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TOPICS ARISING FROM 3×3 BIMATRIX GAMES

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ABSTRACT

This thesis is devoted to the study of bimatrix games, in particular examples of 3×3 bimatrix games governed by the Best Response Dynamics or coupled replicator equations.

In Chapter 2, we present an approach to Hofbauer's conjecture, positing that convergence of bimatrix games to the unique and completely mixed Nash equilibrium under the Best Response Dynamics occurs only for zero-sum games. We present partial results concerning this conjecture and inter alia provide some dynamical characterization of zero-sum games. In section 3.7, we provide a presumptive evidence that Hofbauer's conjecture might be incorrect.

In Chapter 3, we consider an example of a cyclic competition bimatrix game, a Rock-Scissors-Paper game. We investigate shadowing of the non-transverse heteroclinic network naturally appearing in this model, as well as the stability of some peculiar subcycles. Moreover, we present our numerical investigations and provide our presumptions about the way chaos might be born in this model.

In Chapter 4, we consider a toy-model motivated by the Rock-Scissors-Paper game. The toy-model consists of a 2-dimensional center manifold of a saddle-elliptic point in $\mathbb{R}^4$ and 3-dimensional stable/unstable manifolds (of this center manifold) which coincide. We derive the normal form of the ODE-governed toy-model being considered near the center manifold and the normal form of the return map to the vicinity of the center manifold along the non-transverse homoclinic channel. We investigate in detail the dynamics of the return map.

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$(R,S) \to (R,R)$, $(R,S) \to (R,P)$, $(R,P) \to (P,P)$, $(R,P) \to (S,P)$, $(S,P) \to (S,S)$, $(S,P) \to (S,R)$,
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Caesarem vehis, Caesarisque fortunam.

Gaius Iulius Caesar

1

PRELIMINARIES

1.1 - INTRODUCTION TO THIS CHAPTER

In this chapter we give a necessary introduction to the considered models in this thesis, we will also recall some well known results from the literature used throughout the following chapters.

In the section 1.2 we provide historical notes about the Game Theory, Evolutionary Game Theory and learning dynamics. We refer to the following publications and books for more details [33], [34], [93], [118], [111], [56], [57].

Sections 1.2 - 1.9 are based on [105], [109], [82], [84], [80], [52], [3], [86]. These sections summarize the necessary, in the context of this thesis, definitions and results from the literature about the Best Response Dynamics for bimatrix games. We recommend [82] and [105] as the perfect introduction to the theory of Best Response Dynamics and Fictitious Play.

Sections 1.8 and 1.10, summarize the key results from the papers [105], [109], about the existence and uniqueness of the solutions to the Best Response differential inclusion and also about the Shapley system.

Section 1.11 based on [94], [95], [96] contains introduction to the coupled replicator equations along with basic results about it in the context of bimatrix games.

Section 1.12 introduces (after [79]) Rock-Scissors-Paper game governed by the coupled replicator

equations.

Section 1.13 consists of the main results from [54] and [20] describing the relations between the Best Response Dynamics and the coupled replicator equations.

1.2 - Preface

As a new field of mathematics, Game Theory first arose with publication of the paper ”Zur Theorie der Gesellschaftsspiele” in 1928 (by John von Neumann [77]) containing proof of the minimax theorem for the zero-sum games. This article was the basis for the book ”Theory of Games and Economic Behavior” (by von Neumann and Oskar Morgenstern in 1944 [78]), in which they extended the work of J.von Neumann on zero-sum games and laid the foundations for the study of cooperative games. Among the earlier works are those of A. Cournot on quantity competition [23], J. Bertrand on price competition [10], C.L. Dodgson on the application of zero-sum games to the political problem of parliamentary representation [30], E.Zermelo on winning strategies in the game of chess [119].
Soon after the appearance of von Neumann’s and Morgenstern’s book in 1951, John Nash (in his PhD thesis [76]) introduced the notion of an equilibrium (later called after him a ”Nash equilibrium”) for a game which since then has become a crucial object in investigation of Game Theory. Inter alia, Nash has proven that in bimatrix games the Nash equilibrium always exists. For his contributions to Economic Science, John Nash was awarded a Nobel Prize in 1994, together with John C. Harsanyi and Reinhard Selten ”for their pioneering analysis of equilibria in the theory of non-cooperative games”.
In the second half of the 20th century, Game Theory turned into a field, providing essential methods in the analysis of interactions of individuals or populations, and making it possible to formalize decision-making models. Inter alia, Game Theory has become a new language for the description of economics problems. This is mainly due to the work of Harsanyi on modelling games with incomplete information [44] and Reinhard Selten [100] on (sub)game perfect Nash equilibrium. Within the last 40 years, many problems in economics have been translated into the language of Game Theory.

The approach by von Neumann and Morgenstern was a ”static solution concept”, as described in their introduction to [78] and they stressed the need to formulate a ”dynamic approach” towards Game Theory. Nash in [76] sketched a ”mass action approach” to the notion of Nash equilibrium, which later, in the Seventies, gave rise to Evolutionary Game Theory with work [89] on the dynamical changes in population models utilizing the theory of von Neumann and Morgenstern as its main tool. As in

Game Theory, where one can view the setting to consist of players interacting with each other and trying to maximize their own payoff by adjusting their own strategy to their opponents behaviour, in Evolutionary Game Theory one considers groups (populations) formed by players switching between different strategies and adjusting them in a way maximizing the payoff for their group. In Evolutionary Game Theory it is usually assumed that the payoff depends on the proportions of players within each group choosing different strategies and vice versa, proportions of chosen strategies within the groups depending on the received payoffs.

However, at the beginning, finding the Nash equilibria of the games was computationally expensive (though some effective algorithms appeared later, e.g. [65]). Proposing an algorithm for finding equilibria and solutions to the games, G.Brown introduced [12], [13] Fictitious Play dynamics 1.7.2 (which up to the time reparameterization coincides with the Best Response Dynamics 1.7.1). In 1951, J.Robinson [90], proved that in the zero-sum (also called 'fair') games, Best Response Dynamics converge to the set of Nash equilibria. Later on, this convergence property (also known as the Fictitious Play property) was also shown for generic $2 \times n$ non-zero sum games, potential games, dominance-solvable games and, various classes of supermodular games. In these games, however, the Best Response Dynamics converge to the Nash equilibrium on the boundary of the phase space. In the case of non-zero sum games, an example of a bimatrix game converging to the completely mixed (internal) and unique Nash equilibrium (def. 4) has not yet been found.
If the dynamics do not converge to the Nash equilibrium, one can investigate the intermediate, transient dynamics or asymptotic dynamics, as the latter two can be significantly different (see [88], [58]). In 1964, L.Shapley introduced a non-zero sum cyclic bimatrix game (where there is no single best strategy for any player) for which the flow under the Best Response Dynamics possesses a stable periodic orbit. Apart from this, there are more examples of games for which the Best Response Dynamics does not converge to the set of Nash equilibria. For example, in 1993 Jordan [59] constructed a $2 \times 2 \times 2$ game for which the Best Response Dynamics has an asymptotically stable periodic orbit. Also in 1992, Cowan [24] presented an example of a game, together with numerical simulations, which suggested that Best Response Dynamics for this game possesses subshifts of finite type. Within the last decade, the Shapley family was revisited in [105], [109], where it was proven that in the Best Response Dynamics, even for a simple game with cyclic structure (as in the Rock-Scissors-Paper game) a very complicated behaviour can occur, e.g. chaos described by 'random walk - like' dynamics, or convergence to the equilibrium in a finite time.

In the following decades after publication [12], several other variations of classical Best Response Dynamics were introduced, e.g. the payoffs are affected by random perturbations or players instead of playing best responses, chose strategies where probabilities increase with the corresponding payoffs. In addition one can also consider the so-called Better Reply Dynamics [29].

Although initially introduced as an algorithm for finding the solution to games, Fictitious Play (Best Response Dynamics 1.7.1) later on became regarded as a model of a learning algorithm where the players interact with each other. In the case of Best Response Dynamics however, the players chose strategies which maximize their payoff instead of strategies that will lead the game towards the Nash equilibrium. (This was recently investigated in [83] when in the Best Response Dynamics for bimatrix games, payoff for players from playing mixed strategies outperforms payoff from playing the Nash equilibrium.) This kind of learning algorithm is called 'myopic learning' since the players are only interested in maximizing their own payoff without directly influencing the future actions of their opponents [19].

Apart from the Best Response Dynamics, another example of learning in games which we will investigate in this thesis are bimatrix games governed by coupled replicator equations - firstly introduced in [55], [98] in order to study the interaction between two populations of players. It is assumed in this model that there are no internal interactions within the populations.

In recent years, learning dynamics has received a lot of attention due to its application e.g. in biology and economics. There are various models considered since learning procedures differ from one model to another. In addition, an important assumption is about the loss of memory during the game. Due to possible memory loss, recent actions have a bigger impact on players' choices. Many amongst learning processes considered in the literature are similar to Fictitious Play dynamics in the sense of repeating a finite game.

Game Theory, learning dynamics and learning algorithms, etc. from their very beginning turned out to be very successful in their analysis of many problems arising in many different fields of science.

In biology, they are used in modelling ecological and evolutionary interactions of populations, for example analysis of certain predator-prey models that have revealed hidden patterns in large-scale animal behaviour.

In economics, they describe the interactions of market participants.

In psychology, social and political sciences, they describe the behaviour of human individuals, whole societies, and even decision-making processes at an international level.

Finally, in computer science and artificial intelligence they are successfully applied in the: modelling and design of computer networks with complex interaction patterns between large numbers of (electronic) agents, in the analysis and design of automatic electronic trading systems and also the design and management of wireless networks consisting of large numbers of nodes sharing common resources (such as bandwidth). [82]

1.3 - Bimatrix games

This thesis is devoted to the study of the dynamical behaviour of bimatrix games under Best Response Dynamics and coupled replicator equation. For this reason, we need to introduce a mathematical formalism in the context of bimatrix games. Let us start with explaining the necessary terminology used throughout the thesis (see [35]).

A **strategy** is any of the options a player can choose in a setting where the outcome depends not only on his own actions but on the actions of others. A player's strategy will determine the action the player will take at any stage of the game.

A **pure strategy** provides a complete definition of how a player will play a game. In particular, it determines the move a player will make in any situation they could face. A player's **strategy set** is the set of pure strategies available to that player.

In any game, **payoffs** are numbers which represent the motivations of players. Payoffs may represent profit, quantity, utility, or other continuous measures (cardinal payoffs), or may simply rank the desirability of outcomes (ordinal payoffs). In all cases, the payoffs must reflect the motivations of the particular player.

We can now introduce the formal definition of a finite game (see e.g. [66]):

Definition 1. *A finite game in normal (or strategic) form is a tuple* $\Gamma = (I; \{S^i\}_{i\in I}, \{u^i\}_{i\in I})$*, where* $N \in \mathbb{N}$ *and* $I = \{1, ..., N\}$ *is a (finite) collection of players.*

- *For every* $i \in I$*, there exists* $n_i \in \mathbb{N}$ *such that the set* $S^i = \{1, ..., n_i\}$ *is a (finite) collection of pure strategies of player i (the strategy set)*
- *For every* $i \in I$*, the function* $u^i : S^1 \times ... \times S^N \to \mathbb{R}$ *is the payoff function of player i.*

  The space of strategy tuples is denoted by $S = S^1 \times ... \times S^N$ *and an element* $s \in S$ *is called a*

(pure) strategy profile.

One of the most famous examples of a finite game is the Prisoner's dilemma (see [11]):

Example 1. ***(Prisoner's dilemma)***
Two prisoners are accused of a crime. Each prisoner is given the opportunity either to either inform on the other by testifying that the latter committed the crime, or to conspire with the other to remain silent. If one stays silent (conspires) and the other informs, the informer will be released immediately while the conspirer will spend 3 years in prison. If neither informs on the other, each will be held for one year. If both conspire with the other, they will each be jailed for 2 years. They cannot communicate with one another. Given that neither prisoner knows whether the other has informed on them, it is in the self-interest of each to betray. Paradoxically, if each prisoner pursues his self-interest, both end up worse off than they would be if they acted otherwise.
Hence the normal form of this game is $\Gamma = (\{A, B\}, {S^i}_{i\in\{A,B\}}, {u^i}_{i\in\{A,B\}})$, *where* $S^A = S^B = \{1, 2\}$ *after identifying players:* $A = 1$, $B = 2$ *and pure strategies:* $cooperates = 1$ *and* $defects = 2$.
The payoff functions are defined as follows

$$u^A(1,1) = 1, \quad u^A(1,2) = 3, \quad u^A(2,1) = 0, \quad u^A(2,2) = 2$$

$$u^B(1,1) = 1, \quad u^B(1,2) = 0, \quad u^B(2,1) = 3, \quad u^A(2,2) = 2$$

Throughout the whole thesis we will follow two key assumptions:

Assumption 1. *We assume that each player* $i \in I$ *adjusts their strategy (in a specified way) in order to maximize received payoff.*

Assumption 2. *We assume that players do not restrict themselves only to the pure strategies from the set* S^i*, but also use mixed strategies.*

A **mixed strategy** (see [35]) is the assignment of a probability to each pure strategy. Since probabilities are continuous, there are an infinite number of mixed strategies available to a player.
Let us formalize this by defining the following objects:

Definition 2. *We denote by* $\Sigma^i := \{x \in \mathbb{R}^{n_i} \mid x_k \geq 0, \ \Sigma_{k=1}^{n_i} x_k = 1\}$ *- the* $n_i - 1$ *dimensional simplex, to represent probability distributions over the ith player's pure strategies. Each element from* Σ^i *is called a mixed strategy of player i.*

Any vector $e_j = (\delta_{j1}, ..., \delta_{jn_i}) \in \Sigma^i$, where δ_{jk} is a Kronecker's delta, standing for a pure strategy.

A **strategy profile** (sometimes called a strategy combination, see [35]) is a set of strategies for all players which fully specifies all actions in a game. A strategy profile must include one and only one strategy for every player. By $\Sigma := \Sigma^1 \times ... \times \Sigma^N$ we mean the set of **mixed strategy profiles**.

The payoff function u^i for player i can be extended linearly from set S onto the set of mixed strategies Σ. If $\sigma = (\sigma_1, ..., \sigma_N) \in \Sigma$ is a mixed strategy profile with a mixed strategy $\sigma_i = (\sigma_i^1, ..., \sigma_i^{n_i}) \in \Sigma^i$ for each $i \in I$, then the payoff for the player i can be taken as:

$$\tilde{u}^i(\sigma) = \tilde{u}^i(\sigma_1, ..., \sigma_N) := \Sigma_{k_1=1}^{n_1} ... \Sigma_{k_N=1}^{n_N} \sigma_1^{k_1} ... \sigma_N^{k_N} u^i(k_1, ..., k_N) \tag{1.3.1}$$

This thesis concentrates on **bimatrix games**, with a particular interest in the 3×3 bimatrix games. To introduce the notion of a bimatrix game we need to make an additional assumption about the payoff functions of the game

Assumption 3. *The payoff functions u^1, u^2 are defined for each pair of pure strategies $(i, j) \in \{1, ..., m\} \times \{1, ..., n\}$ and extended linearly onto Σ.*

So for $(i, j) \in \{1, ..., m\} \times \{1, ..., n\}$ let us denote:

$$a_{ij} := u^1(i, j), \quad b_{ij} := u^2(i, j) \tag{1.3.2}$$

and

$$A := (a_{ij}), \quad B := (b_{ij}) \in \mathbb{R}^{m \times n} \tag{1.3.3}$$

So there are two players competing (in due course called agents X and Y, or players A and B, etc.) and the payoff function of each of them is decoded by the payoff matrix (see [110], [107]), i.e. denoted by $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{m \times n}$, with $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$. So player A can choose among m pure strategies, B among n. We will also denote $\Sigma_A := \Sigma^1$ and $\Sigma_B := \Sigma^2$.

By the definition (1.3.3) of matrices A and B we have that:

$$u^1(i, j) = e_i^T A e_j, \quad u^2(i, j) = e_i^T B e_j \tag{1.3.4}$$

for $e_i \in \Sigma^1$ and $e_j \in \Sigma^2$. So, the (i, j) entry of the matrix A (matrix B) is the payoff given to player A (resp. B), when the players are playing strategy profile (i, j).

Assumption 4. *As is usually assumed, we take vector $p \in \Sigma_A$ as a row vector in $\mathbb{R}^{1 \times m}$, in order to*

simplify the notation.

By the formula (1.3.1), we obtain directly that for $(p, q) \in \Sigma_A \times \Sigma_B$

$$u^A(p,q) := u^1(p,q) = pAq, \quad u^B(p,q) := u^2(p,q) = pBq \tag{1.3.5}$$

With such encoding, the actions of the players can be identified as follows: player A chooses a row from the matrices A and B and player B selects a column from A and B.

Example 2. *An example of a (cyclic competition) bimatrix game which we will deeply investigate in chapters 3 and 4 is the Rock-Scissors-Paper game* (A, B^T)*, with* A, B *the matrices*

$$A = \begin{pmatrix} \epsilon_X & -1 & 1 \\ 1 & \epsilon_X & -1 \\ -1 & 1 & \epsilon_X \end{pmatrix} \qquad B = \begin{pmatrix} \epsilon_Y & -1 & 1 \\ 1 & \epsilon_Y & -1 \\ -1 & 1 & \epsilon_Y \end{pmatrix} \tag{1.3.6}$$

with $\epsilon_X, \epsilon_Y \in (-1, 1)$*, the rewards for ties.*
Since Rock beats Scissors, Scissors beat Paper and Paper beats Rock, the payoffs for the 1st player A are: $u^1(i, j) = \epsilon_X$ *if* $i = j$*,* $u^1(i, j) = -1$ *if* $j - i \equiv 1 \ (mod\ 3)$*,* $u^1(i, j) = 1$ *if* $j - i \equiv 2 \ (mod\ 3)$*, for* $i, j = 1, 2, 3$*. We identify strategies:* $Rock = 1$*,* $Scissors = 2$*,* $Paper = 3$*.*

Example 3. *A variant of a Rock-Scissors-Paper game which we introduce in section 1.10 of this chapter is the Shapley family of one parameter cyclic matrices* (A_β, B_β)

$$A_\beta = \begin{pmatrix} 1 & 0 & \beta \\ \beta & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix} \quad B_\beta = \begin{pmatrix} -\beta & 1 & 0 \\ 0 & -\beta & 1 \\ 1 & 0 & -\beta \end{pmatrix} \tag{1.3.7}$$

for $\beta \in (-1, 1)$*.*

1.4 - Best Response

In the setup of bimatrix games, we define the notion of **Best Response** which corresponds to the strategy maximizing the player's payoff (see [25]), i.e.:

Definition 3. *For $(p, q) \in \Sigma$, the following set-valued functions*

$$BR_A(q) := \arg\max_{\bar{p}\in\Sigma_A} \bar{p}Aq \quad \text{and} \quad BR_B(p) := \arg\max_{\bar{q}\in\Sigma_B} pB\bar{q} \tag{1.4.1}$$

are called the best responses to strategies $q \in \Sigma_B$ and $p \in \Sigma_A$, respectively.

Example 4. *In the bimatrix game given by the Shapley family of matrices (3) we have that for $(p, q) \in \Sigma$:*

$$BR_{A_\beta}(q) = \arg\max_{\bar{p}\in\Sigma_A} \bar{p}A_\beta q =$$

$$\{\bar{p} = (\bar{p}_1, \bar{p}_2, \bar{p}_3) \mid \bar{p} \text{ maximizes } \bar{p}A_\beta q = \bar{p}_1 \cdot (q_1 + \beta q_3) + \bar{p}_2 \cdot (q_2 + \beta q_1) + \bar{p}_3 \cdot (q_3 + \beta q_2)\}$$

So

$$BR_{A_\beta}(q) = \begin{cases} e_1 = (1,0,0), & \text{if} \quad q_1 + \beta q_3 \geq max(q_2 + \beta q_1, q_3 + \beta q_2) \\ e_2 = (0,1,0), & \text{if} \quad q_2 + \beta q_1 \geq max(q_1 + \beta q_3, q_3 + \beta q_2) \\ e_3 = (0,0,1), & \text{if} \quad q_3 + \beta q_2 \geq max(q_1 + \beta q_3, q_2 + \beta q_1) \\ \{\lambda e_1 + (1-\lambda)e_2 = (\lambda, 1-\lambda, 0), & \text{for any } \lambda \in [0,1]\} \text{ if} \quad q_1 + \beta q_3 = q_2 + \beta q_1 \geq q_3 + \beta q_2 \\ \{\lambda e_2 + (1-\lambda)e_3 = (0, \lambda, 1-\lambda), & \text{for any } \lambda \in [0,1]\} \text{ if} \quad q_2 + \beta q_1 = q_3 + \beta q_2 \geq q_1 + \beta q_3 \\ \{\lambda e_3 + (1-\lambda)e_1 = (1-\lambda, 0, \lambda), & \text{for any } \lambda \in [0,1]\} \text{ if} \quad q_1 + \beta q_3 = q_3 + \beta q_2 \geq q_2 + \beta q_1 \\ \Sigma_A \text{ if} \quad q_1 + \beta q_3 = q_3 + \beta q_2 = q_2 + \beta q_1 \end{cases} \tag{1.4.2}$$

Remark 1. *In general, from the definition 1.4.1 it is easy to see that whenever BR_A (resp. BR_B) is single-valued (i.e. it is a vector), then it is equal to one of the pure strategies $e_i \in \Sigma_A$ (resp. $e_j \in \Sigma_B$). Moreover if $BR_A(q)$ is single-valued then for $q' \in \Sigma_B$ close enough to q, one has $BR_A(q) = BR_A(q')$. So the best response function is locally constant on the set where it is single-valued.*

If the best response is not unique, it is a convex hull of all of the pure strategies belonging to this best response set. In the other words:

Remark 2.

$$BR_A(q) = conv\Big(\{e_k \in \Sigma_A \mid e_k Aq = \max_{i\in\{1,\dots,m\}} (Aq)_i,\ k \in \{1, \dots, m\}\}\Big)$$

$$BR_B(p) = conv\Big(\{e_l \in \Sigma_B \mid pBe_l = \max_{j\in\{1,\dots,n\}} (pB)_j,\ l \in \{1, \dots, n\}\}\Big)$$

In conclusion, BR_A, BR_B are always the faces of the simplices Σ_A and Σ_B, respectively.

Each preimage of a pure strategy by the best response function is either an empty set or a polytope. Hence we can divide Σ_A and Σ_B into polytopes in which the given pure strategies e_i and e_j are preferable by the players A and B, respectively:

$$R_i^A := BR_A^{-1}(e_i) \subset \Sigma_B, \text{ for } i \in \{1, ..., m\}$$

$$R_j^B := BR_B^{-1}(e_j) \subset \Sigma_A, \text{ for } j \in \{1, ..., n\}$$

We define also the sets:

$$R_{ij} := R_j^B \times R_i^A \subset \Sigma_A \times \Sigma_B$$

on which the joint best response (BR_A, BR_B) is single valued. We introduce as well the indifference hyperplanes (compare with [69], [105]) on which the best response function is not defined uniquely

$$\begin{aligned} Z_{ik}^A &:= \{q \in \Sigma^B \mid (Aq)_i = (Aq)_k \geq (Aq)_l, \text{ for } l \in \{1, ..., m\}\} \subset \Sigma^B \\ Z_{jh}^A &:= \{p \in \Sigma^A \mid (pB)_j = (pB)_h \geq (pB)_l, \text{ for } l \in \{1, ..., n\}\} \subset \Sigma^A \end{aligned} \tag{1.4.3}$$

For generic matrices A and B these are codimension one affine subspaces within Σ_B and Σ_A.

Remark 3. *From the definition it follows directly that $e_i, e_k \in BR_A(q)$ for $q \in Z_{ik}^A$, and also $e_j, e_h \in BR_B(p)$ for $p \in Z_{jh}^B$. In the other words:*

$$Z_{ik}^A = R_i^A \cap R_k^A \quad \textit{and} \quad Z_{jh}^A = R_j^B \cap R_h^B$$

Example 5. *Due to example 4 we can easily find implicit formula for the sets R_i^A (for R_j^B it works in analogous way) for the Shapley family with $\beta \in (0, 1)$.*

$$R_1^A = BR_A^{-1}(e_1) = \{q = (q_1, q_2, q_3) \in \Sigma_B \mid q_1 + \beta q_3 \geq max(q_2 + \beta q_1, q_3 + \beta q_2)\}$$

$$R_2^A = BR_A^{-1}(e_2) = \{q = (q_1, q_2, q_3) \in \Sigma_B \mid q_2 + \beta q_1 \geq max(q_1 + \beta q_3, q_3 + \beta q_2)\}$$

$$R_3^A = BR_A^{-1}(e_3) = \{q = (q_1, q_2, q_3) \in \Sigma_B \mid q_3 + \beta q_2 \geq max(q_1 + \beta q_3, q_2 + \beta q_1)\}$$

Moreover we can explicitly write the formulas for the indifference sets Z_{ik}^A:

$$Z_{12}^A = \lambda \cdot (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) + (1 - \lambda) \cdot (\frac{1}{2 - \beta}, \frac{1 - \beta}{2 - \beta}, 0)$$

$$Z_{23}^A = \lambda \cdot (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) + (1-\lambda) \cdot (0, \frac{1}{2-\beta}, \frac{1-\beta}{2-\beta})$$

$$Z_{31}^A = \lambda \cdot (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) + (1-\lambda) \cdot (\frac{1-\beta}{2-\beta}, 0, \frac{1}{2-\beta})$$

where $\lambda \in [0,1]$.

In figure 1.1, the sets R_i^A, R_j^B, Z_{ik}^A, Z_{jh}^B *are presented for the Shapley system with* $\beta \in (0,1)$.

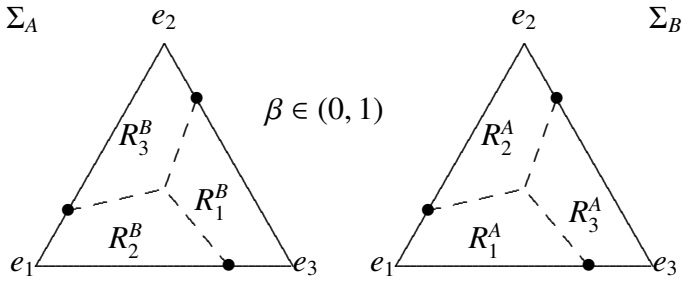


Figure 1.1: The simplices Σ_A and Σ_B and the sets R_i^A, R_j^B, Z_{ik}^A, Z_{jh}^B for the Shapley family with $\beta \in (0,1)$ (see example 5). The sets Z_{ij}^A in Σ_B are connecting the point $E^A = (1/3, 1/3, 1/3)$ to the points $\frac{1}{(2-\beta)}(1, 1-\beta, 0)$ $\frac{1}{(2-\beta)}(1-\beta, 0, 1)$ and $\frac{1}{(2-\beta)}(0, 1, 1-\beta)$. $Z_{i,j}^B$ in Σ_A is connecting the point $(1/3, 1/3, 1/3)$ to the points $\frac{1}{(2+\beta)}(1+\beta, 1, 0)$, $\frac{1}{(2+\beta)}(1, 0, 1+\beta)$ and $\frac{1}{(2+\beta)}(0, 1+\beta, 1)$ (see also[105]).

1.5 - Nash equilibrium

The following object is crucial for investigations in the theory of bimatrix games (see more in [18]):

Definition 4. *Nash equilibrium is a mixed strategy profile* $(\bar{p}, \bar{q}) \in \Sigma$ *such that*

$$\bar{p} \in BR_A(\bar{q}) \quad and \quad \bar{q} \in BR_B(\bar{p}) \tag{1.5.1}$$

Definition 5. *If* (E^A, E^B) *is a Nash Equilibrium and* $(E^A, E^B) \in int(\Sigma)$ *then it is called a completely mixed Nash equilibrium.*

An equivalent definition of the Nash equilibrium would be (see [33]):

Definition 6. (x^*, y^*) *is a Nash equilibrium of the system if and only if* $(x^*)^T A y^* > x A y^*$ *and* $(y^*)^T B x^* > y^T B x^*$, *for all* $x \in \Sigma_A \backslash \{x^*\},\ y \in \Sigma_B \backslash \{y^*\}$.

Remark 4. *The notion of the equilibrium was firstly introduced by John Nash in [76], where he proved also that the Nash equilibrium exists for every bimatrix game. Nash equilibrium, however, does not have to be unique.*

For a bimatrix game, there might exist a finite set of Nash equilibria or it might be a continuum (e.g. in a trivial example when each mixed strategy is a Nash equilibrium).

The idea behind this notion is that none of the players gain profit with change of its strategy given the opponent keeps playing the same strategy. In [83], it was investigated under what assumptions on game (A, B) playing or avoiding Nash equilibrium results in a possibly higher payoff for the players.

Remark 5. *A point (x^*, y^*) is a Nash equilibrium if and only if it satisfies the following minimization problem [68], [86]:*

$$x^T \cdot (A + B) \cdot y - p - q$$

subject to

$$Ay \dot{\geq} p\mathbb{1} \qquad B^T x \dot{\geq} q\mathbb{1}, \quad x \in \Sigma_A, \quad y \in \Sigma_B$$

where $\dot{\geq}$ means the component-wise order and $\mathbb{1}$ is a vector which entries are all equal to 1.

Although, the first parts of the chapter 3, will deal with the general case of $A, B \in \mathbb{R}^{m\times n}$ under the assumption of the existence of completely mixed Nash equilibrium, for the remaining part of the thesis we will restrict ourselves to the case $n = m = 3$.

When $n = m$, we will utilize the following lemma to assure the existence of a unique Nash equilibrium (compare with [105], [109]):

Lemma 1. *A point $(E^A, E^B) \in int(\Sigma)$ is a unique Nash Equilibrium of a bimatrix game (A, B), with $A, B \in \mathbb{R}^{n\times n}$, if and only if for all $i, i_0, j, j_0 \in \{1, ..., n\}$*

$$(AE^B)_i = (AE^B)_{i_0} \quad and \quad (E^A B)_j = (E^A B)_{j_0}$$

Hence $(E^A, E^B) \in R_{ij}$ and $E^A \in Z^B_{ii_0}$, $E^B \in Z^A_{jj_0}$, for all i, i_0, j, j_0.

Lemma 2. ***(The complement of the indifference sets [105])***

Assume that $A, B \in \mathbb{R}^{n\times n}$ are non-singular. Then the set where one player is indifferent between two given strategies (and so where she has a choice of best responses) forms a codimension-one hyperplane and for each $i, j \in \{1, ..., n\}$ the set

$$S_{ij} := \{(p^A, p^B) \in \Sigma_A \times \Sigma_B \mid BR_A(p^B) = e_i \ \ and \ \ BR_B(p^A) = e_j\}$$

is convex. If there exists a Nash equilibrium (E^A, E^B) in the interior of Σ, then all n^2 regions S_{ij} are non-empty, and their closures meet at (E^A, E^B).

Example 6. *From Lemma 1 it follows immediately that for the Shapley family there exists a unique Nash equilibrium -* $(E^A, E^B) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

1.6 - Linear equivalence of bimatrix games

In the next section, we will see that different bimatrix games (A, B) can lead to the same dynamics, i.e. the history of played game. For this reason it is essential to introduce the following equivalence relation (see also [6]):

Definition 7. *We say that two* $m \times n$ *bimatrix games* (A, B) *and* $(\tilde{A}, \tilde{B})$ *are* **(linearly) equivalent**, $(A, B) \sim (\tilde{A}, \tilde{B})$, *if the matrix* $\tilde{A}$ *can be obtained by multiplying* A *with a positive constant* $c > 0$ *and adding constants* $c_1, \ldots, c_n \in \mathbb{R}$ *to the matrix columns, and* $\tilde{B}$ *can be obtained from* B *by multiplication with* $d > 0$ *and addition of* $d_1, \ldots, d_m \in \mathbb{R}$ *to its rows:*

$$\tilde{a}_{ij} = c \cdot a_{ij} + c_j \quad \text{and} \quad \tilde{b}_{ij} = d \cdot b_{ij} + d_i \qquad \text{for } i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\}$$

Example 7. *Let* (A_β, B_β) *be the Shapley family of matrices for* $\beta \in (-1, 1)$. *Let* $\tilde{B}_\beta := -A_\beta$. *Then* $(A_\beta, B_\beta) \sim (A_\beta, \tilde{B}_\beta)$ *if and only if* $\beta = \frac{\sqrt{5}-1}{2}$. *Indeed, let us write:*

$$\begin{pmatrix} -1 & 0 & -\beta \\ -\beta & -1 & 0 \\ 0 & -\beta & -1 \end{pmatrix} = \tilde{B}_\beta = d \cdot B_\beta + \begin{pmatrix} d_1 & d_1 & d_1 \\ d_2 & d_2 & d_2 \\ d_3 & d_3 & d_3 \end{pmatrix} = \begin{pmatrix} -\beta \cdot d + d_1 & d_1 + d & d_1 \\ d_2 & d_2 - \beta \cdot d & d_2 + d \\ d_3 + d & d_3 & d_3 - \beta \cdot d \end{pmatrix} \tag{1.6.1}$$

Hence $-d = d_1 = d_2 = d_3 = -\beta$ *and moreover* $-1 = -\beta \cdot d + d_1 = -\beta^2 - \beta$. *Taking into account* $\beta \in (-1, 1)$, *we find that only for* $\beta = \frac{\sqrt{5}-1}{2}$ *the linear equivalence* $(A_\beta, B_\beta) \sim (A_\beta, \tilde{B}_\beta)$ *is satisfied.*

The following corollary is a direct consequence of the definitions provided in the previous sections:

Corollary 1. *If two* $m \times n$ *bimatrix games* (A, B) *and* $(\tilde{A}, \tilde{B})$ *are (linearly) equivalent, then all their best responses, preference regions and indifference sets coincide, that is,* $BR_A = BR_{\tilde{A}}$, $BR_B = BR_{\tilde{B}}$, $R_i^{\tilde{A}} = R_i^A$, $R_j^{\tilde{B}} = R_j^B$, $Z_{ii_0}^{\tilde{A}} = Z_{ii_0}^A$, $Z_{jj_0}^{\tilde{B}} = Z_{jj_0}^B$, *for all* i, i_0, j, j_0. *Moreover, the set of the Nash equilibria for both games coincide.*

Later on our investigations will be devoted to understanding differences in games separated into two complimentary families (see also [21]):

Definition 8. *We say that a $m \times n$ bimatrix game (A, B) is **zero-sum** if either $A + B = 0$ or there exists a game $(\tilde{A}, \tilde{B})$ linearly equivalent to (A, B) such that $\tilde{A} + \tilde{B} = 0$.*
*A $m \times n$ bimatrix game (A, B) is **not zero-sum** if there is no linearly equivalent game to (A, B) which is zero-sum.*

Remark 6. *In fact in the latter example we have proven that the Shapley family is equivalent to a zero-sum game for $\beta = \frac{\sqrt{5}-1}{2}$.*

1.7 - Fictitious Play and the Best Response Dynamics

We define the continuous **Best Response Dynamics** (shortly BR or BRD) as the following differential inclusion (see also [43], [105], [109], [82]):

$$\begin{cases} \dot{p} \in BR_A(q) - p \\ \dot{q} \in BR_B(p) - q \end{cases} \tag{1.7.1}$$

for $(p, q) \in \Sigma_A \times \Sigma_B$.

Remark 7. *Up to time reparameterization, the Best Response Dynamics (1.7.1) are the same as that of Fictitious Play (FP):*

$$\begin{cases} \dot{p}(t) \in \frac{1}{t}(BR_A(q(t)) - p(t)) \\ \dot{q}(t) \in \frac{1}{t}(BR_B(p(t)) - q(t)) \end{cases} \tag{1.7.2}$$

We introduce Fictitious Play since many results concerning BRD can be found in the literature written in the language of Fictitious Play without mentioning the BRD.

Assumption 5. *By default it is assumed that the Filippov convention (see [32]) is used at the points (p, q) where either the best response $BR_A(q)$ or $BR_B(p)$ is not single-valued.*

Remark 8. *Since BRD is a differential inclusion, uniqueness of solution to it cannot be guaranteed for arbitrarily chosen matrices A, B. The existence of solutions, however, for all initial conditions from Σ follows from the theory of differential inclusions (see [2] [4, Chapter 4, Section 2, Theorem 1]). This is because best response functions are upper semi-continuous set valued functions with image of the singletons being closed and convex sets.*

In the following sections and chapter 3, we will either make use of the known results about the uniqueness of the solutions on $\Sigma \backslash \{(E^A, E^B)\}$ (i.e. everywhere except Nash equilibrium) or assume that

this uniqueness is taken for granted and so develop theory under such an assumption.

From the Nash theorem it follows that BRD possesses at least one equilibrium, since equilibria of BRD are exactly the Nash equilibria of the game (A, B). In the following sections and chapter 3, we will recall well known results from the literature and also provide some new results about convergence to the Nash equilibrium for BRD.

Since on the sets R_i^A and R_j^B (where the best responses are single-valued), the best responses BR_A and BR_B are locally constant, the solutions are locally straight line segments heading towards vertices of Σ. The solution to (1.7.1) is given by

$$\begin{cases} p(t_0 + s) = p(t_0) \cdot (1 - s) + s \cdot BR_A(q(t_0)) \\ q(t_0 + s) = q(t_0) \cdot (1 - s) + s \cdot BR_B(p(t_0)) \end{cases} \tag{1.7.3}$$

for $s \in [0, 1]$.

Remark 9. *The only changes of direction happen when either p or q coordinate of the orbit hits an indifference hyperplane (see figure 1.2), then necessarily* $s < 1$ *and we update* $t_0 := t_0 + s$ *and set* $s = 0$ *in the equation (1.7.3).*

1.8 - Uniqueness of the solutions for BRD

In this section we mention some general results which establish the uniqueness of the solutions of BRD on the subsets of the whole phase space Σ. However in this thesis (especially in chapter 3) we will use more specific results, recalled in section 1.10, which can guarantee the uniqueness of the solutions of the BRD for the small perturbations of the Shapley family on the set $\Sigma \backslash \{E^A, E^B\}$, i.e. everywhere outside the Nash equilibrium.

The obstacle for extending the solutions of BRD (1.7.1) from the sets R_{ij} uniquely on the whole phase space Σ is the existence of the indifference hyperplanes, on which the vector field forming BRD is set-valued (multi-valued). Let us define the following indifference sets:

$$Z^A := \bigcup_{i,i_0} Z^A_{i,i_0} \quad \text{and} \quad Z^B := \bigcup_{j,j_0} Z^B_{j,j_0} \tag{1.8.1}$$

and the set where each of the players is indifferent between at least two strategies

$$Z^* := Z^B \times Z^A \tag{1.8.2}$$

Remark 10. *We emphasize again that the Filippov convention for the vector fields (see [32]) is assumed when treating non-uniqueness of best response function.*

The proof of the following proposition can be found in [105]. It says that if a solution to BRD hits these indifference sets transversally, then it has to leave these sets immediately.

Proposition 1. *[105].*
Let (A, B) be a $m \times n$ bimatrix game. Assume that for all $(p, q) \in \Sigma \backslash Z^*$, if $p \in Z^B$ and $q \notin Z^A$ so that, say, $BR_A(q) = e_k$, then e_k is not parallel to the plane $Z^B \subset \Sigma^A$ at the point p, and similarly for the roles of p and q reversed. Then BRD defines a continuous flow on $\Sigma \backslash Z^*$.

We cite even a stronger result, which later will be useful in the analysis of the Shapley system and its perturbation in chapter 3.

Proposition 2. *[105]*
Assume that the bimatrix game (A, B) satisfies the hypotheses of Proposition 1, and additionally assume that A and B have maximal rank. Then the flow on the interior of the sets R_{ij} has a unique continuous extension everywhere (on Σ), except possibly to points in the subset of Z^* where one of the players is indifferent between at least three of his strategies (and the other between at least two). This remaining set has codimension three.

Remark 11. *For the considered 3×3 Shapley system and its perturbations, the above proposition 2 will lead to the uniqueness of the flow on the set $\Sigma \backslash \{(E^A, E^B)\}$.*

1.9 - Convergence to the Nash equilibrium

Chapter 3 of this thesis will be devoted to the following conjecture, which still remains open:

Conjecture 1. ***(Hofbauer [43])***
A unique Nash equilibrium which is completely mixed for the bimatrix game (A, B) is stable in the sense of Lyapunov for BRD, if and only if (A, B) is a zero-sum game.

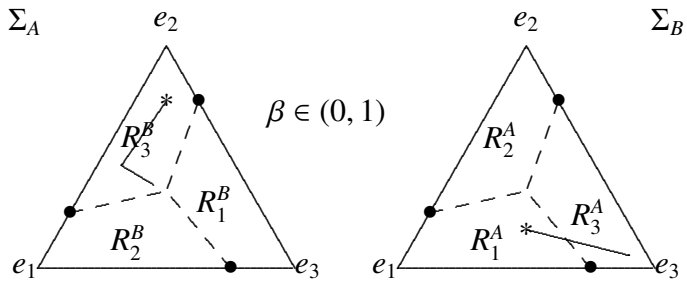


Figure 1.2: An example of a solution of BRD for the Shapley system with $\beta \in (0, 1)$. The solution starts at the point denoted by $*$. When the q coordinate in Σ_B of the solution hits the indifference line Z_{31}^A, the p in Σ_A coordinate changes the direction, it no longer moves towards e_1, but aims at e_3. The q coordinate of the solution hits the indifference line transversally and so the assumption of proposition 1 is satisfied. (see also[105])

Remark 12. *For $n = m$ (the dimensions of Σ_A and Σ_B), lemma 1 guarantees existence and uniqueness of a Nash equilibrium which is also completely mixed.*

The asymptotic stability of the completely mixed (and unique) Nash equilibrium for the zero-sum bimatrix games was proven by Julia Robinson [90]. In chapter 3, the idea of the proof for the zero-sum case will be shown as a direct consequence of the analysis of BRD in a general case.

If the conjecture was true indeed, it would mean that for every non-zero sum game with completely mixed and unique Nash equilibrium, arbitrarily close to Nash equilibrium there exists a point which forward orbit does not converge to the Nash equilibrium and eventually leave some small neighbourhood of the Nash equilibrium.

There are quite a few papers giving sufficient conditions under which fictitious play converges to an equilibrium from all starting points. For example, fictitious play has been shown to converge in: (i) zero-sum games [90, 48, 43] (ii) generic $2 \times n$ nonzero-sum games [72, 70, 99, 8], (iii) potential games [73, 9], (iv) dominance-solvable games [71] and (v) various classes of supermodular games [62, 42, 9]. However, in these cases, the Nash equilibria only lies in the interior when the game is zero-sum.

It turns out that orbits that converge to the Nash equilibrium typically do so in a complicated way: building on earlier work of [87] it is shown in [63] that generically (i.e. for Lebesgue almost all pay-off matrices), fictitious play cannot converge cyclically to a mixed strategy equilibrium if both players use more than two pure strategies. In other words, if fictitious play converges to a mixed strategy equilibrium (i.e. an interior Nash equilibrium) with both players using more than two strategies then the choice of strategies cannot follow a cyclic pattern. Whether one has convergence or not, orbits converge to some equilibrium set, called the coarse correlated equilibrium set (CCE) also called the Hannan set, see [75] and also [83].

There are also quite a few papers devoted to understanding fictitious play in games in which fictitious play does not necessarily converge to an equilibrium. A seminal contribution was made by Shapley [74], who identified a generic 3×3 nonzero-sum game for which fictitious play has a stable limit cycle (corresponding to a Rock-Paper-Scissors game). Other recent contributions are: [59], where a $2 \times 2 \times 2$ game was constructed for which fictitious play has a stable limit cycle; [38] where an interesting relationship between the replicator dynamics and fictitious play was established in three examples in which fictitious play converges to a limit cycle (namely Shapley's example, Jordan's example and the rock-scissors-paper game); Berger [7] who considers a family of symmetric bimatrix games depending on a parameter k such that (i) for $k \in (-2, 0)$ has a Shapley orbit which degenerates as $k \to 0$, and (ii) for $k > 0$, there exists a hexagonal orbit along which both players are indifferent between (at least) 2 strategies. Berger observes similar 'chaotic' numerical phenomena and also shows the existence of an additional periodic orbit of saddle-type; [24], where a very specific numerical example was given in which fictitious play exhibits chaotic behaviour (in the sense that the flow contains a subshift of finite type). Finally, a very detailed analysis of the dynamics of the complicated dynamics that can occur in best response system is given in [105] and [109] by studying a family of 3×3 examples.

In the 3×3 examples where the dynamics of *non-zero* sum games is well-understood, namely those studied in [105], [109], it is known that there are matrices (A, B) for which simultaneously there exist a countable number of distinct orbits which **do** converge to the Nash equilibrium and also a countable number of other orbits **do not** converge to the Nash equilibrium. In fact, the orbits of the non-zero sum games that in [105] were proven to converge to the Nash equilibrium, do so in *finite time*. (Note that the right hand side of (1.7.2) is not Lipschitz, which in particular implies that traveling to an equilibrium in finite time is possible. The same happens for example for the differential equation $x' = -(sign(x))\sqrt{|x|}$.)

From the family of examples studied in [105], [109], it is clear that *it is not true* that *all* orbits starting near a Nash equilibrium of a non-zero-sum game leave a neighbourhood of the Nash equilibrium.

Remark 13. *We should again emphasize that there are several classes of non-zero games which have a unique attracting Nash equilibrium, but in those cases the Nash equilibrium is not in the interior of the phase space.*

1.10 - Shapley family and its perturbation

In chapter 3, we will consider 3×3 matrices:

$$A = \begin{pmatrix} 1+\delta_1 & 0 & \beta-\delta_1 \\ \beta & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix} \quad B = -\begin{pmatrix} 1+\delta_2 & 0 & \beta \\ \beta & 1 & 0 \\ -\delta_2 & \beta & 1 \end{pmatrix} \tag{1.10.1}$$

From lemma 1 if follows immediately that the Nash equilibrium is the center of the phase space $E^A = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $E^B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^{tr}$. For $\delta_1, \delta_2 = 0$, (A, B) is a zero-sum game.

Definition 9. *The Shapley family are the pairs of matrices* (A_β, B_β)

$$A_\beta = \begin{pmatrix} 1 & 0 & \beta \\ \beta & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix} \quad B_\beta = \begin{pmatrix} -\beta & 1 & 0 \\ 0 & -\beta & 1 \\ 1 & 0 & -\beta \end{pmatrix} \tag{1.10.2}$$

for $\beta \in (-1, 1)$

Remark 14. *For* $\delta_1, \delta_2 = 0$ *and* $\beta_\phi = \frac{\sqrt{5}-1}{2}$, (A, B) *is linearly equivalent to* (A_β, B_β).
Indeed, in such a case $A = A_\beta$ *and*

$$\tilde{B} = \frac{1}{\beta_\phi} \cdot B + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -\beta_\phi & 1 & 0 \\ 0 & -\beta_\phi & 1 \\ 1 & 0 & -\beta_\phi \end{pmatrix}$$

is the payoff matrix for the second player, in the Shapley system.

Remark 15. *In the subsequent sections of this chapter we will use the results from [105] and [109], concerning the Shapley system for* $\beta \in (0, 1)$ *and its small perturbations (although their results cover also the case* $\beta \in (-1, 0)$*, discussing them is beyond the scope of this thesis). In chapter 3, we will use the results about the Shapley system and its perturbations for* $\beta = \frac{\sqrt{5}-1}{2}$*, since the starting point for our investigations in the section 3.7 is the BRD with matrices A and B as in (1.10.1).*

The following lemma (see [105]) is essential for establishing the uniqueness of the solutions for BRD for the Shapley family.

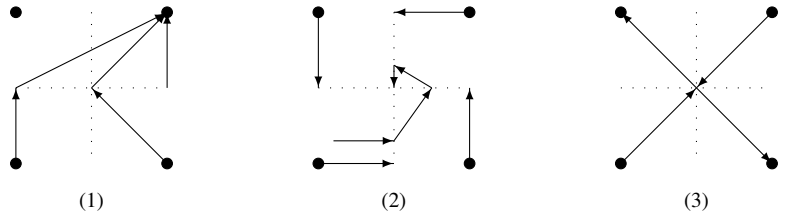


Figure 1.3: The solution of BRD in various cases for 2×2 games. The dotted lines denote where one player in indifferent to two strategies. From left to right cases (1), (2), (3). In case (1), the solutions converge to the singular point on the boundary of the phase space. In case (2) the orbit spirals towards the interior point, while in case (3) the interior point acts as a saddle. (see also [105])

Lemma 3. *(see [105]) (*2×2 ***case)***

Assume that both players have two strategies and that the corresponding 2×2 *matrices are non-singular. Then there are three possibilities:*

(1) there is no interior equilibrium in $\Sigma_A \times \Sigma_B$ *and all orbits tend to a unique singular point on the boundary (this occurs if* Z_A *and or* Z_B *are empty, and is depicted in (1) in figure 1.3).*

(2) there is an interior equilibrium E which is stable, orbits spiral towards it and the flow extends continuously to E (this is depicted in (2) in figure 1.3).

(3) there is an interior equilibrium E which is a saddle point and the flow cannot be extended in a continuous way to E and there is genuine non-uniqueness in the flow near E (this is depicted in (3) in figure 1.3).

For the Shapley family of matrices (A_β, B_β) and matrices which are small enough perturbations of them, the following holds:

Proposition 3. *([105]) (**Tranversality condition implies some continuity**)*

The motion defined by BRD equations (1.7.1) forms a continuous flow on $(\Sigma_A \times \Sigma_B) \backslash Z^*$ *provided the following condition is met: for any point* $(p^A, p^B) \notin Z^*$ *with, say,* $p^A \in Z^B$, $p^B \notin Z^A$ *and strategy* P_k^A *preferable for player A,*

$$\text{the vector } P_k^A \text{ is not parallel to the plane } Z^B \subset \Sigma_A \text{ at } p^A \tag{1.10.3}$$

(and similarly, in the case when the role of p^A *and* p^B *is reversed).*

Proposition 4. *(see [105]) When* $\beta \in (0, 1]$*, the unique flow on the complement of the indifference set Z extends to a flow on Z which is everywhere continuous and unique except at the Nash equilibrium* $E = (E^A, E^B)$*.*

If $\beta > 0$*, then the transversality condition (1.10.3 from proposition 3 holds on all codimension one indifference sets, and all the codimension two points in* Z^* *are in case (1) or (2) from lemma 3. In*

particular, the flow off Z extends continuously to all points except $E = (E^A, E^B)$, and so the differential inclusion 1.7.1 defines a unique and continuous flow everywhere except at the interior equilibrium E.

Remark 16. *Due to the latter proposition 4, we can assume in section 3.7 in chapter 3, that for the 3×3 bimatrix games (A, B) considered there (which are arbitrarily small perturbations of the Shapley family), the flow of BRD extends uniquely to the whole phase space except the Nash equilibrium.*

Remark 17. *Moreover, we should emphasize at this point, that in the chapter 3 we will consider $n\times m$ bimatrix games with unique and completely mixed Nash equilibrium $E = (E^A, E^B)$, with the property that the flow of BRD can be continuously extended to the set $\Sigma\backslash\{(E^A, E^B)\}$. An example of such a game is given by the Shapley system.*

We recall here the main results about Shapley family of bimatrix games (from [105], [109]).

Remark 18. *We will refer to these results when in the subsequent sections of this chapter we will investigate the relation between Shapley family governed by the BRD and Rock-Scissors-Paper game governed by the replicator equations.*

Proposition 5. ***(Existence and stability of the Shapley periodic orbit)***
For $\beta \in (0, \sigma)$ (where $\sigma = \frac{\sqrt{5}-1}{2}$) there exists an attracting periodic orbit in which player 1 and 2 infinitely repeat the strategies

$$\begin{aligned} &\textit{strategy player A } 1\ 2\ 2\ 3\ 3\ 1 \\ &\textit{strategy player B } 2\ 2\ 3\ 3\ 1\ 1 \end{aligned} \tag{1.10.4}$$

(So player A copies B and player B tries to play one ahead of A.) This is a continuous deformation of the orbit that Shapley found for $\beta = 0$, and as $\beta \nearrow \sigma$ this orbit shrinks in diameter to zero and tends to the point (E^A, E^B). For $\beta \in (\sigma, 1)$ this periodic orbit no longer exists.

Proposition 6. ***(Existence and stability of the anti-Shapley periodic orbit)***
For $\beta \in (\sigma, 1]$ there exists a periodic orbit in which player 1 and 2 infinitely repeat the strategies

$$\begin{aligned} &\textit{strategy player A } 1\ 1\ 3\ 3\ 2\ 2 \\ &\textit{strategy player B } 3\ 2\ 2\ 1\ 1\ 3 \end{aligned} \tag{1.10.5}$$

As $\beta \searrow \sigma$ this orbit shrinks in diameter to zero and tends to the point (E^A, E^B). Such an orbit does not exist when $\beta < \sigma$. For $\beta \in (\sigma, \tau)$ this periodic orbit is of saddle-type and is attracting for $\beta \in (\tau, 1]$. Here τ is the root of some specific expression polynomial and is approximately equal to 0.915.

Let us define the so-called Jittering set:

$$J := (Z^B_{1,2} \times Z^A_{3,1}) \cup (Z^B_{1,2} \times Z^A_{1,2}) \cup (Z^B_{2,3} \times Z^A_{1,2}) \cup (Z^B_{1,2} \times Z^A_{2,3}) \cup (Z^B_{3,1} \times Z^A_{2,3}) \cup (Z^B_{2,3} \times Z^A_{3,1}) \quad (1.10.6)$$

Proposition 7. ***(A third periodic which is created by a Hopf-like bifurcation)***
The system has a Hopf-like bifurcation when $\beta = \sigma$ on the topological two-dimensional manifold J. Trajectories on J spiral towards (E^A, E^B) when $\beta \in (0, \sigma]$, while for $\beta \in (\sigma, 1]$ all trajectories in J (except the one stationary at (E^A, E^B)) tend to a periodic orbit Γ in J.

By analyzing the return map (called Jittering map) to a cross section attached to some point of the periodic orbit Γ, in [109] the dithering behaviour was found for Shapley family and its small perturbations:

Theorem 1. ***(An abundance of periodic play)***
For each $\beta \in (0, 1)$ there are infinitely many different orbits γ_s, $s = 1, 2, ...$ of the differential equation 1.7.1 with corresponding cyclic play of period $N_s \to +\infty$ as $s \to +\infty$. Moreover,

- *for $\beta \in (0, \sigma)$, these orbits with cyclic play reach the interior Nash equilibrium (E^A, E^B) in finite time;*
- *for $\beta \in (\sigma, 1)$ these orbits with cyclic play are genuine periodic orbits of (1.7.1)*

Before stating next theorem, we need to introduce the following notions (after [109]):

Definition 10. *To an orbit of BRD let us associate the sequence of the best-response strategies played by the players, let us denote this sequence as $(i_0, j_0), (i_1, j_1), (i_2, j_2), ...$ Given $k \in \mathbb{N}$, we say that players are indecisive at the kth step if $k \geq 3$ and moreover*

$$\{i_{k-3}, i_{k-2}, i_{k-1}, i_k\} \neq \{1, 2, 3\} \text{ and } \{j_{k-3}, j_{k-2}, j_{k-1}, j_k\} \neq \{1, 2, 3\}$$

Example 8. *During the cycle of period* 6

$$(1, 2), (2, 2), (2, 3), (3, 3), (3, 1), (1, 1)$$

the players are never indecisive, whereas during the cycle of period 7

$$(1, 2), (2, 2), (2, 3), (3, 3), (3, 2), (3, 1), (1, 1)$$

the players are indecisive in the 5th step.

Let the orbit of BRD starting at the point $x = (p, q) \in \Sigma$ follow the best response sequence (i_0, j_0), (i_1, j_1), (i_2, j_2), ...

Let us define the sequence of indices $3 \leq N_0(x) < N_1(x) < N_2(x) < N_3(x) < ...$ for which the players are decisive. If $N_{s+1}(x) - N_s(x)$ is large, then we say that the players dither for a long time (as they then each play back and forth between two strategies) (see [109]).

Theorem 2. ***(There is a lot of freedom in the choice of dithering sequences)***
For each $\beta \in (0, 1)$ there exist $N_0 \in \mathbb{N}$, $0 < \lambda < 1 < \mu$ and a compact set $X \in \Sigma$, so that for each sequence $N_0 < N_2 < N_4 < ...$ with $N_0 \geq 3$ and with

$$\lambda \leq \frac{N_{2s+4} - N_{2s+2}}{N_{2s+2} - N_{2s}} \leq \mu \text{ and } N_{2s+2} - N_{2s} \geq N_0 \text{ for all } s \geq 0$$

there exists $p \in X$ such that

$$|N_{2s}(p) - N_{2s}| \leq 4 \text{ for all } s \geq 0$$

Moreover,

- *for $\beta \in (0, \sigma)$ orbits in X converge to E;*
- *for $\beta \in (\sigma, 1)$ orbits starting in X do not converge to E.*

The following theorem is a consequence of the fact that there are orbits starting in X dithering for lengths $N_{2s+2} - N_s$, which can take more or less any value:

Theorem 3. ***(Chaos)***
Take $\beta \in (\sigma, 1)$ and take $\tilde{N} \geq N_0$ so large that $\lambda \leq (\tilde{N} - 1)/\tilde{N} \leq (\tilde{N} + 1)/\tilde{N} \leq \mu$. For each sequence $(\epsilon_{2s})_{s \leq 0}$ with $\epsilon_{2s} \in \{-5, 0, 5\}$ there exists $x \in X$ with $N_{2s+2} - N_{2s} \in [\tilde{N} + \epsilon_{2s}, \tilde{N} + 4 + \epsilon_{2s}]$ for all $s \geq 0$ and so that the orbit through x does not converge to E. So for such a $x \in X$, $N_{2s+2} - N_{2s}$ is in the interval $[\tilde{N} - 5, \tilde{N} - 1]$, $[\tilde{N}, \tilde{N} + 4]$ or $[\tilde{N} + 5, \tilde{N} + 9]$ depending on the parity of ϵ_{2s}. In particular the flow contains subshifts of finite type and has positive topological entropy. The flow also has sensitive dependence on initial conditions.

Remark 19. *We will get back to the above propositions in the last section of this chapter, where we mention relations between BRD and Replicator Dynamics, and corresponding results, for the Rock-Scissors-Paper bimatrix game, to the ones stated above for Shapley system.*

1.11 - Replicator Equations

The following derivation of the replicator equations describing a bimatrix game can be found in [94]. Let us assume that there are two players X, Y, who can choose one out of m and n possible actions, respectively, i.e. from the sets $\{1, ..., m\}$, $\{1, ..., n\}$ at each time step during duration of the game. According to the definition of a finite game 1, in this case $I = 2$ - the number of the players, $S^1 = \{1, ..., m\}$, $S^2 = \{1, ..., n\}$.

By $x_i(t)$, $y_i(t)$ we denote the probability of choosing the strategy i at time t during the game ($t = 0$ is the initial state). Hence $\Sigma x_i(t) = \Sigma y_i(t) = 1$.

Remark 20. *One can look at the probabilities x, y from the point of view of population dynamics or economic models as a distribution of ratio of the population choosing different strategies or different goods.*

Assumption 6. *If $Q_i^X(t)$ and $Q_i^Y(t)$, are the total profits from playing strategy i at the time t, for players X and Y respectively, then we assume that $Q_i^X(0) = Q_i^Y(0) = 0$ and players X, Y update their choice distributions $(x_i(t))_{i\in\{1,...,m\}}$, $(y_i(t))_{i\in\{1,...,n\}}$ according to the distribution of the total profits at the time step k:*

$$x_i(t) = \frac{e^{Q_i^X(t)}}{\Sigma_{j=1}^m e^{Q_j^X(t)}} \quad \text{and} \quad y_i(t) = \frac{e^{Q_i^Y(t)}}{\Sigma_{j=1}^n e^{Q_j^Y(t)}} \tag{1.11.1}$$

From (1.11.1) it follows directly that after the time step dt, the players preferences will change into:

$$\begin{aligned} x_i(t+dt) &= \frac{x_i(t) \cdot e^{Q_i^X(t+dt) - Q_i^X(t)}}{\Sigma_{j=1}^m x_j(t) \cdot e^{Q_j^X(t+dt) - Q_j^X(t)}} \\ y_i(t+dt) &= \frac{y_i(t) \cdot e^{Q_i^Y(t+dt) - Q_i^Y(t)}}{\Sigma_{j=1}^n y_j(t) \cdot e^{Q_j^Y(t+dt) - Q_j^Y(t)}} \end{aligned} \tag{1.11.2}$$

Assumption 7. *Moreover we assume that the probability distributions x and y are constant during the interaction of the players and they change in the process of learning according to the rule (1.11.1).*

Considering the time steps dt to be infinitesimally small, we infer from (1.11.2) (or simply by differentiating (1.11.1)) the continuous time limit dynamics describing the distribution of strategy preferences of X, Y players:

$$\begin{cases} \dot{x}_i = x_i \cdot (\dot{Q}_i^X - \Sigma_{j=1}^m \dot{Q}_j^X \cdot x_j) \\ \dot{y}_i = y_i \cdot (\dot{Q}_i^Y - \Sigma_{j=1}^n \dot{Q}_j^Y \cdot y_j) \end{cases} \tag{1.11.3}$$

Let R^X_{ij}, R^Y_{ij} be the rewards for players X, Y, respectively, provided that X plays strategy i and Y plays strategy j. As well, let R^X_i be the profit gained by the player X for choosing strategy i, averaged over player $Y's$ actions during the time interval $[t, t + dt)$ between learning updates.

Assumption 8. *We assume a simple relation between the rewards $R^X_{ij} = a_{ij}$, $R^Y_{ij} = b_{ij}$ (i.e. X is playing strategy i, player Y chooses j), time independence and independence of distributions x and y.*

Due to the above assumption, we arrive at:

$$\begin{aligned} \dot{Q}^X_i &= R^X_i = \Sigma^m_{j=1} R^X_{ij} y_j = \Sigma^m_{j=1} a_{ij} y_j \\ \dot{Q}^Y_i &= R^Y_i = \Sigma^n_{j=1} R^Y_{ij} x_j = \Sigma^n_{j=1} b_{ij} x_j \end{aligned} \tag{1.11.4}$$

So we can regard the matrices $A = (a_{ij}) \in \mathbb{R}^{n\times m}$, $B = (b_{ji}) \in \mathbb{R}^{m\times n}$, as the payoff matrices for players X and Y, respectively. The equation (1.11.3) together with (1.11.4), simplifies to the following coupled replicator equations:

$$\begin{cases} \dot{x}_i = x_i \cdot ((Ay)_i - x^T Ay) \\ \dot{y}_j = y_j \cdot ((Bx)_j - y^T Bx) \end{cases} \tag{1.11.5}$$

where $(Ay)_i$ is the i-th element of the vector Ay.

For the sake of presenting more information about the coupled replicator equations (1.11.5) and later on - performing more accurate numerical simulations (as it is hard to integrate this system numerically close to the faces of Σ), it is worth introducing the following transformations of coordinates, for $x \in int(\Sigma_A)$, $y \in int(\Sigma_B)$:

$$\begin{aligned} \xi_i &:= -\ln x_i \text{ for } i \in \{1, ..., m\} \\ \eta_j &:= -\ln y_j \text{ for } j \in \{1, ..., n\} \end{aligned} \tag{1.11.6}$$

and

$$\begin{aligned} u_i &:= \xi_i - \frac{1}{m}\Sigma^m_{k=1}\xi_k \\ v_i &:= \eta_j - \frac{1}{n}\Sigma^n_{k=1}\eta_k \end{aligned} \tag{1.11.7}$$

Vectors u and v are called the **normalized (relative to their means) self-information**. The space $int(\Sigma) = int(\Sigma_A) \times int(\Sigma_B)$ in the coordinates (u, v)

$$U := (u, v) := (u_1, ..., u_m) \times (v_1, ..., v_n) \tag{1.11.8}$$

is called the **information space**. The inverse transformation from (u, v) coordinates to $(x, y) \in int(\Sigma)$

coordinates is

$$x_i = \frac{e^{-u_i}}{\Sigma_{k=1}^m e^{-u_k}} \quad \text{and} \quad y_j = \frac{e^{-v_j}}{\Sigma_{k=1}^n e^{-v_k}} \tag{1.11.9}$$

Introducing the notation:

$$e^{-u} := (e^{-u_1}, ..., e^{-u_m}) \quad \text{and} \quad e^{-v} := (e^{-v_1}, ..., e^{-v_n}) \tag{1.11.10}$$

and

$$Z(u) := \Sigma_{k=1}^m e^{-u_k} \quad \text{and} \quad Z(v) := \Sigma_{k=1}^n e^{-v_k} \tag{1.11.11}$$

we arrive with equation (1.11.5) in the (u, v) coordinates

$$\begin{cases} \dot{u} = -Z^{-1}(v) \cdot (A \cdot e^{-v} - \Sigma_{k=1}^m (A \cdot e^{-v})_k) \\ \dot{v} = -Z^{-1}(u) \cdot (B \cdot e^{-u} - \Sigma_{k=1}^n (B \cdot e^{-u})_k) \end{cases} \tag{1.11.12}$$

Remark 21. *If the payoff matrices are normalized, i.e.* $\Sigma_{k=1}^n a_{ik} = \Sigma_{l=1}^m b_{jl} = 0$ *for all* $i \in \{1, ..., m\}$, $j \in \{1, ..., n\}$, *the equation (1.11.12) is simplified to*

$$\begin{cases} \dot{u} = -A \cdot Z^{-1}(v) \cdot e^{-v} \\ \dot{v} = -B \cdot Z^{-1}(u) \cdot e^{-u} \end{cases} \tag{1.11.13}$$

Below we present a proof from [95], that:

Proposition 8. *If* $n = m$ *and* $B = -A^T$, *i.e. the bimatrix game* (A, B) *is zero-sum and* (A, B) *possesses an interior Nash equilibrium* $(E^A, E^B) \in int(\Sigma)$, *then the system (1.11.13) is Hamiltonian.*

Proof. The Hamiltonian is the function:

$$E = -\Sigma_{i=1}^m E_i^A \cdot \ln x_i - \Sigma_{j=1}^n E_j^B \cdot \ln y_j$$

It is a constant of motion since

$$\frac{d}{dt} E = -\Sigma_{i=1}^m E_i^A \cdot \frac{\dot{x}_i}{x_i} - \Sigma_{j=1}^n E_j^B \cdot \frac{\dot{y}_j}{y_j} =$$

$$= -(E^A \cdot A \cdot y - x \cdot A \cdot y) - (E^B \cdot B \cdot x - y \cdot B \cdot x) =$$

$$= (E^A - x) \cdot A \cdot (E^B - y) + (E^B - y) \cdot B \cdot (E^A - x) = 0$$

Moreover

$$\dot{U} = J \cdot \nabla_U E \tag{1.11.14}$$

where J is the Poisson structure

$$J = \begin{pmatrix} 0 & -A \\ A^T & 0 \end{pmatrix} \tag{1.11.15}$$

The $\nabla_U E$ consists of the following coordinates:

$$\frac{\partial E}{\partial u_i} = \frac{\partial}{\partial u_i}\Big(\Sigma_{k=1}^m E_k^A \ln E_k^A + \Sigma_{k=1}^m E_k^B \ln E_k^B$$

$$-(\Sigma_{k=1}^m E_k^A u_k - \ln(\Sigma_{k=1}^m e^{-u_k})) - (\Sigma_{k=1}^m E_k^B v_k - \ln(\Sigma_{k=1}^m e^{-v_k}))\Big) = (E_i^A - \frac{e^{-u_i}}{\Sigma_{k=1}^m e^{-u_k}}) = E_i^A - x_i$$

and

$$\frac{\partial E}{\partial v_j} = E_j^B - y_j$$

As we assumed that (E^A, E^B) is the Nash equilibrium and for the equation (1.11.13), the matrices $A,\ B$ are normalized, then $(Ay^*)_i = (Bx^*)_j = 0$ for all $i, j \in \{1, ..., n\}$. Thus,

$$A \cdot \frac{\partial E}{\partial v} = -Ay \quad \text{and} \quad B \cdot \frac{\partial E}{\partial u} = -Bx$$

which finally gives us

$$J\nabla_U E = \begin{pmatrix} -A\frac{\partial E}{\partial v} \\ A^T\frac{\partial E}{\partial u} \end{pmatrix} = \begin{pmatrix} -Ay \\ -Bx \end{pmatrix} = \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \dot{U} \tag{1.11.16}$$

□

Remark 22. *We find it important to mention that in [1] the generalizations of replicator equations were considered, i.e. multiplayer replicator equations for more than just* 2 *populations interacting. The authors provided there the conditions for existence of local invariant manifolds of the heteroclinic cycles appearing in this multiplayer replicator equations. These results, although very interesting and derived in a very general setup, will not be used in this thesis. This is because the existence of this invariant manifolds in the case of Rock-Scissors-Paper game can be deduced very easily, and moreover we are interested in their global behaviour, i.e. extension of this local manifolds and possible intersections between them, which was beyond the scope of the paper [1].*

1.12 - Rock-Scissors-Paper bimatrix game

In this section we introduce an example of a cyclic competition bimatrix game, which is known as Rock-Scissors-Paper game. The payoff matrices are A and B^T where:

$$A = \begin{pmatrix} \epsilon_X & -1 & 1 \\ 1 & \epsilon_X & -1 \\ -1 & 1 & \epsilon_X \end{pmatrix} \qquad B = \begin{pmatrix} \epsilon_Y & -1 & 1 \\ 1 & \epsilon_Y & -1 \\ -1 & 1 & \epsilon_Y \end{pmatrix} \tag{1.12.1}$$

with $\epsilon_X, \epsilon_Y \in (-1, 1)$, the rewards for ties.

Remark 23. *To be consistent with the notation from the previous sections, note that the Rock-Scissors-Paper bimatrix game is assumed to be the game* (A, B^T).

To be consistent with the previous section, let $x_1, x_2, x_3 \geq 0$ ($y_1, y_2, y_3 \geq 0$) denote the probability that player X (resp. Y) plays Rock, Scissors or Paper. Obviously $x_1 + x_2 + x_3 = 1$, $y_1 + y_2 + y_3 = 1$. For each player, the state space is a two-dimensional simplex Σ_A, Σ_B respectively. Hence the whole state space $\Sigma = \Sigma_A \times \Sigma_B$ is 4-dimensional, however embedded in $\mathbb{R}^3 \times \mathbb{R}^3$. The interaction matrices after normalization are given by:

$$A = \begin{pmatrix} \frac{2}{3}\epsilon_X & 1 - \frac{1}{3}\epsilon_X & -1 - \frac{1}{3}\epsilon_X \\ -1 - \frac{1}{3}\epsilon_X & \frac{2}{3}\epsilon_X & 1 - \frac{1}{3}\epsilon_X \\ 1 - \frac{1}{3}\epsilon_X & -1 - \frac{1}{3}\epsilon_X & \frac{2}{3}\epsilon_X \end{pmatrix} \qquad B = \begin{pmatrix} \frac{2}{3}\epsilon_Y & 1 - \frac{1}{3}\epsilon_Y & -1 - \frac{1}{3}\epsilon_Y \\ -1 - \frac{1}{3}\epsilon_Y & \frac{2}{3}\epsilon_Y & 1 - \frac{1}{3}\epsilon_Y \\ 1 - \frac{1}{3}\epsilon_Y & -1 - \frac{1}{3}\epsilon_Y & \frac{2}{3}\epsilon_Y \end{pmatrix} \tag{1.12.2}$$

Remark 24. *The dynamics are described by the replicator equations (1.11.5) (see [94]). Note that the normalization performed above does not influence the equation (dynamics).*

Remark 25. *After substitutions* $x_3 = 1 - x_1 - x_2, y_3 = 1 - y_1 - y_2$, *we will consider (1.11.5) as a differential equation in* $\mathbb{R}^4$*:*

$$\begin{cases} \dot{x_1} = x_1[(Ay)_1 - x^T Ay] \\ \dot{x_2} = x_2[(Ay)_2 - x^T Ay] \\ \dot{y_1} = y_1[(Bx)_1 - y^T Bx] \\ \dot{y_2} = y_2[(Bx)_2 - y^T Bx] \end{cases} \tag{1.12.3}$$

with constraints $x_1, x_2 \geq 0, x_1 + x_2 \leq 1$, $y_1, y_2 \geq 0, y_1 + y_2 \leq 1$.

Let us denote the canonical basis in $\mathbb{R}^4$ by

$$e_1 = (1,0,0,0), \quad e_2 = (0,1,0,0), \quad e_3 = (0,0,1,0), \quad e_4 = (0,0,0,1)$$

The proofs of the following propositions can be found in the first sections of chapter 4.

Proposition 9. *The system (1.12.3) is invariant under action of the group generated by the cycle σ of order 3:*

$$\sigma(x_1, x_2, x_3, y_1, y_2, y_3) := (x_2, x_3, x_1, y_2, y_3, y_1) \tag{1.12.4}$$

Equation (1.12.3) possesses following equilibria:

Proposition 10. *Nash equilibrium of the system $(E^A, E^B) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ which is of $(2,2)$ saddle type for $\epsilon_X + \epsilon_Y \neq 0$ (or a center otherwise)*

Proposition 11. 6 *equilibria $Z^a, Z^b, Z^c, Z^d, Z^e, Z^f$ of $(1,2,1)$ type (1-dim. stable manifold, 2-dim. center manifold, 1-dim. unstable manifold), which are centers within the 2 dimensional invariant subspaces.*

Proposition 12. 9 *equilibria of the form $\{(x,y) \mid x,y \in \{R,S,P\}\}$, where $R = (1,0)$, $S = (0,1), P = (0,0)$, which are all hyperbolic of $(2,2)$ saddle type.*

Due to the proposition 8 we know that the system (1.11.13), is Hamiltonian for $A = -B^T$, that is when $\epsilon_X + \epsilon_Y = 0$ (see [94], [96]). Moreover it was proven that in this case all trajectories have on average the same payoff as the Nash equilibrium [95]. On the other hand, on any given step huge deviations from the payoff can be observed. Thus, a risk averse agent would prefer staying at the Nash equilibrium to a chaotic orbit. [94]

1.13 - Relations between BRD and Replicator Equations

Since in the chapters 2 and 3 we consider BRD and replicator equations, we would like to recall in this section some of the results from [54], [109], [20] about correspondence between BRD and replicator equations. In particular in these papers, it was investigated what are the corresponding objects for

BRD to those existing in replicator dynamics and vice versa, what are the corresponding objects for replicator equations to those found in [105], [109] for Shapley family in BRD.

We begin with the observation firstly reported in [54] (however stated as in [20]) that the time averages of the solutions of the replicator equations are pseudo-orbits of BRD:

Proposition 13. ***(Time averages of (RE) are pseudo-orbits of BRD)***

Assume that $(x(s), y(s))$ *is a solution to the Replicator Equations (1.11.5) and assume that* $(E^A, E^B) \in \Sigma_A \times \Sigma_B$ *is the interior Nash equilibrium for this bimatrix game* (A, B).

Let

$$X(t) = \frac{1}{t}\int_0^t x(s)ds \quad \textit{and} \quad Y(t) = \frac{1}{t}\int_0^t y(s)ds$$

Then

$$x(t) \in BR_A^{\alpha(t)}(Y(t)) \quad \textit{and} \quad y(t) \in BR_B^{\alpha(t)}(X(t))$$

and

$$\begin{cases} \dot{X}(t) \in \frac{1}{t}(BR_A^{\alpha(t)}(Y(t)) - X(t)) \\ \dot{Y}(t) \in \frac{1}{t}(BR_B^{\alpha(t)}(X(t)) - Y(t)) \end{cases} \tag{1.13.1}$$

where $\alpha(t) \to 0$ *as* $t \to +\infty$, *and* $BR_A^{\alpha(t)}(Y(t)) := \{x \in \Sigma \mid |x - BR_A(Y(t))| < \alpha(t)\}$.

In particular, $(X(t), Y(t))$ *is a* ***pseudo-orbit of the Fictitious Play dynamics*** *and:*

1. When $Y(t)$ *is not close to an indifference line of* BR_A, *then* $x(t)$ *is close to one of the corners of* Σ_A. *Similarly when* $X(t)$ *is not close to indifference line of* BR_B, *then* $y(t)$ *is close to one of the corners of* Σ_B.

2. If $(X(t), Y(t))$ *stays away from the Nash equilibrium* (E^A, E^B) *then* $(x(t), y(t))$ *tends to the boundary of* Σ *as* $t \to +\infty$.

Remark 26. *Moreover, in [54] it was proven that for* $(x_0, y_0) \in \Sigma$, *all accumulation points of* $(X_t(x_0, y_0))_{t\geq 0}$ *are contained in an 'internally chain transitive' invariant subset of the corresponding BRD.*

By 'internally chain transitive' invariant subset we understand the following:

Definition 11. *A nonempty invariant set* $X' \subset X$ *of a flow* ϕ *on a metric space* (X, d) *is said to be internally chain transitive if for any* $x_0, x_m \in X'$ *and for all* $\epsilon > 0$, $T > 0$, *there exist a sequence of points* $x_1, ..., x_{m-1} \in X'$ *and a sequence of times* $t_0, ..., t_{m-1} > T$ *with* $m \geq 1$, *such that* $d(\phi(t_j, x_j), x_{j+1}) < \epsilon$ *for all* $0 \leq j \leq m - 1$.

The following, converse results are due to [20]:

Proposition 14. ***(Periodic orbits of (BRD) correspond to heteroclinic orbits of replicator equations, see [20])***

Assume that $\gamma(s)$ is a hyperbolic periodic orbit of (BRD) and let $\tilde{\gamma}(t) = \gamma(log(t))$, $t > 1$, be the corresponding (FP) orbit. Then, the two-player game has an orbit $(x(s), y(s))$ being a solution to replicator equations (1.11.5) so that for

$$X(t) = \frac{1}{t}\int_0^t x(s)ds \quad and \quad Y(t) = \frac{1}{t}\int_0^t y(s)ds$$

one has

$$d((X(t), Y(t)), \tilde{\gamma}(t)) \to 0 \quad as \quad t \to +\infty$$

and the properties from the previous proposition 13 hold.

Proposition 15. ***(A hyperbolic set for (BRD) correspond to a 'hyperbolic set' of heteroclinic orbits of replicator equations, see [20])***

Assume that the (BRD) has a hyperbolic set Λ. Then for each $\gamma_0 \in \Lambda$ there exists a corresponding replicator equations (1.11.5) orbit $(\psi_t)_{t\geq 0}$ so that

$$d((X(t), Y(t)), \tilde{\gamma}(t)) \to 0 \quad as \quad t \to +\infty$$

where $\tilde{\gamma}(t)$ is the solution of the Fictitious Play through γ_0, $\psi(t) = (x(t), y(t))$, and $X(t) = \frac{1}{t}\int_0^t x(s)ds$, $Y(t) = \frac{1}{t}\int_0^t y(s)ds$.

Remark 27. *In [109], where the 'jittering orbit' and random walk like behaviour was investigated for the BRD Shapley system (see also section 1.10), it was also reported that one should expect some corresponding behaviour in the Rock-Scissor-Paper game governed by replicator equations.*

This is due to the following:

Proposition 16. *(see [109]) The Rock-Scissors-Paper bimatrix game (A, B^T) is linearly equivalent to the Shapley family matrices (A_β, B_β) for $\epsilon_X = \frac{1+3\epsilon_Y}{1-\epsilon_Y}$ and $\beta = \frac{1+\epsilon_X}{2}$.*

Proof. Indeed, due to the $\mathbb{Z}_3$ symmetry we can permute the labelling of the strategies $1 \to 3$, $3 \to 2$,

$2 \to 1$, so that the matrices A, B^T are:

$$A = \begin{pmatrix} 1 & -1 & \epsilon_X \\ \epsilon_X & 1 & -1 \\ -1 & \epsilon_X & 1 \end{pmatrix} \qquad B^T = \begin{pmatrix} -1 & 1 & \epsilon_Y \\ \epsilon_Y & -1 & 1 \\ 1 & \epsilon_Y & -1 \end{pmatrix}$$

Note that for $\beta = \frac{\epsilon_X+1}{2}$ we have:

$$A_\beta = \frac{1}{2} \cdot (A + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix})$$

On the other hand if $\beta = \frac{1+\epsilon_Y}{1-\epsilon_Y}$

$$B_\beta = -\frac{1}{1-\epsilon_Y} \cdot (B^T - \begin{pmatrix} \epsilon_Y & \epsilon_Y & \epsilon_Y \\ \epsilon_Y & \epsilon_Y & \epsilon_Y \\ \epsilon_Y & \epsilon_Y & \epsilon_Y \end{pmatrix})$$

In conclusion for $\beta = \frac{\epsilon_X+1}{2} = \frac{1+\epsilon_Y}{1-\epsilon_Y}$, the bimatrix game (A, B^T) is linearly equivalent to (A_β, B_β). □

Remark 28. *In particular, in the Rock-Scissor-Paper game, described by coupled replicator equations, for the parameter values as in the above proposition it was anticipated in [109] that there should exist switching infinitesimally close to the heteroclinic network conjugated to the subshift of finite type corresponding to the periodic orbits of BRD (periodic points of the return jittering map). In [3] it was claimed, i.e. that infinite switching occurs near the heteroclinic network for all parameter values. We will take a closer look at this problem in chapter 4.*

We have however the following results from [20]:

Proposition 17. *(see [20])* ***(Orbits asymptotic to a heteroclinic cycle with periodic itineraries)***
For $\beta \in (0, \beta_\phi)$ and $\beta \in (\tau, 1)$ there exists an open set of initial conditions $(x(0), y(0))$ whose orbit $(x(t), y(t))$ for replicator equations (1.11.5) has a periodic time average $(X(t), Y(t))$.
1. For $\beta \in (0, \beta_\phi)$ this time average corresponds to a Shapley orbit, which arises from a heteroclinic cycle connecting nodes where the players alternate in winning. Specifically, the orbit $(x(t), y(t))$ of the replicator equations accumulates to a heteroclinic network, periodically visiting the corners of Σ (which are equilibria of the (1.11.5) system) in the following order
$(P, R),\ (P, S),\ (R, S),\ (R, P),\ (S, P),\ (S, P)$.

2. For $\beta \in (\sigma, 1)$ this corresponds to an anti-Shapley orbit, which arises from a heteroclinic cycle connecting nodes where players swap between a win for player B and a draw. Specifically, the orbit $(x(t), y(t))$ of the replicator equations accumulates to a heteroclinic network, visiting the corners of Σ in the following order $(R,R),\ (R,P),\ (P,P),\ (P,S),\ (S,S),\ (S,R)$
Here, $\beta_\phi = \frac{\sqrt{5}-1}{2}$ is the golden ratio, and $\tau \approx 0.915$ is a root of some high order polynomial (see [105]).

Remark 29. *In fact, as we will see in chapter 4, instead of a single orbit (for replicator equations) mentioned in the points 1 and 2 in the proposition above, there exist an open sets of initial conditions which are attracted to the respective cycles for the Rock-Scissor-Paper game.*

Remark 30. *From the proposition 13, the accumulation points of the time averages of the solutions to Rock-Scissors-Paper replicator equations, belong the chain transitive set of the BRD for the Shapley system. This chain transitive set consist of the periodic orbits which are precisely periodic points of the return jittering map.*

The following proposition (from [20]) is a corresponding result to the proposition 3 for the Shapley system under BRD. The subshifts of finite type mentioned in theorem 3, correspond to the random walk behaviour 'between periodic orbits' for the Shapley system.

Proposition 18. *(see [20])* ***(Chaos in (RD))***
For $\beta \in (\sigma, \tau)$ one has the following there exists a subshift Λ of finite type of the full shift space $(\{R,S,P\} \times \{R,S,P\})^{\mathbb{N}}$, so that
1. for each $(\lambda_i)_{i\geq 0}$ there exists an initial condition $(x(0), y(0))$ so that the corresponding (RD) orbit $(x(t), y(t))$ visit the corners $\{R,S,P\} \times \{R,S,P\}$ of Σ in the way prescribed by $(\lambda_i)_{i\geq 0}$
2. there exist times $t_1^- < t_1^+ < t_2^- < t_2^+ < ... < t_i^- < t_i^+$ so that $(x(t), y(t))$ is close to the corner λ_i for all $t \in (t_i^-, t_i^+)$, and so that $\frac{t_i^+ - t_i^-}{t_i^+} \to 1$ and $\frac{t_{i+1}^- - t_i^+}{t_{i+1}^-} \to 0$ as $i \to +\infty$

Remark 31. *However, due to the proposition 19 below, it is unclear how does the subshift of finite type from the proposition 18 looks like, i.e. which transitions in fact are attainable.*

The proposition below specifies only that for certain class of orbits some transitions are not allowed. This problem will be considered in the chapter 4 as well.

Proposition 19. *(see [20])* ***(Allowable orbits)***
Consider a replicator equations (1.11.5) orbit $(x(t), y(t))$ of the Rock-Scissors-Paper bimatrix game,

as in the previous section, so that $(x(0), y(0))$ *is in the interior of* Σ*. Let*

$$X(t) = \frac{1}{t}\int_0^t x(s)ds \quad \textit{and} \quad Y(t) = \frac{1}{t}\int_0^t y(s)ds$$

Assume that there exists $\alpha > 0$ *for which*

$$d((x(t), y(t)), \partial(\Sigma)) \leq exp(-\alpha t) \quad \textit{for all t large enough}$$

Then

1. there exists $\delta > 0$ *so that the distance of* $(X(t), Y(t))$ *to the Nash equilibrium is at least* δ *for all t large enough.*

2. the only possible transitions of replicator equations along the boundary correspond to the transitions available for (BRD) (see figure 7 in [105]).

Volenti non fit iniuria.

Gnaeus Domitius Annius Ulpianus

2

Introduction and statement of the results

2.1 - Introduction to Chapter 2

In this chapter we introduce the problems described in the next chapters and state the main results of each chapter. *

In section 1.2 we give a historical note about Hofbauer's conjecture. Chapter 3 is based on a joint work [80].

Section 1.3 is devoted to the Rock-Scissors-Paper bimatrix game governed by the coupled replicator equations. Chapter 4 is based on the preprint [79]

In section 1.4 we state the results from chapter 4 about non-transverse homoclinic channel of a compact center manifold and motivation, for studying this toy-model, taken from Rock-Scissors-Paper game. Chapter 5 is based on a joint work [81]

*This chapter is based on [79], [81], [80]

2.2 - Introduction to Chapter 3

Best response dynamics is a well-studied process in which two players respond to each other's moves choosing strategies in a very simple way, see equation (1.7.1) below. This dynamics, and the closely related fictitious play dynamics, see equation (1.7.2), was introduced by Brown [12] in 1949, see also [13]. Shortly after, in 1951, Robinson [90] showed that this dynamics converges to the set of Nash equilibria of the game. Since then it was shown that there are also classes of non-zero-sum games (potential games, dominance solvable games, supermodular games), for which one has convergence to the set of Nash equilibria, but in all these classes the convergence is to Nash equilibria which are on the *boundary* of the state space.

Since zero-sum games (or games equivalent to such games) were the only examples known for which one has convergence to a unique completely mixed Nash equilibrium, Hofbauer made the following conjecture, implicitly in [50, on page 87] and explicitly in [48, on page 19] and [51, on page 72] (where Hofbauer refers to the following conjectures as 'one of the main open problems in the evolutionary game dynamics'):

Conjecture. *(Hofbauer Conjecture)*
Assume that (A, B) *is a bimatrix game, with a unique completely mixed Nash equilibrium* NE*. Then this Nash equilibrium is stable (in the sense of Lyapunov) for the best response dynamics if and only if* (A, B) *is equivalent to a zero-sum game.*

Hart and Mas-Colell [45] went even further by showing that there exists a game Γ_0 so that every 'uncoupled dynamic' for any nearby game is not Nash convergent.

Hofbauer's Conjecture is appealing because it would provide a dynamical characterization of the class of zero-sum games. The purpose of chapter 3 is to provide some results which describe the relationship between convergence to Nash for best response and zero-sum games. This relationship will be given in terms of a flow on the level set of a certain energy function K defined by the equation (3.3.4) induced by the best response dynamics. This flow is simply the projection of the original flow on any level set of K and corresponds to studying the BR-dynamics in 'spherical coordinates'. (The proof that show that such a flow is well-defined, and essentially does not depend on the choice of the level set, relies on the fact that the best-response function is constant on half-rays emanating from the Nash equilibrium (E^A, E^B), see Section 3.5.)

One approach in addressing Hofbauer's conjecture is to obtain a result which is more or less a converse of the following result (from [108]):

Theorem. *(Zero-sum games induce volume preserving dynamics, see [108])*
Assume that (A, B) *is a bimatrix game, with a unique interior Nash equilibrium NE.*
(A, B) *is equivalent to a zero-sum game* $\implies$ *the induced dynamics on a level set of K is volume preserving (and in fact Hamiltonian).*

So consider an ensemble of strategies consisting of half-rays emanating from (E^A, E^B). According to this theorem, such sets preserve their volume under the BR-dynamics. Note that Hamiltonian flows usually are associated to system in classical mechanics where there is no friction. In view of this theorem, one could formulate the following version of Hofbauer's conjecture:

Conjecture. *(Alternative Hofbauer Conjecture)*
Assume that (A, B) *is a bimatrix game, with a unique interior Nash equilibrium NE.*
(A, B) *is equivalent to a zero-sum game* $\iff$ *the induced dynamics on a level set of K is volume preserving (and in fact Hamiltonian).*

Since the induced dynamics is about the 'mix of strategies' that players use, this conjecture would say that zero-sum games are precisely those for which an ensemble of half-rays of strategies does not change in volume under the dynamics.

It turns out that convergence of an orbit of the BR dynamics to the NE is completely determined by the volume expansion/contraction along the corresponding orbit. In this thesis we prove the following:

Theorem. *(Th. 6, section 3.5)*
Let (A, B) *be a two player game. Then the following holds: If each orbit of the Best Response Dynamics converges to Nash in infinite time then all attractors of the induced flow have Lyapunov-like exponents which are* ≥ 0*; more precisely,* $\lambda_+ \geq \lambda_- = 0$.

In Section 3.5 we will define the terms used in the statement of this theorem in detail, in particular the notion of the Lyapunov-like exponent (dispersion rate) of an orbit is defined in (3.5.12). In fact, in that section we will state a more complete version of this theorem, see Theorem 6.

Another characterization of the zero-sum games is the following result we prove in section 3.6:

Theorem. *(Th. 23, section 3.6)*
Let (A, B) *be a two player game. Then the following are equivalent:*

1. (A, B) *is equivalent to a zero-sum game;*

2. *the piecewise constant flow*

$$\begin{cases} \dot{p} \in BR_A(q) - E^A \\ \dot{q} \in BR_B(p) - E^B \end{cases}$$

has a continuous first integral whose level sets are hyperplanes.

Theorem 6 (section 3.5) states that each attractor corresponding to a system for which all BR orbits converge to Nash, must be 'neutral' or 'expanding'. Since naively a system which is expanding somewhere, should be contracting somewhere else, it is tempting to conclude that for such systems all orbits should be neutral.

The next theorem shows that having attractors for the induced system of a BR-system, does *not* imply that these attractors necessarily lead to exponential contraction. Indeed, we prove the following theorem (see proposition 24 and the whole construction of the example in the section 3.7), which shows that there are non-zero sum games which have neutral attractors:

Theorem. *(see section 3.7, and proposition 24)*
There exists a family non-zero 3×3 *games so that the flow induced by the best response dynamics has an attractor* Λ *with a forward invariant basin of attraction U (which is mapped compactly inside itself by the time t map of the flow) and so that all orbits in U have zero Lyapunov exponents. In particular, for the original best-response dynamics the orbits starting in the set corresponding to U converge to Nash in infinite time.*

The proof of this theorem describes a very reasonable scenario of a non-zero sum game for which *all* initial conditions for these examples converge to the unique interior Nash equilibrium. Although we do not suggest that this is the case with the examples which we study, we suggest that it might well be possible that this holds for examples in higher dimensions.

The latter theorem suggests that the Hofbauer Conjecture may well be incorrect. Indeed, having zero Lyapunov exponents is in general quite a lot weaker than having a first integral.

2.3 - Introduction to Chapter 4

Chapter 4 will be devoted to a Rock-Scissors-Paper game described by replicator equations, which is a simplified model of a cyclic competition bimatrix game. This model was derived in e.g. [94], [96]. Apart from [79], we will mainly refer here to six other papers on this topic: by Castro et al. [3], [36], [37] and three others by Sato et al. [94], [95], [96].

The first sections of Chapter 4 are devoted to the study of the RSP system on the codimension 1 and codimension 2 boundaries. In addition, we introduce the RSP network naturally arising in this system.

In proposition 42, we compute the linear part of the transition maps in convenient local coordinates. This together with lemma 5, shows that the transition maps are in fact approximated by affine mappings. In proposition 44 we present the approximation of the composition of local maps (proposition 41) with transition maps after rescaling of the coordinates.

In section 4.5, we prove that there is no infinite switching near the RSP network. The first reason for this is the following:

Theorem. *(Th. 9, section 4.5)*
For every $\epsilon_X, \epsilon_Y \in (-1, 1)$ the finite itineraries of the form tie $\to$ win/loss $\to$ loss/win $\to$ tie and tie $\to$ loss/win $\to$ win/loss $\to$ tie are not attainable for h - sufficiently small.

In section 4.5 we will also provide examples showing that attainability of some finite itineraries in the arbitrarily small neighbourhood of the heteroclinic network strongly depends on the parameter values of the system. In section 4.4.1 we investigate the asymptotic dynamics of the specific cycles C_0, C_1, C_2. We prove the following theorems 7 and 8:

Theorem. *(Th. 7, section 4.4.1)*
For $\epsilon_X + \epsilon_Y < 0$, the cycle C_0 is essentially asymptotically stable.

Theorem. *(Th. 8, section 4.4.1)*
For some open regions in parameter space (ϵ_X, ϵ_Y), cycles C_1 and C_2, respectively, are neither completely unstable nor essentially asymptotically stable. For each of them, there exist an open set of points in the phase space which is attracted to the respective cycle.

In papers by Sato et al. [94], [95], [96], the derivation of equations for the RSP game is presented. Further, these equations are transformed into other coordinates which are numerically stable, hence

very useful for the numerical simulations. The authors give a partial description of the global behaviour in the phase space, depending on the parameter values ϵ_X and ϵ_Y based on their numerical investigations. However they mainly concentrate on the case $\epsilon_X + \epsilon_Y = 0$. In Chapter 4, we will also investigate RSP system numerically, concentrating on an explanation of switching behaviour near the RSP network, existence of transient orbits to the cycles and also asymptotic dynamics of the cycles. In the last section of Chapter 4, we will compare our results with others existing in current literature.

2.4 - Introduction to Chapter 5

In Chapter 5 we investigate a toy-model consisting of a saddle-center in $\mathbb{R}^4$ and its 2 - dimensional compact center manifold which 3 - dimensional stable and unstable manifolds coincide, i.e. they create a non-transverse homoclinic channel, later denoted by I (see fig. 5.6). We derive the normal form of the return map to the vicinity of the center manifold along I and under some additional assumptions investigate its dynamics. Although all of the orbits starting in the vicinity of I accumulate on I, the limit dynamics of the return map might be very complicated. Even in a very simple scenario the return map can have the form of the 3-dimensional Henon-like map (in suitable coordinates) possessing the Lorenz-like attractor. It has been proven in [40], [41] that under some additional assumptions such a Henon-like map can be approximated by the time - 1 map of the flow governed by the Shimizu-Morioka system. The existence of the geometric Lorenz attractor for the Shimizu-Morioka system was recently proven in [16]. The return map can also satisfy the so-called 'cone-field condition' and hence be partially hyperbolic in the originally investigated coordinates, which leads to the existence of smooth invariant foliation. Factorizing along the leaves of this foliation reduces the problem to the investigation of 2-dimensional mapping.

The motivation for studying this toy-model comes from the bimatrix Rock-Scissors-Paper game. In [79] the dynamics on the boundary of the codimension 1 were investigated. In the RSP model the existence of 2 - dimensional invariant manifold with a connecting non-transverse homoclinic channel naturally appears for all considered parameter values. In fact, this 2-dimensional invariant manifold has 6 - connected components and they are connected with the corresponding 6 pieces of the non-transverse homoclinic channel. Each of these pieces lies in a different invariant codimension 1 subspace. In Chapter 5 we provide a general picture of the RSP model as an example in the section 5.2.3 as well as its relation to the toy-model we are considering. In particular, we touch upon questions

arising from the RSP model, including the form of the scattering maps defined by the pieces of the non-transverse homoclinic channel and dynamics in the vicinity of this channel, i.e. the possibility of reducing the problem to the study of the scattering maps and infinite shadowing of their composition. Despite the fact that in the RSP model the homoclinic channel is non-transverse and the general theory of scattering maps assumes transversality condition to hold (see [26]), the considered homoclinic channel in fact fulfills the invariant codimension 1 invariant subspaces. Hence, it makes it possible to consider scattering maps in the RSP model. However, in the absence of the transversality condition it is not possible to utilize topological tools [120] to ensure infinite shadowing of the scattering maps as in [39], [27]. Furthermore, comparing our toy-model, we provide an explanation as to why one should not expect infinite shadowing of the scattering maps in the RSP model.

Scattering maps are a powerful geometric tool for investigations, especially in diffusion problems in Hamiltonian systems (a scattering map becomes symplectic in this case). However, they are not defined in the case of intersection of invariant manifolds being robustly non-transverse. The problem of shadowing along such a non-transverse connections with the usage of topological tools was investigated in [28], which is also related to the problems with the RSP model [79], i.e. shadowing of the heteroclinic network (see chapter 4).

Although in theory one gets rid of the issues with the non-transversality via perturbation methods, in practice the problems with non-transversality still have to be dealt with (see [4]). Chapter 5 also addresses these types of questions.

In the remaining part of this section we briefly introduce considered model and state the main results of Chapter 5.

We consider a saddle-center in $\mathbb{R}^4$ and its 2 - dimensional center manifold D foliated by periodic orbits. We assume that the center manifold possesses the 3 - dimensional stable and unstable manifolds which coincide (see fig. 5.6). We prove the following theorem about the normal form of this system near the center manifold:

Theorem. *(see equation 5.3.4, section 5.3)*

There exists local coordinates $(v_1, v_2, r, \phi) \in \mathbb{R}^4$ *near the center manifold D in which* $(0,0,0,0)$ *is the saddle-center and the set* $\{(0,0,r,\phi) \mid r \in [0,1], \phi \in [0,1)\}$ *is the center manifold D. The* (r,ϕ) *are polar coordinates. The coordinates* v_1, v_2 *correspond to the locally unstable and stable directions.*

The normal form of the ODE describing this model, near the center manifold is:

$$\begin{cases} \dot{v_1} = p(r) \cdot v_1 + p_0(v_1, v_2, r, \phi) \cdot {v_1}^2 v_2 \\ \dot{v_2} = \varsigma(r) \cdot v_2 + \varsigma_0(v_1, v_2, r, \phi) \cdot v_1 {v_2}^2 \\ \dot{r} = v_1 v_2 \cdot r_0(v_1, v_2, r, \phi) \\ \dot{\phi} = \omega(r) + v_1 v_2 \cdot \omega_0(v_1, v_2, r, \phi) \end{cases}$$

with p_0, ς_0, r_0, ω_0 being 1 - periodic in ϕ.

We derive the normal form of the return map along the non-transverse homoclinic channel I, formed by the coinciding stable and unstable manifolds of D:

Theorem. *(Th. 10, equation 5.3.37, section 5.3.2)*
The normal form of the return map to the cross section $S_1 := \{v_1 = h\}$ is

$$\begin{pmatrix} v_2 \\ r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} h \cdot \left(\frac{v_2 \cdot a_1(r,\phi)}{h}\right)^{\cdot \frac{-\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))}} + O(v_2 \cdot ln(v_2)) \\ b_0(r, \phi) + O(v_2) \\ c_0(r, \phi) - \frac{\omega(b_0(r,\phi))}{p(b_0(r,\phi))} \cdot ln(\frac{v_2}{h}) + O(v_2 \cdot ln(v_2)) \ \ (mod\ 1) \end{pmatrix}$$

with b_0, c_0, a_1 being 1 - periodic in ϕ.

We investigate the dynamics of the return map under the condition $\varsigma(r) + p(r) < 0$:

Remark. *If $\varsigma(r) + p(r) < 0$, then there exists a neighbourhood of the homoclinic channel I, such that orbit of each point from this neighbourhood accumulates on I.*

Furthermore we investigate the dynamics of the return map after the coordinate transformation $z \approx ln(v_2)$

Theorem. *(Th. 57, section 5.3.2)*
The normal form of the return map along I, in the coordinates (z, r, ϕ) is:

$$\begin{pmatrix} z \\ r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} \Omega(r, \phi) + \Gamma(r, \phi) \cdot z + O(z \cdot e^z) \\ b_0(r, \phi) + O(e^z) \\ c(r, \phi) + z + O(z \cdot e^z) \ \ (mod\ 1) \end{pmatrix} \tag{2.4.1}$$

with Ω, $\Gamma(r, \phi) = -\frac{\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))}$, b_0, c being 1 - periodic in ϕ.

Assuming that $-\frac{\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))} = \Gamma(r,\phi) = \Gamma \in \mathbb{N}$, $\Gamma > 1$, we prove that in the limit with $v_2 \to 0$, the dynamics of the return map are the same as the dynamics of its truncated version:

$$\begin{pmatrix} z \\ r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} \Omega(r,\phi) + \Gamma \cdot z \ (mod\ 1) \\ b_0(r,\phi) \\ c(r,\phi) + z \ \ (mod\ 1) \end{pmatrix} \tag{2.4.2}$$

This is due to the following theorem we prove:

Theorem. *(Th. 11, section 5.3.3)*

For the mapping ϖ:

$$\varpi : \ \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times S^1 \ni \begin{pmatrix} v_2 \\ z \\ r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} h \cdot \left(\frac{a_1(r,\phi)\cdot v_2}{h}\right)^{\frac{-\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))}} + h.o.t.(v_2) \\ \Omega(r,\phi) + \Gamma(r,\phi)\cdot z + O(v_2\cdot\ln(v_2)) \ (mod\ 1) \\ b_0(r,\phi) + O(v_2) \\ c(r,\phi) + z + O(v_2\cdot\ln(v_2)) \ \ (mod\ 1\ ) \end{pmatrix} \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times S^1 \tag{2.4.3}$$

there exists a C^1 and ϖ - invariant foliation $\mathcal{L} = \{h(v_2)\}$ of the space (v_2, z, r, ϕ), with C^2 leaves given by the graphs of the functions $(z, r, \phi) = h(v_2)$.

So there is a one-to-one correspondence between the orbits under the return map (2.4.1) and truncated return map (2.4.2).

It turns out that the coefficients $\Omega(r,\phi), \Gamma, b_0(r,\phi), c(r,\phi)$ can be chosen in such a way that the truncated return map is conjugated to the 3 - dimensional Henon-like map possessing Lorenz-like attractor.

Proposition. *(Prop. 59, section 5.3.3)*

There exist coefficients $\Omega(r,\phi)$, Γ, $b(r,\phi)$, $c(r,\phi)$, such that the mapping

$$T : \ \begin{pmatrix} z \\ r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} \Omega(r,\phi) + \Gamma\cdot z \ \ (mod\ 1) \\ b(r,\phi) \\ c(r,\phi) + z \ \ (mod\ 1) \end{pmatrix} \tag{2.4.4}$$

has a fixed point with a normal form, in a suitable coordinates, given by:

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \mapsto \begin{pmatrix} y \\ w \\ x + y - w + A \cdot y^2 + B \cdot y \cdot w + C \cdot w^2 + h.o.t. \end{pmatrix} \tag{2.4.5}$$

with $(C-A)\cdot(A-B+C) > 0$. *Hence, by [40], [41] and [16] there exists arbitrarily small perturbation of this mapping possessing a Lorenz-like attractor.*

Finally we investigate partial hyperbolicity of the truncated return map:

Proposition. *(Prop. 60, section 5.3.4)*
There exist coefficients $\Omega(r,\phi)$, $\Gamma(r,\phi)$, $b(r,\phi)$, $c(r,\phi)$, *such that the mapping* T *possesses an invariant cone-field*

$$C_{L,(z,r,\phi)} := \left\{ v = v_1 + v_{23} \in T_{(z,r,\phi)}M \mid v_1 = (v_{11}, 0, 0),\ v_{23} = (0, v_2, v_3) \text{ such that } \|v_{23}\| < L \cdot |v_1| \right\}$$

where $M := \mathbb{R} \times [0,1] \times S^1$, $L < 1$ *and* $\|\cdot\|$ *denotes the Euclidean norm.*

Fames est optimus coquus.

Marcus Tullius Cicero

3

On Hofbauer's conjecture

3.1 - Introduction

* In this chapter we investigate the convergence of the solutions of the Best Response Dynamics to the completely mixed Nash equilibrium. The whole analysis performed in the sections (3.2 - 3.6) is valid for bimatrix games of any dimension under the assumption of existence of a completely mixed and unique Nash equilibrium. In the section (3.7), however, we provide a specific 3×3 example of bimatrix game questioning the correctness of Hofbauer's hypothesis.

In the section 3.3 we introduce an energy function K, which for the zero-sum case is a Lyapunov function for the Nash equilibrium. Its non-empty level sets are topologically codimension-one spheres (onto which we will project BRD flow) or a Nash equilibrium. Convergence to the Nash equilibrium is then equivalent to the energy function K decreasing to 0 along the solutions of BRD, what is described in the sections 3.4 - 3.5, along with sufficient conditions for this to happen.

The section 3.6 contains a characterization of the zero-sum bimatrix games in terms of possessing a smooth first integral by the corresponding reduced BRD flow.

In section 3.7, we provide a presumptive evidence that Hofbauer's conjecture might be incorrect. We

*The content of this chapter is mainly based on a joint work: Olszowiec C., van Strien S., Turaev D.V., "Towards a dynamical characterization of zero-sum games and a conjecture by Hofbauer"

consider a 3 parameter family of matrices forming a family of bimatrix games being not equivalent to zero-sum. Within this family we find 2 parameter subfamily of bimatrix games, so that each member of it possesses an open invariant cone of initial conditions converging to the Nash equilibrium.

3.2 - Best response dynamics as averages over actions

Proposition 20. *The following equations hold for best response dynamics*

$$p(t) = e^{-t} \int_0^t BR_A(q(s))e^s ds + e^{-t}p(0)$$

$$q(t) = e^{-t} \int_0^t BR_B(p(s))e^s ds + e^{-t}q(0).$$

(Note that this is simply the frequency of playing the different strategies, the values of BR_A and BR_B, but with respect to reparametrized time.)

Proof. The differential inclusion (1.7.1) immediately implies that if $t_1, \dots, t_{n-1}$ are times $\leq t$ when the best responses are indifferent, i.e. when players switch preference, then we obtain

$$p(t_n) = e^{-t_n}p(t_0) + e^{t_1-t_n}(1 - e^{t_0-t_1})\hat{e}_1^A + e^{t_2-t_n}(1 - e^{t_1-t_2})\hat{e}_2^A + \cdots + (1 - e^{t_{n-1}-t_n})\hat{e}_n^A.$$

$$q(t_n) = e^{-t_n}q(t_0) + e^{t_1-t_n}(1 - e^{t_0-t_1})\hat{e}_1^B + e^{t_2-t_n}(1 - e^{t_1-t_2})\hat{e}_2^B + \cdots + (1 - e^{t_{n-1}-t_n})\hat{e}_n^B.$$

Here $\hat{e}_i^A = BR_A(q(s))$ and $\hat{e}_i^B = BR_B(p(s))$ are the aims players A and B aim for for $s \in (t_{i-1}, t_i)$. This immediately implies the required statement. See also [83]. □

Given a n dimensional row vector $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_m)$ a m dimensional column vector, let $q * p$ be the $m \times n$ matrix with (i, j) entry equal to $q_i p_j$. Using this notation, let us define the joint best response matrix

$$BR(s) := BR_A(q(s)) * BR_B(p(s)).$$

So this is a matrix whose entries are equal to 0 apart from its (i, j) entry where $e'_j = BR_A(q(s))$ and $e_i = BR_B(q(s))$ (so it is measuring the joint best response at time s).

Later on, it is also useful to consider the matrix

$$P(t) = e^{-t} \int_0^t BR(s)e^s ds + e^{-t} q(0) * p(0)$$

3.3 - A Lyapunov function associated to the game

Definition 12. *We say that two $m \times n$ bimatrix games (A, B) and $(\tilde{A}, \tilde{B})$ are* (linearly) equivalent*, $(A, B) \sim (\tilde{A}, \tilde{B})$, if the matrix $\tilde{A}$ can be obtained by multiplying A with a positive constant $c > 0$ and adding constants $c_1, \dots, c_n \in \mathbb{R}$ to the matrix columns, and $\tilde{B}$ can be obtained from B by multiplication with $d > 0$ and addition of $d_1, \dots, d_m \in \mathbb{R}$ to its rows:*

$$\tilde{a}_{ij} = c \cdot a_{ij} + c_j \quad \text{and} \quad \tilde{b}_{ij} = d \cdot b_{ij} + d_i \qquad \text{for } i = 1, \dots, m \text{ and } j = 1, \dots, n.$$

Note that this is equivalent to saying that $(A, B) \sim (\tilde{A}, \tilde{B})$ if and only if their best response maps are the same.

Let us now define the energy function which is associated to the bimatrix game. First, introduce the following functions:

$$\bar{A}(q) \coloneqq \max_{\bar{p} \in \Sigma_A} \bar{p} \cdot A \cdot q \quad \text{and} \quad \bar{B}(p) \coloneqq \max_{\bar{q} \in \Sigma_B} p \cdot B \cdot \bar{q},$$

Observe that $\bar{A}(q) = \max_i (Aq)_i$ and $\bar{B}(p) = \max_j (pB)_j$: the maximal payoff to player A given player B's strategy q is equal to the maximal entry of the vector Aq, and similarly for player B.

The following theorem was proven in [83, Theorem 3.3]

Theorem 4. *Let (A, B) be an $n \times n$ bimatrix game with a unique, completely mixed Nash equilibrium (E^A, E^B). Then there exists a linearly equivalent game (A', B'), for which*

$$\bar{A}'(q) > \bar{A}'(E^B) \text{ and } \bar{B}'(p) > \bar{B}'(E^A) \tag{3.3.1}$$

for all $p \in \Sigma_A \setminus \{E^A\}$ and $q \in \Sigma_B \setminus \{E^B\}$.

From now on we assume that (A, B) satisfies equation (3.3.1) (for (A, B) rather than (A', B')). Since we

consider matrices (A, B) with a unique, completely mixed Nash equilibrium (E^A, E^B), all components of the vectors AE^B and E^AB) are equal. We can therefore also always assume (by replacing (A, B) by an equivalent matrix) that

$$AE^B = 0 \text{ and } E^AB = 0. \tag{3.3.2}$$

which together with (3.3.1) implies

$$\bar{A}(E^B) = \bar{B}(E^A) = 0. \tag{3.3.3}$$

Let us now define

$$K(p, q) := BR_A(q) \cdot A \cdot q + p \cdot B \cdot BR_B(p) \ \ (= \bar{A}(q) + \bar{B}(p)). \tag{3.3.4}$$

Note that it follows from the definition of BR_A, BR_B that this function is single-valued and continuous.

Since we assumed that (3.3.1) and (3.3.2) hold, it follows that $K(E^A, E^B) = 0$ and by the previous theorem $K(p, q) > 0$ except when $(p, q) = (E^A, E^B)$. For this reason, in this chapter, we think of the function $K \colon \Sigma_A \times \Sigma_B \to \mathbb{R}^+$ as measuring distance to the Nash equilibrium, and we will use K to define some kind of spherical coordinates. We shall also think of K as an energy function. Our first new result concerning function K is the following:

Proposition 21. *The function $K : \Sigma_A \times \Sigma_B \to \mathbb{R}$ is continuous and the level set $K_r := \{K(p, q) = r\}$ is empty (when $r < 0$), a Nash equilibrium (when $r = 0$) or is a codimension-one sphere (when $r > 0$).*

Proof. It suffices to check that K increases linearly along every ray starting at the Nash equilibrium:

$$K(p + s(p - E^A), q + s(q - E^B)) =$$

$$BR_A(q + s(q - E^B)) \cdot A \cdot (q + s(q - E^B)) + (p + s(p - E^A)) \cdot B \cdot BR_B(p + s(p - E^A)) =$$

$$BR_A(q) \cdot A \cdot (q + s(q - E^B)) + (p + s(p - E^A)) \cdot B \cdot BR_B(p) =$$

$$(1 + s)\,(BR_A(q) \cdot A \cdot q + p \cdot B \cdot BR_B(p)) + s\left(BR_A(q) \cdot A \cdot E^B + E^A \cdot B \cdot BR_B(p)\right) =$$

$$(1 + s)\,(BR_A(q) \cdot A \cdot q + p \cdot B \cdot BR_B(p)) = (1 + s)K(p, q), \ \ as \ \ s > 0$$

□

3.4 - The value of K along orbits

Proposition 3.4.1.

$$K(p(t), q(t)) = \left[\int_0^t BR_A(q(s)) \cdot [A + B] \cdot BR_B(p(s)) e^s \, ds + K(p(0), q(0)) \right] e^{-t}. \tag{3.4.1}$$

In particular, if (A, B) is a zero-sum bimatrix game, then $K(p(t), q(t)) = K(p(0), q(0)) \cdot e^{-t}$ and so the orbit tends to the Nash equilibrium as $t \to \infty$.

Proof. Since BR_A and BR_B are piecewise constant,

$$\begin{aligned} \frac{dK}{dt} &= BR_A(q) \cdot A \cdot \frac{dq}{dt} + \frac{dp}{dt} \cdot B \cdot BR_B(p) \\ &= BR_A(q) \cdot A \cdot (BR_B(p) - q) + (BR_A(q) - p) \cdot B \cdot BR_B(p) \\ &= -K + BR_A(q) \cdot [A + B] \cdot BR_B(p). \end{aligned} \tag{3.4.2}$$

Of course this implies that when $A + B = 0$ then $\frac{dK}{dt} = -K$ and therefore K decreases exponentially fast to zero along BR-orbits. In particular the Nash equilibrium is asymptotically stable in that case. If $A + B \neq 0$ one obtains from equation (3.4.2) that (3.4.1) holds. □

In particular, orbits converge to the Nash equilibrium for zero-sum games, see [53] and also [48].

Corollary 2. *If an orbit converges to the Nash equilibrium as $t \to \infty$ then*

$$e^{-t} \int_0^t BR_A(q(s)) \cdot [A + B] \cdot BR_B(p(s)) e^s \, ds \to 0. \tag{3.4.3}$$

If an orbit hits the Nash equilibrium at time $t = T$ then

$$\left[\int_0^T BR_A(q(s)) \cdot [A + B] \cdot BR_B(p(s)) e^s \, ds \right] + K(0) = 0. \tag{3.4.4}$$

Remark 32. *Note that if $BR_A(q)$ and $BR_B(p)$ are equal to unit vectors, then $BR_A(q) \cdot [A + B] \cdot BR_B(p)$ corresponds to one of the entries of the matrix $A + B$. In particular, $\frac{dK}{dt}$ is not necessarily small when $(p(t), q(t))$ is close to the Nash equilibrium and $A + B \neq 0$. If $A + B \neq 0$ and (A, B) is equivalent to a*

zero-sum game, then the $(p(t), q(t))$ orbits still converge to the Nash equilibrium, so K tends to zero. It follows that the positive and negative contributions of the integral (3.4.3) cancel in this case.

3.5 - Decomposing the vector field

Following [108], in this section we will consider the projection of the vector field (1.7.1) on the $(n + m - 3)$-dimensional topological sphere $K^{-1}(\rho)$. Consider the map

$$\pi \colon (\Sigma^A \times \Sigma^B) \setminus (E^A, E^B) \to K^{-1}(1)$$

defined by assigning to a point in $(\Sigma^A \times \Sigma^B) \setminus (E^A, E^B)$ the intersection point of the ray through this point emanating from (E^A, E^B) with $K^{-1}(1)$.
For $(p, q) \in (\Sigma^A \times \Sigma^B) \setminus (E^A, E^B)$ let us define

$$(\bar{p}, \bar{q}) = \pi(p, q) \in K^{-1}(1)$$

Note that on each ray through (E^A, E^B), the function K is linear, i.e.,

$$(p, q) - (E^A, E^B) = K(p, q) \cdot \Big((\bar{p}, \bar{q}) - (E^A, E^B)\Big)$$

Also BR_A and BR_B are constant on such rays, and so $BR_B(p) = BR_B(\bar{p})$ and $BR_A(q) = BR_A(\bar{q})$.

Proposition 22. *The map π projects the vector field associated to best response dynamics (1.7.1) onto the vector field*

$$\begin{cases} \dfrac{d\bar{p}}{dt} = (BR_A(\bar{q}) - \bar{p}) + \alpha(\bar{p}, \bar{q}) \cdot (\bar{p} - E^A) \\ \dfrac{d\bar{q}}{dt} = (BR_B(\bar{p}) - \bar{q}) + \alpha(\bar{p}, \bar{q}) \cdot (\bar{q} - E^B) \end{cases} \tag{3.5.1}$$

where $\alpha(\bar{p}, \bar{q})$ is a following piecewise constant function

$$\alpha(\bar{p}, \bar{q}) := 1 - BR_A(\bar{q}) \cdot (A + B) \cdot BR_B(\bar{p}) \tag{3.5.2}$$

Proof. The projection of the vector field is of the form (3.5.1) and so the function α is chosen so that the orbits remain in $K^{-1}(1)$. This means that

$$\frac{dK(p(t), q(t))}{dt} = BR_A(\bar{q}) \cdot A \cdot \big((BR_B(\bar{p}) - \bar{q}) + \alpha(\bar{q} - E^B)\big) + \big((BR_A(\bar{q}) - \bar{p}) + \alpha(\bar{p} - E^A)\big) \cdot B \cdot BR_B(\bar{p}) =$$

$$(\alpha(\bar{p},\bar{q})-1)K(\bar{p},\bar{q})+BR_A(\bar{q})\cdot(A+B)\cdot BR_B(\bar{p})=0$$

This gives the expression in the proposition. □

It follows that equation (1.7.1) can be written in spherical coordinates:

$$\begin{cases}\dfrac{dK}{dt} &= -K+BR_A(q)\cdot[A+B]\cdot BR_B(p)\\[2ex] \dfrac{d\bar{p}}{dt} &= \dfrac{1}{K}[(BR_A(\bar{q})-\bar{p})+\alpha(\bar{p},\bar{q})\cdot(\bar{p}-E^A)]\\[2ex] \dfrac{d\bar{q}}{dt} &= \dfrac{1}{K}[(BR_B(\bar{p})-\bar{q})+\alpha(\bar{p},\bar{q})\cdot(\bar{q}-E^B)]\end{cases} \tag{3.5.3}$$

Applying a time reparameterization

$$dt = Kd\tau$$

we get

$$\begin{cases}\dfrac{dK}{d\tau} &= -K^2+K\cdot BR_A(q)\cdot[A+B]\cdot BR_B(p)\\[2ex] \dfrac{d\bar{p}}{d\tau} &= (BR_A(\bar{q})-\bar{p})+\alpha(\bar{p},\bar{q})\cdot(\bar{p}-E^A)\\[2ex] \dfrac{d\bar{q}}{d\tau} &= (BR_B(\bar{p})-\bar{q})+\alpha(\bar{p},\bar{q})\cdot(\bar{q}-E^B)\end{cases} \tag{3.5.4}$$

Note that the solutions of this system (3.5.4) are defined for all τ. As $t(\tau)=\int_0^\tau K(\tau')\,d\tau'$, it is quite possible that for certain solutions for which $K(\tau)\to 0$ as $\tau\to\infty$ one has $\lim_{\tau\to\infty} t(\tau)<\infty$. So such orbits converge under the best response dynamics to Nash in finite time.

In the remainder of this chapter we shall focus on the flow restricted to the level set $K^{-1}(1)$:

$$\begin{cases}\dfrac{d\bar{p}}{d\tau} &= (BR_A(\bar{q})-\bar{p})+\alpha(\bar{p},\bar{q})\cdot(\bar{p}-E^A)\\[2ex] \dfrac{d\bar{q}}{d\tau} &= (BR_B(\bar{p})-\bar{q})+\alpha(\bar{p},\bar{q})\cdot(\bar{q}-E^B)\end{cases} \tag{3.5.5}$$

Remark 33. *If $A+B=0$ then $\alpha(\bar{p},\bar{q})=1$ and so the r.h.s. of this vector field is piecewise constant. In particular the flow (3.5.5) is volume preserving. It was shown in [108] that in the zero-sum case,*

this flow is even Hamiltonian (w.r.t. a non-standard symplectic form).

Corollary 3. *The divergence of the* $(\bar{p}, \bar{q})$ *part of the vector field (3.5.4) is equal to*

$$div(\bar{p}, \bar{q}) := N \cdot (\alpha(\bar{p}, \bar{q}) - 1) = -N \cdot (BR_A(\bar{q}) \cdot (A + B) \cdot BR_B(\bar{p})) \tag{3.5.6}$$

where N *is equal to the dimension of* $\partial(\Sigma_A \times \Sigma_B)$*, i.e.* $N = n + m - 3$.

Let us denote $\psi(\tau, (\bar{p}, \bar{q}))$ as the $(\bar{p}, \bar{q})$ part of the flow generated by (3.5.4) and relate the behaviour of $K(\tau)$ as $\tau \to \infty$ with the divergence of the vector field (3.5.5). To do this, define

$$a(\tau) = \det\left(\frac{\partial\psi(\tau, (\bar{p}, \bar{q}))}{\partial(\bar{p}, \bar{q})}\right) \tag{3.5.7}$$

Then $a(0) = 1$, $a(\tau) > 0$ for all $\tau > 0$ and from Liouville's formula:

$$a'(\tau) := a(\tau) \cdot div(\bar{p}(\tau), \bar{q}(\tau)) \tag{3.5.8}$$

and therefore

$$a(\tau) = \exp\left(\int_0^\tau div(\bar{p}(s), \bar{q}(s))\, ds\right) = \exp\left(-N \int_0^\tau BR_A(\bar{q}(s)) \cdot (A + B) \cdot BR_B(\bar{p}(s))\, ds\right) \tag{3.5.9}$$

where in the 2nd equality we use the above corollary. Note that $a(\tau)$ is continuous in τ as the integrand is piecewise constant. Using (3.5.6) in the radial component K of (3.5.4) we obtain the Bernoulli equation

$$\frac{dK}{d\tau} + K^2 + \frac{K}{N} \cdot div(\bar{p}(\tau), \bar{q}(\tau)) = 0. \tag{3.5.10}$$

Let us solve this differential equation. Using (3.5.8) and substituting $s = 1/K$ this Bernoulli equation becomes

$$\frac{ds}{d\tau} - 1 - \frac{s(\tau)}{N}\frac{a'(\tau)}{a(\tau)} = 0$$

This is equivalent to

$$a^{-\frac{1}{N}}(\tau)\frac{ds}{d\tau} - \frac{s(\tau)}{N}a^{-\frac{1}{N}-1}(\tau)a'(\tau) - a^{-\frac{1}{N}}(\tau) = 0$$

and therefore to $(a^{-\frac{1}{N}}s)' = a^{-\frac{1}{N}}$ and so $a(\tau)^{-\frac{1}{N}} \cdot s(\tau) = a(0)^{-\frac{1}{N}} \cdot s(0) + \int_0^\tau [a(s)]^{-\frac{1}{N}}\, ds$ which, as $a(0) = 1$, gives

$$K(\tau) = \frac{K(0)}{[a(\tau)]^{\frac{1}{N}} + K(0)\int_0^\tau [a(\tau)/a(s)]^{\frac{1}{N}} ds} \tag{3.5.11}$$

This means that the asymptotic behaviour of $a(\tau)$ as $\tau \to \infty$ determines that of $K(\tau)$. For example, in a zero-sum game, $BR_A(q(t)) \cdot (A + B) \cdot BR_B(p(t)) = 0$, so $div(\bar{p}(\tau), \bar{q}(\tau)) \equiv 1$ and $a(\tau) \equiv 1$. More generally, it is worth defining the following quantities along an orbit $(\bar{p}(\tau), \bar{q}(\tau))$ of the system:

$$\lambda_+(\bar{p}, \bar{q}) := \limsup_{\tau\to\infty} \frac{1}{\tau} \log a(\tau) \quad \text{and} \quad \lambda_-(\bar{p}, \bar{q}) := \liminf_{\tau\to\infty} \frac{1}{\tau} \log a(\tau) \tag{3.5.12}$$

These are the dispersion rates from an orbit of the system induced on the $(\bar{p}, \bar{q})$-space (i.e. the projected flow on the 'spherical' level set $K^{-1}(1)$).

Remark 34. *Some authors may call these dispersion rates also a Lyapounov-like exponents, although the standard definition of the Lyapounov exponents is different from the definition 3.5.12.*

For example, along a periodic orbit in the induced flow, $\lambda_+ = \lambda_-$ is the product of its multipliers. Let $o(\tau), \epsilon(\tau)$ be the functions so that $o(\tau)/\tau \to 0$ and $\epsilon(\tau) \to 0$ as $\tau \to \infty$.

Theorem 5. *If $\int_0^\infty K(\tau)d\tau < \infty$ then the BR-solution tends to the Nash equilibrium in finite time. Moreover, if $\frac{\log K(\tau)}{\tau} \to 0$ and $\frac{1}{\tau}\int_0^\tau K(\tau)\, d\tau \to 0$ as $\tau \to \infty$ then $\lambda_- = \lambda_+ = 0$.*

Proof. Since $dt = Kd\tau$, we have $\lim_{\theta\to\infty} t(\theta) = \int_0^\infty Kd\tau$. To see the 2nd assertion, note that equation (3.5.10) gives

$$\frac{1}{\tau}\int_0^\tau \frac{d\log K}{d\tau} d\tau + \frac{1}{\tau}\int_0^\tau K(\tau)\, d\tau + \frac{1}{\tau N}\int_0^\tau div(\bar{p}(\tau), \bar{q}(\tau))\, d\tau = 0$$

and so

$$\frac{\log K(\tau)}{\tau} + \frac{1}{\tau}\int_0^\tau K(\tau)\, d\tau + \frac{1}{N}\frac{1}{\tau}\log a(\tau) = 0$$

and the proof follows. □

Theorem 6. *For each orbit one has the following:*

1. *If $\lambda_- > 0$ then $K(\tau) \le K_0 e^{-\frac{\lambda_-}{N}\tau + o(\tau)}$ as $\tau \to \infty$, and in the original time t this means $K(t)$ reaches the Nash equilibrium in finite time.*

2. *If $\lambda_- < 0$ then $K(\tau)$ does not tend to zero as $\tau \to \infty$ and*

$$\limsup_{\tau\to\infty} K(\tau) \ge \frac{|\lambda_-|}{N}.$$

In particular the orbit exists for all $t \geq 0$ (where t is the original time). If $\lambda_+ < 0$ holds then it is possible that K oscillates forever between a finite value $\frac{|\lambda_-|}{N}$ and zero.

3. *If $\lambda_+ \geq 0$ then* $\liminf_{\tau\to\infty} K(\tau) = 0$

Proof. 1. If $\lambda_- > 0$ then $a(\tau) \geq \exp\left[(\lambda_- - \epsilon(\tau))\tau\right]$ for all τ large. Hence $\int_0^\tau \left[\frac{1}{a(s)}\right]^{\frac{1}{N}} ds \leq K_0$ for some K_0 and using (3.5.11) we obtain $K(\tau) \leq K_0 e^{-\frac{\lambda_-}{N}\tau + o(\tau)}$. By the previous theorem the first assertion holds.
2. If $\lambda_- < 0$ then there exists τ_n arbitrarily large and function ϵ so that

$$a(\tau_n) = \exp\left[-|\lambda_-|\tau_n + \epsilon(\tau_n)\tau_n\right] \quad \text{and} \quad a(s) \geq \exp\left[-|\lambda_-|s + \epsilon(s)s\right]$$

for all $s \leq \tau_n$. It follows that

$$\int_0^{\tau_n} \left[\frac{a(\tau_n)}{a(s)}\right]^{\frac{1}{N}} ds \leq \int_0^{\tau_n} \exp[-\frac{|\lambda_-|(1+\epsilon(\tau_n))}{N}(\tau_n - s)]\, ds \leq \frac{N(1+\epsilon(\tau_n))}{|\lambda_-|} \cdot (1 - \exp[-\frac{|\lambda_-|(1+\epsilon(\tau_n))}{N} \cdot \tau_n])$$

and using (3.5.11), $\lim_{n\to\infty} K(\tau_n) \geq \dfrac{|\lambda_-|}{N}$. If also $\lambda_+ < 0$ then $a(s) \leq \exp\left[-|\lambda_+|s + \epsilon(s)s\right]$ for all s and if $\int_0^{\tau_n} \left[\dfrac{a(\tau_n)}{a(s)}\right]^{\frac{1}{N}} ds$ converges to ∞ for some subsequence $\tau_n \to \infty$ then $K(\tau)$ oscillates between 0 and $|\lambda_-|/N$.
3. To see the third assertion, note that the 2nd expression in (3.5.11) and therefore $K(\tau)$ can be written as $Q'(\tau)/Q(\tau)$ where $Q(\tau) := 1 + K(0) \cdot \int_0^\tau \left[\frac{1}{a(s)}\right]^{\frac{1}{N}} ds$. So assume by contradiction that $K(\tau) \geq \rho > 0$ for all τ. Then $Q'(\tau)/Q(\tau) \geq \rho$ and so $Q(\tau) \geq \exp(\rho\tau)$ for all τ. It follows that $[a(\tau)]^{-1/N} = Q'(\tau) \geq \rho \exp(\rho\tau)$ and so $\log a(\tau) \leq -N\log(\rho) - N\rho\tau$ and therefore $\lambda_+ = \limsup_{\tau\to\infty} \dfrac{1}{\tau} \log a(\tau) \leq -N\rho < 0$, a contradiction. □

Remark 35. *In the section 3.7 we will describe a concrete scenario where the return map can be either volume preserving or contracting (but the latter with decreasing frequency) and where $\lambda_- = \lambda_+ = 0$.*

3.6 - When is the constant flow integrable?

We will find it convenient to move the Nash equilibrium to $(0,0)$, i.e. consider the coordinates $(p^o, q^o) := (p,q) - (E^A, E^B)$. After substituting this in the BR dynamics (1.7.1), we arrive at the

differential inclusion:

$$\begin{cases} \dot{p}^o \in BR_A(q^o) - p^o - E^A \\ \dot{q}^o \in BR_B(p^o) - q^o - E^B \end{cases} \tag{3.6.1}$$

where $(p^o, q^o) \in \Sigma_A^0 \times \Sigma_B^0$ and $\Sigma_A^0 := \Sigma_A - E^A$, $\Sigma_B^0 := \Sigma_B - E^B$. So $p_1^o + p_2^o + p_3^o = 0$ and $q_1^o + q_2^o + q_3^o = 0$. We note that $BR_A(p) = BR_A(p^0)$ and $BR_B(q) = BR_B(q^0)$.

Given $r > 0$, let $\tilde{\pi}_r \colon (\Sigma_A^0 \times \Sigma_B^0) \backslash \{(0,0)\} \to K^{-1}(r)$ be the map defined by assigning to a point in $(\Sigma_A^0 \times \Sigma_B^0) \backslash \{(0,0)\}$ the intersection point of the ray through this point from $(0,0)$ intersecting $K^{-1}(r)$. For convenience we write $(\tilde{p}, \tilde{q}) = \tilde{\pi}_r(p, q)$.

It turns out that the dynamics projected on a level set of K, is the same as the dynamics of a related piecewise constant system projected on the same level set:

Lemma 1. *The flow of (3.6.1), for any (A, B) with $K(E^A, E^B) = 0$, projected by $\tilde{\pi}_r$ on the level set $K_r = \{K(p,q) = r\} = \{K(p^o, q^o) = r\}$ (for $r > 0$ sufficiently small), is the same as the projection by $\tilde{\pi}_r$ of the truncated dynamics:*

$$\begin{cases} \dot{p}^o \in BR_A(q^o) - E^A \\ \dot{q}^o \in BR_B(p^o) - E^B \end{cases} \tag{3.6.2}$$

Proof. The projected differential inclusions are

$$\begin{cases} \dot{p}^o \in BR_A(q^o) - p^o - E^A - \alpha p^0 \\ \dot{q}^o \in BR_B(p^o) - q^o - E^B - \alpha q^0 \end{cases} \quad \text{resp.} \quad \begin{cases} \dot{p}^o \in BR_A(q^o) - E^A - \tilde{\alpha} p^0 \\ \dot{q}^o \in BR_B(p^o) - E^B - \tilde{\alpha} q^0 \end{cases}$$

where the functions $\alpha, \tilde{\alpha}$ are chosen so that the solutions stay in $K^{-1}(r)$. The lemma follows. □

Remark 36. *If (A, B) is a zero-sum game, then $K(p, q)$ is in fact a first integral of the system (3.6.2). The level sets of K are invariant and are hyperplanes in each region where $(p, q) \mapsto (BR_B(p), BR_A(q))$ is constant.*

Even though system (3.6.2) is piecewise constant, it may not be so easy to determine whether it possesses a first integral. However, we have the following

Proposition 23. *The bimatrix game (A, B) is equivalent to a zero-sum game $\iff$ the reduced piecewise constant flow (3.6.2) has a continuous first integral V whose level sets are hyperplanes.*

Proof. If (A, B) is zero-sum then

$$\frac{dK}{dt} = BR_A(q) \cdot A \cdot \dot{q} + \dot{p} \cdot B \cdot BR_B(p) = BR_A(q) \cdot A \cdot (BR_B(p) - E^A) + (BR_A(q) - E^B) \cdot B \cdot BR_B(q) =$$

$$= -[BR_A(q) \cdot A \cdot E^A + E^B \cdot B \cdot BR_B(p) = 0$$

To prove the reverse, let us write $V(p, q) := v \cdot q + p \cdot w$ where v is a row vector and w a column vector. By assumption v and w are constant on the set where $\Big(BR_B(p), BR_A(q)\Big) = (e_i, e_j)$ and to make this explicit we can write $v_{i,j}$ and $w_{i,j}$. Since $(p, q) \mapsto V(p, q) = v_{ij} \cdot q + p \cdot w_{ij}$ is continuous, considering sequences $p_1^i, p_2^i, \cdots \to E^A$ (corresponding to different i's), the limit value $V(E^A, q)$ does not depend on i. Hence v_{ij} does not depend on i. The same holds for w_{ij}. In particular, we can find matrices $\hat{A}$ and $\hat{B}$ so that $V(p, q) = BR_A(q) \cdot \hat{A} \cdot q + p \cdot \hat{B} \cdot BR_B(p)$. For this to be continuous at E^B and E^A we need that all elements of the vector $\hat{A}E^B$ (and of $E^A\hat{B}$) are the same. This means that the pair $(\hat{A}, \hat{B})$ has Nash equilibrium (E^A, E^B). Similarly, for V to be continuous between the regions where two of the strategies are indifferent (so where $BR_A(p)$ jumps), we need that the corresponding element of $\hat{A}q$ are of equal value. In this way we see that the indifference lines of $\hat{A}$ and $\hat{B}$ would have to be the same. It follows that (A, B) and $(\tilde{A}, \tilde{B})$ are equivalent. For V to be constant under the above piecewise constant flow we need

$$BR_A(q) \cdot \hat{A} \cdot (BR_B(p) - E^B) + (BR_A(q) - E^A) \cdot \hat{B} \cdot BR_B(p) = 0$$

This means that $BR_A(q) \cdot (\hat{A} + \hat{B}) \cdot BR_B(p) = \text{const}$. But then we can replace $\hat{A}, \hat{B}$ by matrices so that $BR_A(q) \cdot (\hat{A} + \hat{B}) \cdot BR_B(p) = 0$. □

3.7 - Examples related to Hofbauer's conjecture

In this section our goal is to present an example of a non-zero sum game for which there is an open set of initial conditions converging to the Nash equilibrium.

The idea of the construction will be discussed in the remark 39, after introducing necessary objects used in this section.

Proposition 24. *1. If $\lambda_+ = \lambda_- = 0$ and $0 < a_- < a(\tau) < a_+ < \infty$ for all τ then $\lim_{\tau\to\infty} K(\tau) = 0$ and the solution exists for all t.*

2. Assume there are constants $T_1 > 0$, $0 < c_- < c_+ < \infty$ and $0 < c_1 < c_2 < 1$ and times

$\tau_0 < \tau_1 < \tau_2 < \dots$ *with* $T_2 > \tau_{n+1} - \tau_n > T_1$*, so that* $c_- \le a(\tau_n)/a(\tau) \le c_+$ *for* $\tau_n \le \tau < \tau_{n+1}$*, so that* $a(\tau_{n+1}) \in (c_1 \cdot a(\tau_n), c_2 \cdot a(\tau_n)) \cup \{a(\tau_n)\}$*. Let* $C = \{n; a(\tau_n) \in (c_1 \cdot a(\tau_{n-1}), c_2 \cdot a(\tau_{n-1}))\}$*. Assume that* $n \in C$ *implies* $n+1 \notin C$*. Let* $C = \{n_1, n_2, \dots\}$*, let* $q_i = n_i - n_{i-1}$ *and assume that* $\lim_{i\to+\infty} q_i = +\infty$*. Then* $K(\tau) \to 0$*, the solution exists for all time t, and* $\lambda_- = \lambda_+ = 0$*.*

Proof. For the first assertion, note that $\tau \cdot \frac{a_-}{a_+} < \int_0^\tau \frac{a(\tau)}{a(s)} ds$, hence $K(\tau) \to 0$.
For the second assertion, let $N \in \mathbb{N}$ and $\tau_N < \tau < \tau_{N+1}$. Then

$$\int_0^\tau \frac{a(\tau)}{a(s)} ds = \int_0^{\tau_0} \frac{a(\tau)}{a(s)} ds + \Sigma_{j=0}^N \int_{\tau_j}^{\tau_{j+1}} \frac{a(\tau)}{a(s)} ds + \int_{\tau_N}^\tau \frac{a(\tau)}{a(s)} ds$$

We will show that

$$\lim_{N\to+\infty} a(\tau) \cdot \Sigma_{j=0}^N \int_{\tau_j}^{\tau_{j+1}} \frac{1}{a(s)} ds = +\infty$$

Let $r := \max\{k \in \mathbb{N} \mid n_k \le N\}$. From the assumptions we get that

$$a(\tau) \cdot \Sigma_{j=0}^N \int_{\tau_j}^{\tau_{j+1}} \frac{1}{a(s)} ds > a(\tau) \cdot \Sigma_{j=0}^N \int_{\tau_j}^{\tau_{j+1}} \frac{c_-}{a(\tau_j)} ds =$$

$$= a(\tau) \cdot \Sigma_{j=0}^N \frac{c_-}{a(\tau_j)} \cdot (\tau_{j+1} - \tau_j) > a(\tau) \cdot T_1 \cdot c_- \cdot \Sigma_{j=0}^N \frac{1}{a(\tau_j)} > T_1 \cdot \frac{c_-}{c_+} \cdot \Sigma_{i=0}^r c_1^{r-i} \cdot q_r$$

Now the divergence of the integral follows from the fact that $\lim_{r\to+\infty} q_r = +\infty$. Hence $K(\tau) \to 0$ as $\tau \to +\infty$. Consequently $\lambda_- \ge 0$.
We will now show now that $\lambda_+ = 0$. Let $\tau \in (\tau_{n_k}, \tau_{n_{k+1}})$, where k and $n_k, n_{k+1} \in C$, are so that

$$a(\tau_{n_k - 1}) > a(\tau_{n_k}) = a(\tau_{n_k + 1}) = \dots = a(\tau_{n_{k+1} - 1}) > a(\tau_{n_{k+1}})$$

By the definition of C we have

$$\frac{1}{\tau} \log a(\tau) = \frac{1}{\tau} \log a(\tau_{n_k}) + \frac{1}{\tau} \left(\int_{\tau_{n_k}}^\tau div(\bar{p}(s), \bar{q}(s))\, ds \right) < \frac{1}{\tau} \cdot \log[c_2^k a(\tau_0)] + \frac{1}{\tau} \left(\int_{\tau_{n_k}}^\tau div(\bar{p}(s), \bar{q}(s))\, ds \right).$$

Since $\tau - \tau_{n_k} \le T_2$, the second term converges to 0 as $\tau \to +\infty$. Since $q_i \to \infty$, it follows that $k/n_k \to 0$ and $\tau \ge \tau_{n_k} \ge n_k T_1$. Hence the first term also converges to 0. □

We will construct the example in a way so that we can make use of the theorem (24). For this reason we will find a local attractor for the induced flow, such that for each initial point $(\bar{p}, \bar{q})$ from its basin of attraction assumptions from the point 2 of the proposition (24) hold.

Let us consider the best response dynamics:

$$\begin{cases} \dot{p} \in BR_A(q) - p \\ \dot{q} \in BR_B(p) - q \end{cases} \tag{3.7.1}$$

with $(p, q) \in \Sigma_A \times \Sigma_B$ and 3×3 matrices:

$$A = \begin{pmatrix} 1+\delta_1 & 0 & \beta-\delta_1 \\ \beta & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix} \quad B = -\begin{pmatrix} 1+\delta_2 & 0 & \beta \\ \beta & 1 & 0 \\ -\delta_2 & \beta & 1 \end{pmatrix} \tag{3.7.2}$$

From lemma 1 it follows immediately that the Nash equilibrium is the center of the phase space $E^A = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $E^B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^{tr}$.

Definition 13. *The Shapley family, considered in [105], [109], consists of the pairs of matrices* (A_β, B_β)

$$A_\beta = \begin{pmatrix} 1 & 0 & \beta \\ \beta & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix} \quad B_\beta = \begin{pmatrix} -\beta & 1 & 0 \\ 0 & -\beta & 1 \\ 1 & 0 & -\beta \end{pmatrix} \tag{3.7.3}$$

for $\beta \in (-1, 1)$

Remark 37. *For* $\delta_1, \delta_2 = 0$ *and* $\beta_\phi = \frac{\sqrt{5}-1}{2}$*, the BRD for* (A, B) *is equivalent to zero-sum game for Shapley system.*

Indeed, in such a case $A = A_\beta$*, the matrix*

$$\tilde{B} = \frac{1}{\beta_\phi} \cdot B + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -\beta_\phi & 1 & 0 \\ 0 & -\beta_\phi & 1 \\ 1 & 0 & -\beta_\phi \end{pmatrix}$$

is the payoff matrix for the second player, in the Shapley system.

Proposition 25. (A, B) *is not equivalent to a zero-sum game for* $(\delta_1, \delta_2) \neq (0, 0)$ *and* $\beta \in (-1, 1)$

Proof. To see this, assume on the contrary that there exist numbers $c, c_1, c_2, c_3, d, d_1, d_2, d_3$, with $c, d \neq$

0, such that

$$0 = c \cdot A + \begin{pmatrix} c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \end{pmatrix} + d \cdot B + \begin{pmatrix} d_1 & d_1 & d_1 \\ d_2 & d_2 & d_2 \\ d_3 & d_3 & d_3 \end{pmatrix} =$$

$$\begin{pmatrix} d_1 + c + c_1 + c\delta_1 - d(1+\delta_2) & d_1 + c_2 & d_1 + \beta(c-d) + c_3 - c\delta_1 \\ d_2 + \beta(c-d) + c_1 & d_2 - d + c + c_2 & d_2 + c_3 \\ d_3 + c_1 + d\delta_2 & d_3 - \beta(c-d) + c_2 & d_3 - d + c + c_3 \end{pmatrix}$$

In particular, from $d_2 + c_3 = 0$, $d_3 - d + c + c_3 = 0$, we get $d_3 - d_2 = d - c$. From $d_2 - d + c + c_2 = 0$ and $d_3 - \beta(c-d) + c_2 = 0$, it follows that $d_3 - d_2 = (1+\beta) \cdot (c-d)$. Together with $d_3 - d_2 = d - c$ it implies $c = d$, for $\beta \neq -2$, and so the last matrix reads as

$$\begin{pmatrix} d_1 + c_1 + c\delta_1 - c\delta_2 & d_1 + c_2 & d_1 + c_3 - c\delta_1 \\ d_2 + c_1 & d_2 + c_2 & d_2 + c_3 \\ d_3 + c_1 + c\delta_2 & d_3 + c_2 & d_3 + c_3 \end{pmatrix}$$

So from the second column, we get $d_1 = d_2 = d_3 = -c_2$, which means $c_1 = c_2 = c_3 = -d_1 = -d_2 = -d_3$. In conclusion $c\delta_2 = 0$, $-c\delta_1 = 0$, which gives $c = d = 0$ for $(\delta_1, \delta_2) \neq (0,0)$. □

Since $A \cdot E^B = \frac{1+\beta}{2}(1,1,1)^T$ and $E^A \cdot B = -\frac{1+\beta}{2}(1,1,1)$, the energy function K satisfies $K(E^A, E^B) = 0$.

Remark 38. *From now on, we consider only the piecewise constant 'truncated' dynamics (3.6.2) on the level set $K_r = \{K(p,q) = r\}$.*

Proposition 26. *For the system (3.6.2), $\beta \approx \frac{\sqrt{5}-1}{2}$ and $\delta_1, \delta_2 \approx 0$ there exists a 2-dimensional global cross-section of the form:*

$$S := K_r \cap \Big((Z_{1,2} \times R_1^A) \cup (Z_{2,3} \times R_1^A) \cup (Z_{2,3} \times R_3^A) \cup (Z_{1,3} \times R_3^A) \Big).$$

So the orbit through any point in S returns infinitely often to S.

Proof. See [109]. □

Note that $K_r \cap (Z_{1,2} \times R_1^A) \cap (Z_{2,3} \times R_1^A) = K_r \cap (E^A \times R_1^A)$.

Remark 39. *We present here the idea of the construction of the example.*

The cross section from the proposition 26 exists for the system (3.6.2), $\beta \approx \frac{\sqrt{5}-1}{2}$ and $\delta_1, \delta_2 \approx 0$.

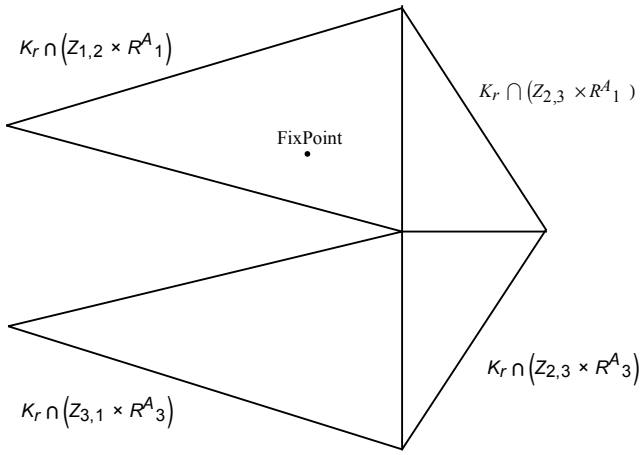


Figure 3.1: Schematic picture of the cross section S from the proposition 26.
S is a global cross section for the system (3.6.2), with matrices (A, B) and $\beta \approx \frac{\sqrt{5}-1}{2}$ and $\delta_1, \delta_2 \approx 0$. S is 2-dimensional and consists of the following pieces:

$$S := K_r \cap \Big((Z_{1,2} \times R_1^A) \cup (Z_{2,3} \times R_1^A) \cup (Z_{2,3} \times R_3^A) \cup (Z_{1,3} \times R_3^A)\Big).$$

On the piece $K_r \cap \Big((Z_{1,2} \times R_1^A)$ there exists a fixed point for the return map to S for $\beta = \frac{\sqrt{5}-1}{2}$ and $\delta_1, \delta_2 = 0$.

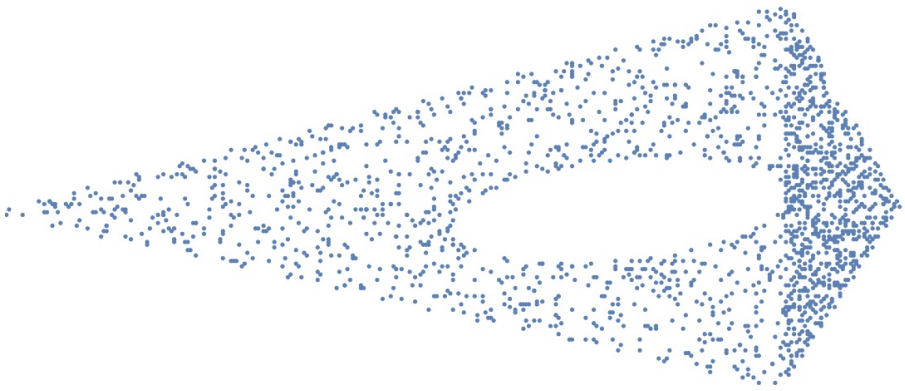


Figure 3.2: Iterations of a single point under the return map to the cross section S. The picture represents the zero-sum case $\beta = \frac{\sqrt{5}-1}{2}$ and $\delta_1, \delta_2 = 0$. Only pieces $K_r \cap (Z_{1,2} \times R_1^A)$, $K_r \cap (Z_{2,3} \times R_1^A)$ are shown. The totally white oval region corresponds to the orbits which are ellipses. The last ellipse, i.e. the boundary of the oval region is tangent to the line $K_r \cap (Z_{1,2} \times R_1^A) \cap (Z_{2,3} \times R_1^A)$.

Schematically it can be represented as in the figure 3.1.

For the zero-sum game when $\beta = \frac{\sqrt{5}-1}{2}$ *and* $\delta_1, \delta_2 = 0$*, the return map to the cross section* S *is area-preserving. There exists a fixed point on the piece* $K_r \cap (Z_{1,2} \times R_1^A)$*, which is surrounded by the ellipses - the orbits under return map from* $K_r \cap (Z_{1,2} \times R_1^A)$ *to* $K_r \cap (Z_{1,2} \times R_1^A)$*. The most outer ellipse is tangent to the line* $K_r \cap (Z_{1,2} \times R_1^A) \cap (Z_{2,3} \times R_1^A) = K_r \cap (E^A \times R_1^A)$*. This is presented in the figure 3.2, the white area is surrounded by the last ellipse. On the cross section* S*, outside the last ellipse, the dynamics seems to be chaotic (see [109]).*

The idea is to find $\delta_1, \delta_2 \neq 0$*, such that there still exists a fixed point in* $K_r \cap (Z_{1,2} \times R_1^A)$ *for the return map, surrounded by the ellipses. The last ellipse should be tangent to* $K_r \cap (Z_{1,2} \times R_1^A) \cap (Z_{2,3} \times R_1^A) = K_r \cap (E^A \times R_1^A)$ *and the absolute value of the Jacobian of the return map at the tangency point should be smaller than* 1*. Lemma 4 will lead to the existence of a local attractor for the return map, i.e. the last ellipse. All of the points from the elliptic region or from the basin of attraction of the last ellipse correspond to the orbits which converge under BRD to the Nash equilibrium, this is the consequence of the lemma 5 and proposition 24. Hence we find an open cone of initial conditions converging to the Nash equilibrium for a non-zero sum game.*

Let R_S be the return map to the section S. Moreover, we distinguish three restrictions of R_S (see figure (3.1)):

- R_{S_1}: its domain is a part of the $K_r \cap (Z_{1,2} \times R_1^A)$ mapped to a subset of $K_r \cap (Z_{1,2} \times R_1^A)$
- R_{S_2}: its domain is a part of the $K_r \cap (Z_{2,3} \times R_1^A)$ mapped to a subset of $K_r \cap (Z_{1,2} \times R_1^A)$
- R_{S_3}: its domain is a part of the $K_r \cap (Z_{1,2} \times R_1^A)$ mapped to a subset of $K_r \cap (Z_{2,3} \times R_1^A)$

Proposition 27. *Return maps* R_{S_i} *are of the form*

$$R_{S_i}(\tilde{p}, \tilde{q}) = \frac{1}{M_i(\tilde{p}, \tilde{q})} T_i(\tilde{p}, \tilde{q})$$

where $T_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ *and* $M_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are affine.*

Proof. See [109]. □

If $(\tilde{p}, \tilde{q}) = (\tilde{p_1}, \tilde{p_2}, \tilde{p_3}, \tilde{q_1}, \tilde{q_2}, \tilde{q_3})$, then we can parametrize the parts $K_r \cap (Z_{1,2} \times R_1^A)$, $K_r \cap (Z_{2,3} \times R_1^A)$ of the section S with $(\tilde{p_2}, \tilde{q_3})$. Let us call these coordinates $S_{(\tilde{p},\tilde{q})}$. In the $S_{(\tilde{p},\tilde{q})}$ coordinates we can write

$$R_{S_1}(\tilde{p_2}, \tilde{q_3}) = \frac{R_b \cdot (\tilde{p_2}, \tilde{q_3}) + b}{f_0' + (f_1', f_2') \cdot (\tilde{p_2}, \tilde{q_3})}$$

for some matrix $R_b \in \mathbb{R}^{2\times 2}$ and $b \in \mathbb{R}^2$ and constants f_0', f_1', f_2'.

Remark 40. *If $\delta_1, \delta_2 = 0$, then R_b is a rotation matrix, and $f_0' = 1$, $f_1' = f_2' = 0$*

Proposition 28. *For any $\delta_1, \delta_2 \approx 0$, there exists a fixed point of the map R_{S_1} which we denote by $FixPoint$, which is a perturbation of the fixed point of the map $(u, v) \mapsto R_b \cdot (u, v) + b$. Moreover, $FixPoint$ is, up to first order expansion in δ_1, δ_2, equal to*

$$FixPoint_T = (-\frac{\beta+1}{\beta^2-\beta+4}, 0) \cdot r + FP_1(\beta) \cdot r\delta_1 + FP_2(\beta) \cdot r\delta_2.$$

Proof. First assertion follows from the implicit function theorem for the function:

$$((\delta_1, \delta_2), (u, v)) \mapsto R_{S_1}(u, v) - (u, v)$$

We derive the formula for $FixPoint_T$ by solving $R_{S_1}(FixPoint) - FixPoint = (0, 0)$, up to the first order terms in δ_1, δ_2. Explicit formula for $FixPoint_T$ can be found in the appendix. □

Let us define the change of the coordinates by the translation map $Tran$, to new coordinates $S_{(x,y)}$:

$$Tran : S_{(\tilde{p},\tilde{q})} \ni (\tilde{p}_2, \tilde{q}_3) \mapsto (\tilde{p}_2, \tilde{q}_3) - FixPoint_T =: (x, y) \in S_{(x,y)}$$

Also define,

$$Ret_1 := Tran \circ R_{S_1} \circ Tran^{-1} : S_{(x,y)} \to S_{(x,y)}$$

$$Ret_2 := Tran \circ R_{S_2} \circ Tran^{-1} : S_{(x,y)} \to S_{(x,y)}$$

$$Ret_3 := Tran \circ R_{S_3} \circ Tran^{-1} : S_{(x,y)} \to S_{(x,y)}$$

Proposition 29. *In the $S_{(x,y)}$ coordinates one can write:*

$$Ret_1(x, y) = \frac{R(x, y) + O(\delta_1^2, \delta_2^2, \delta_1\delta_2) \cdot (x, y)}{f_0 + (f_1, f_2) \cdot (x, y)}$$

with $\det R = 1$, $(f_1, f_2) = O(\delta_1, \delta_2)$ *and*

$$f_0 = f_0(\delta_1, \delta_2) := 1 + (\frac{9(\beta+1)}{2\beta^2-\beta+4}, \frac{9-18\beta}{2\beta^2-2\beta+8}) \cdot (\delta_1, \delta_2) + O(\delta_1^2, \delta_2^2, \delta_1\delta_2).$$

In particular, there exists a C^1 function ϕ such that $\phi(0) = 0$ and $f_0(\delta_1, \phi(\delta_1)) = 1$, for all sufficiently

small δ_1.

Remark 41. *For* δ_1, δ_2 *sufficiently close to zero,* R *is a rotation matrix, i.e. it has complex conjugated eigenvalues and det* $R = 1$.

Consider the truncated version of the return map:

$$Ret_{1T}(x, y) := \frac{R(x, y)}{f_0 + (f_1, f_2) \cdot (x, y)}$$

Proposition 30. *If* $f_0 = 1$*, then there exist real numbers* $L_1, L_2 = O(\delta_1, \delta_2)$ *so that the function*

$$H(z_1, z_2) := \frac{(z_1, z_2)}{1 + (L_1, L_2).(z_1, z_2)},$$

with inverse

$$H^{-1}(z_1, z_2) = \frac{(z_1, z_2)}{1 - (L_1, L_2).(z_1, z_2)},$$

so that

$$H^{-1} \circ Ret_{1T} \circ H = R$$

Proof. If

$$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \tag{3.7.4}$$

then direct calculation shows that

$$(L_1, L_2) = \frac{1}{(1 - r_{11})(1 - r_{22}) - r_{12}r_{21}}((r_{22} - 1)f_1 - f_2 r_{21}, (r_{11} - 1)f_2 - f_1 r_{12})$$

□

Proposition 31. *If* $f_0 = 1$ *then*

$$H^{-1} \circ Ret_1 \circ H = \frac{(R + O(\delta_1^2, \delta_2^2, \delta_1\delta_2)).(x, y)}{1 + O(\delta_1^2, \delta_2^2, \delta_1\delta_2).(x, y)}$$

Remark 42. *As* $(\delta_1, \delta_2) \to (0, 0)$

$$H^{-1} \circ Ret_1 \circ H \to R$$

$$\frac{\partial}{\partial \delta_1}(H^{-1} \circ Ret_1 \circ H) \to \frac{\partial}{\partial \delta_1} R$$

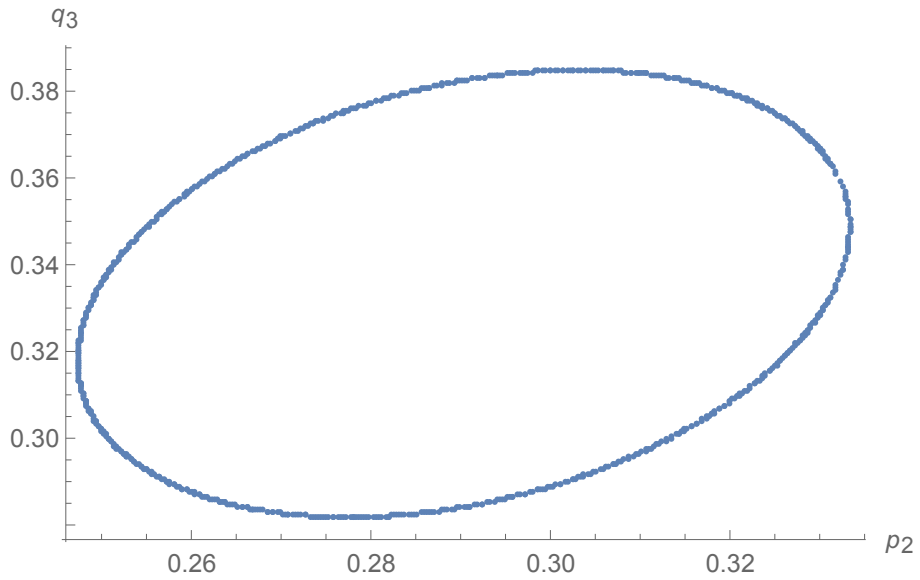


Figure 3.3: Approximately last elliptic orbit (belonging to the piece $K_r \cap (Z_{1,2} \times R_1^A)$) for $\beta \approx \frac{\sqrt{5}-1}{2}$ and $\delta_1, \delta_2 = 0$. The orbit was drawn by iterating 5000 times the return map to the cross section S. Numerically we investigated where is the first tangency point of the last elliptic orbit with the boundary of $K_r \cap (Z_{1,2} \times R_1^A)$.

$$\frac{\partial}{\partial \delta_2}(H^{-1} \circ Ret_1 \circ H) \to \frac{\partial}{\partial \delta_2} R$$

uniformly in $x,\ y$.

Since the mappings Ret_1 and $\frac{(R+O(\delta_1^2,\delta_2^2,\delta_1\delta_2)).(x,y)}{1+O(\delta_1^2,\delta_2^2,\delta_1\delta_2).(x,y)}$ are topologically conjugated, then the tangency point of the last ellipse, for R, with the boundary of $H^{-1} \circ Tran(K_r \cap (Z_{1,2} \times R_1^A))$ (after change of the coordinates) corresponds to the tangency point of the last closed orbit that stays within $K_r \cap (Z_{1,2} \times R_1^A)$ under R_{S1}.

We find numerically (see figure 3.3) that for $\beta \approx \frac{\sqrt{5}-1}{2}$ and $\delta_1, \delta_2 = 0$, this tangency point belongs to the interval $\{(E^A, q_3) \mid q_3 \in (0,1)\}$, i.e. is of the form $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, q_1, 1-q_1-q_3, q_3)$, where q_1 can be found from the condition $K(E^A, (q_1, 1-q_1-q_3, q_3)) = r$.

The lemmas below are helpful in finding analytical formulas for this tangency point.

Lemma 2. *The line* (x_0, y)*, with* x_0 *fixed, is mapped to a line under the map* H^{-1}.

Proof. Write $(X, Y) = H^{-1}(x_0, y) = \frac{(x_0,y)}{1-L_1x_0-L_2y}$, hence

$$y = \frac{1}{L_2}(1 - L_1x_0 - \frac{x_0}{X}) \ \text{ and } \ Y = \frac{X}{x_0}y = \frac{X}{x_0L_2}(1 - L_1x_0 - \frac{x_0}{X})$$

which gives $(L_2x_0) \cdot Y - (1 - L_1x_0)X + x_0 = 0$. □

Lemma 3. *The formula for a point from a line* $\varsigma_1 Y + \varsigma_2 X + \varsigma_3 = 0$ *minimizing the distance from the origin* $(0,0)$ *is given by:* $\left(\frac{-\varsigma_2\varsigma_3}{\varsigma_1^2+\varsigma_2^2}, \frac{-\varsigma_1\varsigma_3}{\varsigma_1^2+\varsigma_2^2}\right)$

Applying the above lemmas to the matrix R in Jordan form and line

$$(x_0, y) = (0, q_3 - E^B) - FixPoint_T$$

we obtain:

Proposition 32. *The tangency point TP of Ret_1 in the coordinates (p_2^o, q_3^o) reads as:*

$$TP = (0, \frac{2\beta - 1}{2\,(\beta^2 - \beta + 1)}) \cdot r + TP_1(\beta) \cdot r \cdot \delta_1 + TP_2(\beta) \cdot r \cdot \delta_2 + O(\delta_1^2, \delta_2^2, \delta_1\delta_2)$$

See appendix for the explicit formulas for $TP_1(\beta),\ TP_2(\beta)$.

Remark 43. *The itineraries consisting of indifference hyperplanes, that are followed in sequence, by the points which come back in the phase space to the section S, for the maps $R_{S_1}, R_{S_2}, R_{S_3}$, respectively:*

$$Z_{1,2} \times R_1^A \to R_2^B \times Z_{1,2} \to Z_{2,3} \times R_2^A \to R_3^B \times Z_{2,3} \to Z_{1,3} \times R_3^A \to R_1^B \times Z_{1,3} \to Z_{1,2} \times R_1^A$$

$$Z_{2,3} \times R_1^A \to R_2^B \times Z_{1,2} \to Z_{2,3} \times R_2^A \to R_3^B \times Z_{2,3} \to Z_{1,3} \times R_3^A \to R_1^B \times Z_{1,3} \to Z_{1,2} \times R_1^A$$

$$Z_{1,2} \times R_1^A \to R_2^B \times Z_{1,2} \to Z_{1,2} \times R_2^A \to Z_{1,3} \times R_2^A \to R_3^B \times Z_{1,2} \to Z_{2,3} \times R_1^A$$

Let us denote:

$$Jac_{32}(\delta_1, \delta_2, \beta, r, h) := Jac(H^{-1} \circ Ret_2 \circ Ret_3 \circ H)\Big((H^{-1} \circ Ret_3 \circ H)^{-1}(H^{-1} \circ Tran(TP + (h, 0)))\Big)$$
$$Jac_2(\delta_1, \delta_2, \beta, r, h) := Jac(H^{-1} \circ Ret_2 \circ H)(TP + (h, 0))$$
$$Jac_3(\delta_1, \delta_2, \beta, r, h) := Jac(H^{-1} \circ Ret_3 \circ H)\Big((H^{-1} \circ Ret_3 \circ H)^{-1}(H^{-1} \circ Tran(TP + (h, 0)))\Big)$$

Note that

$$Jac_{32}(\delta_1, \delta_2, \beta, r, h) = Jac_2(\delta_1, \delta_2, \beta, r, h) \cdot Jac_3(\delta_1, \delta_2, \beta, r, h)$$

Proposition 33. *There exist family of parameters $(\delta_1, \delta_2(\delta_1)) \approx (0, 0)$, with both $\delta_1, \delta_2(\delta_1)$ different from 0, for which $f_0(\delta_1, \delta_2(\delta_1)) = 1$ and $Jac_{32}(\delta_1, \delta_2(\delta_1), r, h) < 1$, with $h > 0$ and $r > 0$ arbitrarily small. Moreover from (3.8.1) it follows that rotation angle of the matrix R from the formula (3.8.1) is not constant along this family of parameters $(\delta_1, \delta_2(\delta_1))$.*

From the Proposition 33 above we conclude that after affine change of coordinates, the assumptions of the Lemma 4 below are fulfilled. Hence the Lemma 4 shows that there exists a local attractor on

the piece of the cross section $K_r \cap (Z_{1,2} \times R_1^A)$, which in fact it is precisely the last orbit tangent to the line $K_r \cap (E^A \times R_1^A)$.

Lemma 4. *Let $\Xi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$\Xi(x,y) := \begin{cases} R(x,y), & for \quad x \leq 1 \\ F(x,y), & for \quad x > 1 \end{cases} \tag{3.7.5}$$

with $R \in \mathbb{R}^{2\times 2}$ an irrational rotation and $F : \mathbb{R}^2 \to \mathbb{R}^2$ a C^1 mapping such that Ξ is continuous. If $Jac(F)(1,0) < 1$, then there exist a neighbourhood U of the closed unit ball B centered at 0 so that each point in U converges to ∂B.

We want to show that for the orbits, of the initial points $(\bar{p}, \bar{q})$, of the flow governed by the equation (3.6.2) and starting from the basin of attraction of the local attractor on the cross-section S, the assumptions of the proposition 24 (point 2) are satisfied. By the virtue of the proposition 24, the corresponding orbits for the original system (3.7.1) converge to the Nash equilibrium.
For this purpose, let us first prove the following:

Lemma 5. *With the assumptions of the previous lemma, let ϕ be the rotation angle of the matrix R, satisfying the Diophantine condition $|\phi - \frac{p}{q}| > \frac{\gamma}{q^{\vartheta+1}}$ with $\gamma > 0$ and $\vartheta > 1$. For the initial condition (x_0, y_0) (with $x_0 > 1$ and $|x_0 - 1|$, $|y_0|$ - small enough, so that (x_0, y_0) belongs to the basin of attraction of the local attractor), denote by*

$$L_n(x_0, y_0) := \#\{\ \Xi^k(x_0, y_0) \mid \quad \pi_x(\Xi^k(x_0, y_0)) > 1,\ k \leq n\}$$

- the number of iterations of (x_0, y_0) through Ξ, among the first n iterations, such that the first (the x) coordinate of these iterations is bigger than 1.
Then

$$\frac{L_n(x_0, y_0)}{n} \to 0 \ \ as \ \ n \to +\infty$$

We explain below why the assumptions of the Theorem 24 (point 2) are satisfied in our particular example for the initial points $(\bar{p}, \bar{q})$ from the basin of attraction.

Remark 44. *Due to the Lemma 4 and Remark 43, we can find a sequence of times $\{\tau_n\}_{n=0}^{+\infty}$ and constants c_1, c_2. The existence of c_-, c_+ follows from the Remark 43 and the definition of $a(\tau)$, i.e. (3.5.7). The existence of set C and $\lim_{r\to+\infty} q_r = +\infty$ is a consequence of Lemma 5. Moreover the*

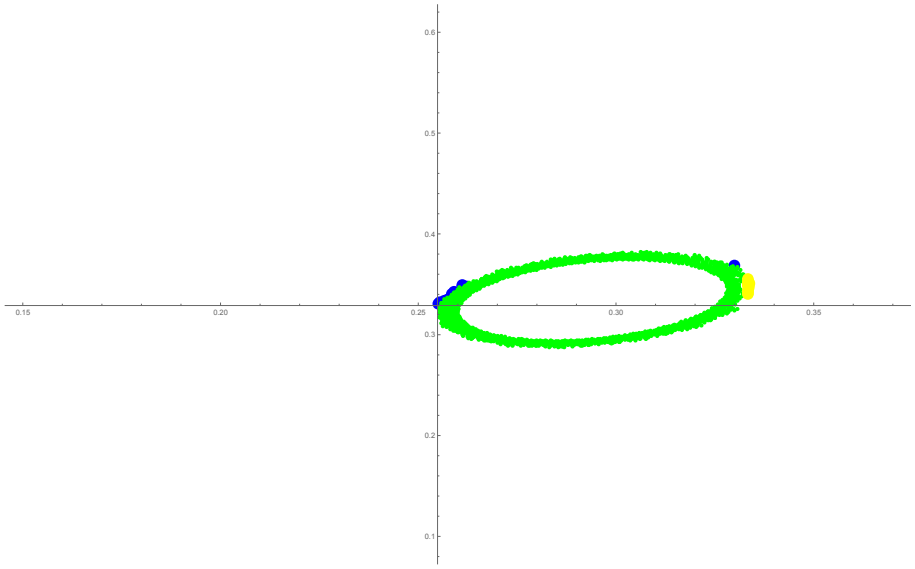


Figure 3.4: This picture represents the case $\delta_1 = 0.02$ and δ_2 such that $f_0(\delta_1, \delta_2) = 1.00001$. Since $f_0 > 1$ then the fixed point of the return map is attracting. In the picture there are shown 3000 iterations of the initial point under the return map to the cross section S. The green and dark dots correspond to the points (iterations) from the piece $K_r \cap (Z_{1,2} \times R_1^A)$ while the yellow dots correspond to the points from the piece $K_r \cap (Z_{2,3} \times R_1^A)$.

constants $T_1, T_2 > 0$ exist due to the Proposition 33, Lemma 1 and easy observation that the return time to the global cross section S along the itineraries (from the Remark 43) is bounded by a constant for the truncated dynamics (3.6.2).

In conclusion, we have found an open set of the initial conditions $(\bar{p}, \bar{q})$ on the cross section S which corresponds to the open and invariant set (a cone) in the phase space for the Best Response Dynamics, where the orbits converge to the Nash equilibrium.

Remark 45. *We point out that it is unlikely for the attractor we have found on the cross section S to be a global attractor for the return map to S. The numerical simulations we have performed for $\delta_1, \delta_2 \approx 0$ (with $\delta_1 < 0.02$) for which the attractor exists and $f_0 = 1.0001,\ 1.00001,\ 1.000001$ suggest that the smaller $f_0 - 1$ is, the longer it takes for randomly drawn points from the cross section to converge to the attractor. For $f_0 = 1.000001$, we do not observe any convergence to the attractor within approximately 3000000 iterations of the return map to S (see also figure 3.4).*

3.8 - Appendix

In this section, we present some of the technical formulas and proofs from the previous section. The calculations were obtained using Wolfram Mathematica.

Proposition (Proposition 28)**.** $FixPoint_T = (-\frac{(\beta+1)r}{\beta^2-\beta+4}, 0) +$

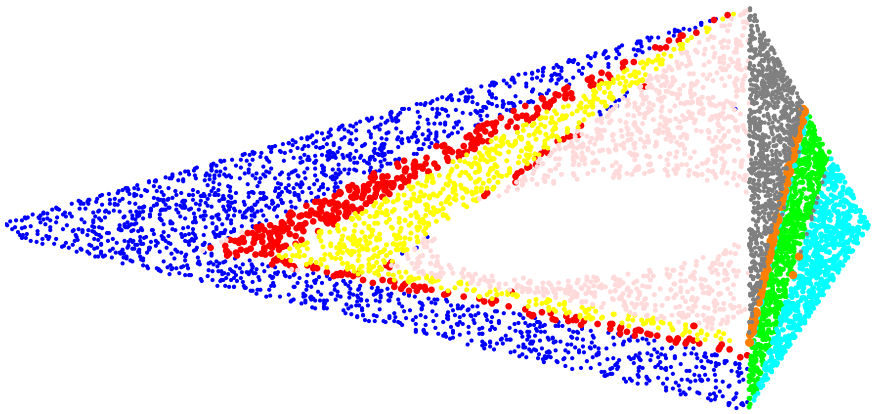


Figure 3.5: Iterations of a single point under the return map to the cross section S. The picture represents the non-zero-sum case $\beta \approx \frac{\sqrt{5}-1}{2}$ and $\delta_1, \delta_2 \approx 0$, such that $f_0(\delta_1, \delta_2) = 1.000001$. Hence the fixed point of the return map is locally attracting. Only pieces $K_r \cap (Z_{1,2} \times R_1^A)$, $K_r \cap (Z_{2,3} \times R_1^A)$ are shown in the picture. Compare this picture with the figure 3.2. Iterations with different itineraries between hitting the cross section S are marked with different colors.

$$\left(\frac{\beta^9-2\beta^8+8\beta^7-7\beta^6+12\beta^5+5\beta^4-7\beta^3+14\beta^2-12\beta-8}{-3\left(\beta^2-\beta+4\right)^2}, \frac{\beta^{11}-6\beta^{10}+26\beta^9-74\beta^8+165\beta^7-275\beta^6+363\beta^5-355\beta^4+273\beta^3-138\beta^2+59\beta+7}{-3\left(\beta^2-\beta+1\right)\left(\beta^2-\beta+4\right)^2}\right) \cdot r\delta_1 +$$
$$\left(\frac{\beta^{11}-8\beta^{10}+34\beta^9-104\beta^8+230\beta^7-400\beta^6+530\beta^5-545\beta^4+409\beta^3-238\beta^2+81\beta-12}{-3\left(\beta^2-\beta+1\right)\left(\beta^2-\beta+4\right)^2},\right.$$
$$\left.\frac{\beta^{10}-9\beta^9+43\beta^8-144\beta^7+350\beta^6-654\beta^5+926\beta^4-988\beta^3+746\beta^2-372\beta+87}{-3(\beta+1)\left(\beta^2-\beta+4\right)^2}\right) \cdot r\delta_2$$

Proposition (Proposition 29)**.** *In the $S_{(x,y)}$ coordinates one can write:*

$$Ret_1(x,y) = \frac{R(x,y) + O(\delta_1^2, \delta_2^2, \delta_1\delta_2) \cdot (x,y)}{f_0 + (f_1, f_2) \cdot (x+y)}$$

with $\det R = 1$, $(f_1, f_2) = O(\delta_1, \delta_2)$ *and*

$$f_0 = f_0(\delta_1, \delta_2) := 1 + \left(\frac{9(\beta+1)}{2\beta^2-\beta+4}, \frac{9-18\beta}{2\beta^2-2\beta+8}\right) \cdot (\delta_1, \delta_2) + O(\delta_1^2, \delta_2^2, \delta_1\delta_2)$$

In particular, there exists a C^1 function ϕ such that $\phi(0) = 0$ and $f_0(\delta_1, \phi(\delta_1)) = 1$, for all sufficiently small δ_1.

Proof. The existence of ϕ follows from the formula above and the implicit function theorem. Note that

$$Ret_1(x,y) = R_{S_1}((x,y) + FixPoint_T) - FixPoint_T =$$
$$\frac{R_b.((x,y) + FixPoint_T) + b - (f_0' + (f_1', f_2').(x,y)) \cdot FixPoint_T}{f_0' + (f_1', f_2').((x,y) + f_0' + (f_1', f_2').(x,y))} =$$

$$\frac{(R_b - (f_1', f_2') * FixPoint_T).(x,y) + O(\delta_1^2, \delta_2^2, \delta_1\delta_2).(x,y)}{f_0' + (f_1', f_2').FixPoint_T + (f_1', f_2').(x,y)}$$

So it suffices to set:

$$\begin{aligned} R &:= \frac{R_b - (f_1', f_2') * FixPoint_T}{\sqrt{det((R_b - (f_1', f_2') * FixPoint_T))}} \\ f_0 &:= \frac{f_0' + (f_1', f_2').FixPoint_T}{\sqrt{det((R_b - (f_1', f_2') * FixPoint_T))}} \\ (f_1, f_2) &:= \frac{(f_1', f_2')}{\sqrt{det((R_b - (f_1', f_2') * FixPoint_T))}} \end{aligned} \tag{3.8.1}$$

□

Proposition (Proposition 31)**.** *If $f_0 = 1$ then $H^{-1} \circ Ret_1 \circ H = \frac{(R+O(\delta_1^2,\delta_2^2,\delta_1\delta_2)).(x,y)}{1+O(\delta_1^2,\delta_2^2,\delta_1\delta_2).(x,y)}$.*

Proof. We have:

$$H^{-1} \circ Ret_1 \circ H(x,y) = H^{-1} \circ Ret_1\Big(\frac{(x,y)}{1 + (L_1, L_2).(x,y)}\Big) = H^{-1}\left(\frac{(R + O(\delta_1^2, \delta_2^2, \delta_1\delta_2)).(x,y)}{1 + (L_1 + f_1, L_2 + f_2).(x,y)}\right) =$$

$$\frac{(R + O(\delta_1^2, \delta_2^2, \delta_1\delta_2)).(x,y)}{1 + (L_1 + f_1, L_2 + f_2).(x,y) - (L_1, L_2).((R + O(\delta_1^2, \delta_2^2, \delta_1\delta_2)).(x,y))} =$$

$$\frac{(R + O(\delta_1^2, \delta_2^2, \delta_1\delta_2)).(x,y)}{1 + (L_1 + f_1, L_2 + f_2).(x,y) - (L_1, L_2).R.(x,y) + O(\delta_1^2, \delta_2^2, \delta_1\delta_2).(x,y)}$$

Since the L_1, L_2 were chosen in a way such that $(L_1 + f_1, L_2 + f_2).(x,y) - (L_1, L_2).R.(x,y) = 0$ for all x, y, the last formula reads as

$$\frac{(R + O(\delta_1^2, \delta_2^2, \delta_1\delta_2)).(x,y)}{1 + O(\delta_1^2, \delta_2^2, \delta_1\delta_2).(x,y)}$$

□

Proposition (Proposition 32)**.** $TP = (0, \frac{(2\beta-1)r}{2(\beta^2-\beta+1)}) + (-\frac{(2\beta^{11}-6\beta^{10}+22\beta^9-34\beta^8+54\beta^7-28\beta^6+43\beta^4-66\beta^3+36\beta^2-8\beta-7)r}{6(\beta^2-\beta+1)(\beta^2-\beta+4)^2},$
$-\frac{(4\beta^{13}-28\beta^{12}+132\beta^{11}-424\beta^{10}+1060\beta^9-2056\beta^8+3212\beta^7-3990\beta^6+4009\beta^5-3190\beta^4+2042\beta^3-904\beta^2+325\beta-8)r}{12(\beta^2-\beta+1)^2(\beta^2-\beta+4)^2}) \cdot \delta_1 +$
$(\frac{(\beta(\beta(\beta(\beta(1081-2\beta(\beta(\beta(\beta(\beta((\beta-8)\beta+34)-104)+230)-400)+530))-809)+485)-162)+33)r}{6((\beta-1)\beta+1)((\beta-1)\beta+4)^2},$
$\frac{(\beta(\beta(\beta(\beta(\beta(2\beta(11283-2\beta(\beta(\beta(\beta(\beta((\beta-11)\beta+64)-259)+786)-1881)+3615))-28587)+29246)-23794)+15143)-7067)+2265)-384)r}{12(\beta+1)((\beta-1)\beta+1)^2((\beta-1)\beta+4)^2}) \cdot \delta_2 +$
$+ O(\delta_1^2, \delta_2^2, \delta_1\delta_2)$

Proposition (Proposition 33)**.** *There exist family of parameters $(\delta_1, \delta_2(\delta_1)) \approx (0,0)$, with both $\delta_1, \delta_2(\delta_1)$ different from 0, for which $f_0(\delta_1, \delta_2(\delta_1)) = 1$ and $Jac_{32}(\delta_1, \delta_2(\delta_1), r, h) < 1$, with $h > 0$ and $r > 0$ arbitrarily small. Moreover from (3.8.1) it follows that rotation angle of the matrix R from the formula (3.8.1) is not constant along this family of parameters $(\delta_1, \delta_2(\delta_1))$.*

Proof. Since $Jac_{32}(0,0,\beta,r,h) = 1$, it suffices to check that its derivative, under the condition $f_0(\delta_1,\delta_2) = 1$, at point $(\delta_1,\delta_2,r,h) = (0,0,r,h)$ is different from 0.

$$\frac{dJac_{32}(\delta_1,\delta_2(\delta_1),\beta,r,h)}{d\delta_1} = \frac{\partial Jac_{32}(\delta_1,\delta_2,\beta,r,h)}{\partial\delta_1} + \frac{\partial Jac_{32}(\delta_1,\delta_2,\beta,r,h)}{\partial\delta_2}\cdot\frac{\partial\delta_2}{\partial\delta_1} =$$

$$/\text{at } \delta_1,\delta_2 = 0 \text{ and } h \to 0^+ / =$$

$$\frac{1}{2(\beta-1)\beta^2(2\beta-1)^3(\beta^2-\beta+4)}\Big(\beta^{17}(20-8r)+2\beta^{16}(34r-79)+\beta^{15}(710-320r)+$$

$$4\beta^{14}(257r-530)-6\beta^{13}(410r-741)+\beta^{12}(4540r-6151)+\beta^{11}(3309-6528r)+$$

$$\beta^{10}(7168r+8954)-\beta^9(5420r+32209)+6\beta^8(240r+10133)+\beta^7(3248r-82961)+$$

$$\beta^6(88001-6422r)+24\beta^5(289r-3085)-6\beta^4(904r-8195)+\beta^3(3310r-25174)+$$

$$\beta^2(9427-1492r)+\beta(448r-2362)-80(r-4)\Big) < 0$$

holds. Indeed, since for $r > 0$ sufficiently small and all $\beta \in \frac{\sqrt{5}-1}{2} + (-0.02, 0.02)$, denominator is always negative and the nominator positive, both being separated from 0. □

Lemma (Lemma 4)**.** *Let* $\Xi : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by*

$$\Xi(x,y) := \begin{cases} R(x,y), & for \quad x \le 1 \\ F(x,y), & for \quad x > 1 \end{cases} \tag{3.8.2}$$

with $R \in \mathbb{R}^{2\times 2}$ *an irrational rotation and* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *a* C^1 *mapping such that* Ξ *is continuous. If* $Jac(F)(1,0) < 1$*, then there exist a neighbourhood* U *of the closed unit ball* B *centered at* 0 *so that each point in* U *converges to* ∂B*.*

Proof. Let us write $F(x,y) = (b_1,b_2) + A.(x-1,y) + (F_1(x,y),F_2(x,y))$, where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad R = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \tag{3.8.3}$$

so that the linear part of F_1, F_2 vanishes at $z_0 = (1,0)$. So F_1, F_2 are the higher order terms in Taylor expansion of F around point $z_0 := (1,0)$.

From the assumption that Ξ is C^1 at $(1,0)$,

$$(\cos\phi + y\cdot\sin\phi, -\sin\phi + y\cdot\cos\phi) = R(1,y) = F(1,y) = (b_1 + a_{12}y + F_1(x,y), b_2 + a_{22}y + F_2(1,y))$$

we find that

$$b_1 = \cos\phi, \;\; b_2 = -\sin\phi, \;\; a_{12} = \sin\phi, \;\; a_{22} = \cos\phi, \;\; F_1(1,y) = F_2(1,y) = 0$$

For $V(x,y) := x^2 + y^2$, we have that

$$V(1,y) - V(F(1,y)) = V(1,y) - V(R(1,y)) = 0$$

and in general:

$$V(x,y) - V(F(x,y)) = x^2 + y^2 - (b_1 + a_{11}(x-1) + a_{12}y + F_1(x,y))^2 - (b_2 + a_{21}(x-1) + a_{22}y + F_2(x,y))^2$$

In particular if $Jac(F)(z_0) < 1$,

$$\frac{\partial(V - V\circ F)}{\partial x}(1,y) = 2 - 2(a_{11}b_1 + a_{21}b_2) - 2y(a_{11}a_{12} + a_{22}a_{21}) = 2 - 2\cdot Jac(F)(z_0) - 2y(a_{11}a_{12} + a_{22}a_{21}) > 0$$

provided $|y|$ sufficiently small. If for simplicity we denote $W := V - V\circ F$, then from the mean value theorem we obtain for $x > 1$ and $|y|$ sufficiently small that there exists $x_1 \in (1,x)$ so that

$$\begin{aligned} V(x,y) - V\circ F(x,y)) = W(x,y) = W(x,y) - W(1,y) = \\ = (x-1)\cdot \frac{\partial(V - V\circ F)}{\partial x}(x_1,y) > 0 \end{aligned}$$

(here we use that $F(x,y)$ is equal to $R(x,y)$ when $x \le 1$ and so $W(1,y) = 0$). Hence V is non-increasing along the orbits of R and F.

Let (x_0,y_0) be any point with $0 < x_0 - 1$ and $|y_0|$, $x_0 - 1$ - sufficiently small. Let $(n_k)_{k\in\mathbb{N}_1}$ be the sequence of natural numbers, such that if we set $F_{n_k} := R^{n_k}\circ F$ and inductively define $(x_{n_k}, y_{n_k}) := F_{n_k}(x_{n_{k-1}}, y_{n_{k-1}})$, then for all k we have $x_{n_k} > 1$ and if $0 < l < n_k$ then $\pi_x(R^l(x_{n_k}, y_{n_k})) < 1$. In order to finish the proof we need to show that there is a constant $c > 0$ and an infinite subsequence $(n_{k_s})_{s\in\mathbb{N}}$ for which

$$\frac{V(x_{n_{k_s}}, y_{n_{k_s}}) - V\circ F(x_{n_{k_s}}, y_{n_{k_s}})}{V(x_{n_{k_s}}, y_{n_{k_s}}) - 1} > c$$

If this does not hold then $V(x_{n_k}, y_{n_k}) \to a$, with $a > 1$. However,

$$\frac{V(x,y) - V(F(x,y))}{V(x,y) - 1} = \frac{2(x-1)(1 - Jac(F)(z_0)) + O((x-1)^2, (x-1)y, y^2)) + h.o.t.}{(x+1)(x-1) + y^2}$$

so the required constant c cannot exist if and only if the iterations converge, i.e. $(x_{n_k}, y_{n_k}) \to (1, Y)$ for some Y.

However this is a contradiction with the fact that the sequence (x_{n_k}, y_{n_k}) cannot have only one accumulation point, since R is an irrational rotation and Ξ is continuous. □

Lemma (Lemma 5)**.** *With the assumptions of the previous lemma, let ϕ be the rotation angle of the matrix R, satisfying the Diophantine condition $|\phi - \frac{p}{q}| > \frac{\gamma}{q^{\vartheta+1}}$ with $\gamma > 0$ and $\vartheta > 1$. For the initial condition (x_0, y_0) (with $x_0 > 1$ and $|x_0 - 1|$, $|y_0|$ - small enough, so that (x_0, y_0) belongs to the basin of attraction of the local attractor), denote by*

$$L_n(x_0, y_0) := \#\{\ \Xi^k(x_0, y_0) \mid \ \ \pi_x(\Xi^k(x_0, y_0)) > 1,\ k \leq n\}$$

- the number of iterations of (x_0, y_0) through Ξ, among the first n iterations, such that the first (the x) coordinate of these iterations is bigger than 1.

Then

$$\frac{L_n(x_0, y_0)}{n} \to 0 \ \ as \ \ n \to +\infty$$

Proof. Let $\{r_n\} \subset \mathbb{N}$ and $q_n := \Sigma_{j=0}^{n} r_j$ be the sequence of natural numbers, such that the iterations $\{(x_{q_n}, y_{q_n})\}$ under Ξ satisfy $x_{q_n} > 1$, i.e. r_{n+1} is the smallest number of iterations under Ξ such that if $x_{q_n} > 1$ then $x_{q_n + r_{n+1}} > 1$. Let ϕ_{q_n} be the angle based at $(0, 0)$ between the points of intersection of the line $\{\ (1, y) \mid y \in \mathbb{R}\ \}$ with the circle $S\Big((0, 0), V(x_{q_n}, y_{q_n})\Big)$. From the proof of the Lemma 4, we have that there exists $c \in (0, 1)$ such that:

$$\frac{V(x_{q_n}, y_{q_n}) - V \circ F(x_{q_n}, y_{q_n})}{V(x_{q_n}, y_{q_n}) - 1} > c$$

which is equivalent to

$$V \circ F(x_{q_n}, y_{q_n}) < V(x_{q_n}, y_{q_n}) \cdot (1 - c) + c$$

and leads us to the following inequality

$$\sqrt{\frac{(V \circ F(x_{q_n}, y_{q_n}))^2 - 1}{V(x_{q_n}, y_{q_n})^2 - 1}} < \sqrt{1 - c}$$

Note that due to the Taylor expansion for any small angle ϕ_{q_n}:

$$\sqrt{(V \circ F(x_{q_n}, y_{q_n}))^2 - 1} = tg(\phi_{q_{n+1}}) = \phi_{q_{n+1}} \cdot (1 + h.o.t.(\phi_{q_{n+1}}))$$

$$\sqrt{V(x_{q_n}, y_{q_n})^2 - 1} = tg(\phi_{q_n}) = \phi_{q_n} \cdot (1 + h.o.t.(\phi_{q_n}))$$

we have

$$\frac{\sqrt{(V \circ F(x_{q_n}, y_{q_n}))^2 - 1}}{\sqrt{V(x_{q_n}, y_{q_n})^2 - 1}} = \frac{\phi_{q_{n+1}} \cdot (1 + h.o.t.(\phi_{q_{n+1}}))}{\phi_{q_n} \cdot (1 + h.o.t.(\phi_{q_n}))}$$

the consequence of this is the convergence

$$\left| \frac{\phi_{q_{n+1}}}{\phi_{q_n}} - \sqrt{\frac{(V \circ F(x_{q_n}, y_{q_n}))^2 - 1}{V(x_{q_n}, y_{q_n})^2 - 1}} \right| \to 0 \text{ as } n \to +\infty$$

Which inductively gives us the estimate on $\phi_{q_{n+1}}$

$$\phi_{q_{n+1}} < \phi_{q_n} \cdot (\sqrt{1 - c} + \epsilon) < const_0 \cdot (\sqrt{1 - c} + \epsilon)^{n+1}$$

where n can be taken so large, so that we can find ϵ satisfying $\sqrt{1 - c} + \epsilon < 1$. From the well known inequality, for $\theta \in (0, \frac{\pi}{2})$:

$$|e^{i\theta} - 1| > \frac{2\theta}{\pi}$$

and the Diophantine condition, we conclude that:

$$|e^{2\pi i(\phi r_n - p_n)} - 1| > const_2 \cdot |r_n|^{-\vartheta}$$

where p_n is an integer such that $2\pi i(\phi r_n - p_n) \in (-\frac{\phi_{q_n}}{2}, \frac{\phi_{q_n}}{2})$. On the other hand, simple geometric argument provides a constant $const_1 > 0$, such that for the angles ϕ_n - small enough, the following holds true:

$$const_1 \cdot \phi_{q_n} > |e^{2\pi i(\phi r_n - p_n)} - 1|$$

If we set $c_1 := (\sqrt{1 - c} + \epsilon)^{-\frac{1}{\vartheta}} > 1$, then:

$$r_n > const_3 \cdot c_1^n$$

Finally, observe that $L_n(x_0, y_0) \to +\infty$ as $n \to +\infty$, since ϕ is irrational and $n \geq \sum_{j=0}^{L_n(x_0,y_0)} r_j$, which gives us:

$$\frac{L_n(x_0, y_0)}{n} \leq \frac{L_n(x_0, y_0)}{\sum_{j=0}^{L_n(x_0,y_0)} r_j} \to 0 \text{ as } n \to +\infty$$

□

Eventus stultorum magister.

4

On the Coupled Replicator Equations for the cyclic competition bimatrix games

4.1 - Introduction

* We consider in this chapter an example of a cyclic competition bimatrix game which is a Rock-Scissors-Paper game with assumption about perfect memory of the playing agents.

In the second section of this chapter we give the introduction to the model, characterize Nash equilibrium and we explain the global behaviour on the boundaries of codimensions 2 and 1.

Following section provides full construction of the heteroclinic network naturally appearing in the system as well as all of the transition maps associated to this network.

Asymptotic analysis, stability of the cycles within the network and bifurcations happening in the system are described in the fourth section.

Section number five, consists of examples showing some phenomenon in the system - attainability of the itineraries along heteroclinic orbits as a dependence of parameter values (ϵ_X, ϵ_Y).

In the sixth section we present our numerical investigations and compare it to Sato's et al simulations

*The content of this chapter is mainly based on preprint: Olszowiec C., (2016). Complex behaviour in cyclic competition bimatrix games. arXiv preprint arXiv:1605.00431

[94], [95], [96]. We present our presumptions about the way the chaos might be born, we explain the obstacles that prevent us from proving our conjectures.

Finally, in the last section we make conclusions, compare our results to those already existing in the literature and discuss open questions and future work.

4.2 - Preliminaries

We consider the Rock-Scissors-Paper game, played by two agents X and Y.

Let $x_1, x_2, x_3 \geq 0$ ($y_1, y_2, y_3 \geq 0$) denote the probability that player X (resp. Y) plays Rock, Scissors or Paper. Obviously $x_1 + x_2 + x_3 = 1$, $y_1 + y_2 + y_3 = 1$. For each player, the state space is a two-dimensional simplex, hence the whole state space for played game is $\Sigma = \Sigma_A \times \Sigma_B$. The interaction matrices are given by:

$$A = \begin{pmatrix} \epsilon_X & 1 & -1 \\ -1 & \epsilon_X & 1 \\ 1 & -1 & \epsilon_X \end{pmatrix} \quad B = \begin{pmatrix} \epsilon_Y & 1 & -1 \\ -1 & \epsilon_Y & 1 \\ 1 & -1 & \epsilon_Y \end{pmatrix} \tag{4.2.1}$$

after normalization they become:

$$A = \begin{pmatrix} \frac{2}{3}\epsilon_X & 1 - \frac{1}{3}\epsilon_X & -1 - \frac{1}{3}\epsilon_X \\ -1 - \frac{1}{3}\epsilon_X & \frac{2}{3}\epsilon_X & 1 - \frac{1}{3}\epsilon_X \\ 1 - \frac{1}{3}\epsilon_X & -1 - \frac{1}{3}\epsilon_X & \frac{2}{3}\epsilon_X \end{pmatrix} \quad B = \begin{pmatrix} \frac{2}{3}\epsilon_Y & 1 - \frac{1}{3}\epsilon_Y & -1 - \frac{1}{3}\epsilon_Y \\ -1 - \frac{1}{3}\epsilon_Y & \frac{2}{3}\epsilon_Y & 1 - \frac{1}{3}\epsilon_Y \\ 1 - \frac{1}{3}\epsilon_Y & -1 - \frac{1}{3}\epsilon_Y & \frac{2}{3}\epsilon_Y \end{pmatrix} \tag{4.2.2}$$

with $\epsilon_X, \epsilon_Y \in (-1, 1)$, the rewards for ties.

The dynamics are described by the coupled replicator equations (see [96]):

$$\begin{cases} \dot{x_i} = x_i[(Ay)_i - x^T Ay] \\ \dot{y_j} = y_j[(Bx)_j - y^T Bx] \end{cases} \tag{4.2.3}$$

with $i, j = 1, 2, 3$.

After substitutions $x_3 = 1 - x_1 - x_2, y_3 = 1 - y_1 - y_2$, we will consider it as a differential equation:

$$\begin{cases} \dot{x_1} = x_1[(Ay)_1 - x^T Ay] \\ \dot{x_2} = x_2[(Ay)_2 - x^T Ay] \\ \dot{y_1} = y_1[(Bx)_1 - y^T Bx] \\ \dot{y_2} = y_2[(Bx)_2 - y^T Bx] \end{cases} \tag{4.2.4}$$

with constraints $x_1, x_2 \geq 0, x_1 + x_2 \leq 1, y_1, y_2 \geq 0, y_1 + y_2 \leq 1$.

Let us denote the canonical basis in $\mathbb{R}^4$ by

$$e_1 = (1,0,0,0), \quad e_2 = (0,1,0,0), \quad e_3 = (0,0,1,0), \quad e_4 = (0,0,0,1)$$

From the form of the equation 4.2.3 the following proposition follows immediately:

Proposition 34. *(see [3]) The system (4.2.3) is invariant under action of the group generated by the cycle σ of order 3:*

$$\sigma(x_1, x_2, x_3, y_1, y_2, y_3) := (x_2, x_3, x_1, y_2, y_3, y_1) \tag{4.2.5}$$

4.2.1 Nash equilibrium

In the bimatrix game in which there is only one Nash equilibrium, it can be found as the intersection of the nullclines (here of the equation (4.2.3)). It is known that system (4.2.3), is Hamiltonian for $A = -B^T$ (see proposition 8), that is when $\epsilon_X + \epsilon_Y = 0$ (see [49], [94]). Moreover it was proven that in this case all trajectories have on average the same payoff as the Nash equilibrium [96]. On the other hand, on any given step huge deviations from the payoff can be observed. Thus, a risk averse agent would prefer the Nash equilibrium to a chaotic orbit [94].

In the sixth section we will make use of the following proposition.

Proposition 35. *Nash equilibrium of the system $(E^A, E^B) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is hyperbolic for $\epsilon_X + \epsilon_Y \neq 0$ (or a center in the Hamiltonian case), with eigenvalues*

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \Big(-\frac{1}{3}\sqrt{-3 + \epsilon_X\epsilon_Y - \sqrt{3}\sqrt{-(\epsilon_X + \epsilon_Y)^2}}, \frac{1}{3}\sqrt{-3 + \epsilon_X\epsilon_Y - \sqrt{3}\sqrt{-(\epsilon_X + \epsilon_Y)^2}},$$

$$-\frac{1}{3}\sqrt{-3+\epsilon_X\epsilon_Y+\sqrt{3}\sqrt{-(\epsilon_X+\epsilon_Y)^2}},\frac{1}{3}\sqrt{-3+\epsilon_X\epsilon_Y+\sqrt{3}\sqrt{-(\epsilon_X+\epsilon_Y)^2}}\Big)$$

where $Re(\lambda_1), Re(\lambda_3) \leq 0$, $Re(\lambda_2), Re(\lambda_4) \geq 0$

Proof. One can see that

$$Ay = \begin{bmatrix} -1+\epsilon_X(-\frac{1}{3}+y_1)+y_1+2y_2 \\ 1-2y_1+\epsilon_X(-\frac{1}{3}+y_2)-y_2 \\ y_1+\epsilon_X(\frac{2}{3}-y_1-y_2)-y_2 \end{bmatrix} \text{ and } (x^TB)^T = \begin{bmatrix} 1+\epsilon_Y(-\frac{1}{3}+x_1)-x_1-2x_2 \\ -1+2x_1+\epsilon_Y(-\frac{1}{3}+x_2)+x_2 \\ -x_1+\epsilon_Y(\frac{2}{3}-x_1-x_2)+x_2 \end{bmatrix}$$

in particular

$A[\frac{1}{3},\frac{1}{3},\frac{1}{3}]^T = [\ 0\ \ 0\ \ 0\ ]^T$ and $[\frac{1}{3},\frac{1}{3},\frac{1}{3}]B = [\ 0\ \ 0\ \ 0\ ]$ which by the definition means that $(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3})$ is a Nash equilibrium. Note that for $z_1 := -3+\epsilon_X\epsilon_Y-\sqrt{3}\sqrt{-(\epsilon_X+\epsilon_Y)^2}$ and $z_2 := -3+\epsilon_X\epsilon_Y+\sqrt{3}\sqrt{-(\epsilon_X+\epsilon_Y)^2}$, we have $Re(z_1) < 0, Im(z_1) \leq 0$, so it follows that $Re(\sqrt{z_1}) \leq 0$. On the other hand $Re(z_2) < 0, Im(z_2) \geq 0$, implies $Re(\sqrt{z_2}) \geq 0$. Hence $Re(\lambda_1), Re(\lambda_3) \leq 0$, $Re(\lambda_2), Re(\lambda_4) \geq 0$. Moreover we see that: $Re(\lambda_1) = 0 \Leftrightarrow Re(z_1) < 0 \Leftrightarrow Im(z_1) = 0 \Leftrightarrow Re(\lambda_1) = Re(\lambda_3) = Re(\lambda_2) = Re(\lambda_4) = 0 \Leftrightarrow Re(z_1) < 0 \Leftrightarrow Im(z_1) = 0 \Leftrightarrow \epsilon_X + \epsilon_Y = 0$. In such a case, the constant of motion E from the proposition 8 is a Lyapunov function for (E^A, E^B), showing that the Nash equilibrium is a center. □

4.2.2 Dynamics on the boundaries of codimensions 2 and 1

If each of the players is restricted to only two strategies, then all orbits are periodic and spiral around one of the 6 equilibria or they converge to draw $(\{R,R\},\{P,P\},\{S,S\})$ with time tending to $\pm\infty$. The latter case occurs for the system restricted to the subspaces $\{x_i = 0,\ y_i = 0\}$, $i = 1,2,3$.

Proposition 36. *There are 6 equilibria $Z^a, Z^b, Z^c, Z^d, Z^e, Z^f$ of (1,2,1) type (1-dim. stable manifold, 2-dim. center manifold, 1-dim. unstable manifold), which are centers within the 2 dimensional invariant subspaces $H^a, H^b, H^c, H^d, H^e, H^f$ (where the system is integrable), and have one dimensional stable and unstable manifolds $W^s_a, W^u_a, W^s_b, W^u_b, W^s_c, W^u_c, W^s_d, W^u_d, W^s_e, W^u_e, W^s_f, W^u_f$:*

$$Z^a = \Big(0, \frac{2}{3-\epsilon_Y}, \frac{1+\epsilon_X}{3+\epsilon_X}, \frac{2}{3+\epsilon_X}\Big), H^a = \{x_1 = 0, y_3 = 0\}, W^s_a \subset \{x_1 = 0\}, W^u_a \subset \{y_3 = 0\}$$

$$Z^b = \Big(\ 0,\ \frac{1+\epsilon_Y}{3+\epsilon_Y},\ \frac{1-\epsilon_X}{3-\epsilon_X},\ 0\ \Big),\ H^b = \{x_1 = 0, y_2 = 0\},\ W^s_b \subset \{y_2 = 0\},\ W^u_b \subset \{x_1 = 0\}$$

$$Z^c = \Big(\frac{1-\epsilon_Y}{3-\epsilon_Y},\ 0,\ 0,\ \frac{1+\epsilon_X}{3+\epsilon_X} \Big),\ H^c = \{x_2 = 0, y_1 = 0\},\ W^s_c \subset \{x_2 = 0\},\ W^u_c \subset \{y_1 = 0\}$$

$$Z^d = \Big(\frac{2}{3+\epsilon_Y}, 0, \frac{2}{3-\epsilon_X}, \frac{1-\epsilon_X}{3-\epsilon_X} \Big), H^d = \{x_2 = 0, y_3 = 0\}, W^s_d \subset \{y_3 = 0\}, W^u_d \subset \{x_2 = 0\}$$

$$Z^e = \Big(\frac{1+\epsilon_Y}{3+\epsilon_Y}, \frac{2}{3+\epsilon_Y}, 0, \frac{2}{3-\epsilon_X} \Big), H^e = \{x_3 = 0, y_1 = 0\}, W^s_e \subset \{y_1 = 0\}, W^u_e \subset \{x_3 = 0\}$$

$$Z^f = \Big(\frac{2}{3-\epsilon_Y}, \frac{1-\epsilon_Y}{3-\epsilon_Y}, \frac{2}{3+\epsilon_X}, 0 \Big), H^f = \{x_3 = 0, y_2 = 0\}, W^s_f \subset \{x_3 = 0\}, W^u_f \subset \{y_2 = 0\}$$

Proof. For Z^a one can compute the eigenvalues of linearized system to be

$$\{\frac{3+{\epsilon_X}^2}{3+\epsilon_X}, -2i\sqrt{\frac{1+\epsilon_X}{3+\epsilon_X}}\sqrt{\frac{1-\epsilon_Y}{3-\epsilon_Y}}, 2i\sqrt{\frac{1+\epsilon_X}{3+\epsilon_X}}\sqrt{\frac{1-\epsilon_Y}{3-\epsilon_Y}}, -\frac{3+{\epsilon_Y}^2}{3-\epsilon_Y}\}$$

Eigenvectors corresponding to purely imaginary eigenvalues lie within subspace $x_1 = 0, y_1 + y_2 = 1$. Restricting our system to this subspace (that is substituting $y_2 = 1 - y_1$ and $x_1 = 0$ in the equation), we find the Lyapunov function for Z^a within subspace $x_1 = 0, y_1 + y_2 = 1$:

$$V_a(x_2, y_1) := (1-\epsilon_Y)\log(1-x_2) + 2\log x_2 + (1+\epsilon_X)\log y_1 + 2\log(1-y_1)$$

One can see that $V_a(x_2, y_1) \leq 0$ for $x_2, y_1 \in (0,1)$ (moreover $V_a(x_2, y_1) = 0 \Leftrightarrow x_2 = \frac{2}{3-\epsilon_Y}, y_1 = \frac{1+\epsilon_X}{3+\epsilon_X}$) and

$$\dot{V}_a(x_2, y_1) = (1-\epsilon_Y)\cdot\frac{-x_2'}{1-x_2} + 2\frac{x_2'}{x_2} + (1+\epsilon_X)\cdot\frac{y_1'}{y_1} + 2\frac{-y_1'}{1-y_1} = (1-\epsilon_Y)\cdot x_2(-1+\epsilon_X(-1+y_1)+3y_1)+$$

$$2(-1+x_2)(-1+\epsilon_X(-1+y_1)+3y_1) + (1+\epsilon_X)(2+(-3+\epsilon_Y)x_2)(-1+y_1) + 2(2+(-3+\epsilon_Y)x_2)y_1 = 0$$

which proves that Z^a is a center within subspace $x_1 = 0, y_1 + y_2 = 1$.

To prove that, for $\epsilon_X + \epsilon_Y = 0$, stable manifold of Z^a is contained in the set $\{x_1 = 0\}$, let us note that the system is Hamiltonian in $int(\Delta \times \Delta)$ so if some part of this stable manifold was included in $int(\Delta \times \Delta)$ then on the whole of unstable manifold the function V, defined in the previous proposition, would be constant which is impossible since $V \to -\infty$ when $x_1 \to 0$.

In general, note that the eigenvector corresponding to the eigenvalue $\frac{3+\epsilon_Y{}^2}{-3+\epsilon_Y}$ is equal to

$$v = (0, v_2, v_3, 1) =$$
$$\Big(0, \frac{(-1+\epsilon_X)(3+\epsilon_X)^2(-1+\epsilon_Y)(3+\epsilon_Y^2)}{(3-\epsilon_Y)(-3(-5+\epsilon_X)(3+2\epsilon_X)+8(3-\epsilon_X)(1+\epsilon_X)\epsilon_Y+2(-12+(-5+\epsilon_X)\epsilon_X)\epsilon_Y^2+(3+\epsilon_X)\epsilon_Y^4)},$$
$$-\frac{(1+\epsilon_X)(3(3+\epsilon_Y^2)^2+\epsilon_X(33+\epsilon_Y(-32+14\epsilon_Y+\epsilon_Y^3)))}{6(-5+\epsilon_X)(3+2\epsilon_X)-16(-3+\epsilon_X)(1+\epsilon_X)\epsilon_Y+4(-12+(-5+\epsilon_X)\epsilon_X)\epsilon_Y^2-2(3+\epsilon_X)\epsilon_Y^4}, 1\Big)$$

Let us write

$$x_1 = 0 + h_1, x_2 = \frac{2}{3-\epsilon_Y} + h_2, y_1 = \frac{1+\epsilon_X}{3+\epsilon_X} + s_1, y_2 = \frac{2}{3+\epsilon_X} + s_2$$

It is easy to see that for all $\epsilon_X, \epsilon_Y \in (-1, 1)$ we have $v_2, v_3 \in (0, 1)$. So for a point (x_1, x_2, y_1, y_2) which belongs to the stable manifold of the point Z^a and is very close to Z^a, we may assume that

$$sign(h_2) = sign(v_2) > 0,\ sign(s_1) = sign(v_3) > 0,\ sign(s_2) = sign(1) > 0$$

(the second case when $sign(h_2) = -sign(v_2) < 0$, $sign(s_1) = -sign(v_3) < 0$, $sign(s_2) = -sign(1) < 0$ can be treated in the same way). We want to prove that projection of the stable manifold on the X -coordinate lies in the set $x_1 = 0$, which is invariant, so it suffices to prove that with the above assumptions on h_2, s_1, s_2 and with $h_1 = 0$ we have $h_2', s_1', s_2' < 0$ (or $h_2', s_1', s_2' > 0$ if we change the signs of h_2, s_1, s_2). Obviously $h_1' = 0$ on the subset $h_1 = 0$ and

$$x_2' = h_2' = (1+\epsilon_Y(-1+h_2)-3h_2)(2+(3-\epsilon_Y)h_2)\frac{(-3+\epsilon_X)s_1+2\epsilon_X s_2}{(-3+\epsilon_Y)^2}$$

Since $h_2 > 0$ and is small enough then

$$(1+\epsilon_Y(-1+h_2)-3h_2)(2+(3-\epsilon_Y)h_2) > 0$$

If $\epsilon_X \leq 0$ then obviously

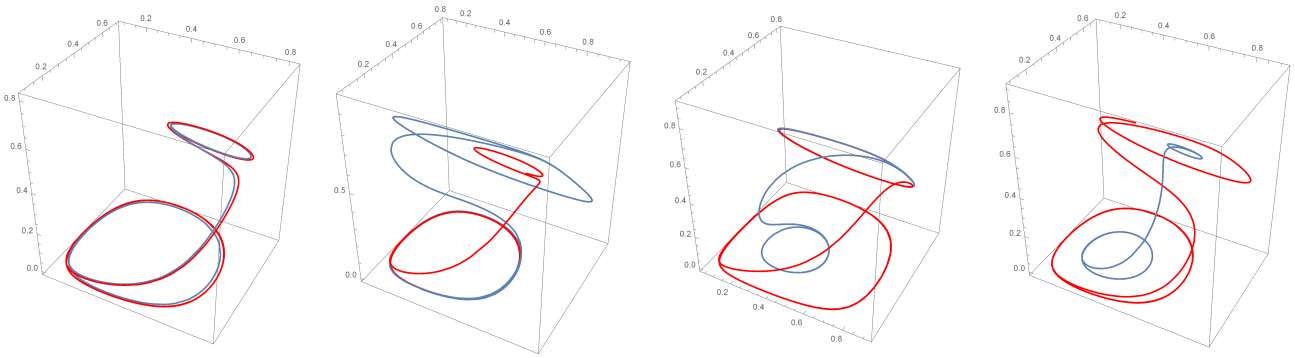
$$(-3+\epsilon_X)s_1 + 2\epsilon_X s_2 < 0$$

since $s_1, s_2 > 0$, otherwise we have

$$(-3+\epsilon_X)s_1 + 2\epsilon_X s_2 < 0 \Leftrightarrow \frac{2\epsilon_X}{3-\epsilon_X} < \frac{s_1}{s_2}$$

Since for any fixed $\epsilon_X, \epsilon_Y \in (-1, 1)$ the following inequality is satisfied $\frac{2\epsilon_X}{3-\epsilon_X} < v_3$, we can choose s_1, s_2

so small such that $\frac{s_1}{s_2}$ is close enough to v_3 and then $\frac{2\epsilon_X}{3-\epsilon_X} < \frac{s_1}{s_2}$, which proves that projection of stable manifold on the X -coordinate lies in the set $x_1 = 0$ □



(a) $\epsilon_X = -0.09,\ \epsilon_Y = -0.79$ (b) $\epsilon_X = 0.81,\ \epsilon_Y = 0.11$ (c) $\epsilon_X = 0.41,\ \epsilon_Y = 0.81$ (d) $\epsilon_X = -0.79,\ \epsilon_Y = -0.29$

Figure 4.1: Generic types of behaviour of the flow on the boundary $\{x_1 = 0\}$, the red curve is a backward orbit of a starting point close to the stable manifold of Z^a, while the blue curve is a forward orbit of a starting point close to the unstable manifold of Z^b. The curves stay either (a) close to each other for all times, (b) close to the subspace $\{y_2 = 0\}$, (c) close to $\{y_3 = 0\}$, (d) or are faraway from each other.

We find numerically that on the boundary of codimension 1, the trajectories of the game (4.2.3), travel from one invariant hyperplane of dimension 2 to another one.

Proposition 37. *For all $\epsilon_X \in (-1, 1), \epsilon_Y \in (0, 1)$, the following inclusions hold:*

$$\alpha(W^s_{Z^a}) \subset \{x_1 = 0,\ y_2 = 0\},\ \omega(W^u_{Z^b}) \subset \{x_1 = 0,\ y_3 = 0\}$$

Proof. For $W^u_{Z^b}$, note that with $x_1 = 0$ in the primary equation (4.2.3) we have: $\dot{y_2} > 0$ if $x_2 \in (0, \frac{1}{2}]$ and $\dot{y_3} < 0$ if $x_2 \in [\frac{1}{3}, 1)$. Moreover $\dot{x_2} > 0$ holds when either ($\epsilon_X < 0$ and $0 < y_1 < \frac{1+\epsilon_X}{3+\epsilon_X}$) or ($\epsilon_X > 0$ and $0 < y_1 < \frac{1-\epsilon_X}{3-\epsilon_X}$). For $\epsilon_Y \geq 0$ the exit subset from the set $\{y_3' \leq 0\}$ is empty. Since $\frac{2}{3-\epsilon_Y} > \frac{1}{2}$, $\frac{1+\epsilon_Y}{3+\epsilon_Y} < \frac{1}{2}$ and the points on $W^u_{Z_b}$ from the small neighbourhood of Z_b belong to the set $\{y_3' < 0\}$. Thus, it means that the forward orbit of a point from $W^u_{Z^b}$ will accumulate on the hyperplane $\{x_1 = 0,\ y_3 = 0\}$. analogously the backward orbit of a point from $W^s_{Z^a}$ will accumulate on the hyperplane $\{x_1 = 0,\ y_2 = 0\}$, where the motion is periodic. □

Remark 46. *Analogously one can state the similar proposition for other $\epsilon_X, \epsilon_Y \in (-1, 1)$ and stable/unstable manifolds of the points Z^a, Z^b, Z^c, Z^d, Z^e, Z^f.*

We believe that the following conjecture is worth further investigation.

Conjecture 2. *As one can check, there exist parameter values (e.g. $\epsilon_X = -0.09$, $\epsilon_Y = -0.79$) for which stable manifold of Z^a and unstable of Z^b seem to be very close. This observation motivates the following question (compare with [112], [114], [115], [117]):*

The set of pairs (ϵ_X, ϵ_Y) for which there is a heteroclinic orbit from Z^b to Z^a is non-empty and isolated in the parameter space. The same question remains open for the existence of other heteroclinic orbits, that is from: Z^a to Z^d, Z^d to Z^c, Z^c to Z^e, Z^e to Z^f, Z^f to Z^b

4.3 - Heteroclinic network and its transition maps

In this section we show how a heteroclinic network naturally appears in our system. We describe all of the maps associated with this network and the process of deriving it.

Remark 47. *(see [3])*
One can check that the system is C^1 linearizable for all $\epsilon_X, \epsilon_Y \in (-1, 1)$ except when $\epsilon_X = \epsilon_Y$. The whole forthcoming analysis is conducted for $\epsilon_X \neq \epsilon_Y$.

Proposition 38. *(compare with [3])*
The equation (4.2.4), possesses 9 equilibria of the form $\{(x, y) \mid x, y \in \{R, S, P\}\}$, where $R = (1, 0)$, $S = (0, 1)$, $P = (0, 0)$. They are all hyperbolic of $(2, 2)$ type with real eigenvalues (and corresponding eigenvectors) and stable/unstable manifolds W^s / W^u (since the system is invariant under the symmetry σ it suffices to provide the description for points (P, P), (P, S), (S, P)):

$$for\ (P,P),\ eigenbasis\ (e_1, e_2, e_3, e_4):\ (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-1-\epsilon_X, 1-\epsilon_X, -1-\epsilon_Y, 1-\epsilon_Y)$$
$$W^s_{(P,P)} = \{(x_1, 0, y_1, 0) \mid x_1 \neq 1, y_1 \neq 1\} \cap \Delta,\ W^u_{(P,P)} = \{(0, x_2, 0, y_2) \mid x_2 \neq 1, y_2 \neq 1\} \cap \Delta$$

$$for\ (S,P),\ eigenbasis\ (e_1 - e_2, e_2, e_3, e_4)): (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-2, -1+\epsilon_X, 2, 1+\epsilon_Y)$$
$$W^s_{(S,P)} = \{(x_1, x_2, 0, 0)\} \cap \Delta \backslash \{x_2 = 0\},\ W^u_{(S,P)} = \{(0, 1, y_1, y_2)\} \cap \Delta \backslash \{y_1 + y_2 = 1\}$$

$$for\ (P,S),\ eigenbasis\ (e_1, e_2, e_3 - e_4, e_4): (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (2, 1+\epsilon_X, 2, -2, -1+\epsilon_Y)$$
$$W^s_{(P,S)} = \{(0, 0, y_1, y_2)\} \cap \Delta \backslash \{y_2 = 0\},\ W^u_{(P,S)} = \{(x_1, x_2, 0, 1)\} \cap \Delta \backslash \{x_1 + x_2 = 0\}$$

Proof. We find that the linearized system near (P,S) is given by:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{y_1} \\ \dot{y_2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1+\epsilon_X & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1+\epsilon_Y & -1+\epsilon_Y \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 - 1 \end{bmatrix}$$

Hence, the searched eigenvalues are $\{2, 1+\epsilon_X, -2, -1+\epsilon_Y\}$ with the corresponding eigenvectors $\{(1,0,0,0),(0,1,0,0),(0,0,1,-1),(0,0,0,1)\}$.

$W^s_{(P,S)}$ is tangent at (P,S) to the 2-dimensional subspace $(P,S)+$ lin$\{(0,0,1,-1),(0,0,0,1)\}$. Note that the subset $\{x_1 = 0, x_2 = 0\}$ is invariant, and moreover, when $y_1 \neq 0$, $y_2 \neq 0$, we have $\dot{y_1} < 0$, $\dot{y_2} > 0$. Furthermore the subset $\{x_1 = 0, x_2 = 0, y_1 = 0\}$ is invariant and $\dot{y_2} > 0$ there. So the whole set $\{(0,0,y_1,y_2) \mid y_2 \neq 0\}$ is attracted to points $(P,S) = (0,0,0,1)$. From uniqueness of the existing stable manifold we have that $W^s_{(P,S)} = \{(0,0,y_1,y_2)\} \cap \Delta \backslash \{y_2 = 0\}$. analogously we prove $W^u_{(P,S)} = \{(x_1,x_2,0,1)\} \cap \Delta \backslash \{x_1 + x_2 = 0\}$ □

Remark 48. *There are no other equilibria for (4.2.4) than described so far.*

Let us denote $[T_1,T_2] := \{(1-\lambda)T_1 + \lambda T_2 \mid \lambda \in (0,1)\}$. Due to the above proposition, there exist a heteroclinic network Σ consisting of these 9 equilibria $\{(x,y) \mid x,y \in \{R,S,P\}\}$ and heteroclinic orbits written explicitly:

$[(P,P),(P,S)]$, $[(P,P),(S,P)]$, $[(P,S),(S,S)]$, $[(P,S),(R,S)]$, $[(S,S),(S,R)]$, $[(S,S),(R,S)]$, $[(R,S),(R,R)]$, $[(R,S),(R,P)]$, $[(R,P),(P,P)]$, $[(R,P),(S,P)]$, $[(S,P),(S,S)]$, $[(S,P),(S,R)]$, $((S,R),(R,R)]$, $[(S,R),(P,R)]$, $[(R,R),(R,P)]$, $[(R,R),(P,R)]$, $[(P,R),(P,P)]$, $[(P,R),(P,S)]$

Remark 49. *Heteroclinic orbits in this network are formed from the non-transverse intersections of stable and unstable manifolds of the appropriate equilibria.*

Because of the above remark and hyperbolicity of the equilibria from the network, one does not necessarily expect chaotic switching infinitely close to the heteroclinic network as indeed we show. In fact we will discuss this matter more closely in sections 5 and 7.

Proposition 39. *The following matrices $MacVU$ (for $V,U \in \{R,P,S\}$) are the transformation matrices from the eigenbasis at the point (V,U) to the canonical basis (e_1,e_2,e_3,e_4) of $\mathbb{R}^4$:*

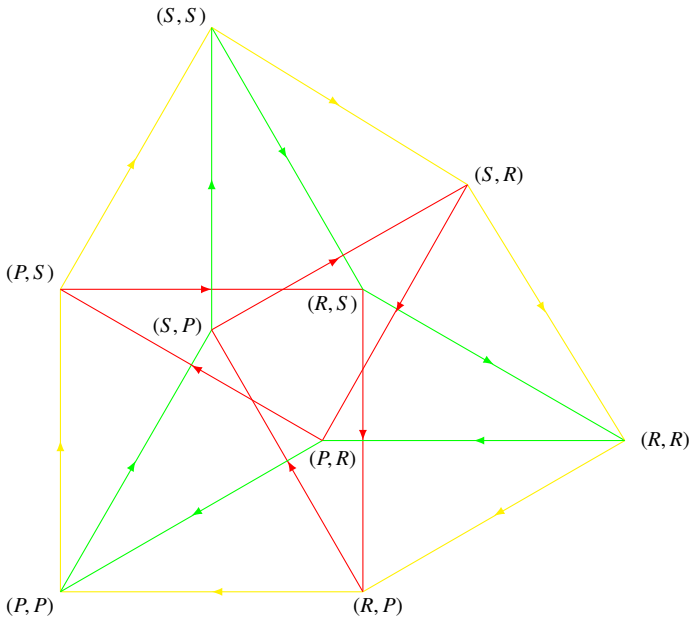


Figure 4.2: RSP network consisting of 9 hyperbolic equilibria $\{(x, y) \mid x, y \in \{R, S, P\}\}$ and connections between them:
$(P,P) \to (P,S)$, $(P,P) \to (S,P)$, $(P,S) \to (S,S)$, $(P,S) \to (R,S)$, $(S,S) \to (S,R)$, $(S,S) \to (R,S)$,
$(R,S) \to (R,R)$, $(R,S) \to (R,P)$, $(R,P) \to (P,P)$, $(R,P) \to (S,P)$, $(S,P) \to (S,S)$, $(S,P) \to (S,R)$,
$(S,R) \to (R,R)$, $(S,R) \to (P,R)$, $(R,R) \to (R,P)$, $(R,R) \to (P,R)$, $(P,R) \to (P,P)$, $(P,R) \to (P,S)$
Red/green/yellow heteroclinic connections correspond to the C_0, C_1, C_2 cycles, respectively. All of the heteroclinic connections within this network are formed by the non-transverse intersections of the stable/unstable manifolds of the corresponding hyperbolic equilibria. Compare with [95].

$$MacRR = [e_1, e_2 - e_1, e_3, e_4 - e_3] = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$MacRS = [e_1, e_2 - e_1, e_3 - e_4, e_4],\ MacSP = [e_1 - e_2, e_2, e_3 - e_4, e_4],\ MacPP = [e_1, e_2, e_3, e_4]$$

$$MacRP = [e_1, e_2 - e_1, e_3, e_4],\ MacPR = [e_1, e_2, e_3, e_4 - e_3],\ MacPS = [e_1, e_2, e_3 - e_4, e_4]$$

$$MacSR = [e_1 - e_2, e_2, e_3, e_4 - e_3],\ MacSS = [e_1 - e_2, e_2, e_3 - e_4, e_4]$$

Proof. $MacVU$ consists of the eigenvectors of the linearized system at the point (V, U). □

Proposition 40. *At each of the points (V, U) (with $V, U \in \{R, P, S\}$) from the heteroclinic network, the local coordinates (e_1', e_2', e_3', e_4') of affine space $\mathbb{R}^4$ in which this point is an origin, and the linearized system at this point is represented by the diagonal matrix (consisting of eigenvalues), are given by:*

$$(e_1', e_2', e_3', e_4') = MacVU^{-1}((e_1, e_2, e_3, e_4) - (V, U))$$

Proof. Note that the transformation matrix from the eigenbasis (that is $(e_1, e_2, e_3 - e_4, e_4)$) at the point (P, S) to the canonical basis of $\mathbb{R}^4$ is given by transformation matrix:

$$MacPS = [e_1, e_2, e_3 - e_4, e_4] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Moreover one can check that $\mathrm{MacVU}(V, U) = (V, U)$, hence the coordinates (e'_1, e'_2, e'_3, e'_4) satisfying statements above should be given by condition: $MacPS((e'_1, e'_2, e'_3, e'_4) + (P, S)) = (e_1, e_2, e_3, e_4)$. This gives the thesis. □

Remark 50. *Near each of the equilibrium from the heteroclinic network, we set four cross-sections to catch the orbits passing by close to this equilibrium. By introducing these cross-sections we can derive local maps near equilibria and global maps from the neighbourhood of one equilibrium to another one along heteroclinic orbit and in consequence discretize system in the neighbourhood of the heteroclinic network. Hence we can also check what kind of switching is possible and whether some particular trajectories infinitely close to the network are attainable.*

Lemma 4. *The in/out sections near each of the points from the heteroclinic network, associated with the heteroclinic orbits between two points might be chosen as follows ($h > 0$ is arbitrarily small):*

In (P, P) local coordinates:

$$\Sigma^{out,(P,P)}_{(P,P)\to(P,S)} = \{(z_1, z_2, z_3, h) \mid |z_1|, |z_2|, |z_3| < h\},\ \Sigma^{in,(P,P)}_{(P,P)\to(P,S)} = \{(z_1, z_2, z_3, 1-h) \mid |z_1|, |z_2|, |z_3| < h\},$$

$$\Sigma^{out,(P,P)}_{(P,P)\to(S,P)} = \{(z_1, h, z_3, z_4) \mid |z_1|, |z_3|, |z_4| < h\},\ \Sigma^{in,(P,P)}_{(P,P)\to(S,P)} = \{(z_1, 1-h, z_3, z_4) \mid |z_1|, |z_3|, |z_4| < h\}$$

In (P, S) local coordinates:

$$\Sigma^{out,(P,S)}_{(P,S)\to(R,S)} = \{(h, z_2, z_3, z_4) \mid |z_2|, |z_3|, |z_4| < h\},\ \Sigma^{in,(P,S)}_{(P,S)\to(R,S)} = \{(1-h, z_2, z_3, z_4) \mid |z_2|, |z_3|, |z_4| < h\},$$

$$\Sigma^{out,(P,S)}_{(P,S)\to(S,S)} = \{(z_1, h, z_3, z_4) \mid |z_1|, |z_3|, |z_4| < h\},\ \Sigma^{in,(P,S)}_{(P,S)\to(S,S)} = \{(z_1, 1-h, z_3, z_4) \mid |z_1|, |z_3|, |z_4| < h\}$$

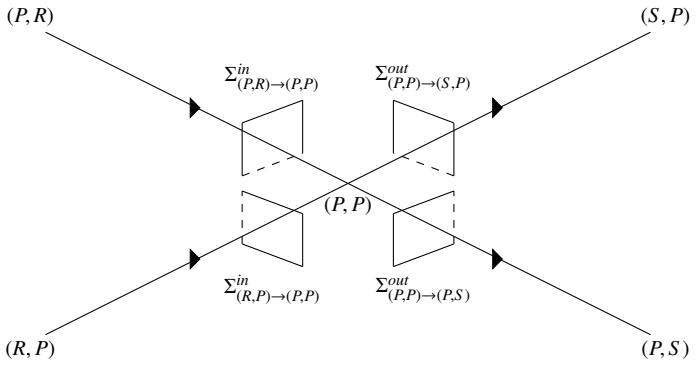


Figure 4.3: Schematic picture of the cross sections near equilibrium (P, P) (lemma 4). The sections are 3-dimensional cubes of size h (the size of a side segment). "The distance from the equilibrium (P, P) to the section" also equals h (on the coordinate which is fixed).

In (S, P) *local coordinates:*

$$\Sigma^{out,(S,P)}_{(S,P)\to(S,S)} = \{(z_1, z_2, z_3, h) \mid |z_1|, |z_2|, |z_3| < h\},\ \Sigma^{in,(S,P)}_{(S,P)\to(S,S)} = \{(z_1, z_2, z_3, 1-h) \mid |z_1|, |z_2|, |z_3| < h\},$$

$$\Sigma^{out,(S,P)}_{(S,P)\to(S,R)} = \{(z_1, z_2, h, z_4) \mid |z_1|, |z_2|, |z_4| < h\},\ \Sigma^{in,(S,P)}_{(S,P)\to(S,R)} = \{(z_1, z_2, 1-h, z_4) \mid |z_1|, |z_2|, |z_4| < h\}$$

Proof. Recall that in (R, R) coordinates we have: $(R, R) = (0, 0, 0, 0)$ and $(R, P) = MacRR^{-1}((1, 0, 0, 0) - (1, 0, 1, 0)) = (0, 0, -1, 0)$ hence the heteroclinic orbit from (R, R) to (R, P) is parameterized by $(0, 0, -\lambda, 0)$ for $\lambda \in (0, 1)$. So in (R, R) coordinates, we define the "out" section $\Sigma^{out,(R,R)}_{(R,R)\to(R,P)}$ from point (R, R) towards point (R, P), to be the set $\{(z_1, z_2, -h, z_4) \mid |z_1|, |z_2|, |z_4| < h\}$ for $h > 0$ chosen arbitrarily small. Analogously, we define the "in" section $\Sigma^{in,(R,R)}_{(R,R)\to(R,P)}$ to the point (R, P) from the point (R, R), to be the set $\{(z_1, z_2, -1 + h, z_4) \mid |z_1|, |z_2|, |z_4| < h\}$ □

Remark 51. *In order to define global and local maps, we need to represent each of the "in cross-sections" in two different coordinates. For the derivation of global maps we will use the representation from the proposition 4.*

The following proposition, enables us to compose local maps with global maps along cycles in the network.

Lemma 5. *Representations of the "in" sections in the local coordinates of the points near which they are.*

In (P,P) *local coordinates:*

$$\Sigma^{in,(P,P)}_{(P,R)\to(P,P)} = \{Tran_{(P,R)\to(P,P)}(z_1,z_2,z_3,z_4) = (z_1,z_2,1+z_3-z_4,z_4) \mid (z_1,z_2,z_3,z_4) \in \Sigma^{in,(P,R)}_{(P,R)\to(P,P)}\},$$
$$\Sigma^{in,(P,P)}_{(R,P)\to(P,P)} = \{Tran_{(R,P)\to(P,P)}(z_1,z_2,z_3,z_4) = (1+z_1-z_2,z_2,z_3,z_4) \mid (z_1,z_2,z_3,z_4) \in \Sigma^{in,(R,P)}_{(R,P)\to(P,P)}\},$$

In (P,S) *local coordinates:*

$$\Sigma^{in,(P,S)}_{(P,P)\to(P,S)} = \{Tran_{(P,P)\to(P,S)}(z_1,z_2,z_3,z_4) = (z_1,z_2,z_3,-1+z_3+z_4) \mid (z_1,z_2,z_3,z_4) \in \Sigma^{in,(P,P)}_{(P,P)\to(P,S)}\},$$
$$\Sigma^{in,(P,S)}_{(P,R)\to(P,S)} = \{Tran_{(P,R)\to(P,S)}(z_1,z_2,z_3,z_4) = (z_1,z_2,1+z_3-z_4,z_3) \mid (z_1,z_2,z_3,z_4) \in \Sigma^{in,(P,R)}_{(P,R)\to(P,S)}\},$$

In (S,P) *local coordinates:*

$$\Sigma^{in,(S,P)}_{(P,P)\to(S,P)} = \{Tran_{(P,P)\to(S,P)}(z_1,z_2,z_3,z_4) = (z_1,-1+z_1+z_2,z_3,z_4) \mid (z_1,z_2,z_3,z_4) \in \Sigma^{in,(P,P)}_{(P,P)\to(S,P)}\},$$
$$\Sigma^{in,(S,P)}_{(R,P)\to(S,P)} = \{Tran_{(R,P)\to(S,P)}(z_1,z_2,z_3,z_4) = (1+z_1-z_2,z_1,z_3,z_4) \mid (z_1,z_2,z_3,z_4) \in \Sigma^{in,(R,P)}_{(R,P)\to(S,P)}\}$$

Proof. Since the transformation from the (P,S) coordinates to canonical ones is given by the mapping

$$z \mapsto MacPS(z+e_4)$$

and the transformation from canonical to (S,S) is by

$$z \mapsto MacSS^{-1}(z) - (e_2+e_4)$$

So the transformation from (P,S) to (S,S) coordinates is given by the mapping

$$z \mapsto -(e_2+e_4) + (MacSS)^{-1} \cdot MacPS(z+e_4)$$

Hence

$$\Sigma^{in,(S,S)}_{(P,S)\to(S,S)} = \{-(e_2+e_4) + (MacSS)^{-1} \cdot MacPS(z+e_4) \mid z \in \Sigma^{in,(P,S)}_{(P,S)\to(S,S)}\} =$$
$$= \{(z_1,-1+z_1+z_2,z_3,z_4) \mid (z_1,z_2,z_3,z_4) \in \Sigma^{in,(R,S)}_{(R,S)\to(R,R)}\}$$

□

We find local maps, near each of the equilibrium from the heteroclinic network, in a standard way by

linearizing the system near the equilibrium.

Proposition 41. *(compare with [3])*

Local maps near each point of the heteroclinic network are given by the following formulas:

In (P,S) *coordinates:*

- *from* $\Sigma^{in}_{(P,S)\to(R,S)}$ *to* $\Sigma^{out}_{(P,S)\to(R,S)}$, $\Phi_{(P,S)\to(R,S)}(z_1,z_2,z_3,z_4) := \left(h, (\frac{h}{z_1})^{\frac{1+\epsilon_X}{2}} z_2, \frac{z_1 z_3}{h}, (\frac{z_1}{h})^{\frac{1-\epsilon_Y}{2}} z_4\right)$
- *from* $\Sigma^{in}_{(P,S)\to(S,S)}$ *to* $\Sigma^{out}_{(P,S)\to(S,S)}$, $\Phi_{(P,S)\to(S,S)}(z_1,z_2,z_3,z_4) := \left((\frac{h}{z_2})^{\frac{2}{1+\epsilon_X}} z_1, h, (\frac{z_2}{h})^{\frac{2}{1+\epsilon_X}} z_3, (\frac{z_2}{h})^{\frac{1-\epsilon_Y}{1+\epsilon_X}} z_4\right)$

In (P,P) *coordinates:*

- *from* $\Sigma^{in}_{(P,P)\to(P,S)}$ *to* $\Sigma^{out}_{(P,P)\to(P,S)}$, $\Phi_{(P,P)\to(P,S)}(z_1,z_2,z_3,z_4) := \left(z_1(\frac{z_4}{h})^{\frac{1+\epsilon_X}{1-\epsilon_Y}}, z_2(\frac{h}{z_4})^{\frac{1-\epsilon_X}{1-\epsilon_Y}}, z_3(\frac{z_4}{h})^{\frac{1-\epsilon_X}{1-\epsilon_Y}}, h\right)$
- *from* $\Sigma^{in}_{(P,P)\to(S,P)}$ *to* $\Sigma^{out}_{(P,P)\to(S,P)}$, $\Phi_{(P,P)\to(S,P)}(z_1,z_2,z_3,z_4) := \left((\frac{z_2}{h})^{\frac{1+\epsilon_X}{1-\epsilon_X}} z_1, h, (\frac{z_2}{h})^{\frac{1+\epsilon_Y}{1-\epsilon_X}} z_3, (\frac{h}{z_2})^{\frac{1-\epsilon_Y}{1-\epsilon_X}} z_4\right)$

In (S,P) *coordinates:*

- *from* $\Sigma^{in}_{(S,P)\to(S,R)}$ *to* $\Sigma^{out}_{(S,P)\to(S,R)}$, $\Phi_{(S,P)\to(S,R)}(z_1,z_2,z_3,z_4) := \left(\frac{z_1 z_3}{h}, (\frac{z_3}{h})^{\frac{1-\epsilon_X}{2}} z_2, h, (\frac{h}{z_3})^{\frac{1+\epsilon_Y}{2}} z_4\right)$
- *from* $\Sigma^{in}_{(S,P)\to(S,S)}$ *to* $\Sigma^{out}_{(S,P)\to(S,S)}$, $\Phi_{(S,P)\to(S,S)}(z_1,z_2,z_3,z_4) := \left((\frac{z_4}{h})^{\frac{2}{1+\epsilon_Y}} z_1, (\frac{z_4}{h})^{\frac{1-\epsilon_X}{1+\epsilon_Y}} z_2, (\frac{h}{z_4})^{\frac{2}{1+\epsilon_Y}} z_3, h\right)$

Proof. We approximate the system near (P,S) by its linearization, in (P,S) coordinates already:

$$\begin{bmatrix} \dot{z_1} \\ \dot{z_2} \\ \dot{z_3} \\ \dot{z_4} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1+\epsilon_X & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1+\epsilon_Y \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

The solution to it with initial condition $(z_1(0), z_2(0), z_3(0), z_4(0)) = (z_1, z_2, z_3, z_4)$ is given by

$$(z_1(t), z_2(t), z_3(t), z_4(t)) = (e^{2t} z_1, e^{t(1+\epsilon_X)} z_2, e^{-2t} z_3, e^{t(-1+\epsilon_Y)} z_4)$$

For the point $(z_1(t), z_2(t), z_3(t), z_4(t))$ to belong to the out section $\Sigma^{out}_{(P,S)\to(R,S)}$, it is needed to have $z_1(t) = h \Leftrightarrow e^t = (\frac{h}{z_1})^{\frac{1}{2}}$. Hence the local map to section $\Sigma^{out}_{(P,S)\to(R,S)}$ is given by mapping

$$\Phi_{(P,S)\to(R,S)}(z_1,z_2,z_3,z_4) := \left(z_1((\frac{h}{z_1})^{\frac{1}{2}})^2, z_2((\frac{h}{z_1})^{\frac{1}{2}})^{1+\epsilon_X}, z_3((\frac{h}{z_1})^{\frac{1}{2}})^{-2}, z_4((\frac{h}{z_1})^{\frac{1}{2}})^{-1+\epsilon_Y}\right) = \left(h, (\frac{h}{z_1})^{\frac{1+\epsilon_X}{2}} z_2, \frac{z_1 z_3}{h}, (\frac{z_1}{h})^{\frac{1-\epsilon_Y}{2}} z_4\right)$$

□

In order to order to introduce global maps from "out cross-sections" to "in cross-sections", we linearize the system along heteroclinic orbits.

Proposition 42. *Global maps along each heteroclinic orbit in the network are given by:*

In (P,P) *coordinates:*

$$\Psi_{(P,P)\to(P,S)}(z_1,z_2,z_3,h) := \Big((\frac{1-h}{h})^{\frac{1-\epsilon_X}{1-\epsilon_Y}}z_1, (\frac{1-h}{h})^{\frac{2}{1-\epsilon_Y}}z_2, (\frac{1-h}{h})^{-\frac{3+\epsilon_Y}{1-\epsilon_Y}}z_3, 1-h\Big)$$

$$\Psi_{(P,P)\to(S,P)}(z_1,h,z_3,z_4) := \Big((\frac{1-h}{h})^{-\frac{3+\epsilon_X}{1-\epsilon_X}}z_1, 1-h, (\frac{1-h}{h})^{\frac{1-\epsilon_Y}{1-\epsilon_X}}z_3, (\frac{1-h}{h})^{-\frac{2}{1-\epsilon_X}}z_4\Big)$$

In (P,S) *coordinates:*

$$\Psi_{(P,S)\to(R,S)}(h,z_2,z_3,z_4) := \Big(1-h, z_2(\frac{1-h}{h})^{\epsilon_X}, z_3(\frac{1-h}{h})^{-\frac{1-\epsilon_Y}{2}}, z_4(\frac{1-h}{h})^{\frac{1+\epsilon_Y}{2}}\Big)$$

$$\Psi_{(P,S)\to(S,S)}(z_1,h,z_3,z_4) := \Big(z_1(\frac{1-h}{h})^{-\frac{-3+\epsilon_X}{1+\epsilon_X}}, 1-h, z_3(\frac{1-h}{h})^{-\frac{1+\epsilon_Y}{1+\epsilon_X}}, z_4(\frac{1-h}{h})^{-\frac{2}{1+\epsilon_X}}\Big)$$

In (S,P) *coordinates:*

$$\Psi_{(S,P)\to(S,S)}(z_1,z_2,z_3,h) := \Big(z_1(\frac{1-h}{h})^{-\frac{1+\epsilon_X}{1+\epsilon_Y}}, z_2(\frac{1-h}{h})^{-\frac{2}{1+\epsilon_Y}}, z_3(\frac{1-h}{h})^{-1+\frac{4}{1+\epsilon_Y}}, 1-h\Big)$$

$$\Psi_{(S,P)\to(S,R)}(z_1,z_2,h,z_4) := \Big(z_1(\frac{1-h}{h})^{-\frac{1-\epsilon_X}{2}}, z_2(\frac{1-h}{h})^{\frac{1+\epsilon_X}{2}}, 1-h, z_4(\frac{1-h}{h})^{\epsilon_Y}\Big)$$

Proof. We compute global maps as shown below in the case of global map $\Psi_{(P,S)\to(R,S)}$. In the initial equation (4.2.4), let us change the coordinates from the canonical basis to local (P,S) coordinates (z_1,z_2,z_3,z_4) in which (P,S) is an origin, and $(R,S)_{(P,S)} = (1,0,0,0)$. Let us rewrite the right-hand side of the system (after changing the coordinates) as

$$G(z_1,z_2,z_3,z_4) = (\zeta(z_1,z_2,z_3,z_4), g(z_1,z_2,z_3,z_4)) \in \mathbb{R}\times\mathbb{R}^3$$

and the heteroclinic orbit from (P,S) to (R,S) by $\alpha = \{(z_1,0,0,0) \mid z_1 \in (0,1)\}$. Near α one can write $(z_1,z_2,z_3,z_4) = (z_1,z)$ with $z \in \mathbb{R}^3$ -small. Then

$$z'(t) = g(z_1(t),z(t)) = g(z_1(t),0) + g_z(z_1(t),0).z(t) + h.o.t.$$

Since $g(z_1(t), 0) = 0$ (because of the invariance of α) and by the chain rule

$$z'(t) = \frac{\partial z}{\partial z_1} \cdot z_1'(t)$$

we have that

$$\frac{\partial z}{\partial z_1} = \frac{1}{z_1'} \cdot z' = \frac{1}{\zeta(z_1,0,0,0) + \zeta_z(z_1,0,0,0).z + h.o.t.(z)} \cdot (g_z(z_1,0).z + h.o.t.(z)) \approx \frac{1}{\zeta(z_1,0,0,0)} \cdot g_z(z_1,0).z$$
$$= \frac{1}{2(1-z_1)z_1} \cdot \begin{bmatrix} 1+\epsilon_X - 2z_1 & 0 & 0 \\ 0 & -2 + (3+\epsilon_Y)z_1 & 0 \\ 0 & 0 & -1 + \epsilon_Y(1-z_1) + 3z_1 \end{bmatrix} \cdot \begin{bmatrix} z_2 \\ z_3 \\ z_4 \end{bmatrix} \quad (4.3.1)$$

Imposing initial condition $z(h) = z_0$,

$$z(\tau) = Exp\Big(\int_h^\tau \frac{1}{\zeta(z_1,0)} g_z(z_1,0) dz_1 \Big).z_0$$

is the solution to the latter equation (4.3.1). The global map $\Psi_{(P,S)\to(R,S)}$ will be given by the mapping

$$(h, z(h)) \longmapsto (1-h, z(1-h))$$

Hence in (P,S) coordinates global map to (R,S) has the form:

$$\Psi_{(P,S)\to(R,S)}(h, z_2, z_3, z_4) := \Big(1-h, z_2(\frac{1-h}{h})^{\epsilon_X}, z_3(\frac{1-h}{h})^{-\frac{1-\epsilon_Y}{2}}, z_4(\frac{1-h}{h})^{\frac{1+\epsilon_Y}{2}}\Big)$$

with $z_1 = h$ fixed. □

4.4 - Asymptotic dynamics near the network

In this section we will provide analytical proofs of numerical findings by Sato et al. [95] and proper analysis of the behaviour close to the heteroclinic network.

Recall that:

$$C_0\ cycle = PS \to RS \to RP \to SP \to SR \to PR \to PS$$

$$C_1\ cycle = PP \to SP \to SS \to RS \to RR \to PR \to PP$$

$$C_2\ cycle = PP \to PS \to SS \to SR \to RR \to RP \to PP$$

4.4.1 Dynamics near C_0, C_1, C_2 cycles

We start with an easy observation, that symmetry appearing in the system simplifies our computations along C_0, C_1, C_2 cycles, so instead of composing maps along six heteroclinic orbits, we can investigate part of it along two heteroclinic orbits.

Proposition 43. *The return maps to the sections $\Sigma^{out}_{(P,P)\to(S,P)}$, $\Sigma^{out}_{(P,P)\to(P,S)}$, $\Sigma^{out}_{(P,S)\to(R,S)}$ along the C_1, C_2, C_0 cycles, respectively, are given by:*

$$Ret_{(P,P),C_1} = (\sigma^{-1} \circ Tran_{(S,S)\to(P,P)} \circ \Phi_{(S,S)\to(R,S)} \circ$$

$$\circ\ Tran_{(S,P)\to(S,S)} \circ \Psi_{(S,P)\to(S,S)} \circ \Phi_{(S,P)\to(S,S)} \circ Tran_{(P,P)\to(S,P)} \circ \Psi_{(P,P)\to(S,P)})^3 \tag{4.4.1}$$

$$Ret_{(P,P),C_2} = (\sigma^{-1} \circ Tran_{(S,S)\to(P,P)} \circ \Phi_{(S,S)\to(S,R)} \circ$$

$$\circ\ Tran_{(P,S)\to(S,S)} \circ \Psi_{(P,S)\to(S,S)} \circ \Phi_{(P,S)\to(S,S)} \circ Tran_{(P,P)\to(P,S)} \circ \Psi_{(P,P)\to(P,S)})^3 \tag{4.4.2}$$

$$Ret_{(P,S),C_0} = (Tran_{(P,P)\to(P,S)} \circ \sigma \circ Tran_{(R,P)\to(P,P)} \circ \Phi_{(R,P)\to(S,P)} \circ$$

$$\circ\ Tran_{(R,S)\to(R,P)} \circ \Psi_{(R,S)\to(R,P)} \circ \Phi_{(R,S)\to(R,P)} \circ Tran_{(P,S)\to(R,S)} \circ \Psi_{(P,S)\to(R,S)})^3 \tag{4.4.3}$$

Proof. For C_2 cycle, let

$$T_{(P,P)\to(S,S)} := \Phi_{(R,P)\to(S,P)} \circ Tran_{(R,S)\to(R,P)} \circ \Psi_{(R,S)\to(R,P)} \circ \Phi_{(R,S)\to(R,P)} \circ Tran_{(P,S)\to(R,S)} \circ \Psi_{(P,S)\to(R,S)}$$

then

$$Ret_{(P,P),C_2} = T_{(R,R)\to(P,P)} \circ T_{(S,S)\to(R,R)} \circ T_{(P,P)\to(S,S)}$$

with $T_{(R,R)\to(P,P)}$, $T_{(S,S)\to(R,R)}$ defined analogously (along C_2 cycle). Due to the symmetry given by the action σ one can write:

$$Ret_{(P,P),C_2} = (T_{(R,R)\to(P,P)} \circ Tran_{(P,P)\to(R,R)}) \circ (Tran_{(R,R)\to(P,P)} \circ T_{(S,S)\to(R,R)} \circ Tran_{(P,P)\to(S,S)}) \circ$$

$$\circ \left(Tran_{(S,S)\to(P,P)} \circ T_{(P,P)\to(S,S)}\right) = \left(\sigma^2 \circ Tran_{(S,S)\to(P,P)} \circ T_{(P,P)\to(S,S)} \circ \sigma^{-2}\right) \circ$$

$$\circ \left(\sigma \circ Tran_{(S,S)\to(P,P)} \circ T_{(P,P)\to(S,S)} \circ \sigma^{-1}\right) \circ \left(Tran_{(S,S)\to(P,P)} \circ T_{(P,P)\to(S,S)}\right) =$$

$$= \left(\sigma^{-1} \circ Tran_{(S,S)\to(P,P)} \circ T_{(P,P)\to(S,S)}\right)^3$$

□

Remark 52. *Let us identify, thanks to the $\mathbb{Z}_3$ symmetry σ:*

$$tie := \{(P,P),(S,S),(R,R)\}/\sigma$$

$$win/loss := \{(S,P),(P,R),(R,S)\}/\sigma$$

$$loss/win := \{(P,S),(S,R),(R,P)\}/\sigma$$

Hence we can enumerate "out cross-sections" as in the figure 4.4. Now, if $h << 1$ is set to be a constant we can find approximations of the maps between the sections 1-6 in the limit of coordinates on the cross-sections tending to zero.

Proposition 44. *(compare with [3])*
With $h << 1$ fixed and $0 < s_1, s_2, s_3, s_4 < \epsilon << h$, it is possible to rescale coordinates linearly at the cross sections near: tie, loss/win, win/loss, so that the mappings are of the "constant-free" form
$\vartheta_{12}(s_1, s_2, s_3, h) = \left(s_1 s_2^{-\frac{2}{1+\epsilon_X}}, h, s_2^{\frac{2}{1+\epsilon_X}} s_3, s_2^{\frac{1-\epsilon_Y}{1+\epsilon_X}}\right) =$

$$= \Phi_{(P,S)\to(S,S)} \circ Tran_{(P,P)\to(P,S)} \circ \Phi_{(P,P)\to(P,S)}(s_1, s_2, s_3, h)$$

$\vartheta_{14}(s_1, s_2, s_3, h) = \left(h, s_1^{-\frac{1+\epsilon_X}{2}} s_2, s_1 s_3, s_1^{\frac{1-\epsilon_Y}{2}}\right) =$

$$= \Phi_{(P,S)\to(R,S)} \circ Tran_{(P,P)\to(P,S)} \circ \Phi_{(P,P)\to(P,S)}(s_1, s_2, s_3, h)$$

$\vartheta_{21}(s_1, h, s_3, s_4) = \left(s_3^{\frac{1+\epsilon_X}{1-\epsilon_Y}}, s_1 s_3^{-\frac{1-\epsilon_X}{1-\epsilon_Y}}, s_3^{\frac{1+\epsilon_Y}{1-\epsilon_Y}} s_4, h\right) =$

$$= \sigma^{-1} \circ Tran_{(S,S)\to(P,P)} \circ \Phi_{(S,S)\to(S,R)} \circ Tran_{(P,S)\to(S,S)} \circ \Phi_{(P,S)\to(S,S)}(s_1, h, s_3, s_4)$$

$\vartheta_{26}(s_1, h, s_3, s_4) = \left(s_1^{\frac{1+\epsilon_X}{1-\epsilon_X}}, h, s_1^{\frac{1+\epsilon_Y}{1-\epsilon_X}} s_4, s_1^{-\frac{1-\epsilon_Y}{1-\epsilon_X}} s_3\right) =$

$$= \sigma^{-1} \circ Tran_{(S,S)\to(P,P)} \circ \Phi_{(S,S)\to(R,S)} \circ Tran_{(P,S)\to(S,S)} \circ \Phi_{(P,S)\to(S,S)}(s_1, h, s_3, s_4)$$

$$\vartheta_{32}(s_1, s_2, h, s_4) = (s_1^{-\frac{2}{1+\epsilon_X}} s_2, h, s_1^{\frac{2}{1+\epsilon_X}}, s_1^{\frac{1-\epsilon_Y}{1-\epsilon_X}} s_4) =$$

$$= Tran_{(P,P)\to(P,S)} \circ \sigma^{-1} \circ Tran_{(S,R)\to(P,P)} \circ \Phi_{(S,R)\to(R,R)} \circ Tran_{(S,P)\to(S,R)} \circ \Phi_{(S,P)\to(S,R)}(s_1, s_2, h, s_4)$$

$$\vartheta_{34}(s_1, s_2, h, s_4) = (h, s_1 s_2^{-\frac{1+\epsilon_X}{2}}, s_2, s_2^{\frac{1-\epsilon_Y}{2}} s_4) =$$

$$= Tran_{(P,P)\to(P,S)} \circ \sigma^{-1} \circ Tran_{(S,R)\to(P,P)} \circ \Phi_{(S,R)\to(P,R)} \circ Tran_{(S,P)\to(S,R)} \circ \Phi_{(S,P)\to(S,R)}(s_1, s_2, h, s_4)$$

$$\vartheta_{43}(h, s_2, s_3, s_4) = (s_4, s_2 s_4^{\frac{1-\epsilon_X}{2}}, h, s_3 s_4^{-\frac{1+\epsilon_Y}{2}}) =$$

$$= Tran_{(P,P)\to(S,P)} \circ \sigma^{-1} \circ Tran_{(R,S)\to(P,P)} \circ \Phi_{(R,S)\to(R,P)} \circ Tran_{(P,S)\to(R,S)} \circ \Phi_{(P,S)\to(R,S)}(h, s_2, s_3, s_4)$$

$$\vartheta_{45}(h, s_2, s_3, s_4) = (s_3^{\frac{2}{1+\epsilon_Y}}, s_2 s_3^{\frac{1-\epsilon_X}{1+\epsilon_Y}}, s_3^{-\frac{2}{1+\epsilon_Y}} s_4, h) =$$

$$= Tran_{(P,P)\to(S,P)} \circ \sigma^{-1} \circ Tran_{(S,R)\to(P,P)} \circ \Phi_{(R,S)\to(R,R)} \circ Tran_{(P,S)\to(R,S)} \circ \Phi_{(P,S)\to(R,S)}(h, s_2, s_3, s_4)$$

$$\vartheta_{51}(s_1, s_2, s_3, h) = (s_2 s_3^{\frac{1+\epsilon_X}{1-\epsilon_Y}}, s_1 s_3^{-\frac{1-\epsilon_X}{1-\epsilon_Y}}, s_3^{\frac{1+\epsilon_Y}{1-\epsilon_Y}}, h) =$$

$$= \sigma^{-1} \circ Tran_{(S,S)\to(P,P)} \circ \Phi_{(S,S)\to(S,R)} \circ Tran_{(S,P)\to(S,S)} \circ \Phi_{(S,P)\to(S,S)}(s_1, s_2, s_3, h)$$

$$\vartheta_{56}(s_1, s_2, s_3, h) = (s_2 s_3^{\frac{1+\epsilon_X}{1-\epsilon_Y}}, s_1 s_3^{-\frac{1-\epsilon_X}{1-\epsilon_Y}}, s_3^{\frac{1+\epsilon_Y}{1-\epsilon_Y}}, h) =$$

$$= \sigma^{-1} \circ Tran_{(S,S)\to(P,P)} \circ \Phi_{(S,S)\to(R,S)} \circ Tran_{(S,P)\to(S,S)} \circ \Phi_{(S,P)\to(S,S)}(s_1, s_2, s_3, h)$$

$$\vartheta_{63}(s_1, h, s_3, s_4) = (s_1 s_3, s_3^{\frac{1-\epsilon_X}{2}}, h, s_3^{-\frac{1+\epsilon_Y}{2}} s_4) =$$

$$= \Phi_{(S,P)\to(S,R)} \circ Tran_{(P,P)\to(S,P)} \circ \Phi_{(P,P)\to(S,P)}(s_1, h, s_3, s_4)$$

$$\vartheta_{65}(s_1, h, s_3, s_4) = (s_1 s_4^{\frac{2}{1+\epsilon_Y}}, s_4^{\frac{1-\epsilon_X}{1+\epsilon_Y}}, s_3 s_4^{-\frac{2}{1+\epsilon_Y}}, h) =$$

$$= \Phi_{(S,P)\to(S,S)} \circ Tran_{(P,P)\to(S,P)} \circ \Phi_{(P,P)\to(S,P)}(s_1, h, s_3, s_4)$$

Proof. For the points (with $h \to 0$):

(z_1, h, z_3, z_4) from section 1 we substitute $(s_1, h, s_3, s_4) = (h^{-\frac{2}{1+\epsilon_X}} z_1, h, h^{\frac{2}{1+\epsilon_X}} z_2, h^{\frac{1-\epsilon_Y}{1+\epsilon_X}} z_4)$

(z_1, z_2, z_3, h) from section 2 we substitute $(s_1, s_2, s_3, h) = (h^{\frac{1+\epsilon_X}{-1+\epsilon_Y}} z_1, h^{\frac{1-\epsilon_X}{-1+\epsilon_Y}} z_2, h^{\frac{1+\epsilon_Y}{1-\epsilon_Y}} z_3, h)$

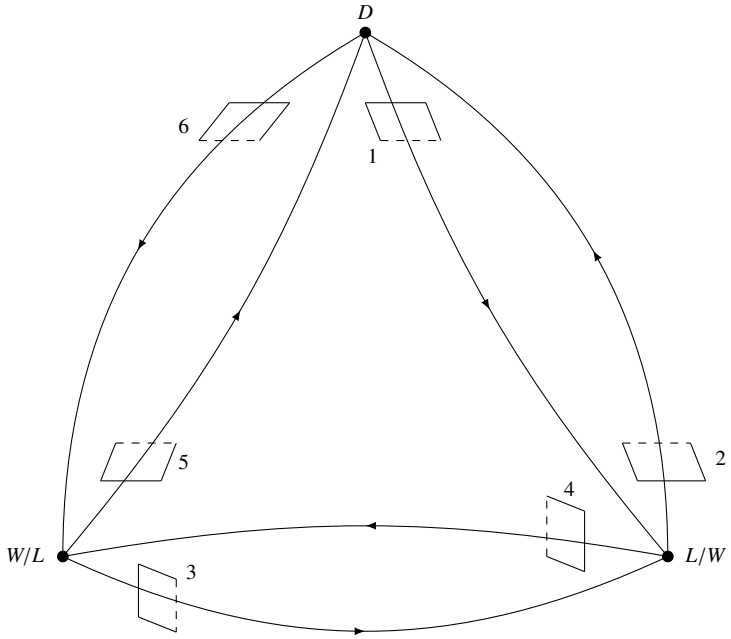


Figure 4.4: Schematic picture of the quotient network together with cross-sections (1-6) after identification of equilibria by the $\mathbb{Z}_3$ symmetry σ. Recall that $tie := \{(P,P),(S,S),(R,R)\}/\sigma$, $W/L := \{(S,P),(P,R),(R,S)\}/\sigma$, $L/W := \{(P,S),(S,R),(R,P)\}/\sigma$. So the $C_0,\ C_1,\ C_2$ cycles are respectively $W/L \to L/W \to W/L$, $W/L \to tie \to W/L$, $L/W \to tie \to L/W$.

(h, z_2, z_3, z_4) from section 3 we substitute $(h, s_2, s_3, s_4) = (h, h^{-\frac{1+\epsilon_X}{2}} z_2, h z_3, h^{\frac{1-\epsilon_Y}{2}} z_4)$

(z_1, z_2, h, z_4) from section 4 we substitute $(s_1, s_2, h, s_4) = (h z_1, h^{\frac{1-\epsilon_X}{2}} z_2, h, h^{-\frac{1+\epsilon_Y}{2}} z_4)$

(z_1, h, z_3, z_4) from section 5 we substitute $(s_1, h, s_3, s_4) = (h^{\frac{1+\epsilon_Y}{-1+\epsilon_X}} z_1, h, h^{\frac{1+\epsilon_X}{1-\epsilon_X}} z_3, h^{\frac{1-\epsilon_Y}{-1+\epsilon_X}} z_4)$

(z_1, z_2, z_3, h) from section 6 we substitute $(s_1, s_2, s_3, h) = (h^{-\frac{2}{1+\epsilon_Y}} z_1, h^{\frac{1-\epsilon_X}{1+\epsilon_Y}} z_2, h^{\frac{2}{1+\epsilon_Y}} z_2, h)$

□

Before stating next proposition, we need to recall some definitions from [64] (see also [85] and [67]).

Definition 14. *A compact invariant set X is called:*

- asymptotically stable relative to N if for every neighbourhood U of X there is a neighbourhood V of X such that for all $x \in V \cap N$ we have $\omega(x) \subset X$ and $\phi_t(x) \in U \cap N$ for all $t > 0$.

- completely unstable relative to N if there is a neighbourhood U of X such that for all $x \in U \cap N$ there is $t_0 > 0$ with $\phi_{t_0}(x) \notin U$.

Definition 15. *A compact invariant set $X \subset \mathbb{R}^n$ is called essentially asymptotically stable (e.a.s.) if it is asymptotically stable relative to a set $N \subset \mathbb{R}^n$ with the property that:*

$$\lim_{\epsilon \to 0} \frac{\mathcal{L}^n(\{x \in \mathbb{R}^n \mid dist(x, X) < \epsilon\} \cap N)}{\mathcal{L}^n(\{x \in \mathbb{R}^n \mid dist(x, X) < \epsilon\})} = 1 \tag{4.4.4}$$

Definition 16. *A compact invariant set X is almost completely unstable (a.c.u.) if it is completely*

unstable relative to a set $N \subset \mathbb{R}^n$ with property (4.4.4).

Definition 17. *For $x \in X$, and small enough $\epsilon, \delta > 0$ define*

$$\Sigma_{\epsilon,\delta}(x) := \frac{\mathcal{L}^n(B(x,\epsilon) \cap \mathcal{B}_\delta(X))}{\mathcal{L}^n(B(x,\epsilon))}$$

where $\mathcal{B}_\delta(X) = \left\{x \in \mathcal{B}(X) \mid \forall t > 0 : \phi_t(x) \in \{z \in \mathbb{R}^n \mid dist(z,X) < \delta\}\right\}$ and $\mathcal{B}(X)$ is the basin of attraction of X, i.e. the set of points $x \in \mathbb{R}^n$ with $\omega(x) \subset X$.

Definition 18. *Local stability index at $x \in X$ is defined to be:*

$$\sigma_{loc}(x) := \sigma_{loc,+}(x) - \sigma_{loc,-}(x)$$

where

$$\sigma_{loc,-}(x) := \lim_{\delta\to 0}\lim_{\epsilon\to 0} \frac{\ln(\Sigma_{\epsilon,\delta}(x))}{\ln(\epsilon)} \qquad \sigma_{loc,+}(x) := \lim_{\delta\to 0}\lim_{\epsilon\to 0} \frac{\ln(1-\Sigma_{\epsilon,\delta}(x))}{\ln(\epsilon)}$$

We will make use of the following theorem (Theorem 3.5 in [67]), in order to prove, for some parameter values ϵ_X, ϵ_Y, essential asymptotic stability of C_0 and relative asymptotic stability of C_1, C_2.

Theorem 4.4.1. *(see [67])*
Let $X \subset \mathbb{R}^n$ be a heteroclinic cycle or network with finitely many equilibria and connecting trajectories. Suppose that $\mathcal{L}^1(X) < +\infty$ and that the local stability index $\sigma_{loc}(x)$ exists and is not equal to zero for all $x \in X$. Then generically we have:
(i) X is e.a.s. $\Leftrightarrow \sigma_{loc} > 0$ along all connecting trajectories
(i) X is a.c.u. $\Leftrightarrow \sigma_{loc} < 0$ along all connecting trajectories

Our first result concerning stability of the cycles within the Rock-Scissors-Paper game is:

Theorem 7. *(compare with [3], [36], [37])*
For $\epsilon_X + \epsilon_Y < 0$, the cycle C_0 is essentially asymptotically stable.

Proof. To prove that C_0 is e.a.s., one has to show that all of the local indices along the C_0 cycle are positive. From propositions 43 and 44, in the limit with $h << 1$ and $s_1, s_2, s_3, s_4 < \epsilon << h$, the formulas for the maps along C_0 cycle simplify to the formulas from [3]. Although the domains of these maps differ by constants from the domains of maps in [3], however for the computation of indices the only thing that counts, in this example, is the relationship between the exponents of coordinates in the

maps and their domains as well - and this remains the same while taking $s_1, s_2, s_3, s_4 < \epsilon << h$ and passing to the limit $\epsilon \to 0$.

Furthermore, note that in the presence of symmetry σ in the system and formula (4.4.3), it suffices to compute only 2 indices for the map

$$Tran_{(P,P)\to(P,S)} \circ \sigma \circ Tran_{(R,P)\to(P,P)} \circ \Phi_{(R,P)\to(S,P)} \circ Tran_{(R,S)\to(R,P)} \circ \Psi_{(R,S)\to(R,P)} \circ$$

$$\circ \Phi_{(R,S)\to(R,P)} \circ Tran_{(P,S)\to(R,S)} \circ \Psi_{(P,S)\to(R,S)}$$

instead of 6 indices for the return map $Ret_{(P,S),C_0}$.

These local indices were computed independently in [36] for the model from [3] and they are all positive for $\epsilon_X + \epsilon_Y < 0$. Hence from the above reasoning and by theorem 4.4.1, C_0 is an e.a.s cycle. □

Because of the Proposition 44, we can introduce logarithmic coordinates, useful for description of stability of the cycles and bifurcations happening in the system with ϵ_X, ϵ_Y varying.

Proposition 45. *The map from (P,S) to (P,S) through (R,S) (and after identifying (R,P) with (P,S) by symmetry σ) :*

$$Tran_{(P,P)\to(P,S)} \circ \sigma \circ Tran_{(R,P)\to(P,P)} \circ T_{(P,S)\to(R,S)\to(R,P)}$$

in logarithmic coordinates $w_i := -\ln(s_i)$, $i = 2,3,4$ has the form:

$$(w_2, w_3, w_4)^T = A_{C_0} \cdot (w_2, w_3, w_4)^T$$

where

$$A_{C_0} = \begin{bmatrix} -\frac{1+\epsilon_X}{2} & 0 & \frac{3+\epsilon_X^2}{4} \\ 1 & 0 & \frac{1-\epsilon_X}{2} \\ \frac{1-\epsilon_Y}{2} & 1 & -\frac{1+\epsilon_Y}{2} + \frac{(1-\epsilon_X)(1-\epsilon_Y)}{4} \end{bmatrix}$$

The eigenvalues λ_i and corresponding eigenvectors $v_i = (v_{i1}, v_{i2}, v_{i3})$, $i = 1,2,3$, of matrix A_{C_0} satisfy the following conditions:

- $\lambda_1 > 0$, *while* λ_2, λ_3 *are complex conjugated and* $Re(\lambda_2) < 0$, $Re(\lambda_3) < 0$, $Im(\lambda_2) < 0$, $Im(\lambda_3) > 0$
- $v_{13} = v_{23} = v_{33} = 1$
- $v_{11} > 0, v_{12} > 0$
- $Re(v_{21}) < 0, Re(v_{22}) < 0$, $Im(v_{21}) > 0$, $Im(v_{22}) < 0$

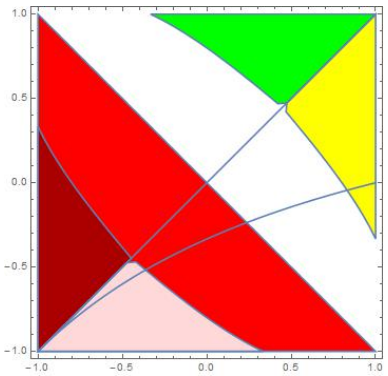


Figure 4.5: Bifurcation diagram.
The picture shows regions O_1, O_2, $\epsilon_X + \epsilon_Y < 0$, and β -curve $\epsilon_X = \frac{1+3\epsilon_Y}{1-\epsilon_Y}$ (where $\beta = \frac{1+\epsilon_X}{2}$). The β-curve intersects boundary of the yellow region O_2 at point $\beta = \tau \approx 0.915$.
In the red regions $\epsilon_X + \epsilon_Y < 0$ the cycle C_0 is essentially asymptotically stable (Theorem 7). In the yellow region the cycle C_2 is relatively asymptotically stable and there exists an open set attracted to the cycle C_2 (Theorem 45). In the green region the cycle C_1 is relatively asymptotically stable and there exists an open set attracted to the cycle C_1. In the white region with $\epsilon_Y < \epsilon_X$ there exists a 3-dimensional set of initial conditions attracted to the cycle C_2. For $\epsilon_X + \epsilon_Y > 0$, there exists an open set of initial conditions which converge backward in time to the cycle C_0.

- $Re(v_{31}) < 0, Re(v_{32}) < 0$, $Im(v_{31}) < 0$, $Im(v_{32}) > 0$

- *for* $\epsilon_X + \epsilon_Y < 0$*:*

$$\lambda_1 > 1 > Abs(\lambda_2) = Abs(\lambda_3)$$

- *for* $\epsilon_X + \epsilon_Y > 0$*:*

$$\lambda_1 < 1 < Abs(\lambda_2) = Abs(\lambda_3)$$

Proposition 46. *The map from* (P, P) *to* (P, P) *through* (P, S) *(and after identifying* (S, S) *with* (P, P) *by symmetry* σ*):*

$$\sigma^{-1} \circ Tran_{(S,S)\to(P,P)} \circ T_{(P,P)\to(P,S)\to(S,S)}$$

in logarithmic coordinates $w_i := -\ln(s_i)$, $i = 1, 2, 3$ *is given by:*

$$(w_1, w_2, w_3)^T = A_{C_2} \cdot (w_1, w_2, w_3)^T$$

where

$$A_{C_2} = \begin{bmatrix} 0 & \frac{2}{1-\epsilon_Y} & \frac{1+\epsilon_X}{1-\epsilon_Y} \\ 1 & -\frac{2}{1+\epsilon_X} - \frac{2(1-\epsilon_X)}{(1+\epsilon_X)(1-\epsilon_Y)} & -\frac{(1-\epsilon_X)}{(1-\epsilon_Y)} \\ 0 & \frac{(1-\epsilon_Y)}{(1+\epsilon_X)} + \frac{2(1+\epsilon_Y)}{(1+\epsilon_X)(1-\epsilon_Y)} & \frac{(1+\epsilon_Y)}{(1-\epsilon_Y)} \end{bmatrix}$$

The eigenvalues λ_i *and corresponding eigenvectors* $v_i = (v_{i1}, v_{i2}, v_{i3})$, $i = 1, 2, 3$, *of matrix* A_{C_2} *satisfy the following conditions:*

- $v_{13} = v_{23} = v_{33} = 1$

- $v_{21} < 0, v_{22} < 0$

- $v_{31} > 0, v_{32} < 0$

- *for* $\epsilon_X > \epsilon_Y$*:*

$$v_{11} > 0, \quad v_{12} > 0$$

- *for* $\epsilon_X < \epsilon_Y$*:*

$$v_{11} > 0, \quad v_{12} < 0$$

- *for* $\epsilon_X > \epsilon_Y$*:*

$$\lambda_2 \in (-1, 0), \quad \lambda_3 < -1, \quad \lambda_1 > 0$$

- *in the yellow region*

$$\lambda_1 > max(1, |\lambda_2|, |\lambda_3|)$$

- *in the white region where* $\epsilon_X > \epsilon_Y$

$$1 < \lambda_1 < |\lambda_3|$$

- *in the red region where* $\epsilon_X > \epsilon_Y$

$$\lambda_1 < 1$$

- *in the light red region where* $\epsilon_X > \epsilon_Y$

$$\lambda_1 < |\lambda_2|$$

Remark 53. *We obtain the same proposition, as Prop. 46, for cycle* C_1 *by swapping* ϵ_X *with* ϵ_Y*.*

We are ready to prove our next result, i.e. that C_2 and C_1 are asymptotically stable relatively to some open set, however they cannot be essentially asymptotically stable according to the Theorem 4.4.1.

Theorem 8. *For some open regions* O_1 *(green),* O_2 *(yellow) (see figure 4.5) in the parameter space* ϵ_X, ϵ_Y*, cycles* C_1 *and* C_2*, respectively, are neither almost completely unstable nor essentially asymptotically stable. For each of them, there exist an open set of points in the phase space which is attracted to the respective cycle.*

Proof. By propositions 43 and 46, the return map to the cross section $\Sigma^{out}_{(P,P)\to(P,S)}$ through the whole C_2 cycle is given by

$$(w_1, w_2, w_3)^T = A^3_{C_2} \cdot (w_1, w_2, w_3)^T$$

Let

$$v_t := t_1 v_1 + t_2 v_2 + t_3 v_3$$

where (v_1, v_2, v_3 - eigenvectors from proposition 46). It suffices to show that for every ϵ_X, ϵ_Y from the yellow region, every compact interval $I_2, I_3 \subset \mathbb{R}$ and $t_2 \in I_2, t_3 \in I_3$, there exist $T_{I_2,I_3} > 0$, big enough, such that for every $t \in (T_{I_2,I_3}, +\infty)$, we have convergence to $+\infty$ on all of the coordinates of the vectors $A^n_{C_2}(v_t)$ and $\vartheta_{26}(A^n_{C_2}(v_t))$ (as well for $n = 1$ their coordinates have to be big enough). Now

$$A^n_{C_2}(v_t) = \lambda_1^n t_1 v_1 + \lambda_2^n t_2 v_2 + \lambda_3^n t_3 v_3$$

and in logarithmic coordinates

$$\vartheta_{26}(A^n_{C_2}(v_t)) = \vartheta_{26}(\lambda_1^n t_1 v_1 + \lambda_2^n t_2 v_2 + \lambda_3^n t_3 v_3) =$$

$$= \lambda_1^n t_1 (v_{13}, v_{11} + \frac{1-\epsilon_X}{2} v_{13}, v_{12} - \frac{1+\epsilon_Y}{2} v_{13})^T + \lambda_2^n t_2 \vartheta_{26}(v_2) + \lambda_3^n t_3 \vartheta_{26}(v_3)$$

Since for ϵ_X, ϵ_Y from the yellow region the following inequalities hold:

$$v_{11},\ v_{12},\ v_{13},\ v_{12} - \frac{1+\epsilon_Y}{2} v_{13},\ v_{11} + \frac{1-\epsilon_X}{2} v_{13} > 0$$

and

$$\lambda_1 > max(1, |\lambda_2|, |\lambda_3|)$$

then while passing to the limit with $n \to +\infty$, the leading terms are those multiplied by $\lambda_1^n t_1$.
Hence, cycle C_2 is not almost completely unstable, however it cannot be essentially asymptotically stable since the domain of every second local map associated to C_2 cycle is a thin cusp, which means that the signs of the local indices alternate through the C_2 cycle. □

Remark 54. *In the same way we can prove about cycle C_0 weaker statement than the one from theorem 7.*

From the above propositions we can conclude that:

Proposition 47.

1. *for $\epsilon_X > \epsilon_Y$ in the yellow region: $dim(W^s(C_2)) = 4$ and $dim(W^u(C_0)) = 4$*
2. *for $\epsilon_X > \epsilon_Y$ in the white region: $dim(W^s(C_2)) = 3$ and $dim(W^u(C_0)) = 4$*
3. *for $\epsilon_X > \epsilon_Y$ in the red region: $dim(W^u(C_2)) = 3$ and $dim(W^s(C_0)) = 4$*

4. *for $\epsilon_X > \epsilon_Y$ in the light red region: $dim(W^u(C_2)) = 4$ and $dim(W^s(C_0)) = 4$*

Note that for $\epsilon_X < \epsilon_Y$ there are no stable/unstable manifolds of C_2, since $v_{12} < 0$.

Remark 55. *Analogous conclusions follow for C_1 cycle after swapping ϵ_X with ϵ_Y.*

We end this subsection with a concluding remark.

Remark 56. *With parameters $\epsilon_X > \epsilon_Y$ passing from the red region to white region and finally to the yellow region, respectively - we observe a transition from essential asymptotic stability of C_0 through the case of Hamiltonian dynamics to the total disappearance of C_0 attraction properties and appearance of the 3-dimensional set attracted by C_2 cycle, then appearance of 4-dimensional set attracted by C_2 cycle.*
When crossing the line $\epsilon_X = \epsilon_Y$ within white region or from the yellow to green region we observe that the set being attracted to C_2 cycle disappear and a new one attracted by C_1 cycle is born.

4.5 - Shadowing between the ties, lack of infinite switching

As in [3] one can easily prove that there is switching along the heteroclinic orbits in the heteroclinic network. We provide some examples demonstrating that even attainability of some finite itineraries being infinitely close to the heteroclinic network depends strongly on parameter values (ϵ_X, ϵ_Y).
Following [3], we introduce the notion of finite switching:

Definition 19. *We say that there is switching along the heteroclinic orbit - from equilibrium X to equilibrium Y, if for all equilibria, say Z_1,...,Z_k, for which there is a heteroclinic orbit from Y to Z_1,...,Z_k, and for all equilibria $Q_1, ..., Q_m$ from which there is a heteroclinic orbit to X, there exist a point $q_{i,j} \in \Sigma^{in}_{Q_i \to X}$ such that:*

$$\Phi_{Y \to Z_j} \circ Tran_{X \to Y} \circ \Psi_{X \to Y} \circ \Phi_{X \to Y}(q_{i,j}) \in \Sigma^{out}_{Y \to Z_j}$$

for all $i = 1, ..., m$ and $j = 1, ..., k$.

Definition 20. *If the process of switching along the heteroclinic orbit can be iterated infinitely many times then we say that there is **infinite switching** near the heteroclinic network.*

Remark 57. *In generic case of heteroclinic network where heteroclinic orbits are non-transverse intersections of stable/unstable manifolds, one does not necessarily expect infinite switching to happen arbitrarily close to the heteroclinic network. We refer here to the work by Simon, Delshams and Zgliczyński [28], where they describe possible scenarios of switching along heteroclinic orbits in the non-transverse case and give explanation what one should expect at most from switching and about length of the itinerary, depending e.g. on the unstable dimensions of the equilibria in the network. From their results, based on the method of dropping dimensions in the covering relations, one can conclude that in the RSP network the finite switching of length 2 (involving 2 heteroclinic orbits) is always possible since each hyperbolic equilibrium of RSP network possesses 2- dimensional unstable manifold.*

Proposition 48. *Finite itineraries following consecutively arbitrarily small neighbourhoods of the points:* $(P,P) \to (S,P) \to (S,S)$ *and* $(P,P) \to (S,P) \to (S,R) \to (P,R) \to (P,P)$
are attainable for all $\epsilon_X, \epsilon_Y \in (-1,1)$

The proposition above is a matter of calculations performed in the same fashion as the proof of the following proposition which presents a counterexample to the conjecture stating that there exist infinite switching near the RSP network.

Theorem 9. *For every* $\epsilon_X, \epsilon_Y \in (-1,1)$ *the finite itineraries of the form tie* $\to$ *win/loss* $\to$ *loss/win* $\to$ *tie and tie* $\to$ *loss/win* $\to$ *win/loss* $\to$ *tie are not attainable for h - sufficiently small. Hence there is no infinite switching near the RSP network.*

Proof. Without loss of generality consider an itinerary $(P,P) \to (S,P) \to (S,R) \to (R,R)$ take a point

$$(z_1, 1-h, z_3, z_4) \in \Sigma^{in}_{(P,P)\to(S,P)} \cap Image(\Psi_{(P,P)\to(S,P)})$$

Hence $|z_1| < h \cdot (\frac{h}{1-h})^{\frac{3+\epsilon_X}{1-\epsilon_X}}$, $|z_4| < h$. The necessary condition for the point

$$Tran_{(S,P)\to(S,R)} \circ \Psi_{(S,P)\to(S,R)} \circ \Phi_{(S,P)\to(S,R)} \circ Tran_{(P,P)\to(S,P)}(z_1, 1-h, z_3, z_4)$$

to lie in the domain of $\Phi_{(S,R)\to(R,R)}$ is the following inequality:

$$(\frac{1-h}{h})^{\frac{1+\epsilon_X}{2}} (\frac{z_3}{h})^{\frac{1-\epsilon_X}{2}} |h - z_1| \leq h^{-\frac{1-\epsilon_X}{1+\epsilon_X}} (\frac{h}{1-h})^{\frac{1-\epsilon_X}{1+\epsilon_X}} (\frac{z_1 z_3}{h})^{\frac{2}{1+\epsilon_X}}$$

which is equivalent to

$$LHS := \frac{|h - z_1|}{h} \leq h^{-\frac{1-\epsilon_X}{1+\epsilon_X}-1}(\frac{h}{1-h})^{\frac{1-\epsilon_X}{1+\epsilon_X}+\frac{1+\epsilon_X}{2}}(\frac{z_3}{h})^{\frac{2}{1+\epsilon_X}-\frac{1-\epsilon_X}{2}} z_1^{\frac{2}{1+\epsilon_X}} =: RHS$$

Note that

$$LHS \in \left(1 - (\frac{h}{1-h})^{\frac{3+\epsilon_X}{1-\epsilon_X}}, 1\right)$$

and

$$RHS \in \left(0, h^{-\frac{1-\epsilon_X}{1+\epsilon_X}-1+\frac{2}{1+\epsilon_X}}(\frac{h}{1-h})^{\frac{1-\epsilon_X}{1+\epsilon_X}+\frac{1+\epsilon_X}{2}+\frac{3+\epsilon_X}{1-\epsilon_X}\cdot\frac{2}{1+\epsilon_X}}\right) =$$

$$\left(0, (\frac{h}{1-h})^{\frac{1-\epsilon_X}{1+\epsilon_X}+\frac{1+\epsilon_X}{2}+\frac{3+\epsilon_X}{1-\epsilon_X}\cdot\frac{2}{1+\epsilon_X}})\right)$$

To see that this inequality cannot be satisfied it suffices to say that

$$(\frac{h}{1-h})^{\frac{1-\epsilon_X}{1+\epsilon_X}+\frac{1+\epsilon_X}{2}+\frac{3+\epsilon_X}{1-\epsilon_X}\cdot\frac{2}{1+\epsilon_X}} < 1 - (\frac{h}{1-h})^{\frac{3+\epsilon_X}{1-\epsilon_X}}$$

for h-small enough. □

Proposition 49. *For every $(\epsilon_X, \epsilon_Y) \in (-1, 0) \times (-1, 1)$, the finite itinerary*

$$(P, P) \to (P, S) \to (R, S) \to (R, P) \to (S, P) \to (S, R) \to (R, R)$$

is not attainable for sufficiently small h.

Proof. For

$$(z_1, z_2, z_3, 1 - h) \in \Sigma^{in}_{(P,P)\to(P,S)} \cap \Psi_{(P,P)\to(P,S)}(\Sigma^{out}_{(P,P)\to(P,S)})$$

we have $|z_1| < h$, $|z_2| < h$, $|z_3| < h\,(\frac{h}{1-h})^{\frac{3+\epsilon_Y}{1-\epsilon_Y}}$. If

$$\begin{aligned}(w_1, w_2, w_3, w_4) = Tran_{(S,P)\to(S,R)} \circ \Psi_{(S,P)\to(S,R)} \circ \Phi_{(S,P)\to(S,R)} \circ Tran_{(R,P)\to(S,P)} \circ \Psi_{(R,P)\to(S,P)} \circ \\ \circ \Phi_{(R,P)\to(S,P)} \circ Tran_{(R,S)\to(R,P)} \circ \Psi_{(R,S)\to(R,P)} \circ \Phi_{(R,S)\to(R,P)} \circ Tran_{(P,S)\to(R,S)} \circ \Psi_{(P,S)\to(R,S)} \circ \\ \circ \Phi_{(P,S)\to(R,S)} \circ Tran_{(P,P)\to(P,S)}(z_1, z_2, z_3, 1 - h)\end{aligned}$$

then the necessary condition for existence of this connection is $|w_2| < h(\frac{|w_1|}{h})^{\frac{2}{1+\epsilon_X}}$, which is equivalent

to

$$\Big(1-(\frac{1-h}{h})^{\epsilon_X}(\frac{h}{|z_1|})^{\frac{1+\epsilon_X}{2}}\frac{|z_2|}{h}\Big)^{p_1}\cdot\Big(1-\frac{|z_3|}{h}\Big)^{p_2}<$$
$$<(1-h)^{\frac{\epsilon_X^2(-5+8\epsilon_Y-3\epsilon_Y^2)+\epsilon_X^3(15-8\epsilon_Y+\epsilon_Y^2)+\epsilon_X(13-24\epsilon_Y+3\epsilon_Y^2)-3(5-8\epsilon_Y+3\epsilon_Y^2)}{16(1+\epsilon_X)}}\cdot h^{-\frac{(17+\epsilon_X-\epsilon_X^3(-3+\epsilon_Y)+9\epsilon_Y-3\epsilon_X\epsilon_Y+3\epsilon_X^2(1+\epsilon_Y)}{8(1+\epsilon_X))}}\cdot$$
$$|z_1|^{\frac{5-3\epsilon_X(-3+\epsilon_Y)^2-\epsilon_X^3(-3+\epsilon_Y)^2+6\epsilon_Y+9\epsilon_Y^2+\epsilon_X^2(-9+2\epsilon_Y+3\epsilon_Y^2)}{16(1+\epsilon_X)}}\cdot|z_2|^{\frac{5+4\epsilon_X-\epsilon_X^2(-3+\epsilon_Y)-3\epsilon_Y}{4(1+\epsilon_X)}}|z_3|^{\frac{3+\epsilon_X^2}{2+2\epsilon_X}}$$

All of the powers of z_1, z_2, z_3 on the RHS are positive for $(\epsilon_X, \epsilon_Y) \in (-1,0)\times(-1,1)$, and for these parameters p_1, p_2 belong to some compact interval in $\mathbb{R}$. Hence RHS attains maximum for $z_1 = h$, $z_2 = h$ and $z_3 = h(\frac{h}{1-h})^{\frac{3+\epsilon_Y}{1-\epsilon_Y}}$ and is equal to

$$(\frac{h}{1-h})^{-\frac{87-15\epsilon_Y+33\epsilon_Y^2-9\epsilon_Y^3+\epsilon_X^2(29-5\epsilon_Y+11\epsilon_Y^2-3\epsilon_Y^3)+\epsilon_X^3(-15+23\epsilon_Y-9\epsilon_Y^2+\epsilon_Y^3)+\epsilon_X(-13+37\epsilon_Y-27\epsilon_Y^2+3\epsilon_Y^3)}{16(1+\epsilon_X)(-1+\epsilon_Y)}}$$

The power is bigger than 0 for $(\epsilon_X, \epsilon_Y) \in (-1,0)\times(-1,1)$, and hence for h small enough, the RHS is always smaller than LHS which is bounded from 0. □

In the same way we investigate the attainability of connections between ties, that is itineraries of the form: *tie* → (part of C_0 cycle) → *tie*

4.5.1 Chaotic switching

We have already proven that there is no infinite switching near the RSP network. In [20], it was proven that the switching can be described by a subshift of finite type for parameter values on β-curve (see proposition 18). However it is still an open question for the parameter values from the white region (see bifurcation diagram, figure 4.5), and as well how does the form of the subshift depend on the parameter values.

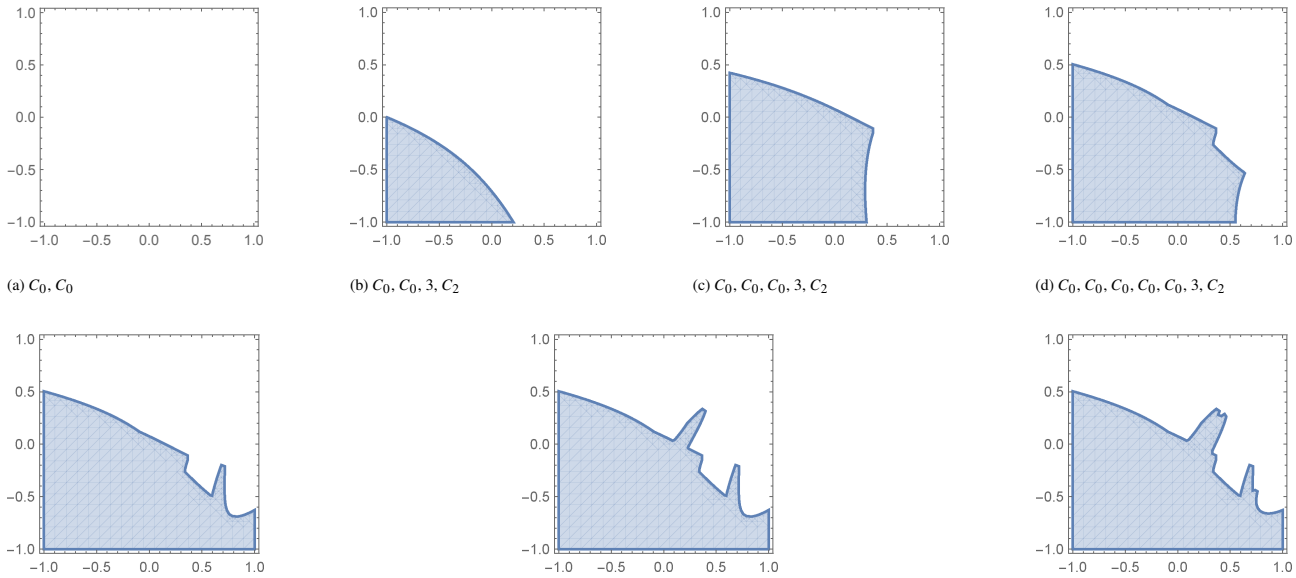
We have already mentioned results by Delshams, Simon and Zgliczyński [28], which convince us that there has to exist some phenomenon in the system which forces existence of chaotic switching. In the sixth section we will provide numerical investigation in order to support our presumptions and intuition about what stays behind the irregular behaviour in the system (4.2.3).

Typically, examples of infinite switching that appear in the literature are associated to the existence of suspended horseshoes in the neighbourhood of the network (see e.g. [92]), in particular if there is an equilibrium in the network with complex eigenvalues - this can lead to the situation similar as in

Shilnikov homoclinic bifurcation i.e. spiraling chaos. Although it is worth noting that A.Rodrigues [91] has given an example of an attracting homoclinic network exhibiting sensitive dependence on initial conditions, where the existence of infinite switching is not associated with suspended horseshoes. However, due to aforementioned results in propositions 15, 19, 17, 18, we believe that there exist chaotic switching topologically conjugated to a subshift of finite type for all parameter values ϵ_X, ϵ_Y from the white region.

4.5.2 Finite switching

One can hope that it would be possible to determine exactly the form of subshift of finite type describing switching near the system for all parameter values (ϵ_X, ϵ_Y) from the white region. In fact, it appears that it is hard to say precisely, for given parameter values (ϵ_X, ϵ_Y), which finite itineraries are forbidden for orbits staying arbitrarily close to the network. In the figure 4.6 we present analytical results, depicting some examples of finite itineraries about which we can say only when for sure they are not attainable.



(a) C_0, C_0 (b) $C_0, C_0, 3, C_2$ (c) $C_0, C_0, C_0, 3, C_2$ (d) $C_0, C_0, C_0, C_0, C_0, 3, C_2$

(e) $C_0, C_0, C_0, C_0, C_0, C_0, 3, C_2, C_2, 4, C_0, 3$ (f) $C_0, C_0, C_0, C_0, C_0, C_0, 3, C_2, C_2, 4, C_0, 3, C_2$ (f) $C_0, C_0, C_0, C_0, C_0, C_0, 3, C_2, C_2, 4, C_0, 3, C_2, C_2, C_2$

Figure 4.6: In these diagrams the parameter space $(\epsilon_X, \epsilon_Y) \in (-1, 1)^2$ and various itineraries are included. Shaded regions in the parameter space indicate parameter values (ϵ_X, ϵ_Y) for which the finite itinerary (written under the picture) consisting of numbered cross-sections in the quotient space (see figure 4.4), are not attainable. For simplicity, we denote sequence (3, 4) as C_0 and (2,1) as C_2. See also the main text from section 3.5.2.

4.6 - Numerical investigation

4.6.1 Sato's et al. simulations

In this subsection we recall some of the numerical observations done by Y. Sato et al. (see [94], [95], [96]) and in the following subsection we compare it to our findings.

1. With $\epsilon_X = \epsilon_Y = 0$ only quasiperiodic tori are observed. For some initial conditions, tori are knotted and form a trefoil.

2. With $\epsilon_X = -\epsilon_Y > 0$ Hamiltonian chaos can occur with positive-negative pairs of Lyapunov exponents. The dynamics is very rich in the sense that there are infinitely many distinct behaviors near the Nash equilibrium and a periodic orbit arbitrarily close to any chaotic one. (see [22])

3. When the game is not zero-sum ($\epsilon_X + \epsilon_Y \neq 0$), transients to heteroclinic cycles are observed. On the one hand, there are intermittent behaviors in which the time spent near pure strategies increases subexponentially with $\epsilon_X + \epsilon_Y < 0$ and on the other hand, with $\epsilon_X + \epsilon_Y > 0$, chaotic transients persist.

4. When $\epsilon_X + \epsilon_Y < 0$, the behavior is intermittent and orbits are guided by the flow on edges of Δ, which describes a network of possible heteroclinic cycles. Since action ties are not rewarded there is only one such cycle - C_0. During the cycle each agent switches between almost deterministic actions in the order $R \to S \to P$. The agents are out of phase with respect to each other and they alternate winning each turn.

5. When $\epsilon_X + \epsilon_Y > 0$, however, the orbit is an infinitely persistent chaotic transient. Since, in this case, agent X can choose a tie, the cycles are not closed. For example, with $\epsilon_X > 0$, at (R, P), agent X has the option of moving to (P, P) instead of (S, P) with a positive probability. This embeds an instability along the heteroclinic cycle and so orbits are chaotic.

6. In [95], Lyapunov spectra, for different parameter values satisfying $\epsilon_X + \epsilon_Y = 0$ and different initial conditions, were investigated. In conclusion, as an evidence for Hamiltonian chaos for $(\epsilon_X, \epsilon_Y) = (0.5, -0.5)$ was presented the Lyapunov spectrum with the biggest Lyapunov exponent of order $4 \cdot 10^{-2}$, and corresponding the second and third were of order $6 \cdot 10^{-4}$.

4.6.2 Our numerical observations

Let us change the coordinates to the ones from the information space (described in the section 1.11), which are useful since being more numerically stable. If $x_j, y_k \neq 0$, we substitute:

$$u_i := \ln\left(\frac{x_{i+1}}{x_1}\right) \quad v_i := \ln\left(\frac{y_{i+1}}{y_1}\right) \tag{4.6.1}$$

for $i = 1, 2$. Then the equation (4.2.3) reduces to:

$$\begin{cases} \dot{u_1} = \frac{-(1+\epsilon_X)-(1-\epsilon_X)e^{v_1}+2e^{v_2}}{1+e^{v_1}+e^{v_2}} \\ \dot{u_2} = \frac{-2+(1-\epsilon_X)e^{v_1}+(1+\epsilon_X)e^{v_2}}{1+e^{v_1}+e^{v_2}} \\ \dot{v_1} = \frac{-(1+\epsilon_Y)-(1-\epsilon_Y)e^{u_1}+2e^{u_2}}{1+e^{u_1}+e^{u_2}} \\ \dot{v_2} = \frac{-2+(1-\epsilon_Y)e^{u_1}+(1+\epsilon_Y)e^{u_2}}{1+e^{u_1}+e^{u_2}} \end{cases} \tag{4.6.2}$$

The reverse transformation is given by:

$$x_1 = \frac{1}{1+e^{u_1}+e^{u_2}} \quad x_2 = \frac{e^{u_1}}{1+e^{u_1}+e^{u_2}} \quad y_1 = \frac{1}{1+e^{v_1}+e^{v_2}} \quad y_2 = \frac{e^{v_1}}{1+e^{v_1}+e^{v_2}}$$

Sato's et al. observations were based on simulations done by drawing points randomly from the phase space $\Sigma_A \times \Sigma_B$ and integrating their orbits using either low order Runge-Kutta method or some symplectic integrator. Our simulations were based on the Taylor method (with order ranging from 10 to 30), and with the time step fixed = 0.01 or adjusted after each iteration, in order to minimize the error.

4.6.3 Spectra of Lyapunov exponents

In this section we want to determine Lyapunov exponents of the system and compare our findings with the results by Sato et al. In [94], [95], [96] it was claimed that for $\epsilon_X + \epsilon_Y > 0$ the system is chaotic.

In order to compute the Lyapunov exponents $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$, we utilized the algorithm of Ruelle and Eckmann based on the QR-decomposition method [31]. As the Lyapunov exponents change slightly (difference for the consecutive time steps is of order 10^{-6}) for the integration time longer than $t_0 = 2000.0$, we decided to estimate them at time t_0.

1. For the Hamiltonian case, i.e. $\epsilon_X + \epsilon_Y = 0$, we observe that the biggest Lyapunov exponent $\lambda_1 > 0$ is of order 10^{-4}. As expected for Hamiltonian systems $\lambda_2 \approx -\lambda_3$ are small (of order 10^{-6}) comparing to $\lambda_4 = -\lambda_1$. These results are not compatible with Sato's et al. numerics as we could not find for $\epsilon_X = -\epsilon_Y = 0.5$ any point with the biggest Lyapunov exponent of order 10^{-2}. In our simulations it was of order 10^{-2}, in most cases, for the integration time from $t = 200.0$ until $t = 400.0$. However, for these times, the conditions $\lambda_2 \approx -\lambda_3$, $\lambda_4 = -\lambda_1$ were not satisfied even with the precision 10^{-4}.

2. For parameter values $\epsilon_X > \epsilon_Y$ from the red and light red region we can observe two different patterns exhibited by the system for various initial conditions:
   - either $\lambda_1 > 0,\ 0 > \lambda_3,\ \lambda_4$ are of order 10^{-4} and $\lambda_2 \approx 0$ is of order 10^{-5} or 10^{-6}, small comparing to $\lambda_1, |\lambda_3|, |\lambda_4|$. For the initial point
   $(u_1, u_2, v_1, v_2) = (-55.6888, 36.3836, -49.0596, -26.4498)$ and parameter values
   $(\epsilon_X, \epsilon_Y) = (-0.8, -0.9)$, the Lyapunov spectrum is
   $(2.94135 \cdot 10^{-4}, 2.2451 \cdot 10^{-5}, -1.4441 \cdot 10^{-4}, -1.99052 \cdot 10^{-4})$.
   - or $\lambda_1 > 0$, $\lambda_4 \approx -\lambda_1$ are of order 10^{-4} and $\lambda_2, \lambda_3 \approx 0$ are of order 10^{-6}, 10^{-5}, respectively (small comparing to $\lambda_1, |\lambda_4|$). For the initial point
   $(u_1, u_2, v_1, v_2) = (-2.93175, -71.3114, -53.5168, -87.164)$ and parameter values
   $(\epsilon_X, \epsilon_Y) = (-0.8, -0.9)$, the Lyapunov spectrum is
   $(2.10125 \cdot 10^{-4}, -3.8856 \cdot 10^{-6}, -1.86203 \cdot 10^{-5}, -2.114478 \cdot 10^{-4})$. Two of the Lyapunov exponents being equal to zero, might indicate that some quantity is preserved by the system.

3. For parameter values $\epsilon_X > \epsilon_Y$ from the white and yellow regions we can observe three patterns exhibited by the system for various initial conditions. Two of them are identical with those described above, in point 2, another one is significantly different:
   - $\lambda_1 > 0$, $\lambda_4 < 0$ are of order 10^{-3}, $\lambda_2 > 0$ of order 10^{-4} and $\lambda_3 \approx 0$ of order 10^{-5} or 10^{-6}, which is small comparing to $\lambda_1, \lambda_2, |\lambda_4|$. In the white region, for the initial point $(u_1, u_2, v_1, v_2) = (6.755, -27.374, 58.7936, -12.1726)$ and parameter values $(\epsilon_X, \epsilon_Y) = (0.7, -0.6)$, the Lyapunov spectrum is $(1.69506 \cdot 10^{-3}, 2.13178 \cdot 10^{-4}, 1.20503 \cdot 10^{-5}, -1.94714 \cdot 10^{-3})$. In the yellow region, for the initial point $(u_1, u_2, v_1, v_2) = (1.19127, -44.7834, 99.1464, 75.3413)$ and parameter values $(\epsilon_X, \epsilon_Y) = (0.9, 0.4)$, the Lyapunov spectrum is $(1.60342 \cdot 10^{-3}, 1.19112 \cdot 10^{-4}, -6.95594 \cdot 10^{-6}, -1.74244 \cdot 10^{-3})$.

Remark 58. *At this place we want to thank Yuzuru Sato for his suggestion, that for the parameters*

from the white region, the biggest Lyapunov exponent is positive and the system (4.2.3) numerically appears to be chaotic.

4.6.4 Invariant manifolds of the Nash equilibrium

Remark 59. *Note that the transformation* $(t, \epsilon_X, \epsilon_Y) \longmapsto (-t, -\epsilon_X, -\epsilon_Y)$ *of the system (4.2.3), reverses the time and ordering of the cycles* C_0*,* C_1 *and* C_2*. As well it swaps* C_1 *and* C_2 *cycles. Due to this fact, it suffices to conduct forthcoming analysis only for the parameter values* (ϵ_X, ϵ_Y) *from the yellow region and white region below the diagonal (i.e.* $\epsilon_X > \epsilon_Y$*).*

Let us denote L_1, L_2, L_3, L_4 - normalized eigenvectors corresponding to the eigenvalues $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, from proposition (35), for the Nash equilibrium of the system (4.2.4). For $\epsilon_X + \epsilon_Y \neq 0$, vectors having real coefficients:

$$w_1 := L_1 + L_3, \;\; w_2 := i(L_1 - L_3), \;\; w_3 := L_2 + L_4, \;\; w_4 := i(L_2 - L_4) \tag{4.6.3}$$

span the tangent spaces to stable and unstable manifolds, i.e. $span(w_1, w_3)$ and $span(w_2, w_4)$ respectively.

For the numeric simulations, in order to observe some irregular behaviour or convergence of orbits either to the cycles C_1, C_2, or C_0, we set Poincare sections $S_1 := \{x_2 + y_2 = \frac{e^{u_1}}{1+e^{u_1}+e^{u_2}} + \frac{e^{v_1}}{1+e^{v_1}+e^{v_2}} = 1\}$, $S_2 := \{x_1 + y_1 = \frac{1}{1+e^{u_1}+e^{u_2}} + \frac{1}{1+e^{v_1}+e^{v_2}} = 1\}$ and we:

- iterate backward the points from the stable tangent eigenspace $p := (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) + \alpha_1 w_1 + \alpha_3 w_3$ (in (x_1, x_2, y_1, y_2) coordinates) with $\alpha_1, \alpha_3 \in (-\frac{1}{1000}, \frac{1}{1000})$, via the Poincare map associated to the sections S_1, S_2
- iterate forward the points from the unstable tangent eigenspace $q := (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}) + \alpha_2 w_2 + \alpha_4 w_4$ (in (x_1, x_2, y_1, y_2) coordinates) with $\alpha_2, \alpha_4 \in (-\frac{1}{1000}, \frac{1}{1000})$, via the Poincare map associated to the sections S_1, S_2

Observations on forward integration of points q:

1. for ϵ_X, ϵ_Y from the white region,

   - if $\epsilon_X + \epsilon_Y$ is small but not ≈ 0, the motion is irregular which is consistent with Sato's results - for (ϵ_X, ϵ_Y) being close to the boundary with yellow region, we observe fast convergence to the C_2 cycle of some of the drawn points. Vast majority of the drawn points have irregular orbits.

2. for ϵ_X, ϵ_Y from the yellow region,

   - we do not observe any symptoms of irregular behaviour of the iterations of points q, all of the drawn points converge fast to the C_2 cycle

Observations on backward integration of points p are obviously similar in few aspects:

1. for ϵ_X, ϵ_Y from the white region,

   - if $\epsilon_X + \epsilon_Y \approx 0$, we observe irregular behaviour, we have not observed any symptoms of convergence of the iterations

   - for $\epsilon_X + \epsilon_Y$ away from 0, we can observe only irregular behaviour

   - we do not observe any symptoms of irregular behaviour of the iterations of points p, almost all of them converge fast to the C_0 cycle

We do not say precisely what do we mean by "fast convergence", $\epsilon_X + \epsilon_Y \approx 0$ or $\epsilon_X + \epsilon_Y$ far away from 0, as this is beyond our goals and scope of this thesis. The purpose of conducting these numerical simulations was to explain: what kind of phenomenon can cause chaotic behaviour in the system or even near the heteroclinic network (switching possibly)? After introduction to this question in the subsection (4.5.1), we can now pose our questions and conjectures.

4.7 - Conclusions, open questions and relations to other papers

4.7.1 Summary

In this chapter we approached the following questions:
1. What is the stability of cycles C_0, C_1, C_2 depending on the parameter values?
2. Does infinite switching exist near the RSP network? If not, which heteroclinic chains within this network can be shadowed (possibly infinitely)? Can the switching be described by the subshift of finite type?
3. What phenomena lie behind the chaotic behaviour of the system? What is the reason for the existence of the transient orbits to the heteroclinic cycles (observed by Sato et al [94], [95], [96])?

Our work presented in this chapter answers some of these questions and provides intuition about those which remain unsolved.

1. In the theorem 7 we proved that C_0 cycle is essentially asymptotically stable for $\epsilon_X + \epsilon_Y < 0$. Theorem 8 says that for the parameter values (ϵ_X, ϵ_Y) from the yellow region (see figure 4.5), the C_2 cycle is relatively asymptotically stable, i.e. the basin of attraction of this cycle is a (locally thin) open set. For the parameter values from the white region where $\epsilon_Y < \epsilon_X$, there exists a 3-dimensional set of initial conditions attracted to the C_2 cycle.

2. Our analytical investigations show that there is no infinite switching near the RSP network (see proposition 9). Moreover, if there are infinite heteroclinic chains which can be shadowed and the switching can be described by the subshift of the finite type it is hard to say how does this form of subshift depend on the parameter values (see section 4.5.1 and figure 4.6). The question about switching near the RSP network, for the parameter values from the white region, remains open.

3. We have investigated numerically spectra of Lyapunov exponents for different parameter values. We found examples of orbits with significantly different behaviour (e.g. having one positive and two negative exponents, or having two positive and one negative exponent). Moreover, we have investigated numerically behaviour of the invariant manifolds of the Nash equilibrium, i.e. for parameter values from the yellow region, the numerical simulations suggest that there exist 2-dimensional heteroclinic connections from the C_0 cycle to the Nash equilibrium, and from the Nash equilibrium to the C_2 cycle. We conjecture that these connections do indeed exist and that this results in the existence of the transient dynamics.

Remark 60. *We emphasize again that to obtain results about C_1 one needs to swap ϵ_X with ϵ_Y in the statement of the results about C_2 cycle.*

4.7.2 Comparison with other papers on the same topic

We refer here to three papers [3], [36], [37], investigating analytically the same system as we do in this chapter (preprints [36], [37], appeared later than preprint [79] on which this chapter is based).
In these three papers, asymptotic analysis of the cycles and infinite switching near the quotient RSP network were investigated.
The authors of [3], [36], [37], used different coordinates near the hyperbolic equilibria from those that we used. Although the method of deriving of local maps near equilibria is the same, the local maps they derived differ from ours due to the different coordinate systems used. In [3], the global transition maps between different equilibria are assumed to be 'identity maps' (compare with proposition 42). In [36], [37], these maps are assumed to be linear. In our model these maps are not linear, they are affine

(as a composition of linear global map and transition map between different coordinate systems, i.e. $Tran \circ glob$, see lemma 5). We find that (see proposition 44) in the limit (with variables infinitesimally small) the global transition maps can be rescaled in a way so that they are of "constant-free" form. Up to the permutation of the coordinates, these formulas resemble those from [3], [36], [37].

In [3] it was claimed that C_0 is relatively stable (but not essentially asymptotically stable) for $\epsilon_X + \epsilon_Y < 0$. Our result and those from [36], [37] state that C_0 is essentially asymptotically stable for $\epsilon_X + \epsilon_Y < 0$. Propositions 8, 46 and 45 provide more detailed stability analysis of the C_0 and C_2 cycles than those performed in [3], [36], [37].

In [3] it was claimed that infinite switching (described by the full shift dynamics) near the RSP network occurs for all parameter values ϵ_X, ϵ_Y (in [36], [37] this problem was not investigated). In section 4.5 we concluded from our analysis that infinite switching cannot occur in the RSP network (for more details see section 4.5).

4.7.3 Discussion

For (ϵ_X, ϵ_Y) from the yellow (resp. red) region (see figure 4.5) away from the anti-diagonal almost all of the points from the stable (unstable) manifold after backward (forward) iteration by the Poincare map converge quickly to the cycles C_0 or C_2, which would suggest that there are not only heteroclinic connections Nash equilibrium – cycles, but that the intersections of stable (unstable) manifolds of the cycles with unstable (stable) manifolds of the Nash equilibrium are 2-dimensional. Do these intersections persist (presumably losing 1 dimension) when parameter values vary, especially when they are in the white region?

Although essential asymptotic stability of C_0 cycle for $\epsilon_X + \epsilon_Y < 0$ does not exclude the possibility of irregular behaviour in the system, it means that close to the equilibria $loss/win$, $win/loss$ there is not much free space for switching from C_0 to the other cycles. This might be the reason, why Sato et al. did not observe numerically irregular behaviour (for $\epsilon_X + \epsilon_Y < 0$ they mentioned only orbits guided by the C_0 cycle).

The situation is different, however, for the C_2 cycle and parameter values in the yellow region, since basins of attraction at every second equilibrium in this cycle are thin cusps, hence there is a lot of space to switch between the cycles. This, together with the fact that the stable manifold of the C_2 cycle is 3-dimensional for ϵ_X, ϵ_Y in the white region, would explain why Sato et al. concluded that for $\epsilon_X + \epsilon_Y > 0$ observed orbits are chaotic and that the cycles are not closed, even for the ϵ_X, ϵ_Y in the

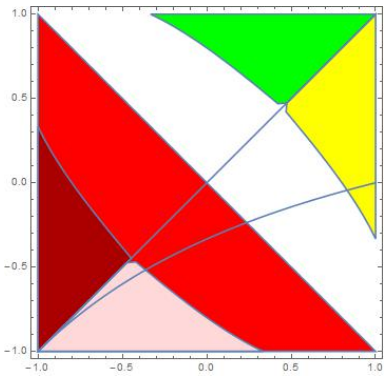


Figure 4.7: Bifurcation diagram.
The picture shows regions O_1, O_2, $\epsilon_X + \epsilon_Y < 0$, and β -curve $\epsilon_X = \frac{1+3\epsilon_Y}{1-\epsilon_Y}$ (where $\beta = \frac{1+\epsilon_X}{2}$). The β-curve intersects boundary of the yellow region O_2 at point $\beta = \tau \approx 0.915$.
In the red regions $\epsilon_X + \epsilon_Y < 0$ the cycle C_0 is essentially asymptotically stable (Theorem 7). In the yellow region the cycle C_2 is relatively asymptotically stable and there exists an open set attracted to the cycle C_2 (Theorem 45). In the green region the cycle C_1 is relatively asymptotically stable and there exists an open set attracted to the cycle C_1. In the white region with $\epsilon_Y < \epsilon_X$ there exists a 3-dimensional set of initial conditions attracted to the cycle C_2. For $\epsilon_X + \epsilon_Y > 0$, there exists an open set of initial conditions which converge backward in time to the cycle C_0.

yellow region.

We must point out that propositions 45, 46, guarantee locally strong contraction of the attracted open sets of points towards cycles C_0 and C_2, for $\epsilon_X + \epsilon_Y$ being very small or very large, respectively. Hence it seems plausible that the iterations of points from the tangent eigenspaces of the Nash equilibrium are converging quickly to the cycles - where we would look for the forward and backward heteroclinics from the Nash equilibrium. On the other hand, for $\epsilon_X + \epsilon_Y$ close to 0, this local contraction is weak (for the C_2 cycle - of the 3-dimensional set of points), so our numerical observations about lack of convergence of points p and q towards the cycles, do not stay in the contradiction with e.g. the essential asymptotic stability of the C_0 cycle. The irregular behaviour which we observe for these parameter values might come from the fact that the system then is close to the Hamiltonian one (the case $\epsilon_X + \epsilon_Y = 0$), where we observe (and it was broadly described in [94], [95], [96], [22]) Hamiltonian chaos.

The irregular behaviour we observe for (ϵ_X, ϵ_Y) from the white region and the possible switching we expect to happen, might also be a consequence of the existence of a homoclinic orbit to the cycles C_0 and C_2 (superhomoclinic orbit [104]), which might not pass very close to the Nash equilibrium, so we did not have the opportunity to detect it.

On the other hand, our numerical experiments suggest, that for some parameter values ϵ_X, ϵ_Y, in the white region, far away from the anti-diagonal and close to the bifurcation curve between the white and yellow regions, there exists a forward heteroclinic connection from the neighbourhood of the Nash equilibrium to the C_2 cycle, and backward heteroclinic to the C_0 cycle. Unfortunately, a huge obstacle for performing a computer-assisted proof of such connections (even for one chosen pair of parameter

values (ϵ_X, ϵ_Y)) is the parametrization and analytical estimations of the stable and unstable manifolds of these cycles (see e.g. [5], [14], [15], [120], [121], [61], [114], [115], [117]). Existence of these heteroclinics together with possibility of switching from C_2 cycle to the C_0 at any equilibrium $loss/win$ (so where the first agent loses), would result in chaotic behaviour possibly described by subshift of finite type, i.e. a point starting close to the Nash equilibrium can follow the heteroclinic, go around C_2 cycle for some time, switch and encircles C_0 and then come back to the neighbourhood of Nash equilibrium and repeat.

If such a connection exists , i.e. superhomoclinic orbit, passing by close to the Nash equilibrium, for the transportation of chaos we would expect it to be a transverse intersection of unstable and stable manifolds of the C_0 and C_2 cycles, respectively. Generically, it should be a 3-dimensional set for the parameter values (ϵ_X, ϵ_Y) from the white region. Unfortunately this is another obstacle which leads us to believe that it must be very difficult to prove its existence or even compute it numerically (see [60]).

Our belief is that the transverse intersection of unstable and stable manifold of the C_0 and C_2 cycles, respectively, exists for parameter values (ϵ_X, ϵ_Y) from the white region and with (ϵ_X, ϵ_Y) crossing the bifurcation curve towards the yellow region, it bifurcates to two 2-dimensional surfaces which are intersections of unstable (resp. stable) manifold of cycle C_0 (C_2) with stable (unstable) manifold of Nash equilibrium. During this bifurcation, the 3-dimensional set would disappear, while C_2 gains one dimension of stability and two 2-dimensional invariant sets would be born.

It seems that due to the nature of the problem (nonlinear vector field, phase space being a cartesian product of simplices, the dependence of the system on the parameters), the open questions and presumptions stated above are not easy to tackle analytically nor numerically (when one wants to provide the dependence of the picture for the whole parameter space). Although the description of the heteroclinic network provided in this chapter seems to give a good insight into local behaviour near the network, it might actually turn out to be not very helpful in dispelling all doubts about chaotic switching. As mentioned in the fifth section of this chapter, the switching relies strongly on the parameter values. Nevertheless, in our considered judgement - from the application point of view -there are far more important questions concerning global behaviour in the system - than local behaviour near the heteroclinic network. For this reason, we would like to point out that it should not be difficult to perform computer-assisted proof, for example, of the existence of chaos (if chaos does actually appear), for an open set of parameter values (ϵ_X, ϵ_Y) in the white region (see e.g. [116]), especially for the parameters lying on the β-curve. However, this might not give a full picture of the switching

happening on the β-curve and an explanation of how it is born near the heteroclinic network.

If we know what we are doing, it wouldn't be called science.

A. Einstein

5

On the non-transverse homoclinic channel of a compact center manifold

5.1 - Introduction

* In the section 5.2, we describe a motivating example related to the considered later toy-model, i.e. aforementioned non-transverse homoclinic channel in the Rock-Scissors-Paper model.

In the section 5.3.1 we derive the normal form of the ODE, governing considered toy-model (see section 5.3 and fig. 5.6), near the center manifold.

Section 5.3.2 contains derivation of the normal form of the return map to the cross section in the vicinity of the center manifold.

In theorem 11, under some additional assumption we prove the existence of the foliation theorem resulting in the description of the limit dynamics of the return map. In this case we prove the existence of a Lorenz-like attractor for the return map.

We show in section 5.3.4 that the return map satisfy a 'cone-field condition' (for certain choice of coefficients), resulting in its partial hyperbolicity.

*The content of this chapter is mainly based on a joint work: Olszowiec C., Turaev D.V., "On the non-transverse homoclinic channel of a center manifold"

5.2 - Non-transverse homoclinic channel in the Rock-Scissors-Paper game

5.2.1 Basic facts about scattering maps for autonomous flows

In this section, as a background to the following ones, we briefly introduce (after [26]) the notion of the scattering map for autonomous flows.

For our purposes, let us assume that Φ is a C^∞ smooth flow on a manifold M and $\Lambda \subset M$ let be a normally hyperbolic invariant manifold.

First of all, let us recall the definitions of the stable and unstable manifolds of Λ and of $x \in \Lambda$

$$W^s_\Lambda := \Big\{ y \in M \mid d(\Phi_t(y), \Lambda) \le C_y e^{-\beta \cdot t},\ t \ge 0 \Big\}, \quad W^u_\Lambda := \Big\{ y \in M \mid d(\Phi_t(y), \Lambda) \le C_y e^{-\beta \cdot |t|},\ t \le 0 \Big\}$$

$$W^s_x := \Big\{ y \in M \mid d(\Phi_t(y), \Phi_t(x)) \le C_{x,y} e^{-\beta \cdot t},\ t \ge 0 \Big\}, \quad W^u_x := \Big\{ y \in M \mid d(\Phi_t(y), \Phi_t(x)) \le C_{x,y} e^{-\beta \cdot |t|},\ t \le 0 \Big\}$$

The constant $\beta > 0$ is the expansion rate from Λ.

Assume that there exists a manifold $\Gamma \subset W^s_\Lambda \cap W^u_\Lambda$, along which W^s_Λ, W^u_Λ intersect transversally, i.e. for every $x \in \Gamma$:

$$T_x M = T_x W^s_\Lambda + T_x W^u_\Lambda, \qquad T_x W^s_\Lambda \cap T_x W^u_\Lambda = T_x \Gamma$$

For the definition of the scattering map, it is also required to assume that Γ is transverse to the W^s_x, W^u_x foliations, i.e. for every $x \in \Gamma$ we have

$$T_x \Gamma \oplus T_x W^s_{x_+} = T_x W^s_\Lambda\,, \qquad T_x \Gamma \oplus T_x W^u_{x_-} = T_x W^u_\Lambda$$

The wave operator is defined as the mapping:

$$\Omega_\pm : W^{s,u}_\Lambda \ni x \mapsto x_\pm \in \Lambda$$

where $x_\pm$ is such that

$$|\Phi_t(x) - \Phi_t(x_\pm)| \le C_{x,x_\pm} e^{-\beta \cdot |t|}, \quad \text{as} \quad t \to \pm\infty$$

We say that Γ is a homoclinic channel if the intersection of W^s_Λ, W^u_Λ is transverse along Γ and that Γ is transverse to the W^s_x, W^u_x foliations. Moreover we require from the wave operators

$$\Omega^\Gamma_\pm := (\Omega_\pm)\big|_\Gamma : \Gamma \to \Omega^\Gamma_\pm\big(\Gamma\big) \subset \Lambda$$

to be C^{l-1} diffeomorphisms. Here l is the smoothness of the normally hyperbolic invariant manifold Λ.

Definition 21. *Assume that Γ is a homoclinic channel. The scattering map is defined as*

$$\sigma^\Gamma := \Omega^\Gamma_+ \circ \left(\Omega^\Gamma_-\right)^{-1}$$

5.2.2 Rock-Scissors-Paper game

Let us recall the basic facts about the Rock-Scissors-Paper game. We are considering a bimatrix game with payoff matrices (A, B^T), where:

$$A = \begin{pmatrix} \epsilon_X & 1 & -1 \\ -1 & \epsilon_X & 1 \\ 1 & -1 & \epsilon_X \end{pmatrix} \quad B = \begin{pmatrix} \epsilon_Y & 1 & -1 \\ -1 & \epsilon_Y & 1 \\ 1 & -1 & \epsilon_Y \end{pmatrix}$$

$\epsilon_X, \epsilon_Y \in (-1, 1)$ are the rewards for ties.

Assuming perfect memory of players X and Y, the dynamics are governed by the coupled replicator equations (see [96]):

$$\begin{cases} \dot{x_i} = x_i[(Ay)_i - x^T Ay] \\ \dot{y_j} = y_j[(Bx)_j - y^T Bx] \end{cases}$$

with $i, j = 1, 2, 3$, and $x_1 + x_2 + x_3 = 1$, $y_1 + y_2 + y_3 = 1$, where x_i, y_i denote the probabilities of playing strategy i by players X and Y, respectively. We investigate these equations as a system of ODE's in $\mathbb{R}^4$:

$$\begin{cases} \dot{x_1} = x_1[(Ay)_1 - x^T Ay] \\ \dot{x_2} = x_2[(Ay)_2 - x^T Ay] \\ \dot{y_1} = y_1[(Bx)_1 - y^T Bx] \\ \dot{y_2} = y_2[(Bx)_2 - y^T Bx] \end{cases}$$

with substitutions $x_3 = 1 - x_1 - x_2, y_3 = 1 - y_1 - y_2$ and constraints $x_1, x_2 \geq 0, x_1 + x_2 \leq 1$, $y_1, y_2 \geq 0, y_1 + y_2 \leq 1$.

Remark 61. *If each of the players is restricted to only two strategies, then either all orbits are periodic and spiral around one of the* 6 *equilibria or they converge to one of the draw equilibrium* $(R,R),(P,P),(S,S)$ *with time tending to* $\pm\infty$. *The latter case occurs for the system restricted to the subspaces* $\{x_i = 0,\ y_i = 0\}$, $i = 1,2,3$.

The following was already investigated in [79]:

Proposition 50. *There are 6 equilibria* $Z^a, Z^b, Z^c, Z^d, Z^e, Z^f$ *of (1,2,1) type (1-dim. stable manifold, 2-dim. center manifold, 1-dim. unstable manifold), which are centers within the 2 dimensional invariant subspaces* $H^a, H^b, H^c, H^d, H^e, H^f$ *(where the system is integrable), and have one dimensional stable and unstable manifolds* $W^s_a, W^u_a, W^s_b, W^u_b, W^s_c, W^u_c, W^s_d, W^u_d, W^s_e, W^u_e, W^s_f, W^u_f$:

$$Z^a = \Big(0, \frac{2}{3-\epsilon_Y}, \frac{1+\epsilon_X}{3+\epsilon_X}, \frac{2}{3+\epsilon_X}\Big), H^a = \{x_1 = 0, y_3 = 0\}, W^s_a \subset \{x_1 = 0\}, W^u_a \subset \{y_3 = 0\}$$

$$Z^b = \Big(0,\ \frac{1+\epsilon_Y}{3+\epsilon_Y},\ \frac{1-\epsilon_X}{3-\epsilon_X},\ 0\Big),\ H^b = \{x_1 = 0, y_2 = 0\},\ W^s_b \subset \{y_2 = 0\},\ W^u_b \subset \{x_1 = 0\}$$

$$Z^c = \Big(\frac{1-\epsilon_Y}{3-\epsilon_Y},\ 0,\ 0,\ \frac{1+\epsilon_X}{3+\epsilon_X}\Big),\ H^c = \{x_2 = 0, y_1 = 0\},\ W^s_c \subset \{x_2 = 0\},\ W^u_c \subset \{y_1 = 0\}$$

$$Z^d = \Big(\frac{2}{3+\epsilon_Y}, 0, \frac{2}{3-\epsilon_X}, \frac{1-\epsilon_X}{3-\epsilon_X}\Big), H^d = \{x_2 = 0, y_3 = 0\}, W^s_d \subset \{y_3 = 0\}, W^u_d \subset \{x_2 = 0\}$$

$$Z^e = \Big(\frac{1+\epsilon_Y}{3+\epsilon_Y}, \frac{2}{3+\epsilon_Y}, 0, \frac{2}{3-\epsilon_X}\Big), H^e = \{x_3 = 0, y_1 = 0\}, W^s_e \subset \{y_1 = 0\}, W^u_e \subset \{x_3 = 0\}$$

$$Z^f = \Big(\frac{2}{3-\epsilon_Y}, \frac{1-\epsilon_Y}{3-\epsilon_Y}, \frac{2}{3+\epsilon_X}, 0\Big), H^f = \{x_3 = 0, y_2 = 0\}, W^s_f \subset \{x_3 = 0\}, W^u_f \subset \{y_2 = 0\}$$

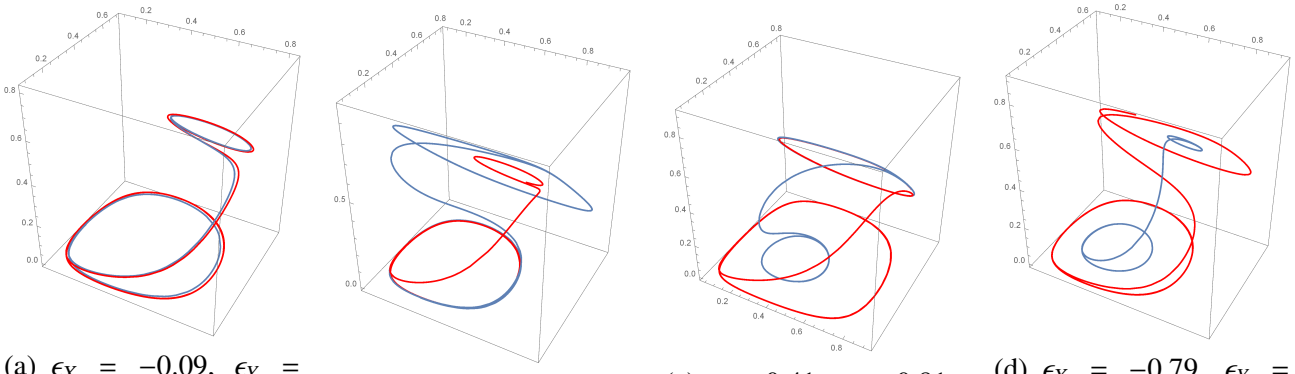
Remark 62. *On the boundary of codimension 1, the orbits travel from one invariant hyperplane* H^σ *(with* $\sigma \in \{a,\ b,\ c,\ d,\ e,\ f\}$*) of dimension 2 to another one.*

The following observation is based on the numerical simulations performed within the invariant subspaces $\{x_i = 0\}$, $\{y_j = 0\}$ and as well in the 4 dimensional neighbourhood of the subspaces H^a,...,H^f.

Remark 63. *For all $\epsilon_X, \epsilon_Y \in (-1, 1)$, the following inclusions hold:*

$$W^s(H^a) = W^u(H^b) \subset \{x_1 = 0\},\ W^s(H^b) = W^u(H^f) \subset \{y_2 = 0\},\ W^s(H^f) = W^u(H^e) \subset \{x_3 = 0\}$$

$$W^s(H^e) = W^u(H^c) \subset \{y_1 = 0\},\ W^s(H^c) = W^u(H^d) \subset \{x_2 = 0\},\ W^s(H^d) = W^u(H^a) \subset \{y_3 = 0\}$$



(a) $\epsilon_X = -0.09,\ \epsilon_Y = -0.79$ (b) $\epsilon_X = 0.81,\ \epsilon_Y = 0.11$ (c) $\epsilon_X = 0.41,\ \epsilon_Y = 0.81$ (d) $\epsilon_X = -0.79,\ \epsilon_Y = -0.29$

Figure 5.1: Generic types of behaviour of the flow on the boundary $\{x_1 = 0\}$, the red curve is a backward orbit of a starting point close to the stable manifold of Z^a, while the blue curve is a forward orbit of a starting point close to the unstable manifold of Z^b. The curves stay either (a) close to each other for all times, (b) close to the subspace $\{y_2 = 0\}$, (c) close to $\{y_3 = 0\}$, (d) or are faraway from each other.

5.2.3 Dynamics in the neighbourhood of the non-transverse homoclinic channel in the Rock-Scissors-Paper game

In this subsection we describe the non-transverse homoclinic channel appearing in the Rock-Scissors-Paper game, and also investigate the dynamics in its neighbourhood.

To start with, note that from the latter remark 63, it follows that there exists a homoclinic channel consisting of invariant subspaces H^a,...,H^f and its invariant manifolds, which are connected in sequence $H^a \to H^d \to H^c \to H^e \to H^f \to H^b \to H^a$. Here $H^a \to H^d$ means that $W^u(H^a) = W^s(H^d)$ (analogously for other connections). Note that this homoclinic channel, which we denote by I, is non-transverse, since corresponding stable/unstable manifolds coincide.

We want to investigate the dynamics near the channel I.

Remark 64. *Since for any point in I, its omega limit set is the subset of one of the invariant subspaces H^a,...,H^f (the same holds for alpha limit set), our idea would be to reduce the study of the dynamics in the vicinity of channel I to the composition of* 6 *scattering maps $H^a \mapsto H^d$, $H^d \mapsto H^c$, $H^c \mapsto H^e$, $H^e \mapsto H^f$, $H^f \mapsto H^b$, $H^b \mapsto H^a$. For this purpose, it would be necessary to establish (infinite) shadowing of the composition of the scattering maps. Since the homoclinic channel I is non-transverse, we cannot*

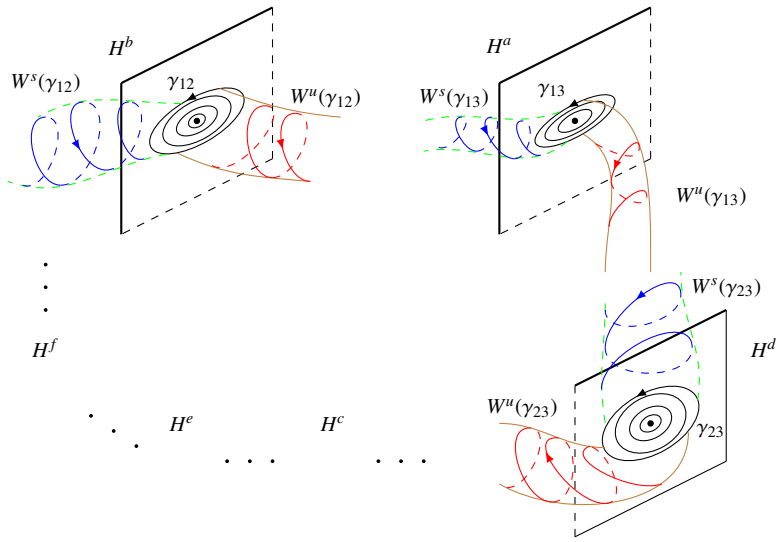


Figure 5.2: Six two-dimensional invariant squares described in the propositions (50) and (63). Red/blue curves relate to the fibers of the unstable/stable manifolds of the periodic orbits lying within invariant squares, respectively.

follow the approach presented in [26], [27], [39], [120]. Note however that these 6 *scattering maps are well defined since these six pieces of homoclinic channel are in fact invariant subspaces (either* $\{x_i = 0\}$ *or* $\{y_i = 0\}$*). Instead, in the remaining part of this section we want to verify whether channel I is locally attracting, i.e. if there exists a neighbourhood of the channel I so that each orbit from this neighbourhood accumulates on I.*

Note that due to the $\mathbb{Z}_3$ symmetry of the Rock-Scissors-Paper model, it suffices to investigate only subspaces $H^b = \{x_1 = 0,\ y_2 = 0\}$ and $H^c = \{x_2 = 0,\ y_1 = 0\}$.

For H^b, we consider the (x_1, y_2) part of the vector field of the original system

$$\begin{cases} \dot{x_1} = x_1[(Ay)_1 - x^T Ay] \\ \dot{x_2} = x_2[(Ay)_2 - x^T Ay] \\ \dot{y_1} = y_1[(Bx)_1 - y^T Bx] \\ \dot{y_2} = y_2[(Bx)_2 - y^T Bx] \end{cases}$$

with substitutions $x_3 = 1 - x_1 - x_2, y_3 = 1 - y_1 - y_2$ and constraints $x_1, x_2 \geq 0, x_1 + x_2 \leq 1$, $y_1, y_2 \geq 0, y_1 + y_2 \leq 1$.

That is

$$\begin{cases} \dot{x_1} = x_1[(Ay)_1 - x^T Ay] \\ \dot{y_2} = y_2[(Bx)_2 - y^T Bx] \end{cases}$$

and corresponding linearized vector field at $x_1 = 0,\ y_2 = 0$, with $x_3 = 1 - x_2, y_3 = 1 - y_1$.

$$V(x_2, y_1) = \begin{pmatrix} V_1(x_2, y_1) \\ V_2(x_2, y_1) \end{pmatrix} := \begin{pmatrix} (Ay)_1 - x^T Ay \\ (Bx)_2 - y^T Bx \end{pmatrix}$$

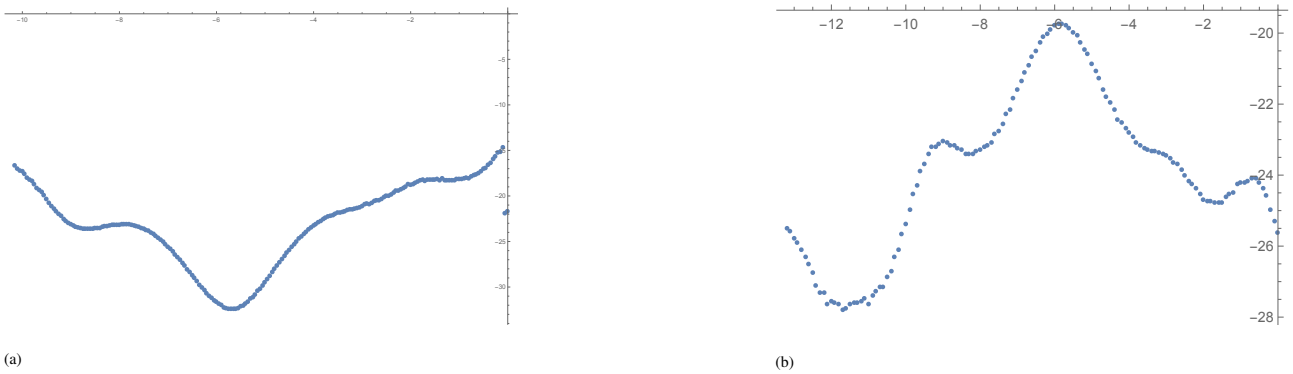
If γ is a periodic solution of the (original) system reduced to the subspace H^b (5.2.1 below), with substitutions $x_3 = 1 - x_2, y_3 = 1 - y_1$.

$$\begin{cases} \dot{x_2} = x_2[(Ay)_2 - x^T Ay] \\ \dot{y_1} = y_1[(Bx)_1 - y^T Bx] \end{cases} \tag{5.2.1}$$

then the rate of attraction/repulsion to/from the periodic orbit $\gamma \subset H^b$ is a number:

$$\int_\gamma (V_1(x_2, y_1) + V_2(x_2, y_1)) = \int_\gamma ((Ay)_1 - x^T Ay + (Bx)_2 - y^T Bx)$$

The corresponding rates for hyperplane H^c can be computed analogously.



(a) (b)

Figure 5.3: Image of two different periodic orbits γ_1, γ_2 from H^b under the same scattering map $H^b \to H^a$. The horizontal axis parameterizes the time, ranging from 0 to the period of the orbit, and the vertical axis parameterizes periodic orbits in H^a.
This figure suggests that the unstable manifold of γ_1 and of γ_2 intersect transversally (within the invariant subspace $\{x_1 = 0\}$) stable manifolds of the nontrivial family of periodic orbits from H^a.

Remark 65. *For all parameter values ϵ_X, ϵ_Y, for which we computed these rates of attraction/repulsion to/from periodic orbits, we do not find any reason for the total sum of these rates, for the whole loop consisting of six hyperplanes H^a, H^b, H^c, H^d, H^e, H^f, to be negative. This explains the figures (5.5), where we investigate numerically shadowing of composition of* 6 *scattering maps corresponding to* 6 *pieces of the non-transverse homoclinic channel.*

In conclusion, in finite time the orbit of an initial point being close to the unstable manifold (but not belonging to it) of any of these invariant 2 dimensional hyperplanes (H^a,...,H^f), will eventually stop

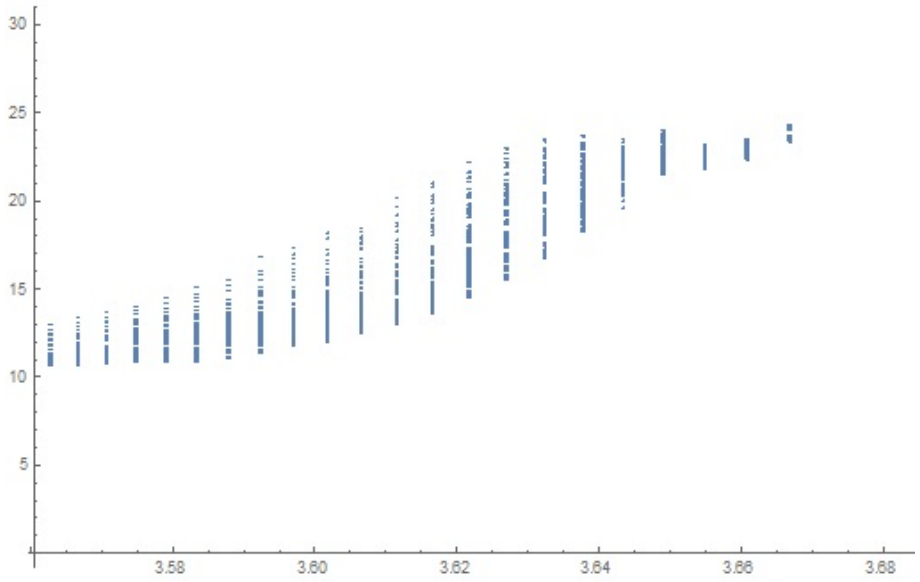
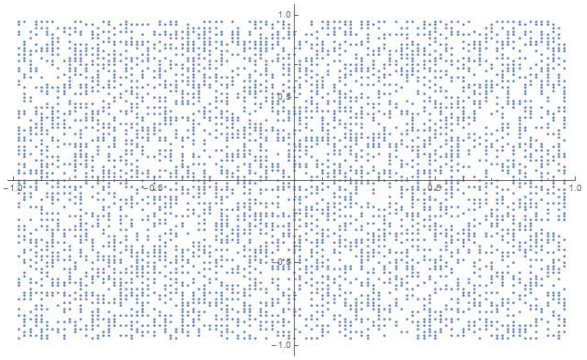
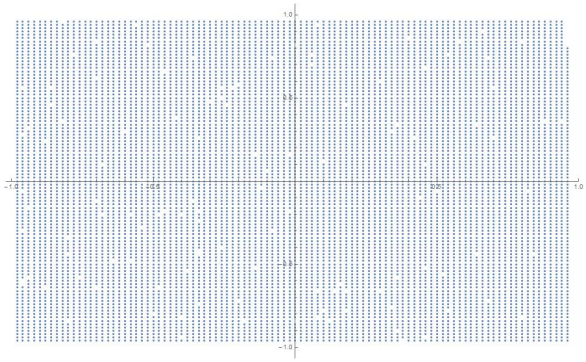


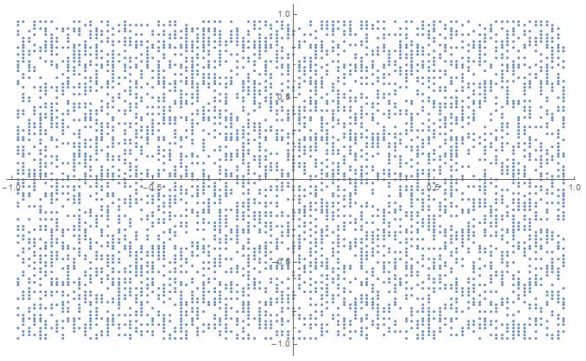
Figure 5.4: Picture presenting a joint image of periodic orbits under the scattering map. Horizontal axis parameterizes (by energy function) periodic orbits in H^b, vertical axis parameterizes periodic orbits in H^a



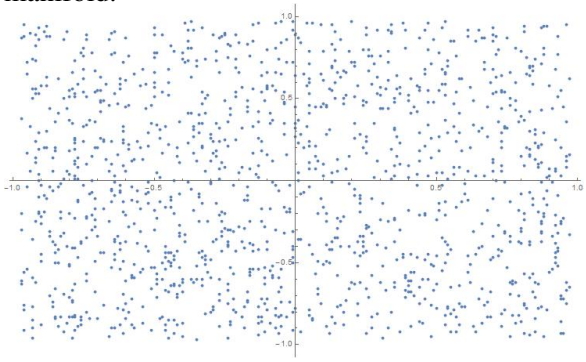
(a) Parameter values for which at least 1% of the points drawn close to the one of the center manifolds visit in sequence neighbourhoods of 60 consecutive center manifolds.



(b) Parameter values for which at least 1% of the points drawn close to the one of the center manifolds visit in sequence neighbourhoods of 12 consecutive center manifolds, i.e. comes back two times to the neighbourhood of the initial center manifold.



(c) Parameter values for which at least 10% of the points drawn close to the one of the center manifolds visit in sequence neighbourhoods of 6 consecutive center manifolds, i.e. comes back to the neighbourhood of the initial center manifold.



(d) Parameter values for which at least 25% of the points drawn close to the one of the center manifolds visit in sequence neighbourhoods of 6 consecutive center manifolds, i.e. comes back to the neighbourhood of the initial center manifold.

Figure 5.5: Numerical investigation of the repulsion from the homoclinic channel in the Rock-Scissors-Paper game

shadowing the sequence of hyperplanes and leave the neighbourhood of this non-transverse homoclinic channel I.

Corollary 4. *The homoclinic channel I is repelling and so the dynamics near it cannot be fully described by the composition of the aforementioned 6 scattering maps.*

In fact, the calculated ratios of attraction/repulsion to the orbits tell us that, when considering the flow backward in time, all of the orbits from some neighbourhood of channel I accumulate on I.

Remark 66. *In the section 5.3 we will consider a toy-model motivated by the example of a non-transverse homoclinic channel I from Rock-Scissors-Paper. We will identify invariant subspaces H^a,...,H^f with one center manifold D foliated with periodic orbits and assume that the stable and unstable manifolds of this center manifold D coincide. However we will assume that this homoclinic channel is locally attracting, so the toy-model will resemble a homoclinic channel I from the Rock-Scissors-Paper game with reversed time.*

5.2.4 Discussion

What we were actually interested in was the form of the scattering maps between the invariant squares $H^a, ..., H^f$ and what information they provide about the dynamics near the non-transverse homoclinic channel I consisting of invariant squares and their stable/unstable manifolds (see figure 5.2).
In the figure (5.3), there are two examples of the images of periodic orbits (from invariant squares) under the scattering map.
Figure (5.4) presents a joint image of the family of parameterized periodic orbits from one invariant square under the scattering map. See the captions under the figures for details and explanations.
However, the conclusion of this section is that although the images of the scattering maps look very complicated, due to lack of attraction to the homoclinic channel and lack of infinite shadowing of the scattering maps (compare with [26], [39], [27] and [120], where the transversality condition for the homoclinic channel is essential), one cannot reduce the study of the behaviour near this non-transverse homoclinic channel to the study of the scattering maps between the invariant squares.
All of the numerical computations and simulations were performed using tools from the CAPD Library [17].

5.3 - The toy-model

5.3.1 Normal form of the ODE near the center manifold

Motivated by the example of a non-transverse homoclinic channel I in the Rock-Scissors-Paper game, we consider in this section the following ODE in $\mathbb{R}^4$:

$$\dot{x} = F(x) \tag{5.3.1}$$

with F - real analytic, $F(0) = 0$ and the point spectrum $\sigma(dF(0)) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, where $\lambda_1 > 0$, $\lambda_2 < 0$, $|\lambda_3| = 1$, $\lambda_3 = \bar{\lambda}_4$, $Im(\lambda_3) \neq 0$.

Assume that the fixed point $x = 0$ possesses a 2-dimensional compact center manifold $W^c(0)$ on which the flow is periodic (i.e. the center manifold is foliated with periodic orbits). After straightening $W^c(0)$, so that it becomes a 2-dimensional disc D and introducing the polar coordinates $(r, \phi) \in [0, 1) \times [0, 1)$ on it, we perform the change of the coordinates $(x_1, x_2, x_3, x_4) \mapsto (v_1, v_2, r, \phi)$ so that we can rewrite (5.3.1) in the following form:

$$\begin{cases} \dot{v_1} = f_1(v_1, v_2, r, \phi) \\ \dot{v_2} = f_2(v_1, v_2, r, \phi) \\ \dot{r} = f_3(v_1, v_2, r, \phi) \\ \dot{\phi} = f_4(v_1, v_2, r, \phi) \end{cases} \tag{5.3.2}$$

with $\frac{\partial f_1}{\partial v_1}(0,0,0,0) = \lambda_1$, $\frac{\partial f_2}{\partial v_2}(0,0,0,0) = \lambda_2$, $\frac{\partial f_1}{\partial v_2}(0,0,0,0) = \frac{\partial f_2}{\partial v_1}(0,0,0,0) = 0$.

Moreover assume that 3 - dimensional stable and unstable manifolds (i.e. $W^s(D)$, $W^u(D)$) of the center manifold $D = W^c(0)$ coincide.

Remark 67. *In particular their intersection is non-transverse and forms a non-transverse homoclinic channel which we denote by I. Because of the lack of the transversality condition, I does not fit into the definition of a homoclinic channel from [26]. We emphasize it by calling I a non-transverse homoclinic channel.*

After expanding f_1, f_2, f_3, f_4 in terms of (v_1, v_2) in the vicinity of $(0, 0)$, for any $(r, \phi) \in [0, 1] \times [0, 1)$ and straightening local unstable and stable manifolds of $D = W^c(0)$ to be locally the 3-dimensional

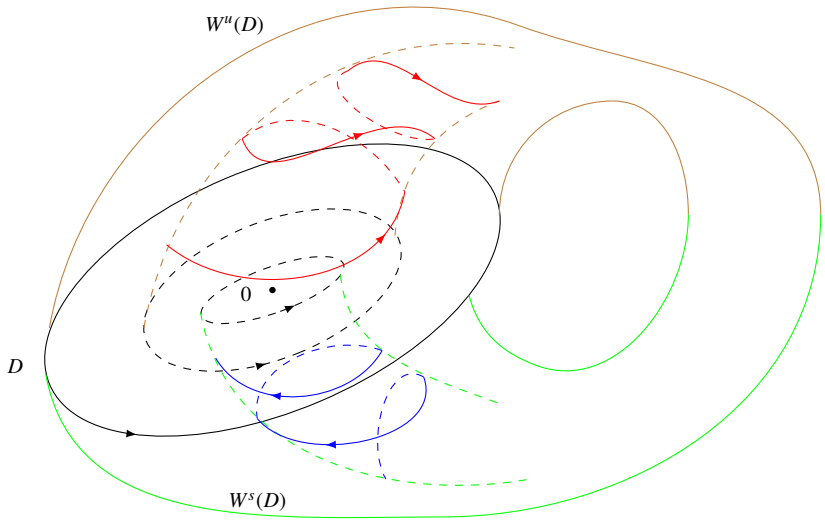


Figure 5.6: 2 - dimensional center manifold D, foliated with periodic orbits, with its coinciding stable and unstable manifolds $W^s(D) = W^u(D)$

tubes $(v_1, 0, r, \phi)$ and $(0, v_2, r, \phi)$ we arrive at the simplified system of equations given by the analytic vector field:

$$\begin{cases} \dot{v_1} = p(r,\phi) \cdot v_1 + p_{00}(v_1, v_2, r, \phi) \cdot v_1 \\ \dot{v_2} = \varsigma(r,\phi) \cdot v_2 + \varsigma_{00}(v_1, v_2, r, \phi) \cdot v_2 \\ \dot{r} = r_{00}(r,\phi) + v_1 v_2 \cdot r_0(v_1, v_2, r, \phi) \\ \dot{\phi} = \omega_{00}(r,\phi) + v_1 v_2 \cdot \omega_0(v_1, v_2, r, \phi) \end{cases} \tag{5.3.3}$$

with $r_{00}, \omega_{00}, r_0, \omega_0, p, p_{00}, \varsigma, \varsigma_{00}$ being 1-periodic with respect to ϕ.

Assumption 9. *Since v_1, v_2 are respectively unstable and stable directions, then $p(r,\phi) > 0$ and $\varsigma(r,\phi) < 0$, for all r, phi. Moreover we assume that $p(r,\phi) + \varsigma(r,\phi) < 0$, which will eventually result in channel I being locally attracting, i.e. orbits starting near the channel I will accumulate on I.*

Until the end of this section we will perform analytic transformations of the coordinates leading us to the analytic and convergent normal form of the ODE describing this model near the center manifold:

$$\begin{cases} \dot{v_1} = p(r) \cdot v_1 + p_0(v_1, v_2, r, \phi) \cdot {v_1}^2 v_2 \\ \dot{v_2} = \varsigma(r) \cdot v_2 + \varsigma_0(v_1, v_2, r, \phi) \cdot v_1 {v_2}^2 \\ \dot{r} = v_1 v_2 \cdot r_0(v_1, v_2, r, \phi) \\ \dot{\phi} = \omega(r) + v_1 v_2 \cdot \omega_0(v_1, v_2, r, \phi) \end{cases} \tag{5.3.4}$$

with p_0, ς_0, r_0, ω_0 being 1 - periodic in ϕ.

We start with simplifications of (r, ϕ) coordinates in (5.3.3), for this we need the following lemma about the action-angle coordinates in the integrable Hamiltonian system:

Lemma 6. *Under the above assumptions, there exists an analytic and convergent change of coordinates that transforms system*

$$\begin{cases} \dot{r} = r_{00}(r, \phi) \\ \dot{\phi} = \omega_{00}(r, \phi) \end{cases} \tag{5.3.5}$$

into the action-angle coordinates

$$\begin{cases} \dot{\xi} = 0 \\ \dot{\zeta} = \omega(\xi) \end{cases} \tag{5.3.6}$$

with $0 \le r < 1$, $0 \le \xi < 1$ *and* $\omega(r) := \int_0^1 \omega_{00}(r, \phi')d\phi'$

Proof. Let us write

$$\xi(t) := r(t) \cdot \Psi(t), \quad \zeta(t) := r(t) \cdot \epsilon(t)$$

Then

$$0 = \dot{\xi}(t) = \Psi(\phi) \cdot \dot{r} + \frac{d\Psi}{dt} \cdot r = \Psi(\phi) \cdot r_{00}(r, \phi) + \frac{d\Psi}{dt} \cdot r$$

and

$$\omega(\xi) = \dot{\zeta}(t) = \epsilon(\phi) \cdot \dot{r} + \frac{d\epsilon}{dt} \cdot r = \epsilon(\phi) \cdot r_{00}(r, \phi) + \frac{d\epsilon}{dt} \cdot r$$

Hence

$$\Psi(t) = exp\left(- \int_0^t \frac{r_{00}(r(s), \phi(s))}{r(s)}\, ds \right)$$

under assumption $\Psi(0) = 1$. In the same way we find that

$$\epsilon(t) = exp\left(- \int_0^t \frac{r_{00}(r(s), \phi(s))}{r(s)}\, ds \right) \cdot \int_0^t \omega(r(t')) \cdot exp\left(\int_0^{t'} \frac{r_{00}(r(s), \phi(s))}{r(s)}\, ds \right) dt'$$

□

Due to the above lemma (6), we can assume that the system (5.3.3) is governed by the equations:

$$\begin{cases} \dot{v_1} = p(r,\phi)\cdot v_1 + p_{00}(v_1,v_2,r,\phi)\cdot v_1 \\ \dot{v_2} = \varsigma(r,\phi)\cdot v_2 + \varsigma_{00}(v_1,v_2,r,\phi)\cdot v_2 \\ \dot{r} = v_1 v_2 \cdot r_0(v_1,v_2,r,\phi) \\ \dot{\phi} = \omega(r) + v_1 v_2 \cdot \omega_0(v_1,v_2,r,\phi) \end{cases} \tag{5.3.7}$$

In order to reduce the above system 5.3.7 by coordinates transformation into 5.3.4, we follow the steps of reduction as in [102]. Let us introduce:

$$\begin{cases} p_1(r,\phi,v_1) := p_{00}(v_1,0,r,\phi) \\ p_2(r,\phi,v_2) := p_{00}(0,v_2,r,\phi) \\ p_3(r,\phi,v_1,v_2) := p_{00}(v_1,v_2,r,\phi) - p_1(r,\phi,v_1) - p_2(r,\phi,v_2) \\ \varsigma_1(r,\phi,v_1) := \varsigma_{00}(v_1,0,r,\phi) \\ \varsigma_2(r,\phi,v_2) := \varsigma_{00}(0,v_2,r,\phi) \\ \varsigma_3(r,\phi,v_1,v_2) := \varsigma_{00}(v_1,v_2,r,\phi) - \varsigma_1(r,\phi,v_1) - \varsigma_2(r,\phi,v_2) \end{cases} \tag{5.3.8}$$

In the system (5.3.7) we want to get rid of the terms $p_1,\ p_2,\ \varsigma_1,\ \varsigma_2$, by subsequent analytic and convergent changes of coordinates. Next four propositions will consequently deal with the terms p_2, then ς_1, p_1, ς_2 (see [102], [103]).

Proposition 51. *Killing the term p_2 with substitution $\xi_1 := v_1 + h_1(r,\phi,v_2)\cdot v_1$*

Proof. We have

$$\dot{\xi}_1 = \dot{v}_1\cdot(1+h) + v_1\cdot\Big(\frac{\partial h_1}{\partial r}\dot{r} + \frac{\partial h_1}{\partial \phi}\dot{\phi} + \frac{\partial h_1}{\partial v_2}\dot{v}_2\Big) =$$

$$= v_1\cdot(1+h)\cdot\Big(p(r,\phi) + p_1(r,\phi,v_1) + p_2(r,\phi,v_2) + p_3(r,\phi,v_1,v_2)\Big) + v_1\cdot\Big(\frac{\partial h_1}{\partial r}v_1v_2\cdot r_0(v_1,v_2,r,\phi)$$

$$+\frac{\partial h_1}{\partial \phi}(\omega(r) + v_1v_2\cdot\omega_0(v_1,v_2,r,\phi)) + \frac{\partial h_1}{\partial v_2}\cdot v_2\cdot(\varsigma(r,\phi) + \varsigma_1(r,\phi,v_1) + \varsigma_2(r,\phi,v_2) + \varsigma_3(r,\phi,v_1,v_2))\Big)$$

$$= \xi_1\cdot\Big(p(r,\phi) + p_1(r,\phi,v_1) + p_2(r,\phi,v_2) + p_3(r,\phi,v_1,v_2)\Big)$$

$$+v_1\cdot\Big(\frac{\partial h_1}{\partial \phi}\omega(r) + \frac{\partial h_1}{\partial v_2}\cdot v_2\cdot(\varsigma(r,\phi) + \varsigma_2(r,\phi,v_2) + \varsigma_3(r,\phi,v_1,v_2))\Big) + h.o.t.$$

So in order to kill the term p_2, it suffices to find the function h_1 such that $h_1(r, \phi, 0) = 0$ and:

$$-(1 + h_1) \cdot p_2 = \frac{\partial h_1}{\partial \phi}\omega(r) + \frac{\partial h_1}{\partial v_2} \cdot v_2 \cdot \Big(\varsigma(r, \phi) + \varsigma_2(r, \phi, v_2)\Big)$$

Let us take $h_1(r, \phi, v_2) = h_1(r, v_2)$, i.e. $\frac{\partial h_1}{\partial \phi} = 0$. Consider the system:

$$\begin{cases} \dot{X} = -(1 + X) \cdot p_2(r, \phi, v_2) \\ \dot{v}_2 = v_2 \cdot (\varsigma(r, \phi) + \varsigma_2(r, \phi, v_2)) \end{cases} \tag{5.3.9}$$

Due to $p_2(r, \phi, 0) = 0$, the point $(X, v_2) = (0, 0)$ is a fixed point. The linearized system at $(0, 0)$ is:

$$\begin{pmatrix} 0 & -\frac{\partial p_2}{\partial v_2}(r, \phi, 0) \\ 0 & \varsigma(r, \phi) \end{pmatrix} \circ \begin{pmatrix} X \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{\partial p_2}{\partial v_2}(r, \phi, 0)v_2 \\ \varsigma(r, \phi)v_2 \end{pmatrix} \tag{5.3.10}$$

Note that we assume $\varsigma(r, \phi) < 0$ for all r and ϕ. As this is the only non-zero eigenvalue of the matrix governing linearized system above, then from the strong stable manifold theorem for fixed points, follows the existence of h_1 which is its parameterization. □

Proposition 52. *Killing the term ς_1 with substitution $\xi_2 := v_2 + h_2(r, \phi, v_1) \cdot v_2$*

Proof. The only difference with respect to the previous proposition is that we need to solve the following equation for h_2 in order to kill the term ς_1:

$$h_2(r, \phi, 0) = 0 \quad \text{and} \quad -(1 + h_2) \cdot \varsigma_1 = \frac{\partial h_2}{\partial \phi}\omega(r) + \frac{\partial h_2}{\partial v_1} \cdot v_1 \cdot p(r, \phi)$$

Let $h_2(r, \phi, v_1) = h_2(r, v_1)$, i.e. $\frac{\partial h_2}{\partial \phi} = 0$.

Consider the system:

$$\begin{cases} \dot{X} = -(1 + X) \cdot \varsigma_1(r, \phi, v_1) \\ \dot{v}_1 = v_1 \cdot p(r, \phi) \end{cases} \tag{5.3.11}$$

Due to $\varsigma_1(r, \phi, 0) = 0$, the point $(X, v_1) = (0, 0)$ is a fixed point. The linearized system at $(0, 0)$ is:

$$\begin{pmatrix} 0 & -\frac{\partial \varsigma_1}{\partial v_1}(r, \phi, 0) \\ 0 & p(r, \phi) \end{pmatrix} \circ \begin{pmatrix} X \\ v_1 \end{pmatrix} = \begin{pmatrix} -\frac{\partial \varsigma_1}{\partial v_1}(r, \phi, 0)v_1 \\ p(r, \phi)v_1 \end{pmatrix} \tag{5.3.12}$$

As we assume $p(r, \phi) > 0$ for all r and ϕ, from the strong unstable manifold theorem for fixed points,

follows the existence of required h_2 which is its the parameterization. □

Proposition 53. *Killing the term p_1 with substitution $\xi_1 := v_1 + h_3(r, \phi, v_1) \cdot v_1$*

Proof. Here in order to kill the term p_1, we have to look for h_3 such that:

$$h_3(r, \phi, 0) = 0 \quad \text{and} \quad -(1 + h_3) \cdot p_1 = \frac{\partial h_3}{\partial \phi}\omega(r) + \frac{\partial h_3}{\partial v_1} \cdot v_1 \cdot \Big(p(r, \phi) + p_1(r, \phi, v_1)\Big)$$

Let $h_3(r, \phi, v_1) = h_3(r, v_1)$, i.e. $\frac{\partial h_3}{\partial \phi} = 0$. Consider the system:

$$\begin{cases} \dot{X} = -(1 + X) \cdot p_1(r, \phi, v_1) \\ \dot{v}_1 = v_1 \cdot \Big(p(r, \phi) + p_1(r, \phi, v_1)\Big) \end{cases} \tag{5.3.13}$$

Due to $p_1(r, \phi, 0) = 0$, the point $(X, v_1) = (0, 0)$ The linearized system at $(0, 0)$ is:

$$\begin{pmatrix} 0 & -\frac{\partial p_1}{\partial v_1}(r, \phi, 0) \\ 0 & p(r, \phi) + p_1(r, \phi, 0) \end{pmatrix} \circ \begin{pmatrix} X \\ v_1 \end{pmatrix} = \begin{pmatrix} -\frac{\partial p_1}{\partial v_1}(r, \phi, 0)v_1 \\ p(r, \phi)v_1 \end{pmatrix} \tag{5.3.14}$$

As in the previous propositions, the strong unstable manifold theorem for fixed points, guarantees the existence of h_3. □

Proposition 54. *Killing the term ς_2 with substitution $\xi_2 := v_2 + h_4(r, \phi, v_2) \cdot v_2$*

Proof. In order to kill the term ς_2, we look for h_4 satisfying the following conditions:

$$h_4(r, \phi, 0) = 0 \quad \text{and} \quad -(1 + h_4) \cdot \varsigma_2 = \frac{\partial h_4}{\partial \phi}\omega(r) + \frac{\partial h_4}{\partial v_2} \cdot v_2 \cdot \Big(\varsigma(r, \phi) + \varsigma_2(r, \phi, v_2)\Big)$$

Let $h_4(r, \phi, v_2) = h_4(r, v_2)$, i.e. $\frac{\partial h_4}{\partial \phi} = 0$. Consider the system:

$$\begin{cases} \dot{X} = -(1 + X) \cdot \varsigma_2(r, \phi, v_2) \\ \dot{v}_2 = v_2 \cdot \Big(\varsigma(r, \phi) + \varsigma_2(r, \phi, v_2)\Big) \end{cases} \tag{5.3.15}$$

Due to $\varsigma_2(r, \phi, 0) = 0$, the point $(X, v_2) = (0, 0)$ is a fixed point. The linearized system at $(0, 0)$ is:

$$\begin{pmatrix} 0 & -\frac{\partial \varsigma_2}{\partial v_2}(r, \phi, 0) \\ 0 & \varsigma(r, \phi) + \varsigma_2(r, \phi, 0) \end{pmatrix} \circ \begin{pmatrix} X \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{\partial \varsigma_2}{\partial v_2}(r, \phi, 0)v_2 \\ \varsigma(r, \phi)v_2 \end{pmatrix} \tag{5.3.16}$$

Hence from the strong stable manifold theorem for fixed point, follows the existence of h_4 which is its the parameterization. □

In order to obtain the normal form (5.3.4) of the ODE near the center manifold D, we need to utilize the lemmas above along with the following:

Lemma 7. *There exists an analytic and convergent change of the coordinates that transforms system*

$$\begin{cases} \dot{v_1} = p(r,\phi)\cdot v_1 + \Theta(v_1, v_2, r, \phi)\cdot v_1 \\ \dot{v_2} = \varsigma(r,\phi)\cdot v_2 + \Lambda(v_1, v_2, r, \phi)\cdot v_2 \\ \dot{r} = v_1 v_2 \cdot r_{0v}(v_1, v_2, r, \phi) \\ \dot{\phi} = \omega(r) + v_1 v_2 \cdot \omega_{0v}(v_1, v_2, r, \phi) \end{cases} \tag{5.3.17}$$

into

$$\begin{cases} \dot{z_1} = p_0(r)\cdot z_1 + \theta(z_1, z_2, r, \phi)\cdot z_1 \\ \dot{z_2} = \varsigma_0(r)\cdot z_2 + \lambda(z_1, z_2, r, \phi)\cdot z_2 \\ \dot{r} = z_1 z_2 \cdot r_{0z}(z_1, z_2, r, \phi) \\ \dot{\phi} = \omega(r) + z_1 z_2 \cdot \omega_{0z}(z_1, z_2, r, \phi) \end{cases} \tag{5.3.18}$$

with

$$p_0(r) := \int_0^1 p(r,\phi)\, d\phi, \quad \varsigma_0(r) := \int_0^1 \varsigma(r,\phi)\, d\phi$$

and θ, λ being 1 *-periodic in ϕ*

Proof. Let us write

$$v_1 := z_1 \cdot (1 + w_1(r,\phi)), \quad v_2 := z_2 \cdot (1 + w_2(r,\phi))$$

with $w_1(r,0) = w_2(r,0) = 0$. Then

$$\begin{aligned} \dot{v}_1 &= \dot{z}_1 \cdot \Big(1 + w_1(r,\phi)\Big) + z_1 \cdot \Big(\frac{\partial w_1}{\partial r}\dot{r} + \frac{\partial w_1}{\partial \phi}\dot{\phi}\Big) = \\ &= \Big(p_0(r) z_1 + \theta(z_1, z_2, r, \phi) z_1\Big) \cdot \Big(1 + w_1(r,\phi)\Big) \\ &+ z_1 \cdot \Big(\frac{\partial w_1}{\partial r} \cdot z_1 z_2 \cdot r_0(z_1, z_2, r, \phi)(1 + w_1(r,\phi))(1 + w_2(r,\phi)) + \frac{\partial w_1}{\partial \phi}\omega(r) \\ &+ \frac{\partial w_1}{\partial \phi} \cdot z_1 z_2 \cdot \omega_0(z_1, z_2, r, \phi)(1 + w_1(r,\phi))(1 + w_2(r,\phi))\Big) \end{aligned} \tag{5.3.19}$$

On the other hand

$$\begin{aligned}\dot{v}_1 &= p(r,\phi)v_1 + \Theta(v_1, v_2, r, \phi)v_1 \\ &= p(r,\phi)z_1(1 + w_1(r,\phi)) + \Theta(v_1, v_2, r, \phi)z_1(1 + w_1(r,\phi))\end{aligned} \tag{5.3.20}$$

Hence we need to solve for w_1 the following equation:

$$p_0(r)(1 + w_1(r,\phi)) + \frac{\partial w_1}{\partial \phi}\omega(r) = p(r,\phi)(1 + w_1(r,\phi))$$

which gives

$$\frac{1}{1 + w_1(r,\phi)} \cdot \frac{\partial w_1}{\partial \phi} = \frac{1}{\omega(r)}(p(r,\phi) - p_0(r))$$

and so

$$1 + w_1(r,\phi) = \exp\Big(\int_0^{\phi} \frac{p(r,\phi') - p_0(r)}{\omega(r)} d\phi' \Big)$$

Analogously we find that

$$1 + w_2(r,\phi) = \exp\Big(\int_0^{\phi} \frac{\varsigma(r,\phi') - \varsigma_0(r)}{\omega(r)} d\phi' \Big)$$

By comparing expressions for $\dot{v}_1$, $\dot{v}_2$ and using formulas for w_1, w_2, we get that θ and λ are 1 - periodic in ϕ. □

Hence, the above lemma and propositions, give us the analytic and convergent normal form:

$$\begin{cases} \dot{v_1} = p(r) \cdot v_1 + p_0(v_1, v_2, r, \phi) \cdot {v_1}^2 v_2 \\ \dot{v_2} = \varsigma(r) \cdot v_2 + \varsigma_0(v_1, v_2, r, \phi) \cdot v_1 {v_2}^2 \\ \dot{r} = v_1 v_2 \cdot r_0(v_1, v_2, r, \phi) \\ \dot{\phi} = \omega(r) + v_1 v_2 \cdot \omega_0(v_1, v_2, r, \phi) \end{cases} \tag{5.3.21}$$

with p_0, ς_0, r_0, ω_0 being 1 - periodic in ϕ.

5.3.2 Normal form of the return map

In this section we solve the system of equations (5.3.21) near the center manifold $(v_1, v_2, r, \phi) = (0, 0, r, \phi)$, find the local - Shilnikov map (proposition 55), and the return map to the neigbourhood of

$W^c(0)$ along the non-transverse homoclinic channel (proposition 10).

Proposition 55. *The local Shilnikov map from the cross section $S_2 = \{v_2 = h\}$ to cross section $S_1 = \{v_1 = h\}$ is given by :*

$$\begin{cases} \bar{v}_2 = h \cdot \left(\frac{v_1(0)}{h\cdot(1-hv_1(0)\cdot O(1))}\right)^{\frac{-\varsigma(r(0))}{p(r(0))}} (1 - h^2 \cdot \left(\frac{v_1(0)}{h\cdot(1-hv_1(0)\cdot O(1))}\right)^{\frac{-\varsigma(r(0))}{p(r(0))}} \cdot O(1)) + h.o.t \\ \bar{r} = r(0) + \frac{v_1(0)h}{(1-hv_1(0)\cdot O(1))} \cdot O(1) + h.o.t. \\ \bar{\phi} = \phi(0) + \frac{-1}{p(r(0))} \ln\left(\frac{v_1(0)}{h\cdot(1-hv_1(0)\cdot O(1))}\right) \cdot \left(\omega(r(0)) + \omega'(r(0)) \cdot \frac{v_1(0)h\cdot O(1)}{(1-hv_1(0)\cdot O(1))}\right) + \frac{v_1(0)h\cdot O(1)}{(1-hv_1(0)\cdot O(1))} + h.o.t. \end{cases} \tag{5.3.22}$$

Proof. Assume that $v_2(0) = h$ and $v_1(\tau) = h$, we want to find expression for $v_2(\tau)$ in terms of $v_1(0)$. Denote by

$$P(s,t) := \frac{1}{-t+s} \int_s^t p(r(s'))ds'$$

and

$$G(s) := \frac{1}{s} \int_0^s \varsigma(r(s'))ds'$$

then

$$\int_t^\tau e^{(-\tau+s)\cdot P(s,\tau)} \dot{v}_1(s)ds = v_1(\tau) - v_1(t) \cdot e^{(-\tau+t)\cdot P(t,\tau)} + \int_t^\tau p(r(s)) \cdot e^{(-\tau+s)\cdot P(s,\tau)} v_1(s)ds =$$

$$\int_t^\tau p(r(s)) \cdot e^{(-\tau+s)\cdot P(s,\tau)} v_1(s)ds + \int_t^\tau p_0\Big(r(s), \phi(s), v_1(s), v_2(s)\Big) \cdot v_1^2(s)v_2(s) \cdot e^{(-\tau+s)\cdot P(s,\tau)} ds$$

after rearrangement of the terms

$$v_1(t) = v_1(\tau)e^{(\tau-t)\cdot P(t,\tau)} - \int_t^\tau p_0\Big(r(s), \phi(s), v_1(s), v_2(s)\Big) \cdot v_1^2(s)v_2(s) \cdot e^{(s-t)\cdot P(t,s)} ds$$

By solving analogously the rest of the equations, we arrive with the system:

$$\begin{cases} v_1(t) = v_1(\tau)e^{(\tau-t)\cdot P(t,\tau)} - \int_t^\tau p_0\Big(r(s), \phi(s), v_1(s), v_2(s)\Big) \cdot v_1^2(s)v_2(s) \cdot e^{(s-t)\cdot P(t,s)} ds \\ v_2(t) = v_2(0)e^{t\cdot G(t)} - e^{t\cdot G(t)} \int_0^t \varsigma_0\Big(r(s), \phi(s), v_1(s), v_2(s)\Big) \cdot v_1(s)v_2^2(s) \cdot e^{-s\cdot G(s)} ds \\ r(t) = r(0) + \int_0^t v_1(s)v_2(s) \cdot r_0\Big(r(s), \phi(s), v_1(s), v_2(s)\Big) ds \\ \phi(t) = \phi(0) + \int_0^t \omega\Big(r(s)\Big) ds + \int_0^t v_1(s)v_2(s) \cdot \omega_0\Big(r(s), \phi(s), v_1(s), v_2(s)\Big) ds \end{cases} \tag{5.3.23}$$

We utilize the method of the consecutive approximations of the solution with the first one taken to be:

$$\begin{cases} v_1(t) := v_1(\tau)e^{(-\tau+t)\cdot p(r(0))} \\ v_2(t) := v_2(0)e^{t\cdot\varsigma(r(0))} \\ r(t) := r(0) \\ \phi(t) := \phi(0) + \int_0^t \omega(r(s))ds \end{cases} \tag{5.3.24}$$

Plugging this into the latter system (5.3.23), we obtain the second approximation of the solution:

$$\begin{cases} v_1(t) = v_1(\tau)\cdot e^{(-\tau+t)\cdot p(r(0))} - v_1^2(\tau)v_2(0)\cdot e^{(-2\tau+t)\cdot p(r(0))}\cdot (e^{\tau\cdot(p(r(0))+\varsigma(r(0))} - e^{t\cdot(p(r(0))+\varsigma(r(0))})\cdot O(1) \\ v_2(t) = v_2(0)\cdot e^{t\cdot\varsigma(r(0))} - v_1(\tau)v_2^2(0)\cdot e^{t\cdot\varsigma(r(0))}e^{-\tau\cdot p(r(0))}\cdot (e^{\tau\cdot(p(r(0))+\varsigma(r(0))} - e^{t\cdot(p(r(0))+\varsigma(r(0))})\cdot O(1) \\ r(t) = r(0) + v_1(\tau)v_2(0)\cdot e^{-\tau\cdot p(r(0))}\cdot (e^{t\cdot(p(r(0))+\varsigma(r(0))} - 1)\cdot O(1) \\ \phi(t) = \phi(0) + \int_0^t \omega(r(s))ds + v_1(\tau)v_2(0)\cdot e^{-\tau\cdot p(r(0))}\cdot (e^{t\cdot(p(r(0))+\varsigma(r(0))} - 1)\cdot O(1) \end{cases} \tag{5.3.25}$$

One can prove by induction that this formulas hold for all of the successive iterations. In conclusion, the true solution of the system (5.3.23) satisfies:

$$\begin{cases} v_1(t) = v_1(\tau)\cdot e^{(-\tau+t)\cdot p(r(0))} - v_1^2(\tau)v_2(0)\cdot e^{(-2\tau+t)\cdot p(r(0))}\cdot O(1) \\ v_2(t) = v_2(0)\cdot e^{t\cdot\varsigma(r(0))} - v_1(\tau)v_2^2(0)\cdot e^{-\tau\cdot p(r(0))+t\cdot\varsigma(r(0))}\cdot O(e^{t\cdot(p(r(0))+\varsigma(r(0))}) \\ r(t) = r(0) + v_1(\tau)v_2(0)\cdot O(e^{-\tau\cdot p(r(0))}) \\ \phi(t) = \phi(0) + \int_0^t \omega(r(s))ds + v_1(\tau)v_2(0)\cdot e^{-\tau\cdot p(r(0))}\cdot O(1) \end{cases} \tag{5.3.26}$$

This means that (5.3.24), i.e. the solution of the truncated system:

$$\begin{cases} \dot{v_1} = p(r)\cdot v_1 \\ \dot{v_2} = \varsigma(r)\cdot v_2 \\ \dot{r} = 0 \\ \dot{\phi} = \omega(r) \end{cases} \tag{5.3.27}$$

approximates the solution of the primary system (5.3.21) at time τ with the error $v_1(\tau)v_2(0)\cdot O(e^{-\tau\cdot p(r(0))})$.

Note that we assumed $p(r) > 0$. The solution to the latter system (5.3.27) at time τ is given by:

$$\begin{cases} v_1(0) = v_1(\tau)e^{-\tau \cdot p(r(0))} \\ v_2(\tau) = v_2(0)e^{\tau \cdot \varsigma(r(0))} \\ r(\tau) = r(0) \\ \phi(\tau) = \phi(0) + \tau \cdot \omega(r(0)) \end{cases} \tag{5.3.28}$$

We find the solution to (5.3.26) in the following way. The first equation gives:

$$v_1(t) = v_1(\tau)e^{(-\tau+t)\cdot p(r(0))}(1 - v_1(\tau)v_2(0) \cdot e^{-\tau \cdot p(r(0))} \cdot O(1))$$

from which we find

$$v_1(0) = v_1(\tau)e^{-\tau \cdot p(r(0))}(1 - v_1(\tau)v_2(0) \cdot e^{-\tau \cdot p(r(0))} \cdot O(1))$$

and so

$$e^{-\tau \cdot p(r(0))} = \frac{v_1(0)}{v_1(\tau) \cdot (1 - v_1(\tau)v_2(0) \cdot e^{-\tau \cdot p(r(0))} \cdot O(1))}$$

which can be substituted once again in the denominator. Taking into account that $\tau \to +\infty$ as $v_1(0) \to 0$:

$$e^{-\tau \cdot p(r(0))} = \frac{v_1(0)}{v_1(\tau) \cdot (1 - v_1(\tau)v_2(0) \cdot \frac{v_1(0)}{v_1(\tau)\cdot(1-v_1(\tau)v_2(0)e^{-\tau\cdot p(r(0))}\cdot O(1))} \cdot O(1))}$$

In consequence one can rewrite the latter equation up to higher order terms:

$$e^{-\tau \cdot p(r(0))} = \frac{v_1(0)}{v_1(\tau) \cdot (1 - v_2(0)v_1(0) \cdot O(1))}$$

From which we can read the approximated formula for τ:

$$\tau = \frac{-1}{p(r(0))} \cdot \ln\Big(\frac{v_1(0)}{v_1(\tau) \cdot (1 - v_2(0)v_1(0) \cdot O(1))}\Big)$$

which we plug into the equations for $v_2(\tau)$, $r(\tau)$, $\phi(\tau)$

$$v_2(\tau) = v_2(0) \cdot e^{\tau \cdot \varsigma(r(0))} \cdot (1 - v_1(\tau)v_2(0) \cdot e^{\tau \cdot \varsigma(r(0))} \cdot O(1))$$

and obtain

$$\begin{cases} v_2(\tau) = v_2(0) \cdot \left(\frac{v_1(0)}{v_1(\tau)\cdot(1-v_2(0)v_1(0)\cdot O(1))}\right)^{\frac{-\varsigma(r(0))}{p(r(0))}} \cdot (1 - v_1(\tau)v_2(0) \cdot \left(\frac{v_1(0)}{v_1(\tau)\cdot(1-v_2(0)v_1(0)\cdot O(1))}\right)^{\frac{-\varsigma(r(0))}{p(r(0))}} O(1)) + h.o.t \\ r(\tau) = r(0) + \frac{v_1(0)v_2(0)}{(1-v_2(0)v_1(0)\cdot O(1))} \cdot O(1) \\ \phi(\tau) = \phi(0) + \int_0^\tau \omega(r(s))ds + \frac{v_1(0)v_2(0)}{(1-v_2(0)v_1(0)\cdot O(1))} \cdot O(1) + h.o.t. \end{cases} \tag{5.3.29}$$

After expanding $\int_0^\tau \omega(r(s))ds$ with the use of approximation $r(t) \approx r(0) + v_1(\tau)v_2(0) \cdot O(e^{-\tau\cdot p(r(0))})$ (taken from the equation (5.3.26)), we find that

$$\phi(\tau) = \phi(0)+\tau\cdot(\omega(r(0))+\omega'(r(0))\cdot\frac{v_1(0)v_2(0)}{(1-v_2(0)v_1(0)\cdot O(1))}\cdot O(1)+h.o.t.)+\frac{v_1(0)v_2(0)}{(1-v_2(0)v_1(0)\cdot O(1))}\cdot O(1)$$

which finally gives us

$$\begin{cases} v_2(\tau) = v_2(0) \cdot \left(\frac{v_1(0)}{v_1(\tau)\cdot(1-v_2(0)v_1(0)\cdot O(1))}\right)^{\frac{-\varsigma(r(0))}{p(r(0))}} (1 - v_1(\tau)v_2(0) \cdot \left(\frac{v_1(0)}{v_1(\tau)\cdot(1-v_2(0)v_1(0)\cdot O(1))}\right)^{\frac{-\varsigma(r(0))}{p(r(0))}} \cdot O(1)) + h.o.t \\ r(\tau) = r(0) + \frac{v_1(0)v_2(0)}{(1-v_2(0)v_1(0)\cdot O(1))} \cdot O(1) + h.o.t. \\ \phi(\tau) = \phi(0) + \frac{-1}{p(r(0))} \ln\left(\frac{v_1(0)}{v_1(\tau)\cdot(1-v_2(0)v_1(0)\cdot O(1))}\right) \cdot \left(\omega(r(0)) + \frac{\omega' g(r(0))v_1(0)v_2(0)\cdot O(1)}{(1-v_2(0)v_1(0)\cdot O(1))} + h.o.t.\right) + \frac{v_1(0)v_2(0)\cdot O(1)}{(1-v_2(0)v_1(0)\cdot O(1))} \end{cases} \tag{5.3.30}$$

□

Remark 68. *Note that all of the performed transformations were analytic. The local Shilnikov map obviously is analytic as well.*

Assumption 10. *We assume that the global map from the cross section $S_1 = \{v_1 = h\}$ to $S_2 = \{v_2 = h\}$ along the non-transverse homoclinic channel I is analytic.*

Proposition 56. *The truncated global map "glob" from the cross section $S_1 = \{v_1 = h\}$ to $S_2 = \{v_2 = h\}$ along the non-transverse homoclinic channel is given by:*

$$\begin{pmatrix} v_2 \\ r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} a_1(r,\phi)\cdot v_2 \\ b_0(r,\phi) + b_1(r,\phi)\cdot v_2 \\ c_0(r,\phi) + c_1(r,\phi)\cdot v_2) \end{pmatrix} \tag{5.3.31}$$

Proof. Let us expand the map

$$glob : S_1 \ni (v_2, r, \phi) \mapsto (glob_1(v_2, r, \phi),\ glob_2(v_2, r, \phi),\ glob_3(v_2, r, \phi)) \in S_2$$

in terms of v_2 - being very small, that is:

$$glob(v_2, r, \phi) = \begin{pmatrix} a_0(r,\phi) + a_1(r,\phi)\cdot v_2 + h.o.t. \\ b_0(r,\phi) + b_1(r,\phi)\cdot v_2 + h.o.t. \\ c_0(r,\phi) + c_1(r,\phi)\cdot v_2 + h.o.t. \end{pmatrix} \tag{5.3.32}$$

Due to the existence of the homoclinic channel I we have $a_0 = 0$. □

In the following theorem we derive the formula for the return map along the homoclinic channel I:

Theorem 10. *The return map $loc \circ glob : S_1 \to S_1$, in suitable coordinates (z, r, ϕ), with $z := ln(\frac{v_2}{h})$ is given by:*

$$\begin{pmatrix} z \\ r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} \Big(\ln(a_1(r,\phi)) + z\Big)\cdot \frac{-\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))} + O(z\cdot e^z) \\ b_0(r,\phi) + O(e^z) \\ (c_0(r,\phi) - \frac{\omega(b_0(r,\phi))}{p(b_0(r,\phi))}\cdot \ln(a_1(r,\phi))) + \frac{-\omega(b_0(r,\phi))}{p(b_0(r,\phi))}\cdot z + O(z\cdot e^z) \ \ (mod 1) \end{pmatrix} \tag{5.3.33}$$

Proof. In order to compose the local map with the global return map between the sections S_1 and S_2 we need to perform the following substitutions

$$\begin{cases} v_1(0) = a_1(r,\phi)\cdot v_2 + h.o.t. \\ r(0) = b_0(r,\phi) + b_1(r,\phi)\cdot v_2 + h.o.t. \\ \phi(0) = c_0(r,\phi) + c_1(r,\phi)\cdot v_2 + h.o.t. \end{cases} \tag{5.3.34}$$

in the equations (5.3.30), which brings us the following form of the return map:

$$\begin{cases} \bar{v}_2 = h\Big(\frac{a_1(r,\phi)\cdot v_2 + h.o.t.}{h\cdot(1-h(a_1(r,\phi)\cdot v_2)\cdot O(1))}\Big)^{\frac{-\varsigma(b_0(r,\phi)+b_1(r,\phi)\cdot v_2)}{p(b_0(r,\phi)+b_1(r,\phi)\cdot v_2)}}\cdot\Big(1 - h^2(\frac{a_1(r,\phi)\cdot v_2}{h\cdot(1-h(a_1(r,\phi)\cdot v_2)\cdot O(1))})^{\frac{-\varsigma(b_0(r,\phi)+b_1(r,\phi)\cdot v_2)}{p(b_0(r,\phi)+b_1(r,\phi)\cdot v_2)}}\cdot O(1)\Big) + h.o.t \\ \bar{r} = b_0(r,\phi) + b_1(r,\phi)\cdot v_2 + \frac{h\cdot(a_1(r,\phi)\cdot v_2)}{(1-h(a_1(r,\phi)\cdot v_2)\cdot O(1))}\cdot O(1) + h.o.t. \\ \bar{\phi} = c_0(r,\phi) + c_1(r,\phi)\cdot v_2 + \frac{-1}{p(b_0(r,\phi)+b_1(r,\phi)\cdot v_2)}\cdot\ln\Big(\frac{a_1(r,\phi)\cdot v_2}{h\cdot(1-h(a_1(r,\phi)\cdot v_2)\cdot O(1))}\Big)\cdot\omega(b_0(r,\phi)+b_1(r,\phi)\cdot v_2) + \\ \qquad \Big(\frac{-\ln\Big(\frac{a_1(r,\phi)\cdot v_2}{h\cdot(1-h(a_1(r,\phi)\cdot v_2)\cdot O(1))}\Big)}{p(b_0(r,\phi)+b_1(r,\phi)\cdot v_2)}\cdot\omega'\Big(b_0(r,\phi)+b_1(r,\phi)\cdot v_2\Big) + 1\Big)\cdot\frac{h(a_1(r,\phi)\cdot v_2)}{(1-h(a_1(r,\phi)\cdot v_2)\cdot O(1))}\cdot O(1) + ... + h.o.t. \end{cases} \tag{5.3.35}$$

which can be further simplified to

$$\begin{cases} \bar{v}_2 = h\Big(\frac{a_1(r,\phi)\cdot v_2 + h.o.t.}{h\cdot(1-h(a_1(r,\phi)\cdot v_2)\cdot O(1))}\Big)^{\frac{-\varsigma(b_0(r,\phi)+b_1(r,\phi)\cdot v_2)}{p(b_0(r,\phi)+b_1(r,\phi)\cdot v_2)}} + h.o.t \\ \bar{r} = b_0(r,\phi) + b_1(r,\phi)\cdot v_2 + \frac{h\cdot(a_1(r,\phi)\cdot v_2)}{(1-h(a_1(r,\phi)\cdot v_2)\cdot O(1))}\cdot O(1) + h.o.t. \\ \bar{\phi} = c_0(r,\phi) + c_1(r,\phi)\cdot v_2 + \frac{-1}{p(b_0(r,\phi)+b_1(r,\phi)\cdot v_2)}\cdot \ln\Big(\frac{a_1(r,\phi)\cdot v_2}{h\cdot(1-h(a_1(r,\phi)\cdot v_2)\cdot O(1))}\Big)\cdot \omega(b_0(r,\phi)) + h.o.t. \end{cases} \tag{5.3.36}$$

the substitution

$$z := ln\Big(\frac{v_2}{h}\Big), \quad \text{so:} \quad v_2 = h\cdot e^z \quad \text{and} \quad z \approx -\infty \ \text{ for } \quad v_2 \approx 0$$

performed in the equation (5.3.36), together with simplifications, leads us to:

$$\begin{cases} \bar{v}_2 = h\cdot\Big(a_1(r,\phi)\cdot e^z\Big)^{\cdot\frac{-\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))}+O(z\cdot e^z)} + h.o.t \\ \bar{r} = b_0(r,\phi) + b_1(r,\phi)\cdot O(e^z) + he^z\cdot O(1) + h.o.t. \\ \bar{\phi} = c_0(r,\phi) + c_1(r,\phi)\cdot he^z - \frac{\omega(b_0(r,\phi))}{p(b_0(r,\phi))}\cdot\Big(z + \ln(a_1(r,\phi))\Big) + O(z\cdot e^z) + h.o.t. \end{cases} \tag{5.3.37}$$

In conclusion:

$$\begin{cases} \bar{z} = \Big(\ln(a_1(r,\phi)) + z\Big)\cdot\frac{-\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))} + O(z\cdot e^z) \\ \bar{r} = b_0(r,\phi) + O(e^z) \\ \bar{\phi} = \Big(c_0(r,\phi) - \frac{\omega(b_0(r,\phi))}{p(b_0(r,\phi))}\cdot\ln(a_1(r,\phi))\Big) + \frac{-\omega(b_0(r,\phi))}{p(b_0(r,\phi))}\cdot z + O(z\cdot e^z) \quad (\ mod\ 1\ ) \end{cases} \tag{5.3.38}$$

□

We are able to write the return map along the homoclinic channel I in a general and concise way due to the following

Proposition 57. *After rescaling variable z, the return map takes the form:*

$$\begin{pmatrix} z \\ r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} \Omega(r,\phi) + \Gamma(r,\phi)\cdot z + O(z\cdot e^z) \\ b_0(r,\phi) + O(e^z) \\ c(r,\phi) + z + O(z\cdot e^z) \ \ (mod\ 1) \end{pmatrix} \tag{5.3.39}$$

with

$$\Omega(r,\phi) := \frac{\omega(b_0(r,\phi))}{p(b_0(r,\phi))}\cdot\ln(a_1(r,\phi))\cdot\frac{-\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))}$$

$$\Gamma(r,\phi) := \frac{-\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))}$$

$$c(r,\phi) := c_0(r,\phi) - \frac{\omega(b_0(r,\phi))}{p(b_0(r,\phi))} \cdot \ln(a_1(r,\phi))$$

Proof. If we now denote $\alpha(r) := -\frac{\omega(r)}{p(r)}$ and $H(z,r,\phi) := (z \cdot \alpha(r),\ r,\ \phi)$ then

$$H \circ loc \circ glob \circ H^{-1}(z,r,\phi) = \Bigg(\Big(\ln(a_1(r,\phi)) \cdot \alpha(b_0(r,\phi)) + z \Big) \cdot \frac{-\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))} + O(z \cdot e^z),$$

$$b_0(r,\phi) + O(e^z),\ \Big(c_0(r,\phi) - \frac{\omega(b_0(r,\phi))}{p(b_0(r,\phi))} \cdot \ln(a_1(r,\phi)) \Big) + z + O(z \cdot e^z)\ \ (mod\ 1) \Bigg) =$$

$$\Big(\Omega(r,\phi) + \Gamma(r,\phi) \cdot z + O(z \cdot e^z),\ b_0(r,\phi) + O(e^z),\ c(r,\phi) + z + O(z \cdot e^z)\ \ (mod\ 1) \Big)$$

□

In the following two sections, we will investigate more carefully its truncated version.

$$\begin{pmatrix} z \\ r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} \Omega(r,\phi) + \Gamma(r,\phi) \cdot z \\ b_0(r,\phi) \\ c(r,\phi) + z\ \ (mod\ 1) \end{pmatrix} \tag{5.3.40}$$

Proposition 58. *The original return map satisfies condition:*

$$Ret_{S_1}(z,r,\phi) = loc \circ glob(z,r,\phi) + O(z \cdot e^{-|z|})$$

Remark 69. *All of the performed transformations were analytic, so the return map in the* (z,r,ϕ) *coordinates is analytic.*

5.3.3 The limit dynamics of the return map

In this section we will prove that in the special case $-\frac{\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))} = \Gamma(r,\phi) = \Gamma \in \mathbb{N}$, $\Gamma > 1$, the dynamics of the return map are in the limit $v_2 \to 0$ the same as the dynamics of its truncated version. This follows immediately from the theorem 11. Before stating it we need to introduce the objects which we will use in this section.

Let us consider the following mapping ϖ:

$$\varpi:\ \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times S^1 \ni \begin{pmatrix} v_2 \\ z \\ r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} h\cdot\left(\frac{a_1(r,\phi)\cdot v_2}{h}\right)^{\frac{-\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))}} + h.o.t.(v_2) \\ \Omega(r,\phi)+\Gamma(r,\phi)\cdot z + O(v_2\cdot\ln(v_2))\ (mod\ 1) \\ b_0(r,\phi)+O(v_2) \\ c(r,\phi)+z+O(v_2\cdot\ln(v_2))\ \ (mod\ 1\ ) \end{pmatrix} \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times S^1 \tag{5.3.41}$$

Remark 70. *In the definition of the mapping ϖ, the variables v_2 and z are independent.*

Observe that: $\varpi(0,z,r,\phi) = (0,\ loc\circ glob(z,r,\phi))$.

On the other hand, $\varpi(v_2,\ln(\frac{v_2}{h}),r,\phi) = (h\cdot\left(\frac{a_1(r,\phi)\cdot v_2}{h}\right)^{\frac{-\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))}} + h.o.t.(v_2), Ret_{S_1}(\ln(\frac{v_2}{h}),r,\phi))$.

Definition 22. *We say that a foliation $\mathcal{L} = \{h(y)\}$ with C^m leaves ($m \geq 0$) given by the graphs of functions $x = h(y)$ is given in a domain $W = \{(x,y) \mid y \geq 0,\ x \in S^1\times[0,1]\times S^1\}$ if the following conditions are satisfied:*

1. the domain of any function $h(y) \in \mathcal{L}$ is an open connected set in $\mathbb{R}$ and its graph lies entirely in W

2. for any point $(x^0,y^0) \in W$ there exists a unique function $h(y) \in \mathcal{L}$ such that the domain of $h(y)$ contains x^0 and $h(y^0) = x^0$ (we denote this function by $h(y;x^0,y^0)$)

3. the function $h(y;x^0,y^0)$ is a C^m function of y for fixed x^0 and y^0

The graphs of the functions $h(y)$ are called the leaves of $\mathcal{L}$ and are denoted by the same letter h.

Definition 23. *A foliation $\mathcal{L}$ is said to be C^l-smooth ($l \geq 0$) if $h(y;x^0,y^0)$ is a C^l function of $(y;x^0,y^0)$, i.e., all partial derivatives of order $\leq l$ are continuous and uniformly bounded.*

Definition 24. *A foliation $\mathcal{L}$ of W is said to be ϖ- invariant if for any leaf $h \in \mathcal{L}$, $h \neq W_0$, there exists a leaf $\bar{h} \in \mathcal{L}$ such that $\varpi(h) \subseteq \bar{h}$*

Definition 25. *An arbitrary vector function $\mu(x^0,y^0)$ on W is called a field of hyperplanes. If $\mu(x^0,y^0) = \frac{\partial h}{\partial y}(y;x^0,y^0)|_{y=y^0}$ for some $h \in \mathcal{L}$, then the field $\mu(x^0,y^0)$ is called a field of tangent hyperplanes to $\mathcal{L}$.*

Theorem 11. *Let us rewrite*

$$x := (z,r,\phi),\quad y := v_2,\quad (F,G) := \varpi$$

so that:

$$\begin{cases} \bar{x} = F(x,y) \\ \bar{y} = G(x,y) \end{cases} \tag{5.3.42}$$

Assume that $|det(\frac{\partial F}{\partial x}(x,y))|$ *is bounded from below by a positive constant independent of* x *and* y*. Then there exists a* ϖ *- invariant* C^1 *foliation* $\mathcal{L} = \{h(y)\}$ *(satisfying the above definitions) of the space* $W = \{(x,y) \mid y \geq 0,\ x \in S^1 \times [0,1] \times S^1\}$*, with* C^2 *leaves given by the graphs of the functions* $x = h(y)$ *(hence* $(z, r, \phi) = h(v_2)$*).*

Proof. We utilize the method described inter alia in [101], i.e. we find the field of tangent hyperplanes to $\mathcal{L}$. Let us denote the following derivatives

$$\begin{cases} A(x,y) := \frac{\partial F}{\partial x}(x,y) \\ B(x,y) := \frac{\partial F}{\partial y}(x,y) \\ C(x,y) := \frac{\partial G}{\partial x}(x,y) \\ D(x,y) := \frac{\partial G}{\partial y}(x,y) \end{cases} \tag{5.3.43}$$

All of the entries of A are continuous and bounded as a functions r and ϕ, moreover by assumption $|det(A)|$ is bounded from below by a positive constant independent of r, ϕ. Then one can easily find constants $A_3, B_2, C_2, D_2 > 0$ such that the following holds for any x and for any y sufficiently small:

$$\begin{aligned} &||A^{-1}(x,y)|| \leq A_3 \\ &||B(x,y)|| \leq B_2 \cdot |\ln y| \\ &||C(x,y)|| \leq C_2 \cdot |y|^{-\frac{\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))}} \cdot |\ln y| \\ &||D(x,y)|| \leq D_2 \cdot |y|^{-\frac{\varsigma(b_0(r,\phi))+p(b_0(r,\phi))}{p(b_0(r,\phi))}} \end{aligned} \tag{5.3.44}$$

Notice that for any invertible square matrix P, we have $||P^{-1}|| = (\min_{||w||=1} ||Pw||)^{-1}$.
Let μ be a field of hyperplanes. If the following condition is satisfied, then μ is the field of hyperplanes tangent to an invariant foliation of the form $\{x = h(y)\}$.

$$\begin{cases} \frac{dx}{dy} = \mu(x,y) \\ \frac{d\bar{x}}{d\bar{y}} = \mu(\bar{x},\bar{y}) \end{cases} \tag{5.3.45}$$

From (5.3.42) we obtain after differentiation

$$\begin{cases} d\bar{x} = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = (A(x,y) \cdot \mu(x,y) + B(x,y))dy \\ d\bar{y} = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy = (C(x,y) \cdot \mu(x,y) + D(x,y))dy \end{cases} \tag{5.3.46}$$

which leads us to

$$(A(x,y)\cdot\mu(x,y)+B(x,y))=\mu(\bar{x},\bar{y})\cdot(C\cdot\mu(x,y)+D(x,y))$$

and in consequence

$$\mu(x,y)=(A(x,y)-\mu(\bar{x},\bar{y})\cdot C(x,y))^{-1}\cdot(\mu(\bar{x},\bar{y})\cdot D(x,y)-B(x,y))$$

Where $(\bar{x},\bar{y})=(F,G)(x,y)$ and V is the space of all fields of hyperplanes μ such that

1) $\mu(x,y)$ is a continuous vector function in W

2) $\sup_{\{x\in S^1\times[0,1]\times S^1,\ y>0\}}||\frac{\mu(x,y)}{\ln y}||\leq E$ for some $E>0$

3) $\sup_{\{x\in S^1\times[0,1]\times S^1\}}||\frac{\mu(x,y)}{\ln y}||=0$ as $y\to 0$

Space V equipped with a metric

$$d(\mu_1,\mu_2):=\sup_{\{x\in S^1\times[0,1]\times S^1,\ y>0\}}\Big|\Big|\frac{\mu_1(x,y)-\mu_2(x,y)}{\ln y}\Big|\Big|$$

is a complete metric space. Consider the mapping $\Gamma_V: V\to V$ defined as follows:

$$\Gamma_V(\mu)(x,y):=(A(x,y)-\mu(\bar{x},\bar{y})\cdot C(x,y))^{-1}\cdot(\mu(\bar{x},\bar{y})\cdot D(x,y)-B(x,y)) \tag{5.3.47}$$

The fixed points of Γ_V correspond to the field of tangent hyperplanes to an invariant foliation. Moreover, as shown in [101] (lemma 2 in [101]), if $\mu_1,\ \mu_2$ are the fields of tangent hyperplanes corresponding to invariant foliations $\mathcal{L}_1,\ \mathcal{L}_2$, $\mu_1=\Gamma_V(\mu_2)$ and $h_1\in\mathcal{L}_1$, then there exist $h_2\in\mathcal{L}_2$ such that $(F,G)(h_1)\subset h_2$. Let

$$M(y):=\frac{||A^{-1}||^{-1}-||D||-\sqrt{(||A||^{-1}-||D||)^2-4\cdot||B||\cdot||C||}}{2\cdot||C||}$$

Note that $M(y)<\frac{||A^{-1}||^{-1}}{2\cdot||C||}$ and for $|y|$ sufficiently small:

$$M(y)=\frac{2\cdot||B||}{||A^{-1}||^{-1}-||D||+\sqrt{(||A^{-1}||^{-1}-||D||)^2-4\cdot||B||\cdot||C||}}\leq 4\cdot A_3\cdot B_2|\ln y|$$

Hence the ball $K_M:=\{\mu\in V\mid d(\mu,0)\leq d(0,M(y))\}$ is Γ_V invariant and is a complete metric space

with the metric d. The mapping $\Gamma_V|_{K_M}$ is continuous and also is a contraction since:

$$||\Gamma_V(\mu_1) - \Gamma_V(\mu_2)|| = ||(A - \mu_1 \cdot C)^{-1}(\mu_1 \cdot D - B) - (A - \mu_2 \cdot C)^{-1}(\mu_2 \cdot D - B)|| =$$

$$||(A-\mu_1\cdot C)^{-1}(\mu_1\cdot D-B)-(A-\mu_1\cdot C)^{-1}(\mu_2\cdot D-B)+(A-\mu_1\cdot C)^{-1}(\mu_2\cdot D-B)-(A-\mu_2\cdot C)^{-1}(\mu_2\cdot D-B)|| =$$

$$||(A - \mu_1 \cdot C)^{-1}(\mu_1 - \mu_2) \cdot D + ((A - \mu_1 \cdot C)^{-1} - (A - \mu_2 \cdot C)^{-1})(\mu_2 \cdot D - B)|| \leq$$

$$||(A - \mu_1 \cdot C)^{-1}|| \cdot ||(\mu_1 - \mu_2)|| \cdot ||D|| + ||(A - \mu_1 \cdot C)^{-1} \cdot (\mu_1 - \mu_2) \cdot C \cdot (A - \mu_2 \cdot C)^{-1} \cdot (\mu_2 \cdot D - B)|| \leq$$

$$||(A - \mu_1 \cdot C)^{-1}|| \cdot ||(\mu_1 - \mu_2)|| \cdot ||D|| + ||(A - \mu_1 \cdot C)^{-1}|| \cdot ||\mu_1 - \mu_2|| \cdot ||C|| \cdot ||(A - \mu_2 \cdot C)^{-1}|| \cdot ||(\mu_2 \cdot D - B)|| =$$

$$||(\mu_1 - \mu_2)|| \cdot (||(A - \mu_1 \cdot C)^{-1}|| \cdot ||D|| + ||(A - \mu_1 \cdot C)^{-1}|| \cdot ||C|| \cdot ||(A - \mu_2 \cdot C)^{-1}|| \cdot ||(\mu_2 \cdot D - B)||)$$

$$||(\mu_1 - \mu_2)|| \cdot (||(A - \mu_1 \cdot C)^{-1}|| \cdot ||D|| + ||(A - \mu_1 \cdot C)^{-1}|| \cdot ||C|| \cdot ||(A - \mu_2 \cdot C)^{-1}|| \cdot (||\mu_2|| \cdot ||D|| + ||B||))$$

We can estimate the coefficient $||(A - \mu_1 \cdot C)^{-1}||$ as follows

$$||(A - \mu_1 \cdot C)^{-1}|| = (\min_{||w||=1} ||A \cdot w - \mu_1 \cdot C \cdot w||)^{-1} \leq (\min_{||w||=1}(||A \cdot w|| - ||\mu_1 \cdot C \cdot w||))^{-1}$$

$$\leq (\min_{||w||=1} ||A \cdot w|| + \min_{||w||=1}(-||\mu_1 \cdot C \cdot w||))^{-1} \leq (\min_{||w||=1} ||A \cdot w|| - \max_{||w||=1}(||\mu_1 \cdot C \cdot w||))^{-1}$$

$$\leq (||A^{-1}||^{-1} - \max_{||w||=1}(||\mu_1 \cdot C \cdot w||))^{-1} \leq (||A^{-1}||^{-1} - ||\mu_1|| \cdot ||C||)^{-1}$$

$$\leq (A_3^{-1} - M \cdot C_2 \cdot |y|^{-\frac{\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))}} \cdot |\ln y|)^{-1}$$

Hence, we can provide an estimate on the Lipshitz constant of :

$$||(A - \mu_1 \cdot C)^{-1}|| \cdot ||D|| + ||(A - \mu_1 \cdot C)^{-1}|| \cdot ||C|| \cdot ||(A - \mu_2 \cdot C)^{-1}|| \cdot (||\mu_2|| \cdot ||D|| + ||B||)$$

$$\leq \frac{(D_2|y|^{-\frac{\varsigma(b_0(r,\phi))+p(b_0(r,\phi))}{p(b_0(r,\phi))}} + (A_3^{-1} - MC_2|y|^{-\frac{\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))}} \cdot |\ln y|)^{-1}C_2|y|^{-\frac{\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))}}|\ln y| \cdot (MD_2|y|^{-\frac{\varsigma(b_0(r,\phi))+p(b_0(r,\phi))}{p(b_0(r,\phi))}} + B_2|\ln y|)}{A_3^{-1} - M \cdot C_2 \cdot |y|^{-\frac{\varsigma(b_0(r,\phi))}{p(b_0(r,\phi))}} \cdot |\ln y|}$$

so it is smaller than 1 for y - small enough and all z, r, ϕ. Hence, Γ_V possesses a unique fixed point μ_0 in K_M, so

$$\mu_0(x, y) = (A(x, y) - \mu_0(\bar{x}, \bar{y}) \cdot C(x, y))^{-1} \cdot (\mu_0(\bar{x}, \bar{y}) \cdot D(x, y) - B(x, y)) \tag{5.3.48}$$

In consequence $||\mu_0|| = O(\ln y)$ and moreover $||\frac{\partial \mu_0(x,y)}{\partial(x,y)}|| = O(\frac{1}{y})$.

Furthermore, formal differentiation of the equation $\mu = \Gamma_V(\bar{\mu})$ with respect to (x, y) gives the formula

for

$$\frac{\partial\mu(x,y)}{\partial(x,y)} = \Big(A(x,y)-\bar\mu(\bar x,\bar y)\cdot C(x,y)\Big)^{-1}\cdot\Big(\frac{\partial\bar\mu(\bar x,\bar y)}{\partial(\bar x,\bar y)}\cdot\frac{\partial(F,G)(x,y)}{\partial(x,y)}\cdot D(x,y)+\bar\mu(\bar x,\bar y)\cdot\frac{\partial D(x,y)}{\partial(x,y)}-\frac{B(x,y)}{\partial(x,y)}\Big)+$$

$$\Big(A(x,y)-\bar\mu(\bar x,\bar y)\cdot C(x,y)\Big)^{-1}\cdot\Big(\frac{\partial A(x,y)}{\partial(x,y)}-\frac{\partial\bar\mu(\bar x,\bar y)}{\partial(\bar x,\bar y)}\cdot\frac{\partial(F,G)(x,y)}{\partial(x,y)}\cdot C(x,y)-$$

$$-\bar\mu(\bar x,\bar y)\cdot\frac{\partial C(x,y)}{\partial(x,y)}\Big)\cdot\Big(A(x,y)-\bar\mu(\bar x,\bar y)\cdot C(x,y)\Big)^{-1}\cdot(\bar\mu(\bar x,\bar y)\cdot D(x,y)-B(x,y))$$

So the equation above suggests to consider, for given $\bar\mu$, the following mapping $P^{\bar\mu}: N\to N$

$$P^{\bar\mu}(\bar\eta)(x,y) := \Big(A(x,y)-\bar\mu(\bar x,\bar y)\cdot C(x,y)\Big)^{-1}\cdot\Big(\bar\eta(\bar x,\bar y)\cdot\frac{\partial(F,G)(x,y)}{\partial(x,y)}\cdot D(x,y)+\bar\mu(\bar x,\bar y)\cdot\frac{\partial D(x,y)}{\partial(x,y)}-\frac{B(x,y)}{\partial(x,y)}\Big)+$$

$$\Big(A(x,y)-\bar\mu(\bar x,\bar y)\cdot C(x,y)\Big)^{-1}\cdot\Big(\frac{\partial A(x,y)}{\partial(x,y)}-\bar\eta(\bar x,\bar y)\cdot\frac{\partial(F,G)(x,y)}{\partial(x,y)}\cdot C(x,y)-$$

$$-\bar\mu(\bar x,\bar y)\cdot\frac{\partial C(x,y)}{\partial(x,y)}\Big)\cdot\Big(A(x,y)-\bar\mu(\bar x,\bar y)\cdot C(x,y)\Big)^{-1}\cdot(\bar\mu(\bar x,\bar y)\cdot D(x,y)-B(x,y))$$

where N is a metric space of 1×12 vector functions $\bar\eta$, satisfying

1) $\eta(x,y)$ is a continuous vector function in W

2) $\sup_{\{x\in S^1\times[0,1]\times S^1,\ y>0\}}\|y\cdot\eta(x,y)\|\le J$ for some $J>0$

3) $\sup_{\{x\in S^1\times[0,1]\times S^1\}}\|y\cdot\eta(x,y)\|=0$ as $y\to 0$

For fixed $\bar\eta$ the mapping $P^{\bar\mu}$ is pointwise continuous, i.e. $\bar\mu_n\to\bar\mu$ implies $P^{\bar\mu_n}(\bar\eta)\to P^{\bar\mu}(\bar\eta)$, and also a contraction on N, since

$$\|P^{\bar\mu}(\bar\eta_1)-P^{\bar\mu}(\bar\eta_2)\| =$$

$$\Big\|(A-\bar\mu C)^{-1}\cdot\Big((\bar\eta_1-\bar\eta_2)\cdot\frac{\partial(F,G)(x,y)}{\partial(x,y)}\cdot D\Big)-$$

$$-(A-\bar\mu C)^{-1}\cdot\Big((\bar\eta_1-\bar\eta_2)\cdot\frac{\partial(F,G)(x,y)}{\partial(x,y)}\cdot C\Big)\cdot(A-\bar\mu C)^{-1}\cdot(\mu(\bar x,\bar y)\cdot D(x,y)-B(x,y))\Big\|\le$$

$$\|\bar\eta_1-\bar\eta_2\|\cdot\Big(\|(A-\bar\mu C)^{-1}\|\cdot\|\frac{\partial(F,G)(x,y)}{\partial(x,y)}\|\cdot\|D\|+\|(A-\bar\mu C)^{-1}\|^2\cdot\|\frac{\partial(F,G)(x,y)}{\partial(x,y)}\|\cdot\|C\|\cdot(\|\bar\mu\|\cdot\|D\|+\|B\|)\Big)\le$$

$$\|\bar\eta_1-\bar\eta_2\|\cdot\Big((A_3^{-1}-M\cdot C_2\cdot|y|^{-\frac{S(b_0(r,\phi))}{p(b_0(r,\phi))}}\cdot|\ln y|)^{-1}\cdot\|\frac{\partial(F,G)(x,y)}{\partial(x,y)}\|\cdot\|D\|+$$

$$+(A_3^{-1}-M\cdot C_2\cdot|y|^{-\frac{S(b_0(r,\phi))}{p(b_0(r,\phi))}}\cdot|\ln y|)^{-2}\cdot\|\frac{\partial(F,G)(x,y)}{\partial(x,y)}\|\cdot\|C\|\cdot(\|\bar\mu\|\cdot\|D\|+\|B\|)\Big)$$

Hence, the Lipschitz constant is smaller than 1 for y small enough and moreover is independent of $\bar\mu$. In conclusion, $P^{\bar\mu}$ has a unique fixed point for every μ and this proves as in [101] that μ_0 is a C^1 smooth

field of hyperplanes. This is due to the following lemmas (Lemma 5, 6, 7 from [101]):

Lemma 8. *The space N can be equipped with a norm equivalent to the original norm and such that for all $\mu \in V$, operator P^μ are contraction operators with the same contraction factor q.*

Lemma 9. *Let W_1 and W_2 be the metric spaces, and suppose that W_2 is complete. Let a map $Q : W_1 \to W_1$ have a unique fixed point v^* to which any sequence of the form $v^{n+1} = Q \cdot v^n$ is convergent. In addition, let a contraction map $\Omega^v : W_2 \to W_2$ be associated with any element $v \in W_1$, and let the family of maps Ω^v have the following properties:*

1) there exists a common contraction factor for all Ω^v

2) the family of maps Ω^v depends on v continuously, i.e. if $v^n \to v^*$ as $n \to +\infty$, then $\Omega^{v^n}[w] \to \Omega^{v^*}$ as $n \to +\infty$ for any $w \in W_2$

Then the map $R : W_1 \times W_2 \to W_1 \times W_2$ given by the formula $R(v, w) := (Qv, Q^v[w])$ has a unique fixed point (v^*, w^*) and the sequence $(v^{n+1}, w^{n+1}) = R(v^n, w^n)$ is convergent to (v^*, w^*), for an arbitrary initial condition (v^0, w^0).

Lemma 10. *The field μ_0 is a smooth field o hyperplanes tangent to a C^1 foliation $\mathcal{L}_0$ with C^2 leaves.*

Moreover, as shown in [101], μ_0 corresponds to the ϖ - invariant foliation $\mathcal{L}$ of the (v_1, z, r, ϕ) space (lemma 2 in [101]).

The only missing part in our reasoning is to show that in the limit $y = v_2 \to 0$ different leaves of the foliation correspond to different points $(v_2 = 0, z, r, \phi)$. In order to prove this, we need to show that the flow governed by the equation

$$\frac{dx}{dy} = \mu(x, y), \qquad \text{with} \quad y = v_2 \tag{5.3.49}$$

can be continuously extended to the manifold $\{y = 0\}$. The vector field $\mu(x, y)$ is C^1 for $y \neq 0$ and moreover

1) $\mu_0(x, y)$ is a continuous vector function in W

2) $\sup_{\{x,\ y>0\}} \|\frac{\mu_0(x,y)}{\ln y}\| \leq E$ for some $E > 0$

3) $\sup_x \|\frac{\mu_0(x,y)}{\ln y}\| = 0$ as $y \to 0$

Let us perform the substitution

$$s := -(\ln y)^{-1}$$

then

$$\frac{ds}{dy} = y^{-1} \cdot (\ln y)^{-2}$$

which gives

$$\frac{dx}{ds} = \mu(x,y) \cdot y \cdot (\ln y)^2 = -\mu(x, e^{-\frac{1}{s}}) \cdot e^{-\frac{1}{s}} s^{-2} \tag{5.3.50}$$

now, due to $s \to 0$ as $y \to 0$ and

$$\|\mu(x,y) \cdot y \cdot (\ln y)^2\| = O(y \cdot (\ln y)^3)$$

we can extend smoothly the vector field

$$\tilde{\mu}(x,s) := \mu(x, e^{-\frac{1}{s}}) \cdot e^{-\frac{1}{s}} s^{-2}$$

onto the manifold $\{y = 0\} = \{s = 0\}$. For this it suffices to see that the following definitions provide the required C^1 smooth extension of $\tilde{\mu}$:

$$\tilde{\mu}(x,0) := \lim_{s \to 0} \tilde{\mu}(x,s) = 0$$

$$\frac{\partial \tilde{\mu}}{\partial s}(x,0) := \lim_{s \to 0} \frac{\partial \tilde{\mu}}{\partial s}(x,s) = 0$$

since $\frac{\partial \mu}{\partial y} = O(y^{-1})$ and

$$\frac{\partial \tilde{\mu}}{\partial s}(x,s) = \frac{\partial \tilde{\mu}}{\partial y} \cdot \frac{\partial y}{\partial s} = y \cdot (\ln y)^2 (\frac{\partial \mu}{\partial y} \cdot y \cdot \ln y + \mu \cdot \ln y \cdot (2 + \ln y)) = O(y \cdot (\ln y)^5)$$

Now the extension of the flow onto the manifold $\{y = 0\} = \{s = 0\}$ follows from Picard's Theorem, since $\tilde{\mu}(x,s)$ is locally Lipshitz (in particular for s in the neighbourhood of 0, i.e. for $s < 0$ take $\tilde{\mu}(x,s) = -\tilde{\mu}(x,-s)$). □

Although the truncated return map (5.3.40) does not have a form of a Henon-like map in the original coordinates (z, r, ϕ), but for suitable choice of parameters it is conjugated to the Henon-like map.

Proposition 59. *There exist coefficients $\Omega(r,\phi)$, Γ, $b(r,\phi)$, $c(r,\phi)$, such that the mapping*

$$T : \begin{pmatrix} z \\ r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} \Omega(r,\phi) + \Gamma \cdot z \ \ (mod\ 1) \\ b(r,\phi) \\ c(r,\phi) + z \ \ (mod\ 1) \end{pmatrix} \tag{5.3.51}$$

has a fixed point with a normal form, in a suitable coordinates, given by:

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \mapsto \begin{pmatrix} y \\ w \\ x + y - w + A \cdot y^2 + B \cdot y \cdot w + C \cdot w^2 + h.o.t. \end{pmatrix} \tag{5.3.52}$$

with $(C-A)\cdot(A-B+C) > 0$. *Hence, by [40], [41] and [16] there exists arbitrarily small perturbation of this mapping possessing a Lorenz-like attractor.*

Proof. Let us choose $\Omega(r,\phi)$, $\Gamma(r,\phi)$, $b(r,\phi)$, $c(r,\phi)$ of the special form:

$$\Omega(r,\phi) = a_1 r + a_2\phi \cdot (1-\phi) + a_3 r^2$$

$$\Gamma = constant \in \mathbb{N}$$

$$b(r,\phi) = b_1 r + b_2\phi \cdot (1-\phi) + b_3 r^2 + b_4 r\phi \cdot (1-\phi) + b_5(\phi \cdot (1-\phi))^2$$

$$c(r,\phi) = c(r) = c_1 r$$

Under the above assumptions, the map T possesses a fixed point $(0,0,0)$. Moreover, this allows us to treat z variable $(mod\ 1)$. We perform the change of the coordinates as follows:

$$\begin{cases} x := \phi \\ y := z + c(r) \\ w := \Omega(r,\phi) + \Gamma \cdot z + c(b(r,\phi)) \end{cases} \tag{5.3.53}$$

In the coordinates (x,y,w) the map T reads as:

$$T: \begin{pmatrix} x \\ y \\ w \end{pmatrix} \mapsto \begin{pmatrix} y \\ w \\ \Omega(b(r,\phi), c(r)+z) + \Gamma \cdot z + c(b(b(r,\phi), c(r)+z)) \end{pmatrix} \tag{5.3.54}$$

Utilizing the chain rule, we find the expansion of $\bar{w}$ in terms of (x,y,w) around fixed point $(0,0,0)$:

$$\bar{w} = \Omega(b(r,\phi), c(r)+z) + \Gamma \cdot z + c(b(b(r,\phi), c(r)+z)) = T_3(x,y,w) =$$

$$= T_3(0,0,0) + (T_3(0,0,0))'_x \cdot x + (T_3(0,0,0))'_y \cdot y + (T_3(0,0,0))'_w \cdot w + (T_3(0,0,0))''_{xx} \cdot \frac{x^2}{2} + (T_3(0,0,0))''_{yy} \cdot \frac{y^2}{2} +$$

$$+(T_3(0,0,0))''_{ww}\cdot\frac{w^2}{2}+(T_3(0,0,0))''_{xy}\cdot x\cdot y+(T_3(0,0,0))''_{xw}\cdot x\cdot w+(T_3(0,0,0))''_{yw}\cdot y\cdot w+h.o.t.$$

We have $T_3(0,0,0)=0$ and moreover the following conditions are satisfied:

$$(T_3(0,0,0))'_x=1,\ \ (T_3(0,0,0))'_y=1,\ \ (T_3(0,0,0))'_w=-1$$

$$(T_3(0,0,0))''_{xx}=0,\ \ (T_3(0,0,0))''_{xy}=0,\ \ (T_3(0,0,0))''_{xw}=0$$

provided that

$$a_2=-b_1-2\Gamma$$

$$c_1=\frac{(1-a_2)(1+b_1)}{b_2(1+b_1+2\Gamma)}$$

$$a_1=c_1\frac{2\Gamma-b_1-b_1^2}{1+b_1}$$

$$b_5=-\Big(b_2(a_3b_2^2(1+b_1^2(-1+\Gamma)^2-4\Gamma+2b_1(-1+\Gamma)\Gamma+5\Gamma^2)-(-1+b_1)\Gamma((-1+a_5+b_1-a_5b_1)\Gamma+$$

$$b_4(-1+2\Gamma)(1+b_1+2\Gamma))+b_2^2c_3(b_1^4(-1+\Gamma)^2+2b_1(-1+\Gamma)\Gamma+2b_1^3(-1+\Gamma)\Gamma+\Gamma(-2+9\Gamma-8\Gamma^2)+$$

$$b_1^2(1-2\Gamma+2\Gamma^2))+b_2(a_4\Gamma(-1+b_1+2\Gamma-2b_1\Gamma)+b_3(1+b_1^3(-1+\Gamma)^2-2\Gamma-3\Gamma^2+8\Gamma^3+$$

$$b_1^2(1-4\Gamma+3\Gamma^2)+b_1(1-6\Gamma+7\Gamma^2)))))\Big)\Big/\Big((-1+b_1)^2\Gamma^2(1+b_1+2\Gamma)\Big)$$

$$b_3=\Big(-b_1b_4+b1^3b_4+2b_1^3b_2^2c_3+2b_1^4b_2^2c_3+b_1b_4\Gamma-b_1^3b_4\Gamma-4b_2^2c_3\Gamma-2b_1^2b_2^2c_3\Gamma-4b_1^3b_2^2c_3\Gamma-2b_1^4b_2^2c_3\Gamma-2b_4\Gamma^2+$$

$$2b_1b_4\Gamma^2+8b_2^2c_3\Gamma^2+a_4b_1b_2(-1+b_1+\Gamma-b_1\Gamma)-2a_3b_2^2(-1+b_1^2(-1+\Gamma)+2\Gamma+b1\Gamma)\Big)\Big/\Big(2b_2(-1+$$

$$b_1^3(-1+\Gamma)+4\Gamma^2+b_1^2(-1+2\Gamma)+b_1(-1+3\Gamma))\Big)$$

$$b_4=\Big(-b_2(a_4(-1+b_1)(\Gamma+b_1^4(-1+\Gamma)^2\Gamma+b_1^5(-1+\Gamma)^2\Gamma-6\Gamma^3+8\Gamma^5+b_1^3(-1+4\Gamma-3\Gamma^2-3\Gamma^3+2\Gamma^4)+$$

$$b_1(-1+5\Gamma-5\Gamma^2-6\Gamma^3+6\Gamma^4)+b_1^2(-2+8\Gamma-4\Gamma^2-15\Gamma^3+12\Gamma^4))+2b_2(b_1^7c_3(-1+\Gamma)^2\Gamma-$$

$$\Gamma(1+\Gamma)(-1+2\Gamma)^3(2a_3+c_3-2c_3\Gamma)+b_1^2c_3(1-2\Gamma-2\Gamma^2+5\Gamma^3)+b_1^5c_3(-1+3\Gamma-\Gamma^2-4\Gamma^3+2\Gamma^4)+$$

$$b_1^4c_3(-1+3\Gamma+\Gamma^2-11\Gamma^3+8\Gamma^4)+b_1^3c_3(1-3\Gamma+6\Gamma^3-6\Gamma^4+4\Gamma^5)+b_1\Gamma(a_3(2-6\Gamma+8\Gamma^3)+$$

$$c_3(1-9\Gamma+15\Gamma^2+8\Gamma^3-20\Gamma^4)))\Big)\Big/\Big((-1+b_1)(1+b_1)(\Gamma+b_1^4(-1+\Gamma)^2\Gamma+b_1^5(-1+\Gamma)^2\Gamma-2\Gamma^2+$$

$$b_1^3(-1+4\Gamma-3\Gamma^2-3\Gamma^3+2\Gamma^4)+b_1(-1+5\Gamma-7\Gamma^2-4\Gamma^3+10\Gamma^4)+b_1^2(-2+8\Gamma-4\Gamma^2-15\Gamma^3+12\Gamma^4))\Big)$$

If we choose for example:

$$a_3 = 1, \;\; b_1 = -2, \;\; b_2 = -1, \;\; \Gamma = 5$$

then

$$(T_3(0,0,0))''_{yw} = B, \;\; (T_3(0,0,0))''_{yy} = 2 \cdot A, \;\; (T_3(0,0,0))''_{ww} = 2 \cdot C$$

satisfy

$$(C - A) \cdot (A - B + C) > 0$$

□

5.3.4 Further analysis of the return map

In the latter proposition we assumed that $\Gamma \in \mathbb{N}$ in order to compute normal form of the map T and to treat z variable (*mod* 1). The proposition 60 below, provides conditions under which the truncated return map (5.3.40) is partially hyperbolic.

Proposition 60. *For the coefficients $\Omega(r,\phi)$, $\Gamma(r,\phi)$, $b(r,\phi)$, $c(r,\phi)$, such that $\Omega'_r(r,\phi)$, $\Omega'_\phi(r,\phi)$, $\Gamma'_r(r,\phi)$, $\Gamma'_\phi(r,\phi)$, $b'_r(r,\phi)$, $b'_\phi(r,\phi)$, $c'_r(r,\phi)$, $c'_\phi(r,\phi)$ are small enough (uniformly in (r,ϕ)), the mapping T possesses an invariant cone-field*

$$C_{L,(z,r,\phi)} := \left\{ v = v_1 + v_{23} \in T_{(z,r,\phi)}M \mid v_1 = (v_{11}, 0, 0), \;\; v_{23} = (0, v_2, v_3) \text{ such that } \|v_{23}\| < L \cdot |v_1| \right\}$$

where $M := \mathbb{R} \times [0,1] \times S^1$, $L < 1$ and $\|\cdot\|$ denotes the Euclidean norm.

Proof. Let $\Gamma(r,\phi) = \Gamma \in \mathbb{N}$, $\Gamma > 1$. Hence at point $(z,r,\phi) \in M$ the derivative of T is given by

$$D_{(z,r,\phi)}T = \begin{pmatrix} \Gamma & \Omega'_r(r,\phi) & \Omega'_\phi(r,\phi) \\ 0 & b'_r(r,\phi) & b'_\phi(r,\phi) \\ 1 & c'_r(r,\phi) & c'_\phi(r,\phi) \end{pmatrix} \tag{5.3.55}$$

and so, for the $v = (v_1, v_2, v_3) \in C_{L,(z,r,\phi)}$, we have

$$D_{(z,r,\phi)}T \cdot v = (\Gamma \cdot v_1 + \Omega'_r(r,\phi) \cdot v_2 + \Omega'_\phi(r,\phi) \cdot v_3, \;\; b'_r(r,\phi) \cdot v_2 + b'_\phi(r,\phi) \cdot v_3, \;\; v_1 + c'_r(r,\phi) \cdot v_2 + c'_\phi(r,\phi) \cdot v_3)$$

Now the invariance condition for the cone-field $C_{L,(z,r,\phi)}$ means that:

$$\|(\ b'_r(r,\phi)\cdot v_2 + b'_\phi(r,\phi)\cdot v_3,\ v_1 + c'_r(r,\phi)\cdot v_2 + c'_\phi(r,\phi)\cdot v_3\ )\| < L\cdot|\ \Gamma\cdot v_1 + \Omega'_r(r,\phi)\cdot v_2 + \Omega'_\phi(r,\phi)\cdot v_3\ |$$

From the Cauchy-Schwarz inequality we have:

$$|\ \Omega'_r(r,\phi)\cdot v_2 + \Omega'_\phi(r,\phi)\cdot v_3\ | \le \|(v_2,v_3)\|\cdot\sqrt{(\Omega'_r(r,\phi))^2+(\Omega'_\phi(r,\phi))^2} < L\cdot|v_1|\cdot\sqrt{(\Omega'_r(r,\phi))^2+(\Omega'_\phi(r,\phi))^2}$$

choose

$$c_1\in(0,1),\quad L\in\left(0,\frac{c_1\cdot\Gamma}{\sqrt{(\Omega'_r(r,\phi))^2+(\Omega'_\phi(r,\phi))^2}}\right)$$

then

$$|\ \Omega'_r(r,\phi)\cdot v_2 + \Omega'_\phi(r,\phi)\cdot v_3\ | < L\cdot|v_1|\cdot\sqrt{(\Omega'_r(r,\phi))^2+(\Omega'_\phi(r,\phi))^2} < c_1\cdot\Gamma\cdot|v_1|$$

which leads to

$$(1-c_1)\cdot\Gamma\cdot|v_1| < |\ \Gamma\cdot v_1 + \Omega'_r(r,\phi)\cdot v_2 + \Omega'_\phi(r,\phi)\cdot v_3\ |$$

On the other hand:

$$\left\|\left(\ b'_r(r,\phi)\cdot v_2 + b'_\phi(r,\phi)\cdot v_3,\ v_1 + c'_r(r,\phi)\cdot v_2 + c'_\phi(r,\phi)\cdot v_3\ \right)\right\|^2 =$$

$$= v_1^2 + \left(\ b'_r(r,\phi)\cdot v_2 + b'_\phi(r,\phi)\cdot v_3\right)^2 + \left(c'_r(r,\phi)\cdot v_2 + c'_\phi(r,\phi)\cdot v_3\right)^2 + 2\cdot v_1\cdot\left(c'_r(r,\phi)\cdot v_2 + c'_\phi(r,\phi)\cdot v_3\right)$$

$$\le v_1^2+\left((\ b'_r(r,\phi))^2+(b'_\phi(r,\phi))^2+(\ c'_r(r,\phi))^2+(c'_\phi(r,\phi))^2\right)\cdot(v_2^2+v_3^2)+2\cdot v_1\cdot\sqrt{(c'_r(r,\phi))^2+(c'_\phi(r,\phi))^2}\cdot\sqrt{v_2^2+v_3^2}$$

$$< v_1^2\cdot\left(1+\ \left((\ b'_r(r,\phi))^2+(b'_\phi(r,\phi))^2+(\ c'_r(r,\phi))^2+(c'_\phi(r,\phi))^2\right)\cdot L^2+2\cdot L\cdot\sqrt{(c'_r(r,\phi))^2+(c'_\phi(r,\phi))^2}\right)$$

we want the right hand side of the last inequality to be smaller than

$$L^2\cdot(1-c_1)^2\cdot\Gamma^2\cdot v_1^2$$

which in consequence means:

$$1+\ \left((\ b'_r(r,\phi))^2+(b'_\phi(r,\phi))^2+(\ c'_r(r,\phi))^2+(c'_\phi(r,\phi))^2\right)\cdot L^2+2\cdot L\cdot\sqrt{(c'_r(r,\phi))^2+(c'_\phi(r,\phi))^2} <$$

$$< L^2\cdot(1-c_1)^2\cdot\Gamma^2$$

Since $\Gamma > 1$, the last inequality has a positive solution in

$$c_1 \in (0,1), \quad L \in \left(0, \frac{c_1 \cdot \Gamma}{\sqrt{(\Omega_r'(r,\phi))^2 + (\Omega_\phi'(r,\phi))^2}}\right)$$

for

$$\Omega_r'(r,\phi),\ \Omega_\phi'(r,\phi),\ b_r'(r,\phi),\ b_\phi'(r,\phi),\ c_r'(r,\phi),\ c_\phi'(r,\phi)$$

being sufficiently small (uniformly in (r,ϕ)). □

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