From Dissipativity Property to Data-Driven GAS Certificate of Degree-One Homogeneous Networks with Unknown Topology

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***Abstract*— In this work, we propose a data-driven *divide and conquer* strategy for the stability analysis of interconnected nonlinear homogeneous networks of degree one with *unknown models* and an *unknown topology*. The proposed scheme leverages joint dissipativity-type properties of subsystems described by storage functions, while providing a stability certificate over unknown interconnected networks. In our data-driven framework, we begin by formulating the required conditions for constructing storage functions as a robust convex program (RCP). Given that unknown models of subsystems are integrated into one of the constraints of the RCP, we collect data from trajectories of each unknown subsystem and provide a scenario convex program (SCP) that aligns with the original RCP. We solve the SCP as a linear program and construct a storage function for each subsystem with unknown dynamics. Under some newly developed *data-driven compositionality* conditions, we then construct a Lyapunov function for the unknown interconnected network utilizing storage functions derived from data of individual subsystems. We show that our data-driven divide and conquer strategy provides correctness guarantees (as opposed to probabilistic confidence) while significantly mitigating the sample complexity problem existing in data-driven approaches. To illustrate the effectiveness of our proposed results, we apply our approaches to three different case studies involving interconnected homogeneous (nonlinear) networks with unknown models. We collect data from trajectories of unknown subsystems and verify the global asymptotic stability (GAS) of the interconnected system with a correctness guarantee.**

***Index Terms*— Data-driven GAS certificate, compositional dissipativity approach, storage and Lyapunov functions, robust convex program, scenario convex program, formal methods**

I. Introduction

In recent years, data-driven analysis of large-scale networks has become increasingly prevalent, driven by their extensive applications in real-world safety-critical systems. Notably, many complex and heterogeneous systems lack closed-form mathematical models that are either unavailable or too intricate to be practically useful. As a result, the model-based techniques proposed in relevant literature cannot effectively analyze such complex networks. While there are *indirect* data-driven approaches based on identification techniques to analyze unknown systems by learning their approximate models (see *e.g.,* [1]–[5]), obtaining accurate models for complex systems remains highly challenging, time-consuming, and expensive. Furthermore, in the realm of stability analysis, even if a model can be identified using system identification techniques, there remains the necessity to find a Lyapunov function for the acquired model to demonstrate stability properties. Consequently, computational complexity arises in both stages: identifying the model and constructing a Lyapunov function. Therefore, it is crucial to develop *direct* data-driven approaches to bypass the system identification phase and directly utilize system measurements for conducting the underlying analysis [6] (cf. Fig. 1).

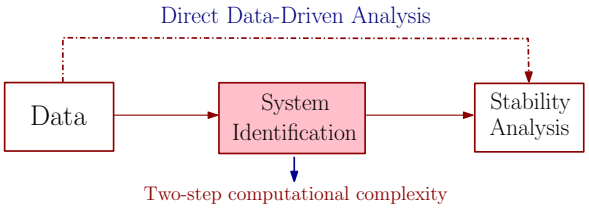


Fig. 1. Indirect (*i.e.,* system identification) vs. direct data-driven techniques.

Over the past two decades, several advancements have emerged in the realm of data-driven optimization techniques. Notably, the scenario approach was initially introduced in [7] for addressing robust control analysis and synthesis problems through semi-infinite convex programming. By introducing a probabilistic relaxation of the worst-case robust paradigm, the robust control problem outlined in [7] can be tackled via random sampling of constraints. Building upon the foundation laid in [7], the work [8] presents a framework for random convex programs, offering a key advantage over [7] by establishing an explicit bound on the upper tail probability of violation. Furthermore, this framework is extended to encompass random convex programs with posteriori violated constraints, aimed at enhancing the optimal objective value while effectively managing the violation probability.

An initial probabilistic connection between the optimal value of scenario convex programs and that of robust convex and chance-constrained programs is introduced in [9], where

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the presented findings are also extended to a specific class of non-convex problems involving binary decision variables. Furthermore, [10] presents a constructive framework for establishing a relationship between an infinite-dimensional linear programming and a manageable finite convex program. In particular, the proposed results in [10] are founded on randomized optimization and first-order methods, offering both a priori and posteriori performance guarantees.

In recent years, there have been notable advancements in the formal stability analysis of unknown dynamical systems through data-driven approaches. In this respect, [11] presents a data-driven stability analysis of black-box switched linear systems by constructing a common Lyapunov function, offering a stability guarantee based on both the number of observations and a required level of confidence. Building upon this work, [12] extends the approach to propose a data-driven stability analysis of switched linear systems using sum of squares Lyapunov functions. Data-driven stability verification of homogeneous systems with a probabilistic level of confidence is proposed in [13]. Additionally, [14] introduces a data-driven approach for the stability analysis of continuous-time unknown systems. In [15], a parameterization scheme for linear-feedback systems utilizing data-dependent linear matrix inequalities is introduced, extending the stabilization problem to include output-feedback control design. Meanwhile, [16] presents a data-enabled predictive control algorithm for unknown stochastic linear systems. This approach utilizes noise-corrupted input/output data to predict future trajectories and compute optimal control policies. In [17], H_∞ performance analysis and distributed controller synthesis for interconnected linear systems were developed using noisy input–state data.

In contrast to existing literature, we introduce a data-driven approach for the stability analysis of *large-scale networks* with unknown models and interconnected topology, employing a divide and conquer strategy. While previous works such as [11]–[14] focus solely on monolithic systems, they encounter significant challenges, known as *sample complexity*, particularly in handling high-dimensional systems. This complexity manifests in the *exponential* growth of the data required to provide stability certificates as the state-space dimension increases. In contrast, utilizing our divide and conquer strategy, we break down the sample complexity to the level of subsystems, where the number of data grows *linearly* with the number of subsystems. Furthermore, while the stability guarantees in [11]–[14] are based on probabilistic confidence levels, our stability results offer a guaranteed confidence of 1.

Our primary contribution is to develop a data-driven compositional approach for analyzing the stability of interconnected homogeneous nonlinear networks, where each subsystem is homogeneous of degree one with unknown models. The proposed framework capitalizes on the joint dissipativity-type properties of subsystems, characterized by homogeneous storage functions of degree 2. Within our data-driven context, we collect data from trajectory of each unknown subsystem and devise a scenario convex program (SCP) to enforce the required conditions of storage functions. By solving the SCP, we construct a data-driven storage function for each unknown subsystem. Subsequently, we construct a Lyapunov function for the unknown interconnected network with a correctness guarantee based on storage functions of individual subsystems, under a newly developed compositionality condition derived from data. We show that our innovative data-driven compositional approach eliminates the necessity to verify the traditional compositional condition based on dissipativity theory that requires the interconnected topology (cf. [18, Proposition 2.1]).

In the scenario where the interconnection topology is known, we provide a sub-result where we initially establish a storage function for each unknown subsystem based on data. This is achieved by solving an SCP for each unknown subsystem with a correctness guarantee. Subsequently, we examine the traditional dissipativity compositionality condition to determine its satisfaction. This condition is represented as a linear matrix inequality (LMI) that interrelates individual storage functions. To avoid potential posteriori checks for this condition, we employ an alternating direction method of multipliers (ADMM) algorithm [19], enabling efficient satisfaction of the compositionality condition in a distributed manner, while searching for storage functions of individual subsystems.

A data-driven stability analysis of homogeneous networks was previously presented in [20]. Our paper offers several distinct advantages compared to [20]. First and foremost, while [20] provides stability guarantees based on ISS properties of subsystems, our focus here is on deriving stability certificates for interconnected networks through the dissipativity properties of individual subsystems. Dissipativity is a broader framework that can simplify the conditions required for stability analysis. Specifically, our approach eliminates the need for the first radially unbounded condition in the ISS Lyapunov function (see [20, condition (6a)]), allowing us to achieve stability with fewer constraints. Additionally, our proposed method involves a convex optimization program. In contrast, the optimization approach used in [20] involves bilinearity, which can be more complex and less tractable. This shift to a convex optimization framework in our work potentially offers computational advantages and makes our method more accessible for practical implementation.

Another key difference is in the treatment of the interconnection topology. While [20] assumes that the interconnection topology is known, our work considers the more challenging scenario where the topology is unknown, a situation common in many real-world applications. Given this unknown topology, one cannot directly verify the compositionality condition in [20, circularity condition (9)]. Instead, we introduce a more relaxed compositionality condition that can be validated during the optimization process using data, thus circumventing the need for prior knowledge of the network topology. Moreover, our approach reduces computational overhead by integrating the verification of the compositionality condition directly into the optimization problem. In contrast, the method in [20] requires posterior checks of the compositionality condition after solving the optimization problem and constructing ISS gains, which can involve computational costs (see [20, circularity condition (9)]). Finally, we employ an algorithm to estimate the Lipschitz constant from data, whereas [20] relies

on analytical computation based on specific system knowledge (see [20, Lemma 5.4]). Our data-driven approach simplifies the process and avoids the need for system knowledge.

While the studies [21]–[23] investigate dissipativity properties based on data, they all focus on *monolithic systems*, which face challenges with sample complexity when applied to large-scale dynamics. In contrast, our approach considers a network of thousands of subsystems, and aims to provide stability guarantees for the overall network by leveraging the dissipativity properties of individual subsystems. In particular, we assume that each subsystem is dissipative with respect to a supply rate (cf. condition (III.4)), meaning that the energy consumed by a subsystem exceeds the energy it generates. We design these supply rates for each subsystem and use them to extend stability guarantees across the interconnected network by verifying the compositionality condition in (V.1b), constructed from the supply rates derived from data.

The remainder of the paper is organized as follows. Section II introduces mathematical notations, and formally defines a discrete-time nonlinear network. In Section III, we present a stability certificate for interconnected networks based on dissipativity-type properties of subsystems described by a notion of storage functions. Section IV outlines our data-driven scheme to construct storage functions for unknown subsystems from data. In Section V, we construct a Lyapunov function for the interconnected system based on storage functions of individual subsystems, derived from data with a correctness guarantee. Section VI discusses the use of an ADMM algorithm to efficiently satisfy the compositionality condition in a distributed manner in the case of known interconnection topologies. In Section VII, we demonstrate our data-driven approaches over different networks with unknown dynamics. Finally, Section VIII concludes the paper.

II. Problem Description

A. Notation

We employ the following notation throughout the paper. Sets of real, positive, and non-negative real numbers are denoted by $\mathbb{R}, \mathbb{R}^+$, and $\mathbb{R}_0^+$, respectively. We denote sets of non-negative and positive integers by $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{N}^+ = \{1, 2, \dots\}$, respectively. Given N vectors $x_i \in \mathbb{R}^{n_i}$, $x = [x_1; \dots; x_N]$ denotes the corresponding column vector of dimension $\sum_i n_i$. We represent the Euclidean norm of a vector $x \in \mathbb{R}^n$ by $\|x\|$, while the absolute value of $a \in \mathbb{R}$ is denoted by $|a|$. The Frobenius norm of a matrix $P \in \mathbb{R}^{n\times m}$ is denoted by $\|P\|_F$. Given sets $X_i, i \in \{1, \dots, N\}$, their Cartesian product is denoted by $\prod_{i=1}^N X_i$. We define an $n \times n$ identity matrix by $\mathbb{I}_n$, while an $n \times m$ zero matrix is denoted by $\mathbf{0}_{n\times m}$. A function $\eta\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to be a class $\mathcal{K}$ function if it is continuous, strictly increasing, and $\eta(0) = 0$. A function $\eta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\eta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed $r > 0$, the map $\eta(r, s)$ is decreasing with respect to s, and $\eta(r, s) \to 0$ as $s \to \infty$.

B. Discrete-Time Nonlinear Systems

In this work, we consider discrete-time nonlinear systems, as formalized in the following definition.

Definition 2.1: A discrete-time nonlinear system (dt-NS) is characterized by the tuple

$$\Sigma = (\mathbb{R}^n, f), \tag{II.1}$$

where:

- $\mathbb{R}^n$ is the state space of the system;
- $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function governing the system's state evolution, assumed to be homogeneous of degree one, *i.e.,* for any $\lambda > 0$ and $x \in \mathbb{R}^n$, $f(\lambda x) = \lambda f(x)$ [24, Definition 1.1]. We presume the map f remains unknown to us.

Given an initial state $x(0) \in \mathbb{R}^n$, the evolution of the state of Σ can be characterized as

$$\Sigma\colon x(k+1) = f(x(k)), \qquad k \in \mathbb{N}. \tag{II.2}$$

In this work, our aim is to verify the global asymptotic stability of Σ concerning its equilibrium point, as defined below.

Definition 2.2: A dt-NS Σ is deemed globally asymptotically stable (GAS) with respect to the equilibrium point $x = 0$ if

$$\|x(k)\| \leq \eta(\|x(0)\|, k),$$

for all $x(0) \in \mathbb{R}^n$ and $\eta \in \mathcal{KL}$, indicating the convergence of all solutions of Σ to the origin as $k \to \infty$.

In the following section, we analyze stability of the interconnected dt-NS by studying dissipativity properties of individual subsystems.

III. Stability Certificate via Dissipativity Approach

Here, we investigate the stability of interconnected networks by considering dissipativity-type properties of subsystems described by storage functions [18]. To achieve this, we represent discrete-time nonlinear *subsystems* as

$$\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{p_i}, f_i), \tag{III.1}$$

where $\mathbb{R}^{n_i}$ is the state set of the subsystem, $\mathbb{R}^{p_i}$ is the *internal input* set of the subsystem (which is used for the sake of interconnection), and $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{p_i} \to \mathbb{R}^{n_i}$ is a continuous function which is homogeneous of degree one, *i.e.,* for any $\lambda > 0$ and $x_i \in \mathbb{R}^{n_i}, w_i \in \mathbb{R}^{p_i}$, $f_i(\lambda x_i, \lambda w_i) = \lambda f_i(x_i, w_i)$. Given an initial state $x_i(0) \in \mathbb{R}^{n_i}$ and input sequence $w_i(\cdot)\colon \mathbb{N} \to \mathbb{R}^{p_i}$, the evolution of the state of Σ_i is characterized as

$$\Sigma_i\colon x_i(k+1) = f_i(x_i(k), w_i(k)), \qquad k \in \mathbb{N}. \tag{III.2}$$

Given $\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{p_i}, f_i), i \in \{1, \dots, M\}$, with a matrix $\mathcal{M}$ being the coupling between subsystems, the interconnection of Σ_i is $\Sigma = (\mathbb{R}^n, f)$ as in (II.1), denoted by $\mathcal{I}(\Sigma_1, \dots, \Sigma_M)$, where $n = \sum_{i=1}^M n_i$ and $f(x) := [f_1(x_1, w_1); \dots; f_M(x_M, w_M)]$, subjected to the following interconnection constraint:

$$\begin{bmatrix} w_1; \cdots; w_M \end{bmatrix} = \mathcal{M} \begin{bmatrix} x_1; \cdots; x_M \end{bmatrix}. \tag{III.3}$$

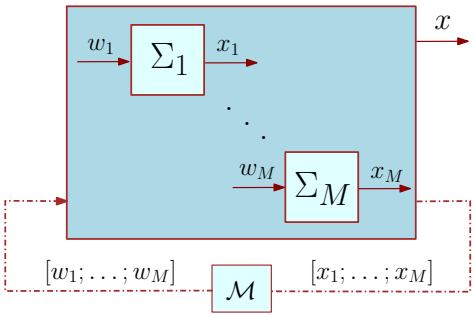


Fig. 2. Interconnected dt-NS $\mathcal{I}(\Sigma_1,\ldots,\Sigma_M)$.

As all subsystems $\Sigma_i, i \in \{1,\ldots,M\}$, are homogeneous of degree one, the interconnected dt-NS Σ maintains homogeneity with the same degree. The schematic representation of an interconnected dt-NS Σ is illustrated in Fig. 2.

To establish the stability certificate of interconnected dt-NS based on dissipativity reasoning, we initially define a concept of storage function for each subsystem.

Definition 3.1: Given a subsystem $\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{p_i}, f_i)$, a positive-definite function $\mathcal{S}_i : \mathbb{R}^{n_i} \to \mathbb{R}_0^+$, *i.e.,* $\mathcal{S}_i(0) = 0$ and $\mathcal{S}_i(x_i) > 0$ for any $x_i \in X_i \backslash 0$, is called a storage function (SF) for Σ_i if there exists a *symmetric* matrix $\mathcal{X}_i$ with partitions $\mathcal{X}_i^{jj'}$, $j, j' \in \{1,2\}$, such that:

$$\forall x_i \in \mathbb{R}^{n_i}, \forall w_i \in \mathbb{R}^{p_i}:$$
$$\mathcal{S}_i(f_i(x_i,w_i)) - \mathcal{S}_i(x_i) \le \begin{bmatrix} w_i \\ x_i \end{bmatrix}^\top \underbrace{\begin{bmatrix} \mathcal{X}_i^{11} & \mathcal{X}_i^{12} \\ \mathcal{X}_i^{21} & \mathcal{X}_i^{22} \end{bmatrix}}_{\mathcal{X}_i} \begin{bmatrix} w_i \\ x_i \end{bmatrix}, \quad \text{(III.4)}$$

with $\mathcal{X}_i^{21} = \mathcal{X}_i^{12\top}$. The expression on the right-hand side of (III.4) is referred to as the *supply rate* of the system. We assume that storage functions $\mathcal{S}_i$ are homogeneous with degree 2, *i.e.,* for any $\lambda > 0$ and $x_i \in \mathbb{R}^{n_i}$, $\mathcal{S}_i(\lambda x_i) = \lambda^2 \mathcal{S}_i(x_i)$.

The following theorem, adapted from [18, Proposition 2.1], offers the stability guarantee for the interconnected dt-NS based on dissipativity reasoning.

Theorem 3.2: Consider an interconnected dt-NS $\Sigma = \mathcal{I}(\Sigma_1,\ldots,\Sigma_M)$ composed of $M \in \mathbb{N}^+$ individual subsystems Σ_i and a coupling matrix $\mathcal{M}$ among them. Suppose that each Σ_i admits an SF $\mathcal{S}_i$, as presented in Definition 3.1. If

$$\begin{bmatrix} \mathcal{M} \\ \mathbb{I} \end{bmatrix}^\top \mathcal{X}_{cmp} \begin{bmatrix} \mathcal{M} \\ \mathbb{I} \end{bmatrix} \preceq 0, \quad \text{(III.5)}$$

$$\text{with} \quad \mathcal{X}_{cmp} := \begin{bmatrix} \mathcal{X}_1^{11} & & & \mathcal{X}_1^{12} & & \\ & \ddots & & & \ddots & \\ & & \mathcal{X}_M^{11} & & & \mathcal{X}_M^{12} \\ \mathcal{X}_1^{21} & & & \mathcal{X}_1^{22} & & \\ & \ddots & & & \ddots & \\ & & \mathcal{X}_M^{21} & & & \mathcal{X}_M^{22} \end{bmatrix},$$

then $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}_0^+$ in the form of

$$\mathcal{V}(x) := \sum_{i=1}^{M} \mathcal{S}_i(x_i) \quad \text{(III.6)}$$

is a Lyapunov function for the interconnected system $\Sigma = \mathcal{I}(\Sigma_1,\ldots,\Sigma_M)$, satisfying the following two conditions:

$$\forall x \in \mathbb{R}^n \backslash \{0\}: \quad \mathcal{V}(x) > 0, \quad \text{(III.7a)}$$
$$\forall x \in \mathbb{R}^n \backslash \{0\}: \quad \mathcal{V}(f(x)) < \mathcal{V}(x). \quad \text{(III.7b)}$$

As a consequence, the interconnected system $\Sigma = \mathcal{I}(\Sigma_1,\ldots,\Sigma_M)$ is GAS with respect to $x = 0$, according to Definition 2.2.

Remark 3.3: Note that if the interconnection topology $\mathcal{M}$ in the model-based analysis is unknown, the only way to verify the compositionality condition in (III.5) is to ensure that the composed supply rate $\mathcal{X}_{\text{cmp}}$ is negative semi-definite, which is a highly conservative requirement. However, as demonstrated later in Section IV, our data-driven approach can handle networks with an unknown topology $\mathcal{M}$ without requiring such a restrictive condition on $\mathcal{X}$ (cf. (V.1b)).

In this work, our focus lies in constructing storage functions $\mathcal{S}_i$ to locally satisfy condition (III.4) over the unit sphere $\|(x_i, w_i)\| = 1, i \in \{1,\ldots,M\}$. We then transfer the results for the interconnected network over the $\|x\| = 1$. Thanks to the homogeneity property of the maps f_i and storage functions $\mathcal{S}_i$, which ensure the homogeneity of the interconnected network, thereby the stability certificate can be globally guaranteed in $\mathbb{R}^n$.

Remark 3.4: To demonstrate that the Lyapunov function $\mathcal{V}$ for the interconnected network can be constructed using the storage functions $\mathcal{S}_i$ of individual subsystems over the unit sphere, it is convenient to ensure that $\|x\| = 1 \Leftrightarrow \|(x_i, w_i)\| = 1$. To this end, we assume that the state measurement of all subsystems in the interconnection topology is available, which is standard in compositional analysis. Now by viewing the global state vector as a coordinate decomposition and considering $w_i = [x_1;\ldots;x_{i-1};x_{i+1};\ldots;x_M]$ for the purpose of analysis, one can readily ensure that $\|(x_i, w_i)\| = 1 \Leftrightarrow \|x\| = 1$. It is important to note that while w_i is expressed in terms of the states of all other subsystems, this construction is only used to characterize the unit sphere and does not redefine the physical interconnection. In particular, the actual interconnection affecting the subsystem dynamics is still governed by the interconnection matrix $\mathcal{M}$ in (III.3), whose nonzero entries determine the interconnection topology.

Remark 3.5: It is worth noting that while the assumption of homogeneity is well-established in control theory and allows for the extension of the stability certificate globally to $\mathbb{R}^n$, it might limit the class of systems considered. Relaxing this condition would enable the consideration of a broader class of systems; however, it would come at the cost of providing only *local*, rather than global, stability certificates. We would like to emphasize that homogeneity of degree one is not restricted only to linear systems, even though all linear systems are homogeneous of degree one. We refer readers to the third case study in Subsection VII-D, where we demonstrate that our framework can successfully handle a nonlinear homogeneous system of degree one.

Given the unknown dynamics of each subsystem f_i in our setting, it is impractical to search for storage functions $\mathcal{S}_i$ that fulfill condition (III.4). Moreover, since the interconnection

topology $\mathcal{M}$ is unknown, directly verifying the compositionality condition in (III.5) is not feasible. To address these challenges, we propose a direct data-driven approach that introduces a novel compositional condition based on data to construct a Lyapunov function for unknown interconnected dt-NS in the form of (III.6) without checking the traditional dissipativity condition (III.5). This Lyapunov function satisfies conditions (III.7a)-(III.7b) and allows us to formally verify the GAS property of unknown interconnected dt-NS.

We now articulate the central problem that this work aims to solve.

Problem 3.6: Consider an interconnected dt-NS $\Sigma = \mathcal{I}(\Sigma_1, \dots, \Sigma_M)$ with an unknown topology, composed of $M \in \mathbb{N}^+$ individual subsystems Σ_i with unknown dynamics. Develop a compositional data-driven scheme to construct storage functions $\mathcal{S}_i$ for unknown subsystems while providing a GAS certificate over interconnected dt-NS with correctness guarantees.

IV. Data-Driven Construction of Storage Functions

In the context of our work, we establish the structure of storage functions as

$$\mathcal{S}_i(q_i, x_i) = \sum_{j=1}^{r} q_i^j p_i^j(x_i), \tag{IV.1}$$

with user-defined homogeneous basis functions $p_i^j(x_i)$ and unknown variables $q_i = [q_i^1; \dots; q_i^r] \in \mathbb{R}^r$.

Assumption 4.1: We assume that the parametric storage function in (IV.1) is continuously differentiable and homogeneous of degree 2. Additionally, the unknown coefficients q_i are assumed to lie within a compact set.

We begin by formulating the required conditions for constructing storage functions in Definition 3.1 as the following robust convex program (RCP):

$$\begin{aligned}
\min_{[g_i;\mu_{R_i};\delta_{R_i}]} \quad & \mu_{R_i} + \delta_{R_i}, \\
\text{s.t.} \quad & \forall x_i \in \mathbb{R}^{n_i}\colon \|x_i\| = 1, \\
& -\mathcal{S}_i(q_i, x_i) \leq \mu_{R_i}, && \text{(IV.2a)} \\
& \forall (x_i, w_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^{p_i}\colon \|(x_i, w_i)\| = 1, \\
& \mathcal{S}_i(q_i, f_i(x_i, w_i)) - \mathcal{S}_i(q_i, x_i) \\
& - \begin{bmatrix} w_i \\ x_i \end{bmatrix}^\top \begin{bmatrix} \mathcal{X}_i^{11} & \mathcal{X}_i^{12} \\ \mathcal{X}_i^{21} & \mathcal{X}_i^{22} \end{bmatrix} \begin{bmatrix} w_i \\ x_i \end{bmatrix} \leq \mu_{R_i}, && \text{(IV.2b)} \\
& \begin{bmatrix} w_i \\ x_i \end{bmatrix}^\top \begin{bmatrix} \mathcal{X}_i^{11} & \mathcal{X}_i^{12} \\ \mathcal{X}_i^{21} & \mathcal{X}_i^{22} \end{bmatrix} \begin{bmatrix} w_i \\ x_i \end{bmatrix} \leq \delta_{R_i}, && \text{(IV.2c)} \\
& g_i = [\mathcal{X}_i^{11}; \mathcal{X}_i^{12}; \mathcal{X}_i^{22}; q_i^1; \dots; q_i^r],\ \mu_{R_i}, \delta_{R_i} \in \mathbb{R}.
\end{aligned}$$

To ensure the well-posedness of the RCP, we assume that all decision variables within vector g_i are subject to compact constraints. We denote the optimal value of RCP by $\mu_{R_i}^* + \delta_{R_i}^*$. If $\mu_{R_i}^* < 0$, a solution to the RCP implies the fulfillment of conditions outlined in Definition 3.1, thus establishing $\mathcal{S}_i$ as a storage function for subsystems (III.2).

Remark 4.2: It is worth noting that since condition (IV.2c) can be rewritten as

$$\begin{bmatrix} w_i \\ x_i \end{bmatrix}^\top \underbrace{\begin{bmatrix} \mathcal{X}_i^{11} & \mathcal{X}_i^{12} \\ \mathcal{X}_i^{21} & \mathcal{X}_i^{22} \end{bmatrix}}_{\mathcal{X}_i} \begin{bmatrix} w_i \\ x_i \end{bmatrix} \leq \delta_{R_i} \underbrace{\begin{bmatrix} w_i \\ x_i \end{bmatrix}^\top \begin{bmatrix} w_i \\ x_i \end{bmatrix}}_{\|(x_i, w_i)\|^2 = 1}, \tag{IV.3}$$

the satisfaction of this condition is equivalent to the satisfaction of $\mathcal{X}_i \leq \delta_{R_i} \mathbb{I}_{p_i+n_i}$. Since δ_{R_i} is a free scalar decision variable, this condition does not impose intrinsic structural restrictions on $\mathcal{X}_i$ at the individual subsystem level. Our compositional data-driven results provide a condition on the summation of δ_{R_i}, ensuring that the supply rates of subsystems can influence one another and compensate for potential adverse effects within the interconnection topology (cf. (V.1b)).

Addressing the proposed RCP in (IV.2) presents two significant challenges. Firstly, this RCP comprises infinitely many constraints due to the continuous nature of the state and internal input of the subsystem. Secondly, and more dauntingly, solving the RCP in (IV.2b) requires precise knowledge of the model f_i, which is unknown in our setting. These challenges motivated us to devise a data-driven approach for constructing storage functions, bypassing the direct solving of the RCP in (IV.2). To accomplish this, we collect a set of two-consecutive sampled data from trajectories of unknown subsystems, represented as pairs $((\tilde{x}_i^z, \tilde{w}_i^z), f_i(\tilde{x}_i^z, \tilde{w}_i^z)), z \in \{1, \dots, \mathcal{N}_i\}$. In our proposed data-driven approach, the first step involves projecting all the data onto a unit sphere by normalizing them with respect to $\|(\tilde{x}_i^z, \tilde{w}_i^z)\|$, denoted as:

$$((\hat{x}_i^z, \hat{w}_i^z), f_i(\hat{x}_i^z, \hat{w}_i^z)) = \frac{((\tilde{x}_i^z, \tilde{w}_i^z), f_i(\tilde{x}_i^z, \tilde{w}_i^z))}{\|(\tilde{x}_i^z, \tilde{w}_i^z)\|}. \tag{IV.4}$$

Remark 4.3: We assume that the collected data satisfy

$$\|\tilde{x}_i^z\| \neq 0 \quad \text{for all } z \in \{1, \dots, \mathcal{N}_i\}. \tag{IV.5}$$

This assumption excludes the origin from the sampling set and ensures that the normalized data $(\hat{x}_i^z, \hat{w}_i^z)$ are well defined, as it guarantees that $\|(\tilde{x}_i^z, \tilde{w}_i^z)\| \neq 0$ in (IV.4). Moreover, it avoids trivial samples with $\hat{x}_i^z = 0$, for which $\mathcal{S}_i(q_i, \hat{x}_i^z) = 0$, and which would otherwise prevent the feasibility of the SCP defined in (IV.7) when aiming for $\mu_{\mathcal{N}_i} < 0$.

According to Remark 3.4 and to ensure the well-posedness of our setting over the unit sphere, we consider $\tilde{w}_i^z = [\tilde{x}_1^z; \dots; \tilde{x}_{i-1}^z; \tilde{x}_{i+1}^z; \dots; \tilde{x}_M^z]$. Subsequently, we compute the maximum distance between any points on the unit sphere and the set of data points as follows:

$$\begin{aligned}
\varepsilon_i = \max_{(x_i, w_i)} \min_{z} & \|(x_i, w_i) - (\hat{x}_i^z, \hat{w}_i^z)\|, \\
& (x_i, w_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^{p_i}\colon \|(x_i, w_i)\| = 1.
\end{aligned} \tag{IV.6}$$

Taking into account $(\hat{x}_i^z, \hat{w}_i^z) \in \mathbb{R}^{n_i} \times \mathbb{R}^{p_i}$, where $z \in \{1, \dots, \mathcal{N}_i\}$, with $\mathcal{N}_i$ being the number of samples, we introduce the following scenario convex program (SCP):

$$
\begin{aligned}
\min_{[g_i;\mu_{\mathcal{N}_i};\delta_{\mathcal{N}_i}]} \quad & \mu_{\mathcal{N}_i}+\delta_{\mathcal{N}_i},\\
\text{s.t.} \quad & \forall(\hat{x}_i^z,\hat{w}_i^z)\in\mathbb{R}^{n_i}\times\mathbb{R}^{p_i},\forall z\in\{1,\dots,\mathcal{N}_i\},\\
& -\mathcal{S}_i(q_i,\hat{x}_i^z)\ \le \mu_{\mathcal{N}_i}, \qquad \text{(IV.7a)}\\
& \mathcal{S}_i(q_i,f_i(\hat{x}_i^z,\hat{w}_i^z))-\mathcal{S}_i(q_i,\hat{x}_i^z)\\
& -\begin{bmatrix}\hat{w}_i^z\\ \hat{x}_i^z\end{bmatrix}^\top\begin{bmatrix}\mathcal{X}_i^{11} & \mathcal{X}_i^{12}\\ \mathcal{X}_i^{21} & \mathcal{X}_i^{22}\end{bmatrix}\begin{bmatrix}\hat{w}_i^z\\ \hat{x}_i^z\end{bmatrix}\le\mu_{\mathcal{N}_i}, \qquad \text{(IV.7b)}\\
& \begin{bmatrix}\hat{w}_i^z\\ \hat{x}_i^z\end{bmatrix}^\top\begin{bmatrix}\mathcal{X}_i^{11} & \mathcal{X}_i^{12}\\ \mathcal{X}_i^{21} & \mathcal{X}_i^{22}\end{bmatrix}\begin{bmatrix}\hat{w}_i^z\\ \hat{x}_i^z\end{bmatrix}\le\delta_{\mathcal{N}_i}, \qquad \text{(IV.7c)}\\
& g_i=[\mathcal{X}_i^{11};\mathcal{X}_i^{12};\mathcal{X}_i^{22};q_i^1;\dots;q_i^r],\mu_{\mathcal{N}_i},\delta_{\mathcal{N}_i}\in\mathbb{R}.
\end{aligned}
$$

It is evident that the proposed SCP in (IV.7) comprises a finite number of constraints of the same form as in (IV.2). Let us denote the optimal value of SCP as $\mu^*_{\mathcal{N}_i}+\delta^*_{\mathcal{N}_i}$.

Remark 4.4: While the unknown dynamics appear in the conditions of the SCP (IV.7), they can be replaced using collected data. In fact, since the dynamics of each subsystem evolve recursively in discrete time, given an input-output data point, the input represents the pair of state and internal input (*i.e.,* $(\hat{x}_i^z,\hat{w}_i^z)$), while the output corresponds to the unknown dynamics (*i.e.,* $f(\hat{x}_i^z,\hat{w}_i^z)$).

Remark 4.5: It is worth mentioning that while (IV.5) excludes the origin from the sampling set and accordingly makes (IV.7a) well defined, this condition is enforced over $(\hat{x}_i^z,\hat{w}_i^z)\in\mathbb{R}^{n_i}\times\mathbb{R}^{p_i}$ satisfying $\|(\hat{x}_i^z,\hat{w}_i^z)\|=1$ (and hence unit ball $\|\hat{x}_i^z\|\le 1$) rather than over the unit sphere $\|\hat{x}_i^z\|=1$, which makes the results slightly conservative.

Remark 4.6: Note that $\begin{bmatrix}\hat{w}_i^z\\ \hat{x}_i^z\end{bmatrix}^\top\mathcal{X}_i^*\begin{bmatrix}\hat{w}_i^z\\ \hat{x}_i^z\end{bmatrix}$ plays a key role in the compositionality condition we introduce in the following section, particularly in Theorem 5.1, which is based on data. This is the primary reason why we have incorporated an additional condition in (IV.2c) and (IV.7c), where the objective of the optimization problems involves the minimization of both $\mu_{\mathcal{N}_i}$ and $\delta_{\mathcal{N}_i}$.

In the next section, we solve the proposed SCP and construct the storage function of unknown Σ_i. Additionally, we formally verify the GAS property of unknown interconnected dt-NS by proposing new compositionality conditions purely based on data.

V. GAS Certificate for Interconnected Network with Unknown Topology

Here, we assume that each unknown subsystem model f_i is Lipschitz continuous with respect to (x_i,w_i). Given that the storage function $\mathcal{S}_i$ is *continuously differentiable* according to Assumption 4.1 and our analysis is on the unit sphere (bounded domain), one can readily conclude that $\mathcal{S}_i(q_i,f_i(x_i,w_i))^*-\mathcal{S}_i(q_i,x_i)^*$ is also Lipschitz continuous with respect to (x_i,w_i) with a Lipschitz constant $\mathscr{L}_i^2$, where $\mathcal{S}_i(q_i,\cdot)^*=\mathcal{S}_i(q_i^*,\cdot)$. Similarly, under the same reasoning, one can readily show that $\mathcal{S}_i(q_i,x_i)^*$ is also Lipschitz continuous with respect to x_i, with a Lipschitz constant $\mathscr{L}_i^1$. Later on, we provide an algorithm to estimate Lipschitz constants $\mathscr{L}_i^1,\mathscr{L}_i^2$ of unknown subsystems from data. We now propose the next theorem to ensure GAS certificate with correctness guarantees over an unknown interconnected network with unknown topology.

Theorem 5.1: Consider an interconnected dt-NS $\Sigma=\mathcal{I}(\Sigma_1,\dots,\Sigma_M)$ composed of $M\in\mathbb{N}^+$ individual subsystems Σ_i with an unknown topology $\mathcal{M}$. Consider subsystems Σ_i and their corresponding SCP in (IV.7) with associated optimal value $\mu^*_{\mathcal{N}_i}+\delta^*_{\mathcal{N}_i}$ and optimal solutions $g_i^*=[\mathcal{X}_i^{11^*};\mathcal{X}_i^{12^*};\mathcal{X}_i^{22^*};q_{1_i}^*;\dots;q_{r_i}^*]$ with $\mathcal{N}_i$ sampled data. If

$$
\mu^*_{\mathcal{N}_i}+\mathscr{L}_i\varepsilon_i<0,\quad \forall i\in\{1,\dots,M\}, \qquad \text{(V.1a)}
$$

$$
\sum_{i=1}^{M}\left(\mu^*_{\mathcal{N}_i}+\delta^*_{\mathcal{N}_i}+\mathscr{L}_i\varepsilon_i\right)<0, \qquad \text{(V.1b)}
$$

with $\mathscr{L}_i=\max\{\mathscr{L}_i^1,\mathscr{L}_i^2\}$, then

$$
\mathcal{V}(q,x)=\sum_{i=1}^{M}\mathcal{S}_i(q_i^*,x_i) \qquad \text{(V.2)}
$$

is a Lyapunov function for the interconnected system $\Sigma=\mathcal{I}(\Sigma_1,\dots,\Sigma_M)$ satisfying conditions (III.7a)-(III.7b). Consequently, unknown interconnected dt-NS Σ is GAS with correctness guarantees, according to Definition 2.2.

Proof: We first show that under condition (V.1a), Lyapunov function $\mathcal{V}$ in (III.6) satisfies condition (III.7a) over the unit sphere, *i.e.,*

$$
\mathcal{V}(q,x)>0,\qquad \forall x\in\mathbb{R}^n\backslash\{0\}\colon\|x\|=1. \qquad \text{(V.3)}
$$

We write down $\mathcal{V}$ based on storage functions of individual subsystems as:

$$
-\mathcal{V}(q,x)=-\sum_{i=1}^{M}\mathcal{S}_i(q_i^*,x_i).
$$

Given x_i and for each subsystem i, with a slight abuse of notation, let $\bar{z}:=\arg\min_z\|x_i-\hat{x}_i^z\|$. By incorporating the terms $\sum_{i=1}^{M}\mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*$ through addition and subtraction, with $\mathcal{S}_i(q_i,\cdot)^*=\mathcal{S}_i(q_i^*,\cdot)$, one has

$$
-\mathcal{V}(x,q)=\sum_{i=1}^{M}\left(-\mathcal{S}_i(q_i^*,x_i)+\mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*-\mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*\right).
$$

Given that $\mathcal{S}_i(q_i,x_i)^*$ is Lipschitz continuous with respect to x_i with a Lipschitz constant $\mathscr{L}_i^1$, and since

$$
\begin{aligned}
\max_{x_i:\,\|x_i\|=1}\|x_i-\hat{x}_i^z\|\ &\le\max_{(x_i,w_i):\,\|(x_i,w_i)\|=1}\|x_i-\hat{x}_i^z\|,\ \text{and}\\
\|x_i-\hat{x}_i^z\|\ &\le\|(x_i,w_i)-(\hat{x}_i^z,\hat{w}_i^z)\|,
\end{aligned}
$$

we have, for any $x\in\mathbb{R}^n$ with $\|x\|=1$ (hence $\|(x_i,w_i)\|=1$ for all i),

$$
\begin{aligned}
-\mathcal{V}(q,x) &\leq \sum_{i=1}^{M}\big(\mathscr{L}_i^1 \min_z \|x_i-\hat{x}_i^z\| - \mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*\big)\\
&\leq \sum_{i=1}^{M}\big(\mathscr{L}_i^1 \max_{\|x_i\|=1}\min_z \|x_i-\hat{x}_i^z\| - \mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*\big)\\
&\leq \sum_{i=1}^{M}\big(\mathscr{L}_i^1 \max_{\|(x_i,w_i)\|=1}\min_z \|x_i-\hat{x}_i^z\| - \mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*\big)\\
&\leq \sum_{i=1}^{M}\big(\mathscr{L}_i^1 \max_{\|(x_i,w_i)\|=1}\min_z \|(x_i,w_i)-(\hat{x}_i^z,\hat{w}_i^z)\| - \mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*\big)\\
&= \sum_{i=1}^{M}\big(\mathscr{L}_i^1\varepsilon_i - \mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*\big) \leq \sum_{i=1}^{M}\big(\mathscr{L}_i\varepsilon_i - \mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*\big).
\end{aligned}
$$

According to condition (IV.7a) of SCP, we have

$$-\mathcal{V}(q,x) \leq \sum_{i=1}^{M}\big(\mu^*_{\mathcal{N}_i} + \mathscr{L}_i\varepsilon_i\big). \tag{V.4}$$

Given our proposed condition in (V.1a), one can conclude that

$$\mathcal{V}(q,x) > 0, \quad \forall x \in \mathbb{R}^n\backslash\{0\}\colon \|x\| = 1.$$

Given that $\mathcal{V}$ in (V.2) is based on the linear combination of $\mathcal{S}_i(q_i,x_i)$ and according to the inequality obtained in (V.4), $\mu^*_{R_i}$ in RCP (IV.2) can be defined as $\mu^*_{R_i} = \mu^*_{\mathcal{N}_i} + \mathscr{L}_i\varepsilon_i$. This expression is negative for all subsystems M under the condition in (V.1a).

We proceed with showing that under condition (V.1b), $\mathcal{V}$ in the form of (III.6) satisfies condition (III.7b), as well, over the unit sphere, *i.e.,*

$$\mathcal{V}(q,f(x)) - \mathcal{V}(q,x) < 0, \qquad \forall x \in \mathbb{R}^n\colon \|x\| = 1.$$

Given the proposed Lyapunov function in (III.6), one has

$$\mathcal{V}(q,f(x)) - \mathcal{V}(q,x) = \sum_{i=1}^{M}\big(\mathcal{S}_i(q_i^*,f_i(x_i,w_i)) - \mathcal{S}_i(q_i^*,x_i)\big)$$

which is well posed over the unit sphere since, by construction of the data normalization and the choice of w_i as the collection of the states of all other subsystems (cf. Remark 3.4), the stacked vector (x_i,w_i) is a permutation of the global state x, and therefore satisfies $\|(x_i,w_i)\| = \|x\|$. Hence, $\|x\| = 1$ if and only if $\|(x_i,w_i)\| = 1$. Given (x_i,w_i) and for each subsystem i, with a slight abuse of notation, let $\bar{z} := \arg\min_z \|(x_i,w_i) - (\hat{x}_i^z,\hat{w}_i^z)\|$. By incorporating the terms $\sum_{i=1}^{M}(\mathcal{S}_i(q_i,f_i(q_i,\hat{x}_i^{\bar{z}},\hat{w}_i^{\bar{z}}))^* - \mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*)$ through addition and subtraction, one has

$$
\begin{aligned}
\mathcal{V}(q,f(x)) - \mathcal{V}(q,x) = \sum_{i=1}^{M}\big(&\mathcal{S}_i(q_i^*,f_i(x_i,w_i)) - \mathcal{S}_i(q_i^*,x_i)\\
&- (\mathcal{S}_i(q_i,f_i(\hat{x}_i^{\bar{z}},\hat{w}_i^{\bar{z}}))^* - \mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*)\\
&+ \ \mathcal{S}_i(q_i,f_i(\hat{x}_i^{\bar{z}},\hat{w}_i^{\bar{z}}))^* - \mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*\big).
\end{aligned}
$$

Given that $\mathcal{S}_i(q_i,f_i(x_i,w_i))^* - \mathcal{S}_i(q_i,x_i)^*$ is Lipschitz continuous with respect to (x_i,w_i) with a Lipschitz constant $\mathscr{L}_i^2$, we have

$$
\begin{aligned}
&\mathcal{V}(q,f(x)) - \mathcal{V}(q,x)\\
&\quad\leq \sum_{i=1}^{M}\big(\mathscr{L}_i^2 \min_z \|(x_i,w_i)-(\hat{x}_i^z,\hat{w}_i^z)\|\\
&\qquad + \mathcal{S}_i(q_i,f_i(\hat{x}_i^{\bar{z}},\hat{w}_i^{\bar{z}}))^* - \mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*\big)\\
&\quad\leq \mathscr{L}_i^2 \max_{(x_i,w_i)}\min_z \|(x_i,w_i)-(\hat{x}_i^z,\hat{w}_i^z)\|\\
&\qquad + \mathcal{S}_i(q_i,f_i(\hat{x}_i^{\bar{z}},\hat{w}_i^{\bar{z}}))^* - \mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*\big)\\
&\quad= \sum_{i=1}^{M}\big(\mathscr{L}_i^2\varepsilon_i + \mathcal{S}_i(q_i,f_i(\hat{x}_i^{\bar{z}},\hat{w}_i^{\bar{z}}))^* - \mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*\big)\\
&\quad\leq \sum_{i=1}^{M}\big(\mathscr{L}_i\varepsilon_i + \mathcal{S}_i(q_i,f_i(\hat{x}_i^{\bar{z}},\hat{w}_i^{\bar{z}}))^* - \mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*\big).
\end{aligned}
$$

According to condition (IV.7b) of SCP, we have

$$\mathcal{V}(q,f(x)) - \mathcal{V}(q,x) \leq \sum_{i=1}^{M}\big(\mu^*_{\mathcal{N}_i} + \delta^*_{\mathcal{N}_i} + \mathscr{L}_i\varepsilon_i\big).$$

Given our proposed condition in (V.1b), one can conclude that

$$\mathcal{V}(q,f(x)) - \mathcal{V}(q,x) < 0, \qquad \forall x \in \mathbb{R}^n\colon \|x\| = 1.$$

Then the unknown interconnected network is GAS with correctness guarantees over the unit sphere. Given the homogeneity property of interconnected network and the constructed Lyapunov function, the GAS certificate can then be globally guaranteed in $\mathbb{R}^n$, which concludes the proof. ■

Remark 5.2: The proposed conditions (V.1a)-(V.1b) introduce a novel notion of compositionality in the dissipativity setting using data, which are proposed in our work for the first time. Given that $\hat{w}_i^z$ captures the interaction effect between subsystems in the unknown interconnected topology, condition (V.1b) allows supply rates of subsystems to be designed in a way that they can compensate the potential undesirable effect of neighboring subsystems while this condition becomes negative. Note that compositionality conditions (V.1a)-(V.1b) do not need to be verified if $\mu^*_{\mathcal{N}_i} + \mathscr{L}_i^1\varepsilon_i$ and $\mu^*_{\mathcal{N}_i} + \delta^*_{\mathcal{N}_i} + \mathscr{L}_i^2\varepsilon_i$ can become negative while solving SCP for each individual subsystem. In this case, the results of the current work can be extended to the stability analysis of unknown networks with an arbitrary number of subsystems.

Remark 5.3: It is worth highlighting that the compositionality conditions in (V.1a)–(V.1b) uniformly use the Lipschitz constant $\mathscr{L}_i = \max\{\mathscr{L}_i^1,\mathscr{L}_i^2\}$, which may introduce conservatism. As a promising future direction, one could allow the right-hand side of (IV.7a) (and its corresponding RCP) to be independent of $\mu_{\mathcal{N}_i}$, say $\mu^{(1)}_{\mathcal{N}_i}$. In this case, the objective value of the RCP would become $\mu^{(1)}_{\mathcal{N}_i} + \mu_{\mathcal{N}_i} + \delta_{\mathcal{N}_i}$, which would allow the compositionality conditions (V.1a)–(V.1b) to employ their own dedicated Lipschitz constants rather than their maximum, thereby reducing conservatism.

In order to verify conditions (V.1a)-(V.1b) in Theorem 5.1, the computation of $\mathscr{L}_i^1, \mathscr{L}_i^2$ is a prerequisite. To achieve this, we present Algorithm 1, which enables the estimation of $\mathscr{L}_i^1, \mathscr{L}_i^2$ using a finite set of data. Within this algorithm,

Algorithm 1 Estimation of $\mathscr{L}_i^1, \mathscr{L}_i^2$ via data

Require: $\mathcal{S}_i(q_i, \hat{x}_i^z)^*, \mathcal{S}_i(q_i, f_i(\hat{x}_i^z, \hat{w}_i^z))^*$

1: Choose $\rho, \sigma \in \mathbb{N}^+$ and $\alpha \in \mathbb{R}^+$
2: **for** $i = 1 : \rho$
3: Select sampled pairs $(\hat{x}_i^z, \hat{w}_i^z), (\hat{x}_i^{z'}, \hat{w}_i^{z'})$ from $\mathbb{R}^{n_i} \times \mathbb{R}^{p_i}$ such that $\|(\hat{x}_i^z, \hat{w}_i^z) - (\hat{x}_i^{z'}, \hat{w}_i^{z'})\| \leq \alpha$
4: Compute the slope θ_i as

$$\theta_i = \frac{\|h_i(\hat{x}_i^z, \hat{w}_i^z) - h_i(\hat{x}_i^{z'}, \hat{w}_i^{z'})\|}{\|(\hat{x}_i^z, \hat{w}_i^z) - (\hat{x}_i^{z'}, \hat{w}_i^{z'})\|}$$

with $h_i(\hat{x}_i^z, \hat{w}_i^z) = \mathcal{S}_i(q_i, f_i(\hat{x}_i^z, \hat{w}_i^z))^* - \mathcal{S}_i(q_i, \hat{x}_i^z)^*$
5: **end**
6: Compute the maximum slope as

$$\Theta = \max\{\theta_1, \ldots, \theta_\rho\}$$

7: Repeat Steps 2-6, σ times and acquire $\Theta_1, \ldots, \Theta_\sigma$
8: Apply *Reverse Weibull distribution* [25] to $\Theta_1, \ldots, \Theta_\sigma$ which provides us with, so-called, location, scale, and shape parameters
9: The obtained *location* parameter is the estimated $\mathscr{L}_i^2$
10: Repeat Steps 1-9 with $h_i = \mathcal{S}_i(q_i, \hat{x}_i^z)^*$ to estimate $\mathscr{L}_i^1$

Ensure: $\mathscr{L}_i^1, \mathscr{L}_i^2$, and $\mathscr{L}_i = \max\{\mathscr{L}_i^1, \mathscr{L}_i^2\}$

Algorithm 2 GAS certificate of the interconnected dt-NS with an unknown topology

Require: Collect data-points $((\tilde{x}_i^z, \tilde{w}_i^z), f_i(\tilde{x}_i^z, \tilde{w}_i^z)), z \in \{1, \ldots, \mathcal{N}_i\}$

1: Project all data-points onto unit sphere by normalizing them as $((\hat{x}_i^z, \hat{w}_i^z), f_i(\hat{x}_i^z, \hat{w}_i^z)) = \frac{((\tilde{x}_i^z, \tilde{w}_i^z), f_i(\tilde{x}_i^z, \tilde{w}_i^z))}{\|(\tilde{x}_i^z, \tilde{w}_i^z)\|}$
2: Estimate Lipschitz constants $\mathscr{L}_i^1, \mathscr{L}_i^2$ according to Algorithm 1
3: Compute ε_i as the maximum distance between any points in the unit sphere and the set of data points, *i.e.,* $\varepsilon_i = \max_{(x_i, w_i)} \min_z \|(x_i, w_i) - (\hat{x}_i^z, \hat{w}_i^z)\|, \ \forall (x_i, w_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^{p_i} : \|(x_i, w_i)\| = 1$
4: Solve SCP in (IV.7) with the normalized data and obtain $\mu_{\mathcal{N}_i}^*$ and $\delta_{\mathcal{N}_i}^*$
5: If $\mu_{\mathcal{N}_i}^* + \mathscr{L}_i \varepsilon_i < 0, \forall i \in \{1, \ldots, M\}$, and $\sum_{i=1}^M \left(\mu_{\mathcal{N}_i}^* + \delta_{\mathcal{N}_i}^* + \mathscr{L}_i \varepsilon_i\right) < 0$, with $\mathscr{L}_i = \max\{\mathscr{L}_i^1, \mathscr{L}_i^2\}$, then unknown interconnected dt-NS is GAS with Lyapunov function $\mathcal{V}(q, x) := \sum_{i=1}^M \mathcal{S}_i(q_i, x_i)$
6: Otherwise, repeat Steps 2-5 with additional collected data $\mathcal{N}_i$ to reduce ε_i, or increase the terms of the homogeneous basis functions, potentially designing other $\mu_{\mathcal{N}_i}^*$ and $\delta_{\mathcal{N}_i}^*$ that satisfy conditions (V.1a)-(V.1b)

we rely on Lemma 5.4, adopted from [25], to guarantee the convergence of the estimated $\mathscr{L}_i^1, \mathscr{L}_i^2$ towards their actual values.

Lemma 5.4: Under Algorithm 1, the estimated $\mathscr{L}_i^1, \mathscr{L}_i^2$ converge to their actual values if and only if α approaches zero, and both ρ and σ go to infinity.

Remark 5.5: While the results of Lemma 5.4 provide estimation guarantees in the limit, the amount of data required for Lipschitz constant estimation in practice does not necessarily need to approach infinity. Specifically, in some scenarios, we observed that the estimated Lipschitz constant remains unchanged after a certain number of data points, as it matches the exact Lipschitz constant derived from the underlying mathematical model. Accordingly, in this work we assume that the Lipschitz constants are correctly estimated and available, without incorporating confidence levels into the analysis.

Remark 5.6: Given that unknown coefficients q_i are required for estimating the Lipschitz constants $\mathscr{L}_i^1, \mathscr{L}_i^2$ in Algorithm 1, and subsequently verifying conditions (V.1a)-(V.1b), it is imperative to first solve the proposed SCP in (IV.7). To avoid a posteriori verification of conditions (V.1a)-(V.1b), one can initially consider unknown coefficients of q_i to exist within a certain range and estimate the Lipschitz constants $\mathscr{L}_i^1$ and $\mathscr{L}_i^2$ before solving the SCP. Subsequently, the underlying ranges should be enforced during the solution of SCP. For instance, one could assume $-2 \leq q_i \leq 2$ and estimate the Lipschitz constants $\mathscr{L}_i^1$ and $\mathscr{L}_i^2$ based on any q_i within that range. Then, when solving the SCP in (IV.7), this range should be enforced as an additional constraint to ensure that q_i remains within the specified range. This ensures that the a-priori computation of the Lipschitz constants based on unknown q_i is valid.

Remark 5.7: Note that while the interconnection topology is unknown, the full-state measurement of the subsystems within $\hat{w}_i^z$ should be used for estimating the Lipschitz constants $\mathscr{L}_i^1$ and $\mathscr{L}_i^2$, *i.e.,* $\hat{w}_i^z = [\hat{x}_1^z; \ldots; \hat{x}_{i-1}^z; \hat{x}_{i+1}^z; \ldots; \hat{x}_M^z]$ (cf. Remark 3.4).

We now propose Algorithm 2 to describe the required steps for ensuring the GAS certificate over unknown interconnected dt-NS with an unknown topology.

Remark 5.8: The proposed data-driven findings can be generalized by assuming that each unknown subsystem is noisy, treating the stochasticity as part of the unknown dynamics. In this case, one can explore the probabilistic stability of the system by introducing an expected value in the second condition of the Lyapunov function in (III.7b) and the storage function in (III.4) for the individual subsystems. In such a case, since the expected value must be first approximated empirically using Chebyshev's inequality [26] and then used in the SCP in (IV.7), the final guarantee comes with a confidence level that has a closed-form relationship with the required number of noise realizations for the empirical computation.

Remark 5.9: Note that Algorithm 1 is specifically for the estimation of the Lipschitz constant and always terminates after completing the required steps. Algorithm 2 is a pseudocode outlining the required steps for verifying the GAS property over the interconnected network. For instance, if the network is GAS, the algorithm returns this outcome. However, Step 6 of the algorithm implements an iterative procedure aimed at verifying the GAS property of the network through successive refinements. It is important to note that since Lyapunov functions provide sufficient conditions for nonlinear systems, if Algorithm 2 cannot successfully terminated, this does not imply that the network is unstable.

VI. Unknown Interconnected Network with Known topology

In the scenario where subsystems are unknown but the interconnection topology $\mathcal{M}$ is known, we provide here our sub-result where we initially construct a storage function for each unknown subsystem based on data. This is achieved by solving an SCP for each unknown subsystem with a correctness guarantee over the SF construction. In this case, we show that the data-driven conditions (V.1a)-(V.1b) are reduced to $\mu^*_{\mathcal{N}_i}+\mathscr{L}_i\varepsilon_i<0$ with $\mathscr{L}_i=\max\{\mathscr{L}_i^1,\mathscr{L}_i^2\}$ in the level of subsystems. Subsequently, we examine the traditional dissipativity compositionality condition (III.5) to determine its satisfaction. To avoid potential posteriori checks for this condition, we employ an alternating direction method of multipliers (ADMM) algorithm [19], enabling efficient satisfaction of the compositionality condition in a distributed manner while searching for storage functions of individual subsystems.

We initially propose an alternative SCP that does not necessitate condition (IV.7c), outlined as follows:

$$
\begin{aligned}
\min_{[g_i;\mu_{\mathcal{N}_i}]}\quad & \mu_{\mathcal{N}_i},\\
\text{s.t.}\quad & \forall(\hat{x}_i^z,\hat{w}_i^z)\in\mathbb{R}^{n_i}\times\mathbb{R}^{p_i},\forall z\in\{1,\dots,\mathcal{N}_i\},\\
& -\mathcal{S}_i(q_i,\hat{x}_i^z)\le\mu_{\mathcal{N}_i}, && \text{(VI.1a)}\\
& \mathcal{S}_i(q_i,f_i(\hat{x}_i^z,\hat{w}_i^z))-\mathcal{S}_i(q_i,\hat{x}_i^z)\\
& -\begin{bmatrix}\hat{w}_i^z\\ \hat{x}_i^z\end{bmatrix}^\top\begin{bmatrix}\mathcal{X}_i^{11} & \mathcal{X}_i^{12}\\ \mathcal{X}_i^{21} & \mathcal{X}_i^{22}\end{bmatrix}\begin{bmatrix}\hat{w}_i^z\\ \hat{x}_i^z\end{bmatrix}\le\mu_{\mathcal{N}_i}, && \text{(VI.1b)}\\
& g_i=[\mathcal{X}_i^{11};\mathcal{X}_i^{12};\mathcal{X}_i^{22};q_i^1;\dots;q_i^r],\ \mu_{\mathcal{N}_i}\in\mathbb{R}.
\end{aligned}
$$

Remark 6.1: Note that in this section, it is not required to consider the entire state measurement of subsystems within w_i, as described in Remark 3.4, during the analysis. In fact, the primary objective here is to demonstrate that each subsystem individually exhibits dissipativity over the unit sphere with respect to a supply rate via a storage function $\mathcal{S}_i$, as specified in the SCP (VI.1). Subsequently, this result can be extended globally to $\mathbb{R}^{n_i}$ by leveraging the homogeneity property of the maps f_i and storage functions $\mathcal{S}_i$. Ultimately, using the dissipativity compositional condition (III.5) and incorporating the knowledge of the interconnection matrix $\mathcal{M}$, the GAS certificate is ensured for the entire network over $\mathbb{R}^n$.

We now raise the following theorem, enabling the construction of a storage function for each unknown subsystem through solving the SCP in (VI.1), while ensuring guaranteed confidence based on collected data.

Theorem 6.2: Consider individual subsystems in (III.2) and their corresponding SCP in (VI.1) with associated optimal value $\mu^*_{\mathcal{N}_i}$ and solutions $g_i^*=[\mathcal{X}_i^{11^*};\mathcal{X}_i^{12^*};\mathcal{X}_i^{22^*};q_{1_i}^*;\dots;q_{r_i}^*]$ with $\mathcal{N}_i$ sampled data. If

$$\mu^*_{\mathcal{N}_i}+\mathscr{L}_i\varepsilon_i<0,\quad\forall i\in\{1,\dots,M\},\qquad\text{(VI.2)}$$

with $\mathscr{L}_i=\max\{\mathscr{L}_i^1,\mathscr{L}_i^2\}$, then the constructed $\mathcal{S}_i$ obtained by solving SCP (VI.1) serve as storage functions for unknown subsystems Σ_i globally over $\mathbb{R}^{n_i}$, with guarantees of correctness.

Proof: Initially, we demonstrate that the constructed $\mathcal{S}_i$ obtained by solving SCP (VI.1) satisfy condition (III.4) across the entire state and disturbance spaces, with $\mu_{\mathcal{N}_i}<0$, *i.e.,*

$$
\begin{aligned}
&\mathcal{S}_i(q_i^*,f_i(x_i,w_i))-\mathcal{S}_i(q_i^*,x_i)-\begin{bmatrix}w_i\\ x_i\end{bmatrix}^\top\mathcal{X}_i^*\begin{bmatrix}w_i\\ x_i\end{bmatrix}\le 0, \quad\text{(VI.3)}\\
&\forall(x_i,w_i)\in\mathbb{R}^{n_i}\times\mathbb{R}^{p_i}:\|(x_i,w_i)\|=1.
\end{aligned}
$$

Given the Lipschitz continuity of $f_i(x_i,w_i)$, it is readily demonstrated that the left-hand side of (VI.3) remains Lipschitz continuous with respect to (x_i,w_i), with a Lipschitz constant $\mathscr{L}_i^2\in\mathbb{R}^+$ (with a slight abuse of notation), since the storage function $\mathcal{S}_i$ is continuously differentiable, and our analysis is conducted on the unit sphere (bounded domain). Given (x_i,w_i) and for each subsystem i, with a slight abuse of notation, let $\bar{z}:=\arg\min_z\|(x_i,w_i)-(\hat{x}_i^z,\hat{w}_i^z)\|$. By incorporating the terms $\mathcal{S}_i(q_i,f_i(q_i,\hat{x}_i^{\bar{z}},\hat{w}_i^{\bar{z}}))^*-\mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*-\begin{bmatrix}\hat{w}_i^{\bar{z}}\\ \hat{x}_i^{\bar{z}}\end{bmatrix}^\top\mathcal{X}_i^*\begin{bmatrix}\hat{w}_i^{\bar{z}}\\ \hat{x}_i^{\bar{z}}\end{bmatrix}$ through addition and subtraction, one has

$$
\begin{aligned}
&\mathcal{S}_i(q_i^*,f_i(x_i,w_i))-\mathcal{S}_i(q_i^*,x_i)-\begin{bmatrix}w_i\\ x_i\end{bmatrix}^\top\mathcal{X}_i^*\begin{bmatrix}w_i\\ x_i\end{bmatrix}\\
&=\mathcal{S}_i(q_i^*,f_i(x_i,w_i))-\mathcal{S}_i(q_i^*,x_i)-\begin{bmatrix}w_i\\ x_i\end{bmatrix}^\top\mathcal{X}_i^*\begin{bmatrix}w_i\\ x_i\end{bmatrix}\\
&\quad-(\mathcal{S}_i(q_i,f_i(q_i,\hat{x}_i^{\bar{z}},\hat{w}_i^{\bar{z}}))^*-\mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*-\begin{bmatrix}\hat{w}_i^{\bar{z}}\\ \hat{x}_i^{\bar{z}}\end{bmatrix}^\top\mathcal{X}_i^*\begin{bmatrix}\hat{w}_i^{\bar{z}}\\ \hat{x}_i^{\bar{z}}\end{bmatrix})\\
&\quad+\mathcal{S}_i(q_i,f_i(q_i,\hat{x}_i^{\bar{z}},\hat{w}_i^{\bar{z}}))^*-\mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*-\begin{bmatrix}\hat{w}_i^{\bar{z}}\\ \hat{x}_i^{\bar{z}}\end{bmatrix}^\top\mathcal{X}_i^*\begin{bmatrix}\hat{w}_i^{\bar{z}}\\ \hat{x}_i^{\bar{z}}\end{bmatrix},
\end{aligned}
$$

with $\mathcal{S}_i(q_i,\cdot)^*=\mathcal{S}_i(q_i^*,\cdot)$. Given the Lipschitz continuity of (VI.3) with a Lipschitz constant $\mathscr{L}_i^2$, we have

$$
\begin{aligned}
&\mathcal{S}_i(q_i^*,f_i(x_i,w_i))-\mathcal{S}_i(q_i^*,x_i)-\begin{bmatrix}w_i\\ x_i\end{bmatrix}^\top\mathcal{X}_i^*\begin{bmatrix}w_i\\ x_i\end{bmatrix}\\
&\le\mathscr{L}_i^2\min_z\|(x_i,w_i)-(\hat{x}_i^z,\hat{w}_i^z)\|\\
&\quad+\mathcal{S}_i(q_i,f_i(q_i,\hat{x}_i^{\bar{z}},\hat{w}_i^{\bar{z}}))^*-\mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*-\begin{bmatrix}\hat{w}_i^{\bar{z}}\\ \hat{x}_i^{\bar{z}}\end{bmatrix}^\top\mathcal{X}_i^*\begin{bmatrix}\hat{w}_i^{\bar{z}}\\ \hat{x}_i^{\bar{z}}\end{bmatrix}\\
&\le\mathscr{L}_i^2\max_{(x_i,w_i)}\min_z\|(x_i,w_i)-(\hat{x}_i^z,\hat{w}_i^z)\|\\
&\quad+\mathcal{S}_i(q_i,f_i(q_i,\hat{x}_i^{\bar{z}},\hat{w}_i^{\bar{z}}))^*-\mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*-\begin{bmatrix}\hat{w}_i^{\bar{z}}\\ \hat{x}_i^{\bar{z}}\end{bmatrix}^\top\mathcal{X}_i^*\begin{bmatrix}\hat{w}_i^{\bar{z}}\\ \hat{x}_i^{\bar{z}}\end{bmatrix}\\
&=\mathscr{L}_i^2\varepsilon_i+\mathcal{S}_i(q_i,f_i(q_i,\hat{x}_i^{\bar{z}},\hat{w}_i^{\bar{z}}))^*-\mathcal{S}_i(q_i,\hat{x}_i^{\bar{z}})^*-\begin{bmatrix}\hat{w}_i^{\bar{z}}\\ \hat{x}_i^{\bar{z}}\end{bmatrix}^\top\mathcal{X}_i^*\begin{bmatrix}\hat{w}_i^{\bar{z}}\\ \hat{x}_i^{\bar{z}}\end{bmatrix}.
\end{aligned}
$$

According to condition (VI.1b) of SCP, we have

$$
\begin{aligned}
&\mathcal{S}_i(q_i^*,f_i(x_i,w_i))-\mathcal{S}_i(q_i^*,x_i)-\begin{bmatrix}w_i\\ x_i\end{bmatrix}^\top\mathcal{X}_i^*\begin{bmatrix}w_i\\ x_i\end{bmatrix}\\
&\le\mu^*_{\mathcal{N}_i}+\mathscr{L}_i\varepsilon_i,
\end{aligned}
$$

with $\mathscr{L}_i=\max\{\mathscr{L}_i^1,\mathscr{L}_i^2\}$. Given our proposed condition in (VI.2), one can conclude that

$$
\begin{aligned}
&\mathcal{S}_i(q_i^*,f_i(x_i,w_i))-\mathcal{S}_i(q_i^*,x_i)-\begin{bmatrix}w_i\\ x_i\end{bmatrix}^\top\mathcal{X}_i^*\begin{bmatrix}w_i\\ x_i\end{bmatrix}\le 0\\
&\forall(x_i,w_i)\in\mathbb{R}^{n_i}\times\mathbb{R}^{p_i}:\|(x_i,w_i)\|=1.
\end{aligned}
$$

This result can subsequently be extended globally to $\mathbb{R}^{n_i}\times\mathbb{R}^{p_i}$, thanks to the homogeneity property of maps f_i and storage functions $\mathcal{S}_i$. By employing a similar reasoning, it can be demonstrated that under condition (VI.2), the $\mathcal{S}_i$ constructed through solving SCP in (VI.1) satisfies $\mathcal{S}_i(q_i^*, x_i) > 0$ across the entire state space. Then the $\mathcal{S}_i$ obtained by solving SCP in (VI.1) serve as storage functions for unknown subsystems $\Sigma_i, i \in \{1,\dots,M\}$, globally over $\mathbb{R}^{n_i}$, with correctness guarantees, thereby concluding the proof. ■

In order to construct the Lyapunov function for the interconnected system based on storage functions of individual subsystems derived from data, one needs to satisfy the traditional compositionality condition in (III.5). Here, we employ the ADMM algorithm [19] to efficiently satisfy condition (III.5) by decomposing it into local sub-problems. To do so, we define the following local $\mathcal{G}_i = \big\{\mathcal{X}_i \,\big|\, \mathcal{X}_i = \mathcal{X}_i^\top, \exists \text{ SF } \mathcal{S}_i \text{ w.r.t. SCP (VI.1)}\big\}$. In addition, we define the global constraint as

$$\mathcal{G} = \Big\{(\mathcal{X}_1,\dots,\mathcal{X}_M) \,\Big|\, \text{condition (III.5) holds}\Big\}. \tag{VI.4}$$

We now recast applicability of Theorem 3.2 as a feasibility problem in the following lemma.

Lemma 6.3: Consider an interconnected dt-NS $\Sigma = \mathcal{I}(\Sigma_1,\dots,\Sigma_M)$. If there exist matrices $\mathcal{X}_1,\dots,\mathcal{X}_M$ such that

$$\begin{aligned} &\mathcal{X}_i \in \mathcal{G}_i, \quad \forall i \in \{1,\dots,M\},\\ &(\mathcal{X}_1,\dots,\mathcal{X}_M) \in \mathcal{G}, \end{aligned} \tag{VI.5}$$

then $\mathcal{V}(x) := \sum_{i=1}^M \mathcal{S}_i(x_i)$ is a Lyapunov function for the interconnected system $\Sigma = \mathcal{I}(\Sigma_1,\dots,\Sigma_M)$.

To reformulate the feasibility problem in Lemma 6.3 as a standard ADMM problem, we first define the following indicator functions:

$$\mathbb{1}_{\mathcal{G}_i}(\mathcal{X}_i) = \begin{cases} 0, & \mathcal{X}_i \in \mathcal{G}_i,\\ \infty, & \text{otherwise,}\end{cases}$$

$$\mathbb{1}_{\mathcal{G}}(\mathcal{X}_1,\dots,\mathcal{X}_M) = \begin{cases} 0, & (\mathcal{X}_1,\dots,\mathcal{X}_M) \in \mathcal{G},\\ \infty, & \text{otherwise.}\end{cases}$$

Now by introducing an auxiliary variable $\mathcal{Z}_i \in \mathbb{R}^{(n_i+p_i)\times(n_i+p_i)}$ for each subsystem, we rewrite (VI.5) as an optimization problem in the ADMM form as:

$$\text{ADMM}: \begin{cases} \min\limits_{\{\mathcal{X}_i,\mathcal{Z}_i\}_{i=1}^M} & \sum_{i=1}^M(\mathbb{1}_{\mathcal{G}_i}(\mathcal{X}_i)) + \mathbb{1}_{\mathcal{G}}(\mathcal{Z}_1,\dots,\mathcal{Z}_M),\\ \text{s.t.} & \mathcal{X}_i - \mathcal{Z}_i = 0, \quad \forall i \in \{1,\dots,M\}. \end{cases} \tag{VI.6}$$

Since the first part of the objective function in (VI.6), *i.e.,* $\mathbb{1}_{\mathcal{G}_i}$, is separable by subsystems, one can find a solution for (VI.6) parallelly by iterating over $\mathcal{X}_i$ and $\mathcal{Z}_i$, alternately, while leveraging a new dual variable Λ_i. In summary, the ADMM algorithm [19] updates the essential variables at each iteration k as follows:

- For each $i \in \{1,\dots,M\}$, solve the following local problem:

$$\mathcal{X}_i^{k+1} \in \underset{\mathcal{X}^*\in\mathcal{G}_i}{\operatorname{argmin}} \ \|\mathcal{X}_i^* - \mathcal{Z}_i^k + \Lambda_i^k\|_F^2.$$

- If $(\mathcal{X}_1^{k+1},\dots,\mathcal{X}_M^{k+1}) \in \mathcal{G}$, the algorithm is successfully terminated. Otherwise, solve the following global problem:

$$(\mathcal{Z}_1^{k+1},\dots,\mathcal{Z}_M^{k+1}) \in \underset{(\mathcal{Z}_1^*,\dots,\mathcal{Z}_M^*)\in\mathcal{G}}{\operatorname{argmin}} \sum_{i=1}^M \|\mathcal{X}_i^{k+1} - \mathcal{Z}_i^* + \Lambda_i^k\|_F^2.$$

- Update the dual variable Λ_i as

$$\Lambda_i^{k+1} = \mathcal{X}_i^{k+1} - \mathcal{Z}_i^{k+1} + \Lambda_i^k,$$

and return to the first step until a possible convergence.

Remark 6.4: The data-driven results proposed in this work have the potential to be extended for designing a *control* Lyapunov function (CLF) for unknown homogeneous interconnected systems. In such a case, the optimization programs are no longer convex due to the bilinearity between the unknown coefficients of the CLF and the controller. One possible solution is to treat the control inputs as discrete values (similar to switched systems), transforming the non-convex optimization problem into a mixed-integer linear program.

Remark 6.5: Note that in the case of an unknown interconnection topology, while considering the entire state measurement of subsystems within w_i, as described in Remark 3.4, does not necessitate full interconnection, it potentially increases the computational complexity. Specifically, this assumption increases the effective dimension of w_i, which raises the sample complexity in (IV.4) and accordingly affects the computation of the maximum distance between points on the unit sphere and the set of data points, as described in (IV.6). Additionally, this assumption impacts the estimation of the Lipschitz constant in Algorithm 1, as $\hat{w}_i^z$ also appears in this computation. In contrast, the results in Section VI do not involve these complexities but require knowledge of the interconnection topology. This highlights a trade-off: if the topology is known, it is advisable to use the results in Section VI (cf. the third case study). Otherwise, the results for an unknown topology offer a valuable solution to the problem, albeit with increased computational complexity (cf. the second case study).

VII. Case Study

In this section, we begin by applying our results to a network composed of only two subsystems. We illustrate, utilizing both model-based techniques and our data-driven methods, that if both subsystems are dissipative in accordance with Definition 3.1, the interconnected network exhibits GAS under suitable compositionality conditions. Subsequently, to demonstrate the effect of our data-driven method over networks with unknown interconnection topology, we extend our analysis to a room temperature network with an unknown topology. Finally, to demonstrate that our data-driven results can be applied to *nonlinear* dynamics with homogeneity, we present a nonlinear homogeneous network of degree one, consisting of $10,000$ subsystems (totaling $20,000$ dimensions), to highlight the applicability of our approach.

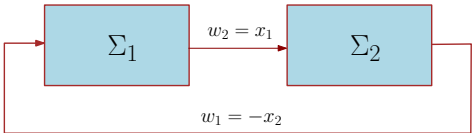


Fig. 3. Interconnected dt-NS $\mathcal{I}(\boldsymbol{\Sigma_1, \Sigma_2})$ with a coupling matrix $\boldsymbol{\mathcal{M}}$ as in (VII.2).

A. Illustrative Example

Consider two subsystems with the following dynamics:

$$\Sigma_i\colon x_i(k+1) = A_i x_i(k) + B_i w_i(k), \quad i \in \{1,2\}, \quad \text{(VII.1)}$$

with system matrices

$$A_1 = \begin{bmatrix} -0.1 & -0.2 \\ -0.2 & -0.1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -0.5 & 1.8 \\ -0.2 & 1.01 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

These two subsystems are interconnected through negative feedback, as illustrated in Fig. 3. It is evident that one can obtain the interconnected system $\Sigma := \mathcal{I}(\Sigma_1, \Sigma_2)$ as

$$\Sigma\colon x(k+1) = Ax(k),$$

with the block matrices

$$A = \begin{bmatrix} A_1 & B_1 \\ B_2 & A_2 \end{bmatrix},$$

and the following interconnection matrix

$$\mathcal{M} = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}. \quad \text{(VII.2)}$$

We initially presume access to the model dynamics and demonstrate that each subsystem Σ_i displays dissipative behavior, fulfilling condition (III.4), through the utilization of a positive-definite storage function $\mathcal{S}_i$. We then show that under the model-based compositionality condition in (III.5), the interconnected system $\mathcal{I}(\Sigma_1, \Sigma_2)$ is GAS in the sense of Definition 2.2 and according to Theorem 3.2.

We first fix a candidate positive-definite storage function $\mathcal{S}_i$ as

$$\mathcal{S}_i(q_i, x_i) = x_i^\top P_i x_i, \quad \text{(VII.3)}$$

with P being a (2×2) symmetric positive-definite matrix. We now aim at showing condition (III.4). Given that we are aware of the model at this stage, we can derive the following model-based analysis:

$$\begin{aligned} \mathcal{S}_i(q_i, f_i(x_i, w_i)) &= (A_i x_i + B_i w_i)^\top P_i (A_i x_i + B_i w_i) \\ &= x_i^\top A_i^\top P_i A_i x_i + w_i^\top B_i^\top P_i B_i w_i + 2 x_i^\top A_i^\top P_i B_i w_i \\ &= \begin{bmatrix} x_i \\ w_i \end{bmatrix}^\top \begin{bmatrix} A_i^\top P_i A_i & A_i^\top P_i B_i \\ B_i^\top P_i A_i & B_i^\top P_i B_i \end{bmatrix} \begin{bmatrix} x_i \\ w_i \end{bmatrix}. \end{aligned}$$

By assuming the following LMI,

$$\begin{bmatrix} A_i^\top P_i A_i & A_i^\top P_i B_i \\ B_i^\top P_i A_i & B_i^\top P_i B_i \end{bmatrix} \leq \begin{bmatrix} P_i + \mathcal{X}_i^{22} & \mathcal{X}_i^{21} \\ \mathcal{X}_i^{12} & \mathcal{X}_i^{11} \end{bmatrix}, \quad \text{(VII.4)}$$

we have

$$\begin{aligned} \mathcal{S}_i(q_i, f_i(x_i, w_i)) &= \begin{bmatrix} x_i \\ w_i \end{bmatrix}^\top \begin{bmatrix} A_i^\top P_i A_i & A_i^\top P_i B_i \\ B_i^\top P_i A_i & B_i^\top P_i B_i \end{bmatrix} \begin{bmatrix} x_i \\ w_i \end{bmatrix} \\ &\leq \begin{bmatrix} x_i \\ w_i \end{bmatrix}^\top \begin{bmatrix} P_i + \mathcal{X}_i^{22} & \mathcal{X}_i^{21} \\ \mathcal{X}_i^{12} & \mathcal{X}_i^{11} \end{bmatrix} \begin{bmatrix} x_i \\ w_i \end{bmatrix} \\ &= \underbrace{x_i^\top P_i x_i}_{\mathcal{S}_i(q_i, x_i)} + \begin{bmatrix} w_i \\ x_i \end{bmatrix}^\top \begin{bmatrix} \mathcal{X}_i^{11} & \mathcal{X}_i^{12} \\ \mathcal{X}_i^{21} & \mathcal{X}_i^{22} \end{bmatrix} \begin{bmatrix} w_i \\ x_i \end{bmatrix}, \end{aligned}$$

which is exactly the satisfaction of condition (III.4). Hence, to demonstrate the satisfaction of condition (III.4) by the candidate storage function $\mathcal{S}_i$ in (VII.3), it is sufficient to solve the constructed LMI in (VII.4) for each subsystem. This involves the design of positive-definite matrices P_i and storage functions $\mathcal{X}_i$ for both subsystems.

We use the semidefinite programming (SDP) solver SeDuMi [27] and fulfill the LMI in (VII.4) for both subsystems with the following matrices:

$$P_1 = \begin{bmatrix} 1.7779 & -0.0290 \\ -0.0290 & 1.9546 \end{bmatrix}, \ \mathcal{X}_1^{11} = \begin{bmatrix} 0.3372 & -0.1620 \\ -0.1620 & 0.5128 \end{bmatrix},$$
$$\mathcal{X}_1^{12} = \mathcal{X}_1^{21} = \begin{bmatrix} -0.0250 & -0.0486 \\ -0.0486 & 0.0923 \end{bmatrix}, \ \mathcal{X}_1^{22} = \begin{bmatrix} -1.0748 & 0.0579 \\ 0.0579 & -1.1826 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 1.1742 & -1.4717 \\ -1.4717 & 5.8613 \end{bmatrix}, \ \mathcal{X}_2^{11} = \begin{bmatrix} 0.5509 & -0.0283 \\ -0.0283 & 0.6199 \end{bmatrix},$$
$$\mathcal{X}_2^{12} = \mathcal{X}_2^{21} = \begin{bmatrix} -0.0147 & -0.0256 \\ -0.0256 & 0.1947 \end{bmatrix}, \ \mathcal{X}_2^{22} = \begin{bmatrix} -0.6036 & 0.3231 \\ 0.3231 & -0.9405 \end{bmatrix},$$

certifying that each subsystem is dissipative in the sense of Definition 3.1. Now, we look at $\Sigma = \mathcal{I}(\Sigma_1, \Sigma_2)$ with the coupling matrix $\mathcal{M}$ as in (VII.2) aiming to verify compositionality condition (III.5). Given the obtained supply rates $\mathcal{X}_i$, one can construct matrix $\mathcal{X}_{cmp}$ in (III.5) as

$$\mathcal{X}_{cmp} = \begin{bmatrix} \mathcal{X}_1^{11} & \mathbf{0}_{2\times 2} & \mathcal{X}_1^{12} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \mathcal{X}_2^{11} & \mathbf{0}_{2\times 2} & \mathcal{X}_2^{12} \\ \mathcal{X}_1^{21} & \mathbf{0}_{2\times 2} & \mathcal{X}_1^{22} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \mathcal{X}_2^{21} & \mathbf{0}_{2\times 2} & \mathcal{X}_2^{22} \end{bmatrix}.$$

The compositionality condition (III.5) is then satisfied as

$$\begin{bmatrix} \mathcal{M} \\ \mathbb{I}_4 \end{bmatrix}^\top \mathcal{X}_{cmp} \begin{bmatrix} \mathcal{M} \\ \mathbb{I}_4 \end{bmatrix} = \begin{bmatrix} -0.5238 & 0.0297 & 0.0103 & 0.0230 \\ 0.0297 & -0.5627 & 0.0230 & 0.1024 \\ 0.0103 & 0.0230 & -0.2664 & 0.1611 \\ 0.0230 & 0.1024 & 0.1611 & -0.4277 \end{bmatrix} \preceq 0, \quad \text{(VII.5)}$$

concluding that the interconnected system $\Sigma = \mathcal{I}(\Sigma_1, \Sigma_2)$ is GAS in the sense of Definition 2.2 and according to Theorem 3.2.

We now assume that we are not aware of the model of subsystems in (VII.1) and employ our data-driven techniques to tackle the problem. We fix the structure of our storage functions as $\mathcal{S}_i(q_i, x_i) = q_i^1 x_{1_i}^2 + q_i^2 x_{2_i}^2$ for both $i \in \{1,2\}$. We employ Algorithm 2 as our proposed data-driven scheme. We collect samples from trajectories of each unknown subsystem and normalize them to be projected onto unit sphere. We

now solve the SCP (IV.7) with collected data and compute coefficients of storage functions together with other decision variables as

$$\mathcal{S}_1(q_1,x_1)=0.1x_{1_1}^2+0.2x_{2_1}^2,\ \mu^*_{\mathcal{N}_1}=-0.0607,\ \delta^*_{\mathcal{N}_1}=0.039,$$
$$\mathcal{X}_1^{*11}=0.0001\mathbb{I}_2,\ \mathcal{X}_1^{*12}=\mathcal{X}_1^{*21}=\mathbf{0}_{2\times2},\ \ \mathcal{X}_1^{*22}=\begin{bmatrix}-0.02 & 0\\ 0 & -0.01\end{bmatrix},$$
$$\mathcal{S}_2(q_2,x_2)=0.01x_{1_2}^2+0.4x_{2_2}^2,\ \mu^*_{\mathcal{N}_2}=-0.0108,\ \delta^*_{\mathcal{N}_2}=0.03,$$
$$\mathcal{X}_2^{*11}=0.0001\mathbb{I}_2,\ \mathcal{X}_2^{*12}=\mathcal{X}_2^{*21}=\mathbf{0}_{2\times2},\ \ \mathcal{X}_2^{*22}=-0.001\mathbb{I}_2.$$

We compute $\varepsilon_1=\varepsilon_2=0.0041, \mathscr{L}_1^1=0.1714, \mathscr{L}_1^2=0.1631$ (with $\mathscr{L}_1=\max\{\mathscr{L}_1^1,\mathscr{L}_1^2\}=0.1714$), and $\mathscr{L}_2^1=0.3906$, $\mathscr{L}_2^2=0.1454$ (with $\mathscr{L}_2=\max\{\mathscr{L}_2^1,\mathscr{L}_2^2\}=0.3906$) according to Steps 2 and 3 in Algorithm 2. Since

$$\begin{aligned}&\mu^*_{\mathcal{N}_1}+\mathscr{L}_1\varepsilon_1=-0.06<0,\\ &\mu^*_{\mathcal{N}_2}+\mathscr{L}_2\varepsilon_2=-0.0092<0,\\ &\sum_{i=1}^{2}\left(\mu^*_{\mathcal{N}_i}+\delta^*_{\mathcal{N}_i}+\mathscr{L}_i\varepsilon_i\right)=-0.0210+0.0208\\ &\qquad\qquad=-0.0002<0,\end{aligned}$$

according to Theorem 5.1, one can certify that the interconnected dt-NS $\Sigma=\mathcal{I}(\Sigma_1,\Sigma_2)$ is GAS with respect to $x=0$, and

$$\mathcal{V}(q,x)=\sum_{i=1}^{2}\mathcal{S}_i(q_i,x_i)=0.1x_{1_1}^2+0.2x_{2_1}^2+0.01x_{1_2}^2+0.4x_{2_2}^2$$

is a Lyapunov function for the interconnected dt-NS with correctness guarantees. This is compatible with the results of the model-based approach that we acquired in the first step of this illustrative example.

B. Room Temperature Network with Unknown Topology

To demonstrate the scalability of our method, we apply our data-driven findings on a room temperature network with an unknown topology. The evolution of the temperature $T(\cdot)$ can be described by the following interconnected network [28]:

$$\Sigma\colon T(k+1)=AT(k),$$

where A is a matrix with diagonal elements $a_{ii}=1-2\varphi-\theta$, $i\in\{1,\dots,M\}$, and off-diagonal elements $a_{ij}=\varphi$, $i,j\in\{1,\dots,M\}, i\neq j$, depending on the unknown interconnection topology. In addition, $T(k)=[T_1(k);\dots;T_M(k)]$, and φ, θ are thermal factors between rooms i and j, and the external environment and the room i, respectively. Now by characterizing each individual room as

$$\Sigma_i\colon T_i(k+1)=a_{ii}T_i(k)+\varphi w_i(k),$$

with w_i being defined according to (III.3) with an unknown interconnection topology $\mathcal{M}$, one has $\Sigma=\mathcal{I}(\Sigma_1,\dots,\Sigma_M)$. We assume that the underlying model is unknown to us.

Let $M=8$. Our primary objective is to systematically construct a Lyapunov function for the unknown interconnected dt-NS by leveraging data-driven storage functions of individual subsystems through solving the SCP (IV.7). Subsequently, we ensure that the interconnected network achieves GAS with respect to its equilibrium point $x=0$, ensuring a correctness guarantee.

Initially, we establish the structure of our storage functions as $\mathcal{S}_i(q_i,x_i)=q_ix_i^2$ for all $i\in\{1,\dots,8\}$. We employ our proposed data-driven scheme, outlined in Algorithm 2. We normalize the samples collected from the trajectories of each unknown room to project them onto the unit sphere. Subsequently, we solve the SCP (IV.7), computing coefficients of storage functions alongside other decision variables as follows:

$$\mathcal{S}_i^*(q_i,x_i)=1.7x_i^2,\ \mu^*_{\mathcal{N}_i}=-0.1013,\ \delta^*_{\mathcal{N}_i}=0.02,$$
$$\mathcal{X}_i^{*11}=\begin{bmatrix}0.0001 & 0 & \cdots & \cdots & 0\\ 0 & 0.0001 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & & \ddots & \ddots & \vdots\\ 0 & \cdots & \cdots & 0 & 0\end{bmatrix}_{7\times7},$$
$$\mathcal{X}_i^{*12}=\mathbf{0}_{7\times1},\ \mathcal{X}_i^{*21}=\mathcal{X}_i^{*12^\top},\ \mathcal{X}_i^{*22}=-0.001.$$

We compute $\varepsilon_i=0.0091$, $\mathscr{L}_i^1=1.3317$, and $\mathscr{L}_i^2=0.3617$ (with $\mathscr{L}_i=\max\{\mathscr{L}_i^1,\mathscr{L}_i^2\}=1.3317$) according to Steps 3, 4 in Algorithm 2. Since

$$\mu^*_{\mathcal{N}_i}+\mathscr{L}_i\varepsilon_i=-892\times10^{-4}<0,\quad\forall i\in\{1,\dots,8\},$$
$$\sum_{i=1}^{8}(\mu^*_{\mathcal{N}_i}+\delta^*_{\mathcal{N}_i}+\mathscr{L}_i\varepsilon_i)=\sum_{i=1}^{8}(-692\times10^{-4})_i=-1.38<0,$$

according to Theorem 5.1, one can verify that unknown interconnected dt-NS $\Sigma=\mathcal{I}(\Sigma_1,\dots,\Sigma_8)$ is GAS with respect to $x=0$, and

$$\mathcal{V}(q,x)=\sum_{i=1}^{8}\mathcal{S}_i(q_i,x_i)=\sum_{i=1}^{8}1.7\,x_i^2$$

is a valid Lyapunov function for the interconnected dt-NS with a correctness guarantee.

Remark 7.1: Note that certain physical insights about unknown models can help guide the selection of homogeneous basis functions. For instance, since room temperature models are typically polynomial due to their underlying physics, we chose the basis functions to be monomials based on the state. Regarding the number of monomial elements, one can include all combinations of state variables (while preserving homogeneity), and the scenario optimization problem will automatically eliminate some of them during the solution of the SCP by assigning zero to their coefficients.

Remark 7.2: The term "unknown topology" in our work does not imply that our data-driven technique can handle any arbitrary interconnection topology. In fact, for model-based dissipativity reasoning, knowledge of the topology is required to verify the compositionality condition in (III.5). However, since our data-driven conditions in (V.1a)-(V.1b) are different from model-based techniques, the interconnection topology does not require to be known. While the unknown topology only influences w_i in the SCP during the solution process, this does not imply that if the SCP is solvable, it will be so for any arbitrary interconnection topology. For each new unknown topology, a new set of data would need to be collected, followed by an attempt to solve the SCP.

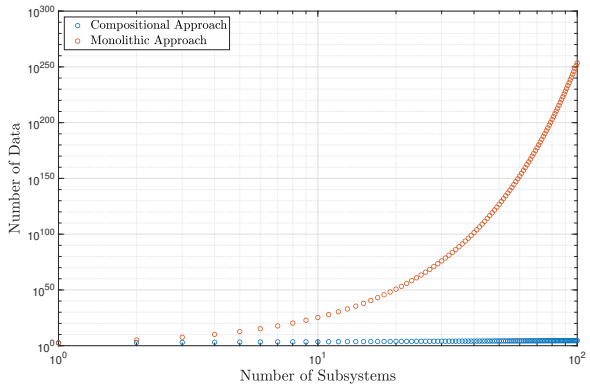


Fig. 4. Sample complexity analysis: Required amount of data vs. number of scalar subsystems in monolithic and compositional approaches. Plot is in logarithmic scale.

C. Sample Complexity Analysis

In this subsection, we perform a sample complexity analysis for the room temperature network within the frameworks of both monolithic (*i.e.,* solving the problem directly over the network) and compositional approaches. To do so, we have visually represented the relationship between the required amount of data and the number of subsystems for both methodologies in Fig. 4. It becomes evident that our data-driven compositional approach significantly mitigates sample complexity by aligning it with the granularity of subsystems. Consequently, as the number of subsystems increases, the growth in required data remains *linear*. Conversely, in the monolithic approach, sample complexity escalates *exponentially* with the number of scalar subsystems, making it operationally unfeasible.

D. Nonlinear Homogeneous Network

To demonstrate that our data-driven results can be applied to nonlinear dynamics with homogeneity, we present here a *nonlinear homogeneous* network of degree one, consisting of $10,000$ subsystems (totaling $20,000$ dimensions), thereby illustrating the scalability of the proposed approach to high-dimensional nonlinear systems. We assume that the interconnection topology is known as a unidirectional ring to use the results of Section VI.

The evolution of each subsystem can be described by the following nonlinear homogeneous system:

$$\Sigma_i\colon x_i(k+1)=H_i\mathcal{F}_i(x_i(k))+0.1w_i(k),$$

where $x_i=[x_{1_i};x_{2_i}]$ and

$$H_i=\begin{bmatrix}-0.1 & 0.2\\ 0.15 & 0.12\end{bmatrix},$$
$$\mathcal{F}_i(x_i(k))=\begin{bmatrix}\sqrt{|x_{1_i}(k)x_{2_i}(k)|+\gamma(x_{1_i}^2(k)+x_{2_i}^2(k))}\\ x_{2_i}\end{bmatrix}, \tag{VII.6}$$

where $\gamma=0.1$, and w_i is defined according to (III.3) for a unidirectional ring interconnection topology, *i.e.,* $w_i=x_{i-1}$ with $x_0=x_M$.

It is clear that each subsystem Σ_i is homogeneous of degree one, *i.e.,* for any $\lambda>0$ and $x_i\in\mathbb{R}^{n_i}, w_i\in\mathbb{R}^{p_i}$, $f_i(\lambda x_i,\lambda w_i)=\lambda f_i(x_i,w_i)$:

$$\begin{aligned}
&f_i(\lambda x_i,\lambda w_i)\\
&\quad=H_i\begin{bmatrix}\sqrt{|\lambda x_{1_i}(k)\lambda x_{2_i}(k)|+\gamma(\lambda^2x_{1_i}^2(k)+\lambda^2x_{2_i}^2(k))}\\ \lambda x_{2_i}\end{bmatrix}\\
&\qquad+0.1(\begin{bmatrix}\lambda w_{1_i}(k)\\ \lambda w_{2_i}(k)\end{bmatrix})\\
&\quad=\lambda H_i\begin{bmatrix}\sqrt{|x_{1_i}(k)x_{2_i}(k)|+\gamma(x_{1_i}^2(k)+x_{2_i}^2(k))}\\ x_{2_i}\end{bmatrix}\\
&\qquad+0.1\lambda(\begin{bmatrix}w_{1_i}(k)\\ w_{2_i}(k)\end{bmatrix})\\
&\quad=\lambda f_i(x_i,w_i).
\end{aligned}$$

Now the dynamics of the interconnected network $\Sigma=\mathcal{I}(\Sigma_1,\dots,\Sigma_M)$ with $M=10000$ can be established as

$$\Sigma\colon x(k+1)=H\mathcal{F}(x(k))+0.1\mathcal{M}x(k),$$

where H is a matrix with block diagonal elements H_i as in (VII.6) and block off-diagonal zero matrices, while $\mathcal{M}$ is a unidirectional ring interconnection, with elements $m_{i+1,i}=\mathbb{I}_2, i\in\{1,\dots,M-1\}$, $m_{1,M}=\mathbb{I}_2$, and all other elements are $\mathbf{0}_{2\times2}$. In addition, $x(k)=[x_1(k);\dots;x_M(k)]$, and $\mathcal{F}(x(k))=[\mathcal{F}_1(x_1(k));\dots;\mathcal{F}_M(x_M(k))]$. Given that each subnetwork is homogeneous of degree one, it is clear that the interconnected network also remains homogeneous of degree one, *i.e.,* for any $\lambda>0$ and $x\in\mathbb{R}^n$, $f(\lambda x)=\lambda f(x)$.

Initially, we establish the structure of our storage functions as $\mathcal{S}_i(q_i,x_i)=q_{1_i}x_{1_i}^2+q_{2_i}x_{1_i}x_{2_i}+q_{3_i}x_{2_i}^2$ for all $i\in\{1,\dots,10000\}$. It is clear that $\mathcal{S}_i(q_i,x_i)$ is a homogeneous function of degree 2, *i.e.,* for any $\lambda>0$ and $x_i\in\mathbb{R}^{n_i}$, $\mathcal{S}_i(\lambda x_i)=\lambda^2\mathcal{S}_i(x_i)$. We gather 1296 samples from trajectories of each unknown subsystem (denoted as $\mathcal{N}_i=1296$) and normalize them for projection onto a unit sphere. Subsequently, we solve the SCP (VI.1) with $\mathcal{N}_i=1296$, computing coefficients of storage functions alongside other decision variables as follows, for any $i\in\{1,\dots,10000\}$:

$$\mathcal{S}_i^*(q_i,x_i)=1.6x_{1_i}^2+1.2x_{1_i}x_{2_i}+1.9x_{2_i}^2,\mu_{\mathcal{N}_i}^*=-0.777,$$
$$\mathcal{X}_i^{*11}=0.0001\mathbb{I}_2,\ \mathcal{X}_i^{*12}=\mathbf{0}_{2\times2},\ \mathcal{X}_i^{*21}=\mathcal{X}_i^{*12^\top},\ \mathcal{X}_i^{*22}=-0.001\mathbb{I}_2.$$

We compute $\varepsilon_i=0.0155$, $\mathscr{L}_i^1=2.3436$, and $\mathscr{L}_i^2=2.3533$ (with $\mathscr{L}_i=\max\{\mathscr{L}_i^1,\mathscr{L}_i^2\}=2.3533$). Since

$$\mu_{\mathcal{N}_i}^*+\mathscr{L}_i\varepsilon_i=-7405\times10^{-4}<0,\quad\forall i\in\{1,\dots,10000\},$$

according to Theorem 6.2, the constructed $\mathcal{S}_i$ obtained by solving SCP (VI.1) serve as storage functions for unknown subsystems Σ_i globally over $\mathbb{R}^{n_i}$, with guarantees of correctness. We also ensure that the dissipativity compositionality condition in (III.5) is satisfied. Then one can verify that unknown interconnected dt-NS $\Sigma=\mathcal{I}(\Sigma_1,\dots,\Sigma_{10000})$ is GAS with respect to $x=0$, and

$$\mathcal{V}(q,x)=\sum_{i=1}^{10000}\mathcal{S}_i(q_i,x_i)=\sum_{i=1}^{10000}1.6x_{1_i}^2+1.2x_{1_i}x_{2_i}+1.9x_{2_i}^2$$

is a Lyapunov function for the interconnected dt-NS with a correctness guarantee. The computation of storage functions took 2.14 seconds for each subsystem (21400 seconds for all 10000 subsystems in a serial computation) on a MacBook Pro (Apple M2Max with 32 GB RAM Memory).

VIII. Conclusion

In this work, we proposed a data-driven *divide and conquer* approach for verifying the GAS property of unknown interconnected networks with an unknown topology, while providing concreteness guarantees. The proposed framework utilized dissipativity-type properties of subsystems, which are characterized by the notion of storage functions. In our data-driven setting, we collected data from trajectories of each unknown subsystem and formulated a scenario convex program (SCP) to enforce the required conditions of storage functions. By solving the SCP using the collected data, we built a Lyapunov function for the interconnected system based on storage functions of individual subsystems, derived from data, by offering newly developed data-driven compositionality conditions in the dissipativity setting. To demonstrate the effectiveness of our data-driven approaches, we applied them to three unknown interconnected networks. As part of our future work, we aim to develop a data-driven approach for constructing *control* Lyapunov functions for unknown interconnected systems. In addition, generalizing our data-driven approach to a broader class of nonlinear systems beyond homogeneous is being investigated as future work.

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