\mathcal{H}; \mathbb{R}^n)$, given by:

$$(u, p) \mapsto u\theta^\top \sigma + x_0 p_0 \sigma_y.$$

Let $\mathcal{I}$ denote the following space:

$$L^2(-d, 0; \mathbb{R}) \times (L^2(-d, 0; \mathbb{R}) \times L(L^2(-d, 0; \mathbb{R}); L^2(-d, 0; \mathbb{R}))).$$

Therefore³ $L(\mathcal{H}, \mathcal{H}) \cong \mathcal{N} := \mathcal{H} \times \mathcal{H} \times \mathcal{I}$, and given an element $P \in \mathcal{N}$, we can index its entries as P_{ij} , where i denotes

³ Indeed a linear bounded operator $Q \in L(\mathcal{H}, \mathcal{H})$ must have three components: $Q_1 : \mathcal{H} \rightarrow \mathbb{R}$, $Q_2 : \mathcal{H} \rightarrow \mathbb{R}$, $Q_3 : \mathcal{H} \rightarrow L^2(-d, 0; \mathbb{R})$. The first two components must be in $L(\mathcal{H}, \mathbb{R}) \cong \mathcal{H}$, while the third component is characterized by three linear operators $Q_{31} : \mathbb{R} \rightarrow L^2(-d, 0; \mathbb{R})$, $Q_{32} : \mathbb{R} \rightarrow L^2(-d, 0; \mathbb{R})$, $Q_{33} : \mathbb{R} \rightarrow L^2(-d, 0; \mathbb{R})$. Hence $Q_{31} \in L(\mathbb{R}, L^2(-d, 0; \mathbb{R})) \cong L^2(-d, 0; \mathbb{R})$, the same for Q_{32} , while $Q_{33} \in L(L^2(-d, 0; \mathbb{R}), L^2(-d, 0; \mathbb{R}))$.

the spaces $\mathcal{H}, \mathcal{H}, \mathcal{I}$ (in this order) and j the components in each space (hence $(P_{11}, P_{12}, P_{13}) \in \mathcal{H}$). Through this interpretation we can define the space of symmetric elements in $\mathcal{N}$ as $\mathcal{N}_{sym} := \{P \in \mathcal{N} : P_{ij} = P_{ji}, i, j = 1, \dots, 3\}$. By simple computations we then have, for any $P \in \mathcal{N}_{sym}$ and $(w, x, c, B, \theta) \in \mathcal{H} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$,

$$\begin{aligned} \text{Tr}[PS(w, x, c, B, \theta)S(w, x, c, B, \theta)^*] = \\ = \theta^\top \sigma \sigma^\top \theta P_{11} + 2\theta^\top \sigma \sigma_y x_0 P_{12} + \sigma_y^\top \sigma_y x_0^2 P_{22}. \end{aligned} \quad (51)$$

Hence, for any given function $f : [0, \tau_R] \times \mathcal{H} \rightarrow \mathbb{R}$, its second Fréchet derivative, w.r.t. the variable in $\mathcal{H}$, at a given point (t, w, x) is an element $\nabla^2 f(t, w, x) \in \mathcal{N}_{sym}$. Formula (51) then provides then the second order term that will appear in our HJB equation. Notice that, since the L^2 -component of $\mathcal{S}$ is zero, such second order term turns out to be finite dimensional: it depends only on P_{11}, P_{12} and P_{22} , that is only on the derivative w.r.t. the real components (w, x_0) . This is reasonable since the noise we consider affects explicitly only the dynamics of W and X_0 , but not that of X_1 . Let $t \in [0, \tau_R]$ and $\pi(\cdot) = (c(\cdot), B(\cdot), \theta(\cdot)) \in \Pi(t, w, x)$. The state equation (48) is then rewritten as

$$\begin{cases} d\mathcal{X}(s) = \mathcal{B}(\mathcal{X}(s), \pi(s)) ds + \mathcal{S}(\mathcal{X}(s), \pi(s)) dZ(s), \\ \mathcal{X}(t) = (w, x). \end{cases} \quad (52)$$

By [32, Chapter 5.6] and Proposition 5 there is a unique mild solution to system (52). We will denote the solution at time $t \in [0, \tau_R]$ by $\mathcal{X}^{w,x}(s; t, \pi) = (W^{w,x}(s; t, \pi), X^x(s; t))$. As usual, dependence on initial conditions will be at times subsumed in the following.

6.1 The HJB equation and its explicit solution

The Hamiltonian of our Problem 22 is

$$\tilde{\mathbb{H}}^{fh} : [0, \tau_R] \times (\mathbb{R} \times D(A)) \times \mathcal{H} \times \mathcal{N}_{sym} \rightarrow \mathbb{R} \cup \{\pm\infty\},$$

$$\begin{aligned} \tilde{\mathbb{H}}^{fh}(t, (w, x), p, P) = \sup_{\pi \in \mathbb{R}_+^2 \times \mathbb{R}^n} \left[\langle \mathcal{B}(w, x, \pi), p \rangle_{\mathcal{H}} \right. \\ \left. + \frac{1}{2} \text{Tr}[PS(w, x, c, B, \theta)S(w, x, c, B, \theta)^*] + l(t, \pi) \right] \end{aligned}$$

where

$$l(t, \pi) := e^{-(\rho+\delta)t} \left(\frac{c^{1-\gamma}}{1-\gamma} + \delta \frac{k^{1-\gamma} B^{1-\gamma}}{1-\gamma} \right).$$

It is however convenient (see Remark 25 below) to rewrite the Hamiltonian in a more explicit way using the definitions of $\mathcal{B}$ and $\mathcal{S}$ and exploiting the definition of A^* . Separating the part that depends on the controls from the rest, the Hamiltonian for Problem 22 is the function:

$$\mathbb{H}^{fh} : [0, \tau_R] \times \mathcal{H} \times (\mathbb{R} \times D(A^*)) \times \mathcal{N}_{sym} \rightarrow \mathbb{R} \cup \{\pm\infty\},$$

decomposed as:

$$\begin{aligned} \mathbb{H}^{fh}(t, (w, x), p, P) := \mathbb{H}_0^{fh}((w, x), p, P_{22}) + \\ + \mathbb{H}_{\max}^{fh}(t, x_0, p_0, P_{11}, P_{12}), \end{aligned} \quad (53)$$

where:

$$\begin{aligned} \mathbb{H}_0^{fh}((w, x), p, P_{22}) &:= (r + \delta)wp_0 + x_0p_0 + \langle x, A^*p_1 \rangle_{M_2} + \\ &+ \frac{1}{2}\sigma_y^\top \sigma_y x_0^2 P_{22}, \end{aligned} \quad (54)$$

$$\mathbb{H}_{\max}^{fh}(t, x_0, p_0, P_{11}, P_{12}) := \sup_{(c, B, \theta) \in \mathbb{R}_+^2 \times \mathbb{R}^n} \mathbb{H}_{cv}^{fh}(t, x_0, p_0, P_{11}, P_{12}; \pi), \quad (55)$$

$$\begin{aligned} \mathbb{H}_{cv}^{fh}(t, x_0, p_0, P_{11}, P_{12}; \pi) &:= l(t, \pi) + [\theta^\top (\mu - r\mathbf{1}) - c - \delta B]p_0 + \\ &\frac{1}{2}\theta^\top \sigma \sigma^\top \theta P_{11} + \theta^\top \sigma \sigma_y x_0 P_{12}. \end{aligned}$$

Reordering the terms in the above definition of $\mathbb{H}_{cv}^{fh}$ we can write

$$\begin{aligned} \mathbb{H}_{cv}^{fh}(t, x_0, p_0, P_{11}, P_{12}, \pi) &= e^{-(\rho+\delta)t} \frac{c^{1-\gamma}}{1-\gamma} - cp_0 \\ &+ \delta \left[e^{-(\rho+\delta)t} \frac{k^{1-\gamma} B^{1-\gamma}}{1-\gamma} - Bp_0 \right] \\ &+ \left[\frac{1}{2} \left| \theta^\top \sigma \right|^2 P_{11} + \theta^\top \sigma \sigma_y x_0 P_{12} + \theta^\top (\mu - r\mathbf{1})p_0 \right], \end{aligned} \quad (56)$$

from which we easily see that for each $x_0 \in \mathbb{R}$ and $P_{12} \in \mathbb{R}$ there are three possible cases:

- (1) if $p_0 > 0$ and $P_{11} < 0$ the sup in (55) is achieved at (θ^*, c^*, B^*) , where (here $b = 1 - \gamma^{-1}$),

$$\begin{aligned} \theta^* &= -\frac{(\sigma \sigma^\top)^{-1}}{P_{11}} [(\mu - r\mathbf{1})p_0 + \sigma \sigma_y x_0 P_{12}], \\ c^* &= e^{-\frac{(\rho+\delta)t}{\gamma}} p_0^{-\frac{1}{\gamma}}, \\ B^* &= k^{-b} e^{-\frac{(\rho+\delta)t}{\gamma}} p_0^{-\frac{1}{\gamma}} \end{aligned} \quad (57)$$

- (2) if $p_0 < 0$ or $P_{11} > 0$ then the supremum in (55) is $+\infty$;

- (3) if $p_0 P_{11} = 0$ the supremum in (55) can be finite or infinite depending on γ and on the sign of the other terms involved.

The HJB equation associated with Problem 22 is the following PDE in the unknown $v : [0, \tau_R] \times \mathcal{H} \rightarrow \mathbb{R}$, with terminal condition (here $\hat{\eta}$ as in (46)):

$$\begin{cases} -\partial_t v(t, w, x) = \mathbb{H}^{fh}(t, (w, x), \nabla v(t, w, x), \nabla^2 v(t, w, x)) \\ v(\tau_R, w, x) = e^{-(\rho+\delta)\tau_R} \hat{\eta}^\gamma \frac{w^{1-\gamma}}{1-\gamma}. \end{cases} \quad (58)$$

Definition 24 A function $v : \mathcal{H}_{++}^{0, \tau_R} \rightarrow \mathbb{R}$ is a classical solution of the HJB equation (58) if:

- (1) v is continuously Fréchet differentiable and its second Fréchet derivatives with respect to the couple (w, x_0) exist and are continuous in $\mathbb{H}_{++}^{0, \tau_R}$.
(2) $\partial_x v(t, w, x) \in D(A^*)$ for all $(t, w, x) \in \mathcal{H}_{++}^{0, \tau_R}$ and $A^* \partial_x v(t, w, x)$ is continuous in $\mathcal{H}_{++}^{0, \tau_R}$.

- (3) v satisfies (58) for every $(t, w, x) \in \mathcal{H}_{++}^{0, \tau_R}$.

Remark 25 The difference between $\tilde{\mathbb{H}}^{fh}$ and $\mathbb{H}^{fh}$ lies in the term involving A , that appears as $\langle Ax, p_1 \rangle$ in the former but as $\langle x, A^*p_1 \rangle$ in the latter. This choice makes $\mathbb{H}^{fh}$ defined on the whole $\mathcal{H}_{++}^{0, \tau_R}$ instead than only on $\mathcal{H}_{++}^{0, \tau_R} \cap (\mathbb{R} \times \mathcal{D}(A))$, at the price of requiring further regularity of the solution, as specified in Definition 24-2. In next Proposition 26 we provide an explicit solution that satisfies the required properties.

If a solution v to (58) satisfies $\partial_w v > 0$ and $\partial_{ww}^2 v < 0$ uniformly in (t, w, x) , then we fall in case 1 above and, plugging θ^*, c^*, B^* in the definition of $\mathbb{H}^{fh}$, we find the PDE for v to take the form

$$\begin{aligned} -\partial_t v &= [(r + \delta)w + x_0] \partial_w v + \langle x, A^* \partial_x v \rangle_{M_2} + \\ &\frac{1}{2} \sigma_y^\top \sigma_y x_0^2 \partial_{x_0 x_0}^2 v + \frac{\gamma}{1-\gamma} e^{-\frac{\rho+\delta}{\gamma} t} (1 + \delta k^{-b}) (\partial_w v)^b \\ &- \frac{1}{2} \frac{1}{\partial_{ww}^2 v} ((\mu - r\mathbf{1}) \partial_w v + \sigma \sigma_y x_0 \partial_{wx_0}^2 v)^\top * \\ &(\sigma \sigma^\top)^{-1} ((\mu - r\mathbf{1}) \partial_w v + \sigma \sigma_y \partial_{wx_0}^2 v), \end{aligned} \quad (59)$$

with the same terminal condition of (58). We provide an explicit solution to (58).

Proposition 26 Let Hypothesis 2 hold true. Let for $(t, w, x) \in \mathcal{H}_{++}^{0, \tau_R}$,

$$\bar{v}(t, w, x) := F(t)^\gamma \frac{\Gamma^{1-\gamma}(t, w, x)}{1-\gamma}. \quad (60)$$

The function $\bar{v}$ is a classical solution of the HJB equation (58), where

$$\begin{aligned} F(t) &:= e^{-\frac{(\rho+\delta)t}{\gamma}} f(t), \quad f(t) := (\hat{\eta} - \eta) \exp\left(-\frac{\tau_R - t}{\nu}\right) + \eta, \\ \eta &:= (1 + \delta k^{-b})\nu, \end{aligned} \quad (61)$$

$\hat{\eta}, \nu, b$ are given in (46), and $\Gamma := \Gamma(t, w, x)$ given in (34).

PROOF. Recall that, by definition, $\mathcal{H}_{++}^{0, \tau_R}$ is the set where Γ is strictly positive. Thanks to the linearity of Γ , the function $\bar{v}$ is twice continuously differentiable in (w, x) and continuously differentiable in t (see Lemma 9-(iii)). The derivatives that appear in the Hamiltonian are easily computed:

$$\begin{aligned} \partial_t \bar{v}(t, w, x) &= \frac{\gamma}{1-\gamma} \Gamma^{1-\gamma} F'(t) F^{\gamma-1}(t) + g'(t) x_0 F(t)^\gamma \Gamma^{-\gamma} + \\ &+ \langle h'(t), x_1 \rangle F(t)^\gamma \Gamma^{-\gamma}, \\ \partial_w \bar{v}(t, w, x) &= F(t)^\gamma \Gamma^{-\gamma}, \quad \partial_{x_0} \bar{v}(t, w, x) = F(t)^\gamma \Gamma^{-\gamma} g(t), \\ \partial_{x_1} \bar{v}(t, w, x) &= F(t)^\gamma \Gamma^{-\gamma} h(t), \\ \partial_{ww} \bar{v}(t, w, x) &= -\gamma F(t)^\gamma \Gamma^{-\gamma-1}, \\ \partial_{vwx_0} \bar{v}(t, w, x) &= -\gamma F(t)^\gamma \Gamma^{-\gamma-1} g(t), \\ \partial_{vxx_0} \bar{v}(t, w, x) &= -\gamma F(t)^\gamma \Gamma^{-\gamma-1} g(t)^2. \end{aligned}$$

Thanks to Lemma 9-(iv), also requirement 2 in Definition 24 is satisfied. Moreover, by Hypothesis 2 we get $\partial_w \bar{v} > 0$ and $\partial_{ww} \bar{v} < 0$ on $\mathcal{H}_{++}^{0, \tau_R}$. Therefore we can consider the simplified form (59) for the HJB equation. Using Proposition 6 we compute:

$$\langle x, A^* \partial_x \bar{v}(t, w, x) \rangle_{M_2} = F(t)^\gamma \Gamma^{-\gamma} \langle x, A^*(g(t), h(t, \cdot)) \rangle_{M_2}$$

Substitute the above expression for $\langle x, A^* \partial_x \bar{v}(t, w, x) \rangle_{M_2}$ and $\bar{v}$ with all its derivatives in (59). Now, after multiplying both terms by $F(t)^{-\gamma} \Gamma^\gamma$, we get:

$$\begin{aligned} & -\frac{\gamma}{1-\gamma} \Gamma \frac{F'(t)}{F(t)} - \langle (g'(t), h'(t)), x \rangle_{M_2} = (r + \delta)w + x_0 \\ & + \langle x, A^*(g(t), h(t)) \rangle_{M_2} + \frac{\gamma}{1-\gamma} e^{-\frac{\rho+\delta}{\gamma}t} F(t)^{-1} \Gamma (1 + \delta k^{-b}) \\ & + \frac{\kappa^\top \kappa}{2\gamma} \Gamma - x_0 g(t) \kappa^\top \sigma_y. \end{aligned} \quad (62)$$

Thanks to (18), equality (62) can be rewritten as

$$\begin{aligned} & \frac{\gamma}{1-\gamma} \Gamma \frac{F'(t)}{F(t)} + ((\beta + \mu_y)g(t) - 1)x_0 + (r + \delta)\langle h(t), x_1 \rangle \\ & + (r + \delta)w + x_0 + \frac{\gamma}{1-\gamma} e^{-\frac{\rho+\delta}{\gamma}t} F(t)^{-1} \Gamma (1 + \delta k^{-b}) + \\ & \frac{\kappa^\top \kappa}{2\gamma} \Gamma - x_0 g(t) \kappa^\top \sigma_y = 0, \end{aligned}$$

which, using also the definition of β in (11) simplifies as

$$\begin{aligned} & \left(\frac{\gamma}{1-\gamma} \frac{F'(t)}{F(t)} + (r + \delta) + \frac{\gamma}{1-\gamma} e^{-\frac{\rho+\delta}{\gamma}t} F(t)^{-1} (1 + \delta k^{-b}) + \right. \\ & \left. + \frac{\kappa^\top \kappa}{2\gamma} \right) \Gamma = 0, \end{aligned} \quad (63)$$

Recalling (46) and (61) we compute the derivative of F and get that (63) is satisfied in $\mathcal{H}_{++}^{0, \tau_R}$.

Remark 27 The function $\bar{v}$ can be defined also in $\mathcal{H}_+^{0, \tau_R}$ by setting, on $\partial \mathcal{H}_+^{0, \tau_R} = \{\Gamma(t, w, x) = 0\}$, $\bar{v}(t, w, x) = 0$, when $\gamma \in (0, 1)$, and $\bar{v}(t, w, x) = -\infty$, when $\gamma \in (1, +\infty)$. From now on we will consider $\bar{v}$ defined on $\mathcal{H}_+^{0, \tau_R}$ with the above values at the boundary.

Next, there is the key step to get the optimal feedback strategies.

6.2 The fundamental identity

We start with the following lemma.

Lemma 28 Let $\bar{v} : \mathcal{H}_+^{0, \tau_R} \rightarrow \bar{\mathbb{R}}$ be the classical solution to the HJB (58) given in Proposition 26. Take any initial condition $(t, w, x) \in \mathcal{H}_+^{0, \tau_R}$ and take $\pi \in \Pi(t, w, x)$. Then:

$$\mathbb{E} \left[\sup_{s \in [t, \tau_R]} \bar{v}(s, (W^{w, x}(s; t, \pi), X^x(s; t))) \right] < \infty.$$

PROOF. The result is trivial when $\gamma \in (1, \infty)$. Let then take $\gamma \in (0, 1)$. We have, by (60) and (38),

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [t, \tau_R]} \bar{v}(s, (W^{w, x}(s; t, \pi), X^x(s; t))) \right] \\ & = \mathbb{E} \left[\sup_{s \in [t, \tau_R]} F^\gamma(s) \frac{\bar{\Gamma}^{1-\gamma}(s)}{1-\gamma} \right] \\ & \leq \max_{s \in [t, \tau_R]} F^\gamma(s) \mathbb{E} \left[\sup_{s \in [t, \tau_R]} \frac{\bar{\Gamma}^{1-\gamma}(s)}{1-\gamma} \right] < \infty. \end{aligned}$$

As from Lemma 17, the time evolution of the total wealth process $\bar{\Gamma}(\cdot)$, on $[t, \tau_R]$, is described by the SDE (40). Theorem 1.130 in [22] implies $\mathbb{E} \left[\sup_{s \in [t, \tau_R]} \bar{\Gamma}(s) \right] < \infty$ which gives the claim.

Proposition 29 Let $\bar{v} : \mathcal{H}_+^{0, \tau_R} \rightarrow \bar{\mathbb{R}}$ be the classical solution to the HJB (58) given in Proposition 26. Let $(t, w, x) \in \mathcal{H}_{++}^{0, \tau_R}$ and take any $\pi \in \Pi(t, w, x)$ such that, when $\gamma > 1$, $J^{fh}(t, w, x; \pi) > -\infty$. Let $\bar{\tau}_t := \tau_t \wedge \tau_R$ (where τ_t is defined in (39)). Let $\mathcal{X}(\cdot; t, \pi) = (W(\cdot; t, \pi), X(\cdot; t))$ be the trajectory corresponding to initial state (t, w, x) and strategy π . The following identity holds:

$$\begin{aligned} & \bar{v}(t, w, x) = J^{fh}(t, w, x; \pi) \\ & + \mathbb{E} \int_t^{\bar{\tau}_t} \left[\mathbb{H}_{\max}^{fh}(s, X_0(s), \partial_w \bar{v}, \partial_{ww} \bar{v}, \partial_{wx_0} \bar{v}) \right. \\ & \left. - \mathbb{H}_{cv}^{fh}(s, X_0(s), \partial_w \bar{v}, \partial_{ww} \bar{v}, \partial_{wx_0} \bar{v}; \pi(s)) \right] ds. \end{aligned} \quad (64)$$

where we set

$$\partial_w \bar{v} := \partial_w \bar{v}(s, \mathcal{X}(s; t, \pi)), \quad \partial_{ww} \bar{v} := \partial_{ww} \bar{v}(s, \mathcal{X}(s; t, \pi)),$$

$$\partial_{wx_0} \bar{v} := \partial_{wx_0} \bar{v}(s, \mathcal{X}(s; t, \pi))$$

PROOF. The proof goes along the same line of the proof of Proposition 4.11 of [7] hence we only provide a sketch of it. Recalling notation (38), let us call

$$\tau_N = \inf \{s \geq t : \bar{\Gamma}(s) \leq 1/N\}. \quad (65)$$

For N sufficiently large we have $\tau_N > 0$. Moreover notice that $\tau_N \rightarrow \tau_t$, $\mathbb{P}$ -a.s., so, still $\mathbb{P}$ -a.s., $\tau_N \wedge \tau_R \rightarrow \bar{\tau}_t$. Let us fix any such N . We now apply, with slight adaptation described in the proof of [7, Proposition 4.11], the Itô's formula of [22, Proposition 1.164] to the process $\bar{v}(s, \mathcal{X}(s; \pi))$ for $s \in [t, \tau_N \wedge \tau_R]$. This is possible since, in this interval, we know that the process $(s, \mathcal{X}(s; t, \pi))$ belongs to the set $\{\Gamma(s, w, x) \geq 1/N\}$ where $\bar{v}$, its derivatives and $A^* \partial_x \bar{v}$ are continuous and bounded on bounded sets, thanks to Definition 24 and Proposition 26.

We set, for brevity, $\mathcal{X}(\cdot; t, \pi) := \mathcal{X}(\cdot)$, $W(\cdot; t, \pi) := W(\cdot)$, $X(\cdot; t) := X(\cdot)$, and we apply [22, Proposition 1.164] to obtain

$$\begin{aligned} & \bar{v}(\tau_N \wedge \tau_R, \mathcal{X}(\tau_N \wedge \tau_R)) - \bar{v}(t, w, x) = \\ & = \int_t^{\tau_N \wedge \tau_R} ds \left\{ \partial_s \bar{v}(s, \mathcal{X}(s)) + \partial_{x_0} \bar{v}(s, \mathcal{X}(s)) X_0(s) \theta^\top(s) \sigma \sigma_y \right. \\ & + \partial_w \bar{v}(s, \mathcal{X}(s)) [(r + \delta)W(s) + \theta^\top(s)(\mu - r\mathbf{1}) + \\ & X_0(s) - c(s) - \delta B(s)] + \langle X(s), A^* \partial_x \bar{v}(s, \mathcal{X}(s)) \rangle_{M_2} + \\ & \left. \frac{1}{2} \partial_{ww} \bar{v}(s, \mathcal{X}(s)) \theta^\top \sigma \sigma^\top \theta + \frac{1}{2} \partial_{x_0 x_0} \bar{v}(s, \mathcal{X}(s)) \sigma_y^\top \sigma_y X_0^2(s) \right\} + \\ & \int_t^{\tau_N \wedge \tau_R} \left[\partial_w \bar{v}(s, \mathcal{X}(s)) \theta^\top(s) \sigma + \partial_{x_0} \bar{v}(s, \mathcal{X}(s)) X_0(s) \sigma_y^\top \right] dZ(s). \end{aligned}$$

Since the function $\bar{v}$ solves the HJB equation (58), by the definition of $\mathbb{H}_{cv}^{fh}$, $\mathbb{H}_{\max}^{fh}$ in (56) and (55) respectively, we can rewrite the above equality as

$$\begin{aligned}
& \bar{v}(\tau_N \wedge \tau_R, \mathcal{X}(\tau_N \wedge \tau_R)) - \bar{v}(t, w, x) + \\
& + \int_t^{\tau_N \wedge \tau_R} e^{-(\rho+\delta)s} \left(\frac{c(s)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB(s))^{1-\gamma}}{1-\gamma} \right) ds = \\
& \int_t^{\tau_N \wedge \tau_R} \left[-\mathbb{H}_{\max}^{fh}(s, X_0(s), \partial_w \bar{v}, \partial_{ww} \bar{v}, \partial_{wx_0} \bar{v}) \right. \\
& + \mathbb{H}_{cv}^{fh}(s, X_0(s), \partial_w \bar{v}, \partial_{ww} \bar{v}, \partial_{wx_0} \bar{v}; \pi(s)) \left. \right] ds \\
& + \int_t^{\tau_N \wedge \tau_R} \left[(\partial_w \bar{v}) \theta^\top(s) \sigma + (\partial_{x_0} \bar{v}) X_0(s) \sigma_y^\top \right] dZ(s). \tag{66}
\end{aligned}$$

Let us now take the expected value on both sides of equation (66). The stochastic integral is a martingale (by the definition of τ_N in (65), by the explicit form of $\bar{v}$ and its derivatives, and by the integrability of θ and X) so its expectation is 0. The integrand at the left hand side has always finite expectation: when $\gamma \in (0, 1)$ by the integrability properties of c and B , while for $\gamma > 1$, since we assumed $J^{fh}(t, w, x; \pi) > -\infty$. We then get that also the expectation of the first term of the right hand side (which exists since the integrand is negative) is finite. Hence, using the same notation as in the statement:

$$\begin{aligned}
& \mathbb{E} [\bar{v}(\tau_N \wedge \tau_R, \mathcal{X}(\tau_N \wedge \tau_R))] - \bar{v}(t, w, x) \tag{67} \\
& + \mathbb{E} \left[\int_t^{\tau_N \wedge \tau_R} e^{-(\rho+\delta)s} \left(\frac{c(s)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB(s))^{1-\gamma}}{1-\gamma} \right) ds \right] \\
& = -\mathbb{E} \left[\int_t^{\tau_N \wedge \tau_R} \left[\mathbb{H}_{\max}^{fh}(s, X_0(s), \partial_w \bar{v}, \partial_{ww} \bar{v}, \partial_{wx_0} \bar{v}) \right. \right. \\
& \left. \left. - \mathbb{H}_{cv}^{fh}(s, X_0(s), \partial_w \bar{v}, \partial_{ww} \bar{v}, \partial_{wx_0} \bar{v}; \pi(s)) \right] ds \right].
\end{aligned}$$

Now let $N \rightarrow \infty$. The first term of the left hand side converges thanks to Lemma 28, the continuity of $\bar{v}$, and dominated convergence. The last term of the left hand side converge to a finite number: when $\gamma \in (0, 1)$ by the integrability properties of c and B , while for $\gamma > 1$, since we assumed $J^{fh}(t, w, x; \pi) > -\infty$. Moreover the right hand side converge thanks to the constant sign of the integrands and monotone convergence: the limit is finite since the left hand side is finite. Hence

$$\begin{aligned}
& \mathbb{E} [\bar{v}(\bar{\tau}_t, \mathcal{X}(\bar{\tau}_t))] - \bar{v}(t, w, x) + \tag{68} \\
& \mathbb{E} \left[\int_t^{\bar{\tau}_t} e^{-(\rho+\delta)s} \left(\frac{c(s)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB(s))^{1-\gamma}}{1-\gamma} \right) ds \right] \\
& = -\mathbb{E} \left[\int_t^{\bar{\tau}_t} \left[\mathbb{H}_{\max}^{fh}(s, X_0(s), \partial_w \bar{v}, \partial_{ww} \bar{v}, \partial_{wx_0} \bar{v}) \right. \right. \\
& \left. \left. - \mathbb{H}_{cv}^{fh}(s, X_0(s), \partial_w \bar{v}, \partial_{ww} \bar{v}, \partial_{wx_0} \bar{v}; \pi(s)) \right] ds \right]. \tag{69}
\end{aligned}$$

The claim now follows rearranging the terms and observing that

$$\begin{aligned}
& J^{fh}(t, w, x; \pi) = \\
& \mathbb{E} \left[\int_t^{\bar{\tau}_t} e^{-(\rho+\delta)s} \left(\frac{c(s)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB(s))^{1-\gamma}}{1-\gamma} \right) ds \right] + \\
& \mathbb{E} [\bar{v}(\bar{\tau}_t, \mathcal{X}(\bar{\tau}_t))].
\end{aligned}$$

The above is obvious when $\bar{\tau}_t = \tau_R$ by the form of $\bar{v}$ in Proposition 26. When $\bar{\tau}_t < \tau_R$ (which can happen only for $\gamma \in (0, 1)$ due to our assumptions), it must be, by (60), $\mathbb{E} [\bar{v}(\bar{\tau}_t, \mathcal{X}(\bar{\tau}_t))] = 0$, and, by Lemma 17-(ii),

$$\mathbb{I}_{\{\bar{\tau} < s\}}(\omega) c(s, \omega) = 0, \quad \mathbb{I}_{\{\bar{\tau} < s\}}(\omega) B(s, \omega) = 0,$$

holds $ds \otimes \mathbb{P} - a.e.$ in $[0, \tau_R] \times \Omega$, and the claim still follows.

As a consequence of Proposition 29 we get the following.

Corollary 30 *The value function V^{fh} given in (50) is finite on $\mathcal{H}_{++}^{0, \tau_R}$ and $V^{fh}(t, w, x) \leq \bar{v}(t, w, x)$ for every $(t, w, x) \in \mathcal{H}_{++}^{0, \tau_R}$.*

PROOF. For $\gamma \in (0, 1)$ the positivity of the integrand in (64) implies, for every $(t, w, x) \in \mathcal{H}_{++}^{0, \tau_R}$ and $\pi \in \Pi(t, w, x)$, $\bar{v}(t, w, x) \geq J^{fh}(t, w, x; \pi)$. Then the claim follows by the definition of V^{fh} . The same argument works for $\gamma > 1$ if we prove that there exists a strategy in $\Pi(t, w, x)$ such that $J^{fh}(t, w, x; \pi) > -\infty$. Such strategy is the one given in (75) below.

6.3 Verification Theorem and optimal feedback strategies

Here we show that $\bar{v} = V^{fh}$ in $\mathcal{H}_{++}^{0, \tau_R}$ also finding the optimal feedback strategies. First, we provide the following definitions.

Definition 31 *Fix $(t, w, x) \in \mathcal{H}_{++}^{0, \tau_R}$. A strategy $\bar{\pi} := (\bar{c}, \bar{B}, \bar{\theta})$ is called an optimal strategy if $(\bar{c}, \bar{B}, \bar{\theta}) \in \Pi(t, w, x)$, and $V^{fh}(t, w, x) = J(t, w, x; \bar{\pi})$.*

Note that if $(t, w, x) \in \partial \mathcal{H}_{++}^{0, \tau_R}$, By Lemma 17 there is only one admissible strategy which is also the unique optimal one.

Definition 32 *We say that a function $(C, B, \Theta) : \mathcal{H}_{++}^{0, \tau_R} \rightarrow \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$ is an optimal feedback map if for any initial datum $(t, w, x) \in \mathcal{H}_{++}^{0, \tau_R}$, the closed loop equation*

$$\begin{cases} dW(s) = [(r + \delta)W(s) + \Theta^\top(s, W(s), X(s))(\mu - r) + \\ + X_0(s) - C(s, W(s), X(s)) + \delta B(s, W(s), X(s))] ds + \\ + \Theta^\top(s, W(s), X(s)) \sigma dZ(s), \\ dX(s) = AX(s) + (CX(s))^\top dZ(s), \\ (W(t), X(t)) = (w, x). \end{cases} \tag{70}$$

has a unique solution $(W^, X) := \mathcal{X}^*$ and the associated control strategy $(\bar{c}, \bar{B}, \bar{\theta})$*

$$\bar{c}(s) := C(s, W^*(s), X(s)), \quad \bar{B}(s) := B(s, W^*(s), X(s)), \tag{71}$$

$$\bar{\theta}(s) := \Theta(s, W^*(s), X(s)) \tag{72}$$

is an optimal strategy.

As usual, the candidate optimal feedback map is given by the maximum points of the Hamiltonian. In our case these are given by (57) so, putting there $\partial_w \bar{v}, \partial_{ww} \bar{v}, \partial_{wx_0} \bar{v}$ in place of p_0, P_{11}, P_{12} we get the map:

$$\begin{cases} C_f(t, w, x) := f(t)^{-1} \Gamma(t, w, x) \\ B_f(t, w, x) := k^{-b} f(t)^{-1} \Gamma(t, w, x) \\ \Theta_f(t, w, x) := (\sigma \sigma^\top)^{-1} (\mu - r \mathbf{1}) \gamma^{-1} \Gamma(t, w, x) - (\sigma^\top)^{-1} \sigma_y g(t) x_0. \end{cases} \tag{73}$$

We prove that this is an optimal feedback map. For $(t, w, x) \in \mathcal{H}_+^{0, \tau_R}$, denote with $W_f^*(s)$ the unique solution of the closed loop equation (70) with $(\mathbf{C}_f, \mathbf{B}_f, \Theta_f)$ in place of $(\mathbf{C}, \mathbf{B}, \Theta)$ and set

$$\Gamma^*(s) = \Gamma(s, W_f^*(s), X(s)) = W_f^*(s) + g(s)X_0(s) + \langle h(s), X_1(s) \rangle. \quad (74)$$

The control strategy associated with (73) is then

$$\begin{aligned} \bar{c}_f(s) &:= \mathbf{C}_f(s, W_f^*(s), X(s)) = f(s)^{-1} \Gamma^*(s), \\ \bar{B}_f(s) &:= \mathbf{B}_f(s, W_f^*(s), X(s)) = k^{-b} f(s)^{-1} \Gamma^*(s), \\ \bar{\theta}_f(s) &:= \Theta_f(s, W_f^*(s), X(s)) = (\sigma \sigma^\top)^{-1} (\mu - r \mathbf{1}) \gamma^{-1} \Gamma^*(s) \\ &\quad - (\sigma^\top)^{-1} \sigma_y g(s) X_0(s). \end{aligned} \quad (75)$$

The next Lemma ensures that this strategy is admissible.

Lemma 33 *Let $(t, w, x) \in \mathcal{H}_+^{0, \tau_R}$. The process Γ^* defined in (74) satisfies the SDE*

$$\begin{aligned} d\Gamma^*(s) &= \Gamma^*(s) \left(r + \delta + \frac{1}{\gamma} |\kappa|^2 - f(s)^{-1} (1 + \delta k^{-b}) \right) ds \\ &\quad + \Gamma^*(s) \kappa^\top \sigma dZ(s). \end{aligned} \quad (76)$$

PROOF. Substituting (75) into the first equation in (70) we obtain

$$\begin{aligned} dW_f^*(s) &= \left[\Gamma^*(s) \left[\frac{|\kappa|^2}{\gamma} - f(s)^{-1} (1 + \delta k^{-b}) \right] + \right. \\ &\quad W_f^*(s) (r + \delta) - \kappa^\top \sigma_y g(s) X_0(s) + \\ &\quad \left. X_0(s) \right] ds + \left[\frac{\Gamma^*(s)}{\gamma} \kappa^\top - g(s) X_0(s) \sigma_y^\top \right] dZ(s). \end{aligned} \quad (77)$$

Since $d\Gamma^*(s) = dW_f^*(s) + d(g(s)X_0(s) + \langle h(s), X_1(s) \rangle)$, recalling (22) and the definition of β in (11), we immediately get the claim.

Theorem 34 (Verification Theorem and Optimal feedback Map) *The equality $V^{fh} = \bar{v}$ in $\mathcal{H}_+^{0, \tau_R}$ holds true and the function $(\mathbf{C}_f, \mathbf{B}_f, \Theta_f)$ defined in (73) is an optimal feedback map. Finally, for every $(t, w, x) \in \mathcal{H}_+^{0, \tau_R}$ the strategy $\bar{\pi}_f := (\bar{c}_f, \bar{B}_f, \bar{\theta}_f)$ is the unique optimal strategy.*

PROOF. First take $(t, w, x) \in \partial \mathcal{H}_+^{0, \tau_R}$. By equation (76) we thus have that for every $s \in [t, \tau_R]$, $\Gamma^*(s) = 0$, $\mathbb{P}$ -a.s.. This in turns implies, by (75), that

$$\bar{c}_f \equiv 0, \quad \bar{B}_f \equiv 0, \quad \bar{\theta}_f \equiv -g(t)X_0(s)(\sigma^\top)^{-1} \sigma_y.$$

It follows from Lemma 17-(ii) that this is the only admissible strategy, hence it must be optimal and, clearly, $V^{fh} = \bar{v}$ in such boundary points.

Now take $(t, w, x) \in \mathcal{H}_{++}^{0, \tau_R}$. First we observe that, by Lemma 33, $\Gamma^*(\cdot)$ is a stochastic exponential, thus $\mathbb{P}$ -a.s. strictly positive for any strictly positive initial condition $\Gamma^*(t) = \Gamma(t, w, x)$. Hence, the constraint in (27) is always satisfied with strict inequality and that $(\bar{c}_f, \bar{B}_f)$ are strictly positive by (75). This implies that $\bar{\pi}_f$ is admissible and that $J^{fh}(t, w, x; \bar{\pi}_f) > -\infty$ when $\gamma > 1$ and the fundamental identity (64) can be used.

Now, since the feedback map (73) is obtained taking the maximum points of the Hamiltonian the fundamental identity (64) becomes

$$\bar{v}(t, w, x) = J^{fh}(t, w, x; \bar{\pi}_f).$$

Hence, Corollary 30 and the definition of the value function yields

$$V^{fh}(t, w, x) \leq \bar{v}(t, w, x) = J^{fh}(t, w, x; \bar{\pi}_f) \leq V^{fh}(t, w, x)$$

which immediately gives $V^{fh}(t, w, x) = J^{fh}(t, w, x; \bar{\pi}_f)$, hence optimality of $\bar{\pi}_f$.

We prove uniqueness. When $(t, w, x) \in \partial \mathcal{H}_+^{0, \tau_R}$ the claim follows from Lemma 17-(ii). Let $(t, w, x) \in \mathcal{H}_{++}^{0, \tau_R}$. Since $\bar{v} = V^{fh}$, an optimal strategy π at (t, w, x) must satisfy $\bar{v}(t, w, x) = J^{fh}(t, w, x; \pi)$, which implies, substituting in (64), that the integral in (64) is zero. This implies that, on $[t, \bar{\tau}_t]$ we have $\pi = \bar{\pi}_f$, $dt \otimes \mathbb{P}$ -a.e.. This provides uniqueness, as, for $\bar{\pi}_f$ we have $\bar{\tau}_t = \tau_R$.

Corollary 35 *For any $(t, w, x) \in \mathcal{H}_+^{0, \tau_R}$ the value function V defined in (36) is equal to V^{fh} .*

PROOF. Lemma 17 implies that equality holds on $\partial \mathcal{H}_+^{0, \tau_R}$. Let $(t, w, x) \in \mathcal{H}_{++}^{0, \tau_R}$. From the definition of V , breaking the integral at τ_R we see that $V \leq V^{fh}$. Moreover, the strategy found concatenating the optimal strategies of the Problems 3 and 4 is admissible for Problem 2, hence $V \geq V^{fh}$.

7 Stating and interpreting of the main result

In the present Section we state and comment the main result of the paper.

Theorem 36 *Assume that Hypotheses 2 holds.*

- The value function of Problem 14 (and hence of Problem 4 when $t = 0$) is

$$V(t, w, x) = F(t)^\gamma \frac{\Gamma(t, w, x)^{1-\gamma}}{1-\gamma}, \quad (t, w, x) \in \mathcal{H}_+^{0, \infty}, \quad (78)$$

where

$$\Gamma(t, w, x) := w + g(t)x_0 + \langle h(t), x_1 \rangle.$$

with

$$\begin{cases} g(t) = \mathbb{I}_{\{t < \tau_R\}} \int_t^{\tau_R} e^{-\beta(\tau-t)} (h(\tau, 0) + 1) d\tau, \\ h(t, s) = \mathbb{I}_{\{t < \tau_R\}} \int_{(s+t-\tau_R) \vee (-d)}^s e^{-(r+\delta)(s-\tau)} * \\ \quad * g(t+s-\tau) \phi(\tau) d\tau, \end{cases}$$

$$f(t) = \left((\hat{\eta} - \eta) \exp \left(-\frac{(\tau_R - t)^+}{\nu} \right) + \eta \right),$$

$$F(t) := e^{-\frac{(\rho+\delta)t}{\gamma}} f(t),$$

and

$$\begin{aligned} \eta &:= (1 + \delta k^{-b})\nu & \hat{\eta} &:= (K^{-b} + \delta k^{-b})\nu, \\ \nu &:= \frac{\gamma}{\rho + \delta - (1 - \gamma)(r + \delta + \frac{\kappa^\top \kappa}{2\gamma})}, & b &= 1 - \frac{1}{\gamma}. \end{aligned}$$

- The optimal strategies of Problem 4 (hence where $t = 0$) are, for all $s \geq 0$,

$$\begin{aligned}\bar{c}_f(s) &:= K^{-bR(s)} f(s)^{-1} \Gamma^*(s), & \bar{B}_f(t) &:= k^{-b} f(s)^{-1} \Gamma^*(s), \\ \bar{\theta}_f(s) &:= (\sigma \sigma^\top)^{-1} \frac{\kappa}{\gamma} \Gamma^*(s) - g(s) y(s) (\sigma^\top)^{-1} \sigma_y\end{aligned}\quad (79)$$

where

$$\Gamma^*(s) := W^*(s) + g(s) y(s) + \int_{-d}^0 h(s, \zeta) y(s + \zeta) d\zeta, \quad s \geq 0 \quad (80)$$

denotes the optimal total wealth, with financial wealth $W^*(\cdot)$ given by the solution of equation (31) with controls defined in (79) above, and with labor income $y(\cdot)$ given by the solution of the second equation in (4).

- The optimal total wealth process has dynamics

$$\begin{aligned}d\Gamma^*(s) &= \Gamma^*(s) \kappa^\top \sigma dZ(s) + \\ &\Gamma^*(s) \left(r + \delta + \frac{1}{\gamma} \kappa^\top \kappa - f(s)^{-1} (K^{-bR(s)} + \delta k^{-b}) \right) ds\end{aligned}\quad (81)$$

PROOF. The result is a direct consequence of Proposition 20, Theorem 34 and Corollary 35.

From the results above we see that the post-retirement problem preserves the original structure of Merton's solution. Indeed, as g and h are null after τ_R , the agent chooses fixed fractions of financial wealth W to consume, leave as bequest, and invest in the risky asset. The pre-retirement solution departs from the Merton baseline in several ways. First, as in [8,9], financial wealth is replaced by total wealth, as the agent capitalizes the value of future wages and treat it as a traded asset (human capital). Second, the optimal fractions of total wealth consumed and left as bequest are time varying, as they reflect the residual time to retirement (e.g., [18]). Third, the allocation to risky assets features a negative hedging demand arising from the exposure to market risk channelled by the labor income process (e.g., [9]). In our model, however, this hedging demand presents novel features and gives rise to more articulate trade-offs. This is for two main reasons. First, only the 'future' component of human capital drives hedging demand, but not its 'past' component (see [6]). Second, the annuity factor appearing in the hedging demand, $g = g_1 + g_2$ (see (29), (30)), takes into account not only the market risk channelled by future wages (g_1), but also the compounding effect of their delayed contribution to labor income (g_2). As a reviewer insightfully points out, human capital has here the ability to shape consumption in a way that can be related to the literature on "habit formation", which recongizes that individuals develop consumption habits based on their own past behavior or reference levels (see [51,43] and references therein). Although these features do not enter our preferences directly, our labor income dynamics introduces a path dependent component in the optimal consumption process via the human capital process, as seen from $\bar{c}_f$ in (79).

We can analyse these trade-offs more precisely by exploiting the closed form solutions obtained in Theorem 36. Let us first compare the optimal strategies above with the case in which

labor income does not present any delay in its drift ($\phi = 0$). As in [7], we observe that the dynamics of total wealth, Γ^* , is not influenced by the path dependent component of the model, in the sense that, *ceteris paribus*, changing ϕ (and hence y) leaves the dynamics of total wealth unchanged at each point in time. Therefore, denoting by $\tilde{\Gamma}$ the total wealth in case of $\phi = 0$, we have that, for any initial point (w, x) the following holds:

$$\begin{aligned}\Gamma^*(0) - \tilde{\Gamma}(0) &= x_0 \left(g(0) - \frac{1 - e^{-\beta\tau_R}}{\beta} \right) + \\ &\langle h(0, \cdot), x_1(0) \rangle = x_0 g_2(0) + \langle h(0, \cdot), x_1(0) \rangle,\end{aligned}$$

with g_2 defined in (30). For any $t \geq 0$, we can therefore write

$$\begin{aligned}\Gamma^*(t) - \tilde{\Gamma}(t) &= (\Gamma^*(0) - \tilde{\Gamma}(0)) \\ &e^{\left(r + \delta + \frac{1}{\gamma} \kappa^\top \kappa - f(t)^{-1} (K^{-bR(t)} + \delta k^{-b})\right)t + \kappa^\top \sigma Z(t)},\end{aligned}$$

showing that any difference in total wealth, and hence optimal consumption level c^* and bequest target B^* , is shaped by the quantity $x_0 g_2(0) + \langle h(0, \cdot), x_1(0) \rangle$. Let us now focus on the risky asset allocation, and denote by $\Theta_{f,\phi}$ the feedback map introduced in (73). We can then write

$$\begin{aligned}\Theta_{f,\phi}(t, w, x) - \Theta_{f,0}(t, w, x) &= \\ (\sigma \sigma^\top)^{-1} \left[\left(\frac{\kappa}{\gamma} - \sigma_y \right) g_2(t) x_0 + \frac{\kappa}{\gamma} \langle h(t, \cdot), x_1(t) \rangle \right].\end{aligned}$$

This result offers a rich set of empirical predictions for stock market participation, as path dependency of labor income is shown to have two complementary effects. On the one hand, it improves the predictability of labor income and increases the demand for risky assets via the 'past' component of human capital, $\langle h(t), x_1(t) \rangle$. On the other hand, exposure to market risk is compounded by the delayed contribution of future wages, as quantified by the annuity factor g_2 . If the sensitivity of labor income to market shocks is high enough (i.e., $\sigma_y > \gamma^{-1} \kappa$), a negative hedging demand arises, but is counterbalanced by the positive contribution of the 'past' component of human capital to the demand for risky assets. Moreover, as the retirement date approaches, the annuity factor g_2 shrinks whereas the 'past' component does not. The model can therefore produce a rich variety of stock market participation patterns. To conclude, Figure 1 offers some examples of risky asset allocations based on different parameter configurations consistent with the extant literature. We consider a single risky asset and two types of labor income dynamics, one without delay (cases a1 and a2), and one with delay (cases b1 and b2). For the latter we assume $\phi = 0.75\%$ and $d = 5$. We then consider two values for the labor income volatility parameter σ_y : a higher value of 10% gives rise to a negative hedging demand (cases a1 and b1), whereas a lower value of 6% is such that the condition $\sigma_y < \gamma^{-1} \kappa$ is satisfied. The values of the market and preference parameters are based on the contributions of [39], [24], and [4]. The examples depicted in Figure 1 shows that when a negative hedging demand is material (cases a1 and b1), the agent is initially short the risky asset and then gradually increases to a positive position as the bond-like nature of human capital becomes more pronounced with the approaching of the retirement date. A delay in labor income dynamics makes human capital more bond-like due to its predictable 'past' component and raises the allocation to the

risky asset. When the sensitivity of labor income to market shocks is small enough (cases a2 and b2), the allocation to the risky asset is positive and increasing throughout the agent's working life. Again, delayed labor income dynamics boost the allocation to the risky asset.

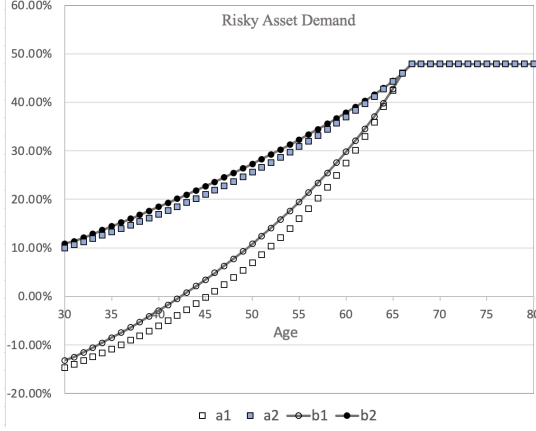


Figure 1. Examples of risky asset demand $\bar{\theta}_f$ as a fraction of Γ^* . See discussion in the text for parameterization details.

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A Proof of Lemma 9

PROOF. Proof of (i). By a change of variables let us rewrite (16), equivalently, as

$$\begin{cases} g(t) = \mathbb{I}_{\{t < \tau_R\}} \int_t^{\tau_R} e^{-\beta(\tau-t)} (h(\tau, 0) + 1) d\tau, & t \in [0, \infty), \\ h(t, \zeta) = \mathbb{I}_{\{t < \tau_R\}} \int_0^{(\tau_R-t) \wedge (\zeta+d)} e^{-(r+\delta)\tau} g(t+\tau) \phi(\zeta-\tau) d\tau, \\ (t, \zeta) \in [0, \infty) \times [-d, 0]. \end{cases} \quad (\text{A.1})$$

We apply the contraction principle depending on a parameter (see [13, Proposition C.11]). Since g and h are identically zero for $t \geq \tau_R$ we work in the space $\mathcal{M} := C([0, \tau_R]; \mathbb{R}) \times C([0, \tau_R] \times [-d, 0]; \mathbb{R})$, endowed with the norm⁴

$$\|(\xi, \psi)\|_{\mathcal{M}} = \left(\sup_{t \in [0, \tau_R]} |\xi(t)|^2 + \sup_{(t, \zeta) \in [0, \tau_R] \times [-d, 0]} |\psi(t, \zeta)|^2 \right)^{\frac{1}{2}}.$$

We introduce the mapping

$$\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) : L^2(-d, 0; \mathbb{R}) \times \mathcal{M} \rightarrow \mathcal{M} \quad (\text{A.2})$$

$$(\phi, (g, h)) \mapsto \left(\int_t^{\tau_R} e^{-\beta(\tau-t)} (h(\tau, 0) + 1) d\tau, \int_0^{(\tau_R-t) \wedge (\zeta+d)} e^{-(r+\delta)\tau} g(t+\tau) \phi(\zeta-\tau) d\tau \right).$$

Notice that the term $\mathcal{F}_2(\phi, (g, h))$ after a change of variables, is equal, on $[0, \tau_R]$, to the r.h.s. of the second equation in (16). According to [13, Proposition C.11], if prove that

⁴ We endow $\mathcal{M}$ with this norm in order to make computations easier.

- (a) there exists $\alpha \in (0, 1)$ such that for $(g, h), (\bar{g}, \bar{h}) \in \mathcal{M}$ the following inequality holds:

$$\|\mathcal{F}(\phi, (g, h)) - \mathcal{F}(\phi, (\bar{g}, \bar{h}))\|_{\mathcal{M}} \leq \alpha \|(g, h) - (\bar{g}, \bar{h})\|_{\mathcal{M}},$$

- (b) and $\mathcal{F}$ is a continuous map,

then there exists a unique map $G \in C(L^2(-d, 0; \mathbb{R}); \mathcal{M})$ such that

$$\mathcal{F}(\phi, G(\phi)) = G(\phi), \quad \phi \in L^2(-d, 0; \mathbb{R})$$

and statement (i) immediately follows.

Let us prove (a). Take (g, h) and $(\bar{g}, \bar{h})$ in $\mathcal{M}$. We start with the following estimates: for $t \in [0, \tau_R]$, by means of Hölder's inequality, we infer

$$\begin{aligned} & |\mathcal{F}_1(\phi, (g, h)) - \mathcal{F}_1(\phi, (\bar{g}, \bar{h}))|^2 = \\ & \left| \int_t^{\tau_R} e^{-\beta(\tau-t)} (h(\tau, 0) - \bar{h}(\tau, 0)) d\tau \right|^2 \leq \\ & \int_t^{\tau_R} e^{2\beta(\tau-t)} d\tau \int_t^{\tau_R} |h(\tau, 0) - \bar{h}(\tau, 0)|^2 d\tau \leq \\ & C \sup_{t \in [0, \tau_R]} \sup_{\zeta \in [-d, 0]} |h(t, \zeta) - \bar{h}(t, \zeta)|^2, \end{aligned}$$

where C is a positive constant depending on β and τ_R . Let $(t, \zeta) \in [0, \tau_R] \times [-d, 0]$. Thanks to Hölder's inequality we get

$$\begin{aligned} & |\mathcal{F}_2(\phi, (g, h)) - \mathcal{F}_2(\phi, (\bar{g}, \bar{h}))|^2 = \\ & \left| \int_0^{(\tau_R-t) \wedge (\zeta+d)} e^{-(r+\delta)\tau} (g(t+\tau) - \bar{g}(t+\tau)) \phi(\zeta-\tau) d\tau \right|^2 \leq \\ & \sup_{\tau \in [0, (\tau_R-t) \wedge (\zeta+d)]} e^{2(r+\delta)\tau} \int_0^{(\tau_R-t) \wedge (\zeta+d)} |\phi(\zeta-\tau)|^2 d\tau \\ & \times \int_0^{(\tau_R-t) \wedge (\zeta+d)} |g(t+\tau) - \bar{g}(t+\tau)|^2 d\tau \leq \\ & \tilde{C} \sup_{t \in [0, \tau_R]} |g(t) - \bar{g}(t)|^2, \end{aligned}$$

where $\tilde{C}$ is a positive constant depending on τ_R, d, r, δ and $\|\phi\|_{L^2}$. If we now denote by $\bar{C} = C \vee \tilde{C}$, putting together the last two estimates we obtain, for every $(t, \zeta) \in [0, \tau_R] \times [-d, 0]$,

$$\begin{aligned} & |\mathcal{F}_1(\phi, (g, h)) - \mathcal{F}_1(\phi, (\bar{g}, \bar{h}))|^2 + |\mathcal{F}_2(\phi, (g, h)) - \mathcal{F}_2(\phi, (\bar{g}, \bar{h}))|^2 \\ & \leq \bar{C} \left(\sup_{t \in [0, \tau_R]} |g(t) - \bar{g}(t)|^2 + \sup_{t \in [0, \tau_R]} \sup_{\zeta \in [-d, 0]} |h(t, \zeta) - \bar{h}(t, \zeta)|^2 \right). \end{aligned}$$

Taking the supremum on $[0, \tau_R] \times [-d, 0]$ in the l.h.s. we get

$$\|\mathcal{F}(\phi, (g, h)) - \mathcal{F}(\phi, (\bar{g}, \bar{h}))\|_{\mathcal{M}}^2 \leq \bar{C} \|(g, h) - (\bar{g}, \bar{h})\|_{\mathcal{M}}^2.$$

One can easily verify that, if τ_R is small enough, then $\bar{C} < 1$ and then (a) is verified. On the other hand, if τ_R is such that $\bar{C} \geq 1$, one can iterate the argument in the intervals $[0, T]$, $[T, 2T]$, etc, with $0 < T < \tau_R$ such that $\bar{C}(T) < 1$. Thus statement (a) is proved.

Let us now prove statement (b). Let $(\phi, (g, h))$ and $(\bar{\phi}, (\bar{g}, \bar{h}))$ belong to $L^2(-d, 0; \mathbb{R}) \times \mathcal{M}$. With the same computations as before we write

$$\begin{aligned} & \|\mathcal{F}(\phi, (h, g)) - \mathcal{F}(\bar{\phi}, (\bar{g}, \bar{h}))\|_{\mathcal{M}}^2 \leq \\ & c_1 \sup_{t \in [0, \tau_R]} \sup_{\zeta \in [-d, 0]} |h(t, \zeta) - \bar{h}(t, \zeta)|^2 + \\ & \sup_{t \in [0, \tau_R]} \sup_{\zeta \in [-d, 0]} \left| \int_0^{(\tau_R - t) \wedge (\zeta + d)} e^{-(r+\delta)\tau} \times \right. \\ & \left. (g(t + \tau)\phi(\zeta - \tau) - \bar{g}(t + \tau)\bar{\phi}(\zeta - \tau)) d\tau \right|^2. \end{aligned}$$

Adding and subtracting the term $\int_0^{(\tau_R - t) \wedge (\zeta + d)} e^{-(r+\delta)\tau} g(t + \tau)\bar{\phi}(\zeta - \tau) d\tau$, by means again of Hölder's inequality, we can estimate (here and below $\|g\|_\infty := \sup_{t \in [0, \tau_R]} |g(t)|$)

$$\begin{aligned} & \left| \int_0^{(\tau_R - t) \wedge (\zeta + d)} e^{-(r+\delta)\tau} (g(t + \tau)\phi(\zeta - \tau) - \bar{g}(t + \tau)\bar{\phi}(\zeta - \tau)) d\tau \right|^2 \\ & \leq 2 \left(\left| \int_0^{(\tau_R - t) \wedge (\zeta + d)} e^{-(r+\delta)\tau} g(t + \tau)(\phi - \bar{\phi})(\zeta - \tau) d\tau \right|^2 + \right. \\ & \left. \left| \int_0^{(\tau_R - t) \wedge (\zeta + d)} e^{-(r+\delta)\tau} (g - \bar{g})(t + \tau)\bar{\phi}(\zeta - \tau) d\tau \right|^2 \right) \\ & \leq c_2 \|g\|_\infty \|\phi - \bar{\phi}\|_{L^2(-d, 0; \mathbb{R})}^2 + c_3 \|\bar{\phi}\|_{L^2(-d, 0; \mathbb{R})}^2 \sup_{t \in [0, \tau_R]} |g(t) - \bar{g}(t)|^2. \end{aligned}$$

If we now denote by $\tilde{C}_1$ the maximum between c_1 , $c_2 \|g\|_\infty$ and $c_3 \|\bar{\phi}\|_{L^2(-d, 0; \mathbb{R})}^2$, collecting the above estimates, we obtain

$$\begin{aligned} & \|\mathcal{F}(\phi, (h, g)) - \mathcal{F}(\bar{\phi}, (\bar{g}, \bar{h}))\|_{\mathcal{M}}^2 \leq \\ & \tilde{C}_1 \|\phi, (g, h) - (\bar{\phi}, (\bar{g}, \bar{h}))\|_{L^2(-d, 0; \mathbb{R}) \times \mathcal{M}}^2, \end{aligned}$$

and statement (b) then follows from straightforward considerations.

Assume now that ϕ is a.e. positive. Then, calling

$$\mathcal{M}^+ := C([0, \tau_R]; \mathbb{R}_+) \times C([0, \tau_R] \times [-d, 0]; \mathbb{R}_+) \subseteq \mathcal{M}$$

it is immediate to see, from its definition, that the map $\mathcal{F}(\phi, \cdot)$ brings $\mathcal{M}^+$ into itself. Since the contraction principle holds also in metric complete spaces (see e.g. [26, Thm 1.1]), then, from the proof above, $\mathcal{F}(\phi, \cdot)$ must have a unique fixed point in $\mathcal{M}^+$, which, by uniqueness, must coincide with the fixed point found above in $\mathcal{M}$.

Proof of (ii). The fact that $g \in C^1([0, \tau_R]; \mathbb{R})$ immediately follows from the fact that h is continuous, the first of (16), and the Torricelli Theorem. It is clear, by differentiating, that the function g solves the first equation in (17). The last three equation of (17) are obviously true by (16). Let us then verify that h solves the second equation in (17). Let $\tau_R - t < \zeta + d$. Differentiating the second of (A.1) and using that g is continuous and $g(\tau_R) = 0$ we have

$$\frac{\partial h(t, \zeta)}{\partial t} = \int_0^{\tau_R - t} e^{-(r+\delta)\tau} g'(t + \tau)\phi(\zeta - \tau) d\tau \quad (\text{A.3})$$

and

$$\frac{\partial h(t, \zeta)}{\partial \zeta} = \int_0^{\tau_R - t} e^{-(r+\delta)\tau} g(t + \tau)\phi'(\zeta - \tau) d\tau. \quad (\text{A.4})$$

Integration by parts formula yields

$$\begin{aligned} & - (r + \delta) \int_0^{\tau_R - t} e^{-(r+\delta)\tau} g(t + \tau)\phi(\zeta - \tau) d\tau = \\ & e^{-(r+\delta)(\tau_R - t)} g(\tau_R)\phi(\zeta - \tau_R + t) - g(t)\phi(\zeta) - \\ & \int_0^{\tau_R - t} e^{-(r+\delta)\tau} g'(t + \tau)\phi(\zeta - \tau) d\tau + \\ & \int_0^{\tau_R - t} e^{-(r+\delta)\tau} g(t + \tau)\phi'(\zeta - \tau) d\tau. \end{aligned}$$

Recalling the definition of h in (A.1) and substituting (A.3) and (A.4) in the above expression, we get

$$- (r + \delta)h(t, \zeta) = - \frac{\partial h(t, \zeta)}{\partial t} + \frac{\partial h(t, \zeta)}{\partial \zeta} - g(t)\phi(\zeta).$$

Thus we immediately see that for $\tau_R - t < \zeta + d$ the function h solves the second equation of (17). Let $\tau_R - t > \zeta + d$. We have

$$\frac{\partial h(t, \zeta)}{\partial t} = \int_0^{\zeta + d} e^{-(r+\delta)\tau} g'(t + \tau)\phi(\zeta - \tau) d\tau, \quad (\text{A.5})$$

$$\begin{aligned} \frac{\partial h(t, \zeta)}{\partial \zeta} &= -e^{-(r+\delta)(\zeta + d)} g(t + \zeta + d)\phi(-d) + \\ & \int_0^{\zeta + d} e^{-(r+\delta)\tau} g(t + \tau)\phi'(\zeta - \tau) d\tau. \end{aligned} \quad (\text{A.6})$$

Integrating by parts as in the previous case shows that, also for $\tau_R - t > \zeta + d$ the function h solves the second equation of (17). Moreover, since $\phi \in C^1([-d, 0]; \mathbb{R})$ with $\phi(-d) = 0$ we immediately see that, also when $\tau_R - t = \zeta + d$ the function h is C^1 and satisfies the second equation of (17).

Proof of (iii). Let now $\phi \in L^2(-d, 0; \mathbb{R})$. The function g belongs to $C^1([0, \tau_R]; \mathbb{R})$ by the same argument of point (ii). First we recall that, by point (i) h is continuous, hence bounded in $[0, \tau_R] \times [-d, 0]$, so, in particular, we have $h(t, \cdot) \in L^2(-d, 0; \mathbb{R})$ for all $t \in [0, \tau_R]$. We observe that the derivative $\frac{\partial h(t, s)}{\partial t}$, defined above in the proof of (ii), makes still sense and defines a function

$$\begin{aligned} & [0, \tau_R] \rightarrow L^2(-d, 0; \mathbb{R}), \\ & t \mapsto \bar{h}(t, \cdot) := \int_0^{(\tau_R - t) \wedge (\cdot + d)} e^{-(r+\delta)\tau} g'(t + \tau)\phi(\cdot - \tau) d\tau. \end{aligned}$$

Using the fact that translations are continuous in the L^2 norm we get that such function is continuous. It remains to prove that it is the derivative of the map $[0, \tau_R] \rightarrow L^2(-d, 0; \mathbb{R})$, $t \mapsto h(t, \cdot)$, i.e. that the following limit holds in the L^2 norm

$$\lim_{\alpha \rightarrow 0} \frac{h(t + \alpha, \cdot) - h(t, \cdot)}{\alpha} = \frac{\partial h(t, \cdot)}{\partial t}.$$

Let $\zeta \in [-d, 0]$, $t \in (0, \tau_R)$ and $\alpha > 0$. Since α must tend to 0 we can suppose, without loss of generality that $t + \alpha \in (0, \tau_R)$. We have to study two cases.

First we suppose that $(\tau_R - t) < (\zeta + d)$ and $(\tau_R - (t + \alpha)) < (\zeta + d)$. Then

$$\frac{h(t+\alpha, \zeta) - h(t, \zeta)}{\alpha} = \quad (\text{A.7})$$

$$= \frac{1}{\alpha} \left[\int_0^{\tau_R - (t+\alpha)} e^{-(r+\delta)\tau} g(t+\alpha+\tau) \phi(\zeta - \tau) d\tau - \int_0^{\tau_R - t} e^{-(r+\delta)\tau} g(t+\tau) \phi(\zeta - \tau) d\tau \right] = \quad (\text{A.8})$$

$$= \frac{1}{\alpha} \int_{\tau_R - (t+\alpha)}^{\tau_R - t} e^{-(r+\delta)\tau} g(t+\alpha+\tau) \phi(\zeta - \tau) d\tau + \int_0^{\tau_R - (t+\alpha)} e^{-(r+\delta)\tau} \frac{g(t+\alpha+\tau) - g(t+\tau)}{\alpha} \phi(\zeta - \tau) d\tau. \quad (\text{A.9})$$

Recalling that g is continuous and $g(t) = 0$ for $t \geq \tau_R = 0$, the first integral above is 0.

Moreover, since g is continuous differentiable, we get

$$\lim_{\alpha \rightarrow 0} \int_0^{\tau_R - (t+\alpha)} e^{-(r+\delta)\tau} \frac{g(t+\alpha+\tau) - g(t+\tau)}{\alpha} \phi(\zeta - \tau) d\tau = \int_0^{\tau_R - t} e^{-(r+\delta)\tau} g'(t+\tau) \phi(\zeta - \tau) d\tau.$$

Now we study the case when $(\tau_R - t) > (\zeta + d)$ and $(\tau_R - (t + \alpha)) > (\zeta + d)$. We have

$$\frac{h(t+\alpha, \zeta) - h(t, \zeta)}{\alpha} = \quad (\text{A.10})$$

$$= \frac{1}{\alpha} \left[\int_0^{\zeta+d} e^{-(r+\delta)\tau} g(t+\alpha+\tau) \phi(\zeta - \tau) d\tau - \int_0^{\zeta+d} e^{-(r+\delta)\tau} g(t+\tau) \phi(\zeta - \tau) d\tau \right] = \quad (\text{A.11})$$

$$= \int_0^{\zeta+d} e^{-(r+\delta)\tau} \frac{g(t+\alpha+\tau) - g(t+\tau)}{\alpha} \phi(\zeta - \tau) d\tau.$$

Recalling that g is continuous differentiable we get

$$\lim_{\alpha \rightarrow 0} \int_0^{\zeta+d} e^{-(r+\delta)\tau} \frac{g(t+\alpha+\tau) - g(t+\tau)}{\alpha} \phi(\zeta - \tau) d\tau = \int_0^{\zeta+d} e^{-(r+\delta)\tau} g'(t+\tau) \phi(\zeta - \tau) d\tau.$$

Combining the two cases above we obtain, for a.e. $\zeta \in [-d, 0]$,

$$\lim_{\alpha \rightarrow 0} \frac{h(t+\alpha, \zeta) - h(t, \zeta)}{\alpha} = \int_0^{(\tau_R - t) \wedge (\zeta + d)} e^{-(r+\delta)\tau} g'(t+\tau) \phi(\zeta - \tau) d\tau, \quad (\text{A.12})$$

The claim follows if we prove that the limit above holds in $L^2(-d, 0; \mathbb{R})$. To do this we apply dominated convergence theorem. Indeed, calling $\|g'\|_\infty := \sup_{t \in [0, \tau_R]} |g'|$, we get, from (A.7)-(A.10),

$$\frac{h(t+\alpha, \zeta) - h(t, \zeta)}{\alpha} \leq \int_0^{(\tau_R - t) \wedge (\zeta + d)} e^{-(r+\delta)\tau} \left| \frac{g(t+\alpha+\tau) - g(t+\tau)}{\alpha} \right| |\phi(\zeta - \tau)| d\tau$$

$$\leq \|g'\|_\infty \int_0^{\zeta+d} |\phi(\zeta - \tau)| d\tau.$$

Since $\phi \in L^2(-d, 0; \mathbb{R})$ the map $\zeta \mapsto \int_0^{\zeta+d} |\phi(\zeta - \tau)| d\tau$ belongs to $L^2(-d, 0; \mathbb{R})$ and the claim follows.

Proof of (iv). First, by the second of (16) we have $h(t, -d) = 0$ for all $t \geq 0$. Thus, to prove that, given any $t \geq 0$, $(g(t), h(t, \cdot))$ is in $\mathcal{D}(A^*)$. it is enough to show that $h(t, \cdot) \in \mathcal{W}^{1,2}([-d, 0]; \mathbb{R})$. For $t \geq \tau_R$ this is clear since in this case $g = h = 0$. Let then $t \in [0, \tau_R]$. We take $\{\phi_n\} \subset C^1([-d, 0]; \mathbb{R})$ such that $\phi_n(-d) = 0$ for all $n \in \mathbb{N}$ and $\phi_n \rightarrow \phi$ in $L^2(-d, 0; \mathbb{R})$ as $n \rightarrow \infty$. Let us define

$$h_n(t, \zeta) := \int_0^{(\tau_R - t) \wedge (\zeta + d)} e^{-(r+\delta)\tau} g(t+\tau) \phi_n(\zeta - \tau) d\tau,$$

for $(t, \zeta) \in [0, \infty) \times [-d, 0]$. It is clear that $h_n(t, \cdot) \rightarrow h(t, \cdot)$ in $L^2(-d, 0; \mathbb{R})$. Moreover in step (ii) we proved that

$$\frac{\partial h_n(t, \zeta)}{\partial \zeta} = \int_0^{(\tau_R - t) \wedge (\zeta + d)} e^{-(r+\delta)\tau} g(t+\tau) \phi_n'(\zeta - \tau) d\tau.$$

By applying integration by parts to the above formula we get

$$\frac{\partial h_n(t, \zeta)}{\partial \zeta} = g(t) \phi_n(\zeta) + \int_0^{(\tau_R - t) \wedge (\zeta + d)} e^{-(r+\delta)\tau} g'(t+\tau) \phi_n(\zeta - \tau) d\tau - (r + \delta) h_n(t, \zeta).$$

By the same computations of point (iii) we obtain, using the closeness of the derivative operator,

$$L^2 - \lim_{n \rightarrow \infty} \frac{\partial h_n(t, \cdot)}{\partial \zeta} = g(t) \phi(\cdot) + \int_0^{(\tau_R - t) \wedge (\zeta + d)} e^{-(r+\delta)\tau} g'(t+\tau) \phi(\cdot - \tau) d\tau - (r + \delta) h(t, \cdot) = \frac{\partial h(t, \cdot)}{\partial \zeta}.$$

This implies that $h(t, \cdot) \in \mathcal{W}^{1,2}(-d, 0; \mathbb{R})$, which gives the first part of the claim. For the second part, by the definition of A^* and by the equation just above,

$$\|A^*(g(t), h(t))\|_{M_2} \leq |\mu g(t) + h(t, 0)| + \left\| -\frac{\partial h(t, \cdot)}{\partial \zeta} + g(t) \phi(\cdot) \right\|_{L^2}$$

$$\leq \mu \|g\|_\infty + C (\|h\|_\infty + \|g'\|_\infty \|\phi\|_{L^2})$$

for some $C > 0$. This immediately gives:

$$\sup_{t \in [0, \tau_R]} \|(g(t), h(t, \cdot))\|_{\mathcal{D}(A^*)} < \infty.$$

Finally, the fact that (g, h) satisfy (17) simply follows by the above computations and by the definition of A^* .

Proof of (v). By (17) and (16) we have

$$0 = g'(\tau_R) \neq \lim_{t \rightarrow \tau_R} [\beta g(t) - h(t, 0) + 1] = 1.$$

References

- [1] Aksamit, A. and Jeanblanc M. (2017). *Enlargement of Filtration with Finance in View*, Springer Briefs in Quantitative Finance, Springer.
- [2] Barucci, E., Biffis, E., Marazzina, D. (2023). Health insurance, portfolio choice, and retirement incentives. *European Journal of Operational Research*, 307(2), 910-921.
- [3] Bensoussan, A., Da Prato, G., Delfour, M.C., and Mitter, S.K. (2007) *Representation and Control of Infinite Dimensional Systems*, Second Edition, Birkhauser
- [4] Benzoni, L., Collin-Dufresne, P., and R.S. Goldstein (2007). Portfolio choice over the life-cycle when the stock and labor markets are cointegrated. *The Journal of Finance*, 62(5), pp. 2123-2167.

- [5] Biagini, S., Gozzi, F., and M. Zanella (2022). Robust portfolio choice with sticky wages. *SIAM Journal on Financial Mathematics*, 13(3), 1004-1039.
- [6] Biffis, E., Goldys, B., C. Prosdociimi and Zanella M. (2023). A pricing formula for delayed claims: Appreciating the past to value the future. *Mathematics and Financial Economics*, vol. 17, 175–202.
- [7] Biffis, E., Gozzi F. and Prosdociimi C. (2020). Optimal portfolio choice with path dependent labor income: the infinite horizon case. *SIAM Journal on Control and Optimization*, 58(4), 1906-1938.
- [8] Bodie, Z., and Merton, R.C., Samuelson, W.F, (1992). Labor supply flexibility and portfolio choice in a life cycle model. *Journal of Economic Dynamics and Control*, 16(3), 427-449.
- [9] Campbell, J.Y. and Viceira, L.M. (2002). *Strategic asset allocation: portfolio choice for long-term investors*. Oxford University Press, USA.
- [10] Chen, L. and Z. Wu, Maximum principle for the stochastic optimal control problem with delay and application, *Automatica*, 46(6), 2010, 1074-1080.
- [11] Choi, K. J., Shim, G., Shin, Y. H. (2008). Optimal portfolio, consumption-leisure and retirement choice problem with CES utility. *Mathematical Finance*, 18(3), 445-472.
- [12] Chojnowska-Michalik A. (1978), *Representation Theorem for General Stochastic Delay Equations*. in Bull. Acad. Polon. Sci.Sér. Sci. Math. Astronom. Phys., **26** 7, pp. 635-642
- [13] Da Prato, G., Zabczyk, J. (1996), *Ergodicity for Infinite Dimensional Systems* (London Mathematical Society Lecture Note Series). Cambridge University Press.
- [14] Da Prato, G. and Zabczyk, J. (2014), *Stochastic equations in Infinite Dimensions*. Cambridge University Press, Second Edition.
- [15] B. Djeiche, F. Gozzi, G. Zanco and M. Zanella (2022), *Optimal portfolio choice with path dependent benchmarked labor income: a mean field model*. Stochastic Processes and their Applications, 145 pp.48-85.
- [16] Dunsmuir, W.T., Goldys, B., and C.V. Tran (2016). Stochastic delay differential equations as weak limits of autoregressive moving average time series. Working paper, University of New South Wales.
- [17] Dybvig, P.H. and Liu, H.(2010) Dybvig, P. H., and H. Liu (2011). Verification theorems for models of optimal consumption and investment with retirement and constrained borrowing. *Mathematics of Operations Research*, 36(4), 620-635.
- [18] Dybvig, P.H. and Liu, H. (2010). Lifetime consumption and investment: retirement and constrained borrowing. *Journal of Economic Theory*, 145, 885-907
- [19] El Karoui, N., and M., Jeanblanc-Picqué (1998). Optimization of consumption with labor income. *Finance and Stochastics*, 2(4), 409-440.
- [20] Exarchos, I. and E.A. Theodorou (2018). Stochastic optimal control via forward and backward stochastic differential equations and importance sampling, *Automatica*, 87, 159-165.
- [21] Fabbri, G. and S. Federico (2014). On the infinite-dimensional representation of stochastic controlled systems with delayed control in the diffusion term. *Mathematical Economics Letters*, 2(3-4), 33-43.
- [22] Fabbri, G., Gozzi, F. and Swiech, A. (2017) *Stochastic Optimal Control in Infinite Dimensions - Dynamic Programming and HJB Equations*, Springer.
- [23] Freni G., Gozzi F., Salvadori N. (2006). Existence of Optimal Strategies in linear Multisector Models. *Economic Theory*, 29 (1), 25-48.
- [24] Gomes, F. and A. Michaelides (2005). Optimal lifecycle allocation: Understanding the empirical evidence. *Journal of Finance*, 60(2), 869-904.
- [25] Gozzi, F. and Marinelli, C. (2006) *Stochastic optimal control of delay equations arising in advertising models* in *Stochastic PDE's and Applications VII*. Leviso, Terme, Italy, January 5-10, 2004., vol. 245, pp. 133-148
- [26] Granas, A. and Dugundji, J. (2003) *Fixed Point Theory*, Springer.
- [27] Guvenen, F. (2007). Learning your earning: Are labor income shocks really very persistent?. *The American economic review*, 687-712.
- [28] Guvenen, F.(2009). An empirical investigation of labor income processes. *Review of Economic dynamics*, 12(1), 58-79.
- [29] Hale, J.K. and Verduyn Lunel, S.M. (1993) *Introduction to Functional Differential Equations*, Springer-Verlag
- [30] Hubbard R.G., Skinner J., Zeldes S.P. (1995). Precautionary Saving and Social Insurance. *Journal of Political Economy*, 103(21).
- [31] Jeanblanc, M., Yor, M., Chesney, (2009). *Mathematical Methods for Financial Markets*, Springer-Verlag.
- [32] Karatzas, I. and Shreve, S.E. (1991). *Brownian Motion and Stochastic Calculus*, Springer-Verlag.
- [33] Keynes, J.M. (1936). *The General Theory of Employment, Interest, and Money*, Macmillan & Co., London.
- [34] Li, N., Wang, G. and Z. Wu, (2020) Linear–quadratic optimal control for time-delay stochastic system with recursive utility under full and partial information. *Automatica*, 121, 109-169.
- [35] Lorenz, R. (2006) *Weak Approximation of Stochastic Delay Differential Equations with Bounded Memory by Discrete Time Series*. PhD dissertation, Humboldt University.
- [36] MacCurdy, T. E. (1982). The use of time series processes to model the error structure of earnings in a longitudinal data analysis. *Journal of econometrics*, 18(1), 83-114.
- [37] K.J. McLaughling, *Wage rigidity?* (1993), Journal of Monetary Economics, 34(3), pp. 383–414.
- [38] Meghir, C., Pistaferri, L. (2004). *Income variance dynamics and heterogeneity*. *Econometrica*, 72(1), 1-32.
- [39] Mehra, T., Prescott, E.C. (1985). The equity premium: A puzzle. *Journal of Monetary Economics*, 15, 145-161.
- [40] Moffitt, R.A., Gottschalk, P. (2002). Trends in the transitory variance of earnings in the United States. *The Economic Journal*, 112(478), C68-C73.

- [41] Mohammed, S.-E. A. (1998). *Stochastic Differential Systems With Memory: Theory, Examples and Applications*. Stochastic Analysis and Related Topics VI, 1-77.
- [42] Mostovyi, O. (2017). Optimal investment with intermediate consumption and random endowment. *Mathematical Finance*, vol. 27(1), pp. 96-114.
- [43] Park, K. and Chen, K. and Wong, H.Y. (2023). Irreversible Consumption Habit under Ambiguity: Singular Control and Optimal G-Stopping Time. SSRN Working Paper <https://dx.doi.org/10.2139/ssrn.4384326>.
- [44] Protter, P.E. (2005) *Stochastic Integration and Differential Equations*, Springer-Verlag Berlin.
- [45] Reiß, M. (2002). *Nonparametric estimation for stochastic delay differential equations*. PhD dissertation, Humboldt University.
- [46] Rogers L.C.G., (2013) *Optimal Investment*, Springer-Berlin.
- [47] C.V. Tran (2016). *Convergence of Time Series Processes to Continuous Time Limits*. PHD dissertation, University of New South Wales. Available at "<http://unsworks.unsw.edu.au/fapi/datastream/unsworks:11500/SOURCE01?view=true>"
- [48] Viceira, L. M. (2001) *Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income*. *The Journal of Finance*, LVI, no. 2 pp. 433-470.
- [49] Vinter, R. B. (1975) *A representation of solution to stochastic delay equations*, Imperial College, Report of the Department of Computing and Control.
- [50] Wang, H. and H. Zhang (2013). LQ control for Itô-type stochastic systems with input delays, *Automatica*, 49(12), 3538-3549.
- [51] Yu, Xiang (2017). Optimal consumption under habit formation in markets with transaction costs and random endowments. *Annals of Applied Probability*, 27(2), 960–1002.
- [52] Yan, T., Chiu, M. C., Wong, H. Y. (2022). Pairs trading under delayed cointegration. *Quantitative Finance*, 22(9), 1627-1648.
- [53] Yan, T., Wong, H. Y. (2022). Equilibrium pairs trading under delayed cointegration. *Automatica*, 144, 110498.
- [54] Yu, Z. (2012) The stochastic maximum principle for optimal control problems of delay systems involving continuous and impulse controls, *Automatica*, 48(10), 2420-2432.