

Nonlinear analytical modelling of flat and hyperbolic paraboloidal panels under shear.

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Abstract

The hyperbolic paraboloid (hypar) form has been widely used in long-span roof structures and the subject of much research under out-of-plane loading. However, the behaviour of hypars under in-plane loading has been less keenly studied and there is no suitable guidance for their design in current codes of practice. A nonlinear analytical model treating the hypar as a deliberate imperfection applied to a flat plate is presented. A Rayleigh–Ritz formulation using appropriate shape functions is developed and the resulting equations are solved using numerical continuation techniques. The results are verified with nonlinear finite element models, showing good correlation across a range of thicknesses and degrees of initial curvature. Key modal contributions that influence the behaviour of the hypar are identified, providing insight into the nonlinear behaviour of hypars subject to in-plane shear. The main differences in behaviour between the flat plate and the hypar panel are shown to be most prevalent in the early stages of loading, where the influence of the initial geometry is at its greatest.

Keywords: Buckling, instability, hypar, shear, Rayleigh–Ritz method, , nonlinear mechanics

1. Introduction

Plated elements are often used in engineering structures to form complex and bespoke structural members. This is clearly exemplified in the construction of bridge decks, where plated elements simultaneously define the architectural expression of the solution while acting as the primary load-bearing elements in the structure. Advances in digital and computational design alongside fabrication have facilitated the visualization and development of complex architectural concepts that are driving equally novel structural solutions. This is demonstrated in the design of the Inner Harbour Bridge in Copenhagen shown in Figure 1, the original architectural intent for which was to use warped steel plates in the form of hyperbolic paraboloids (hypars) for the deck girder webs.

The hypar has frequently been used on an architectural scale as long-span roof structures [3, 4]. At this scale, the out-of-plane loads on the roof are considered to be transferred to the edge supports by membrane shear, leading to a whole body of research concerning the buckling of such structures [5], accounting for different loading conditions [6], presenting physical experiments [7], studying parallelogram projected-plan forms [8], with bending effects occurring at the edge regions near the supporting beams [9, 10, 11, 12] and for hypars constructed from orthotropic steel decks [13]. Recently, the nonlinear vibration response of heterogeneous orthotropic shallow shells [14] has also been studied. However, hypars utilized to form bridge deck girder webs would be primarily loaded by a combination of in-plane shear and bending, as described in Figure 1(c).

The analysis and design of flat deck girder webs loaded with in-plane shear loads has been thoroughly researched, as outlined in a critical review by Hingnekar and Vyavahare [15], and are subject to the design rules established in codes of practice such as the current European design code for plated steel structures, EN1993-1-5 [16]. Research on the behaviour of singly-curved plates under various load conditions has also been performed, as outlined by Martins et al. [17]. However, to the authors’ best knowledge, there is limited

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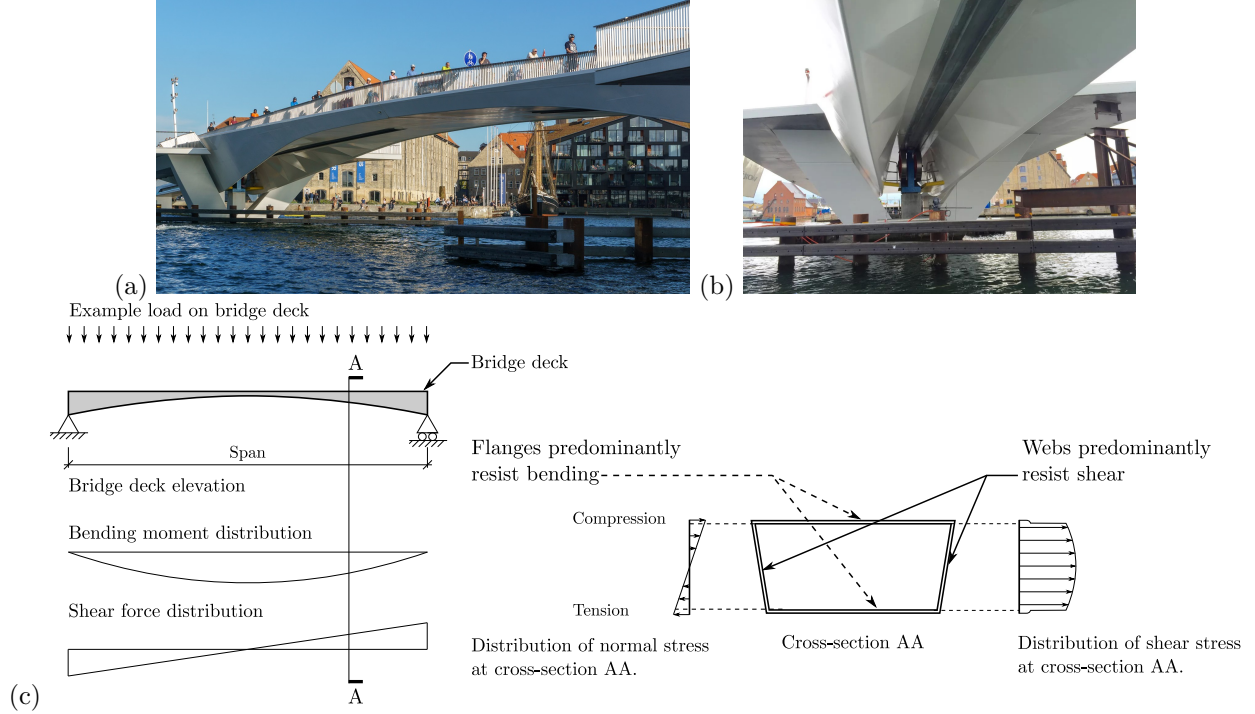


Figure 1: (a) Inner Harbour Bridge in Copenhagen [1]; (b) underside of the deck near abutments composed of flat plates whereas the original intention of the designers was to employ warped steel plates in the form of hypars [2]; (c) schematic of the bridge subject to a uniformly distributed load, depicting the bending moment and shear force distributions along the span as well as the distribution of normal and shear stresses through the cross-section AA, highlighting how the web resists predominantly shear.

research available that considers the behaviour of hypar panels under such loading. Consequently, EN1993-1-5 does not provide any guidance for the structural design of such warped panels and pragmatism led the designers to select a different arrangement for the Inner Harbour Bridge, as shown in Figure 1(b).

A mechanical model of the hypar, for which the flat plate is a special case, is developed in §2 to facilitate the eventual formulation of design guidance. Previous analytical models on doubly curved elements have derived complex kinematic relationships, linking the out-of-plane displacement to unknown stress functions that result in equally complex eighth-order partial differential equations [18, 19, 20, 21]. The current model simplifies matters considerably by utilizing a Rayleigh–Ritz approach for a flat plate and treating the double curvature as a deliberately imposed ‘imperfection’ to the flat geometry. The parameters of the nonlinear finite element (FE) models are introduced in §3, which are used to verify the results of the analytical model. This verification is presented in §4, showing very good correlation, demonstrating that a simplified modelling approach yields commendable results for hypars with moderate, yet practical, initial warping curvature. Finally, conclusions are drawn in §5.

2. Analytical formulation

A formulation based on the principle of minimum total potential energy that describes a hypar as an initial imperfection applied to the flat plate is presented in the current section. Consider a simply-supported hypar with a constant projected-plan of length a , width b , thickness t and offset c that is restrained against rigid body motion at the origin, as depicted in Figure 2, the equation of which is given by the expression:

$$w_0(x, y) = k(x - a/2)(y - b/2). \quad (1)$$

Here, $k = 4c/(ab)$, which defines the degree of warping per unit length of the panel, while x , y and z represent the Cartesian coordinate axes originating at one corner of the panel.

The panel is loaded along all its edges by a uniform shear stress resultant, $N_{xy} = N_{yx}$, defined as:

$$N_{xy} = \tau_{xy}t, \quad (2)$$

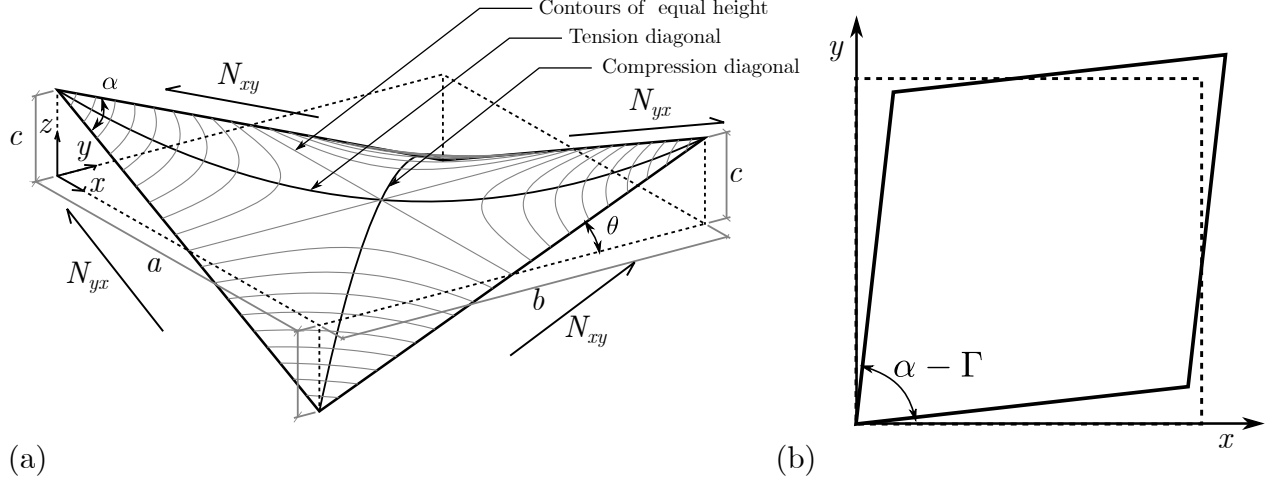


Figure 2: (a) Origin and coordinate system for a hypar of length a , width b , thickness t and offset c with initial angle α , indicating the loading direction, while dashed lines indicate diagonals of ‘compression’ and ‘tension’ (b) deformed shape of the panel under pure in-plane shear indicating the change of the in-plane angle Γ .

with τ_{xy} being the applied shear stress within the xy plane. The panel is restrained in a manner such that it can deform into a pure shear mode, as shown in Figure 2(b), by restraining the corner at $x = a$ and $y = b$, along the line defined by this point and the origin. Presently, the in-plane (membrane) boundary conditions constrain the panel edges to remain straight under deformation but are free to change in length. The study assumes the panel material to be linearly elastic, homogeneous and isotropic with Young’s modulus, E , Poisson’s ratio ν and, through the well-known shear modulus relationship, $G = E / [2(1 + \nu)]$.

2.1. Total potential energy

The total potential energy V of the system comprises the difference between the strain energy stored U and the work done by the external load $N_{xy}\Gamma$:

$$V = U - N_{xy}\Gamma, \quad (3)$$

where the total strain energy U for plane stress problems with linearly elastic materials is given by the well known expression [22]:

$$U = \frac{E}{2(1 - \nu^2)} \int_{-t/2}^{t/2} \int_0^b \int_0^a \left[\varepsilon_x^2 + \varepsilon_y^2 + 2\nu\varepsilon_x\varepsilon_y + \frac{(1 + \nu)}{2}\gamma_{xy}^2 \right] dx dy dz. \quad (4)$$

The derivations of the individual strain terms are presented in §2.1.1(2.1.1), while the resultant displacement from the shear load (Γ) is defined herein as the reduction in the angle at the origin of the panel and is presented in §2.1.1(2.1.2).

2.1.1. Kinematics with initial imperfections

In the present study, rather than considering complex kinematic relationships for shallow shells [19, 23], the membrane and bending strain energy contributions are derived by considering a flat plate (Figure 3(a)) that is intentionally deformed into an initial geometry that is subsequently stress-relieved (Figure 3(b)). These imposed displacements are described by $u_0(x, y)$, $v_0(x, y)$ and $w_0(x, y)$ and are measured as deviations from the flat plate in the x , y and z axes respectively. Shape functions u_T , v_T and w_T are assumed for the overall displacements in the x , y and z directions respectively, which account for the initial geometry (Figure 3(c)). The actual displacement shape functions u, v, w are then retrieved in the post-processing stage by subtracting the initial shape function from this original solution (Figure 3(d)) thus:

$$u = u_T - u_0, \quad (5)$$

$$v = v_T - v_0, \quad (6)$$

$$w = w_T - w_0. \quad (7)$$

For a hypar with constant projected plan dimensions and origin, as shown in Figure 2, $u_0 = v_0 = 0$ and w_0 is given in Equation (1).

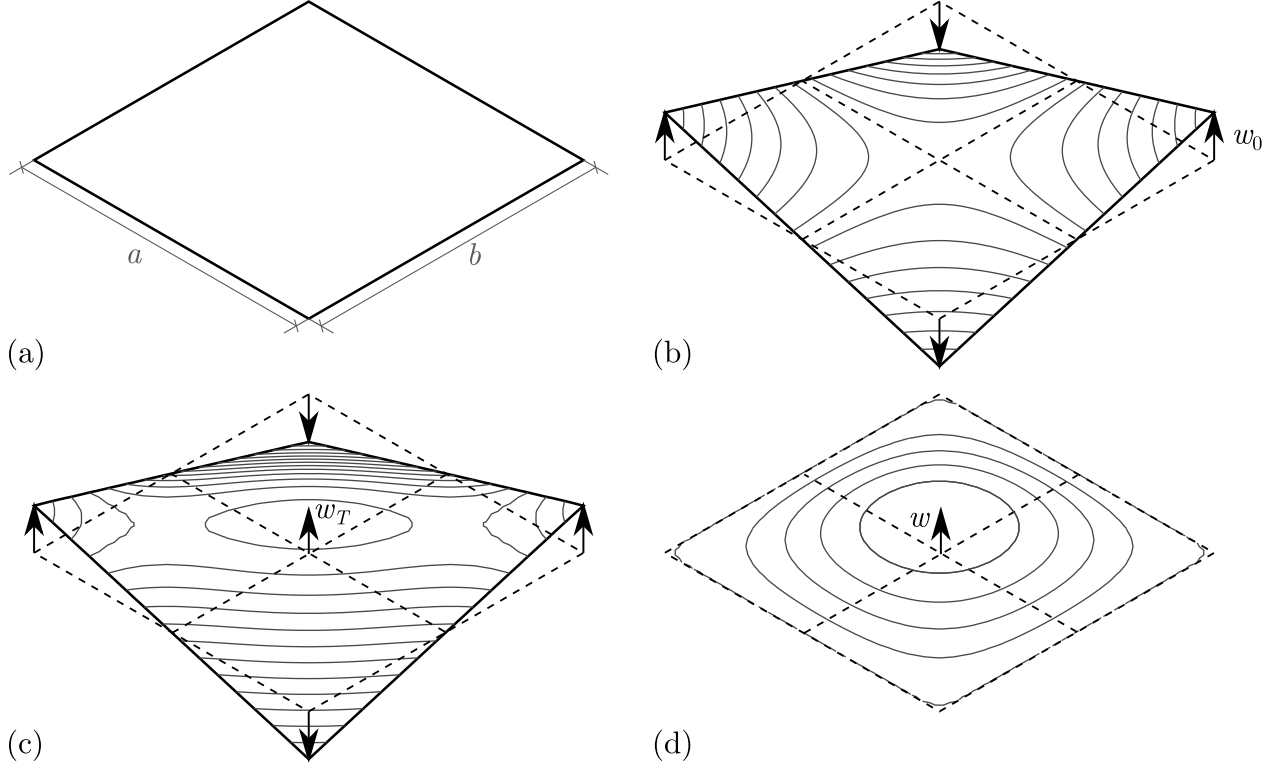


Figure 3: (a) Flat plate with projected plan of dimensions a and b ; (b) transformation of the flat plate into a hypar by the introduction of the shape function w_0 ; (c) total deflection function w_T , including the initial shape w_0 ; (d) actual displacement function w , evaluated by subtracting w_0 from w_T .

This approach has the benefit of providing a relatively straightforward framework to determine the behaviour of panels with an arbitrary but shallow initial geometry under any loading conditions. This is presently exemplified through the loading of a hypar under in-plane shear where even approximate methods to determine the governing equations, such as the Rayleigh–Ritz approach, show very good correlation with the results from more exacting finite element (FE) models. However, unlike the FE method, this approach has the added advantage of allowing direct extraction of individual functional components, such that their influence and that of their interaction on the nonlinear behaviour of the panels can be identified. In the present formulation, it is assumed that:

1. all points lying on a normal to the middle surface before deformation remain normal to the middle surface after loading (akin to the classical Kirchhoff plate bending theory),
2. all through-thickness stresses (τ_{xz} , τ_{yz} & σ_z) are assumed to be negligibly small in comparison with the in-plane stresses (σ_x , σ_y & τ_{xy}). Therefore the panel thickness is unaffected by the system deformation.
3. the panel is stress-relieved after the forming process, i.e. residual stresses from the forming process are not considered.

The derivation closely follows Flügge’s derivation of the kinematic relations of shallow shells [24].

Consider three points A , B and C , lying on the mid-surface of a panel with an initial Euclidean geometry such that it can be kinematically described by the behaviour of a flat plate having a rectangular projected-plan form as shown in Figure 4. Points B and C are respectively located at a distance dx and dy from point A , such that lines AC and AB are orthogonal in the xy -plane. Let u , v and w be defined as displacements that occur after the application of a yet undefined load while orientated in the same direction as the x , y and z coordinate axes respectively.

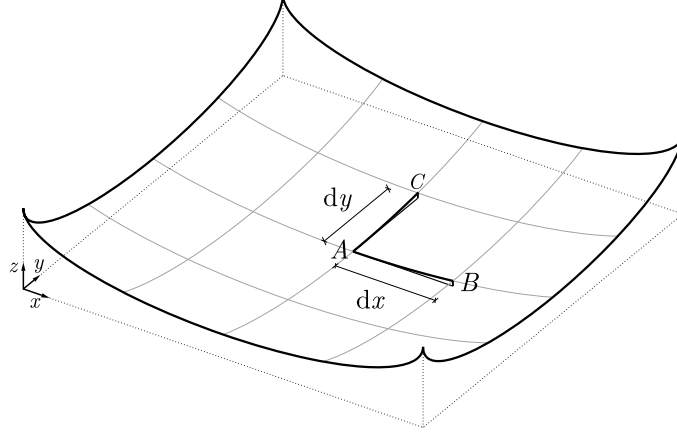


Figure 4: A panel with an initial shallow Euclidean geometry considering points A, B and C lying its mid-surface before the application of a yet undefined load.

Consider also the point A' that lies on the outer surface of the panel at a distance ζ on the normal to the mid-surface at A , as shown in Figure 5; upon deformation of the panel, the points A and A' move to A_1 and A'_1 respectively. Figure 5 also shows a section parallel to the xz -plane of the panel before and after application of the load, together with the associated u and w displacements. Since the thickness of the panel

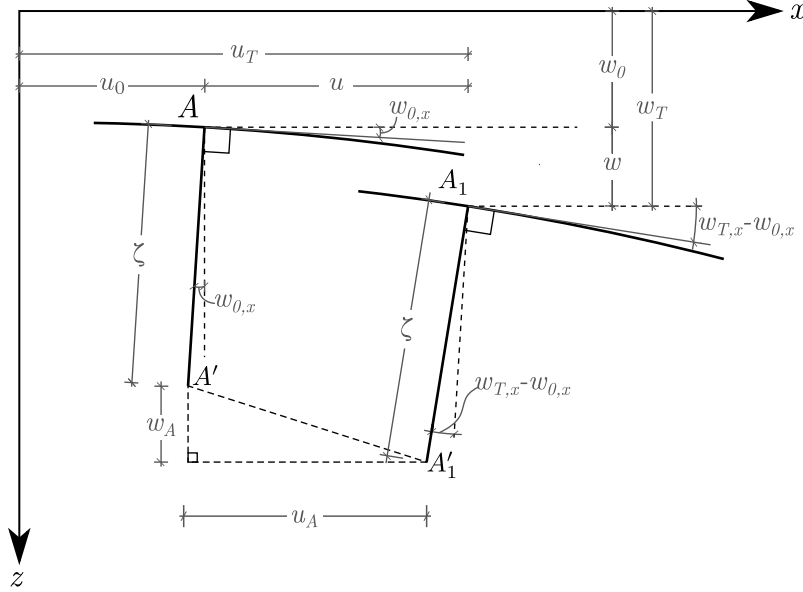


Figure 5: Displacement of point A in the xz -plane after application of an arbitrary load.

is assumed to be constant during deformation, the following relationship results:

$$w_A = w. \quad (8)$$

However, while point A is displaced by a distance u , point A' is displaced by u_A defined thus:

$$u_A = (u_T - u_0) - \zeta(w_{T,x} - w_{0,x}), \quad (9)$$

where a subscript after a comma represents a partial derivative of the function with respect to the defined variable (in this case x).

Similarly, the normal AA' also deviates in the yz -plane by the small angle $w_{0,y}$ during the deformation. Thus, the following additional relationship is derived for the displacement in the y -direction at A :

$$v_A = (v_T - v_0) - \zeta(w_{T,y} - w_{0,y}). \quad (10)$$

The strains ε_x and ε_y are respectively defined as the stretching of the elements AB and AC defined in the xy -plane, while γ_{xy} is the reduction of the angle BAC in the xy -plane. These are defined in terms of the components u_A , v_A and w_A thus:

$$\varepsilon_x = u_{A,x}, \quad \varepsilon_y = v_{A,y}, \quad \gamma_{xy} = u_{A,y} + v_{A,x}. \quad (11)$$

Substituting Equations (8)–(10) into Equation (11) gives the strains at A as:

$$\begin{aligned} \varepsilon_x &= u_{T,x} - u_{0,x} - \zeta(w_{T,xx} - w_{0,xx}), \\ \varepsilon_y &= v_{T,y} - v_{0,y} - \zeta(w_{T,yy} - w_{0,yy}), \\ \gamma_{xy} &= (u_{T,y} - u_{0,y}) + (v_{T,x} - v_{0,x}) - 2\zeta(w_{T,xy} - w_{0,xy}), \end{aligned} \quad (12)$$

which are the kinematic relations for the panel, considering the initial geometry effectively as an initial imperfection to the flat plate.

Substituting the kinematic relations given in Equation (12) into Equation (4) and integrating with respect to the z -axis, gives the strain energy stored in the panel U , which may be decomposed into familiar terms, U_b and U_m , representing the influence of the bending and membrane strain energies respectively:

$$U_b = \frac{D}{2} \int_0^b \int_0^a \left[(w_{,xx} + w_{,yy})^2 - 2(1 - \nu)(w_{,xx}w_{,yy} - w_{,xy}^2) \right] dx dy, \quad (13)$$

$$U_m = \frac{6D}{t^2} \int_0^b \int_0^a \left[\varepsilon_{mx}^2 + \varepsilon_{my}^2 + 2\nu\varepsilon_{mx}\varepsilon_{my} + \frac{(1 - \nu)}{2}\gamma_{mxy}^2 \right] dx dy, \quad (14)$$

where D is the flexural rigidity of the panel, defined as:

$$D = \frac{Et^3}{12(1 - \nu^2)}. \quad (15)$$

If second order effects of the out-of-plane displacement in w are considered, the resulting nonlinear relationships for ε_{mx} , ε_{my} and γ_{mxy} are:

$$\varepsilon_{mx} = u_{,x} + \frac{1}{2}(w_{T,x}^2 - w_{0,x}^2), \quad (16)$$

$$\varepsilon_{my} = v_{,y} + \frac{1}{2}(w_{T,y}^2 - w_{0,y}^2), \quad (17)$$

$$\gamma_{mxy} = u_{,y} + v_{,x} + w_{T,x}w_{T,y} - w_{0,x}w_{0,y}, \quad (18)$$

which are frequently referred to as the von Kármán strains [22]. Note that terms in w_0 would customarily represent plate initial imperfections [25, 26], whereas presently they refer to the initial geometry of the panel.

2.1.2. Work done by load

The shape functions for u_T and v_T represent the global in-plane displacements of the panel due to loading and inherently account for any such deformations that occur due to the inextensional component of displacements in w . Thus the work done by load term is defined:

$$N_{xy}\Gamma = N_{xy} \left[\int_0^b \int_0^a (u_{T,y} + v_{T,x}) dx dy \right], \quad (19)$$

where the double integral represents Γ , the reduction in the corner angle resulting from the application of the external load.

2.2. Shape functions and boundary conditions

A key component of any energy-based model, which is subsequently solved using the Rayleigh–Ritz method, is the accuracy of the shape functions; presently, this refers to the assumptions made for the panel displacements u_T , v_T and w_T under loading. The shape functions must be sufficient to model boundary conditions that satisfy both the flexural and membrane boundary conditions of the panel that are consistent with the applied loads, as highlighted in [27]. Shape functions incorporating appropriate trigonometric series are currently applied to satisfy the boundary conditions shown in Figure 2, and are presented thus:

$$u_T = y \sin \frac{q_{uv}}{2} - 2x \sin^2 \frac{q_{uv}}{4} + Q_{uv1} \left(\frac{y}{b} - \frac{x}{a} \right) + \sum_{g=1}^g \sum_{h=1}^h Q_{ugh} \sin \frac{g\pi x}{a} \cos \frac{h\pi y}{b}, \quad (20)$$

$$v_T = x \sin \frac{q_{uv}}{2} - 2y \sin^2 \frac{q_{uv}}{4} - Q_{uv1} \left(\frac{y}{b} - \frac{x}{a} \right) + \sum_{i=1}^i \sum_{j=1}^j Q_{vij} \cos \frac{i\pi x}{a} \sin \frac{j\pi y}{b}, \quad (21)$$

$$w_T = w_0 + \sum_{m=1}^m \sum_{n=1}^n Q_{wmn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad (22)$$

where:

$$q_{uv} = Q_{uv} \frac{\pi}{2}, \quad (23)$$

and terms beginning with Q are generalized coordinates with g, h, i, j, m and n being integers for the respective series summations. In the solution to the linear buckling equations for flat plates subject to in-plane shear, it was identified that the shape function in Equation (22) can be divided into two groups that can be solved separately, one where $m+n$ is even and another where $m+n$ is odd. When $m+n$ is even, the buckled shape is symmetric about the tension diagonal, whereas if $m+n$ is odd, the buckled shape is anti-symmetric about the tension diagonal [28], as depicted in Figure 6. For a square panel, results for the nonlinear analysis with all

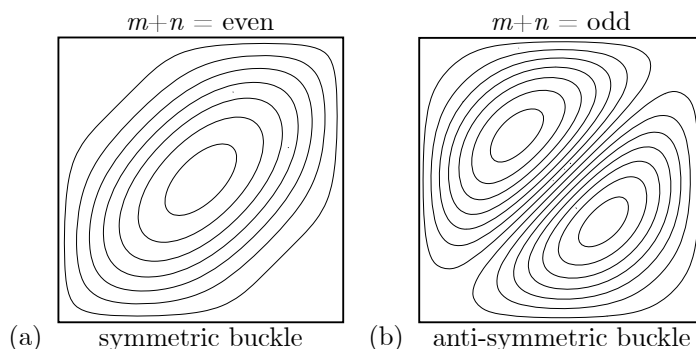


Figure 6: Out-of-plane (a) symmetric and (b) anti-symmetric buckling modes of a square panel subject to shear load.

possible combinations of m and n show no influence from terms where $m+n$ is odd. Therefore, the symmetric mode is dominant in the post-buckling range through all stages of loading and hence, to reduce the required computational effort, the integer pairs: (g, h) , (i, j) and (m, n) , can be limited to those where the sum of the indices that comprise each pair is even, without any loss of accuracy. However, for non-square panels this simplification would not necessarily apply and all integer pairs may need to be included to maintain model consistency.

In the above shape functions, Q_{uv} represents a change in Γ caused by the application of the edge shear, where the panel origin is coincident with the coordinate axes and the panel edge length remains equal to the original length throughout deformation. Meanwhile, a positive value for Q_{uv1} represents a shortening of the compression diagonal and the panel edges remaining straight but also shortened, thereby also contributing to a change in Γ .

2.3. Numerical solution to the analytical model

Once the shape functions are substituted into the total potential energy function V , the Rayleigh–Ritz method is implemented to obtain the governing equilibrium equations. These are established by setting the

partial derivatives of V with respect to each of the generalized coordinates Q_r , equal to zero, since V is stationary at equilibrium [29]:

$$\frac{\partial V}{\partial Q_r} = 0, \quad (24)$$

where r represents any subscript of a previously presented generalized coordinate.

The result is a set of nonlinear algebraic equations, the number of which is equal to the number of independent generalized coordinates, which are solved within the numerical continuation package AUTO-07p [30]. The numerical results are established using the example material and panel properties presented in Table 1.

Table 1: Model parameters of the example material and panel used in the numerical simulations.

Parameter	Value
Young’s Modulus, E	210 kN/mm ²
Poisson’s ratio, ν	0.30
Panel width, a	1000 mm
Panel depth, b	1000 mm
Panel aspect ratio, ϕ	1.0
Panel thickness, t	{4, 6, 8, 10, 15, 20} mm
Offset, c	{0, 1, 5, 10, 25, 50} mm

3. Finite element model

A finite element (FE) model of the panel is developed within the commercial package ABAQUS [31], using the same modelling parameters in Table 1, to verify the numerical solutions to the analytical model. Following a mesh convergence study, the general-purpose four-noded shell element comprising six degrees of freedom per node with reduced integration (S4R) is selected, and 100 elements per panel edge are implemented. The chosen mesh density provides sufficient accuracy without excessive computational effort, with the maximum out-of-plane displacement at the panel centre deviating by a maximum of 0.2% when compared with the same deflections from FE models with 400 elements per panel edge.

3.1. Load and boundary conditions

The membrane boundary conditions shown in Figure 2 are specified such that the panel edges are constrained to remain straight during loading, but, as indicated earlier, the length of these edges may change. This is achieved in ABAQUS through the use of the “SLIDER” multi-point constraint (MPC), which modifies the degrees of freedom of the secondary nodes (the internal nodes along the panel edge) to lie on a straight line defined by the two corners demarcating that edge, while allowing the edge to change in length. Additionally, an “Equation” constraint is imposed on the deformations of the node in corner C3, such that the u and v displacements are equal to achieve the deformation shown in Figure 2(b). Since the MPC definition constrains the deformation of all the nodes along the edges as described, boundary conditions need only be specified at the corners. These boundary conditions are presented in Table 2 and depicted in Figure 7. The

Table 2: Boundary conditions for the FE model as shown in Figure 7; U1–U3 represent the translational degrees of freedom in the coordinate axes, while UR1–UR3 represent the rotational degrees of freedom. Labels ‘F’ or ‘R’ denote that the degree of freedom is free or restrained respectively.

Corner	U1	U2	U3	UR1	UR2	UR3
C1	R	R	R	F	F	F
C2–C4	F	F	R	F	F	F

load is applied as concentrated loads, proportionally distributed at each node along the edges and at the corners of the panel in the directions shown in Figure 7.

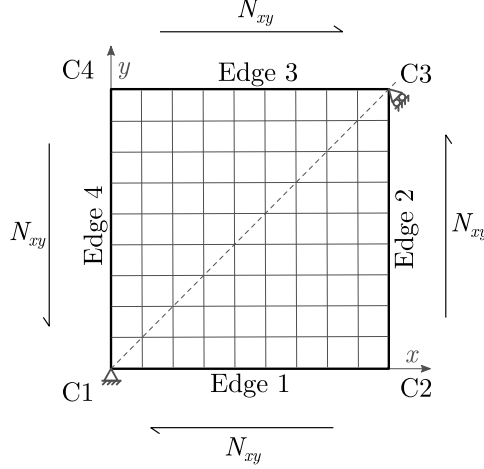


Figure 7: Orientation of the panel and associated boundary conditions in the FE model.

3.2. Analysis procedures

Two sets of analyses are conducted on the FE models. The first is a linear buckling analysis (LBA) on the flat plate, to determine the linear eigenvalues and eigenmodes. The critical buckling loads from the numerical model $N_{xy,FE}^{C,0}$ are compared with the classical results for the critical shear buckling load for flat plates, $N_{xy}^{C,0}$ defined as:

$$N_{xy}^{C,0} = k_\tau \frac{\pi^2 E t^3}{12(1 - \nu^2)b^2} \quad (25)$$

where k_τ is the shear buckling coefficient that is equal to 9.35 for a square plate [28], in Table 3, where very

Table 3: Comparison of the buckling loads from the FE model $N_{xy,FE}^{C,0}$ and the classical expression $N_{xy}^{C,0}$.

t [mm]	4	6	8	10	15	20
$N_{xy}^{C,0}$ [N/mm]	113.45	382.91	907.64	1772.73	5982.97	14181.86
$N_{xy,FE}^{C,0}$ [N/mm]	113.81	384.82	914.01	1789.05	6074.15	14495.99
$N_{xy}^{C,0} / N_{xy,FE}^{C,0}$	1.00	1.00	0.99	0.99	0.98	0.98

good agreement is consistently observed.

The second analysis is geometrically nonlinear (GNA) and uses the Riks algorithm [32], as implemented within ABAQUS. In the nonlinear analysis of the flat plate, a small perturbation is added in the form of the first eigenmode scaled by $10^{-4}t$ such that the nonlinear procedure can bypass the bifurcation point without convergence problems while practically simulating the near-perfect case. In contrast, the hyper under loading generates a combination of membrane and bending stresses from the outset without the need for any additionally imposed geometric perturbations.

4. Numerical studies and discussion

4.1. Analytical model verification

The accuracy of the Rayleigh–Ritz method is highly dependent on the ability of the prescribed shape functions to model the shape of deformed panel without violating the boundary conditions. Shape functions that achieve this are presented in Equations (20)–(22). The shape functions include trigonometric series that can generally yield accurate results provided that a suitable number of terms is utilized, i.e. a sufficiently large value for g and m .

This is observed in the equilibrium paths of the normalized load $\tilde{N}_{xy}$, defined thus:

$$\tilde{N}_{xy} = \frac{N_{xy}}{N_{xy}^{C,0}}, \quad (26)$$

against the central out-of-plane displacement w and the corner angle Γ , shown in Figure 8, comparing the

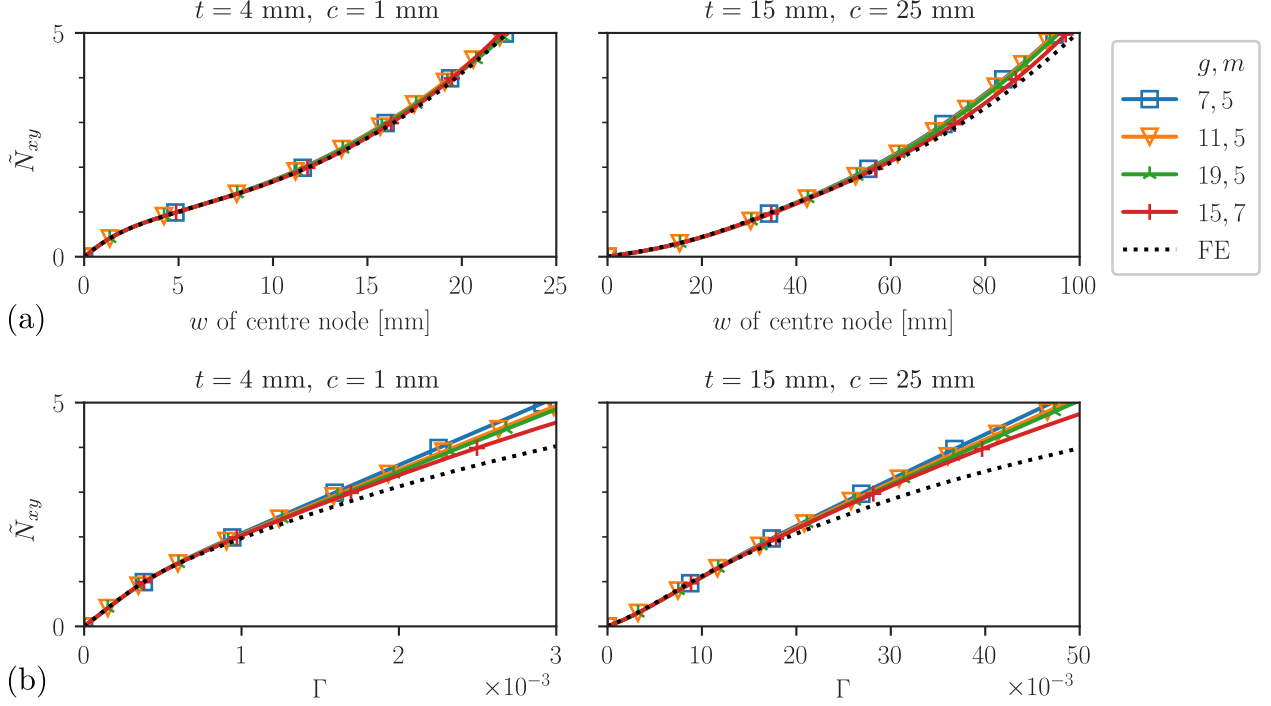


Figure 8: Equilibrium paths for the applied load against (a) change in shear angle Γ and (b) w at the central node of the panel, comparing analytical models with different model fidelities, represented by various g and m values, against the FE results, for $t = 4$ mm and 15 mm and $c = 1$ mm and 25 mm.

results of the analytical model with different combinations of g and m versus the FE results for the cases of $t = \{4, 15\}$ mm and $c = \{1, 25\}$ mm. Shape functions with $m = 5$ are sufficient for modelling out-of-plane deflections up to $5\tilde{N}_{xy}$, with higher values of m achieving better accuracy in the predictions at higher load levels. In terms of the in-plane deflections, an increase in m also improves the prediction but at the expense of a significant increase in computational effort. Increases in g and m also improve the prediction for the change in shear angle Γ , as observed in Figure 8. However, when the load is between 2–3 of $\tilde{N}_{xy}$, there is a drop in stiffness detected in the FE models, as shown by the shallower slope in Figure 8(b), which is not observed within the analytical model results.

The contour plots of the in-plane displacement u over the hyper panel with $t = 15$ mm and $c = 25$ mm at different stages of loading in Figure 9 explain the reasons for this shortcoming. When $\tilde{N}_{xy} = 1.50$ in Figure 9(a), there are barely perceptible differences between the deformed shapes from the FE model (left) and the analytical model (right). However, at higher loads, shown in Figure 9(b), the FE results depict localized deformations at the top-left and bottom-right corners of the panel. The analytical shape functions in their current formulation are not well-suited to detect such localized deformations to the same degree as in the FE model. Such localized corner deformations are also observed in the out-of-plane displacements at higher loads. This is shown in Figure 10 where the deflected profile for w along the principal diagonals of the panel is plotted for both the FE and analytical models. The blue curves show the deflected profile of the tension diagonal and the red curves show those of the compression diagonal, while FE results are distinguished by dashed lines with the same corresponding colours. At higher loads, although the overall deflection at mid-span is accurately predicted by the analytical model, the corner regions exhibit a localized wavelength reduction that the current shape functions are unable to handle with the same level of precision. Such differences between the analytical model and FE results are more noticeable as the load increases.

The trigonometric series terms in Equations (20)–(21) have the effect of introducing the ripples within the boundary of the panel, as observed in Figure 9. Such ripples account for the in-plane deformations of the panel due to the out-of-plane displacements in w , as well as any displacements due to the direct stresses induced in the panel. Therefore, the inclusion of additional terms in g is necessary to predict additional

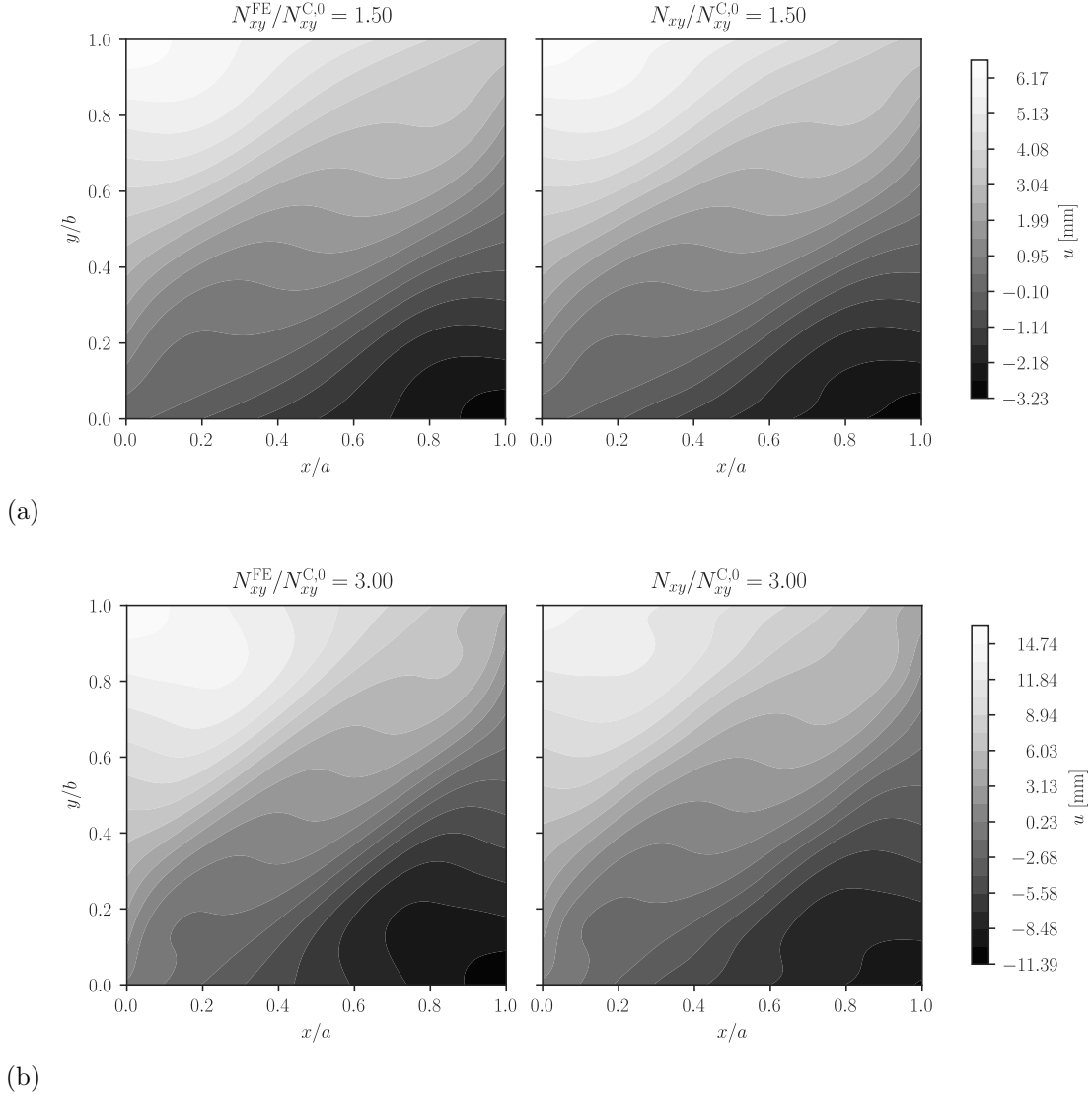


Figure 9: Contour plots of the results for u for a hypar panel with $t = 15$ mm and $c = 25$ mm at different loading levels: (a) $\tilde{N}_{xy} = 1.50$; (b) $\tilde{N}_{xy} = 3.00$. The contour plots on the left are the results from the FE analyses and the right-hand plots show the analytical results for the model with $g = 15$ and $m = 7$.

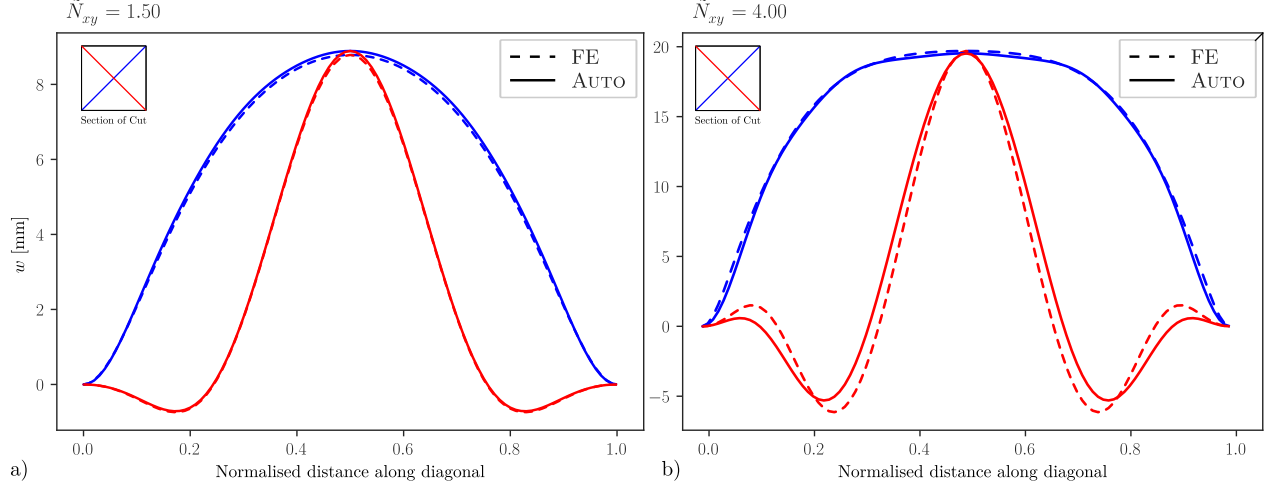


Figure 10: Deflected profile w across principal diagonals of a hypan panel with $t = 4$ mm and $c = 1$ mm at different loading levels: (a) $\tilde{N}_{xy} = 1.50$; (b) $\tilde{N}_{xy} = 4.00$. The solid lines show the analytical results for the model with $g = 15$ and $m = 7$, while the dashed lines are the results from the FE model.

softening in the post-buckling response.

At present, the results are limited to the range of $\tilde{N}_{xy} = [0, 5]$, as the results from geometric and material nonlinear analyses (GMNA) reveal that panel failure would typically occur within this region. This is depicted in Figure 11 for panels with $t = \{4, 15\}$ mm, $c = \{0, 1, 5, 10, 25, 50\}$ mm and $f_y = 690$ N/mm², where f_y is the

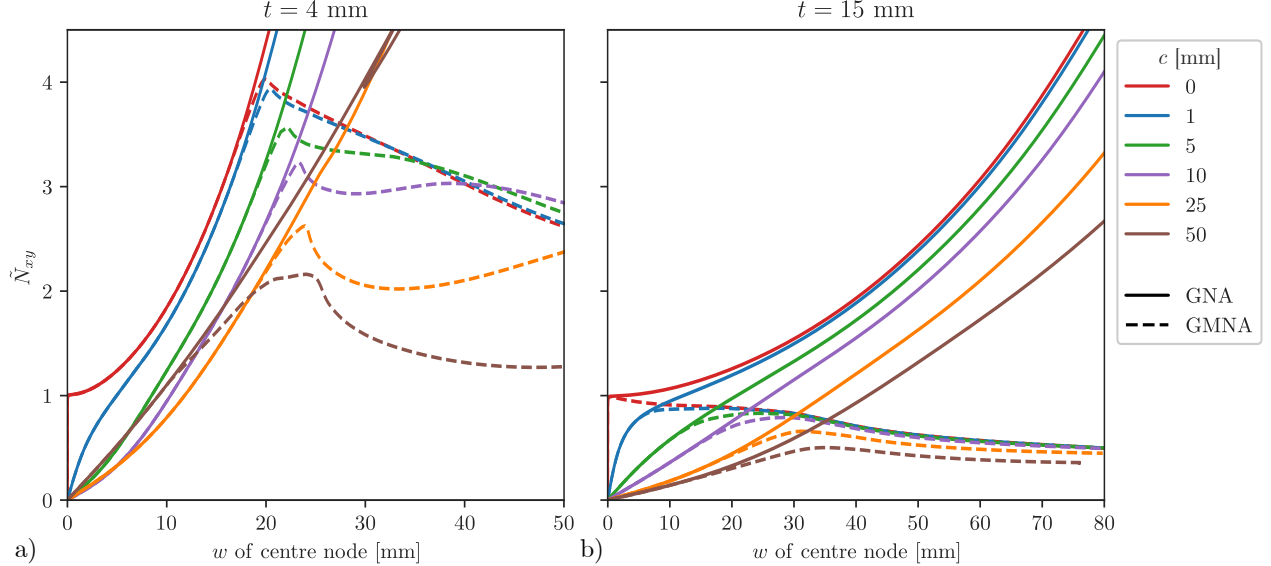


Figure 11: Equilibrium paths for the applied load against w of the central node of the panels modelled within ABAQUS for (a) $t = 4$ mm and (b) $t = 15$ mm, comparing the results of a GNA (solid line) and GMNA with $f_y = 690$ N/mm² (dashed line) for offsets $c = \{0, 1, 5, 10, 25, 50\}$ mm.

material yield stress. Within this range of parameters, the analytical model is noted to be both quantitatively and qualitatively accurate for $c \leq 25$ mm, but valuable qualitative insight can still be drawn from the approach at higher offsets. Should a more sophisticated shape function be implemented into the Rayleigh–Ritz model, especially one that is capable of replicating the localized deformations described above at higher loads, then the analytical model would naturally be able to predict the behaviour at higher loads better. It is further acknowledged that at higher loads and larger offsets, a more accurate kinematic relationship that couples the nonlinear in-plane and out-of-plane displacements may be necessary for enhanced accuracy. Nevertheless,

the present shape function has been demonstrated to be adequate for modelling the salient features of the elastic flat plate and elastic hypar subject to in-plane shear within the loading range of interest.

4.2. Behaviour of hypar panels subject to shear

In the current section, the behaviour of the hypar panel when subject to shear loads is analysed using the results from the analytical model with $g = 15$ and $m = 7$. This behaviour is best demonstrated by the equilibrium paths shown in Figure 12, depicting how the panel centre deflects in w with increasing load and

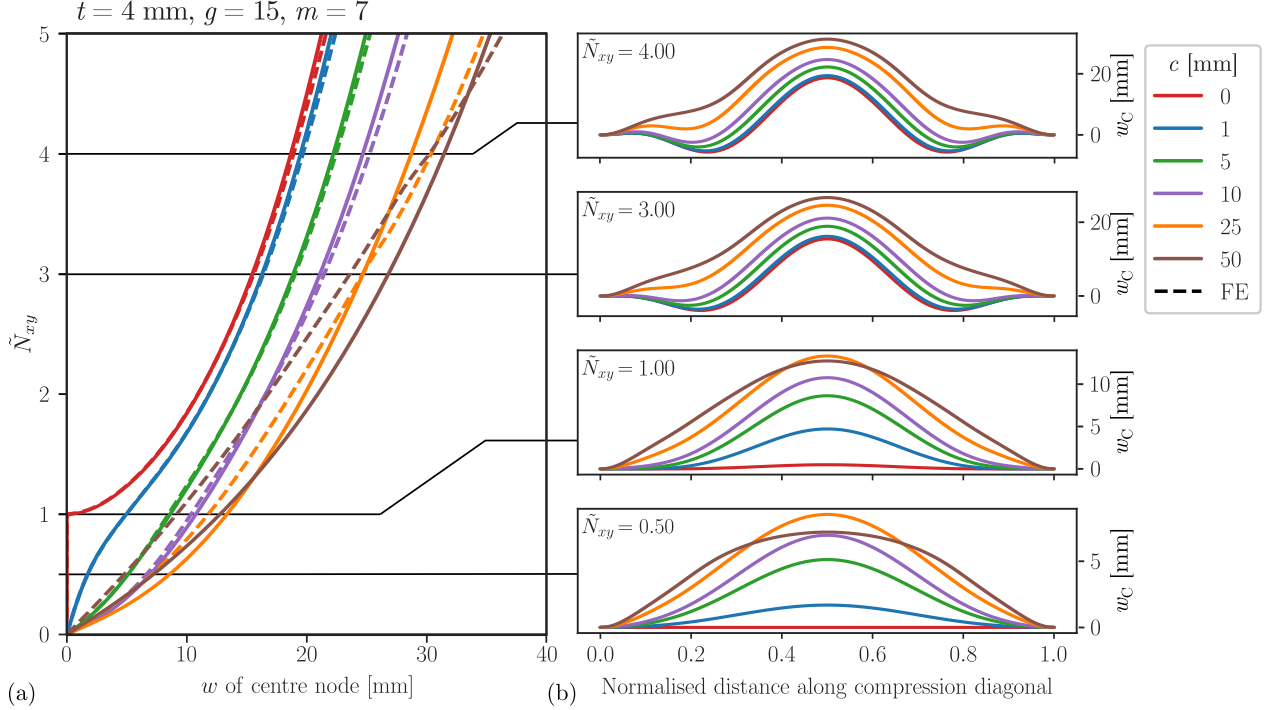


Figure 12: (a) Applied load against w for the range of offsets $c = \{0, 1, 10, 25, 50\}$ mm and $t = 4$ mm, (b) the deflected profile from the analytical model across the compression diagonal at indicated load levels. Solid lines show the analytical model results, where $g = 15$ and $m = 7$, with dashed lines showing the FE results.

varying offsets in comparison with the FE results. Since the out-of-plane deflection of the panel centre does not capture the behaviour over the entirety of the panel, an additional comparison is presented showing the deflected profile w along the compression diagonal for hypars with varying offsets at different load levels.

It is immediately evident that both the elastic hypar and the elastic flat plate have equilibrium paths with monotonically positive stiffnesses. In addition, as expected, the flat plate exhibits a trivial fundamental path in w , detecting a supercritical pitchfork bifurcation point at $\tilde{N}_{xy} = 1$ followed by a stable post-buckling path [29]. On the other hand, the hypars undergo significant nonlinear deformation from the onset of loading, but the response is very similar to that of the flat plate even after only moderate levels of displacements are established.

In the early stage of loading, the buckled shape across all offsets is that of a single bulge with a peak at mid-span. Additional peaks and troughs begin to appear as the loading increases further into the post-buckling range, as seen in Figure 12. The shape functions introduced in Equation (22) can be presented along the compression diagonal as w_C , by substituting in the expression $y = b(1 - x/a)$, which defines the compression diagonal as shown in Figure 13, and noting that for a square panel $a = b$, such that:

$$w_C = \sum_{m=1}^m \sum_{n=1}^n W_{mn}, \quad (27)$$

where:

$$W_{mn} = -Q_{wmn} \sin \frac{m\pi x}{a} \sin \frac{n\pi x}{a} \cos n\pi. \quad (28)$$

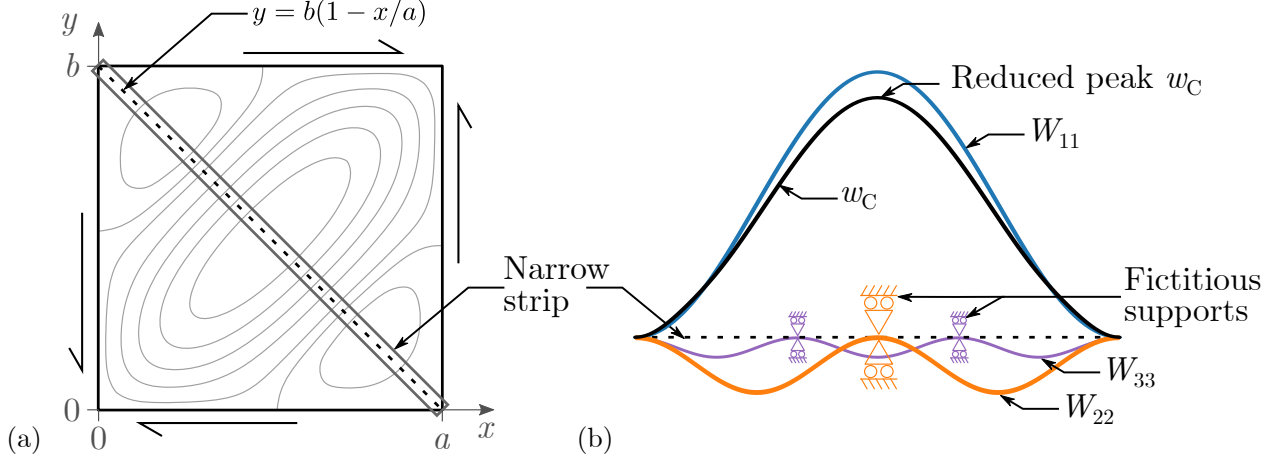


Figure 13: Representation of (a) a narrow strip along the compression diagonal, which is defined by the expression $y = b(1 - x/a)$, and (b) the fictitious restraint to out-of-plane displacements along the compression diagonal introduced by higher harmonics, resulting in the observed reduced peaks in w_C for hypars with higher levels of offset c .

This explains why the deflected shape across the compression diagonal has an initially flat slope at each end, as the function maps onto $\sin^2(\pi x/a)$ rather than $\sin(\pi x/a)$. Each combination of (m, n) in W_{mn} represents a discrete trigonometric component along the compression diagonal, for example W_{11} represents a single peak at the middle of the diagonal, the summation of which results in the overall deflected shape, w_C .

An analysis of the contributions of the fundamental modes W_{mn} to the overall deflected shape w_C provides insight into why the response of the hypar is similar to that of the flat plate at higher loads. Figure 14 depicts plots of w_C and W_{mn} for the $t = 4$ mm panel, where the offset $c = \{0, 1, 25\}$ mm in each column of the plot respectively and each row of the plot represents different stages of the loading process $\tilde{N}_{xy} = \{0.50, 1.00, 2.00, 4.00\}$ respectively. For a square panel, it is observed that $W_{mn} = W_{nm}$. In Figure 14, W_{mn} combinations are limited to those providing a more significant contribution to w_C . Considering first the trigonometric component contributions in the flat plate, the trivial fundamental path in w implies that all W_{mn} terms are zero prior to buckling, while non-zero values for $W_{11}, W_{22}, W_{13}, W_{31}$ and W_{33} are all simultaneously triggered just as the load increases beyond $\tilde{N}_{xy} = 1$. Although higher trigonometric components are also activated upon buckling, their influence on w_C is less pronounced. This is expected, as otherwise an increase in the number of terms within the trigonometric series would not have an effect on the value of k_τ as observed in [28, 33]. In the flat plate, the contributions from W_{11} and W_{22} remain dominant throughout loading, with the influence of higher harmonics increasing as loading progresses deeper into the post-buckling region. The introduction of such higher harmonics explains the reduction in the wavelength towards the edges of the panel, as the amplitudes of W_{11} and W_{22} become less dominant within these regions.

In the hypar, the initial geometry is conducive to a significant contribution from W_{11} across all offsets right from the onset of loading, as shown in Figure 14. The link between pressure loads and membrane shear of a hypar is well-documented [5, 6, 7], and causes the influence of W_{11} from the onset of loading. As loading progresses beyond $\tilde{N}_{xy} = 1.0$, the W_{mn} component contributions in the hypar begin to resemble those of the flat plate, but with a marginally more significant contribution from higher harmonics such as W_{33} and W_{55} .

In hypars with larger offsets, higher harmonics contribute more substantially to w_C from the onset of loading, as observed for the hypar with $c = 25$ mm in Figure 14, and coincides with a stiffer response to out-of-plane deflections with increasing load. The cause of this stiffer response is examined by considering a narrow strip from the panel along the compression diagonal, depicted in Figure 13. Combinations of W_{mn} harmonics where m and n are both even integers do not contribute to the deformation at mid-span of the strip, thus providing a fictitious support at this location. Higher harmonics further divide the span of the narrow strip, resulting in more pronounced deflections towards the ends of the strip and contribute to a stiffer response to out-of-plane deflections. This explains why in Figure 12, the maximum central displacement of the hypar with offset $c = 50$ mm is less than that for lower offsets since even (m, n) harmonics are more prevalent from the onset of loading.

The demonstrated influence of higher order harmonics further establishes the importance of additional

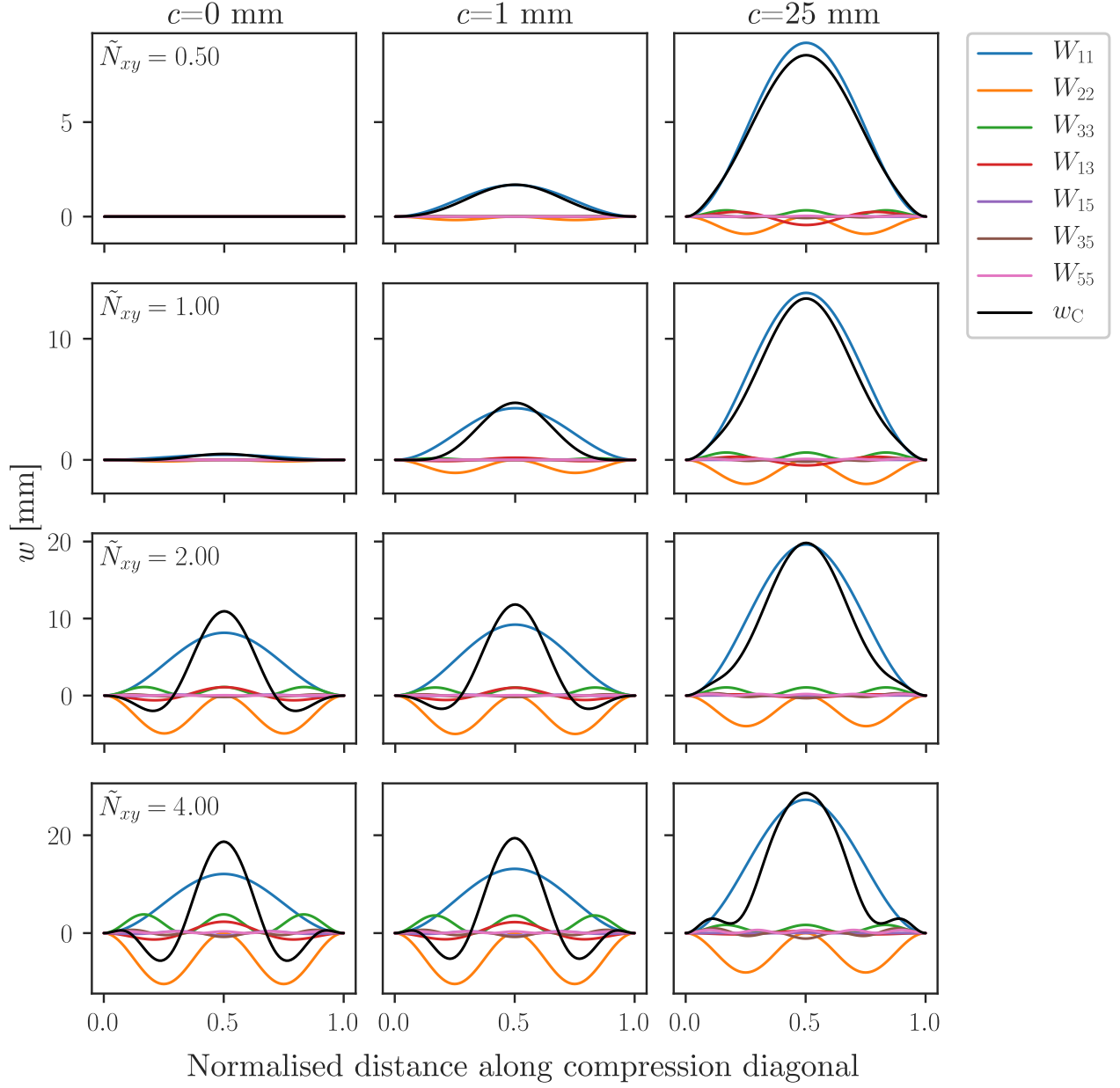


Figure 14: Plots showing the contributions of W_{mn} components to w_C for panels with $t = 4$ mm and $c = \{0, 1, 25\}$ mm at $\tilde{N}_{xy} = \{0.50, 1.00, 2.00, 4.00\}$. The results are from the analytical model with $g = 15$ and $m = 7$. Note that for the case of $\tilde{N}_{xy} = 1.00$, the results are marginally within the post-buckling range, very slightly above the critical buckling load, and hence a small deflection is also detected in the flat plate ($c = 0$ mm).

terms in the trigonometric series to produce better comparisons with the FE results at higher loads, as discussed in §4.1 of the current section.

4.2.1. Rate of change of generalized coordinates

The results in Figure 14 indicate that the relative contribution of different modes changes throughout the loading process. To understand this better, plots of load against generalized coordinates as well as the rate of change of these generalized coordinates with respect to the load level for $t = 4$ mm are shown in Figures 15–16.

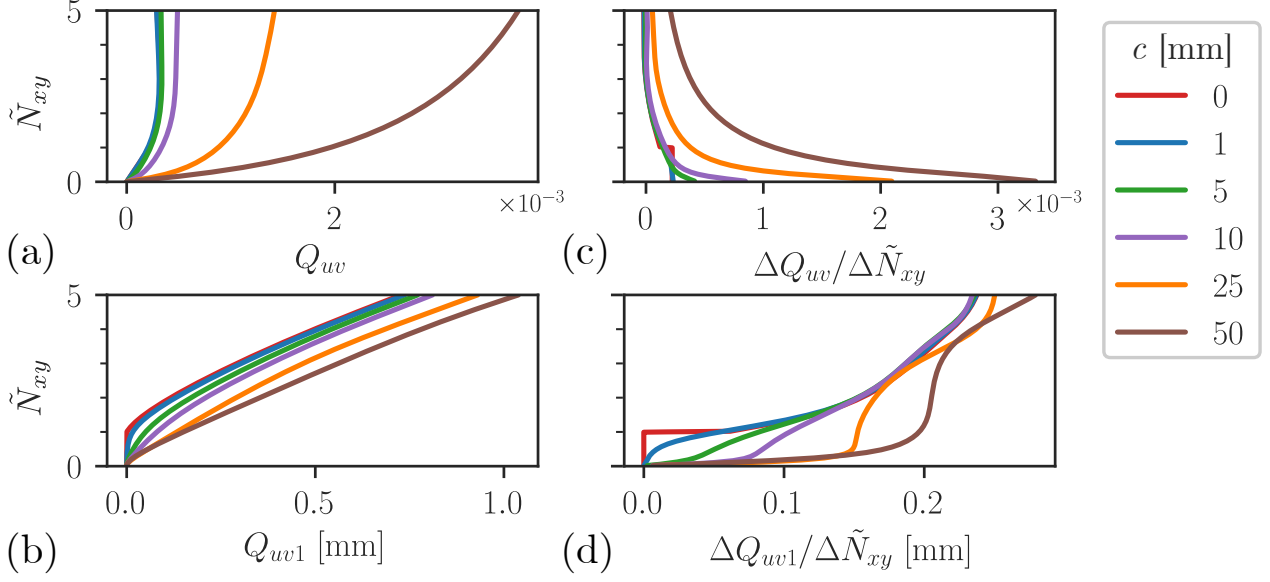


Figure 15: Plots of applied load against (a) Q_{uv} , (b) Q_{uv1} and rate of change of (c) Q_{uv} , (d) Q_{uv1} with respect to load, for panels with $t = 4$ mm and $c = \{0, 1, 5, 10, 25, 50\}$ mm.

In Figure 15, the nonlinear equilibrium path in Q_{uv} is immediately apparent from the constant non-zero rate of change in Figure 15(c) and across all offsets. In the flat plate, the rate is initially constant and reduces sharply at $\tilde{N}_{xy} = 1$. This reduction in rate indicates that the plate is more resistant to a change in Γ , implying a stiffer response as the load increases which seemingly contradicts the post-buckling reduction in stiffness normally expected for plates under shear [34]; this feature is also found in the FE results. However, in the analytical model, the loss of stiffness is instead represented by the increased influence of Q_{uv1} since this term reduces the length of the edges thereby increasing the angle Γ more rapidly. The effects from both Q_{uv} and Q_{uv1} result in a reduced post-buckling stiffness, without violating the membrane boundary conditions.

In the hypar, Q_{uv} changes with load in a similar manner as in the flat plate for lower offsets, but at higher offsets the rate does not reduce below zero. Considering Q_{uv1} , recalling that the rate is initially zero for the flat plate until $\tilde{N}_{xy} = 1$, the rate increases from the onset of loading in the hypar and closely follows the behaviour of the flat plate for offsets up to $c = 10$ mm at higher loads. For offsets between $c = 10$ mm and $c = 50$ mm, there is a stiffening of the behaviour that occurs before $\tilde{N}_{xy} = 1$. The rate plateaus at this value until the load reaches $\tilde{N}_{xy} \approx 4$, at which point it begins to regain influence. This increasing influence of Q_{uv1} accounts for the reason for the drop in stiffness observed in the FE models in Figure 8.

In Figure 16, the behaviour of the more significant generalized coordinates in Equation (22) are presented for the $t = 4$ mm panel with offsets $c = \{0, 1, 25\}$ mm. The trivial fundamental path in w is once again evident for the flat plate, with deformations being triggered at the buckling load in Figure 16, further depicted by the curves of Figure 16(d).

In the hypar cases, Q_{w11} is the dominant coordinate from the onset of loading for all offsets. At lower offsets, the rate of change of Q_{wmn} peaks at $\tilde{N}_{xy} = 1$, showing how the behaviour of the hypar matches closely the behaviour of the flat plate. As the offset increases however, the initial influence of Q_{w11} is dominant to the point that all other generalized coordinates have negligible influence until the load increases. Figure 16 also provides insight as to when additional harmonics are introduced along the compression diagonal. For instance,

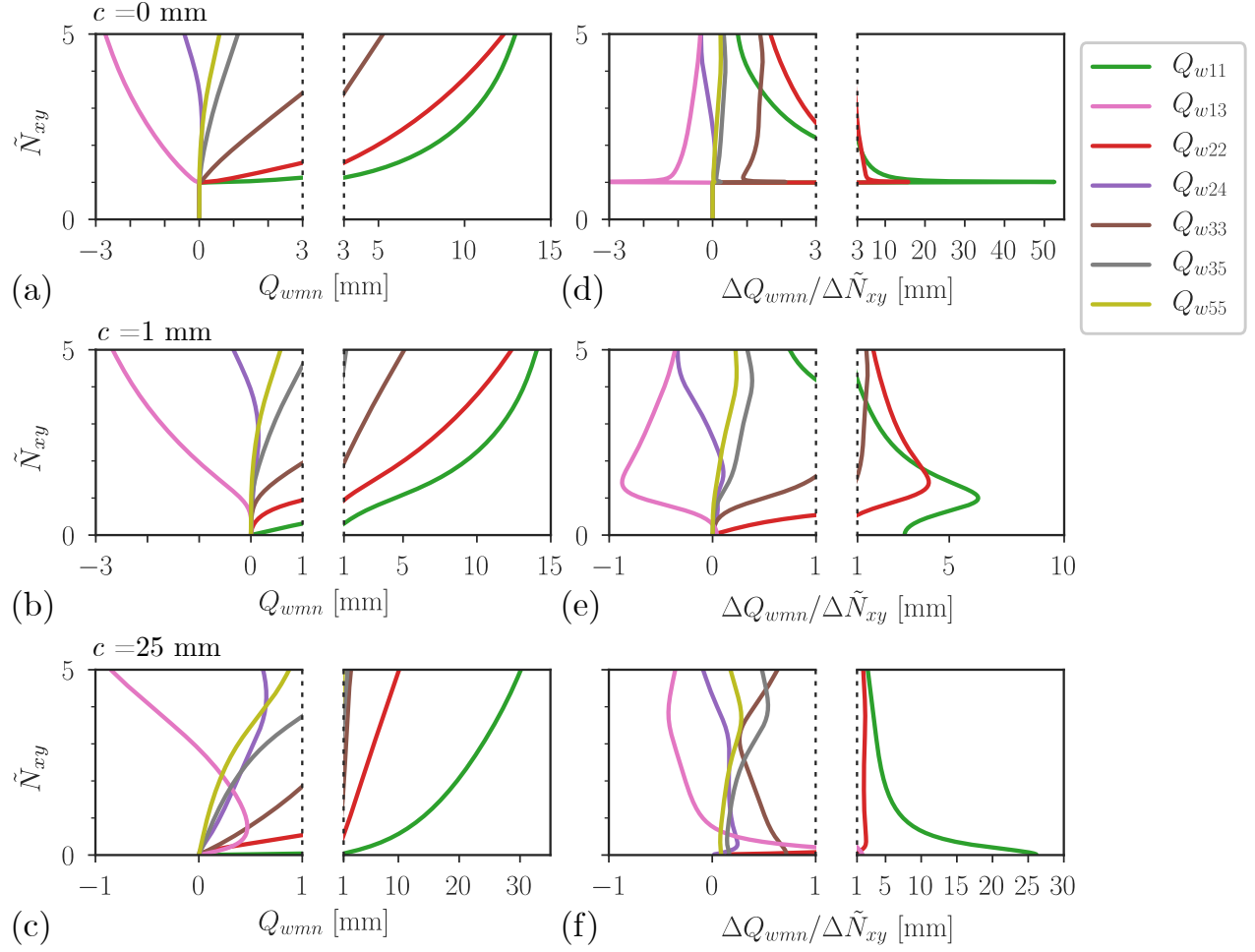


Figure 16: Plots of applied load against Q_{wmn} for $c = 0, 1, 25$ mm in (a)–(c) respectively and load against rate of change of Q_{wmn} with respect to load for $c = 0, 1, 25$ mm in (d)–(f) respectively. All panels have thickness $t = 4$ mm.

when $\tilde{N}_{xy} \approx 4$, the rates of increase in Q_{w35} and Q_{w55} reach a peak simultaneously with a decreased rate in Q_{w24} , signalling the introduction of an additional wave along the compression diagonal. For higher offsets, the modes with even (m, n) combinations are more prevalent, thereby indicating an increased resistance to out-of-plane loading as discussed in the immediately preceding section.

The main differences between the flat plate and the hypar panel are thus most prevalent in the initial stages, where the influence of the initial geometry is at its greatest. At this stage, the application of shear loads along the edges causes an imbalance in the internal forces that result in an out-of-plane deformation from the onset of loading. As loading increases, the hypar response becomes more akin to that of the initially flat plate under in-plane shear indicating that the effects of the shear load overcome the initial imbalance. This suggests that for design purposes, an approach similar to that for designing flat plates under shear may well be suitable, but further research is clearly needed to establish whether any subtle divergences in approach would be necessary.

5. Conclusions

The current study has demonstrated that the behaviour of square shallow doubly-curved panels that are simply-supported along their edges can be determined by considering the initial geometry to be a flat plate with an imposed initial deformation that is subsequently stress-relieved. A nonlinear model based on energy principles to show this has been formulated and compared with results from FE models developed in ABAQUS. The analytical model is evidently capable of accurately predicting the load–deflection behaviour of the panel, as demonstrated by the load case of hyperbolic paraboloidal (hypar) panels of various thicknesses and offsets subject to in-plane shear. In the present load case, a reasonable number of generalized coordinates achieve good correlation of the buckling load and the mechanical response within the elastic post-buckling range. The hypar subject to shear closely follows the behaviour of the flat plate subject to shear, with similar mode contributions taking effect as the load increases. The principal differences lie in the initial stages of loading, where the effects of the initial geometry are most prevalent. Further work is currently being conducted, which builds upon the insights provided by the presented analytical model, to develop practical design guidance for hyperbolic paraboloidal panels. The ultimate aim is for such guidance to be implemented in future revisions of structural design codes.

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References

- [1] Wikimedia Commons / User: Colin / CC BY-SA 4.0. 2017 Inner Harbour Bridge, Copenhagen [Internet]. [cited 08/04/2021]. Available from https://commons.wikimedia.org/wiki/File:Inner_Harbour_Bridge,_Copenhagen_2017-08-16.jpg.
- [2] Alter LCB. 2018 Roller under bridge [Internet]. [cited 08/04/2021]. Available from <http://www.treehugger.com/copenhagens-kissing-bridge-still-cant-kiss-4857203>.
- [3] Candela F. 1958a Understanding the Hyperbolic Paraboloid. *Archit. Rec.* **123**, 191–195.
- [4] Candela F. 1958b Understanding the Hyperbolic Paraboloid: Part II. *Archit. Rec.* **124**, 205–215.
- [5] Reissner E. 1955 On some aspects of the theory of thin elastic shells. *J. Boston Soc. Civil Engineers* **42**, 100–133.
- [6] Ralston A. 1956 On the problem of buckling of a hyperbolic paraboloidal shell loaded by its own weight. *J. Math. Phys.* **35**, 53–59. DOI: 10.1002/sapm195635153.
- [7] Leet KM. 1966 Study of stability in the hyperbolic paraboloid. *J. Eng. Mech.* **92**, 121–142. DOI: 10.1061/JMCEA3.0000707.

- [8] Hees G. 1982 Buckling of hyperbolic paraboloid shells. In *Buckling of Shells. A State-of-the-Art Colloquium. Universität Stuttgart. Institut für Baustatik* pp. 18.1–18.11 Berlin, Heidelberg. Springer. DOI: 10.1007/978-3-642-49334-8_17.
- [9] Leone D, Wang L. 1975 Buckling of hyperbolic paraboloid shells. *J. Eng. Mech.* **101**, 295–299. DOI: 10.1061/JMCEA3.0002022.
- [10] Croll J, Scrivener J. 1969 Edge effects in hyperbolic paraboloid shells. *J. Struct. Div.* **95**, 457–478. DOI: 10.1061/JSDEAG.0002214.
- [11] Janko L. 1991 Buckling of the edge beams of a hyperbolic paraboloid shell supported along the generatrices. *Acta Technica Acad. Sci. Hung.* **104**, 159–185.
- [12] Omurtag M, Aköz A. 1994 Hyperbolic paraboloid shell analysis via mixed finite element formulation. *Int. J. Numer. Methods Eng.* **37**, 3037–3056. DOI: 10.1002/nme.1620371803.
- [13] Gergely P. 1972 Buckling of orthotropic hyperbolic paraboloid shells. *J. Struct. Div.* **98**, 395–399. DOI: 10.1061/JSDEAG.0003139.
- [14] Sofiyev AH, Turan F. 2021 On the nonlinear vibration of heterogenous orthotropic shallow shells in the framework of the shear deformation shell theory. *Thin-Walled Struct.* **161**, 107181. DOI: 10.1016/j.tws.2020.107181.
- [15] Hingnekar DR, Vyavahare AY. 2020 Mechanics of shear resistance in steel plate girder: critical review. *J. Struct. Eng.* **146**, 03120001–1–03120001–16. DOI: 10.1061/(ASCE)ST.1943-541X.0002484.
- [16] EN1993-1-5. 2006 Eurocode 3 - Design of steel structures - Part 1-5: General rules. .
- [17] Martins JP, Ljubinkovic F, Simões da Silva L, Gervásio H. 2018 Behaviour of thin-walled curved steel plates under generalised in-plane stresses: A review. *J. Constr. Steel. Res.* **140**, 191–207. DOI: 10.1016/j.jcsr.2017.10.018.
- [18] Marguerre K. 1938 Zur theorie der gekrümmten platte grosser formänderung. *Proc. 5th Int. Congress Appl. Mech.* pp. 93–101.
- [19] Sanders JL. 1963 Nonlinear theories for thin shells. *Quart. Appl. Math.* **21**, 21–36. DOI: 10.1090/qam/147023.
- [20] Knowles JK, Reissner E. 1960 Torsion and extension of helicoidal shells. *Quart. Appl. Math.* **17**, 409–422. DOI: /10.1090/qam/128149.
- [21] Reissner E, Wan F. 1971 On stretching, twisting, pure bending and flexure of pretwisted elastic plates. *Int. J. Solids Struct.* **7**, 625–637. DOI: 10.1016/0020-7683(71)90118-1.
- [22] Timoshenko SP, Woinowsky-Krieger S. 1959 *Theory of plates and shells*. New York: McGraw-Hill second edition edition.
- [23] Reissner E. 1969 On the Equations of Nonlinear Shallow Shell Theory. *Stud. Appl. Math.* **48**, 171–175. DOI: 10.1002/sapm1969482171.
- [24] Flügge W. 1973 *Stresses in shells*. Berlin, Heidelberg: Springer-Verlag second edition edition. DOI: 10.1007/978-3-662-01028-0.
- [25] Wadee MA, Farsi M. 2014 Local–global mode interaction in stringer-stiffened plates. *Thin-Walled Struct.* **85**, 419–430. DOI: 10.1016/j.tws.2014.09.012.
- [26] Bai L, Wadee MA. 2015 Imperfection sensitivity of thin-walled I-section struts susceptible to cellular buckling. *Int. J. Mech. Sci.* **104**, 162–173. DOI: 10.1016/j.ijmecsci.2015.10.010.
- [27] Williams DG, Aalami B. 1979 *Thin plate design for in-plane loading*. London: Granada.

- [28] Stein M, Neff J. 1947 Buckling stresses of simply supported rectangular flat plates in shear. *NACA Tech. Note 1222*.
- [29] Thompson JMT, Hunt GW. 1973 *A general theory of elastic stability*. London: Wiley.
- [30] Doedel EJ, Oldeman BE. 2009 AUTO-07p: Continuation and bifurcation software for ordinary differential equations. Technical report. Montreal, Canada: Department of Computer Science, Concordia University.
- [31] Dassault Systèmes Simulia Corp. 2019 ABAQUS/Standard, Version 2019. Providence, USA.
- [32] Riks E. 1979 An incremental approach to the solution of snapping and buckling problems. *Int. J. Solids Struct.* **15**, 529–551. DOI: 10.1016/0020-7683(79)90081-7.
- [33] Timoshenko SP, Gere JM. 1961 *Theory of elastic stability*. New York: McGraw-Hill.
- [34] Koiter WT. 1963 Introduction to the post-buckling behaviour of flat plates. *Colloque sur le Comportement Postcritique des utilisées en Construction Metallique, Mémoires de la société Royale des Sciences de Liege, tome VIII, Liege* **8**, 17–35.